GEOMETRIC INVARIANT THEORY, ARTIN L-FUNCTIONS, AND SURFACE ALGEBRAS

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Abstract. We continue the work initiated in [AS1, AS2] on *surface algebras* and *surface orders* and ellaborate on some of the relations to Artin L-functions and Dedekind ζ-functions. In particular, we use the results of [LeBruyn-Procesi] and [**?**] in order to describe the invariant theory of representations of surface algebras in arbitrary characteristic. We show how one can realize Artin L-functions as invariants under the action of a certain algebraic group, and we also show how the action of the Galois group of number fields $K/\mathbb{Q}$ can be understood in these terms as well. We then use this description to show that the exponents in Brauer's $\mathbb{Z}$-linear factorization

$$L(s,\rho) = \prod_{i=1}^{n} L(s,\chi_i)^{n_i}$$

of Artin L-functions into Dirichlet L-functions are positive. By the results of Brauer, this provides some useful results on Artin L-functions. We give a description of L-functions are the Hilbert-Poincare series for projective resolutions of modules over surface algebras (as graded vector spaces). Along the way, we will provide a description of the K-theory of surface algebras, as well as their Leavitt path algebras and graph C^*-algebras. Since relative (semi)invariants for surface algebras can be defined in terms of characters given by maps between projective modules, using the Serre-Swan Theorem gives us a way of interpreting characters in terms of the K-theory, and thus a new way of approaching Beilinson's Conjectures and special values of L-functions, since we identify the character theory and invariant theory with the local factors of L-functions. We also provide a bridge between the theory of surface algebras and surface orders to the theory of noncommutative arithmetic geometry, spectral triples of Mumford curves, and relations to Bruhat-Tits buildings as developed by Marcolli, Connes, Consani, and Manin. This confirms a suspicion that [S4] was of importance and related to the universal cover of surface algebras as was mentioned in [AS1].

Contents

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Part 1. Generating Series, C^*-Algebras from Surface Algebras, and Trace Class Operators

1. Limits of Cyclotomic Polynomials as Hilbert Series

Let $p_n(t) = t^n - 1 \in \mathbb{Q}[t]$ be the polynomial corresponding to the cyclic extension $\mathbb{Q}(\zeta_n)$, where

$$\zeta_n = e^{\frac{2\pi i}{n}} \in S^1 \subset \mathbb{C}^*$$

Let $K_n = \mathbb{Q}(\zeta_n)$ be the splitting field for $p_n(t)$. Next, over $\mathbb{C}$ or K_n, we may identify the solution set (zeros) of $p_n(t) = 0$, with the n^{th} roots of unity $\{\zeta_n^j\}_{j=1}^n$. Moreover, over any field F,

$$K_n \subset K \subset \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

the polynomial splits completely and this identification can also be made over F. For the moment, let us just work over $\mathbb{C}$ and choose some embedding of K_n into $\mathbb{C}$. Then

$$\frac{p_n(t)}{(1-t)} = (t-\zeta_n)(t-\zeta_n^2)\cdots(t-\zeta_n^{n-2})(t-\zeta_n^{n-1}) = \prod_{j=1}^{n-1}(t-\zeta_n^j).$$

Notice

$$\zeta_{rn} = e^{\frac{2\pi i}{rn}} = (e^{\frac{2\pi i}{n}})^{1/r} = \zeta_n^{1/r}.$$

Now, let $r \in \mathbb{Z}_{>0}$. Then we have

$$\begin{aligned} p_{rn}(t)/(1-t) = {} & (t-\zeta_n^{1/r})(t-(\zeta_n^{2/r})\cdots(t-\zeta_n^{(n-1)/r}) \\ & \times \ (t-\zeta_n^{n/r})(t-\zeta_n^{(n+1)/r})\cdots(t-\zeta_n^{(2n-1)/r}) \\ & \times \ (t-\zeta_n^{2n/r})(t-\zeta_n^{(2n+1)/r})\cdots(t-\zeta_n^{(3n-1)/r}) \\ & \vdots \\ & \times \ (t-\zeta_n^{(r-1)n/r})(t-\zeta_n^{((r-1)n+1)/r})\cdots(t-\zeta_n^{(rn-1)/r}) \\ & = (1-t)\prod_{m=1}^{rn-1}(t-\zeta_n^{m/r}) \end{aligned}$$

We might think of these products as product of the columns in the lattice,

$$\begin{tikzcd}
(t-\zeta_n^{1/r}) \arrow[r] & (t-\zeta_n^{2/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{(n-1)/r}) \arrow[dlll] \\
(t-\zeta_n^{n/r}) \arrow[r] & (t-\zeta_n^{(n+1)/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{(2n-1)/r}) \arrow[dlll] \\
(t-\zeta_n^{2n/r}) \arrow[r] & (t-\zeta_n^{(2n+1)/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{(3n-1)/r}) \arrow[dlll] \\
\vdots & \vdots & \ddots & \vdots \arrow[dlll] \\
(t-\zeta_n^{(r-2)n/r}) \arrow[r] & (t-\zeta_n^{((r-2)n+1)/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{((r-1)n-1)/r}) \arrow[dlll] \\
(t-\zeta_n^{(r-1)n/r}) \arrow[r] & (t-\zeta_n^{((r-1)n+1)/r}) \arrow[r] & \cdots \arrow[r] & (t-\zeta_n^{(rn-1)/r})
\end{tikzcd}$$

If we wish to obtain infinite product formulas we might want to organize by columns,

$$\begin{tikzcd}
(t-\zeta_n^{1/r}) \arrow[d] & (t-\zeta_n^{2/r}) \arrow[d] & \cdots & (t-\zeta_n^{(n-1)/r}) \arrow[d] \\
(t-\zeta_n^{n/r}) \arrow[d] & (t-\zeta_n^{(n+1)/r}) \arrow[d] & \cdots & (t-\zeta_n^{(2n-1)/r}) \arrow[d] \\
(t-\zeta_n^{2n/r}) \arrow[d] & (t-\zeta_n^{(2n+1)/r}) \arrow[d] & \cdots & (t-\zeta_n^{(3n-1)/r}) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] \\
(t-\zeta_n^{(r-2)n/r}) \arrow[d] & (t-\zeta_n^{((r-2)n+1)/r}) \arrow[d] & \cdots & (t-\zeta_n^{((r-1)n-1)/r}) \arrow[d] \\
(t-\zeta_n^{(r-1)n/r}) \arrow[uuuuur, dotted, bend left] & (t-\zeta_n^{((r-1)n+1)/r}) \arrow[uuuuur, dotted, bend left] & \cdots \arrow[uuuuur, dotted, bend left] & (t-\zeta_n^{(rn-1)/r})
\end{tikzcd}$$

In which case we can write,

$$p_{rn}(t) = (1-t)\prod_{j=1}^{(r-1)}(1-\zeta_n^{jn/r})\prod_{j=1}^{r-1}(1-\zeta_n^{(jn+1)/r})\cdots\prod_{j=1}^{r}(1-\zeta_n^{(jn-1)/r})$$

Next, notice for each column we have something of the form

$$\prod_m (1-\zeta_n^{m/r}), \quad m = (jn+k),\ j = 0,1,2,....,r-1,\ k = 1,2,...,n-1.$$

$$\begin{aligned}
\det(I_n - \rho(g)) &= (1-\chi_1(g))\cdot(1-\chi_2(g))\cdots(1-\chi_n(g)) \\
&= \prod_{i=1}^{n}(1-\chi_i(g)) \\
&= \sum_{i=1}^{n}(-1)^i\,\mathbf{Tr}\left(\bigwedge^i A\right)
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbf{Tr}\left(\bigwedge^i (I_n-\rho(g))\right) &= \mathbf{Tr}\left(\bigwedge^i A\right) \\
&= \sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} a_{j(1),j(1)}a_{j(2),j(2)}\cdots a_{j(i),j(i)} \\
&= \sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i} a_{j(k),j(k)} \\
&= \sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i}(1-\chi_{j(k)}(g))
\end{aligned}$$

where $a_{j,j} = (1-\chi_j(g))$, and $A = (I_n - \rho(g))$.

So that

$$\begin{aligned}
\det(I_n-\rho(g)) &= \det(A) \\
&= \sum_{i=1}^{n}(-1)^i\left[\sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i} a_{j(k),j(k)}\right] \\
&= \sum_{i=1}^{n}(-1)^i\left[\sum_{\{1\le j(1)<j(2)<\cdots<j(i)\le n\}} \prod_{k=1}^{i}(1-\chi_{j(k)}(g))\right]
\end{aligned}$$

Moreover, the characteristic polynomial then gives us the L-function,

$$\begin{aligned}
L_p(s,\rho) &= \frac{1}{P_p\left(\nu(p)^s\right)} \\
&= \det\left(1-\frac{\rho(\sigma)}{\nu(p)^s}\bigg|_V\right) = \prod_{i=1}^{\dim_{\mathbb{C}} V}\left(1-\frac{\chi_i(g)}{\nu(p)^s}\right),\ \chi_i(g)\in\mathbb{C}^*
\end{aligned}$$

and

$$L(s,\rho) = \prod L_p(s,\rho).$$

Now, we would like to think again in terms of combinatorial commutative algebra for a moment. Notice, the ring

$$\mathbb{Z}[[x_0, x_1, ..., x_{n-1}]]$$

of formal power series is the ring in which Hilbert series of multigraded rings (i.e. $\mathbb{N}^n$-graded) live. Further, the element $1-x_i$ has inverse

$$\frac{1}{1-x_i} = 1 + x_i + x_i^2 + x_i^3 + \cdots$$

The multigraded module $M = \bigoplus_{(a_0,\dots,a_n)\in\mathbb{N}^n} M_{a_0,a_1,\dots,a_{n-1}}$ over some commutative ring R has Hilbert series

$$H(M, x_0, x_1, ..., x_{n-1}) = \sum_{(a_0,a_1,\dots,a_n)\in\mathbb{N}} \dim_K(M_{a_0,a_1,\dots,a_{n-1}})x_0^{a_0}x_1^{a_1}\cdots x_{n-1}^{a_{n-1}}$$

If we have $x_0 = x_1 = \cdots = x_{n-1} = t$, then the definition reduces to a $\mathbb{Z}$-grading.

Now, we noticed any

$$f_\alpha(t) = \frac{p_j(t)}{(\alpha - t)}$$

over $\mathbb{C}$ is holomorphic (for $\alpha = a\zeta^s_{n(j)}$) with removable singularity at $t = \alpha$. We also have that

$$\lambda f_\alpha(t) \cdot \frac{\alpha^s}{p_j(\alpha)} = t^s$$

is of course also holomorphic. Moreover, with the basis $\{1, \alpha, \alpha^2, ..., \alpha^n\}$ and its dual basis as described in the previous sections (see also [Lang] pg. 213), provide us with a description through the trace and determinant maps of the torus and its dual, i.e. we have a descriptions of the characters and cocharacters

$$X^*(T^n) : \mathbb{C}^* \to T^n, \quad X_*(T^n) : T^n \to \mathbb{C}^*$$

which are given in the usual way.

Moreover, we can now identify the action of the torus on roots of unity (or more generally on roots of the form $\alpha_j\zeta^s_{n(j)}$) with an action of the torus

$$\mathbb{C}^{Q_0} \cong (\mathbb{C}^{n(j)})^* \subset \mathfrak{gl}_{n(j)}(\mathbb{C})$$

on the root system for the Cartan subalgebra

$$\mathfrak{h}_{n(j)} = \begin{pmatrix} \mathbb{C} & & & \\ & \mathbb{C} & & \\ & & \ddots & \\ & & & \mathbb{C} \end{pmatrix} \cong \mathbb{C}^{Q_1}$$

where $Q = (Q_0, Q_1)$ is a cyclic quiver (we will have one of size $n(j)$ for each cyclic extension $K_j = \mathbb{Q}(\alpha_j)$. This can of course be understood geometrically as an action on an algebraic curve

$$C(K_j) = \mathbf{Spec}\left(K_j[t]/(t^n - 1)\right)$$

which is given by a determinant polynomial from a diagonal element in $T^n \otimes K_j[t]$. Since we have this identification, we can now describe the determinants giving the characteristic polynomials,

$$L_p(s, \rho) = \frac{1}{P_p\left(\nu(p)^s\right)},$$

where $\nu(p)$ is the cardinality of the residue field at the prime p.

$$L(s, \rho) = \prod L_p(s, \rho).$$

as invariant polynomials under the action of the torus action on a quiver (here $P_p(t)$ is a characteristic polynomial defined in the beginning of the section on Artin L-functions).

2. C^*-Algebras and the Gel'fand-Naĭmark Representation Theorem

2.1. $*$-Algebras.

Definition 2.1. We will define a **Banach algebra** A to be an associative algebra over either $\mathbb{R}$, $\mathbb{C}$, or a normed complete non-Archimedean field, with a norm $\|\cdot\|$, such that A is a complete (linear) space with respect to $\|\cdot\|$, and such that

$$\|xy\| \leq \|x\| \cdot \|y\|.$$

If A has an involution $x \mapsto x^*$, and if $\|x^*x\| = \|x\|^2$, then we call the norm $\|\cdot\|$, $*$-**quadratic**, and we call A a C^*-**algebra**.

Example 2.2. If $\mathbb{H}$ is a separable Hilbert space with orthonormal basis $\{e_j\}_{j=1}^{\infty}$, and $\mathcal{L}(\mathbb{H})$ is the algebra of linear operators (endomorphisms) $X : \mathbb{H} \to \mathbb{H}$, then any norm closed $*$-subalgebra of $\mathcal{L}(\mathbb{H})$ is a C^*-algebra.

Example 2.3. Let X be a compact Hausdorff space, and let $C(X)$ be the complex valued continuous functions on X. Define a *commutative* C^*-algebra with unity structure by letting $f^*(x) = (f(x))^*$ be given by complex conjugation, and let $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ be the norm.

2.2. **UHF Algebras.** Let $\mathcal{L}(\mathbb{H})$ be the algebra of bounded linear operators on a separable Hilbert space $\mathbb{H}$. Consider the $*$-subalgebra $A \subset \mathcal{L}(\mathbb{H})$ given by the direct limit of the inductive system,

$$A = \varinjlim_{\phi} A_n = \varinjlim_{\phi} \mathbf{Mat}_{n\times n}(\mathbb{C}),$$

where $\phi : \mathbf{Mat}_{n\times n}(\mathbb{C}) \to \mathbf{Mat}_{(n+1)\times(n+1)}(\mathbb{C})$. We will let ϕ be the embedding given by

$$X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{Mat}_{(n+1)\times(n+1)}(\mathbb{C}).$$

We will use the notation

$$A = \mathbf{Mat}_\infty(\mathbb{C})$$

to denote this algebra.

2.3. **Compact Operators, $\ell^1(\mathbb{Z})$, and Wiener's Theorem.** The setup here draws heavily on the exposition of §0, pg.1-10 of [Berberian]. In later sections we will see that the following *"graph C^*-algebra* is associated to compact operators,

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

This directed graph represents the algebra

$$C^*(\Gamma) = \varinjlim_{\phi} \mathbf{Mat}_{n\times n}(\mathbb{C}),$$

the algebra of compact operators.

This algebra will be important for the following construction, which we will related to the direct limit of the diagonal matrices $\mathbf{Mat}_{n\times n}(\langle\zeta_{rn}\rangle) \subset \mathbf{Mat}_{n\times n}(S^1)$.

Now, let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers and assume

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

Then

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \quad t \in \mathbb{R},$$

gives a continuous, 2π-periodic, uniformly and absolutely convergent $f \in C(\mathbb{R}, \mathbb{C})$. Furthermore we have

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt, \quad n \in \mathbb{Z}.$$

Let A be the set of all such functions. Then for

$$u_n(t) = e^{int}$$

if $u = \sum_{n \in I} a_n u_n$, with $I \subset \mathbb{Z}$ finite, then $u \in A$. Additionally,

(1) $\langle u_n, u_m \rangle = \int_0^{2\pi} u_n(t) u_m(t)^* dt = 2pi\delta_{m,n}$,
(2) $u_{n+m}(t) = u_n(t) u_m(t)$,
(3) $u_{-n}(t) = \frac{1}{u_n(t)} = u_n(t)^*$.

gives an orthonormal set of functions. We call any $u = \sum_{n \in I} a_n u_n$ a **trigonometric polynomial**[1]. Now, let

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt,$$

and assume

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

Then we also have

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}.$$

Let $\ell^1(\mathbb{Z})$ be all sequences $(a_n)_{n \in \mathbb{Z}}$ such that $a_n \in \mathbb{C}$ and

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

We will treat $\ell^1(\mathbb{Z})$ as the space of functions

$$x : \mathbb{Z} \to \mathbb{C}, \quad x(n) = a_n,$$

so that

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty.$$

We have for $x, y \in \ell^1(\mathbb{Z})$ that $x = y \iff x(n) = y(n)\ \forall\ n \in \mathbb{Z}$. Define the ℓ^1-norm

$$\|x\|_1 = \sum_{n=-\infty}^{\infty} |x(n)| = \sup_I \sum_{n \in I} |x(n)|,$$

where $I \subset \mathbb{Z}$, $I \neq \emptyset$, ranges over all finite subsets of $\mathbb{Z}$. Define a metric on $\ell^1(\mathbb{Z})$ by

$$d(x, y) = \|x - y\|_1.$$

[1] We would like to show that for any $f \in A$ such that *f does not vanish*, we have $1/f \in A$. The significance of this will become clear when we study *holomorphic semigroups* and the semigroup rings of Artin L-functions in the next two sections.

Let
$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n)e^{int}, \quad (t \in \mathbb{R}),$$
be the **Fourier transform**[2] of $x \in \ell^1(\mathbb{Z})$. The map
$$\ell^1(\mathbb{Z}) \to A, \quad x \mapsto \hat{x},$$
give an isomorphism of vector spaces. Next define a function $e_n \in \ell^1(\mathbb{Z})$[3] by
$$e_n(m) = \delta_{m,n}, \ \forall\, m, n \in \mathbb{Z},$$
so that
$$\hat{e_n} = u_n.$$
We have
$$c_{00}(\mathbb{Z}) := \mathbf{Span}_{\mathbb{C}}\{e_n\}_{n\in\mathbb{Z}} = \{x : \mathbb{Z} \to \mathbb{C} \mid x(n) = 0 \ a.e.\},$$
to be the span of the e_n giving functions with finite support. We have that
$$x = \sum_{n\in\mathbb{Z}} x(n)e_n,$$
and
$$\hat{x} = \sum_{n\in\mathbb{Z}} x(n)u_n.$$
Since $\hat{x}$ in this case gives a trigonometric polynomial, we get an isomorphism
$$c_{00}(\mathbb{Z}) \to T,$$
where T is the space of all trigonometric polynomials. We have that $\{e_n\}$ is an orthonormal basis for the separable Hilbert space $\ell^1(\mathbb{Z})$, and so $c_{00}(\mathbb{Z})$ is dense in $\ell^1(\mathbb{Z})$. The isomorphism
$$\hat{x} \mapsto x \in c_{00}(\mathbb{Z}),$$
then induces an algebra structure on c_{00} since $u_{m+n} = u_n u_m \in T$ gives T the structure of an algebra. The multiplication is defined by **convolution**
$$(xy)(m) = \sum_{n=-\infty}^{\infty} x(m-n)y(n).$$

Now, we have two very important results which will be generalized and applied in later sections.

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[2]Since $t \in \mathbb{R}$ it is arbitrary as to whether we define the Fourier transform with the exponent int or $-int$. We may also restrict the Fourier transform $\hat{x}$ to S^1 so that
$$\hat{x}(\zeta) = \sum_{n=-\infty}^{\infty} x(n)\zeta^{-n}.$$

[3]One might wish to think of the e_n as the elementary operators in the direct limit
$$\mathbf{Mat}_\infty(\mathbb{Z}) = \varinjlim_{\phi} \mathbf{Mat}_{n\times n}(\mathbb{Z})$$
given by the elementary matrices $E_{n,n}$ with a 1 at the (n, n) entry and zero elsewhere. One might also wish to think of the $u_n = \hat{e_n}$ at the elementary operators in $\mathbf{Mat}_\infty(\mathbb{C})$, but with coefficient $\zeta = e^{int}$.

Theorem 2.4. *(Wiener's Theorem): If $x \in \ell^1(\mathbb{Z})$ so that*

$$\|x\|_1 = \sum_{n=-\infty}^{\infty} |x(n)| < \infty,$$

and if

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n)e^{int}, \quad (t \in \mathbb{R})$$

does not vanish, then there exists a $y \in ell^1(\mathbb{Z})$ such that

$$\frac{1}{\hat{x}(t)} = \sum_{n=-\infty}^{\infty} y(n)e^{int}.$$

Theorem 2.5. *(Gel'fand's Theorem for $\ell^1(\mathbb{Z})$): Let $\ell^1(\mathbb{Z})$ have the convolution product*

$$xy(m) = \sum_{n=-\infty}^{\infty} x(m-n)y(n)$$

giving it the structure of a unital commutative associative algebra. Suppose $f \in A$ and that f is nowhere vanishing so that $1/f \in A$ (i.e. the principle ideal $(f) = A$) by Wiener's Theorem. Let $P_t : A \to \mathbb{C}$ be the algebra epimorphism given by evaluation $g \mapsto g(t)$. Denote by

$$\mathcal{M}_t = \ker(P_t) = \{g \in A : \ g(t) = 0\},$$

the maximal ideal given by the kernel under the projection P_t (so $A/\mathcal{M}_t = \mathbb{C}$). We have then that

(1) *$f \in \mathcal{M}_t$ for some $t \in \mathbb{R}$, and*
(2) *every maximal ideal of A is of the form $\mathcal{M}_t$.*
(3) *$M_t = \{x \in \ell^1(\mathbb{Z}) : \ \hat{x} \in \mathcal{M}_t\}$ is a maximal ideal of $\ell^1(\mathbb{Z})$ for any maximal ideal $\mathcal{M}_t$ of A.*
(4) *$\ell^1(\mathbb{Z})/M_t = \mathbb{C}$.*

Now, from this, and from the description of the previous section, evalutation of the characteristic polynomials at $t = \nu$, as in [Milne1], defining Artin L-functions for cyclic subgroups of a Galois group $\mathcal{G}(K : \mathbb{Q})$, can then be understood in terms of the algebra $\ell^1(\mathbb{Z})$. In particular, the Hilbert series from the next section converge and by the determinant trace formula and Wiener's Theorem we get some very nice results on Artin L-functions. So our next task is to use the limit

$$\varinjlim_{\phi} \mathbf{Mat}_{n\times n}(\mathbb{C}) = \mathbf{Mat}_{\infty}(\mathbb{C}),$$

to show that the characteristic polynomials of the previous section converge to Hilbert series related to projective modules of the surface algebra (or surface order). Then, we can identify these Hilbert series with series in $\ell^1(\mathbb{Z})$ and use Wiener's Theorem to show for a nonvanishing Hilbert series $f \in A$, we have $1/f \in A$. Once we have this, we will want to understand the zeros (i.e. vanishing sets or support) of such Hilbert series. To do this we need the theory of holomorphic semigroups and semigroup rings. We also may use the fact that for a meromorphic functions f (as Artin L-functions are), away from poles we know f may be uniformly approximated by polynomials (partial sums of the power series representation), and on a punctured disk around a pole f may be uniformly approximated by rational functions (f is represented by a Laurent series on $\mathbb{D}^{\times}$). So we may use local factors of L-functions f to approximate f and understand its zeros and poles as they are uniformly approximated by zeros of polynomials and zeros and poles of rational functions.

Example 2.6. The algebra of continuous functions on S^1, $C^*(\Gamma) = C(S^1)$, can be obtained from the following graph and is the completion of the Leavitt path algebra $K[x, x^{-1}]$. This fact is proven using the *Stone-Weierstrass theorem*.

$$\begin{tikzcd} v_1 \arrow[loop above, "e"] \arrow[loop below, dotted, "e^*"] \end{tikzcd}$$

3. L-functions as Hilbert Series of Projective Resolutions of Simple Modules

4. The Theory of Holomorphic Semigroups

4.1. **Semigroup Rings of L-Functions.**

4.2. **Convergence of Semigroups.**

5. Character Groups and Locally Compact Abelian Groups

6. Leavitt Path Algebras

6.1. **Universal Localization of Path Algebras.**

6.2. **Examples.**

Example 6.1. The path algebra $KQ := K\mathbb{A}_n$, given by the quiver,

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_n \longrightarrow \cdots\cdots \bullet_{n-1} \longrightarrow \bullet_n$$

can be identified with the matrix algebra of lower triangular matrices,

$$K\mathring{A}_n \cong \begin{pmatrix} K & 0 & 0 & \cdots & 0 & 0 \\ K & K & 0 & \cdots & 0 & 0 \\ K & K & K & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K & K & K & \cdots & K & 0 \\ K & K & K & \cdots & K & K \end{pmatrix}$$

by defining the map

$$e_j \mapsto E_{j,j}, \quad a_{j,j+1} \mapsto E_{j+1,j},$$

where $E_{i,j}$ are the matrix units in $\mathbf{Mat}_{n\times n}(K)$. The algebra of $n \times n$ matrices corresponds to the universal localization of this quiver path algebra, which has graph,

$$\begin{tikzcd} \bullet_1 \arrow[r, bend right] & \bullet_2 \arrow[l, bend right, dotted] \arrow[r, bend right] & \bullet_3 \arrow[l, bend right, dotted] & \cdots \arrow[r, bend right] & \bullet_{n-1} \arrow[l, bend right, dotted] \arrow[r, bend right] & \bullet_n \arrow[l, bend right, dotted] \end{tikzcd}$$

The map $L(Q) \to \mathbf{Mat}_{n\times n}(K)$ is then just given by extending the map of $K\mathbb{A}_n$ into $\mathfrak{n}$ (lower triangular matrices in $\mathfrak{gl}_n(K) = \mathbf{Mat}_{n\times n}(K)$), by

$$a^*_{j+1,j} \mapsto E_{j,j+1},$$

for the formal inverse arrows $a_{j+1,j}$.

Example 6.2. Let $\mathfrak{D}_3$ be the matrix algebra generated by matrices of the form

$$\begin{pmatrix} 0 & 0 & a_3 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}$$

with each $a_i \in K$ for some number field K. Let Q be the following quiver:

$$\begin{tikzcd} & v_1 \arrow[dl, "a_1"'] & \\ v_2 \arrow[rr, "a_2"'] & & v_3 \arrow[ul, "a_3"'] \end{tikzcd}$$

Then $KQ \cong \mathfrak{D}_3 \otimes_K K[x]$. To see this, define a map $f : KQ \to \mathfrak{D}_3 \otimes_K K[x]$ by:

$$\begin{aligned} v_1 &\mapsto E_{11} \otimes_K 1 & \qquad a_1 &\mapsto E_{21} \otimes 1 \\ v_2 &\mapsto E_{22} \otimes 1 & \qquad a_2 &\mapsto E_{32} \otimes 1 \\ v_3 &\mapsto E_{33} \otimes 1 & \qquad a_3 &\mapsto E_{13} \otimes x \end{aligned}$$

Where E_{ij} are the matrix units in $\mathbf{Mat}_{3\times 3}(K)$. We see $a_3a_2a_1 \mapsto E_{11} \otimes x$, and powers

$$\begin{aligned} f((a_3a_2a_1)^n) = ((E_{13}E_{32}E_{21}) \otimes x)^n &= ((E_{13}E_{32}E_{21})^n \otimes x^n \\ &= \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)^n \otimes x^n \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^n \otimes x^n \\ &= f(v_1)^n \otimes f(a_3)^n \end{aligned}$$

This map f can be defined similarly for any $n \in \mathbb{Z}_{\geq 0}$, and we have that in general $KQ \cong \mathfrak{D}_n \otimes K[x]$, where

$$\mathfrak{D}_n = \begin{pmatrix} 0 & & & & & & a_n \\ a_1 & 0 & & & & & \\ & a_2 & 0 & & & & \\ & & a_3 & 0 & & & \\ & & & \ddots & \ddots & & \\ & & & & a_{n-2} & 0 & \\ & & & & & a_{n-1} & 0 \end{pmatrix}$$

and $f(v_j) = E_{jj} \otimes 1$, $f(a_n) = E_{1,n} \otimes x$, and $f(a_j) = E_{j+1,j} \otimes 1$ for all $j \neq n$ defines the map

$$f : KQ \to \mathfrak{D}_n \otimes K[x].$$

7. Affine Lie Algebras, Characters, Witt Groups, and Fourier Transforms

Let $\mathfrak{g}_K = \mathfrak{gl}_n(K) = \mathbf{Mat}_{n\times n}(K)$ be the lie algebra of the general linear group over a field K. From the previous section we have the Leavitt path algebra,

$$\mathfrak{g}_K \otimes K[t, t^{-1}] = L(\tilde{\mathbb{A}}(n)),$$

where $\tilde{\mathbb{A}}(n)$ is the cyclic quiver with n vertices. We also had the path algebra of the cyclic quiver

$$
\begin{aligned}
\mathfrak{n}_-(t) &= \mathfrak{n}_- \otimes K[t] \\
&= \begin{pmatrix} K[t] & (t) & (t) & \cdots & (t) & (t) \\ K[t] & K[t] & (t) & \cdots & (t) & (t) \\ K[t] & K[t] & K[t] & \cdots & (t) & (t) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K[t] & K[t] & K[t] & \cdots & K[t] & (t) \\ K[t] & K[t] & K[t] & \cdots & K[t] & K[t] \end{pmatrix} \\
&= \begin{pmatrix} 0 & & & & & & a_n \\ a_1 & 0 & & & & & \\ & a_2 & 0 & & & & \\ & & a_3 & 0 & & & \\ & & & \ddots & \ddots & & \\ & & & & a_{n-2} & 0 & \\ & & & & & a_{n-1} & 0 \end{pmatrix} \otimes K[t] \\
&= \mathfrak{D}_n \otimes K[t]
\end{aligned}
$$

We define the affine Lie algebra $\widehat{\mathfrak{g}_K}$ as the vector space,

$$\widehat{\mathfrak{g}_K} = \mathfrak{g}_K \otimes K[t, t^{-1}] \oplus Kz$$

given by the central extension

$$c \to \widehat{\mathfrak{g}_K} \to \mathfrak{g}_K$$

with Lie bracket,

$$\gamma_1 \otimes t^n + \lambda_1 z, \gamma_2 \otimes t^m + \lambda_2 z] = [\gamma_1, \gamma_2] \otimes t^{n+m} + \langle \gamma_1 \, | \gamma_2 \rangle \cdot n\delta_{n+m,0} z,$$

with $\gamma_j \in \mathfrak{g}_K$, $\lambda_j \in K$, and $n, m \in \mathbb{Z}$, and for $\langle \cdot \, | \cdot \rangle$ is the Cartan-Killing form on $\mathfrak{g}_K$.

the complex **Witt algebra** is the Lie algebra of *meromorphic vector fields* defined on the *Riemann sphere* that are holomorphic *except at two fixed points*. It is also the complexification of the Lie algebra of *polynomial vector fields on $S^1 \subset \mathbb{C}$*, and the Lie algebra of *derivations of the ring* $\mathbb{C}[z, z^{-1}]$ of Laurent series.

The **Virasoro algebra** is a complex Lie algebra given by the unique *central extension of the Witt algebra*[4]. To define it, we let L_n, $n \in \mathbb{Z}$ be generators and let c be a **central charge**[5]. We require $[c, L_n] = 0$ and

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}.$$

[4]The Virasoro algebra is important for two-dimensional conformal field theory and string theory.

[5]Recall a **central charge** c is an operator in a Lie algebra commuting with the symmetry operators. We note in [AS1] §"*Basic Invariants of $\mathcal{G}(\overline{\mathbb{Q}}/\mathbb{Q})$*" (the absolute Galois group), see also §**??** in Part I, we found that central elements were of the form $A_0 \cdot k[[z_1, z_2, ..., z_r]]/(z_i z_j)_{i \neq j}$, where A_0 was the commutative algebra generated by the primitive orthogonal idempotents of the surface algebra.

7.1. **Witt Groups, Witt Vectors, Witt Rings, and Isogenies.** For information on Witt groups we refer the reader to [S1] §VII.8. The Witt group of dimension n, denoted by W_n is the group of elements $(x_0, x_1, ..., x_{n-1})$ with three operations

(1) Frobenius map, $F : W_n \to W_n$, given by $(x_0, x_1, ..., x_{n-1}) \mapsto (x_0^p, x_1^p, ..., x_{n-1}^p)$.
(2) The shift $U : W_n \to W_{n+1}$, given by $(x_0, x_1, ..., x_{n-1}) \mapsto (0, x_0, x_1, ..., x_{n-1})$.
(3) Restriction, $R : W_{n+1} \to W_n$ given by $(x_0, x_1, ..., x_n) \mapsto (x_0, x_1, ..., x_{n-1})$.

We note that our definition of the "*generalized Frobenius*" which is closely related to the shift operator, we have a connection between Witt groups, Witt vectors, and their Witt rings and with Artin-Schreier Theory, via the *necklace algebra* (see [LeBruyn2] pg.18 and [Metropolis-Rota]). Furthermore, we note that isogenies as defined in [S3] §VI.2 (7)-(12) might lead one to follow one's nose to the investigation of the pullbacks defining surface orders and surface algebras. We also note, using our definition of the generalized Frobenius, along with the quadratic form of the quiver, we can interpret the character theory of interest for this article in terms of representation theory via [Milne2] pg. 226.

7.2. **The Fourier Transform.** For the Lie algebra $\mathfrak{g}_K = \mathfrak{gl}_n(K) = \mathbf{Mat}_{n\times n}(K)$ we let

$$\mathfrak{g}_K(S^1) = \mathfrak{g} \otimes C(S^1)$$

be the tensor product of $\mathfrak{g}_K$ with the algebra of smooth complex functions on the circle. It is infinite dimensional and has a Lie bracket

$$[\gamma_1 \otimes f_1, \gamma_2 \otimes f_2] = [\gamma_1, \gamma_2] \otimes f_1 f_2$$

We may define a **Fourier transform** as follows. Let f be an integrable function. Let T_f be the corresponding distribution (generalized function)

$$T_f(\phi) = \int f(x)\phi(x)dx$$

for any Schwarz function ϕ.
[Gilkey], [Su], [], [], [], []

8. Graph C^*-Algebras

In this section we build the other half of the bridge between surface algebras and surface orders to the work of Matilde Marcolli, and her collaborators. For this material we refer primarily to [Marcolli], [Consani-Marcolli1, Consani-Marcolli2], [Connes-Marcolli1, Connes-Marcolli2].

8.1. **Analytic Completion of Leavitt Path Algebras.**

8.2. **Examples.**

Example 8.1. The algebra of $n \times n$ matrices corresponds to,

$$\begin{tikzcd} \bullet_1 \arrow[r, bend right] & \bullet_2 \arrow[l, dotted, bend right] \arrow[r, bend right] & \bullet_3 \arrow[l, dotted, bend right] & \cdots & \bullet_{n-2} \arrow[r, bend right] & \bullet_{n-1} \arrow[l, dotted, bend right] \arrow[r, bend right] & \bullet_n \arrow[l, dotted, bend right] \end{tikzcd}$$

Since the completion of this algebra is just the algebra itself we have that $C^*(Q) = \mathbf{Mat}_{n\times}(\mathbb{C}) = L(Q)$.

Example 8.2. Let $\hat{Q}$ be the localized (infinite) quiver,

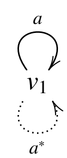
$$\begin{tikzcd} \cdots & \bullet_1 \arrow[r, bend right] & \bullet_2 \arrow[l, dotted, bend right] \arrow[r, bend right] & \bullet_3 \arrow[l, dotted, bend right] & \cdots & \bullet_{n-2} \arrow[r, bend right] & \bullet_{n-1} \arrow[l, dotted, bend right] \arrow[r, bend right] & \bullet_n \arrow[l, dotted, bend right] & \cdots \end{tikzcd}$$

This directed graph represents the algebra

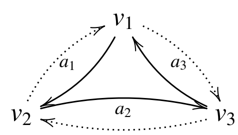
$$C^*(Q) = \bigoplus_{n=1}^{\infty} \mathbf{Mat}_{n\times n}(\mathbb{C}),$$

the algebra of compact operators.

Example 8.3. From the following graph we get the completion of the Leavitt path algebra $L(Q) = K[x, x^{-1}]$, is $C^*(Q) = C(S^1)$, the algebra continuous functions on $S^1 \subset \mathbb{C}$. This fact is proven using the Stone-Weierstrass theorem.



Example 8.4. The algebra $\mathbf{Mat}_{n\times n}(K) \otimes C(S^1)$ can be obtained as follows. Let Q be the following graph:



Let $L(Q)$ be the Leavitt path algebra for the algebra $\mathbf{Mat}_{n\times n}(K) \otimes K[x]$. Then

$$L(Q) \cong \mathbf{Mat}_{n\times n}(K) \otimes K[x, x^{-1}].$$

For this example, define the map $f : KQ \to \mathbf{Mat}_{3\times 3}(K) \otimes K[x]$ as before by:

$$\begin{aligned} v_1 &\mapsto E_{11} \otimes 1 & a_1 &\mapsto E_{21} \otimes 1 \\ v_2 &\mapsto E_{22} \otimes 1 & a_2 &\mapsto E_{32} \otimes 1 \\ v_3 &\mapsto E_{33} \otimes 1 & a_3 &\mapsto E_{13} \otimes x \end{aligned}$$

and the maps for a_i^* of

$$f : L(Q) \to \mathbf{Mat}_{n\times n}(K) \otimes K[x, x^{-1}],$$

are

$$\begin{aligned} a_1^* &\mapsto E_{12} \otimes 1 \\ a_2^* &\mapsto E_{23} \otimes 1 \\ a_3^* &\mapsto E_{31} \otimes x^{-1} \end{aligned}$$

Since $C(S^1)$ is the analytic completion of the algebra $K[x, x^{-1}]$ we have that for the algebra generated by the Cuntz-Krieger Q-family ($L(Q) \cong \mathbf{Mat}_{3\times 3} \otimes K[x, x^{-1}]$), the completion is

$$\overline{L(Q)} = C^*(Q) \cong \mathbf{Mat}_{n\times n} \otimes C(S^1)$$

as desired. We can see that the C^*-algebra obtained from a cycle on n vertices is $C^*(Q) = \mathbf{Mat}_{n\times n} \otimes C(S^1)$.

8.3. **The C^* Algebras of Cyclic Quivers $\mathbf{Mat}_{n\times n}(K) \otimes C(S^1)$ and K-theory.** So, beginning with a cyclic quiver Q, we get a path algebra $KQ \cong \mathbf{Mat}_{n\times n}(K) \otimes K[x]$. The Leavitt path algebras is then given by localization and is

$$L(Q) = \mathbf{Mat}_{n\times n}(K) \otimes K[x, x^{-1}].$$

Finally, taking the analytic completion of this algebra to obtain the graph C^*-algebra, we get

$$C^*(Q) = \mathbf{Mat}_{n\times n}(K) \otimes C(S^1).$$

Now, we should mention a few basic facts about the algebra $C^*(Q)$ and its relevance to K-theory.

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