GEOMETRIC INVARIANT THEORY, ARTIN L-FUNCTIONS, AND SURFACE ALGEBRAS

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Abstract. We continue the work initiated in [AS1, AS2] on *surface algebras* and *surface orders* and ellaborate on some of the relations to Artin L-functions and Dedekind ζ-functions. In particular, we use the results of [LeBruyn-Procesi] and [**?**] in order to describe the invariant theory of representations of surface algebras in arbitrary characteristic. We show how one can realize Artin L-functions as invariants under the action of a certain algebraic group, and we also show how the action of the Galois group of number fields $K/\mathbb{Q}$ can be understood in these terms as well. We then use this description to show that the exponents in Brauer's $\mathbb{Z}$-linear factorization

$$L(s,\rho) = \prod_{i=1}^{n} L(s,\chi_i)^{n_i}$$

of Artin L-functions into Dirichlet L-functions are positive. By the results of Brauer, this provides some useful results on Artin L-functions. We give a description of L-functions are the Hilbert-Poincare series for projective resolutions of modules over surface algebras (as graded vector spaces). Along the way, we will provide a description of the K-theory of surface algebras, as well as their Leavitt path algebras and graph C^*-algebras. Since relative (semi)invariants for surface algebras can be defined in terms of characters given by maps between projective modules, using the Serre-Swan Theorem gives us a way of interpreting characters in terms of the K-theory, and thus a new way of approaching Beilinson's Conjectures and special values of L-functions, since we identify the character theory and invariant theory with the local factors of L-functions. We also provide a bridge between the theory of surface algebras and surface orders to the theory of noncommutative arithmetic geometry, spectral triples of Mumford curves, and relations to Bruhat-Tits buildings as developed by Marcolli, Connes, Consani, and Manin. This confirms a suspicion that [S4] was of importance and related to the universal cover of surface algebras as was mentioned in [AS1].

Contents

Date: July 24, 2021.

2010 *Mathematics Subject Classification.* Primary 11M06 11R42 13A50 11G32 Secondary 11R33 14L24 14L30 .

Key words and phrases. Artin L-functions, Dedekind ζ-functions, Dirichlet L-functions, Geometric Invariant Theory, dessin d'enfant, dessin orders, surface algebra, Semisimple Representations of Quivers, Noncommutative Geometry, Absolute Galois Group.

1. Historical Background and Motivations

1.1. **Dynamical Flows on Riemann Surfaces.** In [] and [] Maryam Mirzakhani studied dynamical flows on Riemann surfaces[1]. This dynamical flow problem is an older problem which leads one to a more general notion of noncommutative geometry and foliations. The theory of surface algebras gives us a way of modeling these dynamical flows in a discrete way via module categories of surface algebras. In particular, we can develop a model from the work of [**?**, **?**, Baur-Marsh] which gives us a way of understanding modules as paths on the Riemann surface. We have a correspondence of band modules with periodic closed orbits with a unique primitive element (primitive band modules). String modules correspond to homotopy classes of open paths with fixed endpoints. Using this identification we should also be able to get ζ-functions of graphs embedded in a Riemann surface via a dessin d'enfant.

[1] Also mentioned in [Connes1].

1.2. **Vector Bundles on Curves and Beilinson's Conjectures.** Using the identification of modules with path on the corresponding Riemann surface for a surface algebra, we can think of representations of the quiver of the surface algebras as vector bundles on the Riemann surface. In particular, we note for any path $\gamma : [0,1] \to S$ on a dessin d'enfant, if we have a vector bundle B on the corresponding Riemann surface, the endpoints of the path have local vector spaces lying over them, and the path between them gives a homotopy class of linear maps between these to local vector spaces. Thinking of projective modules over the surface algebra as vector bundles via the Serre-Swan Theorem, we obtain a way of approaching Beilinson's conjectures. This opens up tools of K-theory of vector bundles coming from surface algebras and gives us more ways of understanding our next line of investigation.

1.3. **L-functions.** In [Sarnak] pp. 2-3, we find the following statement,

> "Let Å be the ring of adeles of $\mathbb{Q}$ and let π be an automorphic cuspidal representation of $\mathbf{GL}_m(\text{Å})$, with central character χ[2]. The representation π is equivalent to $\bigotimes_\nu \pi_\nu$[3], with $\nu = \infty(\mathbb{Q}_\infty = \mathbb{R})$ or $\nu = p$, and π_ν an irreducible unitary representation of $\mathbf{GL}_m(\mathbb{Q}_\nu)$. Corresponding to each prime p, and local representation π_p one forms the local factor $L(s, \pi_p)$ which takes the form,
>
> $$L(s, \pi_p) = \prod_{j=1}^{m} \left(1 - \frac{\alpha_{j,\pi}(p)}{p^s}\right)^{-1}$$
>
> for m complex parameters $\alpha_{j,\pi}(p)$ determined by π_p. Similarly for $\nu = \infty$, π_∞ determines parameters $\mu_{j,\pi}(\infty)$ such that
>
> $$L(s, \pi_\infty) = \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s - \mu_{j,\pi}(\infty)).$$
>
> The standard L-function associated to π is defined using this data by
>
> $$L(s, \pi) = \prod_p L(s, \pi_p).$$
>
> The complete L-function is given as before by
>
> $$\Lambda(s, \pi) = L(s, \pi_\infty) L(s, \pi).$$
>
> From the automorphy of π one can show that $\Lambda(s, \pi)$ is entire and satisfies a Functional Equation[4]
>
> $$\Lambda(s, \pi) = \epsilon_\pi N_\pi^{\frac{1}{2} - s} \Lambda(1 - s, \tilde{\pi}),$$
>
> where $N_\pi \in \mathbb{Z}_{\geq 1}$ is called the **conductor** of π, ϵ_π is of modulus 1 and is computable in terms of Gauss sums and $\tilde{\pi}$ is the contragradient representation $\tilde{\pi}(g) = \pi({}^t g^{-1})$.
>
> General conjectures of Langlands assert these standard L-functions multiplicatively generate all L-functions (in particular Dedekind Zeta Functions, Artin L-functions, Hasse-Weil Zeta Functions, etc.). So at least conjecturally one is reduced to the study of these."

So in this paper, we will embark on a deeper study of L-functions. Most of our attention will be focused on Artin L-functions in particular, and thus as a special case Dirichlet L-functions and Dedekind ζ-functions of number fields. We will attempt to understand these L-functions in terms of Mumford's *Geometric Invariant Theory* (GIT for short). In particular, we will look

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3 Here ν can be thought of as coming from a valuation on a complete discrete valuation domain in more general contexts.

4 This seems to be related to the equation () in [Connes1] and the quantization property found in [Steinacker] in §3.2.

at the polynomial and rational invariants for actions of certain reductive algebraic groups on representations of quivers. We will then show how one identifies the local factors of Artin L-functions with these polynomial invariants. It is important that we remark on the fact that the quotient varieties given by GIT quotients are toric varieties, which are spherical varieties. This means that in the limit (i.e. at the level of the infinite product of the local factors of an L-function), we obtain spherical invariant functions[5].

1.4. **Noncommutative Geometry and other Connes Großartigkeit.** The work of Alain Connes is unarguably some of the most ingenious mathematics to date. In addition, the work of Matilde Marcolli on noncommutative arithmetic geometry, for our purposes, seems to be a significant piece of the picture. In [] and [] we see a close connection of the work thus far on surface algebras in the following way. In [**?**], the localization of associative algebras is given in terms of inverting maps between projective modules. Further work in [**?**] explains how one can understand the invariant theory of group actions on representations of associative algebras can be phrased in terms of maps between projective modules. This construction can be used to define a *Leavitt path algebra*, which is a particular kind of localization of a path algebra of a quiver (see [], []). Moreover, analytic completion of Leavitt path algebras leads on to *Graph C^*-algebras*. The graph C^* algebras in particular, contain the class of AF-algebras[6]. Thus, one may pass from surface algebras to graph C^*-algebras, and to related noncommutative geometry.

In addition, in [] and [] we get a definition of a "quantization" of functions on the torus. This can be related to the surface algebras via the description of the action of the "*generalized Frobenius*" (see §8 and the description of tori and the shift operator therein). We note that the geometric invariant theory we use here, and the Mumford curves studied by Marcolli imply some close connections to noncommutative arithmetic geometry. This seems to lead us to our next line of investigation.

1.5. **Spectral Theory on Separable Hilbert Space and Fredholm Determinants of Trace Class Operators.** Near the end of the first part of this paper, we will see how one can understand certain semi-infinite matrices as being an r-fold "wrapping" of a larger semi-infinite matrix. In the limit these matrices become infinite in both directions and then we want to study the spectral theory of these infinite matrices in order to understand the zeros and geometry of Artin L-functions. This limiting behavior is not difficult or obtuse in the least and we give some very "down to earth" visual examples to help throughout the first part of the paper. In brief, we take cyclic extensions $\mathbb{Q}(\zeta_{rn})$ by a primitive rn^{th} root of unity, and allow $r \to \infty$. Then we "wrap" the matrix representation of the (generalized) Frobenius up on itself to for r-layers for each $r < \infty$. Then we treat this as a semi-infinite matrix in the limit as in [Marcolli]. We then need the theory of Fredholm determinants for trace class operators (since these matrices are all trace class, being diagonal matrices with unitary characters $\chi : \mathcal{G}(K : \mathbb{Q}) \to \mathbb{C}^*$ as the diagonal, for $K = \mathbb{Q}(\zeta_{rn})$ a cyclic extension). The Fredholm determinant gives a way of understanding Connes [] via the infinite version of the determinant-trace formula.

1.6. **Quantum Gravity and Physics.** Next via [Steinacker], [**?**], [**?**, **?**] we are lead to the connections to quantum gravity models and their connections to Riemann surfaces. In particular, our eventual goal is to use some of the ideas developed so far to construct a Riemann surface model of quantum gravity as a quantum surface code (in the sense of quantum computing). For this we will likely need [], [], [], [], []. We also mention, noncommutative localization as defined

[5] See for example [], [], [], [].

[6] "AF" stands for almost finite, and the most important example for us is infinite matrix algebras.

in [**?**], and use of graph C^*-algebras seems to be the appropriate approach to understanding how to localize theories of gravity and deal with the noncommutative behavior of space-time at small scales. We also note that loop quantum gravity seems to benefit from this approach by understanding surface algebras in terms of loop groups and loop algebras. We also remark that the Geometric Invariant Theory developed here will be useful in the study of stability conditions and wall-crossing phenomena.

Part 1. **Relating L-functions to the Geometric Invariant Theory of Surface Algebras**

2. Cyclic Subgroups of Galois Groups

Let $\mathcal{G} = \mathcal{G}(K : \mathbb{Q})$ be the Galois group of some finite extension $K/\mathbb{Q}$. Let

$$\{C_{n(1)}, C_{n(2)}, ..., C_{n(r)}\} = \{\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_r\}$$

Be the cyclic subgroups of $\mathcal{G}$. Let $\{K_1, K_2, ..., K_r\}$ be the corresponding field extensions,

$$\begin{tikzcd}[arrows=dash]
 & & K \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
K_1 \arrow[drr] & K_2 \arrow[dr] & \cdots & K_{r-1} \arrow[dl] & K_r \arrow[dll] \\
 & & \mathbb{Q} & &
\end{tikzcd}$$

So, $\mathcal{G}(K_i : \mathbb{Q}) = \mathcal{G}_i = C_{n(i)}$ in our notation.

Now, we know that if $[K_i : \mathbb{Q}] = n(i)$ is the degree of the extension, then

$$K_i \cong \mathbb{Q}^{n(i)}$$

as a $\mathbb{Q}$-vector space.

Theorem 2.1. *(**Normal Basis Theorem**)*[7] *Let $F/\mathbb{Q}$ be a finite Galois extension of degree n. Let $\{\sigma_1, \sigma_2, ..., \sigma_n\}$ be the elements of the Galois group $\mathcal{G}(F : \mathbb{Q})$. Then there exists an element $\omega \in F$ such that*

$$\mathbf{Span}_{\mathbb{Q}}\{\sigma_1\omega, \sigma_2\omega, ..., \sigma_n\omega\} = F$$

gives a $\mathbb{Q}$ basis of F.

From this we may define a $\mathbb{Q}$-linear representation

$$\rho : \mathcal{G}(K_i : \mathbb{Q}) \to \mathfrak{gl}_{n(i)}(\mathbb{Q})$$

of the cyclic Galois group $\mathcal{G}_i$, which factors through the permutation group,

$$\mathcal{G}_i \xrightarrow{\rho_1} S_{n(i)} \xrightarrow{\rho_2} \mathfrak{gl}_{n(i)}(\mathbb{Q})$$

by sending

$$\sigma \mapsto (1, 2, ..., n(i)) \mapsto \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

[7]For details see for example [Lang].

Suppose first that $K_i = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi\sqrt{-1}/n(i)}$ (or some other primitive $n(i)^{th}$-root of 1. Then, for $\zeta \in K_i$ we have that $\{\sigma\zeta, \sigma^2\zeta, ..., \sigma^{n(i)}\zeta = 1\}$ gives a $\mathbb{Q}$-basis of the cyclotomic extension K_i. There are of course other generators of $\mathcal{G}_i$, and other $n(i)$-cycles in the symmetric group $S_{n(i)}$ we could choose. For this choice of ρ, we will abuse notation a little and use σ for $\rho(\sigma)$ and $\rho_1(\sigma)$ so long as it causes no severe confusion.

Now, suppose K_i is not necessarily cyclotomic (but that it is still cyclic). For what follows, we refer the reader to [Milne1] pg. 71.

Suppose F is some field containing $\zeta = e^{2\pi\sqrt{-1}/n(i)}$ (or some other primitive root), with $n(i) > 1$. Let $\mu_{n(i)}$ be the group of $n(i)^{th}$ roots of unity in F. Then

$$\mu_{n(i)} \leq F^{\times}$$

is a cyclic subgroup of the multiplicative group of F. It has order $|\mu_{n(i)}| = n(i)$, and generator ζ. The most basic nontrivial example we might use could be,

$$F = \mathbb{Q}(\sqrt{-1}) = \mathbb{Q}(\zeta)$$

Now, suppose that $E = F[\alpha]$ is a field generated by α, with α a root of

$$P_\alpha(X) = X^n - \alpha.$$

The remaining roots of $P_\alpha(X)$ are of the form $\zeta^j\alpha$, so that the fully split polynomial is

$$\begin{aligned} X^n - \alpha &= (X-\alpha)(X-\zeta\alpha)(X-\zeta^2\alpha)\cdots(X-\zeta^{n(i)-1}\alpha) \\ &= \prod_{j=0}^{n(i)-1}(1-\zeta^j\alpha) \end{aligned}$$

All such roots are in E, so $\mathcal{G}[E:F] \cong \mu_n$, and E/F is a Galois extension. In particular, If $F = \mathbb{Q}$ (so that $\{\sqrt{1}\} = \{\pm 1\}$), and $E = \mathbb{Q}(\zeta)$ with $\zeta = e^{2\pi\sqrt{-1}/n(i)}$, We have for any permutation $\sigma \in \mathcal{G}_i$, that $\sigma\alpha = \zeta^j\alpha$ for some $j \in \{0, 1, 2, ..., n(i)-1\}$. Then we also must have

$$\frac{\sigma\alpha}{\alpha} \in \mu_{n(i)} = \mathcal{G}_i$$

Then the map

$$\mathcal{G}_i \to \mu_{n(i)}$$

given by

$$\sigma \mapsto \frac{\sigma\alpha}{\alpha}$$

is stable under conjugation of α, and in the case $E = K_i = \mathbb{Q}(\zeta)$, we just have

$$\mathcal{G}_i \to \mu_{n(i)}$$

given by

$$\begin{aligned} \sigma \mapsto \frac{\sigma\zeta}{\zeta} &= \frac{e^{\ell 2\pi\sqrt{-1}/n(i)}}{e^{2\pi\sqrt{-1}/n(i)}} \\ &= e^{(\ell-1)2\pi\sqrt{-1}/n(i)}, \end{aligned}$$

Note, if $\gcd(\ell, n(i)) = 1$, then $\sigma\zeta$ is another primitive root. Moreover, by pg. 71 [Milne1], we have

$$\frac{\sigma\tau\alpha}{\alpha} = \frac{\sigma(\tau\alpha)}{\tau\alpha}\frac{\tau\alpha}{\alpha}$$

and the map $\mathcal{G}(E : F) \to \mu_{n(i)}$ is an injective group homomorphism since α generates E over F. If the map is not surjective, then $\mathcal{G}(E : F) \cong \mu_{d(i)}$, where $d(i)|n(i)$ with $d(i) < n(i)$ so that $\mu_{d(i)} \leq \mu_{n(i)}$ is a subgroup. Then we have

$$\begin{aligned}\left(\frac{\sigma\alpha}{\alpha}\right)^{d(i)} = 1 &\implies \sigma\alpha^{d(i)} = \alpha^{d(i)}, \quad \forall\, \sigma \in \mathcal{G}(E : F)\\ &\implies \alpha^{d(i)} \in F\\ &\implies \lightning \text{ (Widerspruch! Wir haben } \nexists\, d(i) \text{ mit } \alpha^{d(i)} = 1 \text{ angennomen.)}\end{aligned}$$

Theorem 2.2. *([Milne1] pg. 71): Let F be a field containing a primitive n^{th} root of unity. Let $E = F[\alpha]$ and*

$$\alpha^n \in F$$

with n minimal with this property. Then E/F is Galois with $\mathcal{G}(E : F) \cong \mu_n$, and conversely if $\mathcal{G}(E : F) \cong \mu_n$ is cyclic, then $E = F[\alpha]$ for some α such that $\alpha^n \in F$.

Proof. Let $\{1, \sigma, \sigma^2, ..., \sigma^{n-1}\} = \mathcal{G}$. Then each $\sigma^j : F^\times \to F^\times$ give a distinct homomorphisms, using Dedekind's Theorem we have

$$\sum_{j=0}^{n-1} \zeta^j \sigma^j = f : F^\times \to F^\times$$

is not the zero map. Therefore there is some γ with

$$\alpha := \sum_{j=0}^{n-1} \zeta^j \sigma^j \gamma = f(\gamma) \neq 0 \in F^\times.$$

Further, $\sigma\alpha = \zeta^{-1}\alpha$. Since σ generates $\mathcal{G} = \mathcal{G}(E : F)$, and ζ generates μ_n, finding such an α proves the claim. □

2.1. **Artin L-functions and Artin's Conjecture.** Following [Milne1] pg. 258, let $K/\mathbb{Q}$ be a finite Galois extension, and let V be a $\mathbb{C}$-vector space. Let

$$\rho : \mathcal{G}(K : \mathbb{Q}) \to \mathbf{GL}(V)$$

be a complex representation of the Galois group. Define the **trace**

$$\chi(\sigma) = \mathbf{Tr}(\rho(\sigma))$$

for $\sigma \in \mathcal{G}(K : \mathbb{Q})$. Denote the **characteristic polynomial** by

$$P_\sigma(t) = \det\left(1 - \rho(\sigma)t\Big|_V\right) = \prod_{i=1}^{\dim_{\mathbb{C}} V} (1 - \lambda_{i,i}t)\,,\ \lambda_{i,i} \in \mathbb{C},$$

where $\{\lambda_{i,i}\}_{i=1}^{\dim_{\mathbb{C}} V}$ are the diagonal elements of $\rho(\sigma)$. This polynomial is a polynomial invariant under the conjugation action by $\mathbf{GL}(V)$. Now, for any **unramified prime** $p \in \mathbb{Q}$, we have

$$P_p(t) = P_\sigma(t), \quad \sigma = (\mathfrak{p}, K/\mathbb{Q})\,, \text{ for some } \mathfrak{p}\,|p.$$

Milne defines

$$L_p(s, \rho) = \frac{1}{P_p\left(\nu(p)^s\right)},$$

where $\nu(p)$ is the cardinality of the residue field at the prime p.

$$L(s, \rho) = \prod L_p(s, \rho).$$

It is well known that for abelian $K/\mathbb{Q}$, i.e. $\mathcal{G}(K:\mathbb{Q})$ is an abelian group, the representation is diagonalizable and we have

$$\rho(g) = \begin{pmatrix} \chi_1(g) & & \\ & \ddots & \\ & & \chi_n(g) \end{pmatrix}, \quad \forall\, g \in \mathcal{G}(K:\mathbb{Q}),\ n = \dim_{\mathbb{C}} V.$$

Then we have,

$$\begin{aligned} \det(I_n - \rho(g)) &= (1-\chi_1(g))\cdot(1-\chi_2(g))\cdots(1-\chi_n(g)) \\ &= \prod_{i=1}^{n}(1-\chi_i(g)) \\ &= \sum_{i=1}^{n}(-1)^i\,\mathbf{Tr}\left(\bigwedge^i A\right) \end{aligned}$$

Now,

$$\begin{aligned} \mathbf{Tr}\left(\bigwedge^i (I_n-\rho(g))\right) &= \mathbf{Tr}\left(\bigwedge^i A\right) \\ &= \sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} a_{j(1),j(1)}a_{j(2),j(2)}\cdots a_{j(i),j(i)} \\ &= \sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i} a_{j(k),j(k)} \\ &= \sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i}(1-\chi_{j(k)}(g)) \end{aligned}$$

where $a_{j,j} = (1-\chi_j(g))$, and $A = (I_n - \rho(g))$.

So that

$$\begin{aligned} \det(I_n-\rho(g)) &= \det(A) \\ &= \sum_{i=1}^{n}(-1)^i\left[\sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i} a_{j(k),j(k)}\right] \\ &= \sum_{i=1}^{n}(-1)^i\left[\sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i}(1-\chi_{j(k)}(g))\right] \end{aligned}$$

Moreover, the characteristic polynomial then gives us the L-function,

$$\begin{aligned} L_p(s,\rho) &= \frac{1}{P_p\left(\nu(p)^s\right)} \\ &= \det\left(1 - \frac{\rho(\sigma)}{\nu(p)^s}\bigg|_V\right) = \prod_{i=1}^{\dim_{\mathbb{C}} V}\left(1-\frac{\chi_i(g)}{\nu(p)^s}\right),\ \chi_i(g)\in\mathbb{C}^* \end{aligned}$$

and

$$L(s,\rho) = \prod L_p(s,\rho).$$

From this we get

$$\rho \cong \bigoplus_{i=1}^{n} \chi_i$$

is a direct sum of one-dimensional representations $\chi_i : \mathcal{G}(K : \mathbb{Q}) \to \mathbb{C}^*$. Composing with the **Artin map**[8] we get a **Dirichlet character** $\chi_i \circ f = \chi_i' : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^*$. In this case, we get that the Artin L-function is a product of Dirichlet L-functions $L(s, \chi_i')$,

$$L(s,\rho) = \prod_{i=1}^{n} L(s,\chi_i)^{n_i}, \ n_i \in \mathbb{Z},$$

by results of Artin and Brauer[9]. Artin proved that every character of a finite group G is a $\mathbb{Q}$-linear combination of characters induced from cyclic subgroups, and the results of Brauer showed that the above factorization of Artin L-functions actually holds over $\mathbb{Z}$. Artin then made the following,

Conjecture 2.3. (Artin's Conjecture): If a representation ρ as above does not contain the trivial representation, then $L(s,\rho)$ extends to a holomorphic function on the whole complex plane.

Remark 2.4. Brauer's results imply that if the n_i in the factorization

$$L(s,\rho) = \prod_{i=1}^{n} L(s,\chi_i)^{n_i},$$

are in fact all positive integers, then *Artin's Conjecture* holds.

The main goal of the following paper is to understand the Artin L-functions in terms of surface algebras in order to obtain results on Artin's Conjecture. In particular, the plan is as follows,

(1) Define an action of the Galois group $\mathcal{G}(K : \mathbb{Q})$ on a "*surface order*", giving a representation of the Galois group for a finite Galois extension K.
(2) Restrict this action to cyclic subgroups.
(3) Explain how to define the representation as a gluing of these restrictions, and in so doing obtain an understanding of the behavior of the *ramified primes* as well as the *unramified primes*.
(4) Use this construction to identify Artin L-functions with polynomial invariant functions on a parametrizing variety of quiver representations (for the surface algebra).
(5) Use the results of [LeBruyn-Procesi] (and [Domokos-Zubkov1] for characteristic $p > 0$) to give the generators and relations of the rings of polynomial invariants.
(6) Describe these rings as coordinate rings of varieties appearing in the Geometric Invariant Theory (GIT) for the semi-simple representations of the surface algebras.

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[8] Let $K = \mathbb{Q}(\zeta_n)$ be a cyclotomic extension by an n^{th} root of unity. There is an isomorphism, $f : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathcal{G}(K : \mathbb{Q})$, which maps the equivalence class of $p \in \mathbb{Z}$ to $(p, K/\mathbb{Q})$ (see [Milne1] pg. 9). From this we get a character $\chi' = \chi \circ f$,

$$\begin{tikzcd} (\mathbb{Z}/n\mathbb{Z})^\times \arrow[rr, "\chi'"] \arrow[dr, "f"'] & & \mathbb{C}^* \\ & \mathcal{G}(K : \mathbb{Q}) \arrow[ur, "\chi"'] & \end{tikzcd}$$

So, $L(s,\chi') = L(s,\chi)$, and the Artin L-function is a Dirichlet L-function. The above factorization of characters defines the **Artin map**.

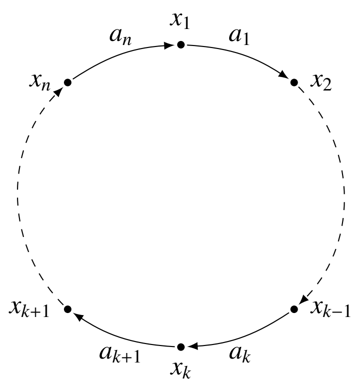
[9] See [**?**] and [Brauer1].

(7) Use the geometric description of these rings given by GIT to completely understand the zeros of Artin L-functions and the singularities of the varieties.
(8) Use the description of the generators of the rings of polynomial invariants given in [LeBruyn-Procesi] (and [Domokos-Zubkov1] for characteristic $p > 0$), the prove Artin's conjecture.
(9) Observe that by [Domokos-Zubkov1], these results hold in arbitrary characteristic, and can thus be transferred to the local field case verbatim (i.e. for $K_{\mathfrak{p}}/\mathbb{Q}_p$).

3. Cyclic Quivers of Type $\tilde{A}(n)$

Let us give some basic background information from the representation theory of quivers. $A(n) = F\tilde{\mathbb{A}}_n^{eq}$ be the hereditary path algebra of the quiver,

$\tilde{\mathbb{A}}_n^{eq}$:



It has primitive orthogonal idempotents which can be identified with the vertices

$$e_i \leftrightarrow x_i$$

and an **arrow ideal** or **Jacobson radical** J generated by the arrows.

3.1. **Loop Algebras.** For any field F, the path algebra $A(n)$ from the previous section can be identified with the matrix algebra[10]

$$\mathfrak{n}_n(F[t]) = \mathfrak{n}_n(t) := \begin{pmatrix} F[t] & (t) & (t) & \cdots & (t) & (t) \\ F[t] & F[t] & (t) & \cdots & (t) & (t) \\ F[t] & F[t] & F[t] & \cdots & (t) & (t) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F[t] & F[t] & F[t] & \cdots & F[t] & (t) \\ F[t] & F[t] & F[t] & \cdots & F[t] & F[t] \end{pmatrix} \subset \mathbf{Mat}_n(F[t]) = \mathfrak{gl}_n(F[t])$$

If the base field F is clear, or is of no consequence, we will simply use the notation $\mathfrak{n}_n(t) \subset \mathfrak{gl}_n(t)$. The indecomposable projective modules can be identified with

$$\left\{ \begin{pmatrix} F[t] \\ F[t] \\ \vdots \\ F[t] \\ F[t] \end{pmatrix}, \begin{pmatrix} (t) \\ F[t] \\ \vdots \\ F[t] \\ F[t] \end{pmatrix}, \cdots, \begin{pmatrix} (t) \\ (t) \\ \vdots \\ F[t] \\ F[t] \end{pmatrix}, \begin{pmatrix} (t) \\ (t) \\ \vdots \\ (t) \\ F[t] \end{pmatrix} \right\}$$

[10] $\mathfrak{n}_n(F[t])$ is an F-algebra, and an $F[t]$-algebra. Both structures will be important at various points.

There is an action by the **shift operator**,

$$\sigma_{n,t} = \sigma := \begin{pmatrix} 0 & & & & & & t \\ 1 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & 1 & 0 \end{pmatrix}$$

by conjugation. The shift operator cyclically permutes the indecomposable projective modules. We may think of the above matrix algebra as an infinitely tall matrix with shifted copies. Multiplication by the shift operator can be thought of as producing and infinite cyclic orbit of the elements in the matrix. For example, the diagonal matrix

$$M = \mathbf{diag}(t, 0, ..., 0)$$

generates an infinite cyclic orbit,

$$\begin{pmatrix} t & & & \\ & t^2 & & \\ & & \ddots & \\ & & & t^n \\ t^{n+1} & & & \\ & t^{n+2} & & \\ & & \ddots & \\ & & & t^{2n} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

Now, suppose we want to study $\pi_n(F[t]/(t^m))$. It is well known[11] that a module over the algebra $F[t]/(t^m)$,

$$M \in \mathbf{Mod}(F[t]/(t^m)),$$

is indecomposable if and only if it is similar to the $m \times m$ matrix with Jordan form

$$\begin{pmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & 1 & 0 \end{pmatrix} \in \mathbf{Mat}_{m\times m}(F)$$

Thus, there are exactly m indecomposable $F[t]/(t^m)$ modules. If for example, $m = rn$ for some $r \in \mathbb{Z}_{\geq 1}$, then the shift operator $\sigma_{n,t}$ generates a cyclic matrix group of order $m = rn$. In

[11]See [**?**] §I.4, pg. 9 for example.

particular, the j^{th} "layer" is generated by multiplying by powers $1, 2, ..., n$ of the matrix for σ and,

$$\begin{pmatrix} 0 & & & & & & t^s \\ t^{s-1} & 0 & & & & & \\ & t^{s-1} & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & t^{s-1} & 0 & & \\ & & & & t^{s-1} & 0 & \\ & & & & & t^{s-1} & 0 \end{pmatrix}$$

which just reduces to a cyclic permutation group of order n over $(t^s)/(t^{s+1})$ isomorphic to the cyclic group generated by $\sigma = (1, 2, 3, ..., n) \in S_n$. The n^{th} power of σ multiplied by the above matrix is the matrix,

$$\begin{pmatrix} t^s & & & \\ & t^s & & \\ & & \ddots & \\ & & & t^s \end{pmatrix}$$

Put another way, taking $t^{s-1} \cdot \{\sigma_{n,t}, \sigma^2_{n,t}, \sigma^3_{n,t}, ..., \sigma^n_{n,t}\}$, gives the s^{th} layer. So, for example, if F contains the $r \cdot n^{th}$ roots of unity, we may study the algebra $\mathfrak{n}_n(F[t]/(t^{rn})$, and the modules over $F[t]/(t^{rn})$. Since

$$F[t]/(t^{rn}) = (F[t]/(t^n))\,/(t^r) = (F[t]/(t^r))\,/(t^n),$$

is we may study the category of $F[t]/(t^r)$ or $F[t]/(t^n)$ modules first, then take quotients of these modules to get the inclusions of module categories,

$$\begin{tikzcd}
 & M \in \mathbf{Mod}(F[t]) & \\
M/(t^n) \in \mathbf{Mod}\,((F[t]/(t^n)) \arrow[ur] & & M/(t^r) \in \mathbf{Mod}\,(F[t]/(t^r)) \arrow[ul] \\
\mathbf{Mod}\,(F[t]/(t^{rn+q}) \arrow[uur, dotted] & & \\
 & M/(t^{rn}) \in \mathbf{Mod}\,(F[t]/(t^{rn}) \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

We may evaluate t at any primitive root ζ of 1 that generates cyclic group μ_{rm}. If we wish to study modules over $F[t]/(t^{rn+q})$ for some $\gcd(q, n) = 1$ we get the first r layers, plus a partial layer generated by σ. In this case we can think of the r block layers of $F[t]/(t^{rn})$ as being diagonally embedded in $\mathbf{Mat}_{rn\times rn}(F) = \mathbf{Mat}_{m\times m}(F)$, with coefficients in F now, rather than in $F[t]$ or $F[t]/(t^m)$.

If F does not contain all of the $r \cdot n^{th}$ roots of 1, then the matrices we obtain by evaluation of t at a primitive $r \cdot n^{th}$ root of 1 is a matrix in

$$\mathbf{Mat}_{n\times n}(F) \otimes_F \mathbb{Q}(\zeta),$$

where $\mathbb{Q}(\zeta)$ is the cyclotomic field given by extending by a primitive rn^{th} root of 1, ζ.

Now, It is important to notice that we have an inclusion

$$\mathbf{Mat}_{n\times n}(F) \subset \mathbf{Mat}_{rn\times rn}(F)$$

given by diagonal embeddings. There is an action of

$$\mathbf{GL}_n(F) \times \mathbf{GL}_n(F) \times \cdots \times \mathbf{GL}_n(F) = \prod_{j=1}^{r} \mathbf{GL}_n(F) \subset \mathbf{GL}_{rn}(F) = \mathbf{GL}_m(F),$$

on the blocks on the diagonal.

3.2. **Lattices and Affine Permutations.**

3.3. **Lusztig's Isomorphism and Affine Schubert Varieties.** In this section we will borrow primarily from the exposition given in [Marcolli] on Lusztig's isomorphism. Other reference material which could be helpful for the reader is [**?**] for affine Coxeter groups, [**?**] for more information on affine Schubert varieties, [MS] Part III and the endless references therein, and [Knutson-Miller-Shimozono] for some background on Matrix Schubert varieties and a description of Zelevinsky's map (which Lusztig's map generalizes). The papers [Lusztig1, Lusztig2] are the papers where Lusztig's map was first defined.

Let $\mathfrak{gl}_n(K) = \mathbf{Mat}_{n\times n}(K)$ be the Lie algebra for $\mathbf{GL}_n(K)$, of $n \times n$-matrices with coefficients in some field K. Let $\mathcal{N} \subset \mathfrak{gl}_n(K)$ be the set of nilpotent matrices. For now, we will assume that $\mathbb{C} \supset \overline{\mathbb{Q}} \supset K \supset \mathbb{Q}$ is a number field given by some finite Galois extension, or $\overline{\mathbb{Q}}$ or $\mathbb{C}$.

Definition 3.1. Let V be a K-vector space (think K^n, $K[t]$, or $K^n \otimes_K K[t]$). Denote by $\mathrm{Gr}(V)$ the Grassmannian of V. Define a map

$$\varphi_N : V \to V,$$

$$\begin{aligned} \varphi_N(v) &:= \frac{t^{n-1}}{1-\frac{N}{t}}(v) \\ &= t^{n-1}v + t^{n-2}N(v) + t^{n-3}N^2(v) + \cdots + N^{n-1}(v) \qquad = \sum_{s=1}^{n} t^{n-s}N^{s-1}(v) \end{aligned}$$

where $N \in \mathcal{N}$. Define **Lusztig's Isomorphism** by

$$\Phi : \mathcal{N} \to \mathrm{Gr}(V), \quad N \mapsto \varphi_N(E_1).$$

The map Φ is $\mathbf{GL}_n(K)$-equivariant.

4. Arithmetic Schemes and Some Examples

4.1. **Arithmetic Schemes.** One reference followed closely in this part is [**?**].

Another more combinatorial and algebro-geometric way to look at this would be as follows. Take $R = F[t]$ with maximal ideal $\mathfrak{m} = (t)$. Let $\mu_n : R \to \{a \in R : a^n = 1\}$ be the representable functor of points for the group of n^{th} roots of unity contained in F. It is represented by $F[t]/(1 - t^n)$. Observe,

(1) If F contains a primitive n^{th} root of 1[12] then

$$F[t]/(1-t^n) \cong \bigotimes_{j=1}^{n} F[t]/(1-\zeta^j),$$

where ζ^j are the n^{th} roots of $1 \in F$. In this case

$$\mu_n \cong \mathbf{Spec}_{max}(F) \times \cdots \times \mathbf{Spec}_{max}(F) = \prod_{j=1}^{n} \mathbf{Spec}_{max}(F).$$

[12]For example cyclotomic extensions $\mathbb{Q}(\zeta)$ and algebraically closed fields of characteristic $\mathbf{char}(F) \neq n$

where $\mathbf{Spec}_{max}(F)$ is the maximal spectrum.

(2) If F *does not* contain a primitive n^{th} root of 1 and $\mathbf{char}(F) \neq n$, then we get

$$F[t]/(1-t^n) \cong \bigotimes_{j=1}^{r} F[t]/p_j(t)$$

where $p_j(t)^{m_j}$ is some factor of

$$p_j(t) \otimes_F F(\zeta) = (1-t)(1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{n-1})$$

and each $p_j(t)$ is a factor of $p_j(t) \otimes_F F(\zeta)$ which does not split completely over F. If F contains an n^{th} root of 1 which is not primitive, ζ^m for example, where $m|n$ and $m < n$, then the polynomial partially splits over F into cyclotomic polynomials according to the roots of 1 that F does contain. In this case μ_n is a product of the spectrum of F, extended by the various $p_j(t)$ which are of the form $F(\zeta^{m_j})$ and $\gcd(m_i, m_j) = 1$ for $i \neq j$.

(3) If on the other hand F contains no roots of 1 other than $\{\pm 1\}$, then the polynomial above does not split at all.

(4) Finally, if $\mathbf{char}(F) = 3$, then $1 - t^n = (1-t)^n$, yielding a non-reduced μ_n. The scheme (as a functor of points) μ_n may be nontrivial when looking at its S-points for some commutative ring S with nilpotents.

Now, let us give some combinatorial gadgets which will help us visualize what is happening. Let

$$R = F[t]/(1-t^n)$$

and suppose $(1-t^n)$ splits completely over F. Each ideal $(1-\zeta^j)$ is maximal, and

$$R_j :== R/(1-\zeta^j) \cong F$$

So, let e_j be a basis vector for the field $R_j \cong F$ as a 1-dimensional $\mathbb{Q}$ vector space. Then we have a sequence of inclusions of $F[t]$-modules

$$\begin{pmatrix} (1-t) \\ \frac{1-t^n}{(1-\zeta)(1-\zeta^2)\cdots(1-\zeta^{n-1})} \\ \vdots \\ \frac{1-t^n}{(1-\zeta)(1-\zeta^2)} \\ \frac{1-t^n}{(1-\zeta)} \\ (1-t^n) \end{pmatrix} = \begin{pmatrix} (1-t) \\ (1-t)(1-\zeta_n) \\ (1-t)(1-\zeta_n)(1-\zeta_n^2) \\ \vdots \\ (1-t)(1-\zeta_n)\cdots(1-\zeta_n^{n-2}) \\ (1-t)(1-\zeta_n)\cdots(1-\zeta_n^{n-2})(1-\zeta_n^{n-1}) \end{pmatrix}$$

This can be thought of as a semisimple representation of the cyclic quiver. In the language we used earlier, this would be the orbit of

$$\begin{pmatrix} (1-t) \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

given by multiplication by

$$I_n - \chi(\sigma) = \begin{pmatrix} (1-\zeta_n) & & & \\ & (1-\zeta_n) & & \\ & & \ddots & \\ & & & (1-\zeta_n) \end{pmatrix}$$

Where $\chi(\sigma)$ is the map

$$\chi : \mathcal{G}(\mathbb{Q}(\zeta_n) : \mathbb{Q}) \to T^n = (\mathbb{C}^*)^n \subset \mathbf{GL}_n(\mathbb{C})$$

on some generator σ of $\mathcal{G}(\mathbb{Q}(\zeta_n) : \mathbb{Q})$. Now, suppose we let K be the extension of $\mathbb{Q}$ by all roots of unity. We can think of this as the column infinite version of the above column of inclusions of $F[t]$-modules, where we take larger and larger values of n to get $F = \mathbb{Q}(\zeta_n)$[13]. Then, the above columns of inclusions will provide us with a list of projective modules over the algebra

$$\mathfrak{n}_n(F[t]) = \mathfrak{n}_n(t) := \begin{pmatrix} F[t] & (t) & (t) & \cdots & (t) & (t) \\ F[t] & F[t] & (t) & \cdots & (t) & (t) \\ F[t] & F[t] & F[t] & \cdots & (t) & (t) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F[t] & F[t] & F[t] & \cdots & F[t] & (t) \\ F[t] & F[t] & F[t] & \cdots & F[t] & F[t] \end{pmatrix} \subset \mathbf{Mat}_n(F[t]) = \mathfrak{gl}_n(F[t])$$

The indecomposable projective modules can be identified with as before.

$$\left\{ \begin{pmatrix} F[t] \\ F[t] \\ \vdots \\ F[t] \\ F[t] \end{pmatrix}, \begin{pmatrix} (t) \\ F[t] \\ \vdots \\ F[t] \\ F[t] \end{pmatrix}, \cdots, \begin{pmatrix} (t) \\ (t) \\ \vdots \\ F[t] \\ F[t] \end{pmatrix}, \begin{pmatrix} (t) \\ (t) \\ \vdots \\ (t) \\ F[t] \end{pmatrix} \right\}$$

4.2. **Applications to Cyclic Extensions.** Now, suppose we have a cyclic extension $K_i/\mathbb{Q}$. For simplicity, suppose $K_i = \mathbb{Q}(\zeta)$ is cyclotomic with $\zeta = e^{2\pi i/n}$. Then we have

$$\mathbf{Span}_{\mathbb{Q}}\{1, \zeta, \zeta^2, ..., \zeta^{n-1}\} = K_i \cong \mathbb{Q}^n$$

We also may identify $\mathcal{G}_i = \mathcal{G}(K_i : \mathbb{Q}) \cong \langle \zeta \rangle = \mu_n$. Then $\mathcal{G}_i$ acts on K_i via $\mu_n \subset K^\times$.

Now, let

$$\mathbb{C}^{A(n)_1} \cong \mathbb{C}^n,$$

be the $\mathbb{C}$-linear span of the arrows $A(n)_1$ of the algebra $A(n)$. Let

$$(\mathbb{C}^*)^{A(n)_0} \cong T^n = (\mathbb{C}^*)^n$$

be the n-dimensional torus in $\mathbf{GL}_n(\mathbb{C})$. We would like to identify $\mathbb{C}^{A(n)}$ with K_i via embeddings of K_i into $\mathbb{C}$. We would then like to let $\mathcal{G}_i$ act as T^n by defining a group homomorphism

$$\mathcal{G}_i \to T^n$$

[13]In other words, we want to adjoin all n^{th} roots of unity to $\mathbb{Q}$ by taking the projective limit

$$\varprojlim_n \mathbb{Q}(\zeta_n)$$

.

given by

$$g \mapsto \begin{pmatrix} \chi_1(g) & & \\ & \ddots & \\ & & \chi_n(g) \end{pmatrix},$$

just as in the previous sections. Once we have established such a morphism, we have a complex representation of the Galois group $\rho : \mathcal{G}_i \to \mathbf{GL}_n(\mathbb{C})$. We might also like to have a permutation representation

$$\varrho : \mathcal{G}_i \to \mathbf{GL}_n(\mathbb{Q})$$

For the moment, take ρ as our representation. Since $\mathcal{G}_i$ is cyclic, all of the $\chi_j(g)$ for $j \in \{1, ..., n\}$ must be n^{th}-roots of unity. So, we are working in the situation described in previous sections. Moreover, from the definition of local Euler factors of Artin L-functions as characteristic polynomials $P_p(t)$, we know that all such polynomials are $\mathbf{GL}_n(\mathbb{C})$-invariants. Thus, we would like to compute the polynomial invariants of the representation spaces of the algebra $A(n)$. In the case of such cyclic quivers, it is not hard to see that each cyclic permutation of the minimal length cycle $a_n a_{n-1} \cdots a_2 a_1$ yields a monomial generator of the ring of invariant polynomials.

In other words, following [MS] Example 10.13, pg. 197, let $(\mathbb{C}^*)^{A(n)_0} \cong T^n$ act on $\mathbb{C}^{A(n)_1} \cong \mathbb{C}^n$ as follows. Coordinatize T^n with $\{z_i\}_{i=1}^n$. Let $x_{i,i+1}$ be the coordinates for $\mathbb{C}^{A(n)_1}$, with $i = 1, 2, ..., n$ modulo n, i.e.

$$\{x_{1,2}, x_{2,3}, ..., x_{n-2,n-1}, x_{n-1,1}\}.$$

One might wish to simple list the arrows instead,

$$\{a_1, a_2, ..., a_{n-1}, a_n\}$$

In either case, the action is give by

$$x_{i,i+1} \mapsto z_i z_{i+1}^{-1} x_{i,i+1}, \text{ or } a_i \mapsto z_{ta} z_{ha}^{-1} a_i$$

whatever strikes your fancy. The grading group is the codimension 1 sublattice of $\mathbb{Z}^{A(n)_0}$ consisting of vectors with coordinate sum zero. Each directed cycle

$$\{a_n a_{n-1} \cdots a_2 a_1, a_1 a_n \cdots a_3 a_2, ..., a_{n-1} a_{n-1} \cdots a_1 a_n\}$$

gives a degree zero monomial

$$x_{i,i-1} x_{i-1,i-2} \cdots x_{i-n+2} x_{i-n+1}$$

which minimally generate a semigroup ring R_0. The affine quotient

$$X = \mathbb{C}^{A(n)_1} // (\mathbb{C}^*)^{A(n)_0} \cong \mathbb{C}^n // T^n$$

is the affine variety given by

$$X = \mathbf{Spec}(R_0).$$

The algebra $R_{(\mathbf{m})}$ is generated over R_0, by monomials of degree $\mathbf{m} = (m_1, m_2, ..., m_n)$ with support given by forests of the quiver. For $\mathbf{m}$ sufficiently generic the projective quotient

$$\tilde{X} = \mathbb{C}^{A(n)_1} //_{\mathbf{m}} (\mathbb{C}^*)^{A(n)_0} \cong \mathbb{C}^n //_{\mathbf{m}} T^n$$

is a smooth projective variety.

Example 4.1. Let us take the quiver

$$\begin{tikzcd} z_1 \arrow[r, bend right, "x_{1,2}=a_2"'] & z_2 \arrow[l, bend right, "x_{2,1}=a_1"'] \end{tikzcd}$$

Then the generators for R_0 are $\{a_2a_1, a_1a_2\} = \{x_{2,1}x_{1,2}, x_{1,2}x_{2,1}\}$ so that

$$R_0 = \mathbb{C}[(x_{2,1}x_{1,2}), (x_{1,2}x_{2,1})] \cong \mathbb{C}[u, v].$$

We have $\mathbb{C}^{A(2)_1} = \mathbb{C}^2$ and $(\mathbb{C}^*)^{A(n)_0} = (\mathbb{C}^*)^2 = T^2$. Notice,

$$x_{1,2} \mapsto z_1z_2^{-1}x_{1,2} = z_{ta_1}z_{ha_1}^{-1}a_1$$

and

$$x_{2,1} \mapsto z_2z_1^{-1}x_{2,1} = z_{ta_2}z_{ha_2}^{-1}a_2$$

under the torus action. Which means

$$x_{2,1}x_{1,2} \mapsto (z_2 \cdot x_{2,1} \cdot z_1^{-1})(z_1 \cdot x_{1,2} \cdot z_2^{-1}) = x_{2,1}x_{1,2}$$

and similarly for $x_{1,2}x_{2,1}$.

This quiver can be used to understand the field extension $K = \mathbb{Q}(i) = \mathbb{Q}(\sqrt{-1})$ as follows[14].

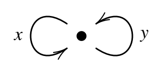
The above path algebra is isomorphic to

$$\mathfrak{n}_-(t) = \begin{pmatrix} \mathbb{C}[t] & (t) \\ \mathbb{C}[t] & \mathbb{C}[t] \end{pmatrix} \subset \mathfrak{gl}_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[t].$$

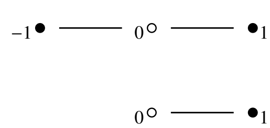
The algebra

$$\mathbb{C}\langle x, y\rangle / \langle xy, yx\rangle \cong \mathbb{C}[x, y]/(xy)$$

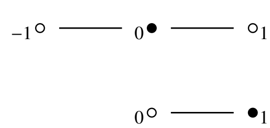
is isomorphic to the Gel'fand-Ponomarev path algebra (with ideal $I = \langle xy, yx\rangle$, $\Lambda = kQ/I$),



Now, taking the following dessin covering



We get the polynomial $p_1(x) = x^2$. If we instead take $p_2(x) = (x-1)(x+1) = x^2 - 1$, we get,



Notice, $\mathbb{Q}$ is the splitting field of both, but $p_2(x)$ gives a nontrivial group action of $\{\pm 1\}$ on the roots. Now, if we extend the dessin corresponding to $p_2(x)$ via the polynomial $p_3(t) = x^4 + 1 = t^2 + 1$, where $t = x^2$, we get the dessin,

[14]Note there are two Dirichlet L-functions for this extension.

$$\begin{tikzcd} & i \arrow[d, dash] & \\ -1 \arrow[r, dash] & 0 \arrow[r, dash] \arrow[d, dash] & 1 \\ & -i & \end{tikzcd}$$

$$\begin{tikzcd} {}_{-1}\circ \arrow[r, dash] & {}_{0}\bullet \arrow[r, dash] & \circ_{1} \end{tikzcd}$$

We get a tower of groups

$$\mu_4 \supset \{\pm 1\} \supset 1$$

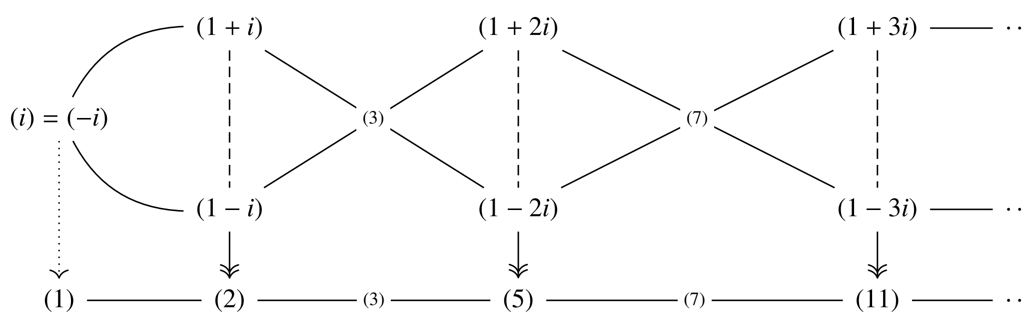
We also get a tower of algebras,

$$\begin{tikzcd} 1 \arrow[r] & i \arrow[d] & & -1 \arrow[r, bend left] & 1 \arrow[l, bend left] & & 1 \\ -i \arrow[u] & -1 \arrow[l] & & & & & \end{tikzcd}$$

Which also corresponds to the group algebras,

$$\mathbb{Q}[x]/(x^4) \cong \mathbb{Q}\langle \mu_4 \rangle \supset \mathbb{Q}[x]/(x^2) \cong \mathbb{Q}\langle \mu_2 \rangle \supset \mathbb{Q}\,.$$

Notice, we have a splitting of primes given by the map $\mathbf{Spec}\,\mathbb{Z}[i] \to \mathbf{Spec}\,\mathbb{Z}$,



In the case of μ_4 we have the invariant ring $\mathbb{C}[a, b, c, d]$ under the action of the torus $(\mathbb{C}^*)^{Q_0} \cong (\mathbb{C}^*)^4$ on the affine space $\mathbb{C}^{Q_1} \cong \mathbb{C}^4$. Now, we have for any quadratic extension

$$K = \mathbb{Q}(\sqrt{\chi(-1)q} = \mathbb{Q}(\alpha)$$

the L-function[15],

$$L(\chi_2, s) = \prod_{p \text{ split}} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \text{ inert}} \left(1 + \frac{1}{p^s}\right)^{-1}$$

with $\chi_2(\sigma_p) = +1$ if p splits, and $\chi_2(\sigma_p) = -1$ if p is inert, for the quadratic character χ_2. Moreover, there is a relation between Dedekind ζ-functions and Dirichlet L-functions,

$$\zeta(s)L(\chi_2, s) = \zeta_K(s).$$

[15] For information on L-functions relevant to this example see [Bump-Cogdell-Gaitsgory-de Shalit-Kowalski-Kudla] Chapter 1.

Next, we know

$$\zeta_K(s) = \prod_{\rho \in \widehat{\mathcal{G}_K}} L(\rho, s),$$

where $\widehat{\mathcal{G}_K}$ is the dual character group to the Galois group $\mathcal{G}(K : \mathbb{Q})$, and

$$L(\rho, s) = \prod_{p \in \mathbb{Q}} \left(1 - \frac{\rho(\sigma_p)}{\nu(p)^s}\right)^{-1}$$

is the Hecke L-function. By the Kronecker-Weber Theorem, for a representation (Galois character),

$$\rho : \mathcal{G}(K : \mathbb{Q}) \to \mathbb{C}^*$$

with Hecke L-function $L(\rho, s)$, there is a unique Dirichlet character χ modulo q for some $q \geq 1$, such that

$$L(\rho, s) = L(\chi, s).$$

In particular, we have for $K = \mathbb{Q}(i)$, $\mathcal{G}(K : \mathbb{Q}) = (\mathbb{Z}/4\mathbb{Z})^\times$, and

$$\begin{aligned}
\zeta_{\mathbb{Q}(i)}(s) &= \prod_p \frac{1}{1 - \frac{1}{\nu(p)^s}} \\
&= \frac{1}{1 - 2^{-s}} \prod_{p \text{ split}} \frac{1}{(1 - p^{-s})^2} \prod_{p \text{ inert}} \frac{1}{1 - p^{-2s}} \\
&= \prod_p \frac{1}{1 - p^{-s}} \prod_{\substack{p \equiv 1 \\ \text{mod } 4}} \frac{1}{1 - p^{-s}} \prod_{\substack{p \equiv 3 \\ \text{mod } 4}} \frac{1}{1 + p^{-s}} \\
&= \zeta(s) L(s, \chi_2)
\end{aligned}$$

for the character $\chi_2(g) = \mathbf{sign}(g) \in \{\pm 1\}$ for $g \in (\mathbb{Z}/4\mathbb{Z})^\times$.

In section 11 where we show how to understand Artin L-functions in terms of invariant polynomial functions under the action of an algebraic group, we will see more details on how this can be interpreted in terms of Geometric Invariant Theory. For the time being, we simple note that

$$(\mathbb{C})^{Q_1} // (\mathbb{C}^*)^{Q_0}$$

has corresponding coordinate ring $\mathbb{C}[a, b, c, d]$ as above corresponding to the invariants of the torus $(\mathbb{C}^*)^{Q_0} \cong (\mathbb{C}^*)^4$, and that the local Euler factors of L-functions can be written as reciprocals of these polynomial invariants.

5. Gluing Cyclic Extensions

Now, suppose we have two cyclic extensions $K_1 = \mathbb{Q}(\alpha)$ and $K_2 = \mathbb{Q}(\beta)$, with $\mathcal{G}(K_1 : \mathbb{Q}) = \mathcal{G}_1 \cong \mu_r$ and $\mathcal{G}(K_2 : \mathbb{Q}) = \mathcal{G}_2 \cong \mu_s$. Now, we know that we can obtain bases,

$$\{\alpha, \alpha \cdot \zeta_r, \alpha \cdot \zeta_r^2, ..., \alpha \cdot \zeta_r^{r-1}\}$$

and,

$$\{\beta, \beta \cdot \zeta_s, \beta \cdot \zeta_s^2, ..., \beta \cdot \zeta_s^{s-1}\}$$

for K_1 and K_2 respectively, as $\mathbb{Q}$ vector spaces. Next, we can define a pullback,

$$\begin{tikzcd} \mathbb{Q}[x,y]/(xy) \arrow[r] \arrow[d] & \mathbb{Q}[x] \arrow[d] \\ \mathbb{Q}[y] \arrow[r] & \mathbb{Q}_y \leftrightarrow \mathbb{Q}_x \end{tikzcd}$$

Where $\mathbb{Q}_y = \mathbb{Q}[y]/(1 - y\zeta_s^j)$ with ζ_s^j primitive, and $\mathbb{Q}_x = \mathbb{Q}[x]/(1 - x\zeta_r^l)$ with ζ_s^l primitive. We define an isomorphism

$$\mathbb{Q}_y \leftrightarrow \mathbb{Q}_x, \quad y\zeta_s^j \mapsto x\zeta_r^l$$

This provides us with a pullback of the two cyclic quivers corresponding to the cyclic extensions $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$, which identifies the two vertices corresponding to $y\zeta_s^j \mapsto x\zeta_r^l$.

5.1. **Gluing Dessin Orders.** Now, let us generalize this procedure. Suppose $K/\mathbb{Q}$ is a finite Galois extension of $\mathbb{Q}$ with Galois group $\mathcal{G} = \mathcal{G}(K : \mathbb{Q})$, and suppose $K_1, K_2, ..., K_r$ are the subfields corresponding to the cyclic subgroups

$$\mathcal{G}_1 = \mathcal{G}(K_1 : \mathbb{Q}),\ \mathcal{G}_2 = \mathcal{G}(K_2 : \mathbb{Q}), ..., \mathcal{G}_r = \mathcal{G}(K_r : \mathbb{Q}).$$

Now, each $\mathcal{G}_j$ is cyclic, i.e. $\mathcal{G}_j \cong \mu_{n(j)}$, and the splitting polynomial is,

$$P_{\alpha_j}(t) = (1 - t)(1 - \alpha_j\zeta_{n(j)})(1 - \alpha_j\zeta_{n(j)}^2) \cdots (1 - \alpha_j\zeta_{n(j)}^{n(i)-1}$$

so that each $K_j = \mathbb{Q}(\alpha_j\zeta_{n(j)})$, and $n(j) = [K_j : \mathbb{Q}]$. Now, we have an affine subalgebra

$$\mathcal{O}_j = \mathcal{O}_{K_j} = \mathbb{Q}[\alpha_j\zeta_{n(j)}] \subset \mathbb{Q}(\alpha_j\zeta_{n(j)}) = K_j,$$

and we have that

$$\mathbb{Q}[t]/(t^{n(j)} - 1) \cong \mathbb{Q}[\alpha_j\zeta_{n(j)}],$$

and

6. Graphs on Surfaces, Dessins d'Enfants, and Number Fields

7. Abelian Covers, Isogenies, Abelian Extensions, and Cyclic Extensions

Serre I.2 Algebraic Groups and Class Fields

7.1. Correspondence between Algebraic Curves and Field Extensions.

7.2. Every Abelian Extension is Contained in a Cyclic Extension.

7.3. Every Abelian Covering of an Algebraic Group is a Pullback of an Isogeny.

8. Reminders on Surface Algebras and Surface Orders

8.1. The Modular Group, Hecke Groups, and Representations of Galois Groups.

8.2. **Triangle Groups.**

Definition 8.1. Suppose $\Delta(p,q,r)$, with $p,q,r \in \mathbb{Z}_{>2} \cup \infty$ be a group with the following properties.

(1) $\Delta(p,q,r)$ is generated by three reflections a,b,c of the Euclidean plane, the Riemann sphere $\mathbb{P}_{\mathbb{Q}}$, or the hyperbolic (upper half) plane (equiv. the unit disk).
(2) $a^2 = b^2 = c^2 = \mathbf{id}$.
(3) $(ab)^p = \sigma^p = \mathbf{id}$.
(4) $(bc)^q = \alpha^q = \mathbf{id}$.
(5) $(ca)^r = \phi^r = \mathbf{id}$.

Then we have a group presentation

$$\Delta(p,q,r) = \langle a,b,c |\, a^2 = b^2 = c^2 = \sigma^p = \alpha^r = \phi^q = \mathbf{id}\rangle$$

making $\Delta(p,q,r)$ into a **Coxeter group**[16] with three generators, which will be called a **triangle group**.

We will need some information about some special triangle groups.

Definition 8.2. The group $H_q = \Delta(2,q,\infty) = \langle \sigma,\alpha |\, \sigma^2 = (\sigma\alpha)^q = \mathbf{id}\rangle$, will be call a **Hecke group**. It maps onto any $\Delta(2,q,r)$ by adding the relations $\alpha^q = \mathbf{id}$. The group $H_3 = \Delta(2,3,\infty) = \langle \sigma,\alpha |\, \sigma^2 = (\sigma\alpha)^3 = \mathbf{id}\rangle$ will be called the **modular group**. It maps onto all triangle groups $\Delta(2,3,r)$ by adding the relation $\alpha^r = \mathbf{id}$. It is a presentation of $\mathbb{P}\mathbf{SL}_2(\mathbb{Z})$, the **projective special linear group**. The generators can then be identified with the two Möbius transformations on the hyperbolic upper half plane $\mathbb{H}_{\mathbb{C}}$ in the complex plane. They are

$$\sigma(z) = -\frac{1}{z}, \quad \text{and} \quad \alpha(z) = z+1.$$

It is well known that there are group isomorphisms,

$$\Delta(2,3\infty) \cong \mathbb{P}\mathbf{SL}_2(\mathbb{Z}) \cong C_2 * C_3$$

where C_2 and C_3 are cyclic groups of order 2 and 3, and "$*$" denoted the free product. The second isomorphism is given by using the generators σ and $\phi^{-1} = \sigma\alpha$. It is also known that the modular group is isomorphic to the quotient of the **Artin braid group** B_3 by its center, or equivalently by all inner automorphisms.

8.3. **Hypermaps and 3-Constellations.**

8.4. **The Galois Groups.**

8.5. **Surface Algebras and Central Elements.** Suppose

$$A = KQ/I = A_0\langle A_1\rangle = \bigoplus_{d=0}^{\infty} A^{\otimes d}$$

is the path algebra of the constellation $C = [\sigma,\alpha,\phi]$, where $\sigma = \sigma_1\sigma_2\cdots\sigma_r$. It has primitive orthogonal idempotents $\{e_i\}_{i\in Q_0}$. Note that

$$A_0 = K[e_1,e_2,....,e_N] \cong K[x_1,x_2,...,x_N]/(x_ix_j, x_i^2-1)$$

for $i \neq j$, is a commutative algebra over the idempotents of $A = KQ/I$.

[16] We note here that this provides one way of making a connection to Bruhat-Tits buildings, and to their use by Marcolli et. al. in the study of spectral triples from Mumford curves and noncommutative arithmetic geometry. We refer the reader to [**?**] for background on Coxeter groups, and [**?**] for triangle groups.

Let $A_{i,j} = e_j A e_i$ be the K-linear span of paths in Q, from vertex i to j. Let $\mathfrak{m} = \prod_{d=1}^{\infty} A^{\otimes d}$ denote the *arrow ideal* of Q, generated by the arrows $a \in Q_1$. We have then that the **complete path algebra** is

$$\widehat{A} = A_0\langle\langle A_1 \rangle\rangle = \prod_{d=0}^{\infty} A^{\otimes d}$$

Then

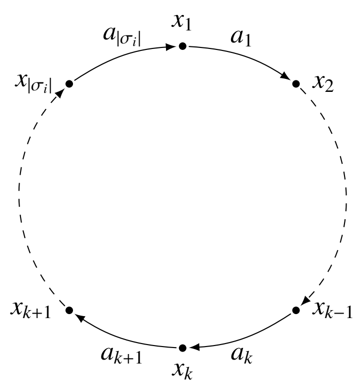
$$\mathcal{Z}(\widehat{A}) \cong A_0 \cdot K[[z_1, z_2, ..., z_r]]/(z_i z_j)_{i \neq j}$$

and

$$\mathcal{N}(\widehat{A}) \cong \prod_{\substack{\sigma_i \\ i=1,...,r}} \mathcal{A}(\sigma_i)$$

where $\mathcal{A}(\sigma_i)$ is the completion of the hereditary algebra $K\tilde{A}(\sigma_i)$ given by the quiver path algebra,

$K\tilde{A}(\sigma_i)$:



Let σ_i be the cycle of σ associated to the i^{th} nonzero cycle in the quiver Q/I. Say σ_i corresponds to the arrows $\{a_1, a_2, ..., a_{n(i)}\}$, where $n_i = |\sigma_i|$. Choosing a distinguished arrow, say a_1, let σ_i^k be identified with $\mathfrak{c}_k = a_k a_{k-1} \cdots a_1 a_n a_{n-1} \cdots a_{k+1}$, the cyclic permutation of the arrows in the cycle of σ_i. Let $z_i = \sum_{k=1}^{n_i} \mathfrak{c}_k$. Then z_i commutes with any arrow $b \in Q_1$. Indeed,

$$\begin{aligned} bz_i &= b(\mathfrak{c}_1 + \mathfrak{c}_2 + \cdots + \mathfrak{c}_{n_i}) \\ &= b\mathfrak{c}_1 + b\mathfrak{c}_2 + \cdots + b\mathfrak{c}_{n_i} \end{aligned}$$

and

$$\begin{aligned} b\mathfrak{c}_k &= ba_k a_{k-1} \cdots a_1 a_n a_{n-1} \cdots a_{k+1} \\ &\neq 0 \iff ha_k = tb, b \in \sigma_i \\ &\iff b = a_{k+1}. \end{aligned}$$

From this we gather $b\mathfrak{c}_k = \mathfrak{c}_k b$ and therefore $bz_i = z_i b$. Thus, the ideal

$$k\langle\langle z_1, z_2, ..., z_r \rangle\rangle \subset \Lambda,$$

is commutative with all paths in Q/I, $z_i z_j = 0$ if and only if $i \neq j$, and so

$$\mathcal{Z}(\widehat{A}) \cong A_0 \cdot K[[z_1, z_2, ..., z_r]]/(z_i z_j)_{i \neq j}.$$

8.6. **Orders.**

Definition 8.3. An R-**order** is a subring $\mathcal{O} \subset R$ of a ring R with the following properties,

(1) R is a ring which is a finite-dimensional algebra over $\mathbb{Q}$.
(2) $\mathcal{O}$ spans R over $\mathbb{Q}$, i.e. $\mathbb{Q}\mathcal{O} = R$.
(3) $\mathcal{O}$ is a $\mathbb{Z}$-lattice in R.

Example 8.4. An important class of examples is that of integral group rings in group algebras over fields. So for example, if

$$\mathcal{G} = \mathcal{G}(K : \mathbb{Q}) = \mu_n = \langle \zeta_n \rangle,$$

is the Galois group of n^{th} roots of unity in the cyclic field extension $K = \mathbb{Q}(\zeta_n) = \mathbb{Q}[\mathcal{G}]$, then K is a finite dimensional $\mathbb{Q}$-algebra of $\mathbb{Q}$-dimension $[K : \mathbb{Q}] = n = |\mathcal{G}|$, and $\mathbb{Z}[\mathcal{G}] = \mathbb{Z}[\zeta_n]$ is the integral group ring. In particular, if we take $K = \mathbb{Q}(i)$, we get a $\mathbb{Z}$-lattice of Gaussian integers $\mathbb{Z}[i] \subset \mathbb{Q}(i)$.

Example 8.5. If R is an integral domain and $K/\mathbb{Q}$ a finite separable extension of$\mathbb{Q}$, then the integral closure $\mathcal{O}$ of R in K is an R-order in K.

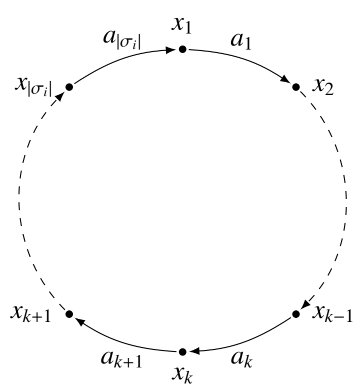
Example 8.6. If $M_n(K) := \mathbf{Mat}_{n\times n}(K)$ is the matrix ring over the number field K, then the matrix ring $M_n(\mathcal{O}_K) := \mathbf{Mat}_{n\times n}(\mathcal{O}_K)$ over the ring of integers $\mathcal{O}_K$ in K, is a $\mathcal{O}_K$-order in $M_n(K)$.

Example 8.7. The polynomial ring $\mathbb{Z}[x]$ is a $\mathbb{Z}$-order in the algebra $\mathbb{Q}[x]$

8.7. **Ramification Orders.**

The algebra $K\tilde{A}(n)$ given by the path algebra of the circular quivers,

$K\tilde{A}(\sigma_i)$:



are isomorphic to the matrix algebras,

$$\mathfrak{n}_-(x) = \mathfrak{n}_-(K) \otimes_K K[x] := \begin{pmatrix} K[x] & (x) & (x) & \cdots & (x) & (x) \\ K[x] & K[x] & (x) & \cdots & (x) & (x) \\ K[x] & K[x] & K[x] & \cdots & (x) & (x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K[x] & K[x] & K[x] & \cdots & K[x] & (x) \\ K[x] & K[x] & K[x] & \cdots & K[x] & K[x] \end{pmatrix}.$$

These are the lower triangular algebras in

$$\mathfrak{gl}_n(x) := \mathfrak{gl}_n(K) \otimes_K K[x] = \mathbf{Mat}_{n\times n}(K[x]),$$

8.8. **Surface Orders as Pullbacks of Ramification Orders.**

8.9. **Tori.** [**?**, **?**] [Steinacker],

Let T_n^2 be defined in terms of the two operators,

$$U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & & & \\ & e^{2\pi i/n} & & \\ & & \ddots & \\ & & & e^{2\pi i(n-1)/n} \end{pmatrix}.$$

We define q in terms of the relations $UV = qVU$[17] and $q^n = 1$ (so q can be chosen to be any primitive n^{th} root of unity $\zeta_n = e^{2\pi ik/n}$ with $\gcd(k, n) = 1$). The operators U and V generate

$$\mathbb{C}\langle U, V\rangle = \mathbf{Mat}_{n\times n}(\mathbb{C}),$$

which can be viewed as the "quantization" of the function algebra $C(T^2)$. We have a decomposition,

$$\mathbf{Mat}_{n\times n}(\mathbb{C}) = \bigoplus_{r,s=0}^{n-1} U^r V^s$$

into irreducible representations (harmonics) in terms of the actions,

$$\langle\zeta_n\rangle \times \mathbf{Mat}_{n\times n}(\mathbb{C}) \to \mathbf{Mat}_{n\times n}(\mathbb{C}), \quad (\zeta_n^r, Y) \mapsto U^r Y U^{-r},$$

and

$$\mathbf{Mat}_{n\times n}(\mathbb{C}) \times \langle\zeta_n\rangle \to \mathbf{Mat}_{n\times n}(\mathbb{C}), \quad (X, \zeta_n^s) \mapsto V^s X V^{-s}.$$

Then we follow [Steinacker] §3.2 and define,

$$I : C(T^2) \to \mathbf{Mat}_{n\times n}(\mathbb{C})$$

by

$$e^{2r\phi}e^{2s\psi} \mapsto \begin{cases} q^{-rs/2}U^r V^s, & |r|,\ |s| < n/2, \\ 0, & \text{otherwise.} \end{cases}$$

The underlying Poisson structure on T^2 is given by

$$\{e^{i\phi}, e^{i\psi}\} = \frac{2\pi}{n} e^{i\phi}e^{i\psi},$$

or equivalently,

$$\{\phi, \psi\} = -\frac{2\pi}{n}.$$

We also have,

$$2\pi\, \mathbf{Tr}(I(f)) = \int_{T^2} \omega_n f, \quad \omega_n = \frac{n}{2\pi} d\phi d\psi,$$

making T_n^2 the "quantization" of the symplectic space (T^2, ω_n).

[17]This and the equation given for $I : C(T^2) \to \mathbf{Mat}_{n\times n}(\mathbb{C})$ seems to be related to the functional equation for L-functions.

8.10. **Cartan Matrices, Bundles, and Characters.** Let $A = KQ/I$ be a surface algebra, and let $\mathfrak{g}_A$ be the corresponding pullback of matrix algebras. Let $|Q_0| = N$ be the number of vertices in the quiver $Q = (Q_0, Q_1)$. Denote by $e_iA = P_i$ the indecomposable projective module corresponding to the vertex $i \in Q_0$.

Let t be an indeterminate and define the **Cartan matrix**,

$$C_A = \begin{pmatrix} \sum_{n=0}^{\infty} \dim_K(e_1Ae_1)t^n & \sum_{n=0}^{\infty} \dim_K(e_2Ae_1)t^n & \cdots & \sum_{n=0}^{\infty} \dim_K(e_NAe_1)t^n \\ \sum_{n=0}^{\infty} \dim_K(e_1Ae_2)t^n & \sum_{n=0}^{\infty} \dim_K(e_2Ae_2)t^n & \cdots & \sum_{n=0}^{\infty} \dim_K(e_NAe_2)t^n \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{n=0}^{\infty} \dim_K(e_1Ae_N)t^n & \sum_{n=0}^{\infty} \dim_K(e_2Ae_N)t^n & \cdots & \sum_{n=0}^{\infty} \dim_K(e_NAe_N)t^n \end{pmatrix}$$

where

$$\sum_{n=0}^{\infty} \dim_K(e_jAe_i)t^n$$

is the $(i, j)^{th}$ entry of C_A, denoted $C_A(i, j)$. We note that since the algebra $A = KQ/I$ is graded by path length as described in this section, the spaces

$$\mathbf{Hom}_A(P_j, P_i) = \mathbf{Hom}_A(e_jA, e_iA) = \bigoplus_{n=0}^{\infty} \mathbf{Hom}_A(e_jA, e_iA)_n$$

are all graded vector spaces, and the K-dimension of the n^{th} graded component (paths of length n) is

$$\dim_K \mathbf{Hom}_A(e_jA, e_iA)_n = \dim_K(e_jAe_i)_n.$$

This means that the entries of the Cartan matrix of a surface algebra have entries given by the Hilbert-Poincare series for graded vector spaces.

Now, we note that if the indecomposable projective modules, P_i and P_j, corresponding to vertices $i, j \in Q_0$ do not lie on the same non-zero cycle (of which there are at most 2), then the $\mathbf{Hom}_A$-space is zero, which simplifies the Cartan matrix for surface algebras.

9. Describing Indecomposable Modules of Surface Algebras

In this section we follow [Crawley-Boevey4] closely. The definitions are more or less standard to those working on representation theory of quivers. For more background on module theoretic properties of so-called *string algebras* we refer to *some* of the extensive literature [**?**], [Butler-Ringel], [Crawley-Boevey1, Crawley-Boevey2, Crawley-Boevey3, Crawley-Boevey4], [Assem-Simson-Skowronski].

9.1. **Indecomposable Projective Modules.**

9.2. **Words.** Let $A = KQ/I$ be a surface algebra. Let Q_0 be the vertices of the quiver Q, and let $Q_1 \amalg Q_1^{-1}$ be the disjoint union of the arrows and their formal inverses. We call elements of Q_1 **direct arrows** and we call Q_1^{-1} **inverse arrows**. We have the maps $h, t : Q_1 \to Q_0$ taking an arrow $a \in Q_1$ to its **head** "ha", and its **tail** "ta". Define $ta^{-1} = ha$ and $ha^{-1} = ta$, for all $a^{-1} \in Q_1^{-1}$. Now, let I be an index set coming from the following list:

- $\{0, 1, 2, ..., n\}$ for $n \in \mathbb{N}$,
- $\mathbb{N} = \{0, 1, 2, ...\}$,
- $-\mathbb{N} = \{0, -1, -2, ...\}$,
- $\mathbb{Z}$.

In [Crawley-Boevey4] an I-**word** is defined as follows: If $I \neq \{0\}$, an I-word w is a sequence of letters ℓ_i for all $i \in I$, with $i-1 \in I$, of the form

$$\begin{cases} \ell_1\ell_2\cdots\ell_n & \text{if } I = \{0,1,2,...,n\} \\ \ell_1\ell_2\ell_3\cdots & \text{if } I = \mathbb{N} \\ \cdots\ell_{-2}\ell_{-1}\ell_0 & \text{if } I = -\mathbb{N} \\ \cdots\ell_{-2}\ell_{-1}\ell_0|\ell_1\ell_2\cdots & \text{if } I = \mathbb{Z} \end{cases}$$

where the letters are in $Q_1 \amalg Q_1^{-1}$, and they satisfy three properties,

(1) If ℓ_i, ℓ_{i+1} are consecutive letters, then $t\ell_i = h\ell_{i+1}$.
(2) If ℓ_i, ℓ_{i+1} are consecutive letters, then $\ell_i^{-1} \neq \ell_{i+1}$ (i.e. there is no subword of w of the form aa^{-1} or $a^{-1}a$ for $a \in Q_1$, $a^{-1} \in Q_1^{-1}$.
(3) No sequence of letters may be a zero relation in I of the algebra KQ/I.

For $I = \{0\}$, there are two **trivial words**, $1_{x,\epsilon}$, for each vertex $x \in Q_0$, and $\epsilon = \pm 1$. For every I-word there is a vertex $x_i(w) = t\ell_i = h\ell_{i+1}$, or just x for $1_{x,\epsilon}$. Any word entirely from Q_1 is called **direct**, and words entirely from Q_1^{-1} are **inverse**. The **inverse word** w^{-1}, of the word w, is obtained by simply inverting every letter of w, and reversing the order of the inverted letters. So, for example if $w = \ell_1\ell_2\ell_3$, then w^{-1} is just $\ell_3^{-1}\ell_2^{-1}\ell_1^{-1}$. Define $1_{x,\epsilon}^{-1} = 1_{x,-\epsilon}$.

Let w be a $\mathbb{Z}$-word. Define $w[n]$ the **shift** of w by $n \in \mathbb{Z}$ by

$$\cdots\ell_{-2}\ell_{-1}\ell_0|\ell_1\ell_2\cdots \mapsto \cdots\ell_{n-2}\ell_{n-1}\ell_n|\ell_{n+1}\ell_{n+2}\cdots$$

If a $\mathbb{Z}$-word $w = w[n]$ for some $n \in \mathbb{N}$, we call w **periodic**. In [Crawley-Boevey4] the shift is defined to be trivial ($w = w[n]$ is just the identity for all n) on non-$\mathbb{Z}$-words. Define an equivalence relation on the set of all words by

$$\ell \sim \ell' \iff \ell' = \ell[m], \text{ or } \ell' = \ell^{-1}[m],$$

for some $m \in \mathbb{N}$.

9.3. **String Modules.** Let w be an I-word. Let $A = KQ/I$. Define a left A-module $M(w)$ as follows,

(1) $M(w)$ has basis $\{e_i : i \in I\}$ as a K-vector space.
(2) For any $x \in Q_0$,

$$x \cdot e_i = \begin{cases} e_i & \text{if } x_i(w) = x \\ 0 & \text{otherwise} \end{cases}$$

(3) For any $\ell \in Q_1$,

$$\ell e_i = \begin{cases} e_{i-1} & \text{if } i-1 \in I, \ell_i = \ell \\ e_{i+1} & \text{if } i+1 \in I, \ell_{i+1} = \ell^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Under the equivalence relation on words it can be shown that $M(w) \cong M(w^{-1})$, and that $M(w) \cong M(w[n])$, as left A-modules (see [Crawley-Boevey4]). Any module $M(w)$, where w is not periodic, is called a **string module**.

Example 9.1. Let $A = KQ/I$ be given by the torus constellation

$$\sigma = (1,2,3)(4,5,6), \quad \alpha = (1,4)(2,5)(3,6)$$

which is obtained from the following quiver,

$$\begin{tikzcd}
& x_1 \arrow[dl, dashed, "a_1"] \arrow[dr, bend left, "b_2"] & \\
x_3 \arrow[ur, bend left, "b_1"] \arrow[rr, dashed, "a_2"] & & x_2 \arrow[ul, dashed, "a_3"'] \arrow[ll, bend left, "b_3"]
\end{tikzcd}$$

with relations $a_i b_j = 0$ for $i, j \in \{1, 2, 3\}$, i.e. the nonzero paths are the three cycles, one may travel around the "a" cycle, or the "b" cycle, but switching between the two is a zero relation. We can define a string module

$$\begin{tikzcd}
& x_1 \arrow[dl, "b_2"] \arrow[dr, dashed, "a_3"] & & & \\
x_2 & & x_2 \arrow[dr, dashed, "a_2"] & & x_2 \arrow[dl, "b_3"] \\
& & & x_3 &
\end{tikzcd}$$

In this case, the module $M(w)$ is a 5-dimensional K-vector space, with basis $\{e_1, e_2, ..., e_5\}$, and with the action of A on the basis of $V \cong K^5$ given by

$$\begin{tikzcd}
e_1 & e_1 \\
e_2 \arrow[ur] \arrow[dr, dashed] & e_2 \\
e_3 \arrow[dr, dashed] & e_3 \\
e_4 & e_4 \\
e_5 \arrow[ur] & e_5
\end{tikzcd}$$

9.4. **Band Modules.** Now, if w is an n-periodic word ($w = w[n]$), the $M(w)$ is a bimodule

$$M(w) =_A M(w)_{K[t,t^{-1}]} =_A M(w)_R,$$

with the commutative local ring $R = K[t, t^{-1}]$ acting via the shift $t \cdot M(w) = M(w[n])$. Then,

$$M(w, V) = M(w) \otimes_R V$$

for any R-module V. $M(w)$ is then a free rank n, R-module, and thus $M(w) \cong_R R^n$, as R-modules. From this we gather that $M(w, V)$ is finite dimensional iff V is. If $w \sim s'$, then $M(w) = M(w')$. In particular, from [Crawley-Boevey4],

$$M(w, V) \cong M(w[n], V) \cong M(w^{-1}[n], \mathbf{res}_\iota V),$$

where ι is the involution (automorphism) $t \leftrightarrow t^{-1}$ on R, and $\mathbf{res}_\iota$ is restriction, in the usual representation theoretic sense, of the $A - R$-bimodule V. Any module of the form $M(w, V)$ is called a **band module**.

9.5. **Cycles of a Surface Algebra.** We note come to the description of the non-zero cycles of the surface algebras. Note, one way of defining a surface algebra is in terms of a graph, cellularly embedded in a Riemann surface. Such graphs were taken to be "clean dessins" and we defined a "non-commutative normalization" describing the surface algebras as a pullback of the infinite dimensional algebras of type $\tilde{A}(n(j))$, where $n(j)$ is the ramification index over a vertex of a dessin $[\sigma, \alpha, \phi]$, or equivalently, the order of the cyclic Galois subgroups of the Galois group corresponding a finite field extension $K/\mathbb{Q}$. The pullback introduced relations which might be described as "crossing a gluing". These relations we called "gentle relations", as described in [Crawley-Boevey4] for example. So to put it simple, the only non-zero cycles in the pullback are the ones corresponding to the cycles of the non-commutative normalization. This will be important in identifying the generators of the rings of invariant polynomial functions under the action of a certain algebraic group on the surface algebra. One thing we should note at this point which will be explained in §**??**

10. $\mathbf{GL}(\mathbf{d})$-Invariant Functions

10.1. **Some References to the Literature.** The invariant theory for a surface algebra $A = KQ/I$ for the action of the algebraic group

$$\mathbf{GL}(\mathbf{d}) = \prod_{x \in Q_0} \mathbf{GL}_{\mathbf{d}(x)}(K) \cong \prod_{x \in Q_0} \mathbf{GL}\,(V(x))$$

on the space of representations

$$\mathbf{rep}_K(A, \mathbf{d}) = \mathbf{rep}_K(KQ/I, \mathbf{d}) = \left\{ (V(a))_{a \in Q_1} \bigoplus_{a \in Q_1} \mathbf{Hom}_K\,(V(ta), V(ha)) \,\middle|\, V(\rho) = 0,\ \forall\, \rho \in I \right\}$$

can be deduced from [LeBruyn-Procesi] in characteristic zero, and [Domokos-Zubkov1] for base fields of arbitrary characteristic. Other references for the combinatorics involved and the relation to toric varieties can be found in Chapter 10 of [MS] and in particular, Example 10.13 pg. 197, as well as the paper [LeBruyn1] and the book [LeBruyn2]. An in depth treatment of a more general invariant theory and "*Cayley-Smooth Orders*" can be found in [LeBruyn2] as well.

10.2. **Cycles and Invariants.** Using the results of [LeBruyn-Procesi] and [Domokos-Zubkov1], a description of the ring of invariant polynomial functions

$$\mathcal{I}(A, \mathbf{d}) = K[\mathbf{rep}(A, \mathbf{d})]^{\mathbf{GL}(\mathbf{d})}$$

on the affine variety $\mathbf{rep}(A, \mathbf{d})$ parametrizing $\mathbf{d}$-dimensional representations of the surface algebra $A = KQ/I$, can be deduced.

Definition 10.1. A quiver Q is said to be **strongly connected** if any pair of its vertices lie on some cycle in the quiver. The cycle may have self crossings, and need not be simple. A subquiver of Q is said to be of **type** $\tilde{\mathbb{A}}_n$ if it is a simple cycle. For a dimension vector $\mathbf{d}$, the **support**, $\mathbf{supp}(\mathbf{d})$, is the full subquiver spanned by vertices $x \in Q_0$ such that $\mathbf{d}(x) > 0$. Define the simple dimension vectors ϵ_x by,

$$\text{For all vertices } x, y \in Q_0, \quad \epsilon_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Define the (bilinear) **Euler form** or **Ringel form** on dimension vectors $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{Z}_{\geq 0}^{Q_0}$, by

$$\langle \mathbf{d}_1, \mathbf{d}_2 \rangle_Q = \sum_{x \in Q_0} \mathbf{d}_1(x)\,\mathbf{d}_2(x) = \sum_{a \in Q_1} \mathbf{d}_1(ta)\,\mathbf{d}_2(ha).$$

Theorem 10.2. *([LeBruyn-Procesi], [Domokos-Zubkov1]): There is a simple representation of Q of dimension vector $\mathbf{d}$ if and only if either $\mathbf{supp}(\mathbf{d})$ is $\tilde{\mathbb{A}}_n$ for some n and $\mathbf{d}\,|\mathbf{supp}(\mathbf{d}) = (1, ..., 1)$, or $\mathbf{supp}(\mathbf{d})$ is strongly connected, different from $\tilde{\mathbb{A}}_n$ for all n, and $\langle \mathbf{d}, \epsilon_x\rangle_Q \leq 0$, $\langle \epsilon_x, \mathbf{d}\rangle_Q \leq 0$ for all $x \in Q_0$.*

We may state this in slightly different language which may be preferable for some following Example 10.13 pg 197 of [MS].

Let Q be a quiver, $Q_0 = \{1, 2, 3, ..., d\}$ the vertices, and Q_1 the arrows. We allow loops and multiple arrows between vertices. The torus $(K^\times)^{Q_0} \cong (K^\times)^d$ with coordinates $\{z_i\}_{i=1}^d$, acts on the vector space $KQ_1 \cong K^{Q_1}$, treating each $a \in Q_1$ as a basis vector. The action is given by

$$a \mapsto z_{ta} z_{ha}^{-1} a, \quad a \in Q_1.$$

So, let us define an embedding

$$\mathbf{rep}_K(A, \mathbf{d}) \hookrightarrow (\mathbf{Mat}_{N\times N}(K))^{Q_1}$$

where $N = \sum_{x\in Q_0} \mathbf{d}(x)$, for some fixed dimension vector $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$. If $V \in \mathbf{rep}_K(A, \mathbf{d})$ is a representation and $V(a) : V(ta) \to V(ha)$ is the matrix associated to the arrow $a \in Q_1$, then $V(a)$ is mapped to the (ha, ta) position in a block matrix with $|Q_0| \times |Q_0| = d \times d$ many blocks, and zeros everywhere except at the (ha, ta) block. The size of the various blocks of a matrix $M \in (\mathbf{Mat}_{N\times N}(K))^{Q_1}$ is determined by the dimension vector $\mathbf{d}$.

We also have a block-diagonal subgroup

$$\mathbf{GL}(\mathbf{d}) \hookrightarrow \mathbf{GL}_N(K)$$

11. Invariant Theory for L-functions

11.1. **Actions of Fundamental Groups of Dessins.** Much of the material in this section can be found in [Lang] VIII, especially §5-§6.

Returning to the case where $K/\mathbb{Q}$ is an arbitrary finite Galois extension with Galois group $\mathcal{G}(K:\mathbb{Q}) = \mathcal{G}$, we again let

$$\begin{tikzcd}
 & & K & & \\
K_0 \arrow[urr] & K_1 \arrow[ur] & \cdots & K_{r-1} \arrow[ul] & K_r \arrow[ull] \\
 & & \mathbb{Q} \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

be the subfields of K corresponding to all of the cyclic subgroups of $\mathcal{G}$,

$$\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_{r-1}, \mathcal{G}_r \subset \mathcal{G}.$$

Now, each $K_j = \mathbb{Q}(\alpha_j)$ is a cyclic extension and is Galois. We have the polynomials

$$p_j(t) = (1-t)(\alpha_j - t)(\alpha_j^2 - t)\cdots(\alpha_j^{n(j)-1} - t)$$

where $\alpha_j = a_j \zeta_{n(j)}$, and $\zeta_{n(j)}$ is a primitive $n(j)^{th}$ root of unity. Now, each

$$\mathcal{G}_j \cong \langle \zeta_{n(j)} \rangle = \mu_{n(j)}$$

is isomorphic to the cyclic group $\mu_{n(j)}$, and we may define the action of a primitive root by

$$\sigma \cdot \alpha_j^r = \alpha^{\sigma \cdot r}$$

given by a permutation $\sigma \in S_{n(j)}$ acting on $\{0, 1, 2, ..., n(j) - 1\}$. Note the fixed field K_j of $\mathcal{G}_j$ is a $\mathbb{Q}$-vector space of dimension $\dim_{\mathbb{Q}} K_j = [K_j : \mathbb{Q}] = n(j)$. Let V_j be the vector space given by sending

$$\zeta_{n(j)}^s \mapsto e_s$$

to the s^{th} unit vector in V_j, with $s = 0, 1, 2, ..., n(j) - 1$. Then we can identify the permutation σ with some permutation matrix, for example,

$$\rho_1(\sigma) = \rho((1, 2, ..., n(j) - 1)) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

This gives a $\mathbb{Q}$-linear representation

$$\rho_1 : \mathcal{G}_j \to \mathbf{GL}_{n(j)}(\mathbb{Q})$$

Obviously, $p_j(t) \in K[t] \subset \overline{\mathbb{Q}}[t] \subset \mathbb{C}[t]$, and we have two maps,

$$\begin{aligned} N_{\mathbb{Q}}^{K_j}(\alpha_j) &= \prod_{s=0}^{n(j)-1} \sigma_s \alpha \\ &= \det \begin{pmatrix} \alpha_j & & & \\ & \sigma\alpha_j & & \\ & & \ddots & \\ & & & \sigma^{n(j)-1}\alpha_j \end{pmatrix} \\ &= \det \begin{pmatrix} \alpha_j \zeta_{n(j)}^0 & & & \\ & \alpha_j \zeta_{n(j)}^1 & & \\ & & \ddots & \\ & & & \alpha_j \zeta_{n(j)}^{n(j)-1} \end{pmatrix} \\ &= \prod_{s=0}^{n(j)-1} \alpha_j \zeta_{n(j)}^s \end{aligned}$$

(which is defined for any $\alpha \in K_j$), called the **norm**, and

$$\begin{aligned}
\mathbf{Tr}_{\mathbb{Q}}^{K_j}(\alpha_j) &= \sum_{s=0}^{n(j)-1} \sigma_s \alpha_j \\
&= \mathbf{Tr} \begin{pmatrix} \alpha_j & & & \\ & \sigma\alpha_j & & \\ & & \ddots & \\ & & & \sigma^{n(j)-1}\alpha_j \end{pmatrix} \\
&= \mathbf{Tr} \begin{pmatrix} \alpha_j \zeta_{n(j)}^0 & & & \\ & \alpha_j \zeta_{n(j)}^1 & & \\ & & \ddots & \\ & & & \alpha_j \zeta_{n(j)}^{n(j)-1} \end{pmatrix} \\
&= \sum_{s=0}^{n(j)-1} \alpha_j \zeta_{n(j)}^s.
\end{aligned}$$

called the **trace**. We also note that

$$\frac{p_j(t)}{(\sigma^s \alpha_j - t)} = \beta_0 + \beta_1 t + \cdots + \beta_{n(j)-1} t^{n(j)-1}$$

with $\beta_r \in K_j$. Moreover, the dual basis to $\{1, \alpha_j, \sigma\alpha_j, ..., \sigma^{n(j)-1}\alpha_j\}$ is,

$$\frac{\beta_0}{p_j'(t)}, \frac{\beta_1}{p_j'(t)}, ..., \frac{\beta_{n(j)-1}}{p_j'(t)},$$

and that

$$\begin{aligned}
\sum_{s=0}^{n(j)-1} \frac{p_j(t)}{(\sigma^s \alpha_j - t)} \cdot \frac{\sigma^s \alpha_j}{p_j'(\sigma^s \alpha_j)} &= \sum_{s=0}^{n(j)-1} \frac{p_j(t)}{(\alpha_j^s - t)} \cdot \frac{\alpha_j^s}{p_j'(\alpha_j^s)} \\
&= t^s
\end{aligned}$$

and every polynomial of this form (for $\alpha_j^s = \alpha_j \zeta_{n(j)}^s$, $s = 0, 1, ..., n(j) - 1$), are all conjugate to one another. Then we have

$$\mathbf{Tr}\left(\frac{p_j(t)}{(\alpha_j^s - t)} \cdot \frac{\alpha_j^s}{p_j'(\alpha_j^s)} \right) = t^s.$$

From this we get

$$\mathbf{Tr}\left(\alpha^m \frac{\beta_s}{p_j'(\alpha_j)} \right) = \delta_{i,j}$$

The two maps are related of course by

$$\det(I_{n(j)} - A_j) = \sum_{s=1}^{n(j)} (-1)^s \, \mathbf{Tr}\left(\bigwedge^s A_j \right).$$

Now, any representation

$$\rho : \mathcal{G}_j \to \mathbf{GL}_{n(j)}(\mathbb{Q})$$

can be extended by K_j,

$$\rho_K : \mathcal{G}_j \to \mathbf{GL}_{n(j)}(\mathbb{Q}) \otimes_{\mathbb{Q}} K_j = \mathbf{GL}_{n(j)}(K_j)$$

in which case we may define a representation

$$\zeta_{n(j)}^s \mapsto \zeta_{n(j)}^s e_s$$

11.2. **Realizing L-functions through Geometric Invariant Theory.** Let us now define a representation

$$\rho_t : \mathcal{G}_j \to \mathfrak{gl}_{n(j)}(\mathbb{Q}) \otimes_{\mathbb{Q}} K_j[t] := \mathfrak{g}_j(t),$$

by the usual

$$\rho_t(\sigma) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & t \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Now, notice that over

$$\mathfrak{g}_j(t^m) := \mathfrak{gl}_{n(j)}(\mathbb{Q}) \otimes_{\mathbb{Q}} K_j[t]/(t^m)$$

this as before generates a cyclic group of order $n(j) \cdot m$. Moreover, evaluation at a primitive root of unity $t = \zeta_{n(j)}$, gives us a way of realizing the "layered" products given for extensions $Q(\zeta_{n(j)\cdot m})$, (which are cyclic extensions of K_j). In particular, this gives us a **nilpotent representation** (of order $m \cdot n$),

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & t \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

If we take the tensor product with the quotient $K_j[t]/(t) \cong K_j$ then the matrix is just the subdiagonal of ones, and any "layer" given by $(t^l)/(t^{l+1})$ can be identified with the nilpotent representation,

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \zeta_{n(j)}^l & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \zeta_{n(j)}^l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \zeta_{n(j)}^l & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \zeta_{n(j)}^l & 0 \end{pmatrix}$$

For increasing $m \in \mathbb{N}$, these representations for extensions of the form $\mathbb{Q}(\zeta_{n(j)\cdot m}$ behave like representations over $\mathfrak{gl}_{n(j)}(\mathbb{Q}) \otimes_{\mathbb{Q}} K_j[t]/(t^m)$.

Part 2. Generating Series, C^*-Algebras from Surface Algebras, and Trace Class Operators

12. Limits of Cyclotomic Polynomials as Hilbert Series

Let $p_n(t) = t^n - 1 \in \mathbb{Q}[t]$ be the polynomial corresponding to the cyclic extension $\mathbb{Q}(\zeta_n)$, where

$$\zeta_n = e^{\frac{2\pi i}{n}} \in S^1 \subset \mathbb{C}^*$$

Let $K_n = \mathbb{Q}(\zeta_n)$ be the splitting field for $p_n(t)$. Next, over $\mathbb{C}$ or K_n, we may identify the solution set (zeros) of $p_n(t) = 0$, with the n^{th} roots of unity $\{\zeta_n^j\}_{j=1}^n$. Moreover, over any field F,

$$K_n \subset K \subset \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

the polynomial splits completely and this identification can also be made over F. For the moment, let us just work over $\mathbb{C}$ and choose some embedding of K_n into $\mathbb{C}$. Then

$$\frac{p_n(t)}{(1-t)} = (t-\zeta_n)(t-\zeta_n^2)\cdots(t-\zeta_n^{n-2})(t-\zeta_n^{n-1}) = \prod_{j=1}^{n-1}(t-\zeta_n^j).$$

Notice

$$\zeta_{rn} = e^{\frac{2\pi i}{rn}} = (e^{\frac{2\pi i}{n}})^{1/r} = \zeta_n^{1/r}.$$

Now, let $r \in \mathbb{Z}_{>0}$. Then we have

$$\begin{aligned} p_{rn}(t)/(1-t) &= (t-\zeta_n^{1/r})(t-(\zeta_n^{2/r})\cdots(t-\zeta_n^{(n-1)/r}) \\ &\times\ (t-\zeta_n^{n/r})(t-\zeta_n^{(n+1)/r})\cdots(t-\zeta_n^{(2n-1)/r}) \\ &\times\ (t-\zeta_n^{2n/r})(t-\zeta_n^{(2n+1)/r})\cdots(t-\zeta_n^{(3n-1)/r}) \\ &\vdots \\ &\times\ (t-\zeta_n^{(r-1)n/r})(t-\zeta_n^{((r-1)n+1)/r})\cdots(t-\zeta_n^{(rn-1)/r}) \\ &= (1-t)\prod_{m=1}^{rn-1}(t-\zeta_n^{m/r}) \end{aligned}$$

We might think of these products as product of the columns in the lattice,

$$\begin{tikzcd}
(t-\zeta_n^{1/r}) \arrow[r] & (t-\zeta_n^{2/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{(n-1)/r}) \arrow[dlll] \\
(t-\zeta_n^{n/r}) \arrow[r] & (t-\zeta_n^{(n+1)/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{(2n-1)/r}) \arrow[dlll] \\
(t-\zeta_n^{2n/r}) \arrow[r] & (t-\zeta_n^{(2n+1)/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{(3n-1)/r}) \arrow[dlll] \\
\vdots & \vdots & \ddots & \vdots \arrow[dlll] \\
(t-\zeta_n^{(r-2)n/r}) \arrow[r] & (t-\zeta_n^{((r-2)n+1)/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{((r-1)n-1)/r}) \arrow[dlll] \\
(t-\zeta_n^{(r-1)n/r}) \arrow[r] & (t-\zeta_n^{((r-1)n+1)/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{(rn-1)/r})
\end{tikzcd}$$

If we wish to obtain infinite product formulas we might want to organize by columns,

$$\begin{tikzcd}
(t-\zeta_n^{1/r}) \arrow[d] & (t-\zeta_n^{2/r}) \arrow[d] & \cdots & (t-\zeta_n^{(n-1)/r}) \arrow[d] \\
(t-\zeta_n^{n/r}) \arrow[d] & (t-\zeta_n^{(n+1)/r}) \arrow[d] & \cdots & (t-\zeta_n^{(2n-1)/r}) \arrow[d] \\
(t-\zeta_n^{2n/r}) \arrow[d] & (t-\zeta_n^{(2n+1)/r}) \arrow[d] & \cdots & (t-\zeta_n^{(3n-1)/r}) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] \\
(t-\zeta_n^{(r-2)n/r}) \arrow[d] & (t-\zeta_n^{((r-2)n+1)/r}) \arrow[d] & \cdots & (t-\zeta_n^{((r-1)n-1)/r}) \arrow[d] \\
(t-\zeta_n^{(r-1)n/r}) \arrow[uuuuur, dotted, bend left] & (t-\zeta_n^{((r-1)n+1)/r}) \arrow[uuuuur, dotted, bend left] & \cdots \arrow[uuuuur, dotted, bend left] & (t-\zeta_n^{(rn-1)/r})
\end{tikzcd}$$

In which case we can write,

$$p_{rn}(t) = (1-t)\prod_{j=1}^{(r-1)}(1-\zeta_n^{jn/r})\prod_{j=1}^{r-1}(1-\zeta_n^{(jn+1)/r})\cdots\prod_{j=1}^{r}(1-\zeta_n^{(jn-1)/r})$$

Next, notice for each column we have something of the form

$$\prod_m (1-\zeta_n^{m/r}), \quad m = (jn+k),\ j = 0,1,2,....,r-1,\ k = 1,2,...,n-1.$$

$$\begin{aligned}\det(I_n - \rho(g)) &= (1-\chi_1(g))\cdot(1-\chi_2(g))\cdots(1-\chi_n(g))\\ &= \prod_{i=1}^{n}(1-\chi_i(g))\\ &= \sum_{i=1}^{n}(-1)^i\,\mathbf{Tr}\left(\bigwedge^{i} A\right)\end{aligned}$$

Now,

$$\begin{aligned}\mathbf{Tr}\left(\bigwedge^{i}(I_n-\rho(g))\right) &= \mathbf{Tr}\left(\bigwedge^{i} A\right)\\ &= \sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} a_{j(1),j(1)}a_{j(2),j(2)}\cdots a_{j(i),j(i)}\\ &= \sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i} a_{j(k),j(k)}\\ &= \sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i}(1-\chi_{j(k)}(g))\end{aligned}$$

where $a_{j,j} = (1-\chi_j(g))$, and $A = (I_n - \rho(g))$.

So that

$$\begin{aligned}\det(I_n-\rho(g)) &= \det(A)\\ &= \sum_{i=1}^{n}(-1)^i\left[\sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i} a_{j(k),j(k)}\right]\\ &= \sum_{i=1}^{n}(-1)^i\left[\sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i}(1-\chi_{j(k)}(g))\right]\end{aligned}$$

Moreover, the characteristic polynomial then gives us the L-function,

$$\begin{aligned}L_p(s,\rho) &= \frac{1}{P_p\left(\nu(p)^s\right)}\\ &= \det\left(1-\left.\frac{\rho(\sigma)}{\nu(p)^s}\right|_V\right) = \prod_{i=1}^{\dim_{\mathbb{C}} V}\left(1-\frac{\chi_i(g)}{\nu(p)^s}\right),\ \chi_i(g)\in\mathbb{C}^*\end{aligned}$$

and

$$L(s,\rho) = \prod L_p(s,\rho).$$

Now, we would like to think again in terms of combinatorial commutative algebra for a moment. Notice, the ring

$$\mathbb{Z}[[x_0, x_1, ..., x_{n-1}]]$$

of formal power series is the ring in which Hilbert series of multigraded rings (i.e. $\mathbb{N}^n$-graded) live. Further, the element $1 - x_i$ has inverse

$$\frac{1}{1-x_i} = 1 + x_i + x_i^2 + x_i^3 + \cdots$$

The multigraded module $M = \bigoplus_{(a_0,...,a_n)\in\mathbb{N}^n} M_{a_0,a_1,...,a_{n-1}}$ over some commutative ring R has Hilbert series

$$H(M, x_0, x_1, ..., x_{n-1}) = \sum_{(a_0,a_1,...,a_n)\in\mathbb{N}} \dim_K(M_{a_0,a_1,...,a_{n-1}})x_0^{a_0}x_1^{a_1}\cdots x_{n-1}^{a_{n-1}}$$

If we have $x_0 = x_1 = \cdots = x_{n-1} = t$, then the definition reduces to a $\mathbb{Z}$-grading.

Now, we noticed any

$$f_\alpha(t) = \frac{p_j(t)}{(\alpha - t)}$$

over $\mathbb{C}$ is holomorphic (for $\alpha = a\zeta_{n(j)}^s$) with removable singularity at $t = \alpha$. We also have that

$$\lambda f_\alpha(t)\cdot\frac{\alpha^s}{p_j(\alpha)} = t^s$$

is of course also holomorphic. Moreover, with the basis $\{1, \alpha, \alpha^2, ..., \alpha^n\}$ and its dual basis as described in the previous sections (see also [Lang] pg. 213), provide us with a description through the trace and determinant maps of the torus and its dual, i.e. we have a descriptions of the characters and cocharacters

$$X^*(T^n) : \mathbb{C}^* \to T^n, \quad X_*(T^n) : T^n \to \mathbb{C}^*$$

which are given in the usual way.

Moreover, we can now identify the action of the torus on roots of unity (or more generally on roots of the form $\alpha_j\zeta_{n(j)}^s$) with an action of the torus

$$\mathbb{C}^{Q_0} \cong (\mathbb{C}^{n(j)})^* \subset \mathfrak{gl}_{n(j)}(\mathbb{C})$$

on the root system for the Cartan subalgebra

$$\mathfrak{h}_{n(j)} = \begin{pmatrix} \mathbb{C} & & & \\ & \mathbb{C} & & \\ & & \ddots & \\ & & & \mathbb{C} \end{pmatrix} \cong \mathbb{C}^{Q_1}$$

where $Q = (Q_0, Q_1)$ is a cyclic quiver (we will have one of size $n(j)$ for each cyclic extension $K_j = \mathbb{Q}(\alpha_j)$. This can of course be understood geometrically as an action on an algebraic curve

$$C(K_j) = \mathbf{Spec}\left(K_j[t]/(t^n - 1)\right)$$

which is given by a determinant polynomial from a diagonal element in $T^n \otimes K_j[t]$. Since we have this identification, we can now describe the determinants giving the characteristic polynomials,

$$L_p(s, \rho) = \frac{1}{P_p\left(\nu(p)^s\right)},$$

where $\nu(p)$ is the cardinality of the residue field at the prime p.

$$L(s, \rho) = \prod L_p(s, \rho).$$

as invariant polynomials under the action of the torus action on a quiver (here $P_p(t)$ is a characteristic polynomial defined in the beginning of the section on Artin L-functions).

13. C^*-Algebras and the Gel'fand-Naĭmark Representation Theorem

13.1. $*$-Algebras.

Definition 13.1. We will define a **Banach algebra** A to be an associative algebra over either $\mathbb{R}$, $\mathbb{C}$, or a normed complete non-Archimedean field, with a norm $||\cdot||$, such that A is a complete (linear) space with respect to $||\cdot||$, and such that

$$||xy|| \leq ||x|| \cdot ||y||.$$

If A has an involution $x \mapsto x^*$, and if $||x^*x|| = ||x||^2$, then we call the norm $||\cdot||$, $*$-**quadratic**, and we call A a C^*-**algebra**.

Example 13.2. If $\mathbb{H}$ is a separable Hilbert space with orthonormal basis $\{e_j\}_{j=1}^{\infty}$, and $\mathcal{L}(\mathbb{H})$ is the algebra of linear operators (endomorphisms) $X : \mathbb{H} \to \mathbb{H}$, then any norm closed $*$-subalgebra of $\mathcal{L}(\mathbb{H})$ is a C^*-algebra.

Example 13.3. Let X be a compact Hausdorff space, and let $C(X)$ be the complex valued continuous functions on X. Define a *commutative* C^*-algebra with unity structure by letting $f^*(x) = (f(x))^*$ be given by complex conjugation, and let $||f||_\infty = \sup\{|f(x)| : \ x \in X\}$ be the norm.

13.2. **UHF Algebras.** Let $\mathcal{L}(\mathbb{H})$ be the algebra of bounded linear operators on a separable Hilbert space $\mathbb{H}$. Consider the $*$-subalgebra $A \subset \mathcal{L}(\mathbb{H})$ given by the direct limit of the inductive system,

$$A = \varinjlim_{\phi} A_n = \varinjlim_{\phi} \mathbf{Mat}_{n\times n}(\mathbb{C}),$$

where $\phi : \mathbf{Mat}_{n\times n}(\mathbb{C}) \to \mathbf{Mat}_{(n+1)\times(n+1)}(\mathbb{C})$. We will let ϕ be the embedding given by

$$X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{Mat}_{(n+1)\times(n+1)}(\mathbb{C}).$$

We will use the notation

$$A = \mathbf{Mat}_\infty(\mathbb{C})$$

to denote this algebra.

13.3. **Compact Operators, $\ell^1(\mathbb{Z})$, and Wiener's Theorem.** The setup here draws heavily on the exposition of §0, pg.1-10 of [Berberian]. In later sections we will see that the following *"graph C^*-algebra* is associated to compact operators,

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

This directed graph represents the algebra

$$C^*(\Gamma) = \varinjlim_{\phi} \mathbf{Mat}_{n\times n}(\mathbb{C}),$$

the algebra of compact operators.

This algebra will be important for the following construction, which we will related to the direct limit of the diagonal matrices $\mathbf{Mat}_{n\times n}(\langle\zeta_{rn}\rangle) \subset \mathbf{Mat}_{n\times n}(S^1)$.

Now, let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers and assume

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

Then

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \quad t \in \mathbb{R},$$

gives a continuous, 2π-periodic, uniformly and absolutely convergent $f \in C(\mathbb{R}, \mathbb{C})$. Furthermore we have

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt, \quad n \in \mathbb{Z}.$$

Let A be the set of all such functions. Then for

$$u_n(t) = e^{int}$$

if $u = \sum_{n \in I} a_n u_n$, with $I \subset \mathbb{Z}$ finite, then $u \in A$. Additionally,

(1) $\langle u_n, u_m \rangle = \int_0^{2\pi} u_n(t) u_m(t)^* dt = 2pi\delta_{m,n}$,
(2) $u_{n+m}(t) = u_n(t) u_m(t)$,
(3) $u_{-n}(t) = \frac{1}{u_n(t)} = u_n(t)^*$.

gives an orthonormal set of functions. We call any $u = \sum_{n \in I} a_n u_n$ a **trigonometric polynomial**[18]. Now, let

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt,$$

and assume

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

Then we also have

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}.$$

Let $\ell^1(\mathbb{Z})$ be all sequences $(a_n)_{n \in \mathbb{Z}}$ such that $a_n \in \mathbb{C}$ and

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

We will treat $\ell^1(\mathbb{Z})$ as the space of functions

$$x : \mathbb{Z} \to \mathbb{C}, \quad x(n) = a_n,$$

so that

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty.$$

We have for $x, y \in \ell^1(\mathbb{Z})$ that $x = y \iff x(n) = y(n)\ \forall\ n \in \mathbb{Z}$. Define the ℓ^1-norm

$$\|x\|_1 = \sum_{n=-\infty}^{\infty} |x(n)| = \sup_I \sum_{n \in I} |x(n)|,$$

where $I \subset \mathbb{Z}$, $I \neq \emptyset$, ranges over all finite subsets of $\mathbb{Z}$. Define a metric on $\ell^1(\mathbb{Z})$ by

$$d(x, y) = \|x - y\|_1.$$

[18]We would like to show that for any $f \in A$ such that f *does not vanish*, we have $1/f \in A$. The significance of this will become clear when we study *holomorphic semigroups* and the semigroup rings of Artin L-functions in the next two sections.

Let

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n)e^{int}, \quad (t \in \mathbb{R}),$$

be the **Fourier transform**[19] of $x \in \ell^1(\mathbb{Z})$. The map

$$\ell^1(\mathbb{Z}) \to A, \quad x \mapsto \hat{x},$$

give an isomorphism of vector spaces. Next define a function $e_n \in \ell^1(\mathbb{Z})$[20] by

$$e_n(m) = \delta_{m,n}, \ \forall\, m, n \in \mathbb{Z},$$

so that

$$\hat{e_n} = u_n.$$

We have

$$c_{00}(\mathbb{Z}) := \mathbf{Span}_{\mathbb{C}}\{e_n\}_{n\in\mathbb{Z}} = \{x : \mathbb{Z} \to \mathbb{C} \mid x(n) = 0 \ a.e.\},$$

to be the span of the e_n giving functions with finite support. We have that

$$x = \sum_{n\in\mathbb{Z}} x(n)e_n,$$

and

$$\hat{x} = \sum_{n\in\mathbb{Z}} x(n)u_n.$$

Since $\hat{x}$ in this case gives a trigonometric polynomial, we get an isomorphism

$$c_{00}(\mathbb{Z}) \to T,$$

where T is the space of all trigonometric polynomials. We have that $\{e_n\}$ is an orthonormal basis for the separable Hilbert space $\ell^1(\mathbb{Z})$, and so $c_{00}(\mathbb{Z})$ is dense in $\ell^1(\mathbb{Z})$. The isomorphism

$$\hat{x} \mapsto x \in c_{00}(\mathbb{Z}),$$

then induces an algebra structure on c_{00} since $u_{m+n} = u_n u_m \in T$ gives T the structure of an algebra. The multiplication is defined by **convolution**

$$(xy)(m) = \sum_{n=-\infty}^{\infty} x(m-n)y(n).$$

Now, we have two very important results which will be generalized and applied in later sections.

---

[19]Since $t \in \mathbb{R}$ it is arbitrary as to whether we define the Fourier transform with the exponent int or $-int$. We may also restrict the Fourier transform $\hat{x}$ to S^1 so that

$$\hat{x}(\zeta) = \sum_{n=-\infty}^{\infty} x(n)\zeta^{-n}.$$

[20]One might wish to think of the e_n as the elementary operators in the direct limit

$$\mathbf{Mat}_\infty(\mathbb{Z}) = \varinjlim_{\phi} \mathbf{Mat}_{n\times n}(\mathbb{Z})$$

given by the elementary matrices $E_{n,n}$ with a 1 at the (n, n) entry and zero elsewhere. One might also wish to think of the $u_n = \hat{e_n}$ at the elementary operators in $\mathbf{Mat}_\infty(\mathbb{C})$, but with coefficient $\zeta = e^{int}$.

Theorem 13.4. *(Wiener's Theorem): If $x \in \ell^1(\mathbb{Z})$ so that*

$$\|x\|_1 = \sum_{n=-\infty}^{\infty} |x(n)| < \infty,$$

and if

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n)e^{int}, \quad (t \in \mathbb{R})$$

does not vanish, then there exists a $y \in ell^1(\mathbb{Z})$ such that

$$\frac{1}{\hat{x}(t)} = \sum_{n=-\infty}^{\infty} y(n)e^{int}.$$

Theorem 13.5. *(Gel'fand's Theorem for $\ell^1(\mathbb{Z})$): Let $\ell^1(\mathbb{Z})$ have the convolution product*

$$xy(m) = \sum_{n=-\infty}^{\infty} x(m-n)y(n)$$

giving it the structure of a unital commutative associative algebra. Suppose $f \in A$ and that f is nowhere vanishing so that $1/f \in A$ (i.e. the principle ideal $(f) = A$) by Wiener's Theorem. Let $P_t : A \to \mathbb{C}$ be the algebra epimorphism given by evaluation $g \mapsto g(t)$. Denote by

$$\mathcal{M}_t = \ker(P_t) = \{g \in A : \ g(t) = 0\},$$

the maximal ideal given by the kernel under the projection P_t (so $A/\mathcal{M}_t = \mathbb{C}$). We have then that

1. *$f \in \mathcal{M}_t$ for some $t \in \mathbb{R}$, and*
2. *every maximal ideal of A is of the form $\mathcal{M}_t$.*
3. *$M_t = \{x \in \ell^1(\mathbb{Z}) : \ \hat{x} \in \mathcal{M}_t\}$ is a maximal ideal of $\ell^1(\mathbb{Z})$ for any maximal ideal $\mathcal{M}_t$ of A.*
4. *$\ell^1(\mathbb{Z})/M_t = \mathbb{C}$.*

Now, from this, and from the description of the previous section, evalutation of the characteristic polynomials at $t = v$, as in [Milne1], defining Artin L-functions for cyclic subgroups of a Galois group $\mathcal{G}(K : \mathbb{Q})$, can then be understood in terms of the algebra $\ell^1(\mathbb{Z})$. In particular, the Hilbert series from the next section converge and by the determinant trace formula and Wiener's Theorem we get some very nice results on Artin L-functions. So our next task is to use the limit

$$\varinjlim_{\phi} \mathbf{Mat}_{n\times n}(\mathbb{C}) = \mathbf{Mat}_{\infty}(\mathbb{C}),$$

to show that the characteristic polynomials of the previous section converge to Hilbert series related to projective modules of the surface algebra (or surface order). Then, we can identify these Hilbert series with series in $\ell^1(\mathbb{Z})$ and use Wiener's Theorem to show for a nonvanishing Hilbert series $f \in A$, we have $1/f \in A$. Once we have this, we will want to understand the zeros (i.e. vanishing sets or support) of such Hilbert series. To do this we need the theory of holomorphic semigroups and semigroup rings. We also may use the fact that for a meromorphic functions f (as Artin L-functions are), away from poles we know f may be uniformly approximated by polynomials (partial sums of the power series representation), and on a punctured disk around a pole f may be uniformly approximated by rational functions (f is represented by a Laurent series on $\mathbb{D}^{\times}$). So we may use local factors of L-functions f to approximate f and understand its zeros and poles as they are uniformly approximated by zeros of polynomials and zeros and poles of rational functions.

Example 13.6. The algebra of continuous functions on S^1, $C^*(\Gamma) = C(S^1)$, can be obtained from the following graph and is the completion of the Leavitt path algebra $K[x, x^{-1}]$. This fact is proven using the *Stone-Weierstrass theorem*.

$$\begin{tikzcd} v_1 \arrow[loop above, "e"] \arrow[loop below, dotted, "e^*"] \end{tikzcd}$$

14. L-functions as Hilbert Series of Projective Resolutions of Simple Modules

15. The Theory of Holomorphic Semigroups

15.1. **Semigroup Rings of L-Functions.**

15.2. **Convergence of Semigroups.**

16. Character Groups and Locally Compact Abelian Groups

17. Leavitt Path Algebras

17.1. **Universal Localization of Path Algebras.**

17.2. **Examples.**

Example 17.1. The path algebra $KQ := K\mathbb{A}_n$, given by the quiver,

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_n \longrightarrow \cdots\cdots \bullet_{n-1} \longrightarrow \bullet_n$$

can be identified with the matrix algebra of lower triangular matrices,

$$K\mathring{A}_n \cong \begin{pmatrix} K & 0 & 0 & \cdots & 0 & 0 \\ K & K & 0 & \cdots & 0 & 0 \\ K & K & K & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K & K & K & \cdots & K & 0 \\ K & K & K & \cdots & K & K \end{pmatrix}$$

by defining the map

$$e_j \mapsto E_{j,j}, \;\; a_{j,j+1} \mapsto E_{j+1,j},$$

where $E_{i,j}$ are the matrix units in $\mathbf{Mat}_{n\times n}(K)$. The algebra of $n \times n$ matrices corresponds to the universal localization of this quiver path algebra, which has graph,

$$\begin{tikzcd} \bullet_1 \arrow[r, bend right] & \bullet_2 \arrow[l, bend right, dotted] \arrow[r, bend right] & \bullet_3 \arrow[l, bend right, dotted] & \cdots \arrow[r, bend right] & \bullet_{n-1} \arrow[l, bend right, dotted] \arrow[r, bend right] & \bullet_n \arrow[l, bend right, dotted] \end{tikzcd}$$

The map $L(Q) \to \mathbf{Mat}_{n\times n}(K)$ is then just given by extending the map of $K\mathbb{A}_n$ into $\mathfrak{n}$ (lower triangular matrices in $\mathfrak{gl}_n(K) = \mathbf{Mat}_{n\times n}(K)$), by

$$a^*_{j+1,j} \mapsto E_{j,j+1},$$

for the formal inverse arrows $a_{j+1,j}$.

Example 17.2. Let $\mathfrak{D}_3$ be the matrix algebra generated by matrices of the form

$$\begin{pmatrix} 0 & 0 & a_3 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}$$

with each $a_i \in K$ for some number field K. Let Q be the following quiver:

$$\begin{tikzcd} & v_1 \arrow[dl, "a_1"'] & \\ v_2 \arrow[rr, "a_2"'] & & v_3 \arrow[ul, "a_3"'] \end{tikzcd}$$

Then $KQ \cong \mathfrak{D}_3 \otimes_K K[x]$. To see this, define a map $f : KQ \to \mathfrak{D}_3 \otimes_K K[x]$ by:

$$\begin{aligned} v_1 &\mapsto E_{11} \otimes_K 1 & a_1 &\mapsto E_{21} \otimes 1 \\ v_2 &\mapsto E_{22} \otimes 1 & a_2 &\mapsto E_{32} \otimes 1 \\ v_3 &\mapsto E_{33} \otimes 1 & a_3 &\mapsto E_{13} \otimes x \end{aligned}$$

Where E_{ij} are the matrix units in $\mathbf{Mat}_{3\times 3}(K)$. We see $a_3a_2a_1 \mapsto E_{11} \otimes x$, and powers

$$\begin{aligned} f((a_3a_2a_1)^n) = ((E_{13}E_{32}E_{21}) \otimes x)^n &= ((E_{13}E_{32}E_{21})^n \otimes x^n \\ &= \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)^n \otimes x^n \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^n \otimes x^n \\ &= f(v_1)^n \otimes f(a_3)^n \end{aligned}$$

This map f can be defined similarly for any $n \in \mathbb{Z}_{\geq 0}$, and we have that in general $KQ \cong \mathfrak{D}_n \otimes K[x]$, where

$$\mathfrak{D}_n = \begin{pmatrix} 0 & & & & & & a_n \\ a_1 & 0 & & & & & \\ & a_2 & 0 & & & & \\ & & a_3 & 0 & & & \\ & & & \ddots & \ddots & & \\ & & & & a_{n-2} & 0 & \\ & & & & & a_{n-1} & 0 \end{pmatrix}$$

and $f(v_j) = E_{jj} \otimes 1$, $f(a_n) = E_{1,n} \otimes x$, and $f(a_j) = E_{j+1,j} \otimes 1$ for all $j \neq n$ defines the map

$$f : KQ \to \mathfrak{D}_n \otimes K[x].$$

18. Affine Lie Algebras, Characters, Witt Groups, and Fourier Transforms

Let $\mathfrak{g}_K = \mathfrak{gl}_n(K) = \mathbf{Mat}_{n\times n}(K)$ be the lie algebra of the general linear group over a field K. From the previous section we have the Leavitt path algebra,

$$\mathfrak{g}_K \otimes K[t, t^{-1}] = L(\tilde{\mathbb{A}}(n)),$$

where $\tilde{\mathbb{A}}(n)$ is the cyclic quiver with n vertices. We also had the path algebra of the cyclic quiver

$$\begin{aligned}
\mathfrak{n}_-(t) &= \mathfrak{n}_- \otimes K[t] \\
&= \begin{pmatrix} K[t] & (t) & (t) & \cdots & (t) & (t) \\ K[t] & K[t] & (t) & \cdots & (t) & (t) \\ K[t] & K[t] & K[t] & \cdots & (t) & (t) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K[t] & K[t] & K[t] & \cdots & K[t] & (t) \\ K[t] & K[t] & K[t] & \cdots & K[t] & K[t] \end{pmatrix} \\
&= \begin{pmatrix} 0 & & & & & & a_n \\ a_1 & 0 & & & & & \\ & a_2 & 0 & & & & \\ & & a_3 & 0 & & & \\ & & & \ddots & \ddots & & \\ & & & & a_{n-2} & 0 & \\ & & & & & a_{n-1} & 0 \end{pmatrix} \otimes K[t] \\
&= \mathfrak{D}_n \otimes K[t]
\end{aligned}$$

We define the affine Lie algebra $\widehat{\mathfrak{g}_K}$ as the vector space,

$$\widehat{\mathfrak{g}_K} = \mathfrak{g}_K \otimes K[t, t^{-1}] \oplus Kz$$

given by the central extension

$$c \to \widehat{\mathfrak{g}_K} \to \mathfrak{g}_K$$

with Lie bracket,

$$\gamma_1 \otimes t^n + \lambda_1 z, \gamma_2 \otimes t^m + \lambda_2 z] = [\gamma_1, \gamma_2] \otimes t^{n+m} + \langle \gamma_1 \, | \gamma_2 \rangle \cdot n\delta_{n+m,0} z,$$

with $\gamma_j \in \mathfrak{g}_K$, $\lambda_j \in K$, and $n, m \in \mathbb{Z}$, and for $\langle \cdot \, | \cdot \rangle$ is the Cartan-Killing form on $\mathfrak{g}_K$.

the complex **Witt algebra** is the Lie algebra of *meromorphic vector fields* defined on the *Riemann sphere* that are holomorphic *except at two fixed points*. It is also the complexification of the Lie algebra of *polynomial vector fields on* $S^1 \subset \mathbb{C}$, and the Lie algebra of *derivations of the ring* $\mathbb{C}[z, z^{-1}]$ of Laurent series.

The **Virasoro algebra** is a complex Lie algebra given by the unique *central extension of the Witt algebra*[21]. To define it, we let L_n, $n \in \mathbb{Z}$ be generators and let c be a **central charge**[22]. We require $[c, L_n] = 0$ and

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}.$$

[21]The Virasoro algebra is important for two-dimensional conformal field theory and string theory.

[22]Recall a **central charge** c is an operator in a Lie algebra commuting with the symmetry operators. We note in [AS1] §"*Basic Invariants of* $\mathcal{G}(\overline{\mathbb{Q}}/\mathbb{Q})$" (the absolute Galois group), see also §8.5 in Part I, we found that central elements were of the form $A_0 \cdot k[[z_1, z_2, ..., z_r]]/(z_i z_j)_{i \neq j}$, where A_0 was the commutative algebra generated by the primitive orthogonal idempotents of the surface algebra.

18.1. **Witt Groups, Witt Vectors, Witt Rings, and Isogenies.** For information on Witt groups we refer the reader to [S1] §VII.8. The Witt group of dimension n, denoted by W_n is the group of elements $(x_0, x_1, ..., x_{n-1})$ with three operations

(1) Frobenius map, $F : W_n \to W_n$, given by $(x_0, x_1, ..., x_{n-1}) \mapsto (x_0^p, x_1^p, ..., x_{n-1}^p)$.
(2) The shift $U : W_n \to W_{n+1}$, given by $(x_0, x_1, ..., x_{n-1}) \mapsto (0, x_0, x_1, ..., x_{n-1})$.
(3) Restriction, $R : W_{n+1} \to W_n$ given by $(x_0, x_1, ..., x_n) \mapsto (x_0, x_1, ..., x_{n-1})$.

We note that our definition of the "*generalized Frobenius*" which is closely related to the shift operator, we have a connection between Witt groups, Witt vectors, and their Witt rings and with Artin-Schreier Theory, via the *necklace algebra* (see [LeBruyn2] pg.18 and [Metropolis-Rota]). Furthermore, we note that isogenies as defined in [S3] §VI.2 (7)-(12) might lead one to follow one's nose to the investigation of the pullbacks defining surface orders and surface algebras. We also note, using our definition of the generalized Frobenius, along with the quadratic form of the quiver, we can interpret the character theory of interest for this article in terms of representation theory via [Milne2] pg. 226.

18.2. **The Fourier Transform.** For the Lie algebra $\mathfrak{g}_K = \mathfrak{gl}_n(K) = \mathbf{Mat}_{n\times n}(K)$ we let

$$\mathfrak{g}_K(S^1) = \mathfrak{g} \otimes C(S^1)$$

be the tensor product of $\mathfrak{g}_K$ with the algebra of smooth complex functions on the circle. It is infinite dimensional and has a Lie bracket

$$[\gamma_1 \otimes f_1, \gamma_2 \otimes f_2] = [\gamma_1, \gamma_2] \otimes f_1 f_2$$

We may define a **Fourier transform** as follows. Let f be an integrable function. Let T_f be the corresponding distribution (generalized function)

$$T_f(\phi) = \int f(x)\phi(x)dx$$

for any Schwarz function ϕ.
[Gilkey], [Su], [], [], [], []

19. Graph C^*-Algebras

In this section we build the other half of the bridge between surface algebras and surface orders to the work of Matilde Marcolli, and her collaborators. For this material we refer primarily to [Marcolli], [Consani-Marcolli1, Consani-Marcolli2], [Connes-Marcolli1, Connes-Marcolli2].

19.1. **Analytic Completion of Leavitt Path Algebras.**

19.2. **Examples.**

Example 19.1. The algebra of $n \times n$ matrices corresponds to,

$$\begin{tikzcd} \bullet_1 \arrow[r, bend right] & \bullet_2 \arrow[l, bend right, dotted] \arrow[r, bend right] & \bullet_3 \arrow[l, bend right, dotted] & \cdots & \bullet_{n-2} \arrow[r, bend right] & \bullet_{n-1} \arrow[l, bend right, dotted] \arrow[r, bend right] & \bullet_n \arrow[l, bend right, dotted] \end{tikzcd}$$

Since the completion of this algebra is just the algebra itself we have that $C^*(Q) = \mathbf{Mat}_{n\times}(\mathbb{C}) = L(Q)$.

Example 19.2. Let $\hat{Q}$ be the localized (infinite) quiver,

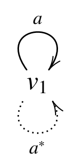
$$\begin{tikzcd} \cdots & \bullet_1 \arrow[r, bend right] & \bullet_2 \arrow[l, bend right, dotted] \arrow[r, bend right] & \bullet_3 \arrow[l, bend right, dotted] & \cdots & \bullet_{n-2} \arrow[r, bend right] & \bullet_{n-1} \arrow[l, bend right, dotted] \arrow[r, bend right] & \bullet_n \arrow[l, bend right, dotted] & \cdots \end{tikzcd}$$

This directed graph represents the algebra

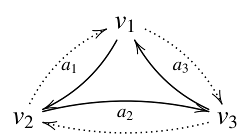
$$C^*(Q) = \bigoplus_{n=1}^{\infty} \mathbf{Mat}_{n\times n}(\mathbb{C}),$$

the algebra of compact operators.

Example 19.3. From the following graph we get the completion of the Leavitt path algebra $L(Q) = K[x, x^{-1}]$, is $C^*(Q) = C(S^1)$, the algebra continuous functions on $S^1 \subset \mathbb{C}$. This fact is proven using the Stone-Weierstrass theorem.



Example 19.4. The algebra $\mathbf{Mat}_{n\times n}(K) \otimes C(S^1)$ can be obtained as follows. Let Q be the following graph:



Let $L(Q)$ be the Leavitt path algebra for the algebra $\mathbf{Mat}_{n\times n}(K) \otimes K[x]$. Then

$$L(Q) \cong \mathbf{Mat}_{n\times n}(K) \otimes K[x, x^{-1}].$$

For this example, define the map $f : KQ \to \mathbf{Mat}_{3\times 3}(K) \otimes K[x]$ as before by:

$$\begin{aligned} v_1 &\mapsto E_{11} \otimes 1 & a_1 &\mapsto E_{21} \otimes 1 \\ v_2 &\mapsto E_{22} \otimes 1 & a_2 &\mapsto E_{32} \otimes 1 \\ v_3 &\mapsto E_{33} \otimes 1 & a_3 &\mapsto E_{13} \otimes x \end{aligned}$$

and the maps for a_i^* of

$$f : L(Q) \to \mathbf{Mat}_{n\times n}(K) \otimes K[x, x^{-1}],$$

are

$$\begin{aligned} a_1^* &\mapsto E_{12} \otimes 1 \\ a_2^* &\mapsto E_{23} \otimes 1 \\ a_3^* &\mapsto E_{31} \otimes x^{-1} \end{aligned}$$

Since $C(S^1)$ is the analytic completion of the algebra $K[x, x^{-1}]$ we have that for the algebra generated by the Cuntz-Krieger Q-family ($L(Q) \cong \mathbf{Mat}_{3\times 3} \otimes K[x, x^{-1}]$), the completion is

$$\overline{L(Q)} = C^*(Q) \cong \mathbf{Mat}_{n\times n} \otimes C(S^1)$$

as desired. We can see that the C^*-algebra obtained from a cycle on n vertices is $C^*(Q) = \mathbf{Mat}_{n\times n} \otimes C(S^1)$.

19.3. **The C^* Algebras of Cyclic Quivers $\mathbf{Mat}_{n\times n}(K) \otimes C(S^1)$ and K-theory.** So, beginning with a cyclic quiver Q, we get a path algebra $KQ \cong \mathbf{Mat}_{n\times n}(K) \otimes K[x]$. The Leavitt path algebras is then given by localization and is

$$L(Q) = \mathbf{Mat}_{n\times n}(K) \otimes K[x, x^{-1}].$$

Finally, taking the analytic completion of this algebra to obtain the graph C^*-algebra, we get

$$C^*(Q) = \mathbf{Mat}_{n\times n}(K) \otimes C(S^1).$$

Now, we should mention a few basic facts about the algebra $C^*(Q)$ and its relevance to K-theory.

20. Trace Class Operators, Fredholm Determinants, and Noncommutative Arithmetic Geometry

We now want to understand the above constructions in the limit as we extend $Q(\zeta_{rn})$ as $r \to \infty$. In particular, by our description, the products from the previous section are a kind of "layering" given by "wrapping" very large matrices up on themselves (taking an $rn \times rn$ matrix to an $n \times n$ matrix with r layers. So, we are lead to the conclusion that we should study "large matrices". In particular, we should look at the relations to the work of [Connes1], [Steinacker], and [**?**]. Thus, we turn now to some spectral theory of trace class operators on a separable Hilbert space and Fredholm determinants.

Let $\mathbb{H}$ be a separable Hilbert space and let $\mathcal{L}(\mathbb{H})$ be the space of linear operators on $\mathbb{H}$. We define a **trace class operator** $T \in \mathcal{L}(\mathbb{H})$ to be a bounded operator such that

$$\mathbf{Tr}\,|A| = \sum_{n\geq 0} \langle (A^*A)^{1/2} e_n, e_n \rangle$$

converges, i.e. $\mathbf{Tr}\,|A| < \infty$. For a trace class operator A on a separable Hilbert space $\mathbb{H}$, we define a **Fredholm determinant** of $(I - A)$ by

$$\det(I - A) = \sum_{r=0}^{\infty} \mathbf{Tr}\left(\bigwedge^{r} A \right).$$

As in [**?**, **?**] and [Steinacker]

21. Automorphic Cuspidal Representations and Generalized Cusp Forms

21.1. **The Adelic General Linear Groups.** Let $G = \mathbf{GL}_n$ be the reductive algebraic group, treated as a **functor of points**[23], or as a locally ringed space with structure sheaf $\mathcal{O}(G)$. If $L/\mathbb{Q}$ is any number field given by a finite Galois extension of $\mathbb{Q}$, we can *complete* the L-points $G(L)$ to $G(L_p) = \mathbf{GL}_n(L_p)$, the L_p-points.

Next, let

$$G_p = \mathbf{GL}_n(\mathbb{Q}_p), \quad K_p = \mathbf{GL}_n(\mathcal{O}_p),$$

with $\mathcal{O}_p = \{q \in \mathbb{Q}_p \mid |q|_p \leq 1\}$. Define the **adelic group**

$$G_{\mathbb{A}} = \mathbf{GL}_n(\mathbb{A}) = \prod_{p \text{ prime}}{}' G_p \times G_\infty.$$

[23] Recall, a **functor of points** for a scheme is a representable functor is...
For example, the functor F describing projective n-space $\mathbb{P}^n$, is given on fields K by

$$F(K) = \{\text{lines through the origin in } K^{n+1} \cong (K^{n+1} - \{0\})/K^\times = (K^{n+1} - \{0\})/\,\mathbf{GL}_1\}.$$

The product is a **restricted product**, so that for an element (infinite sequence)

$$(g_p)_{p\leq\infty} \in G_{\mathbb{A}}$$

all but finitely many $g_p \in G_p$ are in fact in K_p. We embed

$$G_{\mathbb{Q}} = \mathbf{GL}_n(\mathbb{Q}) \xrightarrow{\Delta} G_{\mathbb{A}}$$

diagonally. Similarly, define

$$\mathfrak{g}_p = \mathfrak{gl}_n(\mathbb{Q}_p), \quad \mathfrak{n}_p = \mathfrak{gl}_n(O_p),$$

and define

$$\mathfrak{g}_{\mathbb{A}} = \mathfrak{gl}_n(\mathbb{A}) = \prod_{p \text{ prime}}' \mathfrak{g}_p \times \mathfrak{g}_\infty,$$

where $\mathfrak{g}_\infty = \mathfrak{gl}_n(\mathbb{R})$, and embed $\mathfrak{gl}_n(\mathbb{Q})$ diagonally into $\mathfrak{g}_{\mathbb{A}}$. In general, we regard $G(F)$ as a discrete subgroup of $G_{\mathbb{A}}$ via the diagonal embedding.

21.2. **The Regular Representation.** For a group G, let $L^2(G)$ be the square integrable functions on G. Let $L^2_0 = L^2_0(G_{\mathbb{A}}/G_F)$ be square integrable functions satisfying the **cuspidal condition**[24]. Let $R_0 = R_0(G_{\mathbb{A}})$ be the **right regular representation** defined by,

$$R_0(g) \cdot f(h) = f(hg).$$

Recall 21.1. Let $C_0(G, \mathbb{R})$ be the continuous real valued functions with compact support, on a locally compact group G. Let $C_0(G)$ the complex valued continuous functions with compact support, and let $C(G)$ be all continuous complex valued functions on G. We define an **integral**[25] on G to be a function

$$\int_G : C_0(G, \mathbb{R}) \to \mathbb{R},$$

such that for all $f \in C_0(G, \mathbb{R})$ and $f(g) \geq 0$ for all $g \in G$ we also have that $\int_G f \geq 0$. We say the integral $\int_G$ is **left invariant** if for all $f \in C_0(G, \mathbb{R})$ and for every $g \in G$ we have

$$\int_G f(h) = \int_G f(g^{-1}h).$$

The measure dx associated to such an integral is the **left invariant Haar measure**[26]. The left invariant Haar measure is *unique up to scalar*. Thus for any such left invariant integral we get

$$\int_G f(x) = \chi(g) \int_G f(xg)$$

for $g \in G$. The function $\chi : G \to \mathbb{R}_{\geq 0}$ is a multiplicative character on G.

Example 21.2. The irreducible representations for the (compact) complex torus

$$T_0^n = \{(\lambda_1, ..., \lambda_n) : |\lambda_j| = 1\} = (S^1)^n \subset U_n(\mathbb{C}),$$

[24]The **cuspidal condition** is...

[25]See for example [Pontrjagin] Chapter IV.

[26]We are dropping the dx in the integral $\int_G f(x)dx$ and writing $\int_G f$ for brevity. For the important background material on invariant integrals, characters, and especially the applications to $\mathbf{GL}_n$, $\mathbf{SL}_n$, compact groups such as $\mathbf{SU}_n$ and finite groups we refer the reader to [Adams], [S1] Chapter 4, [Procesi] Chapter 8.

are all one-dimensional[27], and the irreducible characters form an orthonormal basis of $L^2(T^n)$ given by monomials

$$\lambda_1^{a_1}\lambda_2^{a_2}\cdots\lambda_n^{a_n}$$

and for $\lambda_j = e^{2\pi i t_j}$ we get Fourier series. Further, we have that

$$T^n = (\mathbb{C}^*)^n \cong U_1(\mathbb{C})^n \times (\mathbb{R}_{\geq 0})^n$$

where $U_1(\mathbb{C}) = S^1$, and there is a (logarithmic) isomorhpism of the multiplicative group and the additive group

$$\log : ((\mathbb{R}_{\geq 0})^n, \cdot) \to (\mathbb{R}^n, +).$$

The compact torus T_0^n is the unique maximal compact subgroup of T^n.

Now, from [Gelbart] pg. 198-200, we have

(1) Every irreducible unitary representation $(\pi, \mathbb{H})$ of $G_{\mathbb{A}}$ on a separable Hilbert space is of the form
$$\pi = \bigotimes_p \pi_p$$
with $(\pi_p, \mathbb{H}_p)$ an irreducible unitary representation of G_p.
(2) For all but finitely many p, the representation (π_p, V_p) has a fixed vector (i.e. an invariant one-dimensional subspace) under the action of $\pi_p(K_p)$. This means π_p is **unramified** for almost every p.

We also get

$$\pi(g) \cdot u_v = \prod_v \pi_v(g_v) \cdot u_v,$$

with $\pi_v(g_v) \cdot u_v \in K_v$ for almost every v.

Definition 21.3. An irreducible unitary representation $(\pi, \mathbb{H})$ of $G_{\mathbb{A}} = \mathbf{GL}_n(\mathbb{A})$ is **cuspidal automorphic** is it appear in the regular representation

$$R_0 = R_0(G_{\mathbb{A}}) \subset L_0^2 = L_0^2(G_{\mathbb{A}}).$$

At times it will be useful to restrict to representations $(\pi, \mathbb{H})$ which are invariant under the action of the center

$$Z(G_{\mathbb{A}}) = \left\{ \begin{pmatrix} z & & & \\ & z & & \\ & & \ddots & \\ & & & z \end{pmatrix}, \quad \text{with } z \in \mathbb{A} \right\}.$$

Let Z be the corresponding algebraic group so that $Z(\mathbb{A})$ are the adelic points, and note $Z \subset T^n$ is contained in the torus of $\mathbf{GL}_n$ (so that we may treat L points and L_p points with equal emphasis anytime it is useful to do so). Now, for F a number field and v a finite place of F, let $(\pi_v, \mathbb{H}_v)$ be an unramified representation of $G_v = \mathbf{GL}_n(F_v)$. Let $T^n(\mathbb{C})$ be the complex torus. Let $W_n = S_n$ be the symmetric group on $[n]$ (i.e. the Weyl group).

[27]Note that $U_n(\mathbb{C}) = \{A \in \mathbf{GL}_n(\mathbb{C}) : AA^* = 1\}$ is Zariski dense in $\mathbf{GL}_n(\mathbb{C})$, and is given by a deformation retract of $\mathbf{GL}_n(\mathbb{C})$. Moreover, $\mathbf{GL}_n(\mathbb{C})$ is diffeomorphic to $U_n(\mathbb{C}) \times \mathbb{R}^{n^2}$. Notice, in [AS2] §"Some Geometric Interpretations of The Commutative Algebra", we saw the use of deformation retracts of punctured disks to T^n as an interpretation of the gluing of continuous characters for pullbacks of matrix algebras defining the *"surface orders"*, which were completions of the surface algebras.

Part 3. A Prelude to Beilinson's Conjectures, Ergodic Geodesic Flows as Indecomposable Modules and Indecomposables in the Derived Category, and the Atiyah-Bott-Lefschetz F.P.T. for Elliptic Operators

22. Elementary K-theory, Characters, Characteristic Classes, and Vector Bundles

22.1. $K_0(A)$ of Associative Algebras (Grothendieck Groups). Let A be an associative algebra over a field K. Denote by $\mathbf{Mod}(A)$ the category of left A-modules. Let $\mathcal{F}$ be the free abelian group with basis $[M]$ the isomorphism classes of A-modules in $\mathbf{Mod}(A)$. Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence in $\mathbf{Mod}(A)$. Let $[M] - [M'] - [M'']$ be a subgroup of $\mathcal{F}'$ generated by such elements corresponding to the short exact sequences. We define the **Grothendieck group** to be

$$K_0(A) = \mathcal{F}/\mathcal{F}'.$$

Equivalently, if we let $\mathbf{Proj}(A)$ be the category of projective modules over A, and let $[P]$ denote the isomorphism class for a projective module P. Then we quotient out by all

$$[P_1 \oplus P_2] - [P_1] - [P_2]$$

and one may define $K_0(A)$ as this quotient.

22.2. $K_1(A)$ for Associative Algebras (Whitehead Groups).

22.3. Characters, Representation Rings, and Characteristic Classes for Surface Algebras.

22.4. K-Theory for Graph C^*-Algebras.

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