GRAPH C^*-ALGEBRAS AND RELATIVE TENSOR PRODUCTS

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Abstract.

Contents

1. Introduction

While trying to understand the rôle tensor products play in defining entanglement of two quantum systems it became clear to me that almost no attention has been given to what one might call a **relative tensor product** by anyone outside of a few individuals who are predominately interested in the very abstract mathematical foundations of the quantum, those studying *higher category theory* and *higher algebra*. Taking inspiration from [Grothendieck] we know that for two *affine schemes* X and Y with morphisms $f : X \to Z$ and $g : Y \to Z$ to a third scheme Z we can define a fibre product via a pullback diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

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that *glues* the schemes X and Y along the scheme Z (often a subscheme of X and Y). We then know that there is a *dual construction* on the *structure sheaves* (or coordinate rings) that involves a *relative tensor product*. In particular, the structure sheaf of the fibre product space is

$$\mathcal{O}(X) \otimes_{\mathcal{O}(Z)} \mathcal{O}(Y).$$

This is standard information in most introductions to algebraic geometry, but most often Gel'fand duality between (locally) compact Hausdorff spaces and commutative C^*-algebras (with unit if the space is compact) is encountered in first by those studying quantum theory in the context of functional analysis. Since the work of [Connes], we know this duality can be extended to noncommutative C^*-algebras thought of as the algebras of functions on some noncommutative spaces. Quantum theory is fundamentally a noncommutative theory, so this generalization of the duality between a space and some class of functions on that space from the commutative C^*-algebras to noncommutative C^*-algebras is fundamental, but the corresponding relative tensor product of two noncommutative algebras is needed when gluing noncommutative spaces along some subspace via a pullback construction. So, we need an appropriate relative tensor product for the gluing construction if we want a complete mathematical theory. Additionally, when one first studies the notion of *entanglement* one learns that entanglement is encoded via a tensor product of two algebras of operators

$$\mathcal{A}(\mathcal{H}_1) \otimes \mathcal{A}(\mathcal{H}_2).$$

on (complex) Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If the Hilbert spaces are finite dimensional (say using qubits or qudits from quantum computing), then they are just finite dimensional complex vector spaces, $\mathcal{H}_1 \cong \mathbb{C}^n$ and $\mathcal{H}_2 \cong \mathbb{C}^m$. Otherwise, they are usually assumed to be separable, implying they are isomorphic to $\ell^2(\mathbb{Z}) \cong L^2([0,1])$. Notice, in the above tensor product, no mention of what ring or field the tensor product is defined over is mentioned. This is because in the context of quantum theory, it is almost always simply assumed to be over the complex numbers $\mathbb{C}$.

In my opinion, this is an unnecessary restriction and directs attention away from a geometrically fundamental construction encoded by tensor products of algebras thought of as algebras of functions on (commutative or noncommutative) spaces. Moreover, ignoring the more general relative tensor product of algebras may be causing us to miss opportunities as reducing the computational complexity of many if not all quantum systems. In particular, it is well known among those studying quantum theory, as well as those working on neural networks that "the curse of dimensionality" is a huge obstacle when trying to actually compute things. Tensor products of n-copies of $\mathbb{C}^2$ (for n-qubits) in quantum computing, the dimension of the *"global"* Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, describing the quantum computer increases exponentially in n. In other words, as the number of qubits n increases, the dimension of the Hilbert space is 2^n, an exponential function of the number of qubits.

What if the tensor product and the algebras of operators operating on our quantum state space are not of this form, and what if other kinds of algebras of operators and their relative tensor products are required for an accurate description of quantum mechanics? How should we think of *"dimension"* of a quantum system if we use the more general construction? What kinds of algebras and tensor products should we consider? Finally, and perhaps most importantly, how should we think about entanglement between subsystems of a quantum system so that we are following natures lead and not simply developing a well rounded mathematical theory?

I will try to give a very concise answer to these questions here with some explicit examples that will hopefully be helpful to anyone working at the intersection of neural networks and quantum theory as this is likely the group of people ”in the trenches” trying to solve and the problem of error correction in quantum computing by implementing it as apposed to simply writing theoretical results about it. My personal preference is to think of everything *”locall”*, and *”patch”* or *”glue”* local things together to build a *”global”* object (such as an ideal ”isolated” quantum processor, a quantum hardware stack with ”classical” components, an arbitrary *”isolated”* quantum system, or the something like Everett’s universal wavefunction for the entire universe). This is very much informed and influenced by thinking in terms of ”ringed spaces” and sheaves of algebras or modules over a ringed space, and the construction of arbitrary schemes as fibre products of affine schemes developed by Grothendieck. The mathematics involved are well developed by those studying higher category theory and higher algebra. They also show up in things like homotopy type theory, and more recently Connes and various coauthors have introduced Grothendieck toposes into (arithmetic) noncommutative geometry. I mention this as an indicator for people who aren’t uncomfortable with such words and phrases. If toposes and homotopy type theory aren’t in your vocabulary, don’t worry!

2. What Algebras Should We Consider?

To answer this, we should first think of what kinds of spaces we want to consider. Since we want to study quantum theory, we should probably include noncommutative spaces. The simplest noncommutative spaces are directed discrete sets. These are not necessarily posets (partially ordered sets), but do include posets. The example I want to focus on here the most is a *cyclic directed graph*. We will look at a few other loosely related examples so that the reader can feel a little more familiar with the concepts and ideas. In particular, I’ll also give some examples of algebras that are given by n-loop directed graphs, and an *”equioriented linear”* directed graph. There are several reasons for focusing on the oriented cycle graph though. First, such a directed cyclic graph is used in Gerard t’Hooft’s work [t’Hooft] on *”The Cellular Automaton Interpretation of Quantum Mechanics”* in the *”cogwheel”* construction of *cyclic harmonic oscillators*. This makes it a fundamental combinatorial object for understanding quantum mechanics and quantum information. Second, it allows us to construct something called a *”groupoid”* and a Grothendieck topos, important in homotopy type theory and arithmetic noncommutative geometry being actively developed by in the work of Connes, Consani, Marcolli, and others. Third, the path algebra of such a direct graph can be completed to a graph C^*-algebra. Fourth, once we have constructed certain matrix algebras corresponding to such a cyclic graph we can glue many of them together using a fibre product analogous to the fibre product of affine schemes and this will inform our definition of relative tensor products of algebras of functions on these noncommutative spaces. This construction allows us to construct certain functions called Artin L-functions used to determine thermodynamic and information theoretic properties of the quantum system such as phase transitions, k-local Hamiltonian ground state problems, and quenching in Ising type models. Since all computations must ultimately be encoded on some substrate, and information and complexity theory tend to be the fundamental languages of quantum thermodynamics this was my initial motivation for attempting to understand the questions mentioned at the end of the introduction.

3. : Introduction

Leavitt path algebras and graph C^*-algebras are relatively new areas of study in mathematics. The two can be thought of as two sides of the same coin, Leavitt path algebras the more algebraic side, and graph C^*-algebras being the analytic side. The two have influenced each other heavily and results from one sometimes has implications for the other, but often analogous results must be proven in two very different ways for each. Leavitt path algebras can be thought of as an extension of the usual path algebra on a quiver. Quivers have been very important in representation theory, and were popularized by P. Gabriel, who classified all quivers of finite type as those quivers with underlying graphs being Dynkin diagrams of type ADE $(A_n, D_n, E_6, E_7, E_8)$. We get a quiver representation by assigning vector spaces to the vertices of the directed graph, and linear maps to the arrows. A path algebra is the free algebra on the paths in the graph. The Leavitt path algebra is an extension of the definition of the Quiver path algebra by introducing formal symbols $\{e^*\}$ for each edge e and by taking the quotient of the standard path algebra by certain relations known as the *Cuntz-Krieger relations*. For vertices we assign othogonal projections and for edges we assign partial isometries. For graph C^*-algebras, we move to the analytic version and require each vertex to be assigned a collection of mutually orthogonal subspaces of some Hilbert space $\mathcal{H}$, and to the arrows we assign partial isometries between them.

Quivers and their representation theory show up in many areas of math and physics, such as in the study of quantum groups and quantum spaces, in string theory, quantum physics, and in the theory of representations of finite dimensonsal algebras over an algebraically closed field. Leavitt path algebras can be thought of an extension of these ideas, but they can also be thought of as a generalization of the algebras constructed by W. G. Leavitt in [13]. These algebras, now denoted $L(m,n)$ and called *the classical Leavitt algebras,* are universal algebras with respect to an isomorphism property between finite rank free modules [1]. Since the graph C^*-algebras are the analytic completion of the Leavitt path algebras in an appropriate norm, they can also be viewed as an extension of quiver path algebras. They are the analytic counterparts to the Leavitt path algebras, and can be linked back to the algebras investigated by Cuntz and Krieger in [7]. The two considered what we now call the *Cuntz-Krieger algebra* O_A, which is the universal algebra generated by certain matrices over $\mathbb{Z}/2\mathbb{Z}$.

In in each case we require these spaces and maps assigned to the graphs to satisfy certain relations, and from these relations we obtain an algebra. The following is an introduction to these ideas. We begin by introducing the Leavitt path algebra as an extension of the path algebra of a directed graph, then give some examples of algebras obtained fram various kinds of graphs. We then move on to the graph C^*-algebra and we draw attention to the similarities between the Leavitt path algebra and graph C^*-algebras. Some examples of graph C^*-algebras follow. We then briefly discuss some background material on C^*-algebras in general, and introduce some of the basic K-theory of the graph C^*-algebras, with some examples of computations.

4. : Leavitt Path Algebras from Path Algebras

It has been shown that to each finite dimensional algebra over a field of characteristic zero, that there is a corresponding graphical structure, known as a quiver, and conversely, there is an associative K-algebra corresponding to quivers with certain properties, and these algebras are unital. The quiver was popularized in the seventies by P. Gabriel and this was the starting point of the modern theory of representations of associative algebras.

Definition 4.1. Let Γ be a directed graph. We denote the vertices of Γ by Γ^0, the set of edges by Γ^1. We call any vertex that only emits arrows a **source** and a vertex that only receives arrows a **sink**.

Definition 4.2. A **quiver** Q, is a pair $Q = (Q^0, Q^1)$, where Q^0 denotes a set of vertices, and Q^1 is a set of arrows connecting them. In other words, Q is a directed digraph, where we allow multiple arrows between vertices, as well as loops and cycles.

Definition 4.3. A **quiver representation** is the assignment of vector spaces K^{n_i} to each vertex $v_i \in Q^0$, and linear maps A_i to each $e_i \in Q^1$.

The set of vertices Q^0 is the set of all paths of length zero, and the set Q^1 is the set of all paths of length one.

4.1. **Path Algebras.**

Definition 4.4. Let K be a field and Γ be a graph. The *path K-algebra of* Γ is defined to be the free K-algebra $K[\Gamma^0 \cup \Gamma^1]$ with relations:

(1) $v_i v_j = \delta_{ij} v_i, \forall v_i, v_j \in \Gamma^0$

(2) $e_i = e_i r(e_i) = s(e_i) e_i, \forall e_i \in \Gamma^1$.

Where $r : Q^1 \to Q^0$ and $s : Q^1 \to Q^0$ are the range and source functions on the arrows of Q, respectively. We denote this algebra by KQ.

The basis of this algebra is the set of all paths in Q, and the multiplication of paths described above is simply concatenation of the paths if such a concatenation exists, and 0 otherwise. The elements $v_i \in Q^0$ are *mutually orthogonal idempotents*.

Claim 4.5. *If the quiver is finite, then it has an identity element, namely*

$$\sum_i v_i : v_i \in Q^0.$$

Proof. Let $v = \sum_i v_i : v_i \in Q^0$. It needs to be shown that $\omega v = \omega = v\omega$ for any path ω. From this it follows that $\Omega v = \Omega = v\Omega$, for any linear combination of paths $\Omega = \sum_i c_i \omega_i$. Let ω be a path with tail v_i and head v_j, then $v_i \omega = \omega$ and $v_j \omega = 0$ for all $j \neq i$. Thus, $v\omega = v_i\omega + \sum_{i \neq j} e_j \omega = \omega$. Similarly for ωe_i. □

Definition 4.6. A ring R has **sufficiently many idempotents**, if there exists a set of mutually orthogonal idempotents $v_i \in R$, indexed by the set I, such that for any element $r \in R$, there exists some finite subset $I' \subseteq I$ such that $(\sum_{i \in I'} v_i) r = r = r(\sum_{i \in I'} v_i)$.

In the case of quivers we let $I = Q^0$. In general, if Q is not a finite quiver, the path algebra has *sufficienlty many idempotents*. This means Q is an algebra (thus a ring) without identity. These sets of idempotents are often termed *local units*.

$K\Gamma$ is an $\mathbb{N}$-graded algebra, with grading induced by,

$$deg(v_i) = 0, \forall v_i \in E_0$$
$$deg(e_i) = 1, \forall e_i \in \Gamma_1$$

In other words, $L(\Gamma) = \bigoplus_{n \in \mathbb{N}} K\Gamma_n$, with $K\Gamma_n$ the set of all paths of length n, and linear combinations of such paths.

Definition 4.7. For a quiver with n vertices, we define the **adjacency matrix** $A_Q \in M_n(K)$ of Q to be the the matrix with element a_{ij} equal to the number of arrows from v_i to v_j.

The entry a'_{ij} in A_Q^n is the number of paths of length n from v_i to v_j.

Theorem A. Let Q be a quiver with n vertices. If A_Q^n has a nonzero entry, then the path algebra KQ is infinite.

Proof. Since the entries of A_Q^n is the number of entries from vertices v_i and v_j, and by the fact that if Q has no oriented cycles then the maximum length of a path is $n-1$, we see Q must contain an oriented cycle. Since oriented cycles allow one to obtain paths of arbitrary length, we have that KQ must be infinite. □

4.2. **The Algebra of Upper Triangular Matrices.**

Theorem B. [2] Let K be a field and Q be the following quiver:

$$v_1 \xrightarrow{\alpha} v_2 \xrightarrow{\beta} v_3$$

Then $KQ \cong UT_3(K)$, where $UT_3(K)$ is the algebra of upper triangular 3×3 matrices.

Proof. Define a map $\rho : KQ \to UT_3(K)$ be given by,

$$\sum_{i=1}^{3} a_i v_i + a_4\alpha + a_5\beta + a_6\alpha\beta \mapsto \begin{pmatrix} a_1 & a_4 & a_6 \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{pmatrix}$$

Let $x = \sum_{i=1}^{3} a_i v_i + a_4\alpha + a_5\beta + a_6\alpha\beta$ and $y = \sum_{i=1}^{3} b_i v_i + b_4\alpha + b_5\beta + b_6\alpha\beta$. Then

$$\rho(x+y) = \begin{pmatrix} a_1 & a_4 & a_6 \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{pmatrix} + \begin{pmatrix} b_1 & b_4 & b_6 \\ 0 & b_2 & b_5 \\ 0 & 0 & b_3 \end{pmatrix} = \begin{pmatrix} a_1+b_1 & a_4+b_4 & a_6+b_6 \\ 0 & a_2+b_2 & a_5+b_5 \\ 0 & 0 & a_3+b_3 \end{pmatrix} = \rho(x) + \rho(y)$$

Next,

$$\rho(x)\rho(y) = \begin{pmatrix} a_1 & a_4 & a_6 \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{pmatrix}\begin{pmatrix} b_1 & b_4 & b_6 \\ 0 & b_2 & b_5 \\ 0 & 0 & b_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_4 + a_4b_2 & a_1b_6 + a_4b_5 + a_6b_3 \\ 0 & a_2b_2 & a_2b_5 + a_5b_3 \\ 0 & 0 & a_3b_3 \end{pmatrix} = \rho(xy)$$

Next, ρ is injective, since if $\rho(x) = \rho(y)$ then the matrices are equal in each component and thus equal. Finally, ρ is onto since every element of KQ is of the form $\sum_{i=1}^{3} a_i v_i + a_4\alpha + a_5\beta + a_6\alpha\beta$, so there exists an $x \in KQ$ such that $\rho(x) = A$ for any matrix $A \in UT_3(K)$. Therefore we have an isomorphism of algebras. □

Theorem C. [2] Let Q be the following quiver:

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \cdots\cdots \bullet_{n-1} \longrightarrow \bullet_n$$

When the direction of the arrows is forgotten, the underlying graph is a Dynkin diagram of type A_n. The path algebra of Q, where the underlying graph is a Dynkin diagram of type A_n is isomorphic to $UT_n(K)$, the algebra of all upper triangular matrices of dimension n, over K, where n is the number of vertices.

Proof. Q has $\frac{n(n-1)}{2}$ non-trivial paths. Thus the size of the basis $\mathcal{B}$ for the path algebra KQ has $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ elements. Define the map

$$\rho : KQ \to UT_n(K)$$

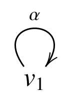
by

$$\sum_{i=1}^{n} a_i v_i + \sum_{i=n+1}^{\frac{n(n+1)}{2}} \alpha_i \mapsto \begin{pmatrix} a_1 & \cdots & a_{\frac{n(n+1)}{2}} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{pmatrix}$$

The fact that this is an algebra ismorphism follows directly from the previous theorem and induction on n, the number of vertices. $\square$

4.3. **The Ring $\langle x_1, x_2, ..., x_n \rangle$.**

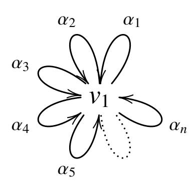
Theorem D. Let Q be the following quiver:



Then $KQ \cong K[x]$.

Proof. The isomorphism is straightforward, simply define the map $\rho : KQ \to K[x]$ to be $\sum_i c_i \alpha^i \mapsto \sum_i c_i x^i$, taking linear combinations of paths (powers of α) to the linear combination of powers of x, with the same coefficients and powers in the polynomial ring $K[x]$, and $cv_1 \mapsto cx^0$. Checking that it is an algebra homomorphism and that it is injective and surjective is left to the reader. $\square$

Theorem E. Let Q be the following quiver:



with n loops. Then the path algebra $KQ \cong K\langle x_1, x_2, ..., x_n \rangle$.

Proof. The map is again straightforward and follows by induction on n, where one takes each $\alpha_i \mapsto x_i$, giving the ring of n non-commuting variables. $\square$

4.4. **The Algebra $M' \otimes K[x]$.** Recall the definition of the tensor product of vector spaces,

Definition 4.8. The **tensor product** $V \otimes W$ of vector spaces V and W over a field K is the quotient of free product space $V \times W$, whose basis is given by the formal symbols $v \otimes w, v \in V, w \in W$, by the relations:

$$\begin{aligned} (v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w \\ \otimes(w_1 + w_2) - v \otimes w_1 - v \otimes w_2 \\ av \otimes w - a(v \otimes w) \\ v \otimes aw - a(v \otimes w) \end{aligned}$$

This definition is easily extended to algebras since algebras are simply vector spaces with an additional bilinear map $V \times V \to V; (v_1, v_2) \mapsto v_1 v_2$.

Theorem F. Let M' be the matrix algebra generated by matrices of the form

$$\begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ a_3 & 0 & 0 \end{pmatrix}$$

with each $a_i \in K$. Let Q be the following quiver:

$$\begin{tikzcd} & v_1 \arrow[dl, "\alpha"'] & \\ v_2 \arrow[rr, "\beta"'] & & v_3 \arrow[ul, "\gamma"'] \end{tikzcd}$$

Then $KQ \cong M' \otimes K[x]$.

Proof. Define a map $\rho : KQ \to M' \otimes K[x]$ by:

$$\begin{aligned} v_1 &\mapsto E_{11} \otimes 1 \qquad & \alpha &\mapsto E_{12} \otimes x \\ v_2 &\mapsto E_{22} \otimes 1 \qquad & \beta &\mapsto E_{23} \otimes 1 \\ v_3 &\mapsto E_{33} \otimes 1 \qquad & \gamma &\mapsto E_{31} \otimes 1 \end{aligned}$$

Where E_{ij} are the matrix units in $M_3(K)$. We see $\alpha\beta\gamma \mapsto E_{11} \otimes x$, and powers

$$\begin{aligned} \rho((\alpha\beta\gamma)^n) = ((E_{12}E_{23}E_{31}) \otimes x)^n &= (E_{12}E_{23}E_{31})^n \otimes x^n \\ &= \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right)^n \otimes x^n \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^n \otimes x^n \\ &= \rho(v_1)^n \otimes \rho(\alpha)^n \end{aligned}$$

□

This map ρ can be defined similarly for any $n \geq 3$, and we have that in general $K\Gamma \cong M'_n \otimes K[x]$.

4.5. **Extending the Path Algebra to Obtain the Leavitt Path Algebra.** We are interested in functors

D-Graphs → Algebras

from the category of directed graphs, or *quivers*, to the category of algebras. Since the algebras generated by graphs are universal, thus unique up to isomorphism, these functors can be constructed. A Leavitt path algebra is obtained from an extended directed graph. In essence, we simply add a 'reverse' arrow, for every arrow, going from the range of an arrow to its source. Each of these new arrows are often termed *ghost arrows* and the paths obtained from concatenation of them are known as *ghost paths*. To each arrow $e \in \Gamma^1$ we have a map α, and we thus assign a map α^* to each ghost arrow $e^* \in (\Gamma^1)^*$, satisfying certain relations. We also require, as before, that the vertices be mutually orthogonal idempotents. From this extended graph, and assignment of maps and spaces, we get a new path algebra known as the Leavitt path algebra.

More formally,

Definition 4.9. Given a graph Γ, the extended graph $\hat{\Gamma} = (\Gamma_0, \Gamma_1 \cup (\Gamma_1)^*, r', s')$ where $(\Gamma_1)^* = \{e_i^* : e_i \in \Gamma_1\}$, and the functions r' and s' are defined as

$$r'|_{\Gamma_1} = r, \quad s'|_{\Gamma_1} = s, \quad r'(e_i^*) = s(e_i), \quad s'(e_i^*) = r(e_i)$$

Definition 4.10. Let K be a field and Γ be row finite. The *Leavitt path algebra of* Γ *with coefficients in* K is defined to be the path algebra over $\hat{\Gamma}$ subject to the following relations,

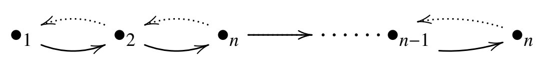
$$(CK1): \ e_i^* e_j = \delta_{ij} r(e_j), \forall e_j \in \Gamma_1, e_i^* \in (\Gamma_1)^* (CK2): \ v_i = \sum_{e_j \in \Gamma_1 : s(e_j) = v_i} e_j e_j^*, \forall v_i \in \Gamma_0, \text{ which is not a sink.}$$

The relations $(CK1)$ and $(CK2)$, are known as the *Cuntz-Krieger relations*. Note that if v_i is a sink, there is no relation $(CK2)$ at v_i. Note also that if the graph Γ is not row finite, then this relation cannot be defined. This algebra will be denoted by $L_K(\Gamma)$, or simply $L(\Gamma)$. It is exactly the path algebra $\frac{K\hat{\Gamma}}{\langle (CK1),(CK2) \rangle}$, i.e. the path algebra modulo the ideal generated by the Cuntz-Krieger realtions. This means in $K\hat{\Gamma}$, the path e^*e is a path of length two, while in $L_K(\Gamma)$ is is just $r(e)$. In the future the map $\pi_{K,\Gamma} : K\hat{\Gamma} \to L_K(\Gamma)$ will denote the quotient map.

Now, using the CK-relations, we can reduce any path in $L_K(\Gamma)$ that is the product of a ghost path and a real path. This means we can write paths in $L_K(\Gamma)$ as K-linear combinations of the form $\alpha\beta^*$, such that α and β are real paths in Γ.

4.6. **The Algebra of** $n \times n$ **Matrices.**

Theorem G. Given the following extended graph $\hat{\Gamma}$



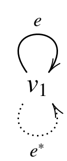
we have that the Leavitt path algebra $L(\Gamma)$ is isomorphic to $M_n(K)$, the algebra of $n \times n$ matrices over K.

Proof. Define a map ρ by

$$v_i \mapsto E_{i,i}, \quad e_i \mapsto E_{i,i+1}, \quad e_i^* \mapsto E_{i+1,i}$$

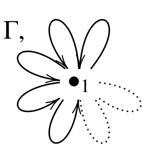
where $E_{i,j}$ are matrix units in $M_n(K)$. □

4.7. **The Ring** $[x, x^{-1}]$**.** Given $\hat{\Gamma}$:



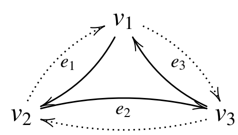
we clearly have that $L(\Gamma) \cong K[x, x^{-1}]$ via the map ρ, such that $e \mapsto x, e^* \mapsto x^{-1}, v_1 \mapsto 1$.

4.8. **L(1,n).** Given the following graph Γ,



We have that $L(\Gamma) \cong L(1, n)$, where $L(1, n)$ is the classical Leavitt algebra $(n \geq 2)$. Note, we have not drawn in the elements of the extended graph for sake of clarity.

4.9. **The Algebra $M_n \otimes [x, x^{-1}]$.**



Theorem H. Then $L(\Gamma) \cong M_n(K) \otimes K[x]$.

Proof. Define a map $\rho : KQ \to M' \otimes K[x]$ by:

$$\begin{aligned} v_1 &\mapsto E_{11} \otimes 1 & e_1 &\mapsto E_{12} \otimes x \\ v_2 &\mapsto E_{22} \otimes 1 & e_2 &\mapsto E_{23} \otimes 1 \\ v_3 &\mapsto E_{33} \otimes 1 & e_3 &\mapsto E_{31} \otimes 1 \end{aligned}$$

Where E_{ij} are the matrix units in $M_3(K)$. This implies the maps for e_i^* are:

$$\begin{aligned} e_1^* &\mapsto E_{21} \otimes x^{-1} \\ e_2^* &\mapsto E_{32} \otimes 1 \\ e_3^* &\mapsto E_{13} \otimes 1 \end{aligned}$$

□

By induction on the number of vertices, $n = |\Gamma^0|$, we get the a cycle on n vertices is in fact $M_n \otimes K[x, x^{-1}]$.

4.10. **General Properties of Leavitt Path Algebras.** Every monomial in $L(\Gamma)$ is of the form:

(1) $k_i v_i : k_i \in K, v_i \in \Gamma_0$.
(2) $ke_{i_1} \cdots e_{i_\sigma} e^*_{j_1} \cdots e^*_{j_\tau}$ with $k \in K; \sigma, \tau \geq 0, \sigma + \tau > 0, e_{i_s} \in \Gamma_1, e_{j_t} \in (\Gamma_1)^*$ for $0 \leq s \leq \sigma, 0 \leq t \leq \tau$.

Lemma 4.11. *If Γ_0 is finite then $L(\Gamma)$ is a unital K-algebra. If Γ_0 is infinite, then $L(\Gamma)$ is an algebra with local units (specifically, the set generated by finite sums of distinct elements of Γ_0).*

Proof. First, let Γ be finite. Then $(\sum_{i=1}^n v_i)v_j = \sum_{i=1}^n \delta_{ij} v_j = v_j$. Now if $e_j \in \Gamma_1$, then using equation (2) in the definition of the path algebra together with the previous computation, we get $(\sum_{i=1}^n v_i)e_j = (\sum_{i=1}^n v_i)s(e_j)e_j = s(e_j)e_j = e_j$. Similarly we see that $(\sum_{i=1}^n v_i)e_j^* = e_j^*$. Now, $L(\Gamma)$ is generated by $\Gamma_0 \cup \Gamma_1 \cup (\Gamma_1)^*$, so it must be that $(\sum_{i=1}^n v_i)\alpha = \alpha$ for an path α in $L(\Gamma)$. Also, $\alpha(\sum_{i=1}^n v_i) = \alpha$. Thus the unite is precisely $1_{L(\Gamma)} = \sum_{i=1}^n v_i$, for finite Γ.

Now suppose that Γ is not finite. Let $\{\alpha_i\}_{i=1}^t \in L(\Gamma)$ be a finite set of elements of the algebra. Then $\alpha_i = \sum_{s=1}^{n_i} k_s^i v_s^i + \sum_{l=1}^{m_i} c_l^i p_l^i$ with $k_s^i, c_l^i \in K - \{0\}$, and p_l^i are monomials of the second kind in the remark about the form of monomial in $L(\Gamma)$. Then for

$$V = \bigcup_{i=1}^{t} \{v_s^i, s(p_l^i), r(p_l^i) : s = 1, ..., n_i;\ l = 1, ..., m_i\}$$

which is a finite subgraph of $\hat{\Gamma}$, we use the exact same argument as for the finite case, and for each $V \subset \hat{\Gamma}$ we have $\iota_V = \sum_{v \in V} v$ is a finite sum of vertices and $\iota_V \alpha_i = \alpha_i \iota_V = \alpha_i$ for every i. Thus ι_V is a local unit for V. [1] □

$L(\Gamma)$ is a $\mathbb{Z}$-graded algebra, with grading induced by,

$$deg(v_i) = 0, \forall v_i \in E_0$$
$$deg(e_i) = 1, \quad deg(e_i^*) = -1, \forall e_i \in \Gamma_1, e_i^* \in (\Gamma_1)^*$$

In other words, $L(\Gamma) = \bigoplus_{n\in\mathbb{Z}} L(\Gamma)_n$, with $L(\Gamma)_0 = K\Gamma_0 + A_0, L(\Gamma)_n = A_n$ for $n \neq 0$ where,

$$A_n = \sum\{ke_{i_1} \cdots e_{i_\sigma} e^*_{j_1} \cdots e^*_{j_\tau}\}$$

where $k \in K; \sigma + \tau > 0, e_{i_s} \in \Gamma_1, e_{j_t} \in (\Gamma_1)^*, \sigma - \tau = n$. So, each real edge $e \in \Gamma^1$ has degree 1, and each ghost edge in $(\Gamma^1)^*$ has degree -1. Therefore any real path α will be homogeneous of degree $l(\alpha)$, where l denotes the *length* of the path α, and similarly $deg(\beta^*) = -deg(\beta)$ for ghost paths β. Since the Cuntz-Krieger relations give homogeneous degree 0 to any path in the ideal.

Theorem I. Let K be a field and Γ a graph. The quotient map $\pi_{K,\Gamma} : K\hat{\Gamma} \to L_K(\Gamma)$ sends vertices from Γ^0 to K-linearly independent elements of $L_K(\Gamma)$.

Proof. We build a representation of $L_K(E)$ as linear transformations on a vector space. Let $\aleph$ be some infinite cardinal at least as large as $(\Gamma^0 \sqcup \Gamma^1)$. Let X be a $\aleph$-dimensional vector space over K, and let $R = End_K(X)$. Since $\aleph \cdot |\Gamma^0| = \aleph$, we can choose a decomposition $X = \bigoplus_{v\in\Gamma^0} X_v$ with $dim(X_v) = \aleph, \forall v$. For each $v \in \Gamma^0$, let $p_v \in R$ denote the projection of X onto X_v with kernel $\bigoplus_{w\neq v} X_w$. For each $v \in \Gamma^0$ which is not a sink, we can choose a decomposition $X_v = \bigoplus_{e\in\Gamma^1, s(e)=v} Y_e$, with $dim(Y_e) = \aleph, \forall e$. For $e \in \Gamma^1$, let $q_e \in R$ denote the projection of X onto Y_e with kernel

$$(1 - p_{s(e)})X \oplus \bigoplus_{\substack{f\in\Gamma^1, f\neq e\\ s(f)=s(e)}} Y_f.$$

Finally, for each $e \in \Gamma^1$, choose $\alpha_e \in q_e R p_{r(e)}$ and $\alpha_e^* \in p_{r(e)} R q_e$ such that α_e restricts to an isomorphism $X_{r(e)} \to Y_e$ and α_e^* restricts to the inverse isomorphism.

Observe, $p_v^2 = p_v$ and $p_v p_w = 0, \forall v \neq w \in \Gamma^0$. For all $e \in \Gamma^1$ we have $\alpha_e = p_{s(e)}\alpha_e = \alpha_e p_{r(e)}$ and $\alpha_e^* = p_{s(e^*)}\alpha_e^* = \alpha_e^* p_{r(e^*)}$. Thus, there is a unique K-algebra homomorphism $\phi : K\hat{\Gamma} \to R$ such that

(1) $\phi(v) = p_v, \forall v \in \Gamma^0$
(2) $\phi(e) = \alpha_e$ and $\phi(e^*) = \alpha_e^*, \forall e \in \Gamma^1$.

We now check that the kernel of ϕ contains the CK-Relations. First, given any $e \in \Gamma^1$, we have $\alpha_e^*\alpha_f = \alpha_e^* q_e q_f \alpha_f = 0$, therefore $e^* f \in ker(\phi)$. Finally, if $v \in \Gamma^0$ is not a sink or an infinite emitter, then

$$p_v = \sum_{\substack{e\in\Gamma^1\\ s(e)=v}} q_e = \sum_{\substack{e\in\Gamma^1\\ s(e)=r}} \alpha_e \alpha_e^*,$$

Thus, $v - \sum_{\substack{e\in\Gamma^1\\ s(e)=v}} ee^* \in ker(\phi)$. Therefore, ϕ induces a unique K-algebra homomorphism $\psi : L_K(\Gamma) \to R$ such that $\psi\pi = \phi$. By construction the projections p_v for $v \in \Gamma^0$ are pair-wise orthogonal nonzero idempotents, and hence K-linearly independent elements of R. Since $\psi\pi(v) = p_v, \forall v$, we have that π maps the elements of Γ^0 to K-linearly independent elements of $L_K(\Gamma)$. [11] $\square$

5. : Graph C^*-algebras

5.1. **Completion of the Leavitt Path Algebra.** Here we are interested in the functor from graphs to C^*-algebras

D-Graphs → C^*-Algebras

Again, this functor is only constructible if the algebras generated by the graphs are unique up to isomorphism. For a proof of this fact and a universal construction of the graph C^*-algebras see [15] **chapter 2**, and [17] **section 4: Categorical aspects of graph algebras**. As we will see, the graph C^*-algebra is the analytic completion of the Leavitt path algebra. Here we will show some of the details of how the graph C^*-algebra is related to the Leavitt path algebra. It is important to note, that many results about one can be transferred to the other, or have an analogous result. One example is the process of *desingularisation* of a directed graph.

Definition 5.1. A graph Γ is row finite if the adjacency matrix of the vertices Γ^0 defined by

$$A^{\Gamma}_{i,j} = |\{e \in \Gamma^1 : \ r(e) = v_i, \ s(e) = v_j\}|$$

where $v_i, v_j \in \Gamma^1$. In other words the (i, j) entry of the matrix is the number of edges from vertex i to vertex j. This matrix is of dimension $|\Gamma^0|^2$. We say a graph Γ is **row-finite** if each row of A^Γ has a finite sum when all entries of that row are summed together.

Definition 5.2. A **partial isometry** is an operator S_e such that $S_e = S_e S_e^* S_e$. Associated to every partial isometry S is its **initial projection** $P_{s(e)} = S_e^* S_e$ and its **range projection** $P_{r(e)} = S_e S_e^*$. The partial isometry S_e maps $P_{s(e)}\mathcal{H}$ isometrically onto $P_{r(e)}\mathcal{H}$ and vanishes on $P^{\perp}_{s(e)}\mathcal{H}$. [9]

Definition 5.3. Let Γ be a row finite directed graph. A **Cuntz-Krieger Γ-family** $\{S, P\}$ on a Hilbert space $\mathcal{H}$ consists of a set $\{S_e : \ e \in \Gamma^1\}$ of partial isometries on $\mathcal{H}$, and a set $\{P_v : v \in \Gamma^0\}$ of mutually orthogonal projections on $\mathcal{H}$, such that the Cuntz-Krieger relations $(CK1)$ and $(CK2)$, which were previously defined in §2.5, hold. This gives a **representation of** Γ, by operators on the hilber space $\mathcal{H}$, analogous to the *quiver representations* previously mentioned in **definition 2.3**.

The projections P_v are mutually orthogonal, thus the ranges $P_v\mathcal{H}$ are all mutually orthogonal subspaces of the Hilbert space $\mathcal{H}$. The Cuntz-Krieger relation $(CK1)$ means that each S_e is a partial isometry with initial space $P_{s(e)}\mathcal{H}$. $(CK2)$ says that the range projection $S_e S_e^*$ is dominated by $P_{r(e)}$. Additionally, we see $S_e\mathcal{H} \subset P_{r(e)}\mathcal{H}$, and S_e is an isometry f $P_{s(e)}\mathcal{H}$ onto a closed subspace of $P_{r(e)}\mathcal{H}$., and we have the following relation

$$S_e = P_{r(e)} S_e = S_e P_{s(e)}.$$

By the relation $(CK2)$ we have

$$P_v\mathcal{H} = \bigoplus_{e \in \Gamma^1 : \ r(e) = v} S_e\mathcal{H}.$$

Further we have that the Hilbert space $\mathcal{H} = \bigoplus_{v \in \Gamma^0} P_v\mathcal{H}$ due to the fact that [15]

Theorem J. For any row-finite directed graph Γ, there is a C^*-algebra $C(\Gamma)$ generated by a Cuntz-Krieger Γ-family $\{s, p\}$ such that for every Cuntz-Krieger Γ-family $\{T, Q\}$ in a C^*-algebra A, there is a homomorphism $\pi_{T,Q}$ of $C^*(\Gamma)$ into A satisfying $\pi_{T,Q}(s_e) = T_e$ for every $e \in \Gamma^1$ and $\pi_{T,Q}(p_v) = Q_v$ for every $v \in \Gamma^0$.

Proof. For every Cuntz-Krieger family $\{S, P\}$ on $\mathcal{H}$, the operators $\{S_\mu S_\nu^*\}$ satisfy the relations imposed on the $\{d_{\mu,\nu}$, and hence there is a $*$-representation $\pi_{S,P}$ of V, on $\mathcal{H}$ such that $\pi_{S,P}(d_{\mu,\nu}) = S)\mu S_\nu^*$. Since the norm of a projection P satisfies $\|P\|^2 = \|P^*P\| = \|P\|$, every non-zero projection has norm 1, and thus for every non-zero partial isometry W, we have $\|W\| = \|W^*W\| = 1$. Thus,

$$\|\pi_{S,P}(\sum z_{\mu,\nu} d_{\mu,\nu})\| \leq \sum |z_{\mu,\nu}| \, \|S_\mu S_\nu^*\| \leq \sum |z_{\mu,\nu}|.$$

It follows that

$$\|a\|_1 := \sup\{\|\pi_{S,p}(a)\| : \{S, P\} \text{ is a Cuntz-Krieger } \Gamma\text{-family }\}$$

is finite for every $v \in V$, and $\|\cdot\|_1$ is an algebra seminorm satisfying $\|a^*a\|_1 = \|a\|_1^2$. Let I be the $*$-ideal $\{u \in V : \|u\|_1 = 0\}$. Then $V_0 = V/I$ is a $*$-algebra, and the quotient norm $\|\cdot\|_0$ defined by $\|v + I\|_0 = \inf\{\|u + j\|_1 : j \in I\}$ is a C^*-norm, so the completion $\overline{V}_0$ is a C^*-algebra. Each $\pi_{S,P}$ is $\|\cdot\|_0$-continuous, and hence extends uniquely to a representation of $\overline{V}_0$. Now, take $C^*(\Gamma) = \overline{V}_0$ and check that $s_e := d_{e,s(e)}$, $p_v := d_{v,v}$ form a Cuntz-Krieger Γ-family which generates V_0. To get $\pi_{T,Q}$, choose a faithful representation $\rho : A \to B(\mathbb{H})$, and take $\pi_{T,Q} = \rho^{-1} \circ \pi_{\rho(T),\rho(Q)}$. [15] $\square$

5.2. **The Algebra of $n \times n$ Matrices.**

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_n \longrightarrow \cdots\cdots \bullet_{n-1} \longrightarrow \bullet_n$$

Since the completion of this algebra is just the algebra itself we have that $C^*(\Gamma) = M_n(\mathbb{C})$.

5.3. **The Algebra of Compact Operators.**

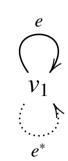
$$\cdots\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots\cdots$$

This directed graph represents the algebra

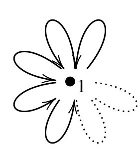
$$C^*(\Gamma) = \bigoplus_{i=1}^{\infty} M_i(\mathbb{C}),$$

the algebra of compact operators.

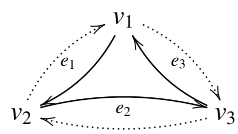
5.4. **The Algebra Continuous Functions on S^1.** From this graph we get the completion of the Leavitt path algebra $K[x, x^{-1}]$, which gives $C^*(\Gamma) = C(S^1)$. This fact is proven using the Stone-Weierstrass theorem.



5.5. **The Cuntz Algebras $\mathcal{O}_n$.** In this case $C^*(\Gamma) = \mathcal{O}_n$, the classical Cuntz algebras. When $n = \infty$ we get $\mathcal{O}_\infty$.



5.6. **The Algebra $M_n \otimes C(S^1)$.** Let Γ be the following graph:



Theorem K. $L(\Gamma) \cong M_n(K) \otimes K[x]$.

Proof. As in the proof of the algebra $M_3 \otimes K[x, x^{-1}$ we define a map $\rho : KQ \to M' \otimes K[x]$ by:

$$\begin{aligned} v_1 &\mapsto E_{11} \otimes 1 & e_1 &\mapsto E_{12} \otimes x \\ v_2 &\mapsto E_{22} \otimes 1 & e_2 &\mapsto E_{23} \otimes 1 \\ v_3 &\mapsto E_{33} \otimes 1 & e_3 &\mapsto E_{31} \otimes 1 \end{aligned}$$

Where E_{ij} are the matrix units in $M_3(K)$. This implies the maps for e_i^* are:

$$\begin{aligned} e_1^* &\mapsto E_{21} \otimes x^{-1} \\ e_2^* &\mapsto E_{32} \otimes 1 \\ e_3^* &\mapsto E_{13} \otimes 1 \end{aligned}$$

Since $C(S^1)$ is the analytic completion of the algebra $K[x, x^{-1}]$ as shown in *Example 3.4*, we have that the completion of the algebra generated by the Cuntz-Krieger Γ-family is $M_n \otimes K[x, x^{-1}]$, as it was in the Leavitt path algebra $L_K(\Gamma)$, and the completion $\overline{L_K(\Gamma)}$ is $M_n \otimes C(S^1)$ as desired. $\square$

Via an induction argument on $n = |\Gamma^0|$ we can see that the algebra obtained from a cycle on n vertices is indeed $M_n \otimes C(S^1)$.

6. Computation of K-groups of a Graph Algebra

In this section we are interested in functors

Graphs $\to$ Algebras $\to$ Abelian Groups

We are associating to directed graphs an algebra (either the Leavitt path algebra, or the graph C^*-algebra), then to these algebras we are associating the two abelian groups K_0 and K_1. The K-groups $K_0(\Gamma)$ and $K_1(\Gamma)$ of a graph algebra $K\Gamma$ can be computed using only the properties of the graph Γ. These groups are very useful and provide a lot of information about the graph algebras. Here we remind the reader of the notion of a presentation matrix of a module. We then give a general formula for computing $K_0(\Gamma)$ and $K_1(\Gamma)$ of graphs with sources and countably many vertices found in **Proposition 2** [18], and apply the formula in the computation of several familiar examples of graphs, as well as some we have yet to present.

The association of K-groups to graph C^*-algebras is a functor **Graph**C^* $\to$ **AbGrp** from the category of graph C^*-algebras to the category of abelian groups. If two C^*-algebras are isomorphic, then they have the same K-groups. If two C^*-algebras have different K-groups, then they cannot be isomorphic. For a more thorough discussion of the K-theory of graph algebras and C^*-algebras in general, we refer the reader to [15] and [14].

6.1. **Modules and Presentations.** Recall,

Definition 6.1. A **left R-module** M, over a ring R is an abelian group such that

(1) $r(m_1 + m_2) = rm_1 + rm_2, \forall r \in R, m_1, m_2 \in M$
(2) $m(r + s) = mr + ms, \forall r, s \in R, m \in M$
(3) $m(rs) = (mr)s, \forall r, s \in R, m \in M$
(4) $m1_R = m$

Definition 6.2. We call M a **free module** if it has a finite basis $\mathcal{B} = \{m_1, m_2, ..., m_k\}$, i.e. $M = R\mathcal{B}$.

Any finitely generated R-module M, over a principal ideal domain is particularly nice, and many theorems from linear algebra have analogues in the theory of modules over a P.I.D. For our purposes, these are the only modules we will deal with in this section. In particular we will deal with the category of free $\mathbb{Z}$-modules, which is the same as the category of abelian groups. The free abelian groups/free $\mathbb{Z}$-modules will be the K-groups of graph C^*-algebras.

Definition 6.3. A **module homomorphism** $\phi : M \to N$ is a map of (left) R-modules such that

(1) $\phi(rm) = r\phi(m)$, and
(2) $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$

Definition 6.4. If $\phi : R^n \to R^m$ is a left R-module homomorphism given by right multiplication by some matrix A_ϕ, and the image of ϕ is denoted $R^n A_\phi$, then the quotient module $M = \mathbb{R}^m/R^n A_\phi$ is called **module presented by** A_ϕ, and A_ϕ is called the **presentation matrix** of the quotient module $R^m/R^n A_\phi$.

(1) The $\mathbb{Z}$-module $\mathbb{Z}/k\mathbb{Z}$ is presented by

$$\mathbb{Z} \xrightarrow{(k)} \mathbb{Z}$$

(2) The module presented by

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}$$

is isomorphic to $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$.

[6]

6.2. **The Map Δ_Γ.**

Definition 6.5. Let Γ be a directed graph, and Γ^0 the set of vertices. By V_Γ we denote the set of all vertices which emit at least one, but only finitely many vertices. We then denote the free $\mathbb{Z}$-module generated by V_Γ by $\mathbb{Z}V_\Gamma$, and the free $\mathbb{Z}$-module generated by Γ^0 by $\mathbb{Z}\Gamma^0$.

Definition 6.6. Define $\Delta_\Gamma : V_\Gamma \to \Gamma^0$ by the following formula:

$$\Delta_\Gamma(v) = \left(\sum_{e \in \Gamma^1 : s(e) = v} r(e) \right) - v$$

Computing this for each generator of $\mathbb{Z}V_\Gamma$ gives a set of elements in $\mathbb{Z}\Gamma^0$ which spans the image of Δ_Γ. The Codomain mod this submodule, i.e. $\mathbb{Z}\Gamma^0/im(\Delta_\Gamma)$ is then the module presented by a matrix obtained by coordinatizing the generators of $\mathbb{Z}V_\Gamma$. We then have for countable directed graphs Γ with finitely many vertices, such that V_Γ is the set of vertices in Γ which emit at least, but only finitely many edges, that the K-group $K_0(C^*(\Gamma)) \cong \mathbb{Z}\Gamma^0/im(\Delta_\Gamma) = coker(\Delta_\Gamma)$, and the K-group $K_1(C^*(\Gamma)) \cong ker(\Delta_\Gamma)$. A proof of this result is given in **Proposition 2** [18].

6.3. $M_n(K)$**.** Let Γ be the following directed graph:

$$v_1 \xrightarrow{e} v_2$$

We already know that the Leavitt path algebra is just $M_2(K)$. We have $\mathbb{Z}\Gamma^0 = \langle v_1, v_2 \rangle$ and $\mathbb{Z}V_\Gamma = \langle v_1 \rangle$. Thus,

$$\Delta_\Gamma(v_1) = \left(\sum_{e \in \Gamma^1 : s(e) = v_1} r(e) \right) - v_1 = v_2 - v_1$$

Coordinatizing in the free $\mathbb{Z}$-modules $\mathbb{Z}\Gamma^0$ and $\mathbb{Z}V_\Gamma$ this equation can be written as,

$$\Delta_\Gamma((1)) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus we have a presentation matrix

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}$$

and the module presented by $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is $\mathbb{Z} \oplus \mathbb{Z}/(-1, 1)\mathbb{Z} \cong \mathbb{Z}$, and thus the cokernel is isomorphic to $\mathbb{Z}$, and so $K_0(C^*(\Gamma)) \cong \mathbb{Z}$ as well. The kernel of the presentation matrix is just 0, thus $K_1(C^*(\Gamma)) = 0$.

Similarly let Γ be

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3$$

We now have the free $\mathbb{Z}$-modules $\mathbb{Z}\Gamma^0 = \langle v_1, v_2, v_3 \rangle$, and $\mathbb{Z}V_\Gamma = \langle v_1, v_2 \rangle$. Thus we must compute the map Δ_Γ on the generators v_1 and v_2 of $\mathbb{Z}V_\Gamma$.

$$\Delta_\Gamma(v_1) = \left(\sum_{e \in \Gamma^1 : s(e) = v_1} r(e) \right) - v_1 = v_2 - v_1$$

and

$$\Delta_\Gamma(v_2) = \left(\sum_{e \in \Gamma^1 : s(e) = v_2} r(e) \right) - v_2 = v_3 - v_2$$

Again, we coordinatize and get the two equations,

$$\Delta_\Gamma((1, 0)) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Delta_\Gamma((0, 1)) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

and we obtain a presentation matrix

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}} \mathbb{Z}^3$$

Now, $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = A'$, and thus the module presented by A is isomorphic to $\mathbb{Z}^3/((1,0,0)\mathbb{Z} \oplus (0,1,0)\mathbb{Z}) \cong \mathbb{Z} \cong K_0(C^*(\Gamma))$. Again, $ker(\Delta_\Gamma) \cong ker(A) \cong ker(A') = 0 \implies K_1(C^*(\Gamma)) = 0$.

It's clear now by an inductive argument on the number of vertices in Γ of this form, that we have Δ_Γ is presented by the $n \times n-1 = |\Gamma^0| \times |V_\Gamma|$ matrix

$$\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Which by similarity of matrices gives an isomorphic module presentation:

$$\mathbb{Z}^{n-1} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}} \mathbb{Z}^n$$

where $coker(\Delta_\Gamma) \cong \mathbb{Z}^n/\mathbb{Z}^{n-1} \cong \mathbb{Z} \cong K_0(C^*(\Gamma))$ and $ker(\Delta_\Gamma) = 0 = K_1(C^*(\Gamma))$. For such Γ the map Δ_Γ is always injective, and thus $K_1(C^*(\Gamma))$ is always trivial, and the image is always isomorphic to $\mathbb{Z}^{n-1}$, thus we always have $K_0(C^*(\Gamma)) \cong \mathbb{Z}\Gamma^0/im(\Delta_\Gamma) \cong \mathbb{Z}$.

6.4. $A \oplus M_n(K)$.

6.5. $C(S^1)$ **and** O_n. For the graph Γ:

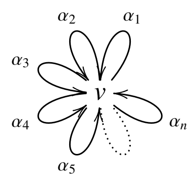
$$\begin{tikzcd} v \arrow[loop above, "\alpha"] \end{tikzcd}$$

We see $\Delta_\Gamma(v) = v - v = 0$, thus we have a presentation

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 \end{pmatrix}} \mathbb{Z}$$

This means $K_0(C^*(\Gamma)) \cong \mathbb{Z}\Gamma^0/im(\Delta_\Gamma) \cong \mathbb{Z}$, and $K_1(C^*(\Gamma)) \cong ker(\Delta_\Gamma) = \mathbb{Z}$.

Now let Γ be



Then we have

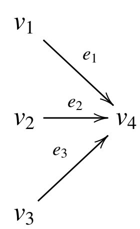
$$\Delta_\Gamma(v) = \left(\sum_{e \in \Gamma^1 : s(e) = v} r(e) \right) = nv - v = (n-1)v$$

Thus we have a presentation

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} n \end{pmatrix}} \mathbb{Z}$$

and we get $im(\Delta_\Gamma(v)) = n\mathbb{Z} \implies K_0(C^*(\Gamma)) \cong coker(\Delta_\Gamma) = \mathbb{Z}\Gamma^0/\mathbb{Z}V_\Gamma = \mathbb{Z}/n\mathbb{Z}$, and $K_1(C^*(\Gamma)) \cong ker(\Delta_\Gamma) = 0$. We know from §3.5 that the algebra $C^*(\Gamma) = \mathcal{O}_n$, the Cuntz algebra.

6.6. **The n-clock Receiver.** Let Γ be the following graph:



Then coordinatizing we have $\mathbb{Z}\Gamma^0 \cong \mathbb{Z}^4$ and $\mathbb{Z}V_\Gamma \cong \mathbb{Z}^3$, and

$$\Delta_\Gamma(v_1) = \Delta_\Gamma\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Similarly

$$\Delta_\Gamma(v_2) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \Delta_\Gamma(v_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

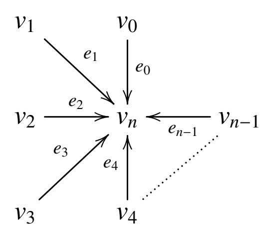
Thus we have a presentation matrix and a presentation

$$\mathbb{Z}^3 \xrightarrow{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}} \mathbb{Z}^4$$

which is isomorphic to the presentation

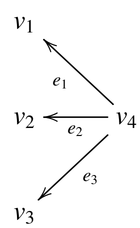
$$\mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} \mathbb{Z}^4$$

and thus we have $K_0(C^*(\Gamma)) \cong coker(\Delta_\Gamma)/im(\Delta_\Gamma) \cong \mathbb{Z}^4/\mathbb{Z}^3 \cong \mathbb{Z}$, and $K_1(C^*(\Gamma)) \cong ker(\Delta_\Gamma) = 0$. Thus we deduce for the graph Γ:



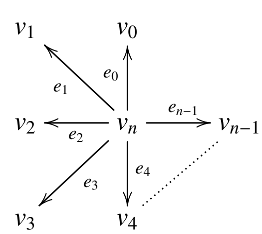
that $K_0(C^*(\Gamma)) \cong \mathbb{Z}$ and $K_1(C^*(\Gamma)) = 0$.

6.7. **The *n*-clock Emitter.** Suppose now that Γ is the following graph:



Then $\mathbb{Z}V_\Gamma = \langle v_4 \rangle \cong \mathbb{Z}$ and $\mathbb{Z}\Gamma^0 = \langle v_1, v_2, v_3, v_4 \rangle \cong \mathbb{Z}^4$. Then $\Delta_\Gamma(v_4) = v_1 + v_2 + v_3 - v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

So, $K_0(C^*(\Gamma)) \cong \mathbb{Z}^4/\mathbb{Z} \cong \mathbb{Z}^3$ and $K_1(C^*(\Gamma)) \cong ker(\Delta_\Gamma)0$. Then by induction on $|\Gamma^0 - V_\Gamma| = n - 1$ we have $K_0(C^*(\Gamma)) \cong \mathbb{Z}^{n-1}$ and $K_1(C^*(\Gamma)) = 0$, for the n-clock emitter Γ:



6.8. **The Non-commutative Sphere $C(S_q^7)$.** Let Γ be the following graph:

$$\begin{tikzcd}
v_1 \arrow[loop above, "e_1"] & v_2 \arrow[l, "e_5"'] \arrow[loop above, "e_2"] & v_3 \arrow[l, "e_6"'] \arrow[loop above, "e_3"] \arrow[ll, bend left, "e_{10}"] & v_4 \arrow[l, "e_7"'] \arrow[loop above, "e_4"] \arrow[ll, bend left, "e_9"] \arrow[lll, bend left=50, "e_8"]
\end{tikzcd}$$

Then $\mathbb{Z}V_\Gamma = \langle v_1, v_2, v_3, v_4 \rangle \cong \mathbb{Z}^4$ and $\mathbb{Z}\Gamma^0 = \langle v_1, v_2, v_3, v_4 \rangle \cong \mathbb{Z}^4$. Now, we give the values of Δ_Γ on each generator of $\mathbb{Z}V_\Gamma$:

$$\Delta_\Gamma(v_1) = v_1 - v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Delta_\Gamma(v_2) = (v_2 + v_1) - v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Delta_\Gamma(v_3) = (v_3 + v_2 + v_1) - v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Delta_\Gamma(v_3) = (v_4 + v_3 + v_2 + v_1) - v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Which gives a presentation matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Which gives a presentation isomorphic to

$$\mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \mathbb{Z}^4$$

Thus $K_0(C^*(\Gamma)) \cong coker(\Delta_\Gamma)/im(\Delta_\Gamma \cong \mathbb{Z}^4/\mathbb{Z}^3 \cong \mathbb{Z}$ and $K_1(C^*(\Gamma)) \cong ker(\Delta_\Gamma) \cong \mathbb{Z}$, and we have computed the K-groups of the non-commutative sphere $C(S_q^7)$. The graph C^*-algebra of this graph was proven in [12] to be the non-commutative sphere $C(S_q^7)$ in **Theorem 4.4**.

7. Drinen-Tomforde Desingularisation of Graphs

The following is an explanation of the desingularization process for graph algebras that are not row finite. The process for Leavitt Path Algebras is the same as the process for graph C^*-algebras. In the case of Leavitt path algebras, the desingularization yields a new graph which gives a new row finite Leavitt path algebra which is Morita equivalent to the original. In the graph C^*-algebra case we can extend results of row finite graphs to graphs that are not row finite using the desingularization process.

Thus far we have only dealt with examples of graphs which are row finite. The problem we run into when considering graphs that are not row finite is of course the relation $(CK2)$:

$$P_v = \sum_{e \in \Gamma^1:\ r(e) = v} S_e S_e^*$$

where $r^{-1}(v)$ is infinite, i.e. v receives infinitely many edges. To resolve this issue we define

Definition 7.1. [8] If Γ is a countable graph a **desingularization** of Γ is a graph G obtained from Γ with no singular vertices. The graph G is obtained by adding a tail to each sink and infinite emitter in Γ^0. If v is a sink then we attach an infinite line:

$$\begin{tikzcd} v \arrow[r] & v_1 \arrow[r] & v_2 \arrow[r] & \cdots \end{tikzcd}$$

If v is an infinite emitter then we first list the edges $e_1, e_2, ... \in s^{-1}(v_0)$ which can be done due to the countability assumption on Γ. Then we attach an infinite line at v:

$$\begin{tikzcd} v \arrow[r, "f_1"] & v_1 \arrow[r, "f_2"] & v_2 \arrow[r, "f_3"] & \cdots \end{tikzcd}$$

We now remove the edges $e_i \in s^{-1}(v)$, and add an edge g_j from v_{j-1} in the newly constructed infinite line, to $r(e_j)$ for every $e_j \in s^{-1}(v)$. We are thus removing each e_j and replacing it with a path $f_1 f_2 \cdots f_{j-1} g_j$ of length j, which has the same source and range as e_j.

$$\begin{tikzcd} u \arrow[r, "\infty"] & v \end{tikzcd}$$

Gives the desingularization:

$$\begin{tikzcd} u \arrow[r] \arrow[d] & u_1 \arrow[r] \arrow[dl] & u_2 \arrow[r] \arrow[dll] & \cdots \\ v \arrow[r] & v_1 \arrow[r] & v_2 \arrow[r] & \cdots \end{tikzcd}$$

An alternate construction is given by [15] in the following way:

Definition 7.2. Let Γ be a directed graph. An infinite path $e \in \Gamma^\infty$ is collapsible if e has no exits except at $r(e)$, $r^{-1}(r(e_i))$ is finite for ever $e_i \in e$, and $r^{-1}(r(e)) = \{e_1\}$.

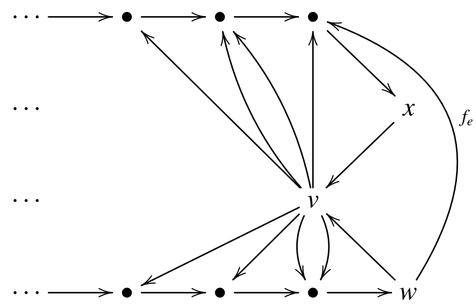
We collapse the infinite path e to a single vertex.

Definition 7.3. A **Drinen-Tomforde desingularization** of a graph Γ is a row finite graph G with a collection M of disoint collapsible paths such that upon collapsing these paths yield the graph Γ. Every graph has a Drinen-Tomforde desingularization via adding a head e to each infinite receiver v in Γ, and replace each edge $e \in \Gamma^1$ with $r(e) = v$ by an edge $f_e \in G$ with $s_G(f_e) = s(e)$ and range some $s(e_i)$ in such a way that each $s(e_i)$ receives only finitely many of these new edges. The new graph G will be row finite, each new path will be collapsible, and collapsing them recovers the original graph Γ.

Let Γ be the following graph:

$$\begin{tikzcd} & v_1 \arrow[dl, bend left] \\ v_2 \arrow[ur, bend left, "\infty"] \arrow[dr, bend left, "\infty"] & \\ & v_3 \arrow[ul, bend left] \arrow[uu, "e"'] \end{tikzcd}$$

A desingularization might look like



8. Relative Tensor Products of Algebras

Say X and Y are two (quantum) noncommutative spaces that we wish to glue along a space Z via a pullback construction. This should geometrically encode the idea of entanglement. Algebraically we encode this with a relative tensor product of noncommutative algebras of functions on each of these three spaces. Naively we might think of the following idea: Tensor products of $C(X) \otimes_{C(Z)} C(Y)$ matrix algebras over $C(X) \otimes_{C(Z)} C(Y)$, or tensor products of $C(X \times_Z Y)$-algebras of the form $M_n(C(X)) \otimes_R M_n(C(Y))$ where $R \subseteq M_n(C(X))$ and $R \subseteq M_n(C(Y))$. We might allow R to change depending on our notion of entanglement. We might also consider tensor products of $C(X \times_Z Y)$-bundles.

9. Appendix

9.1. Associative Algebras and C^*-algebras.

Definition 9.1. An **associative algebra** is a vector space A with a bilinear map $A \times A \to A$ such that $(a, b) \mapsto ab$ and $a(bc) = (ab)c$. A **subalgebra** B is a vector subspace such that $b, b' \in B \implies bb' \in B$. An alternate definition is, if R is an associative ring and K a field, then R is a finitely generated K-algebra if if R as a ring is generated by K together with a finite set $X = \{r_1, r_2, ..., r_n\}$ of elements of R. If R is a K-algebra, then it is a ring and a vector space over K. Although R is finitely generated, it may be infinite dimensional, for example if $R = K[x_1, x_2, ..., x_n]$ is the polynomial ring in n variable over K, it is finitely generated, but an infinite dimensional K-vector space.

Definition 9.2. A **normed algebra** is an Algebra with a norm $(A, \|\cdot\|)$ such that $\|ab\| \leq \|a\| \, |b\|$ for all $a, b \in A$, i.e. the norm is *submultiplicative*. If A has a unit $1_A \in A |\, a1_A = 1_A a = a, \forall a \in A$, then A is called a **unital normed algebra**.

Definition 9.3. A **Banach algebra** is a complete normed algebra.

Since $\|ab - a'b'\| \leq \|a\| \, \|b - b'\| + \|a - a'\| \, \|b'\|$, we have that the multiplication in A is continuous in both a and b in the product ab.

A closed subalgebra of a Banach algebra is a Banach algebra.

Definition 9.4. An **involution** on an algebra A is a conjugate-linear map $a \mapsto a^*$ on A such that $a^{**} = a$ and $(ab)^* = b^* a^*$ for all $a, b \in A$. The pair $(A, *)$ is called an **involutive algebra**, or a **$*$-algebra**.

Definition 9.5. An element $p \in A$ is a **projection** if $p = p^* = p^2$.

Definition 9.6. An element $u \in A$ is **unitary** if $uu^* = u^*u = 1$. If $u^*u = 1$ then u is an **isometry** and if $uu^* = 1$ then u is a **co-isometry**.

9.2. **Elementary Spectral Theory.** [14]

9.2.1. *Exercise 1.1 (a).* Let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of Banach algebras. The *direct sum* $A = \bigoplus_\lambda A_\lambda$ is the set of all $(a_\lambda) \in \Pi_\lambda A_\lambda$ such that $\|(a_\lambda)\| = \sup_\lambda \|a_\lambda\| < \infty$. A is a Banach algebra under the pointwise defined operations:

$$\begin{aligned}(a_\lambda) + (b_\lambda) &= (a_\lambda + b_\lambda)\\ \mu(_\lambda) &= (\mu a_\lambda)\\ (a_\lambda)(b_\lambda) &= (a_\lambda b_\lambda)\end{aligned}$$

and norm given by $(a_\lambda) \mapsto \|(a_\lambda)\|$. If for all $\lambda \in \Lambda$, A_λ is unital or abelian, then so is A.

Proof. Verifying the vector space axioms is straightforward given the properties above.

(1)

$$\begin{aligned}((a_\lambda) + (b_\lambda)) + (c_\lambda) &= (a_\lambda + b_\lambda) + (c_\lambda)\\ &= (a_\lambda + b_\lambda + c_\lambda)\\ &= (a_\lambda) + (b_\lambda + c_\lambda)\\ &= (a_\lambda) + ((b_\lambda) + (c_\lambda))\end{aligned}$$

(2) $(a_\lambda) + (b_\lambda) = (a_\lambda + b_\lambda) = (b_\lambda + a_\lambda) = (b_\lambda) + (a_\lambda)$
(3) $0_A = (0_\lambda)$
(4) $-(a_\lambda) = (-a_\lambda)$
(5) $\gamma(\mu a_\lambda) = \gamma\mu(a_\lambda)$
(6) For $1 \in \mathbf{F}$ we have $1(a_\lambda) = (1a_\lambda) = (a_\lambda)$
(7) $\mu((a_\lambda) + (b_\lambda)) = \mu(a_\lambda + b_\lambda) = (\mu(a_\lambda + b_\lambda)) = (\mu a_\lambda + \mu b_\lambda) = (\mu a_\lambda) + (\mu b_\lambda) = \mu(a_\lambda + \mu(b_\lambda)$.
(8) $(\mu + \gamma)(a_\lambda) = ((\mu + \gamma)a_\lambda) = (\mu a_\lambda + \gamma a_\lambda) = (\mu a_\lambda) + (\gamma a_\lambda) = \mu(a_\lambda) + \gamma(a_\lambda)$

Now, let $a_{\lambda,n} \to a_\lambda$ be a Cauchy sequence in each A_λ. Since A_λ is Banach for all $\lambda \in \Lambda$, we have that $a_\lambda \in A_\lambda, \forall\lambda \in \Lambda$. For each λ and for any $\epsilon > 0$ there is an $N_\lambda \in \mathbb{N} :\ \forall n, m \geq N$, $\|a_{\lambda,n} - a_{\lambda,m}\| < \epsilon/2$ and $\|a_{\lambda,m} - a_\lambda\| < \epsilon/2$. Choose $N = \max\{N_\lambda\}$. Then

$$\begin{aligned}\|(a_\lambda)_n - (a_\lambda)\| &= \sup_\lambda \|a_{\lambda,n} - a_\lambda\|\\ &= \sup_\lambda \|a_{\lambda,n} - a_{\lambda,m} + a_{\lambda,m} - a_\lambda\|\\ &\leq \sup_\lambda \|a_{\lambda,n} - a_{\lambda,m}\| + \|a_{\lambda,m} - a_\lambda\|\\ &\leq \epsilon/2 + \epsilon/2\\ &= \epsilon\\ &\implies \|(a_\lambda)\| \leq \|(a_\lambda)_n\| + \epsilon\\ &\implies \|(a_\lambda)\| < \infty \implies (a_\lambda) \in A\end{aligned}$$

Thus A is Banach as well. Also, if A_λ is unital for all $\lambda \in \Lambda$, then there is a $1_\lambda \in A_\lambda$ such that $1_\lambda a_\lambda = a_\lambda 1_\lambda = a_\lambda$, thus setting $1_A = (1_\lambda)$ makes A unital as well. If every A_λ is abelian, then $(a_\lambda)(b_\lambda) = (a_\lambda b_\lambda) = (b_\lambda a_\lambda) = (b_\lambda)(a_\lambda)$ and A is also abelian. ∎

9.2.2. *Exercise 1.1 (b).* The restricted sum $\bigoplus_{\lambda}^{c_0} A_\lambda$, is defined as the set of all $(a_\lambda) \in A$ such that for a given $\epsilon > 0$, there is a finite subset $F \subseteq \Lambda$ such that $\|a_\lambda\| < \epsilon$ if $\lambda \in \Lambda - F$. B is a closed ideal of A.

Proof. Let $(b_\lambda) \in \bigoplus_{\lambda}^{c_0} A_\lambda = B$ and $(a_\lambda) \in A$. We want to first show that $(a_\lambda b_\lambda) \in B$. We have $\|(a_\lambda)\| = \sup \|a_\lambda\| = R$. Now let $\epsilon > 0$ be given. There is a finite set $F \subseteq \Lambda$ such that $\|b_\lambda\| < \epsilon/R, \forall \lambda \in \Lambda - F$. If $\lambda \in \Lambda - F$ then $\|a_\lambda b_\lambda\| \leq \|a_\lambda\| \cdot \|b_\lambda\| < R\frac{\epsilon}{R} = \epsilon$. Therefore, $(a_\lambda b_\lambda) = (a_\lambda)(b_\lambda) \in B$. Now, let $(b_\lambda)_n$ be a sequence in B with $(b_\lambda)_n \to (b_\lambda)$. We need $(b_\lambda) \in B$. For this, there must exist an $F \subseteq \Lambda$ such that $|F| < \infty$ and $\|b_\lambda\| < \epsilon, \forall \lambda \in \Lambda - F$. We have for any $\epsilon > 0$, there is an N such that for any $\geq N$, $\|(b_\lambda)_n - (b_\lambda)\| = \sup_\lambda \|b_{\lambda,n} - b_\lambda\| < \epsilon/2$. Now, there is an $F \subseteq \Lambda$ such that $|F| < \infty$ and $\|b_{\lambda,n}\| < \epsilon/2$ if $\lambda \in \Lambda - F$. If $\lambda \in \Lambda - F$ then $\|b_\lambda\| = \|b_\lambda - b_{\lambda,n} + b_{\lambda,n}\| \leq \|b_\lambda - b_{\lambda,n}\| + \|b_{\lambda,n}\| < \epsilon$, thus the ideal is closed. □

9.2.3. *Proof of claim pg. 6.* Let A be a unital abelian Banach algebra. **Claim:** The set of inverses of A is denoted

$$Inv(A) = \{a \in A | a \text{ is invertible}\}$$

and is a group under multiplication.

Proof. Clearly $\mathbf{1} \in Inv(A)$. By associativity of the multiplication of the algebra we have associativity of the elements of $Inv(A)$. Finally, let $a, b \in Inv(A)$. By definition of $Inv(A)$ there are elements $a^{-1}, b^{-1} \in A$ such that $aa^{-1} = a^{-1}a = \mathbf{1} = b^{-1}b = bb^{-1}$. Using the multiplication of the algebra we have that $(ab)^{-1} = b^{-1}a^{-1}$, and we therefore have inverses for all elements by definition and closure of the group under multiplication. □

9.2.4. *Exercise 1.5 (a).*

Definition 9.7. The *spectrum* $\sigma(a)$, of an element a of an algebra A is defined to be

$$\sigma(a) = \{\lambda \in \mathbb{C} | \lambda\mathbf{1} - a \notin Inv(A)\}$$

Claim: $\sigma(a + b) \subseteq \sigma(a) + \sigma(b)$

Proof. □

Claim: $\sigma(ab) \subseteq \sigma(a)\sigma(b)$ for all $a, b \in A$.

Proof. □

Theorem L. Let $I \subseteq R$ be an ideal in a unital ring R.

- $I = R \iff 1 \in I$
- If R is commutative, then R is a field if and only if the only ideals in R are 0 and R.

Proof. (1) If $I = R$ then $1 \in R$, since R is assumed to be unital. Conversely, if *uinI* is a unit with inverse v then for all $r \in R$, $r = r1 = r(uv) = (rv)u \in I$, thus $R = I$. (2) R is a field if and only if every nonzero $r \in R$ is a unit. If R is a field then every nonzero ideal contains a unit, so by (1) R is the only nonzero ideal. Conversely, if 0 and R are the only ideals of R, let u be any nonzero element of R. By hypothesis $(u) = R$, and so $1 \in (u)$. Then, there is a $v \in R$ such that $1 = vu$, i.e. u is a unit, which implies every $r \in R$ is a unit and R is therefore a field. □

Theorem M. If R is commutative, then an ideal M is maximal if and only if R/M is a field.

Proof. By definition, if $M \subset I \subset R$ then either $M = I$ or $I = R$. By the lattice isomorphism theorem, we have that ideals containing M in R are isomorphic to the ideals of R/M. Since there are no ideals other than $0_{R/M} = M \subset R/M$ and R/M itself, we have that R/M must be a field by the previous theorem. □

If I is a modular maximal ideal of a unital abelian algebra A, then A/I is a field.

Theorem N. Let A be a unital abelian Banach algebra.

- If $\tau \in \Omega(A)$, then $\|\tau\| = 1$.
- The set $\Omega(A)$ is nonempty and the map

$$\phi : \Omega \to \{I \subseteq A | I \text{ is a maximal ideal}\}; \tau \mapsto ker(\tau)$$

is a bijection.

Proof. If $\tau \in \Omega(A)$ and $a \in A$ then $\tau(a) \in \sigma(a)$. To see this assume $\tau(a) \notin \sigma(a)$. Then $\tau(a)1 - a$ is invertible. So, $\tau(\tau(a)1 - a) = \tau(\tau(a)1) - \tau(a) = \tau(a)\tau(1) - \tau(a) = \tau(a)1 - \tau(a) = 0$, which is not invertible. Thus $\tau(a) \in \sigma(a)$. By theorem 1.2.6, we have that $r(a) = \sup_{\lambda \in \sigma(a)} |\lambda| \leq \|a\| \leq 1$. Also, $\tau(1) = 1$, so $\|\tau\| = 1$.

□

Theorem O. Let $C()$ be the algebra of continuous functions $\{f :\to \mathbb{C} | \ f \text{ continuous}\}$, with operations defined pointwise. If $\alpha \in$, then the maps $\{\tau_\alpha\}_{\alpha \in}$ defined by:

$$\tau_\alpha : C() \to \mathbb{C}; \ t_\alpha(f) = f(\alpha)$$

are algebra homomorphisms.

Proof. Let $\alpha \in$. Then

$$\tau_\alpha(fg) = (fg)(\alpha) = f(\alpha)g(\alpha) = \tau_\alpha(f)\tau_\alpha(g).$$

Similarly,

$$\tau_\alpha(f + g) = (f + g)(\alpha) = f(\alpha) + g(\alpha) = \tau_\alpha(f) + \tau_\alpha(g)$$

and,

$$\tau_\alpha(\lambda f) = (\lambda f)(\alpha) = \lambda f(\alpha) = \lambda \tau_\alpha(f).$$

Therefore τ_α is an algebra homomorphism for any $\alpha \in$. □

The following chart summarizes some main ideas:

$$\begin{tikzcd}
 & & C(\Omega(C())) & \\
 {} \arrow[r, squiggly] & C() \arrow[ur, "\phi:f\mapsto\hat{f}"] \arrow[r, squiggly] \arrow[dr, "\tau_\alpha"] & \Omega(C()) \arrow[u, squiggly] \arrow[d, "\hat{f}:\tau\mapsto\tau(f)"] \arrow[dr, bend left, "\hat{f}_{\cong}"] & \\
 & & \mathbb{C} \arrow[r, dotted, no head] \arrow[d] & \supseteq \sigma(f) = \{\tau(f) \in \mathbb{C} | \tau \in \Omega(C())\} \\
 & & \mathbb{C} - \sigma(f) \arrow[uul, "\varphi:\lambda\mapsto(f-\lambda)^{-1}"] &
\end{tikzcd}$$

$$\tau_\alpha \in \Omega(C()),\ \forall \alpha \in$$

$$\hat{f} \in C(\Omega(C())),\ \forall f \in C()$$

Here the homomorphism $\phi : C() \to C(\Omega(C())); f \mapsto \hat{f}$, is the the Gelfand representation of the algebra of continuous functions on the space , by the algebra of continuous functions on the compact Hausdorff space $\Omega(C())$. In general, the representation is of a commutative Banach algebra A as an algebra of continuous functions on a *locally* compact space $\Omega(A)$, called $C_0(\Omega(A))$.

In this particular case, we have by lemma 1.2.4, that $\sigma(a)$ is a closed subset of the disk in $\mathbb{C}$, and the map $\varphi : \mathbb{C} - \sigma(a) \to C(); \lambda \mapsto (f - \lambda)^{-1}$, is differentiable.
0.5pt

9.3. **Categories and Functors.** We now introduce the language of categories and functors, which will become a useful tool later on when we talk about representations of algebras and the Gelfand-Naimark construction. Similar to linear algebra, where one must first define vector spaces in order to study the important notion of linear transformations, one must define categories before one can study the notion of a functor.

Definition 9.8. A class is called **small** if it has a cardinal number, and a class is a set if and only if it is small. A class that is not a set is called a **proper class**.

Definition 9.9. A **category** consists of three ingredients: a class $obj(C)$ of **objects**, a set of **morphisms** $Hom(A, B)$ for every ordered pair (A, B) of objects, and composition $Hom(A, B) \times Hom(B, C) \to Hom(A, C)$, denoted by $(f, g) \mapsto gf$, for every ordered triple A, B, C of objects. These ingredients are subject to the following axioms:

(1) the Hom sets are pairwise disjoint; that is, each $f \in Hom(A, B)$ has a unique domain A and a unique target B.
(2) for each object A, there is an identity morphism $1_A \in Hom(A, A)$ such that $f 1_A = f$ and $1_B f = f$ for all $f : A \to B$;

(3) composition is associative: given a morphism

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we have $h(gf) = (gh)f$.

Sets. The objects in this category are sets (not proper classes), morphisms are functions, and composition is the usual composition of functions. An axiom of set theory says that if A and B are sets, then the class $Hom(A, B)$ is also a set. Two functions are equal if they have the same domain, the same target, and the same graphs, thus morphisms are pairwise disjoint. For example, if $U \subset X$ is a proper subset, then 1_U and the inclusion map $\iota : U \to X$ are distinct since they each have different targets. We also say that is $f : A \to B$ and $g : C \to D$ are functions, then $gf : A \to D$ is defined if $B = C$, rather than if $B \subseteq C$. If we merely have $B \subseteq C$, then we make the composition $g\iota f$.

Groups. Objects are groups, morphisms are group homomorphisms, and composition is the usual composition. Since 1_G is a groups homomorphism and since if $\theta : G \to H$ and $\phi : H \to K$ are groups homomorphisms, then $\phi\theta : G \to K$ is also a group homomorphism (i.e. $\phi\theta \in Hom(G, K)$) we have that this is indeed a category.

A partially ordered set X can be regarded as a category with objects being elements of X, and with Hom sets either empty or having only one element:

$$Hom(x, y) = \begin{cases} \emptyset & \text{if } x \nleq y \\ \{\iota_y^x\} & \text{if } x \leq y \end{cases}$$

The symbol ι_y^x is the unique element in the Hom set when $x \leq y$. The composition is given by $\iota_z^y \iota_y^x = \iota_z^x$. Note, $1_x = \iota_x^x$ and since $\leq$ is transitive, composition is well defined.

Let X be a topological space and let $\mathcal{U}$ denote its topology. $\mathcal{U}$ is a poset under inclusion. We realize the the morphism ι_V^U as the inclusion map $\iota_V^U : U \to V$.

Top. Objects are all topological spaces, morphisms are continuous functions, and compositions are the usual composition of functions. Since the idenentity functions are continuous and since compositions of continuous functions are continuous, we have a category.

Definition 9.10. A category $\mathcal{S}$ is a **subcategory** of a category $\mathcal{C}$ if

(1) $obj(\mathcal{S}) \subseteq obj(\mathcal{C})$
(2) $Hom_{\mathcal{S}}(A, B) \subseteq Hom_{\mathcal{C}}(A, B) \forall A, B \in obj(\mathcal{S})$
(3) if $f \in Hom_{\mathcal{S}}(A, B)$ and $g \in Hom_{\mathcal{S}}(B, C)$ then $gf \in Hom_{\mathcal{S}}(A, C)$ and gf is equal to the composite $gf \in Hom_{\mathcal{C}}(A, C)$.
(4) If $A \in obj(\mathcal{S})$, then $1_A \in Hom_{\mathcal{S}}(A, A)$ is equal to the identity $1_A \in Hom_{\mathcal{C}}(A, A)$.

A subcategory $\mathcal{S}$ is a **full subcategory** of $\mathcal{C}$ if for all $A, B \in obj(\mathcal{S})$ we have $Hom_{\mathcal{S}}(A, B) = Hom_{\mathcal{C}}(A, B)$.

For example **Ab** the category of abelian groups is a full subcategory of the category **Groups**. The homotopy category **Htp** is not a subcategory of **Top** even though $obj(\mathbf{Htp}) = obj(\mathbf{Top})$, since the morphisms are not coninuous functions.

Definition 9.11. If $\mathcal{C}$ is any category and $\mathcal{S} \subseteq obj(\mathcal{C})$, then the full **subcategory generated by** $\mathcal{S}$ is the category with $obj(\mathcal{S}) = \mathcal{S}$ and $Hom_{\mathcal{S}}(A, B) = Hom_{\mathcal{C}}(A, B)$ for all $A, B \in \mathcal{S}$. For example we define the category $\mathbf{Top}_2$ to be the full subcategory of **Top** generated by all Hausdorff spaces.

Definition 9.12. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a **functor** $T : \mathcal{C} \to \mathcal{D}$ is a function such that

(1) if $A \in obj(\mathcal{C})$ then $T(A) \in obj(\mathcal{D})$

(2) if $f : A \to A'$ in $\mathcal{C}$ then $T(f) : T(A) \to T(A')$ in $\mathcal{D}$
(3) if
$$A \xrightarrow{f} A' \xrightarrow{g} A''$$
in $\mathcal{C}$ then
$$T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$$
in $\mathcal{D}$ and $T(gf) = T(g)T(f)$.
(4) $T(1_A) = 1_{T(A)} \forall A \in obj(\mathcal{C})$.

(1) If $\mathcal{C}$ is a category then the **identity functor** $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined by $1_{\mathcal{C}}(A) = A, \forall\, A \in obj(\mathcal{C})$ and $1_{\mathcal{C}}(f) = f$ for all morphisms f.
(2) If $\mathcal{C}$ is a category and $A \in obj(\mathcal{C})$, then the **Hom functor** $T_A : \mathcal{C} \to$ **Sets**, usually denoted $Hom(A, \bullet)$ is defined by
$$T_A(B) = Hom(A, B) \forall\, B \in obj(\mathcal{C})$$
and if $f : B \to B'$ in $\mathcal{C}$, then $T_A(f) : Hom(A, B) \to Hom(A, B')$ is given by
$$T_A(f) : h \mapsto fh.$$
We call $T_A(f)$ the **induced map**, and denote it by f_*; thus,
$$f_* : h \mapsto fh.$$
We now verify the parts of the definition. First, the definition of category says $Hom(A, B)$ is a set. The composition of fh makes sense:
$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "fh"] & B \arrow[r, "h"'] & B' \end{tikzcd}$$
Suppose now that $g : B' \to B''$. Let us compare the functions
$$(gf)_*, g_*f_* : Hom(A, B) \to Hom(A, B'').$$

[16]

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