A SKETCH OF AN AMALGAM OF PROGRAMS

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Contents

1. Historical Background and Motivations

1.1. L-Functions and ζ-functions.

In [Bombieri], it is stated,

*"The Riemann hypothesis has become a central problem of pure mathematics, and not just because of its fundamental consequences for the law of distribution of prime numbers. One reason is that **the Riemann zeta function is not an isolated object, rather is the prototype of a general class of functions, called L-functions,** associated with algebraic (automorphic representations) or arithmetical objects (arithmetic varieties); we shall refer to them as **global L-functions**. They are Dirichlet series with a suitable Euler product, and are expected to satisfy an appropriate functional equation and a Riemann hypothesis. **The factors of the Euler product may also be considered as some kind of zeta functions of a local nature,** which also should satisfy an appropriate Riemann hypothesis (the so-called Ramanujan property). **The most important properties of the algebraic or arithmetical objects underlying an L-function***

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can or should be described in terms of the location of its zeros and poles, and values at special points."

They go on to say,

"Not a single example of validity or failure of a Riemann hypothesis for an L-function is known up to this date. The Riemann hypothesis for $\zeta(s)$ does not seem to be any easier than for Dirichlet L-functions (except possibly for non-trivial real zeros), leading to the view that its solution may require attacking much more general problems, by means of entirely new ideas."

Definition 1.1. Dirichlet L-function

Definition 1.2. Artin L-function

Definition 1.3. automorphic L-function

1.2. **Spectral Approaches.**

Definition 1.4. trace class operator

Definition 1.5. Fredholm determinant

2. Background on Representation Theory

2.1. **The Left Regular Representation on $L^2(G)$.** Main references [Ge], [H3], [BCGSKK]

In this section we let $G = \mathbf{SL}_2(\mathbb{R})$, and let the maximal compact subgroup $K = \mathbf{SO}_2(\mathbb{R})$. Complexify the Lie algebras to obtain the $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, and the Cartan subalgebra $\mathfrak{h}$, respectively. Let $L^2(G)$ be all square (Lebesgue) integrable functions on G, with respect to the Haar measure.

2.2. **Harish-Chandra Modules.** A complex Lie group G, having complex Lie algebra $\mathfrak{g}$, is treated as a real Lie group, then the corresponding Lie algebra is complexified, and turns out to be $\mathfrak{g}\times\mathfrak{g}$. The envoloping algebra is then $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ with center $Z(\mathfrak{g}) \otimes Z(\mathfrak{g})$ ($Z(\mathfrak{g})$ being the center of $U(\mathfrak{g})$). *Central* characters are then put into correspondence with pairs of central characters of $Z(\mathfrak{g})$ and are of the form

$$(\chi_1, \chi_2),$$

for $\chi \in X(Z)$ a character of $Z(\mathfrak{g})$, $\tau : U \to U$ an *anti*-involution of $\mathfrak{g}$ which is extended to an anti-automorphism of $U(\mathfrak{g})$ leaving $Z(\mathfrak{g})$ pointwise fixes. We then realize $\mathfrak{h}$ as the diagonally embedded

$$\mathfrak{g} \to \mathfrak{g}\times\mathfrak{g}\,.$$

Definition 2.1. admissible representation

Definition 2.2. Harish-Chandra pair

Definition 2.3. Harish-Chandra module

2.3. **Surface Orders.** In general everything in this paper will work for R a complete discrete valuation ring, or a Dedekind domain. We will need R to be a commutative ring from the following list,

(1) $\mathbf{A}_{\mathbb{Q}}$ or $\mathbf{A}_K$, the adeles of a number field;
(2) K or $\mathbb{Q}$, the global number fields;
(3) $\mathbb{C}$ or $\mathbb{R}$, the infinite places;
(4) $\mathbb{Z}$ or O_K, the ring of integers;
(5) $\mathbb{Z}_p$ or $O_{\mathfrak{p}}$, the completion at a prime to a local ring;
(6) $\mathbb{Q}_p$ or $K_v = K_{\mathfrak{p}}$ the corresponding local fields for finite places;

Definition 2.4. An R-**lattice** is a finitely generated projective module over R. In particular, if R is a Dedekind domain, every R-lattice is finitely generated and torsion free.

Definition 2.5. An R-**Order** Ω in a K-algebra Λ is a unital subring of Λ such that

(1) $\Omega \otimes_R K = \Lambda$, and
(2) Ω is finitely generated as an R-module.

The σ-**order** Ω, associated to a maximal length n-cycle $\sigma \in S_n$ in the symmetric group, will be given by a matrix R-subalgebra of $\mathbf{Mat}_{n\times n}(R)$.

$$\Omega = \begin{pmatrix} R & \mathfrak{m} & \mathfrak{m} & \cdots & \mathfrak{m} & \mathfrak{m} \\ R & R & \mathfrak{m} & \cdots & \mathfrak{m} & \mathfrak{m} \\ R & R & R & \cdots & \mathfrak{m} & \mathfrak{m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R & R & R & \cdots & R & \mathfrak{m} \\ R & R & R & \cdots & R & R \end{pmatrix}_n \subset \mathbf{Mat}_{n\times n}(R) \subset \mathbf{Mat}_{n\times n}(K)$$

In this case, $\Omega \otimes_R K$ is the K-algebra of lower triangular matrices over K. Important examples will be orders such as $R = \mathscr{O}_K$ the ring of integers of the (global) number field $K/\mathbb{Q}$, and $R = \mathscr{O}_\mathfrak{p} \subset K_\mathfrak{p}$ for a local field at a prime ideal for the completion of the ring of integers. The "base" example is of course $K = \mathbb{Q}$, $R = \mathbb{Z}$ and $K = \mathbb{Q}_p$ and $R = \mathbb{Z}_p$.

3. The Gel'fand-Ponomarev Algebra and Representations of $\mathfrak{sl}_2$

In the highly influential paper [GP], Gel'fand and Ponomarev investigated the indecomposable representations of the Lorentz group, which is equivalent to classifying the "*Harish-Chandra modules*" of the Lie algebra $\mathfrak{sl}_2$. They used methods now standard in the representation theory of so-called *string algebras*. Since $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of both the Lorentz group and $\mathbf{SL}_2(\mathbb{C})$, they study representations of $\mathfrak{sl}_2(\mathbb{C})$, as well as the Lie algebra $\mathfrak{su}_2$ of the maximal compact subgroup $\mathbf{SU}_2 \subseteq \mathbf{SL}_2(\mathbb{C})$. Note, $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_2 \otimes \mathbb{C}$. In [FH] one sees the following Diagram 3 for $\mathfrak{sl}_2(\mathbb{C})$[1]

$$\begin{tikzcd}
\cdots \arrow[r, shift right, "H_+"'] & V_{\alpha-4} \arrow[l, shift right, "H_-"'] \arrow[loop above, "H_3"] \arrow[r, shift right, "H_+"'] & V_{\alpha-2} \arrow[l, shift right, "H_-"'] \arrow[loop above, "H_3"] \arrow[r, shift right, "H_+"'] & V_{\alpha} \arrow[l, shift right, "H_-"'] \arrow[loop above, "H"] \arrow[r, shift right, "H_+"'] & V_{\alpha+2} \arrow[l, shift right, "H_-"'] \arrow[loop above, "H_3"] \arrow[r, shift right, "H_+"'] & \cdots \arrow[l, shift right, "H_-"']
\end{tikzcd}$$

with the typical basis,

$$H_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In their study of Harish-Chandra modules[2], they asked the following question:

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[1] This is more in line with the notation of [GP]. In [FH] $H_+ = x$, $H_- = Y$, $H_3 = H$.

[2] It seems very likely under the construction given here that the Bruhat-Tits Building for $\mathbf{SL}_2$, described by Serre in [**?**], in II §1.1, is closely related to the universal cover of the surface algebras being a directed four regular tree, The Gel'fand-Ponomarev algebra being related to an amalgam of two free (noncommutative) associative algebras on two generators, and being closely related to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ seems to further suggest a connection. It seems like the work of Matilde Marcolli suggests how one might understand this connection.

Let K, be a field and let P_1 and P_2 be finite dimensional K-vector spaces.[3] Suppose we have nilpotent operators

$$H_+ : P_1 \to P_2, \quad H_- : P_2 \to P_1, \quad H_3 : P_2 \to P_2$$

such that $YZ = 0 = ZX$. Fix a basis of P_1 and P_2 and classify all canonical forms of $H_\pm$ and H_3.

If we take instead, as suggested in [GP],

$$P = P_1 \oplus P_2, \quad X = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & H_3 \\ 0 & 0 \end{pmatrix}$$

then this is equivalent to classifying all representations of the completion of the following path algebra: $I = \langle xy, yx \rangle$, $\Lambda = KQ/I$,

$$\begin{tikzcd} \bullet \arrow[loop left, "x"] \arrow[loop right, "y"] \end{tikzcd}$$

The completion of this path algebra is $k\langle\langle x, y\rangle\rangle/\langle xy, yx\rangle \cong K[[x, y]]/(xy)$. One should note, this is a **"gluing"** of sorts, identifying the vector spaces P_1 and P_2 by something that looks like an extension of a matrix algebra given by H_3.

4. The Quadratic Extension $\mathbb{Q}(i)/\,\mathbb{Q}$

Let $\mathbb{Q}(i)/\,\mathbb{Q}$ be the quadratic extension complexifying $\mathbb{Q}$, given by the polynomial $p(x) = x^2 + 1 = 0$ over $\mathbb{Q}$. This corresponds to the covering of the trivial dessin d'enfant,

$$\begin{tikzcd} \bullet_{-i} \arrow[r, dash] & \circ_0 \arrow[r, dash] \arrow[d, squiggly] & \bullet_i \\ & \circ_0 \arrow[r, dash] & \bullet_1 \end{tikzcd}$$

This implies we should study the corresponding surface algebra KQ/I, via representations of the quiver Q of the following form,

$$\begin{tikzcd} P_1 \arrow[loop left, "H_3^\vee"] \arrow[r, bend left, "H_+"] & P_2 \arrow[l, bend left, "H_-"] \arrow[loop right, "H_3"] \end{tikzcd}$$

The ideal is

$$I = \langle H_+ H_3^\vee, H_3 H_+, H_- H_3, H_3^\vee H_- \rangle.$$

The surface orders[4] are of the following form

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[3] It seems likely that the notation P_1 and P_2 in [GP] suggests thinking of these vector spaces as projective modules over $K[x, y]$.

[4] Recall, these are orders with respect to the completions Λ_K of the surface algebra $A = KQ/I$, over the field K.

$$\begin{tikzcd}
R_- \arrow[r, dash, "\alpha_-"] & R_1 \arrow[dr, "\sigma_1"] & \mathfrak{m} & \\
& R_1 & R_1 \arrow[r, dash, "\alpha_+"] & R_+
\end{tikzcd}$$

The (noncommutative) normalization of the surface order is,

$$R_- \times \begin{pmatrix} R_1 & \mathfrak{m}_1 \\ R_1 & R_1 \end{pmatrix} \times R_+$$

and corresponds to the quiver with no relations giving a *"hereditary"* path algebra,

$$\begin{tikzcd}
\bullet_{2^-} \arrow[loop left, "H_3^\vee"] & & \bullet_{2^+} \arrow[r, bend left, "H_+"] & \bullet_{1^+} \arrow[l, bend left, "H_-"] & & \bullet_{1_-} \arrow[loop right, "H_3"]
\end{tikzcd}$$

For the middle 2×2 R_1-order, we can think of this in the following way. Take the 3×3 R_1-order,

$$\tilde{\Omega} = \begin{pmatrix} R_1 & \mathfrak{m}_1 & \mathfrak{m}_1 \\ R_1 & R_1 & \mathfrak{m}_1 \\ R_1 & R_1 & R_1 \end{pmatrix}$$

Now, quotient out by the condition $\tilde{\Omega}_{1,1} = \tilde{\Omega}_{3,3}$, modulo the maximal ideal $\mathfrak{m}_1$. This identifies the radicals of the two projective modules

$$\tilde{P}_1 = \begin{pmatrix} R_1 \\ R_1 \\ R_1 \end{pmatrix} \quad \text{and} \quad \tilde{P}_3 = \begin{pmatrix} \mathfrak{m}_1 \\ \mathfrak{m}_1 \\ R_1 \end{pmatrix}.$$

Here

$$\mathbf{rad}(P_1) = \begin{pmatrix} \mathfrak{m}_1 \\ R_1 \\ R_1 \end{pmatrix}, \qquad \mathbf{rad}(P_3) = \begin{pmatrix} \mathfrak{m}_1 \\ \mathfrak{m}_1 \\ \mathfrak{m}_1 \end{pmatrix}$$

One should be thinking of this as a gluing of the lower triangular matrix algebra

$$\begin{pmatrix} R_1/\,\mathfrak{m}_1\,/ & \mathfrak{m}_1\,/\,\mathfrak{m}_1 & \mathfrak{m}_1\,/\,\mathfrak{m}_1 \\ R_1/\,\mathfrak{m}_1 & R_1/\,\mathfrak{m}_1 & \mathfrak{m}_1\,/\,\mathfrak{m}_1 \\ R_1/\,\mathfrak{m}_1 & R_1/\,\mathfrak{m}_1 & R_1/\,\mathfrak{m}_1 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix} \cong kQ$$

isomorphic the the path algebra of the quiver $1 \to 2 \to 3$, of type $\mathbf{A}_3$ over the residue field $R_1/\,\mathfrak{m}_1 = k$. When we identify the radicals by setting the entry $a_{1,1} \in k$ equal to the entry $a_{3,3} \in k$ in the matrix algebra kQ, we are reduced to the case of the 2×2 matrix ring

$$\Omega_\pm = \begin{pmatrix} R_1 & \mathfrak{m}_1 \\ R_1 & R_1 \end{pmatrix}$$

for the glued 2-cycle quiver,

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "H_+"] & \bullet \arrow[l, bend left, "H_-"]
\end{tikzcd}$$

Similarly, the 2×2 matrix rings,

$$\tilde{\Omega}_- = \begin{pmatrix} R_- & \mathfrak{m}_- \\ R_- & R_- \end{pmatrix} \mapsto R_- \quad \text{and} \quad \tilde{\Omega}_+ = \begin{pmatrix} R_+ & \mathfrak{m}_+ \\ R_+ & R_+ \end{pmatrix} \mapsto R_+$$

each corresponding to a gluing of type $\mathbf{A}_2$ quivers

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet & \bullet \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[loop left, "H_3^\vee"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[loop right, "H_3"] \end{tikzcd}$$

after gluing[5] reduce to $\Omega_- = R_-$ and $\Omega_+ = R_+$, respectively.

Note, for the surface order Ω_1, we have an infinite chain of embedding maps of projective modules which is 2-periodic,

$$P_1 \xrightarrow{\mathfrak{m}_1} P_2 \xrightarrow{\mathfrak{m}_1} P_1 \xrightarrow{\mathfrak{m}_1} P_2 \xrightarrow{\mathfrak{m}_1} P_1 \xrightarrow{\mathfrak{m}_1} P_2 \xrightarrow{\mathfrak{m}_1} \cdots$$

This can be identified with the infinite chain of free (and therefore projective) modules over R_1,

$$R_1 \to \mathfrak{m}_1 \to \mathfrak{m}_1^2 \to \mathfrak{m}_1^3 \to \mathfrak{m}_1^4 \to \cdots$$

which is the *"top"* of the image under multiplication by $\mathfrak{m}_1$.[6] This is a 2-periodic R_1 lattice. For Ω_- and Ω_+ the chain of projectives is equal to the chain of R_- and R_+ modules, i.e.

$$R_- \to \mathfrak{m}_- \to \mathfrak{m}_-^2 \to \mathfrak{m}_-^3 \to \cdots$$

and similarly for R_+. Now, the pullback defining the surface order Ω which glues the projectives

$$R_- \leftrightarrow P_1 \qquad P_2 \leftrightarrow R_+$$

and we obtain the new projective modules over Ω as

$$P_{-,1} = \begin{pmatrix} & R_{-,1} & \\ R_{-,1} & & R_{1,+} \\ R_{-,1} & & R_{-,1} \\ R_{-,1} & & R_{1,+} \\ R_{-,1} & & R_{-,1} \\ \vdots & & \vdots \end{pmatrix}, \qquad P_{2,+} = \begin{pmatrix} & R_{1,+} & \\ R_{-,1} & & R_{1,+} \\ R_{1,+} & & R_{1,+} \\ R_{-,1} & & R_{1,+} \\ R_{1,+} & & R_{1,+} \\ \vdots & & \vdots \end{pmatrix}$$

where $R_{-,1}$ is given by the pullback,

$$\begin{tikzcd} R_{-,1} \arrow[r] \arrow[d] & R_- \arrow[d] \\ R_1 \arrow[r] & R_1/\mathfrak{m}_1 \cong_\lambda R_-/\mathfrak{m}_- \end{tikzcd}$$

and $R_{1,+}$ is given by

$$\begin{tikzcd} R_{1,+} \arrow[r] \arrow[d] & R_+ \arrow[d] \\ R_1 \arrow[r] & R_1/\mathfrak{m}_1 \cong_\mu R_+/\mathfrak{m}_+ \end{tikzcd}$$

This can be visualized in terms of quivers as follows,

[5]This is the gluing of the lower triangular matrix algebras of type $\mathbf{A}_2$, over the residue field k, where will will assume for the time being for simplicity, that $R_1/\mathfrak{m}_1 = R_-/\mathfrak{m}_- = R_+/\mathfrak{m}_+ = k$ all have the same residue field. However, this assumption is not necessary.

[6]Recall the **"top"** of a module M is define to be $M/\,\mathbf{rad}(M)$. For the indecomposable projectives P_1, P_2 for Ω_1, this is just a copy of $R_1 = \mathbf{top}(P_i)$ for $i = 1, 2$.

$$
\begin{tikzcd}
& P_{-,1} \arrow[dl] \arrow[dr] & & & & P_{2,+} \arrow[dl] \arrow[dr] & \\
P_{-,1} \arrow[d] & & P_{2,+} \arrow[d] & & P_{-,1} \arrow[d] & & P_{2,+} \arrow[d] \\
P_{-,1} \arrow[d] & & P_{-,1} \arrow[d] & & P_{2,+} \arrow[d] & & P_{2,+} \arrow[d] \\
P_{-,1} & & P_{2,+} & & P_{-,1} & & P_{2,+} \\
\vdots & & \vdots & & \vdots & & \vdots
\end{tikzcd}
$$

Where we glue the indecomposable projective lattices P_- and P_1, and the two projective lattices P_2 and P_+.

Here we are assuming $R_1/\mathfrak{m}_1 \cong_\lambda R_-/\mathfrak{m}_- = k$, is given by some automorphism of the residue field,

$$\lambda \in k^\times = \mathbf{GL}_1(k),$$

and similarly that $R_1/\mathfrak{m}_1 \cong_\mu R_+/\mathfrak{m}_+ = k$, is given by some automorphism of the residue field,

$$\mu \in k^\times = \mathbf{GL}_1(k),$$

Now, this pair of automorphisms will be in

$$(\lambda, \mu) \in k^\times \times k^\times = \mathbf{GL}_1(k) \times \mathbf{GL}_1(k)$$

which is isomorphic to the torus T in $\mathbf{SL}_2(k)$ given by the diagonal matrices in $\mathbf{SL}_2(k)$.

Now, the reason for doing all of this is of course to realize this as a kind of *"folded version"* of the Diagram 3 given in [FH], but we need a kind of *"splitting"* of the operator H_3, in the basis

$$H_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $\mathfrak{sl}_2(\mathbb{R})$ (or $\mathfrak{sl}_2(\mathbb{C})$). We want to understand the new pair of operators $H_3, H_3^\vee$ for the new path algebra and surface order. Let us recall the construction given by Humphreys in [H3] for Harish-Chandra modules.

The general idea is as follows. If G is a semisimple complex Lie group, K a maximal compact subgroup. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{k}$ be the Lie algebra of K. View G as a *real* Lie group, then its complexified Lie algebra is $\mathfrak{g} \times \mathfrak{g}$. The enveloping algebra is $U \otimes U$ ($U = U(\mathfrak{g})$). The center of $U \otimes U$ is then $Z \otimes Z$ ($Z = Z(\mathfrak{g})$). Central characters are then of the form

$$\chi = (\chi_1, \chi_2)$$

with χ_1, χ_2 being characters of Z. The anti-involution τ of $\mathfrak{g}$ extends to an anti-automorphism of U leaving Z pointwise fized. Then $\mathfrak{k}$ can be realized as a copy of $\mathfrak{g}$ embedded diagonally into $\mathfrak{g} \times \mathfrak{g}$ via $x \mapsto (x, -\tau(x))$. Then a **Harish-Chandra module** for $\mathfrak{g} \times \mathfrak{g}$ is a finitely generated module such that the restriction of a module to $\mathfrak{k}$ decomposes into a direct sum of finite dimensional simple module.

For the current context, let us take the Lie group $\mathbf{SL}_2(\mathbb{C})$, viewed as a real Lie group, we get the complexified Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$. Now, a maximal compact subgroup K of $\mathbf{SL}_2(\mathbb{C})$ could be taken to be $\mathbf{SU}_2(\mathbb{R})$. Its Lie algebra complexifies to $\mathfrak{su}_2 \otimes \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$. We will call

$(\mathfrak{g}, K)$ a **Harish-Chandra pair**, and a module for such a pair is a Harish-Chandra module over $\mathfrak{g} \times \mathfrak{g}$, such that restructed to $\mathfrak{k}$ it is of the form

$$V = \bigoplus_{\chi} V_{\chi}$$

where each V_{χ} is a finite dimensional K module. Observe the following diagrams,

$$\begin{tikzcd} P_1 \arrow[r, bend left, "H_+"] \arrow[d, dotted, no head] & P_2 \arrow[l, bend left, "H_-"] \arrow[loop right, "H_3"] \arrow[d, dotted, no head] & & P \arrow[loop left, "X^{-1}"] \arrow[loop right, "Y"] \\ P_1^{\vee} \arrow[r, bend left, "H_+^{\vee}"] \arrow[loop left, "H_3^{\vee}"] & P_2^{\vee} \arrow[l, bend left, "H_-^{\vee}"] & & P^{\vee} \arrow[loop left, "X"] \arrow[loop right, "Y^{-1}"] \end{tikzcd}$$

where $P = P_1 \oplus P_2$. Now, as described by Gel'fand and Ponomarev, one may classify all Harish-Chandra modules via so-called *"string and band modules"* in modern terminology.[7] Now, suppose we take the $\mathfrak{su}_2 \otimes_{\mathbb{Z}} K$ module

$$K[s,t] \cong \mathbf{Sym}(K^2) = \bigoplus_{n \geq 0} \mathbf{Sym}^n(K^2)$$

Here, $\mathbf{Sym}^n(K^2)$ has basis $\{s^p t^q : \; p + q = n$. For example, $\mathbf{Sym}^3(K^2)$ has basis,

$$\{s^3, s^2t, st^2, t^3\}$$

It is well known that $\mathbf{Sym}^n(K^2)$ comprise all of the invariant subspaces of the module $\mathbf{Sym}(K^2)$ with respect to the $\mathbf{GL}_2(K)$ action given by $g \cdot f((s,t)) = f(g^{-1}(s,t))$. Restricting the action of $\mathbf{GL}_2(\mathbb{C})$ to the actions of $\mathbf{SL}_2(\mathbb{C})$ and $\mathbf{SU}_2(\mathbb{R})$, these subspaces remain invariant, and we may define a Harish-Chandra pair

$$(\mathfrak{g}, K) = (\mathfrak{sl}_2(\mathbb{C}), \mathbf{SU}_2(\mathbb{R}))$$

with the infinite dimensional $(\mathfrak{g}, K)$-bimodule $V = \mathbf{Sym}(\mathbb{C}^2)$ having a decomposition given by

$$V = \bigoplus_{\chi} V_{\chi}$$

where each $V_{\chi} \cong \mathbf{Sym}^n(\mathbb{C}^2)$ and as an $\mathbf{SU}_2(\mathbb{R})$-module, V_{χ} becomes a finite direct sum of simple $\mathbf{SU}_2(\mathbb{R})$ modules of the form

$$V_{\chi} \cong \bigoplus_{a+b=n} \langle x^a y^b \rangle$$

where $\langle x^a y^b \rangle$ can be identified with the free module $\mathbb{C}[x,y](-a,-b)$ given by the bi-graded shifts of the polynomial ring. Or, equivalently

$$\langle x^a y^b \rangle = \mathbf{Sym}^{(a,b)}(\mathbb{C}^2)$$

is the bi-graded components of $\mathbf{Sym}^n(\mathbb{C}^2) = \bigoplus_{a+b=n} \mathbf{Sym}^{(a,b)}(\mathbb{C}^2)$.

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[7] See for example [CB4] or [**?**].

5. Characters, Weight Vectors, Root Systems, and Semi-invariants for Quivers

6. Quadratic Extensions and Combinatoral Quadratic Tensor Flows on Moduli Spaces

6.1. **Quadratic Tensors.** In this section, let us set $V \cong K \cong W$, to be one dimensional K-vector spaces. We will for simplicity assume we are working over $K = \mathbb{Z}$ or K an algebraically closed field of arbitrary characteristic unless otherwise specified. Now, let us recall some basics on quadratic tensors and set out notation.

For V (and similarly for W), we have

$$\mathbf{Sym}(V) = \bigoplus_{n=0}^{\infty} \mathbf{Sym}^n(V) \cong K[x] = \bigoplus_{n=0}^{\infty} K[x]_n$$

where $x \in K^{\times}$ is a basis vector. Here $\mathbf{Sym}^n(V)$ and $K[x]_n$ are the homogeneous graded components. Next, we have

$$\mathbf{Sym}(K^2) \cong \mathbf{Sym}(V \oplus W) = \bigoplus_{\substack{a+b=n \\ n \geq 0}} \mathbf{Sym}^{(a,b)}(V \oplus W).$$

Here $\mathbf{Sym}^n(V \oplus W) = \bigoplus_{a+b=n} \mathbf{Sym}^{(a,b)}(V \oplus W)$ has a multigraded basis

$$\{v^a \cdot w^b\}_{a+b=n}$$

where the symmetric tensor $v^p \cdot w^q$ can be identified with $x^p y^q \in K[x, y]$. Now, denote by $V^{\vee} = \mathbf{Hom}_K(V, K)$ and $W^{\vee} = \mathbf{Hom}_K(W, K)$ the dual spaces. Define four isomorphic vector spaces via the following diagram

$$\begin{tikzcd}
V^{\vee} \otimes W \arrow[r, "(\langle\cdot,v\rangle \mapsto v)\otimes \mathbf{id}_W"] & V \otimes W \arrow[d, "\mathbf{id}_V \otimes (w \mapsto \langle\cdot,w\rangle)"] \\
V^{\vee} \otimes W^{\vee} \arrow[u, "\mathbf{id}_{V^{\vee}} \otimes (\langle\cdot,w\rangle \mapsto w)"] & V \otimes W^{\vee} \arrow[l, "(\langle\cdot,v\rangle \mapsto v)\otimes \mathbf{id}_{W^{\vee}}"]
\end{tikzcd}$$

If the in context in which we are working, a choice of bilinear forms

$$\langle \cdot, \cdot \rangle : V \otimes V^{\vee} \to K, \quad \text{and } \langle \cdot, \cdot \rangle : W \otimes W^{\vee} \to K$$

is unimportant, we will simply assume to take the maps above with the maps being natural with respect to the standard basis (i.e. $v^{\vee}(v) = 1 = w^{vee}(w)$). Now, identify the symmetric tensors

$$v \cdot w \leftrightarrow v \otimes w + w \otimes v \in T(V \oplus W)$$

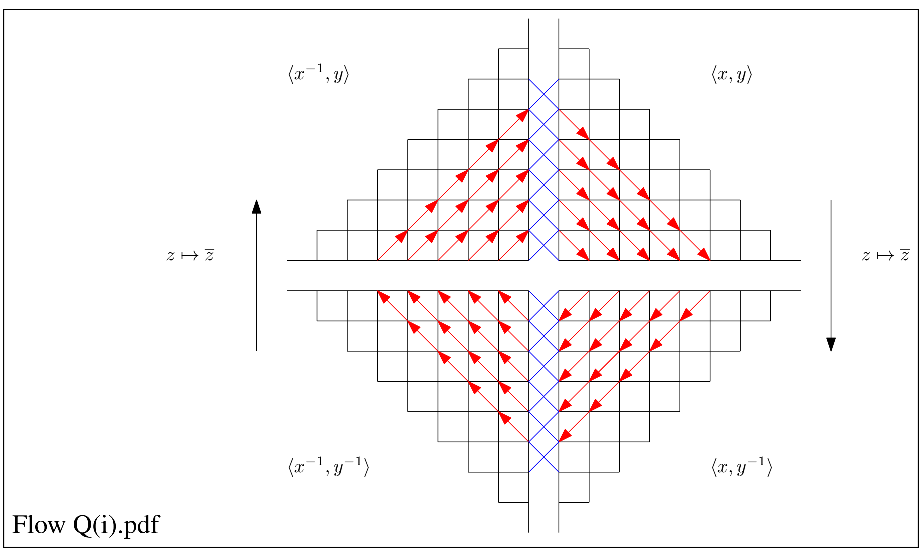
where $T(V \oplus W)$ is the tensor algebra. This induces similar identifications on the duals, and we want to study the following four symmetric algebras, (all being isomorphic)

$$\mathbf{Sym}(V \oplus W), \quad \mathbf{Sym}(V \oplus W^{\vee}), \quad \mathbf{Sym}(V^{\vee} \oplus W), \quad \mathbf{Sym}(V^{\vee} \oplus W^{\vee}.$$

We will identify the integral points with respect to the standard basis of K^2 of each of the above symmetric algebras with a copy of the lattice $\mathbb{Z}^2$, allowing bigraded shifts of free modules as is typical for polynomial rings in commutative algebra. Now, Let $\mathfrak{sl}_2(V \oplus W)$ act on a copy of affine 2-space $\mathbf{A}^2_{\mathbb{Z}} \subset \mathbf{A}^2_K$, after a choice of base point for 0, as follows,

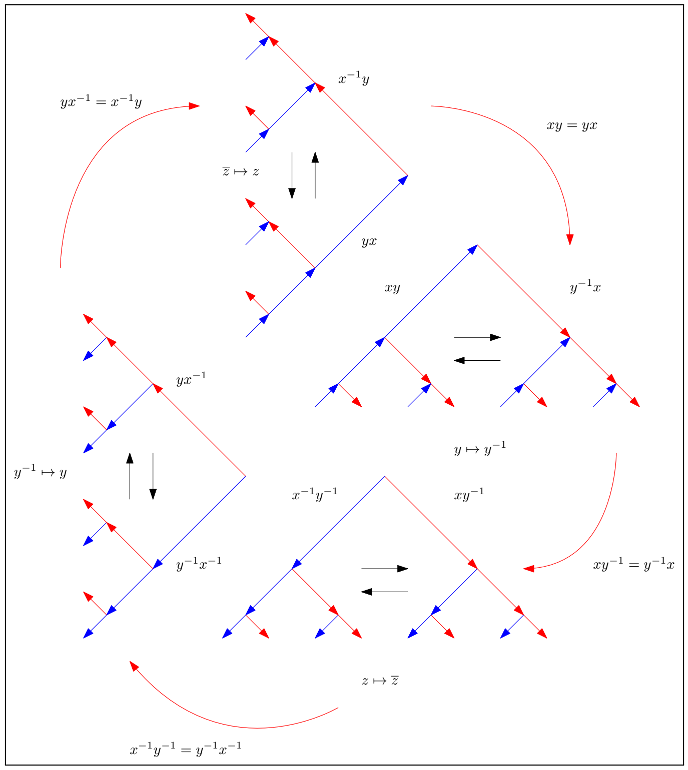
$$\begin{tikzcd}
(s^3) \arrow[dr, "H_-"] & & & \\
(s^2) \arrow[dr, "H_-"] & (s^2t) \arrow[dr, "H_-"] & & \\
(s) \arrow[dr, "H_-"] & (st) \arrow[dr, "H_-"] & (st^2) \arrow[dr, "H_-"] & \\
(1) & (t) & (t^2) & (t^3)
\end{tikzcd}
\qquad
\begin{tikzcd}
(s^3) & & & \\
(s^2) & (s^2t) \arrow[ul, "H_+"'] & & \\
(s) & (st) \arrow[ul, "H_+"'] & (st^2) \arrow[ul, "H_+"'] & \\
(1) & (t) \arrow[ul, "H_+"'] & (t^2) \arrow[ul, "H_+"'] & (t^3) \arrow[ul, "H_+"']
\end{tikzcd}$$

Now, continuing with this reasoning, let us call the following diagram for $\mathbb{Q}(i)/\,\mathbb{Q}$ its **combinatorial flow** diagram.

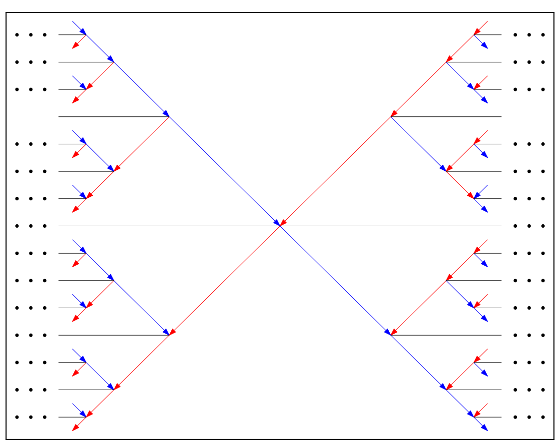


Now, observe the following diagram of four copies of Cantor space

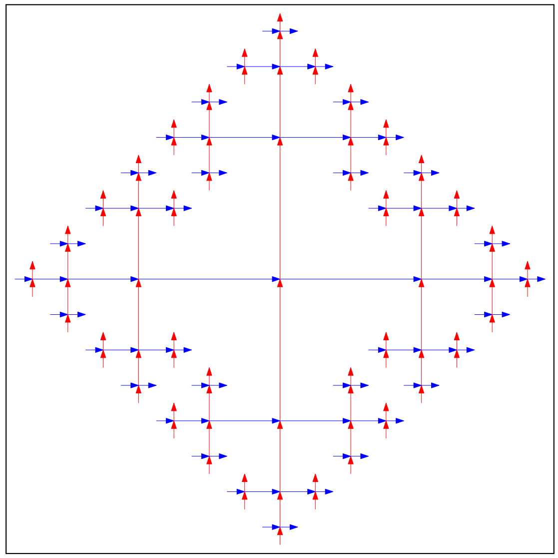
$$\{x, y^{-1}\}^{\omega} \amalg \{y^{-1}, x^{-1}\}^{\omega} \amalg \{x^{-1}, y\}^{\omega} \amalg \{x, y\}$$



Each copy $\{x^\epsilon, y^\delta\}^\omega$, with $\epsilon, \delta \in \{\pm 1\}$, is the space of infinite sequences in two of the *”formal* letters x, y, x^{-1}, y^{-1}. We consider these as free semigroups on two generators. If we quotient this space by the relations $xx^{-1} = 1 = x^{-1}x$ and $yy^{-1} = 1 = y^{-1}y$, we get to copies of the following four regular tree,



Or, oriented a slightly different way,



7. More General Cyclotomic Extensions, Artin L-Functions, and Semisimple Representations of Surface Algebras

7.1. **Notation and Conventions.** Throughout this section, we will say that a bounded linear operator $U \in \mathcal{U}(\mathbb{H}) \subset \mathcal{L}(\mathbb{H})$, on the Hilbert space $\mathbb{H}$ is **unitary** if $UU^* = U^*U = \mathbf{id}_{\mathbb{H}}$. A **unitary representation** of a locally compact topological group

$$\rho : G \to \mathcal{U}(\mathbb{H}),$$

is a *continuous* group homomorphism for whatever topology we have on G, and the *"strong operator topology"* on unitary operators $\mathcal{U}(\mathbb{H})$, so that $g \to h$ in G implies $\rho(g) \to \rho(h)$ strongly.

7.2. **Cyclotomic Extensions and Global Surface Orders.** Given a cyclotomic (Galois) extension $\mathbb{Q}(\zeta)/\,\mathbb{Q}$, such that $\zeta = e^{2\pi i/n}$ is a primitive n^{th} root of unity, we have the splitting polynomial of the form

$$P_\zeta(x) = (x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{n-1})(x-1) = \prod_{j=1}^{n}(x-\zeta^j).$$

This corresponds to the determinant

$$\det(1 = t_\zeta) = \det\begin{pmatrix} 1-\zeta & & & \\ & 1-\zeta^2 & & \\ & & \ddots & \\ & & & 1-\zeta^n \end{pmatrix} \in \mathbb{T}^n = (K^\times)^n$$

where $\mathbb{T}^n$ is the torus in $\mathbf{GL}_n(K)$ and $K = \mathbb{Q}(\zeta)$ is the extension of $\mathbb{Q}$. Even better, $t_\zeta \in \mathbb{T}^n_0$ has determinant $|\det(t_\zeta)| = 1$, as it is a product of roots of $1 \in \mathbb{C}$. In particular, $\rho : \langle\zeta\rangle \to \mathbf{GL}_n(K) \subset \mathbf{GL}_n(\mathbb{C})$ is a unitary representation on the (complex multiplicative) cyclic group

$$\langle\zeta\rangle = \langle e^{2\pi i/n}\rangle,$$

and $\rho(\zeta)$ is a *trace class operator*. In the finite dimensional case $\det(1 - t_\zeta)$ always exists, and the coefficients of the splitting polynomial

$$P_\zeta(x) = a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

are given by

$$a_r = \mathbf{Tr}\left(\bigwedge^r (1 - t_\zeta)\right)$$

where the entries of $\bigwedge^r(1 - t_\zeta)$ are given by the $r \times r$ minors of $(1 - t_\zeta)$. Namely, take the polynomial,

$$\sum_I x_{i(1),i(2),\ldots,i(r)}$$

where $\{1 \leq i(1) < i(2) < \cdots < i(r) \leq n\}$ and there are $\binom{n}{r}$ monomials in the sum. Then evalutaion of $x_{i(1),i(1)} x_{i(2),i(2)} \cdots x_{i(r)i(r)} = x_{i(1),i(2),\ldots,i(r)}$ on the diagonal entries of $1 - t_\zeta$ gives the corresponding indexed entry of $\bigwedge^r(1 - t_\zeta)$.

Recall, a **Dirichlet L-function**

$$L_\chi = L(\chi, s) = \sum_{n-1}^{\infty} \frac{\chi(n)}{n^s}$$

is a complex valued function, which can be extended via analytic continuation to a meromorphic function $L_\chi \in \mathcal{M}(\mathbb{C})$, on the whole complex plane. In particular, L_χ has finitely many poles $\{x_1, ..., x_r\} \subset \mathbb{C}$, and thus is locally holomorphic on any open subset of the punctured Riemann $\mathbb{P}_\mathbb{C} - \{x_1, ..., x_r\}$. Moreover, it can be uniformly approximated by a uniformly convergent sequence of polynomial functions $f_n \to L_\chi$ on any open set U away from the punctures; it can also be approximated by a uniformly convergent sequence of rational functions $h_n \to L_\chi$ on any open (punctured) set $U(x_i)$.[8]

Now, the *Dirichlet character* $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}$ has the following properties,

(1) χ is *completely multiplicative*, and thus $\chi(l + m) = \chi(l)\chi(m)$.
(2) For all integers $l \in \mathbb{Z}$, there exists some $n \in \mathbb{Z}_{>0}$ such that $\chi(l + m) = \chi(l)$.
(3) If $(l, m) \neq 1$, i.e. there is a prime $p \in \mathbb{Z}$ which is a factor of l and m, then $\chi(l) = 0$. Otherwise $\chi(l) \neq 0$.
(4) $\chi(1) = 1$.
(5) χ is periodic with some finite period $m \in \mathbb{Z}$.

Now, let us write the following cyclic matrix group,

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \zeta^\ell \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathbf{Mat}_{n\times n}(\mathbb{C}).$$

where ζ^ℓ is in the image of the morphism

$$e_\ell : \mathbb{Z}/n\mathbb{Z}\, to \langle\zeta\rangle$$

[8] Here, we are implicitely applying some biholomorphism using the Riemann mapping Theorem, to think of L_χ as being defined on the punctured Riemann sphere.

restricted to

$$e_\ell : (\mathbb{Z}/n\mathbb{Z})^\times \to \langle\zeta\rangle$$

Note, the number of such morphisms is exactly $\phi(n)$, the number of integers $1 \leq \ell \leq n-1$, relatively prime to n, i.e. $|(\mathbb{Z}/n\mathbb{Z})^\times|$. Next, let σ act on t_ζ by cyclically permuting the diagonal entries given by n^{th} roots of 1. This then defines an action of the Galois group $\mathcal{G}(K/\mathbb{Q}) \cong$, on the n^{th} roots of unity. Here ℓ is relatively prime to n, so that it generates $(\mathbb{Z}/n\mathbb{Z})^\times$.

Remark 7.1. Note, the map $e_\ell : \mathbb{Z}/n\mathbb{Z} \to \langle\zeta\rangle$ gives an identification of permutations of the n^{th} roots of unity $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ given by the cyclotomic extension $K/\mathbb{Q}$ with a permutation matrix in the **Weyl group** S_n of $\mathbf{GL}_n$. In particular, a generator of $\sigma \in \mathcal{G}(K/\mathbb{Q})$ is mapped to a maximal length cycle of S_n, which is then mapped to the corresponding permutation matrix, which we also denote by σ. From this observation, we have the following,

Theorem 7.2. *The representation e_ℓ defines a cyclic permutation in the Weyl group S_n of $\mathbf{GL}_n$ identified with the corresponding permutation matrices. Moreover, this defines an action on the root system*

$$\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$$

of $\mathbf{GL}_n$ and therefore on the torus and on its Cartan subalgebra. In particular, We may identify the root system of $\mathbf{GL}_n(\mathbb{Q})$ with the n^{th} roots of unity, which can be ordered via a choice of basis for the n-dimensional $\mathbb{Q}$-vector space K so that there is an identification of $\mathbf{GL}(K)$ with $\mathbf{GL}_n(\mathbb{Q})$ by treating the (ordered) $\mathbb{Q}$-basis

$$\{\zeta_1, \zeta_2, ..., \zeta_n\}$$

as the standard basis for the $\mathbb{Q}$-vector space K.

Now, We have a representation

$$\begin{tikzcd}
\mathcal{G}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r, "\sigma"] \arrow[dr, "\rho"'] & \mathbf{GL}_n(\mathbb{Q}) \cong \mathbf{GL}(K) \arrow[d] \\
& \mathbf{GL}_n(\mathbb{C})
\end{tikzcd}$$

Where there are $\phi(n)$ choices of embeddings $\mathbf{GL}(K) \hookrightarrow \mathbf{GL}_n(\mathbb{C})$ given by the $\phi(n)$ automorphisms of the *orthonormal basis*

$$\{\zeta_1, ..., \zeta_m\} \to \{\zeta_1, ..., \zeta_n\}$$

corresponding to the elements of $(\mathbb{Z}/n\mathbb{Z})^\times$.

7.3. **Arbitrary Extensions and Local Surface Orders.** Next, let us look at local fields K_v for K corresponding to arbitrary extensions of $\mathbb{Q}$. In particular, we wish to localize $\mathbb{Z}$ at a prime p to get the p-adic integers $\mathbb{Z}_p$, and the p-adic numbers $\mathbb{Q}_p$. The we wish to take a cyclotomic extension $\mathbb{Q}_p(\zeta)$, by the nth roots of unity $\langle\zeta\rangle$. We then define the surface order

$$\begin{pmatrix} \mathbb{Q}_p & p & \cdots & p & p \\ \mathbb{Q}_p & \mathbb{Q}_p & \cdots & p & p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{Q}_p & \mathbb{Q}_p & \cdot & \mathbb{Q}_p & p \\ \mathbb{Q}_p & \mathbb{Q}_p & \cdot & \mathbb{Q}_p & \mathbb{Q}_p \end{pmatrix} \subset \mathbf{Mat}_{n\times n}(\mathbb{Q}_p)$$

and the corresponding lower triangular matrix ring over the local ring $\mathbb{Z}_p$ living inside of $\mathbf{Mat}_{n\times n}(\mathbb{Z}_p)$.

Now, at first, we assume that $p\mathcal{O}_K$ is *not* ramified. In particular, it factors complete. Then we define everything as before for the global case. If on the other hand $\{p_1, p_2, ..., p_r\}$ is the set of ramified primes in $\mathbb{Q}$, we will let n_i be the number of factors in the factorizations

$$p_i\mathcal{O}_K = \mathfrak{p}_1^{a_1}\,\mathfrak{p}_2^{a_2}\cdots\mathfrak{p}_{n_i}^{a_{n_i}}$$

Then over each $p = p_i$, we define an $n_i \times n_i$ surface order,

$$\begin{pmatrix} \mathbb{Q}_p & p & \cdots & p & p \\ \mathbb{Q}_p & \mathbb{Q}_p & \cdots & p & p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{Q}_p & \mathbb{Q}_p & \cdot & \mathbb{Q}_p & p \\ \mathbb{Q}_p & \mathbb{Q}_p & \cdot & \mathbb{Q}_p & \mathbb{Q}_p \end{pmatrix} \subset \mathbf{Mat}_{n\times n}(\mathbb{Q}_p)$$

and the corresponding lower triangular matrix ring over the local ring $\mathbb{Z}_p$ living inside of $\mathbf{Mat}_{n\times n}(\mathbb{Z}_p)$.

Now, for each ramified prime p_i, there is an $n_i \times n_i$ surface order, where now

$$n = n_1 + n_2 + \cdots n_r$$

In particular, Let Ω_i be the surface order over the ramified prime p_i. Then there is a block diagonal embedding

$$\begin{pmatrix} \Omega_1 & & & \\ & \Omega_2 & & \\ & & \ddots & \\ & & & \Omega_r \end{pmatrix} \subset \mathbf{Mat}_{n\times n}(\mathbb{Q})$$

Now, for each block, everything is constructed as before for global surface orders of cyclotomic extensions. Moreover, the Galois group $\mathcal{G}(\overline{Q}/\mathbb{Q})$ restricted to any of the above blocks behaves as before as a cyclic subgroup $\mathcal{G}(K_v/\mathbb{Q}_p)$ of the Galois group $\mathcal{G}(K/\mathbb{Q})$. Moreover, as in [?], we define a pullback of the matrix algebras Ω_i, defined inside $\mathbf{Mat}_{n\times n}(\overline{\mathbb{Q}})$ which give a *"gluing"* of the cyclic Galois subgroups

$$\begin{tikzcd} \mathcal{G}(K_v/\mathbb{Q}_p) \arrow[r] \arrow[dr] & \mathcal{G}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[d] \\ & \mathcal{G}(K/\mathbb{Q}) \end{tikzcd}$$

This gluing is given by a radical embedding of the surface algebra into the product of the cyclic path algebras of the **noncommutative normalization** of the surface algebra (or surface order given by completion of the surface algebra).

To understand this a little more clearly, we note that each completion gives rise to a surface order defined as a pullback of a block diagonal matrix subalgebra of the square $n \times n$ matrix algebra over the completion of $\mathbb{Q}$ (either a finite or infinite place). Now we have the following,

Theorem 7.3. *Given any finite extension $K/\mathbb{Q}$, there is a corresponding surface global surface order defined over $\mathbf{A}_{\mathbb{Q}}$, which localizes to a local surface order over $\mathbb{Q}$ or $\mathbb{Q}_p$. The local surface orders defined over $\mathbb{Q}$ correspond to the completions of the form $\mathbb{R}$ or $\mathbb{C}$ as described for cyclotomic field extensions. The local surface orders over $\mathbb{Q}_p$ are either of the form $\mathbf{Mat}_{n\times n}(\mathbb{Q}_p)$ for unramified primes p, or they are smaller algebras of size $n_i \times n_i$, which can be embedded in the matrix algebra $\mathbf{Mat}_{n\times n}(\mathbf{A}_{\mathbb{Q}})$ via the pullback construction given in* [?].

From this we a complete description of the local and global Langlands correspondence for $\mathbf{GL}_n$. Moreover, this provides an understanding of how various root systems of Lie algebras $\mathfrak{g}_i$ for $\mathbf{GL}(n_i)$ are glued together inside of $\mathbf{GL}_n(\mathbf{A}_{\mathbb{Q}})$ via the pullback construction and the identification of the absolute Galois groups $\mathcal{G}(\overline{\mathbb{Q}}/\mathbb{Q})$ with a subgroup of the Weyl group of $\mathbf{GL}_n$. In particular, we have an embedding $\mathbf{Mat}_{n\times n}(\mathbb{Q}) \to \mathbf{Mat}_{n\times n}(\mathbb{C})$ via the identification of the basis $\{\zeta_1, ..., \zeta_n\}$ of the $\mathbb{Q}$-vector space $K = \mathbb{Q}(\zeta)$, with a root system for $\mathbf{GL}_n(\mathbb{C})$ given by the orthonormal basis,

$$\begin{pmatrix}\zeta\\0\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\\zeta^2\\0\\\vdots\\0\end{pmatrix}, \cdots, \begin{pmatrix}0\\0\\0\\\vdots\\\zeta^n\end{pmatrix}$$

of $\mathbb{C}^n$. Now, from this construction we see more eplicitely how to use Artin's Theorem for cyclic subgroups mentioned in [**?**] in order to compute characters of Artin representations ρ of Galois groups for extensions of $\mathbb{Q}$. Moreover, we have an explicit way of computing the Artin L-functions.

8. Artin's L-functions

Definition 8.1. A **character**, χ, of an algebraic group G over a field k is a homomorphism

$$G \to \mathbb{G}_m$$

where $\mathbb{G}_m$ is the **multiplicative group** in the center of $\mathbf{GL}(V)$, represented by $\mathcal{O}(\mathbb{G}_m) = k[t, t^{-1}] \subset k(t)$. Any character defines a representation of G on a vector space V by defining eigenspaces for the action $\rho(g) \cdot v = \chi(g)v$. This gives

$$G \xrightarrow{\chi} \mathbb{G}_m \longrightarrow \mathbf{GL}(V)$$

via the map

$$g \mapsto \begin{pmatrix}\chi(g) & & 0\\ & \ddots & \\ 0 & & \chi(g)\end{pmatrix}.$$

We can define such actions on subspaces $W \subset V$ if W is stable under the G-action. We can define the **product** of two characters via $(\chi_1\chi_2)(g) \cdot v = \chi_1(g)\chi_2(g)v$. So the set of all characters $\mathfrak{X}(G)$ is a commutative group. Let

$$V_\chi = \{v \in V : \ \rho(g) \cdot v = \chi(g)v \ \forall \ g \in G\}$$

be the G-stable subspace of **semi-invariants** of **weight** χ.

Note, any representation

$$\rho : G \to \mathbf{GL}(V)$$

induces a map of characters given by the commutative diagram,

$$\begin{tikzcd} \mathfrak{X}(\rho(G)) \arrow[r] \arrow[dr] & \mathfrak{X}(\mathbf{GL}(V)) \arrow[d] \\ & \mathfrak{X}(G) \end{tikzcd}$$

with $\mathfrak{X}(\rho(G)) \hookrightarrow \mathfrak{X}(G)$ being injective. If the representation ρ is *faithful* (i.e. injective) then all characters of the group G may be identified with characters of $\rho(G)$. Generally speaking, given a linear representation (V, ρ) of G, characters are obtained via taking the trace of $\rho(g) \in \mathbf{GL}(V)$.

8.1. **Artin L-Functions.** Let us recall some information on Artin's L-functions. First, let $K/\mathbb{Q}$ be a finite Galois extension, $\mathscr{O}_K$ the ring of integers, and $p \in \mathbb{Z}$ a ramified prime. Let

$$\mathfrak{p}_1^{a_1}\, \mathfrak{p}_2^{a_2} \cdots \mathfrak{p}_{n_i}^{a_{n_i}}$$

be the primes lying over p. Since $K/\mathbb{Q}$ is Galois we have

$$a_1 = a_2 = \cdots = a_{n_i}.$$

Choose any of the primes $\mathfrak{p}_i$ lying over p (since the choice is well defined up to conjugation and determinants and traces are invariant under base change by $\mathbf{GL}_n$). We have the following data

1. $\mathfrak{D}_i = \mathfrak{D}_{\mathfrak{p}_i}$, the **decomposition group**, defined as
$$\mathfrak{D}_i := \{g \in \mathcal{G}(K/\mathbb{Q}) : \; g(\mathfrak{p}_i) = \mathfrak{p}_i\}.$$
2. $\mathfrak{I}_i$, the **Inertia subgroup**, define as
$$\mathfrak{I}_i := \{g \in \mathfrak{D}_i : \; g(x) = x (\mathrm{mod}\;\; \mathfrak{p}_i)\; \forall\, x \in \mathscr{O}_K\}.$$
3. The **Frobenius automorphism** $\sigma_i = \sigma_{\mathfrak{p}_i}$ which generates the cyclic group
$$\mathfrak{D}_i/\mathfrak{I}_i \cong \mathcal{G}\left(\mathscr{O}_K \,/\, \mathfrak{p}_i \Big/ \mathbb{Z} \,/\, p\,\mathbb{Z}\right),$$
where $O_K \,/\, \mathfrak{p}_i \cong \mathbb{F}_{p^{n_i}} = \mathbb{Z} \,/\, p^{n_i}\, \mathbb{Z}$.

Given a linear representation of the Galois group,

$$\rho : \mathcal{G}(K/\mathbb{Q}) \hookrightarrow \mathbf{GL}(V)$$

let $V^{\mathfrak{I}_i} \subset V$ be the invariant subspace under the action of $\mathfrak{I}_i$. Then $\rho(\sigma_i)$ is well define on $V^{\mathfrak{I}_i}$. Define an **Euler factor** by

$$L_p(\rho, s) := det\left(I - \rho(\sigma_i) N\, \mathfrak{p}_i^{-s}\Big|_{V^{\mathfrak{I}_i}}\right).$$

Again, this depends only on the prime $p \in \mathbb{Z}$ and not on the choice of prime $\mathfrak{p}_i$ over p. Now, define an **Artin L-function** associated to the representation $\rho : \mathcal{G}(K/\mathbb{Q}) \hookrightarrow \mathbf{GL}(V)$ by

$$L(\rho, s) = L_{K/\mathbb{Q}}(\rho, s) = \prod_{p \in \mathbb{Z} \;\; \text{prime}} L_p(\rho, s)^{-1}.$$

Now, at **unramified primes** the definition simplifies since $\mathfrak{I}_{\mathfrak{p}} = \{1\}$ and $V^{\mathfrak{I}_{\mathfrak{p}}} = V$.

Theorem 8.2. *(Artin's Theorem, see* [**?**] *pg. 70): Let G be a family of subgroups of the finite group $\mathcal{G}$. Let*

$$\mathbf{Ind} : \bigoplus_{H \in G} R(H) \to R(\mathcal{G})$$

be the homomorphism defined by the family of $\mathbf{Ind}_H^{\mathcal{G}}$*. Then the following are equivalent:*

1. *$\mathcal{G}$ is the union of the conjugates of the subgroups belonging to G.*
2. *The cokernel of $\mathbb{I}\ltimes : \bigoplus_{H \in G} R(H) \to R(\mathcal{G})$ is finite.*
3. *For each character χ of $\mathcal{G}$, there exist virtual characters $\chi_H \in R(H)$, $H \in G$, and an integer $d \geq 1$ such that*
$$d_\chi = \sum_{H \in G} \mathbf{Ind}_H^{\mathcal{G}}(\chi_H).$$

and since the family of cyclic subgroups of $\mathcal{G}$ satisfies the first of these properties we have that each character is a linear combination with rational coefficients of the characters induced by characters of cyclic subgroups.

It is desirable to have an $\mathbb{Z}$ linear combination of characters of cyclic groups as apposed to a $\mathbb{Q}$-linear combination as this has applications to Artin L-functions. This can in fact be achieved. By the construction of the pullback defining surface orders, we may compute characteristic polynomials for all factors of the Artin L-functions, including the ramified primes. Moreover, as the characteristic polynomials are invariant under base change, if we are able to completely understand the polynomial invariants and obtain an explicit description of them, the geometry of the rings of invariants, and a detailed description of how this is related to the representation theory of the Galois group and the algebraic group, then we are able to obtain a much deeper understanding of the Artin L-functions. Further, if we are able to understand the *"semi-invariants"*, i.e. polynomial invariants under the action of special linear groups (see for example [H1] IV §11.4), then we can completely understand moduli stacks of the automorphic representations. This will be completed in a forthcoming paper on the invariant theory of surface algebras. In particular one can show the following:

Theorem 8.3. (1) *The rings of polynomial semi-invariants (under an action of special linear groups), and therefore of the polynomial invariants under arbitrary base change are all semi-group rings and are the coordinate rings of affine toric varieties.*
(2) *Moreover the parametrizing varieties of representations, i.e. the "representation varieties," of fixed dimension for a given surface algebra are normal, Cohen-Macaulay, and have rational singularities.*
(3) *By results of* [CCKW] *one may deduce that the moduli spaces of the semi-stable "regular modules" are in fact isomorphic to projective lines. It turns out that the invariant rings (as apposed to semi-invariant) provide such moduli spaces and the Artin L-functions are defined in terms of these invariants.*

The interested read can refer to [MS] pg. 197, [**?**], [LP], [C], and [D] for the background theory necessary to prove this. In particular, one must generalize the methods of [LP], similar to what is presented in [Do1, Do2, Do3, Do4] to the case of surface algebras. We may also treat the field extensions which are *not* Galois using the decomposition on pg. 5 of [LP] and the generalizations given in [Do1, Do2, Do3, Do4]. In particular, we have the following:

Corollary 8.4. *Fix any character χ of the group $G(E/F)$. The Artin L-functions for any finite field extension K/F of number fields can be realized as*

$$L(s,\chi) = m_1^{a_1} m_2^{a_2} \cdots m_r^{a_r}$$

a product of monomials corresponding to characters of cyclic subgroups of the automorphism group of the field extension, corresponding to the local orders of a surface order. Moreover, we have that

$$\chi = a_1\chi_1 + a_2\chi_2 + \cdots + a_r\chi_r$$

where the χ_i are characters of the cyclic subgroups of $G(E/F)$ corresponding to the local orders and $a_i \in \mathbb{Z}$.

Some of the properties of the corresponding semigroup rings have been recently observed in [Ci1], [CN1, CN2, CN3] and [N1, N2, N3], for *Galois extensions $K/\mathbb{Q}$*. It is stated there that Artin's conjecture is equivalent to various other statements, for example those given in Corollary

1.7 of [Ci1]. One of the equivalent conditions given there is

$$k[H(s_0)] = k[x_1, x_2, ..., x_r]$$

for all $s_0 \in \mathbb{C}\setminus\{0\}$, where $\mathbb{C} \subset k \subset \mathcal{M}_{<1}$, $\mathcal{M}_{<1}$ the field of meromorphic functions of order < 1, $H(s_0)$ is the semigroup corresponding to Artin L-functions holomorphic at $s_0 \in \mathbb{C}\setminus\{0\}$, and $k[x_1, ..., x_r]$ is the affine polynomial ring. In other words, if $\mathbf{Proj}\,(k[x_1, x_2, ..., x_r]) \cong \mathbb{P}^r_K$. By results of [CCKW], this is always true for rational invariants corresponding to "regular components" of module varieties, given by a sum of "band modules".

With a complete understanding of the polynomial (semi)invariants, one can verify many of these conditions explicitely now since an explicit description of the generators and relations of the invariant rings and semi-invariant rings can be computed. This can be done for any Artin L-functions, not just those corresponding to automorphic representations. Moreover, the case of arbitrary extensions of number fields (not necessarily Galois) can be treated in detail, and it can be shown that the equations for induced characters are $\mathbb{Z}$-linear. All of these properties may be deduced for example from from [**?**], [D], [DW], [C], and [Do1, Do2, Do3, Do4]. Moreover, the behavior of the *"restricted partition functions"* mentioned in [CN1, CN2, CN3] can be explained via the results of [Do1, Do2, Do3, Do4].

Setting up the background material on representation varieties, Schofield semi-invariants, the equivalent determinantal semi-invariants, and the Geometric Invariant Theory needed to describe the moduli stacks here would be potentially suicidal, so this is deferred to the next paper [**?**], which uses methods of [**?**] to describe the indecomposable representations and the geometry of the representation varieties, and provides a complete description of all polynomial (semi)invariant functions on the representation varieties under the action of a connected reductive algebraic group whose Lie algebra is given by the noncommutative normalization of the surface order given by the completion of the surface algebra.

9. Recalling some Preliminaries from Complex Analysis

The previous two sections on the combinatorial commutative algebra and representation theory constructions will be of significance for what is contained in this section for very fundamental reasons. First, recall,

Theorem 9.1. *(**Huritz-Runge**[9] Let G be some region and suppose the sequence $\{f_n\}$ in the holomorphic functions $\mathcal{H}(G)$, on G, converges*

$$f_n \to f.$$

If $f \not\equiv 0$, $\overline{B}(a, R) \subset G$, and $f(z) \neq 0$ for $|z - a| = R$, then there exists an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a, R)$. Moreover, if U is a simply connected open subset such as an open disk $\mathbb{D}_a$, and f is holomorphic on U, then f may be uniformly approximated by polynomials in the local coordinate z of U. In particular, there is some sequence of polynomials $f_n \to f$, converging uniformly to f, and f is represented a a power series in $\mathbb{C}[[z]]$ on U. Finally, if f is meromorphic on U, then on any punctured disk in U, with the puncture at one of the finitely many poles of f in U, and containing no other poles of f, then f may be uniformly approximated by a uniformly convergent sequence of rational functions $f_n \to f$. In particular, on such punctured disks f may be represented by a Laurent series.

[9] See for example [**?**] VII §2 and VIII §1.

Now, suppose $f = L(\rho, s)$ is an Artin L-function. Suppose further we have in hand some sequence of functions $\{f_n\}$ approximating f uniformly on some simply connected open subset U_a of $\mathbb{C}$, say $U_a = B(a, R)$, for example. Then the uniform convergence of $f_n \to f$ allows us to determine the number of zeros of f inside U_a using the approximations. In later sections, we will define such a sequence f_n, after determining on which U_a the Artin L-function is holomorphic, i.e. when $f \in \mathcal{H}(U_a)$.

10. Schofield Semi-invariants and Rational Invariants for Surface Algebras

In this section we follow [**?**] and [**?**] closely in order to define the *"Schofield semi-invariants"*. These are invariant polynomial functions defined on a represenatation variety of some quiver Q, with dimension vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_r) \in \mathbb{N}^r_{\geq 0}$. We will then follow Ringel in [**?**] and introduce *"rational invariants"*, which are shown to be quotients of Schofield semi-invariant. Next, we will follow [LP] and ellaborate on the semi-simple invariant polynomials and their reciprocals. In particular, we will define such polynomials in terms of determinantal invariants. This will give us a way of defining local Euler factors of Artin's L-functions in terms of such invariants.

10.1. **Linear Categories.** We may give any category, in particular "*linear categories*", the structure of a *directed graph* with *path multiplication*, which is *extended linearly* (i.e. a *quiver path algebra*)

Example 10.1. As an example, say we identify some category with two objects and a single morphism C, with the quiver Q,

$$x \xrightarrow{a} y$$

and the paths of Q are $\{e_x, e_y, ae_x = a = e_y a\}$, where we read paths from right to left like composition of linear maps, and e_x, e_y are the **trivial paths**. The path category $\mathbf{P}(Q)$ has objects $Q(x, x) = \{e_x\}$, $Q(y, y) = \{e_y\}$, and $Q(x, y) = \{a\}$.

Objects are vertices, and morphisms are arrows. This allows us to set up much of the material in terms of combinatorial algebraic topology, and gives us a way of encoding the ideas into intuitive pictures with an aesthetic that might intrigue even the most formal and serious reader.

Under this construction, we may define the **category of small categories**, denoted by $\mathfrak{C}$. We may consider concatenation of two consecutive arrows in the quiver Q of the category C as a composition of two morphisms. In this way, we may define a category as a **path category** $\mathbf{P}(Q)$, of a quiver Q. In particular, $Q(x, y)$ will denote all paths from vertex x to vertex y, and "multiplication" is given by concatenation of two consecutive arrows (or else is zero if this is not possible). This is an associative operation, and the collection of all paths in the quiver Q identified with the category C will be what we have called path category $\mathbf{P}(Q)$, with Q_0 as objects, and the collection of all $Q(x, y)$ the morphisms. The morphisms in the category of all categories $\mathfrak{C}$, will be called **functors**, and will be given by directed graph maps (i.e. quiver maps) which preserve products (of arrows) and the *local identities* $e_i, i \in Q_0$, at each vertex $i \in Q_0$. We will call a category a **linear category**[10], or a $\mathbb{K}$-**category**, if all morphism sets $C(A, B)$ have a $\mathbb{K}$-linear structure, and all composition maps are $\mathbb{K}$-bilinear, for some field $\mathbb{K}$.

[10]So, we define the linear structure on Q to be "formal linear combinations of paths". If Q is

$$x \xrightarrow{a} y$$

$\mathbb{K}\,Q(x, y) = \mathbb{K}\,a \cong \mathbb{K}$, is a one dimensional vector space. Similarly $\mathbb{K}\,Q(x, x) = \mathbb{K}\,x \cong \mathbb{K}\,y = \mathbb{K}\,Q(y, y)$. We may define a vector space $\mathbb{K}\,Q$ with basis $\{x, y, a\}$, which is isomorphic to $\mathbb{K}^3$ as a vector space.

A **linear functor** $\theta : \mathcal{A} \to \mathcal{B}$ between two $\mathbb{K}$-categories if the associated graph maps are $\mathbb{K}$-linear.

Example 10.2. Take Q to be the category with two objects and one morphism,

$$x \xrightarrow{a} y\ .$$

For a second category, say

$$x' \xrightarrow{a'} y' \xrightarrow{b'} z'$$

we have $\mathbb{K}\,Q'$ has basis $\{x', y', z', a', b'b'a'\}$ and is isomorphic to $\mathbb{K}^6$. Then we must define a $\mathbb{K}$-linear map $V : \mathbb{K}^3 \to \mathbb{K}^6$ and then we must impose some "graph map structure" onto it that makes sense. On basis elements we could order the bases as above, then define V to be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This would take $(x \to y) \mapsto (x' \to y')$ mapping the arrow $a \mapsto a'$. We could also choose $Va = b'$ which would be given by the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

In pictures, this means all arrows in the following diagram are $\mathbb{K}$-linear,

$$\begin{tikzcd} v = ta \arrow[r, "a"] \arrow[d, "\theta_0"'] & w = ha \arrow[d, "\theta_0"] \\ \theta_0(v) = t(\theta_1 a) \arrow[r, "\theta_1(a)"'] & \theta_0(w) = h(\theta_1 a) \end{tikzcd}$$

We define an **ideal** $\mathcal{I}$ of a $\mathbb{K}$-category C, to be a family of subspaces $\mathcal{I}(v, w) \subset C(v, w)$, such that if $f \in C(u, v)$ and $g \in C(w, x)$, then $g \cdot \mathcal{I}(v, w) \cdot f \subset \mathcal{I}(u, x)$. In terms of quivers, we may visualize this as follows. Suppose we have the following diagram, with $a \in \mathcal{I}(v, w)$:

$$u \xrightarrow{f} v \xrightarrow{a} w \xrightarrow{g} x$$

Then $a \cdot f \in \mathcal{I}(u, w)$, $g \cdot a \in \mathcal{I}(v, x)$, and $g \cdot a \cdot f \in \mathcal{I}(u, x)$. We will define the **quotient category** $C/\mathcal{I}$ to be the category with the same objects as C, but with morphisms $(C/\mathcal{I})(v, w) = C(v, w)/\mathcal{I}(v, w)$, and composition of morphisms will be the residue class of the composition of some chosen representative morphisms. We will define the $\mathbb{K}$-linear path category $\mathbb{K}\,Q$, of a quiver Q, to have objects Q_0, the vertices of the quiver, morphisms all paths $p \in Q(u, v)$ endowed with a linear vector space structure $\mathbb{K}\,Q(v, w)$, treating the paths as formal basis elements. In other words, $\mathbb{K}\,Q(u, v)$ is the vector space of formal linear combinations of paths from u to v, for vertices $u, v \in Q_0$. All ideals we will deal with will be generated by linear combinations of paths in various $Q(u, v)$.

11. Artin L-functions as Fredholm Determinants and Traces

11.1. **Fredholm Determinant.** *"Fredholm determinants"* are *"infinite dimensional determinants"* which one can define for any bounded operator on a Hilbert space,

$$A = (I - T) \in \mathcal{B}(\mathbb{H})$$

where I is the identity operator on $\mathbb{H}$ and T is a *trace class operator*[11] For our purposes, the most important Hilbert spaces we will need are all separable, and thus have a very nice spectral theory for the linear operators on them in terms of some orthonormal basis $\{e_i\}_i$.

(1) $\mathbf{Sym}(V) \cong K[x_1, ..., x_n]$, the symmetric algebra on a vector space V.
(2) $K(x_1, x_2, ..., x_n)$, the rational functions obtained from $\mathbf{Sym}(V)$.
(3) $K[[x_1, ..., x_n]]$, the (convergent) power series.
(4) $K((x_1, ..., x_n))$, the (convergent) Laurent series.
(5) $\mathcal{H}(X)$, the sheaf of locally holomorphic functions on a marked Riemann surface (or projective algebraic curve).
(6) $\mathcal{M}(X)$, the sheaf of locally meromorphic functions on a marked Riemann surface (or projective algebraic curve).
(7) $C^\infty(G, K)$ for $K = \mathbb{R}$ or $\mathbb{C}$, and some of its subalgebras.

Now, the goal here is to exploit the following fact discussed previously,

Recall 11.1. Suppose $f \in \mathcal{H}(U_a)$ is a holomorphic function on some open disk $U \subset X$. Then f may be uniformly approximated by polynomial functions. In particular, there is a sequence $f_n \to f$, converging uniformly on U, given by $f_n \in \mathbb{C}[z]$. $\{f_n\}$ converges to a power series representation of f on U, i.e. f may be written on U as an element of $\mathbb{C}((z))$, where z is some local coordinate on $U \subset X$. Furthermore, the same statement is true for a meromorphic function $f \in \mathcal{M}(U^\times)$, and a uniformly convergent sequence $f_n \to f$, of rational functions $f_n \in \mathbb{C}(z))$. Here $U^\times$ is a punctured disk, f is represented as a Laurent series on $U^\times$ with the only possible pole of f on $U^\times$ at the puncture.

Now, suppose $f = L(\rho, s)$ is an Artin L-function for a representation of the absolute Galois group

$$\begin{tikzcd} \mathcal{G}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[rr] \arrow[dr] & & \mathbf{GL}_n \\ & \mathcal{G}(K/\mathbb{Q}) \arrow[ur] & \end{tikzcd}$$

for some completion of $\mathbb{Q}$ factoring through the finite field extension $K/\mathbb{Q}$. Then restricted to a cyclic subgroup $C_i = \mathcal{G}(K_i/\mathbb{Q}) \subset \mathcal{G}(K/\mathbb{Q})$, for some cyclotomic field extension $K_i = \mathbb{Q}(\zeta_i)$, the representation $\rho|_{C_i} : C_i \to \mathbf{rep}(Q_i)$ is a representation of the cyclic quiver Q_i as in Section 7. I particular, with respect to the noncommutative normalization of the surface algebra Λ for the

[11] Recall an operator on a Hilber space $\mathbb{H}$ is called **trace class** if it is compact and one can define a trace on it.

extension $K/\mathbb{Q}$, we have the algebras of cyclic quivers in the following diagram

$$\begin{tikzcd}
 & & \Lambda \arrow[dll, "\mathbf{Res}"'] \arrow[dl, "\mathbf{Res}"] \arrow[dr, "\mathbf{Res}"'] \arrow[drr, "\mathbf{Res}"] & & \\
\Lambda_1 & \Lambda_2 & \cdots & \Lambda_{r-1} & \Lambda_r \\
 & & 1 \arrow[ull, "\mathbf{Ind}"] \arrow[ul, "\mathbf{Ind}"'] \arrow[ur, "\mathbf{Ind}"] \arrow[urr, "\mathbf{Ind}"'] & &
\end{tikzcd}$$

Now, any representation of a cyclic group (in this case given by restriction $\rho|_{C_i} = \rho_i$,

$$\rho_i : C_i \to \mathbf{GL}_n(\mathbb{C})$$

is necessarily of the form

$$\rho_i(g) = \begin{pmatrix} \omega_1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_{n_i} \end{pmatrix}$$

where each ω_j is a primitive n_i^{th} root of 1 for any choice of generator g for C_i. This was shown to have meaning in terms of actions of the Galois group as a subgroup of the Weyl group, and the corresponding cyclotomic extensions K_i/QQ. Of course this means any character for ρ_i is of the form

$$\chi_i = \chi_{i,1} + \chi_{i,2} + \cdots + \chi_{i,n_i}$$

where $\chi_{i,j} : C_i \to S^1$ are one dimensional characters. Moreover, we have that

$$\begin{aligned} \det(\rho_i(g)) &= \prod_{j=1}^{n_i} \omega_j \\ &= \sum_{j=1}^{n_i} \omega_j + \sum_{i \neq j} \omega_i \omega_j + \cdots + \omega_1 \omega_2 \cdots \omega_{n_i} \\ &= \sum_{r=1}^{n_i} \mathbf{Tr}\left(\bigwedge^{r} \rho_i(g) \right). \end{aligned}$$

The important fact to realize at this point is that each of these representations for the cyclic subgroups C_i must be glued together in an appropriate way according to the pullback construction for surface algebras, and thus, the corresponding determinant is just the determinant of a diagonal matrix inside $\mathbf{GL}_n(\mathbb{C})$, where $n = \sum_i n_i$ is the order of the extension $K/\mathbb{Q}$, i.e. the dimension of the $\mathbb{Q}$-vector space K.

Now, for every place v of K, we have this construction, where if the prime $p \in \mathbb{Q}$ is unramified, then there is a single cyclic subgroup of $\mathcal{G}(K_v/\mathbb{Q}_p)$. Next, we have that since the local factors of the Artin L-function $f = L(\rho, s)$ are defined by determinants of diagonal (unitary) matrices with entries in $S^1 \subset \mathbb{C}^{\times}$, i.e.

$$L_p(\rho, s) = 1 - \left(\frac{\rho(\sigma_p)}{p^s}\right)$$

with $\rho(\sigma_p)$ the (semisimple unitary) representation of the Frobenius at p, then the product formula

$$L(\rho, s) = \prod_{p \text{ prime}} L_p(\rho, s)^{-1} = \prod_{p \text{ prime}} \left[1 - \left(\frac{\rho(\sigma_p)}{p^s}\right)\right]^{-1},$$

Moreover, as $p \to \infty$, $\|\det(1 - \rho(\sigma_p)/p^s)\| \to 1$. In particular, the diagonal entries of each $\rho(\sigma_p)$ are in S^1, and thus $\|\omega/p^s\| \to 0$, so long as the real part of s is > 1. Moreover, we actually have that each local factor is a product of Dirichlet local factors giving the global L-functions $f = L(\rho, s)$ the analytic properties of a product of finitely many Dirichlet L-functions as,

$$L(\rho, s) = L(\chi_1, s)L(\chi_2, s) \cdots L(\chi_r, s)$$

where the $L(\chi_i, s)$ are Dirichlet L-functions for cyclic groups. Each Dirichlet L-functions can be extended to a holomorphic function $L_{\chi_i} = f_i$ on the complex plane, with the only pole being at $1 \in \mathbb{C}$. Being a product of such functions, this much hald true for $f = L_\rho = \prod L_{\chi_i} = f_1 f_2 \cdots f_r$ as well. Thus we have the following,

Theorem 11.2. *Artin L-functions $L(\rho, s)$ can be anayltically continued to functions holomorphic on $\mathbb{C} - \{1\}$.*

12. Connes Trace and the Conclusion

Now, we have shown that Artin's conjecture indeed is true, but moreover, we have given an explicit construction of them, and have a way of computing them explicitely. This of course has consequence for ζ-functions via the spectral theory of the operators given by the Artin representations ρ. In particular, from this, we may conclude that the trace formula and results from [**?**] are verifiably correct. To be precise, Theorem 1 and Corollary 2 of [**?**] hold.

12.1. **Sketch of How to Proceed.**

12.2. **Combinatorial Approximations of Geodesic Flows.** Here we will use the work of Karin Baur, to realize paths on surfaces as the indecomposable string and band modules of surface algebras as defined in [CB4].

13. The Linear Path Category of a Medial Quiver

Now, let us suppose that $Q = Q(\Sigma)$ for some combinatorially embedded graph $\Sigma = [\sigma, \alpha, \phi]$. We will define a representation V of Q to be:

(1) For every vertex $x \in Q_0$ of the quiver, an assignment of some $\mathbb{K}$-vector space $V(x)$, and
(2) for every arrow $a \in Q_1$ the assignment of some linear map $V(a)$.

This can be thought of as a linear functor [12] from a small linear category $\mathbb{K}\,Q$, to the category of $\mathbb{K}$-vector spaces $\mathbf{Mod}(\mathbb{K})$, so that for each vertex and arrow we have the following diagram,

$$\begin{tikzcd} x \arrow[r, "a"] \arrow[d, "V_0"'] & y \arrow[d, "V_0"] \\ V_0(x) \arrow[r, "V_1(a)"'] & V_0(y) \end{tikzcd}$$

Here we have $V_0 : Q_0 \to Q_0'$ is the assignment of the vector spaces to the vertices, and $V_1 : Q_1 \to Q_1'$ is the assignment of a linear map. The functor $V = (V_0, V_1)$ is extended to a linear functor in the obvious way. Representation of a quiver Q are often identified with "*modules over the path algebra*", which we will discuss next.

13.1. **Approximating Flows with Quivers.** The combinatorial flows as defined in Section **??**, can be extended via the work of Karin Baur on the geometric models of the module category of a gentle algebra. Her work is on finite dimensional gentle algebras, but the same string and band combinatorics work for the surface algebras. Extending this realization of modules over the surface algebra as paths on a Riemann surface with marked points leads directly to the work of geodesic flows on moduli spaces of Maryam Mirzakhani. This, along with the work of Matilde Marcolli, especially the work on graph C^*-algebras related to Mumford curves will have direct connections to the current work on surface algebras. In particular, it is known,

Theorem 13.1. *Via a universal noncommutative localization of a surface (path) algebra, one obtains a Leavitt path algebra. Via the analytic completion of the Leavitt path algebra, one obtains a graph C^*-algebra. Thus, there is a Leavitt path algebra, and graph C^*-algebra given by localization and completion of a surface algebra. This closely mirrors the procedure of localization and completion of number fields.*

It is thought that this is the next most fruitful direction to head in for a deeper understanding of the L-functions and ζ-functions. Understanding the geometry of the L-functions in terms of determinantal (rational) invariants using Schofield semi-invariants is closely linked to Schofiled's universal noncommutative localization. In particular, when a nontrivial semi-invariant exists, one may invert the corresponding map of projective modules over the path algebra, giving the localizations.

14. Jump to refs

[12] i.e. a map of quivers (equivalently a directed graph map), $V : Q \to Q'$ in the category of categories $\mathfrak{C}$, "extended linearly". Here, Q can be thought of as a subcategory of the category $\mathcal{D}$ of all directed graphs. Thinking of the category $\mathbf{Mod}(\mathbb{K})$ of all $\mathbb{K}$-vector spaces as a directed graph can be more intuitive, but after a moment of thought it is potentially a bit unsettling considering the absurdity of vertices and arrows that follow. Try to use the picture as a mnemonic tool, and a way of encoding a large amount of abstract information into a picture which can then be filtered by all of the formalisms later on.

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