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Andrzej Skowroński
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Frobenius Algebras II

Tilted and Hochschild Extension Algebras

European Mathematical Society

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To our wifes
Mira and Taeko

and children
Magda, Akiko, Ikuo and Taketo

Contents

Introduction

The main topic of this book is the representation theory of finite dimensional associative algebras with an identity over a field, which currently can be regarded as the study of the categories of their finite dimensional modules and the associated combinatorial and homological invariants. A prominent role in the representation theory of finite dimensional algebras over fields is played by Frobenius algebras. The Frobenius algebras originated in the 1903 papers by Frobenius and received modern characterizations in a series of papers by Brauer, Nesbitt and Nakayama from 1937–1941. In particular, we may say that a finite dimensional algebra A over a field K is a Frobenius algebra if there exists a nondegenerate K-bilinear form $(-,-)\colon A \times A \to K$ which is associative, in the sense that $(ab, c) = (a, bc)$ for all elements a, b, c of A. Frobenius algebras are selfinjective algebras (projective and injective modules coincide), and the module category of every finite dimensional selfinjective algebra over a field is equivalent to the module category of a Frobenius algebra.

The book is divided into three volumes and its main aim is to provide a comprehensive introduction to the modern representation theory of finite dimensional algebras over fields, with special attention devoted to the representation theory of Frobenius algebras, or more generally selfinjective algebras. The book is primarily addressed to graduate students starting research in the representation theory of algebras, as well as to mathematicians working in other related fields. It is hoped that the book will provide a friendly access to the representation theory of finite dimensional algebras, as the only prerequisite is a basic knowledge of linear algebra. We present complete proofs of all results stated in the book. Moreover, a rich supply of examples and exercises will help the reader understand and master the theory presented in the book.

In the first volume of the book, “Frobenius Algebras I. Basic Representation Theory” [SY2], divided into six chapters, we provided a general introduction to basic results and techniques of the modern representation theory of finite dimensional algebras over fields, including the Morita equivalences and the Morita–Azumaya dualities for the module categories, and the Auslander–Reiten theory of irreducible homomorphisms and almost split sequences. The heart of the first volume is devoted to presenting fundamental classical as well as recent results concerning the selfinjective algebras and their module categories. Moreover, two chapters of the first volume are devoted to basic properties of two classical classes of Frobenius algebras formed by the Hecke algebras of finite Coxeter groups and the finite dimensional Hopf algebras.

In the second volume of the book we continue to present basic results and techniques of the modern representation theory of finite dimensional algebras over fields as well as exhibit a new wide class of selfinjective algebras and describe their representation theory.

The second volume of the book is divided into four chapters, each of which is subdivided into sections. We start with Chapter VII presenting a rather detailed representation theory of finite dimensional hereditary algebras over fields. The indecomposable finite dimensional hereditary algebras over a field are divided into three disjoint subclasses (hereditary algebras of Dynkin type, of Euclidean type, and of wild type) according to the behaviour of the associated Euler quadratic form on their Grothendieck group. Chapter VIII is devoted to introducing the tilting theory of finite dimensional algebras over fields and describing basic properties of the tilted algebras, which are the endomorphism algebras of tilting modules over finite dimensional hereditary algebras. In Chapter IX we introduce a functorial approach to the representation theory of finite dimensional algebras over fields and elements of the theory of degrees of irreducible homomorphisms in module categories. As an application, we obtain a complete description of the shapes of infinite stable components in the Auslander–Reiten quivers of finite dimensional selfinjective algebras over fields. The final Chapter X is devoted to the theory of Hochschild extensions of finite dimensional algebras over fields by duality bimodules, which form a prominent wide class of finite dimensional selfinjective algebras. In particular, we describe completely the representation theory of arbitrary Hochschild extensions of finite dimensional hereditary algebras over fields by means of duality bimodules.

The third volume of the book, “Frobenius Algebras III. Orbit Algebras”, will be devoted to the study of Frobenius algebras as the orbit algebras of repetitive categories of finite dimensional algebras over fields with respect to actions of admissible automorphism groups. In particular, we will introduce the covering techniques which frequently allow to reduce the representation theory of Frobenius algebras to the representation theory of algebras of small homological dimension. A prominent role in these investigations will be played by the tilting theory and the authors’ theory of selfinjective algebras with deforming ideals.

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Chapter VII

Hereditary algebras

This chapter is devoted to a comprehensive introduction to the representation theory of finite dimensional hereditary algebras over a field, which is one of the most extensively studied and best understood representation theories developed so far. The theory will serve as the starting point to create more complicated representation theories of Frobenius algebras of tilted type, which are the orbit algebras of the repetitive categories of the endomorphism algebras of tilting modules over hereditary algebras.

We start by introducing the quiver of a finite dimensional algebra over a field and prove its useful description in terms of the extension spaces of simple modules. Then we introduce a wide class of finite dimensional hereditary algebras formed by the tensor algebras of systems of bimodules over finite dimensional division algebras, containing the class of path algebras of finite acyclic quivers over fields as a prominent subclass. Next we prove several results on exact sequences of modules which play a fundamental role in the further considerations. The main concepts introduced in this section are the homomological nonsymetric bilinear form and its quadratic form on the Grothendieck group of a finite dimensional hereditary algebra over a field K, called the Euler forms. This allows to divide the class of indecomposable finite dimensional hereditary algebras over a field into three disjoint subclasses: the hereditary algebras of Dynkin type, the hereditary algebras of Euclidean type, and the hereditary algebras of wild type, given by the shape of the quivers of algebras for which the Euler quadratic form is respectively positive definite, positive semidefinite with nonzero radical, and indefinite. Moreover, we introduce the Coxeter transformation on the Grothendieck group of a finite dimensional hereditary algebra over a field and use it to describe the structure of postprojective and preinjective components of its Auslander–Reiten quiver. The next three sections of this chapter are devoted to the representation theories of hereditary algebras of Dynkin type, Euclidean type, and wild type, and to the description of their Auslander–Reiten quivers. In the final section we introduce matrix algebras of bimodules over finite dimensional algebras over a field and describe their module categories.

The starting point of the modern representation theory of finite dimensional hereditary algebras over a field was the pioneering paper by P. Gabriel [Ga1] from 1972, where the hereditary algebras of finite representation type over an algebraically closed field K and their indecomposable modules are described. This was done by showing that these algebras are exactly the path algebras of Dynkin quivers of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$, in which case there is a bijection between the isomorphism classes of the indecomposable representations of the

Dynkin quiver and the positive roots of the associated Tits quadratic form of the quiver. One year later P. Donovan and M. R. Freislich [DF], and independently L. A. Nazarova [N], provided descriptions of the indecomposable finite dimensional representations of Euclidean quivers of type $\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$ over an algebraically closed field as well as of their relations with the positive roots and positive radical vectors of the associated Tits quadratic form. This was extended to similar results for finite dimensional hereditary algebras over a field in the series of papers [DR1], [DR2], [DR3] by V. Dlab and C. M. Ringel. The structure of the category of finite dimensional modules over a hereditary algebra of Euclidean type over an algebraically closed field was described in the book [R3] by C. M. Ringel (see also the books [SS1], [SS2]). The shapes of components of the Auslander–Reiten quivers of the remaining finite dimensional hereditary algebras of wild type over a field were described by C. M. Ringel [R1], [R2], M. Auslander and M. I. Platzeck [AP], and M. Auslander, R. Bautista, M. I. Platzeck, I. Reiten and S. O. Smalø [ABPRS]. Important properties of finite dimensional regular modules over wild hereditary algebras over a field were exhibited by D. Baer [Ba2], O. Kerner [K1], [K2], [K3], O. Kerner and F. Lukas [KL1], [KL2], O. Kerner and A. Skowroński [KS2], and Y. Zhang [Z1]. We refer also to the papers [Ba1], [BBL], [BGL], [BL], [Ge1], [Ge2], by D. Baer, W. Geigle, H. Brune and H. Lenzing, for homological properties of module categories of finite dimensional hereditary algebras.

1 The quiver of an algebra

In this section we associate to any finite dimensional algebra over a field a valued quiver, playing a prominent role in further considerations.

Let A be a finite dimensional K-algebra over a field K. Then it follows from Corollary I.5.9 that there is a decomposition

$$1_A = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$$

of the identity 1_A of A into a sum of pairwise orthogonal primitive idempotents of A such that

$$\begin{aligned} e_{ij}A &\cong e_{il}A && \text{for } j,l \in \{1,\dots,m_A(i)\}, i \in \{1,\dots,n_A\}, \\ e_{ij}A &\ncong e_{kl}A && \text{for } i,k \in \{1,\dots,m_A(i)\} \text{ with } i \neq k \text{ and} \\ & && \text{all } j \in \{1,\dots,m_A(i)\}, l \in \{1,\dots,m_A(k)\}. \end{aligned}$$

We call such a decomposition of 1_A a canonical decomposition of 1_A. The idempotents $e_i = e_{i1}$, $i \in \{1,\dots,n_A\}$, are called basic primitive idempotents of A

(see Section IV.6). Then it follows from Propositions I.8.2 and I.8.19, Corollary I.8.6, and Lemma I.8.22 that

- $P_i = e_i A$, $i \in \{1, \dots, n_A\}$, is a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} A$;
- $I_i = D(Ae_i)$, $i \in \{1, \dots, n_A\}$, is a complete set of pairwise nonisomorphic indecomposable injective modules in $\operatorname{mod} A$;
- $S_i = \operatorname{top}(P_i) = e_i A / e_i \operatorname{rad} A$, $i \in \{1, \dots, n_A\}$, is a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$;
- $S_i \cong \operatorname{soc}(I_i)$, for any $i \in \{1, \dots, n_A\}$.

The idempotent

$$e_A = \sum_{i=1}^{n_A} e_{i1} = \sum_{i=1}^{n_A} e_i$$

is called a basic idempotent of A. Further, the finite dimensional K-algebra $A^{\mathrm{b}} = e_A A e_A$, with the identity $1_{A^{\mathrm{b}}} = e_A$, is called the basic algebra of A. Moreover, A and A^{b} are Morita equivalent (Theorem II.6.16). In fact, $e_A A$ is a minimal progenerator in $\operatorname{mod} A$ and the functors

$$\operatorname{res}_{e_A} = (-)e_A \colon \operatorname{mod} A \longrightarrow \operatorname{mod} A^{\mathrm{b}},$$

isomorphic to $\operatorname{Hom}_A(e_A A, -)$, and

$$- \otimes e_A A \colon A^{\mathrm{b}} \longrightarrow \operatorname{mod} A,$$

isomorphic to $\operatorname{Hom}_{A^{\mathrm{b}}}(\operatorname{Hom}_A(e_A A, A), -)$, define a Morita equivalence between the categories $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{b}}$ (see Theorem II.6.7). For each $i \in \{1, \dots, n_A\}$, $F_i = \operatorname{End}_A(S_i)$ is a finite dimensional division K-algebra (Lemma I.5.1), which is isomorphic to the K-algebra $e_i A e_i / e_i (\operatorname{rad} A) e_i$ (see Lemma I.11.2). We will identify F_i with $e_i A e_i / e_i (\operatorname{rad} A) e_i$ for all $i \in \{1, \dots, n_A\}$. Finally, $\operatorname{rad} A / (\operatorname{rad} A)^2$ is a semisimple left and right A-module, and, for any $i, j \in \{1, \dots, n_A\}$,

$$e_i (\operatorname{rad} A) e_j / e_i (\operatorname{rad} A)^2 e_j = e_i \big(\operatorname{rad} A / (\operatorname{rad} A)^2 \big) e_j$$

is an (F_i, F_j)-bimodule with the left F_i-module and the right F_j-bimodule structures given by

$$\big(e_i a e_i + e_i (\operatorname{rad} A) e_i\big)\big(e_i x e_j + e_i (\operatorname{rad} A)^2 e_j\big) = e_i a e_i x e_j + e_i (\operatorname{rad} A)^2 e_j,$$
$$\big(e_i x e_j + e_i (\operatorname{rad} A)^2 e_j\big)\big(e_j a e_j + e_j (\operatorname{rad} A) e_j\big) = e_i x e_j a e_j + e_i (\operatorname{rad} A)^2 e_j,$$

for $a \in A$ and $x \in \operatorname{rad} A$.

The *quiver* Q_A of A is the valued quiver defined as follows:

(a) The vertices of Q_A are the indices $1, \dots, n_A$ of a complete set of basic primitive idempotents of A.

(b) For two vertices i and j in Q_A, there exists an arrow $i \to j$ if and only if $e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2 e_j \neq 0$. Moreover, we associate to an arrow $i \to j$ of Q_A the valuation (d_{ij}, d'_{ij}), so we have in Q_A the valued arrow

$$i \xrightarrow{(d_{ij}, d'_{ij})} j,$$

where

$$\begin{aligned} d_{ij} &= \dim_{F_j} e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2 e_j, \\ d'_{ij} &= \dim_{F_i} e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2 e_j. \end{aligned}$$

Instead of an arrow $i \xrightarrow{(1,1)} j$ of Q_A we will write simply $i \to j$. We note that we have the following equalities of quivers

$$Q_{A^{\flat}} = Q_A = Q_{A/(\operatorname{rad} A)^2}.$$

For $i \in \{1, \dots, n_A\}$, we set $f_i = \dim_K F_i$.

The following lemma shows that the valuations of arrows of the quiver Q_A are symmetrizable.

Lemma 1.1. *Let A be a finite dimensional K-algebra over a field K and $i \xrightarrow{(d_{ij}, d'_{ij})} j$ an arrow of Q_A. Then $f_i d'_{ij} = d_{ij} f_j$.*

Proof. Since K acts centrally on A, we have the equalities

$$\begin{aligned} f_i d'_{ij} &= (\dim_K F_i)\big(\dim_{F_i} e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2 e_j\big) \\ &= \dim_K e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2 e_j \\ &= \big(\dim_{F_j} e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2 e_j\big)(\dim_K F_j) \\ &= d_{ij} f_j. \end{aligned}$$ □

Corollary 1.2. *Let A be a finite dimensional K-algebra over an algebraically closed field K and $i \xrightarrow{(d_{ij}, d'_{ij})} j$ an arrow of Q_A. Then $d_{ij} = d'_{ij}$.*

Proof. Since K is algebraically closed, $f_i = \dim_K F_i = \dim_K K = 1$ for any $i \in \{1, \dots, n_A\}$, by Corollary I.5.2, and the claim follows. □

In the representation theory of finite dimensional algebras over an algebraically closed field K, instead of a valued arrow $i \xrightarrow{(m,m)} j$ of the quiver of a K-algebra A, usually one writes a multiple arrow

$$\begin{tikzcd} i \arrow[r, shift left=4] \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=4] & j \end{tikzcd}$$

consisting of m arrows from i to j (see [ASS], [SS1], [SS2]).

Examples 1.3. (a) Let A be the following $\mathbb{R}$-subalgebra of the matrix algebra $M_2(\mathbb{C})$:

$$\begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \in M_2(\mathbb{C}) \;\middle|\; a \in \mathbb{R}, b, c \in \mathbb{C} \right\}.$$

Then A has the standard basic primitive idempotents

$$e_1 = \begin{bmatrix} 1_{\mathbb{R}} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathbb{C}} \end{bmatrix},$$

with $1_A = e_1 + e_2$, and

$$\operatorname{rad} A = \begin{bmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{bmatrix}$$

(see Example III.10.6(a)). Further, we have $(\operatorname{rad} A)^2 = 0$ and $e_1(\operatorname{rad} A)e_1 = 0$, $e_2(\operatorname{rad} A)e_2 = 0$, $e_1(\operatorname{rad} A)e_2 = 0$, $e_2(\operatorname{rad} A)e_1 \cong \mathbb{C}$. Moreover, $F_1 = e_1Ae_1/e_1(\operatorname{rad} A)e_1 = \mathbb{R}$ and $F_2 = e_2Ae_2/e_2(\operatorname{rad} A)e_2 = \mathbb{C}$. Hence, the quiver Q_A of A is of the form

$$2 \xrightarrow{(2,1)} 1,$$

because $2 = \dim_{\mathbb{R}} \mathbb{C} = \dim_{F_1} e_2(\operatorname{rad} A)e_1 = d_{21}$ and $1 = \dim_{\mathbb{C}} \mathbb{C} = \dim_{F_2} e_2(\operatorname{rad} A)e_1 = d'_{21}$.

(b) Let $\mathbb{H}$ be the $\mathbb{R}$-algebra of quaternions and A the following $\mathbb{R}$-subalgebra of the matrix algebra $M_4(\mathbb{H})$:

$$\begin{bmatrix} \mathbb{R} & 0 & 0 & 0 \\ \mathbb{R} & \mathbb{R} & 0 & 0 \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & 0 \\ \mathbb{H} & 0 & 0 & \mathbb{H} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ x & b & 0 & 0 \\ y & z & c & 0 \\ h & 0 & 0 & d \end{bmatrix} \in M_4(\mathbb{H}) \;\middle|\; \begin{matrix} a, b, c, x, y, z \in \mathbb{R} \\ d, h \in \mathbb{H} \end{matrix} \right\}.$$

Then A has the standard basic primitive idempotents

$$e_1 = \begin{bmatrix} 1_{\mathbb{R}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{\mathbb{R}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{\mathbb{R}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{\mathbb{H}} \end{bmatrix},$$

with $1_A = e_1 + e_2 + e_3 + e_4$. We claim that

$$\operatorname{rad} A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbb{R} & 0 & 0 & 0 \\ \mathbb{R} & \mathbb{R} & 0 & 0 \\ \mathbb{H} & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, the right side J of the required equality is a two-sided ideal of A with $J^3 = 0$ and A/J is isomorphic to the product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}$ of division $\mathbb{R}$-algebras (see Lemma I.3.5). Further, we obtain

$$\begin{gathered} e_1 \operatorname{rad} A = 0, \ \ e_2 \operatorname{rad} A = e_2(\operatorname{rad} A)e_1 = \mathbb{R}, \\ e_3 \operatorname{rad} A = e_3(\operatorname{rad} A)e_1 \oplus e_3(\operatorname{rad} A)e_2 = \mathbb{R} \oplus \mathbb{R}, \\ e_4 \operatorname{rad} A = e_4(\operatorname{rad} A)e_1 = \mathbb{H}, \\ (\operatorname{rad} A)^2 = e_3(\operatorname{rad} A)^2 e_1 = e_3(\operatorname{rad} A)e_1 = \mathbb{R}. \end{gathered}$$

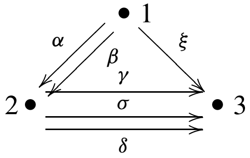
Moreover, we have

$$\begin{aligned} F_1 &= e_1Ae_1/e_1(\operatorname{rad} A)e_1 = \mathbb{R}, & F_2 &= e_2Ae_2/e_2(\operatorname{rad} A)e_2 = \mathbb{R}, \\ F_3 &= e_3Ae_3/e_3(\operatorname{rad} A)e_3 = \mathbb{R}, & F_4 &= e_4Ae_4/e_4(\operatorname{rad} A)e_4 = \mathbb{H}. \end{aligned}$$

Therefore, the quiver Q_A of A is of the form

$$3 \xrightarrow{(1,1)} 2 \xrightarrow{(1,1)} 1 \xleftarrow{(1,4)} 4.$$

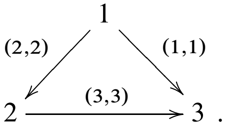
(c) Let K be a field, Q the quiver



and $A = KQ$ the path algebra of Q over K. Since Q is an acyclic quiver, A is a finite dimensional K-algebra (Lemma I.1.3). Further, it follows from Corollary I.3.7 that $\operatorname{rad} A$ coincides with the two-sided ideal R_Q of A generated by all arrows of Q. Hence $(\operatorname{rad} A)^2 = R_Q^2$ is the 6-dimensional subspace of A generated by the paths $\alpha\gamma, \alpha\sigma, \alpha\delta, \beta\gamma, \beta\sigma, \beta\delta$. Let $e_1 = \varepsilon_1, e_2 = \varepsilon_2, e_3 = \varepsilon_3$ be the primitive idempotents of A given by the trivial paths at the vertices 1, 2, and 3. Observe also that $e_i(\operatorname{rad} A)e_i = 0$ and $F_i = e_iAe_i/e_i(\operatorname{rad} A)e_i = e_iAe_i = K$ for any $i \in \{1, 2, 3\}$. Finally, we conclude that

$$\begin{gathered}
e_1(\operatorname{rad} A)e_2/e_1(\operatorname{rad} A)^2e_2 = e_1(\operatorname{rad} A)e_2 = K\alpha \oplus K\beta,\\
e_2(\operatorname{rad} A)e_3/e_2(\operatorname{rad} A)^2e_3 = e_2(\operatorname{rad} A)e_3 = K\gamma \oplus K\sigma \oplus K\delta,\\
e_1(\operatorname{rad} A)e_3/e_1(\operatorname{rad} A)^2e_3 = K\xi,\\
e_2(\operatorname{rad} A)e_1 = 0,\;\; e_3(\operatorname{rad} A)e_2 = 0,\;\; e_3(\operatorname{rad} A)e_1 = 0.
\end{gathered}$$

Therefore, the quiver Q_A of A is of the form



We note that Q_A is different from Q.

Lemma 1.4. *Let K be a field, Q a finite quiver, I an admissible ideal of KQ, and $A = KQ/I$ the associated bound quiver algebra. Assume that, for any two vertices i and j of Q, there is at most one arrow from i to j in Q. Then $Q_A = Q$.*

Proof. Let Q_0 and Q_1 be the sets of vertices and arrows of Q, respectively. It follows from Lemmas I.1.5 and I.3.6 that A is a finite dimensional K-algebra and $\operatorname{rad} A = R_Q/I$, where R_Q is the arrow ideal of KQ. Moreover, by Lemma II.6.17, A is a basic algebra and the classes $e_a = \varepsilon_a + I$ of the trivial paths ε_a at the vertices $a \in Q_0$ form a set of basic primitive idempotents of A. Then, for any two vertices $a, b \in Q_0$, we have isomorphisms of K-vector spaces

$$e_a(\operatorname{rad} A)e_b/e_a(\operatorname{rad} A)^2e_b \cong \varepsilon_a R_Q \varepsilon_b/\varepsilon_a R_Q^2 \varepsilon_b,$$

and hence $\dim_K e_a(\operatorname{rad} A)e_b/e_a(\operatorname{rad} A)^2e_b$ is the number of arrows $\alpha \in Q_1$ with $s(\alpha) = a$ and $t(\alpha) = b$. Observe also that $F_a = e_aAe_a/e_a(\operatorname{rad} A)e_a = K$ for any $a \in Q_0$. Therefore, we have $Q_A = Q$, by the assumption imposed on Q. □

We exhibit now the following characterization of semisimple algebras by their quivers.

Proposition 1.5. *Let A be a finite dimensional K-algebra over a field K. The following conditions are equivalent:*

(i) *A is a semisimple algebra.*

(ii) *Q_A consists of isolated vertices.*

Proof. Let $1_A = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$ be a canonical decomposition of 1_A into a sum of pairwise orthogonal primitive idempotents of A, so $e_i = e_{i1}, i \in \{1, \dots, n_A\}$, form a set of basic primitive idempotents of A.

Assume A is a semisimple algebra. Then $\operatorname{rad} A = 0$, and consequently $e_i(\operatorname{rad} A)e_j = 0$ for any $i, j \in \{1, \dots, n_A\}$. This shows that Q_A consists of isolated vertices $1, \dots, n_A$, and hence (i) implies (ii).

Assume Q_A consists of isolated vertices. Consider the basic algebra $A^{\mathrm{b}} = e_A A e_A$ of A, where $e_A = e_{11} + \dots + e_{n_A 1}$. Since $Q_A = Q_{A^{\mathrm{b}}}$, we infer that $Q_{A^{\mathrm{b}}}$ consists of isolated vertices $1, \dots, n_A$. Clearly, then $\operatorname{rad} A^{\mathrm{b}} = 0$, and consequently A^{b} is a semisimple algebra. In particular, applying Theorem I.6.3, we conclude that every module in $\operatorname{mod} A^{\mathrm{b}}$ is semisimple. Invoking now the Morita equivalence functor $- \otimes_{A^{\mathrm{b}}} e_A A \colon \operatorname{mod} A^{\mathrm{b}} \to \operatorname{mod} A$ we infer from Proposition II.6.6 that every module in $\operatorname{mod} A$ is semisimple. Applying Theorem I.6.3 again, we infer that A is a semisimple algebra, and hence (ii) implies (i). $\square$

Proposition 1.6. *Let A be a finite dimensional K-algebra over a field K, $e_1, \dots, e_n$ basic primitive idempotents of A, $P_1 = e_1 A, \dots, P_n = e_n A$ the associated indecomposable projective modules, $I_1 = D(Ae_1), \dots, I_n = D(Ae_n)$ the associated indecomposable injective modules, and $i_1, \dots, i_m$ vertices of Q_A (not necessarily pairwise different). The following conditions are equivalent:*

(i) *Q_A admits a path from i_1 to i_m of the form*

$$i_1 \longrightarrow \cdots \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_{m-1} \longrightarrow \cdots \longrightarrow i_m.$$

(ii) *There is a sequence of nonzero nonisomorphisms between indecomposable projective modules in the family $P_1, \dots, P_n$ of the form*

$$P_{i_m} \longrightarrow \cdots \longrightarrow P_{i_{m-1}} \longrightarrow \cdots \longrightarrow P_{i_2} \longrightarrow \cdots \longrightarrow P_{i_1}.$$

(iii) *There is a sequence of nonzero nonisomorphisms between indecomposable injective modules in the family $I_1, \dots, I_n$ of the form*

$$I_{i_m} \longrightarrow \cdots \longrightarrow I_{i_{m-1}} \longrightarrow \cdots \longrightarrow I_{i_2} \longrightarrow \cdots \longrightarrow I_{i_i}.$$

Proof. Since $Q_A = Q_{A^{\mathrm{b}}}$, we may assume (without loss of generality) that A is a basic algebra.

For (i) ⇒ (ii), observe that, for any arrow $i \xrightarrow{(d_{ij}, d'_{ij})} j$ in Q_A, we have $e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j \neq 0$, and hence $e_i(\operatorname{rad} A)e_j \neq 0$. Moreover, we have isomorphisms of K-vector spaces

$$e_i(\operatorname{rad} A)e_j \cong \operatorname{Hom}_A(e_j A, e_i \operatorname{rad} A) = \operatorname{Hom}_A(e_j A, \operatorname{rad} e_i A) = \operatorname{rad}_A(e_j A, e_i A).$$

Hence, $e_i(\operatorname{rad} A)e_j \neq 0$ implies the existence of a nonzero nonisomorphism from $P_j = e_j A$ to $P_i = e_i A$. Therefore, (i) implies (ii).

For (ii) $\Rightarrow$ (i), it is enough to show that, if $\operatorname{rad}_A(P_r, P_s) \neq 0$ for $r, s \in \{1, \dots, n\}$, then Q_A admits a path from s to r. Assume $\operatorname{rad}_A(P_r, P_s) \neq 0$ with $r, s \in \{1, \dots, n\}$. Then

$$\begin{aligned} e_s(\operatorname{rad} A)e_r &\cong \operatorname{Hom}_A(e_r A, e_s \operatorname{rad} A) = \operatorname{Hom}_A(e_r A, \operatorname{rad} e_s A) \\ &= \operatorname{rad}_A(P_r, P_s) \neq 0. \end{aligned}$$

Consider the radical series

$$e_s A \supset e_s \operatorname{rad} A \supset e_s(\operatorname{rad} A)^2 \supset \cdots \supset e_s(\operatorname{rad} A)^l \supset e_s(\operatorname{rad} A)^{l+1} = 0$$

of $P_s = e_s A$. Take the maximal p in $\{1, \dots, l\}$ such that $e_s(\operatorname{rad} A)^p e_r \neq 0$, so $e_s(\operatorname{rad} A)^{p+1} e_r = 0$. Then there exist elements $a_1, \dots, a_p \in \operatorname{rad} A$ such that $e_s a_1 \dots a_p e_r \neq 0$. Hence there exist $j_1, \dots, j_{p-1} \in \{1, \dots, n\}$ such that $e_s a_1 e_{j_1} a_2 e_{j_2} \dots e_{j_{p-1}} a_p e_r \neq 0$. Since $e_s(\operatorname{rad} A)^{p+1} e_r = 0$, we have $e_{j_{q-1}} a_q e_{j_q} \in e_{j_{q-1}}(\operatorname{rad} A)e_{j_q} \setminus e_{j_{q-1}}(\operatorname{rad} A)^2 e_{j_q}$ for all $q \in \{1, \dots, p\}$, where $j_0 = s$ and $j_p = r$. This shows that Q_A admits a path of the form

$$s = j_0 \xrightarrow{(d_{j_0 j_1}, d'_{j_0 j_1})} j_1 \xrightarrow{(d_{j_1 j_2}, d'_{j_1 j_2})} j_2 \to \cdots \to j_{p-1} \xrightarrow{(d_{j_{p-1} j_p}, d'_{j_{p-1} j_p})} j_p = r.$$

Therefore, (ii) implies (i).

We prove now that (i) and (iii) are equivalent. Since $Q_{A^{\mathrm{op}}} = Q_A^{\mathrm{op}}$, we infer that Q_A admits a path

$$i_1 \longrightarrow \cdots \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_{m-1} \longrightarrow \cdots \longrightarrow i_m$$

if and only if $Q_{A^{\mathrm{op}}}$ admits a path

$$i_m \longrightarrow \cdots \longrightarrow i_{m-1} \longrightarrow \cdots \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_1.$$

Applying the equivalence (i) $\Leftrightarrow$ (ii) to A^{op}, we conclude that the existence of a path from i_m to i_1 in $Q_{A^{\mathrm{op}}}$ is equivalent to the existence of a sequence

$$Ae_{i_1} \longrightarrow \cdots \longrightarrow Ae_{i_2} \longrightarrow \cdots \longrightarrow Ae_{i_{m-1}} \longrightarrow \cdots \longrightarrow Ae_{i_m}$$

of nonzero nonisomorphisms between indecomposable projective modules in $\operatorname{mod} A^{\mathrm{op}}$ (hence in the family $Ae_1, \dots, Ae_n$), or equivalently, to the existence of a sequence

$$I_{i_m} \longrightarrow \cdots \longrightarrow I_{i_{m-1}} \longrightarrow \cdots \longrightarrow I_{i_2} \longrightarrow \cdots \longrightarrow I_{i_1}$$

of nonzero nonisomorphisms between indecomposable injective modules in the family $I_1, \dots, I_n$. Therefore, (i) and (iii) are equivalent. □

Corollary 1.7. *Let A be a finite dimensional K-algebra over a field K. The following conditions are equivalent:*

(i) *A is an indecomposable algebra.*

(ii) *The quiver Q_A is connected.*

Proof. Let $1_A = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$ be a canonical decomposition of 1_A into a sum of pairwise orthogonal primitive idempotents of A, $e_i = e_{i1}$, $i \in \{1, \dots, n_A\}$, the associated basic primitive idempotents of A. Then we have isomorphisms of K-vector spaces

$$e_i A e_k \cong \operatorname{Hom}_A(e_k A, e_i A) \cong \operatorname{Hom}_A(e_{kl} A, e_{ij} A) \cong e_{ij} A e_{kl},$$

for all $i, k \in \{1, \dots, n_A\}$ and $j \in \{1, \dots, m_A(i)\}$, $l \in \{1, \dots, m_A(k)\}$. Moreover, for $i \neq k$ in $\{1, \dots, n_A\}$, we have

$$\begin{aligned} e_i A e_k &= \operatorname{Hom}_A(e_k A, e_i A) = \operatorname{Hom}_A(e_k A, \operatorname{rad} e_i A) \\ &= \operatorname{Hom}_A(e_k A, e_i \operatorname{rad} A) \cong e_i (\operatorname{rad} A) e_k. \end{aligned}$$

Hence, A is an indecomposable algebra if and only if for any two different vertices $i, k \in \{1, \dots, n_A\}$ of Q_A there is a sequence of vertices $i = j_0, j_1, \dots, j_t = k$ of Q_A such that

$$\operatorname{rad}_A(e_{j_{r-1}} A, e_{j_r} A) \neq 0 \quad \text{or} \quad \operatorname{rad}_A(e_{j_r} A, e_{j_{r-1}} A) \neq 0$$

for any $r \in \{1, \dots, t\}$. Then the equivalence of (i) and (ii) follows from the equivalence of (i) and (ii) in Proposition 1.6. □

Corollary 1.8. *Let A be a finite dimensional hereditary K-algebra over a field K. Then the quiver Q_A is acyclic.*

Proof. Assume Q_A admits an oriented cycle of the form

$$i_1 \xrightarrow{(d_{i_1 i_2}, d'_{i_1 i_2})} i_2 \xrightarrow{(d_{i_2 i_3}, d'_{i_2 i_3})} i_3 \longrightarrow \cdots \longrightarrow i_{m-1} \xrightarrow{(d_{i_{m-1} i_m}, d'_{i_{m-1} i_m})} i_m = i_1.$$

Then it follows from Proposition 1.6 that there is a sequence of nonzero nonisomorphisms

$$P_{i_1} = P_{i_m} \xrightarrow{f_{m-1}} P_{i_{m-1}} \xrightarrow{f_{m-2}} \dots \longrightarrow P_{i_2} \xrightarrow{f_1} P_{i_1}.$$

Applying Corollary I.9.4, we obtain that $f_1, f_2, \dots, f_{m-1}$ are proper monomorphisms, and consequently we obtain the inequalities

$$\ell(P_{i_1}) = \ell(P_{i_m}) < \ell(P_{i_{m-1}}) < \cdots < \ell(P_{i_2}) < \ell(P_{i_1}),$$

a contradiction. Therefore, the quiver Q_A is acyclic. □

We will provide an alternative description of the quiver Q_A of an algebra A invoking the extension spaces of simple modules in $\operatorname{mod} A$.

Let A be a finite dimensional K-algebra over a field K. Then it follows from Propositions III.3.8 and III.3.10 that, for any modules L and N in $\operatorname{mod} A$, there exist natural isomorphisms

$$\operatorname{Ext}_A^1(N, L) \cong \mathcal{E}\mathrm{xt}_A^1(N, L) \cong \widetilde{\operatorname{Ext}}_A^1(N, L)$$

of $(\operatorname{End}_A(L), \operatorname{End}_A(N))$-bimodules. In particular, for any vertices i and j in Q_A, we have natural isomorphisms

$$\operatorname{Ext}_A^1(S_i, S_j) \cong \mathcal{E}\mathrm{xt}_A^1(S_i, S_j) \cong \widetilde{\operatorname{Ext}}_A^1(S_i, S_j)$$

of (F_j, F_i)-bimodules, under the fixed identifications $F_i = \operatorname{End}_A(S_i)$ and $F_j = \operatorname{End}_A(S_j)$.

Theorem 1.9. *Let A be a finite dimensional K-algebra over a field K and $i, j \in \{1, \dots, n_A\}$. Then*

(i) $\dim_{F_j} \operatorname{Ext}_A^1(S_i, S_j) = \dim_{F_j} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j$.

(ii) $\dim_{F_i} \operatorname{Ext}_A^1(S_i, S_j) = \dim_{F_i} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j$.

Proof. (i) We set $n = n_A$. It follows from Proposition I.5.13 and Corollary I.5.15 that

$$\operatorname{rad} P_i / \operatorname{rad}^2 P_i = e_i \operatorname{rad} A / e_i(\operatorname{rad} A)^2 = e_i\big(\operatorname{rad} A/(\operatorname{rad} A)^2\big)$$

is a semisimple module in $\operatorname{mod} A$, and hence we have an isomorphism $\operatorname{rad} P_i / \operatorname{rad}^2 P_i \cong \bigoplus_{k=1}^n S_k^{r_{ik}}$ for some nonnegative integers r_{ik}, $k \in \{1, \dots, n\}$. Further, applying Lemma I.8.7, we obtain isomorphisms of right F_j-spaces

$$\begin{aligned} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j &\cong \operatorname{Hom}_A\Big(e_j A, e_i\big(\operatorname{rad} A/(\operatorname{rad} A)^2\big)\Big) \\ &\cong \operatorname{Hom}_A\Big(e_j A / e_j \operatorname{rad} A, e_i\big(\operatorname{rad} A/(\operatorname{rad} A)^2\big)\Big) \\ &\cong \operatorname{Hom}_A\left(S_j, \bigoplus_{k=1}^n S_k^{r_{ik}}\right) \\ &\cong \operatorname{End}_A(S_j)^{r_{ij}}. \end{aligned}$$

Hence,

$$\dim_{F_j} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j = r_{ij}.$$

We will show now that r_{ij} is the dimension of the left F_j-space $\operatorname{Ext}_A^1(S_i, S_j)$. Observe first that

$$\operatorname{top}(\operatorname{rad} P_i) = \operatorname{rad} P_i / \operatorname{rad}(\operatorname{rad} P_i) = \operatorname{rad} P_i / \operatorname{rad}^2 P_i \cong \bigoplus_{k=1}^n S_k^{r_{ik}}.$$

Thus, applying Theorem I.8.4, we conclude that there is a projective cover

$$u_1\colon \bigoplus_{k=1}^{n} P_k^{r_{ik}} \longrightarrow \operatorname{rad} P_i$$

of $\operatorname{rad} P_i$ in $\operatorname{mod} A$. Therefore, S_i admits a minimal projective resolution in $\operatorname{mod} A$

$$\ldots \longrightarrow \mathbb{P}_3 \xrightarrow{u_3} \mathbb{P}_2 \xrightarrow{u_2} \mathbb{P}_1 \xrightarrow{u_1} \mathbb{P}_0 \xrightarrow{u_0} S_i \longrightarrow 0$$

with $\mathbb{P}_0 = P_i$, $\mathbb{P}_1 = \bigoplus_{k=1}^{n} P_k^{r_{ik}}$ and $\operatorname{Im} u_2 \subseteq \operatorname{rad} \mathbb{P}_1$. Applying the contravariant functor $\operatorname{Hom}_A(-, S_j)$ we obtain the sequence of K-vector spaces

$$\operatorname{Hom}_A(\mathbb{P}_0, S_j) \xrightarrow{\operatorname{Hom}_A(u_1,S_j)} \operatorname{Hom}_A(\mathbb{P}_1, S_j) \xrightarrow{\operatorname{Hom}_A(u_2,S_j)} \operatorname{Hom}_A(\mathbb{P}_2, S_j)$$

and

$$\operatorname{Ext}_A^1(S_i, S_j) = \operatorname{Ker} \operatorname{Hom}_A(u_2, S_j) / \operatorname{Im} \operatorname{Hom}_A(u_1, S_j).$$

Observe that $\operatorname{Hom}_A(u_1, S_j) = 0$ and $\operatorname{Hom}_A(u_2, S_j) = 0$, because S_j is a simple module, $\operatorname{Im} u_1 = \operatorname{rad} \mathbb{P}_0$, and $\operatorname{Im} u_2 \subseteq \operatorname{rad} \mathbb{P}_1$. Hence we get isomorphisms of left F_j-spaces

$$\operatorname{Ext}_A^1(S_i, S_j) = \operatorname{Hom}_A(\mathbb{P}_1, S_j) \cong \operatorname{Hom}_A\left(\bigoplus_{k=1}^{n} S_k^{r_{ik}}, S_j\right) = \operatorname{End}_A(S_j)^{r_{ij}}.$$

Summing up, we obtain the required equality

$$\dim_{F_j} \operatorname{Ext}_A^1(S_i, S_j) = \dim_{F_j} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j.$$

(ii) Since there is an isomorphism $\operatorname{Ext}_A^1(S_i, S_j) \cong \widetilde{\operatorname{Ext}}_A^1(S_i, S_j)$ of (F_j, F_i)-bimodules, it is enough to show the equality

$$\dim_{F_i} \widetilde{\operatorname{Ext}}_A^1(S_i, S_j) = \dim_{F_i} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j.$$

Consider the left part of a minimal injective resolution of S_j in $\operatorname{mod} A$

$$0 \longrightarrow S_j \xrightarrow{v^0} \mathbb{I}_0 \xrightarrow{v^1} \mathbb{I}_1 \xrightarrow{v^2} \mathbb{I}_2 \xrightarrow{v^3} \mathbb{I}_3 \longrightarrow \cdots$$

Then $\mathbb{I}_0 = I_j$ and v^0 is a monomorphism $S_j \to I_j$, v^1 is the composition of the canonical epimorphism $I_j \to I_j/S_j$ with an injective envelope $I_j/S_j \to \mathbb{I}_1$ of I_j/S_j in $\operatorname{mod} A$, and v^2 is the composition of the canonical epimorphism $\mathbb{I}_1 \to \operatorname{Coker} v^1$ with an injective envelope $\operatorname{Coker} v^1 \to \mathbb{I}_2$ of $\operatorname{Coker} v^1$ in $\operatorname{mod} A$. Applying the covariant functor $\operatorname{Hom}_A(S_i, -)$, we obtain the sequence of K-vector spaces

$$\operatorname{Hom}_A(S_i, \mathbb{I}_0) \xrightarrow{\operatorname{Hom}_A(S_i,v^1)} \operatorname{Hom}_A(S_i, \mathbb{I}_1) \xrightarrow{\operatorname{Hom}_A(S_i,v^2)} \operatorname{Hom}_A(S_i, \mathbb{I}_2),$$

and

$$\widetilde{\operatorname{Ext}}_A^1(S_i, S_j) = \operatorname{Ker}\operatorname{Hom}_A(S_i, v^2)/\operatorname{Im}\operatorname{Hom}_A(S_i, v^1).$$

Observe that $\operatorname{Hom}_A(S_i, v^1) = 0$ and $\operatorname{Hom}_A(S_i, v^2) = 0$, because $v^1(S_i) = 0$ and $v^2(\operatorname{soc}\mathbb{I}_1) = 0$. Hence we get

$$\widetilde{\operatorname{Ext}}_A^1(S_i, S_j) = \operatorname{Hom}_A(S_i, \mathbb{I}_1).$$

Let $\mathbb{I}_1 \cong \bigoplus_{k=1}^n I_k^{t_{kj}}$ be a decomposition of $\mathbb{I}_1$ into a direct sum of indecomposable injective modules $I_1, \dots, I_n$, with t_{kj} nonnegative integers. Then we have isomorphisms of right F_i-spaces

$$\operatorname{Hom}_A(S_i, \mathbb{I}_1) \cong \operatorname{Hom}_A\left(S_i, \bigoplus_{k=1}^{n} I_k^{t_{kj}}\right) \cong \operatorname{End}_A(S_i)^{t_{ij}},$$

and consequently $\dim_{F_i} \operatorname{Ext}_A^1(S_i, S_j) = t_{ij}$. On the other hand, the canonical exact sequences of left A-modules

$$0 \longrightarrow \operatorname{rad} Ae_j \longrightarrow Ae_j \longrightarrow Ae_j/\operatorname{rad} Ae_j \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{rad}^2 Ae_j \longrightarrow \operatorname{rad} Ae_j \longrightarrow \operatorname{rad} Ae_j/\operatorname{rad}^2 Ae_j \longrightarrow 0,$$

induce the exact sequences of right A-modules

$$0 \longrightarrow D(Ae_j/\operatorname{rad} Ae_j) \longrightarrow D(Ae_j) \longrightarrow D(\operatorname{rad} Ae_j) \longrightarrow 0,$$
$$0 \longrightarrow D(\operatorname{rad} Ae_j/\operatorname{rad}^2 Ae_j) \longrightarrow D(\operatorname{rad} Ae_j) \longrightarrow D(\operatorname{rad}^2 Ae_j) \longrightarrow 0.$$

Since $D(Ae_j) = I_j$ and $D(Ae_j/\operatorname{rad} Ae_j) \cong S_j$, we conclude that $D(\operatorname{rad} Ae_j) \cong I_j/S_j$ in $\operatorname{mod} A$. Further, we have isomorphisms of right A-modules

$$D(\operatorname{rad} Ae_j/\operatorname{rad}^2 Ae_j) = D(\operatorname{top}(\operatorname{rad} Ae_j)) \cong \operatorname{soc} D(\operatorname{rad} Ae_j) \cong \operatorname{soc}(I_j/S_j),$$

by Proposition I.8.16. Further, we have also isomorphisms of right A-modules

$$\operatorname{soc}(I_j/S_j) \cong \operatorname{soc}(\mathbb{I}_1) \cong \operatorname{soc}\left(\bigoplus_{k=1}^{n} I_k^{t_{kj}}\right) \cong \bigoplus_{k=1}^{n} S_k^{t_{kj}}.$$

This leads to isomorphisms of right F_i-spaces

$$\begin{aligned} D\left(e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2e_j\right) &\cong D\left(e_i(\operatorname{rad} Ae_j/\operatorname{rad}^2 Ae_j)\right) \\ &\cong D\operatorname{Hom}_{A^{\mathrm{op}}}\left(Ae_i, \operatorname{rad} Ae_j/\operatorname{rad}^2 Ae_j\right) \\ &\cong D\operatorname{Hom}_{A^{\mathrm{op}}}\left(Ae_i/\operatorname{rad} Ae_i, \operatorname{rad} Ae_j/\operatorname{rad}^2 Ae_j\right) \\ &\cong D\operatorname{Hom}_{A^{\mathrm{op}}}\left(\operatorname{rad} Ae_j/\operatorname{rad}^2 Ae_j, Ae_i/\operatorname{rad} Ae_i\right) \\ &\cong \operatorname{Hom}_A\left(D\left(Ae_i/\operatorname{rad} Ae_i\right),\right. \\ &\qquad\qquad \left.D\left(\operatorname{rad} Ae_j/\operatorname{rad}^2 Ae_j\right)\right) \\ &\cong \operatorname{Hom}_A\left(S_i, \operatorname{soc}(I_j/S_j)\right) \\ &\cong \operatorname{Hom}_A\left(S_i, \bigoplus_{k=1}^{n} S_k^{t_{kj}}\right) \\ &\cong \operatorname{End}_A(S_i)^{t_{ij}} \\ &\cong \operatorname{Hom}_A(S_i, \mathbb{I}_1), \end{aligned}$$

because $Ae_i/\operatorname{rad} Ae_i$ is simple and $\operatorname{rad} Ae_i/\operatorname{rad}^2 Ae_j$ is semisimple. Hence, we conclude that

$$\begin{aligned} \dim_{F_i} \widetilde{\operatorname{Ext}}_A^1(S_i, S_j) &= \dim_{F_i} \operatorname{Hom}_A(S_i, \mathbb{I}_1) \\ &= \dim_{F_i} D\left(e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2e_j\right) \\ &= \dim_{F_i} \left(e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2e_j\right). \qquad \square \end{aligned}$$

It follows from Theorem 1.9 that the quiver Q_A of a finite dimensional K-algebra A over a field K can be defined as follows:

(a) The vertices of Q_A are the indices $1, \dots, n = n_A$ of a complete set $S_1, \dots, S_n$ of pairwise nonisomorphic simple modules in $\operatorname{mod} A$.

(b) For two vertices i and j in Q_A, there exists an arrow $i \to j$ if and only if $\operatorname{Ext}_A^1(S_i, S_j) \neq 0$. Moreover, an arrow $i \to j$ of Q_A has the valuation (d_{ij}, d'_{ij}), where

$$\begin{aligned} d_{ij} &= \dim_{F_j} \operatorname{Ext}_A^1(S_i, S_j), \\ d'_{ij} &= \dim_{F_i} \operatorname{Ext}_A^1(S_i, S_j), \end{aligned}$$

so we have in Q_A the valued arrow

$$i \xrightarrow{\left(\dim_{F_j} \operatorname{Ext}_A^1(S_i,S_j),\dim_{F_i} \operatorname{Ext}_A^1(S_i,S_j)\right)} j.$$

This is the reason the quiver Q_A of A is frequently called the Ext-*quiver* of A.

The next aim is to present connections between the quiver Q_A and the Auslander–Reiten quiver Γ_A of a finite dimensional K-algebra A over a field K.

Proposition 1.10. *Let A be a finite dimensional K-algebra over a field K, $e_1, \dots, e_n$ basic primitive idempotents of A, and $P_1 = e_1 A, \dots, P_n = e_n A$ the associated set of pairwise nonisomorphic indecomposable projective right A-modules. Assume that the Auslander–Reiten quiver Γ_A contains an arrow*

$$P_j \xrightarrow{(d_{P_j P_i}, d'_{P_j P_i})} P_i .$$

Then the quiver Q_A contains an arrow

$$i \xrightarrow{(d_{ij}, d'_{ij})} j,$$

where $d_{ij} = d'_{P_j P_i}$ and $d'_{ij} = d_{P_j P_i}$.

Proof. It follows from the assumption and Corollary III.9.4 that

$$\begin{aligned} \operatorname{irr}_A(P_j, P_i) &= \operatorname{rad}_A(P_j, P_i) / \operatorname{rad}_A^2(P_j, P_i) \neq 0 \quad \text{and} \\ d_{P_j P_i} &= \dim_{F_{P_i}} \operatorname{irr}_A(P_j, P_i), \\ d'_{P_j P_i} &= \dim_{F_{P_j}} \operatorname{irr}_A(P_j, P_i), \end{aligned}$$

where $F_{P_i} = \operatorname{End}_A(P_i) / \operatorname{rad} \operatorname{End}_A(P_i)$ and $F_{P_j} = \operatorname{End}_A(P_j) / \operatorname{rad} \operatorname{End}_A(P_j)$. Further, by Lemma I.11.2, we have isomorphisms of K-algebras $F_{P_i} \cong F_i$ and $F_{P_j} \cong F_j$. Since $P_i = e_i A$ and $P_j = e_j A$, applying Lemmas I.8.7 and I.11.2, we obtain isomorphisms of K-vector spaces

$$\begin{aligned} \operatorname{rad}_A(P_j, P_i) &= \operatorname{rad}_A(e_j A, e_i A) \cong \operatorname{Hom}_A\big(e_j A, \operatorname{rad}(e_i A)\big) \\ &= \operatorname{Hom}_A(e_j A, e_i \operatorname{rad} A) \cong e_i (\operatorname{rad} A) e_j, \\ \operatorname{rad}_A^2(P_j, P_i) &= \operatorname{rad}_A^2(e_j A, e_i A) \cong \operatorname{Hom}_A\big(e_j A, \operatorname{rad}^2(e_i A)\big) \\ &= \operatorname{Hom}_A\big(e_j A, e_i (\operatorname{rad} A)^2\big) \cong e_i (\operatorname{rad} A)^2 e_j, \end{aligned}$$

and hence an isomorphism

$$\operatorname{irr}_A(P_j, P_i) \cong e_i (\operatorname{rad} A) e_j / e_i (\operatorname{rad} A)^2 e_j .$$

This leads to the equalities

$$\begin{aligned} d_{P_j P_i} &= \dim_{F_{P_i}} \operatorname{irr}_A(P_j, P_i) = \dim_{F_i} e_i (\operatorname{rad} A) e_j / e_i (\operatorname{rad} A)^2 e_j = d'_{ij}, \\ d'_{P_j P_i} &= \dim_{F_{P_j}} \operatorname{irr}_A(P_j, P_i) = \dim_{F_j} e_i (\operatorname{rad} A) e_j / e_i (\operatorname{rad} A)^2 e_j = d_{ij}. \end{aligned}$$

Therefore, Q_A contains the required arrow $i \xrightarrow{(d_{ij}, d'_{ij})} j$. □

Lemma 1.11. *Let A be a finite dimensional K-algebra over a field K, $e_1, \dots, e_n$ basic primitive idempotents of A, and $S_i = e_i A / e_i \operatorname{rad} A$ the simple module in* $\operatorname{mod} A$ *associated to a vertex i of Q_A. The following conditions are equivalent:*

(i) *i is a sink of the quiver Q_A.*

(ii) *S_i is a projective module in* $\operatorname{mod} A$.

(iii) *S_i is a source in the quiver Γ_A.*

Proof. Assume i is a sink in Q_A. This means that Q_A has no arrow with the source at i, or equivalently, $e_i (\operatorname{rad} A) e_j / e_i (\operatorname{rad} A)^2 e_j = 0$ for any $j \in \{1, \dots, n\}$. Then $(e_i \operatorname{rad} A) e_j = e_i (\operatorname{rad} A)^2 e_j$ for any $j \in \{1, \dots, n\}$, and hence

$$e_i \operatorname{rad} A = e_i (\operatorname{rad} A)^2 = (e_i \operatorname{rad} A) \operatorname{rad} A.$$

Applying Lemma I.3.3, we obtain $e_i \operatorname{rad} A = 0$. Therefore, $S_i = e_i A / e_i \operatorname{rad} A = e_i A = P_i$ is a projective module. Hence (i) implies (ii).

Assume now that S_i is a projective module. Then $\operatorname{rad} S_i = 0$ and there is no irreducible homomorphism in $\operatorname{mod} A$ with codomain S_i (see Lemma III.7.6), and so S_i is a source in Γ_A. Thus (ii) implies (iii).

Let S_i be a source in Γ_A. Then, by Theorem III.8.4, S_i is projective and we obtain $e_i A / e_i \operatorname{rad} A = S_i = P_i = e_i A$. Hence $e_i \operatorname{rad} A = 0$. This gives $e_i (\operatorname{rad} A) e_j = 0$ for any $j \in \{1, \dots, n\}$, and hence i is a sink of Q_A. This shows that (iii) implies (i). □

Proposition 1.12. *Let A be a finite dimensional K-algebra over a field K, $e_1, \dots, e_n$ basic primitive idempotents of A, and $I_1 = D(Ae_1), \dots, I_n = D(Ae_n)$ the associated set of pairwise nonisomorphic indecomposable injective right A-modules. Assume that the Auslander–Reiten quiver Γ_A contains an arrow*

$$I_j \xrightarrow{(d_{I_j I_i}, d'_{I_j I_i})} I_i.$$

Then the quiver Q_A contains an arrow

$$i \xrightarrow{(d_{ij}, d'_{ij})} j,$$

where $d_{ij} = d'_{I_j I_i}$ and $d'_{ij} = d_{I_j I_i}$.

Proof. We have in $\operatorname{mod} A$ a left minimal almost split homomorphism $I_j \to I_i^{d_{I_j I_i}} \oplus M$, with M without a direct summand isomorphic to I_i, and a right minimal almost split homomorphism $I_j^{d'_{I_j I_i}} \oplus N \to I_i$, with N without a direct summand isomorphic to I_j. Applying the duality functor $D \colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$, we obtain in mod A^{op} a right minimal almost split homomorphism $D(I_i)^{d_{I_j I_i}} \oplus D(M) \to D(I_j)$, with $D(M)$ without a direct summand isomorphic to $D(I_i)$, and a left minimal almost split homomorphism $D(I_i) \to D(I_j)^{d'_{I_j I_i}} \oplus D(N)$, with $D(N)$ without a direct summand isomorphic to $D(I_j)$. Since $D(I_i) = DD(Ae_i) \cong Ae_i$ and $D(I_j) = DD(Ae_j) \cong Ae_j$ are indecomposable projective modules in mod A^{op}, we conclude that the Auslander–Reiten quiver $\Gamma_{A^{\mathrm{op}}}$ of A^{op} contains an arrow

$$Ae_i \xrightarrow{(d'_{I_j I_i}, d_{I_j I_i})} Ae_j.$$

Applying now Proposition 1.10 to the algebra A^{op}, we obtain that the quiver $Q_{A^{\mathrm{op}}}$ contains an arrow

$$j \xrightarrow{(d_{I_j I_i}, d'_{I_j I_i})} i.$$

Since $Q_{A^{\mathrm{op}}}$ is the opposite quiver to Q_A, we infer that Q_A contains an arrow

$$i \xrightarrow{(d_{ij}, d'_{ij})} j,$$

where $d_{ij} = d'_{I_j I_i}$ and $d'_{ij} = d_{I_j I_i}$. □

Lemma 1.13. *Let A be a finite dimensional K-algebra over a field K, $e_1, \dots, e_n$ basic primitive idempotents of A, and $S_i = e_i A / e_i \operatorname{rad} A$ the simple module in* mod A *associated to a vertex i of Q_A. The following conditions are equivalent:*

(i) *i is a source of the quiver Q_A.*

(ii) *S_i is an injective module in* mod A.

(iii) *S_i is a sink in the quiver Γ_A.*

Proof. We have the equivalences:

(a) i is a source in Q_A if and only if i is a sink in $Q_{A^{\mathrm{op}}}$;

(b) S_i is injective in mod A if and only if $D(S_i)$ is projective in mod A^{op};

(c) S_i is a sink in Γ_A if and only if $D(S_i)$ is a source in $\Gamma_{A^{\mathrm{op}}}$.

Moreover, $D(S_i) = D(e_i A / e_i \operatorname{rad} A) \cong Ae_i / (\operatorname{rad} A)e_i$, by Lemma I.8.22. Then the equivalence of (i), (ii), (iii) follows from Lemma 1.11 and the equalities $\Gamma_{A^{\mathrm{op}}} = \Gamma_A^{\mathrm{op}}$, $Q_{A^{\mathrm{op}}} = Q_A^{\mathrm{op}}$. □

We present an example showing that in general the arrows of the quiver Q_A do not correspond to arrows of Γ_A.

Example 1.14. Let K be a field, Q the quiver

$$\begin{tikzcd}
 & \overset{2}{\bullet} \arrow[dl, "\alpha"'] & \\
1\ \bullet & & \bullet\ 4, \arrow[ul, "\beta"'] \arrow[dl, "\sigma"] \\
 & \underset{3}{\bullet} \arrow[ul, "\gamma"] &
\end{tikzcd}$$

KQ the path algebra of Q over K, I the ideal in KQ generated by $\beta\alpha - \sigma\gamma$, and $A = KQ/I$ the associated bound quiver algebra. Then A is a basic algebra (see Lemma II.6.17) and the classes $e_1 = \varepsilon_1 + I, e_2 = \varepsilon_2 + I, e_3 = \varepsilon_3 + I, e_4 = \varepsilon_4 + I$ of the trivial paths $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ of Q at the vertices $1, 2, 3, 4$ form a set of basic primitive idempotents of A with $1_A = e_1 + e_2 + e_3 + e_4$. We identify $\operatorname{mod} A$ with the category $\operatorname{rep}_K(Q, I)$ of finite dimensional K-linear representations of Q bound by I (see Theorem I.2.10). Then, by Proposition I.8.27, the representations

$$P(1):\quad \begin{tikzcd}
 & 0 \arrow[dl] & \\
K & & 0, \arrow[ul] \arrow[dl] \\
 & 0 \arrow[ul] &
\end{tikzcd}
\qquad
P(2):\quad \begin{tikzcd}
 & K \arrow[dl, "1"'] & \\
K & & 0, \arrow[ul] \arrow[dl] \\
 & 0 \arrow[ul] &
\end{tikzcd}$$

$$P(3):\quad \begin{tikzcd}
 & 0 \arrow[dl] & \\
K & & 0, \arrow[ul] \arrow[dl] \\
 & K \arrow[ul, "1"] &
\end{tikzcd}
\qquad
P(4):\quad \begin{tikzcd}
 & K \arrow[dl, "1"'] & \\
K & & K, \arrow[ul, "1"'] \arrow[dl, "1"] \\
 & K \arrow[ul, "1"] &
\end{tikzcd}$$

form a complete set of pairwise nonisomorphic indecomposable projective representations in $\operatorname{rep}_K(Q, I)$, corresponding to the indecomposable projective right A-modules $P_1 = e_1 A$, $P_2 = e_2 A$, $P_3 = e_3 A$, $P_4 = e_4 A$, and the representations

$$I(1):\quad \begin{tikzcd}
 & K \arrow[dl, "1"'] & \\
K & & K, \arrow[ul, "1"'] \arrow[dl, "1"] \\
 & K \arrow[ul, "1"] &
\end{tikzcd}
\qquad
I(2):\quad \begin{tikzcd}
 & K \arrow[dl] & \\
0 & & K, \arrow[ul, "1"'] \arrow[dl] \\
 & 0 \arrow[ul] &
\end{tikzcd}$$

$$I(3):\quad \begin{tikzcd}
 & 0 \arrow[dl] & \\
0 & & K, \arrow[ul] \arrow[dl, "1"] \\
 & K \arrow[ul] &
\end{tikzcd}
\qquad
I(4):\quad \begin{tikzcd}
 & 0 \arrow[dl] & \\
0 & & K \arrow[ul] \arrow[dl] \\
 & 0 \arrow[ul] &
\end{tikzcd}$$

form a complete set of pairwise nonisomorphic indecomposable injective representations in $\operatorname{rep}_K(Q, I)$, corresponding to the indecomposable injective right A-modules $I_1 = D(Ae_1)$, $I_2 = D(Ae_2)$, $I_3 = D(Ae_3)$, $I_4 = D(Ae_4)$. In particular, we conclude that $P(1)$ and $I(4)$ are simple representations (see Lemma I.8.25) in $\operatorname{rep}_K(Q, I)$, while $P(4) = I(1)$ is a projective-injective representations in $\operatorname{rep}_K(Q, I)$. This gives that $P_4 = e_4 A$ is a projective-injective module in $\operatorname{mod} A$, isomorphic to the injective module $I_1 = D(Ae_1)$. Applying Proposition III.8.6, we obtain in $\operatorname{mod} A$ an almost split sequence of the form

$$0 \longrightarrow \operatorname{rad} P_4 \longrightarrow \big(\operatorname{rad} P_4/\operatorname{soc}(P_4)\big) \oplus P_4 \longrightarrow P_4/\operatorname{soc}(P_4) \longrightarrow 0.$$

Then it follows from Theorems III.7.11 and III.7.12 that Γ_A admits a unique arrow $\operatorname{rad} P_4 \xrightarrow{(1,1)} P_4$ with target at P_4 and a unique arrow $P_4 \xrightarrow{(1,1)} P_4/\operatorname{soc}(P_4)$ with source P_4. Observe also that $\operatorname{rad} P_4$ and $P_4/\operatorname{soc}(P_4)$ correspond to the representations $\operatorname{rad} P(4)$ and $P(4)/\operatorname{soc}(P(4))$ in $\operatorname{rep}_K(Q, I)$ of the form

$$\begin{tikzcd}
 & & K \arrow[dl, "1"'] & & & & & K \arrow[dl] & \\
\operatorname{rad} P(4): & K & & 0, \arrow[ul] \arrow[dl] & & P(4)/\operatorname{soc}\big(P(4)\big): & 0 & & K, \arrow[ul, "1"'] \arrow[dl, "1"] \\
 & & K \arrow[ul, "1"] & & & & & K \arrow[ul] &
\end{tikzcd}$$

which are neither projective nor injective. This shows that there are no arrows $P_2 \to P_4$ and $P_3 \to P_4$ in Γ_A corresponding to the arrows $2 \overset{\beta}{\leftarrow} 4$ and $3 \overset{\sigma}{\leftarrow} 4$ in Q_A. Similarly, there are no arrows $I_1 \to I_2$ and $I_1 \to I_3$ in Γ_A corresponding to the arrows $1 \overset{\alpha}{\leftarrow} 2$ and $1 \overset{\gamma}{\leftarrow} 3$ in Q_A.

We end this section with the following useful information that is a direct consequence of the description of the quiver of an algebra via the extension spaces of simple modules.

Corollary 1.15. *Let A be a finite dimensional K-algebra over a field such that the quiver Q_A is acyclic. Then* $\operatorname{Ext}^1_A(S, S) = 0$ *for any simple module S in* $\operatorname{mod} A$.

2 The tensor algebras of species

In this section we introduce a wide class of hereditary algebras, describe their quivers, and give a useful description of their module categories.

Let A be a finite dimensional K-algebra over a field K. Then A is called *right hereditary* if any right ideal of A is projective as a right A-module, and *left hereditary* if any left ideal of A is projective as left A-module. Naturally, A is called *hereditary* if it is both left and right hereditary (see Section I.9). In fact, we proved in Theorem I.9.3 that A is left hereditary if and only if A is right hereditary. Then it follows from Theorems I.9.1 and I.9.2 that A is a hereditary algebra if and only if the radical $\operatorname{rad} P$ of any indecomposable projective module P in $\operatorname{mod} A$ is projective, and if and only if the socle factor $E/\operatorname{soc}(E)$ of any indecomposable injective module E in $\operatorname{mod} A$ is injective. We also proved in Theorem I.9.6 that if $A = KQ/I$, for a finite quiver Q and an admissible ideal I in KQ, then A is a hereditary algebra if and only if $I = 0$ and the quiver Q is acyclic.

Let K be a field. A K-*species* is a system (see [DR2] and [Ga2])

$$\mathbb{M} = (F_i, {}_iM_j)_{1\le i,j\le n},$$

where $F_1, \dots, F_n$ are finite dimensional division K-algebras and, for each pair $i, j \in \{1, \dots, n\}$, ${}_iM_j$ is an (F_i, F_j)-bimodule on which K acts centrally (that is $\lambda x = (\lambda 1_{F_i})x = x(1_{F_i}\lambda) = x\lambda$ for any $\lambda \in K$ and $x \in {}_iM_j$) and $\dim_K {}_iM_j$ is finite. We may associate to such a K-species $\mathbb{M}$ the valued quiver $Q_{\mathbb{M}}$ defined as follows:

(a) The vertices of $Q_{\mathbb{M}}$ are the numbers $1, 2, \dots, n$.

(b) For two vertices i and j in $Q_{\mathbb{M}}$, there exists an arrow $i \to j$ if and only if ${}_iM_j \neq 0$. Moreover, we associate to an arrow $i \to j$ in $Q_{\mathbb{M}}$ the valuation (d_{ij}, d'_{ij}), so we have in $Q_{\mathbb{M}}$ the valued arrow

$$i \xrightarrow{(d_{ij},d'_{ij})} j,$$

where $d_{ij} = \dim_{F_j} {}_iM_j$ and $d'_{ij} = \dim_{F_i} {}_iM_j$.

Observe that the valuation (d_{ij}, d'_{ij}) of an arrow $i \to j$ in $Q_{\mathbb{M}}$ satisfies the condition $d_{ij} f_j = f_i d'_{ij}$ with the positive integers $f_i = \dim_K F_i$ and $f_j = \dim_K F_j$.

A K-species $\mathbb{M} = (F_i, {}_iM_j)_{1\le i,j\le n}$ is said to be *acyclic* if the associated valued graph $Q_{\mathbb{M}}$ is acyclic. Observe that then ${}_iM_i = 0$ for any $i \in \{1, \dots, n\}$, and ${}_iM_j \neq 0$ forces ${}_jM_i = 0$ for any $i, j \in \{1, \dots, n\}$.

Let $\mathbb{M} = (F_i, {}_iM_j)_{1\le i,j\le n}$ be a K-species over a field K. Consider the K-vector spaces

$$R = R_{\mathbb{M}} = \prod_{i=1}^{n} F_i \quad \text{and} \quad M = M_{\mathbb{F}} = \bigoplus_{i,j=1}^{n} {}_iM_j.$$

Then R is a K-algebra and M is an R-bimodule with the left R-module and the right R-module actions on M given by

$$(\lambda_1, \dots, \lambda_n)\left(\sum_{i,j=1}^{n} m_{ij}\right) = \sum_{i,j=1}^{n} \lambda_i m_{ij},$$

$$\left(\sum_{i,j=1}^{n} m_{ij}\right)(\lambda_1, \dots, \lambda_n) = \sum_{i,j=1}^{n} m_{ij}\lambda_j,$$

for $\lambda_i \in F_i$ and $m_{ij} \in {}_iM_j$, $i, j \in \{1, \ldots, n\}$. Then we can define the tensor algebra

$$T(\mathbb{M}) = T_R(M) = \bigoplus_{n=0}^{\infty} T_R^n(M)$$

of M over R (see Example II.3.6(b)), where $T_R^0(M) = R$, $T_R^1(M) = M$, and $T_R^n(M) = M \otimes_R M \otimes_R \cdots \otimes_R M$ is the tensor product of n copies of the R-bimodule M for $n \geq 2$. Recall that the multiplication in $T_R(M)$ is given by

$$(x_1 \otimes \cdots \otimes x_m)(y_1 \otimes \cdots \otimes y_n) = x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n,$$

for $x_1 \otimes \cdots \otimes x_m \in T_R^m(M)$, $y_1 \otimes \cdots \otimes y_n \in T_R^n(M)$, with $m, n \geq 1$, and

$$a(x_1 \otimes \cdots \otimes x_m) = (ax_1) \otimes \cdots \otimes x_m, \ (x_1 \otimes \cdots \otimes x_m)a = x_1 \otimes \cdots \otimes (x_m a),$$

for $a \in R = T_R^0(M)$, $x_1 \otimes \cdots \otimes x_m \in T_R^m(M)$, $m \geq 1$, and the K-algebra structure of R. In particular, $1_R = (1_{F_1}, \ldots, 1_{F_n})$ is the identity $1_{T_R(M)}$ of $T_R(M)$. The algebra $T(\mathbb{M}) = T_R(M)$ is said to be the *tensor algebra of the K-species* $\mathbb{M}$.

Lemma 2.1. *Let $\mathbb{M} = (F_i, {}_iM_j)_{1 \leq i,j \leq n}$ be a K-species over a field K. Then $T(\mathbb{M})$ is a finite dimensional K-algebra if and only if $\mathbb{M}$ is acyclic.*

Proof. Assume $\mathbb{M}$ is acyclic, and let p be the length of the longest path in $Q_{\mathbb{M}}$. Then we have $T_R^m(M) = 0$ for all $m \geq p + 1$, and consequently

$$T(\mathbb{M}) = \bigoplus_{m=0}^{\infty} T_R^m(M) = \bigoplus_{m=0}^{p} T_R^m(M)$$

is a finite dimensional K-algebra. Conversely, assume $Q_{\mathbb{M}}$ contains an oriented cycle. Then there are vertices $i_1, i_2, \ldots, i_r, i_{r+1} = i_1$ in $Q_{\mathbb{M}}$ such that ${}_{i_s}M_{j_{s+1}} \neq 0$ for any $s \in \{1, \ldots, r\}$. Now for any nonzero elements $x_s \in {}_{i_s}M_{j_{s+1}}$, $s \in \{1, \ldots, r\}$, we have $(x_1 \otimes \cdots \otimes x_r)^l \neq 0$ in $T_R(M)$ for any $l \geq 1$, and so $T(\mathbb{M})$ is of infinite dimension over K. □

The following theorem provides a wide class of hereditary algebras over an arbitrary field.

Theorem 2.2. *Let $\mathbb{M} = (F_i, {}_iM_j)_{1 \leq i,j \leq n}$ be an acyclic K-species over a field K. Then the following statements hold:*

(i) *$T(\mathbb{M})$ is a finite dimensional hereditary K-algebra.*

(ii) $Q_{T(\mathbb{M})} = Q_{\mathbb{M}}$.

Proof. Since $\mathbb{M}$ is acyclic, $T(\mathbb{M})$ is a finite dimensional K-algebra. Observe also that $e_i = (0, \dots, 0, 1_{F_i}, 0, \dots, 0), i \in \{1, \dots, n\}$, form a set of basic primitive idempotents of $T(\mathbb{M})$ such that $1_{T(\mathbb{M})} = e_1 + \dots + e_n$. We claim that

$$\operatorname{rad} T(\mathbb{M}) = \bigoplus_{m=1}^{\infty} T_R^m(M).$$

There exists a positive integer p such that $T_R^m(M) = 0$ for all $m \geq p+1$, because $T_{\mathbb{M}}(R)$ is finite dimensional. Consider $I = \bigoplus_{m=1}^{p} T_R^m(M)$. Then I is a two-sided ideal of $T(\mathbb{M})$ with $I^{p+1} = 0$ and $T(\mathbb{M})/I \cong R = \prod_{i=1}^{n} F_i$ is a product of finite dimensional division K-algebras. Then, applying Lemma I.3.5, we conclude that $\operatorname{rad} A = I$. We will show now that $T(\mathbb{M})$ is a hereditary algebra. In view of Theorems I.9.1 and I.9.3 it is enough to prove that the radical $\operatorname{rad} P$ of every indecomposable projective module P in $\operatorname{mod} A$ is also projective.

Fix $i \in \{1, \dots, n\}$. Let $j_1, \dots, j_r$ be all elements in $\{1, \dots, n\}$ such that ${}_iM_{j_t} \neq 0$ for $t \in \{1, \dots, r\}$. Then, for any $t \in \{1, \dots, r\}$, we have $e_{j_t} T_R^p(M) = 0$, because otherwise $0 \neq e_i M e_{j_t} T_R^p(M) \subseteq T_R^{p+1}(M)$. Observe also that

$$\operatorname{rad} e_i T(\mathbb{M}) = e_i \operatorname{rad} T(\mathbb{M}) = \bigoplus_{m=1}^{p} e_i T_R^m(M).$$

Further, for any fixed $m \in \{1, \dots, p\}$, we have isomorphisms of K-vector spaces

$$\begin{aligned}
e_i T_R^m(M) &= e_i(M \otimes_R M \otimes_R \dots \otimes_R M) = e_i M \otimes_R M \otimes_R \dots \otimes_R M \\
&= \left(\bigoplus_{t=1}^{r} {}_iM_{j_t} \right) \otimes_R M \otimes_R \dots \otimes_R M \\
&\cong \bigoplus_{t=1}^{r} \left({}_iM_{j_t} \otimes_R M \otimes_R \dots \otimes_R M \right) \\
&= \bigoplus_{t=1}^{r} \left({}_iM_{j_t} e_{j_t} \otimes_R M \otimes_R \dots \otimes_R M \right) \\
&= \bigoplus_{t=1}^{r} \left({}_iM_{j_t} \otimes_R e_{j_t} M \otimes_R \dots \otimes_R M \right) \\
&\cong \bigoplus_{t=1}^{r} \left(F_{j_t}^{d_{ij_t}} \otimes_R e_{j_t} M \otimes_R \dots \otimes_R M \right) \\
&\cong \bigoplus_{t=1}^{r} \left(F_{j_t} \otimes_R e_{j_t} M \otimes_R \dots \otimes_R M \right)^{d_{ij_t}} \\
&= \bigoplus_{t=1}^{r} \left(F_{j_t} \otimes_R e_{j_t} T_R^{m-1}(M) \right)^{d_{ij_t}} \cong \bigoplus_{t=1}^{r} \left(e_{j_t} T_R^{m-1}(M) \right)^{d_{ij_t}}.
\end{aligned}$$

Hence we obtain isomorphisms of right $T(\mathbb{M})$-modules

$$\begin{aligned}\operatorname{rad} e_i T(\mathbb{M}) &= \bigoplus_{m=1}^{p} e_i T_R^m(M) \cong \bigoplus_{t=1}^{r} \left(\bigoplus_{m=1}^{p} e_{j_t} T_R^{m-1}(M) \right)^{d_{ij_t}} \\ &= \bigoplus_{t=1}^{r} \left(e_{j_t} \left(\bigoplus_{m=1}^{p} T_R^{m-1}(M) \right) \right)^{d_{ij_t}} = \bigoplus_{t=1}^{r} \left(e_{j_t} \bigoplus_{l=0}^{p-1} T_R^{l}(M) \right)^{d_{ij_t}} \\ &= \bigoplus_{t=1}^{r} \left(e_{j_t} T(\mathbb{M}) \right)^{d_{ij_t}},\end{aligned}$$

because $e_{j_t} T_R^p(M) = 0$ for $t \in \{1, \dots, r\}$. This proves that $\operatorname{rad} e_i T(\mathbb{M})$ is a projective right $T(\mathbb{M})$-module. Since $e_i T(\mathbb{M})$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic indecomposable projective right $T(\mathbb{M})$-modules, we conclude that $T(\mathbb{M})$ is a hereditary algebra. Hence (i) holds.

For (ii), observe that

$$\operatorname{rad} T(\mathbb{M}) = \bigoplus_{m=1}^{p} T_R^m(M) \quad \text{and} \quad \left(\operatorname{rad} T(\mathbb{M})\right)^2 = \bigoplus_{m=2}^{p} T_R^m(M),$$

so

$$\operatorname{rad} T(\mathbb{M}) / \left(\operatorname{rad} T(\mathbb{M})\right)^2 \cong T_R^1(M) = M.$$

Hence, for any $i, j \in \{1, \dots, n\}$, we have

$$e_i \operatorname{rad} T(\mathbb{M}) e_j / e_i \left(\operatorname{rad} T(\mathbb{M})\right)^2 e_j \cong e_i M e_j = {}_iM_j$$

as (F_i, F_j)-bimodules. This implies the required equality $Q_{T(\mathbb{M})} = Q_{\mathbb{M}}$ of valued quivers. □

Corollary 2.3. *Let K be a field and Q a finite acyclic quiver. Then there is a K-species $\mathbb{M} = \mathbb{M}_Q$ such that the path algebra KQ of Q over K is isomorphic to the tensor algebra $T(\mathbb{M})$ of $\mathbb{M}$.*

Proof. Let Q_0 and Q_1 be the sets of vertices and arrows of Q. We may assume that $Q_0 = \{1, \dots, n\}$. We set $F_i = K$ for any $i \in Q_0$. Moreover, for any $i, j \in Q_0$, we define ${}_iM_j$ to be the K-vector space whose basis is formed by all arrows $\alpha \in Q_1$ with $s(\alpha) = i$ and $t(\alpha) = j$. Clearly, ${}_iM_j$ is an (F_i, F_j)-bimodule and $d_{ij} = \dim_{F_j} {}_iM_j = \dim_K {}_iM_j = \dim_{F_i} {}_iM_j = d'_{ij}$ is the number of arrows from i to j in Q. Hence $\mathbb{M} = \mathbb{M}_Q = (F_i, {}_iM_j)_{1 \le i, j \le n}$ is an acyclic K-species, because the quiver Q is acyclic. We note also that $R = R_{\mathbb{M}} = \prod_{i=1}^{n} F_i =$

$\prod_{i=1}^{n} K\varepsilon_i$, where $\varepsilon_1, \dots, \varepsilon_n$ are the trivial paths at the vertices $1, \dots, n$ of Q. Then we have the isomorphisms of K-algebras

$$\begin{aligned} T(\mathbb{M}) &= \bigoplus_{m=0}^{\infty} T_R^m(M) = R \oplus \left(\bigoplus_{m=1}^{\infty} T_R^m(M) \right) \\ &\cong \left(\prod_{i=1}^{n} K\varepsilon_i \right) \oplus \left(\bigoplus_{m=1}^{\infty} R_Q^m(M) \right) = KQ, \end{aligned}$$

where R_Q is the arrow ideal of KQ. □

Let $\mathbb{M} = (F_i, {}_iM_j)_{1 \le i,j \le n}$ be a K-species over a field K. A *representation of the K-species* $\mathbb{M}$ is a system $X = (X_i, {}_j\varphi_i)_{1 \le i,j \le n}$, briefly $X = (X_i, {}_j\varphi_i)$, consisting of right F_i-spaces X_i, $i \in \{1, \dots, n\}$, and F_j-linear homomorphisms

$${}_j\varphi_i \colon X_i \otimes_{F_i} {}_iM_j \longrightarrow X_j,$$

for all $i, j \in \{1, \dots, n\}$ with ${}_iM_j \neq 0$. The representation X is said to be finite dimensional if each F_i-space X_i is finite dimensional. Let $X = (X_i, {}_j\varphi_i)$ and $Y = (Y_i, {}_j\psi_i)$ be representations of the K-species $\mathbb{M}$. A *morphism* (*of representations*) $f \colon X \to Y$ is a family $f = (f_i)$ of homomorphisms $f_i \colon X_i \to Y_i$ in $\operatorname{mod} F_i$, for $i \in \{1, \dots, n\}$, such that ${}_j\psi_i(f_i \otimes 1) = f_j\, {}_j\varphi_i$, or equivalently the square of F_j-linear homomorphisms

$$\begin{tikzcd} X_i \otimes_{F_i} {}_iM_j \arrow[r, "{}_j\varphi_i"] \arrow[d, "f_i \otimes 1"] & X_j \arrow[d, "f_j"] \\ Y_i \otimes_{F_i} {}_iM_j \arrow[r, "{}_j\psi_i"] & Y_j \end{tikzcd}$$

is commutative for all $i, j \in \{1, \dots, n\}$ with ${}_iM_j \neq 0$. A morphism $f = (f_i) \colon X \to Y$ of representations of $\mathbb{M}$ is called an isomorphism if all F_i-linear homomorphisms $f_i \colon X_i \to Y_i$ are isomorphisms. We denote by $\operatorname{Hom}_{\mathbb{M}}(X, Y)$ the set of all morphisms of representations from X to Y. Observe that then $\operatorname{Hom}_{\mathbb{M}}(X, Y)$ has a K-vector space structure given by $f + g = (f_i + g_i)$ and $f\lambda = (f_i\lambda)$, for $f = (f_i)$, $g = (g_i)$ in $\operatorname{Hom}_{\mathbb{M}}(X, Y)$ and $\lambda \in K$. Moreover, for any triple X, Y, Z of representations of $\mathbb{M}$ the composition map

$$\operatorname{Hom}_{\mathbb{M}}(Y, Z) \times \operatorname{Hom}_{\mathbb{M}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathbb{M}}(X, Z),$$

which assigns to $h = (h_i) \in \operatorname{Hom}_{\mathbb{M}}(Y, Z)$ and $g = (g_i) \in \operatorname{Hom}_{\mathbb{M}}(X, Y)$ the morphism $hg = (h_i g_i) \in \operatorname{Hom}_{\mathbb{M}}(X, Z)$ is K-bilinear.

Let $f = (f_i) \in \operatorname{Hom}_{\mathbb{M}}(X, Y)$ for representations $X = (X_i, {}_j\varphi_i)$ and $Y = (Y_i, {}_j\psi_i)$ of a K-species $\mathbb{M} = (F_i, {}_iM_j)_{1 \le i,j \le n}$ over K. Then the kernel of f is defined as $\operatorname{Ker} f = (\operatorname{Ker} f_i, {}_j\varphi_i')$, where ${}_j\varphi_i' \colon \operatorname{Ker} f_i \otimes_{F_i} {}_iM_j \to \operatorname{Ker} f_j$ denotes the restriction of ${}_j\varphi_i \colon X_i \otimes_{F_i} {}_iM_j \to X_j$ to $\operatorname{Ker} f_i \otimes_{F_i} {}_iM_j$, the image of f is defined as $\operatorname{Im} f = (\operatorname{Im} f_i, {}_j\psi_i')$, where ${}_j\psi_i'\colon \operatorname{Im} f_i \otimes_{F_i} {}_iM_j \to \operatorname{Im} f_j$ is defined as the restriction of ${}_j\psi_i\colon Y_i \otimes_{F_i} {}_iM_j \to Y_j$ to $\operatorname{Im} f_i \otimes_{F_i} {}_iM_j$, and the cokernel of f is defined as $\operatorname{Coker} f = (\operatorname{Coker} f_i, {}_j\bar{\psi}_i)$, where ${}_j\bar{\psi}_i\colon \operatorname{Coker} f_i \otimes_{F_i} {}_iM_j \to \operatorname{Coker} f_j$ is given by ${}_j\bar{\psi}_i((y_i + \operatorname{Im} f_i) \otimes m_{ij}) = {}_j\psi_i(y_i \otimes m_{ij}) + \operatorname{Im} f_j$ for $y_i \in Y_i$ and $m_{ij} \in {}_iM_j$. Observe that, if X and Y are finite dimensional, then the representations $\operatorname{Ker} f$, $\operatorname{Im} f$ and $\operatorname{Coker} f$ are also finite dimensional. Given two representations $X = (X_i, {}_j\varphi_i)$ and $Y = (Y_i, {}_j\psi_i)$ of $\mathbb{M}$ their direct sum is the representation

$$X \oplus Y = \left(X_i \oplus Y_i, \begin{bmatrix} {}_j\varphi_i & 0 \\ 0 & {}_j\psi_i \end{bmatrix} \right).$$

A representation X of $\mathbb{M}$ is said to be *indecomposable* if X is nonzero and not isomorphic (as a representation of $\mathbb{M}$) to a direct sum $Y \oplus Z$ of two nonzero representations of $\mathbb{M}$.

We denote by $\operatorname{Rep}(\mathbb{M})$ the category of all representations of a K-species $\mathbb{M}$ and the morphisms of representations, and by $\operatorname{rep}(\mathbb{M})$ the full subcategory of $\operatorname{Rep}(\mathbb{M})$ consisting of all finite dimensional representations.

The following proposition summarizes our discussion above.

Proposition 2.4. *Let $\mathbb{M}$ be a K-species over a field K. Then* $\operatorname{Rep}(\mathbb{M})$ *and* $\operatorname{rep}(\mathbb{M})$ *are abelian K-categories.*

In fact, we have the following theorem.

Theorem 2.5. *Let $\mathbb{M}$ be a K-species over a field K and $T(\mathbb{M})$ the tensor algebra of $\mathbb{M}$. Then there exists a K-linear equivalence of categories*

$$F\colon \operatorname{Mod} T(\mathbb{M}) \longrightarrow \operatorname{Rep}(\mathbb{M})$$

which restricts to a K-linear equivalence of categories

$$F\colon \operatorname{mod} T(\mathbb{M}) \longrightarrow \operatorname{rep}(\mathbb{M}).$$

Proof. We construct a K-linear functor $F\colon \operatorname{Mod} T(\mathbb{M}) \to \operatorname{Rep}(\mathbb{M})$ and its quasi-inverse functor $G\colon \operatorname{Rep}(\mathbb{M}) \to \operatorname{Mod} T(\mathbb{M})$. Let $\mathbb{M} = (F_i, {}_iM_j)_{1\le i,j\le n}$. Recall that

$$T(\mathbb{M}) = \bigoplus_{m=0}^{\infty} T_R^m(\mathbb{M}),$$

where $T_R^0(\mathbb{M}) = R = \prod_{i=1}^n F_i$, $T_R^1(\mathbb{M}) = M = \bigoplus_{i,j=1}^n {}_iM_j$, and $T_R^m(\mathbb{M}) = M \otimes_R M \otimes_R \cdots \otimes_R M$ is the tensor product of m copies of the R-bimodule M for $m \ge 2$. In particular,

$$1_{T(\mathbb{M})} = 1_R = e_1 + \cdots + e_n,$$

where $e_i = (0, \ldots, 0, 1_{F_i}, 0, \ldots, 0)$, for $i \in \{1, \ldots, n\}$, is a complete set of primitive idempotents of $T(\mathbb{M})$.

Let X be a module in $\operatorname{Mod} T(\mathbb{M})$. We associate to X the representation $F(X) = (X_i, {}_j\varphi_i)$ of $\mathbb{M}$ as follows. For each $i \in \{1, \dots, n\}$, we set $X_i = Xe_i$ and note that X_i is a module in $\operatorname{Mod} F_i$. Further, for $i, j \in \{1, \dots, n\}$ with ${}_iM_j \neq 0$, the homomorphism

$${}_j\varphi_i \colon X_i \otimes_{F_i} {}_iM_j \longrightarrow X_j$$

of right F_j-spaces is defined by ${}_j\varphi_i(x_i \otimes m_{ij}) = x_i m_{ij}$ for $x_i \in X_i$ and $m_{ij} \in {}_iM_j$.

Let $f \colon X \to Y$ be a homomorphism in $\operatorname{Mod} T(\mathbb{M})$, and $F(X) = (X_i, {}_j\varphi_i)$, $F(Y) = (Y_i, {}_j\psi_i)$. We define the morphism $F(f) \colon F(X) \to F(Y)$ of representations of $\mathbb{M}$. For $x = xe_i \in Xe_i = X_i$, we have $f(x) = f(xe_i) = f(x)e_i \in Ye_i = Y_i$, because f is a homomorphism of right $T(\mathbb{M})$-modules. Hence, the restriction f_i of f to $X_i = Xe_i$ gives a homomorphism $f_i \colon X_i \to Y_i$ of right F_i-modules. We set $F(f) = (f_i)$. Observe that, for any $i, j \in \{1, \dots, n\}$ with ${}_iM_j \neq 0$, the square of F_j-linear homomorphisms

$$\begin{tikzcd}
X_i \otimes_{F_i} {}_iM_j \arrow[r, "{}_j\varphi_i"] \arrow[d, "f_i \otimes 1"] & X_j \arrow[d, "f_j"] \\
Y_i \otimes_{F_i} {}_iM_j \arrow[r, "{}_j\psi_i"] & Y_j
\end{tikzcd}$$

is commutative. Indeed, for $x_i \in X_i$ and $m_{ij} \in {}_iM_j$, we have

$$\begin{aligned}
\big({}_j\psi_i(f_i \otimes 1)\big)(x_i \otimes m_{ij}) &= {}_j\psi_i\big(f_i(x_i) \otimes m_{ij}\big) = f_i(x_i)m_{ij} \\
&= f(x_i e_i)m_{ij} = f(x_i m_{ij}) \\
&= (f_j\, {}_j\varphi_i)(x_i \otimes m_{ij}),
\end{aligned}$$

and hence $F(f)$ is a morphism in $\operatorname{Rep}(\mathbb{M})$. Therefore, $F \colon \operatorname{Mod} T(\mathbb{M}) \to \operatorname{Rep}(\mathbb{M})$ is a K-linear functor which restricts to a K-linear functor $F \colon \operatorname{mod} T(\mathbb{M}) \to \operatorname{rep}(\mathbb{M})$.

We define now a K-linear functor $G \colon \operatorname{Rep}(\mathbb{M}) \to \operatorname{Mod} T(\mathbb{M})$ which is a quasi-inverse to F. Let $X = (X_i, {}_j\varphi_i)$ be a representation in $\operatorname{Rep}(\mathbb{M})$. Consider the K-vector space

$$G(X) = \bigoplus_{i=1}^{n} X_i .$$

We define a structure of right $T(\mathbb{M})$-module on $G(X)$. Observe that $G(X)$ is a right module over $R = \prod_{i=1}^{n} F_i$ by

$$(x_1, \dots, x_n)(\lambda_1, \dots, \lambda_n) = (x_1\lambda_1, \dots, x_n\lambda_n) \quad \text{for } x_i \in X_i \quad \text{and } \lambda_i \in F_i,$$

$i \in \{1, \dots, n\}$. For any $i, j \in \{1, \dots, n\}$ with ${}_iM_j \neq 0$, $x = (x_1, \dots, x_n) \in G(X)$ and $m_{ij} \in {}_iM_j$, we set

$$x\, m_{ij} = {}_j\varphi_i(x_i \otimes m_{ij}) \in X_j \subseteq G(X).$$

More generally, if $i_1, i_2, \dots, i_{r-1}, i_r$ is a sequence from $\{1, \dots, n\}$ such that ${}_{i_1}M_{i_2} \neq 0$, ${}_{i_2}M_{i_3} \neq 0$, …, ${}_{i_{r-1}}M_{i_r} \neq 0$, and $x = (x_1, \dots, x_n) \in G(X)$, $m_{i_1 i_2} \in {}_{i_1}M_{i_2}, m_{i_2 i_3} \in {}_{i_2}M_{i_3}, \dots, m_{i_{r-1} i_r} \in {}_{i_{r-1}}M_{i_r}$, then we set

$$x(m_{i_1 i_2} \otimes m_{i_2 i_3} \otimes \dots \otimes m_{i_{r-1} i_r}) = \big(\dots\big((x\, m_{i_1 i_2})m_{i_2 i_3}\big)\dots\big)\, m_{i_{r-1} i_r}.$$

Clearly, this defines on $G(X)$ a structure of right $T(\mathbb{M})$-module.

Let $f = (f_i)$ be a morphism from $X = (X_i, {}_j\varphi_i)$ to $Y = (Y_i, {}_j\psi_i)$ in $\operatorname{Rep}(\mathbb{M})$. Consider the K-linear homomorphism

$$G(f) = \bigoplus_{i=1}^{n} f_i \colon \bigoplus_{i=1}^{n} X_i \longrightarrow \bigoplus_{i=1}^{n} Y_i.$$

We show that $G(f)$ is a homomorphism of right $T(\mathbb{M})$-modules. Indeed, take $x = x_i \in X_i$. Then, for any $r = (\lambda_1, \dots, \lambda_n) \in R$, we have

$$G(f)(xr) = G(f)(x_i\lambda_i) = f_i(x_i\lambda_i) = f_i(x_i)\lambda_i = G(f)(x)r,$$

because $f_i \colon X_i \to Y_i$ is a homomorphism of right F_i-modules. Further, if ${}_iM_j \neq 0$ and $m_{ij} \in {}_iM_j$, then

$$\begin{aligned} G(f)(xm_{ij}) = G(f)(x_i m_{ij}) &= f_j\big({}_j\varphi_i(x_i \otimes m_{ij})\big) = {}_j\psi_i\big((f_i \otimes 1)(x_i \otimes m_{ij})\big) \\ &= {}_j\psi_i\big(f_i(x_i) \otimes m_{ij}\big) = f_i(x_i)m_{ij} = G(f)(x)m_{ij}. \end{aligned}$$

More generally, if ${}_{i_1}M_{i_2} \neq 0$, ${}_{i_2}M_{i_3} \neq 0$, …, ${}_{i_{r-1}}M_{i_r} \neq 0$, and $m_{i_1 i_2} \in {}_{i_1}M_{i_2}$, $m_{i_2 i_3} \in {}_{i_2}M_{i_3}, \dots, m_{i_{r-1} i_r} \in {}_{i_{r-1}}M_{i_r}$, for some $i_1, i_2, \dots, i_r \in \{1, \dots, n\}$, then

$$\begin{aligned} &G(f)\big(x(m_{i_1 i_2} \otimes m_{i_2 i_3} \otimes \dots \otimes m_{i_{r-1} i_r})\big) \\ &\quad = G(f)\Big(\big(\dots\big((x_{i_1} m_{i_1 i_2})m_{i_2 i_3}\big)\dots\big)m_{i_{r-1} i_r}\Big) \\ &\quad = f_{i_r}\Big(\big(\dots\big((x_{i_1} m_{i_1 i_2})m_{i_2 i_3}\big)\dots\big)m_{i_{r-1} i_r}\Big) \\ &\quad = \Big(\big(\dots\big((f_{i_1}(x_{i_1}) m_{i_1 i_2})m_{i_2 i_3}\big)\dots\big)m_{i_{r-1} i_r}\Big) \\ &\quad = G(f)(x)(m_{i_1 i_2} \otimes m_{i_2 i_3} \otimes \dots \otimes m_{i_{r-1} i_r}), \end{aligned}$$

because we have $f_{i_s}\, {}_{i_s}\varphi_{i_{s-1}} = {}_{i_s}\psi_{i_{s-1}}(f_{i_{s-1}} \otimes 1)$ for any $s \in \{2, \dots, r\}$.

Hence, $G(f) \colon G(X) \to G(Y)$ is a homomorphism in $\operatorname{Mod} T(\mathbb{M})$. Finally, observe that, if X belongs to $\operatorname{rep}(\mathbb{M})$, then $G(X)$ belongs to $\operatorname{mod} T(\mathbb{M})$. Therefore, $G \colon \operatorname{Rep}(\mathbb{M}) \to \operatorname{Mod} T(\mathbb{M})$ is a K-linear functor which restricts to a K-linear functor $G \colon \operatorname{rep}(\mathbb{M}) \to \operatorname{mod} T(\mathbb{M})$.

A standard checking shows that we have equivalences of functors $GF \xrightarrow{\sim} \mathbf{1}_{\operatorname{Mod} T(\mathbb{M})}$ and $FG \xrightarrow{\sim} \mathbf{1}_{\operatorname{Rep}(\mathbb{M})}$. □

Example 2.6. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ be the field of complex numbers. Consider the $\mathbb{R}$-species $\mathbb{M} = (F_i, {}_iM_j)_{1 \le i,j \le 4}$ defined as follows: $F_1 = F_2 = \mathbb{C}$, $F_3 = F_4 = \mathbb{R}$, ${}_2M_1 = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$, ${}_2M_3 = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}}$, ${}_4M_3 = {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}}$, and ${}_iM_j = 0$, for the remaining (i, j) with $i, j \in \{1, 2, 3, 4\}$. Then $Q_{\mathbb{M}}$ is the acyclic valued quiver of the form

$$1 \longleftarrow 2 \xrightarrow{(2,1)} 3 \longleftarrow 4\,,$$

and consequently the associated tensor algebra $T(\mathbb{M})$ is a finite dimensional hereditary $\mathbb{R}$-algebra with $Q_{T(\mathbb{M})} = Q_{\mathbb{M}}$, by Lemma 2.1 and Theorem 2.2. In fact, a simple checking shows that $T(\mathbb{M})$ is isomorphic to the following $\mathbb{R}$-subalgebra A of the matrix algebra $M_4(\mathbb{C})$:

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{R} & 0 \\ 0 & 0 & \mathbb{R} & \mathbb{R} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ x & b & y & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & z & d \end{bmatrix} \in M_4(\mathbb{C}) \;\middle|\; \begin{matrix} a, b, x, y \in \mathbb{C} \\ c, d, z \in \mathbb{R} \end{matrix} \right\}.$$

We will show later (Example 7.10) that A is of finite representation type and there are 24 isomorphism classes of indecomposable modules in $\operatorname{mod} A$, and consequently $T(\mathbb{M})$ is of finite representation type with 24 isomorphism classes of finite dimensional indecomposable modules. It follows also from Theorem 2.5 that $\operatorname{mod} T(\mathbb{M})$ is equivalent to the category $\operatorname{rep}(\mathbb{M})$ of finite dimensional representations of the $\mathbb{R}$-species $\mathbb{M}$. We look more closely to the category $\operatorname{rep}(\mathbb{M})$. A representation X in $\operatorname{rep}(\mathbb{M})$ is a system

$$X = (X_1, X_2, X_3, X_4, {}_1\varphi_2, {}_3\varphi_2, {}_3\varphi_4),$$

where X_1, X_2 are finite dimensional $\mathbb{C}$-vector spaces, X_3, X_4 are finite dimensional $\mathbb{R}$-vector spaces, and

$${}_1\varphi_2 \colon X_2 \otimes_{\mathbb{C}} \mathbb{C} \longrightarrow X_1 \text{ is a } \mathbb{C}\text{-linear homomorphism,}$$
$${}_3\varphi_2 \colon X_2 \otimes_{\mathbb{C}} \mathbb{C} \longrightarrow X_3 \text{ is an } \mathbb{R}\text{-linear homomorphism,}$$
$${}_3\varphi_4 \colon X_4 \otimes_{\mathbb{R}} \mathbb{R} \longrightarrow X_3 \text{ is an } \mathbb{R}\text{-linear homomorphism.}$$

Since $X_2 \otimes_{\mathbb{C}} \mathbb{C} \cong X_2$ in $\operatorname{mod} \mathbb{C}$ and $X_4 \otimes_{\mathbb{R}} \mathbb{R} \cong X_4$ in $\operatorname{mod} \mathbb{R}$ (see Lemma II.3.5), we may regard the representations in $\operatorname{rep}(\mathbb{M})$ as diagrams

$$X \colon X_1 \xleftarrow{{}_1\varphi_2} X_2 \xrightarrow{{}_3\varphi_2} X_3 \xleftarrow{{}_3\varphi_4} X_4\,,$$

where X_1, X_2 are finite dimensional $\mathbb{C}$-vector spaces, X_3, X_4 are finite dimensional $\mathbb{R}$-vector spaces, ${}_1\varphi_2 \colon X_2 \to X_1$ is a $\mathbb{C}$-linear homomorphism, and ${}_3\varphi_2 \colon X_2 \to X_3$ and ${}_3\varphi_4 \colon X_4 \to X_3$ are $\mathbb{R}$-linear homomorphisms. Further, for another diagram

$$Y \colon Y_1 \xleftarrow{{}_1\psi_2} Y_2 \xrightarrow{{}_3\psi_2} Y_3 \xleftarrow{{}_3\psi_4} Y_4$$

describing a representation Y in $\operatorname{rep}(\mathbb{M})$, a morphism $f\colon X \to Y$ in $\operatorname{rep}(\mathbb{M})$ is a collection $f = (f_1, f_2, f_3, f_4)$, where $f_1\colon X_1 \to Y_1$, $f_2\colon X_2 \to Y_2$ are $\mathbb{C}$-linear homomorphisms and $f_3\colon X_3 \to Y_3$, $f_4\colon X_4 \to Y_4$ are $\mathbb{R}$-linear homomorphisms such that the following diagram is commutative:

$$\begin{tikzcd}
X_1 \arrow[d, "f_1"] & X_2 \arrow[l, "{}_1\varphi_2"'] \arrow[r, "{}_3\varphi_2"] \arrow[d, "f_2"] & X_3 \arrow[d, "f_3"] & X_4 \arrow[l, "{}_4\varphi_3"'] \arrow[d, "f_4"] \\
Y_1 & Y_2 \arrow[l, "{}_1\psi_2"'] \arrow[r, "{}_3\psi_2"] & Y_3 & Y_4 \arrow[l, "{}_4\psi_3"'] .
\end{tikzcd}$$

Moreover, f is an isomorphism in $\operatorname{rep}(\mathbb{M})$ if and only if f_1, f_2, f_3, f_4 are isomorphisms.

3 Exact sequences

In this section we prove several results on the existence of exact sequences of modules, playing a fundamental role in further considerations. Moreover, we provide a criterion for a family of indecomposable modules to be the mouth modules of a generalized standard stable tube.

We start considerations with the known *Snake Lemma*.

Lemma 3.1. *Let A be a finite dimensional K-algebra over a field K and*

$$\begin{tikzcd}
 & M_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & M_3 \arrow[r] \arrow[d, "h_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 &
\end{tikzcd}$$

a commutative diagram of homomorphisms in $\operatorname{mod} A$ *with exact rows. Then there exists in* $\operatorname{mod} A$ *an exact sequence of the form*

$$\begin{tikzcd}
\operatorname{Ker} h_1 \arrow[r, "u_1"] & \operatorname{Ker} h_2 \arrow[r, "u_2"] & \operatorname{Ker} h_3 \arrow[dll, rounded corners, "\delta"'] \\
\operatorname{Coker} h_1 \arrow[r, "v_1"] & \operatorname{Coker} h_2 \arrow[r, "v_2"] & \operatorname{Coker} h_3 .
\end{tikzcd}$$

Moreover, if f_1 is a monomorphism, then u_1 is a monomorphism. Similarly, if g_2 is an epimorphism, then v_2 is an epimorphism.

Proof. Observe that $h_2 f_1(\operatorname{Ker} h_1) = g_1 h_1(\operatorname{Ker} h_1) = 0$ and $h_3 f_2(\operatorname{Ker} h_2) = g_2 h_2(\operatorname{Ker} h_2) = 0$, and hence $f_1(\operatorname{Ker} h_1) \subseteq \operatorname{Ker} h_2$ and $f_2(\operatorname{Ker} h_2) \subseteq \operatorname{Ker} h_3$. We define $u_1\colon \operatorname{Ker} h_1 \to \operatorname{Ker} h_2$ as the restriction of f_1 to $\operatorname{Ker} h_1$ and $u_2\colon \operatorname{Ker} h_2 \to \operatorname{Ker} h_3$ as the restriction of f_2 to $\operatorname{Ker} h_2$. Further, the equalities $g_1 h_1 = h_2 f_1$ and $g_2 h_2 = h_3 f_2$ force the inclusions $g_1(\operatorname{Im} h_1) \subseteq \operatorname{Im} h_2$ and $g_2(\operatorname{Im} h_2) \subseteq \operatorname{Im} h_3$. Since $\operatorname{Coker} h_1 = N_1/\operatorname{Im} h_1$ and $\operatorname{Coker} h_2 = N_2/\operatorname{Im} h_2$, we define $v_1\colon \operatorname{Coker} h_1 \to \operatorname{Coker} h_2$ and $v_2\colon \operatorname{Coker} h_2 \to \operatorname{Coker} h_3$ by $v_1(n_1 + \operatorname{Im} h_1) = g_1(n_1) + \operatorname{Im} h_2$ and $v_2(n_2 + \operatorname{Im} h_2) = g_2(n_2) + \operatorname{Im} h_3$, for $n_1 \in N_1$ and $n_2 \in N_2$. We define now the connecting homomorphism $\delta\colon \operatorname{Ker} h_3 \to \operatorname{Coker} h_1$. Let $m_3 \in \operatorname{Ker} h_3$. Then $m_3 = f_2(m_2)$ for some $m_2 \in M_2$ and $g_2(h_2(m_2)) = h_3(f_2(m_2)) = h_3(m_3) = 0$. Hence $h_2(m_2) \in \operatorname{Ker} g_2 = \operatorname{Im} g_1$, and so there exists exactly one element $n_1 \in N_1$ such that $h_2(m_2) = g_1(h_1)$, because g_1 is a monomorphism. We define $\delta(m_3) = n_1 + \operatorname{Im} h_1 \in \operatorname{Coker} h_1$. We claim that δ is well defined. Let $m_2' \in M_2$ be an element with $m_3 = f_2(m_2')$ and $h_2(m_2') = g_1(n_1')$ for $n_1' \in N_1$. Then $m_2 - m_2' \in \operatorname{Ker} f_2 = \operatorname{Im} f_1$, and so $m_2 - m_2' = f_1(m_1)$ for an element $m_1 \in M_1$. This implies that

$$\begin{aligned} g_1(h_1(m_1)) &= h_2(f_1(m_1)) = h_2(m_2 - m_2') = h_2(m_2) - h_2(m_2') \\ &= g_1(n_1) - g_1(n_1') = g_1(n_1 - n_1'), \end{aligned}$$

and hence $n_1 - n_1' = h_1(m_1) \in \operatorname{Im} h_1$. This shows that $n_1 + \operatorname{Im} h_1 = n_1' + \operatorname{Im} h_1$. Therefore δ is well defined, as claimed.

We will prove now that the sequence presented above formed by $u_1, u_2, \delta, v_1, v_2$ is exact. Observe that $f_2 f_1 = 0$ implies $u_2 u_1 = 0$, and so $\operatorname{Im} u_1 \subseteq \operatorname{Ker} u_2$. Assume $x_2 \in \operatorname{Ker} u_2$. Then $x_2 \in \operatorname{Ker} f_2 = \operatorname{Im} f_1$, and hence $x_2 = f_1(x_1)$ for some $x_1 \in M_1$. Applying $g_1 h_1$ to x_1 we obtain $g_1(h_1(x_1)) = h_2(f_1(x_1)) = h_2(x_2) = 0$, which gives $h_1(x_1) = 0$, because g_1 is a monomorphism. This shows that $x_1 \in \operatorname{Ker} h_1$, and consequently $x_2 = u_1(x_1)$. Thus $\operatorname{Im} u_1 = \operatorname{Ker} u_2$.

Let us show that $\operatorname{Im} u_2 = \operatorname{Ker} \delta$ and $\operatorname{Im} \delta = \operatorname{Ker} v_1$. For $x_2 \in \operatorname{Ker} h_2$, we have $u_2(x_2) \in \operatorname{Ker} h_3$ and $h_2(x_2) = 0 = g_1(0)$, so $\delta(u_2(x_2)) = 0$, by the definition of δ. Hence $\operatorname{Im} u_2 \subseteq \operatorname{Ker} \delta$. Assume now that $x_3 \in \operatorname{Ker} \delta$. Then it follows from the definition of δ that $x_3 = f_2(m_2)$ for $m_2 \in M_2$ satisfying $h_2(m_2) = g_1(h_1(m_1))$, with $m_1 \in M_1$. Thus $h_2(m_2) = h_2 f_1(m_1)$ and hence $m_2 - f_1(m_1) \in \operatorname{Ker} h_2$. But then $u_2(m_2 - f_1(m_1)) = f_2(m_2 - f_1(m_1)) = f_2(m_2) - f_2(f_1(m_1)) = f_2(m_2) = x_3$, which shows that $\operatorname{Ker} \delta \subseteq \operatorname{Im} u_2$. This proves that $\operatorname{Im} u_2 = \operatorname{Ker} \delta$. Take now an element $x \in \operatorname{Im} \delta$. Then $x = n_1 + \operatorname{Im} h_1$ for $n_1 \in N_1$ satisfying $g_1(n_1) = h_2(m_2)$ and $x = \delta(f_2(m_2))$, for some $m_2 \in M_2$. Applying v_1 to x we obtain $v_1(x) = g_1(n_1) + \operatorname{Im} h_2 = h_2(m_2) + \operatorname{Im} h_2 = 0 + \operatorname{Im} h_2$, and so $\operatorname{Im} \delta \subseteq \operatorname{Ker} v_1$. Conversely, take $y \in \operatorname{Ker} v_1$. Hence $y = n_1 + \operatorname{Im} h_1$ for $n_1 \in N_1$ and $g_1(n_1) \in \operatorname{Im} h_2$. Thus we have $g_1(n_1) = h_2(m_2)$ for some $m_2 \in M_2$, $h_3(f_2(m_2)) = g_2(h_2(m_2)) = g_2(g_1(n_1)) = 0$, and $y = \delta(f_2(m_2))$. This shows that $\operatorname{Ker} v_1 \subseteq \operatorname{Im} \delta$, and consequently $\operatorname{Im} \delta = \operatorname{Ker} v_1$.

We claim now that $\operatorname{Im} v_1 = \operatorname{Ker} v_2$. Clearly, $g_2 g_1 = 0$ implies $v_2 v_1 = 0$, and so $\operatorname{Im} v_1 \subseteq \operatorname{Ker} v_2$. Take $z \in \operatorname{Ker} v_2$. Then $z = n_2 + \operatorname{Im} h_2$ with $n_2 \in N_2$ such that $g_2(n_2) \in \operatorname{Im} h_3$. Hence, there exist $m_3 \in M_3$ and $m_2 \in M_2$ such that $g_2(n_2) = h_3(m_3)$ and $m_3 = f_2(m_2)$. We conclude that $g_2(n_2) = h_3(f_2(m_2)) =$

$g_2(h_2(m_2))$, and hence $n_2 - h_2(m_2) \in \operatorname{Ker} g_2 = \operatorname{Im} g_1$. Thus $n_2 - h_2(m_2) = g_1(n_1)$ for some $n_1 \in N_1$. This leads to the equalities in $\operatorname{Coker} h_2 = N_2/\operatorname{Im} h_2$

$$\begin{aligned} z &= n_2 + \operatorname{Im} h_2 = (n_2 - h_2(m_2)) + \operatorname{Im} h_2 \\ &= g_1(n_1) + \operatorname{Im} h_2 = v_1(n_1 + \operatorname{Im} h_1), \end{aligned}$$

and so $\operatorname{Ker} v_2 \subseteq \operatorname{Im} v_1$. This shows that $\operatorname{Im} v_1 = \operatorname{Ker} v_2$.

Finally, we note that if f_1 is a monomorphism, then its restriction u_1 is also a monomorphism. Moreover, if g_2 is an epimorphism, then the induced homomorphism v_2 is an epimorphism. □

Theorem 3.2. *Let A be a finite dimensional K-algebra over a field K, N a module in* mod A*, and*

$$\mathbb{E}\colon \qquad 0 \longrightarrow L_1 \xrightarrow{f_1} L_2 \xrightarrow{f_2} L_3 \longrightarrow 0$$

an exact sequence in mod A. *Moreover, let $\chi_{N,L_1}\colon \mathcal{E}\mathrm{xt}^1_A(N,L_1) \to \operatorname{Ext}^1_A(N,L_1)$ be the K-linear isomorphism defined in* Section III.3. *Then there exists an exact sequence in* mod K *of the form*

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(N,L_1) \arrow[r, "\operatorname{Hom}_A(N,f_1)"] & \operatorname{Hom}_A(N,L_2) \arrow[r, "\operatorname{Hom}_A(N,f_2)"] & \operatorname{Hom}_A(N,L_3) \arrow[dll, "\delta^N_{L_3,L_1}"] \\
& \operatorname{Ext}^1_A(N,L_1) \arrow[r, "\operatorname{Ext}^1_A(N,f_1)"] & \operatorname{Ext}^1_A(N,L_2) \arrow[r, "\operatorname{Ext}^1_A(N,f_2)"] & \operatorname{Ext}^1_A(N,L_3)
\end{tikzcd}$$

such that the following statements hold:

(i) *If* $\operatorname{pd}_A N \le 1$, *then* $\operatorname{Ext}^1_A(N,f_2)$ *is an epimorphism.*

(ii) $\delta^N_{L_3,L_1}(v) = \chi_{N,L_1}([\mathbb{E}v])$ *for any v in* $\operatorname{Hom}_A(N,L_3)$.

Proof. It follows from Lemma II.2.5 that the covariant functor $\operatorname{Hom}_A(N,-)$: mod $A \to$ mod K and the contravariant functors $\operatorname{Hom}_A(-,L_i)\colon \operatorname{mod} A \to K$, $i = 1,2,3$, are left exact. Further, the module N admits a minimal projective resolution in mod A of the form

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} N \longrightarrow 0.$$

Moreover, by Proposition II.2.6(i), the covariant functors $\operatorname{Hom}_A(P_i,-)\colon \operatorname{mod} A \to \operatorname{mod} K$, $i = 0,1,2$, are exact. Then we obtain the following commutative diagram in mod K with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_A(N,L_1) \arrow[r, "\mathrm{Hom}_A(N,f_1)"] \arrow[d, "\mathrm{Hom}_A(d_0,L_1)"] & \mathrm{Hom}_A(N,L_2) \arrow[r, "\mathrm{Hom}_A(N,f_2)"] \arrow[d, "\mathrm{Hom}_A(d_0,L_2)"] & \mathrm{Hom}_A(N,L_3) \arrow[d, "\mathrm{Hom}_A(d_0,L_3)"] & \\
0 \arrow[r] & \mathrm{Hom}_A(P_0,L_1) \arrow[r, "\mathrm{Hom}_A(P_0,f_1)"] \arrow[d, "\mathrm{Hom}_A(d_1,L_1)"] & \mathrm{Hom}_A(P_0,L_2) \arrow[r, "\mathrm{Hom}_A(P_0,f_2)"] \arrow[d, "\mathrm{Hom}_A(d_1,L_2)"] & \mathrm{Hom}_A(P_0,L_3) \arrow[r] \arrow[d, "\mathrm{Hom}_A(d_1,L_3)"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(P_1,L_1) \arrow[r, "\mathrm{Hom}_A(P_1,f_1)"] \arrow[d, "\mathrm{Hom}_A(d_2,L_1)"] & \mathrm{Hom}_A(P_1,L_2) \arrow[r, "\mathrm{Hom}_A(P_1,f_2)"] \arrow[d, "\mathrm{Hom}_A(d_2,L_2)"] & \mathrm{Hom}_A(P_1,L_3) \arrow[r] \arrow[d, "\mathrm{Hom}_A(d_2,L_3)"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(P_2,L_1) \arrow[r, "\mathrm{Hom}_A(P_2,f_1)"] & \mathrm{Hom}_A(P_2,L_2) \arrow[r, "\mathrm{Hom}_A(P_2,f_2)"] & \mathrm{Hom}_A(P_2,L_3) \arrow[r] & 0 .
\end{tikzcd}$$

Moreover, we have $\mathrm{Im}\,\mathrm{Hom}_A(d_0, L_i) = \mathrm{Ker}\,\mathrm{Hom}_A(d_1, L_i)$ for any $i \in \{1, 2, 3\}$.

Further, since $\mathrm{Hom}_A(d_2, L_i)\,\mathrm{Hom}_A(d_1, L_i) = \mathrm{Hom}_A(d_1 d_2, L_i) = 0$ for $i \in \{1, 2, 3\}$, we obtain the following commutative diagram in mod K with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(P_0,L_1) \arrow[r, "\mathrm{Hom}_A(P_0,f_1)"] \arrow[d, "t_1"] & \mathrm{Hom}_A(P_0,L_2) \arrow[r, "\mathrm{Hom}_A(P_0,f_2)"] \arrow[d, "t_2"] & \mathrm{Hom}_A(P_0,L_3) \arrow[r] \arrow[d, "t_3"] & 0 \\
0 \arrow[r] & \mathrm{Ker}\,\mathrm{Hom}_A(d_2,L_1) \arrow[r, "w_1"] & \mathrm{Ker}\,\mathrm{Hom}_A(d_2,L_2) \arrow[r, "w_2"] & \mathrm{Ker}\,\mathrm{Hom}_A(d_2,L_3) . &
\end{tikzcd}$$

where t_i is induced by $\mathrm{Hom}_A(d_1, L_i)$, for $i \in \{1, 2, 3\}$, and w_j is given by the restriction of $\mathrm{Hom}_A(P_1, f_j)$ to $\mathrm{Ker}\,\mathrm{Hom}_A(d_2, L_j)$, for $j \in \{1, 2\}$.

Applying Lemma 3.1 we conclude that there is a K-linear homomorphism $\delta\colon \mathrm{Ker}\,t_3 \to \mathrm{Coker}\,t_1$ such that the sequence in mod K

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\,t_1 \arrow[r, "u_1"] & \mathrm{Ker}\,t_2 \arrow[r, "u_2"] & \mathrm{Ker}\,t_3 \arrow[dll, "\delta"'] \\
 & \mathrm{Coker}\,t_1 \arrow[r, "v_1"] & \mathrm{Coker}\,t_2 \arrow[r, "v_2"] & \mathrm{Coker}\,t_3 ,
\end{tikzcd}$$

where u_j is the restriction of $\mathrm{Hom}_A(P_0, f_j)$ to $\mathrm{Ker}\,t_j = \mathrm{Ker}\,\mathrm{Hom}_A(d_1, L_j)$ and v_j is induced by w_j, for $j \in \{1, 2\}$, is exact. Observe also that we have in mod K the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(N,L_1) \arrow[r, "\mathrm{Hom}_A(N,f_1)"] \arrow[d, "s_1"] & \mathrm{Hom}_A(N,L_2) \arrow[r, "\mathrm{Hom}_A(N,f_2)"] \arrow[d, "s_2"] & \mathrm{Hom}_A(N,L_3) \arrow[d, "s_3"] \\
0 \arrow[r] & \mathrm{Ker}\,t_1 \arrow[r, "u_1"] & \mathrm{Ker}\,t_2 \arrow[r, "u_2"] & \mathrm{Ker}\,t_3 .
\end{tikzcd}$$

where s_i are isomorphisms induced by the monomorphisms $\mathrm{Hom}_A(d_0, L_i)$ and the equalities $\mathrm{Im}\,\mathrm{Hom}_A(d_0, L_i) = \mathrm{Ker}\,\mathrm{Hom}_A(d_1, L_i)$, for $i \in \{1, 2, 3\}$. Moreover, we have

$$\mathrm{Coker}\, t_i = \mathrm{Ker}\,\mathrm{Hom}_A(d_2, L_i)/\,\mathrm{Im}\,\mathrm{Hom}_A(d_1, L_i) = \mathrm{Ext}^1_A(N, L_i)$$

for any $i \in \{1, 2, 3\}$.

Taking $\delta^N_{L_3,L_1}\colon \mathrm{Hom}_A(N, L_3) \to \mathrm{Ext}^1_A(N, L_1)$ as the composed homomorphism

$$\mathrm{Hom}_A(N, L_3) \xrightarrow{s_3} \mathrm{Ker}\, t_3 \xrightarrow{\delta} \mathrm{Ext}^1_A(N, L_1)$$

we obtain the required exact sequence in $\mathrm{mod}\, K$.

(i) Assume now that $\mathrm{pd}_A N \leq 1$. Then $P_2 = 0$, and hence $\mathrm{Ker}\,\mathrm{Hom}_A(d_2, L_i) = \mathrm{Hom}_A(P_1, L_i)$ and

$$\mathrm{Ext}^1_A(N, L_i) = \mathrm{Hom}_A(P_1, L_i)/\,\mathrm{Im}\,\mathrm{Hom}_A(d_1, L_i) = \mathrm{Coker}\,\mathrm{Hom}_A(d_1, L_i)$$

for any $i \in \{1, 2, 3\}$. Since $\mathrm{Hom}_A(P_1, f_2)$ is an epimorphism, we conclude that $\mathrm{Ext}^1_A(N, f_2)$ is an epimorphism.

(ii) Let $v\colon N \to L_3$ be a homomorphism in $\mathrm{mod}\, A$. Then the extension $\mathbb{E}v$ is the upper short exact sequence in the following commutative diagram in $\mathrm{mod}\, A$

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r, "i"] \arrow[d, "\mathrm{id}_{L_1}"] & L_2 \times_{L_3} N \arrow[r, "v'"] \arrow[d, "f_2'"] & N \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "f_1"] & L_2 \arrow[r, "f_2"] & L_3 \arrow[r] & 0,
\end{tikzcd}$$

where $L_2 \times_{L_3} N$ is the fibered product of L_2 and N over L_3, via f_2 and v (see Section III.3). Consider the homomorphism $vd_0\colon P_0 \to L_3$. Then there exists a homomorphism $g\colon P_0 \to L_2$ such that $vd_0 = f_2 g$, by the projectivity of P_0 in $\mathrm{mod}\, A$. Using the universality property of the fibered product (see Exercise I.12.18) we conclude that there is exactly one homomorphism $\varphi_0\colon P_0 \to L_2 \times_{L_3} N$ such that $g = f_2'\varphi_0$ and $d_0 = v'\varphi_0$. Observe that $v'\varphi_0 d_1 = d_0 d_1 = 0$, and hence $\mathrm{Im}\,\varphi_0 d_1 \subseteq \mathrm{Ker}\, v' = \mathrm{Im}\, i$. Thus there exists a homomorphism $\varphi_1\colon P_1 \to L_1$ such that $i\varphi_1 = \varphi_0 d_1$. Moreover, $i\varphi_1 d_2 = \varphi_0 d_1 d_2 = 0$, and hence $\varphi_1 d_2 = 0$, because i is a monomorphism. Therefore, we obtain the following commutative diagram in $\mathrm{mod}\, A$

$$\begin{tikzcd}
P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "\varphi_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "\varphi_0"] & N \arrow[r] \arrow[d, "\mathrm{id}_N"] & 0 \\
0 \arrow[r] & L_1 \arrow[r, "i"] & L_2 \times_{L_3} N \arrow[r, "v'"] & N \arrow[r] & 0.
\end{tikzcd}$$

We also note that φ_1 is uniquely determined by φ_0, and consequently by v, again because i is a monomorphism. This shows that the K-linear isomorphism

$$\chi_{N,L_1} \colon \mathcal{E}\mathrm{xt}_A^1(N, L_1) \longrightarrow \mathrm{Ext}_A^1(N, L_1)$$

assigns to the element $[\mathbb{E}v]$ in $\mathcal{E}\mathrm{xt}_A^1(N, L_1)$ the element $\varphi_1 + \operatorname{Im}\mathrm{Hom}_A(d_1, L_1)$ in $\mathrm{Ext}_A^1(N, L_1)$ (see Theorem III.3.5 and Corollary III.3.6). We will show that $\delta_{L_3,L_1}^{N}(v) = \varphi_1 + \operatorname{Im}\mathrm{Hom}_A(d_1, L_1)$. Observe that $\varphi_1 \in \operatorname{Ker}\mathrm{Hom}_A(d_2, L_1)$. Moreover, we have the equalities

$$\begin{aligned} w_1(\varphi_1) &= \mathrm{Hom}_A(P_1, f_1)(\varphi_1) = f_1\varphi_1 = f_2' i \varphi_1 = f_2'\varphi_0 d_1 \\ &= \mathrm{Hom}_A(d_1, L_2)(f_2'\varphi_0) = t_2(f_2'\varphi_0), \end{aligned}$$

$$\mathrm{Hom}_A(P_0, f_2)(f_2'\varphi_0) = f_2 f_2'\varphi_0 = v v'\varphi_0 = v d_0 = \mathrm{Hom}_A(d_0, L_3)(v) = s_3(v).$$

Hence, we obtain the required equality

$$\delta_{L_3,L_1}^{N}(v) = \delta\big(s_3(v)\big) = \varphi_1 + \operatorname{Im}\mathrm{Hom}_A(d_1, L_3).$$

Summing up, we have $\delta_{L_3,L_1}^{N}(v) = \chi_{N,L_1}([\mathbb{E}v])$. □

Theorem 3.3. *Let A be a finite dimensional K-algebra over a field K, L a module in* mod A*, and*

$$\mathbb{E}\colon \qquad 0 \longrightarrow N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \longrightarrow 0$$

an exact sequence in mod A*. Then there exists an exact sequence in* mod K *of the form*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(N_3, L) \arrow[r, "\mathrm{Hom}_A(f_2, L)"] & \mathrm{Hom}_A(N_2, L) \arrow[r, "\mathrm{Hom}_A(f_1, L)"] & \mathrm{Hom}_A(N_1, L) \arrow[dll, "\delta_L^{N_1,N_3}"] \\
 & \mathrm{Ext}_A^1(N_3, L) \arrow[r, "\mathrm{Ext}_A^1(f_2, L)"] & \mathrm{Ext}_A^1(N_2, L) \arrow[r, "\mathrm{Ext}_A^1(f_1, L)"] & \mathrm{Ext}_A^1(N_1, L)
\end{tikzcd}$$

such that the following statements hold:

(i) *If* $\mathrm{id}_A L \leq 1$, *then* $\mathrm{Ext}_A^1(f_1, L)$ *is an epimorphism.*

(ii) $\delta_L^{N_1,N_3}(u) = \chi_{N_3,L}([u\mathbb{E}])$ *for any u in* $\mathrm{Hom}_A(N_1, L)$.

Proof. We will apply Theorem 3.2 and the duality functors

$$\begin{tikzcd} \operatorname{mod} A \arrow[r, shift left, "D"] & \operatorname{mod} A^{\mathrm{op}} \arrow[l, shift left, "D"] \end{tikzcd}.$$

Consider the module $D(L)$ and the short exact sequence

$$D(\mathbb{E})\colon \qquad 0 \longrightarrow D(N_3) \xrightarrow{D(f_2)} D(N_2) \xrightarrow{D(f_1)} D(N_1) \longrightarrow 0$$

in $\operatorname{mod} A^{\mathrm{op}}$, induced by the short exact sequence $\mathbb{E}$ in $\operatorname{mod} A$. Applying Theorem 3.2, we conclude that there exists an exact sequence in $\operatorname{mod} K$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{A^{\mathrm{op}}}(D(L), D(N_3)) \arrow[r, "\operatorname{Hom}_{A^{\mathrm{op}}}(D(L),D(f_2))"] & \operatorname{Hom}_{A^{\mathrm{op}}}(D(L), D(N_2)) \arrow[r, "\operatorname{Hom}_{A^{\mathrm{op}}}(D(L),D(f_1))"] & \operatorname{Hom}_{A^{\mathrm{op}}}(D(L), D(N_1)) \arrow[dll, "\delta^{D(L)}_{D(N_1),D(N_3)}"'] \\
& \operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), D(N_3)) \arrow[r, "\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L),D(f_2))"] & \operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), D(N_2)) \arrow[r, "\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L),D(f_1))"] & \operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), D(N_1))
\end{tikzcd}$$

such that

(i′) if $\operatorname{pd}_{A^{\mathrm{op}}} D(L) \leq 1$, then $\operatorname{Ext}^1_A(D(L), D(f_1))$ is an epimorphism;

(ii′) $\delta^{D(L)}_{D(N_1),D(N_3)}(v) = \chi_{D(L),D(N_3)}([D(\mathbb{E})v])$ for any v in $\operatorname{Hom}_{A^{\mathrm{op}}}(D(L), D(N_1))$.

Further, the duality functor D induces an isomorphism of contravariant functors

$$\operatorname{Hom}_A(-, L) \xrightarrow{\sim} \operatorname{Hom}_{A^{\mathrm{op}}}(D(L), -)D$$

from $\operatorname{mod} A$ to $\operatorname{mod} K$. Hence there exists a commutative diagram in $\operatorname{mod} K$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(N_3, L) \arrow[r, "\operatorname{Hom}_A(f_2,L)"] \arrow[d, "\eta_3"] & \operatorname{Hom}_A(N_2, L) \arrow[r, "\operatorname{Hom}_A(f_1,L)"] \arrow[d, "\eta_2"] & \operatorname{Hom}_A(N_1, L) \arrow[d, "\eta_1"] \\
0 \arrow[r] & \operatorname{Hom}_{A^{\mathrm{op}}}(D(L), D(N_3)) \arrow[r, "\operatorname{Hom}_{A^{\mathrm{op}}}(D(L),D(f_2))"] & \operatorname{Hom}_{A^{\mathrm{op}}}(D(L), D(N_2)) \arrow[r, "\operatorname{Hom}_{A^{\mathrm{op}}}(D(L),D(f_1))"] & \operatorname{Hom}_{A^{\mathrm{op}}}(D(L), D(N_1)),
\end{tikzcd}$$

where η_1, η_2, η_3 are isomorphisms and the rows are exact.

Similarly, the duality functor D induces an isomorphism of contravariant functors

$$\mathcal{E}\mathrm{xt}^1_A(-, L) \xrightarrow{\sim} \mathcal{E}\mathrm{xt}^1_{A^{\mathrm{op}}}(D(L), -)D$$

from $\operatorname{mod} A$ to $\operatorname{mod} K$. Moreover, it follows from Proposition III.3.7 that there are natural isomorphisms of contravariant functors

$$\chi_{D(L),D(-)} \colon \mathcal{E}\mathrm{xt}^1_{A^{\mathrm{op}}}(D(L), -)D \xrightarrow{\sim} \operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), -)D,$$
$$\chi_{-,L} \colon \mathcal{E}\mathrm{xt}^1_A(-, L) \xrightarrow{\sim} \operatorname{Ext}^1_A(-, L)$$

from $\operatorname{mod} A$ to $\operatorname{mod} K$. For each $i \in \{1, 2, 3\}$, we denote by $\xi_i \colon \mathcal{E}\mathrm{xt}^1_{A^{\mathrm{op}}}(D(L), D(N_i)) \to \operatorname{Ext}^1_A(N_i, L)$ the following composition of K-linear isomorphisms $\chi_{N_i,L}(\chi_{D(L),D(N_i)}D)^{-1}$. Then there exists a commutative diagram in $\operatorname{mod} K$ of the form

$$\begin{tikzcd}
\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), D(N_3)) \arrow[r, "\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L),D(f_2))"] \arrow[d, "\xi_3"] & \operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), D(N_2)) \arrow[r, "\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L),D(f_1))"] \arrow[d, "\xi_2"] & \operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), D(N_1)) \arrow[d, "\xi_1"] \\
\operatorname{Ext}^1_A(N_3, L) \arrow[r, "\operatorname{Ext}^1_A(f_2,L)"] & \operatorname{Ext}^1_A(N_2, L) \arrow[r, "\operatorname{Ext}^1_A(f_1,L)"] & \operatorname{Ext}^1_A(N_1, L).
\end{tikzcd}$$

Hence, since ξ_1, ξ_2, ξ_3 are isomorphisms and the upper sequence is exact, we conclude that the lower sequence is also exact. We define the homomorphism $\delta_L^{N_1,N_3}\colon \operatorname{Hom}_A(N_1, L) \to \operatorname{Ext}_A^1(N_3, L)$ as the composed homomorphism

$$\begin{aligned}\operatorname{Hom}_A(N_1, L) &\xrightarrow{\eta_1} \operatorname{Hom}_{A^{\mathrm{op}}}(D(L), D(N_1)) \xrightarrow{\delta^{D(L)}_{D(N_1),D(N_3)}} \operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), D(N_3))\\ &\xrightarrow{\xi_3} \operatorname{Ext}^1_A(N_3, L).\end{aligned}$$

Since $\eta_1, \eta_2, \xi_3, \xi_2$ are isomorphisms, we obtain the equalities

$$\begin{aligned}\eta_1\big(\operatorname{Im}\operatorname{Hom}_A(f_1, L)\big) &= \operatorname{Im}\operatorname{Hom}_{A^{\mathrm{op}}}\big(D(L), D(f_1)\big) = \operatorname{Ker}\delta^{D(L)}_{D(N_1),D(N_3)}\\ &= \operatorname{Ker}\left(\xi_3\delta^{D(L)}_{D(N_1),D(N_3)}\right) = \eta_1\big(\operatorname{Ker}\delta_L^{N_1,N_3}\big),\end{aligned}$$

and hence $\operatorname{Im}\operatorname{Hom}_A(f_1, L) = \operatorname{Ker}\delta_L^{N_1,N_3}$, and

$$\begin{aligned}\operatorname{Im}\delta_L^{N_1,N_3} &= \operatorname{Im}\left(\xi_3\delta^{D(L)}_{D(N_1),D(N_3)}\right) = \xi_3\left(\operatorname{Im}\delta^{D(L)}_{D(N_1),D(N_3)}\right)\\ &= \xi_3\big(\operatorname{Ker}\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), D(f_2))\big) = \operatorname{Ker}\operatorname{Ext}^1_A(f_2, L).\end{aligned}$$

Summing up, we proved that the sequence

$$\begin{tikzcd}0 \arrow[r] & \operatorname{Hom}_A(N_3, L) \arrow[r, "\operatorname{Hom}_A(f_2{,}L)"] & \operatorname{Hom}_A(N_2, L) \arrow[r, "\operatorname{Hom}_A(f_1{,}L)"] & \operatorname{Hom}_A(N_1, L) \arrow[dll, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- node[near start, below]{\scriptstyle \delta_L^{N_1,N_3}} ([yshift=-3ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & \operatorname{Ext}^1_A(N_3, L) \arrow[r, "\operatorname{Ext}^1_A(f_2{,}L)"] & \operatorname{Ext}^1_A(N_2, L) \arrow[r, "\operatorname{Ext}^1_A(f_1{,}L)"] & \operatorname{Ext}^1_A(N_1, L)\end{tikzcd}$$

in $\operatorname{mod} K$ is exact.

(i) Assume now that $\operatorname{id}_A L \leq 1$. Then $\operatorname{pd}_{A^{\mathrm{op}}} D(L) \leq 1$ and it follows from (i$'$) that $\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(L), D(f_1))$ is an epimorphism. Since $\operatorname{Ext}^1_A(f_1, L)\xi_2 = \xi_1 \operatorname{Ext}^1_{A^{\mathrm{op}}} (D(L), D(f_1))$ with ξ_1 and ξ_2 isomorphisms, we conclude that $\operatorname{Ext}^1_A(f_1, L)$ is an epimorphism.

(ii) Let $u\colon N_1 \to L$ be a homomorphism in $\operatorname{mod} A$. Then the extension $u\mathbb{E}$ is the bottom short exact sequence in the following commutative diagram in $\operatorname{mod} A$:

$$\begin{tikzcd}0 \arrow[r] & N_1 \arrow[r, "f_1"] \arrow[d, "u"] & N_2 \arrow[r, "f_2"] \arrow[d, "f_1'"] & N_3 \arrow[r] \arrow[d, "\mathrm{id}_{N_3}"] & 0\\ 0 \arrow[r] & L \arrow[r, "u'"] & L \oplus_{N_1} N_2 \arrow[r, "p"] & N_3 \arrow[r] & 0\,,\end{tikzcd}$$

where $L \oplus_{N_1} N_2$ is the fibered sum of L and N_2 over N_1, via u and f_1 (see Section III.3). Applying the duality functor $D\colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$, we obtain the commutative diagram in $\operatorname{mod} A^{\mathrm{op}}$ of the form

$$\begin{tikzcd}
0 \arrow[r] & D(N_3) \arrow[r, "D(p)"] \arrow[d, "\mathrm{id}_{D(N_3)}"] & D(L \oplus_{N_1} N_2) \arrow[r, "D(u')"] \arrow[d, "D(f_1')"] & D(L) \arrow[r] \arrow[d, "D(u)"] & 0 \\
0 \arrow[r] & D(N_3) \arrow[r, "D(f_2)"] & D(N_2) \arrow[r, "D(f_1)"] & D(N_1) \arrow[r] & 0 .
\end{tikzcd}$$

Using the universality properties of the fibered sums and products (see Exercises I.12.18 and I.12.20), we conclude that $D(L \oplus_{N_1} N_2)$ is isomorphic to the fibered product $D(N_2) \times_{D(N_1)} D(L)$ of $D(N_2)$ and $D(L)$ over $D(N_1)$, via $D(f_1)$ and $D(u)$. This shows that $[D(u\mathbb{E})] = [D(\mathbb{E})D(u)]$ in $\mathcal{E}\mathrm{xt}^1_{A^{\mathrm{op}}}(D(L), D(N_3))$. Then using (ii$'$) we obtain the qualities

$$\begin{aligned}
\delta_L^{N_1,N_3}(u) &= \Big(\xi_3 \delta^{D(L)}_{D(N_1),D(N_3)} \eta_1\Big)(u) = \xi_3\Big(\delta^{D(L)}_{D(N_1),D(N_3)}\big(D(u)\big)\Big) \\
&= \xi_3\Big(\chi_{D(L),D(N_3)}\big([D(\mathbb{E})D(u)]\big)\Big) = \xi_3\Big(\chi_{D(L),D(N_3)}\big([D(u\mathbb{E})]\big)\Big) \\
&= \chi_{N_3,L}\big([u\mathbb{E}]\big),
\end{aligned}$$

as required. □

Let A be a finite dimensional hereditary K-algebra over a field K. Recall that $\operatorname{mod}_{\mathcal{P}} A$ denotes the full subcategory of $\operatorname{mod} A$ consisting of all modules without nonzero projective direct summands and $\operatorname{mod}_{\mathcal{I}} A$ the full subcategory of $\operatorname{mod} A$ consisting of all modules without nonzero injective direct summands. Then we know from Corollary III.4.11 that the Auslander–Reiten translations $\tau_A = D\operatorname{Tr}$ and $\tau_A^{-1} = \operatorname{Tr} D$ induce the mutually inverse equivalences of categories

$$\begin{tikzcd}
\operatorname{mod}_{\mathcal{P}} A \arrow[r, shift left, "\tau_A"] & \operatorname{mod}_{\mathcal{I}} A^{\mathrm{op}} \arrow[l, shift left, "\tau_A^{-1}"] .
\end{tikzcd}$$

The following proposition establishes the exactness of these two functors.

Proposition 3.4. *Let A be a finite dimensional hereditary K-algebra over a field K. Then the functors*

$$\tau_A\colon \operatorname{mod}_{\mathcal{P}} A \longrightarrow \operatorname{mod}_{\mathcal{I}} A \qquad \textit{and} \qquad \tau_A^{-1}\colon \operatorname{mod}_{\mathcal{I}} A \longrightarrow \operatorname{mod}_{\mathcal{P}} A$$

are exact.

Proof. Let $0 \to N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \to 0$ be an exact sequence in $\operatorname{mod} A$ with N_1, N_2, N_3 in $\operatorname{mod}_{\mathcal{P}} A$. In particular, we have $\operatorname{Hom}_A(N_1, A) = 0$. Since

$\mathrm{id}_A A \leq 1$, applying Theorem 3.3, we obtain the exact sequence in $\mathrm{mod}\, K$

$$0 \longrightarrow \mathrm{Ext}_A^1(N_3, A) \xrightarrow{\mathrm{Ext}_A^1(f_2, A)} \mathrm{Ext}_A^1(N_2, A) \xrightarrow{\mathrm{Ext}_A^1(f_1, A)} \mathrm{Ext}_A^1(N_1, A) \longrightarrow 0,$$

and hence in $\mathrm{mod}\, A^{\mathrm{op}}$. Applying the duality functor $D\colon \mathrm{mod}\, A^{\mathrm{op}} \to \mathrm{mod}\, A$, we obtain the exact sequence in $\mathrm{mod}\, A$

$$0 \to D\,\mathrm{Ext}_A^1(N_1, A) \xrightarrow{D\,\mathrm{Ext}_A^1(f_1, A)} D\,\mathrm{Ext}_A^1(N_2, A) \xrightarrow{D\,\mathrm{Ext}_A^1(f_2, A)} D\,\mathrm{Ext}_A^1(N_3, A) \to 0.$$

On the other hand, it follows from Theorem III.4.10 that the covariant functors $\tau_A = D\,\mathrm{Tr}$ and $D\,\mathrm{Ext}_A^1(-, A)$ from $\mathrm{mod}\, A$ to $\mathrm{mod}\, A$ are naturally isomorphic. Therefore, we obtain the exact sequence in $\mathrm{mod}\, A$

$$0 \longrightarrow \tau_A N_1 \xrightarrow{\tau_A f_1} \tau_A N_2 \xrightarrow{\tau_A f_2} \tau_A N_3 \longrightarrow 0.$$

This shows that the functor $\tau_A\colon \mathrm{mod}_{\mathcal{P}}\, A \to \mathrm{mod}_{\mathcal{I}}\, A$ is exact. The proof that the functor $\tau_A^{-1}\colon \mathrm{mod}_{\mathcal{I}}\, A \to \mathrm{mod}_{\mathcal{P}}\, A$ is exact is similar and uses the fact that the covariant functors $\tau_A^{-1} = \mathrm{Tr}\, D$ and $\mathrm{Ext}_{A^{\mathrm{op}}}^1(D(-), A)$ from $\mathrm{mod}\, A$ to $\mathrm{mod}\, A$ are naturally isomorphic, again by Theorem III.4.10. □

Let A be a finite dimensional K-algebra over a field K. A module E in $\mathrm{mod}\, A$ is said to be a *brick* if $\mathrm{End}_A(E)$ is a division K-algebra. Observe that a brick E in $\mathrm{mod}\, A$ is necessarily indecomposable, since its endomorphism algebra is a local algebra (see Lemma I.4.4). Two bricks E an E' in $\mathrm{mod}\, A$ are called *orthogonal* if $\mathrm{Hom}_A(E, E') = 0$ and $\mathrm{Hom}_A(E', E) = 0$. For a family $E_1, \dots, E_r$ of pairwise orthogonal bricks in $\mathrm{mod}\, A$, we denote by $\mathcal{EXT}_A(E_1, \dots, E_r)$ the full subcategory of $\mathrm{mod}\, A$ whose nonzero objects are modules M such that there exists a chain of right A-submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{t-1} \subset M_t = M,$$

for some $t \geq 1$, with M_i/M_{i-1} isomorphic to one of the bricks $E_1, \dots, E_r$, for any $i \in \{1, \dots, t\}$. Then $\mathcal{EXT}_A(E_1, \dots, E_r)$ is called the *extension category* of $E_1, \dots, E_r$ in $\mathrm{mod}\, A$, and is the smallest additive subcategory of $\mathrm{mod}\, A$ containing the bricks $E_1, \dots, E_r$ and closed under extensions. A nonzero object S in $\mathcal{EXT}_A(E_1, \dots, E_r)$ is said to be *simple*, if any nonzero subobject of S in $\mathcal{EXT}_A(E_1, \dots, E_r)$ equals S. We say that $\mathcal{EXT}_A(E_1, \dots, E_r)$ is an *exact subcategory* of $\mathrm{mod}\, A$ if the inclusion functor $\mathcal{EXT}_A(E_1, \dots, E_r) \hookrightarrow \mathrm{mod}\, A$ is exact.

Lemma 3.5. *Let A be a finite dimensional K-algebra over a field K and $E_1, \dots, E_r$ a family of pairwise orthogonal bricks in* $\mathrm{mod}\, A$*. Then the extension category $\mathcal{EXT}_A(E_1, \dots, E_r)$ is an exact abelian subcategory of* $\mathrm{mod}\, A$*, and $E_1, \dots, E_r$ is a complete set of pairwise nonisomorphic simple objects in $\mathcal{EXT}_A(E_1, \dots, E_r)$.*

Proof. We abbreviate $\mathcal{E} = \mathcal{EXT}_A(E_1, \dots, E_r)$. Let M and N be modules in $\mathcal{E}$ and $f\colon M \to N$ a homomorphism in $\operatorname{mod} A$. We claim that the modules $\operatorname{Ker} f$, $\operatorname{Im} f$, and $\operatorname{Coker} f$ are in $\mathcal{E}$. Assume that

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{m-1} \subset M_m = M,$$
$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_{n-1} \subset N_n = N$$

are chains of submodules of M and N in $\operatorname{mod} A$ such that the quotient modules $M_i/M_{i-1}, i \in \{1, \dots, m\}$, and N_j/N_{j-1}, $j \in \{1, \dots, n\}$, are isomorphic to modules in $\{E_1, \dots, E_r\}$. We prove our claim by induction on $m+n$. For $m+n \leq 1$, we have that $\operatorname{Ker} f$, $\operatorname{Im} f$, or $\operatorname{Coker} f$ is either zero or is isomorphic to one of the modules $E_1, \dots, E_r$. Therefore, we may assume that $m+n \geq 2$.

(1) Assume first that $f(M_1) = 0$. Then f induces the homomorphism $\bar{f}\colon M/M_1 \to N$ given by $\bar{f}(m + M_1) = f(m)$. Observe that we have $\operatorname{Im} \bar{f} = \operatorname{Im} f$, $\operatorname{Ker} \bar{f} \cong \operatorname{Ker} f/M_1$, and $\operatorname{Coker} \bar{f} \cong \operatorname{Coker} f$. Moreover, we have the chain of submodules of M/M_1 in $\operatorname{mod} A$

$$0 = \overline{M_1} \subset \overline{M_2} \subset \cdots \subset \overline{M_{m-1}} \subset \overline{M_m} = M/M_1$$

with $\overline{M_i} = M_i/M_1$ for $i \in \{1, \dots, m\}$, and $\overline{M_j}/\overline{M_{j-1}} \cong M_j/M_{j-1}$ for $j \in \{2, \dots, m\}$, which shows that M/M_1 belongs to $\mathcal{E}$. Then it follows from our induction hypothesis that $\operatorname{Ker} \bar{f}$, $\operatorname{Im} \bar{f}$, and $\operatorname{Coker} \bar{f}$ belong to $\mathcal{E}$. In particular, we obtain that $\operatorname{Ker} f/M_1$, $\operatorname{Im} f$, and $\operatorname{Coker} f$ belong to $\mathcal{E}$. Let

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{t-1} \subset X_t = \operatorname{Ker} f/M_1$$

be a sequence of submodules of $\operatorname{Ker} f/M_1$ in $\operatorname{mod} A$ such that X_i/X_{i-1}, $i \in \{1, \dots, t\}$, are isomorphic to modules in $\{E_1, \dots, E_r\}$. Consider the canonical epimorphism $\pi\colon \operatorname{Ker} f \to \operatorname{Ker} f/M_1$ such that $\pi(x) = x + M_1$ for any $x \in \operatorname{Ker} f$. Then we obtain the chain of submodules of $\operatorname{Ker} f$ in $\operatorname{mod} A$

$$0 = X'_{-1} \subset X'_0 \subset X'_1 \subset \cdots \subset X'_{t-1} \subset X'_t = \operatorname{Ker} f$$

with $X'_i = \pi^{-1}(X_i)$ for any $i \in \{0, 1, \dots, t\}$. Observe that $X'_0/X'_{-1} = X'_0 = M_1$ and $X'_i/X'_{i-1} \cong X_i/X_{i-1}$ for $i \in \{1, \dots, t\}$. This shows that $\operatorname{Ker} f$ belongs to $\mathcal{E}$.

(2) Assume that $f(M_1) \neq 0$. Then there exists the smallest j in $\{1, \dots, n\}$ such that $f(M_1) \subseteq N_j$; let $f'\colon M_1 \to N_j$ be the homomorphism induced by f. Consider the canonical epimorphism $p\colon N_j \to N_j/N_{j-1}$, given by $p(x) = x + N_{j-1}$ for $x \in N_j$. The composition $pf'\colon M_1 \to N_j/N_{j-1}$ is nonzero, by the choice of j. Since M_1 and N_j/N_{j-1} are isomorphic to some bricks in $\{E_1, \dots, E_r\}$, we conclude that pf' is an isomorphism. Then there exists a homomorphism $u\colon N_j/N_{j-1} \to M_1$ such that $u(pf') = \mathrm{id}_{M_1}$ and $(pf')u = \mathrm{id}_{N_j/N_{j-1}}$. Hence $f'\colon M_1 \to N_j$ is a section and $p\colon N_j \to N_j/N_{j-1}$ a retraction. Applying Lemma I.4.2, we obtain

$$N_j = \operatorname{Ker} p \oplus \operatorname{Im} f'u = N_{j-1} \oplus \operatorname{Im} f' = N_{j-1} \oplus f(M_1).$$

We define $N'_k = N_{k-1} \oplus f(M_1)$ for $k \in \{1, \dots, j\}$. Then we obtain a chain of submodules of N in $\operatorname{mod} A$

$$0 = N_0 \subset N'_1 \subset N'_2 \subset \cdots \subset N'_j = N_j \subset \cdots \subset N_{n-1} \subset N_n = N,$$

where the quotients of two consecutive terms are isomorphic to modules in $\{E_1, \dots, E_r\}$. Observe that $N'_1 = N_0 \oplus f(M_1) = f(M_1)$. This shows that we may assume additionally that $f(M_1) = N_1$. Since M_1 and N_1 are pairwise orthogonal brics, we infer that f induces an isomorphism $f''\colon M_1 \to N_1$. Moreover, we have a commutative diagram in $\operatorname{mod} A$ with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, hook] \arrow[d, "f''"] & M \arrow[r] \arrow[d, "f"] & M/M_1 \arrow[r] \arrow[d, "\bar{f}"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, hook] & N \arrow[r] & N/N_1 \arrow[r] & 0,
\end{tikzcd}$$

where $\bar{f}$ is given by $\bar{f}(m + M_1) = f(m) + N_1$ for $m \in M$. Since Ker $f'' = 0$ and Coker $f'' = 0$, Lemma 3.1 yields isomorphisms Ker $f \cong$ Ker $\bar{f}$ and Coker $f \cong$ Coker $\bar{f}$. Observe that we have in $\operatorname{mod} A$ the chains of submodules

$$0 = \overline{M_1} \subset \overline{M_2} \subset \cdots \subset \overline{M_{m-1}} \subset \overline{M_m} = M/M_1,$$
$$0 = \overline{N_1} \subset \overline{N_2} \subset \cdots \subset \overline{N_{n-1}} \subset \overline{N_n} = N/N_1,$$

with $\overline{M_i} = M_i/M_1$, for $i \in \{1, \dots, m\}$, and $\overline{N_j} = N_j/N_1$, for $j \in \{1, \dots, n\}$. Obviously, we have isomorphisms $\overline{M_i}/\overline{M_{i-1}} \cong M_i/M_{i-1}$ for $i \in \{2, \dots, m\}$, and $\overline{N_j}/\overline{N_{j-1}} \cong N_j/N_{j-1}$ for $j \in \{2, \dots, n\}$. This shows that M/M_1 and N/N_1 belong to the category $\mathcal{E}$. Then the induction hypothesis yields that Ker $\bar{f}$, Im $\bar{f}$, and Coker $\bar{f}$ are in $\mathcal{E}$. In particular, we get that Ker f and Coker f are in $\mathcal{E}$. Observe also that Im $\bar{f}$ = Im f/N_1 with Im $\bar{f}$ and N_1 in $\mathcal{E}$. Since $\mathcal{E}$ is closed under extensions, we conclude that Im f also belongs to $\mathcal{E}$. This shows that $\mathcal{E} = \mathcal{EXT}_A(E_1, \dots, E_r)$ is an exact abelian subcategory of $\operatorname{mod} A$. □

Let A be a finite dimensional K-algebra over a field K, $E_1, \dots, E_r$ a family of pairwise orthogonal bricks in $\operatorname{mod} A$, and $\mathcal{E} = \mathcal{EXT}_A(E_1, \dots, E_r)$ the extension category of $E_1, \dots, E_r$. It follows from Lemma 3.5 that $\mathcal{E}$ is an exact abelian subcategory of $\operatorname{mod} A$. An object U of the category $\mathcal{E}$ is said to be *uniserial*, if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$, for each pair of subobjects U_1, U_2 of U in $\mathcal{E}$. Observe that an object U in $\mathcal{E}$ is uniserial if all its subobjects in $\mathcal{E}$ form a chain with respect to inclusion. The length of the chain of subobjects of a uniserial object U in $\mathcal{E}$ will be denoted by $\ell_{\mathcal{E}}(U)$. We say that the family of bricks $E_1, \dots, E_r$ is *hereditary* if $\operatorname{pd}_A E_i \leq 1$ and $\operatorname{id}_A E_i \leq 1$ for any $i \in \{1, \dots, r\}$.

Proposition 3.6. *Let A be a finite dimensional K-algebra over a field K and $E_1, \dots, E_r$ a hereditary family of pairwise orthogonal bricks in* $\operatorname{mod} A$ *such that*

$\tau_A E_{i+1} = E_i$ for all $i \in \{1, \dots, r\}$, and $E_{r+1} = E_1$. Then the extension category

$$\mathcal{E} = \mathcal{EXT}_A(E_1, \dots, E_r)$$

has the following properties:

(i) *For each pair $(i, j) \in \{1, \dots, r\} \times \mathbb{N}^+$, there exist a uniserial object $E_i[j]$ in $\mathcal{E}$ with $\ell_{\mathcal{E}}(E_i[j]) = j$ and homomorphisms*

$$u_{ij} \colon E_i[j-1] \longrightarrow E_i[j] \quad \text{and} \quad p_{ij} \colon E_i[j] \longrightarrow E_{i+1}[j-1],$$

for $j \geq 2$, such that we have in mod *A two exact sequences*

$$0 \longrightarrow E_i[j-1] \xrightarrow{u_{ij}} E_i[j] \xrightarrow{p'_{ij}} E_{i+j-1}[1] \longrightarrow 0,$$

$$0 \longrightarrow E_i[1] \xrightarrow{u'_{ij}} E_i[j] \xrightarrow{p_{ij}} E_{i+1}[j-1] \longrightarrow 0,$$

where $p'_{ij} = p_{i+j-2\,2} \dots p_{i+1\,j-1} p_{ij}$ and $u'_{ij} = u_{ij} u_{i\ j-1} \dots u_{i2}$.

(ii) *For each $j \geq 2$, there exists in* mod *A an almost split sequence of the form*

$$0 \longrightarrow E_i[j-1] \xrightarrow{\begin{bmatrix} p_{i\,j-1} \\ u_{ij} \end{bmatrix}} E_{i+1}[j-2] \oplus E_i[j] \xrightarrow{\begin{bmatrix} u_{i+1\,j-1} & p_{ij} \end{bmatrix}} E_{i+1}[j-1] \longrightarrow 0,$$

where we set $E_i[0] = 0$ and $E_{r+1}[m] = E_1[m]$ for $i \in \{1, \dots, r\}$ and $m \in \mathbb{N}^+$.

(iii) *Every indecomposable object M of $\mathcal{E}$ is uniserial and isomorphic to a module $E_i[j]$ for some $i \in \{1, \dots, r\}$ and $j \in \mathbb{N}^+$.*

Proof. We set $E_i[1] = E_i$ for each $i \in \{1, \dots, r\}$. Assume $j = 2$. For each $i \in \{1, \dots, r\}$, there exists in mod A an almost split sequence

$$\mathbb{E}_i \colon \qquad 0 \longrightarrow E_i[1] \xrightarrow{u_{i2}} E_i[2] \xrightarrow{p_{i2}} E_{i+1}[1] \longrightarrow 0,$$

because $E_i = \tau_A E_{i+1}$. Since by assumption $\operatorname{pd}_A E_{i+1} = 1$, applying Corollary III.6.4, we obtain K-linear isomorphisms

$$\begin{aligned} \operatorname{Ext}^1_A(E_{i+1}[1], E_i[1]) &\cong D \operatorname{Hom}_A(E_i[1], \tau_A E_{i+1}[1]) \\ &\cong D \operatorname{Hom}_A(E_i[1], E_i[1]) \\ &\cong \operatorname{End}_A(E_i). \end{aligned}$$

On the other hand, by Proposition III.3.8, there is an isomorphism

$$\chi_{E_{i+1}[1],E_i[1]}\colon \mathcal{E}\mathrm{xt}^1_A(E_{i+1}[1], E_i[1]) \longrightarrow \mathrm{Ext}^1_A(E_{i+1}[1], E_i[1])$$

of $(\mathrm{End}(E_i), \mathrm{End}(E_{i+1}))$-bimodules. Therefore, $\mathcal{E}\mathrm{xt}^1_A(E_{i+1}[1], E_i[1])$ is generated as a left $\mathrm{End}_A(E_i[1])$-module and as a right $\mathrm{End}_A(E_{i+1}[1])$-module by the class $[\mathbb{E}_i]$ of the almost split sequence $\mathbb{E}_i$, for any $i \in \{1, \dots, r\}$. Observe also that for any homomorphism $f\colon E_i[2] \to E_k$, with $k \in \{1, \dots, r\}$, we have $f u_{i2} = 0$. Indeed, if $f u_{i2} \neq 0$, then $f u_{i2}$ is an isomorphism of the bricks $E_i = E_i[1]$ and E_k, and consequently u_{i2} is a section, a contradiction. This shows that, for each $i \in \{1, \dots, r\}$, $u_{i2}(E_i[1])$ is a unique proper subobject of $E_i[2]$ in $\mathcal{E}$, and consequently $E_i[2]$ is a uniserial object of $\mathcal{E}$ with $\ell_{\mathcal{E}}(E_i[2]) = 2$.

Assume now that $k \geq 3$ and we have constructed all the required modules $E_i[j]$ and homomorphisms u_{ij}, p_{ij}, with $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, k-1\}$. In particular, we have in mod A exact sequences

$$0 \longrightarrow E_{i+1}[k-2] \xrightarrow{u_{i+1\,k-1}} E_{i+1}[k-1] \xrightarrow{p'_{i+1\,k-1}} E_{i+k-1}[1] \longrightarrow 0, \qquad \text{(a)}$$

$$0 \longrightarrow E_i[1] \xrightarrow{u'_{i\,k-1}} E_i[k-1] \xrightarrow{p_{i\,k-1}} E_{i+1}[k-2] \longrightarrow 0. \qquad \text{(b)}$$

Since $\mathrm{id}_A E_i = 1$, applying Theorem 3.3 to the exact sequence (a), we obtain an epimorphism in mod K

$$\mathrm{Ext}^1_A(E_{i+1}[k-1], E_i[1]) \xrightarrow{\mathrm{Ext}^1_A(u_{i+1\,k-1}, E_i[1])} \mathrm{Ext}^1_A(E_{i+1}[k-2], E_i[1]).$$

Then we obtain an epimorphism in mod K of the form

$$\mathcal{E}\mathrm{xt}^1_A(E_{i+1}[k-1], E_i[1]) \xrightarrow{\mathcal{E}\mathrm{xt}^1_A(u_{i+1\,k-1}, E_i[1])} \mathcal{E}\mathrm{xt}^1_A(E_{i+1}[k-2], E_i[1]),$$

since the contravariant functors $\mathcal{E}\mathrm{xt}^1_A(-, E_i[1])$ and $\mathrm{Ext}^1_A(-, E_i[1])$ from mod A to mod K are naturally isomorphic (see Proposition III.3.7). Observe also that $\mathcal{E}\mathrm{xt}^1_A(E_{i+1}[k-2], E_i[1])$ contains the class of the exact sequence (b). Therefore, it follows from definition of $\mathcal{E}\mathrm{xt}^1_A(u_{i+1\,k-1}, E_i[1])$ that there exist a module $E_i[k]$, homomorphisms u_{ik}, p_{ik}, $u'_{ik} = u_{ik} u'_{i\,k-1}$, and $p'_{ik} = p'_{i\,k-1} p_{ik}$ such that we have in mod A a commutative diagram with exact rows and colums

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E_i[1] \arrow[r, "u'_{i\,k-1}"] \arrow[d, "\mathrm{id}_{E_i[1]}"] & E_i[k-1] \arrow[r, "p_{i\,k-1}"] \arrow[d, "u_{ik}"] & E_{i+1}[k-2] \arrow[r] \arrow[d, "u_{i+1\,k-1}"] & 0 \\
0 \arrow[r] & E_i[1] \arrow[r, "u'_{ik}"] & E_i[k] \arrow[r, "-p_{ik}"] \arrow[d, "-p'_{ik}"] & E_{i+1}[k-1] \arrow[r] \arrow[d, "p'_{i+1\,k-1}"] & 0 \\
 & & E_{i+k-1}[1] \arrow[r, "\mathrm{id}_{E_{i+k-1}[1]}"] \arrow[d] & E_{i+k-1}[1] \arrow[d] & \\
 & & 0 & 0\,. &
\end{tikzcd}$$

In particular, we obtain the two exact sequences required in (i) for $j = k$ and any $i \in \{1, \dots, r\}$. Further, the upper right square of the above diagram gives the exact sequence

$$\mathbb{E}_{i,k}\colon 0 \to E_i[k-1] \xrightarrow{\begin{bmatrix} p_{i\,k-1} \\ u_{ik} \end{bmatrix}} E_{i+1}[k-2] \oplus E_i[k] \xrightarrow{\begin{bmatrix} u_{i+1\,k-1} & p_{ik} \end{bmatrix}} E_{i+1}[k-1] \to 0,$$

because, by the definition of $\mathcal{E}\mathrm{xt}^1_A(u_{i+1\,k-1}, E_i[1])$, $E_i[k-1]$ is the fibered product of $E_i[k]$ and $E_{i+1}[k-2]$ over $E_{i+1}[k-1]$, via $-p_{ik}$ and $u_{i+1\,k-1}$. We will prove now that $\operatorname{Im} u_{ik} = u_{ik}(E_i[k-1])$ is a unique maximal subobject of $E_i[k]$ in $\mathcal{E}$. Since $u_{ik}\colon E_i[k-1] \to \operatorname{Im} u_{ij}$ is an isomorphism and, by the induction hypothesis, $E_i[k-1]$ is a uniserial object of $\mathcal{E}$ with $\ell_{\mathcal{E}}(E_i[k-1]) = k-1$, this will imply that $E_i[k]$ is a uniserial object of $\mathcal{E}$ with $\ell_{\mathcal{E}}(E_i[k]) = k$. Let $f\colon E_i[k] \to E_s$ be a nonzero homomorphism in mod A, for some $s \in \{1, \dots, r\}$. We claim that $f u_{ik} = 0$. Since f is a nonzero homomorphism in $\mathcal{E}$ and E_s is a simple object of $\mathcal{E}$, we infer that f is an epimorphism. Observe that $f u_{ik} u_{i\,k-1} = 0$, because $\operatorname{Im} u_{i\,k-1} \cong E_i[k-2]$ is the unique maximal subobject of the uniserial object $E_i[k-1]$ of $\mathcal{E}$, and the codomain of $f u_{ik}\colon E_i[k-1] \to E_s$ is a simple object of $\mathcal{E}$. Observe that we have the exact sequence

$$0 \longrightarrow E_i[k-2] \xrightarrow{u_{i\,k-1}} E_i[k-1] \xrightarrow{p'_{i\,k-1}} E_{i+k-2}[1] \longrightarrow 0.$$

Hence, there exists a homomorphism $g\colon E_{i+k-2}[1] \to E_s$ such that $f u_{ik} = g p'_{i\,k-1} = g p_{i+k-2\,2} \dots p_{i\,k-1}$. Let

$$f' = g p_{i+k-2\,2} \dots p_{i+1\,k-2}\colon E_{i+1}[k-2] \longrightarrow E_s.$$

Then we get $f u_{ik} = f' p_{i\,k-1}$, or equivalently $\begin{bmatrix} -f' & f \end{bmatrix} \begin{bmatrix} p_{i\,k-1} \\ u_{ik} \end{bmatrix} = 0$. Using the exact sequence $\mathbb{E}_{i,k}$ we infer that there is a homomorphism $f''\colon E_{i+1}[k-1] \to E_s$ such that $\begin{bmatrix} -f' & f \end{bmatrix} = f'' \begin{bmatrix} u_{i+1\,k-1} & p_{ik} \end{bmatrix}$. We note that $-f' = f'' u_{i+1\,k-1} = 0$, because, by the induction hypothesis, $\operatorname{Im} u_{i+1\,k-1}$ is the unique maximal subobject of $E_{i+1}[k-1]$ in $\mathcal{E}$. Hence, $f u_{ik} = f' p_{i\,k-1} = 0$. This shows that $\operatorname{Im} u_{ik}$ is the unique maximal subobject of $E_i[k]$ in $\mathcal{E}$, and consequently $E_i[k]$ is a uniserial object in $\mathcal{E}$ with $\ell_{\mathcal{E}}(E_i[k]) = k$.

We will prove now that the exact sequence $\mathbb{E}_{i,k}$ is an almost split sequence in $\operatorname{mod} A$. Since $E_i[k-1]$, $E_{i+1}[k-2]$, $E_i[k]$, $E_{i+1}[k-1]$ are uniserial (hence indecomposable) objects in $\mathcal{E}$ with $\ell_{\mathcal{E}}(E_i[k-1]) = k-1$, $\ell_{\mathcal{E}}(E_{i+1}[k-2]) = k-2$, $\ell_{\mathcal{E}}(E_i[k]) = k$, and $\ell_{\mathcal{E}}(E_{i+1}[k-1]) = k-1$, we conclude that the exact sequence $\mathbb{E}_{i,k}$ does not split, by Lemmas III.3.1, I.4.2, and Theorem I.4.6. In particular, we deduce that $E_{i+1}[k-1]$ is not a projective module in $\operatorname{mod} A$, and hence $\tau_A E_{i+1}[k-1] \neq 0$. We claim that $\tau_A E_{i+1}[k-1] \cong E_i[k-1]$. It follows from the induction hypothesis that we have in $\operatorname{mod} A$ an almost split sequence of the form

$$0 \to E_i[k-2] \xrightarrow{\begin{bmatrix} p_{i\,k-2} \\ u_{i\,k-1} \end{bmatrix}} E_{i+1}[k-3] \oplus E_i[k-1] \xrightarrow{\begin{bmatrix} u_{i+1\,k-2} & p_{i\,k-1} \end{bmatrix}} E_{i+1}[k-2] \to 0$$

for any $i \in \{1, \dots, r\}$. Hence, replacing i with $i+1$, we conclude that $u_{i+1\,k-1}\colon E_{i+1}[k-2] \to E_{i+1}[k-1]$ is an irreducible homomorphism in $\operatorname{mod} A$. Since $E_{i+1}[k-1]$ is not projective, there is in $\operatorname{mod} A$ an irreducible homomorphism $\tau_A E_{i+1}[k-1] \to E_{i+1}[k-2]$. We have $E_{i+1}[k-3] = 0$ for $k = 3$. For $k > 3$, the induction hypothesis yields $\tau_A^{-1} E_{i+1}[k-3] \cong E_{i+2}[k-3] \ncong E_{i+1}[k-1]$. Therefore, applying Theorem III.7.12, we conclude that $\tau_A E_{i+1}[k-1] = E_i[k-1]$, as required. It follows from the proof of the Auslander–Reiten theorem (Theorem III.8.4) that an extension

$$\mathbb{E}\colon \qquad 0 \longrightarrow \tau_A E_{i+1}[k-1] \longrightarrow E \longrightarrow E_{i+1}[k-1] \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod} A$ if and only if $\chi_{E_{i+1}[k-1], \tau_A E_{i+1}[k-1]}(\mathbb{E})$ is a nonzero element of the socle of the right $\operatorname{End}_A(E_{i+1}[k-1])$-module $\operatorname{Ext}^1_A(E_{i+1}[k-1], \tau_A E_{i+1}[k-1])$. Since $\tau_A E_{i+1}[k-1] \cong E_i[k-1]$, we have $\operatorname{Ext}^1_A(E_{i+1}[k-1], \tau_A E_{i+1}[k-1]) \cong \operatorname{Ext}^1_A(E_{i+1}[k-1], E_i[k-1])$ as right $\operatorname{End}_A(E_{i+1}[k-1])$-modules. Then it remains to show that, for any $h \in \operatorname{rad} \operatorname{End}_A(E_{i+1}[k-1])$, $\operatorname{Ext}^1_A(h, E_i[k-1])(\chi_{E_{i+1}[k-1], E_i[k-1]}([\mathbb{E}_{i,k}])) = 0$ holds. Observe that $h \in \operatorname{End}_A(E_{i+1}[k-1])$ belongs to $\operatorname{rad} \operatorname{End}_A(E_{i+1}[k-1])$ if and only if h is not an isomorphism.

Let $h\colon E_{i+1}[k-1] \to E_{i+1}[k-1]$ be a homomorphism in $\operatorname{mod} A$ which is not an isomorphism. Then h is not an epimorphism, and hence $\operatorname{Im} h$ is contained in the unique maximal subobject $\operatorname{Im} u_{i+1\,k-1} \cong E_{i+1}[k-2]$ of $E_{i+1}[k-1]$.

Consider the exact sequence

$$0 \longrightarrow E_{i+1}[k-2] \xrightarrow{u_{i+1\,k-1}} E_{i+1}[k-1] \xrightarrow{p'_{i+1\,k-1}} E_{i+k-1}[1] \longrightarrow 0.$$

Then $p'_{i+1\,k-1}h = 0$, and hence there exists a homomorphism $g\colon E_{i+1}[k-1] \to E_{i+1}[k-2]$ such that $h = u_{i+1\,k-1}g$. We claim that

$$\mathcal{E}\mathrm{xt}^1_A(u_{i+1\,k-1}, E_i[k-1])([\mathbb{E}_{i,k}]) = [\mathbb{E}_{i,k}u_{i+1\,k-1}] = [\mathcal{O}_{E_{i+1}[k-2],E_i[k-1]}],$$

being the zero element in $\mathcal{E}\mathrm{xt}^1_A(E_i[k-2], E_i[k-1])$. The extension $\mathbb{E}_{i,k}u_{i+1\,k-1}$ is given by the upper sequence of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_i[k-1] \arrow[r, "\gamma"] \arrow[d, "\mathrm{id}_{E_i[k-1]}"'] & M \arrow[r, "\beta"] \arrow[d, "\alpha"] & E_{i+1}[k-2] \arrow[r] \arrow[d, "u_{i+1\,k-1}"] & 0 \\
0 \arrow[r] & E_i[k-1] \arrow[r, "\begin{bmatrix} p_{i\,k-2} \\ u_{ik} \end{bmatrix}"] & E_{i+1}[k-2] \oplus E_i[k] \arrow[r, "\begin{bmatrix} u_{i+1\,k-1} & p_{ik} \end{bmatrix}"] & E_{i+1}[k-1] \arrow[r] & 0,
\end{tikzcd}$$

where $M = (E_{i+1}[k-2] \oplus E_i[k]) \times_{E_{i+1}[k-1]} E_{i+1}[k-2]$ is the fibered product of $E_{i+1}[k-2] \oplus E_i[k]$ and $E_{i+1}[k-2]$ over $E_{i+1}[k-1]$, via the homomorphisms $\begin{bmatrix} u_{i+1\,k-1} & p_{ik} \end{bmatrix}$ and $u_{i+1\,k-1}$. Recall that

$$M = \Big\{(x, y, z) \in E_{i+1}[k-2] \oplus E_i[k] \oplus E_{i+1}[k-2] \\ \Big|\, u_{i+1\,k-1}(x) + p_{ik}(y) = u_{i+1\,k-1}(z)\Big\},$$

and $\alpha((x, y, z)) = (x, y)$ and $\beta((x, y, z)) = z$ for any $(x, y, z) \in M$. Consider the homomorphism $\varrho\colon E_{i+1}[k-2] \to M$ in $\operatorname{mod} A$ given by $\varrho(z) = (z, 0, z)$. Then $\beta\varrho = \mathrm{id}_{E_{i+1}[k-2]}$, and so β is a retraction. Applying Lemma III.3.1, we conclude that $[\mathbb{E}_{i,k}u_{i+1\,k-1}] = [\mathcal{O}_{E_{i+1}[k-2],E_i[k-1]}]$. Then we obtain

$$\begin{aligned}
\mathcal{E}\mathrm{xt}^1_A(h, E_i[k-1])([\mathbb{E}_{i,k}]) &= \mathcal{E}\mathrm{xt}^1_A(u_{i+1\,k-1}g, E_i[k-1])([\mathbb{E}_{i,k}]) \\
&= [\mathbb{E}_{i,k}(u_{i+1\,k-1}g)] = [(\mathbb{E}_{i,k}u_{i+1\,k-1})g] \\
&= [\mathcal{O}_{E_{i+1}[k-2],E_i[k-1]}g] \\
&= [\mathcal{O}_{E_{i+1}[k-1],E_i[k-1]}].
\end{aligned}$$

Finally, it follows from Theorem III.3.5 that there is a commutative diagram of K-vector spaces

$$\begin{tikzcd}
\mathcal{E}\mathrm{xt}^1_A(E_{i+1}[k-1], E_i[k-1]) \arrow[r, "\chi_{E_{i+1}[k-1],E_i[k-1]}"] \arrow[d, "\mathcal{E}\mathrm{xt}^1_A(h{,}E_i[k-1])"] & \mathrm{Ext}^1_A(E_{i+1}[k-1], E_i[k-1]) \arrow[d, "\mathrm{Ext}^1_A(h{,}E_i[k-1])"] \\
\mathcal{E}\mathrm{xt}^1_A(E_{i+1}[k-1], E_i[k-1]) \arrow[r, "\chi_{E_{i+1}[k-1],E_i[k-1]}"] & \mathrm{Ext}^1_A(E_{i+1}[k-1], E_i[k-1]),
\end{tikzcd}$$

where the horizontal homomorphisms are isomorphisms, and

$$\chi_{E_{i+1}[k-1],E_i[k-1]}([\mathcal{O}_{E_{i+1}[k-1],E_i[k-1]}]) = 0 \quad \text{in} \quad \operatorname{Ext}^1_A(E_{i+1}[k-1], E_i[k-1]).$$

Therefore,

$$\begin{aligned}\operatorname{Ext}^1_A(h, E_i[k-1])\big(\chi_{E_{i+1}[k-1],E_i[k-1]}([\mathbb{E}_{i,k}])\big) & \\ &= \chi_{E_{i+1}[k-1],E_i[k-1]}\big(\mathcal{E}\mathrm{xt}^1_A(h, E_i[k-1])([\mathbb{E}_{i,k}])\big) \\ &= \chi_{E_{i+1}[k-1],E_i[k-1]}\big([\mathcal{O}_{E_{i+1}[k-1],E_i[k-1]}]\big) \\ &= 0,\end{aligned}$$

as required. This shows that $\mathbb{E}_{i,k}$ is an almost split sequence in $\operatorname{mod} A$, for any $i \in \{1, \dots, r\}$, which finishes the proof of (i) and (ii).

(iii) Let M be an indecomposable object of the category $\mathcal{E}$. We will prove that M is isomorphic to a module $E_i[j]$, for some $i \in \{1, \dots, r\}$ and $j \in \mathbb{N}^+$. It follows from the definition of $\mathcal{E}$ that M contains a subobject of the form $E_i = E_i[1]$ for some $i \in \{1, \dots, r\}$. Let j be the maximal positive integer such that there is a monomorphism $f\colon E_i[j] \to M$ for some $i \in \{1, \dots, r\}$. We claim that f is an isomorphism. Assume that this is not the case. Consider the almost split sequence

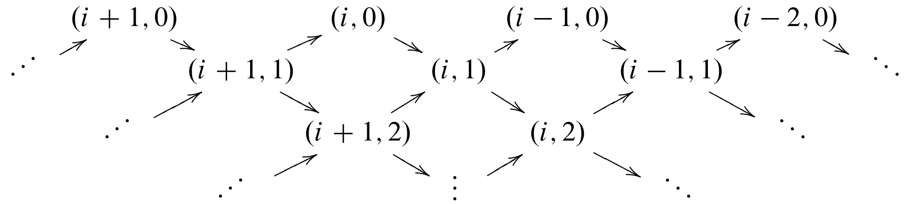
$$0 \longrightarrow E_i[j] \xrightarrow{\begin{bmatrix} p_{ij} \\ u_{i\,j+1} \end{bmatrix}} E_{i+1}[j-1] \oplus E_i[j+1] \xrightarrow{\begin{bmatrix} u_{i+1\,j} & p_{i\,j+1} \end{bmatrix}} E_{i+1}[j] \longrightarrow 0.$$

Since f is not a section, it follows from the property of almost split sequences that there exists a homomorphism $[f', f'']\colon E_{i+1}[j-1] \oplus E_i[j+1] \to M$ such that $f = f'p_{ij} + f''u_{i\,j+1}$. Observe that $\operatorname{Im} u'_{ij} \cong E_i[1]$ is the unique simple subobject of $E_i[j]$ in $\mathcal{E}$. Then $fu'_{ij} \neq 0$, because f is a monomorphism. Hence $p_{ij}u'_{ij} = 0$ yields

$$f''u'_{i\,j+1} = f'p_{ij}u'_{ij} + f''u_{i\,j+1}u'_{ij} = fu'_{ij} \neq 0.$$

Observe now that $\operatorname{Im} u'_{i\,j+1} \cong E_i[1]$ is the unique simple subobject of the uniserial object $E_i[j+1]$ in $\mathcal{E}$. This implies that $f''\colon E_i[j+1] \to M$ is a monomorphism, which contradicts the choice of j. Therefore, $f\colon E_i[j] \to M$ is an isomorphism. Clearly, then M is a uniserial object of $\mathcal{E}$. □

Recall that $\mathbb{Z}\mathbb{A}_\infty$ is the translation quiver of the form (see Section III.9)



with the translation τ defined by $\tau(i,j) = (i+1,j)$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$. Then, for an integer $r \geq 1$, $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is the orbit translation quiver obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex x of $\mathbb{Z}\mathbb{A}_\infty$ with $\tau^r x$ and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with $\tau^r x \to \tau^r y$, called a *stable tube of rank* r. The set of vertices of a stable tube $\mathcal{T} = \mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ having exactly one immediate predecessor (equivalently, exactly one immediate successor) is called *mouth* of $\mathcal{T}$.

Theorem 3.7. *Let A be a finite dimensional K-algebra over a field K and $E_1, \dots, E_r$ a hereditary family of pairwise orthogonal bricks in* mod A *such that $\tau_A E_{i+1} = E_i$ for all $i \in \{1, \dots, r\}$, and $E_{r+1} = E_1$. Then the abelian category*

$$\mathcal{E} = \mathcal{EXT}_A(E_1, \dots, E_r)$$

has the following properties:

(i) *Every indecomposable object of $\mathcal{E}$ is uniserial.*

(ii) *The simple objects $E_1, \dots, E_r$ of $\mathcal{E}$ form the mouth of a stable tube $\mathcal{T}_\mathcal{E}$ of Γ_A of rank r.*

(iii) $\mathcal{E} = \operatorname{add} \mathcal{T}_\mathcal{E}$.

Our next aim is to prove that the stable tube $\mathcal{T}_\mathcal{E}$ given by the extension category $\mathcal{E} = \mathcal{EXT}_A(E_1, \dots, E_r)$ of a hereditary family $E_1, \dots, E_r$ of pairwise orthogonal bricks of a module category mod A is generalized standard in the following sense (see [S2]).

Let A be a finite dimensional K-algebra over a field K. A connected component $\mathcal{C}$ of Γ_A is said to be *generalized standard* if $\operatorname{rad}_A^\infty(X,Y) = 0$ for all indecomposable modules X and Y in $\mathcal{C}$ (see Section III.1).

Proposition 3.8. *Let A be a finite dimensional K-algebra over a field K, M and N indecomposable modules in* mod A, *and let $f \in \operatorname{rad}_A^m(M,N)$ with $m \geq 2$. Then the following statements hold:*

(i) *There exist a positive integer s, indecomposable modules $X_1, \dots, X_s$ in* mod A, *homomorphisms $f_i \in \operatorname{rad}_A(M, X_i)$, and homomorphisms $g_i \colon X_i \to N$ with each g_i a finite sum of compositions of $m-1$ irreducible homomorphisms between indecomposable modules in* mod A, *such that $f = \sum_{i=1}^s g_i f_i$.*

(ii) *There exist a positive integer t, indecomposable modules $Y_1, \dots, Y_t$ in* mod A, *homomorphisms $u_j \in \operatorname{rad}_A(Y_j, N)$, and homomorphisms $v_j \colon M \to Y_j$ with each v_j a finite sum of compositions of $m-1$ irreducible homomorphisms between indecomposable modules in* mod A, *such that $f = \sum_{j=1}^t u_j v_j$.*

Proof. We prove the statement (i) by induction on m.

Assume $m = 2$. Let $g: X \to N$ be a minimal right almost split homomorphism in $\operatorname{mod} A$, and $X = X_1 \oplus \cdots \oplus X_s$ a decomposition of X into a direct sum of indecomposable modules in $\operatorname{mod} A$. Let $g_i: X_i \to N$ be the restrictions of g to X_i, for $i \in \{1, \dots, s\}$. Then $g_1, \dots, g_s$ are irreducible homomorphisms in $\operatorname{mod} A$, by Theorem III.7.12. Since $f \in \operatorname{rad}_A^2(M, N)$, we have $f = uv$ for some module L in $\operatorname{mod} A$ and homomorphisms $v \in \operatorname{rad}_A(M, L)$ and $u \in \operatorname{rad}_A(L, N)$. We conclude from Lemma III.1.5 that u is not a retraction. Then there exists a homomorphism $h: L \to X$ such that $u = gh$. For each $i \in \{1, \dots, s\}$, denote by $f_i: M \to X_i$ the composition $\pi_i h v$, where $\pi_i: X \to X_i$ is the canonical projection. Observe that $v \in \operatorname{rad}_A(M, L)$ implies $f_i \in \operatorname{rad}_A(M, X_i)$ for any $i \in \{1, \dots, s\}$. Therefore, we obtain the required presentation $f = uv = ghv = \sum_{i=1}^{s} g_i f_i$.

Assume now $m \geq 3$. Then there exist a module L in $\operatorname{mod} A$ and homomorphisms $v \in \operatorname{rad}_A(M, L)$ and $u \in \operatorname{rad}_A^{m-1}(L, N)$ such that $f = uv$. Let $L = L_1 \oplus \cdots \oplus L_t$ be a decomposition of L into a direct sum of indecomposable modules, and let $v_i: M \to L_i$ and $u_i: L_i \to N$ be the homomorphisms induced by v and u, respectively. Since v_i is the composition of v with the projection of L onto L_i and u_i is the composition of u with the inclusion homomorphism of L_i into L, we conclude that $v_i \in \operatorname{rad}_A(M, L_i)$ and $u_i \in \operatorname{rad}_A^{m-1}(L_i, N)$ for any $i \in \{1, \dots, t\}$. Moreover, $f = \sum_{i=1}^{t} u_i v_i$. We apply now the induction assumption for $m - 1 \geq 2$ and the homomorphisms $u_1, \dots, u_t$. For each $i \in \{1, \dots, t\}$, there exist a positive integer s_i, indecomposable modules Z_{ij} with $j \in \{1, \dots, s_i\}$, homomorphisms $h_{ij} \in \operatorname{rad}_A(L_i, Z_{ij})$, and homomorphisms $w_{ij}: Z_{ij} \to N$, where each w_{ij} is a finite sum of compositions of $m - 2$ irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$, such that $u_i = \sum_{j=1}^{s_i} w_{ij} h_{ij}$. Further, for each $i \in \{1, \dots, t\}$ and $j \in \{1, \dots, s_i\}$, we have $h_{ij} v_i \in \operatorname{rad}_A^2(M, Z_{ij})$ and, by induction, there exist a positive integer q_{ij}, indecomposable modules $R_{ij1}, \dots, R_{ijq_{ij}}$ in $\operatorname{mod} A$, homomorphisms $\varphi_{ijp} \in \operatorname{rad}_A(M, R_{ijp})$, and irreducible homomorphisms $\psi_{ijp}: R_{ijp} \to Z_{ij}$, such that $h_{ij} v_i = \sum_{p=1}^{q_{ij}} \psi_{ijp} \varphi_{ijp}$. Then we obtain the equalities

$$
\begin{aligned}
f &= \sum_{i=1}^{t} u_i v_i = \sum_{i=1}^{t} \left(\sum_{j=1}^{s_i} w_{ij} h_{ij} \right) v_i = \sum_{i=1}^{t} \left(\sum_{j=1}^{s_i} w_{ij} (h_{ij} v_i) \right) \\
&= \sum_{i=1}^{t} \sum_{j=1}^{s_i} w_{ij} \left(\sum_{p=1}^{q_{ij}} \psi_{ijp} \varphi_{ijp} \right) = \sum_{i=1}^{t} \sum_{j=1}^{s_i} \sum_{p=1}^{q_{ij}} (w_{ij} \psi_{ijp}) \varphi_{ijp},
\end{aligned}
$$

where each homomorphism φ_{ijp} is in $\operatorname{rad}_A(M, R_{ijp})$ and each homomorphism $w_{ij} \psi_{ijp}$ is a finite sum of compositions of $m - 1$ irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$. Therefore, f has the required presentation.

The proof of (ii) is similar. □

Proposition 3.9. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ a generalized standard component of Γ_A. Then any nonzero nonisomorphism $f : M \to N$ in* mod A *with M and N indecomposable modules in $\mathcal{C}$ is a finite sum of compositions of irreducible homomorphisms between indecomposable modules in $\mathcal{C}$.*

Proof. Let M and N be indecomposable modules in $\mathcal{C}$ and $f : M \to N$ a nonzero nonisomorphism in mod A. We have $\operatorname{rad}_A^\infty(M, N) = 0$, because $\mathcal{C}$ is generalized standard. In fact, there exists a minimal positive natural number n such that $\operatorname{rad}_A^n(M, N) = \operatorname{rad}_A^\infty(M, N) = 0$ (see Lemma III.1.6). Since $0 \neq f \in \operatorname{rad}_A(M, N)$, there exists a positive integer m such that $f \in \operatorname{rad}_A^m(M, N) \setminus \operatorname{rad}_A^{m+1}(M, N)$. We proceed by induction on $n - m$, starting from $n - m = 1$. Applying Proposition 3.8, we conclude that there exist a positive integer s, indecomposable modules $X_1, \dots, X_s$, homomorphisms $f_i \in \operatorname{rad}_A(M, X_i)$, and homomorphisms $g_i : X_i \to N$ with each g_i a finite sum of compositions of $m-1$ irreducible homomorphisms between indecomposable modules in $\mathcal{C}$, such that $f = \sum_{i=1}^s g_i f_i$. Since $f \notin \operatorname{rad}_A^{m+1}(M, N)$, there exists at least one $i \in \{1, \dots, s\}$ such that $f_i \notin \operatorname{rad}_A^2(M, X_i)$, or equivalently, f_i is irreducible, by Lemma III.7.8. Let f' be the sum of all compositions $g_i f_i$ with f_i irreducible and $f'' = f - f'$. Observe that $f'' \in \operatorname{rad}_A^{m+1}(M, N)$, and $n - (m + 1) < n - m$. If $f'' = 0$, then $f = f'$ has the required presentation. Observe that it is the case for $n - m = 1$. If $f'' \neq 0$, then, by induction, f'' is a finite sum of compositions of irreducible homomorphisms between indecomposable modules in $\mathcal{C}$. Hence, $f = f' + f''$ has the same property. □

Proposition 3.10. *Let A be a finite dimensional K-algebra over a field K and $E_1, \dots, E_r$ a hereditary family of pairwise orthogonal bricks in* mod A *such that $\tau_A E_{i+1} = E_i$, for $i \in \{1, \dots, r\}$, and $E_{r+1} = E_1$. Then, in the notation of Proposition 3.6, the following statements hold:*

(i) *For any irreducible homomorphism $v : E_i[j-1] \to E_i[j]$ in* mod A, *there exist automorphisms f of $E_i[j]$ and g of $E_i[j-1]$ such that $f u_{ij} = v = u_{ij} g$.*

(ii) *For any irreducible homomorphism $w : E_i[j] \to E_{i+1}[j-1]$ in* mod A, *there exist automorphisms f of $E_i[j]$ and h of $E_{i+1}[j-1]$ such that $h p_{ij} = w = p_{ij} f$.*

Proof. (i) Let $v : E_i[j-1] \to E_i[j]$ be an irreducible homomorphism in mod A. Consider the exact sequence

$$0 \longrightarrow E_i[j-1] \xrightarrow{u_{ij}} E_i[j] \xrightarrow{p'_{ij}} E_{i+1}[1] \longrightarrow 0.$$

Since $v \in \operatorname{rad}_A(E_i[j-1], E_i[j])$, we infer that the image of v is contained in the unique maximal subobject $\operatorname{Im} u_{ij}$ of $E_i[j]$ in the extension category $\mathcal{E} =$

$\mathcal{EXT}_A(E_1, \dots, E_r)$, and hence $p'_{ij} v = 0$. Then there exists $g \in \operatorname{End}_A(E_i[j-1])$ such that $v = u_{ij} g$. It follows from Lemma III.7.8 that $v \in \operatorname{rad}_A(E_i[j-1], E_i[j]) \setminus \operatorname{rad}^2_A(E_i[j-1], E_i[j])$. This implies that $g \notin \operatorname{rad}_A(E_i[j-1], E_i[j-1]) = \operatorname{rad}\operatorname{End}_A(E_i[j-1])$, because $u_{ij} \in \operatorname{rad}_A(E_i[j-1], E_i[j])$. Therefore, g is an automorphism of $E_i[j-1]$, since $\operatorname{End}_A(E_i[j-1])$ is a local K-algebra.

Consider the almost split sequence

$$0 \to E_i[j-1] \xrightarrow{\begin{bmatrix} p_{i\,j-1} \\ u_{ij} \end{bmatrix}} E_{i+1}[j-2] \oplus E_i[j] \xrightarrow{\begin{bmatrix} u_{i+1\,j-1} & p_{ij} \end{bmatrix}} E_{i+1}[j-1] \to 0.$$

Since $v\colon E_i[j-1] \to E_i[j]$ is not a section in $\operatorname{mod} A$, there exists a homomorphism $[\varphi, \psi]\colon E_{i+1}[j-2] \oplus E_i[j] \to E_i[j]$ such that

$$v = \begin{bmatrix} \varphi, \psi \end{bmatrix} \begin{bmatrix} p_{i\,j-1} \\ u_{ij} \end{bmatrix} = \varphi p_{i\,j-1} + \psi u_{ij}.$$

We have $\varphi \in \operatorname{rad}_A(E_{i+1}[j-2], E_i[j])$, because $E_{i+1}[j-2]$ and $E_i[j]$ are nonisomorphic indecomposable modules (see Lemma III.1.4). This shows that $\varphi p_{i\,j-1} \in \operatorname{rad}^2_A(E_i[j-1], E_i[j])$. Since $v \in \operatorname{rad}_A(E_i[j-1], E_i[j]) \setminus \operatorname{rad}^2_A(E_i[j-1], E_i[j])$, we conclude that ψ is an automorphism of $E_i[j]$. If $\varphi = 0$, we set $f = \psi$. Assume $\varphi \neq 0$. We claim that $\varphi = \varphi' u_{i+1\,j-1}$ for a homomorphism $\varphi'\colon E_{i+1}[j-1] \to E_i[j]$. Observe first that

$$u_{i+t+1\,j-t-1} p_{i+t\,j-t-1} + p_{i+t\,j-t} u_{i+t\,j-t} = 0, \text{ for } t \in \{1, \dots, j-3\}.$$

Moreover, every path in the stable tube $\mathcal{T}_{\mathcal{E}}$ from $E_{i+1}[j-2]$ to $E_i[j]$ passes through an indecomposable module of the form $E_{i+s}[j-s]$ for $s \in \{1, \dots, j-2\}$. Using now the left almost split homomorphisms

$$\begin{bmatrix} p_{i+t\,j-t-1} \\ u_{i+t\,j-t} \end{bmatrix} : E_{i+t}[j-t-1] \longrightarrow E_{i+t+1}[j-t-2] \oplus E_{i+t}[j-t],$$

for $t \in \{1, \dots, j-3\}$, we conclude that $\varphi = \varphi' u_{i+1\,j-1}$ for a homomorphism $\varphi'\colon E_{i+1}[j-1] \to E_i[j]$. Therefore, we obtain

$$v = \varphi p_{i\,j-1} + \psi u_{ij} = \varphi' u_{i+1\,j-1} p_{i\,j-1} + \psi u_{ij} = -\varphi' p_{ij} u_{ij} + \psi u_{ij} = f u_{ij}$$

for $f = -\varphi' p_{ij} + \psi \in \operatorname{End}_A(E_i[j])$. Since $\operatorname{End}_A(E_i[j])$ is a local K-algebra, $-\varphi' p_{ij} \in \operatorname{rad}\operatorname{End}_A(E_i[j]) = \operatorname{rad}_A(E_i[j], E_i[j])$, and ψ is an automorphism of $E_i[j]$, we conclude that f is an automorphism of $E_i[j]$.

The proof of (ii) is similar and left to the reader. □

Theorem 3.11. *Let A be a finite dimensional K-algebra over a field K and $E_1, \dots, E_r$ a hereditary family of pairwise orthogonal bricks in* mod A *such that $\tau_A E_{i+1} = E_i$ for $i \in \{1, \dots, r\}$, and $E_{r+1} = E_1$. Then the stable tube $\mathcal{T}_{\mathcal{E}}$ given by the extension category $\mathcal{E} = \mathcal{EXT}_A(E_1, \dots, E_r)$ is generalized standard.*

Proof. Let $E_i[j]$ and $E_k[l]$, for $i, k \in \{1, \dots, r\}$ and $j, l \in \mathbb{N}^+$, be indecomposable modules in the stable tube $\mathcal{T}_{\mathcal{E}}$. We will prove that $\operatorname{rad}_A^m(E_i[j], E_k[l]) = 0$ for $m \geq j + l - 1$, and consequently $\operatorname{rad}_A^{\infty}(E_i[j], E_k[l]) = 0$. Observe that this is the case for $j + l = 2$. Indeed, then $E_i[j] = E_i$, $E_k[l] = E_k$, and $\operatorname{rad}_A(E_i, E_k) = 0$, because $E_1, \dots, E_r$ are pairwise orthogonal bricks. Hence we may assume that $j+l \geq 3$. Let $m \geq j+l-1 \geq 2$. Take a nonzero homomorphism $f \in \operatorname{rad}_A^m(E_i[j], E_k[l])$. It follows from Proposition 3.8 that there exist a positive integer s, indecomposable modules $X_1, \dots, X_s$ in mod A, homomorphisms $f_t \in \operatorname{rad}_A(E_i[j], X_t)$, and homomorphisms $g_t \colon X_t \to E_k[l]$ with each g_t a finite sum of compositions of $m - 1$ irreducible homomorphisms between indecomposable modules in $\mathcal{T}_{\mathcal{E}}$, such that $f = \sum_{t=1}^{s} g_t f_t$. Applying Propositions 3.6 and 3.10, we infer that $g_t = h_t \varphi_t$, where φ_t is an automorphism of X_t and h_t is a composition of $m-1$ irreducible homomorphisms of the form $u_{ab} \colon E_a[b-1] \to E_a[b]$ or $p_{ab} \colon E_a[b] \to E_{a+1}[b-1]$. Moreover, $f = \sum_{t=1}^{s} h_t(\varphi_t f_t)$. We claim that $h_t = 0$ for any $t \in \{1, \dots, s\}$. Recall that we have the relations, given by the almost split sequences established in Proposition 3.6(ii),

$$u_{a+1,b-1} p_{a,b-1} + p_{ab} u_{ab} = 0, \text{ for } a \in \{1, \dots, r\}, b \geq 2,$$
$$p_{a2} u_{a2} = 0, \text{ for } a \in \{1, \dots, r\}.$$

Since $m \geq j + l - 1$, invoking these relations we conclude that each homomorphism h_t may be presented in the form $h_t = \varepsilon h'_t p_{a2} u_{a2} h''_t$, for $\varepsilon \in \{-1, 1\}$, $a \in \{1, \dots, r\}$, and h'_t, h''_t compositions of irreducible homomorphisms of the form u_{cd} or p_{cd}, and consequently $h_t = 0$. Therefore, we get $f = 0$. □

We end this section with three useful results.

Lemma 3.12. *Let A be a finite dimensional K-algebra over a field K, and*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

a nonsplittable sequence in mod A. *Then*

$$\dim_K \operatorname{End}_A(M) < \dim_K \operatorname{End}_A(L \oplus N).$$

Proof. It follows from Lemma II.2.5 that, for any module X in mod A, the functors $\operatorname{Hom}_A(X, -)$ and $\operatorname{Hom}_A(-, X)$ from mod A to mod K are left exact. Therefore, we obtain the following commutative diagram in mod K with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_A(N, L) \arrow[r] \arrow[d] & \operatorname{Hom}_A(N, M) \arrow[r] \arrow[d] & \operatorname{Hom}_A(N, N) \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_A(M, L) \arrow[r] \arrow[d, "\operatorname{Hom}_A(f,L)"] & \operatorname{Hom}_A(M, M) \arrow[r] \arrow[d] & \operatorname{Hom}_A(M, N) \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_A(L, L) \arrow[r] & \operatorname{Hom}_A(L, M) \arrow[r] & \operatorname{Hom}_A(L, N) .
\end{tikzcd}$$

Note that $\operatorname{Hom}_A(f, L)\colon \operatorname{Hom}_A(M, L) \to \operatorname{Hom}_A(L, L)$ is not an epimorphism. Indeed, if $\operatorname{Hom}_A(f, L)$ is an epimorphism, then there is a $h \in \operatorname{Hom}_A(M, L)$ such that $\mathrm{id}_L = \operatorname{Hom}_A(f, L)(h) = hf$, which shows that f is a section, a contradiction (see Lemma III.3.1). Therefore, we obtain the inequalities

$$\begin{aligned}\dim_K \operatorname{Hom}_A(M, L) &= \dim_K \operatorname{Hom}_A(N, L) + \dim_K \operatorname{Im} \operatorname{Hom}_A(f, L)\\ &< \dim_K \operatorname{Hom}_A(N, L) + \dim_K \operatorname{Hom}_A(L, L).\end{aligned}$$

This gives

$$\begin{aligned}\dim_K \operatorname{End}_A(M) &\le \dim_K \operatorname{Hom}_A(M, L) + \dim_K \operatorname{Hom}_A(M, N)\\ &< \dim_K \operatorname{Hom}_A(N, L) + \dim_K \operatorname{Hom}_A(L, L)\\ &\quad + \dim_K \operatorname{Hom}_A(N, N) + \dim_K \operatorname{Hom}_A(L, N)\\ &= \dim_K \operatorname{End}_A(L \oplus N).\end{aligned}$$ □

Lemma 3.13. *Let A be a finite dimensional K-algebra over a field K and $\mathbf{x}$ a vector in $K_0(A)$. Then there exist semisimple modules M and N in* mod A *satisfying the following conditions:*

(i) $\mathbf{x} = [M] - [N]$.

(ii) $\mathbf{x} = [M]$ *if* $\mathbf{x}$ *is a positive vector.*

(iii) $\operatorname{Hom}_A(M, N) = 0$ *and* $\operatorname{Hom}_A(N, M) = 0$.

Proof. Let $S_1, \dots, S_n$ be a complete set of pairwise nonisomorphic simple modules in mod A. Since $[S_1], \dots, [S_n]$ form a $\mathbb{Z}$-basis of $K_0(A)$, by Theorem I.11.1, we have an expression $\mathbf{x} = \sum_{i=1}^{n} x_i [S_i]$ with $x_1, \dots, x_n \in \mathbb{Z}$. Let M be the direct sum of the semisimple modules $S_i^{x_i}$ for all $i \in \{1, \dots, n\}$ with $x_i \ge 0$, and N the direct sum of the semisimple modules $S_j^{|x_j|}$ for all $j \in \{1, \dots, n\}$ with $x_j \le 0$. Then the required conditions (i), (ii), (iii) are satisfied. □

Lemma 3.14. *Let A be a finite dimensional K-algebra over a field K, and $\mathbf{x}$ a positive vector in $K_0(A)$. Then there exist indecomposable modules $X_1, \ldots, X_r$ in* mod A *such that the following statements hold:*

(i) $\mathbf{x} = [X_1] + \cdots + [X_r] = [X_1 \oplus \cdots \oplus X_r]$.
(ii) $\operatorname{Ext}_A^1(X_i, X_j) = 0$ *for any* $i \neq j$ *in* $\{1, \ldots, r\}$.

Proof. By Lemma 3.13, $\mathbf{x} = [M]$ for a nonzero module M in mod A. We may choose a module X in mod A with $\mathbf{x} = [X]$ and $\dim_k \operatorname{End}_A(X) \leq \dim_K \operatorname{End}_A(Y)$ for any module Y in mod A with $[Y] = \mathbf{x}$. Let $X = X_1 \oplus \cdots \oplus X_r$ be a decomposition of X into direct sum of indecomposable modules in mod A. We claim that $\operatorname{Ext}_A^1(X_i, X_j) = 0$ for all $i \neq j$ in $\{1, \ldots, r\}$. Suppose that $\operatorname{Ext}_A^1(X_i, X_j) \neq 0$ for some $i \neq j$ in $\{1, \ldots, r\}$. Then $\operatorname{Ext}_A^1(X_i, \hat{X}_i) \neq 0$, where $\hat{X}_i$ is the direct sum of all modules $X_1, \ldots, X_r$ except X_i. It follows from Theorem III.3.5 and Corollary III.3.6 that $\mathcal{E}\mathrm{xt}_A^1(X_i, \hat{X}_i) \neq 0$, and hence there exists in mod A a nonsplittable exact sequence

$$0 \longrightarrow \hat{X}_i \longrightarrow M \longrightarrow X_i \longrightarrow 0.$$

Applying Lemma 3.12, we obtain

$$\dim_K \operatorname{End}_A(M) < \dim_K \operatorname{End}_A(\hat{X}_i \oplus X_i) = \dim_K \operatorname{End}_A(X).$$

Moreover, $[M] = [\hat{X}_i] + [X_i] = [X] = \mathbf{x}$. This contradicts the choice of X. □

4 The Euler forms

In this section we introduce a homological nonsymmetric bilinear form on the Grothendieck group $K_0(A)$ of a finite dimensional hereditary K-algebra A over a field K and the associated quadratic from on $K_0(A)$, both called the Euler forms of A. Moreover, we determine the quivers of indecomposable finite dimensional hereditary K-algebras over a field for which the Euler quadratic forms are positive definite and positive semidefinite with nonzero radical.

Theorem 4.1. *Let A be a finite dimensional hereditary K-algebra over a field K. Then there exists a unique $\mathbb{Z}$-bilinear form*

$$\langle -, - \rangle_A \colon K_0(A) \times K_0(A) \longrightarrow \mathbb{Z}$$

such that

$$\langle [M], [N] \rangle_A = \dim_K \operatorname{Hom}_A(M, N) - \dim_K \operatorname{Ext}_A^1(M, N)$$

for any modules M and N in mod A.

Proof. Let M and N be modules in $\operatorname{mod} A$. Then for any exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\operatorname{mod} A$ we have, by Theorems 3.2 and 3.3, the exact sequences in $\operatorname{mod} K$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(M,X) \arrow[r] & \operatorname{Hom}_A(M,Y) \arrow[r] & \operatorname{Hom}_A(M,Z) \arrow[dll, rounded corners] & \\
 & \operatorname{Ext}_A^1(M,X) \arrow[r] & \operatorname{Ext}_A^1(M,Y) \arrow[r] & \operatorname{Ext}_A^1(M,Z) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(Z,N) \arrow[r] & \operatorname{Hom}_A(Y,N) \arrow[r] & \operatorname{Hom}_A(X,N) \arrow[dll, rounded corners] & \\
 & \operatorname{Ext}_A^1(Z,N) \arrow[r] & \operatorname{Ext}_A^1(Y,N) \arrow[r] & \operatorname{Ext}_A^1(X,N) \arrow[r] & 0\,.
\end{tikzcd}$$

This leads to the equalities

$$\begin{aligned}
&\dim_K \operatorname{Hom}_A(M,Y) - \dim_K \operatorname{Ext}_A^1(M,Y)\\
&\quad = \dim_K \operatorname{Hom}_A(M,X) - \dim_K \operatorname{Ext}_A^1(M,X)\\
&\qquad + \dim_K \operatorname{Hom}_A(M,Z) - \dim_K \operatorname{Ext}_A^1(M,Z)
\end{aligned}$$

and

$$\begin{aligned}
&\dim_K \operatorname{Hom}_A(Y,N) - \dim_K \operatorname{Ext}_A^1(Y,N)\\
&\quad = \dim_K \operatorname{Hom}_A(X,N) - \dim_K \operatorname{Ext}_A^1(X,N)\\
&\qquad + \dim_K \operatorname{Hom}_A(Z,N) - \dim_K \operatorname{Ext}_A^1(Z,N).
\end{aligned}$$

Therefore, the assignment to the classes $[M], [N] \in K_0(A)$ of modules M, N in $\operatorname{mod} A$ of the integer

$$\langle [M],[N]\rangle_A = \dim_K \operatorname{Hom}_A(M,N) - \dim_K \operatorname{Ext}_A^1(M,N)$$

is well defined. Moreover, it follows from Lemma 3.13 that any vector $\mathbf{x} \in K_0(A)$ is of the form $\mathbf{x} = [U] - [V]$ for some modules U and V in $\operatorname{mod} A$. Hence there exists a $\mathbb{Z}$-bilinear form

$$\langle -,-\rangle_A \colon K_0(A) \times K_0(A) \longrightarrow \mathbb{Z}$$

given for $\mathbf{x}$ and $\mathbf{y}$ in $K_0(A)$ by

$$\langle \mathbf{x},\mathbf{y}\rangle_A = \langle [U],[R]\rangle_A - \langle [U],[W]\rangle_A - \langle [V],[R]\rangle_A + \langle [V],[W]\rangle_A$$

for any presentations $\mathbf{x} = [U] - [V]$ and $\mathbf{y} = [R] - [W]$ with U, V, W, R modules in $\operatorname{mod} A$. □

The $\mathbb{Z}$-bilinear form $\langle -,- \rangle_A \colon K_0(A) \times K_0(A) \to \mathbb{Z}$ of a finite dimensional hereditary K-algebra over a field K is called the *Euler bilinear form of* A. The associated quadratic form $\chi_A \colon K_0(A) \to \mathbb{Z}$ given by $\chi_A(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle_A$ for any $\mathbf{x} \in K_0(A)$ is called the *Euler quadratic form*, or *Euler characteristic*, of A. These two Euler forms of a hereditary algebra have been proposed by C. M. Ringel in [R1]. We note also that the $\mathbb{Z}$-bilinear form $\langle -,- \rangle_A$ is nonsymmetric.

Let A be a finite dimensional hereditary K-algebra over a field K and $S_1, \dots, S_n$ a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$. For each $i \in \{1, \dots, n\}$ and a module M in $\operatorname{mod} A$, we denote by $c_i(M) = c_{S_i}(M)$ the composition multiplicity of S_i in $\operatorname{mod} A$, that is, the number of simple composition factors M_j/M_{j-1} of M, $j \in \{1, \dots, m\}$, isomorphic to S_i in a composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of M (see Theorem I.7.5). It follows from Theorem I.11.1 that $[S_1], \dots, [S_n]$ form a $\mathbb{Z}$-basis of $K_0(A)$ and there is an isomorphism of abelian groups $c \colon K_0(A) \to \mathbb{Z}^n$ such that $c([M]) = (c_1(M), \dots, c_n(M))$ for any module M in $\operatorname{mod} A$. Moreover, $[M] = \sum_{i=1}^{n} c_i(M)[S_i]$ in $K_0(A)$. This allows us to identify canonically $K_0(A)$ with $\mathbb{Z}^n$. Then we obtain an alternative description of the Euler forms of A.

Theorem 4.2. *Let A be a finite dimensional hereditary K-algebra over a field K and $S_1, \dots, S_n$ a complete set of pairwise nonisomorphic simple modules in* $\operatorname{mod} A$ *and $K_0(A) = \mathbb{Z}^n$. Moreover, let $f_i = \dim_K \operatorname{End}_A(S_i)$ and $f_{ij} = \dim_K \operatorname{Ext}^1_A(S_i, S_j)$ for $i, j \in \{1, \dots, n\}$. Then the following statements hold:*

(i) *The bilinear Euler form $\langle -,- \rangle_A \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ is defined by*

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \sum_{i=1}^{n} f_i x_i y_i - \sum_{i,j=1}^{n} f_{ij} x_i y_j$$

for $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{Z}^n$.

(ii) *The quadratic Euler form $\chi_A \colon \mathbb{Z}^n \to \mathbb{Z}$ is defined by*

$$\chi_A(\mathbf{x}) = \sum_{i=1}^{n} f_i x_i^2 - \sum_{i,j=1}^{n} f_{ij} x_i x_j$$

for $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathbb{Z}^n$.

Proof. Let M and N be modules in $\operatorname{mod} A$. Then we have

$$[M] = \sum_{i=1}^{n} c_i(M)[S_i] \quad \text{and} \quad [N] = \sum_{j=1}^{n} c_j(N)[S_j].$$

Hence,

$$\begin{aligned}
\langle [M],[N]\rangle_A &= \left\langle \sum_{i=1}^{n} c_i(M)[S_i], \sum_{j=1}^{n} c_j(N)[S_j] \right\rangle_A \\
&= \sum_{i,j=1}^{n} c_i(M)c_j(N)\big\langle [S_i],[S_j]\big\rangle_A \\
&= \sum_{i,j=1}^{n} c_i(M)c_j(N)\left(\dim_K \operatorname{Hom}_A(S_i,S_j) - \dim_K \operatorname{Ext}_A^1(S_i,S_j)\right) \\
&= \sum_{i=1}^{n} \dim_K \operatorname{End}_A(S_i)c_i(M)c_j(N) \\
&\quad - \sum_{i,j=1}^{n} \dim_K \operatorname{Ext}_A^1(S_i,S_j)c_i(M)c_j(N) \\
&= \sum_{i=1}^{n} f_i c_i(M)c_i(N) - \sum_{i,j=1}^{n} f_{ij} c_i(M)c_j(N).
\end{aligned}$$

Therefore, under the identification $K_0(A) = \mathbb{Z}^n$ by means of the isomorphism $c\colon K_0(A) \to \mathbb{Z}^n$ described above, the Euler forms $\langle -,-\rangle_A$ and χ_A have the presentations stated in (i) and (ii), respectively. □

Let A be a finite dimensional hereditary K-algebra over a field K, $e_1,\dots,e_n$ a complete set of basic primitive idempotents of A, and $S_1 = e_1A/e_1 \operatorname{rad} A,\dots,$ $S_n = e_nA/e_n \operatorname{rad} A$ the associated complete set of pairwise nonisomorphic simple modules in mod A. Recall that we identify $F_i = \operatorname{End}_A(S_i)$ with $e_iAe_i/e_i(\operatorname{rad} A)e_i$ for any $i \in \{1,\dots,n\}$. Further, it follows from Theorem 1.9 that

$$\begin{aligned}
d_{ij} &= \dim_{F_j} e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2e_j = \dim_{F_j} \operatorname{Ext}_A^1(S_i,S_j), \\
d'_{ij} &= \dim_{F_i} e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2e_j = \dim_{F_i} \operatorname{Ext}_A^1(S_i,S_j),
\end{aligned}$$

for any $i,j \in \{1,\dots,n\}$. Therefore (see Lemma 1.1),

$$d_{ij}f_j = f_{ij} = f_id'_{ij} \quad \text{for } i,j \in \{1,\dots,n\}.$$

This shows that the Euler forms $\langle -,-\rangle_A$ and χ_A are uniquely determined by the quiver Q_A of A and the positive integers $f_i = \dim_K F_i$, $i \in \{1,\dots,n\}$. In fact, since the quiver Q_A of A is acyclic, by Corollary 1.8, we conclude that $f_{ii} = 0$ for all $i \in \{1,\dots,n\}$, and $f_{ij} \neq 0$ implies $f_{ji} = 0$. In particular, we may order the vertices $1,\dots,n$ of Q_A in such a way that $f_{ij} \neq 0$ implies $i > j$, for all

$i, j \in \{1, \dots, n\}$. Then the Euler forms $\langle -, - \rangle_A \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ and $\chi_A \colon \mathbb{Z}^n \to \mathbb{Z}$ are given by

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \sum_{i=1}^{n} f_i x_i y_i - \sum_{1 \le j < i \le n} f_{ij} x_i y_j,$$

$$\chi_A(\mathbf{x}) = \sum_{i=1}^{n} f_i x_i^2 - \sum_{1 \le j < i \le n} f_{ij} x_i x_j,$$

for all $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{Z}^n$.

Let K be a field, Q a finite acyclic quiver and $A = KQ$ the path algebra of Q over K. Let Q_0 be the set of vertices of Q, Q_1 be the set of arrows of Q, and $s, t \colon Q_1 \to Q_0$ associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and target $t(\alpha) \in Q_0$. It follows from Theorem I.9.6 and Corollary II.6.18 that $A = KQ$ is a finite dimensional basic hereditary K-algebra, and the trivial paths ε_i, $i \in Q_0$, of Q form a complete set of basic primitive idempotents of KQ, $F_i = K$, and so $f_i = 1$ for any $i \in \{1, \dots, n\}$. Moreover, for any $i, j \in Q_0$, we have, by Theorem 1.9, that $d_{ij} = d'_{ij}$ and is the number of arrows from i to j in Q. Observe also that $K_0(A) = \mathbb{Z}^{Q_0}$. Then we obtain the following consequence of Theorem 4.2.

Corollary 4.3. *Let K be a field, Q a finite acyclic quiver, and $A = KQ$ the path algebra of Q over K. Then the following statements hold:*

(i) *The bilinear Euler form $\langle -, - \rangle_A \colon \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is given by*

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}$$

for all $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ in $\mathbb{Z}^{Q_0}$.

(ii) *The quadratic Euler form $\chi_A \colon \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is given by*

$$\chi_A(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$$

for all $\mathbf{x} = (x_i)$ in $\mathbb{Z}^{Q_0}$.

Our next aim is to describe the quivers Q_A of indecomposable finite dimensional hereditary K-algebras A over a field K for which the Euler quadratic form $\chi_A \colon K_0(A) \to \mathbb{Z}$ is positive semidefinite.

Let A be an indecomposable finite dimensional hereditary K-algebra over a field K, and $S_1, \dots, S_n$ a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$. We identify $K_0(A)$ with $\mathbb{Z}^n$ and the $\mathbb{Z}$-linear basis $[S_1], \dots, [S_n]$ of $K_0(A)$ with the canonical basis $e_1, \dots, e_n$ of $\mathbb{Z}^n$. Recall that then the Euler quadratic form $\chi_A\colon K_0(A) \to \mathbb{Z}$ may be viewed as the quadratic form $\chi_A\colon \mathbb{Z}^n \to \mathbb{Z}$ given for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$ by

$$\chi_A(\mathbf{x}) = \sum_{i=1}^{n} f_i x_i^2 - \sum_{i,j=1}^{n} f_{ij} x_i x_j,$$

where $f_i = \dim_K \operatorname{End}_A(S_i)$ and $f_{ij} = \dim_K \operatorname{Ext}_A^1(S_i, S_j)$ for $i, j \in \{1, \dots, n\}$. Moreover, $f_{ij} \neq 0$ if and only if the quiver Q_A of A contains a valued arrow $i \xrightarrow{(d_{ij}, d'_{ij})} j$, and then $f_{ji} = 0$. Further, if $f_{ij} \neq 0$, then $d_{ij} f_j = f_{ij} = f_i d'_{ij}$. By a *proper restriction* of χ_A we mean a quadratic form $q\colon \mathbb{Z}^m \to \mathbb{Z}$, with $m \leq n$, of the form

$$q(\mathbf{x}) = \sum_{i=1}^{m} f_i^* x_i^2 - \sum_{i,j=1}^{m} f_{ij}^* x_i x_j$$

for $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ and for which there is an injection $\sigma\colon \{1, \dots, m\} \to \{1, \dots, n\}$ such that the following conditions are satisfied:

- $f_i^* = f_{\sigma(i)}$ for any $i \in \{1, \dots, m\}$;
- f_{ij}^* is an integer with $0 \leq f_{ij}^* \leq f_{\sigma(i)\sigma(j)}$ for any $i, j \in \{1, \dots, m\}$;
- if $m = n$, then $f_{ij}^* < f_{\sigma(i)\sigma(j)}$ for some $i, j \in \{1, \dots, m\}$.

An integral quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is called *positive definite* if $q(\mathbf{x}) > 0$ for all nonzero vectors $\mathbf{x}$ in $\mathbb{Z}^n$, *positive semidefinite* if $q(\mathbf{x}) \geq 0$ for all vectors $\mathbf{x}$ in $\mathbb{Z}^n$, and *indefinite* if $q(\mathbf{x}) < 0$ for some vector $\mathbf{x} \in \mathbb{Z}^n$.

Proposition 4.4. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K with positive semidefinite Euler form $\chi_A\colon \mathbb{Z}^n \to \mathbb{Z}$, and let $q\colon \mathbb{Z}^m \to \mathbb{Z}$ be a proper restriction of χ_A. Then q is positive definite.*

Proof. Let $\sigma\colon \{1, \dots, m\} \to \{1, \dots, n\}$ be an injection such that quadratic form q is given by

$$q(\mathbf{x}) = \sum_{i=1}^{m} f_i^* x_i^2 - \sum_{i,j=1}^{m} f_{ij}^* x_i x_j$$

for $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$, with $f_i^* = f_{\sigma(i)}$ and $0 \leq f_{ij}^* \leq f_{\sigma(i)\sigma(j)}$ for $i, j \in \{1, \dots, m\}$. For any $\mathbf{x} \in \mathbb{Z}^m$, let $\mathbf{x}^\sigma = (x_j^\sigma)$ be the element of $\mathbb{Z}^n$ such that $x_j^\sigma = x_i$ if $j = \sigma(i)$ with $i \in \{1, \dots, m\}$, and $x_j^\sigma = 0$ otherwise. Then we have

$$\begin{aligned} q(\mathbf{x}) &= \sum_{i=1}^{m} f_i^* x_i^2 - \sum_{i,j=1}^{m} f_{ij}^* x_i x_j \geq \sum_{i=1}^{m} f_{\sigma(i)} \big(x_{\sigma(i)}^\sigma\big)^2 - \sum_{i,j=1}^{m} f_{\sigma(i)\sigma(j)} x_{\sigma(i)}^\sigma x_{\sigma(j)}^\sigma \\ &= \chi_A(\mathbf{x}^\sigma) \geq 0 \end{aligned}$$

for any $\mathbf{x} \in \mathbb{Z}^m$. Hence, q is positive semidefinite. Suppose that q is not positive definite. Then there exists a nonzero vector $\mathbf{y}$ in $\mathbb{Z}^m$ such that $q(\mathbf{y}) = 0$. We may assume that $\mathbf{y} \in \mathbb{N}^m$. Indeed, consider the vector $|\mathbf{y}| \in \mathbb{N}^m$ such that $|\mathbf{y}|_i = |y_i|$ for any $i \in \{1, \dots, m\}$. Then

$$\begin{aligned} 0 = q(\mathbf{y}) = \sum_{i=1}^{m} f_i^* y_i^2 - \sum_{i,j=1}^{m} f_{ij}^* y_i y_j \geq \sum_{i=1}^{m} f_i^* |y_i|^2 - \sum_{i,j=1}^{m} f_{ij}^* |y_i| |y_j| \\ = q(|\mathbf{y}|) \geq 0, \end{aligned}$$

and so $q(|\mathbf{y}|) = 0$. We may also assume that $y_i > 0$ for any $i \in \{1, \dots, m\}$, by further restricting (if necessary) q to $\mathbb{Z}^r$, where r is the number of nonzero coordinates of $\mathbf{y}$. We have two cases to consider.

(1) Assume $m = n$. Since q is a proper restriction of χ_A, we have $f_{ij}^* < f_{\sigma(i)\sigma(j)}$ for some $i, j \in \{1, \dots, m\}$. Then we have

$$\begin{aligned} \chi_A(\mathbf{y}^\sigma) &= \sum_{k=1}^{n} f_k (y_k^\sigma)^2 - \sum_{k,l=1}^{n} f_{kl} y_k^\sigma y_l^\sigma \\ &= \sum_{i=1}^{n} f_{\sigma(i)} \big(y_{\sigma(i)}^\sigma\big)^2 - \sum_{i,j=1}^{n} f_{\sigma(i)\sigma(j)} y_{\sigma(i)}^\sigma y_{\sigma(j)}^\sigma \\ &< \sum_{i=1}^{n} f_i^* y_i^2 - \sum_{i,j=1}^{n} f_{ij}^* y_i y_j = q(\mathbf{y}) = 0. \end{aligned}$$

This leads to contradiction, because $\mathbf{y}^\sigma \neq 0$ and χ_A is positive semidefinite.

(2) Assume $m < n$. Since A is an indecomposable algebra, the quiver Q_A is connected, by Corollary 1.7. Then there exist vertices r and s in Q_A such that $r = \sigma(j)$ for some $j \in \{1, \dots, m\}$, $s \notin \operatorname{Im}\sigma$ and $f_{rs} \neq 0$ (equivalently, there is an arrow from r to s in Q_A). Consider the vector $\mathbf{z} \in \mathbb{N}^n$ defined as follows

$$z_l = \begin{cases} 2y_i, & \text{for } l = \sigma(i) \text{ with } i \in \{1, \dots, m\}, \\ y_j, & \text{for } l = s, \\ 0, & \text{otherwise}. \end{cases}$$

Then we obtain the inequalities

$$\begin{aligned} \chi_A(\mathbf{z}) &= \sum_{l=1}^{n} f_l z_l^2 - \sum_{k,l=1}^{n} f_{kl} z_k z_l \\ &= \sum_{i=1}^{m} f_{\sigma(i)} (2y_i)^2 - \sum_{i,j=1}^{m} f_{\sigma(i)\sigma(j)} (2y_i)(2y_j) + f_s y_j^2 - f_{\sigma(j)s} (2y_j) y_j \\ &\leq 4 \left(\sum_{i=1}^{m} f_i^* y_i^2 - \sum_{i,j=1}^{m} f_{ij}^* y_i y_j \right) + \big(f_s - 2 f_{\sigma(j)s}\big) y_j^2 \\ &= 4q(\mathbf{y}) + \big(f_s - 2 d_{\sigma(j)s} f_s\big) y_j^2 = f_s \big(1 - 2 d_{\sigma(j)s}\big) y_j^2 < 0, \end{aligned}$$

because $2d_{\sigma(j)s} \geq 2$. This contradicts the positive semidefiniteness of χ_A. □

Let Q be a finite valued quiver. For a proper subset Σ of vertices of Q, we denote by $Q^{(\Sigma)}$ the full valued subquiver of Q obtained by removing in Q all vertices from Σ and the arrows attached to them.

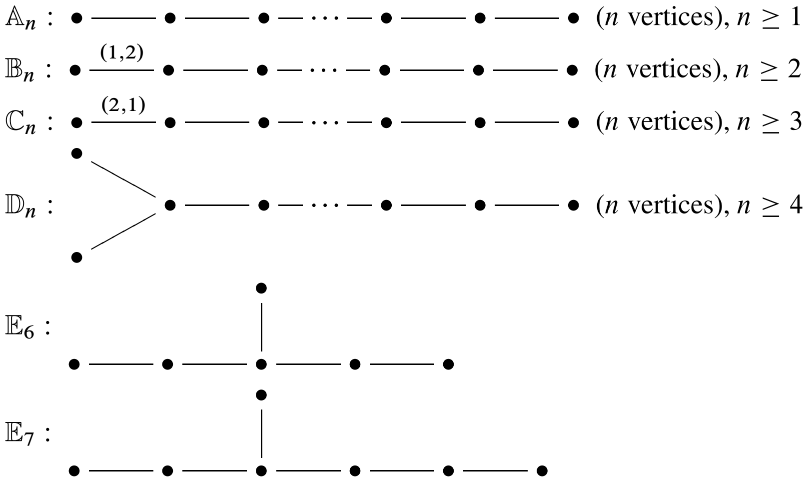
Lemma 4.5. *Let A be a finite dimensional hereditary K-algebra over a field K and Σ be a proper subset of the set of vertices of the quiver Q_A of A. Then there exists a finite dimensional hereditary K-algebra H such that $Q_H = Q_A^{(\Sigma)}$ and the Euler form χ_H of H is a proper restriction of the Euler form χ_A of A.*

Proof. Let $e_1, \dots, e_n$ be a complete set of basic pairwise orthogonal primitive idempotents of A. Moreover, let $S_1 = e_1 A / e_1 \operatorname{rad} A, \dots, S_n = e_n A / e_n \operatorname{rad} A$ be the associated complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$. We identify $F_i = \operatorname{End}_A(S_i)$ with $e_i A e_i / e_i (\operatorname{rad} A) e_i$ for any $i \in \{1, \dots, n\}$. Then for any pair i, j of vertices in Q_A we have the (F_i, F_j)-bimodule ${}_iM_j = e_i (\operatorname{rad} A) e_i / e_i (\operatorname{rad} A)^2 e_i$. We may assume (without loss of generality) that $\{1, \dots, m\}$, for some $m < n$, is the set of vertices of $Q_A^{(\Sigma)}$. Then we may consider the K-species $\mathbb{M}_\Sigma$ given by the vertices of the valued subquiver $Q_A^{(\Sigma)}$ of Q_A, that is,

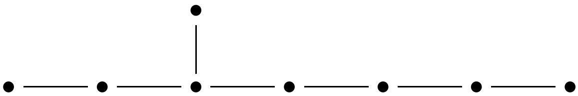
$$\mathbb{M}_\Sigma = (F_i, {}_iM_j)_{1 \le i,j \le m}.$$

Observe that $Q_{\mathbb{M}_\Sigma} = Q_A^{(\Sigma)}$ and is an acyclic quiver, because the quiver Q_A of A is acyclic, by Corollary 1.8. Let H be the tensor algebra $T(\mathbb{M}_\Sigma)$ of the K-species $\mathbb{M}_\Sigma$. Then it follows from Theorem 2.2 that $H = T(\mathbb{M}_\Sigma)$ is a finite dimensional hereditary K-algebra over K with $Q_H = Q_{T(\mathbb{M}_\Sigma)} = Q_{\mathbb{M}_\Sigma} = Q_A^{(\Sigma)}$. Clearly, then the Euler form χ_H of H is a proper restriction of the Euler form χ_A of A via the canonical injection $\{1, \dots, m\} \to \{1, \dots, n\}$. □

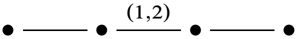
A valued quiver Δ is called a *Dynkin quiver* if its underlying graph $\bar{\Delta}$ is one of the Dynkin graphs



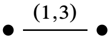
$\mathbb{E}_8$:



$\mathbb{F}_4$:



$\mathbb{G}_2$:



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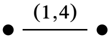
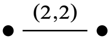
A valued quiver Δ is called a *Euclidean quiver* if its underlying graph $\bar{\Delta}$ is one of the Euclidean graphs

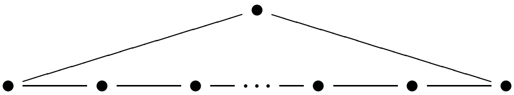
$\widetilde{\mathbb{A}}_{11}$:



$\widetilde{\mathbb{A}}_{12}$:



$\widetilde{\mathbb{A}}_n$:



($n + 1$ vertices), $n \geq 2$

$\widetilde{\mathbb{B}}_n$:

($n + 1$ vertices), $n \geq 2$

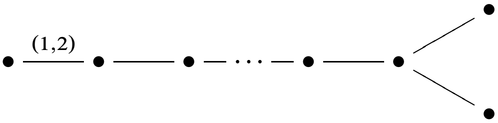
$\widetilde{\mathbb{C}}_n$:

($n + 1$ vertices), $n \geq 2$

$\widetilde{\mathbb{BC}}_n$:

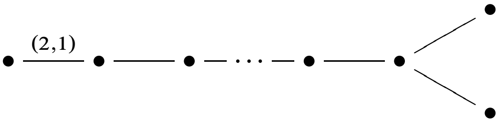
($n + 1$ vertices), $n \geq 2$

$\widetilde{\mathbb{BD}}_n$:



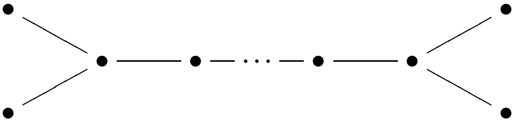
($n + 1$ vertices), $n \geq 3$

$\widetilde{\mathbb{CD}}_n$:



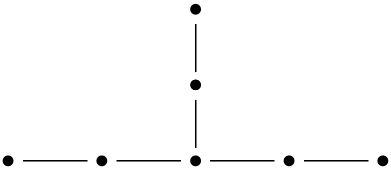
($n + 1$ vertices), $n \geq 3$

$\widetilde{\mathbb{D}}_n$:

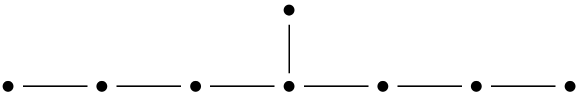


($n + 1$ vertices), $n \geq 4$

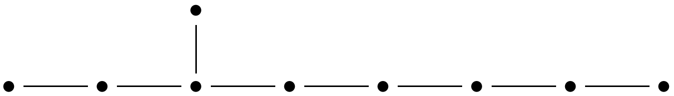
$\widetilde{\mathbb{E}}_6$:



$\widetilde{\mathbb{E}}_7$:



$\widetilde{\mathbb{E}}_8$:

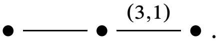


$\widetilde{\mathbb{F}}_{41}$:

$\widetilde{\mathbb{F}}_{42}$:

$\widetilde{\mathbb{G}}_{21}$:

$\widetilde{\mathbb{G}}_{22}$:



A finite connected valued quiver Δ, without loops and multiple arrows, is said to be a *wild quiver* if Δ is neither a Dynkin quiver, nor a Euclidean quiver.

Let A be an indecomposable finite dimensional hereditary K-algebra over a field K and the Euler quadratic form $\chi_A \colon K_0(A) \to \mathbb{Z}$ is positive semidefinite. Then

$$\operatorname{rad} \chi_A = \big\{\mathbf{x} \in K_0(A) \mid \chi_A(\mathbf{x}) = 0\big\}$$

is said to be the *radical* of χ_A. We note that $\operatorname{rad} \chi_A$ is a subgroup of $K_0(A)$. Indeed, for $\mathbf{x}, \mathbf{y} \in \operatorname{rad} \chi_A$, we have

$$\begin{aligned}\chi_A(\mathbf{x}+\mathbf{y}) + \chi_A(\mathbf{x}-\mathbf{y}) &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle_A + \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle_A \\ &= 2\langle \mathbf{x}, \mathbf{x}\rangle_A + 2\langle \mathbf{y}, \mathbf{y}\rangle_A = 2\big(\chi_A(\mathbf{x}) + \chi_A(\mathbf{y})\big) \\ &= 0,\end{aligned}$$

and hence $\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y} \in \operatorname{rad} \chi_A$, because $\chi_A(\mathbf{x}+\mathbf{y}) \geq 0$ and $\chi_A(\mathbf{x}-\mathbf{y}) \geq 0$. Moreover, since $K_0(A) = \mathbb{Z}^n$, the group $\operatorname{rad} \chi_A$ is isomorphic to $\mathbb{Z}^r$, for some $r \in \{0, 1, \dots, n-1\}$, and r is called the *corank* of χ_A. Observe that χ_A is positive definite if and only if $\operatorname{rad} \chi_A = 0$, or equivalently, χ_A is of corank 0.

Proposition 4.6. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K such that Q_A is a Euclidean quiver. The following statements hold:*

(i) *χ_A is positive semidefinite of corank* 1.

(ii) $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$ *for a vector* $\mathbf{h}_A$ *in* $K_0(A)$ *with all coordinates positive and at least one equal to* 1.

Proof. Let $S_1, \dots, S_n$ be a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$ and $F_1 = \operatorname{End}_A(S_1), \dots, F_n = \operatorname{End}_A(S_n)$. Then, under the identification of $K_0(A)$ with $\mathbb{Z}^n$ and the canonical $\mathbb{Z}$-bases $[S_1], \dots, [S_n]$ of $K_0(A)$ and $e_1, \dots, e_n$ of $\mathbb{Z}^n$, we have $\chi_A \colon \mathbb{Z}^n \to \mathbb{Z}$ defined by

$$\chi_A(\mathbf{x}) = \sum_{i=1}^{n} f_i x_i^2 - \sum_{i,j=1}^{n} f_{ij} x_i x_j ,$$

with $f_i = \dim_K F_i$ and $f_{ij} = \dim_K \operatorname{Ext}_A^1(S_i, S_j)$ for $i, j \in \{1, \dots, n\}$, and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$. Moreover, the quiver Q_A of A admits a valued arrow $i \xrightarrow{(d_{ij}, d'_{ij})} j$ if and only if $f_{ij} \neq 0$, and then $d_{ij} f_j = f_{ij} = f_i d'_{ij}$. This shows that the Euler form χ_A does not depend on the orientation of Q, that is, χ_A is uniquely determined by the numbers $f_1, \dots, f_n$ and the valuations of the edges $i \xrightarrow{(d_{ij}, d'_{ij})} j$ of the underlying valued graph $\bar{Q}_A$ of Q_A. Therefore, it is enough to show the statements (i) and (ii) for one chosen orientation of the Euclidean graph $\bar{Q}_A$ of Q_A. We analyze all Euclidean graphs case by case.

(1) Assume Q_A is the quiver

$$\underset{1}{\bullet} \xrightarrow{(1,4)} \underset{2}{\bullet}$$

of type $\widetilde{\mathbb{A}}_{11}$. Then $d_{12} = 1$, $d'_{12} = 4$, $f_2 = d_{12} f_2 = f_1 d'_{12} = 4 f_1$. Hence, for $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$, we have

$$\begin{aligned} \chi_A(\mathbf{x}) &= f_1 x_1^2 + f_2 x_2^2 - f_2 x_1 x_2 = f_1 x_1^2 + 4 f_1 x_2^2 - 4 f_1 x_1 x_2 \\ &= f_1 \big(x_1^2 - 4 x_1 x_2 + 4 x_2^2\big) = f_1 (x_1 - 2 x_2)^2. \end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (2, 1)$.

(2) Assume Q_A is the quiver

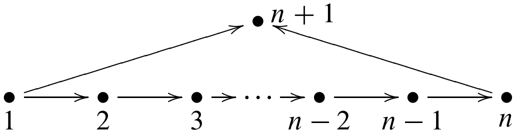
$$\underset{1}{\bullet} \xrightarrow{(2,2)} \underset{2}{\bullet}$$

of type $\widetilde{\mathbb{A}}_{12}$. Then $d_{12} = 2$, $d'_{12} = 2$, $2 f_2 = d_{12} f_2 = f_1 d'_{12} = 2 f_1$, and so $f_1 = f_2$. Hence, for $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$, we have

$$\chi_A(\mathbf{x}) = f_1 x_1^2 + f_2 x_2^2 - 2 f_2 x_1 x_2 = f_1 \big(x_1^2 - 2 x_1 x_2 + x_2^2\big) = f_1 (x_1 - x_2)^2.$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (1, 1)$.

(3) Assume Q_A is the quiver



of type $\widetilde{\mathbb{A}}_n$, with $n \geq 2$. Then we have $d_{i\,i+1} = 1 = d'_{i\,i+1}$ for $i \in \{1, \dots, n\}$, $d_{1\,n+1} = 1 = d'_{1\,n+1}$, and $f_j = f_1$ for all $j \in \{1, \dots, n+1\}$. Hence, for

$\mathbf{x} = (x_i) \in \mathbb{Z}^{n+1}$,

$$\begin{aligned}\chi_A(\mathbf{x}) &= \sum_{i=1}^{n+1} f_i x_i^2 - \sum_{i=1}^{n} f_{i+1} x_i x_{i+1} - f_{n+1} x_1 x_{n+1} \\ &= f_1 \left(\sum_{i=1}^{n+1} x_i^2 - \sum_{i=1}^{n} x_i x_{i+1} - x_1 x_{n+1} \right) \\ &= \frac{1}{2} f_1 \left(\sum_{i=1}^{n} (x_i - x_{i+1})^2 + (x_1 - x_{n+1})^2 \right).\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = \overset{1}{111\cdots111}$.

(4) Assume Q_A is the quiver

$$\underset{1}{\bullet} \xrightarrow{(1,2)} \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \rightarrow \cdots \rightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet} \xrightarrow{(2,1)} \underset{n+1}{\bullet}$$

of type $\widetilde{\mathbb{B}}_n$, with $n \geq 2$. Then $d_{12} = 1$, $d'_{12} = 2$, $d_{n\,n+1} = 2$, $d'_{n\,n+1} = 1$, and $d_{i\,i+1} = 1 = d'_{i\,i+1}$ for $i \in \{2, \ldots, n-1\}$. This leads to the equalities $f_2 = d_{12} f_2 = f_1 d'_{12} = 2f_1$, $2f_{n+1} = d_{n\,n+1} f_{n+1} = f_n d'_{n\,n+1} = f_n$, and $f_{i+1} = d_{i\,i+1} f_{i+1} = f_i d'_{i\,i+1} = f_i$ for $i \in \{2, \ldots, n-1\}$. Hence $f_1 = f_{n+1}$ and $f_i = 2f_1$ for $i \in \{2, \ldots, n\}$. Thus, for $\mathbf{x} = (x_i) \in \mathbb{Z}^{n+1}$,

$$\begin{aligned}\chi_A(\mathbf{x}) &= \sum_{i=1}^{n+1} f_i x_i^2 - \sum_{i=1}^{n} d_{i\,i+1} f_{i+1} x_i x_{i+1} \\ &= f_1 \left(x_1^2 + \sum_{i=1}^{n} 2x_i^2 + x_{n+1}^2 - \sum_{i=1}^{n} 2x_i x_{i+1} \right) \\ &= f_1 \left(\sum_{i=1}^{n} (x_i - x_{i+1})^2 \right).\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (1, 1, \ldots, 1, 1)$.

(5) Assume Q_A is the quiver

$$\underset{1}{\bullet} \xrightarrow{(2,1)} \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \rightarrow \cdots \rightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet} \xrightarrow{(1,2)} \underset{n+1}{\bullet}$$

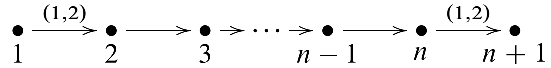
of type $\widetilde{\mathbb{C}}_n$, with $n \geq 2$. Then $d_{12} = 2$, $d'_{12} = 1$, $d_{n\,n+1} = 1$, $d'_{n\,n+1} = 2$, and $d_{i\,i+1} = 1 = d'_{i\,i+1}$ for $i \in \{2, \ldots, n-1\}$. This gives the equalities $2f_2 =$

$d_{12}f_2 = f_1 d'_{12} = f_1$, $f_{n+1} = d_{n\,n+1} f_{n+1} = f_n d'_{n\,n+1} = 2f_n$, and $f_{i+1} = d_{i\,i+1} f_{i+1} = f_i d'_{i\,i+1} = f_i$ for $i \in \{2, \dots, n-1\}$. Hence, for any $\mathbf{x} = (x_i) \in \mathbb{Z}^{n+1}$,

$$\begin{aligned}
\chi_A(\mathbf{x}) &= \sum_{i=1}^{n+1} f_i x_i^2 - \sum_{i=1}^{n} d_{i\,i+1} f_{i+1} x_i x_{i+1} \\
&= f_2\left(2x_1^2 + \sum_{i=2}^{n} x_i^2 + 2x_{n+1}^2 - 2x_1x_2 - \sum_{i=2}^{n-1} x_i x_{i+1} - 2x_n x_{n+1}\right) \\
&= f_2\left(2\left(x_1 - \frac{1}{2}x_2\right)^2 + \sum_{i=2}^{n-1} \frac{1}{2}(x_i - x_{i+1})^2 + 2\left(x_{n+1} - \frac{1}{2}x_n\right)^2\right).
\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (1, 2, 2, \dots, 2, 2, 1)$.

(6) Assume Q_A is the quiver

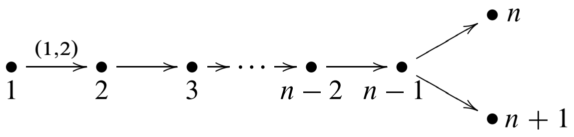


of type $\widetilde{\mathbb{BC}}_n$, with $n \geq 2$. Then $d_{12} = 1$, $d'_{12} = 2$, $d_{n\,n+1} = 1$, $d'_{n\,n+1} = 2$, and $d_{i\,i+1} = 1 = d'_{i\,i+1}$ for $i \in \{2, \dots, n-1\}$. Hence we obtain the equalities $f_2 = d_{12} f_2 = f_1 d'_{12} = 2f_1$, $f_{n+1} = d_{n\,n+1} f_{n+1} = f_n d'_{n\,n+1} = 2f_n$, and $f_{i+1} = d_{i\,i+1} f_{i+1} = f_i d'_{i\,i+1} = f_i$ for $i \in \{2, \dots, n-1\}$. Thus, for $\mathbf{x} = (x_i) \in \mathbb{Z}^{n+1}$,

$$\begin{aligned}
\chi_A(\mathbf{x}) &= \sum_{i=1}^{n+1} f_i x_i^2 - \sum_{i=1}^{n} d_{i\,i+1} f_{i+1} x_i x_{i+1} \\
&= f_1\left(x_1^2 + \sum_{i=2}^{n} 2x_i^2 + 4x_{n+1}^2 - \sum_{i=1}^{n-1} 2x_i x_{i+1} - 4x_n x_{n+1}\right) \\
&= f_1\left(\sum_{i=1}^{n-1} (x_i - x_{i+1})^2 + (x_n - 2x_{n+1})^2\right).
\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (2, 2, 2, \dots, 2, 2, 1)$.

(7) Assume Q_A is the quiver

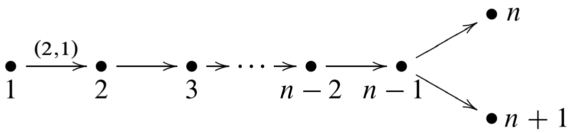


of type $\widetilde{\mathbb{BD}}_n$, with $n \geq 3$. Then $d_{12} = 1$, $d'_{12} = 2$, $d_{n-1\,n+1} = 1 = d'_{n-1\,n+1}$, and $d_{i\,i+1} = 1 = d'_{i\,i+1}$ for $i \in \{2, \dots, n-1\}$. This leads to the equalities $f_2 = d_{12} f_2 = f_1 d'_{12} = 2 f_1$, $f_{n+1} = d_{n-1\,n+1} f_{n+1} = f_{n-1} d'_{n-1\,n+1} = f_{n-1}$, and $f_{i+1} = d_{i\,i+1} f_{i+1} = f_i d'_{i\,i+1} = f_i$ for $i \in \{2, \dots, n-1\}$. Then, for $\mathbf{x} = (x_i) \in \mathbb{Z}^{n+1}$,

$$\begin{aligned}
\chi_A(\mathbf{x}) &= \sum_{i=1}^{n+1} f_i x_i^2 - \sum_{i=1}^{n-1} d_{i\,i+1} f_{i+1} x_i x_{i+1} - d_{n-1\,n+1} f_{n+1} x_{n-1} x_{n+1} \\
&= f_1 \left(x_1^2 + \sum_{i=2}^{n+1} 2 x_i^2 - \sum_{i=1}^{n-1} 2 x_i x_{i+1} - 2 x_{n-1} x_{n+1} \right) \\
&= f_1 \left(\sum_{i=1}^{n-2} (x_i - x_{i+1})^2 + \frac{1}{2}(x_{n-1} - 2x_n)^2 + \frac{1}{2}(x_{n-1} - 2x_{n+1})^2 \right).
\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = 222 \cdots 22{}^{1}_{1}$.

(8) Assume Q_A is the quiver

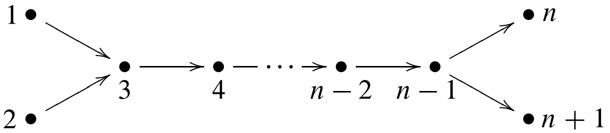


of type $\widetilde{\mathbb{CD}}_n$, with $n \geq 3$. Then $d_{12} = 2$, $d'_{12} = 1$, $d_{n-1\,n+1} = 1 = d'_{n-1\,n+1}$, and $d_{i\,i+1} = 1 = d'_{i\,i+1}$ for $i \in \{2, \dots, n-1\}$. This gives $2 f_2 = d_{12} f_2 = f_1 d'_{12} = f_1$, $f_{n+1} = d_{n-1\,n+1} f_{n+1} = f_{n-1} d'_{n-1\,n+1} = f_{n-1}$, and $f_{i+1} = d_{i\,i+1} f_{i+1} = f_i d'_{i\,i+1} = f_i$ for $i \in \{2, \dots, n-1\}$. Hence, for $\mathbf{x} = (x_i) \in \mathbb{Z}^{n+1}$,

$$\begin{aligned}
\chi_A(\mathbf{x}) &= \sum_{i=1}^{n+1} f_i x_i^2 - \sum_{i=1}^{n-1} d_{i\,i+1} f_{i+1} x_i x_{i+1} - d_{n-1\,n+1} f_{n+1} x_{n-1} x_{n+1} \\
&= f_2 \left(2 x_1^2 + \sum_{i=2}^{n+1} x_i^2 - 2 x_1 x_2 - \sum_{i=2}^{n-1} x_i x_{i+1} - x_{n-1} x_{n+1} \right) \\
&= f_2 \left(\frac{1}{2}(2x_1 - x_2)^2 + \sum_{i=2}^{n-2} \frac{1}{2}(x_i - x_{i+1})^2 + \left(\frac{1}{2} x_{n-1} - x_n \right)^2 \right. \\
&\qquad \left. + \left(\frac{1}{2} x_{n-1} - x_{n+1} \right)^2 \right).
\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = 122 \cdots 22{}^{1}_{1}$.

(9) Assume Q_A is the quiver

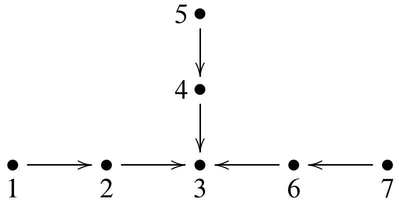


of type $\widetilde{\mathbb{D}}_n$, with $n \geq 4$. Then $f_i = f_1$ for all $i \in \{2, \dots, n+1\}$. Hence, for $\mathbf{x} = (x_i) \in \mathbb{Z}^{n+1}$,

$$\begin{aligned}
\chi_A(\mathbf{x}) &= \sum_{i=1}^{n+1} f_i x_i^2 - f_3 x_1 x_3 - \sum_{i=2}^{n-1} f_{i+1} x_i x_{i+1} - f_{n+1} x_{n-1} x_{n+1} \\
&= f_1 \left(\sum_{i=2}^{n+1} x_i^2 - x_1 x_3 - x_2 x_3 - \sum_{i=3}^{n-2} x_i x_{i+1} - x_{n-1} x_n - x_{n-1} x_{n+1} \right) \\
&= f_1 \left(\left(x_1 - \frac{1}{2} x_3 \right)^2 + \left(x_2 - \frac{1}{2} x_3 \right)^2 + \sum_{i=3}^{n-2} \frac{1}{2} (x_i - x_{i+1})^2 \right. \\
&\qquad \left. + \left(\frac{1}{2} x_{n-1} - x_n \right)^2 + \left(\frac{1}{2} x_{n-1} - x_{n+1} \right)^2 \right).
\end{aligned}$$

Thus χ_A is positive semidefinite of corank 1 and rad $\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = {}^1_1 22 \cdots 22 ^1_1$.

(10) Assume Q_A is the quiver

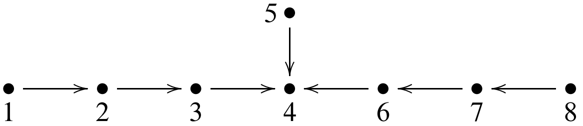


of type $\widetilde{\mathbb{E}}_6$. Then, for any $\mathbf{x} = (x_i) \in \mathbb{Z}^7$,

$$\begin{aligned}
\chi_A(\mathbf{x}) &= \sum_{i=1}^{7} f_i x_i^2 - f_2 x_1 x_2 - f_3 x_2 x_3 - f_3 x_4 x_3 - f_4 x_5 x_4 - f_3 x_6 x_3 - f_6 x_7 x_6 \\
&= f_3 \left(\left(x_1 - \frac{1}{2} x_2 \right)^2 + \left(x_5 - \frac{1}{2} x_4 \right)^2 + \left(x_7 - \frac{1}{2} x_6 \right)^2 \right) \\
&\quad + 3 f_3 \left(\left(\frac{1}{2} x_2 - \frac{1}{3} x_3 \right)^2 + \left(\frac{1}{2} x_4 - \frac{1}{3} x_3 \right)^2 + \left(\frac{1}{2} x_6 - \frac{1}{3} x_3 \right)^2 \right).
\end{aligned}$$

Hence, χ_A is positive semidefinite of corank 1 and rad $\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = \begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$.

(11) Assume Q_A is the quiver

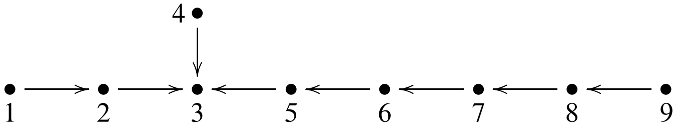


of type $\widetilde{\mathbb{E}}_7$. Then, for $\mathbf{x} = (x_i) \in \mathbb{Z}^8$,

$$\begin{aligned}\chi_A(\mathbf{x}) &= \sum_{i=1}^{8} f_i x_i^2 - f_2x_1x_2 - f_3x_2x_3 - f_4x_3x_4 - f_4x_5x_4 - f_4x_6x_4 \\ &\quad - f_6x_7x_6 - f_7x_8x_7 \\ &= f_4\left(\left(x_1 - \frac{1}{2}x_2\right)^2 + \left(x_8 - \frac{1}{2}x_7\right)^2 + \frac{3}{4}\left(x_2 - \frac{2}{3}x_3\right)^2\right) \\ &\quad + f_4\left(\frac{3}{4}\left(x_7 - \frac{2}{3}x_6\right)^2 + \left(x_5 - \frac{1}{2}x_4\right)^2 + \frac{2}{3}\left(x_3 - \frac{3}{4}x_4\right)^2\right) \\ &\quad + \frac{2}{3}f_4\left(x_6 - \frac{3}{4}x_4\right)^2.\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = 123\overset{2}{4}321$.

(12) Assume Q_A is the quiver

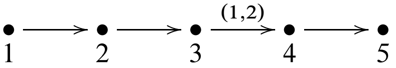


of type $\widetilde{\mathbb{E}}_8$. Then, for $\mathbf{x} = (x_i) \in \mathbb{Z}^9$,

$$\begin{aligned}\chi_A(\mathbf{x}) &= \sum_{i=1}^{9} f_i x_i^2 - f_2x_1x_2 - f_3x_2x_3 - f_3x_4x_3 - f_3x_5x_3 - \sum_{i=5}^{8} f_i x_{i+1}x_i \\ &= f_3\left(\left(x_9 - \frac{1}{2}x_8\right)^2 + \frac{3}{4}\left(x_8 - \frac{2}{3}x_7\right)^2 + \frac{2}{3}\left(x_7 - \frac{3}{4}x_6\right)^2\right) \\ &\quad + f_3\left(\frac{5}{8}\left(x_6 - \frac{4}{5}x_5\right)^2 + \left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{4}\left(x_2 - \frac{2}{3}x_3\right)^2\right) \\ &\quad + f_3\left(\left(x_4 - \frac{1}{2}x_3\right)^2 + \frac{3}{5}\left(x_5 - \frac{5}{6}x_3\right)^2\right).\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = {}_{24\overset{3}{6}54321}\cdot$

(13) Assume Q_A is the quiver

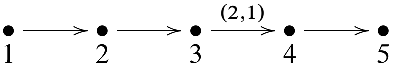


of type $\widetilde{\mathbb{F}}_{41}$. Then we have $d_{12} = 1 = d'_{12}$, $d_{23} = 1 = d'_{23}$, $d_{34} = 1$, $d'_{34} = 2$, and $d_{45} = 1 = d'_{45}$. Hence we conclude that $f_1 = f_2 = f_3$, $f_4 = f_5$, and $f_4 = d_{34}f_4 = f_3d'_{34} = 2f_3$. Then, for $\mathbf{x} = (x_i) \in \mathbb{Z}^5$,

$$\begin{aligned}
\chi_A(\mathbf{x}) &= \sum_{i=1}^{5} f_i x_i^2 - \sum_{i=1}^{4} d_{i\,i+1} f_{i+1} x_i x_{i+1} \\
&= f_1\big(x_1^2 + x_2^2 + x_3^2 + 2x_4^2 + 2x_5^2 - x_1x_2 - x_2x_3 - 2x_3x_4 - 2x_4x_5\big) \\
&= f_1\left(\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{4}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{2}{3}\left(x_3 - \frac{3}{2}x_4\right)^2\right) \\
&\quad + \frac{1}{2} f_1 (x_4 - 2x_5)^2.
\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (1, 2, 3, 2, 1)$.

(14) Assume Q_A is the quiver



of type $\widetilde{\mathbb{F}}_{42}$. Then $d_{12} = 1 = d'_{12}$, $d_{23} = 1 = d'_{23}$, $d_{34} = 2$, $d'_{34} = 1$, and $d_{45} = 1 = d'_{45}$. Hence we conclude that $f_1 = f_2 = f_3$, $f_4 = f_5$, and $2f_4 = d_{34}f_4 = f_3d'_{34} = f_3$. Thus, for $\mathbf{x} = (x_i) \in \mathbb{Z}^5$,

$$\begin{aligned}
\chi_A(\mathbf{x}) &= \sum_{i=1}^{5} f_i x_i^2 - \sum_{i=1}^{4} d_{i\,i+1} f_{i+1} x_i x_{i+1} \\
&= f_5\big(2x_1^2 + 2x_2^2 + 2x_3^2 + x_4^2 + x_5^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_4 - x_4x_5\big) \\
&= f_5\left(\frac{1}{2}(2x_1 - x_2)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}\left(x_3 - \frac{3}{4}x_4\right)^2\right) \\
&\quad + f_5\left(\frac{1}{2}x_4 - x_5\right)^2.
\end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (1,2,3,4,2)$.

(15) Assume Q_A is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r, "{(1,3)}"] & \underset{3}{\bullet} \end{tikzcd}$$

of type $\widetilde{\mathbb{G}}_{21}$. Then $d_{12} = 1 = d'_{12}$, $d_{23} = 1$, and $d'_{23} = 3$. Hence we get $f_1 = f_2$ and $f_3 = d_{23}f_3 = f_2 d'_{23} = 3f_2$. Consequently,

$$\begin{aligned} \chi_A(\mathbf{x}) &= f_1x_1^2 + f_2x_2^2 + f_3x_3^2 - f_2x_1x_2 - f_3x_2x_3 \\ &= f_1\big(x_1^2 + x_2^2 + 3x_3^2 - x_1x_2 - 3x_2x_3\big) \\ &= f_1\left(\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{4}(x_2 - 2x_3)^2\right), \end{aligned}$$

for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$. This shows that χ_A is positive semidefinite of corank 1 and $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (1,2,1)$.

(16) Assume Q_A is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r, "{(3,1)}"] & \underset{3}{\bullet} \end{tikzcd}$$

of type $\widetilde{\mathbb{G}}_{22}$. Then $d_{12} = 1 = d'_{12}$, $d_{23} = 3$, and $d'_{23} = 1$. This gives $f_1 = f_2$ and $3f_3 = d_{23}f_3 = f_2d'_{23} = f_2$. Hence, for $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$,

$$\begin{aligned} \chi_A(\mathbf{x}) &= f_1x_1^2 + f_2x_2^2 + f_3x_3^2 - f_2x_1x_2 - 3f_3x_2x_3 \\ &= f_3\big(3x_1^2 + 3x_2^2 + x_3^2 - 3x_1x_2 - 3x_2x_3\big) \\ &= f_3\left(3\left(x_1 - \frac{1}{2}x_2\right)^2 + \left(\frac{3}{2}x_2 - x_3\right)^2\right). \end{aligned}$$

Therefore, χ_A is positive semidefinite of corank 1 and $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (1,2,3)$. □

Proposition 4.7. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K such that Q_A is a wild quiver. Then χ_A is indefinite.*

Proof. Let $S_1, \dots, S_n$ be a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$ and $F_1 = \operatorname{End}_A(S_1), \dots, F_n = \operatorname{End}_A(S_n)$. Then, under the identification of $K_0(A)$ with $\mathbb{Z}^n$ and of the canonical $\mathbb{Z}$-bases $[S_1], \dots, [S_n]$ of $K_0(A)$ and $e_1, \dots, e_n$ of $\mathbb{Z}^n$, the Euler form $\chi_A\colon \mathbb{Z}^n \to \mathbb{Z}$ is defined by

$$\chi_A(\mathbf{x}) = \sum_{i=1}^{n} f_i x_i^2 - \sum_{i,j=1}^{n} f_{ij} x_i x_j,$$

with $f_i = \dim_K F_i$ and $f_{ij} = \dim_K \mathrm{Ext}^1_A(S_i, S_j)$ for $i, j \in \{1, \dots, n\}$, and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$. Moreover, we have $d_{ij} f_j = f_{ij} = f_i d'_{ij}$ for any valued arrow

$$i \xrightarrow{(d_{ij}, d'_{ij})} j$$

of Q_A. We consider first several cases when the quiver Q_A has a small number of points and every quiver $Q_A^{(\Sigma)}$, for a proper set Σ of vertices of Q_A, is a Dynkin quiver.

(1) Assume Q_A is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{(1,d'_{12})}"] & \underset{2}{\bullet} \end{tikzcd}$$

with $d'_{12} \geq 5$. Hence $f_2 = d_{12} f_2 = f_1 d'_{12} \geq 5 f_1$, because $d_{12} = 1$. Then, for $\mathbf{x} = (2, 1) \in \mathbb{Z}^2$, we have

$$\chi_A(\mathbf{x}) = 4f_1 + f_2 - 2d_{12} f_2 = 4f_1 + f_2 - 2f_2 = 4f_1 - f_2 < 0,$$

and hence χ_A is indefinite.

(2) Assume Q_A is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{(d_{12},d'_{12})}"] & \underset{2}{\bullet} \end{tikzcd}$$

with $d_{12} = d'_{12} \geq 3$. Then $d_{12} f_2 = f_1 d'_{12}$ implies $f_1 = f_2$. Taking $\mathbf{x} = (1, 1) \in \mathbb{Z}^2$, we obtain

$$\chi_A(\mathbf{x}) = f_1 + f_2 - d_{12} f_2 = f_2(2 - d_{12}) < 0,$$

because $d_{12} \geq 3$, and hence χ_A is indefinite.

(3) Assume Q_A is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{(d_{12},d'_{12})}"] & \underset{2}{\bullet} \end{tikzcd}$$

with $d'_{12} > d_{12} \geq 2$. Then $d_{12} f_2 = f_{12} = f_1 d'_{12}$ implies $f_2 > f_1$. Taking again $\mathbf{x} = (1, 1) \in \mathbb{Z}^2$, we obtain

$$\chi_A(\mathbf{x}) = f_1 + f_2 - d_{12} f_2 < f_1 - f_2 < 0,$$

and hence χ_A is indefinite.

(4) Assume Q_A is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{(1,2)}"] & \underset{2}{\bullet} \arrow[r, "{(1,3)}"] & \underset{3}{\bullet} \end{tikzcd}$$

Then we have $f_2 = d_{12}f_2 = f_1d'_{12} = 2f_1$ and $f_3 = d_{23}f_3 = f_2d'_{23} = 3f_2 = 6f_1$. Taking $\mathbf{x} = (1, 2, 1) \in \mathbb{Z}^3$, we obtain that

$$\chi_A(\mathbf{x}) = f_1 + 4f_2 + f_3 - 2f_2 - 6f_2 = -f_1 < 0,$$

and hence χ_A is indefinite.

(5) Assume Q_A is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{(1,2)}"] & \underset{2}{\bullet} \arrow[r, "{(3,1)}"] & \underset{3}{\bullet} \end{tikzcd}$$

Then we have $f_2 = d_{12}f_2 = f_1d'_{12} = 2f_1$ and $f_2 = f_2d'_{23} = d_{23}f_3 = 3f_3$, and so $f_3 < f_1$. Taking $\mathbf{x} = (1, 1, 1) \in \mathbb{Z}^3$, we obtain

$$\chi_A(\mathbf{x}) = f_1 + f_2 + f_3 - 2f_1 - f_2 = f_3 - f_1 < 0,$$

and hence χ_A is indefinite.

(6) Assume Q_A is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{(1,3)}"] & \underset{2}{\bullet} \arrow[r, "{(1,3)}"] & \underset{3}{\bullet} \end{tikzcd}$$

Then we have $f_2 = d_{12}f_2 = f_1d'_{12} = 3f_1$ and $f_3 = d_{23}f_3 = f_2d'_{23} = 3f_2 = 9f_1$. Hence, taking $\mathbf{x} = (2, 2, 1) \in \mathbb{Z}^3$, we obtain

$$\chi_A(\mathbf{x}) = 4f_1 + 4f_2 + f_3 - 4f_2 - 2f_3 = 4f_1 - f_3 < 0,$$

and so χ_A is indefinite.

(7) Assume Q_A is the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{(1,3)}"] & \underset{2}{\bullet} \arrow[r, "{(3,1)}"] & \underset{3}{\bullet} \end{tikzcd}$$

Then we have $f_2 = d_{12}f_2 = f_1d'_{12} = 3f_1$ and $f_2 = f_2d'_{23} = d_{23}f_3 = 3f_3$, and so $f_1 = f_3$. Taking $\mathbf{x} = (2, 1, 2) \in \mathbb{Z}^3$, we obtain

$$\chi_A(\mathbf{x}) = 4f_1 + f_2 + 4f_3 - 2f_2 - 2f_2 = 8f_1 - 3f_2 = -f_1 < 0,$$

and hence χ_A is indefinite.

(8) Assume Q_A is the quiver

$$\begin{tikzcd} & \overset{1}{\bullet} \arrow[dl, "{(d_{13},d'_{13})}"'] \arrow[dr, "{(d_{12},d'_{12})}"] & \\ 2\,\bullet \arrow[rr, "{(d_{23},d'_{23})}"] & & \bullet\,3 \end{tikzcd}$$

such that the quiver obtained from Q_A by removing the arrow from 1 to 3 is one of the quivers considered in (4), (5), (6), or (7). Then, for the vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$ chosen in (4), (5), (6), (7), respectively, we obtain the inequalities

$$\begin{aligned}\chi_A(\mathbf{x}) &= f_1x_1^2 + f_2x_2^2 + f_3x_3^2 - f_{12}x_1x_2 - f_{23}x_2x_3 - f_{13}x_1x_3\\ &< f_1x_1^2 + f_2x_2^2 + f_3x_3^2 - f_{12}x_1x_2 - f_{23}x_2x_3 < 0,\end{aligned}$$

and hence $\chi_A(\mathbf{x})$ is indefinite.

(9) Assume Q_A is the quiver

$$\begin{tikzcd} 1\bullet \arrow[r, "{(1,2)}"] \arrow[d] & \bullet 2 \arrow[d] \\ 4\bullet & \bullet 3 \arrow[l, "{(1,2)}"'] \end{tikzcd}$$

Observe that the quiver obtained from Q_A by removing the arrows from 1 to 4 is a Euclidean quiver of type $\widetilde{\mathbb{B}}_3$. Then, for the vector $\mathbf{x} = (1, 1, 1, 1) \in \mathbb{Z}^4$, we obtain the inequalities

$$\begin{aligned}\chi_A(\mathbf{x}) &= \sum_{i=1}^{4} f_ix_i^2 - f_{12}x_1x_2 - f_{23}x_2x_3 - f_{34}x_3x_4 - f_{14}x_1x_4\\ &= f_1(x_1^2 + 2x_2^2 + 2x_3^2 + x_4^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_4) - f_1x_1x_4\\ &= -f_1 < 0,\end{aligned}$$

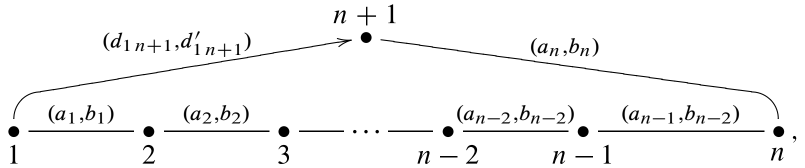
and hence $\chi_A(\mathbf{x})$ is indefinite.

We also note that if Q_A is a quiver whose underlying graph $\bar{Q}_A$ coincides with the underlying graph of one of the quivers considered in (1)–(9), then the Euler form χ_A of A is again indefinite, because the Euler form χ_A does not depend on the orientation of arrows in Q_A.

Assume now that there is a proper subset Σ of the set of vertices of Q_A such that the quiver $Q_A^{(\Sigma)}$ is either a Euclidean quiver, or a wild quiver whose underlying graph is the same as the underlying graph of one of the wild quivers considered in (1)–(9). Then it follows from Lemma 4.5 that there exists a finite dimensional hereditary K-algebra H such that $Q_H = Q_A^{(\Sigma)}$ and the Euler form χ_H of H is a proper restriction of the Euler form χ_A of A. Applying now Propositions 4.4 and 4.6, we conclude that χ_A is indefinite.

Therefore, we may assume that Q_A does not contain a proper set Σ of vertices such that $Q_A^{(\Sigma)}$ is a Euclidean quiver as well as a quiver whose underlying graph coincides with the underlying graph of one of the quivers considered in (1)–(9). We will show that this leads to a contradiction.

Assume Q_A contains a cycle Δ. We may take Δ with the minimal number of vertices among all cycles contained in Q_A. Then Δ is of the form



where $i \overset{(a_i,b_i)}{\overline{\qquad}} i+1$ means $i \xrightarrow{(d_{i\,i+1},d'_{i\,i+1})} i+1$ or $i \xleftarrow{(d'_{i+1\,i},d_{i+1\,i})} i+1$, for some integer $n \geq 2$. Moreover, there are no more arrows between the vertices $1, 2, \dots, n, n+1$ in Q_A, by the minimality assumption imposed on Δ. Further, at least one of the arrows of Δ has a nontrivial valuation, because otherwise Δ is a quiver of Euclidean type $\widetilde{\mathbb{A}}_n$ and of the form $Q_A^{(\Sigma)}$ for a proper set Σ of vertices of Q_A. We claim that in fact at least two arrows of Δ have nontrivial valuations. Indeed, suppose that exactly one arrow of Δ has a nontrivial valuation. Without loss of generality, we may assume that it is the arrow from 1 to $n+1$. Then $(d_{1\,n+1}, d'_{1\,n+1}) \neq (1,1)$ and $(a_i, b_i) = (1,1)$ for any $i \in \{1 \dots, n\}$. But then we infer that $f_1 = f_2 = \cdots = f_n = f_{n+1}$, and consequently $d_{1\,n+1} f_{n+1} = f_1 d'_{1\,n+1}$ forces $d_{1\,n+1} = d'_{1\,n+1}$. This shows that for the set Σ consisting of all vertices of Q_A except 1 and $n+1$, the quiver $Q_A^{(\Sigma)}$ is either a Euclidean quiver of type $\widetilde{\mathbb{A}}_{12}$, or a wild quiver considered in (2), a contradiction to the assumption made on Q_A. We note also that every valued arrow of Δ has one of the valuations $(1,1)$, $(1,2)$, $(2,1)$, $(1,3)$, or $(3,1)$. Using now the assumption that no quiver of the forms $Q_A^{(\Sigma)}$ is a Euclidean quiver, we conclude that Δ is a quiver whose underlying graph coincides with the underlying graph of one of the quivers considered in (8) or (9), again a contradiction.

Finally, assume that Q_A is a tree, that is, the underlying graph $\bar{Q}_A$ of Q_A is a tree. Then we easily conclude that there is a proper set Σ of vertices of Q_A such that the quiver $Q_A^{(\Sigma)}$ is either a Euclidean quiver, or a quiver with 2 or 3 vertices whose underlying graph $\bar{Q}_A^{(\Sigma)}$ of $Q_A^{(\Sigma)}$ coincides with the underlying graph of one of the quivers considered in (1)–(7), again contradicting the assumption made on Q_A.

Summing up, we proved that the Euler form χ_A of A is indefinite. □

The following theorem divides the class of indecomposable finite dimensional hereditary K-algebras over a field K into three disjoint subclasses.

Theorem 4.8. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K. Then the following equivalences hold:*

(i) *Q_A is a Dynkin quiver if and only if χ_A is positive definite.*

(ii) *Q_A is a Euclidean quiver if and only if χ_A is positive semidefinite, but not positive definite.*

(iii) *Q_A is a wild quiver if and only if χ_A is indefinite.*

Proof. If Q_A is a wild quiver, then it follows from Proposition 4.7 that χ_A is indefinite. Further, if Q_A is a Dynkin quiver, then there is a finite dimensional hereditary algebra H such that Q_H is a Euclidean quiver and Q_A is obtained from Q_H by removing one vertex and the arrows attached to it, and consequently χ_A is a proper restriction of χ_H. Hence, it follows from Propositions 4.4 and 4.6 that if Q_A is a Dynkin quiver, then χ_A is positive definite. Moreover, by Proposition 4.6, if Q_A is a Euclidean quiver, then χ_A is positive semidefinite of corank 1. Summing up, we proved that the equivalences (i), (ii), (iii) hold. $\square$

Let A be an indecomposable finite dimensional hereditary K-algebra over a field K. Then A is said to be

- a *hereditary algebra of Dynkin type* if Q_A is a Dynkin quiver;
- a *hereditary algebra of Euclidean type* if Q_A is a Euclidean quiver;
- a *hereditary algebra of wild type* if Q_A is a wild quiver.

5 The Coxeter transformation

In this section we introduce the Coxeter transformation on the Grothendieck group $K_0(A)$ of a hereditary algebra A and show that it is closely related to the Auslander–Reiten translation $\tau_A = D\,\mathrm{Tr}$ on $\operatorname{mod} A$.

Recall from Section I.11 that the Grothendieck group $K_0(A)$ of a finite dimensional K-algebra over a field K is the abelian group $\mathcal{F}/\mathcal{F}'$, where $\mathcal{F}$ is the free abelian group having as $\mathbb{Z}$-basis the set of isomorphism classes $\{M\}$ of modules M in $\operatorname{mod} A$, and $\mathcal{F}'$ is the subgroup of $\mathcal{F}$ generated by the elements $\{M\} - \{L\} - \{N\}$ given by all short exact sequences

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\operatorname{mod} A$. Moreover, we denote by $[M]$ the image of the isomorphism class $\{M\}$ of a module M in $\operatorname{mod} A$ via the canonical group epimorphism $\mathcal{F} \to \mathcal{F}/\mathcal{F}' = K_0(A)$. Then it follows from Theorem I.11.1 that if $S_1, \dots, S_n$ is a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$, then $[S_1], \dots, [S_n]$ form a $\mathbb{Z}$-basis of $K_0(A)$, and hence $K_0(A)$ is a free abelian group of rank n.

The following lemma shows that the Grothendieck groups of hereditary algebras admit also $\mathbb{Z}$-bases given by the classes of indecomposable projective modules and indecomposable injective modules.

Lemma 5.1. *Let A be a finite dimensional hereditary K-algebra over a field K, $P_1, \ldots, P_n$ be a complete set of pairwise nonisomorphic indecomposable projective modules in* $\operatorname{mod} A$*, and $I_1, \ldots, I_n$ be a complete set of pairwise nonisomorphic indecomposable injective modules in* $\operatorname{mod} A$*. Then:*

(i) $[P_1], \ldots, [P_n]$ *is a $\mathbb{Z}$-basis of $K_0(A)$.*

(ii) $[I_1], \ldots, [I_n]$ *is a $\mathbb{Z}$-basis of $K_0(A)$.*

Proof. We may assume that $\operatorname{top}(P_i) = S_i = \operatorname{soc}(I_i)$ for any $i \in \{1, \ldots, n\}$. Then $S_1, \ldots, S_n$ are pairwise nonisomorphic simple modules in $\operatorname{mod} A$, and $[S_1], \ldots, [S_n]$ form a $\mathbb{Z}$-basis of $K_0(A)$. Since A is a hereditary algebra, every simple module S_i has a minimal projective resolution in $\operatorname{mod} A$ of the form

$$0 \longrightarrow \operatorname{rad} P_i \longrightarrow P_i \longrightarrow S_i \longrightarrow 0$$

where $\operatorname{rad} P_i \cong \bigoplus_{j=1}^{n} P_j^{d'_{P_j P_i}}$, and a minimal injective resolution in $\operatorname{mod} A$ of the form

$$0 \longrightarrow S_i \longrightarrow I_i \longrightarrow I_i/S_i \longrightarrow 0,$$

where $I_i/S_i \cong \bigoplus_{j=1}^{n} I_j^{d_{I_i I_j}}$ (see Section III.9). Hence we obtain the equalities in $K_0(A)$

$$[S_i] = [P_i] - [\operatorname{rad} P_i] = [P_i] - \sum_{j=1}^{n} d'_{P_j P_i} [P_j]$$

and

$$[S_i] = [I_i] - [I_i/S_i] = [I_i] - \sum_{j=1}^{n} d_{I_i I_j} [I_j],$$

for any $i \in \{1, \ldots, n\}$. This shows that $[P_1], \ldots, [P_n]$ (respectively $[I_1], \ldots, [I_n]$) generate the group $K_0(A)$. Since $K_0(A)$ is a free abelian group of rank n, we conclude that $[P_1], \ldots, [P_n]$ (respectively $[I_1], \ldots, [I_n]$) form a $\mathbb{Z}$-basis of $K_0(A)$. □

Let A be a finite dimensional hereditary K-algebra over a field K, $e_1, \ldots, e_n$ basic primitive idempotents of A, and $P_i = e_i A$, $I_i = D(Ae_i)$, for $i \in \{1, \ldots, n\}$. It follows from Lemma 5.1 that $[P_1], \ldots, [P_n]$ and $[I_1], \ldots, [I_n]$ form two $\mathbb{Z}$-bases of $K_0(A)$. Hence we may consider the $\mathbb{Z}$-linear isomorphism

$$\varphi_A \colon K_0(A) \longrightarrow K_0(A)$$

such that $\varphi_A([P_i]) = -[I_i]$ for any $i \in \{1, \ldots, n\}$, which we call the *Coxeter transformation* of A. Observe that

$$I_i = D(Ae_i) \cong D \operatorname{Hom}_A(e_i A, A) = D \operatorname{Hom}_A(P_i, A)$$

and hence $\varphi_A([P_i]) = -[I_i] = -[D\operatorname{Hom}_A(e_iA, A)] = -[D\operatorname{Hom}_A(P_i, A)]$ for any $i \in \{1, \dots, n\}$. Since every projective module P in $\operatorname{mod} A$ is isomorphic to a direct sum of modules of the form $P_1, \dots, P_n$, we conclude that

$$\varphi_A([P]) = -[D\operatorname{Hom}_A(P, A)]$$

for any projective module P in $\operatorname{mod} A$.

We provide now general formula for the Coxeter transformation φ_A and its inverse φ_A^{-1}.

Proposition 5.2. *Let A be a finite dimensional hereditary K-algebra over a field K and M a module in* $\operatorname{mod} A$. *Then:*

(i) $\varphi_A([M]) = [D\operatorname{Ext}_A^1(M, A)] - [D\operatorname{Hom}_A(M, A)]$.

(ii) $\varphi_A^{-1}([M]) = [\operatorname{Ext}_{A^{\mathrm{op}}}^1(D(M), A)] - [\operatorname{Hom}_{A^{\mathrm{op}}}(D(M), A)]$.

Proof. (i) Consider a minimal projective resolution

$$0 \longrightarrow P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

of M in $\operatorname{mod} A$. Applying Theorem 3.3, we obtain an exact sequence in $\operatorname{mod} A^{\mathrm{op}}$ of the form

$$0 \to \operatorname{Hom}_A(M, A) \to \operatorname{Hom}_A(P_0, A) \to \operatorname{Hom}_A(P_1, A) \to \operatorname{Ext}_A^1(M, A) \to 0,$$

because $\operatorname{Ext}_A^1(P_0, A) = 0$. Applying the duality functor $D\colon \operatorname{mod} A^{\mathrm{op}} \to \operatorname{mod} A$ we obtain the exact sequence in $\operatorname{mod} A$

$$\begin{aligned} 0 \to D\operatorname{Ext}_A^1(M, A) &\to D\operatorname{Hom}_A(P_1, A) \to D\operatorname{Hom}_A(P_0, A) \\ &\to D\operatorname{Hom}_A(M, A) \to 0. \end{aligned}$$

This leads to the equalities in $K_0(A)$

$$\begin{aligned} \varphi_A([M]) = \varphi_A([P_0] - [P_1]) &= \varphi_A([P_0]) - \varphi_A([P_1]) \\ &= -[D\operatorname{Hom}_A(P_0, A)] + [D\operatorname{Hom}_A(P_1, A)] \\ &= [D\operatorname{Ext}_A^1(M, A)] - [D\operatorname{Hom}_A(M, A)]. \end{aligned}$$

(ii) Observe that the duality functor $D = \operatorname{Hom}_K(-, K)$ induces the following commutative diagram of $\mathbb{Z}$-linear isomorphisms

$$\begin{tikzcd} K_0(A) \arrow[r, "\varphi_A"] \arrow[d, "D"] & K_0(A) \arrow[d, "D"] \\ K_0(A^{\mathrm{op}}) & K_0(A^{\mathrm{op}}) \arrow[l, "\varphi_{A^{\mathrm{op}}}"'] \end{tikzcd}.$$

Indeed, for any $i \in \{1, \dots, n\}$, we have

$$\begin{aligned}(\varphi_{A^{\mathrm{op}}} D \varphi_A)[P_i] &= (\varphi_{A^{\mathrm{op}}} D)(\varphi_A[P_i]) = \varphi_{A^{\mathrm{op}}} D(-[I_i]) \\ &= \varphi_{A^{\mathrm{op}}}\big(-[D(I_i)]\big) = -\varphi_{A^{\mathrm{op}}}([Ae_i]) \\ &= [D(e_i A)] = D([P_i]).\end{aligned}$$

Applying now (i) to $\varphi_{A^{\mathrm{op}}}$ and $D(M)$, we have

$$\varphi_{A^{\mathrm{op}}}([D(M)]) = [D\,\mathrm{Ext}^1_{A^{\mathrm{op}}}(D(M), A)] - [D\,\mathrm{Hom}_{A^{\mathrm{op}}}(D(M), A)].$$

Observe that $D\colon K_0(A^{\mathrm{op}}) \to K_0(A)$ is the inverse of $D\colon K_0(A) \to K_0(A^{\mathrm{op}})$. Then $\varphi_{A^{\mathrm{op}}} D \varphi_A = D$ leads to the equality $\varphi_A^{-1} = D\varphi_{A^{\mathrm{op}}} D$. Therefore,

$$\begin{aligned}\varphi_A^{-1}([M]) &= (D\varphi_{A^{\mathrm{op}}} D)([M]) = D\big(\varphi_{A^{\mathrm{op}}}([D(M)])\big) \\ &= D\big([D\,\mathrm{Ext}^1_{A^{\mathrm{op}}}(D(M), A)] - [D\,\mathrm{Hom}_{A^{\mathrm{op}}}(D(M), A)]\big) \\ &= [\mathrm{Ext}^1_{A^{\mathrm{op}}}(D(M), A)] - [\mathrm{Hom}_{A^{\mathrm{op}}}(D(M), A)],\end{aligned}$$

as required. □

Corollary 5.3. *Let A be a finite dimensional hereditary K-algebra over a field K and M an indecomposable module in* mod A. *Then:*

(i) *If M is nonprojective, then $\varphi_A([M]) = [\tau_A M]$.*
(ii) *M is projective if and only if $\varphi_A([M]) < 0$.*
(iii) *$\varphi_A([M]) > 0$ or $\varphi_A([M]) < 0$.*

Proof. (i) Assume M is not projective. We claim that then $\mathrm{Hom}_A(M, A) = 0$. Indeed, suppose $f\colon M \to A$ is a nonzero homomorphism in mod A. Then Im f is a nonzero submodule of the right projective module A, and hence Im f is projective, because A is hereditary. Then the induced epimorphism $M \to \mathrm{Im}\, f$ is a retraction (Lemma I.8.1), and consequently M has a decomposition $M = X \oplus Q$, where Q is isomorphic to Im f, by Lemma I.4.2. Since M is indecomposable, we conclude that $M = Q \cong \mathrm{Im}\, f$ and is projective, a contradiction. Further, we known from Theorem III.4.10 that τ_A is a functor from mod A to mod A naturally isomorphic to the functor $D\,\mathrm{Ext}^1_A(-, A)$. Applying Proposition 5.2 (i), we obtain

$$\begin{aligned}\varphi_A([M]) &= [D\,\mathrm{Ext}^1_A(M, A)] - [D\,\mathrm{Hom}_A(M, A)] \\ &= [D\,\mathrm{Ext}^1_A(M, A)] = [\tau_A M] > 0.\end{aligned}$$

(ii) Assume that M is projective. Then $\mathrm{Hom}_A(M, A) \neq 0$, because M is isomorphic to a direct summand of A in mod A. Moreover, $\mathrm{Ext}^1_A(M, A) = 0$. Hence,

$$\begin{aligned}\varphi_A([M]) &= [D\,\mathrm{Ext}^1_A(M, A)] - [D\,\mathrm{Hom}_A(M, A)] \\ &= -[D\,\mathrm{Hom}_A(M, A)] < 0.\end{aligned}$$

Using (i) we conclude that M is projective if and only if $\varphi_A([M]) < 0$.

Clearly, (iii) is a direct consequence of (i) and (ii). ☐

Corollary 5.4. *Let A be a finite dimensional hereditary K-algebra over a field K and M an indecomposable module in* mod A. *Then:*

(i) *If M is noninjective, then $\varphi_A^{-1}([M]) = [\tau_A^{-1}M]$.*

(ii) *M is injective if and only if $\varphi_A^{-1}([M]) < 0$.*

(iii) *$\varphi_A^{-1}([M]) > 0$ or $\varphi_A^{-1}([M]) < 0$.*

Proof. (i) Assume M is not injective. Then $D(M)$ is not projective in $\operatorname{mod} A^{\mathrm{op}}$ and $\operatorname{Hom}_{A^{\mathrm{op}}}(D(M), A) = 0$. Further, it follows from Theorem III.4.10 that $\tau_A^{-1} = \operatorname{Tr} D$ is a functor from $\operatorname{mod} A$ to $\operatorname{mod} A$ and naturally isomorphic to the functor $\operatorname{Ext}_{A^{\mathrm{op}}}(D(-), A)$. Applying Proposition 5.2 (ii), we obtain

$$\begin{aligned}\varphi_A^{-1}([M]) &= [\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(M), A)] - [\operatorname{Hom}_{A^{\mathrm{op}}}(D(M), A)]\\ &= [\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(M), A)] = [\tau_A^{-1}M] > 0.\end{aligned}$$

(ii) Assume that M is injective. Then $D(M)$ is projective in $\operatorname{mod} A^{\mathrm{op}}$, and hence $\operatorname{Hom}_{A^{\mathrm{op}}}(D(M), A) \neq 0$. Moreover, $\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(M), A) = 0$. Therefore,

$$\begin{aligned}\varphi_A^{-1}([M]) &= [\operatorname{Ext}^1_{A^{\mathrm{op}}}(D(M), A)] - [\operatorname{Hom}_{A^{\mathrm{op}}}(D(M), A)]\\ &= -[\operatorname{Hom}_{A^{\mathrm{op}}}(D(M), A)] < 0.\end{aligned}$$

The statement (iii) is a direct consequence of (i) and (ii). ☐

The next aim of this section is to prove that the Euler forms $\langle -, - \rangle_A$ and χ_A of a finite dimensional hereditary algebra A over a field are invariant under the action of the Coxeter transformation φ_A. We need some preliminary facts.

Proposition 5.5. *Let A be a finite dimensional hereditary K-algebra over a field K, and M, N be indecomposable modules in* mod A. *The following statements hold:*

(i) *If M is nonprojective, then there exists a K-linear isomorphism*

$$\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_A(\tau_A M, \tau_A N).$$

(ii) *If N is noninjective, then there exists a K-linear isomorphism*

$$\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_A(\tau_A^{-1} M, \tau_A^{-1} N).$$

(iii) *If N is nonprojective, then there exists a K-linear isomorphism*

$$\operatorname{Ext}^1_A(M, N) \cong \operatorname{Ext}^1_A(\tau_A M, \tau_A N).$$

(iv) *If M is noninjective, then there exists a K-linear isomorphism*

$$\operatorname{Ext}_A^1(M,N) \cong \operatorname{Ext}_A^1(\tau_A^{-1}M, \tau_A^{-1}N).$$

Proof. Since A is a hereditary algebra, applying Theorems I.9.1, I.9.2 and I.9.3, we conclude that $\operatorname{pd}_A X \leq 1$ and $\operatorname{id}_A X \leq 1$ for any module X in $\operatorname{mod} A$. Then it follows from Corollary III.6.4 that there exist K-linear isomorphisms

$$D\operatorname{Hom}_A(N, \tau_A M) \cong \operatorname{Ext}_A^1(M,N) \cong D\operatorname{Hom}_A(\tau_A^{-1}N, M).$$

(i) Assume that M is nonprojective. Then $M \cong \tau_A^{-1}(\tau_A M)$ and we have K-linear isomorphisms

$$\begin{aligned}\operatorname{Hom}_A(M,N) &\cong D\operatorname{Hom}_A(M,N) \cong D\operatorname{Hom}_A\big(\tau_A^{-1}(\tau_A M), N\big)\\ &\cong \operatorname{Ext}_A^1(N, \tau_A M) \cong D\operatorname{Hom}_A(\tau_A M, \tau_A N)\\ &\cong \operatorname{Hom}_A(\tau_A M, \tau_A N).\end{aligned}$$

(ii) Assume that N is noninjective. Then $N \cong \tau_A(\tau_A^{-1}N)$ and we have K-linear isomorphisms

$$\begin{aligned}\operatorname{Hom}_A(M,N) &\cong D\operatorname{Hom}_A(M,N) \cong D\operatorname{Hom}_A\big(M, \tau_A(\tau_A^{-1}N)\big)\\ &\cong \operatorname{Ext}_A^1(\tau_A^{-1}N, M) \cong D\operatorname{Hom}_A(\tau_A^{-1}M, \tau_A^{-1}N)\\ &\cong \operatorname{Hom}_A(\tau_A^{-1}M, \tau_A^{-1}N).\end{aligned}$$

(iii) Assume that N is nonprojective. Then $N \cong \tau_A^{-1}(\tau_A N)$ and we have K-linear isomorphisms

$$\begin{aligned}\operatorname{Ext}_A^1(M,N) &\cong D\operatorname{Hom}_A(N, \tau_A M) \cong D\operatorname{Hom}_A\big(\tau_A^{-1}(\tau_A N), \tau_A M\big)\\ &\cong \operatorname{Ext}_A^1(\tau_A M, \tau_A N).\end{aligned}$$

(iv) Assume that M is noninjective. Then $M \cong \tau_A(\tau_A^{-1}M)$ and we have K-linear isomorphisms

$$\begin{aligned}\operatorname{Ext}_A^1(M,N) &\cong D\operatorname{Hom}_A(\tau_A^{-1}N, M) \cong D\operatorname{Hom}_A\big(\tau_A^{-1}N, \tau_A(\tau_A^{-1}M)\big)\\ &\cong \operatorname{Ext}_A^1(\tau_A^{-1}M, \tau_A^{-1}N).\end{aligned}$$ □

Lemma 5.6. *Let A be a finite dimensional hereditary K-algebra over a field K, $P_1, \dots, P_n$ a complete set of pairwise nonisomorphic indecomposable projective modules in* $\operatorname{mod} A$, *and $I_1, \dots, I_n$ a complete set of pairwise nonisomorphic indecomposable injective modules in* $\operatorname{mod} A$ *such that* $\operatorname{top}(P_i) = S_i = \operatorname{soc}(I_i)$ *for any $i \in \{1, \dots, n\}$. Then for any module M in* $\operatorname{mod} A$,

$$\dim_K \operatorname{Hom}_A(P_i, M) = c_i(M) \dim_K \operatorname{End}_A(S_i) = \dim_K \operatorname{Hom}_A(M, I_i).$$

Proof. Let M be a module in $\operatorname{mod} A$. We prove the claimed equalities by induction on the length $\ell(M)$ of M. For $\ell(M) = 0$, $M = 0$ and the equalities hold. Assume $\ell(M) \geq 1$. Then we have, by Proposition I.7.4, two exact sequences in $\operatorname{mod} A$ of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow S \longrightarrow 0$$

and

$$0 \longrightarrow T \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

where S and T are simple modules. Since the functors $\operatorname{Hom}_A(P_i, -)$ and $\operatorname{Hom}_A(-, I_i)$ from $\operatorname{mod} A$ to $\operatorname{mod} K$ are exact, by Proposition II.2.6, we obtain the induced exact sequences in $\operatorname{mod} K$

$$0 \longrightarrow \operatorname{Hom}_A(P_i, M') \longrightarrow \operatorname{Hom}_A(P_i, M) \longrightarrow \operatorname{Hom}_A(P_i, S) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Hom}_A(M'', I_i) \longrightarrow \operatorname{Hom}_A(M, I_i) \longrightarrow \operatorname{Hom}_A(T, I_i) \longrightarrow 0,$$

for any $i \in \{1, \ldots, n\}$. Then we have

$$\dim_K \operatorname{Hom}_A(P_i, M) = \dim_K \operatorname{Hom}_A(P_i, M') + \dim_K \operatorname{Hom}_A(P_i, S)$$

and

$$\dim_K \operatorname{Hom}_A(M, I_i) = \dim_K \operatorname{Hom}_A(M'', I_i) + \dim_K \operatorname{Hom}_A(T, I_i).$$

Observe that $\operatorname{Hom}_A(P_i, S) \neq 0$ if and only if $S \cong \operatorname{top}(P_i) = S_i$, and we have $\operatorname{Hom}_A(P_i, S_i) \cong \operatorname{End}_A(S_i)$. Similarly, $\operatorname{Hom}_A(T, I_i) \neq 0$ if and only if $T \cong S_i = \operatorname{soc}(I_i)$, and we have $\operatorname{Hom}_A(S_i, I_i) \cong \operatorname{End}_A(S_i)$. Further, we have $c_i(M') + c_i(S) = c_i(M) = c_i(T) + c_i(M'')$. Obviously, $c_i(S) = 1$ if $S \cong S_i$, and $c_i(S) = 0$ if $S \ncong S_i$. Similarly, $c_i(T) = 1$ if $T \cong S_i$, and $c_i(T) = 0$ if $T \ncong S_i$. Finally, we have $\ell(M') = \ell(M) - 1 = \ell(M'')$, so the required equalities hold by induction assumption. □

Proposition 5.7. *Let A be a finite dimensional hereditary K-algebra over a field K. Then*

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = -\langle \mathbf{y}, \varphi_A(\mathbf{x}) \rangle_A = \langle \varphi_A(\mathbf{x}), \varphi_A(\mathbf{y}) \rangle_A$$

for all $\mathbf{x}, \mathbf{y} \in K_0(A)$.

Proof. Let M and N be indecomposable modules in $\operatorname{mod} A$. We will prove that $\langle [M], [N] \rangle_A = -\langle [N], \varphi_A([M]) \rangle_A$. Assume first that M is nonprojective. Then, applying Corollaries III.6.4 and 5.3, Proposition 5.5, we obtain the equalities

$$\begin{aligned}
\langle [M], [N] \rangle_A &= \dim_K \operatorname{Hom}_A(M, N) - \dim_K \operatorname{Ext}^1_A(M, N) \\
&= \dim_K \operatorname{Hom}_A(\tau_A M, \tau_A N) - \dim_K D \operatorname{Hom}_A(N, \tau_A M) \\
&= -\dim_K \operatorname{Hom}_A(N, \tau_A M) + \dim_K D \operatorname{Hom}_A(\tau_A M, \tau_A N) \\
&= -\big(\dim_K \operatorname{Hom}_A(N, \tau_A M) - \dim_K \operatorname{Ext}^1_A(N, \tau_A M) \big) \\
&= -\langle [N], [\tau_A M] \rangle_A = -\langle [N], \varphi_A([M]) \rangle_A.
\end{aligned}$$

Assume M is projective, say $M \cong P_i = e_i A$ for some basic primitive idempotent e_i of A. Moreover, let $S_i = e_i A / e_i \operatorname{rad} A$ and $I_i = D(Ae_i)$. Applying Lemma 5.6 we obtain the equalities

$$\begin{aligned}
\langle [M], [N] \rangle_A &= \dim_K \operatorname{Hom}_A(P_i, N) - \dim_K \operatorname{Ext}_A^1(P_i, N) \\
&= \dim_K \operatorname{Hom}_A(P_i, N) = \dim_K \operatorname{Hom}_A(N, I_i) \\
&= \dim_K \operatorname{Hom}_A(N, I_i) - \dim_K \operatorname{Ext}_A^1(N, I_i) \\
&= \langle [N], [I_i] \rangle_A = \langle [N], -\varphi_A([P_i]) \rangle_A \\
&= -\langle [N], \varphi_A([M]) \rangle_A.
\end{aligned}$$

Let M and N be arbitrary modules in $\operatorname{mod} A$. Then $M = M_1 \oplus \cdots \oplus M_r$ and $N = N_1 \oplus \cdots \oplus N_s$ for some indecomposable modules $M_1, \dots, M_r$ and $N_1, \dots, N_s$ in $\operatorname{mod} A$, and $[M] = [M_1] + \cdots + [M_r]$ and $[N] = [N_1] + \cdots + [N_s]$ in $K_0(A)$. Then $\langle [M], [N] \rangle_A = -\langle [N], \varphi_A([M]) \rangle_A$ by the first part of the proof, the $\mathbb{Z}$-bilinearity of $\langle -, - \rangle_A$, and $\mathbb{Z}$-linearity of φ_A.

Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary elements of $K_0(A)$. Then we have $\mathbf{x} = [M] - [N]$ and $\mathbf{y} = [Q] - [R]$ for some (even semisimple) modules M, N, R, Q in $\operatorname{mod} A$, by Lemma 3.13. This leads to the equalities

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle_A &= \langle [M] - [N], [Q] - [R] \rangle_A \\
&= \langle [M], [Q] \rangle_A - \langle [M], [R] \rangle_A - \langle [N], [Q] \rangle_A + \langle [N], [R] \rangle_A \\
&= -\langle [Q], \varphi_A([M]) \rangle_A + \langle [R], \varphi_A([M]) \rangle_A \\
&\quad + \langle [Q], \varphi_A([N]) \rangle_A - \langle [R], \varphi_A([N]) \rangle_A \\
&= -\langle [Q] - [R], \varphi_A([M]) - \varphi_A([N]) \rangle_A \\
&= -\langle [Q] - [R], \varphi_A([M] - [N]) \rangle_A \\
&= -\langle \mathbf{y}, \varphi_A(\mathbf{x}) \rangle_A.
\end{aligned}$$

Clearly, we have also $-\langle \mathbf{y}, \varphi_A(\mathbf{x}) \rangle_A = \langle \varphi_A(\mathbf{x}), \varphi_A(\mathbf{y}) \rangle_A$. □

6 Postprojective and preinjective components

In this section we begin the study of the Auslander–Reiten quiver of a finite dimensional hereditary algebra over a field. We recall that if A is a finite dimenional K-algebra over a field K and $A = A_1 \times A_2 \times \cdots \times A_r$ is a decomposition of A into a product of indecomposable K-algebras (blocks), then the Auslander–Reiten quiver Γ_A of A is the disjoint union $\Gamma_A = \Gamma_{A_1} \cup \Gamma_{A_2} \cup \cdots \cup \Gamma_{A_r}$ of the Auslander–Reiten quivers of $A_1, A_2, \dots, A_r$ (see Proposition III.9.12). Moreover, for a simple algebra A, Q_A and Γ_A consist of a single isolated vertex. Therefore, we may devote our considerations to nonsimple indecomposable hereditary algebras.

Theorem 6.1. *Let A be a nonsimple indecomposable finite dimensional hereditary K-algebra over a field K. Then the Auslander–Reiten quiver Γ_A of A contains a connected component $\mathcal{P}(A)$ with the following properties:*

(i) *$\mathcal{P}(A)$ contains all projective vertices of Γ_A.*

(ii) *The opposite quiver Q_A^{op} of Q_A is isomorphic to a full valued subquiver of $\mathcal{P}(A)$.*

(iii) *Every indecomposable module X in $\mathcal{P}(A)$ is of the form $X = \tau_A^{-m_X} P_X$ for an indecomposable projective module P_X and a nonnegative integer m_X, both uniquely determined by X.*

(iv) *$\mathcal{P}(A)$ is an acyclic quiver.*

Proof. Let $e_1, \ldots, e_n$ be a complete set of basic primitive idempotents of A and $P_i = e_i A$, $I_i = D(Ae_i)$ and $S_i = e_i A/e_i \operatorname{rad} A$, $i \in \{1, \ldots, n\}$, the associated pairwise nonisomorphic indecomposable projective, indecomposable injective and simple right A-modules, respectively.

It follows from Lemma III.7.6 and Theorem III.7.12 that, for an indecomposable module X and an indecomposable projective module P in mod A, there is an irreducible homomorphism from X to P if and only if X is a direct summand of rad P. Since A is a hereditary algebra, the radical rad P of any indecomposable projective module P in mod A is also projective (see Theorem I.9.1). Hence we conclude that, for any path in Γ_A

$$X_1 \xrightarrow{\left(d_{X_1X_2}, d'_{X_1X_2}\right)} X_2 \longrightarrow \ldots \longrightarrow X_{m-1} \xrightarrow{\left(d_{X_{m-1}X_m}, d'_{X_{m-1}X_m}\right)} X_m = P$$

with P projective, all the indecomposable modules $X_1, X_2, \ldots, X_m$ are projective. Further, we know from Proposition 1.10 that, if Γ_A contains an arrow

$$P_j \xrightarrow{\left(d_{P_jP_i}, d'_{P_jP_i}\right)} P_i,$$

then the quiver Q_A of A contains an arrow

$$i \xrightarrow{\left(d_{ij}, d'_{ij}\right)} j,$$

where $d_{ij} = d'_{P_jP_i}$ and $d'_{ij} = d_{P_jP_i}$. Conversely, assume that Q_A contains an arrow

$$i \xrightarrow{\left(d_{ij}, d'_{ij}\right)} j,$$

for some $i, j \in \{1, \ldots, n\}$. Then $e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2 e_j \neq 0$. Observe that $\operatorname{top}(\operatorname{rad} P_i) = \operatorname{rad} P_i / \operatorname{rad}^2 P_i = e_i \operatorname{rad} A/e_i(\operatorname{rad} A)^2$. Hence, applying Lemma

I.8.7, we conclude that

$$\begin{aligned}\operatorname{Hom}_A\big(P_j, \operatorname{top}(\operatorname{rad} P_i)\big) &= \operatorname{Hom}_A\big(e_j A, e_i \operatorname{rad} A/e_i(\operatorname{rad} A)^2\big)\\ &\cong \big(e_i \operatorname{rad} A/e_i(\operatorname{rad} A)^2\big)e_j\\ &\cong e_i(\operatorname{rad} A)e_j/e_i(\operatorname{rad} A)^2 e_j \neq 0.\end{aligned}$$

This shows that P_j occurs in the projective cover $P(\operatorname{rad} P_i)$ of $\operatorname{rad} P_i$ in $\operatorname{mod} A$ (see Theorem I.8.4). Since A is a hereditary algebra, $\operatorname{rad} P_i$ is projective, and so P_j is an indecomposable direct summand of $\operatorname{rad} P_i$. Thus Γ_A contains an arrow

$$P_j \xrightarrow{\left(d_{P_j P_i}, d'_{P_j P_i}\right)} P_i,$$

and, as mentioned above, $d_{P_j P_i} = d'_{ij}$ and $d'_{P_j P_i} = d_{ij}$. Summing up, we proved that the opposite quiver Q_A^{op} of Q_A is isomorphic to the full valued subquiver of Γ_A whose vertices are the indecomposable projective modules $P_1, \dots, P_n$. Moreover, since A is an indecomposable algebra, the quiver Q_A is connected, by Corollary 1.7. This shows that Γ_A admits a connected component $\mathcal{P}(A)$ containing all the modules $P_1, \dots, P_n$, hence all projective vertices of Γ_A, and Q_A^{op} is isomorphic to a full subquiver of $\mathcal{P}(A)$. Therefore, the properties (i) and (ii) hold.

We prove now that (iii) also holds. Let X be an indecomposable nonprojective module in $\mathcal{P}(A)$. Since $\mathcal{P}(A)$ is a connected quiver, there is a finite sequence of arrows in $\mathcal{P}(A)$ connecting X to an indecomposable projective module $P = P_i$ (in fact any of the modules $P_1, \dots, P_n$). Hence there is a shortest sequence of arrows in $\mathcal{P}(A)$ of the form

$$P_i = Z_0 \text{ --- } Z_1 \text{ --- } \cdots \text{ --- } Z_{r-1} \longrightarrow Z_r = X$$

with Z_0 indecomposable projective and $Z_1, \dots, Z_r$ indecomposable nonprojective modules, where $Z_{i-1} \text{ --- } Z_i$ means

$$Z_{i-1} \xrightarrow{\left(d_{Z_{i-1} Z_i}, d'_{Z_{i-1} Z_i}\right)} Z_i \quad \text{or} \quad Z_i \xrightarrow{\left(d_{Z_i Z_{i-1}}, d'_{Z_i Z_{i-1}}\right)} Z_{i-1},$$

for any $i \in \{1, \dots, r\}$. Observe also that $Z_0 \text{ --- } Z_1$ is an arrow from Z_0 to Z_1, because Z_0 is projective and Z_1 is nonprojective. Applying τ_A to the sequence $Z_1 \text{ --- } Z_2 \text{ --- } \cdots \text{ --- } Z_{r-1} \text{ --- } Z_r = X$ we obtain the sequence of arrows in $\mathcal{P}(A)$ of the form

$$\tau_A Z_1 \text{ --- } \tau_A Z_2 \text{ --- } \cdots \text{ --- } \tau_A Z_r = \tau_A X,$$

where $\tau_A Z_j \text{ --- } \tau_A Z_{j+1}$ means

$$\tau_A Z_j \xrightarrow{\left(d_{\tau_A Z_j \tau_A Z_{j+1}}, d'_{\tau_A Z_j \tau_A Z_{j+1}}\right)} \tau_A Z_{j+1}$$

or

$$\tau_A Z_{j+1} \xrightarrow{\left(d_{\tau_A Z_{j+1}\tau_A Z_j}, d'_{\tau_A Z_{j+1}\tau_A Z_j}\right)} \tau_A Z_j$$

for any $j \in \{1, \ldots, r-1\}$. We also note that we have in $\mathcal{P}(A)$ an arrow $\tau_A Z_1 \xrightarrow{\left(d'_{Z_0 Z_1}, d_{Z_0 Z_1}\right)} Z_0$ (see Lemma III.9.1 and Proposition III.9.6). Since $Z_0 = P$ is projective, we conclude that $\tau_A Z_1$ is also projective. This shows that there is in $\mathcal{P}(A)$ a sequence of $r-1$ arrows connecting $\tau_A X = \tau_A Z_r$ to an indecomposable projective module $\tau_A Z_1 = P_j$ for some $j \in \{1, \ldots, n\}$. Hence we obtain by induction that $\tau_A X = \tau_A^{-s} P$ for an indecomposable projective module P and a nonnegative integer s. But then $X = \tau_A^{-m} P$ with $m = s + 1$. We claim that m and P are uniquely determined by X. Indeed, suppose that $X = \tau_A^{-t} Q$ for some nonnegative integer t and an indecomposable projective module Q. We may assume without loss of generality that $t \geq m$. Then we obtain $P = \tau_A^m X = \tau_A^{-(t-m)} Q$, and hence $t = m$ and $P = Q$, by Corollary III.4.9. Therefore, we may write $m = m_X$ and $P = P_X$.

We claim now that $\mathcal{P}(A)$ is acyclic. Indeed, suppose that $\mathcal{P}(A)$ contains an oriented cycle

$$Z_1 \xrightarrow{\left(d_{Z_1 Z_2}, d'_{Z_1 Z_2}\right)} Z_2 \longrightarrow \cdots \longrightarrow Z_{t-1} \xrightarrow{\left(d_{Z_{t-1} Z_t}, d'_{Z_{t-1} Z_t}\right)} Z_t \xrightarrow{\left(d_{Z_t Z_1}, d'_{Z_t Z_1}\right)} Z_1.$$

It follows from (iii) that, for any $s \in \{1, \ldots, t\}$, there are an indecomposable projective module $P^{(s)}$ in $\mathcal{P}(A)$ and a nonnegative integer $m_s = m_{Z_s}$ such that $Z_s = \tau_A^{-m_s} P^{(s)}$. Let m be the minimal integer in the set $\{m_1, \ldots, m_t\}$. Then, applying τ_A^m to the above cycle, we obtain an oriented cycle in $\mathcal{P}(A)$ of the form

$$\tau_A^m Z_1 \longrightarrow \cdots \longrightarrow \tau_A^m Z_{t-1} \xrightarrow{\left(d_{Z_{t-1} Z_t}, d'_{Z_{t-1} Z_t}\right)} \tau_A^m Z_t \xrightarrow{\left(d_{Z_t Z_1}, d'_{Z_t Z_1}\right)} \tau_A^m Z_1$$

containing at least one projective module $P^{(r)}$ for $r \in \{1, \ldots, t\}$ with $m_r = m$. Since every predecessor of $P^{(r)}$ in $\mathcal{P}(A)$ is projective, we conclude that $m = m_s$ such and $\tau_A^m Z_s = P^{(s)}$ for any $s \in \{1, \ldots, t\}$. But then, applying Proposition 1.10, we conclude that the quiver Q_A of A contains an oriented cycle, a contradiction to Corollary 1.8. Therefore, the component $\mathcal{P}(A)$ is acyclic, and (iv) holds. □

The unique connected component $\mathcal{P}(A)$ of Γ_A described in Theorem 6.1 is called the *postprojective component* of Γ_A. The name is justified by the fact that every indecomposable module in $\mathcal{P}(A)$ is a successor of an indecomposable projective module. An indecomposable module X in mod A which belongs to $\mathcal{P}(A)$ is said to be an *indecomposable postprojective module*. Moreover, a module M in mod A is said to be a *postprojective module* if M is a direct sum of indecomposable postprojective modules.

Theorem 6.2. *Let A be a nonsimple indecomposable finite dimensional hereditary K-algebra over a field K. Then the Auslander–Reiten quiver Γ_A of A contains a connected component $\mathcal{Q}(A)$ with the following properties:*

(i) *$\mathcal{Q}(A)$ contains all injective vertices of Γ_A.*

(ii) *The opposite quiver Q_A^{op} of Q_A is isomorphic to a full valued subquiver of $\mathcal{Q}(A)$.*

(iii) *Every indecomposable module Y in $\mathcal{Q}(A)$ is of the form $Y = \tau_A^{n_Y} I_Y$ for an indecomposable injective module I_Y and a nonnegative integer n_Y, both uniquely determined by Y.*

(iv) *$\mathcal{Q}(A)$ is an acyclic quiver.*

Proof. The proof is entirely similar to the proof of Theorem 6.1 and is left to the reader. □

The unique connected component $\mathcal{Q}(A)$ of Γ_A described in Theorem 6.2 is called the *preinjective component* of Γ_A. The name is justified by the fact that every indecomposable module in $\mathcal{Q}(A)$ is a predecessor of an indecomposable injective module. An indecomposable module Y in mod A which belongs to $\mathcal{Q}(A)$ is said to be an *indecomposable preinjective module*. Moreover, a module N in mod A is said to be a *preinjective module* if N is a direct sum of indecomposable preinjective modules.

Recall that a finite dimensional K-algebra A over a field K is called of *finite representation type* if the number of isomorphism classes of indecomposable modules in mod A is finite.

Corollary 6.3. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K. The following conditions are equivalent:*

(i) *A is of finite representation type.*

(ii) *$\mathcal{P}(A) = \mathcal{Q}(A)$.*

Proof. Observe that if $\mathcal{P}(A) \neq \mathcal{Q}(A)$, then the postprojective component $\mathcal{P}(A)$ does not contain an injective module. Then, for any indecomposable projective module P, the modules $\tau_A^{-m} P$, $m \geq 0$, form an infinite family of pairwise nonisomorphic indecomposable modules in $\mathcal{P}(A)$, because $\mathcal{P}(A)$ is acyclic. Therefore, (i) implies (ii).

Assume that $\mathcal{P}(A) = \mathcal{Q}(A)$. Then $\mathcal{P}(A)$ is a finite connected component of Γ_A. Applying the Auslander theorem (Theorem III.10.2), we conclude that A is of finite representation type (see Corollary III.10.3). Hence (ii) implies (i). □

For a locally finite valued quiver Δ without loops and multiple arrows, we introduced in Section III.9 the stable valued translation quiver $\mathbb{Z}\Delta$ and its full translation subquivers $\mathbb{N}\Delta$ and $(-\mathbb{N})\Delta$. Then we have the following direct consequences of Theorems 6.1 and 6.2.

Corollary 6.4. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K. Then the postprojective component $\mathcal{P}(A)$ of Γ_A is isomorphic to the valued translation quiver $(-\mathbb{N})Q_A^{\mathrm{op}}$.*

Corollary 6.5. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K. Then the preinjective component $\mathcal{Q}(A)$ of Γ_A is isomorphic to the valued translation quiver $\mathbb{N}Q_A^{\mathrm{op}}$.*

We will present now some properties of indecomposable postprojective and indecomposable preinjective modules over finite dimensional hereditary algebras, important for our further considerations.

Proposition 6.6. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K and X be an indecomposable module in $\mathcal{P}(A)$. The following statements hold:*

- (i) $\mathrm{End}_A(X)$ *is a division K-algebra, isomorphic to the endomorphism algebra* $\mathrm{End}_A(P)$ *of an indecomposable projective module P in* $\mathrm{mod}\, A$.
- (ii) $\mathrm{Ext}_A^1(X, X) = 0$.
- (iii) *For every indecomposable module M in* $\mathrm{mod}\, A$ *with* $\mathrm{Hom}_A(M, X) \neq 0$, *$M$ is a predecessor of X in $\mathcal{P}(A)$.*
- (iv) *For an indecomposable module Z in* $\mathrm{mod}\, A$, *$[X] = [Z]$ in $K_0(A)$ if and only if $X \cong Z$.*

Proof. We may assume that A is nonsimple. It follows from Theorem 6.1 that $X = \tau_A^{-m_X} P_X$ for an indecomposable projective module P_X and a nonnegative integer m_X. Then, by Proposition 5.5, $\mathrm{End}_A(X) \cong \mathrm{End}_A(\tau_A^{m_X} X) \cong \mathrm{End}_A(P_X)$ as K-algebras and $\mathrm{Ext}_A^1(X, X) \cong \mathrm{Ext}_A^1(\tau_A^{m_X} X, \tau_A^{m_X} X) = \mathrm{Ext}_A^1(P_X, P_X) = 0$ as K-vector spaces. Moreover, $\mathrm{End}_A(P_X)$ is a division K-algebra, by Corollary I.9.4. This proves the statements (i) and (ii).

For (iii), let M be an indecomposable module in $\mathrm{mod}\, A$ with $\mathrm{Hom}_A(M, X) \neq 0$. Suppose M is not a predecessor of X in $\mathcal{P}(A)$. Then there is no finite path of irreducible homomorphisms between indecomposable modules in $\mathrm{mod}\, A$ from M to X. It follows from Proposition III.10.1 that there exist an infinite path of irreducible homomorphisms in $\mathrm{mod}\, A$

$$\dots \longrightarrow X_t \xrightarrow{g_t} X_{t-1} \longrightarrow \dots \longrightarrow X_2 \xrightarrow{g_2} X_1 \xrightarrow{g_1} X_0 = X$$

between indecomposable modules and homomorphisms $h_t\colon M \to X_t$ in $\mathrm{mod}\, A$ such that $g_1 g_2 \dots g_t h_t \neq 0$, for any $t \in \mathbb{N}^+$. Hence, X admits infinitely many predecessors in $\mathcal{P}(A)$, which is impossible because $\mathcal{P}(A)$ is acyclic with finitely many τ_A-orbits. Therefore, M is a predecessor of X in $\mathcal{P}(A)$.

For (iv), let Z be an indecomposable module in $\operatorname{mod} A$. Clearly, if $X \cong Z$ then $[X] = [Z]$ in $K_0(A)$. Assume $[X] = [Z]$ in $K_0(A)$. We claim that $X \cong Z$. We have

$$\begin{aligned}\dim_K \operatorname{Hom}_A(X, Z) - \dim_K \operatorname{Ext}^1_A(X, Z) &= \langle [X], [Z] \rangle_A = \langle [X], [X] \rangle_A \\ &= \dim_K \operatorname{End}_A(X) - \dim_K \operatorname{Ext}^1_A(X, X) \\ &= \dim_K \operatorname{End}_A(X), \\ \dim_K \operatorname{Hom}_A(Z, X) - \dim_K \operatorname{Ext}^1_A(Z, X) &= \langle [Z], [X] \rangle_A = \langle [X], [X] \rangle_A \\ &= \dim_K \operatorname{End}_A(X) - \dim_K \operatorname{Ext}^1_A(X, X) \\ &= \dim_K \operatorname{End}_A(X),\end{aligned}$$

by (ii). Since $\dim_K \operatorname{End}_A(X) > 0$, we conclude that $\operatorname{Hom}_A(X, Z) \neq 0$ and $\operatorname{Hom}_A(Z, X) \neq 0$. Applying (iii) we obtain that Z is a predecessor of X in $\mathcal{P}(A)$ and X is a predecessor of Z in $\mathcal{P}(A)$. Then $X \cong Z$, because $\mathcal{P}(A)$ is acyclic. □

Proposition 6.7. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K and Y be an indecomposable module in $\mathcal{Q}(A)$. The following statements hold:*

(i) $\operatorname{End}_A(Y)$ *is a division K-algebra, isomorphic to the endomorphism algebra* $\operatorname{End}_A(I)$ *of an indecomposable injective module I in* $\operatorname{mod} A$.

(ii) $\operatorname{Ext}^1_A(Y, Y) = 0$.

(iii) *For every indecomposable module N in* $\operatorname{mod} A$ *with* $\operatorname{Hom}_A(Y, N) \neq 0$, *$N$ is a successor of Y in $\mathcal{Q}(A)$.*

(iv) *For an indecomposable module Z in* $\operatorname{mod} A$, *$[Y] = [Z]$ in $K_0(A)$ if and only if $Y \cong Z$.*

Proof. The proof is similar to the proof of Proposition 6.6 and is left to the reader. □

The following proposition describes the values of the Euler quadratic form on the classes of indecomposable postprojective and indecomposable preinjective modules over a finite dimensional hereditary algebra.

Proposition 6.8. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K and $S_1, \dots, S_n$ a complete set of pairwise nonisomorphic simple modules in* $\operatorname{mod} A$. *Then we have the equalities of sets*

$$\begin{aligned}\{\chi_A([X]) \mid X \in \mathcal{P}(A)\} &= \{\dim_K \operatorname{End}_A(S_i) \mid i \in \{1, \dots, n\}\} \\ &= \{\chi_A([Y]) \mid Y \in \mathcal{Q}(A)\}.\end{aligned}$$

Proof. Let $P_1, \dots, P_n$ (respectively, $I_1, \dots, I_n$) be complete sets of pairwise nonisomorphic indecomposable projective (respectively, injective) modules such that $\operatorname{top}(P_i) = S_i = \operatorname{soc}(I_i)$ for any $i \in \{1, \dots, n\}$. For an indecomposable module X in $\mathcal{P}(A)$, applying Proposition 6.6, we obtain

$$\begin{aligned}\chi_A([X]) &= \dim_K \operatorname{End}_A(X) - \dim_K \operatorname{Ext}^1_A(X, X)\\ &= \dim_K \operatorname{End}_A(X) = \dim_K \operatorname{End}_A(P_i)\end{aligned}$$

for some $i \in \{1, \dots, n\}$. Similarly, for an indecomposable module Y in $\mathcal{Q}(A)$, applying Proposition 6.7, we obtain

$$\begin{aligned}\chi_A([Y]) &= \dim_K \operatorname{End}_A(Y) - \dim_K \operatorname{Ext}^1_A(Y, Y)\\ &= \dim_K \operatorname{End}_A(Y) = \dim_K \operatorname{End}_A(I_j)\end{aligned}$$

for some $j \in \{1, \dots, n\}$. We claim that there are canonical isomorphisms of K-algebras

$$\operatorname{End}_A(P_i) \xrightarrow{\sim} \operatorname{End}_A(S_i) \qquad \text{and} \qquad \operatorname{End}_A(S_i) \xrightarrow{\sim} \operatorname{End}_A(I_i),$$

for all $i \in \{1, \dots, r\}$. Fix $i \in \{1, \dots, n\}$. For any $f \in \operatorname{End}_A(P_i)$ we have a commutative diagram in $\operatorname{mod} A$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{rad} P_i \arrow[r, hook] \arrow[d, "f'"] & P_i \arrow[r, "\pi_i"] \arrow[d, "f"] & S_i \arrow[r] \arrow[d, "\bar{f}"] & 0 \\
0 \arrow[r] & \operatorname{rad} P_i \arrow[r, hook] & P_i \arrow[r, "\pi_i"] & S_i \arrow[r] & 0 ,
\end{tikzcd}$$

where f' is the restriction of f to $\operatorname{rad} P_i$, and $\bar{f}$ is given by $\bar{f}(x + \operatorname{rad} P_i) = f(x) + \operatorname{rad} P_i$ for any $x \in P_i$. Observe that $\bar{f} = 0$ implies $\operatorname{Im} f \subseteq \operatorname{rad} P_i$, and hence $f = 0$, because $\operatorname{End}_A(P_i)$ is a division K-algebra. Moreover, since P_i is projective, for any $g \in \operatorname{End}_A(S_i)$ there exists $f \in \operatorname{End}_A(P_i)$ such that $g = \bar{f}$. Therefore, the map $\operatorname{End}_A(P_i) \to \operatorname{End}_A(S_i)$ defined above is a K-algebra isomorphism. Dually, for any $g \in \operatorname{End}_A(S_i)$ we have a commutative diagram in $\operatorname{mod} A$

$$\begin{tikzcd}
0 \arrow[r] & S_i \arrow[r, hook] \arrow[d, "g"] & I_i \arrow[r, "p_i"] \arrow[d, "\tilde{g}"] & I_i/S_i \arrow[r] \arrow[d, "\bar{g}"] & 0 \\
0 \arrow[r] & S_i \arrow[r, hook] & I_i \arrow[r, "p_i"] & I_i/S_i \arrow[r] & 0 ,
\end{tikzcd}$$

where $\tilde{g}$ is the extension of g (due to the injectivity of I_i), and $\bar{g}$ is given by $\bar{g}(y + S_i) = \tilde{g}(y) + S_i$. Observe that $\tilde{g}$ is uniquely determined by g, because $S_i = \operatorname{soc}(I_i)$, so $h \in \operatorname{End}_A(I_i)$ is nonzero if and only if $h(S_i) \neq 0$. Clearly, for any $h \in \operatorname{End}_A(I_i)$ its restriction to $S_i = \operatorname{soc}(I_i)$ defines a homomorphism

$g \in \operatorname{End}_A(S_i)$ such that $h = \tilde{g}$. Therefore, the map $\operatorname{End}_A(S_i) \to \operatorname{End}_A(I_i)$ defined above is a K-algebra isomorphism. Finally, we note that $P_1, \ldots, P_n$ lie in $\mathcal{P}(A)$ and $I_1, \ldots, I_n$ lie in $\mathcal{Q}(A)$. This finishes the proof. □

Proposition 6.9. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K. Then the following statements hold:*

(i) *$\mathcal{P}(A)$ is a generalized standard component.*

(ii) *$\mathcal{Q}(A)$ is a generalized standard component.*

Proof. (i) Assume that there exist indecomposable modules M and N in $\mathcal{P}(A)$ such that $\operatorname{rad}_A^\infty(M,N) \neq 0$. Since $\mathcal{P}(A)$ is an acyclic component with finitely many τ_A-orbits, we conclude that N has only finitely many indecomposable predecessors in $\mathcal{P}(A)$. Let r be the maximal length of path in $\mathcal{P}(A)$ with the target in N. Take an integer $m > r+1$. Then $\operatorname{rad}_A^m(M,N) \neq 0$, because $\operatorname{rad}_A^\infty(M,N) \neq 0$. Let f be a nonzero homomorphism in $\operatorname{rad}_A^m(M,N)$. Applying Proposition 3.8 (ii), we infer that there exist a positive integer t, indecomposable modules $Y_1, \ldots, Y_t$ in $\operatorname{mod} A$, homomorphisms $u_j \in \operatorname{rad}_A(Y_j, N)$, and homomorphisms $v_j \colon M \to Y_j$ with each v_j a finite sum of compositions of $m-1$ irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$, such that $f = \sum_{j=1}^t u_j v_j$. Clearly, $f \neq 0$ implies that $u_j v_j \neq 0$ for some $j \in \{1, \ldots, t\}$. Then, by Proposition 6.6 (iii), $u_j \neq 0$ implies that Y_j is a predecessor of N in $\mathcal{P}(A)$. Moreover, since $v_j \neq 0$, there is a nonzero composition of $m-1$ irreducible homomorphisms between indecomposable modules leading from M to Y_j, and then all these indecomposable modules belong to $\mathcal{P}(A)$. This shows that $\mathcal{P}(A)$ admits a path from M to N of length at least $r+1$, a contradiction with the choice of r.

The proof of (ii) is similar. □

Let A be a finite dimensional K-algebra over a field K. A module M in $\operatorname{mod} A$ is said to be *sincere* if every simple module in $\operatorname{mod} A$ is a composition factor of M. Our next aim is to prove that all but finitely many isomorphism classes of indecomposable modules in the postprojective and preinjective components of hereditary algebras are sincere modules. We need the following preliminary results.

Proposition 6.10. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, and $P_1, \ldots, P_n$ and $I_1, \ldots, I_n$ be complete sets of pairwise nonisomorphic indecomposable projective and indecomposable injective modules in* $\operatorname{mod} A$*, respectively. The following statements hold:*

(i) *For any $i, j \in \{1, \ldots, n\}$, the sequence* $\dim_K \operatorname{Hom}_A(P_i, \tau_A^{-m} P_j)$*, with $m \geq 1$, is unbounded.*

(ii) *For any $i, j \in \{1, \ldots, n\}$, the sequence* $\dim_K \operatorname{Hom}_A(\tau_A^m I_i, I_j)$*, with $m \geq 1$, is unbounded.*

Proof. Since A is of infinite representation, we know from Corollary 6.3 that $\mathcal{P}(A) \neq \mathcal{Q}(A)$, and consequently $\mathcal{P}(A)$ contains all modules $P_1, \dots, P_n$, but not an injective module, and $\mathcal{Q}(A)$ contains all modules $I_1, \dots, I_n$, but not a projective module. It follows also from Lemma 5.6 that, for a module M in mod A and $i \in \{1, \dots, n\}$, we have

$$\dim_K \operatorname{Hom}_A(P_i, M) = c_i(M) \dim_K \operatorname{End}_A(S_i) = \dim_K \operatorname{Hom}_A(M, I_i),$$

where $c_i(M)$ is the multiplicity of the simple module $S_i = \operatorname{top}(P_i) = \operatorname{soc}(I_i)$ as a composition factor of M.

For $i, j \in \{1, \dots, n\}$, we have the equalities

$$\begin{aligned}\dim_K \operatorname{Hom}_A(P_i, \tau_A^{-m} P_j) &= c_i\big(\tau_A^{-m} P_j\big) \dim_K \operatorname{End}_A(S_i)\\ &= \dim_K \operatorname{Hom}_A(\tau_A^{-m} P_j, I_i)\\ &= \dim_K \operatorname{Hom}_A(P_j, \tau_A^{m} I_i)\\ &= c_j\big(\tau_A^{m} I_i\big) \dim_K \operatorname{End}_A(S_j)\\ &= \dim_K \operatorname{Hom}_A(\tau_A^{m} I_i, I_j).\end{aligned}$$

For any $i, j \in \{1, \dots, n\}$, we set

$$r_{ij} = \overline{\lim_{m\to\infty}} \dim_K \operatorname{Hom}_A(P_i, \tau_A^{-m} P_j) = \overline{\lim_{m\to\infty}} \dim_K \operatorname{Hom}_A(\tau_A^{m} I_i, I_j).$$

Recall also that every indecomposable module M in $\mathcal{P}(A)$ is of the form $M = \tau_A^{-m} P_j$ for some $m \geq 0$ and $j \in \{1, \dots, n\}$. Further, by Auslander's theorem (Theorem III.10.2), there is no common bound of the dimensions of indecomposable modules in the infinite component $\mathcal{P}(A)$. Consequently, not all r_{ij} are finite. We claim that in fact all r_{ij} are infinite. Let

$$b \xrightarrow{(d_{ab}, d'_{ab})} a$$

be an arrow of the valued quiver Q_A of A. Then it follows from Theorem 6.1 that the postprojective component $\mathcal{P}(A)$ contains an arrow

$$P_a \xrightarrow{(d'_{ab}, d_{ab})} P_b.$$

Hence, we have in mod A almost split sequences of the form

$$\begin{gathered}0 \longrightarrow P_a \longrightarrow P_b^{d'_{ab}} \oplus X \longrightarrow \tau_A^{-1} P_a \longrightarrow 0,\\ 0 \longrightarrow P_b \longrightarrow \big(\tau_A^{-1} P_a\big)^{d_{ab}} \oplus Y \longrightarrow \tau_A^{-1} P_b \longrightarrow 0,\end{gathered}$$

with all indecomposable direct summands of X and Y in $\mathcal{P}(A)$ (see Lemma III.9.1 and Proposition III.9.6). Applying now the exact functor $\tau_A^{-1}\colon \operatorname{mod}_{\mathcal{I}} A \to$

$\operatorname{mod}_{\mathcal{P}} A$ (see Proposition 3.4), we obtain almost split sequences in $\operatorname{mod} A$ of the form

$$\begin{aligned} 0 \longrightarrow \tau_A^{-m} P_a \longrightarrow \left(\tau_A^{-m} P_b\right)^{d'_{ab}} \oplus \tau_A^{-m} X \longrightarrow \tau_A^{-m-1} P_a \longrightarrow 0, \\ 0 \longrightarrow \tau_A^{-m} P_b \longrightarrow \left(\tau_A^{-m-1} P_a\right)^{d_{ab}} \oplus \tau_A^{-m} Y \longrightarrow \tau_A^{-m-1} P_b \longrightarrow 0, \end{aligned}$$

for each $m \geq 1$. Applying the exact functor $\operatorname{Hom}_A(P_i, -)\colon \operatorname{mod} A \to \operatorname{mod} K$, we obtain the exact sequences in $\operatorname{mod} K$

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_A\left(P_i, \tau_A^{-m} P_a\right) \arrow[r] & \operatorname{Hom}_A\left(P_i, \tau_A^{-m} P_b\right)^{d'_{ab}} \oplus \operatorname{Hom}_A\left(P_i, \tau_A^{-m} X\right) \arrow[d] & \\ & & \operatorname{Hom}_A\left(P_i, \tau_A^{-m-1} P_a\right) \arrow[r] & 0\,, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_A\left(P_i, \tau_A^{-m} P_b\right) \arrow[r] & \operatorname{Hom}_A\left(P_i, \tau_A^{-m-1} P_a\right)^{d_{ab}} \oplus \operatorname{Hom}_A\left(P_i, \tau_A^{-m} Y\right) \arrow[d] & \\ & & \operatorname{Hom}_A\left(P_i, \tau_A^{-m-1} P_b\right) \arrow[r] & 0\,, \end{tikzcd}$$

for each $m \geq 1$. This gives the inequalities

$$d'_{ab} r_{ib} \leq 2 r_{ia} \qquad \text{and} \qquad d_{ab} r_{ia} \leq 2 r_{ib}.$$

Therefore, r_{ia} is infinite if and only if r_{ib} is infinite.

It follows from Theorem 6.2 that the preinjective component $\mathcal{Q}(A)$ contains an arrow

$$I_a \xrightarrow{(d'_{ab}, d_{ab})} I_b.$$

Then we have in $\operatorname{mod} A$ almost split sequences of the form

$$\begin{aligned} 0 \longrightarrow \tau_A I_b \longrightarrow I_a^{d_{ab}} \oplus U \longrightarrow I_b \longrightarrow 0, \\ 0 \longrightarrow \tau_A I_a \longrightarrow \left(\tau_A I_b\right)^{d'_{ab}} \oplus V \longrightarrow I_a \longrightarrow 0, \end{aligned}$$

with all indecomposable direct summands of U and V in $\mathcal{Q}(A)$ (see Lemma III.9.1 and Proposition III.9.6). Applying now the exact functor $\tau_A\colon \operatorname{mod}_{\mathcal{P}} A \to \operatorname{mod}_{\mathcal{I}} A$ (see Proposition 3.4), we obtain almost split sequences in $\operatorname{mod} A$ of the form

$$\begin{aligned} 0 \longrightarrow \tau_A^{m+1} I_b \longrightarrow \left(\tau_A^m I_a\right)^{d_{ab}} \oplus \tau_A^m U \longrightarrow \tau_A^m I_b \longrightarrow 0, \\ 0 \longrightarrow \tau_A^{m+1} I_a \longrightarrow \left(\tau_A^{m+1} I_b\right)^{d'_{ab}} \oplus \tau_A^m V \longrightarrow \tau_A^m I_a \longrightarrow 0, \end{aligned}$$

for each $m \geq 1$. Further, applying the exact functor $\operatorname{Hom}_A(-, I_j)\colon \operatorname{mod} A \to \operatorname{mod} K$, we obtain the exact sequences in $\operatorname{mod} K$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A\left(\tau_A^m I_b, I_j\right) \arrow[r] & \operatorname{Hom}_A\left(\tau_A^m I_a, I_j\right)^{d_{ab}} \oplus \operatorname{Hom}_A\left(\tau_A^m U, I_j\right) \arrow[dl, rounded corners] \\
& \operatorname{Hom}_A\left(\tau_A^{m+1} I_b, I_j\right) \arrow[r] & 0,
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A\left(\tau_A^m I_a, I_j\right) \arrow[r] & \operatorname{Hom}_A\left(\tau_A^{m+1} I_b, I_j\right)^{d'_{ab}} \oplus \operatorname{Hom}_A\left(\tau_A^m V, I_j\right) \arrow[dl, rounded corners] \\
& \operatorname{Hom}_A\left(\tau_A^{m+1} I_a, I_j\right) \arrow[r] & 0,
\end{tikzcd}$$

for each $m \geq 1$. This gives the inequalities

$$d_{ab} r_{aj} \leq 2 r_{bj} \qquad \text{and} \qquad d'_{ab} r_{bj} \leq 2 r_{aj}.$$

Therefore, r_{aj} is infinite if and only if r_{bj} is infinite. Since A is an indecomposable algebra, the quiver Q_A of A is connected (Corollary 1.7), and so the statements (i) and (ii) follow. □

Theorem 6.11. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K. The following statements hold:*

(i) *All but finitely many indecomposable modules in $\mathcal{P}(A)$ are sincere.*
(ii) *All but finitely many indecomposable modules in $\mathcal{Q}(A)$ are sincere.*

Proof. (i) Since A is of infinite representation type, the postprojective component $\mathcal{P}(A)$ is infinite, and different from $\mathcal{Q}(A)$, by Corollary 6.3. Let $P_1, \ldots, P_n$ be a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} A$, $S_1 = \operatorname{top}(P_1), \ldots, S_n = \operatorname{top}(P_n)$ the associated simple modules, and $f = \max\{\dim_K \operatorname{End}_K(S_i) \mid i \in \{1, \ldots, n\}\}$. It follows from Lemma 5.6 that, for any indecomposable module M in $\mathcal{P}(A)$, we have $\dim_K \operatorname{Hom}_A(P_i, M) = c_i(M) \dim_K \operatorname{End}_A(S_i)$.

Assume $\mathcal{P}(A)$ contains infinitely many nonsincere indecomposable modules. Then there is $a \in \{1, \ldots, n\}$ such that $\operatorname{Hom}_A(P_a, M) = 0$ for infinitely many indecomposable modules M in $\mathcal{P}(A)$. Invoking Theorem VIII.8.1 and Lemma VIII.8.3, we may assume that a is a source of the quiver Q_A of A. Let $H = (1_A - e_a) A (1_A - e_a)$. Then it follows from Example 10.6 that H is a finite dimensional hereditary K-algebra whose quiver Q_H is obtained from Q_A by removing the source a and the valued arrows attached to a. In fact, H is the quotient algebra A/Ae_aA, and $\operatorname{mod} H$ is the full subcategory of $\operatorname{mod} A$ given by all modules N with $\operatorname{Hom}_A(P_a, N) = 0$. Clearly, the algebra H is not necessarily indecomposable. But we may write $H = B \times C$, where B is an indecomposable finite dimensional hereditary K-algebra such that the postprojective component $\mathcal{P}(A)$ contains infinitely many indecomposable modules M_i, $i \in \mathbb{N}$, from $\operatorname{mod} B$. Observe that then B is of infinite representation type, and consequently the postprojective component $\mathcal{P}(B)$ is infinite, and different from $\mathcal{Q}(B)$. We claim that all the modules M_i, $i \in \mathbb{N}$, belong to $\mathcal{P}(B)$. We note first that every indecomposable module from $\operatorname{mod} B$ is an indecomposable module in $\operatorname{mod} A$. Hence it follows from Proposition 6.6 (iii) that every indecomposable module N in $\operatorname{mod} B$ such that $\operatorname{Hom}_B(N, M_i) \neq 0$, for some $i \in \mathbb{N}$, is a predecessor of M_i in $\mathcal{P}(A)$. In particular, for any $i \in \mathbb{N}$, there is a common bound on the length of paths of irreducible homomorphisms

$$Y_r \xrightarrow{f_r} Y_{r-1} \xrightarrow{f_{r-1}} \dots \longrightarrow Y_1 \xrightarrow{f_1} Y_0 = M_i$$

between indecomposable modules in $\operatorname{mod} B$ with $f_1 \dots f_r \neq 0$. Suppose that there is $i \in \mathbb{N}$ such that M_i does not lie in $\mathcal{P}(B)$. Take an indecomposable projective right B-module P with $\operatorname{Hom}_B(P, M_i) \neq 0$. Since P is in $\mathcal{P}(B)$, there is no finite path of irreducible homomorphisms in $\operatorname{mod} B$ from P to M_i. Then, applying Proposition III.10.1, we conclude that, for any positive integer t, there exist a path of irreducible homomorphisms

$$N_t \xrightarrow{g_t} N_{t-1} \xrightarrow{g_{t-1}} \dots \longrightarrow N_2 \xrightarrow{g_2} N_1 \xrightarrow{g_1} Y_0 = M_i$$

and a homomorphism $v_t \colon P \to N_t$ in $\operatorname{mod} B$ such that $g_1 \dots g_t v_t \neq 0$, a contradiction. Therefore, all the modules M_i, $i \in \mathbb{N}$, lie in $\mathcal{P}(B)$. Since the postprojective component $\mathcal{P}(B)$ is acyclic with finitely many τ_B-orbits, we conclude that every indecomposable module in $\mathcal{P}(B)$ is a predecessor of a module M_i. Then we infer that all indecomposable modules from $\mathcal{P}(B)$ lie in $\mathcal{P}(A)$, because $\mathcal{P}(A)$ is closed under predecessors in Γ_A.

Let $\operatorname{rad} P_a = R \oplus T$ be a decomposition in $\operatorname{mod} A$ with R a module in $\operatorname{mod} B$ and T a module in $\operatorname{mod} C$. Since Q_A is connected, we conclude that R is a nonzero projective module in $\operatorname{mod} B$. Since B is an indecomposable hereditary K-algebra of infinite representation type, it follows from Proposition 6.10 that there is an indecomposable module X in $\mathcal{P}(B)$ such that $\dim_K \operatorname{Hom}_A(R, X) \geq 3f$. Consider the canonical exact sequence

$$0 \longrightarrow \operatorname{rad} P_a \longrightarrow P_a \longrightarrow S_a \longrightarrow 0.$$

Applying Theorem 3.2 to the indecomposable A-module $\tau_B^{-1} X$, we obtain an exact sequence in $\operatorname{mod} K$ of the form

$$\begin{tikzcd}
\operatorname{Hom}_A\left(\tau_B^{-1}X, S_a\right) \arrow[r] & \operatorname{Ext}_A^1\left(\tau_B^{-1}X, \operatorname{rad} P_a\right) \arrow[dl, rounded corners] & & \\
& \operatorname{Ext}_A^1\left(\tau_B^{-1}X, P_a\right) \arrow[r] & \operatorname{Ext}_A^1\left(\tau_B^{-1}X, S_a\right) \arrow[r] & 0 .
\end{tikzcd}$$

Since a is a source of Q_A, we know from Lemma 1.13 that S_a is an injective module in mod A, and hence $\operatorname{Ext}_A^1(\tau_B^{-1}X, S_a) = 0$. Moreover, by Lemma 5.6, we have

$$\begin{aligned}\dim_K \operatorname{Hom}_A\left(\tau_B^{-1}X, S_a\right) &= \dim_K \operatorname{Hom}_A(\tau_B^{-1}X, I_a)\\ &= c_a\left(\tau_B^{-1}X\right)\dim_K \operatorname{End}_A(S_a) = 0,\end{aligned}$$

because S_a is not a composition factor of the B-module $\tau_B^{-1}X$. Hence $\operatorname{Ext}_A^1(\tau_B^{-1}X, P_a) \cong \operatorname{Ext}_A^1(\tau_B^{-1}X, \operatorname{rad} P_a)$ in mod K. Further, we have isomorphisms in mod K

$$\begin{aligned}\operatorname{Ext}_A^1\left(\tau_B^{-1}X, P_a\right) &\cong \operatorname{Ext}_A^1\left(\tau_B^{-1}X, R \oplus T\right) = \operatorname{Ext}_B^1\left(\tau_B^{-1}X, R\right)\\ &\cong D\operatorname{Hom}_B\left(R, \tau_B(\tau_B^{-1}X)\right) = D\operatorname{Hom}_B(R, X).\end{aligned}$$

This gives $\dim_K \operatorname{Ext}_A^1(\tau_B^{-1}X, P_a) = \dim_K D\operatorname{Hom}_B(R, X) \geq 3f$. Let

$$0 \longrightarrow P_a \xrightarrow{u} Z \xrightarrow{v} \tau_B^{-1}X \longrightarrow 0.$$

be a nonsplittable exact sequence in mod A. We claim that $\operatorname{End}_A(Z)$ is a division K-algebra, and consequently Z is an indecomposable module. Since the above exact sequence is nonsplittable, we have $\operatorname{rad}_A(Y, \tau_B^{-1}X) \neq 0$ for any indecomposable direct summand Y of Z, and hence Y is a proper predecessor of $\tau_B^{-1}X$ in $\mathcal{P}(A)$, by Proposition 6.6. In particular, we conclude that $\operatorname{Hom}_A(\tau_B^{-1}X, Z) = 0$, because the component $\mathcal{P}(A)$ is acyclic. Observe also that $\operatorname{Hom}_A(Z, P_a) = 0$. Indeed, if V is an indecomposable direct summand of Z with $\operatorname{Hom}_A(V, P_a) \neq 0$, then we have a cycle $P_a \xrightarrow{g} V \xrightarrow{h} P_a$ of nonzero homomorphisms, with g the composition of u with the projection homomorphism from Z on V. Since $\mathcal{P}(A)$ is an acyclic generalized standard component and V lies in $\mathcal{P}(A)$, we obtain that hg is an isomorphism. But then g is a section, which is not possible because the above exact sequence is nonsplittable. Take now a nonzero endomorphism $\varphi \in \operatorname{End}_A(Z)$. Then there is a commutative diagram in mod A with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P_a \arrow[r, "u"] \arrow[d, "\varphi'"] & Z \arrow[r, "v"] \arrow[d, "\varphi"] & \tau_B^{-1}X \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & P_a \arrow[r, "u"] & Z \arrow[r, "v"] & \tau_B^{-1}X \arrow[r] & 0\,.
\end{tikzcd}$$

The existence of the homomorphism $\varphi' \colon P_a \to P_a$ that satisfies $u\varphi' = \varphi u$ follows from the fact that $v\varphi u = 0$, because

$$\dim_K \operatorname{Hom}_A(P_a, \tau_B^{-1}X) = c_a(\tau_B^{-1}X)\dim_K \operatorname{End}_A(S_a) = 0$$

due to the fact that $\tau_B^{-1}X$ is a B-module. Clearly, $v\varphi u = 0$ implies that there exists a homomorphism $\varphi''\colon \tau_B^{-1}X \to \tau_B^{-1}X$ such that $v\varphi = \varphi'' v$. We prove now that $\varphi \neq 0$ forces $\varphi' \neq 0$ and $\varphi'' \neq 0$. Indeed, if $\varphi' = 0$, then $\varphi u = 0$ and hence $\varphi = \psi v$ for some homomorphism $\psi\colon\colon \tau_B^{-1}X \to Z$, which is impossible because $\varphi \neq 0$ and $\operatorname{Hom}_A(\tau_B^{-1}X, Z) = 0$. Similarly, if $\varphi'' = 0$, then $v\varphi = 0$, and so $\varphi = u\omega$ for some homomorphism $\omega\colon Z \to P_a$, which is again impossible, because $\varphi \neq 0$ and $\operatorname{Hom}_A(Z, P_a) = 0$. It follows now from Proposition 6.6 that $\operatorname{End}_A(P_a)$ and $\operatorname{End}_A(\tau_B^{-1}X)$ are division K-algebras, because P_a and $\tau_B^{-1}X$ are in $\mathcal{P}(A)$. This shows that φ' and φ'' are isomorphisms, and hence φ is an isomorphism, by Lemma 3.1. Therefore, $\operatorname{End}_A(Z)$ is a division K-algebra, and hence Z is an indecomposable module in $\mathcal{P}(A)$, which is a proper successor of P_a and a proper predecessor of $\tau_B^{-1}X$ in $\mathcal{P}(A)$. Moreover, $\chi_A([Z]) = \dim_K \operatorname{End}_A(Z) > 0$. Further, $\chi_A([P_a]) = \dim_K \operatorname{End}_A(P_a) = \dim_K \operatorname{End}_A(S_a)$ and $\chi_A([\tau_B^{-1}Z]) = \dim_K \operatorname{End}_A(S_b)$ for some $b \in \{1, \dots, n\}$, by Proposition 6.8. This leads to the inequalities

$$\begin{aligned}
\chi_A([Z]) &= \langle [Z], [Z] \rangle_A \\
&= \big\langle [P_a] + [\tau_B^{-1}X], [P_a] + [\tau_B^{-1}X] \big\rangle_A \\
&= \chi_A([P_a]) + \chi_A([\tau_B^{-1}X]) + \langle [P_a], [\tau_B^{-1}X] \rangle_A + \langle [\tau_B^{-1}X], [P_a] \rangle_A \\
&= \chi_A([P_a]) + \chi_A([\tau_B^{-1}X]) + \dim_K \operatorname{Hom}_A(P_a, \tau_B^{-1}X) \\
&\quad - \dim_K \operatorname{Ext}_A^1(P_a, \tau_B^{-1}X) + \dim_K \operatorname{Hom}_A(\tau_B^{-1}X, P_a) \\
&\quad - \dim_K \operatorname{Ext}_A^1(\tau_B^{-1}X, P_a) \\
&= \chi_A([P_a]) + \chi_A([\tau_B^{-1}X]) - \dim_K \operatorname{Ext}_A^1(\tau_B^{-1}X, P_a) \\
&\leq f + f - 3f = -f,
\end{aligned}$$

a contradiction.

The proof of (ii) is similar. □

7 Hereditary algebras of Dynkin type

In this section we will prove that an indecomposable finite dimensional hereditary algebra A over a field K is of finite representation type if and only if A is a hereditary algebra of Dynkin type.

Proposition 7.1. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K. Then the Euler form χ_A is not positive definite.*

Proof. Since A is of infinite representation type, it follows from Corollary 6.3 that $\mathcal{P}(A)$ and $\mathcal{Q}(A)$ are disjoint infinite components of Γ_A. In particular, all

τ_A-orbits in $\mathcal{P}(A)$ are infinite. In fact, by Proposition 6.6, for any indecomposable projective module P in $\mathcal{P}(A)$, the indecomposable modules $\tau_A^{-m}P$, $m \geq 0$, have pairwise different images $[\tau_A^{-m}P]$ in the Grothendieck group $K_0(A)$, and $\chi_A([\tau_A^{-m}P]) = \dim_K \operatorname{End}_A(P)$ for any $m \geq 0$. Hence, the Euler form χ_A takes the value $\dim_K \operatorname{End}_A(P)$ on infinitely many pairwise different positive vectors $[\tau_A^{-m}P]$, $m \geq 0$, of $K_0(A)$, identified with $\mathbb{Z}^n$ (in the canonical way).

Assume $\chi_A \colon \mathbb{Z}^n \to \mathbb{Z}$ is positive definite, so $\chi_A(X) > 0$ for any nonzero vectors $\mathbf{x} \in \mathbb{Z}^n$. Consider the natural extension $\chi_A^{\mathbb{Q}} \colon \mathbb{Q}^n \to \mathbb{Q}$ of χ_A to $\mathbb{Q}^n$. Then for any vector $\mathbf{y} \in \mathbb{Q}^n$ there exists a positive integer m such that $\mathbf{y} = \frac{1}{m}\mathbf{x}$ for a vector $\mathbf{x} \in \mathbb{Z}^n$, and

$$\chi_A^{\mathbb{Q}}(\mathbf{y}) = \chi_A^{\mathbb{Q}}\left(\frac{1}{m}\mathbf{x}\right) = \frac{1}{m^2}\chi_A^{\mathbb{Q}}(\mathbf{x}) = \frac{1}{m^2}\chi_A(\mathbf{x}).$$

Therefore, the quadratic form $\chi_A^{\mathbb{Q}} \colon \mathbb{Q}^n \to \mathbb{Q}$ is positive definite. Further, consider the natural extension of $\chi_A^{\mathbb{Q}}$ (and hence of χ_A) to the quadratic form $\chi_A^{\mathbb{R}} \colon \mathbb{R}^n \to \mathbb{R}$. Since $\mathbb{Q}^n$ is a dense subset of $\mathbb{R}^n$, $\chi_A^{\mathbb{Q}}$ is positive definite, and $\chi_A^{\mathbb{R}}$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$, we conclude that $\chi_A^{\mathbb{R}}(\mathbf{z}) \geq 0$ for any vector $\mathbf{z} \in \mathbb{Z}^n$. We claim that the quadratic form $\chi_A^{\mathbb{R}}$ is positive definite. Suppose $\chi_A^{\mathbb{R}}(\mathbf{z}^*) = 0$ for a nonzero vector $\mathbf{z}^* \in \mathbb{R}^n$. Then $\chi_A^{\mathbb{R}}$ attains a local minimum at $\mathbf{z}^*$, and consequently the partial derivatives $\frac{\partial \chi_A^{\mathbb{R}}}{\partial x_i}$, $0 \leq i \leq n$, vanish on $\mathbf{z}^*$. This implies that the real vector space

$$V = \left\{ \mathbf{z} \in \mathbb{R}^n \;\middle|\; \frac{\partial \chi_A^{\mathbb{R}}}{\partial x_i}(\mathbf{z}) = 0 \text{ for } i = 1, \ldots, n \right\}$$

is nonzero. Hence the rank of the $n \times n$ matrix (with rational coefficients) determining this system of linear equations is smaller than n. Then the rational vector space

$$U = \left\{ \mathbf{y} \in \mathbb{Q}^n \;\middle|\; \frac{\partial \chi_A^{\mathbb{Q}}}{\partial x_i}(\mathbf{y}) = 0 \text{ for } i = 1, \ldots, n \right\}$$

is nonzero. Thus there exists a nonzero vector $\mathbf{y}^* \in \mathbb{Q}^n$ such that $\frac{\partial \chi_A^{\mathbb{Q}}}{\partial x_i}(\mathbf{y}^*) = 0$ for all $i \in \{1, \ldots, n\}$. Clearly, for any $i \in \{1, \ldots, n\}$, we have $\frac{\partial \chi_A^{\mathbb{Q}}}{\partial x_i} = \frac{\partial \chi_A}{\partial x_i}$. Then there exist a positive integer m such that $\mathbf{x}^* = m\mathbf{y}^* \in \mathbb{Z}^n$ and $\frac{\partial \chi_A}{\partial x_i}(\mathbf{x}^*) = 0$ for any $i \in \{1, \ldots, n\}$. On the other hand, χ_A is given by

$$\chi_A(\mathbf{x}) = \sum_{i=1}^{n} f_i x_i^2 - \sum_{i,j=1}^{n} f_{ij} x_i x_j$$

for $\mathbf{x} \in \mathbb{Z}^n$. Since Q_A is acyclic, we may order a complete set of pairwise nonisomorphic simple modules $S_1 \ldots, S_n$ in $\operatorname{mod} A$ such that the numbers $f_{ij} =$

$\dim_K \operatorname{Ext}^1_A(S_i, S_j)$ satisfy the condition: $f_{ij} \neq 0$ implies $i > j$. Then χ_A is given by

$$\chi_A(\mathbf{x}) = \sum_{i=1}^{n} f_i x_i^2 - \sum_{1 \le j < i \le n} f_{ij} x_i x_j$$

for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$. Moreover, the associated Euler bilinear form $\langle -, - \rangle_A : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ is then given by

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \sum_{i=1}^{n} f_i x_i y_i - \sum_{1 \le j < i \le n} f_{ij} x_i y_j$$

for $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{Z}^n$. Then, for $i \in \{1, \dots, n\}$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{Z}^n$,

$$\langle e_i, \mathbf{y} \rangle_A + \langle \mathbf{y}, e_i \rangle_A = 2 f_i y_i - \sum_{1 \le j < i \le n} f_{ij} y_j - \sum_{1 \le i < j \le n} f_{ji} y_j = \frac{\partial \chi_A}{\partial x_i}(\mathbf{y}).$$

Hence, $\frac{\partial \chi_A}{\partial x_i}(\mathbf{x}^*) = 0$ implies $\langle e_i, \mathbf{x}^* \rangle_A + \langle \mathbf{x}^*, e_i \rangle_A = 0$ for any $i \in \{1, \dots, n\}$. Then we conclude that $2\chi_A(\mathbf{x}^*) = 2\langle \mathbf{x}^*, \mathbf{x}^* \rangle_A = 0$, a contradiction because $\mathbf{x}^*$ is a nonzero vector in $\mathbb{Z}^n$ and χ_A is assumed to be positive definite. Therefore, the quadratic form $\chi_A^{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}$ is positive definite. Let P be an indecomposable projective module in $\mathcal{P}(A)$ and

$$C_P = \Big\{ \mathbf{z} \in \mathbb{R}^n \mid \chi_A^{\mathbb{R}}(\mathbf{z}) \le \dim_K \operatorname{End}_A(P) \Big\}.$$

Observe that C_P is the preimage of the interval $[0, \dim_k \operatorname{End}_A(P)]$ in $\mathbb{R}$ under the continuous function $\chi_A^{\mathbb{R}}$, and consequently C_P is a compact subset of $\mathbb{R}^n$. But then the intersection $C_P \cap \mathbb{Z}^n$ is a finite set. On the other hand, $C_P \cap \mathbb{Z}^n$ contains the pairwise different vectors $[\tau_A^{-m} P]$, $m \ge 0$. This contradiction shows that χ_A is not positive definite. ☐

Lemma 7.2. *Let A be a finite dimensional hereditary K-algebra over a field K. The following conditions are equivalent:*

(i) *χ_A is positive definite.*

(ii) *$\chi_A([M]) > 0$ for all indecomposable modules M in* $\operatorname{mod} A$.

Proof. Clearly (i) implies (ii). We prove the converse implication. Take a nonzero vector $\mathbf{x}$ in $K_0(A)$. Then $\mathbf{x}$ has the expression $\mathbf{x} = [M] - [N]$, where M and N are semisimple modules in mod A with $\operatorname{Hom}_A(M,N) = 0$ and $\operatorname{Hom}_A(N,M) = 0$, by Lemma 3.13. It follows that

$$\begin{aligned}\chi_A(\mathbf{x}) &= \langle \mathbf{x}, \mathbf{x}\rangle_A = \langle [M]-[N], [M]-[N]\rangle_A \\ &= \langle [M],[M]\rangle_A - \langle [M],[N]\rangle_A - \langle [N],[M]\rangle_A + \langle [N],[N]\rangle_A \\ &= \chi_A([M]) + \chi_A([N]) + \dim_K \operatorname{Ext}^1_A(M,N) + \dim_K \operatorname{Ext}^1_A(N,M) \\ &\geq \chi_A([M]) + \chi_A([N]).\end{aligned}$$

Since $\mathbf{x} \neq 0$, we have $M \neq 0$ or $N \neq 0$, and hence $\chi_A([M]) > 0$ or $\chi_A([N]) > 0$. Thus $\chi_A(\mathbf{x}) > 0$. □

Proposition 7.3. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K such that χ_A is not positive definite. Then there exists an indecomposable module X in* mod A *such that* $\operatorname{Ext}^1_A(X,X) \neq 0$.

Proof. Since χ_A is not positive definite, we conclude from Lemma 7.2 that there exists a module M in mod A such that $\chi_A([M]) \leq 0$. We may take such a module M in mod A with $\dim_K \operatorname{End}_A(M)$ minimal. We claim that M admits an indecomposable direct summand X with $\operatorname{Ext}^1_A(X,X) \neq 0$.

Assume that $\operatorname{Ext}^1_A(N,N) = 0$ for every indecomposable direct summand N of M in mod A. Observe that

$$\dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M,M) = \chi_A([M]) \leq 0$$

forces $\operatorname{Ext}^1_A(M,M) \neq 0$. Then there exists an indecomposable direct summand X of M such that $\operatorname{Ext}^1_A(X,M) \neq 0$. Let $M = X \oplus Y$ for a module Y in mod A. Then

$$\operatorname{Ext}^1_A(X,M) = \operatorname{Ext}^1_A(X,X) \oplus \operatorname{Ext}^1_A(X,Y) = \operatorname{Ext}^1_A(X,Y).$$

Hence $\operatorname{Ext}^1_A(X,Y) \neq 0$, and so $\mathcal{E}\mathrm{xt}^1_A(X,Y) \neq 0$ (see Corollary III.3.6). Therefore, there exists a nonsplittable exact sequence

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

in mod A. Then $\dim_K \operatorname{End}_A(Z) < \dim_K \operatorname{End}_A(X \oplus Y) = \dim_K \operatorname{End}_A(M)$, by Lemma 3.12. This contradicts the choice of M, because $[Z] = [X] \oplus [Y] = [M]$. Thus $\operatorname{Ext}^1_A(X,X) \neq 0$ for some indecomposable summand X of M. □

Theorem 7.4. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K. The following conditions are equivalent:*

(i) *A is of finite representation type.*

(ii) *A is of Dynkin type.*

Proof. Assume A is of finite representation type. Then $\mathcal{P}(A) = \Gamma_A = \mathcal{Q}(A)$, by Corollary 6.3 and Theorem III.10.2. In particular, it follows from Proposition 6.6 that $\mathrm{Ext}_A^1(X, X) = 0$ for any indecomposable module X in $\operatorname{mod} A$. Hence, χ_A is positive definite, by Proposition 7.3. Applying now Theorem 4.8, we conclude that Q_A is a Dynkin quiver, and so A is a hereditary algebra of Dynkin type. This shows that (i) implies (ii).

Assume A is of Dynkin type. Then, applying Theorem 4.8, we conclude that χ_A is positive definite. Hence implication (ii) $\Rightarrow$ (i) follows from Proposition 7.1. □

We also note the following direct consequence of Propositions 6.6, 6.7, 7.3 and Theorems 4.8, 7.4.

Corollary 7.5. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K. Then Γ_A admits a connected component different from $\mathcal{P}(A)$ and $\mathcal{Q}(A)$.*

Theorem 7.6. *Let A be a finite dimensional hereditary K-algebra over a field K of one of the Dynkin types $\mathbb{A}_n (n \geq 1)$, $\mathbb{D}_n (n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$. Then there is a positive integer f such that the assignment $X \mapsto [X]$ induces a bijection between the isomorphism classes of indecomposable modules in* $\operatorname{mod} A$ *and the elements of the set*

$$\big\{\mathbf{x} \in K_0(A) \,\big|\, \mathbf{x} > 0 \text{ and } \chi_A(\mathbf{x}) = f\big\}.$$

Proof. Let $S_1, \dots, S_n$ be a complete set of pairwise nonisomorphic simple modules, and $f_i = \dim_K \mathrm{End}_K(S_i)$ for $i \in \{1, \dots, n\}$. Since Q_A is a Dynkin quiver of one of the Dynkin types $\mathbb{A}_n (n \geq 1)$, $\mathbb{D}_n (n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$, the valuations of arrows in Q_A are $(1, 1)$, and consequently we have $f_1 = \dots = f_n$ (see Lemma 1.1). Take $f = f_1$. Further, by Theorem 7.4 and Corollary 6.3, we have $\Gamma_A = \mathcal{P}(A)$. Then it follows from Proposition 6.8 that, for any indecomposable module X in $\operatorname{mod} A$, we have $\chi_A([X]) = f$. Moreover, by Proposition 6.6, for two indecomposable modules X and Y in $\mathcal{P}(A)$, we have $[X] = [Y]$ if and only if $X \cong Y$. Therefore, it remains to show that, for any positive vector $\mathbf{x}$ in $K_0(A)$ with $\chi_A(\mathbf{x}) = f$, there exists an indecomposable module X in $\operatorname{mod} A$ such that $\mathbf{x} = [X]$. Take $\mathbf{x} > 0$ in $K_0(A)$ with $\chi_A(\mathbf{x}) = f$. Applying Lemma 3.14, we conclude that there exist indecomposable modules $X_1, \dots, X_r$ in $\operatorname{mod} A$ such that

(1) $\mathbf{x} = [X]$ for $X = X_1 \oplus \dots \oplus X_r$;
(2) $\mathrm{Ext}_A^1(X_i, X_j) = 0$ for any $i \neq j$ in $\{1, \dots, r\}$.

It follows that

$$\begin{aligned}
f = \chi_A(\mathbf{x}) &= \chi_A([X]) = \langle [X],[X]\rangle_A \\
&= \left\langle \sum_{i=1}^{r}[X_i], \sum_{j=1}^{r}[X_j] \right\rangle_A = \sum_{i,j=1}^{r} \langle [X_i],[X_j]\rangle_A \\
&= \sum_{i,j=1}^{r} \big(\dim_K \operatorname{Hom}_A(X_i,X_j) - \dim_K \operatorname{Ext}_A^1(X_i,X_j)\big) \\
&= \sum_{i=1}^{r} \big(\dim_K \operatorname{End}_A(X_i) - \dim_K \operatorname{Ext}_A^1(X_i,X_i)\big) \\
&\quad + \sum_{i\neq j} \dim_K \operatorname{Hom}_A(X_i,X_j) \\
&\geq \sum_{i=1}^{r} \chi_A([X_i]) = rf > 0.
\end{aligned}$$

This implies $r = 1$, and consequently X is an indecomposable module in $\operatorname{mod} A$ with $[X] = \mathbf{x}$. □

Recall that every finite dimensional division K-algebra over an algebraically closed field K is isomorphic to K. Then we obtain the following direct consequence of Theorems 7.4 and 7.6, which is an essential part of the Gabriel's theorem from [Ga1].

Theorem 7.7. *Let A be an indecomposable finite dimensional hereditary K-algebra of finite representation type over an algebraically closed field K. Then the assignment $X \mapsto [X]$ induces a bijection between the isoclasses of indecomposable modules in* mod A *and the elements of the set*

$$R^+(A) = \big\{\mathbf{x} \in K_0(A) \,\big|\, \mathbf{x} > 0 \text{ and } \chi_A(\mathbf{x}) = 1\big\}.$$

The set $R^+(A)$ is called the set of *positive roots* of χ_A and its cardinality $|R^+(A)|$ is

$$\frac{n(n+1)}{2}, \quad n(n-1), \quad 36, \quad 63, \quad 120$$

if Q_A is of the Dynkin type $\mathbb{A}_n (n \geq 1)$, $\mathbb{D}_n (n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, respectively (see [ASS, Theorem VII.5.10]).

Example 7.8. Let A be the following $\mathbb{Q}$-subalgebra of the matrix algebra $M_2(\mathbb{R})$,

$$\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \in M_2(\mathbb{R}) \,\middle|\, a \in \mathbb{Q}, b, c \in \mathbb{Q}(\sqrt[3]{2}) \right\}.$$

The $\mathbb{Q}$-algebra A has the standard idempotents

$$e_1 = \begin{bmatrix} 1_{\mathbb{Q}} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathbb{Q}(\sqrt[3]{2})} \end{bmatrix}$$

such that $e_1 + e_2 = 1_A$. Hence we have in $\operatorname{mod} A$ two indecomposable projective modules $P_1 = e_1 A$, $P_2 = e_2 A$ and two indecomposable injective modules $I_1 = D(Ae_1)$, $I_2 = D(Ae_2)$, where $D = \operatorname{Hom}_{\mathbb{Q}}(-, \mathbb{Q})$ is the standard duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$. Further, we have $\dim_{\mathbb{Q}} P_1 = 1$, $\dim_{\mathbb{Q}} P_2 = 6$, $\dim_{\mathbb{Q}} I_1 = 4$, $\dim_{\mathbb{Q}} I_2 = 3$, and there are isomorphisms of $\mathbb{Q}$-vector spaces

$$\begin{aligned}
&\operatorname{Hom}_A(P_1, P_2) \cong e_2 A e_1 \cong \mathbb{Q}(\sqrt[3]{2}),\\
&\operatorname{Hom}_A(P_2, P_1) \cong e_1 A e_2 = 0,\\
&\operatorname{Hom}_A(I_1, I_2) \cong \operatorname{Hom}_{A^{\mathrm{op}}}(Ae_2, Ae_1) \cong e_2 A e_1 \cong \mathbb{Q}(\sqrt[3]{2}), \text{ and}\\
&\operatorname{Hom}_A(I_2, I_1) \cong \operatorname{Hom}_{A^{\mathrm{op}}}(Ae_1, Ae_2) \cong e_1 A e_2 = 0.
\end{aligned}$$

Moreover, $\operatorname{End}_A(P_1) \cong e_1 A e_1 \cong \mathbb{Q}$ and $\operatorname{End}_A(P_2) \cong e_2 A e_2 \cong \mathbb{Q}(\sqrt[3]{2})$. We also note that

$$\operatorname{rad} A = \begin{bmatrix} 0 & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & 0 \end{bmatrix},$$

and $\operatorname{rad} P_1 = e_1 \operatorname{rad} A = 0$, $\operatorname{rad} P_2 = e_2 \operatorname{rad} A = \mathbb{Q}(\sqrt[3]{2})$. In particular, for the simple modules S_1 and S_2 in $\operatorname{mod} A$ with $\operatorname{top}(P_1) = S_1 = \operatorname{soc}(I_1)$ and $\operatorname{top}(P_2) = S_2 = \operatorname{soc}(I_2)$, we have

$$S_1 = e_1 A / e_1 \operatorname{rad} A = e_1 A = P_1, \quad S_2 = e_2 A / e_2 \operatorname{rad} A,$$

and the division $\mathbb{Q}$-algebras

$$F_1 = \operatorname{End}_A(S_1) = \mathbb{Q} \quad \text{and} \quad F_2 = \operatorname{End}_A(S_2) = \mathbb{Q}(\sqrt[3]{2}).$$

Since $(\operatorname{rad} A)^2 = 0$, $e_1(\operatorname{rad} A)e_2 = 0$, $\dim_{F_1} e_2(\operatorname{rad} A)e_1 = \dim_{\mathbb{Q}} \mathbb{Q}(\sqrt[3]{2}) = 3$, and $\dim_{F_2} e_2(\operatorname{rad} A)e_1 = \dim_{\mathbb{Q}(\sqrt[3]{2})} \mathbb{Q}(\sqrt[3]{2}) = 1$, the quiver Q_A of A is of the form

$$1 \xleftarrow{(1,3)} 2\,,$$

so $d_{21} = 3$ and $d'_{21} = 1$. Therefore, A is a hereditary $\mathbb{Q}$-algebra of Dynkin type $\mathbb{G}_2$ and $\dim_{\mathbb{Q}} A = 7$. We identify the basis $[S_1], [S_2]$ of $K_0(A)$ with the standard basis $e_1 = (1, 0)$, $e_2 = (0, 1)$ of $\mathbb{Z}^2 = K_0(A)$. We note also $S_1 = P_1$ and $I_2 = S_2$, because 1 is a sink of Q_A and 2 is a source of Q_A (see Lemmas 1.11 and 1.13). Observe also that $\dim_{\mathbb{Q}} S_1 = 1$, $\dim_{\mathbb{Q}} S_2 = 3$, and $f_1 = \dim_{\mathbb{Q}} F_1 = 1$, $f_2 = \dim_{\mathbb{Q}} F_2 = 3$. Then the Euler quadratic form $\chi_A \colon \mathbb{Z}^2 \to \mathbb{Z}$ is given by

$$\chi_A(\mathbf{x}) = x_1^2 + 3x_2^2 - 3x_1 x_2$$

for $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$.

We determine now the Coxeter transformation $\varphi_A\colon K_0(A) \to K_0(A)$, in the standard basis e_1, e_2 of $K_0(A) = \mathbb{Z}^2$. By definition, φ_A is given by $\varphi_A([P_1]) = -[I_1]$ and $\varphi_A([P_2]) = -[I_2]$. Since $[P_1] = (1,0)$, $[P_2] = (3,1)$, $[I_1] = (1,1)$, $[I_2] = (0,1)$, we conclude that

$$\varphi_A(e_1) = -e_1 - e_2 \qquad \text{and} \qquad \varphi_A(e_2) = 3e_1 + 2e_2,$$

and consequently φ_A is given by

$$\varphi_A(\mathbf{x}) = (-x_1 + 3x_2, -x_1 + 2x_2)$$

for any $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$. Similarly, for the inverse Coxeter transformation $\varphi_A^{-1}\colon K_0(A) \to K_0(A)$, we have $\varphi_A^{-1}([I_1]) = -[P_1]$ and $\varphi_A^{-1}([I_2]) = -[P_2]$. Then

$$\varphi_A^{-1}(e_1) = 2e_1 + e_2 \qquad \text{and} \qquad \varphi_A^{-1}(e_2) = -3e_1 - e_2,$$

and so φ_A^{-1} is given by

$$\varphi_A^{-1}(\mathbf{x}) = (2x_1 - 3x_2, x_1 - x_2)$$

for $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$.

Recall that, for an indecomposable module M in mod A, we have

- $\varphi_A([M]) = [\tau_A M]$ if M is nonprojective;
- $\varphi_A([M]) < 0$ if and only if M is projective;
- $\varphi_A^{-1}([M]) = [\tau_A^{-1} M]$ if M is noninjective;
- $\varphi_A^{-1}([M]) < 0$ if and only if M is injective.

In particular, we obtain

$$\begin{array}{rrr}
\varphi_A^{-1}([P_1]) = (2,1), & \varphi_A^{-2}([P_1]) = (1,1), & \varphi_A^{-3}([P_1]) = (-1,0), \\
\varphi_A^{-1}([P_2]) = (3,2), & \varphi_A^{-2}([P_2]) = (0,1), & \varphi_A^{-3}([P_2]) = (-3,-1), \\
\varphi_A([I_1]) = (2,1), & \varphi_A^{2}([I_1]) = (1,0), & \varphi_A^{3}([I_1]) = (-1,-1), \\
\varphi_A([I_2]) = (3,2), & \varphi_A^{2}([I_2]) = (3,1), & \varphi_A^{3}([I_2]) = (0,-1).
\end{array}$$

Hence, we conclude that

$$\begin{array}{rrr}
[P_1] = (1,0), & [\tau_A^{-1} P_1] = (2,1), & [\tau_A^{-2} P_1] = (1,1) = [I_1], \\
[P_2] = (3,1), & [\tau_A^{-1} P_1] = (3,2), & [\tau_A^{-2} P_2] = (0,1) = [I_2], \\
[I_1] = (1,1), & [\tau_A I_1] = (2,1), & [\tau_A^{2} I_1] = (1,0) = [P_1], \\
[I_2] = (0,1), & [\tau_A I_2] = (3,2), & [\tau_A^{2} I_2] = (3,1) = [P_2].
\end{array}$$

Since every indecomposable module M in $\mathcal{P}(A) = \mathcal{Q}(A)$ is uniquely determined (up to isomorphism) by its class $[M]$ in $K_0(A)$, we obtain that

$$P_1, \quad P_2, \quad \tau_A^{-1}P_1 = \tau_A I_1, \quad \tau_A^{-1}P_2 = \tau_A I_2, \quad I_1, \quad I_2$$

form a complete family of pairwise nonisomorphic indecomposable modules in $\operatorname{mod} A$, and the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
 & P_2 \arrow[dr, "(3{,}1)"] & & \tau_A^{-1}P_2 \arrow[dr, "(3{,}1)"] & & \tau_A^{-2}P_2 = I_2 = S_2 \\
S_1 = P_1 \arrow[ur, "(1{,}3)"] & & \tau_A^{-1}P_1 \arrow[ur, "(1{,}3)"] & & \tau_A^{-2}P_1 = I_1 \arrow[ur, "(1{,}3)"] &
\end{tikzcd}$$

Example 7.9. Let K be a field, Q the quiver

$$\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\alpha"'] & \underset{3}{\bullet} \arrow[l, "\beta"'] \arrow[r, "\gamma"] & \underset{4}{\bullet}
\end{tikzcd}$$

and $A = KQ$ the path algebra of Q over K. Then A is a hereditary K-algebra of Dynkin type $\mathbb{A}_4$, with $\dim_K A = 8$, and Q is the quiver Q_A of A. Let P_1, P_2, P_3, P_4 and I_1, I_2, I_3, I_4 be the indecomposable projective modules and the indecomposable injective modules in $\operatorname{mod} A$, respectively, associated to the vertices $1, 2, 3, 4$ of Q. Moreover, let S_1, S_2, S_3, S_4 be the simple modules given by the vertices $1, 2, 3, 4$ of Q, so we have $\operatorname{top}(P_i) = S_i = \operatorname{soc}(I_i)$ for any $i \in \{1, 2, 3, 4\}$. We identify the basis $[S_1], [S_2], [S_3], [S_4]$ with the standard basis e_1, e_2, e_3, e_4 of $\mathbb{Z}^4$. Then we have

$$[P_1] = (1,0,0,0), \ [P_2] = (1,1,0,0), \ [P_3] = (1,1,1,1), \ [P_4] = (0,0,0,1),$$
$$[I_1] = (1,1,1,0), \ [I_2] = (0,1,1,0), \ [I_3] = (0,0,1,0), \ [I_4] = (0,0,1,1).$$

We determine the inverse Coxeter transformation φ_A^{-1} of A in the standard basis of $K_0(A) = \mathbb{Z}^4$. By definition,

$$\varphi_A^{-1}([I_1]) = -[P_1], \quad \varphi_A^{-1}([I_2]) = -[P_2],$$
$$\varphi_A^{-1}([I_3]) = -[P_3], \quad \varphi_A^{-1}([I_4]) = -[P_4].$$

This leads to the equalities

$$\varphi_A^{-1}(e_1) + \varphi_A^{-1}(e_2) + \varphi_A^{-1}(e_3) = -e_1, \quad \varphi_A^{-1}(e_2) + \varphi_A^{-1}(e_3) = -e_1 - e_2,$$
$$\varphi_A^{-1}(e_3) = -e_1 - e_2 - e_3 - e_4, \quad \varphi_A^{-1}(e_3) + \varphi_A^{-1}(e_4) = -e_4.$$

Hence we easily deduce that $\varphi_A^{-1}\colon \mathbb{Z}^4 \to \mathbb{Z}^4$ is given by

$$\varphi_A^{-1}(\mathbf{x}) = (-x_3 + x_4, x_1 - x_3 + x_4, x_2 - x_3 + x_4, x_2 - x_3)$$

for any $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$. Then we obtain the sequences of vectors in $K_0(A) = \mathbb{Z}^4$

$$\begin{aligned}
&\varphi_A^{-1}([P_1]) = (0,1,0,0), && \varphi_A^{-2}([P_1]) = (0,0,1,1),\\
&\varphi_A^{-3}([P_1]) = (0,0,0,-1), && \varphi_A^{-1}([P_2]) = (0,1,1,1),\\
&\varphi_A^{-2}([P_2]) = (0,0,1,0), && \varphi_A^{-3}([P_2]) = (-1,-1,-1,-1),\\
&\varphi_A^{-1}([P_3]) = (0,1,1,0), && \varphi_A^{-2}([P_3]) = (-1,-1,0,0),\\
&\varphi_A^{-1}([P_4]) = (1,1,1,0), && \varphi_A^{-2}([P_4]) = (-1,0,0,0).
\end{aligned}$$

Recall that, for an indecomposable module M in $\operatorname{mod} A$, we have

- $\varphi_A^{-1}([M]) = [\tau_A^{-1}M]$ if M is noninjective;
- $\varphi_A^{-1}([M]) < 0$ if and only if M is injective.

Hence we conclude that

$$\begin{gathered}
[P_1] = (1,0,0,0),\ [\tau_A^{-1}P_1] = (0,1,0,0) = [S_2],\ [\tau_A^{-2}P_1] = (0,0,1,1) = [I_4],\\
[P_2] = (1,1,0,0), \quad [\tau_A^{-1}P_2] = (0,1,1,1), \quad [\tau_A^{-2}P_2] = (0,0,1,0) = [I_3],\\
[P_3] = (1,1,1,1), \quad [\tau_A^{-1}P_3] = (0,1,1,0) = [I_2],\\
[P_4] = (0,0,0,1), \quad [\tau_A^{-1}P_4] = (1,1,1,0) = [I_1].
\end{gathered}$$

Since every indecomposable module M in $\mathcal{P}(A)$ is uniquely determined by its class $[M]$ in $K_0(A)$, it follows that

$$\begin{gathered}
S_1 = P_1, \quad \tau_A^{-1}P_1 = S_2, \quad \tau_A^{-2}P_1 = I_4, \quad P_2, \quad \tau_A^{-1}P_2 = P_3/S_1,\\
\tau_A^{-2}P_2 = I_3 = S_3, \quad P_3, \quad \tau_A^{-1}P_3 = I_2, \quad P_4 = S_4, \quad \tau_A^{-1}P_4 = I_1
\end{gathered}$$

form a complete set of pairwise nonisomorphic indecomposable modules in $\operatorname{mod} A$, and the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
 & S_4 = P_4 \arrow[dr] & & \tau_A^{-1}P_4 = I_1 \arrow[dr] & & \\
 & & P_3 \arrow[ur] \arrow[dr] & & \tau_A^{-1}P_3 = I_2 \arrow[dr] & \\
 & P_2 \arrow[ur] \arrow[dr] & & \tau_A^{-1}P_2 = P_3/S_1 \arrow[ur] \arrow[dr] & & \tau_A^{-2}P_2 = I_3 = S_3 \\
S_1 = P_1 \arrow[ur] & & \tau_A^{-1}P_1 = S_2 \arrow[ur] & & \tau_A^{-2}P_1 = I_4 \arrow[ur] &
\end{tikzcd} .$$

Observe that the full subquiver of Γ_A given by the indecomposable projective modules P_1, P_2, P_3, P_4 (respectively, the indecomposable injective modules I_1, I_2, I_3, I_4) is isomorphic to the opposite quiver $Q_A^{\mathrm{op}} = Q^{\mathrm{op}}$ of Q.

Example 7.10. Let A be the following $\mathbb{R}$-subalgebra of the matrix algebra $M_4(\mathbb{C})$

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{R} & 0 \\ 0 & 0 & \mathbb{R} & \mathbb{R} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ x & b & y & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & z & d \end{bmatrix} \in M_4(\mathbb{C}) \;\middle|\; \begin{matrix} a, b, x, y \in \mathbb{C} \\ c, d, z \in \mathbb{R} \end{matrix} \right\}.$$

Then A has the standard basic primitive idempotents

$$e_1 = \begin{bmatrix} 1_{\mathbb{C}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1_{\mathbb{C}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{\mathbb{R}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{\mathbb{R}} \end{bmatrix}$$

with $1_A = e_1 + e_2 + e_3 + e_4$. We claim that

$$\operatorname{rad} A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{C} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{R} & 0 \end{bmatrix}.$$

Indeed, the right side J of the above equality is a two-sided ideal of A with $J^2 = 0$ and A/J isomorphic to the product $\mathbb{C} \times \mathbb{C} \times \mathbb{R} \times \mathbb{R}$ of division $\mathbb{R}$-algebras (see Lemma I.3.5). Further, we obtain

$$\begin{aligned} &e_1 \operatorname{rad} A = 0, \quad && e_2 \operatorname{rad} A = e_2(\operatorname{rad} A)e_1 \oplus e_2(\operatorname{rad} A)e_3 = \mathbb{C} \oplus \mathbb{C}, \\ &e_3 \operatorname{rad} A = 0, \quad && e_4 \operatorname{rad} A = e_4(\operatorname{rad} A)e_3 = \mathbb{R}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &F_1 = e_1Ae_1/e_1(\operatorname{rad} A)e_1 = \mathbb{C}, \quad && F_2 = e_2Ae_2/e_2(\operatorname{rad} A)e_2 = \mathbb{C}, \\ &F_3 = e_3Ae_3/e_3(\operatorname{rad} A)e_3 = \mathbb{R}, \quad && F_4 = e_4Ae_4/e_4(\operatorname{rad} A)e_4 = \mathbb{R}. \end{aligned}$$

Therefore, the quiver Q_A of A is of the form

$$1 \longleftarrow 2 \xrightarrow{(2,1)} 3 \longleftarrow 4.$$

Let $P_1 = e_1A$, $P_2 = e_2A$, $P_3 = e_3A$, $P_4 = e_4A$ be the indecomposable projective modules and $I_1 = D(Ae_1)$, $I_2 = D(Ae_2)$, $I_3 = D(Ae_3)$, $I_4 = D(Ae_4)$ the indecomposable injective modules in mod A given by the idempotents e_1, e_2, e_3, e_4. Observe that P_1 and P_3 are simple modules, given by the sinks 1 and 3 of Q_A, and $\operatorname{rad} P_2 = P_1 \oplus P_3 \oplus P_3$, $\operatorname{rad} P_4 = P_3$, and hence A is a hereditary $\mathbb{R}$-algebra of Dynkin type $\mathbb{F}_4$ and $\dim_{\mathbb{R}} A = 11$. We note also that I_2 and I_4 are simple injective modules, given by the sources 2 and 4 of Q_A, and $I_1/S_1 = I_2 = S_2$, $I_3/S_3 = I_2 \oplus I_4 = S_2 \oplus S_4$. We identify $K_0(A)$ with $\mathbb{Z}^4$ and the basis $[S_1], [S_2], [S_3], [S_4]$ of $K_0(A)$ with the standard basis e_1, e_2, e_3, e_4 of $\mathbb{Z}^4$. Then we have

$$[P_1] = (1,0,0,0),\ \ [P_2] = (1,1,2,0),\ \ [P_3] = (0,0,1,0),\ \ [P_4] = (0,0,1,1),$$
$$[I_1] = (1,1,0,0),\ \ [I_2] = (0,1,0,0),\ \ [I_3] = (0,1,1,1),\ \ [I_4] = (0,0,0,1).$$

We determine the inverse Coxeter transformation φ_A^{-1} of A. Since $\varphi_A^{-1}([I_i]) = -[P_i]$ for any $i \in \{1,2,3,4\}$, we obtain the equalities

$$\varphi_A^{-1}(e_1) + \varphi_A^{-1}(e_2) = -e_1, \quad \varphi_A^{-1}(e_2) = -e_1 - e_2 - 2e_3,$$
$$\varphi_A^{-1}(e_2) + \varphi_A^{-1}(e_3) + \varphi_A^{-1}(e_4) = -e_3, \quad \varphi_A^{-1}(e_4) = -e_3 - e_4.$$

Hence we deduce that $\varphi_A^{-1} \colon \mathbb{Z}^4 \to \mathbb{Z}^4$ is given by

$$\varphi_A^{-1}(\mathbf{x}) = (-x_2 + x_3, x_1 - x_2 + x_3, 2x_1 - 2x_2 + 2x_3 - x_4, x_3 - x_4)$$

for any $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$. This yields the sequences of vectors in $K_0(A) = \mathbb{Z}^4$

$$\varphi_A^{-1}([P_1]) = (0,1,2,0), \quad \varphi_A^{-2}([P_1]) = (1,1,2,2),$$
$$\varphi_A^{-3}([P_1]) = (1,2,2,0), \quad \varphi_A^{-4}([P_1]) = (0,1,2,2),$$
$$\varphi_A^{-5}([P_1]) = (1,1,0,0), \quad \varphi_A^{-6}([P_1]) = (-1,0,0,0),$$
$$\varphi_A^{-1}([P_2]) = (1,2,4,2), \quad \varphi_A^{-2}([P_2]) = (2,3,4,2),$$
$$\varphi_A^{-3}([P_2]) = (1,3,4,2), \quad \varphi_A^{-4}([P_2]) = (1,2,2,2),$$
$$\varphi_A^{-5}([P_2]) = (0,1,0,0), \quad \varphi_A^{-6}([P_2]) = (-1,-1,-2,0),$$
$$\varphi_A^{-1}([P_3]) = (1,1,2,1), \quad \varphi_A^{-2}([P_3]) = (1,2,3,1),$$
$$\varphi_A^{-3}([P_3]) = (1,2,3,2), \quad \varphi_A^{-4}([P_3]) = (1,2,2,1),$$
$$\varphi_A^{-5}([P_3]) = (0,1,1,1), \quad \varphi_A^{-6}([P_3]) = (0,0,-1,0),$$
$$\varphi_A^{-1}([P_4]) = (1,1,1,0), \quad \varphi_A^{-2}([P_4]) = (0,1,2,1),$$
$$\varphi_A^{-3}([P_4]) = (1,1,1,1), \quad \varphi_A^{-4}([P_4]) = (0,1,1,0),$$
$$\varphi_A^{-5}([P_4]) = (0,0,0,1), \quad \varphi_A^{-6}([P_4]) = (0,0,-1,-1).$$

Therefore, the indecomposable modules

$$\tau_A^{-r} P_i, \;\; i \in \{1,2,3,4\}, \;\; r \in \{0,1,2,3,4,5\},$$

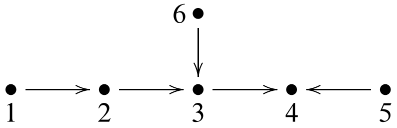
form a complete set of pairwise nonisomorphic indecomposable modules in mod A, and their composition vectors

$$[\tau_A^{-r} P_i] = \varphi_A^{-r}([P_i]), \;\; i \in \{1,2,3,4\}, \;\; r \in \{0,1,2,3,4,5\},$$

are described above. Moreover, the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
 & P_4 \arrow[dr] & & \tau_A^{-1} P_4 \arrow[dr] & & \tau_A^{-2} P_4 \arrow[dr] & & \tau_A^{-3} P_4 \arrow[dr] & & \tau_A^{-4} P_4 \arrow[dr] & & \tau_A^{-5} P_4 = I_4 \\
P_3 \arrow[ur] \arrow[dr, "(1,2)"'] & & \tau_A^{-1} P_3 \arrow[ur] \arrow[dr, "(1,2)"'] & & \tau_A^{-2} P_3 \arrow[ur] \arrow[dr, "(1,2)"'] & & \tau_A^{-3} P_3 \arrow[ur] \arrow[dr, "(1,2)"'] & & \tau_A^{-4} P_3 \arrow[ur] \arrow[dr, "(1,2)"'] & & \tau_A^{-5} P_3 = I_3 \arrow[ur] \arrow[dr, "(1,2)"] & \\
 & P_2 \arrow[ur, "(2,1)"'] \arrow[dr] & & \tau_A^{-1} P_2 \arrow[ur, "(2,1)"'] \arrow[dr] & & \tau_A^{-2} P_2 \arrow[ur, "(2,1)"'] \arrow[dr] & & \tau_A^{-3} P_2 \arrow[ur, "(2,1)"'] \arrow[dr] & & \tau_A^{-4} P_2 \arrow[ur, "(2,1)"'] \arrow[dr] & & \tau_A^{-5} P_2 = I_2 \\
P_1 \arrow[ur] & & \tau_A^{-1} P_1 \arrow[ur] & & \tau_A^{-2} P_1 \arrow[ur] & & \tau_A^{-3} P_1 \arrow[ur] & & \tau_A^{-4} P_1 \arrow[ur] & & \tau_A^{-5} P_1 = I_1 \, . \arrow[ur] &
\end{tikzcd}$$

Example 7.11. Let K be a field, F a finite dimensional division K-algebra, and Q the Dynkin quiver



of type $\mathbb{E}_6$. Consider the K-species $\mathbb{M} = (F_i, {}_iM_j)_{1 \le i,j \le 6}$ defined as follows: $F_i = F$ for any $i \in \{1, \dots, 6\}$, and

$${}_1M_2 = F, \;\; {}_2M_3 = F, \;\; {}_3M_4 = F, \;\; {}_5M_4 = F, \;\; {}_6M_3 = F$$

are the unique nonzero F_i-F_j-bimodules ${}_iM_j$. Clearly, Q is the quiver $Q_{\mathbb{M}}$ of $\mathbb{M}$. It follows from Theorem 2.2 that the tensor algebra $A = T(\mathbb{M})$ of the K–species $\mathbb{M}$ is a finite dimensional hereditary K-algebra such that $Q_A = Q_{T(\mathbb{M})} = Q_{\mathbb{M}}$. In particular, A is a hereditary K-algebra of Dynkin type $\mathbb{E}_6$. We will describe the composition vectors of indecomposable modules in mod A as well as the Auslander–Reiten quiver Γ_A of A. We denote by P_i, I_i, and S_i the indecomposable projective modules, the indecomposable injective modules, and the simple modules in mod A associated to the vertices $i \in \{1, \dots, 6\}$ of $Q = Q_A$, respectively. Hence we have $\mathrm{top}(P_i) = S_i = \mathrm{soc}(I_i)$ for any $i \in \{1, \ldots, 6\}$ Moreover, we identify $K_0(A)$ with $\mathbb{Z}^6$ and the canonical basis $[S_1], \ldots, [S_6]$ of $K_0(A)$ with the standard basis $e_1, \ldots, e_6$ of $\mathbb{Z}^6$. Then we have

$$\begin{aligned} [P_1] &= (1,1,1,1,0,0), & [P_2] &= (0,1,1,1,0,0), & [P_3] &= (0,0,1,1,0,0),\\ [P_4] &= (0,0,0,1,0,0), & [P_5] &= (0,0,0,1,1,0), & [P_6] &= (0,0,1,1,0,1),\\ [I_1] &= (1,0,0,0,0,0), & [I_2] &= (1,1,0,0,0,0), & [I_3] &= (1,1,1,0,0,1),\\ [I_4] &= (1,1,1,1,1,1), & [I_5] &= (0,0,0,0,1,0), & [I_6] &= (0,0,0,0,0,1). \end{aligned}$$

Observe that $P_4 = S_4$, $I_1 = S_1$, $I_5 = S_5$ and $I_6 = S_6$.

We determine the inverse Coxeter transformation φ_A^{-1} of A in the standard basis $e_1, \ldots, e_6$ of $K_0(A) = \mathbb{Z}^6$. By definition, we have $\varphi_A^{-1}([I_i]) = -[P_i]$ for any $i \in \{1, \ldots, 6\}$. This leads to the equalities

$$\varphi_A^{-1}(e_1) = -e_1 - e_2 - e_3 - e_4, \quad \varphi_A^{-1}(e_1) + \varphi_A^{-1}(e_2) = -e_2 - e_3 - e_4,$$
$$\varphi_A^{-1}(e_1) + \varphi_A^{-1}(e_2) + \varphi_A^{-1}(e_3) + \varphi_A^{-1}(e_6) = -e_3 - e_4,$$
$$\varphi_A^{-1}(e_1) + \varphi_A^{-1}(e_2) + \varphi_A^{-1}(e_3) + \varphi_A^{-1}(e_4) + \varphi_A^{-1}(e_5) + \varphi_A^{-1}(e_6) = -e_4,$$
$$\varphi_A^{-1}(e_5) = -e_4 - e_5, \quad \varphi_A^{-1}(e_6) = -e_3 - e_4 - e_6.$$

Then we easily calculate that $\varphi_A^{-1}\colon \mathbb{Z}^6 \to \mathbb{Z}^6$ is given by $\varphi_A^{-1}(\mathbf{x}) = (-x_1 + x_2, -x_1 + x_3, -x_1 + x_3 + x_4 - x_6, -x_1 + x_3 + x_4 - x_5 - x_6, x_4 - x_5, x_3 - x_6)$ for any $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6$. We obtain the sequences of vectors in $K_0(A) = \mathbb{Z}^6$

$$\begin{aligned}
&\varphi_A^{-1}([P_1]) = (0,0,1,1,1,1), && \varphi_A^{-2}([P_1]) = (0,1,1,0,0,0),\\
&\varphi_A^{-3}([P_1]) = (1,1,1,1,0,1), && \varphi_A^{-4}([P_1]) = (0,0,0,0,1,0),\\
&\varphi_A^{-5}([P_1]) = (0,0,0,-1,-1,0), && \\
&\varphi_A^{-1}([P_2]) = (1,1,2,2,1,1), && \varphi_A^{-2}([P_2]) = (0,1,2,1,1,1),\\
&\varphi_A^{-3}([P_2]) = (1,2,2,1,0,1), && \varphi_A^{-4}([P_2]) = (1,1,1,1,1,1),\\
&\varphi_A^{-5}([P_2]) = (0,0,0,-1,0,0), && \\
&\varphi_A^{-1}([P_3]) = (0,1,2,2,1,1), && \varphi_A^{-2}([P_3]) = (1,2,3,2,1,1),\\
&\varphi_A^{-3}([P_3]) = (1,2,3,2,1,2), && \varphi_A^{-4}([P_3]) = (1,2,2,1,1,1),\\
&\varphi_A^{-5}([P_3]) = (1,1,1,0,0,1), && \varphi_A^{-6}([P_3]) = (0,0,-1,-1,0,0),\\
&\varphi_A^{-1}([P_4]) = (0,0,1,1,1,0), && \varphi_A^{-2}([P_4]) = (0,1,2,1,0,1),\\
&\varphi_A^{-3}([P_4]) = (1,2,2,2,1,1), && \varphi_A^{-4}([P_4]) = (1,1,2,1,1,1),\\
&\varphi_A^{-5}([P_4]) = (0,1,1,0,0,1), && \varphi_A^{-6}([P_4]) = (1,1,0,0,0,0),\\
&\varphi_A^{-7}([P_4]) = (0,-1,-1,-1,0,0), &&
\end{aligned}$$

$$\begin{aligned}
&\varphi_A^{-1}([P_5]) = (0,0,1,0,0,0), && \varphi_A^{-2}([P_5]) = (0,1,1,1,0,1),\\
&\varphi_A^{-3}([P_5]) = (1,1,1,1,1,0), && \varphi_A^{-4}([P_5]) = (0,0,1,0,0,1),\\
&\varphi_A^{-5}([P_5]) = (0,1,0,0,0,0), && \varphi_A^{-6}([P_5]) = (1,0,0,0,0,0),\\
&\varphi_A^{-7}([P_5]) = (-1,-1,-1,-1,0,0), &&\\
&\varphi_A^{-1}([P_6]) = (0,1,1,1,1,0), && \varphi_A^{-2}([P_6]) = (1,1,2,1,0,1),\\
&\varphi_A^{-3}([P_6]) = (0,1,1,1,1,1), && \varphi_A^{-4}([P_6]) = (1,1,1,0,0,0),\\
&\varphi_A^{-5}([P_6]) = (0,0,0,0,0,1), && \varphi_A^{-6}([P_6]) = (0,0,-1,-1,0,-1),
\end{aligned}$$

Therefore, the vectors

$$\begin{aligned}
&[\varphi_A^{-r} P_1] = \varphi_A^{-r}([P_1]),\quad [\varphi_A^{-r} P_2] = \varphi_A^{-r}([P_2]), \quad \text{for } r \in \{0,1,2,3,4\},\\
&[\varphi_A^{-r} P_3] = \varphi_A^{-r}([P_3]),\quad [\varphi_A^{-r} P_6] = \varphi_A^{-r}([P_6]), \quad \text{for } r \in \{0,1,2,3,4,5\},\\
&[\varphi_A^{-r} P_4] = \varphi_A^{-r}([P_4]),\quad [\varphi_A^{-r} P_5] = \varphi_A^{-r}([P_5]), \quad \text{for } r \in \{0,1,2,3,4,5,6\},
\end{aligned}$$

form a complete family of composition vectors of indecomposable modules in mod A and the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}[column sep=tiny]
 & & & P_1 \arrow[dr] & & \tau_A^{-1}P_1 \arrow[dr] & & \tau_A^{-2}P_1 \arrow[dr] & & \tau_A^{-3}P_1 \arrow[dr] & & \tau_A^{-4}P_1 = I_5 & & \\
 & & P_2 \arrow[ur] \arrow[dr] & & \tau_A^{-1}P_2 \arrow[ur] \arrow[dr] & & \tau_A^{-2}P_2 \arrow[ur] \arrow[dr] & & \tau_A^{-3}P_2 \arrow[ur] \arrow[dr] & & \tau_A^{-4}P_2 = I_4 \arrow[ur] \arrow[dr] & & I_3 & \\
 & P_3 \arrow[r] \arrow[ur] \arrow[dr] & P_6 \arrow[r] & \tau_A^{-1}P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A^{-1}P_6 \arrow[r] & \tau_A^{-2}P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A^{-2}P_6 \arrow[r] & \tau_A^{-3}P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A^{-3}P_6 \arrow[r] & \tau_A^{-4}P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A^{-4}P_6 \arrow[r] & \tau_A^{-5}P_3 \arrow[r] \arrow[ur, equal] \arrow[dr] & \tau_A^{-5}P_6 = I_6 & \\
P_4 \arrow[ur] \arrow[dr] & & \tau_A^{-1}P_4 \arrow[ur] \arrow[dr] & & \tau_A^{-2}P_4 \arrow[ur] \arrow[dr] & & \tau_A^{-3}P_4 \arrow[ur] \arrow[dr] & & \tau_A^{-4}P_4 \arrow[ur] \arrow[dr] & & \tau_A^{-5}P_4 \arrow[ur] \arrow[dr] & & \tau_A^{-6}P_4 = I_2 \arrow[dr] & \\
 & P_5 \arrow[ur] & & \tau_A^{-1}P_5 \arrow[ur] & & \tau_A^{-2}P_5 \arrow[ur] & & \tau_A^{-3}P_5 \arrow[ur] & & \tau_A^{-4}P_5 \arrow[ur] & & \tau_A^{-5}P_5 \arrow[ur] & & \tau_A^{-6}P_5 = I_1
\end{tikzcd}$$

In particular, we conclude that there are 36 isomorphism classes of indecomposable modules in mod A. We also note that the Euler form $\chi_A\colon \mathbb{Z}^6 \to \mathbb{Z}$ is given by

$$\begin{aligned}
\chi_A(\mathbf{x}) &= \sum_{i=1}^{6} f x_i^2 - f x_1x_2 - f x_2x_3 - f x_3x_4 - f x_4x_5 - f x_3x_6\\
&= f\left(\sum_{i=1}^{6} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_3x_6\right)\\
&= f\chi_{KQ}(\mathbf{x}),
\end{aligned}$$

where $f = \dim_K F$ and χ_{KQ} is the Euler form of the path algebra KQ of the quiver Q over K. Hence

$$\{\mathbf{x} \in \mathbb{N}^6 \mid \chi_A(\mathbf{x}) = f\} = \{\mathbf{x} \in \mathbb{N}^6 \mid \chi_{KQ}(\mathbf{x}) = 1\} = R^+(A),$$

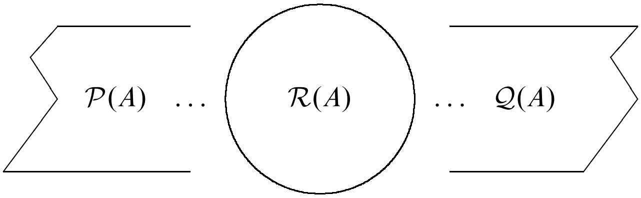
and $R^+(A)$ consists of the composition vectors of indecomposable modules in $\operatorname{mod} A$, described above.

8 Hereditary algebras of Euclidean type

Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K. It follows from Theorems 6.1, 6.2, 7.4 and Corollaries 6.3, 7.5 that the Auslander–Reiten quiver Γ_A of A has the disjoint union form

$$\Gamma_A = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A),$$

where $\mathcal{P}(A)$ is the unique postprojective component containing all indecomposable projective right A-modules, $\mathcal{Q}(A)$ is the unique preinjective component containing all the indecomposable injective right A-modules, and $\mathcal{R}(A)$ is the (non-empty) family of the remaining connected components, which we call *regular components* of Γ_A. Hence the shape of Γ_A may be visualised as follows



where $\mathcal{P}(A) \cong (-\mathbb{N})Q_A^{\mathrm{op}}$ and $\mathcal{Q}(A) \cong \mathbb{N}Q_A^{\mathrm{op}}$ (see Corollaries 6.4 and 6.5). Furthermore, we have

$$\operatorname{Hom}_A\big(\mathcal{R}(A), \mathcal{P}(A)\big) = 0, \quad \operatorname{Hom}_A\big(\mathcal{Q}(A), \mathcal{P}(A)\big) = 0,$$
$$\operatorname{Hom}_A\big(\mathcal{Q}(A), \mathcal{R}(A)\big) = 0,$$

(see Propositions 6.6 and 6.7).

The main aim of this section is to describe the structure of the additive category $\operatorname{add}\mathcal{R}(A)$ of $\mathcal{R}(A)$, called the category of *regular modules* over A, in the case where Q_A is a Euclidean quiver. Then we describe the structure of the module category $\operatorname{mod} A$ of a hereditary algebra A of Euclidean type.

Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. We know from Proposition 4.6 that the Euler quadratic form $\chi_A\colon K_0(A) \to \mathbb{Z}$ is positive semidefinite of corank 1 and

$$\operatorname{rad}\chi_A = \{\mathbf{x} \in K_0(A) \mid \chi_A(\mathbf{x}) = 0\} = \mathbb{Z}\mathbf{h}_A$$

for a unique vector $\mathbf{h}_A$ in $K_0(A)$ having all coordinates positive and at least one equal 1, in the canonical basis of $K_0(A)$.

Lemma 8.1. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then there exists an indecomposable module X in $\mathcal{R}(A)$ with $[X] = \mathbf{h}_A$.*

Proof. It follows from Lemma 3.14 that there exist indecomposable modules $X_1, \dots, X_r$ in mod A such that

(1) $\mathbf{h}_A = [X]$ for $X = X_1 \oplus \cdots \oplus X_r$;
(2) $\operatorname{Ext}^1_A(X_i, X_j) = 0$ for all $i \neq j \in \{1, \dots, r\}$.

Then we obtain the equalities

$$\begin{aligned}
0 &= \chi_A(\mathbf{h}_A) = \chi_A([X]) = \langle [X], [X] \rangle_A \\
&= \left\langle \sum_{i=1}^{r} [X_i], \sum_{j=1}^{r} [X_j] \right\rangle_A = \sum_{i,j=1}^{r} \langle [X_i], [X_j] \rangle_A \\
&= \sum_{i,j=1}^{r} \big(\dim_K \operatorname{Hom}_A(X_i, X_j) - \dim_K \operatorname{Ext}^1_A(X_i, X_j) \big) \\
&= \sum_{i=1}^{r} \big(\dim_K \operatorname{End}_A(X_i) - \dim_K \operatorname{Ext}^1_A(X_i, X_i) \big) + \sum_{i \neq j} \dim_K \operatorname{Hom}_A(X_i, X_j) \\
&= \sum_{i=1}^{r} \chi_A([X_i]) + \sum_{i \neq j} \dim_K \operatorname{Hom}_A(X_i, X_j).
\end{aligned}$$

Since χ_A is positive semidefinite, we have $\chi_A([X_i]) \geq 0$ for any $i \in \{1, \dots, r\}$. This leads to $\chi_A([X_i]) = 0$ for any $i \in \{1, \dots, r\}$. Since $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$, there exist positive integers $m_1, \dots, m_r$ such that $[X_i] = m_i \mathbf{h}_A$ for any $i \in \{1, \dots, r\}$. Hence,

$$\mathbf{h}_A = [X] = \sum_{i=1}^{r} [X_i] = \sum_{i=1}^{r} m_i \mathbf{h}_A = \left(\sum_{i=1}^{r} m_i \right) \mathbf{h}_A.$$

But then $r = 1$ and $m_1 = 1$, because at least one coordinate of $\mathbf{h}_A$ is equal 1. Therefore, X is an indecomposable module in $\mathcal{R}(A)$ with $[X] = \mathbf{h}_A$. □

Lemma 8.2. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K and $\varphi_A \colon K_0(A) \to K_0(A)$ the Coxeter transformation of A. Then $\varphi_A(\operatorname{rad} \chi_A) = \operatorname{rad} \chi_A$ and $\varphi_A(\mathbf{h}_A) = \mathbf{h}_A$.*

Proof. It follows from Proposition 5.7 that $\chi_A(\varphi_A(\mathbf{x})) = \chi_A(\mathbf{x})$ for any vector $\mathbf{x} \in K_0(A)$. Hence $\chi_A(\varphi_A(\mathbf{x})) = \chi_A(\mathbf{x}) = 0$ for any $\mathbf{x} \in \operatorname{rad} \chi_A$, and so

$\varphi_A(\operatorname{rad}\chi_A) \subseteq \operatorname{rad}\chi_A$. Similarly, $\varphi_A^{-1}(\operatorname{rad}\chi_A) \subseteq \operatorname{rad}\chi_A$, because $\chi_A(\varphi_A^{-1}(\mathbf{x})) = \chi_A(\varphi_A(\varphi_A^{-1}(\mathbf{x}))) = \chi_A(\mathbf{x}) = 0$ for any $\mathbf{x} \in \operatorname{rad}\chi_A$. This gives $\operatorname{rad}\chi_A = \varphi_A(\varphi_A^{-1}(\operatorname{rad}\chi_A)) \subseteq \varphi_A(\operatorname{rad}\chi_A)$. Thus we have $\varphi_A(\operatorname{rad}\chi_A) = \operatorname{rad}\chi_A$. Since $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$, we obtain $\mathbb{Z}\mathbf{h}_A = \varphi_A(\mathbb{Z}\mathbf{h}_A) = \mathbb{Z}\varphi_A(\mathbf{h}_A)$. Hence, there exist nonzero integers l and m such that $\mathbf{h}_A = l\varphi_A(\mathbf{h}_A)$ and $\varphi_A(\mathbf{h}_A) = m\mathbf{h}_A$, and consequently $\mathbf{h}_A = lm\mathbf{h}_A$. Using now the fact that at least one of the coordinates of $\mathbf{h}_A$ is 1 we obtain $lm = 1$. This shows that $m \in \{-1, 1\}$. On the other hand, it follows from Lemma 8.1 that $\mathbf{h}_A = [X]$ for an indecomposable module $[X]$ in $\mathcal{R}(A)$. Then, applying Corollary 5.3, we conclude that $\varphi_A(\mathbf{h}_A) = \varphi_A([X]) = [\tau_A X] > 0$ in $K_0(A)$. Therefore, $\varphi_A(\mathbf{h}_A) = \mathbf{h}_A$. □

Proposition 8.3. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then there exists a positive integer d such that $\varphi_A^d(\mathbf{x}) - \mathbf{x} \in \operatorname{rad}\chi_A$ for each $\mathbf{x} \in K_0(A)$.*

Proof. Let $S_1, \dots, S_n$ be a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$. We identify $K_0(A)$ with $\mathbb{Z}^n$ and the $\mathbb{Z}$-basis $[S_1], \dots, [S_n]$ of $K_0(A)$ with the standard $\mathbb{Z}$-basis $e_1, \dots, e_n$ of $\mathbb{Z}^n$. Consider the quotient group $\overline{K_0(A)} = K_0(A)/\operatorname{rad}\chi_A$ of $K_0(A)$ modulo the subgroup $\operatorname{rad}\chi_A = \mathbb{Z}\mathbf{h}_A$. Let $\pi\colon K_0(A) \to \overline{K_0(A)}$ be the canonical epimorphism defined by $\pi(\mathbf{x}) = \bar{\mathbf{x}} = \mathbf{x} + \operatorname{rad}\chi_A$, for each $\mathbf{x} \in K_0(A)$. Since $e_1, \dots, e_n$ generate the group $K_0(A)$, the residue classes $\bar{e}_1, \dots, \bar{e}_n$ generate the quotient group $\overline{K_0(A)}$. Observe also that $\overline{K_0(A)}$ is a torsion-free group. Indeed, if $\mathbf{x} \in K_0(A)$ and $m\bar{\mathbf{x}} = 0$ in $\overline{K_0(A)}$, then $m\mathbf{x} \in \operatorname{rad}\chi_A$ and $0 = \chi_A(m\mathbf{x}) = m^2\chi_A(\mathbf{x})$, which forces $\chi_A(\mathbf{x}) = 0$ and $\mathbf{x} \in \operatorname{rad}\chi_A$, or equivalently $\bar{\mathbf{x}} = 0$. Therefore, $\overline{K_0(A)}$ is a torsion-free group of rank $n-1$, and hence isomorphic to $\mathbb{Z}^{n-1}$.

Consider the Euler $\mathbb{Z}$-bilinear form $\langle -, - \rangle_A\colon K_0(A) \times K_0(A) \to \mathbb{Z}$. For every $\mathbf{x}, \mathbf{y} \in \operatorname{rad}\chi_A$, we have $\mathbf{x} = r\mathbf{h}_A$ and $\mathbf{y} = s\mathbf{h}_A$ for some $r, s \in \mathbb{Z}$, and hence $\langle \mathbf{x}, \mathbf{y} \rangle_A = \langle r\mathbf{h}_A, s\mathbf{h}_A \rangle_A = rs\langle \mathbf{h}_A, \mathbf{h}_A \rangle_A = rs\chi_A(\mathbf{h}_A) = 0$. This allows us to define the $\mathbb{Z}$-bilinear form

$$\overline{\langle -, - \rangle}_A\colon \overline{K_0(A)} \times \overline{K_0(A)} \longrightarrow \mathbb{Z}$$

such that $\overline{\langle \bar{\mathbf{x}}, \bar{\mathbf{y}} \rangle}_A = \langle \mathbf{x}, \mathbf{y} \rangle_A$ for all $\mathbf{x}, \mathbf{y} \in K_0(A)$, and consequently, the quadratic form

$$\overline{\chi}_A\colon \overline{K_0(A)} \longrightarrow \mathbb{Z}$$

given by $\overline{\chi}_A(\bar{\mathbf{x}}) = \chi_A(\mathbf{x})$, for each $\mathbf{x} \in K_0(A)$. Recall also that χ_A is given, for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n = K_0(A)$, by

$$\chi_A(\mathbf{x}) = \sum_{i=1}^{n} f_i x_i^2 - \sum_{i,j=1}^{n} f_{ij} x_i x_j,$$

where $f_i = \dim_K \operatorname{End}_A(S_i)$ and $f_{ij} = \dim_K \operatorname{Ext}_A^1(S_i, S_j)$, for $i, j \in \{1, \ldots, n\}$. Let $f = \max\{f_i \mid i = 1, \ldots, n\}$ and

$$M = \{\mathbf{x} \in K_0(A) \mid \chi_A(\mathbf{x}) \leq f\}.$$

Since Q_A is an acyclic quiver, we have $\operatorname{Ext}_A^1(S_i, S_i) = 0$ for any $i \in \{1, \ldots, n\}$. This gives

$$\chi_A(e_i) = \chi_A([S_i]) = \dim_K \operatorname{End}_A(S_i) - \dim_K \operatorname{Ext}_A^1(S_i, S_i) = f_i$$

for any $i \in \{1, \ldots, n\}$. Hence the basis vectors $e_1, \ldots, e_n$ of $K_0(A) = \mathbb{Z}^n$ belong to M. Consider also the quotient set

$$\overline{M} = \pi(M) = \{\bar{\mathbf{x}} \in \overline{K_0(A)} \mid \overline{\chi}_A(\bar{\mathbf{x}}) \leq f\}.$$

Then $\overline{M}$ contains the generators $\bar{e}_1, \ldots, \bar{e}_n$ of $\overline{K_0(A)}$, because $\overline{\chi}_A(\bar{e}_i) = \chi_A(e_i) = f_i \leq f$ for $i \in \{1, \ldots, n\}$. We claim that $\overline{M}$ is a finite set.

Consider the natural extension

$$\chi_A^{\mathbb{Q}} \colon \mathbb{Q}^n \longrightarrow \mathbb{Q}$$

of $\chi_A \colon \mathbb{Z}^n \to \mathbb{Z}$ to $\mathbb{Q}^n$. Clearly, $\chi_A^{\mathbb{Q}}$ is a positive semidefinite quadratic form of corank 1 and $\operatorname{rad} \chi_A^{\mathbb{Q}} = \mathbb{Q}\mathbf{h}_A$. Then there exists an automorphism $T \colon \mathbb{Q}^n \to \mathbb{Q}^n$ of the $\mathbb{Q}$-vector space $\mathbb{Q}^n$ such that

$$\chi_A^{\mathbb{Q}}(\mathbf{x}) = \sum_{i=1}^{n-1} \lambda_i y_i^2$$

for some positive rational numbers $\lambda_1, \ldots, \lambda_{n-1}$, all $\mathbf{x} \in \mathbb{Q}^n$ and $\mathbf{y} = T(\mathbf{x})$. Observe that then

$$T\big(\operatorname{rad} \chi_A^{\mathbb{Q}}\big) = \{\mathbf{y} \in \mathbb{Q}^n \mid y_1 = \cdots = y_{n-1} = 0\}.$$

Let m be the least common multiplicity of the denominators of the coefficients of the transformation matrix of $T \colon \mathbb{Q}^n \to \mathbb{Q}^n$ in the standard basis $e_1, \ldots, e_n$ of $\mathbb{Q}^n$ over $\mathbb{Q}$. Then, for every $\mathbf{x} \in \mathbb{Z}^n$ and $\mathbf{y} = T(\mathbf{x})$, we have $m\mathbf{y} = mT(\mathbf{x}) \in \mathbb{Z}^n$. Observe also that, if $\mathbf{x} \in M$ and $\mathbf{y} = T(\mathbf{x})$, then

$$\sum_{i=1}^{n-1} \lambda_i y_i^2 = \chi_A^{\mathbb{Q}}(\mathbf{x}) = \chi_A(\mathbf{x}) \leq f,$$

and hence $|y_i| \leq \sqrt{f/\lambda_i}$ for each $i \in \{1, \ldots, n-1\}$. Then we conclude that $T(M)$ is a subset of $\mathbb{Q}^n$ with

$$\Big(T(M) + T\big(\operatorname{rad} \chi_A^{\mathbb{Q}}\big)\Big)/T\big(\operatorname{rad} \chi_A^{\mathbb{Q}}\big) = \big(T(M) + \mathbb{Q}e_n\big)/\mathbb{Q}e_n$$

a finite set. This implies that

$$\overline{M} = \pi(M) = (M + \operatorname{rad} \chi_A)/\operatorname{rad} \chi_A$$

is a finite set.

It follows from Lemma 8.2 that $\varphi(\operatorname{rad} \chi_A) = \operatorname{rad} \chi_A$. Hence the Coxeter transformation $\varphi_A \colon K_0(A) \to K_0(A)$ induces the isomorphism

$$\overline{\varphi}_A \colon \overline{K_0(A)} \longrightarrow \overline{K_0(A)}$$

of abelian groups such that $\overline{\varphi}_A(\bar{\mathbf{x}}) = \overline{\varphi_A(\mathbf{x})}$ for each $\mathbf{x} \in K_0(A)$. Further, for any element $\mathbf{x} \in M$, we have

$$\overline{\chi}_A\big(\overline{\varphi}_A(\bar{\mathbf{x}})\big) = \chi_A\big(\varphi_A(\mathbf{x})\big) = \chi_A(\mathbf{x}) \leq f,$$

by Proposition 5.7. Hence, $\overline{\varphi}_A(\overline{M}) \subseteq \overline{M}$, and so $\overline{\varphi}_A(\overline{M}) = \overline{M}$, because $\overline{\varphi}_A$ is a bijection and $\overline{M}$ is a finite set. Since $\overline{\varphi}_A$ is a group automorphism of $\overline{K_0(A)}$ and $\overline{M}$ contains the generators $\bar{e}_1, \dots, \bar{e}_n$ of the group $\overline{K_0(A)}$, we infer that $\overline{\varphi}_A$ is of finite order. Therefore, there exists a positive integer d such that $(\overline{\varphi}_A)^d = \mathrm{id}_{\overline{K_0(A)}}$. Then, for any $\mathbf{x} \in K_0(A)$, we have $\overline{\varphi_A^d(\mathbf{x})} = (\overline{\varphi}_A)^d(\bar{\mathbf{x}}) = \bar{\mathbf{x}}$, or equivalently, $\varphi_A^d(\mathbf{x}) - \mathbf{x} \in \operatorname{rad} \chi_A$. □

Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then it follows from Proposition 8.3 that there exists a least positive integer d_A such that $\varphi_A^{d_A}(\mathbf{x}) - \mathbf{x} \in \operatorname{rad} \chi_A$ for each element $\mathbf{x} \in K_0(A)$. Since $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$, for every element $\mathbf{x} \in K_0(A)$, we have

$$\varphi_A^{d_A}(\mathbf{x}) = \mathbf{x} + \partial_A(\mathbf{x})\mathbf{h}_A$$

for an integer $\partial_A(\mathbf{x})$, uniquely determined by $\mathbf{x}$. For $\mathbf{x}, \mathbf{y} \in K_0(A)$, we have

$$\begin{aligned}
\varphi_A^{d_A}(\mathbf{x}+\mathbf{y}) &= \varphi_A^{d_A}(\mathbf{x}) + \varphi_A^{d_A}(\mathbf{y}) = \big(\mathbf{x} + \partial_A(\mathbf{x})\mathbf{h}_A\big) + \big(\mathbf{y} + \partial_A(\mathbf{y})\mathbf{h}_A\big) \\
&= (\mathbf{x}+\mathbf{y}) + \big(\partial_A(\mathbf{x}) + \partial_A(\mathbf{y})\big)\mathbf{h}_A, \\
\varphi_A^{d_A}(-\mathbf{x}) &= -\varphi_A^{d_A}(\mathbf{x}) = -\big(\mathbf{x} + \partial_A(\mathbf{x})\mathbf{h}_A\big) = (-\mathbf{x}) + \big(-\partial_A(\mathbf{x})\big)\mathbf{h}_A, \\
\varphi_A^{d_A}(0) &= 0 = 0 + 0\mathbf{h}_A.
\end{aligned}$$

Therefore, we obtain the group homomorphism

$$\partial_A \colon K_0(A) \longrightarrow \mathbb{Z},$$

called the *defect* of A.

The following lemma shows that the defect of a hereditary algebra of Euclidean type is invariant on the action of its Coxeter transformation.

Lemma 8.4. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then*

$$\partial_A\big(\varphi_A(\mathbf{x})\big) = \partial_A(\mathbf{x})$$

for any $\mathbf{x} \in K_0(A)$.

Proof. Let $\mathbf{x} \in K_0(A)$. Then we have the equalities

$$\begin{aligned}\varphi_A^{d_A}\big(\varphi_A(\mathbf{x})\big) &= \varphi_A^{d_A+1}(\mathbf{x}) = \varphi_A\big(\varphi_A^{d_A}(\mathbf{x})\big) = \varphi_A\big(\mathbf{x} + \partial_A(\mathbf{x})\mathbf{h}_A\big)\\ &= \varphi_A(\mathbf{x}) + \partial_A(\mathbf{x})\varphi_A(\mathbf{h}_A) = \varphi_A(\mathbf{x}) + \partial_A(\mathbf{x})\mathbf{h}_A,\end{aligned}$$

and hence $\partial_A\big(\varphi_A(\mathbf{x})\big) = \partial_A(\mathbf{x})$. □

Proposition 8.5. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, and M be an indecomposable module in* mod A. *Then the following equivalences hold:*

(i) $M \in \mathcal{P}(A)$ *if and only if* $\partial_A([M]) < 0$.

(ii) $M \in \mathcal{R}(A)$ *if and only if* $\partial_A([M]) = 0$.

(iii) $M \in \mathcal{Q}(A)$ *if and only if* $\partial_A([M]) > 0$.

Proof. Assume M is not postprojective. Then, for any nonnegative integer m, $\tau_A^m M$ is a nonprojective indecomposable module in mod A with $[\tau_A^m M] = \varphi_A^m([M])$, by Corollary 5.3. We claim that $\partial_A([M]) \geq 0$. Suppose $\partial_A([M]) < 0$. Then we obtain

$$\partial_A([\tau_A^m M]) = \partial_A\big(\varphi_A^m([M])\big) = \partial_A([M]) < 0$$

for any nonnegative integer m. Observe also that, for a positive integer s, we have

$$[\tau_A^{sd_A} M] = \varphi_A^{sd_A}([M]) = [M] + s\partial_A([M])\mathbf{h}_A.$$

This leads to an infinite sequence of inequalities

$$[M] > [\tau_A^{d_A} M] > \dots > [\tau_A^{sd_A} M] > [\tau_A^{(s+1)d_A} M] > \dots$$

and hence to a contradiction, because $[\tau_A^{sd_A} M] > 0$ for any $s \geq 0$. Therefore, $\partial_A([M]) \geq 0$. Moreover, $\partial_A([M]) > 0$ if M belongs to $\mathcal{Q}(A)$. Indeed, suppose that $M \in \mathcal{Q}(A)$ and $\partial_A([M]) = 0$. Then we obtain

$$[\tau_A^{d_A} M] = \varphi_A^{d_A}([M]) = [M] + \partial_A([M])\mathbf{h}_A = [M].$$

Applying Proposition 6.7, we conclude that $M \cong \tau_A^{d_A} M$, a contradiction, because $d_A \geq 1$.

Similarly, we prove that, if M is not preinjective, then $\partial_A([M]) \leq 0$, and $\partial_A([M]) < 0$ if M belongs to $\mathcal{P}(A)$.

Summing up, the required equivalences (i), (ii), (iii) hold. □

Theorem 8.6. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then the additive category* $\operatorname{add}\mathcal{R}(A)$ *of* $\mathcal{R}(A)$ *is an abelian category and closed under extensions.*

Proof. Let M and N be modules in $\operatorname{add}\mathcal{R}(A)$ and $f\colon M \to N$ a nonzero homomorphism in $\operatorname{mod} A$. We prove first that $\operatorname{Im} f$ is in $\operatorname{add}\mathcal{R}(A)$. Let X be an indecomposable direct summand of $\operatorname{Im} f$. Then there exist indecomposable direct summands U of M and V of N such that $\operatorname{Hom}_A(U, X) \neq 0$ and $\operatorname{Hom}_A(X, V) \neq 0$. Then it follows from Propositions 6.6 and 6.7 that X belongs to $\mathcal{R}(A)$, and hence $\partial_A([X]) = 0$. This shows that $\operatorname{Im} f$ belongs to $\operatorname{add}\mathcal{R}(A)$ and $\partial_A([\operatorname{Im} f]) = 0$.

Consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow M \longrightarrow \operatorname{Im} f \longrightarrow 0$$

in $\operatorname{mod} A$. Then $[\operatorname{Ker} f] = [M] - [\operatorname{Im} f]$, and hence we get $\partial_A([\operatorname{Ker} f]) = \partial_A([M]) - \partial_A([\operatorname{Im} f]) = 0$. Assume $\operatorname{Ker} f \neq 0$, and let Y be an indecomposable direct summand of $\operatorname{Ker} f$. Then $\operatorname{Hom}_A(Y, M) \neq 0$, and, applying Proposition 6.7, we conclude that $Y \in \mathcal{P}(A)$ or $Y \in \mathcal{R}(A)$. Then $\partial_A([Y]) \leq 0$, by Proposition 8.5. Thus $\partial_A([\operatorname{Ker} f]) = 0$ implies that $\partial_A([Y]) = 0$ for any indecomposable direct summand Y of $\operatorname{Ker} f$, and consequently $\operatorname{Ker} f$ belongs to $\operatorname{add}\mathcal{R}(A)$. Similarly, the exact sequence

$$0 \longrightarrow \operatorname{Im} f \longrightarrow N \longrightarrow \operatorname{Coker} f \longrightarrow 0,$$

with $\operatorname{Im} f$ and N in $\operatorname{add}\mathcal{R}(A)$, gives that $\operatorname{Coker} f$ is in $\operatorname{add}\mathcal{R}(A)$. Summing up, we proved that $\operatorname{add}\mathcal{R}(A)$ is an abelian category. Finally, let

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

be an exact sequence in $\operatorname{mod} A$ with U and W from $\operatorname{add}\mathcal{R}(A)$. We claim that V belongs to $\operatorname{add}\mathcal{R}(A)$. If the above exact sequence is splittable, then $V \cong U \oplus W$, and V belongs to $\operatorname{add}\mathcal{R}(A)$. Assume the exact sequence is not splittable. Then for any indecomposable direct summand Y of V there exist indecomposable direct summands X of U and Z of W such that $\operatorname{Hom}_A(X, Y) \neq 0$ and $\operatorname{Hom}_A(Y, Z) \neq 0$. Then it follows from Propositions 6.6 and 6.7 that Y belongs to $\mathcal{R}(A)$. This shows that V is in $\operatorname{add}\mathcal{R}(A)$. Therefore, $\operatorname{add}\mathcal{R}(A)$ is closed under extensions. □

Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. It follows from Theorem 8.6 that the category $\operatorname{add}\mathcal{R}(A)$ of regular modules in $\operatorname{mod} A$ is an abelian category and closed under extensions. A nonzero regular module in $\operatorname{mod} A$ having no proper regular submodules is said to be a *simple regular module*. A chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{m-1} \subset M_m = M$$

of regular right A-submodules of a nonzero module M in $\operatorname{add}\mathcal{R}(A)$ is said to be a *regular composition series* of M if M_i/M_{i-1} is a simple regular A-module for any $i \in \{1, \dots, m\}$. If this is the case, then $M_1/M_0, M_2/M_1, \dots, M_m/M_{m-1}$ are called *simple regular composition factors* of M.

Proposition 8.7. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K and M be a nonzero module in* $\operatorname{add} \mathcal{R}(A)$. *Then M admits a regular composition series*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M.$$

Proof. Since $\dim_K M$ is finite, we conclude that M contains a simple regular submodule M_1, for example, a regular submodule of minimal dimension. Moreover, M/M_1 is a regular module with $\dim_K(M/M_1) < \dim_K M$. Then the existence of a regular composition series of M follows by induction on $\dim_K M$. □

Moreover, we have the following Jordan–Hölder theorem for regular modules over a hereditary algebra of Euclidean type.

Theorem 8.8. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, M be a nonzero module in* $\operatorname{add} \mathcal{R}(A)$, *and*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{m-1} \subset M_m = M,$$
$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_{r-1} \subset N_r = M$$

two regular composition series of M. Then $m = r$ and there exists a permutation σ of $\{1, \dots, m\}$ such that $M_i/M_{i-1} \cong N_{\sigma(i)}/N_{\sigma(i)-1}$ for any $i \in \{1, \dots, m\}$.

Proof. Repeat the arguments from the proof of Theorem I.7.5 for the abelian category $\operatorname{add} \mathcal{R}(A)$ of regular A-modules. □

Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. It follows from Theorem 8.8 that the number m of modules in a regular composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M$ of a nonzero module M in $\operatorname{add} \mathcal{R}(A)$ depends only on M; it is called the *regular length* of M and is denoted by $\operatorname{rl}(M)$. Moreover, for a simple regular module E in $\operatorname{add} \mathcal{R}(A)$, the number of simple regular factors M_i/M_{i-1} of M isomorphic to E also depends only on M; it is called the *regular composition multiplicity* of E in M and is denoted by $\operatorname{rc}_E(M)$. The regular length $\operatorname{rl}(0)$ of the zero module 0 in $\operatorname{add} \mathcal{R}(A)$ is defined to be 0.

Proposition 8.9. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then the following statements hold:*

(i) *The Auslander–Reiten translations $\tau_A = D \operatorname{Tr}$ and $\tau_A^{-1} = \operatorname{Tr} D$ induce the mutually inverse equivalences of categories*

$$\begin{tikzcd} \operatorname{add} \mathcal{R}(A) \arrow[r, shift left, "\tau_A"] & \operatorname{add} \mathcal{R}(A) . \arrow[l, shift left, "\tau_A^{-1}"] \end{tikzcd}$$

(ii) *The functors* τ_A *and* τ_A^{-1} *from* add $\mathcal{R}(A)$ *to* add $\mathcal{R}(A)$ *are exact.*

(iii) *The functors* τ_A *and* τ_A^{-1} *preserve the regular length of modules in* add $\mathcal{R}(A)$.

Proof. The statement (i) is a direct consequence of Corollary III.4.11. The statement (ii) follows from Proposition 3.4. Finally, the statement (iii) follows from (i) and (ii). ☐

Proposition 8.10. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, and E a simple regular module in* mod A. *The following statements hold:*

(i) $\operatorname{End}_A(E)$ *is a division K-algebra.*

(ii) *The τ_A-orbit $\mathcal{O}(E) = \{\tau_A^m E \mid m \in \mathbb{Z}\}$ of E is finite.*

Proof. (i) Let $f : E \to E$ be a nonzero homomorphism in mod A. Then $\operatorname{Im} f$ is a nonzero regular submodule of E, by Theorem 8.6, and so $\operatorname{Im} f = E$, because E is simple regular. Hence f is an isomorphism. This shows that $\operatorname{End}_A(E)$ is a division K-algebra.

(ii) Assume $\mathcal{O}(E)$ is infinite, that is, the modules $\tau_A^m E, m \in \mathbb{Z}$, are pairwise nonisomorphic. Then we have $\operatorname{Hom}_A(\tau_A^r E, \tau_A^s E) = 0$ for $r \neq s$ in $\mathbb{Z}$, because then $\tau_A^r E$ and $\tau_A^s E$ are nonisomorphic simple objects of the abelian category add $\mathcal{R}(A)$. Let n be the rank of $K_0(A)$, and consider the module

$$M = E \oplus \tau_A^2 E \oplus \tau_A^4 E \oplus \cdots \oplus \tau_A^{2n} E.$$

Then $\operatorname{Hom}_A(M, \tau_A M) = 0$ and M is the direct sum of $n+1$ pairwise nonisomorphic indecomposable modules in mod A. This contradicts Lemma VIII.7.4. ☐

Theorem 8.11. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, E a simple regular module in* mod A, *and $\mathcal{T}_E$ the connected component of Γ_A containing the module E. Then the following statements hold:*

(i) *$\mathcal{T}_E$ is a stable tube of rank $r_E = |\mathcal{O}(E)|$ whose mouth consists of the simple regular modules from the τ_A-orbit $\mathcal{O}(E)$ of E.*

(ii) *$\mathcal{T}_E$ is a generalized standard component of Γ_A.*

(iii) *The additive category* add $\mathcal{T}_E$ *of $\mathcal{T}_E$ is a uniserial abelian exact subcategory of* mod A *and is closed under extensions.*

(iv) *Every indecomposable module X in $\mathcal{T}_E$ with* $\operatorname{rl}(X) \geq r_E$ *is a sincere A-module.*

Proof. It follows from Proposition 8.10 that the τ_A-orbit $\mathcal{O}(E)$ of E is finite and consists of pairwise nonisomorphic simple regular A-modules. Assume $\mathcal{O}(E)$ is formed by the modules $E_1, \ldots, E_r$ and $E_i = \tau_A E_{i+1}$, for $i \in \{1, \ldots, r\}$, with

$E = E_1 = E_{r+1}$. Since A is a hereditary algebra, we conclude that $E_1, \dots, E_r$ is a hereditary family of pairwise orthogonal bricks in $\operatorname{mod} A$. Therefore, applying Lemma 3.5, Proposition 3.6 and Theorem 3.7, we obtain that the extension category $\mathcal{E} = \mathcal{EXT}_A(E_1, \dots, E_r)$ of $E_1, \dots, E_r$ is a uniserial abelian subcategory of $\operatorname{mod} A$ with $E_1, \dots, E_r$ forming a complete set of pairwise nonisomorphic simple objects of $\mathcal{E}$, as well as the mouth of a stable tube $\mathcal{T}_{\mathcal{E}}$ of Γ_A of rank r, with $\mathcal{E} = \operatorname{add} \mathcal{T}_{\mathcal{E}}$. Clearly, then $\mathcal{T}_{\mathcal{E}} = \mathcal{T}_E$. Moreover, $\mathcal{E}$ is closed under extensions in $\operatorname{mod} A$ and every indecomposable object of $\mathcal{E}$ is isomorphic to an object of the form $E_i[j]$, for some $i \in \{1, \dots, r\}$ and $j \in \mathbb{N}^+$. In particular, the indecomposable modules $E_i[j]$, $i \in \{1, \dots, r\}$, $j \in \mathbb{N}^+$, form the vertices of the stable tube $\mathcal{T}_E$, and its rank r is the cardinality $|\mathcal{O}(E)|$ of the τ_A-orbit $\mathcal{O}(E)$ of E. Further, by Theorem 3.11, $\mathcal{T}_E$ is a generalized standard component of Γ_A. Therefore, (i), (ii) and (iii) hold.

For (iv), consider the module $M = E_1 \oplus \cdots \oplus E_r$. Observe that $M = \tau_A M$. Then we obtain

$$\begin{aligned}\chi_A([M]) &= \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M, M)\\ &= \dim_K \operatorname{End}_A(M) - \dim_K D \operatorname{Hom}_A(M, \tau_A M)\\ &= \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{End}_A(M) = 0.\end{aligned}$$

This shows that $[M] \in \operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$, where $\mathbf{h}_A$ is a vector in $K_0(A)$ with all its coordinates positive and at least one equal 1, in the canonical $\mathbb{Z}$-basis of $K_0(A) = \mathbb{Z}^n$. This shows that $[M] = m\mathbf{h}_A$ for a positive integer m, and hence M is a sincere A-module. We note now that $[M] = [E_1] + \cdots + [E_r] = \big[E_i[r]\big]$ for any $i \in \{1, \dots, r\}$, and so $E_1[r], \dots, E_r[r]$ are sincere indecomposable modules. Take now an arbitrary indecomposable module X in $\mathcal{T}_E$ with $\operatorname{rl}(X) \geq r = r_E$. Then $X = E_i[j]$ for some $i \in \{1, \dots, r\}$ and a positive integer $j \geq r$. Thus there exists a chain of monomorphisms in $\operatorname{mod} A$ from $E_i[r]$ to $E_i[j]$, and consequently $\big[E_i[r]\big] \leq \big[E_i[j]\big]$ in the canonical order of $K_0(A)$. This proves that $X = E_i[j]$ is a sincere A-module. □

Let A be a finite dimensional K-algebra over a field K. A component $\mathcal{C}$ in the Auslander–Reiten quiver Γ_A of A is said to be *sincere* if every simple module in $\operatorname{mod} A$ is a composition factor of a module M in $\mathcal{C}$. A family $\mathcal{C} = \{\mathcal{C}_i\}_{i \in \Lambda}$ of components in Γ_A is said to be a *separating family* if the indecomposable modules in $\operatorname{mod} A$ outside of the family $\mathcal{C}$ fall into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that the following conditions are satisfied:

(1) $\mathcal{C}$ is a family of pairwise orthogonal generalized standard sincere components.
(2) $\operatorname{Hom}_A(\mathcal{C}, \mathcal{P}) = 0$, $\operatorname{Hom}_A(\mathcal{Q}, \mathcal{C}) = 0$ and $\operatorname{Hom}_A(\mathcal{Q}, \mathcal{P}) = 0$.
(3) For each $i \in \Lambda$, any homomorphism $f\colon P \to Q$, with P in $\mathcal{P}$ and Q in $\mathcal{Q}$, factors through a module from $\operatorname{add} \mathcal{C}_i$.

In this case, we say that $\mathcal{C}$ *separates* $\mathcal{P}$ from $\mathcal{Q}$.

For a finite dimensional hereditary K-algebra A of Euclidean type over a field K, we denote by $\Lambda(A)$ the set of all τ_A-orbits of simple regular modules in mod A. It follows from Theorem 8.11 that for any $\lambda \in \Lambda(A)$, the Auslander–Reiten quiver Γ_A of A contains a stable tube $\mathcal{T}_\lambda^A$ whose mouth is formed by modules of the τ_A-orbit $\lambda = \{E_1^{(\lambda)}, \ldots, E_{r_\lambda^A}^{(\lambda)}\}$ and r_λ^A is the rank of $\mathcal{T}_\lambda^A$.

Theorem 8.12. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, and n be the rank of $K_0(A)$. Then the regular part $\mathcal{R}(A)$ of Γ_A has the disjoint decomposition*

$$\mathcal{R}(A) = \mathcal{T}^A = \bigcup_{\lambda \in \Lambda(A)} \mathcal{T}_\lambda^A$$

into a sum of stable tubes, and separates the postprojective component $\mathcal{P}(A)$ from the preinjective component $\mathcal{Q}(A)$. Moreover, we have

$$\sum_{\lambda \in \Lambda(A)} \left(r_\lambda^A - 1\right) \le n.$$

Proof. Consider the family of stable tubes

$$\mathcal{T}^A = \bigcup_{\lambda \in \Lambda(A)} \mathcal{T}_\lambda^A$$

in $\mathcal{R}(A)$. We claim that $\mathcal{R}(A) = \mathcal{T}^A$. Let M be an indecomposable module in $\mathcal{R}(A)$. It follows from Proposition 8.7 that M contains a simple regular submodule E. Moreover, by Theorem 8.11, the module E lies on the mouth of a stable tube $\mathcal{T}_E$, which is clearly a stable tube $\mathcal{T}_\lambda^A$, for some $\lambda \in \Lambda(A)$. Let j be the maximal positive integer such that there is a monomorphism $f: E[j] \to M$. Then, applying arguments as in the proof of the statement (iii) of Proposition 3.6, we conclude that f is an isomorphism. Since $E[j]$ belong to $\mathcal{T}_E = \mathcal{T}_\lambda^A$, we obtain that M belongs to $\mathcal{T}_\lambda^A$. Hence, $\mathcal{R}(A) = \mathcal{T}^A$.

It follows also from Theorem 8.11 that each stable tube $\mathcal{T}_\lambda^A$, for $\lambda \in \Lambda(A)$, is sincere and a generalized standard component of Γ_A. We claim that the stable tubes $\mathcal{T}_\lambda^A, \lambda \in \Lambda(A)$, are pairwise orthogonal. Indeed, let λ and μ be different elements of $\Lambda(A)$. Moreover, by Theorems 3.7 and 8.11, add $\mathcal{T}_\lambda^A$ is the extension category $\mathcal{E}_\lambda = \mathcal{EXT}_A(E_1^{(\lambda)}, \ldots, E_{r_\lambda}^{(\lambda)})$ of the τ_A-orbit λ and add $\mathcal{T}_\mu^A$ is the extension category $\mathcal{E}_\mu = \mathcal{EXT}_A(E_1^{(\mu)}, \ldots, E_{r_\mu}^{(\mu)})$ of the τ_A-orbit μ. Take now indecomposable modules X in $\mathcal{T}_\lambda^A$ and Y in $\mathcal{T}_\mu^A$. Suppose there exists a nonzero homomorphism $f: X \to Y$ in mod A. Then Im f is a nonzero quotient regular module of X with the simple regular composition factors from the τ_A-orbit λ, and also a nonzero regular submodule of Y with the simple regular composition factors from the τ_A-orbit μ. This leads to a contradiction since the τ_A-orbits λ and

μ are disjoint. Therefore, $\operatorname{Hom}_A(X, Y) = 0$. This shows that the stable tubes $\mathcal{T}_\lambda^A$ and $\mathcal{T}_\mu^A$ are orthogonal. We will prove now that

$$\sum_{\lambda \in \Lambda(A)} \left(r_\lambda^A - 1\right) \leq n.$$

Let Λ_0 be the subset of $\Lambda(A)$ consisting of all $\lambda \in \Lambda(A)$ with $r_\lambda^A \geq 2$. We note first that $|\Lambda_0| \leq n$. Indeed, suppose $|\Lambda_0| \geq n + 1$, and let $\lambda_1, \dots, \lambda_n, \lambda_{n+1}$ be pairwise distinct elements of Λ_0. Consider the module

$$M = E_1^{(\lambda_1)} \oplus E_1^{(\lambda_2)} \oplus \cdots \oplus E_1^{(\lambda_n)} \oplus E_1^{(\lambda_{n+1})},$$

which is a direct sum of simple regular modules lying in the distinct stable tubes $\mathcal{T}_{\lambda_1}^A, \mathcal{T}_{\lambda_2}^A, \dots, \mathcal{T}_{\lambda_n}^A, \mathcal{T}_{\lambda_{n+1}}^A$. Because each stable tube $\mathcal{T}_{\lambda_i}^A$ with $\lambda_i \in \Lambda_0$ is of rank $r_{\lambda_i}^A \geq 2$,

$$\operatorname{Hom}_A\left(E_1^{(\lambda_i)}, \tau_A E_1^{(\lambda_i)}\right) = \operatorname{Hom}_A\left(E_1^{(\lambda_i)}, E_{r_{\lambda_i}^A}^{(\lambda_i)}\right) = 0.$$

Moreover, $\operatorname{Hom}_A\left(E_1^{(\lambda_i)}, \tau_A E_1^{(\lambda_j)}\right) = 0$ for $i \neq j$ in $\{1, \dots, n+1\}$, because the stable tubes $\mathcal{T}_{\lambda_1}^A, \dots, \mathcal{T}_{\lambda_n}^A, \mathcal{T}_{\lambda_{n+1}}^A$ are pairwise orthogonal. Therefore, we obtain $\operatorname{Hom}_A(M, \tau_A M) = 0$. This leads to a contradiction with Lemma VIII.7.4. This proves that $|\Lambda_0| \leq n$. For each $\lambda \in \Lambda_0$, consider the family of indecomposable modules $E_1^{(\lambda)}[1], E_1^{(\lambda)}[2], \dots, E_1^{(\lambda)}[r_\lambda^A - 1]$. Observe that the simple regular composition factors of these modules belong to the family $E_1^{(\lambda)}[1], E_2^{(\lambda)}[1], \dots, E_{r_\lambda^A - 1}^{(\lambda)}[1]$. On the other hand,

$$\tau_A E_1^{(\lambda)}[1] = E_{r_\lambda^A}^{(\lambda)}[1], \tau_A E_1^{(\lambda)}[2] = E_{r_\lambda^A}^{(\lambda)}[2], \dots, \tau_A E_1^{(\lambda)}[r_\lambda^A - 1] = E_{r_\lambda^A}^{(\lambda)}[r_\lambda^A - 1],$$

and the unique simple regular submodule of each of those modules is isomorphic to $E_{r_\lambda^A}^{(\lambda)}[1]$. This implies that $\operatorname{Hom}_A\left(E_1^{(\lambda)}[k], \tau_A E_1^{(\lambda)}[l]\right) = 0$ for all $k, l \in \{1, \dots, r_\lambda^A - 1\}$. Further, we have $\operatorname{Hom}_A\left(E_1^{(\lambda)}[k], \tau_A E_1^{(\mu)}[l]\right) = 0$ for all $\lambda \neq \mu$ in Λ_0 and $k, l \geq 1$. Then for the module

$$N = \bigoplus_{\lambda \in \Lambda_0} \bigoplus_{j=1}^{r_\lambda^A - 1} E_1^{(\lambda)}[j]$$

we have

$$\operatorname{Hom}_A(N, \tau_A N) = \operatorname{Hom}_A\left(\bigoplus_{\lambda \in \Lambda_0} \bigoplus_{j=1}^{r_\lambda^A - 1} E_1^{(\lambda)}[j], \bigoplus_{\lambda \in \Lambda_0} \bigoplus_{j=1}^{r_\lambda^A - 1} \tau_A E_1^{(\lambda)}[j]\right) = 0.$$

Then, applying Lemma VIII.7.4, we conclude that

$$\sum_{\lambda \in \Lambda(A)} (r_\lambda^A - 1) = \sum_{\lambda \in \Lambda_0} (r_\lambda^A - 1) \le n.$$

It remains to show that the family $\mathcal{T}^A = (\mathcal{T}_\lambda^A)_{\lambda \in \Lambda(A)}$ separates $\mathcal{P}(A)$ from $\mathcal{Q}(A)$. We fix $\lambda \in \Lambda(A)$. Let P be a nonzero module in add $\mathcal{P}(A)$. We claim that there is a monomorphism $j \colon P \to E$ for a module E in add $\mathcal{T}_\lambda^A$. We note first that $\operatorname{Hom}_A(P, X) \neq 0$ for a module X in $\mathcal{T}_\lambda^A$. Indeed, it follows from Theorem 6.1 that P has an indecomposable direct summand of the form $\tau_A^{-m} P_i$ for some indecomposable projective A-module P_i and some nonnegative integer m. Further, by Theorem 8.11, the stable tube $\mathcal{T}_\lambda^A$ contains a sincere indecomposable module Y, and consequently the simple top $S_i = \operatorname{top}(P_i)$ of P_i occurs as a simple composition factor of Y. Then we have $\dim_K \operatorname{Hom}_A(P_i, Y) = c_{S_i}(Y) \dim_K \operatorname{End}_A(S_i) \neq 0$, by Lemma 5.6. Hence, applying Proposition 5.5, we obtain $\operatorname{Hom}_A(\tau_A^{-m} P_i, \tau_A^{-m} Y) \cong \operatorname{Hom}_A(P_i, Y) \neq 0$, and so $\operatorname{Hom}_A(P, X) \neq 0$ for $X = \tau_A^{-m} Y$ in $\mathcal{T}_\lambda^A$. Let $u \colon P \to X$ be a nonzero homomorphism in mod A with X in $\mathcal{T}_\lambda^A$, and $P' = \operatorname{Ker} u$. If $P' = 0$, then u is a monomorphism, and the claim follows. Assume $P' \neq 0$. Since u is nonzero, $\dim_K P' < \dim_K P$, so we may assume by induction that there is a monomorphism $u' \colon P' \to X'$ with X' in add $\mathcal{T}_\lambda^A$. Then we have in mod A a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r, hook, "\omega"] \arrow[d, "u'"] & P \arrow[r, "\pi"] \arrow[d, "v'"] & P'' \arrow[r] \arrow[d, "\mathrm{id}_{P''}"] & 0 \\
0 \arrow[r] & X' \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & P'' \arrow[r] & 0
\end{tikzcd}$$

with $P'' = P/P'$, where M is the fibered sum of X' and P over P', via u' and ω. It follows from Lemma 3.1 that v' is a monomorphism. Assume $P'' \in \operatorname{add} \mathcal{T}_\lambda^A$. Then M belongs to add $\mathcal{T}_\lambda^A$, because $\operatorname{add} \mathcal{T}_\lambda^A = \mathcal{EXT}_A(E_1^{(\lambda)}, \dots, E_{r_\lambda^A}^{(\lambda)})$ is closed under extensions in mod A, by Theorem 8.11, and so $v' \colon P \to M$ is a required monomorphism of P into a module in add $\mathcal{T}_\lambda^A$. Observe that, if $u \colon P \to X$ is an epimorphism, then $P'' \cong \operatorname{Im} u = X$ belongs to add $\mathcal{T}_\lambda^A$. Therefore, assume that u is not an epimorphism and P'' is not in add $\mathcal{T}_\lambda^A$. Observe that $\operatorname{Im} u$ is a submodule of X from $\mathcal{T}_\lambda^A$. Since $\operatorname{Hom}_A(\mathcal{Q}(A), \mathcal{T}_\lambda^A) = 0$ and $\operatorname{Hom}_A(\mathcal{T}_\mu^A, \mathcal{T}_\lambda^A) = 0$ for any $\mu \neq \lambda$ in $\Lambda(A)$, we conclude that $P'' = U \oplus V$ with U a nonzero module from add $\mathcal{P}(A)$ and V a module from add $\mathcal{T}_\lambda^A$. Applying Corollary III.6.4, we obtain $\operatorname{Ext}_A^1(U, X') \cong D \operatorname{Hom}_A(X', \tau_A U) = 0$, because $X' \in \operatorname{add} \mathcal{T}_\lambda^A$, $\tau_A U \in \operatorname{add} \mathcal{P}(A)$, and $\operatorname{Hom}_A(\mathcal{T}_\lambda^A, \mathcal{P}(A)) = 0$. Let $i_U \colon U \to P''$ be the inclusion homomorphism and $\pi_U \colon P'' \to U$ the projection homomorphism, so

$\pi_U i_U = \mathrm{id}_U$. We have a commutative diagram in $\operatorname{mod} A$ with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "\delta"] \arrow[d, "\mathrm{id}_{X'}"] & N \arrow[r, "\gamma"] \arrow[d, "\sigma"] & U \arrow[r] \arrow[d, "i_U"] & 0 \\
0 \arrow[r] & X' \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & P'' \arrow[r] & 0,
\end{tikzcd}$$

where N is the fibered product of M and U over P'', via β and i_U. Observe that σ is a monomorphism, by Lemma 3.1. Since $\mathcal{E}\mathrm{xt}^1_A(U, X') \cong \operatorname{Ext}^1_A(U, X') = 0$, we infer that γ is a retraction in $\operatorname{mod} A$ (see Lemma III.3.1). Hence there exists a homomorphism $\varrho\colon U \to N$ in $\operatorname{mod} A$ with $\gamma\varrho = \mathrm{id}_U$. In particular, applying Lemma I.4.2, we obtain $N = \operatorname{Im}\varrho \oplus \operatorname{Ker}\gamma = \operatorname{Im}\varrho \oplus \operatorname{Im}\delta$ with $\operatorname{Im}\varrho \cong U$ and $\operatorname{Im}\delta \cong X'$. Let $N' = \operatorname{Im}\varrho$, and $\sigma'\colon N' \to M$ be the restriction of σ to N'. We claim that σ' is a section. Indeed, for $\pi' = \varrho\pi_U\beta\colon M \to N'$ and $n' = \varrho(x)$, with $x \in U$, we have $\sigma(\pi'\sigma'(n')) = \sigma(\varrho\pi_U\beta\sigma(\varrho(x)) = \sigma\varrho(\pi_U i_U)(\gamma\varrho)(x) = \sigma\varrho(x) = \sigma(n')$, and hence $\pi'\sigma'(n') = n'$, because σ is a monomorphism. Therefore, applying Lemma I.4.2 again, we obtain that $M = \operatorname{Im}\sigma' \oplus \operatorname{Ker}\pi'$ with $\operatorname{Im}\sigma' \cong U$ and $\operatorname{Ker}\pi' = \operatorname{Ker}\varrho\pi_U\beta = \operatorname{Ker}\pi_U\beta$, since ϱ is a monomorphism. Moreover, U is a nonzero module in $\operatorname{add}\mathcal{P}(A)$ with $\dim_K U \le \dim_K P'' < \dim_K P$, and so, by induction, there is a monomorphism $u''\colon U \to X''$ with X'' in $\operatorname{add}\mathcal{T}^A_\lambda$. We prove now that $\operatorname{Ker}\pi' = \operatorname{Ker}\pi_U\beta$ is a module in $\operatorname{add}\mathcal{T}^A_\lambda$. Let $i_V\colon V \to P''$ be the inclusion homomorphism. Then there is a commutative diagram in $\operatorname{mod} A$ with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "\xi"] \arrow[d, "\mathrm{id}_{X'}"] & L \arrow[r, "\eta"] \arrow[d, "\varphi"] & V \arrow[r] \arrow[d, "i_V"] & 0 \\
0 \arrow[r] & X' \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & P'' \arrow[r] & 0,
\end{tikzcd}$$

where L is the fibered product of M and V over P'', via β and i_V. It follows from Lemma 3.1 that φ is a monomorphism and $\operatorname{Coker}\varphi \cong \operatorname{Coker} i_V = U$. Further, $\pi_U\beta\varphi = \pi_U i_V\eta = 0$, and hence $\operatorname{Im}\varphi \subseteq \operatorname{Ker}\pi_U\beta = \operatorname{Ker}\pi'$. We have the equalities

$$\begin{aligned}\dim_K \operatorname{Im}\varphi = \dim_K M - \dim_K \operatorname{Coker}\varphi &= \dim_K M - \dim_K U \\ = \dim_K M - \dim_K \operatorname{Im}\sigma' &= \dim_K \operatorname{Ker}\pi'.\end{aligned}$$

This shows that $\operatorname{Im}\varphi = \operatorname{Ker}\pi'$, and hence $L \cong \operatorname{Ker}\pi'$. Finally, since $\operatorname{add}\mathcal{T}^A_\lambda$ is closed under extensions and X', V are in $\operatorname{add}\mathcal{T}^A_\lambda$, we conclude that L belongs to $\operatorname{add}\mathcal{T}^A_\lambda$. Thus $\operatorname{Ker}\pi'$ is a module in $\operatorname{add}\mathcal{T}^A_\lambda$. This leads to a monomorphism $M = \operatorname{Im}\sigma' \oplus \operatorname{Ker}\pi' \to X'' \oplus \operatorname{Ker}\pi'$ with $X'' \oplus \operatorname{Ker}\pi'$ in $\operatorname{add}\mathcal{T}^A_\lambda$. Composing it with the monomorphism $v'\colon P \to M$ we get a required monomorphism $j\colon P \to E$ with $E = X'' \oplus \operatorname{Ker}\pi'$ from $\operatorname{add}\mathcal{T}^A_\lambda$.

Let now $f: P \to Q$ be a nonzero homomorphism in mod A with P in $\mathcal{P}(A)$ and Q in $\mathcal{Q}(A)$. We have shown that there is a monomorphism $j: P \to E$ with E in add $\mathcal{T}_\lambda^A$. Hence, if Q is injective, then there exists a homomorphism $g: E \to Q$ such that $f = gj$, and we are done in this case. Assume Q is not injective. Then it follows from Theorem 6.2 that $Q = \tau_A^m I_i$ for a positive integer m and an indecomposable injective module I_i. Then the homomorphism $\tau_A^{-m} f: \tau_A^{-m} P \to I_i$ factors through a module R in add $\mathcal{T}_\lambda^A$. Clearly, then $f: P \to Q$ factors through the module $\tau_A^m R$ in add $\mathcal{T}_\lambda^A$. This finishes the proof. □

It is known that for any finite dimensional hereditary K-algebra A of Euclidean type over a field K, the set $\Lambda(A)$ is infinite.

Let A be a finite dimensional K-algebra over a field K. For an indecomposable module M in mod A, $\operatorname{End}_A(M)$ is a local K-algebra, and hence $F_M = \operatorname{End}_A(M)/\operatorname{rad}\operatorname{End}_A(M) = \operatorname{End}_A(M)/\operatorname{rad}_A(M, M)$ is a finite dimensional division K-algebra. Moreover, for indecomposable modules X and Y in mod A, and a positive integer m, the finite dimensional K-vector space $\operatorname{rad}_A^m(X, Y)/\operatorname{rad}_A^{m+1}(X, Y)$ is an (F_Y, F_X)-bimodule by

$$\big(h + \operatorname{rad}_A(Y, Y)\big)\big(f + \operatorname{rad}_A^{m+1}(X, Y)\big) = hf + \operatorname{rad}_A^{m+1}(X, Y),$$
$$\big(f + \operatorname{rad}_A^{m+1}(X, Y)\big)\big(g + \operatorname{rad}_A(X, X)\big) = fg + \operatorname{rad}_A^{m+1}(X, Y),$$

for $f \in \operatorname{rad}_A^m(X, Y)$, $g \in \operatorname{End}_A(X)$, $h \in \operatorname{End}_A(Y)$.

Proposition 8.13. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, M an indecomposable regular module in* mod A, *$\mathcal{T}_\lambda^A$ the stable tube of Γ_A containing M, and $r = r_\lambda^A$ the rank of $\mathcal{T}_\lambda^A$. The following statements hold:*

(i) $\dim_K \operatorname{End}_A(M) = (s + 1) \dim_K F_M$, *where $s \geq 0$ is such that* $sr < \operatorname{rl}(M) \leq (s + 1)r$.

(ii) $\dim_K \operatorname{Ext}_A^1(M, M) = s \dim_K F_M$, *where $s \geq 0$ is such that* $sr \leq \operatorname{rl}(M) < (s + 1)r$.

Proof. Let $E_1 = E_1^{(\lambda)}, \dots, E_r = E_r^{(\lambda)}$ be the simple regular modules on the mouth of $\mathcal{T}_\lambda^A$ such that $\tau_A E_{i+1}^{(\lambda)} = E_i^{(\lambda)}$ for $i \in \{1, \dots, r\}$ and $\tau_A E_{r+1}^{(\lambda)} = E_1^{(\lambda)}$. Then M is a module $E_i[j]$ for some $i \in \{1, \dots, r\}$ and a positive integer j. Clearly, we have $\operatorname{rl}(M) = j$. Since $\mathcal{T}_\lambda^A$ is a generalized standard component of Γ_A, it follows from Proposition 3.9 that every nonzero nonisomorphism between two indecomposable modules in $\mathcal{T}_\lambda^A$ is a finite direct sum of compositions of irreducible homomorphisms between indecomposable modules in $\mathcal{T}_\lambda^A$. Further, it follows from Propositions 3.6 and 3.10 that

(a) every irreducible homomorphism in $\mathcal{T}_\lambda^A$ is either of the form $v\colon E_k[l-1] \to E_k[l]$ or of the form $w\colon E_k[l] \to E_{k+1}[l-1]$;

(b) for any irreducible homomorphism $v\colon E_k[l-1] \to E_k[l]$ there exist automorphisms f of $E_k[l]$ and g of $E_k[l-1]$ such that $fu_{kl} = v = u_{kl}g$;

(c) for any irreducible homomorphism $w\colon E_k[l] \to E_{k+1}[l-1]$ there exist automorphisms f of $E_k[l]$ and h of $E_{k+1}[l-1]$ such that $hp_{kl} = w = p_{kl}f$.

Recall also that we have in mod A almost split sequences

$$0 \longrightarrow E_k[l-1] \xrightarrow{\begin{bmatrix} p_{k\,l-1} \\ u_{kl} \end{bmatrix}} E_{k+1}[l-2] \oplus E_k[l] \xrightarrow{\begin{bmatrix} u_{k+1\,l-1} & p_{kl} \end{bmatrix}} E_{k+1}[l-1] \longrightarrow 0,$$

with $l \geq 3$, $k \in \{1, \ldots, r\}$, and

$$0 \longrightarrow E_k[1] \xrightarrow{u_{k2}} E_k[2] \xrightarrow{p_{k2}} E_{k+1}[1] \longrightarrow 0,$$

with $k \in \{1, \ldots, r\}$, and hence the relations

(d) $u_{k+1\,l-1}p_{k\,l-1} + p_{kl}u_{kl} = 0$, $p_{k2}u_{k2} = 0$,

with $l \geq 3$, $k \in \{1, \ldots, r\}$.

(i) Let s be the nonnegative integer such that $sr < j \leq (s+1)r$. Assume first that $s = 0$, that is, $\mathrm{rl}(M) = j \leq r$. Then the simple regular composition factors of $M = E_i[j]$ are $E_i[1], E_{i+1}[1], \ldots, E_{i+j-1}[1]$, and consequently are pairwise nonisomorphic. Since $E_i[j]$ is a uniserial object of the category add $\mathcal{T}_\lambda^A$, we infer that $\operatorname{rad}\operatorname{End}_A(E_i[j]) = 0$. Therefore, $\operatorname{End}_A(M) = F_M$.

Assume $s \geq 1$. Consider the homomorphism $\varphi_{ij} \in \operatorname{rad}\operatorname{End}_A(E_i[j])$ of the form

$$\varphi_{ij} = u_{ij}u_{i\,j-1}\cdots u_{i\,j-r+1}p_{i+r-1\,j-r+1}\cdots p_{i+1\,j-1}p_{ij}.$$

Observe that the shortest path in $\mathcal{T}_\lambda^A$ from $E_i[j]$ to $E_i[j]$ is of length $2r$, and hence $\operatorname{rad}_A(E_i[j], E_i[j]) = \operatorname{rad}_A^{2r}(E_i[j], E_i[j])$. In fact, every nontrivial path in $\mathcal{T}_\lambda^A$ from $E_i[j]$ to $E_i[j]$ is of length $2rt$ for some $t \geq 1$. This implies that $\operatorname{rad}_A^{2rt+1}(E_i[j], E_i[j]) = \operatorname{rad}_A^{2r(t+1)}(E_i[j], E_i[j])$ for any $t \geq 1$. It follows from (a)–(d) that, for $t \in \{1, \ldots, s\}$, any composition of $2rt$ irreducible homomorphisms between indecomposable modules in $\mathcal{T}_\lambda^A$ leading from $E_i[j]$ to $E_i[j]$ is of the form $\varphi_{ij}^t g_t$, where g_t is an automorphism of $E_i[j]$. On the other hand, for any $t \geq s+1$, we have $\varphi_{ij}^t = 0$ and any composition of t irreducible homomorphisms between indecomposable modules in $\mathcal{T}_\lambda^A$ leading from $E_i[j]$ to $E_i[j]$ is zero. Summing up, every homomorphism $f \in \operatorname{rad}_A(E_i[j], E_i[j])$ is of the form

$$f = \sum_{t=1}^{s} \varphi_{ij}^t g_t,$$

where each $g_t\colon E_i[j] \to E_i[j]$ is either zero or an automorphism. We note also that, for each $t \in \{1, \dots, s\}$, $\operatorname{Im} \varphi^t_{i,j}$ is the unique regular submodule of $E_i[j]$ isomorphic to $E_{i+rt}[j-rt]$. Therefore, the K-vector space $\operatorname{Hom}_A(M, M)$ admits the chain of subspaces

$$0 = \operatorname{rad}^{2r(s+1)}_A(M, M) \subset \operatorname{rad}^{2rs}_A(M, M) \subset \dots \subset \operatorname{rad}^{2r}_A(M, M) \subset \operatorname{Hom}_A(M, M)$$

such that for each $t \in \{0, 1, \dots, s\}$,

$$\operatorname{rad}^{2rt}_A(M, M)/\operatorname{rad}^{2r(t+1)}_A(M, M) = \operatorname{rad}^{2rt}_A(M, M)/\operatorname{rad}^{2rt+1}_A(M, M)$$

is a right F_M-module generated by $\varphi^t_{i,j} + \operatorname{rad}^{2rt+1}_A(M, M)$, where $\varphi^0_{ij} = \mathrm{id}_M$. Hence, we have $\dim_K \operatorname{End}_A(M) = (s+1)\dim_K F_M$.

(ii) It follows from Corollary III.6.4 that there is a K-linear isomorphism $\operatorname{Ext}^1_A(M, M) \cong D\operatorname{Hom}_A(M, \tau_A M)$, and hence $\dim_K \operatorname{Ext}^1_A(M, M) = \dim_K \operatorname{Hom}_A(M, \tau_A M)$. Hence, if $M = \tau_A M$, then the required formula for $\dim_K \operatorname{Ext}^1_A(M, M)$ follows from (i). Therefore, we assume that $M \neq \tau_A M$, or equivalently, $r \geq 2$. Then $\operatorname{Hom}_A(M, \tau_A M) = \operatorname{rad}_A(M, \tau_A M)$. Let s be the nonnegative integer such that $sr \leq j < (s+1)r$.

Let first that $s = 0$, that is, $\operatorname{rl}(M) = j < r$. Then the simple regular composition factors of $M = E_i[j]$ are $E_i[1], E_{i+1}[1], \dots, E_{i+j-1}[1]$, and the simple regular composition factors of $\tau_A M = E_{i-1}[j]$ are $E_{i-1}[1], E_i[1], \dots, E_{i+j-2}[1]$. Observe that $r \geq 2$ implies $E_i[j] \ncong E_{i-1}[j]$. On the other hand, every regular quotient of $M = E_i[j]$ is isomorphic to $E_{i+t}[j-t]$, for some $t \in \{0, 1, \dots, j-1\}$, and so its regular top is isomorphic to $E_{i+j-1}[1]$. Since the module $E_{i+j-1}[1]$ is not a simple regular composition factor of $\tau_A M = E_{i-1}[j]$, we infer that $\operatorname{Hom}_A(M, \tau_A M) = 0$, and consequently $\operatorname{Ext}^1_A(M, M) = 0$.

Assume $s \geq 1$. Consider the homomorphism $\psi_{ij} \in \operatorname{rad}_A(E_i[j], E_{i-1}[j])$ of the form

$$\psi_{ij} = u_{i-1\,j} u_{i-1\,j-1} \cdots u_{i-1\,j-r+2} p_{i+r-2\,j-r+2} \cdots p_{i+1\,j-1} p_{ij},$$

and note that $E_{i+r-1}[l] = E_{i-1}[l]$ for $l \geq 1$. Since the shortest path in $\mathcal{T}^A_\lambda$ from $E_i[j]$ to $E_{i-1}[j]$ is of length $2r-2$, we have $\operatorname{rad}_A(E_i[j], E_{i-1}[j]) = \operatorname{rad}^{2r-2}_A(E_i[j], E_{i-1}[j])$. Further, every nontrivial path in $\mathcal{T}^A_\lambda$ from $E_i[j]$ to $E_{i-1}[j]$ is of length $2rt-2$ for some $t \geq 1$. This implies that

$$\operatorname{rad}^{2rt-1}_A(E_i[j], E_{i-1}[j]) = \operatorname{rad}^{2r(t+1)-2}_A(E_i[j], E_{i-1}[j]).$$

It follows from (a)–(d) that, for $t \in \{1, \dots, s\}$, any composition of $2rt-2$ irreducible homomorphisms between indecomposable modules in $\mathcal{T}^A_\lambda$ leading from $E_i[j]$ to $E_{i-1}[j]$ is of the form $\psi_{ij}\varphi^{t-1}_{i,j} g_t$, where g_t is an automorphism of $E_i[j]$ and $\varphi^0_{i,j} = \mathrm{id}_{E_i[j]}$. On the other hand, for any $t \geq s+1$, we have

$\psi_{ij}\varphi_{i,j}^{t-1} = 0$. Hence, for $t \geq 2r(s+1)-2$, any composition of t irreducible homomorphisms between indecomposable modules in $\mathcal{T}_\lambda^A$ leading from $E_i[j]$ to $E_i[j-1]$ is zero. Summing up, we conclude that every homomorphism $f \in \mathrm{Hom}_A(E_i[j], E_{i-1}[j]) = \mathrm{rad}_A(E_i[j], E_{i-1}[j])$ is of the form

$$f = \sum_{t=1}^{s} \psi_{ij}\varphi_{i,j}^{t-1} g_t,$$

where each $g_t\colon E_i[j] \to E_i[j]$ is either zero or an automorphism. We note also that, for each $t \in \{1,\ldots,s\}$, $\mathrm{Im}\,\psi_{ij}\varphi_{i,j}^{t-1}$ is the unique regular submodule of $E_{i-1}[j]$ isomorphic to $E_{i+rt-1}[j-rt+1]$. Therefore, the K-vector space $\mathrm{Hom}_A(M,\tau_A M) = \mathrm{rad}_A(M,\tau_A M)$ admits the chain of subspaces

$$\begin{aligned}0 = \mathrm{rad}_A^{2r(s+1)-2}(M,\tau_A M) \subset \mathrm{rad}_A^{2rs-2}(M,\tau_A M) \subset \ldots \\ \subset \mathrm{rad}_A^{2r-2}(M,\tau_A M) \subset \mathrm{Hom}_A(M,\tau_A M)\end{aligned}$$

such that for each $t \in \{1,\ldots,s\}$,

$$\begin{aligned}&\mathrm{rad}_A^{2rt-2}(M,\tau_A M)/\,\mathrm{rad}_A^{2r(t+1)-2}(M,\tau_A M) \\ &\qquad = \mathrm{rad}_A^{2rt-2}(M,\tau_A M)/\,\mathrm{rad}_A^{2rt-1}(M,\tau_A M)\end{aligned}$$

is a right F_M-module generated by $\psi_{ij}\varphi_{i,j}^{t-1} + \mathrm{rad}_A^{2rt-1}(M,\tau_A M)$, where $\varphi_{ij}^0 = \mathrm{id}_M$. Hence, we obtain

$$\dim_K \mathrm{Ext}_A^1(M,M) = \dim_K \mathrm{Hom}_A(M,\tau_A M) = s\dim_K F_M. \qquad \square$$

Proposition 8.13 has the following immediate corollary.

Corollary 8.14. *Let A be an indecomposable finite dimensional hereditary K-algebra of Euclidean type over a field K, $\lambda \in \Lambda(A)$, and M be an indecomposable module in the stable tube $\mathcal{T}_\lambda^A$. Then the following statements hold:*

(i) $\dim_K \mathrm{End}_A(M) \geq \dim_K \mathrm{Ext}_A^1(M,M)$.

(ii) $\dim_K \mathrm{End}_A(M) = \dim_K \mathrm{Ext}_A^1(M,M)$ *if and only if* r_λ^A *divides* $\mathrm{rl}(M)$.

We will determine now the values of the Euler quadratic form χ_A of a finite dimensional hereditary algebra A of Euclidean type on the classes $[M]$ of indecomposable modules M in $\mathrm{mod}\,A$.

We need a preliminary lemma.

Lemma 8.15. *Let A be an indecomposable finite dimensional hereditary K-algebra of Euclidean type over a field K, $\lambda \in \Lambda(A)$, and M, N indecomposable modules in the stable tube $\mathcal{T}_\lambda^A$. Then the division K-algebras F_M and F_N are isomorphic.*

Proof. Let $E_1 = E_1^{(\lambda)}, \dots, E_r = E_r^{(\lambda)}$) with $r = r_\lambda^A$ be the simple regular modules lying on the mouth of $\mathcal{T}_\lambda^A$ such that $\tau_A E_{i+1} = E_i$ for $i \in \{1, \dots, r\}$ and $E_{r+1} = E_1$. It follows from the general fact (Lemma III.9.5) that, for any indecomposable module X in the stable tube $\mathcal{T}_\lambda^A$, the Auslander–Reiten translation τ_A induces an isomorphism of division K-algebras F_X and $F_{\tau_A X}$, and τ_A^{-1} induces an isomorphism of division K-algebras F_X and $F_{\tau_A^{-1} X}$. Take now an irreducible monomorphism $u_{ij}\colon E_i[j-1] \to E_i[j]$ for some $i \in \{1, \dots, r\}$ and $j \in \mathbb{N}^+$. Then it follows from Lemma III.9.3 and Corollary III.9.4 that the space of irreducible homomorphisms

$$\operatorname{irr}_A(E_i[j-1], E_i[j]) = \operatorname{rad}_A(E_i[j-1], E_i[j]) / \operatorname{rad}_A^2(E_i[j-1], E_i[j])$$

from $E_i[j-1]$ to $E_i[j]$ is the left $F_{E_i[j]}$-module generated by $\bar{u}_{ij} = u_{ij} + \operatorname{rad}_A^2(E_i[j-1], E_i[j])$ and the right $F_{E_i[j-1]}$-module generated by $\bar{u}_{ij}$. Then assigning to any element $x \in F_{E_i[j-1]}$ the unique element $\varphi(x) \in F_{E_i[j]}$ such that $\bar{u}_{ij} x = \varphi(x) \bar{u}_{ij}$ one defines an isomorphism $\varphi\colon F_{E_i[j-1]} \to F_{E_i[j]}$ of division K-algebras. This implies that, for any $i \in \{1, \dots, r\}$ and $j \in \mathbb{N}^+$, the division K-algebra $F_{E_i[j]}$ is isomorphic to the the division K-algebra $F_{E_i[1]}$ of the simple regular module $E_i[1] = E_i$. Finally, we note that every indecomposable module in $\mathcal{T}_\lambda^A$ belongs to the τ_A-orbit of a module $E_i[j]$ for a fixed $i \in \{1, \dots, r\}$ and some $j \in \mathbb{N}^+$, Therefore, we conclude that, for any indecomposable modules M and N in $\mathcal{T}_\lambda^A$, the division K-algebras F_M and F_N are isomorphic. □

Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, and $\Lambda(A)$ the set of τ_A-orbits of simple regular modules in $\operatorname{mod} A$. It follows from Lemma 8.15 that, for any $\lambda \in \Lambda(A)$, there exists a finite dimensional division K-algebra $F^{(\lambda)}$ which is isomorphic to the division K-algebra F_M of any indecomposable module M in the stable tube $\mathcal{T}_\lambda^A$. We define $f_\lambda^A = \dim_K F^{(\lambda)}$. Let $S_1, \dots, S_n$ be a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$, and $f_i^A = \dim_K \operatorname{End}_A(S_i)$. Then we have the following description of the values of the Euler quadratic form χ_A on the classes $[M]$ of indecomposable modules M in $\operatorname{mod} A$.

Theorem 8.16. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, and M be an indecomposable module in* mod *A. Then*

$$\chi_A([M]) \in \{0\} \cup \left\{f_1^A, \dots, f_n^A\right\} \cup \left\{f_\lambda^A \mid \lambda \in \Lambda(A)\right\}.$$

Proof. It follows from Proposition 6.8 that, if M belongs to $\mathcal{P}(A) \cup \mathcal{Q}(A)$, then $\chi_A([M]) = f_i^A$ for some $i \in \{1, \dots, r\}$. Therefore, assume that M is a regular module. Hence, M belongs to a stable tube $\mathcal{T}_\lambda^A$, with $\lambda \in \Lambda(A)$, by Theorem 8.12. Applying Proposition 8.13, we conclude that

(1) $\chi_A([M]) = \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M,M) = 0$, if the rank $r = r^A_\lambda$ of $\mathcal{T}^A_\lambda$ divides $\operatorname{rl}(M)$;

(2) $\chi_A([M]) = \dim_K \operatorname{End}_A(M) - \dim_K \operatorname{Ext}^1_A(M,M) = \dim_K F_M = f^A_\lambda$, if the rank $r = r^A_\lambda$ of $\mathcal{T}^A_\lambda$ does not divide $\operatorname{rl}(M)$. □

We note that for any nonnegative number m in the set $\{0\} \cup \{f^A_1, \dots, f^A_n\} \cup \{f^A_\lambda \mid \lambda \in \Lambda(A)\}$ there exists an indecomposable module M in mod A such that $\chi_A([M]) = m$.

Corollary 8.17. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over an algebraically closed field K, and M be an indecomposable module in* mod A. *Then $\chi_A([M]) \in \{0, 1\}$.*

Proof. Since K is an algebraically closed field, every finite dimensional division K-algebra F is isomorphic to K. Then we have $f^A_i = 1$ for $i \in \{1, \dots, n\}$ and $f^A_\lambda = 1$ for $\lambda \in \Lambda(A)$. □

It follows from Propositions 6.6 and 6.7 that if M is an indecomposable postprojective or preinjective module over a finite dimensional hereditary K-algebra A over a field K, then M is uniquely determined (up to isomorphism) by its class $[M]$ in $K_0(A)$. This is not the case for arbitrary indecomposable regular modules over hereditary algebras of infinite representation type. Our next aim is to determine the classes $[M]$ of indecomposable regular modules M over indecomposable hereditary algebras of Euclidean type, and provide a criterion for M to be uniquely determined by $[M]$ (see [S4]).

Theorem 8.18. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, and $\mathcal{T}^A_\lambda$ is a stable tube in Γ_A of rank $r^A_\lambda \geq 2$. Assume that M and N are nonisomorphic indecomposable modules in $\mathcal{T}^A_\lambda$. Then $[M] = [N]$ if and only if* $\operatorname{rl}(M) = c r^A_\lambda = \operatorname{rl}(N)$ *for some positive integer c.*

Proof. Let $r = r^A_\lambda$ and $E_1 = E^{(\lambda)}_1, \dots, E_r = E^{(\lambda)}_r$ be the simple regular modules lying on the mouth of $\mathcal{T}^A_\lambda$ such that $\tau_A E_{i+1} = E_i$ for $i \in \{1, \dots, r\}$ and $E_{r+1} = E_1$. We put

$$e = \sum_{t=1}^{r} \ell(E_t).$$

Let Z be an indecomposable module in $\mathcal{T}^A_\lambda$. Then $Z = E_i[m]$ for some $i \in \{1, \dots, r\}$ and $m \geq 1$, and $\operatorname{rl}(Z) = m = kr + s$ for $k \geq 0$ and $0 \leq s < r$. For $m \geq 2$, we have in $\mathcal{T}^A_\lambda$ a sectional path

$$E_i[1] \longrightarrow E_i[2] \longrightarrow \dots \longrightarrow E_i[m-1] \longrightarrow E_i[m] = Z$$

and the exact sequences

$$0 \longrightarrow E_i[j] \xrightarrow{u_{i\,j+1}} E_i[j+1] \xrightarrow{p'_{i\,j+1}} E_{i+j}[1] \longrightarrow 0,$$

for $j \in \{1, \dots, m-1\}$. Hence we get

$$[Z] = \sum_{t=0}^{m-1} \big[E_{i+t}[1]\big] = \sum_{t=0}^{m-1} [E_{i+t}],$$

and so

$$[Z] = \begin{cases} k\Big(\sum\limits_{t=1}^{r}[E_t]\Big), & \text{if } s = 0\,, \\ k\Big(\sum\limits_{t=1}^{r}[E_t]\Big) + \sum\limits_{p=0}^{s-1}[E_{i+p}], & \text{if } s \geq 1\,. \end{cases}$$

In particular, $ke \leq \ell(Z) < (k+1)e$ and $ke = \ell(Z)$ if and only if $s = 0$. Assume $\mathrm{rl}(M) = cr = \mathrm{rl}(N)$ for some $c \geq 1$. Then

$$[M] = c\left(\sum_{t=1}^{r}[E_t]\right) = [N].$$

Conversely, assume that $[M] = [N]$. Then clearly $\ell(M) = \ell(N)$, and, by the above remarks, $\mathrm{rl}(M) = cr + a$ and $\mathrm{rl}(N) = cr + b$ for some $c \geq 0$ with $0 \leq a < r$ and $0 \leq b < r$. We shall show that $a = 0$ and $b = 0$. First observe that $a = 0$ if and only if $b = 0$. Therefore, assume that $a \neq 0$ and $b \neq 0$. We claim that there exist nonisomorphic indecomposable modules X and Y in $\mathcal{T}_\lambda^A$ such that $[X] = [Y]$ and $\mathrm{rl}(X) < r$, $\mathrm{rl}(Y) < r$. Assume $c \geq 1$. Observe that $M = E_i[cr + a]$ and $N = E_k[cr + b]$ for some $i, k \in \{1, \dots, r\}$. Obviously, $i \neq k$, because M and N are nonisomorphic modules in $\mathcal{T}_\lambda^A$ of the same length. Invoking now the almost split sequences in add $\mathcal{T}_\lambda^A$ described by Proposition 3.6, we conclude that

$$\big[E_i[r]\big] + \big[E_{i+a}[cr]\big] = \big[E_i[cr+a]\big] + \big[E_{i+a}[r-a]\big],$$
$$\big[E_k[r]\big] + \big[E_{k+b}[cr]\big] = \big[E_k[cr+b]\big] + \big[E_{k+b}[r-b]\big].$$

Since $\big[E_{i+a}[cr]\big] = c\big(\sum_{t=1}^{r}[E_t]\big) = \big[E_{k+b}[cr]\big]$ and $\big[E_i[cr+a]\big] = [M] = [N] = \big[E_k[cr+b]\big]$, we obtain $\big[E_{i+a}[r-a]\big] = \big[E_{k+b}[r-b]\big]$. Moreover, $0 < r - a < r$ and $0 < r - b < r$, because $0 < a < r$ and $0 < b < r$. Hence we may take $X = E_{i+a}[r-a]$ and $Y = E_{k+b}[r-b]$. Observe that $X \ncong Y$. Indeed, if $X \cong Y$, then $r - a = \mathrm{rl}(X) = \mathrm{rl}(Y) = r - b$, and so $a = b$. Moreover, then $i + a = k + b$, and hence $i = k$, which is impossible by our assumption that $M \ncong N$ with $[M] = [N]$. Therefore, X

and Y are nonisomorphic indecomposable modules in $\mathcal{T}_\lambda^A$ with $[X] = [Y]$ and $\mathrm{rl}(X) < r$, $\mathrm{rl}(Y) < r$. In particular, we conclude from Proposition 8.13 that $\mathrm{Ext}_A^1(X, X) = 0$ and $\mathrm{Ext}_A^1(Y, Y) = 0$. Invoking now the Euler bilinear form $\langle -, - \rangle_A \colon K_0(A) \times K_0(A) \to \mathbb{Z}$, we obtain

$$\begin{aligned}
\dim_K \mathrm{Hom}_A(X, Y) - \dim_K \mathrm{Ext}_A^1(X, Y) &= \big\langle [X], [Y] \big\rangle_A = \big\langle [X], [X] \big\rangle_A \\
&= \dim_K \mathrm{End}_A(X) - \dim_K \mathrm{Ext}_A^1(X, X) \\
&= \dim_K \mathrm{End}_A(X) > 0, \\
\dim_K \mathrm{Hom}_A(Y, X) - \dim_K \mathrm{Ext}_A^1(Y, X) &= \big\langle [Y], [X] \big\rangle_A = \big\langle [Y], [Y] \big\rangle_A \\
&= \dim_K \mathrm{End}_A(Y) - \dim_K \mathrm{Ext}_A^1(Y, Y) \\
&= \dim_K \mathrm{End}_A(Y) > 0.
\end{aligned}$$

Hence we conclude that $\mathrm{Hom}_A(X, Y) \neq 0$ and $\mathrm{Hom}_A(Y, X) \neq 0$. Assume $X = E_d[j]$ and $Y = E_q[t]$ with $d, q \in \{1, \dots, r\}$ and $j, t \in \{1, \dots, r-1\}$. Then $d \neq q$, because $X \ncong Y$ and $[X] = [Y]$. We may assume (without loss of generality) that $1 \leq d < q \leq r$. Since $\mathrm{Hom}_A(X, Y) \neq 0$, we have $q \leq d + j - 1$ and $j - (q - d) \leq t$. Observe also that $t > j - (q - d)$. Indeed, if $t = j - (q - d)$, then we have the epimorphism $p_{q-1\, j-(q-d)+1} \cdots p_{d+1\, j-1} p_{d\, j} \colon X \to Y$, because $X = E_d[j]$ and $Y = E_q[j - (q - d)]$, and hence a contradiction with $X \ncong Y$ and $[X] = [Y]$. Further, $\mathrm{Hom}_A(X, Y) = \mathrm{rad}_A^{2(q-d)+t-j}(X, Y)$, $\mathrm{rad}_A^{2(q-d)+t-j+1}(X, Y) = 0$, and consequently, $\mathrm{Hom}_A(X, Y)$ is a right F_X-module generated by the composition of irreducible homomorphisms

$$u_{q\,t} \cdots u_{q\, j-(q-d)+1}\, p_{q-1\, j-(q-d)+1} \cdots p_{d+1\, j-1}\, p_{d\, j} \colon X \longrightarrow Y.$$

This shows that $\dim_K \mathrm{Hom}_A(X, Y) = \dim_K F_X$. Moreover, it follows from Proposition 8.13 that $\dim_K \mathrm{End}_A(X) = \dim_K F_X$, because $\mathrm{rl}(X) < r$. Therefore, applying the equality established above, we obtain that

$$\dim_K \mathrm{Ext}_A^1(X, Y) = \dim_K \mathrm{Hom}_A(X, Y) - \dim_K \mathrm{End}_A(X) = 0.$$

We also note that $\mathrm{Hom}_A(X, \tau_A Y) \neq 0$, because $X = E_d[j]$, $\tau_A Y = E_{q-1}[t]$, $d \leq q - 1$, and $t > j - (q - d)$. Then, applying Corollary III.6.4, we obtain $\dim_K \mathrm{Ext}_A^1(Y, X) = \dim_K D\,\mathrm{Hom}_A(X, \tau_A Y) = \dim_K \mathrm{Hom}_A(X, \tau_A Y) \neq 0$. Since $\mathrm{Hom}_A(Y, X) \neq 0$, similar arguments show that $\dim_K \mathrm{Hom}_A(Y, X) = \dim_K F_Y$, $\dim_K \mathrm{Ext}_A^1(Y, X) = 0$, and $\dim_K \mathrm{Ext}_A^1(X, Y) \neq 0$. This contradiction proves that $a = 0 = b$, and consequently $\mathrm{rl}(M) = cr = \mathrm{rl}(N)$, as required. □

We are now in position to prove the announced criterion (see [S4]).

Theorem 8.19. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, and M be an indecomposable module in* mod A *with $\chi_A([M]) > 0$. Then for any indecomposable module N in* mod A, *$[M] = [N]$ if and only if $M \cong N$.*

Proof. For M lying in $\mathcal{P}(A)$ or $\mathcal{Q}(A)$, the equivalence follows from Propositions 6.6 and 6.7. Therefore, assume that M belongs to a stable tube $\mathcal{T}_\lambda^A$, for some $\lambda \in \Lambda(A)$. Let N be an indecomposable module in mod A with $[M] = [N]$. Then we have

$$\begin{aligned}\dim_K \operatorname{Hom}_A(M,N) - \dim_K \operatorname{Ext}_A^1(M,N) &= \langle [M],[N]\rangle_A = \langle [M],[M]\rangle_A \\ &= \chi_A([M]) > 0, \\ \dim_K \operatorname{Hom}_A(N,M) - \dim_K \operatorname{Ext}_A^1(N,M) &= \langle [N],[M]\rangle_A = \langle [M],[M]\rangle_A \\ &= \chi_A([M]) > 0,\end{aligned}$$

and consequently $\operatorname{Hom}_A(M,N) \neq 0$ and $\operatorname{Hom}_A(N,M) \neq 0$. Since $\operatorname{Hom}_A(\mathcal{T}_\lambda^A, \mathcal{P}(A)) = 0$, $\operatorname{Hom}_A(\mathcal{Q}(A), \mathcal{T}_\lambda^A) = 0$, and $\operatorname{Hom}_A(\mathcal{T}_\lambda^A, \mathcal{T}_\mu^A) = 0$, $\operatorname{Hom}_A(\mathcal{T}_\mu^A, \mathcal{T}_\lambda^A) = 0$, for $\mu \neq \lambda$ in $\Lambda(A)$, we infer that N belongs to $\mathcal{T}_\lambda^A$. Further, $\chi_A([M]) > 0$ implies that the rank r_λ^A of $\mathcal{T}_\lambda^A$ does not divide the regular length $\operatorname{rl}(M)$ of M, by Corollary 8.14, and so $r_\lambda^A \geq 2$. Now it follows from Theorem 8.18 that $M \cong N$. Obviously, $M \cong N$ implies $[M] = [N]$. □

Proposition 8.20. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Assume that $|\Lambda(A)| \geq 2$, or $r_\lambda^A \geq 2$ for some $\lambda \in \Lambda(A)$. Then there exist nonisomorphic indecomposable modules M and N in* mod A *with $[M] = [N]$.*

Proof. Assume there is $\lambda \in \Lambda(A)$ with $r_\lambda^A \geq 2$. Then the stable tube $\mathcal{T}_\lambda^A$ admits nonisomorphic indecomposable modules M and N with $\operatorname{rl}(M) = cr_\lambda^A = \operatorname{rl}(N)$, and hence with $[M] = [N]$ (see Theorem 8.18). Assume $|\Lambda(A)| \geq 2$. It is enough to consider the case where $r_\lambda^A = 1 = r_\mu^A$ for $\mu \neq \lambda$ in $\Lambda(A)$. Let $E^{(\lambda)}$ and $E^{(\mu)}$ be the simple regular modules lying on the mouth of stable tubes $\mathcal{T}_\lambda^A$ and $\mathcal{T}_\mu^A$, respectively. Then it follows from Proposition 8.13, that

$$\begin{aligned}\chi_A\big([E^{(\lambda)}]\big) &= \dim_K \operatorname{End}_A\big(E^{(\lambda)}\big) - \dim_K \operatorname{Ext}_A^1\big(E^{(\lambda)}, E^{(\lambda)}\big) = 0, \\ \chi_A\big([E^{(\mu)}]\big) &= \dim_K \operatorname{End}_A\big(E^{(\mu)}\big) - \dim_K \operatorname{Ext}_A^1\big(E^{(\mu)}, E^{(\mu)}\big) = 0,\end{aligned}$$

and hence $[E^{(\lambda)}], [E^{(\mu)}] \in \operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$. Hence there are positive integers m_λ and m_μ such that $[E^{(\lambda)}] = m_\lambda \mathbf{h}_A$ and $[E^{(\mu)}] = m_\mu \mathbf{h}_A$. Let $M = E^{(\lambda)}[m_\mu]$ and $N = E^{(\mu)}[m_\lambda]$. Then M and N are indecomposable modules that

$$\begin{aligned}[M] &= \big[E^{(\lambda)}[m_\mu]\big] = m_\mu\big[E^{(\lambda)}\big] = m_\mu m_\lambda \mathbf{h}_A \\ &= m_\lambda\big[E^{(\mu)}\big] = \big[E^{(\mu)}[m_\lambda]\big] = [N].\end{aligned}$$

Moreover, M and N are nonisomorphic, because they belong to different stable tubes $\mathcal{T}_\lambda^A$ and $\mathcal{T}_\mu^A$, respectively. □

We have also the following proposition showing that all positive vectors of the radical of the Euler form of a finite dimensional hereditary algebra of Euclidean type occur as composition factors of indecomposable modules.

Proposition 8.21. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then there exists $\lambda \in \Lambda(A)$ such that, for any positive integer m and indecomposable module Z in the stable tube $\mathcal{T}_\lambda^A$, we have*

$$\mathrm{rl}(Z) = m r_\lambda^A \text{ if and only if } [Z] = m\mathbf{h}_A.$$

Proof. It follows from Lemma 8.1 that there exists an indecomposable module X in $\mathcal{R}(A)$ such that $[X] = \mathbf{h}_A$. Hence, by Theorem 8.12, X belongs to a stable tube $\mathcal{T}_\lambda^A$ with $\lambda \in \Lambda(A)$. Moreover, we have

$$\dim_K \mathrm{End}_A(X) - \dim_K \mathrm{Ext}_A^1(X, X) = \chi_A([X]) = \chi_A(\mathbf{h}_A) = 0,$$

so r_λ^A divides $\mathrm{rl}(X)$, by Corollary 8.14. Take now a module Y in $\mathcal{T}_\lambda^A$ with $\mathrm{rl}(Y) = r_\lambda^A$. Applying again Corollary 8.14, we conclude that $\chi_A([Y]) = 0$, and consequently $[Y] \in \mathrm{rad}\,\chi_A = \mathbb{Z}\mathbf{h}_A$. Thus there exists a positive integer p such that $[Y] = p\mathbf{h}_A$. On the other hand, for any indecomposable module Z in $\mathcal{T}_\lambda^A$ with $\mathrm{rl}(Z) = m r_\lambda^A$, we have

$$[Z] = m \left(\sum_{t=1}^{r_\lambda^A} \left[E_t^{(\lambda)} \right] \right),$$

where $E_1^{(\lambda)}, \dots, E_{r_\lambda^A}^{(\lambda)}$ are the simple regular modules forming the mouth of $\mathcal{T}_\lambda^A$. This implies that, if $\mathrm{rl}(Z) = m r_\lambda^A$, then $[Z] = mp\mathbf{h}_A$. Hence, since X is an indecomposable module in $\mathcal{T}_\lambda^A$ with $[X] = \mathbf{h}_A$, we conclude that $p = 1$ and $\mathrm{rl}(X) = r_\lambda^A$. In particular, for any indecomposable module Z in $\mathcal{T}_\lambda^A$ with $\mathrm{rl}(Z) = m r_\lambda^A$, we have $[Z] = m\mathbf{h}_A$. Conversely, if Z is an indecomposable module in $\mathcal{T}_\lambda^A$ with $[Z] = m\mathbf{h}_A$ then $\chi_A([Z]) = \chi_A(m\mathbf{h}_A) = 0$. Then r_λ^A divides $\mathrm{rl}(Z)$, by Corollary 8.14. This leads to $\mathrm{rl}(Z) = m r_\lambda^A$, by the above discussion. □

Theorem 8.22. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then all but finitely many isomorphism classes of indecomposable modules in* mod A *are sincere modules.*

Proof. It follows from Theorem 6.11, that all but finitely many isomorphism classes of indecomposable modules in $\mathcal{P}(A) \cup \mathcal{Q}(A)$ are sincere modules. Further, by Theorem 8.11, for any $\lambda \in \Lambda(A)$, all indecomposable modules X in $\mathcal{T}_\lambda^A$ with

$\mathrm{rl}(X) \geq r_\lambda^A$ are sincere modules, and hence all but finitely many indecomposable modules in $\mathcal{T}_\lambda^A$ are sincere. Moreover, if $r_\lambda^A = 1$, then all indecomposable modules in $\mathcal{T}_\lambda^A$ are sincere. Finally, we know from Theorem 8.12 that

$$\sum_{\lambda \in \Lambda(A)} \left(r_\lambda^A - 1\right) \leq n,$$

where n is the rank of $K_0(A)$, and hence all but finitely many stable tubes $\mathcal{T}_\lambda^A$, $\lambda \in \Lambda(A)$, have rank r_λ^A equal 1. Summing up, we conclude that all but finitely many isomorphism classes of indecomposable modules in mod A are sincere. □

We present also the following immediate consequence of Proposition 6.9 and Theorems 8.11, 8.12.

Theorem 8.23. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then every component of Γ_A is generalized standard.*

Let A be a finite dimensional hereditary K-algebra over a field K. An indecomposable module X in mod A is said to be *rigid* if $\mathrm{Ext}_A^1(X, X) = 0$, and *stone* if it is rigid and brick. We will prove in Corollary 9.16 that, if A is a hereditary algebra, then every rigid module in mod A is a stone. It follows from Propositions 6.6 and 6.7 that every indecomposable module in a postprojective component $\mathcal{P}(A)$ and the preinjective component $\mathcal{Q}(A)$ of the Auslander–Reiten quiver Γ_A of a hereditary algebra A is a stone.

The following proposition provides a characterization of bricks and stones in the regular parts of hereditary algebras of Euclidean type.

Proposition 8.24. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, E a simple regular module, and*

$$E = E[1] \longrightarrow E[2] \longrightarrow \cdots \longrightarrow E[m-1] \longrightarrow E[m]$$

a sectional path in Γ_A with $m \geq 2$ and source E. Then the following statements are equivalent:

(i) *$E[m]$ is a brick.*

(ii) *$E[m-1]$ is a stone.*

(iii) *$E[1], \dots, E[m-1]$ are stones.*

(iv) *$E, \tau_A^{-1}E, \dots, \tau_A^{-(m-1)}E$ are pairwise orthogonal stones.*

(v) $\mathrm{Hom}_A\left(\bigoplus_{i=1}^{m-1} E[i], \bigoplus_{i=1}^{m-1} \tau_A E[i]\right) = 0.$

Proof. This follows from Propositions 8.10 and 8.13, Theorems 8.11 and 8.12, and their proofs. □

In particular, we obtain the following consequences of Proposition 8.24 and Lemma VIII.7.4.

Corollary 8.25. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K, n the rank of $K_0(A)$, and X be an indecomposable regular module in* $\operatorname{mod} A$. *Then the following statements hold:*

(i) *If X is a stone, then* $\operatorname{rl}(X) \leq n$.
(ii) *If X is a brick, then* $\operatorname{rl}(X) \leq n+1$.

Moreover, we have the following fact.

Corollary 8.26. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then the number of isomorphism classes of regular stones in* $\operatorname{mod} A$ *is finite.*

Proof. This follows from Corollary 8.25 and the fact that the number of stable tubes in $\mathcal{R}(A)$ of rank at least 2 is bounded by the rank of $K_0(A)$ (see Theorem 8.12). ☐

Example 8.27. Let K be a field, Q the *Kronecker quiver*

$$\begin{tikzcd} 1 \bullet & \bullet\, 2 \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\beta"] \end{tikzcd},$$

and $A = KQ$ the path algebra of Q over K. Then A is isomorphic to the matrix K-algebra

$$\begin{bmatrix} K & 0 \\ K^2 & K \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ (b,c) & d \end{bmatrix} \,\middle|\, a,b,c,d \in K \right\},$$

called the *Kronecker algebra* over K (see Example I.1.4(b)). Since Q is an acyclic quiver, A is a 4-dimensional hereditary K-algebra (Theorem I.9.6). Moreover, there is a K-linear equivalence $\operatorname{mod} A \xrightarrow{\sim} \operatorname{rep}_K(Q)$ of $\operatorname{mod} A$ with the category of finite dimensional K-linear representations of Q. We identify $\operatorname{mod} A$ with $\operatorname{rep}_K(Q)$ as well as the Grothendieck group $K_0(A)$ with $\mathbb{Z}^2$. Then the indecomposable projective modules in $\operatorname{mod} A = \operatorname{rep}_K(Q)$ associated to the vertices 1 and 2 are of the form

$$\begin{tikzcd} P_1\colon K & 0 \arrow[l, shift right] \arrow[l, shift left] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P_2\colon K^2 & K \arrow[l, shift right, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"'] \arrow[l, shift left, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] \end{tikzcd},$$

and the indecomposable injective modules in $\operatorname{mod} A$ associated to the vertices 1 and 2 are of the form

$$\begin{tikzcd} I_1\colon K & K^2 \arrow[l, shift right, "{[1,0]}"'] \arrow[l, shift left, "{[0,1]}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} I_2\colon 0 & K \arrow[l, shift right] \arrow[l, shift left] \end{tikzcd}.$$

Clearly, then $S_1 = P_1$ and $S_2 = I_2$ are the simple modules associated to 1 and 2. Further, the valued quiver Q_A of A is of the form

$$1 \xleftarrow{(2,2)} 2,$$

and hence of Euclidean type $\widetilde{\mathbb{A}}_{12}$. Moreover, $f_1 = \dim_K \operatorname{End}_A(S_1) = 1$ and $f_2 = \dim_K \operatorname{End}_A(S_2) = 1$, and the Euler quadratic form $\chi_A\colon \mathbb{Z}^2 \to \mathbb{Z}$ is given by

$$\chi_A(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1x_2 = (x_1 - x_2)^2,$$

for any vector $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$. Observe also that χ_A is positive semidefinite of corank 1, with $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$ for $\mathbf{h}_A = (1,1)$. We calculate now the Coxeter transformations $\varphi_A, \varphi_A^{-1}\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ and the defect $\partial_A\colon \mathbb{Z}^2 \to \mathbb{Z}$ of A. By definition, we have $\varphi_A([P_1]) = -[I_1]$ and $\varphi_A([P_2]) = -[I_2]$, where $[P_1] = (1,0)$, $[P_2] = (2,1)$, $[I_1] = (1,2)$, and $[I_2] = (0,1)$. Hence,

$$\varphi_A(e_1) = -e_1 - 2e_2 \quad \text{and} \quad \varphi_A(2e_1 + e_2) = -e_2,$$

and consequently $\varphi_A\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by

$$\varphi_A(\mathbf{x}) = (-x_1 + 2x_2, -2x_1 + 3x_2)$$

for any $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$. Similarly, $\varphi_A^{-1}([I_1]) = -[P_1]$ and $\varphi_A^{-1}([I_2]) = -[P_2]$ lead to the equalities

$$\varphi_A^{-1}(e_1 + 2e_2) = -e_1 \quad \text{and} \quad \varphi_A^{-1}(e_2) = -2e_1 - e_2,$$

and hence $\varphi_A^{-1}\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by

$$\varphi_A^{-1}(\mathbf{x}) = (3x_1 - 2x_2, 2x_1 - x_2)$$

for any $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$. Observe also that for any $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$, we have

$$\begin{aligned} \varphi_A(\mathbf{x}) &= (-x_1 + 2x_2, -2x_1 + 3x_2) = (x_1, x_2) + (-2x_1 + 2x_2)(1,1) \\ &= \mathbf{x} + 2(x_2 - x_1)\mathbf{h}_A. \end{aligned}$$

Hence the defect $\partial_A\colon \mathbb{Z}^2 \to \mathbb{Z}$ is given by

$$\partial_A(\mathbf{x}) = 2(x_2 - x_1)$$

for any $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$.

It follows from Theorem 6.1 that the postprojective component $\mathcal{P}(A)$ of Γ_A is of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
 & P_2 \arrow[dr, "(2,2)"] & & \tau_A^{-1}P_2 \arrow[dr, dash] & & {} \arrow[dr] & & \tau_A^{-m}P_2 \arrow[dr, "(2,2)"] & & {} & \\
P_1 \arrow[ur, "(2,2)"] & & \tau_A^{-1}P_1 \arrow[ur, "(2,2)"] & & {} & \cdots & \tau_A^{-m}P_1 \arrow[ur, "(2,2)"] & & \tau_A^{-m-1}P_1 \arrow[ur, dash] & & \cdots
\end{tikzcd}$$

and consists of the modules $\tau_A^{-m} P_1$ and $\tau_A^{-m} P_2$, $m \geq 0$, having the composition vectors

$$[\tau_A^{-m} P_1] = \varphi_A^{-m}([P_1]) = \varphi_A^{-m}(1,0) = (2m+1, 2m),$$
$$[\tau_A^{-m} P_2] = \varphi_A^{-m}([P_2]) = \varphi_A^{-m}(2,1) = (2m+2, 2m+1).$$

For each $r \geq 0$, consider the module in $\operatorname{mod} A = \operatorname{rep}_K(Q)$ of the form

$$P^{(r)}\colon\ K^{r+1} \underset{B^{(r)}}{\overset{A^{(r)}}{\leftleftarrows}} K^r\ ,$$

where $A^{(r)}$ and $B^{(r)}$ are the matrices

$$A^{(r)} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad B^{(r)} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

A simple calculation shows that $\operatorname{End}_A(P^{(r)}) \cong K$, and hence $P^{(r)}$ is an indecomposable module (representation) for any $r > 0$. Moreover, we have

$$[P^{(2m)}] = (2m+1, 2m) = [\tau_A^{-m} P_1],$$
$$[P^{(2m+1)}] = (2m+2, 2m+1) = [\tau_A^{-m} P_2],$$

for all $m \geq 0$. Hence, applying Proposition 6.6, we conclude that

$$\tau_A^{-m} P_1 \cong P^{(2m)} \quad \text{and} \quad \tau_A^{-m} P_2 \cong P^{(2m+1)}$$

in $\operatorname{mod} A = \operatorname{rep}_K(Q)$, for any $m \geq 0$.

It follows from Theorem 6.2 that the preinjective component $\mathcal{Q}(A)$ of Γ_A is of the form $\mathbb{N}Q_A^{\mathrm{op}}$

$$\begin{tikzcd}[column sep=small]
 & \tau_A^{m+1} I_2 \arrow[dr, "{(2,2)}"] & & \tau_A^m I_2 \arrow[dr] & & {} \arrow[dr] & & \tau_A I_2 \arrow[dr, "{(2,2)}"] & & I_2 \\
\cdots \arrow[ur] & & \tau_A^m I_1 \arrow[ur, "{(2,2)}"'] & & \cdots & & \tau_A I_1 \arrow[ur, "{(2,2)}"'] & & I_1 \arrow[ur, "{(2,2)}"'] &
\end{tikzcd}$$

and consists of the modules $\tau_A^m I_1$ and $\tau_A^m I_2$, $m \geq 0$, having the composition vectors

$$[\tau_A^m I_1] = \varphi_A^m([I_1]) = \varphi_A^m(1,2) = (2m+1, 2m+2),$$
$$[\tau_A^m I_2] = \varphi_A^m([I_2]) = \varphi_A^m(0,1) = (2m, 2m+1).$$

For each $r \geq 0$, consider the module in $\operatorname{mod} A = \operatorname{rep}_K(Q)$ of the form

$$Q^{(r)}\colon\ K^r \underset{D^{(r)}}{\overset{C^{(r)}}{\leftleftarrows}} K^{r+1}\ ,$$

where $C^{(r)}$ and $D^{(r)}$ are the matrices

$$C^{(r)} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix} \text{ and } D^{(r)} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

A simple calculation shows that $\operatorname{End}_A(Q^{(r)}) \cong K$, and hence $Q^{(r)}$ is an indecomposable module (representation) for any $r \geq 0$. Moreover, we have

$$[Q^{(2m+1)}] = (2m+1, 2m+2) = [\tau_A^m I_1],$$
$$[Q^{(2m)}] = (2m, 2m+1) = [\tau_A^m I_2],$$

for all $m \geq 0$. Hence, applying Proposition 6.7, we conclude that

$$\tau_A^m I_1 \cong Q^{(2m+1)} \quad \text{and} \quad \tau_A^m I_2 \cong Q^{(2m)}$$

in $\operatorname{mod} A = \operatorname{rep}_K(Q)$, for any $m \geq 0$.

We also note that for any indecomposable modules M in $\mathcal{P}(A)$ and N in $\mathcal{Q}(A)$ we have

$$\partial_A([M]) = -2 < 0 \quad \text{and} \quad \partial_A([N]) = 2 > 0.$$

Now we will consider indecomposable modules in the regular part $\mathcal{R}(A)$ of Γ_A.

Consider the polynomial algebra $K[x]$ in one variable x over K and the category $\operatorname{mod} K[x]$ of finite dimensional right (left) $K[x]$-modules. It is well known that every indecomposable module in $\operatorname{mod} K[x]$ is isomorphic to a module of the form $K[x]/(f^m)$, for a monic irreducible polynomial $f \in K[x]$ and a positive integer m. In particular, a module S in $\operatorname{mod} K[x]$ is simple if and only if S is isomorphic to a module $S_f = K[x]/(f)$ for a monic irreducible polynomial $f \in K[x]$. We denote by $\operatorname{irr}(K[x])$ the set of all monic irreducible polynomials f in $K[x]$. Let $f \in \operatorname{irr}(K[x])$ and

$$f = x^d + \lambda_{d-1}x^{d-1} + \dots + \lambda_1 x + \lambda_0,$$

with $d \geq 1$, $\lambda_0, \lambda_1, \dots, \lambda_{d-1} \in K$. Then $\dim_K S_f = d$ and we may consider the (representation)

$$E^{(f)}\colon\ K^d \underset{\varphi^{(f)}}{\overset{\mathrm{id}_{K^d}}{\leftleftarrows}} K^d$$

in $\operatorname{mod} A = \operatorname{rep}_K(Q)$, where $\varphi^{(f)}\colon K^d \to K^d$ is given in the standard basis $e_1, \dots, e_d$ of K^d by the matrix

$$A^{(f)} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -\lambda_0 \\ 1 & 0 & 0 & \dots & 0 & -\lambda_1 \\ 0 & 1 & 0 & \dots & 0 & -\lambda_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & -\lambda_{d-2} \\ 0 & 0 & 0 & \dots & 1 & -\lambda_{d-1} \end{bmatrix}.$$

Observe that, if $f = x$, then $d = 1$, $\lambda_0 = 0$, and $\varphi^{(f)} = 0$. On the other hand, if $f \neq x$, then $\lambda_0 \neq 0$ and $\varphi^{(f)}$ is an isomorphism for any $f \in \operatorname{irr}(K[x])$, because $\det A^{(f)} = (-1)^d \lambda_0 \neq 0$. We claim that $E^{(f)}$ is a simple regular module in $\operatorname{mod} A$. This is clear for $d = 1$. Assume that $d \geq 2$ and

$$V\colon\ V_1 \underset{\varphi_\beta}{\overset{\varphi_\alpha}{\leftleftarrows}} V_2$$

is a nonzero regular submodule (subrepresentation) of $E^{(f)}$. Then we have the following commutative diagram in $\operatorname{mod} K$:

$$\begin{tikzcd} V_1 \arrow[d, hook] & V_2 \arrow[l, shift right, "\varphi_\alpha"'] \arrow[l, shift left, "\varphi_\beta"] \arrow[d, hook] \\ K^d & K^d \arrow[l, shift right, "\mathrm{id}_{K^d}"'] \arrow[l, shift left, "\varphi^{(f)}"] \end{tikzcd},$$

where V_1, V_2 are K-subspaces of K^d, φ_α and φ_β are the restrictions of id_{K^d} and $\varphi^{(f)}$ to V_2, and consequently $V_1 = V_2$ and $\varphi^{(f)}(V_1) = V_1$. But then we obtain a nonzero $K[x]$-submodule of $S_f = K[x]/(f)$ of dimension $\dim_K V_1$. Since S_f is a simple module in $\operatorname{mod} K[x]$, we get $d = \dim_K V_1$, and hence $V = E^{(f)}$. In particular, $E^{(f)}$ is an indecomposable module in $\operatorname{mod} A$, and consequently a simple regular module, because $\partial_A([E^{(f)}]) = 2(d - d) = 0$. Now we show that $\tau_A E^{(f)} \cong E^{(f)}$ for any $f \in \operatorname{irr}(K[x])$. Since $\tau_A\colon \operatorname{add} \mathcal{R}(A) \to \operatorname{add} \mathcal{R}(A)$ is exact and an equivalence of categories, $\tau_A E^{(f)}$ is a simple regular module in $\operatorname{mod} A$. On the other hand, we have

$$\dim_K \operatorname{End}_A\big(E^{(f)}\big) - \dim_K \operatorname{Ext}^1_A\big(E^{(f)}, E^{(f)}\big) = \chi_A\big([E^{(f)}]\big) = \chi_A(d\mathbf{h}_A) = 0.$$

Then we conclude that

$$\begin{aligned} \dim_K \operatorname{Hom}_A\big(E^{(f)}, \tau_A E^{(f)}\big) &= \dim_K D \operatorname{Hom}_A\big(E^{(f)}, \tau_A E^{(f)}\big) \\ &= \dim_K \operatorname{Ext}^1_A\big(E^{(f)}, E^{(f)}\big) \\ &= \dim_K \operatorname{End}_A\big(E^{(f)}\big) > 0. \end{aligned}$$

Then $\operatorname{Hom}_A\left(E^{(f)}, \tau_A E^{(f)}\right) \neq 0$ forces $E^{(f)} \cong \tau_A E^{(f)}$, because $E^{(f)}$ and $\tau_A E^{(f)}$ are simple regular modules in $\operatorname{mod} A$. We also observe that, for $f, g \in \operatorname{irr}(K[x])$, we have $E^{(f)} \cong E^{(g)}$ in $\operatorname{mod} A$ if and only if $S_f \cong S_g$ in $\operatorname{mod} K[x]$, or equivalently, $f = g$.

Consider also the indecomposable module in $\operatorname{mod} A$ of the form

$$E^{(\infty)}\colon\ K \underset{1}{\overset{0}{\leftleftarrows}} K\ .$$

Observe that $\partial_A([E^{(\infty)}]) = 2(1-1) = 0$, and hence $E^{(\infty)}$ is a regular module. Moreover, $E^{(\infty)}$ is a simple regular module. Indeed, let X be a nonzero regular submodule of $E^{(\infty)}$. Then $[X] = (x_1, x_2)$ with $x_1 = x_2$, because $0 = \partial_A([X]) = 2(x_2 - x_1)$, and hence $x_1 = x_2 = 1$. This shows that $X = E^{(\infty)}$. We claim that $\tau_A E^{(\infty)} \cong E^{(\infty)}$. As above, we have

$$\dim_K \operatorname{End}_A\left(E^{(\infty)}\right) - \dim_K \operatorname{Ext}^1_A\left(E^{(\infty)}, E^{(\infty)}\right) = \chi_A\left(\left[E^{(\infty)}\right]\right) = \chi_A(\mathbf{h}_A) = 0.$$

Hence we get

$$\begin{aligned}\dim_K \operatorname{Hom}_A\left(E^{(\infty)}, \tau_A E^{(\infty)}\right) &= \dim_K D \operatorname{Hom}_A\left(E^{(\infty)}, \tau_A E^{(\infty)}\right)\\ &= \dim_K \operatorname{Ext}^1_A\left(E^{(\infty)}, E^{(\infty)}\right)\\ &= \dim_K \operatorname{End}_A\left(E^{(\infty)}\right) > 0.\end{aligned}$$

Then $\operatorname{Hom}_A\left(E^{(\infty)}, \tau_A E^{(\infty)}\right) \neq 0$ implies $E^{(\infty)} \cong \tau_A E^{(\infty)}$, because $E^{(\infty)}$ and $\tau_A E^{(\infty)}$ are simple regular modules in $\operatorname{mod} A$. Observe that $E^{(\infty)} \ncong E^{(f)}$ in $\operatorname{mod} A$ for any $f \in \operatorname{irr}(K[x])$.

We will prove now that the modules

$$\left\{E^{(f)} \mid f \in \operatorname{irr}(K[x])\right\} \cup \left\{E^{(\infty)}\right\}$$

form a complete family of pairwise nonisomorphic simple regular modules in $\operatorname{mod} A$.

Let M be an indecomposable module in $\mathcal{R}(A)$. Since $\partial_A([M]) = 0$, we have $[M] = (n, n)$ for some positive integer n, and so M is of the form

$$K^n \underset{\varphi_\beta}{\overset{\varphi_\alpha}{\leftleftarrows}} K^n\ .$$

We have two cases to consider.

Assume first that $\varphi_\alpha\colon K^n \to K^n$ is not an isomorphism. Take a nonzero vector $\mathbf{z}$ in $\operatorname{Ker} \varphi_\alpha$. Then we obtain a nonzero homomorphism $h\colon E^{(\infty)} \to M$ in

$\operatorname{rep}_K(Q) = \operatorname{mod} A$ given by the commutative diagram of K-linear homomorphisms

$$\begin{tikzcd}
K \arrow[d, "h_1"] & K \arrow[l, shift right, "0"'] \arrow[l, shift left, "1"] \arrow[d, "h_2"] \\
K^n & K^n \arrow[l, shift right, "\varphi_\alpha"'] \arrow[l, shift left, "\varphi_\beta"]
\end{tikzcd},$$

where h_1 and h_2 are defined by $h_1(1_K) = \varphi_\beta(\mathbf{z})$ and $h_2(1_K) = \mathbf{z}$. Observe that $\varphi_\beta(\mathbf{z}) \neq 0$, because M is an indecomposable module in $\mathcal{R}(A)$, and hence $h = (h_1, h_2)$ is a monomorphism. In particular, M belongs to the stable tube $\mathcal{T}_\infty^A$ of rank 1 in Γ_A having $E^{(\infty)}$ on the mouth.

Assume now that $\varphi_\alpha: K^n \to K^n$ is an isomorphism. Let N be the module in $\operatorname{mod} A$ of the form

$$N:\ \begin{tikzcd}
K^n & K^n \arrow[l, shift right, "\mathrm{id}_{K^n}"'] \arrow[l, shift left, "\varphi_\beta\varphi_\alpha^{-1}"]
\end{tikzcd}.$$

Then we have the following commutative diagram of K-linear homomorphisms

$$\begin{tikzcd}
K^n \arrow[d, "\mathrm{id}_{K^n}"] & K^n \arrow[l, shift right, "\varphi_\alpha"'] \arrow[l, shift left, "\varphi_\beta"] \arrow[d, "\varphi_\alpha"] \\
K^n & K^n \arrow[l, shift right, "\mathrm{id}_{K^n}"'] \arrow[l, shift left, "\varphi_\beta\varphi_\alpha^{-1}"]
\end{tikzcd},$$

and hence $M \cong N$ in $\operatorname{mod} A$. The endomorphism $\varphi_\beta\varphi_\alpha^{-1}$ of K^n defines the structure of $K[x]$-module R on K^n by

$$xy = \big(\varphi_\beta\varphi_\alpha^{-1}\big)(y)$$

for any $y \in K^n$. Since N is indecomposable in $\operatorname{mod} A$, we conclude that R is an indecomposable $K[x]$-module of dimension n. Therefore, there exists a monic irreducible polynomial f in $K[x]$ such that R is isomorphic to $K[x]/(f^m)$ for some positive integer m. We note that $K[x]/(f^m)$ is a uniserial $K[x]$-module of dimension $n = md$, where d is the degree of f, whose all simple composition factors are isomorphic to $S_f = K[x]/(f)$. In particular, $K[x]/(f^m)$ admits a unique simple $K[x]$-submodule, isomorphic to S_f. We conclude that there is a commutative diagram of K-linear homomorphisms

$$\begin{tikzcd}
K^d \arrow[d, "\psi"] & K^d \arrow[l, shift right, "\mathrm{id}_{K^d}"'] \arrow[l, shift left, "\varphi^{(f)}"] \arrow[d, "\psi"] \\
K^n & K^n \arrow[l, shift right, "\mathrm{id}_{K^n}"'] \arrow[l, shift left, "\varphi_\beta\varphi_\alpha^{-1}"]
\end{tikzcd},$$

with ψ a monomorphism. This shows that there is a monomorphism $\psi\colon E^{(f)} \to M$ in mod A, because $M \cong N$. In particular, we conclude that M belongs to the stable tube $\mathcal{T}_f^A$ of rank 1 in Γ_A having $E^{(f)}$ on the mouth.

Summing up, we have proved that

$$\Lambda(A) = \operatorname{irr}(K[x]) \cup \{\infty\},$$

which is an infinite set. Therefore, it follows from Theorems 6.1, 6.2 and 8.12 that the Auslander–Reiten quiver Γ_A of A has the disjoint decomposition

$$\Gamma_A = \mathcal{P}(A) \cup \left(\bigcup_{\lambda \in \Lambda(A)} \mathcal{T}_\lambda^A \right) \cup \mathcal{Q}(A),$$

where $\mathcal{T}_\lambda^A$, $\lambda \in \Lambda(A)$, is a family of stable tubes of rank 1 having the modules $E^{(\lambda)}$ on the mouth, separating $\mathcal{P}(A)$ from $\mathcal{Q}(A)$.

We mention that the classification of all simple regular modules in mod A, described above, leads to a complete classification of all indecomposable modules in $\mathcal{R}(A)$, up to isomorphisms. Namely, it follows from Proposition 3.6 and Theorem 3.7 that, for any $\lambda \in \Lambda(A)$, the indecomposable modules lying in the stable tube $\mathcal{T}_\lambda^A$ are the modules $E^{(\lambda)}[j]$ of a unique infinite sequence of irreducible monomorphisms

$$E^{(\lambda)} = E^{(\lambda)}[1] \xrightarrow{u_2^{(\lambda)}} E^{(\lambda)}[2] \xrightarrow{u_3^{(\lambda)}} \dots \longrightarrow E^{(\lambda)}[j-1] \xrightarrow{u_j^{(\lambda)}} E^{(\lambda)}[j] \longrightarrow \dots$$

and there are in mod A canonical exact sequences

$$0 \longrightarrow E^{(\lambda)}[j-1] \xrightarrow{u_j^{(\lambda)}} E^{(\lambda)}[j] \xrightarrow{p_j^{(\lambda)}} E^{(\lambda)}[1] \longrightarrow 0,$$

and $E^{(\lambda)}[j]$, $j \geq 1$, are uniserial objects of the extension abelian subcategory add $\mathcal{T}_\lambda^A = \mathcal{E}\mathrm{xt}(E^{(\lambda)})$ of mod A.

Finally, we consider special fields K.

(1) Assume K is an algebraically closed field. Then

$$\operatorname{irr}(K[x]) = \left\{ x - \lambda \mid \lambda \in K \right\},$$

and, for any $\lambda \in K$, we have

$$E^{(\lambda)} = E^{(x-\lambda)}\colon\ K \overset{1}{\underset{\lambda}{\leftleftarrows}} K\ .$$

Therefore, we may identify $\Lambda(A)$ with the projective line $\mathbb{P}_1(K) = K \cup \{\infty\}$, where $K = \{(1 : \lambda) \in \mathbb{P}_1(K)\}$ and $\infty = (0 : 1)$. We also note that, for any

$\lambda \in \Lambda(A)$, we have $\operatorname{End}_A(E^{(\lambda)}) \cong K$. Moreover, for $\lambda \in \Lambda(A) = K \cup \{\infty\}$, the indecomposable modules in the stable tube $\mathcal{T}_\lambda^A$ are (up to isomorphism) of the form

$$E^{(\lambda)}[m]\colon\ K^m \underset{J_m(\lambda)}{\overset{I_m}{\leftleftarrows}} K^m\ , \quad m \geq 1,$$

if $\lambda \in K$, and

$$E^{(\infty)}[m]\colon\ K^m \underset{I_m}{\overset{J_m(0)}{\leftleftarrows}} K^m\ , \quad m \geq 1,$$

where I_m is the identity matrix in $M_m(K)$, and $J_m(\mu)$ is the *Jordan block*

$$\begin{bmatrix} \mu & 1 & 0 & \dots & 0 & 0 \\ 0 & \mu & 1 & \dots & 0 & 0 \\ 0 & 0 & \mu & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & \mu & 1 \\ 0 & 0 & 0 & \dots & 0 & \mu \end{bmatrix}$$

in $M_m(K)$ having $\mu \in K$ on the diagonal.

(2) Assume $K = \mathbb{R}$ is the field of real numbers. Then we have

$$\operatorname{irr}(\mathbb{R}[x]) = \big\{x - \lambda \in \mathbb{R}[x] \mid \lambda \in \mathbb{R}\big\} \cup \big\{x^2 + \lambda_1 x + \lambda_0 \in \mathbb{R}[x] \mid 4\lambda_0 > \lambda_1^2\big\}.$$

Therefore, we have two types of simple regular modules in $\operatorname{mod} A$.

For $\lambda \in \mathbb{R}$, we have the simple regular module

$$E^{(\lambda)} = E^{(x-\lambda)}\colon\ \mathbb{R} \underset{\lambda}{\overset{1}{\leftleftarrows}} \mathbb{R}\ .$$

Moreover, we have also the simple regular module

$$E^{(\infty)}\colon\ \mathbb{R} \underset{1}{\overset{0}{\leftleftarrows}} \mathbb{R}\ .$$

Hence, we have an infinite family of pairwise nonisomorphic simple regular modules $E^{(\lambda)}$, $\lambda \in \mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, with $\operatorname{End}_A(E^{(\lambda)}) \cong \mathbb{R}$. Moreover, for $\lambda \in \mathbb{P}_1(\mathbb{R})$, the indecomposable modules in the stable tube $\mathcal{T}_\lambda^A$ are (up to isomorphism) of the form

$$E^{(\lambda)}[m]\colon\ \mathbb{R}^m \underset{J_m(\lambda)}{\overset{I_m}{\leftleftarrows}} \mathbb{R}^m\ , \quad m \geq 1,$$

if $\lambda \in \mathbb{R}$, and

$$\begin{tikzcd}E^{(\infty)}[m]\colon \mathbb{R}^m & \mathbb{R}^m \arrow[l, shift right, "J_m(0)"'] \arrow[l, shift left, "I_m"'] , \quad m \geq 1.\end{tikzcd}$$

We look now more closely at the simple regular modules $E^{(f)}$ in mod A given by the monic irreducible polynomials f in $\mathbb{R}[x]$ of degree 2. Let $f = x^2 + \lambda_1 x + \lambda_0 \in \mathbb{R}[x]$ be an irreducible polynomial. Then we have $4\lambda_0 > \lambda_1^2$, because f has no real roots. The simple regular representation $E^{(f)}$ associated to f is of the form

$$\begin{tikzcd}E^{(f)}\colon \mathbb{R}^2 & \mathbb{R}^2 \arrow[l, shift right, "I_2"'] \arrow[l, shift left, "A^{(f)}"']\end{tikzcd}$$

with $I_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $A^{(f)} = \left[\begin{smallmatrix} 0 & -\lambda_0 \\ 1 & -\lambda_1 \end{smallmatrix}\right]$. Observe that f is the characteristic polynomial

$$\det\left(xI_2 - A^{(f)}\right) = \det\begin{bmatrix} x & \lambda_0 \\ -1 & x + \lambda_1 \end{bmatrix} = x^2 + \lambda_1 x + \lambda_0$$

of the matrix $A^{(f)}$. Now it follows from the classification of endomorphisms of finite dimensional real vector spaces that there is an invertible matrix B in $M_2(\mathbb{R})$ such that $BA^{(f)} = J_2(a,b)B$, where $J_2(a,b)$ is a generalized Jordan matrix of order 2,

$$J_2(a,b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

with $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$. Moreover, we have $\lambda_1 = -2a$ and $\lambda_0 = a^2 + b^2$, since the characteristic polynomials of the matrices $A^{(f)}$ and $J_2(a,b)$ coincide. In fact, we have a bijection between the sets

$$\mathbb{R} \times \mathbb{R}^+ = \left\{(a,b) \in \mathbb{R}^2 \mid b > 0\right\} \xrightarrow{\sim} \left\{(\lambda_0, \lambda_1) \in \mathbb{R}^2 \mid 4\lambda_0 > \lambda_1^2\right\}$$

given by $(a,b) \mapsto (-2a, a^2 + b^2)$. In particular, we conclude that the family of modules (representations) in mod $A = \operatorname{rep}_{\mathbb{R}}(Q)$

$$\begin{tikzcd}E^{(a,b)}\colon \mathbb{R}^2 & \mathbb{R}^2 \arrow[l, shift right, "I_2"'] \arrow[l, shift left, "J_2(a,b)"']\end{tikzcd}$$

with $(a,b) \in \mathbb{R} \times \mathbb{R}^+$, forms a complete set of representatives of the isomorphism classes of simple regular representations $E^{(f)}$ in mod A given by the irreducible polynomials in $\mathbb{R}[x]$ of degree 2. We note also that, for $(a,b) \in \mathbb{R} \times \mathbb{R}^+$, we have $[E^{(a,b)}] = (2,2) = 2\mathbf{h}_A$ and $\operatorname{End}_A(E^{(a,b)})$ is a 2-dimensional division $\mathbb{R}$-algebra, and hence is isomorphic to the $\mathbb{R}$-algebra $\mathbb{C}$ of complex numbers. Moreover, the indecomposable modules in a stable tube $\mathcal{T}^A_{(a,b)}$, with $(a,b) \in \mathbb{R} \times \mathbb{R}^+$, are (up to isomorphism) of the form

$$E^{(a,b)}[m]\colon\ \mathbb{R}^{2m} \overset{I_{2m}}{\underset{J_{2m}(a,b)}{\leftleftarrows}} \mathbb{R}^{2m}\ , \quad m \geq 1,$$

where I_{2m} is the identity matrix in $M_{2m}(\mathbb{R})$, and $J_{2m}(a,b)$ is the *generalized Jordan block* of even order $2m$ in $M_{2m}(\mathbb{R})$ of the form

$$\begin{bmatrix} J_2(a,b) & I_2 & 0 & \dots & 0 & 0 \\ 0 & J_2(a,b) & I_2 & \dots & 0 & 0 \\ 0 & 0 & J_2(a,b) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & J_2(a,b) & I_2 \\ 0 & 0 & 0 & \dots & 0 & J_2(a,b) \end{bmatrix}.$$

Observe also that $\big[E^{(a,b)}[m]\big] = m\big[E^{(a,b)}\big] = 2m\mathbf{h}_A$ for any $m \geq 1$. In particular, the stable tube $\mathcal{T}^A_{(a,b)}$ does not contain an indecomposable module X with $[X] = \mathbf{h}_A$.

Finally, consider the closed real upper half-plane

$$H^+\big(\mathbb{R}^2\big) = \big\{(a,b) \in \mathbb{R}^2 \mid b \geq 0\big\}.$$

Then we may identify $\Lambda(A)$ with one-point compactification

$$H^+\big(\mathbb{R}^2\big) \cup \{\infty\}.$$

of $H^+(\mathbb{R}^2)$ such that the boundary of $\Lambda(A)$ is the projective line $\mathbb{P}_1(\mathbb{R}) = \{(a,0) \mid a \in \mathbb{R}\} \cup \{\infty\}$ and $\mathbb{R} \times \mathbb{R}^+$ is the interior of $\Lambda(A)$. Then the endomorphism algebra of a simple regular module corresponding to a point on the boundary of $\Lambda(A)$ is the field $\mathbb{R}$, and a simple regular module corresponding to a point in the interior of $\Lambda(A)$ is $\mathbb{C}$.

(3) Assume $K = \mathbb{Q}$ is the field of rational numbers. Then $\operatorname{irr}(\mathbb{Q}[x])$ is a countable set containing polynomials of arbitrary positive degree. Let $f = x^d + \lambda_{d-1}x^{d-1} + \cdots + \lambda_1 x + \lambda_0$ be a monic irreducible polynomial in $\mathbb{Q}[x]$ of degree d. It follows from the above considerations that $\operatorname{mod} A$ contains a simple regular module of the form

$$E^{(f)}\colon\ \mathbb{Q}^d \overset{\mathrm{id}_{\mathbb{Q}^d}}{\underset{\varphi^{(f)}}{\leftleftarrows}} \mathbb{Q}^d$$

which lies on the mouth of a stable tube $\mathcal{T}^A_f$ of Γ_A of rank 1. Since $[E^{(f)}] = (d,d) = d\mathbf{h}_A$, we conclude that, for any indecomposable module $E^{(f)}[m]$, with $m \geq 1$, in the stable tube $\mathcal{T}^A_f$, we have $\big[E^{(f)}[m]\big] = m[E^{(f)}] = md\mathbf{h}_A$.

Example 8.28. Let A be the following $\mathbb{Q}$-subalgebra of the matrix algebra $M_3(\mathbb{R})$:

$$\begin{aligned}&\begin{bmatrix} \mathbb{Q} & 0 & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) \end{bmatrix} \\ &= \left\{ \begin{bmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{bmatrix} \in M_3(\mathbb{R}) \;\middle|\; \begin{matrix} a \in \mathbb{Q}, \\ b, c, x, y, z \in \mathbb{Q}(\sqrt[3]{2}) \end{matrix} \right\}.\end{aligned}$$

Then A has the standard basic primitive idempotents

$$e_1 = \begin{bmatrix} 1_{\mathbb{Q}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1_{\mathbb{Q}(\sqrt[3]{2})} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_{\mathbb{Q}(\sqrt[3]{2})} \end{bmatrix}$$

such that $1_A = e_1 + e_2 + e_3$. Moreover,

$$\operatorname{rad} A = \begin{bmatrix} 0 & 0 & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & 0 & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) & 0 \end{bmatrix}, \quad (\operatorname{rad} A)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & 0 & 0 \end{bmatrix},$$

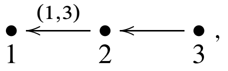
and $(\operatorname{rad} A)^3 = 0$. Further, we obtain

$$\begin{gathered} e_1 \operatorname{rad} A = 0, \quad e_2 \operatorname{rad} A = e_2(\operatorname{rad} A)e_1 = \mathbb{Q}(\sqrt[3]{2}), \quad e_2(\operatorname{rad} A)^2 = 0, \\ e_3 \operatorname{rad} A = e_3(\operatorname{rad} A)e_1 \oplus e_3(\operatorname{rad} A)e_2 = \mathbb{Q}(\sqrt[3]{2}) \oplus \mathbb{Q}(\sqrt[3]{2}), \\ e_3(\operatorname{rad} A)^2 = e_3(\operatorname{rad} A)^2 e_1 = \mathbb{Q}(\sqrt[3]{2}), \quad e_3(\operatorname{rad} A)^2 e_2 = 0. \end{gathered}$$

Moreover, we get

$$\begin{aligned} F_1 &= e_1 A e_1 / e_1(\operatorname{rad} A)e_1 = \mathbb{Q}, \\ F_2 &= e_2 A e_2 / e_2(\operatorname{rad} A)e_2 = \mathbb{Q}(\sqrt[3]{2}), \\ F_3 &= e_3 A e_3 / e_3(\operatorname{rad} A)e_3 = \mathbb{Q}(\sqrt[3]{2}). \end{aligned}$$

Therefore, the quiver Q_A of A is of the form



and so is of Euclidean type $\widetilde{\mathbb{G}}_{22}$. We claim that A is a hereditary algebra.

Let $P_1 = e_1 A$, $P_2 = e_2 A$, $P_3 = e_3 A$ be the indecomposable projective modules and $I_1 = D(Ae_1)$, $I_2 = D(Ae_2)$, $I_3 = D(Ae_3)$ the indecomposable injective modules in $\operatorname{mod} A$ given by the idempotents e_1, e_2, e_3. Then for the simple modules $S_i = \mathrm{top}(P_i)$ we have $S_i = \mathrm{soc}(I_i)$, for any $i \in \{1, 2, 3\}$. Moreover, we have $P_1 = S_1$, $\mathrm{rad}\, P_2 = S_1 \oplus S_1 \oplus S_1$, $\mathrm{rad}\, P_3 = P_2$, and $\mathrm{rad}^2 P_3 = S_1 \oplus S_1 \oplus S_1$. In particular, A is a hereditary algebra, since the radical of every indecomposable projective module in $\mathrm{mod}\, A$ is projective. We also note that $\dim_{\mathbb{Q}} S_1 = 1$, $\dim_{\mathbb{Q}} S_2 = 3$, and $\dim_{\mathbb{Q}} S_3 = 3$. We identify $K_0(A)$ with $\mathbb{Z}^3$ and the basis $[S_1], [S_2], [S_3]$ of $K_0(A)$ with the standard basis e_1, e_2, e_3 of $\mathbb{Z}^3$. Then we have

$$[P_1] = (1, 0, 0), \quad [P_2] = (3, 1, 0), \quad [P_3] = (3, 1, 1).$$

We claim that

$$[I_1] = (1, 1, 1), \quad [I_2] = (0, 1, 1), \quad [I_3] = (0, 0, 1).$$

Indeed, it follows Theorem 6.2 that the full valued subquiver of the preinjective component $\mathcal{Q}(A)$ of Γ_A given by the indecomposable injective modules I_1, I_2, I_3 is the opposite quiver Q_A^{op} of Q_A, and hence we have in $\mathcal{Q}(A)$ the arrows

$$I_1 \xrightarrow{(1,3)} I_2 \qquad \text{and} \qquad I_2 \longrightarrow I_3 \;,$$

and $I_3 = S_3$, since 3 is a source of Q_A. This gives $I_1/S_1 = I_2$ and $I_2/S_2 = I_3 = S_3$, and so $[I_1], [I_2], [I_3]$ are as described above. We also note that the Euler quadratic form $\chi_A : \mathbb{Z}^3 \to \mathbb{Z}$ is given by

$$\begin{aligned} \chi_A(\mathbf{x}) &= f_1 x_1^2 + f_2 x_2^2 + f_3 x_3^2 - 3 f_1 x_1 x_2 - f_2 x_2 x_3 \\ &= x_1^2 + 3x_2^2 + 3x_3^2 - 3x_1 x_2 - 3x_2 x_3, \end{aligned}$$

for $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$, because $f_1 = \dim_{\mathbb{Q}} F_1 = 1$, $f_2 = \dim_{\mathbb{Q}} F_2 = 3$, $f_3 = \dim_{\mathbb{Q}} F_3 = 3$, $d_{21} = 3$, and $d_{32} = 1$. Moreover, χ_A is positive semidefinite with $\mathrm{rad}\, \chi_A = \mathbb{Z}\mathbf{h}_A$, for $\mathbf{h}_A = (3, 2, 1)$, by Proposition 4.6.

We determine now the Coxeter transformation $\varphi_A : K_0(A) \to K_0(A)$ and its inverse transformation $\varphi_A^{-1} : K_0(A) \to K_0(A)$. Since $\varphi_A([P_1]) = -[I_1]$, $\varphi_A([P_2]) = -[I_2]$, $\varphi_A([P_3]) = -[I_3]$, we obtain

$$\varphi_A(e_1) = -e_1 - e_2 - e_3, \quad 3\varphi_A(e_1) + \varphi_A(e_2) = -e_2 - e_3,$$
$$3\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) = -e_3,$$

and consequently φ_A is given by

$$\varphi_A(\mathbf{x}) = (-x_1 + 3x_2, -x_1 + 2x_2 + x_3, -x_1 + 2x_2)$$

for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$. Similarly, $\varphi_A^{-1}([I_1]) = -[P_1]$, $\varphi_A^{-1}([I_2]) = -[P_2]$, $\varphi_A^{-1}([I_3]) = -[P_3]$, lead to the equalities

$$\varphi_A^{-1}(e_1) + \varphi_A^{-1}(e_2) + \varphi_A^{-1}(e_3) = -e_1, \quad \varphi_A^{-1}(e_2) + \varphi_A^{-1}(e_3) = -3e_1 - e_2,$$
$$\varphi_A^{-1}(e_3) = -3e_1 - e_2 - e_3,$$

and hence φ_A^{-1} is given by

$$\varphi_A^{-1}(\mathbf{x}) = (2x_1 - 3x_3, x_1 - x_3, x_2 - x_3)$$

for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$. Further, for $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$, we have

$$\begin{aligned}\varphi_A^2(\mathbf{x}) &= (-2x_1 + 3x_2 + 3x_3, -2x_1 + 3x_2 + 2x_3, -x_1 + x_2 + 2x_3)\\ &= (x_1, x_2, x_3) + (-x_1 + x_2 + x_3)(3, 2, 1)\\ &= \mathbf{x} + (-x_1 + x_2 + x_3)\mathbf{h}_A.\end{aligned}$$

Observe also that $\varphi_A(\mathbf{x}) - \mathbf{x} = (-2x_1 + 3x_2, -x_1 + x_2 + x_3, -x_1 + 2x_2 - x_3)$ for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$. In particular, for $\mathbf{x} = (1, 1, 1)$, we have $\varphi_A(\mathbf{x}) - \mathbf{x} = (1, 1, 0) \notin \operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$. Hence, the defect $\partial_A : \mathbb{Z}^3 \to \mathbb{Z}$ of A is given by

$$\partial_A(\mathbf{x}) = -x_1 + x_2 + x_3$$

for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$.

It follows from Theorem 6.1 that the postprojective component $\mathcal{P}(A)$ of Γ_A is of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
 & & P_3 \arrow[dr] & & \tau_A^{-1}P_3 \arrow[dr] & & \tau_A^{-2}P_3 \arrow[dr] & & \tau_A^{-3}P_3 \arrow[dr, dash] & & \\
 & P_2 \arrow[ur] \arrow[dr, "{(3,1)}"] & & \tau_A^{-1}P_2 \arrow[ur] \arrow[dr, "{(3,1)}"] & & \tau_A^{-2}P_2 \arrow[ur] \arrow[dr, "{(3,1)}"] & & \tau_A^{-3}P_2 \arrow[ur] \arrow[dr, dash] & & & \cdots \\
P_1 \arrow[ur, "{(1,3)}"] & & \tau_A^{-1}P_1 \arrow[ur, "{(1,3)}"] & & \tau_A^{-2}P_1 \arrow[ur, "{(1,3)}"] & & \tau_A^{-3}P_1 \arrow[ur, "{(1,3)}"] & & & &
\end{tikzcd}$$

and consists of the modules $\tau_A^{-m}P_i$, with the composition vectors $[\tau_A^{-m}P_i] = \varphi_A^{-m}([P_i])$, for $i \in \{1, 2, 3\}$ and $m \in \mathbb{N}$. Similarly, by Theorem 6.2, the preinjective component $\mathcal{Q}(A)$ of Γ_A is of the form $\mathbb{N}Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
 & & {} \arrow[dr] & & \tau_A^3 I_3 \arrow[dr] & & \tau_A^2 I_3 \arrow[dr] & & \tau_A I_3 \arrow[dr] & & I_3 \\
\cdots & {} \arrow[dr] & & \tau_A^3 I_2 \arrow[ur] \arrow[dr, "{(3,1)}"] & & \tau_A^2 I_2 \arrow[ur] \arrow[dr, "{(3,1)}"] & & \tau_A I_2 \arrow[ur] \arrow[dr, "{(3,1)}"] & & I_2 \arrow[ur] & \\
 & & \tau_A^3 I_1 \arrow[ur, "{(1,3)}"] & & \tau_A^2 I_1 \arrow[ur, "{(1,3)}"] & & \tau_A I_1 \arrow[ur, "{(1,3)}"] & & I_1 \arrow[ur, "{(1,3)}"] & &
\end{tikzcd}$$

and consists of the modules $\tau_A^m I_i$, with the composition vectors $[\tau_A^m I_i] = \varphi_A^m([I_i])$, for $i \in \{1, 2, 3\}$ and $m \in \mathbb{N}$.

We will show that the regular part $\mathcal{R}(A)$ of Γ_A contains a stable tube of rank 2. Consider the module $E = \operatorname{rad} I_1$. Then E is an indecomposable module of length 2, with the simple socle $\operatorname{soc}(E) = S_1$ and the simple top $\operatorname{top}(E) = S_2$, and so $[E] = (1,1,0)$. In particular, we have $\partial_A([E]) = 0$, and hence E is an indecomposable regular module. Since the unique proper submodule of E is the simple projective module $S_1 = P_1$, lying in $\mathcal{P}(A)$, we conclude that E is a simple regular module in $\operatorname{mod} A$. We determine the τ_A-orbit $\mathcal{O}(E)$ of E. We have

$$[\tau_A E] = \varphi_A([E]) = \varphi_A\big((1,1,0)\big) = (2,1,1),$$
$$[\tau_A^2 E] = \varphi_A^2([E]) = \varphi_A\big((2,1,1)\big) = (1,1,0).$$

In particular, we conclude that $\tau_A E \ncong E$, and hence the stable tube $\mathcal{T}_E$ of Γ_A containing E is of rank $r_E = |\mathcal{O}(E)| \geq 2$ (see Theorem 8.11). Then it follows from Theorem 8.18 that $[E] = [\tau_A^2 E]$ forces $E \cong \tau_A^2 E$, because $\operatorname{rl}(E) = 1$ and $\operatorname{rl}(\tau_A^2 E) = 1$ are not divisible by r_E. Therefore, $\mathcal{O}(E)$ consists of E and $\tau_A E$, and hence $\mathcal{T}_E$ is a stable tube of rank 2. We note that $\tau_A E$ is isomorphic to P_3/S_1, for any embedding of S_1 into P_3. Indeed, for any such embedding of S_1 into P_3, we have $[P_3/S_1] = (2,1,1)$ and P_3/S_1 is indecomposable, because the $\operatorname{top}(P_3/S_1) = S_3$ is simple. On the other hand, we have $\chi_A([\tau_A E]) = \chi_A([P_3/S_1]) > 0$, because

$$\chi_A\big((2,1,1)\big) = 4 + 3 + 3 - 6 - 3 = 1 > 0.$$

Hence, $\tau_A E \cong P_3/S_1$, by Theorem 8.19. Finally, we note that, for any indecomposable module X in $\mathcal{T}_E$ with $\operatorname{rl}(X) = 2$,

$$[X] = [E] + [\tau_A E] = (1,1,0) + (2,1,1) = (3,2,1) = \mathbf{h}_A.$$

One can show that the stable tube $\mathcal{T}_E$ is the unique stable tube of rank at least 2 in $\mathcal{R}(A)$.

Example 8.29. Let $\mathbb{M} = (F_i, {}_iM_j)_{1 \leq i,j \leq 5}$ be the $\mathbb{R}$-species defined as follows:

$$F_1 = F_2 = \mathbb{C}, \quad F_3 = F_4 = F_5 = \mathbb{R},$$
$${}_2M_1 = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}, \quad {}_2M_3 = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{R}}, \quad {}_4M_3 = {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}}, \quad {}_5M_4 = {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}},$$

and ${}_iM_j = 0$ for the remaining (i,j) with $i,j \in \{1,2,3,4,5\}$. Then $Q_{\mathbb{M}}$ is the acyclic valued quiver of the form

$$\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l] \arrow[r, "{(2,1)}"] & \underset{3}{\bullet} & \underset{4}{\bullet} \arrow[l] & \underset{5}{\bullet} \arrow[l]
\end{tikzcd},$$

and consequently the tensor algebra $A = T(\mathbb{M})$ of $\mathbb{M}$ is a finite dimensional hereditary $\mathbb{R}$-algebra of Euclidean type $\widetilde{\mathbb{F}}_{41}$, because $Q_A = Q_{T(\mathbb{M})} = Q_{\mathbb{M}}$, by

Lemma 2.1 and Theorem 2.2. In fact, a simple checking shows that A is isomorphic to the following $\mathbb{R}$-subalgebra of the matrix algebra $M_5(\mathbb{C})$:

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 & 0 \\ 0 & 0 & \mathbb{R} & 0 & 0 \\ 0 & 0 & \mathbb{R} & \mathbb{R} & 0 \\ 0 & 0 & \mathbb{R} & \mathbb{R} & \mathbb{R} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ x & b & y & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & z & d & 0 \\ 0 & 0 & u & v & e \end{bmatrix} \in M_5(\mathbb{C}) \;\middle|\; \begin{matrix} a, b, x, y \in \mathbb{C} \\ c, d, e, z, u, v \in \mathbb{R} \end{matrix} \right\}.$$

Let P_1, P_2, P_3, P_4, P_5 be the indecomposable projective modules, I_1, I_2, I_3, I_4, I_5 the indecomposable injective modules, and S_1, S_2, S_3, S_4, S_5 the simple modules in mod A associated to the vertices $1, 2, 3, 4, 5$ of $Q_A = Q_{\mathbb{M}}$, respectively. It follows from Theorems 6.1 and 6.2 that the full valued subquiver of the postprojective component $\mathcal{P}(A)$ of Γ_A given by the projective modules P_1, P_2, P_3, P_4, P_5, and the full valued subquiver of the preinjective component $\mathcal{Q}(A)$ of Γ_A given by the injective modules I_1, I_2, I_3, I_4, I_5, are isomorphic to the opposite quiver Q_A^{op} of Q_A. We identify $K_0(A)$ with $\mathbb{Z}^5$ and the basis $[S_1], [S_2], [S_3], [S_4], [S_5]$ of $K_0(A)$ with the standard basis e_1, e_2, e_3, e_4, e_5 of $\mathbb{Z}^5$. Then we obtain

$$\begin{gathered} [P_1] = (1,0,0,0,0), \quad [P_2] = (1,1,2,0,0), \quad [P_3] = (0,0,1,0,0), \\ [P_4] = (0,0,1,1,0), \quad [P_5] = (0,0,1,1,1), \\ [I_1] = (1,1,0,0,0), \quad [I_2] = (0,1,0,0,0), \quad [I_3] = (0,1,1,1,1), \\ [I_4] = (0,0,0,1,1), \quad [I_5] = (0,0,0,0,1). \end{gathered}$$

Moreover, the Euler quadratic form $\chi_A \colon \mathbb{Z}^5 \to \mathbb{Z}$ is given by

$$\begin{aligned} \chi_A(\mathbf{x}) &= \sum_{i=1}^{5} f_i x_i^2 - f_1 x_1 x_2 - 2 f_3 x_2 x_3 - f_3 x_3 x_4 - f_4 x_4 x_5 \\ &= 2x_1^2 + 2x_2^2 + x_3^2 + x_4^2 + x_5^2 - 2x_1x_2 - 2x_2x_3 - x_3x_4 - x_4x_5, \end{aligned}$$

where $f_i = \dim_{\mathbb{R}} F_i$, for $i \in \{1,2,3,4,5\}$. By Proposition 4.6, χ_A is positive semidefinite with rad $\chi_A = \mathbb{Z}\mathbf{h}_A$, for $\mathbf{h}_A = (1,2,3,2,1)$.

We determine now the Coxeter transformation $\varphi_A \colon K_0(A) \to K_0(A)$ and its inverse transformation $\varphi_A^{-1} \colon K_0(A) \to K_0(A)$. Since $\varphi_A([P_i]) = -[I_i]$ for any $i \in \{1,2,3,4,5\}$, we obtain

$$\begin{gathered} \varphi_A(e_1) = -e_1 - e_2, \quad \varphi_A(e_1) + \varphi_A(e_2) + 2\varphi_A(e_3) = -e_2, \\ \varphi_A(e_3) = -e_2 - e_3 - e_4 - e_5, \quad \varphi_A(e_3) + \varphi_A(e_4) = -e_4 - e_5, \\ \varphi_A(e_3) + \varphi_A(e_4) + \varphi_A(e_5) = -e_5. \end{gathered}$$

Hence $\varphi_A \colon \mathbb{Z}^5 \to \mathbb{Z}^5$ is given by

$$\varphi_A(\mathbf{x}) = (-x_1 + x_2, -x_1 + 2x_2 - x_3 + x_4, 2x_2 - x_3 + x_4, 2x_2 - x_3 + x_5, 2x_2 - x_3)$$

for any $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}^5$. Similarly, $\varphi_A^{-1}([I_i]) = -[P_i]$ for $i \in \{1, 2, 3, 4, 5\}$ lead to the equalities

$$\begin{gathered}
\varphi_A^{-1}(e_1) + \varphi_A^{-1}(e_2) = -e_1, \quad \varphi_A^{-1}(e_2) = -e_1 - e_2 - 2e_3, \\
\varphi_A^{-1}(e_2) + \varphi_A^{-1}(e_3) + \varphi_A^{-1}(e_4) + \varphi_A^{-1}(e_5) = -e_3, \\
\varphi_A^{-1}(e_4) + \varphi_A^{-1}(e_5) = -e_3 - e_4, \quad \varphi_A^{-1}(e_5) = -e_3 - e_4 - e_5.
\end{gathered}$$

Then $\varphi_A^{-1} \colon \mathbb{Z}^5 \to \mathbb{Z}^5$ is given by

$$\varphi_A^{-1}(\mathbf{x}) = (-x_2 + x_3, x_1 - x_2 + x_3, 2x_1 - 2x_2 + 2x_3 - x_5, x_3 - x_5, x_4 - x_5)$$

for any $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}^5$. A simple calculation shows that $\varphi_A^r(\mathbf{x}) - \mathbf{x} \notin \operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$ for $r \in \{1, 2, 3, 4, 5\}$, and

$$\varphi_A^6(\mathbf{x}) = \mathbf{x} + (-2x_1 + 4x_2 - 3x_3 + x_4 + x_5)\mathbf{h}_A$$

for any $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}^5$. Hence the defect $\partial_A \colon \mathbb{Z}^5 \to \mathbb{Z}$ of A is given by

$$\partial_A(\mathbf{x}) = -2x_1 + 4x_2 - 3x_3 + x_4 + x_5$$

for any $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}^5$.

It follows from Theorem 6.1 that the postprojective component $\mathcal{P}(A)$ of Γ_A is of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
& & P_5 \arrow[dr] & & \tau_A^{-1}P_5 \arrow[dr] & & \tau_A^{-2}P_5 \arrow[dr] & & \tau_A^{-3}P_5 \arrow[dr, dash] & & \\
& P_4 \arrow[ur] \arrow[dr] & & \tau_A^{-1}P_4 \arrow[ur] \arrow[dr] & & \tau_A^{-2}P_4 \arrow[ur] \arrow[dr] & & \tau_A^{-3}P_4 \arrow[ur] \arrow[dr, dash] & & & \\
P_3 \arrow[ur] \arrow[dr, "{(1,2)}"] & & \tau_A^{-1}P_3 \arrow[ur] \arrow[dr, "{(1,2)}"] & & \tau_A^{-2}P_3 \arrow[ur] \arrow[dr, "{(1,2)}"] & & \tau_A^{-3}P_3 \arrow[ur] \arrow[dr, dash] & & & & \cdots \\
& P_2 \arrow[ur, "{(2,1)}"] \arrow[dr] & & \tau_A^{-1}P_2 \arrow[ur, "{(2,1)}"] \arrow[dr] & & \tau_A^{-2}P_2 \arrow[ur, "{(2,1)}"] \arrow[dr, dash] & & & & & \\
P_1 \arrow[ur] & & \tau_A^{-1}P_1 \arrow[ur] & & \tau_A^{-2}P_1 \arrow[ur] & & & & & &
\end{tikzcd}$$

and consists of the modules $\tau_A^{-m}P_i$, with the composition vectors $[\tau_A^{-m}P_i] = \varphi_A^{-m}([P_i])$, for $i \in \{1, 2, 3, 4, 5\}$ and $m \in \mathbb{N}$. Similarly, by Theorem 6.2, the preinjective component $\mathcal{Q}(A)$ of Γ_A is of the form $\mathbb{N}Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
 & & & {} \arrow[dr] & & \tau_A^3 I_5 \arrow[dr] & & \tau_A^2 I_5 \arrow[dr] & & \tau_A I_5 \arrow[dr] & & I_5 \\
 & & {} \arrow[dr] & & \tau_A^3 I_4 \arrow[ur] \arrow[dr] & & \tau_A^2 I_4 \arrow[ur] \arrow[dr] & & \tau_A I_4 \arrow[ur] \arrow[dr] & & I_4 \arrow[ur] & \\
\cdots & {} \arrow[dr] & & \tau_A^3 I_3 \arrow[ur] \arrow[dr, "{(1,2)}"] & & \tau_A^2 I_3 \arrow[ur] \arrow[dr, "{(1,2)}"] & & \tau_A I_3 \arrow[ur] \arrow[dr, "{(1,2)}"] & & I_3 \arrow[ur] \arrow[dr, "{(1,2)}"] & & \\
{} \arrow[dr] & & \tau_A^4 I_2 \arrow[ur, "{(2,1)}"] \arrow[dr] & & \tau_A^3 I_2 \arrow[ur, "{(2,1)}"] \arrow[dr] & & \tau_A^2 I_2 \arrow[ur, "{(2,1)}"] \arrow[dr] & & \tau_A I_2 \arrow[ur, "{(2,1)}"] \arrow[dr] & & I_2 & \\
 & \tau_A^4 I_1 \arrow[ur] & & \tau_A^3 I_1 \arrow[ur] & & \tau_A^2 I_1 \arrow[ur] & & \tau_A I_1 \arrow[ur] & & I_1 \arrow[ur] & &
\end{tikzcd}$$

and consists of the modules $\tau_A^m I_i$, with the composition vectors $[\tau_A^m I_i] = \varphi_A^m([I_i])$, for $i \in \{1,2,3,4,5\}$ and $m \in \mathbb{N}$.

We will show that the regular part $\mathcal{R}(A)$ of Γ_A contains a stable tube of rank 2 and a stable tube of rank 3. We will use the canonical equivalence of the module category $\operatorname{mod} A$ with the category $\operatorname{rep}(\mathbb{M})$ of representations of the $\mathbb{R}$-species $\mathbb{M}$, established in Theorem 2.5. Moreover, we may consider (see Example 2.6) the representations in $\operatorname{rep}(\mathbb{M})$ as the diagrams

$$X\colon X_1 \xleftarrow{{}_1\varphi_2} X_2 \xrightarrow{{}_3\varphi_2} X_3 \xleftarrow{{}_3\varphi_4} X_4 \xleftarrow{{}_4\varphi_5} X_5$$

where X_1, X_2 are in $\operatorname{mod}\mathbb{C}$, X_3, X_4, X_5 are in $\operatorname{mod}\mathbb{R}$, ${}_1\varphi_2\colon X_2 \to X_1$ is a $\mathbb{C}$-linear homomorphism, and ${}_3\varphi_2\colon X_2 \to X_3$, ${}_3\varphi_4\colon X_4 \to X_3$, ${}_4\varphi_5\colon X_5 \to X_4$ are $\mathbb{R}$-linear homomorphisms. Further, for a representation

$$Y\colon Y_1 \xleftarrow{{}_1\psi_2} Y_2 \xrightarrow{{}_3\psi_2} Y_3 \xleftarrow{{}_3\psi_4} Y_4 \xleftarrow{{}_4\psi_5} Y_5$$

in $\operatorname{rep}(\mathbb{M})$, a homomorphism $f\colon X \to Y$ in $\operatorname{rep}(\mathbb{M})$ is a collection $f = (f_1, f_2, f_3, f_4, f_5)$, where $f_1\colon X_1 \to Y_1$, $f_2\colon X_2 \to Y_2$ are $\mathbb{C}$-linear homomorphisms and $f_3\colon X_3 \to Y_3$, $f_4\colon X_4 \to Y_4$, $f_5\colon X_5 \to Y_5$ are $\mathbb{R}$-linear homomorphisms such that the following diagram is commutative

$$\begin{tikzcd}
X_1 \arrow[d, "f_1"] & X_2 \arrow[l, "{}_1\varphi_2"'] \arrow[r, "{}_3\varphi_2"] \arrow[d, "f_2"] & X_3 \arrow[d, "f_3"] & X_4 \arrow[l, "{}_3\varphi_4"'] \arrow[d, "f_4"] & X_5 \arrow[l, "{}_4\varphi_5"'] \arrow[d, "f_5"] \\
Y_1 & Y_2 \arrow[l, "{}_1\psi_2"'] \arrow[r, "{}_3\psi_2"] & Y_3 & Y_4 \arrow[l, "{}_3\psi_4"'] & Y_5 . \arrow[l, "{}_4\psi_5"']
\end{tikzcd}$$

Moreover, f is an isomorphism in $\operatorname{rep}(\mathbb{M})$ if and only if f_1, f_2, f_3, f_4, f_5 are isomorphisms. Observe also that the indecomposable projective and the indecomposable injective modules of mod A are given in rep($\mathbb{M}$) by the diagrams

$$\begin{aligned}
&P_1\colon \mathbb{C} \longleftarrow 0 \longrightarrow 0 \longleftarrow 0 \longleftarrow 0, && P_2\colon \mathbb{C} \xleftarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} \longleftarrow 0 \longleftarrow 0,\\
&P_3\colon 0 \longleftarrow 0 \longrightarrow \mathbb{R} \longleftarrow 0 \longleftarrow 0, && P_4\colon 0 \longleftarrow 0 \longrightarrow \mathbb{R} \xleftarrow{1} \mathbb{R} \longleftarrow 0,\\
&P_5\colon 0 \longleftarrow 0 \longrightarrow \mathbb{R} \xleftarrow{1} \mathbb{R} \xleftarrow{1} \mathbb{R}, && I_1\colon \mathbb{C} \xleftarrow{1} \mathbb{C} \longrightarrow 0 \longleftarrow 0 \longleftarrow 0,\\
&I_2\colon 0 \longleftarrow \mathbb{C} \longrightarrow 0 \longleftarrow 0 \longleftarrow 0, && I_3\colon 0 \longleftarrow \mathbb{C} \xrightarrow{\varepsilon} \mathbb{R} \xleftarrow{1} \mathbb{R} \xleftarrow{1} \mathbb{R},\\
&I_4\colon 0 \longleftarrow 0 \longrightarrow 0 \longleftarrow \mathbb{R} \xleftarrow{1} \mathbb{R}, && I_5\colon 0 \longleftarrow 0 \longrightarrow 0 \longleftarrow 0 \longleftarrow \mathbb{R},
\end{aligned}$$

where $\mathbb{C}$ has the canonical $\mathbb{R}$-vector space structure $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ and the $\mathbb{R}$-linear homomorphism ε from $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ to $\mathbb{R}$ is given by $\varepsilon(a + bi) = a$ for $a + bi \in \mathbb{R} \oplus \mathbb{R}i = \mathbb{C}$. We identify mod A with rep($\mathbb{M}$).

Consider the indecomposable modules in mod A = rep($\mathbb{M}$)

$$E_1\colon 0 \longleftarrow \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{\omega} \mathbb{R} \xleftarrow{1} \mathbb{R}, \quad E_2\colon \mathbb{C} \xleftarrow{1} \mathbb{C} \xrightarrow{\varepsilon} \mathbb{R} \xleftarrow{1} \mathbb{R} \longleftarrow 0,$$

where the $\mathbb{R}$-linear homomorphism $\omega\colon \mathbb{R} \to \mathbb{C}$ is given by $\omega(a) = a + ai$ for $a \in \mathbb{R}$. Then we have

$$[E_1] = (0, 1, 2, 1, 1), \quad [E_2] = (1, 1, 1, 1, 0),$$

and $\partial_A([E_1]) = 0$, $\partial_A([E_2]) = 0$, and hence E_1, E_2 are indecomposable regular modules in mod A. In fact, E_1 and E_2 are simple regular modules. Indeed, the indecomposable proper submodules of E_1 are isomorphic to one of the indecomposable modules

$$\begin{aligned}
&X\colon 0 \longleftarrow \mathbb{C} \xrightarrow{1} \mathbb{C} \longleftarrow 0 \longleftarrow 0,\\
&Y\colon 0 \longleftarrow \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{\omega} \mathbb{R} \longleftarrow 0, \quad P_3, P_4, P_5,
\end{aligned}$$

which have nonzero defect. Similarly, the indecomposable proper submodules of E_2 are isomorphic to one of the indecomposable modules

$$Z\colon \mathbb{C} \xleftarrow{1} \mathbb{C} \xrightarrow{\varepsilon} \mathbb{R} \longleftarrow 0 \longleftarrow 0, \quad P_1, P_3, P_4,$$

which also have nonzero defects. Further, using the Coxeter transformation φ_A, we obtain

$$\begin{aligned}
[\tau_A E_2] &= \varphi_A([E_2]) = \varphi_A\big((1, 1, 1, 1, 0)\big) = (0, 1, 2, 1, 1) = [E_1],\\
[\tau_A E_1] &= \varphi_A([E_1]) = \varphi_A\big((0, 1, 2, 1, 1)\big) = (1, 1, 1, 1, 0) = [E_2].
\end{aligned}$$

Since $[E_1]$, $[E_2]$ are not in rad $\chi_A = \mathbb{Z}\mathbf{h}_A$, we have $\chi_A([E_1]) > 0$, $\chi_A([E_2]) > 0$, and hence $E_1 = \tau_A E_2$, $E_2 = \tau_A E_1$, by Theorem 8.19. Therefore, E_1 and E_2

form the mouth of a stable tube $\mathcal{T}^A_{\mathcal{O}(E_1)} = \mathcal{T}^A_{\mathcal{O}(E_2)}$ of rank 2 in Γ_A. We also note that $[E_1] + [E_2] = \mathbf{h}_A$.

Consider now the following indecomposable modules in $\operatorname{mod} A = \operatorname{rep}(\mathbb{M})$

$$E_1' \colon \mathbb{C} \xleftarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{1} \mathbb{C} \xleftarrow{1} \mathbb{C},$$

$$E_2' \colon \mathbb{C} \xleftarrow{(1,1)} \mathbb{C}^2 \xrightarrow{\left[\begin{smallmatrix} \varepsilon & \sigma \\ 0 & \varepsilon \end{smallmatrix}\right]} \mathbb{R}^2 \longleftarrow 0 \longleftarrow 0,$$

$$E_3' \colon 0 \longleftarrow \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{1} \mathbb{C} \longleftarrow 0,$$

where $\sigma \colon \mathbb{C} \to \mathbb{R}$ is the $\mathbb{R}$-linear map given by $\sigma(a + bi) = a + b$ for $a + bi \in \mathbb{R} \oplus \mathbb{R}i = \mathbb{C}$. Then we have

$$[E_1'] = (1,1,2,2,2), \quad [E_2'] = (1,2,2,0,0), \quad [E_3'] = (0,1,2,2,0),$$

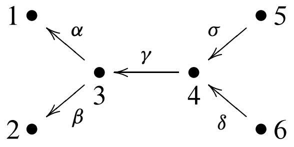
and hence $\partial_A([E_1']) = 0$, $\partial_A([E_2']) = 0$, $\partial_A([E_3']) = 0$. Thus E_1', E_2', E_3' are indecomposable regular modules in $\operatorname{mod} A$. Further, we have

$$\begin{aligned} [\tau_A E_1'] &= \varphi_A([E_1']) = \varphi_A\big((1,1,2,2,2)\big) = (0,1,2,2,0) = [E_3'], \\ [\tau_A E_2'] &= \varphi_A([E_2']) = \varphi_A\big((1,2,2,0,0)\big) = (1,1,2,2,2) = [E_1'], \\ [\tau_A E_3'] &= \varphi_A([E_3]) = \varphi_A\big((0,1,2,2,0)\big) = (1,2,2,0,0) = [E_2']. \end{aligned}$$

Since $[E_1'], [E_2'], [E_3']$ are not in $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A = \mathbb{Z}(1,2,3,2,1)$, we have $\chi_A([E_1']) > 0$, $\chi_A([E_2']) > 0$, $\chi_A([E_3']) > 0$, and consequently $\tau_A E_1' = E_3'$, $\tau_A E_2' = E_1'$, $\tau_A E_3' = E_2'$, again by Theorem 8.19. Observe also that, for every indecomposable proper submodule R of E_3', $[R]$ is one of the vectors $(0,1,2,0,0)$, $(0,1,2,1,0)$, $(0,0,1,1,0)$, $(0,0,1,0,0)$, and hence $\partial_A([R]) < 0$. This shows that E_3' is a simple regular module, and consequently $E_2' = \tau_A E_3'$ and $E_1' = \tau_A^2 E_3'$ are simple regular modules. Therefore, E_1', E_2', E_3' form the mouth of a stable tube $\mathcal{T}^A_{\mathcal{O}(E_1')} = \mathcal{T}^A_{\mathcal{O}(E_2')} = \mathcal{T}^A_{\mathcal{O}(E_3')}$ of rank 3 in Γ_A. We also note that $[E_1'] + [E_2'] + [E_3'] = 2\mathbf{h}_A$.

One can show that the two stable tubes described above exhaust all stable tube of rank at least 2 in $\mathcal{R}(A)$.

Example 8.30. Let K be a field, Q the quiver



and $A = KQ$ the path algebra of Q over K. Then A is a hereditary K-algebra of Euclidean type $\mathbb{D}_5$, and Q is the quiver Q_A of A. Let $P_1, P_2, P_3, P_4, P_5, P_6$

and $I_1, I_2, I_3, I_4, I_5, I_6$ be the indecomposable projective modules and the indecomposable injective modules in $\operatorname{mod} A$, respectively, associated to the vertices $1, 2, 3, 4, 5, 6$ of Q. Moreover, let $S_1, S_2, S_3, S_4, S_5, S_6$ be the simple modules given by the vertices $1, 2, 3, 4, 5, 6$ of Q, so we have $\operatorname{top}(P_i) = S_i = \operatorname{soc}(I_i)$ for any $i \in \{1, 2, 3, 4, 5, 6\}$. We identify the basis $[S_1], [S_2], [S_3], [S_4], [S_5], [S_6]$ of $K_0(A)$ with the standard basis $e_1, e_2, e_3, e_4, e_5, e_6$ of $\mathbb{Z}^6$. Then we have

$$\begin{aligned}
[P_1] &= (1,0,0,0,0,0), & [P_2] &= (0,1,0,0,0,0), & [P_3] &= (1,1,1,0,0,0),\\
[P_4] &= (1,1,1,1,0,0), & [P_5] &= (1,1,1,1,1,0), & [P_6] &= (1,1,1,1,0,1),\\
[I_1] &= (1,0,1,1,1,1), & [I_2] &= (0,1,1,1,1,1), & [I_3] &= (0,0,1,1,1,1),\\
[I_4] &= (0,0,0,1,1,1), & [I_5] &= (0,0,0,0,1,0), & [I_6] &= (0,0,0,0,0,1).
\end{aligned}$$

We determine now the Coxeter transformation φ_A of A in the standard basis $e_1, e_2, e_3, e_4, e_5, e_6$ of $K_0(A) = \mathbb{Z}^6$. By definition, we have $\varphi_A([P_i]) = -[I_i]$ for any $i \in \{1, 2, 3, 4, 5, 6\}$. This leads to the equalities

$$\begin{gathered}
\varphi_A(e_1) = -e_1 - e_3 - e_4 - e_5 - e_6, \quad \varphi_A(e_2) = -e_2 - e_3 - e_4 - e_5 - e_6,\\
\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) = -e_3 - e_4 - e_5 - e_6,\\
\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) + \varphi_A(e_4) = -e_4 - e_5 - e_6,\\
\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) + \varphi_A(e_4) + \varphi_A(e_5) = -e_5,\\
\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) + \varphi_A(e_4) + \varphi_A(e_6) = -e_6.
\end{gathered}$$

Hence we conclude that $\varphi_A\colon \mathbb{Z}^6 \to \mathbb{Z}^6$ is given by the matrix

$$\Phi_A = \begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0\\
0 & -1 & 1 & 0 & 0 & 0\\
-1 & -1 & 1 & 1 & 0 & 0\\
-1 & -1 & 1 & 0 & 1 & 1\\
-1 & -1 & 1 & 0 & 0 & 1\\
-1 & -1 & 1 & 0 & 1 & 0
\end{bmatrix}.$$

Moreover, the Euler quadratic form $\chi_A\colon \mathbb{Z}^6 \to \mathbb{Z}$ is given by

$$\chi_A(\mathbf{x}) = \sum_{i=1}^{6} x_i^2 - x_1x_3 - x_2x_3 - x_3x_4 - x_4x_5 - x_4x_6$$

for $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6$, and χ_A is positive semidefinite with $\operatorname{rad} \chi_A = \mathbb{Z}\mathbf{h}_A$, for $\mathbf{h}_A = (1, 1, 2, 2, 1, 1)$, by Proposition 4.6. It follows from Theorem 6.1

that the postprojective component $\mathcal{P}(A)$ of Γ_A is of the form $(-\mathbb{N})Q_A^{\mathrm{op}} = (-\mathbb{N})Q^{\mathrm{op}}$

$$\begin{tikzcd}
P_1 \arrow[dr] & & \tau_A^{-1}P_1 \arrow[dr] & & \tau_A^{-2}P_1 \arrow[dr] & & & & & \\
P_2 \arrow[r] & P_3 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-1}P_2 \arrow[r] & \tau_A^{-1}P_3 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-2}P_2 \arrow[r] & \tau_A^{-2}P_3 \arrow[ur, dash] \arrow[dr] & & & & \cdots \\
& & P_4 \arrow[ur] \arrow[r] \arrow[dr] & P_5 \arrow[r] & \tau_A^{-1}P_4 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-1}P_5 \arrow[r] & \tau_A^{-2}P_4 \arrow[ur, dash] \arrow[r] \arrow[dr] & \tau_A^{-2}P_5 & & \\
& & & P_6 \arrow[ur] & & \tau_A^{-1}P_6 \arrow[ur] & & \tau_A^{-2}P_6 \arrow[ur, dash] & &
\end{tikzcd}$$

and consists of the modules $\tau_A^{-m}P_i$, with the composition vectors $[\tau_A^{-m}P_i] = \varphi_A^{-m}([P_i])$, for $i \in \{1,2,3,4,5,6\}$ and $m \in \mathbb{N}$. Similarly, by Theorem 6.2, the preinjective component $\mathcal{Q}(A)$ of Γ_A is of the form $\mathbb{N}Q_A^{\mathrm{op}} = \mathbb{N}Q^{\mathrm{op}}$

$$\begin{tikzcd}
& \tau_A^2 I_1 \arrow[dr] & & \tau_A I_1 \arrow[dr] & & I_1 \arrow[dr] & & & \\
{} \arrow[ur] \arrow[r] & \tau_A^2 I_2 \arrow[r] & \tau_A^2 I_3 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A I_2 \arrow[r] & \tau_A I_3 \arrow[ur] \arrow[r] \arrow[dr] & I_2 \arrow[r] & I_3 \arrow[dr] & & \\
\cdots & {} \arrow[ur] & {} \arrow[r] & \tau_A^2 I_4 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^2 I_5 \arrow[r] & \tau_A I_4 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A I_5 \arrow[r] & I_4 \arrow[r] \arrow[dr] & I_5 \\
& & {} \arrow[ur] & & \tau_A^2 I_6 \arrow[ur] & & \tau_A I_6 \arrow[ur] & & I_6
\end{tikzcd}$$

and consists of the modules $\tau_A^m I_i$, with the composition vectors $[\tau_A^m I_i] = \varphi_A^m([I_i])$, for $i \in \{1,2,3,4,5,6\}$ and $m \in \mathbb{N}$. Consider the indecomposable modules R, U_1, U_2, V_1, V_2 in $\operatorname{mod} A$ with $[R] = (1,1,1,1,1,1)$, $[U_1] = (1,0,1,1,1,0)$, $[U_2] = (0,1,1,1,0,1)$, $[V_1] = (1,0,1,1,0,1)$, $[V_2] = (0,1,1,1,1,0)$. Then we obtain the equalities in $K_0(A)$

$$\varphi_A\big([R]\big) = [S_4], \quad \varphi_A\big([S_4]\big) = [S_3], \quad \varphi_A\big([S_3]\big) = [R],$$
$$\varphi_A\big([U_1]\big) = [U_2], \quad \varphi_A\big([U_2]\big) = [U_1], \quad \varphi_A\big([V_1]\big) = [V_2], \quad \varphi_A\big([V_2]\big) = [V_1].$$

Observe also that the indecomposable modules of R, U_1, U_2, V_1, V_2 are simple regular modules. We have also the equalities $[S_3]+[S_4]+[R] = \mathbf{h}_A$, $[U_1]+[U_2] = \mathbf{h}_A$, and $[V_1] + [V_2] = \mathbf{h}_A$. Hence, applying Corollary 5.3 and Theorem 8.19, we conclude that

$$\tau_A R = S_4, \qquad \tau_A S_4 = S_3, \qquad \tau_A S_3 = R,$$
$$\tau_A U_1 = U_2, \qquad \tau_A U_2 = U_1, \qquad \tau_A V_1 = V_2, \qquad \tau_A V_2 = V_1.$$

Therefore, we conclude that the regular part $\mathcal{R}(A)$ of Γ_A contains a stable tube $\mathcal{T}^A_{\lambda_0}$ of rank 3 with the simple regular modules S_3, S_4, R, a stable tube $\mathcal{T}^A_{\lambda_1}$ of rank 2 with the simple regular modules U_1, U_2, and a stable tube $\mathcal{T}^A_{\lambda_2}$ of rank 2 with the simple regular modules V_1, V_2. One can show that these three stable tubes exhaust all stable tubes of rank at least 2 in $\mathcal{R}(A)$.

9 Hereditary algebras of wild type

The main aim of this section is to describe the shape and basic properties of the connected components of the Auslander–Reiten quivers of finite dimensional hereditary algebras of wild type over a field.

Proposition 9.1. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, and M be an indecomposable module in $\mathcal{R}(A)$. Then $[M] \neq [\tau^m M]$ for any nonzero integer m.*

Proof. Let $S_1, \dots, S_n$ be a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$. Let

$$f_i = \dim_K \operatorname{End}_A(S_i) \quad \text{and} \quad f_{ij} = \dim_K \operatorname{Ext}^1_A(S_i, S_j)$$

for $i, j \in \{1, \dots, n\}$. Recall that for a valued arrow

$$i \xrightarrow{\left(d_{ij}, d'_{ij}\right)} j$$

in Q_A we have $d_{ij} f_j = f_{ij} = f_i d'_{ij}$. Further, under the canonical identification $K_0(A) = \mathbb{Z}^n$, the bilinear Euler form

$$\langle -,- \rangle_A \colon K_0(A) \times K_0(A) \longrightarrow \mathbb{Z}$$

is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \sum_{i=1}^{n} f_i x_i y_i - \sum_{i,j=1}^{n} f_{ij} x_i y_j,$$

for $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{Z}^n$. We consider also the Euler symmetric bilinear form

$$(-,-)_A \colon K_0(A) \times K_0(A) \longrightarrow \mathbb{Q},$$

associated to $\langle -,- \rangle_A$, given by

$$(\mathbf{x}, \mathbf{y})_A = \frac{1}{2}\big(\langle \mathbf{x}, \mathbf{y} \rangle_A + \langle \mathbf{y}, \mathbf{x} \rangle_A\big)$$

for $\mathbf{x}, \mathbf{y} \in K_0(A)$. Hence we have

$$2(\mathbf{x}, \mathbf{y})_A = \langle \mathbf{x}, \mathbf{y} \rangle_A + \langle \mathbf{y}, \mathbf{x} \rangle_A$$

for $\mathbf{x}, \mathbf{y} \in K_0(A)$. We note also that, for $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{Z}^n = K_0(A)$, we have

$$\langle \mathbf{x}, \mathbf{y} \rangle_A = \sum_{i=1}^{n} f_i x_i y_i - \sum_{i \to j} d_{ij} f_j x_i y_j,$$

and hence

$$2(\mathbf{x}, \mathbf{y})_A = \sum_{i=1}^{n} 2 f_i x_i y_i - \sum_{i \to j} d_{ij} f_j x_i y_j - \sum_{j \to i} d_{ji} f_i x_i y_j,$$

where the arrows in the above sums are in Q_A.

Assume now that $[M] = [\tau_A^m M]$ for a nonzero integer m. We may assume that $m \geq 1$ (exchanging eventually M with $\tau_A^m M$). Consider the vector

$$\mathbf{d} = \sum_{i=0}^{m-1} \left[\tau_A^i M\right] = \left[\bigoplus_{i=0}^{m-1} \tau_A^i M\right].$$

Then $\mathbf{d} \in \mathbb{N}^n$ and $\mathbf{d} \neq 0$. Since the modules $\tau_A^i M$, $i \in \{0, \dots, m-1\}$, are nonprojective, applying Corollary 5.3, we obtain

$$\begin{aligned} \varphi_A(\mathbf{d}) &= \varphi_A\left(\sum_{i=0}^{m-1} [\tau_A^i M]\right) = \sum_{i=0}^{m-1} \varphi_A\big([\tau_A^i M]\big) \\ &= \sum_{i=0}^{m-1} \left[\tau_A^{i+1} M\right] = \sum_{j=0}^{m-1} \left[\tau_A^j M\right] = \mathbf{d}, \end{aligned}$$

where $\varphi_A \colon K_0(A) \to K_0(A)$ is the Coxeter transformation of A. Then it follows from Proposition 5.7 that

$$\langle \mathbf{d}, \mathbf{x} \rangle_A = -\langle \mathbf{x}, \varphi_A(\mathbf{d}) \rangle_A = -\langle \mathbf{x}, \mathbf{d} \rangle_A,$$

and hence

$$2(\mathbf{d}, \mathbf{x})_A = \langle \mathbf{d}, \mathbf{x} \rangle_A + \langle \mathbf{x}, \mathbf{d} \rangle_A = 0$$

for any $\mathbf{x} \in K_0(A)$. In particular, we have $2(\mathbf{d}, \mathbf{e}_k)_A = 0$ for the standard basis vectors $\mathbf{e}_k$, $k \in \{1, \dots, n\}$, of $\mathbb{Z}^n = K_0(A)$. Therefore, we have the equalities

$$2 f_k d_k - \sum_{i \to k} d_{ik} f_k d_i - \sum_{k \to i} d_{ki} f_i d_i = 0$$

for all $k \in \{1, \dots, n\}$. We claim now that $d_i > 0$ for any $i \in \{1, \dots, n\}$. Indeed, suppose this is not the case. Then, since Q_A is a connected quiver, we have $d_k = 0$ and $d_l > 0$ for some vertices k and l in Q_A connected by an arrow. This leads to

$$2f_k d_k - \sum_{i \to k} d_{ik} f_k d_i - \sum_{k \to i} d_{ki} f_i d_i = -\sum_{i \to k} d_{ik} f_k d_i - \sum_{k \to i} d_{ki} f_i d_i < 0,$$

a contradiction. Consider the symmetric matrix $G = (g_{ij}) \in M_n(\mathbb{Z})$ defined as follows:

- $g_{ii} = 2f_i$ for any $i \in \{1, \dots, n\}$;
- $g_{ij} = -d_{ij} f_j = -f_{ij}$, if Q_A admits an arrow from i to j;
- $g_{ij} = -d_{ji} f_i = -f_{ji}$, if Q_A admits an arrow from j to i;
- $g_{ij} = 0$, if i and j are not connected by an arrow in Q_A.

Observe that then we have $\mathbf{d}G = 0$, because

$$(\mathbf{d}G)_k = 2f_k d_k - \sum_{i \to k} d_{ik} f_k d_i - \sum_{k \to i} d_{ki} f_i d_i = 0$$

for any $k \in \{1, \dots, n\}$.

Since Q_A is a connected wild valued quiver, there exists an indecomposable finite dimensional hereditary K-algebra B of Euclidean type over K, for which the Euler form χ_B is a proper restriction of the Euler form χ_A (see Lemma 4.5). In particular, there exist a positive integer $r \leq n$ and an injection $\sigma\colon \{1, \dots, r\} \to \{1, \dots, n\}$ such that the following statements hold:

- $f_i^* := \dim_K \operatorname{End}_B(S_i) = \dim_K \operatorname{End}_A(S_{\sigma(i)}) = f_{\sigma(i)}$, for any $i \in \{1, \dots, r\}$;
- $f_{ij}^* := \dim_K \operatorname{Ext}_B^1(S_i, S_j)$ is an integer with $0 \leq f_{ij}^* \leq f_{\sigma(i)\sigma(j)}$, for any $i, j \in \{1, \dots, r\}$;
- if $r = n$, then $f_{ij}^* < f_{\sigma(i)\sigma(j)}$, for some $i, j \in \{1, \dots, r\}$.

Moreover, for any valued arrow

$$i \xrightarrow{\left(d_{ij}^*, (d_{ij}^*)'\right)} j$$

in Q_B, we have $d_{ij}^* f_j^* = f_{ij}^* = f_i^* (d_{ij}^*)'$. Then the Euler bilinear form

$$\langle -, - \rangle_B \colon K_0(B) \times K_0(B) \longrightarrow \mathbb{Z}$$

is given, under the canonical identification $K_0(B) = \mathbb{Z}^r$, by

$$\langle \mathbf{x}, \mathbf{y} \rangle_B = \sum_{i=1}^{r} f_i^* x_i y_i - \sum_{i,j=1}^{r} f_{ij}^* x_i y_j$$

for $\mathbf{x} = (x_1, \ldots, x_r), \mathbf{y} = (y_1, \ldots, y_r) \in \mathbb{Z}^r$. We may also consider the Euler symmetric bilinear form

$$(-,-)_B \colon K_0(B) \times K_0(B) \longrightarrow \mathbb{Q}$$

associated to $\langle -,- \rangle_B$, given by

$$(\mathbf{x}, \mathbf{y})_B = \frac{1}{2}\big(\langle \mathbf{x}, \mathbf{y} \rangle_B + \langle \mathbf{y}, \mathbf{x} \rangle_B\big)$$

for $\mathbf{x}, \mathbf{y} \in K_0(B)$. Hence we have

$$2(\mathbf{x}, \mathbf{y})_B = \langle \mathbf{x}, \mathbf{y} \rangle_B + \langle \mathbf{y}, \mathbf{x} \rangle_B$$

for any $\mathbf{x}, \mathbf{y} \in K_0(B)$. Clearly, we have also

$$2(\mathbf{x}, \mathbf{y})_B = \sum_{i=1}^{r} 2 f_i^{\,*} x_i y_i - \sum_{i \to j} d_{ij}^* f_j^{\,*} x_i y_j - \sum_{j \to i} d_{ji}^* f_i^{\,*} x_i y_j,$$

for $\mathbf{x} = (x_1, \ldots, x_r), \mathbf{y} = (y_1, \ldots, y_r) \in \mathbb{Z}^r$, where the arrows in the above sums are in Q_B. Further, we have that χ_B is positive semidefinite of corank one with $\operatorname{rad} \chi_B = \mathbb{Z}\mathbf{h}_B$ for a vector $\mathbf{h}_B$ in $K_0(B) = \mathbb{Z}^r$ having all coordinates positive and at least one equal to 1. It follows from Lemma 8.2 that, for the Coxeter transformation $\varphi_B \colon K_0(B) \to K_0(B)$ of B, we have $\varphi_B(\mathbf{h}_B) = \mathbf{h}_B$. Then, applying Proposition 5.7, we obtain

$$\langle \mathbf{h}_B, \mathbf{x} \rangle_B = -\langle \mathbf{x}, \varphi_B(\mathbf{h}_B) \rangle_B = -\langle \mathbf{x}, \mathbf{h}_B \rangle_B,$$

and hence

$$2(\mathbf{h}_B, \mathbf{x})_B = \langle \mathbf{h}_B, \mathbf{x} \rangle_B + \langle \mathbf{x}, \mathbf{h}_B \rangle_B = 0,$$

for any $\mathbf{x} \in K_0(B)$. In particular, we have $2(\mathbf{h}_B, \mathbf{e}_k)_B = 0$ for the standard basis vectors $\mathbf{e}_k$, $k \in \{1, \ldots, r\}$, of $\mathbb{Z}^r = K_0(B)$. Therefore, for $\mathbf{h} = \mathbf{h}_B$, we obtain the equalities

$$2 f_k^{\,*} h_k - \sum_{i \to k} d_{ik}^* f_k^{\,*} h_i - \sum_{k \to i} d_{ki}^* f_i^{\,*} h_i = 0,$$

for all $k \in \{1, \ldots, r\}$. Consider also the symmetric matrix $G^* = (g_{ij}^*) \in M_r(\mathbb{Z})$ defined as follows:

- $g_{ii}^* = 2 f_i^{\,*}$ for any $i \in \{1, \ldots, r\}$;
- $g_{ij}^* = -d_{ij}^* f_j^{\,*} = -f_{ij}^*$, if Q_B admits an arrow from i to j;
- $g_{ij}^* = -d_{ji}^* f_i^{\,*} = -f_{ji}^*$, if Q_B admits an arrow from j to i;
- $g_{ij}^* = 0$, if i and j are not connected by an arrow in Q_B.

Observe that then we have $\mathbf{h}G^* = 0$, because

$$(\mathbf{h}G^*)_k = 2f_k^* h_k - \sum_{i\to k} d_{ik}^* f_k^* h_i - \sum_{k\to i} d_{ki}^* f_i^* h_i = 0,$$

for any $k \in \{1,\dots,r\}$.

Let $\mathbf{d}^* \in \mathbb{Z}^r$ be the restriction of $\mathbf{d} \in \mathbb{Z}^n$ to $\mathbb{Z}^r$ via the injection σ, that is, $d_i^* = d_{\sigma(i)}$ for any $i \in \{1,\dots,r\}$. Observe that $d_i^* > 0$ for any $i \in \{1,\dots,r\}$. We claim that $\mathbf{d}^*G^* \in \mathbb{N}^r$. Indeed, for each $k \in \{1,\dots,r\}$, it holds that

$$\begin{aligned}(\mathbf{d}^*G^*)_k &= 2f_k^* d_k^* - \sum_{i\to k} d_{ik}^* f_k^* d_i^* - \sum_{k\to i} d_{ki}^* f_i^* d_i^* \\ &\geq 2f_{\sigma(k)} d_{\sigma(k)} - \sum_{i\to\sigma(k)} d_{i\sigma(k)} f_{\sigma(k)} d_i - \sum_{\sigma(k)\to i} d_{\sigma(k)i} f_i d_i \\ &= (\mathbf{d}G)_{\sigma(k)} = 0,\end{aligned}$$

where the arrows in the last summation are in Q_A. Observe that $(\mathbf{d}^*G^*)_l > 0$ for some $l \in \{1,\dots,r\}$. Indeed, if $r = n$, then $f_{ij}^* < f_{\sigma(i)\sigma(j)}$, for some $i, j \in \{1,\dots,r\}$, and so either there is no arrow from i to j in Q_B, or there is a valued arrow $i \xrightarrow{\left(d_{ij}^*,(d_{ij}^*)'\right)} j$ in Q_B with $1 \leq d_{ij}^* f_j^* = f_{ij}^* < f_{\sigma(i)\sigma(j)}$. In the case $r < n$, the connectedness of Q_B implies that there exist two vertices $k \in \sigma(\{1,\dots,r\})$ and $l \in \{1,\dots,r\} \setminus \sigma(\{1,\dots,r\})$ which are connected in Q_A by an arrow, $d_i^* = d_{\sigma(i)}$ for all $i \in \{1,\dots,r\}$, and $d_j^* > 0$ for any $j \in \{1,\dots,r\}$. Therefore, $\mathbf{d}^*G^*$ is a nonzero vector in $\mathbb{N}^r$. On the other hand, we have the equalities

$$\mathbf{h}_B(\mathbf{d}^*G^*)^t = \mathbf{h}_B\big((G^*)^t(\mathbf{d}^*)^t\big) = \mathbf{h}_B\big(G^*(\mathbf{d}^*)^t\big) = (\mathbf{h}_B G^*)\big(\mathbf{d}^*\big)^t = 0,$$

because $\mathbf{h}_B G^* = 0$. This is a contradiction, because $\mathbf{h}_B$ has all coordinates positive and $\mathbf{d}^*G^*$ is a nonzero vector in $\mathbb{N}^r$.

Summing up, we proved that $[M] \neq [\tau^m M]$ for any nonzero integer m. □

Theorem 9.2. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K and*

$$0 \longrightarrow X \longrightarrow \bigoplus_{i=1}^{r} Y_i \longrightarrow Z \longrightarrow 0$$

be an almost split sequence in mod A *with $X, Y_1, \dots, Y_r, Z$ indecomposable modules from $\mathcal{R}(A)$. Then the following statements hold:*

(i) $r \leq 2$.

(ii) *If* $r = 2$ *and* $\dim_K Y_1 \leq \dim_K Y_2$, *then* $\dim_K Y_1 < \dim_K X < \dim_K Y_2$ *and* $\dim_K Y_1 < \dim_K Z < \dim_K Y_2$.

Proof. (i) We divide the proof into several steps.

(1) Assume $\bigoplus_{i=1}^{r} Y_i = Y' \oplus Y''$ is a decomposition in $\operatorname{mod} A$ such that $\dim_K X \leq \dim_K Y'$. We claim that then $\dim_K Y'' < \dim_K X$. Suppose that $\dim_K X \leq \dim_K Y''$. Then $Y' \neq 0 \neq Y''$ and there are irreducible monomorphisms $X \to Y'$ and $X \to Y''$ in $\operatorname{mod} A$, which are proper monomorphisms (see Lemma III.7.5 and Theorem III.7.11). There exists a nonnegative integer m such that $\dim_K \tau_A^m X \leq \dim_K \tau_A^{m+1} X$, because $\dim_K X$ is finite. Further, since $\tau_A\colon \operatorname{add} \mathcal{R}(A) \to \operatorname{add} \mathcal{R}(A)$ is an exact functor preserving almost split sequences, there exists in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow \tau_A^{m+1} X \longrightarrow \tau_A^{m+1} Y' \oplus \tau_A^{m+1} Y'' \longrightarrow \tau_A^m X \longrightarrow 0,$$

where $\tau_A^{m+1} X \to \tau_A^{m+1} Y'$ and $\tau_A^{m+1} X \to \tau_A^{m+1} Y''$ are proper monomorphisms. Then we obtain

$$\begin{aligned} \dim_K \tau_A^{m+1} Y' + \dim_K \tau_A^{m+1} Y'' &= \dim_K \tau_A^{m+1} X + \dim_K \tau_A^m X \\ &\leq 2 \dim_K \tau_A^{m+1} X \\ &< \dim_K \tau_A^{m+1} Y' + \dim_K \tau_A^{m+1} Y'', \end{aligned}$$

and hence a contradiction.

(2) For each $i \neq j$ in $\{1, \dots, r\}$, we claim that $\dim_K X < \dim_K Y_i + \dim_K Y_j$. Suppose $\dim_K X \geq \dim_K Y_i + \dim_K Y_j$ for some $i \neq j$ in $\{1, \dots, r\}$. Then the induced irreducible homomorphism $X \to Y_i \oplus Y_j$ is an epimorphism. Since $\tau_A^{-1} X$ is finite dimensional over K, we may take a positive integer s such that $\dim_K \tau_A^{-s} X \leq \dim_K \tau_A^{-s-1} X$. Since the functor $\tau_A^{-1}\colon \operatorname{add} \mathcal{R}(A) \to \operatorname{add} \mathcal{R}(A)$ is exact and preserves irreducible homomorphisms, we obtain two irreducible epimorphisms in $\operatorname{mod} A$

$$\tau_A^{-s} X \longrightarrow \tau_A^{-s} Y_i \oplus \tau_A^{-s} Y_j \quad \text{and} \quad \tau_A^{-s-1} X \longrightarrow \tau_A^{-s-1} Y_i \oplus \tau_A^{-s-1} Y_j,$$

which are proper epimorphisms (Lemma III.7.5). Observe that we have in $\operatorname{mod} A$ almost split sequences

$$\begin{gathered} 0 \longrightarrow \tau_A Y_i \longrightarrow X \oplus V_i \longrightarrow Y_i \longrightarrow 0, \\ 0 \longrightarrow \tau_A Y_j \longrightarrow X \oplus V_j \longrightarrow Y_j \longrightarrow 0, \end{gathered}$$

with the right terms Y_i and Y_j, because there exist irreducible homomorphisms $X \to Y_i$ and $X \to Y_j$. Applying the functor τ_A^{-s-1} we obtain the almost split sequences

$$0 \longrightarrow \tau_A^{-s} Y_i \longrightarrow \tau_A^{-s-1} X \oplus \tau_A^{-s-1} V_i \longrightarrow \tau_A^{-s-1} Y_i \longrightarrow 0,$$
$$0 \longrightarrow \tau_A^{-s} Y_j \longrightarrow \tau_A^{-s-1} X \oplus \tau_A^{-s-1} V_j \longrightarrow \tau_A^{-s-1} Y_j \longrightarrow 0.$$

This leads to the inequalities

$$\begin{aligned} 2 \dim_K \tau_A^{-s-1} X &\le \dim_K \tau_A^{-s} Y_i + \dim_K \tau_A^{-s-1} Y_i + \dim_K \tau_A^{-s} Y_j \\ &\quad + \dim_K \tau_A^{-s-1} Y_j \\ &< \dim_K \tau_A^{-s} X + \dim_K \tau_A^{-s-1} X \\ &\le 2 \dim_K \tau_A^{-s-1} X, \end{aligned}$$

and we get a contradiction.

(3) We have $r \le 3$. Indeed, suppose that $r \ge 4$. Then it follows from (2) that

$$\dim_K X < \dim_K (Y_1 \oplus Y_2) \quad \text{and} \quad \dim_K X < \dim_K (Y_3 \oplus Y_4),$$

and this contradicts (1).

(4) We have $r \le 2$. Assume, to the contrary, that $r = 3$. We claim that $\dim_K Y_i < \dim_K X$ for any $i \in \{1,2,3\}$. Suppose that $\dim_K X \le \dim_K Y_i$ for some $i \in \{1,2,3\}$. Let $\{1,2,3\} \setminus \{i\} = \{k,l\}$. Then, applying (2), we obtain $\dim_K X < \dim_K (Y_k \oplus Y_l)$, and again a contradiction with (1). Therefore, for each $i \in \{1,2,3\}$, there exists an irreducible epimorphism $X \to Y_i$ in $\operatorname{mod} A$. Since $X \cong \tau_A Z$, we obtain a composite epimorphism h

$$\tau_A Z \oplus \tau_A Z \oplus \tau_A Z \longrightarrow Y_1 \oplus Y_2 \oplus Y_3 \longrightarrow Z$$

in $\operatorname{mod} A$. Applying now the exact functor $\tau_A \colon \operatorname{add} \mathcal{R}(A) \to \operatorname{add} \mathcal{R}(A)$ we obtain an epimorphism

$$\tau_A^m h \colon \tau_A^{m+1} Z \oplus \tau_A^{m+1} Z \oplus \tau_A^{m+1} Z \longrightarrow \tau_A^m Z$$

for any positive integer m. We define now inductively a chain of homomorphisms

$$\ldots \longrightarrow \tau_A^{m+1} Z \xrightarrow{f_{m+1}} \tau_A^m Z \xrightarrow{f_m} \ldots \longrightarrow \tau_A^2 Z \xrightarrow{f_2} \tau_A Z \xrightarrow{f_1} Z$$

in $\operatorname{mod} A$ such that $g_m = f_1 \cdots f_m \ne 0$ for any $m \ge 1$. We define $f_1 \colon \tau_A Z \to Z$ as a nonzero restriction of the epimorphism h to a direct summand $\tau_A Z$. Assume that, for each $p \in \{1, \ldots, m\}$, we have defined a homomorphism $f_p \colon \tau_A^p Z \to \tau_A^{p-1} Z$ such that $g_p = f_1 \cdots f_p \ne 0$. Consider now the epimorphism

$$\tau_A^m h \colon \tau_A^{m+1} Z \oplus \tau_A^{m+1} Z \oplus \tau_A^{m+1} Z \longrightarrow \tau_A^m Z.$$

Then $f_1 \cdots f_m \tau_A^m h \neq 0$, and we take as f_{m+1} a restriction of $\tau_A^m h$ to a direct summand $\tau_A^{m+1} Z$ such that $f_1 \cdots f_m f_{m+1} \neq 0$. For each $m \geq 1$, we set $L_m = \operatorname{Im} f_1 \cdots f_m$, and observe that L_m is a module in $\operatorname{add} \mathcal{R}(A)$, since the category $\operatorname{add} \mathcal{R}(A)$ is closed under images. Therefore, we obtain an infinite chain

$$L_1 \supseteq L_2 \supseteq \cdots \supseteq L_m \supseteq L_{m+1} \supseteq \cdots$$

of regular submodules of the module Z. Since $\dim_K L_1$ is finite, there exists a positive integer m_0 such that $L_s = L_t$ for all $s, t \geq m_0$. This also shows that $g_m = f_1 \cdots f_m$ is an epimorphism to $L = L_{m_0}$ for any $m \geq m_0$. Applying the exact functor $\tau_A^{-m}\colon \operatorname{add} \mathcal{R}(A) \to \operatorname{add} \mathcal{R}(A)$, for $m \geq m_0$, we obtain an epimorphism $\tau_A^{-m} g_m\colon Z \to \tau_A^{-m} L$. Hence, for each $m \geq m_0$, we have $[\tau_A^{-m} L] \leq [Z]$, in the canonical order of $K_0(A)$. Let N be an indecomposable direct summand of L. Then, for $m \geq m_0$, $\tau_A^{-m} N$ is a direct summand of $\tau_A^{-m} L$, and hence $[\tau_A^{-m} N] \leq [Z]$. Thus we have $[\tau_A^{-m} N] = [\tau_A^{-s} N]$ for some $m > s \geq m_0$. This gives $[M] = [\tau_A^t M]$ for the indecomposable module $M = \tau_A^t N$ in $\mathcal{R}(A)$ and $t = m - s \geq 1$, contrary to Proposition 9.1. Therefore, $r \leq 2$ holds.

(ii) Assume $r = 2$ and $\dim_K Y_1 \leq \dim_K Y_2$. It follows from Lemma III.7.5 that every irreducible homomorphism in $\operatorname{mod} A$ is either a proper monomorphism or a proper epimorphism. Then, using (1), we conclude that

$$\text{either } \dim_K Y_1 \leq \dim_K Y_2 < \dim_K X, \text{ or } \dim_K Y_1 < \dim_K X < \dim_K Y_2.$$

Dually, we have

$$\text{either } \dim_K Y_1 \leq \dim_K Y_2 < \dim_K Z, \text{ or } \dim_K Y_1 < \dim_K Z < \dim_K Y_2.$$

Moreover, since

$$\dim_K X + \dim_K Z = \dim_K Y_1 + \dim_K Y_2,$$

the inequalities

$$\dim_K Y_1 < \dim_K X \quad \text{and} \quad \dim_K Y_2 < \dim_K Z$$

and

$$\dim_K Y_2 < \dim_K X \quad \text{and} \quad \dim_K Y_1 < \dim_K Z$$

do not hold simultaneously. It follows also that

$$\dim_K Y_1 < \dim_K X < \dim_K Y_2 \text{ if and only if } \dim_K Y_1 < \dim_K Z < \dim_K Y_2.$$

This implies the statement (ii). □

Let A be a finite dimensional hereditary K-algebra of wild type over a field K. An indecomposable module X in $\mathcal{R}(A)$ is said to be *quasi-simple* if the middle term M in the almost split sequence

$$0 \longrightarrow \tau_A X \longrightarrow M \longrightarrow X \longrightarrow 0$$

in $\operatorname{mod} A$ is indecomposable.

The following theorem established by Ringel [R2] describes the shape of the regular components of the Auslander–Reiten quivers of hereditary algebras of wild type.

Theorem 9.3. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K and $\mathcal{C}$ a component in $\mathcal{R}(A)$. Then the following statements hold:*

(i) *$\mathcal{C}$ contains a quasi-simple module X, and the τ_A-orbit $\mathcal{O}(X) = \{\tau_A^m X \mid m \in \mathbb{Z}\}$ of X consists of all quasi-simple modules in $\mathcal{C}$.*

(ii) *There exist an infinite chain of irreducible monomorphisms*

$$X = X[1] \longrightarrow X[2] \longrightarrow \cdots \longrightarrow X[j] \longrightarrow X[j+1] \longrightarrow \cdots$$

and an infinite chain of irreducible epimorphisms

$$\cdots \longrightarrow [j+1]X \longrightarrow [j]X \longrightarrow \cdots \longrightarrow [2]X \longrightarrow [1]X = X,$$

where $X[j]$ and $[j]X$ are indecomposable modules, for all $j \in \mathbb{N}^+$.

(iii) *For each integer $m \in \mathbb{Z}$, there exist isomorphisms in* mod A

$$\tau_A^m\bigl(X[j]\bigr) \cong \bigl(\tau_A^m X\bigr)[j] \quad \textit{and} \quad \tau_A^m\bigl([j]X\bigr) \cong [j]\bigl(\tau_A^m X\bigr)$$

for all $j \in \mathbb{N}^+$.

(iv) *$\mathcal{C}$ consists of the indecomposable modules $\tau_A^m(X[j])$ (respectively, the indecomposable modules $\tau_A^m([j]X)$), for $m \in \mathbb{Z}$, $j \in \mathbb{N}^+$.*

(v) *$\mathcal{C}$ is isomorphic to the translation quiver $\mathbb{Z}\mathbb{A}_\infty$.*

Proof. Let X be an indecomposable module in $\mathcal{C}$ of minimal dimension over K among all indecomposable modules from $\mathcal{C}$. Then it follows from Theorem 9.2 (ii) that X is a quasi-simple module of $\mathcal{C}$. We recall that the functors

$$\tau_A, \tau_A^{-1}\colon \operatorname{add} \mathcal{R}(A) \longrightarrow \operatorname{add} \mathcal{R}(A)$$

are mutually inverse equivalences of categories, exact, preserve the irreducibility of homomorphisms, and map almost split sequences to almost split sequences. Then it follows that the τ_A-orbit $\mathcal{O}(X) = \{\tau_A^m X \mid m \in \mathbb{Z}\}$ consists of quasi-simple modules of $\mathcal{C}$. Moreover, by Proposition 9.1, the modules $\tau_A^m X$, $m \in \mathbb{Z}$, are pairwise nonisomorphic. Further, it follows from Theorem 9.2 that every indecomposable module M in $\mathcal{C}$ has exactly two immediate predecessors and two immediate successors, or is quasi-simple. In particular, we conclude that there exist an infinite chain of irreducible monomorphisms

$$X = X[1] \longrightarrow X[2] \longrightarrow \cdots \longrightarrow X[j] \longrightarrow X[j+1] \longrightarrow \cdots$$

and an infinite chain of irreducible epimorphisms

$$\cdots \longrightarrow [j+1]X \longrightarrow [j]X \longrightarrow \cdots \longrightarrow [2]X \longrightarrow [1]X = X$$

where $X[j]$ and $[j]X$, $j \in \mathbb{N}^+$, are indecomposable modules in $\mathcal{C}$. Observe that then all the modules $X[j]$ and $[j]X$, with $j \geq 2$, are not quasi-simple. Invoking now Theorem 9.2 and the aforementioned properties of the functors $\tau_A, \tau_A^{-1}\colon \operatorname{add}\mathcal{R}(A) \to \operatorname{add}\mathcal{R}(A)$, we infer that $\mathcal{C}$ consists of the modules $\tau_A^m(X[j])$ (respectively, $\tau_A^m([j]X)$) for $m \in \mathbb{Z}$ and $j \in \mathbb{N}^+$, and is isomorphic to the translation quiver $\mathbb{Z}\mathbb{A}_\infty$. Clearly, the τ_A-orbit $\mathcal{O}(X)$ of X consists of all quasi-simple modules of $\mathcal{C}$, and $\tau_A^m(X[j]) \cong (\tau_A^m X)[j]$ and $\tau_A^m([j]X) \cong [j](\tau_A^m X)$ in $\operatorname{mod} A$ for all $m \in \mathbb{Z}$, $j \in \mathbb{N}^+$. Summing up, we conclude that statements (i)–(v) hold. □

Let A be a finite dimensional hereditary K-algebra of wild type over a field K. By analogy with the Euclidean case, we denote by $\Lambda(A)$ the set of all τ_A-orbits of quasi-simple modules in $\mathcal{R}(A)$. It is known that $\Lambda(A)$ is an infinite set. For a quasi-simple module X in $\mathcal{R}(A)$, we have in Γ_A two infinite sectional paths

$$\begin{tikzcd}
 & & & & & {[1]X = X = X[1]} \arrow[dr] & & & & & \\
 & & & & {[2]X} \arrow[ur] & & X[2] \arrow[dr] & & & & \\
 & & & \cdots \arrow[ur] & & & & \cdots \arrow[dr] & & & \\
 & & {[j]X} \arrow[ur] & & & & & & X[j] \arrow[dr] & & \\
 & {[j+1]X} \arrow[ur] & & & & & & & & X[j+1] \arrow[dr] & \\
\cdots \arrow[ur] & & & & & & & & & & \cdots
\end{tikzcd}$$

with $[j]X = \tau_A^{j-1} X[j]$ for any $j \in \mathbb{N}^+$. Following Ringel [R2], the number j in $[j]X$ and $X[j]$ is called the *quasi-length* of $[j]X$ and $X[j]$, and usually denoted by $\operatorname{ql}([j]X) = j = \operatorname{ql}(X[j])$. We also note that, in contrast to the Euclidean case, the quasi-simple modules in $\mathcal{R}(A)$ may contain proper submodules and proper quotient modules from $\mathcal{R}(A)$.

We exhibit an immediate consequence of Theorem 9.3.

Corollary 9.4. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K. Then the regular part $\mathcal{R}(A)$ of Γ_A has the disjoint decomposition*

$$\mathcal{R}(A) = \bigcup_{\lambda \in \Lambda(A)} \mathcal{C}_\lambda^A,$$

where each $\mathcal{C}_\lambda^A$ is a component of type $\mathbb{Z}\mathbb{A}_\infty$ whose upper border is formed by the quasi-simple modules of the τ_A-orbit $\lambda \in \Lambda(A)$.

The following theorem was proved by Y. Zhang in [Z1].

Theorem 9.5. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, $\mathcal{C}$ a component in $\mathcal{R}(A)$, and M, N nonisomorphic modules in $\mathcal{C}$. Then $[M] \neq [N]$.*

Proof. Assume $[M] = [N]$. Let X be a quasi-simple module in $\mathcal{C}$ such that $M = X[r]$ for some $r \geq 1$. Then it follows from Theorem 9.3 that $N = \tau_A^m(X[r+i])$ for some $m \in \mathbb{Z}$ and $i > -r$. Clearly, $m \neq 0$, because $[M] = [N]$. Moreover, $i \neq 0$, by Proposition 9.1. We may assume, exchanging if necessary M and N, that $m \geq 1$. We have two cases to consider.

(1) Assume $i > 0$. Then, applying Corollary 5.3, we obtain in $K_0(A)$ the equalities

$$\begin{aligned}\big[\tau_A^{sm} X[r]\big] &= \varphi_A^{sm}\big(\big[X[r]\big]\big) = \varphi_A^{sm}\big(\big[\tau_A^m X[r+i]\big]\big) = \varphi_A^{sm}\big(\varphi_A^m(X[r+i])\big)\\ &= \varphi_A^{(s+1)m}\big(\big[X[r+i]\big]\big) = \big[\tau_A^{(s+1)m} X[r+i]\big]\end{aligned}$$

for any $s \geq 0$. Since $i > 0$, there is a sequence of irreducible monomorphisms

$$X[r] \longrightarrow X[r+1] \longrightarrow \cdots \longrightarrow X[r+i-1] \longrightarrow X[r+i],$$

and hence a sequence of irreducible monomorphisms

$$\tau_A^{sm} X[r] \longrightarrow \tau_A^{sm} X[r+1] \longrightarrow \cdots \longrightarrow \tau_A^{sm} X[r+i-1] \longrightarrow \tau_A^{sm} X[r+i],$$

for any $s \geq 0$. This gives $\big[\tau_A^{sm} X[r]\big] < \big[\tau_A^{sm} X[r+i]\big]$ for any $s \geq 0$. Then we obtain the infinite sequence of inequalities in $K_0(A)$

$$\big[X[r]\big] > \big[\tau_A^m X[r]\big] > \cdots > \big[\tau_A^{sm} X[r]\big] > \big[\tau_A^{(s+1)m} X[r]\big] > \cdots ,$$

a contradiction.

(2) Assume $i < 0$. Then, applying Corollaries 5.3 and 5.4, we obtain

$$\begin{aligned}\big[\tau_A^{-sm} X[r+i]\big] &= \varphi_A^{-sm}\big(\big[X[r+i]\big]\big) = \varphi_A^{-(s+1)m}\big(\varphi_A^m(X[r+i])\big)\\ &= \varphi_A^{-(s+1)m}\big(\big[\tau_A^m X[r+i]\big]\big) = \varphi_A^{-(s+1)m}\big(\big[X[r]\big]\big)\\ &= \big[\tau_A^{-(s+1)m} X[r]\big]\end{aligned}$$

for any $s \geq 0$. Since $i < 0$, there is a sequence of irreducible monomorphisms

$$X[r+i] \longrightarrow X[r+i+1] \longrightarrow \cdots \longrightarrow X[r-1] \longrightarrow X[r],$$

and hence a sequence of irreducible monomorphisms

$$\tau_A^{-sm} X[r+i] \longrightarrow \tau_A^{-sm} X[r+i+1] \longrightarrow \cdots \longrightarrow \tau_A^{-sm} X[r-1] \longrightarrow \tau_A^{-sm} X[r],$$

for any $s \geq 0$. This gives $\big[\tau_A^{-sm} X[r+i]\big] < \big[\tau_A^{-sm} X[r]\big]$ for any $s \geq 0$. Then we obtain the infinite sequence of inequalities in $K_0(A)$

$$\big[X[r+i]\big] > \big[\tau_A^{-m} X[r+i]\big] > \cdots > \big[\tau_A^{-sm} X[r+i]\big] > \big[\tau_A^{-(s+1)m} X[r+i]\big] > \cdots,$$

a contradiction.

Therefore, we have $[M] \neq [N]$. □

Lemma 9.6. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, M a nonzero module in* $\operatorname{add} \mathcal{R}(A)$*, m a positive integer, and $f : M \to \tau_A^{-m} M$ a homomorphism. Then f is neither a monomorphism, nor an epimorphism.*

Proof. We first show that f is not an isomorphism. Suppose that $M \cong \tau_A^{-m} M$, and let $M = M_1 \oplus \cdots \oplus M_r$ be a decomposition of M as a direct sum of indecomposable modules. Then $M \cong \tau_A^m M$, $\tau_A^m M = \tau_A^m M_1 \oplus \cdots \oplus \tau_A^m M_r$, and, applying Theorem I.4.6, we conclude that there is a permutation σ of $\{1, \dots, r\}$ such that $\tau_A^m M_i = M_{\sigma(i)}$ for any $i \in \{1, \dots, r\}$. Hence there exists a positive integer s such that $\tau_A^{sm} M_i = M_i$ for any $i \in \{1, \dots, r\}$. Since $M_1, \dots, M_r$ belong to $\mathcal{R}(A)$, we get a contradiction with Proposition 9.1.

Assume $f : M \to \tau_A^{-m} M$ is a proper monomorphism. Since $\tau_A^m : \operatorname{add} \mathcal{R}(A) \to \operatorname{add} \mathcal{R}(A)$ is an exact functor, we obtain the infinite sequence of proper monomorphisms

$$\cdots \longrightarrow \tau_A^{(s+1)m} M \xrightarrow{\tau_A^{(s+1)m} f} \tau_A^{sm} M \longrightarrow \cdots \longrightarrow \tau_A^{2m} M \xrightarrow{\tau_A^{2m} f} \tau_A^m M \xrightarrow{\tau_A^m f} M;$$

a contradiction, because $\dim_K M$ is finite.

Assume $f : M \to \tau_A^{-m} M$ is a proper epimorphism. Then, applying the exact functor $\tau_A^{-m} : \operatorname{add} \mathcal{R}(A) \to \operatorname{add} \mathcal{R}(A)$, we get the infinite sequence of proper epimorphisms

$$M \xrightarrow{f} \tau_A^{-m} M \xrightarrow{\tau_A^{-m} f} \tau_A^{-2m} M \longrightarrow \cdots \longrightarrow \tau_A^{-sm} M \xrightarrow{\tau_A^{-sm} f} \tau_A^{-(s+1)m} \longrightarrow \dots,$$

again a contradiction. □

Lemma 9.7. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K. Then the following statements hold:*

(i) *Let X be a nonzero module in* $\operatorname{add}\mathcal{R}(A)$ *without nontrivial quotient modules from* $\operatorname{add}\mathcal{R}(A)$. *Then there exist a positive integer m and an exact sequence*

$$0 \longrightarrow X \longrightarrow \tau_A^m X \longrightarrow Q \longrightarrow 0$$

in $\operatorname{mod} A$, *with Q a nonzero module in* $\operatorname{add}\mathcal{Q}(A)$.

(ii) *Let Y be a nonzero module in* $\operatorname{add}\mathcal{R}(A)$ *without nontrivial submodules from* $\operatorname{add}\mathcal{R}(A)$. *Then there exist a positive integer m and an exact sequence*

$$0 \longrightarrow P \longrightarrow \tau_A^{-m} Y \longrightarrow Y \longrightarrow 0$$

in $\operatorname{mod} A$, *with P a nonzero module in* $\operatorname{add}\mathcal{P}(A)$.

Proof. We prove only (i), because the proof of (ii) is dual. Observe that the assumption imposed on X implies that X is an indecomposable module from $\mathcal{R}(A)$. Let n be the rank of the Grothendieck group $K_0(A)$. Consider the module

$$M = \bigoplus_{i=0}^{n} \tau_A^{2i} X.$$

It follows from Theorem 9.3 that X lies in an acyclic component $\mathcal{C}$ of Γ_A of type $\mathbb{Z}\mathbb{A}_\infty$. Hence M is a direct sum of $n+1$ pairwise nonisomorphic indecomposable modules. Applying Lemma VIII.7.4, we conclude that $\operatorname{Hom}_A(M, \tau_A M) \neq 0$. Then there exist $i \neq j$ in $\{0, 1 \dots, n\}$ such that

$$\operatorname{Hom}_A\left(\tau_A^{2i} X, \tau_A^{2j+1} X\right) = \operatorname{Hom}_A\left(\tau_A^{2i} X, \tau_A \tau_A^{2j} X\right) \neq 0.$$

Since $\tau_A^{-1}\colon \operatorname{add}\mathcal{R}(A) \to \operatorname{add}\mathcal{R}(A)$ is an equivalence of categories, we get $\operatorname{Hom}_A(X, \tau_A^{2(j-i)+1} X) \neq 0$. We set $m = 2(j-i)+1$ and take a nonzero homomorphism $f\colon X \to \tau_A^m X$. Observe that f is a monomorphism. Indeed, if $\operatorname{Ker} f \neq 0$, then $\operatorname{Im} f \cong X/\operatorname{Ker} f$ is a proper quotient module of X in $\operatorname{add}\mathcal{R}(A)$, because $\operatorname{add}\mathcal{R}(A)$ is closed under images, which contradicts the assumption imposed on X. Moreover, f is a proper monomorphism due to $m \neq 0$ and Proposition 9.1. Further, by Lemma 9.6, we have $m \geq 1$. Therefore, we obtain an exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow X \xrightarrow{f} \tau_A^m X \xrightarrow{g} Q \longrightarrow 0,$$

where $Q = \operatorname{Coker} f$. Clearly, $Q \neq 0$, since f is not an isomorphism. Since $\tau_A^{-m}\colon \operatorname{mod} A \to \operatorname{mod} A$ is a right exact functor (see Theorems 3.3 and III.4.10), we get an epimorphism $\tau_A^{-m} g\colon X \to \tau_A^{-m} Q$. Moreover, $\tau_A^{-m} g$ is not an isomorphism, because otherwise $\tau_A^{-m} Q$ belongs to $\operatorname{add}\mathcal{R}(A)$, and consequently g is an isomorphism. Then it follows from our assumption on X that $\tau_A^{-m}Q$ has no indecomposable direct summand from $\mathcal{R}(A)$. But then $\operatorname{Hom}_A(\mathcal{R}(A), \mathcal{P}(A)) = 0$ implies that $\tau_A^{-m}Q$ belongs to $\operatorname{add}\mathcal{Q}(A)$. Obviously, then $Q = \tau_A^{m}(\tau_A^{-m}Q)$ belongs to $\operatorname{add}\mathcal{Q}(A)$. □

We note that the above lemma says that the category $\operatorname{add}\mathcal{R}(A)$ of regular modules over a hereditary algebra A of wild type is not closed under kernels and cokernels, and consequently is not an abelian subcategory of $\operatorname{mod} A$ (in contrast to the Euclidean case).

Proposition 9.8. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, and M a nonzero module in* $\operatorname{add}\mathcal{R}(A)$. *Then there exists a positive integer m_0 such that for every module N in* $\operatorname{add}\mathcal{R}(A)$ *and any integer $m \geq m_0$ the following statements hold:*

(i) *For every homomorphism $f\colon \tau_A^{m}M \to N$ in* $\operatorname{mod} A$, $\operatorname{Ker} f$ *belongs to* $\operatorname{add}\mathcal{R}(A)$.

(ii) *For every homomorphism $g\colon N \to \tau_A^{-m}M$ in* $\operatorname{mod} A$, $\operatorname{Coker} g$ *belongs to* $\operatorname{add}\mathcal{R}(A)$.

Proof. We prove only (i), because the proof of (ii) is dual. It follows from Proposition 6.10 that there exists a positive integer m_0 such that $\dim_K \tau_A^{-m}P > \dim_K M$ for any nonzero module P in $\operatorname{add}\mathcal{P}(A)$ and integers $m \geq m_0$. Let N be a module in $\operatorname{add}\mathcal{R}(A)$ and $f\colon \tau_A^{m}M \to N$ a homomorphism in $\operatorname{mod} A$, with $m \geq m_0$. Then we have in $\operatorname{mod} A$ the exact sequence

$$0 \longrightarrow L \longrightarrow \tau_A^{m}M \longrightarrow R \longrightarrow 0,$$

where $L = \operatorname{Ker} f$ and $R = \operatorname{Coker} f$. Observe that R is in $\operatorname{add}\mathcal{R}(A)$, because R is a quotient of $\tau_A^{m}M$ and a submodule of N. Moreover, since L is a submodule of $\tau_A^{m}M$, L has no indecomposable direct summand from $\mathcal{Q}(A)$, because $\operatorname{Hom}_A(\mathcal{Q}(A), \mathcal{R}(A)) = 0$. Applying the exact functor $\tau_A^{-m}\colon \operatorname{mod}_{\mathcal{I}} A \to \operatorname{mod}_{\mathcal{P}} A$ (see Corollary III.4.11), we obtain the exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow \tau_A^{-m}L \longrightarrow M \longrightarrow \tau_A^{-m}N \longrightarrow 0,$$

and hence $\dim_K \tau_A^{-m}L \leq \dim_K M$. Then it follows from our choice of m_0 that L has no indecomposable direct summand from $\mathcal{P}(A)$. Therefore, L is in $\operatorname{add}\mathcal{R}(A)$. □

Proposition 9.9. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K and X an indecomposable module in* $\operatorname{add}\mathcal{R}(A)$. *Then all but finitely many modules $\tau_A^{i}X$, $i \in \mathbb{Z}$, are sincere.*

Proof. We prove first that all but finitely many modules $\tau_A^i X$, with $i \geq 0$, are sincere. We have two cases to consider.

(1) Assume X has no nontrivial quotient module from $\operatorname{add} \mathcal{R}(A)$. Then it follows from Lemma 9.7 (i) that there exist a positive integer m and an exact sequence

$$0 \longrightarrow X \longrightarrow \tau_A^m X \longrightarrow Q \longrightarrow 0$$

in $\operatorname{mod} A$, with Q a nonzero module in $\operatorname{add} \mathcal{Q}(A)$. Since the functor $\tau_A \colon \operatorname{add}(\mathcal{R}(A) \cup \mathcal{Q}(A)) \to \operatorname{add}(\mathcal{R}(A) \cup \mathcal{Q}(A))$ is exact, we obtain the induced exact sequence

$$0 \longrightarrow \tau_A^r X \longrightarrow \tau_A^{m+r} X \longrightarrow \tau_A^r Q \longrightarrow 0$$

in $\operatorname{mod} A$, for any nonnegative integer r. Further, by Theorem 6.11 (ii), all but finitely many indecomposable modules in $\mathcal{Q}(A)$ are sincere. Hence there exists a positive integer s such that all modules $\tau_A^r Q$, $r \geq s$, are sincere. Then all modules $\tau_A^{m+r} X$, $r \geq s$, are sincere.

(2) Assume X is an arbitrary indecomposable module in $\operatorname{add} \mathcal{R}(A)$. It follows from Proposition 9.8 (i) that there is a positive integer m_0 such that, for every module N in $\operatorname{add} \mathcal{R}(A)$ and any integer $m \geq m_0$, all homomorphisms $f \colon \tau_A^m X \to N$ in $\operatorname{mod} A$ have the kernel $\operatorname{Ker} f$ in $\operatorname{add} \mathcal{R}(A)$. Take $m \geq m_0$. Let Y be a quotient module of $\tau_A^m X$ in $\operatorname{add} \mathcal{R}(A)$ without nontrivial quotient module from $\operatorname{add} \mathcal{R}(A)$. Then we have the exact sequence

$$0 \longrightarrow L \longrightarrow \tau_A^m X \longrightarrow Y \longrightarrow 0$$

with L in $\operatorname{add} \mathcal{R}(A)$, by the choice of m. Applying the exact functor $\tau_A \colon \operatorname{add} \mathcal{R}(A) \to \operatorname{add} \mathcal{R}(A)$, we obtain the induced exact sequence

$$0 \longrightarrow \tau_A^t L \longrightarrow \tau_A^{m+t} X \longrightarrow \tau_A^t Y \longrightarrow 0$$

in $\operatorname{mod} A$, for any nonnegative integer t. It follows from (1) that all but finitely many modules $\tau_A^t Y$, $t \geq 0$, are sincere. Therefore, we conclude that all but finitely many modules $\tau_A^i X$, $i \geq 0$, are sincere.

Applying Theorem 6.11 (i) and Proposition 9.8 (ii), we prove similarly that all but finitely many modules $\tau_A^i X$, with $i \leq 0$, are sincere. □

Theorem 9.10. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, and $\mathcal{C}$ be a component in $\mathcal{R}(A)$. Then all but finitely many modules in $\mathcal{C}$ are sincere.*

Proof. It follows from Theorem 9.3 that $\mathcal{C}$ contains a quasi-simple module X such that, for any $i \in \mathbb{Z}$, there exists an infinite chain of irreducible monomorphisms

$$\tau_A^i X = \tau_A^i X[1] \longrightarrow \tau_A^i X[2] \longrightarrow \cdots \longrightarrow \tau_A^i X[j] \longrightarrow \tau_A^i X[j+1] \longrightarrow \cdots$$

between indecomposable modules from $\mathcal{C}$, and the modules of these chains exhaust all indecomposable modules of $\mathcal{C}$. Moreover, the τ_A-orbit $\mathcal{O}(X) = \{\tau_A^i X \mid i \in \mathbb{Z}\}$ of X consists of all quasi-simple modules of $\mathcal{C}$. Further, by Theorem 9.3, there exists also an infinite chain of irreducible epimorphisms

$$\cdots \longrightarrow \tau_A^i[j+1]X \longrightarrow \tau_A^i[j]X \longrightarrow \cdots \longrightarrow \tau_A^i[2]X \longrightarrow \tau_A^i[1]X = \tau_A^i X,$$

for any $i \in \mathbb{Z}$. Moreover, for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}^+$, we have $\tau_A^i[j]X = \tau_A^i(\tau_A^{j-1}X[j]) = \tau_A^{j-1}(\tau_A^i X[j])$. Observe that if a quasi-simple module $\tau_A^i X$ is sincere, then all modules $\tau_A^i X[j]$ and $\tau_A^i[j]X$, $j \in \mathbb{N}^+$, are sincere. Finally, it follows from Proposition 9.9 that all but finitely many modules $\tau_A^i X$, $i \in \mathbb{Z}$, are sincere. We conclude that all but finitely many indecomposable modules in $\mathcal{C}$ are sincere. □

Corollary 9.11. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, $\mathcal{C}$ a component in $\mathcal{R}(A)$, P an indecomposable module in $\mathcal{P}(A)$, and Q an indecomposable module in $\mathcal{Q}(A)$. Then, for all but finitely many modules X in $\mathcal{C}$, we have* $\operatorname{Hom}_A(P, X) \neq 0$ *and* $\operatorname{Hom}_A(X, Q) \neq 0$.

Proof. Let $P_1, \dots, P_n$ be a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} A$, $I_1, \dots, I_n$ a complete set of pairwise nonisomorphic indecomposable injective modules in $\operatorname{mod} A$, and $\operatorname{top}(P_i) = S_i = \operatorname{soc}(I_i)$ for any $i \in \{1, \dots, n\}$. It follows from Theorems 6.1 and 6.2 that $P = \tau_A^{-m_P} P_i$ and $Q = \tau_A^{m_Q} I_j$ for some $i, j \in \{1, \dots, n\}$ and some nonnegative integers m_P, m_Q. Applying Proposition 5.5, we obtain, for any indecomposable module X in $\mathcal{C}$, isomorphisms of K-vector spaces

$$\operatorname{Hom}_A(P, X) = \operatorname{Hom}_A(\tau_A^{-m_P} P_i, X) \cong \operatorname{Hom}_A(P_i, \tau_A^{m_P} X),$$
$$\operatorname{Hom}_A(X, Q) = \operatorname{Hom}_A(X, \tau_A^{m_Q} I_j) \cong \operatorname{Hom}_A(\tau_A^{-m_Q} X, I_j).$$

On the other hand, for any module M in $\operatorname{mod} A$, we have

$$\dim_K \operatorname{Hom}_A(P_i, M) = c_i(M) \dim_K \operatorname{End}_A(S_i),$$
$$\dim_K \operatorname{Hom}_A(M, I_j) = c_j(M) \dim_K \operatorname{End}_A(S_j),$$

by Lemma 5.6. We know from Theorem 9.10 that all but finitely many modules M in $\mathcal{C}$ are sincere, that is, $c_k(M) \neq 0$ for any $k \in \{1, \dots, n\}$. Therefore, we conclude that for all but finitely many indecomposable modules X in $\mathcal{C}$ we have $\operatorname{Hom}_A(P, X) \neq 0$ and $\operatorname{Hom}_A(X, Q) \neq 0$. □

The following theorem was proved by O. Kerner in [K1].

Theorem 9.12. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, and X, Y be nonzero modules in* $\operatorname{add} \mathcal{R}(A)$. *Then there exists a positive integer m_0 such that* $\operatorname{Hom}_A(X, \tau_A^{-m} Y) = 0$ *for all integers $m \geq m_0$.*

Proof. Let n be the rank of $K_0(A)$, and $K_0(A) = \mathbb{Z}^n$. For each vector $\mathbf{x} \in \mathbb{Z}^n$, we set $|\mathbf{x}| = \sum_{i=1}^{n} x_i$. Consider the subset of $K_0(A)$ of the form

$$\mathcal{R}_Y = \Big\{[M] \,\Big|\, M \in \operatorname{add}\mathcal{R}(A) \text{ with } \big|[M]\big| \leq \dim_K Y\Big\}.$$

Observe that $\mathcal{R}_Y$ is a finite set. Let $\mathcal{R}_Y = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)}\}$, with $\mathbf{x}^{(i)} = [X^{(i)}]$ and $X^{(i)} \in \mathcal{R}(A)$ for any $i \in \{1, \dots, r\}$. Moreover, let $\varphi_A \colon K_0(A) \to K_0(A)$ be the Coxeter transformation of A. For any nonnegative integer m, we have

$$\varphi_A^{-m}(\mathbf{x}^{(i)}) = \varphi_A^{-m}\big([X^{(i)}]\big) = \big[\tau_A^{-m} X^{(i)}\big].$$

It follows from Theorem 9.5 that there exists a positive integer m_0 such that $|\varphi_A^{-m}(\mathbf{x}^{(i)})| > \dim_K X$ for any integer $m \geq m_0$ and $i \in \{1, \dots, r\}$. Take now an integer $m \geq m_0$. Suppose that there is a nonzero homomorphism $f \colon X \to \tau_A^{-m} Y$. Then there is a nonzero homomorphism $\tau_A^m f \colon \tau_A^m X \to Y$. Consider the exact sequence

$$0 \longrightarrow L \longrightarrow \tau_A^m X \longrightarrow Z \longrightarrow 0$$

where $L = \operatorname{Ker} \tau_A^m f$ and $Z = \operatorname{Im} \tau_A^m f$. Observe that Z is a quotient module of $\tau_A^m X$ and a submodule of Y, and so Z belongs to $\operatorname{add}\mathcal{R}(A)$. Moreover, we have $|[Z]| \leq |[Y]| \leq \dim_K Y$. Hence $[Z] = \mathbf{x}^{(j)}$ for some $j \in \{1, \dots, r\}$. Applying the functor τ_A^{-m} to the above exact sequence, we obtain the exact sequence

$$0 \longrightarrow \tau_A^{-m} L \longrightarrow X \longrightarrow \tau_A^{-m} Z \longrightarrow 0$$

in $\operatorname{mod} A$. On the other hand,

$$\dim_K \tau_A^{-m} Z \geq \big|[\tau_A^{-m} Z]\big| = \big|\varphi_A^{-m}([Z])\big| = \Big|\varphi_A^{-m}\big(\mathbf{x}^{(j)}\big)\Big| > \dim_K X.$$

Therefore, $\operatorname{Hom}_A(X, \tau_A^{-m} Y) = 0$ for any integer $m \geq m_0$. □

The following theorem was proved by D. Baer in [Ba2].

Theorem 9.13. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, and X, Y be nonzero modules in* $\operatorname{add}\mathcal{R}(A)$. *Then there is a positive integer n_0 such that* $\operatorname{Hom}_A(X, \tau_A^n Y) \neq 0$ *for any integer $n \geq n_0$.*

Proof. We may assume (without loss of generality) that X has no nontrivial quotient modules from $\operatorname{add}\mathcal{R}(A)$. If follows from Lemma 9.7 that there exist a positive integer m and an exact sequence

$$0 \longrightarrow X \longrightarrow \tau_A^m X \longrightarrow Q \longrightarrow 0$$

in $\operatorname{mod} A$, with Q a nonzero module in $\operatorname{add}\mathcal{Q}(A)$. Since Y is a nonzero module in $\operatorname{add}\mathcal{R}(A)$, we conclude that the indecomposable direct summands of Y belong to a finite number of components of $\mathcal{R}(A)$. Then, applying Corollaries III.6.4 and 9.11, we conclude that there is a positive integer n_1 such that

$$\operatorname{Ext}_A^1\left(Q, \tau_A^s Y\right) \cong D\operatorname{Hom}_A\left(\tau_A^{s-1} Y, Q\right) \neq 0$$

for any integer $s \geq n_1$. On the other hand, it follows from Theorem 9.12 that there exists a positive integer n_2 such that

$$\begin{aligned}\operatorname{Ext}_A^1\left(\tau_A^m X, \tau_A^s Y\right) &\cong D\operatorname{Hom}_A\left(\tau_A^s Y, \tau_A^{m+1} X\right)\\ &\cong D\operatorname{Hom}_A\left(Y, \tau_A^{-s}(\tau_A^{m+1} X)\right) = 0\end{aligned}$$

for any integer $s \geq n_2$. Let $n_0 = \max(n_1, n_2)$. For any integer $n \geq n_0$, applying Theorem 3.3 to the above exact sequence, we obtain the following exact sequence in mod K:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(Q, \tau_A^n Y) \arrow[r] & \operatorname{Hom}_A(\tau_A^m X, \tau_A^n Y) \arrow[r] & \operatorname{Hom}_A(X, \tau_A^n Y) \arrow[dll, rounded corners] & \\
 & \operatorname{Ext}_A^1(Q, \tau_A^n Y) \arrow[r] & \operatorname{Ext}_A^1(\tau_A^m X, \tau_A^n Y) \arrow[r] & \operatorname{Ext}_A^1(X, \tau_A^n Y) \arrow[r] & 0\,.
\end{tikzcd}$$

Since $n \geq n_1$ and $n \geq n_2$, we have $\operatorname{Ext}_A^1(Q, \tau_A^n Y) \neq 0$ and $\operatorname{Ext}_A^1(\tau_A^m X, \tau_A^n Y) = 0$. Clearly, then $\operatorname{Hom}_A(X, \tau_A^n Y) \neq 0$, for any $n \geq n_0$, as required. □

The following corollary shows that, in contrast to the Euclidean case, the regular components $\mathcal{C}_\lambda^A$, $\lambda \in \Lambda(A)$, of a finite dimensional hereditary K-algebra of wild type over a field K are neither generalized standard, nor pairwise orthogonal.

Corollary 9.14. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K. The following statements hold:*

(i) *For each $\lambda \in \Lambda(A)$, the component $\mathcal{C}_\lambda^A$ is not generalized standard.*

(ii) *For $\lambda \neq \mu$ in $\Lambda(A)$, we have* $\operatorname{Hom}_A(\mathcal{C}_\lambda^A, \mathcal{C}_\mu^A) \neq 0$.

Proof. (i) Let $\lambda \in \Lambda(A)$, and X be an indecomposable module in the component $\mathcal{C}_\lambda^A$. It follows from Theorem 9.13 that there exists a positive integer n_0 such that $\operatorname{Hom}_A(X, \tau_A^n X) \neq 0$, for all integers $n \geq n_0$. Since $\mathcal{C}_\lambda^A$ is an acyclic component of type $\mathbb{Z}\mathbb{A}_\infty$, there is no path in $\mathcal{C}_\lambda^A$ from X to a module $\tau_A^n X$ with $n \geq 1$. In particular, for any $n \geq n_0$, there is no path of irreducible homomorphisms between indecomposable modules in mod A leading from X to $\tau_A^n X$. Then it follows from Proposition 3.9 that $\mathcal{C}_\lambda^A$ is not a generalized standard component of Γ_A.

(ii) Let λ and μ be disjoint elements of $\Lambda(A)$, and take indecomposable modules X in $\mathcal{C}_\lambda^A$ and Y in $\mathcal{C}_\mu^A$. Then, by Theorem 9.13, that there exists a positive integer m such that $\operatorname{Hom}_A(X, \tau_A^m Y) \neq 0$. Since $\tau_A^m Y$ is in $\mathcal{C}_\mu^A$, we get $\operatorname{Hom}_A(\mathcal{C}_\lambda^A, \mathcal{C}_\mu^A) \neq 0$. □

Lemma 9.15. *Let A be a finite dimensional hereditary K-algebra over a field K, and M, N indecomposable modules in* $\operatorname{mod} A$ *such that* $\operatorname{Ext}^1_A(N, M) = 0$. *Then any nonzero homomorphism $f : M \to N$ in* $\operatorname{mod} A$ *is a monomorphism or an epimorphism.*

Proof. Let $f : M \to N$ be a nonzero homomorphism in $\operatorname{mod} A$. Assume that f is neither a monomorphism, nor an epimorphism. Take $X = \operatorname{Im} f$. Then we can factor f as $f = gh$, where $h : M \to X$ is the canonical epimorphism induced by f and $g : X \to N$ is the inclusion homomorphism. It follows from our assumption on f that $\dim_K X < \dim_K M$ and $\dim_K X < \dim_K N$. In particular, we get $X \ncong M$ and $X \ncong N$. Consider the canonical exact sequence

$$0 \longrightarrow L \overset{u}{\hookrightarrow} M \xrightarrow{h} X \longrightarrow 0$$

in $\operatorname{mod} A$ with $L = \operatorname{Ker} h$, and the quotient module $Y = N/X$. Applying Theorem 3.2 to Y and the above exact sequence, we obtain the exact sequence

$$\operatorname{Ext}^1_A(Y, L) \xrightarrow{\operatorname{Ext}^1_A(Y,u)} \operatorname{Ext}^1_A(Y, M) \xrightarrow{\operatorname{Ext}^1_A(Y,h)} \operatorname{Ext}^1_A(Y, X) \longrightarrow 0$$

in $\operatorname{mod} K$, and hence $\operatorname{Ext}^1_A(Y, h) : \operatorname{Ext}^1_A(Y, M) \to \operatorname{Ext}^1_A(Y, X)$ is an epimorphism. Since the covariant functors $\mathcal{E}\mathrm{xt}^1_A(Y, -)$ and $\operatorname{Ext}^1_A(Y, -)$ from $\operatorname{mod} A$ to $\operatorname{mod} K$ are naturally isomorphic, by Proposition III.3.7, we conclude that $\mathcal{E}\mathrm{xt}^1_A(Y, h)$: $\mathcal{E}\mathrm{xt}^1_A(Y, M) \to \mathcal{E}\mathrm{xt}^1_A(Y, X)$ is an epimorphism. Then it follows that there exists a commutative diagram in $\operatorname{mod} A$ with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "g'"] \arrow[d, "h"] & Z \arrow[r, "\pi'"] \arrow[d, "h'"] & Y \arrow[r] \arrow[d, "\mathrm{id}_Y"] & 0 \\
0 \arrow[r] & X \arrow[r, hook, "g"] & N \arrow[r, "\pi"] & Y \arrow[r] & 0.
\end{tikzcd}$$

Observe that N is isomorphic to the fibered sum of X and Z over M, via h and g'. Hence, we have the exact sequence

$$0 \longrightarrow M \xrightarrow{\begin{bmatrix} h \\ -g' \end{bmatrix}} X \oplus Z \xrightarrow{\begin{bmatrix} g \ h' \end{bmatrix}} N \longrightarrow 0$$

in $\operatorname{mod} A$. Because $\operatorname{Ext}^1_A(N, M) = 0$ by the assumption, the above sequence splits. Therefore, we obtain an isomorphism $X \oplus Z \cong M \oplus N$ in $\operatorname{mod} A$. Since M and N are indecomposable modules, applying Theorem I.4.6 we obtain that X and Z are indecomposable modules, and M is isomorphic to one of the modules X or Z. But it is not possible, because $\dim_K X < \dim_K M < \dim_K Z$. Therefore, $f : M \to N$ is a monomorphism or an epimorphism. □

Corollary 9.16. *Let A be a finite dimensional hereditary K-algebra over a field K, and M an indecomposable module in* $\operatorname{mod} A$ *such that* $\operatorname{Ext}^1_A(M,M) = 0$. *Then* $\operatorname{End}_A(M)$ *is a division K-algebra.*

Proof. It follows from Lemma 9.15 that every nonzero homomorphism $f \in \operatorname{End}_A(M)$ is a monomorphism or an epimorphism, and consequently an isomorphism. This shows that $\operatorname{End}_A(M)$ is a division K-algebra. □

In connection to Theorem 9.12, we have also the following fact.

Proposition 9.17. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, X be an indecomposable module in $\mathcal{R}(A)$ such that* $\operatorname{Hom}_A(X, \tau_A^{-m}X) \neq 0$ *for a positive integer m. Then* $\operatorname{Hom}_A(X, \tau_A^{-i}X) \neq 0$ *for all $i \in \{1,\dots,m\}$.*

Proof. Assume $m \geq 2$ and $\operatorname{Hom}_A(X, \tau_A^{-i}X) = 0$ for some $i \in \{1,\dots,m-1\}$. Applying Corollary III.6.4, we obtain

$$\operatorname{Ext}^1_A\left(\tau_A^{-(i+1)}X, X\right) \cong D\operatorname{Hom}_A\left(X, \tau_A^{-i}X\right) = 0.$$

Then it follows from Lemma 9.15 that every nonzero homomorphism $f\colon X \to \tau_A^{-(i+1)}X$ is a monomorphism or an epimorphism. But this contradicts Lemma 9.6. □

The following proposition provides a characterization of quasi-simple bricks over hereditary algebras of wild type.

Proposition 9.18. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K and X be a brick in $\mathcal{R}(A)$. The following statements are equivalent:*

(i) *X is quasi-simple.*

(ii) $\operatorname{Hom}_A(X, \tau_A^{-1}X) = 0$.

(iii) $\operatorname{Hom}_A(X, \tau_A^{-m}X) = 0$ *for all positive integers m.*

Proof. We prove first that (i) implies (ii). Assume that $\operatorname{Hom}_A(X, \tau_A^{-1}X) \neq 0$, and let $f\colon X \to \tau_A^{-1}X$ be a nonzero homomorphism in $\operatorname{mod} A$. It follows from Lemma 9.6 that f is neither a monomorphism, nor an epimorphism. In particular, we have in $\operatorname{mod} A$ the exact sequence

$$0 \longrightarrow L \overset{u}{\hookrightarrow} X \xrightarrow{g} M \longrightarrow 0,$$

where $L = \operatorname{Ker} f$, $M = \operatorname{Im} f$, and g is the epimorphism induced by f. Moreover, M is a proper submodule of $\tau_A^{-1}X$ and hence $Y = \tau_A^{-1}X/M$ is a nonzero module. Applying Theorem 3.2 to Y and the above exact sequence, we obtain the exact sequence

$$\operatorname{Ext}_A^1(Y, L) \xrightarrow{\operatorname{Ext}_A^1(Y,u)} \operatorname{Ext}_A^1(Y, X) \xrightarrow{\operatorname{Ext}_A^1(Y,g)} \operatorname{Ext}_A^1(Y, M) \longrightarrow 0$$

in $\operatorname{mod} K$, and hence $\operatorname{Ext}_A^1(Y, g)\colon \operatorname{Ext}_A^1(Y, X) \to \operatorname{Ext}_A^1(Y, M)$ is an epimorphism. Further, since the covariant functors $\mathcal{E}\mathrm{xt}_A^1(Y, -)$ and $\operatorname{Ext}_A^1(Y, -)$ from $\operatorname{mod} A$ to $\operatorname{mod} K$ are naturally isomorphic, by Proposition III.3.7, we conclude that $\mathcal{E}\mathrm{xt}_A^1(Y, g)\colon \mathcal{E}\mathrm{xt}_A^1(Y, X) \to \mathcal{E}\mathrm{xt}_A^1(Y, M)$ is an epimorphism. Then we obtain a commutative diagram in $\operatorname{mod} A$ with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "h'"] \arrow[d, "g"] & N \arrow[r, "v'"] \arrow[d, "g'"] & Y \arrow[r] \arrow[d, "\mathrm{id}_Y"] & 0 \\
0 \arrow[r] & M \arrow[r, hook, "h"] & \tau_A^{-1}X \arrow[r, "v"] & Y \arrow[r] & 0,
\end{tikzcd}$$

where h is the inclusion homomorphism and $v\colon \tau_A^{-1}X \to Y = \tau_A^{-1}X/M$ the canonical epimorphism. Moreover, $\tau_A^{-1}X$ is the fibered sum of M and N over X, via g and h'. Hence, we get the exact sequence

$$0 \longrightarrow X \xrightarrow{\begin{bmatrix} g \\ -h' \end{bmatrix}} M \oplus N \xrightarrow{[h\ g']} \tau_A^{-1}X \longrightarrow 0$$

in $\operatorname{mod} A$. Observe that this exact sequence is not splittable, because $\dim_K M < \dim_K X$ and $\dim_K M < \dim_K \tau_A^{-1}X$. Summing up, we have proved that there is a nonsplittable exact sequence

$$\mathbb{V}\colon \quad 0 \longrightarrow X \xrightarrow{\varphi} V \xrightarrow{\psi} \tau_A^{-1}X \longrightarrow 0$$

in $\operatorname{mod} A$ with V a decomposable module. Consider an almost split sequence in $\operatorname{mod} A$ with the left term X

$$\mathbb{E}\colon \quad 0 \longrightarrow X \xrightarrow{\alpha} E \xrightarrow{\beta} \tau_A^{-1}X \longrightarrow 0.$$

It follows from Proposition III.3.8 that there is an isomorphism $\chi_{\tau_A^{-1}X,X}\colon \mathcal{E}\mathrm{xt}_A^1(\tau_A^{-1}X, X) \to \operatorname{Ext}_A^1(\tau_A^{-1}X, X)$ of $(\operatorname{End}_A(X), \operatorname{End}_A(\tau_A^{-1}X))$-bimodules. Further, because X is a brick, $\operatorname{End}_A(X)$ is the division K-algebra $F_X = \operatorname{End}_A(X)/\operatorname{rad}\operatorname{End}_A(X)$. Moreover, the equivalence $\tau_A^{-1}\colon \operatorname{add}\mathcal{R}(A) \to \operatorname{add}\mathcal{R}(A)$ induces an isomorphism of K-algebras $\operatorname{End}_A(X) \to \operatorname{End}_A(\tau_A^{-1}X)$, and so $\operatorname{End}_A(\tau_A^{-1}X)$ is the division K-algebra $F_{\tau_A^{-1}X}$ associated to $\tau_A^{-1}X$. Since $\operatorname{Ext}_A^1(X, \tau_A^{-1}X) \cong D\operatorname{End}_A(X)$ in $\operatorname{mod} A$, by Corollary III.6.4, we conclude that $\mathcal{E}\mathrm{xt}_A^1(\tau_A^{-1}X, X)$ is a one-dimensional left F_X-module and one-dimensional right $F_{\tau_A^{-1}X}$-module. In particular, since $[\mathbb{V}]$ and $[\mathbb{E}]$ are nonzero elements of $\mathcal{E}\mathrm{xt}_A^1(\tau_A^{-1}X, X)$, we conclude that there is a nonzero homomorphism $\gamma \in \mathrm{End}_A(X)$ such that $[\gamma\mathbb{V}] = [\mathbb{E}]$. Therefore, there exists a commutative diagram in $\operatorname{mod} A$ of the form

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\varphi"] \arrow[d, "\gamma"] & V \arrow[r, "\psi"] \arrow[d, "\sigma"] & \tau_A^{-1}X \arrow[r] \arrow[d, "\mathrm{id}_{\tau_A^{-1}X}"] & 0 \\
0 \arrow[r] & X \arrow[r, "\alpha"] & E \arrow[r, "\beta"] & \tau_A^{-1}X \arrow[r] & 0,
\end{tikzcd}$$

and σ is an isomorphism. This implies that $E \cong V$ is a decomposable module, and consequently X is not quasi-simple. This shows that (i) implies (ii).

The equivalence (ii) $\Leftrightarrow$ (iii) follows from Proposition 9.17. For (ii) $\Rightarrow$ (i), we note that, if X is not quasi-simple, then $\mathrm{Hom}_A(X, \tau_A^{-1}X) \neq 0$. Indeed, consider the component $\mathcal{C}$ of Γ_A containing the module X. Then there is a quasi-simple module Z in $\mathcal{C}$ such that $X \cong Z[j]$ with $j = \mathrm{ql}(X)$. For $j \geq 2$, we have in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow Z[j] \longrightarrow \tau_A^{-1}Z[j-1] \oplus Z[j+1] \longrightarrow \tau_A^{-1}Z[j] \longrightarrow 0,$$

and we have a nonzero composition of an irreducible epimorphism $Z[j] \to \tau_A^{-1}Z[j-1]$ with an irreducible monomorphism $\tau_A^{-1}Z[j-1] \to \tau_A^{-1}Z[j]$. Since $X \cong Z[j]$ and $\tau_A^{-1}X \cong \tau_A^{-1}Z[j]$, we get $\mathrm{Hom}_A(X, \tau_A^{-1}X) \neq 0$. Hence (ii) implies (i). □

Proposition 9.19. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K and X be an indecomposable regular module in $\mathcal{R}(A)$ such that* $\mathrm{Ext}_A^1(X, X) \neq 0$. *Then* $\mathrm{Hom}_A(X, \tau_A^m X) \neq 0$ *for any positive integer m.*

Proof. It follows from Corollary III.6.4 that

$$\mathrm{Hom}_A(X, \tau_A X) \cong D\,\mathrm{Ext}_A^1(X, X) \neq 0,$$
$$\mathrm{Hom}_A(\tau_A^{-1}X, X) \cong D\,\mathrm{Ext}_A^1(X, X) \neq 0.$$

Suppose there exists an integer $m \geq 1$ such that $\mathrm{Hom}_A(X, \tau_A^m X) \neq 0$ and $\mathrm{Hom}_A(X, \tau_A^{m+1}X) = 0$. Applying Corollary III.6.4 again, we obtain that

$$\mathrm{Ext}_A^1(\tau_A^m X, X) \cong D\,\mathrm{Hom}_A(X, \tau_A^{m+1}X) = 0.$$

Since $\mathrm{Hom}_A(X, \tau_A^m X) \neq 0$, we may take a nonzero homomorphism $f : X \to \tau_A^m X$. It follows from Lemma 9.15 that f is a monomorphism or an epimorphism. Assume f is a monomorphism. Then composing f with a nonzero homomorphism $g : \tau_A^{-1}X \to X$ we obtain a nonzero homomorphism $fg \in \mathrm{Hom}_A(\tau_A^{-1}X, \tau_A^m X)$.

Hence $\mathrm{Hom}_A(X, \tau_A^{m+1}X) \cong \mathrm{Hom}_A(\tau_A^{-1}X, \tau_A^m X) \neq 0$, which contradicts the imposed assumption. Now assume that f is an epimorphism. Observe that $\mathrm{Hom}_A(\tau_A^m X, \tau_A^{m+1}X) \cong \mathrm{Hom}_A(X, \tau_A X) \neq 0$, so we may take a nonzero homomorphism $h\colon \tau_A^m X \to \tau_A^{m+1}X$. But then hf is a nonzero homomorphism in $\mathrm{Hom}_A(X, \tau_A^{m+1}X)$, again a contradiction. □

Let A be a finite dimensional K-algebra over a field K. A *cycle* in mod A is a sequence

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = X$$

with $X_1, \dots, X_t$ indecomposable modules $f_1, \dots, f_t$ are nonzero homomorphisms from rad_A (equivalently, nonzero nonisomorphisms) (see Lemma III.1.4).

The following theorem provides a useful characterization of indecomposable modules in the regular components of finite dimensional hereditary algebras of infinite representation type.

Theorem 9.20. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, and X be an indecomposable module in* mod A. *The following conditions are equivalent:*

(i) *X belongs to $\mathcal{R}(A)$.*

(ii) *X lies on a cycle in* mod A.

Proof. We first prove that (i) implies (ii).

Assume that X belongs to the regular part $\mathcal{R}(A)$ of Γ_A. If A is of Euclidean type, then it follows from Theorem 8.12 that X belongs to a stable tube $\mathcal{T}_\lambda^A$, for some $\lambda \in \Lambda(A)$, and hence X lies on a cycle

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = X,$$

where $X_1, \dots, X_t$ are indecomposable modules in $\mathcal{T}_\lambda^A$ and $f_1, \dots, f_t$ are irreducible homomorphisms in mod A. Assume now that A is of wild type. Then it follows from Theorem 9.13 that there is a positive integer n such that $\mathrm{Hom}_A(X, \tau_A^n X) \neq 0$. Clearly, then $\mathrm{rad}_A(X, \tau_A^n X) \neq 0$, by Lemma III.1.4. Since there exists in Γ_A a path from $\tau_A^n X$ to X, we conclude that X lies on a cycle

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{2n-1} \xrightarrow{f_{2n}} X_{2n} = X,$$

where $X_1 = \tau_A^n X, X_2, \dots, X_{2n} = X$ are indecomposable modules from the same component of the regular part $\mathcal{R}(A)$ of Γ_A, and $f_2, \dots, f_{2n}$ are irreducible homomorphisms in mod A. Therefore, indeed (i) implies (ii).

Assume now that X lies on a cycle

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = X$$

in mod A. We know that the Auslander–Reiten quiver Γ_A of A has the disjoint union form

$$\Gamma_A = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A),$$

where $\mathcal{P}(A)$ is the unique postprojective component containing all indecomposable projective A-modules, $\mathcal{Q}(A)$ is the unique preinjective component containing all indecomposable preinjective A-modules, and $\mathcal{P}(A) \neq \mathcal{Q}(A)$. Moreover, $\mathcal{P}(A)$ and $\mathcal{Q}(A)$ are acyclic components. Suppose that X belongs to $\mathcal{P}(A)$. Then it follows from Proposition 6.6 that all the modules $X_t, X_{t-1}, \ldots, X_2, X_1$ belong to $\mathcal{P}(A)$ and lie on an oriented cycle in $\mathcal{P}(A)$, which is a contradiction. Similarly, if X belongs to $\mathcal{Q}(A)$, then it follows from Proposition 6.7 that all the modules $X_0, X_1, \ldots, X_{t-1}$ belong to $\mathcal{Q}(A)$ and lie on an oriented cycle in $\mathcal{Q}(A)$, which is again a contradiction. Hence X belongs to $\mathcal{R}(A)$. Therefore, (ii) implies (i). ☐

We will establish now the analogues of Proposition 8.24 and Corollary 8.25 for wild hereditary algebras. We need a preparatory lemma.

Lemma 9.21. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, $\mathcal{C}$ a regular component of Γ_A, and X a quasi-simple module in $\mathcal{C}$. Then there exist irreducible monomorphisms $f_{r,j}\colon \tau_A^r X[j] \to \tau_A^r X[j+1]$ and irreducible epimorphisms $g_{r,j}\colon \tau_A^r X[j+1] \to \tau_A^{r-1} X[j]$, for $r \in \mathbb{Z}$ and $j \in \mathbb{N}^+$, in* mod A *such that we have almost split sequences*

$$0 \to \tau_A^s X[1] \xrightarrow{f_{s,1}} \tau_A^s X[2] \xrightarrow{g_{s,1}} \tau_A^{s-1} X[1] \to 0,$$

$$0 \to \tau_A^s X[j] \xrightarrow{\left[\begin{smallmatrix} g_{s,j} \\ f_{s,j} \end{smallmatrix}\right]} \tau_A^{s-1} X[j-1] \oplus \tau_A^s X[j+1] \xrightarrow{[f_{s-1,j-1}\ g_{s,j+1}]} \tau_A^{s-1} X[j] \to 0,$$

for $s \in \mathbb{Z}$ and $j \in \mathbb{N} \setminus \{0, 1\}$.

Proof. It follows from Theorem 9.3 that $\mathcal{C}$ is an acyclic component of the form $\mathbb{Z}\mathbb{A}_\infty$ whose vertices are the modules $\tau_A^r X[j]$, $r \in \mathbb{Z}$, $j \in \mathbb{N}^+$. Let Δ be the full subquiver of $\mathcal{C}$ given by the arrows

$$\begin{tikzcd}
\tau_A^r X[2r+1] \arrow[rd] & \\
& \tau_A^r X[2r+2] \\
\tau_A^{r+1} X[2r+3] \arrow[ru] &
\end{tikzcd}$$

for all $r \in \mathbb{N}$. Then Δ is a connected subquiver of $\mathbb{Z}\mathbb{A}_\infty$ such that there is an isomorphism of translation quivers $\mathcal{C} \xrightarrow{\sim} \mathbb{Z}\Delta$.

Observe now that for any irreducible homomorphism $f\colon \tau_A^t X[1] \to \tau_A^t X[2]$ in $\operatorname{mod} A$ with $t \in \mathbb{Z}$ there is an almost split sequence in $\operatorname{mod} A$ of the form

$$0 \longrightarrow \tau_A^t X[1] \xrightarrow{f} \tau_A^t X[2] \xrightarrow{f'} \tau_A^{t-1} X[1] \longrightarrow 0.$$

Similarly, for any irreducible homomorphisms $u\colon \tau_A^t X[j] \to \tau_A^{t-1} X[j-1]$ and $v\colon \tau_A^t X[j] \to \tau_A^t X[j+1]$ in $\operatorname{mod} A$, for $t \in \mathbb{Z}$ and $j \in \mathbb{N} \setminus \{0,1\}$, there is an almost split sequence in $\operatorname{mod} A$ of the form

$$0 \longrightarrow \tau_A^t X[j] \xrightarrow{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]} \tau_A^{t-1} X[j-1] \oplus \tau_A^t X[j+1] \xrightarrow{[u'\ v']} \tau_A^{t-1} X[j] \longrightarrow 0.$$

Dually, for any irreducible homomorphism $h\colon \tau_A^{t+1} X[2] \to \tau_A^t X[1]$ in $\operatorname{mod} A$ with $t \in \mathbb{Z}$, there is an almost split sequence in $\operatorname{mod} A$ of the form

$$0 \longrightarrow \tau_A^{t+1} X[1] \xrightarrow{h'} \tau_A^{t+1} X[2] \xrightarrow{h} \tau_A^t X[1] \longrightarrow 0.$$

Moreover, for any irreducible homomorphisms $p\colon \tau_A^t X[j-1] \to \tau_A^t X[j]$ and $q\colon \tau_A^{t+1} X[j+1] \to \tau_A^t X[j]$ in $\operatorname{mod} A$, for $t \in \mathbb{Z}$ and $j \in \mathbb{N} \setminus \{0,1\}$, there is an almost split sequence in $\operatorname{mod} A$ of the form

$$0 \longrightarrow \tau_A^{t+1} X[j] \xrightarrow{\left[\begin{smallmatrix} p' \\ q' \end{smallmatrix}\right]} \tau_A^t X[j-1] \oplus \tau_A^{t+1} X[j+1] \xrightarrow{[p\ q]} \tau_A^t X[j] \longrightarrow 0.$$

Choose now arbitrary irreducible monomorphisms

$$f_{r,2r+1}\colon \tau_A^r X[2r+1] \longrightarrow \tau_A^r X[2r+2]$$

and irreducible epimorphisms

$$g_{r+1,2r+2}\colon \tau_A^{r+1} X[2r+3] \longrightarrow \tau_A^r X[2r+2]$$

in $\operatorname{mod} A$, for $r \in \mathbb{N}$, corresponding to all arrows of Δ. Then we find the required irreducible monomorphisms $f_{r,j}\colon \tau_A^r X[j] \to \tau_A^r X[j+1]$ and irreducible epimorphisms $g_{r,j}\colon \tau_A^r X[j+1] \to \tau_A^{r-1} X[j]$, for $r \in \mathbb{Z}$ and $j \in \mathbb{N}^+$, inductively passing from Δ to all meshes of $(-\mathbb{N})\Delta$ and passing from Δ to all meshes of $\mathbb{N}\Delta$. □

Proposition 9.22. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, X a quasi-simple module in $\mathcal{R}(A)$, and*

$$X = X[1] \longrightarrow X[2] \longrightarrow \cdots \longrightarrow X[m-1] \longrightarrow X[m]$$

a sectional path in Γ_A with $m \geq 2$ and source X. Then the following statements are equivalent:

(i) *$X[m]$ is a brick.*

(ii) *$X[m-1]$ is a stone.*

(iii) *$X[1], \dots, X[m-1]$ are stones.*

(iv) *$X, \tau_A^{-1}X, \dots, \tau_A^{-(m-1)}X$ are pairwise orthogonal stones.*

(v) $\operatorname{Hom}_A\left(\bigoplus_{i=1}^{m-1} X[i], \bigoplus_{i=1}^{m-1} \tau_A X[i]\right) = 0.$

Proof. Recall that the functors $\tau_A, \tau_A^{-1}\colon \operatorname{add}\mathcal{R}(A) \to \operatorname{add}\mathcal{R}(A)$ are mutually inverse equivalences of categories, exact, preserve the irreducibilily of homomorphisms, and take almost split sequences into almost split sequences. Let $\mathcal{C}$ be the regular component in Γ_A containing the module X. Then $\mathcal{C}$ is an acyclic component of the form $\mathbb{Z}\mathbb{A}_\infty$ whose vertices are the modules $\tau_A^r X[i]$, $r \in \mathbb{Z}$, $i \in \mathbb{N}^+$. We choose irreducible monomorphisms $f_{r,j}\colon \tau_A^r X[j] \to \tau_A^r X[j+1]$ and irreducible epimorphisms $g_{r,j}\colon \tau_A^r X[j+1] \to \tau_A^{r-1} X[j]$, for $r \in \mathbb{Z}$ and $j \in \mathbb{N}^+$, as in Lemma 9.21.

(i) ⇒ (ii) Suppose $X[m-1]$ is not a stone. Then, by Corollary 9.16, we have $\operatorname{Ext}_A^1(X[m-1], X[m-1]) \neq 0$, and consequently $\operatorname{Hom}_A(\tau_A^{-1}X[m-1], X[m-1]) \neq 0$. Take a nonzero homomorphism $h\colon \tau_A^{-1}X[m-1] \to X[m-1]$ in $\operatorname{mod} A$. Observe that we have an irreducible monomorphism $f\colon X[m-1] \to X[m]$ and an irreducible epimorphism $g\colon X[m] \to \tau_A^1 X[m-1]$. Hence fhg is a nonzero homomorphism in $\operatorname{rad}_A(X[m], X[m])$, and consequently $X[m]$ is not a brick. This shows that (i) implies (ii).

(ii) ⇒ (iii) Assume that $X[m-1]$ is a stone. Then applying the above arguments we conclude inductively that $X[m-1], X[m-2], \dots, X[1]$ are stones.

(iii) ⇒ (iv) Assume that $X[1], \dots, X[m-1]$ are stones. Since $X = X[1]$ is a stone, applying Proposition 9.18, we conclude that $\operatorname{Hom}_A(X, \tau_A^{-s}X) = 0$ for all positive integers s. Hence, we obtain that $\operatorname{Hom}_A(\tau_A^{-i}X, \tau_A^{-j}X) = 0$ for all $0 \leq i < j \leq m-1$. Moreover, the modules $X, \tau_A^{-1}X, \dots, \tau_A^{-(m-1)}X$ are stones. It remains to show that $\operatorname{Hom}_A(\tau_A^{-i}X, \tau_A^{-j}X) = 0$ for all $0 \leq j < i \leq m-1$. Assume that $\operatorname{Hom}_A(\tau_A^{-p}X, \tau_A^{-q}X) \neq 0$ for some $p > q$ in $\{0, 1, \dots, m-1\}$. Take $r = (m-1) - p$ and $s = q + r$. Then we have an isomorphism in $\operatorname{mod} K$

$$\operatorname{Hom}_A(\tau_A^{-(m-1)}X, \tau_A^{-s}X) \cong \operatorname{Hom}_A(\tau_A^{-p}X, \tau_A^{-q}X),$$

and hence $\operatorname{Hom}_A(\tau_A^{-(m-1)}X, \tau_A^{-s}X) \neq 0$, with $0 \leq s < m-1$. Then there exist nonzero homomorphisms $u\colon \tau_A^{-(m-1)}X \to X[s+1]$ and $v\colon X[s+1] \to \tau_A^{-s}X$ with $h = vu$, where v is the identity homomorphism (if $s = 0$) or the composition of the chosen irreducible epimorphisms for all arrows of the sectional path $X[s+1] \to \tau_A^{-1}X[s] \to \cdots \to \tau_A^{-(s-1)}X[2] \to \tau_A^{-s}X$ (if $s \geq 1$). Since $s+1 \leq m-1$, there exists the monomorphism $w\colon X[s+1] \to X[m-1]$ in $\operatorname{mod} A$, which is the identity homomorphism (if $s+1=m-1$) or the composition of the chosen monomorphisms corresponding to arrows of the sectional path $X[s+1] \to X[s+2] \to \cdots \to X[m-1]$ (if $s+1<m-1$). Observe also that there is an epimorphism $\pi\colon \tau_A^{-1}X[m-1] \to \tau_A^{-(m-1)}X$ in $\operatorname{mod} A$, which is the identity (if $m=2$) or the composition of the chosen irreducible epimorphisms for all arrows of the sectional path $\tau_A^{-1}X[m-1] \to \tau_A^{-2}X[m-2] \to \cdots \to \tau_A^{-(m-1)}X$ (if $m \geq 3$). Then we obtain a nonzero homomorphism $wu\pi\colon \tau_A^{-1}X[m-1] \to X[m-1]$ in $\operatorname{mod} A$. But then $\operatorname{Ext}_A^1(X[m-1], X[m-1]) \cong \operatorname{Hom}_A(\tau_A^{-1}X[m-1], X[m-1])$ is nonzero, which contradicts the assumption that $X[m-1]$ is a stone. Hence (iii) implies (iv).

(iv) $\Rightarrow$ (v) Let $\operatorname{Hom}_A(X[i], \tau_A X[j]) \neq 0$ for some $i, j \in \{1, \dots, m-1\}$. Clearly, then $\operatorname{Hom}_A(\tau_A^{-1}X[i], X[j]) \neq 0$. We claim that $\operatorname{Hom}_A(\tau_A^{-(r-1)}X, X[j]) \neq 0$ for some $r \in \{1, \dots, m\}$. We may assume $i \geq 2$. Take a nonzero homomorphism $h\colon \tau_A^{-1}X[i] \to X[j]$ in $\operatorname{mod} A$. Consider the chain of chosen irreducible monomorphisms

$$\tau_A^{-1}X[1] \xrightarrow{f_1} \tau_A^{-1}X[2] \xrightarrow{f_2} \cdots \longrightarrow \tau_A^{-1}X[i-1] \xrightarrow{f_{i-1}} \tau_A^{-1}X[i],$$

where $f_k = f_{-1,k}$ for $k \in \{1, \dots, i-1\}$. If $hf_{i-1} \dots f_1 \neq 0$, then we have $\operatorname{Hom}_A(\tau_A^{-1}X, X[j]) \neq 0$, and the claim follows. Assume $hf_{i-1} \cdots f_1 = 0$. There exists a minimal $r \in \{2, \dots, i\}$ such that $hf_{i-1} \cdots f_r \neq 0$ and $hf_{i-1} \cdots f_r f_{r-1} = 0$, where $f_i = \mathrm{id}_{\tau_A^{-1}X[i]}$. Let $p\colon \tau_A^{-1}X[r] \to \tau_A^{-(r-1)}X$ be the composition of the chosen irreducible epimorphisms for the arrows of the sectional path $\tau_A^{-1}X[r] \to \tau_A^{-2}X[r-1] \to \cdots \to \tau_A^{-(r-1)}X$. Then we have an exact sequence in $\operatorname{mod} A$ of the form (see Exercise 11.33)

$$0 \longrightarrow \tau_A^{-1}X[r-1] \xrightarrow{f_{r-1}} \tau_A^{-1}X[r] \xrightarrow{p} \tau_A^{-(r-1)}X \longrightarrow 0.$$

Since $(hf_{i-1} \cdots f_r)f_{r-1} = 0$, we conclude that there is a homomorphism $h'\colon \tau_A^{-(r-1)}X \to X[j]$ such that $hf_{i-1} \cdots f_r = h'p$. Clearly, $h' \neq 0$, and hence $\operatorname{Hom}_A(\tau_A^{-(r-1)}X, X[j]) \neq 0$. We note that then $j \geq 2$, because X and $\tau_A^{-(r-1)}X$ are nonisomorphic, and hence orthogonal. Consider now the chain of the chosen irreducible epimorphisms

$$X[j] \xrightarrow{g_{j-1}} \tau_A^{-1}X[j-1] \longrightarrow \cdots \longrightarrow \tau_A^{-(j-2)}X[2] \xrightarrow{g_1} \tau_A^{-(j-1)}X[1],$$

corresponding to the arrows of the sectional path from $X[j]$ to $\tau_A^{-(j-1)}X[1]$. Take a nonzero homomorphism $u\colon \tau_A^{-(r-1)}X \to X[j]$. Observe that $g_1 \dots g_{j-1}u \in \operatorname{rad}_A(\tau_A^{-(r-1)}X, \tau_A^{-(j-1)}X)$, and hence $g_1 \dots g_{j-1}u = 0$. Let $t \in \{2, \dots, j\}$ be minimal such that $g_{j-t} \cdots g_{j-1}u \neq 0$ and $g_{j-(t+1)}g_{j-t} \cdots g_{j-1}u = 0$, where

$g_j = \mathrm{id}_{X[j]}$. We have in $\operatorname{mod} A$ an exact sequence of the form (see Exercise 11.33)

$$0 \longrightarrow \tau_A^{-t} X \xrightarrow{v} \tau_A^{-t} X[j-t] \xrightarrow{g_{j-(t+1)}} \tau_A^{-t-1} X[j-t-1] \longrightarrow 0,$$

where v is the composition in the irreducible chosen monomorphisms for the arrows of the sectional path from $\tau_A^{-t} X$ to $\tau_A^{-t} X[j-t]$. Since $g_{t-1}(g_t \cdots g_{j-1} u) = 0$, we conclude that there is a nonzero homomorphism $u' \colon \tau_A^{-(r-1)} X \to \tau_A^{-t} X$ in $\operatorname{mod} A$ such that $g_{j-t} \cdots g_{j-1} u = v u'$. This implies that u' is an isomorphism, and hence $v = g_{j-t} \cdots g_{j-1} u w$, where w is the inverse of u'. On the other hand, there is no path in $\mathcal{C}$ from $\tau_A^{-(r-1)} X$ to $X[j]$, because $r \in \{2, \ldots, i\}$. Then it follows from Proposition 3.8 that $\operatorname{Hom}_A(\tau_A^{-(r-1)} X, X[j]) = \operatorname{rad}_A^{\infty}(\tau_A^{-r-1} X, X[j])$. Hence we conclude that v belongs to $\operatorname{rad}_A^{\infty}(\tau_A^{-t} X, \tau_A^{-t} X[j-t])$. This contradicts the Igusa–Todorov Theorem IX.2.4. Therefore, (iv) implies (v).

Since the implication (v) $\Rightarrow$ (ii) is obvious, we conclude that the statements (ii), (iii), (iv), and (v) are equivalent. Assume that the statements (ii), (iii), (iv), and (v) hold. We claim that they imply (i). Suppose that $X[m]$ is not a brick. Take a nonzero homomorphism $h \in \operatorname{rad}_A(X[m], X[m])$. Consider the canonical exact sequences in $\operatorname{mod} A$ (see Exercise 11.33)

$$0 \longrightarrow X[1] \xrightarrow{f} X[m] \xrightarrow{g} \tau_A^{-1} X[m-1] \longrightarrow 0,$$
$$0 \longrightarrow X[m-1] \xrightarrow{u} X[m] \xrightarrow{v} \tau_A^{-(m-1)} X[1] \longrightarrow 0,$$

where $f = f_{0,m-1} \cdots f_{0,1}$, $g = g_{0,m-1}$, $u = f_{0,m-1}$, $v = g_{-(m-2),1} \cdots g_{0,m-1}$. Assume first that $hf \neq 0$. Observe that $vhf = 0$, because $X[1]$ and $\tau_A^{-(m-1)} X[1]$ are orthogonal. Then there exists a nonzero homomorphism $w \colon X[1] \to X[m-1]$ such that $hf = uw$. Since u induces an isomorphism $u' \colon X[m-1] \to \operatorname{Im} u$, we obtain a nonzero homomorphism $f' = u'' h f \colon X[1] \to X[m-1]$, where u'' is the inverse of u'. This provides a nonzero homomorphism $h' = u'' h f_{0,m-1} \colon X[m-1] \to X[m-1]$, which clearly belongs to $\operatorname{rad}_A(X[m-1], X[m-1])$. This contradicts the fact that $X[m-1]$ is a stone, and hence a brick. Assume now that $hf = 0$. Then there is a homomorphism $e \colon \tau_A^{-1} X[m-1] \to X[m]$ such that $h = eg$. Observe that $\tau_A^{-1} X[m-1]$ is a stone, and hence a brick, because $X[m-1]$ is a stone. Hence $ge = 0$. Then there exists a homomorphism $e' \colon \tau_A^{-1} X[m-1] \to X[1]$ such that $e = fe'$. But then $e'' = f_{0,m-2} \cdots f_{0,1} e'$ is a nonzero homomorphism in $\operatorname{Hom}_A(\tau_A^{-1} X[m-1], X[m-1])$, where $f_{0,m-2} = \mathrm{id}_{X[1]}$ if $m = 2$. Since $\operatorname{Ext}_A^1(X[m-1], X[m-1]) \cong \operatorname{Hom}_A(\tau_A^{-1} X[m-1], X[m-1])$ in $\operatorname{mod} K$, this contradicts the fact that $X[m-1]$ is a stone. Therefore, $X[m]$ is a brick. □

In particular, we obtain the following consequences of Proposition 9.22 and Lemma VIII.7.4.

Corollary 9.23. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, n the rank of $K_0(A)$, X a quasi-simple module in $\mathcal{R}(A)$, and*

$$X = X[1] \longrightarrow X[2] \longrightarrow \cdots \longrightarrow X[m-1] \longrightarrow X[m]$$

a sectional path in Γ_A with $m \geq 2$ and source X. Then the following statements hold:

(i) *If $X[m]$ is a stone then $m \leq n$.*

(ii) *If $X[m]$ is a brick then $m \leq n+1$.*

Example 9.24. Let A be the following $\mathbb{R}$-subalgebra of the matrix algebra $M_2(\mathbb{H})$:

$$\begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ x & b \end{bmatrix} \in M_2(\mathbb{H}) \;\middle|\; a \in \mathbb{R}, b \in \mathbb{C}, x \in \mathbb{H} \right\},$$

where $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ is the division $\mathbb{R}$-algebra of quaternions. Then A is the tensor algebra $T(\mathbb{M})$ of the $\mathbb{R}$-species $\mathbb{M} = (F_1, F_2, {}_1M_2, {}_2M_1)$ such that $F_1 = \mathbb{R}$, $F_2 = \mathbb{C}$, ${}_1M_2 = 0$, and ${}_2M_1 = {}_{\mathbb{C}}\mathbb{H}_{\mathbb{R}}$. Hence, according to Theorem 2.5, $\operatorname{mod} A$ is equivalent to the category $\operatorname{rep}(\mathbb{M})$ of finite dimensional representations of the $\mathbb{R}$-species $\mathbb{M}$. Recall that the objects of $\operatorname{rep}(\mathbb{M})$ are the triples $X = (X_1, X_2, {}_1\varphi_2)$, where X_1 is a finite dimensional $\mathbb{R}$-vector space, X_2 is a finite dimensional $\mathbb{C}$-vector space, and ${}_1\varphi_2 \colon X_2 \otimes_{\mathbb{C}} \mathbb{H} \to X_1$ is an $\mathbb{R}$-linear homomorphism. We identify $\operatorname{mod} A$ with $\operatorname{rep}(\mathbb{M})$. Observe that the quiver $Q_{\mathbb{M}}$ of $\mathbb{M}$ is of the form

$$2 \xrightarrow{(4,2)} 1$$

because $d_{21} = \dim_{F_1} {}_2M_1 = \dim_{\mathbb{R}} \mathbb{H} = 4$ and $d'_{21} = \dim_{F_2} {}_2M_1 = \dim_{\mathbb{C}} \mathbb{H} = 2$. Hence if follows from Theorem 2.2 that $A = T(\mathbb{M})$ is a 7-dimensional hereditary $\mathbb{R}$-algebra with $Q_A = Q_{\mathbb{M}}$, and so of wild type. Further, the algebra A has the standard basic primitive idempotents

$$e_1 = \begin{bmatrix} 1_{\mathbb{R}} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathbb{C}} \end{bmatrix}$$

with $1_A = e_1 + e_2$,

$$\operatorname{rad} A = \begin{bmatrix} 0 & 0 \\ \mathbb{H} & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \in M_2(\mathbb{H}) \;\middle|\; x \in \mathbb{H} \right\},$$

and $(\operatorname{rad} A)^2 = 0$. Moreover, we have $e_1 \operatorname{rad} A = 0$ and $e_2 \operatorname{rad} A = e_2(\operatorname{rad} A)e_1 \cong \mathbb{H}$.

Let $P_1 = e_1A$, $P_2 = e_2A$ be the indecomposable projective modules and $I_1 = D(Ae_1)$, $I_2 = D(Ae_2)$ the indecomposable injective modules in $\operatorname{mod} A$ given by the idempotents e_1 and e_2, and hence $\operatorname{top}(P_1) = S_1 = \operatorname{soc}(I_1)$, $\operatorname{top}(P_2) = S_2 =$

$\mathrm{soc}(I_2)$, with $\dim_{\mathbb{R}} S_1 = \dim_{\mathbb{R}} \mathbb{R} = 1$, $\dim_{\mathbb{R}} S_2 = \dim_{\mathbb{R}} \mathbb{C} = 2$. Since $f_1 = \dim_{\mathbb{R}} \mathrm{End}_A(S_1) = 1$ and $f_2 = \dim_{\mathbb{R}} \mathrm{End}_A(S_2) = 2$, the Euler form $\chi_A \colon K_0(A) \to \mathbb{Z}$, with $K_0(A) = \mathbb{Z}^2$, is given by

$$\begin{aligned}\chi_A(\mathbf{x}) &= f_1 x_1^2 + f_2 x_2^2 - d_{21} f_1 x_1 x_2 \\ &= x_1^2 + 2x_2^2 - 4x_1x_2\end{aligned}$$

for $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$. Clearly, χ_A is not positive semidefinite, because for $\mathbf{x} = (1,1)$ we have $\chi_A(\mathbf{x}) = 1 + 2 - 4 = -1 < 0$.

We determine now the Coxeter transformations $\varphi_A, \varphi_A^{-1} \colon K_0(A) \to K_0(A)$ of A. We identify the basis $[S_1], [S_2]$ of $K_0(A)$ with the standard basis e_1, e_2 of $\mathbb{Z}^2$. Observe that

$$[P_1] = (1,0), \quad [P_2] = (4,1).$$

We claim that

$$[I_1] = (1,2), \quad [I_2] = (0,1).$$

Indeed, by Theorem 6.2, the full valued subquiver of the preinjective component $\mathcal{Q}(A)$ of Γ_A given by I_1 and I_2 is the opposite quiver Q_A^{op} of Q_A, so we have $I_1/S_1 = S_2 \oplus S_2$. Moreover, $I_2 = S_2$, because 2 is a source of Q_A. Now the equalities $\varphi_A([P_1]) = -[I_1]$ and $\varphi_A([P_2]) = -[I_2]$ lead to

$$\varphi_A(e_1) = -e_1 - 2e_2, \quad 4\varphi_A(e_1) + \varphi_A(e_2) = -e_2.$$

Hence $\varphi_A \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by

$$\varphi_A(\mathbf{x}) = (-x_1 + 4x_2, -2x_1 + 7x_2)$$

for $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$. Similarly, the equalities $\varphi_A^{-1}([I_1]) = -[P_1]$ and $\varphi_A^{-1}([I_2]) = -[P_2]$ lead to

$$\varphi_A^{-1}(e_1) + 2\varphi_A^{-1}(e_2) = -e_1, \quad \varphi_A^{-1}(e_2) = -4e_1 - e_2.$$

Hence $\varphi_A^{-1} \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by

$$\varphi_A^{-1}(\mathbf{x}) = (7x_1 - 4x_2, 2x_1 - x_2)$$

for $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$.

It follows from Theorem 6.1 that the postprojective component $\mathcal{P}(A)$ of Γ_A is of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
 & P_2 \arrow[dr, "(4,2)"] & & \tau_A^{-1}P_2 \arrow[dr, "(4,2)"] & & \tau_A^{-2}P_2 \arrow[dr, "(4,2)"] & & \tau_A^{-3}P_2 \arrow[dr, dash] & & \cdots \\
P_1 \arrow[ur, "(2,4)"] & & \tau_A^{-1}P_1 \arrow[ur, "(2,4)"] & & \tau_A^{-2}P_1 \arrow[ur, "(2,4)"] & & \tau_A^{-3}P_1 \arrow[ur, "(2,4)"] & & {}
\end{tikzcd}$$

and consists of the modules $\tau_A^{-m} P_i$, with the composition vectors $[\tau_A^{-m} P_i] = \varphi_A^{-m}([P_i])$, for $i \in \{1, 2\}$ and $m \in \mathbb{N}$. Similarly, by Theorem 6.2, the preinjective component $\mathcal{Q}(A)$ of Γ_A is of the form $\mathbb{N}Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
& & \tau_A^3 I_2 \arrow[dr, "(4{,}2)"] & & \tau_A^2 I_2 \arrow[dr, "(4{,}2)"] & & \tau_A I_2 \arrow[dr, "(4{,}2)"] & & I_2 \\
\cdots \arrow[r, phantom] & \tau_A^3 I_1 \arrow[ur, "(2{,}4)"] & & \tau_A^2 I_1 \arrow[ur, "(2{,}4)"] & & \tau_A I_1 \arrow[ur, "(2{,}4)"] & & I_1 \arrow[ur, "(2{,}4)"] &
\end{tikzcd}$$

and consists of the modules $\tau_A^m I_i$, with the composition vectors $[\tau_A^m I_i] = \varphi_A^m([I_i])$, for $i \in \{1, 2\}$ and $m \in \mathbb{N}$.

Since A is a hereditary algebra of wild type, it follows from Corollary 9.4 that every component in the regular part $\mathcal{R}(A)$ of Γ_A is of the form $\mathbb{Z}\mathbb{A}_\infty$. We know from Corollary 7.5 that $\mathcal{R}(A)$ contains at least one component of type $\mathbb{Z}\mathbb{A}_\infty$. We claim that there are infinitely many such components in $\mathcal{R}(A)$. We will construct a family $X^{(\lambda)}$, $\lambda \in \mathbb{R}$, of pairwise nonisomorphic indecomposable modules in $\operatorname{mod} A = \operatorname{rep}(\mathbb{M})$ with $[X^{(\lambda)}] = (1, 1)$. We note that for such an indecomposable module $X^{(\lambda)}$ we have $\chi_A([X^{(\lambda)}]) = -1$, and hence $X^{(\lambda)}$ is a module in $\mathcal{R}(A)$, due to Proposition 6.8. Moreover, it follows from Theorem 9.5 that every such module $X^{(\lambda)}$ determines a unique component $\mathcal{C}_\lambda$ of type $\mathbb{Z}\mathbb{A}_\infty$ in $\mathcal{R}(A)$, containing $X^{(\lambda)}$. In fact, $[X^{(\lambda)}] = (1, 1)$ implies that $X^{(\lambda)}$ is a module of length 2 having the simple projective module $P_1 = S_1$ as the socle, and hence $X^{(\lambda)}$ is a quasi-simple module of the component $\mathcal{C}_\lambda$.

Let X be an indecomposable module in $\operatorname{mod} A = \operatorname{rep}(\mathbb{M})$ with $[X] = (1, 1)$. Then X is up to isomorphism of the form $X = (X_1, X_2, {}_1\varphi_2)$ with $X_1 = \mathbb{R}$, $X_2 = \mathbb{C}$, and ${}_1\varphi_2 \colon \mathbb{C} \otimes_{\mathbb{C}} {}_2M_1 \to \mathbb{R}$ is a nonzero $\mathbb{R}$-linear homomorphism. Since $\mathbb{C} \otimes_{\mathbb{C}} {}_2M_1 = \mathbb{C} \otimes_{\mathbb{C}} \mathbb{H} \cong \mathbb{H}$ as $(\mathbb{C}, \mathbb{R})$-bimodule, we may consider X as a nonzero $\mathbb{R}$-linear map

$$\mathbb{H} \xrightarrow{\varphi} \mathbb{R}.$$

Moreover, for another module Y in $\operatorname{mod} A = \operatorname{rep}(\mathbb{M})$ with $[Y] = (1, 1)$ of the form

$$\mathbb{H} \xrightarrow{\psi} \mathbb{R},$$

an isomorphism from X to Y in $\operatorname{rep}(\mathbb{M})$ is a pair $(z, r) \in \mathbb{C} \times \mathbb{R}$ with $z \neq 0$ and $r \neq 0$ such that $\psi(xz) = \varphi(x)r = \varphi(xr)$ for any $x \in \mathbb{H}$. For $\lambda \in \mathbb{R}$, consider the indecomposable module

$$X^{(\lambda)} \colon \mathbb{H} \xrightarrow{\varphi^{(\lambda)}} \mathbb{R}$$

in $\operatorname{mod} A$, where the $\mathbb{R}$-linear homomorphism $\varphi^{(\lambda)}$ is given by $\varphi^{(\lambda)}(a + bi + cj + dk) = a + \lambda c$ for $a + bi + cj + dk \in \mathbb{H}$. Take now $\lambda \neq \mu$ in $\mathbb{R}$, and suppose that $X^{(\lambda)} \cong X^{(\mu)}$. Then there exist a nonzero complex number $z = \alpha + \beta i \in$

$\mathbb{R} \oplus \mathbb{R}i = \mathbb{C}$ and a nonzero real number r such that $\varphi^{(\mu)}(xz) = \varphi^{(\lambda)}(xr)$ for any $x \in \mathbb{H}$. Then we obtain the equalities

$$\begin{aligned}
\alpha &= \varphi^{(\mu)}(\alpha + \beta i) = \varphi^{(\mu)}\big(1(\alpha + \beta i)\big) = \varphi^{(\lambda)}(1r) = r,\\
-\beta &= \varphi^{(\mu)}(-\beta + \alpha i) = \varphi^{(\mu)}\big(i(\alpha + \beta i)\big) = \varphi^{(\lambda)}(ir) = 0,\\
\mu\alpha &= \varphi^{(\mu)}(\alpha j - \beta k) = \varphi^{(\mu)}\big(j(\alpha + \beta i)\big) = \varphi^{(\lambda)}(jr) = \lambda r,\\
\mu\beta &= \varphi^{(\mu)}(\beta j + \alpha k) = \varphi^{(\mu)}\big(k(\alpha + \beta i)\big) = \varphi^{(\lambda)}(kr) = 0.
\end{aligned}$$

This gives $\lambda r = \mu\alpha = \mu r$, and so $\lambda = \mu$, because $r \in \mathbb{R} \setminus \{0\}$. Therefore, $X^{(\lambda)}$, $\lambda \in \mathbb{R}$, is the required family of pairwise nonisomorphic indecomposable regular modules in $\operatorname{mod} A = \operatorname{rep}(\mathbb{M})$ with $[X^{(\lambda)}] = (1,1)$. We also have

$$\dim_K \operatorname{End}_A\big(X^{(\lambda)}\big) - \dim_K \operatorname{Ext}^1_A\big(X^{(\lambda)}, X^{(\lambda)}\big) = \chi_A\big([X^{(\lambda)}]\big) = -1 < 0,$$

and hence $\operatorname{Ext}^1_A(X^{(\lambda)}, X^{(\lambda)}) \neq 0$ for any $\lambda \in \mathbb{R}$. Then, applying Proposition 9.19, we get $\operatorname{Hom}_A(X^{(\lambda)}, \tau_A^m X^{(\lambda)}) \neq 0$ for any positive integer m and $\lambda \in \mathbb{R}$. Finally, we note that $[\tau_A X] = \varphi_A([X])$ and $[\tau_A^{-1} X] = \varphi_A^{-1}([X])$ for any indecomposable module X in $\mathcal{R}(A)$. Then the indecomposable modules around the module $X^{(\lambda)}$ in the component $\mathcal{C}_\lambda$ have the composition vectors as follows

$$\begin{tikzcd}[column sep=tiny]
{} & (99,169) \arrow[dr] & & (17,29) \arrow[dr] & & (3,5) \arrow[dr] & & (1,1) \arrow[dr] & & (3,1) \arrow[dr] & & (17,5) \arrow[dr] & & (99,29) \arrow[dr] & {} \\
\cdots \arrow[ur] \arrow[dr] & & (116,198) \arrow[ur] \arrow[dr] & & (20,34) \arrow[ur] \arrow[dr] & & (4,6) \arrow[ur] \arrow[dr] & & (4,2) \arrow[ur] \arrow[dr] & & (20,6) \arrow[ur] \arrow[dr] & & (116,34) \arrow[ur] \arrow[dr] & & \cdots \\
{} & (693,1183) \arrow[ur] \arrow[dr] & & (119,203) \arrow[ur] \arrow[dr] & & (21,35) \arrow[ur] \arrow[dr] & & (7,7) \arrow[ur] \arrow[dr] & & (21,7) \arrow[ur] \arrow[dr] & & (119,35) \arrow[ur] \arrow[dr] & & (693,203) \arrow[ur] \arrow[dr] & {} \\
\cdots \arrow[ur] & & (696,1188) \arrow[ur] \arrow[dr] & & (120,204) \arrow[ur] \arrow[dr] & & (24,36) \arrow[ur] \arrow[dr] & & (24,12) \arrow[ur] \arrow[dr] & & (120,36) \arrow[ur] \arrow[dr] & & (696,204) \arrow[ur] \arrow[dr] & & \cdots \\
{} & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots & {}
\end{tikzcd}$$

Example 9.25. Let K be a field, Q the quiver

$$\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\beta"] & \underset{3}{\bullet} \arrow[l, "\gamma"']
\end{tikzcd},$$

and $A = KQ$ the path algebra of Q over K. Then A is isomorphic to the matrix K-algebra

$$\begin{bmatrix} K & 0 & 0 \\ K^2 & K & 0 \\ K^2 & K & K \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 \\ (x,y) & b & 0 \\ (z,t) & u & c \end{bmatrix} \;\middle|\; \begin{matrix} a,b,c \in K \\ x,y,z,t,u \in K \end{matrix} \right\}.$$

Further, since Q is an acyclic quiver, A is an 8-dimensional hereditary K-algebra (Theorem I.9.6). Moreover, there is a K-linear equivalence $\operatorname{mod} A \overset{\sim}{\longrightarrow} \operatorname{rep}_K(Q)$ of $\operatorname{mod} A$ with the category $\operatorname{rep}_K(Q)$ of finite dimensional K-linear representations of Q. We will identify $\operatorname{mod} A$ with $\operatorname{rep}_K(Q)$ and $K_0(A)$ with $\mathbb{Z}^3$. Then the indecomposable projective modules in $\operatorname{mod} A = \operatorname{rep}_K(Q)$ associated to the vertices $1, 2, 3$ of Q are of the form

$$\begin{tikzcd} P_1\colon K & 0 \arrow[l, shift left] \arrow[l, shift right] & 0\ , \arrow[l] & P_2\colon K^2 & K \arrow[l, shift right, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"'] \arrow[l, shift left, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & 0\ , \arrow[l] & P_3\colon K^2 & K \arrow[l, shift right, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"'] \arrow[l, shift left, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & K\ , \arrow[l, "1"'] \end{tikzcd}$$

and the indecomposable injective modules in $\operatorname{mod} A$ associated to the vertices $1, 2, 3$ of Q are of the form

$$\begin{tikzcd} I_1\colon K & K^2 \arrow[l, shift right, "{[1,0]}"'] \arrow[l, shift left, "{[0,1]}"] & K^2\ , \arrow[l, "{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]}"'] & I_2\colon 0 & K \arrow[l, shift left] \arrow[l, shift right] & K\ , \arrow[l, "1"'] & I_3\colon 0 & 0 \arrow[l, shift left] \arrow[l, shift right] & K\ . \arrow[l] \end{tikzcd}$$

The simple modules in $\operatorname{mod} A$ associated to the vertices $1, 2, 3$ of Q are of the form

$$\begin{tikzcd} S_1\colon K & 0 \arrow[l, shift left] \arrow[l, shift right] & 0\ , \arrow[l] & S_2\colon 0 & K \arrow[l, shift left] \arrow[l, shift right] & 0\ , \arrow[l] & S_3\colon 0 & 0 \arrow[l, shift left] \arrow[l, shift right] & K\ , \arrow[l] \end{tikzcd}$$

and so $S_1 = P_1$ and $S_3 = I_3$. In particular, we have $f_i = \dim_K \operatorname{End}_A(S_i) = 1$ for $i \in \{1, 2, 3\}$, and the Euler quadratic form $\chi_A\colon \mathbb{Z}^3 \to \mathbb{Z}$ is given by

$$\chi_A(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - x_2x_3$$

for $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$. Observe that $\chi_A((2, 2, 1)) = -1$, and hence χ_A is not positive semidefinite. Clearly, the quiver Q_A of A is of the form

$$1 \xleftarrow{(2,2)} 2 \longleftarrow 3,$$

and A is a hereditary algebra of wild type.

We determine now the Coxeter transformations φ_A and φ_A^{-1} of A. We have

$$[P_1] = (1, 0, 0), \quad [P_2] = (2, 1, 0), \quad [P_3] = (2, 1, 1),$$
$$[I_1] = (1, 2, 2), \quad [I_2] = (0, 1, 1), \quad [I_3] = (0, 0, 1).$$

Then $\varphi_A([P_1]) = -[I_1]$, $\varphi_A([P_2]) = -[I_2]$, $\varphi_A([P_3]) = -[I_3]$ lead to the equalities

$$\varphi_A(e_1) = -e_1 - 2e_2 - 2e_3, \quad 2\varphi_A(e_1) + \varphi_A(e_2) = -e_2 - e_3,$$
$$2\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) = -e_3,$$

and hence $\varphi_A\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ is given by

$$\varphi_A(\mathbf{x}) = (-x_1 + 2x_2, -2x_1 + 3x_2 + x_3, -2x_1 + 3x_2)$$

for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$. Further, $\varphi_A^{-1}([I_1]) = -[P_1]$, $\varphi_A^{-1}([I_2]) = -[P_2]$, $\varphi_A^{-1}([I_3]) = -[P_3]$ lead to the equalities

$$\varphi_A^{-1}(e_1) + 2\varphi_A^{-1}(e_2) + 2\varphi_A^{-1}(e_3) = -e_1, \quad \varphi_A^{-1}(e_2) + \varphi_A^{-1}(e_3) = -2e_1 - e_2,$$
$$\varphi_A^{-1}(e_3) = -2e_1 - e_2 - e_3,$$

and hence $\varphi_A^{-1}\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ is given by

$$\varphi_A^{-1}(\mathbf{x}) = (3x_1 - 2x_3, 2x_1 - x_3, x_2 - x_3)$$

for $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$.

It follows from Theorem 6.1 that the postprojective component $\mathcal{P}(A)$ of Γ_A is of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$

$$\begin{tikzcd}[column sep=small]
 & & P_3 \arrow[dr] & & \tau_A^{-1}P_3 \arrow[dr] & & \tau_A^{-2}P_3 \arrow[dr] & & \tau_A^{-3}P_3 \arrow[dr, dash] & & \\
 & P_2 \arrow[ur] \arrow[dr, "{(2,2)}"] & & \tau_A^{-1}P_2 \arrow[ur] \arrow[dr, "{(2,2)}"] & & \tau_A^{-2}P_2 \arrow[ur] \arrow[dr, "{(2,2)}"] & & \tau_A^{-3}P_2 \arrow[ur] \arrow[dr, dash] & & & \cdots \\
P_1 \arrow[ur, "{(2,2)}"] & & \tau_A^{-1}P_1 \arrow[ur, "{(2,2)}"] & & \tau_A^{-2}P_1 \arrow[ur, "{(2,2)}"] & & \tau_A^{-3}P_1 \arrow[ur, "{(2,2)}"] & & & &
\end{tikzcd}$$

and consists of the modules $\tau_A^{-m}P_i$, with the composition vectors $[\tau_A^{-m}P_i] = \varphi_A^{-m}([P_i])$, for $i \in \{1, 2, 3\}$ and $m \in \mathbb{N}$. In particular, the composition vectors of modules in the left part of $\mathcal{P}(A)$ are as follows

$$\begin{tikzcd}[column sep=small]
 & & (2,1,1) \arrow[dr] & & (4,3,0) \arrow[dr] & & (12,8,3) \arrow[dr] & & (30,21,5) \arrow[dr, dash] & & \\
 & (2,1,0) \arrow[ur] \arrow[dr, "{(2,2)}"] & & (6,4,1) \arrow[ur] \arrow[dr, "{(2,2)}"] & & (16,11,3) \arrow[ur] \arrow[dr, "{(2,2)}"] & & (42,29,8) \arrow[ur] \arrow[dr, dash] & & & \cdots \\
(1,0,0) \arrow[ur, "{(2,2)}"] & & (3,2,0) \arrow[ur, "{(2,2)}"] & & (9,6,2) \arrow[ur, "{(2,2)}"] & & (23,16,4) \arrow[ur, "{(2,2)}"] & & & &
\end{tikzcd}$$

Similarly, it follows from Theorem 6.2 that the preinjective component $\mathcal{Q}(A)$ of Γ_A is of the form $\mathbb{N}Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
 & & {} \arrow[dr] & & \tau_A^3 I_3 \arrow[dr] & & \tau_A^2 I_3 \arrow[dr] & & \tau_A I_3 \arrow[dr] & & I_3 \\
\cdots & {} \arrow[dr] & & \tau_A^3 I_2 \arrow[ur] \arrow[dr, "{(2,2)}"] & & \tau_A^2 I_2 \arrow[ur] \arrow[dr, "{(2,2)}"] & & \tau_A I_2 \arrow[ur] \arrow[dr, "{(2,2)}"] & & I_2 \arrow[ur] & \\
 & & \tau_A^3 I_1 \arrow[ur, "{(2,2)}"] & & \tau_A^2 I_1 \arrow[ur, "{(2,2)}"] & & \tau_A I_1 \arrow[ur, "{(2,2)}"] & & I_1 \arrow[ur, "{(2,2)}"] & &
\end{tikzcd}$$

and consists of the modules $\tau_A^m I_i$, with the composition vectors $[\tau_A^m I_i] = \varphi_A^m([I_i])$, for $i \in \{1, 2, 3\}$ and $m \in \mathbb{N}$. In particular, the composition vectors of modules in the right part of $\mathcal{Q}(A)$ are as follows

$$\begin{tikzcd}
 & & {} \arrow[dr] & & (4,8,5) \arrow[dr] & & (2,3,3) \arrow[dr] & & (0,1,0) \arrow[dr] & & (0,0,1) \\
\cdots & {} \arrow[dr] & & (16,29,21) \arrow[ur] \arrow[dr, "{(2,2)}"] & & (6,11,8) \arrow[ur] \arrow[dr, "{(2,2)}"] & & (2,4,3) \arrow[ur] \arrow[dr, "{(2,2)}"] & & (0,1,1) \arrow[ur] & \\
 & & (23,42,30) \arrow[ur, "{(2,2)}"] & & (9,16,12) \arrow[ur, "{(2,2)}"] & & (3,6,4) \arrow[ur, "{(2,2)}"] & & (1,2,2) \arrow[ur, "{(2,2)}"] & &
\end{tikzcd}$$

We also note that, by Propositions 6.6 and 6.7, every indecomposable module X in $\mathcal{P}(A) \cup \mathcal{Q}(A)$ is uniquely determined (up to isomorphism) by its composition vector $[X]$ in $K_0(A)$.

Consider now the quotient algebra $B = A/Ae_3A$ of A by the two-sided ideal Ae_3A of A generated by e_3. Then B is the Kronecker algebra investigated in Example 8.27. In particular, we have the canonical embedding of categories $\operatorname{mod} B \hookrightarrow \operatorname{mod} A$, and all indecomposable modules in $\operatorname{mod} B$ are nonsincere indecomposable modules in $\operatorname{mod} A$. It follows from Theorems 6.11 and 9.10 that, for every component $\mathcal{C}$ of Γ_A, all but finitely many indecomposable modules in $\mathcal{C}$ are sincere A-modules. In particular, we conclude that every component $\mathcal{C}$ of Γ_A contains at most finitely many indecomposable B-modules. On the other hand, it follows from the above description of $\mathcal{P}(A)$ and $\mathcal{Q}(A)$ that $\mathcal{P}(A)$ contains exactly 4 indecomposable B-modules and $\mathcal{Q}(A)$ contains exactly 3 indecomposable B-modules. This shows that the regular part $\mathcal{R}(A)$ of Γ_A contains an infinite countable number of pairwise different component containing the indecomposable modules from the postprojective component $\mathcal{P}(B)$ and the preinjective component $\mathcal{Q}(B)$ of Γ_B. In particular, the indecomposable injective B-module I_1' with $\operatorname{soc}(I_1') = S_1$ is the indecomposable module in $\operatorname{mod} A$ with $[I_1'] = (1, 2, 0)$, and is neither in $\mathcal{P}(A)$ nor in $\mathcal{Q}(A)$. In fact, I_1' is a quasi-simple module in $\operatorname{mod} A$, because its unique proper submodule S_1 is located in $\mathcal{P}(A)$. Applying the formulae for φ_A and φ_A^{-1}, we infer that the indecomposable modules around I_1' in the component $\mathcal{C}'$ of Γ_A containing I_1' have the composition vectors as follows:

$$\begin{tikzcd}
 & (5,10,6) \arrow[dr] & & (3,4,4) \arrow[dr] & & (1,2,0) \arrow[dr] & & (3,2,2) \arrow[dr] & & (5,4,0) \arrow[dr, dash] & \\
\cdots \arrow[ur] \arrow[dr] & & (8,14,10) \arrow[ur] \arrow[dr] & & (4,6,4) \arrow[ur] \arrow[dr] & & (4,4,2) \arrow[ur] \arrow[dr] & & (8,6,2) \arrow[ur] \arrow[dr] & & \cdots \\
 & (23,40,30) \arrow[ur] \arrow[dr] & & (9,16,10) \arrow[ur] \arrow[dr] & & (7,8,6) \arrow[ur] \arrow[dr] & & (9,8,2) \arrow[ur] \arrow[dr] & & (23,16,6) \arrow[ur, dash] \arrow[dr, dash] & \\
\cdots \arrow[ur] & & (24,42,30) \arrow[ur] \arrow[dr, dash] & & (12,18,12) \arrow[ur] \arrow[dr, dash] & & (12,12,6) \arrow[ur] \arrow[dr, dash] & & (24,18,6) \arrow[ur] \arrow[dr, dash] & & \cdots \\
 & {} \arrow[ur] & & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots &
\end{tikzcd}$$

We note that the indecomposable module $\tau_A^{-2} I_1'$ of the above component with $[\tau_A^{-2} I_1'] = (5,4,0)$ is a module from the postprojective component $\mathcal{P}(B)$ of Γ_B. This shows that the components in $\mathcal{R}(A)$ may contain indecomposable modules both from $\mathcal{P}(B)$ and from $\mathcal{Q}(B)$. Observe also that $\operatorname{End}_A(I_1') = \operatorname{End}_B(I_1') = K$ and $\operatorname{Ext}_A^1(I_1', I_1') = \operatorname{Ext}_B^1(I_1', I_1') = 0$.

In particular, we have $\operatorname{Hom}_A(I_1', \tau_A I_1') = 0$, because

$$D \operatorname{Hom}_A(I_1', \tau_A I_1') \cong \operatorname{Ext}_A^1(I_1', I_1') = 0.$$

Moreover, it follows from Proposition 9.18 that $\operatorname{Hom}_A(I_1', \tau_A^{-m} I_1') = 0$ for any $m \in \mathbb{N}^+$.

We have proved in Example 8.27 that all simple regular modules in mod B are of the form $E^{(f)}$, for some monic irreducible polynomial f in $K[X]$, or of the form $E^{(\infty)}\colon K \overset{0}{\underset{1}{\leftleftarrows}} K$, and all these modules lie on the mouth of a stable tube of rank 1 in Γ_B. Then it follows that all these simple regular B-modules are nonsincere quasi-simple modules of the corresponding components of type $\mathbb{Z}\mathbb{A}_\infty$ in $\mathcal{R}(A)$. We also mention that, by Theorem 9.5, for two nonisomorphic indecomposable modules M and N in a component $\mathcal{C}$ of $\mathcal{R}(A)$, we have $[M] \neq [N]$. Moreover, by Theorem 9.10, all but finitely many indecomposable modules in a fixed component $\mathcal{C}$ of $\mathcal{R}(A)$ are sincere A-modules, and hence may contain at most finitely many B-modules. This shows that the indecomposable regular modules in mod B are distributed in many components of type $\mathbb{Z}\mathbb{A}_\infty$ in $\mathcal{R}(A)$. In particular, we have in $\mathcal{R}(A)$, a family $\mathcal{C}_\lambda$ of components containing the simple regular B-modules

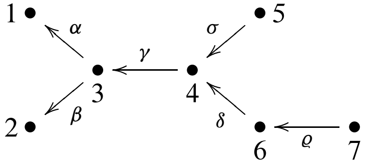
$$E^{(\lambda)}\colon\ K \overset{1}{\underset{\lambda}{\leftleftarrows}} K \longleftarrow 0\ ,$$

for $\lambda \in K$. These components $\mathcal{C}_\lambda$, $\lambda \in K$, are pairwise distinct, but the composition vectors of modules in all these components $\mathcal{C}_\lambda$ are as follows

$$\begin{tikzcd}
& (1,2,1) \arrow[dr] & & (1,1,1) \arrow[dr] & & (1,1,0) \arrow[dr] & & (3,2,1) \arrow[dr] & & (7,5,1) \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & (2,3,2) \arrow[ur] \arrow[dr] & & (2,2,1) \arrow[ur] \arrow[dr] & & (4,3,1) \arrow[ur] \arrow[dr] & & (10,7,2) \arrow[ur] \arrow[dr] & & \cdots \\
& (5,8,6) \arrow[ur] \arrow[dr] & & (3,4,2) \arrow[ur] \arrow[dr] & & (5,4,2) \arrow[ur] \arrow[dr] & & (11,8,2) \arrow[ur] \arrow[dr] & & (29,20,6) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] & & (6,9,6) \arrow[ur] \arrow[dr] & & (6,6,3) \arrow[ur] \arrow[dr] & & (12,9,3) \arrow[ur] \arrow[dr] & & (30,21,6) \arrow[ur] \arrow[dr] & & \cdots \\
& {} \arrow[ur] & & {} \arrow[ur] & & {} \arrow[ur] & & {} \arrow[ur] & & {} & \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

and hence are independent of the choice of $\lambda \in K$.

Example 9.26. Let K be a field, Δ the quiver



and $H = K\Delta$ the path algebra of Δ over K. Then H is a hereditary K-algebra of wild type, and Δ is the quiver Q_H of H. We note that the path algebra KQ of the subquiver Q of Δ given by the vertices $1, 2, 3, 4, 5, 6$ is the hereditary algebra A considered in Example 8.30. In particular, we have the canonical embedding $\operatorname{mod} A \to \operatorname{mod} H$ of module categories. For $i \in \{1, 2, 3, 4, 5, 6, 7\}$, we denote by P_i, I_i, S_i the indecomposable projective module, the indecomposable injective module, the simple module, respectively, in $\operatorname{mod} H$ associated to the vertex i of Δ. We identify the basis $[S_1], [S_2], [S_3], [S_4], [S_5], [S_6], [S_7]$ of $K_0(A)$ with the standard basis $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ of $\mathbb{Z}^7$. Then we have

$$\begin{aligned}
&[P_1] = (1,0,0,0,0,0,0), && [P_2] = (0,1,0,0,0,0,0), && [P_3] = (1,1,1,0,0,0,0),\\
&[P_4] = (1,1,1,1,0,0,0), && [P_5] = (1,1,1,1,1,0,0), && [P_6] = (1,1,1,1,0,1,0),\\
&[P_7] = (1,1,1,1,0,1,1), && [I_1] = (1,0,1,1,1,1,1), && [I_2] = (0,1,1,1,1,1,1),\\
&[I_3] = (0,0,1,1,1,1,1), && [I_4] = (0,0,0,1,1,1,1), && [I_5] = (0,0,0,0,1,0,0),\\
&[I_6] = (0,0,0,0,0,1,1), && [I_7] = (0,0,0,0,0,0,1).
\end{aligned}$$

We determine now the Coxeter transformation $\varphi_A\colon K_0(A) \to K_0(A)$ of A in the standard basis $e_1, \dots, e_7$ of $K_0(A)$. By definition, $\varphi_A([P_i]) = -[I_i]$ for any

$i \in \{1,2,3,4,5,6,7\}$. This leads to the equalities

$$\begin{aligned}
\varphi_A(e_1) &= -e_1 - e_3 - e_4 - e_5 - e_6 - e_7,\\
\varphi_A(e_2) &= -e_2 - e_3 - e_4 - e_5 - e_6 - e_7,\\
\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) &= -e_3 - e_4 - e_5 - e_6 - e_7,\\
\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) + \varphi_A(e_4) &= -e_4 - e_5 - e_6 - e_7,\\
\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) + \varphi_A(e_4) + \varphi_A(e_5) &= -e_5,\\
\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) + \varphi_A(e_4) + \varphi_A(e_6) &= -e_6 - e_7,\\
\varphi_A(e_1) + \varphi_A(e_2) + \varphi_A(e_3) + \varphi_A(e_4) + \varphi_A(e_6) + \varphi_A(e_7) &= -e_7.
\end{aligned}$$

Hence, $\varphi_A\colon \mathbb{Z}^7 \to \mathbb{Z}^7$ is given by the matrix

$$\Phi_A = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0 & 0\\ -1 & -1 & 1 & 1 & 0 & 0 & 0\\ -1 & -1 & 1 & 0 & 1 & 1 & 0\\ -1 & -1 & 1 & 0 & 0 & 1 & 0\\ -1 & -1 & 1 & 0 & 1 & 0 & 1\\ -1 & -1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The inverse Coxeter transformation $\varphi_A^{-1}\colon \mathbb{Z}^7 \to \mathbb{Z}^7$ is given by the matrix

$$\Phi_A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 1 & -1 & 0 & -1\\ 1 & 0 & 0 & 1 & -1 & 0 & -1\\ 1 & 1 & 0 & 1 & -1 & 0 & -1\\ 0 & 0 & 1 & 1 & -1 & 0 & -1\\ 0 & 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

It follows from Theorem 6.1 that the postprojective component $\mathcal{P}(H)$ of Γ_H is of the form $(-\mathbb{N})Q_H^{\mathrm{op}} = (-\mathbb{N})\Delta^{\mathrm{op}}$

$$\begin{tikzcd}
P_1 \arrow[dr] & & \tau_H^{-1}P_1 \arrow[dr] & & \tau_H^{-2}P_1 \arrow[dr] & & {} & & & \\
P_2 \arrow[r] & P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^{-1}P_2 \arrow[r] & \tau_H^{-1}P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^{-2}P_2 \arrow[r] & \tau_H^{-2}P_3 \arrow[ur, no head] \arrow[dr] & & {} & & \cdots \\
& & P_4 \arrow[r] \arrow[ur] \arrow[dr] & P_5 \arrow[r] & \tau_H^{-1}P_4 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^{-1}P_5 \arrow[r] & \tau_H^{-2}P_4 \arrow[r] \arrow[ur, no head] \arrow[dr] & \tau_H^{-2}P_5 & {} & \\
& & & P_6 \arrow[ur] \arrow[dr] & & \tau_H^{-1}P_6 \arrow[ur] \arrow[dr] & & \tau_H^{-2}P_6 \arrow[ur, no head] \arrow[dr] & & {} \\
& & & & P_7 \arrow[ur] & & \tau_H^{-1}P_7 \arrow[ur] & & \tau_H^{-2}P_7 \arrow[ur, no head] &
\end{tikzcd}$$

and consists of the modules $\tau_H^{-m} P_i$, with the composition vectors $[\tau_H^{-m} P_i] = \varphi_H^{-m}([P_i])$, for $i \in \{1, 2, \dots, 7\}$ and $m \in \mathbb{N}$. Similarly, by Theorem 6.2, the preinjective component $\mathcal{Q}(H)$ of Γ_H is of the form $\mathbb{N}Q_H^{\mathrm{op}} = \mathbb{N}\Delta^{\mathrm{op}}$

$$\begin{tikzcd}
 & \tau_H^2 I_1 \arrow[dr] & & \tau_H I_1 \arrow[dr] & & I_1 \arrow[dr] & & & & \\
{} \arrow[r] \arrow[ur] & \tau_H^2 I_2 \arrow[r] & \tau_H^2 I_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H I_2 \arrow[r] & \tau_H I_3 \arrow[r] \arrow[ur] \arrow[dr] & I_2 \arrow[r] & I_3 \arrow[dr] & & & \\
 & {} \arrow[ur] & {} \arrow[r, tail] & \tau_H^2 I_4 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^2 I_5 \arrow[r] & \tau_H I_4 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H I_5 \arrow[r] & I_4 \arrow[r] \arrow[dr] & I_5 & \\
\cdots & & {} \arrow[ur] & & \tau_H^2 I_6 \arrow[ur] \arrow[dr] & & \tau_H I_6 \arrow[ur] \arrow[dr] & & I_6 \arrow[dr] & \\
 & & & {} \arrow[ur] & & \tau_H^2 I_7 \arrow[ur] & & \tau_H I_7 \arrow[ur] & & I_7
\end{tikzcd}$$

and consists of the modules $\tau_H^m I_i$, with the composition vectors $[\tau_H^m I_i] = \varphi_H^m([I_i])$, for $i \in \{1, 2, \dots, 7\}$ and $m \in \mathbb{N}$. Let R be the indecomposable module in mod H with $[R] = (1, 1, 1, 1, 1, 1, 0)$, which is also the indecomposable module in mod A, considered in Example 8.30. We note that R is a module in $\mathcal{R}(H)$ because there is an oriented cycle of nonzero homomorphisms

$$R \longrightarrow X \longrightarrow S_3 \longrightarrow Y \longrightarrow S_4 \longrightarrow Z \longrightarrow R$$

in mod A, and hence in mod H (see Theorem 9.20). Clearly, then S_3 and S_4 also belong to $\mathcal{R}(H)$. We have the equalities in $K_0(H)$

$$\begin{aligned}
[\tau_H R] &= \varphi_H([R]) = (0, 0, 0, 1, 0, 0, 0) = [S_4], \\
[\tau_H^2 R] &= \varphi_H^2([R]) = (0, 0, 1, 0, 0, 0, 0) = [S_3], \\
[\tau_H^3 R] &= \varphi_H^3([R]) = (1, 1, 1, 1, 1, 1, 1), \\
[\tau_H^{-1} R] &= \varphi_H^{-1}([R]) = (1, 1, 2, 1, 0, 1, 1), \\
[\tau_H^{-2} R] &= \varphi_H^{-2}([R]) = (1, 1, 2, 2, 1, 0, 0).
\end{aligned}$$

Hence $S_3 = \tau_H S_4$, $S_4 = \tau_H R$, R are quasi-simple regular modules in mod H lying in a component $\mathcal{C}$ of Γ_H of type $\mathbb{Z}\mathbb{A}_\infty$, and the indecomposable modules in $\mathcal{C}$ around S_3, S_4, R have the composition vectors

$$\begin{tikzcd}[column sep=tiny]
(1,1,1,1,1,1,1) \arrow[dr] & & (0,0,1,0,0,0,0) \arrow[dr] & & (0,0,0,1,0,0,0) \arrow[dr] & & (1,1,1,1,1,1,0) \arrow[dr] & & (1,1,2,1,0,1,1) \\
 & (1,1,2,1,1,1,1) \arrow[ur] \arrow[dr] & & (0,0,1,1,0,0,0) \arrow[ur] \arrow[dr] & & (1,1,1,2,1,1,0) \arrow[ur] \arrow[dr] & & (2,2,3,2,1,2,1) \arrow[ur] \arrow[dr, dash] & \\
{} \arrow[ur] & & (1,1,2,2,1,1,1) \arrow[ur] \arrow[dr, dash] & & (1,1,2,2,1,1,0) \arrow[ur] \arrow[dr, dash] & & (2,2,3,3,1,2,1) \arrow[ur] \arrow[dr, dash] & & {} \\
 & {} \arrow[ur] & & {} \arrow[ur] & & {} \arrow[ur] & & {} &
\end{tikzcd}$$

10 Representations of bimodules

In this section we introduce matrix algebras of bimodules over finite dimensional algebras over a field and describe their module categories.

Let B and C be finite dimensional K-algebras over a field K, ${}_BM_C$ a (B, C)-bimodule, and assume that K acts centrally on $M = {}_BM_C$ with $\dim_K {}_BM_C$ finite. Then we may consider the finite dimensional K-algebra

$$A = \begin{bmatrix} B & M \\ 0 & C \end{bmatrix} = \left\{ \begin{bmatrix} b & m \\ 0 & c \end{bmatrix} \,\middle|\, b \in B,\ c \in C,\ m \in M \right\}.$$

with the multiplication given by

$$\begin{bmatrix} b & m \\ 0 & c \end{bmatrix} \begin{bmatrix} b' & m' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} bb' & bm' + mc' \\ 0 & cc' \end{bmatrix}$$

for $b, b' \in B$, $c, c' \in C$, $m, m' \in M$. Clearly, the identity 1_A of A has the decomposition $1_A = e_B + e_C$, where $e_B = \begin{bmatrix} 1_B & 0 \\ 0 & 0 \end{bmatrix}$ and $e_C = \begin{bmatrix} 0 & 0 \\ 0 & 1_C \end{bmatrix}$. We denote by $\operatorname{rep}({}_BM_C)$ the category of finite dimensional representations of the (B, C)-bimodule ${}_BM_C$, defined as follows. The objects of $\operatorname{rep}({}_BM_C)$ are the triples (X, Y, φ), where X is a module in $\operatorname{mod} B$, Y is a module in $\operatorname{mod} C$, and $\varphi\colon X \otimes_B M \to Y$ is a homomorphism in $\operatorname{mod} C$. A morphism from (X, Y, φ) to (X', Y', φ') in $\operatorname{rep}({}_BM_C)$ is a pair (f, g), where $f\colon X \to X'$ is a homomorphism in $\operatorname{mod} B$ and $g\colon Y \to Y'$ is a homomorphism in $\operatorname{mod} C$, making the following diagram in $\operatorname{mod} C$ commutative:

$$\begin{tikzcd}
X \otimes_B M \arrow[r, "\varphi"] \arrow[d, "f \otimes M"] & Y \arrow[d, "g"] \\
X' \otimes_B M \arrow[r, "\varphi'"] & Y'.
\end{tikzcd}$$

The composition of morphisms and the direct sum in $\operatorname{rep}({}_BM_C)$ are defined componentwise. We also note that $\operatorname{rep}({}_BM_C)$ is a K-category. In fact, we will show below that $\operatorname{rep}({}_BM_C)$ is an abelian K-category. We define a K-linear functor

$$F\colon \operatorname{mod} A \longrightarrow \operatorname{rep}({}_BM_C)$$

as follows. For each module Z in $\operatorname{mod} A$, we set $F(Z) = (X, Y, \varphi)$, where $X = Ze_B$, $Y = Ze_C$, and $\varphi\colon X \otimes_B M \to Y$ is the homomorphism in $\operatorname{mod} C$ defined by $\varphi(x \otimes m) = x \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = x \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} e_C$ for all $x \in X$ and $m \in M$. If $h\colon Z \to Z'$ is a homomorphism in $\operatorname{mod} A$, we define $F(h)$ to be the pair (f, g), where $f: Ze_B \to Z'e_B$ is the homomorphism in $\operatorname{mod} B$ defined by $f(ze_B) = h(ze_B) = h(ze_B)e_B$, and $g: Ze_C \to Z'e_C$ is the homomorphism in $\operatorname{mod} C$ defined by $g(ze_C) = h(ze_C) = h(ze_C)e_C$, for all $z \in Z$. An easy checking shows that $g\varphi = \varphi'(f \otimes M)$, so $F(h)$ is a morphism in $\operatorname{rep}({}_B M_C)$.

Lemma 10.1. *The functor* $F: \operatorname{mod} A \to \operatorname{rep}({}_B M_C)$ *is a* K*-linear equivalence of categories.*

Proof. According to Proposition II.6.1 we have to prove that F is faithful, full, and dense. Observe that F is faithful. Indeed, let $h: Z \to Z'$ be a homomorphism in $\operatorname{mod} A$ with $F(h) = (f, g) = 0$. Then, using the decomposition $1_A = e_B + e_C$, we obtain $h(z) = h(ze_B + ze_C) = h(ze_B) + h(ze_C) = f(ze_B) + g(ze_C) = 0$, for any $z \in Z$. Hence, $h = 0$. Now, let (X, Y, φ) be an object in $\operatorname{rep}({}_B M_C)$. Take $Z = X \oplus Y$ as K-vector space and define the structure of right A-module on Z by

$$(x, y)\begin{bmatrix} b & m \\ 0 & c \end{bmatrix} = \big(xb, \varphi(x \otimes m) + yc\big)$$

for $x \in X$, $y \in Y$, $b \in B$, $c \in C$, $m \in M$. Then we have $F(Z) = (X, Y, \varphi)$. Hence, the functor F is dense. Finally, let $(f, g): (X, Y, \varphi) \to (X', Y', \varphi')$ be a morphism in $\operatorname{rep}({}_B M_C)$. Consider the modules $Z = X \oplus Y$ and $Z' = X' \oplus Y'$ in $\operatorname{mod} A$ with $F(Z) = (X, Y, \varphi)$ and $F(Z') = (X', Y', \varphi')$ defined above. Then a simple checking shows that the K-linear map $h = \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} : X \oplus Y \to: X' \oplus Y'$ is a homomorphism in $\operatorname{mod} A$. Clearly, $F(h) = (f, g)$. This shows that F is full. □

Therefore, we may identify the modules Z in $\operatorname{mod} A$ with the associated triples $F(Z) = (X, Y, \varphi)$ in $\operatorname{rep}({}_B M_C)$, and the homomorphisms $h: Z \to Z'$ in $\operatorname{mod} A$ with the associated pairs $F(h) = (f, g)$ in $\operatorname{rep}({}_B M_C)$. Moreover, by Theorem II.4.3, for any homomorphism $\varphi: X \otimes_B M \to Y$ in $\operatorname{mod} C$, we have the uniquely determined homomorphism $\bar{\varphi}: X \to \operatorname{Hom}_C(M, Y)$ in $\operatorname{mod} B$ such that $\bar{\varphi}(x)(m) = \varphi(x \otimes m)$ for $x \in X$, $m \in M$. Hence, we may also identify a module Z in $\operatorname{mod} A$ with the triple $(X, Y, \bar{\varphi})$, where $X = Ze_B$, $Y = Ze_C$, and $\bar{\varphi}: X \to \operatorname{Hom}_C(M, Y)$ is the homomorphism in $\operatorname{mod} B$, corresponding to the homomorphism $\varphi: X \otimes_B M \to Y$ in $\operatorname{mod} C$.

In fact, we may consider the category $\overline{\operatorname{rep}}({}_B M_C)$ defined as follows. The objects of $\overline{\operatorname{rep}}({}_B M_C)$ are triples (X, Y, ψ), where X is a module in $\operatorname{mod} B$, Y is a module in $\operatorname{mod} C$, and $\psi: X \to \operatorname{Hom}_C(M, Y)$ is a homomorphism in $\operatorname{mod} B$. A morphism from (X, Y, ψ) to (X', Y', ψ') in $\overline{\operatorname{rep}}({}_B M_C)$ is a pair (f, g), where $f: X \to X'$ is a homomorphism in $\operatorname{mod} B$ and $g: Y \to Y'$ is a homomorphism in

$\operatorname{mod} C$, making the following diagram in $\operatorname{mod} B$ commutative:

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "f"] & \mathrm{Hom}_C(M, Y) \arrow[d, "\mathrm{Hom}_C(M,g)"] \\
X' \arrow[r, "\psi'"] & \mathrm{Hom}_C(M, Y') .
\end{tikzcd}$$

The composition of morphisms and the direct sum in $\overline{\mathrm{rep}}({}_B M_C)$ are defined componentwise. Clearly, $\overline{\mathrm{rep}}({}_B M_C)$ is a K-category. Moreover, by the adjoint Theorem II.4.3, there is a canonical K-linear equivalence of categories

$$H\colon \mathrm{rep}({}_B M_C) \longrightarrow \overline{\mathrm{rep}}({}_B M_C)$$

such that $H((X, Y, \varphi)) = (X, Y, \bar{\varphi})$ for any object (X, Y, φ) in $\mathrm{rep}({}_B M_C)$, and $H((f, g)) = (f, g)$ for any morphism (f, g) in $\mathrm{rep}({}_B M_C)$. In particular, we obtain the following fact.

Lemma 10.2. *The functor* $\bar{F} = HF\colon \operatorname{mod} A \to \overline{\mathrm{rep}}({}_B M_C)$ *is a* K*-linear equivalence of categories.*

We will consider now special cases.

Example 10.3. Let F and G be finite dimensional division K-algebras over a field K, ${}_F M_G$ an (F, G)-bimodule on which K acts centrally and for which $\dim_K {}_F M_G$ is finite. Then we have the finite dimensional K-algebra

$$A = A({}_F M_G) = \begin{bmatrix} F & {}_F M_G \\ 0 & G \end{bmatrix}$$

whose module category $\operatorname{mod} A$ is equivalent to the category $\mathrm{rep}({}_F M_G)$ of finite dimensional representations of the (F, G)-bimodule ${}_F M_G$. Consider the species

$$\mathbb{M} = \big(F_i, {}_i M_j\big)_{1 \le i, j \le 2}$$

where $F_1 = F$, $F_2 = G$, ${}_1 M_2 = {}_F M_G$, and ${}_2 M_1 = 0$. The quiver $Q_{\mathbb{M}}$ of $\mathbb{M}$ is of the form

$$1 \xrightarrow{(d,d')} 2$$

where $d = \dim_G {}_F M_G$ and $d' = \dim_F {}_F M_G$. Hence, the K-species $\mathbb{M}$ is acyclic and the tensor algebra $T(\mathbb{M})$ is a finite dimensional hereditary K-algebra, by Lemma 2.1. Observe now that the K-algebras $A({}_F M_G)$ and $T(\mathbb{M})$ are isomorphic. In particular, we conclude that $A({}_F M_G)$ is a finite dimensional hereditary K-algebra whose quiver coincides with $Q_{\mathbb{M}}$. Hence we obtain that

- $A({}_F M_G)$ is of Dynkin type if and only if $dd' \leq 3$;
- $A({}_F M_G)$ is of Euclidean type if and only if $dd' = 4$;
- $A({}_F M_G)$ is of wild type if and only if $dd' \geq 5$.

Example 10.4. Let K be a field, F a finite dimensional division K-algebra, C a finite dimensional K-algebra, and $M = {}_F M_C$ an (F, C)-bimodule on which K acts centrally with $\dim_K M$ finite. Then the associated matrix algebra

$$C[M] = \begin{bmatrix} F & {}_F M_C \\ 0 & C \end{bmatrix}$$

is said to be the *one-point extension algebra* of C by the (F, C)-bimodule M. We note that $C[M]$ is a hereditary algebra if and only if C is a hereditary algebra and M_C is a projective right C-module (see Exercise 11.30).

Example 10.5. Let C be a finite dimensional K-algebra over a field K, M a module in $\operatorname{mod} C$, and F a division K-subalgebra of the endomorphism K-algebra $\operatorname{End}_C(M)$. Then M is an (F, C)-bimodule with the central action of K and $\dim_K M$ finite. Hence, we may consider the one-point extension algebra

$$C[M] = \begin{bmatrix} F & {}_F M_C \\ 0 & C \end{bmatrix}.$$

Example 10.6. Let A be a nonsimple basic finite dimensional K-algebra over a field K, $e_1, \ldots, e_n$ a complete set of pairwise orthogonal primitive idempotents of A with $1_A = e_1 + \cdots + e_n$, and $P_1 = e_1 A, \ldots, P_n = e_n A$ the associated indecomposable projective modules in $\operatorname{mod} A$. Assume that 1 is a source vertex of the quiver Q_A of A. Then it follows that $e_i(\operatorname{rad} A)e_1 = 0$ for $i \in \{2, \ldots, n\}$ and $F = e_1 A e_1$ is a finite dimensional division K-algebra isomorphic to the endomorphism algebra $\operatorname{End}_A(P_1)$ (see Lemma I.8.7). Let $P = P_1$ and $P' = P_2 \oplus \cdots \oplus P_n$. Then $A_A = P \oplus P'$ in $\operatorname{mod} A$. Moreover, by Lemma I.6.1, there is a canonical isomorphism of K-algebras $A \xrightarrow{\sim} \operatorname{End}_A(A_A)$. Since $\operatorname{Hom}_A(P', P) = 0$, we conclude that A is isomorphic to the one-point extension algebra

$$C[M] = \begin{bmatrix} F & {}_F M_C \\ 0 & C \end{bmatrix},$$

where $C = \operatorname{End}_A(P')$ and ${}_F M_C = \operatorname{Hom}_A(P', P)$. We also note that the quiver Q_C of C is the full valued subquiver of Q_A given by the vertices $2, \ldots, n$.

Example 10.7. Let K be a field, F a finite dimensional division K-algebra, C a finite dimensional K-algebra, and $M = {}_F M_C$ an (F, C)-bimodule on which K acts centrally and with $\dim_K M$ is finite. Then $D(M) = \operatorname{Hom}_K(M, K)$ is a

(C, F)-bimodule on which K acts centrally and $\dim_K D(M) = \dim_K M$ is finite. Then we may consider the matrix algebra

$$[M]C = \begin{bmatrix} C & D(M) \\ 0 & F \end{bmatrix}$$

which is called the *one-point coextension algebra* of C by the (F, C)-bimodule $M = {}_F M_C$. We note that $[M]C$ is a hereditary algebra if and only if C is a hereditary algebra and $D(M)$ is an injective module in $\operatorname{mod} C^{\mathrm{op}}$, or equivalently, M is a projective module in $\operatorname{mod} C$ (see Exercise 11.31).

Let B and C be finite dimensional K-algebras over a field K, $M = {}_B M_C$ a (B, C)-bimodule, and assume that K acts centrally on M with $\dim_K M$ finite. Consider also the associated matrix algebra

$$A = \begin{bmatrix} B & M \\ 0 & C \end{bmatrix}.$$

Then there are K-linear equivalences of categories

$$\operatorname{mod} A \xrightarrow{F} \operatorname{rep}({}_B M_C) \xrightarrow{H} \overline{\operatorname{rep}}({}_B M_C).$$

For the purposes of the remaining part of this section the category $\overline{\operatorname{rep}}({}_B M_C)$ is more suited, and we will identify $\operatorname{mod} A$ with $\overline{\operatorname{rep}}({}_B M_C)$ via the functor $\bar{F} = HF$. There are two essentially distinct full and faithful embeddings of the category $\operatorname{mod} C$ inside $\operatorname{mod} A = \overline{\operatorname{rep}}(M) = \overline{\operatorname{rep}}({}_B M_C)$ preserving the indecomposability of modules:

(1) the standard embedding which associates to a module X in $\operatorname{mod} C$ the triple $(0, X, 0)$, which we simply identify with X;
(2) the functor associating to a module X in $\operatorname{mod} C$ the triple

$$\bar{X} = (\operatorname{Hom}_C(M, X), X, \operatorname{id}_{\operatorname{Hom}_C(M,X)}).$$

We use the above embeddings to describe almost split sequences in $\operatorname{mod} A$ whose right terms are modules $X = (0, X, 0)$ from $\operatorname{mod} C$.

Proposition 10.8. *Let B and C be finite dimensional K-algebras over a field K, $M = {}_B M_C$ a (B, C)-bimodule such that K acts centrally on M and $\dim_K M$ is finite, and $A = \begin{bmatrix} B & M \\ 0 & C \end{bmatrix}$ be the associated matrix algebra. The following statements hold:*

(i) *Let $f: X \to Y$ be a left minimal almost split homomorphism in* $\operatorname{mod} C$. *Then*

$$\left(\operatorname{id}_{\operatorname{Hom}_C(M,X)}, f\right): \bar{X} \longrightarrow \left(\operatorname{Hom}_C(M, X), Y, \operatorname{Hom}_C(M, f)\right)$$

is a left minimal almost split homomorphism in $\operatorname{mod} A$.

(ii) *Let* $g\colon Y \to Z$ *be a right minimal almost split homomorphism in* $\operatorname{mod} C$ *and* $j\colon \operatorname{Ker}\operatorname{Hom}_C(M,g) \to \operatorname{Hom}_C(M,Y)$ *denotes the inclusion homomorphism. Then*

$$(0,g)\colon \big(\operatorname{Ker}\operatorname{Hom}_C(M,g), Y, j\big) \longrightarrow Z = (0,Z,0)$$

is a right minimal almost split homomorphism in $\operatorname{mod} A$.

Proof. We abbreviate $\mathrm{id} = \mathrm{id}_{\operatorname{Hom}_C(M,X)}$. We have the commutative diagram in $\operatorname{mod} B$

$$\begin{tikzcd}
\operatorname{Hom}_C(M,X) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \operatorname{Hom}_C(M,X) \arrow[d, "\operatorname{Hom}_C(M,f)"] \\
\operatorname{Hom}_C(M,X) \arrow[r, "\operatorname{Hom}_C(M,f)"] & \operatorname{Hom}_C(M,Y)
\end{tikzcd}$$

and hence (id, f) is a morphism in $\overline{\operatorname{rep}}(M) = \operatorname{mod} A$. Moreover, (id, f) is not a section in $\operatorname{mod} A$, because f is not a section in $\operatorname{mod} C$. Assume now that $(u,v)\colon \bar{X} \to (U,V,\varphi)$ is a homomorphism in $\operatorname{mod} A = \overline{\operatorname{rep}}(M)$ which is not a section. We claim that $v\colon X \to V$ is not a section in $\operatorname{mod} C$. Suppose that there exists a homomorphism $v'\colon V \to X$ in $\operatorname{mod} C$ with $v'v = \mathrm{id}_X$. We note that $\varphi\colon U \to \operatorname{Hom}_C(M,V)$ is a homomorphism in $\operatorname{mod} B$. Then we obtain the homomorphism $u' = \operatorname{Hom}_C(M,v')\varphi\colon U \to \operatorname{Hom}_C(M,X)$ in $\operatorname{mod} B$. Moreover, we have the commutative diagram in $\operatorname{mod} B$

$$\begin{tikzcd}
U \arrow[r, "\varphi"] \arrow[d, "u'"] & \operatorname{Hom}_C(M,V) \arrow[d, "\operatorname{Hom}_C(M,v')"] \\
\operatorname{Hom}_C(M,X) \arrow[r, "\mathrm{id}"] & \operatorname{Hom}_C(M,X)\,,
\end{tikzcd}$$

and hence the pair (u',v') is a morphism from (U,V,φ) to $\bar{X}$ in $\overline{\operatorname{rep}}(M)$, which satisfies $(u',v')(u,v) = \mathrm{id}_{\bar{X}}$, because

$$u'u = \operatorname{Hom}_C(M,v')\varphi u = \operatorname{Hom}_C(M,v')\operatorname{Hom}_C(M,v) = \operatorname{Hom}_C(M,v'v) = \mathrm{id}\,.$$

This contradicts the assumption that (u,v) is not a section. Therefore, indeed $v\colon X \to V$ is not a section in $\operatorname{mod} C$. Since $f\colon X \to Y$ is a left almost split homomorphism in $\operatorname{mod} C$, there exists a homomorphism $w\colon Y \to V$ such that $v = wf$. Observe that we have the commutative diagram in $\operatorname{mod} B$

$$\begin{tikzcd}
\operatorname{Hom}_C(M,X) \arrow[r, "\operatorname{Hom}_C(M,f)"] \arrow[d, "u"] & \operatorname{Hom}_C(M,Y) \arrow[d, "\operatorname{Hom}_C(M,w)"] \\
U \arrow[r, "\varphi"] & \operatorname{Hom}_C(M,V)\,,
\end{tikzcd}$$

and hence $(u, w)\colon (\mathrm{Hom}_C(M, X), Y, \mathrm{Hom}_C(M, f)) \to (U, V, \varphi)$ is a morphism in $\overline{\mathrm{rep}}(M)$. Moreover, we have the equality $(u, w)(\mathrm{id}, f) = (u, v)$. Therefore, (id, f) is a left almost split homomorphism in $\operatorname{mod} A = \overline{\mathrm{rep}}(M)$. Finally, we show that (id, f) is a left minimal homomorphism. Assume that (p, h) is an endomorphism of $(\mathrm{Hom}_C(M, X), Y, \mathrm{Hom}_C(M, f))$ in $\overline{\mathrm{rep}}(M)$ such that $(p, h)(\mathrm{id}, f) = (\mathrm{id}, f)$. Then $hf = f$, and hence h is an automorphism of Y, because f is left minimal in $\operatorname{mod} C$. Clearly, $p = \mathrm{id}$. Hence (g, h) is an isomorphism in $\overline{\mathrm{rep}}(M)$. Therefore, (id, f) is a left minimal almost split homomorphism in $\operatorname{mod} A = \overline{\mathrm{rep}}(M)$.

(ii) Observe that $(0, g)$ is a morphism from $(\mathrm{Ker}\,\mathrm{Hom}_C(M, g), Y, j)$ to $Z = (0, Z, 0)$, because we have the commutative diagram in $\operatorname{mod} B$

$$\begin{tikzcd}
\mathrm{Ker}\,\mathrm{Hom}_C(M, g) \arrow[r, "j"] \arrow[d] & \mathrm{Hom}_C(M, Y) \arrow[d, "\mathrm{Hom}_C(M,g)"] \\
0 \arrow[r] & \mathrm{Hom}_C(M, Z)\,.
\end{tikzcd}$$

Clearly, $(0, g)$ is not a retraction in $\overline{\mathrm{rep}}(M)$, because g is not a retraction in $\operatorname{mod} C$. Assume now that $(u, v)\colon (U, V, \varphi) \to (0, Z, 0) = Z$ is a morphism in $\overline{\mathrm{rep}}(M)$ which is not a retraction. Then $u = 0$ and $v\colon V \to Z$ is not a retraction in $\operatorname{mod} C$. Since by the assumption $g\colon Y \to Z$ is a right almost split homomorphism in $\operatorname{mod} C$, there exists a homomorphism $v'\colon V \to Y$ such that $v = gv'$. Observe that $\mathrm{Hom}_C(M, g)\,\mathrm{Hom}_C(M, v')\varphi = \mathrm{Hom}_C(M, v)\varphi = 0$, and hence there exists a homomorphism $u'\colon U \to \mathrm{Ker}\,\mathrm{Hom}_C(M, g)$ such that $ju' = \mathrm{Hom}_C(M, v')\varphi$. This shows that the pair (u', v') is a morphism from (U, V, φ) to $(\mathrm{Ker}\,\mathrm{Hom}_C(M, g), Y, j)$ in $\overline{\mathrm{rep}}(M)$. Further, $(0, g)(u', v') = (0, v) = (u, v)$. Hence $(0, g)$ is a right almost split morphism in $\overline{\mathrm{rep}}(M)$. It remains to show that $(0, g)$ is a right minimal in $\overline{\mathrm{rep}}(M)$. Assume that (p, h) is an endomorphism of $(\mathrm{Ker}\,\mathrm{Hom}_C(M, g), Y, j)$ in $\overline{\mathrm{rep}}(M)$ such that $(0, g)(p, h) = (0, g)$. Then $gh = g$, and hence h is an automorphism in $\operatorname{mod} C$, because g is a right minimal homomorphism in $\operatorname{mod} C$. Further, we have the commutative diagram in $\operatorname{mod} B$

$$\begin{tikzcd}
\mathrm{Ker}\,\mathrm{Hom}_C(M, g) \arrow[r, hook, "j"] \arrow[d, "p"] & \mathrm{Hom}_C(M, Y) \arrow[d, "\mathrm{Hom}_C(M,h)"] \\
\mathrm{Ker}\,\mathrm{Hom}_C(M, g) \arrow[r, hook, "j"] & \mathrm{Hom}_C(M, Y)\,,
\end{tikzcd}$$

and hence p is the restriction of $\mathrm{Hom}_C(M, h)$ to $\mathrm{Ker}\,\mathrm{Hom}_C(M, g)$. Since $\mathrm{Hom}_C(M, h)$ is an automorphism of $\mathrm{Hom}_C(M, Y)$ in $\operatorname{mod} B$, we conclude that p is also an automorphism in $\operatorname{mod} B$. Hence (p, h) is an automorphism in $\overline{\mathrm{rep}}(M)$. Therefore, $(0, g)$ is a right minimal almost split homomorphism in $\operatorname{mod} A = \overline{\mathrm{rep}}(M)$. □

Theorem 10.9. *Let B and C be finite dimensional K-algebras over a field K, $M = {}_B M_C$ a (B, C)-bimodule such that K acts centrally on M and* $\dim_K M$ *is finite, and* $A = \begin{bmatrix} B & M \\ 0 & C \end{bmatrix}$ *be the associated matrix algebra. Assume that*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod} C$. *Then*

$$0 \longrightarrow \bar{X} \xrightarrow{(\mathrm{id}, f)} \big(\mathrm{Hom}_C(M, X), Y, \mathrm{Hom}_C(M, f)\big) \xrightarrow{(0,g)} Z \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod} A$.

Proof. We note first that $\mathrm{Hom}_C(M, -)\colon \operatorname{mod} C \to \operatorname{mod} B$ is a left exact functor (see Lemma II.2.5). Then we obtain the commutative diagram in $\operatorname{mod} B$ with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_C(M, X) \arrow[r, "\mathrm{Hom}_C(M, \mathrm{id}_X)"] \arrow[d, "\mathrm{Hom}_C(M, \mathrm{id}_X)"] & \mathrm{Hom}_C(M, X) \arrow[r] \arrow[d, "\mathrm{Hom}_C(M, f)"] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_C(M, X) \arrow[r, "\mathrm{Hom}_C(M, f)"] & \mathrm{Hom}_C(M, Y) \arrow[r, "\mathrm{Hom}_C(M, g)"] & \mathrm{Hom}_C(M, Z) .
\end{tikzcd}$$

This shows that the sequence in $\operatorname{mod} A$

$$0 \longrightarrow \bar{X} \xrightarrow{(\mathrm{id}, f)} \big(\mathrm{Hom}_C(M, X), Y, \mathrm{Hom}_C(M, f)\big) \xrightarrow{(0,g)} Z$$

is exact. We claim that $(0, g)$ is an epimorphism. Indeed, we have the equalities

$$\begin{aligned}
\dim_K \mathrm{Coker}(\mathrm{id}, f) &= \dim_K \big(\mathrm{Hom}_C(M, X), Y, \mathrm{Hom}_C(M, f)\big) - \dim_K \bar{X} \\
&= \dim_K \mathrm{Hom}_C(M, X) + \dim_K Y - \dim_K \mathrm{Hom}_C(M, X) \\
&\quad - \dim_K X \\
&= \dim_K Y - \dim_K X = \dim_K Z,
\end{aligned}$$

and hence $\dim_K \mathrm{Im}(0, g) = \dim_K \mathrm{Coker}(\mathrm{id}, f) = \dim_K Z$.

Observe also that $(\mathrm{Hom}_C(M, X), Y, \mathrm{Hom}_C(M, f))$ is isomorphic to $(\mathrm{Ker}\,\mathrm{Hom}_C(M, g), Y, j)$ in $\operatorname{mod} A = \overline{\mathrm{rep}}(M)$, where $j\colon \mathrm{Ker}\,\mathrm{Hom}_C(M, g) \to \mathrm{Hom}_C(M, Y)$ is the inclusion homomorphism. It follows from Proposition 10.8 that (id, f) is a left minimal almost split homomorphism and $(0, g)$ is a right minimal almost split homomorphism in $\operatorname{mod} A$. □

We have also the following useful consequence of Theorem 10.9.

Corollary 10.10. *Let B and C be finite dimensional K-algebras over a field K, $M = {}_BM_C$ a (B, C)-bimodule such that K acts centrally on M and* $\dim_K M$ *is finite, and* $A = \begin{bmatrix} B & M \\ 0 & C \end{bmatrix}$ *be the associated matrix algebra. Assume that*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod} C$*. Then the sequence remains an almost split sequence in* $\operatorname{mod} A$ *under the standard embedding if and only if* $\operatorname{Hom}_C(M, X) = 0$.

11 Exercises

1. Let A be the following $\mathbb{R}$-subalgebra of the matrix $\mathbb{R}$-algebra $M_4(\mathbb{H})$

$$\begin{bmatrix} \mathbb{H} & 0 & 0 & 0 \\ \mathbb{H} & \mathbb{H} & 0 & 0 \\ \mathbb{H} & \mathbb{H} & \mathbb{R} & \mathbb{C} \\ 0 & 0 & 0 & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ x & b & 0 & 0 \\ y & z & c & u \\ 0 & 0 & 0 & d \end{bmatrix} \middle| \begin{matrix} a, b, x, y, z \in \mathbb{H} \\ c \in \mathbb{R},\ d, u \in \mathbb{C} \end{matrix} \right\}.$$

(a) Prove that A is a finite dimensional hereditary $\mathbb{R}$-algebra.
(b) Describe the quiver Q_A of A.

2. Let A be the following $\mathbb{R}$-subalgebra of the matrix algebra $M_4(\mathbb{C})$

$$\begin{bmatrix} \mathbb{R} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} r & 0 & 0 & 0 \\ x & a & 0 & 0 \\ y & z & b & u \\ 0 & 0 & 0 & c \end{bmatrix} \middle| \begin{matrix} r \in \mathbb{R}, \\ a, b, c \in \mathbb{C} \\ x, y, z, u \in \mathbb{C} \end{matrix} \right\}.$$

(a) Prove that A is a finite dimensional hereditary $\mathbb{R}$-algebra of Dynkin type.
(b) Determine the Euler form χ_A in the standard basis of $K_0(A) = \mathbb{Z}^4$.
(c) Determine the Coxeter transformations φ_A and φ_A^{-1} on $K_0(A) = \mathbb{Z}^4$.
(d) Determine the Auslander–Reiten quiver Γ_A of A.

3. Let A be the following $\mathbb{R}$-subalgebra of the matrix $\mathbb{R}$-algebra $M_5(\mathbb{C})$:

$$\begin{bmatrix} \mathbb{R} & 0 & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 & 0 & \mathbb{R} \end{bmatrix} = \left\{ \begin{bmatrix} r & 0 & 0 & 0 & 0 \\ x & a & 0 & 0 & 0 \\ y & z & b & u & v \\ 0 & 0 & 0 & c & w \\ 0 & 0 & 0 & 0 & s \end{bmatrix} \middle| \begin{matrix} r, s \in \mathbb{R}, \\ a, b, c \in \mathbb{C} \\ x, y, z \in \mathbb{C} \\ u, v, w \in \mathbb{C} \end{matrix} \right\}.$$

(a) Prove that A is a finite dimensional hereditary $\mathbb{R}$-algebra of Euclidean type.

(b) Describe the Euler form χ_A and its radical $\operatorname{rad}\chi_A$ in the standard basis of $K_0(A) = \mathbb{Z}^5$.

(c) Describe the Coxeter transformations φ_A and φ_A^{-1} on $K_0(A) = \mathbb{Z}^5$.

(d) Describe the defect $\partial_A \colon K_0(A) \to \mathbb{Z}$.

4. Let K be a field and A the K-subalgebra of the matrix K-algebra $M_5(K)$:

$$\begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & K & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & K & K & K & K \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ x & b & 0 & 0 & 0 \\ y & z & c & 0 & 0 \\ r & 0 & 0 & d & 0 \\ s & u & v & w & e \end{bmatrix} \;\middle|\; \begin{matrix} a, b, c, d, e \in K \\ x, y, z, r, s \in K \\ u, v, w \in K \end{matrix} \right\}.$$

(a) Prove that A is a finite dimensional K-algebra.

(b) Prove that A is not a hereditary K-algebra.

(c) Determine the quiver Q_A of A.

5. Let Q be the valued quiver

$$\begin{tikzcd}[row sep=small] 1\,\bullet \arrow[dr] & & & & \\ & \underset{3}{\bullet} \arrow[r] \arrow[dl] & \underset{4}{\bullet} \arrow[r] & \underset{5}{\bullet} & \underset{6}{\bullet} \arrow[l, "{(1,2)}"'] \, . \\ 2\,\bullet & & & & \end{tikzcd}$$

Describe an $\mathbb{R}$-species $\mathbb{M}$ with $Q_{\mathbb{M}} = Q$ and calculate the dimension $\dim_{\mathbb{R}} T(\mathbb{M})$ of the tensor algebra $T(\mathbb{M})$ of $\mathbb{M}$.

6. Let Q be the valued quiver

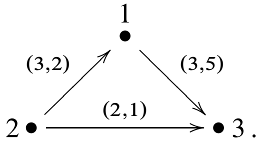
$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "{(1,2)}"'] & \underset{3}{\bullet} \arrow[l, "{(2,3)}"'] \arrow[r] & \underset{4}{\bullet} & \underset{5}{\bullet} \arrow[l, "{(3,5)}"'] \arrow[r, "{(2,1)}"] & \underset{6}{\bullet} \, . \end{tikzcd}$$

Describe a $\mathbb{Q}$-species $\mathbb{M}$ with $Q_{\mathbb{M}} = Q$ and calculate the dimension $\dim_{\mathbb{Q}} T(\mathbb{M})$ of the tensor algebra $T(\mathbb{M})$ of $\mathbb{M}$.

7. Let a and b be positive integers. Prove that there is a finite dimensional K-algebra A over a field K whose quiver Q_A is of the form

$$\begin{tikzcd} 1\,\bullet \arrow[r, "{(m,n)}"] & \bullet\,2 \, . \end{tikzcd}$$

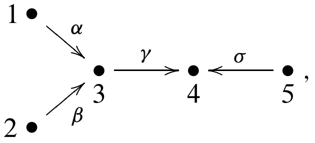
8. Let Q be the valued quiver



Prove that Q is not the quiver Q_A of a finite dimensional K-algebra A over a field K.

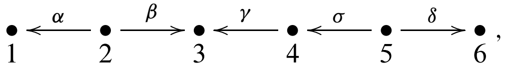
9. Let Q be a finite valued quiver whose underlying graph is a tree. Prove that there exists a finite dimensional hereditary K-algebra A over a field K such that $Q_A = Q$.

10. Let K be a field, Q the quiver



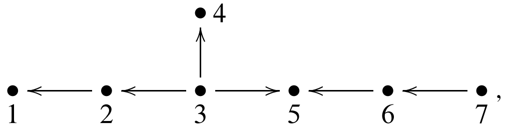
and $A = KQ$ the path algebra of Q over K. Determine the Auslander–Reiten quiver Γ_A of A.

11. Let K be a field, Q the quiver



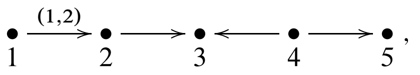
and $A = KQ$ the path algebra of Q over K. Determine the Auslander–Reiten quiver Γ_A of A.

12. Let K be a field, Q the quiver



and $A = KQ$ the path algebra of Q over K. Determine the Auslander–Reiten quiver Γ_A of A.

13. Let K be a field, $\mathbb{M}$ a K-species with the quiver $Q_{\mathbb{M}}$ of the form



and $A = T(\mathbb{M})$ the tensor algebra of $\mathbb{M}$. Determine the Auslander–Reiten quiver Γ_A of A.

14. Let A be the $\mathbb{R}$-subalgebra of the matrix algebra $M_2(\mathbb{H})$:

$$\begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{H} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \in M_2(\mathbb{H}) \;\middle|\; a \in \mathbb{R},\, b, c \in \mathbb{H} \right\}.$$

(a) Prove that A is a finite dimensional hereditary $\mathbb{R}$-algebra of Euclidean type $\widetilde{\mathbb{A}}_{11}$.

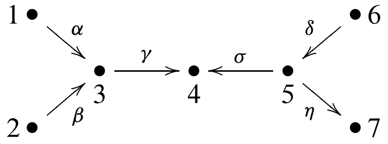
(b) Describe the simple regular modules in Γ_A.

(c) Describe the postprojective component $\mathcal{P}(A)$ of Γ_A.

(d) Describe the preinjective component $\mathcal{Q}(A)$ of Γ_A.

(e) Prove that the regular part $\mathcal{R}(A)$ of Γ_A consists of infinitely many stable tubes of rank one.

15. Let $A = \mathbb{C}Q$ be the path algebra of the quiver Q of the form

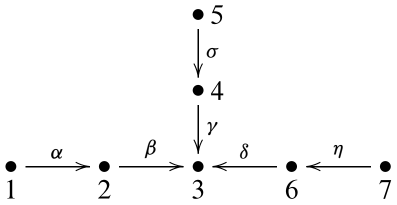


over the field $\mathbb{C}$.

(a) Describe the simple regular modules in mod A.

(b) Prove that the regular part $\mathcal{R}(A)$ of Γ_A consists of two stable tubes of rank 2, one stable tube of rank 4, and infinitely many stable tubes of rank 1.

16. Let $A = \mathbb{R}Q$ be the path algebra of the quiver Q of the form

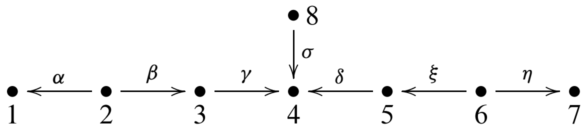


over the field $\mathbb{R}$.

(a) Describe the simple regular modules in mod A.

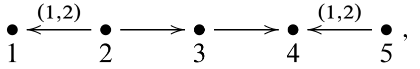
(b) Prove that the regular part $\mathcal{R}(A)$ of Γ_A consists of one stable tube of rank 2, two stable tubes of rank 3, and infinitely many stable tubes of rank 1.

17. Let $A = KQ$ be the path algebra of the quiver Q of the form



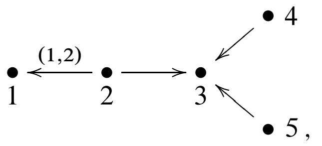
over a field K. Prove that the regular part $\mathcal{R}(A)$ of Γ_A contains stable tubes of ranks $2, 3, 4$, and the remaining stable tubes in $\mathcal{R}(A)$ are of rank 1.

18. Let $\mathbb{M}$ be an $\mathbb{R}$-species with the quiver $Q_{\mathbb{M}}$ of the form



and $A = T(\mathbb{M})$ the tensor algebra of $\mathbb{M}$. Prove that the regular part $\mathcal{R}(A)$ of Γ_A consists of one stable tube of rank 4 and infinitely many stable tubes of rank 1.

19. Let $\mathbb{M}$ be a $\mathbb{C}$-species with the quiver $Q_{\mathbb{M}}$ of the form



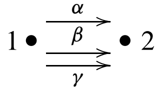
and $A = T(\mathbb{M})$ the tensor algebra of $\mathbb{M}$. Prove that the regular part $\mathcal{R}(A)$ of Γ_A consists of one stable tube of rank 2, one stable tube of rank 3, and infinitely many stable tubes of rank 1.

20. Let A be the $\mathbb{R}$-subalgebra of the matrix algebra $M_3(\mathbb{H})$

$$\begin{bmatrix} \mathbb{R} & 0 & 0 \\ \mathbb{H} & \mathbb{H} & 0 \\ \mathbb{H} & \mathbb{H} & \mathbb{H} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{bmatrix} \in M_3(\mathbb{H}) \;\middle|\; \begin{matrix} a \in \mathbb{R}, \\ b, c, x, y, z \in \mathbb{H} \end{matrix} \right\}.$$

(a) Prove that A is a finite dimensional hereditary $\mathbb{R}$-algebra of wild type.
(b) Describe the postprojective component $\mathcal{P}(A)$ of Γ_A.
(c) Describe the preinjective component $\mathcal{Q}(A)$ of Γ_A.
(d) Prove that the regular part $\mathcal{R}(A)$ of Γ_A consists of infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty$.

21. Let $A = KQ$ be the path algebra of the quiver Q of the form



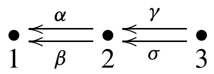
over a field K.

(a) Describe the postprojective component $\mathcal{P}(A)$ of Γ_A.

(b) Describe the preinjective component $\mathcal{Q}(A)$ of Γ_A.

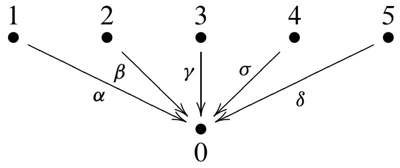
(c) Prove that the regular part $\mathcal{R}(A)$ of Γ_A consists of infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty$.

22. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K. Prove that the simple module S_2 in $\operatorname{mod} A$ at the vertex 2 is a quasi-simple regular module.

23. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K.

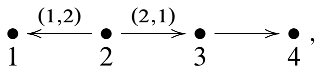
(a) Describe the postprojective component $\mathcal{P}(A)$ of Γ_A.

(b) Describe the preinjective component $\mathcal{Q}(A)$ of Γ_A.

(c) Prove that the regular part $\mathcal{R}(A)$ of Γ_A consists of infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty$.

(d) Describe a component $\mathcal{C}$ in $\mathcal{R}(A)$ whose all indecomposable modules are sincere.

24. Let $\mathbb{M}$ be an $\mathbb{R}$-species with the quiver $Q_{\mathbb{M}}$ of the form



and $A = T(\mathbb{M})$ the tensor algebra of $\mathbb{M}$.

(a) Describe the postprojective component $\mathcal{P}(A)$ of Γ_A.

(b) Describe the preinjective component $\mathcal{Q}(A)$ of Γ_A.

(c) Prove that the regular part $\mathcal{R}(A)$ of Γ_A consists of infinitely many components of the form $\mathbb{Z}\mathbb{A}_\infty$.

(d) Describe a component $\mathcal{C}$ in $\mathcal{R}(A)$ whose all indecomposable modules are sincere.

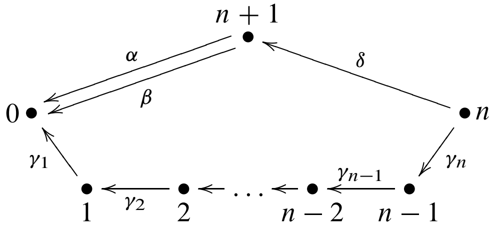
25. Let A be a finite dimensional hereditary K-algebra of Euclidean type and with $K_0(A)$ of rank 2. Prove that every stone in mod A is either postprojective or preinjective.

26. Let A be a finite dimensional hereditary K-algebra of Euclidean type and with $K_0(A)$ of rank at least 3. Prove the following assertions:

(a) There is a regular stone in mod A.

(b) The Auslander–Reiten quiver Γ_A of A contains a stable tube of rank at least 2.

27. Let K be a field, $n \geq 2$ an integer, Q the quiver



and $A = KQ$ the path algebra of Q over K. Let $S_1, \dots, S_n$ be the simple modules in mod A corresponding to the vertices $1, \dots, n$, and X the indecomposable module in mod A with $[X] = [S_1] + \dots + [S_n]$ in $K_0(A)$. Prove the following assertions:

(a) $S_1, \dots, S_n$ are quasi-simple regular modules in mod A such that $\tau_A S_i = S_{i-1}$ for all $i \in \{2, \dots, n\}$.

(b) $S_1, \dots, S_n$ are stones in mod A.

(c) X is a stone in mod A.

(d) There is a sectional path in Γ_A of the form

$$S_1 = S_1[1] \longrightarrow S_1[2] \longrightarrow \cdots \longrightarrow S_1[n-1] \longrightarrow S_1[n] = X.$$

28. Let A be a finite dimensional hereditary K-algebra of wild type over a field K, X a quasi-simple regular module, $m \geq 2$ an integer, and $\mathcal{C}(X[m])$ the full translation subquiver of Γ_A of the form

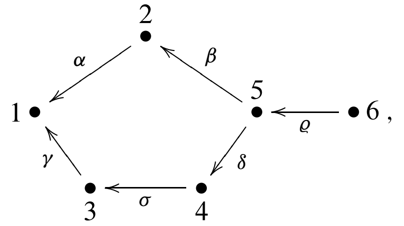
$$\begin{tikzcd}
X[1] \arrow[dr] & & \tau_A^{-1}X[1] \arrow[dr] & & \tau_A^{-2}X[1] \arrow[dr] & & & & \tau_A^{-(m-2)}X[1] \arrow[dr] & & \tau_A^{-(m-1)}X[1] \\
& X[2] \arrow[ur] \arrow[dr] & & \tau_A^{-1}X[2] \arrow[ur] \arrow[dr] & & \ddots \arrow[dr] & & \iddots \arrow[ur] & & \tau_A^{-(m-2)}X[2] \arrow[ur] & \\
& & X[3] \arrow[ur] \arrow[dr] & & \ddots \arrow[dr] & & \tau_A^{-2}X[m-3] \arrow[ur] \arrow[dr] & & \iddots \arrow[ur] & & \\
& & & \ddots \arrow[dr] & & \tau_A^{-1}X[m-2] \arrow[ur] \arrow[dr] & & \tau_A^{-2}X[m-2] \arrow[ur] & & & \\
& & & & X[m-1] \arrow[ur] \arrow[dr] & & \tau_A^{-1}X[m-1] \arrow[ur] & & & & \\
& & & & & X[m] \arrow[ur] & & & & &
\end{tikzcd}$$

called the *cone* in Γ_A at $X[m]$. Prove that the following statements are equivalent:

(a) $X[m]$ is a brick.

(b) $\mathcal{C}(X[m])$ is generalized standard in $\operatorname{mod} A$, that is, $\operatorname{rad}_A^{\infty}(U, V) = 0$ for all indecomposable modules U and V in $\mathcal{C}(X[m])$.

29. Let K be a field, Q the quiver



and $A = KQ$ the path algebra of Q over K. Prove the following assertions:

(a) The simple module S_2 in $\operatorname{mod} A$ at the vertex 2 is a quasi-simple module of a regular component $\mathcal{C}$ of Γ_A.

(b) The simple modules S_3 and S_4 in $\operatorname{mod} A$ at the vertices 3 and 4 are quasi-simple modules of a regular component $\mathcal{D}$ of Γ_A.

(c) The components $\mathcal{C}$ and $\mathcal{D}$ are different.

30. Let K be a field, F a finite dimensional division K-algebra, C a finite dimensional K-algebra, M an (F, C)-bimodule on which K acts centrally with $\dim_K M$ finite, and

$$A = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix}$$

the associated matrix algebra. Prove that A is a hereditary algebra if and only if C is a hereditary algebra and M is a projective module in $\operatorname{mod} C$.

31. Let K be a field, F a finite dimensional division K-algebra, B a finite dimensional K-algebra, N a (B, F)-bimodule on which K acts centrally with $\dim_K N$ finite, and

$$A = \begin{bmatrix} B & N \\ 0 & F \end{bmatrix}$$

the associated matrix algebra. Prove that A is a hereditary algebra if and only if B is a hereditary algebra and N is an injective module in $\operatorname{mod} B^{\mathrm{op}}$.

32. Let A be a basic finite dimensional hereditary K-algebra over a field K whose quiver Q_A is of the form

$$\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "{(1,2)}"'] & \underset{3}{\bullet} \arrow[l] \arrow[r, "{(3,1)}"] & \underset{4}{\bullet} & \underset{5}{\bullet} \arrow[l, "{(2,5)}"']
\end{tikzcd}.$$

Prove the following assertions:

(a) A is isomorphic to a matrix algebra

$$\begin{bmatrix} F & M \\ 0 & C \end{bmatrix},$$

where F is a finite dimensional division K-algebra, C is an indecomposable finite dimensional hereditary K-algebra, and M is an (F, C)-bimodule on which K acts centrally.

(b) A is isomorphic to a matrix algebra

$$\begin{bmatrix} G & N \\ 0 & D \end{bmatrix},$$

where G is a finite dimensional division K-algebra, D is a decomposable finite dimensional hereditary K-algebra, and N is a (G, D)-bimodule on which K acts centrally.

33. Let A be a basic finite dimensional K-algebra over a field K and

$$\begin{tikzcd}
N_1 \arrow[rr, "u_2"] & & N_2 \arrow[rr, "v_2"] & & N_3 \\
& P_1 \arrow[ur, "h_1"] & & P_2 \arrow[ur, "h_2"] & \\
M_1 \arrow[uu, "f_1"] \arrow[ur, "g_1"] \arrow[rr, "u_1"] & & M_2 \arrow[uu, "f_2"] \arrow[ur, "g_2"] \arrow[rr, "v_1"] & & M_3 \arrow[uu, "f_3"]
\end{tikzcd}$$

homomorphisms in $\operatorname{mod} A$. Assume that

$$0 \longrightarrow M_1 \xrightarrow{\begin{bmatrix} f_1 \\ u_1 \\ g_1 \end{bmatrix}} N_1 \oplus M_2 \oplus P_1 \xrightarrow{[u_2\ f_2\ h_1]} N_2 \longrightarrow 0,$$

$$0 \longrightarrow M_2 \xrightarrow{\begin{bmatrix} f_2 \\ v_1 \\ g_2 \end{bmatrix}} N_2 \oplus M_3 \oplus P_2 \xrightarrow{[v_2\ f_3\ h_2]} N_3 \longrightarrow 0$$

are exact sequences in $\operatorname{mod} A$. Prove that the sequence

$$0 \longrightarrow M_1 \xrightarrow{\begin{bmatrix} f_1 \\ v_1 u_1 \\ g_1 \\ g_2 u_1 \end{bmatrix}} N_1 \oplus M_3 \oplus P_1 \oplus P_2 \xrightarrow{[-v_2 u_2\ \ f_3\ \ -v_2 h_1\ \ h_2]} N_3 \longrightarrow 0$$

in $\operatorname{mod} A$ is exact. In particular, the following statements hold:

(a) If $N_1 = 0$, $P_1 = 0$, and $P_2 = 0$, we have in $\operatorname{mod} A$ the exact sequence of the form

$$0 \longrightarrow M_1 \xrightarrow{v_1 u_1} M_3 \xrightarrow{f_3} N_3 \longrightarrow 0.$$

(b) If $P_1 = 0$, $P_2 = 0$, and $M_3 = 0$, we have in $\operatorname{mod} A$ the exact sequence of the form

$$0 \longrightarrow M_1 \xrightarrow{f_1} N_1 \xrightarrow{v_2 u_2} N_3 \longrightarrow 0.$$

(c) If $P_1 = 0$ and $P_2 = 0$, we have in $\operatorname{mod} A$ the exact sequence of the form

$$0 \longrightarrow M_1 \xrightarrow{\begin{bmatrix} f_1 \\ v_1 u_1 \end{bmatrix}} N_1 \oplus M_3 \xrightarrow{[-v_2 u_2\ \ f_3]} N_3 \longrightarrow 0.$$

34. Let H a finite dimensional hereditary K-algebra of wild type over a field K, M a regular brick in $\operatorname{mod} H$, and A be the one-point extension algebra

$$A = \begin{bmatrix} F & M \\ 0 & H \end{bmatrix},$$

where $F = \operatorname{End}_H(M)$. Moreover, let $\mathcal{C}$ be a regular component of Γ_H. Prove that there is a quasi-simple module X in $\mathcal{C}$ such that the full translation subquiver $\mathcal{C}(X \rightarrow)$ of $\mathcal{C}$ formed by all successors of X in $\mathcal{C}$ is a full translation subquiver of Γ_A.

35. Let H a finite dimensional hereditary K-algebra of wild type over a field K, M a regular brick in $\operatorname{mod} H$, and A be the one-point coextension algebra

$$A = \begin{bmatrix} H & D(M) \\ 0 & F \end{bmatrix},$$

where $F = \operatorname{End}_H(M)$. Moreover, let $\mathcal{C}$ be a regular component of Γ_H. Prove that there is a quasi-simple module Y in $\mathcal{C}$ such that the full translation subquiver $\mathcal{C}(\rightarrow Y)$ of $\mathcal{C}$ formed by all predecessors of Y in $\mathcal{C}$ is a full translation subquiver of Γ_A.

Chapter VIII
Tilted algebras

This chapter is devoted to presenting background material on the tilting theory of finite dimensional algebras over a field, which is one of the main tools of the modern representation theory of algebras. The second objective of this chapter is to introduce the tilted algebras and prove their characteristic properties, essential for further considerations.

The tilting theory originates with the study of reflection functors by I. N. Bernstein, I. M. Gelfand, and V. A. Ponomarev [BGP] and the Coxeter functors defined by M. Auslander, M. I. Platzeck, and I. Reiten [APR]. The first set of axioms of a tilting module is due to S. Brenner and M. C. R. Butler [BB], and was subsequently formulated in its current form by D. Happel and C. M. Ringel in [HR], with an essential completion by K. Bongartz [Bo]. The main idea of the tilting theory is to present a finite dimensional algebra B over a field K as the endomorphism algebra $\operatorname{End}_A(T)$ of a finite dimensional right module T over another finite dimensional algebra A over K, called a tilting module, satisfying some homological conditions. One of the main results proved in this chapter is the tilting theorem of Brenner and Butler, which relates two torsion pairs $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ via naturally defined homological functors. In the important case when A is a finite dimensional hereditary algebra over a field and T is a tilting module in $\operatorname{mod} A$, this allows to describe the representation theory of the associated tilted algebra $B = \operatorname{End}_A(T)$ using the known representation theory of A. We will prove that the tilted algebras form a distinguished class of finite dimensional algebras of global dimension at most two, for which the Auslander–Reiten quiver admits an acyclic component containing a canonical section, connecting the torsion-free part and the torsion part of the torsion pair defined by a given tilting module. In the final part of this chapter we prove a handy criterion for a finite dimensional algebra over a field to be a tilted algebra of a finite dimensional hereditary algebra, established independently by S. Liu [L3] and A. Skowroński [S1]. Moreover, an important theorem by C. M. Ringel [R4] on the existence of regular tilting modules in the module categories of finite dimensional hereditary algebras will be proved in the final section.

The representation theory of tilted algebras of finite dimensional hereditary algebras is currently one of the best developed and understood representation theories. For further important results concerning this class of algebras we refer to the papers by: D. Baer [Ba2], [Ba3], O. Kerner [K1], [K2], [K3], [K4], O. Kerner and A. Skowroński [KS1], S. Liu [L4], A. Skowroński [S1], [S2], [S3], [S5], and H. Strauss [St]. We refer also to a recent characterization of tilted algebras in terms of short chains of modules established by A. Jaworska, P. Malicki, and A. Skowroński in [JMS].

1 Torsion pairs

In this section we introduce the concept of a torsion pair in a module category, which plays a fundamental role in our further considerations.

Let A be a finite dimensional K-algebra over a field K. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\operatorname{mod} A$ is called a *torsion pair* if the following conditions are satisfied:

(a) $\operatorname{Hom}_A(M, N) = 0$ for all modules $M \in \mathcal{T}$ and $N \in \mathcal{F}$.

(b) $\operatorname{Hom}_A(M, Y) = 0$ for all $Y \in \mathcal{F}$ implies $M \in \mathcal{T}$.

(c) $\operatorname{Hom}_A(X, N) = 0$ for all $X \in \mathcal{T}$ implies $N \in \mathcal{F}$.

The subcategory $\mathcal{T}$ is called the *torsion class*, and the modules in $\mathcal{T}$ are called *torsion modules*, while the subcategory $\mathcal{F}$ is called the *torsion-free class*, and the modules in $\mathcal{F}$ are called *torsion-free modules*. It follows from the definitions that the torsion class and the torsion-free class of a torsion pair in $\operatorname{mod} A$ determine uniquely each other.

Examples 1.1. (a) Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ a full subcategory of $\operatorname{mod} A$. Then $\mathcal{C}$ induces the torsion pair $(\mathcal{T}, \mathcal{F})$ in $\operatorname{mod} A$, defined as follows:

$$\begin{aligned} \mathcal{F} &= \{N \in \operatorname{mod} A \mid \operatorname{Hom}_A(X, N) = 0 \text{ for all } X \in \mathcal{C}\}, \\ \mathcal{T} &= \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M, Y) = 0 \text{ for all } Y \in \mathcal{F}\}. \end{aligned}$$

We note that $\mathcal{T}$ is the smallest torsion class in $\operatorname{mod} A$ containing $\mathcal{C}$.

(b) Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ a full subcategory of $\operatorname{mod} A$. Then $\mathcal{C}$ induces the torsion pair $(\mathcal{T}^*, \mathcal{F}^*)$ in $\operatorname{mod} A$, defined as follows:

$$\begin{aligned} \mathcal{T}^* &= \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M, Y) = 0 \text{ for all } Y \in \mathcal{C}\}, \\ \mathcal{F}^* &= \{N \in \operatorname{mod} A \mid \operatorname{Hom}_A(X, N) = 0 \text{ for all } X \in \mathcal{T}^*\}. \end{aligned}$$

Then $\mathcal{F}^*$ is the smallest torsion-free class in $\operatorname{mod} A$ containing $\mathcal{C}$.

(c) Let A be a finite dimensional K-algebra over a field K and $(\mathcal{T}, \mathcal{F})$ a torsion pair in $\operatorname{mod} A$. Then, for the standard duality $D = \operatorname{Hom}_K(-, K)\colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$, the pair $(D(\mathcal{F}), D(\mathcal{T}))$ is a torsion pair in $\operatorname{mod} A^{\mathrm{op}}$.

(d) Let $A = KQ$ be the path algebra of the quiver Q of the form

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l, "\beta"'] \end{tikzcd}$$

over a field K. Then A is a hereditary algebra of Dynkin type $\mathbb{A}_3$ and its Auslander–Reiten quiver Γ_A is of the form

$$\begin{tikzcd}
 & P_1 \arrow[dr] & & I_3 = S_3 \\
S_2 = P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & \\
 & P_3 \arrow[ur] & & I_1 = S_1 ,
\end{tikzcd}$$

where P_i, I_i, and S_i are the indecomposable projective modules, indecomposable injective modules, and the simple modules associated to the vertices $i \in \{1, 2, 3\}$ of Q, respectively. Then we have in $\operatorname{mod} A$ the torsion pair $(\mathcal{T}, \mathcal{F})$ with $\mathcal{T} = \operatorname{add}(P_1 \oplus I_2 \oplus S_1 \oplus S_3)$ and $\mathcal{F} = \operatorname{add}(P_2 \oplus P_3)$.

Let $\mathcal{C}$ be the additive category $\operatorname{add}(S_2)$ of the simple projective module S_2 in $\operatorname{mod} A$. According to (a), we may associate to $\mathcal{C}$ the torsion pair $(\bar{\mathcal{T}}, \bar{\mathcal{F}})$ in $\operatorname{mod} A$, with

$$\begin{aligned}
\bar{\mathcal{F}} &= \big\{N \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(X, N) = 0 \text{ for all } X \in \mathcal{C}\big\} = \operatorname{add}(S_1 \oplus S_3), \\
\bar{\mathcal{T}} &= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(M, Y) = 0 \text{ for all } Y \in \bar{\mathcal{F}}\big\} = \operatorname{add}(S_2) = \mathcal{C}.
\end{aligned}$$

Similarly, according to (b), we may associate to $\mathcal{C}$ the torsion pair $(\mathcal{T}^*, \mathcal{F}^*)$ in $\operatorname{mod} A$, with

$$\begin{aligned}
\mathcal{T}^* &= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(M, Y) = 0 \text{ for all } Y \in \mathcal{C}\big\} \\
&= \operatorname{add}(P_1 \oplus P_3 \oplus I_1 \oplus I_2 \oplus I_3), \\
\mathcal{F}^* &= \big\{N \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(X, N) = 0 \text{ for all } X \in \mathcal{T}^*\big\} = \operatorname{add}(S_2) = \mathcal{C}.
\end{aligned}$$

We will establish intrinsic characterizations of torsion and torsion-free classes in module categories.

Let A be a finite dimensional K-algebra over a field K. A subfunctor of the identity functor $\mathbf{1}_{\operatorname{mod} A}\colon \operatorname{mod} A \to \operatorname{mod} A$ is a functor $t\colon \operatorname{mod} A \to \operatorname{mod} A$ that assigns to each module M in $\operatorname{mod} A$ a right A-submodule $tM \subseteq M$ such that each homomorphism $M \to N$ in $\operatorname{mod} A$ restricts to a homomorphism $tM \to tN$. A subfunctor t of the identity functor $\mathbf{1}_{\operatorname{mod} A}$ on $\operatorname{mod} A$ is called an *idempotent radical* if, for every module M in $\operatorname{mod} A$, we have $t(tM) = tM$ and $t(M/tM) = 0$.

Proposition 1.2. *Let A be a finite dimensional K-algebra over a field K, and $\mathcal{T}$ a full subcategory of* $\operatorname{mod} A$. *The following conditions are equivalent:*

(i) *$\mathcal{T}$ is the torsion class of a torsion pair $(\mathcal{T}, \mathcal{F})$ in* $\operatorname{mod} A$.

(ii) *$\mathcal{T}$ is closed under images, direct sums, and extensions in* $\operatorname{mod} A$.

(iii) *There exists an idempotent radical t in* $\operatorname{mod} A$ *such that* $\mathcal{T} = \{M \in \operatorname{mod} A \mid tM = M\}$.

Proof. (i) $\Rightarrow$ (ii). Obviously, $\mathcal{T}$ is closed under direct sums in $\operatorname{mod} A$. Let $f: M \to N$ be a homomorphism in $\operatorname{mod} A$ with $M \in \mathcal{T}$. Then we have an epimorphism $M \to \operatorname{Im} f$ induced by f, and, for any module Y in $\mathcal{F}$, $\operatorname{Hom}_A(M, Y) = 0$ forces $\operatorname{Hom}_A(\operatorname{Im} f, Y) = 0$, and hence $\operatorname{Im} f \in \mathcal{T}$. Assume now that

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is an exact sequence in $\operatorname{mod} A$ with L and N from $\mathcal{T}$. For any module Y in $\operatorname{mod} A$ we have, by Lemma II.2.5, an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(N, Y) \longrightarrow \operatorname{Hom}_A(M, Y) \longrightarrow \operatorname{Hom}_A(L, Y)$$

in $\operatorname{mod} K$. In particular, if $Y \in \mathcal{F}$, then $\operatorname{Hom}_A(N, Y) = 0$ and $\operatorname{Hom}_A(L, Y) = 0$, and so $\operatorname{Hom}_A(M, Y) = 0$. This shows that M belongs to $\mathcal{T}$.

(ii) $\Rightarrow$ (iii). Let M be a module in $\operatorname{mod} A$. We define tM to be the trace of $\mathcal{T}$ in M, that is, the sum of the images of all homomorphisms $f: X \to M$ in $\operatorname{mod} A$ with X in $\mathcal{T}$. Observe that, for two homomorphisms $f: X \to M$ and $g: Y \to M$ in $\operatorname{mod} A$, we have $\operatorname{Im} f + \operatorname{Im} g = \operatorname{Im}(f, g)$ with $(f, g): X \oplus Y \to M$ induced by f and g. Since $\mathcal{T}$ is closed under images and direct sums, we conclude that tM is the largest right A-submodule of M that belongs to $\mathcal{T}$. In particular, for any homomorphism $f: M \to N$ in $\operatorname{mod} A$, we have $f(tM) \subseteq tN$. Hence the trace defines a subfunctor t of the identity functor $\mathbf{1}_{\operatorname{mod} A}$. Clearly, $t(tM) = tM$ for any module M in $\operatorname{mod} A$, and $M \in \mathcal{T}$ if and only if $tM = M$. Finally, let M be an arbitrary module in $\operatorname{mod} A$. Then $t(M/tM) = M'/tM$ for a right A-submodule M' of M with $tM \subseteq M'$. Then we have the short exact sequence

$$0 \longrightarrow tM \longrightarrow M' \longrightarrow M'/tM \longrightarrow 0$$

in $\operatorname{mod} A$ with tM and $M'/tM = t(M/tM)$ from $\mathcal{T}$. Since $\mathcal{T}$ is closed under extensions, we obtain $M' \in \mathcal{T}$. Hence, $M' = tM$ and $t(M/tM) = M'/tM = 0$.

(iii) $\Rightarrow$ (i). Assume t is an idempotent radical in $\operatorname{mod} A$ with $\mathcal{T} = \{M \in \operatorname{mod} A \mid tM = M\}$. Let $\mathcal{F} = \{N \in \operatorname{mod} A \mid tN = 0\}$. Obviously, we have $\operatorname{Hom}_A(M, Y) = 0$ for all modules $M \in \mathcal{T}$ and $Y \in \mathcal{F}$. Let M be a module in $\operatorname{mod} A$ such that $\operatorname{Hom}_A(M, Y) = 0$ for any module $Y \in \mathcal{F}$. We claim that $M \in \mathcal{T}$. Indeed, $t(M/tM) = 0$ implies $M/tM \in \mathcal{F}$, and hence the canonical epimorphism $M \to M/tM$ is the zero homomorphism. Therefore, we obtain $M/tM = 0$, or equivalently, $M = tM \in \mathcal{T}$. Assume now that N is a module in $\operatorname{mod} A$ such that $\operatorname{Hom}_A(X, N) = 0$ for any module $X \in \mathcal{T}$. We have the inclusion homomorphism $tN \to N$ with $tN = t(tN) \in \mathcal{T}$. Hence, $tN = 0$ and $N \in \mathcal{F}$. □

We have the following dual proposition, whose proof is left to the reader.

Proposition 1.3. *Let A be a finite dimensional K-algebra over a field K, and $\mathcal{F}$ a full subcategory of* $\operatorname{mod} A$. *The following conditions are equivalent:*

(i) *$\mathcal{F}$ is the torsion-free class of a torsion pair $(\mathcal{T}, \mathcal{F})$ in* mod A.

(ii) *$\mathcal{F}$ is closed under submodules, direct sums, and extensions in* mod A.

(iii) *There exists an idempotent radical t in* mod A *such that $\mathcal{F} = \{N \in \operatorname{mod} A \mid tN = 0\}$.*

The idempotent radical t associated to a given torsion pair $(\mathcal{T}, \mathcal{F})$ of a module category mod A is frequently called the *torsion radical* of mod A.

Proposition 1.4. *Let A be a finite dimensional K-algebra over a field K, $(\mathcal{T}, \mathcal{F})$ a torsion pair in* mod A, *and M a module in* mod A. *The following statements hold:*

(i) *There exists in* mod A *a short exact sequence*

$$0 \longrightarrow tM \xrightarrow{i} M \xrightarrow{\pi} M/tM \longrightarrow 0$$

with $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$.

(ii) *For any short exact sequence in* mod A

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

with $L \in \mathcal{T}$ and $N \in \mathcal{F}$, there exists a commutative diagram in mod A *of the form*

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "u"] & M \arrow[r, "g"] \arrow[d, "\mathrm{id}_M"] & N \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & tM \arrow[r, "i"] & M \arrow[r, "\pi"] & M/tM \arrow[r] & 0,
\end{tikzcd}$$

where u and v are isomorphisms.

Proof. (i) It follows from Propositions 1.2 and 1.3 that there exists an idempotent radical t in mod A such that $\mathcal{T} = \{N \in \operatorname{mod} A \mid tN = N\}$ and $\mathcal{F} = \{N \in \operatorname{mod} A \mid tN = 0\}$. Then we have the canonical exact sequence in mod A

$$0 \longrightarrow tM \xrightarrow{i} M \xrightarrow{\pi} M/tM \longrightarrow 0,$$

where i is the inclusion homomorphism and π is the canonical epimorphism, and $tM \in \mathcal{T}$, $M/tM \in \mathcal{F}$, because $tM = t(tM)$ and $t(M/tM) = 0$.

(ii) Assume we have in mod A a short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

with $L \in \mathcal{T}$ and $N \in \mathcal{F}$. Since $\mathcal{T}$ is closed under images, we conclude that $\operatorname{Im} f \subseteq tM = i(tM)$, and consequently there exists a homomorphism $u\colon L \to tM$ such that $iu = f$. Then $\pi f = 0$ and there exists a homomorphism $v\colon N \to M/tM$ such that $vg = \pi$. Now it follows from Lemma VII.3.1 that u is a monomorphism, v is an epimorphism, and there exists an isomorphism $\delta\colon \operatorname{Ker} v \to \operatorname{Coker} u$. Moreover, $\operatorname{Ker} v \in \mathcal{F}$ and $\operatorname{Coker} u \in \mathcal{T}$, because $\mathcal{F}$ is closed under submodules and $\mathcal{T}$ is closed under images. Therefore, $\delta = 0$, and so $\operatorname{Ker} v = 0$, $\operatorname{Coker} u = 0$. This shows that u an v are isomorphisms. This completes the proof of (ii). □

The short exact sequence associated in the above proposition to a module M is called the *canonical sequence* for M, with respect to a given torsion pair $(\mathcal{T}, \mathcal{F})$ of $\operatorname{mod} A$.

Corollary 1.5. *Let A be a finite dimensional K-algebra over a field K, and $(\mathcal{T}, \mathcal{F})$ a torsion pair in* $\operatorname{mod} A$. *Then every simple module S in* $\operatorname{mod} A$ *belongs either to $\mathcal{F}$, or to $\mathcal{T}$.*

Proof. Let S be a simple module in $\operatorname{mod} A$. Consider the canonical exact sequence of S with respect to $(\mathcal{T}, \mathcal{F})$

$$0 \longrightarrow tS \longrightarrow S \longrightarrow S/tS \longrightarrow 0.$$

Since S is simple, we have either $tS = 0$ or $tS = S$, and consequently S belongs either to $\mathcal{F}$ or to $\mathcal{T}$. □

Let A be a finite dimensional K-algebra over a field K. A torsion pair $(\mathcal{T}, \mathcal{F})$ in $\operatorname{mod} A$ is said to be a *splitting torsion pair* if every indecomposable module X in $\operatorname{mod} A$ belongs either to $\mathcal{T}$ or to $\mathcal{F}$. Observe that for the torsion pairs considered in Example 1.1 (d), $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}^*, \mathcal{F}^*)$ are splitting torsion pairs, but $(\bar{\mathcal{T}}, \bar{\mathcal{F}})$ is not a splitting torsion pair. The following proposition provides a characterization of splitting torsion pairs in a module category.

Proposition 1.6. *Let A be a finite dimensional K-algebra over a field K, and $(\mathcal{T}, \mathcal{F})$ a torsion pair in* $\operatorname{mod} A$. *The following conditions are equivalent:*

(i) *$(\mathcal{T}, \mathcal{F})$ is a splitting torsion pair.*

(ii) *For each module M in* $\operatorname{mod} A$, *the canonical exact sequence of M splits.*

(iii) $\operatorname{Ext}_A^1(N, M) = 0$ *for all modules $M \in \mathcal{T}$ and $N \in \mathcal{F}$.*

(iv) *For each module M in $\mathcal{T}$, we have $\tau_A^{-1} M \in \mathcal{T}$.*

(v) *For each module N in $\mathcal{F}$, we have $\tau_A N \in \mathcal{F}$.*

Proof. (i) ⇒ (ii). Let M be a module in $\operatorname{mod} A$. It follows from (i) that $M = M' \oplus M''$, where M' is a direct sum of indecomposable modules from $\mathcal{T}$ and M''

is a direct sum of indecomposable modules from $\mathcal{F}$. Moreover, by Proposition 1.4, there exists a commutative diagram in mod A

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "u"] & M \arrow[r, "g"] \arrow[d, "\mathrm{id}_M"] & M'' \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & tM \arrow[r, "i"] & M \arrow[r, "\pi"] & M/tM \arrow[r] & 0,
\end{tikzcd}$$

where the lower sequence is the canonical exact sequence for M, u and v are isomorphisms, f is the canonical section, and g is the canonical retraction. Hence it follows that the canonical exact sequence for M splits.

(ii) $\Rightarrow$ (iii). Take modules $M \in \mathcal{T}$ and $N \in \mathcal{F}$. We have a canonical K-linear isomorphism $\chi_{N,M}: \mathcal{E}\mathrm{xt}_A^1(N, M) \to \mathrm{Ext}_A^1(N, M)$ (see Proposition III.3.7). It follows from (ii) and Proposition 1.4(ii) that $\mathcal{E}\mathrm{xt}_A^1(N, M) = 0$, and hence $\mathrm{Ext}_A^1(N, M) = 0$.

(iii) $\Rightarrow$ (i). Let M be an indecomposable module in mod A. Consider the canonical exact sequence

$$0 \longrightarrow tM \xrightarrow{i} M \xrightarrow{\pi} M/tM \longrightarrow 0$$

for M, with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. Since $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$, it follows from (iii) and Proposition III.3.7 that $\mathcal{E}\mathrm{xt}_A^1(M/tM, tM) = 0$, and hence $M = \mathrm{Im}\, i \oplus \mathrm{Ker}\, \pi \cong tM \oplus M/tM$ (see Lemmas I.4.2 and III.3.1). Then the indecomposability of M implies that either $M = tM \in \mathcal{T}$ or $M = M/tM \in \mathcal{F}$.

(i) $\Rightarrow$ (iv). Let M be an indecomposable module in $\mathcal{T}$. We may assume that $\tau_A^{-1} M \neq 0$. Consider an almost split sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow \tau_A^{-1} M \longrightarrow 0$$

and a decomposition $E = \bigoplus_{i=1}^r E_i$ of E into a direct sum of indecomposable modules in mod A. Since the above exact sequence is not splitting, we have $\mathrm{Hom}_A(M, E_i) \neq 0$ for any $i \in \{1, \dots, r\}$. Then the assumption (i) and $M \in \mathcal{T}$ imply that $E_1, \dots, E_r$ belong to $\mathcal{T}$. Hence $E \in \mathcal{T}$, and consequently $\tau_A^{-1} M \in \mathcal{T}$, because $\mathcal{T}$ is closed under images in mod A.

The proof that (i) implies (v) is similar.

(iv) $\Rightarrow$ (iii). Take modules $M \in \mathcal{T}$ and $N \in \mathcal{F}$. Theorem III.6.3 provides an isomorphism of K-vector spaces $\mathrm{Ext}_A^1(N, M) \cong D\, \underline{\mathrm{Hom}}_A(\tau_A^{-1} M, N)$. It follows from the assumption (iv) and $M \in \mathcal{T}$ that $\tau_A^{-1} M \in \mathcal{T}$. Then $\mathrm{Hom}_A(\tau_A^{-1} M, N) = 0$ because $N \in \mathcal{F}$. This shows that $\mathrm{Ext}_A^1(N, M) = 0$.

The proof that (v) implies (iii) is similar. □

Let A be a finite dimensional K-algebra over a field K and T a module in $\operatorname{mod} A$. We define $\operatorname{Gen} T$ to be the full subcategory of $\operatorname{mod} A$ formed by all modules M in $\operatorname{mod} A$ generated by T, that is, the modules M for which there exists an epimorphism $T^d \to M$ for some positive integer d. Dually, we define $\operatorname{Cogen} T$ to be the full subcategory of $\operatorname{mod} A$ formed by all modules N cogenerated by T, that is, the modules N in $\operatorname{mod} A$ for which there exists a monomorphism $N \to T^d$ for some positive integer d. Consider the endomorphism algebra $B = \operatorname{End}_A(T)$ and observe that T is a (B, A)-bimodule with the left B-module structure given by $\varphi t = \varphi(t)$ for any $\varphi \in B$ and $t \in T$. For a module M in $\operatorname{mod} A$, we define the canonical homomorphisms of right A-modules

$$\varepsilon_M \colon \operatorname{Hom}_A(T, M) \otimes_B T \longrightarrow M,$$

given by $\varepsilon_M(f \otimes t) = f(t)$ for $f \in \operatorname{Hom}_A(T, M)$ and $t \in T$, and

$$\eta_M \colon M \longrightarrow \operatorname{Hom}_{B^{\mathrm{op}}}\big(\operatorname{Hom}_A(M, T), T\big),$$

given by $\eta_M(m)(g) = g(m)$ for any $m \in M$ and $g \in \operatorname{Hom}_A(M, T)$.

We present now necessary and sufficient conditions for a module M from $\operatorname{mod} A$ to belong to $\operatorname{Gen} T$ (respectively, $\operatorname{Cogen} T$).

Lemma 1.7. *Let A be a finite dimensional K-algebra over a field K, T a module in* $\operatorname{mod} A$*, and $B = \operatorname{End}_A(T)$. For a module M in* $\operatorname{mod} A$*, the following statements are equivalent:*

(i) $M \in \operatorname{Gen} T$.

(ii) *The canonical homomorphism $\varepsilon_M \colon \operatorname{Hom}_A(T, M) \otimes_B T \to M$ in* $\operatorname{mod} A$ *is an epimorphism.*

Proof. Assume that M belongs to $\operatorname{Gen} T$. Let $f_1, \dots, f_d$ be a basis of the K-vector space $\operatorname{Hom}_A(T, M)$. We claim that $f = [f_1, \dots, f_d] \colon T^d \to M$ is an epimorphism. Indeed, since $M \in \operatorname{Gen} T$, there exists an epimorphism $g \colon T^m \to M$ in $\operatorname{mod} A$ for some positive integer m. Then it follows from our choice of $f_1, \dots, f_d$ that there exists a homomorphism $h \colon T^m \to T^d$ in $\operatorname{mod} A$ such that $g = fh$, and hence f is an epimorphism. Let $m \in M$. Then there exists $t = (t_1, \dots, t_d) \in T^d$ such that $m = f(t)$. Then $\varepsilon_M(\sum_{i=1}^d f_i \otimes t_i) = \sum_{i=1}^d f_i(t_i) = f(t) = m$. This shows that ε_M is an epimorphism. Hence (i) implies (ii).

Conversely, assume that $\varepsilon_M \colon \operatorname{Hom}_A(T, M) \otimes_B T \to M$ is an epimorphism. Since $\operatorname{Hom}_A(T, M)$ is a module in $\operatorname{mod} B$, there exists an epimorphism $g \colon B^m \to \operatorname{Hom}_A(T, M)$ in $\operatorname{mod} B$ for some positive integer m. Then we obtain the composed epimorphism in $\operatorname{mod} A$

$$T^m \xrightarrow{\sim} B^m \otimes_B T \xrightarrow{g \otimes T} \operatorname{Hom}_A(T, M) \otimes_B T \xrightarrow{\varepsilon_M} M,$$

and hence $M \in \operatorname{Gen} T$. Thus (ii) implies (i). □

Lemma 1.8. *Let A be a finite dimensional K-algebra over a field K, T a module in* $\operatorname{mod} A$*, and $B = \operatorname{End}_A(T)$. For a module N in* $\operatorname{mod} A$*, the following statements are equivalent.*

(i) $N \in \operatorname{Cogen} T$.

(ii) *The canonical homomorphism $\eta_N : N \to \operatorname{Hom}_{B^{\mathrm{op}}}(\operatorname{Hom}_A(N, T), T)$ in* $\operatorname{mod} A$ *is a monomorphism.*

Proof. Assume that $N \in \operatorname{Cogen} T$. Let $f_1, \ldots, f_d$ be a K-basis of the K-vector space $\operatorname{Hom}_A(N, T)$. We claim that the induced homomorphism

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_d \end{bmatrix} : N \longrightarrow T^d$$

is a monomorphism. Indeed, since $N \in \operatorname{Cogen} T$, there exists a monomorphism $g: N \to T^m$ in $\operatorname{mod} A$ for some positive integer m. Then it follows from our choice of $f_1, \ldots, f_d$ that there exists a homomorphism $h: T^d \to T^m$ in $\operatorname{mod} A$ such that $g = hf$, and hence f is a monomorphism. Let $0 \neq u \in N$. Since f is a monomorphism and $f(u) = (f_1(u), \ldots, f_d(u))$, we have $f_i(u) \neq 0$ for some $i \in \{1, \ldots, d\}$. Then $\eta_N(u)(f_i) = f_i(u) \neq 0$, and hence $\eta_N(u) \neq 0$. This shows that η_N is an monomorphism. Hence (i) implies (ii).

Conversely, assume that $\eta_N : \operatorname{Hom}_{B^{\mathrm{op}}}(\operatorname{Hom}_A(N, T), T)$ is a monomorphism in $\operatorname{mod} A$. Since $\operatorname{Hom}_A(N, T)$ is a module in $\operatorname{mod} B^{\mathrm{op}}$, we have in $\operatorname{mod} B^{\mathrm{op}}$ an epimorphism $g: B^n \to \operatorname{Hom}_A(N, T)$ for some positive integer n. Then we obtain the composed monomorphism in $\operatorname{mod} A$

$$N \xrightarrow{\eta_N} \operatorname{Hom}_{B^{\mathrm{op}}}(\operatorname{Hom}_A(N, T), T) \xrightarrow{\operatorname{Hom}_{B^{\mathrm{op}}}(g, T)} \operatorname{Hom}_{B^{\mathrm{op}}}(B^n, T) \xrightarrow{\sim} T^n,$$

and hence $N \in \operatorname{Cogen} T$. Thus (ii) implies (i). □

2 Tilting modules

Let A be a finite dimensional K-algebra over a field K. A module T in $\operatorname{mod} A$ is called a *tilting module* if the following three conditions are satisfied:

(T1) $\operatorname{pd}_A T \leq 1$.

(T2) $\operatorname{Ext}_A^1(T, T) = 0$.

(T3) There exists an exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with T' and T'' in $\operatorname{add} T$.

A module T in $\operatorname{mod} A$ satisfying the conditions (T1) and (T2) is called a *partial tilting module*. It follows from (T3) and Lemma II.5.5 that every tilting module T in $\operatorname{mod} A$ is a faithful module, that is, its right annihilator $r_A(T) = \{a \in A \mid Ta = 0\}$ is zero. We also note that then $D(T)$ is a faithful module in $\operatorname{mod} A^{\mathrm{op}}$, and hence there exists a monomorphism $A \to D(T)^r$ in $\operatorname{mod} A^{\mathrm{op}}$, for some positive integer r, again by Lemma II.5.5. But then there exists an epimorphism $T^r \to D(A)$ in $\operatorname{mod} A$. Therefore, if T is a tilting module in $\operatorname{mod} A$, then $A \in \operatorname{Cogen} T$ and $D(A) = \operatorname{Hom}_K(A, K) \in \operatorname{Gen} T$.

Examples 2.1. (a) Let A be a finite dimensional K-algebra over a field K and P a projective module in $\operatorname{mod} A$. Then $\operatorname{pd}_A P = 0$ and $\operatorname{Ext}^1_A(P, P) = 0$, and hence P is a partial tilting module. Moreover, P is a tilting module in $\operatorname{mod} A$ if and only if P is a generator of $\operatorname{mod} A$, or equivalently, every indecomposable projective module in $\operatorname{mod} A$ is a direct summand of P (see Section II.5).

(b) Let A be a finite dimensional hereditary K-algebra over a field K and I an injective module in $\operatorname{mod} A$. Then $\operatorname{pd}_A I \leq 1$ and $\operatorname{Ext}^1_A(I, I) \cong \widetilde{\operatorname{Ext}}^1_A(I, I) = 0$, and hence I is a partial tilting module. Moreover, I is a tilting module in $\operatorname{mod} A$ if and only if I is a cogenerator of $\operatorname{mod} A$, or equivalently, every indecomposable injective module in $\operatorname{mod} A$ is a direct summand of I (see Section II.7).

(c) Let $A = KQ$ be the path algebra of the quiver Q of the form

$$\begin{tikzcd}\underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l, "\beta"']\end{tikzcd}$$

over a field K, considered in Example 1.1 (d). Consider the module $T = P_1 \oplus I_2 \oplus P_3$ in $\operatorname{mod} A$. Then $\operatorname{pd}_A T \leq 1$ since A is a hereditary algebra. Moreover, applying Corollary III.6.4, we obtain isomorphisms

$$\operatorname{Ext}^1_A(T, T) \cong D \operatorname{Hom}_A(T, \tau_A T) = D \operatorname{Hom}_A(P_1 \oplus I_2 \oplus P_3, P_2) = 0,$$

and hence T is a partial tilting module. Further, since $A = P_1 \oplus P_2 \oplus P_3$, we have in $\operatorname{mod} A$ an exact sequence

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with $T' = P_1 \oplus P_1 \oplus P_3 \oplus P_3$ and $T'' = I_2$ in $\operatorname{add} T$, so the condition (T3) is also satisfied. Therefore, T is a tilting module in $\operatorname{mod} A$.

Let A be a finite dimensional K-algebra over a field K and T a module in $\operatorname{mod} A$. We introduce two full subcategories of $\operatorname{mod} A$,

$$\mathcal{T}(T) = \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Ext}^1_A(T, M) = 0\big\},$$
$$\mathcal{F}(T) = \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(T, M) = 0\big\}.$$

Proposition 2.2. *Let A be a finite dimensional K-algebra over a field K and T a partial tilting module in* $\operatorname{mod} A$. *Then the following statements hold:*

(i) $(\operatorname{Gen} T, \mathcal{F}(T))$ *is a torsion pair in* $\operatorname{mod} A$.

(ii) $(\mathcal{T}(T), \operatorname{Cogen} \tau_A T)$ *is a torsion pair in* $\operatorname{mod} A$.

(iii) $\operatorname{Gen} T \subseteq \mathcal{T}(T)$ *and* $\operatorname{Cogen} \tau_A T \subseteq \mathcal{F}(T)$.

Proof. We first show that $\operatorname{Gen} T \subseteq \mathcal{T}(T)$. Let M be a module in $\operatorname{Gen} T$. Then there is an exact sequence in $\operatorname{mod} A$ of the form

$$0 \longrightarrow L \xrightarrow{u} T^m \xrightarrow{v} M \longrightarrow 0,$$

for some positive integer m. Since $\operatorname{pd}_A T \leq 1$, applying Theorem VII.3.2 we obtain an exact sequence in $\operatorname{mod} K$

$$\operatorname{Ext}_A^1(T, L) \xrightarrow{\operatorname{Ext}_A^1(T,u)} \operatorname{Ext}_A^1(T, T^m) \xrightarrow{\operatorname{Ext}_A^1(T,v)} \operatorname{Ext}_A^1(T, M) \longrightarrow 0,$$

where $\operatorname{Ext}_A^1(T, T^m) = 0$, because $\operatorname{Ext}_A^1(T, T) = 0$. Hence $\operatorname{Ext}_A^1(T, M) = 0$. This shows that $\operatorname{Gen} T \subseteq \mathcal{T}(T)$.

We prove now that $\operatorname{Gen} T$ is a torsion class in $\operatorname{mod} A$. Observe that $\operatorname{Gen} T$ is closed under images and direct sums in $\operatorname{mod} A$. Let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an exact sequence in $\operatorname{mod} A$ with L and N in $\operatorname{Gen} T$. Applying Theorem VII.3.2 again, we obtain an exact sequence in $\operatorname{mod} B$

$$0 \longrightarrow \operatorname{Hom}_A(T, L) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T, M) \xrightarrow{\operatorname{Hom}_A(T,g)} \operatorname{Hom}_A(T, N) \longrightarrow 0,$$

because $L \in \operatorname{Gen} T \subseteq \mathcal{T}(T)$ gives $\operatorname{Ext}_A^1(T, L) = 0$. Applying the right exact functor $- \otimes_B T \colon \operatorname{mod} B \to \operatorname{mod} A$ we conclude that the upper row of the commutative diagram in $\operatorname{mod} A$

$$\begin{tikzcd}
 & \operatorname{Hom}_A(T, L) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_L"] & \operatorname{Hom}_A(T, M) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_M"] & \operatorname{Hom}_A(T, N) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_N"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

is exact. Since L and N are in $\operatorname{Gen} T$, applying Lemma 1.7 we obtain that ε_L and ε_N are epimorphisms. Then, applying Lemma VII.3.1 we conclude that ε_M is an epimorphism. Thus $M \in \operatorname{Gen} T$, by Lemma 1.7. Therefore, $\operatorname{Gen} T$ is also closed under extensions in $\operatorname{mod} A$, and consequently $\operatorname{Gen} T$ is a torsion class in $\operatorname{mod} A$ (see Proposition 1.2).

Let $\mathcal{F}$ be the full subcategory of mod A such that $(\operatorname{Gen} T, \mathcal{F})$ is a torsion pair in mod A (see Example 1.1 (a)). We claim that $\mathcal{F} = \mathcal{F}(T)$. For a module N in $\mathcal{F}$ we have $\operatorname{Hom}_A(T, N) = 0$, because $T \in \operatorname{Gen} T$, and hence $N \in \mathcal{F}(T)$. This shows that $\mathcal{F} \subseteq \mathcal{F}(T)$. Conversely, let N be a module in $\mathcal{F}(T)$ and X a module in $\operatorname{Gen} T$. Then there exists an epimorphism $T^r \to X$ in mod A, for some positive integer r. Hence $\operatorname{Hom}_A(T, N) = 0$ forces $\operatorname{Hom}_A(X, N) = 0$. Then $N \in \mathcal{F}$, because $(\operatorname{Gen} T, \mathcal{F})$ is a torsion pair in mod A. This proves that $\mathcal{F}(T) \subseteq \mathcal{F}$.

Therefore, we have $\mathcal{F} = \mathcal{F}(T)$.

Our next aim is to show that $\operatorname{Cogen} \tau_A T$ is a torsion-free class in mod A. Observe that $\operatorname{Cogen} \tau_A T$ is closed under submodules and direct sums in mod A. Let

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

be an exact sequence in mod A with U and W in $\operatorname{Cogen} \tau_A T$. Applying Theorem VII.3.3, we obtain the following exact sequence in mod K, and hence in mod B^{op} with $B = \operatorname{End}_A(\tau_A T)$:

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_A(W, \tau_A T) &\longrightarrow \operatorname{Hom}_A(V, \tau_A T) \\ &\longrightarrow \operatorname{Hom}_A(U, \tau_A T) \longrightarrow \operatorname{Ext}^1_A(W, \tau_A T). \end{aligned}$$

We claim that $\operatorname{Ext}^1_A(W, \tau_A T) = 0$. Applying Theorem III.6.3, we obtain isomorphisms of K-vector spaces

$$\operatorname{Ext}^1_A(W, \tau_A T) \cong D\,\underline{\operatorname{Hom}}_A\left(\tau_A^{-1}(\tau_A T), W\right) = \underline{\operatorname{Hom}}_A(T, W).$$

Further, since $\operatorname{pd}_A T \leq 1$, we have isomorphisms in mod K

$$\operatorname{Hom}_A(T, \tau_A T) \cong D \operatorname{Hom}_A(T, \tau_A T) \cong \operatorname{Ext}^1_A(T, T) = 0,$$

by Corollary III.6.4. Finally, we observe that $\operatorname{Hom}_A(T, \tau_A T) = 0$ and $W \in \operatorname{Cogen} \tau_A T$ imply $\operatorname{Hom}_A(T, W) = 0$. Hence the claim follows, and we have in mod B^{op} the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(W, \tau_A T) \longrightarrow \operatorname{Hom}_A(V, \tau_A T) \longrightarrow \operatorname{Hom}_A(U, \tau_A T) \longrightarrow 0.$$

Applying the left exact functor $\operatorname{Hom}_{B^{\mathrm{op}}}(-, \tau_A T)\colon \operatorname{mod} B^{\mathrm{op}} \to \operatorname{mod} A$, we obtain the following commutative diagram in mod A with exact rows

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, "\eta_U"] & V \arrow[r, "g"] \arrow[d, "\eta_V"] & W \arrow[r] \arrow[d, "\eta_W"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_{B^{\mathrm{op}}}(\operatorname{Hom}_A(U, \tau_A T), \tau_A T) \arrow[r] & \operatorname{Hom}_{B^{\mathrm{op}}}(\operatorname{Hom}_A(V, \tau_A T), \tau_A T) \arrow[r] & \operatorname{Hom}_{B^{\mathrm{op}}}(\operatorname{Hom}_A(W, \tau_A T), \tau_A T) \, . &
\end{tikzcd}$$

Since U and W belong to $\operatorname{Cogen} \tau_A T$, we conclude by Lemma 1.8 that η_U and η_W are monomorphisms. Then it follows from Lemma VII.3.1 that η_V is a monomorphism. Applying Lemma 1.8 again, we obtain that $V \in \operatorname{Cogen} \tau_A T$. Therefore, $\operatorname{Cogen} \tau_A T$ is closed under extensions in $\operatorname{mod} A$. This shows that $\operatorname{Cogen} \tau_A T$ is a torsion-free class in $\operatorname{mod} A$, by Proposition 1.3.

Let $\mathcal{T}$ be the full subcategory of $\operatorname{mod} A$ such that $(\mathcal{T}, \operatorname{Cogen} \tau_A T)$ is a torsion pair in $\operatorname{mod} A$ (see Example 1.1 (b)). We claim that $\mathcal{T} = \mathcal{T}(T)$. Observe that

$$\mathcal{T} = \left\{ M \in \operatorname{mod} A \,\middle|\, \operatorname{Hom}_A(M, X) = 0 \text{ for any } X \in \operatorname{Cogen} \tau_A T \right\}.$$

On the other hand, since $\operatorname{pd}_A \leq 1$, we have

$$\begin{aligned} \mathcal{T}(T) &= \left\{ M \in \operatorname{mod} A \,\middle|\, \operatorname{Ext}^1_A(T, M) = 0 \right\} \\ &= \left\{ M \in \operatorname{mod} A \,\middle|\, \operatorname{Hom}_A(M, \tau_A T) = 0 \right\}, \end{aligned}$$

by Corollary III.6.4. Finally, we note that $X \in \operatorname{Cogen} \tau_A T$ if and only if there exists a monomorphism $X \to (\tau_A T)^d$ for some positive integer d. Then the equality $\mathcal{T} = \mathcal{T}(T)$ follows.

Summing up, we proved that $(\operatorname{Gen} T, \mathcal{F}(T))$ and $(\mathcal{T}(T), \operatorname{Cogen} \tau_A T)$ are torsion pairs in $\operatorname{mod} A$, and $\operatorname{Gen} T \subseteq \mathcal{T}(T)$. Now the inclusion $\operatorname{Cogen} \tau_A T \subseteq \mathcal{F}(T)$ follows. $\square$

Example 2.3. Let $A = KQ$ be the path algebra of the quiver Q of the form

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l, "\beta"'] \end{tikzcd}$$

over a field K, considered in Example 1.1 (d). Recall that the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd} & P_1 \arrow[dr] & & I_3 = S_3 \\ S_2 = P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & \\ & P_3 \arrow[ur] & & I_1 = S_1\,. \end{tikzcd}$$

Consider the tilting module $T = P_1 \oplus I_2 \oplus P_3$ from Example 2.1 (c) and its direct summand $T^* = P_1 \oplus I_2$. Obviously, T^* is a partial tilting module in $\operatorname{mod} A$, as a direct summand of T. Observe that T^* is not a tilting module in $\operatorname{mod} A$. Indeed, for any short exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with T' in add T^*, the simple module S_1 is a direct summand of T'', and hence T'' is not in add T^*. A simple checking shows that

$$\begin{aligned} \operatorname{Gen} T^* &= \operatorname{add}(P_1 \oplus I_2 \oplus S_1 \oplus S_3), \\ \operatorname{Cogen} \tau_A T^* &= \operatorname{Cogen} S_2 = \operatorname{add}(S_2), \\ \mathcal{T}(T^*) &= \operatorname{add}(P_1 \oplus I_2 \oplus P_3 \oplus S_1 \oplus S_3), \\ \mathcal{F}(T^*) &= \operatorname{add}(S_2 \oplus P_3). \end{aligned}$$

Therefore, $\operatorname{Gen} T^* \neq \mathcal{T}(T^*)$ and $\operatorname{Cogen} \tau_A T^* \neq \mathcal{F}(T^*)$. On the other hand, for the tilting module T,

$$\begin{aligned} \operatorname{Gen}(T) &= \operatorname{add}(P_1 \oplus I_2 \oplus P_3 \oplus S_1 \oplus S_3) = \mathcal{T}(T) \\ \operatorname{Cogen}(\tau_A T) &= \operatorname{add}(S_2) = \mathcal{F}(T). \end{aligned}$$

The following lemma, known as *Bongartz's lemma*, justifies the name of partial tilting module.

Lemma 2.4. *Let A be a finite dimensional K-algebra over a field K and T be a partial tilting module in* mod A. *Then there exists a module E in* mod A *such that $T \oplus E$ is a tilting module in* mod A.

Proof. Let $d = \dim_K \mathcal{E}\mathrm{xt}_A^1(T, A)$ and

$$\mathbb{E}_i\colon \qquad 0 \longrightarrow A \xrightarrow{f_i} E_i \xrightarrow{g_i} T \longrightarrow 0,$$

$i \in \{1, \dots, d\}$, be exact sequences in mod A such that $[\mathbb{E}_1], \dots, [\mathbb{E}_d]$ form a K-basis of $\mathcal{E}\mathrm{xt}_A^1(T, A)$. Consider the commutative diagram in mod A with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A^d \arrow[r, "f"] \arrow[d, "h"] & \bigoplus_{i=1}^{d} E_i \arrow[r, "g"] \arrow[d, "u"] & T^d \arrow[r] \arrow[d, "\mathrm{id}_{T^d}"] & 0 \\ 0 \arrow[r] & A \arrow[r, "v"] & E \arrow[r, "w"] & T^d \arrow[r] & 0, \end{tikzcd}$$

where the upper exact sequence is the direct sum $\mathbb{E} = \bigoplus_{i=1}^{d} \mathbb{E}_i$ of the exact sequences $\mathbb{E}_1, \dots, \mathbb{E}_d$, $h\colon A^d \to A$ is the codiagonal homomorphism given by $h(a_1, \dots, a_d) = a_1 + \cdots + a_d$ for $a_1, \dots, a_d \in A$, and the lower exact sequence is the sequence $\mathbb{F} = h\mathbb{E}$ (see Section III.3). We note that E is the fibered sum of A and $\bigoplus_{i=1}^{d} E_i$ over A^d, via h and f. Let $w_i\colon T \to T^d$, $i \in \{1, \dots, d\}$, be the inclusion homomorphisms in the i-th coordinates. We claim that $\mathbb{E}_i \cong \mathbb{F} w_i$ for any $i \in \{1, \dots, d\}$. Indeed, consider the commutative diagram in mod A with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f_i"] \arrow[d, "\varphi_i"] & E_i \arrow[r, "g_i"] \arrow[d, "\psi_i"] & T \arrow[r] \arrow[d, "w_i"] & 0 \\
0 \arrow[r] & A^d \arrow[r, "f"] \arrow[d, "h"] & \bigoplus_{i=1}^{d} E_i \arrow[r, "g"] \arrow[d, "u"] & T^d \arrow[r] \arrow[d, "\mathrm{id}_{T^d}"] & 0 \\
0 \arrow[r] & A \arrow[r, "v"] & E \arrow[r, "w"] & T^d \arrow[r] & 0\,,
\end{tikzcd}$$

where $\varphi_i\colon A \to A^d$ and $\psi_i\colon E_i \to \bigoplus_{i=1}^d E_i$ are the inclusion homomorphisms into the i-th coordinate. Since $h\varphi_i = \mathrm{id}_A$ and $w_i = \mathrm{id}_{T^d}\, w_i$, we obtain the commutative diagram in mod A with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f_i"] \arrow[d, "\mathrm{id}_A"] & E_i \arrow[r, "g_i"] \arrow[d, "u\psi_i"] & T \arrow[r] \arrow[d, "w_i"] & 0 \\
0 \arrow[r] & A \arrow[r] & E \arrow[r, "w"] & T^d \arrow[r] & 0\,,
\end{tikzcd}$$

and hence $\mathbb{E}_i \cong \mathbb{F}w_i$. We prove now that $\mathrm{Ext}_A^1(T \oplus E, T \oplus E) = 0$. Applying Proposition VII.3.2 to the exact sequence $\mathbb{F}$, we obtain an exact sequence in mod K of the form

$$\cdots \longrightarrow \mathrm{Hom}_A(T, T^d) \xrightarrow{\delta} \mathrm{Ext}_A^1(T, A) \longrightarrow \mathrm{Ext}_A^1(T, E) \longrightarrow \mathrm{Ext}_A^1(T, T^d),$$

where $\delta = \delta^T_{T^d,A}$ is the connecting homomorphism. Observe that $\mathrm{Ext}_A^1(T, T^d) = 0$, because T is a partial tilting module. We claim that δ is an epimorphism, and consequently $\mathrm{Ext}_A^1(T, E) = 0$. Indeed, by Theorem VII.3.2, we have $\delta(w_i) = \chi_{T,A}([\mathbb{F}w_i]) = \chi_{T,A}([\mathbb{E}_i])$ for any $i \in \{1, \dots, d\}$. Hence $\delta(w_1), \dots, \delta(w_d)$ form a K-basis of the K-vector space $\mathrm{Ext}_A^1(T, A)$, because $[\mathbb{E}_1], \dots, [\mathbb{E}_d]$ form a K-basis of $\mathcal{E}\mathrm{xt}_A^1(T, A)$ and $\chi_{T,A}$ is a K-linear isomorphism. Therefore, δ is an epimorphism. Applying Theorem VII.3.3, we obtain exact sequences in mod K of the forms

$$0 = \mathrm{Ext}_A^1(T^d, T) \longrightarrow \mathrm{Ext}_A^1(E, T) \longrightarrow \mathrm{Ext}_A^1(A, T) = 0,$$
$$0 = \mathrm{Ext}_A^1(T^d, E) \longrightarrow \mathrm{Ext}_A^1(E, E) \longrightarrow \mathrm{Ext}_A^1(A, E) = 0,$$

because A is projective, $\mathrm{Ext}_A^1(T, T) = 0$, and $\mathrm{Ext}_A^1(T, E) = 0$. Summing up, we proved that $\mathrm{Ext}_A^1(T \oplus E, T \oplus E) = 0$, so $T \oplus E$ satisfies the condition (T2). Clearly, $T \oplus E$ satisfies also (T3), because we have the exact sequence

$$\mathbb{F}\colon \qquad 0 \longrightarrow A \xrightarrow{v} E \xrightarrow{w} T^d \longrightarrow 0$$

with E and T^d in $\mathrm{add}(T \oplus E)$.

Finally, we prove that $\operatorname{pd}_A E \le 1$, and hence $\operatorname{pd}_A(T \oplus E) \le 1$, because $\operatorname{pd}_A T \le 1$. Since $\operatorname{pd}_A T^d \le 1$, T^d admits a minimal projective resolution in $\operatorname{mod} A$ of the form

$$0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} T^d \longrightarrow 0.$$

Invoking the projectivity of P_0 in $\operatorname{mod} A$, we conclude that there is in $\operatorname{mod} A$ a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & P_0 \arrow[r, "\mathrm{id}_{P_0}"] \arrow[d, "f_0"] & P_0 \arrow[r] \arrow[d, "d_0"] & 0 \\
0 \arrow[r] & A \arrow[r, "v"] & E \arrow[r, "w"] & T^d \arrow[r] & 0,
\end{tikzcd}$$

with exact rows. It follows from the proof of Lemma III.3.1 that there is a homomorphism $\delta\colon P_1 \to A$ in $\operatorname{mod} A$ such that $v\delta = f_0 d_1$. Then we obtain a short exact sequence in $\operatorname{mod} A$ of the form

$$0 \longrightarrow P_1 \xrightarrow{\left[\begin{smallmatrix} -\delta \\ d_1 \end{smallmatrix}\right]} A \oplus P_0 \xrightarrow{[v\ f_0]} E \longrightarrow 0,$$

and hence $\operatorname{pd}_A E \le 1$. This shows that $T \oplus E$ satisfies also the condition (T1), and hence is a tilting module in $\operatorname{mod} A$. □

The following theorem provides a characterization of tilting modules.

Theorem 2.5. *Let A be a finite dimensional K-algebra over a field K and T a partial tilting module in* mod A. *Then the following conditions are equivalent:*

(i) *T is a tilting module.*

(ii) $\operatorname{Gen} T = \mathcal{T}(T)$.

(iii) *For every module $M \in \mathcal{T}(T)$ there exists a short exact sequence in* mod A

$$0 \longrightarrow L \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with $T_0 \in \operatorname{add} T$ and $L \in \mathcal{T}(T)$.

(iv) *For a module X in* mod A, *$X \in \operatorname{add} T$ if and only if $X \in \mathcal{T}(T)$ and $\operatorname{Ext}^1_A(X, Y) = 0$ for any module $Y \in \mathcal{T}(T)$.*

(v) $\operatorname{Cogen} \tau_A T = \mathcal{F}(T)$.

Proof. The equivalence of (ii) and (v) follows from Proposition 2.2. Hence, it suffices to prove the equivalence of the first four conditions.

(i) ⇒ (ii). Assume T is a tilting module in $\operatorname{mod} A$. It follows from Proposition 2.2 that $\operatorname{Gen} T$ and $\mathcal{T}(T)$ are torsion classes in $\operatorname{mod} A$ with $\operatorname{Gen} T \subseteq$

$\mathcal{T}(T)$. Let t be the torsion radical in $\operatorname{mod} A$ with respects to the torsion pair $(\operatorname{Gen} T, \mathcal{F}(T))$. Take a module $M \in \mathcal{T}(T)$. We will show that $M = tM$, and consequently $M \in \operatorname{Gen} T$. Since $\operatorname{pd}_A T \leq 1$, the canonical sequence

$$0 \longrightarrow tM \longrightarrow M \longrightarrow M/tM \longrightarrow 0$$

for M induces an epimorphism $\operatorname{Ext}^1_A(T, M) \to \operatorname{Ext}^1_A(T, M/tM)$ in $\operatorname{mod} K$, by Theorem VII.3.2. Then $M \in \mathcal{T}(T)$ implies $\operatorname{Ext}^1_A(T, M/tM) = 0$. Moreover, we have $\operatorname{Hom}_A(T, M/tM) = 0$, because $T \in \operatorname{Gen} T$ and $M/tM \in \mathcal{F}(T)$. Further, since T is a tilting module, we have in $\operatorname{mod} A$ an exact sequence

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with T', T'' in $\operatorname{add} T$. Applying Theorem VII.3.3 to this sequence, we obtain an exact sequence in $\operatorname{mod} K$ of the form

$$0 = \operatorname{Hom}_A(T', M/tM) \longrightarrow \operatorname{Hom}_A(A, M/tM) \longrightarrow \operatorname{Ext}^1_A(T'', M/tM) = 0,$$

because $T', T'' \in \operatorname{add} T$. Hence, we get $M/tM \cong \operatorname{Hom}_A(A, M/tM) = 0$. This shows that $M = tM$. Therefore, $\mathcal{T}(T) \subseteq \operatorname{Gen} T$, and so $\operatorname{Gen} T = \mathcal{T}(T)$.

(ii) $\Rightarrow$ (iii). Let $\operatorname{Gen} T = \mathcal{T}(T)$ and M be a module in $\mathcal{T}(T)$. Choose a basis $f_1, \dots, f_d$ of the K-vector space $\operatorname{Hom}_A(T, M)$. Because $M \in \operatorname{Gen} T$, the induced homomorphism $f = [f_1, \dots, f_d]\colon T^d \to M$ is an epimorphism (see the proof of Lemma 1.7). Then we have the exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow L \xrightarrow{u} T^d \xrightarrow{f} M \longrightarrow 0$$

with $L = \operatorname{Ker} f$ and u the inclusion homomorphism. Applying Theorem VII.3.2, we obtain an exact sequence in $\operatorname{mod} K$ of the form

$$\operatorname{Hom}_A(T, T^d) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T, M) \xrightarrow{\delta} \operatorname{Ext}^1_A(T, L) \xrightarrow{\operatorname{Hom}_A(T,u)} \operatorname{Ext}^1_A(T, T^d),$$

where $\operatorname{Ext}^1_A(T, T^d) = 0$ and $\operatorname{Hom}_A(T, f)$ is an epimorphism, by the construction of f. Hence $\operatorname{Ext}^1_A(T, L) = 0$, which shows that $L \in \mathcal{T}(T)$. Letting $T_0 = T^d \in \operatorname{add} T$, we have the required exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow L \xrightarrow{u} T_0 \xrightarrow{f} M \longrightarrow 0.$$

(iii) $\Rightarrow$ (iv). Let X be a module in $\operatorname{mod} A$. Obviously, if $X \in \operatorname{add} T$, then $\operatorname{Ext}^1_A(X, Y) = 0$ for any module $Y \in \mathcal{T}(T)$. Conversely, assume that $X \in \mathcal{T}(T)$ and $\operatorname{Ext}^1_A(X, Y) = 0$ for any module $Y \in \mathcal{T}(T)$. It follows from the assumption (iii) that there exists an exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow L \xrightarrow{u} T_0 \xrightarrow{f} X \longrightarrow 0$$

with $T_0 \in \operatorname{add} T$ and $L \in \mathcal{T}(T)$. Hence $\mathcal{E}\mathrm{xt}_A^1(X, L) \cong \operatorname{Ext}_A^1(X, L) = 0$, and consequently the above exact sequence splits (see Corollary III.3.6). Then it follows from Lemma III.3.1 that f is a retraction and u is a section. But then $T_0 \cong L \oplus X$, by Lemma I.4.2, and so $X \in \operatorname{add} T$.

(iv) ⇒ (i). Consider the Bongartz exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow T^d \longrightarrow 0$$

constructed in the proof of Lemma 2.4. Then $T \oplus E$ is a tilting module in $\operatorname{mod} A$. In order to prove that T is a tilting module, it suffices to show that $E \in \operatorname{add} T$. Observe that $\operatorname{Ext}_A^1(T \oplus E, T \oplus E) = 0$ implies $\operatorname{Ext}_A^1(T, E) = 0$, and hence $E \in \mathcal{T}(T)$. Let Y be a module in $\mathcal{T}(T)$. Then applying Theorem VII.3.3 to the module Y and the Bongartz exact sequence, we obtain an exact sequence in $\operatorname{mod} K$ of the form

$$0 = \operatorname{Ext}_A^1(T^d, Y) \longrightarrow \operatorname{Ext}_A^1(E, Y) \longrightarrow \operatorname{Ext}_A^1(A, Y) = 0,$$

because $Y \in \mathcal{T}(T)$ and A is projective. Then $\operatorname{Ext}_A^1(E, Y) = 0$. Now the assumption (iv) gives $E \in \operatorname{add} T$. □

Let A be a finite dimensional K-algebra over a field K and T a tilting module in $\operatorname{mod} A$. Then the torsion pair $(\operatorname{Gen} T, \mathcal{F}(T)) = (\mathcal{T}(T), \operatorname{Cogen} \tau_A T)$ will be denoted by $(\mathcal{T}(T), \mathcal{F}(T))$ and called the *torsion pair induced* by the tilting module T.

We have the following important consequence of Theorem 2.5.

Corollary 2.6. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$, *and M a module in $\mathcal{T}(T)$. Then there is an exact sequence in* $\operatorname{mod} A$ *of the form*

$$\cdots \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with all modules T_i, $i \in \mathbb{N}$, from $\operatorname{add} T$.

Let A be a finite dimensional K-algebra over a field K, T a tilting module in $\operatorname{mod} A$, and M a module in $\mathcal{T}(T)$. An exact sequence in $\operatorname{mod} A$ of the form

$$\cdots \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with all modules T_i, $i \in \mathbb{N}$, in $\operatorname{add} T$ is called a *torsion resolution* of M, with respect to the tilting module T.

The following proposition is another important consequence of Theorem 2.5.

Proposition 2.7. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$, *and* $B = \operatorname{End}_A(T)$. *For a module M in* $\operatorname{mod} A$, *the following statements are equivalent:*

(i) $M \in \mathcal{T}(T)$.

(ii) *The canonical homomorphism* $\varepsilon_M \colon \operatorname{Hom}_A(T, M) \otimes_B T \to M$ *in* $\operatorname{mod} A$ *is an isomorphism.*

Proof. The implication (ii) ⇒ (i) is a direct consequence of Lemma 1.7 and Theorem 2.5. Let M be a module in $\mathcal{T}(T)$. Then we have in $\operatorname{mod} A$ two exact sequences

$$0 \longrightarrow L_0 \longrightarrow T_0 \longrightarrow M \longrightarrow 0,$$
$$0 \longrightarrow L_1 \longrightarrow T_1 \longrightarrow L_0 \longrightarrow 0,$$

with $T_0, T_1 \in \operatorname{add} T$ and $L_0, L_1 \in \operatorname{add} \mathcal{T}(T)$, by Theorem 2.5. Then, applying Theorem VII.3.2, we obtain exact sequences in $\operatorname{mod} K$, and hence in $\operatorname{mod} B$, of the forms

$$0 \longrightarrow \operatorname{Hom}_A(T, L_0) \longrightarrow \operatorname{Hom}_A(T, T_0) \longrightarrow \operatorname{Hom}_A(T, M) \longrightarrow \operatorname{Ext}^1_A(T, L_0) = 0,$$
$$0 \longrightarrow \operatorname{Hom}_A(T, L_1) \longrightarrow \operatorname{Hom}_A(T, T_1) \longrightarrow \operatorname{Hom}_A(T, L_0) \longrightarrow \operatorname{Ext}^1_A(T, L_1) = 0,$$

because $L_0, L_1 \in \mathcal{T}(T)$. In particular, we obtain an exact sequence in $\operatorname{mod} B$ of the form

$$\operatorname{Hom}_A(T, T_1) \longrightarrow \operatorname{Hom}_A(T, T_0) \longrightarrow \operatorname{Hom}_A(T, M) \longrightarrow 0.$$

Applying the right exact functor $- \otimes_B T \colon \operatorname{mod} B \to \operatorname{mod} A$, we obtain the following commutative diagram in $\operatorname{mod} A$ with exact rows:

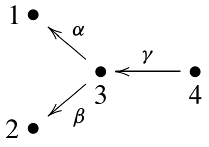
$$\begin{tikzcd}
\operatorname{Hom}_A(T, T_1) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_{T_1}"] & \operatorname{Hom}_A(T, T_0) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_{T_0}"] & \operatorname{Hom}_A(T, M) \otimes_B T \arrow[r] \arrow[d, "\varepsilon_M"] & 0 \\
T_1 \arrow[r] & T_0 \arrow[r] & M \arrow[r] & 0 .
\end{tikzcd}$$

Because ε_T is the composition of the canonical isomorphisms

$$\operatorname{Hom}_A(T, T) \otimes_B T \xrightarrow{\sim} B \otimes_B T \xrightarrow{\sim} T$$

and $T_0, T_1 \in \operatorname{add} T$, we conclude that $\varepsilon_{T_0}, \varepsilon_{T_1}$ are isomorphisms. Then ε_M is also an isomorphism. Therefore, (i) implies (ii). □

Example 2.8. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K. The Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
S_1 = P_1 \arrow[dr] & & P_3/S_1 \arrow[dr] & & I_1 \arrow[dr] & & \\
& P_3 \arrow[ur] \arrow[r] \arrow[dr] & P_4 \arrow[r] & M \arrow[ur] \arrow[r] \arrow[dr] & S_3 \arrow[r] & I_3 \arrow[r] & I_4 = S_4 , \\
S_2 = P_2 \arrow[ur] & & P_3/S_2 \arrow[ur] & & I_2 \arrow[ur] & &
\end{tikzcd}$$

where S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} A$ given by the vertex $i \in \{1, 2, 3, 4\}$ of Q, and M is the indecomposable module with the composition vector

$$c(M) = \big(c_1(M), c_2(M), c_3(M), c_4(M)\big) = (1, 1, 2, 1).$$

Consider the module $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4$ in $\operatorname{mod} A$ with $T_1 = S_1$, $T_2 = P_4$, $T_3 = P_3/S_2$, $T_4 = I_1$. We claim that T is a tilting module in $\operatorname{mod} A$. Clearly, $\operatorname{pd}_A T \leq 1$, because A is a hereditary K-algebra. Further, applying Corollary III.6.4, we conclude that

$$\operatorname{Ext}^1_A(T, T) \cong D \operatorname{Hom}_A(T, \tau_A T) = D \operatorname{Hom}_A \big(T, S_2 \oplus (P_3/S_1)\big) = 0.$$

Observe also that we have in $\operatorname{mod} A$ short exact sequences

$$0 \longrightarrow P_2 \longrightarrow P_4 \longrightarrow I_1 \longrightarrow 0,$$
$$0 \longrightarrow P_3 \longrightarrow P_4 \oplus (P_3/S_2) \longrightarrow I_1 \longrightarrow 0,$$

and consequently an exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with $T' = P_1 \oplus P_4 \oplus P_4 \oplus (P_3/S_2) \oplus P_4$ and $T'' = I_1 \oplus I_1$ in $\operatorname{add} T$, because $A = P_1 \oplus P_2 \oplus P_3 \oplus P_4$ in $\operatorname{mod} A$. Therefore, T is indeed a tilting module in $\operatorname{mod} A$. We determine now the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ induced by T. Since $\operatorname{pd}_A T \leq 1$, applying Corollary III.6.4, we conclude that

$$\begin{aligned} \mathcal{T}(T) &= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Ext}^1_A(T, M) = 0\big\} \\ &= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(M, \tau_A T) = 0\big\}. \end{aligned}$$

Moreover, $\tau_A T = S_2 \oplus (P_3/S_1)$. Hence,

$$\mathcal{T}(T) = \operatorname{add} \big(S_1 \oplus P_4 \oplus (P_3/S_2) \oplus M \oplus S_3 \oplus I_1 \oplus I_2 \oplus I_3 \oplus S_4\big).$$

On the other hand, we have

$$\mathcal{F}(T) = \{N \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, N) = 0\} = \operatorname{add}\big(S_2 \oplus (P_3/S_1)\big).$$

Observe that the projective module P_3 is neither in $\mathcal{T}(T)$, nor in $\mathcal{F}(T)$. We also note that the endomorphism algebra $B = \operatorname{End}_A(T)$ of T is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver

$$\begin{tikzcd}
& \overset{2}{\bullet} \arrow[dl, "\alpha"'] & \\
1\ \bullet & & \bullet\ 4 \arrow[ul, "\beta"'] \arrow[dl, "\sigma"] \\
& \underset{3}{\bullet} \arrow[ul, "\gamma"] &
\end{tikzcd}$$

and J is the ideal in the path algebra $K\Delta$ of Δ generated by $\beta\alpha - \sigma\gamma$, and hence B is not a hereditary K-algebra (see Theorem I.9.6). Here, the vertices $1, 2, 3, 4$ of the quiver Δ correspond to the indices of the indecomposable direct summands T_1, T_2, T_3, T_4 of T.

Example 2.9. Let K be a field, Q the quiver

$$\begin{tikzcd}
& \overset{2}{\bullet} \arrow[dl, "\alpha"'] & \\
1\ \bullet & & \bullet\ 5 \arrow[ul, "\beta"'] \arrow[dl, "\sigma"] \\
& 3\ \bullet \arrow[ul, "\gamma"] & \\
& & \bullet\ 4, \arrow[ul, "\delta"]
\end{tikzcd}$$

I the ideal in the path algebra KQ of Q generated by $\beta\alpha - \sigma\gamma$, and $A = KQ/I$ the associated bound quiver algebra. Then the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}[column sep=small]
& P_2 \arrow[dr] & & S_3 \arrow[dr] & & I_1 \arrow[dr] & & & \\
S_1 = P_1 \arrow[ur] \arrow[dr] & & \operatorname{rad} P_5 \arrow[ur] \arrow[r] \arrow[dr] & P_5 \arrow[r] & M \arrow[ur] \arrow[r] \arrow[dr] & P_4/S_1 \arrow[r] & I_1/S_1 \arrow[r] \arrow[dr] & I_2 \arrow[r] & I_5 = S_5 \\
& P_3 \arrow[ur] \arrow[dr] & & R \arrow[ur] \arrow[dr] & & P_5/S_1 \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & \\
& & P_4 \arrow[ur] & & S_2 \arrow[ur] & & P_5/P_2 \arrow[ur] & & I_4 = S_4
\end{tikzcd}$$

where S_i, P_i, I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} A$ given by the vertex $i \in \{1,2,3,4,5\}$ of Q, and M, R are the indecomposable modules with the composition vectors

$$\begin{aligned} c(M) &= \big(c_1(M), c_2(M), c_3(M), c_4(M), c_5(M)\big) = (1,1,2,1,1), \\ c(R) &= \big(c_1(R), c_2(R), c_3(R), c_4(R), c_5(R)\big) = (1,1,1,1,0). \end{aligned}$$

We note that A is not a hereditary K-algebra (see Theorem I.9.6). Let $T_1 = I_1$, $T_2 = P_4/S_1$, $T_3 = P_5/S_1$, $T_4 = I_1/S_1$, $T_5 = P_5/P_2$, and $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$. We claim that T is a tilting module in $\operatorname{mod} A$. Observe that we have in $\operatorname{mod} A$ the following minimal projective resolutions of the modules T_1, T_2, T_3, T_4, T_5:

$$\begin{gathered} 0 \longrightarrow P_3 \longrightarrow P_4 \oplus P_5 \longrightarrow T_1 \longrightarrow 0, \\ 0 \longrightarrow P_1 \longrightarrow P_4 \longrightarrow T_2 \longrightarrow 0, \\ 0 \longrightarrow P_1 \longrightarrow P_5 \longrightarrow T_3 \longrightarrow 0, \\ 0 \longrightarrow P_3 \oplus P_1 \longrightarrow P_4 \oplus P_5 \longrightarrow T_4 \longrightarrow 0, \\ 0 \longrightarrow P_2 \longrightarrow P_5 \longrightarrow T_5 \longrightarrow 0, \end{gathered}$$

and hence $\operatorname{pd}_A T = 1$. Then, applying Corollary III.6.4, we conclude that there are isomorphisms of K-vector spaces

$$\begin{aligned} \operatorname{Ext}^1_A(T,T) &\cong D\operatorname{Hom}_A(T, \tau_A T) \\ &\cong D\operatorname{Hom}_A(T, \tau_A T_1 \oplus \tau_A T_2 \oplus \tau_A T_3 \oplus \tau_A T_4 \oplus \tau_A T_5) \\ &= D\operatorname{Hom}_A(T, S_3 \oplus P_5 \oplus R \oplus M \oplus S_2) = 0. \end{aligned}$$

Further, we have in $\operatorname{mod} A$ the exact sequences

$$\begin{gathered} 0 \longrightarrow P_1 \longrightarrow I_1 \longrightarrow I_1/S_1 \longrightarrow 0, \\ 0 \longrightarrow P_2 \longrightarrow I_1 \oplus (P_5/S_1) \longrightarrow (I_1/S_1) \oplus (P_5/P_2) \longrightarrow 0, \\ 0 \longrightarrow P_3 \longrightarrow I_1 \oplus (P_4/S_1) \oplus (P_5/S_1) \longrightarrow (I_1/S_1) \oplus (I_1/S_1) \longrightarrow 0, \\ 0 \longrightarrow P_4 \longrightarrow I_1 \oplus (P_4/S_1) \longrightarrow I_1/S_1 \longrightarrow 0, \\ 0 \longrightarrow P_5 \longrightarrow I_1 \oplus (P_5/S_1) \longrightarrow I_1/S_1 \longrightarrow 0. \end{gathered}$$

Hence, there is in $\operatorname{mod} A$ an exact sequence

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with

$$T' = I_1 \oplus I_1 \oplus (P_5/S_1) \oplus I_1 \oplus (P_4/S_1) \oplus (P_5/S_1) \oplus I_1 \oplus (P_4/S_1) \oplus I_1 \oplus (P_5/S_1)$$

and

$$T'' = (I_1/S_1) \oplus (I_1/S_1) \oplus (P_5/P_2) \oplus (I_1/S_1) \oplus (I_1/S_1) \oplus (I_1/S_1) \oplus (I_1/S_1)$$

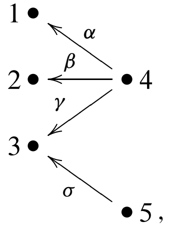
in add T. Therefore, T is indeed a tilting module in mod A. The associated torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A has the torsion class

$$\begin{aligned}\mathcal{T}(T) &= \{X \in \operatorname{mod} A \mid \operatorname{Hom}_A(X, \tau_A T) = 0\} \\ &= \{X \in \operatorname{mod} A \mid \operatorname{Hom}_A(X, S_3 \oplus P_5 \oplus R \oplus M \oplus S_2) = 0\} \\ &= \operatorname{add}(T \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5) = \operatorname{add}(T \oplus D(A)),\end{aligned}$$

and the torsion-free class

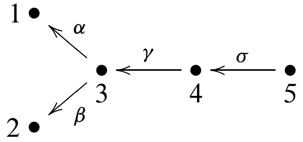
$$\begin{aligned}\mathcal{F}(T) &= \{Y \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, Y) = 0\} \\ &= \operatorname{add}(A \oplus \operatorname{rad} P_5 \oplus S_3 \oplus R \oplus S_2 \oplus M).\end{aligned}$$

Observe that every indecomposable module Z in mod A belongs to $\mathcal{T}(T)$ or $\mathcal{F}(T)$, and hence $(\mathcal{T}(T), \mathcal{F}(T))$ is a splitting torsion pair in mod A. Finally, we note that the endomorphism algebra $B = \operatorname{End}_A(T)$ of T is the path algebra $K\Delta$ of the quiver Δ of the form



and hence B is a hereditary K-algebra.

Example 2.10. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K. The Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
S_1 = P_1 \arrow[dr] & & P_3/S_1 \arrow[dr] & & X \arrow[dr] & & I_2 \arrow[dr] & & & \\
S_2 = P_2 \arrow[r] & P_3 \arrow[r] \arrow[ur] \arrow[dr] & P_3/S_2 \arrow[r] & M \arrow[r] \arrow[ur] \arrow[dr] & Y \arrow[r] & R \arrow[r] \arrow[ur] \arrow[dr] & I_1 \arrow[r] & I_3 \arrow[dr] & & \\
& & P_4 \arrow[ur] \arrow[dr] & & N \arrow[ur] \arrow[dr] & & Z \arrow[ur] \arrow[dr] & & I_4 \arrow[dr] & \\
& & & P_5 \arrow[ur] & & S_3 \arrow[ur] & & S_4 \arrow[ur] & & I_5 = S_5
\end{tikzcd}$$

where S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in mod B given by the vertex $i \in \{1, 2, 3, 4, 5\}$ of Q, and M, N, R, X, Y, Z are the indecomposable modules with the composition vectors

$$\begin{aligned} c(M) &= \big(c_1(M), c_2(M), c_3(M), c_4(M), c_5(M)\big) = (1, 1, 2, 1, 0), \\ c(N) &= \big(c_1(N), c_2(N), c_3(N), c_4(N), c_5(N)\big) = (1, 1, 2, 1, 1), \\ c(R) &= \big(c_1(R), c_2(R), c_3(R), c_4(R), c_5(R)\big) = (1, 1, 2, 2, 1), \\ c(X) &= \big(c_1(X), c_2(X), c_3(X), c_4(X), c_5(X)\big) = (1, 0, 1, 1, 0), \\ c(Y) &= \big(c_1(Y), c_2(Y), c_3(Y), c_4(Y), c_5(Y)\big) = (0, 1, 1, 1, 0), \\ c(Z) &= \big(c_1(Z), c_2(Z), c_3(Z), c_4(Z), c_5(Z)\big) = (0, 0, 1, 1, 0). \end{aligned}$$

Consider the module

$$T = (P_3/S_1) \oplus N \oplus I_2 \oplus I_5.$$

Then $\operatorname{pd}_A T \leq 1$, because A is a hereditary K-algebra. Moreover, applying Corollary III.6.4, we have isomorphisms in mod K

$$\operatorname{Ext}^1_A(T, T) \cong D\operatorname{Hom}_A(T, \tau_A T) = D\operatorname{Hom}_A(T, S_1 \oplus P_4 \oplus X \oplus S_4) = 0,$$

and hence T is a partial tilting module in mod A. It follows from Bongartz construction presented in the proof of Lemma 2.4 that there exists a module E in mod A such that $T \oplus E$ is a tilting module in mod A and there is a nonsplittable short exact sequence in mod A of the form

$$0 \longrightarrow A \longrightarrow E \longrightarrow T^d \longrightarrow 0,$$

where $d = \dim_K \mathcal{E}\mathrm{xt}^1_A(T, A)$. In fact, applying Lemma I.8.7, Proposition III.3.8 and Corollary III.6.4, we obtain the equalities

$$\begin{aligned} d &= \dim_K \operatorname{Ext}^1_A(T, A) = \dim_K \operatorname{Hom}_A(A, \tau_A T) \\ &= \dim_K \operatorname{Hom}_A(A, S_1 \oplus P_4 \oplus X \oplus S_4) \\ &= \dim_K S_1 + \dim_K P_4 + \dim_K X + \dim_K S_4 \\ &= 1 + 4 + 3 + 1 = 9, \end{aligned}$$

so E is a rather large module. We claim that T is not a tilting module in mod A. Indeed, the torsion class $\mathcal{T}(T) = \{L \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(T, L) = 0\} = \{L \in \operatorname{mod} A \mid \operatorname{Hom}_A(L, \tau_A T) = 0\}$ is the additive category

$$\operatorname{add}\big((P_3/S_1) \oplus \oplus N \oplus I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5 \oplus P_5 \oplus S_3\big),$$

while the torsion class $\operatorname{Gen} T$ of modules in mod A generated by T is the additive category

$$\operatorname{add}\big((P_3/S_1) \oplus N \oplus I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5 \oplus S_3\big).$$

Hence, $\operatorname{Gen} T \neq \mathcal{T}(T)$ and T is not a tilting module, by Theorem 2.5. Further, the indecomposable modules V in $\operatorname{mod} A$ with the properties $\operatorname{Hom}_A(V, \tau_A T) = 0$ and $\operatorname{Hom}_A(\tau_A^{-1} T, V) = 0$ are isomorphic to P_5 or S_3. Moreover, $P_5 = \tau_A S_3$ implies that $\operatorname{Ext}_A^1(S_3, P_5) \neq 0$, so P_5 and S_3 cannot be simultaneously direct summands of a tilting module in $\operatorname{mod} A$. Clearly, we have also $\operatorname{Ext}_A^1(P_5, P_5) = 0$ and $\operatorname{Ext}_A^1(S_3, S_3) \cong D \operatorname{Hom}_A(S_3, \tau_A S_3) = 0$.

Summing up, we conclude that the partial tilting module T has exactly two completions (up to isomorphism) to a tilting module in $\operatorname{mod} A$ that is a direct sum of pairwise nonisomorphic indecomposable modules, and these tilting modules are of the forms

$$\begin{aligned} T^{(1)} &= T \oplus P_3 = (P_3/S_1) \oplus N \oplus I_2 \oplus I_5 \oplus P_3, \\ T^{(2)} &= T \oplus S_3 = (P_3/S_1) \oplus N \oplus I_2 \oplus I_5 \oplus S_3. \end{aligned}$$

Finally, we note that $\operatorname{Hom}_A(S_3, T) = 0$, which implies that S_3 cannot be a direct summand of the tilting module $T \oplus E$ obtained by the Bongartz construction. Therefore, we obtain that P_3 is a direct summand of $T \oplus E$, equivalently of E, and hence $T \oplus E$ is a module in the additive category $\operatorname{add} T^{(1)}$ of the tilting module $T^{(1)}$.

Example 2.11. Let A be a finite dimensional K-algebra over a field K and S be a simple projective but not injective module in $\operatorname{mod} A$. Let $P_1, P_2, \dots, P_n$ be a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} A$. Clearly, then $S \cong P_m$ for some $m \in \{1, \dots, n\}$. We may assume that $S = P_1$. Consider the module

$$T_S = \tau_A^{-1} S \oplus \left(\bigoplus_{i=2}^{n} P_i \right).$$

We claim that T_S is a tilting module in $\operatorname{mod} A$. Since S is noninjective we have in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow S \longrightarrow P \longrightarrow \tau_A^{-1} S \longrightarrow 0,$$

where P is a projective module (Corollary III.9.7), and clearly S is not isomorphic to a direct summand of P. Since S is projective, we obtain $\operatorname{pd}_A \tau_A^{-1} S = 1$. This shows that $\operatorname{pd}_A T_S \leq 1$. Moreover, applying Corollary III.6.4, we conclude that

$$\operatorname{Ext}_A^1(T_S, T_S) \cong D \operatorname{Hom}_A(T_S, \tau_A T_S) = D \operatorname{Hom}_A(T_S, S) = 0.$$

Observe also that we have in $\operatorname{mod} A$ an exact sequence

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with T' and T'' in add T_S. Finally, we conclude that

$$\begin{aligned}\mathcal{T}(T_S) &= \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M, \tau_A T_S) = 0\}\\ &= \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M, S) = 0\}\end{aligned}$$

is the additive category given by all indecomposable modules in mod A which are nonisomorphic to S, and

$$\mathcal{F}(T_S) = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T_S, M) = 0\} = \operatorname{add}(S).$$

Clearly, $(\mathcal{T}(T_S), \mathcal{F}(T_S))$ is a splitting torsion pair in mod A.

The tilting module T_S was constructed by M. Auslander, M. I. Platzeck, and I. Reiten in [APR], and is called an APR-*tilting module*.

3 The Brenner–Butler theorem

Let A be a finite dimensional K-algebra over a field K, T a module in mod A, and $B = \operatorname{End}_A(T)$ the associated endomorphism K-algebra. Then T is a (B, A)-bimodule, because, for any $a \in A$, $b \in B$, $t \in T$, we have $b \cdot (ta) = b(ta) = b(t)a = (b \cdot t)a$. Hence we have the K-linear covariant functor

$$\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B.$$

The following simple lemma shows that this functor induces a canonical equivalence of the additive category add T of T and the category proj B of projective modules in mod B.

Lemma 3.1. *Let A be a finite dimensional K-algebra over a field K, T a module in* mod A, *and $B = \operatorname{End}_A(T)$. Then the following statements hold:*

(i) *For each module X in* add T *and each module M in* mod A, *we have a natural K-linear isomorphism*

$$\operatorname{Hom}_A(X, M) \xrightarrow{\sim} \operatorname{Hom}_B\big(\operatorname{Hom}_A(T, X), \operatorname{Hom}_A(T, M)\big)$$

which assigns to $f \in \operatorname{Hom}_A(X, M)$ the homomorphism

$$\operatorname{Hom}_A(T, f) \in \operatorname{Hom}_B\big(\operatorname{Hom}_A(T, X), \operatorname{Hom}_A(T, M)\big).$$

(ii) *The functor* $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ *induces an equivalence of categories*

$$\operatorname{add} T \xrightarrow{\sim} \operatorname{proj} B.$$

Proof. (i) Observe that for $X = T$ the defined K-linear homomorphism is the composition of the canonical isomorphisms

$$\begin{aligned} \operatorname{Hom}_A(T, M) &\xrightarrow{\sim} \operatorname{Hom}_B\big(B, \operatorname{Hom}_A(T, M)\big) \\ &\xrightarrow{\sim} \operatorname{Hom}_B\big(\operatorname{Hom}_A(T, T), \operatorname{Hom}_A(T, M)\big) \end{aligned}$$

Then the claim follows from the additivity of the functor $\operatorname{Hom}_A(T, -)$.

(ii) It follows from Proposition I.8.2 that an indecomposable module P in $\operatorname{mod} B$ is projective if and only if P is an indecomposable direct summand of the right B module $B = \operatorname{End}_A(T) = \operatorname{Hom}_A(T, T)$, and if and only if $P \cong \operatorname{Hom}_A(T, X)$ for some indecomposable direct summand X of T in $\operatorname{mod} A$. This shows that the K-linear functor $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ induces a dense functor

$$\operatorname{Hom}_A(T, -)\colon \operatorname{add} T \longrightarrow \operatorname{proj} B.$$

It follows from (i) that this functor is full and faithful, and consequently is an equivalence of categories. □

Proposition 3.2. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$*, and $B = \operatorname{End}_A(T)$. Then, for any modules M and N in the torsion class $\mathcal{T}(T)$ of* $\operatorname{mod} A$ *induced by T, there are natural K-linear isomorphisms:*

(i) $\operatorname{Hom}_A(M, N) \xrightarrow{\sim} \operatorname{Hom}_B(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, N))$.

(ii) $\operatorname{Ext}^1_A(M, N) \xrightarrow{\sim} \operatorname{Ext}^1_B(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, N))$.

Proof. Since M belongs to $\mathcal{T}(T)$, the module M admits a torsion resolution (see Corollary 2.6) in $\operatorname{mod} A$

$$\cdots \longrightarrow T_n \xrightarrow{d_n} T_{n-1} \longrightarrow \cdots \longrightarrow T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} M \longrightarrow 0$$

with $T_n \in \operatorname{add} T$ for all $n \in \mathbb{N}$. The exact sequence

$$T_1 \xrightarrow{d_1} T_0 \xrightarrow{d_0} M \longrightarrow 0$$

in $\operatorname{mod} A$ induces the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(M, N) \xrightarrow{\operatorname{Hom}_A(d_0, N)} \operatorname{Hom}_A(T_0, N) \xrightarrow{\operatorname{Hom}_A(d_1, N)} \operatorname{Hom}_A(T_1, N)$$

in $\operatorname{mod} K$, because the functor $\operatorname{Hom}_A(-, N)\colon \operatorname{mod} A \to \operatorname{mod} K$ is left exact. We have also the left exact functor $\operatorname{Hom}_B(-, \operatorname{Hom}_A(T, N))\colon \operatorname{mod} B \to \operatorname{mod} K$. Then, applying Lemma 3.1 (i), we obtain a commutative diagram in mod K with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\operatorname{Hom}_A(M,N) \arrow[r, "u"] \arrow[d] & \operatorname{Hom}_B(\operatorname{Hom}_A(T,M),\operatorname{Hom}_A(T,N)) \arrow[d] \\
\operatorname{Hom}_A(T_0,N) \arrow[r, "v"] \arrow[d] & \operatorname{Hom}_B(\operatorname{Hom}_A(T,T_0),\operatorname{Hom}_A(T,N)) \arrow[d] \\
\operatorname{Hom}_A(T_1,N) \arrow[r, "w"] & \operatorname{Hom}_B(\operatorname{Hom}_A(T,T_1),\operatorname{Hom}_A(T,N))
\end{tikzcd}$$

where v and w are isomorphisms. Then u is also an isomorphism. This shows (i).

Let $L_0 = \operatorname{Ker} d_0$. Then we have an exact sequence

$$0 \longrightarrow L_0 \xrightarrow{i_0} T_0 \xrightarrow{d_0} M \longrightarrow 0,$$

where i_0 is the inclusion homomorphism. Applying Theorem VII.3.3 to the module N and the above exact sequence, we obtain in mod K the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(M,N) \arrow[r, "\operatorname{Hom}_A(d_0,N)"] & \operatorname{Hom}_A(T_0,N) \arrow[r, "\operatorname{Hom}_A(i_0,N)"] & \operatorname{Hom}_A(L_0,N) \arrow[dll, rounded corners, "\delta_N^{L_0,M}"] \\
& \operatorname{Ext}_A^1(M,N) \arrow[r, "\operatorname{Ext}_A^1(d_0,N)"] & \operatorname{Ext}_A^1(T_0,N) \arrow[r, "\operatorname{Ext}_A^1(i_0,N)"] & \operatorname{Ext}_A^1(L_0,N)\,,
\end{tikzcd}$$

where $\operatorname{Ext}_A^1(T_0,N) = 0$, because $T_0 \in \operatorname{add} T$ and $N \in \mathcal{T}(T)$. Hence we conclude that $\operatorname{Ext}_A^1(M,N) \cong \operatorname{Coker}\operatorname{Hom}_A(i_0,N)$. We claim that there is a natural isomorphism in mod K

$$\operatorname{Coker}\operatorname{Hom}_A(i_0,N) \xrightarrow{\sim} \operatorname{Ker}\operatorname{Hom}_A(d_2,N)/\operatorname{Im}\operatorname{Hom}_A(d_1,N).$$

We have also in mod A an exact sequence

$$0 \longrightarrow L_1 \xrightarrow{i_1} T_1 \xrightarrow{p_1} L_0 \longrightarrow 0$$

such that $i_0 p_1 = d_1$, $L_1 = \operatorname{Ker} p_1$, and i_1 is the inclusion homomorphism, and an epimorphism $p_2\colon T_2 \to L_1$ with $d_2 = i_1 p_2$. Then $\operatorname{Hom}_A(d_2,N) = \operatorname{Hom}_A(p_2,N)\operatorname{Hom}_A(i_1,N)$ with $\operatorname{Hom}_A(p_2,N)$ a monomorphism, because p_2 is an epimorphism. Hence we obtain that $\operatorname{Ker}\operatorname{Hom}_A(d_2,N) = \operatorname{Ker}\operatorname{Hom}_A(i_1,N)$. Moreover, we have in mod K an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(L_0,N) \xrightarrow{\operatorname{Hom}_A(p_1,N)} \operatorname{Hom}_A(T_1,N) \xrightarrow{\operatorname{Hom}_A(i_1,N)} \operatorname{Hom}_A(L_1,N),$$

and hence $\operatorname{Ker}\operatorname{Hom}_A(i_1, N) = \operatorname{Im}\operatorname{Hom}_A(p_1, N)$. Observe also that $\operatorname{Hom}_A(d_1, N) = \operatorname{Im}\operatorname{Hom}_A(p_1, N)\operatorname{Hom}_A(i_0, N)$. Therefore, we have in mod A a commutative diagram

$$\begin{tikzcd}
 & \operatorname{Hom}_A(T_0, N) \arrow[dl, "\operatorname{Hom}_A(i_0,N)"'] \arrow[dr, "\operatorname{Hom}_A(d_1,N)"] & \\
\operatorname{Hom}_A(L_0, N) \arrow[rr, "\operatorname{Hom}_A(p_1,N)"'] & & \operatorname{Ker}\operatorname{Hom}_A(d_2, N)\,,
\end{tikzcd}$$

where the horizontal homomorphism is an isomorphism. This implies that there exists a required natural isomorphism

$$\operatorname{Coker}\operatorname{Hom}_A(i_0, N) \cong \operatorname{Ker}\operatorname{Hom}_A(d_2, N)/\operatorname{Im}\operatorname{Hom}_A(d_1, N)$$

in mod K.

The torsion resolution of M induces an infinite sequence in mod B of the form

$$\cdots \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} \operatorname{Hom}_A(T, M) \longrightarrow 0,$$

where $P_n = \operatorname{Hom}_A(T, T_n)$ and $\partial_n = \operatorname{Hom}_A(T, d_n)$, for any $n \in \mathbb{N}$. We claim that this is a projective resolution of $\operatorname{Hom}_A(T, M)$ in mod B. It follows from Lemma 3.1 (ii) that P_n, $n \in \mathbb{N}$, are modules in proj B. For any $n \in \mathbb{N}$, we have in mod A a short exact sequence

$$0 \longrightarrow L_n \xrightarrow{i_n} T_n \xrightarrow{p_n} L_{n-1} \longrightarrow 0$$

such that $L_{-1} = M$, $L_n = \operatorname{Ker} p_n = \operatorname{Ker} d_n$, $i_{n-1}p_n = d_n$. Observe that $L_n = \operatorname{Ker} d_n = \operatorname{Im} d_{n+1}$ belongs to $\mathcal{T}(T)$, and hence $\operatorname{Ext}^1_A(T, L_n) = 0$, for any $n \in \mathbb{N}$. Applying Theorem VII.3.2, we obtain in mod B the short exact sequence

$$0 \to \operatorname{Hom}_A(T, L_n) \xrightarrow{\operatorname{Hom}_A(T,i_n)} \operatorname{Hom}_A(T, T_n) \xrightarrow{\operatorname{Hom}_A(T,p_n)} \operatorname{Hom}_A(T, L_{n-1}) \to 0,$$

for any $n \in \mathbb{N}$. Since $\partial_n = \operatorname{Hom}_A(T, d_n) = \operatorname{Hom}_A(T, i_{n-1})\operatorname{Hom}_A(T, p_n)$, we conclude that $\operatorname{Ker}\partial_n = \operatorname{Ker}\operatorname{Hom}_A(T, p_n) = \operatorname{Im}\operatorname{Hom}_A(T, i_n) = \operatorname{Im}\partial_{n+1}$, for any $n \in \mathbb{N}$, as required. Therefore,

$$\begin{aligned}
&\operatorname{Ext}^1_B\big(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, N)\big)\\
&\qquad \cong \operatorname{Ker}\operatorname{Hom}_B\big(\partial_2, \operatorname{Hom}_A(T, N)\big)/\operatorname{Im}\operatorname{Hom}_B\big(\partial_1, \operatorname{Hom}_A(T, N)\big)
\end{aligned}$$

in mod K. On the other hand, it follows from Lemma 3.1 (i) that we have in mod K the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(T_0, N) \arrow[r, "u_0"] \arrow[d, "\operatorname{Hom}_A(d_1,N)"] & \operatorname{Hom}_B(\operatorname{Hom}_A(T, T_0), \operatorname{Hom}_A(T, N)) \arrow[d, "\operatorname{Hom}_B(\partial_1,\operatorname{Hom}_A(T,N))"] \\
\operatorname{Hom}_A(T_1, N) \arrow[r, "u_1"] \arrow[d, "\operatorname{Hom}_A(d_2,N)"] & \operatorname{Hom}_B(\operatorname{Hom}_A(T, T_1), \operatorname{Hom}_A(T, N)) \arrow[d, "\operatorname{Hom}_B(\partial_2,\operatorname{Hom}_A(T,N))"] \\
\operatorname{Hom}_A(T_2, N) \arrow[r, "u_2"] & \operatorname{Hom}_B(\operatorname{Hom}_A(T, T_2), \operatorname{Hom}_A(T, N))\,,
\end{tikzcd}$$

where the horizontal homomorphisms u_0, u_1, u_2 are isomorphisms. Then we obtain a K-linear isomorphism

$$\begin{aligned}&\operatorname{Ker}\operatorname{Hom}_A(d_2, N)/\operatorname{Im}\operatorname{Hom}_A(d_1, N)\\ &\quad\xrightarrow{\sim} \operatorname{Ker}\operatorname{Hom}_B\big(\partial_2, \operatorname{Hom}_A(T, N)\big)/\operatorname{Im}\operatorname{Hom}_B\big(\partial_1, \operatorname{Hom}_A(T, N)\big).\end{aligned}$$

Summing up, we obtain a required natural K-linear isomorphism

$$\operatorname{Ext}_A^1(M, N) \xrightarrow{\sim} \operatorname{Ext}_B^1\big(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, N)\big).$$

This shows (ii). □

Proposition 3.3. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$*, and $B = \operatorname{End}_A(T)$. The following statements hold:*

(i) *There is a canonical isomorphism $D(T) \xrightarrow{\sim} \operatorname{Hom}_A(T, D(A))$ of right B-modules.*

(ii) *T is a tilting module in* $\operatorname{mod} B^{\mathrm{op}}$.

(iii) *There is a canonical isomorphism of K-algebras*

$$\varrho\colon A \xrightarrow{\sim} \operatorname{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$$

such that $\varrho(a)(t) = ta$ for $a \in A$ and $t \in T$.

Proof. (i) Observe first that $T \cong T \otimes_A A$ in $\operatorname{mod} B^{\mathrm{op}}$, by Lemma II.3.5. Then, applying the duality $D = \operatorname{Hom}_K(-, K)\colon \operatorname{mod} B^{\mathrm{op}} \to \operatorname{mod} B$ and the adjoint theorem Theorem II.4.3, we obtain isomorphisms

$$D(T) \xrightarrow{\sim} D(T \otimes_A A) \xrightarrow{\sim} \operatorname{Hom}_A(T, D(A))$$

in $\operatorname{mod} B$.

(ii) We verify the axioms of tilting module for the left B-module T.

(T1) Since T is a tilting module in $\operatorname{mod} A$, there is a short exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow A \longrightarrow T' \longrightarrow T'' \longrightarrow 0$$

with T', T'' in $\operatorname{add} T$. Applying the contravariant functor $\operatorname{Hom}_A(-, T)\colon \operatorname{mod} A \to \operatorname{mod} B^{\mathrm{op}}$, given by the (B, A)-module T, and Theorem VII.3.3, we obtain a short exact sequence in $\operatorname{mod} K$, and hence in $\operatorname{mod} B^{\mathrm{op}}$, of the form

$$0 \longrightarrow \operatorname{Hom}_A(T'', T) \longrightarrow \operatorname{Hom}_A(T', T) \longrightarrow \operatorname{Hom}_A(A, T) \longrightarrow \operatorname{Ext}_A^1(T'', T)$$

where $\operatorname{Ext}_A^1(T'', T) = 0$, because $T'' \in \operatorname{add} T$ and $\operatorname{Ext}_A^1(T, T) = 0$. Further, $\operatorname{Hom}_A(A, T) \cong T$ in $\operatorname{mod} A$, by Lemma I.8.7, and $\operatorname{Hom}_A(T', T)$, $\operatorname{Hom}_A(T'', T)$

are in the additive category of the left B-module $B = \mathrm{End}_A(T)$. Therefore, we obtain a projective resolution

$$0 \longrightarrow \mathrm{Hom}_A(T'', T) \longrightarrow \mathrm{Hom}_A(T', T) \longrightarrow T \longrightarrow 0$$

of T in mod B^{op}, and so $\mathrm{pd}_{B^{\mathrm{op}}} T \leq 1$.

(T2) Observe that $\mathrm{Ext}^1_A(T, D(A)) = 0$, because $D(A)$ is injective in mod A, and hence $D(A)$ belongs to $\mathcal{T}(T)$. Applying now Proposition 3.2 we obtain isomorphisms of K-vector spaces

$$\begin{aligned}\mathrm{Ext}^1_B(D(T), D(T)) &\cong \mathrm{Ext}^1_B\left(\mathrm{Hom}_A(T, D(A)), \mathrm{Hom}_A(T, D(A))\right)\\ &\cong \mathrm{Ext}^1_A(D(A), D(A)) = 0.\end{aligned}$$

Further, applying Corollary III.3.6, we obtain isomorphisms of K-vector spaces

$$\mathrm{Ext}^1_{B^{\mathrm{op}}}(T, T) \cong \mathcal{E}\mathrm{xt}^1_{B^{\mathrm{op}}}(T, T) \cong \mathcal{E}\mathrm{xt}^1_B(D(T), D(T)) \cong \mathrm{Ext}^1_B(D(T), D(T)).$$

Therefore, we conclude that $\mathrm{Ext}^1_{B^{\mathrm{op}}}(T, T) = 0$.

(T3) Since $\mathrm{pd}_A T \leq 1$, we have in mod A a projective resolution of T

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow T \longrightarrow 0.$$

Applying the contravariant functor $\mathrm{Hom}_A(-, T)\colon \mathrm{mod}\, A \to \mathrm{mod}\, B^{\mathrm{op}}$ again, and Theorem VII.3.3, we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(T, T) \longrightarrow \mathrm{Hom}_A(P_0, T) \longrightarrow \mathrm{Hom}_A(P_1, T) \longrightarrow 0$$

in mod B^{op}, because $\mathrm{Ext}^1_A(T, T) = 0$. Observe also that $\mathrm{Hom}_A(T, T) = B$ and $\mathrm{Hom}_A(P_0, T)$, $\mathrm{Hom}_A(P_1, T)$ belong to add T, since P_0 and P_1 are direct summands of free right A-modules A^m and A^n, respectively. Therefore, the right B^{op}-module satisfies the axiom (T3).

Thus, we proved that T is a tilting module in mod B^{op}.

(iii) Consider the K-linear homomorphism

$$\varrho\colon A \longrightarrow \mathrm{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$$

defined by $\varrho(a)(t) = ta$ for $a \in A$ and $t \in T$. Observe that, for $a \in A$, $b \in B$, and $t \in T$, we have $\varrho(a)(bt) = \varrho(a)(b(t)) = b(t)a = b(ta) = (b\varrho(a))(t)$, and hence $\varrho(a) \in \mathrm{End}_{B^{\mathrm{op}}}(T)$. Moreover, for $a_1, a_2 \in A$ and $t \in T$, we have $\varrho(a_1a_2)(t) = t(a_1a_2) = (ta_1)a_2 = \varrho(a_2)(\varrho(a_1)(t))$, and hence $\varrho(a_1a_2) = \varrho(a_1)\varrho(a_2)$ in $\mathrm{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$. Therefore, ϱ is a homomorphism of K-algebras. Further, if $\varrho(a) = 0$ for some $a \in A$, then $Ta = 0$, and consequently $a = 0$, because T is a faithful right A-module. This shows that ϱ is a monomorphism. On the other hand, since the injective module $D(A)$ belongs to $\mathcal{T}(T)$, applying Proposition 3.2 we conclude that there are K-linear isomorphisms

$$\begin{aligned}A \cong A^{\mathrm{op}} &\cong \mathrm{End}_{A^{\mathrm{op}}}(A^{\mathrm{op}}) \cong \mathrm{End}_A(D(A))\\ &\cong \mathrm{End}_B\left(\mathrm{Hom}_A(T, D(A))\right) \cong \mathrm{End}_B(D(T)) \cong \mathrm{End}_{B^{\mathrm{op}}}(T),\end{aligned}$$

and hence $\dim_K A = \dim_K \operatorname{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$. Thus ϱ is an isomorphism of K-algebras. $\square$

Let A be a finite dimensional K-algebra, T a tilting module in $\operatorname{mod} A$, and $B = \operatorname{End}_A(T)$. Since $T = {}_BT$ is, by Proposition 3.3, a tilting module in $\operatorname{mod} B^{\mathrm{op}}$, we have in $\operatorname{mod} B^{\mathrm{op}}$ the associated torsion pair $(\mathcal{T}({}_BT), \mathcal{F}({}_BT))$, where

$$\begin{aligned}\mathcal{T}({}_BT) &= \{U \in \operatorname{mod} B^{\mathrm{op}} \mid \operatorname{Ext}^1_{B^{\mathrm{op}}}({}_BT, U) = 0\} = \operatorname{Gen} {}_BT,\\ \mathcal{F}({}_BT) &= \{V \in \operatorname{mod} B^{\mathrm{op}} \mid \operatorname{Hom}_{B^{\mathrm{op}}}({}_BT, V) = 0\} = \operatorname{Cogen} \tau_{B^{\mathrm{op}}} {}_BT\end{aligned}$$

(see Theorem 2.5). Then we have the induced torsion pair in the category $\operatorname{mod} B$ of finite dimensional right B-modules (see Example 1.1 (c)), $(D(\mathcal{F}({}_BT)), D(\mathcal{T}({}_BT)))$. Observe that

$$\begin{aligned}D(\mathcal{F}({}_BT)) &= \{X \in \operatorname{mod} B \mid D(X) \in \mathcal{F}({}_BT)\}\\ &= \{X \in \operatorname{mod} B \mid \operatorname{Hom}_{B^{\mathrm{op}}}({}_BT, D(X)) = 0\}\\ &= \{X \in \operatorname{mod} B \mid \operatorname{Hom}_B(X, D({}_BT)) = 0\},\\ D(\mathcal{T}({}_BT)) &= \{Y \in \operatorname{mod} B \mid D(Y) \in \mathcal{T}({}_BT)\}\\ &= \{Y \in \operatorname{mod} B \mid \operatorname{Ext}^1_{B^{\mathrm{op}}}({}_BT, D(Y)) = 0\}\\ &= \{Y \in \operatorname{mod} B \mid \operatorname{Ext}^1_B(Y, D({}_BT)) = 0\}.\end{aligned}$$

We will give now an alternative description of the classes $(D(\mathcal{F}({}_BT))$ and $D(\mathcal{T}({}_BT)))$.

Let B be a finite dimensional K-algebra over a field K and N be a module in $\operatorname{mod} B^{\mathrm{op}}$. Then we have the covariant functor

$$- \otimes_B N \colon \operatorname{mod} B \longrightarrow \operatorname{mod} K$$

(see Section II.3). Let M be a module in $\operatorname{mod} B$ and consider a minimal projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

of M in $\operatorname{mod} B$ (see Proposition I.8.30). Application of the functor $- \otimes_B N$ yields the chain of K-vector spaces

$$P_2 \otimes_B N \xrightarrow{d_2 \otimes N} P_1 \otimes_B N \xrightarrow{d_1 \otimes N} P_0 \otimes_B N$$

with $(d_1 \otimes_B N)(d_2 \otimes_B N) = (d_1 d_2) \otimes_B N = 0$. This allows us to define the K-vector space

$$\operatorname{Tor}_1^B(M, N) = \operatorname{Ker}(d_1 \otimes_B N) / \operatorname{Im}(d_2 \otimes_B N).$$

Observe that, by Proposition I.8.30 and Lemma I.8.31, $\mathrm{Tor}_1^B(M,N)$ is well defined (does not depend on the choice of minimal projective resolution). Let $u\colon M \to L$ be a homomorphism in $\mathrm{mod}\,B$. Consider also a minimal projective resolution

$$\cdots \longrightarrow P_n^* \xrightarrow{d_n^*} P_{n-1}^* \longrightarrow \cdots \longrightarrow P_2^* \xrightarrow{d_2^*} P_1^* \xrightarrow{d_1^*} P_0^* \xrightarrow{d_0^*} L \longrightarrow 0$$

of L in $\mathrm{mod}\,B$. Then invoking the projectivity of the modules P_i, $i \geq 0$, and the exactness of the projective resolution of L, we conclude that there exists a commutative diagram in $\mathrm{mod}\,A$ of the form

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r, "d_n"] \arrow[d, "u_n"] & P_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "u_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "u_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "u_0"] & M \arrow[d, "u"] \\
\cdots \arrow[r] & P_n^* \arrow[r, "d_n^*"] & P_{n-1}^* \arrow[r] & \cdots \arrow[r] & P_2^* \arrow[r, "d_2^*"] & P_1^* \arrow[r, "d_1^*"] & P_0^* \arrow[r, "d_0^*"] & L\,.
\end{tikzcd}$$

Then we have the commutative diagram in $\mathrm{mod}\,K$,

$$\begin{tikzcd}
P_2 \otimes N \arrow[r, "d_2\otimes N"] \arrow[d, "u_2\otimes N"] & P_1 \otimes N \arrow[r, "d_1\otimes N"] \arrow[d, "u_1\otimes N"] & P_0 \otimes N \arrow[d, "u_0\otimes N"] \\
P_2^* \otimes N \arrow[r, "d_2^*\otimes N"] & P_1^* \otimes N \arrow[r, "d_1^*\otimes N"] & P_0^* \otimes N\,,
\end{tikzcd}$$

from which we infer that

$$(u_1 \otimes N)(\mathrm{Ker}(d_1 \otimes N)) \subseteq \mathrm{Ker}(d_1^* \otimes N),$$
$$(u_1 \otimes N)(\mathrm{Im}(d_2 \otimes N)) \subseteq \mathrm{Im}(d_2^* \otimes N).$$

We can now define the K-linear homomorphism

$$\mathrm{Tor}_1^B(u,N)\colon \mathrm{Tor}_1^B(M,N) \longrightarrow \mathrm{Tor}_1^B(L,N)$$

by $\mathrm{Tor}_1^B(u,N)((p_1 \otimes n) + \mathrm{Im}(d_2 \otimes N)) = u_1(p_1) \otimes n + \mathrm{Im}(d_2^* \otimes n)$ for $p_1 \otimes n \in P_1 \otimes N$ with $d_1(p_1) \otimes n = (d_1 \otimes N)(p_1 \otimes n) = 0$. The homomorphism $u_1\colon P_1 \to P_1^*$ occurring in the above commutative diagram of projective resolutions is not uniquely determined by the homomorphism $u\colon M \to L$. Assume $u_n' \in \mathrm{Hom}_B(P_n, P_n^*)$, $n \geq 0$, is another family of homomorphisms such that $d_0^* u_0' = u d_0$ and $d_n^* u_n' = u_{n-1}' d_n$ for $n \geq 1$. Then we have $d_0^*(u_0 - u_0') = ud_0 - ud_0 = 0$, and hence $\mathrm{Im}(u_0 - u_0') \subseteq \mathrm{Ker}\, d_0^* = \mathrm{Im}\, d_1^*$. Consequently, by the projectivity of P_0, there exists a homomorphism $s_0 \in \mathrm{Hom}_B(P_0, P_1^*)$ such that $u_0 - u_0' = d_1^* s_0$. Moreover, we have

$$\begin{aligned} d_1^*(u_1 - u_1' - s_0 d_1) &= d_1^* u_1 - d_1^* u_1' - d_1^* s_0 d_1 \\ &= d_1^* u_1 - d_1^* u_1' - u_0 d_1 + u_0' d_1 \\ &= (d_1^* u_1 - u_0 d_1) - (d_1^* u_1' - u_0' d_1) = 0, \end{aligned}$$

and so $\operatorname{Im}(u_1 - u'_1 - s_0 d_1) \subseteq \operatorname{Ker} d_1^* = \operatorname{Im} d_2^*$. Hence, by the projectivity of P_1, there exists a homomorphism $s_1 \in \operatorname{Hom}_B(P_1, P_2^*)$ such that $u_1 - u'_1 - s_0 d_1 = d_2^* s_1$, or equivalently, $u_1 - u'_1 = d_2^* s_1 + s_0 d_1$. Then, for any $p_1 \otimes n \in \operatorname{Ker}(d_1 \otimes N)$, we have

$$\begin{aligned}
&\big(u_1(p_1) \otimes n\big) - \big(u'_1(p_1) \otimes n\big) \\
&\quad = \big(u_1(p_1) - u'_1(p_1)\big) \otimes n = (u_1 - u'_1)(p_1) \otimes n \\
&\quad = (d_2^* s_1 + s_0 d_1)(p_1) \otimes n = (d_2^* s_1)(p_1) \otimes n + (s_0 d_1)(p_1) \otimes n \\
&\quad = (d_2^* \otimes N)\big((s_1 \otimes N)(p_1 \otimes n)\big) + (s_0 \otimes N)\big((d_1 \otimes N)(p_1 \otimes n)\big) \\
&\quad = (d_2^* \otimes N)(s_1(p_1) \otimes n),
\end{aligned}$$

and hence $u_1(p_1) \otimes n + \operatorname{Im}(d_2^* \otimes N) = u'_1(p_1) \otimes n + \operatorname{Im}(d_2^* \otimes N)$. As a consequence, the homomorphism $\operatorname{Tor}_1^B(u, N)$ does not depend on the choice of homomorphisms $u_n \colon P_n \to P_n^*$, $n \geq 0$. Moreover, we have $\operatorname{Tor}_1^B(vu, N) = \operatorname{Tor}_1^B(v, N)\operatorname{Tor}_1^B(u, N)$ for $u \in \operatorname{Hom}_B(M, L)$ and $v \in \operatorname{Hom}_B(L, R)$, and clearly $\operatorname{Tor}_1^B(\mathrm{id}_M, N) = \mathrm{id}_{\operatorname{Tor}_1^B(M,N)}$. This shows that we have the covariant functor

$$\operatorname{Tor}_1^B(-, N) \colon \operatorname{mod} B \longrightarrow \operatorname{mod} K.$$

We also note that, for a module N in $\operatorname{mod} B^{\mathrm{op}}$, $D(N) = \operatorname{Hom}_K(N, K)$ is a module in $\operatorname{mod} B$, and hence we may consider the covariant functor

$$D \operatorname{Ext}_B^1(-, D(N)) \colon \operatorname{mod} B \longrightarrow \operatorname{mod} K.$$

Lemma 3.4. *Let B be a finite dimensional K-algebra over a field K and N be a module in* $\operatorname{mod} B^{\mathrm{op}}$. *Then the functors*

$$\operatorname{Tor}_1^B(-, N),\ D \operatorname{Ext}_B^1(-, D(N)) \colon \operatorname{mod} B \longrightarrow \operatorname{mod} K$$

are naturally isomorphic.

Proof. It follows from the adjoint theorem Theorem II.4.3 that, for any module M in $\operatorname{mod} B$, there exists a natural K-linear isomorphism

$$\varphi_N(M) \colon \operatorname{Hom}_K(M \otimes_B N, K) \longrightarrow \operatorname{Hom}_B(M, \operatorname{Hom}_K(N, K)),$$

and hence a natural K-linear isomorphism

$$D\big(\varphi_N(M)^{-1}\big) \colon M \otimes_B N \longrightarrow D \operatorname{Hom}_B(M, D(N)).$$

This leads to a natural isomorphism of covariant functors

$$\psi_N \colon - \otimes_B N \longrightarrow D \operatorname{Hom}_B(-, D(N))$$

from $\operatorname{mod} B$ to $\operatorname{mod} K$ such that $\psi_N(M) = D(\varphi_N(M)^{-1})$ for any module M in $\operatorname{mod} B$.

Let M be a module in $\operatorname{mod} B$ and

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

be a minimal projective resolution of M in $\operatorname{mod} B$. Then we obtain the commutative diagram in $\operatorname{mod} K$ of the form

$$\begin{tikzcd}
P_2 \otimes_B N \arrow[r, "d_2 \otimes N"] \arrow[d, "\psi_N(P_2)"] & P_1 \otimes_B N \arrow[r, "d_1 \otimes N"] \arrow[d, "\psi_N(P_1)"] & P_0 \otimes_B N \arrow[d, "\psi_N(P_0)"] \\
D\big(\operatorname{Hom}_B(P_2, D(N))\big) \arrow[r, "D \operatorname{Hom}_B(d_2, D(N))"] & D\big(\operatorname{Hom}_B(P_1, D(N))\big) \arrow[r, "D \operatorname{Hom}_B(d_1, D(N))"] & D\big(\operatorname{Hom}_B(P_0, D(N))\big)
\end{tikzcd}$$

where the vertical homomorphisms are isomorphisms. This leads to isomorphisms of K-vector spaces

$$\begin{aligned}
\operatorname{Tor}_1^B(M, N) &\cong \operatorname{Ker}(d_1 \otimes_B N)/\operatorname{Im}(d_2 \otimes_B N) \\
&\cong \operatorname{Ker} D \operatorname{Hom}_B(d_1, D(N))/\operatorname{Im} D \operatorname{Hom}_B(d_2, D(N)) \\
&\cong D \operatorname{Ext}_B^1(M, D(N)),
\end{aligned}$$

because $\operatorname{Ext}_B^1(M, D(N)) \cong \operatorname{Ker} \operatorname{Hom}_B(d_2, D(N))/\operatorname{Im} \operatorname{Hom}_B(d_1, D(N))$. Moreover, for any homomorphism $u\colon M \to L$ in $\operatorname{mod} B$, a minimal projective resolution

$$\cdots \longrightarrow P_n^* \xrightarrow{d_n^*} P_{n-1}^* \longrightarrow \cdots \longrightarrow P_2^* \xrightarrow{d_2^*} P_1^* \xrightarrow{d_1^*} P_0^* \xrightarrow{d_0^*} L \longrightarrow 0$$

of L in $\operatorname{mod} B$, and homomorphisms $u_n \in \operatorname{Hom}_B(P_n, P_n^*)$, $n \geq 0$, such that $d_0^* u_0 = u d_0$ and $d_n^* u_n = u_{n-1} d_n$ for $n \geq 1$, we have the equalities

$$\psi_N(P_i^*)(u_i \otimes N) = D\big(\operatorname{Hom}_B(u_i, D(N))\big)\psi_N(P_i),$$

for $i \in \{0, 1, 2\}$. Therefore, we may assign to each module M in $\operatorname{mod} B$ a K-linear isomorphism

$$\xi_N(M)\colon \operatorname{Tor}_1^B(M, N) \longrightarrow D \operatorname{Ext}_B^1(M, D(N))$$

such that for any homomorphism $u\colon M \to L$ in $\operatorname{mod} B$ the diagram of K-linear homomorphisms

$$\begin{tikzcd}
\operatorname{Tor}_1^B(M, N) \arrow[r, "\xi_N(M)"] \arrow[d, "\operatorname{Tor}_1^B(u, N)"] & D \operatorname{Ext}_B^1(M, D(N)) \arrow[d, "D \operatorname{Ext}_B^1(u, D(N))"] \\
\operatorname{Tor}_1^B(L, N) \arrow[r, "\xi_N(L)"] & D \operatorname{Ext}_B^1(L, D(N))
\end{tikzcd}$$

is commutative. This shows that we have the natural isomorphism

$$\xi_N\colon \operatorname{Tor}_1^B(-,N) \longrightarrow D\operatorname{Ext}_B^1(-,D(N))$$

of covariant functors from $\operatorname{mod} B$ to $\operatorname{mod} K$. □

Proposition 3.5. *Let B be a finite dimensional K-algebra over a field K, N a module in* $\operatorname{mod} B^{\mathrm{op}}$*, and*

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$$

an exact sequence in $\operatorname{mod} B$*. Then there exists an exact sequence in* $\operatorname{mod} K$ *of the form*

$$\begin{tikzcd}
\operatorname{Tor}_1^B(M_1,N) \arrow[r, "\operatorname{Tor}_1^B(f_1,N)"] & \operatorname{Tor}_1^B(M_2,N) \arrow[r, "\operatorname{Tor}_1^B(f_2,N)"] & \operatorname{Tor}_1^B(M_3,N) \arrow[dll, rounded corners, "\sigma_N^{M_3,M_1}"'] \\
M_1 \otimes_B N \arrow[r, "f_1\otimes N"] & M_2 \otimes_B N \arrow[r, "f_2\otimes N"] & M_3 \otimes_B N \arrow[r] & 0\,.
\end{tikzcd}$$

Moreover, if $\operatorname{pd}_{B^{\mathrm{op}}} N \leq 1$*, then* $\operatorname{Tor}_1^B(f_1,N)$ *is a monomorphism.*

Proof. It follows from Theorem VII.3.3 that there exists an exact sequence in $\operatorname{mod} K$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_B(M_3,D(N)) \arrow[r, "\operatorname{Hom}_B(f_2,D(N))"] & \operatorname{Hom}_B(M_2,D(N)) \arrow[r, "\operatorname{Hom}_B(f_1,D(N))"] & \operatorname{Hom}_B(M_1,D(N)) \arrow[dll, rounded corners, "\delta_{D(N)}^{M_1,M_3}"'] \\
& \operatorname{Ext}_B^1(M_3,D(N)) \arrow[r, "\operatorname{Ext}_B^1(f_2,D(N))"] & \operatorname{Ext}_B^1(M_2,D(N)) \arrow[r, "\operatorname{Ext}_B^1(f_1,D(N))"] & \operatorname{Ext}_B^1(M_1,D(N))\,.
\end{tikzcd}$$

Moreover, if $\operatorname{pd}_{B^{\mathrm{op}}} N \leq 1$, then $\operatorname{id}_B D(N) \leq 1$, and hence $\operatorname{Ext}_B^1(f_1,D(N))$ is an epimorphism.

Applying the duality functor $D = \operatorname{Hom}_K(-,K)\colon \operatorname{mod} K \to \operatorname{mod} K$ we obtain the exact sequence of the form

$$\begin{tikzcd}
D\operatorname{Ext}_B^1(M_1,D(N)) \arrow[r, "D\operatorname{Ext}_B^1(f_1,D(N))"] & D\operatorname{Ext}_B^1(M_2,D(N)) \arrow[r, "D\operatorname{Ext}_B^1(f_2,D(N))"] & D\operatorname{Ext}_B^1(M_3,D(N)) \arrow[dll, rounded corners, "D\left(\delta_{D(N)}^{M_1,M_3}\right)"'] \\
D\operatorname{Hom}_B(M_1,D(N)) \arrow[r, "D\operatorname{Hom}_B(f_1,D(N))"] & D\operatorname{Hom}_B(M_2,D(N)) \arrow[r, "D\operatorname{Hom}_B(f_2,D(N))"] & D\operatorname{Hom}_B(M_3,D(N)) \arrow[r] & 0\,.
\end{tikzcd}$$

Using the natural isomorphism of functors $\psi_N\colon - \otimes_B N \to D\operatorname{Hom}_B(-,D(N))$ from $\operatorname{mod} B$ to $\operatorname{mod} K$, we obtain the commutative diagram in $\operatorname{mod} K$

$$\begin{tikzcd}
M_1 \otimes_B N \arrow[r, "f_1\otimes N"] \arrow[d, "\psi_N(M_1)"] & M_2 \otimes_B N \arrow[r, "f_2\otimes N"] \arrow[d, "\psi_N(M_2)"] & M_3 \otimes_B N \arrow[r] \arrow[d, "\psi_N(M_3)"] & 0 \\
D\operatorname{Hom}_B(M_1,D(N)) \arrow[r, "D\operatorname{Hom}_B(f_1,D(N))"] & D\operatorname{Hom}_B(M_2,D(N)) \arrow[r, "D\operatorname{Hom}_B(f_2,D(N))"] & D\operatorname{Hom}_B(M_3,D(N)) \arrow[r] & 0\,,
\end{tikzcd}$$

where the vertical homomorphisms $\psi_N(M_1)$, $\psi_N(M_2)$, and $\psi_N(M_3)$ are isomorphisms, and the lower and upper sequences are exact. Similarly, using the natural isomorphism $\xi_N\colon \operatorname{Tor}_1^B(-,N) \longrightarrow D\operatorname{Ext}_B^1(-,D(N))$ of functors from $\operatorname{mod} B$ to $\operatorname{mod} K$ established in Lemma 3.4, we obtain in $\operatorname{mod} K$ the commutative diagram

$$\begin{tikzcd}
\operatorname{Tor}_1^B(M_1,N) \arrow[r, "\operatorname{Tor}_1^B(f_1,N)"] \arrow[d, "\xi_N(M_1)"] & \operatorname{Tor}_1^B(M_2,N) \arrow[r, "\operatorname{Tor}_1^B(f_2,N)"] \arrow[d, "\xi_N(M_2)"] & \operatorname{Tor}_1^B(M_3,N) \arrow[d, "\xi_N(M_3)"] \\
D\operatorname{Ext}_B^1(M_1,D(N)) \arrow[r, "D\operatorname{Ext}_B^1(f_1,D(N))"] & D\operatorname{Ext}_B^1(M_2,D(N)) \arrow[r, "D\operatorname{Ext}_B^1(f_2,D(N))"] & D\operatorname{Ext}_B^1(M_3,D(N))
\end{tikzcd}$$

where the vertical homomorphisms $\xi_N(M_1)$, $\xi_N(M_2)$, and $\xi_N(M_3)$ are isomorphisms. In particular, we obtain that $\operatorname{Im}\operatorname{Tor}_1^B(f_1,N) = \operatorname{Ker}\operatorname{Tor}_1^B(f_2,N)$, because $\operatorname{Im} D\operatorname{Ext}_B^1(f_1,D(N)) = \operatorname{Ker} D\operatorname{Ext}_B^1(f_2,D(N))$. Finally, there is a unique K-linear isomorphism $\sigma_N^{M_3,M_1}\colon \operatorname{Tor}_1^B(M_3,N) \to M_1 \otimes_B N$ such that the following diagram in $\operatorname{mod} K$ is commutative:

$$\begin{tikzcd}
\operatorname{Tor}_1^B(M_3,N) \arrow[r, "\sigma_N^{M_3,M_1}"] \arrow[d, "\xi(M_3)"] & M_1 \otimes_B N \arrow[d, "\psi(M_1)"] \\
D\operatorname{Ext}_B^1(M_3,D(N)) \arrow[r, "D\left(\delta_{D(N)}^{M_1,M_3}\right)"] & D\operatorname{Hom}_B(M_1,D(N))\,.
\end{tikzcd}$$

Observe that then $\operatorname{Im}\operatorname{Tor}_1^B(f_2,N) = \operatorname{Ker}\sigma_N^{M_3,M_1}$ and $\operatorname{Im}\sigma_N^{M_3,M_1} = \operatorname{Ker}(f_1 \otimes N)$, since

$$\operatorname{Im} D\operatorname{Ext}_B^1(f_2,D(N)) = \operatorname{Ker} D\left(\delta_{D(N)}^{M_1,M_3}\right) \quad \text{and}$$
$$\operatorname{Im} D\left(\delta_{D(N)}^{M_1,M_3}\right) = \operatorname{Ker} D\operatorname{Hom}_B(f_1,D(N)).$$

We also note that, if $\operatorname{pd}_{B^{\mathrm{op}}} N \le 1$, then $D\operatorname{Ext}_B^1(f_1,D(N))$ is a monomorphism, and hence $\operatorname{Tor}_1^B(f_1,N)$ is a monomorphism. □

Let A be a finite dimensional K-algebra over a field K, T a tilting module in $\operatorname{mod} A$, and $B = \operatorname{End}_A(T)$. We define in $\operatorname{mod} B$ the classes of modules

$$\mathcal{X}(T) = \left\{X \in \operatorname{mod} B \,\middle|\, X \otimes_B T = 0\right\},$$
$$\mathcal{Y}(T) = \left\{Y \in \operatorname{mod} B \,\middle|\, \operatorname{Tor}_1^B(Y,T) = 0\right\}.$$

We note that $\mathcal{Y}(T)$ contains the category $\operatorname{proj} B$ of projective modules in $\operatorname{mod} B$.

Lemma 3.6. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$*, and $B = \operatorname{End}_A(T)$. Then $(\mathcal{X}(T),\mathcal{Y}(T))$ is a torsion pair in* $\operatorname{mod} B$ *such that*

$$\mathcal{X}(T) = D(\mathcal{F}({}_BT)) \quad \textit{and} \quad \mathcal{Y}(T) = D(\mathcal{T}({}_BT)).$$

Proof. Since there is a natural isomorphism of covariant functors

$$- \otimes_B T \xrightarrow{\sim} D\operatorname{Hom}_B(-, D(T))$$

from $\operatorname{mod} B$ to $\operatorname{mod} K$, by Theorem II.4.3, we conclude that

$$\mathcal{X}(T) = \{X \in \operatorname{mod} B \mid D\operatorname{Hom}_B(X, D(T)) = 0\} = D(\mathcal{F}({}_B T)).$$

Further, it follows from Lemma 3.4 that there is a natural isomorphism of functors

$$\operatorname{Tor}_1^B(-, T) \xrightarrow{\sim} D\operatorname{Ext}_B^1(-, D(T))$$

from $\operatorname{mod} B$ to $\operatorname{mod} K$. Hence,

$$\mathcal{Y}(T) = \{Y \in \operatorname{mod} B \mid D\operatorname{Ext}_B^1(Y, D(T)) = 0\} = D(\mathcal{T}({}_B T)).$$

Moreover, since $(D(\mathcal{F}({}_B T)), D(\mathcal{T}({}_B T)))$ is a torsion pair in $\operatorname{mod} B$, $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion pair in $\operatorname{mod} B$. □

Proposition 3.7. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$*, $B = \operatorname{End}_A(T)$, and Y a module in $\mathcal{Y}(T)$. Then the following statements hold:*

(i) *There is an exact sequence in* $\operatorname{mod} B$ *of the form*

$$0 \longrightarrow Y \longrightarrow T_0^* \longrightarrow Y_0 \longrightarrow 0$$

with T_0^* in $\operatorname{add} D({}_B T)$ *and Y_0 in $\mathcal{Y}(T)$.*

(ii) *There is a canonical isomorphism of right B-modules*

$$\delta_Y : Y \xrightarrow{\sim} \operatorname{Hom}_A(T, Y \otimes_B T).$$

such that $\delta_Y(y)(t) = y \otimes t$ for $y \in Y$ and $t \in T$.

Proof. (i) Since $\mathcal{Y}(T) = D(\mathcal{T}({}_B T))$ and Y belongs to $\mathcal{Y}(T)$, we conclude that the left B-module $D(Y)$ belongs to the torsion class $\mathcal{T}({}_B T)$ induced by the tilting module ${}_B T$ in $\operatorname{mod} B^{\mathrm{op}}$. Then it follows from Theorem 2.5 that there exists a short exact sequence

$$0 \longrightarrow L_0' \longrightarrow T_0' \longrightarrow D(Y) \longrightarrow 0$$

in $\operatorname{mod} B^{\mathrm{op}}$ with T_0' in $\operatorname{add} {}_B T$ and L_0' in $\mathcal{T}({}_B T)$. Applying now the duality $D\colon \operatorname{mod} B^{\mathrm{op}} \to \operatorname{mod} B$, we obtain the required short exact sequence in $\operatorname{mod} B$

$$0 \longrightarrow Y \longrightarrow T_0^* \longrightarrow Y_0 \longrightarrow 0$$

with $T_0^* = D(T_0')$ in $\operatorname{add} D({}_B T)$ and $Y_0 = D(L_0')$ in $D(\mathcal{T}({}_B T)) = \mathcal{Y}(T)$.

(ii) We note that δ_Y is a homomorphism of right B-modules. Indeed, for $y \in Y$, $b \in B$, $t \in T$, we have $\delta_Y(yb)(t) = yb \otimes t = y \otimes bt = \delta_Y(y)(bt) = (\delta_Y(y)b)(t)$, and hence $\delta_Y(yb) = \delta_Y(y)b$. Observe also that $D(T)$ belongs to $\mathcal{Y}(T)$, since $T \in \mathcal{T}({}_BT)$ and $\mathcal{Y}(T) = D(\mathcal{T}({}_BT))$. We will show now that the homomorphism

$$\delta_{D(T)}\colon D(T) \longrightarrow \operatorname{Hom}_A(T, D(T) \otimes_B T)$$

is an isomorphism. Assume $f \in D(T)$ is such that $\delta_{D(T)}(f) = 0$. Then we have $f \otimes t = \delta_{D(T)}(f)(t) = 0$ for any $t \in T$. But this leads to $f(t) = 0$ for any $t \in T$, because there is a canonical evaluation K-linear homomorphism $D(T) \otimes_B T \to K$. Hence $f = 0$. This shows that $\delta_{D(T)}$ is a monomorphism. We claim that $\dim_K D(T) = \dim_K \operatorname{Hom}_A(T, D(T) \otimes_B T)$. It follows from Proposition 3.3 (i) that there is a canonical isomorphism of right B-modules $D(T) \xrightarrow{\sim} \operatorname{Hom}_A(T, D(A))$. On the other hand, we have, by Theorem II.4.3, the adjoint K-linear isomorphism $D(D(T) \otimes_B T) = \operatorname{Hom}_K(D(T) \otimes_B T, K) \xrightarrow{\sim} \operatorname{Hom}_B(D(T), \operatorname{Hom}_K(T, K)) = \operatorname{Hom}_B(D(T), D(T))$. We have also the canonical isomorphism of K-algebras $\varrho\colon A \to \operatorname{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$ such that $\varrho(a)(t) = ta$ for $a \in A$ and $t \in T$ (see Proposition 3.3 (iii)). This induces the isomorphism of K-algebras $\sigma\colon A \to \operatorname{End}_B(D(T))$ such that $\sigma(a) = D(\varrho(a))$ for any $a \in A$. Observe that then

$$\big(\sigma(a)(f)\big)(t) = \big(D(\varrho(a))(f)\big)(t) = \big(f\varrho(a)\big)(t) = f(ta)$$

for $a \in A$, $f \in D(T)$, $t \in T$. This yields the composite K-linear isomorphism

$$D(A) \xrightarrow{\sim} D\operatorname{End}_A(D(T)) \xrightarrow{\sim} D(T) \otimes_B T,$$

which is in fact an isomorphism of right A-modules. Consequently,

$$\dim_K D(T) = \dim_K \operatorname{Hom}_A(T, D(A)) = \dim_K \operatorname{Hom}_A(T, D(T) \otimes_B T).$$

Therefore, $\delta_{D(T)}$ is an isomorphism in $\operatorname{mod} B$. Obviously, then δ_{T^*} is an isomorphism in $\operatorname{mod} B$ for any module T^* in $\operatorname{add} D(T)$.

It follows now from (i) that there are in $\operatorname{mod} B$ short exact sequences

$$0 \longrightarrow Y \longrightarrow T_0^* \longrightarrow Y_0 \longrightarrow 0,$$
$$0 \longrightarrow Y_0 \longrightarrow T_1^* \longrightarrow Y_1 \longrightarrow 0,$$

with T_0^*, T_1^* in $\operatorname{add} D({}_BT)$ and Y_0, Y_1 in $\mathcal{Y}(T)$. In particular, we have also the induced exact sequence

$$0 \longrightarrow Y \longrightarrow T_0^* \longrightarrow T_1^*.$$

Further, observe that $\operatorname{Tor}_1^B(Y_0, T) = 0$ and $\operatorname{Tor}_1^B(Y_1, T) = 0$. Hence, applying Proposition 3.5, we obtain short exact sequences in $\operatorname{mod} A$

$$\begin{gathered}0 \longrightarrow Y \otimes_B T \longrightarrow T_0^* \otimes_B T \longrightarrow Y_0 \otimes_B T \longrightarrow 0,\\ 0 \longrightarrow Y_0 \otimes_B T \longrightarrow T_1^* \otimes_B T \longrightarrow Y_1 \otimes_B T \longrightarrow 0.\end{gathered}$$

In particular, we obtain a short exact sequence in $\operatorname{mod} A$ of the form

$$0 \longrightarrow Y \otimes_B T \longrightarrow T_0^* \otimes_B T \longrightarrow T_1^* \otimes_B T.$$

Applying now the left exact functor $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ to this short exact sequence, we obtain the exactness of the bottom row of the commutative diagram in $\operatorname{mod} B$

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "\delta_Y"] & T_0^* \arrow[r] \arrow[d, "\delta_{T_0^*}"] & T_1^* \arrow[d, "\delta_{T_1^*}"] \\
0 \arrow[r] & \operatorname{Hom}_A(T, Y \otimes_B T) \arrow[r] & \operatorname{Hom}_A(T, T_0^* \otimes_B T) \arrow[r] & \operatorname{Hom}_A(T, T_1^* \otimes_B T),
\end{tikzcd}$$

where $\delta_{T_0^*}$ and $\delta_{T_1^*}$ are isomorphisms, because T_0^*, T_1^* are in $\operatorname{add} D(T)$. Then δ_Y is also an isomorphism. □

Let A be a finite dimensional K-algebra over a field K, T a tilting module in $\operatorname{mod} A$, and $B = \operatorname{End}_A(T)$. Since T is a (B, A)-bimodule, we have two K-linear covariant functors

$$\operatorname{Hom}_A(T, -), \operatorname{Ext}_A^1(T, -)\colon \operatorname{mod} A \longrightarrow \operatorname{mod} B$$

(see Sections II.2 and III.3), and two K-linear covariant functors

$$- \otimes_B T, \operatorname{Tor}_1^B(-, T)\colon \operatorname{mod} B \longrightarrow \operatorname{mod} A$$

(see Section II.3). Moreover, the tilting module T induces the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$, with

$$\begin{aligned}\mathcal{T}(T) &= \{M \in \operatorname{mod} A \mid \operatorname{Ext}_A^1(T, M) = 0\},\\ \mathcal{F}(T) &= \{N \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, N) = 0\},\end{aligned}$$

and the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$, with

$$\begin{aligned}\mathcal{X}(T) &= \{X \in \operatorname{mod} B \mid X \otimes_B T = 0\},\\ \mathcal{Y}(T) &= \{Y \in \operatorname{mod} B \mid \operatorname{Tor}_1^B(Y, T) = 0\}.\end{aligned}$$

The following result, known as the *Brenner–Butler theorem* or *tilting theorem*, relates these two torsion pairs.

Theorem 3.8. *Let A be a finite dimensional K-algebra over a field K, T be a tilting module in* $\operatorname{mod} A$, $B = \operatorname{End}_A(T)$, *and* $(\mathcal{T}(T), \mathcal{F}(T))$, $(\mathcal{X}(T), \mathcal{Y}(T))$ *the induced torsion pairs in* $\operatorname{mod} A$ *and* $\operatorname{mod} B$, *respectively. Then the following statements hold:*

(i) *The functors* $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ *and* $- \otimes_B T\colon \operatorname{mod} B \to \operatorname{mod} A$ *induce an equivalence of categories* $\mathcal{T}(T)$ *and* $\mathcal{Y}(T)$.

(ii) *The functors* $\operatorname{Ext}_A^1(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ *and* $\operatorname{Tor}_1^B(-, T)\colon \operatorname{mod} B \to \operatorname{mod} A$ *induce an equivalence of categories* $\mathcal{F}(T)$ *and* $\mathcal{X}(T)$.

Proof. (i) Let M be a module in $\mathcal{T}(T)$. Applying the adjoint theorem Theorem II.4.3, we obtain isomorphisms

$$\begin{aligned} D(T \otimes_A D(M)) = \operatorname{Hom}_K(T \otimes_A D(M), K) &\xrightarrow{\sim} \operatorname{Hom}_A(T, \operatorname{Hom}_K(D(M), K)) \\ &\cong \operatorname{Hom}_A(T, M) \end{aligned}$$

in $\operatorname{mod} B$, and hence an isomorphism $D(\operatorname{Hom}_A(T, M)) \xrightarrow{\sim} T \otimes_A D(M)$ in $\operatorname{mod} B^{\mathrm{op}}$. Since $D(M)$ is in $\operatorname{mod} A^{\mathrm{op}}$, we have an epimorphism $A^m \to D(M)$ in $\operatorname{mod} A^{\mathrm{op}}$, for some $m \geq 1$, and hence epimorphisms

$$T^m \xrightarrow{\sim} (T \otimes_A A)^m \xrightarrow{\sim} T \otimes_A A^m \longrightarrow T \otimes_A D(M)$$

in $\operatorname{mod} B^{\mathrm{op}}$. Then we obtain monomorphisms in $\operatorname{mod} B$

$$\operatorname{Hom}_A(T, M) \xrightarrow{\sim} D(T \otimes_A D(M)) \longrightarrow D(T^m) = D(T)^m,$$

and so $\operatorname{Hom}_A(T, M)$ belongs to $\operatorname{Cogen} D({}_BT) = D(\operatorname{Gen} {}_BT) = D(\mathcal{T}({}_BT)) = \mathcal{Y}(T)$. Hence, we have the functor

$$\operatorname{Hom}_A(T, -)\colon \mathcal{T}(T) \longrightarrow \mathcal{Y}(T).$$

Let Y be a module in $\mathcal{Y}(T)$. Then there is an epimorphism $B^n \to Y$ of right B-modules, for some $n \geq 1$, and hence epimorphisms

$$T^n \xrightarrow{\sim} (B \otimes_B T)^n \xrightarrow{\sim} B^n \otimes_B T \longrightarrow Y \otimes_B T$$

in $\operatorname{mod} A$. This shows that $Y \otimes_B T$ belongs to $\operatorname{Gen} T = \mathcal{T}(T)$ (see Theorem 2.5). Hence, we have the functor

$$- \otimes_B T\colon \mathcal{Y}(T) \longrightarrow \mathcal{T}(T).$$

Further, by Proposition 2.7, for any module M in $\mathcal{T}(T)$, the canonical homomorphism

$$\varepsilon_M : \mathrm{Hom}_A(T, M) \otimes_B T \longrightarrow M$$

in $\operatorname{mod} A$ is an isomorphism. Similarly, by Proposition 3.7, for any module Y in $\mathcal{Y}(T)$, the canonical homomorphism

$$\delta_Y : Y \longrightarrow \mathrm{Hom}_A(T, Y \otimes_B T)$$

in $\operatorname{mod} B$ is an isomorphism. Therefore, we have natural isomorphisms of functors

$$\mathrm{Hom}_A(T, -) \otimes_B T \xrightarrow{\sim} \mathbf{1}_{\mathcal{T}(T)} \quad \text{and} \quad \mathbf{1}_{\mathcal{Y}(T)} \xrightarrow{\sim} \mathrm{Hom}_A(T, - \otimes_B T),$$

and so the functors $\mathrm{Hom}_A(T, -)$ and $- \otimes_B T$ induce an equivalence of the categories $\mathcal{T}(T)$ and $\mathcal{Y}(T)$.

(ii) Let N be a module in $\mathcal{F}(T)$. Then there exists a short exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow N \longrightarrow E \longrightarrow L \longrightarrow 0$$

with E an injective module. Since $\mathrm{Ext}_A^1(T, E) = 0$, E belongs to $\mathcal{T}(T)$, and hence L belongs to $\mathcal{T}(T)$, because $\mathcal{T}(T)$ is closed under images. Applying Theorem VII.3.2, we obtain in $\operatorname{mod} B$ an exact sequence of the form

$$0 \longrightarrow \mathrm{Hom}_A(T, E) \longrightarrow \mathrm{Hom}_A(T, L) \longrightarrow \mathrm{Ext}_A^1(T, N) \longrightarrow 0,$$

because $\mathrm{Hom}_A(T, N) = 0$ and $\mathrm{Ext}_A^1(T, E) = 0$. Applying now Propositions 2.7 and 3.5, we obtain in $\operatorname{mod} A$ the commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T, N), T) \arrow[r, "\varepsilon"] \arrow[d] & N \arrow[d] \\
\mathrm{Hom}_A(T, E) \otimes_B T \arrow[r, "\varepsilon_E"] \arrow[d] & E \arrow[d] \\
\mathrm{Hom}_A(T, L) \otimes_B T \arrow[r, "\varepsilon_L"] \arrow[d] & L \arrow[d] \\
\mathrm{Ext}_A^1(T, N) \otimes_B T \arrow[d] & 0 \\
0 &
\end{tikzcd}$$

where ε_E and ε_L are isomorphisms, and hence ε is also an isomorphism. We conclude also that $\mathrm{Ext}_A^1(T, N) \otimes_B T = 0$, which shows that $\mathrm{Ext}_A^1(T, N)$ belongs to $\mathcal{X}(T)$. Hence, we have the functor

$$\mathrm{Ext}_B^1(T, -) : \mathcal{F}(T) \longrightarrow \mathcal{X}(T).$$

Let X be a module in $\mathcal{X}(T)$, and consider a short exact sequence in mod B of the form

$$0 \longrightarrow Y \longrightarrow P \longrightarrow X \longrightarrow 0$$

with P a projective module. Since P belongs to $\mathcal{Y}(T)$ and $\mathcal{Y}(T)$ is closed under submodules, we infer that Y also belongs to $\mathcal{Y}(T)$. Applying Proposition 3.5 we obtain a short exact sequence in mod A of the form

$$0 \longrightarrow \operatorname{Tor}_1^B(X,T) \longrightarrow Y \otimes_B T \longrightarrow P \otimes_B T \longrightarrow 0,$$

because $\operatorname{Tor}_1^B(P,T) = 0$ and $X \otimes_B T = 0$. Applying now Theorem VII.3.2 and Proposition 3.7, we get in mod B the commutative diagram with exact columns

$$\begin{tikzcd}
 & 0 \arrow[d] \\
0 \arrow[d] & \operatorname{Hom}_A(T, \operatorname{Tor}_1^B(X,T)) \arrow[d] \\
Y \arrow[r, "\delta_Y"] \arrow[d] & \operatorname{Hom}_A(T, Y \otimes_B T) \arrow[d] \\
P \arrow[r, "\delta_P"] \arrow[d] & \operatorname{Hom}_A(T, P \otimes_B T) \arrow[d] \\
X \arrow[r, "\delta"] \arrow[d] & \operatorname{Ext}_A^1(T, \operatorname{Tor}_1^B(X,T)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

where δ_Y and δ_P are isomorphisms, and hence δ is also an isomorphism.

Moreover, we infer that $\operatorname{Hom}_A(T, \operatorname{Tor}_1^B(X,T)) = 0$, which shows that $\operatorname{Tor}_1^B(X,T)$ belongs to $\mathcal{F}(T)$. Hence, we have the functor

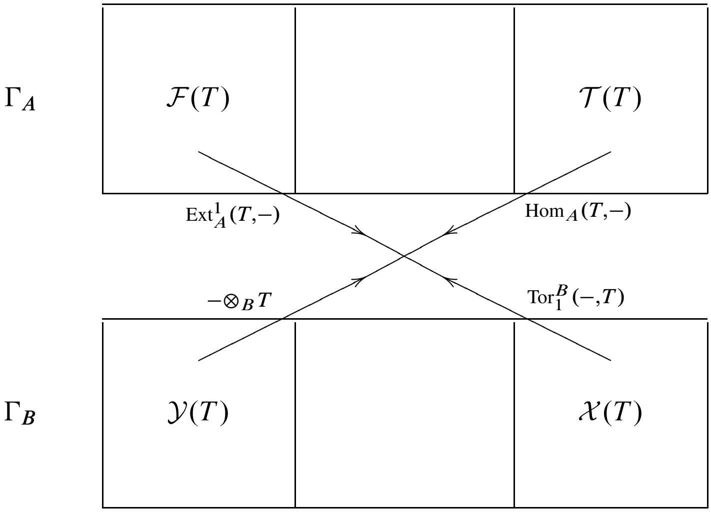
$$\operatorname{Tor}_1^B(-,T)\colon \mathcal{X}(T) \longrightarrow \mathcal{F}(T).$$

Furthermore, there are natural isomorphisms of functors

$$\operatorname{Tor}_1^B(\operatorname{Ext}_A^1(T,-),T) \xrightarrow{\sim} \mathbf{1}_{\mathcal{F}(T)} \quad \text{and} \quad \mathbf{1}_{\mathcal{X}(T)} \xrightarrow{\sim} \operatorname{Ext}_A^1(T, \operatorname{Tor}_1^B(-,T)),$$

and so the functors $\operatorname{Ext}_A^1(T,-)$ and $\operatorname{Tor}_1^B(-,T)$ induce an equivalence of the categories $\mathcal{F}(T)$ and $\mathcal{X}(T)$. □

We may visualize the equivalences established in Theorem 3.8 in the Auslander–Reiten quivers of A and B as follows:



We note that the torsion class $\mathcal{T}(T)$ contains all indecomposable injective modules and thus lies (roughly speaking) at the right side of Γ_A, while the torsion-free class $\mathcal{F}(T)$ lies on the left side of $\mathcal{T}(T)$, because there is no nonzero homomorphism from a module M in $\mathcal{T}(T)$ to a module N in $\mathcal{F}(T)$. Similarly, the torsion-free class $\mathcal{Y}(T)$ contains all indecomposable projective B-modules and hence lies at the left side of Γ_B, while the torsion class $\mathcal{X}(T)$ lies on the right side of $\mathcal{Y}(T)$, because there is no nonzero homomorphism from a module X in $\mathcal{X}(T)$ to a module Y in $\mathcal{Y}(T)$.

The following proposition completes the picture.

Proposition 3.9. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$*, and $B = \mathrm{End}_A(T)$. Then the following statements hold:*

(i) *Let M be a module in* $\operatorname{mod} A$*. Then*

(1) $\mathrm{Tor}_1^B(\mathrm{Hom}_A(T,M),T) = 0$*;*

(2) $\mathrm{Ext}_A^1(T,M) \otimes_B T = 0$*;*

(3) *the canonical sequence for M in* $\operatorname{mod} A$ *with respect to the torsion pair $(\mathcal{T}(T),\mathcal{F}(T))$ is of the form*

$$0 \longrightarrow \mathrm{Hom}_A(T,M) \otimes_B T \xrightarrow{\varepsilon_M} M \longrightarrow \mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T,M),T) \longrightarrow 0.$$

(ii) *Let X be a module in* $\operatorname{mod} B$*. Then*

(1) $\mathrm{Hom}_A(T,\mathrm{Tor}_1^B(X,T)) = 0$*;*

(2) $\mathrm{Ext}_A^1(T, X \otimes_B T) = 0$;

(3) *the canonical sequence for* X *in* mod B *with respect to the torsion pair* $(\mathcal{X}(T), \mathcal{Y}(T))$ *is of the form*

$$0 \longrightarrow \mathrm{Ext}_A^1(T, \mathrm{Tor}_1^B(X, T)) \longrightarrow X \xrightarrow{\delta_X} \mathrm{Hom}_A(T, X \otimes_B T) \longrightarrow 0.$$

Proof. (i) Consider the canonical sequence

$$0 \longrightarrow tM \xrightarrow{i} M \xrightarrow{\pi} M/tM \longrightarrow 0$$

for M in mod A with respect to $(\mathcal{T}(T), \mathcal{F}(T))$. Applying the functor $\mathrm{Hom}_A(T, -)$, we obtain, by Theorem VII.3.2, the exact sequence of the form

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(T, tM) \arrow[r] & \mathrm{Hom}_A(T, M) \arrow[r] & \mathrm{Hom}_A(T, M/tM) \arrow[dll, rounded corners] \\
& \mathrm{Ext}_A^1(T, tM) \arrow[r] & \mathrm{Ext}_A^1(T, M) \arrow[r] & \mathrm{Ext}_A^1(T, M/tM) \arrow[r] & 0,
\end{tikzcd}$$

in mod B, where $\mathrm{Hom}_A(T, M/tM) = 0$ and $\mathrm{Ext}_A^1(T, tM) = 0$, because $\mathrm{pd}_A\, T \leq 1$, M/tM is in $\mathcal{F}(T)$, and tM is in $\mathcal{T}(T)$. Hence $\mathrm{Hom}_A(T, tM) \xrightarrow{\sim} \mathrm{Hom}_A(T, M)$ and $\mathrm{Ext}_A^1(T, M) \xrightarrow{\sim} \mathrm{Ext}_A^1(T, M/tM)$ in mod B. Further, $tM \in \mathcal{T}(T)$ implies $\mathrm{Hom}_A(T, tM) \in \mathcal{Y}(T)$, and then we obtain isomorphisms

$$\mathrm{Tor}_1^B(\mathrm{Hom}_A(T, M), T) \cong \mathrm{Tor}_1^B(\mathrm{Hom}_A(T, tM), T) = 0$$

and

$$tM \cong \mathrm{Hom}_A(T, tM) \otimes_B T \cong \mathrm{Hom}_A(T, M) \otimes_B T.$$

Similarly, $M/tM \in \mathcal{F}(T)$ implies $\mathrm{Ext}_A^1(M/tM, T) \in \mathcal{X}(T)$, and then we obtain isomorphisms

$$\mathrm{Ext}_A^1(T, M) \otimes_B T \cong \mathrm{Ext}_A^1(T, M/tM) \otimes_B T = 0$$

and

$$M/tM \cong \mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T, M/tM), T) \cong \mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T, M), T).$$

Moreover, we have in mod A the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(T, tM) \otimes_B T \arrow[r, "\varepsilon_{tM}"] \arrow[d, "\mathrm{Hom}_A(T,i) \otimes_B T"] & tM \arrow[d, "i"] \\
\mathrm{Hom}_A(T, M) \otimes_B T \arrow[r, "\varepsilon_M"] & M,
\end{tikzcd}$$

where ε_{tM} is an isomorphism (by Proposition 2.7) and $\operatorname{Hom}_A(T,i)\otimes_B T$ is an isomorphism, because $\operatorname{Hom}_A(T,i)\colon \operatorname{Hom}_A(T,tM)\to\operatorname{Hom}_A(T,M)$ is an isomorphism. This shows that the canonical sequence for M in $\operatorname{mod} A$ with respect to the torsion pair $(\mathcal{T}(T),\mathcal{F}(T))$ is of the form

$$0\longrightarrow \operatorname{Hom}_A(T,M)\otimes_B M\xrightarrow{\varepsilon_M} M\longrightarrow \operatorname{Tor}_1^B(\operatorname{Ext}_A^1(T,M),T)\longrightarrow 0,$$

because $M/tM\cong\operatorname{Tor}_1^B(\operatorname{Ext}_A^1(T,M),T)$ in $\operatorname{mod} A$.

(ii) Consider the canonical sequence

$$0\longrightarrow tX\longrightarrow X\xrightarrow{p} X/tX\longrightarrow 0$$

for X in $\operatorname{mod} B$ with respect to the torsion pair $(\mathcal{X}(T),\mathcal{Y}(T))$. Applying the functor $-\otimes_B T$, we obtain, by Proposition 3.5, the exact sequence in $\operatorname{mod} A$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Tor}_1^B(tX,T) \arrow[r] & \operatorname{Tor}_1^B(X,T) \arrow[r] & \operatorname{Tor}_1^B(X/tX,T) \arrow[dlll, rounded corners] & \\
tX\otimes_B T \arrow[r] & X\otimes_B T \arrow[r] & (X/tX)\otimes_B T \arrow[r] & 0,
\end{tikzcd}$$

where $\operatorname{Tor}_1^B(X/tX,T)=0$ and $tX\otimes_B T=0$, because $\operatorname{pd}_{B^{\mathrm{op}}} T\le 1$, $X/tX\in\mathcal{Y}(T)$ and $tX\in\mathcal{X}(T)$. Hence we have isomorphisms $\operatorname{Tor}_1^B(tX,T)\xrightarrow{\sim}\operatorname{Tor}_1^B(X,T)$ and $X\otimes_B T\xrightarrow{\sim}(X/tX)\otimes_B T$ in $\operatorname{mod} A$. Further, $tX\in\mathcal{X}(T)$ implies $\operatorname{Tor}_1^B(tX,T)\in\mathcal{F}(T)$, and then we obtain isomorphisms

$$\operatorname{Hom}_A(T,\operatorname{Tor}_1^B(X,T))\cong\operatorname{Hom}_A(T,\operatorname{Tor}_1^B(tX,T))=0$$

and

$$tX\cong\operatorname{Ext}_A^1(T,\operatorname{Tor}_1^B(tX,T))\cong\operatorname{Ext}_A^1(T,\operatorname{Tor}_1^B(X,T)).$$

Similarly, $X/tX\in\mathcal{Y}(T)$ implies $(X/tX)\otimes_B T\in\mathcal{T}(T)$, and then we get isomorphisms

$$\operatorname{Ext}_A^1(T,X\otimes_B T)\cong\operatorname{Ext}_A^1(T,(X/tX)\otimes_B T)=0$$

and

$$X/tX\cong\operatorname{Hom}_A(T,(X/tX)\otimes_B T)\cong\operatorname{Hom}_A(T,X\otimes_B T).$$

Moreover, we have in $\operatorname{mod} B$ the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\delta_X"] \arrow[d, "p"] & \operatorname{Hom}_A(T,X\otimes_B T) \arrow[d, "\operatorname{Hom}_A(T,p\otimes_B T)"] \\
X/tX \arrow[r, "\delta_{X/tX}"] & \operatorname{Hom}_A(T,(X/tX)\otimes_B T),
\end{tikzcd}$$

where $\delta_{X/tX}$ is an isomorphism (by Proposition 3.7) and $\operatorname{Hom}_A(T, p \otimes_B T)$ is an isomorphism, because $p \otimes_B T \colon X \otimes_B T \to (X/tX) \otimes_B T$ is an isomorphism, due to the equality $tX \otimes_B T = 0$. This shows that the canonical sequence for X in $\operatorname{mod} B$ with respect to the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is of the form

$$0 \longrightarrow \operatorname{Ext}^1_A(T, \operatorname{Tor}^B_1(X, T)) \longrightarrow X \xrightarrow{\delta_X} \operatorname{Hom}_A(T, X \otimes_B T) \longrightarrow 0. \qquad \square$$

Theorem 3.10. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$*, and $B = \operatorname{End}_A(T)$. Then there is an isomorphism $f_T \colon K_0(A) \to K_0(B)$ of Grothendieck groups of A and B such that*

$$f_T([M]) = [\operatorname{Hom}_A(T, M)] - [\operatorname{Ext}^1_A(T, M)]$$

for any module M in $\operatorname{mod} A$.

Proof. Since $\operatorname{pd}_A T \leq 1$, every short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\operatorname{mod} A$ induces an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(T, L) \arrow[r] & \operatorname{Hom}_A(T, M) \arrow[r] & \operatorname{Hom}_A(T, N) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- node[pos=0.75, above]{\scriptstyle \delta^T_{N,L}} ([yshift=-1.5ex]\tikztostart.south -| \tikztotarget.west) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
& \operatorname{Ext}^1_A(T, L) \arrow[r] & \operatorname{Ext}^1_A(T, M) \arrow[r] & \operatorname{Ext}^1_A(T, N) \arrow[r] & 0
\end{tikzcd}$$

in $\operatorname{mod} B$ (see Theorem VII.3.2). Then in the Grothendieck group $K_0(B)$ we have

$$\begin{aligned}
[\operatorname{Hom}_A(T, M)] - [\operatorname{Ext}^1_A(T, M)] = {} & [\operatorname{Hom}_A(T, L)] - [\operatorname{Ext}^1_A(T, L)] \\
& + [\operatorname{Hom}_A(T, N)] - [\operatorname{Ext}^1_A(T, N)].
\end{aligned}$$

Hence there is a well defined group homomorphism $f_T \colon K_0(A) \to K_0(B)$ such that $f_T([M]) = [\operatorname{Hom}_A(T, M)] - [\operatorname{Ext}^1_A(T, M)]$ for any module M in $\operatorname{mod} A$. We will show that f_T is an isomorphism.

Let S be a simple module in $\operatorname{mod} B$. Since $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion pair in $\operatorname{mod} B$, applying Corollary 1.5, we conclude that S belongs to $\mathcal{X}(T)$ or S belongs to $\mathcal{Y}(T)$. Assume S is in $\mathcal{Y}(T)$. Then it follows from Theorem 3.8 that $S \cong \operatorname{Hom}_A(T, S \otimes_B T)$ and $\operatorname{Ext}^1_A(T, S \otimes_B T) = 0$, because $S \otimes_B T$ belongs to $\mathcal{T}(T)$. Hence we obtain

$$[S] = [\operatorname{Hom}_A(T, S \otimes_B T)] = f_T([S \otimes_B T]).$$

Assume now that S belongs to $\mathcal{X}(T)$. Applying again Theorem 3.8, we obtain $S \cong \operatorname{Ext}^1_A(T, \operatorname{Tor}^B_1(S, T))$ and $\operatorname{Hom}_A(T, \operatorname{Tor}^B_1(S, T)) = 0$, because $\operatorname{Tor}^B_1(S, T)$ is in $\mathcal{F}(T)$. It follows that

$$[S] = [\operatorname{Ext}^1_A(T, \operatorname{Tor}^B_1(S, T))] = f_T(-[\operatorname{Tor}^B_1(S, T)]).$$

We know from Theorem I.11.1 that $K_0(B)$ is a free abelian group with a $\mathbb{Z}$-basis formed by the classes $[S_1], \dots, [S_n]$ of a complete set $S_1, \dots, S_n$ of pairwise nonisomorphic simple modules in $\operatorname{mod} B$. In particular, we conclude that f_T is an epimorphism. Since $K_0(A)$ is a free abelian group, we obtain that the rank of $K_0(A)$ is greater than or equal to the rank of $K_0(B)$. On the other hand, by Proposition 3.3, T is a tilting module in $\operatorname{mod} B^{\mathrm{op}}$ and there is an isomorphism of K-algebras $A \xrightarrow{\sim} \operatorname{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$, and hence an isomorphism of K-algebras $A^{\mathrm{op}} \xrightarrow{\sim} \operatorname{End}_{B^{\mathrm{op}}}(T)$, Then it follows from the above arguments, that the rank of $K_0(B^{\mathrm{op}})$ is greater than or equal to the rank of $K_0(A^{\mathrm{op}})$. Since the duality functors $D\colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ and $D\colon \operatorname{mod} B \to \operatorname{mod} B^{\mathrm{op}}$ induce group isomorphisms $K_0(A) \xrightarrow{\sim} K_0(A^{\mathrm{op}})$ and $K_0(B) \xrightarrow{\sim} K_0(B^{\mathrm{op}})$, we conclude that the rank of $K_0(B)$ is greater than or equal to the rank of $K_0(A)$. Therefore, the groups $K_0(A)$ and $K_0(B)$ have the same rank, and $f_T\colon K_0(A) \to K_0(B)$ is an isomorphism. □

The following proposition is a consequence of Theorem 3.10 and Bongartz's Lemma (Lemma 2.4) and provides a very useful criterion for deciding whether a partial tilting module is a tilting module or not.

Proposition 3.11. *Let A be a finite dimensional K-algebra over a field K, and T a partial tilting module in* $\operatorname{mod} A$. *Then T is a tilting module in* $\operatorname{mod} A$ *if and only if the number of pairwise nonisomorphic indecomposable direct summands of T in* $\operatorname{mod} A$ *equals the rank of $K_0(A)$.*

Proof. Let $B = \operatorname{End}_A(T)$ and r be the number of pairwise nonisomorphic indecomposable direct summands of T in $\operatorname{mod} A$.

Assume T is a tilting module in $\operatorname{mod} A$. Then it follows from Lemma 3.1 (ii) that r is the number of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} B$, and hence equals the rank of $K_0(B)$, because $K_0(B)$ has a $\mathbb{Z}$-basis $[S_1], \dots, [S_m]$, for a complete set $S_1, \dots, S_m$ of pairwise nonisomorphic simple modules in $\operatorname{mod} B$ (so $m = r$). Applying Theorem 3.10, we conclude that r is the rank of $K_0(A)$.

Assume now that r is the rank of $K_0(A)$. Since T is a partial tilting module in $\operatorname{mod} A$, it follows from Lemma 2.4 that there exists a module E in $\operatorname{mod} A$ such that $T \oplus E$ is a tilting module in $\operatorname{mod} A$. Then, by the first part of the proof, the number of pairwise nonisomorphic indecomposable direct summands of $T \oplus E$ in $\operatorname{mod} A$ equals the rank of $K_0(A)$, and consequently equals r. This implies that E belongs to the additive category $\operatorname{add} T$ of T, and hence T is a tilting module in $\operatorname{mod} A$. □

Proposition 3.12. *Let A be a finite dimensional K-algebra over a field K, $T = T_1 \oplus \cdots \oplus T_n$ a tilting module in* $\operatorname{mod} A$ *with $T_1, \dots, T_n$ pairwise nonisomorphic indecomposable modules, and $B = \operatorname{End}_A(T)$. For each $i \in \{1, \dots, n\}$, let $e_i \in \operatorname{End}_A(T)$ be the composition $u_i p_i$ of the canonical projection $p_i\colon T \to T_i$ with the canonical injection $u_i\colon T_i \to T$. Then the following statements hold:*

(i) *$e_1, \dots, e_n$ are pairwise orthogonal primitive idempotents of B with $1_B = e_1 + \cdots + e_n$.*

(ii) *For each $i \in \{1, \dots, n\}$, there exists an isomorphism of right B-modules* $\mathrm{Hom}_A(T, T_i) \to e_i B$.

Proof. Recall that, for a module M in $\operatorname{mod} A$, $\mathrm{Hom}_A(T, M)$ has a right B-module structure defined by $fb = f \circ b$ for $f \in \mathrm{Hom}_A(T, M)$ and $b \in B = \mathrm{End}_A(T)$, where $f \circ b$ is the composition of b with f. It follows from Lemma 3.1 (ii), that $\mathrm{Hom}_A(T, T_1), \dots, \mathrm{Hom}_A(T, T_n)$ form a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} B$. Clearly, for $i, j \in \{1, \dots, n\}$, we have $e_i^2 = (u_i p_i)(u_i p_i) = u_i(p_i u_i)p_i = u_i p_i = e_i$ and $e_i e_j = (u_i p_i)(u_j p_j) = u_i(p_i u_j)p_j = 0$ for $i \neq j$. Hence, $e_1, \dots, e_n$ are pairwise orthogonal idempotents of B. Further, for each $i \in \{1, \dots, n\}$, the B-module homomorphism $\varphi_i\colon \mathrm{Hom}_A(T, T_i) \to e_i B$ defined for $f \in \mathrm{Hom}_A(T, T_i)$ by $\varphi_i(f) = u_i f = e_i u_i f$ is an isomorphism. Indeed, the homomorphisms $\varphi_1, \dots, \varphi_n$ are monomorphisms, because the homomorphisms $u_1, \dots, u_n$ are monomorphisms. On the other hand, we have $e_i B = u_i p_i B = \varphi_i(p_i B) = \varphi_i(\mathrm{Hom}_A(T, T_i))$, which shows that φ_i is an epimorphism, and consequently φ_i is an isomorphism, for any $i \in \{1, \dots, n\}$. In particular, we conclude that $e_1 B, \dots, e_n B$ are indecomposable modules, and so $e_1, \dots, e_n$ are primitive idempotents of B, by Corollary I.5.8. □

Let A be a finite dimensional K-algebra over a field K, and $S_1, \dots, S_n$ a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$. Then it follows from Theorem I.11.1 that $K_0(A)$ is a free abelian group and $[S_1], \dots, [S_n]$ form a $\mathbb{Z}$-basis of $K_0(A)$. Moreover, for a module M in $\operatorname{mod} A$, we have the expression

$$[M] = \sum_{i=1}^{n} c_i(M)[S_i],$$

where $c_i(M)$ is the composition multiplicity of the simple module S_i in M. Further, for complete sets $P_1, \dots, P_n$ and $I_1, \dots, I_n$ of pairwise nonisomorphic indecomposable projective and indecomposable injective modules in $\operatorname{mod} A$ with $\mathrm{top}(P_i) \cong S_i \cong \mathrm{soc}(I_i)$ for $i \in \{1, \dots, n\}$, we have the equalities

$$\dim_K \mathrm{Hom}_A(P_i, M) = c_i(M) \dim_K \mathrm{End}_A(S_i) = \dim_K \mathrm{Hom}_A(M, I_i)$$

for any module M in $\operatorname{mod} A$, and $i \in \{1, \dots, n\}$ (see Lemma VII.5.6).

Lemma 3.13. *Let A be a finite dimensional K-algebra over a field K, $T = T_1 \oplus \cdots \oplus T_n$ a tilting module in* $\operatorname{mod} A$ *with $T_1, \dots, T_n$ pairwise nonisomorphic indecomposable modules, and $B =$* $\mathrm{End}_A(T)$. *Moreover, let $e_i = u_i p_i$, $i \in \{1, \dots, n\}$, be the associated complete set of pairwise orthogonal primitive idempotents in B and $S_1 = e_1 B/e_1$* $\operatorname{rad} B$*, $\dots, S_n = e_n B/e_n$* $\operatorname{rad} B$ *the associated complete set of pairwise nonisomorphic simple modules in* $\operatorname{mod} B$. *Then the following statements hold:*

(i) *For any module M in $\mathcal{T}(T)$ and $i \in \{1, \dots, n\}$, we have*

$$c_i(\operatorname{Hom}_A(T, M)) \dim_K \operatorname{End}_B(S_i) = \dim_K \operatorname{Hom}_A(T_i, M).$$

(ii) *For any module N in $\mathcal{F}(T)$ and $i \in \{1, \dots, n\}$, we have*

$$c_i(\operatorname{Ext}_A^1(T, N)) \dim_K \operatorname{End}_B(S_i) = \dim_K \operatorname{Hom}_A(N, \tau_A T_i).$$

Proof. Let $P_i = e_i B$, for any $i \in \{1, \dots, n\}$. It follows from Proposition 3.12, that there is an isomorphism of right B-modules $\operatorname{Hom}_A(T, T_i) \xrightarrow{\sim} e_i B$ for any $i \in \{1, \dots, n\}$.

(i) Let M be a module in $\mathcal{T}(T)$ and $i \in \{1, \dots, n\}$. Then

$$\begin{aligned} c_i\big(\operatorname{Hom}_A(T, M)\big) \dim_K \operatorname{End}_B(S_i) &= \dim_K \operatorname{Hom}_B\big(P_i, \operatorname{Hom}_A(T, M)\big) \\ &= \dim_K \operatorname{Hom}_B\big(\operatorname{Hom}_A(T, T_i), \operatorname{Hom}_A(T, M)\big) \\ &= \dim_K \operatorname{Hom}_A(T_i, M), \end{aligned}$$

by Proposition 3.2 (i), because T_i and M are in $\mathcal{T}(T)$.

(ii) Let N be a module in $\mathcal{F}(T)$ and $i \in \{1, \dots, n\}$. Then

$$\begin{aligned} c_i\big(\operatorname{Ext}_A^1(T, N)\big) \dim_K \operatorname{End}_B(S_i) &= \dim_K \operatorname{Hom}_B\big(P_i, \operatorname{Ext}_A^1(T, N)\big) \\ &= \dim_K \operatorname{Hom}_B\big(e_i B, \operatorname{Ext}_A^1(T, N)\big) = \dim_K \operatorname{Ext}_A^1(T, N)e_i \\ &= \dim_K \operatorname{Ext}_A^1(e_i T, N) = \dim_K \operatorname{Ext}_A^1(e_i(T), N) \\ &= \dim_K \operatorname{Ext}_A^1(T_i, N) = \dim_K \operatorname{Hom}_A(N, \tau_A T_i), \end{aligned}$$

because $\operatorname{pd}_A T_i \leq 1$ (see Corollary III.6.4). □

Example 3.14. Let $A = KQ$ be the path algebra of the quiver Q of the form

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\alpha"'] & \underset{3}{\bullet} \arrow[l, "\beta"'] \end{tikzcd}$$

over a field K. Then A is a hereditary algebra and the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd} & & P_3 = I_1 \arrow[dr] & & \\ & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[dr] & \\ S_1 = P_1 \arrow[ur] & & S_2 \arrow[ur] & & I_3 = S_3\,, \end{tikzcd}$$

where S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} A$ given by the vertex $i \in \{1,2,3\}$ of Q. Let $T_1 = S_1$, $T_2 = P_3$, $T_3 = S_3$, and $T = T_1 \oplus T_2 \oplus T_3$. Clearly, $\operatorname{pd}_A T \leq 1$ since A is a hereditary K-algebra. Moreover, applying Corollary III.6.4, we obtain isomorphisms in $\operatorname{mod} K$

$$\operatorname{Ext}^1_A(T,T) \cong D\operatorname{Hom}_A(T,\tau_A T) = D\operatorname{Hom}_A(S_1 \oplus P_3 \oplus S_3, S_2) = 0,$$

and hence T is a partial tilting module in $\operatorname{mod} A$. Finally, by Proposition 3.11, T is a tilting module in $\operatorname{mod} A$, since $K_0(A)$ is of rank 3. The torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ induced by T is described as follows:

$$\begin{aligned}\mathcal{T}(T) &= \{M \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(T,M) = 0\}\\ &= \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M,\tau_A T) = 0\}\\ &= \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M,S_2) = 0\} = \operatorname{add}(S_1 \oplus P_3 \oplus I_2 \oplus S_3),\\ \mathcal{F}(T) &= \{N \in \operatorname{mod} A \mid \operatorname{Hom}_A(T,N) = 0\} = \operatorname{add}(S_2).\end{aligned}$$

Consider now the endomorphism algebra $B = \operatorname{End}_A(T)$, the canonical pairwise orthogonal primitive idempotents $e_1 = u_1 p_1$, $e_2 = u_2 p_2$, $e_3 = u_3 p_3$ with $1_B = e_1 + e_2 + e_3$, and the associated indecomposable projective B-modules $P_1^B = e_1 B \cong \operatorname{Hom}_A(T,T_1)$, $P_2^B = e_2 B \cong \operatorname{Hom}_A(T,T_2)$, $P_3^B = e_3 B \cong \operatorname{Hom}_A(T,T_3)$ (see Proposition 3.12). Observe that B is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is of the form

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\sigma"'] & \underset{3}{\bullet} \arrow[l, "\mu"'] \end{tikzcd},$$

J is the ideal in $K\Delta$ generated by $\mu\sigma$, and the vertices $1,2,3$ of Δ correspond to the idempotents e_1, e_2, e_3 of B. It follows from Theorem I.9.6 that B is not a hereditary algebra. Further, a direct checking shows that the Auslander–Reiten quiver Γ_B of B is of the form (see Theorem III.10.2)

$$\begin{tikzcd}[column sep=small] & P_2^B = I_1^B \arrow[dr] & & & \\ S_1^B = P_1^B \arrow[ur] & & S_2^B \arrow[dr] & & I_3^B = S_3^B \\ & & & I_2^B = P_3^B \arrow[ur] & \end{tikzcd}$$

where $S_i^B = e_i B / e_i \operatorname{rad} B$ and $I_i^B = D(Be_i)$, $i \in \{1,2,3\}$, are the simple and indecomposable injective modules in $\operatorname{mod} B$, respectively. Observe also that $\operatorname{End}_B(S_i^B) \cong K$ for any $i \in \{1,2,3\}$.

We describe now the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ induced by T. It follows from Theorem 3.8 that we have the equivalences of categories

$$\operatorname{Hom}_A(T, -)\colon \mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T),$$
$$\operatorname{Ext}^1_A(T, -)\colon \mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T).$$

We know that $P_i^B \cong \operatorname{Hom}_A(T, T_i)$, for $i \in \{1, 2, 3\}$, and hence we obtain isomorphisms $P_1^B \cong \operatorname{Hom}_A(T, S_1)$, $P_2^B \cong \operatorname{Hom}_A(T, P_3)$, $P_3^B \cong \operatorname{Hom}_A(T, S_3)$ in $\operatorname{mod} B$. Applying Lemma 3.13, we obtain

$$\begin{aligned} c_1(\operatorname{Hom}_A(T, I_2)) &= \dim_K \operatorname{Hom}_A(T_1, I_2) = 0,\\ c_2(\operatorname{Hom}_A(T, I_2)) &= \dim_K \operatorname{Hom}_A(T_2, I_2) = 1,\\ c_3(\operatorname{Hom}_A(T, I_2)) &= \dim_K \operatorname{Hom}_A(T_3, I_2) = 0, \end{aligned}$$

and hence $\operatorname{Hom}_A(T, I_2) \cong S_2^B$ in $\operatorname{mod} B$. This shows that

$$\mathcal{Y}(T) = \operatorname{Hom}_A(T, \mathcal{T}(T)) = \operatorname{add}(P_1^B \oplus P_2^B \oplus S_2^B \oplus P_3^B).$$

Similarly, applying Lemma 3.13 again, we conclude that

$$\begin{aligned} c_1(\operatorname{Ext}^1_A(T, S_2)) &= \dim_K \operatorname{Hom}_A(S_2, \tau_A T_1) = 0,\\ c_2(\operatorname{Ext}^1_A(T, S_2)) &= \dim_K \operatorname{Hom}_A(S_2, \tau_A T_2) = 0,\\ c_3(\operatorname{Ext}^1_A(T, S_2)) &= \dim_K \operatorname{Hom}_A(S_2, \tau_A T_3) = 1, \end{aligned}$$

and so $\operatorname{Ext}^1_A(T, S_2) \cong S_3^B$ in $\operatorname{mod} B$. Therefore,

$$\mathcal{X}(T) = \operatorname{Ext}^1_A(T, \mathcal{F}(T)) = \operatorname{add}(S_3^B).$$

Example 3.15. Let K be a field, Q the quiver

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\alpha"'] & \underset{3}{\bullet} \arrow[l, "\beta"'] & \underset{4}{\bullet} \arrow[l, "\gamma"'] & \underset{5}{\bullet} \arrow[l, "\sigma"'] \end{tikzcd},$$

I the ideal in the path algebra KQ of Q over K generated by the path $\sigma\gamma\beta\alpha$, and $A = KQ/I$ the associated bound quiver algebra. We note that A is not a hereditary algebra, by Theorem I.9.6. Applying Auslander's Theorem III.10.2, we compute that the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}[column sep=small] S_1 = P_1 \arrow[dr] & & S_2 \arrow[dr] & & S_3 \arrow[dr] & & S_4 \arrow[dr] & & I_5 = S_5 \\ & P_2 \arrow[ur] \arrow[dr] & & P_3/S_1 \arrow[ur] \arrow[dr] & & P_4/P_2 \arrow[ur] \arrow[dr] & & I_4 \arrow[ur] & \\ & & P_3 \arrow[ur] \arrow[dr] & & P_4/S_1 \arrow[ur] \arrow[r] & P_5 \arrow[r] \arrow[d, equal] & I_3 \arrow[ur] & & \\ & & & P_4 = I_1 \arrow[ur] & & I_2 & & & \end{tikzcd}$$

where S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in mod A given by the vertex $i \in \{1, 2, 3, 4, 5\}$ of Q. Let $T_1 = S_1$, $T_2 = P_4$, $T_3 = P_5$, $T_4 = P_4/P_2$, $T_5 = S_3$, and $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$. We claim that T is a tilting module in mod A. Since T_1, T_2, T_3 are projective modules and T_4, T_5 admit minimal projective resolutions in mod A of the forms

$$0 \longrightarrow P_2 \longrightarrow P_4 \longrightarrow T_4 \longrightarrow 0,$$
$$0 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow T_5 \longrightarrow 0,$$

we conclude that $\operatorname{pd}_A T \leq 1$. Moreover, applying Corollary III.6.4, we obtain isomorphisms in mod K

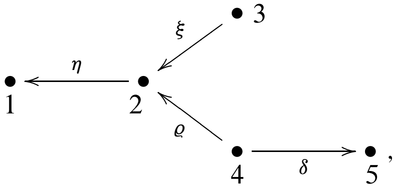
$$\operatorname{Ext}_A^1(T, T) \cong D\operatorname{Hom}_A(T, \tau_A T) = D\operatorname{Hom}_A\big(T, (P_3/S_1) \oplus S_2\big) = 0,$$

and hence T is a partial tilting module in mod A. Since $K_0(A)$ is of rank 5, applying Proposition 3.11, we conclude that indeed T is a tilting module in mod A. Further, the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A induced by T is described as follows:

$$\begin{aligned}
\mathcal{T}(T) &= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Ext}_A^1(T, M) = 0\big\}\\
&= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(M, \tau_A T) = 0\big\}\\
&= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(M, (P_3/S_1) \oplus S_2) = 0\big\}\\
&= \operatorname{add}\big(S_1 \oplus P_4 \oplus (P_4/S_1) \oplus (P_4/P_2) \oplus S_3 \oplus S_4 \oplus P_5 \oplus I_3 \oplus I_4 \oplus S_5\big),\\
\mathcal{F}(T) &= \big\{N \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(T, N) = 0\big\} = \operatorname{add}\big(S_2 \oplus (P_3/S_1)\big).
\end{aligned}$$

We note that the indecomposable modules P_2 and P_3 are neither in $\mathcal{T}(T)$, nor in $\mathcal{F}(T)$.

Consider now the endomorphism algebra $B = \operatorname{End}_A(T)$, the canonical pairwise orthogonal primitive idempotents $e_i = u_i p_i$, $i \in \{1, 2, 3, 4, 5\}$, with $1_B = e_1 + e_2 + e_3 + e_4 + e_5$, and the associated indecomposable projective B-modules $P_j^B = e_j B \cong \operatorname{Hom}_A(T, T_j)$, $j \in \{1, 2, 3, 4, 5\}$, in mod B (see Proposition 3.12). Observe that B is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver



J is the ideal in $K\Delta$ generated by $\xi\eta$ and $\varrho\eta$, and the vertices $1, 2, 3, 4, 5$ of Δ correspond to the idempotents e_1, e_2, e_3, e_4, e_5 of B. We note also that B is not a hereditary algebra, again by Theorem I.9.6. Applying Theorem III.10.2, we compute that the Auslander–Reiten quiver Γ_B of B is of the form

$$\begin{tikzcd}
 & P_2^B = I_1^B \arrow[dr] & & & & & \\
S_1^B = P_1^B \arrow[ur] & & S_2^B \arrow[r] \arrow[dr] & P_3^B \arrow[r] & R \arrow[r] \arrow[dr] & I_5^B \arrow[r] & I_4^B = S_4^B \\
 & & & P_4^B \arrow[ur] \arrow[dr] & & I_2^B \arrow[ur] \arrow[dr] & \\
 & & S_5^B = P_5^B \arrow[ur] & & P_4^B/S_5^B \arrow[ur] & & I_3^B = S_3^B ,
\end{tikzcd}$$

where R is the indecomposable module in mod B with the composition vector

$$c(R) = \big(c_1(R), c_2(R), c_3(R), c_4(R), c_5(R)\big) = (0, 1, 1, 1, 1),$$

and $S_j^B = e_j B/e_j \operatorname{rad} B$ and $I_j^B = D(Be_j)$, for $j \in \{1, 2, 3, 4, 5\}$. Observe also that $\operatorname{End}_B(S_j^B) \cong K$ for any $j \in \{1, 2, 3, 4, 5\}$, because mod B is equivalent to $\operatorname{rep}_K(\Delta, J)$ (see Theorem I.2.10).

We describe now the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B induced by T. It follows from Theorem 3.8 that we have the equivalences of categories

$$\operatorname{Hom}_A(T, -)\colon \mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T),$$
$$\operatorname{Ext}_A^1(T, -)\colon \mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T).$$

Since $P_i^B = \operatorname{Hom}_A(T, T_i)$, for $i \in \{1, 2, 3, 4, 5\}$, we have isomorphisms

$$P_1^B \cong \operatorname{Hom}_A(T, S_1), \quad P_2^B \cong \operatorname{Hom}_A(T, P_4), \quad P_3^B \cong \operatorname{Hom}_A(T, P_5),$$
$$P_4^B \cong \operatorname{Hom}_A(T, P_4/S_2), \quad P_5^B \cong \operatorname{Hom}_A(T, S_3)$$

in mod B. Since $\dim_K \operatorname{End}_B(S_i^B) = 1$ for any $i \in \{1, 2, 3, 4, 5\}$, applying Lemma 3.13, we obtain the equalities

$$c_i(\operatorname{Hom}_A(T, P_4/S_1)) = \dim_K \operatorname{Hom}_A(T_i, P_4/S_1) = 0, \text{ for } i \in \{1, 3, 4, 5\},$$
$$c_2(\operatorname{Hom}_A(T, P_4/S_1)) = \dim_K \operatorname{Hom}_A(T_2, P_4/S_1) = 1,$$

and hence $\operatorname{Hom}_A(T, P_4/S_1) \cong S_2^B$ in mod B. Similarly,

$$c_i(\operatorname{Hom}_A(T, S_4)) = \dim_K \operatorname{Hom}_A(T_i, S_4) = 0, \text{ for } i \in \{1, 3, 5\},$$
$$c_i(\operatorname{Hom}_A(T, S_4)) = \dim_K \operatorname{Hom}_A(T_i, S_4) = 1, \text{ for } i \in \{2, 4\},$$

and hence $\mathrm{Hom}_A(T, S_4) \cong P_4^B/S_5^B$ in mod B. Further,

$$\begin{aligned}
c_i(\mathrm{Hom}_A(T, I_3)) &= \dim_K \mathrm{Hom}_A(T_i, I_3) = 1, \text{ for } i \in \{2,3,4,5\},\\
c_1(\mathrm{Hom}_A(T, I_3)) &= \dim_K \mathrm{Hom}_A(T_1, I_3) = 0,\\
c_i(\mathrm{Hom}_A(T, I_4)) &= \dim_K \mathrm{Hom}_A(T_i, I_4) = 1, \text{ for } i \in \{2,3,4\},\\
c_i(\mathrm{Hom}_A(T, I_4)) &= \dim_K \mathrm{Hom}_A(T_i, I_4) = 0, \text{ for } i \in \{1,5\},\\
c_i(\mathrm{Hom}_A(T, S_5)) &= \dim_K \mathrm{Hom}_A(T_i, S_5) = 0, \text{ for } i \in \{1,2,4,5\},\\
c_3(\mathrm{Hom}_A(T, S_5)) &= \dim_K \mathrm{Hom}_A(T_3, S_5) = 1,
\end{aligned}$$

and hence $\mathrm{Hom}_A(T, I_3) \cong R$, $\mathrm{Hom}_A(T, I_4) \cong I_2^B$, $\mathrm{Hom}_A(T, S_5) \cong S_3^B$ in mod B. This shows that

$$\begin{aligned}
\mathcal{Y}(T) &= \mathrm{Hom}_A(T, \mathcal{T}(T))\\
&= \mathrm{add}\left(\left(\bigoplus_{i=1}^{5} P_i^B\right) \oplus S_2^B \oplus \left(P_4^B/S_5^B\right) \oplus R \oplus I_2^B \oplus S_3^B\right).
\end{aligned}$$

Similarly, applying Lemma 3.13 again, we conclude that

$$\begin{aligned}
c_i(\mathrm{Ext}_A^1(T, S_2)) &= \dim_K \mathrm{Hom}_A(S_2, \tau_A T_i) = 0, \text{ for } i \in \{1,2,3\},\\
c_i(\mathrm{Ext}_A^1(T, S_2)) &= \dim_K \mathrm{Hom}_A(S_2, \tau_A T_i) = 1, \text{ for } i \in \{4,5\},
\end{aligned}$$

and hence $\mathrm{Ext}_A^1(T, S_2) \cong I_5^B$ in mod B. Finally, we have

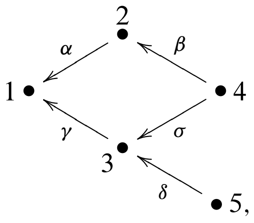
$$\begin{aligned}
c_i(\mathrm{Ext}_A^1(T, P_3/S_1)) &= \dim_K \mathrm{Hom}_A(P_3/S_1, \tau_A T_i) = 0, \text{ for } i \in \{1,2,3,5\},\\
c_4(\mathrm{Ext}_A^1(T, P_3/S_1)) &= \dim_K \mathrm{Hom}_A(P_3/S_1, \tau_A T_4) = 1,
\end{aligned}$$

and hence $\mathrm{Ext}_A^1(T, P_3/S_1) \cong S_4^B$ in mod B. This proves that

$$\mathcal{X}(T) = \mathrm{Ext}_A^1(T, \mathcal{F}(T)) = \mathrm{add}(I_5^B \oplus S_4^B).$$

Observe that every indecomposable module in mod B belongs to $\mathcal{X}(T)$ or $\mathcal{Y}(T)$.

Example 3.16. Let K be a field, Q the quiver



I the ideal in KQ generated by $\beta\alpha - \sigma\gamma$ and $\delta\gamma$, and $A = KQ/I$ the associated bound quiver algebra. Clearly, A is not a hereditary algebra, again by Theorem I.9.6. Applying Theorem III.10.2, we compute that the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
 & & & & P_5 \arrow[dr] & & I_2 \arrow[dr] & \\
 & P_2 \arrow[dr] & & S_3 \arrow[ur] \arrow[dr] & & V \arrow[ur] \arrow[dr] & & I_4 = S_4 \\
S_1 = P_1 \arrow[ur] \arrow[ddr] & & \operatorname{rad} P_4 \arrow[ur] \arrow[r] \arrow[ddr] & P_4 \arrow[r] \arrow[d, equal] & P_4/S_1 \arrow[ur] \arrow[ddr] & & I_3 \arrow[ur] \arrow[ddr] & \\
 & & & I_1 & & & & \\
 & P_3 \arrow[uur] & & S_2 \arrow[uur] & & P_4/P_2 \arrow[uur] & & I_5 = S_5,
\end{tikzcd}$$

where S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} A$ given by the vertex $i \in \{1, 2, 3, 4, 5\}$ of Q, and V is the indecomposable module in $\operatorname{mod} B$ with the composition vector

$$c(V) = \big(c_1(V), c_2(V), c_3(V), c_4(V), c_5(V)\big) = (0, 1, 1, 1, 1).$$

Let $T_1 = P_2$, $T_2 = S_2$, $T_3 = P_4$, $T_4 = V$, $T_5 = I_2$, and $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$. We claim that T is a tilting module in $\operatorname{mod} A$. Since T_1 and T_3 are projective modules and T_2, T_4, T_5 have minimal projective resolutions in $\operatorname{mod} A$ of the forms

$$0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow T_2 \longrightarrow 0,$$
$$0 \longrightarrow P_3 \longrightarrow P_4 \oplus P_5 \longrightarrow T_4 \longrightarrow 0,$$
$$0 \longrightarrow P_3 \longrightarrow P_4 \longrightarrow T_5 \longrightarrow 0,$$

we obtain $\operatorname{pd}_A T \le 1$. Moreover, applying Corollary III.6.4, we get isomorphisms in $\operatorname{mod} K$

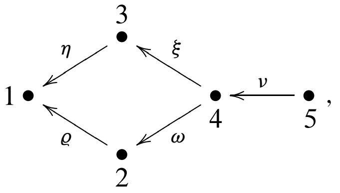
$$\operatorname{Ext}_A^1(T, T) \cong D \operatorname{Hom}_A(T, \tau_A T) = D \operatorname{Hom}_A(T, P_3 \oplus S_3 \oplus P_5) = 0,$$

and so T is a partial tilting module in $\operatorname{mod} A$. Since $K_0(A)$ is of rank 5, it follows from Proposition 3.11 that T is a tilting module in $\operatorname{mod} A$. Further, the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ induced by T is

$$\begin{aligned}
\mathcal{T}(T) &= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Ext}_A^1(T, M) = 0\big\} \\
&= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(M, \tau_A T) = 0\big\} \\
&= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(M, P_3 \oplus S_3 \oplus P_5) = 0\big\} \\
&= \operatorname{add}\big(P_2 \oplus S_2 \oplus P_4 \oplus (P_4/S_1) \oplus V \oplus I_2 \oplus (P_4/P_2) \oplus I_3 \oplus S_4 \oplus S_5\big), \\
\mathcal{F}(T) &= \big\{N \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(T, N) = 0\big\} = \operatorname{add}(S_1 \oplus P_3 \oplus S_3 \oplus P_5).
\end{aligned}$$

Observe that the indecomposable module $\operatorname{rad} P_4$ is neither in $\mathcal{T}(T)$, nor in $\mathcal{F}(T)$.

Consider now the endomorphism algebra $B = \operatorname{End}_A(T)$, the canonical pairwise orthogonal primitive idempotents $e_i = u_i p_i$, $i \in \{1,2,3,4,5\}$, with $1_B = e_1 + e_2 + e_3 + e_4 + e_5$, and the associated indecomposable projective modules $P_j^B = e_j B \cong \operatorname{Hom}_A(T, T_j)$, $j \in \{1,2,3,4,5\}$, in $\operatorname{mod} B$ (see Proposition 3.12). The algebra B is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver



J is the ideal in $K\Delta$ generated by $\xi\eta - \omega\varrho$, and the vertices $1,2,3,4,5$ of Δ correspond to the idempotents e_1, e_2, e_3, e_4, e_5 of B. In particular, B is not a hereditary algebra. Applying Theorem III.10.2, we compute that the Auslander–Reiten quiver Γ_B of B is of the form

$$\begin{tikzcd}
 & P_2^B \arrow[dr] & & S_3^B \arrow[dr] & & P_4^B/P_3^B \arrow[dr] & & I_3^B \arrow[dr] & & \\
S_1^B = P_1^B \arrow[ur] \arrow[r] & P_3^B \arrow[r] & \operatorname{rad} P_4^B \arrow[ur] \arrow[r] \arrow[dr] & S_2^B \arrow[r] & P_4^B/S_1^B \arrow[ur] \arrow[r] \arrow[dr] & P_4^B/P_2^B \arrow[r] & W \arrow[ur] \arrow[r] \arrow[dr] & I_2^B \arrow[r] & I_4^B \arrow[dr] & \\
 & & & P_4^B \arrow[ur] \arrow[dr] & & P_5^B/S_1^B \arrow[ur] & & S_4^B \arrow[ur] & & I_5^B = S_5^B \\
 & & & & P_5^B = I_1^B \arrow[ur] & & & & &
\end{tikzcd}$$

where $S_i^B = e_i B / e_i \operatorname{rad} B$ and $I_i^B = D(Be_i)$, for $i \in \{1,2,3,4,5\}$, and W is the indecomposable module in $\operatorname{mod} B$ with the composition vector

$$c(W) = \big(c_1(W), c_2(W), c_3(W), c_4(W), c_5(W)\big) = (0,1,1,2,1).$$

Since $B \cong K\Delta/J$, we have also $\operatorname{End}_B(S_i^B) \cong K$ for any $i \in \{1,2,3,4,5\}$.

We describe now the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ induced by T, using the equivalences of categories

$$\operatorname{Hom}_A(T,-)\colon \mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T),$$
$$\operatorname{Ext}_A^1(T,-)\colon \mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T),$$

established in Theorem 3.8. Since $P_i^B \cong \operatorname{Hom}_A(T, T_i)$, for $i \in \{1,2,3,4,5\}$, we have isomorphisms in $\operatorname{mod} B$

$$P_1^B \cong \operatorname{Hom}_A(T, P_2), \quad P_2^B \cong \operatorname{Hom}_A(T, S_2), \quad P_3^B \cong \operatorname{Hom}_A(T, P_4),$$
$$P_4^B \cong \operatorname{Hom}_A(T, V), \quad P_5^B \cong \operatorname{Hom}_A(T, I_2).$$

Applying Lemma 3.13, we obtain the equalities

$$\begin{aligned}
c_i(\mathrm{Hom}_A(T, P_4/S_1)) &= \dim_K \mathrm{Hom}_A(T_i, P_4/S_1) = 1, \text{ for } i \in \{1,2,3\},\\
c_i(\mathrm{Hom}_A(T, P_4/S_1)) &= \dim_K \mathrm{Hom}_A(T_i, P_4/S_1) = 0, \text{ for } i \in \{4,5\},\\
c_i(\mathrm{Hom}_A(T, P_4/P_2)) &= \dim_K \mathrm{Hom}_A(T_i, P_4/P_2) = 0, \text{ for } i \in \{1,2,4,5\},\\
c_3(\mathrm{Hom}_A(T, P_4/P_2)) &= \dim_K \mathrm{Hom}_A(T_3, P_4/P_2) = 1,\\
c_i(\mathrm{Hom}_A(T, I_3)) &= \dim_K \mathrm{Hom}_A(T_i, I_3) = 0, \text{ for } i \in \{1,2,5\},\\
c_i(\mathrm{Hom}_A(T, I_3)) &= \dim_K \mathrm{Hom}_A(T_i, I_3) = 1, \text{ for } i \in \{3,4\},\\
c_i(\mathrm{Hom}_A(T, S_4)) &= \dim_K \mathrm{Hom}_A(T_i, S_4) = 0, \text{ for } i \in \{1,2\},\\
c_i(\mathrm{Hom}_A(T, S_4)) &= \dim_K \mathrm{Hom}_A(T_i, S_4) = 1, \text{ for } i \in \{3,4,5\},\\
c_i(\mathrm{Hom}_A(T, S_5)) &= \dim_K \mathrm{Hom}_A(T_i, S_5) = 0, \text{ for } i \in \{1,2,3,5\},\\
c_4(\mathrm{Hom}_A(T, S_5)) &= \dim_K \mathrm{Hom}_A(T_4, S_5) = 1.
\end{aligned}$$

Hence, we have in mod B the isomorphisms

$$\mathrm{Hom}_A(T, P_4/S_1) \cong \mathrm{rad}\, P_4^B, \quad \mathrm{Hom}_A(T, P_4/P_2) \cong S_3^B,$$
$$\mathrm{Hom}_A(T, I_3) \cong P_4^B/P_2^B, \quad \mathrm{Hom}_A(T, S_4) \cong I_3^B, \quad \mathrm{Hom}_A(T, S_5) \cong S_4^B.$$

Therefore,

$$\begin{aligned}
\mathcal{Y}(T) &= \mathrm{Hom}_A(T, \mathcal{T}(T))\\
&= \mathrm{add}\left(\left(\bigoplus_{i=1}^{5} P_i^B\right) \oplus \mathrm{rad}\, P_4^B \oplus S_3^B \oplus \left(P_4^B/P_2^B\right) \oplus I_3^B \oplus S_4^B\right).
\end{aligned}$$

Applying Lemma 3.13 again, we conclude that

$$\begin{aligned}
c_i(\mathrm{Ext}_A^1(T, S_1)) &= \dim_K \mathrm{Hom}_A(S_1, \tau_A T_i) = 0, \text{ for } i \in \{1,3,4,5\},\\
c_2(\mathrm{Ext}_A^1(T, S_1)) &= \dim_K \mathrm{Hom}_A(S_1, \tau_A T_2) = 1,\\
c_i(\mathrm{Ext}_A^1(T, P_3)) &= \dim_K \mathrm{Hom}_A(P_3, \tau_A T_i) = 0, \text{ for } i \in \{1,3\},\\
c_i(\mathrm{Ext}_A^1(T, P_3)) &= \dim_K \mathrm{Hom}_A(P_3, \tau_A T_i) = 1, \text{ for } i \in \{2,4,5\},\\
c_i(\mathrm{Ext}_A^1(T, S_3)) &= \dim_K \mathrm{Hom}_A(S_3, \tau_A T_i) = 0, \text{ for } i \in \{1,2,3\},\\
c_i(\mathrm{Ext}_A^1(T, S_3)) &= \dim_K \mathrm{Hom}_A(S_3, \tau_A T_i) = 1, \text{ for } i \in \{4,5\},\\
c_i(\mathrm{Ext}_A^1(T, P_5)) &= \dim_K \mathrm{Hom}_A(P_5, \tau_A T_i) = 0, \text{ for } i \in \{1,2,3,4\},\\
c_5(\mathrm{Ext}_A^1(T, P_5)) &= \dim_K \mathrm{Hom}_A(P_5, \tau_A T_5) = 1.
\end{aligned}$$

This shows that we have in mod B the isomorphisms

$$\mathrm{Ext}_A^1(T, S_1) \cong S_2^B, \quad \mathrm{Ext}_A^1(T, P_3) \cong I_2^B,$$
$$\mathrm{Ext}_A^1(T, S_3) \cong I_4^B, \quad \mathrm{Ext}_A^1(T, P_5) \cong S_5^B.$$

Therefore,

$$\mathcal{X}(T) = \operatorname{Ext}_A^1(T, \mathcal{F}(T)) = \operatorname{add}(S_2^B \oplus I_2^B \oplus I_4^B \oplus S_5^B).$$

We would like to point out that the indecomposable modules P_4^B/S_1^B, P_5^B/S_1^B, P_4^B/P_3^B, W are neither in $\mathcal{X}(T)$, nor in $\mathcal{Y}(T)$.

Example 3.17. Let A be the $\mathbb{Q}$-subalgebra

$$\begin{bmatrix} \mathbb{Q} & 0 & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{bmatrix} \in M_3(\mathbb{R}) \;\middle|\; \begin{matrix} a \in \mathbb{Q}, \\ b, c, x, y, z \in \mathbb{Q}(\sqrt[3]{2}) \end{matrix} \right\}$$

of the matrix algebra $M_3(\mathbb{R})$. We proved in Example VII.8.28 that A is a hereditary algebra of Euclidean type $\widetilde{\mathbb{G}}_{22}$, with the quiver Q_A of the form

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "(1,3)"'] & \underset{3}{\bullet} \arrow[l] \end{tikzcd},$$

and described the Auslander–Reiten quiver Γ_A of A. In particular, the postprojective component $\mathcal{P}(A)$ of Γ_A is of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$

$$\begin{tikzcd}[column sep=small]
 & & P_3 \arrow[dr] & & \tau_A^{-1}P_3 \arrow[dr] & & \tau_A^{-2}P_3 \arrow[dr] & & \tau_A^{-3}P_3 \arrow[dr, no head] & & \\
 & P_2 \arrow[ur] \arrow[dr, "(3,1)"] & & \tau_A^{-1}P_2 \arrow[ur] \arrow[dr, "(3,1)"] & & \tau_A^{-2}P_2 \arrow[ur] \arrow[dr, "(3,1)"] & & \tau_A^{-3}P_2 \arrow[ur] \arrow[dr, no head] & & & \cdots \\
P_1 \arrow[ur, "(1,3)"] & & \tau_A^{-1}P_1 \arrow[ur, "(1,3)"] & & \tau_A^{-2}P_1 \arrow[ur, "(1,3)"] & & \tau_A^{-3}P_1 \arrow[ur, "(1,3)"] & & & &
\end{tikzcd}$$

Since $P_1 = S_1$ is a simple projective and not injective module in $\operatorname{mod} A$, we may consider the APR-tilting module

$$T = T_{S_1} = \tau_A^{-1}S_1 \oplus P_2 \oplus P_3$$

in $\operatorname{mod} A$ (see Example 2.11). Let $T_1 = \tau_A^{-1}S_1$, $T_2 = P_2$ and $T_3 = P_3$. Then it follows from Example 2.11 that the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ induced by T is given by

$$\mathcal{T}(T) = \operatorname{mod} A \setminus \operatorname{add}(S_1),$$

the additive category given by all indecomposable modules in mod A which are nonisomorphic to S_1, and

$$\mathcal{F}(T) = \operatorname{add}(S_1).$$

Consider the endomorphism algebra $B = \operatorname{End}_A(T)$, the canonical pairwise orthogonal primitive idempotents $e_1 = u_1 p_1$, $e_2 = u_2 p_2$, $e_3 = u_3 p_3$ with $1_B = e_1 + e_2 + e_3$, the associated indecomposable projective B-modules $P_1^B = e_1 B$, $P_2^B = e_2 B$, $P_3^B = e_3 B$ in mod B, and the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B induced by T. Then it follows from Theorem 3.8 that we have the equivalences of categories

$$\operatorname{Hom}_A(T,-)\colon \mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T),$$

$$\operatorname{Ext}_A^1(T,-)\colon \mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T).$$

Moreover, we know from Proposition 3.12 that there are isomorphisms $P_i^B \cong \operatorname{Hom}_A(T, T_i)$ in mod B, for $i \in \{1,2,3\}$. Since P_1^B, P_2^B, P_3^B belong to the torsion-free class $\mathcal{Y}(T)$ and $\mathcal{Y}(T)$ is closed under predecessors in mod B, we conclude that Γ_B contains a full valued subquiver of the form

$$\begin{tikzcd} P_1^B & P_2^B \arrow[l, "{(1,3)}"'] \arrow[r] & P_3^B \end{tikzcd} .$$

Hence, the radical of any indecomposable projective module in mod B is projective, and consequently B is a hereditary algebra (see Theorem I.9.1). Applying now Proposition VII.1.10, we conclude that B is a hereditary algebra of Euclidean type $\widetilde{\mathbb{G}}_{22}$ and the quiver Q_B of B is of the form

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{(1,3)}"] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l] \end{tikzcd} ,$$

where the vertices $1, 2, 3$ correspond to the chosen primitive idempotents e_1, e_2, e_3 of B. Then it follows from Theorems VII.6.1, VII.6.2, VII.7.4, and VII.8.12 that the Auslander–Reiten quiver Γ_B of B has the disjoint union form

$$\Gamma_B = \mathcal{P}(B) \cup \mathcal{R}(B) \cup \mathcal{Q}(B),$$

where $\mathcal{P}(B)$ is the unique postprojective component of the form $(-\mathbb{N})Q_B^{\mathrm{op}}$

$$\begin{tikzcd}[column sep=small]
 & P_3^B \arrow[dr] & & \tau_B^{-1} P_3^B \arrow[dr] & & \tau_B^{-2} P_3 \arrow[dr, dash] & \\
S_2^B = P_2^B \arrow[ur] \arrow[dr, "{(3,1)}"] & & \tau_B^{-1} P_2^B \arrow[ur] \arrow[dr, "{(3,1)}"] & & \tau_B^{-2} P_2^B \arrow[ur] \arrow[dr, "{(3,1)}"] & & \cdots \\
 & P_1^B \arrow[ur, "{(1,3)}"] & & \tau_B^{-1} P_1^B \arrow[ur, "{(1,3)}"] & & \tau_B^{-2} P_1^B \arrow[ur, dash] &
\end{tikzcd}$$

$\mathcal{Q}(B)$ is the unique preinjective component of the form $\mathbb{N}Q_B^{\mathrm{op}}$

$$\begin{tikzcd}
 & \tau_B^3 I_3^B \arrow[dr] & & \tau_B^2 I_3^B \arrow[dr] & & \tau_B I_3^B \arrow[dr] & & I_3^B = S_3^B \\
\cdots \arrow[ur] \arrow[dr] & & \tau_B^2 I_2^B \arrow[ur] \arrow[dr, "{(3,1)}"] & & \tau_B I_2^B \arrow[ur] \arrow[dr, "{(3,1)}"] & & I_2^B \arrow[ur] \arrow[dr, "{(3,1)}"] & \\
 & \tau_B^3 I_1^B \arrow[ur, "{(1,3)}"] & & \tau_B^2 I_1^B \arrow[ur, "{(1,3)}"] & & \tau_B I_1^B \arrow[ur, "{(1,3)}"] & & I_1^B = S_1^B
\end{tikzcd}$$

and $\mathcal{R}(B)$ is a family $\mathcal{T}^B = \bigcup_{\lambda\in\Lambda(B)} \mathcal{T}_\lambda^B$ of stable tubes separating $\mathcal{P}(B)$ from $\mathcal{Q}(B)$. Here $I_i^B = D(Be_i)$, $i \in \{1,2,3\}$, are the indecomposable injective B-modules associated to e_1, e_2, e_3. Observe that $I_1^B = S_1^B$ and $I_3^B = S_3^B$ are the simple injective modules in $\operatorname{mod} B$ corresponding to the sources 1 and 3 of Q_B (see Lemma VII.1.13). We claim that $S_1^B \cong \operatorname{Ext}_B^1(T, S_1)$ in $\operatorname{mod} B$, and consequently $\mathcal{X}(T) = \operatorname{Ext}_A^1(T, \mathcal{F}(T)) = \operatorname{add}(S_1^B)$. Applying Lemma 3.13, we obtain the equalities

$$\begin{aligned}
c_1\big(\operatorname{Ext}_A^1(T,S_1)\big)\dim_{\mathbb{Q}}\operatorname{End}_B(S_1^B) &= \dim_{\mathbb{Q}}\operatorname{Hom}_A(S_1,\tau_A T_1)\\
&= \dim_{\mathbb{Q}}\operatorname{End}_A(S_1) = \dim_{\mathbb{Q}}\mathbb{Q} = 1,\\
c_2\big(\operatorname{Ext}_A^1(T,S_1)\big)\dim_{\mathbb{Q}}\operatorname{End}_B(S_2^B) &= \dim_{\mathbb{Q}}\operatorname{Hom}_A(S_1,\tau_A T_2) = 0,\\
c_3\big(\operatorname{Ext}_A^1(T,S_1)\big)\dim_{\mathbb{Q}}\operatorname{End}_B(S_3^B) &= \dim_{\mathbb{Q}}\operatorname{Hom}_A(S_1,\tau_A T_3) = 0,
\end{aligned}$$

because $T_2 = P_2$ and $T_3 = P_3$ are projective. Therefore, indeed $\operatorname{Ext}_B^1(T, S_1) \cong S_1^B$ in $\operatorname{mod} B$. We show now that every indecomposable module Z in $\operatorname{mod} B$ belongs to $\mathcal{X}(T)$ or $\mathcal{Y}(T)$. Indeed, consider the canonical exact sequence

$$0 \longrightarrow tZ \xrightarrow{i} Z \xrightarrow{\pi} Z/tZ \longrightarrow 0$$

for such a module Z in $\operatorname{mod} B$ with respect to the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$, so tZ belongs to $\mathcal{X}(T)$ and Z/tZ belongs to $\mathcal{Y}(T)$. Since $\mathcal{X}(T)$ is the additive category $\operatorname{add}(S_1^B)$ of the simple injective module S_1^B in $\operatorname{mod} B$, $tZ \neq 0$ forces $tZ \cong Z$, because then i is a section and Z is indecomposable. Clearly, $tZ = 0$ implies $Z = Z/tZ$. This proves the claim.

In particular, we obtain the equalities

$$\begin{aligned}
\mathcal{P}(B) &= \operatorname{Hom}_A(T, \mathcal{P}(A) \setminus \{S_1\}),\\
\mathcal{R}(B) &= \operatorname{Hom}_A(T, \mathcal{R}(A)),\\
\mathcal{Q}(B) \setminus \{S_1^B\} &= \operatorname{Hom}_A(T, \mathcal{Q}(A)).
\end{aligned}$$

Moreover, it follows from Example VII.8.28 that $\mathcal{R}(B)$ contains a stable tube of rank 2 and an infinite family of stable tubes of rank 1 indexed by the irreducible monic polynomials in $\mathbb{Q}(\sqrt[3]{2})[x]$.

Observe also that we have the isomorphisms of $\mathbb{Q}$-algebras

$$\begin{aligned}
&\operatorname{End}_B(P_1^B) \cong \operatorname{End}_B\big(\operatorname{Hom}_B(T, T_1)\big) \cong \operatorname{End}_A(T_1) \cong \operatorname{End}_A(P_1),\\
&\operatorname{End}_B(P_2^B) \cong \operatorname{End}_B\big(\operatorname{Hom}_B(T, T_2)\big) \cong \operatorname{End}_A(T_2) \cong \operatorname{End}_A(P_2),\\
&\operatorname{End}_B(P_3^B) \cong \operatorname{End}_B\big(\operatorname{Hom}_B(T, T_3)\big) \cong \operatorname{End}_A(T_3) \cong \operatorname{End}_A(P_3),\\
&\operatorname{End}_A(P_1) \cong \operatorname{End}_A(S_1) \cong \mathbb{Q},\\
&\operatorname{End}_A(P_2) \cong \operatorname{End}_A(S_2) \cong \mathbb{Q}(\sqrt[3]{2}),\\
&\operatorname{End}_A(P_3) \cong \operatorname{End}_A(S_3) \cong \mathbb{Q}(\sqrt[3]{2}),
\end{aligned}$$

by Proposition VII.6.8 and Theorem 3.8. Further, we have $(\operatorname{rad} B)^2 = 0$, because $e_i(\operatorname{rad} B)^2 e_j = \operatorname{rad}_B^2(P_j^B, P_i^B) = 0$, for all $i, j \in \{1, 2, 3\}$. Applying Corollary III.9.4, we obtain the equalities

$$\begin{aligned}
3 &= d_{P_2^B P_1^B} = \dim_{F_{P_1^B}} \operatorname{irr}(P_2^B, P_1^B)\\
&= \dim_{\mathbb{Q}} \operatorname{rad}_B(P_2^B, P_1^B) = \dim_{\mathbb{Q}} e_1(\operatorname{rad} B)e_2,\\
1 &= d'_{P_2^B P_1^B} = \dim_{F_{P_2^B}} \operatorname{irr}(P_2^B, P_1^B)\\
&= \dim_{\mathbb{Q}(\sqrt[3]{2})} \operatorname{rad}_B(P_2^B, P_1^B) = \dim_{\mathbb{Q}(\sqrt[3]{2})} e_1(\operatorname{rad} B)e_2,\\
1 &= d_{P_2^B P_3^B} = \dim_{F_{P_3^B}} \operatorname{irr}(P_2^B, P_3^B)\\
&= \dim_{\mathbb{Q}(\sqrt[3]{2})} \operatorname{rad}_B(P_2^B, P_3^B) = \dim_{\mathbb{Q}(\sqrt[3]{2})} e_3(\operatorname{rad} B)e_2,\\
1 &= d'_{P_2^B P_3^B} = \dim_{F_{P_2^B}} \operatorname{irr}(P_2^B, P_3^B)\\
&= \dim_{\mathbb{Q}(\sqrt[3]{2})} \operatorname{rad}_B(P_2^B, P_3^B) = \dim_{\mathbb{Q}(\sqrt[3]{2})} e_3(\operatorname{rad} B)e_2,
\end{aligned}$$

Moreover, we have also the equalities

$$\begin{aligned}
0 &= \operatorname{Hom}_B(P_1^B, P_2^B) = \operatorname{rad}_B(P_1^B, P_2^B) = e_2(\operatorname{rad} B)e_1,\\
0 &= \operatorname{Hom}_B(P_1^B, P_3^B) = \operatorname{rad}_B(P_1^B, P_3^B) = e_3(\operatorname{rad} B)e_1,\\
0 &= \operatorname{Hom}_B(P_3^B, P_1^B) = \operatorname{rad}_B(P_3^B, P_1^B) = e_1(\operatorname{rad} B)e_3,\\
0 &= \operatorname{Hom}_B(P_3^B, P_2^B) = \operatorname{rad}_B(P_3^B, P_2^B) = e_2(\operatorname{rad} B)e_3.
\end{aligned}$$

Further, we have the canonical isomorphism of algebras $B \xrightarrow{\sim} \operatorname{End}_B(B_B) = \operatorname{End}_B(P_1^B \oplus P_2^B \oplus P_3^B)$, by Lemma I.6.1. Summing up, we conclude that B is isomorphic to the following $\mathbb{Q}$-subalgebra of the matrix algebra $M_3(\mathbb{R})$

$$\begin{bmatrix} \mathbb{Q} & \mathbb{Q}(\sqrt[3]{2}) & 0 \\ 0 & \mathbb{Q}(\sqrt[3]{2}) & 0 \\ 0 & \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) \end{bmatrix} = \left\{ \begin{bmatrix} a & x & 0 \\ 0 & b & 0 \\ 0 & y & c \end{bmatrix} \in M_3(\mathbb{R}) \;\middle|\; \begin{matrix} a \in \mathbb{Q}, \\ b, c, x, y \in \mathbb{Q}(\sqrt[3]{2}) \end{matrix} \right\}.$$

Example 3.18. Let K be a field, Q the quiver

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\beta"] & \underset{3}{\bullet} \arrow[l, "\gamma"'] \end{tikzcd},$$

and $A = KQ$ the path algebra of Q over K. Then it follows from Example VII.9.25 that A is a hereditary algebra of wild type, the quiver Q_A of A is of the form

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "(2{,}2)"'] & \underset{3}{\bullet} \arrow[l] \end{tikzcd},$$

and the Auslander–Reiten quiver Γ_A of A consists of a unique postprojective component $\mathcal{P}(A)$ of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$

$$\begin{tikzcd}[column sep=small]
 & & P_3 \arrow[dr] & & \tau_A^{-1}P_3 \arrow[dr] & & \tau_A^{-2}P_3 \arrow[dr] & & \tau_A^{-3}P_3 \arrow[dr, dash] & & \\
 & P_2 \arrow[ur] \arrow[dr, "(2{,}2)"] & & \tau_A^{-1}P_2 \arrow[ur] \arrow[dr, "(2{,}2)"] & & \tau_A^{-2}P_2 \arrow[ur] \arrow[dr, "(2{,}2)"] & & \tau_A^{-3}P_2 \arrow[ur] \arrow[dr, dash] & & {} & \cdots \\
P_1 \arrow[ur, "(2{,}2)"] & & \tau_A^{-1}P_1 \arrow[ur, "(2{,}2)"] & & \tau_A^{-2}P_1 \arrow[ur, "(2{,}2)"] & & \tau_A^{-3}P_1 \arrow[ur, "(2{,}2)"] & & {} & &
\end{tikzcd}$$

a unique preinjective component $\mathcal{Q}(A)$ of the form $\mathbb{N}Q_A^{\mathrm{op}}$

$$\begin{tikzcd}[column sep=small]
 & & {} \arrow[dr] & & \tau_A^3 I_3 \arrow[dr] & & \tau_A^2 I_3 \arrow[dr] & & \tau_A I_3 \arrow[dr] & & I_3 \\
\cdots & {} \arrow[dr] & & \tau_A^3 I_2 \arrow[ur] \arrow[dr, "(2{,}2)"] & & \tau_A^2 I_2 \arrow[ur] \arrow[dr, "(2{,}2)"] & & \tau_A I_2 \arrow[ur] \arrow[dr, "(2{,}2)"] & & I_2 \arrow[ur] & \\
 & & \tau_A^3 I_1 \arrow[ur, "(2{,}2)"] & & \tau_A^2 I_1 \arrow[ur, "(2{,}2)"] & & \tau_A I_1 \arrow[ur, "(2{,}2)"] & & I_1 \arrow[ur, "(2{,}2)"] & &
\end{tikzcd}$$

and an infinite family $\mathcal{R}(A)$ of components of the form $\mathbb{Z}\mathbb{A}_\infty$.

Consider the modules $T_1 = P_1$, $T_2 = P_3$, $T_3 = I_3$, and $T = T_1 \oplus T_2 \oplus T_3$. Since A is a hereditary K-algebra, we have $\mathrm{pd}_A T \leq 1$. Moreover, applying Corollary III.6.4, we get isomorphisms in mod K

$$\mathrm{Ext}_A^1(T, T) \cong D\,\mathrm{Hom}_A(T, \tau_A T) = D\,\mathrm{Hom}_A(P_1 \oplus P_3 \oplus I_3, \tau_A I_3) = 0,$$

because $\tau_A I_3 = S_2$. Hence, it follows from Proposition 3.11 that T is a tilting module in $\operatorname{mod} A$. The torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ induced by T is described as follows:

$$\begin{aligned}
\mathcal{T}(T) &= \{M \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(T, M) = 0\} \\
&= \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M, \tau_A T) = 0\} \\
&= \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M, \tau_A I_3) = 0\} \\
&= \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M, S_2) = 0\} \\
&= \{M \in \operatorname{mod} A \mid \operatorname{top}(M) \in \operatorname{add}(S_1 \oplus S_3)\}, \\
\mathcal{F}(T) &= \{N \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, N) = 0\} = \operatorname{add}(S_2),
\end{aligned}$$

by Lemma VII.5.6.

Consider now the endomorphism algebra $B = \operatorname{End}_A(T)$, the canonical pairwise orthogonal primitive idempotents $e_1 = u_1 p_1$, $e_2 = u_2 p_2$, $e_3 = u_3 p_3$ of B with $1_B = e_1 + e_2 + e_3$, and the associated indecomposable projective B-modules $P_1^B = e_1 B$, $P_2^B = e_2 B$, $P_3^B = e_3 B$ in $\operatorname{mod} B$. Then we have in $\operatorname{mod} B$ isomorphisms $P_i^B \cong \operatorname{Hom}_A(T, T_i)$, for $i \in \{1, 2, 3\}$, by Proposition 3.12. We also recall that, under the canonical equivalence $\operatorname{mod} A \xrightarrow{\sim} \operatorname{rep}_K(Q)$, the modules T_1, T_2, T_3 correspond to the following indecomposable representations of Q over K:

$$\begin{tikzcd}
P_1\colon\ K & 0 \arrow[l, shift right] \arrow[l, shift left] & 0 \arrow[l]\ , & P_3\colon\ K^2 & K \arrow[l, shift right, "\begin{bmatrix}1\\0\end{bmatrix}"'] \arrow[l, shift left, "\begin{bmatrix}0\\1\end{bmatrix}"] & K \arrow[l, "1"']\ , & I_3\colon\ 0 & 0 \arrow[l, shift right] \arrow[l, shift left] & K \arrow[l]\ .
\end{tikzcd}$$

Then we easily conclude that B is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver

$$\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, shift right, "\eta"'] \arrow[l, shift left, "\xi"] & \underset{3}{\bullet} \arrow[l, "\varrho"']
\end{tikzcd}$$

and J is the ideal in $K\Delta$ generated by $\varrho\eta$ and $\varrho\xi$, where the vertices $1, 2, 3$ of Δ correspond to the chosen idempotents e_1, e_2, e_3 of B. It also follows from Theorem I.2.10 that there exists a K-linear equivalence of categories

$$\operatorname{mod} B \longrightarrow \operatorname{rep}_K(\Delta, J).$$

Moreover, the path algebra $H = K\Sigma$ of the Kronecker quiver Σ, given by the vertices 1 and 2, is a quotient algebra of B, and hence $\operatorname{mod} H$ is a full subcategory of $\operatorname{mod} B$ consisting of all modules which have no the simple module $S_3^B = e_3 B / e_3 \operatorname{rad} B$ as a composition factor. Since $\operatorname{mod} H \xrightarrow{\sim} \operatorname{rep}_K(\Sigma)$ and $\operatorname{mod} B \cong \operatorname{rep}_K(\Delta, J)$, we infer that P_3^B and S_3 are unique (up to isomorphism) indecomposable modules in $\operatorname{mod} B$ which are not in $\operatorname{mod} H$. We also note that

P_3^B corresponds to the indecomposable representation in $\operatorname{rep}_K(\Delta, J)$ of the form

$$\begin{tikzcd} 0 & K \arrow[l, shift left] \arrow[l, shift right] & K \arrow[l, "1"'] \end{tikzcd},$$

and hence P_3^B is also an injective module in $\operatorname{mod} B$ with $\operatorname{soc}(P_3^B) = S_2^B = e_2B/e_2 \operatorname{rad} B$ and $\operatorname{top}(P_3^B) = S_3^B$. Therefore, it follows from Example VII.8.27 that the Auslander–Reiten quiver Γ_B of B is of the form

$$\Gamma_B = \mathcal{P}(B) \cup \mathcal{T}^B \cup \mathcal{Q}(B),$$

where $\mathcal{P}(B) = \mathcal{P}(H)$ is the postprojective component of the form $(-\mathbb{N})\Sigma$

$$\begin{tikzcd}[column sep=small]
& P_2^B \arrow[dr, "{(2,2)}"] & & \tau_B^{-1} P_2^B \arrow[dr, no head] & & \cdots \arrow[dr] & & \tau_B^{-m} P_2^B \arrow[dr, "{(2,2)}"] & & \cdots \\
P_1^B \arrow[ur, "{(2,2)}"] & & \tau_B^{-1} P_1^B \arrow[ur, "{(2,2)}"] & & {} & & \tau_B^{-m} P_1^B \arrow[ur, "{(2,2)}"] & & \tau_B^{-m-1} P_1^B \arrow[ur, no head] &
\end{tikzcd}$$

$\mathcal{T}^B = \mathcal{T}^H$ is the family $\mathcal{T}_\lambda^B = \mathcal{T}_\lambda^H$ of stable tubes of rank 1 with $\lambda \in \operatorname{irr}(K[x]) \cup \{\infty\}$, and $\mathcal{Q}(B)$ is obtained from the preinjective component $\mathcal{Q}(H)$ of Γ_H, of the form $\mathbb{N}\Sigma$ by adding P_3^B and S_3^B, and is of the form

$$\begin{tikzcd}[column sep=small]
& & & & & & & & & I_2^B = P_3^B \arrow[dr] & \\
& \tau_B^{m+1} I_2^B \arrow[dr, "{(2,2)}"] & & \tau_B^{m} I_2^B \arrow[dr, no head] & \cdots \arrow[dr] & & \tau_B I_2^B \arrow[dr, "{(2,2)}"] & & S_2^B \arrow[ur] & & I_3^B = S_3^B \\
\cdots \arrow[ur] & & \tau_B^{m} I_1^B \arrow[ur, "{(2,2)}"] & & {} & \tau_B I_1^B \arrow[ur, "{(2,2)}"] & & I_1^B \arrow[ur, "{(2,2)}"] & & &
\end{tikzcd}$$

where $I_i^B = D(Be_i)$, $i \in \{1, 2, 3\}$, are the indecomposable injective modules in $\operatorname{mod} B$ given by the vertices $1, 2, 3$ of Δ, respectively.

We determine now the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ induced by T. It follows from Theorem 3.8 that there are the equivalences of categories

$$\operatorname{Hom}_A(T, -)\colon \mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T),$$
$$\operatorname{Ext}_A^1(T, -)\colon \mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T).$$

Since $\mathcal{F}(T) = \operatorname{add}(S_2)$, we obtain that $\mathcal{X}(T) = \operatorname{Ext}_A^1(T, \mathcal{F}(T)) = \operatorname{add}(\operatorname{Ext}_A^1(T, S_2))$. Applying Lemma 3.13, we find that

$$c_1(\operatorname{Ext}_A^1(T, S_2)) = \dim_K \operatorname{Hom}_A(S_2, \tau_A T_1) = 0,$$
$$c_2(\operatorname{Ext}_A^1(T, S_2)) = \dim_K \operatorname{Hom}_A(S_2, \tau_A T_2) = 0,$$
$$c_3(\operatorname{Ext}_A^1(T, S_2)) = \dim_K \operatorname{Hom}_A(S_2, \tau_A T_3) = 1,$$

because $T_1 = P_1$ and $T_2 = P_3$ are projective modules, $\tau_A T_3 = S_2$, and $\operatorname{End}_A(S_i) \cong K$ for any $i \in \{1,2,3\}$. Therefore, we conclude that

$$\mathcal{X}(T) = \operatorname{add}(S_3^B).$$

Observe also that S_3^B is a simple injective module. Then, for any indecomposable module Z in $\operatorname{mod} B$ and the associated canonical exact sequence

$$0 \longrightarrow tZ \longrightarrow Z \longrightarrow Z/tZ \longrightarrow 0$$

in $\operatorname{mod} B$ with respect to the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$, we have $Z = tZ \in \mathcal{X}(T)$ or $Z = Z/tZ \in \mathcal{Y}(T)$. This implies that

$$\mathcal{Y}(T) = \operatorname{add}(\operatorname{mod} H, P_3^B)$$

is the additive category in $\operatorname{mod} B$ given by all modules of $\operatorname{mod} H$ and the projective module P_3^B. This gives also a description of the torsion class $\mathcal{T}(T)$ in $\operatorname{mod} A$ as

$$\mathcal{T}(T) = \mathcal{Y}(T) \otimes_B T = \big\{ Y \otimes_B T \,\big|\, Y \in \mathcal{Y}(T) \big\}.$$

Recall that the center of a finite dimensional K-algebra A over a field K is its K-subalgebra $C(A) = \{a \in A \mid ax = xa \text{ for all } x \in A\}$ (see Section IV.5).

Proposition 3.19. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$, *and $B = \operatorname{End}_A(T)$. Then there is an isomorphism of K-algebras*

$$\sigma \colon C(A) \longrightarrow C(B)$$

such that $\sigma(a)(t) = ta$ for $a \in C(A)$ and $t \in T$.

Proof. Observe first that, for $a \in C(A)$, $a_1, a_2 \in A$ and $t_1, t_2 \in T$, we have $\sigma(a)(t_1a_1 + t_2a_2) = (t_1a_1 + t_2a_2)a = (t_1a_1)a + (t_2a_2)a = t_1(a_1a) + t_2(a_2a) = t_1(aa_1) + t_2(aa_2) = (t_1a)a_1 + (t_2a)a_2 = \sigma(a)(t_1)a_1 + \sigma(a)(t_2)a_2$, and hence $\sigma(a) \in B$. Moreover, for $b \in B$ and $t \in T$, we have $(\sigma(a)b)(t) = \sigma(a)(b(t)) = b(t)a = b(ta) = b(\sigma(a)t) = (b\sigma(a))(t)$, and so $\sigma(a)$ belongs to $C(B)$. Finally, for $a_1, a_2 \in C(A)$ and $t \in T$,

$$\begin{aligned}
\sigma(a_1a_2)(t) &= \sigma(a_2a_1)(t) = t(a_2a_1) = \sigma(a_1)\big(\sigma(a_2)t\big) = \big(\sigma(a_1)\sigma(a_2)\big)(t),\\
\sigma(a_1 + a_2)(t) &= t(a_1 + a_2) = ta_1 + ta_2 = \sigma(a_1)(t) + \sigma(a_2)(t)\\
&= \big(\sigma(a_1) + \sigma(a_2)\big)(t),\\
\sigma(1_A)(t) &= t1_A = t = \operatorname{id}_T(t) = 1_B(t),
\end{aligned}$$

and consequently σ is a homomorphism of K-algebras. In fact, σ is a monomorphism of K-algebras, because $\sigma(a) = 0$ for some $a \in C(A)$ implies $Ta = 0$, and then $a = 0$, since T is a faithful right A-module. On the other hand, it follows from Proposition 3.3 that there is an isomorphism of K-algebras $\varrho: A \to \mathrm{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$ such that $\varrho(a)(t) = ta$ for $a \in A$ and $t \in T$. Take now an element $b \in C(B)$ and consider the K-linear map $\lambda(b): T \to T$ given by $\lambda(b)(t) = bt$ for any $t \in T$. We note that $\lambda(b) \in \mathrm{End}_{B^{\mathrm{op}}}(T)$, because $b \in C(B)$ implies that $\lambda(b)(b_1 t) = b(b_1 t) = b_1(bt) = b_1(\lambda(b)t)$ for $b_1 \in B$ and $t \in T$. Further, for any $f \in \mathrm{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$ and $t \in T$, we have the equalities

$$\big(\lambda(b) \cdot f\big)(t) = f\big(\lambda(b)(t)\big) = f(bt) = bf(t) = \lambda(b)\big(f(t)\big) = \big(f \cdot \lambda(b)\big)(t),$$

and hence $\lambda(b) \cdot f = f \cdot \lambda(b)$, where $\cdot$ is the multiplication in $\mathrm{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$. This shows that $\lambda(b)$ belongs to $C(\mathrm{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}})$. Take now the element $a \in C(A)$ such that $\varrho(a) = \lambda(b)$. Then, for any element $t \in T$, we obtain

$$\sigma(a)(t) = ta = \varrho(a)(t) = \lambda(b)(t) = bt = b(t),$$

and hence $\sigma(a) = b$. Therefore, σ is an isomorphism. □

Corollary 3.20. *Let A be an indecomposable finite dimensional K-algebra over a field K, T a tilting module in* mod A, *and* $B = \mathrm{End}_A(T)$. *Then B is an indecomposable K-algebra.*

Proof. Since A is an indecomposable K-algebra, by Lemma I.3.14, 0_A and 1_A are unique central idempotents of A. Then it follows from Proposition 3.19 that $0_B = \sigma(0_A)$ and $1_B = \sigma(1_A)$ are unique central idempotents of B. Applying Lemma I.3.14 again, we conclude that B is an indecomposable K-algebra. □

Let A be a finite dimensional K-algebra over a field K. Then the *global dimension* of A, denoted by gl. dim A, is defined to be the supremum of the projective dimensions $\mathrm{pd}_A M$ of all modules M in mod A. It follows from Corollary I.8.24 that gl. dim $A = 0$ if and only if A is semisimple algebra. Further, by Theorems I.9.1 and I.9.3, gl. dim $A \leq 1$ if and only if A is a hereditary algebra.

We have also the following useful result.

Proposition 3.21. *Let A be a finite dimensional K-algebra of finite global dimension over a field K, and $T_1, \dots, T_n$ pairwise nonisomorphic indecomposable modules in* mod A *such that $T = T_1 \oplus \dots \oplus T_n$ is a tilting module in* mod A. *Then $[T_1], \dots, [T_n]$ form a $\mathbb{Z}$-basis of $K_0(A)$.*

Proof. It follows from Proposition 3.11 that n is the rank of $K_0(A)$. Let $P_1, \dots, P_n$ be a set of pairwise nonisomorphic indecomposable projective modules in mod A, and $S_1 = \mathrm{top}(P_1), \dots, S_n = \mathrm{top}(P_n)$ the associated set of pairwise nonisomorphic simple modules in mod A. We know from Theorem I.11.1 that $[S_1], \dots, [S_n]$ form a $\mathbb{Z}$-basis of $K_0(A)$. Since A is of finite global dimension, for each $i \in \{1, \dots, n\}$ the simple module S_i admits a minimal projective resolution in mod A

$$0 \longrightarrow \mathbb{P}^{(i)}_{m_i} \longrightarrow \mathbb{P}^{(i)}_{m_i - 1} \longrightarrow \cdots \longrightarrow \mathbb{P}^{(i)}_1 \longrightarrow \mathbb{P}^{(i)}_0 \longrightarrow S_i \longrightarrow 0$$

with $\mathbb{P}_0 = P_i$, and hence we obtain

$$[S_i] = \sum_{j=0}^{m_i} (-1)^j \big[\mathbb{P}_j^{(i)}\big]$$

in $K_0(A)$. This shows that $[P_1], \dots, [P_n]$ generate the free abelian group of $K_0(A)$, and consequently form a $\mathbb{Z}$-basis of $K_0(A)$. Further, since T is a tilting module in $\operatorname{mod} A$, for each $i \in \{1, \dots, n\}$, there exists an exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow P_i \longrightarrow \mathbb{T}_0^{(i)} \longrightarrow \mathbb{T}_1^{(i)} \longrightarrow 0$$

with $\mathbb{T}_0^{(i)}$ and $\mathbb{T}_1^{(i)}$ in $\operatorname{add} T$, and hence

$$[P_i] = \big[\mathbb{T}_0^{(i)}\big] - \big[\mathbb{T}_1^{(i)}\big]$$

in $K_0(A)$. This shows that $[T_1], \dots, [T_n]$ generate the free abelian group of $K_0(A)$, and consequently form a $\mathbb{Z}$-basis of $K_0(A)$. □

4 Connecting sequences

Let A be a finite dimensional K-algebra over a field K, T a tilting module in $\operatorname{mod} A$, $B = \operatorname{End}_A(T)$, and $(\mathcal{X}(T), \mathcal{Y}(T))$ the torsion pair in $\operatorname{mod} B$ induced by T. An almost split sequence

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

in $\operatorname{mod} B$ whose left term Y is in $\mathcal{Y}(T)$ and whose right term X is in $\mathcal{X}(T)$ is called a *connecting sequence*.

The following lemma shows that there are only finitely many connecting sequences.

Lemma 4.1. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$*, $B = \operatorname{End}_A(T)$, and*

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

a connecting sequence in $\operatorname{mod} B$*. Then there exists an indecomposable injective module I in* $\operatorname{mod} A$ *such that $Y \cong \operatorname{Hom}_A(T, I)$ in* $\operatorname{mod} B$.

Proof. Since Y is in $\mathcal{Y}(T)$, it follows from Theorem 3.8 that there exists an indecomposable module M in $\mathcal{T}(T)$ such that $Y \cong \operatorname{Hom}_A(T, M)$ in $\operatorname{mod} B$. Let

$f\colon M \to N$ be an injective envelope of M in $\operatorname{mod} A$. Then we have in $\operatorname{mod} A$ a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0,$$

where $L = \operatorname{Coker} f$. Observe that this is an exact sequence in $\mathcal{T}(T)$, because N belongs to $\mathcal{T}(T)$ and $\mathcal{T}(T)$ is closed under images. Applying now Theorem VII.3.2, we obtain the short exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, M) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T, N) \xrightarrow{\operatorname{Hom}_A(T,g)} \operatorname{Hom}_A(T, L) \longrightarrow 0$$

in $\mathcal{Y}(T)$, because $\operatorname{Ext}^1_A(T, M) = 0$. Moreover, by Theorem III.6.3, we have K-linear isomorphisms

$$\begin{aligned} 0 &= D\,\underline{\operatorname{Hom}}_B\left(\tau_B^{-1}\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, L)\right) \\ &\cong \operatorname{Ext}^1_B(\operatorname{Hom}_A(T, L), \operatorname{Hom}_A(T, M)), \end{aligned}$$

because $\tau_B^{-1}\operatorname{Hom}_A(T, M) \cong \tau_B^{-1} Y = X$ is in $\mathcal{X}(T)$ and $\operatorname{Hom}_A(T, L)$ is in $\mathcal{Y}(T)$. Hence, applying Lemma III.3.1 and Proposition III.3.7, we conclude that $\operatorname{Hom}_A(T, f)$ is a section in $\operatorname{mod} B$. Since the functors $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ and $- \otimes_B T\colon \operatorname{mod} B \to \operatorname{mod} A$ induce an equivalence of the categories $\mathcal{T}(T)$ and $\mathcal{Y}(T)$, we conclude that $f\colon M \to N$ is a section in $\operatorname{mod} A$, and consequently M is isomorphic to an indecomposable direct summand of N (see Lemma I.4.2). Therefore, there exists an indecomposable injective module I in $\operatorname{mod} A$, being a direct summand of N, such that $M \cong I$ in $\operatorname{mod} A$, and hence $Y \cong \operatorname{Hom}_A(T, I)$ in $\operatorname{mod} B$. ☐

The next lemma is known as the *connecting lemma* and provides the connection between the left and the right terms of a connecting sequence given by a tilting module.

Lemma 4.2. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$, *and* $B = \operatorname{End}_A(T)$. *Moreover, let I be an indecomposable injective module in* $\operatorname{mod} A$ *and P an indecomposable projective module in* $\operatorname{mod} A$ *such that* $\operatorname{top}(P) \cong \operatorname{soc}(I)$. *Then*

$$\tau_B^{-1}\operatorname{Hom}_A(T, I) \cong \operatorname{Ext}^1_A(T, P)$$

in $\operatorname{mod} B$. *In particular, P belongs to* $\operatorname{add} T$ *if and only if* $\operatorname{Hom}_A(T, I)$ *is injective in* $\operatorname{mod} B$.

Proof. Let e be a primitive idempotent in A such that $P \cong eA$ and $I \cong D(Ae)$ (see Lemma I.8.22). Then we have natural isomorphisms in $\operatorname{mod} B^{\mathrm{op}}$

$$\begin{aligned} D\operatorname{Hom}_A(T, I) &\cong D\operatorname{Hom}_A(T, D(Ae)) \cong D\operatorname{Hom}_{A^{\mathrm{op}}}(Ae, D(T)) \cong D(eD(T)) \\ &\cong DD(Te) \cong Te \cong \operatorname{Hom}_A(eA, T) \cong \operatorname{Hom}_A(P, T). \end{aligned}$$

We will show that the transpose of the left B-module $\operatorname{Hom}_A(P, T)$ is isomorphic to $\operatorname{Ext}^1_A(T, P)$. Observe that then we will obtain $\tau_B^{-1} \operatorname{Hom}_A(T, I) \cong \operatorname{Tr} D \operatorname{Hom}_A(T, I) \cong \operatorname{Tr} \operatorname{Hom}_A(P, T) \cong \operatorname{Ext}^1_A(T, P)$ in $\operatorname{mod} B$.

Invoking the axiom (T3) for the tilting module T, we conclude that there is in $\operatorname{mod} A$ a short exact sequence of the form

$$0 \longrightarrow P \xrightarrow{f} T' \xrightarrow{g} T'' \longrightarrow 0$$

with T' and T'' from $\operatorname{add} T$. Applying Theorem VII.3.3, we obtain the short exact sequence in $\operatorname{mod} B^{\mathrm{op}}$ of the form

$$0 \longrightarrow \operatorname{Hom}_A(T'', T) \xrightarrow{\operatorname{Hom}_A(g,T)} \operatorname{Hom}_A(T', T) \xrightarrow{\operatorname{Hom}_A(f,T)} \operatorname{Hom}_A(P, T) \longrightarrow 0,$$

because $\operatorname{Ext}^1_A(T'', T) = 0$. Observe that T', T'' are direct summands of the right A-module T^d, for some positive integer d, and hence $\operatorname{Hom}_A(T', T)$, $\operatorname{Hom}_A(T'', T)$ are direct summands of the left B-module $\operatorname{Hom}_A(T^d, T) = B^d$. This shows that $\operatorname{Hom}_A(T', T)$ and $\operatorname{Hom}_A(T'', T)$ are projective modules in $\operatorname{mod} B^{\mathrm{op}}$, and hence the above short exact sequence is a projective resolution of $\operatorname{Hom}_A(P, T)$ in $\operatorname{mod} B^{\mathrm{op}}$. Then the transpose $\operatorname{Tr} \operatorname{Hom}_A(P, T)$ is the cokernel of the homomorphism

$$\operatorname{Hom}_{B^{\mathrm{op}}}\big(\operatorname{Hom}_A(g, T), B\big) : \operatorname{Hom}_{B^{\mathrm{op}}}\big(\operatorname{Hom}_A(T', T), B\big) \longrightarrow \operatorname{Hom}_{B^{\mathrm{op}}}\big(\operatorname{Hom}_A(T'', T), B\big).$$

Observe also that for any T_0 in $\operatorname{add} T$ we have a natural isomorphism

$$\operatorname{Hom}_A(T, T_0) \cong \operatorname{Hom}_{B^{\mathrm{op}}}\big(\operatorname{Hom}_A(T_0, T), \operatorname{Hom}_A(T, T)\big)$$

in $\operatorname{mod} B$. Indeed, for $T_0 = T$, it is the canonical isomorphism $B = \operatorname{End}_A(T, T) \xrightarrow{\sim} \operatorname{Hom}_{B^{\mathrm{op}}}(B, B)$ of right B-modules, which assigns to an element $x \in B$ the homomorphism $\varphi_x \in \operatorname{Hom}_{B^{\mathrm{op}}}(B, B)$ such that $\varphi_x(b) = bx$ for $b \in B$. Then, since T_0 is a direct summand of T^m, for some positive integer m, the claim follows from the additivity of the hom functors. Hence, we obtain a commutative diagram in $\operatorname{mod} B$

$$\begin{tikzcd}
\operatorname{Hom}_A(T, T') \arrow[r, "\sim"] \arrow[d, "\operatorname{Hom}_A(T,g)"] & \operatorname{Hom}_{B^{\mathrm{op}}}\big(\operatorname{Hom}_A(T', T), \operatorname{Hom}_A(T, T)\big) \arrow[d, "\operatorname{Hom}_{B^{\mathrm{op}}}(\operatorname{Hom}_A(g,T),\operatorname{Hom}_A(T,T))"] \\
\operatorname{Hom}_A(T, T'') \arrow[r, "\sim"] & \operatorname{Hom}_{B^{\mathrm{op}}}\big(\operatorname{Hom}_A(T'', T), \operatorname{Hom}_A(T, T)\big),
\end{tikzcd}$$

where the horizontal homomorphisms are isomorphisms. This implies that

$$\operatorname{Coker} \operatorname{Hom}_A(T, g) \cong \operatorname{Coker} \operatorname{Hom}_{B^{\mathrm{op}}}\big(\operatorname{Hom}_A(g, T), \operatorname{Hom}_A(T, T)\big)$$

in $\operatorname{mod} B$, and consequently $\operatorname{Tr}\operatorname{Hom}_A(P,T) \cong \operatorname{Coker}\operatorname{Hom}_A(T,g)$ in $\operatorname{mod} B$. On the other hand, applying Theorem VII.3.2 to the covariant functor $\operatorname{Hom}(T,-)\colon \operatorname{mod} A \to \operatorname{mod} B$ and the first short exact sequence, we obtain the exact sequence in $\operatorname{mod} B$ of the form

$$0 \longrightarrow \operatorname{Hom}_A(T,P) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T,T') \xrightarrow{\operatorname{Hom}_A(T,g)} \operatorname{Hom}_A(T,T'') \longrightarrow \operatorname{Ext}_A^1(T,P) \longrightarrow 0,$$

because $\operatorname{Ext}_A^1(T,T') = 0$. Then $\operatorname{Coker}\operatorname{Hom}_A(T,g) \cong \operatorname{Ext}_A(T,P)$ in $\operatorname{mod} B$, as required.

For the second part, observe that the projective module P is in $\operatorname{add} T$ if and only if P is in $\operatorname{Gen} T = \mathcal{T}(T)$, that is, if and only $\operatorname{Ext}_A^1(T,P) = 0$. This shows that $\tau_B^{-1}\operatorname{Hom}_A(T,I) = 0$ if and only if P is in $\operatorname{add} T$. Hence, we conclude that P is in $\operatorname{add} T$ if and only if $\operatorname{Hom}_A(T,I)$ is an injective module in $\operatorname{mod} B$. □

The following theorem provides a description of the canonical sequence for the middle term of a connecting sequence induced by a tilting module.

Theorem 4.3. *Let A be a finite dimensional K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$, *and* $B = \operatorname{End}_A(T)$. *Moreover, let*

$$0 \longrightarrow \operatorname{Hom}_A(T,I) \xrightarrow{u} E \xrightarrow{v} \operatorname{Ext}_A^1(T,P) \longrightarrow 0$$

be a connecting sequence in $\operatorname{mod} B$, *with I an indecomposable injective module and P an indecomposable projective module in* $\operatorname{mod} A$ *such that* $\operatorname{top}(P) \cong \operatorname{soc}(I)$. *Then the canonical sequence for E in* $\operatorname{mod} B$ *with respect to the torsion pair* $(\mathcal{X}(T),\mathcal{Y}(T))$ *in* $\operatorname{mod} B$ *induced by T is of the form*

$$0 \longrightarrow \operatorname{Ext}_A^1(T,\operatorname{rad} P) \longrightarrow E \longrightarrow \operatorname{Hom}_A(T,I/\operatorname{soc}(I)) \longrightarrow 0.$$

Proof. Since, by the assumption, $\operatorname{Hom}_A(T,I)$ is not an injective module in $\operatorname{mod} B$, we conclude from Lemma 4.2 that P is not in $\operatorname{add} T$. Let S be a simple module in $\operatorname{mod} A$ such that $\operatorname{top}(P) \cong S \cong \operatorname{soc}(I)$. Consider the torsion pair $(\mathcal{T}(T),\mathcal{F}(T))$ in $\operatorname{mod} A$ induced by T. Then it follows from Corollary 1.5 that S is either in $\mathcal{T}(T)$ or in $\mathcal{F}(T)$. Hence we have two cases to consider.

(i) Assume that S belongs to $\mathcal{T}(T)$. Then $\operatorname{Ext}_A^1(T,S) = 0$. Consider the canonical short exact sequence

$$0 \longrightarrow \operatorname{rad} P \xrightarrow{i} P \xrightarrow{p} S \longrightarrow 0$$

in $\operatorname{mod} A$. Then $\operatorname{Hom}_A(T,i)\colon \operatorname{Hom}_A(T,\operatorname{rad} P) \to \operatorname{Hom}_A(T,P)$ is an isomorphism in $\operatorname{mod} B$, because $\operatorname{rad} P$ is a unique maximal right A-submodule of P and there is no epimorphism from T to P. Let t be the torsion radical in $\operatorname{mod} A$ with respect to the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$. Then we have the canonical sequence

$$0 \longrightarrow t \operatorname{rad} P \longrightarrow \operatorname{rad} P \longrightarrow \operatorname{rad} P/t \operatorname{rad} P \longrightarrow 0$$

for $\operatorname{rad} P$, with $t \operatorname{rad} P$ in $\mathcal{T}(T)$ and $\operatorname{rad} P/t \operatorname{rad} P$ in $\mathcal{F}(T)$. Applying Theorem VII.3.2, we obtain an exact sequence in $\operatorname{mod} B$ of the form

$$\operatorname{Ext}_A^1(T, t \operatorname{rad} P) \longrightarrow \operatorname{Ext}_A^1(T, \operatorname{rad} P) \longrightarrow \operatorname{Ext}_A^1(T, \operatorname{rad} P/t \operatorname{rad} P) \longrightarrow 0,$$

because $\operatorname{pd}_A T \leq 1$, with $\operatorname{Ext}_A^1(T, t \operatorname{rad} P) = 0$. Hence we have an isomorphism $\operatorname{Ext}_A^1(T, \operatorname{rad} P) \cong \operatorname{Ext}_A^1(T, \operatorname{rad} P/t \operatorname{rad} P)$ in $\operatorname{mod} B$, and consequently $\operatorname{Ext}_A^1(T, \operatorname{rad} P)$ belongs to $\mathcal{X}(T)$, because $\operatorname{rad} P/t \operatorname{rad} P$ belongs to $\mathcal{F}(T)$, and then $\operatorname{Ext}_A^1(T, \operatorname{rad} P/t \operatorname{rad} P)$ is in $\mathcal{X}(T)$, by Theorem 3.8. Further, applying Theorem VII.3.2 again, we obtain a short exact sequence in $\operatorname{mod} B$ of the form

$$0 \longrightarrow \operatorname{Hom}_A(T, S) \xrightarrow{\delta} \operatorname{Ext}_A^1(T, \operatorname{rad} P) \xrightarrow{h} \operatorname{Ext}_A^1(T, P) \longrightarrow 0,$$

because $\operatorname{Hom}_A(T, i)$ is an isomorphism and $\operatorname{Ext}_A^1(T, S) = 0$. Moreover, since $\operatorname{Ext}_A^1(T, \operatorname{rad} P)$ is in $\mathcal{X}(T)$ and $\operatorname{Hom}_A(T, S)$ is in $\mathcal{Y}(T)$, we conclude that δ is not a section in $\operatorname{mod} B$. Then it follows from Lemma III.3.1 that h is not a retraction in $\operatorname{mod} B$. Since the given connecting sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, I) \xrightarrow{u} E \xrightarrow{v} \operatorname{Ext}_A^1(T, P) \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod} B$, we conclude that there is a homomorphism $g\colon \operatorname{Ext}_A^1(T, \operatorname{rad} P) \to E$ such that $h = vg$. Then we obtain a commutative diagram in $\operatorname{mod} B$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(T, S) \arrow[r, "\delta"] \arrow[d, "f"] & \operatorname{Ext}_A^1(T, \operatorname{rad} P) \arrow[r, "h"] \arrow[d, "g"] & \operatorname{Ext}_A^1(T, P) \arrow[r] \arrow[d, "\operatorname{id}_{\operatorname{Ext}_A^1(T,P)}"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_A(T, I) \arrow[r, "u"] & E \arrow[r, "v"] & \operatorname{Ext}_A^1(T, P) \arrow[r] & 0,
\end{tikzcd}$$

with exact rows, because $vg\delta = h\delta = 0$ implies that $g(\operatorname{Im} \delta) \subseteq \operatorname{Ker} v = \operatorname{Im} u$. Since S and I belong to $\mathcal{T}(T)$, it follows from Proposition 3.2 that the functor $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ induces a K-linear isomorphism

$$\operatorname{Hom}_A(S, I) \xrightarrow{\sim} \operatorname{Hom}_B\big(\operatorname{Hom}_A(T, S), \operatorname{Hom}_A(T, I)\big).$$

Hence we may choose a homomorphism $j\colon S \to I$ in $\operatorname{mod} A$ such that $f = \operatorname{Hom}_A(T, j)$. Observe that $f \neq 0$ and consequently $j \neq 0$. Indeed, $f = 0$ forces $g\delta = 0$, and then $g = \varphi h$ for some homomorphism $\varphi\colon \operatorname{Ext}_A^1(T, P) \to E$ in $\operatorname{mod} B$. But then $h = vg = v\varphi h$, and hence $v\varphi = \operatorname{id}_{\operatorname{Ext}_A^1(T,P)}$, because h

is an epimorphism. This contradicts the fact that v is not a retraction in mod B. Therefore, $j: S \to I$ is a monomorphism, and we have in mod A a short exact sequence

$$0 \longrightarrow S \xrightarrow{j} I \xrightarrow{q} I/S \longrightarrow 0.$$

Applying Theorem VII.3.2, we obtain the short exact sequence in mod B

$$0 \longrightarrow \mathrm{Hom}_A(T,S) \xrightarrow{f} \mathrm{Hom}_A(T,I) \xrightarrow{w} \mathrm{Hom}_A(T,I/S) \longrightarrow 0$$

with $f = \mathrm{Hom}_A(T,j)$ and $w = \mathrm{Hom}_A(T,q)$, because $\mathrm{Ext}^1_A(T,S) = 0$. Then it follows from Lemma VII.3.1 that there is in mod B a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_A(T,S) \arrow[rr, "\delta"] \arrow[d, "f"] & & \mathrm{Ext}^1_A(T,\mathrm{rad}\,P) \arrow[r, "h"] \arrow[d, "g"] & \mathrm{Ext}^1_A(T,P) \arrow[r] \arrow[d, "\mathrm{id}_{\mathrm{Ext}^1_A(T,P)}"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(T,I) \arrow[rr, "u"] \arrow[d, "w"] & & E \arrow[r, "v"] \arrow[d, "r"] & \mathrm{Ext}^1_A(T,P) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(T,I/S) \arrow[rr, "\mathrm{id}_{\mathrm{Hom}_A(T,I/S)}"] \arrow[d] & & \mathrm{Hom}_A(T,I/S) \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \\
 & 0 & & 0 & &
\end{tikzcd}$$

with exact rows and columns. Observe also that $I \in \mathcal{T}(T)$ implies $I/S \in \mathcal{T}(T)$, and hence $\mathrm{Hom}_A(T,I/S) \in \mathcal{Y}(T)$. Moreover, we proved above that $\mathrm{Ext}^1_A(T,\mathrm{rad}\,P)$ belongs to $\mathcal{X}(T)$. Then it follows from Proposition 1.4 that

$$0 \longrightarrow \mathrm{Ext}^1_A(T,\mathrm{rad}\,P) \xrightarrow{g} E \xrightarrow{r} \mathrm{Hom}_A(T,I/S) \longrightarrow 0$$

is the canonical sequence for E with respect to the torsion pair $(\mathcal{X}(T),\mathcal{Y}(T))$ in mod B whose existence is asserted in the theorem.

(ii) Assume $S \in \mathcal{F}(T)$. Then $\mathrm{Hom}_A(T,S) = 0$. Consider a canonical short exact sequence

$$0 \longrightarrow S \xrightarrow{j} I \xrightarrow{q} I/S \longrightarrow 0$$

in mod A. Applying Theorem VII.3.2, we obtain a short exact sequence in mod B of the form

$$0 \longrightarrow \mathrm{Hom}_A(T,I) \xrightarrow{h} \mathrm{Hom}_A(T,I/S) \xrightarrow{\delta} \mathrm{Ext}^1_A(T,S) \longrightarrow 0$$

with $h = \mathrm{Hom}_A(T,q)$, because $\mathrm{Hom}_A(T,S) = 0$ and $\mathrm{Ext}^1_A(T,I) = 0$. Since $S \in \mathcal{F}(T)$, it follows from Theorem 3.8 that $\mathrm{Ext}^1_A(T,S)$ is in $\mathcal{X}(T)$. On the other hand, I/S belongs to $\mathcal{T}(T)$, because I belongs to $\mathcal{T}(T)$ and $\mathcal{T}(T)$ is closed under images. Hence $\operatorname{Hom}_A(T, I/S)$ belongs to $\mathcal{Y}(T)$. In particular, we conclude that δ is not a retraction in mod B, and consequently h is not a section in mod B, by Lemma III.3.1. Since the given connecting sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, I) \xrightarrow{u} E \xrightarrow{v} \operatorname{Ext}_A^1(T, P) \longrightarrow 0$$

is an almost split sequence in mod B, we conclude that there is a homomorphism $g\colon E \to \operatorname{Hom}_A(T, I/S)$ such that $h = gu$. Then we obtain a commutative diagram in mod B of the form

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(T, I) \arrow[r, "u"] \arrow[d, "\mathrm{id}_{\operatorname{Hom}_A(T,I)}"] & E \arrow[r, "v"] \arrow[d, "g"] & \operatorname{Ext}_A^1(T, P) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_A(T, I) \arrow[r, "h"] & \operatorname{Hom}_A(T, I/S) \arrow[r, "\delta"] & \operatorname{Ext}_A^1(T, S) \arrow[r] & 0\,,
\end{tikzcd}$$

with exact rows, because $\delta g u = \delta h = 0$. Observe also that $f \neq 0$. Indeed, $f = 0$ forces $\delta g = f v = 0$, and then $g = h\psi$ for some homomorphism $\psi\colon E \to \operatorname{Hom}_A(T, I)$. But then $h = gu = h\psi u$ and hence $\mathrm{id}_{\operatorname{Hom}_A(T,I)} = \psi u$, because h is a monomorphism. This contradicts the fact that u is not a section. Consider now the canonical sequence

$$0 \longrightarrow tP \xrightarrow{\omega} P \xrightarrow{\pi} P/tP \longrightarrow 0$$

for P, with respect to the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A. Applying Theorem VII.3.2, we obtain the exact sequence

$$\operatorname{Ext}_A^1(T, tP) \xrightarrow{\operatorname{Ext}_A^1(T,\omega)} \operatorname{Ext}_A^1(T, P) \xrightarrow{\operatorname{Ext}_A^1(T,\pi)} \operatorname{Ext}_A^1(T, P/tP) \longrightarrow 0$$

in mod B, because $\operatorname{pd}_A T \le 1$, with $\operatorname{Ext}_A^1(T, tP) = 0$. Hence

$$\operatorname{Ext}_A^1(T, \pi)\colon \operatorname{Ext}_A^1(T, P) \longrightarrow \operatorname{Ext}_A^1(T, P/tP)$$

is an isomorphism in mod B. Moreover, since P/tP and S belong to $\mathcal{F}(T)$, it follows from Theorem 3.8 that the functor $\operatorname{Ext}_A^1(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ induces a K-linear isomorphism

$$\operatorname{Hom}_A(P/tP, S) \xrightarrow{\sim} \operatorname{Hom}_B\left(\operatorname{Ext}_A^1(T, P/tP), \operatorname{Ext}_A^1(T, S)\right).$$

This leads to a commutative diagram in mod K of the form

$$\begin{tikzcd}
\operatorname{Hom}_A(P/tP, S) \arrow[r] \arrow[d, "\operatorname{Hom}_A(\pi,S)"] & \operatorname{Hom}_B\left(\operatorname{Ext}_A^1(T, P/tP), \operatorname{Ext}_A^1(T, S)\right) \arrow[d, "\operatorname{Hom}_B(\operatorname{Ext}_A^1(T,\pi),\operatorname{Ext}_A^1(T,S))"] \\
\operatorname{Hom}_A(P, S) \arrow[r] & \operatorname{Hom}_B\left(\operatorname{Ext}_A^1(T, P), \operatorname{Ext}_A^1(T, S)\right),
\end{tikzcd}$$

where the upper homomorphism and the vertical homomorphisms are isomorphisms, and consequently the lower homomorphism is also an isomorphism. Obviously, the horizontal homomorphisms are induced by the functor $\operatorname{Ext}_A^1(T,-)$: $\operatorname{mod} A \to \operatorname{mod} B$. Therefore, there exists an epimorphism $p\colon P \to S$ in $\operatorname{mod} A$ such that $f = \operatorname{Ext}_A^1(T,p)$. Consider now the exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow \operatorname{rad} P \xrightarrow{i} P \xrightarrow{p} S \longrightarrow 0$$

induced by p. Applying Theorem VII.3.2 again, we obtain the short exact sequence in $\operatorname{mod} B$ of the form

$$0 \longrightarrow \operatorname{Ext}_A^1(T,\operatorname{rad} P) \xrightarrow{w} \operatorname{Ext}_A^1(T,P) \xrightarrow{f} \operatorname{Ext}_A^1(T,S) \longrightarrow 0,$$

with $w = \operatorname{Ext}_A^1(T,i)$ and $f = \operatorname{Ext}_A^1(T,p)$, because $\operatorname{Hom}_A(T,S) = 0$ and $\operatorname{pd}_A T \leq 1$. Then it follows from Lemma VII.3.1 that there is in $\operatorname{mod} B$ a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{Ext}_A^1(T,\operatorname{rad} P) \arrow[r, "\operatorname{id}_{\operatorname{Ext}_A^1(T,\operatorname{rad} P)}"] \arrow[d, "s"] & \operatorname{Ext}_A^1(T,\operatorname{rad} P) \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_A(T,I) \arrow[r, "u"] \arrow[d, "\operatorname{id}_{\operatorname{Hom}_A(T,I)}"] & E \arrow[r, "v"] \arrow[d, "g"] & \operatorname{Ext}_A^1(T,P) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_A(T,I) \arrow[r, "h"] \arrow[d] & \operatorname{Hom}_A(T,I/S) \arrow[r, "\delta"] \arrow[d] & \operatorname{Ext}_A^1(T,S) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. It follows from the first part of the proof that $\operatorname{Ext}_A^1(T,\operatorname{rad} P)$ belongs to $\mathcal{X}(T)$, since $\operatorname{Ext}_A^1(T,\operatorname{rad} P) \cong \operatorname{Ext}_A^1(T,\operatorname{rad} P/t\operatorname{rad} P)$ in $\operatorname{mod} B$ with $\operatorname{rad} P/t\operatorname{rad} P$ in $\mathcal{F}(T)$, due to the fact that P is not in $\mathcal{T}(T)$. Moreover, $I \in \mathcal{T}(T)$ implies that $I/S \in \mathcal{T}(T)$, and hence $\operatorname{Hom}_A(T,I/S)$ belongs to $\mathcal{Y}(T)$. Then it follows from Proposition 1.4 that

$$0 \longrightarrow \operatorname{Ext}_A^1(T,\operatorname{rad} P) \xrightarrow{s} E \xrightarrow{g} \operatorname{Hom}_A(T,I/S) \longrightarrow 0$$

is a required canonical sequence for the module E with respect to the torsion pair $(\mathcal{X}(T),\mathcal{Y}(T))$ in $\operatorname{mod} B$.

Summing up, we proved that the canonical sequence for E in $\operatorname{mod} B$ with respect to the torsion pair $(\mathcal{X}(T),\mathcal{Y}(T))$ is of the required form

$$0 \longrightarrow \operatorname{Ext}_A^1(T,\operatorname{rad} P) \longrightarrow E \longrightarrow \operatorname{Hom}_A(T,I/\operatorname{soc}(I)) \longrightarrow 0. \qquad \square$$

Example 4.4. Let $A = KQ$ be the path algebra of the quiver Q of the form

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\alpha"'] & \underset{3}{\bullet} \arrow[l, "\beta"'] \end{tikzcd}$$

over a field K and $T = S_1 \oplus P_3 \oplus S_3$ the tilting module in mod A, considered in Example 3.14. Then $B = \operatorname{End}_A(T)$ is the bound quiver algebra $K\Delta/J$ given by the quiver Δ of the form

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\sigma"'] & \underset{3}{\bullet} \arrow[l, "\mu"'] \end{tikzcd}$$

and the ideal J in $K\Delta$ generated by $\mu\sigma$. Moreover, the Auslander–Reiten quiver Γ_B of B is of the form

$$\begin{tikzcd} & P_2^B = I_1^B \arrow[dr] & & & \\ S_1^B = P_1^B \arrow[ur] & & S_2^B \arrow[dr] & & I_3^B = S_3^B \\ & & & I_2^B = P_3^B \arrow[ur] & \end{tikzcd}$$

and the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B, induced by T, is

$$\mathcal{X}(T) = \operatorname{add}\left(S_3^B\right) \quad \text{and} \quad \mathcal{Y}(T) = \operatorname{add}\left(P_1^B \oplus P_2^B \oplus S_2^B \oplus P_3^B\right).$$

Since P_2 is the unique indecomposable projective module in mod A which is not in add T, we have in mod B only one connecting sequence

$$0 \longrightarrow S_2^B \longrightarrow I_2^B \longrightarrow S_3^B \longrightarrow 0,$$

where $S_2^B \cong \operatorname{Hom}_A(T, I_2)$, $I_2^B \cong P_3^B \cong \operatorname{Hom}_A(T, S_3)$, and $S_3^B \cong \operatorname{Ext}_A^1(T, S_2)$. Observe that $\operatorname{Ext}_A^1(T, S_2) \cong \operatorname{Ext}_A^1(T, P_2)$, because we have in mod A the canonical short exact sequence

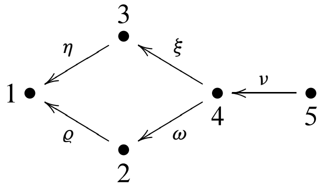
$$0 \longrightarrow S_1 \longrightarrow P_2 \longrightarrow S_2 \longrightarrow 0$$

with $S_1 \in \mathcal{T}(T)$. Moreover, $\operatorname{Hom}_A(T, S_3) = \operatorname{Hom}_A(T, I_2/S_2)$ and $\operatorname{Ext}_A^1(T, \operatorname{rad} P_2) = \operatorname{Ext}_A^1(T, S_1) = 0$.

Example 4.5. Let K be a field, Q the quiver

$$\begin{tikzcd} & \overset{2}{\bullet} \arrow[dl, "\alpha"'] & \\ 1\ \bullet & & \bullet\ 4 \arrow[ul, "\beta"'] \arrow[dl, "\sigma"] \\ & \underset{3}{\bullet} \arrow[ul, "\gamma"] & \\ & & \bullet\ 5, \arrow[ul, "\delta"] \end{tikzcd}$$

I the ideal in KQ generated by $\beta\alpha - \sigma\gamma$ and $\delta\gamma$, $A = KQ/I$, and $T = P_2 \oplus S_2 \oplus P_4 \oplus V \oplus I_2$ the tilting module in mod A, considered in Example 3.16. Then $B = \mathrm{End}_A(T)$ is the bound quiver algebra $K\Delta/J$ given by the quiver Δ of the form



and the ideal J in $K\Delta$ generated by $\xi\eta - \omega\varrho$. Moreover, the Auslander–Reiten quiver Γ_B of B is of the form

$$\begin{tikzcd}
 & P_2^B \arrow[dr] & & S_3^B \arrow[dr] & & P_4^B/P_3^B \arrow[dr] & & I_3^B \arrow[dr] & & \\
S_1^B = P_1^B \arrow[ur] \arrow[r] & P_3^B \arrow[r] & \mathrm{rad}\, P_4^B \arrow[ur] \arrow[r] \arrow[dr] & S_2^B \arrow[r] & P_4^B/S_1^B \arrow[ur] \arrow[r] \arrow[dr] & P_4^B/P_2^B \arrow[r] & W \arrow[ur] \arrow[r] \arrow[dr] & I_2^B \arrow[r] & I_4^B \arrow[dr] & \\
 & & & P_4^B \arrow[ur] \arrow[dr] & & P_5^B/S_1^B \arrow[ur] & & S_4^B \arrow[ur] & & I_5^B = S_5^B \\
 & & & & P_5^B = I_1^B \arrow[ur] & & & & &
\end{tikzcd}$$

and the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B, induced by T, is

$$\mathcal{X}(T) = \mathrm{add}\left(S_2^B \oplus I_2^B \oplus I_4^B \oplus S_5^B\right)$$

$$\mathcal{Y}(T) = \mathrm{add}\left(\left(\bigoplus_{i=1}^{5} P_i^B\right) \oplus \mathrm{rad}\, P_4^B \oplus S_3^B \oplus \left(P_4^B/P_2^B\right) \oplus I_3^B \oplus S_4^B\right).$$

Since P_1, P_3, P_5 are all pairwise nonisomorphic indecomposable projective modules in mod A which are not in add T, we have in mod B exactly three connecting sequences. It follows from the above description of the quiver Γ_B and the classes $\mathcal{X}(T)$ and $\mathcal{Y}(T)$ that these three almost split sequences in mod B are of the forms

$$0 \longrightarrow P_3^B \longrightarrow \mathrm{rad}\, P_4^B \longrightarrow S_2^B \longrightarrow 0,$$
$$0 \longrightarrow P_4^B/P_2^B \longrightarrow W \longrightarrow I_2^B \longrightarrow 0,$$
$$0 \longrightarrow S_4^B \longrightarrow I_4^B \longrightarrow S_5^B \longrightarrow 0,$$

with

$$\begin{aligned} P_3^B &\cong \operatorname{Hom}_A(T, P_4) = \operatorname{Hom}_A(T, I_1), \\ S_2^B &\cong \operatorname{Ext}_A^1(T, S_1) = \operatorname{Ext}_A^1(T, P_1), \\ P_4^B/P_2^B &\cong \operatorname{Hom}_A(T, I_3), \\ I_2^B &\cong \operatorname{Ext}_A^1(T, P_3), \\ S_4^B &\cong \operatorname{Hom}_A(T, S_5) = \operatorname{Hom}_A(T, I_5), \quad \text{and} \\ S_5^B &\cong \operatorname{Ext}_A^1(T, P_5). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \operatorname{rad} P_4^B \cong \operatorname{Hom}_A(T, P_4/S_1) &= \operatorname{Hom}_A(T, I_1/S_1), \\ \operatorname{Ext}_A^1(T, \operatorname{rad} P_1) &= 0, \\ I_4^B \cong \operatorname{Ext}_A^1(T, S_3) &= \operatorname{Ext}_A^1(T, \operatorname{rad} P_5), \\ \operatorname{Hom}_A(T, I_5/S_5) &= 0. \end{aligned}$$

Finally, we note that the middle W of the second connecting sequence is neither in $\mathcal{X}(T)$ nor in $\mathcal{Y}(T)$. But it follows from Theorem 4.3 that the canonical sequence for W in $\operatorname{mod} B$ with respect to the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is of the form

$$0 \longrightarrow S_2^B \longrightarrow W \longrightarrow I_3^B \oplus S_4^B \longrightarrow 0,$$

because $S_2^B \cong \operatorname{Ext}_A^1(T, S_1) = \operatorname{Ext}_A^1(T, \operatorname{rad} P_3)$, $I_3^B \oplus S_4^B \cong \operatorname{Hom}_A(T, S_4) \oplus \operatorname{Hom}_A(T, S_5) \cong \operatorname{Hom}_A(T, S_4 \oplus S_5) = \operatorname{Hom}_A(T, I_3/S_3)$.

5 Splitting tilting modules

Let A be a finite dimensional K-algebra over a field K, T a tilting module in $\operatorname{mod} A$, and $B = \operatorname{End}_A(T)$. Then T is said to be a *splitting tilting module* if the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ induced by T is a splitting torsion pair, that is, every indecomposable module Z in $\operatorname{mod} B$ belongs either to $\mathcal{X}(T)$ or to $\mathcal{Y}(T)$. Observe that, if T is a splitting tilting module, then it follows from Theorem 3.8 that every indecomposable module in $\operatorname{mod} B$ is the image of an indecomposable module in $\operatorname{mod} A$ under one of the functors $\operatorname{Hom}_A(T, -)$ or $\operatorname{Ext}_A^1(T, -)$, so $\operatorname{mod} B$ has fewer indecomposable modules than $\operatorname{mod} A$. In particular, if A is of finite representation type, then B is also of finite representation type. We note that the tilting modules considered in Examples 3.14, 3.15, 3.17, and 3.18 are splitting tilting modules. On the other hand, the tilting module considered in Example 3.16 is not a splitting tilting module.

Let B be a finite dimensional K-algebra over a field K. Then a sequence of homomorphisms in $\operatorname{mod} B$

$$Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} \cdots \longrightarrow Z_{t-1} \xrightarrow{f_t} Z_t,$$

with $t \geq 1$, is said to be a *path* (*of length* t) provided the modules $Z_0, Z_1, \dots, Z_{t-1}, Z_t$ are indecomposable and the homomorphisms $f_1, f_2, \dots, f_{t-1}, f_t$ are nonzero and belong to the radical rad_B of $\operatorname{mod} B$. Moreover, then we say that Z_t is a *successor* of Z_0 and Z_0 is a *predecessor* of Z_t in $\operatorname{mod} B$. Further, a full subcategory $\mathcal{C}$ of $\operatorname{mod} B$ is said to be *closed under successors* (respectively, *closed under predecessors*) in $\operatorname{mod} B$ if, for every path

$$Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} \cdots \longrightarrow Z_{t-1} \xrightarrow{f_t} Z_t$$

in $\operatorname{mod} B$ with Z_0 from $\mathcal{C}$ (respectively, Z_t from $\mathcal{C}$), the module Z_t (respectively, the module Z_0) belongs to $\mathcal{C}$.

Lemma 5.1. *Let A be a finite dimensional K-algebra over a field K, T a splitting tilting module in* $\operatorname{mod} A$*, and* $B = \operatorname{End}_A(T)$. *Then the following statements hold:*

(i) *The torsion class $\mathcal{X}(T)$ is closed under successors in* $\operatorname{mod} B$.

(ii) *The torsion-free class $\mathcal{Y}(T)$ is closed under predecessors in* $\operatorname{mod} B$.

Proof. The statements (i) and (ii) are direct consequences of the fact that $(\mathcal{X}(T), \mathcal{Y}(T))$ is a splitting torsion pair in $\operatorname{mod} B$ and $\operatorname{Hom}_B(X, Y) = 0$ for any modules X in $\mathcal{X}(T)$ and Y in $\mathcal{Y}(T)$. □

The following proposition provides a characterization of almost split sequences in the module categories of the endomorphism algebras of splitting tilting modules.

Proposition 5.2. *Let A be a finite dimensional K-algebra over a field K, T a splitting tilting module in* $\operatorname{mod} A$*, and* $B = \operatorname{End}_A(T)$. *Then any almost split sequence in* $\operatorname{mod} B$ *lies entirely in either $\mathcal{X}(T)$ or $\mathcal{Y}(T)$, or else is of the form*

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_A(T, I) \longrightarrow \operatorname{Hom}_A(T, I/\operatorname{soc}(I)) \oplus \operatorname{Ext}^1_A(T, \operatorname{rad} P) \\ \longrightarrow \operatorname{Ext}^1_A(T, P) \longrightarrow 0, \end{aligned}$$

where P is an indecomposable projective module in $\operatorname{mod} A$ *not in* $\operatorname{add} T$ *and I is an indecomposable injective module in* $\operatorname{mod} A$ *such that* $\operatorname{top}(P) \cong \operatorname{soc}(I)$.

Proof. Let $0 \to M \to E \to N \to 0$ be an almost split sequence in $\operatorname{mod} B$. Then it follows from Lemma 5.1 that either it lies entirely in $\mathcal{X}(T)$, if M belongs to $\mathcal{X}(T)$, or it lies entirely in $\mathcal{Y}(T)$, if N belongs to $\mathcal{Y}(T)$, or we have $M \in \mathcal{Y}(T)$

and $N \in \mathcal{X}(T)$, that is, the almost split sequence is a connecting sequence. In the last case, it follows from Lemmas 4.1 and 4.2 that $M \cong \operatorname{Hom}_A(T, I)$ and $N \cong \operatorname{Ext}_A^1(T, P)$ for an indecomposable projective module P in mod A not lying in the additive category add T of T and an indecomposable injective module I in mod A such that $\operatorname{top}(P) \cong \operatorname{soc}(A)$. Moreover, by Theorem 4.3, the canonical sequence for the module E in mod B with respect to the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is of the form

$$0 \longrightarrow \operatorname{Ext}_A^1(T, \operatorname{rad} P) \longrightarrow E \longrightarrow \operatorname{Hom}_A(T, I/\operatorname{soc}(I)) \longrightarrow 0.$$

Since the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting, applying Proposition 1.6, we conclude that

$$E \cong \operatorname{Hom}_A(T, I/\operatorname{soc}(I)) \oplus \operatorname{Ext}_A^1(T, \operatorname{rad} P)$$

in mod B, as required. □

Proposition 5.3. *Let A be a finite dimensional K-algebra over a field K, T a splitting tilting module in* mod A*, and $B = \operatorname{End}_A(T)$. Moreover, let*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an almost split sequence in mod A*. Then the following statements hold:*

(i) *If L, M, N are in $\mathcal{T}(T)$, then*

$$0 \to \operatorname{Hom}_A(T, L) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T, M) \xrightarrow{\operatorname{Hom}_A(T,g)} \operatorname{Hom}_A(T, N) \to 0$$

is an almost split sequence in mod B *whose all terms are in $\mathcal{Y}(T)$.*

(ii) *If L, M, N are in $\mathcal{F}(T)$, then*

$$0 \to \operatorname{Ext}_A^1(T, L) \xrightarrow{\operatorname{Ext}_A^1(T,f)} \operatorname{Ext}_A^1(T, M) \xrightarrow{\operatorname{Ext}_A^1(T,g)} \operatorname{Ext}_A^1(T, N) \to 0$$

is an almost split sequence in mod B *whose all terms are in $\mathcal{X}(T)$.*

Proof. Assume the modules L, M, N are in $\mathcal{T}(T)$. Applying Theorem VII.3.2, we conclude that

$$0 \longrightarrow \operatorname{Hom}_A(T, L) \xrightarrow{\operatorname{Hom}_A(T,f)} \operatorname{Hom}_A(T, M) \xrightarrow{\operatorname{Hom}_A(T,g)} \operatorname{Hom}_A(T, N) \longrightarrow 0$$

is an exact sequence in mod B, because $\operatorname{Ext}_A^1(T, L) = 0$. Moreover, by Theorem 3.8, the functor $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ induces an equivalence of categories $\mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$. Then the right B-modules $\operatorname{Hom}_A(T, L)$ and $\operatorname{Hom}_A(T, N)$ are indecomposable, because L and N are indecomposable right A-modules.

Hence, in order to prove that the above exact sequence is an almost split sequence in $\operatorname{mod} B$, it suffices to show, by Theorem III.8.3, that the homomorphisms $\operatorname{Hom}_A(T, f)$ and $\operatorname{Hom}_A(T, g)$ are irreducible homomorphisms in $\operatorname{mod} B$. We will show that $\operatorname{Hom}_A(T, f)$ is an irreducible homomorphisms in $\operatorname{mod} B$. The proof that $\operatorname{Hom}_A(T, g)$ is an irreducible homomorphisms in $\operatorname{mod} B$ is similar. First observe that $\operatorname{Hom}_A(T, f)$ is neither a section, nor a retraction in $\operatorname{mod} B$, because f is neither a section, nor a retraction in $\operatorname{mod} A$, $\mathcal{T}(T)$ is a full subcategory of $\operatorname{mod} A$ and $\mathcal{Y}(T)$ is a full subcategory of $\operatorname{mod} B$. Assume there exist homomorphisms $u: \operatorname{Hom}_A(T, L) \to Z$ and $v: Z \to \operatorname{Hom}_A(T, M)$ in $\operatorname{mod} B$ such that $\operatorname{Hom}_A(T, f) = vu$. Clearly, $v \neq 0$ and $u \neq 0$, because $f \neq 0$ forces $\operatorname{Hom}_A(T, f) \neq 0$. Since $\operatorname{Hom}_A(T, M)$ belongs to $\mathcal{Y}(T)$ and $\mathcal{Y}(T)$ is closed under predecessors in $\operatorname{mod} B$ (Lemma 5.1), we conclude that Z belongs to $\mathcal{Y}(T)$, and consequently there exists a module W in $\mathcal{T}(T)$ such that $Z = \operatorname{Hom}_A(T, W)$. Moreover, there exist homomorphisms $u': L \to W$ and $v': W \to M$ such that $u = \operatorname{Hom}_A(T, u')$ and $v = \operatorname{Hom}_A(T, v')$. Then we have $\operatorname{Hom}_A(T, f) = vu = \operatorname{Hom}_A(T, v') \operatorname{Hom}_A(T, u') = \operatorname{Hom}_A(T, v'u')$, and hence $f = v'u'$, because $\operatorname{Hom}_A(T, -): \mathcal{T}(T) \to \mathcal{Y}(T)$ is a faithful functor. Now the irreducibility of f implies that u' is a section or v' is a retraction. Hence u is a section or v is a retraction. This shows that $\operatorname{Hom}_A(T, f)$ is an irreducible homomorphism in $\operatorname{mod} B$.

(ii) Assume the modules L, M, N are in $\mathcal{F}(T)$. Applying Theorem VII.3.2 again, we conclude that

$$0 \longrightarrow \operatorname{Ext}_A^1(T, L) \xrightarrow{\operatorname{Ext}_A^1(T,f)} \operatorname{Ext}_A^1(T, M) \xrightarrow{\operatorname{Ext}_A^1(T,g)} \operatorname{Ext}_A^1(T, N) \longrightarrow 0$$

is an exact sequence in $\operatorname{mod} B$, because $\operatorname{Hom}_A(T, N) = 0$ and $\operatorname{pd}_A T \leq 1$. Moreover, by Theorem 3.8, the functor $\operatorname{Ext}_A^1(T, -): \operatorname{mod} A \to \operatorname{mod} B$ induces an equivalence of categories $\mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T)$. Then $\operatorname{Ext}_A^1(T, L)$ and $\operatorname{Ext}_A^1(T, N)$ are indecomposable modules in $\operatorname{mod} B$, because L and N are indecomposable modules in $\operatorname{mod} A$. Moreover, similarly as in the part (i), we prove that $\operatorname{Ext}_A^1(T, f)$ and $\operatorname{Ext}_A^1(T, g)$ are irreducible homomorphisms in $\operatorname{mod} B$. Therefore, it follows from Theorem III.8.3 that the above exact sequence is an almost split sequence in $\operatorname{mod} B$. □

Let A be a finite dimensional K-algebra over a field K, T a tilting module in $\operatorname{mod} A$, and $B = \operatorname{End}_A(T)$. It follows from Lemma 3.1 that, if $T_1, \dots, T_n$ form a complete set of pairwise nonisomorphic indecomposable direct summands of T, then $\operatorname{Hom}_A(T, T_1), \dots, \operatorname{Hom}_A(T, T_n)$ form a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} B$. It is not easy in general to describe the indecomposable injective modules in $\operatorname{mod} B$. But, if the tilting module T is splitting, this is possible due to the following proposition.

Proposition 5.4. *Let A be a finite dimensional K-algebra over a field K, T be a splitting tilting module in* $\operatorname{mod} A$, *$B = \operatorname{End}_A(T)$, and $T_1, \dots, T_n$ be a complete* *set of pairwise nonisomorphic indecomposable direct summands of* T. *Assume that the modules* $T_1, \dots, T_m$ *are projective, the modules* $T_{m+1}, \dots, T_n$ *are nonprojective, and let* $I_1, \dots, I_m$ *be indecomposable injective modules in* $\operatorname{mod} A$ *such that* $\operatorname{top}(T_j) \cong \operatorname{soc}(I_j)$ *for any* $j \in \{1, \dots, m\}$. *Then the right* B*-modules*

$$\operatorname{Hom}_A(T, I_1), \dots, \operatorname{Hom}_A(T, I_m), \operatorname{Ext}^1_A(T, \tau_A T_{m+1}), \dots, \operatorname{Ext}^1_A(T, \tau_A T_n)$$

form a complete set of pairwise nonisomorphic indecomposable injective modules in $\operatorname{mod} B$.

Proof. By Lemma 4.2, $\operatorname{Hom}_A(T, I_1), \dots, \operatorname{Hom}_A(T, I_m)$ are pairwise nonisomorphic indecomposable injective modules belonging to $\mathcal{Y}(T)$, because the pairwise nonisomorphic injective modules $I_1, \dots, I_m$ belong to $\mathcal{T}(T)$ and the functor $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ induces an equivalence of categories $\mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$. If $m = n$, then the modules $\operatorname{Hom}_A(T, I_1), \dots, \operatorname{Hom}_A(T, I_n)$ form a complete set of pairwise nonisomorphic indecomposable injective modules in $\operatorname{mod} B$, because $\operatorname{Hom}_A(T, T_1), \dots, \operatorname{Hom}_A(T, T_n)$ form a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} B$.

Assume that $m < n$. Observe that, for any $i \in \{m+1, \dots, n\}$, we have $\operatorname{Hom}_A(T, \tau_A T_i) \cong D \operatorname{Ext}^1_A(T_i, T) = 0$, by Corollary III.6.4 and $\operatorname{Ext}^1_A(T, T) = 0$. Hence $\tau_A T_{m+1}, \dots, \tau_A T_n$ is a family of pairwise nonisomorphic indecomposable modules in $\mathcal{F}(T)$. Since the functor $\operatorname{Ext}^1_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ induces an equivalence of categories $\mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T)$, we conclude that $\operatorname{Ext}^1_A(T, \tau_A T_{m+1}), \dots, \operatorname{Ext}^1_A(T, \tau_A T_n)$ form a family of pairwise nonisomorphic indecomposable modules in $\mathcal{X}(T)$. Therefore, it remains to show that the right B-modules $\operatorname{Ext}^1_A(T, \tau_A T_{m+1}), \dots, \operatorname{Ext}^1_A(T, \tau_A T_n)$ are injective. Suppose there exists $i \in \{m+1, \dots, n\}$ such that $\operatorname{Ext}^1_A(T, \tau_A T_i)$ is not injective in $\operatorname{mod} B$. Then $\operatorname{Ext}^1_A(T, \tau_A T_i)$ is the left term of an almost split sequence

$$0 \longrightarrow \operatorname{Ext}^1_A(T, \tau_A T_i) \longrightarrow E \longrightarrow X \longrightarrow 0$$

in $\operatorname{mod} B$. Since T is a splitting tilting module, the torsion class $\mathcal{X}(T)$ is closed under successors in $\operatorname{mod} B$, and hence E and X belong to $\mathcal{X}(T)$. Then there exist a module M and an indecomposable module N in $\mathcal{F}(T)$ such that $E \cong \operatorname{Ext}^1_A(T, M)$ and $X \cong \operatorname{Ext}^1_A(T, N)$ in $\operatorname{mod} B$. We have in $\operatorname{mod} B$ an almost split sequence of the form

$$0 \longrightarrow \operatorname{Ext}^1_A(T, \tau_A T_i) \longrightarrow \operatorname{Ext}^1_A(T, M) \longrightarrow \operatorname{Ext}^1_A(T, N) \longrightarrow 0.$$

Since $\operatorname{Ext}^1_A(T, -)$ induces an equivalence $\mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T)$, we conclude that there exists a nonsplittable short exact sequence in $\operatorname{mod} A$ of the form

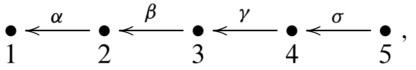
$$0 \longrightarrow \tau_A T_i \longrightarrow M \longrightarrow N \longrightarrow 0$$

and all its terms are in $\mathcal{F}(T)$. On the other hand, applying Theorem III.6.3, we obtain K-linear isomorphisms

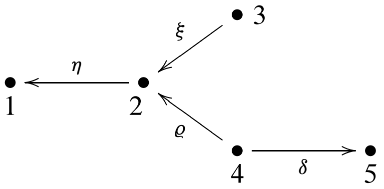
$$\mathrm{Ext}_A^1(N, \tau_A T_i) \cong D\,\underline{\mathrm{Hom}}_A\left(\tau_A^{-1}(\tau_A T_i), N\right) \cong D\,\underline{\mathrm{Hom}}_A(T_i, N) = 0,$$

because T_i belongs to $\mathcal{T}(T)$ and N belongs to $\mathcal{F}(T)$. Therefore, $\mathcal{E}\mathrm{xt}_A^1(N, \tau_A T_i) = 0$, by Proposition III.3.7. Hence, the above sequence is splittable, by Lemma III.3.1, a contradiction. This shows that $\mathrm{Ext}_A^1(T, \tau_A T_{m+1}), \dots, \mathrm{Ext}_A^1(T, \tau_A T_n)$ are injective modules in mod B. ☐

Example 5.5. Let K be a field, Q the quiver



I the ideal in the path algebra KQ of Q over K generated by $\sigma\gamma\beta\alpha$, $A = KQ/I$, and $T = S_1 \oplus P_4 \oplus P_5 \oplus (P_4/P_2) \oplus S_3$ the tilting module in mod A considered in Example 3.15. It follows from Example 3.15 that T is a splitting tilting module in mod A and $B = \mathrm{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver



and J is the ideal in path algebra $K\Delta$ of Δ over K generated by $\xi\eta$ and $\varrho\eta$. Since $S_1 = P_1$, P_4, P_5 are indecomposable projective direct summands of T, it follows from Proposition 5.4 that $\mathrm{Hom}_A(T, I_1)$, $\mathrm{Hom}_A(T, I_4)$, $\mathrm{Hom}_A(T, I_5)$ are injective modules in mod B. In fact, in the notation of Example 3.15, we have isomorphisms in mod B

$$\begin{aligned}\mathrm{Hom}_A(T, I_1) = \mathrm{Hom}_A(T, P_4) &\cong P_2^B = I_1^B,\\ \mathrm{Hom}_A(T, I_4) &\cong I_2^B,\\ \mathrm{Hom}_A(T, I_5) = \mathrm{Hom}_A(T, S_5) &\cong S_3^B = I_3^B.\end{aligned}$$

Further, it follows from Example 3.15 that there are isomorphisms

$$\begin{aligned}\mathrm{Ext}_A^1\left(T, \tau_A(P_4/P_2)\right) = \mathrm{Ext}_A^1(T, P_3/S_1) &\cong S_4^B = I_4^B,\\ \mathrm{Ext}_A^1(T, \tau_A S_3) = \mathrm{Ext}_A^1(T, S_2) &\cong I_5^B\end{aligned}$$

in mod B.

Lemma 5.6. *Let A be a finite dimensional K-algebra over a field K, T a splitting tilting module in* $\operatorname{mod} A$*, and* $B = \operatorname{End}_A(T)$*. Then, for any module X in $\mathcal{X}(T)$, we have* $\operatorname{id}_B X \leq 1$.

Proof. Let X be a module in the torsion class $\mathcal{X}(T)$ of $\operatorname{mod} B$. Since the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting, it follows from Proposition 1.6 that $\tau_B^{-1} X$ belongs to $\mathcal{X}(T)$. On the other hand, since the right B-module $B = \operatorname{Hom}_A(T, T)$ belongs to the torsion-free class $\mathcal{Y}(T)$, we conclude that $\operatorname{Hom}_B(\tau_B^{-1} X, B) = 0$. But then $\operatorname{id}_B X \leq 1$, by Proposition III.5.4. □

Lemma 5.7. *Let A be a finite dimensional hereditary K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$*, and* $B = \operatorname{End}_A(T)$*. Then, for any module Y in $\mathcal{Y}(T)$, we have* $\operatorname{pd}_B Y \leq 1$.

Proof. Let Y be a module in the torsion-free class $\mathcal{Y}(T)$ of $\operatorname{mod} B$. Then it follows from Theorem 3.8 that there is a module M in the torsion class $\mathcal{T}(T)$ of $\operatorname{mod} A$ such that $Y \cong \operatorname{Hom}_A(T, M)$ in $\operatorname{mod} B$. Since A is a hereditary K-algebra, we have $\operatorname{pd}_A M \leq 1$. If M is projective in $\operatorname{mod} A$, then $M \in \mathcal{T}(T) = \operatorname{Gen}(T)$ implies that $M \in \operatorname{add} T$, and consequently $\operatorname{Hom}_A(T, M)$ is projective in $\operatorname{mod} B$, by Lemma 3.1, or equivalently $\operatorname{pd}_B Y = \operatorname{pd}_B \operatorname{Hom}_A(T, M) = 0$. Assume now that $\operatorname{pd}_A M = 1$. Applying Theorem 2.5 we conclude that there is in $\operatorname{mod} A$ a short exact sequence of the form

$$0 \longrightarrow L \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with $T_0 \in \operatorname{add} T$ and $L \in \mathcal{T}(T)$. By Theorem VII.3.2, the induced sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, L) \longrightarrow \operatorname{Hom}_A(T, T_0) \longrightarrow \operatorname{Hom}_A(T, M) \longrightarrow 0$$

in $\operatorname{mod} B$ is exact, because $\operatorname{Ext}_A^1(T, L) = 0$. We note that $\operatorname{Hom}_A(T, T_0)$ is a projective module in $\operatorname{mod} B$. Further, for any module $X \in \mathcal{T}(T)$, we have, by Theorem VII.3.3, an exact sequence

$$\operatorname{Ext}_A^1(M, X) \longrightarrow \operatorname{Ext}_A^1(T_0, X) \longrightarrow \operatorname{Ext}_A^1(L, X) \longrightarrow 0$$

in $\operatorname{mod} B$, because $\operatorname{id}_A X \leq 1$, with $\operatorname{Ext}_A^1(T_0, X) = 0$. Hence, we obtain that $\operatorname{Ext}_A^1(L, X) = 0$ for any module X in $\mathcal{T}(T)$. Then, applying Theorem 2.5 again, we conclude that L belongs to $\operatorname{add} T$. This shows that $\operatorname{Hom}_A(T, L)$ is also a projective module in $\operatorname{mod} B$, by Lemma 3.1. Therefore, we infer that $\operatorname{pd}_B Y = \operatorname{pd}_B \operatorname{Hom}_A(T, M) \leq 1$. □

Theorem 5.8. *Let A be a finite dimensional hereditary K-algebra over a field K and T be a tilting module in* $\operatorname{mod} A$*. Then T is a splitting tilting module.*

Proof. Let $B = \operatorname{End}_A(T)$ and $(\mathcal{T}(T), \mathcal{F}(T))$, $(\mathcal{X}(T), \mathcal{Y}(T))$ be the torsion pairs in $\operatorname{mod} A$ and $\operatorname{mod} B$, respectively, induced by T.

In order to prove that the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ is splitting, it is enough to show that $\operatorname{Ext}^1_B(Y, X) = 0$ for any modules $Y \in \mathcal{Y}(T)$ and $X \in \mathcal{X}(T)$ (see Proposition 1.6). Take a module $Y \in \mathcal{Y}(T)$ and a module $X \in \mathcal{X}(T)$. Then it follows from Theorem 3.8 that there exist modules $M \in \mathcal{T}(T)$ and $N \in \mathcal{F}(T)$ such that $Y \cong \operatorname{Hom}_A(T, M)$ and $X \cong \operatorname{Ext}^1_A(T, N)$ in $\operatorname{mod} B$. Since A is a hereditary algebra, we have in $\operatorname{mod} A$ a short exact sequence of the form

$$0 \longrightarrow N \longrightarrow I \longrightarrow I' \longrightarrow 0,$$

with I and I' injective modules. Applying now Theorem VII.3.2, we obtain a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, I) \longrightarrow \operatorname{Hom}_A(T, I') \longrightarrow \operatorname{Ext}^1_A(T, N) \longrightarrow 0$$

in $\operatorname{mod} B$, because $\operatorname{Hom}_A(T, N) = 0$ and $\operatorname{Ext}^1_A(T, I) = 0$. Further, by Lemma 5.7, $\operatorname{pd}_B \operatorname{Hom}_A(T, M) \leq 1$. Hence, we obtain from Theorem VII.3.2 an exact sequence in $\operatorname{mod} K$ of the form

$$\operatorname{Ext}^1_B\big(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, I)\big) \longrightarrow \operatorname{Ext}^1_B\big(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, I')\big) \\ \longrightarrow \operatorname{Ext}^1_B\big(\operatorname{Hom}_A(T, M), \operatorname{Ext}^1_A(T, N)\big) \longrightarrow 0.$$

Moreover, applying Proposition 3.2, we obtain K-linear isomorphisms

$$\operatorname{Ext}^1_B\big(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, I')\big) \cong \operatorname{Ext}^1_A(M, I') = 0,$$

because M and I' are in $\mathcal{T}(T)$ and I' is injective in $\operatorname{mod} A$. Therefore,

$$\operatorname{Ext}^1_B(Y, X) \cong \operatorname{Ext}^1_B\big(\operatorname{Hom}_A(T, M), \operatorname{Ext}^1_A(T, N)\big) = 0,$$

as required. □

6 Tilted algebras

In this section we describe properties of module categories of the endomorphism algebras of tilting modules over hereditary algebras.

We introduce now the class of tilted algebras playing a fundamental role in further parts of the book. A finite dimensional K-algebra B over a field K is called a *tilted algebra* if there is an indecomposable finite dimensional hereditary K-algebra A and a tilting module T in $\operatorname{mod} A$ such that $B \cong \operatorname{End}_A(T)$. We note that then B is also an indecomposable algebra, by Corollary 3.20.

The following proposition describes some homological properties of tilted algebras.

Proposition 6.1. *Let B be a tilted K-algebra over a field K. Then the following statements hold:*

(i) $\operatorname{gl.dim} B \leq 2$.

(ii) *For any indecomposable module Z in* $\operatorname{mod} B$*, we have* $\operatorname{pd}_B Z \leq 1$ *or* $\operatorname{id}_B Z \leq 1$.

Proof. Let A be an indecomposable finite dimensional hereditary K-algebra and T a tilting module in $\operatorname{mod} A$ such that $B = \operatorname{End}_A(T)$. We know from Theorem 5.8 that T is a splitting tilting module, and hence every indecomposable module Z in $\operatorname{mod} B$ belongs to $\mathcal{X}(T)$ or to $\mathcal{Y}(T)$. Then the statement (ii) follows from Lemmas 5.6 and 5.7. Now let us show that $\operatorname{gl.dim} B \leq 2$.

Let X be a module in $\operatorname{mod} B$. Then we have in $\operatorname{mod} B$ a short exact sequence

$$0 \longrightarrow Y \xrightarrow{i} P \xrightarrow{p} X \longrightarrow 0,$$

where p is a projective cover of X in $\operatorname{mod} B$, $Y = \operatorname{Ker} p$, and i is the canonical inclusion homomorphism. Since P belongs to $\mathcal{Y}(T)$ and $\mathcal{Y}(T)$ is closed under submodules, we conclude that Y belongs to $\mathcal{Y}(T)$. Then, applying Lemma 5.7, we infer that $\operatorname{pd}_B Y \leq 1$, and hence there is a minimal projective resolution

$$0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} Y \longrightarrow 0$$

of Y in $\operatorname{mod} B$. Then the exact sequence

$$0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{p_0} P \xrightarrow{p} X \longrightarrow 0,$$

with $p_0 = i d_0$, is a minimal projective resolution of X in $\operatorname{mod} B$, and so $\operatorname{pd}_B X \leq 2$. Therefore, we get $\operatorname{gl.dim} B \leq 2$. □

It follows from Corollaries VII.1.7 and VII.1.8 that the quiver Q_A of an indecomposable hereditary K-algebra over a field K is a connected and acyclic quiver. Our next aim is to show that the quiver Q_B of any tilted K-algebra B has the same property.

Proposition 6.2. *Let B be a tilted K-algebra over a field K. Then the quiver Q_B of B is connected and acyclic.*

Proof. Let A be an indecomposable finite dimensional hereditary K-algebra over a field K and T a tilting module in $\operatorname{mod} A$ such that $B = \operatorname{End}_A(T)$. It follows from Corollary 3.20 that B is an indecomposable algebra. Hence the quiver Q_B is connected, by Corollary VII.1.7. We will show that Q_B is also acyclic.

Let $T_1, \dots, T_n$ be a complete set of pairwise nonisomorphic indecomposable direct summands of T. Then, by Lemma 3.1, the right B-modules $P_1^B =$

$\mathrm{Hom}_A(T, T_1), \ldots, P_n^B = \mathrm{Hom}_A(T, T_n)$ form a complete set of pairwise nonisomorphic indecomposable projective modules in $\mathrm{mod}\, B$. Assume Q_B is not acyclic. Then, applying Proposition VII.1.6, we conclude that there is a cycle in $\mathrm{mod}\, B$ of the form

$$P_{i_0}^B \xrightarrow{g_1} P_{i_1}^B \xrightarrow{g_2} P_{i_2}^B \longrightarrow \cdots \longrightarrow P_{i_{r-1}}^B \xrightarrow{g_r} P_{i_r}^B = P_{i_0}^B$$

for some $i_0, i_1, \ldots, i_r$ in $\{1, \ldots, n\}$, and $r \geq 1$. Since the functor $\mathrm{Hom}_A(T, -)$ $: \mathrm{mod}\, A \to \mathrm{mod}\, B$ induces an equivalence of categories $\mathrm{add}\, T \xrightarrow{\sim} \mathrm{proj}\, B$, again by Lemma 3.1, we conclude that there is a cycle in $\mathrm{mod}\, A$ of the form

$$T_{i_0} \xrightarrow{f_1} T_{i_1} \xrightarrow{f_2} T_{i_2} \longrightarrow \cdots \longrightarrow T_{i_{r-1}} \xrightarrow{f_r} T_{i_r} = T_{i_0}$$

with $g_j = \mathrm{Hom}_A(T, f_j)$ for any $j \in \{1, \ldots, r\}$. Since $\mathrm{Ext}_A^1(T, T) = 0$, we conclude from Lemma VII.9.15 that, for any $j \in \{1, \ldots, r\}$, f_j is a monomorphism or an epimorphism. Moreover, it follows from the definition of a path in $\mathrm{mod}\, A$ that $f_1, f_2, \ldots, f_r$ are nonzero and nonisomorphisms. Observe also that, for any $j \in \{1, \ldots, r-1\}$ (if $r \geq 2$), we cannot have that f_j is a proper epimorphism and f_{j+1} is a proper monomorphism. Indeed, if this is not the case, then $f_{j+1} f_j : T_{i_{j-1}} \to T_{i_{j+1}}$ is a nonzero homomorphism which is neither a monomorphism nor an epimorphism, a contradiction to Lemma VII.9.15, because $\mathrm{Ext}_A^1(T_{i_{j+1}}, T_{i_{j-1}}) = 0$. This shows that all the homomorphisms $f_1, \ldots, f_r$ are either proper monomorphisms, or proper epimorphisms. Hence the composition $f = f_r \ldots f_1 \in \mathrm{End}_A(T_{i_0})$ is either a proper monomorphism or a proper epimorphism. On the other hand, since $\mathrm{Ext}_A^1(T_{i_0}, T_{i_0}) = 0$, it follows from Corollary VII.9.16 that $\mathrm{End}_A(T_{i_0})$ is a division K-algebra. This implies that either $f = 0$, or f is an isomorphism, a contradiction. Therefore, the quiver Q_B of B is acyclic. □

The next aim is to show that the Auslander–Reiten quiver of a tilted algebra admits a distinguished component, connecting the torsion-free class and the torsion class of the associate torsion pair. We need some preliminary results and concepts.

Proposition 6.3. *Let A be a finite dimensional hereditary K-algebra over a field K, T a tilting module in* $\mathrm{mod}\, A$*, $B = \mathrm{End}_A(T)$. Moreover, let I be an indecomposable injective module in* $\mathrm{mod}\, A$*. Then the following statements hold:*

(i) *Assume Y is an indecomposable module in $\mathcal{Y}(T)$. Then there exists an irreducible homomorphism $\mathrm{Hom}_A(T, I) \to Y$ in $\mathrm{mod}\, B$ if and only if there is an indecomposable direct summand J of $I/\mathrm{soc}\, I$ such that $Y \cong \mathrm{Hom}_A(T, J)$.*

(ii) *Assume X is an indecomposable module in $\mathcal{X}(T)$. Then there exists an irreducible homomorphism $\mathrm{Hom}_A(T, I) \to X$ in $\mathrm{mod}\, B$ if and only if there* *is an indecomposable injective module* J *in* $\operatorname{mod} A$ *such that* I *is a direct summand of* $J/\operatorname{soc} J$ *and* $\tau_B X \cong \operatorname{Hom}_A(T, J)$ *in* $\operatorname{mod} B$. *Further, in this case, we have* $X \cong \operatorname{Ext}_A^1(T, P)$ *in* $\operatorname{mod} B$, *where* P *is an indecomposable projective module in* $\operatorname{mod} A$ *such that* $\operatorname{top}(P) \cong \operatorname{soc}(J)$.

Proof. Since A is a hereditary algebra, it follows from Theorem 5.8 that T is a splitting tilting module. In particular, Lemma 5.1 shows that $\mathcal{Y}(T)$ is closed under predecessors and $\mathcal{X}(T)$ is closed under successors in $\operatorname{mod} B$.

(i) Let Y be an indecomposable module in $\mathcal{Y}(T)$. Assume that there exists an irreducible homomorphism $f\colon \operatorname{Hom}_A(T, I) \to Y$ in $\operatorname{mod} B$. Then, applying Theorem 3.8, we conclude that there exist an indecomposable module L in $\mathcal{T}(T)$ and a homomorphism $g\colon I \to L$ in $\operatorname{mod} A$ such that $Y = \operatorname{Hom}_A(T, L)$ and $f = \operatorname{Hom}_A(T, g)$. We claim that g is an irreducible homomorphism in $\operatorname{mod} A$. Observe first that g is neither a section, nor a retraction in $\operatorname{mod} A$, because the functor $\operatorname{Hom}_A(T, -)$ induces an equivalence of categories $\mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$ and $f = \operatorname{Hom}_A(T, g)$ is neither a section, nor a retraction in $\operatorname{mod} B$. Suppose that $g = vu$ for some homomorphisms $u\colon I \to M$ and $v\colon M \to L$ in $\operatorname{mod} A$. Then $f = \operatorname{Hom}_A(T, g) = \operatorname{Hom}_A(T, v) \operatorname{Hom}_A(T, u)$, and hence either $\operatorname{Hom}_A(T, u)$ is a section, or $\operatorname{Hom}_A(T, v)$ is a retraction in $\operatorname{mod} B$. But then either u is a section, or v is a retraction in $\operatorname{mod} A$, by the equivalence $\mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$ given by $\operatorname{Hom}_A(T, -)$. Therefore, g is indeed an irreducible homomorphism in $\operatorname{mod} A$. Consider now the canonical projection $p\colon I \to I/\operatorname{soc}(I)$. We know from Lemma III.7.7 that p is a left minimal almost split homomorphism in $\operatorname{mod} A$. We also note that $I \neq \operatorname{soc}(I)$, because there is an irreducible homomorphism $g\colon I \to L$. Since g is not a section in $\operatorname{mod} A$, we conclude that there is a homomorphism $h\colon I/\operatorname{soc}(I) \to L$ in $\operatorname{mod} A$ such that $g = hp$. Then the irreducibility of g implies that h is a retraction in $\operatorname{mod} A$, because clearly p is not a section in $\operatorname{mod} A$. Applying Lemma I.4.2, we conclude that L is isomorphic to an indecomposable direct summand J of $I/\operatorname{soc}(I)$. Obviously then $Y = \operatorname{Hom}_A(T, L) \cong \operatorname{Hom}_A(T, J)$ in $\operatorname{mod} B$.

Assume now that there is an indecomposable direct summand J of $I/\operatorname{soc}(I)$ such that $Y \cong \operatorname{Hom}_A(T, J)$ in $\operatorname{mod} B$. Clearly, then $I \neq \operatorname{soc}(I)$, so I is a nonsimple module. Consider the canonical projection $p\colon I \to I/\operatorname{soc}(I)$ and the induced homomorphism $q = \operatorname{Hom}_A(T, p)\colon \operatorname{Hom}_A(T, I) \to \operatorname{Hom}_A(T, I/\operatorname{soc}(I))$ in $\operatorname{mod} B$. We claim that q is an irreducible homomorphism in $\operatorname{mod} B$. Since the functor $\operatorname{Hom}_A(T, -)$ induces an equivalence of categories $\mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$, we conclude that q is neither a section, nor a retraction in $\operatorname{mod} B$. Assume that $q = rs$ for some homomorphisms $s\colon \operatorname{Hom}_A(T, I) \to Z$ and $r\colon Z \to \operatorname{Hom}_A(T, I/\operatorname{soc}(I))$ in $\operatorname{mod} B$. Since $\mathcal{Y}(T)$ is closed under predecessors in $\operatorname{mod} B$, we conclude that Z belongs to $\mathcal{Y}(T)$, and consequently there exists a module M in $\mathcal{T}(T)$ such that $Z = \operatorname{Hom}_A(T, M)$. Moreover, there exist homomorphisms $u\colon I \to M$ and $v\colon M \to I/\operatorname{soc}(I)$ in $\operatorname{mod} A$ such that $s = \operatorname{Hom}_A(T, u)$ and $r = \operatorname{Hom}_A(T, v)$.

Then

$$\mathrm{Hom}_A(T, p) = q = rs = \mathrm{Hom}_A(T, v)\,\mathrm{Hom}_A(T, u) = \mathrm{Hom}_A(T, vu),$$

and hence $p = vu$. Further, it follows from Lemma III.7.7 that p is an irreducible homomorphism in $\mathrm{mod}\,A$. Hence we conclude that u is a section or v is a retraction. This implies that $s = \mathrm{Hom}_A(T, u)$ is a section or $r = \mathrm{Hom}_A(T, v)$ is a retraction. Therefore, indeed q is an irreducible homomorphism in $\mathrm{mod}\,B$. Moreover, since there is a retraction $\pi: I/\operatorname{soc} I \to J$ in $\mathrm{mod}\,A$, we have in $\mathrm{mod}\,B$ the retraction $\mathrm{Hom}_A(T, \pi): \mathrm{Hom}_A(T, I/\operatorname{soc}(I)) \to \mathrm{Hom}_A(T, J)$, and consequently a retraction $\varrho: \mathrm{Hom}_A(T, I/\operatorname{soc}(I)) \to Y$, because $Y \cong \mathrm{Hom}_A(T, J)$ (see Lemma I.4.1). Then it follows from Lemma III.7.11 that $\varrho\,\mathrm{Hom}_A(T, p): \mathrm{Hom}_A(T, I) \to Y$ is an irreducible homomorphism in $\mathrm{mod}\,B$.

(ii) Let X be an indecomposable module in $\mathcal{X}(T)$. Assume that there exists an irreducible homomorphism $f: \mathrm{Hom}_A(T, I) \to X$ in $\mathrm{mod}\,B$. Observe that the module X is not projective, because all indecomposable projective right B-modules are in $\mathcal{Y}(T)$. Hence there is an irreducible homomorphism $g: \tau_B X \to \mathrm{Hom}_A(T, I)$ in $\mathrm{mod}\,B$. Since $\mathrm{Hom}_A(T, I)$ is in $\mathcal{Y}(T)$, we conclude that $\tau_B X$ is also in $\mathcal{Y}(T)$. This shows that we have in $\mathrm{mod}\,B$ a connecting sequence

$$0 \longrightarrow \tau_B X \longrightarrow E \longrightarrow X \longrightarrow 0,$$

with $\mathrm{Hom}_A(T, I)$ being isomorphic to an indecomposable direct summand of E. Then it follows from Lemma 4.1 that $\tau_B X \cong \mathrm{Hom}_A(T, J)$ in $\mathrm{mod}\,B$, for some indecomposable injective module J in $\mathrm{mod}\,A$. Further, applying Lemma 4.2, we conclude that $X \cong \mathrm{Ext}^1_A(T, P)$ in $\mathrm{mod}\,B$, for some projective cover P of the simple module $\operatorname{soc}(J)$ in $\mathrm{mod}\,A$. It follows also from Theorem III.7.11 that we have an irreducible map from $\mathrm{Hom}_A(T, J) \to \mathrm{Hom}_A(T, I)$. Then I is a direct summand of $J/\operatorname{soc}(J)$, by part (i).

Conversely, assume that there exists an indecomposable injective module J in $\mathrm{mod}\,A$ such that I is a direct summand of $J/\operatorname{soc}(J)$ and $\tau_B X \cong \mathrm{Hom}_A(T, J)$ in $\mathrm{mod}\,B$. Then it follows from (i) that there is an irreducible homomorphism $\tau_B X \to \mathrm{Hom}_A(T, I)$ in $\mathrm{mod}\,B$. But then $\mathrm{Hom}_A(T, I)$ is isomorphic to a direct summand of the middle term E of an almost split sequence

$$0 \longrightarrow \tau_B X \longrightarrow E \longrightarrow X \longrightarrow 0$$

in $\mathrm{mod}\,B$, and hence there exists an irreducible homomorphism $\mathrm{Hom}_A(T, I) \to X$ in $\mathrm{mod}\,B$ (see Theorems III.7.11 and III.7.12). ☐

Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a component of the Auslander–Reiten quiver Γ_A of A. Then a full valued subquiver Δ of $\mathcal{C}$ is said to be a *section* of $\mathcal{C}$ if the following conditions are satisfied:

(S1) Δ is an acyclic quiver.

(S2) Δ is a convex subquiver of $\mathcal{C}$, that is, every path in $\mathcal{C}$ with the source and target in Δ lies entirely in Δ.

(S3) For any indecomposable module X in $\mathcal{C}$ there is a unique integer m_X such that $\tau_A^{m_X} X$ belongs to Δ.

Observe that, if A is a hereditary algebra, then the postprojective component $\mathcal{P}(A)$ of Γ_A admits a section formed by a complete set of pairwise nonisomorphic indecomposable projective modules in mod A, and the preinjective component $\mathcal{Q}(A)$ admits a section formed by a complete set of pairwise nonisomorphic indecomposable injective modules in mod A (see Theorems VII.6.1 and VII.6.2).

Lemma 6.4. *Let A be a finite dimensional K-algebra over a field K, $\mathcal{C}$ a component of Γ_A, and Δ a section of $\mathcal{C}$. Then the following statements hold:*

(i) *For any arrow $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ in $\mathcal{C}$ with X in Δ, we have Y in Δ or $\tau_A Y$ in Δ.*

(ii) *For any arrow $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ in $\mathcal{C}$ with Y in Δ, we have X in Δ or $\tau_A^{-1} X$ in Δ.*

Proof. (i) Assume $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ is an arrow in $\mathcal{C}$ with X in Δ. It follows from the condition (S3) that $\tau_A^m Y \in \Delta$ for some integer $m = m_Y$. Assume $m \leq 0$. Then there is a path in $\mathcal{C}$ from X to $\tau_A^m Y$ with both ends in Δ and passing through Y. The condition (S2) implies that Y belongs to Δ. Moreover, $m = 0$, by the condition (S3). Similarly, if $m > 0$, then we infer that there is a path in $\mathcal{C}$ from $\tau_A^m Y$ to X with both ends in Δ and passing through $\tau_A Y$. Then, applying the conditions (S2) and (S3), we infer that $m = 1$ and $\tau_A Y \in \Delta$.

The proof of (ii) is similar. □

Proposition 6.5. *Let A be a finite dimensional K-algebra over a field K, $\mathcal{C}$ a component of Γ_A, and Δ a section of $\mathcal{C}$. Then $\mathcal{C}$ is isomorphic to a full valued translation subquiver of the stable translation quiver $\mathbb{Z}\Delta$. In particular, $\mathcal{C}$ is an acyclic component of Γ_A.*

Proof. Let Δ_0 be the set of indecomposable modules X in $\mathcal{C}$ lying in Δ and Δ_1 the set of all valued arrows $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ in $\mathcal{C}$ with $X, Y \in \Delta_0$. Then $\mathbb{Z}\Delta$ is the stable quiver with the set of vertices

$$(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0 = \big\{(i, X) \,\big|\, i \in \mathbb{Z}, X \in \Delta_0\big\},$$

and the set $(\mathbb{Z}\Delta)_1$ of arrows of $\mathbb{Z}\Delta$ consists of the valued arrows

$$(i, X) \xrightarrow{(d_{XY}, d'_{XY})} (i, Y), \quad (i+1, Y) \xrightarrow{(d'_{XY}, d_{XY})} (i, X), \quad i \in \mathbb{Z},$$

for all valued arrows $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ in Δ. Moreover, the translation $\tau \colon (\mathbb{Z}\Delta)_0 \to (\mathbb{Z}\Delta)_0$ is defined by $\tau(i, X) = (i + 1, X)$ for all $i \in \mathbb{Z}$ and $X \in \Delta_0$ (see Section III.9).

Let Γ be the full valued translation subquiver of $\mathbb{Z}\Delta$ given by all vertices $(i, X) \in (\mathbb{Z}\Delta)_0$ such that $\tau_A^i X$ is a module in $\mathcal{C}$. Then Γ is isomorphic to the full translation subquiver Σ of $\mathcal{C}$ whose set Σ_0 of vertices coincides with the set of all indecomposable modules in $\mathcal{C}$, that is, the set of all vertices of $\mathcal{C}$, and the set Σ_1 of arrows consists of all possible arrows in $\mathcal{C}$ of the forms

$$\tau_A^i X \xrightarrow{(d_{XY}, d'_{XY})} \tau_A^i Y \quad \text{and} \quad \tau_A^{i+1} Y \xrightarrow{(d'_{XY}, d_{XY})} \tau_A^i X,$$

where $i \in \mathbb{Z}$ and $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ are arrows in Δ. We have to show that Σ_1 is the set of all arrows in $\mathcal{C}$.

Let $M \xrightarrow{(d_{MN}, d'_{MN})} N$ be an arrow in $\mathcal{C}$. By (S3) there exist indecomposable modules X, Y in Δ and integers m, n such that $M = \tau_A^m X$ and $N = \tau_A^n Y$. We abbreviate $d = d_{MN}$ and $d' = d'_{MN}$. We have several cases to consider.

Assume $m = 0$. Then $M = X$ is in Δ. Applying Lemma 6.4, we conclude that $N \in \Delta_0$ or $\tau_A N \in \Delta_0$. If $\tau_A N$ belongs to Δ_0, then $\mathcal{C}$ admits the arrow $\tau_A N \xrightarrow{(d', d)} M$ (see Lemma III.9.1 and Proposition III.9.6), which is moreover an arrow in Δ. Hence, in both cases, $M \xrightarrow{(d, d')} N$ is an arrow in Σ_1. Because the case $n = 0$ is similar, we may assume that $m \neq 0$ and $n \neq 0$.

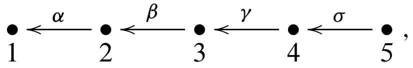
Assume $m > 0$ and $n > 0$. Then the component $\mathcal{C}$ contains the modules $\tau_A^i X$ and $\tau_A^j Y$ for $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$. This implies that $\mathcal{C}$ contains an arrow of one of the forms

$$\tau_A^{m-n} X \xrightarrow{(d, d')} Y \ (\text{for } m \geq n) \quad \text{or} \quad \tau_A^{n-m+1} Y \xrightarrow{(d', d)} X \ (\text{for } m < n).$$

In the first case, it follows from Lemma 6.4 that $\tau_A^{m-n} X \in \Delta_0$ or $\tau_A^{m-n-1} X \in \Delta_0$. Then, by the conditions (S2) and (S3), we conclude that $m = n$ or $m = n + 1$. In the second case, Lemma 6.4 yields that $\tau_A^{n-m+1} Y \in \Delta_0$ or $\tau_A^{n-m} Y \in \Delta_0$, and hence $n = m + 1$ or $m = n$, again by (S2) and (S3). Hence, in both cases, $M \xrightarrow{(d, d')} N$ is an arrow in Σ_1. The case where $m < 0$ and $n < 0$ is similar.

Assume $m > 0$ and $n < 0$. Then $\mathcal{C}$ contains a path from Y to X passing through $\tau_A^n Y = N$ and $\tau_A^{m-1} X = \tau_A^{-1} M$, and hence two indecomposable modules Y and $\tau_A^n Y$ in the τ_A-orbit of Y, by (S2), a contradiction to the condition (S3). We reach a similar contradiction in the case $m < 0$ and $n > 0$. □

Example 6.6. Let K be a field, Q the quiver



,

I the ideal in the path algebra KQ of Q over K generated by the path $\sigma\gamma\beta\alpha$, and $A = KQ/I$ the associated bound quiver algebra, considered in Example 3.15. Then the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
S_1 = P_1 \arrow[dr] & & S_2 \arrow[dr] & & S_3 \arrow[dr] & & S_4 \arrow[dr] & & I_5 = S_5 \\
& P_2 \arrow[ur] \arrow[dr] & & P_3/S_1 \arrow[ur] \arrow[dr] & & P_4/P_2 \arrow[ur] \arrow[dr] & & I_4 \arrow[ur] & \\
& & P_3 \arrow[ur] \arrow[dr] & & P_4/S_1 \arrow[ur] \arrow[r] & P_5 \arrow[r] \arrow[d, equal] & I_3 \arrow[ur] & & \\
& & & P_4 = I_1 \arrow[ur] & & I_2 & & &
\end{tikzcd}$$

where S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in mod A given by the vertex $i \in \{1, 2, 3, 4, 5\}$ of Q. Then the following full subquivers of Γ_A form the complete family of all possible sections in Γ_A:

$$\begin{tikzcd}
S_2 \arrow[dr] & & & \\
& P_3/S_1 \arrow[dr] & & \\
& & P_4/S_1 \arrow[r] & P_5 \\
& P_4 \arrow[ur] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& S_3 & \\
P_3/S_1 \arrow[ur] \arrow[dr] & & \\
& P_4/S_1 \arrow[r] & P_5 \\
P_4 \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
& S_3 \arrow[dr] & & \\
& & P_4/P_2 & \\
& P_4/S_1 \arrow[ur] \arrow[r] & P_5 & \\
P_4 \arrow[ur] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & S_4 \\
& & P_4/P_2 \arrow[ur] & \\
& P_4/S_1 \arrow[ur] \arrow[r] & P_5 & \\
P_4 \arrow[ur] & & &
\end{tikzcd}$$

We note that the projective-injective modules $P_4 = I_1$ and $P_5 = I_2$ have to belong to all sections of Γ_A.

Let A be a finite dimensional K-algebra over a field K, $\mathcal{C}$ a component of the Auslander–Reiten quiver Γ_A of A, and Δ a full valued subquiver of $\mathcal{C}$. A module

M in $\mathcal{C}$ is said to be a *predecessor* of Δ in $\mathcal{C}$ if there is a path in $\mathcal{C}$ from M to a module N in Δ. Moreover, M is said to be a *proper predecessor* of Δ in $\mathcal{C}$ if M is a predecessor of Δ in $\mathcal{C}$ but does not belong to Δ. Dually, a module M in $\mathcal{C}$ is said to be a *successor* of Δ in $\mathcal{C}$ if there is a path in $\mathcal{C}$ from a module N in Δ to M. Finally, M is said to be a *proper successor* of Δ in $\mathcal{C}$ if M is a successor of Δ in $\mathcal{C}$ but does not belong to Δ.

Theorem 6.7. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$, *and $B = \operatorname{End}_A(T)$ the associated tilted algebra. Moreover, let $I_1, \dots, I_n$ be a complete set of pairwise nonisomorphic indecomposable injective modules in* $\operatorname{mod} A$. *Then the following statements hold:*

(i) *The Auslander–Reiten quiver Γ_B of B contains a connected, acyclic component $\mathcal{C}_T$ with a section Δ_T formed by the indecomposable right B-modules* $\operatorname{Hom}_A(T, I_1), \dots, \operatorname{Hom}_A(T, I_n)$.

(ii) *The quiver Δ_T is isomorphic to Q_A^{op}.*

(iii) *Every predecessor of Δ_T in $\mathcal{C}_T$ belongs to $\mathcal{Y}(T)$.*

(iv) *Every proper successor of Δ_T in $\mathcal{C}_T$ belongs to $\mathcal{X}(T)$.*

Proof. It follows from Theorem VII.6.2 that the Auslander–Reiten quiver Γ_A of A contains a unique preinjective component $\mathcal{Q}(A)$ with the section Σ formed by the indecomposable injective modules $I_1, \dots, I_n$, and the valued quiver Σ is isomorphic to the opposite quiver Q_A^{op} of Q_A. Moreover, every indecomposable module in $\mathcal{Q}(A)$ is of the form $\tau_A^j I_i$ for some $j \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. We also note that $I_1, \dots, I_n$ belong to the torsion class $\mathcal{T}(T)$ of mod A induced by T. Let

$$I_i \xrightarrow{\left(d_{I_i I_j}, d'_{I_i I_j}\right)} I_j$$

be a valued arrow in $\mathcal{Q}(A)$ for some $i, j \in \{1, \dots, n\}$. We claim that Γ_B admits a valued arrow

$$\operatorname{Hom}_A(T, I_i) \xrightarrow{\left(d_{I_i I_j}, d'_{I_i I_j}\right)} \operatorname{Hom}_A(T, I_j).$$

Observe that I_i is nonsimple, because I_i is not a sink in Γ_A. Further, it follows from Lemma III.7.7 that the canonical epimorphism $p_i \colon I_i \to I_i/\operatorname{soc}(I_i)$ is a left minimal almost split homomorphism and an irreducible homomorphism in mod A.

The number $d_{I_i I_j}$ is the multiplicity of I_j in the decomposition of $I_i/\operatorname{soc}(I_i)$ into a direct sum of indecomposable modules, and there is an irreducible homomorphism $h_{ij} \colon I_i \to I_j^{d_{I_i I_j}}$ in mod A, by Theorem III.7.11. Now the arguments applied in the proof of Proposition 6.3 show that $\operatorname{Hom}_A(T, h_{ij}) \colon \operatorname{Hom}_A(T, I_i) \to$

$\mathrm{Hom}_A(T, I_j^{d_{I_iI_j}})$ is an irreducible homomorphism in $\mathrm{mod}\, B$. Clearly, we have an isomorphism $\mathrm{Hom}_A(T, I_j^{d_{I_iI_j}}) \cong \mathrm{Hom}_A(T, I_j)^{d_{I_iI_j}}$ in $\mathrm{mod}\, B$. Let now $\varphi_i \colon \mathrm{Hom}_A(T, I_i) \to M$ be a left minimal almost split homomorphism in $\mathrm{mod}\, B$ with the left term $\mathrm{Hom}_A(T, I_i)$. Since there is an irreducible homomorphism $\mathrm{Hom}_A(T, h_{ij})$ with the domain $\mathrm{Hom}_A(T, I_i)$ and the codomain $\mathrm{Hom}_A(T, I_j)^{d_{I_iI_j}}$, we infer from Theorem III.7.11 that $M \cong \mathrm{Hom}_A(T, I_j)^{m_{ij}} \oplus M'$ in $\mathrm{mod}\, B$, for some integer $m_{ij} \geq d_{I_iI_j}$, and M' has no indecomposable direct summand isomorphic to $\mathrm{Hom}_A(T, I_j)$. Applying Theorem III.7.11 again, we see that there exists an irreducible homomorphism in $\mathrm{mod}\, B$ of the form $\mathrm{Hom}_A(T, I_i) \to \mathrm{Hom}_A(T, I_j)^{m_{ij}}$. Since $\mathrm{Hom}_A(T, I_j)^{m_{ij}} \cong \mathrm{Hom}_A(T, I_j^{m_{ij}})$ in $\mathrm{mod}\, B$, we conclude that there exists an irreducible homomorphism in $\mathrm{mod}\, A$ of the form $I_i \to I_j^{m_{ij}}$, and consequently $I_j^{m_{ij}}$ is a direct summand of $I_i/\operatorname{soc}(I_i)$. This leads to the equality $m_{ij} = d_{I_iI_j}$.

The number $d'_{I_iI_j}$ is the multiplicity of I_i in the decomposition of the domain V of a right minimal almost split homomorphism $V \to I_j$ in $\mathrm{mod}\, A$, and there is an irreducible homomorphism $g_{ij} \colon I_i^{d'_{I_iI_j}} \to I_j$ in $\mathrm{mod}\, A$, by Theorem III.7.12. The arguments applied in the proof of Proposition 6.3 show that $\mathrm{Hom}_A(T, g_{ij}) \colon \mathrm{Hom}_A(T, I_i^{d'_{I_iI_j}}) \to \mathrm{Hom}_A(T, I_j)$ is an irreducible homomorphism in $\mathrm{mod}\, B$. Let now $\psi_j \colon N \to \mathrm{Hom}_A(T, I_j)$ be a right minimal almost split homomorphism in $\mathrm{mod}\, B$ with the right term $\mathrm{Hom}_A(T, I_j)$. Then, applying Theorem III.7.12, we conclude that $N \cong \mathrm{Hom}_A(T, I_i)^{n_{ij}} \oplus N'$ in $\mathrm{mod}\, B$, for some integer $n_{ij} \geq d'_{I_iI_j}$, and N' has no indecomposable direct summand isomorphic to $\mathrm{Hom}_A(T, I_i)$. Applying Theorem III.7.12 again, we conclude that there exists an irreducible homomorphism in $\mathrm{mod}\, B$ of the form $\mathrm{Hom}_A(T, I_i)^{n_{ij}} \to \mathrm{Hom}_A(T, I_j)$. Since $\mathrm{Hom}_A(T, I_i)^{n_{ij}} \cong \mathrm{Hom}_A(T, I_i^{n_{ij}})$ in $\mathrm{mod}\, B$, we conclude that there exists an irreducible homomorphism in $\mathrm{mod}\, A$ of the form $I_i^{n_{ij}} \to I_j$, and consequently $I_i^{n_{ij}}$ is a direct summand of V. This leads to the equality $n_{ij} = d'_{I_iI_j}$.

Therefore, we proved that indeed Γ_B admits the valued arrow

$$\mathrm{Hom}_A(T, I_i) \xrightarrow{\left(d_{I_iI_j}, d'_{I_iI_j}\right)} \mathrm{Hom}_A(T, I_j).$$

This shows that Γ_B admits a full valued subquiver Δ_T, isomorphic to the quiver Q_A^{op}, whose vertices are the indecomposable right B-modules $\mathrm{Hom}_A(T, I_1)$, $\dots, \mathrm{Hom}_A(T, I_n)$. Since Δ_T is a connected quiver, we obtain that Δ_T is a full valued subquiver of a connected component $\mathcal{C}_T$ of Γ_A. Further, it follows from Proposition 6.3 that, for any irreducible homomorphism $\mathrm{Hom}_A(T, I_i) \to Y$ in $\mathrm{mod}\, B$ with Y an indecomposable module in $\mathcal{Y}(T)$, there exists an indecomposable direct summand J of $I_i/\operatorname{soc}(I_i)$ such that $Y \cong \mathrm{Hom}_A(T, J)$ in $\mathrm{mod}\, B$.

Moreover, since A is a hereditary algebra, the module $I_i / \operatorname{soc} I_i$ is injective (see Theorems I.9.2 and I.9.3), and hence J is an indecomposable injective module in $\operatorname{mod} A$, so isomorphic to some module I_j. This shows that $Y \cong \operatorname{Hom}_A(T, I_j)$, for some $j \in \{1, \ldots, n\}$, and hence belongs to Δ_T. Observe also that every predecessor of an indecomposable module of Δ_T in $\mathcal{C}_T$ belongs to $\mathcal{Y}(T)$, because Δ_T consists of modules from $\mathcal{Y}(T)$ and $\mathcal{Y}(T)$ is closed under predecessors in $\operatorname{mod} B$. On the other hand, if there exists an irreducible homomorphism $Y \to X$ in $\operatorname{mod} B$ with Y an indecomposable module in Δ_T and X an indecomposable module not in Δ_T, then X belongs to $\mathcal{X}(T)$, by Proposition 6.3. Since $\mathcal{X}(T)$ is closed under successors in $\operatorname{mod} B$, we conclude that every proper successor of Δ_T in $\mathcal{C}_T$ is in $\mathcal{X}(T)$. Therefore, it remains to show that Δ_T is a section of $\mathcal{C}_T$, because then the acyclicity of $\mathcal{C}_T$ follows from Proposition 6.5.

Observe that Δ_T is an acyclic quiver, because $\Delta_T \cong Q_A^{\mathrm{op}}$ and the quiver Q_A is acyclic, by Corollary VII.1.8. Hence the condition (S1) is satisfied. Assume now that

$$Y_0 \xrightarrow{(d_{Y_0 Y_1}, d'_{Y_0 Y_1})} Y_1 \longrightarrow \cdots \longrightarrow Y_{t-1} \xrightarrow{(d_{Y_{t-1} Y_t}, d'_{Y_{t-1} Y_t})} Y_t$$

is a path in $\mathcal{C}_T$ with Y_0 and Y_t in Δ_T. Then all the modules $Y_0, Y_1, \ldots, Y_t$ are in $\mathcal{Y}(T)$, because $\mathcal{Y}(T)$ is closed under predecessors in $\operatorname{mod} B$. Hence the arrow $Y_0 \xrightarrow{(d_{Y_0 Y_1}, d'_{Y_0 Y_1})} Y_1$ with $Y_0 \in \Delta_T$ and $Y_1 \in \mathcal{Y}(T)$ forces Y_1 to be in Δ_T, again by Proposition 6.3. Then we show inductively that all modules $Y_1, \ldots, Y_t$ belong to Δ_T. This shows that Δ_T is a convex subquiver of $\mathcal{C}_T$, and so the condition (S2) holds.

We will show now that Δ_T satisfies the condition (S3). Observe first that every indecomposable projective right B-module is in $\mathcal{Y}(T)$ and hence cannot be a proper successor of Δ_T in $\mathcal{C}_T$. On the other hand, by Proposition 5.4 and Theorem 5.8, if an indecomposable injective right B-module belongs to $\mathcal{Y}(T)$, then it must lie on Δ_T. We will show now that Δ_T intersects each τ_B-orbit in $\mathcal{C}_T$.

Suppose there is a τ_B-orbit $\mathcal{O}$ in $\mathcal{C}_T$ which does not intersect Δ_T. We may choose such a τ_B-orbit $\mathcal{O}$ which is close to a τ_B-orbit $\mathcal{O}'$ in $\mathcal{C}_T$ intersecting Δ_T. Then there exist a module Y in Δ_T and an integer $n \in \mathbb{Z}$ such that we have an irreducible homomorphism $\tau_B^n Y \to X$ or $X \to \tau_B^n Y$ for some module X in $\mathcal{O}$. We may choose such an orbit $\mathcal{O}$ with $|n|$ being minimal. We claim that then $n = 0$. We have two cases to consider.

(1) Assume $n < 0$. We claim that then $X \in \mathcal{Y}(T)$. Indeed, if $X \in \mathcal{X}(T)$, then X is nonprojective, and hence there exists an irreducible homomorphism $\tau_B^{n+1} Y \to \tau_B X$ or $\tau_B X \to \tau_B^{n+1} Y$ with $\tau_B X$ in $\mathcal{O}$, which contradict with the minimality assumption. Hence, indeed X belongs to $\mathcal{Y}(T)$. Since $\tau_B^n Y$ belongs to $\mathcal{X}(T)$ and $\mathcal{X}(T)$ is closed under successors in $\operatorname{mod} B$, we conclude that there exists an irreducible homomorphism $X \to \tau_B^n Y$, and hence an irreducible ho-

momorphism $\tau_B^{n+1}Y \to X$, because $\tau_B^n Y$ is nonprojective, again a contradiction with the minimality assumption.

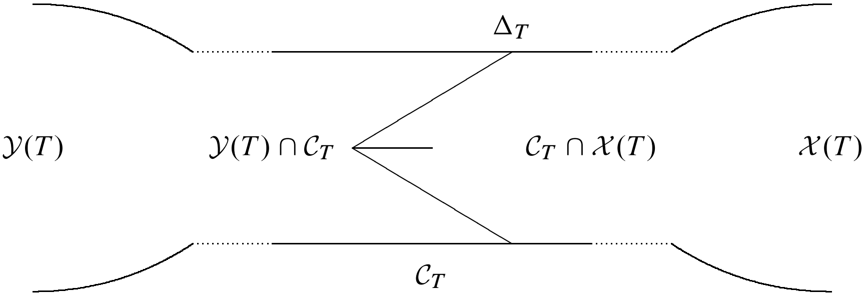
(2) Assume $n > 0$. We claim that either $X \in \Delta_T$ or $X \in \mathcal{X}(T)$. Indeed, suppose that X is neither in Δ_T, nor in $\mathcal{X}(T)$. Then X is noninjective, and hence there exists an irreducible homomorphism $\tau_B^{n-1}Y \to \tau_B^{-1}X$ or $\tau_B^{-1}X \to \tau_B^{n-1}Y$ with $\tau_B^{-1}X$ in $\mathcal{O}$, and this contradicts the minimality assumption. Hence, indeed X is in Δ_T or in $\mathcal{X}(T)$. Since $\tau_B^n Y$ is a proper predecessor of Y, lying on Δ_T, we conclude that there exists an irreducible homomorphism $\tau_B^n Y \to X$, and then X is a predecessor of Y in $\mathcal{C}_T$, and hence belongs to $\mathcal{Y}(T)$, a contradiction. Therefore, we conclude that $X \in \Delta_T$, and hence $\mathcal{O}$ intersects Δ_T, a contradiction to the assumption on $\mathcal{O}$.

Therefore, we have $n = 0$. Then there exists an irreducible homomorphism $Y \to X$ or one $X \to Y$. In the first case, applying Proposition 6.3, we conclude that X or $\tau_B X$ lies in Δ_T. In the second case, we have $X \in \mathcal{Y}(T)$. Since X is in the τ_B-orbit $\mathcal{O}$ not intersecting Δ_T, X is not in Δ_T, and so X is noninjective. But then there exists an irreducible homomorphism $Y \to \tau_B^{-1}X$, and, from the first case, we have that $\tau_B^{-1}X$ or X lies in Δ_T, again a contradiction to the assumption imposed on $\mathcal{O}$.

Summing up, we proved that Δ_T intersects each τ_B-orbit of $\mathcal{C}_T$. Finally, we show that Δ_T intersects each τ_B-orbit on $\mathcal{C}_T$ exactly once. Suppose it is not the case. Then Δ_T contains two modules Y and $\tau_B^{-s}Y$ with $s \geq 1$. But then $\tau_B^{-s}Y \in \mathcal{Y}(T)$ implies that $Y \in \Delta_T$ and $\tau_B^{-1}Y \in \mathcal{Y}(T)$, because $\mathcal{Y}(T)$ is closed under predecessors in mod B. This contradicts the fact that every proper successor of Δ_T in $\mathcal{C}_T$ lies in $\mathcal{X}(T)$. Therefore, Δ_T is a section of $\mathcal{C}_T$. This finishes the proof. □

One may think of the component $\mathcal{C}_T$ of Γ_B as connecting the torsion-free part $\mathcal{Y}(T)$ with the torsion part $\mathcal{X}(T)$ along the section Δ_T. For this reasons, the component $\mathcal{C}_T$ is called the *connecting component* of Γ_B determined by the tilting module T.

We may visualize this situation by the following picture



Example 6.8. Let K be a field, Q the quiver

$$\begin{tikzcd}
1 \bullet & & & \\
& \underset{3}{\bullet} \arrow[lu, "\alpha"'] \arrow[ld, "\beta"] & \underset{4}{\bullet} \arrow[l, "\gamma"'] \arrow[r, "\sigma"] & \underset{5}{\bullet} \\
2 \bullet & & &
\end{tikzcd}$$

and $A = KQ$ the path algebra of Q over K. Then A is a hereditary algebra of Dynkin type $\mathbb{D}_5$, and hence A is of finite representation type, by Theorem VII.7.4. Moreover, the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
S_1 = P_1 \arrow[rd] & & P_3/S_1 \arrow[rd] & & P_4/S_2 \arrow[rd] & & I_2 \arrow[rd] & & \\
S_2 = P_2 \arrow[r] & P_3 \arrow[ru] \arrow[r] \arrow[rd] & P_3/S_2 \arrow[r] & M \arrow[ru] \arrow[r] \arrow[rd] & P_4/S_1 \arrow[r] & V \arrow[ru] \arrow[r] \arrow[rd] & I_1 \arrow[r] & I_3 \arrow[rd] & \\
& & P_4 \arrow[ru] \arrow[rd] & & N \arrow[ru] \arrow[rd] & & P_4/(S_1 \oplus S_2) \arrow[ru] \arrow[rd] & & I_4 = S_4 \\
& S_5 = P_5 \arrow[ru] & & P_4/S_5 \arrow[ru] & & S_3 \arrow[ru] & & I_5 \arrow[ru] &
\end{tikzcd}$$

where S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} A$ given by the vertex $i \in \{1,2,3,4,5\}$ of Q, and M, N, V are the indecomposable modules with the composition vectors

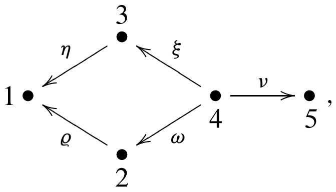
$$\begin{aligned}
c(M) &= \big(c_1(M), c_2(M), c_3(M), c_4(M), c_5(M)\big) = (1,1,2,1,1),\\
c(N) &= \big(c_1(N), c_2(N), c_3(N), c_4(N), c_5(N)\big) = (1,1,2,1,0),\\
c(V) &= \big(c_1(V), c_2(V), c_3(V), c_4(V), c_5(V)\big) = (1,1,2,2,1).
\end{aligned}$$

Let $T_1 = S_5$, $T_2 = P_4/S_1$, $T_3 = P_4/S_2$, $T_4 = P_4/(S_1 \oplus S_2)$, $T_5 = S_3$, and $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$. We claim that T is a tilting module in $\operatorname{mod} A$. Clearly, we have $\operatorname{pd}_A T \le 1$, because A is a hereditary algebra. Moreover, we have K-linear isomorphisms

$$\begin{aligned}
\operatorname{Ext}^1_A(T,T) &\cong D\operatorname{Hom}_A(T, \tau_A T)\\
&\cong D\operatorname{Hom}_A\big(T, (P_3/S_2) \oplus (P_3/S_1) \oplus N \oplus (P_4/S_5)\big) = 0.
\end{aligned}$$

Moreover, $K_0(A)$ is of rank 5 and T_1, T_2, T_3, T_4, T_5 are pairwise nonisomorphic indecomposable modules in $\operatorname{mod} A$. Then it follows from Proposition 3.11 that T is a tilting module. Let $B = \operatorname{End}_A(T)$ be the tilted algebra given by T. Then B

is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver



J is the ideal in the path algebra $K\Delta$ of Δ generated by $\xi\eta - \omega\varrho$, and the vertices $1, 2, 3, 4, 5$ of the quiver Δ correspond to the indices of the modules T_1, T_2, T_3, T_4, T_5. Since T is a splitting tilting module in mod A, by Theorem 5.8, and A is of finite representation type, we infer that B is also of finite representation type. In fact, the Auslander–Reiten quiver Γ_B of B is of the form

$$\begin{tikzcd}
 & P_3^B \arrow[dr] & & S_2^B \arrow[dr] & & P_4^B/P_2^B \arrow[dr] & & I_2^B \arrow[dr] & \\
S_1^B = P_1^B \arrow[r] \arrow[ur] & P_2^B \arrow[r] & \operatorname{rad} I_1^B \arrow[r] \arrow[ur] \arrow[dr] & S_3^B \arrow[r] & P_4^B/S_1^B \arrow[r] \arrow[ur] \arrow[dr] & P_4^B/P_3^B \arrow[r] & R \arrow[r] \arrow[ur] \arrow[dr] & I_3^B \arrow[r] & I_4^B = S_4^B \\
 & & & P_4^B \arrow[ur] \arrow[dr] & & P_4^B/(S_1^B \oplus S_5^B) \arrow[ur] & & I_5^B \arrow[ur] & \\
 & & S_5^B = P_5^B \arrow[ur] & & I_1^B \arrow[ur] & & & &
\end{tikzcd}$$

where S_i^B, P_i^B, and I_i^B are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in mod B given by the vertex $i \in \{1, 2, 3, 4, 5\}$ of Δ, and R is the indecomposable module with the composition vector

$$c(R) = \big(c_1(R), c_2(R), c_3(R), c_4(R), c_5(R)\big) = (0, 1, 1, 2, 1).$$

Obviously, Γ_B is the connecting component $\mathcal{C}_T$ determined by T. Let us describe the section Δ_T of $\mathcal{C}_T$. To this end we compute the composition vectors $c(\operatorname{Hom}_A(T, I_i))$ of the right B-modules $\operatorname{Hom}_A(T, I_i)$ forming the modules of Δ_T. Since $B = K\Delta/J$, we have $\dim_K \operatorname{End}_B(S_i^B) = 1$ for any $i \in \{1, 2, 3, 4, 5\}$. Applying Lemma 3.13, we obtain the equalities

$$\begin{aligned}
c_i(\operatorname{Hom}_A(T, I_1)) &= \dim_K \operatorname{Hom}_A(T_i, I_1) = 0, \text{ for } i \in \{1, 2, 4, 5\},\\
c_3(\operatorname{Hom}_A(T, I_1)) &= \dim_K \operatorname{Hom}_A(T_3, I_1) = 1,\\
c_i(\operatorname{Hom}_A(T, I_2)) &= \dim_K \operatorname{Hom}_A(T_i, I_2) = 0, \text{ for } i \in \{1, 3, 4, 5\},\\
c_2(\operatorname{Hom}_A(T, I_2)) &= \dim_K \operatorname{Hom}_A(T_2, I_2) = 1,\\
c_i(\operatorname{Hom}_A(T, I_3)) &= \dim_K \operatorname{Hom}_A(T_i, I_3) = 1, \text{ for } i \in \{2, 3, 4, 5\},\\
c_1(\operatorname{Hom}_A(T, I_3)) &= \dim_K \operatorname{Hom}_A(T_1, I_3) = 0,
\end{aligned}$$

$$\begin{aligned} c_i(\mathrm{Hom}_A(T, I_4)) &= \dim_K \mathrm{Hom}_A(T_i, I_4) = 0, \text{ for } i \in \{1,5\},\\ c_i(\mathrm{Hom}_A(T, I_4)) &= \dim_K \mathrm{Hom}_A(T_i, I_4) = 1, \text{ for } i \in \{2,3,4\},\\ c_i(\mathrm{Hom}_A(T, I_5)) &= \dim_K \mathrm{Hom}_A(T_i, I_5) = 1, \text{ for } i \in \{1,2,3,4\},\\ c_5(\mathrm{Hom}_A(T, I_5)) &= \dim_K \mathrm{Hom}_A(T_5, I_5) = 0. \end{aligned}$$

Therefore, we obtain isomorphisms in mod B

$$\mathrm{Hom}_A(T, I_1) \cong S_3^B, \quad \mathrm{Hom}_A(T, I_2) \cong S_2^B, \quad \mathrm{Hom}_A(T, I_3) \cong P_4^B/S_1^B,$$
$$\mathrm{Hom}_A(T, I_4) \cong P_4^B/(S_1^B \oplus S_5^B), \quad \mathrm{Hom}_A(T, I_5) \cong I_1^B.$$

Observe that the section Δ_T contains the injective module $\mathrm{Hom}_A(T, I_5)$, because the socle S_5 of I_5 is projective and a direct summand of the tilting module T. Moreover, $\Delta_T \cong Q^{\mathrm{op}} = Q_A^{\mathrm{op}}$. Finally, observe that the torsion-free class $\mathcal{Y}(T)$ is the additive category of the 11 indecomposable predecessors of Δ_T in $\mathcal{C}_T$, while the torsion class $\mathcal{X}(T)$ is the additive category of the 7 indecomposable proper successors of Δ_T in $\mathcal{C}_T$.

Proposition 6.9. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, T a tilting module in* mod A, *$B = \mathrm{End}_A(T)$, and $\mathcal{C}_T$ the connecting component of Γ_B determined by T. Then the following statements hold:*

(i) *$\mathcal{C}_T$ contains a projective module if and only if T has a nonzero preinjective direct summand.*

(ii) *$\mathcal{C}_T$ contains an injective module if and only if T has a nonzero postprojective direct summand.*

Proof. (i) Assume that T has no nonzero preinjective direct summand. We claim that $\mathcal{C}_T$ has no projective module. It follows from Proposition VII.6.7 that, for any indecomposable modules Y and N in mod A with Y in the preinjective component $\mathcal{Q}(A)$, $\mathrm{Hom}_A(Y, N) \neq 0$ implies that N is a successor of Y in $\mathcal{Q}(A)$. Then we conclude that $\mathcal{Q}(A)$ is entirely contained in the torsion class

$$\begin{aligned} \mathcal{T}(T) &= \{M \in \mathrm{mod}\, A \mid \mathrm{Ext}_A^1(T, M) = 0\}\\ &= \{M \in \mathrm{mod}\, A \mid \mathrm{Hom}_A(M, \tau_A T) = 0\}. \end{aligned}$$

Applying now Theorems 5.8, 6.7 and Proposition 5.3, we obtain that the image $\mathrm{Hom}_A(T, \mathcal{Q}(A))$ of $\mathcal{Q}(A)$ under the functor $\mathrm{Hom}_A(T, -)\colon \mathcal{T}(T) \to \mathcal{Y}(T)$ coincides with the torsion-free part $\mathcal{Y}(T) \cap \mathcal{C}_T$ of $\mathcal{C}_T$, formed by all predecessors of modules lying on the section Δ_T, given by the images of the indecomposable injective modules in $\mathcal{Q}(A)$, equivalently in mod A, under the functor $\mathrm{Hom}_A(T, -)$. Since every indecomposable projective module in mod B belongs to $\mathcal{Y}(T)$, we conclude that $\mathcal{C}_T$ does not contain a projective module.

Conversely, assume that T contains a nonzero preinjective direct summand. We will show that $\mathcal{C}_T$ contains a projective module. Since the preinjective component $\mathcal{Q}(A)$ is an acyclic component, we may choose an indecomposable direct summand T_0 of T from $\mathcal{Q}(A)$ such that no proper successor of T_0 in $\mathcal{Q}(A)$ is a direct summand of T. Observe that T_0 belongs to $\mathcal{T}(T)$, because $\operatorname{Ext}_A^1(T, T_0) = 0$. Further, it follows from Theorem VII.6.2, that there exist an indecomposable injective module I in $\mathcal{Q}(A)$ and a nonnegative integer m such that $\tau_A^{-m} T_0 = I$. Since every successor of $T_0 \in \mathcal{T}(T)$ in $\mathcal{Q}(A)$ also belongs to $\mathcal{T}(T)$, applying Proposition 5.3 we conclude that

$$\tau_B^{-m} \operatorname{Hom}_A(T, T_0) \cong \operatorname{Hom}_A(T, \tau_A^{-m} T_0) \cong \operatorname{Hom}_A(T, I)$$

in $\operatorname{mod} B$. Then the projective right B-module $\operatorname{Hom}_A(T, T_0) \cong \tau_B^{m} \operatorname{Hom}_A(T, I)$ belongs to $\mathcal{C}_T$, because $\operatorname{Hom}_A(T, I)$ is in Δ_T.

(ii) Assume T has no nonzero postprojective direct summand. We claim that $\mathcal{C}_T$ has no injective module. It follows from Proposition VII.6.6 that, for any indecomposable modules M and X in $\operatorname{mod} A$ with X in the postprojective component $\mathcal{P}(A)$, $\operatorname{Hom}_A(M, X) \neq 0$ implies that M is a predecessor of X in $\mathcal{P}(A)$. Then we conclude that $\mathcal{P}(A)$ is entirely contained in the torsion-free class $\mathcal{F}(T) = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, M) = 0\}$. Applying now Theorems 5.8, 6.7 and Proposition 5.3, we obtain that the image $\operatorname{Ext}_A^1(T, \mathcal{P}(A))$ of $\mathcal{P}(A)$ under the functor $\operatorname{Ext}_A^1(T, -)\colon \mathcal{F}(T) \to \mathcal{X}(T)$ coincides with the torsion part $\mathcal{X}(T) \cap \mathcal{C}_T$ of $\mathcal{C}_T$, formed by all proper successors of the section Δ_T in $\mathcal{C}_T$. Suppose that the connecting component $\mathcal{C}_T$ contains an injective module E. Clearly, E is a successor of the section Δ_T in $\mathcal{C}_T$. On the other hand, the torsion part $\mathcal{X}(T) \cap \mathcal{C}_T = \operatorname{Ext}_A^1(T, \mathcal{P}(A))$ has no injective module. Hence E lies on Δ_T, and consequently there exists an indecomposable injective module I in $\mathcal{Q}(A)$ such that $E \cong \operatorname{Hom}_A(T, I)$ in $\operatorname{mod} B$. Since $\operatorname{Hom}_A(T, I)$ is injective, it follows from Lemma 4.2 that the projective cover P of the socle $\operatorname{soc}(I)$ of I is a direct summand of T, a contradiction. Therefore, $\mathcal{C}_T$ does not contain an injective module.

Conversely, assume that T contains a nonzero postprojective direct summand. We will show that $\mathcal{C}_T$ contains an injective module. Observe that if an indecomposable projective module P is a direct summand of T, then $\mathcal{C}_T$ contains the injective module $\operatorname{Hom}_A(T, I)$ with I being an injective envelope of the $\operatorname{top}(P)$, by Lemma 4.2. Hence, assume that T has no nonzero projective direct summand. Since $\mathcal{P}(A)$ is an acyclic component, we may choose an indecomposable direct summand T_0 of T in $\mathcal{P}(A)$ such that no proper predecessor of T_0 in $\mathcal{P}(A)$ is a direct summand of T. This implies that all proper predecessors of T_0 in $\mathcal{P}(A)$ are in $\mathcal{F}(T)$, by Proposition VII.6.6. On the other hand, it follows from Theorem VII.6.1 and the assumption made on T_0 that there exist an indecomposable projective module P in $\mathcal{P}(A)$ and a positive integer n such that $\tau_A^n T_0 = P$. Let I be an indecomposable injective module in $\operatorname{mod} A$ with $\operatorname{top}(P) \cong \operatorname{soc}(I)$. Since P is not a direct summand of T, it follows from Lemma 4.2 that $\operatorname{Hom}_A(T, I)$ is noninjective in $\operatorname{mod} B$ and $\tau_B^{-1}\operatorname{Hom}_A(T, I) \cong \operatorname{Ext}_A^1(T, P)$. Further, since all predecessors of $\tau_A T_0$ in $\mathcal{P}(A)$ belong to $\mathcal{F}(T)$, applying Proposition 5.3, we conclude that

$$\tau_B^{-n}\operatorname{Hom}_A(T, I) \cong \tau_B^{-n+1}\operatorname{Ext}_A^1(T, P) \cong \operatorname{Ext}_A^1(T, \tau_A^{-n+1}P) \cong \operatorname{Ext}_A^1(T, \tau_A T_0)$$

in $\operatorname{mod} B$, which shows that $\operatorname{Ext}_A^1(T, \tau_A T_0)$ belongs to $\mathcal{C}_T$, because $\operatorname{Hom}_A(T, I)$ lies on the section Δ_T. Moreover, we infer from Proposition 5.4 and Theorem 5.8 that $\operatorname{Ext}_A^1(T, \tau_A T_0)$ is an injective right B-module. Therefore, the component $\mathcal{C}_T$ contains an injective module. □

Corollary 6.10. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K, T a tilting module in* $\operatorname{mod} A$, $B = \operatorname{End}_A(T)$, *and $\mathcal{C}_T$ the connecting component of Γ_B determined by T. Then the following statements hold:*

(i) *B is of finite representation type if and only if $\mathcal{C}_T$ is both a postprojective component and a preinjective component.*

(ii) *If B is of finite representation type, then T has both a nonzero postprojective and a nonzero preinjective direct summand.*

Proof. (i) Assume B is of finite representation type. Since B is an indecomposable algebra, it follows from Theorem III.10.2 that $\Gamma_B = \mathcal{C}_T$ and is a finite acyclic component. Then for every module Z in $\mathcal{C}_T$ there exist nonnegative integers m, n such that $\tau_B^m Z$ is a projective module and $\tau_B^{-n} Z$ is an injective module. This shows that $\mathcal{C}_T$ is both a postprojective component and a preinjective component.

Conversely, assume that $\mathcal{C}_T$ is both a postprojective component and a preinjective component. Then it follows from Theorem 6.7 that the section Δ_T of $\mathcal{C}_T$ has finitely many predecessors and finitely many successors, and consequently $\mathcal{C}_T$ is a finite component. Since B is an indecomposable algebra, applying Theorem III.10.2 we conclude that B is of finite representation type.

The part (ii) follows from the part (i) and Proposition 6.9. □

We note that there exist indecomposable finite dimensional hereditary algebras A and tilting modules T in $\operatorname{mod} A$ having both nonzero postprojective and nonzero preinjective direct summands such that the associated tilted algebras $B = \operatorname{End}_A(T)$ are of infinite representation type (see Example 3.18). Hence, the converse of the implication in (ii) of Corollary 6.10 is not true in general. On the other hand, we have the following proposition.

Proposition 6.11. *Let A be a hereditary K-algebra of Euclidean type over a field K, and T be a tilting module in* $\operatorname{mod} A$ *having both a nonzero postprojective and a nonzero preinjective direct summand. Then the tilted algebra* $B = \operatorname{End}_A(T)$ *is of finite representation type.*

Proof. We know from Theorem 5.8 that T is a splitting tilting module. Then it suffices to show, by Theorem 3.8, that the classes $\mathcal{T}(T)$ and $\mathcal{F}(T)$ contain at most finitely many pairwise nonisomorphic indecomposable modules, because there are equivalences of categories $\mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$ and $\mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T)$.

Let T_0 be a postprojective indecomposable direct summand of T. We claim that then $\mathcal{F}(T) = \{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(T, M) = 0\}$ contains at most finitely many pairwise nonisomorphic indecomposable modules. Since T_0 belongs to $\mathcal{P}(A)$, there exists a nonnegative integer m such that $T_0 = \tau_A^{-m} P_0$ for some indecomposable projective module P_0 in $\mathcal{P}(A)$. Let M be a module in $\operatorname{mod} A$ with $\operatorname{Hom}_A(T_0, M) = 0$. Then, applying Proposition VII.5.5, we obtain isomorphisms in $\operatorname{mod} K$

$$\operatorname{Hom}_A(P_0, \tau_A^m M) \cong \operatorname{Hom}_A(\tau_A^{-m} P_0, M) \cong \operatorname{Hom}_A(T_0, M) = 0.$$

This shows that the simple module $S_0 = \operatorname{top}(P_0)$ is not a composition factor of the module $\tau_A^m M$ (see Lemma VII.5.6). On the other hand, it follows from Theorem VII.8.22 that all but finitely many isomorphism classes of indecomposable modules in $\operatorname{mod} A$ are sincere modules. This shows that there are at most finitely many isomorphism classes of indecomposable modules M in $\operatorname{mod} A$ with $\operatorname{Hom}_A(P_0, M) = 0$. Therefore, $\mathcal{F}(T)$ contains at most finitely many pairwise nonisomorphic indecomposable modules.

Let T_1 be a preinjective indecomposable direct summand of T. We claim that then $\mathcal{T}(T) = \{N \in \operatorname{mod} A \,|\, \operatorname{Ext}_A^1(T, N) = 0\}$ contains at most finitely many pairwise nonisomorphic indecomposable modules. Since T_1 belongs to $\mathcal{Q}(A)$, there exists a nonnegative integer n such that $T_1 = \tau_A^n I_1$ for some indecomposable injective module I_1 in $\mathcal{Q}(A)$. Let N be an indecomposable module in $\operatorname{mod} A$ with $\operatorname{Ext}_A^1(T_1, N) = 0$. Then, applying Corollary III.6.4 and Proposition VII.5.5, we obtain isomorphisms in $\operatorname{mod} K$

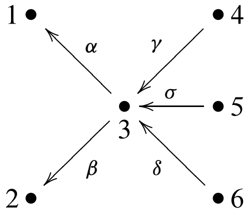
$$\operatorname{Hom}_A(\tau_A^{-n-1} N, I_1) \cong \operatorname{Hom}_A(\tau_A^{-1} N, T_1) \cong D \operatorname{Ext}_A^1(T_1, N) = 0.$$

This shows that the simple module $S_1 = \operatorname{soc}(I_1)$ is not a composition factor of the module $\tau_A^{-n-1} N$ (see Lemma VII.5.6). Since all but finitely many isomorphism classes of indecomposable modules in $\operatorname{mod} A$ are sincere modules (Theorem VII.8.22), we conclude that there are at most finitely many isomorphism classes of indecomposable modules N in $\operatorname{mod} A$ with $\operatorname{Ext}_A^1(T_1, N) = 0$. Therefore, $\mathcal{T}(T)$ contains at most finitely many pairwise nonisomorphic indecomposable modules.

Summing up, we proved that B is of finite representation type. ☐

We will present now an example of a tilted algebra of finite representation type which comes from a hereditary algebra of wild type.

Example 6.12. Let K be a field, Q the quiver



and $A = KQ$ the path algebra of Q over K. Then A is a hereditary algebra of wild type. It follows from Theorems VII.6.1, VII.6.2, VII.9.3 and Corollary VII.7.5 that the Auslander–Reiten quiver Γ_A of A has a disjoint decomposition

$$\Gamma_A = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A),$$

where $\mathcal{P}(A)$ is a postprojective component of the form $(-\mathbb{N})Q^{\mathrm{op}}$

$$\begin{tikzcd}
S_1 = P_1 \arrow[ddr] & & \tau_A^{-1} P_1 \arrow[ddr] & & \tau_A^{-2} P_1 \arrow[ddr] & & & \\
& & P_4 \arrow[dr] & & \tau_A^{-1} P_4 \arrow[dr] & & & \\
& P_3 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & P_5 \arrow[r] & \tau_A^{-1} P_3 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \tau_A^{-1} P_5 \arrow[r] & \tau_A^{-2} P_3 \arrow[uur, no head] \arrow[ur, no head] \arrow[r, no head] \arrow[dr, no head] \arrow[ddr, no head] & & \cdots \\
& & P_6 \arrow[ur] & & \tau_A^{-1} P_6 \arrow[ur] & & & \\
S_2 = P_2 \arrow[uur] & & \tau_A^{-1} P_2 \arrow[uur] & & \tau_A^{-2} P_2 \arrow[uur] & & &
\end{tikzcd}$$

$\mathcal{Q}(A)$ is a preinjective component of the form $\mathbb{N}Q^{\mathrm{op}}$

$$\begin{tikzcd}
& {} \arrow[ddr] & & \tau_A I_1 \arrow[ddr] & & I_1 \arrow[ddr] & & \\
& {} \arrow[dr] & & \tau_A^2 I_4 \arrow[dr] & & \tau_A I_4 \arrow[dr] & & I_4 = S_4 \\
\cdots & {} \arrow[r] & \tau_A^2 I_3 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \tau_A^2 I_5 \arrow[r] & \tau_A I_3 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \tau_A I_5 \arrow[r] & I_3 \arrow[ur] \arrow[r] \arrow[dr] & I_5 = S_5 \\
& {} \arrow[ur] & & \tau_A^2 I_6 \arrow[ur] & & \tau_A I_6 \arrow[ur] & & I_6 = S_6 \\
& {} \arrow[uur] & & \tau_A I_2 \arrow[uur] & & I_2 \arrow[uur] & &
\end{tikzcd}$$

$\mathcal{R}(A)$ is a nonempty family of regular components of the form $\mathbb{Z}\mathbb{A}_\infty$, and S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} A$ given by the vertex $i \in \{1,2,3,4,5,6\}$ of Q. Consider the indecomposable modules in $\operatorname{mod} A$

$$T_1 = S_1, \quad T_2 = S_2, \quad T_3 = R, \quad T_4 = S_4, \quad T_5 = S_5, \quad T_6 = S_6,$$

where R is the unique indecomposable module with the composition vector

$$c(R) = \big(c_1(R), c_2(R), c_3(R), c_4(R), c_5(R), c_6(R)\big) = (1,1,1,1,1,1).$$

We claim that $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6$ is a tilting module in $\operatorname{mod} A$. Since A is a hereditary algebra, we have $\operatorname{pd}_A T \leq 1$. Moreover, the Grothendieck group $K_0(A)$ is of rank 6 and T is a direct sum of 6 pairwise nonisomorphic indecomposable modules in $\operatorname{mod} A$. Hence, it remains to show, by Proposition 3.11, that $\operatorname{Ext}_A^1(T,T) = 0$, or equivalently, $\operatorname{Hom}_A(T, \tau_A T) = 0$. Observe first that S_1, S_2 are projective modules, so $\tau_A T_1 = 0$, $\tau_A T_2 = 0$. Moreover, S_4, S_5, S_5 are simple injective modules for which $\tau_A S_4, \tau_A S_5, \tau_A S_6$ are unique indecomposable modules in $\operatorname{mod} A$ with the composition vectors

$$\begin{aligned} c(\tau_A S_4) &= (0,0,1,0,1,1),\\ c(\tau_A S_5) &= (0,0,1,1,0,1),\\ c(\tau_A S_6) &= (0,0,1,1,1,0). \end{aligned}$$

Then it follows that $\operatorname{Hom}_A(T, \tau_A S_4 \oplus \tau_A S_5 \oplus \tau_A S_6) = 0$. We will show now that R is a quasi-simple module in the regular part $\mathcal{R}(A)$ of Γ_A such that $\tau_A R$ and $\tau_A^{-1} R$ have the composition vectors

$$c(\tau_A R) = (0,0,2,1,1,1) \quad \text{and} \quad c(\tau_A^{-1} R) = (0,0,1,0,0,0),$$

so $\tau_A^{-1} R = S_3$.

We determine first the Coxeter transformation $\varphi_A\colon K_0(A) \to K_0(A)$ and its inverse. We identify $K_0(A)$ with $\mathbb{Z}^6$ and the canonical basis $[S_1], [S_2], [S_3], [S_4], [S_5], [S_6]$ of $K_0(A)$ with the standard basis $e_1, e_2, e_3, e_4, e_5, e_6$ of $\mathbb{Z}^6$. Then we have

$$\begin{aligned} &[P_1] = (1,0,0,0,0,0), \quad [P_2] = (0,1,0,0,0,0), \quad [P_3] = (1,1,1,0,0,0),\\ &[P_4] = (1,1,1,1,0,0), \quad [P_5] = (1,1,1,0,1,0), \quad [P_6] = (1,1,1,0,0,1),\\ &[I_1] = (1,0,1,1,1,1), \quad [I_2] = (0,1,1,1,1,1), \quad [I_3] = (0,0,1,1,1,1),\\ &[I_4] = (0,0,0,1,0,0), \quad [I_5] = (0,0,0,0,1,0), \quad [I_6] = (0,0,0,0,0,1). \end{aligned}$$

By the definition of φ_A, we have $\varphi_A([P_i]) = -[I_i]$ for any $i \in \{1,2,3,4,5,6\}$. Then we obtain the following values of φ_A on the basis vectors of $K_0(A)$

$$\begin{aligned}
\varphi_A(e_1) &= -e_1 - e_3 - e_4 - e_5 - e_6,\\
\varphi_A(e_2) &= -e_2 - e_3 - e_4 - e_5 - e_6,\\
\varphi_A(e_3) &= e_1 + e_2 + e_3 + e_4 + e_5 + e_6,\\
\varphi_A(e_4) &= e_3 + e_5 + e_6,\\
\varphi_A(e_5) &= e_3 + e_4 + e_6,\\
\varphi_A(e_6) &= e_3 + e_4 + e_5.
\end{aligned}$$

Similarly, $\varphi_A^{-1}([I_i]) = -[P_i]$ for any $i \in \{1,2,3,4,5,6\}$. Then we obtain the following values of φ_A^{-1} on the basis vectors of $K_0(A)$

$$\begin{aligned}
\varphi_A^{-1}(e_1) &= e_2 + e_3,\\
\varphi_A^{-1}(e_2) &= e_1 + e_3,\\
\varphi_A^{-1}(e_3) &= 2e_1 + 2e_2 + 2e_3 + e_4 + e_5 + e_6,\\
\varphi_A^{-1}(e_4) &= -e_1 - e_2 - e_3 - e_4,\\
\varphi_A^{-1}(e_5) &= -e_1 - e_2 - e_3 - e_5,\\
\varphi_A^{-1}(e_6) &= -e_1 - e_2 - e_3 - e_6.
\end{aligned}$$

Applying now Corollaries VII.5.3 and VII.5.4, we obtain the following equalities in $K_0(A) = \mathbb{Z}^6$

$$\begin{aligned}
[\tau_A R] &= \varphi_A([R]) = \varphi_A(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) = 2e_3 + e_4 + e_5 + e_6,\\
[\tau_A^{-1} R] &= \varphi_A^{-1}([R]) = \varphi_A^{-1}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) = e_3.
\end{aligned}$$

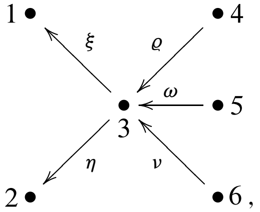
Therefore, $\tau_A R$ and $\tau_A^{-1} R$ are the required indecomposable modules.

In particular, we conclude that $\tau_A^{-1} R = S_3$ is a quasi-simple module in $\mathcal{R}(A)$, and hence $R = \tau_A S_3$ is a quasi-simple module in $\mathcal{R}(A)$, and so lies on the boundary of a component of the form $\mathbb{Z}\mathbb{A}_\infty$ (see Section VII.9). On the other hand, we infer that $\mathrm{Hom}_A(T_1 \oplus T_2, \tau_A T_3) = \mathrm{Hom}_A(S_1 \oplus S_2, \tau_A R) = 0$. Further, applying Corollary III.6.4, we obtain isomorphisms in mod K

$$\begin{aligned}
\mathrm{Hom}_A(T_3, \tau_A T_3) &= \mathrm{Hom}_A(R, \tau_A R) \cong D\,\mathrm{Ext}_A^1(R, R) \cong \mathrm{Hom}_A(\tau_A^{-1} R, R)\\
&= \mathrm{Hom}_A(S_3, R) = 0,
\end{aligned}$$

because $\mathrm{soc}(R) \cong S_1 \oplus S_2$. Moreover, we have $\mathrm{Hom}_A(T_4 \oplus T_5 \oplus T_6, \tau_A T_3) = \mathrm{Hom}_A(S_4 \oplus S_5 \oplus S_6, \tau_A R) = 0$, because $\mathrm{Hom}_A(\mathcal{Q}(A), \mathcal{R}(A)) = 0$, by Proposition VII.6.7. Therefore, we obtain $\mathrm{Hom}_A(T, \tau_A T) = 0$, and hence T is a tilting module in mod A.

Let $B = \operatorname{End}_A(T)$. Then B is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver



J is the ideal in $K\Delta$ of Δ generated by $\varrho\xi$, $\varrho\eta$, $\omega\xi$, $\omega\eta$, $\nu\xi$, $\nu\eta$, and the vertices $1, 2, 3, 4, 5, 6$ of the quiver Δ correspond to the indices of $T_1, T_2, T_3, T_4, T_5, T_6$. Then, applying Theorem III.10.2, we conclude that B is of finite representation type and its Auslander–Reiten quiver Γ_B is of the form

$$\begin{tikzcd}
S_1^B = P_1^B \arrow[dr] & & I_2^B = P_3^B/S_1^B \arrow[dr] & & P_4^B \arrow[dr] & & V/P_4^B \arrow[dr] & & I_4^B = S_4^B \\
& P_3^B \arrow[ur] \arrow[dr] & & S_3^B \arrow[ur] \arrow[r] \arrow[dr] & P_5^B \arrow[r] & V \arrow[ur] \arrow[r] \arrow[dr] & V/P_5^B \arrow[r] & I_3^B \arrow[ur] \arrow[r] \arrow[dr] & I_5^B = S_5^B \\
S_2^B = P_2^B \arrow[ur] & & I_1^B = P_3^B/S_2^B \arrow[ur] & & P_6^B \arrow[ur] & & V/P_6^B \arrow[ur] & & I_6^B = S_6^B ,
\end{tikzcd}$$

where S_i^B, P_i^B, and I_i^B are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} B$ given by the vertex $i \in \{1, 2, 3, 4, 5, 6\}$ of the quiver Δ, and V is the indecomposable module with the composition vector

$$c(V) = \big(c_1(V), c_2(V), c_3(V), c_4(V), c_5(V)\big) = (0, 0, 2, 1, 1, 1).$$

Obviously, Γ_B is the connecting component $\mathcal{C}_T$ determined by T. Observe also that $\mathcal{C}_T = \Gamma_B$ admits exactly one section

$$\begin{tikzcd}
I_2^B \arrow[dr] & & P_4^B \\
& S_3^B \arrow[ur] \arrow[r] \arrow[dr] & P_5^B \\
I_1^B \arrow[ur] & & P_6^B ,
\end{tikzcd}$$

and hence it is the section Δ_T given by the indecomposable right B-modules $\operatorname{Hom}_A(T, I_i)$, $i \in \{1, 2, 3, 4, 5, 6\}$.

We note that in general, if A is an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, and T is a tilting module in $\operatorname{mod} A$ such that the associated tilted algebra $B = \operatorname{End}_A(T)$ is of finite representation type, then the torsion class $\mathcal{T}(T)$ and the torsion-free class $\mathcal{F}(T)$ of $\operatorname{mod} A$ induced by T contain at most finitely many pairwise nonisomorphic indecomposable modules, and hence there is a big difference between the module categories $\operatorname{mod} A$ and $\operatorname{mod} B$. We will present now the classes of tilted algebras of infinite representation type whose module categories are as close as possible to that of the hereditary algebras from which we tilt.

Theorem 6.13. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, T be a postprojective tilting right A-module, and $B = \operatorname{End}_A(T)$. Then the following statements hold:*

(i) *The torsion class $\mathcal{T}(T)$ of $\operatorname{mod} A$ induced by T contains all but finitely many isomorphism classes of indecomposable modules from $\operatorname{mod} A$, and any indecomposable module of $\operatorname{mod} A$ which is not in $\mathcal{T}(T)$ is postprojective.*

(ii) *The torsion-free class $\mathcal{F}(T)$ of $\operatorname{mod} A$ induced by T contains only finitely many pairwise nonisomorphic indecomposable modules from $\operatorname{mod} A$, and all of them are postprojective.*

(iii) *The connecting component $\mathcal{C}_T$ of Γ_B determined by T is a preinjective component $\mathcal{Q}(B)$ of Γ_B, containing all indecomposable injective right B-modules and all indecomposable modules of the torsion class $\mathcal{X}(T)$ in $\operatorname{mod} B$ induced by T, and no projective right B-module.*

(iv) *The image $\operatorname{Hom}_A(T, \mathcal{R}(A))$ of the family $\mathcal{R}(A)$ of regular components of Γ_A under the functor $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ forms a family $\mathcal{R}(B)$ of regular components of Γ_B, which are either stable tubes (if A is of Euclidean type) or components of the form $\mathbb{Z}\mathbb{A}_\infty$ (if A is of wild type).*

(v) *The image $\operatorname{Hom}_A(T, \mathcal{T}(T) \cap \mathcal{P}(A))$ of the torsion part $\mathcal{T}(T) \cap \mathcal{P}(A)$ of $\mathcal{P}(A)$ under the functor $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ forms a postprojective component $\mathcal{P}(B)$ of Γ_B, containing all indecomposable projective right B-modules, but no injective right B-module.*

(vi) *The Auslander–Reiten quiver Γ_B of B has the disjoint union decomposition*

$$\Gamma_B = \mathcal{P}(B) \cup \mathcal{R}(B) \cup \mathcal{Q}(B),$$

and we have $\operatorname{Hom}_B(\mathcal{R}(B), \mathcal{P}(B)) = 0$, $\operatorname{Hom}_B(\mathcal{Q}(B), \mathcal{R}(B)) = 0$, $\operatorname{Hom}_B(\mathcal{Q}(B), \mathcal{P}(B)) = 0$.

(vii) *$\operatorname{pd}_B Z \leq 1$ and $\operatorname{id}_B Z \leq 1$ for all indecomposable modules Z in $\mathcal{R}(B)$ and all but finitely many isomorphism classes of indecomposable modules Z in $\mathcal{P}(B) \cup \mathcal{Q}(B)$.*

Proof. (i) and (ii). Observe that, by Theorem VII.6.1, the postprojective component $\mathcal{P}(A)$ of Γ_A is of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$, and hence $\mathcal{P}(A)$ contains infinitely many sections isomorphic to Q_A^{op}. Since T is a direct sum of finitely many indecomposable modules from $\mathcal{P}(A)$ we conclude that $\mathcal{P}(A)$ contains a section Δ such that the full valued translation subquiver $\mathcal{P}_\Delta$ of $\mathcal{P}(A)$ consisting of all successors of Δ in $\mathcal{P}(A)$ contains no indecomposable direct summand of T. But then, applying Proposition VII.6.6 we conclude that the torsion class $\mathcal{T}(T) = \{M \in \operatorname{mod} A \mid \operatorname{Ext}_A^1(T, M) = 0\} = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M, \tau_A T) = 0\}$ contains all indecomposable modules of $\mathcal{P}_\Delta \cup \mathcal{R}(A) \cup \mathcal{Q}(A)$. In particular, all indecomposable modules in $\operatorname{mod} A$ which are not in $\mathcal{T}(T)$ are postprojective. Moreover, the torsion-free class $\mathcal{F}(T) = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, M) = 0\}$ contains only finitely many pairwise nonisomorphic indecomposable modules, and all of them are proper predecessors of Δ in $\mathcal{P}(A)$.

(iii) Let Δ_T be the section of $\mathcal{C}_T$ determined by T. Since T has no indecomposable preinjective direct summand, it follows from Proposition 6.9 that $\mathcal{C}_T$ does not contain a projective module. Moreover, by Proposition 5.3, the image $\operatorname{Hom}_A(T, \mathcal{Q}(A))$ of $\mathcal{Q}(A)$ under the functor $\operatorname{Hom}_A(T, -)\colon \mathcal{T}(T) \to \mathcal{Y}(T)$ coincides with $\mathcal{Y}(T) \cap \mathcal{C}_T$ and consists of all predecessors of Δ_T in $\mathcal{C}_T$. On the other hand, $\mathcal{X}(T) \cap \mathcal{C}_T$ is formed by all proper successors of Δ_T in $\mathcal{C}_T$. Further, it follows from (ii) that $\mathcal{X}(T)$ contains only finitely many pairwise nonisomorphic indecomposable modules, because the functor $\operatorname{Ext}_A^1(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ induces an equivalence of categories $\mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T)$. In particular, $\mathcal{X}(T) \cap \mathcal{C}_T$ is finite. Then, since $\mathcal{C}_T$ is an acyclic component with the section Δ_T, we conclude that, for any indecomposable module Z in $\mathcal{C}_T$, there exists a nonnegative integer n such that $\tau_B^{-n} Z$ is an injective module. This shows that $\mathcal{C}_T$ is a preinjective component $\mathcal{Q}(B)$ of Γ_B. It remains to show that all indecomposable modules of $\mathcal{X}(T)$ belong to $\mathcal{C}_T$. Let X be an indecomposable module in $\mathcal{X}(T)$. Since the functor $\operatorname{Ext}_A^1(T, -)$ induces an equivalence of categories $\mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T)$ and the indecomposable modules in $\mathcal{F}(T)$ are postprojective, we conclude that there is an indecomposable postprojective module N in $\mathcal{F}(T)$ such that $X \cong \operatorname{Ext}_A^1(T, N)$ in $\operatorname{mod} B$. We may take an indecomposable projective module P in $\mathcal{P}(A)$ such that $\operatorname{Hom}_A(P, N) \neq 0$, because all indecomposable projective right A-modules are in $\mathcal{P}(A)$. Observe that P is not a direct summand of T, since $\operatorname{Hom}_A(T, N) = 0$. Let I be an indecomposable injective module in $\operatorname{mod} A$ such that $\operatorname{top}(P) \cong \operatorname{soc}(I)$. Then it follows from Lemma 4.2 that $\operatorname{Hom}_A(T, I)$ is a noninjective module of the section Δ_T and $\tau_B^{-1}(\operatorname{Hom}_A(T, I)) \cong \operatorname{Ext}_A^1(T, P)$. Moreover, $\operatorname{Hom}_A(P, N) \neq 0$ implies that

$$\begin{aligned}\operatorname{Hom}_B\left(\operatorname{Ext}_A^1(T, P), X\right) &\cong \operatorname{Hom}_B\left(\operatorname{Ext}_A^1(T, P), \operatorname{Ext}_A^1(T, N)\right)\\ &\cong \operatorname{Hom}_A(P, N) \neq 0.\end{aligned}$$

Then, applying Proposition III.10.1(i), we obtain that X is a successor of $\mathrm{Ext}_A^1(T, P)$ in the preinjective component $\mathcal{Q}(B)$, and so X is in $\mathcal{Q}(B)$.

(iv) Since the regular part $\mathcal{R}(A)$ of Γ_A is entirely contained in $\mathcal{T}(T)$, it follows from Proposition 5.3 that its image $\mathrm{Hom}_A(T, \mathcal{R}(A))$ under the equivalence functor $\mathrm{Hom}_A(T, -)\colon \mathcal{T}(T) \to \mathcal{Y}(T)$ is a family $\mathcal{R}(B)$ of regular components of Γ_B. Moreover, by Theorems VII.8.12 and VII.9.3, the components in $\mathcal{R}(B)$ are either stable tubes (if A is of Euclidean type), or of the form $\mathbb{Z}\mathbb{A}_\infty$ (if A is of wild type).

(v) It follows from Proposition 5.3 that the image $\mathcal{P}_\Delta^B = \mathrm{Hom}_A(T, \mathcal{P}_\Delta)$ of the full valued translation subquiver $\mathcal{P}_\Delta$ of $\mathcal{P}(A)$, chosen in the first part of the proof, is an acyclic full valued translation subquiver of a component $\mathcal{P}(B)$ of Γ_B, and is closed under successors in $\mathcal{P}(B)$. Moreover, since the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting and all indecomposable modules of $\mathcal{X}(T)$ lie in the preinjective component $\mathcal{Q}(B)$, we infer that all modules of $\mathcal{P}(B)$ are in the torsion-free class $\mathcal{Y}(T)$. Further, by (iv), we obtain that $\mathcal{P}(B) = \mathrm{Hom}_A(T, \mathcal{T}(T) \cap \mathcal{P}(A))$, and moreover all but finitely many indecomposable modules of $\mathcal{P}(B)$ belong to $\mathcal{P}_\Delta^B$. On the other hand, since all indecomposable direct summands of T belong to $\mathcal{P}(A)$, we conclude that all the indecomposable projective right B-modules are in $\mathcal{P}(B)$. Observe also that $\mathcal{P}(B)$ is an acyclic component, because there are no cycles in $\operatorname{mod} A$ between the indecomposable modules from $\mathcal{P}(A)$. Finally, for any indecomposable module Z in $\mathcal{P}(B)$, there exists a nonnegative integer m such that $\tau_B^m Z$ is a projective module, because there are only finitely many proper predecessors of the image $\mathrm{Hom}_A(T, \Delta)$ of the chosen section Δ of $\mathcal{P}(A)$ via the functor $\mathrm{Hom}_A(T, -)\colon \mathcal{T}(T) \to \mathcal{Y}(T)$. This completes the proof that $\mathcal{P}(B)$ is a postprojective component of Γ_B.

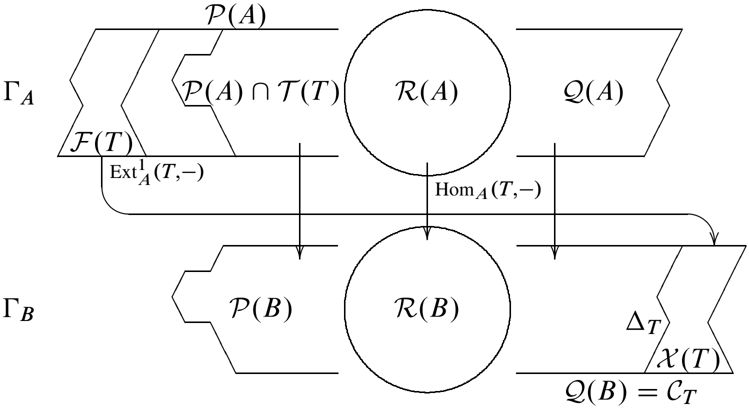
(vi) We know from the parts (iii)–(v) that Γ_B has the disjoint decomposition

$$\Gamma_B = \mathcal{P}(B) \cup \mathcal{R}(B) \cup \mathcal{Q}(B).$$

Further, it follows from Proposition VII.6.6 that $\mathrm{Hom}_A(\mathcal{R}(A), \mathcal{P}(A)) = 0$ and $\mathrm{Hom}_A(\mathcal{Q}(A), \mathcal{P}(A)) = 0$. Similarly, we have $\mathrm{Hom}_A(\mathcal{Q}(A), \mathcal{R}(A)) = 0$, by Proposition VII.6.7. Moreover, $\mathrm{Hom}_B(\mathcal{X}(T), \mathcal{Y}(T)) = 0$, because $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion pair in $\operatorname{mod} B$. Then the required equalities $\mathrm{Hom}_B(\mathcal{R}(B), \mathcal{P}(B)) = 0$, $\mathrm{Hom}_B(\mathcal{Q}(B), \mathcal{R}(B)) = 0$, $\mathrm{Hom}_B(\mathcal{Q}(B), \mathcal{P}(B)) = 0$ follow from the descriptions of $\mathcal{P}(B)$, $\mathcal{R}(B)$, $\mathcal{Q}(B)$ given in the previous parts (iii)–(v).

(vii) Because all the indecomposable projective right B-modules belong to $\mathcal{P}(B)$ and all the indecomposable injective right B-modules belong to $\mathcal{Q}(B)$, it follows from (vi) that we have $\mathrm{Hom}_B(D(B), \tau_B Z) = 0$ and $\mathrm{Hom}_B(\tau_B^{-1} Z, B) = 0$ for all indecomposable modules Z in $\mathcal{R}(B)$ and all but finitely many isomorphism classes of indecomposable modules Z in $\mathcal{P}(B) \cup \mathcal{Q}(B)$. Then the statement (vii) follows from Proposition III.5.4. □

We may visualize the relation between the Auslander–Reiten quivers Γ_A and Γ_B of algebras A and B in Theorem 6.13 by the following picture:



Although in general the tilted algebra $B = \operatorname{End}_A(T)$ given by a postprojective tilting module T in mod A is of global dimension 2, the relation between the module categories mod A and mod B described in Theorem 6.13 as well as the above picture shows that mod B is concealing some hereditary full subcategory involving all but finitely many isomorphism classes of indecomposable modules. This explains the following terminology.

Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, Q_A the quiver of A, and T a postprojective tilting module in mod A. Then the tilted algebra $B = \operatorname{End}_A(T)$ is said to be a *concealed hereditary algebra* of type Q_A. Moreover, B is said to be a *concealed hereditary algebra of Euclidean type*, if Q_A is a Euclidean quiver, and a *concealed hereditary algebra of wild type*, if Q_A is a wild quiver. The difference between two types of concealed hereditary algebras is visible in the shape of their Auslander–Reiten quivers. Namely, for a concealed hereditary algebra B of Euclidean type, the components $\mathcal{P}(B)$ and $\mathcal{Q}(B)$ have sections of Euclidean type and the regular part $\mathcal{R}(B)$ consists of stable tubes. For a concealed hereditary algebra B of wild type, the components $\mathcal{P}(B)$ and $\mathcal{Q}(B)$ have wild sections and the regular part $\mathcal{R}(B)$ consists of components of the form $\mathbb{Z}\mathbb{A}_\infty$.

We have also the following analogue of Theorem 6.13.

Theorem 6.14. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, T be a preinjective tilting right A-module, and $B = \operatorname{End}_A(T)$. Then the following statements hold:*

(i) *The torsion-free class $\mathcal{F}(T)$ of* mod A *induced by T contains all but finitely many isomorphism classes of indecomposable modules from* mod A*, and any indecomposable module of* mod A *which is not in $\mathcal{F}(T)$ is preinjective.*

(ii) *The torsion class* $\mathcal{T}(T)$ *of* $\operatorname{mod} A$ *induced by* T *contains only finitely many pairwise nonisomorphic indecomposable modules from* $\operatorname{mod} A$, *and all of them are preinjective.*

(iii) *The connecting component* $\mathcal{C}_T$ *of* Γ_B *determined by* T *is a postprojective component* $\mathcal{P}(B)$ *of* Γ_B, *containing all indecomposable projective right* B*-modules and all indecomposable modules of the torsion-free class* $\mathcal{Y}(T)$ *in* $\operatorname{mod} B$ *induced by* T, *and no injective right* B*-module.*

(iv) *The image* $\operatorname{Ext}^1_A(T, \mathcal{R}(A))$ *of the family* $\mathcal{R}(A)$ *of regular components of* Γ_A *under the functor* $\operatorname{Ext}^1_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ *forms a family* $\mathcal{R}(B)$ *of regular components of* Γ_B, *which are either stable tubes* (*if* A *is of Euclidean type*) *or components of the form* $\mathbb{Z}\mathbb{A}_\infty$ (*if* A *is of wild type*).

(v) *The image* $\operatorname{Ext}^1_A(T, \mathcal{F}(T) \cap \mathcal{Q}(A))$ *of the torsion-free part* $\mathcal{F}(T) \cap \mathcal{Q}(A)$ *of* $\mathcal{Q}(A)$ *under the functor* $\operatorname{Ext}^1_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ *forms a preinjective component* $\mathcal{Q}(B)$ *of* Γ_B, *containing all indecomposable injective right* B*-modules, but no projective right* B*-module.*

(vi) *The Auslander–Reiten quiver* Γ_B *of* B *has the disjoint union decomposition*

$$\Gamma_B = \mathcal{P}(B) \cup \mathcal{R}(B) \cup \mathcal{Q}(B),$$

and we have $\operatorname{Hom}_B(\mathcal{R}(B), \mathcal{P}(B)) = 0$, $\operatorname{Hom}_B(\mathcal{Q}(B), \mathcal{R}(B)) = 0$, $\operatorname{Hom}_B(\mathcal{Q}(B), \mathcal{P}(B)) = 0$.

(vii) $\operatorname{pd}_B Z \leq 1$ *and* $\operatorname{id}_B Z \leq 1$, *for all indecomposable modules* Z *in* $\mathcal{R}(B)$ *and all but finitely many isomorphism classes of indecomposable modules* Z *in* $\mathcal{P}(B) \cup \mathcal{Q}(B)$.

Proof. (i) and (ii). Observe that, by Theorem VII.6.2, the preinjective component $\mathcal{Q}(A)$ of Γ_A is of the form $\mathbb{N}Q_A^{\mathrm{op}}$, and hence $\mathcal{Q}(A)$ contains infinitely many sections isomorphic to Q_A^{op}. Since T is a direct sum of finitely many indecomposable modules from $\mathcal{Q}(A)$, we conclude that $\mathcal{Q}(A)$ contains a section Δ such that the full valued translation subquiver $\mathcal{Q}_\Delta$ of $\mathcal{Q}(A)$ consisting of all predecessors of Δ in $\mathcal{Q}(A)$ contains no indecomposable direct summand of T. But then, applying Proposition VII.6.7, we conclude that the torsion-free class $\mathcal{F}(T) = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, M) = 0\}$ contains all indecomposable modules of $\mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}_\Delta$. In particular, all indecomposable modules in $\operatorname{mod} A$ which are not in $\mathcal{F}(T)$ are preinjective. Moreover, the torsion class $\mathcal{T}(T) = \{M \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(T, M) = 0\}$ contains only finitely many pairwise nonisomorphic indecomposable modules, and all of them are proper successors of Δ in $\mathcal{Q}(A)$.

(iii) Let Δ_T be the section of $\mathcal{C}_T$ determined by T. Since T has no indecomposable postprojective direct summand, it follows from Proposition 6.9 that $\mathcal{C}_T$ does not contain an injective module. Moreover, by Proposition 5.3, the image $\operatorname{Ext}^1_A(T, \mathcal{P}(A))$ of $\mathcal{P}(A)$ under the functor $\operatorname{Ext}^1_A(T, -)\colon \mathcal{F}(T) \to \mathcal{X}(T)$ coincides with $\mathcal{X}(T) \cap \mathcal{C}_T$ and consists of all proper successors of Δ_T in $\mathcal{C}_T$. On the other hand, $\mathcal{Y}(T) \cap \mathcal{C}_T$ consists of all predecessors of Δ_T in $\mathcal{C}_T$. Further, it follows from (ii) that $\mathcal{Y}(T)$ contains only finitely many pairwise nonisomorphic indecomposable modules, because the functor $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$ induces an equivalence of categories $\mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$. In particular, $\mathcal{Y}(T) \cap \mathcal{C}_T$ is finite. Then, since $\mathcal{C}_T$ is an acyclic component with the section Δ_T, we conclude that, for any indecomposable module Z in $\mathcal{C}_T$, there exists a nonnegative integer m such that $\tau_B^m Z$ is a projective module. This shows that $\mathcal{C}_T$ is a postprojective component $\mathcal{P}(B)$ of Γ_B. It remains to show that all indecomposable modules of $\mathcal{Y}(T)$ belong to $\mathcal{C}_T$. Let Y be an indecomposable module in $\mathcal{Y}(T)$. Since the functor $\operatorname{Hom}_A(T, -)$ induces an equivalence of categories $\mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$ and the indecomposable modules in $\mathcal{T}(T)$ are preinjective, we conclude that there is an indecomposable preinjective module M in $\mathcal{T}(T)$ such that $Y \cong \operatorname{Hom}_A(T, M)$ in $\operatorname{mod} B$. We may take an indecomposable injective module I in $\mathcal{Q}(A)$ such that $\operatorname{Hom}_A(M, I) \neq 0$. Observe that the right B-module $\operatorname{Hom}_A(T, I)$ is on the section Δ_T of the connecting component $\mathcal{C}_T = \mathcal{P}(B)$. Since M and I belong to $\mathcal{T}(T)$, applying the equivalence functor $\operatorname{Hom}_A(T, -)\colon \mathcal{T}(T) \xrightarrow{\sim} \mathcal{Y}(T)$, we obtain that

$$\begin{aligned}\operatorname{Hom}_B\big(Y, \operatorname{Hom}_A(T, I)\big) &\cong \operatorname{Hom}_B\big(\operatorname{Hom}_A(T, M), \operatorname{Hom}_A(T, I)\big)\\ &\cong \operatorname{Hom}_A(M, I) \neq 0.\end{aligned}$$

Then, applying Proposition III.10.1(ii), we obtain that Y is a predecessor of the module $\operatorname{Hom}_A(T, I)$ in $\mathcal{P}(B)$, and so Y belongs to $\mathcal{P}(B)$.

(iv) Since the regular part $\mathcal{R}(A)$ of Γ_A is entirely contained in $\mathcal{F}(T)$, it follows from Proposition 5.3 that its image $\operatorname{Ext}_A^1(T, \mathcal{R}(A))$ under the equivalence functor $\operatorname{Ext}_A^1(T, -)\colon \mathcal{F}(T) \to \mathcal{X}(T)$ is a family $\mathcal{R}(B)$ of regular components of Γ_B. Moreover, by Theorems VII.8.12 and VII.9.3, the components in $\mathcal{R}(B)$ are either stable tubes (if A is of Euclidean type), or of the form $\mathbb{Z}\mathbb{A}_\infty$ (if A is of wild type).

(v) It follows from Proposition 5.3 that the image $\mathcal{Q}_\Delta^B = \operatorname{Ext}_A^1(T, \mathcal{Q}_\Delta)$ of the full valued translation subquiver $\mathcal{Q}_\Delta$ of $\mathcal{Q}(A)$, chosen in the first part of the proof, is an acyclic full valued translation subquiver of a component $\mathcal{Q}(B)$ of Γ_B, and is closed under predecessors in $\mathcal{Q}(B)$. Moreover, since the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ is splitting and all indecomposable modules of $\mathcal{Y}(T)$ lie in the postprojective component $\mathcal{P}(B)$, we infer that all modules of $\mathcal{Q}(B)$ are in the torsion class $\mathcal{X}(T)$. Further, by (iv), we obtain that $\mathcal{Q}(B) = \operatorname{Ext}_A^1(T, \mathcal{F}(T) \cap \mathcal{Q}(A))$, and moreover all but finitely many indecomposable modules of $\mathcal{Q}(B)$ belong to $\mathcal{Q}_\Delta^B$. On the other hand, since all indecomposable direct summands of T belong to $\mathcal{Q}(A)$, we conclude that the indecomposable projective right A-modules are not direct summands of T. Then, applying Proposition 5.4, we obtain that the indecomposable injective right B-modules are isomorphic to the modules $\operatorname{Ext}_A^1(T, \tau_A T')$, for indecomposable direct summands T' of T, and hence belong to $\mathcal{Q}(B)$. Observe also that $\mathcal{Q}(B)$ is an acyclic component, because $\mathcal{Q}_\Delta^B$ is acyclic and there are no cycles in $\operatorname{mod} A$ between the indecomposable modules from $\mathcal{Q}(A)$. Finally, for any indecomposable module Z in $\mathcal{Q}(B)$, there exists a nonnegative integer n such that $\tau_B^{-n} Z$ is an injective module, because there are only finitely many successors of the image $\operatorname{Ext}_A^1(T, \Delta)$ of the chosen section Δ of $\mathcal{Q}(A)$ via the functor $\operatorname{Ext}_A^1(T,-)\colon \mathcal{F}(T) \to \mathcal{X}(T)$. This completes the proof that $\mathcal{Q}(B)$ is a preinjective component of Γ_B.

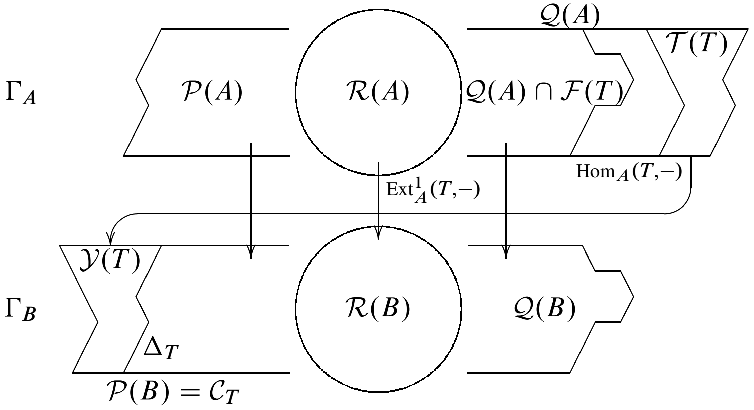
(vi) We know from the parts (iii)–(v) that Γ_B has the disjoint union decomposition

$$\Gamma_B = \mathcal{P}(B) \cup \mathcal{R}(B) \cup \mathcal{Q}(B).$$

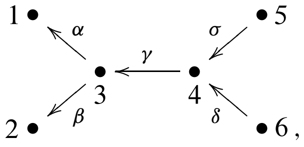
Further, it follows from Proposition VII.6.6 that $\operatorname{Hom}_A(\mathcal{R}(A), \mathcal{P}(A)) = 0$ and $\operatorname{Hom}_A(\mathcal{Q}(A), \mathcal{P}(A)) = 0$. Similarly, we have $\operatorname{Hom}_A(\mathcal{Q}(A), \mathcal{R}(A)) = 0$, by Proposition VII.6.7. Moreover, $\operatorname{Hom}_B(\mathcal{X}(T), \mathcal{Y}(T)) = 0$, because $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion pair in $\operatorname{mod} B$. Then the required equalities $\operatorname{Hom}_B(\mathcal{R}(B), \mathcal{P}(B)) = 0$, $\operatorname{Hom}_B(\mathcal{Q}(B), \mathcal{R}(B)) = 0$, $\operatorname{Hom}_B(\mathcal{Q}(B), \mathcal{P}(B)) = 0$ follow from the descriptions of $\mathcal{P}(B)$, $\mathcal{R}(B)$, $\mathcal{Q}(B)$ given in the previous parts (iii)–(v).

(vii) Because all the indecomposable projective right B-modules belong to $\mathcal{P}(B)$ and all the indecomposable injective right B-modules belong to $\mathcal{Q}(B)$, it follows from (vi) that we have $\operatorname{Hom}_B(D(B), \tau_B Z) = 0$ and $\operatorname{Hom}_B(\tau_B^{-1} Z, B) = 0$ for all indecomposable modules Z in $\mathcal{R}(B)$ and all but finitely many isomorphism classes of indecomposable modules Z in $\mathcal{P}(B) \cup \mathcal{Q}(B)$. Then the statement (vii) follows from Proposition III.5.4. □

We may visualize the relation between the Auslander–Reiten quivers Γ_A and Γ_B of algebras A and B in Theorem 6.14 by the following picture:



Example 6.15. Let K be a field, Q the quiver



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and $A = KQ$ the path algebra of Q over K, considered in Example VII.8.30. Then A is a hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_5$ and the postprojective component $\mathcal{P}(A)$ of Γ_A is of the form $(-\mathbb{N})Q^{\mathrm{op}}$, and has the left part as follows:

$$\begin{tikzcd}
 & & & P_5 \arrow[dr] & & \tau_A^{-1}P_5 \arrow[dr] & & \tau_A^{-2}P_5 \arrow[dr, dash] & \\
 & & P_4 \arrow[ur] \arrow[r] \arrow[dr] & P_6 \arrow[r] & \tau_A^{-1}P_4 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-1}P_6 \arrow[r] & \tau_A^{-2}P_4 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-2}P_6 \arrow[r, dash] & \cdots \\
S_1 = P_1 \arrow[r] & P_3 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-1}P_1 \arrow[r] & \tau_A^{-1}P_3 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-2}P_1 \arrow[r] & \tau_A^{-2}P_3 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-3}P_1 \arrow[r] & \tau_A^{-3}P_3 \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \cdots \\
S_2 = P_2 \arrow[ur] & & \tau_A^{-1}P_2 \arrow[ur] & & \tau_A^{-2}P_2 \arrow[ur] & & \tau_A^{-3}P_2 \arrow[ur] & &
\end{tikzcd}$$

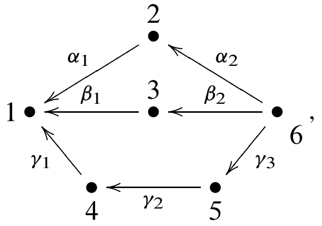
Consider the indecomposable postprojective modules

$$T_1 = P_2, \quad T_2 = P_5, \quad T_3 = P_6, \quad T_4 = \tau_A^{-1}P_1, \quad T_5 = \tau_A^{-2}P_2, \quad T_6 = \tau_A^{-3}P_1,$$

and $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6$. Then $\mathrm{pd}_A\, T \le 1$, because A is a hereditary algebra. Moreover, we have isomorphisms

$$\mathrm{Hom}_A(T, \tau_A T) \cong \mathrm{Hom}_A(T, P_1 \oplus \tau_A^{-1}P_2 \oplus \tau_A^{-2}P_1) = 0,$$

and hence $\mathrm{Ext}_A^1(T, T) \cong D\,\mathrm{Hom}_A(T, \tau_A T) = 0$, by Corollary III.6.4. Since $K_0(A)$ is of rank 6 and $T_1, T_2, T_3, T_4, T_5, T_6$ are pairwise nonisomorphic, we conclude, by Proposition 3.11, that T is a postprojective tilting module in $\mathrm{mod}\,A$. Then $B = \mathrm{End}_A(T)$ is a concealed hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_5$, isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver



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J is the ideal in the path algebra $K\Delta$ of Δ over K generated by the element $\alpha_2\alpha_1 + \beta_2\beta_1 + \gamma_3\gamma_2\gamma_1$, and the vertices $1, 2, 3, 4, 5, 6$ of the quiver Δ correspond to the indices of the indecomposable direct summands $T_1, T_2, T_3, T_4, T_5, T_6$ of T. Observe also that the torsion part $\mathcal{T}(T) \cap \mathcal{P}(A)$ of the postprojective component $\mathcal{P}(A)$ consists of all indecomposable modules of $\mathcal{P}(A)$ except $P_1, P_3, P_4, \tau_A^{-1}P_2, \tau_A^{-1}P_3, \tau_A^{-2}P_1$, the modules $P_1, \tau_A^{-1}P_2, \tau_A^{-2}P_1$ form the complete family of indecomposable modules in the torsion-free part of $\mathcal{F}(T)$, equivalently of $\mathcal{F}(T) \cap \mathcal{P}(A)$, and the indecomposable modules $P_3, P_4, \tau_A^{-1}P_3$ are neither in $\mathcal{T}(T)$ nor in $\mathcal{F}(T)$. We exhibit now the postprojective component $\mathcal{P}(B)$ of Γ_B, being the image $\mathrm{Hom}_A(T, \mathcal{T}(T) \cap \mathcal{P}(A))$ of the torsion part $\mathcal{T}(T) \cap \mathcal{P}(A)$ of $\mathcal{P}(A)$ under the equivalence functor $\mathrm{Hom}_A(T, -)\colon \mathcal{T}(T) \to \mathcal{Y}(T)$

$$\begin{tikzcd}
 & P_2^B \arrow[dr] & & \tau_B^{-1}P_2^B \arrow[dr] & & \tau_B^{-2}P_2^B \arrow[dr] & & \tau_B^{-3}P_2^B \arrow[dr, dash] & \\
P_1^B \arrow[ur] \arrow[r] \arrow[dr] & P_3^B \arrow[r] & \tau_B^{-1}P_1^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B^{-1}P_3^B \arrow[r] & \tau_B^{-2}P_1^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B^{-2}P_3^B \arrow[r] & \tau_B^{-3}P_1^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B^{-3}P_3^B \arrow[r, dash] & \cdots \\
S_1^B \arrow[u, equal] & P_4^B \arrow[ur] \arrow[dr] & & \tau_B^{-1}P_4^B \arrow[r] \arrow[ur] \arrow[dr] & P_6^B \arrow[r] & \tau_B^{-2}P_4^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B^{-1}P_6^B \arrow[r] & \tau_B^{-3}P_4^B \arrow[r, dash] \arrow[ur, dash] \arrow[dr, dash] & \cdots \\
 & & P_5^B \arrow[ur] & & \tau_B^{-1}P_5^B \arrow[ur] & & \tau_B^{-2}P_5^B \arrow[ur] & &
\end{tikzcd}$$

where $P_i^B = \mathrm{Hom}_A(T, T_i)$, $i \in \{1, 2, 3, 4, 5, 6\}$, are the indecomposable projective right B-modules. Finally, the preinjective component $\mathcal{Q}(B)$, being the connecting component $\mathcal{C}_T$ of Γ_B determined by T, is of the form

$$\begin{tikzcd}
 & \tau_B^3 I_2^B \arrow[dr] & & \tau_B^2 I_2^B \arrow[dr] & & \tau_B I_2^B \arrow[dr] & & I_2^B \arrow[dr] & \\
\cdots \arrow[ur] \arrow[r] \arrow[dr] & \tau_B^3 I_3^B \arrow[r] & \tau_B^3 I_6^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B^2 I_3^B \arrow[r] & \tau_B^2 I_6^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B I_3^B \arrow[r] & \tau_B I_6^B \arrow[r] \arrow[ur] \arrow[dr] & I_3^B \arrow[r] & I_6^B \\
\cdots \arrow[r] & \tau_B^3 I_5^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B I_1^B \arrow[r] & \tau_B^2 I_5^B \arrow[r] \arrow[ur] \arrow[dr] & I_1^B \arrow[r] & \tau_B I_5^B \arrow[ur] \arrow[dr] & & I_5^B \arrow[ur] & S_6^B \arrow[u, equal] \\
 {} \arrow[ur] & & \tau_B^2 I_4^B \arrow[ur] & & \tau_B I_4^B \arrow[ur] & & I_4^B \arrow[ur] & &
\end{tikzcd}$$

where I_i^B is the indecomposable injective right B-modules given by the vertex $i \in \{1, 2, 3, 4, 5, 6\}$ of Δ. We note that, according to Proposition 5.4, the indecomposable injective right B-modules are of two types: $I_1^B = \mathrm{Hom}_A(T, I_2)$, $I_2^B = \mathrm{Hom}_A(T, I_5)$, $I_3^B = \mathrm{Hom}_A(T, I_6)$, because $T_1 = P_2$, $T_2 = P_5$, $T_3 = P_6$ are projective direct summands of T, and $I_4^B = \mathrm{Ext}_A^1(T, \tau_A T_4) = \mathrm{Ext}_A^1(T, P_1)$, $I_5^B = \mathrm{Ext}_A^1(T, \tau_A T_5) = \mathrm{Ext}_A^1(T, \tau_A^{-1}P_2)$, $I_6^B = \mathrm{Ext}_A^1(T, \tau_A T_6) = \mathrm{Ext}_A^1(T, \tau_A^{-2}P_1)$, because T_4, T_5, T_6 are the nonprojective direct summands of T. Since the functor $\operatorname{Ext}_A^1(T,-)\colon \operatorname{mod} A \to \operatorname{mod} B$ induces an equivalence of categories $\mathcal{F}(T) \xrightarrow{\sim} \mathcal{X}(T)$, and $P_1, \tau_A^{-1}P_2, \tau_A^{-2}P_1$ form a complete family of pairwise nonisomorphic indecomposable modules in $\mathcal{F}(T)$, we conclude that all indecomposable modules of the preinjective component $\mathcal{Q}(B)$ of Γ_B except the injective modules I_4^B, I_5^B, I_6^B are in the torsion-free part $\mathcal{Y}(T)$. In particular, we conclude that

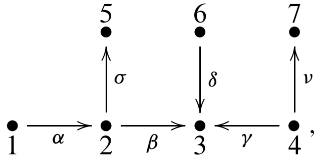
$$\begin{tikzcd}
 & & & I_2^B \\
 & & \tau_B I_6^B \arrow[ur] \arrow[r] & I_3^B \\
I_1^B \arrow[r] & \tau_B I_5^B \arrow[ur] & & \\
\tau_B I_4^B \arrow[ur] & & &
\end{tikzcd}$$

is the section Δ_T formed by the images $\operatorname{Hom}_A(T,T_i)$, $i \in \{1,2,3,4,5,6\}$, of the complete set $I_1, I_2, I_3, I_4, I_5, I_6$ of pairwise nonisomorphic indecomposable injective right A-modules, forming the final section of $\mathcal{Q}(A)$. In particular, we deduce that $\tau_B I_4^B = \operatorname{Hom}_A(T, I_1)$, $\tau_B I_5^B = \operatorname{Hom}_A(T, I_3)$, and $\tau_B I_6^B = \operatorname{Hom}_A(T, I_4)$. In fact, it follows from Lemma 4.2 that there are isomorphisms

$$\begin{aligned}
\operatorname{Ext}_A^1(T, \tau_A^{-1}P_2) &= \tau_B^{-1}(\tau_B I_5^B) = \tau_B^{-1}\operatorname{Hom}_A(T, I_3) \cong \operatorname{Ext}_A^1(T, P_3),\\
\operatorname{Ext}_A^1(T, \tau_A^{-2}P_1) &= \tau_B^{-1}(\tau_B I_6^B) = \tau_B^{-1}\operatorname{Hom}_A(T, I_4) \cong \operatorname{Ext}_A^1(T, P_4)
\end{aligned}$$

in $\operatorname{mod} B$. Finally, we note that the regular part $\mathcal{R}(B) = \operatorname{Hom}_A(T, \mathcal{R}(A))$ of Γ_B is a family of stable tubes, containing a stable tube of rank 3 and two stable tubes of rank 2 (see Example VII.8.30).

Example 6.16. Let K be a field, Q be the quiver



and A the path algebra KQ of Q over K. Then A is a hereditary algebra of wild type. It follows from Theorem VII.6.2 that the preinjective component $\mathcal{Q}(A)$ of A

is of the form $\mathbb{N}Q^{\mathrm{op}}$ and its right part is as follows

$$\begin{tikzcd}
 & \tau_A^2 I_7 \arrow[dr] & & \tau_A I_7 \arrow[dr] & & I_7 \arrow[dr] & & \\
\cdots \arrow[ur] \arrow[dr] & & \tau_A^2 I_4 \arrow[ur] \arrow[dr] & & \tau_A I_4 \arrow[ur] \arrow[dr] & & I_4 = S_4 & \\
\cdots \arrow[r] & \tau_A^2 I_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A^2 I_6 \arrow[r] & \tau_A I_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A I_6 \arrow[r] & I_3 \arrow[r] \arrow[ur] \arrow[dr] & I_6 = S_6 & \\
\cdots \arrow[r] \arrow[ur] \arrow[dr] & \tau_A^3 I_1 \arrow[r] & \tau_A^2 I_2 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A^2 I_1 \arrow[r] & \tau_A I_2 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A I_1 \arrow[r] & I_2 \arrow[r] & I_1 = S_1 \\
 & \tau_A^2 I_5 \arrow[urr] & & \tau_A I_5 \arrow[ur] & & I_5 \arrow[ur] & &
\end{tikzcd}$$

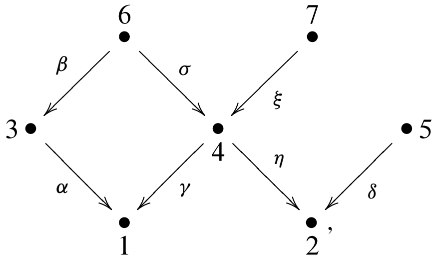
where I_i are the indecomposable injective right A-modules given by the vertices $i \in \{1,2,3,4,5,6,7\}$ of Q. Consider the indecomposable modules in $\mathcal{Q}(A)$

$$T_1 = \tau_A^2 I_1,\ T_2 = \tau_A I_4,\ T_3 = I_5,\ T_4 = I_3,\ T_5 = I_7,\ T_6 = I_1,\ T_7 = I_6,$$

and $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6 \oplus T_7$. Then $\mathrm{pd}_A T \le 1$ and $\mathrm{id}_A T \le 1$, since A is a hereditary algebra. Moreover, applying Corollary III.6.4, we obtain isomorphisms

$$\mathrm{Ext}_A^1(T,T) \cong D\,\mathrm{Hom}_A(\tau_A^{-1}T,T) = D\,\mathrm{Hom}_A(\tau_A I_1 \oplus I_4, T) = 0.$$

Then it follows from Proposition 3.11 that T is a preinjective tilting module in $\mathrm{mod}\,A$. Further, the tilted algebra $B = \mathrm{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver



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J is the ideal in the path algebra $K\Delta$ of Δ over K generated by the element $\beta\alpha - \sigma\gamma$, and the vertices $1,2,3,4,5,6,7$ of the quiver Δ correspond to the indices of the indecomposable direct summands $T_1, T_2, T_3, T_4, T_5, T_6, T_7$ of T. Observe that the indecomposable modules in the torsion class $\mathcal{T}(T)$ of $\mathrm{mod}\,A$ are the indecomposable injective modules $I_1, I_2, I_3, I_4, I_5, I_6, I_7$ and $\tau_A I_4, \tau_A^2 I_1$. Moreover,

$\tau_A I_2$ is the unique indecomposable module in $\mathcal{Q}(A)$ which is neither in $\mathcal{T}(T)$, nor in $\mathcal{F}(T)$. According to Theorem 6.14, the connecting component of $\mathcal{C}_T$ of Γ_B determined by T is the postprojective component $\mathcal{P}(B)$ of the form

$$\begin{tikzcd}
 & P_5^B \arrow[dr] & & \tau_B^{-1}P_5^B \arrow[dr] & & \tau_B^{-2}P_5^B \arrow[dr] & & \tau_B^{-3}P_5^B \arrow[dr, dash] & \\
\begin{matrix} P_2^B \\ \parallel \\ S_2^B \end{matrix} \arrow[ur] \arrow[dr] & & \tau_B^{-1}P_2^B \arrow[ur] \arrow[dr] & & \tau_B^{-2}P_2^B \arrow[ur] \arrow[dr] & & \tau_B^{-3}P_2^B \arrow[ur] \arrow[dr] & & \cdots \\
 & P_4^B \arrow[ur] \arrow[r] \arrow[dr] & P_7^B \arrow[r] & \tau_B^{-1}P_4^B \arrow[ur] \arrow[r] \arrow[dr] & \tau_B^{-1}P_7^B \arrow[r] & \tau_B^{-2}P_4^B \arrow[ur] \arrow[r] \arrow[dr] & \tau_B^{-2}P_7^B \arrow[r] & \tau_B^{-3}P_4^B \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \cdots \\
\begin{matrix} P_1^B \\ \parallel \\ S_1^B \end{matrix} \arrow[ur] \arrow[dr] & & \tau_B^{-1}P_1^B \arrow[ur] \arrow[r] \arrow[dr] & P_6^B \arrow[r] & \tau_B^{-2}P_1^B \arrow[ur] \arrow[r] \arrow[dr] & \tau_B^{-1}P_6^B \arrow[r] & \tau_B^{-3}P_1^B \arrow[ur] \arrow[r] \arrow[dr] & \tau_B^{-2}P_6^B \arrow[r, dash] & \cdots \\
 & P_3^B \arrow[ur] & & \tau_B^{-1}P_3^B \arrow[ur] & & \tau_B^{-2}P_3^B \arrow[ur] & & \tau_B^{-3}P_3^B \arrow[ur, dash] &
\end{tikzcd}$$

where $P_i^B = \operatorname{Hom}_A(T, T_i)$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$, are the indecomposable projective right B-modules given by the vertices of Q, and the section Δ_T of $\mathcal{C}_T$ formed by the modules $\operatorname{Hom}_A(T, I_i)$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$, is of the form

$$\begin{tikzcd}
P_5^B \arrow[dr] & & \\
 & \tau_B^{-1}P_2^B & \\
P_4^B \arrow[ur] \arrow[r] \arrow[dr] & P_7^B & \\
 & \tau_B^{-1}P_1^B \arrow[r] & P_6^B \\
P_5^B \arrow[ur] & &
\end{tikzcd}$$

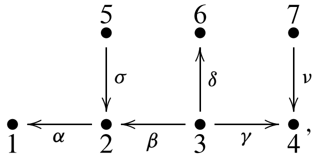
Moreover, the full translation subquiver of $\mathcal{C}_T = \mathcal{P}(B)$ formed by all proper successors of Δ_T in $\mathcal{P}(B)$ is the image $\operatorname{Ext}_A^1(T, \mathcal{P}(A))$ of the postprojective component $\mathcal{P}(A)$ of Γ_A under the functor $\operatorname{Ext}_A^1(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$. Further, the preinjective component $\mathcal{Q}(B)$ of Γ_B is the image $\operatorname{Ext}_A^1(T, \mathcal{F}(T) \cap \mathcal{Q}(A))$ of the torsion-free part $\mathcal{F}(T) \cap \mathcal{Q}(A)$ of $\mathcal{Q}(A)$ under the functor $\operatorname{Ext}_A^1(T, -)$ and its right part is of the form

$$\begin{tikzcd}
{} \arrow[dr] & & \tau_B^2 I_5^B \arrow[dr] & & \tau_B I_5^B \arrow[dr] & & I_5^B = S_5^B & \\
\cdots & \tau_B^2 I_2^B \arrow[ur] \arrow[dr] & & \tau_B I_2^B \arrow[ur] \arrow[dr] & & I_2^B \arrow[ur] \arrow[dr] & & \\
\cdots \arrow[r] \arrow[ur] \arrow[dr] & \tau_B^3 I_7^B \arrow[r] & \tau_B^2 I_4^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B^2 I_7^B \arrow[r] & \tau_B I_4^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B I_7^B \arrow[r] & I_4^B \arrow[r] \arrow[dr] & I_7^B = S_7 \\
\cdots \arrow[r] & \tau_B^3 I_6^B \arrow[r] \arrow[ur] \arrow[dr] & \tau_B I_1^B \arrow[r] & \tau_B^2 I_6^B \arrow[r] \arrow[ur] \arrow[dr] & I_1^B \arrow[r] & \tau_B I_6^B \arrow[ur] \arrow[dr] & & I_6^B = S_6^B \\
{} \arrow[ur] & & \tau_B^2 I_3^B \arrow[ur] & & \tau_B I_3^B \arrow[ur] & & I_3^B \arrow[ur] &
\end{tikzcd}$$

Since the preinjective tilting module T has no indecomposable projective direct summands, we conclude from Proposition 5.4 that the indecomposable injective right B-modules are of the forms $\operatorname{Ext}_A^1(T, \tau_A T_i)$, for $i \in \{1, 2, 3, 4, 5, 6, 7\}$.

Finally, the regular part $\mathcal{R}(B) = \operatorname{Ext}_A^1(T, \mathcal{R}(A))$ of Γ_B consists of components of the form $\mathbb{Z}\mathbb{A}_\infty$.

We will show now that the tilted algebra B is a concealed hereditary algebra. Let H be the opposite algebra of A^{op}. Then H is the path algebra KQ^{op} of the opposite quiver Q^{op} of Q



and hence H is a wild hereditary algebra. Then, according to Theorem VII.6.1, the postprojective component $\mathcal{P}(H)$ of Γ_H is of the form $(-\mathbb{N})(Q^{\mathrm{op}})^{\mathrm{op}} = (-\mathbb{N})Q$, and its left part is

$$\begin{tikzcd}
& & P_5^H \arrow[dr] & & \tau_H^{-1}P_5^H \arrow[dr] & & \tau_H^{-2}P_5^H \arrow[dr, dash] & \\
S_1^H = P_1^H \arrow[r] & P_2^H \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^{-1}P_1^H \arrow[r, tail] & \tau_H^{-1}P_2^H \arrow[r, tail] \arrow[ur] \arrow[dr] & \tau_H^{-2}P_1^H \arrow[r, tail] & \tau_H^{-2}P_2^H \arrow[r, tail] \arrow[ur] \arrow[dr] & \tau_H^{-3}P_1^H \arrow[r, dash] & \cdots \\
& S_6^H = P_6^H \arrow[r] & P_3^H \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^{-1}P_6^H \arrow[r, tail] & \tau_H^{-1}P_3^H \arrow[r, tail] \arrow[ur] \arrow[dr] & \tau_H^{-2}P_6^H \arrow[r, tail] & \tau_H^{-2}P_3^H \arrow[r, dash] \arrow[ur, dash] \arrow[dr, dash] & \cdots \\
& S_4^H = P_4^H \arrow[ur] \arrow[dr] & & \tau_H^{-1}P_4^H \arrow[ur] \arrow[dr] & & \tau_H^{-2}P_4^H \arrow[ur] \arrow[dr] & & \cdots \\
& & P_7^H \arrow[ur] & & \tau_H^{-1}P_7^H \arrow[ur] & & \tau_H^{-2}P_7^H \arrow[ur, dash] &
\end{tikzcd}$$

where P_i^H are the indecomposable projective right H-modules given by the vertices $i \in \{1, 2, 3, 4, 5, 6, 7\}$ of Q^{op}. Consider the following indecomposable modules in $\mathcal{P}(H)$

$$T_1' = P_1^H, T_2' = P_4^H, T_3' = P_5^H, T_4' = P_3^H, T_5' = P_7^H, T_6' = \tau_H^{-2} P_1^H,$$
$$T_7' = \tau_H^{-1} P_6^H,$$

and $T' = T_1' \oplus T_2' \oplus T_3' \oplus T_4' \oplus T_5' \oplus T_6' \oplus T_7'$. Clearly, we have $\mathrm{pd}_H T' \leq 1$, since H is a hereditary algebra. Moreover, we have isomorphisms in $\operatorname{mod} K$

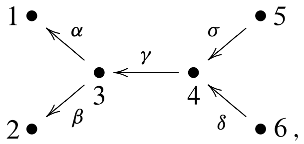
$$\mathrm{Ext}_H^1(T', T') \cong D\,\mathrm{Hom}_H(T', \tau_H T') = D\,\mathrm{Hom}_H(T', \tau_H^{-1} P_1^H \oplus P_6^H) = 0.$$

Then, applying Proposition 3.11, we conclude that T' is a postprojective tilting module in $\operatorname{mod} H$. Finally, we observe that the tilted algebra $B' = \mathrm{End}_H(T')$ is isomorphic to the tilted algebra $B = \mathrm{End}_A(T)$, and hence B is a concealed hereditary algebra of wild type.

In general, an indecomposable finite dimensional K-algebra B over a field K is isomorphic to an algebra of the form $\mathrm{End}_A(T)$ for an indecomposable finite dimensional hereditary K-algebra A and a preinjective tilting module T in $\operatorname{mod} A$ if and only if B is isomorphic to an algebra of the form $\mathrm{End}_{A'}(T')$ for an indecomposable finite dimensional hereditary K-algebra A' and a postprojective tilting module T' in $\operatorname{mod} A'$.

The next two examples show that describing the Auslander–Reiten quiver of a tilted but not concealed hereditary algebra of Euclidean or wild type is a rather complicated matter.

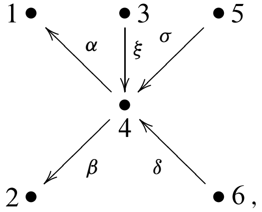
Example 6.17. Let K be a field, Q the quiver



and $A = KQ$ the path algebra of Q over K, considered in Examples VII.8.30 and 6.15. For each $i \in \{1,2,3,4,5,6\}$, let P_i, I_i, S_i be the indecomposable projective module, the indecomposable injective module, and the simple module in mod A, respectively, associated to the vertex i of Q. Consider the indecomposable modules

$$T_1 = P_1, \quad T_2 = P_2, \quad T_3 = S_4, \quad T_4 = P_4, \quad T_5 = P_5, \quad T_6 = P_6,$$

and $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6$. Then $\operatorname{pd}_A T \leq 1$, because A is a hereditary algebra. We know from Example VII.8.30 that Γ_A contains a stable tube $\mathcal{T}^A_{\lambda_0}$ of rank 3 whose mouth is formed by the simple regular modules S_3, S_4, R with $\tau_A R = S_4$, $\tau_A S_4 = S_3$, $\tau_A S_3 = R$, where R is the indecomposable module with $[R] = [S_1] + [S_2] + [S_3] + [S_4] + [S_5] + [S_6]$ in $K_0(A)$. Then, applying Corollary III.6.4, we obtain that $\operatorname{Ext}^1_A(T,T) \cong D\operatorname{Hom}_A(T, \tau_A T) \cong D\operatorname{Hom}_A(T, \tau_A S_4) \cong D\operatorname{Hom}_A(T, S_3) = 0$. Since $K_0(A)$ is of rank 6 and $T_1, T_2, T_3, T_4, T_5, T_6$ are pairwise nonisomorphic, we conclude, by Proposition 3.11, that T is a tilting module in mod A. Let $B = \operatorname{End}_A(T)$ be the associated tilted algebra. Then B is isomorphic to the bound quiver algebra KQ^*/I, where Q^* is the quiver



I is the ideal in the path algebra KQ^* of Q^* over K generated by the elements $\xi\alpha$ and $\xi\beta$, and the vertices $1,2,3,4,5,6$ of the quiver Q^* correspond to the indices of the indecomposable direct summands $T_1, T_2, T_3, T_4, T_5, T_6$ of T. Observe also that the torsion-free class $\mathcal{F}(T) = \{X \in \operatorname{mod} A \mid \operatorname{Hom}_A(T,X) = 0\}$ in mod A determined by T has only one indecomposable module, namely the simple module $S_3 = \tau_A S_4$. We will describe the Auslander–Reiten quiver Γ_B of B. It follows from Lemma 3.1 that the modules $P^B_i = \operatorname{Hom}_A(T, T_i)$, $i \in \{1,2,3,4,5,6\}$, form a complete set of pairwise nonisomorphic indecomposable projective modules in mod B. Since T is a splitting tilting module, applying Proposition 5.4, we conclude that a complete set of pairwise nonisomorphic indecomposable injective modules in mod B is formed by the modules $I^B_i = \operatorname{Hom}_A(T, I_i)$, $i \in$

$\{1, 2, 4, 5, 6\}$, and the module $I_3^B = \operatorname{Ext}_A^1(T, \tau_A T_3) = \operatorname{Ext}_A^1(T, S_3)$. Further, since $\mathcal{F}(T)$ is the additive category of S_3, we have in $\operatorname{mod} B$ only one connecting sequence, which is of the form

$$0 \longrightarrow \operatorname{Hom}_A(T, I_3) \longrightarrow \operatorname{Hom}_A(T, I_4) \longrightarrow \operatorname{Ext}_A^1(T, S_3) \longrightarrow 0,$$

by Proposition 5.2. We note that

$$\begin{aligned} \operatorname{Ext}_A^1(T, \operatorname{rad} P_3) &= \operatorname{Ext}_A^1(T, P_1 \oplus P_2) \\ &\cong D\operatorname{Hom}_A(P_1 \oplus P_2, \tau_A T) = D\operatorname{Hom}_A(P_1 \oplus P_2, S_3) = 0, \end{aligned}$$

and hence $\operatorname{Ext}_A^1(T, P_3) \cong \operatorname{Ext}_A^1(T, S_3)$. Note also that for any indecomposable module X in the preinjective component $\mathcal{Q}(A)$ of Γ_A, we have $\operatorname{Ext}_A^1(T, X) \cong D\operatorname{Hom}_A(X, \tau_A T) = D\operatorname{Hom}_A(X, S_3) = 0$, because S_3 belongs to $\mathcal{R}(A)$. Hence, the component $\mathcal{Q}(A)$ is contained in the torsion class $\mathcal{T}(T)$. Therefore, the connecting component $\mathcal{C}_T$ of Γ_B determined by T is the preinjective component $\mathcal{Q}(B)$

$$\begin{tikzcd}
{} \arrow[dr] & & & \tau_B^2 I_6^B \arrow[drr] & & & & \tau_B I_6^B \arrow[drr] & & & & I_6^B \\
\cdots \arrow[r] & \tau_B^2 I_4^B \arrow[rr] \arrow[urr] \arrow[drr] & & \tau_B^2 I_5^B \arrow[rr] & & \tau_B I_4^B \arrow[rr] \arrow[urr] \arrow[drr] & & \tau_B I_5^B \arrow[rr] & & I_4^B \arrow[rr] \arrow[urr] \arrow[drr] & & I_5^B \\
\cdots \arrow[ur] \arrow[dr] & \tau_B I_2^B \arrow[rr] & & \tau_B \operatorname{Hom}_A(T, I_3) \arrow[rr] \arrow[urr] \arrow[drr] & & I_2^B \arrow[rr] & & \operatorname{Hom}_A(T, I_3) \arrow[rrrr] \arrow[urr] & & & & \operatorname{Ext}_A^1(T, S_3) = S_3^B \\
& \tau_B I_1^B \arrow[urr] & & & & I_1^B \arrow[urr] & & & & & &
\end{tikzcd}$$

obtained from the image $\operatorname{Hom}_A(T, \mathcal{Q}(A))$ of $\mathcal{Q}(A)$ under the functor

$$\operatorname{Hom}_A(T, -) \colon \operatorname{mod} A \longrightarrow \operatorname{mod} B,$$

by adding the simple injective module $S_3^B = \operatorname{Ext}_A^1(T, S_3)$. We note that $\mathcal{Q}(B)$ contains all indecomposable injective right B-modules. Further, by Theorem VII.8.12, the regular part $\mathcal{R}(A)$ of Γ_A is a family $\mathcal{T}^A = (\mathcal{T}_\lambda^A)_{\lambda \in \Lambda(A)}$ of pairwise orthogonal stable tubes separating the postprojective component $\mathcal{P}(A)$ from the preinjective component $\mathcal{Q}(A)$. In particular, we conclude that all stable tubes $\mathcal{T}_\lambda^A$, $\lambda \in \Lambda(A) \setminus \{\lambda_0\}$, are contained in the torsion class $\mathcal{T}(T)$, and then their images $\mathcal{T}_\lambda^B = \operatorname{Hom}(T, \mathcal{T}_\lambda^A)$ are stable tubes of Γ_B. We observe now that the torsion part $\mathcal{T}(T) \cap \mathcal{T}_{\lambda_0}^A$ of the stable tube $\mathcal{T}_{\lambda_0}^A$ consists of the indecomposable modules Z in $\mathcal{T}_{\lambda_0}^A$ such that $\operatorname{Hom}_A(Z, S_3) = 0$, because $\operatorname{Ext}_A^1(T, Z) \cong D\operatorname{Hom}_A(Z, \tau_A S_4) =$

$D\operatorname{Hom}_A(Z, S_3)$. These modules Z are exactly the modules which do not lie on the infinite sectional path in $\mathcal{T}^A_{\lambda_0}$

$$\cdots \longrightarrow [j+1]S_3 \longrightarrow [j]S_3 \longrightarrow \cdots \longrightarrow [2]S_3 \longrightarrow [1]S_3$$

ending at the module $[1]S_3 = S_3$. Then we conclude that the image $\mathcal{T}^B_{\lambda_0} = \operatorname{Hom}_A(T, \mathcal{T}(T) \cap \mathcal{T}^A_{\lambda_0})$ of $\mathcal{T}^A_{\lambda_0}$ under the functor $\operatorname{Hom}_A(T, -)$ is a component of the form

$$\begin{tikzcd}[column sep=tiny]
 & & & \operatorname{Hom}_A(T, S_4) = P^B_3 \arrow[dr] & & \operatorname{Hom}_A(T, R) \\
\operatorname{Hom}_A(T, R) \arrow[dr] & & \operatorname{Hom}_A(T, [2]S_4) \arrow[ur] \arrow[dr] & & \operatorname{Hom}_A(T, [2]R) \arrow[ur] \arrow[dr] & \\
 & \operatorname{Hom}_A(T, [3]S_4) \arrow[ur] \arrow[dr] & & \operatorname{Hom}_A(T, [3]R) \arrow[ur] \arrow[dr] & & \operatorname{Hom}_A(T, [4]S_4) \\
\operatorname{Hom}_A(T, [4]S_4) \arrow[ur] \arrow[dr, no head] & & \operatorname{Hom}_A(T, [4]R) \arrow[ur] \arrow[dr, no head] & & \operatorname{Hom}_A(T, [5]S_4) \arrow[ur] \arrow[dr, no head] & \\
 & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots
\end{tikzcd}$$

called a *ray tube*. We note that every module in $\mathcal{T}^B_{\lambda_0}$ is of the form $\tau_B^{-t} P^B_3$ for some $t \in \mathbb{N}$, and hence $\mathcal{T}^B_{\lambda_0}$ is without τ_B-periodic modules. Therefore, we have in Γ_B a family $\mathcal{T}^B = (\mathcal{T}^B_\lambda)_{\lambda \in \Lambda(A)}$ of tubes. Finally, we observe that the image $\mathcal{P}(B)$ of the torsion part $\mathcal{T}(T) \cap \mathcal{P}(A)$ of the postprojective component $\mathcal{P}(A)$ of Γ_A is a postprojective component of Γ_B of the form

$$\begin{tikzcd}
P_1 \arrow[ddr] & & \tau_B^{-1} P_1 \arrow[ddr] & & \tau_B^{-2} P_1 \arrow[ddr] & & {} \\
P_2 \arrow[dr] & & \tau_B^{-1} P_2 \arrow[dr] & & \tau_B^{-2} P_2 \arrow[dr] & & {} \\
 & P_4 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau_B^{-1} P_4 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau_B^{-2} P_4 \arrow[uur, no head] \arrow[ur, no head] \arrow[dr, no head] \arrow[ddr, no head] & \cdots \\
 & & P_5 \arrow[ur] & & \tau_B^{-1} P_5 \arrow[ur] & & {} \\
 & & P_6 \arrow[uur] & & \tau_B^{-1} P_6 \arrow[uur] & & {}
\end{tikzcd}$$

Summing up, the Auslander–Reiten quiver Γ_B of B has a decomposition

$$\Gamma_B = \mathcal{P}(B) \vee \mathcal{T}^B \vee \mathcal{Q}(B).$$

We also observe that B is isomorphic to the one-point extension algebra

$$C[M] = \begin{bmatrix} K & M \\ 0 & C \end{bmatrix},$$

where C is the path algebra $K\Omega$ of the subquiver Ω of Q^* given by the vertices $1, 2, 4, 5, 6$ over K, M is the simple module S_4^C in $\operatorname{mod} C$, associated to the vertex 4 of Q^*, and $K = \operatorname{End}_C(M) = \operatorname{End}_B(S_4^B) = \operatorname{End}_B(\operatorname{Hom}_A(T, [2]S_4)) \cong \operatorname{End}_A([2]S_4)$. In particular, we have $\operatorname{rad} P_3^B = S_4^B = S_4^C$. According to Theorem VII.8.12, the Auslander–Reiten quiver Γ_C of C has a decomposition

$$\Gamma_C = \mathcal{P}(C) \vee \mathcal{T}^C \vee \mathcal{Q}(C),$$

where $\mathcal{P}(C)$ is the postprojective component containing all indecomposable projective right C-modules, $\mathcal{Q}(C)$ is the preinjective component containing all indecomposable injective right C-modules, and $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda \in \Lambda(C)}$ is a family of pairwise orthogonal stable tubes. In particular, there is a stable tube $\mathcal{T}_{\lambda_0}^C$ of rank 2 having the simple regular modules S_4^C and R with $\tau_C R^C = S_4^C$ and $\tau_C S_4^C = R^C$, where R^C is the indecomposable module in $\operatorname{mod} C$ such that $[R^C] = [S_1^C] + [S_2^C] + [S_4^C] + [S_5^C] + [S_6^C]$ in $K_0(C)$. Here, S_i^C denotes the simple module in $\operatorname{mod} C$ at the vertex i of Q^*. Applying Corollary VII.10.10, we conclude that $\mathcal{P}(C) = \mathcal{P}(B)$ and

$$\bigcup_{\lambda \in \Lambda(A) \setminus \{\lambda_0\}} \mathcal{T}_\lambda^B = \bigcup_{\lambda \in \Lambda(C) \setminus \{\lambda_0\}} \mathcal{T}_\lambda^C.$$

Further, the ray tube $\mathcal{T}_{\lambda_0}^B$ is obtained from the stable tube $\mathcal{T}_{\lambda_0}^C$ by inserting an infinite sectional path (ray) starting from the indecomposable projective module $P_3^B = \bar{S}_4^C$, so it is of the form

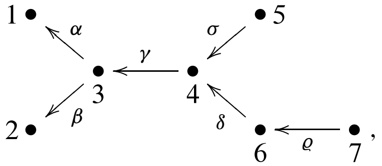
$$\begin{tikzcd}
 & & & \overline{S_4^C} = P_3^B \arrow[dr] & & R^C = R \arrow[dr] & \\
R^C \arrow[dr] & & S_4^C \arrow[ur] \arrow[dr] & & \overline{S_4^C}[2] \arrow[ur] \arrow[dr] & & R^C[2] \\
 & R^C[2] \arrow[ur] \arrow[dr] & & S_4^C[2] \arrow[ur] \arrow[dr] & & \overline{S_4^C}[3] \arrow[ur] \arrow[dr] & \\
\overline{S_4^C}[3] \arrow[ur] \arrow[dr, no head] & & R^C[3] \arrow[ur] \arrow[dr, no head] & & S_4^C[3] \arrow[ur] \arrow[dr, no head] & & \overline{S_4^C}[4] \\
 & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots \arrow[ur] &
\end{tikzcd}$$

where, for a module X in $\operatorname{mod} C$, $\bar{X}$ denotes the right B-module

$$\bar{X} = (\operatorname{Hom}_C(M, X), X, \operatorname{id}_{\operatorname{Hom}_C(M,X)}).$$

Moreover, we observe that all indecomposable modules of the preinjective component $\mathcal{Q}(C)$ of Γ_C are contained in the preinjective component $\mathcal{Q}(B)$ of Γ_B.

Example 6.18. Let K be a field, Δ be the quiver

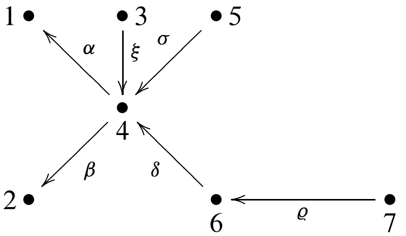


and $H = K\Delta$ the path algebra of Δ over K, considered in Examples VII.9.26 and 6.17. We note also that the path algebra KQ of the subquiver Q of Δ given by all vertices except 7 is the hereditary algebra A considered in Examples VII.8.30 and 6.15, and we have the canonical embedding of the module categories $\operatorname{mod} A \to \operatorname{mod} H$. For each $i \in \{1, 2, 3, 4, 5, 6, 7\}$, let P_i, I_i, S_i be the indecomposable projective module, the indecomposable injective module, and the simple module in $\operatorname{mod} H$, respectively, associated to the vertex i of Δ. Consider the indecomposable modules in $\operatorname{mod} H$

$$T_1 = P_1, \quad T_2 = P_2, \quad T_3 = S_4, \quad T_4 = P_4, \quad T_5 = P_5, \quad T_6 = P_6, \quad T_7 = P_7,$$

and $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6 \oplus T_7$. We know from Example VII.9.26 that the regular part $\mathcal{R}(H)$ of Γ_H contains a component $\mathcal{C}_{\lambda_0}$ having the modules S_3, S_4, R as quasi-simple modules with $\tau_H S_4 = S_3$ and $\tau_H R = S_4$, where R is the indecomposable module in $\operatorname{mod} H$ with $[R] = [S_1] + [S_2] + [S_3] + [S_4] + [S_5] + [S_6]$. Applying arguments as in Example 6.17, we conclude that T is a tilting module in $\operatorname{mod} H$.

Let $D = \operatorname{End}_H(T)$ be the associated tilted algebra. Then D is isomorphic to the bound quiver algebra $K\Delta^*/J$, where Δ^* is the quiver



and J is the ideal in the path algebra $K\Delta^*$ of Δ^* over K generated by the elements $\xi\alpha$ and $\xi\beta$, and the vertices $1, 2, 3, 4, 5, 6, 7$ of the quiver Δ^* correspond to the indices of the indecomposable direct summands $T_1, T_2, T_3, T_4, T_5, T_6, T_7$ of T. Observe also that the torsion-free class $\mathcal{F}(T) = \{X \in \operatorname{mod} H \mid \operatorname{Hom}_A(T, H) = 0\}$

in mod H determined by T has only one indecomposable module, namely the simple module $S_3 = \tau_H S_4$. We will describe the shape of some connected components of Γ_D. It follows from Lemma 3.1 that the modules $P_i^D = \operatorname{Hom}_A(T, T_i)$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$, form a complete set of pairwise nonisomorphic indecomposable projective modules in mod D. Since T is a splitting tilting module, applying Proposition 5.4, we conclude that the modules $I_i^D = \operatorname{Hom}_H(T, I_i)$, $i \in \{1, 2, 4, 5, 6, 7\}$, and the module $I_3^D = \operatorname{Ext}_H^1(T, \tau_H T_3) = \operatorname{Ext}_H^1(T, S_3)$ form a complete set of pairwise nonisomorphic indecomposable injective modules in mod D. Further, since $\mathcal{F}(T)$ is the additive category of S_3, Proposition 5.2 shows that we have in mod D only one connecting sequence

$$0 \longrightarrow \operatorname{Hom}_H(T, I_3) \longrightarrow \operatorname{Hom}_H(T, I_4) \longrightarrow \operatorname{Ext}_H^1(T, S_3) \longrightarrow 0,$$

because of isomorphisms $\operatorname{Ext}_H^1(T, \operatorname{rad} P_3) = \operatorname{Ext}_H^1(T, P_1 \oplus P_2) \cong D \operatorname{Hom}_H(P_1 \oplus P_2, \tau_H T) = D \operatorname{Hom}_H(P_1 \oplus P_2, S_3) = 0$, and hence $\operatorname{Ext}_H^1(T, P_3) \cong \operatorname{Ext}_H^1(T, S_3)$. Further, for any indecomposable module X in the preinjective component $\mathcal{Q}(H)$ of Γ_H, we have $\operatorname{Ext}_H^1(T, X) \cong D \operatorname{Hom}_H(X, \tau_H T) = 0$, and hence $\mathcal{Q}(H)$ is entirely contained in the torsion class $\mathcal{T}(T)$ of mod H. Therefore, the connecting component $\mathcal{C}_T$ of Γ_D determined by T is the preinjective component $\mathcal{Q}(D)$ of the form

$$\begin{tikzcd}
 & \tau_D^3 I_7^D \arrow[dr] & & \tau_D^2 I_7^D \arrow[dr] & & \tau_D I_7^D \arrow[dr] & & I_7^D \\
\cdots \arrow[ur] \arrow[dr] & & \tau_D^2 I_6^D \arrow[ur] \arrow[dr] & & \tau_D I_6^D \arrow[ur] \arrow[dr] & & I_6^D \arrow[ur] & \\
\cdots \arrow[r] & \tau_D^2 I_4^D \arrow[r] \arrow[ur] \arrow[dr] & \tau_D^2 I_5^D \arrow[r] & \tau_D I_4^D \arrow[r] \arrow[ur] \arrow[dr] & \tau_D I_5^D \arrow[r] & I_4^D \arrow[r] \arrow[ur] \arrow[dr] & I_5^D & \\
\cdots \arrow[ur] \arrow[dr] & \tau_D I_2^D \arrow[r] & \tau_D \operatorname{Hom}_H(T, I_3) \arrow[r] \arrow[ur] \arrow[dr] & I_2^D \arrow[r] & \operatorname{Hom}_H(T, I_3) \arrow[rr] \arrow[ur] & & \operatorname{Ext}_H^1(T, S_3) = S_3^D & \\
 & \tau_D I_1^D \arrow[ur] & & I_1^D \arrow[ur] & & & &
\end{tikzcd}$$

obtained from the image $\operatorname{Hom}_H(T, \mathcal{Q}(H))$ of $\mathcal{Q}(H)$ under the functor

$$\operatorname{Hom}_H(T, -) \colon \operatorname{mod} H \longrightarrow \operatorname{mod} D,$$

by adding the simple injective module $S_3^D = \operatorname{Ext}_H^1(T, S_3)$. We note that $\mathcal{Q}(D)$ contains all indecomposable injective right D-modules. Observe now that the subquiver of $\mathcal{C}_{\lambda_0}$

$$\begin{tikzcd}
S_3 \arrow[dr] & & S_4 \arrow[dr] & & R \\
& {[2]S_4} \arrow[ur] \arrow[dr] & & {[2]R} \arrow[ur] & \\
& & {[3]R} \arrow[ur] & &
\end{tikzcd}$$

is the full translation subquiver of the stable tube $\mathcal{T}^A_{\lambda_0}$ of rank 3 in Γ_A, and hence the modules S_4, $[2]S_4$, R, $[2]R$ and $[3]R$ belong to the torsion class $\mathcal{T}(T)$ of $\operatorname{mod} H$. Moreover, we have isomorphisms in $\operatorname{mod} K$

$$\operatorname{Hom}_H\big([3]S_4, S_3\big) \cong \operatorname{Hom}_H\big(\tau_H^{-1}[3]S_4, \tau_H^{-1}S_3\big) = \operatorname{Hom}_H\big([3]R, S_4\big) = 0,$$

and hence $[3]S_4$ belongs to $\mathcal{T}(T)$. Since there is an almost split sequence in $\operatorname{mod} H$ of the form

$$0 \longrightarrow [3]S_4 \longrightarrow [2]S_4 \oplus [4]R \longrightarrow [3]R \longrightarrow 0,$$

and $\mathcal{T}(T)$ is closed under extensions, we conclude that $[4]R$ also belongs to $\mathcal{T}(T)$. Then, applying Proposition 5.3, we conclude that there is a component $\mathcal{C}^D_{\lambda_0}$ in Γ_D containing a full translation subquiver of the form

$$\begin{tikzcd}
& & P_3^B = \operatorname{Hom}_H(T, S_4) \arrow[dr] & & \operatorname{Hom}_H(T, R) \\
& \operatorname{Hom}_H(T, [2]S_4) \arrow[ur] \arrow[dr] & & \operatorname{Hom}_H(T, [2]R) \arrow[ur] & \\
\operatorname{Hom}_H(T, [3]S_4) \arrow[ur] \arrow[dr] & & \operatorname{Hom}_H(T, [3]R) \arrow[ur] & & \\
& \operatorname{Hom}_H(T, [4]R) \arrow[ur] & & &
\end{tikzcd}$$

Let $\mathcal{D}_{\lambda_0}$ be the full translation subquiver of $\mathcal{C}_{\lambda_0}$ formed by all predecessors of $\tau_H S_3$ in $\mathcal{C}_{\lambda_0}$. We claim that $\mathcal{D}_{\lambda_0}$ is contained in the torsion part $\mathcal{T}(T) \cap \mathcal{C}_{\lambda_0}$ of $\mathcal{C}_{\lambda_0}$. Since S_3 is a quasi-simple brick of $\mathcal{C}_{\lambda_0}$, it follows from Proposition VII.9.18 that $\operatorname{Hom}_H(S_3, \tau_H^{-m}S_3) = 0$ for all positive integers m. Then we conclude that $\operatorname{Hom}_H(\tau_H^m S_3, S_3) = 0$ for all positive integers m. Then, applying Lemma VII.9.21 and Proposition 1.2 (see also Exercise VII.11.28), we conclude that $\operatorname{Hom}_H(X, S_3) = 0$ for any module X in $\mathcal{D}_{\lambda_0}$. Then it follows from Proposition 5.3 that the image $\operatorname{Hom}_H(T, \mathcal{D}_{\lambda_0})$ of $\mathcal{D}_{\lambda_0}$ under the functor $\operatorname{Hom}_H(T, -)$:

$\operatorname{mod} H \to \operatorname{mod} D$ is a full translation subquiver $\mathcal{D}_{\lambda_0}^D$ of Γ_D which is closed under predecessors in Γ_D. We also observe that, since $\mathcal{F}(T) = \operatorname{add} S_3$, all almost split sequences in $\operatorname{mod} D$ with the right term different from the simple module $S_3^D = \operatorname{Ext}_H^1(T, S_3)$ are formed by modules from the torsion-free part $\mathcal{Y}(T)$ of $\operatorname{mod} D$, which is the image of $\mathcal{T}(T)$ by the functor $\operatorname{Hom}_H(T, -)\colon \operatorname{mod} H \to \operatorname{mod} D$. Moreover, we have in $\operatorname{mod} H$ an exact sequence

$$0 \longrightarrow \tau_H S_3 \longrightarrow [3]S_4 \longrightarrow [2]S_4 \longrightarrow 0,$$

which induces an almost split sequence in $\operatorname{mod} D$

$$0 \longrightarrow \operatorname{Hom}_H(T, \tau_H S_3) \longrightarrow \operatorname{Hom}_H(T, [3]S_4) \longrightarrow \operatorname{Hom}_H(T, [2]S_4) \longrightarrow 0,$$

because the right homomorphism is a right minimal almost split homomorphism in $\operatorname{mod} D$. Therefore, there is a component $\mathcal{C}_{\lambda_0}^D$ in Γ_D of the form

$$\begin{tikzcd}[column sep=tiny]
 & & & & & & P_3^D \cong \operatorname{Hom}_H(T, S_4) \arrow[dr] & & \operatorname{Hom}_H(T, R) \arrow[dr, dash] & \\
 & \tau_D \operatorname{Hom}_H(T, \tau_H S_3) \arrow[dr] & & \operatorname{Hom}_H(T, \tau_H S_3) \arrow[dr] & & \operatorname{Hom}_H(T, [2]S_4) \arrow[ur] \arrow[dr] & & \operatorname{Hom}_H(T, [2]R) \arrow[ur] \arrow[dr, dash] & & \\
 {} \arrow[ur] & & \operatorname{Hom}_H(T, [2]\tau_H S_3) \arrow[ur] \arrow[dr, dash] & & \operatorname{Hom}_H(T, [3]S_4) \arrow[ur] \arrow[dr, dash] & & \operatorname{Hom}_H(T, [3]R) \arrow[ur] \arrow[dr, dash] & & {} & \\
 & {} \arrow[ur] & & {} \arrow[ur] & & {} \arrow[ur] & & {} & & \\
 & \vdots & & \vdots & & \vdots & & \vdots & &
\end{tikzcd}$$

having $\mathcal{D}_{\lambda_0}^D$ as the full translation subquiver given by all predecessors of the module $\operatorname{Hom}_H(T, \tau_H S_3)$ in Γ_D. We note that the quiver obtained from $\mathcal{C}_{\lambda_0}^D$ by deleting the τ_D-orbit of P_3^D is of the form $\mathbb{Z}\mathbb{A}_\infty$. Since the preinjective component $\mathcal{Q}(D)$ contains all indecomposable injective right D-modules, we deduce now that Γ_D contains a postprojective component $\mathcal{P}(D)$ of the form

$$\begin{tikzcd}[column sep=small]
P_1^D \arrow[ddr] & & \tau_B^{-1} P_1^D \arrow[ddr] & & \tau_B^{-2} P_1^D \arrow[ddr] & & {} & \\
P_2^D \arrow[dr] & & \tau_B^{-1} P_2^D \arrow[dr] & & \tau_B^{-2} P_2^D \arrow[dr] & & {} & \\
 & P_4^D \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau_B^{-1} P_4^D \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau_B^{-2} P_4^D \arrow[uur, dash] \arrow[ur, dash] \arrow[dr, dash] \arrow[ddr, dash] & & \cdots \\
 & & P_5^D \arrow[ur] & & \tau_B^{-1} P_5^D \arrow[ur] & & {} & \\
 & & P_6^D \arrow[uur] \arrow[dr] & & \tau_B^{-1} P_6^D \arrow[uur] \arrow[dr] & & {} & \cdots \\
 & & & P_7^D \arrow[ur] & & \tau_D^{-1} P_7^D \arrow[ur, dash] & &
\end{tikzcd}$$

being the image of the torsion part $\mathcal{T}(T) \cap \mathcal{P}(H)$ of the postprojective component $\mathcal{P}(H)$ of Γ_H under the functor $\operatorname{Hom}_H(T,-)\colon \operatorname{mod} H \to \operatorname{mod} D$. In particular, we conclude that all components of Γ_D different from $\mathcal{P}(D)$, $\mathcal{Q}(D)$, and $\mathcal{C}^D_{\lambda_0}$ do not contain projective modules and injective modules. Let $\mathcal{C}_\lambda$, $\lambda \in \Lambda(H) \setminus \{\lambda_0\}$, be a component in the regular part $\mathcal{R}(H)$ of Γ_H, different from $\mathcal{C}_{\lambda_0}$. Take a quasi-simple module X in $\mathcal{C}_\lambda$. It follows from Theorem VII.9.12 that there exists a positive integer m_0 such that $\operatorname{Hom}_H(X, \tau_H^{-m} S_3) = 0$ for all integers $m \geq m_0$. Then we conclude that $\operatorname{Hom}_H(\tau_H^m X, S_3) = 0$ for all integers $m \geq m_0$. Let $\mathcal{D}_\lambda$ be the full translation subquiver of $\mathcal{C}_\lambda$ given by all predecessors of $\tau_H^{m_0} X$ in $\mathcal{C}_\lambda$. Then it follows, as above, that $\mathcal{D}_\lambda$ is contained in the torsion part $\mathcal{T}(T)$ of $\operatorname{mod} H$. Hence, applying Proposition 5.3 again, we conclude that the image $\mathcal{D}^D_\lambda$ of $\mathcal{D}_\lambda$ under the functor $\operatorname{Hom}_H(T,-)\colon \operatorname{mod} H \to \operatorname{mod} D$ is a full translation subquiver of a component $\mathcal{C}^D_\lambda$ of Γ_D of type $\mathbb{Z}\mathbb{A}_\infty$. Observe that, for $\lambda \neq \mu$ in $\Lambda(H) \setminus \{\lambda_0\}$, we have $\mathcal{C}^D_\lambda \neq \mathcal{C}^D_\mu$. Denote by $\mathcal{R}(D)$ the family of components $\mathcal{C}^D_\lambda$, $\lambda \in \Lambda(H)$. One can prove that the Auslander–Reiten quiver Γ_D of D has a decomposition

$$\Gamma_D = \mathcal{P}(D) \vee \mathcal{R}(D) \vee \mathcal{Q}(D).$$

We also observe that D is isomorphic to the one-point extension algebra

$$E[N] = \begin{bmatrix} K & N \\ 0 & E \end{bmatrix},$$

where E is the path algebra $K\Sigma$ of the subquiver Σ of Δ^* given by the vertices $1, 2, 4, 5, 6, 7$ over K, and N is the simple module S^E_4 in $\operatorname{mod} E$ associated to the vertex 4, and $K = \operatorname{End}_E(N) = \operatorname{End}_D(S^D_4) = \operatorname{End}_D(\operatorname{Hom}_H(T, [2]S_4)) \cong \operatorname{End}_H([2]S_4)$. In particular, $\operatorname{rad} P^D_3 = S^D_4$. According to Theorem VII.9.3, the Auslander–Reiten quiver Γ_E of E has a decomposition

$$\Gamma_E = \mathcal{P}(E) \vee \mathcal{R}(E) \vee \mathcal{Q}(E),$$

where $\mathcal{P}(E)$ is the postprojective component containing all indecomposable projective right E-modules, $\mathcal{Q}(E)$ is the preinjective component containing all indecomposable injective right E-modules, and $\mathcal{R}(E) = (\mathcal{C}^E_\lambda)_{\lambda \in \Lambda(E)}$ is a family of components of the form $\mathbb{Z}\mathbb{A}_\infty$. We note that the module $N = S^E_4$ is a quasi-simple brick of a component $\mathcal{C}^E_{\lambda_0}$. But, in contrast to the Euclidean case, for any module Z in a component $\mathcal{C}^E_\lambda$, there exists an integer m_0 such that $\operatorname{Hom}_E(S^E_4, \tau^m_E Z) \neq 0$ for all integers $m \geq m_0$, by Theorem VII.9.13. Then applying Corollary VII.10.10, we conclude that, for any $\lambda \in \Lambda(E)$, the component $\mathcal{C}^E_\lambda$ is not a component of Γ_D. On the other hand, it follows from Theorem VII.9.12 that for any $\lambda \in \Lambda(E)$ and a quasi-simple module Y in $\mathcal{C}^E_\lambda$, there exists a positive integer n_0 such that $\operatorname{Hom}_E(N, \tau_E^{-n} Y) = 0$ for all integers $n \geq n_0$. We denote by $\mathcal{E}_\lambda$ the full translation subquiver of $\mathcal{C}^E_\lambda$ formed by the successors of the module $\tau_E^{-n_0} Y$ in $\mathcal{C}^E_\lambda$. Then

$\operatorname{Hom}_E(N, V) = 0$ for all indecomposable modules V in $\mathcal{E}_\lambda$, and consequently $\mathcal{E}_\lambda$ is a full translation subquiver of a component $\bar{\mathcal{C}}_\lambda^E$ of Γ_D which is closed under successors, again by Corollary VII.10.10. We also observe that the component $\mathcal{C}_{\lambda_0}^E$ is of the form

$$\begin{tikzcd}[column sep=small]
 & \tau_E S_4^E \arrow[dr] & & S_4^E \arrow[dr] & & R^E \arrow[dr] & & \tau_E^{-1} R \arrow[dr, dash] & \\
{} \arrow[ur] & & \tau_E S_4^E[2] \arrow[ur] \arrow[dr] & & S_4^E[2] \arrow[ur] \arrow[dr] & & R^E[2] \arrow[ur] \arrow[dr] & & {} \\
 & {} \arrow[ur] & & \tau_E S_4^E[3] \arrow[ur] & & S_4^E[3] \arrow[ur] & & R^E[3] &
\end{tikzcd}$$

where R^E is the indecomposable module in $\operatorname{mod} E$ such that $[R^E] = [S_1^E] + [S_2^E] + [S_4^E] + [S_5^E] + [S_6^E]$ in $K_0(E)$. Since S_4^E is a quasi-simple brick, it follows from Proposition VII.9.18 that $\operatorname{Hom}_E(S_4^E, \tau_E^{-m} S_4^E) = 0$ for all positive integers m. Hence, we may take as $\mathcal{E}_{\lambda_0}$ the full translation subquiver of $\mathcal{C}_{\lambda_0}^E$ formed by all successors of R^E in $\mathcal{C}_{\lambda_0}^E$. Applying Theorem VII.10.9, we conclude that $\bar{\mathcal{C}}_{\lambda_0}^D$ coincides with the component $\mathcal{C}_{\lambda_0}^D$ containing the projective module P_3^D. In particular, all the components $\bar{\mathcal{C}}_\lambda^D$, $\lambda \in \Lambda(E) \setminus \{\lambda_0\}$, are components of the form $\mathbb{Z}\mathbb{A}_\infty$. In fact, one can show that

$$\bigcup_{\lambda \in \Lambda(H)} \mathcal{C}_\lambda^D = \bigcup_{\lambda \in \Lambda(E)} \bar{\mathcal{C}}_\lambda^D .$$

We also mention that all indecomposable modules from the preinjective component $\mathcal{Q}(E)$ of Γ_E are contained in the preinjective component $\mathcal{Q}(D)$ of Γ_D.

7 The criterion of Liu and Skowroński

The aim of this section is to provide a very handy criterion to decide whether a given finite dimensional K-algebra over a field K is a tilted algebra, obtained independently by S. Liu [L3] and A. Skowroński [S1].

Let A be a finite dimensional K-algebra over a field K. Recall that a module M in $\operatorname{mod} A$ is called faithful if $Ma \neq 0$ for any nonzero element $a \in A$. We already mentioned in Section 2 that, if T is a tilting module in $\operatorname{mod} A$, then it follows from the condition (T3) that T is a faithful module. In fact, we showed in Section II.5 that a module M in $\operatorname{mod} A$ is faithful if and only if the module A_A is cogenerated by M (Lemma II.5.5), and if and only if the module $D(A)$ is generated by M (Corollary II.5.7).

Lemma 7.1. *Let A be a finite dimensional K-algebra over a field K and M be a faithful module in* $\operatorname{mod} A$. *Then the following statements hold:*

(i) *If* $\operatorname{Hom}_A(M, \tau_A M) = 0$, *then* $\operatorname{pd}_A M \leq 1$.

(ii) *If* $\operatorname{Hom}_A(\tau_A^{-1} M, M) = 0$, *then* $\operatorname{id}_A M \leq 1$.

Proof. Since M is a faithful module, it follows from Lemma II.5.5 and Corollary II.5.7 that there exist a monomorphism $A \to M^r$ and an epimorphism $M^s \to D(A)$ in $\operatorname{mod} A$, for some positive integers r and s. Then $\operatorname{Hom}_A(M, \tau_A M) = 0$ forces $\operatorname{Hom}_A(D(M), \tau_A M) = 0$, and hence $\operatorname{pd}_A M \leq 1$, by Proposition III.5.4(i). Similarly, $\operatorname{Hom}_A(\tau_A^{-1} M, M) = 0$ implies $\operatorname{Hom}_A(\tau_A^{-1} M, A) = 0$, and hence $\operatorname{id}_A M \leq 1$, by Proposition III.5.4(ii). Therefore, the statements (i) and (ii) hold. □

Corollary 7.2. *Let A be a finite dimensional K-algebra over a field K and M be a faithful module in* $\operatorname{mod} A$ *such that* $\operatorname{Hom}_A(M, \tau_A M) = 0$. *Then M is a partial tilting module.*

Proof. It follows from Lemma 7.1 that $\operatorname{pd}_A M \leq 1$. Moreover, by Corollary III.6.4, we have $\operatorname{Ext}_A^1(M, M) \cong D \operatorname{Hom}_A(M, \tau_A M) = 0$. □

Proposition 7.3. *Let A be a finite dimensional K-algebra over a field K, I a two-sided ideal of A, and $B = A/I$. Then for any module M in* $\operatorname{mod} B$ *its Auslander–Reiten translate $\tau_B M$ in* $\operatorname{mod} B$ *is isomorphic to a right A-submodule of its Auslander–Reiten translate $\tau_A M$ in* $\operatorname{mod} A$.

Proof. We consider the module category $\operatorname{mod} B$ as a full subcategory of $\operatorname{mod} A$ consisting of all modules M with $MI = 0$, so we have the canonical fully faithful embedding of categories $e_I \colon \operatorname{mod} B \to \operatorname{mod} A$. Conversely, for any module M in $\operatorname{mod} A$, we may consider the right B-module $t_I(M) = \{m \in M \mid mI = 0\}$. Observe that, for each homomorphism $f \colon M \to N$ in $\operatorname{mod} A$, we have $f(t_I(M)) \subseteq t_I(N)$, and hence the restriction $t_I(f) \colon t_I(M) \to t_I(N)$ of f to $t_I(M)$ is a homomorphism in $\operatorname{mod} B$. Therefore, we have the covariant functor $t_I \colon \operatorname{mod} A \to \operatorname{mod} B$ such that $t_I e_I = \mathbf{1}_{\operatorname{mod} B}$.

Let M be a module in $\operatorname{mod} A$. We may assume, without loss of generality, that M is indecomposable. Assume first that M is projective in $\operatorname{mod} A$. We claim that then M is projective in $\operatorname{mod} B$. Indeed, if $h \colon X \to Y$ is an epimorphism in $\operatorname{mod} B$, then h is an epimorphism in $\operatorname{mod} A$, and hence the induced homomorphism $\operatorname{Hom}_B(M, h) \colon \operatorname{Hom}_B(M, X) \to \operatorname{Hom}_B(M, Y)$ is an epimorphism, because $\operatorname{Hom}_B(M, Z) = \operatorname{Hom}_A(M, Z)$ for any module Z in $\operatorname{mod} B$ and M is projective in $\operatorname{mod} A$. Therefore, if $\tau_A M = 0$ then $\tau_B M = 0$, and the claim follows.

Assume now that M is not a projective module in $\operatorname{mod} B$. Then M is not projective in $\operatorname{mod} A$ and we have an almost split sequence

$$0 \longrightarrow \tau_A M \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

in mod A. Then we obtain an exact sequence in mod B of the form

$$0 \longrightarrow t_I(\tau_A M) \xrightarrow{t_I(f)} t_I(E) \xrightarrow{t_I(g)} M \longrightarrow 0,$$

because $t_I(M) = M$. We claim that $t_I(g)$ is a right almost split homomorphism in mod B. Clearly, $t_I(g)$ is not a retraction because $t_I(E) \subseteq E$ and $g\colon E \to M$ is not a retraction in mod A. Let $u\colon X \to M$ be a homomorphism in mod B which is not a retraction. Then u is not a retraction in mod A, because mod B is a full subcategory of mod A. Since g is a right almost split homomorphism in mod A, we conclude that there is a homomorphism $v\colon X \to E$ in mod A such that $u = gv$. But $XI = 0$ implies that $\operatorname{Im} v \subseteq t_I(E)$, and hence v is a homomorphism of right B-modules from X to $t_I(E)$. Thus we obtain that $u = t_I(g)v$. This shows that $t_I(g)$ is a right almost split homomorphism in mod B.

We may also consider an almost split sequence

$$0 \longrightarrow \tau_B M \xrightarrow{f'} E' \xrightarrow{g'} M \longrightarrow 0$$

in mod B, because M is assumed to be nonprojective in mod B. Now, using the fact that $t_I(g)$ is a right almost split homomorphism and g' is not a retraction in mod B, we conclude that there exists a homomorphism $w\colon E' \to t_I(E)$ in mod B such that $g' = t_I(g)w$. Observe that g' is an irreducible homomorphism in mod B, because g' is a minimal right almost split homomorphism in mod B. Since $t_I(g)$ is not a retraction in mod B, we conclude that w is a section. Then we obtain a commutative diagram in mod A of the form

$$\begin{tikzcd}
0 \arrow[r] & \tau_B M \arrow[r, "f'"] \arrow[d, "w'"] & E' \arrow[r, "g'"] \arrow[d, "w"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
0 \arrow[r] & t_I(\tau_A M) \arrow[r, "t_I(f)"] \arrow[d, hook] & t_I(E) \arrow[r, "t_I(g)"] \arrow[d, hook] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
0 \arrow[r] & \tau_A M \arrow[r, "f"] & E \arrow[r, "g"] & M \arrow[r] & 0,
\end{tikzcd}$$

because $t_I(g)wf' = g'f' = 0$, whose all vertical homomorphisms are monomorphisms. Therefore, $\tau_B M$ is isomorphic to a right A-submodule of $\tau_A M$. □

The following lemma, obtained in [S3], will be crucial for several further considerations, and have been already applied in the proof of Theorem VII.8.12.

Lemma 7.4. *Let A be a finite dimensional K-algebra over a field K and n be the rank of the Grothendieck group $K_0(A)$. Assume that M is a module in* mod A *which is a direct sum of m pairwise nonisomorphic indecomposable modules and* $\operatorname{Hom}_A(M, \tau_A M) = 0$. *Then $m \leq n$.*

Proof. Consider the right annihilator $r_A(M) = \{a \in A \mid Ma = 0\}$ of M. Then $r_A(M)$ is a two-sided ideal of A, and let $B = A/r_A(M)$. Observe that M is a faithful right B-module. Moreover, it follows from Proposition 7.3 that $\tau_B M$ is isomorphic to a right A-submodule of $\tau_A M$. Hence, $\mathrm{Hom}_A(M, \tau_A M) = 0$ implies $\mathrm{Hom}_B(M, \tau_B M) = 0$. Then it follows from Corollary 7.2 that M is a partial tilting module in mod B. Applying Lemma 2.4, we conclude that there exists a module N in mod B such that $T = M \oplus N$ is a tilting module in mod B. Then it follows from Proposition 3.11 that the number of pairwise nonisomorphic indecomposable direct summands of T equals the rank of $K_0(B)$. Clearly, the rank of $K_0(B)$ is less than or equal to the rank of $K_0(A)$. Therefore, we conclude that $m \leq n$. □

Lemma 7.5. *Let B be a finite dimensional K-algebra over a field K, $\mathcal{C}$ be a connected component of Γ_B, and Σ a finite and acyclic full valued subquiver of $\mathcal{C}$. Then the following statements hold:*

(i) *Every homomorphism $f: Y \to V$ between indecomposable modules in* mod B, *with V on Σ and Y not on Σ, factors through a direct sum of modules from $\tau_B \Sigma = \{\tau_B Z \mid Z \in \Sigma\}$.*

(ii) *Every homomorphism $g: V \to X$ between indecomposable modules in* mod B, *with V on Σ and X not on Σ, factors through a direct sum of modules from $\tau_B^{-1} \Sigma = \{\tau_B^{-1} Z \mid Z \in \Sigma\}$.*

Proof. (i) For any module U on Σ, we define $m(U)$ to be the maximum of the lengths of paths in Σ with target in U. Hence, $m(U) = 0$ if and only if U is a source of Σ. Let $f: Y \to V$ be a homomorphism between indecomposable modules in mod B with V on Σ and Y not on Σ. Consider a minimal right almost split homomorphism $u: E \to V$ in mod B with the right term V. Since Y is not on Σ, then f is not a retraction, and hence there exists a homomorphism $h: Y \to E$ in mod B such that $f = uh$. If $m(V) = 0$, then E is a direct sum of modules from $\tau_B \Sigma$, and the claim follows. Assume $m(V) \geq 1$. Then we may decompose E as $E = E' \oplus E''$ such that E' is a direct sum of modules from $\tau_B \Sigma$ and E'' is a direct sum of modules from Σ. Moreover, we have

$$h = \begin{bmatrix} u' & u'' \end{bmatrix} : E' \oplus E'' \longrightarrow V \quad \text{and} \quad h = \begin{bmatrix} h' \\ h'' \end{bmatrix} : Y \longrightarrow E' \oplus E'',$$

so $f = uh = u'h' + u''h''$. Observe that, for any indecomposable direct summand W of E'', we have $W \in \Sigma$ and $m(W) < m(V)$. Therefore, by induction, the homomorphism $h'': Y \to E''$ factors through a direct sum of modules from $\tau_B \Sigma$. Hence, the homomorphism $f = u'h' + u''h''$ factors through a direct sum of modules from $\tau_B \Sigma$.

The proof of (ii) is dual. □

Lemma 7.6. *Let B be a finite dimensional K-algebra over a field K, $\mathcal{C}$ a connected component of Γ_B containing a finite section Δ, and T the direct sum of all indecomposable modules of Δ. Then* $\operatorname{Hom}_B(T, \tau_B T) = 0$ *if and only if* $\operatorname{Hom}_B(\tau_B^{-1} T, T) = 0$.

Proof. Let $p\colon P \to \tau_B^{-1} T$ be a projective cover of $\tau_B^{-1} T$ in $\operatorname{mod} B$ and $u\colon \tau_B T \to I$ an injective envelope of $\tau_B T$ in $\operatorname{mod} B$. Applying Lemma 7.5(i) to the quiver $\Sigma = \tau_B^{-1}\Delta$, we conclude that p factors through a direct sum of modules from Δ, and consequently there is an epimorphism $f\colon T^r \to \tau_B^{-1} T$, for some positive integer r. Similarly, applying Lemma 7.5(ii) to the quiver $\Sigma = \tau_B \Delta$, we conclude that u factors through a direct sum of modules from Δ, and consequently there is an monomorphism $g\colon \tau_B T \to T^s$, for some positive integer s.

Assume now that $\operatorname{Hom}_B(T, \tau_B T) \neq 0$, and let $h\colon T \to \tau_B T$ be a nonzero homomorphism in $\operatorname{mod} B$. Applying Lemma 7.5(ii) to Δ, we infer that there exist a positive integer t and a factorization $h = h_2 h_1$, for some homomorphisms $h_1\colon T \to (\tau_B^{-1} T)^t$ and $h_2\colon (\tau_B^{-1} T)^t \to \tau_B T$ in $\operatorname{mod} B$. Then the composite homomorphism $g h_2\colon (\tau_B^{-1} T)^t \to T^s$ is nonzero, and hence $\operatorname{Hom}_B(\tau_B^{-1} T, T) \neq 0$.

Conversely, assume that $\operatorname{Hom}_B(\tau_B^{-1} T, T) \neq 0$, and let $w\colon \tau_B^{-1} T \to T$ be a nonzero homomorphism in $\operatorname{mod} B$. Applying Lemma 7.5(i) to Δ, we conclude that there exist a positive integer m and a factorization $w = w_2 w_1$, for some homomorphisms $w_1\colon \tau_B^{-1} T \to (\tau_B T)^m$ and $w_2\colon (\tau_B T)^m \to T$ in $\operatorname{mod} B$. Then the composed homomorphism $w_1 f\colon T^r \to (\tau_B T)^m$ is nonzero, and hence $\operatorname{Hom}_B(T, \tau_B T) \neq 0$. □

We are now in position to establish the criterion of Liu and Skowroński for a finite dimensional algebra to be a tilted algebra.

Theorem 7.7. *Let B be an indecomposable finite dimensional K-algebra over a field K. Then B is a tilted algebra if and only if Γ_B contains a connected component $\mathcal{C}$ with a faithful section Δ such that* $\operatorname{Hom}_B(X, \tau_B Y) = 0$ *for all modules X and Y lying on Δ. Moreover, in this case, the following statements hold:*

(i) *The direct sum T_Δ^* of all modules lying on Δ is a tilting module in* $\operatorname{mod} B$.

(ii) $A_\Delta = \operatorname{End}_B(T_\Delta^*)$ *is a basic indecomposable finite dimensional hereditary K-algebra.*

(iii) $T_\Delta = D(T_\Delta^*)$ *is a tilting module in* $\operatorname{mod} A_\Delta$.

(iv) *There is a canonical isomorphism of K-algebras*

$$\sigma\colon B \longrightarrow \operatorname{End}_{A_\Delta}(T_\Delta)$$

such that $\sigma(b)(f)(t^*) = f(t^* b)$ for $b \in B$, $f \in T_\Delta$ and $t^* \in T_\Delta^*$.

(v) *The component $\mathcal{C}$ is the connecting component $\mathcal{C}_{T_\Delta}$ of Γ_B and Δ the section Δ_{T_Δ} determined by T_Δ.*

Proof. Let B be a tilted algebra, A an indecomposable finite dimensional hereditary K-algebra, and T a tilting module in $\operatorname{mod} A$ such that $B = \operatorname{End}_A(T)$. Let $\mathcal{C}_T$ be the connecting component of Γ_B determined by T and Δ_T the section of $\mathcal{C}_T$ formed by the images $\operatorname{Hom}_A(T, I_1), \dots, \operatorname{Hom}_A(T, I_n)$ of a complete set of pairwise nonisomorphic indecomposable injective modules $I_1, \dots, I_n$ in $\operatorname{mod} A$, under the functor $\operatorname{Hom}_A(T, -)\colon \operatorname{mod} A \to \operatorname{mod} B$. We claim that the section Δ_T is a faithful right B-module. It follows from Proposition 3.3 that T is a tilting module in $\operatorname{mod} B^{\mathrm{op}}$. Hence there exist a positive integer m and a monomorphism $B \to T^m$ in $\operatorname{mod} B^{\mathrm{op}}$. This induces an epimorpism $D(T)^m \to D(B)$ in $\operatorname{mod} B$, and so $D(T)$ is a faithful module in $\operatorname{mod} B$. Further, it follows also from Proposition 3.3 that there exists a canonical isomorphism $D(T) \xrightarrow{\sim} \operatorname{Hom}_A(T, D(A))$ in $\operatorname{mod} B$. Further, $D(A)$ belongs to $\operatorname{add}(I_1 \oplus \dots \oplus I_n)$, and hence $\operatorname{Hom}_A(T, D(A))$ belongs to $\operatorname{add}(\operatorname{Hom}_A(T, I_1) \oplus \dots \oplus \operatorname{Hom}_A(T, I_n))$. Therefore, the direct sum $\operatorname{Hom}_A(T, I_1) \oplus \dots \oplus \operatorname{Hom}_A(T, I_n)$ of the indecomposable modules lying on the section Δ_T is a faithful right B-module, and so Δ_T is a faithful section. Observe now that, by Lemma 4.2, for any modules X and Y on Δ, we have $Y \in \mathcal{Y}(T)$ and $\tau_B^{-1} X \in \mathcal{X}(T)$, and consequently $\operatorname{Hom}_B(\tau_B^{-1} X, Y) = 0$. Applying now Lemma 7.6, we obtain that $\operatorname{Hom}_B(X, \tau_B Y) = 0$ for any modules X and Y on Δ. This finishes the proof of the necessity part.

Assume now that Γ_B contains a connected component $\mathcal{C}$ with a faithful section Δ such that $\operatorname{Hom}_B(X, \tau_B Y) = 0$ for all modules X and Y lying on Δ. Let us show that the statements (i)–(v) hold.

(i) It follows from the assumption on Δ and Lemma 7.4 that Δ is a finite section of $\mathcal{C}$. Let T_Δ^* be the direct sum of all indecomposable modules lying on Δ. Then T_Δ^* is a faithful right B-module such that $\operatorname{Hom}_B(T_\Delta^*, \tau_B T_\Delta^*) = 0$, and consequently T_Δ^* is a partial tilting module in $\operatorname{mod} B$, by Corollary 7.2. Moreover, by Lemma 7.6, $\operatorname{Hom}_B(T_\Delta^*, \tau_B T_\Delta^*) = 0$ implies $\operatorname{Hom}_B(\tau_B^{-1} T_\Delta^*, T_\Delta^*) = 0$. Hence, we have also $\operatorname{id}_B T_\Delta^* \leq 1$, by Lemma 7.1(ii). Let $f_1, \dots, f_d$ be a K-basis of $\operatorname{Hom}_B(B, T_\Delta^*)$. Then we have the monomorphism

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_d \end{bmatrix} : B \longrightarrow (T_\Delta^*)^d$$

and a short exact sequence

$$0 \longrightarrow B \xrightarrow{f} (T_\Delta^*)^d \xrightarrow{g} V \longrightarrow 0$$

in $\operatorname{mod} B$, where $V = \operatorname{Coker} f$. We claim that $T_\Delta^* \oplus V$ is a tilting module in $\operatorname{mod} B$. We show first that $\operatorname{pd}_B V = 1$. Since $\operatorname{pd}_B T_\Delta^* \leq 1$, $(T_\Delta^*)^d$ admits a minimal projective resolution in $\operatorname{mod} B$ of the form

$$0 \longrightarrow P_1 \xrightarrow{u_1} P_0 \xrightarrow{u_0} (T_\Delta^*)^d \longrightarrow 0.$$

Invoking the projectivity of B in $\operatorname{mod} B$, we conclude that there is in $\operatorname{mod} B$ a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "h"] \arrow[d, "\mathrm{id}_B"] & P_0 \arrow[r, "p"] \arrow[d, "u_0"] & U \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & B \arrow[r, "f"] & (T_\Delta^*)^d \arrow[r, "g"] & V \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Moreover, it follows from Lemma VII.3.1 that q is an epimorphism and $\operatorname{Ker} q \cong \operatorname{Ker} u_0 = \operatorname{Im} u_1 \cong P_1$. Further, $gu_0 \colon P_0 \to V$ is an epimorphism with $\operatorname{Ker} gu_0 = \operatorname{Im} h + \operatorname{Im} u_1$. In fact, we have $\operatorname{Ker} gu_0 = \operatorname{Im} h \oplus \operatorname{Im} u_1$, because

$$\begin{aligned}\operatorname{Ker} gu_0 = \dim_K P_0 - \dim_K V &= \dim_K \operatorname{Im} u_1 + \dim_K (T_\Delta^*)^d - \dim_K V \\ &= \dim_K \operatorname{Im} f + \dim_K \operatorname{Im} u_1 = \dim_K \operatorname{Im} h + \dim_K \operatorname{Im} u_1 .\end{aligned}$$

Hence we obtain in $\operatorname{mod} B$ a short exact sequence

$$0 \longrightarrow B \oplus P_1 \xrightarrow{[h\ u_1]} P_0 \xrightarrow{gu_0} V \longrightarrow 0,$$

and consequently $\operatorname{pd}_B V \leq 1$. This shows that $\operatorname{pd}_B (T_\Delta^* \oplus V) \leq 1$.

Let us prove that $\operatorname{Ext}_A^1(T_\Delta^* \oplus V, T_\Delta^* \oplus V) = 0$. Applying Theorem VII.3.3, we obtain an exact sequence in $\operatorname{mod} K$ of the form

$$\begin{aligned}\operatorname{Hom}_A\left((T_\Delta^*)^d, T_\Delta^*\right) &\xrightarrow{\operatorname{Hom}_A(f, T_\Delta^*)} \operatorname{Hom}_B(B, T_\Delta^*) \\ &\xrightarrow{\delta} \operatorname{Ext}_A^1\left(V, T_\Delta^*\right) \longrightarrow \operatorname{Ext}_A^1\left((T_\Delta^*)^d, T_\Delta^*\right)\end{aligned}$$

with $\operatorname{Ext}_A^1((T_\Delta^*)^d, T_\Delta^*) = 0$. Moreover, it follows from the choice of f that $\operatorname{Hom}_A(f, T_\Delta^*)$ is an epimorphism. This leads to $\operatorname{Ext}_A^1(V, T_\Delta^*) = 0$. Further, applying Theorem VII.3.2, we obtain an exact sequence in $\operatorname{mod} K$ of the form

$$0 = \operatorname{Ext}_B^1\left(V, (T_\Delta^*)^d\right) \longrightarrow \operatorname{Ext}_B^1(V, V) \longrightarrow 0,$$

because $\operatorname{pd}_B V \leq 1$, and hence $\operatorname{Ext}_B^1(V, V) = 0$. Finally, applying Theorem VII.3.2 again, we obtain an exact sequence in $\operatorname{mod} K$ of the form

$$0 \longrightarrow \operatorname{Ext}_A^1\left(T_\Delta^*, (T_\Delta^*)^d\right) \longrightarrow \operatorname{Ext}_B^1(T_\Delta^*, V) \longrightarrow 0,$$

because $\operatorname{pd}_B T_\Delta^* \leq 1$, and hence $\operatorname{Ext}_B^1(T_\Delta^*, V) = 0$. Therefore, we have $\operatorname{Ext}_A^1(T_\Delta^* \oplus V, T_\Delta^* \oplus V) = 0$. Summing up, we conclude that $T_\Delta^* \oplus V$ is a tilting module in $\operatorname{mod} B$.

Let us show that V belongs to $\operatorname{add} T_{\Delta}^{*}$. Suppose V admits an indecomposable direct summand V_0 which is not a direct summand of T_{Δ}^{*}. Then we obtain the epimorphism $\pi g\colon (T_{\Delta}^{*})^{d} \to V_0$, where π is the projection of V on V_0, and hence $\operatorname{Hom}_B(T_{\Delta}^{*}, V_0) \neq 0$. Then it follows from Lemma 7.5(ii) that $\operatorname{Hom}_B(\tau_B^{-1} T_{\Delta}^{*}, V_0) \neq 0$. Since $\operatorname{id}_B T_{\Delta}^{*} \leq 1$, applying Corollary III.6.4, we obtain that $\operatorname{Ext}_B^1(V_0, T) \cong D \operatorname{Hom}_B(\tau_B^{-1} T_{\Delta}^{*}, V_0) \neq 0$, a contradiction to $\operatorname{Ext}_A^1(V, T_{\Delta}^{*}) = 0$. This shows that T_{Δ}^{*} is a tilting module in $\operatorname{mod} B$.

(ii) Since B is an indecomposable algebra, it follows from Corollary 3.20 that $A_{\Delta} = \operatorname{End}_B(T_{\Delta}^{*})$ is an indecomposable algebra. Moreover, T_{Δ}^{*} is a direct sum of pairwise nonisomorphic indecomposable modules lying on the section Δ, and hence $A_{\Delta} = \operatorname{End}_B(T_{\Delta}^{*}) = \operatorname{End}_B(T_{\Delta}^{*}, T_{\Delta}^{*})$ is, by Lemma 3.1, a direct sum of pairwise nonisomorphic indecomposable projective right A_{Δ}-modules, and consequently A_{Δ} is a basic algebra (see Section II.6). We show that A_{Δ} is also a hereditary algebra. Let P be an indecomposable projective module in $\operatorname{mod} A_{\Delta}$, M an indecomposable right A-submodule of P, and $f\colon M \to P$ the inclusion homomorphism. The tilting module T_{Δ}^{*} induces the torsion pair $(\mathcal{T}(T_{\Delta}^{*}), \mathcal{F}(T_{\Delta}^{*}))$ in $\operatorname{mod} B$ and the torsion pair $(\mathcal{X}(T_{\Delta}^{*}), \mathcal{Y}(T_{\Delta}^{*}))$ in $\operatorname{mod} A_{\Delta}$. Since P belongs to $\mathcal{Y}(T_{\Delta}^{*})$ and $\mathcal{Y}(T_{\Delta}^{*})$ is closed under submodules, we obtain that M also belongs to $\mathcal{Y}(T_{\Delta}^{*})$. Since the functor $\operatorname{Hom}_B(T_{\Delta}^{*}, -)\colon \operatorname{mod} B \to \operatorname{mod} A_{\Delta}$ induces an equivalence of categories $\mathcal{T}(T_{\Delta}^{*}) \xrightarrow{\sim} \mathcal{Y}(T_{\Delta}^{*})$, we conclude that there exists a homomorphism $g\colon U \to V$ in $\operatorname{mod} B$, with U, V indecomposable modules from $\mathcal{T}(T_{\Delta}^{*})$, V lying on Δ, $\operatorname{Hom}_B(T_{\Delta}^{*}, U) = M$, $\operatorname{Hom}_B(T_{\Delta}^{*}, V) = P$, and $\operatorname{Hom}_B(T_{\Delta}^{*}, g) = f$. Take now a nonzero homomorphism $h\colon P' \to M$ in $\operatorname{mod} A_{\Delta}$ with P' an indecomposable projective module.

Then there exists an nonzero homomorphism $u\colon V' \to U$ in $\operatorname{mod} B$ such that V' lies on Δ, $\operatorname{Hom}_B(T_{\Delta}^{*}, V') = P'$, and $\operatorname{Hom}_B(T_{\Delta}^{*}, u) = h$. Since f is a monomorphism, we conclude that $fh \neq 0$, and consequently $gu \neq 0$. We claim that U lies in Δ. Assume, to the contrary, that U does not lie on Δ. Then, applying Lemma 7.5(i), we conclude that there exist homomorphisms $p\colon U \to W$ and $q\colon W \to V$ in $\operatorname{mod} B$, with W being a direct sum of modules from $\tau_B \Delta$, such that $g = qp$. But then $qpu = gu \neq 0$ implies that $pu\colon V' \to W$ is a nonzero homomorphism, and hence $\operatorname{Hom}_B(T_{\Delta}^{*}, \tau_B T_{\Delta}^{*}) \neq 0$, a contradiction to the assumption imposed on Δ. Hence, U belongs to Δ, which implies that $\operatorname{Hom}_B(T_{\Delta}^{*}, U)$ is a projective right A_{Δ}-module. This shows that every right A_{Δ}-submodule of P is projective. Therefore, A_{Δ} is a hereditary algebra, by Theorems I.9.1 and I.9.3.

(iii) It follows from Proposition 3.3(ii) that T_{Δ}^{*} is a tilting module in $\operatorname{mod} A_{\Delta}^{\mathrm{op}}$. Moreover, by (ii) and Theorem I.9.3, A_{Δ}^{op} is also a hereditary algebra. Consider now the right A_{Δ}-module $T_{\Delta} = D(T_{\Delta}^{*})$. Then $\operatorname{id}_{A_{\Delta}^{\mathrm{op}}} T_{\Delta}^{*} \leq 1$ implies that $\operatorname{pd}_{A_{\Delta}} T_{\Delta} \leq 1$. Further, we have K-linear isomorphisms

$$\operatorname{Ext}_{A_{\Delta}}^1(T_{\Delta}, T_{\Delta}) \cong \mathcal{E}\mathrm{xt}_{A_{\Delta}}^1(T_{\Delta}, T_{\Delta}) \cong \mathcal{E}\mathrm{xt}_{A_{\Delta}^{\mathrm{op}}}^1(T_{\Delta}^{*}, T_{\Delta}^{*}) \cong \operatorname{Ext}_{A_{\Delta}^{\mathrm{op}}}^1(T_{\Delta}^{*}, T_{\Delta}^{*}) = 0,$$

and hence $\operatorname{Ext}^1_{A_\Delta}(T_\Delta, T_\Delta) = 0$, by Proposition III.3.7. Finally, observe that the Grothendieck groups $K_0(A_\Delta)$ and $K_0(A_\Delta^{\mathrm{op}})$ have the same rank. Since T_Δ^* is a tilting module in $\operatorname{mod} A_\Delta^{\mathrm{op}}$, we conclude that the rank of $K_0(A_\Delta^{\mathrm{op}})$ is the number of pairwise nonisomorphic indecomposable direct summands of T_Δ^*. Then we conclude that the rank of $K_0(A_\Delta)$ is the number of pairwise nonisomorphic indecomposable direct summands of $T_\Delta = D(T_\Delta^*)$. Therefore, T_Δ is a tilting module in $\operatorname{mod} A_\Delta$, by Proposition 3.11.

(iv) It follows from Proposition 3.3(iii) that there exists the canonical isomorphism of K-algebras

$$\varrho\colon B \longrightarrow \operatorname{End}_{A_\Delta^{\mathrm{op}}}(T_\Delta^*)^{\mathrm{op}}$$

such that $\varrho(b)(t^*) = t^*b$ for $b \in B$ and $t^* \in T_\Delta^*$. Moreover, we have the canonical isomorphism of K-algebras

$$\delta\colon \operatorname{End}_{A_\Delta^{\mathrm{op}}}(T_\Delta^*)^{\mathrm{op}} \longrightarrow \operatorname{End}_{A_\Delta}\big(D(T_\Delta^*)\big) = \operatorname{End}_{A_\Delta}(T_\Delta)$$

such that $\delta(g) = D(g)$ for $g\colon T_\Delta^* \to T_\Delta^*$ in $\operatorname{mod} A_\Delta^{\mathrm{op}}$. This leads to the composed isomorphism of K-algebras

$$\sigma = \delta\varrho\colon B \longrightarrow \operatorname{End}_{A_\Delta}(T_\Delta)$$

such that $\sigma(b)(f)(t^*) = D(\varrho(b))(f)(t^*) = f(t^*b)$ for $b \in B$, $f \in T_\Delta$ and $t^* \in T_\Delta^*$.

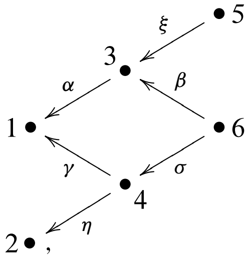
(v) Observe that we have isomorphisms of right B-modules

$$\operatorname{Hom}_{A_\Delta}\big(T_\Delta, D(A_\Delta)\big) \cong \operatorname{Hom}_{A_\Delta}\big(D(T_\Delta^*), D(A_\Delta)\big) \cong \operatorname{Hom}_{A_\Delta^{\mathrm{op}}}(A_\Delta, T_\Delta^*) \cong T_\Delta^*.$$

Since A_Δ is a basic algebra, $D(A_\Delta)$ is the direct sum of a complete set of pairwise nonisomorphic indecomposable injective modules in $\operatorname{mod} A_\Delta$. This shows that the section Δ of $\mathcal{C}$ is the section Δ_{T_Δ} in the connecting component $\mathcal{C}_{T_\Delta}$ of Γ_B determined by the tilting module T_Δ in $\operatorname{mod} A_\Delta$, and in particular we have $\mathcal{C} = \mathcal{C}_{T_\Delta}$, under the identification of the isomorphic algebras B and $\operatorname{End}_{A_\Delta}(T_\Delta)$.

Moreover, it follows from the statements (ii)–(iv) that B is a tilted algebra, which provides the proof of the sufficiency part. □

Example 7.8. Let K be a field, Q the quiver



I the ideal in the path algebra KQ of Q generated by $\beta\alpha - \sigma\gamma$, $\xi\alpha$, $\sigma\eta$, and $B = KQ/I$ the associated bound quiver algebra. Applying Theorem III.10.2, we infer that B is of finite representation type and the Auslander–Reiten quiver Γ_B is of the form

$$\begin{tikzcd}
 & P_3 \arrow[dr] & & I_2 \arrow[dr] & & & & & \\
S_1 = P_1 \arrow[ur] \arrow[dr] & & M \arrow[ur] \arrow[dr] & & S_4 \arrow[dr] & & V \arrow[dr] & & I_5 = S_5 \\
 & P_4 \arrow[ur] \arrow[dr] & & \operatorname{rad} P_6 \arrow[ur] \arrow[r] \arrow[dr] & \begin{array}{c} P_6 \\ \parallel \\ I_1 \end{array} \arrow[r] & P_6/S_1 \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & \\
S_2 = P_2 \arrow[ur] & & P_4/S_2 \arrow[ur] & & S_3 \arrow[ur] \arrow[dr] & & N \arrow[ur] \arrow[dr] & & I_6 = S_6 \\
 & & & & & P_5 \arrow[ur] & & I_4 \arrow[ur] &
\end{tikzcd}$$

where S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} B$ given by the vertex $i \in \{1, 2, 3, 4, 5, 6\}$ of Q, and M, N, V are the indecomposable modules with the composition vectors

$$\begin{aligned}
c(M) &= \big(c_1(M), c_2(M), c_3(M), c_4(M), c_5(M), c_6(M)\big) = (1, 1, 1, 1, 0, 0),\\
c(N) &= \big(c_1(N), c_2(N), c_3(N), c_4(N), c_5(N), c_6(N)\big) = (0, 0, 1, 1, 1, 1),\\
c(V) &= \big(c_1(V), c_2(V), c_3(V), c_4(V), c_5(V), c_6(V)\big) = (0, 0, 1, 0, 0, 1).
\end{aligned}$$

Observe that Γ_B has the section Δ given by the modules $I_2, S_4, \operatorname{rad} P_6, P_6, S_3, P_5$. Moreover, the associated module

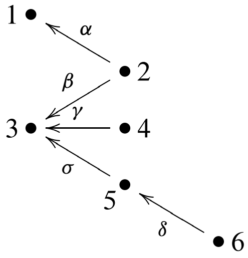
$$T^*_\Delta = I_2 \oplus S_4 \oplus \operatorname{rad} P_6 \oplus P_6 \oplus S_3 \oplus P_5$$

is a faithful right B-module, because there are monomorphisms $P_1 \to P_6$, $P_2 \to I_2$, $P_3 \to P_6$, $P_4 \to I_2 \oplus P_6$, and P_5, P_6 are direct summands of T^*_Δ, and hence there is a monomorphism $B \to (T^*_\Delta)^6$ in $\operatorname{mod} B$ (see Lemma II.5.5). Observe also that

$$\operatorname{Hom}_B(T^*_\Delta, \tau_B T^*_\Delta) = \operatorname{Hom}_B(T^*_\Delta, P_3 \oplus M \oplus P_4 \oplus P_4/S_2) = 0.$$

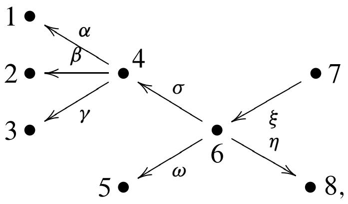
Therefore, applying Theorem 7.7, we conclude that B is a tilted algebra. In fact, the hereditary algebra $A_\Delta = \operatorname{End}_B(T^*_\Delta)$ is the path algebra $K\Omega$ of the quiver Ω

of the form



over K, $\Omega \cong \Delta^{\mathrm{op}}$, and so A_Δ is a hereditary algebra of Dynkin type $\mathbb{E}_6$. Moreover, we have an isomorphism of K-algebras $B \cong \operatorname{End}_{A_\Delta}(T_\Delta)$ for the tilting module $T_\Delta = D(T_\Delta^*)$, established in Theorem 7.7.

Example 7.9. Let K be a field, Q the quiver



I the ideal in the path algebra KQ of Q over K generated by the paths $\sigma\alpha$, $\sigma\beta$, $\sigma\gamma$, and $B = KQ/I$ the associated bound quiver K-algebra. Then we have in the Auslander–Reiten quiver Γ_B of B an acyclic component with the left part of the form

$$\begin{tikzcd}
S_1 = P_1 \arrow[dr] & & P_4/S_1 \arrow[dr] & & I_2 \arrow[dr] & & & & & & \\
S_2 = P_2 \arrow[r] & P_4 \arrow[ur] \arrow[r] \arrow[dr] & P_4/S_2 \arrow[r] & M \arrow[ur] \arrow[r] \arrow[dr] & I_2 \arrow[r] & S_4 \arrow[dddr] & & \tau_A^{-1}S_4 \arrow[dddr] & & \tau_A^{-2}S_4 \arrow[dr, dash] & \\
S_3 = P_3 \arrow[ur] & & P_4/S_3 \arrow[ur] & & I_3 \arrow[ur] & & & & & & \\
& & & & & S_5 = P_5 \arrow[dr] & & \tau_A^{-1}P_5 \arrow[dr] & & \tau_A^{-2}P_5 & \\
& & & & & & P_6 \arrow[uuur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau_A^{-1}P_6 \arrow[uuur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \cdots \\
& & & & & S_8 = P_8 \arrow[ur] & & \tau_A^{-1}P_8 \arrow[ur] & & \tau_A^{-2}P_8 & \\
& & & & & & & P_7 \arrow[uur] & & \tau_A^{-1}P_7 \arrow[ur, dash] &
\end{tikzcd}$$

where S_i, P_i, and I_i are respectively the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} B$ given by the vertex $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ of Q, and M is the indecomposable module with the composition vector $c(M) = (1, 1, 1, 2, 0, 0, 0, 0)$. Let Q' be the full subquiver of Q given by the vertices $1, 2, 3, 4$ and Q'' the full subquiver of Q given by the vertices $4, 5, 6, 7, 8$, and $H' = KQ'$, $H'' = KQ''$ be the path algebras of Q' and Q'' over K, respectively. Then H' is a hereditary algebra of Dynkin type $\mathbb{D}_4$ and H'' is a hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_4$, and the component $\mathcal{C}$ is obtained by gluing the Auslander–Reiten quiver $\Gamma_{H'}$ of H' with the postprojective component $\mathcal{P}(H'')$ of $\Gamma_{H''}$ at the module S_4. Observe that the component $\mathcal{C}$ contains the section Δ given by the modules $I_1, I_2, I_3, S_4, P_5, P_6, P_7, P_8$. Moreover, the module

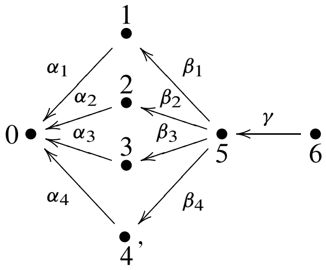
$$T_\Delta^* = I_1 \oplus I_2 \oplus I_3 \oplus S_4 \oplus P_5 \oplus P_6 \oplus P_7 \oplus P_8$$

is a faithful right B-module, because there are monomorphisms $P_1 \to I_1$, $P_2 \to I_2$, $P_3 \to I_3$, $P_4 \to I_1 \oplus I_2 \oplus I_3$, and P_5, P_6, P_7, P_8 are direct summands of T_Δ^*, and consequently there exists a monomorphism $B \to (T_\Delta^*)^2$ (see Lemma II.5.5). Moreover, we have

$$\operatorname{Hom}_B(T_\Delta^*, \tau_B T_\Delta^*) = \operatorname{Hom}_B\big(T_\Delta^*, (P_4/S_1) \oplus (P_4/S_2) \oplus (P_4/S_3) \oplus M\big) = 0.$$

Therefore, applying Theorem 7.7, we conclude that B is a tilted algebra of infinite representation type. Further, the hereditary algebra $A_\Delta = \operatorname{End}_B(T_\Delta^*)$ is the path algebra KQ of the wild quiver $Q \cong \Delta^{\mathrm{op}}$ and there is an isomorphism of K-algebras $B \cong \operatorname{End}_{A_\Delta}(T_\Delta)$ for the tilting module $T_\Delta = D(T_\Delta^*)$ in $\operatorname{mod} A_\Delta$, and $\mathcal{C}$ is the connecting component $\mathcal{C}_{T_\Delta}$ of Γ_B determined by T_Δ. Observe also that $\mathcal{C} = \mathcal{C}_{T_\Delta}$ is a postprojective component with projective and injective modules, and hence the tilting module T_Δ has both a nonzero preinjective and a nonzero postprojective direct summand, by Proposition 6.9. In fact, the section $\Delta = \Delta_{T_\Delta}$ contains both projective modules and injective modules. Finally, we note that the torsion-free class $\mathcal{Y}(T_\Delta)$ in $\operatorname{mod} B$ determined by T_Δ has only 16 pairwise nonisomorphic indecomposable modules, namely the indecomposable modules isomorphic to the predecessors of $\Delta = \Delta_T$ in $\mathcal{C}_T$.

Example 7.10. Let K be a field, Q the quiver



I the ideal in the path algebra KQ of Q over K generated by the elements $\beta_1\alpha_1 - \beta_i\alpha_i$, for $i \in \{2,3,4\}$, and $B = KQ/I$ the associated bound quiver K-algebra. We denote by S_i, P_i, and I_i the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} B$ given by the vertex $i \in \{0,1,2,3,4,5,6\}$ of Q. We know from Theorem I.2.10 that the category $\operatorname{mod} A$ is equivalent with the category $\operatorname{rep}_K(Q,I)$ of the representations of Q over K bound by I. We denote also by H' the path algebra KQ' of the full subquiver Q' of Q given by the vertices 0, 1, 2, 3, 4, and by H'' the path algebra KQ'' of the full subquiver Q'' of Q given by the vertices 1, 2, 3, 4, 5, 6. We note that H' is a hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_4$ and H'' is a hereditary algebra of wild type. Then we conclude that $I_0 = P_6$ is a projective-injective module, and hence we have in $\operatorname{mod} A$ an almost split sequence of the form

$$0 \longrightarrow \operatorname{rad} P_6 \longrightarrow (\operatorname{rad} P_6/S_0) \oplus P_6 \longrightarrow P_6/S_0 \longrightarrow 0$$

(see Proposition III.8.6) with $\operatorname{rad} P_6 = P_5$. Moreover, a direct calculation invoking Proposition III.5.3 shows that $\tau_B(P_5/S_0) = \operatorname{rad} P_5$, and hence we have in $\operatorname{mod} B$ an almost split sequence

$$0 \longrightarrow \operatorname{rad} P_5 \longrightarrow (\operatorname{rad} P_5/S_0) \oplus P_5 \longrightarrow P_5/S_0 \longrightarrow 0$$

with $\operatorname{rad} P_5/S_0 = S_1 \oplus S_2 \oplus S_3 \oplus S_4$. Then, applying Proposition III.5.3 again, we obtain that the Auslander–Reiten quiver Γ_B of B has a connected component $\mathcal{C}$ of the form

$$\begin{tikzcd}[column sep=small]
 & & & & & P_6 \arrow[dr] & & & & & \\
 & & & & P_5 \arrow[ur] \arrow[ddr] & & P_6/S_0 \arrow[ddr] & & \tau_B^{-1}(P_6/S_0) \arrow[ddr] & & \\
 & & \tau_B S_1 \arrow[dr] & & S_1 \arrow[dr] & & \tau_B^{-1} S_1 \arrow[dr] & & \tau_B^{-2} S_1 \arrow[dr] & & {} \\
\cdots \arrow[r] & \tau_B \operatorname{rad} P_5 \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \tau_B S_2 \arrow[r] & \operatorname{rad} P_5 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & S_2 \arrow[r] & P_5/S_0 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \tau_B^{-1} S_2 \arrow[r] & \tau_B^{-1}(P_5/S_0) \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \tau_B^{-2} S_2 \arrow[r] & \tau_B^{-2}(P_5/S_0) \arrow[r, dash] \arrow[ur, dash] \arrow[dr, dash] & \cdots \\
 & & \tau_B S_3 \arrow[ur] & & S_3 \arrow[ur] & & \tau_B^{-1} S_3 \arrow[ur] & & \tau_B^{-2} S_3 \arrow[ur] & & {} \\
 & & \tau_B S_4 \arrow[uur] & & S_4 \arrow[uur] & & \tau_B^{-1} S_4 \arrow[uur] & & \tau_B^{-2} S_4 \arrow[uur] & &
\end{tikzcd}$$

where $\tau_B^i S_j = \tau_{H'}^i S_j$ and $\tau_B^i \operatorname{rad} P_5 = \tau_{H'}^i \operatorname{rad} P_5$, for any $i \in \mathbb{N}$ and $j \in \{1,2,3,4\}$, $\tau_B^{-r} S_j = \tau_{H''}^{-r} S_j$, $\tau_B^{-r}(P_5/S_0) = \tau_{H''}^{-r}(P_5/S_0)$, and $\tau_B^{-r}(P_6/S_0) = \tau_{H''}^{-r}(P_6/S_0)$, for any $r \in \mathbb{N}$ and $j \in \{1,2,3,4\}$, We note also that $\operatorname{rad} P_5$ is the injective envelope of the simple module S_0 in $\operatorname{mod} H'$, and P_5/S_0 and P_6/S_0 are respectively the projective covers fo the simple modules S_5 and S_6 in $\operatorname{mod} H''$. Therefore, $\mathcal{C}$ is the gluing of the preinjective component $\mathcal{Q}(H')$ of $\Gamma_{H'}$ and the postprojective component $\mathcal{P}(H'')$ of $\Gamma_{H''}$ via the modules $S_1, S_2, S_3, S_4, P_5, P_6$.

Observe that the component $\mathcal{C}$ contains the section Δ of the form

$$\begin{tikzcd}
 & P_6 \\
P_5 \arrow[ur] \arrow[ddr] & \\
S_1 \arrow[dr] & \\
S_2 \arrow[r] & P_5/S_0 \\
S_3 \arrow[ur] & \\
S_4 \,. \arrow[uur] &
\end{tikzcd}$$

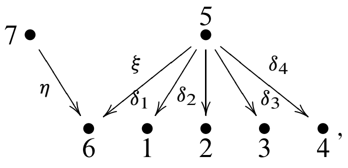
Moreover, Δ is the faithful section, because every indecomposable projective right B-module $P_0, P_1, P_2, P_3, P_4, P_5, P_6$ is a submodule of the projective-injective module P_6. Let

$$T_{\Delta}^{*} = S_1 \oplus S_2 \oplus S_3 \oplus S_4 \oplus (P_5/S_0) \oplus P_5 \oplus P_6$$

be the direct sum of all modules lying on the section Δ. Then we have

$$\operatorname{Hom}_B(T_{\Delta}^{*}, \tau_B T_{\Delta}^{*}) = \operatorname{Hom}_B\big(T_{\Delta}^{*}, \tau_B S_1 \oplus \tau_B S_2 \oplus \tau_B S_3 \oplus \tau_B S_4 \oplus \operatorname{rad} P_5\big) = 0,$$

because $\tau_B S_1, \tau_B S_2, \tau_B S_3, \tau_B S_4, \operatorname{rad} P_5$ are modules from the preinjective component $\mathcal{Q}(H')$ of $\Gamma_{H'}$ and all modules in $\mathcal{Q}(H')$ have no composition factors isomorphic to S_5, S_6, so we have $\operatorname{Hom}_B(P_5 \oplus P_6, \mathcal{Q}(H')) = 0$, by Lemma VII.5.6. Therefore, applying Theorem 7.7, we conclude that B is a tilted algebra isomorphic to $\operatorname{End}_{A_\Delta}(T_\Delta)$, where $A_\Delta = \operatorname{End}_B(T_{\Delta}^{*})$ is the path algebra $K\Omega$ of the wild quiver Ω of the form



where the vertices $1, 2, 3, 4, 5, 6, 7$ of Ω correspond to the summands $S_1, S_2, S_3, S_4, P_5/S_0, P_5, P_6$ of T_{Δ}^{*}, and $T_\Delta = D(T_{\Delta}^{*})$ is the associated tilting module in $\operatorname{mod} A_\Delta$. Moreover, Δ is the section Δ_T given by the images of the indecomposable injective modules in the preinjective component $\mathcal{Q}(A_\Delta)$ of Γ_{A_Δ} under the functor $\operatorname{Hom}_{A_\Delta}(T_\Delta, -)\colon \operatorname{mod} A_\Delta \to \operatorname{mod} B$. We know also that the torsion pair $(\mathcal{X}(T_\Delta), \mathcal{Y}(T_\Delta))$ in $\operatorname{mod} B$ induced by the tilting module T_Δ is splitting, the torsion-free part $\mathcal{Y}(T_\Delta)$ is closed under predecessors in $\operatorname{mod} B$, and the torsion part $\mathcal{X}(T_\Delta)$ is closed under successors in $\operatorname{mod} B$. Moreover, $\mathcal{Y}(T_\Delta) \cap \mathcal{C} =$

$\mathcal{Y}(T_\Delta) \cap \mathcal{C}_{T_\Delta}$ consists of the predecessors of $\Delta = \Delta_{T_\Delta}$ in $\mathcal{C}$, while $\mathcal{X}(T_\Delta) \cap \mathcal{C} = \mathcal{X}(T_\Delta) \cap \mathcal{C}_{T_\Delta}$ consists of the proper successors of $\Delta = \Delta_{T_\Delta}$ in $\mathcal{C}$. Hence, we obtain that the torsion-free part $\mathcal{Y}(T_\Delta) \cap \mathcal{C}$ of $\mathcal{C}$ consists of all indecomposable modules of the preinjective component $\mathcal{Q}(H')$ of $\Gamma_{H'}$ and the three additional modules P_5/S_0, P_5, P_6. On the other hand, the torsion part $\mathcal{X}(T_\Delta) \cap \mathcal{C}$ of $\mathcal{C}$ consists of all the indecomposable modules in the postprojective component $\mathcal{P}(H'')$ of $\Gamma_{H''}$ except the simple modules S_1, S_2, S_3, S_4 and the projective right H''-module P_5/S_0. It follows from Theorems VII.6.1, VII.6.2, VII.8.12, and VII.9.3 that the Auslander–Reiten quiver Γ_B of the tilted algebra B has the disjoint union form

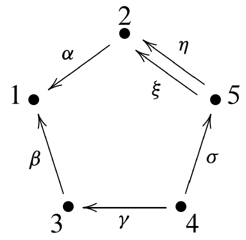
$$\Gamma_B = \mathcal{P}(H') \cup \mathcal{T}^{H'} \cup \mathcal{C} \cup \mathcal{R}(H'') \cup \mathcal{Q}(H''),$$

where:

- $\mathcal{P}(H')$ is the postprojective component of $\Gamma_{H'}$ of the form $(-\mathbb{N})(Q')^{\mathrm{op}}$, with Q' being a Euclidean quiver of type $\widetilde{\mathbb{D}}_4$;
- $\mathcal{T}^{H'}$ is the separating family $(\mathcal{T}^{H'}_\lambda)_{\lambda\in\Lambda(H')}$ of stable tubes of $\Gamma_{H'}$;
- $\mathcal{C} = \mathcal{C}_{T_\Delta}$ is the connecting component, determined by the tilting module T_Δ in $\operatorname{mod} A_\Delta$, described above, having infinite torsion-free and torsion parts;
- $\mathcal{R}(H'')$ is the family of components of $\Gamma_{H''}$ of the form $\mathbb{Z}\mathbb{A}_\infty$;
- $\mathcal{Q}(H'')$ is the preinjective component of $\Gamma_{H''}$ of the form $\mathbb{N}(Q'')^{\mathrm{op}}$, with Q'' being a wild quiver.

We end this section with an example showing the necessity of the assumption imposed in Theorem 7.7.

Example 7.11. Let K be a field, Q the quiver



I the ideal in the path algebra KQ of Q over K generated by the paths $\eta\alpha$, $\xi\alpha$, $\gamma\beta$, $\sigma\eta$, $\sigma\xi$, and $B = KQ/I$ the associated bound quiver K-algebra. Then B is an 11-dimensional K-algebra with $(\operatorname{rad} B)^2 = 0$ and, using the equivalence of categories $\operatorname{mod} B \cong \operatorname{rep}_K(Q, I)$ (see Theorem I.2.10), we conclude that the Auslander–Reiten quiver Γ_B of B has the disjoint union form

$$\Gamma_B = \mathcal{C} \cup \left(\bigcup_{\lambda\in\Lambda(H)} \mathcal{T}^H_\lambda \right),$$

where H is the path algebra $K\Sigma$ of the Kronecker subquiver Σ of Q given by the vertices 2, 5 and the arrows ξ, η, $\Lambda(H) = \operatorname{irr}(K[X]) \cup \{\infty\}$ (see Example VII.8.27), and $\mathcal{T}_\lambda^H$, $\lambda \in \Lambda(H)$, is the family of all stable tubes of rank 1 in Γ_H, and $\mathcal{C}$ is the following acyclic component

$$\begin{tikzcd}
 & \tau_H I_2 \arrow[dr, "{(2,2)}"] & & I_2 \arrow[dr, "{(2,2)}"] & & & & & \\
\cdots \arrow[ur, "{(2,2)}"] & & \tau_H S_5 \arrow[ur, "{(2,2)}"] & & S_5 \arrow[dr] & & I_3 \arrow[dr] & & \\
 & & & & & P_4 \arrow[ur] \arrow[dr] & & S_4 = I_4 & \\
 & & P_2 \arrow[dr] & & S_3 \arrow[ur] & & I_5 \arrow[ur] & & \\
 & S_1 = P_1 \arrow[ur] \arrow[dr] & & I_1 \arrow[ur] \arrow[dr] & & & & & \\
 & & P_3 \arrow[ur] & & S_2 \arrow[dr, "{(2,2)}"] & & \tau_H^{-1} S_2 \arrow[dr, "{(2,2)}"] & & \cdots \\
 & & & & & P_5 \arrow[ur, "{(2,2)}"] & & \tau_H^{-1} P_5 \arrow[ur, "{(2,2)}"] &
\end{tikzcd}$$

where S_i, P_i, and I_i denotes the simple module, the indecomposable projective module, and the indecomposable injective module in $\operatorname{mod} B$ given by the vertex $i \in \{1, 2, 3, 4, 5\}$ of Q. Observe that the postprojective component $\mathcal{P}(H)$ of Γ_H is a full translation subquiver of $\mathcal{C}$ closed under successors while the preinjective component $\mathcal{Q}(H)$ of Γ_H is a full translation subquiver of $\mathcal{C}$ closed under predecessors. Moreover, $\mathcal{C}$ admits a unique section Δ of the from

$$\begin{tikzcd}
I_2 \arrow[dr, "{(2,2)}"] & & \\
 & S_5 \arrow[dr] & \\
 & & P_4 \\
 & S_3 \arrow[ur] & \\
I_1 \arrow[ur] \arrow[dr] & & \\
 & S_2 \arrow[dr, "{(2,2)}"] & \\
 & & P_5
\end{tikzcd}$$

and it is a faithful section of $\mathcal{C}$, because Δ contains P_4, P_5, and P_1, P_2, P_3 are submodules of I_1 lying on Δ. On the other hand, B is not a tilted algebra, because there are nonzero homomorphisms from the proper successors $\tau_H^{-m} S_2, \tau_H^{-m} P_5$, $m \geq 1$, of Δ in $\mathcal{C}$ to the injective module I_2 lying on Δ. We also note that $\operatorname{Hom}_B(P_5, \tau_B S_5) = \operatorname{Hom}_H(P_5, \tau_H S_5) \neq 0$ for P_5 and S_5 lying on Δ.

8 Reflections of hereditary algebras

In this section we introduce reflections of finite dimensional hereditary algebras which allow to reduce the study of their module categories to that for finite dimensional hereditary algebras whose quivers have suitable orientations of arrows.

Let A be a finite dimensional K-algebra over a field K and $e_1, \dots, e_n$ a complete set of pairwise orthogonal basic primitive idempotents of A. Then

- $P_i = e_i A$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} A$;
- $I_i = D(Ae_i)$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic indecomposable injective modules in $\operatorname{mod} A$;
- $S_i = \operatorname{top}(P_i) = e_i A / e_i \operatorname{rad} A$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$;
- $S_i \cong \operatorname{soc}(I_i)$, for any $i \in \{1, \dots, n\}$.

Moreover, for each $i \in \{1, \dots, n\}$, we have the finite dimensional division K-algebra $F_i = \operatorname{End}_A(S_i)$. Recall that the quiver Q_A of A is the valued quiver whose vertices are $1, \dots, n$, and, for two vertices i and j in Q_A, there is an arrow from i to j in Q_A if and only if $e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j \neq 0$. Moreover, if there is an arrow from i to j in Q_A, then we have in Q_A the valued arrow

$$i \xrightarrow{\left(d_{ij}, d'_{ij}\right)} j,$$

where

$$\begin{aligned} d_{ij} &= \dim_{F_j} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j, \\ d'_{ij} &= \dim_{F_i} e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j. \end{aligned}$$

Instead of an arrow $i \xrightarrow{(1,1)} j$ of Q_A we write simply $i \to j$.

Let Δ be a finite valued quiver without loops and multiple arrows, Δ_0 the set of vertices of Δ, Δ_1 the set of arrows of Δ, and $d, d' \colon \Delta_1 \to \mathbb{N}_1$ the valuation maps. Let i be a vertex in Δ_0. We define the valued quiver $\sigma_i \Delta$, called the *reflection of* Δ *at the vertex* i, as follows. The set $(\sigma_i \Delta)_0$ of vertices of $\sigma_i \Delta$ is the set of vertices Δ_0 of Δ. The set $(\sigma_i \Delta)_1$ of arrows of $\sigma_i \Delta$ is obtained from the set Δ_1 of arrows of Δ by reversing the orientations of all arrows in Δ_1 connected to i. Moreover, the valuation functions $e, e' \colon (\sigma_i \Delta)_1 \to \mathbb{N}_1$ are defined such that

- for each valued arrow $i \xrightarrow{\left(d_{ij}, d'_{ij}\right)} j$ in Δ, we have in $\sigma_i \Delta$ the valued arrow $j \xrightarrow{\left(d'_{ij}, d_{ij}\right)} i$,
- for each valued arrow $j \xrightarrow{\left(d_{ji}, d'_{ji}\right)} i$ in Δ, we have in $\sigma_i \Delta$ the valued arrow $i \xrightarrow{\left(d'_{ji}, d_{ji}\right)} j$,

- for each valued arrow $r \xrightarrow{(d_{rs}, d'_{rs})} s$ in Δ with r and s different from i, we have in $\sigma_i \Delta$ the valued arrow $r \xrightarrow{(d_{rs}, d'_{rs})} s$.

Observe that $\sigma_i(\sigma_i \Delta) = \Delta$. We also note that Δ is connected if and only if $\sigma_i \Delta$ is connected.

Let A be a finite dimensional K-algebra over a field K. For an indecomposable module X in mod A, we denote by $\operatorname{mod}_X A$ the full subcategory of mod A formed by all modules without direct summand isomorphic to X.

Let A be a nonsimple indecomposable finite dimensional hereditary K-algebra over a field K. Then it follows from Proposition VII.1.5 and Corollary VII.1.7 that Q_A is a finite connected acyclic valued quiver with at least two vertices, and clearly without multiple arrows. In particular, the quiver Q_A has at least one sink and at least one source. Let i be a sink in Q_A. Consider the set $Q_A(i)$ of all vertices j in Q_A different from i, and the module in mod A

$$T_i = \tau_A^{-1} S_i \oplus \left(\bigoplus_{j \in Q_A(i)} P_j \right).$$

Then the endomorphism algebra $\Sigma_i A = \operatorname{End}_A(T_i)$ is said to be the *reflection of A at the sink i*, and the associated functor

$$\Phi_i = \operatorname{Hom}_A(T_i, -) \colon \operatorname{mod} A \longrightarrow \operatorname{mod} \Sigma_i A$$

the *reflection functor of* mod A *at the sink i*.

Theorem 8.1. *Let A be a nonsimple indecomposable finite dimensional hereditary K-algebra over a field K, and i be a sink in the quiver Q_A of A. Then the following statements hold:*

(i) *$\Sigma_i A$ is a nonsimple indecomposable finite dimensional hereditary K-algebra over K.*

(ii) $Q_{\Sigma_i A} = \sigma_i Q_A$.

(iii) *The modules $P'_i = \Phi_i(\tau_A^{-1} S_i)$ and $P'_j = \Phi_i(P_j)$, for $j \in Q_A(i)$, form a complete set of pairwise nonisomorphic indecomposable projective modules in* mod $\Sigma_i A$.

(iv) *The modules $I'_i = \operatorname{Ext}^1_A(T_i, S_i)$ and $I'_j = \Phi_i(I_j)$, for $j \in Q_A(i)$, form a complete set of pairwise nonisomorphic indecomposable injective modules in* mod $\Sigma_i A$.

(v) *The modules $S'_i = I'_i$ and $S'_j = \Phi_i(S_j)$, for $j \in Q_A(i)$, form a complete set of pairwise nonisomorphic simple modules in* mod $\Sigma_i A$.

(vi) *The reflection functor $\Phi_i \colon \operatorname{mod} A \to \operatorname{mod} \Sigma_i A$ induces an equivalence of categories*

$$\Phi_i \colon \operatorname{mod}_{S_i} A \xrightarrow{\sim} \operatorname{mod}_{S'_i} \Sigma_i A.$$

Proof. It follows from Example 2.11 that T_i is a tilting module in $\operatorname{mod} A$, and hence $\Sigma_i A = \operatorname{End}_A(T_i)$ is the associated tilted algebra. Further, for the associated torsion pair $(\mathcal{T}(T_i), \mathcal{F}(T_i))$ in $\operatorname{mod} A$, we have

$$\begin{aligned}\mathcal{T}(T_i) &= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Ext}^1_A(T_i, M) = 0\big\} = \operatorname{mod}_{S_i} A,\\ \mathcal{F}(T) &= \big\{M \in \operatorname{mod} A \,\big|\, \operatorname{Hom}_A(T_i, M) = 0\big\} = \operatorname{add}(S_i).\end{aligned}$$

Clearly, $(\mathcal{T}(T_i), \mathcal{F}(T_i))$ is a splitting torsion pair in $\operatorname{mod} A$. Moreover, we have $\mathcal{T}(T_i) = \operatorname{Gen}(T_i)$ and $\mathcal{F}(T_i) = \operatorname{Cogen} \tau_A T_i = \operatorname{Cogen} S_i$ (see Theorem 2.5). Consider also the induced torsion pair $(\mathcal{X}(T_i), \mathcal{Y}(T_i))$ in $\operatorname{mod} \Sigma_i A$, where

$$\begin{aligned}\mathcal{X}(T_i) &= \big\{X \in \operatorname{mod} \Sigma_i A \,\big|\, X \otimes_{\Sigma_i A} T_i = 0\big\},\\ \mathcal{Y}(T_i) &= \big\{Y \in \operatorname{mod} \Sigma_i A \,\big|\, \operatorname{Tor}_1^{\Sigma_i A}(Y, T_i) = 0\big\}.\end{aligned}$$

Then it follows from the Brenner–Bulter Theorem 3.8 that

(i) the functors $\operatorname{Hom}_A(T_i, -)\colon \operatorname{mod} A \to \operatorname{mod} \Sigma_i A$ and $-\otimes_{\Sigma_i A} T_i\colon \operatorname{mod} \Sigma_i A \to \operatorname{mod} A$ induce an equivalence of categories $\mathcal{T}(T_i) \xrightarrow{\sim} \mathcal{Y}(T_i)$, and

(ii) the functors $\operatorname{Ext}^1_A(T_i, -)\colon \operatorname{mod} A \to \operatorname{mod} \Sigma_i A$ and $\operatorname{Tor}_1^{\Sigma_i A}(-, T_i)\colon \operatorname{mod} \Sigma_i A \to \operatorname{mod} A$ induce an equivalence of categories $\mathcal{F}(T_i) \xrightarrow{\sim} \mathcal{X}(T_i)$.

Moreover, since A is a hereditary algebra, applying Theorem 5.8, we conclude that $(\mathcal{X}(T_i), \mathcal{Y}(T_i))$ is a splitting torsion pair in $\operatorname{mod} \Sigma_i A$. Therefore, we obtain that

$$\begin{aligned}\mathcal{X}(T_i) &= \operatorname{add}\big(\operatorname{Ext}^1_A(T_i, S_i)\big) = \operatorname{add}(S'_i),\\ \mathcal{Y}(T_i) &= \operatorname{mod}_{S'_i} \Sigma_i A.\end{aligned}$$

In particular, the reflection functor $\Phi_i = \operatorname{Hom}_A(T_i, -)\colon \operatorname{mod} A \to \operatorname{mod} \Sigma_i A$ induces an equivalence of categories $\Phi_i\colon \operatorname{mod}_{S_i} A \to \operatorname{mod}_{S'_i} \Sigma_i A$. Further, it follows from Theorem 4.3 and Proposition 5.2 that any almost split sequence in $\operatorname{mod} \Sigma_i A$ lies entirely in $\mathcal{Y}(T_i) = \operatorname{mod}_{S'_i} \Sigma_i A$ or is the unique connecting sequence of the form

$$0 \longrightarrow \operatorname{Hom}_A(T_i, I_i) \longrightarrow \operatorname{Hom}_A(T_i, I_i/S_i) \longrightarrow \operatorname{Ext}^1_A(T, S_i) \longrightarrow 0.$$

In particular, we know that $\operatorname{Hom}_A(T_i, I_i/S_i)$ is injective in $\operatorname{mod} \Sigma_i A$. In fact, if follows from Proposition 5.4, that the modules $I'_i = \operatorname{Ext}^1_A(T, S_i)$ and $I'_j = \operatorname{Hom}_A(T, I_j) = \Phi_i(I_j)$, for $j \in Q_A(i)$, form a complete set of pairwise nonisomorphic indecomposable injective modules in $\operatorname{mod} \Sigma_i A$. Moreover, I'_i is a sink in the Auslander–Reiten quiver $\Gamma_{\Sigma_i A}$ of $\Sigma_i A$, and hence $I'_i = S'_i$ is a simple injective module (see Lemma VII.1.13). Further, by Lemma 3.1, the modules

$P'_i = \Phi_i(\tau_A^{-1} S_i)$ and $P'_j = \Phi_i(P_j)$, for $j \in Q_A(i)$, form a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod} \Sigma_i A$. We also note that, for $j \in Q_A(i)$, we have the following isomorphisms in $\operatorname{mod} \Sigma_i A$

$$\Phi_i(S_j) = \Phi_i\big(\operatorname{top}(P_j)\big) = \operatorname{top}\big(\Phi_i(P_j)\big) = \operatorname{top}(P'_j),$$

because Φ_i induces an equivalence of categories $\operatorname{mod}_{S_i} A \xrightarrow{\sim} \operatorname{mod}_{S'_i} \Sigma_i A$. In particular, we conclude that $S'_i = \operatorname{Ext}^1_A(T_i, S_i)$ and $S'_j = \Phi_i(S_j)$, for $j \in Q_A(i)$, form a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} \Sigma_i A$. Further, we observe that the Auslander–Reiten quiver $\Gamma_{\Sigma_i A}$ of $\Sigma_i A$ contains a full valued subquiver Δ'_i given by the indecomposable projective modules P'_i and P'_j, $j \in Q_A(i)$, which is isomorphic to the full valued subquiver Δ_i of the the Auslander–Reiten quiver Γ_A of A given by the modules $\tau_A^{-1} S_i$ and P'_j, $j \in Q_A(i)$, and Δ'_i is closed under predecessors in $\Gamma_{\Sigma_i A}$. This implies that $\Sigma_i A$ is a nonsimple indecomposable finite dimensional hereditary K-algebra, and the quiver $Q_{\Sigma_i A}$ of $\Sigma_i A$ is the quiver $(\Delta'_i)^{\mathrm{op}} = \Delta_i^{\mathrm{op}}$. Finally, we note that we have in Γ_A a mesh of the form

$$\begin{tikzcd}
 & P_{j_1} \arrow[ddr, "{(d'_{P_i P_{j_1}}, d_{P_i P_{j_1}})}"] & \\
 & P_{j_2} \arrow[dr, "{(d'_{P_i P_{j_2}}, d_{P_i P_{j_2}})}"'] & \\
S_i = P_i \arrow[uur, "{(d_{P_i P_{j_1}}, d'_{P_i P_{j_1}})}"] \arrow[ur, "{(d_{P_i P_{j_2}}, d'_{P_i P_{j_2}})}"'] \arrow[dr, "{(d_{P_i P_{j_r}}, d'_{P_i P_{j_r}})}"'] & \vdots & \tau_A^{-1} S_i \\
 & P_{j_r} \arrow[ur, "{(d'_{P_i P_{j_r}}, d_{P_i P_{j_r}})}"'] &
\end{tikzcd}$$

which corresponds to the full valued subquiver of Q_A

$$\begin{tikzcd}
j_1 \arrow[ddr, "{(d_{j_1 i}, d'_{j_1 i})}"] & \\
j_2 \arrow[dr, "{(d_{j_2 i}, d'_{j_2 i})}"'] & \\
\vdots & i \\
j_t \arrow[ur, "{(d_{j_t i}, d'_{j_t i})}"'] &
\end{tikzcd}$$

with $d_{j_k i} = d'_{P_i P_{j_k}}$ and $d'_{j_k i} = d_{P_i P_{j_k}}$, for $k \in \{2, \dots, r\}$, given by all arrows in Q_A connected to i (see Proposition VII.1.10 and Theorem VII.6.1). This shows that the quiver $\Gamma_{\Sigma_i A}$ of $\Sigma_i A$ is the reflection $\sigma_i Q_A$ of the quiver Q_A of A at the sink i. □

We obtain the following direct consequence of Theorem 8.1.

Corollary 8.2. *Let A be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, and i be a sink of the quiver Q_A of A. Then the reflection functor $\Phi_i\colon \operatorname{mod} A \to \operatorname{mod} \Sigma_i A$ induces an equivalence of categories of regular modules*

$$\Phi_i\colon \operatorname{add} \mathcal{R}(A) \xrightarrow{\sim} \operatorname{add} \mathcal{R}\big(\Sigma_i A\big).$$

In the remaining part of this section we present some technical results concerning reflections, which will be applied in the next section.

Let Δ be a finite acyclic valued quiver without multiple arrows. An *admissible sequence of sinks* in Δ is a sequence $i_1, \dots, i_r$ of vertices in Δ such that

- i_1 is a sink of Δ;
- i_k is a sink of $\sigma_{i_{k-1}} \dots \sigma_{i_1} Q_A$ for every $k \in \{2, \dots, r\}$.

Lemma 8.3. *Let A be a basic nonsimple indecomposable finite dimensional hereditary K-algebra over a field K, and ω a vertex in Q_A which is not a source. Then there is an admissible sequence of sinks $i_1, \dots, i_r$ in Q_A such that ω is a source of the valued quiver $\sigma_{i_r} \dots \sigma_{i_1} Q_A$.*

Proof. Since ω is not a source in Q_A, there is a path in Q_A from ω to a sink, because Q_A is an acyclic quiver. We denote by $p_A(\omega)$ the length of the longest path in Q_A from ω to a sink in Q_A. Let $j_1, \dots, j_t$ be the set of all sinks in Q_A which are targets of paths in Q_A of length $p_A(\omega)$ with source ω. Clearly, $j_1, \dots, j_t$ form an admissible sequence of sinks in Q_A. Then it follows from Theorem 8.1 that $B = \Sigma_{j_t} \dots \Sigma_{j_1} A$ is a nonsimple indecomposable finite dimensional hereditary K-algebra over K with $Q_B = \sigma_{j_t} \dots \sigma_{j_1} Q_A$ and $p_B(\omega) = p_A(\omega) - 1$. Hence the claim follows by induction on $p_A(\omega)$. □

Proposition 8.4. *Let A be a basic nonsimple indecomposable finite dimensional hereditary K-algebra over a field K. Then there is an admissible sequence of sinks $i_1, \dots, i_r$ in Q_A (possibly empty) such that the associated hereditary algebra $B = \Sigma_{i_r} \dots \Sigma_{i_1} A$ is a one-point extension algebra*

$$B = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix},$$

where C is a basic indecomposable finite dimensional hereditary K-algebra, F is a finite dimensional division K-algebra, and M an (F, C)-bimodule, being a projective right C-module.

Proof. Consider the graph G_A obtained from the underlying valued graph $\bar{Q}_A$ of Q_A by removing the valuations of all edges. Since A is an indecomposable algebra, the graph G_A is connected. Applying the depth-first search algorithm to the graph G_A (see [CLRS, Section VI.22.3] for more details), we conclude that there is a vertex ω in G_A such that the graph G_A^ω obtained from G_A by removing ω and all edges in G_A connected to ω is a connected graph. More precisely, take a vertex u of G_A and construct a depth-first tree $T_u(G_A)$ rooted at u. Since G_A is a connected graph, the set of vertices of $T_u(G_A)$ contains the set of all vertices of G_A. Take as ω a vertex of degree one in the tree $T_u(G_A)$. Then it follows that ω is a requested vertex in G_A. In particular, we obtain that ω is a vertex in Q_A such that the full subquiver Q_A^ω obtained from Q_A by removing ω and all valued arrows of Q_A attached to ω is a connected valued quiver. If ω is a source of Q_A, we take $B = A$. Assume that ω is not a source of Q_A. Then, applying Lemma 8.3, we conclude that there is an admissible sequence of sinks $i_1, \dots, i_r$ in Q_A such that ω is a source of the valued quiver $\sigma_{i_r} \dots \sigma_{i_1} Q_A$, and we take $B = \Sigma_{i_r} \dots \Sigma_{i_1} Q_A$. Let e_ω be the primitive idempotent of B associated to the source ω of Q_B. Then $F = e_\omega B e_\omega$ is a finite dimensional division K-algebra, $C = (1_B - e_\omega)B(1 - e_\omega)$ is a finite dimensional hereditary K-algebra, and B is a one-point extension algebra

$$B = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix},$$

where $M = e_\omega B(1 - e_\omega)$ is an (F, C)-bimodule, being a right projective C-module (see Example VII.10.6). Moreover, it follows from the choice of source ω in Q_B that C is an indecomposable algebra. □

9 The theorem of Ringel on regular tilting modules

The aim of this section is to provide a proof of an important theorem by C. M. Ringel on the existence of regular tilting modules in the module categories of finite dimensional hereditary algebras of wild type over a field, and derive some consequences.

Proposition 9.1. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K, and T be a regular tilting module in* $\operatorname{mod} A$. *Then $K_0(A)$ is of rank at least three.*

Proof. Assume to the contrary that $K_0(A)$ is of rank two. We may assume that T is a multiplicity-free tilting module, and hence $T = X \oplus Y$ for two nonisomorphic indecomposable modules in $\mathcal{R}(A)$ (see Proposition 3.11). Consider the associated tilted algebra $B = \operatorname{End}_A(T)$. Then it follows from Theorem 3.10 and Corollary 3.20 that B is an indecomposable finite dimensional K-algebra with $K_0(B)$ of rank two. Moreover, B is a basic algebra, by the assumption imposed on T (see Proposition 3.12). Let $F = \operatorname{End}_A(X)$ and $G = \operatorname{End}_A(Y)$. Since $\operatorname{Ext}_A^1(M, M) =$

0 and $\operatorname{Ext}_A^1(N,N) = 0$, applying Corollary VII.9.16, we conclude that F and G are finite dimensional division K-algebras. Further, by Proposition 6.2, the quiver Q_B of B is acyclic. Hence, either $\operatorname{Hom}_A(X,Y) = 0$, or $\operatorname{Hom}_A(Y,X) = 0$. We may assume (without loss of generality) that $\operatorname{Hom}_A(X,Y) = 0$. Then $M = \operatorname{Hom}_A(Y,X)$ is a nonzero (F,G)-bimodule on which K acts centrally. Summing up, we conclude that B is isomorphic to the matrix algebra

$$\begin{bmatrix} F & M \\ 0 & G \end{bmatrix},$$

which is clearly a hereditary K-algebra. Therefore, B is a hereditary K-algebra.

It follows from Proposition 3.3 that T is a tilting module in $\operatorname{mod} B^{\mathrm{op}}$ and there is a canonical isomorphism of K-algebras $\varrho\colon A \to \operatorname{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$ such that $\varrho(a)(t) = ta$ for $a \in A$ and $t \in T$. Since B is a hereditary algebra, applying Proposition 3.11, we obtain that $T^* = D(T)$ is a tilting module in $\operatorname{mod} B$. Moreover, there is a canonical isomorphism of K-algebras $\varphi\colon A \to \operatorname{End}_B(T^*)$ such that $(\varphi(a)(f))(t) = f(ta)$ for $a \in A$, $t \in T$, and $f \in T^* = \operatorname{Hom}_K(T,K)$. Consider now the torsion pair $(\mathcal{X}(T^*), \mathcal{Y}(T^*))$ in $\operatorname{mod} A$ induced by T^*. Recall that $\mathcal{X}(T^*)$ is the torsion class in $\operatorname{mod} A$ of the form

$$\mathcal{X}(T^*) = \bigl\{U \in \operatorname{mod} A \bigm| U \otimes_A T^* = 0\bigr\}$$

and $\mathcal{Y}(T^*)$ is the torsion-free class in $\operatorname{mod} A$ of the form

$$\mathcal{Y}(T^*) = \bigl\{V \in \operatorname{mod} A \bigm| \operatorname{Tor}_1^A(V,T^*) = 0\bigr\}.$$

Moreover, since B is a hereditary algebra, it follows from Theorem 5.8 that the torsion pair $(\mathcal{X}(T^*), \mathcal{Y}(T^*))$ is splitting. On the other hand, by Lemma 3.6, we have the equalities of full subcategories in $\operatorname{mod} A$

$$\mathcal{X}(T^*) = D\bigl(\mathcal{F}({}_AT^*)\bigr) \qquad \text{and} \qquad \mathcal{Y}(T^*) = D\bigl(\mathcal{T}({}_AT^*)\bigr).$$

Then we conclude that

$$\begin{aligned} \mathcal{X}(T^*) &= \bigl\{U \in \operatorname{mod} A \bigm| \operatorname{Hom}_A(U,T) = 0\bigr\}, \\ \mathcal{Y}(T^*) &= \bigl\{V \in \operatorname{mod} A \bigm| \operatorname{Ext}_A^1(U,T) = 0\bigr\}. \end{aligned}$$

We also note that the torsion class $\mathcal{X}(T^*)$ is closed under successors in $\operatorname{mod} A$, and the torsion-free class $\mathcal{Y}(T^*)$ is closed under predecessors in $\operatorname{mod} A$ (see Lemma 5.1).

Take now the indecomposable direct summand X of T. Then X belongs to $\mathcal{Y}(T^*)$, because $\operatorname{Ext}_A^1(X,T) = 0$. On the other hand, $\operatorname{Ext}_A^1(T,X) = 0$ implies that $\operatorname{Hom}_A(\tau_A^{-1}X, T) = 0$, and hence $\tau_A^{-1}X \in \mathcal{X}(T^*)$. Further, it follows from Theorem VII.9.13 that there exists a positive integer m such that

$\mathrm{Hom}_A(\tau_A^{-1}X, \tau_A^m X) \neq 0$. Observe also that there is a path of irreducible homomorphisms in $\operatorname{mod} A$ from $\tau_A^m X$ to X. This shows that X is a successor of $\tau_A^{-1}X$ in $\operatorname{mod} A$, and consequently X belongs to $\mathcal{X}(T^*)$. This is a contradiction, because X belongs also to $\mathcal{Y}(T^*)$.

Therefore, the Grothendieck group $K_0(A)$ of A is of rank at least three. □

Proposition 9.2. *Let A be a finite dimensional hereditary K-algebra of Euclidean type over a field K. Then* $\operatorname{mod} A$ *does not contain a regular tilting module.*

Proof. Suppose that there exists a regular tilting module T in $\operatorname{mod} A$. Let $T_1, \dots, T_n$ be a complete set of pairwise nonisomorphic indecomposable direct summands of T. Since A is of global dimension one, it follows from Proposition 3.21 that $[T_1], \dots, [T_n]$ form a $\mathbb{Z}$-basis of the Grothendieck group $K_0(A)$ of A. Consider the defect homomorphism

$$\partial_A \colon K_0(A) \longrightarrow \mathbb{Z},$$

defined in Section VII.8. It follows from Proposition VII.8.5 that the kernel $\operatorname{Ker} \partial_A$ of ∂_A is a proper subgroup of $K_0(A)$ containing the classes $[T_1], \dots, [T_n]$, because $T_1, \dots, T_n$ are regular modules. This contradicts the fact that the classes $[T_1], \dots, [T_n]$ generate the group $K_0(A)$. Therefore, $\operatorname{mod} A$ does not contain a regular tilting module. □

We will apply the following variant of the Bongartz lemma (Lemma 2.4).

Lemma 9.3. *Let C be a finite dimensional K-algebra over a field K, M a nonzero projective module in* $\operatorname{mod} C$*, and F be a division K-subalgebra of* $\mathrm{End}_C(M)$*. Moreover, let*

$$A = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix}$$

be the associated one-point extension algebra, S the unique simple module in $\operatorname{mod} A$ *which is not a right C-module, and T be a tilting module in* $\operatorname{mod} C$*. Then there exists an indecomposable module X in* $\operatorname{mod} A$ *such that the following statements hold:*

(i) *$T \oplus X$ is a tilting module in* $\operatorname{mod} A$.

(ii) *There is an exact sequence in* $\operatorname{mod} A$

$$0 \longrightarrow T' \longrightarrow X \longrightarrow S \longrightarrow 0,$$

where T' is a nonzero module from $\operatorname{add} T$.

Proof. Let P be the indecomposable projective module in $\operatorname{mod} A$ with $\operatorname{top}(P) = S$. Then we have the canonical exact sequence in $\operatorname{mod} A$ of the form

$$0 \longrightarrow M \longrightarrow P \longrightarrow S \longrightarrow 0.$$

Applying Theorem VII.3.3 we obtain an exact sequence in mod K

$$\operatorname{Hom}_A(M,T) \longrightarrow \operatorname{Ext}_A^1(S,T) \longrightarrow \operatorname{Ext}_A^1(P,T) = 0.$$

Moreover, $\operatorname{Hom}_A(M,T) = \operatorname{Hom}_C(M,T) \neq 0$, because M is projective in C and T is a faithful right C-module, by (T3) and Lemma II.5.5. Hence, $\operatorname{Ext}_A^1(S,T) \neq 0$, and consequently $\mathcal{E}\mathrm{xt}_A^1(S,T) \neq 0$, by Corollary III.3.6. Let $d = \dim_K \mathcal{E}\mathrm{xt}_A^1(S,T)$ and

$$\mathbb{E}_i\colon \quad 0 \longrightarrow T \longrightarrow E_i \longrightarrow S \longrightarrow 0,$$

$i \in \{1,\dots,d\}$, be exact sequences in mod A such that $[\mathbb{E}_1],\dots,[\mathbb{E}_d]$ form a K-basis of $\mathcal{E}\mathrm{xt}_A^1(S,T)$. Consider the commutative diagram in mod A with exact rows

$$\begin{tikzcd}
0 \arrow[r] & T^d \arrow[r, "f"] \arrow[d, "\mathrm{id}_{T^d}"] & E \arrow[r, "g"] \arrow[d, "u"] & S \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & T^d \arrow[r, "v"] & \bigoplus_{i=1}^{d} E_i \arrow[r, "w"] & S^d \arrow[r] & 0,
\end{tikzcd}$$

where the lower exact sequence is the direct sum $\mathbb{E} = \bigoplus_{i=1}^d \mathbb{E}_i$ of the exact sequences $\mathbb{E}_1,\dots,\mathbb{E}_d$, $h\colon S \to S^d$ is the diagonal homomorphism given by $h(x) = (x,\dots,x)$ for $x \in S$, and the upper exact sequence is the sequence $\mathbb{F} = \mathbb{E}h$ (see Section III.3). We note that E is the fibered product of $\bigoplus_{i=1}^d E_i$ and S over S^d, via w and h. Let $w_i\colon S^d \to S$, for $i \in \{1,\dots,d\}$, be the projection homomorphisms given by $w_i(x_1,\dots,x_d) = x_i$ for $(x_1,\dots,x_d) \in S^d$. We claim that the upper exact sequence $\mathbb{F}$ of the above diagram is not splittable. Indeed, applying similar arguments as in the proof of Lemma 2.4, we conclude that $\mathbb{E}_i \cong w_i\mathbb{F}$ for any $i \in \{1,\dots,d\}$. In particular, we conclude that $[\mathbb{F}] \neq 0$, or equivalently, the exact sequence $\mathbb{F}$ is not splitting (see Lemma III.3.1). We prove that $\operatorname{Ext}_A^1(T \oplus E, T \oplus E) = 0$. Applying Theorem VII.3.3 to the exact sequence $\mathbb{F}$, we obtain an exact sequence in mod K of the form

$$\cdots \longrightarrow \operatorname{Hom}_A(T^d,T) \xrightarrow{\delta} \operatorname{Ext}_A^1(S,T) \longrightarrow \operatorname{Ext}_A^1(E,T) \longrightarrow \operatorname{Ext}_A^1(T^d,T),$$

where $\delta = \delta_T^{T^d,S}$ is the connecting homomorphism. Observe that $\operatorname{Ext}_A^1(T,T) = \operatorname{Ext}_C^1(T,T) = 0$, because T is a tilting module in mod C. We claim that δ is an epimorphism, and consequently $\operatorname{Ext}_A^1(E,T) = 0$. Indeed, by Theorem VII.3.3, we have $\delta(w_i) = \chi_{S,T^d}([w_i\mathbb{F}]) = \chi_{S,T^d}([\mathbb{E}_i])$ for any $i \in \{1,\dots,d\}$. Hence, $\delta(w_1),\dots,\delta(w_d)$ form a K-basis of the K-vector space $\operatorname{Ext}_A^1(S,T)$, because $[\mathbb{E}_1],\dots,[\mathbb{E}_d]$ form a K-basis of $\mathcal{E}\mathrm{xt}_A^1(S,T)$ and χ_{S,T^d} is a K-linear isomorphism. Therefore, δ is an epimorphism. Applying Theorem VII.3.2, we obtain exact sequences in mod K of the forms

$$0 = \operatorname{Ext}_A^1(T, T^d) \longrightarrow \operatorname{Ext}_A^1(T, E) \longrightarrow \operatorname{Ext}_A^1(T, S) = 0,$$
$$0 = \operatorname{Ext}_A^1(E, T^d) \longrightarrow \operatorname{Ext}_A^1(E, E) \longrightarrow \operatorname{Ext}_A^1(E, S) = 0,$$

because S is injective in mod A, $\operatorname{Ext}_A^1(T, T) = 0$ and $\operatorname{Ext}_A^1(E, T) = 0$. Summing up, we have proved that $\operatorname{Ext}_A^1(T \oplus E, T \oplus E) = 0$, so $T \oplus E$ satisfies (T2). Since we have in mod A an exact sequence of the form

$$0 \longrightarrow M \longrightarrow P \longrightarrow S \longrightarrow 0,$$

we obtain $\operatorname{pd}_A S = 1$, because M is projective in mod C, and hence projective in mod A. Moreover, we have $\operatorname{pd}_A T = \operatorname{pd}_C T \leq 1$, because T is a tilting module in mod C. Now invoking the exact sequence $\mathbb{F}$ we conclude that $\operatorname{pd}_A E \leq 1$. Hence, $\operatorname{pd}_A(T \oplus E) \leq 1$, so $T \oplus E$ satisfies (T1). Therefore, $T \oplus E$ is a partial tilting module in mod A. Let m be the rank of $K_0(C)$ and n be the rank of $K_0(A)$. Clearly, we have $n = m + 1$. In particular, it follows from Proposition 3.11 that m is the number of pairwise nonisomorphic indecomposable direct summands of T in mod C. The exact sequence $\mathbb{F}$ provides also the equality $[T^d] = [E] - [S]$ in $K_0(A)$. Hence, the module E has a decomposition $E = X \oplus Y$ in mod A, where X is an indecomposable module having S as its composition factor and Y is a direct sum of indecomposable modules in mod C. In particular, we conclude that $T \oplus X$ is a partial tilting module in mod A whose number of pairwise nonisomorphic indecomposable direct summands is equal to $m + 1 = n$. Applying Proposition 3.11 again, we obtain that $T \oplus X$ is a tilting module in mod A. We also note that Y belongs to add T. Finally, since S is a direct summand of $\operatorname{top}(X)$, we may consider the commutative diagram in mod A

$$\begin{tikzcd}
0 \arrow[r] & T' \arrow[r, "f'"] \arrow[d, "j'"] & X \arrow[r, "g'"] \arrow[d, "j"] & S \arrow[r] \arrow[d, "\mathrm{id}_S"] & 0 \\
0 \arrow[r] & T^d \arrow[r, "f"] & E \arrow[r, "g"] & S \arrow[r] & 0,
\end{tikzcd}$$

induced by the canonical embedding $j\colon X \to E$. Observe that $T' \neq 0$, because the lover exact sequence is not splitting. Applying Lemma VII.3.1, we conclude that there exists an exact sequence in mod A of the form

$$0 \longrightarrow T' \xrightarrow{j'} T^d \xrightarrow{p'} Y \longrightarrow 0,$$

where p' is the composition of f with the canonical projection $p\colon E \to Y$. Finally, let $s\colon Y \to E$ be the section with $ps = \mathrm{id}_Y$. Since $gs = 0$, there exists a homomorphism $s'\colon Y \to T^d$ such that $s = fs'$. Then we obtain $p's' = pfs' = ps = \mathrm{id}_Y$, and so the above exact sequence is splitting. Therefore, T' is a direct summand of T^d, and hence T' belongs to add T. □

Lemma 9.4. *Let C be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, M be a nonzero projective module in* $\operatorname{mod} C$*, and F be a division K-subalgebra of* $\operatorname{End}_C(M)$*. Moreover, let*

$$A = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix}$$

be the associated one-point extension algebra, and T be a postprojective tilting module in $\operatorname{mod} C$*, which is a regular module in* $\operatorname{mod} A$*. Then there exists an indecomposable module X in* $\operatorname{mod} A$ *such that $T \oplus X$ is a regular tilting module in* $\operatorname{mod} A$.

Proof. Observe first that A is an indecomposable finite dimensional hereditary K-algebra of infinite representation type. We shall identify $\operatorname{mod} A$ with the category $\overline{\operatorname{rep}}({}_F M_C)$, via the functor $HF\colon \operatorname{mod} A \to \overline{\operatorname{rep}}({}_F M_C)$ defined in Section VII.10. Then we have the full and faithful embedding $\bar{\ }\colon \operatorname{mod} C \to \operatorname{mod} A$ preserving the indecomposablility of modules, which associates to a module Z in $\operatorname{mod} C$ the triple $\overline{Z} = (\operatorname{Hom}_C(M, Z), Z, \operatorname{id}_{\operatorname{Hom}_C(M,Z)})$. It follows from Lemma 9.3 that there exists an indecomposable module X in $\operatorname{mod} A$ such that $T \oplus X$ is a tilting module in $\operatorname{mod} A$ and there exists an exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow T' \xrightarrow{f'} X \xrightarrow{g'} S \longrightarrow 0,$$

where S is the simple injective module in $\operatorname{mod} A$ which is not a right C-module, and T' is a nonzero module from $\operatorname{add} T$. Since T is a regular module in $\operatorname{mod} A$, T' is also a regular module in $\operatorname{mod} A$, and hence X does not belong to the postprojective component $\mathcal{P}(A)$ of Γ_A. We claim that X does not belong to the preinjective component $\mathcal{Q}(A)$ of Γ_A, and hence belongs to the regular part $\mathcal{R}(A)$ of Γ_A. Since T' belongs to $\operatorname{add} T$ and T is a postprojective right C-module, we conclude that T' is a postprojective right C-module. Let $T' = Z_1 \oplus \cdots \oplus Z_r$ be a decomposition of T' into a direct sum of indecomposable modules in $\operatorname{mod} C$. Take $Z = Z_i$ for some $i \in \{1, \dots, r\}$. Since Z is an indecomposable postprojective module in $\operatorname{mod} C$, there exists an infinite family U_n, $n \in \mathbb{N}$, of pairwise nonisomorphic indecomposable postprojective modules in $\operatorname{mod} C$ such that $\operatorname{Hom}_C(Z, U_n) \neq 0$ for any $n \in \mathbb{N}$. Then $\overline{U}_n$, $n \in \mathbb{N}$, is an infinite family of pairwise nonisomorphic indecomposable modules in $\operatorname{mod} A$ such that $\operatorname{Hom}_A(\overline{Z}, \overline{U}_n) \neq 0$ for any $n \in \mathbb{N}$. Hence $\overline{Z}$ does not belong to the preinjective component $\mathcal{Q}(A)$ of Γ_A. Since $\overline{T'} = \overline{Z}_1 \oplus \cdots \oplus \overline{Z}_r$ is a decomposition of $\overline{T'}$ into a direct sum of indecomposable modules in $\operatorname{mod} A$, we conclude that $\overline{T'}$ does not admit an indecomposable preinjective direct summand in $\operatorname{mod} A$. Observe also that the monomorphism $f'\colon T' \to X$ in $\operatorname{mod} A$ induces an isomorphism $f''\colon T' \to Y$ in $\operatorname{mod} C$, where Y is the largest right C-submodule of X. Let $h'\colon Y \to T'$ be the inverse of f'' in $\operatorname{mod} C$. Under the identification $\operatorname{mod} A = \overline{\operatorname{rep}}({}_F M_C)$, the module X is a triple $X = (F, Y, \psi)$, for a homomorphism $\psi\colon F \to \operatorname{Hom}_C(M, Y)$ of right F-modules.

Then we have in $\operatorname{mod} F$ the commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\psi"] \arrow[d, "u'"] & \operatorname{Hom}_C(M, Y) \arrow[d, "\operatorname{Hom}_C(M,h')"] \\
\operatorname{Hom}_C(M, T') \arrow[r, "\mathrm{id}_{\operatorname{Hom}_C(M,T')}"] & \operatorname{Hom}_C(M, T'),
\end{tikzcd}$$

which shows that there is a monomorphism $\varphi\colon X \to \overline{T'}$ in $\operatorname{mod} A$, given by the pair (u', h'). Since $\overline{T'}$ does not admit an indecomposable preinjective direct summand in $\operatorname{mod} A$, we conclude that X does not belong to $\mathcal{Q}(A)$. Hence, X is a module in $\mathcal{R}(A)$. Therefore, $T \oplus X$ is a regular tilting module in $\operatorname{mod} A$. □

We obtain the following direct consequence of Lemma 9.4.

Corollary 9.5. *Let C be an indecomposable finite dimensional hereditary K-algebra of infinite representation type over a field K, M a nonzero projective module in* $\operatorname{mod} C$, *F a division K-subalgebra of* $\operatorname{End}_C(M)$, *and*

$$A = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix}$$

the associated one-point extension algebra. Then there exists a positive integer n_0 such that, for any integer $n \geq n_0$, there exists an indecomposable module X_n in $\operatorname{mod} A$ *such that $\tau_C^{-n} C \oplus X_n$ is a regular tilting module in* $\operatorname{mod} A$.

Proof. It follows from Theorem VII.6.11 that all but finitely many indecomposable modules in the components $\mathcal{P}(A)$ and $\mathcal{Q}(A)$ are sincere modules. Hence there exists a positive integer n_0 such that, for any integer $n \geq n_0$, the right postprojective C-module $\tau_C^{-n} C$ is a regular module in $\operatorname{mod} A$. It follows from Lemma 9.4 that there exist indecomposable modules X_n, $n \geq n_0$, in $\operatorname{mod} A$ such that $\tau_C^{-n} C \oplus X_n$ are regular tilting modules in $\operatorname{mod} A$. □

Lemma 9.6. *Let C be an indecomposable finite dimensional hereditary K-algebra over a field K, M be a nonzero projective but noninjective module in* $\operatorname{mod} C$, *F a division K-subalgebra of* $\operatorname{End}_C(M)$, *and*

$$A = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix}$$

the associated one-point extension algebra. Then there exists an indecomposable module X in $\operatorname{mod} A$ *such that $D(C) \oplus X$ is a tilting module in* $\operatorname{mod} A$ *with* $\operatorname{Hom}_A(D(C), X) \neq 0$ *and* $\operatorname{Hom}_A(X, D(C)) \neq 0$. *In particular, if $D(C)$ is a regular module in* $\operatorname{mod} A$, *then $D(C) \oplus X$ is a regular tilting module in* $\operatorname{mod} A$.

Proof. Let P be the indecomposable projective module in $\operatorname{mod} A$ which is not a right C-module and $S = \operatorname{top}(P)$ the associated simple module, which is an injective module in $\operatorname{mod} A$. We abbreviate $T = D(C)$. Since T is a tilting module in $\operatorname{mod} C$, applying Lemma 9.3, we conclude that there is an indecomposable module X in $\operatorname{mod} A$ such that $T \oplus X$ is a tilting module in $\operatorname{mod} A$ and there is an exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow T' \longrightarrow X \longrightarrow S \longrightarrow 0$$

with T' a nonzero module from $\operatorname{add} T$. Then T' is a nonzero injective module in $\operatorname{mod} C$. Moreover, we have $\operatorname{Hom}_A(T, X) \neq 0$. We will prove that also $\operatorname{Hom}_A(X, T) \neq 0$. Observe that it is enough to show that X has nonsimple top. Indeed, if it is the case, then there is a nonzero homomorphism from X to a simple factor module S' of T', because T' is isomorphic to the largest right C-submodule of X. But then S' is a simple injective module in $\operatorname{mod} C$, and hence a direct summand of T. This shows that $\operatorname{Hom}_A(X, T) \neq 0$.

Suppose that X has simple top. Then $\operatorname{top}(X) \cong S$ in $\operatorname{mod} A$, and hence $X = P/U$ for a proper right A-submodule U of P. Clearly, U is a right C-submodule of M, because we have the canonical exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow M \longrightarrow P \longrightarrow S \longrightarrow 0.$$

Assume first that $U = 0$. Then $X = P$ and $T \oplus P = T \oplus X$ is a tilting module in $\operatorname{mod} A$. Applying Theorem VII.3.2 to the above exact sequence, we obtain an exact sequence in $\operatorname{mod} K$

$$0 = \operatorname{Hom}_A(T, S) \longrightarrow \operatorname{Ext}_A^1(T, M) \longrightarrow \operatorname{Ext}_A^1(T, P) = 0,$$

because T is a C-module and $\operatorname{Ext}_A^1(T, P) = \operatorname{Ext}_A^1(T, X) = 0$. Hence $\operatorname{Ext}_A^1(T, M) = 0$, and consequently $\operatorname{Ext}_C^1(T, M) = 0$. Consider now a minimal injective resolution of M in $\operatorname{mod} C$

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow 0.$$

Since E_0 and E_1 belong to $\operatorname{add} T = \operatorname{add} D(C)$, we conclude that this exact sequence is splitting, and then M is an injective module in $\operatorname{mod} C$. This contradicts the assumption imposed on M. Assume now that $U \neq 0$. Then we have the nonsplittable exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow U \longrightarrow P \longrightarrow X \longrightarrow 0.$$

Applying Theorem VII.3.3 to this exact sequence, we obtain an exact sequence in $\operatorname{mod} K$

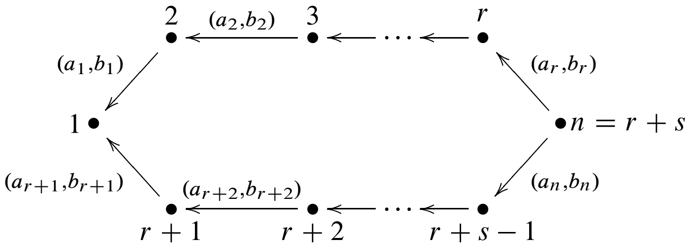
$$0 = \operatorname{Hom}_A(P, T) \longrightarrow \operatorname{Hom}_A(U, T) \longrightarrow \operatorname{Ext}_A^1(X, T) = 0,$$

and hence $\operatorname{Hom}_A(U, T) = 0$. But $\operatorname{Hom}_A(U, T) = \operatorname{Hom}_C(U, T) \neq 0$, because U is a nonzero module in $\operatorname{mod} C$ and $T = D(C)$ is an injective cogenerator in $\operatorname{mod} C$. Summing up, we proved that X has nonsimple top.

Assume now that $T = D(C)$ is a regular module in $\operatorname{mod} A$. Since we have $\operatorname{Hom}_A(T, X) \neq 0$ and $\operatorname{Hom}_A(X, T) \neq 0$, it follows that X is a regular module in $\operatorname{mod} A$. Therefore, $T \oplus X = D(C) \oplus X$ is a regular tilting module in $\operatorname{mod} A$. ☐

Proposition 9.7. *Let A be a basic finite dimensional hereditary K-algebra of wild type whose quiver Q_A is a simple cycle. Then there exists a regular tilting module in* $\operatorname{mod} A$.

Proof. Taking a sequence of reflections given by an admissible sequence of sinks in Q_A (if it is necessary), and using Corollary 8.2, we may assume that Q_A is of the form



with $r \geq 2$, $s \geq 1$, and $b_1 \geq 2$. For each $i \in \{1, \dots, n\}$, we denote by P_i the indecomposable projective module in $\operatorname{mod} A$ corresponding to i, and let $S_i = \operatorname{top}(P_i)$. It follows from Theorem VII.6.1 that the indecomposable projective modules $P_1, \dots, P_n$ form a section of the postprojective component $\mathcal{P}(A)$ of A, isomorphic to the opposite quiver Q_A^{op} of Q_A. In particular, we have in $\mathcal{P}(A)$ the arrow

$$P_1 \xrightarrow{(a_1,b_1)} P_2,$$

and hence $P_1 = S_1$ occurs with multiplicity $b_1 \geq 2$ in $\operatorname{rad} P_2$. Similarly, we have in $\mathcal{P}(A)$ the arrow

$$P_r \xrightarrow{(a_r,b_r)} P_n,$$

and hence P_r occurs with multiplicity $b_r \geq 1$ in $\operatorname{rad} P_n$. Moreover, by Theorem III.11.2, there is a nonzero homomorphism $f : P_2 \to P_r$, which is a monomorphism, because A is a hereditary algebra. Hence, the socle of P_r is a direct sum of at least two copies of S_1. Let U be the right A-submodule of P_n isomorphic to the direct sum of b_r copies of P_r, and $V = P_n/U$. Then V is an indecomposable module in $\operatorname{mod} A$ such that $\operatorname{Hom}_A(P_i, V) = 0$ for all $i \in \{2, \dots, r\}$. Further, the simple modules $S_2, \dots, S_r$ are pairwise orthogonal bricks with $\operatorname{Ext}_A^1(S_{i+1}, S_i) \neq 0$ for $i \in \{2, \dots, r-1\}$, by Theorem VII.1.9. We claim that $\operatorname{Ext}_A^1(V, S_r) \neq 0$ and

$\operatorname{Ext}^1_A(S_2, V) \neq 0$. Consider the exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow U \longrightarrow P_n \longrightarrow V \longrightarrow 0.$$

Applying Theorem VII.3.3, we obtain an exact sequence in $\operatorname{mod} K$

$$\operatorname{Hom}_A(U, S_r) \longrightarrow \operatorname{Ext}^1_A(V, S_r) \longrightarrow \operatorname{Ext}^1_A(P_n, V) = 0,$$

with $\operatorname{Hom}_A(U, S_r) \neq 0$, because U is a direct sum of copies of P_r. Then $\operatorname{Ext}^1_A(V, S_r) \neq 0$. Applying Theorem VII.3.3 again to the canonical exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow \operatorname{rad} P_2 \longrightarrow P_2 \longrightarrow S_2 \longrightarrow 0,$$

we obtain an exact sequence in $\operatorname{mod} K$

$$\operatorname{Hom}_A(\operatorname{rad} P_2, V) \longrightarrow \operatorname{Ext}^1_A(S_2, V) \longrightarrow \operatorname{Ext}^1_A(P_2, V) = 0,$$

with $\operatorname{Hom}_A(\operatorname{rad} P_2, V) \neq 0$, because $\operatorname{rad} P_2$ is a direct sum of b_1 copies of $S_1 = P_1$ and $\operatorname{soc} V$ is a direct sum of copies of S_1. Then $\operatorname{Ext}^1_A(S_2, V) \neq 0$. Summing up, we obtain a cycle in $\operatorname{mod} A$ of the form

$$V \longrightarrow V_2 \longrightarrow S_2 \longrightarrow V_3 \longrightarrow S_3 \longrightarrow \cdots \longrightarrow V_r \longrightarrow S_r \longrightarrow V_{r+1} \longrightarrow V,$$

for some indecomposable modules $V_2, V_3, \ldots, V_r, V_{r+1}$ in $\operatorname{mod} A$. Then it follows from Theorem VII.9.20 that the modules $S_2, \ldots, S_r$ belong to $\mathcal{R}(A)$. Similarly, if $s \geq 2$, we prove that the modules $S_{r+1}, \ldots, S_{r+s-1}$ belong to $\mathcal{R}(A)$.

Since n is a source of Q_A, the algebra A has the one-point extension presentation

$$A = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix},$$

where C is the hereditary algebra $C = \operatorname{End}_A(\bigoplus_{i=1}^{n-1} P_i)$, $M = \operatorname{rad} P_n$, and $F = \operatorname{End}_A(P_n)$. Observe that F is a division K-subalgebra of $\operatorname{End}_A(M)$. We note that C is possibly of Dynkin type. Clearly, the indecomposable projective modules $P_i, i \in \{1, \ldots, n-1\}$, are indecomposable projective modules in $\operatorname{mod} C$. For each $i \in \{1, \ldots, n-1\}$, we denote by E_i the indecomposable injective module in $\operatorname{mod} C$ with $\operatorname{soc}(E_i) = S_i$. Then

$$\bigoplus_{i=1}^{n-1} E_i = D(C)$$

is a minimal injective congenerator in $\operatorname{mod} C$, which is a tilting module in $\operatorname{mod} C$. We will show that $D(C)$ is a regular module in $\operatorname{mod} A$.

The quiver Q_C is obtained from Q_A by removing the vertex n and the two arrows starting from n. Further, it follows from Theorem VII.6.2 that the preinjective component $\mathcal{Q}(C)$ of Γ_C has the section given by the indecomposable injective modules $E_i, i \in \{1, \ldots, r+s-1\}$, isomorphic to the opposite quiver Q_C^{op}

of Q_C. Hence, applying Theorem III.11.2, we conclude that there is a nonzero homomorphism in $\operatorname{mod} C$, and hence in $\operatorname{mod} A$, from E_i to $E_r = S_r$, for any $i \in \{2, \ldots, r-1\}$, and from E_j to $E_{r+s-1} = S_{r+s-1}$ (if $s \geq 2$), for any $j \in \{r+1, \ldots, r+s-1\}$. Since S_r and S_{r+s-1} (if $s \geq 2$) are regular right A-modules, we conclude that the modules E_t, $t \in \{2, \ldots, r+s-1\}$, do not belong to $\mathcal{Q}(A)$. Observe also that there is a nonzero homomorphism in $\operatorname{mod} C$, and hence in $\operatorname{mod} A$, from E_1 to E_t, for any $t \in \{2, \ldots, r+s-1\}$, again by Theorem III.11.2. In particular, E_1 does not belong to $\mathcal{Q}(A)$. Hence, in order to prove that $D(C) = \bigoplus_{i=1}^{r+s-1} E_i$ is a regular module in $\operatorname{mod} A$, it is enough to show that E_1 does not belong to $\mathcal{P}(A)$. Since S_1 is a direct summand of $\operatorname{soc}(P_r)$ and $\operatorname{soc}(V)$, we may take nonzero homomorphisms $f\colon S_1 \to P_r$ and $g\colon S_1 \to V$, and consider the induced exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow S_1 \xrightarrow{\begin{bmatrix} f \\ g \end{bmatrix}} P_r \oplus V \xrightarrow{h} W \longrightarrow 0.$$

We note that S_1 is the unique common simple factor of P_r and V, and consequently W is an indecomposable module. Observe also that W admits a minimal projective resolution in $\operatorname{mod} A$ of the form

$$0 \longrightarrow U \xrightarrow{d_1} P_r \oplus P_n \xrightarrow{d_0} W \longrightarrow 0,$$

where $d_1 = \begin{bmatrix} 0 \\ j \end{bmatrix}$ and $d_0 = [u,\ v]$, for the inclusion homomorphism $j\colon U \to P_n$, the canonical monomorphism $u\colon P_r \to W$, and a homomorphism $v\colon P_n \to W$ with $\operatorname{Ker} v = \operatorname{Im} j$. We have an isomorphism of K-vector spaces

$$\operatorname{Ext}_A^1(W, W) \cong \operatorname{Coker} \operatorname{Hom}_A(d_1, W),$$

and $\operatorname{Hom}_A(d_1, W)\colon \operatorname{Hom}_A(P_n \oplus P_n, W) \to \operatorname{Hom}_A(U, W)$ is not an epimorphism, because there is a nonzero homomorphism $w\colon U \to W$ with $\operatorname{Im} w = \operatorname{Im} u$. Therefore, $\operatorname{Ext}_A^1(W, W) \neq 0$. Hence, W is an indecomposable module in $\mathcal{R}(A)$. Further, W has a right A-submodule V' isomorphic to V and a right A-submodule P' isomorphic to P_r, with $V' + P' = W$ and $V' \cap P'$ isomorphic to S_1. We note that $W/V' \cong P'/P' \cap V'$ in $\operatorname{mod} A$. Moreover, $P'/P' \cap V'$ is a right C-module. Hence, W/V' is a right C-module whose socle contains the simple module S_1, because S_1 occurs in $\operatorname{soc}(P_r)$ with multiplicity ≥ 2. Then we obtain that $\operatorname{Hom}_C(W/V', E_1) \neq 0$, and consequently $\operatorname{Hom}_A(W, E_1) \neq 0$. Since W belongs to $\mathcal{R}(A)$, we conclude that E_1 does not belong to $\mathcal{P}(A)$.

Summing up, we have proved that $D(C)$ is a regular module in $\operatorname{mod} A$. Now, applying Lemma 9.6, we conclude that there is an indecomposable module X in $\operatorname{mod} A$ such that $D(C) \oplus X$ is a regular tilting module in $\operatorname{mod} A$. □

Proposition 9.8. *Let A be a finite dimensional hereditary K-algebra over a field K with Q_A of the form*

$$\underset{1}{\bullet} \xleftarrow{(a,b)} \underset{2}{\bullet} \xleftarrow{(c,d)} \underset{3}{\bullet}\,,$$

$ab = 3$, and $2 \leq cd \leq 3$. Then there exists a regular tilting module in mod A.

Proof. Let e_1, e_2, e_3 be the primitive idempotents of A with $1_A = e_1 + e_2 + e_3$, given by the vertices $1, 2, 3$ of Q_A. For each $i \in \{1, 2, 3\}$, let $P_i = e_i A$, $S_i = e_i A / \operatorname{rad} e_i A$, and $I_i = D(Ae_i)$. Further, let $e = e_1 + e_2$ and $C = eAe$. Moreover, let $F = e_3 A e_3$ and $M = e_3 A e = \operatorname{rad} P_3$. Then C is a finite dimensional hereditary K-algebra with Q_C of the form

$$1 \xleftarrow{(a,b)} 2,$$

M is an (F, C)-bimodule on which K acts centrally and $\dim_K M = \dim_K \operatorname{rad} P_3$ is finite, F is a division K-subalgebra of $\operatorname{End}_C(M)$, and A is the one-point extension algebra

$$A = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix}.$$

Clearly, M is a nonzero projective module in mod C. Observe also that M is not injective in mod C, because C is a hereditary algebra of Dynkin type $\mathbb{G}_2$ (see Example VII.7.8). We will prove that the injective cogenerator $D(C)$ of mod C is a regular module in mod A. Then Lemma 9.6 will imply that there is an indecomposable module X in mod A such that $D(C) \oplus X$ is a regular tilting module in mod A.

We identify the $\mathbb{Z}$-basis $[S_1], [S_2], [S_3]$ of $K_0(A)$ with the standard basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ of $\mathbb{Z}^3 = K_0(A)$. We determine now the Coxeter transformation $\varphi_A \colon K_0(A) \to K_0(A)$ in the basis e_1, e_2, e_3. By definition, φ_A is given by $\varphi_A([P_1]) = -[I_1]$, $\varphi_A([P_2]) = -[I_2]$, $\varphi_A([P_3]) = -[I_3]$ (see Section VII.5 for details). It follows from Theorems VII.6.1 and VII.6.2 that the postprojective component $\mathcal{P}(A)$ of Γ_A has the section

$$P_1 \xrightarrow{(a,b)} P_2 \xrightarrow{(c,d)} P_3$$

and the preinjective component $\mathcal{Q}(A)$ of Γ_A has the section

$$I_1 \xrightarrow{(a,b)} I_2 \xrightarrow{(c,d)} I_3.$$

Hence, we have $P_1 = S_1$, $\operatorname{rad} P_2 = P_1^b$, $\operatorname{rad} P_3 = P_2^d$, and $I_1/S_1 = I_2^a$, $I_2/S_2 = I_3^c$, $I_3 = S_3$. Then we conclude that

$$\begin{aligned} &[P_1] = (1, 0, 0), && [P_2] = (b, 1, 0), && [P_3] = (bd, d, 1), \\ &[I_1] = (1, a, ac), && [I_2] = (0, 1, c), && [I_3] = (0, 0, 1), \end{aligned}$$

in the basis e_1, e_2, e_3 of $\mathbb{Z}^3 = K_0(A)$. Then we obtain that

$$\begin{aligned}\varphi_A(e_1) &= -e_1 - ae_2 - ace_3,\\ \varphi_A(e_2) &= be_1 + (ab-1)e_2 + (ab-1)ce_3 = be_1 + 2e_2 + 2ce_3,\\ \varphi_A(e_3) &= de_2 + (cd-1)e_3,\end{aligned}$$

because $ab = 3$. Therefore, φ_A is given by

$$\varphi_A(\mathbf{x}) = \big(-x_1 + bx_2, -ax_1 + 2x_2 + dx_3, -acx_1 + 2cx_2 + (cd-1)x_3\big)$$

for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$.

We denote by E_1 and E_2 the indecomposable injective modules in mod C such that $S_1 = \operatorname{soc}(E_1)$ and $S_2 = \operatorname{soc}(E_2) = E_2$. Since the preinjective component $\mathcal{Q}(C)$ of Γ_C has the section

$$E_1 \xrightarrow{(a,b)} E_2,$$

we conclude that $E_1/S_1 = E_2^a$. Then we obtain

$$[E_1] = (1, a, 0) \quad \text{and} \quad [E_2] = (0, 1, 0)$$

in $K_0(A)$. Since E_1 is not projective in mod A, applying Corollary VII.5.3, we obtain that

$$[\tau_A E_1] = \varphi_A\big([E_1]\big) = (2, a, ac).$$

Then $\tau_A E_1$ is nonprojective in mod A, and hence

$$[\tau_A^2 E_1] = \varphi_A\big([\tau_A E_1]\big) = \big(1, acd, (cd-1)ac\big),$$

again by Corollary VII.5.3. In particular, we conclude that $\tau_A^2 E_1$ has S_1 as a simple composition factor, with multiplicity 1, which is contained in $\operatorname{soc}(\tau_A^2 E_1)$. Then there is a nonzero homomorphism $f: \tau_A^2 E_1 \to I_1$ in mod A whose kernel has no simple composition factor isomorphic to S_1. Hence $[\operatorname{Ker} f] = (0, r, s)$ in $K_0(A)$, for some nonnegative integers r, s. On the other hand, we have $[\operatorname{Ker} f] = [\tau_A^2 E_1] - [\operatorname{Im} f]$, and $\operatorname{Im} f$ is a right A-submodule of I_1. Then we conclude that $r \geq acd - a = a(cd-1) \geq 1$, because $cd \geq 2$. Thus the simple module $S_2 = E_2$ is contained in $\operatorname{Ker} f$. Summing up, we obtain in mod A a cycle of the form

$$E_2 \longrightarrow \tau_A^2 E_1 \longrightarrow U \longrightarrow \tau_A E_1 \longrightarrow V \longrightarrow E_1 \longrightarrow E_2,$$

for some indecomposable modules U and V in mod A. Since A is a hereditary algebra of wild type, applying Theorem VII.9.20, we conclude that E_1 and E_2 are regular modules in mod A. Therefore, $D(C) = E_1 \oplus E_2$ is a regular module in mod A. □

We are now able to prove the following theorem established by Ringel in [R4].

Theorem 9.9. *Let A be an indecomposable finite dimensional hereditary K-algebra over a field K. The following statements are equivalent:*

(i) *There is a regular tilting module in* mod A.

(ii) *A is of wild type and $K_0(A)$ is of rank at least three.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Propositions 9.1 and 9.2, Theorem VII.7.4, and Corollary VII.7.5. We will prove that (ii) implies (i).

Assume that A is of wild type and $K_0(A)$ is of rank at least three. Without loss of generality, we may assume that A is basic (see Theorem II.6.16). It follows from Proposition 8.4 that there is an admissible sequence of sinks $i_1, \dots, i_r$ in Q_A (possibly empty) such that the associated hereditary algebra $B = \Sigma_{i_r} \dots \Sigma_{i_1} A$ is a one-point extension algebra

$$B = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix},$$

where C is a basic indecomposable finite dimensional hereditary K-algebra, F is a finite dimensional division K-algebra, and M is an (F, C)-bimodule, which is a projective right C-module. Further, the quiver Q_B of B is the quiver $\sigma_{i_r} \dots \sigma_{i_1} Q_A$, and hence a wild quiver. Hence, B is a hereditary algebra of wild type. Moreover, the composition $\Phi = \Phi_{i_r} \cdots \Phi_{i_1}$ of the reflection functors $\Phi_{i_1}, \dots, \Phi_{i_r}$ induces an equivalence of categories $\operatorname{add}\mathcal{R}(A) \xrightarrow{\sim} \operatorname{add}\mathcal{R}(B)$ (see Corollary 8.2). Therefore, we may assume that $B = A$. In particular, Q_A admits a source ω and the quiver Q_C of C is obtained from Q_A by removing the vertex ω and the arrows attached to ω. If C is of infinite representation type, then it follows Corollary 9.5 that there is a regular tilting module in mod A. Hence, we may assume that C is of Dynkin type. In particular, Q_C is a Dynkin quiver. Suppose Q_A contains a cycle Δ. Observe that then Δ contains the source ω and is the unique cycle of Q_A. Moreover, if $Q_A \neq \Delta$, then applying (if necessary) a sequence of reflections with respect to an admissible sequence $j_1, \dots, j_t$ of sinks of Q_A, we conclude that $\Sigma_{j_t} \dots \Sigma_{j_1} A$ is a hereditary algebra

$$\begin{bmatrix} G & N \\ 0 & D \end{bmatrix},$$

where D is an indecomposable finite dimensional hereditary K-algebra whose quiver Q_D contains Δ, G is a finite dimensional division K-algebra, and N is a (G, D)-bimodule. Then D is of infinite representation type and, applying Corollary 9.5 again, we conclude that there is a regular tilting module in mod $\Sigma_{j_t} \dots \Sigma_{j_1} A$. Hence, there is a regular tilting module in mod A, by Corollary 8.2. Therefore, we may assume that Q_A is a cycle. But then, applying Proposition 9.7, we conclude there is a regular tilting module in mod A.

Summing up, it remains to consider the case when Q_A is a tree and Q_C is a Dynkin quiver. In fact, applying (if necessary) reflections, we may assume that, for any external vertex (having only one neighbour) i of Q_A, the quiver obtained from Q_A by removing the vertex i and the arrows attached to i is a Dynkin quiver. Since Q_A is a wild quiver with at least three vertices, we may assume that Q_A is a quiver of the form

$$\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "{(a,b)}"'] & \underset{3}{\bullet} \arrow[l, "{(c,d)}"']
\end{tikzcd}$$

with $ab = 3$ and $2 \leq cd \leq 3$. Then, applying Proposition 9.8, we conclude there is a regular tilting module in mod A.

Therefore, there is a regular tilting module in mod A. □

Corollary 9.10. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K and with $K_0(A)$ of rank at least three. Then there are infinitely many pairwise nonisomorphic quasi-simple regular stones.*

Proof. It follows from Theorem 9.9 that there is a regular tilting module in mod A, and consequently a stone in the regular part $\mathcal{R}(A)$ of Γ_A. Then it follows from Proposition VII.9.22 that there is a quasi-simple stone X in $\mathcal{R}(A)$. Clearly, then the modules $\tau_A^m X$, $m \in \mathbb{Z}$, form an infinite family of pairwise nonisomorphic quasi-simple regular stones. □

Corollary 9.11. *Let A be a finite dimensional hereditary K-algebra of wild type over a field K and with $K_0(A)$ of rank at least three. Then there is a finite dimensional tilted K-algebra B such that the Auslander–Reiten quiver Γ_B of B contains a component $\mathcal{C}$ of the form $\mathbb{Z}Q_A$.*

Proof. It follows from Theorem 9.9 that there is a regular module T in mod A. Let $B = \operatorname{End}_A(T)$ be the associated tilted algebra. Then, by Theorem 6.7 and Propositions 6.5, and 6.9, the connecting component $\mathcal{C} = \mathcal{C}_T$ of Γ_B determined by T is a component of the form $\mathbb{Z}Q_A^{\text{op}}$. We also note that $\mathbb{Z}Q_A^{\text{op}} = \mathbb{Z}Q_A$. □

Example 9.12. Let K be a field, Q the quiver

$$\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\beta"] & \underset{3}{\bullet} \arrow[l, "\gamma"'] ,
\end{tikzcd}$$

and $A = KQ$ the path algebra of Q over K considered in Example VII.9.25; and we use the notation introduced there. Consider the indecomposable module X in mod A of the form

$$\begin{tikzcd}
K & K^2 \arrow[l, shift right, "{(1,0)}"'] \arrow[l, shift left, "{(0,1)}"] & 0 \arrow[l] ,
\end{tikzcd}$$

which is the injective module at the vertex 1 over the path algebra $H = K\Delta$ of the Kronecker subquiver Δ of Q given by the vertices 1 and 2. Then we have $\operatorname{Ext}_A^1(X, X) = \operatorname{Ext}_H^1(X, X) = 0$, and hence X is a stone. It was shown in

Example VII.9.25 that X is a quasi-simple regular module in $\operatorname{mod} A$ such that $[\tau_A X] = (3,4,4)$. We will construct a regular tilting module $T = T_1 \oplus T_2 \oplus T_3$ in $\operatorname{mod} A$ with $T_3 = X$, applying the Bongartz lemma (Lemma 2.4).

Let P_1, P_2, P_3 be the indecomposable projective modules in $\operatorname{mod} A$ associated to the vertices $1,2,3$ of Q. Then we have $[P_1] = (1,0,0)$, $[P_2] = (2,1,0)$, $[P_3] = (2,1,1)$ in $K_0(A) = \mathbb{Z}^3$. Applying Corollary III.6.4, we obtain

$$\begin{aligned}
\dim_K \operatorname{Ext}^1_A(X, P_1) &= \dim_K D \operatorname{Hom}_A(P_1, \tau_A X) = 3,\\
\dim_K \operatorname{Ext}^1_A(X, P_2) &= \dim_K D \operatorname{Hom}_A(P_2, \tau_A X) = 4,\\
\dim_K \operatorname{Ext}^1_A(X, P_3) &= \dim_K D \operatorname{Hom}_A(P_3, \tau_A X) = 4.
\end{aligned}$$

Since $A = P_1 \oplus P_2 \oplus P_3$ in $\operatorname{mod} A$, we conclude that $d = \dim_K \mathcal{E}\mathrm{xt}^1_A(X, A) = \dim_K \operatorname{Ext}^1_A(X, A) = 3 + 4 + 4 = 11$. It follows from the proof of Lemma 2.4 that there exists a short exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow A \longrightarrow Y \longrightarrow X^d \longrightarrow 0,$$

which is the direct sum of short exact sequences in $\operatorname{mod} A$

$$\begin{aligned}
0 \longrightarrow P_1 \longrightarrow Y_1 \longrightarrow X^3 \longrightarrow 0,\\
0 \longrightarrow P_2 \longrightarrow Y_2 \longrightarrow X^4 \longrightarrow 0,\\
0 \longrightarrow P_3 \longrightarrow Y_3 \longrightarrow X^4 \longrightarrow 0,
\end{aligned}$$

and $X \oplus Y$ is a tilting module in $\operatorname{mod} A$. Observe that we have the following equalities in $K_0(A)$:

$$\begin{aligned}
[Y_1] &= [P_1] + 3[X] = (4,6,0),\\
[Y_2] &= [P_2] + 4[X] = (6,9,0),\\
[Y_3] &= [P_3] + 4[X] = (6,9,1).
\end{aligned}$$

It follows from the description of the composition vectors of indecomposable modules over the Kronecker algebra $H = K\Delta$ (see Example VII.8.27) that $Y_1 \cong Z^2$ and $Y_2 \cong Z^3$ in $\operatorname{mod} H$, for the indecomposable preinjective H-module Z with $[Z] = (2,3)$ in $K_0(H) = \mathbb{Z}^2$. Moreover, there exists in $\operatorname{mod} H$ an almost split sequence of the form

$$0 \longrightarrow Z \longrightarrow X^2 \longrightarrow S_2 \longrightarrow 0.$$

The module Z, considered as a module in $\operatorname{mod} A$, is of the form

$$\begin{tikzcd}
K^2 & K^3 \arrow[l, shift right, "\left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right]"'] \arrow[l, shift left, "\left[\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right]"] & 0 \arrow[l]
\end{tikzcd} .$$

Further, it follows from the description of the postprojective component $\mathcal{P}(A)$ and the preinjective component $\mathcal{Q}(A)$ of Γ_A, given in Example VII.9.25, that Z is a regular module in $\operatorname{mod} A$. We claim that Y_3 is an indecomposable regular module in $\operatorname{mod} A$. Recall that the Euler quadratic form $\chi_A\colon K_0(A) \to \mathbb{Z}$ is given by

$$\chi_A(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - x_2x_3$$

for any vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$. Hence, we conclude that $\chi_A([Y_3]) = 1$. On the other hand, applying Theorems VII.4.1 and VII.4.2, we obtain

$$\chi_A\big([Y_3]\big) = \dim_K \operatorname{End}_A(Y_3) - \dim_K \operatorname{Ext}_A^1(Y_3, Y_3).$$

Since Y_3 is a direct summand of the tilting module $X \oplus Y$ in $\operatorname{mod} A$, we have $\operatorname{Ext}_A^1(Y_3, Y_3) = 0$. Hence, $\dim_K \operatorname{End}_A(Y_3) = 1$, and so Y_3 is a brick. Now it follows from the description of the composition vectors of indecomposable modules in $\mathcal{P}(A)$ and $\mathcal{Q}(A)$ that Y_3 is an indecomposable regular module in $\operatorname{mod} A$. Observe also that Z^3 is the largest right H-submodule of Y_3. Then we conclude that Y_3 is of the form

$$\begin{tikzcd} K^6 & K^9 \arrow[l, shift right, "U"'] \arrow[l, shift left, "V"] & K \arrow[l, "W"'] \end{tikzcd},$$

where U, V, W are the matrices

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

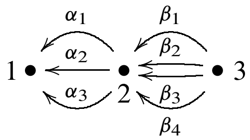
Let $T_1 = Z$, $T_2 = V_3$, $T_3 = X$, and $T = T_1 \oplus T_2 \oplus T_3$. Then T is a regular tilting module in $\operatorname{mod} A$. Observe that we have $\operatorname{End}_A(T_i) \cong K$, for $i \in \{1, 2, 3\}$, and

$\operatorname{Hom}_A(T_j, T_i) \cong K$, for $i < j$ in $\{1, 2, 3\}$. Moreover, applying Theorems VII.4.1 and VII.4.2 again, we conclude that

$$\begin{aligned}
\dim_K \operatorname{Hom}_A(T_1, T_2) &= \big\langle [T_1], [T_2] \big\rangle_A = 3,\\
\dim_K \operatorname{Hom}_A(T_2, T_3) &= \big\langle [T_2], [T_3] \big\rangle_A = 4,\\
\dim_K \operatorname{Hom}_A(T_1, T_3) &= \big\langle [T_1], [T_3] \big\rangle_A = 2,
\end{aligned}$$

because $\operatorname{Ext}^1_A(T_1, T_2) = 0$, $\operatorname{Ext}^1_A(T_2, T_3) = 0$, $\operatorname{Ext}^1_A(T_1, T_3) = 0$.

Let $B = \operatorname{End}_A(T)$ be the associated tilted algebra. Then B is isomorphic to the bound quiver algebra KQ_B/I, where Q_B is the quiver



and I is the ideal in the path algebra KQ_B of Q_B over K generated by the elements

$$\begin{gathered}
\beta_2\alpha_1 + \beta_2\alpha_2, \quad \beta_2\alpha_2 + \beta_3\alpha_2, \quad \beta_3\alpha_2 - \beta_3\alpha_1, \quad \beta_3\alpha_1 + \beta_4\alpha_3,\\
\beta_1\alpha_1 + \beta_2\alpha_3, \quad \beta_2\alpha_3 + \beta_3\alpha_2, \quad \beta_3\alpha_2 + \beta_3\alpha_3,\\
\beta_1\alpha_2, \quad \beta_1\alpha_3, \quad \beta_4\alpha_1, \quad \beta_4\alpha_2,
\end{gathered}$$

where the vertices $1, 2, 3$ of Q_B correspond to the indices of the indecomposable direct summands T_1, T_2, T_3 of T. It follows from Theorem 6.7 and Proposition 6.9 that the Auslander–Reiten quiver Γ_B of B admits the connecting component $\mathcal{C}_T$, determined by T, without projective modules and injective modules, and having the canonical section Δ_T

$$\begin{tikzcd}
 & & \operatorname{Hom}_A(T, T_3) \\
 & \operatorname{Hom}_A(T, T_2) \arrow[ur] & \\
\operatorname{Hom}_A(T, T_1) \arrow[ur, "{(2,2)}"] & &
\end{tikzcd}$$

of type Q_A^{op}, where Q_A is the valued quiver of A. Hence, $\mathcal{C}_T$ is isomorphic to the translation quiver $\mathbb{Z}Q_A^{\mathrm{op}} = \mathbb{Z}Q_A$. We also note that the modules forming the canonical section Δ_T of $\mathcal{C}_T$ have the following classes in $K_0(B)$:

$$\begin{aligned}
&[\mathrm{Hom}_A(T, I_1)] \\
&\quad = (\dim_K \mathrm{Hom}_A(T_1, I_1), \dim_K \mathrm{Hom}_A(T_2, I_1), \dim_K \mathrm{Hom}_A(T_3, I_1)) \\
&\quad = (2, 6, 1), \\
&[\mathrm{Hom}_A(T, I_2)] \\
&\quad = (\dim_K \mathrm{Hom}_A(T_1, I_2), \dim_K \mathrm{Hom}_A(T_2, I_2), \dim_K \mathrm{Hom}_A(T_3, I_2)) \\
&\quad = (3, 9, 2), \\
&[\mathrm{Hom}_A(T, I_3)] \\
&\quad = (\dim_K \mathrm{Hom}_A(T_1, I_3), \dim_K \mathrm{Hom}_A(T_2, I_3), \dim_K \mathrm{Hom}_A(T_3, I_3)) \\
&\quad = (0, 1, 0),
\end{aligned}$$

by Lemma 3.13. In particular, we conclude that the simple module S_2^B in mod B associated to the vertex 2 of Q_B lies in $\mathcal{C}_T$.

10 Exercises

1. Let $A = KQ$ be the path algebra of the quiver Q of the form

$$\begin{tikzcd}
\underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l, "\beta"'] & \underset{4}{\bullet} \arrow[l, "\gamma"']
\end{tikzcd}$$

over a field K and S_3 the simple module in mod A given by the vertex 3 of Q.

(a) Describe the minimal torsion class $\mathcal{T}$ in mod A containing S_3 and the associated torsion pair $(\mathcal{T}, \mathcal{F})$.

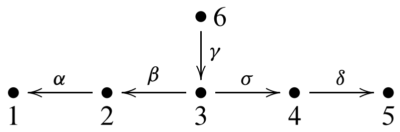
(b) Describe the minimal torsion-free class $\mathcal{F}^*$ in mod A containing S_3 and the associated torsion pair $(\mathcal{T}^*, \mathcal{F}^*)$.

2. Let $A = KQ$ be the path algebra of the quiver Q of the form

$$\begin{tikzcd}
1\,\bullet & & \\
& \underset{3}{\bullet} \arrow[ul, "\alpha"'] \arrow[dl, "\beta"] & \underset{4}{\bullet} \arrow[l, "\gamma"'] \\
2\,\bullet & &
\end{tikzcd}$$

over a field K. Describe all torsion pairs $(\mathcal{T}, \mathcal{F})$ in mod A.

3. Let $A = KQ$ be the path algebra of the quiver Q of the form



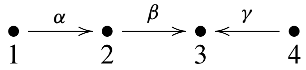
over a field K and S_3 the simple module in mod A given by the vertex 3 of Q.

(a) Describe the minimal torsion class $\mathcal{T}$ in mod A containing S_3 and the associated torsion pair $(\mathcal{T}, \mathcal{F})$.

(b) Describe the canonical sequences for the indecomposable projective modules in mod A with respect to $(\mathcal{T}, \mathcal{F})$.

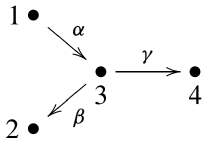
(c) Describe the canonical sequences for the indecomposable injective modules in mod A with respect to $(\mathcal{T}, \mathcal{F})$.

4. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K and I_3 the indecomposable injective module in mod A given by the vertex 3 of Q. Describe all tilting modules T in mod A having I_3 as a direct summand.

5. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K, $T_1 = S_3$ the simple module in mod A at the vertex 3, $T_2 = I_2$ the indecomposable injective module in mod A at the vertex 2, and $T_3 = \operatorname{rad} I_2$.

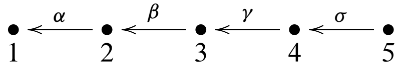
(a) Prove that $T = T_1 \oplus T_2 \oplus T_3$ is a partial tilting module in mod A.

(b) Describe the torsion pair $(\operatorname{Gen} T, \mathcal{F}(T))$ in mod A.

(c) Describe the torsion pair $(\mathcal{T}(T), \operatorname{Cogen} \tau_A T)$ in mod A.

(d) Show that there is a unique (up to isomorphism) indecomposable module M in mod A such that $T \oplus M$ is a tilting module in mod A.

6. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K. For each vertex i of Q, let P_i be the indecomposable projective module in mod A at i, and $S_i = P_i / \operatorname{rad} P_i$ the associated simple module. Let $T_1 = S_1, T_2 = P_3, T_3 = P_5, T_4 = S_3, T_5 = S_5$, and $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$.

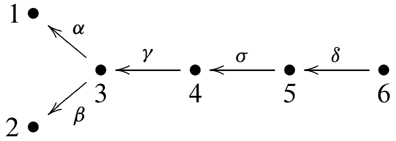
(a) Prove that T is a tilting module in mod A.

(b) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A induced by T.

(c) Describe the endomorphism algebra $B = \operatorname{End}_A(T)$.

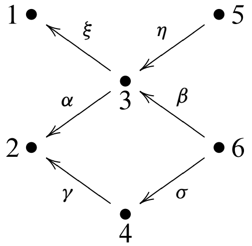
(d) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B induced by T.

7. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K.

(a) Describe a tilting module T in mod A such that $B = \operatorname{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta / J$, where Δ is the quiver

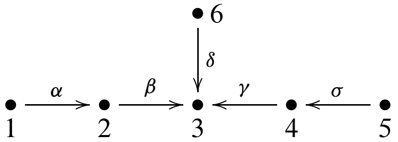


and J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\beta\alpha - \sigma\gamma$ and $\eta\xi$.

(b) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A induced by T.

(c) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B induced by T.

8. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K.

(a) Describe a tilting module T in mod A such that $B = \operatorname{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver

$$\begin{tikzcd}
\overset{4}{\bullet} \arrow[r, "\xi"] \arrow[d, "\alpha"'] & \overset{5}{\bullet} \arrow[d, "\beta"] & \overset{6}{\bullet} \arrow[l, "\eta"'] \arrow[d, "\gamma"] \\
\underset{1}{\bullet} \arrow[r, "\sigma"'] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l, "\delta"]
\end{tikzcd}$$

and J is the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\sigma - \xi\beta$ and $\eta\beta - \gamma\delta$.

(b) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A induced by T.

(c) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B induced by T.

9. Let $A = KQ$ be the path algebra of the quiver Q of the form

$$\begin{tikzcd}
1\,\bullet & & & & \bullet\,6 \arrow[dl, "\delta"'] \\
& \underset{3}{\bullet} \arrow[ul, "\alpha"'] \arrow[dl, "\beta"] & \underset{4}{\bullet} \arrow[l, "\gamma"'] & \underset{5}{\bullet} \arrow[l, "\sigma"'] & \\
2\,\bullet & & & & \bullet\,7 \arrow[ul, "\omega"]
\end{tikzcd}$$

over a field K.

(a) Describe a postprojective tilting module T in mod A such that $B = \operatorname{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver

$$\begin{tikzcd}
& \overset{6}{\bullet} \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & & \overset{7}{\bullet} \arrow[dl, "\xi"'] \arrow[dr, "\nu"] & \\
3\,\bullet \arrow[dr, "\beta"'] & & \underset{4}{\bullet} \arrow[dl, "\sigma"] \arrow[dr, "\eta"'] & & \bullet\,5 \arrow[dl, "\omega"] \\
& \underset{1}{\bullet} & & \underset{2}{\bullet} &
\end{tikzcd}$$

and J is the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta - \gamma\sigma$ and $\xi\eta - \nu\omega$.

(b) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A induced by T.

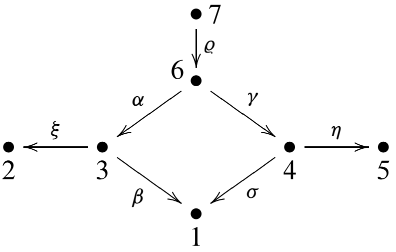
(c) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B induced by T.

10. Let $A = KQ$ be the path algebra of the quiver Q of the form

$$\begin{tikzcd}
& & \bullet\,7 \arrow[d, "\delta"] & & \\
& & \bullet\,6 \arrow[d, "\omega"] & & \\
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\alpha"'] \arrow[r, "\beta"] & \underset{3}{\bullet} & \underset{4}{\bullet} \arrow[l, "\gamma"'] \arrow[r, "\sigma"] & \underset{5}{\bullet}
\end{tikzcd}$$

over a field K.

(a) Describe a postprojective tilting module T in $\operatorname{mod} A$ such that $B = \operatorname{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver

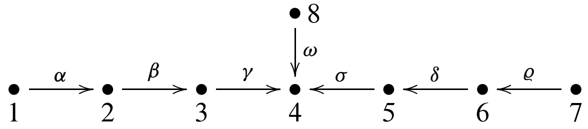


and J is the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta - \gamma\sigma$.

(b) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ induced by T.

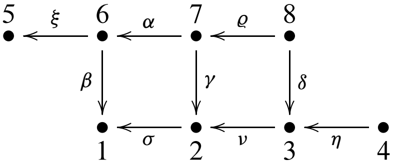
(c) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ induced by T.

11. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K.

(a) Describe a postprojective tilting module T in $\operatorname{mod} A$ such that $B = \operatorname{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver

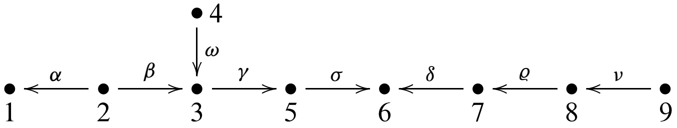


and J is the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta - \gamma\sigma$ and $\varrho\gamma - \delta\nu$.

(b) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ induced by T.

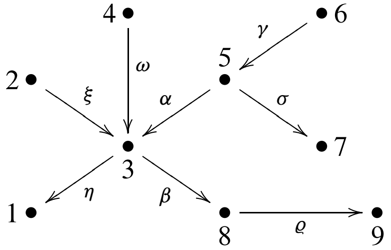
(c) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ induced by T.

12. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K.

(a) Describe a postprojective tilting module T in mod A such that $B = \operatorname{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver

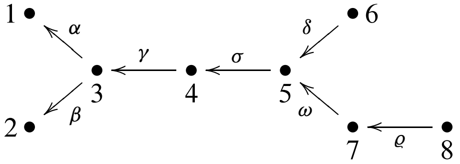


and J is the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta$, $\gamma\sigma$, $\xi\eta$.

(b) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A induced by T.

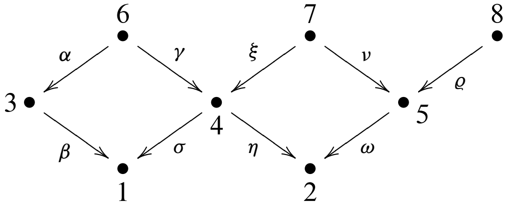
(c) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B induced by T.

13. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K.

(a) Describe a postprojective tilting module T in mod A such that $B = \operatorname{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver

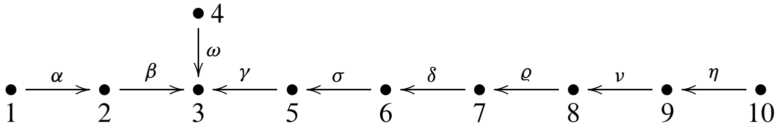


and J is the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta - \gamma\sigma$ and $\xi\eta - \nu\omega$.

(b) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod A induced by T.

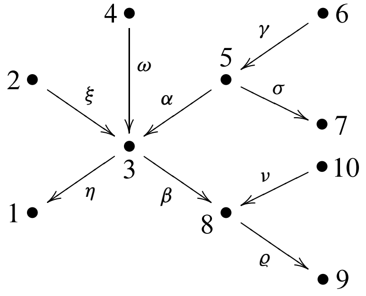
(c) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in mod B induced by T.

14. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K.

(a) Describe a postprojective tilting module T in $\operatorname{mod} A$ such that $B = \operatorname{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver

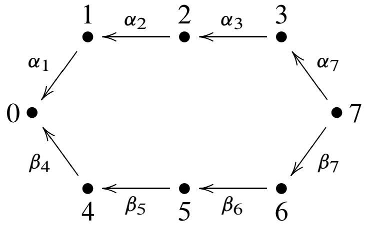


and J is the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta$, $\gamma\sigma$, $\xi\eta$, $\nu\varrho$.

(b) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ induced by T.

(c) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ induced by T.

15. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K. For each vertex i of Q, let S_i be the simple module in $\operatorname{mod} A$ at i. Moreover, let P_7 be the projective cover of S_7 in $\operatorname{mod} A$. Consider the indecomposable modules T_1, T_2, T_3, T_4, T_5, T_6 in $\operatorname{mod} A$ with the composition vectors

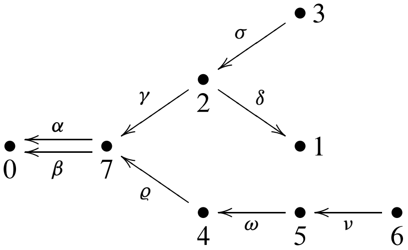
$$\begin{aligned} c(T_1) &= (0,0,1,0,0,0,0,0), & c(T_2) &= (1,0,1,1,1,1,1,1), \\ c(T_3) &= (1,0,0,0,1,1,1,1), & c(T_4) &= (1,1,1,1,0,1,1,1), \\ c(T_5) &= (1,1,1,1,0,0,1,1), & c(T_6) &= (1,1,1,1,0,0,0,1), \end{aligned}$$

in the basis $[S_0]$, $[S_1]$, $[S_2]$, $[S_3]$, $[S_4]$, $[S_5]$, $[S_6]$, $[S_7]$ of $K_0(A)$.

(a) Prove that $T_1 \oplus T_2 \oplus T_3$ is a partial tilting module from the additive category $\operatorname{add} \mathcal{T}_0^A$ of a stable tube $\mathcal{T}_0^A$ of rank 3 in Γ_A.

(b) Prove that $T_3 \oplus T_4 \oplus T_5$ is a partial tilting module from the additive category $\operatorname{add} \mathcal{T}_\infty^A$ of a stable tube $\mathcal{T}_\infty^A$ of rank 3 in Γ_A.

(c) Prove that $T = S_0 \oplus T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6 \oplus P_7$ is a tilting module in $\operatorname{mod} A$ such that $B = \operatorname{End}_A(T)$ is isomorphic to the bound quiver algebra $K\Delta/J$, where Δ is the quiver



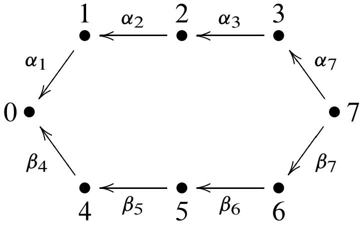
and J is the ideal in the path algebra $K\Delta$ of Δ over K generated by $\gamma\alpha$, $\sigma\delta$, $\varrho\beta$.

(d) Describe the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ induced by T.

(e) Describe the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ induced by T.

(f) Describe the Auslander–Reiten quiver Γ_B of B.

16. Let $A = KQ$ be the path algebra of the quiver Q of the form



over a field K. For each vertex i of Q, let S_i be the simple module in $\operatorname{mod} A$ at i. Moreover, let I_0 be the injective envelope of S_0 in $\operatorname{mod} A$. Consider the indecomposable modules T_1^*, T_2^*, T_3^*, T_4^*, T_5^*, T_6^* in $\operatorname{mod} A$ with the composition vectors

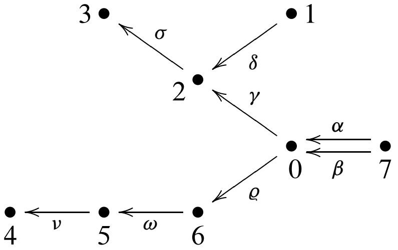
$$\begin{aligned} c(T_1^*) &= (0,0,1,0,0,0,0,0), & c(T_2^*) &= (1,1,1,0,1,1,1,1), \\ c(T_3^*) &= (1,0,0,0,1,1,1,1), & c(T_4^*) &= (1,1,1,1,0,0,0,1), \\ c(T_5^*) &= (1,1,1,1,1,0,0,1), & c(T_6^*) &= (1,1,1,1,1,1,0,1), \end{aligned}$$

in the basis $[S_0]$, $[S_1]$, $[S_2]$, $[S_3]$, $[S_4]$, $[S_5]$, $[S_6]$, $[S_7]$ of $K_0(A)$.

(a) Prove that $T_1^* \oplus T_2^* \oplus T_3^*$ is a partial tilting module in the additive category $\operatorname{add} \mathcal{T}_0^A$ of a stable tube $\mathcal{T}_0^A$ of rank 3 in Γ_A.

(b) Prove that $T_3^* \oplus T_4^* \oplus T_5^*$ is a partial tilting module in the additive category $\operatorname{add} \mathcal{T}_\infty^A$ of a stable tube $\mathcal{T}_\infty^A$ of rank 3 in Γ_A.

(c) Prove that $T^* = I_0 \oplus T_1^* \oplus T_2^* \oplus T_3^* \oplus T_4^* \oplus T_5^* \oplus T_6^* \oplus S_7$ is a tilting module in $\operatorname{mod} A$ such that $B^* = \operatorname{End}_A(T^*)$ is isomorphic to the bound quiver algebra $K\Delta^*/J^*$, where Δ^* is the quiver



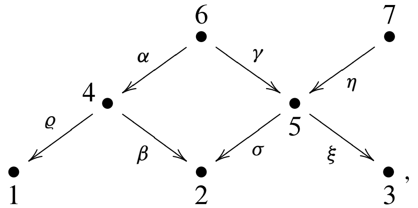
and J^* is the ideal in the path algebra $K\Delta^*$ of Δ^* over K generated by $\alpha\gamma$, $\delta\sigma$, $\beta\varrho$.

(d) Describe the torsion pair $(\mathcal{T}(T^*), \mathcal{F}(T^*))$ in $\operatorname{mod} A$ induced by T^*.

(e) Describe the torsion pair $(\mathcal{X}(T^*), \mathcal{Y}(T^*))$ in $\operatorname{mod} B^*$ induced by T^*.

(f) Describe the Auslander–Reiten quiver Γ_{B^*} of B^*.

17. Let K be a field, Δ the quiver

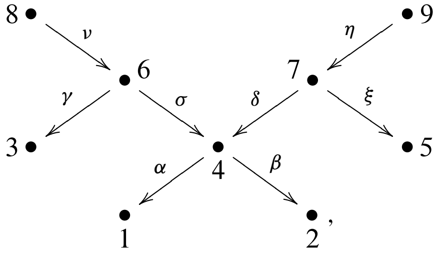


J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta - \gamma\sigma$ and $\eta\xi$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) B is of finite representation type.

(b) There exist a hereditary algebra A of Dynkin type $\mathbb{E}_6$ and a tilting module T in $\operatorname{mod} A$ such that B is isomorphic to $\operatorname{End}_A(T)$.

18. Let K be a field, Δ the quiver

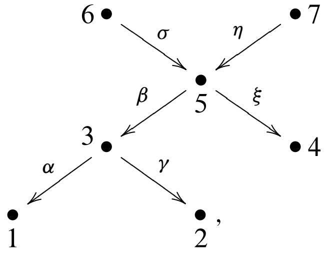


J the ideal in the path algebra $K\Delta$ of Δ over K generated by the paths $\sigma\alpha$, $\delta\beta$, $\nu\gamma$, $\eta\xi$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) B is of finite representation type.

(b) There exist a hereditary algebra A of Dynkin type $\mathbb{A}_9$ and a tilting module T in mod A such that B is isomorphic to $\operatorname{End}_A(T)$.

19. Let K be a field, Δ the quiver

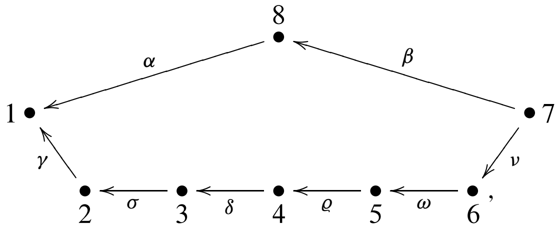


J the ideal in the path algebra $K\Delta$ of Δ over K generated by the paths $\beta\gamma$, $\sigma\beta$, $\eta\xi$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) B is of finite representation type.

(b) B is not a tilted algebra.

20. Let K be a field, Δ the quiver

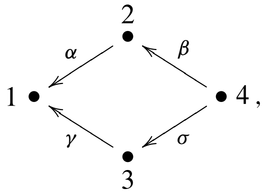


J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\beta\alpha - \nu\omega\varrho\delta\sigma\gamma$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) B is of finite representation type.

(b) There exist a hereditary algebra A of Dynkin type $\mathbb{D}_8$ and a tilting module T in $\operatorname{mod} A$ such that B is isomorphic to $\operatorname{End}_A(T)$.

21. Let K be a field, Δ the quiver



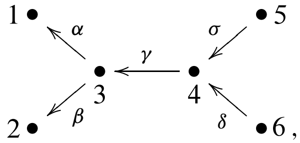
J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\beta\alpha$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) B is of finite representation type.

(b) B is not a tilted algebra.

(c) The Auslander–Reiten quiver Γ_B of B contains an oriented cycle.

22. Let K be a field, Δ the quiver



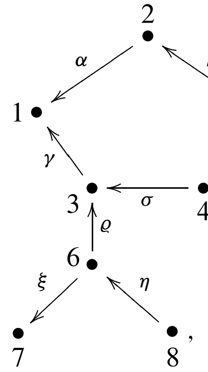
J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\gamma\alpha$, $\gamma\beta$, $\sigma\gamma$, $\delta\gamma$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) B is of finite representation type.

(b) B is not a tilted algebra.

(c) The Auslander–Reiten quiver Γ_B of B is acyclic.

23. Let K be a field, Δ the quiver



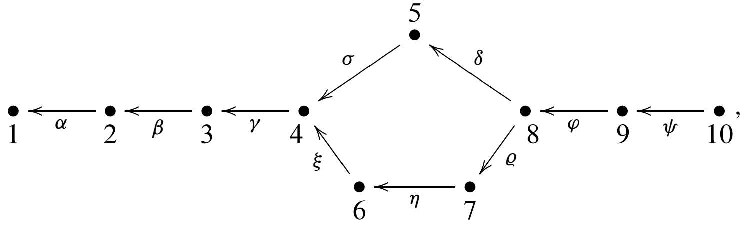
J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\beta\alpha$, $\sigma\gamma$, $\eta\xi$, J^* the ideal in $K\Delta$ generated by $\beta\alpha$, $\sigma\gamma$, $\eta\xi$, $\delta\sigma$, and $B = K\Delta/J$, $B^* = K\Delta/J^*$ the associated bound quiver algebras. Prove the following statements:

(a) B is of finite representation type.

(b) There exist a hereditary algebra A of Euclidean type $\widetilde{\mathbb{A}}_6$ and a tilting module T in mod A such that B is isomorphic to $\operatorname{End}_A(T)$.

(c) B^* is a quotient algebra of B but is not a tilted algebra.

24. Let K be a field, Δ the quiver

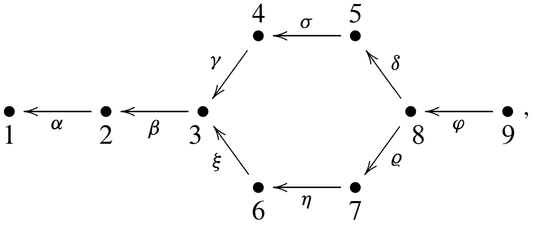


J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\delta\sigma - \varrho\eta\xi$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) B is of finite representation type.

(b) There exist a hereditary algebra A of wild type and a tilting module T in mod A such that B is isomorphic to $\operatorname{End}_A(T)$.

25. Let K be a field, Δ the quiver



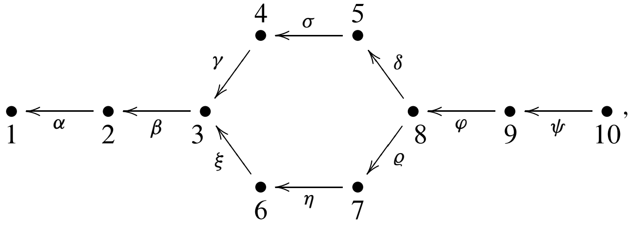
J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\delta\sigma\gamma - \varrho\eta\xi$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove that there exist a hereditary algebra A of wild type and a tilting module T in mod A such that the following statements hold:

(a) B is isomorphic to $\operatorname{End}_A(T)$.

(b) The connecting component $\mathcal{C}_T$ of Γ_B determined by T has an infinite torsion-free part $\mathcal{Y}(T) \cap \mathcal{C}_T$ and a finite torsion part $\mathcal{X}(T) \cap \mathcal{C}_T$.

(c) The Auslander–Reiten quiver Γ_B of B consists of $\mathcal{C}_T$, one postprojective component, and stable tubes.

26. Let K be a field, Δ the quiver



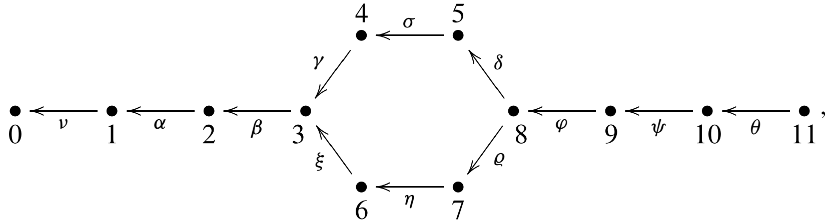
J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\delta\sigma\gamma - \varrho\eta\xi$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove that there exist a hereditary algebra A of wild type and a tilting module T in mod A such that the following statements hold:

(a) B is isomorphic to $\operatorname{End}_A(T)$.

(b) The connecting component $\mathcal{C}_T$ of Γ_B determined by T has an infinite torsion-free part $\mathcal{Y}(T) \cap \mathcal{C}_T$ and an infinite torsion part $\mathcal{X}(T) \cap \mathcal{C}_T$.

(c) The Auslander–Reiten quiver Γ_B of B consists of $\mathcal{C}_T$, one postprojective component, one preinjective component, and stable tubes.

27. Let K be a field, Δ the quiver



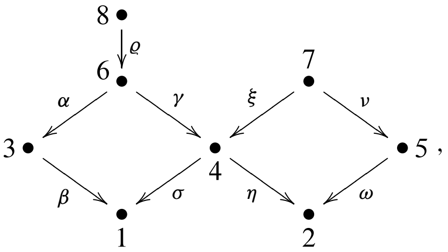
J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\delta\sigma\gamma - \varrho\eta\xi$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove that there exist a hereditary algebra A of wild type and a tilting module T in $\operatorname{mod} A$ such that the following statements hold:

(a) B is isomorphic to $\operatorname{End}_A(T)$.

(b) The connecting component $\mathcal{C}_T$ of Γ_B determined by T has an infinite torsion-free part $\mathcal{Y}(T) \cap \mathcal{C}_T$ and an infinite torsion part $\mathcal{X}(T) \cap \mathcal{C}_T$.

(c) The Auslander–Reiten quiver Γ_B of B consists of $\mathcal{C}_T$, one postprojective component, one preinjective component, and regular components of the form $\mathbb{Z}\mathbb{A}_\infty$.

28. Let K be a field, Δ the quiver



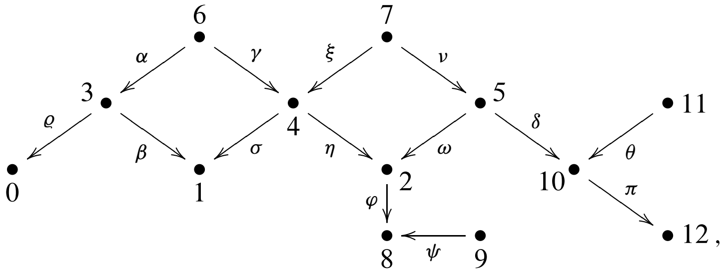
J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta - \gamma\sigma$, $\xi\eta - \nu\omega$, $\varrho\gamma\eta$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) The Auslander–Reiten quiver Γ_B of B contains a preinjective component containing all indecomposable injective right B-modules.

(b) There exist a hereditary K-algebra A of Euclidean type $\widetilde{\mathbb{E}}_7$ and a tilting module T in $\operatorname{mod} A$ such that B is isomorphic to $\operatorname{End}_A(T)$.

(c) B is not a concealed hereditary algebra.

29. Let K be a field, Δ the quiver



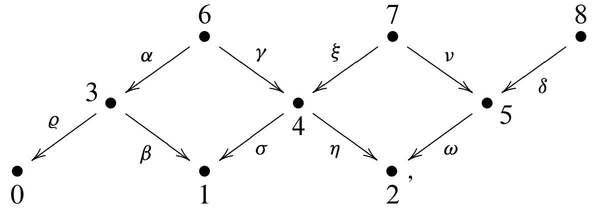
J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta - \gamma\sigma$, $\xi\eta - \nu\omega$, $\alpha\varrho$, $\omega\varphi$, $\nu\delta$, $\theta\pi$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) The Auslander–Reiten quiver Γ_B of B contains a postprojective component containing all indecomposable projective right B-modules.

(b) There exist a hereditary K-algebra A of Euclidean type $\widetilde{\mathbb{D}}_{12}$ and a tilting module T in mod A such that B is isomorphic to $\mathrm{End}_A(T)$.

(c) B is not a concealed algebra.

30. Let K be a field, Δ the quiver



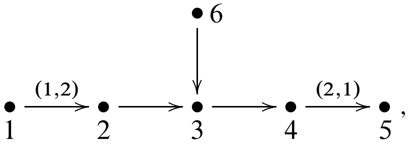
J the ideal in the path algebra $K\Delta$ of Δ over K generated by $\alpha\beta - \gamma\sigma$, $\xi\eta - \nu\omega$, $\alpha\varrho$, $\delta\omega$, and $B = K\Delta/J$ the associated bound quiver algebra. Prove the following statements:

(a) The Auslander–Reiten quiver Γ_B of B contains a postprojective component having a section of Euclidean type $\widetilde{\mathbb{D}}_7$.

(b) The Auslander–Reiten quiver Γ_B of B contains a preinjective component having a section of Euclidean type $\widetilde{\mathbb{D}}_7$.

(c) B is not a tilted algebra.

31. Let K be a field, $\mathbb{M}$ a K-species with the quiver $Q_{\mathbb{M}}$ of the form



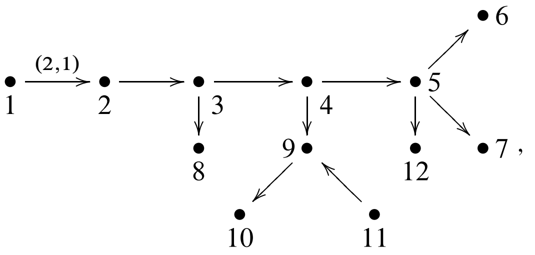
$H = \mathrm{T}(\mathbb{M})$ the tensor algebra of $\mathbb{M}$, J the ideal He_6He_4H of H, and $B = H/J$ the associated quotient algebra. Prove the following statements:

(a) The Auslander–Reiten quiver Γ_B of B contains a preinjective component containing all indecomposable injective right B-modules.

(b) There exist a hereditary K-algebra A of Euclidean type $\widetilde{\mathbb{B}}_5$ and a tilting module T in mod A such that B is isomorphic to $\mathrm{End}_A(T)$.

(c) B is not a concealed algebra.

32. Let K be a field, $\mathbb{M}$ a K-species with the quiver $Q_{\mathbb{M}}$ of the form



$H = \mathrm{T}(\mathbb{M})$ the tensor algebra of $\mathbb{M}$, J the ideal

$$He_2He_8H + He_3He_9H + He_{11}He_{10}H + He_4He_{12}H$$

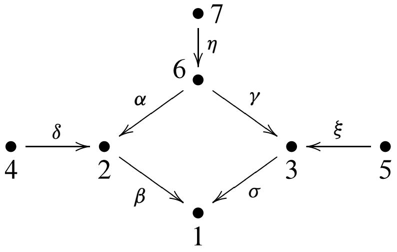
of H, and $B = H/J$ the associated quotient algebra. Prove the following statements:

(a) The Auslander–Reiten quiver Γ_B of B contains a postprojective component containing all indecomposable projective right B-modules.

(b) There exist a hereditary K-algebra A of Euclidean type $\widetilde{\mathbb{CD}}_{11}$ and a tilting module T in mod A such that B is isomorphic to $\mathrm{End}_A(T)$.

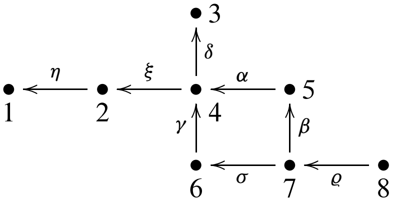
(c) B is not a concealed algebra.

33. Let K be a field, Q the quiver



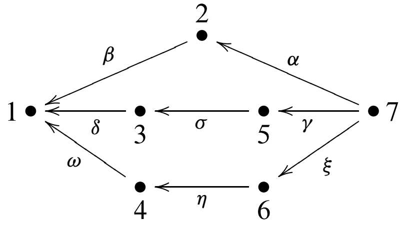
and I the ideal in KQ generated by $\alpha\beta - \gamma\sigma$. Prove that the bound quiver algebra $B = KQ/I$ is a concealed hereditary algebra of Euclidean type $\widetilde{\mathbb{E}}_6$.

34. Let K be a field, Q the quiver



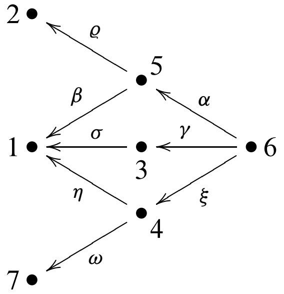
and I the ideal in KQ generated by $\beta\alpha - \sigma\gamma$ and $\alpha\delta$. Prove that the bound quiver algebra $B = KQ/I$ is a concealed hereditary algebra of Euclidean type $\widetilde{\mathbb{E}}_7$.

35. Let K be a field, Q the quiver



and I the ideal in KQ generated by $\alpha\beta + \gamma\sigma\delta + \xi\eta\omega$. Prove that the bound quiver algebra $B = KQ/I$ is a concealed hereditary algebra of Euclidean type $\widetilde{\mathbb{E}}_6$.

36. Let K be a field, Q the quiver



and I the ideal in KQ generated by $\alpha\beta + \gamma\sigma + \xi\eta$, $\alpha\varrho$ and $\xi\omega$, and $B = KQ/I$ the associated bound quiver algebra. Prove the following statements:

(a) There exist a hereditary K-algebra A of Euclidean type $\widetilde{\mathbb{E}}_6$ and a tilting module T in mod A such that B is isomorphic to $\operatorname{End}_A(T)$.

(b) B is not a concealed hereditary algebra.

(c) The Auslander–Reiten quiver Γ_B of B contains a postprojective component containing all indecomposable projective right B-modules.

(d) The Auslander–Reiten quiver Γ_B of B contains a preinjective component containing five indecomposable injective right B-modules.

37. Let K be a field, Q the quiver

$$\begin{tikzcd}
 & \overset{6}{\bullet} \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \overset{7}{\bullet} \arrow[d, "\omega"] & \overset{8}{\bullet} \arrow[dl, "\xi"'] \arrow[dr, "\delta"] & \\
3\,\bullet \arrow[dr, "\beta"'] & & \underset{4}{\bullet} \arrow[dl, "\sigma"] \arrow[dr, "\eta"'] & & \bullet\,5 \arrow[dl, "\varrho"] \\
 & \underset{1}{\bullet} & & \underset{2}{\bullet} &
\end{tikzcd}$$

and I the ideal in KQ generated by $\alpha\beta - \gamma\sigma$ and $\xi\eta - \delta\varrho$. Prove that the bound quiver algebra $B = KQ/I$ is a concealed hereditary algebra of wild type.

38. Let K be a field, Q the quiver

$$\begin{tikzcd}
 & & \overset{1}{\bullet} \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & \overset{2}{\bullet} \arrow[r, "\delta"] \arrow[d, "\sigma"] & \overset{3}{\bullet} \arrow[r, "\varrho"] & \overset{4}{\bullet} & \overset{5}{\bullet} \arrow[l, "\omega"'] \\
\underset{6}{\bullet} \arrow[r, "\xi"'] & \underset{7}{\bullet} \arrow[r, "\eta"'] & \underset{8}{\bullet} \arrow[r, "\beta"'] & \underset{9}{\bullet} & & &
\end{tikzcd}$$

and I the ideal in KQ generated by $\alpha\beta - \gamma\sigma$. Prove that the bound quiver algebra $B = KQ/I$ is a concealed hereditary algebra of wild type.

39. Let K be a field, Q the quiver

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[r, "\eta"] & \overset{2}{\bullet} \arrow[r, "\xi"] & \overset{3}{\bullet} \arrow[dd, "\beta"'] & \overset{4}{\bullet} \arrow[l, "\alpha"'] \arrow[dr, "\sigma"] & \\
 & & & & \bullet\,8 \arrow[dl, "\delta"] \\
 & \underset{5}{\bullet} \arrow[r, "\varrho"'] & \underset{6}{\bullet} \arrow[r, "\gamma"'] & \underset{7}{\bullet} &
\end{tikzcd}$$

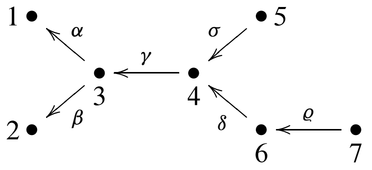
and I the ideal in KQ generated by $\alpha\beta\gamma - \sigma\delta$. Prove that the bound quiver algebra $B = KQ/I$ is a concealed hereditary algebra of wild type.

40. Let K be a field, Q the quiver

$$\begin{tikzcd}
 & \overset{1}{\bullet} \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & \overset{2}{\bullet} \arrow[r, "\varrho"] \arrow[d, "\sigma"] & \overset{3}{\bullet} \arrow[d, "\omega"] & \overset{4}{\bullet} \arrow[l, "\nu"'] \arrow[d, "\theta"] & \overset{5}{\bullet} \arrow[l, "\pi"'] \\
\underset{6}{\bullet} \arrow[r, "\eta"'] & \underset{7}{\bullet} \arrow[r, "\beta"'] & \underset{8}{\bullet} \arrow[r, "\delta"'] & \underset{9}{\bullet} & \underset{10}{\bullet} \arrow[l, "\xi"] &
\end{tikzcd}$$

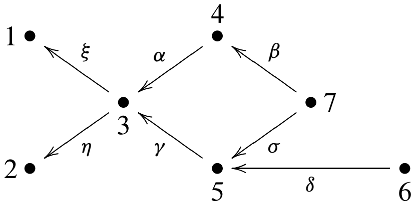
and I the ideal in KQ generated by $\alpha\beta - \gamma\sigma$, $\sigma\delta - \varrho\omega$ and $\nu\omega - \theta\xi$. Prove that the bound quiver algebra $B = KQ/I$ is a concealed hereditary algebra of wild type.

41. Let K be a field, Δ the quiver



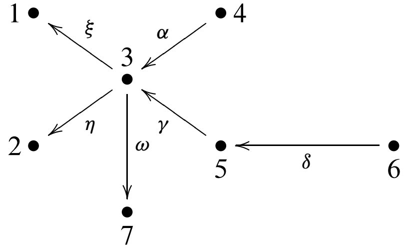
and $H = K\Delta$ the path algebra of Δ over K. Prove the following statements:

(a) There is a tilting module T in $\operatorname{mod} H$ such that $B = \operatorname{End}_H(T)$ is isomorphic to the bound quiver algebra KQ/I, where Q is the quiver of the form



and I the ideal in KQ generated by $\beta\alpha - \sigma\gamma$.

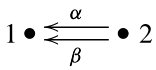
(b) There is a tilting module T^* in $\operatorname{mod} H$ such that $B^* = \operatorname{End}_H(T^*)$ is isomorphic to the bound quiver algebra KQ^*/I^*, where Q^* is the quiver of the form



and I^* the ideal in KQ^* generated by $\alpha\omega$ and $\gamma\omega$.

(c) B and B^* are not concealed hereditary algebras.

42. Let K be a field, Δ the Kronecker quiver



and $H = K\Delta$ the path algebra of Δ over K. Prove that there exist a hereditary K-algebra A and a tilting module T in $\operatorname{mod} A$ such that the tilted algebra $B = \operatorname{End}_A(T)$ has the following properties:

(a) The quiver Q_B of B is of the form

$$\begin{tikzcd}\underset{0}{\bullet} & \underset{1}{\bullet} \arrow[l, "(4{,}4)"'] & \underset{2}{\bullet} \arrow[l, "(2{,}2)"']\end{tikzcd}.$$

(b) The connecting component $\mathcal{C}_T$ of Γ_B determined by T has one injective module, but no projective modules.

(c) The postprojective component $\mathcal{P}(H)$ of Γ_H is a full translation subquiver of $\mathcal{C}_T$.

(d) The preinjective component $\mathcal{Q}(H)$ of Γ_H is a component of Γ_B.

43. Let K be a field, Δ the Kronecker quiver

$$\begin{tikzcd}1 \bullet & \bullet\, 2 \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\beta"]\end{tikzcd}$$

and $H = K\Delta$ the path algebra of Δ over K. Prove that there exist a hereditary K-algebra A and a tilting module T in $\operatorname{mod} A$ such that the tilted algebra $B = \operatorname{End}_A(T)$ has the following properties:

(a) The quiver Q_B of B is of the form

$$\begin{tikzcd}\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "(2{,}2)"'] & \underset{3}{\bullet} \arrow[l, "(3{,}3)"']\end{tikzcd}.$$

(b) The connecting component $\mathcal{C}_T$ of Γ_B determined by T has one projective module, but no injective modules.

(c) The preinjective component $\mathcal{Q}(H)$ of Γ_H is a full translation subquiver of $\mathcal{C}_T$.

(d) The postprojective component $\mathcal{P}(H)$ of Γ_H is a component of Γ_B.

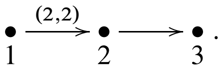
44. Let K be a field. Prove that there exist a hereditary K-algebra A and a tilting module T in $\operatorname{mod} A$ such that the tilted algebra $B = \operatorname{End}_A(T)$ has the following properties:

(a) The quiver Q_B of B is of the form

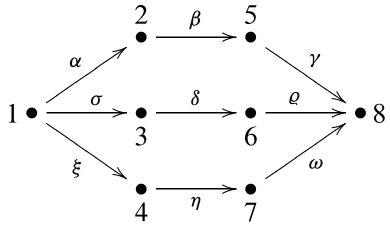
$$\begin{tikzcd}\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "(3{,}3)"'] & \underset{3}{\bullet} \arrow[l, "(4{,}4)"'] & \underset{4}{\bullet} \arrow[l, "(2{,}2)"']\end{tikzcd}.$$

(b) The connecting component $\mathcal{C}_T$ of Γ_B determined by T contains the indecomposable projective-injective module P whose socle is the unique simple projective module and top is the unique simple injective module in $\operatorname{mod} B$.

(c) The translation quiver $\mathcal{C}_T^s$ obtained from $\mathcal{C}$ by removing the module P is isomorphic to the stable translation quiver $\mathbb{Z}\Delta$, where Δ is the valued quiver

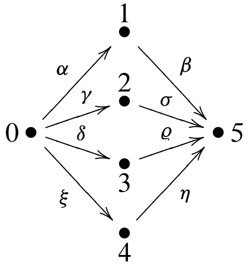


45. Let K be a field, Q the quiver



and I the ideal in KQ generated by $\alpha\beta\gamma + \sigma\delta\varrho + \xi\eta\omega$. Prove that the bound quiver algebra $B = KQ/I$ is not a tilted algebra.

46. Let K be a field, Q the quiver



and I the ideal in KQ generated by $\alpha\beta - \gamma\sigma$ and $\delta\varrho - \xi\eta$. Prove that the bound quiver algebra $B = KQ/I$ is not a tilted algebra.

47. Let A be a finite dimensional hereditary K-algebra of wild type over a field K and X a quasi-simple regular stone in $\operatorname{mod} A$. Prove that there exists a regular module Y in $\operatorname{mod} A$ such that $T = X \oplus Y$ is a tilting module in $\operatorname{mod} A$.

Chapter IX

Auslander–Reiten components

This chapter is devoted to presenting results on the shape and structure of connected components of the Auslander–Reiten quivers of finite dimensional algebras over a field. A prominent role is played by the functorial approach to the representation theory of finite dimensional algebras over a field, proposed by M. Auslander and I. Reiten in [AR2].

We start with characterizations of almost split sequences in module categories of finite dimensional algebras over a field in terms of minimal projective resolutions of simple functors in associated functor categories. As an application, we obtain the theorem due to K. Igusa and G. Todorov [IT] describing the radical level of the composition of irreducible homomorphisms between indecomposable modules forming a sectional path. Next we present elements of the theory of degrees of irreducible homomorphisms in module categories developed by S. Liu in [L1] and [L2]. This is applied in the next section to describe the shape of the connected components of the stable Auslander–Reiten quiver of a finite dimensional algebra over a field, proved independently by S. Liu [L1], [L2] and Y. Zhang [Z2]. In particular, the shape of regular components of Auslander–Reiten quivers of finite dimensional algebras over a field is described completely. Moreover, we prove the theorem due to A. Skowroński [S2] asserting that the generalized standard acyclic regular components of Auslander–Reiten quivers of finite dimensional algebras are the connecting components of tilted algebras determined by the regular tilting modules over hereditary algebras of wild type. The final section is devoted to results on stable equivalences on module categories of finite dimensional algebras over a field, established by M. Auslander and I. Reiten in [AR1], needed for further considerations.

1 Functors on module categories

The aim of this section is to present interpretations of almost split homomorphisms and almost split sequences in the module categories of finite dimensional algebras in terms of projective presentations of simple functors in associated functor categories.

Let A be a finite dimensional K-algebra over a field K and mod A the category of finite dimensional right A-modules. We denote by $\mathcal{F}(A)$ the category of all covariant functors from mod A to the category mod K of finite dimensional K-vector spaces. Recall that a functor F in $\mathcal{F}(A)$ assigns to a module X in mod A a vector space $F(X)$ in $\operatorname{mod} K$, and to a homomorphism $f\colon X \to Y$ in $\operatorname{mod} A$ a K-linear homomorphism $F(f)\colon F(X) \to F(Y)$ such that the following conditions are satisfied:

(1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any module X in $\operatorname{mod} A$;

(2) for each pair of homomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\operatorname{mod} A$, the equality $F(gf) = F(g)F(f)$ holds in $\operatorname{mod} K$.

For two functors F and G in $\mathcal{F}(A)$, the set $\operatorname{Hom}_{\mathcal{F}(A)}(F, G)$ of morphisms in $\mathcal{F}(A)$ from F to G consists of all natural transformations of functors $\varphi\colon F \to G$, which assign to modules X in $\operatorname{mod} A$ homomorphisms $\varphi_X\colon F(X) \to G(X)$ in $\operatorname{mod} K$ such that, for every homomorphism $f\colon X \to Y$ in $\operatorname{mod} A$, one has $\varphi_Y F(f) = G(f)\varphi_X$. We note that every module M in $\operatorname{mod} A$ provides the hom functor $\operatorname{Hom}_A(M, -)$ in $\mathcal{F}(A)$. Moreover, $\mathcal{F}(A)$ is a K-category with the K-vector space structures on the morphism sets $\operatorname{Hom}_{\mathcal{F}(A)}(F, G)$ given by the K-vector spaces of $\operatorname{Hom}_K(F(X), G(X))$, for all modules X in $\operatorname{mod} A$.

We denote by $\mathcal{F}(A)^{\mathrm{o}}$ the category of all contravariant functors from $\operatorname{mod} A$ to $\operatorname{mod} K$. Recall that a functor F in $\mathcal{F}(A)^{\mathrm{o}}$ assigns to a module X in $\operatorname{mod} A$ a vector space $F(X)$ in $\operatorname{mod} K$, and to a homomorphism $f\colon X \to Y$ in $\operatorname{mod} A$ a K-linear homomorphism $F(f)\colon F(Y) \to F(X)$ such that the following conditions are satisfied:

(1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any module X in $\operatorname{mod} A$;

(2) for each pair of homomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\operatorname{mod} A$, the equality $F(gf) = F(f)F(g)$ holds in $\operatorname{mod} K$.

For two functors F and G in $\mathcal{F}(A)^{\mathrm{o}}$, the set $\operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}(F, G)$ of morphisms in $\mathcal{F}(A)^{\mathrm{o}}$ from F to G consists of all natural transformations of functors $\varphi\colon F \to G$, which assign to modules X in $\operatorname{mod} A$ homomorphisms $\varphi_X\colon F(X) \to G(X)$ in $\operatorname{mod} K$ such that, for every homomorphism $f\colon X \to Y$ in $\operatorname{mod} A$, the equality $\varphi_X F(f) = G(f)\varphi_Y$ holds. We note that every module M in $\operatorname{mod} A$ provides the hom functor $\operatorname{Hom}_A(-, M)$ in $\mathcal{F}(A)^{\mathrm{o}}$. Similarly, as above, $\mathcal{F}(A)^{\mathrm{o}}$ is a K-category with the K-vector space structures on the morphism sets $\operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}(F, G)$ given by the K-vector spaces of $\operatorname{Hom}_K(F(X), G(X))$, for all modules X in $\operatorname{mod} A$.

Let F' and F be functors in $\mathcal{F}(A)$ (respectively, in $\mathcal{F}(A)^{\mathrm{o}}$). Then F' is said to be a *subfunctor* of F if, for every module X in $\operatorname{mod} A$, $F'(X)$ is a K-vector subspace of $F(X)$, and $F'(f)$ is the restriction of $F(f)$ to $F'(X)$ (respectively, to $F'(Y)$) for any homomorphism $f\colon X \to Y$ in $\operatorname{mod} A$. In this case, we may define the *quotient functor* F/F' such that $(F/F')(X) = F(X)/F'(X)$ for any module X in $\operatorname{mod} A$.

Let $\varphi\colon F \to G$ be a morphism in $\mathcal{F}(A)$ (respectively, in $\mathcal{F}(A)^{\mathrm{o}}$). Then the *kernel* $\operatorname{Ker}\varphi$ of φ is the subfunctor of F with $(\operatorname{Ker}\varphi)(X) = \operatorname{Ker}\varphi_X$ for any module X in $\operatorname{mod} A$. Further, the *image* of φ is the subfunctor $\operatorname{Im}\varphi$ of G with $(\operatorname{Im}\varphi)(X) = \operatorname{Im}\varphi_X$ for any module X in $\operatorname{mod} A$. Then the *cokernel* of φ is the functor $\operatorname{Coker}\varphi$ such that $(\operatorname{Coker}\varphi)(X) = \operatorname{Coker}\varphi_X$ for any module X in $\operatorname{mod} A$. There is a canonical isomorphism of functors $\psi\colon F/\operatorname{Ker}\varphi \to \operatorname{Im}\varphi$ such that, for any module X in $\operatorname{mod} A$, $\psi_X\colon (F/\operatorname{Ker}\varphi)(X) \to (\operatorname{Im}\varphi)(X)$ is the canonical K-linear isomorphism $\psi_X\colon F(X)/\operatorname{Ker}\varphi_X \to \operatorname{Im}\varphi_X$ given by $\psi_X(m + \operatorname{Ker}\varphi_X) = \varphi_X(m)$ for $m \in F(X)$.

The following proposition summarizes the properties of the categories $\mathcal{F}(A)$ and $\mathcal{F}(A)^{\mathrm{o}}$ described above.

Proposition 1.1. *Let A be a finite dimensional K-algebra over a field K. Then $\mathcal{F}(A)$ and $\mathcal{F}(A)^{\mathrm{o}}$ are abelian K-categories.*

The categories $\mathcal{F}(A)$ and $\mathcal{F}(A)^{\mathrm{o}}$ admit exact sequences. A sequence (finite or infinite)

$$\cdots \longrightarrow F_{n+1} \xrightarrow{\varphi_n} F_n \xrightarrow{\varphi_{n-1}} F_{n-1} \longrightarrow \cdots$$

in $\mathcal{F}(A)$ (respectively, in $\mathcal{F}(A)^{\mathrm{o}}$) is said to be *exact* if $\operatorname{Ker}\varphi_{n-1} = \operatorname{Im}\varphi_n$ for all n. We note that such a sequence is exact if and only if, for every module X in $\operatorname{mod} A$, the induced sequence of K-vector spaces

$$\cdots \longrightarrow F_{n+1}(X) \xrightarrow{(\varphi_n)_X} F_n(X) \xrightarrow{(\varphi_{n-1})_X} F_{n-1}(X) \longrightarrow \cdots$$

is exact. An exact sequence in $\mathcal{F}(A)$ (respectively, in $\mathcal{F}(A)^{\mathrm{o}}$) of the form

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0$$

is called a short exact sequence.

An important role in our considerations will be played by the following special case of the classical *Yoneda's lemma*.

Theorem 1.2. *Let A be a finite dimensional K-algebra over a field K and M a module in* mod A. *Then the following statements hold:*

(i) *For any functor F in $\mathcal{F}(A)^{\mathrm{o}}$, there is a bijection*

$$\pi\colon \operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\operatorname{Hom}_A(-, M), F\big) \longrightarrow F(M)$$

given by $\pi(\varphi) = \varphi_M(\mathrm{id}_M)$.

(ii) *For any functor F in $\mathcal{F}(A)$, there is a bijection*

$$\pi\colon \operatorname{Hom}_{\mathcal{F}(A)}\big(\operatorname{Hom}_A(M, -), F\big) \longrightarrow F(M)$$

given by $\pi(\varphi) = \varphi_M(\mathrm{id}_M)$.

Proof. We prove only the statement (i), because the proof of (ii) is similar.

Observe that, for any morphism $\varphi\colon \mathrm{Hom}_A(-, M) \to F$ in $\mathcal{F}(A)^{\mathrm{o}}$, $\pi(\varphi) = \varphi_M(\mathrm{id}_M)$ belongs to $F(M)$. We define now its inverse

$$\varrho\colon F(M) \longrightarrow \mathrm{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\mathrm{Hom}_A(-, M), F\big).$$

Let a be an element of $F(M)$ and X a module in $\operatorname{mod} A$. We define the map $\varrho(a)_X\colon \mathrm{Hom}_A(X, M) \to F(X)$ by $\varrho(a)_X(f) = F(f)(a)$ for $f \in \mathrm{Hom}_A(X, M)$. We note that $F(f)\colon F(M) \to F(X)$, because F is a contravariant functor. We will show now that $\varrho(a)\colon \mathrm{Hom}_A(-, M) \to F$ is a morphism in $\mathcal{F}(A)^{\mathrm{o}}$. Take a homomorphism $g\colon X \to Y$ in $\operatorname{mod} A$. We have to show that the following diagram in $\operatorname{mod} K$ is commutative

$$\begin{tikzcd}
\mathrm{Hom}_A(Y, M) \arrow[r, "\varrho(a)_Y"] \arrow[d, "\mathrm{Hom}_A(g,M)"'] & F(Y) \arrow[d, "F(g)"] \\
\mathrm{Hom}_A(X, M) \arrow[r, "\varrho(a)_X"] & F(X)\,.
\end{tikzcd}$$

Indeed, for any $f \in \mathrm{Hom}_A(Y, M)$, we have the equalities

$$\begin{aligned}\big(\varrho(a)_X \,\mathrm{Hom}_A(g, M)\big)(f) &= \varrho(a)_X(fg) = F(fg)(a) \\ &= F(g)\big(F(f)(a)\big) = \big(F(g)\varrho(a)_X\big)(f).\end{aligned}$$

We will show now that π and ϱ are mutually inverse maps.

Let $a \in F(M)$. Then

$$(\pi\varrho)(a) = \varrho(a)_M(\mathrm{id}_M) = F(\mathrm{id}_M)(a) = \mathrm{id}_{F(M)}(a) = a,$$

hence $\pi\varrho = \mathrm{id}_{F(M)}$.

Let $\varphi \in \mathrm{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}(\mathrm{Hom}_A(-, M), F)$. In order to show that $(\varrho\pi)(\varphi) = \varphi$, it is enough to verify that $(\varrho\pi)(\varphi)_X = \varphi_X$ for any module X in $\operatorname{mod} A$. Take a homomorphism $f \in \mathrm{Hom}_A(X, M)$. Since φ is a morphism in $\mathcal{F}(A)^{\mathrm{o}}$, the following diagram in $\operatorname{mod} K$ is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_A(M, M) \arrow[r, "\varphi_M"] \arrow[d, "\mathrm{Hom}_A(f,M)"'] & F(M) \arrow[d, "F(f)"] \\
\mathrm{Hom}_A(X, M) \arrow[r, "\varphi_X"] & F(X)\,.
\end{tikzcd}$$

Then we obtain the equalities

$$\begin{aligned}(\varrho\pi)(\varphi)_X(f) &= \varrho\big(\pi(\varphi)\big)_X = \varrho\big(\varphi_M(\mathrm{id}_M)\big)_X(f) = F(f)\big(\varphi_M(\mathrm{id}_M)\big) \\ &= \big(F(f)\varphi_M\big)(\mathrm{id}_M) = \big(\varphi_X \,\mathrm{Hom}_A(f, M)\big)(\mathrm{id}_M) = \varphi_X(f).\end{aligned}$$

Hence, $(\varrho\pi)(\varphi)_X = \varphi_X$. □

Corollary 1.3. *Let A be a finite dimensional K-algebra over a field K and M a module in* mod A*. Then the following statements hold:*

(i) *Let F be a subfunctor of the functor* $\operatorname{Hom}_A(-, M)$*. Then there is a bijection*

$$\varrho\colon F(M) \longrightarrow \operatorname{Hom}_{\mathcal{F}(A)^{o}}\big(\operatorname{Hom}_A(-, M), F\big)$$

given by $\varrho(f) = \operatorname{Hom}_A(-, f)$ *for* $f \in F(M)$.

(ii) *For any module N in* mod A*, there is a bijection*

$$\varrho\colon \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_{\mathcal{F}(A)^{o}}\big(\operatorname{Hom}_A(-, M), \operatorname{Hom}_A(-, N)\big)$$

given by $\varrho(f) = \operatorname{Hom}_A(-, f)$ *for* $f \in \operatorname{Hom}_A(M, N)$.

(iii) *Let F be a subfunctor of the functor* $\operatorname{Hom}_A(M, -)$*. Then there is a bijection*

$$\varrho\colon F(M) \longrightarrow \operatorname{Hom}_{\mathcal{F}(A)}\big(\operatorname{Hom}_A(M, -), F\big)$$

given by $\varrho(f) = \operatorname{Hom}_A(f, -)$ *for* $f \in F(M)$.

(iv) *For any module N in* mod A*, there is a bijection*

$$\varrho\colon \operatorname{Hom}_A(N, M) \longrightarrow \operatorname{Hom}_{\mathcal{F}(A)}\big(\operatorname{Hom}_A(M, -), \operatorname{Hom}_A(N, -)\big)$$

given by $\varrho(f) = \operatorname{Hom}_A(f, -)$ *for* $f \in \operatorname{Hom}_A(N, M)$.

Proof. We prove (i). Let $f \in F(M) \subseteq \operatorname{Hom}_A(M, M)$. Then it follows from Theorem 1.2 that we have the morphism $\varrho(f)\colon \operatorname{Hom}_A(-, M) \to F$. We claim that $\varrho(f) = \operatorname{Hom}_A(-, f)$. Let X be a module in mod A and $g \in \operatorname{Hom}_A(X, M)$. Then

$$\varrho(f)_X(g) = F(g)(f) = fg = \operatorname{Hom}_A(X, f)(g),$$

because $F(g) \in F(X) \subseteq \operatorname{Hom}_A(X, M)$. Hence, $\varrho(f) = \operatorname{Hom}_A(-, f)$, as claimed.

The statement (ii) is a direct consequence of (i). The proofs of (iii) and (iv) are similar. □

Corollary 1.4. *Let A be a finite dimensional K-algebra over a field K and M, N two modules in* mod A*. Then the following statements hold:*

(i) $M \cong N$ *in* mod A *if and only if* $\operatorname{Hom}_A(-, M) \cong \operatorname{Hom}_A(-, N)$ *in* $\mathcal{F}(A)^{o}$.

(ii) $M \cong N$ *in* mod A *if and only if* $\operatorname{Hom}_A(M, -) \cong \operatorname{Hom}_A(N, -)$ *in* $\mathcal{F}(A)$.

Proof. We prove only (i), since the proof of (ii) is similar. Obviously, if $M \cong N$ in mod A, then $\operatorname{Hom}_A(-, M) \cong \operatorname{Hom}_A(-, N)$ in $\mathcal{F}(A)^{o}$. Assume now that $\operatorname{Hom}_A(-, M) \cong \operatorname{Hom}_A(-, N)$ in $\mathcal{F}(A)^{o}$. Then it follows from Corollary 1.3 (ii) that there exist homomorphisms $f: M \to N$ and $g: N \to M$ in $\operatorname{mod} A$ such that the induced natural transformations of functors

$$\operatorname{Hom}_A(-, f): \operatorname{Hom}_A(-, M) \longrightarrow \operatorname{Hom}_A(-, N)$$

and

$$\operatorname{Hom}_A(-, g): \operatorname{Hom}_A(-, N) \longrightarrow \operatorname{Hom}_A(-, M)$$

are mutually inverse isomorphisms in $\mathcal{F}(A)^{\mathrm{o}}$. Hence we obtain the following equalities of morphisms in $\mathcal{F}(A)^{\mathrm{o}}$:

$$\operatorname{Hom}_A(-, \mathrm{id}_M) = \mathrm{id}_{\operatorname{Hom}_A(-,M)} = \operatorname{Hom}_A(-, g)\operatorname{Hom}_A(-, f) = \operatorname{Hom}_A(-, gf),$$
$$\operatorname{Hom}_A(-, \mathrm{id}_N) = \mathrm{id}_{\operatorname{Hom}_A(-,N)} = \operatorname{Hom}_A(-, f)\operatorname{Hom}_A(-, g) = \operatorname{Hom}_A(-, fg).$$

Applying Corollary 1.3 (ii), we obtain $\mathrm{id}_M = gf$, $\mathrm{id}_N = fg$, and consequently $M \cong N$ in $\operatorname{mod} A$. ☐

Let A be a finite dimensional K-algebra. A functor P in $\mathcal{F}(A)^{\mathrm{o}}$ (respectively, in $\mathcal{F}(A)$) is said to be *projective* if, for any epimorphism $\varphi: F \to G$ of functors in $\mathcal{F}(A)^{\mathrm{o}}$ (respectively, in $\mathcal{F}(A)$), the induced homomorphism of K-vector spaces

$$\operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}(P, \varphi): \operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}(P, F) \longrightarrow \operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}(P, G)$$

(respectively, $\operatorname{Hom}_{\mathcal{F}(A)}(P, \varphi): \operatorname{Hom}_{\mathcal{F}(A)}(P, F) \to \operatorname{Hom}_{\mathcal{F}(A)}(P, G)$) is an epimorphism. A functor F in $\mathcal{F}(A)^{\mathrm{o}}$ (respectively, in $\mathcal{F}(A)$) is said to be *finitely generated* if F is isomorphic to a quotient of a functor of the form $\operatorname{Hom}_A(-, M)$ (respectively, $\operatorname{Hom}_A(M, -)$), for a module M in $\operatorname{mod} A$.

We will show now that Theorem 1.2 provides also a description of the finitely generated projective functors in $\mathcal{F}(A)^{\mathrm{o}}$ and $\mathcal{F}(A)$.

Corollary 1.5. *Let A be a finite dimensional K-algebra over a field K and M a module in* $\operatorname{mod} A$. *Then the following statements hold:*

(i) *The functor* $\operatorname{Hom}_A(-, M)$ *is a projective object in* $\mathcal{F}(A)^{\mathrm{o}}$.

(ii) *The functor* $\operatorname{Hom}_A(M, -)$ *is a projective object in* $\mathcal{F}(A)$.

Proof. We prove only the statement (i), because the proof of (ii) is similar. Let $\varphi: F \to G$ be an epimorphism in $\mathcal{F}(A)^{\mathrm{o}}$. We must show that the induced homomorphism of K-vector spaces

$$\operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\operatorname{Hom}_A(-, M), \varphi\big): \operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\operatorname{Hom}_A(-, M), F\big) \longrightarrow \operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\operatorname{Hom}_A(-, M), G\big)$$

is an epimorphism. It follows from Theorem 1.2 that we have the bijections

$$\pi^F: \operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\operatorname{Hom}_A(-, M), F\big) \longrightarrow F(M),$$
$$\pi^G: \operatorname{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\operatorname{Hom}_A(-, M), G\big) \longrightarrow G(M).$$

We claim that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\mathrm{Hom}_A(-,M),F\big) \arrow[r, "\mathrm{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\left(\mathrm{Hom}_A(-,M),\varphi\right)"] \arrow[d, "\pi^F"'] & \mathrm{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\mathrm{Hom}_A(-,M),G\big) \arrow[d, "\pi^G"] \\
F(M) \arrow[r, "\varphi_M"] & G(M)
\end{tikzcd}$$

in mod K is commutative. Indeed, let $\psi\colon \mathrm{Hom}_A(-,M) \to F$ be a morphism in $\mathcal{F}(A)^{\mathrm{o}}$. Then

$$\begin{aligned}
\big(\varphi_M \pi^F\big)(\psi) &= \varphi_M\big(\psi_M(\mathrm{id}_M)\big) = (\varphi\psi)_M(\mathrm{id}_M) \\
&= \pi^G(\varphi\psi) = \Big(\pi^G\,\mathrm{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}\big(\mathrm{Hom}_A(-,M),\varphi\big)\Big)(\psi).
\end{aligned}$$

Moreover, φ_M is an epimorphism, because φ is an epimorphism. Since π^F and π^G are isomorphisms, we infer that $\mathrm{Hom}_{\mathcal{F}(A)^{\mathrm{o}}}(\mathrm{Hom}_A(-,M),\varphi)$ is also an epimorphism. □

Proposition 1.6. *Let A be a finite dimensional K-algebra over a field K. The following equivalences hold:*

(i) *A functor F in $\mathcal{F}(A)^{\mathrm{o}}$ is finitely generated projective if and only if F is isomorphic to a functor of the form* $\mathrm{Hom}_A(-,M)$*, for some module M in* mod A.

(ii) *A functor F in $\mathcal{F}(A)$ is finitely generated projective if and only if F is isomorphic to a functor of the form* $\mathrm{Hom}_A(M,-)$*, for some module M in* mod A.

Proof. We prove only (i), since the proof of (ii) is similar. The projectivity of a finitely generated functor $\mathrm{Hom}_A(-,M)$ in $\mathcal{F}(A)^{\mathrm{o}}$ follows from Corollary 1.5 (i). Conversely, assume that F is a finitely generated projective functor in $\mathcal{F}(A)^{\mathrm{o}}$. Then there is an epimorphism of functors $\varphi\colon \mathrm{Hom}_A(-,N) \to F$, for a module N in mod A. Since F is a projective functor in $\mathcal{F}(A)^{\mathrm{o}}$, there exists a morphism $\psi\colon F \to \mathrm{Hom}_A(-,N)$ in $\mathcal{F}(A)^{\mathrm{o}}$ such that $\varphi\psi = \mathrm{id}_F$. Let $\sigma = \psi\varphi$. It follows from Corollary 1.3 (ii) that there exists $f \in \mathrm{End}_A(N)$ such that $\sigma = \mathrm{Hom}_A(-,f)$. Moreover, we have the equalities

$$\mathrm{Hom}_A(-,f^2) = \mathrm{Hom}_A(-,f)^2 = \sigma^2 = \sigma = \mathrm{Hom}_A(-,f),$$

and hence $f^2 = f$. Now it follows from Lemma I.4.3 that $N = \mathrm{Ker}\, f \oplus \mathrm{Im}\, f$. Let $M = \mathrm{Im}\, f$. We note that ψ is a monomorphism in $\mathcal{F}(A)^{\mathrm{o}}$, because $\varphi\psi = \mathrm{id}_F$. Hence, F is isomorphic to $\mathrm{Im}\,\psi$. Moreover, we have in $\mathcal{F}(A)^{\mathrm{o}}$ isomorphisms

$$\mathrm{Im}\,\psi = \mathrm{Im}\,\psi\varphi = \mathrm{Im}\,\sigma = \mathrm{Im}\,\mathrm{Hom}_A(-,f) \cong \mathrm{Hom}_A(-,\mathrm{Im}\, f) = \mathrm{Hom}_A(-,M),$$

because φ is an epimorphism. Therefore, F is isomorphic to $\mathrm{Hom}_A(-,M)$. □

Let A be a finite dimensional K-algebra over a field K and M be a module in $\operatorname{mod} A$. Then we may consider the *radical functors* $\operatorname{rad}_A(-, M)$ in $\mathcal{F}(A)$ and $\operatorname{rad}_A(M, -)$ in $\mathcal{F}(A)^{\mathrm{o}}$ such that, for any module X in $\operatorname{mod} A$, $\operatorname{rad}_A(X, M)$ is the Jacobson radical of $\operatorname{Hom}_A(M, X)$ and $\operatorname{rad}_A(M, X)$ is the Jacobson radical of $\operatorname{Hom}_A(M, X)$. We refer to Section III.1 for more details.

Lemma 1.7. *Let A be a finite dimensional K-algebra over a field K and M be an indecomposable module in* $\operatorname{mod} A$. *The following statements hold:*

(i) *The functor* $\operatorname{rad}_A(-, M)$ *is the unique maximal subfunctor of the functor* $\operatorname{Hom}_A(-, M)$.

(ii) *The functor* $\operatorname{rad}_A(M, -)$ *is the unique maximal subfunctor of the functor* $\operatorname{Hom}_A(M, -)$.

Proof. We prove only (i), since the proof of (ii) is similar. Assume F is a proper subfunctor of $\operatorname{Hom}_A(-, M)$. We will prove that F is contained in $\operatorname{rad}_A(-, M)$. It suffices to show that $F(N) \subseteq \operatorname{rad}_A(N, M)$ for any indecomposable module N in $\operatorname{mod} A$. If N is not isomorphic to M, then $\operatorname{rad}_A(N, M) = \operatorname{Hom}_A(N, M)$, by Lemma III.1.4, and the claim follows. Assume $N \cong M$. Take a homomorphism $f: M \to M$ from $F(M)$. Then it follows from Corollary 1.3 (i) that the induced morphism $\operatorname{Hom}_A(-, f): \operatorname{Hom}_A(-, M) \to \operatorname{Hom}_A(-, M)$ maps $\operatorname{Hom}_A(-, M)$ to F. Since F is a proper subfunctor of $\operatorname{Hom}_A(-, M)$, we conclude that f is not an isomorphism. Therefore, $f \in \operatorname{rad}_A(M, M)$, by Lemma III.1.4. □

Corollary 1.8. *Let A be a finite dimensional K-algebra over a field K and M a module in* $\operatorname{mod} A$. *Then the following statements hold:*

(i) *The functor* $\operatorname{Hom}_A(-, M)$ *is indecomposable in* $\mathcal{F}(A)^{\mathrm{o}}$ *if and only if M is indecomposable in* $\operatorname{mod} A$.

(ii) *The functor* $\operatorname{Hom}_A(M, -)$ *is indecomposable in* $\mathcal{F}(A)$ *if and only if M is indecomposable in* $\operatorname{mod} A$.

Proof. Observe that, if $M = X \oplus Y$ in $\operatorname{mod} A$, then we have isomorphisms $\operatorname{Hom}_A(-, M) \cong \operatorname{Hom}_A(-, X) \oplus \operatorname{Hom}_A(-, Y)$ in $\mathcal{F}(A)^{\mathrm{o}}$ and $\operatorname{Hom}_A(M, -) \cong \operatorname{Hom}_A(X, -) \oplus \operatorname{Hom}_A(Y, -)$ in $\mathcal{F}(A)$. Then the required equivalences follow from Lemma 1.7. □

A nonzero functor S in $\mathcal{F}(A)^{\mathrm{o}}$ (respectively, in $\mathcal{F}(A)$) is said to be *simple* if it has no proper nonzero subfunctor. For an indecomposable module M in $\operatorname{mod} A$, we define $S^M = \operatorname{Hom}_A(-, M)/\operatorname{rad}_A(-, M)$ and $S_M = \operatorname{Hom}_A(M, -)/\operatorname{rad}_A(M, -)$. We note that for a module X in $\operatorname{mod} A$, we have $S^M(X) \neq 0$ (respectively, $S_M(X) \neq 0$) if and only if M is isomorphic to a direct summand of X, by Lemmas III.1.3 and III.1.4.

We have also the following direct consequence of Lemma 1.7.

Corollary 1.9. *Let A be a finite dimensional K-algebra over a field K and M be an indecomposable module in* mod A. *Then the following statements hold:*

(i) *The functor S^M is a simple functor in $\mathcal{F}(A)^{\circ}$.*

(ii) *The functor S_M is a simple functor in $\mathcal{F}(A)$.*

Proposition 1.10. *Let A be a finite dimensional K-algebra over a field K. The following equivalences hold:*

(i) *A functor S in $\mathcal{F}(A)^{\circ}$ is simple if and only if S is isomorphic to the functor S^M, for a unique indecomposable module M in* mod A.

(ii) *A functor S in $\mathcal{F}(A)$ is simple if and only if S is isomorphic to the functor S_M, for a unique indecomposable module M in* mod A.

Proof. We prove only (i), since the proof of (ii) is similar. The sufficiency part of (i) follows from Corollary 1.9 (i). Let S be a simple functor in $\mathcal{F}(A)^{\circ}$. Because S is a nonzero functor, there exists an indecomposable module M in mod A such that $S(M) \neq 0$. Then, by Theorem 1.2, there is a nonzero morphism $\theta^M \colon \mathrm{Hom}_A(-, M) \to S$ in $\mathcal{F}(A)^{\circ}$. Clearly, θ^M is necessarily an epimorphism of functors, because S is a simple functor and $\mathrm{Im}\,\theta^M \neq 0$. We claim that $\theta^M(\mathrm{rad}_A(-, M)) = 0$, and consequently θ^M induces an isomorphism $S^M \xrightarrow{\sim} S$ of functors in $\mathcal{F}(A)^{\circ}$. It suffices to show that, if $S(X) \neq 0$ for an indecomposable module X in mod A, then $X \cong M$. Suppose that $S(X) \neq 0$ for some indecomposable module X in mod A. Then, applying Theorem 1.2 again, we conclude that there is an epimorphism of functors $\theta^X \colon \mathrm{Hom}_A(-, X) \to S$ in $\mathcal{F}(A)^{\circ}$. Then it follows from Corollaries 1.3 and 1.5 that there is a commutative diagram in $\mathcal{F}(A)^{\circ}$

$$\begin{tikzcd}
\mathrm{Hom}_A(-, M) \arrow[r, "\theta^M"] \arrow[d, "\mathrm{Hom}_A(-,f)"'] & S \arrow[d, "\mathrm{id}_S"] \\
\mathrm{Hom}_A(-, X) \arrow[r, "\theta^X"] \arrow[d, "\mathrm{Hom}_A(-,g)"'] & S \arrow[d, "\mathrm{id}_S"] \\
\mathrm{Hom}_A(-, M) \arrow[r, "\theta^M"] & S
\end{tikzcd}$$

for some homomorphisms $f \colon M \to X$ and $g \colon X \to M$ in mod A. Since M is indecomposable, $\mathrm{End}_A(M)$ is a local algebra, by Lemma I.4.4. Then it follows from Corollary I.3.9 that the endomorphism gf of M is nilpotent or invertible. However, if $(gf)^m = 0$ for some positive integer m, then $\theta^M = \theta^M\,\mathrm{Hom}_A(-, (gf)^m) = 0$, a contradiction. Hence, gf is invertible. In particular, $hgf = \mathrm{id}_M$ for some $h \in \mathrm{End}_A(M)$. It follows from Lemma I.4.2 that $X = \mathrm{Im}\, f \oplus \mathrm{Ker}(hg)$, with $\mathrm{Im}\, f \cong M$, because f is a section. Then the indecomposability of X forces $X = \mathrm{Im}\, f$, and so $X \cong M$. □

Let A be a finite dimensional K-algebra. For an indecomposable module M in $\operatorname{mod} A$, we denote by $\pi^M\colon \operatorname{Hom}_A(-, M) \to S^M$ the canonical epimorphism of functors such that $(\pi^M)_M(\mathrm{id}_M) = \mathrm{id}_M + \operatorname{rad}_A(M, M)$, and by $\pi_M\colon \operatorname{Hom}_A(M, -) \to S_M$ the canonical epimorphism of functors such that $(\pi_M)_M(\mathrm{id}_M) = \mathrm{id}_M + \operatorname{rad}_A(M, M)$. An epimorphism $\varphi\colon F \to G$ in $\mathcal{F}(A)^{\mathrm{o}}$ (respectively, in $\mathcal{F}(A)$) is said to be *minimal* if, for each morphism $\psi\colon H \to F$ in $\mathcal{F}(A)^{\mathrm{o}}$ (respectively, in $\mathcal{F}(A)$), the composition $\varphi\psi$ is an epimorphism if and only if ψ is an epimorphism. A minimal epimorphism $\varphi\colon F \to G$ in $\mathcal{F}(A)^{\mathrm{o}}$ (respectively, in $\mathcal{F}(A)$) with F a projective functor is said to be a *projective cover* of G.

Lemma 1.11. *Let A be a finite dimensional K-algebra over a field K and M be an indecomposable module in* $\operatorname{mod} A$. *Then the following statements hold:*

(i) *The epimorphism $\pi^M\colon \operatorname{Hom}_A(-, M) \to S^M$ in $\mathcal{F}(A)^{\mathrm{o}}$ is a projective cover of S^M.*

(ii) *The epimorphism $\pi_M\colon \operatorname{Hom}_A(M, -) \to S_M$ in $\mathcal{F}(A)$ is a projective cover of S_M.*

Proof. We prove only (i), the proof of (ii) being similar. Since the functor $\operatorname{Hom}_A(-, M)$ is projective in $\mathcal{F}(A)^{\mathrm{o}}$, it remains to show that the epimorphism π^M is minimal. Let $\psi\colon H \to \operatorname{Hom}_A(-, M)$ be a morphism in $\mathcal{F}(A)^{\mathrm{o}}$ such that $\pi^M\psi$ is an epimorphism in $\mathcal{F}(A)^{\mathrm{o}}$. We first show that $\psi_M\colon H(M) \to \operatorname{Hom}_A(M, M)$ is an epimorphism. Observe that $\operatorname{rad}_A(M, M)$ is the Jacobson radical $\operatorname{rad}\operatorname{End}_A(M)$ of the endomorphism algebra $\operatorname{End}_A(M)$, which is a local algebra, and $S^M(M)$ is the quotient algebra $\operatorname{End}_A(M)/\operatorname{rad}\operatorname{End}_A(M)$. Further, $H(M)$ is a right $\operatorname{End}_A(M)$-module, by $xf = H(f)(x)$ for any $x \in H(M)$ and $f \in \operatorname{End}_A(M)$, and $\psi_M\colon H(M) \to \operatorname{End}_A(M)$ is a homomorphism of right $\operatorname{End}_A(M)$-modules. Since $\operatorname{rad}\operatorname{End}_A(M)$ is a unique maximal right ideal of the algebra $\operatorname{End}_A(M)$ and $(\pi^M)_M\psi_M = (\pi^M\psi)_M\colon H(M) \to S^M(M)$ is an epimorphism, we conclude that $\operatorname{Im}\psi_M = \operatorname{End}_A(M)$, and so ψ_M is an epimorphism. We may then choose an element $a \in H(M)$ such that $\psi_M(a) = \mathrm{id}_M$. Let X be a module in $\operatorname{mod} A$ and $f\colon X \to M$ a homomorphism in $\operatorname{mod} A$. Then we have the commutative diagram of K-vector spaces

$$\begin{tikzcd}
H(M) \arrow[r, "\psi_M"] \arrow[d, "H(f)"'] & \operatorname{Hom}_A(M, M) \arrow[d, "\operatorname{Hom}_A(f, M)"] \\
H(X) \arrow[r, "\psi_X"] & \operatorname{Hom}_A(X, M) .
\end{tikzcd}$$

In particular, we obtain that

$$f = \operatorname{Hom}_A(f, M)(\mathrm{id}_M) = \operatorname{Hom}_A(f, M)\big(\psi_M(a)\big) = \psi_M\big(H(f)a\big).$$

This shows that ψ_X is an epimorphism. Therefore, ψ is an epimorphism in $\mathcal{F}(A)^{\mathrm{o}}$. □

Let A be a finite dimensional K-algebra. An exact sequence $F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} G \to 0$ in $\mathcal{F}(A)^{\mathrm{o}}$ (respectively, in $\mathcal{F}(A)$), with F_0 and F_1 projective functors, is said to be a *projective presentation* of the functor G. Moreover, if in addition, $\varphi_0\colon F_0 \to G$ and $\varphi_1\colon F_1 \to \operatorname{Im}\varphi_1$ are projective covers, the sequence is called a *minimal projective presentation* of G.

The next lemma shows that the almost split homomorphisms in $\operatorname{mod} A$ correspond to the projective presentations of simple functors in $\mathcal{F}(A)^{\mathrm{o}}$ and $\mathcal{F}(A)$.

Lemma 1.12. *Let A be a finite dimensional K-algebra over a field K and N, L be indecomposable modules in* $\operatorname{mod} A$. *The following equivalences hold:*

(i) *A homomorphism $g\colon M \to N$ in* $\operatorname{mod} A$ *is a right almost split homomorphism if and only if the induced sequence of functors*

$$\operatorname{Hom}_A(-, M) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{Hom}_A(-, N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a projective presentation of S^N in $\mathcal{F}(A)^{\mathrm{o}}$.

(ii) *A homomorphism $f\colon L \to M$ in* $\operatorname{mod} A$ *is a left almost split homomorphism if and only if the induced sequence of functors*

$$\operatorname{Hom}_A(M, -) \xrightarrow{\operatorname{Hom}_A(f,-)} \operatorname{Hom}_A(L, -) \xrightarrow{\pi_L} S_L \longrightarrow 0$$

is a projective presentation of S_L in $\mathcal{F}(A)$.

Proof. We prove only (i), since the proof of (ii) is similar.

Let $g\colon M \to N$ be a homomorphism in $\operatorname{mod} A$. We must show that g is a right almost split homomorphism if and only if $\operatorname{Im}\operatorname{Hom}_A(-, g) = \operatorname{Ker}\pi^N = \operatorname{rad}_A(-, N)$, or equivalently, that $\operatorname{Im}\operatorname{Hom}_A(X, g) = \operatorname{rad}_A(X, N)$ for any indecomposable module X in $\operatorname{mod} A$.

Assume g is a right almost split homomorphism, and X an indecomposable module in $\operatorname{mod} A$. Take $h \in \operatorname{rad}_A(X, N)$. Then h is not an isomorphism, and so is not a retraction. Hence there exists a homomorphism $s\colon X \to M$ such that $h = gs = \operatorname{Hom}_A(X, g)(s)$. Thus $\operatorname{rad}_A(X, N) \subseteq \operatorname{Im}\operatorname{Hom}_A(X, g)$. Observe also that if X is not isomorphic to N, then $\operatorname{rad}_A(X, N) = \operatorname{Hom}_A(X, N)$, and then $\operatorname{Im}\operatorname{Hom}_A(X, g) \subseteq \operatorname{rad}_A(X, N)$. Assume $X \cong N$. Suppose that there exists $r \in \operatorname{Hom}_A(X, M)$ with $gr = \operatorname{Hom}_A(X, g)(r)$ not in $\operatorname{rad}_A(X, N)$. But then gr is an isomorphism, which implies that g is a retraction. This is a contradiction because g is a right almost split homomorphism. This shows that $\operatorname{rad}_A(X, N) \subseteq \operatorname{Im}\operatorname{Hom}_A(X, g)$. Summing up, we have $\operatorname{Im}\operatorname{Hom}_A(X, g) = \operatorname{rad}_A(X, N)$ for any indecomposable module X in $\operatorname{mod} A$, and hence $\operatorname{Im}\operatorname{Hom}_A(-, g) = \operatorname{rad}_A(-, N)$.

Assume now that $\operatorname{Im}\operatorname{Hom}_A(-, g) = \operatorname{rad}_A(-, N)$. We will show that g is a right almost split homomorphism in $\operatorname{mod} A$. We prove first that g is not a retraction. Suppose $gh = \mathrm{id}_N$ for some homomorphism $h\colon N \to M$ in $\operatorname{mod} A$. Then for any $t \in \operatorname{End}_A(N)$ we have $t = ght = \operatorname{Hom}_A(N,g)(ht) \in \operatorname{Ker}(\pi^N)_N$, and consequently $S^N(N) = 0$, a contradiction. Hence g is not a retraction. Now let X be an indecomposable module and $u\colon X \to N$ a homomorphism in $\operatorname{mod} A$ which is not a retraction. Then u is not an isomorphism, and hence $u \in \operatorname{rad}_A(X,N)$. Since $\operatorname{rad}_A(X,N) = \operatorname{Ker}(\pi^N)_X = \operatorname{Im}\operatorname{Hom}_A(X,g)$, we conclude that there exists a homomorphism $v\colon X \to M$ in $\operatorname{mod} A$ such that $u = \operatorname{Hom}_A(X,g)(v) = gv$. Therefore, g is a right almost split homomorphism in $\operatorname{mod} A$. □

The next lemma shows that the minimal almost split homomorphisms in $\operatorname{mod} A$ correspond to the minimal projective presentations of simple functors in $\mathcal{F}(A)^{\mathrm{o}}$ and $\mathcal{F}(A)$.

Lemma 1.13. *Let A be a finite dimensional K-algebra over a field K and N, L be indecomposable modules in* $\operatorname{mod} A$. *The following equivalences hold:*

(i) *A homomorphism $g\colon M \to N$ in* $\operatorname{mod} A$ *is a right minimal almost split homomorphism if and only if the induced sequence of functors*

$$\operatorname{Hom}_A(-,M) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective presentation of S^N in $\mathcal{F}(A)^{\mathrm{o}}$.

(ii) *A homomorphism $f\colon L \to M$ in* $\operatorname{mod} A$ *is a left minimal almost split homomorphism if and only if the induced sequence of functors*

$$\operatorname{Hom}_A(M,-) \xrightarrow{\operatorname{Hom}_A(f,-)} \operatorname{Hom}_A(L,-) \xrightarrow{\pi_L} S_L \longrightarrow 0$$

is a minimal projective presentation of S_L in $\mathcal{F}(A)$.

Proof. We prove only (i), since the proof of (ii) is similar.

Assume g is a right minimal almost split sequence homomorphism in $\operatorname{mod} A$. Then it follows from Lemma 1.12 (i) that the induced sequence of functors is a projective presentation of S^N in $\mathcal{F}(A)^{\mathrm{o}}$. Moreover, π^N is a projective cover of S^N, by Lemma 1.11 (i). We have to show that the epimorphism of functors $\operatorname{Hom}_A(-,g)\colon \operatorname{Hom}_A(-,M) \to \operatorname{rad}_A(-,N)$ is minimal. Let $\varphi\colon F \to \operatorname{Hom}_A(-,M)$ be a morphism in $\mathcal{F}(A)^{\mathrm{o}}$ such that $\operatorname{Hom}_A(-,g)\varphi$ is an epimorphism. In particular, we have an epimorphism $\operatorname{Hom}_A(M,g)\varphi_M\colon F(M) \to \operatorname{rad}_A(M,N)$ of K-vector spaces. Hence, there exists an element $a \in F(M)$ such that $g = (\operatorname{Hom}_A(M,g)\varphi_M)(a) = g\varphi_M(a)$, where $\varphi_M(a) \in \operatorname{Hom}_A(M,M)$. Since g is right minimal, we infer that $\varphi_M(a)$ is an isomorphism. Let $u \in \operatorname{End}_A(M)$ be the homomorphism such that $\varphi_M(a)u = \mathrm{id}_M$. Take $b = F(u)(a) \in F(M)$. Then

$$\varphi_M(b) = \varphi_M\big(F(u)(a)\big) = \operatorname{Hom}_A(u,M)\big(\varphi_M(a)\big) = \varphi_M(a)u = \mathrm{id}_M\,.$$

Moreover, let $h: X \to M$ be a homomorphism in $\operatorname{mod} A$. Then we have the commutative diagram of K-vector spaces

$$\begin{tikzcd}
F(M) \arrow[r, "\varphi_M"] \arrow[d, "F(h)"'] & \operatorname{Hom}_A(M, M) \arrow[d, "\operatorname{Hom}_A(h,M)"] \\
F(X) \arrow[r, "\varphi_X"] & \operatorname{Hom}_A(X, M)\,,
\end{tikzcd}$$

and hence $h = \operatorname{Hom}_A(h, M)(\varphi_M(b)) = \varphi_X(F(h)(b))$. This shows that φ_X is an epimorphism. Hence, $\varphi: F \to \operatorname{Hom}_A(-, M)$ is an epimorphism of functors, and consequently $\operatorname{Hom}(-, g)$ is minimal.

Assume now the sequence of functors presented in (i) is a minimal projective presentation of S^N in $\mathcal{F}(A)^{\mathrm{o}}$. Then it follows from Lemma 1.12 (i) that g is a right almost split homomorphism in $\operatorname{mod} A$. We show that g is also right minimal. Let $g = gh$ for a homomorphism $h \in \operatorname{End}_A(M)$. Then we have a commutative diagram in $\mathcal{F}(A)^{\mathrm{o}}$ with exact rows

$$\begin{tikzcd}
\operatorname{Hom}_A(-, M) \arrow[r, "\operatorname{Hom}_A(-,g)"] \arrow[d, "\operatorname{Hom}_A(-,h)"'] & \operatorname{rad}_A(-, M) \arrow[r] \arrow[d, "\operatorname{rad}_A(-,\mathrm{id}_M)"] & 0 \\
\operatorname{Hom}_A(-, M) \arrow[r, "\operatorname{Hom}_A(-,g)"] & \operatorname{rad}_A(-, M) \arrow[r] & 0\,.
\end{tikzcd}$$

Since $\operatorname{Hom}_A(-, g) \operatorname{Hom}_A(-, h) = \operatorname{Hom}_A(-, g)$ is a minimal epimorphism, $\operatorname{Hom}_A(-, h)$ is an epimorphism. Then, for any module X in $\operatorname{mod} A$, the induced K-linear homomorphism $\operatorname{Hom}_A(X, h): \operatorname{Hom}_A(X, M) \to \operatorname{Hom}_A(X, M)$ is an epimorphism, and hence an isomorphism, because $\dim_K \operatorname{Hom}_A(X, M)$ is finite.

Thus $\operatorname{Hom}_A(-, h)$ is an isomorphism in $\mathcal{F}(A)^{\mathrm{o}}$. Then, applying Corollary 1.3 (i), we obtain that h is an isomorphism. Therefore, g is right minimal. ☐

We will prove now the main theorems of this section, describing minimal projective resolutions of simple functors.

Theorem 1.14. *Let A be a finite dimensional K-algebra over a field K and N an indecomposable module in* mod A. *Then the following equivalences hold:*

(i) *N is a projective module and $g: M \to N$ is a right minimal almost split homomorphism in* mod A *if and only if the induced sequence of functors*

$$0 \longrightarrow \operatorname{Hom}_A(-, M) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{Hom}_A(-, N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective resolution of S^N in $\mathcal{F}(A)^{\mathrm{o}}$.

(ii) *N is a nonprojective module and*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is an almost split sequence in mod A *if and only if the induced sequence of functors*

$$0 \longrightarrow \operatorname{Hom}_A(-,L) \xrightarrow{\operatorname{Hom}_A(-,f)} \operatorname{Hom}_A(-,M) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective resolution of S^N *in* $\mathcal{F}(A)^{\mathrm{o}}$.

Proof. (i) Assume that N is projective and $g\colon M \to N$ is a right minimal almost split homomorphism in mod A. Then, by Lemma III.7.6, g is a monomorphism and its image $\operatorname{Im} g$ is isomorphic to the radical $\operatorname{rad} N$ of N. Moreover, by the left exactness of the hom functors (Lemma II.2.5), $\operatorname{Hom}_A(-,g)\colon \operatorname{Hom}_A(-,M) \to \operatorname{Hom}_A(-,N)$ is a monomorphism in $\mathcal{F}(A)^{\mathrm{o}}$. Therefore, Lemma 1.13 shows that the sequence of functors

$$0 \longrightarrow \operatorname{Hom}_A(-,M) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective resolution of S^N in $\mathcal{F}(A)^{\mathrm{o}}$. Conversely, assume that the above sequence of functors is a minimal projective resolution of S^N in $\mathcal{F}(A)^{\mathrm{o}}$. It follows from Lemma 1.13 that g is a right minimal almost split homomorphism in mod A. Moreover, by Lemma I.8.7, there is a commutative diagram in mod A

$$\begin{tikzcd}
\operatorname{Hom}_A(A_A,M) \arrow[r, "\operatorname{Hom}_A(A,g)"] \arrow[d, "\theta_M"'] & \operatorname{Hom}_A(A_A,N) \arrow[d, "\theta_N"] \\
M \arrow[r, "g"] & N\,,
\end{tikzcd}$$

where θ_M and θ_N are isomorphisms in mod A given by $\theta_M(u) = u(1_A)$ and $\theta_N(v) = v(1_A)$ for $u \in \operatorname{Hom}_A(A_A,M)$ and $v \in \operatorname{Hom}_A(A_A,N)$. Since $\operatorname{Hom}_A(A,g)$ is a monomorphism, we conclude that g is a monomorphism. Then it follows from Theorem III.8.4 that N is projective.

(ii) Assume that N is nonprojective, and let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an almost split sequence in mod A. By the left exactness of the hom functors, we obtain the exact sequence of functors in $\mathcal{F}(A)^{\mathrm{o}}$

$$0 \longrightarrow \operatorname{Hom}_A(-,L) \xrightarrow{\operatorname{Hom}_A(-,f)} \operatorname{Hom}_A(-,M) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{Hom}_A(-,N).$$

Moreover, since g is a right minimal almost split homomorphism in mod A, it follows from Lemma 1.13 that

$$\operatorname{Hom}_A(-,M) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective presentation of S^N in $\mathcal{F}(A)^{\text{o}}$. Clearly, the isomorphism of functors $\operatorname{Hom}_A(-, f)\colon \operatorname{Hom}_A(-, L) \to \operatorname{Im}\operatorname{Hom}_A(-, f)$ is a projective cover of $\operatorname{Im}\operatorname{Hom}_A(-, f) = \operatorname{Ker}\operatorname{Hom}_A(-, g)$ in $\mathcal{F}(A)^{\text{o}}$. Therefore, the sequence of functors in $\mathcal{F}(A)^{\text{o}}$

$$0 \longrightarrow \operatorname{Hom}_A(-, L) \xrightarrow{\operatorname{Hom}_A(-,f)} \operatorname{Hom}_A(-, M) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{Hom}_A(-, N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective resolution of S^N in $\mathcal{F}(A)^{\text{o}}$. Conversely, assume that the above sequence of functors is a minimal projective resolution of S^N in $\mathcal{F}(A)^{\text{o}}$. We claim that the module N is nonprojective. Suppose N is projective. Then it follows from (i) that the simple simple functor S^N has a minimal projective resolution of the form

$$0 \longrightarrow \operatorname{Hom}_A(-, \operatorname{rad} N) \longrightarrow \operatorname{Hom}_A(-, N) \xrightarrow{\pi^N} S^N \longrightarrow 0,$$

induced by the canonical inclusion homomorphism $\operatorname{rad} N \hookrightarrow N$. Then we conclude that $\operatorname{rad}_A(-, N) \cong \operatorname{Hom}_A(-, \operatorname{rad} N)$ in $\mathcal{F}(A)^{\text{o}}$, and hence $\operatorname{rad}_A(-, N) = \operatorname{Ker}\pi^N$ is a projective functor in $\mathcal{F}(A)^{\text{o}}$. Thus we obtain in $\mathcal{F}(A)$ the splittable exact sequence of functors

$$0 \longrightarrow \operatorname{Hom}_A(-, L) \xrightarrow{\operatorname{Hom}_A(-,f)} \operatorname{Hom}_A(-, M) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{rad}_A(-, N) \longrightarrow 0,$$

which contradicts the form of the minimal projective resolution of S^N in $\mathcal{F}(A)^{\text{o}}$. Therefore, N is nonprojective. But then we have $S^N(A_A) = 0$. Applying Lemma I.8.7 again, we obtain the commutative diagram in $\operatorname{mod} A$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_A(A_A, L) \arrow[r, "\operatorname{Hom}_A(A_A,f)"] \arrow[d, "\theta_L"'] & \operatorname{Hom}_A(A_A, M) \arrow[r, "\operatorname{Hom}_A(A_A,g)"] \arrow[d, "\theta_M"'] & \operatorname{Hom}_A(A_A, N) \arrow[r] \arrow[d, "\theta_N"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0,
\end{tikzcd}$$

with exact upper row and $\theta_L, \theta_M, \theta_N$ being isomorphisms. Hence, the lower sequence is exact. It follows from Lemma 1.13 that g is a right minimal almost split homomorphism in $\operatorname{mod} A$. Then, by Theorem III.8.3, the sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod} A$. □

We end this section with the covariant version of the above theorem.

Theorem 1.15. *Let A be a finite dimensional K-algebra over a field K and L be an indecomposable module in* mod A. *Then the following equivalences hold:*

(i) *L is an injective module and $f: L \to M$ is a left minimal almost split homomorphism in* mod A *if and only if the induced sequence of functors*

$$0 \longrightarrow \operatorname{Hom}_A(M,-) \xrightarrow{\operatorname{Hom}_A(f,-)} \operatorname{Hom}_A(L,-) \xrightarrow{\pi_L} S_L \longrightarrow 0$$

is a minimal projective resolution of S_L in $\mathcal{F}(A)$.

(ii) *L is a noninjective module and*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is an almost split sequence in mod A *if and only if the induced sequence of functors*

$$0 \longrightarrow \operatorname{Hom}_A(N,-) \xrightarrow{\operatorname{Hom}_A(g,-)} \operatorname{Hom}_A(M,-) \xrightarrow{\operatorname{Hom}_A(f,-)} \operatorname{Hom}_A(L,-) \xrightarrow{\pi_L} S_L \longrightarrow 0$$

is a minimal projective resolution of S_L in $\mathcal{F}(A)$.

2 The Igusa–Todorov theorem

The aim of this section is to prove a theorem due to K. Igusa and G. Todorov in [IT], describing the radical level of the composition of irreducible homomorphisms between indecomposable modules forming a sectional path.

We present first some characterizations of irreducible homomorphisms.

Proposition 2.1. *Let A be a finite dimensional K-algebra over a field K, $g: E \to N$ a right minimal almost split homomorphism in* mod A, *and $f: X \to N$ a nonzero homomorphism in* mod A. *The following statements are equivalent:*

(i) *f is an irreducible homomorphism in* mod A.

(ii) *$f = gh$ for a section $h: X \to E$ in* mod A.

(iii) *$f \in \operatorname{rad}_A(X,N)$ and the induced morphism of functors in $\mathcal{F}(A)^{\mathrm{o}}$*

$$\operatorname{Hom}_A(-,X)/\operatorname{rad}_A(-,X) \longrightarrow \operatorname{rad}_A(-,N)/\operatorname{rad}_A^2(-,N)$$

is a monomorphism.

Proof. It follows from Lemma III.7.2 (ii) that N is an indecomposable module.

(i) $\Rightarrow$ (ii) Assume f is an irreducible homomorphism. Then f is not a retraction. Since g is a right almost split homomorphism, we have $f = gh$ for some homomorphism $h: X \to E$ in mod A. But then h is a section, because f is irreducible.

(ii) ⇒ (iii) Assume $f = gh$ for a section $h\colon X \to E$ in $\operatorname{mod} A$. Observe also that $f \in \operatorname{rad}_A(X, N)$, because $g \in \operatorname{rad}_A(E, N)$ by Lemma III.1.5. Consider the morphism of functors in $\mathcal{F}(A)^{\mathrm{o}}$

$$\varphi\colon \operatorname{Hom}_A(-, X)/\operatorname{rad}_A(-, X) \longrightarrow \operatorname{Hom}_A(-, E)/\operatorname{rad}_A(-, E)$$

induced by h. We claim that φ is a monomorphism. Indeed, suppose that $hu \in \operatorname{rad}_A(M, E)$ for some $u \in \operatorname{Hom}_A(M, X)$. Since h is a section, there exists a homomorphism $h'\colon E \to X$ in $\operatorname{mod} A$ such that $h'h = \mathrm{id}_X$. Then we obtain $u = (h'h)u = h'(hu) \in \operatorname{rad}_A(M, X)$, which proves the claim. Since g is a right minimal almost split homomorphism in $\operatorname{mod} A$, it follows from Lemma 1.13 (i) that the induced sequence of functors

$$\operatorname{Hom}_A(-, E) \xrightarrow{\operatorname{Hom}_A(-,g)} \operatorname{Hom}_A(-, N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective presentation of S^N in $\mathcal{F}(A)^{\mathrm{o}}$, and hence

$$\operatorname{Hom}_A(-, g)\colon \operatorname{Hom}_A(-, E) \longrightarrow \operatorname{rad}_A(-, N)$$

is a projective cover of $\operatorname{rad}_A(-, N)$ in $\mathcal{F}(A)^{\mathrm{o}}$. Further, $\operatorname{Hom}_A(-, g)$ induces the commutative diagram of functors in $\mathcal{F}(A)^{\mathrm{o}}$

$$\begin{tikzcd}
\operatorname{Hom}_A(-, E) \arrow[r, "\operatorname{Hom}_A(-,g)"] \arrow[d, "\pi"'] & \operatorname{rad}_A(-, N) \arrow[d, "\omega"] \\
\operatorname{Hom}_A(-, E)/\operatorname{rad}_A(-, E) \arrow[r, "\psi"] & \operatorname{rad}_A(-, N)/\operatorname{rad}_A^2(-, N)\,,
\end{tikzcd}$$

where π and ω are the canonical epimorphisms of functors. Observe that ψ is an epimorphism, since $\operatorname{Hom}_A(-, g)$ is an epimorphism. We claim that ψ is also a monomorphism, and hence an isomorphism. Let L be a module in $\operatorname{mod} A$. We have to show that $\psi_L\colon \operatorname{Hom}_A(L, E)/\operatorname{rad}_A(L, E) \to \operatorname{rad}_A(L, N)/\operatorname{rad}_A^2(L, N)$ is a monomorphism in $\operatorname{mod} K$. Assume first that L is indecomposable and take $v \in \operatorname{Hom}_A(L, E) \setminus \operatorname{rad}_A(L, E)$. It follows from Lemma III.1.5 that v is a section in $\operatorname{mod} A$. Then, applying Theorem III.7.12, we conclude that gv is an irreducible homomorphism in $\operatorname{mod} A$. Since L and N are indecomposable, we obtain from Lemma III.7.8 that $gv \in \operatorname{rad}_A(L, N) \setminus \operatorname{rad}_A^2(L, N)$, and hence $\psi_L(v) = gv + \operatorname{rad}_A^2(L, N)$ is nonzero. Hence, ψ_L is a monomorphism. Assume that $L = L_1 \oplus \cdots \oplus L_r$ in $\operatorname{mod} A$, where $L_1, \dots, L_r$ are indecomposable modules. Then it follows from Lemma III.1.3 that $\psi_L = \psi_{L_1} \oplus \cdots \oplus \psi_{L_r}$, and hence is a monomorphism. Therefore, ψ is a monomorphism. Finally, observe that the morphism of functors

$$\operatorname{Hom}_A(-, X)/\operatorname{rad}_A(-, X) \longrightarrow \operatorname{rad}_A(-, N)/\operatorname{rad}_A^2(-, N)$$

induced by f is the composition $\psi\varphi$, and consequently is a monomorphism.

(iii) ⇒ (i) Assume that $f \in \operatorname{rad}_A(X, N)$ and the induced morphism of functors in $\mathcal{F}(A)^{\mathrm{o}}$

$$\operatorname{Hom}_A(-, X)/\operatorname{rad}_A(-, X) \longrightarrow \operatorname{rad}_A(-, N)/\operatorname{rad}_A^2(-, N)$$

is a monomorphism. In particular, we conclude from Lemma III.1.5 (ii) that f is not a retraction. Since N is indecomposable, Lemma I.4.2 shows that f is not a section. Suppose there exists in mod A a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & & N \\ & M . \arrow[ur, "v"'] & \end{tikzcd}$$

Assume v is not a retraction. Let us show that then u is a section. Applying again Lemma III.1.5 (ii), we infer that $v \in \operatorname{rad}_A(M, N)$. We obtain the commutative diagram in $\mathcal{F}(A)^{\mathrm{o}}$

$$\begin{tikzcd} \operatorname{Hom}_A(-, X)/\operatorname{rad}_A(-, X) \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & \operatorname{rad}_A(-, N)/\operatorname{rad}_A^2(-, N) \\ & \operatorname{Hom}_A(-, M)/\operatorname{rad}_A^2(-, M) , \arrow[ur, "\gamma"'] & \end{tikzcd}$$

where α is induced by f, β is induced by u, and γ is induced by v. Since by our assumption α is a monomorphism, we obtain that β is a monomorphism. Let $X = X_1 \oplus \cdots \oplus X_s$ be a decomposition of X into a sum of indecomposable modules in mod A. Then

$$\operatorname{Hom}_A(-, X)/\operatorname{rad}_A(-, X) = \bigoplus_{i=1}^{s} \operatorname{Hom}_A(-, X_i)/\operatorname{rad}_A(-, X_i),$$

by Lemma III.1.3. Thus we have

$$\alpha = [\alpha_1, \dots, \alpha_s]\colon \bigoplus_{i=1}^{s} \big(\operatorname{Hom}_A(-, X_i)/\operatorname{rad}_A(-, X_i)\big) \\ \longrightarrow \operatorname{rad}_A(-, N)/\operatorname{rad}_A^2(-, N),$$

$$\beta = [\beta_1, \dots, \beta_s]\colon \bigoplus_{i=1}^{s} \big(\operatorname{Hom}_A(-, X_i)/\operatorname{rad}_A(-, X_i)\big) \\ \longrightarrow \operatorname{rad}_A(-, N)/\operatorname{rad}_A^2(-, N),$$

with $\alpha_i = \alpha\omega_i$, $\beta_i = \beta\omega_i$, and ω_i the morphism

$$\operatorname{Hom}_A(-, X_i)/\operatorname{rad}_A(-, X_i) \longrightarrow \operatorname{Hom}_A(-, X)/\operatorname{rad}_A(-, X)$$

induced by the canonical monomorphism $w_i\colon X_i \to X$, for any $i \in \{1, \dots, s\}$. Observe that $\beta_1, \dots, \beta_s$ are monomorphisms, because β is a monomorphism. Hence, $u_i = uw_i \notin \operatorname{rad}_A(X_i, M)$ for any $i \in \{1, \dots, s\}$. Then it follows from Lemma III.1.5 (i) that $u_1, \dots, u_s$ are sections. For each $i \in \{1, \dots, s\}$, take a homomorphism $p_i\colon M \to X_i$ such that $p_i u_i = \mathrm{id}_{X_i}$. Then for the homomorphism

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_s \end{bmatrix} : M \longrightarrow X,$$

we have $pu = \mathrm{id}_X$, and hence u is a section. Therefore, f is an irreducible homomorphism in $\operatorname{mod} A$. □

Similarly, one proves the following proposition.

Proposition 2.2. *Let A be a finite dimensional K-algebra over a field K, $g\colon M \to E$ a left minimal almost split homomorphism in* $\operatorname{mod} A$*, and $f\colon M \to Y$ a nonzero homomorphism in* $\operatorname{mod} A$*. The following statements are equivalent:*

(i) *f is an irreducible homomorphism in* $\operatorname{mod} A$.

(ii) *$f = hg$ for a retraction $h\colon E \to Y$ in* $\operatorname{mod} A$.

(iii) *$f \in \operatorname{rad}_A(M, Y)$ and the induced morphism of functors in $\mathcal{F}(A)$*

$$\operatorname{Hom}_A(Y, -)/\operatorname{rad}_A(Y, -) \longrightarrow \operatorname{rad}_A(M, -)/\operatorname{rad}_A^2(M, -)$$

is a monomorphism.

Let A be a finite dimensional K-algebra over a field K. Recall that a path of irreducible homomorphisms

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \longrightarrow M_{t-1} \xrightarrow{f_t} M_t$$

between indecomposable modules in $\operatorname{mod} A$ is called *sectional* if $\tau_A M_i \ncong M_{i-2}$ for all $i \in \{2, \dots, t\}$.

Lemma 2.3. *Let A be a finite dimensional K-algebra over a field K and*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = Y$$

be a sectional path of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$*. Then there do not exist homomorphisms $g\colon X \to Z$ and $f_n'\colon Z \to Y$ in* $\operatorname{mod} A$ *such that $\begin{bmatrix} f_n' & f_n \end{bmatrix} : Z \oplus X_{n-1} \to Y$ is a right minimal almost split homomorphism in* $\operatorname{mod} A$ *and the image of the induced morphism of functors*

$$\operatorname{Hom}_A(-, f_n' g + f_n \dots f_1)\colon \operatorname{Hom}_A(-, X) \longrightarrow \operatorname{Hom}_A(-, Y)$$

in $\mathcal{F}(A)^{\circ}$ is contained in $\operatorname{rad}_A^{n+1}(-, Y)$.

Proof. We proceed by induction on n.

Let $n = 1$. Suppose that there exist homomorphisms $g: X \to Z$ and $f_1': Z \to Y$ in $\operatorname{mod} A$ such that $\begin{bmatrix} f_1' & f_1 \end{bmatrix}: Z \oplus X \to Y$ is a right minimal almost split homomorphism in $\operatorname{mod} A$ and the image of the morphism

$$\operatorname{Hom}_A(-, f_1'g + f_1): \operatorname{Hom}_A(-, X) \longrightarrow \operatorname{Hom}_A(-, Y)$$

in $\mathcal{F}(A)^{\mathrm{o}}$ is contained in $\operatorname{rad}_A^2(-, Y)$. We have the commutative diagram in $\operatorname{mod} A$ of the form

$$\begin{tikzcd}
X \arrow[rr, "f_1'g+f_1"] \arrow[dr, "\begin{bmatrix} g \\ \mathrm{id}_X \end{bmatrix}"'] & & Y \\
& Z \oplus X . \arrow[ur, "\begin{bmatrix} f_1' & f_1 \end{bmatrix}"'] &
\end{tikzcd}$$

Observe that $\begin{bmatrix} g \\ \mathrm{id}_X \end{bmatrix}$ is a section. Since $\begin{bmatrix} f_1' & f_1 \end{bmatrix}$ is a right minimal almost split homomorphism, it follows from Proposition 2.1 that $f_1'g + f_1: X \to Y$ is an irreducible homomorphism. On the other hand, by our assumption, we have $f_1'g + f_1 = (f_1'g + f_1)\,\mathrm{id}_X \in \operatorname{rad}_A^2(X, Y)$. This contradicts Lemma III.7.8.

Let $n \geq 2$. Assume that the claim holds for all sectional paths of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ of length $n - 1$. Suppose that there exist in $\operatorname{mod} A$ a sectional path

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = Y$$

of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ and homomorphisms $g: X \to Z$ and $f_n': Z \to Y$ such that $\begin{bmatrix} f_n' & f_n \end{bmatrix}: Z \oplus X_{n-1} \to Y$ is a right minimal almost split homomorphism in $\operatorname{mod} A$ and the image of the induced morphism of functors

$$\operatorname{Hom}_A(-, f_n'g + f_n \cdots f_1): \operatorname{Hom}_A(-, X) \longrightarrow \operatorname{Hom}_A(-, Y)$$

in $\mathcal{F}(A)^{\mathrm{o}}$ is contained in $\operatorname{rad}_A^{n+1}(-, Y)$. Let $L = \operatorname{Ker}\begin{bmatrix} f_n' & f_n \end{bmatrix}$. Then we have in $\operatorname{mod} A$ an exact sequence

$$0 \longrightarrow L \xrightarrow{\begin{bmatrix} u \\ v \end{bmatrix}} Z \oplus X_{n-1} \xrightarrow{\begin{bmatrix} f_n' & f_n \end{bmatrix}} X_n,$$

where $L = \tau_A X_n$. We note that $L = 0$, if X_n is projective. Since, by the assumption, the path $X_{n-2} \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n$ is sectional, we have $L \not\cong X_{n-2}$. Assume $L \neq 0$. Then we have two irreducible homomorphisms $v = f_{n-1}': L \to X_{n-1}$ and $f_{n-1}: X_{n-2} \to X_{n-1}$, with nonisomorphic domains, and the irreducible homomorphism $\begin{bmatrix} f'_{n-1} & f_{n-1} \end{bmatrix} : L \oplus X_{n-2} \to X_{n-1}$, by Theorem III.7.12. For $L = 0$, we set $f'_{n-1} = v = 0$. Moreover, by our assumption, $f'_n g + f_n \cdots f_1 = (f'_n g + f_n \cdots f_1)\,\mathrm{id}_X \in \mathrm{rad}_A^{n+1}(X, Y)$. Then there exist a module M in $\mathrm{mod}\, A$ and homomorphisms $\alpha \in \mathrm{rad}_A^n(X, M)$ and $\beta \in \mathrm{rad}_A(M, Y)$ such that $f'_n g + f_n \cdots f_1 = \beta\alpha$. We note that $Y = X_n$ is indecomposable. Then it follows from Lemma III.1.5 (ii) that β is not a retraction. Because $\begin{bmatrix} f'_n & f_n \end{bmatrix} : Z \oplus X_{n-1} \to Y$ is a right minimal almost split homomorphism in $\mathrm{mod}\, A$, we conclude that there exists a homomorphism $\begin{bmatrix} h' \\ h \end{bmatrix} : M \to Z \oplus X_{n-1}$ in $\mathrm{mod}\, A$ such that $\beta = \begin{bmatrix} f'_n & f_n \end{bmatrix} \begin{bmatrix} h' \\ h \end{bmatrix}$. We have also the equality $f'_n g + f_n \cdots f_1 = \begin{bmatrix} f'_n & f_n \end{bmatrix} \begin{bmatrix} g \\ f_{n-1} \cdots f_1 \end{bmatrix}$. Consequently

$$\begin{bmatrix} f'_n & f_n \end{bmatrix} \left(\begin{bmatrix} h'\alpha \\ h\alpha \end{bmatrix} - \begin{bmatrix} g \\ f_{n-1} \cdots f_1 \end{bmatrix} \right) = 0.$$

But then there exists a homomorphism $g' : X \to L$ in $\mathrm{mod}\, A$ such that

$$\begin{bmatrix} h'\alpha \\ h\alpha \end{bmatrix} - \begin{bmatrix} g \\ f_{n-1} \cdots f_1 \end{bmatrix} = \begin{bmatrix} u \\ f'_{n-1} \end{bmatrix} g'.$$

In particular, $f'_{n-1} g' + f_{n-1} \cdots f_1 = h\alpha$. Since $\alpha \in \mathrm{rad}_A^n(X, M)$, we infer that $f'_{n-1} g' + f_{n-1} \cdots f_1 \in \mathrm{rad}_A^n(X, X_{n-1})$. Therefore, the image of the morphism in $\mathcal{F}(A)^{\mathrm{o}}$

$$\mathrm{Hom}_A(-, f'_{n-1} g' + f_{n-1} \cdots f_1) : \mathrm{Hom}_A(-, X) \longrightarrow \mathrm{Hom}_A(-, X_{n-1})$$

is contained in $\mathrm{rad}_A^n(-, X_{n-1})$, which contradicts the inductive assumption. □

We are now able to prove the Igusa–Todorov theorem.

Theorem 2.4. *Let A be a finite dimensional K-algebra over a field K and*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n = Y$$

be a sectional path of irreducible homomorphisms between indecomposable modules in $\mathrm{mod}\, A$*. The following statements hold:*

(i) $\mathrm{Im}\,\mathrm{Hom}_A(-, f_n \cdots f_1) \subseteq \mathrm{rad}_A^n(-, Y)$.

(ii) $\mathrm{Im}\,\mathrm{Hom}_A(-, f_n \cdots f_1) \nsubseteq \mathrm{rad}_A^{n+1}(-, Y)$.

In particular, $f_n \cdots f_1 \in \mathrm{rad}_A^n(X, Y) \setminus \mathrm{rad}_A^{n+1}(X, Y)$.

Proof. (i) Since $X_0, X_1, \dots, X_n$ are indecomposable modules, we have $f_i \in \operatorname{rad}_A(X_{i-1}, X_i)$ for $i \in \{1, \dots, n\}$, and hence $f_n \cdots f_1 \in \operatorname{rad}_A^n(X, Y)$. Therefore, $\operatorname{Im} \operatorname{Hom}_A(-, f_n \cdots f_1) \subseteq \operatorname{rad}_A^n(-, Y)$.

(ii) Suppose that $\operatorname{Im} \operatorname{Hom}_A(-, f_n \cdots f_1) \subseteq \operatorname{rad}_A^{n+1}(-, Y)$. Since $f_n \colon X_{n-1} \to X_n$ is an irreducible homomorphism, it follows from Theorem III.7.12 that there is in mod A a right minimal almost split homomorphism of the form

$$\begin{bmatrix} f_n' & f_n \end{bmatrix} \colon Z \oplus X_{n-1} \longrightarrow X_n.$$

Then, for the homomorphism $g = 0$ from X to Z, we conclude that

$$\operatorname{Im} \operatorname{Hom}_A(-, f_n' g + f_n \cdots f_1) = \operatorname{Im} \operatorname{Hom}_A(-, f_n \cdots f_1) \subseteq \operatorname{rad}_A^{n+1}(-, Y),$$

which contradicts Lemma 2.3. Therefore, $\operatorname{Im} \operatorname{Hom}_A(-, f_n \cdots f_1)$ is not contained in $\operatorname{rad}_A^{n+1}(-, Y)$. □

3 Degrees of irreducible homomorphisms

In this section we present some results on left and right degrees of irreducible homomorphisms on module categories, established by S. Liu in [L1].

Let A be a finite dimensional K-algebra over a field K and $f \colon X \to Y$ an irreducible homomorphism in mod A. For each nonnegative integer n, the homomorphism f induces the morphisms of functors

$$l_n(f) \colon \operatorname{rad}_A^n(Y, -)/\operatorname{rad}_A^{n+1}(Y, -) \longrightarrow \operatorname{rad}_A^{n+1}(X, -)/\operatorname{rad}_A^{n+2}(X, -)$$

in $\mathcal{F}(A)$, and

$$r_n(f) \colon \operatorname{rad}_A^n(-, X)/\operatorname{rad}_A^{n+1}(-, X) \longrightarrow \operatorname{rad}_A^{n+1}(-, Y)/\operatorname{rad}_A^{n+2}(-, Y)$$

in $\mathcal{F}(A)^{\circ}$. Following S. Liu, we define the *left degree* $d_{\mathrm{l}}(f)$ of f to be ∞ if $l_n(f)$ is a monomorphism for all integers $n \geq 0$, and otherwise to be the least integer m such that $l_m(f)$ is not a monomorphism. Dually, we define the *right degree* $d_{\mathrm{r}}(f)$ of f to be ∞ if $r_n(f)$ is a monomorphism for all integers $n \geq 0$, and otherwise to be the least integer m such that $r_m(f)$ is not a monomorphism.

Lemma 3.1. *Let A be a finite dimensional K-algebra over a field K, and $f \colon X \to Y$ an irreducible homomorphism in* mod A. *The following statements hold:*

(i) *Let $p \colon Y \to V$ be a retraction in* mod A. *Then $d_{\mathrm{r}}(pf) \leq d_{\mathrm{r}}(f)$.*

(ii) *Let $s \colon U \to X$ be a section in* mod A. *Then $d_{\mathrm{l}}(fs) \leq d_{\mathrm{l}}(f)$.*

Proof. We will prove only (i), since the proof of (ii) is dual.

We may assume that $n = d_{\mathrm{r}}(f)$ is finite. We have the commutative diagram of morphisms in $\mathcal{F}(A)^{\mathrm{o}}$

$$\begin{tikzcd}
\operatorname{rad}_A^{n+1}(-,Y)/\operatorname{rad}_A^{n+2}(-,Y) \arrow[rr, "\varphi"] & & \operatorname{rad}_A^{n+1}(-,V)/\operatorname{rad}_A^{n+2}(-,V) \\
& \operatorname{rad}_A^{n}(-,X)/\operatorname{rad}_A^{n+1}(-,X) \arrow[ul, "r_n(f)"] \arrow[ur, "r_n(pf)"'] &
\end{tikzcd}$$

where φ is induced by the morphism of functors

$$\operatorname{rad}_A^{n+1}(-,p)\colon \operatorname{rad}_A^{n+1}(-,Y) \longrightarrow \operatorname{rad}_A^{n+1}(-,V).$$

Since $r_n(f)$ is not a monomorphism, we conclude that $r_n(pf)$ is not a monomorphism. Hence, the inequality $d_{\mathrm{r}}(pf) \le d_{\mathrm{r}}(f)$ holds. □

Lemma 3.2. *Let A be a finite dimensional K-algebra over a field K, and*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

be a sequence of irreducible homomorphisms in $\operatorname{mod} A$ *such that $n \ge 2$ and $f_n \cdots f_1 \in \operatorname{rad}_A^{n+1}(X_0, X_n)$. Then there exist $i, j \in \{1, \dots, n\}$ such that $d_{\mathrm{r}}(f_i) < \infty$ and $d_{\mathrm{l}}(f_j) < \infty$.*

Proof. Since $f_1, \dots, f_n$ are irreducible, we have $f_t \notin \operatorname{rad}_A^2(X_{t-1}, X_t)$ for any $t \in \{1, \dots, n\}$. Then $f_n \cdots f_1 \in \operatorname{rad}_A^{n+1}(X_0, X_n)$ implies that there is a minimal $m \in \{1, \dots, n-1\}$ such that $f_m \cdots f_1 \notin \operatorname{rad}_A^{m+1}(X_0, X_m)$ and $f_{m+1} f_m \cdots f_1 \in \operatorname{rad}_A^{m+2}(X_0, X_{m+1})$. Consider the morphism of functors $r_m(f_{m+1})$:

$$\operatorname{rad}_A^{m}(-,X_m)/\operatorname{rad}_A^{m+1}(-,X_m) \to \operatorname{rad}_A^{m+1}(-,X_{m+1})/\operatorname{rad}_A^{m+2}(-,X_{m+1}).$$

Then $f_m \cdots f_1 + \operatorname{rad}_A^{m+1}(X_0, X_1)$ is nonzero in $\operatorname{rad}_A^{m}(X_0, X_m)/\operatorname{rad}_A^{m+1}(X_0, X_m)$. On the other hand, we have

$$\begin{aligned}
&r_m(f_{m+1})_{X_0}\big(f_m \cdots f_1 + \operatorname{rad}_A^{m+1}(X_0, X_m)\big)\\
&\quad = f_{m+1} f_m \cdots f_1 + \operatorname{rad}_A^{m+2}(X_0, X_{m+1})\\
&\quad = 0 + \operatorname{rad}_A^{m+2}(X_0, X_{m+1}).
\end{aligned}$$

Hence $r_m(f_{m+1})$ is not a monomorphism, and so $d_{\mathrm{r}}(f_{m+1}) \le m \le n-1$. Therefore $d_{\mathrm{r}}(f_i) < \infty$ for some $i \in \{1, \dots, n\}$. Similarly, we prove that $d_{\mathrm{l}}(f_j) < \infty$ for some $j \in \{1, \dots, n\}$. □

Lemma 3.3. *Let A be a finite dimensional K-algebra over a field K, $p\colon X \to Y$ a homomorphism in* $\operatorname{mod} A$*, and $f\colon Y \to Z$ an irreducible homomorphism in* $\operatorname{mod} A$ *with Z an indecomposable module. Assume that $p \notin \operatorname{rad}_A^{m+1}(X,Y)$ and $fp \in \operatorname{rad}_A^{m+2}(X,Z)$ for some $m \geq 1$. Then the following statements hold:*

(i) *Z is not projective.*

(ii) *If $0 \to \tau_A Z \xrightarrow{\begin{bmatrix} g \\ g' \end{bmatrix}} Y \oplus Y' \xrightarrow{\begin{bmatrix} f & f' \end{bmatrix}} Z \to 0$ is an almost split sequence in* $\operatorname{mod} A$*, then there exists a homomorphism $q\colon X \to \tau_A Z$ in* $\operatorname{mod} A$ *such that $q \notin \operatorname{rad}_A^m(X,\tau_A Z)$, $p + gq \in \operatorname{rad}_A^{m+1}(X,Y)$, $g'q \in \operatorname{rad}_A^{m+1}(X,Y')$.*

Proof. Since $fp \in \operatorname{rad}_A^{m+2}(X,Z)$, we have $fp = ts$ for some homomorphisms $s \in \operatorname{rad}_A^{m+1}(X,W)$, $t \in \operatorname{rad}_A(W,Z)$, and a module W in $\operatorname{mod} A$. Moreover, there exists a right minimal almost split homomorphism in $\operatorname{mod} A$ of the form

$$Y \oplus Y' \xrightarrow{\begin{bmatrix} f & f' \end{bmatrix}} Z$$

because Z is indecomposable and $f\colon Y \to Z$ is an irreducible homomorphism (see Lemmas III.7.4, III.7.6 and Theorems III.7.12, III.8.3). Observe also that $t \in \operatorname{rad}_A(W,Z)$ is not a retraction, because Z is indecomposable (Lemma III.1.5). Then there exists a homomorphism

$$\begin{bmatrix} u \\ u' \end{bmatrix} \colon W \longrightarrow Y \oplus Y'$$

such that $t = \begin{bmatrix} f & f' \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix} = fu + f'u'$. Then we obtain

$$\begin{bmatrix} f & f' \end{bmatrix} \begin{bmatrix} us - p \\ u's \end{bmatrix} = fus - fp + f'u's = ts - fp = 0.$$

Suppose now that Z is projective. Then it follows from Lemma III.7.6 that $\begin{bmatrix} f & f' \end{bmatrix}$ is a monomorphism, and consequently $us = p$ and $u's = 0$. In particular, we conclude that $p = us \in \operatorname{rad}_A^{m+1}(X,Y)$, which contradicts our assumption. Therefore, Z is not projective, so (i) holds. Then we have in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow \tau_A Z \xrightarrow{\begin{bmatrix} g \\ g' \end{bmatrix}} Y \oplus Y' \xrightarrow{\begin{bmatrix} f & f' \end{bmatrix}} Z \longrightarrow 0.$$

Since $\operatorname{Ker} \begin{bmatrix} f & f' \end{bmatrix} = \operatorname{Im} \begin{bmatrix} g \\ g' \end{bmatrix}$, there exists a homomorphism $q\colon X \to \tau_A Z$ in $\operatorname{mod} A$ such that

$$\begin{bmatrix} us - p \\ u's \end{bmatrix} = \begin{bmatrix} g \\ g' \end{bmatrix} q.$$

Consequently, $p + gq = us \in \operatorname{rad}_A^{m+1}(X, Y)$ and $g'q = u's \in \operatorname{rad}_A^{m+1}(X, Y')$. Moreover, $p \notin \operatorname{rad}_A^{m+1}(X, Y)$ and $g \in \operatorname{rad}_A(\tau_A Z, Y)$ imply that $q \notin \operatorname{rad}_A^{m}(X, \tau_A Z)$. Therefore, the statement (ii) holds. ☐

Corollary 3.4. *Let A be a finite dimensional K-algebra over a field K and $f: Y \to Z$ be an irreducible homomorphism in* $\operatorname{mod} A$ *with $d_{\mathrm{r}}(f) < \infty$. Then every indecomposable direct summand of Z is nonprojective.*

Proof. Let $d_{\mathrm{r}}(f) = m < \infty$. Then there exists a homomorphism $p: X \to Y$ in $\operatorname{mod} A$ such that $p \notin \operatorname{rad}_A^{m+1}(X, Y)$ and $fp \in \operatorname{rad}_A^{m+2}(X, Z)$. Suppose P is an indecomposable projective direct summand of Z, and $v: Z \to P$ the associated retraction. Let $u = vf$. Then u is irreducible and $up = vfp \in \operatorname{rad}_A^{m+2}(X, P)$. This contradicts Lemma 3.3. ☐

Lemma 3.5. *Let A be a finite dimensional K-algebra over a field K and $f: Y \to Z$ an irreducible homomorphism in* $\operatorname{mod} A$ *such that $d_{\mathrm{r}}(f) < \infty$ and Z is indecomposable. Moreover, let*

$$\begin{tikzcd} 0 \arrow[r] & \tau_A Z \arrow[r, "\begin{bmatrix} g \\ g' \end{bmatrix}"] & Y \oplus Y' \arrow[r, "\begin{bmatrix} f & f' \end{bmatrix}"] & Z \arrow[r] & 0 \end{tikzcd}$$

be an almost split sequence in $\operatorname{mod} A$ *with $Y' \neq 0$. Then $d_{\mathrm{r}}(g') < d_{\mathrm{r}}(f)$.*

Proof. Let $d_{\mathrm{r}}(f) = m < \infty$. Then there exists a homomorphism $p: X \to Y$ in $\operatorname{mod} A$ such that $p \notin \operatorname{rad}_A^{m+1}(X, Y)$ and $fp \in \operatorname{rad}_A^{m+2}(X, Z)$. Then it follows from Lemma 3.3 that there exists a homomorphism $q: X \to \tau_A Z$ in $\operatorname{mod} A$ such that $q \notin \operatorname{rad}_A^{m}(X, \tau_A Z)$ and $g'q \in \operatorname{rad}_A^{m+1}(X, Y')$. This implies that $d_{\mathrm{r}}(g') \le m - 1 < d_{\mathrm{r}}(f)$. ☐

Corollary 3.6. *Let A be a finite dimensional K-algebra over a field K and $f: Y \to Z$ be an irreducible homomorphism in* $\operatorname{mod} A$ *with $d_{\mathrm{r}}(f) < \infty$ and Z is indecomposable. Assume that U is a nonzero module in* $\operatorname{mod} A$ *such that $Y \oplus U$ is a direct summand of the middle term of an almost split sequence in* $\operatorname{mod} A$ *with the right term Z. Then there exists an irreducible homomorphism $h: \tau_A Z \to U$ such that $d_{\mathrm{r}}(h) < d_{\mathrm{r}}(f)$.*

Proof. There exists an almost split sequence in $\operatorname{mod} A$

$$\begin{tikzcd} 0 \arrow[r] & \tau_A Z \arrow[r, "\begin{bmatrix} g \\ g' \end{bmatrix}"] & Y \oplus Y' \arrow[r, "\begin{bmatrix} f & f' \end{bmatrix}"] & Z \arrow[r] & 0 \end{tikzcd}$$

such that U is a direct summand of Y'. Let $u: Y' \to U$ be the associated retraction, and set $h = ug': \tau_A Z \to U$. Applying Lemma 3.1, we conclude that $d_{\mathrm{r}}(h) \le d_{\mathrm{r}}(g')$. Further, it follows from Lemma 3.5 that $d_{\mathrm{r}}(g') < d_{\mathrm{r}}(f)$. Therefore, the required inequality $d_{\mathrm{r}}(h) < d_{\mathrm{r}}(f)$ holds. ☐

We have also the following direct consequence of the above result.

Corollary 3.7. *Let A be a finite dimensional K-algebra over a field K and $f\colon Y \to Z$ be an irreducible homomorphism in* $\operatorname{mod} A$ *such that $d_{\mathrm{r}}(f) = 1$ and Z is indecomposable. Then f is an epimorphism and a right minimal almost split homomorphism in* $\operatorname{mod} A$.

The following results are dual to Lemma 3.3, Corollary 3.4, Lemma 3.5, Corollary 3.6, and Corollary 3.7.

Lemma 3.8. *Let A be a finite dimensional K-algebra over a field K, $p\colon Y \to Z$ a homomorphism in* $\operatorname{mod} A$, *and $g\colon X \to Y$ an irreducible homomorphism in* $\operatorname{mod} A$ *with X an indecomposable module. Assume that $p \notin \operatorname{rad}_A^{m+1}(Y, Z)$ and $pg \in \operatorname{rad}_A^{m+2}(X, Z)$ for some $m \geq 1$. Then the following statements hold:*

(i) *X is not injective.*

(ii) *If* $0 \longrightarrow X \xrightarrow{\begin{bmatrix} g \\ g' \end{bmatrix}} Y \oplus Y' \xrightarrow{\begin{bmatrix} f & f' \end{bmatrix}} \tau_A^{-1} X \longrightarrow 0$ *is an almost split homomorphism in* $\operatorname{mod} A$, *then there exists a homomorphism $q\colon \tau_A^{-1} X \to Z$ such that $q \notin \operatorname{rad}_A^{m}(\tau_A^{-1} X, Z)$, $p + qf \in \operatorname{rad}_A^{m+1}(Y, Z)$, $qf' \in \operatorname{rad}_A^{m+1}(Y', Z)$.*

Corollary 3.9. *Let A be a finite dimensional K-algebra over a field K and $g\colon X \to Y$ an irreducible homomorphism in* $\operatorname{mod} A$ *with $d_{\mathrm{l}}(g) < \infty$. Then every indecomposable direct summand of X is noninjective.*

Lemma 3.10. *Let A be a finite dimensional K-algebra over a field K and $g\colon X \to Y$ an irreducible homomorphism in* $\operatorname{mod} A$ *such that $d_{\mathrm{l}}(g) < \infty$ and X is indecomposable. Moreover, let*

$$0 \longrightarrow X \xrightarrow{\begin{bmatrix} g \\ g' \end{bmatrix}} Y \oplus Y' \xrightarrow{\begin{bmatrix} f & f' \end{bmatrix}} \tau_A^{-1} X \longrightarrow 0$$

be an almost split sequence in $\operatorname{mod} A$ *with $Y' \neq 0$. Then $d_{\mathrm{l}}(f') < d_{\mathrm{l}}(g)$.*

Corollary 3.11. *Let A be a finite dimensional K-algebra over a field K and $g\colon X \to Y$ an irreducible homomorphism in* $\operatorname{mod} A$ *with $d_{\mathrm{l}}(g) < \infty$ and X is indecomposable. Assume that U is a nonzero module in* $\operatorname{mod} A$ *such that $Y \oplus U$ is a direct summand of the middle term of an almost split sequence in* $\operatorname{mod} A$ *with the left term X. Then there exists an irreducible homomorphism $h\colon U \to \tau_A^{-1} X$ such that $d_{\mathrm{l}}(h) < d_{\mathrm{l}}(g)$.*

Corollary 3.12. *Let A be a finite dimensional K-algebra over a field K and $g\colon X \to Y$ an irreducible homomorphism in* $\operatorname{mod} A$ *such that $d_{\mathrm{l}}(g) = 1$ and*

X is indecomposable. Then g is a monomorphism and a left minimal almost split homomorphism in mod A.

Let A be finite dimensional K-algebra over a field K. Recall that we do not distinguish between an indecomposable module X in mod A and the corresponding vertex $\{X\}$ of the Auslander–Reiten quiver Γ_A. A path in Γ_A

$$Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n$$

is said to be *presectional* if, for each $i \in \{1, \dots, n-1\}$, $Y_{i-1} = \tau_A Y_{i+1}$ implies that $Y_{i-1} \oplus \tau_A Y_{i+1}$ is a direct summand of the domain of a right minimal almost split homomorphism in mod A with the codomain Y_i, or equivalently, $Y_{i+1} = \tau_A^{-1} Y_{i-1}$ implies that $\tau_A^{-1} Y_{i-1} \oplus Y_{i+1}$ is a direct summand of the codomain of a left minimal almost split homomorphism in mod A with the domain Y_i. Observe that every sectional path in Γ_A is presectional.

Lemma 3.13. *Let A be a finite dimensional K-algebra over a field K and*

$$Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n$$

a presectional path in Γ_A. The following paths in Γ_A are presectional (*if they are defined*)*:*

$$\begin{aligned} \tau_A^n Y_n &\longrightarrow \tau_A^{n-1} Y_{n-1} \longrightarrow \cdots \longrightarrow \tau_A Y_1 \longrightarrow Y_0, \\ Y_n &\longrightarrow \tau_A^{-1} Y_{n-1} \longrightarrow \cdots \longrightarrow \tau_A^{-n+1} Y_1 \longrightarrow \tau_A^{-n} Y_0, \\ \tau_A^m Y_0 &\longrightarrow \tau_A^m Y_1 \longrightarrow \cdots \longrightarrow \tau_A^m Y_{n-1} \longrightarrow \tau_A^m Y_n, \quad m \in \mathbb{Z}. \end{aligned}$$

Proof. Follows easily from the above definition. □

Proposition 3.14. *Let A be a finite dimensional K-algebra over a field K and $f: X \to Y$ be an irreducible homomorphism in* mod A *with $d_r(f) < \infty$ and Y is indecomposable. Assume that there is in Γ_A a presectional path*

$$Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = Y,$$

for some integer $n \geq 1$, such that $X \oplus Y_1$ is a direct summand of the middle term of an almost split sequence in mod A *with the right term Y. Then there exist irreducible homomorphisms $f_i: \tau_A Y_{i-1} \to Y_i$, $i \in \{1, \dots, n\}$, in* mod A *such that*

$$d_r(f_n) < d_r(f_{n-1}) < \cdots < d_r(f_1) < d_r(f).$$

In particular, $d_r(f) > n$.

Proof. It follows from Corollaries 3.4 and 3.6 that Y_0 is not projective and there exists an irreducible homomorphism $f_1: \tau_A Y_0 \to Y_1$ such that $d_r(f_1) < d_r(f)$.

Suppose that, for some $m \in \{1, \dots, n-1\}$, there exist irreducible homomorphisms $f_i : \tau_A Y_{i-1} \to Y_i$, $i \in \{1, \dots, m\}$, such that $d_{\mathrm{r}}(f_m) < \cdots < d_{\mathrm{r}}(f_1) < d_{\mathrm{r}}(f)$. Then $d_{\mathrm{r}}(f_m) < \infty$ and, applying Corollary 3.4, we conclude that Y_m is not projective. Since the considered path is presectional, we obtain that $\tau_A Y_{m-1} \oplus Y_{m+1}$ is a direct summand of the middle term of an almost split sequence in mod A with the right term Y_m. Applying now Corollary 3.6 to the irreducible homomorphism $f_m : \tau_A Y_{m-1} \to Y_m$, we infer that there exists an irreducible homomorphism $f_{m+1} : \tau_A Y_m \to Y_{m+1}$ in mod A such that $d_{\mathrm{r}}(f_{m+1}) < d_{\mathrm{r}}(f_m)$. Hence the claim follows by induction. □

We have the following direct consequence of the above proposition.

Corollary 3.15. *Let A be a finite dimensional K-algebra over a field K and $f : X \to Y$ an irreducible homomorphism in* mod A *and Y is indecomposable. Assume that there is in Γ_A an infinite presectional path*

$$\cdots \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 = Y$$

such that $X \oplus Y_1$ is a direct summand of the middle term of an almost split sequence in mod A *with the right term Y. Then $d_{\mathrm{r}}(f) = \infty$.*

The following dual facts to Proposition 3.14 and Corollary 3.15 are also of importance.

Proposition 3.16. *Let A be a finite dimensional K-algebra over a field K and $g : X \to Y$ an irreducible homomorphism in* mod A *such that $d_{\mathrm{l}}(g) < \infty$ and Y is indecomposable. Assume that there is in Γ_A a presectional path*

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$$

for some integer $n \geq 1$, such that $X_1 \oplus Y$ is a direct summand of the middle term of an almost split sequence in mod A *with the left term X. Then there exist irreducible homomorphisms $g_i : X_i \to \tau_A^{-1} X_{i-1}$, $i \in \{1, \dots, n\}$, in* mod A *such that*

$$d_{\mathrm{l}}(g_n) < d_{\mathrm{l}}(g_{n-1}) < \cdots < d_{\mathrm{l}}(g_1) < d_{\mathrm{l}}(g).$$

In particular, we have $d_{\mathrm{l}}(g) > n$.

Corollary 3.17. *Let A be a finite dimensional K-algebra over a field K and $g : X \to Y$ an irreducible homomorphism in* mod A *and X is indecomposable. Assume that there is in Γ_A an infinite presectional path*

$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow \cdots$$

such that $X_1 \oplus Y$ is a direct summand of the middle term of an almost split sequence in mod A *with the left term X. Then $d_{\mathrm{l}}(g) = \infty$.*

Let A be a finite dimensional K-algebra over a field K. Recall that a valued arrow

$$X \xrightarrow{(d_{XY}, d'_{XY})} Y$$

in Γ_A means that d_{XY} is the multiplicity of Y in the codomain of a left minimal almost split homomorphism $X \to M$ with the domain X and d'_{XY} is the multiplicity of X in the domain N of a right minimal almost split homomorphism $N \to Y$ with the domain Y (see Section III.9). It follows from Corollary III.9.4 that

$$d_{XY} = \dim_{F_Y} \operatorname{irr}_A(X, Y) \quad \text{and} \quad d'_{XY} = \dim_{F_X} \operatorname{irr}_A(X, Y),$$

where $\operatorname{irr}_A(X, Y) = \operatorname{rad}_A(X, Y)/\operatorname{rad}^2_A(X, Y)$, $F_X = \operatorname{End}_A(X)/\operatorname{rad}\operatorname{End}_A(X)$, $F_Y = \operatorname{End}_A(Y)/\operatorname{rad}\operatorname{End}_A(Y)$. We also recall (see Lemma III.9.1 and Proposition III.9.6) that if Y is not projective, then $d_{\tau_A Y\, X} = d'_{XY}$ and $d'_{\tau_A Y\, X} = d_{XY}$. Similarly, if X is not injective, then $d_{Y\, \tau_A^{-1} X} = d'_{XY}$ and $d'_{Y\, \tau_A^{-1} X} = d_{XY}$.

Lemma 3.18. *Let A be a finite dimensional K-algebra over a field K and $f: X \to Y$ be an irreducible homomorphism between indecomposable modules in* mod A *such that $d_{\mathrm{l}}(f) < \infty$ or $d_{\mathrm{r}}(f) < \infty$. Then $d_{XY} = 1$ or $d'_{XY} = 1$.*

Proof. Assume that $d_{\mathrm{r}}(f) < \infty$. We use induction on $d_{\mathrm{r}}(f)$. Let $d_{\mathrm{r}}(f) = 1$. Then it follows from Corollary 3.7 that f is an epimorphism and a right minimal almost split homomorphisms in mod A. But then the indecomposable module X is the middle term of an almost split sequence in mod A with the right term Y. Hence $d'_{XY} = 1$.

Let $d_{\mathrm{r}}(f) = m \geq 2$. Suppose $d'_{XY} > 1$. Then $X \oplus X$ is a direct summand of the middle term of an almost split sequence in mod A with the right term Y. Applying Corollary 3.6 we conclude that there exists an irreducible homomorphism $h: \tau_A Y \to X$ such that $d_{\mathrm{r}}(h) < m$. Then it follows from the inductive assumption that $d_{\tau_A Y\, X} = 1$ or $d'_{\tau_A Y\, X} = 1$. Since $d'_{\tau_A Y\, X} = d_{XY}$ and $d_{\tau_A Y\, X} = d'_{XY} > 1$, we conclude that $d_{XY} = 1$.

The proof for $d_{\mathrm{l}}(f) < \infty$ is similar. □

Let A be a finite dimensional K-algebra over a field K, and $f: X \to Y$, $g: Y \to Z$ be irreducible homomorphisms in mod A. Then the pair $\{f, g\}$ is said to be a *component of an almost split sequence* if there exists an almost split sequence in mod A of the form

$$0 \longrightarrow X \xrightarrow{\begin{bmatrix} f \\ f' \end{bmatrix}} Y \oplus Y' \xrightarrow{\begin{bmatrix} g & g' \end{bmatrix}} Z \longrightarrow 0.$$

In particular, X is an indecomposable noninjective module and Z is an indecomposable nonprojective module. Let Y_1, Y_2 be indecomposable nonprojective modules in mod A and

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} : \tau_A Y_1 \oplus \tau_A Y_2 \longrightarrow X, \qquad \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} : X \longrightarrow Y_1 \oplus Y_2$$

irreducible homomorphisms in mod A. If the pairs $\{f_1, g_1\}$ and $\{f_2, g_2\}$ are components of almost split sequences in mod A with the right terms Y_1 and Y_2, respectively, then $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is said to be a *left neighbour* of $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, and $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ is said to be a *right neighbour* of $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$.

Lemma 3.19. *Let A be a finite dimensional K-algebra over a field K and*

$$0 \longrightarrow X \xrightarrow{\begin{bmatrix} e \\ f \end{bmatrix}} U \oplus V \xrightarrow{\begin{bmatrix} g & h \end{bmatrix}} Z \longrightarrow 0,$$

$$0 \longrightarrow X \xrightarrow{\begin{bmatrix} e' \\ f' \end{bmatrix}} U \oplus V \xrightarrow{\begin{bmatrix} g' & h' \end{bmatrix}} Z \longrightarrow 0$$

be almost split sequences in mod A *such that the modules U and V have no common indecomposable direct summands. Then the following statements are equivalent:*

(i) *The classes $\bar{e} = e + \mathrm{rad}_A^2(X, U)$ and $\bar{e}' = e' + \mathrm{rad}_A^2(X, U)$ in $\mathrm{irr}_A(X, U)$ are linearly independent over F_X.*

(ii) *The classes $\bar{g} = g + \mathrm{rad}_A^2(U, Z)$ and $\bar{g}' = g' + \mathrm{rad}_A^2(U, Z)$ in $\mathrm{irr}_A(U, Z)$ are linearly independent over F_Z.*

Proof. Suppose that $\bar{e}$ and $\bar{e}'$ are linearly dependent over F_X. Then there is an automorphism $a \in \mathrm{End}_A(X)$ such that $e = e'a - p$ for some $p \in \mathrm{rad}_A^2(X, U)$. Since p is not a section and $\begin{bmatrix} e \\ f \end{bmatrix}$ is a left almost split homomorphism, we conclude that there exist homomorphisms $u : U \to U$ and $v : V \to U$ such that $p = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = ue + vf$. Further, $f'a \in \mathrm{rad}_A(X, V)$, because $f' \in \mathrm{rad}_A(X, V)$, and hence $f'a$ is not a section. Then there exist homomorphisms $u' : U \to V$ and $v' : V \to V$ such that $f'a = \begin{bmatrix} u' & v' \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = u'e + v'f$. Observe that $v \in \mathrm{rad}_A(V, U)$ and $u' \in \mathrm{rad}_A(U, V)$, because U and V have no common indecomposable direct summands, by Lemmas III.1.3 and III.1.4. Moreover, we have $v' \notin \mathrm{rad}_A(V, V)$, because $u'ea^{-1} \in \mathrm{rad}_A^2(X, V)$, $fa^{-1} \in \mathrm{rad}_A(X, V)$, and $u'ea^{-1} + v'fa^{-1} = f' \notin \mathrm{rad}_A^2(X, V)$. We have the equalities

$$\begin{bmatrix} e' \\ f' \end{bmatrix} a = \begin{bmatrix} e'a \\ f'a \end{bmatrix} = \begin{bmatrix} e + p \\ f'a \end{bmatrix} = \begin{bmatrix} e + ue + vf \\ u'e + v'f \end{bmatrix} = \begin{bmatrix} 1 + u & v \\ u' & v' \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}.$$

Then there exists a homomorphism $b\colon Z \to Z$ such that the following diagram in $\operatorname{mod} A$ is commutative:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\begin{bmatrix} e \\ f \end{bmatrix}"] \arrow[d, "a"] & U \oplus V \arrow[r, "\begin{bmatrix} g & h \end{bmatrix}"] \arrow[d, "\begin{bmatrix} 1+u & v \\ u' & v' \end{bmatrix}"] & Z \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & X \arrow[r, "\begin{bmatrix} e' \\ f' \end{bmatrix}"] & U \oplus V \arrow[r, "\begin{bmatrix} g' & h' \end{bmatrix}"] & Z \arrow[r] & 0\,.
\end{tikzcd}$$

We claim that b is an automorphism. Suppose that $b \in \operatorname{rad}\operatorname{End}_A(Z) = \operatorname{rad}_A(Z, Z)$. Then the equality

$$\begin{bmatrix} g' & h' \end{bmatrix} \begin{bmatrix} 1+u & v \\ u' & v' \end{bmatrix} = b \begin{bmatrix} g & h \end{bmatrix}$$

gives $g' + (g'u + h'u') = bg \in \operatorname{rad}^2_A(UZ)$, and hence $g' \in \operatorname{rad}^2_A(U, Z)$, because $g'u + h'u' \in \operatorname{rad}^2_A(U, Z)$. This is a contradiction, because g' is irreducible. Thus b is an automorphism. Observe also that $bg - g' = g'u + h'u' \in \operatorname{rad}^2_A(UZ)$. Therefore, the classes $\bar{g} = g + \operatorname{rad}^2_A(U, Z)$ and $\bar{g}' = g' + \operatorname{rad}^2_A(U, Z)$ are linearly dependent over $F_Z = \operatorname{End}_Z(Z)/\operatorname{rad}\operatorname{End}_A(A)$. This shows that (ii) implies (i). The proof that (i) implies (ii) is similar. ☐

The following lemma is the converse of Lemma III.9.2.

Lemma 3.20. *Let A be a finite dimensional K-algebra over a field K, X, Y indecomposable modules in* $\operatorname{mod} A$, *$f_1, \ldots, f_r \in \operatorname{rad}_A(X, Y)$, and $\bar{f_1} = f_1 + \operatorname{rad}^2_A(X, Y), \ldots, \bar{f_r} = f_r + \operatorname{rad}^2_A(X, Y)$ are the associated classes in* $\operatorname{irr}_A(X, Y)$. *The following statements hold:*

(i) *If $\bar{f_1}, \ldots, \bar{f_r}$ are linearly independent vectors of the left F_Y-space* $\operatorname{irr}_A(X, Y)$, *then*

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix} : X \longrightarrow Y^r$$

is an irreducible homomorphism in $\operatorname{mod} A$.

(ii) *If $\bar{f_1}, \ldots, \bar{f_r}$ are linearly independent vectors of the right F_X-space* $\operatorname{irr}_A(X, Y)$, *then*

$$f' = \begin{bmatrix} f_1 & \ldots & f_r \end{bmatrix} : X^r \longrightarrow Y$$

is an irreducible homomorphism in $\operatorname{mod} A$.

Proof. We prove only (i), since the proof of (ii) is similar. Assume $\bar{f}_1, \dots, \bar{f}_r$ are linearly independent vectors of the left F_Y-space $\operatorname{irr}_A(X, Y)$. Consider the morphism of functors

$$\ell_0(f) \colon \operatorname{Hom}_A(Y^r, -)/\operatorname{rad}_A(Y^r, -) \longrightarrow \operatorname{rad}_A(X, -)/\operatorname{rad}_A^2(X, -)$$

in $\mathcal{F}(A)$ induced by f. In order to prove that f is an irreducible homomorphism in $\operatorname{mod} A$, it is enough to show that $\ell_0(f)$ is a monomorphism in $\mathcal{F}(A)$, by Proposition 2.2. Observe that, for an indecomposable module Z in $\operatorname{mod} A$, we have $\operatorname{Hom}_A(Y^r, Z)/\operatorname{rad}_A(Y^r, Z) \neq 0$ if and only if $Z \cong Y$. On the other hand, $\operatorname{Hom}_A(Y^r, Y)/\operatorname{rad}_A(Y^r, Y) = F_Y^r$ as left F_Y-spaces. Further, the homomorphism of left F_Y-spaces

$$\ell_0(f)_Y \colon \operatorname{Hom}_A(Y^r, Y)/\operatorname{rad}_A(Y^r, Y) \longrightarrow \operatorname{rad}_A(X, Y)/\operatorname{rad}_A^2(X, Y)$$

assigns to $(a_1, \dots, a_r) \in F_Y^r$ the element $a_1 \bar{f}_1 + \cdots + a_r \bar{f}_r$. Therefore, the assumption imposed on $\bar{f}_1, \dots, \bar{f}_r$ forces $\ell_0(f)$ to be a monomorphism. Hence f is an irreducible homomorphism. □

Lemma 3.21. *Let A be a finite dimensional K-algebra over a field K and $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ be an arrow in Γ_A such that Y is not projective and $d_{XY} \geq 2$, $d'_{XY} = 1$. Then the following statements hold:*

(i) *Every irreducible homomorphism $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \colon X \longrightarrow Y \oplus Y$ in $\operatorname{mod} A$ has a left neighbour.*

(ii) *Every irreducible homomorphism $\begin{bmatrix} f_1 & f_2 \end{bmatrix} \colon \tau_A Y_1 \oplus \tau_A Y_2 \longrightarrow X$ in $\operatorname{mod} A$ has a right neighbour.*

Proof. We prove only (i), since the proof of (ii) is similar. Assume $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \colon X \to Y \oplus Y$ is an irreducible homomorphism in $\operatorname{mod} A$. We note that the existence of such a homomorphism is a consequence of $d_{XY} \geq 2$. Applying Lemma III.9.2, we conclude that the classes $\bar{g}_1 = g_1 + \operatorname{rad}_A^2(X, Y)$ and $\bar{g}_2 = g_2 + \operatorname{rad}_A^2(X, Y)$ are linearly independent vectors of the left F_Y-space $\operatorname{irr}_A(X, Y) = \operatorname{rad}_A(X, Y)/\operatorname{rad}_A^2(X, Y)$. Further, the imposed assumption $d'_{XY} = 1$ means that $\operatorname{irr}_A(X, Y)$ is a one-dimensional right F_X-space. Then it follows from Theorem III.7.12 that we have in $\operatorname{mod} A$ almost split sequences

$$0 \longrightarrow \tau_A Y \xrightarrow{\begin{bmatrix} f_1 \\ h_1 \end{bmatrix}} X \oplus Z \xrightarrow{\begin{bmatrix} g_1 & v_1 \end{bmatrix}} Y \longrightarrow 0,$$

$$0 \longrightarrow \tau_A Y \xrightarrow{\begin{bmatrix} f_2 \\ h_2 \end{bmatrix}} X \oplus Z \xrightarrow{\begin{bmatrix} g_2 & v_2 \end{bmatrix}} Y \longrightarrow 0,$$

where X is not a direct summand of Z. In particular, $\{f_1, g_1\}$ and $\{f_2, g_2\}$ are components of almost split sequences in $\operatorname{mod} A$. Since $\bar{g}_1$ and $\bar{g}_2$ are linearly independent vectors of the left F_Y-space $\operatorname{irr}_A(X, Y)$, applying Lemma 3.19, we conclude that the classes $\bar{f}_1 = f_1 + \operatorname{rad}_A^2(\tau_A Y, X)$ and $\bar{f}_2 = f_2 + \operatorname{rad}_A^2(\tau_A Y, X)$ are linearly independent vectors of the right $F_{\tau_A Y}$-space $\operatorname{irr}_A(\tau_A Y, X)$. Then it follows from Lemma 3.20 that $\begin{bmatrix} f_1 & f_2 \end{bmatrix} : \tau_A Y \oplus \tau_A Y \to X$ is an irreducible homomorphism in $\operatorname{mod} A$. Therefore, $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is a left neighbour of $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$. □

Lemma 3.22. *Let A be a finite dimensional K-algebra over a field K and $f : X \to Y$ an irreducible homomorphism in* $\operatorname{mod} A$. *The following statements hold:*

(i) *If $d_r(f) < \infty$ and $Y = Y_1 \oplus Y_2$ for some indecomposable modules Y_1 and Y_2 in* $\operatorname{mod} A$, *then f is a right neighbour of an irreducible homomorphism $g = \begin{bmatrix} g_1 & g_2 \end{bmatrix} : \tau_A Y_1 \oplus \tau_A Y_2 \to X$ such that $d_r(g) < d_r(f)$.*

(ii) *If $d_l(f) < \infty$ and $X = X_1 \oplus X_2$ for some indecomposable modules X_1 and X_2 in* $\operatorname{mod} A$, *then f is a left neighbour of an irreducible homomorphism $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} : Y \to \tau_A^{-1} X_1 \oplus \tau_A^{-1} X_2$ such that $d_l(g) < d_l(f)$.*

Proof. We prove only (i), since the proof of (ii) is similar.

Assume that $d_r(f) = m < \infty$. Moreover, let $Y = Y_1 \oplus Y_2$ for some indecomposable modules Y_1 and Y_2 in $\operatorname{mod} A$. Then there exists a module M in $\operatorname{mod} A$ such that the homomorphism

$$r_m(f)_M : \operatorname{rad}_A^m(M, X)/\operatorname{rad}_A^{m+1}(M, X) \longrightarrow \operatorname{rad}_A^{m+1}(M, Y)/\operatorname{rad}_A^{m+2}(M, Y)$$

induced by f is not a monomorphism. Hence there exists $p \in \operatorname{rad}_A^m(M, X) \setminus \operatorname{rad}_A^{m+1}(M, X)$ such that $fp \in \operatorname{rad}_A^{m+2}(M, Y)$. Let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ be homomorphisms in $\operatorname{mod} A$ such that $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$. Clearly, we have $f_1 p \in \operatorname{rad}_A^{m+2}(M, Y_1)$ and $f_2 p \in \operatorname{rad}_A^{m+2}(M, Y_2)$. Note also that Y_1 and Y_2 are nonprojective, by Corollary 3.4.

Assume first that $Y_1 \ncong Y_2$. Then we have in $\operatorname{mod} A$ almost split sequences

$$0 \longrightarrow \tau_A Y_1 \xrightarrow{\begin{bmatrix} g_1 \\ h_1 \end{bmatrix}} X \oplus Z_1 \xrightarrow{\begin{bmatrix} f_1 & v_1 \end{bmatrix}} Y_1 \longrightarrow 0,$$

$$0 \longrightarrow \tau_A Y_2 \xrightarrow{\begin{bmatrix} g_2 \\ h_2 \end{bmatrix}} X \oplus Z_2 \xrightarrow{\begin{bmatrix} f_2 & v_2 \end{bmatrix}} Y_2 \longrightarrow 0,$$

where $\tau_A Y_1 \ncong \tau_A Y_2$, so there is an irreducible homomorphism $\begin{bmatrix} g_1 & g_2 \end{bmatrix} : \tau_A Y_1 \oplus \tau_A Y_2 \longrightarrow X$ in $\operatorname{mod} A$. Hence $\begin{bmatrix} g_1 & g_2 \end{bmatrix}$ is a left neighbour of $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

Assume now that $Y_1 \cong Y_2$. Then $d_{XY_1} \geq 2$. Since $d_r(f) < \infty$, it follows from Lemma 3.18, that $d'_{XY_1} = 1$. Applying Lemma 3.21, we infer that f has a left neighbour, say $\begin{bmatrix} g_1 & g_2 \end{bmatrix}: \tau_A Y_1 \oplus \tau_A Y_2 \to X$.

In any case, $\{g_1, f_1\}$ and $\{g_2, f_2\}$ are components of almost split sequences in $\operatorname{mod} A$. Applying now Lemma 3.3, we conclude that there exist homomorphisms $q_1: M \to \tau_A Y_1$ and $q_2: M \to \tau_A Y_2$ in $\operatorname{mod} A$ such that $q_1 \notin \operatorname{rad}_A^m(M, \tau_A Y_1)$, $q_2 \notin \operatorname{rad}_A^m(M, \tau_A Y_2)$, and $p + g_1 q_1 \in \operatorname{rad}_A^{m+1}(M, X)$, $p + g_2 q_2 \in \operatorname{rad}_A^{m+1}(M, X)$. Hence $g_1 q_1 - g_2 q_2 = (p + g_1 q_1) - (p + g_2 q_2) \in \operatorname{rad}_A^{m+1}(M, X)$. Then we have $\begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{bmatrix} q_1 \\ -q_2 \end{bmatrix} \in \operatorname{rad}_A^{m+1}(M, X)$. On the other hand, since $q_1 \notin \operatorname{rad}_A^m(M, \tau_A Y_1)$ and $q_2 \notin \operatorname{rad}_A^m(M, \tau_A Y_2)$, we have $\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix} \notin \operatorname{rad}_A^m(M, \tau_A Y_1 \oplus \tau_A Y_2)$. This shows that $d_r(g) \leq m - 1 < d_r(f)$. □

Theorem 3.23. *Let A be a finite dimensional K-algebra over a field K and $f: X \to Y$ an irreducible homomorphism in* $\operatorname{mod} A$*, where X or Y is indecomposable. Then the following equivalences hold:*

(i) $d_r(f) = 1$ *if and only if f is a right minimal almost split epimorphism in* $\operatorname{mod} A$.

(ii) $d_l(f) = 1$ *if and only if f is a left minimal almost split monomorphism in* $\operatorname{mod} A$.

Proof. We prove only (i), since the proof of (ii) is similar. Assume that $d_r(f) = 1$. If Y is indecomposable, then, by Corollary 3.7, f is an epimorphism and right minimal almost split homomorphism in $\operatorname{mod} A$. Hence, assume that Y is a decomposable module. Then it follows from the assumption that X is indecomposable. Let Z and Z' be indecomposable modules in $\operatorname{mod} A$ such that $Z \oplus Z'$ is a direct summand of Y, and let $p: Y \to Z \oplus Z'$ be the canonical retraction. Then $pf: X \to Z \oplus Z'$ is an irreducible homomorphism in $\operatorname{mod} A$, by Lemma III.7.4. Hence, using Lemma 3.1, we obtain the inequalities $1 \leq d_r(pf) \leq d_r(f) = 1$, and so $d_r(pf) = 1$. By Lemma 3.22, pf has a left neighbour $g: \tau_A Z \oplus \tau_A Z' \to X$ such that $d_r(g) < 1$, a contradiction.

Assume now that $f: X \to Y$ is a right minimal almost split epimorphism in $\operatorname{mod} A$. Then we have in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow \tau_A Y \xrightarrow{g} X \xrightarrow{f} Y \longrightarrow 0.$$

Since g is an irreducible homomorphism, $g \in \operatorname{rad}_A(\tau_A Y, X) \setminus \operatorname{rad}_A^2(\tau_A Y, X)$. Hence, $\bar{g} = g + \operatorname{rad}_A^2(\tau_A Y, X)$ is a nonzero element of $\operatorname{rad}_A(\tau_A Y, X)/\operatorname{rad}_A^2(\tau_A Y, X)$ such that $r_1(f)(\bar{g}) = fg + \operatorname{rad}_A^3(\tau_A Y, Y) = 0 + \operatorname{rad}_A^3(\tau_A Y, Y)$, because $fg = 0$. Hence, $r_1(f)$ is not a monomorphism in $\mathcal{F}(A)^{\mathrm{o}}$. Therefore, $d_r(f) = 1$. □

Proposition 3.24. *Let A be a finite dimensional K-algebra over a field K and $f: X \to Y$, $g: X \to Y$ irreducible homomorphisms between indecomposable modules in* $\operatorname{mod} A$. *Then $d_{\mathrm{l}}(f) = d_{\mathrm{l}}(g)$ and $d_{\mathrm{r}}(f) = d_{\mathrm{r}}(g)$.*

Proof. It follows that we have in Γ_A an arrow $X \xrightarrow{(d_{XY}, d'_{XY})} Y$. We prove that $d_{\mathrm{r}}(f) = d_{\mathrm{r}}(g)$. If $d_{\mathrm{r}}(f) = \infty$ and $d_{\mathrm{r}}(g) = \infty$, there is nothing to show. We may assume (without loss of generality) that $d_{\mathrm{r}}(f) < \infty$. It follows from Lemma 3.18 that $d_{XY} = 1$ or $d'_{XY} = 1$.

For any nonnegative integer n, consider the morphisms of functors in $\mathcal{F}(A)^{\mathrm{o}}$

$$r_n(f), r_n(g) \colon \operatorname{rad}_A^n(-, X)/\operatorname{rad}_A^{n+1}(-, X) \longrightarrow \operatorname{rad}_A^{n+1}(-, Y)/\operatorname{rad}_A^{n+2}(-, Y),$$

induced by f and g.

Let $d_{XY} = 1$. Then $g + \operatorname{rad}_A^2(X, Y)$ is a generator of the one-dimensional left F_Y-space $\operatorname{irr}_A(X, Y) = \operatorname{rad}_A(X, Y)/\operatorname{rad}_A^2(X, Y)$, and hence $f + \operatorname{rad}_A^2(X, Y) = \alpha(g + \operatorname{rad}_A^2(X, Y))$ for a nonzero element $\alpha \in F_Y$. Clearly, $\alpha = a + \operatorname{rad}_A(Y, Y)$ for an endomorphism $a \in \operatorname{End}_A(Y) \setminus \operatorname{rad}_A(Y, Y)$, which is an automorphism of Y. Then $f - ag \in \operatorname{rad}_A^2(X, Y)$. Hence, $r_n(f) = a_{n+1} r_n(g)$, where a_{n+1} is the isomorphism of functors

$$a_{n+1} \colon \operatorname{rad}_A^{n+1}(-, Y)/\operatorname{rad}_A^{n+2}(-, Y) \longrightarrow \operatorname{rad}_A^{n+1}(-, Y)/\operatorname{rad}_A^{n+2}(-, Y),$$

induced by the automorphism a. In particular, we obtain that $r_n(f)$ is a monomorphism if and only if $r_n(g)$ is a monomorphism. This shows that $d_{\mathrm{r}}(f) = d_{\mathrm{r}}(g)$.

Assume that $d'_{XY} = 1$. Then $g + \operatorname{rad}_A^2(X, Y)$ is a generator of the one-dimensional right F_X-space $\operatorname{irr}_A(X, Y)$. Then there exists an automorphism $b \in \operatorname{End}_A(X)$ such that $f - gb \in \operatorname{rad}_A^2(X, Y)$. Hence, for any nonnegative integer n, we have $r_n(f) = r_n(g) b_n$, where b_n is the isomorphism of functors

$$b_n \colon \operatorname{rad}_A^n(-, Y)/\operatorname{rad}_A^{n+1}(-, Y) \longrightarrow \operatorname{rad}_A^n(-, Y)/\operatorname{rad}_A^{n+1}(-, Y),$$

induced by b. Therefore, $r_n(f)$ is a monomorphism if and only if $r_n(g)$ is a monomorphism. This shows that $d_{\mathrm{r}}(f) = d_{\mathrm{r}}(g)$.

The proof that $d_{\mathrm{l}}(f) = d_{\mathrm{l}}(g)$ is similar. □

The above proposition allows to define the left degree and the right degree of an arrow $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ of the Auslander–Reiten quiver Γ_A of a finite dimensional K-algebra A as the left degree $d_{\mathrm{l}}(f)$ and the right degree $d_{\mathrm{r}}(f)$ of an arbitrary irreducible homomorphism f from X to Y in $\operatorname{mod} A$.

We end the section with the following fact.

Proposition 3.25. *Let A be a finite dimensional K-algebra over a field K. Then every oriented cycle in Γ_A contains both an arrow of finite left degree and an arrow of finite right degree.*

Proof. Let $X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n = X_0$ be an oriented cycle in Γ_A. We choose irreducible homomorphisms $f_i \colon X_{i-1} \to X_i$, $i \in \{1, \dots, n\}$. Then $f_i \in \operatorname{rad}_A(X_{i-1}, X_i)$, for $i \in \{1, \dots, n\}$, and hence $f_n \cdots f_1 \in \operatorname{rad}_A(X_0, X_0) = \operatorname{rad} \operatorname{End}_A(X_0)$. Since $\operatorname{rad} \operatorname{End}_A(X_0)$ is a nilpotent ideal of $\operatorname{End}_A(X_0)$, we conclude that $(f_n \cdots f_1)^n = 0$ for some positive integer r. Observe that then $(f_n \cdots f_1)^r \in \operatorname{rad}_A^{rn+1}(X_0, X_0)$. Then it follows from Lemma 3.2 that there exist $i, j \in \{1, \dots, n\}$ such that $d_{\mathrm{r}}(f_i) < \infty$ and $d_{\mathrm{l}}(f_j) < \infty$. Therefore, the arrow $X_{i-1} \to X_i$ has finite right degree, and the arrow $X_{j-1} \to X_j$ has finite left degree. □

4 Stable Auslander–Reiten components

Let A be a finite dimensional K-algebra over a field K. An indecomposable module X in $\operatorname{mod} A$ is said to be *left stable* if $\tau_A^n X \neq 0$ for any integer $n \geq 0$, *right stable* if $\tau_A^n X \neq 0$ for any integer $n \leq 0$, and *stable* if $\tau_A^n X \neq 0$ for any integer n.

Proposition 4.1. *Let A be a finite dimensional K-algebra over a field K and*

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$

be a presectional path in Γ_A, where all the modules X_i, $i \in \mathbb{N}$, are left stable. Assume that there exists a positive integer n such that one of the following conditions is satisfied:

(1) *the arrow $X_{n+1} \to X_n$ has nontrivial valuation;*
(2) *the middle term of an almost split sequence in* $\operatorname{mod} A$

$$0 \longrightarrow \tau_A X_n \longrightarrow E_n \longrightarrow X_n \longrightarrow 0$$

has at least three left stable indecomposable direct summands.

Then all arrows in Γ_A of the forms

$$\tau_A^j X_{i+1} \longrightarrow \tau_A^j X_i \quad \text{and} \quad \tau_A^{j+1} X_i \longrightarrow \tau_A^j X_{i+1},$$

for $i > n+1$ and $j \geq 0$, have infinite right degree.

Proof. It follows from our assumption that, for any $i \geq 0$, $\tau_A X_i \oplus X_{i+2}$ is a direct summand of the middle term of an almost split sequence in $\operatorname{mod} A$ with the right term X_{i+1}. Then, for any $j \geq 0$, there is in Γ_A the presectional path of the form (see Lemma 3.13)

$$\cdots \longrightarrow \tau_A^j X_{i-1} \longrightarrow \tau_A^j X_i \longrightarrow \cdots \longrightarrow \tau_A^j X_1 \longrightarrow \tau_A^j X_0$$

whose all modules are left stable. Hence, for any $i \geq 0$, $\tau_A^{j+1} X_i \oplus \tau_A^j X_{i+2} = \tau_A(\tau_A^j X_i) \oplus \tau_A^j X_{i+2}$ is a direct summand of the middle term of an almost split sequence in $\operatorname{mod} A$ with the right term $\tau_A^j X_{i+1}$. Applying Corollary 3.15 and Proposition 3.24, we conclude that all arrows $\tau_A^{j+1} X_i \longrightarrow \tau_A^j X_{i+1}$, for $i, j \geq 0$, have infinite right degree.

We will prove now that all arrows $\tau_A^j X_{i+1} \longrightarrow \tau_A^j X_i$, for all $i > n+1$ and $j \geq 0$, have infinite right degree.

Assume first that there is in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow \tau_A X_n \longrightarrow E_n \longrightarrow X_n \longrightarrow 0,$$

where $E_n = X_{n+1} \oplus Y_n \oplus Z_n \oplus E_n'$ and X_{n+1}, Y_n, Z_n are left stable indecomposable modules. Suppose that, for some $i > n$, there exists in $\operatorname{mod} A$ an irreducible homomorphism $f \colon X_{i+1} \to X_i$ with $d_{\mathrm{r}}(f) < \infty$. Observe that we have in Γ_A a presectional path

$$\tau_A^{i-n} X_n \longrightarrow \tau_A^{i-n-1} X_{n+1} \longrightarrow \cdots \longrightarrow \tau_A X_{i-1} \longrightarrow X_i$$

such that $X_{i+1} \oplus \tau_A X_{i-1}$ is a direct summand of the middle term E_i of an almost split sequence in $\operatorname{mod} A$

$$0 \longrightarrow \tau_A X_i \longrightarrow E_i \longrightarrow X_i \longrightarrow 0$$

with the right term X_i. Then it follows from Proposition 3.14 that there exists an irreducible homomorphism $f' \colon \tau_A^{i-n} X_{n+1} \to \tau_A^{i-n} X_n$ such that $d_{\mathrm{r}}(f') < d_{\mathrm{r}}(f)$, because $\tau_A^{i-n} X_{n+1} = \tau_A(\tau_A^{i-n-1} X_{n+1})$. Further, the middle term of an almost split sequence in $\operatorname{mod} A$ with the right term $\tau_A^{i-n} X_n$ has a direct summand of the form $\tau_A^{i-n} X_{n+1} \oplus \tau_A^{i-n} Y_n \oplus \tau_A^{i-n} Z_n$. Moreover, applying Corollary 3.6, we conclude that there exists an irreducible homomorphism $h \colon \tau_A^{i-n+1} X_n \to \tau_A^{i-n} Y_n \oplus \tau_A^{i-n} Z_n$ such that $d_{\mathrm{r}}(h) < d_{\mathrm{r}}(f')$. Hence, by Lemma 3.22, there exists an irreducible homomorphism $g \colon \tau_A^{i-n+1} Y_n \oplus \tau_A^{i-n+1} Z_n \to \tau_A^{i-n+1} X_n$ such that $d_{\mathrm{r}}(g) < d_{\mathrm{r}}(h)$. In particular, we obtain that $d_{\mathrm{r}}(g) < \infty$. On the other hand, we have in Γ_A a presectional path

$$\cdots \to \tau_A^{i-n+1} X_{n+t} \to \tau_A^{i-n+1} X_{n+t-1} \to \cdots \to \tau_A^{i-n+1} X_{n+1} \to \tau_A^{i-n+1} X_n$$

such that $\tau_A^{i-n+1} X_{n+1} \oplus \tau_A^{i-n+1} Y_n \oplus \tau_A^{i-n+1} Z_n$ is a direct summand of the middle term of an almost split sequence in $\operatorname{mod} A$ with the right term $\tau_A^{i-n+1} X_n$, and hence $d_{\mathrm{r}}(g) = \infty$, by Corollary 3.15. Therefore, every homomorphism $f_i \colon X_{i+1} \to X_i$, for $i > n$, is of infinite right degree. Similarly, we conclude that, for any $i > n$ and $j \geq 0$, an arbitrary irreducible homomorphism from $\tau_A^j X_{i+1}$ to $\tau_A^j X_i$ in $\operatorname{mod} A$ is of infinite right degree, because all modules X_i, $i \in \mathbb{N}$, are left stable.

Assume now that the arrow $X_{n+1} \to X_n$ has a nontrivial valuation. We abbreviate $d_n = d_{X_{n+1}X_n}$ and $d'_n = d'_{X_{n+1}X_n}$. Hence, $d_n > 1$ or $d'_n > 1$.

We consider first the case when $d'_n > 1$. Then $X_{n+1} \oplus X_{n+1}$ is a direct summand of the middle term of an almost split sequence in mod A with the right term X_n, and the left term $\tau_A X_n$. Since the modules X_i, $i \geq 1$, are left stable, we conclude that, for any $i > n$, the module

$$\tau_A^{i-n} X_{n+1} \oplus \tau_A(\tau_A^{i-n-1} X_{n+1}) = \tau_A^{i-n} X_{n+1} \oplus \tau_A^{i-n} X_{n+1}$$

is a direct summand of the middle term of an almost split sequence in mod A with the right term $\tau_A^{i-n} X_n$, and the left term $\tau_A^{i-n+1} X_n$. Then we conclude that, for any $i > n$, there exists an infinite presectional path in Γ_A of the form

$$\begin{tikzcd}
\cdots \arrow[r] & \tau_A^{i-n} X_{n+t+1} \arrow[r] & \tau_A^{i-n} X_{n+t} \arrow[r] & \cdots \arrow[r] & \tau_A^{i-n} X_{n+1} \arrow[r] & \tau_A^{i-n} X_n \arrow[dllll, rounded corners] \\
& \tau_A^{i-n-1} X_{n+1} \arrow[r] & \cdots \arrow[r] & \tau_A X_{i-1} \arrow[r] & X_i &
\end{tikzcd}$$

such that $X_{i+1} \oplus \tau_A X_{i-1}$ is a direct summand of the middle term of an almost split sequence in mod A with the right term X_i, and consequently the arrow $X_{i+1} \to X_i$ is of infinite right degree, by Corollary 3.15 and Proposition 3.24. Similarly, we conclude that, for any $i > n$ and $j \geq 1$, the arrow $\tau_A^j X_{i+1} \to \tau_A^j X_i$ is of infinite right degree, because all modules X_i, $i \in \mathbb{N}$, are left stable.

Assume now that $d_n > 1$. Then we have in Γ_A the valued arrows

$$\tau_A X_n \xrightarrow{(d'_n, d_n)} X_{n+1} \xrightarrow{(d_n, d'_n)} X_n,$$

by Lemma III.9.1 and Proposition III.9.6. If $X_{n+2} \neq \tau_A X_n$, then there is in mod A an almost split sequence

$$0 \longrightarrow \tau_A X_{n+1} \longrightarrow E_{n+1} \longrightarrow X_{n+1} \longrightarrow 0$$

such that $X_{n+2} \oplus \tau_A X_n \oplus \tau_A X_n$ is a direct summand of E_{n+1}. Hence it follows from the first part of the proof that all arrows $X_{i+1} \to X_i$, for $i > n+1$, have infinite right degree. If $X_{n+2} = \tau_A X_n$, then the arrow $X_{n+2} \to X_{n+1}$ has the valuation $(d_{n+1}, d'_{n+1}) = (d'_n, d_n)$ with $d'_{n+1} = d_n \geq 2$, so we are in the previous case where n is replaced by $n+1$. Therefore, we conclude that (in this case) all arrows $X_{i+1} \to X_i$, for $i > n+1$, have infinite right degree. Similarly, we conclude that, for any $i > n+1$ and $j \geq 1$, the arrow $\tau_A^j X_{i+1} \to \tau_A^j X_i$ is of infinite right degree, again because all modules X_i, $i \in \mathbb{N}$, are left stable. □

We have also the following dual proposition.

Proposition 4.2. *Let A be a finite dimensional K-algebra over a field K and*

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots$$

be a presectional path in Γ_A, where all the modules X_i, $i \in \mathbb{N}$, are right stable. Assume that there exists a positive integer n such that one of the following conditions are satisfied:

(i) *the arrow $X_n \to X_{n+1}$ has nontrivial valuation;*

(ii) *the middle term of an almost split sequence in* mod A

$$0 \longrightarrow X_n \longrightarrow E_n \longrightarrow \tau_A^{-1} X_n \longrightarrow 0$$

has at least three right stable indecomposable direct summands.

Then all arrows in Γ_A of the forms

$$\tau_A^j X_i \longrightarrow \tau_A^j X_{i+1} \quad \textit{and} \quad \tau_A^j X_{i+1} \longrightarrow \tau_A^{j-1} X_i,$$

for $i > n+1$ and $j \leq 0$, have infinite left degree.

Proposition 4.3. *Let A be a finite dimensional K-algebra over a field K and*

$$\cdots \longrightarrow X_{-n} \longrightarrow \cdots \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots$$

be a presectional path in Γ_A. Then the following statements hold:

(i) *If all modules X_i, $i \in \mathbb{Z}$, are left stable, then all arrows*

$$\tau_A^j X_i \longrightarrow \tau_A^j X_{i+1} \quad \textit{and} \quad \tau_A^{j+1} X_{i+1} \longrightarrow \tau_A^j X_i,$$

for $i \in \mathbb{Z}$ and $j \geq 0$, have infinite right degree.

(ii) *If all modules X_i, $i \in \mathbb{Z}$, are right stable, then all arrows*

$$\tau_A^j X_i \longrightarrow \tau_A^j X_{i+1} \quad \textit{and} \quad \tau_A^j X_{i+1} \longrightarrow \tau_A^{j-1} X_i,$$

for $i \in \mathbb{Z}$ and $j \leq 0$, have infinite left degree.

Proof. We prove only (i), since the proof of (ii) is dual.

Assume that all modules X_i, $i \in \mathbb{Z}$, are left stable. Then, for all $i \in \mathbb{Z}$ and $j \geq 0$, we have in Γ_A infinite presectional paths

$$\cdots \longrightarrow \tau_A^j X_{i-2} \longrightarrow \tau_A^j X_{i-1} \longrightarrow \tau_A^j X_i,$$
$$\cdots \longrightarrow \tau_A^{j+2} X_{i+3} \longrightarrow \tau_A^{j+1} X_{i+2} \longrightarrow \tau_A^j X_{i+1},$$

such that $\tau_A^{j+1}X_{i+1} \oplus \tau_A^j X_{i-1}$ is a direct summand of the middle term of an almost split sequence in $\operatorname{mod} A$ with the right term $\tau_A^j X_i$, and $\tau_A^{j+1}X_{i+2} \oplus \tau_A^j X_i$ is a direct summand of the middle term of an almost split sequence in $\operatorname{mod} A$ with the right term $\tau_A^j X_{i+1}$. Then it follows from Corollary 3.15 that the arrows $\tau_A^{j+1}X_{i+1} \to \tau_A^j X_i$ and $\tau_A^j X_i \to \tau_A^j X_{i+1}$, for $i \in \mathbb{Z}$ and $j \geq 0$, have infinite right degree. □

Let A be a finite dimensional K-algebra over a field K. The *stable Auslander–Reiten quiver* Γ_A^{s} of A is the translation quiver obtained from the Auslander–Reiten quiver Γ_A of A by removing all nonstable modules and the arrows attached to them, and with the induced Auslander–Reiten translations τ_A and τ_A^{-1}. Then a connected component of the quiver Γ_A^{s} is said to be a *stable Auslander–Reiten component* of Γ_A^{s}.

Lemma 4.4. *Let A be a finite dimensional K-algebra over a field K, $\mathcal{C}$ a component of Γ_A^{s}, and X, Y modules in $\mathcal{C}$. Assume that there exists in $\mathcal{C}$ a path of length ≥ 1 from X to Y. Then either $X = \tau_A^r Y$ for some integer $r \geq 1$, or there exists a sectional path in $\mathcal{C}$ from X to $\tau_A^t Y$ for some integer $t \geq 0$.*

Proof. Let $X = Y_0 \to Y_1 \to \cdots \to Y_{p-1} \to Y_p = Y$ be a path in $\mathcal{C}$ from X to Y with $p \geq 1$. We use induction on p.

Let $p = 1$. Then $X = Y_0 \to Y_1 = Y$ is a sectional path in $\mathcal{C}$.

Assume that $p \geq 2$ and either $X = \tau_A^m Y_{p-1}$ for some integer $m \geq 1$, or there exists in $\mathcal{C}$ a sectional path

$$X = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_{q-1} \longrightarrow Z_q = \tau_A^s Y_{p-1}$$

for some integer $s \geq 0$. In the first case, we have the sectional path $X = \tau_A^m Y_{p-1} \to \tau_A^m Y_p = \tau_A^m Y$. Assume that the second case holds. We have in $\mathcal{C}$ the arrows $Z_{q-1} \to \tau_A^s Y_{p-1}$ and $\tau_A^s Y_{p-1} \to \tau_A^s Y_p$. If $Z_{q-1} \neq \tau_A^{s+1} Y_p$, then we have in $\mathcal{C}$ a sectional path

$$X = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_{q-1} \longrightarrow Z_q \longrightarrow \tau_A^s Y_p = \tau_A^s Y.$$

Assume $Z_{q-1} = \tau_A^{s+1} Y_p$. Then either the path

$$X = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_{q-1} = \tau_A^{s+1} Y$$

is sectional, or $q = 1$ and $X = \tau_A^{s+1} Y$, with $s + 1 \geq 1$. □

Let A be a finite dimensional K-algebra over a field K. An indecomposable module X in $\operatorname{mod} A$ is called *τ_A-periodic* if $X \cong \tau_A^m X$ for some integer $m \geq 1$.

Lemma 4.5. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ a component of Γ_A^{s} containing a τ_A-periodic module in $\mathcal{C}$. Then all modules in $\mathcal{C}$ are τ_A-periodic.*

Proof. Assume that $\tau_A^m X = X$ for a module X in $\mathcal{C}$ and some integer $m \geq 1$. Observe that, if $X \to Y$ is an arrow in $\mathcal{C}$, then we have in $\mathcal{C}$ an arrow $X = \tau_A^m X \to \tau_A^m Y$. Similarly, if $Z \to X$ is an arrow in $\mathcal{C}$, then we have in $\mathcal{C}$ an arrow $\tau_A^m Z \to \tau_A^m X = X$. Since X has only finitely many neighbours in $\mathcal{C}$, we conclude that, if M is a neighbour of X in $\mathcal{C}$, then there is a positive integer r such that $M = \tau_A^{rm} M$. Let N be a module in $\mathcal{C}$ different from X. Since $\mathcal{C}$ is connected, there is in $\mathcal{C}$ a walk

$$X = X_0 — X_1 — \cdots — X_{s-1} — X_s = N,$$

where $s \geq 1$ and $X_i — X_{i+1}$ means either $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$, for $i \in \{0, \dots, s-1\}$. Then it follows from the above arguments that the modules $X_1, \dots, X_s = N$ are τ_A-periodic. Therefore, every module in $\mathcal{C}$ is τ_A-periodic. □

Proposition 4.6. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ a component of Γ_A^s containing an oriented cycle. Then every module in $\mathcal{C}$ is τ_A-periodic.*

Proof. Suppose that $\mathcal{C}$ contains a module which is not τ_A-periodic. Then it follows from Lemma 4.5 that $\mathcal{C}$ does not contain a τ_A-periodic module. Take a module X in $\mathcal{C}$ lying on an oriented cycle. Then we have in $\mathcal{C}$ a path of length ≥ 1 from X to X. Applying Lemma 4.4, we conclude that there is in $\mathcal{C}$ a sectional path from X to $\tau_A^t X$ for some integer $t \geq 0$. Let

$$Y_1 \longrightarrow Y_2 \longrightarrow \cdots \longrightarrow Y_s \longrightarrow Y_{s+1} \tag{$*$}$$

be a sectional path in $\mathcal{C}$ of minimal length such that $Y_{s+1} = \tau_A^r Y_1$ for some integer $r \geq 0$. We claim that $r \geq 1$. Suppose that $r = 0$. Hence $Y_{s+1} = Y_1$. Since $\mathcal{C}$ does not contain a sectional cycle (see Corollary III.11.3), we have $s \geq 2$ and $\tau_A Y_1 = \tau_A Y_{s+1} = Y_{s-1}$. Moreover, if $s = 2$, then we have a sectional path

$$Y_1 \longrightarrow Y_2 \longrightarrow Y_3 = Y_1$$

with $\tau_A Y_1 = \tau_A Y_3 \neq Y_1$, because Y_1 is not τ_A-periodic, and hence again a contradiction with Corollary III.11.3. For $s \geq 3$, we have in $\mathcal{C}$ a sectional path $Y_1 \to \cdots \to Y_{s-1}$ of length $s - 1$ with $Y_{s-1} = \tau_A Y_1$, which contradicts the minimality of ($*$). Therefore, indeed $r \geq 1$. It follows from the minimality of the sectional path ($*$) that there is in $\mathcal{C}$ an infinite sectional path of the form

$$\begin{tikzcd}
\cdots \arrow[r] & \tau_A^{-2r} Y_{s-1} \arrow[r] & \tau_A^{-2r} Y_s \arrow[r] & \tau_A^{-r} Y_1 \arrow[r] & \tau_A^{-r} Y_2 \arrow[r] & \cdots \arrow[r] & \tau_A^{-r} Y_{s-1} \arrow[r] & \tau_A^{-r} Y_s \arrow[dllllll] \\
& Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_{s-1} \arrow[r] & Y_s \arrow[dllll] & & \\
& \tau_A^r Y_1 \arrow[r] & \tau_A^r Y_2 \arrow[r] & \cdots \arrow[r] & \tau_A^r Y_{s-1} \arrow[r] & \tau_A^r Y_s \arrow[r] & \cdots &
\end{tikzcd}$$

Observe also that there is an oriented cycle in $\mathcal{C}$ of the form

$$\begin{tikzcd}
Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_{s-1} \arrow[r] & Y_s \arrow[r] & \tau_A^{-1} Y_{s-1} \arrow[r] & \tau_A^{-1} Y_s \arrow[dllllll, rounded corners] \\
\tau_A^{-2} Y_{s-1} \arrow[r] & \tau_A^{-2} Y_s \arrow[r] & \cdots \arrow[r] & \tau_A^{-r} Y_{s-1} \arrow[r] & \tau_A^{-r} Y_s \arrow[r] & Y_1 \,.
\end{tikzcd}$$

Then it follows from Proposition 3.25 that there are integers $i \in \{1, \dots, s\}$ and $j \in \{0, \dots, r\}$ such that either $\tau_A^{-j} Y_i \to \tau_A^{-j} Y_{i+1}$ or $\tau_A^{-j} Y_{i+1} \to \tau_A^{-j-1} Y_i$ has finite left degree. But this contradicts Proposition 4.3 (ii). □

We have also the following direct consequence of the above proposition.

Corollary 4.7. *Let A be a finite dimensional K-algebra over a field K and $X_0 \to X_1 \to \cdots \to X_{n-1} \to X_0$ an oriented cycle in Γ_A consisting of stable modules. Then the modules $X_0, X_1, \dots, X_{n-1}$ are τ_A-periodic.*

Proof. Observe that there is a component $\mathcal{C}$ of Γ_A^{s} containing the considered cycle. Then the claim follows from Proposition 4.6. □

Lemma 4.8. *Let A be a finite dimensional K-algebra over a field K, $\mathcal{C}$ a component of Γ_A^{s}, and X, Y modules in $\mathcal{C}$ lying in different τ_A-orbits. Then there is in $\mathcal{C}$ a sectional path from $\tau_A^r X$ to Y for some integer r.*

Proof. Since $\mathcal{C}$ is connected, there is a walk in $\mathcal{C}$

$$X = X_0 \,—\, X_1 \,—\, \cdots \,—\, X_{s-1} \,—\, X_s = Y$$

connecting X and Y. We will show first that there is a path in $\mathcal{C}$ from $\tau_A^m X$ to Y, for some integer m. We use the induction on s.

Let $s = 1$. Then we have in $\mathcal{C}$ an arrow $X \to Y$ or an arrow $Y \to X$. In the second case, we have an arrow $\tau_A X \to Y$, and so the claim follows.

Let $s \geq 2$. Assume that there is a path in $\mathcal{C}$ from $\tau_A^t X$ to X_{s-1} for some integer t. We have in $\mathcal{C}$ an arrow $X_{s-1} \to X_s$ or an arrow $X_s \to X_{s-1}$. In the first case, we obtain a path from $\tau_A^t X$ to $X_s = Y$. In the second case, we have in $\mathcal{C}$ an arrow $\tau_A X_{s-1} \to X_s$, and consequently a path from $\tau_A^{t+1} X = \tau_A(\tau_A^t X)$ to $X_s = Y$.

Therefore, we do indeed have in $\mathcal{C}$ a path from $\tau_A^m X$ to Y for some integer m. Moreover, since by the assumption that the modules X and Y lie in different τ_A-orbits of $\mathcal{C}$, we conclude that $\tau_A^m X \neq \tau_A^p Y$ for any integer $p \geq 0$. Then it follows from Lemma 4.4 that there is in $\mathcal{C}$ a sectional path from $\tau_A^m X$ to $\tau_A^q Y$ for some integer $q \geq 0$. But then there is a sectional path from $\tau_A^r X$ to Y for the integer $r = m - q$. □

Recall that a stable tube is a translation quiver of the for $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some positive integer r, called the rank of the tube.

We are now in position to prove the announced theorem describing the shape of infinite stable Auslander–Reiten components containing oriented cycles, established by S. Liu [L2] and Y. Zhang [Z2].

Theorem 4.9. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be an infinite component of Γ_A^s. Then the following statements are equivalent:*

(i) *$\mathcal{C}$ contains an oriented cycle.*

(ii) *$\mathcal{C}$ is a stable tube.*

Proof. The implication (ii) ⇒ (i) is obvious. We prove that (i) implies (ii).

Assume that $\mathcal{C}$ contains an oriented cycle. Then it follows from Proposition 4.6 that every module in $\mathcal{C}$ is τ_A-periodic. In particular, $\mathcal{C}$ admits infinitely many pairwise different τ_A-orbits, because $\mathcal{C}$ is assumed to be infinite. Take a module X_0 in $\mathcal{C}$. Applying Lemma 4.8, we conclude that $\mathcal{C}$ admits infinitely many finite sectional paths ending at X_0. Since $\mathcal{C}$ is a locally finite quiver, König's lemma implies that there is an infinite sectional path in $\mathcal{C}$ of the form

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0. \qquad (**)$$

For each integer $i \geq 0$, there is an oriented cycle in $\mathcal{C}$ of the form

$$X_{i+1} = \tau_A^{n_i} X_{i+1} \longrightarrow \tau_A^{n_i} X_i \longrightarrow \tau_A^{n_i-1} X_{i+1} \longrightarrow \cdots \longrightarrow \tau_A X_i \longrightarrow X_{i+1},$$

for some integer $n_i \geq 1$, because X_{i+1} is τ_A-periodic. Then it follows from Proposition 3.25 that, for each integer $i \geq 0$, there is some integer $j_i \geq 0$ such that either $\tau_A^{j_i} X_{i+1} \to \tau_A^{j_i} X_i$ or $\tau_A^{j_i+1} X_i \to \tau_A^{j_i} X_{i+1}$ has finite right degree. Then it follows from Proposition 4.1 that each arrow of the path $(**)$ has trivial valuation and, for each integer $i \geq 1$, X_i has exactly two immediate predecessors X_{i+1} and $\tau_A X_{i-1}$ in $\mathcal{C}$ and exactly two immediate successors X_{i-1} and $\tau_A^{-1} X_{i+1}$ in $\mathcal{C}$. Moreover, by Proposition 4.3, the sectional path $(**)$ is not a subpath of the sectional path of the form

$$\cdots \to X_{i+1} \to X_i \to \cdots \to X_1 \to X_0 \to Y_1 \to \cdots \to Y_j \to Y_{j+1} \to \cdots$$

Hence we may assume (without loss of generality) that $(**)$ is a maximal sectional path in $\mathcal{C}$. Then X_0 has only one immediate predecessor X_1 in $\mathcal{C}$ and one immediate successor $\tau_A^{-1} X_1$ in $\mathcal{C}$. Since $\mathcal{C}$ is a connected stable translation quiver, we conclude also that the path $(**)$ intersects each τ_A-orbit in $\mathcal{C}$ at least once. Moreover, if r is the τ_A-period of X_0, then, using induction on $i \geq 0$, we easily obtain that every module X_i of the path $(**)$ has the τ_A-period equal to r.

It remains to show that the modules X_i, $i \geq 0$, belong to pairwise different τ_A-orbits. Since the module X_0 has only one immediate predecessor in $\mathcal{C}$, for each integer $i \geq 1$, X_0 and X_i cannot lie in the same τ_A-orbit. Assume now that, for some integer $m \geq 1$ and arbitrary $i \in \{0, \dots, m-1\}$, the τ_A-orbit of X_i does not contain a module X_j for any positive integer $j \neq i$. Suppose that the τ_A-orbit of X_m contains a module X_j for some positive integer $j \neq m$. Clearly, by the imposed assumption, we have $j \geq m+1$. Let $X_j = \tau_A^t X_m$ for some positive integer t. Then $\tau_A^{t+1} X_{m-1}$ is an immediate predecessor of X_j in $\mathcal{C}$. We know that X_j has exactly two immediate predecessors X_{j+1} and $\tau_A X_{j-1}$ in $\mathcal{C}$. Hence, $\tau_A^{t+1} X_{m-1} = X_{j+1}$ or $\tau_A^{t+1} X_{m-1} = \tau_A X_{j-1}$. In the second case, we have $\tau_A^t X_{m-1} = X_{j-1}$. In both cases, we reach a contradiction with inductive assumption.

Summing up, we proved that $\mathcal{C}$, as a translation quiver, is isomorphic to the stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ of rank r. □

Let A be a finite dimensional K-algebra over a field K. A component $\mathcal{C}$ of Γ_A is called *regular* if $\mathcal{C}$ contains neither a projective module nor an injective module. Observe that every regular component $\mathcal{C}$ of Γ_A is a component of the stable Auslander–Reiten quiver Γ_A^{s}. We have also the following fact.

Lemma 4.10. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a regular component of Γ_A. Then $\mathcal{C}$ is an infinite component.*

Proof. We may assume that A is an indecomposable algebra. Suppose that $\mathcal{C}$ is a finite component. Then it follows from Theorem III.10.2 that $\mathcal{C} = \Gamma_A$. However, this is impossible because Γ_A contains projective modules. □

The following consequence of Theorem 4.9 and Lemma 4.10 describes the shape of regular Auslander–Reiten components containing oriented cycles.

Theorem 4.11. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a regular component of Γ_A. The following statements are equivalent:*

(i) *$\mathcal{C}$ contains an oriented cycle.*

(ii) *$\mathcal{C}$ is a stable tube.*

Let A be a finite dimensional selfinjective K-algebra over a field K. Then the stable Auslander–Reiten quiver Γ_A^{s} is obtained from Γ_A by removing the indecomposable projective-injective modules and the arrows attached to them. For a component $\mathcal{C}$ of Γ_A we denote by $\mathcal{C}^{\mathrm{s}}$ the stable part of $\mathcal{C}$, obtained by removing in $\mathcal{C}$ the indecomposable projective-injective modules and the arrows attached to them. Clearly, if $\mathcal{C}$ is infinite, then $\mathcal{C}^{\mathrm{s}}$ is a component of Γ_A^{s}. A component $\mathcal{C}$ of Γ_A is called a *quasi-tube* if its stable part $\mathcal{C}^{\mathrm{s}}$ is a stable tube.

The following consequence of Theorem 4.9 provides a description of all infinite components of the Auslander–Reiten quivers of selfinjective algebras containing oriented cycles.

Theorem 4.12. *Let A be a finite dimensional selfinjective K-algebra over a field K and $\mathcal{C}$ be an infinite component of Γ_A. The following statements are equivalent:*

(i) *$\mathcal{C}$ contains an oriented cycle.*

(ii) *$\mathcal{C}$ is a quasi-tube.*

The final aim of this section is to describe the shape of acyclic stable Auslander–Reiten components. We start with a simple preparatory lemma.

Lemma 4.13. *Let A be a finite dimensional K-algebra over a field K, $\mathcal{C}$ an acyclic component of Γ^{s}_{A}, and X, Y different modules in $\mathcal{C}$. The following statements hold:*

(i) *There exists a path in $\mathcal{C}$ from $\tau_A^n X$ to Y for some integer $n \geq 0$.*

(ii) *There exists an integer $m \geq 0$ such that there is no path in $\mathcal{C}$ from $\tau_A^{-m} X$ to Y.*

Proof. (i) Since $\mathcal{C}$ is a connected quiver, there is a walk in $\mathcal{C}$

$$X = Z_0 \,\text{—}\, Z_1 \,\text{—} \cdots \text{—}\, Z_{r-1} \,\text{—}\, Z_r = Y,$$

with $r \geq 1$, where $Z_{i-1} \,\text{—}\, Z_i$ means $Z_{i-1} \to Z_i$ or $Z_{i-1} \leftarrow Z_i$, for any $i \in \{1, \dots, r\}$. We use induction on r.

Let $r = 1$. Then we have in $\mathcal{C}$ an arrow $X \to Y$ or an arrow $Y \to X$. In the second case, we have an arrow $\tau_A X \to Y$, since X is a stable module.

Let $r \geq 2$, and assume that there is a path in $\mathcal{C}$ from $\tau_A^n X$ to Z_{r-1} for some integer $n \geq 0$. If $Z_{r-1} \,\text{—}\, Z_r$ is the arrow $Z_{r-1} \to Z_r$, then we have in $\mathcal{C}$ a path from $\tau_A^n X$ to $Z_r = Y$. Assume that $Z_{r-1} \,\text{—}\, Z_r$ is the arrow $Z_{r-1} \leftarrow Z_r$. Then we have in $\mathcal{C}$ an arrow $\tau_A Z_{r-1} \to Z_r$. Further, since $\mathcal{C}$ is stable and there exists a path in $\mathcal{C}$ from $\tau_A^n X$ to Z_{r-1}, we conclude that there is also in $\mathcal{C}$ a path from $\tau_A^{n+1} X$ to $\tau_A Z_{r-1}$, and consequently a path in $\mathcal{C}$ from $\tau_A^{n+1} X$ to $Z_r = Y$. Hence, the statement (i) follows.

(ii) It follows from (i) that there exists an integer $m \geq 0$ such that $\mathcal{C}$ contains a path from $\tau_A^m Y$ to X. Since $\mathcal{C}$ is a stable translation quiver, we obtain that $\mathcal{C}$ contains a path from Y to $\tau_A^{-m} X$. Then there is no path in $\mathcal{C}$ from $\tau_A^{-m} X$ to Y, because $\mathcal{C}$ is an acyclic quiver. □

Proposition 4.14. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ a component of Γ^{s}_{A} with a section Δ. Then $\mathcal{C}$ is isomorphic to the translation quiver $\mathbb{Z}\Delta$.*

Proof. This follows from the proof of Proposition VIII.6.5. □

The following theorem due to S. Liu [L2] and Y. Zhang [Z2] describes the shape of acyclic stable Auslander–Reiten components.

Theorem 4.15. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be an acyclic component of Γ_A^{s}. Then there exists an acyclic locally finite valued quiver Δ such that $\mathcal{C}$ is isomorphic to the translation quiver $\mathbb{Z}\Delta$.*

Proof. Thanks to Proposition 4.14 it is enough to show that $\mathcal{C}$ admits a section Δ. We fix a module M in $\mathcal{C}$. It follows from Lemma 4.13 that, for any module X in $\mathcal{C}$, there exist nonnegative integers m and n such that there is a path in $\mathcal{C}$ from $\tau_A^n X$ to M and there is no path in $\mathcal{C}$ from $\tau_A^{-m} X$ to M. Hence, the τ_A-orbit $\mathcal{O}(X)$ of a module X in $\mathcal{C}$ contains a unique module Y such that there is a path in $\mathcal{C}$ from Y to M and there is no path in $\mathcal{C}$ from $\tau_A^{-1} Y$ to M. Observe that, for any positive integer p, there exists a path in $\mathcal{C}$ from $\tau_A^p Y$ to M and there is no path in $\mathcal{C}$ from $\tau_A^{-p} Y$ to M.

Let $\mathcal{O}_i$, $i \in I$, be set of all τ_A-orbits in $\mathcal{C}$. For each $i \in I$, we denote by Y_i the unique module in $\mathcal{O}_i$ such that there is a path in $\mathcal{C}$ from Y_i to M and there is no path in $\mathcal{C}$ from $\tau_A^{-1} Y_i$ to M. Let Δ be the full valued subquiver of $\mathcal{C}$ given by all modules Y_i, $i \in I$. Since $\mathcal{C}$ is acyclic, we conclude that Δ is an acyclic quiver. Moreover, it follows from the definition of Δ that the module M is a unique sink of Δ. In particular, Δ is a connected quiver. Moreover, Δ intersects each τ_A-orbit in $\mathcal{C}$ exactly once. We will show that Δ is a convex subquiver of $\mathcal{C}$. Let

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{t-1} \longrightarrow X_t$$

be a path in $\mathcal{C}$ with X_0 and X_t from Δ. We may assume that $t \geq 2$. Since $\mathcal{C}$ is a stable quiver, we have in $\mathcal{C}$ a path

$$\tau_A^{-1} X_0 \longrightarrow \tau_A^{-1} X_1 \longrightarrow \cdots \longrightarrow \tau_A^{-1} X_{t-1} \longrightarrow \tau_A^{-1} X_t.$$

Fix $r \in \{1, \dots, t\}$. Then there exists a path in $\mathcal{C}$ from X_r to M, because there is a path in $\mathcal{C}$ from X_t to M. Similarly, there is no path in $\mathcal{C}$ from $\tau_A^{-1} X_r$ to M, because there is no path in $\mathcal{C}$ from $\tau_A^{-1} X_0$ to M. Hence X_r belongs to Δ, and so Δ is a section of $\mathcal{C}$. □

Theorem 4.16. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ an acyclic regular component of Γ_A. Then there exists an acyclic locally finite valued quiver Δ such that $\mathcal{C}$ is isomorphic to the translation quiver $\mathbb{Z}\Delta$.*

We proved in Corollary VIII.9.11 that the connecting components of tilted algebras determined by regular tilting modules over hereditary algebras of wild type provide examples of regular acyclic components with finitely many orbits with respect to the Auslander–Reiten translation. We also know that the Auslander–Reiten quivers of hereditary algebras of wild type have components of type $\mathbb{Z}\mathbb{A}_\infty$ (see Theorem VII.9.3 and Corollary VII.9.4). This is also the case for the Auslander–Reiten quivers of concealed hereditary algebras of wild type (see Theorem VIII.6.13). We refer to the interesting article [CBR] by W. Crawley-Boevey and C. M. Ringel providing constructions of finite dimensional algebras over fields whose Auslander–Reiten quivers contain large acyclic regular components of types $\mathbb{Z}\Delta$ with Δ infinite acyclic locally finite valued quivers having only finitely many vertices of valency ≥ 3.

Note also that the shapes of infinite components of the stable Auslander–Reiten quivers of selfinjective algebras were described for the blocks of group algebras of finite groups over fields (see [E1], [E2], [ES], [W]) and for the tame selfinjective algebras of polynomial growth over algebraically closed fields (see [S8]). Finally, we refer to the recent article by O. Kerner and D. Zacharia [KZ] describing the shapes of all stable Auslander–Reiten components of finite dimensional selfinjective algebras over algebraically closed fields containing modules of finite complexity.

5 Generalized standard Auslander–Reiten components

Let A be a finite dimensional K-algebra over a field K. A family $\mathcal{C}$ of indecomposable modules in $\operatorname{mod} A$ is said to be *generalized standard* if $\operatorname{rad}_A^{\infty}(X, Y) = 0$ for all modules X and Y in $\mathcal{C}$ (see Sections III.1 and VII.3 for the special case). We note the following property of generalized standard families of modules.

Lemma 5.1. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a generalized standard family of indecomposable modules in* $\operatorname{mod} A$. *Then any nonzero nonisomorphism $f : M \to N$ in* $\operatorname{mod} A$ *with M and N indecomposable modules in $\mathcal{C}$ is a finite sum of compositions of irreducible homomorphisms between indecomposable modules in* $\operatorname{mod} A$.

Proof. This follows from the proof of Proposition VII.3.9. □

Proposition 5.2. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be an acyclic generalized standard component of Γ_A^{s}. Then there exists a finite acyclic valued quiver Δ such that $\mathcal{C}$ is isomorphic to the translation quiver $\mathbb{Z}\Delta$.*

Proof. Since $\mathcal{C}$ is an acyclic component of Γ_A^{s}, it follows from Theorem 4.15 that there exists an acyclic locally finite valued quiver Δ such that $\mathcal{C}$ is isomorphic to the translation quiver $\mathbb{Z}\Delta$. Hence, it remains to show that Δ is finite. Fix a copy of Δ in $\mathcal{C}$. Take two modules X and Y in $\mathcal{C}$ lying on Δ. Then there is no path in $\mathcal{C}$ from X to $\tau_A Y$, and hence there is no path of irreducible homomorphisms

$$X = Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2 \longrightarrow \cdots \longrightarrow Z_{t-1} \xrightarrow{f_t} Z_t = \tau_A Y$$

between indecomposable modules in $\operatorname{mod} A$. Since X and $\tau_A Y$ belong to $\mathcal{C}$ and $\mathcal{C}$ is generalized standard, Lemma 5.1 shows that $\operatorname{Hom}_A(X, \tau_A Y) = 0$. Then it follows from Lemma VIII.7.4 that the number of indecomposable modules lying on Δ is less than or equal to the rank of the Grothendieck group $K_0(A)$ of A. Therefore, Δ is a finite quiver. □

The following direct consequence of Theorem 4.9 and Proposition 5.2 describes the shape of generalized standard infinite stable Auslander–Reiten components.

Theorem 5.3. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a generalized standard infinite component of Γ_A^{s}. Then $\mathcal{C}$ is either a stable tube, or a component of the form $\mathbb{Z}\Delta$ for a finite acyclic valued quiver Δ.*

The following special case, established in [S2], is of particular interest.

Theorem 5.4. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a generalized standard regular component of Γ_A. Then $\mathcal{C}$ is either a stable tube, or a component of the form $\mathbb{Z}\Delta$ for a finite acyclic valued quiver Δ.*

The next theorem, proved also in [S2], provides information on the general shape of generalized standard Auslander–Reiten components.

Theorem 5.5. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a generalized standard component of Γ_A. Then $\mathcal{C}$ admits at most finitely many nonperiodic τ_A-orbits.*

Proof. Suppose that the number of nonperiodic τ_A-orbits in $\mathcal{C}$ is infinite. Consider that stable part $\mathcal{C}^{\mathrm{s}}$ of $\mathcal{C}$. Observe that $\mathcal{C}^{\mathrm{s}}$ is obtained from $\mathcal{C}$ by removing a finite number of τ_A-orbits containing projective modules and injective modules. Hence there exists a connected component $\mathcal{D}$ of $\mathcal{C}^{\mathrm{s}}$, and hence a component of Γ_A^{s}, containing infinitely many nonperiodic τ_A-orbits. Moreover, $\mathcal{D}$ is a generalized standard subquiver of Γ_A, since $\mathcal{C}$ is a generalized standard component. Clearly, $\mathcal{D}$ is acyclic, by Theorem 4.9. This contradicts Proposition 5.2. □

Let A be a finite dimensional K-algebra over a field K. For a component $\mathcal{C}$ of Γ_A, we denote by $\operatorname{ann}_A(\mathcal{C})$ the intersection of the annihilators $\operatorname{ann}_A(X) = \{a \in A \mid Xa = 0\}$ of all indecomposable modules X in $\mathcal{C}$, and call it the *annihilator* of $\mathcal{C}$. Observe that $\operatorname{ann}_A(\mathcal{C})$ is a two-sided ideal of A. Then we may consider the quotient algebra $B(\mathcal{C}) = A/\operatorname{ann}_A(\mathcal{C})$, which we call the *faithful algebra* of $\mathcal{C}$. Observe that $\mathcal{C}$ is a component of $\Gamma_{B(\mathcal{C})}$.

Lemma 5.6. *Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a component of Γ_A. Then $\operatorname{ann}_A(\mathcal{C}) = \operatorname{ann}_A(M)$ for a direct sum $M = M_1 \oplus \cdots \oplus M_r$ of indecomposable modules $M_1, \ldots, M_r$ in $\mathcal{C}$.*

Proof. Observe that, if $X_1, \dots, X_s$ are indecomposable modules in $\mathcal{C}$, then $\operatorname{ann}_A(X_1 \oplus \cdots \oplus X_s) = \operatorname{ann}_A(X_1) \cap \cdots \cap \operatorname{ann}_A(X_s)$. Since A is a finite dimensional K-algebra, we conclude that $\operatorname{ann}_A(\mathcal{C}) = \operatorname{ann}_A(M_1) \cap \cdots \cap \operatorname{ann}_A(M_r)$ for a finite family $M_1, \dots, M_r$ of indecomposable modules in $\mathcal{C}$. Hence, $\operatorname{ann}_A(\mathcal{C}) = \operatorname{ann}_A(M)$ for $M = M_1 \oplus \cdots \oplus M_r$. □

The following theorem describes the structure of generalized standard acyclic regular Auslander–Reiten components.

Theorem 5.7. *Let A be an indecomposable finite dimensional K-algebra over a field K and $\mathcal{C}$ be a generalized standard acyclic regular component of Γ_A. Then the following statements hold:*

(i) *$B(\mathcal{C})$ is a tilted algebra of the form $\operatorname{End}_H(T)$, for a finite dimensional hereditary K-algebra H of wild type with $K_0(H)$ of rank at least three and a regular tilting module T in $\operatorname{mod} H$.*

(ii) *$\mathcal{C}$ is the connecting component $\mathcal{C}_T$ of $\Gamma_{B(\mathcal{C})}$ determined by T.*

Proof. It follows from Theorem 5.4 that $\mathcal{C}$ is a component of the form $\mathbb{Z}\Delta$ for a finite acyclic valued quiver Δ. Moreover, by Lemma 5.6, $\operatorname{ann}_A(\mathcal{C}) = \operatorname{ann}_A(U)$, where U is a direct sum of a finite number of indecomposable modules in $\mathcal{C}$. Take a copy of Δ in $\mathcal{C}$ such that every indecomposable direct summand of U is a successor in $\mathcal{C}$ of a module lying on Δ. Let M be the direct sum of all indecomposable modules lying on Δ. We claim that $\operatorname{ann}_A(\mathcal{C}) = \operatorname{ann}_A(M)$. Clearly, $\operatorname{ann}_A(\mathcal{C}) \subseteq \operatorname{ann}_A(M)$. Consider a projective cover $f: P(U) \to U$ of U in $\operatorname{mod} A$. Since $\mathcal{C}$ is a regular component, it follows from the choice of section Δ in $\mathcal{C}$ that there exist homomorphisms $g: P(U) \to M^r$ and $h: M^r \to U$ in $\operatorname{mod} A$ with $f = hg$, for some positive integer r. Then $U = h(M^r)$, and so $\operatorname{ann}_A(M) = \operatorname{ann}_A(M^r) \subseteq \operatorname{ann}_A(U) = \operatorname{ann}_A(\mathcal{C})$. Hence, indeed $\operatorname{ann}_A(\mathcal{C}) = \operatorname{ann}_A(M)$, and consequently M is a faithful module in $\operatorname{mod} B(\mathcal{C})$. Observe also that $\operatorname{Hom}_{B(\mathcal{C})}(M, \tau_{B(\mathcal{C})} M) = \operatorname{Hom}_A(M, \tau_A(M)) = 0$, because $\tau_{B(\mathcal{C})} M = \tau_A M$ and $\mathcal{C}$ is a generalized standard acyclic component with M being the direct sum of all indecomposable modules lying on the section Δ. Then, applying the criterion of Liu and Skowroński (Theorem VIII.7.7), we conclude that $B(\mathcal{C})$ is a tilted algebra of the form $\operatorname{End}_H(T)$ for an indecomposable finite dimensional hereditary K-algebra H and a tilting module T in $\operatorname{mod} H$, and $\mathcal{C}$ is the connecting component $\mathcal{C}_T$ of $\Gamma_{B(\mathcal{C})}$ determined by T. Moreover, since $\mathcal{C}_T = \mathcal{C}$ is a regular component of $\Gamma_{B(\mathcal{C})}$, it follows from Proposition VIII.6.9 that T is a regular tilting module in $\operatorname{mod} H$. Finally, applying the theorem of Ringel (Theorem VIII.9.9), we conclude that H is of wild type and $K_0(H)$ is of rank at least three. □

We would like to mention that the problem of describing of all finite dimensional K-algebras A over a field K such that the Auslander–Reiten quiver Γ_A admits a generalized standard stable tube is difficult, as the following results established in [S6], [S7] show.

Theorem 5.8. *Let B be an arbitrary finite dimensional K-algebra over a field K and M be a nonzero module in* $\operatorname{mod} B$. *Let $r_1, \ldots, r_n$ be an arbitrary sequence of positive integers. Then there exists a finite dimensional K-algebra A such that the following statements hold:*

(i) *B is a quotient algebra of A.*

(ii) *For each $i \in \{1, \ldots, n\}$, Γ_A admits a generalized standard stable tube $\mathcal{T}_i$ of rank r_i such that M is a subfactor of all but finitely many indecomposable modules in $\mathcal{T}_i$.*

(iii) *For each $i \in \{1, \ldots, n\}$, A is the faithful algebra $B(\mathcal{T}_i)$ of $\mathcal{T}_i$.*

(iv) *The stable tubes $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are pairwise orthogonal.*

(v) *The stable tubes $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are generalized standard stable tubes of the Auslander–Reiten quiver $\Gamma_{\mathrm{T}(A)}$ of the trivial extension algebra* $\mathrm{T}(A) = T \ltimes D(A)$.

For a finite dimensional K-algebra A over a field K and nonzero modules M and N in $\operatorname{mod} A$, the module M is said to be a *subfactor* of N if there exist right A-submodules U and V of N such that $U \subset V$ and V/U is isomorphic to M in $\operatorname{mod} A$.

6 Stable equivalence

Let A be a finite dimensional K-algebra over a field K. We introduced in Chapter III.4 the projectively stable category

$$\underline{\operatorname{mod}}\, A = \operatorname{mod} A/\mathcal{P}_A,$$

which we call the *stable module category* of A. Recall that the objects of $\underline{\operatorname{mod}}\, A$ are the modules in $\operatorname{mod} A$, the K-vector space of morphisms from M to N in $\underline{\operatorname{mod}}\, A$ is the quotient space

$$\underline{\operatorname{Hom}}_A(M, N) = \operatorname{Hom}_A(M, N)/\mathcal{P}_A(M, N),$$

and the composition of morphisms in $\underline{\operatorname{mod}}\, A$ is induced by the composition of homomorphisms in $\operatorname{mod} A$. For a homomorphism $f \in \operatorname{Hom}_A(M, N)$, we denote by $\underline{f} = f + \mathcal{P}_A(M, N)$ its class in $\underline{\operatorname{Hom}}_A(M, N)$. We denote by $\operatorname{mod}_{\mathcal{P}} A$ the full subcategory of $\operatorname{mod} A$ consisting of all modules without nonzero projective direct summands.

Two finite dimensional K-algebras A and B over a field K are said to be *stably equivalent* if the associated stable module categories $\underline{\operatorname{mod}}\, A$ and $\underline{\operatorname{mod}}\, B$ are equivalent, and a functor $F\colon \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ defining such an equivalence is called a *stable equivalence*. We will investigate the behaviour of irreducible homomorphisms and almost split sequences under stable equivalence.

For a finite dimensional K-algebra A over a field K and indecomposable modules X and Y in $\operatorname{mod} A$, we introduced in Chapter III.9 the finite dimensional K-vector space

$$\operatorname{irr}_A(X,Y) = \operatorname{rad}_A(X,Y)/\operatorname{rad}_A^2(X,Y),$$

called the *space of irreducible homomorphisms* from X to Y. Recall that $\operatorname{irr}_A(X,Y)$ is an F_Y-F_X-bimodule, where $F_X = \operatorname{End}_A(X)/\operatorname{rad}_A(X,X)$ and $F_Y = \operatorname{End}_A(Y)/\operatorname{rad}_A(Y,Y)$ are the associated finite dimensional division K-algebras. Moreover, $\dim_{F_Y} \operatorname{irr}_A(X,Y)$ is the multiplicity d_{XY} of Y in the codomain M of a left minimal almost split homomorphism $X \to M$ in $\operatorname{mod} A$, and $\dim_{F_X} \operatorname{irr}_A(X,Y)$ is the multiplicity d'_{XY} of X in the domain N of a right minimal almost split homomorphism $N \to Y$ in $\operatorname{mod} A$.

Lemma 6.1. *Let* $F\colon \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ *be a stable equivalence between finite dimensional K-algebras A and B over a field K, and M be a nonzero module in* $\operatorname{mod}_{\mathcal{P}} A$. *Then F induces an isomorphism of K-algebras*

$$\operatorname{End}_A(M)/\operatorname{rad}\operatorname{End}_A(M) \xrightarrow{\sim} \operatorname{End}_B(F(M))/\operatorname{rad}\operatorname{End}_B(F(M)).$$

In particular, M is indecomposable in $\operatorname{mod} A$ *if and only if $F(M)$ is indecomposable in* $\operatorname{mod} B$.

Proof. Since M is in $\operatorname{mod}_{\mathcal{P}} A$, for any projective module P in $\operatorname{mod} A$ we have $\operatorname{Hom}_A(P,M) = \operatorname{rad}_A(P,M)$, by Lemmas III.1.3 and III.1.4. Hence, we conclude that $\mathcal{P}_A(M,M) \subseteq \operatorname{rad}_A(M,M) = \operatorname{rad}\operatorname{End}_A(M)$. Moreover, F induces an isomorphism of K-algebras $\underline{\operatorname{End}}_A(M) \xrightarrow{\sim} \underline{\operatorname{End}}_B(F(M))$. Then we obtain that F induces an isomorphism of K-algebras $\operatorname{End}_A(M)/\operatorname{rad}\operatorname{End}_A(M) \xrightarrow{\sim} \operatorname{End}_B(F(M))/\operatorname{rad}\operatorname{End}_B(F(M))$. Therefore, by Lemma I.3.8, $\operatorname{End}_A(M)$ is a local K-algebra if and only if $\operatorname{End}_B(F(M))$ is a local K-algebra. Applying Lemma I.4.4, we infer that M is indecomposable in $\operatorname{mod} A$ if and only if $F(M)$ is indecomposable in $\operatorname{mod} B$. □

Lemma 6.2. *Let* $F\colon \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ *be a stable equivalence between finite dimensional K-algebras A and B over a field K, and X, Y modules in* $\operatorname{mod}_{\mathcal{P}} A$. *Then the following statements hold:*

(i) *Let $f\colon X \to Y$ be a homomorphism in* $\operatorname{mod} A$ *and $f'\colon F(X) \to F(Y)$ a homomorphism in* $\operatorname{mod} B$ *such that* $F(\underline{f}) = \underline{f'}$. *Then* $f \in \operatorname{rad}_A(X,Y)$ *if and only if* $f' \in \operatorname{rad}_B(F(X),F(Y))$.

(ii) $F(\operatorname{rad}_A^n(X,Y) + \mathcal{P}_A(X,Y)) = \operatorname{rad}_B^n(F(X),F(Y)) + \mathcal{P}_B(F(X),F(Y))$ *for any* $n \geq 1$.

(iii) *If X and Y are indecomposable modules, then F induces an isomorphism of K-vector spaces* $\operatorname{irr}_A(X,Y) \xrightarrow{\sim} \operatorname{irr}_B(F(X),F(Y))$.

Proof. Assume that $f \notin \mathrm{rad}_A(X,Y)$. Then it follows from Lemmas III.1.3, III.1.4, and III.1.5 that there exist an indecomposable direct summand Z of X and homomorphisms $u\colon Z \to X$ and $v\colon Y \to Z$ in $\mathrm{mod}\,A$ such that $\mathrm{id}_Z = vfu$. Clearly, Z belongs to $\mathrm{mod}_{\mathcal{P}}\,A$. Then $\underline{\mathrm{id}}_{F(Z)} = F(\underline{\mathrm{id}}_Z) = F(\underline{vfu}) = F(\underline{v})F(\underline{f})F(\underline{u})$, and hence $\mathrm{id}_{F(Z)} = v'f'u' + p$ for some homomorphisms $p \in \mathcal{P}_B(F(X),F(Y))$, $u' \in \mathrm{Hom}_B(F(Z),F(X))$, $v' \in \mathrm{Hom}_B(F(Y),F(Z))$, with $F(\underline{u}) = \underline{u'}$ and $F(\underline{v}) = \underline{v'}$. Since $\mathcal{P}_B(F(X),F(Y)) \subseteq \mathrm{rad}_B(F(X),F(Y))$, we conclude that $f' \notin \mathrm{rad}_B(F(X),F(Y))$. Applying the inverse functor $G\colon \underline{\mathrm{mod}}\,B \to \underline{\mathrm{mod}}\,A$ of F, we conclude that, if $f' \notin \mathrm{rad}_B(F(X),F(Y))$, then $f \notin \mathrm{rad}_A(X,Y)$.

(ii) This follows from (i) and the definitions of rad_A^n and rad_B^n.

(iii) Assume that X and Y are indecomposable modules. Observe that $\mathcal{P}_A(X,Y) \subseteq \mathrm{rad}_A^2(X,Y)$ and $\mathcal{P}_B(F(X),F(Y)) \subseteq \mathrm{rad}_B^2(F(X),F(Y))$. Moreover, the stable equivalence functor F induces an isomorphism of K-vector spaces $\underline{\mathrm{Hom}}_A(X,Y) \xrightarrow{\sim} \underline{\mathrm{Hom}}_B(F(X),F(Y))$. Then (i) and (ii) imply that F induces an isomorphism of K-vector spaces $\mathrm{irr}_A(X,Y) \xrightarrow{\sim} \mathrm{irr}_B(F(X),F(Y))$. □

Lemma 6.3. *Let* $F\colon \underline{\mathrm{mod}}\,A \to \underline{\mathrm{mod}}\,B$ *be a stable equivalence between finite dimensional* K*-algebras* A *and* B *over a field* K. *Let* $f\colon X \to Y$ *be a homomorphism in* $\mathrm{mod}_{\mathcal{P}}\,A$ *with* X *and* Y *indecomposable, and* $f'\colon F(X) \to F(Y)$ *a homomorphism in* $\mathrm{mod}\,B$ *such that* $F(\underline{f}) = \underline{f'}$. *Then* f *is irreducible in* $\mathrm{mod}\,A$ *if and only if* f' *is irreducible in* $\mathrm{mod}\,B$.

Proof. This follows from Lemma 6.2 and Lemmas III.1.3, III.1.4, and III.7.8. □

Proposition 6.4. *Let* $F\colon \underline{\mathrm{mod}}\,A \to \underline{\mathrm{mod}}\,B$ *be a stable equivalence between finite dimensional* K*-algebras* A *and* B *over a field* K, $f\colon X \to Y$ *a homomorphism in* $\mathrm{mod}_{\mathcal{P}}\,A$ *with* X *indecomposable, and* $f'\colon F(X) \to F(Y)$ *a homomorphism in* $\mathrm{mod}\,B$ *such that* $F(\underline{f}) = \underline{f'}$. *Then the following statements are equivalent:*

(i) *There is a homomorphism* $g\colon X \to P$ *in* $\mathrm{mod}\,A$ *with* P *projective such that*

$$\begin{bmatrix} f \\ g \end{bmatrix} : X \longrightarrow Y \oplus P$$

is a left minimal almost split homomorphism in $\mathrm{mod}\,A$.

(ii) *There is a homomorphism* $h\colon F(X) \to Q$ *in* $\mathrm{mod}\,B$ *with* Q *projective such that*

$$\begin{bmatrix} f' \\ h \end{bmatrix} : F(X) \longrightarrow F(Y) \oplus Q$$

is a left minimal almost split homomorphism in $\mathrm{mod}\,B$.

Proof. We will prove only that (i) implies (ii), because the proof of the converse implication is similar. Assume that there is a homomorphism $g\colon X \to P$ in $\mathrm{mod}\,A$

with P projective such that

$$\begin{bmatrix} f \\ g \end{bmatrix} : X \longrightarrow Y \oplus P$$

is a left minimal almost split homomorphism in $\operatorname{mod} A$. It follows from Lemmas 6.1 and 6.3 that $F(X)$ is an indecomposable module in $\operatorname{mod} B$ and $f'\colon F(X) \to F(Y)$ an irreducible homomorphism in $\operatorname{mod} B$. Applying Theorem III.7.11, we conclude that there exist a module N in $\operatorname{mod} B$ and a homomorphism $h\colon F(X) \to M$ such that

$$\begin{bmatrix} f' \\ h \end{bmatrix} : F(X) \longrightarrow F(Y) \oplus M$$

is a left minimal almost split homomorphism in $\operatorname{mod} B$. Let $M = N \oplus Q$ be a decomposition of M in $\operatorname{mod} B$ with N in $\operatorname{mod}_{\mathcal{P}} B$ and Q projective. We show that $N = 0$. Take an indecomposable direct summand Z of Y in $\operatorname{mod} A$. Then $Y = Z^{d_{XZ}} \oplus Y'$ in $\operatorname{mod} A$, with Y' without a direct summand isomorphic to Z. Since Y belongs to $\operatorname{mod}_{\mathcal{P}} A$, we have $F(Y) = F(Z)^{d_{XZ}} \oplus F(Y')$ in $\operatorname{mod} B$. Further, by Lemmas 6.1 and 6.2, the functor F induces an isomorphism of division K-algebras $F_Z = \operatorname{End}_A(Z)/\operatorname{rad}\operatorname{End}_A(Z) \xrightarrow{\sim} \operatorname{End}_B(F(Z))/\operatorname{rad}\operatorname{End}_B(F(Z)) = F_{F(Z)}$ and an isomorphism of K-vector spaces $\operatorname{irr}_A(X, Z) \xrightarrow{\sim} \operatorname{irr}_B(F(X), F(Z))$. Hence we obtain that

$$d_{XZ} = \dim_{F_Z} \operatorname{irr}_A(X, Z) = \dim_{F_{F(Z)}} \operatorname{irr}_B(F(X), F(Z)) = d_{F(X)F(Z)}.$$

This implies that the indecomposable right B-module $F(Z)$ is not a direct summand of M, and hence is not a direct summand of N, because $F(Z)$ belongs to $\operatorname{mod}_{\mathcal{P}} B$. Suppose that $N \neq 0$, and let V be an indecomposable direct summand of N. Then there is an irreducible homomorphism $v\colon F(X) \to V$ in $\operatorname{mod} B$, again by Theorem III.7.11. Since V belongs to $\operatorname{mod}_{\mathcal{P}} B$, there exists a homomorphism $u\colon X \to U$ in $\operatorname{mod} A$ such that $F(U) = V$ and $F(\underline{u}) = \underline{v}$. Moreover, by Lemmas 6.1 and 6.3, U is an indecomposable module in $\operatorname{mod}_{\mathcal{P}} A$ and u is an irreducible homomorphism. But then U is a direct summand of Y, and hence $V = F(U)$ is a direct summand of $F(Y)$, a contradiction. Therefore, $N = 0$, as claimed. □

Proposition 6.5. *Let $F\colon \underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} B$ be a stable equivalence between finite dimensional K-algebras A and B over a field K, $f\colon X \to Y$ a homomorphism in* $\operatorname{mod}_{\mathcal{P}} A$ *with Y indecomposable, and $f'\colon F(X) \to F(Y)$ a homomorphism in* $\operatorname{mod} B$ *such that $F(\underline{f}) = \underline{f'}$. Then the following statements are equivalent:*

(i) *There is a homomorphism $g\colon P \to Y$ in* $\operatorname{mod} A$ *with P projective such that*

$$\begin{bmatrix} f & g \end{bmatrix} : X \oplus P \longrightarrow Y$$

is a right minimal almost split homomorphism in $\operatorname{mod} A$.

(ii) *There is a homomorphism* $h\colon Q \to F(Y)$ *in* $\operatorname{mod} B$ *with* Q *projective such that*
$$\begin{bmatrix} f' & h \end{bmatrix} : F(X) \oplus Q \longrightarrow F(Y)$$
is a right minimal almost split homomorphism in $\operatorname{mod} B$.

Proof. Similar to the proof of Proposition 6.4. □

Lemma 6.6. *Let* A *be a finite dimensional* K*-algebra over a field* K, *and* $f\colon L \to M$ *and* $g\colon M \to N$ *irreducible homomorphisms in* $\operatorname{mod}_{\mathcal{P}} A$, *with* L *and* N *indecomposable and* $\underline{gf} = 0$. *Then the following statements hold:*

(i) $L \cong \tau_A N$ *in* $\operatorname{mod} A$.

(ii) *There are a projective module* Q *and irreducible homomorphisms* $u\colon L \to Q$ *and* $v\colon Q \to N$ *in* $\operatorname{mod} A$ *and a homomorphism* $w \in \mathcal{P}_A(L, M)$ *such that*
$$0 \longrightarrow L \xrightarrow{\begin{bmatrix} f+w \\ u \end{bmatrix}} M \oplus Q \xrightarrow{\begin{bmatrix} g & v \end{bmatrix}} N \longrightarrow 0,$$
is an almost split sequence in $\operatorname{mod} A$.

Proof. Since $g\colon M \to N$ is irreducible and N is indecomposable nonprojective, it follows from Theorems III.7.12 and III.8.4 that there is an almost split sequence in $\operatorname{mod} A$ of the form
$$0 \longrightarrow \tau_A N \xrightarrow{\begin{bmatrix} i \\ j \end{bmatrix}} M \oplus Q \xrightarrow{\begin{bmatrix} g & v \end{bmatrix}} N \longrightarrow 0.$$
By the assumption $\underline{gf} = 0$, there exist a projective module P and homomorphisms $f'\colon L \to P$ and $g'\colon P \to N$ in $\operatorname{mod} A$ such that $gf = g'f'$. Moreover, since $[g \;\; v]$ is an epimorphism, there exists a homomorphism $\begin{bmatrix} h \\ h' \end{bmatrix}\colon P \to M \oplus Q$ in $\operatorname{mod} A$ such that $g' = [g \;\; v]\begin{bmatrix} h \\ h' \end{bmatrix}$. Then $gf = [g \;\; v]\begin{bmatrix} h \\ h' \end{bmatrix} f'$. Consider the homomorphisms $w = -hf'\colon L \to M$ and $u = -h'f'\colon L \to Q$. Then we have the equalities
$$[g \;\; v]\begin{bmatrix} f+w \\ u \end{bmatrix} = gf + gw + vu = gf - ghf' - vh'f' = 0.$$
Hence there exists a homomorphism $s\colon L \to \tau_A N$ such that $\begin{bmatrix} f+w \\ u \end{bmatrix} = \begin{bmatrix} i \\ j \end{bmatrix} s$. Observe that the homomorphism $f + w\colon L \to M$ is irreducible, because $f\colon L \to M$ is irreducible and $w \in \mathcal{P}_A(L, M) \subseteq \operatorname{rad}_A^2(L, M)$. Since $f + w = is$, we obtain that s is a section and consequently an isomorphism. Therefore, the sequence
$$0 \longrightarrow L \xrightarrow{\begin{bmatrix} f+w \\ u \end{bmatrix}} M \oplus Q \xrightarrow{\begin{bmatrix} g & v \end{bmatrix}} N \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod} A$ and $L \cong \tau_A N$. Further, $u = -h' f' \colon L \to Q$ is an irreducible homomorphism and $f' \colon L \to P$ is not a section, because L is not projective. Hence $-h' \colon P \to Q$ is a retraction, and so Q is projective. $\square$

Proposition 6.7. *Let $F \colon \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ be a stable equivalence between finite dimensional K-algebras A and B over a field K. Let*

$$0 \longrightarrow L \xrightarrow{\begin{bmatrix} f \\ r \end{bmatrix}} M \oplus P \xrightarrow{[g \;\; s]} N \longrightarrow 0$$

be an almost split sequence in $\operatorname{mod} A$*, where L, M, N are modules in* $\operatorname{mod}_{\mathcal{P}} A$*, M is nonzero, and P is projective. Then for any homomorphism $g' \colon F(M) \to F(N)$ in* $\operatorname{mod} B$ *with $F(\underline{g}) = \underline{g'}$ there exists an almost split sequence in* $\operatorname{mod} B$ *of the form*

$$0 \longrightarrow F(L) \xrightarrow{\begin{bmatrix} f' \\ u \end{bmatrix}} F(M) \oplus Q \xrightarrow{[g' \;\; v]} F(N) \longrightarrow 0,$$

where Q is projective and $F(\underline{f}) = \underline{f'}$.

Proof. Let $g' \colon F(M) \to F(N)$ and $f'' \colon F(L) \to F(M)$ be homomorphisms in $\operatorname{mod} B$ such that $F(\underline{g}) = \underline{g'}$ and $F(\underline{f}) = \underline{f''}$. It follows from Lemma 6.3 that g' and f'' are irreducible homomorphisms in $\operatorname{mod} B$. Moreover, we have $gf = -sr \in \mathcal{P}_A(L, N)$, and hence $\underline{gf} = 0$. But then $\underline{g' f''} = \underline{g'}\,\underline{f''} = F(\underline{g}) F(\underline{f}) = F(\underline{gf}) = 0$. Since $F(L)$ nd $F(N)$ are indecomposable modules in $\operatorname{mod}_{\mathcal{P}} B$, applying Lemma 6.6, we conclude that there exists an almost split sequence in $\operatorname{mod} B$ of the form

$$0 \longrightarrow F(L) \xrightarrow{\begin{bmatrix} f'' + w \\ u \end{bmatrix}} F(M) \oplus Q \xrightarrow{[g' \;\; v]} F(N) \longrightarrow 0,$$

where Q is projective and $w \in \mathcal{P}_B(F(L), F(M))$. We take $f' = f'' + w$. Observe that $\underline{f'} = \underline{f''}$, and hence $F(\underline{f}) = \underline{f'}$. $\square$

Let A be a finite dimensional selfinjective K-algebra over a field K. Then the injective modules in $\operatorname{mod} A$ are projective and hence the stable category $\underline{\operatorname{mod}}\, A$ of A coincides with the injectively stable category $\overline{\operatorname{mod}}\, A = \operatorname{mod} A / \mathcal{J}_A$, where $\mathcal{J}_A$ is the ideal of $\operatorname{mod} A$ consisting of all homomorphisms which factor through an injective module. In particular, the Auslander–Reiten translation $\tau_A = D \operatorname{Tr}$ and $\tau_A^{-1} = \operatorname{Tr} D$ induce the mutually inverse equivalences of categories

$$\begin{tikzcd} \underline{\operatorname{mod}}\, A \arrow[r, shift left, "\tau_A"] & \underline{\operatorname{mod}}\, A \arrow[l, shift left, "\tau_A^{-1}"] \end{tikzcd}$$

(see Corollary III.4.8). Further, it follows from Theorem IV.8.4 that the syzygy operators Ω_A and Ω_A^{-1} in mod A induce the mutually inverse equivalences of categories

$$\begin{tikzcd} \underline{\operatorname{mod}}\, A \arrow[r, shift left, "\Omega_A"] & \underline{\operatorname{mod}}\, A \arrow[l, shift left, "\Omega_A^{-1}"] \end{tikzcd} .$$

We also note that the Nakayama functors $\mathcal{N}_A = D\operatorname{Hom}_A(-, A)$ and $\mathcal{N}_A^{-1} = \operatorname{Hom}_A(-, A)D$ induce the mutually inverse equivalences of categories

$$\begin{tikzcd} \underline{\operatorname{mod}}\, A \arrow[r, shift left, "\mathcal{N}_A"] & \underline{\operatorname{mod}}\, A \arrow[l, shift left, "\mathcal{N}_A^{-1}"] \end{tikzcd} ,$$

by the Morita–Azumaya Theorem II.7.11. Finally, it follows from Theorem IV.8.5 that the functors

$$\tau_A, \Omega_A^2\mathcal{N}_A, \mathcal{N}_A\Omega_A^2 \colon \underline{\operatorname{mod}}\, A \longrightarrow \underline{\operatorname{mod}}\, A$$

and the functors

$$\tau_A^{-1}, \Omega_A^{-2}\mathcal{N}_A^{-1}, \mathcal{N}_A^{-1}\Omega_A^{-2} \colon \underline{\operatorname{mod}}\, A \longrightarrow \underline{\operatorname{mod}}\, A$$

are naturally isomorphic. The stable Auslander–Reiten quiver Γ_A^s of A is obtained from the Auslander–Reiten quiver Γ_A of A by removing the projective-injective vertices and the arrows attached to them. Moreover, we have the following description of almost split sequences in mod A whose middle term admits a nonzero projective direct summand.

Proposition 6.8. *Let A be a finite dimensional selfinjective K-algebra over a field K and*

$$\mathbb{E}: \quad 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an almost split sequence in mod A *such that M admits a nonzero projective direct summand. Then $\mathbb{E}$ is isomorphic to an almost split sequence in* mod A *of the form*

$$0 \longrightarrow \operatorname{rad} P \xrightarrow{\begin{bmatrix} q \\ u \end{bmatrix}} (\operatorname{rad} P/\operatorname{soc} P) \oplus P \xrightarrow{[-j \ \ v]} P/\operatorname{soc} P \longrightarrow 0,$$

where P is an indecomposable projective module of length at least two, u and j are the inclusion homomorphisms, and q and v are the canonical epimorphisms.

Proof. Let P be an indecomposable projective direct summand of M. Then the restriction $g' \colon P \to N$ of g to P is an irreducible homomorphism in mod A. Since P is an indecomposable injective module in mod A, we conclude that P

is of length at least two. Moreover, it follows from Lemma III.7.7 that there is a commutative diagram in mod A

$$\begin{tikzcd} & N \\ P \arrow[ur, "g'"] \arrow[dr, "v"'] & \\ & P/\operatorname{soc} P \arrow[uu, "w"'] \end{tikzcd}$$

where v is the canonical projection and w is an isomorphism. Similarly, the composition $f'\colon L \to P$ of f with the projection of M on P is an irreducible homomorphisms in mod A. Hence it follows from Lemma III.7.6 that there is a commutative diagram in mod A

$$\begin{tikzcd} L \arrow[dd, "r"'] \arrow[dr, "f'"] & \\ & P \\ \operatorname{rad} P \arrow[ur, "u"'] & \end{tikzcd}$$

where u is the inclusion homomorphism and r is an isomorphism. We also note that M has only one indecomposable projective direct summand. Indeed, if Q is an indecomposable projective module in mod A such that $P \oplus Q$ is a direct summand of M, then $P/\operatorname{soc} P \cong N \cong Q/\operatorname{soc} Q$, and so $P \cong Q$, by Corollary I.8.5. Then $\ell(L) + \ell(N) < \ell(P) + \ell(Q) \leq \ell(M)$, a contradiction. Therefore, P is a unique indecomposable projective direct summand of M, and the required claim follows from Proposition III.8.6. □

The syzygy operators Ω_A and Ω_A^{-1} in the module category mod A of a finite dimensional selfinjective K-algebra over a field K allow to construct new indecomposable nonprojective modules from a given indecomposable nonprojective module in mod A (see Proposition IV.8.3). The next proposition shows that Ω_A and Ω_A^{-1} allow also to construct new almost split sequences in mod A from a given almost split sequence.

Proposition 6.9. *Let A be a finite dimensional selfinjective K-algebra over a field K and*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an almost split sequence in mod A *with M nonprojective. Then the following statements hold:*

(i) *There exists in* mod A *an almost split sequence*

$$0 \longrightarrow \Omega_A(L) \xrightarrow{\begin{bmatrix} f' \\ u \end{bmatrix}} \Omega_A(M) \oplus Q \xrightarrow{[g' \;\; v]} \Omega_A(N) \longrightarrow 0,$$

where $\underline{f'} = \Omega_A(\underline{f})$, $\underline{g'} = \Omega_A(\underline{g})$, and Q is projective.

(ii) *There exists in* mod A *an almost split sequence*

$$0 \longrightarrow \Omega_A^{-1}(L) \xrightarrow{\begin{bmatrix} f'' \\ u' \end{bmatrix}} \Omega_A^{-1}(M) \oplus Q' \xrightarrow{[g'' \;\; v']} \Omega_A^{-1}(N) \longrightarrow 0,$$

where $\underline{f''} = \Omega_A^{-1}(\underline{f})$, $\underline{g''} = \Omega_A^{-1}(\underline{g})$, *and* Q' *is projective.*

Proof. We note that $\Omega_A\colon \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ and $\Omega_A^{-1}\colon \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ are stable equivalences. Moreover, by the assumption, we have $M = M' \oplus P$, where M' is a nonzero module in $\mathrm{mod}_{\mathcal{P}}\, A$ and P is projective. Then the statements (i) and (ii) follow from Proposition 6.7. □

The following proposition shows that the finite dimensional selfinjective algebras which admit an almost split sequence with projective middle term are very special.

Proposition 6.10. *Let A be a nonsimple indecomposable finite dimensional selfinjective K-algebra over a field K. The following statements are equivalent:*

(i) *There is an almost split sequence in* mod A *with projective middle term.*

(ii) *All almost split sequences in* mod A *have projective middle terms.*

(iii) *A is a Nakayama algebra of Loewy length* 2.

(iv) *A is of Loewy length* 2.

Proof. Assume that A is of Loewy length 2. Since A is nonsimple indecomposable and selfinjective, every indecomposable projective module P in mod A has rad P = soc P simple, and hence A is a Nakayama algebra of Loewy length 2. Moreover, every indecomposable module in mod A is either simple or projective. Then it follows from Theorem III.8.7 that every almost split sequence in mod A is isomorphic to an almost split sequence of the form

$$0 \longrightarrow \mathrm{rad}\, P \xrightarrow{u} P \xrightarrow{v} P/\,\mathrm{rad}\, P \longrightarrow 0,$$

where P is an indecomposable projective module, rad P = soc P, u is the inclusion homomorphism, and v is the canonical epimorphism. Therefore, the implications (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) hold. Hence, it remains to show that (i) implies (iv), or equivalently (iii). Assume that

$$0 \longrightarrow L \xrightarrow{f} P \xrightarrow{g} N \longrightarrow 0$$

is an almost split sequence in mod A with P projective. Then it follows from Proposition 6.8 that P is indecomposable, $\mathrm{rad}\, P/\,\mathrm{soc}\, P = 0$, $L \cong \mathrm{rad}\, P$, and $N \cong P/\,\mathrm{soc}\, P$. In particular, we conclude that P is uniserial of length 2, $N \cong$

top(P), g is a projective cover of N in mod A, and hence $L \cong \Omega_A(N)$. Consider now an almost split sequence in mod A

$$0 \longrightarrow V \longrightarrow M \longrightarrow L \longrightarrow 0$$

with the right term L. We claim that M is a projective module. Suppose that M is nonprojective. Then, applying Proposition 6.9(ii), we obtain that there is in mod A an almost split sequence of the form

$$0 \longrightarrow \Omega_A^{-1}(V) \longrightarrow \Omega_A^{-1}(M) \oplus Q' \longrightarrow \Omega_A^{-1}(L) \longrightarrow 0$$

with $\Omega_A^{-1}(M)$ nonzero and nonprojective. On the other hand, we have $\Omega_A^{-1}(L) \cong \Omega_A^{-1}(\Omega_A(N)) \cong N$, by Proposition IV.8.3. Then it follows from Lemma III.8.2 that $\Omega_A^{-1}(M) \oplus Q'$ is isomorphic to P, a contradiction. Hence, M is indeed projective. But then M is an indecomposable projective module, and we conclude as above that M is uniserial of length 2, $V \cong \operatorname{rad} M = \operatorname{soc} M$, $L \cong \operatorname{top}(M)$, and $V \cong \Omega_A(L)$. Continuing the procedure we obtain an algebra summand of A where all indecomposable projective modules have length 2. Since A is indecomposable, we obtain that all indecomposable projective modules in mod A have length 2, and consequently A is a Nakayama algebra of length 2. ☐

Proposition 6.11. *Let $F\colon \underline{\text{mod}}\, A \to \underline{\text{mod}}\, B$ be a stable equivalence between indecomposable finite dimensional selfinjective K-algebras A and B over a field K, and assume that A is of Loewy length at least* 3. *Then the following statements hold:*

(i) *B is of Loewy length at least* 3.

(ii) *For every indecomposable nonprojective module M in* mod A, *there is an isomorphism $F(\tau_A M) \cong \tau_B F(M)$ in* mod B.

Proof. Consider an almost split sequence in mod A

$$0 \longrightarrow \tau_A M \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

with the right term M. Since A is indecomposable of Loewy length at least 3, it follows from Proposition 6.10 that E is not projective. Then Proposition 6.7 shows that there is in mod B an almost split sequence of the form

$$0 \longrightarrow F(\tau_A M) \longrightarrow F(E) \oplus Q \longrightarrow F(M) \longrightarrow 0,$$

for some projective module Q. Hence, there is an isomorphism $F(\tau_A M) \cong \tau_B F(M)$ in mod B. Moreover, $F(E)$ is a nonzero nonprojective module in mod B. Clearly, B is not simple, because the category $\underline{\text{mod}}\, B$ is nontrivial. Now, applying Proposition 6.10 again, we conclude that B is of Loewy length at least 3. ☐

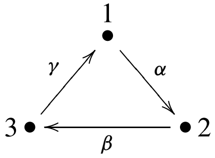
Corollary 6.12. *Let $F\colon \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, B$ be a stable equivalence between indecomposable finite dimensional selfinjective K-algebras A and B of Loewy length at least 3 over a field K. Then F induces an isomorphism $\Gamma^{s}_{A} \to \Gamma^{s}_{B}$ of the stable Auslander–Reiten quivers of A and B.*

Proof. This follows from Lemma 6.2 and Propositions 6.7 and 6.11. □

Corollary 6.13. *Let A be an indecomposable finite dimensional selfinjective K-algebra of Loewy length at least 3 over a field K. Then the syzygy operators Ω_A and Ω_A^{-1} induce the mutually inverse isomorphisms of the stable Auslander–Reiten quivers*

$$\begin{tikzcd}\Gamma^{s}_{A} \arrow[r, shift left, "\Omega_A"] & \Gamma^{s}_{A} \arrow[l, shift left, "\Omega_A^{-1}"] \end{tikzcd}.$$

Example 6.14. Let K be a field and $A = KQ/I$ be the bound quiver algebra given by the quiver Q of the form



and the ideal I in the path algebra KQ of Q over K generated by $\alpha\beta$, $\beta\gamma$ and $\gamma\alpha$. Then A is a selfinjective Nakayama algebra of Loewy length 2 and there are only six pairwise nonisomorphic indecomposable modules in mod A: the indecomposable projective-injective modules $P_1 = e_1A$, $P_2 = e_2A$, $P_3 = e_3A$, and the simple modules $S_1 = e_1A/e_1 \operatorname{rad} A$, $S_2 = e_2A/e_2 \operatorname{rad} A$, $S_3 = e_3A/e_3 \operatorname{rad} A$, given by the primitive idempotents e_1, e_2, e_3 in A associated to the vertices 1, 2, 3 of Q. Then $S_1 = \operatorname{soc} P_3$, $S_2 = \operatorname{soc} P_1$, $S_3 = \operatorname{soc} P_2$, and the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
 & P_1 \arrow[dr] & \\
S_2 \arrow[ur] & & S_1 \arrow[d] \\
P_2 \arrow[u] & & P_3 \arrow[dl] \\
 & S_3 \arrow[ul] &
\end{tikzcd}$$

with $\tau_A S_1 = S_2$, $\tau_A S_2 = S_3$, and $\tau_A S_3 = S_1$. We also note that $\Omega_A(S_1) = S_2$, $\Omega_A(S_2) = S_3$, and $\Omega_A(S_3) = S_1$. The stable Auslander–Reiten quiver Γ^{s}_{A}

consists of the three isolated vertices

$$\begin{tikzcd} S_2 & & S_1 \\ & S_3 & \end{tikzcd}$$

with $\tau_A S_1 = S_2$, $\tau_A S_2 = S_3$, and $\tau_A S_3 = S_1$. We may define the stable equivalence $F\colon \underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} A$ by $F(S_1) = S_1$, $F(S_2) = S_3$, and $F(S_3) = S_2$. Then we obtain that $F(\tau_A S_i) \not\cong \tau_A F(S_i)$ and $F(\Omega_A S_i) \not\cong \Omega_A F(S_i)$ for any $i \in \{1, 2, 3\}$.

The aim of the final part of this section is to show that the stable equivalences between indecomposable finite dimensional selfinjective algebras of Loewy length at least 3 commute with the syzygy functors.

We observe first a characterization of almost split sequences.

Lemma 6.15. *Let A be a finite dimensional selfinjective K-algebra over a field K and*

$$\mathbb{E}: \quad 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a nonsplittsable sequence in $\operatorname{mod} A$ *with L and N indecomposable modules. The following statements are equivalent:*

(i) $\mathbb{E}$ *is an almost split sequence in* $\operatorname{mod} A$.

(ii) *For any homomorphism $h\colon X \to N$ in* $\operatorname{mod} A$ *with X indecomposable and h nonisomorphism, the induced exact sequence $\mathbb{E}h$ is splittable.*

Proof. By Theorem III.8.3, the statement (i) is equivalent to the fact that g is a right almost split homomorphism in $\operatorname{mod} A$. Let $h\colon X \to N$ be a homomorphism in $\operatorname{mod} A$ with X indecomposable and h nonisomorphism. Then the induced exact sequence $\mathbb{E}h$ is the upper sequence in the commutative diagram in $\operatorname{mod} A$

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "\mathrm{id}_L"'] & M' \arrow[r, "h'"] \arrow[d, "g'"] & X \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0,
\end{tikzcd}$$

given by the fiber product $M' = M \times_N X$ of M and X over N, via g and h. Assume that $\mathbb{E}h$ is splittable. Then h' is a retraction, and hence there is a homomorphism $h''\colon X \to M'$ in $\operatorname{mod} A$ such that $h'h'' = \mathrm{id}_X$ (see Lemma III.3.1). Then, for $v = g'h''$, we have $gv = gg'h'' = hh'h'' = h$. Therefore, (ii) implies that g is a right almost split sequence in $\operatorname{mod} A$. Conversely, assume that g is a right almost split homomorphism in $\operatorname{mod} A$. Then, for any homomorphism

$h\colon X \to N$ with X indecomposable and h nonisomorphism, there exists a homomorphism $v\colon X \to M$ such that $gv = h = h\,\mathrm{id}_X$. Then, by the fiber product property of the middle term M' of the short exact sequence $\mathbb{E}h$, there exists a homomorphism $h''\colon X \to M'$ such that $v = g'h''$ and $h'h'' = \mathrm{id}_X$. Hence, h' is a retraction, and consequently the induced exact sequence $\mathbb{E}h$ is splittable, as required. □

We need also the following lemma.

Lemma 6.16. *Let A be a finite dimensional selfinjective K-algebra over a field K and L, N indecomposable nonprojective modules in* $\operatorname{mod} A$. *The following statements are equivalent:*

(i) *N is isomorphic to $\tau_A^{-1}\Omega_A(L)$ in* $\operatorname{mod} A$.

(ii) *There is a nonzero homomorphism $f\colon N \to L$ in* $\operatorname{mod} A$ *with* $\underline{f} \neq 0$ *such that, if X is an indecomposable module and $h\colon X \to N$ is a homomorphism, but not an isomorphism in* $\operatorname{mod} A$, *then* $\underline{fh} = 0$.

Proof. Let X be an indecomposable module in $\operatorname{mod} A$ and $h\colon X \to N$ a homomorphism in $\operatorname{mod} A$ such that h is not an isomorphism, or equivalently, $\underline{h}$ is not an isomorphism in $\underline{\operatorname{mod}}\, A$, because N is nonprojective (see also Lemma III.4.3). Applying Proposition III.3.7 and Theorem IV.9.9, we conclude that there exists a commutative diagram in $\operatorname{mod} K$ of the form

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_A(N,L) \arrow[r, "\delta_N"] \arrow[d, "\underline{\mathrm{Hom}}_A(h,L)"] & \mathrm{Ext}_A^1\big(N,\Omega_A(L)\big) \arrow[r, "\eta_N"] \arrow[d, "\mathrm{Ext}_A^1(h,\Omega_A(L))"] & \mathcal{E}\mathrm{xt}_A^1\big(N,\Omega_A(L)\big) \arrow[d, "\mathcal{E}\mathrm{xt}_A^1(h,\Omega_A(L))"] \\
\underline{\mathrm{Hom}}_A(X,L) \arrow[r, "\delta_X"] & \mathrm{Ext}_A^1\big(X,\Omega_A(L)\big) \arrow[r, "\eta_X"] & \mathcal{E}\mathrm{xt}_A^1\big(X,\Omega_A(L)\big),
\end{tikzcd}$$

where the horizontal homomorphisms are isomorphisms.

Assume that $N \cong \tau_A^{-1}\Omega_A(L)$ in $\operatorname{mod} A$. Then there is in $\operatorname{mod} A$ an almost split sequence

$$\mathbb{E}\colon\quad 0 \longrightarrow \Omega_A(L) \longrightarrow E \longrightarrow N \longrightarrow 0\,.$$

It now follows from Lemma 6.15 that the induced exact sequence $\mathbb{E}h$ is splittable, and consequently we obtain $\mathcal{E}\mathrm{xt}_A^1(h,\Omega_A(L))([\mathbb{E}]) = [\mathbb{E}h] = [\mathcal{O}_{\Omega_A(L),X}]$, which is the zero element of $\mathcal{E}\mathrm{xt}_A^1(X,\Omega_A(L))$. Clearly, $[\mathbb{E}]$ is a nonzero element of $\mathcal{E}\mathrm{xt}_A^1(N,\Omega_A(L))$, because $\mathbb{E}$ is nonsplittable. Then there is a nonzero element $\underline{f} \in \underline{\mathrm{Hom}}(N,L)$ such that $[\mathbb{E}] = \eta_N(\delta_N(\underline{f}))$. Then $\eta_X\delta_X\,\underline{\mathrm{Hom}}(h,L)(\underline{f}) = [\mathbb{E}h]$ is zero, and so $\underline{fh} = \underline{\mathrm{Hom}}(h,L)(\underline{f}) = 0$. Hence (i) implies (ii).

Conversely, assume that there is a nonzero homomorphism $\underline{f}\colon N \to L$ satisfying the condition (ii). Consider a short exact sequence

$$\mathbb{E}\colon\quad 0 \longrightarrow \Omega_A(L) \longrightarrow E \longrightarrow N \longrightarrow 0$$

in $\mathrm{mod}\, A$ such that $[\mathbb{E}] = \eta_N(\delta_N(\underline{f}))$. Clearly, $\mathbb{E}$ is not splittable because $\underline{f} \neq 0$ and η_X, δ_X are isomorphisms. Take a homomorphism $h: X \to N$ in $\mathrm{mod}\, A$ with X indecomposable and h not an isomorphism. Then, by our assumption, $\underline{\mathrm{Hom}}(h, L)(\underline{f}) = \underline{fh} = 0$, and hence $[\mathbb{E}h] = \mathrm{Ext}^1_A(h, \Omega_A(L))[\mathbb{E}] = \eta_N\delta_N(\underline{fh}) = 0$. Therefore, the exact sequence $\mathbb{E}h$ is splittable, and so N is isomorphic to $\tau_A^{-1}\Omega_A(L)$, by Lemma 6.15. This shows that (ii) implies (i). □

Now we are able to prove the announced fact.

Proposition 6.17. *Let $F: \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, B$ be a stable equivalence between indecomposable finite dimensional selfinjective K-algebras A and B of Loewy length at least* 3 *over a field K. Then for any indecomposable nonprojective module M in* $\mathrm{mod}\, A$ *there exists an isomorphism $F(\Omega_A(M)) \cong \Omega_B(F(M))$ in* $\mathrm{mod}\, B$.

Proof. We prove that $F(\tau_A^{-1}\Omega_A(M))$ and $\tau_B^{-1}\Omega_B(F(M))$ are isomorphic in $\mathrm{mod}\, B$. Observe that $F(\tau_A^{-1}\Omega_A(M)) \cong \tau_B^{-1}F(\Omega_A(M))$ in $\mathrm{mod}\, B$, by Proposition 6.11. Hence, we will obtain a required isomorphism $F(\Omega_A(M)) \cong \Omega_B(F(M))$ in $\mathrm{mod}\, B$.

Let $N = \tau_A^{-1}\Omega_A(M)$. Then it follows from Lemma 6.16 that there is a nonzero homomorphism $f: N \to M$ in $\mathrm{mod}\, A$ with $\underline{f} \neq 0$ such that, if X is an indecomposable module and $h: X \to N$ is a homomorphism, but not an isomorphism in $\mathrm{mod}\, A$, then $\underline{fh} = 0$. Choose a homomorphism $f': F(N) \to F(M)$ in $\mathrm{mod}\, B$ such that $F(\underline{f}) = \underline{f'}$. Clearly, $\underline{f'} \neq 0$, because $\underline{f} \neq 0$. Let Y be an indecomposable module and $h': Y \to F(N)$ a homomorphism, but not a isomorphism in $\mathrm{mod}\, B$. We claim that $\underline{f'h'} = 0$. We may assume that $\underline{h'} \neq 0$, and hence Y is nonprojective. Then there exist an indecomposable nonprojective module X and a homomorphism $h: X \to N$ in $\mathrm{mod}\, A$ such that $F(X) = Y$ and $F(\underline{h}) = \underline{h'}$. Since h' is a nonisomorphism in $\mathrm{mod}\, B$, applying Lemma III.4.3 we conclude that $\underline{h'}$ is a nonisomorphism in $\underline{\mathrm{mod}}\, B$, and hence $\underline{h}$ is a nonisomorphism in $\underline{\mathrm{mod}}\, A$. Then, applying Lemma III.4.3 again, we obtain that h is a nonisomorphism in $\mathrm{mod}\, A$. Hence, $\underline{fh} = 0$. This implies $\underline{f'h'} = F(\underline{fh}) = 0$. Therefore, it follows from Lemma 6.16 that $F(N) = F(\tau_A^{-1}\Omega_A(M))$ is isomorphic to $\tau_B^{-1}\Omega_B(F(M))$ in $\mathrm{mod}\, B$. □

7 Exercises

1. Let A be a finite dimensional K-algebra over a field K, and F a nonzero functor in $\mathcal{F}(A)$ (respectively, $\mathcal{F}(A)^{\mathrm{o}}$). We say that F is of finite length if there is a chain of subfunctors

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = F$$

of F in $\mathcal{F}(A)$ (respectively, in $\mathcal{F}(A)^{\mathrm{o}}$) such that all quotient functors F_i/F_{i-1}, $i \in \{1, \dots, m\}$, are simple. Prove that a functor F is of finite length if and only if $F(X) = 0$ for all but finitely many (up to isomorphism) indecomposable modules X in $\operatorname{mod} A$.

2. Let A be a finite dimensional K-algebra over a field K. Prove that the following statements are equivalent:

(a) A is of finite representation type.

(b) Every functor in $\mathcal{F}(A)$ has finite length.

(c) Every functor in $\mathcal{F}(A)^{\mathrm{o}}$ has finite length.

3. Let A be a finite dimensional K-algebra over a field K, $\mathcal{C}$ a component of Γ_A^{s}, and X a module in $\mathcal{C}$. Assume that there is a sectional path from X to $\tau_A^r X$ in $\mathcal{C}$, for some integer r. Prove that $\mathcal{C}$ is finite.

4. Let A be a finite dimensional K-algebra over a field K and

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n$$

a presectional path in Γ_A. Set $\tau_A X_j = 0$ if $j = n + 1$ or X_j is projective.

(a) Prove that, for any $j \in \{1, \dots, n\}$, there exists an irreducible homomorphism $\begin{bmatrix} f_j & g_j \end{bmatrix} : X_{j-1} \oplus \tau_A X_{j+1} \to X_j$ such that $f_j \cdots f_1 + g_j p_{j-1} \notin \operatorname{rad}_A^{j+1}(X_0, X_j)$ for any homomorphism $p_{j-1} : X_0 \to \tau_A X_{j+1}$ in $\operatorname{mod} A$.

(b) Prove that there exist irreducible homomorphisms $f_i : X_{i-1} \to X_i$, $i \in \{1, \dots, n\}$, in $\operatorname{mod} A$ such that $f_n \cdots f_1 \notin \operatorname{rad}_A^n(X_0, X_n)$.

5. Let A be a finite dimensional K-algebra over a field K and $X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n$ a presectional path in Γ_A. Then $X_0 \neq X_{n-1}$ or $X_1 \neq X_n$.

6. Let A be a finite dimensional K-algebra over a field K, $\mathcal{C}$ a component of Γ_A, and $B = A/\operatorname{ann}_A(\mathcal{C})$.

(a) Prove that $\mathcal{C}$ is a component of Γ_B.

(b) Prove that $\mathcal{C}$ is a generalized standard component of Γ_A if and only if $\mathcal{C}$ is a generalized standard component of Γ_B.

7. Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ a component of Γ_A with a section Δ. Prove that $\mathcal{C}$ is generalized standard if and only if $\operatorname{Hom}_A(X, \tau_A Y) = 0$ for all modules X and Y in $\mathcal{C}$.

8. Let A be a finite dimensional K-algebra over a field K. An indecomposable module X in $\operatorname{mod} A$ is said to be *directing* if there is no oriented cycle

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = X$$

of nonzero nonisomorphisms between indecomposable modules in $\operatorname{mod} A$. Assume that $\mathcal{C}$ is a regular component of Γ_A containing a directing module X. Prove that $\mathcal{C}$ is a generalized standard component isomorphic to the translation quiver $\mathbb{Z}\Delta$, for a finite acyclic valued quiver Δ (see [S3]).

9. Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ a regular component of Γ_A isomorphic to the translation quiver $\mathbb{Z}\mathbb{A}_\infty$. Prove that there is an indecomposable module X in $\mathcal{C}$ such that $\operatorname{rad}_A^\infty(X, X) \neq 0$ (see [S5]).

10. Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ a stable tube of Γ_A. Prove that we may choose irreducible homomorphisms $f_\sigma : U \to V$ in $\operatorname{mod} A$, for all arrows $U \overset{\sigma}{\to} V$ in $\mathcal{C}$, such that the following statements hold:

(i) $f_\eta f_\xi \in \operatorname{rad}_A^3(\tau_A W, W)$ for any mesh in $\mathcal{C}$ of the form

$$\begin{tikzcd}
\tau_A W \arrow[dr, "\xi"'] & & W \\
& T \arrow[ur, "\eta"'] &
\end{tikzcd}$$

(ii) $f_\alpha f_\beta - f_\gamma f_\delta \in \operatorname{rad}_A^3(\tau_A W, W)$ for any mesh in $\mathcal{C}$ of the form

$$\begin{tikzcd}
& U \arrow[dr, "\alpha"] & \\
\tau_A W \arrow[ur, "\beta"] \arrow[dr, "\delta"'] & & W \\
& V \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

11. Let A be a finite dimensional K-algebra over a field K, $\mathcal{C}$ a generalized standard stable tube of rank r in Γ_A, and $E_1, \ldots, E_r$ the modules forming the mouth of $\mathcal{C}$. Moreover, let $B(\mathcal{C}) = A/\operatorname{ann}_A(\mathcal{C})$ be the faithful algebra of $\mathcal{C}$. Prove that $E_1, \ldots, E_r$ is a hereditary family of pairwise orthogonal bricks in $\operatorname{mod} B(\mathcal{C})$.

12. Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be an acyclic component of Γ_A without projective modules. Prove that there is an acyclic locally finite valued quiver Δ such that $\mathcal{C}$ is isomorphic to a full valued translation subquiver of the translation quiver $\mathbb{Z}\Delta$ which is closed under predecessors (see [L2]).

13. Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be an acyclic component of Γ_A without injective modules. Prove that there is an acyclic locally finite valued quiver Δ such that $\mathcal{C}$ is isomorphic to a full valued translation subquiver of the translation quiver $\mathbb{Z}\Delta$ which is closed under successors (see [L2]).

14. Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a generalized standard acyclic component of Γ_A without projective modules, and $B(\mathcal{C})$ the faithful algebra of $\mathcal{C}$. Prove the following assertions (established in [S2]):

(a) $B(\mathcal{C})$ is a tilted algebra of the form $\operatorname{End}_H(T)$, for an indecomposable finite dimensional hereditary K-algebra H of infinite representation type and a tilting module T in mod H without nonzero preinjective direct summands.

(b) $\mathcal{C}$ is the connecting component $\mathcal{C}_T$ of $\Gamma_{B(\mathcal{C})}$ determined by T.

15. Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a generalized standard acyclic component of Γ_A without injective modules, and $B(\mathcal{C})$ the faithful algebra of $\mathcal{C}$. Prove the following assertions (established in [S2]):

(a) $B(\mathcal{C})$ is a tilted algebra of the form $\operatorname{End}_H(T)$, for an indecomposable finite dimensional hereditary K-algebra H of infinite representation type and a tilting module T in mod H without nonzero postprojective direct summands.

(b) $\mathcal{C}$ is the connecting component $\mathcal{C}_T$ of $\Gamma_{B(\mathcal{C})}$ determined by T.

16. Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a component of Γ_A containing an oriented cycle and a projective module, but without injective modules. Prove the following assertions (see [L2]):

(a) The arrows in $\mathcal{C}$ have trivial valuations.

(b) Every module in $\mathcal{C}$ belongs to the τ_A-orbit of a projective module.

(c) For any projective module P in $\mathcal{C}$, there exists an infinite sectional path in Γ_A of the form

$$P = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots$$

(d) All but finitely many modules in $\mathcal{C}$ lie on an oriented cycle.

17. Let A be a finite dimensional K-algebra over a field K and $\mathcal{C}$ be a component of Γ_A containing an oriented cycle and an injective module, but without projective modules. Prove the following assertions (see [L2]):

(a) The arrows in $\mathcal{C}$ have trivial valuations.

(b) Every module in $\mathcal{C}$ belongs to the τ_A-orbit of an injective module.

(c) For any injective module I in $\mathcal{C}$, there exists an infinite sectional path in Γ_A of the form

$$\cdots \longrightarrow Y_{j+1} \longrightarrow Y_j \longrightarrow \cdots \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow Y_0 = I.$$

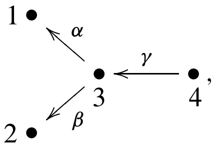
(d) All but finitely many modules in $\mathcal{C}$ lie on an oriented cycle.

18. Let $n \geq 4$ be a natural number and A a finite dimensional K-algebra of Euclidean type $\widetilde{\mathbb{D}}_n$ over a field K. Prove that the Auslander–Reiten quiver Γ_A of A contains a generalized standard stable tube of rank $n-2$.

19. Let $n \geq 2$ be a natural number and A a finite dimensional K-algebra of Euclidean type $\widetilde{\mathbb{A}}_{2n}$ over a field K. Prove that the Auslander–Reiten quiver Γ_A of A contains a generalized standard stable tube of rank at least n.

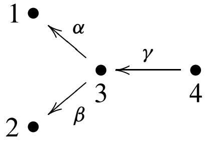
20. Let A be a finite dimensional selfinjective K-algebra over a field K and $f\colon M \to N$ be a homomorphism in $\operatorname{mod} A$ between indecomposable nonprojective modules. Assume that f is a monomorphism or an epimorphism. Then $\underline{f} \neq 0$ in the stable category $\underline{\operatorname{mod}}\, A$.

21. Let K be a field, Q the quiver



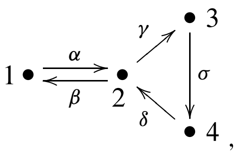
I the ideal in KQ generated by $\gamma\alpha$ and $\gamma\beta$, and $A = KQ/I$ the associated bound quiver algebra. Prove that A is stably equivalent to the path algebra $B = K\Delta$ of the quiver Δ of the form

22. Let K be a field, Q the quiver



and Q^{op} the opposite quiver of Q. Prove that the path algebras $A = KQ$ and $A^{\mathrm{op}} = KQ^{\mathrm{op}}$ are not stably equivalent.

23. Let K be a field, Q the quiver



I the ideal in KQ generated by the elements $\alpha\gamma$, $\delta\beta$, $\beta\alpha - \gamma\sigma\delta$, $\sigma\delta\gamma\sigma$, and $A = KQ/I$ the associated bound quiver algebra. Prove the following assertions:

(a) A is a finite dimensional symmetric K-algebra of finite representation type, having 20 pairwise nonisomorphic indecomposable finite dimensional right A-modules.

(b) A is stably equivalent to the symmetric Nakayama algebra $B = K\Delta/J$, where Δ is the quiver

$$\begin{tikzcd}
1 \bullet \arrow[r, "\alpha_1"] & \bullet 2 \arrow[d, "\alpha_2"] \\
4 \bullet \arrow[u, "\alpha_4"] & \bullet 3 \arrow[l, "\alpha_3"]
\end{tikzcd}$$

and J is the ideal generated by the elements $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_1$, $\alpha_2\alpha_3\alpha_4\alpha_1\alpha_2$, $\alpha_3\alpha_4\alpha_1\alpha_2\alpha_3$, $\alpha_4\alpha_1\alpha_2\alpha_3\alpha_4$.

24. Let K be a field, Q the quiver

$$\begin{tikzcd}
 & & \bullet 2 \arrow[dl, shift left, "\alpha_2"] \\
1 \bullet \arrow[r, shift left, "\alpha_1"] & \underset{0}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[ur, shift left, "\beta_2"] \arrow[dr, shift left, "\beta_3"] & \\
 & & \bullet 3 \arrow[ul, shift left, "\alpha_3"]
\end{tikzcd},$$

I the ideal in KQ generated by the elements

$$\beta_1\alpha_1 - \beta_2\alpha_2, \quad \beta_2\alpha_2 - \beta_3\alpha_3, \quad \alpha_1\beta_2, \quad \alpha_1\beta_3, \quad \alpha_2\beta_1, \quad \alpha_2\beta_3, \quad \alpha_3\beta_1, \quad \alpha_3\beta_2,$$

J the ideal in KQ generated by the elements

$$\beta_1\alpha_1 - \beta_2\alpha_2, \quad \beta_2\alpha_2 - \beta_3\alpha_3, \quad \alpha_1\beta_1, \quad \alpha_1\beta_2, \quad \alpha_2\beta_1, \quad \alpha_2\beta_3, \quad \alpha_3\beta_2, \quad \alpha_3\beta_3,$$

and $A = KQ/I$, $B = KQ/J$ the associated bound quiver algebras. Prove the following assertions:

(a) A and B are finite dimensional selfinjective K-algebras of finite representation type, having 24 pairwise nonisomorphic indecomposable finite dimensional right A-modules and right B-modules, respectively.

(b) A and B are not stably equivalent.

25. Let K be a field and $\Lambda = K[X,Y]/(X^2, XY, Y^2)$ the quotient algebra of the polynomial algebra $K[X,Y]$ in two commuting variables over K by the ideal generated by the monomials X^2, XY, Y^2 of degree 2. Prove that Λ is stably equivalent to the path algebra $K\Delta$ of the Kronecker quiver

$$\Delta\colon \qquad \begin{tikzcd}
1 \bullet & \bullet 2 \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\beta"]
\end{tikzcd}.$$

26. Let A be a finite dimensional K-algebra of Loewy length 2 over a field K, and let $H(A)$ be the matrix algebra

$$\begin{bmatrix} A/\operatorname{rad} A & \operatorname{rad} A \\ 0 & A/\operatorname{rad} A \end{bmatrix}.$$

We identify the category $\operatorname{mod} H(A)$ with the category $\operatorname{rep}({}_{A/\operatorname{rad} A} \operatorname{rad} A_{A/\operatorname{rad} A})$ of finite dimensional representations of the $(A/\operatorname{rad} A, A/\operatorname{rad} A)$-bimodule $\operatorname{rad} A$ (see Lemma VI.10.1). Prove the following statements (see [ARS]):

(a) $H(A)$ is a finite dimensional hereditary K-algebra.

(b) There is a covariant functor $\Sigma\colon \operatorname{mod} A \to \operatorname{mod} H(A)$ which assigns to a module M in $\operatorname{mod} A$ the triple $\Sigma(M) = (M/\operatorname{rad} M, \operatorname{rad} M, f)$, where $f\colon (M/\operatorname{rad} M) \otimes_{A/\operatorname{rad} A} \operatorname{rad} A \to \operatorname{rad} M$ is given by $f((m + \operatorname{rad} M) \otimes a) = ma$ for $m \in M$ and $a \in \operatorname{rad} A$.

(c) The functor Σ is full, but not faithful.

(d) The functor Σ induces a stable equivalence $\Sigma\colon \underline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, H(A)$.

(e) An exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\operatorname{mod} A$, with X and Z indecomposable and X nonsimple, is an almost split sequence in $\operatorname{mod} A$ if and only if the induced sequence $0 \to \Sigma(X) \xrightarrow{\Sigma(f)} \Sigma(Y) \xrightarrow{\Sigma(g)} \Sigma(Z) \to 0$ is an almost split sequence in $\operatorname{mod} H(A)$.

(f) Every component $\mathcal{C}$ of Γ_A without simple modules is either a stable tube, or a component of the form $\mathbb{Z}\mathbb{A}_\infty$.

27. Let A be a finite dimensional selfinjective K-algebra of Loewy length 3 over a field K, and let $\mathcal{C}$ be a regular component of Γ_A. Prove that $\mathcal{C}$ is either a stable tube, or a component of the form $\mathbb{Z}\mathbb{A}_\infty$.

Chapter X

Selfinjective Hochschild extension algebras

The aim of this chapter is to introduce a new class of finite dimensional selfinjective algebras over a field, which are obtained as Hochschild extension algebras of finite dimensional algebras over a field by duality bimodules. We shall present basic properties of selfinjective Hochschild extensions of algebras by duality bimodules and investigate the structure of their finite dimensional modules. Most of the results presented in the first part of this chapter were established by K. Yamagata in the series of papers [Y1], [Y2], [Y3], [Y5] (see also [Y4]). In Section 4 we present also very recent results of the authors on non-Frobenius selfinjective Hochschild extension algebras. Moreover, Section 5 contains results concerning Hochschild extension algebras of finite field extensions proved by Y. Ohnuki, K. Takeda and K. Yamagata in [OTY]. In the second part of this chapter, applying results of Chapter VII, we describe in detail the structure of the Auslander–Reiten quiver of an indecomposable Hochschild extension algebra of a finite dimensional hereditary algebra over a field by a duality bimodule.

1 Hochschild cohomology spaces

Let A be a finite dimensional K-algebra over a field K and $M = {}_AM_A$ be a finite dimensional A-bimodule. Following [H], the *Hochschild* (*cochain*) *complex* of the algebra A with coefficients in the A-bimodule M is a sequence $C^\bullet = C^\bullet(A, M)$ of K-linear maps of finite dimensional K-vector spaces

$$C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \longrightarrow \cdots \longrightarrow C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \cdots$$

where $C^0 = C^0(A, M) = M$, $C^i = C^i(A, M) = \operatorname{Hom}_K(A^{\otimes i}, M)$, with $A^{\otimes i}$ the tensor product $A \otimes_K A \otimes_K \cdots \otimes_K A$ of i-copies of A, for $i \geq 1$, and the K-linear maps $d^i : C^i \to C^{i+1}$ are defined as follows:

$$d^0 : M \longrightarrow \operatorname{Hom}_K(A, M)$$

is given by $d^0(m)(a) = am - ma$ for $a \in A$ and $m \in M$, and

$$d^i : C^i = \operatorname{Hom}_K(A^{\otimes i}, M) \longrightarrow \operatorname{Hom}_K(A^{\otimes(i+1)}, M) = C^{i+1}, \ i \geq 1,$$

is given by

$$\begin{aligned} d^i(f)(a_1 \otimes \cdots \otimes a_{i+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) \\ + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) \\ + (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i) a_{i+1}, \end{aligned}$$

for $f \in \operatorname{Hom}_K(A^{\otimes i}, M)$ and $a_1, \dots, a_{i+1} \in A$.

Then a direct checking shows that $d^{i+1} d^i = 0$ for all $i \geq 0$. This allows to define, for each $i \geq 0$, the K-vector space

$$H^i(A, M) = \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$$

(with $d^{-1} = 0$), called the i-th *Hochschild cohomology space* of the K-algebra A with coefficients in the A-bimodule M.

We give now interpretations of the lower Hochschild cohomology spaces $H^0(A, M)$ and $H^1(A, M)$.

Observe that

$$H^0(A, M) = \operatorname{Ker} d^0 = \{m \in M \mid am = ma \text{ for all } a \in A\}.$$

In particular, for ${}_A M_A = {}_A A_A$, $H^0(A, A)$ is the center $C(A)$ of the algebra A.

Consider the K-vector spaces

$$\operatorname{Der}_K(A, M) = \{\delta \in \operatorname{Hom}_K(A, M) \mid \delta(ab) = a\delta(b) + \delta(a)b \text{ for all } a, b \in A\},$$

called the *space of derivations* from A to M, and

$$\operatorname{Der}_K^0(A, M) = \left\{ \delta_m \in \operatorname{Hom}_K(A, M) \;\middle|\; \begin{array}{c} \delta_m(a) = am - ma \\ \text{for all } m \in M \text{ and } a \in A \end{array} \right\},$$

called the *space of inner derivations* from A to M. Then $\operatorname{Der}_K^0(A, M)$ is a K-subspace of $\operatorname{Der}_K(A, M)$, and

$$H^1(A, M) = \operatorname{Ker} d^1 / \operatorname{Im} d^0 = \operatorname{Der}_K(A, M) / \operatorname{Der}_K^0(A, M)$$

is called the *space of outer derivations* from A to M. Observe that $d^0(m) = \delta_m$, for $m \in M$. Moreover, for $f \in C^1(A, M) = \operatorname{Hom}_K(A, M)$, $d^1(f) \in C^2(A, M) = \operatorname{Hom}_K(A \otimes_K A, M)$ is defined by $d^1(f)(a \otimes b) = af(b) - f(ab) + f(a)b$, for $a, b \in A$, and so indeed $\operatorname{Im} d^0 = \operatorname{Der}_K^0(A, M)$ and $\operatorname{Ker} d^1 = \operatorname{Der}_K(A, M)$.

2 Hochschild extension algebras

In this section we provide an interpretation of the second Hochschild cohomology space of an algebra A in terms of its extensions by A-bimodules (see [CE], [H]).

Let A and Λ be finite dimensional K-algebras and $\varrho\colon \Lambda \to A$ be an epimorphism of K-algebras with $(\operatorname{Ker}\varrho)^2 = 0$. Then $M = \operatorname{Ker}\varrho$ is a two-sided ideal of Λ and naturally a (Λ/M)-bimodule, and hence an A-bimodule by the canonical isomorphism of K-algebras $\bar{\varrho}\colon \Lambda/M \to A$ induced by ϱ, that is, $\bar{\varrho}(\lambda + M) = \varrho(\lambda)$, for $\lambda \in \Lambda$. This leads to the following definition.

For a finite dimensional K-algebra A over a field K and a finite dimensional A-bimodule M, a short exact sequence of K-vector spaces

$$0 \longrightarrow M \xrightarrow{\omega} \Lambda \xrightarrow{\varrho} A \longrightarrow 0$$

is said to be a *Hochschild extension* of A by M if the following conditions are satisfied:

(i) Λ is a K-algebra and ϱ is an epimorphism of K-algebras;

(ii) ω is a monomorphism of Λ-bimodules, where the structure of Λ-bimodule on M is induced from the A-bimodule structure on M by ϱ.

Observe that then $\omega(M)$ is a two-sided ideal of Λ and $\omega(M)^2 = 0$, because, for $x, y \in M$, we have $\omega(x)\omega(y) = \omega(x\omega(y)) = \omega(x\varrho(\omega(y))) = \omega(x0) = 0$.

Two Hochschild extensions

$$\mathbb{E}\colon \quad 0 \longrightarrow M \xrightarrow{\omega} \Lambda \xrightarrow{\varrho} A \longrightarrow 0,$$
$$\mathbb{E}'\colon \quad 0 \longrightarrow M \xrightarrow{\omega'} \Lambda' \xrightarrow{\varrho'} A \longrightarrow 0$$

are said to be *equivalent* if there is a homomorphism of K-algebras $\varphi\colon \Lambda \to \Lambda'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\omega"] \arrow[d, "\mathrm{id}_M"] & \Lambda \arrow[r, "\varrho"] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
0 \arrow[r] & M \arrow[r, "\omega'"] & \Lambda' \arrow[r, "\varrho'"] & A \arrow[r] & 0,
\end{tikzcd}$$

is commutative, and then we write $\mathbb{E} \cong \mathbb{E}'$. Observe that then φ is an isomorphism of K-algebras. Indeed, for $\lambda' \in \Lambda'$, we have $\varrho'(\lambda') = \varrho(\lambda)$ for some $\lambda \in \Lambda$, and then $\varrho'(\lambda' - \varphi(\lambda)) = \varrho'(\lambda') - \varrho'\varphi(\lambda) = \varrho'(\lambda') - \varrho(\lambda) = 0$, and so $\lambda' - \varphi(\lambda) \in \operatorname{Ker}\varrho' = \operatorname{Im}\omega'$. Hence, $\lambda' = \omega'(y) + \varphi(\lambda) = \varphi(\omega(y) + \lambda)$ for some $y \in M$. Therefore, φ is an epimorphism of K-algebras, and hence an isomorphism of K-algebras, because $\dim_K \Lambda' = \dim_K M + \dim_K A = \dim_K \Lambda$.

For a finite dimensional K-algebra A over a field K and a finite dimensional A-bimodule M, we denote by $\mathcal{HE}(A,M)$ the set of all Hochschild extensions of A by M. Then $\cong$ is an equivalence relation on the set $\mathcal{HE}(A,M)$, and we may consider the set

$$\mathcal{H}\,\mathcal{E}\mathrm{xt}(A,M) = \mathcal{HE}(A,M)/\cong$$

of the equivalence classes $[\mathbb{E}] = \mathbb{E}/\cong$ of extensions $\mathbb{E}$ in $\mathcal{HE}(A,M)$.

We will show that, for a finite dimensional K-algebra A over a field K and a finite dimensional A-bimodule M, there is a bijection between the elements of the second Hochschild cohomology space $H^2(A,M)$ and the elements of $\mathcal{H}\,\mathcal{E}\mathrm{xt}(A,M)$.

Let A be a finite dimensional K-algebra over a field K and M be a finite dimensional A-bimodule. For an element $f \in C^2(A,M) = \mathrm{Hom}_K(A \otimes_K A, M)$, we denote by $A \ltimes_f M$ the K-vector space $A \oplus M$ equipped with the multiplication given by

$$(a,x)(b,y) = \big(ab, ay + xb + f(a \otimes b)\big)$$

for $a, b \in A$ and $x, y \in M$.

Proposition 2.1. *Let A be a finite dimensional K-algebra over a field K, M a finite dimensional A-bimodule, and $f \in C^2(A,M)$ satisfies $d^2(f) = 0$. Then $A \ltimes_f M$ is a finite dimensional K-algebra.*

Proof. It follows from the definition of $d^2\colon C^2(A,M) \to C^3(A,M)$ that the K-linear map $d^2(f)\colon A \otimes_K A \otimes_K A \to M$ is given by

$$d^2(a \otimes b \otimes c) = af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c$$

for all $a, b, c \in A$. Hence $d^2(f) = 0$ is equivalent to the following property of the K-linear map $f\colon A \otimes_K A \to K$:

$$af(b \otimes c) + f(a \otimes bc) = f(ab \otimes c) + f(a \otimes b)c$$

for all $a, b, c \in A$. In particular, we obtain the equalities

$$\begin{aligned} f(1 \otimes bc) &= f(1 \otimes b)c, \text{ for } b, c \in A, \\ af(1 \otimes c) &= f(a \otimes 1)c, \text{ for } a, c \in A, \\ af(b \otimes 1) &= f(ab \otimes 1), \text{ for } a, b \in A. \end{aligned}$$

Obviously, $A \ltimes_f M = A \oplus M$ is a finite dimensional K-vector space. For $a, b, c \in A$ and $x, y, z \in M$, we have

$$\begin{aligned}
&\big((a,x)(b,y)\big)(c,z) \\
&\quad = \big(ab, ay + xb + f(a \otimes b)\big)(c,z) \\
&\quad = \big((ab)c, (ab)z + (ay + xb + f(a \otimes b))c + f(ab \otimes c)\big) \\
&\quad = \big((ab)c, (ab)z + (ay)c + (xb)c + f(a \otimes b)c + f(ab \otimes c)\big), \\
&(a,x)\big((b,y)(c,z)\big) \\
&\quad = (a,x)\big(bc, bz + yc + f(b \otimes c)\big) \\
&\quad = \big(a(bc), a(bz + yc + f(b \otimes c)) + x(bc) + f(a \otimes bc)\big) \\
&\quad = \big(a(bc), a(bz) + a(yc) + af(b \otimes c) + x(bc) + f(a \otimes bc)\big) \\
&\quad = \big(a(bc), a(bz) + a(yc) + x(bc) + af(b \otimes c) + f(a \otimes bc)\big),
\end{aligned}$$

and $(ab)z = a(bz)$, $(ay)c = a(yc)$, $(xb)c = x(bc)$, because M is an A-bimodule, and $f(a \otimes b)c + f(ab \otimes c) = af(b \otimes c) + f(a \otimes bc)$, since $d^2(f) = 0$, and consequently we obtain that

$$\big((a,x)(b,y)\big)(c,z) = (a,x)\big((b,y)(c,z)\big).$$

Hence, the multiplication in $A \ltimes_f M = A \oplus M$ is associative.

We show now that $(1, -f(1 \otimes 1))$, with $1 = 1_A$ the identity of A, is the identity $1_{A \ltimes_f M}$ of $A \ltimes_f M$ with respect to multiplication. Indeed, for $a \in A$ and $x \in M$, the following equalities are consequences of the properties of f described above:

$$\begin{aligned}
\big(1, -f(1 \otimes 1)\big)(a,x) &= \big(1a, 1x - f(1 \otimes 1)a + f(1 \otimes a)\big) = (a,x), \\
(a,x)\big(1, -f(1 \otimes 1)\big) &= \big(a1, -af(1 \otimes 1) + x1 + f(a \otimes 1)\big) = (a,x).
\end{aligned}$$

Further, for $\lambda \in K$, $a, b \in A$, and $x, y \in M$, we have

$$\begin{aligned}
\lambda\big((a,x)(b,y)\big) &= \lambda\big(ab, ay + xb + f(a \otimes b)\big) \\
&= \big(\lambda(ab), \lambda(ay) + \lambda(xb) + \lambda f(a \otimes b)\big), \\
\big(\lambda(a,x)\big)(b,y) &= (\lambda a, \lambda x)(b,y) = \big((\lambda a)b, (\lambda a)y + (\lambda x)b + f(\lambda a \otimes b)\big), \\
(a,x)\big(\lambda(b,y)\big) &= (a,x)(\lambda b, \lambda y) = \big(a(\lambda b), a(\lambda y) + x(\lambda b) + f(a \otimes \lambda b)\big),
\end{aligned}$$

and hence $\lambda((a,x)(b,y)) = (\lambda(a,x))(b,y) = (a,x)(\lambda(b,y))$, because A is a K-algebra, M is an A-bimodule, f is a K-linear map, and $\lambda(a \otimes b) = \lambda a \otimes b = a \otimes \lambda b$ by the K-vector space structure of $A \otimes_K A$.

Finally, we have also $1_K 1_{A \ltimes_f M} = (1_K 1_A, -1_K f(1_A \otimes 1_A)) = (1_A, -f(1_A \otimes 1_A)) = 1_{A \ltimes_f M}$, because $1_K 1_A = 1_A$.

Therefore, $A \ltimes_f M$ is a finite dimensional K-algebra. □

Observe that, for $f \in C^2(A, M)$ with $d^2(f) = 0$, we have an exact sequence of K-vector spaces

$$\mathbb{E}_f\colon \quad 0 \longrightarrow M \xrightarrow{\omega} A \ltimes_f M \xrightarrow{\varrho} A \longrightarrow 0,$$

where $\omega(x) = (0, x)$ and $\varrho(a, x) = a$ for $a \in A$, $x \in M$. We also note that $\mathbb{E}_f$ is a Hochschild extension of A by M, because ϱ is an epimorphism of K-algebras and ω is a monomorphism of $(A \ltimes_f M)$-bimodules. Indeed, we have

$$\begin{aligned}\omega\big((a, x)m(b, y)\big) &= \omega(amb) = (0, amb)\\ &= (a, x)(0, m)(b, y) = (a, x)\omega(m)(b, y)\end{aligned}$$

for all $a, b \in A$ and $x, y, m \in M$. The Hochschild extension $\mathbb{E}_f$ is said to be the *Hochschild extension of the algebra A* by the A-bimodule M with respect to $f \in C^2(A, M)$ satisfying $d^2(f) = 0$.

For an element $f \in C^2(A, M)$ with $d^2(f) = 0$, we denote by $\bar{f}$ its residue class $f + \operatorname{Im} d^1 \in H^2(A, M)$.

Proposition 2.2. *Let A be a finite dimensional K-algebra over a field K, M be a finite dimensional A-bimodule, and f, f' be two elements of $C^2(A, M)$ with $d^2(f) = 0$, $d^2(f') = 0$. Then the Hochschild extensions*

$$0 \longrightarrow M \xrightarrow{\omega} A \ltimes_f M \xrightarrow{\varrho} A \longrightarrow 0,$$

$$0 \longrightarrow M \xrightarrow{\omega'} A \ltimes_{f'} M \xrightarrow{\varrho'} A \longrightarrow 0$$

are equivalent if and only if $\bar{f} = \bar{f}'$.

Proof. Assume that the Hochschild extensions induced by f and f' are equivalent. Let $\varphi\colon A \ltimes_f M \to A \ltimes_f M$ be an isomorphism of K-algebras such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\omega"] \arrow[d, "\mathrm{id}_M"] & A \ltimes_f M \arrow[r, "\varrho"] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
0 \arrow[r] & M \arrow[r, "\omega'"] & A \ltimes_{f'} M \arrow[r, "\varrho'"] & A \arrow[r] & 0
\end{tikzcd}$$

is commutative. Let $a \in A$. Then $\varphi((a, 0)) - (a, 0) \in \operatorname{Ker} \varrho' = \operatorname{Im} \omega'$, and hence there exists exactly one element $x_a \in M$ such that $\varphi((a, 0)) - (a, 0) = \omega'(x_a) = (0, x_a)$. We define the map $g\colon A \to M$ by $g(a) = x_a$ for any $a \in A$. We claim that $g \in \operatorname{Hom}_K(A, M)$. Observe that, for $a, b \in A$, we have

$$\begin{aligned}\varphi\big((a + b, 0)\big) - (a + b, 0) &= \big(\varphi((a, 0)) - (a, 0)\big) + \big(\varphi((b, 0)) - (b, 0)\big)\\ &= \omega'(x_a) + \omega'(x_b) = \omega'(x_a + x_b),\end{aligned}$$

and consequently $x_{a+b} = x_a + x_b$, which shows that $g(a + b) = g(a) + g(b)$.

Similarly, for $\lambda \in K$ and $a \in A$, we have

$$\begin{aligned}\varphi\big((\lambda a,0)\big)-(\lambda a,0) &= \lambda\varphi\big((a,0)\big)-\lambda(a,0)=\lambda\big(\varphi((a,0))-(a,0)\big)\\ &= \lambda\omega'(x_a)=\omega'(\lambda x_a),\end{aligned}$$

and consequently $x_{\lambda a} = \lambda x_a$, which shows that $g(\lambda a) = \lambda g(a)$.

We will show now that $f - f' = d^1(g)$. Clearly, then $\bar{f} = \bar{f}'$. Observe that the homomorphism $\varphi: A \ltimes_f M \to A \ltimes_{f'} M$ is given by the formula

$$\varphi\big((a,x)\big) = (a, x + g(a)), \text{ for } a \in A,\ x \in M.$$

Indeed, $\varphi((a,x)) = \varphi((a,0) + (0,x)) = \varphi((a,0)) + \varphi((0,x)) = (a,0) + (0,g(a)) + (0,x) = (a, x + g(a))$, because $\varphi((0,x)) = \varphi(\omega(x)) = \omega'(x) = (0,x)$, for $a \in A$, $x \in M$. Take elements $a, b \in A$. Then we have in $A \ltimes_{f'} M$ the equalities

$$\begin{aligned}\varphi\big((a,0)(b,0)\big) &= \varphi\big((ab, f(a\otimes b))\big) = \big(ab, f(a\otimes b) + g(ab)\big),\\ \varphi\big((a,0)\big)\varphi\big((b,0)\big) &= \big(a, g(a))(b, g(b)\big) = \big(ab, ag(a) + g(a)b + f'(a\otimes b)\big).\end{aligned}$$

Since $\varphi((a,0)(b,0)) = \varphi((a,0))\varphi((b,0))$, we obtain

$$f(a\otimes b) + g(ab) = ag(a) + g(a)b + f'(a\otimes b),$$

or equivalently,

$$f(a\otimes b) - f'(a\otimes b) = ag(a) - g(ab) + g(a)b = d^1(g)(a\otimes b).$$

Therefore, $f - f' = d^1(g)$.

Conversely, assume that $f - f' = d^1(g)$ for some $g \in \operatorname{Hom}_K(A, M)$. We define the K-linear map $\varphi: A \ltimes_f M \to A \ltimes_{f'} M$ by $\varphi((a,x)) = (a, x + g(a))$ for $a \in A$ and $x \in M$. Then, for $a, b \in A$ and $x, y \in M$, we have

$$\begin{aligned}\varphi\big((a,x)(b,y)\big) &= \varphi\big((ab, ay + xb + f(a\otimes b))\big)\\ &= \big(ab, ay + xb + f(a\otimes b) + g(ab)\big)\\ &= \big(ab, ay + xb + ag(b) + g(a)b + f'(a\otimes b)\big)\\ &= \big(a, x + g(a)\big)\big(b, y + g(b)\big)\\ &= \varphi\big((a,x)\big)\varphi\big((b,y)\big).\end{aligned}$$

Moreover, $\varphi(1_{A\ltimes_f M}) = \varphi(1, -f(1\otimes 1)) = (1, -f(1\otimes 1) + g(1)) = (1, -f'(1\otimes 1)) = 1_{A\ltimes_{f'} M}$, because $g(1) = 1g(1) - g(1) + g(1)1 = d^1(g)(1\otimes 1)$. Therefore, φ is a homomorphism of K-algebras. It follows also from definition of φ that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\omega"] \arrow[d, "\mathrm{id}_M"] & A \ltimes_f M \arrow[r, "\varrho"] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0\\
0 \arrow[r] & M \arrow[r, "\omega'"] & A \ltimes_{f'} M \arrow[r, "\varrho'"] & A \arrow[r] & 0
\end{tikzcd}$$

is commutative. This shows that the Hochschild extensions given by the rows of the above diagram are equivalent. □

Therefore, we have a well defined map

$$\mu_{A,M}\colon H^2(A,M) \longrightarrow \mathscr{H}\mathscr{E}\mathrm{xt}(A,M)$$

which assigns to an element $\bar{f} \in H^2(A,M)$ with $f \in C^2(A,M)$ satisfying $d^2(f) = 0$ the equivalence class $[\mathbb{E}_f]$ of the Hochschild extension

$$\mathbb{E}_f\colon \quad 0 \longrightarrow M \xrightarrow{\omega} A \ltimes_f M \xrightarrow{\varrho} A \longrightarrow 0$$

of A by M with respect to f. Moreover, it follows from Proposition 2.2 that $\mu_{A,M}$ is an injection.

The next proposition shows that $\mu_{A,M}$ is also a surjection.

Proposition 2.3. *Let A be a finite dimensional K-algebra over a field K, M be a finite dimensional A-bimodule, and*

$$\mathbb{E}\colon \quad 0 \longrightarrow M \xrightarrow{\eta} \Lambda \xrightarrow{\pi} A \longrightarrow 0$$

be a Hochschild extension of A by M. Then there is an element $f \in C^2(A,M)$ with $d^2(f) = 0$ such that $\mathbb{E} \cong \mathbb{E}_f$.

Proof. Let $r\colon A \to \Lambda$ be a K-linear map such that $r(1_A) = 1_\Lambda$ and $\pi r = \mathrm{id}_A$. Consider the K-linear map

$$\varphi\colon A \oplus M \longrightarrow \Lambda$$

such that $\varphi((a,x)) = r(a) + \eta(x)$ for all $a \in A$ and $x \in M$. Then we obtain in $\operatorname{mod} K$ the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\omega"] \arrow[d, "\mathrm{id}_M"] & A \oplus M \arrow[r, "\varrho"] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
0 \arrow[r] & M \arrow[r, "\eta"] & \Lambda \arrow[r, "\pi"] & A \arrow[r] & 0,
\end{tikzcd}$$

where $\omega(x) = (0,x)$ and $\varrho((a,x)) = a$ for $a \in A$ and $x \in M$. We will define $f \in C^2(A,M)$ with $d^2(f) = 0$ such that φ is a K-algebra homomorphism from $A \ltimes_f M$ to Λ.

Consider the map $h\colon A \times A \to \Lambda$ given by $h(a,b) = r(a)r(b) - r(ab)$. Then it is easily seen that h is K-bilinear. Moreover, for $a,b \in A$, we have $\pi h(a,b) = \pi(r(a)r(b) - r(ab)) = \pi(r(a)r(b)) - \pi r(ab) = \pi(r(a))\pi(r(b)) - ab = ab - ab = 0$, since π is a homomorphism of K-algebras, and hence

$h(a,b) \in \operatorname{Ker}\pi = \operatorname{Im}\eta$. Therefore, $h(a,b) = \eta(x)$ for a uniquely determined element $x \in M$, because η is a monomorphism. This allows to define the K-bilinear map $g\colon A \times A \to M$ such that $h(a,b) = \eta(g(a,b))$ for all $a,b \in A$. Then we obtain the K-linear map

$$f\colon A \otimes_K A \longrightarrow M$$

such that $f(a \otimes b) = g(a,b)$ for all $a,b \in A$. We claim that $d^2(f) = 0$. Indeed, for $a,b,c \in A$, we have the equalities

$$\begin{aligned}
\eta\big(d^2(f)(a \otimes b \otimes c)\big) &= \eta\big(af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c\big) \\
&= \eta\big(ag(b,c) - g(ab,c) + g(a,bc) - g(a,b)c\big) \\
&= a\eta\big(g(b,c)\big) - \eta\big(g(ab,c\big) + \eta\big(g(a,bc)\big) - \eta\big(g(a,b)\big)c \\
&= ah(b,c) - h(ab,c) + h(a,bc) - h(a,b)c \\
&= a\big(r(b)r(c) - r(bc)\big) - \big(r(ab)r(c) - r((ab)c)\big) \\
&\quad + \big(r(a)r(bc) - r(a(bc))\big) - \big(r(a)r(b) - r(ab)\big)c \\
&= r(a)\big(r(b)r(c)\big) - r(a)r(bc) - r(ab)r(c) + r\big((ab)c\big) \\
&\quad + r(a)r(bc) - r\big(a(bc)\big) - r(a)r(b)r(c) + r(ab)r(c) \\
&= 0,
\end{aligned}$$

because $dx = r(d)x$ and $xd = xr(d)$, for all $d \in A$ and $x \in M$. Since η is a monomorphism, we obtain $d^2(f)(a \otimes b \otimes c) = 0$ for all $a,b,c \in A$, and consequently $d^2(f) = 0$. This allows to define, by Proposition 2.1, the finite dimensional K-algebra $A \ltimes_f M$ having $A \oplus M$ as the underlying K-vector space, the multiplication

$$(a,x)(b,y) = \big(ab, ay + xb + f(a \otimes b)\big)$$

for $a,b \in A$ and $x,y \in M$, and the identity $1_{A \ltimes_f M} = (1_A, -f(1_A \otimes 1_A))$.

We prove now that the K-linear map $\varphi\colon A \ltimes_f M \to \Lambda$, defined above, is a homomorphism of K-algebras. For $a,b \in A$ and $x,y \in M$, we have the equalities

$$\begin{aligned}
\varphi((a,x)(b,y)) &= \varphi\big((ab, ay + xb + f(a \otimes b)\big) \\
&= r(ab) + \eta\big(ay + xb + f(a \otimes b)\big) \\
&= r(ab) + a\eta(y) + \eta(x)b + \eta\big(f(a \otimes b)\big) \\
&= r(ab) + r(a)\eta(y) + \eta(x)r(b) + \eta\big(g(a,b)\big) \\
&= r(ab) + r(a)\eta(y) + \eta(x)r(b) + h(a,b) \\
&= r(ab) + r(a)\eta(y) + \eta(x)r(b) + r(a)r(b) - r(ab) \\
&= \big(r(a) + \eta(x)\big)\big(r(b) + \eta(y)\big) \\
&= \varphi\big((a,x)\big)\varphi\big((b,y)\big).
\end{aligned}$$

Moreover,

$$\begin{aligned}\varphi(1_{A\ltimes_f M}) &= \varphi\big(1_A, -f(1_A \otimes 1_A)\big) = r(1_A) - \eta\big(f(1_A \otimes 1_A)\big)\\ &= r(1_A) - h(1_A \otimes 1_A) = r(1_A) - r(1_A)r(1_A) + r(1_A1_A)\\ &= 1_\Lambda,\end{aligned}$$

because $r(1_A) = 1_\Lambda$ by the choice of $r\colon A \to \Lambda$. Therefore, $\varphi\colon A \ltimes_f M \to \Lambda$ is a homomorphism of K-algebras with $\varphi\omega = \eta$ and $\pi\varphi = \varrho$, and hence the Hochschild extensions $\mathbb{E}_f$ and $\mathbb{E}$ are equivalent. □

The following theorem summarizes the above discussion.

Theorem 2.4. *Let A be a finite dimensional K-algebra over a field K and M be a finite dimensional A-bimodule. Then the map $\mu_{A,M}\colon H^2(A, M) \to \mathcal{H}\,\mathcal{E}\mathrm{xt}(A, M)$ is a bijection.*

Let A be a finite dimensional K-algebra over a field K and M be a finite dimensional A-bimodule. A Hochschild extension

$$0 \longrightarrow M \xrightarrow{\omega} \Lambda \xrightarrow{\varrho} A \longrightarrow 0$$

of A by M is said to be a *splittable Hochschild extension* if ϱ is a K-algebra retraction, that is, there is a homomorphism of K-algebras $\varrho'\colon A \to \Lambda$ such that $\varrho\varrho' = \mathrm{id}_A$. Observe also that, for the zero homomorphism $0 = f \in C^2(A, M)$, the Hochschild extension $\mathbb{E}_f = \mathbb{E}_0$ is the *trivial extension* of A by M,

$$0 \longrightarrow M \xrightarrow{\omega_0} \mathrm{T}_M(A) \xrightarrow{\varrho_0} A \longrightarrow 0,$$

where $\mathrm{T}_M(A) = A \oplus M$ as K-vector space and the multiplication in $\mathrm{T}_M(A)$ is given by

$$(a, x)(b, y) = (ab, ay + xb)$$

for $a, b \in A$ and $x, y \in M$. The following lemma describes the equivalence class $[\mathbb{E}_0] = \mu_{A,M}(\bar{0})$, where $\bar{0} = 0 + \operatorname{Im} d^1$ is the zero element of $H^2(A, M)$.

Lemma 2.5. *Let A be a finite dimensional K-algebra over a field K, M a finite dimensional A-bimodule, and*

$$\mathbb{E}\colon\quad 0 \longrightarrow M \xrightarrow{\omega} \Lambda \xrightarrow{\varrho} A \longrightarrow 0$$

a Hochschild extension of A by M. The following conditions are equivalent:

(i) $\mathbb{E} \cong \mathbb{E}_0$.

(ii) $\mathbb{E}$ *is a splittable Hochschild extension.*

Proof. Assume $\mathbb{E} \cong \mathbb{E}_0$. Then there is a homomorphism of K-algebras $\varphi\colon \mathrm{T}_M(A) \to \Lambda$ such that the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\omega_0"] \arrow[d, "\mathrm{id}_M"] & \mathrm{T}_M(A) \arrow[r, "\varrho_0"] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
0 \arrow[r] & M \arrow[r, "\omega"] & \Lambda \arrow[r, "\varrho"] & A \arrow[r] & 0 .
\end{tikzcd}$$

Observe that the trivial extension $\mathbb{E}_0$ is a splittable Hochschild extension, and $\varrho_0\varrho_0' = \mathrm{id}_A$ for the homomorphism of K-algebras $\varrho_0'\colon A \to \mathrm{T}_M(A)$ given by $\varrho_0(a) = (a, 0)$ for $a \in A$. Then $\varphi\varrho_0'\colon A \to \Lambda$ is a homomorphism of K-algebras such that $\varrho(\varphi\varrho_0') = (\varrho\varphi)\varrho_0' = \varrho_0\varrho_0' = \mathrm{id}_A$, and consequently $\mathbb{E}$ is a splittable Hochschild extension.

Conversely, assume that $\mathbb{E}$ is a splittable Hochschild extension. Let $\varrho'\colon A \to \Lambda$ be a homomorphism of K-algebras with $\varrho\varrho' = \mathrm{id}_A$. Then it follows from the proof of Proposition 2.3 that $\mathbb{E}$ is equivalent to the Hochschild extension $\mathbb{E}_f$, where $f \in C^2(A, M)$ with $d^2(f) = 0$ is given by $f(a \otimes b) = \varrho'(a)\varrho'(b) - \varrho'(ab) = 0$ for all $a, b \in A$. Therefore, $\mathbb{E} \cong \mathbb{E}_0$. □

We end this section with some definitions which will simplify our further considerations.

Let A be a finite dimensional K-algebra over a field K and M be a finite dimensional A-bimodule. Then a K-linear map $\alpha\colon A \times A \to M$ satisfying the condition

$$a\alpha(b,c) - \alpha(ab,c) = \alpha(a,b)c - \alpha(a,bc),$$

for all $a, b, c \in A$, is said to be a 2-*cocycle*. A 2-cocycle $\alpha\colon A \times A \to M$ is said to be *splittable* (*inessential* in the sense of H. Cartan and S. Eilenberg [CE]) if there exists a K-linear map $\gamma\colon A \to M$ such that

$$\alpha(a,b) = a\gamma(b) - \gamma(a)b$$

for $a, b \in A$ (α is a 2-*coboundary* of γ). Observe that $\alpha\colon A \times A \to M$ is a 2-cocycle if and only if the associated K-linear map $f_\alpha\colon A \otimes_K A \to M$ ($f_\alpha(a \otimes b) = \alpha(a,b)$ for $a, b \in A$) is an element of $C^2(A, M)$ with $d^2(f_\alpha) = 0$. Moreover, α is splittable if and only if $f_\alpha = d^1(\gamma)$ for some $\gamma \in C^1(A, M)$. For a 2-cocycle $\alpha\colon A \times A \to M$, we denote by $\mathrm{T}_M(A, \alpha)$ the finite dimensional K-algebra such that $\mathrm{T}_M(A, \alpha) = A \oplus M$ as K-vector space and the multiplication in $\mathrm{T}_M(A, \alpha)$ is given by

$$(a, x)(b, y) = \big(ab, ay + xb + \alpha(a, b)\big)$$

for $a, b \in A$ and $x, y \in M$. Therefore, $\mathrm{T}_M(A, \alpha) = A \ltimes_{f_\alpha} M$ for the map $f_\alpha \in C^2(A, M)$ associated to α. Hence the K-algebra $\mathrm{T}_M(A, \alpha)$ is said to be the *Hochschild extension algebra* (or *Hochschild extension*) of the algebra A by the A-bimodule M given by the 2-cocycle α. Observe also that $\mathrm{T}_M(A, 0)$ is the *trivial extension algebra* $\mathrm{T}_M(A)$ (or *trivial extension*) of A by M. We recall that $\mathrm{T}_{\mathrm{Hom}_K(A,K)}(A)$ is simply denoted by $\mathrm{T}(A)$ in Example IV.2.7.

3 Hochschild extensions by duality modules

For a finite dimensional K-algebra A over a field K, by specializing the A-bimodule M to a selfduality module, the Hochschild extensions of A by M will give us a wide class of selfinjective K-algebras. In this section we will show some basic properties of these selfinjective Hochschild extension algebras. Throughout this chapter, since we will consider only selfduality modules as duality modules, by a duality module we understand a selfduality module. We refer to Section II.7 for details on duality modules.

Let A and Λ be finite dimensional K-algebras over a field K. Assume that there is a surjective homomorphism $\varrho\colon \Lambda \to A$ of K-algebras, and let $I = \operatorname{Ker}\varrho$. Then the factor algebra Λ/I is regarded as an A-bimodule by the rule $b\bar{t}a = \bar{t}_2\bar{t}\,\bar{t}_1$ for all $a, b \in A$ and $t_1, t_2 \in \Lambda$ with $\varrho(t_1) = a$, $\varrho(t_2) = b$, where $\bar{t} = t + I \in \Lambda/I$ for $t \in \Lambda$. In particular, any left, right, or two-sided ideal of Λ/I becomes a left, right A-module, or A-bimodule, respectively. On the other hand, A is regarded as a Λ-bimodule by the rule $t_2at_1 = \varrho(t_2)a\varrho(t_1)$ for all $a \in A$ and $t_1, t_2 \in \Lambda$. In particular, any left, right, or two-sided ideal of A becomes a left, right Λ-module, or Λ-bimodule, respectively. Under these conventions the canonical isomorphism $\bar{\varrho}\colon \Lambda/I \to A$ of K-algebras with $\bar{\varrho}(\bar{t}) = \varrho(t)$ for $t \in \Lambda$ becomes an isomorphism of A-bimodules, and of Λ-bimodules. Indeed, for $a, b \in A$ and $t_1, t_2 \in \Lambda$ with $\varrho(t_1) = a$ and $\varrho(t_2) = b$, we have the equalities $\bar{\varrho}(b\bar{t}a) = \bar{\varrho}(\bar{t}_2\bar{t}\,\bar{t}_1) = \varrho(\bar{t}_2)\bar{\varrho}(\bar{t})\varrho(\bar{t}_1) = b\bar{\varrho}(\bar{t})a$, for any $t \in \Lambda$.

Assume now that $I = \operatorname{Ker}\varrho$ is contained in $\operatorname{rad}\Lambda$. Let $1_\Lambda = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n$ be a decomposition of 1_Λ into a sum of pairwise orthogonal primitive idempotents $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of Λ, and let $e_i = \varrho(\boldsymbol{e}_i)$ for $i \in \{1, \dots, n\}$. It is then clear that $e_1, \dots, e_n$ are pairwise orthogonal idempotents of A and $1_A = e_1 + \cdots + e_n$. Further, by Lemma I.3.17, each $\bar{e}_i = \boldsymbol{e}_i + \operatorname{rad} A$ is a primitive idempotent of $\bar{A} = A/\operatorname{rad} A$, because so is $\bar{\boldsymbol{e}}_i = \boldsymbol{e}_i + \operatorname{rad}\Lambda$ in $\Lambda/\operatorname{rad}\Lambda$. Then Proposition I.5.16 ensures that e_i is primitive for any $i \in \{1, \dots, n\}$. Thus we know that $e_1, \dots, e_n$ form a set of pairwise orthogonal primitive idempotents of A. Conversely, for a decomposition $1_A = e_1 + \cdots + e_n$ of 1_A into a sum of pairwise orthogonal primitive idempotents $e_1, \dots, e_n$ of A, Lemma I.3.12 provides pairwise orthogonal primitive idempotents $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ of Λ with $\varrho(\boldsymbol{e}_1) = e_1, \dots \varrho(\boldsymbol{e}_n) = e_n$.

From now on we will consider a Hochschild extension of a finite dimensional K-algebra A over a field K by the duality module Q, which will be denoted by

$$0 \longrightarrow Q \xrightarrow{\omega} T \xrightarrow{\varrho} A \longrightarrow 0.$$

Let

$$1_A = \sum_{i=1}^{n_A} \sum_{j=1}^{m_A(i)} e_{ij}$$

be a canonical decomposition of 1_A into a sum of pairwise orthogonal primitive idempotents of A, that is,

$$\begin{aligned}e_{ij}A &\cong e_{il}A \text{ for } j,l \in \{1,\dots,m_A(i)\}, i \in \{1,\dots,n_A\},\\ e_{ij}A &\ncong e_{kl}A \text{ for } i,k \in \{1,\dots,n_A\},\ \text{with } i \neq k \text{ and all}\\ &\qquad j \in \{1,\dots,m_A(i)\},\ l \in \{1,\dots,m_A(k)\}.\end{aligned}$$

Let $e_i = e_{i1}$, $i \in \{1,\dots,n_A\}$, be the basic primitive idempotents of A. We abbreviate $n = n_A$ and $m_i = m_A(i)$ for $i \in \{1,\dots,n\}$.

Since $A = \operatorname{End}_A(Q)$ by the definition of a duality module, it follows from Lemma II.2.1 that each $e_{ij}Q$ is indecomposable. Hence, by Corollary I.8.21, $\operatorname{soc}(e_{ij}Q)$ is a simple right A-module, which implies that, for each $i \in \{1,\dots,n\}$, there is an element i' in $\{1,\dots,n\}$ such that $\operatorname{soc}(e_iQ) \xrightarrow{\sim} \operatorname{top}(e_{i'}A)$. Moreover, $\operatorname{soc}(e_iQ) \ncong \operatorname{soc}(e_jQ)$ for $i,j \in \{1,\dots,n\}$, $i \neq j$, and so $\operatorname{top}(e_{i'}A) \ncong \operatorname{top}(e_{j'}A)$. Thus the correspondence $i \mapsto i'$ is a permutation ν_Q on $\{1,\dots,n\}$ such that we have an isomorphism

$$\operatorname{soc}\big(e_{\nu_Q(i)}Q\big) \cong \operatorname{top}(e_iA)$$

in $\operatorname{mod} A$ for all $i \in \{1,\dots,n\}$. We call ν_Q the *Nakayama permutation* of Q. In case A is selfinjective, the A-bimodule ${}_AA_A$ is a duality A-bimodule and ν_Q for $Q = A$ is nothing other than the Nakayama permutation defined in Section IV.6.

Lemma 3.1. *Let A be a finite dimensional K-algebra over a field K and*

$$0 \longrightarrow Q \xrightarrow{\omega} T \xrightarrow{\varrho} A \longrightarrow 0$$

be a Hochschild extension of A by a duality A-bimodule Q. Let $\boldsymbol{e}$ be an idempotent of T and $\varrho(\boldsymbol{e}) = e$. Then there are canonical isomorphisms in $\operatorname{mod} T$*:*

(i) $\boldsymbol{e}T/\boldsymbol{e}\,\omega(Q) \xrightarrow{\sim} eA$.

(ii) $\operatorname{top}(\boldsymbol{e}T) \xrightarrow{\sim} \operatorname{top}(eA)$.

(iii) $\operatorname{soc}(\boldsymbol{e}T_T) = \operatorname{soc}\big(\boldsymbol{e}\,\omega(Q)_T\big) \xrightarrow{\sim} \operatorname{soc}(eQ_A)$.

(iv) $T\boldsymbol{e}/\omega(Q)\boldsymbol{e} \xrightarrow{\sim} Ae$.

(v) $\operatorname{top}(T\boldsymbol{e}) \xrightarrow{\sim} \operatorname{top}(Ae)$.

(vi) $\operatorname{soc}({}_TT\boldsymbol{e}) = \operatorname{soc}\big(\omega(Q)\boldsymbol{e}\big) \xrightarrow{\sim} \operatorname{soc}(Qe)$.

Proof. (i) It is clear from the definition of e.

(ii) By Lemma I.3.17, there is an isomorphism $T/\operatorname{rad} T \cong A/\operatorname{rad} A$ in $\operatorname{mod} T$ which is naturally induced by ϱ. Hence, it implies an isomorphism $\boldsymbol{e}T/\boldsymbol{e}\operatorname{rad} T \cong eA/e\operatorname{rad} A$, or equivalently, $\operatorname{top}(\boldsymbol{e}T) \cong \operatorname{top}(eA)$ in $\operatorname{mod} T$.

(iii) Let $Q' = \omega(Q)$. Then $\boldsymbol{e}Q \xrightarrow{\sim} \boldsymbol{e}Q'$ as right T-modules. By the definition of Q as a T-module, T-submodules of Q and A-submodules of Q coincide. Hence, $\operatorname{soc}(\boldsymbol{e}Q_T) = \operatorname{soc}(\boldsymbol{e}Q_A)$ and then $\operatorname{soc}(\boldsymbol{e}Q') \cong \operatorname{soc}(\boldsymbol{e}Q) \cong \operatorname{soc}(eQ)$. Next we show that $\operatorname{soc}(\boldsymbol{e}T) = \operatorname{soc}(\boldsymbol{e}Q')$. Obviously, $\operatorname{soc}(\boldsymbol{e}Q') \subseteq \operatorname{soc}(\boldsymbol{e}T)$. Let S be a simple right T-submodule of $\boldsymbol{e}T$. We claim that $S \subseteq \operatorname{soc}(\boldsymbol{e}Q')$. Let $S = sT$ for some $0 \neq \boldsymbol{e}s = s \in S$. Since ${}_AQ$ is faithful, $\varrho(s)Q \neq 0$, while $sQ = \varrho(s)Q$ by definition of left T-module Q. Hence $0 \neq \omega(sQ) = sQ' \subseteq sT = S$. This implies that $S = sQ'$, because S is simple, and so $S \subseteq \boldsymbol{e}Q'$ as claimed.

The isomorphisms (iv), (v), and (vi) are shown similarly. □

The following theorem from [Y2] is essential in this chapter and will be used freely without mention.

Theorem 3.2. *Let A be a finite dimensional K-algebra over a field K and Q a duality A-bimodule. Then any Hochschild extension algebra T of A by Q is a selfinjective algebra, and the Nakayama permutation of T coincides with the Nakayama permutation of Q.*

Proof. Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be a complete set of basic pairwise orthogonal primitive idempotents of T. Then $e_1 = \varrho(\boldsymbol{e}_1), \dots, e_n = \varrho(\boldsymbol{e}_n)$ is a complete set of basic pairwise orthogonal primitive idempotents of A. In order to show that T is selfinjective, by Theorem IV.6.1, it is enough to show that there is a permutation ν_T on $\{1, \dots, n\}$ such that $\operatorname{top}(\boldsymbol{e}_i T) \cong \operatorname{soc}(\boldsymbol{e}_{\nu_T(i)}T)$ for any $i \in \{1, \dots, n\}$. In fact, this follows from Lemma 3.1, because letting $\nu = \nu_Q$, we have isomorphisms $\operatorname{top}(\boldsymbol{e}_i T) \cong \operatorname{top}(e_i A)$ and $\operatorname{soc}(\boldsymbol{e}_{\nu(i)}T) \cong \operatorname{soc}(e_{\nu(i)}Q)$ in $\operatorname{mod} T$. Further, $\operatorname{top}(e_i A) \cong \operatorname{soc}(e_{\nu(i)}Q)$ in $\operatorname{mod} A$, because Q is a duality module. Hence, $\operatorname{top}(\boldsymbol{e}_i T) \cong \operatorname{soc}(\boldsymbol{e}_{\nu(i)}T)$ for any $i \in \{1, \dots, n\}$. Therefore, T is selfinjective and $\nu_T = \nu_Q$, and so ν_T is the Nakayama permutation of T. □

Lemma 3.3. *Let A be a finite dimensional K-algebra over a field K and Q be a duality A-bimodule. Then the following conditions are equivalent:*

(i) $\operatorname{soc}(Q_A) \cong \operatorname{top}(A_A)$ *in* $\operatorname{mod} A$.

(ii) $\operatorname{soc}({}_AQ) \cong \operatorname{top}({}_AA)$ *in* $\operatorname{mod} A^{\mathrm{op}}$.

(iii) $Q \cong D(A)$ *in* $\operatorname{mod} A$.

(iv) $Q \cong D(A)$ *in* $\operatorname{mod} A^{\mathrm{op}}$.

Proof. Since, by Lemma I.8.22, we have $\operatorname{top}(A_A) \cong \operatorname{soc} D(A)_A$ in $\operatorname{mod} A$ and $\operatorname{top}({}_AA) \cong \operatorname{soc}({}_AD(A))$ in $\operatorname{mod} A^{\mathrm{op}}$, the equivalences (i) ⇔ (iii) and (i) ⇔ (iv) hold, because Q and $D(A)$ are injective as left and right A-modules.

(iii) ⇒ (iv) Assume that $Q \cong D(A)$ in $\operatorname{mod} A$. Then there is an automorphism σ of the algebra A such that $Q \cong {}_{\sigma}D(A)$ as A-bimodules, and hence $Q \cong D(A)_{\sigma^{-1}}$ as A-bimodules (see Section II.7). In particular, we have an isomorphism $Q \cong D(A)$ in $\operatorname{mod} A^{\mathrm{op}}$.

(iv) $\Rightarrow$ (iii) Assume that $Q \cong D(A)$ in mod A^{op}. Then there is an algebra automorphism σ of A such that $Q \cong D(A)_\sigma$ as A-bimodules. Then $Q \cong {}_{\sigma^{-1}}D(A)$ as right A-modules. Hence, there is an isomorphism $Q \cong D(A)$ in mod A^{op}. ☐

We note that an algebra A is Frobenius if and only if $A \cong D(A)$ in mod A, or in mod A^{op}, by the Brauer–Nesbitt–Nakayama Theorem IV.2.1, and in this case, $A \cong D(A)_{\nu_A}$ as A-bimodules for a Nakayama automorphism ν_A of A (see Proposition IV.3.15). In view of this fact, we call Q satisfying the equivalent conditions in Lemma 3.3 a *Frobenius A-bimodule*.

Now let Q be a Frobenius A-bimodule. Then there is an automorphism σ of the algebra A such that there are isomorphisms $Q \cong D(A)_\sigma \cong {}_{\sigma^{-1}}D(A)$ as A-bimodules (see Section II.7). In particular, for the basic primitive idempotents $e_i = e_{i1}$, $i \in \{1, \dots, n\}$, we have a permutation π on $\{1, \dots, n_A\}$ with $\sigma(e_i) = e_{\pi(i)}$, and hence $e_{\pi(i)}Q \cong e_{\pi(i)}D(A_{\sigma^{-1}}) = D(A\sigma^{-1}(e_{\pi(i)})) = D(Ae_i)$ in mod A. It follows that there are isomorphisms $\mathrm{soc}(e_{\pi(i)}Q) \cong \mathrm{soc}(D(Ae_i)) \cong \mathrm{top}(e_iA)$ in mod A. Thus π is the Nakayama permutation ν_Q of Q. We denote σ by ν_Q again, so we have

$$Q \cong D(A)_{\nu_Q}$$

as A-bimodules. Finally, it should be noted that an algebra automorphism σ of A with $Q \cong D(A)_\sigma$ is uniquely determined by Q up to inner automorphism, by Proposition II.7.16.

The property of the Nakayama automorphism of a Frobenius algebra A that it induces a bimodule isomorphism between the duality modules A and $D(A)$ (Proposition IV.3.15) was first observed by Morita [M]. Lemma 3.3 is its generalization from algebras to bimodules.

A Hochschild extension of a basic finite dimensional K-algebra A over a field K by a duality A-bimodule Q is said to be *basic*. In this case, by Proposition II.7.16, Q is a Frobenius A-bimodule. The following proposition shows that any Hochschild extension has a basic Hochschild extension.

Proposition 3.4. *Let* $0 \to Q \xrightarrow{\omega} T \xrightarrow{\rho} A \to 0$ *be a Hochschild extension of a finite dimensional K-algebra A over a field K by a duality A-bimodule Q. Then, for a basic idempotent $\boldsymbol{e}$ of T and $e = \rho(\boldsymbol{e})$ of A, eQe is a duality eAe-bimodule and the canonical exact sequence* $0 \to eQe \xrightarrow{\omega'} \boldsymbol{e}T\boldsymbol{e} \xrightarrow{\rho'} eAe \to 0$ *is a Hochschild extension of eAe by eQe, where ω' is the restriction of ω to eQe and ρ' is the restriction of ρ to $\boldsymbol{e}T\boldsymbol{e}$.*

Proof. Note that $\boldsymbol{e}T\boldsymbol{e}$ and eAe are the basic algebras of T and A, respectively. Obviously the sequence $0 \to eQe \xrightarrow{\omega'} \boldsymbol{e}T\boldsymbol{e} \xrightarrow{\rho'} eAe \to 0$ is exact, ρ' is a surjective algebra homomorphism and ω' is an eAe-bimodule homomorphism with $\omega'(eQe)^2 = 0$. Hence, it suffices to show that eQe is a duality eAe-bimodule. For this, we shall show that the eAe-bimodule eQe has the double centralizer property. That is, the correspondences $r_{eAe}\colon eAe \to \mathrm{End}_{(eAe)^{\mathrm{op}}}(eQe)^{\mathrm{op}}$

and $\ell_{eAe}\colon eAe \to \operatorname{End}_{eAe}(eQe)$ are bijective, where $r_{eAe}(eae)$ and $\ell_{eAe}(eae)$ are right and left multiplication maps such that $r_{eAe}(eae)(u) = ueae$ and $\ell_{eAe}(eae)(u) = eaeu$ for $u \in eQe$.

Consider the functors $F = - \otimes_{eAe} eA\colon \operatorname{mod} eAe \to \operatorname{mod} A$ and $G = \operatorname{Hom}_{A^{\mathrm{op}}}(Ae, -)\colon \operatorname{mod} A^{\mathrm{op}} \to \operatorname{mod}(eAe)^{\mathrm{op}}$, which are equivalences by Theorems II.6.8 and II.6.16. The composition $H = G \circ \operatorname{Hom}_A(-, Q) \circ F\colon \operatorname{mod} eAe \to \operatorname{mod}(eAe)^{\mathrm{op}}$ is then a duality, because $\operatorname{Hom}_A(-, Q)\colon \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ is a duality. Observe that we have isomorphisms

$$H(eAe) \cong \big(G \circ \operatorname{Hom}_A(-, Q)\big)(eA) \cong G\big(\operatorname{Hom}_A(eA, Q)\big) \cong G(Qe) \cong eQe$$

in $\operatorname{mod}(eAe)^{\mathrm{op}}$. Hence, we obtain isomorphisms

$$eAe \cong \operatorname{End}_{eAe}(eAe) \cong \big(\operatorname{End}_{(eAe)^{\mathrm{op}}}(eQe)\big)^{\mathrm{op}}$$

of K-algebras, and so $\dim_K eAe = \dim_K \operatorname{End}_{(eAe)^{\mathrm{op}}}(eQe)^{\mathrm{op}}$. On the other hand, eQe is a faithful eAe-module, because so is Q as a right A-module, or equivalently, the correspondence $r_{eAe}\colon eAe \to \operatorname{End}_{(eAe)^{\mathrm{op}}}(eQe)^{\mathrm{op}}$ is injective. Consequently, we conclude that r_{eAe} is bijective. Similarly, by using the functors $F' = \operatorname{Hom}_A(eA, -)\colon \operatorname{mod} A \to \operatorname{mod} eAe$, $G' = Ae \otimes_{eAe} -\colon \operatorname{mod}(eAe)^{\mathrm{op}} \to \operatorname{mod} A^{\mathrm{op}}$ and the induced duality $H' = F' \circ \operatorname{Hom}_{A^{\mathrm{op}}}(-, Q) \circ G'\colon \operatorname{mod}(eAe)^{\mathrm{op}} \to \operatorname{mod} eAe$, we prove that the correspondence $\ell_{eAe}\colon eAe \to \operatorname{End}_{eAe}(eQe)$ is bijective. Thus we have proved that the eAe-bimodule eQe has the double centralizer property. Moreover, since $H(eAe) \cong eQe$ in $\operatorname{mod}(eAe)^{\mathrm{op}}$ and $H'(eAe) \cong eQe$ in $\operatorname{mod} eAe$, the functors H and H' are naturally isomorphic to the functors $\operatorname{Hom}_{eAe}(-, eQe)$ and $\operatorname{Hom}_{(eAe)^{\mathrm{op}}}(-, eQe)$. Therefore, by the Morita–Azumaya Duality Theorem II.7.12, eQe is a duality eAe-bimodule. □

Lemma 3.5. *Let A be a finite dimensional K-algebra over a field K and Q a duality A-bimodule. The following statements are equivalent:*

- (i) *All Hochschild extension algebras of A by Q are Frobenius algebras.*
- (ii) *There is a Hochschild extension Frobenius algebra of A by Q.*
- (iii) *Q is a Frobenius A-bimodule.*

Proof. (i) ⇒ (ii) Take the trivial extension algebra $\mathrm{T}(A) = A \ltimes Q$, which is a Frobenius algebra by assumption.

(ii) ⇒ (iii) Take a Hochschild extension Frobenius algebra T of A by Q. Since there is an isomorphism $T \xrightarrow{\sim} D(T)$ in $\operatorname{mod} T$, by Theorem IV.2.1, we have isomorphisms $\operatorname{soc}(T_T) \cong D(\operatorname{top}({}_T T)) \cong \operatorname{top}(T_T)$ in $\operatorname{mod} T$, and hence in $\operatorname{mod} A$. Thus $\operatorname{soc}(Q_A) \cong \operatorname{soc}(T_T) \cong \operatorname{top}(A_A)$ in $\operatorname{mod} A$, by Lemma 3.1. It follow from Lemma 3.3 that Q is a Frobenius A-bimodule.

(iii) ⇒ (i) Assume that Q is a Frobenius A-bimodule and $\theta\colon Q \to D(A)$ is an isomorphism in $\operatorname{mod} A$. Let T be a Hochschild extension algebra of A by Q.

Then $\operatorname{soc}(Q_A) \cong \operatorname{soc} D(A)_A \cong \operatorname{top}(A_A)$ in $\operatorname{mod} T$, while $\operatorname{soc}(T_T) \cong \operatorname{soc}(Q_A)$ and $\operatorname{top}(T_T) \cong \operatorname{top}(A_A)$ in $\operatorname{mod} T$. Therefore, we have $\operatorname{soc}(T_T) \cong \operatorname{top}(T_T)$ in $\operatorname{mod} T$, and it follows from Lemma 3.3 that $T \cong D(T)$ in $\operatorname{mod} T$, that is, T is a Frobenius algebra, by Theorem IV.2.1. $\square$

Since any duality bimodule over a basic algebra is Frobenius, the following assertion is an immediate consequence of Lemma 3.5.

Corollary 3.6. *Let A be a finite dimensional basic K-algebra over a field K. Then any Hochschild extension algebra of A by a duality A-bimodule is a Frobenius algebra.*

Let A and Λ be finite dimensional K-algebras over a field K and $\rho\colon \Lambda \to A$ a homomorphism of K-algebras. By regarding $D(A)$ as a Λ-bimodule through ρ, that is, $D(A) = {}_{\rho}D(A)_{\rho}$, the K-linear map $D(\rho)\colon D(A) \to D(\Lambda)$ is a homomorphism of Λ-bimodules. In fact, for $x, y, z \in \Lambda$ and $u \in D(A)$, we have $D(\rho)(yuz)(x) = (\rho(y)u\rho(z))(\rho(x)) = u(\rho(z)\rho(x)\rho(y)) = u(\rho(zxy)) = (y(u\rho)z)(x)$, which implies that $D(\rho)(yuz) = y(D(\rho)(u))z$. Further, $D(\rho)$ has the following property.

Lemma 3.7. *Let A and Λ be finite dimensional K-algebras over a field K and $\rho\colon \Lambda \to A$ a K-algebra homomorphism. Moreover, let σ be a K-algebra automorphism of A and τ a K-algebra automorphism of Λ. The following conditions are equivalent:*

(i) *$D(\rho)\colon D(A)_{\sigma} \to D(\Lambda)_{\tau}$ is a homomorphism in* $\operatorname{mod} \Lambda$.

(ii) *$\sigma\rho = \rho\tau$.*

Proof. For any $u \in D(A)$, $x, y \in \Lambda$, we have

$$\begin{aligned}\big(D(\rho)(ux)\big)(y) &= \big(D(\rho)u\sigma(\rho(x))\big)(y) = \big(u\sigma(\rho(x))\rho\big)(y) = u\big(\sigma(\rho(x))\rho(y)\big),\\ \big(D(\rho)(u)x\big)(y) &= \big(u\rho\big)(\tau(x)y\big) = u\big(\rho(\tau(x)y)\big) = u\big(\rho(\tau(x))\rho(y)\big).\end{aligned}$$

Hence, $D(\rho)$ is a homomorphism from $D(A)_{\sigma}$ to $D(\Lambda)_{\tau}$ in $\operatorname{mod} \Lambda$ if and only if $u(\sigma(\rho(x))\rho(y)) = u(\rho(\tau(x))\rho(y))$ for all $u \in D(A)$, $x, y \in \Lambda$. This is equivalent to saying that $\sigma(\rho(x)\rho(y)) = \rho(\tau(x))\rho(y)$ for all $x, y \in \Lambda$, that is, $\sigma(\rho(x)) = \rho(\tau(x))$ for all $x \in \Lambda$, or equivalently, $\sigma\rho = \rho\tau$. $\square$

Let $0 \to Q \xrightarrow{\omega} T \xrightarrow{\rho} A \to 0$ be a Hochschild extension of a finite dimensional K-algebra A over a field K by a duality A-bimodule Q. Assume that Q is a Frobenius A-bimodule with a Nakayama automorphism ν_Q. Then T is a Frobenius algebra, by Lemma 3.5. A Nakayama automorphism ν_T of T is said to be an *extension* of ν_Q if there is a commutative diagram of T-bimodule homomorphisms

$$\begin{tikzcd}
Q \arrow[r, "\omega"] \arrow[d, "\theta_Q"'] & T \arrow[d, "\theta_T"] \\
D(A)_{\nu_Q} \arrow[r, "D(\rho)"] & D(T)_{\nu_T} ,
\end{tikzcd}$$

where θ_Q and θ_T are T-bimodule isomorphisms.

Lemma 3.8. *Let* $0 \to Q \overset{\omega}{\to} T \overset{\rho}{\to} A \to 0$ *be a Hochschild extension of a finite dimensional* K*-algebra* A *over a field* K *by a Frobenius* A*-bimodule* Q *with a Nakayama automorphism* ν_Q*. Then* ν_Q *extends to a Nakayama automorphism* ν_T *of* T*. Moreover, the following statements hold:*

(i) $\nu_Q \rho = \rho \nu_T$.

(ii) ν_Q *is inner, if so is* ν_T.

Proof. Let $\theta_Q \colon Q \to D(A)_{\nu_Q}$ be an A-bimodule isomorphism. By the injectivity of $D(T)$ in $\operatorname{mod} T^{\mathrm{op}}$, $D(\rho)\theta_Q \colon Q \to D(T)$ extends through ω to a homomorphism $\theta_T \colon T \to D(T)$ in $\operatorname{mod} T^{\mathrm{op}}$, that is, $D(\rho)\theta_Q = \theta_T \omega$. First, we claim that θ_T is an isomorphism in $\operatorname{mod} T^{\mathrm{op}}$. Indeed, restricting ω and $D(\rho)$ to the socles of the domains, in view of Lemma 3.1(iii), we have isomorphisms $\operatorname{soc}({}_A Q) \to \operatorname{soc}({}_T T)$ and $\operatorname{soc}({}_A D(A)) \to \operatorname{soc}({}_T D(T))$ in $\operatorname{mod} T^{\mathrm{op}}$. Hence, the restriction of θ_T to $\operatorname{soc}({}_T T)$ is an isomorphism $\operatorname{soc}({}_T T) \overset{\sim}{\to} \operatorname{soc}({}_T D(T))$ in $\operatorname{mod} T^{\mathrm{op}}$. It follows that θ_T is a monomorphism, so an isomorphism, because $\dim_K T = \dim_K D(T)$. The isomorphism θ_T induces an automorphism ν_T of T, a Nakayama automorphism, such that there is a T-bimodule isomorphism $\theta_T \colon T \to D(T)_{\nu_T}$.

Next, we shall show that $D(\rho) \colon D(A)_{\nu_Q} \to D(T)_{\nu_T}$ is a homomorphism in $\operatorname{mod} T$. To say that θ_T is a right T-module homomorphism is equivalent to saying that $\theta_T r_t = r_{\nu_T(t)} \theta_T$ for all $t \in T$, where r_t and $r_{\nu_T(t)}$ are right multiplication maps of t and $r_{\nu_T(t)}$. Also, from the right A-module homomorphism $\theta_Q \colon Q \to D(A)_{\nu_Q}$, we have $\theta_Q r_a = r_{\nu_Q(a)} \theta_Q$ for all $a \in A$. Now, consider the following diagram, for $t \in T$ and $a = \rho(t) \in A$:

$$\begin{tikzcd}
Q \arrow[rr, "\omega"] \arrow[dr, "\theta_Q"] \arrow[dd, "r_a"'] & & T \arrow[dr, "\theta_T"] \arrow[dd, "r_t"] & \\
& D(A)_{\nu_Q} \arrow[rr, "D(\rho)"] \arrow[dd, "r_{\nu_Q(a)}"'] & & D(T)_{\nu_T} \arrow[dd, "r_{\nu_T(t)}"] \\
Q \arrow[rr, "\omega"] \arrow[dr, "\theta_Q"] & & T \arrow[dr, "\theta_T"] & \\
& D(A)_{\nu_Q} \arrow[rr, "D(\rho)"] & & D(T)_{\nu_T} .
\end{tikzcd}$$

Observe that all squares of the above diagram, except the front square, are commutative. Consequently,

$$r_{\nu_T(t)}D(\rho)\theta_Q = D(\rho)r_{\nu_Q(a)}\theta_Q,$$

which implies that $r_{\nu_T(t)}D(\rho) = D(\rho)r_{\nu_Q(a)}$, because θ_Q is an isomorphism. Thus we conclude that $D(\rho)\colon D(A)_{\nu_Q} \to D(T)_{\nu_T}$ is a T-bimodule isomorphism. The statement (i) follows now from Lemma 3.7. In order to show (ii), assume that ν_T is an inner automorphism of T, that is, $\nu_T(x) = txt^{-1}$ for all $x \in T$, and some invertible element $t \in T$. Let $a = \rho(t) \in A$. Then a is invertible in A. Moreover, every element $y \in A$ is of the form $\varrho(x)$ for some $x \in T$. Applying (i), we have $\nu_Q(y) = \nu_Q(\rho(x)) = \rho\nu_T(x) = \rho(txt^{-1}) = aya^{-1}$. This shows that ν_Q is an inner automorphism of A. □

We have shown in Example IV.2.7 that the trivial extension algebra $\mathrm{T}(A)$ of a finite dimensional K-algebra A is a symmetric Hochschild extension algebra of A by the standard duality module $\mathrm{Hom}_K(A, K)$. The following theorem asserts that the symmetric Hochschild extension algebras occur only for the Hochschild extension algebras by the standard duality module.

Theorem 3.9. *Let A be a finite dimensional K-algebra over a field K and Q a duality A-bimodule. Then the following statements are equivalent:*

(i) *There is a symmetric Hochschild extension algebra of A by Q.*

(ii) *The trivial extension algebra $\mathrm{T}_Q(A) = A \ltimes Q$ is symmetric.*

(iii) $\mathrm{Hom}_A(-, Q) \cong \mathrm{Hom}_K(-, K)$ *as contravariant functors between* mod A *and* mod A^{op}.

(iv) $Q \cong \mathrm{Hom}_K(A, K)$ *as A-bimodules.*

Proof. The implication (ii) ⇒ (i) is trivial.

(i) ⇒ (iv) Let $0 \to Q \xrightarrow{\omega} T \xrightarrow{\rho} A \to 0$ be a Hochschild extension of A by Q, and assume that T is symmetric. Since T is Frobenius, by Lemma 3.8, there are Nakayama automorphisms ν_T of T and ν_Q of Q such that ν_T is an extension of ν_Q. By the assumption on T, ν_T is inner, and hence it follows from Lemma II.7.15 that $Q \cong D(A)_{\nu_Q} \cong D(A)$ as A-bimodules.

(iv) ⇒ (iii) Assume that $u\colon Q \to \mathrm{Hom}_K(A, K)$ is an isomorphism of A-bimodules. By Corollary II.7.14, there is an isomorphism

$$\psi\colon \mathrm{Hom}_A\big(-, \mathrm{Hom}_K(A, K)\big) \longrightarrow \mathrm{Hom}_K(-, K)$$

of contravariant functors between mod A and mod A^{op}. Hence, the composition $\psi\,\mathrm{Hom}_K(-, u)\colon \mathrm{Hom}_A(-, Q) \to \mathrm{Hom}_K(-, K)$ is a natural isomorphism of contravariant functors between mod A and mod A^{op}.

(iii) $\Rightarrow$ (ii) Assume that there is a natural isomorphism $\operatorname{Hom}_A(-, Q) \to \operatorname{Hom}_K(-, K)$ of contravariant functors between $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$. Since the contravariant functor $\operatorname{Hom}_K(-, K)$ is naturally isomorphic to the contravariant functor $\operatorname{Hom}_A(-, \operatorname{Hom}_K(A, K))$, we have a natural isomorphism $\operatorname{Hom}_A(-, Q) \cong \operatorname{Hom}_A(-, \operatorname{Hom}_K(A, K))$ of contravariant functors between $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$. Hence, by Proposition II.7.13, there is an A-bimodule isomorphism $u \colon Q \to \operatorname{Hom}_K(A, K)$. Let $\Phi \colon A \ltimes Q \to \mathrm{T}(A) = A \ltimes \operatorname{Hom}_K(A, K)$ be the K-linear map defined by $\Phi(a, q) = (a, u(q))$ for $a \in A$, $q \in Q$. Obviously, Φ is bijective. We claim that Φ is an algebra homomorphism. Indeed, for all $a, b \in A$ and $q, r \in Q$, we have that

$$\begin{aligned}\Phi\big((a,q)(b,r)\big) &= \Phi(ab, ar + qb) = \big(ab, u(ar+qb)\big)\\ &= \big(ab, au(r) + u(q)b\big) = \big(a, u(q)\big)\big(b, u(r)\big) = \Phi(a,q)\Phi(b,r),\end{aligned}$$

and

$$\Phi(1_A, 0_Q) = \big(1_A, u(0_Q)\big) = (1_A, 0) = 1_{\mathrm{T}(A)}.$$

Therefore, the algebra $A \ltimes Q$ is isomorphic to the trivial extension algebra $\mathrm{T}(A)$, and hence is a symmetric algebra. □

Corollary 3.10. *Let A be a finite dimensional K-algebra over a field K and σ be a K-algebra automorphism of A. Then the following statements are equivalent:*

(i) *The algebra $A \ltimes D(A)_\sigma$ is symmetric.*

(ii) *$A \ltimes D(A)_\sigma \cong A \ltimes D(A)$ as K-algebras.*

(iii) *σ is a inner automorphism.*

Proof. Consider the case when $Q = D(A)_\sigma$ and $\mathrm{T}_Q(A) = A \ltimes D(A)_\sigma$. Then the equivalence (i) $\Leftrightarrow$ (ii) follows from the equivalence (ii) $\Leftrightarrow$ (iii) in Theorem 3.9. Moreover, by Lemma II.7.15, the statement (iii) is equivalent to the statement (iv) in Theorem 3.9. □

As we will see later, the Hochschild extension algebras of a K-algebra A by $\operatorname{Hom}_K(A, K)$ are not always symmetric. But this is the case when A is a hereditary algebra over an algebraically closed field, which will follow from the following stronger statement.

Proposition 3.11. *Let A be the path algebra KQ of a finite acyclic quiver Q over an algebraically closed field K. Then $H^2(A, M) = 0$ for all finite dimensional A-bimodules M.*

Proof. Let $Q = (Q_0, Q_1, s, t)$. Let M be a finite dimensional A-bimodule and T a Hochschild extension algebra of A by M defined by a 2-cocycle $\alpha \colon A \times A \to M$, that is, $T = A \ltimes_\alpha M$. Let $\varrho \colon T \to A$ be the canonical surjection, that is, $\varrho(a, m) =$

a for all $(a, m) \in A \ltimes_\alpha M$. We want to find a homomorphism $\kappa\colon A \to T$ of K-algebras with $\varrho\kappa = 1_A$. Let e_i be a primitive idempotent of A corresponding to a vertex $i \in Q_0$. Since M is nilpotent in T, any set of pairwise orthogonal primitive idempotents of A is lifted to a set of pairwise orthogonal primitive idempotents of T, by Lemma I.3.12. Hence, there is a complete set of pairwise orthogonal primitive idempotents of T, say $\{\boldsymbol{e}_i \mid i \in Q_0\}$, such that $\varrho(\boldsymbol{e}_i) = e_i$. Let $\boldsymbol{e}_i = (e_i, m_i)$, for $i \in Q_0$. Remember that the set of all paths (including the paths of length 0) of Q forms a basis of A as a K-vector space. To find the homomorphism $\kappa\colon A \to T$, we first define a K-linear map κ from A to T. We put $\kappa(e_i) = \boldsymbol{e}_i$ for $i \in Q_0$ and $\kappa(p) = \boldsymbol{e}_{s(p)}(p, 0)\boldsymbol{e}_{t(p)}$ for any path p of length 1 with the source $s(p)$ and the target $t(p)$. For a path p of Q of length $l \geq 2$, say $p\colon a = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \to \ldots \to a_{l-1} \xrightarrow{\alpha_l} a_l = b$, we put

$$\kappa(p) = \kappa(\alpha_1)\kappa(\alpha_2)\cdots\kappa(\alpha_l).$$

Moreover, let $\kappa\left(\sum_{i=1}^m \lambda_i p_i\right) = \sum_{i=1}^m \lambda_i \kappa(p_i)$ for $\lambda_i \in K$ and paths p_i in Q. Then κ is a well defined mapping from A to T and becomes a K-linear map, because any element $a \in A$ is uniquely written as a K-linear combination of paths in Q. To show that κ preserves the multiplication in A it is enough to show that $\kappa(pq) = \kappa(p)\kappa(q)$ for any paths p and q in Q. This is clear in the case when $t(p) \neq s(q)$, because $pq = 0$, $\boldsymbol{e}_{t(p)} \neq \boldsymbol{e}_{s(q)}$, and $\kappa(p)\kappa(q) = \kappa(p)\boldsymbol{e}_{t(p)}\boldsymbol{e}_{s(q)}\kappa(q) = 0$ by definition. In the case when $t(p) = s(q)$, pq is a path of Q and hence the equality $\kappa(pq) = \kappa(p)\kappa(q)$ is obvious by the definition of $\kappa(pq)$. Moreover, $\kappa(1_A) = \kappa(e_1) + \cdots + \kappa(e_n) = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n = 1_T$. Thus we know that κ is a homomorphism of K-algebras. Finally, we have to show that $\varrho\kappa = 1_A$. For this, we have only to show that $\varrho\kappa(p) = p$ for all paths p of length 1. But if p is an arrow in Q with $i = s(p)$ and $j = t(p)$, then

$$\begin{aligned}\varrho\kappa(p) &= \varrho\big(\boldsymbol{e}_i(p, 0)\boldsymbol{e}_j\big) = \varrho\big((e_i, m_i)(p, 0)(e_j, m_j)\big)\\ &= \varrho(e_i p e_j, m) = e_i p e_j = p,\end{aligned}$$

for some $m \in M$. □

Corollary 3.12. *Let K be an algebraically closed field and A be a finite dimensional path algebra KQ of a finite acyclic quiver Q over K. Then any Hochschild extension algebra of A by* $\mathrm{Hom}_K(A, K)$ *is symmetric.*

Proof. Let $M = \mathrm{Hom}_K(A, K)$, and

$$\mathbb{E}\colon \quad 0 \longrightarrow M \xrightarrow{\omega} T \xrightarrow{\varrho} A \longrightarrow 0$$

be a Hochschild extension of A by M. By Proposition 3.11, we have $H^2(A, M) = 0$. Then it follows from Lemma 2.5 that $\mathbb{E}$ is equivalent to the trivial extension of A by M,

$$\mathbb{E}_0\colon \quad 0 \longrightarrow M \longrightarrow \mathrm{T}_M(A) \longrightarrow A \longrightarrow 0.$$

In particular, there is an isomorphism of K-algebras $T \cong \mathrm{T}_M(A)$. Therefore, T is a symmetric algebra. □

If a Hochschild extension T of a finite dimensional K-algebra A over a field K by a duality A-bimodule Q is symmetric, it follows from Theorem 3.9 that $T \cong A \ltimes D(A)$. This implies obviously that the indecomposability of T is passed to A. But this is not the case when T is not symmetric. In the sequel we shall study the indecomposability of Hochschild extension algebras. We start by recalling the definition of a (finite) direct product of algebras.

Let $\{A_i\}$, $i \in \{1, \dots, m\}$, be a finite family of K-algebras over a field K. Its *direct product* is the K-algebra

$$A = \prod_{i=1}^{m} A_i = A_1 \times \cdots \times A_m,$$

which is the Cartesian product with operations defined componentwise:

$$(x_i) + (y_i) = (x_i + y_i), (x_i)(y_i) = (x_i y_i), (x_i)\lambda = (x_i \lambda)$$

for $(x_i), (y_i) \in A$ and $\lambda \in K$. Note that $1_A = (1_{A_i})$. The natural projections $\xi_i \colon A \to A_i$, $i \in \{1, \dots, m\}$, are homomorphisms of K-algebras, while the natural injections $\mu_i \colon A_i \to A$, with $\xi_i(\mu_i(a_i)) = a_i$ and $\xi_j(\mu_i(a_i)) = 0$ for $j \neq i$, are not homomorphisms of K-algebras (in case $m > 1$), as $\mu_i(1_{A_i}) \neq 1_A$. The image of any μ_i is however an ideal of A and, for $I_i = \operatorname{Im} \mu_i$, we have the induced decomposition

$$A = I_1 \oplus \cdots \oplus I_m$$

of A into a direct sum of ideals. Conversely, a K-algebra A being a finite direct sum of ideals, say $A = I_1 \oplus \cdots \oplus I_m$, can be expressed as a direct product of m K-algebras. Explicitly, let $1_A = c_1 + \cdots + c_m$, for $c_i \in I_i$. Then each c_i is a central idempotent of A and I_i becomes a K-algebra A_i with $1_{A_i} = c_i$, for any $i \in \{1, \dots, m\}$. Moreover, there is an isomorphism

$$f_A \colon A \longrightarrow \prod_{i=1}^{m} A_i$$

of K-algebras such that $f_A\left(\sum_i a_i\right) = (a_i)$ for all $\sum_i a_i \in \bigoplus_{i=1}^{m} I_i$. Thanks to this isomorphism, we freely identify I_i with A_i, and write $A = A_1 \oplus \cdots \oplus A_m$, unless any confusion may arise, so that $1_A = 1_{A_1} + \cdots + 1_{A_m}$. Recall that there is a category equivalence $\operatorname{mod} A \to \operatorname{mod} A_1 \times \cdots \times \operatorname{mod} A_m$ by the natural correspondence of objects: $M \mapsto M1_{A_1} \times \cdots \times M1_{A_m}$ for $M \in \operatorname{mod} A$ (see Exercise II.8.11).

Lemma 3.13. *Let A and B be finite dimensional K-algebras over a field K. Assume that A and B are Morita equivalent. Then A is indecomposable if and only if B is indecomposable.*

Proof. Assume that A is decomposable, say $A = A_1 \times A_2$. We shall show that B is decomposable. It follows from Theorem II.6.8 that there is a projective generator P in $\operatorname{mod} A$ with $\operatorname{End}_A(P) \cong B$. Regarding A_1 and A_2 as ideals of A, we have $P = P_1 \oplus P_2$ in $\operatorname{mod} A$, where $P_1 = P1_{A_1}$ and $P_2 = P1_{A_2}$. Then $B \cong \operatorname{End}_A(P) \cong \operatorname{End}_A(P_1) \times \operatorname{End}_A(P_2)$ as K-algebras, which shows that B is decomposable. □

Lemma 3.14. *Assume that a finite dimensional K-algebra A over a field K is a product of K-algebras $A_1, \dots, A_m$. Then the following statements hold:*

(i) *A is a selfinjective algebra if and only if $A_1, \dots, A_m$ are selfinjective algebras.*

(ii) *A is a Frobenius algebra if and only if $A_1, \dots, A_m$ are Frobenius algebras.*

Proof. We leave to the readers the proof of (i).

(ii) By Theorem IV.2.1, an algebra Λ is Frobenius if and only if there is a K-linear map $\lambda\colon \Lambda \to K$ whose kernel does not contain a nonzero right ideal (equivalently, nonzero left ideal). For simplicity, we call such a K-linear map λ a *regular* map. By $\xi_i\colon A \to A_i$ and $\mu_i\colon A_i \to A$, for $i \in \{1, \dots, m\}$, we denote the natural projection and injection, respectively.

Assume that A is a Frobenius algebra and $\lambda\colon A \to K$ is a regular K-linear map. Define K-linear maps $\lambda_i = \lambda\mu_i\colon A_i \to K$, $i \in \{1, \dots, m\}$. Our aim is to show that all λ_i are regular. For this, take a right ideal $L_i \subseteq \operatorname{Ker} \lambda_i$ of A_i, for each $i \in \{1, \dots, m\}$. Then $L = \sum_{i=1}^m \mu_i(L_i)$ is a direct sum of right ideals of A, and applying λ, we obtain $\lambda(L) = \lambda\left(\sum_{i=1}^m \mu_i(L_i)\right) = \sum_{i=1}^m \lambda(\mu_i(L_i)) = \sum_{i=1}^m \lambda_i(L_i) = 0$. By the regularity of λ, it follows that $L = 0$. Hence, $\mu_i(L_i) = 0$, and consequently $L_i = 0$, for any $i \in \{1, \dots, m\}$. Thus all K-linear maps λ_i are regular. Therefore, $A_1, \dots, A_m$ are Frobenius algebras.

Conversely, assume that the K-algebras $A_1, \dots, A_m$ are Frobenius algebras and take regular K-linear maps $\lambda_i\colon A_i \to K$. We define the K-linear map $\lambda\colon A \to K$ such that $\lambda_i = \lambda\mu_i$ for $i \in \{1, \dots, m\}$. Let L be a right ideal of A contained in $\operatorname{Ker} \lambda$. We show that $L = 0$. Note that $L_i = \xi_i(L) \subseteq A_i$ is a right ideal of A_i and $\mu_i(L_i) = L1_{A_i}$. Then $\lambda_i(L_i) = \lambda\mu_i(L_i) = \lambda(L1_{A_i}) = 0$, because $L1_{A_i} \subseteq L$ and $\lambda(L) = 0$. Hence, by the regularity of λ_i, we have $L_i = 0$, for any $i \in \{1, \dots, m\}$, and so $L = \sum_{i=1}^m \mu_i(L_i) = 0$, as required. □

Assume that a finite dimensional K-algebra A over a field K is a direct product $A = \prod_{i=1}^m A_i$ of indecomposable K-algebras A_i, $i \in \{1, \dots, m\}$. Such a decomposition is called the *block decomposition* of A and each A_i is called a *block* of the algebra A (see Section I.3). In this case, A is a direct sum of indecomposable ideals, say $A = I_1 \oplus \dots \oplus I_m$. Hence, for an automorphism $\sigma \in \operatorname{Aut}(A)$, $A = \sigma(I_1) \oplus \dots \oplus \sigma(I_m)$ and all $\sigma(I_i)$ are indecomposable ideals of A. Since the set of those indecomposable ideals I_i is uniquely determined (Proposition I.3.16), it follows that there exists a permutation π_σ on $\{1, \dots, m\}$

such that $\sigma(I_i) = I_{\pi_\sigma(i)}$ for $i \in \{1, \dots, m\}$, which is uniquely determined by σ up to inner automorphism, because π_σ is identity if σ is inner. By identification of I_i with A_i, we may write

$$\sigma(A_i) = A_{\pi_\sigma(i)}, \text{ for } i \in \{1, \dots, m\}.$$

We call π_σ the *block permutation* of σ. Moreover, π_σ is written as a product $\pi_\sigma = \pi_1 \cdots \pi_r$ of pairwise disjoint cyclic permutations $\pi_1, \dots, \pi_r$ on $\{1, \dots, m\}$. This means that, for some pairwise disjoint subsets $\Omega_1, \dots, \Omega_r$ of $\{1, \dots, m\}$, π_i is a cyclic permutation on Ω_i. We denote $\Omega_i = \operatorname{supp}(\pi_i)$ and call Ω_i the support of π_i.

Now consider a Hochschild extension of $A = \prod_{i=1}^m A_i$ by a Frobenius A-bimodule Q,

$$0 \longrightarrow Q \xrightarrow{\omega} T \xrightarrow{\rho} A \longrightarrow 0,$$

and take a complete set of pairwise orthogonal idempotents c_i of T with $\rho(c_i) = 1_{A_i}$ for $i \in \{1, \dots, m\}$, where $1_A = 1_{A_1} + \cdots + 1_{A_m}$ in the direct sum of right ideals $A = A_1 \oplus \cdots \oplus A_m$. We set

$$T_{\pi_i} = \sum_{k \in \Omega_i} c_k T \quad \text{and} \quad A_{\pi_i} = \prod_{k \in \Omega_i} A_k,$$

for any $i \in \{1, \dots, r\}$. Any algebra A_{π_i} is regarded as an ideal of A, and we have $A = A_{\pi_i} \oplus \cdots \oplus A_{\pi_r}$. Obviously, $T = T_{\pi_1} \oplus \cdots \oplus T_{\pi_r}$ as right T-modules, because the permutations π_i, $i \in \{1, \dots, r\}$, have disjoint supports. In the case when all T_{π_i} are ideals of T, each T_{π_i} is regarded as a K-algebra. The next lemma shows that this is always the case.

Lemma 3.15. *Let* $0 \to Q \overset{\omega}{\to} T \overset{\rho}{\to} A \to 0$ *be a Hochschild extension of a finite dimensional K-algebra A over a field K by a Frobenius A-bimodule Q, with a Nakayama automorphism ν_Q, and let π_{ν_Q} be its block permutation. Moreover, let $\pi_{\nu_Q} = \pi_1 \cdots \pi_r$ be a decomposition of π_{ν_Q} as a product of pairwise disjoint cyclic permutations, and $Q_{\pi_i} = 1_{A_{\pi_i}} Q 1_{A_{\pi_i}}$, for $i \in \{1, \dots, r\}$. Then the following statements hold:*

(i) *T_{π_i} is an indecomposable ideal of T, for any $i \in \{1, \dots, r\}$.*

(ii) *$Q = Q_{\pi_1} \oplus \cdots \oplus Q_{\pi_r}$ and each Q_{π_i} is a duality A_{π_i}-bimodule, for any $i \in \{1, \dots, r\}$.*

(iii) *For any $i \in \{1, \dots, r\}$, the sequence*

$$0 \longrightarrow Q_{\pi_i} \xrightarrow{\omega_i} T_{\pi_i} \xrightarrow{\rho_i} A_{\pi_i} \longrightarrow 0$$

is a Hochschild extension of A_{π_i} by Q_{π_i}, where ω_i and ρ_i are the restrictions of ω and ρ, and there exists a K-linear isomorphism $f_Q \colon Q \to$

$\prod_{i=1}^{r} Q_{\pi_i}$ with $f_Q(txs) = f_T(t)f_Q(x)f_T(s)$, for all $x \in Q$ and $s, t \in T$, such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "\omega"] \arrow[d, "f_Q"] & T \arrow[r, "\rho"] \arrow[d, "f_T"] & A \arrow[r] \arrow[d, "f_A"] & 0 \\
0 \arrow[r] & \prod_{i=1}^{r} Q_{\pi_i} \arrow[r, "\prod \omega_i"] & \prod_{i=1}^{r} T_{\pi_i} \arrow[r, "\prod \rho_i"] & \prod_{i=1}^{r} A_{\pi_i} \arrow[r] & 0.
\end{tikzcd}$$

Proof. Take a set of pairwise orthogonal idempotents $c_1, \dots, c_m$ of T such that $1_T = c_1 + \cdots + c_m$ and $\rho(c_i) = 1_{A_i}$, and we write $T_i = c_i T$, $\nu = \nu_Q$ and $\pi = \pi_{\nu_Q}$. For simplicity of notation, we may identify Q with $\omega(Q)$ and assume that ω is the inclusion homomorphism.

(i) First we shall show that all T_{π_i} are ideals of T. It follows from Lemma 3.1 that, for each $i \in \{1, \dots, m\}$,

$$\begin{aligned}
& c_i T / c_i Q \cong A_i, \\
& \quad c_i Q \cong c_i ({}_{\nu^{-1}} \operatorname{Hom}_K(A, K)) \\
& \qquad = \operatorname{Hom}_K \left(A\nu^{-1}(\rho(c_i))\right., K) = \operatorname{Hom}_K \left(A\nu^{-1}(1_{A_i}), K\right) \quad \text{and} \\
& \nu^{-1}(1_{A_i}) = 1_{A_{\pi^{-1}(i)}},
\end{aligned}$$

and so $c_i Q \cong \operatorname{Hom}_K(A_{\pi^{-1}(i)}, K)$ in $\operatorname{mod} A$. Therefore, $c_i T c_j \neq 0$ for some $i, j \in \{1, \dots, m\}$ implies that

$$\begin{aligned}
& 0 \neq A_i c_j = A_i 1_{A_j} \quad \text{or} \\
& 0 \neq c_i Q c_j \cong \operatorname{Hom}_K(c_j A_{\pi^{-1}(i)}, K) = \operatorname{Hom}_K(1_{A_j} A_{\pi^{-1}(i)}, K),
\end{aligned}$$

so that $j = i$ or $j = \pi^{-1}(i)$. Hence, $(c_j t c_k) c_i T \neq 0$ for some $t \in T$ implies that $k = i$ and $j = \pi(i)$, as $c_j T c_i \neq 0$. This proves that $T_{\pi_i} = \bigoplus_{k \in \pi_i} c_k T$ is a left ideal of T and so an ideal of T.

In order to show indecomposability of all T_{π_i}, we consider without loss of generality, only the case T_{π_1}. Assume, to the contrary, that $T_{\pi_1} = X \oplus Y$ is a direct sum of nonzero ideals of T_{π_1}. Clearly, $T_{\pi_1}/T_{\pi_1}Q \cong X/XQ \oplus Y/YQ$, and so $A_{\pi_1} = \rho(T_{\pi_1}) = \rho(X) \oplus \rho(Y)$. Since $\rho(X)$ and $\rho(Y)$ are nonzero ideals of A_{π_1}, by the uniqueness of the block decomposition of A, $\rho(X)$ and $\rho(Y)$ are direct sums of some of $A_i, i \in \operatorname{supp}(\pi_1)$. Then there exist some direct summands A_i of $\rho(X)$ and A_j of $\rho(Y)$ such that $j = \pi_1(i)$ or $i = \pi_1(j)$. We assume $j = \pi_1(i)$, that is, $\nu_Q(A_i) = A_j$. Remember that T_i and T_j are then direct summands of X and Y, respectively. By Lemma 3.8, we can take a Nakayama automorphism ν_T of T as an extension of ν_Q satisfying $\nu_Q \rho = \rho \nu_T$. We claim that $\nu_T(T_i) \cong T_j$. Indeed, $\rho(\nu_T(T_i)) = \nu_Q \rho(T_i) = \nu_Q(A_i) = A_j$ and $\rho(T_j) = A_j$, which implies that $\rho^{-1}(A_j) = \nu_T(T_i) + Q = T_j + Q$, for $\operatorname{Ker} \rho = Q$. Moreover, we have $\nu_T(T_i) \cap Q = \nu_T(T_i)Q$ and $T_j \cap Q = T_j Q$, because $\nu_T(T_i)$ and T_j are projective right ideals of T, and hence $\rho^{-1}(A_j)/Q \cong \nu_T(T_i)/\nu_T(T_i)Q$ and $\rho^{-1}(A_j)/Q \cong T_j/T_jQ$. The obtained isomorphism $\nu_T(T_i)/\nu_T(T_i)Q \cong T_j/T_jQ$, by Theorem I.8.4, lifts to an isomorphism between $\nu_T(T_i)$ and T_j, because they are projective in $\operatorname{mod} T$ and $\nu_T(T_i)Q \subseteq \operatorname{rad} \nu_T(T_i)$ and $T_jQ \subseteq \operatorname{rad} T_j$, as claimed. Now, by the definition of the Nakayama automorphism of T, the isomorphism $\nu_T(T_i) \cong T_j$ forces the isomorphisms

$$\operatorname{top}(T_i) \cong \operatorname{soc}\big(\nu_T(T_i)\big) \cong \operatorname{soc}(T_j)$$

in $\operatorname{mod} T$ (see Section IV.6). In particular, we have a nonzero homomorphism $f: T_i \to T_j$, whose image is $\operatorname{soc}(T_j)$. Since T_i and T_j are direct summands of T in $\operatorname{mod} T$, there exists an element $t \in T$ such that $f(x) = tx$ for all $x \in T_i$. Hence, $0 \neq tT_i \subseteq X \cap Y$, which contradicts the assumption that $X \cap Y = 0$. Thus we conclude that T_{π_1} is indecomposable as an ideal.

(ii) Multiplying the Hochschild extension $0 \to Q \xrightarrow{\omega} T \xrightarrow{\rho} A \to 0$ from the left and the right by $1_{T_{\pi_i}} = \sum_{k \in \pi_i} c_k$, we obtain an exact sequence $0 \to Q_{\pi_i} \xrightarrow{\omega_i} T_{\pi_i} \xrightarrow{\rho_i} A_{\pi_i} \to 0$ in $\operatorname{mod} K$, and clearly ρ_i is a homomorphism of algebras and ω_i is a monomorphism (inclusion homomorphism by our assumption) of T_{π_i}-bimodules. Clearly, $\omega_i(Q_{\pi_i})^2 \subseteq \omega(Q)^2 = 0$ in T, hence $\omega_{\pi_i}(Q_{\pi_i})^2 = 0$. Moreover, $Q = (1_{T_{\pi_1}} + \cdots + 1_{T_{\pi_r}})Q(1_{T_{\pi_1}} + \cdots + 1_{T_{\pi_r}}) = \bigoplus_{i=1}^r (1_{T_{\pi_i}} Q 1_{T_{\pi_i}}) = \bigoplus_{i=1}^r Q_{\pi_i}$, because $1_{T_{\pi_i}} Q 1_{T_{\pi_j}} \subseteq 1_{T_{\pi_i}} T 1_{T_{\pi_j}} = 0$ for all $i \neq j$. Thus each Q_{π_i} is an A_{π_i}-bimodule and $Q = Q_{\pi_1} \oplus \cdots \oplus Q_{\pi_r}$ as A-bimodules. All what we have to prove now is that Q_{π_i} is a duality A_{π_i}-bimodule, for any $i \in \{1, \dots, r\}$. Since $\operatorname{mod} A \cong \operatorname{mod} A_{\pi_1} \times \cdots \times \operatorname{mod} A_{\pi_r}$, the functor $\operatorname{Hom}_A(-, Q): \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ is a product of the functors $\operatorname{Hom}_{A_{\pi_i}}(-, Q_{\pi_i}): \operatorname{mod} A_{\pi_i} \to \operatorname{mod} A_{\pi_i}^{\mathrm{op}}$, $i \in \{1, \dots, r\}$. Hence, the duality of $\operatorname{Hom}_A(-, Q)$ ensures that $\operatorname{Hom}_{A_{\pi_i}}(-, Q_{\pi_i})$ is a duality on $\operatorname{mod} A_{\pi_i}$. Thus, by Proposition II.7.13, we conclude that Q_{π_i} is a duality A_{π_i}-bimodule.

(iii) The required canonical K-linear isomorphism

$$f_Q: Q = \sum_{i=1}^r Q_{\pi_i} \longrightarrow \prod_{i=1}^r Q_{\pi_i}$$

is defined by $f_Q\left(\sum_{i=1}^r x_i\right) = (x_i)$, for $x_1 \in Q_{\pi_1}, \dots, x_r \in Q_{\pi_r}$. □

We remark that, since $T = \prod_{i=1}^r T_{\pi_i}$ is the block decomposition of T, the algebras T_{π_i} are uniquely determined (Proposition I.3.16), so the choice of central idempotents $c_1, \dots, c_r$ is unique. Moreover, for any decomposition $T = T^{(1)} \times T^{(2)}$ as a product of algebras, the Hochschild extension $0 \to Q \xrightarrow{\omega} T \xrightarrow{\rho} A \to 0$ is isomorphic to a product of some Hochschild extensions $0 \to Q^{(i)} \xrightarrow{\omega_i} T^{(i)} \xrightarrow{\rho_i} A^{(i)} \to 0$, $i \in \{1, 2\}$. Indeed, by the uniqueness of the block decomposition of an algebra, each of $T^{(1)}$ and $T^{(2)}$ is isomorphic to a product of some of $T_{\pi(1)}, \dots, T_{\pi(r)}$, and then the assertion follows from Lemma 3.15.

As a particular case of Lemma 3.15 we have the following important result.

Theorem 3.16. *Let A be a finite dimensional K-algebra over a field K and $\sigma \in$ $\mathrm{Aut}(A)$. Then the following statements are equivalent:*

(i) *The block permutation of σ is cyclic.*

(ii) *There is an indecomposable Hochschild extension algebra of A by a Frobenius A-bimodule Q with $\nu_Q = \sigma$.*

(iii) *All Hochschild extension algebras of A by a Frobenius A-bimodule Q with $\nu_Q = \sigma$ are indecomposable.*

Proof. Let π be the block permutation of σ and $\pi = \pi_1 \cdots \pi_r$ a disjoint cyclic decomposition of π into cyclic permutations.

(i) $\Rightarrow$ (iii) Assume that π is cyclic, that is, $r = 1$. We may assume that $Q = D(A)_\sigma$, without loss of generality. Take an arbitrary Hochschild extension algebra T of A by Q. Then, by Lemma 3.15, $T = T_1$ is an indecomposable K-algebra.

(iii) $\Rightarrow$ (ii) Let $Q = D(A)_\sigma$. Then $\sigma = \nu_Q$ by definition. Take the trivial extension algebra $\mathrm{T}(A) = A \ltimes Q$, which is a Hochschild extension algebra of A by Q.

(ii) $\Rightarrow$ (i) Let T be an indecomposable Hochschild extension algebra of A by Q. By Lemma 3.15, we have a block decomposition $T = T_1 \oplus \cdots \oplus T_r$. Hence the indecomposability of T forces $r = 1$, that is, $\pi = \pi_1$ a cyclic permutation. ☐

Let A be a finite dimensional K-algebra over a field K and T an indecomposable Hochschild extension algebra of A by a Frobenius A-bimodule Q. As seen from Theorem 3.16 and Lemma 3.8, A has a block decomposition $A = A_1 \times \cdots \times A_m$ such that $\nu_Q(A_i) = A_{i+1}$, $i \in \{1, \ldots, m-1\}$, $\nu_Q(A_m) = A_1$, for a Nakayama automorphism ν_Q of Q. By stressing the relation between ν_T and $A_1, \ldots, A_m$, this block decomposition of A is also called the *Nakayama block decomposition* of A with respect to T.

Corollary 3.17. *Assume that a finite dimensional K-algebra A over a field K is a product of pairwise isomorphic indecomposable K-algebras $A_1, \ldots, A_m$. Then there are a Frobenius A-bimodule Q and an indecomposable Hochschild extension algebra of A by Q.*

Proof. Take an automorphism σ of A such that $\sigma(A_i) = A_{i+1}$, for any $i \in \{1, \ldots, m-1\}$, and $\sigma(A_m) = A_1$. Then the block permutation of σ is cyclic. Hence, for $Q = D(A)_\sigma$, the assertion follows from Theorem 3.16. ☐

Since the block permutation of an inner automorphism of A is identity, the followings corollaries are immediate consequences of Theorem 3.16.

Corollary 3.18. *Let A be a finite dimensional K-algebra over a field K. Assume that $A = A_1 \times \cdots \times A_m$ is a block decomposition of A and σ is an inner automorphism of A. Then any Hochschild extension algebra T of A by $D(A)_\sigma$ is a product of indecomposable Hochschild extension algebras T_i of A_i by $D(A)_{\sigma_i}$, $i \in \{1, \dots, m\}$, where $\sigma_i \in \operatorname{Aut}(A_i)$ is the restriction of σ to A_i.*

Corollary 3.19. *Let A be a finite dimensional K-algebra over a field K. Then the algebra A is indecomposable if and only if there is an indecomposable Hochschild extension algebra of A by $D(A)$, and if and only if any Hochschild extension algebra of A by $D(A)$ is indecomposable.*

Let A be a finite dimensional K-algebra over a field K and Q a duality A-bimodule. Take a basic idempotent e of A. Then it follows from Proposition 3.4 that eQe is a duality eAe-bimodule. Moreover, applying Proposition II.7.16, we conclude that there is an automorphism $\sigma(e)$ of eAe such that $eQe \cong D(eAe)_{\sigma(e)}$ as eAe-bimodules, and $\sigma(e)$ is uniquely determined up to inner automorphism.

Proposition 3.20. *Let A be a finite dimensional K-algebra over a field K and $0 \to Q \overset{\omega}{\to} T \overset{\rho}{\to} A \to 0$ a Hochschild extension of A by a duality A-bimodule Q. Let $\boldsymbol{e}$ be a basic idempotent of T and $e = \rho(\boldsymbol{e})$. Then the following statements are equivalent:*

(i) *T is an indecomposable K-algebra.*

(ii) *The automorphism $\sigma(e)$ of the basic algebra eAe of A induces a cyclic permutation on the blocks of eAe.*

Proof. We set $T^{\mathrm{b}} = \boldsymbol{e}T\boldsymbol{e}$ and $A^{\mathrm{b}} = eAe$. By Proposition 3.4, the sequence

$$0 \longrightarrow eQe \longrightarrow T^{\mathrm{b}} \longrightarrow A^{\mathrm{b}} \longrightarrow 0$$

is a Hochschild extension of A^{b} by eQe. Since T and T^{b} are Morita equivalent (see Theorem II.6.16), by Lemma 3.13, (i) is equivalent to saying that T^{b} is indecomposable. Therefore, Theorem 3.16 implies the required equivalence of the conditions (i) and (ii). □

We note that every nonzero projective module over a Hochschild extension algebra T of a finite dimensional K-algebra A over a field K by a duality A-bimodule Q is of Loewy length at least 2 (see Lemma 3.1).

Proposition 3.21. *Let A be a finite dimensional K-algebra over a field K and*

$$0 \longrightarrow Q \xrightarrow{\omega} T \xrightarrow{\varrho} A \longrightarrow 0$$

a Hochschild extension of A by a duality A-bimodule Q. Then the following statements are equivalent:

(i) *A is a semisimple algebra.*

(ii) *T is of Loewy length* 2.

Proof. (i) $\Rightarrow$ (ii). Assume that A is a semisimple algebra, $\boldsymbol{e}$ a primitive idempotent of T and $e = \varrho(\boldsymbol{e})$. It follows from Lemma 3.1 (i) that $\boldsymbol{e}T/\boldsymbol{e}\omega(Q) \cong eA$ in $\operatorname{mod} A$. Clearly, eA is an indecomposable module, and hence a simple module in $\operatorname{mod} A$. Then $\boldsymbol{e}T/\boldsymbol{e}\omega(Q)$ is a simple module in $\operatorname{mod} A$, and hence in $\operatorname{mod} T$. On the other hand, $\boldsymbol{e}\omega(Q)$ is a semisimple module in $\operatorname{mod} A$, and there are isomorphisms $\boldsymbol{e}\omega(Q) = \operatorname{soc}(\boldsymbol{e}\omega(Q)) \cong \operatorname{soc}(eQ) = eQ$ in $\operatorname{mod} A$, by Lemma 3.1 (iii). Since e is a primitive idempotent of A, eQ is a simple module in $\operatorname{mod} A$, and hence $\boldsymbol{e}\omega(Q)$ is a simple module in $\operatorname{mod} A$, and so a simple module in $\operatorname{mod} T$. Thus $\boldsymbol{e}T$ is an indecomposable projective module in $\operatorname{mod} T$ of Loewy length 2. Therefore, T is an algebra of Loewy length 2.

(ii) $\Rightarrow$ (i). Assume that T is of Loewy length 2. Let e be a primitive idempotent of A. Then $e = \varrho(\boldsymbol{e})$ for a primitive idempotent $\boldsymbol{e}$ of T, by Lemma I.3.12. By the imposed assumption, T is of Loewy length 2. Since $\boldsymbol{e}\omega(Q) \neq 0$, we conclude that $\boldsymbol{e}T/\boldsymbol{e}\omega(Q)$ is a simple module in $\operatorname{mod} T$, and hence a simple module in $\operatorname{mod} A$. Moreover, by Lemma 3.1 (i), we have $eA \cong \boldsymbol{e}T/\boldsymbol{e}\omega(Q)$ in $\operatorname{mod} A$, and so eA is a simple module in $\operatorname{mod} A$. Therefore, A is a semisimple algebra. ☐

4 Non-Frobenius selfinjective Hochschild extensions

We shall describe a construction of Hochschild extension algebras that are non-Frobenius selfinjective algebras. For a Hochschild extension

$$0 \longrightarrow Q \xrightarrow{\omega} T \xrightarrow{\varrho} A \longrightarrow 0$$

of a finite dimensional K-algebra A over a field K by a duality A-bimodule Q, we freely identify Q with $\omega(Q)$, so that Q may be regarded as an ideal of T. Let P be a projective module in $\operatorname{mod} T$ and

$$0 \longrightarrow PQ \xrightarrow{\omega_0} P \xrightarrow{\rho_0} P/PQ \longrightarrow 0$$

the canonical exact sequence in $\operatorname{mod} T$, with the inclusion homomorphism ω_0 and the natural surjective homomorphism ρ_0. Consider the induced exact sequence

$$0 \longrightarrow PQ \oplus Q \xrightarrow{\omega_0 \oplus \omega} P \oplus T \xrightarrow{\rho_0 \oplus \rho} P/PQ \oplus A \longrightarrow 0$$

in $\operatorname{mod} K$. Let $\tau_0 \colon P/PQ \otimes_A Q \to PQ$ be the canonical homomorphism in $\operatorname{mod} A$ defined by $\tau_0(\bar{x} \otimes q) = xq$, for $\bar{x} = x + PQ \in P/PQ$, $q \in Q$. Note that τ_0 is well defined and surjective, because $Q^2 = 0$ as an ideal of T.

Before going to the next lemma, we shall give a remark that $X \otimes_A Y = X \otimes_T Y$ for right T-modules X and left T-modules Y with $XQ = 0$ and $QY = 0$. Indeed, consider the K-vector space $F = K\{X \times Y\}$ with the basis consisting of all $(x, y) \in X \times Y$, and the K-vector subspace F_0^A (respectively, F_0^T) of F generated by all elements of the form $(x + x', y) - (x, y) - (x', y)$, $(x, y + y') - (x, y) - (x, y')$, $\lambda(x, y) - (\lambda x, y)$ and $(xa, y) - (x, ay)$ (respectively, $(xt, y) - (x, ty)$) for all $x, x' \in X$, $y, y' \in Y$, $\lambda \in K$, $a \in A$, $t \in T$. Notice that $F_0^A = F_0^T$, because $xa = xt, ay = ty$ for all $x \in X$, $y \in Y$, $t \in T$, and $a = \rho(t) \in A$. Hence, we have $X \otimes_A Y = F/F_0^A = F/F_0^T = X \otimes_T Y$, by the definition of the tensor product (see Section II.3).

Lemma 4.1. *The homomorphism* $\tau_0\colon P/PQ \otimes_A Q \to PQ$ *is an isomorphism.*

Proof. By the above remark, we have $P/PQ \otimes_A Q = P/PQ \otimes_T Q$. Since τ_0 is surjective, it is enough to show the injectivity of τ_0. We consider the homomorphisms

$$\begin{tikzcd}
P \otimes_T Q \arrow[r, "\rho_0 \otimes Q"] \arrow[d, "P \otimes \omega"'] & P/PQ \otimes_T Q \arrow[r] & 0 \\
P \otimes_T T \arrow[r, "\xi"] & P ,
\end{tikzcd}$$

where $\xi\colon P \otimes_T T \to P$ is a canonical isomorphism (see Lemma II.3.5) and $P \otimes \omega$ is a monomorphism, because P is projective (see Exercise II.8.8). Now, take any $z \in \operatorname{Ker} \tau_0$ and write $z = \sum \bar{p}_i \otimes q_i \in P/PQ \otimes_T Q$. Let $z' = \sum p_i \otimes q_i \in P \otimes_T Q$, so that $(\rho_0 \otimes Q)(z') = z$. Then, $(\xi(P \otimes \omega))(z') = \xi(\sum p_i \otimes \omega(q_i)) = \sum p_i q_i$ by the identification of $\omega(Q)$ with Q. On the other hand, $0 = \tau_0(z) = \tau_0(\sum \bar{p}_i \otimes q_i) = \sum p_i q_i$. Hence, $(\xi(P \otimes \omega))(z') = 0$ and $z' = 0$, because $\xi(P \otimes \omega)$ is a monomorphism. Thus $z = (\rho_0 \otimes Q)(z') = 0$. □

We set

$$\tilde{T} = \operatorname{End}_T(P \oplus T) \quad \text{and} \quad \tilde{A} = \operatorname{End}_T(P/PQ \oplus A).$$

Then $(P/PQ \oplus A) \otimes_A Q$ becomes an $(\tilde{A}, A)$-bimodule naturally induced by the left $\tilde{A}$-module $P/PQ \oplus A$ and the right A-module Q, that is, $u(m \otimes q)a = um \otimes qa$ for $u \in \tilde{A}$, $m \otimes q \in (P/PQ \oplus A) \otimes_A Q$, and $a \in A$.

Lemma 4.2. *There are* K*-algebra homomorphisms*

$$\tilde{T} \xrightarrow{\tilde{\rho}} \tilde{A} \xrightarrow{\theta} \operatorname{End}_T(PQ \oplus Q),$$

where $\tilde{\rho}$ *is a surjective homomorphism satisfying* $\tilde{\rho}(f)(\rho_0 \oplus \rho) = (\rho_0 \oplus \rho)f$ *for all* $f \in \tilde{T}$, *and* θ *is an isomorphism such that, for any* $u \in \tilde{A}$, *there is a* *commutative diagram in* mod T

$$\begin{tikzcd}
0 \arrow[r] & PQ \oplus Q \arrow[r, "\omega_0 \oplus \omega"] \arrow[d, "\theta(u)"] & P \oplus T \arrow[r, "\rho_0 \oplus \rho"] \arrow[d, "f"] & P/PQ \oplus A \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & PQ \oplus Q \arrow[r, "\omega_0 \oplus \omega"] & P \oplus T \arrow[r, "\rho_0 \oplus \rho"] & P/PQ \oplus A \arrow[r] & 0
\end{tikzcd}$$

for some homomorphism $f \in \tilde{T}$.

Proof. We will consider commutative diagrams of the form

$$\begin{tikzcd}
0 \arrow[r] & PQ \oplus Q \arrow[r, "\omega'"] \arrow[d, "f''"] & P \oplus T \arrow[r, "\rho'"] \arrow[d, "f"] & P/PQ \oplus A \arrow[r] \arrow[d, "f'"] & 0 \\
0 \arrow[r] & PQ \oplus Q \arrow[r, "\omega'"] & P \oplus T \arrow[r, "\rho'"] & P/PQ \oplus A \arrow[r] & 0
\end{tikzcd}$$

in mod T, where $\omega' = \omega_0 \oplus \omega$ and $\rho' = \rho_0 \oplus \rho$.

First, we define an algebra homomorphism $\tilde{\rho}\colon \tilde{T} \to \tilde{A}$. Take any $f \in \tilde{T}$. Then $\operatorname{Im} f\omega' = f(PQ \oplus TQ) = f(P \oplus T)Q$, and hence $\operatorname{Im} f\omega' \subseteq PQ \oplus TQ = \operatorname{Im} \omega' = \operatorname{Ker} \rho'$ and $(\rho' f)\omega' = \rho'(f\omega') = 0$. It follows that there is $f' \in \tilde{A}$ with $f'\rho' = \rho' f$, which is uniquely determined by f, because of the surjectivity of ρ'. Thus we have a K-linear map $\tilde{\rho}\colon \tilde{T} \to \tilde{A}$ with $\tilde{\rho}(f) = f'$ for $f \in \tilde{T}$. Clearly, $\tilde{\rho}$ preserves the identity and multiplication, so is a K-algebra homomorphism. Moreover, the surjectivity of $\tilde{\rho}$ follows from the projectivity of $P \oplus T$ in mod T.

Next, we define a homomorphism $\theta\colon \tilde{A} \to \operatorname{End}_T(PQ \oplus Q)$. Take any $f' \in \tilde{A}$ and $f \in \tilde{T}$ with $\tilde{\rho}(f) = f'$. Then there is a homomorphism $f'' \in \operatorname{End}_T(PQ \oplus Q)$ with $\omega' f'' = f\omega'$, because $\rho'(f\omega') = 0$. We shall show that the correspondence $\theta\colon f' \mapsto f''$ is well defined. For this, take homomorphisms $g \in \tilde{T}$ and $g'' \in \operatorname{End}_T(PQ \oplus Q)$ with $\rho' g = f'\rho'$ and $\omega' g'' = g\omega'$. Then $\rho'(f - g) = f'\rho' - f'\rho' = 0$, which ensures the existence of a homomorphism $h \in \operatorname{Hom}_T(P \oplus T, PQ \oplus Q)$ with $\omega' h = f - g$. Observe that $\operatorname{Im} \omega' h\omega' = \omega' h(PQ \oplus TQ) = \omega'(h(P \oplus T))Q \subseteq \omega'(PQ \oplus Q)Q = 0$, because $\omega'(PQ \oplus Q) = PQ \oplus Q \subseteq P \oplus T$ and $Q^2 = 0$ in T. This implies that $(f - g)\omega' = 0$, and hence $\omega'(f'' - g'') = 0$. Since ω' is a monomorphism, we get $f'' = g''$. Thus $\theta(f') = f''$ is independent of the choice of f. Then the surjectivity of θ follows from the injectivity of $P \oplus T$ in mod T. In fact, for any $f'' \in \operatorname{End}_T(PQ \oplus Q)$, $\omega' f''$ extends to a homomorphism $f \in \operatorname{End}_T(P \oplus T)$ along ω', that is, $f\omega' = \omega' f''$. Then $f' = \tilde{\rho}(f)$ satisfies $\theta(f') = f''$.

Finally, to show the injectivity of θ, assume that $\theta(f') = f'' = 0$, and take a homomorphism $f \in \tilde{T}$ such that $f\omega' = \omega' f''$ and $\rho' f = f'\rho'$. Then there is a homomorphism $h'\colon P/PQ \oplus A \to P \oplus T$ with $f = h'\rho'$, because $f\omega' = \omega' f'' = 0$. Observe that $\operatorname{Im} h' \subseteq \operatorname{Ker} \rho'$. In fact, $\operatorname{Im} \omega' = PQ \oplus TQ$ is the left annihilator of Q in $P \oplus T$ (see Lemma 7.3), $P/PQ \oplus A$ is annihilated by Q as a right T-module, and hence $(\operatorname{Im} h')Q = h'(P/PQ \oplus A)Q = h'((P/PQ \oplus A)Q) = 0$. Therefore, $f'\rho' = \rho'(h'\rho') = (\rho'h')\rho' = 0$, which implies $f' = 0$ because of the surjectivity of ρ'. Thus we conclude that the K-algebra homomorphism θ is an isomorphism. $\square$

The $(\tilde{A}, A)$-bimodule $P/PQ \oplus A$ induces the left $\tilde{A}$-module structure on $(P/PQ \oplus A) \otimes_A Q$ given by $f'(m \otimes q) = (f'm) \otimes q$, for $f' \in \tilde{A}, m \otimes q \in (P/PQ \oplus A) \otimes_A Q$, and we have a left multiplication map

$$(-)_L \colon \tilde{A} \longrightarrow \operatorname{End}_A\big((P/PQ \oplus A) \otimes_A Q\big),$$

where $(-)_L(f') = f'_L$ and $f'_L(m \otimes q) = f'(m \otimes q)$. We further define the K-linear map

$$\tilde{\tau} \colon \operatorname{End}_A\big((P/PQ \oplus A) \otimes_A Q\big) \longrightarrow \operatorname{End}_A(PQ \oplus Q)$$

by $\tilde{\tau}(f) = \tau f \tau^{-1}$ for all $f \in \operatorname{End}_A((P/PQ \oplus A) \otimes_A Q)$, where τ is the composition of the isomorphisms ξ and $\tau_0 \oplus Q$,

$$\tau \colon (P/PQ \oplus_A A) \otimes_A Q \xrightarrow{\xi} (P/PQ \otimes_A Q) \oplus Q \xrightarrow{\tau_0 \oplus Q} PQ \oplus Q,$$

such that $\xi((\bar{p}, a) \otimes q) = (\bar{p} \otimes q, aq)$, for $\bar{p} \in P/PQ$, $a \in A$, and $q \in Q$.

Lemma 4.3. *$(-)_L$ and $\tilde{\tau}$ are isomorphisms of K-algebras and $\theta = \tilde{\tau}(-)_L$.*

Proof. We set $M = P/PQ \oplus A$, which is a projective generator in $\operatorname{mod} A$, by Lemma 7.3. Then A and $\tilde{A}$ are Morita equivalent, by Theorem II.6.7. It follows from Propositions II.6.9 and II.6.11 that A^{op} and $\tilde{A}^{\mathrm{op}}$ are Morita equivalent, and the functor $F = M \otimes_A - \colon \operatorname{mod} A^{\mathrm{op}} \to \operatorname{mod} \tilde{A}^{\mathrm{op}}$ is an equivalence of categories.

(i) Clearly, $(-)_L$ is a homomorphism of K-algebras. We shall show that $(-)_L$ is an isomorphism. The equivalence of module categories $F = M \otimes_A - \colon \operatorname{mod} A^{\mathrm{op}} \to \operatorname{mod} \tilde{A}^{\mathrm{op}}$ forces that $F(Q)$ is faithful in $\operatorname{mod} \tilde{A}$, since Q is faithful in $\operatorname{mod} A^{\mathrm{op}}$ and the Morita equivalence preserves faithfulness of modules. This is equivalent to saying that $(-)_L \colon \tilde{A} \to \operatorname{End}_A(M \otimes_A Q)$ is a monomorphism. Thus we can conclude that $(-)_L$ is an isomorphism if we can show that $\dim_K \operatorname{End}_A(M \otimes_A Q) = \dim_K \tilde{A}$. But this equality follows from the isomorphisms $M \otimes_A Q \cong (P/PQ \otimes_A Q) \oplus Q \cong PQ \oplus Q$ in $\operatorname{mod} A$, which imply isomorphisms $\tilde{A} \xrightarrow{\theta} \operatorname{End}_T(PQ \oplus Q) = \operatorname{End}_A(PQ \oplus Q) \cong \operatorname{End}_A(M \otimes_A Q)$ of K-algebras.

(ii) The isomorphism of $(-)_L$ asserts that any homomorphism in $\operatorname{End}_A(M \otimes_A Q)$ is expressed as $f' \otimes Q$ for some $f' \in \tilde{A}$. With the same notations as in the proof of Lemma 4.2, consider the following diagram in mod T

$$\begin{tikzcd}
(P \oplus T) \otimes_T Q \arrow[r, "\rho' \otimes Q"] \arrow[d, "f \otimes Q"] & (P/PQ \oplus A) \otimes_A Q \arrow[r, "\tau"] \arrow[d, "f' \otimes Q"] & PQ \oplus Q \arrow[d, "f''"] \\
(P \oplus T) \otimes_T Q \arrow[r, "\rho' \otimes Q"] & (P/PQ \oplus A) \otimes_A Q \arrow[r, "\tau"] & PQ \oplus Q
\end{tikzcd}$$

where $\rho' = \rho_0 \oplus \rho$, $f'' = \theta(f')$, and the left square is commutative. If we can show that the right square is commutative, then $\tilde{\tau}(f' \otimes Q) = \tau(f' \otimes Q)\tau^{-1} = f''$, so that $\theta(f') = f'' = \tilde{\tau}(f' \otimes Q) = \tilde{\tau}(-)_L(f')$ for all $f' \in \tilde{A}$. Hence, $\theta = \tilde{\tau}(-)_L$ and $\tilde{\tau} = \theta(-)_L^{-1}$ is an isomorphism. Now, to show the commutativity of the right square, take any $f \in \tilde{A}$ and $(\bar{p}, a) \otimes q \in M \otimes_A Q$ with $a = \rho(t)$ for $t \in T$. Let $f(p,t) = \sum(p_i, t_i) \in P \oplus T$. Then

$$\begin{aligned}
(f' \otimes Q)\big((\bar{p}, a) \otimes q\big) &= (f' \otimes Q)(\rho' \otimes Q)\big((p,t) \otimes q\big) \\
&= (\rho' \otimes Q)(f \otimes Q)\big((p,t) \otimes q\big) = \rho'\big(f(p,t)\big) \otimes q \\
&= \sum \big(\bar{p}_i, \rho(t_i)\big) \otimes q,
\end{aligned}$$

and hence

$$\begin{aligned}
\big(\omega'\tau(f' \otimes Q)\big)\big((\bar{p}, a) \otimes q\big) &= \omega'\tau\Big(\sum \big(\bar{p}_i, \rho(t_i)\big) \otimes q\Big) \\
&= \omega'\Big(\sum \big(p_i q, \rho(t_i) q\big)\Big) \\
&= \sum (p_i q, t_i q).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\omega' f''\tau\big((\bar{p}, a) \otimes q\big) &= \omega' f''(pq, aq) = f\omega'(pq, aq) = f(pq, tq) = f(p,t)q \\
&= \sum (p_i q, t_i q).
\end{aligned}$$

As a consequence, we have $\omega'\tau(f' \otimes Q) = \omega' f''\tau$, and hence the required equality $\tau(f' \otimes Q) = f''\tau$. ☐

The $(\tilde{A}, A)$-bimodules $P/PQ \oplus A$ and $(P/PQ \oplus A) \otimes_A Q$ naturally induce the $\tilde{A}$-bimodule structure on

$$\tilde{Q} = \operatorname{Hom}_A\big(P/PQ \oplus A, (P/PQ \oplus A) \otimes_A Q\big).$$

Further, $\tilde{Q}$ is regarded as a $\tilde{T}$-bimodule through $\tilde{\rho}$, namely, as the $\tilde{T}$-bimodule ${}_{\tilde{\rho}}\tilde{Q}_{\tilde{\rho}}$. Also we have the $(\operatorname{End}_T(PQ \oplus Q), \tilde{A})$-bimodule

$$\tilde{Q}_0 = \operatorname{Hom}_T(P/PQ \oplus A, PQ \oplus Q).$$

Lemma 4.4. *The K-linear map* $\operatorname{Hom}(P/PQ \oplus A, \tau)\colon \tilde{Q} \to {}_{\theta}\tilde{Q}_0$ *is an $\tilde{A}$-bimodule isomorphism.*

Proof. We set $\varphi = \operatorname{Hom}(P/PQ \oplus A, \tau)$. Obviously, φ is a homomorphism of right $\tilde{A}$-modules. To show that φ belongs to $\operatorname{mod} \tilde{A}^{\mathrm{op}}$, recall that, by Lemma 4.2, any element of $\tilde{A}$ can be written as $f' = \tilde{\rho}(f)$, for some $f \in \tilde{T}$. Now take any $f \in \tilde{T}$ and $u \in \tilde{Q}$. Then, by Lemma 4.3,

$$\begin{aligned} f'\varphi(u) = \theta(f')\tau u = \tilde{\tau}(f' \otimes Q)\tau u = \tau(f' \otimes Q)\tau^{-1}\tau u \\ = \tau(f' \otimes Q)u = \varphi\big((f' \otimes Q)u\big) = \varphi(f'u), \end{aligned}$$

which ensures that φ is a homomorphism of left $\tilde{A}$-modules. □

Now consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {}_{\theta}\tilde{Q}_0 \arrow[r, "\tilde{\omega}_1"] \arrow[d, "{\operatorname{Hom}(\rho', PQ\oplus Q)}"'] & \tilde{T} \arrow[r, "\tilde{\rho}"] \arrow[d, "id"] & \tilde{A} \arrow[r] \arrow[d, "{\operatorname{Hom}(\rho', P/PQ\oplus A)}"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_T(P \oplus T, PQ \oplus Q) \arrow[r, "\omega''"] & \tilde{T} \arrow[r, "\rho''"] & \operatorname{Hom}_T(P \oplus Q, P/PQ \oplus A) \arrow[r] & 0
\end{tikzcd}$$

with exact rows in $\operatorname{mod} K$, where $\tilde{\omega}_1(u) = (\omega_0 \oplus \omega)u(\rho_0 \oplus \rho) = \omega' u \rho'$ for all $u \in \tilde{Q}_0$, $\omega'' = \operatorname{Hom}_T(P \oplus T, \omega')$, and $\rho'' = \operatorname{Hom}_T(P \oplus T, \rho')$. It should be noted that $\tilde{\omega}_1$ is a $\tilde{T}$-bimodule homomorphism. Indeed, for any $u \in \tilde{Q}_0$, $f, g \in \tilde{T}$, we have $\tilde{\omega}_1(guf) = \tilde{\omega}_1(\theta(g')uf') = \omega'\theta(g')uf'\rho' = g\omega' u\rho' f = g\tilde{\omega}_1(u)f$, where $f' = \tilde{\rho}(f)$ and $g' = \tilde{\rho}(g)$.

We define the monomorphism $\tilde{\omega} = \tilde{\omega}_1 \operatorname{Hom}(P/PQ \oplus A, \tau)\colon \tilde{Q} \to \tilde{T}$. Then we have the following fact.

Proposition 4.5. *Let A be a finite dimensional K-algebra over a field K, Q a duality A-bimodule, and $0 \to Q \xrightarrow{\omega} T \xrightarrow{\rho} A \to 0$ a Hochschild extension of A by Q. Then, for a projective module P in* $\operatorname{mod} T$, *the following statements hold:*

(i) *$\tilde{Q}$ is a duality $\tilde{A}$-bimodule.*

(ii) *The exact sequence $0 \longrightarrow \tilde{Q} \xrightarrow{\tilde{\omega}} \tilde{T} \xrightarrow{\tilde{\rho}} \tilde{A} \longrightarrow 0$ is a Hochschild extension of $\tilde{A}$ by $\tilde{Q}$.*

Proof. (i) We set $M = P/PQ \oplus A$. Since M is a projective generator in $\operatorname{mod} A$, the functor

$$F = \operatorname{Hom}_A(M, -)\colon \operatorname{mod} A \longrightarrow \operatorname{mod} \tilde{A}$$

is an equivalence, by Theorem II.6.7. It follows that $\tilde{Q} = F(M \otimes_A Q)$ is an injective cogenerator in $\operatorname{mod} \tilde{A}$, because $M \otimes_A Q$ is an injective cogenerator in mod A. We remark that a Morita equivalence preserves cogenerators and injectivity of modules. In fact, by Proposition II.7.3, a cogenerator M in mod A is characterized as a module with the property that any indecomposable injective module is isomorphic to a submodule of M, and, by Proposition II.6.6, this property is preserved by a Morita equivalence. Moreover, F induces an algebra isomorphism

$$\zeta\colon \operatorname{End}_A(M \otimes_A Q) \longrightarrow \operatorname{End}_{\tilde{A}}\big(F(M \otimes_A Q)\big).$$

Observe that, for any $u \in \operatorname{End}_A(M \otimes_A Q)$, $\zeta(u)\colon F(M \otimes_A Q) \to F(M \otimes_A Q)$ is a left multiplication map, that is, $\zeta(u)(v) = uv$ for all $v \in F(M \otimes_A Q)$, and hence the composed isomorphism $\zeta(-)_L\colon \tilde{A} \to \operatorname{End}_A(M \otimes_A Q) \to \operatorname{End}_{\tilde{A}}(\tilde{Q})$ also maps $f' \in \tilde{A}$ to the left multiplication by f' on $\tilde{Q}$. Thus we have proved that $\tilde{A} \cong \operatorname{End}_{\tilde{A}}(\tilde{Q})$ as K-algebras, and consequently $\tilde{Q}$ is a duality $\tilde{A}$-bimodule, by the Morita–Azumaya duality theorem (Theorem II.7.11).

(ii) Since $\tilde{\rho}$ is a surjective algebra homomorphism and $\tilde{\omega}$ is a $\tilde{T}$-bimodule monomorphism, it remains to prove that $\tilde{\omega}(\tilde{Q})^2 = 0$. This follows from the equalities $\tilde{\omega}(\tilde{Q}) = \tilde{\omega}_1(\tilde{Q}_0)$ and $\tilde{\omega}_1(y)\tilde{\omega}_1(x) = (\omega' y \rho')(\omega' x \rho') = 0$ for all $x, y \in \tilde{Q}_0$, because $\rho'\omega' = 0$ for $\omega' = \omega_0 \oplus \omega$ and $\rho' = \rho_0 \oplus \rho$. □

Corollary 4.6. *Let A be a finite dimensional K-algebra over a field K and Q a duality A-bimodule. Then, for a projective module M in* mod A*,*

$$\operatorname{Hom}_A\big(M \oplus A, (M \oplus A) \otimes_A Q\big)$$

is a duality $\operatorname{End}_A(M \oplus A)$*-bimodule.*

Proof. Let $T = A \ltimes Q$ be the trivial extension algebra of A by Q with canonical epimorphism $\rho\colon T \to A$. We take a complete set $e_1A, \dots, e_nA$ of pairwise nonisomorphic indecomposable projective modules in mod A, with $e_1, \dots, e_n$ primitive idempotents of A. Let $\boldsymbol{e}_i = (e_i, 0) \in T$, for $i \in \{1, \dots, n\}$. We may write $M = (e_1A)^{\ell_1} \oplus \cdots \oplus (e_nA)^{\ell_n}$ for some nonnegative integers $\ell_1, \dots, \ell_n$, where $(e_iA)^{\ell_i}$ stands for 0 in case $\ell_i = 0$ for some i. Then, for the right T-module $P = (\boldsymbol{e}_1T)^{\ell_1} \oplus \cdots \oplus (\boldsymbol{e}_nT)^{\ell_n}$, we have $P = (e_1A \oplus e_1Q)^{\ell_1} \oplus \cdots \oplus (e_nA \oplus e_nQ)^{\ell_n} \cong M \oplus PQ$ and $P/PQ \cong M$ as right A-modules. The assertion now follows from Proposition 4.5. □

We proved in Section IV.7 that every non-Frobenius selfinjective algebra is constructed from a basic Frobenius algebra. We will show that any non-Frobenius Hochschild extension algebra is constructed from a basic Frobenius Hochschild extension algebra.

Let us recall notations introduced in Section IV.7. Let Λ be a finite dimensional K-algebra over a field K and $P_1, \dots, P_n$ be a complete set of pairwise nonisomorphic indecomposable projective modules in $\operatorname{mod}\Lambda$. For a sequence $m(1), \dots, m(n)$ of positive integers, we write

$$\Lambda\big(m(1), \dots, m(n)\big) = \operatorname{End}_\Lambda\left(\bigoplus_{i=1}^{n} P_i^{m(i)}\right).$$

The following theorem corresponds to Theorem IV.7.3.

Theorem 4.7. *Let A be a finite dimensional K-algebra over a field K and $A = (e_1A)^{m(1)} \oplus \cdots \oplus (e_nA)^{m(n)}$, where $e_1, \dots, e_n$ is a complete set of basic primitive idempotents of A and $m(1), \dots, m(n)$ are positive integers. Then, for a duality A-bimodule Q with ν_Q its Nakayama permutation, the following statements are equivalent:*

- (i) *There exists a non-Frobenius Hochschild extension algebra of A by Q.*
- (ii) *All Hochschild extension algebras of A by Q are non-Frobenius.*
- (iii) *$m(i) \neq m(\nu_Q(i))$ for some $i \in \{1, \dots, n\}$.*

Proof. Let $0 \to Q \xrightarrow{\omega} T \xrightarrow{\rho} A \to 0$ be a Hochschild extension of A by Q. We have a canonical decomposition

$$1_T = \sum_{i=1}^{n} \sum_{j=1}^{m(i)} \boldsymbol{e}_j$$

of the identity 1_T of T into a sum of pairwise orthogonal primitive idempotents $\boldsymbol{e}_{ij} \in T$, such that $\boldsymbol{e}_{ij}T \cong \boldsymbol{e}_{ij'}T$ for $i \in \{1, \dots, n\}$ and $j, j' \in \{1, \dots, m(i)\}$, and $\boldsymbol{e}_{ij}T \ncong \boldsymbol{e}_{i'j'}T$ for $i \neq i'$ in $\{1, \dots, n\}$, $j \in \{1, \dots, m(i)\}$, $j' \in \{1, \dots, m(i')\}$. We may assume that $\rho(\boldsymbol{e}_{i1}) = e_i$ for $i \in \{1, \dots, n\}$. By Theorem IV.7.3, T is not a Frobenius algebra if and only if $T \cong \Lambda(m(1), \dots, m(n))$ for some basic selfinjective K-algebra Λ such that $m(i) \neq m(\nu_\Lambda(i))$ for some i, where ν_Λ is the Nakayama permutation of Λ on the set $\{1, \dots, n\}$, and coincides with the Nakayama permutation ν_T of T (see Proposition IV.7.1). On the other hand, we know that $\nu_T = \nu_Q$, by Theorem 3.2. Hence, T is a non-Frobenius algebra if and only if $m(i) \neq m(\nu_Q(i))$ for some $i \in \{1, \dots, n\}$. Since the condition (iii) does not depend on any Hochschild extension of A by Q, we therefore conclude the equivalence of all statements (i), (ii), and (iii). □

Corollary 4.8. *Let Λ be a basic finite dimensional K-algebra over a field K and $\Lambda = P_1 \oplus \cdots \oplus P_n$ a direct sum decomposition of Λ into indecomposable modules in* $\operatorname{mod}\Lambda$. *Let $A = \Lambda(m(1), \dots, m(n))$ for some positive integers $m(1), \dots, m(n)$, and take the Λ-module $M = P_1^{\ell_1} \oplus \cdots \oplus P_n^{\ell_n}$, with $\ell_i = m(i) - 1$ for any $i \in \{1, \dots, n\}$. Assume that there is an automorphism $\sigma \in \operatorname{Aut}(\Lambda)$ such* *that* $\sigma(P_i) \cong (P_j)_\sigma$ *in* $\operatorname{mod}\Lambda$ *and* $m(i) \neq m(j)$, *for some* $i, j \in \{1, \dots, n\}$. *Then* $A = \operatorname{End}_\Lambda(M \oplus \Lambda)$, *the* A*-bimodule*

$$Q = \operatorname{Hom}_A\big(M \oplus \Lambda, (M \oplus \Lambda) \otimes_\Lambda D(\Lambda)_\sigma\big)$$

is a non-Frobenius duality A*-bimodule, and any Hochschild extension algebra of* A *by* Q *is not a Frobenius algebra.*

Proof. Obviously, $A = \operatorname{End}_\Lambda(M \oplus \Lambda)$ by definition, and $M \oplus \Lambda$ is an (A, Λ)-bimodule. Moreover, by Corollary 4.6, Q is a duality A-bimodule. Let ν_Q be its Nakayma permutation. For each $r \in \{1, \dots, n\}$, let $P_r = \varepsilon_r\Lambda$ for a primitive idempotent ε_r in Λ, and define the idempotent

$$e_r \colon M \oplus \Lambda \longrightarrow \varepsilon_r\Lambda \longrightarrow M \oplus \Lambda$$

of A as the composition of the canonical projection and injection. We have only to show that $\nu_Q(i) = j$. In fact, then $m(i) \neq m(\nu_Q(i))$ by the assumption, and the claims of the corollary follow from Theorem 4.7 and Lemma 3.5.

We have the following equalities for $\nu = \nu_Q$:

$$\begin{aligned} e_{\nu(i)}Q &= \operatorname{Hom}_\Lambda\big(M \oplus \Lambda, e_{\nu(i)}(M \oplus \Lambda) \otimes_\Lambda D(\Lambda)_\sigma\big) \\ &= \operatorname{Hom}_\Lambda\big(M \oplus \Lambda, \varepsilon_{\nu(i)}D(\Lambda)_\sigma\big) = \operatorname{Hom}_\Lambda\big(M \oplus \Lambda, D({}_\sigma\Lambda\varepsilon_{\nu(i)})\big). \end{aligned}$$

Since $F = \operatorname{Hom}_\Lambda(M \oplus \Lambda, -)\colon \operatorname{mod}\Lambda \to \operatorname{mod}A$ is an equivalence of categories, it follows that $\operatorname{soc}(e_{\nu(i)}Q) \cong F(\operatorname{soc}D({}_\sigma\Lambda\varepsilon_{\nu(i)})) \cong F(\operatorname{top}(\varepsilon_{\nu(i)}\Lambda_\sigma))$. The isomorphism $\operatorname{soc}(e_{\nu(i)}Q) \cong \operatorname{top}(e_iA)$ implies $\operatorname{soc}(e_{\nu(i)}Q)e_i \neq 0$, while

$$\begin{aligned} F\big(\operatorname{top}(\varepsilon_{\nu(i)}\Lambda)_\sigma\big)e_i &= \operatorname{Hom}_\Lambda\big(e_i(M \oplus \Lambda), \operatorname{top}(\varepsilon_{\nu(i)}\Lambda)_\sigma\big) \\ &= \operatorname{Hom}_\Lambda\big(\varepsilon_i\Lambda, \operatorname{top}(\varepsilon_{\nu(i)}\Lambda)_\sigma\big) \\ &\cong \operatorname{top}(\varepsilon_{\nu(i)}\Lambda)_\sigma\varepsilon_i = \operatorname{top}(\varepsilon_{\nu(i)}\Lambda)\sigma(\varepsilon_i). \end{aligned}$$

Hence, we conclude that $\operatorname{top}(\varepsilon_{\nu(i)}\Lambda)\sigma(\varepsilon_i) \neq 0$, which implies that $\sigma(\varepsilon_i)\Lambda = \sigma(\varepsilon_i\Lambda)_{\sigma^{-1}}$. Thus we have proved that $P_{\nu(i)} \cong \sigma(P_j)_{\sigma^{-1}} \cong P_j$, so that $\nu(i) = j$, as required. □

In the case when A and Q are the K-algebra and A-bimodule in Corollary 4.8, we know the existence of an indecomposable non-Frobenius Hochschild extension algebra. In fact, take any Hochschild extension $0 \to Q \to T \to A \to 0$, and let $T = T_1 \times \cdots \times T_k$ be the block decomposition of T, and assume that T is non-Frobenius. It follows from Lemma 3.15 that each T_i is a Hochschild extension algebra of some direct summand, say A_i, of A. Further, since T is not a Frobenius algebra, by Lemma 3.14, at least one of the blocks T_i is non-Frobenius. In this case, since T_i is indecomposable, it follows from Theorem 3.16 that any two blocks of A_i are isomorphic as K-algebras.

The next theorem gives a general construction of non-Frobenius Hochschild extension algebras.

Theorem 4.9. *Let A be a basic finite dimensional K-algebra over a field K, and assume that $A = \prod_{i=1}^{m} A_i$ is a product of pairwise isomorphic K-algebras $A_1, \dots, A_m$. Let σ_A be an automorphism of A with $\sigma_A(A_i) = A_{i+1}$ for $i \in \{1, \dots, m\}$ such that $\sigma_m \cdots \sigma_1 = \mathrm{id}_{A_1}$, where $A_{m+1} = A_1$ and $\sigma_i \colon A_i \to A_{i+1}$ is the restriction of σ to A_i. Take a family of projective modules P_i in* mod A_i, *$i \in \{1, \dots, m\}$, with $(P_r)_{\sigma_A} \ncong P_{r-1}$ in* mod A *for some r. Moreover, let $M = P_1 \oplus \cdots \oplus P_m$ in* mod A, *$\tilde{A} = \mathrm{End}_A(M \oplus A)$, and $\tilde{Q} = \mathrm{Hom}_A(M \oplus A, (M \oplus A) \otimes_A D(A)_{\sigma_A})$. Then $\tilde{Q}$ is a non-Frobenius duality $\tilde{A}$-bimodule and any Hochschild extension algebra of $\tilde{A}$ by $\tilde{Q}$ is an indecomposable non-Frobenius algebra.*

Proof. In the proof, $i + 1$ stands for 1 if $i = m$, when i runs through $\{1, \dots, m\}$, and we write $\sigma = \sigma_A$. Note that each A_i is a basic (not necessarily indecomposable) K-algebra. We may take a complete set of basic pairwise orthogonal primitive idempotents $e_1(i), \dots, e_n(i)$ of A_i such that

$$\sigma\big(e_1(i)\big) = e_1(i+1), \dots, \sigma(e_n(i)) = e_n(i+1)$$

for any $i \in \{1, \dots, m\}$. Then the collection of all these idempotents $e_j(i)$ forms a complete set of basic pairwise orthogonal primitive idempotents of A. By naturally regarding A_i as an ideal of A, we may write $A = A_1 \oplus \cdots \oplus A_m$, and $A_i = e_1(i)A_i \oplus \cdots \oplus e_n(i)A_i = e_1(i)A \oplus \cdots \oplus e_n(i)A$ as a right ideal of A, for $i \in \{1, \dots, m\}$. Each (possibly zero) projective module P_i in mod A_i is then written as

$$P_i \cong \big(e_1(i)A\big)^{m(1,i)} \oplus \cdots \oplus \big(e_n(i)A\big)^{m(n,i)}$$

for some nonnegative integers $m(1,i), \dots, m(n,i)$. Because of the decomposition $A = \bigoplus_{i=1}^{m}(e_i(i)A \oplus \cdots \oplus e_n(i)A)$, we have

$$M \oplus A \cong \bigoplus_{i=1}^{m} \big(e_i(i)A^{m(1,i)+1} \oplus \cdots \oplus e_n(i)A^{m(n,i)+1}\big)$$

in mod A. Observe that then

$$\tilde{A} \cong A\big(m(j,i) + 1 \mid 1 \le i \le m, 1 \le j \le n\big).$$

Further, the assumption $(P_r)_\sigma \ncong P_{r-1}$ implies that $m(j,r) \neq m(j, r-1)$ for some $j \in \{1, \dots, n\}$, because σ induces an isomorphism $\sigma(e_j(i)A) \xrightarrow{\sim} \sigma(e_j(i)A)\sigma(A)_\sigma = e_j(i+1)A_\sigma$ in mod A, for all $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$. As a result, we have

$$\sigma\big(e_j(r-1)A\big) \cong e_j(r)A_\sigma, \quad m(j,r) + 1 \neq m(j, r-1) + 1.$$

It therefore follows from Corollary 4.8 that $\tilde{Q}$ is a non-Frobenius duality $\tilde{A}$-bimodule.

Finally, we have to show that any Hochschild extension algebra of $\tilde{A}$ by $\tilde{Q}$ is indecomposable. Let $0 \to \tilde{Q} \xrightarrow{\omega} T \xrightarrow{\rho} \tilde{A} \to 0$ be a Hochschild extension, and take $e\colon M \oplus A \to A \to M \oplus A$ as the composition of the natural projection and injection maps. Observe that e is a basic idempotent of $\tilde{A}$, because A is basic and M is isomorphic to a direct summand of a free A-module in $\operatorname{mod} A$. We can take an idempotent $\boldsymbol{e}$ of T with $\rho(\boldsymbol{e}) = e$, because the ideal $\omega(\tilde{Q})$ is nilpotent (see Lemma I.3.12). By Proposition 3.4, we have a Hochschild extension

$$0 \longrightarrow e\tilde{Q}e \xrightarrow{\omega'} \boldsymbol{e}T\boldsymbol{e} \xrightarrow{\rho'} e\tilde{A}e \longrightarrow 0,$$

where ω' and ρ' are the restrictions of ω and ρ, respectively. Obviously, $e\tilde{A}e \cong A$ as K-algebras, and $e\tilde{Q}e = \operatorname{Hom}_A(e(M \oplus A), e(M \oplus A) \otimes_A D(A)_\sigma)$, where $e(M \oplus A) = A$ and $e(M \oplus A) \otimes_A D(A)_\sigma = A \otimes_A D(A)_\sigma \cong D(A)_\sigma$ as $(e\tilde{A}e, A)$-bimodules. Hence, $e\tilde{Q}e \cong D(A)_\sigma$ as $\boldsymbol{e}T\boldsymbol{e}$-bimodules. Thus, letting $T^{\mathrm{b}} = \boldsymbol{e}T\boldsymbol{e}$, we have a Hochschild extension

$$0 \longrightarrow D(A)_\sigma \longrightarrow T^{\mathrm{b}} \longrightarrow A \longrightarrow 0.$$

Since the block permutation of σ is cyclic, it follows from Theorem 3.16 that the algebra T^{b} is indecomposable, so T is also indecomposable, because T and T^{b} are Morita equivalent algebras and the indecomposability of an algebra is Morita invariant. □

It should be noted that in the above theorem $\tilde{A}$ is not necessarily indecomposable as an algebra, though all Hochschild extension algebras of $\tilde{A}$ by $\tilde{Q}$ are indecomposable. Indeed, there is an isomorphism

$$\tilde{A} \cong \operatorname{End}_{A_1}(P_1 \oplus A_1) \times \cdots \times \operatorname{End}_{A_m}(P_m \oplus A_m)$$

of K-algebras.

Example 4.10. Let A_1 and A_2 be the path algebras $K\Delta_1$ and $K\Delta_2$ of the quivers Δ_1 and Δ_2 of the forms

over a field K. Let $A = A_1 \times A_2$ be the direct product of A_1 and A_2, and σ the automorphism of the K-algebra A with $\sigma(e_1) = e_3$, $\sigma(e_2) = e_4$, $\sigma(e_3) = e_1$, $\sigma(e_4) = e_2$. Then $\sigma(A_1) = A_2$ and $\sigma(A_2) = A_1$, and hence the block permutation of σ is cyclic.

For nonnegative integers $\ell_1, \ell_2, \ell_3, \ell_4$, let

$$P_1 = (e_1 A_1)^{\ell_1} \oplus (e_2 A_1)^{\ell_2}, \quad P_2 = (e_3 A_2)^{\ell_3} \oplus (e_4 A_2)^{\ell_4},$$

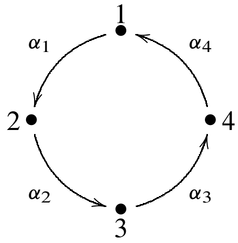
and $M = P_1 \oplus P_2$, by regarding P_1, P_2 as right A-modules. Further, let $\tilde{A}_i = \operatorname{End}_A(P_i \oplus A_i)$, for $i \in \{1, 2\}$, and $\tilde{A} = \operatorname{End}_A(M \oplus A)$.

(1) Assume $(\ell_1, \ell_2) \neq (\ell_3, \ell_4)$. Then, by Theorem 4.9, any Hochschild extension algebra of $\tilde{A}$ by $\tilde{Q} = \operatorname{Hom}_A(M \oplus A, (M \oplus A) \otimes D(A)_\sigma)$ (for example, the trivial extension algebra $T = \tilde{A} \ltimes \tilde{Q}$) is an indecomposable non-Frobenius algebra. On the other hand, $\mathrm{T}(\tilde{A}) = \tilde{A} \ltimes D(\tilde{A}) \cong \mathrm{T}(\tilde{A}_1) \times \mathrm{T}(\tilde{A}_2)$, which is a decomposable Frobenius algebra.

(2) Assume $(\ell_1, \ell_2) = (\ell_3, \ell_4)$. We note that $\tilde{A}_1 \cong \tilde{A}_2$ as K-algebras. In fact, the induced equivalence $F_\sigma \colon \operatorname{mod} A \to \operatorname{mod} A$, $X \mapsto X_\sigma$, assigns $P_2 \oplus A_2$ to $P_1 \oplus A_1$ (see Lemma II.6.3). Hence $\tilde{A}_1 \cong \operatorname{End}_A(F_\sigma(P_1 \oplus A_1)) = \tilde{A}_2$ as K-algebras. Observe that $\tilde{A}_1 / \operatorname{rad}(\tilde{A}_1) \cong M_{\ell_1}(K) \times M_{\ell_2}(K) \times K \times K$ as K-algebras, where $M_\ell(K)$ denotes the K-algebra of $\ell \times \ell$ matrices over the field K.

We consider now the following two special cases.

(i) Assume $(\ell_1, \ell_2) = (0, 0)$, that is, $P_1 = 0 = P_2$ and $\tilde{A} = A$, $Q = D(A)_\sigma$. Then, by Theorem 3.16, any Hochschild extension algebra of $\tilde{A}$ by $D(\tilde{A})_\sigma$ is an indecomposable Frobenius algebra. In particular, $\mathrm{T}_\sigma(A) = A \ltimes D(A)_\sigma$ is isomorphic to the Nakayama algebra with Loewy length 3, defined by the cyclic quiver with 4 vertices



Indeed, let $T = \mathrm{T}_\sigma(A)$, and let $\boldsymbol{e}_i = (e_i, 0) \in T$, for $i \in \{1, 2, 3, 4\}$. By the definition of the trivial extension algebra, $T = A \oplus D(A)_\sigma$ and $\operatorname{rad} T = \operatorname{rad} A \oplus D(A)_\sigma$, as right A-modules. Observe that, for $i \in \{1, 2, 3, 4\}$, $\boldsymbol{e}_i T = e_i A \oplus e_i D(A)_\sigma$, $\boldsymbol{e}_i \operatorname{rad} T = e_i \operatorname{rad} A \oplus e_i D(A)_\sigma$, $\boldsymbol{e}_i (\operatorname{rad} T)^2 = e_i (\operatorname{rad} A)^2 \oplus e_i D(A)_\sigma \operatorname{rad} A$. Hence, we have $\boldsymbol{e}_1 \operatorname{rad} T / \boldsymbol{e}_1 (\operatorname{rad} T)^2 \cong e_1 \operatorname{rad} A / e_1 (\operatorname{rad} A)^2 \cong \operatorname{top}(e_2 A) \cong \operatorname{top}(\boldsymbol{e}_2 T)$ and $\boldsymbol{e}_2 \operatorname{rad} T / \boldsymbol{e}_2 (\operatorname{rad} T)^2 \cong e_2 D(A)_\sigma / e_2 D(A)_\sigma \operatorname{rad} A$ in $\operatorname{mod} T$. Since $e_2 D(A)_\sigma \operatorname{rad} A = (e_2 D(A) \operatorname{rad} A)_\sigma$ for $\sigma(\operatorname{rad} A) = \operatorname{rad} A$, we have isomorphisms in $\operatorname{mod} T$

$$
\begin{aligned}
e_2 D(A)_\sigma / e_2 D(A)_\sigma \operatorname{rad} A &\cong \big(e_2 D(A) / e_2 D(A) \operatorname{rad} A\big)_\sigma \\
&\cong (\operatorname{top} e_1 A)_\sigma \cong \operatorname{top} e_3 A \cong \operatorname{top}(\boldsymbol{e}_3 T).
\end{aligned}
$$

Then we obtain that

$$
\boldsymbol{e}_1 \operatorname{rad} T / \boldsymbol{e}_1 (\operatorname{rad} T)^2 \cong \operatorname{top}(\boldsymbol{e}_2 T) \text{ and } \boldsymbol{e}_2 \operatorname{rad} T / \boldsymbol{e}_2 (\operatorname{rad} T)^2 \cong \operatorname{top}(\boldsymbol{e}_3 T).
$$

Similarly, we conclude that

$$
\boldsymbol{e}_3 \operatorname{rad} T / \boldsymbol{e}_3 (\operatorname{rad} T)^2 \cong \operatorname{top}(\boldsymbol{e}_4 T) \text{ and } \boldsymbol{e}_4 \operatorname{rad} T / \boldsymbol{e}_4 (\operatorname{rad} T)^2 \cong \operatorname{top}(\boldsymbol{e}_1 T).
$$

On the other hand, the trivial extension algebra $\mathrm{T}(\tilde{A}) = \tilde{A} \ltimes D(\tilde{A}) \cong \mathrm{T}(\tilde{A}_1) \times \mathrm{T}(\tilde{A}_2)$ is a decomposable Frobenius algebra.

(ii) Assume $(\ell_1, \ell_2) \neq (0,0)$. By using the natural equivalence $F_\sigma \colon \operatorname{mod} A \to \operatorname{mod} A$ induced by σ, we can take an automorphism $\tilde{\sigma}$ of $\tilde{A}$ such that $\tilde{\sigma}(\tilde{A}_1) = \tilde{A}_2$ and $\tilde{\sigma}(\tilde{A}_2) = \tilde{A}_1$. Then, by Theorem 3.16, all Hochschild extensions of $\tilde{A}$ by $D(\tilde{A})_{\tilde{\sigma}}$ (for example, $\mathrm{T}_{\tilde{\sigma}}(\tilde{A}) = \tilde{A} \ltimes D(\tilde{A})_{\tilde{\sigma}}$) are indecomposable Frobenius algebras, while $\mathrm{T}(\tilde{A}) = \tilde{A} \ltimes D(\tilde{A}) \cong \tilde{A}_1 \times \tilde{A}_1$ is a decomposable Frobenius algebra.

5 Hochschild extension algebras of finite field extensions

The aim of this section is to show the Ohnuki–Takeda–Yamagata theorems from [OTY], which will give us many concrete examples of symmetric or nonsymmetric Hochschild extension algebras of finite dimensional K-algebras A being finite field extensions of a field K.

Let A be a finite dimensional K-algebra over a field K. A K-linear homomorphism $\varphi\colon A \to K$ is said to be *regular* if $\operatorname{Ker}\varphi$ does not contain a nonzero one-sided ideal of A, and *symmetric* if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$. It follows from Theorem IV.2.2 that A is a symmetric algebra if and only if there is a regular symmetric K-linear homomorphism $\varphi\colon A \to K$.

Throughout this section, we treat a Hochschild extension as the extension of an algebra A by a duality A-bimodule Q determined by a 2-cocycle α

$$0 \longrightarrow Q \xrightarrow{\omega} \mathrm{T}_Q(A,\alpha) \xrightarrow{\varrho} A \longrightarrow 0,$$

where $\mathrm{T}_Q(A,\alpha) = A \ltimes_\alpha Q$, ω and ϱ are respectively the canonical monomorphism of T-bimodules and the canonical surjective homomorphism of K-algebras such that $\omega(x) = (0,x)$ and $\varrho(a,x) = a$ for all $a \in A$, $x \in Q$.

Let A be a finite dimensional K-algebra over a field K. Assume that L is a finite field extension of K such that L is contained in the center of A. By Lemma II.7.17, $\operatorname{Hom}_K(A,K)$ and $\operatorname{Hom}_L(A,L)$ are isomorphic as A-bimodules, and let $\gamma\colon \operatorname{Hom}_K(A,K) \to \operatorname{Hom}_L(A,L)$ be an A-bimodule isomorphism. For simplicity, we sometimes denote the A-bimodules $\operatorname{Hom}_K(A,K)$ and $\operatorname{Hom}_L(A,L)$ by $D_K(A)$ and $D_L(A)$, respectively. For $\alpha_K\colon A \times A \to \operatorname{Hom}_K(A,K)$ a 2-cocycle of A by $\operatorname{Hom}_K(A,K)$, we let $\alpha_L = \gamma\alpha_K\colon A \times A \to \operatorname{Hom}_L(A,L)$. Then the following statements hold:

Lemma 5.1. (i) *α_L is a 2-cocycle of A by* $\mathrm{Hom}_L(A, L)$.

(ii) *There is a K-algebra isomorphism $\theta\colon \mathrm{T}_{D_K(A)}(A, \alpha_K) \to \mathrm{T}_{D_L(A)}(A, \alpha_L)$ such that the following diagram is commutative:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_K(A, K) \arrow[r, "\omega_K"] \arrow[d, "\gamma"] & \mathrm{T}_{D_K(A)}(A, \alpha_K) \arrow[r, "\varrho_K"] \arrow[d, "\theta"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_L(A, L) \arrow[r, "\omega_L"] & \mathrm{T}_{D_L(A)}(A, \alpha_L) \arrow[r, "\varrho_L"] & A \arrow[r] & 0 .
\end{tikzcd}$$

Proof. (i) Clearly α_L is K-bilinear. We claim that the 2-cocycle condition

$$a\alpha_L(b, c) - \alpha_L(ab, c) = \alpha_L(a, b)c - \alpha_L(a, bc)$$

holds for all $a, b, c \in A$. Indeed, since α_K is a 2-cocycle, we have the equality

$$a\alpha_K(b, c) - \alpha_K(ab, c) = \alpha_K(a, b)c - \alpha_K(a, bc)$$

for all $a, b, c \in A$. Applying the A-bimodule isomorphism γ to the above equality, we obtain

$$a\big(\gamma\alpha_K(b, c)\big) - \gamma\alpha_K(ab, c) = \big(\gamma\alpha_K(a, b)\big)c - \gamma\alpha_K(a, bc),$$

which is nothing else than the required 2-cocycle condition for α_L.

(ii) Let $\theta\colon A \ltimes_{\alpha_K} \mathrm{Hom}_K(A, K) \to A \ltimes_{\alpha_L} \mathrm{Hom}_L(A, L)$ be the map defined by $\theta(a, u) = (a, \gamma(u))$ for all $a \in A$, $u \in \mathrm{Hom}_K(A, K)$. Then for any $u \in \mathrm{Hom}_K(A, K)$, we have

$$\big(\theta\omega_K\big)(u) = \theta(0, u) = \big(0, \gamma(u)\big) \text{ and } (\omega_L\gamma)(u) = \omega_L\big(\gamma(u)\big) = (0, \gamma(u)).$$

Hence $\theta\omega_K = \omega_L\gamma$. The other equality $\varrho_K = \varrho_L\theta$ follows from the equalities $\varrho_L\theta(a, u) = \varrho_L(a, \gamma(u)) = a$ and $\varrho_K(a, u) = a$, for any $a \in A$. □

By Lemma 5.1, when we consider a Hochschild extension algebra of A by $\mathrm{Hom}_K(A, K)$, it is enough to consider a Hochschild extension algebra of A by $\mathrm{Hom}_L(A, L)$ defined by a 2-cocycle K-linear map $A \times A \to \mathrm{Hom}_L(A, L)$. Let $\alpha\colon A \times A \to \mathrm{Hom}_L(A, L)$ be a 2-cocycle of A by $\mathrm{Hom}_L(A, L)$. For calculations related to 2-cocycles, it is convenient to introduce the notation $[\ ,\ ]_\alpha$:

$$[a, b]_\alpha = \alpha(a, b) - \alpha(b, a)$$

for all $a, b \in A$, where $[a, b]_\alpha$ is also written simply as $[a, b]$, when there is no danger of confusion, and $[A, A]_\alpha$ or $[A, A]$ denotes the K-subspace of $\mathrm{Hom}_L(A, L)$ spanned by $[a, b]$ for all $a, b \in A$. Next two lemmas proved in [OTY] provide useful criteria, for some class of algebras, to know if a Hochschild extension is splittable or a Hochschild extension algebra is symmetric.

Lemma 5.2. *Let A be a finite dimensional commutative K-algebra over a field K and $\alpha\colon A \times A \to \mathrm{Hom}_K(A, K)$ be a 2-cocycle. Then $[A, A]_\alpha = 0$ if the Hochschild extension*

$$0 \longrightarrow \mathrm{Hom}_K(A, K) \xrightarrow{\omega} \mathrm{T}_{D_K(A)}(A, \alpha) \xrightarrow{\varrho} A \longrightarrow 0$$

is splittable.

Proof. Let $T = \mathrm{T}_{D_K(A)}(A, \alpha)$, and assume that there is a homomorphism $\eta\colon A \to T$ of K-algebras with $\varrho\eta = \mathrm{id}_A$. For any $a \in A$, we write $\eta(a) = (a, a')$ for some $a' \in D_K(A)$. For any $a, b \in A$, it then holds that

$$\eta(ab) = \eta(a)\eta(b) = (a, a')(b, b') = \big(ab, ab' + a'b + \alpha(a, b)\big),$$

and similarly

$$\eta(ba) = \big(ba, ba' + b'a + \alpha(b, a)\big).$$

Here, since A is commutative by assumption, it holds that $\eta(ab) = \eta(ba), ab' = b'a, a'b = ba'$. Consequently, we have $\alpha(a, b) = \alpha(b, a)$ for all $a, b \in A$. This means that $[A, A]_\alpha = 0$. □

A finite field extension L of K is regarded as a finite dimensional K-algebra. By Lemma 5.1, any Hochschild extension algebra T of the K-algebra L by $\mathrm{Hom}_K(L, K)$ is isomorphic to a Hochschild extension algebra of L by the K-bimodule $\mathrm{Hom}_L(L, L)$. Hence, using the canonical isomorphism $L \to \mathrm{Hom}_L(L, L)$ of L-bimodules, we conclude that T is isomorphic to a Hochschild extension K-algebra $\mathrm{T}_L(L, \alpha)$ by a K-linear 2-cocycle $\alpha\colon L \times L \to L$ of the K-algebra L by the K-bimodule L. We often abbreviate as $\mathrm{T}(L, \alpha)$ the Hochschild extension K-algebra $\mathrm{T}_L(L, \alpha) = L \ltimes_\alpha L$ defined by a K-linear 2-cocycle $\alpha\colon L \times L \to L$.

Lemma 5.3. *Let L be a finite field extension of K and $\alpha\colon L \times L \to L$ be a K-bilinear 2-cocycle. Then $\mathrm{T}_L(L, \alpha)$ is symmetric if and only if $[L, L]_\alpha \subset L$.*

Proof. Let $T = \mathrm{T}_L(L, \alpha)$. Assume that T is symmetric, and let $\lambda\colon T \to K$ be a regular symmetric K-linear homomorphism. Then, for any $(a, 0), (b, 0) \in T = L \ltimes_\alpha L$, we have

$$\begin{aligned} 0 &= \lambda\big((a,0)(b,0)\big) - \lambda\big((b,0)(a,0)\big) = \lambda\big(ab, \alpha(a,b)\big) - \lambda\big(ba, \alpha(b,a)\big) \\ &= \lambda\big((ab, \alpha(a,b)) - (ba, \alpha(b,a))\big) = \lambda(0, [a,b]), \end{aligned}$$

because $ab = ba \in L$. Hence $\lambda(0, [L, L]) = 0$. On the other hand, $\lambda(0, L) = \lambda(T(0, 1)) \neq 0$ thanks to the regularity of λ. Hence $[L, L] \subset L$.

Conversely, assume that $[L, L] \subset L$, and take an element $c \in L \setminus [L, L]$. Let L' be a K-vector subspace of L such that $L = Kc \oplus L'$ and $[L, L] \subseteq L'$. We define the K-linear homomorphism $\lambda\colon T \to K$ by $\lambda(L, L') = 0$ and $\lambda(0, kc) = k$ for all $k \in K$. We claim that λ is regular and symmetric.

In order to show the regularity, take a nonzero element $x = (a, b) \in T$. If $a \neq 0$, then $\lambda((0, ca^{-1})x) = \lambda(0, c + \alpha(0, a)) = \lambda(0, c)$ and $\lambda(0, c) = 1$ by definition. If $a = 0$, then $b \neq 0$ and $\lambda((cb^{-1}, 0)x) = \lambda((cb^{-1}, 0)(0, b)) = \lambda(0, c + \alpha(cb^{-1}, 0)) = \lambda(0, c) = 1$. Thus we conclude that $\lambda(Ax) \neq 0$ for all $0 \neq x \in T$.

Next, to show that λ is symmetric, take any $x = (a, a')$, $y = (b, b')$ in T. Then we have the equalities

$$\begin{aligned}\lambda(xy) - \lambda(yx) &= \lambda(xy - yx)\\ &= \lambda\big((ab, ab' + a'b + \alpha(a, b)) - (ba, ba' + b'a + \alpha(b, a))\big)\\ &= \lambda(0, [a, b]),\end{aligned}$$

because L is commutative. Since $[a, b] \in L'$, it follows that $\lambda(0, [a, b]) = 0$ by the definition of λ, and hence $\lambda(xy) = \lambda(yx)$ for all $x, y \in A$. □

Example 5.4. Let $K = \mathbb{Z}_2(u)$ be the field of rational functions in one variable u over the field $\mathbb{Z}_2$, which denotes the residue field $\mathbb{Z}/2\mathbb{Z}$. Let $K[X]$ be the polynomial algebra in one variable X over K and

$$L = K[X]/\big(X^2 - u\big)$$

be the quotient algebra of $K[X]$ by the ideal generated by $X^2 - u$. We denote by x the residue class $X + (X^2 - u) \in L$. Notice that $1_L = 1_K + (X^2 - u)$, and hence L has a basis $\{1_L, x\}$ as a K-vector space, and $x^2 = u \in L$. We define a K-bilinear map $\alpha\colon L \times L \to L$ naturally by

$$\alpha\big(x^l, x^m\big) = x^{l+m}, \quad \text{for } l, m \in \{0, 1\}.$$

Then α is a 2-cocycle of the K-algebra L by the L-bimodule L. In fact, it is enough to show the 2-cocycle condition for the elements x^l, x^m, x^n, for $l, m, n \in \{0, 1\}$,

$$x^l\alpha\big(x^m, x^n\big) - \alpha\big(x^{l+m}, x^n\big) = \alpha\big(x^l, x^m\big)x^n - \alpha\big(x^l, x^{m+n}\big),$$

which however follows by the definition of α.

We claim that $\mathrm{T}(L, \alpha)$ is symmetric. Indeed, $\alpha(x^l, x^m) = x^{l+m} = \alpha(x^m, x^l)$ for all $m, l \in \{0, 1\}$, which implies that $[L, L] = 0 \subset L$. Hence, it follows from Lemma 5.3 that $\mathrm{T}(L, \alpha)$ is symmetric. Next, we claim that the canonical surjective homomorphism $\varrho\colon \mathrm{T}(L, \alpha) \to L$ of K-algebras is not splittable. For this, by Lemma 2.5, it suffices to show that α is not splittable. Suppose now, on the contrary, that α is splittable, and let $\gamma\colon L \to L$ be a K-linear map with $\alpha(a, b) = a\gamma(b) - \gamma(a)b$ for all $a, b \in L$. Then, for $a = x$ and $b = x$, it holds that $\alpha(a, b) = x^2 = u$ and $a\gamma(b) - \gamma(a)b = x\gamma(x) - \gamma(x)x = 0$, which implies $u = 0$, a contradiction.

We notice that the above example also shows that the converse of Lemma 5.2 is not true in general.

Example 5.5. Let $K = \mathbb{Z}_2(u, v)$ be the field of rational functions in two variables u, v over the field $\mathbb{Z}_2$. Let $K[X, Y]$ be the polynomial algebra in two variables X, Y over K and

$$L = K[X, Y]/\big(X^2 - u, Y^2 - v\big)$$

the quotient algebra of $K[X, Y]$ by the ideal generated by $X^2 - u$ and $Y^2 - v$. Let $x = X + (X^2 - u, Y^2 - v)$ and $y = Y + (X^2 - u, Y^2 - v)$ in L. Observe that the set $\{1, x, y, xy\}$ is a basis of L as K-vector space. Hence $L = K \oplus Kx \oplus Ky \oplus Kxy$, and $x^2 = u$, $y^2 = v$, $xy = yx$. Now let us define a K-bilinear map $\alpha\colon L \times L \to L$ by

$$\alpha\big(x^l y^m, x^p y^q\big) = lqx^{l+p} y^{m+q}$$

for $l, m, p, q \in \{0, 1\}$. To show that α is a 2-cocycle it is enough to check the 2-cocycle condition for the elements $x^l y^m$, $x^p y^q$, $x^r y^s$, with $l, m, p, q, r, s \in \{0, 1\}$. In fact, letting $a = x^l y^m$, $b = x^p y^q$ and $c = x^r y^s$, we have that

$$\begin{aligned} a\alpha(b, c) - \alpha(ab, c) &= x^l y^m \big(psx^{p+r} y^{q+s}\big) - \alpha\big(x^{l+p} y^{m+q}, x^r y^s\big) \\ &= ps\, x^{l+(p+r)} y^{m+(q+s)} - (l + p)s\, x^{(l+p)+r} y^{(m+q)+s} \\ &= -ls\, x^{l+p+r} y^{m+q+s} \end{aligned}$$

and

$$\begin{aligned} \alpha(a, b)c - \alpha(a, bc) &= \big(lqx^{l+p} y^{m+q}\big)x^r y^s - \alpha\big(x^l y^m, x^{p+r} y^{q+s}\big) \\ &= lq\, x^{(l+p)+r} y^{(m+q)+s} - l(q + s)x^{l+(p+r)} y^{m+(q+s)} \\ &= -ls\, x^{l+p+r} y^{m+q+s}. \end{aligned}$$

Hence, $a\alpha(b, c) - \alpha(ab, c) = \alpha(a, b)c - \alpha(a, bc)$.

Next we claim that $0 \neq [L, L] \subset L$. It suffices to show that $[L, L] = Kx + Ky + Kxy$. We have

$$\begin{aligned} \big[x^l y^m, x^p y^q\big] &= \alpha\big(x^l y^m, x^p y^q\big) - \alpha\big(x^p y^q, x^l y^m\big) \\ &= lqx^{l+p} y^{m+q} - pmx^{p+l} y^{q+m} = (lq - pm)x^{l+p} y^{m+q}, \end{aligned}$$

for $l, m, p, q \in \{0, 1\}$. By choosing suitable values for l, m, p, q in these equalities, it is easily seen that

$$[1, 1] = 0,\ [x, x] = 0,\ [y, y] = 0,\ [xy, xy] = 0,$$
$$[x, y] = xy,\ [x, xy] = x^2 y = uy,\ [y, xy] = -xy^2 = -vx.$$

Moreover $[z_1, z_2] = -[z_2, z_1]$ for all $z_1, z_2 \in L$. Using the K-basis $1, x, y, xy$ of L, those equalities imply that $[L, L] = Kx + Ky + Kxy$, as desired. Hence, applying Lemmas 5.2 and 5.3, we conclude that the extension $\mathrm{T}(L, \alpha)$ is not splittable, but is symmetric.

In the above two examples, the K-algebras $\mathrm{T}(L,\alpha)$ are symmetric and the finite field extensions L have at most two generators as K-algebras. We shall show that this phenomenon is true for any finite field extensions. We start by proving some formulas related to the operator $[\ ,\ ] = [\ ,\ ]_\alpha$ associated with a 2-cocycle α.

Lemma 5.6. *Let $\alpha\colon L \times L \to L$ be a 2-cocycle of the K-algebra L. Then the following statements hold for $a, b \in L$.*

(i) $a\alpha(1,1) = \alpha(a,1) = \alpha(1,a)$.

(ii) $[a, bc] = [a,b]c + [a,c]b$, $[ab,c] = a[b,c] + b[a,c]$.

In particular, we have $[a,1] = 0$ *and* $[1,a] = 0$.

Proof. (i) Recall the 2-cocycle condition for the triple $a,b,c \in L$: $a\alpha(b,c) - \alpha(ab,c) = \alpha(a,b)c - \alpha(a,bc)$. Applying it to the triple $a,1,1$, we obtain the equality $a\alpha(1,1) - \alpha(a1,1) = \alpha(a,1)1 - \alpha(a,1\cdot 1)$, which implies that $a\alpha(1,1) = \alpha(a,1)$. Similarly, we get $\alpha(1,1)a = \alpha(1,a)$ for the triple $1,1,a$. Since L is commutative, it follows that $\alpha(a,1) = a\alpha(1,1) = \alpha(1,a)$.

(ii) Since $[ab,c] = -[c,ab]$, we only show the first equality. By the definition of the 2-cocycle α,

$$\begin{aligned}
[a,bc] &= \alpha(a,bc) - \alpha(bc,a) \\
&= \big(\alpha(ab,c) + \alpha(a,b)c - a\alpha(b,c)\big) - \big(b\alpha(c,a) + \alpha(b,ca) - \alpha(b,c)a\big) \\
&= \big(\alpha(ab,c) + \alpha(a,b)c - b\alpha(c,a)\big) - \alpha(b,ac) \\
&= \big(\alpha(ab,c) + \alpha(a,b)c - b\alpha(c,a)\big) - \big(\alpha(ba,c) + \alpha(b,a)c - b\alpha(a,c)\big) \\
&= \alpha(a,b)c - b\alpha(c,a) - \alpha(b,a)c + b\alpha(a,c) \\
&= \big(\alpha(a,b) - \alpha(b,a)\big)c + \big(\alpha(a,c) - \alpha(c,a)\big)b \\
&= [a,b]c + [a,c]b.
\end{aligned}$$

In particular, taking $b = c = 1$, we get $[a,1] = [a,1] + [a,1]$, and so $[a,1] = 0$. The relation $[1,a] = 0$ is also true, because $[1,a] = -[a,1]$ by definition. □

Lemma 5.7. *For any $a,b \in L$ and $i,j,i',j' \in \mathbb{Z}$, the following statements hold:*

(i) $[a^i, b^j] = ija^{i-1}b^{j-1}[a,b]$ *and* $[a^i, a^j] = 0$.

(ii) $[a^ib^j, a^{i'}b^{j'}] = (ij' - ji')a^{i+i'-1}b^{j+j'-1}[a,b]$.

(iii) $[f(a)b^j, a] = jf(a)b^{j-1}[b,a]$ *for any polynomial* $f(X) \in K[X]$.

Proof. (i) The second relation $[a^i, a^j] = 0$ follows from the first one, by taking $b = a$, because $[a,a] = 0$ by definition. To show the first equality, we first show that $[a^i, c] = ia^{i-1}[a,c]$ for all $a, c \in L$. This obviously holds for $i = 0, 1$, because $[1, c] = 0$, by Lemma 5.6. Let $i \geq 1$ and assume that the equality holds for $i - 1$. By Lemma 5.6 (ii), $\left[a \cdot a^{i-1}, c\right] = a\left[a^{i-1}, c\right] + a^{i-1}[a, c]$ and, by the inductive hypothesis, $a\left[a^{i-1}, c\right] = a\big((i-1)a^{i-2}[a, c]\big) = (i-1)a^{i-1}[a, c]$. Hence $[a^i, c] = [a \cdot a^{i-1}, c] = ia^{i-1}[a, c]$ for all $a, c \in L$ and $i \geq 0$. Next, we consider the case when $i < 0$. Since $a^{-1} \in L$ and $-i > 0$, we have the equalities $[a^i, c] = [(a^{-1})^{-i}, c] = (-i)(a^{-1})^{-i-1}[a^{-1}, c] = -ia^{i+1}[a^{-1}, c]$. Moreover, $[a^{-1}, c] = -a^{-2}[a, c]$. For, by Lemma 5.6 (ii), we obtain

$$\begin{aligned}[a^{-1}, c] &= a^{-1}(a[a^{-1}, c]) = a^{-1}\big([aa^{-1}, c] - a^{-1}[a, c]\big)\\ &= a^{-1}[1, c] - a^{-2}[a, c] = -a^{-2}[a, c].\end{aligned}$$

Therefore, $\left[a^i, c\right] = -ia^{i+1}\left(-a^{-2}[a, c]\right) = ia^{i-1}[a, c]$ for $i < 0$. Summing up, we obtain that $\left[a^i, c\right] = ia^{i-1}[a, c]$ for all integers i.

Now, for all $a, b \in L$ and $i, j \in \mathbb{Z}$, applying the equalities shown above, we obtain that $[a^i, b^j] = ia^{i-1}[a, b^j] = -ia^{i-1}[b^j, a]$ and $[b^j, a] = jb^{j-1}[b, a] = -jb^{j-1}[a, b]$. Hence, $[a^i, b^j] = ija^{i-1}b^{j-1}[a, b]$ for all integers i, j, as required.

(ii) By Lemma 5.6 and (i),

$$\begin{aligned}\left[a^i b^j, a^{i'} b^{j'}\right] &= a^i\left[b^j, a^{i'} b^{j'}\right] + b^j\left[a^i, a^{i'} b^{j'}\right]\\ &= \left(a^{i+i'}\left[b^j, b^{j'}\right] + a^i b^{j'}\left[b^j, a^{i'}\right]\right)\\ &\quad + \left(a^{i'} b^j\left[a^i, b^{j'}\right] + b^{j+j'}\left[a^i, a^{i'}\right]\right)\\ &= a^i b^{j'}\left[b^j, a^{i'}\right] + a^{i'} b^j\left[a^i, b^{j'}\right]\\ &= a^i b^{j'}\big(ji'b^{j-1}a^{i'-1}\big)[b, a] + a^{i'} b^j\big(ij'a^{i-1}b^{j'-1}\big)[a, b]\\ &= (ij' - ji')a^{i+i'-1}b^{j+j'-1}[a, b],\end{aligned}$$

where $[b, a] = -[a, b]$ by definition.

(iii) Let $f(X) = c_n X^n + c_{n-1}X^{n-1} + \cdots + c_1 X + c_0$ be a polynomial in $K[X]$. Then $[f(a)b^j, a] = \sum_{i=0}^{n} c_i [a^i b^j, a]$, because $[\ ,\ ]$ is K-bilinear, where by (i) $[a^i b^j, a] = [a^i b^j, a^1 b^0] = -ja^i b^{j-1}[a, b] = ja^i b^{j-1}[b, a]$. Hence,

$$\left[f(a)b^j, a\right] = \sum_{i=0}^{n} \left(c_i a^i\right)\left(jb^{j-1}\right)[b, a] = f(a)\left(jb^{j-1}\right)[b, a]. \qquad \square$$

Let p be a positive integer. We let $\varrho(n)$ denote the residue of an integer n by p, that is, $n = pm + \varrho(n)$ for some integer m and $0 \leq \varrho(n) < p$. Notice that $\varrho(0) = 0$. We do not append p in the notation ϱ, but there will be no danger of confusion in the sequel.

Let L be a finite field extension of K. Take two elements $a, b \in L$, and let $K(a)$ and $K(a, b)$ be the subfields of L generated by a and by a, b, respectively.

Let $d_1 = \dim_K K(a)$ and $d_2 = \dim_{K(a)} K(a,b)$. Then the set $\{1, a, \dots, a^{d_1-1}\}$ is a basis of $K(a)$ as K-vector space, and the set $\{1, b, \dots, b^{d_2-1}\}$ is a basis of $K(a,b)$ as $K(a)$-vector space. Further, the set $\{a^l b^m \mid 0 \le l < d_1, 0 \le m < d_2\}$ is a basis of the K-vector space $K(a,b)$. Let

$$\chi = \left\{a^l b^m \,\middle|\, 0 \le l < d_1,\, 0 \le m < d_2\right\}$$

and $\chi_{i,j} = \{a^l b^m \in \chi \mid \varrho(l) = i, \varrho(m) = j\}$ for $0 \le i, j < p$. Obviously, χ is the disjoint union of $\chi_{i,j}$ for $0 \le i, j < p$. Let $\bar{\chi}$ and $\bar{\chi}_{i,j}$ be the K-subspaces of L spanned by χ and $\chi_{i,j}$, respectively. Notice that $\chi_{i,0}$ and $\chi_{0,j}$ consist of the elements $a^l b^m \in \chi$ with m multiple of p and l multiple of p, respectively. We let $\bar{\chi}_{i,0}\bar{\chi}_{0,j}$ denote the K-subspace of L spanned by xy for all $x \in \bar{\chi}_{i,0}$ and $y \in \bar{\chi}_{0,j}$.

Lemma 5.8. *Assume that* $\operatorname{char} K = p > 0$. *Let α be a 2-cocycle of L and a, b two elements of L such that $[a,b] \ne 0$. Let $d_1 = \dim_K K(a)$ and $d_2 = \dim_{K(a)} K(a,b)$. Then $\varrho(d_1) = 0$, $\varrho(d_2) = 0$, and the following assertions hold for all nonnegative integers l, m:*

(i) $a^l = \sum_{0 \le i < d_1, \varrho(i) = \varrho(l)} \lambda_i a^i$ *for some* $\lambda_i \in K$.

(ii) $a^l b^m \in \bar{\chi}_{\varrho(l),\varrho(m)}$.

(iii) $\bar{\chi}_{i,j} = \bar{\chi}_{i,0}\bar{\chi}_{0,j} \ne 0$ *for* $i, j \in \{0, 1, \dots, p-1\}$.

Proof. To simplify notations, for nonnegative integers i, m, we consider the sets

$$\varrho_1\{m\} = \{i \mid 0 \le i < d_1, \varrho(i) = \varrho(m)\},$$
$$\varrho_2\{m\} = \{i \mid 0 \le i < d_2, \varrho(i) = \varrho(m)\}.$$

The equalities $\varrho(d_1) = 0$ and $\varrho(d_2) = 0$ will be shown in the proofs of (i) and (ii), respectively.

(i) There is nothing to prove for $0 \le l < d_1$. For $l = d_1$, let

$$a^{d_1} = \sum_{0 \le i < d_1} k_i a^i$$

for some $k_i \in K$, where $K(a) = K[a]$ and $K(a)$ has the K-basis $\{1, a, \dots, a^{d_1-1}\}$. It follows from Lemma 5.7 that

$$[a^{d_1}, b] = d_1 a^{d_1-1}[a,b],$$
$$[a^{d_1}, b] = \sum_{0 \le i < d_1} k_i [a^i, b] = \sum_{0 < i < d_1} i k_i a^{i-1}[a,b],$$

where $[a^0, b] = [1, b] = 0$. Since $[a,b] \ne 0$ by assumption, comparing the right-hand sides of the above equalities, we see that $d_1 a^{d_1-1} - \sum_{0 < i < d_1} i k_i a^{i-1} = 0$, which implies that $d_1 1_K = 0$, $ik_i = (i1_K)k_i = 0$, for $i \in \{1, 2, \dots, d_1 - 1\}$, because the set $\{1, a, \dots, a^{d_1-1}\}$ is a K-basis of $K(a)$. Since $\operatorname{char} K = p \neq 0$, this is equivalent to saying that d_1 is divisible by p and $k_i = 0$ for all $0 < i < d_1$ with $i1_K \neq 0$, or equivalently, that

$$\varrho(d_1) = 0$$

and $k_i = 0$ for all $0 < i < d_1$ with $\varrho(i) \neq 0$. Hence, we have the required form for some $k_0, \dots, k_{d_1-1} \in K$:

$$a^{d_1} = \sum_{i \in \varrho_1\{d_1\}} k_i a^i.$$

So far, we have proved the assertion (i) for $0 \leq l \leq d_1$.

Next, let $r > d_1$, and assume that (i) is true for all $0 \leq l < r$. We shall show (i) for r by induction. Let $r' = r - d_1 > 0$. Multiplying the above equality by $a^{r'}$, we obtain

$$a^r = a^{d_1} a^{r'} = \sum_{i \in \varrho_1\{d_1\}} k_i a^{i+r'}.$$

By the inequality $i + r' < d_1 + r' = r$ for $0 \leq i < d_1$ and inductive hypothesis, each $a^{i+r'}$ can be written as

$$a^{i+r'} = \sum_{j \in \varrho_1\{i+r'\}} l_j a^j = \sum_{j \in \varrho_1\{r\}} l_j a^j,$$

for some $l_j \in K$, where $\varrho(i + r') = \varrho(r')$, because $\varrho(i) = 0$, and $\varrho(r') = \varrho(r - d_1) = \varrho(r)$, because $\varrho(d_1) = 0$ as shown above, and hence $\varrho(i + r') = \varrho(r)$. Hence, letting $k'_j = \sum_{i \in \varrho_1\{d_1\}} k_i l_j$ for $0 \leq j < d_1$, a^r is written in the form $a^r = \sum_{j \in \varrho_1\{r\}} k'_j a^j$, which is a desired expression of a^r.

(ii) First we shall show that $b^{d_2} \in \bar{\chi}_{0,\varrho(d_2)}$. Since $K(a, b) = K(a)[b]$, we may write $b^{d_2} = \sum_{0 \leq j < d_2} f_j(a) b^j$ for some $f_j(a) \in K[a]$. Let $f_j(a) = \sum_{0 \leq i < d_1} k_{ij} a^i$ for some $k_{ij} \in K$. It follows from Lemma 5.7 that

$$[b^{d_2}, a] = d_2 b^{d_2-1}[b, a],$$

$$[b^{d_2}, a] = \sum_{0 \leq j < d_2} [f_j(a) b^j, a] = \sum_{0 \leq j < d_2} j f_j(a) b^{j-1}[b, a].$$

Since $[b, a] \neq 0$ by assumption, these two equalities show that $d_2 b^{d_2-1} - \sum_{0 \leq j < d_2} j f_j(a) b^{j-1} = 0$, which implies that $d_2 1_K = 0$ and $j f_j(a) = 0$ for $0 \leq j < d_2$, because $K(a, b)$ has a basis $\{1, b, \dots, b^{d_2-1}\}$ as a $K(a)$-vector space. Since $\operatorname{char} K = p \neq 0$, this is equivalent to saying that d_2 is divisible by p

and $f_j(a) = 0$ for $0 \leq j < d_2$ with $j1_K \neq 0$, or equivalently, $\varrho(d_2) = 0$ and $f_j(a) = 0$ for $0 \leq j < d_2$ with $\varrho(j) \neq 0$. Therefore,

$$\varrho(d_2) = 0$$

and $b^{d_2} = \sum_{j \in \varrho_2\{d_2\}} f_j(a) b^j$. On the other hand, $[b^{d_2}, b] = 0$ by Lemma 5.7 (i), and Lemma 5.7 (ii) shows that

$$\begin{aligned}
[b^{d_2}, b] &= \sum_{j \in \varrho_2\{d_2\}} [f_j(a) b^j, b] \\
&= \sum_{j \in \varrho_2\{d_2\}} \sum_{0 \leq i < d_1} k_{ij} [a^i b^j, b] = \sum_{j \in \varrho_2\{d_2\}} \sum_{0 \leq i < d_1} i k_{ij} a^{i-1} b^j [a, b].
\end{aligned}$$

Hence, as $[a, b] \neq 0$, we get

$$\sum_{j \in \varrho_2\{d_2\}} \sum_{0 \leq i < d_1} i k_{ij} a^{i-1} b^j = 0.$$

Because the set χ is K-linearly independent, it follows that $i k_{ij} = 0$ for $0 \leq i < d_1$, $0 \leq j < d_2$ with $\varrho(j) = 0$. Since $i k_{ij} = (i1_K) k_{ij}$, this implies that $k_{ij} = 0$ in case $i1_K \neq 0$, that is, $k_{ij} = 0$ for $i \notin \varrho_1\{d_1\}$ and $0 \leq j < d_2$, $\varrho(j) = 0$. Therefore, we have $f_j(a) = \sum_{i \in \varrho_1\{d_1\}} k_{ij} a^i$ for $0 \leq j < d_2$, $\varrho(j) = 0$, and we conclude that

$$b^{d_2} = \sum_{j \in \varrho_2\{d_2\}} f_j(a) b^j = \sum_{i \in \varrho_1\{d_1\}} \sum_{j \in \varrho_2\{d_2\}} k_{ij} a^i b^j \in \bar{\chi}_{0, \varrho(d_2)},$$

where $\varrho(d_2) = 0$.

Now we shall show that $a^l b^m \in \bar{\chi}_{\varrho(l), \varrho(m)}$. In case $0 \leq m < d_2$, it follows from (i) that $a^l b^m$ is spanned by the elements $a^i b^m$ with $i \in \varrho_1\{l\}$, where $a^i b^m \in \bar{\chi}_{\varrho(l), \varrho(m)}$. Hence, $a^l b^m \in \bar{\chi}_{\varrho(l), \varrho(m)}$.

Next, let $m \geq d_2$, and assume that $a^l b^t \in \bar{\chi}_{\varrho(l), \varrho(t)}$ for all nonnegative integers l and t with $0 \leq t < m$. Let $m' = m - d_2 \geq 0$. Then, using the above equality for b^{d_2} and (i), we obtain $a^l b^m = a^l b^{d_2} b^{m'}$ and

$$a^l b^{d_2} b^{m'} = \sum_{i \in \varrho_1\{d_1\}} \sum_{j \in \varrho_2\{d_2\}} k_{ij} a^{i+l} b^{j+m'} = \sum_{i \in \varrho_1\{l\}} \sum_{j \in \varrho_2\{d_2\}} k'_{ij} a^i b^{j+m'},$$

for some $k'_{ij} \in K$. Here, we have used the fact that, for each i with $0 \leq i < d_1$ and $\varrho(i) = 0$, by applying (i), a^{i+l} is written as a K-linear combination of the elements a^r with $0 \leq r < d_1$ and $\varrho(r) = \varrho(i+l) = \varrho(l)$. Further, $0 \leq m' \leq j + m' < d_2 + m' = m$, $\varrho(j + m') = \varrho(m')$ for $\varrho(j) = 0$, and $\varrho(m') = \varrho(m - d_2) = \varrho(m)$, because $\varrho(d_2) = 0$. Hence, we have $\varrho(j + m') = \varrho(m)$. Since $j + m' < m$, it follows by the inductive hypothesis that $a^i b^{j+m'} \in \bar{\chi}_{\varrho(i),\varrho(j+m')} = \bar{\chi}_{\varrho(i),\varrho(m)}$, so we conclude that $a^l b^m \in \bar{\chi}_{\varrho(l),\varrho(m)}$.

(iii) By (ii), $a^i b^j$ is contained in $\bar{\chi}_{i,j}$ for all $0 \leq i, j < p$, and hence $\bar{\chi}_{i,j} \neq 0$. Next we claim that $\chi_{i,j} \subseteq \chi_{i,0}\chi_{0,j}$, and so $\bar{\chi}_{i,j} \subseteq \bar{\chi}_{i,0}\bar{\chi}_{0,j}$. Indeed, for $a^l b^m \in \chi_{i,j}$, we have

$$a^l b^m = \big(a^l b^0\big)\big(a^0 b^m\big) \in \bar{\chi}_{\varrho(l),0}\bar{\chi}_{0,\varrho(m)} = \bar{\chi}_{i,0}\bar{\chi}_{0,j},$$

and hence $\chi_{i,j} \subseteq \bar{\chi}_{i,0}\bar{\chi}_{0,j}$ and $\bar{\chi}_{i,j} \subseteq \bar{\chi}_{i,0}\bar{\chi}_{0,j}$. On the other hand, for $x = a^l b^{pt} \in \chi_{i,0}$ and $y = a^{ps} b^m \in \chi_{0,j}$, we have $\varrho(l+ps) = \varrho(l) = i$ and $\varrho(m+pt) = \varrho(m) = j$, and hence $xy = a^{l+ps} b^{m+pt} \in \chi_{i,j}$. This implies that $\bar{\chi}_{i,0}\bar{\chi}_{0,j} \subseteq \bar{\chi}_{i,j}$, and consequently $\bar{\chi}_{i,j} = \bar{\chi}_{i,0}\bar{\chi}_{0,j}$. ☐

Theorem 5.9. *Let L be an algebraic extension of a field K generated by at most two elements. Then all Hochschild extension algebras of the K-algebra L by the L-bimodule L are symmetric.*

Proof. By Lemma 5.3, we have only to show that $[L, L] \subset L$. First, consider the case when L is a simple extension field of K. In case $L = K$, this is clear, because $[K, K] = K[1, 1] = 0 \subset L$ (remember that $[\ ,\ ]$ is K-bilinear). In case $K \subset L$, let $L = K(a)$ for some element $a \in L \setminus K$ and $d = \dim_K L > 0$. Then all elements of L are written in the form $\sum_{i=0}^{d-1} k_i a^i$. Since $[a^l, a^m] = 0$ for all integers l, m, by Lemma 5.7, it then follows that $[x, y] = 0$ for all $x, y \in L$, and so $[L, L] = 0 \subset L$. Thus, to complete the proof, we consider the case when L is not a simple extension of K, and then $\operatorname{char} K = p > 0$ (see [Coh, Theorem 7.9.2]) and $L = K(a, b)$ for some elements $a, b \in L \setminus K$, by assumption. Let $d_1 = \dim_K K(a)$ and $d_2 = \dim_{K(a)} L$.

Then $\{1, a, \ldots, a^{d_1-1}\}$ forms a K-basis of $K(a)$, and $\{1, b, \ldots, b^{d_2-1}\}$ forms a $K(a)$-basis of $L = K(a, b)$. Further, $\chi = \{a^i b^i \mid 0 \leq i < d_1, 0 \leq j < d_2\}$ is a K-basis of L. Let I be the K-vector subspace of L spanned by

$$(ij' - ji')a^{i+i'-1} b^{j+j'-1},$$

for all $0 \leq i < d_1$, $0 \leq j < d_2$. Then $[L, L] = I[a, b]$, by Lemma 5.7. Since the proper inclusion $[L, L] \subset L$ is clear if $[a, b] = 0$, we consider the case when $[a, b] \neq 0$. Then it is enough to show that $I \subset L$. Indeed, in this case, $x[a, b] \notin I[a, b]$ for an element $x \in L \setminus I$, which ensures that $I[a, b] \subset L$.

Take any nonnegative integers l, m, l', m', and let

$$c = (lm' - ml')a^{l+l'-1} b^{m+m'-1},$$
$$r_1 = \varrho(l + l' - 1) \text{ and } r_2 = \varrho(m + m' - 1),$$

for any $0 \leq l, l' < d_1$, $0 \leq m, m' < d_2$. We claim that the K-subspace I spanned by all those elements c is properly contained in L. Assume that $c \neq 0$.

Then, $l + l' \geq 1$ and $m + m' \geq 1$, because if $l + l' = 0$ or $m + m' = 0$, then $l = l' = 0$ or $m = m' = 0$, which implies that $c = 0$, a contradiction. Moreover, $l + l'$ or $m + m'$ does not belong to $p\mathbb{Z}$. Indeed, if $l + l' \in p\mathbb{Z}$ and $m + m' \in p\mathbb{Z}$, then $l' = ps - l$ and $m' = pt - m$ for some integers s, t. Hence $lm' - ml' = (lt - ms)p \in p\mathbb{Z}$, and so $(lm' - ml')1_K = 0$, because char $K = p$. This implies that $c = 0$, a contradiction. Now, in case $l + l' \notin p\mathbb{Z}$, we may write $l + l' = ps + q$ for some integers s, q with $0 < q < p$. Hence $l + l' - 1 = ps + (q - 1)$ and $0 \leq q - 1 < p - 1$, that is, $r_1 = q - 1$ and $0 \leq r_1 < p - 1$. Similarly, we have $0 \leq r_2 < p - 1$ in case $m + m' \notin p\mathbb{Z}$. Therefore, since $c \in \bar{\chi}_{r_1,r_2}$, by Lemma 5.8 (ii), I is contained in the K-subspace J spanned by the elements of all $\bar{\chi}_{i,j}$ with $0 \leq i < p-1$ or $0 \leq j < p-1$. On the other hand, $J \cap \bar{\chi}_{p-1,p-1} = 0$ by definition, and $\bar{\chi}_{p-1,p-1} \neq 0$, by Lemma 5.8, which implies that $J \subset L$. Hence we conclude that $I \subseteq J \subset L$, as required. ☐

We obtain the following direct consequence of the above theorem.

Corollary 5.10. *Let L be a finite algebraic extension of a field K of characteristic 0. Then all Hochschild extension algebras of the K-algebra L by the L-bimodule L are symmetric.*

The next example shows that the assumption on the number of generators of L over K is essential for the validity of Theorem 5.9.

Example 5.11. Let $K = \mathbb{Z}_2(u, v, w)$ be the field of rational functions in three variables u, v, w over the field $\mathbb{Z}_2$. Let $K[X, Y, Z]$ be the polynomial algebra in three variables X, Y, Z over K and

$$L = K[X, Y, Z]/\big(X^2 - u, Y^2 - v, Z^2 - w\big)$$

the quotient algebra of $K[X, Y, Z]$ by the ideal $(X^2 - u, Y^2 - v, Z^2 - w)$. Then L has a basis $\{1, x, y, z, xy, yz, zx, xyz\}$ as a K-vector space, with $x^2 = u$, $y^2 = v$, $z^2 = w$ in L. Let us define a K-bilinear map $\alpha\colon L \times L \to L$ of the K-algebra L by the K-bimodule L by

$$\alpha\big(x^l y^m z^n, x^p y^q z^r\big) = x^{l+p-1} y^{m+q-1} z^{n+r-1}(lqz + mrxy),$$

for $l, m, n, p, q, r \in \{0, 1\}$.

First of all, we claim that α is a 2-cocycle. For this, we have to show the 2-cocycle condition:

$$f\alpha(g, h) - \alpha(fg, h) = \alpha(f, g)h - \alpha(f, gh)$$

for f, g, h from the K-basis $\big\{x^l y^m z^n \mid l, m, n \in \{0, 1\}\big\}$ of L. This is a direct consequence of the definition of α, and we leave it to the readers as an exercise.

Next we claim that $\mathrm{T}(L, \alpha)$ is a nonsplittable extension algebra and nonsymmetric. Observe that

$$\begin{gathered}[x, y] = 1,\ [x, xy] = x,\ [y, z] = y,\ [x, yz] = z,\ [xy, z] = xy, \\ [yz, z] = yz,\ [x, xyz] = zx,\ [xyz, z] = xyz.\end{gathered}$$

It follows that the K-vector space $[L, L]$ contains the K-subspace generated by $\{1, x, y, z, xy, yz, zx, xyz\}$, and hence $L = [L, L] \neq 0$. By Lemmas 5.2 and 5.3, we conclude that $\mathrm{T}_L(L, \alpha)$ is a nonsplittable extension algebra and nonsymmetric.

6 Hochschild extension algebras of path algebras

In this section we show a way to construct symmetric Hochschild extension algebras of the path algebras of finite acyclic quivers over a field.

Let L be a finite field extension of a field K and $Q = (Q_0, Q_1, s, t)$ a finite connected acyclic quiver. We denote by Q_+ the set of all paths in Q of length ≥ 1. Let A be the path algebra LQ of Q over L, considered as a K-algebra. The set $\mathcal{B} = \{e_i, p \mid i \in Q_0, p \in Q_+\}$ forms an L-basis of A. By e_i^*, p^* we denote the L-linear maps from A to L such that

$$e_i^*(e_j) = \delta_{ij} 1_L,\ e_i^*(q) = 0, \text{ and } p^*(e_j) = 0,\ p^*(q) = \delta_{pq} 1_L$$

for all $j \in Q_0$ and $q \in Q_+$, where 1_L is the identity of L.

The set $\mathcal{B}^* = \{e_i^*, p^* \mid e_i, p \in \mathcal{B}\}$ forms an L-basis of $\mathrm{Hom}_L(A, L)$, called the *dual basis* of $\mathcal{B}$. Indeed, for any $f \in \mathrm{Hom}_L(A, L)$, it is easily seen that

$$f = \sum_{i \in Q_0} f(e_i) e_i^* + \sum_{p \in Q_+} f(p) p^*,$$

by comparing the value of both sides at each element of $\mathcal{B}$. This shows that $\mathcal{B}^*$ is a generating set of the L-vector space $\mathrm{Hom}_L(A, L)$. To show that $\mathcal{B}^*$ is an independent set over L, consider $x = \sum_i \lambda_i e_i^* + \sum_p \mu_p p^* \in \mathrm{Hom}_L(A, L)$ for some $\lambda_i, \mu_p \in L$. Then, for any $j \in Q_0$ and $q \in Q_+$, we have

$$\begin{aligned} x(e_j) &= \sum_i \lambda_i e_i^*(e_j) + \sum_p \mu_p p^*(e_j) = \lambda_j e_j^*(e_j) = \lambda_j, \\ x(q) &= \sum_i \lambda_i e_i^*(q) + \sum_p \mu_p p^*(q) = \mu_q q^*(q) = \mu_q. \end{aligned}$$

Hence, if $x = 0$, then all $\lambda_i = 0$ and $\mu_p = 0$, which shows that $\mathcal{B}^*$ is independent over L.

We need four technical lemmas to prove the main theorems presented in this section. The first two are about some properties of elements of LQ. The next two are related to a criterion for a Hochschild extension algebra to be symmetric.

Lemma 6.1. *The following equalities hold in* LQ*:*

(i) $e_i(LQ_+)e_i = 0$ *and* $e_i(LQ)e_i = Le_i$*, for all* $i \in Q_0$.

(ii) $e_i xye_i = e_i xe_i ye_i$*, for all* $i \in Q_0$*,* $x, y \in LQ$.

(iii) $e_i^* = e_i e_i^* e_i$ *and* $xe_i^* = e_i^* x = (e_i xe_i)e_i^* = e_i^*(e_i xe_i)$*, for all* $i \in Q_0$*,* $x \in LQ$.

Proof. (i) For $x \in LQ$, let $x = \sum_{i\in Q_0} \lambda_i e_i + \sum_{p\in Q_+} \mu_p p$ with $\lambda_i, \mu_p \in L$. Then $e_i xe_i = \lambda_i e_i + \sum_p \mu_p e_i pe_i$ for $i \in Q_0$, where $e_i pe_i = e_i e_{s(p)} pe_{t(p)} e_i$. If $e_i pe_i \neq 0$, then $e_i e_{s(p)} = e_i$ and $e_{t(p)} e_i = e_i$, and hence $i = s(p)$ and $i = t(p)$, which implies that p is a cycle of positive length. This however contradicts the assumption that Q has no oriented cycles. Hence, $e_i pe_i = 0$ and $e_i xe_i = \lambda_i e_i$ for all $i \in Q_0$, $p \in Q_+$.

(ii) For any $x, y \in LQ$, let

$$x = \sum_{i\in Q_0} \lambda_i e_i + \sum_{p\in Q_+} \mu_p p \quad \text{and} \quad y = \sum_{i\in Q_0} \lambda'_i e_i + \sum_{p\in Q_+} \mu'_p p$$

with $\lambda_i, \mu_p, \lambda'_i, \mu'_p \in L$. It follows that $xy = \sum_i (\lambda_i \lambda'_i)e_i + z$, where z belongs to LQ_+. Hence, by the fact shown above, $e_i xe_i = \lambda_i e_i$, $e_i ye_i = \lambda'_i e_i$ and $e_i xye_i = \lambda_i \lambda'_i e_i$ for $i \in Q_0$, which implies that $e_i xye_i = e_i xe_i ye_i$.

(iii) We shall show the equality $xe_i^* = e_i^* x$ for $x \in LQ$. First, it is easily observed that $e_i^* = e_i e_i^* e_i$ for $i \in Q_0$. Now, for any $w \in LQ$, we have $xe_i^*(w) = e_i^*(wx)$ by the left LQ-module structure of $\operatorname{Hom}_L(LQ, L)$. Moreover, by the LQ-bimodule structure of $\operatorname{Hom}_L(LQ, L)$,

$$\begin{aligned} e_i^*(wx) &= e_i e_i^* e_i(wx) = e_i^*(e_i wxe_i) \\ &= e_i^*\big((e_i we_i)(e_i xe_i)\big) = (e_i xe_i e_i^*)(e_i we_i), \end{aligned}$$

where the third equality follows from (ii). Further, we have the equalities

$$(e_i xe_i e_i^*)(e_i we_i) = \big(e_i(e_i xe_i e_i^*)e_i\big)(w) = (e_i xe_i e_i^*)(w) = \lambda_i e_i^*(w).$$

Hence, we obtain the equality $xe_i^*(w) = \lambda_i e_i^*(w)$ for all $w \in LQ$, which implies that $xe_i^* = \lambda_i e_i^*$. Next, we conclude that the following equalities hold

$$\begin{aligned} e_i^* x(w) &= e_i^*(xw) = (e_i e_i^* e_i)(xw) = e_i^*(e_i xwe_i) = e_i^*\big((e_i xe_i)(e_i we_i)\big) \\ &= (e_i^* e_i xe_i)(e_i we_i) = (\lambda_i e_i^* e_i)(w) = \lambda_i e_i^*(w). \end{aligned}$$

This implies that $e_i^* x = \lambda_i e_i^*$, and consequently $xe_i^* = e_i^* x$ as required. Moreover, since $e_i^* = e_i e_i^* = e_i^* e_i$, it holds that $e_i xe_i e_i^* = e_i(xe_i^*) = e_i(e_i^* x) = e_i^* x$ and $e_i^* e_i xe_i = (e_i^* x)e_i = (xe_i^*)e_i = xe_i^*$, and hence $xe_i^* = e_i^* x = (e_i xe_i)e_i^*$. This proves (iii). □

By using Hochschild extensions of a finite field extension L of K, we will study Hochschild extensions of hereditary algebras over L.

Let A be the path algebra LQ of a finite connected acyclic quiver Q over a finite field extension L of K. Let $\beta\colon A \times A \to D(A) = \operatorname{Hom}_L(A, L)$ be a 2-cocycle of the K-algebra A. Let $\beta_i\colon L \times L \to L$ be the K-bilinear map defined by

$$\beta_i(a,b) = \beta(ae_i, be_i)(e_i)$$

for all $i \in Q_0$ and $a, b \in L$. Then β_i is a 2-cocycle of the K-algebra L by the L-bimodule L. Indeed, for $a, b, c \in L$, we have the equalities

$$\begin{aligned} a\beta_i(b,c) &= a\big(\beta(be_i, ce_i)(e_i)\big) = \beta(be_i, ce_i)(ae_i) = (ae_i)\beta(be_i, ce_i)(e_i), \\ \beta_i(a,b)c &= \big(\beta(ae_i, be_i)(e_i)\big)c = (\beta(ae_i, be_i)(ce_i) = \big(\beta(ae_i, be_i)ce_i\big)(e_i), \end{aligned}$$

because of $e_i^2 = e_i$ and the A-bimodule structure of $\operatorname{Hom}_L(A, L)$. Hence we obtain that

$$\begin{aligned} &a\beta_i(b,c) - \beta_i(ab,c) \\ &\quad= \big(ae_i\beta(be_i, ce_i)\big)(e_i) - \beta(abe_i, ce_i)(e_i) \\ &\quad= \big(ae_i\beta(be_i, ce_i) - \beta(ae_ibe_i, ce_i)\big)(e_i) \\ &\quad= \big(\beta(ae_i, be_i)ce_i - \beta(ae_i, be_ice_i)\big)(e_i) \\ &\quad= \big(\beta(ae_i, be_i)ce_i\big)(e_i) - \beta(ae_i, bce_i)(e_i) \\ &\quad= \beta_i(a,b)c - \beta_i(a,bc), \end{aligned}$$

where the middle equality follows by the 2-cocycle condition of β.

Let $[\ ,\ ]_i$ denote the K-bilinear map $[\ ,\ ]_{\beta_i}\colon L \times L \to L$ associated with β_i, that is, $[a,b]_i = \beta_i(a,b) - \beta_i(b,a)$ for $a, b \in L$. For a K-subspace M of L, we denote by $[M, M]$ the K-subspace of L spanned by $[x, y]$ for all $x, y \in M$, and set $[M, M](a) = \{u(a) \mid u \in [M, M]\}$ for any element $a \in L$.

Lemma 6.2. *For any $i, j, k \in Q_0$, it holds that $[Le_i, Le_j]_\beta(e_k) = 0$ unless $i = j = k$.*

Proof. Assume that some of two elements of $\{i, j, k\}$ are different. Consider the following 2-cocycle conditions for $a, b \in L$:

$$\begin{aligned} f &:= e_i\beta(ae_i, be_j) - \beta(ae_i, be_j) + \beta(e_i, abe_ie_j) - \beta(e_i, ae_i)be_j = 0, \\ g &:= e_j\beta(be_j, ae_i) - \beta(be_j, ae_i) + \beta(e_j, bae_je_i) - \beta(e_j, be_j)ae_i = 0. \end{aligned}$$

We shall prove the lemma in each possible case for i, j, k.

(i) Let $i \neq j$, $i \neq k$ and $j \neq k$. Then $e_ie_j = e_je_k = e_ke_i = 0$. Hence $0 = f(e_k) = -\beta(ae_i, be_j)(e_k)$ and $0 = g(e_k) = -\beta(be_j, ae_i)(e_k)$. It follows that $[ae_i, be_j]_\beta(e_k) = 0$.

(ii) Let $i = j \neq k$. Then $e_i e_k = 0 = e_j e_k$. It follows that

$$\begin{aligned} 0 = f(e_k) &= \big(-\beta(ae_i, be_j) + \beta(e_i, abe_i)\big)(e_k), \\ 0 = g(e_k) &= \big(-\beta(be_j, ae_i) + \beta(e_i, abe_i)\big)(e_k), \end{aligned}$$

hence $\beta(ae_i, be_j)(e_k) = \beta(be_j, ae_i)(e_k)$. Thus we have $[ae_i, be_j]_\beta(e_k) = 0$.

(iii) Let $i \neq j = k$. Then $e_i e_j = 0$ and $e_i e_k = 0$. Hence

$$\begin{aligned} 0 = f(e_k) &= \big(-\beta(ae_i, be_j) - \beta(e_i, ae_i)be_j\big)(e_k), \\ 0 = g(e_k) &= \big(e_j\beta(be_j, ae_i) - \beta(be_j, ae_i)\big)(e_k). \end{aligned}$$

It follows that

$$\begin{aligned} [ae_i, be_j]_\beta(e_k) &= \big(-\beta(e_i, ae_i)be_j - e_j\beta(be_j, ae_i)\big)(e_k) \\ &= -\beta(e_i, ae_i)(be_j) - \beta(be_j, ae_i)(e_j). \end{aligned}$$

On the other hand, by the 2-cocycle condition for β, we have

$$h := be_j\beta(e_i, ae_i) - \beta(be_je_i, ae_i) + \beta(be_j, ae_i) - \beta(be_j, e_i)ae_i = 0.$$

Hence, we obtain that $[ae_i, be_j]_\beta(e_k) = 0$, because

$$\begin{aligned} 0 = h(e_k) &= \big(be_j\beta(e_i, ae_i) + \beta(be_j, ae_i)\big)(e_k) \\ &= \beta(e_i, ae_i)(be_j) + \beta(be_j, ae_i)(e_j). \end{aligned}$$

(iv) Let $k = i \neq j$. Then, by changing i and j in (iii), we have $[be_j, ae_i]_\beta(e_k) = 0$, so $[ae_i, be_j]_\beta(e_k) = -[be_j, ae_i]_\beta(e_k) = 0$. □

Since a Hochschild extension algebra $\mathrm{T}_{D_L(LQ)}(LQ, \beta)$ is the direct sum of LQ and $D_L(LQ) = \mathrm{Hom}_L(LQ, L)$ as a K-vector space, an element x of $\mathrm{T}_{D_L(LQ)}(LQ, \beta)$ is written as a pair (a, q), for $a \in LQ$ and $q \in \mathrm{Hom}_L(LQ, L)$. For a K-linear map $\lambda\colon \mathrm{T}_{D_L(LQ)}(LQ, \alpha) \to K$, the image $\lambda(x)$ of x is then simply denoted by $\lambda(a, q)$.

The criterion in Lemma 5.3 on the K-algebra L is generalized to the K-algebra LQ in the following way.

Lemma 6.3. *Let $\mathrm{T}_{D_L(LQ)}(LQ, \beta)$ be symmetric and let $\lambda\colon \mathrm{T}_{D_L(LQ)}(LQ, \beta) \to K$ be a symmetric regular K-linear form. Then the following statements hold:*

(i) $\lambda(0, ae_i^*) = \lambda(0, ae_j^*)$ *and* $\lambda(0, ap^*) = 0$ *for* $i, j \in Q_0$, $a \in L$, *and* $p \in Q_+$.

(ii) $\lambda(0, [Le_i, Le_j]_\beta) = 0$ *for* $i, j \in Q_0$.

(iii) $\sum_{i \in Q_0}[L, L]_i \subset L$.

Proof. We abbreviate $[\ ,\ ]_\beta$ by $[\ ,\ ]$.

(i) First we observe that $pp^* = e_i^*$ and $p^*p = e_j^*$ for any path $p \in Q_+$ with the starting point i and terminal point j. Indeed, $p = e_i p e_j$ as an element of $A = LQ$, and $p^* = e_j p^* e_i$ as an element of the LQ-bimodule $\mathrm{Hom}_L(LQ, L)$. Hence $(pp^*)(e_j) = p^*(e_j p) = p^*(e_j e_i p)$, which is 0 if $j \neq i$, and 1 if $j = i$. Therefore, $(pp^*)(e_j) = e_i^*(e_j)$ for all $j \in Q_0$. Further, by definition, $(pp^*)(q) = p^*(qp) = 0$ and $e_i^*(q) = 0$ for all $q \in Q_+$, because $qp \neq p$ and $q \neq e_i$ obviously. Thus we have $pp^* = e_i^*$. Similarly, one shows that $p^*p = e_j^*$.

Now, in order to show that $\lambda(0, ae_i^*) = \lambda(0, ae_j^*)$ for all $i, j \in Q_0$, it is enough to consider the case where i and j are different and connected by a path p, because Q is connected. So, assume that $i \neq j$ and $p \in Q_+$ with $s(p) = i$ and $t(p) = j$. Since λ is symmetric,

$$\lambda\big((ap,0)(0,p^*) - (0,p^*)(ap,0)\big) = 0,$$

where the left-hand side is equal to

$$\begin{aligned}\lambda\big((0,app^*) - (0,p^*ap)\big) &= \lambda\big((0,app^*) - (0,ap^*p)\big)\\ &= \lambda\big((0,ae_i^*) - (0,ae_j^*)\big).\end{aligned}$$

Hence, $\lambda(0, ae_i^*) = \lambda(0, ae_j^*)$ for all $i, j \in Q_0$ and $a \in L$. Further, as λ is symmetric, it holds that

$$\begin{aligned}0 &= \lambda\big((ae_j^*,0)(0,p^*) - (0,p^*)(ae_j^*,0)\big)\\ &= \lambda\big((0,ae_jp^*) - (0,ap^*e_j)\big) = \lambda(0,ap^*),\end{aligned}$$

because $p^* = e_j p^* e_i$ and $i \neq j$, and hence $ae_j p^* = ap^*$ and $ap^* e_j = 0$. Thus $\lambda(0, ap^*) = 0$ for $a \in L$ and $p \in Q_+$.

(ii) In the K-algebra $\mathrm{T}_L(LQ, \beta)$, for $a, b \in L$ and $i, j \in Q_0$, we have the equalities

$$\begin{aligned}&(ae_i,0)(be_j,0) - (be_j,0)(ae_i,0)\\ &\qquad = \big(abe_ie_j, \alpha(ae_i,be_j)\big) - \big(bae_je_i, \alpha(be_j,ae_i)\big) = (0,[ae_i,be_j]),\end{aligned}$$

because $abe_ie_j = bae_je_i$. Applying here λ, we obtain that

$$\lambda(0,[ae_i,be_j]) = \lambda\big((ae_i,0)(be_j,0) - (be_j,0)(ae_i,0)\big) = 0,$$

for all $a, b \in L$, and hence $\lambda(0, [Le_i, Le_j]) = 0$.

(iii) We recall that $u = \sum_{j \in Q_0} u(e_j)e_j^* + \sum_{p \in Q_+} u(p)p^*$ for $u \in \mathrm{Hom}_L(A, L)$, and we write $A = LQ$.

First we claim that $\lambda(0,u) = \lambda(0,u(1_A)e_i^*)$ for all $i \in Q_0$ and $u \in \operatorname{Hom}_L(A,L)$. For an arbitrarily fixed $i \in Q_0$, it follows from (i) that $\lambda(0,u(e_j)e_j^*) = \lambda(0,u(e_j)e_i^*)$ for $j \in Q_0$, and hence

$$\lambda\left(0,\sum_j u(e_j)e_j^*\right) = \lambda\left(0,\sum_j u(e_j)e_i^*\right) = \lambda\left(0,u\left(\sum_j e_j\right)e_i^*\right) \\ = \lambda\big(0,u(1_A)e_i^*\big).$$

Since $\lambda(0,u(p)p^*) = 0$ by (i), it therefore follows that

$$\lambda(0,u) = \lambda\left(0,\sum_j u(e_j)e_j^*\right) + \lambda\left(0,\sum_j u(p)p^*\right) \\ = \lambda\big(0,u(1_A)e_i^*\big).$$

Next, suppose that $\sum_{i\in Q_0}[L,L]_i = L$. We show that this leads to a contradiction. Indeed, we have $\lambda(0,Le_j^*) = \lambda(0,\sum_i[L,L]_i e_j^*) = \sum_i \lambda(0,[L,L]_i e_j^*)$. On the other hand, $[L,L]_i = [Le_i,Le_i](e_i)$, by definition, and $[Le_i,Le_i](e_i) = [Le_i,Le_i](1_A)$, by Lemma 6.2. Hence,

$$\lambda(0,Le_j^*) = \sum_i \lambda\left(0,[Le_i,Le_i](1_A)e_j^*\right),$$

and it follows from the equalities in the first step that $\lambda\big(0,[Le_i,Le_i](1_A)e_j^*\big) = \lambda(0,[Le_i,Le_i])$, and $\lambda(0,[Le_i,Le_i]) = 0$, by (ii). Thus we get the equalities $\lambda(0,Le_j^*) = \sum_i \lambda(0,[Le_i,Le_i]) = 0$. Observe that $\mathrm{T}_{D_L(A)}(A,\beta)(0,Le_j^*) = (0,LQe_j^*)$ and $LQe_j^* = Le_j^*$. Therefore, $\lambda(\mathrm{T}_{D_L(A)}(A,\beta)(0,Le_j^*)) = 0$, which implies that $\mathrm{T}_{D_L(A)}(A,\beta)(0,Le_j^*) = 0$ by the regularity of λ. This contradicts the fact that $Le_j^* \neq 0$. □

The following lemma is related to the converse of Lemma 6.3.

Lemma 6.4. *If $\sum_{i\in Q_0}[L,L]_i \subset L$, then there exist symmetric regular K-linear maps $\lambda_i\colon \mathrm{T}_L(L,\beta_i) \to K$ satisfying $\lambda_i = \lambda_j$ for all $i,j \in Q_0$.*

Proof. Take an element $c \in L \setminus \sum_i[L,L]_i$ and a K-subspace L' of L such that $\sum_i[L,L]_i \subseteq L'$ and $L = Kc \oplus L'$. Let $\lambda_i\colon \mathrm{T}_L(L,\beta_i) \to K$ be the K-linear map with $\lambda_i(L,0) = 0$, $\lambda_i(0,L') = 0$ and $\lambda_i(0,c) = 1_K$. Then $\lambda_i = \lambda_j$ for all $i,j \in Q_0$ as K-linear maps from $L \oplus L$ to K. The proof of Lemma 5.3 establishing that λ is regular and symmetric is then valid in this case, because $[L,L]_{\beta_i} \subseteq \sum_{i\in Q_0}[L,L]_i \subset L$. Thus we conclude the lemma. □

In the above lemmas we observed some properties of the 2-cocycles of L associated with a 2-cocycle of LQ. Conversely, we may define a 2-cocycle of LQ using of a 2-cocycle α of L as follows.

Let $\alpha\colon L\times L \to L$ be a 2-cocycle of the K-algebra L by the L-bimodule L. For a finite connected acyclic quiver Q, we define a K-bilinear map $\hat{\alpha}\colon LQ \times LQ \to \mathrm{Hom}_L(LQ, L)$ by

$$\hat{\alpha}(x, y) = \sum_{i \in Q_0} \alpha(x_i, y_i)e_i^*$$

for all $x, y \in LQ$, where $x_i, y_i \in L$ and $e_i x e_i = x_i e_i$, $e_i y e_i = y_i e_i$.

Lemma 6.5. *The K-bilinear map $\hat{\alpha}$ is a 2-cocycle of the K-algebra LQ by the LQ-bimodule* $\mathrm{Hom}_L(LQ, L)$.

Proof. Let $x, y, z \in LQ$. By the definition of $\hat{\alpha}$, we have

$$x\hat{\alpha}(y, z) = \sum_i x\alpha(y_i, z_i)e_i^* = \sum_i \alpha(y_i, z_i)xe_i^*.$$

It follows from Lemma 6.1 that $xe_i^* = (e_i x e_i)e_i^* = x_i e_i e_i^* = x_i e_i^*$, and so $x\hat{\alpha}(y, z) = \sum_i \alpha(y_i, z_i)x_i e_i^* = \sum_i x_i\alpha(y_i, z_i)e_i^*$. Therefore, applying Lemma 6.1 again, we get

$$\begin{aligned} x\hat{\alpha}(y, z) - \hat{\alpha}(xy, z) &= \sum_i x_i\alpha(y_i, z_i)e_i^* - \sum_i \alpha(x_i y_i, z_i)e_i^* \\ &= \sum_i \big(x_i\alpha(y_i, z_i) - \alpha(x_i y_i, z_i)\big)e_i^*. \end{aligned}$$

On the other hand, $\hat{\alpha}(x, y)z - \hat{\alpha}(x, yz) = \sum_i \alpha(x_i, y_i)e_i^* z - \sum_i \alpha(x_i, y_i z_i)e_i^*$, where, by Lemma 6.1, $e_i^* z = z_i e_i^*$ and $e_i(yz)e_i = (e_i y e_i)(e_i z e_i) = (y_i e_i)(z_i e_i) = (y_i z_i)e_i$. Hence, we have

$$\hat{\alpha}(x, y)z - \hat{\alpha}(x, yz) = \sum_i \big(\alpha(x_i, y_i)z_i - \alpha(x_i, y_i z_i)\big)e_i^*.$$

Now, applying the 2-cocycle condition of α for the triples x_i, y_i, z_i, we conclude that $x\hat{\alpha}(y, z) - \hat{\alpha}(xy, z) = \hat{\alpha}(x, y)z - \hat{\alpha}(x, yz)$ for all x, y, z. ☐

Now we are in a position to provide the announced construction of symmetric Hochschild extension algebras (see [OTY]).

Theorem 6.6. *Let L be a finite field extension of a field K and $\alpha\colon L\times L \to L$ be a 2-cocycle of the K-algebra L by the L-bimodule L. Then the following statements are equivalent:*

(i) $\mathrm{T}_L(L,\alpha)$ *is symmetric.*

(ii) $\mathrm{T}_{D_L(LQ)}(LQ,\hat{\alpha})$ *is symmetric for some finite connected acyclic quiver* Q.

(iii) $\mathrm{T}_{D_L(LQ)}(LQ,\hat{\alpha})$ *is symmetric for any finite connected acyclic quiver* Q.

Proof. Let Q be a finite connected acyclic quiver and $\beta = \hat{\alpha}$. For any vertex $i \in Q_0$, let $\beta_i : L \times L \to L$ be the K-bilinear map defined by

$$\beta_i(x,y) = \hat{\alpha}(xe_i, ye_i)(e_i),$$

for $x, y \in L$. It follows from the definition of $\hat{\alpha}$ that, for any $x, y \in L$,

$$\hat{\alpha}(xe_i, ye_i) = \alpha(x,y)e_i^*,$$

because $e_j(xe_i)e_j = 0$ for $j \neq i$. Hence, $\beta_i(x,y) = \alpha(x,y)e_i^*(e_i) = \alpha(x,y)$, and so $\beta_i(x,y) = \alpha(x,y)$ for all $x, y \in L$, that is, $\beta_i = \alpha$ for all $i \in Q_0$. Consequently, $\mathrm{T}_L(L,\beta_i) = \mathrm{T}_L(L,\alpha)$ for all $i \in Q_0$. Now we prove the equivalence of the statements (i), (ii) and (iii).

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i) Let $[\ ,\ ]_i = [\ ,\ ]_{\beta_i}$. Assume that $\mathrm{T}_{D_L(LQ)}(LQ,\hat{\alpha})$ is a symmetric K-algebra, for some finite connected acyclic quiver Q. Then $\sum_{i \in Q_0}[L,L]_i \subset L$ by Lemma 6.3, and hence it follows from Lemma 6.4 that there are regular symmetric K-linear maps $\lambda_i : \mathrm{T}_L(L,\beta_i) \to K$ for all $i \in Q_0$. By the above observation, λ_i is a regular symmetric K-bilinear map from $\mathrm{T}_L(L,\alpha)$ to K, which implies that $\mathrm{T}_L(L,\alpha)$ is a symmetric K-algebra.

(i) $\Rightarrow$ (iii) Let $T_0 = \mathrm{T}_L(L,\alpha)$, and assume that T_0 is a symmetric K-algebra, and let $\lambda_0 : T_0 \to K$ be a regular symmetric K-linear map. Let Q be any finite connected acyclic quiver and $T = \mathrm{T}_{D_L(LQ)}(LQ,\hat{\alpha})$. We define a K-linear map $\lambda : T \to K$ by

$$\lambda(a,u) = \lambda_0\big(0, u(1_{LQ})\big),$$

for $a \in LQ$ and $u \in \mathrm{Hom}_L(LQ,K)$. We shall show that λ is regular and symmetric. For the regularity, observe that

$$\lambda\big(T(a,u)\big) \supseteq \lambda_0(0,L) = \lambda_0\big(T_0(0,1_K)\big),$$

for all $0 \neq (a,u) \in T = LQ \ltimes_\alpha \mathrm{Hom}_L(LQ,L)$, and $\lambda_0(T_0(0,1_K)) \neq 0$ by the regularity of λ_0. This implies that $\lambda(T(a,u)) \neq 0$, that is, λ is regular.

In order to prove that λ is symmetric, it suffices to show that $\lambda_0(0,[a,b](1_{LQ})) = 0$ for $a, b \in LQ$. We recall that $[\ ,\ ] = [\ ,\ ]_\beta = [\ ,\ ]_{\hat{\alpha}}$. In fact, for $x = (a,u), y = (b,v) \in T$, we have

$$\begin{aligned}\lambda(xy - yx) &= \lambda\big((ab, av + ub + \alpha(a,b)) - (ba, bu + va + \alpha(b,a))\big)\\ &= \lambda\big(ab - ba, av + ub - bu - va + [a,b]\big),\end{aligned}$$

where $(av+ub-bu-va)(1_{LQ}) = v(1_{LQ}a)+u(b1_{LQ})-u(1_{LQ}b)-v(a1_{LQ}) = 0$. It then follows from the definition of λ that

$$\lambda(xy-yx) = \lambda_0\big(0, [a,b](1_{LQ})\big).$$

Let us calculate $[a,b](1_{LQ})$. Take $a = \sum_{i\in Q_0} a_i e_i + a'$, $b = \sum_{i\in Q_0} b_i e_i + b'$, where $a_i, b_i \in L$ and $a', b' \in LQ_+$. Then

$$[a,b] = \sum_{i,j}[a_ie_i, b_je_j] + \sum_i \Big([a_ie_i, b'] + [a', b_ie_i]\Big) + [a', b'].$$

Note that $\hat{\alpha}(x,y) = 0$ if x or y belongs to LQ_+. Indeed, $\hat{\alpha}(x,y) = \sum_{i\in Q_0}\alpha(x_i, y_i)e_i^*$, where $x_ie_i = e_ixe_i = 0$ or $y_ie_i = e_iye_i = 0$ according to $x \in LQ_+$ or $y \in LQ_+$. Hence $[x,y] = \hat{\alpha}(x,y) - \hat{\alpha}(y,x) = 0$ if $x \in LQ_+$ or $y \in LQ_+$. Then we conclude that $[a,b] = \sum_{i,j}[a_ie_i, b_je_j]$. Moreover, it follows from Lemma 6.2 that

$$[a,b](1_{LQ}) = \sum_{i,j,k\in Q_0} [a_ie_i, b_je_j](e_k) = \sum_{i\in Q_0} [a_ie_i, b_ie_i](e_i),$$

where $[a_ie_i, b_ie_i](e_i) = [a_i, b_i]_{\alpha_i} = [a_i, b_i]_\alpha$, because $\hat{\alpha}(a_ie_i, b_ie_i)(e_i) = \beta_i(a_i, b_i)$ and $\beta_i = \alpha$, as observed in the first part of the proof. Consequently, we obtain that

$$\lambda_0\big(0, [a,b](1_{LQ})\big) = \lambda_0\left(0, \sum_{i\in Q_0}[a_i, b_i]_\alpha\right) = \sum_{i\in Q_0}\lambda_0\big(0, [a_i, b_i]_\alpha\big).$$

Since $\lambda_0\colon T_0 \to K$ is symmetric, it holds that

$$\lambda_0\big(0, [a_i, b_i]_\alpha\big) = \lambda_0\big((a_i, 0)(b_i, 0) - (b_i, 0)(a_i, 0)\big) = 0,$$

because $a_ib_i = b_ia_i$ in L. Thus we have proved that $\lambda_0(0, [a_i, b_i](1_{LQ})) = 0$, as required. □

Finally, we turn our attention to the splittability of Hochschild extensions of LQ. The following result was proved in [SY1].

Theorem 6.7. *Let L be a finite field extension of a field K, and $\alpha\colon L\times L \to L$ be a 2-cocycle of the K-algebra L. Then, for a finite connected acyclic quiver Q, the Hochschild extension of the K-algebra LQ by the LQ-bimodule $D_L(LQ) =$* $\mathrm{Hom}_L(LQ, L)$ *determined by the induced 2-cycle $\hat{\alpha}$,*

$$\mathbb{E}\colon \quad 0 \longrightarrow \mathrm{Hom}_L(LQ, L) \xrightarrow{\omega} \mathrm{T}_{D_L(LQ)}(LQ, \hat{\alpha}) \xrightarrow{\varrho} LQ \longrightarrow 0,$$

is splittable if and only if the 2-cocycle α is splittable, where ϱ and ω are the canonical surjective homomorphism of K-algebras and canonical monomorphism, respectively.

Proof. Let $A = LQ$ and $T = \mathrm{T}_{D_L(LQ)}(LQ, \hat{\alpha})$. For each $i \in Q_0$, we denote by $\hat{e}_i$ the element $(e_i, -\alpha(1,1)e_i^*)$ of $T = A \oplus D(A)$, considered as K-vector space. We claim that $\hat{e}_i$ is an idempotent. Since $e_i e_i^* = e_i^* e_i = e_i^*$ and $\hat{\alpha}(e_i, e_i) = \alpha(1,1)e_i^*$ by definition, we have $\hat{e}_i^2 = (e_i, -\alpha(1,1)e_i^*)^2 = (e_i, x)$, where

$$x = -\alpha(1,1)e_i e_i^* - \alpha(1,1)e_i^* e_i + \hat{\alpha}(e_i, e_i) = -\alpha(1,1)e_i^*.$$

Hence, $\hat{e}_i^2 = \hat{e}_i$.

Assume that the Hochschild extension $\mathbb{E}$ is splittable, and let $\kappa\colon A \to T$ be a K-algebra homomorphism with $\varrho\kappa = \mathrm{id}_A$. For $i \in Q_0$, we shall define K-algebra homomorphisms $\varrho_i\colon \hat{e}_i T \hat{e}_i \to e_i A e_i$ and $\kappa_i\colon e_i A e_i \to \hat{e}_i T \hat{e}_i$ such that $\varrho_i \kappa_i = \mathrm{id}_{e_i A e_i}$. For this, let ϱ_i be the restriction of ϱ to $\hat{e}_i T \hat{e}_i$, and let $\kappa_i(e_i a e_i) = \hat{e}_i \kappa(e_i a e_i)\hat{e}_i$, for all $a \in A$. Then ϱ_i and κ_i are K-linear maps with $\varrho_i \kappa_i = \mathrm{id}_{e_i A e_i}$, because

$$\begin{aligned} \varrho_i \kappa_i(e_i a e_i) &= \varrho_i\big(\hat{e}_i \kappa(e_i a e_i)\hat{e}_i\big) = \varrho(\hat{e}_i)\varrho\big(\kappa(e_i a e_i)\big)\varrho(\hat{e}_i) \\ &= e_i(e_i a e_i)e_i = e_i a e_i, \end{aligned}$$

for all $a \in A$. Obviously, ϱ_i is a K-algebra homomorphism, because so is ϱ. To show that κ_i is a K-algebra homomorphism, we verify that $\kappa_i(e_i) = \hat{e}_i$ and $\kappa_i(ab) = \kappa_i(a)\kappa_i(b)$ for all $a, b \in e_i A e_i$. First, we show that $\kappa_i(e_i) = \hat{e}_i$ for any $i \in Q_0$. Since $\varrho(\kappa(e_i)) = e_i$ and $\varrho(\hat{e}_i) = e_i$, we may write $\kappa(e_i) = \hat{e}_i + \omega(f_i)$, for some $f_i \in \mathrm{Hom}_L(LQ, L)$. Since κ is a K-algebra homomorphism, we get $(\hat{e}_i + \omega(f_i))^2 = \hat{e}_i + \omega(f_i)$. On the other hand,

$$\big(\hat{e}_i + \omega(f_i)\big)^2 = \hat{e}_i^2 + \hat{e}_i\omega(f_i) + \omega(f_i)\hat{e}_i + \omega(f_i)^2 = \hat{e}_i + \omega(\hat{e}_i f_i + f_i \hat{e}_i),$$

because ω is a T-bimodule homomorphism and $\omega(D(A))^2 = 0$. It follows that $\omega(f_i) = \omega(\hat{e}_i f_i + f_i \hat{e}_i)$, and so $\hat{e}_i f_i + f_i \hat{e}_i = f_i$, because ω is a monomorphism. Hence, multiplying the above equality by $\hat{e}_i$ from the right, we obtain $\hat{e}_i f_i \hat{e}_i + f_i \hat{e}_i = f_i \hat{e}_i$, which implies that $\hat{e}_i f_i \hat{e}_i = 0$. Then we conclude that

$$\kappa_i(e_i) = \hat{e}_i \kappa(e_i)\hat{e}_i = \hat{e}_i\big(\hat{e}_i + \omega(f_i)\big)\hat{e}_i = \hat{e}_i^2 + \omega(\hat{e}_i f_i \hat{e}_i) = \hat{e}_i.$$

Next we show that $\kappa_i(ab) = \kappa_i(a)\kappa_i(b)$ for all $a, b \in e_i A e_i$. We recall that $e_i A e_i = L e_i$, by Lemma 6.1. Then, letting $\kappa(re_i) = (re_i, f) \in A \ltimes_{\hat{\alpha}} D(A)$ for $r \in L$ and $f \in D(A)$, we observe that $\hat{e}_i \kappa(re_i) = (re_i, e_i f)$ and $\kappa(re_i)\hat{e}_i = (re_i, fe_i)$. Indeed,

$$\hat{e}_i \kappa(re_i) = \big(e_i, -\alpha(1,1)e_i^*\big)(re_i, f) = \big(re_i, e_i f - r\alpha(1,1)e_i^* + \hat{\alpha}(e_i, re_i)\big),$$

and $\hat{\alpha}(e_i, re_i) = \alpha(1, r)e_i^*$ by definition. Further, since $\alpha(1, r) = r\alpha(1,1)$, by Lemma 5.6, it follows that $\hat{e}_i \kappa(re_i) = (re_i, e_i f)$. Similarly, by using $\alpha(r, 1) =$

$r\alpha(1,1)$, it is seen that $\kappa(re_i)\hat{e}_i = (re_i, fe_i)$. Now, for any $s \in L$, let $\kappa(se_i) = (se_i, g) \in A \ltimes_{\hat{\alpha}} D(A)$. Then, from the above observation, we obtain the equalities

$$\begin{aligned}\kappa_i(re_i)\kappa_i(se_i) &= \big(\hat{e}_i\kappa(re_i)\hat{e}_i\big)\big(\hat{e}_i\kappa(se_i)\hat{e}_i\big) = (re_i, e_i fe_i)(se_i, e_i ge_i)\\ &= \big(rse_i, se_i fe_i + re_i ge_i + \hat{\alpha}(re_i, se_i)\big),\end{aligned}$$

$$\begin{aligned}\kappa_i(re_i se_i) &= \hat{e}_i\kappa(re_i se_i)\hat{e}_i = \hat{e}_i\kappa(re_i)\kappa(se_i)\hat{e}_i = (re_i, e_i f)(se_i, ge_i)\\ &= \big(rse_i, se_i fe_i + re_i ge_i + \hat{\alpha}(re_i, se_i)\big).\end{aligned}$$

Consequently, $\kappa_i(re_i)\kappa_i(se_i) = \kappa_i(re_i se_i)$. Thus we have proved that $\kappa_i\colon e_iAe_i \to \hat{e}_iT\hat{e}_i$ is a K-algebra homomorphism with $\varrho_i\kappa_i = \mathrm{id}_{e_iAe_i}$. In other words, the Hochschild extension

$$\mathbb{E}_i\colon \quad 0 \longrightarrow \hat{e}_i\big(D(A)\big)\hat{e}_i \xrightarrow{\omega_i} \hat{e}_iT\hat{e}_i \xrightarrow{\varrho_i} e_iAe_i \longrightarrow 0,$$

where ω_i is the restriction of ω, is splittable. In order to conclude that $\mathbb{E}_i$ is isomorphic to the Hochschild extension of L by L,

$$\mathbb{E}\colon \quad 0 \longrightarrow L \xrightarrow{\omega'} \mathrm{T}_L(L,\alpha) \xrightarrow{\varrho'} L \longrightarrow 0,$$

we have to define an isomorphism $\psi\colon \hat{e}_iT\hat{e}_i \to T_L(L,\alpha)$. First note that $\hat{e}_i(D(A))\hat{e}_i = Le_i^*$. Remember that the operation of elements $t \in T$ on the A-module $D(A)$ is by definition induced by ϱ, and we have $\hat{e}_i f\hat{e}_i = \varrho(\hat{e}_i)f\varrho(\hat{e}_i) = e_i fe_i$ for $f \in D(A)$. On the other hand, $e_i fe_i = f(e_i)e_i^*$, because

$$f = \sum_{i\in Q_0} f(e_i)e_i^* + \sum_{p\in Q_+} f(p)p^*,$$

and $e_ip^*e_i = 0$, by Lemma 6.1. Hence $\hat{e}_i f\hat{e}_i \in Le_i^*$, so $\hat{e}_i(D(A))\hat{e}_i \subseteq Le_i^*$. Moreover, $e_i^* = e_ie_i^*e_i \in e_i(D(A))e_i = \hat{e}_i(D(A))\hat{e}_i$, and hence $Le_i^* \subseteq \hat{e}_i(D(A))\hat{e}_i$. Thus $\hat{e}_i(D(A))\hat{e}_i = Le_i^*$. Now let $\psi_1\colon e_iAe_i = Le_i \to L$ with $\psi_1(re_i) = r$ and $\psi_2\colon \hat{e}_i(D(A))\hat{e}_i \to L$ with $\psi_2(re_i^*) = r$, for all $r \in L$. Then it is easy to see that ψ_1 is a K-algebra homomorphism and ψ_2 is a K-linear isomorphism such that $\psi_2(axb) = \psi_1(a)\psi_2(x)\psi_1(b)$ for $a, b \in e_iAe_i$ and $x \in \hat{e}_i(D(A))\hat{e}_i$, where $\hat{e}_i(D(A))\hat{e}_i$ is naturally regarded as an $\hat{e}_iA\hat{e}_i$-bimodule, and hence as an e_iAe_i-bimodule. Consider the K-linear map $\psi\colon \hat{e}_iT\hat{e}_i \to L \ltimes_\alpha L$ given by

$$\psi(e_iae_i, \hat{e}_i f\hat{e}_i) = \big(\psi_1(e_iae_i), \psi_2(\hat{e}_i f\hat{e}_i)\big)$$

for $a \in A$ and $f \in D(A)$. Then an easy checking shows that the following diagram of K-linear homomorphisms is commutative

$$\begin{tikzcd}
\mathbb{E}_i\colon & 0 \arrow[r] & \hat{e}_i\big(D(A)\big)\hat{e}_i \arrow[r, "\omega_i"] \arrow[d, "\psi_2"] & \hat{e}_iT\hat{e}_i \arrow[r, "\varrho_i"] \arrow[d, "\psi"] & e_iAe_i \arrow[r] \arrow[d, "\psi_1"] & 0 \\
\mathbb{E}\colon & 0 \arrow[r] & L \arrow[r, "\omega'"] & \mathrm{T}_L(L,\alpha) \arrow[r, "\varrho'"] & L \arrow[r] & 0\,.
\end{tikzcd}$$

Therefore, we conclude that $\mathbb{E}$ is splittable, and hence α is splittable, by Lemma 2.5.

Conversely, assume that the 2-cycle α is splittable. Then α is the coboundary of a K-linear map $\gamma\colon L \to L$, that is, $\alpha(a,b) = a\gamma(b) - \gamma(a)b$ for all $a, b \in L$. Define the K-linear map $\hat{\gamma}\colon LQ \to \operatorname{Hom}_L(LQ, L)$ by

$$\hat{\gamma}(x) = \sum_{i \in Q} \gamma(x_i)e_i^*,$$

for all $x \in LQ$, where $e_i x e_i = x_i e_i$ with $x_i \in L$. We claim that $\hat{\alpha}(x,y) = x\hat{\gamma}(y) - \hat{\gamma}(x)y$ for all $x, y \in LQ$. In fact, we have

$$x\hat{\gamma}(y) - \hat{\gamma}(x)y = \sum_i \gamma(y_i)xe_i^* - \sum_i \gamma(x_i)e_i^* y,$$

where $xe_i^* = x_i e_i^*$ and $e_i^* y = y_i e_i^*$, by Lemma 6.1. Hence, we obtain that

$$x\hat{\gamma}(y) - \hat{\gamma}(x)y = \sum_i \gamma(y_i)x_i e_i^* - \sum_i \gamma(x_i)y_i e_i^* = \sum_i \big(x_i\gamma(y_i) - \gamma(x_i)y_i\big)e_i^*,$$

which is equal to $\sum_{i \in Q_0} \alpha(x_i, y_i)e_i^*$. It follows that $x\hat{\gamma}(y) - \hat{\gamma}(x)y = \hat{\alpha}(x,y)$. Therefore, $\hat{\alpha}$ is the coboundary of $\hat{\gamma}$. This shows that the Hochschild extension $\mathbb{E}$ is splittable. ☐

Example 6.8. Let K and L be the fields from Example 5.5, and $\alpha\colon L \times L \to L$ be the nonsplittable 2-cocycle given there. Then it follows from Theorem 6.7 that, for any finite connected acyclic quiver Q, the Hochschild extension

$$0 \longrightarrow \operatorname{Hom}_L(LQ, L) \xrightarrow{\omega} \mathrm{T}(LQ, \hat{\alpha}) \xrightarrow{\varrho} LQ \longrightarrow 0$$

is not splittable.

7 Hochschild extension algebras of hereditary algebras

In this section we are concerned with modules over Hochschild extension algebras. We will present some relations between modules over a hereditary algebra A and modules over a Hochschild extension algebra T of A by a duality A-bimodule Q, aiming to determine the Auslander–Reiten quiver Γ_T. They were obtained in [Y2].

Let A be a finite dimensional K-algebra over a field K. For a module M in mod A, a module N in mod A^{op}, and a subset X of A, we define the left annihilator of X in M as the subset $\ell_M(X) = \{m \in M \mid mX = 0\}$ and the right annihilator of X in N as the subset $r_N(X) = \{n \in N \mid Xn = 0\}$.

Throughout this section we will assume that A is a finite dimensional K-algebra over a field K, Q is a duality A-bimodule, and

$$\mathbb{E}\colon \quad 0 \longrightarrow Q \xrightarrow{\omega} T \xrightarrow{\varrho} A \longrightarrow 0$$

is a Hochschild extension of A by Q. Since $\mathbb{E}$ is isomorphic to the extension

$$\mathbb{E}_0\colon \quad 0 \longrightarrow Q \xrightarrow{\omega_0} A \ltimes_\alpha Q \xrightarrow{\varrho_0} A \longrightarrow 0$$

with $\varrho_0(a,q) = a$ and $\omega_0(q) = (0,q)$ for all $a \in A$, $q \in Q$ and $\alpha\colon A \times A \to Q$ a 2-cycle, we may identify $\mathbb{E}$ with $\mathbb{E}_0$, and moreover we may identify $q \in Q$ with $(0,q) \in A \ltimes_\alpha Q$. Thus Q is also regarded as an ideal of T such that $Q^2 = 0$ in T and ω is an inclusion homomorphism.

We present first two illustrating examples.

Example 7.1. Let K be a field and T be the bound quiver algebra $K\Delta/J$ defined by the following quiver Δ

$$\begin{tikzcd} 1 \bullet \arrow[r, shift left, "\alpha"] & \bullet\, 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

and the ideal J of $K\Delta$ generated by $(\alpha\beta)^2$ and $(\beta\alpha)^2$. Then T is a selfinjective algebra whose Nakayama permutation is $\nu_T = \left(\begin{smallmatrix}1 & 2\\ 2 & 1\end{smallmatrix}\right)$, and each e_iT has a unique composition series. The composition length of e_iT is equal to its Loewy length 4. Let $Q = \operatorname{rad}^2 T$, and let $A = T/Q$ be the quotient algebra of T by Q. Observe that $Q^2 = 0$ in T and $e_1Q \cong e_1A$, $e_2Q \cong e_2A$ as right A-modules. Hence, the A-bimodule Q is a minimal injective cogenerator in $\operatorname{mod} A$, and so a duality A-bimodule. Let $\omega\colon Q \to T$ be the inclusion homomorphism and $\varrho\colon T \to A$ the canonical surjection. Then ϱ is a K-algebra homomorphism and the sequence $0 \to Q \overset{\omega}{\to} T \overset{\varrho}{\to} A \to 0$ is a Hochschild extension of A by the duality bimodule Q. We notice that T is not isomorphic to the trivial extension algebra $\mathrm{T}(A) = A \ltimes \operatorname{Hom}_K(A, K)$, because $\mathrm{T}(A)$ is a symmetric algebra, and hence its Nakayama permutation is the identity, while ν_T is not the identity.

The algebra A is isomorphic to the bound quiver algebra $K\Delta'/J'$ of the quiver Δ'

$$\begin{tikzcd} 1 \bullet \arrow[r, shift left, "\alpha"] & \bullet\, 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

and the ideal J' of $K\Delta'$ generated by $\alpha\beta$ and $\beta\alpha$. Then $\mathrm{T}(A)$ is isomorphic to the bound quiver algebra $K\Delta''/I$ defined by the quiver Δ'' of the form

$$\begin{tikzcd}
1 \bullet \arrow[r, shift left, "\alpha"] \arrow[r, bend right=60, swap, "\beta^*"] & \bullet 2 \arrow[l, shift left, "\beta"] \arrow[l, bend right=60, swap, "\alpha^*"]
\end{tikzcd}$$

and the ideal I of $K\Delta''$ generated by $\alpha\beta, \beta\alpha, \alpha^*\beta^*, \beta^*\alpha^*, \alpha\alpha^*-\beta^*\beta, \beta\beta^*-\alpha^*\alpha$. Moreover, for the K-algebra automorphism σ of A such that $\sigma(e_1) = e_2$, $\sigma(e_2) = e_1$, $\sigma(\alpha) = \beta$, and $\sigma(\beta) = \alpha$, there exists a K-algebra isomorphism $\mathrm{T}(A) \cong A \ltimes Q_\sigma$.

Example 7.2. Let K be a field and T be the bound quiver algebra KQ/J defined by the quiver Δ of the form

$$\begin{tikzcd}
\underset{1}{\bullet} \arrow[r, shift right, swap, "\gamma"] & \underset{2}{\bullet} \arrow[l, shift right, swap, "\alpha"] \arrow[r, shift left, "\beta"] & \underset{3}{\bullet} \arrow[l, shift left, "\delta"]
\end{tikzcd}$$

and the ideal J of $K\Delta$ generated by $\gamma\alpha$, $\delta\beta$, $\alpha\gamma - \beta\delta$. Then T is a selfinjective algebra with the Nakayama permutation $\nu_T = \left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)$. We denote by $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$, the classes of $\alpha, \beta, \gamma, \delta$ in T, respectively. Let I be the ideal of T generated by $\bar{\gamma}$ and $\bar{\delta}$. As a right T-module, $I = e_1 \operatorname{rad} T \oplus e_2 \operatorname{rad}^2 T \oplus e_3 \operatorname{rad} T$, where

$$\begin{aligned}
e_1 \operatorname{rad} T &= K\bar{\gamma} \oplus K\bar{\gamma}\bar{\beta} = \bar{\gamma}(Ke_2 \oplus K\bar{\beta}),\\
e_2 \operatorname{rad}^2 T &= K\bar{\alpha}\bar{\gamma},\\
e_3 \operatorname{rad} T &= K\bar{\delta} \oplus K\bar{\delta}\bar{\alpha} = \bar{\delta}(Ke_2 \oplus K\bar{\alpha}).
\end{aligned}$$

There are isomorphisms $\gamma_L \colon Ke_2 \oplus K\beta \to e_1 \operatorname{rad} T$, $(\alpha\gamma)_L \colon Ke_2 \to e_2 \operatorname{rad}^2 T$, $\delta_L \colon Ke_2 \oplus K\alpha \to e_3 \operatorname{rad} T$ in $\operatorname{mod} T$, given by left multiplications by $\bar{\gamma}, \bar{\alpha}\bar{\gamma}, \bar{\delta}$, respectively. Let $A = T/I$ be the quotient algebra of T by I. Then T is the hereditary K-algebra $K\Delta'$ with the quiver Δ' of the form

$$\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, swap, "\alpha"] \arrow[r, "\beta"] & \underset{3}{\bullet}
\end{tikzcd}.$$

Observe that $I^2 = 0$, and, as a right A-module, I is isomorphic to $I_A(S_1) \oplus I_A(S_2) \oplus I_A(S_3)$, where S_i are the simple modules in $\operatorname{mod} A$ isomorphic to $\operatorname{top}(e_i A)$, for $i \in \{1, 2, 3\}$. In fact, $I_A(S_1) = Ke_1 \oplus K\alpha$, $I_A(S_2) = Ke_2$, and $I_A(S_3) = Ke_3 \oplus K\beta$. This implies that I is a minimal injective cogenerator in $\operatorname{mod} A$. Hence, I is a duality module, and T is a Hochschild extension algebra of the hereditary algebra A by I. Notice again that the Nakayama permutation ν_T is not the identity, hence T is not isomorphic to $A \ltimes D(A)$ as a K-algebra.

The following lemma is a generalization of Lemma 3.1 from direct summands of T_T to projective T-modules.

Lemma 7.3. *Let P be a nonzero projective module in* $\operatorname{mod} T$. *The following statements hold:*

(i) $\ell_P(Q) = PQ$, *and* $\ell_M(Q) = M \cap PQ$ *for a submodule M of P.*

(ii) PQ *is a nonzero injective module in* $\operatorname{mod} A$.

(iii) P/PQ *is a nonzero projective module in* $\operatorname{mod} A$ *and* $\operatorname{top}(P) \cong \operatorname{top}(P/PQ)$.

(iv) $\operatorname{soc}(P_T) = \operatorname{soc}(PQ_T) = \operatorname{soc}(PQ_A)$.

Proof. Let $P = P_1 \oplus \cdots \oplus P_m$ be a decomposition of P as a direct sum of indecomposable projective right A-submodules $P_1, \ldots, P_m$ of P, and $P_i \cong \boldsymbol{e}_i T$ for a primitive idempotent $\boldsymbol{e}_i$ of T. Let $\varrho(\boldsymbol{e}_i) = e_i$ for $i \in \{1, \ldots, m\}$.

(i) Clearly $PQ \subseteq \ell_P(Q)$, because $PQ^2 = 0$. Conversely, observe that $PQ \cong \boldsymbol{e}_1 Q \oplus \cdots \oplus \boldsymbol{e}_m Q$ and $\ell_P(Q) \cong \ell_{\boldsymbol{e}_1 T}(Q) \oplus \cdots \oplus \ell_{\boldsymbol{e}_m T}(Q)$. Hence, it suffices to show that $\ell_{\boldsymbol{e}_i T}(Q) \subseteq \boldsymbol{e}_i Q$ for all $i \in \{1, \ldots, m\}$. Take $i \in \{1, \ldots, m\}$ and $t \in \boldsymbol{e}_i T$ with $tQ = 0$, and let $t = (a_i, r_i) \in T = A \ltimes_\alpha Q$. Then, for all $q \in Q$, we have $tq = (a_i, r_i)(0, q) = (0, a_i q)$ and $tq = 0$, which implies $a_i Q = 0$. This forces $a_i = 0$, because Q is a faithful module in $\operatorname{mod} A^{\mathrm{op}}$. Thus $t = (0, r_i) \in \boldsymbol{e}_i T \cap Q = \boldsymbol{e}_i Q$. Hence, $\ell_{\boldsymbol{e}_i T}(Q) \subseteq \boldsymbol{e}_i Q$ for all $i \in \{1, \ldots, m\}$. For a submodule M of P, $M \cap PQ = M \cap \ell_P(Q)$ and $M \cap \ell_P(Q) = \ell_M(Q)$ obviously. Thus $M \cap PQ = \ell_M(Q)$.

(ii) Since $\operatorname{soc}(P) \subseteq \ell_P(Q)$, it follows from (i) that $\operatorname{soc}(P) \subseteq PQ$, and hence $PQ \neq 0$, because $\operatorname{soc}(P) \neq 0$. Each $\boldsymbol{e}_i Q$ is an injective module in $\operatorname{mod} A$, because $\boldsymbol{e}_i Q = e_i Q$ and $e_i Q$ is a direct summand of Q. Hence PQ is an injective module in $\operatorname{mod} A$.

(iii) Since $PQ \subseteq \operatorname{rad} P$ and $PQ \subset P$, we have $P/PQ \neq 0$. Clearly $P/PQ \cong \boldsymbol{e}_1 T/\boldsymbol{e}_1 Q \oplus \cdots \oplus \boldsymbol{e}_m T/\boldsymbol{e}_m Q$, and ϱ induces an isomorphism $\boldsymbol{e}_i T/\boldsymbol{e}_i Q \cong e_i A$ in $\operatorname{mod} A$. Hence, we conclude that P/PQ is a projective module in $\operatorname{mod} A$, because so is $e_i A$. The isomorphism $\operatorname{top}(P) \cong \operatorname{top}(P/PQ)$ follows from the fact that $Q \subseteq \operatorname{rad} T$, because $Q^2 = 0$ in T, and $PQ \subseteq \operatorname{rad} P$.

(iv) It follows from Lemma 3.1 that $\operatorname{soc}(\boldsymbol{e}_i T) = \operatorname{soc}(\boldsymbol{e}_i Q)$ for all i. This implies that $\operatorname{soc}(P_i) = \operatorname{soc}(P_i Q)$ and hence $\operatorname{soc}(P) = \operatorname{soc}(P_1 Q) \oplus \cdots \oplus \operatorname{soc}(P_m Q) = \operatorname{soc}(PQ)$. The equality $\operatorname{soc}(PQ_T) = \operatorname{soc}(PQ_A)$ follows from the fact that $xt = x\varrho(t)$ for all $x \in PQ$ and $t \in T$. □

Lemma 7.4. *Let M and N be modules in* $\operatorname{mod} T$ *without nonzero projective direct summands, and let*

$$0 \longrightarrow N \xrightarrow{v} P \xrightarrow{u} M \longrightarrow 0$$

be an exact sequence in $\operatorname{mod} T$ *such that u is a projective cover of M (equivalently, v is an injective envelope of N). Then the following statements hold:*

(i) *PQ is an injective envelope of $\ell_N(Q)$ in* $\operatorname{mod} A$ *and we have an exact sequence*

$$0 \longrightarrow \ell_N(Q) \xrightarrow{v'} PQ \xrightarrow{u'} MQ \longrightarrow 0$$

in $\operatorname{mod} A$*, where u' and v' are the canonical homomorphisms induced by u and v.*

(ii) *P/PQ is a projective cover of M/MQ in* $\operatorname{mod} A$ *and we have an exact sequence*

$$0 \longrightarrow N/\ell_N(Q) \xrightarrow{\bar{v}} P/PQ \xrightarrow{\bar{u}} M/MQ \longrightarrow 0$$

in $\operatorname{mod} A$*, where $\bar{u}$ and $\bar{v}$ are the canonical homomorphisms induced by u and v.*

Proof. (i) It is clear that $0 \neq \operatorname{soc}(N) \subseteq \operatorname{soc}(\ell_N(Q))$, because $Q \subseteq \operatorname{rad} T$, and so $(\operatorname{soc}(N))Q = 0$. Further, v induces an isomorphism $\operatorname{soc}(N) \cong \operatorname{soc}(P)$. Hence it follows from Lemma 7.3 that $\operatorname{soc}(PQ) \subseteq \operatorname{soc}(\ell_N(Q))$, while $\operatorname{soc}(\ell_N(Q)) \xrightarrow{\sim} \operatorname{soc}(PQ)$ by v, because $\ell_N(Q) \xrightarrow{\sim} v(\ell_N(Q)) \subseteq \ell_P(Q) = PQ$. Therefore, the restriction of v induces an isomorphism $\operatorname{soc}(\ell_N(Q)) \xrightarrow{\sim} \operatorname{soc}(PQ)$. This ensures that PQ is an injective envelope of $\ell_N(Q)$, because PQ is injective in $\operatorname{mod} A$ by Lemma 7.3. On the other hand, $\operatorname{Ker} u' = \operatorname{Ker} u \cap PQ = v(N) \cap \ell_P(Q) = \ell_{v(N)}(Q) = v(\ell_N(Q))$, and hence we have the required exact sequence

$$0 \longrightarrow \ell_N(Q) \xrightarrow{v'} PQ \xrightarrow{u'} MQ \longrightarrow 0$$

in $\operatorname{mod} A$.

(ii) Since $Q \subseteq \operatorname{rad} T$, we have $\operatorname{top}(M) \cong \operatorname{top}(M/MQ)$ and $\operatorname{top}(P) \cong \operatorname{top}(P/PQ)$. Hence, $\operatorname{top}(P/PQ) \cong \operatorname{top}(M/MQ)$, because $\operatorname{top}(M) \cong \operatorname{top}(P)$. Since P/PQ is projective in $\operatorname{mod} A$, by Lemma 7.3, it follows that the canonical epimorphism $\bar{u}\colon P/PQ \to M/MQ$ must be a projective cover of M/MQ in $\operatorname{mod} A$. As for the kernel of $\bar{u}$, let X be a submodule of P such that $X/PQ = \operatorname{Ker} \bar{u}$. Then $X = u^{-1}(MQ)$ and $u(PQ) = MQ = u(X)$. Hence $X = PQ + \operatorname{Ker} u$. Thus $X/PQ = (PQ + \operatorname{Ker} u)/PQ \cong \operatorname{Ker} u/(\operatorname{Ker} u \cap PQ) \cong N/\ell_N(Q)$. Consequently, we get the required exact sequence

$$0 \longrightarrow N/\ell_N(Q) \xrightarrow{\bar{v}} P/PQ \xrightarrow{\bar{u}} M/MQ \longrightarrow 0$$

in $\operatorname{mod} A$. □

Lemma 7.5. *Let M and N be modules in* $\operatorname{mod} T$*. Then the following equivalences hold:*

(i) *$MQ = 0$ and M is projective in* $\operatorname{mod} A$ *if and only if $M \cong P/PQ$ for a projective cover P of M in* $\operatorname{mod} T$.

(ii) *$NQ = 0$ and N is injective in* $\operatorname{mod} A$ *if and only if $N \cong PQ$ for an injective envelope P of N in* $\operatorname{mod} T$.

Proof. (i) Let $u: P \to M$ be a projective cover of M in $\operatorname{mod} T$. Assume that $MQ = 0$ and M is projective in $\operatorname{mod} A$. Then it follows from Lemma 7.4 that the natural epimorphism $\bar{u}: P/PQ \to M$ induced by u is a projective cover of M in $\operatorname{mod} A$. Hence $\bar{u}$ is an isomorphism, because M is projective by assumption. Conversely, if $M \cong P/PQ$ for some projective module P in $\operatorname{mod} T$, then $MQ = 0$ clearly, and M is projective in $\operatorname{mod} A$, by Lemma 7.3.

(ii) Let $v: N \to P$ be an injective envelope of N in $\operatorname{mod} T$. Assume that $NQ = 0$ and N is injective in $\operatorname{mod} A$. Then $N = \ell_N(Q)$ and, by Lemma 7.4, PQ is an injective envelope of N in $\operatorname{mod} A$. Hence the natural monomorphism $v': N \to PQ$ induced by v is an isomorphism, because N is injective by assumption. Conversely, if $N \cong PQ$ for some injective module P in $\operatorname{mod} T$, then clearly $NQ = 0$, and N is injective in $\operatorname{mod} A$, by Lemma 7.3. □

The following two propositions are basic and will be often used in the sequel.

Proposition 7.6. *Let M be a module in* $\operatorname{mod} T$ *without nonzero projective direct summands and*

$$0 \longrightarrow N \xrightarrow{q} P \xrightarrow{p} M \longrightarrow 0$$

be an exact sequence in $\operatorname{mod} T$ *with p a projective cover of M. Then the following statements hold:*

(i) $MQ = 0$ *if and only if* $\ell_N(Q)$ *is injective in* $\operatorname{mod} A$.

(ii) $NQ = 0$ *if and only if* M/MQ *is projective in* $\operatorname{mod} A$.

(iii) *If* $MQ = 0$ *and* $NQ = 0$*, then* $M \cong P/PQ$*,* $N \cong PQ$*,* M *is projective and* N *is injective in* $\operatorname{mod} A$.

(iv) $MQ = 0$ *and* M *is projective in* $\operatorname{mod} A$ *if and only if* $NQ = 0$ *and* N *is injective in* $\operatorname{mod} A$.

Proof. The statements (i) and (ii) follow from Lemma 7.4.

(iii) Assume that $MQ = 0$ and $NQ = 0$. Then, by (i) and (ii), $\ell_N(Q) = N$ is injective in $\operatorname{mod} A$ and $M/MQ = M$ is projective in $\operatorname{mod} A$. Moreover, it follows from Lemma 7.5 that $N \cong PQ$ and $M \cong P/PQ$.

(iv) Assume that $MQ = 0$ and M is projective in $\operatorname{mod} A$. Then $NQ = 0$ by (ii), and hence N is injective in $\operatorname{mod} A$ by (iii). Conversely, if $NQ = 0$ and N is injective in $\operatorname{mod} A$, then $N = \ell_N(Q)$ and so $MQ = 0$ by (i). Hence $M = M/MQ$ is projective in $\operatorname{mod} A$ by (ii). □

Proposition 7.7. *Let A be a finite dimensional K-algebra over a field K, and T be a Hochschild extension algebra of A by a duality A-bimodule Q. The following statements are equivalent:*

(i) M/MQ *is projective in* $\operatorname{mod} A$ *for any indecomposable module M in* $\operatorname{mod} T$ *with* $MQ \neq 0$.

(ii) $\ell_N(Q)$ *is injective in* $\operatorname{mod} A$ *for any indecomposable module N in* $\operatorname{mod} T$ *with* $NQ \neq 0$.

(iii) $\Omega_T(M)Q = 0$ *for any indecomposable module* M *in* mod T *with* $MQ \neq 0$.

(iv) $\Omega_T^{-1}(N)Q = 0$ *for any indecomposable module* N *in* mod T *with* $NQ \neq 0$.

Proof. If X is an indecomposable projective module in mod T, then clearly $XQ \neq 0$, $\Omega_T(X) = 0$, $\Omega_T^{-1}(X) = 0$, and all statements (i)–(iv) are true by Lemma 7.3. Hence, we have only to show the equivalence of (i)–(iv) for nonprojective indecomposable modules X in mod T with $XQ \neq 0$.

(i) $\Leftrightarrow$ (iii) follows from Proposition 7.6 (ii).

(ii) $\Leftrightarrow$ (iv) Let N be an indecomposable nonprojective module in mod T with $NQ \neq 0$. Let $M = \Omega_T^{-1}(N)$. Then, by Proposition 7.6 (i), $MQ = 0$ if and only if $\ell_{\Omega_T(M)}(Q)$ is injective in mod A, which shows the equivalence of (ii) and (iv).

(iv) $\Rightarrow$ (iii) Assume that $\Omega_T(M)Q \neq 0$ for some nonprojective indecomposable module M in mod T with $MQ \neq 0$. Then $\Omega_T^{-1}(\Omega_T(M))Q = 0$ by assumption (iv), and hence $MQ \cong \Omega_T^{-1}(\Omega_T(M))Q = 0$, a contradiction to the imposed assumption $MQ \neq 0$.

(iii) $\Rightarrow$ (iv) Assume that $\Omega_T^{-1}(N)Q \neq 0$ for some nonprojective indecomposable module N in mod T with $NQ \neq 0$. Then $\Omega_T(\Omega_T^{-1}(N))Q = 0$ by assumption (iii), and hence $NQ \cong \Omega_T(\Omega_T^{-1}(N))Q = 0$, contradicting the assumption that $NQ \neq 0$. □

For a finite dimensional K-algebra A over a field K, we denote by ind A the set of isomorphism classes of indecomposable modules in mod A. We recall that the isomorphism class of a module M is denoted by $\{M\}$.

Let T be a Hochschild extension algebra of a finite dimensional K-algebra A over a field K by a duality module Q. The category mod A is regarded as the full subcategory of mod T. Thus ind A is considered as a subclass of ind T. Then ind $T \setminus$ ind A denotes the set of isomorphism classes of indecomposable proper right T-modules, that is, all indecomposable modules in mod T which are not annihilated by Q. We will investigate the difference between the classes ind A and ind $T \setminus$ ind A. Consider the maps

$$\begin{aligned} \Phi_T\colon &\quad \operatorname{ind} A \setminus \operatorname{proj} A \longrightarrow \operatorname{ind} T \setminus (\operatorname{proj} T \cup \operatorname{ind} A), \\ \Phi_T^{-1}\colon &\quad \operatorname{ind} A \setminus \operatorname{inj} A \longrightarrow \operatorname{ind} T \setminus (\operatorname{proj} T \cup \operatorname{ind} A), \end{aligned}$$

defined by $\Phi_T(\{X\}) = \{\Omega_T(X)\}$ for $\{X\} \in$ ind A with X nonprojective in mod A, and $\Phi_T^{-1}(\{X\}) = \{\Omega_T^{-1}(X)\}$ for $\{X\} \in$ ind A with X noninjective in mod A. Notice that, for an indecomposable nonprojective module X in mod A, we have $\Omega_T(X)Q \neq 0$, by Proposition 7.6 (ii), and so $\Phi_T(\{X\})$ belongs to ind $T \setminus$ (proj $T \cup$ ind A). Further, in the case $\{X\} \in$ ind A with X projective in mod A, $\Omega_T(X)$ is injective in mod A. Similarly, for an indecomposable noninjective module X in mod A, we have $\Omega_T^{-1}(X)Q \neq 0$, by Proposition 7.6 (i), and so $\Omega_T^{-1}(\{X\})$ belongs to ind $T \setminus$ (proj $T \cup$ ind A). Moreover, in the case $\{X\} \in$ ind A with X injective in mod A, $\Omega_T^{-1}(X)$ is projective in mod A.

Lemma 7.8. *Φ_T and Φ_T^{-1} are injective maps.*

Proof. Let $\Phi_T(\{X\}) = \Phi_T(\{Y\})$ for $\{X\}, \{Y\} \in \operatorname{ind} A \backslash \operatorname{proj} A$. Then $\{\Omega_T(X)\} = \{\Omega_T(Y)\}$, that is, $\Omega_T(X) \cong \Omega_T(Y)$, which implies that $X \cong Y$, and so $\{X\} = \{Y\}$. The injectivity of the map Φ_T^{-1} follows by a similar argument. □

Lemma 7.9. *Let A be a finite dimensional K-algebra over a field K and T be a Hochschild extension algebra of A by a duality A-bimodule Q. Then the following statements are equivalent:*

(i) *Φ_T is a bijection.*

(ii) *Φ_T^{-1} is a bijection.*

(iii) *T satisfies the equivalent statements of Proposition 7.7.*

Proof. By Lemma 7.8, Φ_T is bijective if and only if Φ_T is surjective. This is equivalent to saying that, for each $\{X\} \in \operatorname{ind} T \backslash(\operatorname{proj} T \cup \operatorname{ind} A)$, there is an isomorphism class $\{Y\} \in \operatorname{ind} A \setminus \operatorname{proj} A$ such that $X \cong \Omega_T(Y)$, that is, $\Omega_T^{-1}(X)$ is a right A-module or, equivalently, $\Omega_T^{-1}(X)Q = 0$. Thus Φ_T is bijective if and only if $\Omega_T^{-1}(X)Q = 0$ for all indecomposable nonprojective modules X in $\operatorname{mod} T$ with $XQ \neq 0$, which is the statement (iv) in Proposition 7.7.

Similarly, we prove that the map Φ_T^{-1} is bijective if and only if $\Omega_T(X)Q = 0$ for all indecomposable nonprojective modules X in $\operatorname{mod} T$ with $XQ \neq 0$, which is statement (iii) in Proposition 7.7. □

In general, for any module M in $\operatorname{mod} T$, we have the inclusion $MQ \subseteq \ell_M(Q)$. The property $MQ = \ell_M(Q)$ will characterize A as being a hereditary algebra. We show this fact by using two preparatory lemmas.

A module M in $\operatorname{mod} A$ is said to be *torsionless* if M is isomorphic to a submodule of a projective module in $\operatorname{mod} A$.

Lemma 7.10. *Let X be a torsionless module in* $\operatorname{mod} A$. *Then there is a module Y in* $\operatorname{mod} T$ *such that $X \cong Y/\ell_Y(Q)$ in* $\operatorname{mod} A$.

Proof. Let L be a projective module in $\operatorname{mod} A$ containing a submodule isomorphic to X, and let $P \to L$ be a projective cover of L in $\operatorname{mod} T$. Then, by Lemma 7.5, $L \cong P/PQ$ and there is a monomorphism $u: X \to P/PQ$ in $\operatorname{mod} A$. Let $u(X) = Y/PQ$ for some submodule Y of P containing PQ. Since $PQ \subseteq \ell_Y(Q) \subseteq \ell_P(Q)$ and $\ell_P(Q) = PQ$, by Lemma 7.3, it holds that $\ell_Y(Q) = PQ$. Thus we have an isomorphism $X \cong Y/\ell_Y(Q)$ in $\operatorname{mod} A$. □

Lemma 7.11. *Let M be a nonprojective torsionless module in* $\operatorname{mod} A$. *Then* $\Omega_T(M)Q \subset \ell_{\Omega_T(M)}(Q)$.

Proof. Let $P \xrightarrow{u} M$ be a projective cover of M in $\operatorname{mod} T$ and $\Omega_T(M) = \operatorname{Ker} u$. Let $P = P_1 \oplus \cdots \oplus P_n$ be a decomposition of P as a direct sum of indecomposable modules in $\operatorname{mod} T$. Since M is an A-module, $MQ = 0$, and so $PQ \subseteq \operatorname{Ker} u$. Hence $\ell_{\Omega_T(M)}(Q) = \Omega_T(M) \cap \ell_P(Q) = \operatorname{Ker} u \cap PQ = PQ$. Then we have $\Omega_T(M)Q \subseteq \ell_{\Omega_T(M)}(Q) = PQ = P_1Q \oplus \cdots \oplus P_nQ$, where the first inclusion follows from $Q^2 = 0$. Hence, it suffices to show that $P_iQ \not\subseteq \Omega_T(M)Q$, for all $i \in \{1, \ldots, n\}$. It follows from Lemma 7.10 that there is an exact sequence

$$0 \longrightarrow \ell_Y(Q) \hookrightarrow Y \xrightarrow{w} M \longrightarrow 0$$

in $\operatorname{mod} T$. By the projectivity of P, there is a homomorphism $v \colon P \to Y$ in $\operatorname{mod} T$ such that $u = wv$. Then $wv(\operatorname{Ker} u) = u(\operatorname{Ker} u) = 0$, and so $v(\operatorname{Ker} u) \subseteq \operatorname{Ker} w = \ell_Y(Q)$. Hence $v(\operatorname{Ker} u)Q \subseteq \ell_Y(Q)Q = 0$, and then $v(\operatorname{Ker} u)Q = 0$. Now, suppose that $P_iQ \subseteq \Omega_T(M)Q$ for some $i \in \{1, \ldots, n\}$. In this case, $v(P_iQ) \subseteq v(\Omega_T(M)Q) = v(\operatorname{Ker} u)Q = 0$, and hence $v(P_i)Q = 0$, which implies that $v(P_i) \subseteq \ell_Y(Q)$. Then, applying w to this inclusion, we obtain $u(P_i) = wv(P_i) \subseteq w(\ell_Y(Q)) = 0$, because $\ell_Y(Q) = \operatorname{Ker} w$. This implies that $u(P_i) = 0$, which contradicts the fact that u is a projective cover. $\square$

Now we may characterize hereditary algebras by some properties of indecomposable modules over Hochschild extension algebras.

Theorem 7.12. *Let A be a finite dimensional K-algebra over a field K and Q a duality A-bimodule. Let T be a Hochschild extension algebra of A by Q. Then the following statements are equivalent:*

(i) *A is a hereditary algebra.*

(ii) *MQ is injective in* $\operatorname{mod} A$ *for any module M in* $\operatorname{mod} T$.

(iii) *MQ is injective in* $\operatorname{mod} A$ *for any indecomposable module M in* $\operatorname{mod} T$.

(iv) *$MQ = \ell_M(Q)$ for any indecomposable module M in* $\operatorname{mod} T$ *with $MQ \neq 0$.*

(v) *$M/\ell_M(Q)$ is projective in* $\operatorname{mod} A$ *for any indecomposable module M in* $\operatorname{mod} T$.

(vi) *$M/\ell_M(Q)$ is projective in* $\operatorname{mod} A$ *for any module M in* $\operatorname{mod} T$.

Proof. (i) $\Rightarrow$ (ii) Assume that A is a hereditary algebra. Take any module M in $\operatorname{mod} T$ and let $u \colon P \to M$ be a projective cover of M in $\operatorname{mod} T$. Then MQ is the homomorphic image $u(PQ)$ of PQ. Since A is hereditary and PQ is injective, by Lemma 7.3 (ii), the homomorphic image MQ of PQ is also injective, by Theorem I.9.2.

(ii) $\Leftrightarrow$ (iii) Obviously, (ii) implies (iii). Conversely, assume that (iii) holds. Let M be any module in $\operatorname{mod} A$ and $M = M_1 \oplus \cdots \oplus M_m$ be a decomposition of M as a direct sum of indecomposable submodules. Then $MQ \cong M_1Q \oplus \cdots \oplus M_mQ$, which is a direct sum of injective modules in mod A, and so MQ is injective. Therefore, (iii) implies (ii).

(ii) ⇒ (i) Assume that (ii) holds. Let I be an injective module in mod A. It is enough to show that, for any epimorphism $f: I \to X$ in mod A, the module X is injective in mod A. Let $u: I \to P$ be an injective envelope of I in mod T. Then, by Lemma 7.5, u induces an isomorphism $I \xrightarrow{\sim} PQ$. Let $M = P/u(\operatorname{Ker} f)$. Then $MQ = PQ/u(\operatorname{Ker} f)$ is injective in mod A by the assumption (ii). Moreover, there are canonical isomorphisms $X \cong I/\operatorname{Ker} f \cong PQ/u(\operatorname{Ker} f) = MQ$ in mod A. Thus X is injective in mod A, as claimed.

(i) ⇒ (iv) Assume that (i) holds. Let M be an indecomposable module in mod T with $MQ \neq 0$, and $j: M \to P$ an injective envelope of M in mod T. Since j induces natural isomorphisms $MQ \xrightarrow{\sim} j(M)Q$ and $\ell_M(Q) \xrightarrow{\sim} \ell_{j(M)}(Q)$, we may assume that M is a submodule of P and j is the inclusion homomorphism $M \hookrightarrow P$. Let $w: M/MQ \to P/PQ$ be the canonical homomorphism induced by j. Since (i) is equivalent to (ii), we conclude that MQ is injective in mod A, and hence $\ell_M(Q) = MQ \oplus X$ for some submodule X of $\ell_M(Q)$ in mod A. There is in mod T the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, "s"] & M \arrow[r] \arrow[d, "t"] & M/X \arrow[r] & 0 \\
0 \arrow[r] & \ell_M(Q)/MQ \arrow[r, "v"] & M/MQ \arrow[r, "w"] & P/PQ\,, &
\end{tikzcd}$$

where s is a canonical isomorphism, t a canonical epimorphism and u, v are inclusion homomorphisms. We observe that the lower horizontal sequence in the above diagram is exact. Indeed, it holds that $\operatorname{Ker} w = (M \cap PQ)/MQ$, where $M \cap PQ = \ell_M(Q)$, by Lemma 7.3 (i). Now P/PQ is projective in mod A, by Lemma 7.3 (iii), and hence its submodule $\operatorname{Im} w$ is also projective, because A is hereditary. This implies that the exact sequence

$$0 \longrightarrow \ell_M(Q)/MQ \xrightarrow{v} M/MQ \longrightarrow \operatorname{Im} w \longrightarrow 0$$

in mod A is splittable, and consequently there is a homomorphism $v': M/MQ \to \ell_M(Q)/MQ$ with $v'v$ the identity on $\ell_M(Q)/MQ$. Hence $s = (v'v)s = (v't)u$, which implies that u is a splittable monomorphism, by Lemma I.4.1, because s is an isomorphism. Therefore $M \cong X \oplus (M/X)$ in mod T. We claim that $X = 0$. Indeed, if $X \neq 0$, then $M \cong X$, because M is indecomposable. This is impossible, because $MQ \neq 0$ and $XQ = 0$, and hence X must be zero. Thus $\ell_M(Q)/MQ = 0$, or equivalently, $MQ = \ell_M(Q)$, which is the required equality. Hence (i) implies (iv).

(iv) ⇒ (v) Assume that (iv) holds. Let M be an indecomposable module in mod T. We may assume that $MQ \neq 0$. Indeed, if $MQ = 0$, then $\ell_M(Q) = M$, and so $M/\ell_M(Q) = 0$ is projective. Let P be an injective envelope of M in $\operatorname{mod} T$, and assume that M is a submodule of P. Then P is projective in $\operatorname{mod} T$ and, by Lemma 7.3, P/PQ is projective in $\operatorname{mod} A$. Further, we have $P/PQ \supseteq (M + PQ)/PQ \cong M/(M \cap PQ) = M/\ell_M(Q)$, where $M \cap PQ = \ell_M(Q)$, by Lemma 7.3. Thus $M/\ell_M(Q)$ is torsionless in $\operatorname{mod} A$. By the assumption (iv), $MQ = \ell_M(Q)$, and therefore M/MQ is torsionless in $\operatorname{mod} A$. Thus our aim is to prove that M/MQ is projective in $\operatorname{mod} A$. For this, we suppose that M/MQ is not projective in $\operatorname{mod} A$, and we will show a contradiction. Let X be a nonprojective indecomposable direct summand of M/MQ in $\operatorname{mod} A$. Then $\Omega_T(X)$ is an indecomposable right T-module, and it follows from Proposition 7.6 (iii) that $\Omega_T(X)Q \neq 0$, because $XQ = 0$, by the assumption that X is not projective in $\operatorname{mod} A$. Hence, $\Omega_T(X)Q = \ell_{\Omega_T(X)}(Q)$ by the assumption (iv). However, X is torsionless in $\operatorname{mod} A$, because so is M/MQ, and hence $\Omega_T(X)Q \subset \ell_{\Omega_T(X)}(Q)$, by Lemma 7.11, a contradiction. Thus (iv) implies (v).

(v) $\Rightarrow$ (vi) Assume (v) holds. Let M be a module in $\operatorname{mod} T$, and $M = M_1 \oplus \cdots \oplus M_n$ be a decomposition of M as a direct sum of indecomposable submodules. Since $\ell_M(Q) = \ell_{M_1}(Q) \oplus \cdots \oplus \ell_{M_n}(Q)$, there is an isomorphism $M/\ell_M(Q) \cong M_1/\ell_{M_1}(Q) \oplus \cdots \oplus M_n/\ell_{M_n}(Q)$ in $\operatorname{mod} A$. By the assumption (v), the summands on the right-hand side of the isomorphism are projective in $\operatorname{mod} A$, and so is $M/\ell_M(Q)$. Hence (v) implies (vi).

(vi) $\Rightarrow$ (i) Assume (vi) holds. Let I be a nonzero right ideal of A. By Lemma 7.10, there is a module M in $\operatorname{mod} T$ such that $I \cong M/\ell_M(Q)$. Further, by the assumption (vi), $M/\ell_M(Q)$ is projective in $\operatorname{mod} A$, and hence so is I. Applying Theorems I.9.1 and I.9.3, we conclude that A is a hereditary algebra. Hence (vi) implies (i). □

We list some characteristic properties of modules in $\operatorname{mod} T$, for a hereditary algebra A, which follow from Theorem 7.12 and Proposition 7.7.

Corollary 7.13. *Let A be a finite dimensional hereditary K-algebra over a field K and T be a Hochschild extension algebra of A by a duality A-bimodule Q. Then the following statements hold for a module M in* $\operatorname{mod} T$ *without nonzero direct summands annihilated by Q:*

(i) *M/MQ is a projective module in* $\operatorname{mod} A$.

(ii) *$\ell_M(Q) = MQ$ is an injective module in* $\operatorname{mod} A$.

(iii) $\Omega_T(M)Q = 0$ *and* $\Omega_T^{-1}(M)Q = 0$.

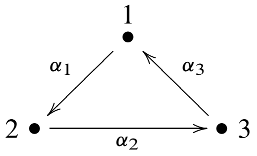
The following theorem is a consequence of Theorem 7.12 and Lemma 7.9.

Theorem 7.14. *Let A be a finite dimensional hereditary K-algebra over a field K and T be a Hochschild extension algebra of A by a duality A-bimodule Q. Then* *the maps*

$$\Phi_T\colon \operatorname{ind} A \setminus \operatorname{proj} A \longrightarrow \operatorname{ind} T \setminus (\operatorname{proj} T \cup \operatorname{inj} A),$$
$$\Phi_T^{-1}\colon \operatorname{ind} A \setminus \operatorname{inj} A \longrightarrow \operatorname{ind} T \setminus (\operatorname{proj} T \cup \operatorname{ind} A)$$

are bijective.

Example 7.15. Let K be a field and T be the bound quiver algebra KQ_T/I_T defined by the quiver Q_T of the form



and the ideal I_T in KQ_T generated by the paths $\alpha_1\alpha_2\alpha_3\alpha_1$, $\alpha_2\alpha_3\alpha_1\alpha_2$, and $\alpha_3\alpha_1\alpha_2\alpha_3$.

We show that T satisfies the equivalent conditions in Proposition 7.7. Observe that T is a selfinjective Nakayama algebra with $1_T = e_1 + e_2 + e_3$, where e_i is the primitive idempotent corresponding to the vertex i of Q_T. Moreover, every indecomposable projective module e_iT in $\operatorname{mod} T$ has Loewy length 4. Let Q be the ideal $\operatorname{rad}^2 T$, which has a basis $\alpha_1\alpha_2$, $\alpha_1\alpha_2\alpha_3$, $\alpha_2\alpha_3$, $\alpha_2\alpha_3\alpha_1$, $\alpha_3\alpha_1$, $\alpha_3\alpha_1\alpha_2$ as a K-vector space. Let $A = T/Q$ be the quotient algebra of T by Q. Then A is isomorphic to the bound quiver algebra KQ_T/J_T of the above quiver Q_T and the ideal J_T in KQ_T generated by the paths $\alpha_1\alpha_2$, $\alpha_2\alpha_3$, $\alpha_3\alpha_1$. Observe that A is a selfinjective Nakayama algebra whose indecomposable projective right modules e_iA have Loewy length 2. A complete set of representatives of isomorphism classes in $\operatorname{ind} A$ is given by the indecomposable modules

$$\begin{aligned} S_1 &= Ke_1, & S_2 &= Ke_2, & S_3 &= Ke_3,\\ X_1 &= Ke_1 + K\alpha_1, & X_2 &= Ke_2 + K\alpha_2, & X_3 &= Ke_3 + K\alpha_3, \end{aligned}$$

while a complete set of representatives of isomorphism classes in $\operatorname{ind} T \setminus \operatorname{ind} A$ is given by the indecomposable modules

$$\begin{aligned} Y_1 &= Ke_1 + K\alpha_1 + K\alpha_1\alpha_2, & Z_1 &= Ke_1 + K\alpha_1 + K\alpha_1\alpha_2 + K\alpha_1\alpha_2\alpha_3,\\ Y_2 &= Ke_2 + K\alpha_2 + K\alpha_2\alpha_3, & Z_2 &= Ke_2 + K\alpha_2 + K\alpha_2\alpha_3 + K\alpha_2\alpha_3\alpha_1,\\ Y_3 &= Ke_3 + K\alpha_3 + K\alpha_3\alpha_1, & Z_3 &= Ke_3 + K\alpha_3 + K\alpha_3\alpha_1 + K\alpha_3\alpha_1\alpha_2. \end{aligned}$$

Hence, it holds that $Y_1Q = K\alpha_1\alpha_2$, $Y_2Q = K\alpha_2\alpha_3$, $Y_3Q = K\alpha_3\alpha_1$, $Z_1Q = K\alpha_1\alpha_2 + K\alpha_1\alpha_2\alpha_3$, $Z_2Q = K\alpha_2\alpha_3 + K\alpha_2\alpha_3\alpha_1$, and $Z_3Q = K\alpha_3\alpha_1 + K\alpha_3\alpha_1\alpha_2$. Observe that $Y_i/Y_iQ \cong e_iA$ and $Z_i/Z_iQ \cong e_iA$, which are projective modules in $\operatorname{mod} A$. Thus T satisfies the condition (i) in Proposition 7.7.

It follows from Example 7.15 that the converse of Corollary 7.13 is not true in general. However, it does hold when T is the trivial extension algebra of A by Q. This is a consequence of the following theorem.

Theorem 7.16. *Let A be a finite dimensional K-algebra over a field K, Q a duality A-bimodule, and $T = A \ltimes Q$ the associated trivial extension algebra. Then the following statements are equivalent:*

(i) *A is a hereditary algebra.*

(ii) *The equivalent conditions in Proposition 7.7 hold.*

(iii) *The map Φ_T is bijective.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Corollary 7.13. Moreover, the equivalence (ii) $\Leftrightarrow$ (iii) is simply Lemma 7.9.

(ii) $\Rightarrow$ (i). In order to prove that A is hereditary, it is enough to show that, for any indecomposable projective right A-module eA with $e^2 = e$, every right A-submodule I of eA is projective in mod A. In order to avoid ambiguity in dealing with A-submodules of T, for K-subspaces $X \subseteq A$ and $Y \subseteq Q$ we denote by (X, Y) the K-subspace $\{(x, y) \in T \mid x \in X, y \in Y\}$ of T, where it should be noticed that $T = A \ltimes Q$ is the direct sum of A and Q as a right A-module. Now, let I be a nonzero right A-submodule of eA and let $M = (I, eQ)$. Then $MT = (I, eQ)(A, Q) = (IA, eQ + IQ)$, where $IA = I$ and $IQ \subseteq eQ$. Hence, $MT = (I, eQ)$, and so $MT = M$, which shows that M is a right T-module. Let $\boldsymbol{e} = (e, 0) \in T$. Since $\boldsymbol{e}$ is a primitive idempotent, $\boldsymbol{e}T$ is indecomposable injective, and hence $\operatorname{soc}(\boldsymbol{e}T)$ is simple. Obviously, $M \subseteq \boldsymbol{e}T$. Hence, $\operatorname{soc}(\boldsymbol{e}T)$ is contained in M and $\operatorname{soc}(\boldsymbol{e}T) = \operatorname{soc}_T(M)$. This implies that M is indecomposable. Moreover, $MQ = (I, eQ)(0, Q) = (0, IQ)$ and $IQ \neq 0$, because Q is faithful as a left A-module. Then $MQ \neq 0$. Thus we can apply the statement (i) of Proposition 7.7 to M, and conclude that M/MQ is projective in mod A. It therefore follows that I is projective in mod A as desired, because there are isomorphisms in mod A

$$M/MQ \cong (I, eQ)/(0, IQ) \cong I \oplus eQ/IQ. \qquad \square$$

The representation theory of the trivial extension algebra $\mathrm{T}(A)$ of a finite-dimensional hereditary K-algebra A over a field K was first given in general form, for any Hochschild extension algebras, by K. Yamagata (see [Y1], [Y2], [Y3]), and then another proof for $\mathrm{T}(A)$ was given by H. Tachikawa [T]. The *Tachikawa–Yamagata theorem* for hereditary algebras A of finite representation type is now obtained as a consequence of Theorem 7.16.

Corollary 7.17. *Let A be a finite dimensional K-algebra over a field K and Q a duality A-bimodule, and assume that A is of finite representation type. Then, if A is a hereditary algebra, any Hochschild extension algebra T of A by Q is of finite* *representation type, and the number of isomorphism classes of indecomposable modules in* mod T *is twice the number of isomorphism classes of indecomposable modules in* mod A*. Moreover, for* $T = A \ltimes Q$*, the converse holds.*

Proof. Since the number of nonisomorphic indecomposable projective modules in mod A is the same as the number of nonisomorphic indecomposable projective modules in mod T, the corollary immediately follows from Theorems 7.14 and 7.16. □

We end this section with the following proposition, needed for further considerations.

Proposition 7.18. *Let* T *be a basic indecomposable Hochschild extension algebra of a finite dimensional hereditary* K*-algebra* A *over a field* K *by a duality* A*-bimodule* Q*, and* $A = A_1 \times \cdots \times A_m$ *be a Nakayama block decomposition of* A *with respect to* T*. Assume that* M *is an indecomposable nonprojective module and* N *is an indecomposable noninjective module in* mod A *such that* $\Omega_T(M) \cong \Omega_T^{-1}(N)$ *in* mod T*. Then there exists* $i \in \{1, \ldots, m\}$ *such that* M *is a module* mod A_i *and* N *is a module in* mod A_{i-1}*, where* $A_0 = A_m$*.*

Proof. Since $A = A_1 \times \cdots \times A_m$ and M, N are indecomposable modules in mod A = mod $A_1 \times \cdots \times$ mod A_m, there exist $i, j \in \{1, \ldots, m\}$ such that M belongs to mod A_i and N belongs to mod A_j. Then we conclude that top$(\Omega_T(M))$ belongs to mod A_i and top$(\Omega_T^{-1}(N))$ belongs to mod A_{j+1}, because the Nakayama automorphism ν_T of T gives $\nu_T(A_j) = A_{j+1}$. Hence, we obtain $j+1 = i$, and then $j = i - 1$, as required. □

8 The Auslander–Reiten quivers of Hochschild extension algebras

Let A be a finite dimensional hereditary K-algebra over a field K. The structure of the Auslander–Reiten quiver Γ_A of A has been described completely in Chapter VII (Theorems VII 6.1, VII 6.2, VII 7.4, VII 8.11, VII 9.3). Our next aim here is to describe the structure of the Auslander–Reiten quiver Γ_T of a Hochschild extension algebra T of A by a duality module Q.

Lemma 8.1. *Let* A *be a finite dimensional hereditary* K*-algebra over a field* K *and* T *a Hochschild extension algebra of* A *by a duality* A*-bimodule* Q*. Then the following statements are equivalent:*

(i) *Every almost split sequence in* mod A *is an almost split sequence in* mod T.

(ii) *For every indecomposable module* M *in* mod T *with* $MQ \neq 0$, $\ell_M(Q)$ *is an injective module in* mod A

Proof. (i) ⇒ (ii) Let M be an indecomposable module in mod T with $MQ \neq 0$. Suppose to the contrary that $\ell_M(Q)$ is not injective in mod A. Take a noninjective indecomposable direct summand X of $\ell_M(Q)$, say $\ell_M(Q) = X \oplus X'$ for some module X' in mod A, and let $u: X \to \ell_M(Q)$ and $v: \ell_M(Q) \to X$ be the canonical injection and canonical projection, respectively. By Theorem III.8.4, there is a left minimal almost split homomorphism $f: X \to Y$ in mod A. To get a contradiction, we shall show that f is a section in mod T. By the definition of f, for the inclusion homomorphism $h: X \to M$, there is a homomorphism $g: Y \to M$ such that $h = gf$. Since $g(Y)Q = g(YQ) = 0$, g factorizes as $g = ig'$, where $i: \ell_M(Q) \to M$ is the inclusion homomorphism and $g': Y \to \ell_M(Q)$ is a homomorphism with $g'(y) = g(y)$ for all $y \in Y$. We have the following diagram in mod T

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[rd, shift left, "u"'] \arrow[rdd, "h"'] & & Y \arrow[ld, "g'"'] \arrow[ldd, "g"] \\
& \ell_M(Q) \arrow[lu, shift left, "v"'] \arrow[d, "i"] & \\
& M &
\end{tikzcd}$$

Then $iu = h = gf = i(g'f)$, and hence $u = g'f$, because i is a monomorphism. It follows that $\mathrm{id}_X = vu = v(g'f) = (vg')f$. This shows that f is a section, as desired.

(ii) ⇒ (i). Let

$$\mathbb{E}: \quad 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an almost split sequence in mod A. By Theorem III.8.3, in order to show that $\mathbb{E}$ is almost split in mod T, it is enough to show that f is a left almost split homomorphism in mod T. It is obvious that f is not a section in mod T, because f is not a section in mod A. Let $u: X \to W$ be a homomorphism in mod T that is not a section. Our aim is to extend u to a homomorphism $w: Y \to W$ through f. Consider a decomposition $W = W_1 \oplus W_2$ in mod T, where W_1 has no nonzero direct summand annihilated by Q, and $W_2Q = 0$. For $i \in \{1, 2\}$, we denote by $u_i: X \to W_i$ the composition p_iu of u with the canonical projection $p_i: W \to W_i$. Then we have $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. Since $u(X)Q = u(XQ) = 0$, we can write $u = jv$, for the inclusion homomorphism $j: \ell_W(Q) \to W$ and a homomorphism $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}: X \to \ell_W(Q) = \ell_{W_1}(Q) \oplus W_2$ in mod T. We have the following diagram in mod T

$$\begin{tikzcd}
& X \arrow[dl, "v=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}"'] \arrow[r, "f"] \arrow[dd, "u=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}"] & Y \\
\ell_W(Q) = \ell_{W_1}(Q) \oplus W_2 \arrow[dr, "j=\begin{bmatrix} j_1 & 0 \\ 0 & \mathrm{id}_{W_2} \end{bmatrix}"'] & & \\
& W = W_1 \oplus W_2 &
\end{tikzcd}$$

where $j_1\colon \ell_{W_1}(Q) \to W_1$ is the inclusion homomorphism. Notice that $v_2 = u_2$ and v_2 is not a section. Indeed, if $v_2' v_2 = \mathrm{id}_X$ for some homomorphism $v_2'\colon W_2 \to X$, then

$$(v_2' p_2)u = v_2' \,[0, \mathrm{id}_{W_2}] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = v_2' u_2 = v_2' v_2 = \mathrm{id}_X \,.$$

This shows that u is a section, a contradiction. Further, we observe that v is not a section. Indeed, if there is a homomorphism $v' = [v_1', v_2']\colon \ell_M(Q) \oplus W_2 \to X$ with $v'v = \mathrm{id}_X$, then $\mathrm{id}_X = v_1' v_1 + v_2' v_2$. Since $\mathrm{End}_A(X)$ is a local algebra, one of $v_1' v_1$ and $v_2' v_2$ should be an isomorphism, and hence $v_1' v_1$ is an isomorphism, because v_2 is not a section. Then X is isomorphic to a direct summand $v_1(X)$ of $\ell_{W_1}(Q)$, which is injective in mod A by assumption. This however contradicts the fact that X is not injective in mod A.

Let $W_1 = \bigoplus_{i=1}^r W_{1,i}$ be a decomposition of W_1 into a direct sum of indecomposable right T-submodules. Then $\ell_{W_1}(Q) = \bigoplus_{i=1}^r \ell_{W_{1,i}}(Q)$ and it is injective in mod A by assumption. Hence $v_1\colon X \to \ell_{W_1}(Q)$ extends to a homomorphism $w_1\colon Y \to \ell_{W_1}(Q)$ through f, that is, $v_1 = w_1 f$. Since v_2 is not a section in mod A and f is a left almost split homomorphism in mod A by assumption, there is a homomorphism $w_2\colon Y \to W_2$ in mod A with $v_2 = w_2 f$. Therefore, letting $w = j\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\colon Y \to W$, we have

$$wf = j\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} f = j\begin{bmatrix} w_1 f \\ w_2 f \end{bmatrix} = j\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = jv = u,$$

which shows that f is a left almost split homomorphism in mod T, as required. □

Theorem 8.2. *Let A be a finite dimensional hereditary K-algebra over a field K and T be a Hochschild extension algebra of A by a duality A-bimodule Q. Then any almost split sequence in* mod A *is an almost split sequence in* mod T.

Proof. This follows from Theorem 7.12, Proposition 7.7, and Lemma 8.1. □

In the case where T is the trivial extension algebra, the converse of Theorem 8.2 is true.

Theorem 8.3. *Let T be the trivial extension algebra $A \ltimes Q$ of a finite dimensional K-algebra A over a field K by a duality A-bimodule Q. Then A is hereditary if and only if all almost split sequences in* $\operatorname{mod} A$ *are almost split sequences in* $\operatorname{mod} T$.

Proof. This is an immediate consequence of Theorem 7.16 and Lemma 8.1. ☐

In the case where A is a hereditary algebra, Theorem 8.2 implies that Γ_A forms a full valued translation subquiver of Γ_T.

From now on, we assume that A is a finite dimensional hereditary K-algebra over a field K and T a Hochschild extension of A by a duality A-bimodule Q. Since T is selfinjective, it follows from Propositions IX.6.8 and IX.6.9 that, for any almost split sequence

$$\mathbb{E}\colon \quad 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

in $\operatorname{mod} T$, there are almost split sequences in $\operatorname{mod} T$ of the forms

$$\Omega_T(\mathbb{E})\colon 0 \longrightarrow \Omega_T(X) \xrightarrow{\begin{bmatrix}\Omega_T(f)\\ f'\end{bmatrix}} \Omega_T(Y) \oplus P \xrightarrow{[\Omega_T(g)\ g']} \Omega_T(Z) \longrightarrow 0,$$

$$\Omega_T^{-1}(\mathbb{E})\colon 0 \longrightarrow \Omega_T^{-1}(X) \xrightarrow{\begin{bmatrix}\Omega_T^{-1}(f)\\ f''\end{bmatrix}} \Omega_T^{-1}(Y) \oplus P' \xrightarrow{[\Omega_T^{-1}(g)\ g'']} \Omega_T^{-1}(Z) \longrightarrow 0,$$

where P and P' are indecomposable projective or zero modules in $\operatorname{mod} T$.

In particular, we may start with $\mathbb{E}$ being an almost split sequence in $\operatorname{mod} A$. Then it follows from Theorem 8.2 that Ω_T or Ω_T^{-1} transfers Γ_A to a full valued subquiver of Γ_T. Moreover, applying Theorem 7.14 we conclude that the stable full valued subquiver Γ_T^{s} of Γ_T is obtained as a union of Γ_A and $\Omega_T(\Gamma_A)$, and also as a union of Γ_A and $\Omega_T^{-1}(\Gamma_A)$, where for a full valued subquiver $\mathcal{C}$ of Γ_T, $\Omega_T(\mathcal{C})$ and $\Omega_T^{-1}(\mathcal{C})$ denote the full valued subquiver of Γ_T^{s} consisting of the isomorphism classes of modules $\Omega_T(X)$ and $\Omega_T^{-1}(X)$, respectively, for all modules X in $\mathcal{C}$. In the following lemmas we shall show how all components of Γ_T are obtained.

Lemma 8.4. *Let X be an indecomposable module in* $\operatorname{mod} A$. *Then $\Omega_T(\tau_A^n X) \cong \tau_T^n(\Omega_T(X))$, if $\tau_A^n X$ is defined for an integer n.*

Proof. There is nothing to prove for $n = 0$.

Assume that X is nonprojective in $\operatorname{mod} A$. Let

$$\mathbb{E}_X\colon \quad 0 \longrightarrow \tau_A X \longrightarrow Y \longrightarrow X \longrightarrow 0$$

be an almost split sequence in $\operatorname{mod} A$ with the right term X. Then, by Theorem 8.3, $\mathbb{E}_X$ is an almost split sequence in $\operatorname{mod} T$, and hence there is an almost split sequence

$$\Omega_T(\mathbb{E}_X)\colon \quad 0 \longrightarrow \Omega_T(\tau_A X) \longrightarrow \Omega_T(Y) \oplus P \longrightarrow \Omega_T(X) \longrightarrow 0$$

in $\operatorname{mod} T$, where P is a projective module in $\operatorname{mod} T$. This shows that $\Omega_T(\tau_A X) \cong \tau_T(\Omega_T(X))$. Then, repeating the argument, we conclude that $\Omega_T(\tau_A^n X) \cong \tau_T^n(\Omega_T(X))$ in $\operatorname{mod} T$, for any integer $n \geq 1$ with $\tau_A^n X \neq 0$.

Assume that X is noninjective in $\operatorname{mod} A$. Let

$${}_X\mathbb{E}\colon \quad 0 \longrightarrow X \longrightarrow Y' \longrightarrow \tau_A^{-1} X \longrightarrow 0$$

be an almost split sequence in $\operatorname{mod} A$ with the left term X. Then, by Theorem 8.3, ${}_X\mathbb{E}$ is an almost split sequence in $\operatorname{mod} T$, and hence there is an almost split sequence

$$\Omega_T({}_X\mathbb{E})\colon \quad 0 \longrightarrow \Omega_T(X) \longrightarrow \Omega_T(Y) \oplus P' \longrightarrow \Omega_T(\tau_A^{-1} X) \longrightarrow 0$$

in $\operatorname{mod} T$, where P' is a projective module in $\operatorname{mod} T$. This implies that $\Omega_T(\tau_A^{-1} X) \cong \tau_T^{-1}(\Omega_T(X))$ in $\operatorname{mod} T$. Repeating similar arguments, we conclude that $\Omega_T(\tau_A^n X) \cong \tau_T^n(\Omega_T(X))$ in $\operatorname{mod} T$, for any integer $n < 0$ with $\tau_A^n X \neq 0$. □

Before stating some lemmas on simple modules, let us recall that a simple module S in $\operatorname{mod} T$ is annihilated by $Q \subseteq \operatorname{rad} T$, and so S is regarded as a module in $\operatorname{mod} A$. Thus, when we consider simple modules, it is not necessary to stress that they belong to $\operatorname{mod} T$ or $\operatorname{mod} A$. Moreover, when T is Frobenius, $\nu_T(S) \cong \operatorname{top}(I_T(S))$ and $\nu_T^{-1}(S) \cong \operatorname{soc}(P_T(S))$ in $\operatorname{mod} T$, where ν_T is a Nakayama automorphism of T, $I_T(S)$ and $P_T(S)$ are an injective envelope and a projective cover of S in $\operatorname{mod} T$, respectively, which are indecomposable projective-injective modules in $\operatorname{mod} T$.

Lemma 8.5. *Let S be a simple module in* $\operatorname{mod} A$. *Then the following equivalences hold:*

(i) *S is projective in* $\operatorname{mod} A$ *if and only if there is no proper epimorphism from an indecomposable injective module to $I_A(S)$ in* $\operatorname{mod} A$.

(ii) *S is injective in* $\operatorname{mod} A$ *if and only if there is no proper monomorphism from $P_A(S)$ to an indecomposable projective module in* $\operatorname{mod} A$.

Proof. The equivalences (i) and (ii) follow directly from Lemmas VII.1.11, VII.1.13 and Theorems VII.6.1, VII.6.2. □

Lemma 8.6. *Let P be an indecomposable projective module in* $\operatorname{mod} T$. *Then the following equivalences hold:*

(i) $\operatorname{soc}(P)$ *is projective in* $\operatorname{mod} A$ *if and only if* $\operatorname{top}(P)$ *is projective in* $\operatorname{mod} A$.

(ii) $\operatorname{soc}(P)$ *is injective in* $\operatorname{mod} A$ *if and only if* $\operatorname{top}(P)$ *is injective in* $\operatorname{mod} A$.

Proof. We note that, in case T is a Frobenius K-algebra, (i) is equivalent to saying that, for a simple module S in $\operatorname{mod} A$, S is projective in $\operatorname{mod} A$ if and only if $\nu_T(S)$ is projective in $\operatorname{mod} A$. Similarly, (ii) is equivalent to saying that, for a simple module S in $\operatorname{mod} A$, S is injective in $\operatorname{mod} A$ if and only if $\nu_T^{-1}(S)$ is injective in $\operatorname{mod} A$.

Let P be an indecomposable projective module in $\operatorname{mod} T$, $S = \operatorname{soc}(P)$, $S' = \operatorname{top}(P)$, and $p\colon P \to S'$ be the canonical epimorphism with $\operatorname{Ker} p = \operatorname{rad} P$. We let $\Omega_T(S') = \operatorname{Ker} p$.

(i) Assume that S' is projective in $\operatorname{mod} A$. Then, by Proposition 7.6, $\Omega_T(S') = \operatorname{rad} P$ is annihilated by Q and is injective in $\operatorname{mod} A$. Hence $\operatorname{rad} P \subseteq \ell_P(Q) = PQ \subseteq \operatorname{rad} P$, and so $PQ = \operatorname{rad} P$. Now in order to show that S is projective in $\operatorname{mod} A$, it suffices to show, by Lemma 8.5, that any epimorphism $u\colon I \to I_A(S) = PQ$ in $\operatorname{mod} A$ is an isomorphism, for any indecomposable injective module I in $\operatorname{mod} A$. Let $j\colon I \to P'$ be an injective envelope of I in $\operatorname{mod} T$. Observe that P' is indecomposable, because $\operatorname{soc}(P') \cong \operatorname{soc}(I)$ is simple. Let $i\colon PQ \to P$ be the inclusion homomorphism. Since P is injective in $\operatorname{mod} T$, there is a nonzero homomorphism $v\colon P' \to P$ such that we have in $\operatorname{mod} T$ a commutative diagram

$$\begin{tikzcd}
I \arrow[r, "u"] \arrow[d, "j"'] & PQ \arrow[d, "i"] \\
P' \arrow[r, "v"] & P\,.
\end{tikzcd}$$

Observe that $j(I) = P'Q$, by Lemma 7.5. We claim that v is surjective. Indeed, if $v(P') \neq P$, then $v(P') \subseteq \operatorname{rad} P$, where $\operatorname{rad} P = PQ$ as shown above, and hence $v(P') \subseteq PQ = iu(I) = v(j(I)) \subseteq v(P')$, because u is surjective. Hence, $v(P') = v(j(I))$, and so $P' = j(I) + \operatorname{Ker} v$. But this implies that $P' = \operatorname{Ker} v$, because $j(I) \subseteq \operatorname{rad} P'$. Thus $v(P') = 0$, a contradiction. Hence v is an epimorphism. Moreover, since P and P' are indecomposable projective modules, the epimorphism v has to be an isomorphism. This ensures that the epimorphism u is also an isomorphism, which is what we needed.

Conversely, assume that S is projective in $\operatorname{mod} A$. Suppose to the contrary that S' is not projective in $\operatorname{mod} A$. Then the canonical homomorphism $p'\colon P/PQ \to S'$ is not an isomorphism, because P/PQ is projective in $\operatorname{mod} A$, and hence $(\operatorname{rad} P)/PQ = \operatorname{Ker} p' \neq 0$. Take an indecomposable projective module P' in $\operatorname{mod} T$ and a nonzero homomorphism $f\colon P' \to P$ such that $f(P') \subseteq \operatorname{rad} P$ and $f(P') \not\subseteq PQ$. Let $f'\colon P'Q \to PQ$ be the restriction of f to $P'Q$. Then f' is nonzero, because, if $f(P'Q) = 0$, then $f(P')Q = 0$ and $f(P') \subseteq \ell_P(Q) = PQ$, a contradiction to the choice of f. Further, observe that f' is an epimorphism. Indeed, $f(P'Q)$ is a nonzero injective module in $\operatorname{mod} A$, because A is hereditary and $f(P'Q)$ is a homomorphic image of the injective module $P'Q$ in $\operatorname{mod} A$. Hence, $f(P'Q)$ is a direct summand of PQ. This implies that $f(P'Q) = PQ$, because PQ is indecomposable with $\operatorname{soc}(PQ) \cong S$. Since S is projective in $\operatorname{mod} A$ and $PQ \cong I_A(S)$, it follows from Lemma 8.5 that f' is an isomorphism, and hence f is a monomorphism, because $f(\operatorname{soc}(P')) = f'(\operatorname{soc}(P')) \neq 0$. This means that f is proper monomorphism and so P' is isomorphic to a direct summand of $\operatorname{rad} P$, a contradiction. Therefore, S' is projective in $\operatorname{mod} A$.

The proof of (ii) is similar and is left to the reader. □

Let $\mathbb{E}\colon 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an almost split sequence in $\operatorname{mod} T$. In the case when Y belongs to $\operatorname{mod} A$, both X and Z belong to $\operatorname{mod} A$ obviously, and hence $\mathbb{E}$ is an almost split sequence in $\operatorname{mod} A$. In the rest of this section we will consider the other case, that is, Y does not belong to $\operatorname{mod} A$, or equivalently $YQ \neq 0$.

Lemma 8.7. *Let X and Y be modules in* $\operatorname{mod} T$ *with $XQ = 0$ and $YQ = 0$. Then any nonzero homomorphism $u\colon X \to Y$ in* $\operatorname{mod} T$ *does not factor through a module Z without indecomposable direct summands annihilated by Q.*

Proof. Let $v\colon X \to Z$ and $w\colon Z \to Y$ be homomorphisms in $\operatorname{mod} T$ such that $Z = Z_1 \oplus \cdots \oplus Z_m$ is a direct sum of indecomposable submodules Z_i with $Z_iQ \neq 0$ for all $i \in \{1, \ldots, m\}$, and $u = wv$. We claim that $u = 0$. Observe that $v(X) \subseteq \ell_Z(Q)$ and $ZQ \subseteq \operatorname{Ker} w$, because $XQ = 0$ and $YQ = 0$ by assumption. On the other hand, $\ell_{Z_i}(Q) = Z_iQ$ for all $i \in \{1, \ldots, m\}$, by Theorem 7.12, and so $\ell_Z(Q) = \ell_{Z_1}(Q) \oplus \cdots \oplus \ell_{Z_m}(Q) = Z_1Q \oplus \cdots \oplus Z_mQ = ZQ$. Hence $v(X) \subseteq \operatorname{Ker} w$, that is, $u = wv = 0$. □

Proposition 8.8. *Let A be a finite dimensional hereditary K-algebra over a field K and T a Hochschild extension algebra of A by a duality A-bimodule Q. Moreover, let*

$$\mathbb{E}\colon \quad 0 \longrightarrow X \xrightarrow{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}} Y_1 \oplus Y_2 \xrightarrow{[g_1\ g_2]} Z \longrightarrow 0$$

be an almost split sequence in $\operatorname{mod} T$*, where $Y_1 \neq 0$, $Y_1Q = 0$, $Y_2 \neq 0$, and Y_2 has no nonzero direct summands annihilated by Q. Then the following statements hold:*

(i) *If $XQ = 0$, then X is an injective module in* $\operatorname{mod} A$ *and $\mathbb{E}$ is isomorphic to an almost split sequence in* $\operatorname{mod} T$ *of the form*

$$\mathbb{E}'\colon \quad 0 \longrightarrow X \xrightarrow{\begin{bmatrix} v \\ f_2 \end{bmatrix}} X/\operatorname{soc} X \oplus Y_2 \xrightarrow{[v'\ g_2]} Z \longrightarrow 0,$$

where $v\colon X \to X/\operatorname{soc} X$ is the canonical epimorphism.

(ii) *If* $ZQ = 0$, *then* Z *is a projective module in* $\operatorname{mod} A$ *and* $\mathbb{E}$ *is isomorphic to an almost split sequence in* $\operatorname{mod} T$ *of the form*

$$\mathbb{E}'' \colon \quad 0 \longrightarrow X \xrightarrow{\left[\begin{smallmatrix} u' \\ f_2 \end{smallmatrix}\right]} \operatorname{rad} Z \oplus Y_2 \xrightarrow{[u \ g_2]} Z \longrightarrow 0,$$

where $u \colon \operatorname{rad} Z \to Z$ *is the canonical monomorphism.*

(iii) *If* $XQ \neq 0$ *and* $ZQ \neq 0$, *then there exist almost split sequences*

$$\mathbb{E}_0 \colon \quad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and

$$\mathbb{E}_1 \colon \quad 0 \longrightarrow L' \longrightarrow M' \longrightarrow N' \longrightarrow 0$$

in $\operatorname{mod} A$ *such that* $\mathbb{E}$ *is isomorphic to* $\Omega_T(\mathbb{E}_0)$ *and* $\Omega_T^{-1}(\mathbb{E}_1)$.

Proof. (i) Assume that $XQ = 0$. Then X is an A-module. If X is not injective in $\operatorname{mod} A$, then there is an almost split sequence in $\operatorname{mod} A$

$${}_X\mathbb{E} \colon \quad 0 \longrightarrow X \longrightarrow X' \longrightarrow X'' \longrightarrow 0.$$

Applying Theorem 8.2, we conclude that ${}_X\mathbb{E}$ is an almost split sequence in $\operatorname{mod} T$. Then it follows from Lemma III.8.2 that $\mathbb{E}$ is isomorphic to ${}_X\mathbb{E}$, and hence $Y_1 \oplus Y_2 \cong X'$, which is a contradiction, because $Y_2 Q \neq 0$ and $X'Q = 0$. Hence, X is an injective module in $\operatorname{mod} A$. We will show that $f_1 \colon X \to Y_1$ is a left minimal almost split homomorphism in $\operatorname{mod} A$. It follows from Theorem III.7.11 that f_1 is an irreducible homomorphism in $\operatorname{mod} T$, and hence f_1 is an irreducible homomorphism in $\operatorname{mod} A$. Let $h \colon X \to W$ be a homomorphism in $\operatorname{mod} A$ which is not a section. Since $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$ is a left almost split homomorphism in $\operatorname{mod} T$, there is a homomorphism $w = [w_1 \, w_2] \colon Y_1 \oplus Y_2 \to W$ in $\operatorname{mod} T$ such that $h = [w_1 \, w_2]\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] = w_1 f_1 + w_2 f_2$. Observe that $XQ = 0$ and $WQ = 0$. Then it follows from Lemma 8.7 and the assumption made on Y_2 that $w_2 f_2 = 0$, and consequently $h = w_1 f_1$. Hence f_1 is a left almost split homomorphism in $\operatorname{mod} A$. Let u be a homomorphism in $\operatorname{mod} A$ such that $u f_1 = f_1$. Then u is a section, because f_1 is irreducible, and so u is an isomorphism. Therefore, f_1 is a left minimal almost split homomorphism in $\operatorname{mod} A$. On the other hand, since X is indecomposable injective in $\operatorname{mod} A$, Lemma III.7.7 shows that the canonical projection $v \colon X \to X/\operatorname{soc} X$ is a left minimal almost split homomorphism in $\operatorname{mod} A$. Then, applying Theorem III.7.11, we conclude that there is an isomorphism $s \colon Y_1 \to X/\operatorname{soc} X$ in $\operatorname{mod} A$ such that $v = s f_1$. Summing up, we obtain that $\mathbb{E}$ is isomorphic to an almost split sequence in $\operatorname{mod} T$ of the form

$$\mathbb{E}' \colon \quad 0 \longrightarrow X \xrightarrow{\left[\begin{smallmatrix} v \\ f_2 \end{smallmatrix}\right]} X/\operatorname{soc} X \oplus Y_2 \xrightarrow{[v' \ g_2]} Z \longrightarrow 0.$$

(ii) Assume that $ZQ = 0$. Consider the standard duality $D = \operatorname{Hom}_K(-, K)\colon \operatorname{mod} T \to \operatorname{mod} T^{\mathrm{op}}$, which induces a duality $\operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$. The almost split sequence $\mathbb{E}$ gives in $\operatorname{mod} T^{\mathrm{op}}$ the almost split sequence

$$D(\mathbb{E})\colon \quad 0 \longrightarrow D(Z) \xrightarrow{\begin{bmatrix} D(g_1) \\ D(g_2) \end{bmatrix}} D(Y_1) \oplus D(Y_2) \xrightarrow{[D(f_1)\ D(f_2)]} D(X) \longrightarrow 0,$$

where $QD(Z) = D(ZQ) = 0$, $QD(Y_1) = D(Y_1 Q) = 0$ and $D(Y_2)$ has no nonzero direct summands annihilated by Q. Hence, by (i), $D(Z)$ is an injective A^{op}-module and $D(g_1)\colon D(Z) \to D(Y_1)$ is a left minimal almost split homomorphism in $\operatorname{mod} A^{\mathrm{op}}$. Hence, applying the duality $D = \operatorname{Hom}_K(-, K)\colon \operatorname{mod} A^{\mathrm{op}} \to \operatorname{mod} A$, we conclude that Z is a projective A-module and $g_1\colon Y_1 \to Z$ is a right minimal almost split homomorphism in $\operatorname{mod} A$. Moreover, it follows from Lemma III.7.6 that the canonical monomorphism $u\colon \operatorname{rad} Z \to Z$ is a right minimal almost split homomorphism in $\operatorname{mod} A$. Then, applying Theorem III.7.12, we conclude that there is an isomorphism $r\colon Z \to \operatorname{rad} Z$ in $\operatorname{mod} A$ such that $g_1 = ru$. Therefore, we obtain that $\mathbb{E}$ is isomorphic to an almost split sequence in $\operatorname{mod} T$ of the form

$$\mathbb{E}''\colon \quad 0 \longrightarrow X \xrightarrow{\begin{bmatrix} u' \\ f_2 \end{bmatrix}} \operatorname{rad} Z \oplus Y_2 \xrightarrow{[u\ g_2]} Z \longrightarrow 0.$$

(iii) Assume $XQ \neq 0$ and $ZQ \neq 0$. Let $Y = Y_1 \oplus Y_2$. By Corollary 7.13, we have $\Omega_T^{-1}(Z)Q = 0$, and hence $\Omega_T^{-1}(Z)Q$ belongs to $\operatorname{mod} A$. Further, $\Omega_T^{-1}(Z)$ is not projective in $\operatorname{mod} A$. Indeed, if $\Omega_T^{-1}(Z)$ is projective in $\operatorname{mod} A$, then Lemma 7.5 implies that $ZQ \cong \Omega_T(\Omega_T^{-1}(Z))Q = 0$, contradicting the assumption $ZQ \neq 0$. Hence, putting $N = \Omega_T^{-1}(Z)$, we obtain, by Theorem III.8.4, that there is an almost split sequence

$$\mathbb{E}\colon \quad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in $\operatorname{mod} A$. Then the almost split sequences $\mathbb{E}$ and $\Omega_T(\mathbb{E}_0)$ in $\operatorname{mod} T$ are isomorphic, because their right terms Z and $\Omega_T(N)$ are isomorphic.

Next, we consider the module $\Omega_T(X)$. Since $\Omega_T(X)Q = 0$, by Corollary 7.13, $\Omega_T(X)$ belongs to $\operatorname{mod} A$. Moreover, $\Omega_T(X)$ is not injective in $\operatorname{mod} A$, by Proposition 7.6 (i), because $XQ \neq 0$ and $\ell_{\Omega_T(X)}(Q) = \Omega_T(X)$. Hence, by Theorem III.8.4, there is an almost split sequence

$$\mathbb{E}_1\colon \quad 0 \longrightarrow L' \longrightarrow M' \longrightarrow N' \longrightarrow 0$$

in $\operatorname{mod} A$ with the left term $L' = \Omega_T(X)$. Then the almost split sequences $\mathbb{E}$ and $\Omega_T^{-1}(\mathbb{E}_1)$ in $\operatorname{mod} T$ are isomorphic, because their left terms X and $\Omega_T^{-1}(L')$ are isomorphic. □

We will discuss now more precisely the structure of almost split sequences in $\operatorname{mod} T$ whose left terms are indecomposable injective modules from $\operatorname{mod} A$ and almost split sequences in $\operatorname{mod} T$ whose right terms are indecomposable projective modules from $\operatorname{mod} A$. For a module X in $\operatorname{mod} T$, we denote by $P_T(X)$ a projective cover of X in $\operatorname{mod} T$ and by $I_T(X)$ an injective envelope of X in $\operatorname{mod} T$. Moreover, if $XQ = 0$, we denote by $P_A(X)$ a projective cover of X in $\operatorname{mod} A$ and by $I_A(X)$ an injective envelope of X in $\operatorname{mod} A$.

Proposition 8.9. *Let A be a finite dimensional hereditary K-algebra over a field K and T a Hochschild extension algebra of A by a duality A-bimodule Q. Moreover, let I be an indecomposable injective module in* $\operatorname{mod} A$ *and $S = \operatorname{soc}(I)$. Assume that*

$$\mathbb{E}\colon\quad 0 \longrightarrow I \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod} T$ *with the left term I. Then the following statements hold:*

(i) *In case S is projective in* $\operatorname{mod} A$*, $\mathbb{E}$ can be written in the form*

$$\mathbb{E}\colon\quad 0 \longrightarrow I \xrightarrow{\begin{bmatrix} v \\ v' \end{bmatrix}} I/S \oplus I_T(S) \xrightarrow{[w' \ w]} \Omega_T^{-1}(S) \longrightarrow 0,$$

where $v\colon I \to I/S$ and $w\colon I_T(S) \to I_T(S)/\operatorname{soc}(I_T(S)) \xrightarrow{\sim} \Omega_T^{-1}(S)$ are the canonical epimorphisms.

(ii) *In case S is nonprojective in* $\operatorname{mod} A$*, $\mathbb{E}$ can be written in the form*

$$\mathbb{E}\colon\quad 0 \longrightarrow I \xrightarrow{\begin{bmatrix} v \\ v' \end{bmatrix}} I/S \oplus \Omega_T^{-1}(\operatorname{rad} P_A(S)) \xrightarrow{[u' \ \Omega_T^{-1}(u)]} \Omega_T^{-1}(P_A(S)) \longrightarrow 0,$$

where $u\colon \operatorname{rad} P_A(S) \to P_A(S)$ is the inclusion homomorphism and $v\colon I \to I/S$ is the canonical epimorphism.

Proof. We abbreviate $\Omega = \Omega_T$ and $\Omega^{-1} = \Omega_T^{-1}$.

(i) Assume that S is projective in $\operatorname{mod} A$. By Lemma 8.6, $I_T(S)$ is an injective envelope of I in $\operatorname{mod} T$ whose top is projective in $\operatorname{mod} A$. Hence, by Lemma 7.5, $I \cong I_T(S)Q$ and $I_T(S)/\operatorname{rad} I_T(S) \cong I_T(S)/I_T(S)Q$, so that $\operatorname{rad} I_T(S) = I_T(S)Q$ and $I \cong \operatorname{rad} I_T(S)$. Therefore, $I/S \cong \operatorname{rad} I_T(S)/S$ in $\operatorname{mod} T$. Since $\Omega^{-1}(S) \cong I_T(S)/S$ by definition, the canonical exact sequence

$$\begin{aligned}\mathbb{E}_0\colon\quad 0 \longrightarrow \operatorname{rad} I_T(S) \longrightarrow \big(\operatorname{rad} I_T(S)/\operatorname{soc}(I_T(S))\big) \oplus I_T(S) \\ \longrightarrow I_T(S)/\operatorname{soc}(I_T(S)) \longrightarrow 0\end{aligned}$$

is isomorphic to the exact sequence

$$\mathbb{E}\colon\quad 0 \longrightarrow I \xrightarrow{\begin{bmatrix} v \\ v' \end{bmatrix}} I/S \oplus I_T(S) \xrightarrow{[w' \ w]} \Omega^{-1}(S) \longrightarrow 0,$$

where $v: I \to I/S$ is the canonical epimorphism and $w: I_T(S) \to \Omega^{-1}(S)$ is the composition of the canonical epimorphism $I_T(S) \to I_T(S)/\operatorname{soc}(I_T(S))$ and an isomorphism $I_T(S)/\operatorname{soc}(I_T(S)) \xrightarrow{\sim} \Omega^{-1}(S)$.

(ii) Assume that S is not projective in $\operatorname{mod} A$. Since I is injective in $\operatorname{mod} A$ by assumption, it follows from Proposition 8.8 that the almost split sequence $\mathbb{E}$ is isomorphic to the exact sequence of the form, denoted again by $\mathbb{E}$,

$$\mathbb{E}: \quad 0 \longrightarrow I \xrightarrow{\begin{bmatrix} v \\ v' \end{bmatrix}} I/S \oplus Y_2 \xrightarrow{[w' \ w]} Z \longrightarrow 0,$$

where Y_2 is nonzero and has no nonzero direct summands annihilated by Q. We claim that Y_2 has no indecomposable projective direct summands in $\operatorname{mod} T$. Indeed, if P is an indecomposable projective direct summand of Y_2 in $\operatorname{mod} T$, then $I \cong \operatorname{rad} P$, by Proposition IX.6.8, which implies that $\operatorname{rad} P = \ell_P(Q) = PQ$. Hence $\operatorname{top}(P) = P/PQ$, which is projective in $\operatorname{mod} A$, by Lemma 7.3. Moreover, because of the isomorphism $I \cong \operatorname{rad} P$, we have $S \cong \operatorname{soc}(P)$, which implies that $\operatorname{top}(I_T(S)) \cong \operatorname{top}(P)$. Then $\operatorname{top}(I_T(S))$ is projective in $\operatorname{mod} A$ and hence it follows from Lemma 8.6 that S is also projective in $\operatorname{mod} A$, a contradiction.

We consider the almost split sequence

$$\Omega(\mathbb{E}): \quad 0 \longrightarrow \Omega(I) \xrightarrow{\begin{bmatrix} \Omega(v) \\ \Omega(v') \\ u' \end{bmatrix}} \Omega(I/S) \oplus \Omega(Y_2) \oplus R \xrightarrow{[\Omega(w') \ \Omega(w) \ u]} \Omega(Z) \longrightarrow 0,$$

where R is an indecomposable projective or zero module in $\operatorname{mod} T$.

(a) Assume that S is injective in $\operatorname{mod} A$. Then $I = S$, and $\Omega(I) = \Omega(S) \cong \operatorname{rad} P_T(S)$ by definition. On the other hand, the almost split sequence in $\operatorname{mod} T$ with the left term $\operatorname{rad} P_T(S)$ is isomorphic to the canonical almost split sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{rad} P_T(S) &\longrightarrow \big(\operatorname{rad} P_T(S)/\operatorname{soc}(P_T(S))\big) \oplus P_T(S) \\ &\longrightarrow P_T(S)/\operatorname{soc}(P_T(S)) \longrightarrow 0. \end{aligned}$$

Here, $\operatorname{soc}(P_T(S)) \cong \nu_T^{-1}(S)$ by definition, which is injective in $\operatorname{mod} A$, by Lemma 8.6. Hence, $\operatorname{soc}(P_T(S)) = P_T(S)Q$, by Lemma 7.5, and then we have $P_T(S)/\operatorname{soc}(P_T(S)) = P_T(S)/P_T(S)Q$, which is a projective cover of S in $\operatorname{mod} A$, by Lemma 7.4. Thus, $P_T(S)/\operatorname{soc}(P_T(S)) \cong P_A(S)$ and $\operatorname{rad} P_T(S)/\operatorname{soc}(P_T(S)) \cong \operatorname{rad} P_A(S)$. Consequently, $\Omega(\mathbb{E})$ is isomorphic to the exact sequence

$$0 \longrightarrow \Omega(I) \longrightarrow \operatorname{rad} P_A(S) \oplus P_T(S) \longrightarrow P_A(S) \longrightarrow 0.$$

Hence, we have an almost split sequence in $\operatorname{mod} T$ of the form

$$0 \longrightarrow I \longrightarrow \Omega^{-1}(\operatorname{rad} P_A(S)) \oplus R' \longrightarrow \Omega^{-1}(P_A(S)) \longrightarrow 0,$$

where R' is projective in $\operatorname{mod} T$. However, R' should be zero, because we have an isomorphism $\Omega^{-1}(\operatorname{rad} P_A(S)) \oplus R' \cong Y_2$ and Y_2 has no nonzero projective direct summands in $\operatorname{mod} T$. Therefore, the almost split sequence $\mathbb{E}$ is of the required form.

(b) Assume that S is noninjective in $\operatorname{mod} A$, so $I/S \neq 0$ and $\Omega(I/S) \neq 0$. Since I is indecomposable, I/S has no nonzero projective direct summands in $\operatorname{mod} A$. Hence, by Proposition 7.6 (ii), $\Omega(I/S)$ has no nonzero direct summands annihilated by Q. Moreover, $\Omega(Z)Q = 0$. In fact, in the case $ZQ \neq 0$, this is a consequence of Corollary 7.13 (iii). In the case $ZQ = 0$, it follows from Proposition 8.8 that Z is projective in $\operatorname{mod} A$, and hence $\Omega(Z)Q = 0$, by Proposition 7.6 (iv). Therefore, it follows from Proposition 8.8 that $\Omega(Z)$ is projective in $\operatorname{mod} A$, say $P = \Omega(Z)$. Then $\Omega(\mathbb{E})$ is of the form

$$0 \longrightarrow \Omega(I) \longrightarrow \operatorname{rad} P \oplus Y_2' \longrightarrow P \longrightarrow 0,$$

where Y_2' has no nonzero direct summands annihilated by Q. Thus we have an isomorphism $\Omega(I/S) \oplus \Omega(Y_2) \oplus R \cong \operatorname{rad} P \oplus Y_2'$ in $\operatorname{mod} T$. Since $\Omega(I/S)$ has no nonzero direct summands annihilated by Q and $\Omega(Y_2)Q = 0$, it follows from Theorem I.4.6 that $\Omega(Y_2) \cong \operatorname{rad} P$. Consequently, we conclude that $\Omega(\mathbb{E})$ is isomorphic to an exact sequence in $\operatorname{mod} T$ of the form

$$\Omega(\mathbb{E})\colon \quad 0 \longrightarrow \Omega(I) \xrightarrow{\begin{bmatrix} \Omega(v) \\ \Omega(v') \\ u' \end{bmatrix}} \Omega(I/S) \oplus \operatorname{rad} P \oplus R \xrightarrow{[\Omega(w') \ \Omega(w) \ u]} P \longrightarrow 0,$$

where $\Omega(w)$ is a monomorphism. Applying Ω^{-1}, we conclude that $\mathbb{E}$ is isomorphic to an almost split sequence

$$\mathbb{E}\colon \quad 0 \longrightarrow I \xrightarrow{\begin{bmatrix} v \\ v' \end{bmatrix}} I/S \oplus \Omega^{-1}(\operatorname{rad} P) \xrightarrow{[w' \ w]} \Omega^{-1}(P) \longrightarrow 0$$

in $\operatorname{mod} T$. Hence, it remains is to show that $P \cong P_A(S)$ in $\operatorname{mod} A$.

Let us consider the following commutative diagram in $\operatorname{mod} T$:

$$\begin{tikzcd}
 & & P_T(S) \arrow[r, "p''"] \arrow[d, "r"] & S \arrow[d, "j"] & \\
0 \arrow[r] & \Omega(I) \arrow[r, "q"] \arrow[d, "\Omega(v)"] & P_T(I) \arrow[r, "p"] \arrow[d, "\mathrm{id}"] & I \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & \Omega(I/S) \arrow[r, "q'"] & P_T(I) \arrow[r, "p'"] & I/S \arrow[r] & 0,
\end{tikzcd}$$

where the horizontal sequences are exact and p, p', p'' are projective covers of I, I/S and S in $\operatorname{mod} T$, respectively. Since $p'r = vjp'' = 0$, there is a homomorphism $f\colon P_T(S) \to \Omega(I/S)$ with $r = q'f$. We claim that the homomorphism $\Omega(w')f\colon P_T(S) \to P$ is an epimorphism, which implies that $\operatorname{top}(P) \cong$

$\operatorname{top}(P_T(S)) \cong S$, as desired. For this, suppose that $\Omega(w')f$ is not an epimorphism. Then $(\Omega(w')f)(P_T(S)) \subseteq \operatorname{rad} P$, because P is indecomposable projective and so $\operatorname{rad} P$ is a unique maximal submodule of P. Let $f_0\colon P_T(S) \to \operatorname{rad} P$ be a homomorphism in $\operatorname{mod} T$ such that $-\Omega(w)f_0 = \Omega(w')f$. Consider the homomorphism

$$\tilde{f} = \begin{bmatrix} f \\ f_0 \\ 0 \end{bmatrix} : P_T(S) \longrightarrow \Omega(I/S) \oplus \operatorname{rad} P \oplus R.$$

Then $[\Omega(w') \ \ \Omega(w) \ \ u]\,\tilde{f} = \Omega(w')f + \Omega(w)f_0 = 0$, and hence there exists a homomorphism $g\colon P_T(S) \to \Omega(I)$ in $\operatorname{mod} T$ such that

$$\tilde{f} = \begin{bmatrix} \Omega(v) \\ \Omega(v') \\ u' \end{bmatrix} g.$$

In particular, we obtain that $f = \Omega(v)g$. Hence, we conclude that $jp'' = pr = pq'f = pq'\Omega(v)g = pqg = 0$, a contradiction. This completes the proof. □

We observe that Proposition 8.9 is equivalent to saying that an almost split sequence in $\operatorname{mod} T$ with the left term an indecomposable injective module I from $\operatorname{mod} A$ is of the form

$$0 \longrightarrow I \longrightarrow I/S \oplus \big(I_T(P_A(S))/\operatorname{rad} P_A(S)\big) \longrightarrow \Omega_T^{-1}(P_A(S)) \longrightarrow 0.$$

Proposition 8.10. *Let A be a finite dimensional hereditary K-algebra over a field K and T a Hochschild extension algebra of A by a duality A-bimodule Q. Moreover, let P be an indecomposable projective module in* $\operatorname{mod} A$ *and* $S' = \operatorname{top}(P)$. *Assume that*

$$\mathbb{E}'\colon \quad 0 \longrightarrow X \longrightarrow Y \longrightarrow P \longrightarrow 0$$

is an almost split sequence in $\operatorname{mod} T$ *with the right term P. Then the following statements hold:*

(i) *In case S' is injective in* $\operatorname{mod} A$, *$\mathbb{E}'$ can be written in the form*

$$\mathbb{E}'\colon \quad 0 \longrightarrow \Omega_T(S') \xrightarrow{\left[\begin{smallmatrix} w \\ w' \end{smallmatrix}\right]} P_T(S') \oplus \operatorname{rad} P \xrightarrow{[u' \ u]} P \longrightarrow 0,$$

where $w\colon \Omega_T(S') \to P_T(S')$ is a canonical monomorphism and $u\colon \operatorname{rad} P \hookrightarrow P$ is the inclusion homomorphism.

(ii) *In case S' is not injective in* $\operatorname{mod} A$, *$\mathbb{E}'$ can be written in the form*

$$\mathbb{E}'\colon\ 0 \longrightarrow \Omega_T\big(I_A(S')\big) \xrightarrow{\left[\begin{smallmatrix} \Omega_T(v) \\ v' \end{smallmatrix}\right]} \Omega_T\big(I_A(S')/S'\big) \oplus \operatorname{rad} P \xrightarrow{[u' \ u]} P \longrightarrow 0,$$

where $v\colon I_A(S') \to I_A(S')/S'$ is the canonical epimorphism.

Proof. Consider the standard duality $D\colon \operatorname{mod} T \to \operatorname{mod} T^{\mathrm{op}}$, and let $I = D(P)$ and $S = D(S')$. Then I is an indecomposable injective module in $\operatorname{mod} A^{\mathrm{op}}$ and S is the socle of I.

(i) Assume that S' is injective in $\operatorname{mod} A$. Then S is projective in $\operatorname{mod} A^{\mathrm{op}}$. Hence, there is an almost split sequence $\mathbb{E}$ in $\operatorname{mod} T^{\mathrm{op}}$ described in Proposition 8.9 (i). Applying D, we obtain an almost split sequence in $\operatorname{mod} T$

$$D(\mathbb{E})\colon \quad 0 \longrightarrow D\big(\Omega^{-1}_{T^{\mathrm{op}}}(S)\big) \longrightarrow D\big(I_{T^{\mathrm{op}}}(S)\big) \oplus D(I/S) \longrightarrow P \longrightarrow 0,$$

where $D(\Omega^{-1}_{T^{\mathrm{op}}}(S)) \cong \Omega_T(D(S)) \cong \Omega_T(S')$ and $D(I_{T^{\mathrm{op}}}(S)) \cong P_T(S)$. Thus we have in $\operatorname{mod} T$ a desired almost split sequence $\mathbb{E}'$ isomorphic to $D(D(\mathbb{E}))$, and hence isomorphic to $\mathbb{E}$.

(ii) Assume that S' is not injective in $\operatorname{mod} A$. Then S is not projective in $\operatorname{mod} A^{\mathrm{op}}$. Hence, there is an almost split sequence $\mathbb{E}$ in $\operatorname{mod} T^{\mathrm{op}}$ described in Proposition 8.9 (ii). Applying D, we obtain an almost split sequence in $\operatorname{mod} T$

$$\begin{aligned} D(\mathbb{E})\colon \quad 0 &\longrightarrow D\big(\Omega^{-1}_{T^{\mathrm{op}}}(P_{A^{\mathrm{op}}}(S))\big) \longrightarrow D\big(\Omega^{-1}_{T^{\mathrm{op}}}(\operatorname{rad} P_{A^{\mathrm{op}}}(S))\big) \oplus D(I/S) \\ &\longrightarrow P \longrightarrow 0, \end{aligned}$$

where

$$\begin{aligned} D\big(\Omega^{-1}_{T^{\mathrm{op}}}(P_{A^{\mathrm{op}}}(S))\big) &\cong \Omega_T\big(D(P_{A^{\mathrm{op}}}(S))\big) \cong \Omega_T\big(I_A(S')\big), \\ D\big(\Omega^{-1}_{T^{\mathrm{op}}}(\operatorname{rad} P_{A^{\mathrm{op}}}(S))\big) &\cong \Omega_T\big(D(\operatorname{rad} P_{A^{\mathrm{op}}}(S))\big) \cong \Omega_T\big(I_A(S')/S'\big), \end{aligned}$$

and $D(I/S) \cong \operatorname{rad} D(I) = \operatorname{rad} P$. Therefore, we have a desired almost split sequence $\mathbb{E}'$ in $\operatorname{mod} T$. ☐

Let Λ be a finite dimensional K-algebra over a field K. A valued subquiver of the Auslander–Reiten quiver Γ_Λ of Λ

$$\begin{tikzcd}
 & \{Y_1\} \arrow[ddr, "{(d'_1,d_1)}"] & \\
 & \{Y_2\} \arrow[dr, "{(d'_2,d_2)}"'] & \\
\{X\} \arrow[uur, "{(d_1,d'_1)}"] \arrow[ur, "{(d_2,d'_2)}"'] \arrow[dr, "{(d_r,d'_r)}"'] & \vdots & \{Z\} \\
 & \{Y_r\} \arrow[ur, "{(d'_r,d_r)}"'] &
\end{tikzcd}$$

is said to be *valued mesh* of Γ_Λ if there exists in $\operatorname{mod} \Lambda$ an almost split sequence

$$0 \longrightarrow X \longrightarrow \bigoplus_{i=1}^{r} Y_i^{d_i} \longrightarrow Z \longrightarrow 0.$$

We note that then $\{X\} = \{\tau_\Lambda Z\}$, $\{Z\} = \{\tau_\Lambda^{-1} Z\}$, and $d_i = d_{X Y_i} = d'_{Y_i X}$ and $d'_i = d'_{X Y_i} = d_{Y_i X}$ for all $i \in \{1, \dots, r\}$, by Lemma III.9.1 and Proposition III.9.6. Moreover, if Λ is a selfinjective algebra, then by a valued mesh of the stable Auslander–Reiten quiver $\Gamma_\Lambda^{\mathrm{s}}$ of Λ, we mean a valued subquiver of $\Gamma_\Lambda^{\mathrm{s}}$ obtained from a valued mesh of Γ_Λ by deleting the projective vertices and the arrows attached to them.

Corollary 8.11. *Let A be a finite dimensional hereditary K-algebra over a field K and T a Hochschild extension algebra of A by a duality A-bimodule Q. Moreover, let*

$$\begin{tikzcd}
 & \{Y_1\} \arrow[rdd, "{(d'_1,d_1)}"] & \\
 & \{Y_2\} \arrow[rd, "{(d'_2,d_2)}"'] & \\
\{X\} \arrow[ruu, "{(d_1,d'_1)}"] \arrow[ru, "{(d_2,d'_2)}"'] \arrow[rd, "{(d_r,d'_r)}"'] & \vdots & \{Z\} \\
 & \{Y_r\} \arrow[ru, "{(d'_r,d_r)}"'] &
\end{tikzcd}$$

be a valued subquiver of the Auslander–Reiten quiver Γ_A of A. Then the following statements hold:

(i) *The above valued mesh of Γ_A is a valued mesh of the Auslander–Reiten quiver Γ_T of T.*

(ii) *The stable Auslander–Reiten quiver Γ_T^{s} of T contains a valued mesh of the form*

$$\begin{tikzcd}
 & \{\Omega_T(Y_1)\} \arrow[rdd, "{(d'_1,d_1)}"] & \\
 & \{\Omega_T(Y_2)\} \arrow[rd, "{(d'_2,d_2)}"'] & \\
\{\Omega_T(X)\} \arrow[ruu, "{(d_1,d'_1)}"] \arrow[ru, "{(d_2,d'_2)}"'] \arrow[rd, "{(d_r,d'_r)}"'] & \vdots & \{\Omega_T(Z)\} \\
 & \{\Omega_T(Y_r)\} \arrow[ru, "{(d'_r,d_r)}"'] &
\end{tikzcd}$$

(iii) *The stable Auslander–Reiten quiver Γ_T^{s} of T contains a valued mesh of the form*

$$\begin{tikzcd}
 & \{\Omega_T^{-1}(Y_1)\} \arrow[rdd, "{(d'_1,d_1)}"] & \\
 & \{\Omega_T^{-1}(Y_2)\} \arrow[rd, "{(d'_2,d_2)}"'] & \\
\{\Omega_T^{-1}(X)\} \arrow[ruu, "{(d_1,d'_1)}"] \arrow[ru, "{(d_2,d'_2)}"'] \arrow[rd, "{(d_r,d'_r)}"'] & \vdots & \{\Omega_T^{-1}(Z)\} \\
 & \{\Omega_T^{-1}(Y_r)\} \arrow[ru, "{(d'_r,d_r)}"'] &
\end{tikzcd}$$

(iv) *Every valued mesh in* Γ_T^{s} *of the form*

$$\begin{tikzcd}
& \{V_1\} \arrow[rddd, "{(m_1',m_1)}"] & \\
& \{V_2\} \arrow[rdd, "{(m_2',m_2)}"'] & \\
\{M\} \arrow[ruu, "{(m_1,m_1')}"] \arrow[ru, "{(m_2,m_2')}"'] \arrow[rd, "{(m_t,m_t')}"'] & \vdots & \{N\} \\
& \{V_t\} \arrow[ru, "{(m_t',m_t)}"'] &
\end{tikzcd}$$

with $MQ \neq 0$ *and* $NQ \neq 0$ *is of the forms* (ii) *and* (iii).

Proof. We have in mod A an almost split sequence

$$\mathbb{E}\colon \quad 0 \longrightarrow X \longrightarrow \bigoplus_{i=1}^{r} Y_i^{d_i} \longrightarrow Z \longrightarrow 0.$$

Then it follows from Theorem 8.2 that $\mathbb{E}$ is an almost split sequence in mod T, and the statement (i) holds. Further, applying Propositions IX.6.8 and IX.6.9, we conclude that there are in mod T almost split sequences of the forms

$$\Omega_T(\mathbb{E})\colon \quad 0 \longrightarrow \Omega_T(X) \longrightarrow \bigoplus_{i=1}^{r} \Omega_T(Y_i)^{d_i} \oplus P \longrightarrow \Omega_T(Z) \longrightarrow 0,$$

$$\Omega_T^{-1}(\mathbb{E})\colon \quad 0 \longrightarrow \Omega_T^{-1}(X) \longrightarrow \bigoplus_{i=1}^{r} \Omega_T^{-1}(Y_i)^{d_i} \oplus P' \longrightarrow \Omega_T^{-1}(Z) \longrightarrow 0,$$

for some projective modules P and P' in mod T, and clearly $\Omega_T(Y_i)$ and $\Omega_T^{-1}(Y_i)$ are nonprojective modules in mod T, for $i \in \{1, \dots, r\}$. Hence, the statements (ii) and (iii) follow from Lemma III.9.1 and Proposition III.9.6. The statement (iv) follows from Proposition 8.8 (iii) and the statements (ii), (iii). □

Let A be a finite dimensional hereditary K-algebra over a field K and T be a Hochschild extension algebra of A by a duality A-bimodule Q. Assume that the stable Auslander–Reiten quiver Γ_T^{s} of T admits a valued mesh

$$\begin{tikzcd}
& \{V_1\} \arrow[rddd, "{(m_1',m_1)}"] & \\
& \{V_2\} \arrow[rdd, "{(m_2',m_2)}"'] & \\
\{M\} \arrow[ruu, "{(m_1,m_1')}"] \arrow[ru, "{(m_2,m_2')}"'] \arrow[rd, "{(m_t,m_t')}"'] & \vdots & \{N\} \\
& \{V_t\} \arrow[ru, "{(m_t',m_t)}"'] &
\end{tikzcd}$$

which is not a valued mesh of the Auslander–Reiten quiver Γ_T of T. Then there exists an indecomposable projective module P in $\operatorname{mod} T$ such that $\{M\} = \{\operatorname{rad} P\}$, $\{N\} = \{P/\operatorname{soc}(P)\}$, and Γ_T admits a valued mesh ${}_M\mathfrak{M} = \mathfrak{M}_N$ of the form

$$\begin{tikzcd}
 & \{V_1\} \arrow[ddr, "(m'_1,m_1)"] & \\
 & \{V_2\} \arrow[dr, "(m'_2,m_2)"] & \\
\{M\} \arrow[uur, "(m_1,m'_1)"] \arrow[ur, "(m_2,m'_2)"'] \arrow[dr, "(m_t,m'_t)"] \arrow[ddr, "(1,1)"'] & \vdots & \{N\} \\
 & \{V_t\} \arrow[ur, "(m'_t,m_t)"] & \\
 & P \arrow[uur, "(1,1)"'] &
\end{tikzcd}$$

and there is in $\operatorname{mod} T$ an isomorphism

$$\operatorname{rad} P/\operatorname{soc}(P) \cong \bigoplus_{i=1}^{t} V_i^{m_i}.$$

We note that it occurs in the following particular situation. Assume that S is a simple module in $\operatorname{mod} A$ which is neither projective nor injective. Then there are in $\operatorname{mod} A$ almost split sequences

$$0 \longrightarrow S \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$
$$0 \longrightarrow L \longrightarrow M \longrightarrow S \longrightarrow 0.$$

Then $\Omega_T(S) \xrightarrow{\sim} \operatorname{rad} P_T(S)$ and $\Omega_T^{-1}(S) \cong I_T(S)/S$ in $\operatorname{mod} T$, for a projective cover $P_T(S)$ of S and an injective envelope $I_T(S)$ of S in $\operatorname{mod} T$. Hence, by Propositions III.8.6, IX.6.8 and IX.6.9, there are in $\operatorname{mod} T$ almost split sequences of the forms

$$0 \longrightarrow \Omega_T(S) \longrightarrow \Omega_T(Y) \oplus P_T(S) \longrightarrow \Omega_T(Z) \longrightarrow 0,$$
$$0 \longrightarrow \Omega_T^{-1}(L) \longrightarrow \Omega_T^{-1}(M) \oplus I_T(S) \longrightarrow \Omega_T^{-1}(S) \longrightarrow 0.$$

Observe also that $I_T(S) = P_T(\operatorname{top}(I_T(S)))$ and $P_T(S) = I_T(\operatorname{soc}(P_T(S)))$.

Let A be a finite dimensional hereditary K-algebra over a field K. Then, for each nonsimple indecomposable projective module P in $\operatorname{mod} A$, the quiver Γ_A contains a valued subquiver of the form (see Lemma III.7.6 and Theorems I.9.1, I.9.3)

$$\begin{tikzcd}
\{R_1\} \arrow[ddr, "(d_{R_1P},d'_{R_1P})"] & \\
\{R_2\} \arrow[dr, "(d_{R_2P},d'_{R_2P})"'] & \\
\vdots & \{P\} \\
\{R_t\} \arrow[ur, "(d_{R_tP},d'_{R_tP})"'] &
\end{tikzcd}$$

such that $R_1, R_2, \dots, R_t$ are indecomposable projective modules and there is in $\operatorname{mod} A$ an isomorphism

$$\operatorname{rad} P \cong \bigoplus_{i=1}^{t} R_i^{d'_{R_i P}}.$$

Dually, for each nonsimple indecomposable injective module I in $\operatorname{mod} A$, the quiver Γ_A contains a valued subquiver of the form (see Lemma III.7.7 and Theorems I.9.2, I.9.3)

$$\begin{tikzcd}
 & \{U_1\} \\
 & \{U_2\} \\
\{I\} \arrow[ruu, "{(d_{IU_1},d'_{IU_1})}"] \arrow[ru, "{(d_{IU_2},d'_{IU_2})}"'] \arrow[rd, "{(d_{IU_m},d'_{IU_m})}"'] & \vdots \\
 & \{U_m\}
\end{tikzcd}$$

such that $U_1, U_2, \dots, U_m$ are indecomposable injective modules and there is in $\operatorname{mod} A$ an isomorphism

$$I/\operatorname{soc}(I) \cong \bigoplus_{j=1}^{m} U_j^{d_{IU_j}}.$$

Let T be a Hochschild extension algebra of A by a duality A-bimodule Q. A valued mesh in the Auslander–Reiten quiver Γ_T

$$\begin{tikzcd}
 & \{V_1\} \arrow[rdd, "{(m'_1,m_1)}"] & \\
 & \{V_2\} \arrow[rd, "{(m'_2,m_2)}"'] & \\
\{M\} \arrow[ruu, "{(m_1,m'_1)}"] \arrow[ru, "{(m_2,m'_2)}"'] \arrow[rd, "{(m_t,m'_t)}"'] & \vdots & \{N\} \\
 & \{V_t\} \arrow[ru, "{(m'_t,m_t)}"'] &
\end{tikzcd}$$

is said to be a *connecting valued mesh* if either M is an injective module in $\operatorname{mod} A$, or N is a projective module in $\operatorname{mod} A$. Observe that then we have in $\operatorname{mod} T$ an almost split sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_{i=1}^{r} V_i^{m_i} \longrightarrow N \longrightarrow 0,$$

which is not an almost split sequence in $\operatorname{mod} A$. Hence, if M is injective (respectively, N is projective) in $\operatorname{mod} A$, then $NQ \neq 0$ (respectively, $MQ \neq 0$). Let us describe the connecting valued meshes in Γ_T more precisely.

Corollary 8.12. *Let A be a finite dimensional hereditary K-algebra over a field K and T a Hochschild extension algebra of A by a duality A-bimodule Q. Moreover, let P be an indecomposable projective module and I an indecomposable injective module in* $\operatorname{mod} A$ *with* $\operatorname{top}(P) \cong \operatorname{soc}(I)$*. Then the connecting valued mesh $\mathfrak{M}_P$ in Γ_T with the right term $\{P\}$ is of one of the forms*

(i)

$$\begin{tikzcd}
 & \{R_1\} \arrow[ddr, "{(a_1,a_1')}"] & \\
 & \{R_2\} \arrow[dr, "{(a_2,a_2')}"] & \\
\{\Omega_T(I)\} \arrow[uur, "{(a_1',a_1)}"] \arrow[ur, "{(a_2',a_2)}"'] \arrow[dr, "{(a_t',a_t)}"] \arrow[ddr, "{(1,1)}"'] & \vdots & \{P\} \\
 & \{R_t\} \arrow[ur, "{(a_t,a_t')}"] & \\
 & P_T(I) \arrow[uur, "{(1,1)}"'] &
\end{tikzcd}$$

if I is simple, where $a_i = d_{R_i P}$ and $a_i' = d_{R_i P}'$ for $i \in \{1, \dots, t\}$, or

(ii)

$$\begin{tikzcd}
 & \{R_1\} \arrow[ddddr, "{(a_1,a_1')}"] & \\
 & \{R_2\} \arrow[dddr, "{(a_2,a_2')}"'] & \\
 & \vdots & \\
 & \{R_t\} \arrow[dr, "{(a_t,a_t')}"] & \\
\{\Omega_T(I)\} \arrow[uuuur, "{(a_1',a_1)}"] \arrow[uuur, "{(a_2',a_2)}"'] \arrow[ur, "{(a_t',a_t)}"'] \arrow[dr, "{(b_1,b_1')}"] \arrow[ddr, "{(b_2,b_2')}"] \arrow[ddddr, "{(b_m,b_m')}"'] & & \{P\} \\
 & \{\Omega_T(U_1)\} \arrow[ur, "{(b_1',b_1)}"] & \\
 & \{\Omega_T(U_2)\} \arrow[uur, "{(b_2',b_2)}"'] & \\
 & \vdots & \\
 & \{\Omega_T(U_m)\} \arrow[uuuur, "{(b_m',b_m)}"'] &
\end{tikzcd}$$

if I is nonsimple, where $a_i = d_{R_i P}$, $a_i' = d_{R_i P}'$, for $i \in \{1, \dots, t\}$, and $b_j = d_{I U_j}$, $b_j' = d_{I U_j}'$, for $j \in \{1, \dots, m\}$.

Proof. It is a direct consequence of Proposition 8.10. □

Corollary 8.13. *Let A be a finite dimensional hereditary K-algebra over a field K and T a Hochschild extension algebra of A by a duality A-bimodule Q. Moreover, let I be an indecomposable injective module and P be an indecomposable* *projective module in* mod A *with* $\operatorname{soc}(I) \cong \operatorname{top}(P)$. *Then the connecting valued mesh* ${}_I\mathfrak{M}$ *in* Γ_T *with the left term* $\{I\}$ *is of one of the forms*

(i)

$$\begin{tikzcd}
 & \{U_1\} \arrow[ddr, "{(b'_1,b_1)}"] & \\
 & \{U_2\} \arrow[dr, "{(b'_2,b_2)}"'] & \\
\{I\} \arrow[uur, "{(b_1,b'_1)}"] \arrow[ur, "{(b_2,b'_2)}"'] \arrow[dr, "{(b_m,b'_m)}"] \arrow[ddr, "{(1,1)}"'] & \vdots & \{\Omega_T^{-1}(P)\} \\
 & \{U_m\} \arrow[ur, "{(b'_m,b_m)}"] & \\
 & I_T(P) \arrow[uur, "{(1,1)}"'] &
\end{tikzcd}$$

if P *is simple, where* $b_j = d_{I\,U_j}$ *and* $b'_j = d'_{I\,U_j}$*, for* $j \in \{1, \dots, m\}$*, or*

(ii)

$$\begin{tikzcd}
 & \{U_1\} \arrow[ddddr, "{(b'_1,b_1)}"] & \\
 & \{U_2\} \arrow[dddr, "{(b'_2,b_2)}"'] & \\
 & \vdots & \\
 & \{U_m\} \arrow[dr, "{(b'_m,b_m)}"'] & \\
\{I\} \arrow[uuuur, "{(b_1,b'_1)}"] \arrow[uuur, "{(b_2,b'_2)}"'] \arrow[ur, "{(b_m,b'_m)}"'] \arrow[dr, "{(a'_1,a_1)}"] \arrow[ddr, "{(a'_2,a_2)}"'] \arrow[ddddr, "{(a'_t,a_t)}"'] & & \{\Omega_T^{-1}(P)\} \\
 & \{\Omega_T^{-1}(R_1)\} \arrow[ur, "{(a_1,a'_1)}"] & \\
 & \{\Omega_T^{-1}(R_2)\} \arrow[uur, "{(a_2,a'_2)}"'] & \\
 & \vdots & \\
 & \{\Omega_T^{-1}(R_t)\} \arrow[uuuur, "{(a_t,a'_t)}"'] &
\end{tikzcd}$$

if P *is nonsimple, where* $b_j = d_{I\,U_j}$, $b'_j = d'_{I\,U_j}$*, for* $j \in \{1, \dots, m\}$*, and* $a_i = d_{R_i\,P}$, $a'_i = d'_{R_i\,P}$*, for* $i \in \{1, \dots, t\}$.

Proof. This is a direct consequence of Proposition 8.9. □

Let A be a nonsimple indecomposable finite dimensional hereditary K-algebra over a field K and Γ_A the Auslander–Reiten quiver of A. Then it follows from Theorem VII.6.1 that Γ_A contains a postprojective component $\mathcal{P}(A)$ with the following properties:

(1) $\mathcal{P}(A)$ contains all projective vertices of Γ_A;
(2) the opposite quiver Q_A^{op} of the quiver Q_A of A is isomorphic to the full valued subquiver $\Delta(\operatorname{proj} A)$ of $\mathcal{P}(A)$ given by all projective vertices;
(3) every indecomposable module X in $\mathcal{P}(A)$ is of the form $X = \tau_A^{-m_X} P_X$ for an indecomposable projective module P_X and a nonnegative integer m_X, both uniquely determined by X;
(4) $\mathcal{P}(A)$ is an acyclic quiver.

Dually, it follows from Theorem VII.6.2 that Γ_A contains a preinjective component $\mathcal{Q}(A)$ with the following properties:

(1) $\mathcal{Q}(A)$ contains all injective vertices of Γ_A;
(2) the opposite quiver Q_A^{op} of the quiver Q_A of A is isomorphic to the full valued subquiver $\Delta(\operatorname{inj} A)$ of $\mathcal{Q}(A)$ given by all injective vertices;
(3) every indecomposable module Y in $\mathcal{Q}(A)$ is of the form $Y = \tau_A^{n_Y} I_Y$ for an indecomposable injective module I_Y and a nonnegative integer n_Y, both uniquely determined by Y;
(4) $\mathcal{Q}(A)$ is an acyclic quiver.

We note that $\Delta(\operatorname{proj} A)$ is a finite section of $\mathcal{P}(A)$ and $\Delta(\operatorname{inj} A)$ is a finite section of $\mathcal{Q}(A)$, defined in Section VIII.6.

We shall denote by $\mathcal{P}(A)^*$ the full translation subquiver of $\mathcal{P}(A)$ given by all nonprojective vertices, and by $\mathcal{Q}(A)^*$ the full translation subquiver of $\mathcal{Q}(A)$ given by all noninjective vertices. Recall also that A is called of Dynkin type, Euclidean type, or wild type if the quiver Q_A of A is a Dynkin quiver, Euclidean quiver, or wild quiver, respectively. Then it follows from Corollary VII.6.3 and Theorem VII.7.4 that A is of finite representation type if and only if A is of Dynkin type, and if and only if $\mathcal{P}(A) = \Gamma_A = \mathcal{Q}(A)$. Further, if A is of Euclidean type, then Γ_A has a disjoint decomposition

$$\Gamma_A = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A),$$

where the regular part $\mathcal{R}(A)$ is a disjoint union

$$\mathcal{R}(A) = \bigcup_{\lambda \in \Lambda(A)} \mathcal{T}_\lambda^A$$

of a family $\mathcal{T}_\lambda^A$, $\lambda \in \Lambda(A)$, of stable tubes, all but finitely many being stable tubes of rank 1 (see Theorem VII.8.12). Finally, if A is of wild type, then Γ_A has a disjoint decomposition

$$\Gamma_A = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A),$$

where the regular part $\mathcal{R}(A)$ is a disjoint union

$$\mathcal{R}(A) = \bigcup_{\lambda \in \Lambda(A)} \mathcal{C}_\lambda^A$$

of a family $\mathcal{C}_\lambda^A$, $\lambda \in \Lambda(A)$, of components of type $\mathbb{Z}\mathbb{A}_\infty$ (see Theorem VII.9.3 and Corollary VII.9.4).

Theorem 8.14. *Let T be a basic indecomposable Hochschild extension algebra of a finite dimensional hereditary K-algebra A over a field K by a duality A-bimodule Q, and $A = A_1 \times \cdots \times A_m$ be a Nakayama block decomposition of A with respect to T. Assume that T is of finite representation type and of Loewy length at least* 3*. Then the Auslander–Reiten quiver Γ_T of T is obtained from the disjoint union of translation quivers*

$$\bigcup_{i=1}^{m} \big(\mathcal{P}(A_i) \cup \Omega_T\big(\mathcal{P}(A_i)^*\big)\big) = \bigcup_{i=1}^{m} \big(\mathcal{Q}(A_i) \cup \Omega_T^{-1}\big(\mathcal{Q}(A_i)^*\big)\big)$$

by adding the connecting valued meshes

(1) $\mathfrak{M}_P$, *for all indecomposable projective modules P in* mod A,

(2) ${}_I\mathfrak{M}$, *for all indecomposable injective modules I in* mod A, *and completing the meshes in*

$$\bigcup_{i=1}^{m} \Omega_T\big(\mathcal{P}(A_i)^*\big) = \bigcup_{i=1}^{m} \Omega_T^{-1}\big(\mathcal{Q}(A_i)^*\big)$$

by the arrows

(3)
$$\{\operatorname{rad} P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)/\operatorname{soc}(P_T(S))\},$$

for all nonprojective and noninjective simple modules S in mod A.

Proof. It follows from the assumption imposed on T that $A_1, \dots, A_m$ are isomorphic nonsimple basic indecomposable hereditary algebras of Dynkin type, and $\nu_T(A_i) = A_{i+1}$ for $i \in \{1, \dots, m\}$, where $A_{m+1} = A_1$. In particular, we have $\mathcal{P}(A_i) = \Gamma_{A_i} = \mathcal{Q}(A_i)$ for any $i \in \{1, \dots, m\}$. Moreover, by Corollary 8.11 (i), the valued meshes of $\mathcal{P}(A_i) = \mathcal{Q}(A_i)$, $i \in \{1, \dots, m\}$, are valued meshes of Γ_T. We claim now that, for any $i \in \{1, \dots, m\}$, we have $\Omega_T(\mathcal{P}(A_{i+1})^*) = \Omega_T^{-1}(\mathcal{Q}(A_i)^*)$, where $A_{m+1} = A_1$. Fix $i \in \{1, \dots, m\}$. Consider a mesh

$$\begin{tikzcd}
 & \{Y_1\} \arrow[ddr, "{(d_1',d_1)}"] & \\
 & \{Y_2\} \arrow[dr, "{(d_2',d_2)}"'] & \\
\{X\} \arrow[uur, "{(d_1,d_1')}"] \arrow[ur, "{(d_2,d_2')}"'] \arrow[dr, "{(d_r,d_r')}"'] & \vdots & \{Z\} \\
 & \{Y_r\} \arrow[ur, "{(d_r',d_r)}"'] &
\end{tikzcd}$$

contained entirely in $\mathcal{P}(A_{i+1})^*$. Then, by Corollary 8.11 (ii), the stable Auslander–Reiten quiver Γ_T^s of T contains a valued mesh of the form

$$\begin{tikzcd}
 & \{\Omega_T(Y_1)\} \arrow[ddr, "{(d_1',d_1)}"] & \\
 & \{\Omega_T(Y_2)\} \arrow[dr, "{(d_2',d_2)}"'] & \\
\{\Omega_T(X)\} \arrow[uur, "{(d_1,d_1')}"] \arrow[ur, "{(d_2,d_2')}"'] \arrow[dr, "{(d_r,d_r')}"'] & \vdots & \{\Omega_T(Z)\}\,. \\
 & \{\Omega_T(Y_r)\} \arrow[ur, "{(d_r',d_r)}"'] &
\end{tikzcd}$$

Observe that the modules $\Omega_T(X), \Omega_T(Y_1), \ldots, \Omega_T(Y_r), \Omega_T(Z)$ belong to $\operatorname{ind} T \setminus \{\operatorname{proj} T \cup \{\operatorname{ind} A\}$. Moreover, by Theorem 7.14, the inverse syzygy Ω_T^{-1} induces a bijection $\Phi_T^{-1} \colon \operatorname{ind} A \setminus \operatorname{inj} A \to \operatorname{ind} \setminus \{\operatorname{proj} T \cup \operatorname{ind} A\}$. Hence there exist modules $L, M_1, \ldots, M_r, N$ in $\operatorname{ind} A \setminus \operatorname{inj} A$ such that $\{\Omega_T^{-1}(L)\} = \{\Omega_T(X)\}$, $\{\Omega_T^{-1}(M_1)\} = \{Y_1\}, \ldots, \{\Omega_T^{-1}(M_r)\} = \{Y_r\}$, $\{\Omega_T^{-1}(N)\} = \{\Omega_T(Z)\}$. Since T is a selfinjective algebra of Loewy length at least 3, Proposition IX.6.11 shows that there are isomorphisms

$$\Omega_T^{-1}(\tau_T N) \cong \tau_T \Omega_T^{-1}(N) \cong \tau_T \Omega_T(Z) \cong \Omega_T(\tau_T Z) \cong \Omega_T(X) \cong \Omega_T^{-1}(L)$$

in $\operatorname{mod} T$, and consequently $\tau_T N \cong L$ in $\operatorname{mod} T$. Since L is an indecomposable noninjective module in $\operatorname{mod} A$, there is in $\operatorname{mod} A$ an almost split sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow \tau_A^{-1} L \longrightarrow 0,$$

which is an almost split sequence in $\operatorname{mod} T$, by Lemma 8.1. In particular, we have isomorphisms $\tau_A^{-1} L \cong \tau_T^{-1} L \cong N$ in $\operatorname{mod} T$, and hence in $\operatorname{mod} A$. Hence, $L \cong \tau_T N$ in $\operatorname{mod} A$. Therefore, we have in Γ_A a valued mesh with the left term $\{L\}$ and the right term $\{N\}$. Moreover, by Corollary 8.11 (iii), the image of this mesh under Ω_T^{-1} is a valued mesh in Γ_T^s with the left term $\{\Omega_T^{-1}(L)\} = \{\Omega_T(X)\}$ and the right term $\{\Omega_T^{-1}(N)\} = \{\Omega_T(Z)\}$. Summing up, we obtain that there is a valued mesh in Γ_A of the form

$$\begin{tikzcd}
 & \{M_1\} \arrow[ddr, "{(d_1',d_1)}"] & \\
 & \{M_2\} \arrow[dr, "{(d_2',d_2)}"'] & \\
\{L\} \arrow[uur, "{(d_1,d_1')}"] \arrow[ur, "{(d_2,d_2')}"'] \arrow[dr, "{(d_r,d_r')}"'] & \vdots & \{N\}\,, \\
 & \{M_r\} \arrow[ur, "{(d_r',d_r)}"'] &
\end{tikzcd}$$

which is clearly a valued mesh of $\Gamma_{A_j} = \mathcal{Q}(A_j)$, contained entirely in $\mathcal{Q}(A_j)^*$, for some $j \in \{1, \ldots, m\}$. Moreover, by Proposition 7.18, we have $j = i$. Therefore, indeed we have $\Omega_T(\mathcal{P}(A_{i+1})^*) = \Omega_T^{-1}(\mathcal{Q}(A_i)^*)$, for any $i \in \{1, \ldots, m\}$.

It follows from Lemma 7.9 and Theorem 7.16 that

$$\bigcup_{i=1}^{m} \big(\mathcal{P}(A_i) \cup \Omega_T\big(\mathcal{P}(A_i)^*\big)\big) = \bigcup_{i=1}^{m} \big(\mathcal{Q}(A_i) \cup \Omega_T^{-1}\big(\mathcal{Q}(A_i)^*\big)\big)$$

contains all indecomposable nonprojective modules in $\operatorname{mod} T$. Fix $i \in \{1, \dots, m\}$ again. Then for any indecomposable projective module P in $\operatorname{mod} A_i$ and the indecomposable injective module I in $\operatorname{mod} A_i$ with $\operatorname{top}(P) \cong \operatorname{soc}(I)$, the connecting mesh $\mathfrak{M}_P$, described in Corollary 8.12, has the left term $\{\Omega_T(I)\}$ in $\Omega_T(\mathcal{P}(A_i)^*)$ and the right term $\{P\}$ in $\mathcal{P}(A_i)$, so it connects $\Omega_T(\mathcal{P}(A_i)^*) = \Omega_T^{-1}(\mathcal{Q}(A_{i-1}^*))$ with $\mathcal{P}(A_i) = \mathcal{Q}(A_i)$, where $A_0 = A_m$. Moreover, by Proposition 8.10, if $\operatorname{top}(P)$ is injective in $\operatorname{mod} A_i$, then the mesh $\mathfrak{M}_P$ contains the projective module $P_T(P)$. Similarly, for any indecomposable injective module I in $\operatorname{mod} A_i$ and the indecomposable projective module P in $\operatorname{mod} A_i$ with $\operatorname{soc}(I) \cong \operatorname{top}(P)$, the connecting mesh ${}_I\mathfrak{M}$, described in Corollary 8.13, has the left term $\{I\}$ in $\mathcal{Q}(A_i)$ and the right term $\{\Omega_T^{-1}(P)\}$ in $\Omega_T^{-1}(\mathcal{Q}(A_i)^*)$, so it connects $\mathcal{P}(A_i) = \mathcal{Q}(A_i)$ with $\Omega_T^{-1}(\mathcal{Q}(A_i)^*) = \Omega_T(\mathcal{P}(A_{i+1}^*))$. Next, by Proposition 8.9, if $\operatorname{soc}(I)$ is projective in $\operatorname{mod} A_i$, then the mesh ${}_I\mathfrak{M}$ contains the projective module $I_T(I)$. Finally, if S is a simple module in $\mathcal{P}(A_i)^* \cap \mathcal{Q}(A_i)^*$, then we have in Γ_T the arrows

$$\{\operatorname{rad} P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)/\operatorname{soc}(P_T(S))\},$$

with

$$\begin{gathered}\{\operatorname{rad} P_T(S)\} = \{\Omega_T(S)\} \quad \text{and} \\ \{P_T(S)/\operatorname{soc}(P_T(S))\} = \{\tau_T^{-1}\Omega_T(S)\} = \{\Omega_T(\tau_{A_i}^{-1} S)\} \quad \text{in} \quad \Omega_T(\mathcal{P}(A_i)^*).\end{gathered}$$

Observe also that, by Lemma 8.6, the simple module $\nu_T(S)$ belongs to $\mathcal{P}(A_{i+1})^* \cap \mathcal{Q}(A_{i+1})^*$ and the simple module $\nu_T^{-1}(S)$ belongs to $\mathcal{P}(A_{i-1})^* \cap \mathcal{Q}(A_{i-1})^*$. In particular,

$$\begin{aligned}\{P_T(S)\} &= \{I_T(\nu_T^{-1}(S))\}, \quad \text{and} \\ \{P_T(S)/\operatorname{soc}(P_T(S))\} &= \{\Omega_T^{-1}(\nu_T^{-1}(S))\}, \quad \text{so} \\ \{\operatorname{rad} P_T(S)\} &= \{\Omega_T^{-1}(\tau_{A_{i-1}}\nu_T^{-1}(S))\}.\end{aligned}$$

Thus the above pair of arrows coincides with the pair of arrows

$$\{\Omega_T^{-1}(\tau_{A_{i-1}}\nu_T^{-1}(S))\} \xrightarrow{(1,1)} \{I_T(\nu_T^{-1}(S))\} \xrightarrow{(1,1)} \{\Omega_T^{-1}(\nu_T^{-1}(S))\} \qquad \square$$

We have the following direct consequence of Proposition 3.21, Corollary 7.17, and Theorem 8.14.

Corollary 8.15. *Let T be a basic indecomposable Hochschild extension algebra of a nonsimple indecomposable finite dimensional hereditary K-algebra A of finite representation type over a field K by a duality A-bimodule Q. Then the Auslander–Reiten quiver Γ_T of T is obtained from the disjoint union of translation quivers*

$$\Omega_T\big(\mathcal{P}(A)^*\big) \cup \mathcal{P}(A) = \mathcal{Q}(A) \cup \Omega_T^{-1}\big(\mathcal{Q}(A)^*\big)$$

by adding the connecting valued meshes

(1) $\mathfrak{M}_P$*, for all indecomposable projective modules P in* $\operatorname{mod} A$*,*

(2) ${}_I\mathfrak{M}$*, for all indecomposable injective modules I in* $\operatorname{mod} A$*, and completing the meshes in $\Omega_T\big(\mathcal{P}(A)^*\big) = \Omega_T^{-1}(\mathcal{Q}(A)^*)$ by the arrows*

(3)

$$\{\operatorname{rad} P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)/\operatorname{soc}(P_T(S))\},$$

for all nonprojective and noninjective simple modules S in $\operatorname{mod} A$.

Example 8.16. Let K be a field, Δ the quiver

$$\begin{tikzcd}
1 \bullet \arrow[r, "\alpha"] & \bullet 2 \arrow[d, "\beta"] \\
4 \bullet \arrow[u, "\delta"] & \bullet 3 \arrow[l, "\gamma"]
\end{tikzcd},$$

I the ideal in $K\Delta$ generated by $\alpha\beta\gamma$, $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$, and $T = K\Delta/I$ the associated bound quiver algebra. Then T is an indecomposable 12-dimensional selfinjective Nakayama K-algebra. Hence, T is of finite representation type and a complete description of the indecomposable modules and almost split sequences in $\operatorname{mod} T$ is known, by Theorems I.10.5 and III.8.7. Let Q be the two-sided ideal of T generated by the elements $\bar{\delta} = \delta + I$ and $\bar{\beta} = \beta + I$. The quotient algebra $A = T/Q$ is the product $A = A_1 \times A_2$ of the path algebras $A_1 = K\Delta^{(1)}$ and $A_2 = K\Delta^{(2)}$ of the quivers

$$\Delta^{(1)}\colon 1 \bullet \xrightarrow{\alpha} \bullet 2, \qquad \Delta^{(2)}\colon 4 \bullet \xleftarrow{\gamma} \bullet 3.$$

Observe that the canonical exact sequence of K-vector spaces

$$0 \longrightarrow Q \xrightarrow{\omega} T \xrightarrow{\varrho} A \longrightarrow 0$$

is a Hochschild extension of the hereditary K-algebra A by the duality A-bimodule Q, and $A = A_1 \times A_2$ is a Nakayama block decomposition of A with respect to T. For each vertex i of $Q_A = \Delta^{(1)} \cup \Delta^{(2)}$, we denote by P_i the indecomposable projective module, by I_i the indecomposable injective module, and by S_i the simple module in mod A, associated to i. The Auslander–Reiten quiver Γ_A of A is a disjoint union $\Gamma_A = \Gamma_{A_1} \cup \Gamma_{A_2}$ of the Auslander–Reiten quivers Γ_{A_1} and Γ_{A_2}, and hence is of the form

$$\begin{tikzcd}
& P_1 = I_2 \arrow[dr] & & & P_3 = I_4 \arrow[dr] & \\
S_2 = P_2 \arrow[ur] & & S_1 = I_1 & S_4 = P_4 \arrow[ur] & & S_3 = I_3 .
\end{tikzcd}$$

In particular, we have $\mathcal{P}(A_1)^* = \{S_1\}$, $\mathcal{Q}(A_1)^* = \{S_2\}$, $\mathcal{P}(A_2)^* = \{S_3\}$, $\mathcal{Q}(A_2)^* = \{S_4\}$. Then

$$\begin{aligned}
\Omega_T\big(\mathcal{P}(A_1)^*\big) &= \Omega_T(S_1) = \operatorname{rad} P_T(S_1), \\
\Omega_T\big(\mathcal{P}(A_2)^*\big) &= \Omega_T(S_3) = \operatorname{rad} P_T(S_3), \\
\Omega_T^{-1}\big(\mathcal{Q}(A_1)^*\big) &= \Omega_T^{-1}(S_2) = I_T(S_2)/S_2 = \operatorname{rad} P_T(S_3), \\
\Omega_T^{-1}\big(\mathcal{Q}(A_2)^*\big) &= \Omega_T^{-1}(S_4) = I_T(S_4)/S_4 = \operatorname{rad} P_T(S_1).
\end{aligned}$$

Further, it follows from Corollaries 8.12 and 8.13 that the connecting meshes in Γ_T are of the forms

$$\begin{tikzcd}
& P_T(S_1) \arrow[dr] & \\
\mathfrak{M}_{P_1}\colon \quad \Omega_T(I_1) \arrow[ur] \arrow[dr] & & P_1 , \\
& P_2 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \Omega_T^{-1}(P_2) \arrow[dr] & \\
{}_{I_1}\mathfrak{M}\colon \quad I_1 \arrow[ur] & & P_1 ,
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{M}_{P_2}\colon \quad \Omega_T(I_2) \arrow[dr] & & P_2 , \\
& \Omega_T(I_1) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& I_T(S_2) = P_T(S_4) \arrow[dr] & \\
{}_{I_2}\mathfrak{M}\colon \quad I_2 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_2) , \\
& I_1 \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P_T(S_3) \arrow[dr] & \\
\mathfrak{M}_{P_3}: & \Omega_T(I_3) \arrow[ur] \arrow[dr] & & P_3\,, \\
 & & P_4 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Omega_T^{-1}(P_4) \arrow[dr] & \\
{}_{I_3}\mathfrak{M}: & I_3 \arrow[ur] & & \Omega_T^{-1}(P_3)\,,
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Omega_T(I_4) \arrow[dr] & & P_4\,, \\
\mathfrak{M}_{P_4}: & & & \\
 & & \Omega_T(I_3) \arrow[uur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & I_T(S_4) = P_T(S_2) \arrow[dr] & \\
{}_{I_4}\mathfrak{M}: & I_4 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_4)\,. \\
 & & I_3 \arrow[ur] &
\end{tikzcd}$$

Then, applying Theorem 8.14, we conclude that the Auslander–Reiten quiver Γ_T of T is of the form

$$\begin{tikzcd}[column sep=small]
P_T(S_2) \arrow[dr] & & P_T(S_1) \arrow[dr] & & P_T(S_4) \arrow[dr] & & P_T(S_3) \arrow[dr] & & P_T(S_2) \\
 & \Omega_T(S_1) = \Omega_T^{-1}(S_4) \arrow[ur] \arrow[dr] & & P_1 \arrow[ur] \arrow[dr] & & \Omega_T(S_3) = \Omega_T^{-1}(S_2) \arrow[ur] \arrow[dr] & & P_3 \arrow[ur] \arrow[dr] & \\
S_3 \arrow[ur] & & S_2 \arrow[ur] & & S_1 \arrow[ur] & & S_4 \arrow[ur] & & S_3 \\
 & & & \Gamma_{A_1} & & & & \Gamma_{A_2} &
\end{tikzcd}$$

Example 8.17. Let A be the $\mathbb{Q}$-subalgebra of the matrix algebra $M_2(\mathbb{R})$

$$\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \in M_2(\mathbb{R}) \;\middle|\; a \in \mathbb{Q}, b, c \in \mathbb{Q}(\sqrt[3]{2}) \right\}.$$

We proved in Example VII.7.8 that A is a 7-dimensional hereditary $\mathbb{Q}$-algebra of Dynkin type $\mathbb{G}_2$ and the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
 & P_2 \arrow[dr, "{(3,1)}"] & & \tau_A^{-1} P_2 \arrow[dr, "{(3,1)}"] & & \tau_A^{-2} P_2 = I_2 = S_2 \\
S_1 = P_1 \arrow[ur, "{(1,3)}"] & & \tau_A^{-1} P_1 \arrow[ur, "{(1,3)}"] & & \tau_A^{-2} P_1 = I_1 \arrow[ur, "{(1,3)}"] &
\end{tikzcd}$$

In particular, we have $\mathcal{P}(A) = \Gamma_A = \mathcal{Q}(A)$. Observe that every simple module in $\operatorname{mod} A$ is projective or injective. Consider the trivial extension algebra $T = \mathrm{T}(A) = A \ltimes D(A)$. We will determine the Auslander–Reiten quiver Γ_T of T. Corollaries 8.12 and 8.13 show that the connecting valued meshes in Γ_T are of the forms

$$\begin{tikzcd} & & \Omega_T(I_2) \arrow[dr, "{(3,1)}"] & \\ \mathfrak{M}_{P_1}\colon & \Omega_T(I_1) \arrow[ur, "{(1,3)}"] & & P_1\, , \end{tikzcd} \qquad \begin{tikzcd} & & I_2 \arrow[dr, "{(3,1)}"] & \\ {}_{I_1}\mathfrak{M}\colon & I_1 \arrow[ur, "{(1,3)}"] \arrow[dr, "{(1,1)}"'] & & \Omega_T^{-1}(P_1)\, , \\ & & I_T(P_1) = I_T(S_1) \arrow[ur, "{(1,1)}"'] & \end{tikzcd}$$

$$\begin{tikzcd} & & P_T(I_2) = P_T(S_2) \arrow[dr, "{(1,1)}"] & \\ \mathfrak{M}_{P_2}\colon & \Omega_T(I_2) \arrow[ur, "{(1,1)}"] \arrow[dr, "{(3,1)}"'] & & P_2\, , \\ & & P_1 \arrow[ur, "{(1,3)}"'] & \end{tikzcd} \qquad \begin{tikzcd} & I_2 \arrow[dr, "{(3,1)}"'] & & \Omega_T^{-1}(P_2)\, . \\ {}_{I_2}\mathfrak{M}\colon & & \Omega_T^{-1}(P_1) \arrow[ur, "{(1,3)}"'] & \end{tikzcd}$$

Then, applying Corollary 8.15, we obtain that the Auslander–Reiten quiver Γ_T is of the form

$$\begin{tikzcd} & & & & P_T(S_2) \arrow[dr, "{(1,1)}"] & & & & & & & \\ & \Omega_T(\tau_A^{-1}P_2) \arrow[dr, "{(3,1)}"] & & \Omega_T(I_2) \arrow[ur, "{(1,1)}"] \arrow[dr, "{(3,1)}"] & & P_2 \arrow[dr, "{(3,1)}"] & & \tau_A^{-1}P_2 \arrow[dr, "{(3,1)}"] & & I_2 \arrow[dr, "{(3,1)}"] & & \Omega_T^{-1}(P_2) \\ \Omega_T(\tau_A^{-1}P_1) \arrow[ur, "{(1,3)}"] & & \Omega_T(I_1) \arrow[ur, "{(1,3)}"] & & P_1 \arrow[ur, "{(1,3)}"] & & \tau_A^{-1}P_1 \arrow[ur, "{(1,3)}"] & & I_1 \arrow[ur, "{(1,3)}"] \arrow[dr, "{(1,1)}"'] & & \Omega_T^{-1}(P_1) \arrow[ur, "{(1,3)}"] & \\ & & & & & & & & & I_T(S_1) \arrow[ur, "{(1,1)}"'] & & \end{tikzcd}$$

where $\Omega_T(\tau_A^{-1}P_1) = \Omega_T^{-1}(P_1)$, $\Omega_T(\tau_A^{-1}P_2) = \Omega_T^{-1}(P_2)$, $\Omega_T(I_1) = \Omega_T^{-1}(\tau_A^{-1}P_1)$, $\Omega_T(I_2) = \Omega_T^{-1}(\tau_A^{-1}P_2)$. We note also that $I_T(S_1) = P_T(S_1)$ and $P_T(S_2) = I_T(S_2)$, because T is a symmetric algebra.

Example 8.18. Let K be a field, Q the quiver

$$\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, "\alpha"'] & \underset{3}{\bullet} \arrow[l, "\beta"'] \arrow[r, "\gamma"] & \underset{4}{\bullet}
\end{tikzcd},$$

and $A = KQ$ the path algebra of Q over K, considered in Example VII.7.9. Then A is a hereditary K-algebra of Dynkin type $\mathbb{A}_4$, with $\dim_K A = 8$, and the Auslander–Reiten quiver Γ_A of A is of the form

$$\begin{tikzcd}
 & S_4 = P_4 \arrow[dr] & & \tau_A^{-1} P_4 = I_1 \arrow[dr] & & \\
 & & P_3 \arrow[ur] \arrow[dr] & & \tau_A^{-1} P_3 = I_2 \arrow[dr] & \\
 & P_2 \arrow[ur] \arrow[dr] & & \tau_A^{-1} P_2 = P_3/S_1 \arrow[ur] \arrow[dr] & & \tau_A^{-2} P_2 = I_3 = S_3 \\
S_1 = P_1 \arrow[ur] & & \tau_A^{-1} P_1 = S_2 \arrow[ur] & & \tau_A^{-2} P_1 = I_4 \arrow[ur] &
\end{tikzcd}.$$

In particular, we have $\mathcal{P}(A) = \Gamma_A = \mathcal{Q}(A)$. Moreover, S_2 is the unique simple module in $\operatorname{mod} A$ which is neither projective, nor injective. Let $T = \mathrm{T}(A) = A \ltimes D(A)$ be the trivial extension algebra. We will describe the Auslander–Reiten quiver Γ_T of T. Applying Corollaries 8.12 and 8.13, we conclude that the connecting meshes in Γ_T are of the forms

$$\begin{tikzcd}
 & \Omega_T(I_1) \arrow[ddr] & & P_1 , \\
\mathfrak{M}_{P_1}\colon & & & \\
 & & \Omega_T(I_2) \arrow[uur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & I_2 \arrow[dr] & \\
{}_{I_1}\mathfrak{M}\colon & I_1 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_1) , \\
 & & I_T(P_1) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P_1 \arrow[dr] & \\
\mathfrak{M}_{P_2}\colon & \Omega_T(I_2) \arrow[ur] \arrow[dr] & & P_2 , \\
 & & \Omega_T(I_3) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & I_3 \arrow[dr] & \\
{}_{I_2}\mathfrak{M}\colon & I_2 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_2) , \\
 & & \Omega_T^{-1}(P_1) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P_4 \arrow[dr] & \\
\mathfrak{M}_{P_3}\colon & \Omega_T(I_3) \arrow[ur] \arrow[r, tail] \arrow[dr] & P_T(I_3) \arrow[r] & P_3 , \\
 & & P_2 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Omega_T^{-1}(P_2) \arrow[dr] & \\
{}_{I_3}\mathfrak{M}\colon & I_3 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_3) , \\
 & & \Omega_T^{-1}(P_4) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Omega_T(I_4) \arrow[dr] & & P_4 , \\
\mathfrak{M}_{P_4}\colon & & \Omega_T(I_3) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & I_3 \arrow[dr] & \\
{}_{I_4}\mathfrak{M}\colon & I_4 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_4) \\
 & & I_T(P_4) \arrow[ur] &
\end{tikzcd}$$

Then, applying Corollary 8.15, we obtain that the Auslander–Reiten quiver Γ_T is of the form

$$\begin{tikzcd}[column sep=tiny]
 & P_T(S_2) \arrow[dr] & & & & & & I_T(P_1) \arrow[dr] & & P_T(S_2) \arrow[dr] & & \\
\Omega_T(S_2) \arrow[ur] \arrow[dr] & & \Omega_T(I_4) \arrow[dr] & & P_4 \arrow[dr] & & I_1 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_1) \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(S_2) \arrow[dr] & \\
 & \Omega_T(P_3/S_1) \arrow[ur] \arrow[dr] & & \Omega_T(I_3) \arrow[ur] \arrow[r, tail] \arrow[dr] & P_T(I_3) \arrow[r] & P_3 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_2) \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_3/S_1) \\
 & & \Omega_T(I_2) \arrow[ur] \arrow[dr] & & P_2 \arrow[ur] \arrow[dr] & & P_3/S_1 \arrow[ur] \arrow[dr] & & I_3 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_3) \arrow[ur] & \\
 & \Omega_T(I_1) \arrow[ur] & & P_1 \arrow[ur] & & S_2 \arrow[ur] & & I_4 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_4) \arrow[ur] & & \\
 & & & & & & & & I_T(P_4) \arrow[ur] & & &
\end{tikzcd}$$

where $\Omega_T(S_2) = \operatorname{rad} P_T(S_2) = \Omega_T^{-1}(P_1)$, $\Omega_T(I_4) = P_T(S_2)/S_2 = \Omega_T^{-1}(S_2)$, $\Omega_T(P_3/S_1) = \Omega_T^{-1}(P_2)$, $\Omega_T(I_3) = \Omega_T^{-1}(P_3/S_1)$, $\Omega_T(I_2) = \Omega_T^{-1}(P_3)$,

$\Omega_T(I_1) = \Omega_T^{-1}(P_4)$, $P_T(I_3) = P_T(S_3)$, $I_T(P_1) = P_T(S_1)$, and $I_T(P_4) = P_T(S_4)$, because T is a symmetric algebra.

Theorem 8.19. *Let T be a basic indecomposable Hochschild extension algebra of infinite representation type of a finite dimensional hereditary K-algebra A over a field K by a duality A-bimodule Q, and $A = A_1 \times \cdots \times A_m$ be a Nakayama block decomposition of A with respect to T. Then the Auslander–Reiten quiver Γ_T of T has a disjoint union decomposition*

$$\bigcup_{i=1}^{m} \big(\mathcal{R}(A_i)^+ \cup \mathcal{X}(A_i) \cup \mathcal{R}(A_i) \cup \mathcal{Y}(A_i)\big)$$
$$= \bigcup_{i=1}^{m} (\mathcal{X}(A_i) \cup \mathcal{R}(A_i) \cup \mathcal{Y}(A_i) \cup \mathcal{R}(A_i)^-),$$

where, for each $i \in \{1, \dots, m\}$, we have

(1) *$\mathcal{X}(A_i)$ is obtained from the union of translation quivers*

$$\Omega_T\big(\mathcal{Q}(A_i)\big) \cup \mathcal{P}(A_i)$$

by adding the connecting valued meshes $\mathfrak{M}_P$, for all indecomposable projective modules P in $\mathcal{P}(A_i)$, and completing the meshes in $\Omega_T(\mathcal{Q}(A_i))$ by the arrows $\{\Omega_T(S)\} \xrightarrow{(1,1)} \{P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)/\operatorname{soc}(P_T(S))\}$, for all simple noninjective modules S in $\mathcal{Q}(A_i)$;

(2) *$\mathcal{Y}(A_i)$ is obtained from the union of translation quivers*

$$\mathcal{Q}(A_i) \cup \Omega_T^{-1}\big(\mathcal{P}(A_i)\big)$$

by adding the connecting valued meshes ${}_I\mathfrak{M}$, for all indecomposable injective modules I in $\mathcal{Q}(A_i)$, and completing the meshes in $\Omega_T^{-1}(\mathcal{P}(A_i))$ by the arrows $\{\operatorname{rad} I_T(S)\} \xrightarrow{(1,1)} \{I_T(S)\} \xrightarrow{(1,1)} \{\Omega_T^{-1}(S)\}$, for all simple nonprojective modules S in $\mathcal{P}(A_i)$;

(3) *$\mathcal{R}(A_i)$ is the family of all regular components of Γ_{A_i};*

(4) *$\mathcal{R}(A_i)^+$ is the family of components obtained from the family of $\Omega_T(\mathcal{R}(A_i))$ of components of Γ_T^{s} by adding the arrows $\{\Omega_T(S)\} \xrightarrow{(1,1)} \{P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)/\operatorname{soc}(P_T(S))\}$, for all simple modules S in $\mathcal{R}(A_i)$;*

(5) *$\mathcal{R}(A_i)^-$ is the family of components obtained from the family of $\Omega_T^{-1}(\mathcal{R}(A_i))$ of components of Γ_T^{s} by adding the arrows $\{\operatorname{rad} I_T(S)\} \xrightarrow{(1,1)} \{I_T(S)\} \xrightarrow{(1,1)} \{\Omega_T^{-1}(S)\}$, for all simple modules S in $\mathcal{R}(A_i)$;*

(6) *$\mathcal{R}(A_i)^+ = \mathcal{R}(A_{i-1})^-$ and $\mathcal{R}(A_i)^- = \mathcal{R}(A_{i+1})^+$, where $A_0 = A_m$ and $A_{m+1} = A_1$.*

Proof. Since T is a basic indecomposable Frobenius algebra of infinite representation type, it follows from Proposition 3.21 and Theorem 8.14 that $A_1, \dots, A_m$ are isomorphic basic indecomposable hereditary algebras of infinite representation type. In particular, we have $Q_{A_1} = \dots = Q_{A_m}$. Moreover, applying Corollary 8.11 (i), we conclude that, for any $i \in \{1, \dots, m\}$, the Auslander–Reiten quiver

$$\Gamma_{A_i} = \mathcal{P}(A_i) \cup \mathcal{R}(A_i) \cup \mathcal{Q}(A_i)$$

of A_i is a full translation subquiver of Γ_A. Fix $i \in \{1, \dots, m\}$.

There exists a component $\mathcal{X}(A_i)$ of Γ_T such that $\mathcal{P}(A_i)$ is a full translation subquiver of $\mathcal{X}(A_i)$ which is closed under successors. Further, it follows from Theorems IX.4.12 and IX.4.15 that the stable part $\mathcal{X}(A_i)^{\mathrm{s}}$ of $\mathcal{X}(A_i)$ is isomorphic to the acyclic stable translation quiver $\mathbb{Z}Q_{A_i}^{\mathrm{op}}$, because the opposite valued quiver $Q_{A_i}^{\mathrm{op}}$ of the quiver Q_{A_i} of A_i is isomorphic to the full subquiver of $\mathcal{P}(A_i)$ given by the projective modules. On the other hand, the image $\Omega_T(\mathcal{Q}(A_i))$ of $\mathcal{Q}(A_i)$ under Ω_T is a full translation subquiver of the stable Auslander–Reiten quiver Γ_T^{s} which is closed under predecessors, by Corollary 8.11 (ii). Further, by Corollary 8.12, there are in Γ_T the connecting valued meshes $\mathfrak{M}_P$ given by the indecomposable projective modules P in $\mathcal{P}(A_i)$, whose left terms are given by the modules $\Omega_T(I)$ for the indecomposable injective module I in $\mathcal{Q}(A_i)$ with $\operatorname{top}(P) \cong \operatorname{soc}(I)$. This shows that $\Omega_T(\mathcal{Q}(A_i))$ is a full translation subquiver of the stable translation quiver $\mathcal{X}(A_i)^{\mathrm{s}}$. We also note that the indecomposable projective modules in $\mathcal{X}(A_i)$ occur in the connecting meshes $\mathfrak{M}_P$ with $\operatorname{top}(P)$ being injective modules in $\operatorname{mod} A_i$ (by Corollary 8.12) and in the meshes ${}_{\Omega_T(S)}\mathfrak{M}$ given by all simple noninjective modules S in $\mathcal{Q}(A_i)$. Therefore, $\mathcal{X}(A_i)$ is a component of Γ_T described in (1).

Dually, there exists a component $\mathcal{Y}(A_i)$ of Γ_T such that $\mathcal{Q}(A_i)$ is a full translation subquiver of $\mathcal{Y}(A_i)$ which is closed under predecessors. Applying Theorems IX.4.12 and IX.4.15 again, we conclude that the stable part $\mathcal{Y}(A_i)^{\mathrm{s}}$ of $\mathcal{Y}(A_i)$ is isomorphic to the acyclic stable translation quiver $\mathbb{Z}Q_{A_i}^{\mathrm{op}}$, because the quiver $Q_{A_i}^{\mathrm{op}}$ is isomorphic to the full valued subquiver of $\mathcal{Q}(A_i)$ given by the injective modules. It follows also from Corollary 8.11 (iii) that the image $\Omega_T^{-1}(\mathcal{P}(A_i))$ of $\mathcal{P}(A_i)$ via Ω_T^{-1} is a full translation subquiver of the stable Auslander–Reiten quiver Γ_T^{s} which is closed under successors. Further, by Corollary 8.12, there are in Γ_T the connecting valued meshes ${}_I\mathfrak{M}$ given by the indecomposable injective modules I in $\mathcal{Q}(A_i)$, whose right terms are given by the modules $\Omega_T^{-1}(P)$ for the indecomposable projective module P in $\mathcal{P}(A_i)$ with $\operatorname{soc}(I) \cong \operatorname{top}(P)$. This shows that $\Omega_T^{-1}(\mathcal{P}(A_i))$ is a full translation subquiver of the stable translation quiver $\mathcal{Y}(A_i)^{\mathrm{s}}$. Finally, we observe that the indecomposable projective modules in $\mathcal{Y}(A_i)$ occur in the connecting valued meshes ${}_I\mathfrak{M}$ with $\operatorname{soc}(I)$ being projective modules in $\operatorname{mod} A_i$ (by Corollary 8.13) and in the meshes ${}_{\Omega_T^{-1}(S)}\mathfrak{M}$ given by all simple nonprojective modules S in $\mathcal{P}(A_i)$. Therefore, $\mathcal{Y}(A_i)$ is a component of Γ_T described in (2).

It follows from Corollary 8.11 (ii) that the images of all components in $\mathcal{R}(A_i)$ under Ω_T form a family $\Omega_T(\mathcal{R}(A_i))$ of components of Γ_T^{s}. Hence, there is a family of components $\mathcal{R}(A_i)^+$ of Γ_T obtained from $\Omega_T(\mathcal{R}(A_i))$ by adding the arrows

$$\{\operatorname{rad} P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)/\operatorname{soc}(P_T(S))\},$$

for all simple modules S in $\mathcal{R}(A_i)$, as described in (4).

Similarly, it follows from Corollary 8.11 (iii) that the images of all components in $\mathcal{R}(A_i)$ via Ω_T^{-1} form a family $\Omega_T^{-1}(\mathcal{R}(A_i))$ of components of Γ_T^{s}. Hence, there is a family of components $\mathcal{R}(A_i)^-$ of Γ_T obtained from $\Omega_T^{-1}(\mathcal{R}(A_i))$ by adding the arrows

$$\{\operatorname{rad} I_T(S)\} \xrightarrow{(1,1)} \{I_T(S)\} \xrightarrow{(1,1)} \{\Omega_T^{-1}(S)\},$$

for all simple modules S in $\mathcal{R}(A_i)$, as described in (5).

Theorem 7.14 shows that Γ_T has the following disjoint union decompositions

$$\Gamma_A = \bigcup_{i=1}^{m} \big(\mathcal{R}(A_i)^+ \cup \mathcal{X}(A_i) \cup \mathcal{R}(A_i) \cup \mathcal{Y}(A_i)\big),$$

$$\Gamma_A = \bigcup_{i=1}^{m} \big(\mathcal{X}(A_i) \cup \mathcal{R}(A_i) \cup \mathcal{Y}(A_i) \cup \mathcal{R}(A_i)^-\big).$$

Moreover, it follows from Proposition 7.18 that $\mathcal{R}(A_i)^+ = \mathcal{R}(A_{i-1})^-$ and $\mathcal{R}(A_i)^- = \mathcal{R}(A_{i+1})^+$, for any $i \in \{1, \ldots, m\}$. □

We have the following direct consequence of Theorem 8.19.

Corollary 8.20. *Let T be a basic indecomposable Hochschild extension algebra of an indecomposable finite dimensional hereditary K-algebra A of infinite representation type over a field K by a duality A-bimodule Q. Then the Auslander–Reiten quiver Γ_T of T has the disjoint union decompositions*

$$\mathcal{R}(A)^+ \cup \mathcal{X}(A) \cup \mathcal{R}(A) \cup \mathcal{Y}(A) = \mathcal{X}(A) \cup \mathcal{R}(A) \cup \mathcal{Y}(A) \cup \mathcal{R}(A)^-,$$

where

(1) *$\mathcal{X}(A)$ is obtained from the union of translation quivers*

$$\Omega_T\big(\mathcal{Q}(A)\big) \cup \mathcal{P}(A)$$

by adding the connecting valued meshes $\mathfrak{M}_P$, for all indecomposable projective modules P in $\mathcal{P}(A)$, and completing the meshes in $\Omega_T(\mathcal{Q}(A))$ by the arrows $\{\Omega_T(S)\} \xrightarrow{(1,1)} \{P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)/\operatorname{soc}(P_T(S))\}$, for all simple noninjective modules S in $\mathcal{Q}(A)$;

(2) $\mathcal{Y}(A)$ *is obtained from the union of translation quivers*

$$\mathcal{Q}(A) \cup \Omega_T^{-1}\big(\mathcal{P}(A)\big)$$

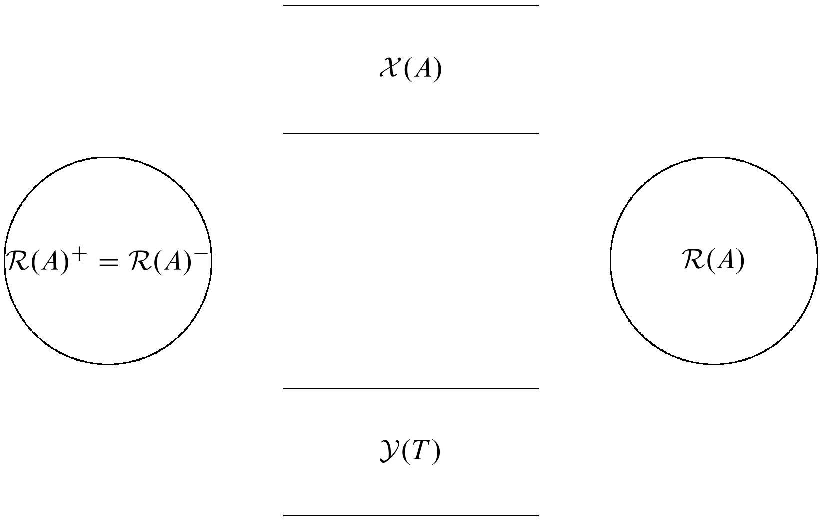
by adding the connecting valued meshes ${}_I\mathfrak{M}$*, for all indecomposable injective modules* I *in* $\mathcal{Q}(A)$*, and completing the meshes in* $\Omega_T^{-1}(\mathcal{P}(A))$ *by the arrows* $\{\operatorname{rad} I_T(S)\} \xrightarrow{(1,1)} \{I_T(S)\} \xrightarrow{(1,1)} \{\Omega_T^{-1}(S)\}$*, for all simple nonprojective modules* S *in* $\mathcal{P}(A)$*;*

(3) $\mathcal{R}(A)$ *is the family of all regular components of* Γ_A*;*

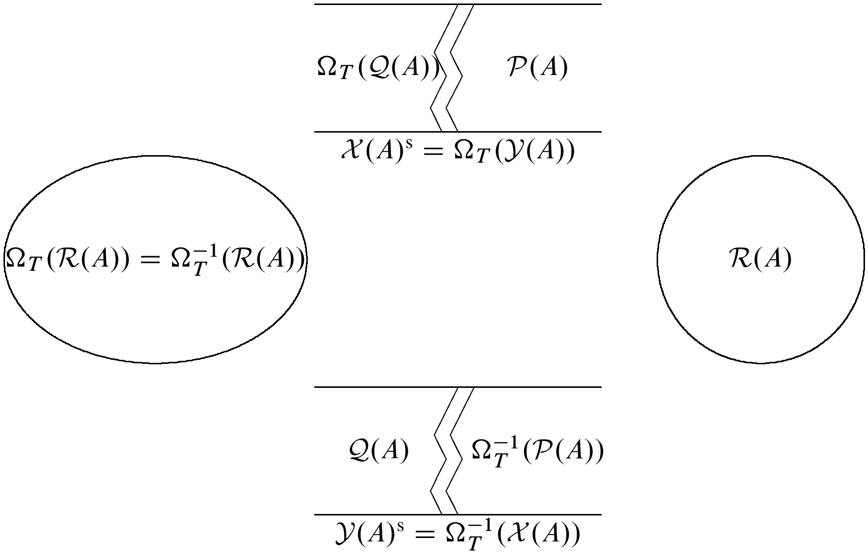
(4) $\mathcal{R}(A)^+$ *is the family of components obtained from the family* $\Omega_T(\mathcal{R}(A))$ *of components of* Γ_T^{s} *by adding the arrows* $\{\Omega_T(S)\} \xrightarrow{(1,1)} \{P_T(S)\} \xrightarrow{(1,1)} \{P_T(S)/\operatorname{soc}(P_T(S))\}$*, for all simple modules* S *in* $\mathcal{R}(A)$*;*

(5) $\mathcal{R}(A)^-$ *is the family of components obtained from the family* $\Omega_T^{-1}(\mathcal{R}(A))$ *of components of* Γ_T^{s} *by adding the arrows* $\{\operatorname{rad} I_T(S)\} \xrightarrow{(1,1)} \{I_T(S)\} \xrightarrow{(1,1)} \{\Omega_T^{-1}(S)\}$*, for all simple modules* S *in* $\mathcal{R}(A)$*.*

We may visualise the Auslander–Reiten quiver Γ_T of the Hochschild extension algebra in Corollary 8.20 as follows:



and the stable Auslander–Reiten quiver Γ_T^{s} of T as follows:



We also note that $\mathcal{X}(A)$ and $\mathcal{Y}(T)$ are acyclic components containing projective-injective modules, and the stable parts $\mathcal{X}(A)^{\mathrm{s}}$ and $\mathcal{Y}(A)^{\mathrm{s}}$ are isomorphic to the translation quiver $\mathbb{Z}Q_A^{\mathrm{op}}$. On the other hand, the family of components $\mathcal{R}(A)^+ = \mathcal{R}(A)^-$ contains a projective-injective module if and only if the regular part $\mathcal{R}(A)$ of Γ_A contains a simple module.

The following examples show that all possible situations occur.

Example 8.21. Let A be the $\mathbb{Q}$-subalgebra of the matrix algebra $M_3(\mathbb{R})$

$$\begin{bmatrix} \mathbb{Q} & 0 & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) \end{bmatrix}$$
$$= \left\{ \begin{bmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{bmatrix} \in M_3(\mathbb{R}) \;\middle|\; \begin{matrix} a \in \mathbb{Q}, \\ b, c, x, y, z \in \mathbb{Q}(\sqrt[3]{2}) \end{matrix} \right\}.$$

We proved in Example VII.8.28 that A is a 16-dimensional hereditary $\mathbb{Q}$-algebra of Dynkin type $\mathbb{G}_{22}$ and the Auslander–Reiten quiver Γ_A of A has a disjoint union decomposition

$$\Gamma_A = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A),$$

where $\mathcal{P}(A)$ is the postprojective component of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
& & P_3 \arrow[dr] & & \tau_A^{-1} P_3 \arrow[dr] & & \tau_A^{-2} P_3 \arrow[dr] & & \tau_A^{-3} P_3 \arrow[dr, dash] & \\
& P_2 \arrow[ur] \arrow[dr, "(3,1)"] & & \tau_A^{-1} P_2 \arrow[ur] \arrow[dr, "(3,1)"] & & \tau_A^{-2} P_2 \arrow[ur] \arrow[dr, "(3,1)"] & & \tau_A^{-3} P_2 \arrow[ur] \arrow[dr, dash] & & \cdots \\
P_1 \arrow[ur, "(1,3)"] & & \tau_A^{-1} P_1 \arrow[ur, "(1,3)"] & & \tau_A^{-2} P_1 \arrow[ur, "(1,3)"] & & \tau_A^{-3} P_1 \arrow[ur, "(1,3)"] & & &
\end{tikzcd}$$

$\mathcal{Q}(A)$ is the preinjective component of the form $\mathbb{N}Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
& & {} \arrow[dr] & & \tau_A^3 I_3 \arrow[dr] & & \tau_A^2 I_3 \arrow[dr] & & \tau_A I_3 \arrow[dr] & & I_3 \\
\cdots & {} \arrow[dr] & & \tau_A^3 I_2 \arrow[ur] \arrow[dr, "(3,1)"] & & \tau_A^2 I_2 \arrow[ur] \arrow[dr, "(3,1)"] & & \tau_A I_2 \arrow[ur] \arrow[dr, "(3,1)"] & & I_2 \arrow[ur] & \\
& & \tau_A^3 I_1 \arrow[ur, "(1,3)"] & & \tau_A^2 I_1 \arrow[ur, "(1,3)"] & & \tau_A I_1 \arrow[ur, "(1,3)"] & & I_1 \arrow[ur, "(1,3)"] & &
\end{tikzcd}$$

and $\mathcal{R}(A)$ is an infinite family of stable tubes, containing a stable tube of rank 2 and infinitely many stable tubes of rank 1. We also note that the simple module $S_1 = P_1$ is projective, the simple module $S_3 = I_3$ is injective, and the simple module $S_2 = \tau_A I_3$ is preinjective but not injective.

Consider the trivial extension algebra $T = \mathrm{T}(A) = A \ltimes D(A)$. We will determine the Auslander–Reiten quiver Γ_T of T, using Corollary 8.20. Corollaries 8.12 and 8.13 show that the connecting valued meshes in Γ_T are of the forms

$$\mathfrak{M}_{P_1}\colon \quad \begin{tikzcd}
& \Omega_T(I_2) \arrow[dr, "(3,1)"] & \\
\Omega_T(I_1) \arrow[ur, "(1,3)"] & & P_1 ,
\end{tikzcd} \qquad {}_{I_1}\mathfrak{M}\colon \quad \begin{tikzcd}
& I_2 \arrow[dr, "(3,1)"] & \\
I_1 \arrow[ur, "(1,3)"] \arrow[dr] & & \Omega_T^{-1}(P_1) , \\
& I_T(P_1) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Omega_T(I_3) \arrow[dr] & \\
\mathfrak{M}_{P_2}\colon \quad \Omega_T(I_2) \arrow[ur] \arrow[dr, "{(3,1)}"'] & & P_2\,, \\
 & P_1 \arrow[ur, "{(1,3)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I_3 \arrow[dr] & \\
{}_{I_2}\mathfrak{M}\colon \quad I_2 \arrow[ur] \arrow[dr, "{(3,1)}"'] & & \Omega_T^{-1}(P_2)\,, \\
 & \Omega_T^{-1}(P_1) \arrow[ur, "{(1,3)}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & P_T(I_3) \arrow[dr] & \\
\mathfrak{M}_{P_3}\colon \quad \Omega_T(I_3) \arrow[ur] \arrow[dr] & & P_3\,, \\
 & P_2 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
{}_{I_3}\mathfrak{M}\colon \quad I_3 \arrow[dr] & & \Omega_T^{-1}(P_3) \\
 & \Omega_T^{-1}(P_2) \arrow[ur] &
\end{tikzcd}$$

Then the acyclic component $\mathcal{X}(A)$ of Γ_T, obtained by gluing $\Omega_T(\mathcal{Q}(A))$ and $\mathcal{P}(A)$, is of the form

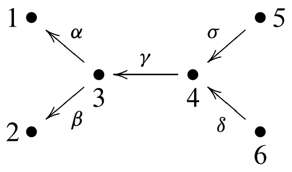
$$\begin{tikzcd}[column sep=tiny]
 & & & & P_T(S_2) \arrow[dr] & & P_T(I_3) = P_T(S_3) \arrow[dr] & & & & \\
 & {} \arrow[dr] & & \Omega_T(\tau_A I_3) \arrow[ur] \arrow[dr] & & \Omega_T(I_3) \arrow[ur] \arrow[dr] & & P_3 \arrow[dr] & & \tau_A^{-1} P_3 \arrow[dr, dash] & \\
\cdots \arrow[dr] & & \Omega_T(\tau_A I_2) \arrow[ur] \arrow[dr, "{(3,1)}"'] & & \Omega_T(I_2) \arrow[ur] \arrow[dr, "{(3,1)}"'] & & P_2 \arrow[ur] \arrow[dr, "{(3,1)}"'] & & \tau_A^{-1} P_2 \arrow[ur] \arrow[dr, dash] & & \cdots \\
 & \Omega_T(\tau_A I_1) \arrow[ur, "{(1,3)}"] & & \Omega_T(I_1) \arrow[ur, "{(1,3)}"] & & P_1 \arrow[ur, "{(1,3)}"] & & \tau_A^{-1} P_1 \arrow[ur, "{(1,3)}"] & & {} &
\end{tikzcd}$$

because $\Omega_T(\tau_A I_3) = \Omega_T(S_2) = \operatorname{rad} P_T(S_3)$, and the acyclic component $\mathcal{Y}(T)$ of Γ_T, obtained by gluing $\mathcal{Q}(A)$ and $\Omega_T^{-1}(\mathcal{P}(A))$, is of the form

$$\begin{tikzcd}
 & & {} \arrow[dr] & & \tau_A I_3 \arrow[dr] & & I_3 \arrow[dr] & & \Omega_T^{-1}(P_3) \arrow[dr] & & \Omega_T^{-1}(\tau_A^{-1} P_3) \arrow[dr, dash] & & \\
\cdots & {} \arrow[dr] & & \tau_A I_2 \arrow[ur] \arrow[dr, "{(3,1)}"] & & I_2 \arrow[ur] \arrow[dr, "{(3,1)}"] & & \Omega_T^{-1}(P_2) \arrow[ur] \arrow[dr, "{(3,1)}"] & & \Omega_T^{-1}(\tau_A^{-1} P_2) \arrow[ur] \arrow[dr, dash] & & {} & \cdots \\
 & & \tau_A I_1 \arrow[ur, "{(1,3)}"] & & I_1 \arrow[ur, "{(1,3)}"] \arrow[dr] & & \Omega_T^{-1}(P_1) \arrow[ur, "{(1,3)}"] & & \Omega_T^{-1}(\tau_A^{-1} P_1) \arrow[ur, "{(1,3)}"] & & {} & & \\
 & & & & & I_T(P_1) = P_T(S_1) \arrow[ur] & & & & & & &
\end{tikzcd}$$

Moreover, since the regular part $\mathcal{R}(A)$ of Γ_A does not contain a simple module, we have $\mathcal{R}(A)^+ = \Omega_T(\mathcal{R}(A))$ and $\mathcal{R}(A)^- = \Omega_T^{-1}(\mathcal{R}(A))$. In particular, we conclude that $\mathcal{R}(A)^+ = \mathcal{R}(A)^-$ is an infinite family of stable tubes, containing a stable tube of rank 2 and infinitely many stable tubes of rank 1.

Example 8.22. Let K be a field, Q the quiver



and $A = KQ$ the path algebra of Q over K. We proved in Example VII.8.30 that A is a hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_5$ and the Auslander–Reiten quiver Γ_A of A has a disjoint union decomposition

$$\Gamma_A = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A),$$

where $\mathcal{P}(A)$ is the postprojective component of the form $(-\mathbb{N})Q^{\mathrm{op}}$

$$\begin{tikzcd}
P_1 \arrow[dr] & & \tau_A^{-1}P_1 \arrow[dr] & & \tau_A^{-2}P_1 \arrow[dr] & & & & \\
P_2 \arrow[r] & P_3 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-1}P_2 \arrow[r] & \tau_A^{-1}P_3 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-2}P_2 \arrow[r] & \tau_A^{-2}P_3 \arrow[ur, dash] \arrow[r, dash] \arrow[dr] & & & \cdots \\
& & P_4 \arrow[r] \arrow[ur] \arrow[dr] & P_5 \arrow[r] & \tau_A^{-1}P_4 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^{-1}P_5 \arrow[r] & \tau_A^{-2}P_4 \arrow[ur, dash] \arrow[r] \arrow[dr] & \tau_A^{-2}P_5 \arrow[r, dash] & \\
& & & P_6 \arrow[ur] & & \tau_A^{-1}P_6 \arrow[ur] & & \tau_A^{-2}P_6 \arrow[ur, dash] &
\end{tikzcd}$$

$\mathcal{Q}(A)$ is the preinjective component of the form $\mathbb{N}Q^{\mathrm{op}}$

$$\begin{tikzcd}
& \tau_A^2 I_1 \arrow[dr] & & \tau_A I_1 \arrow[dr] & & I_1 \arrow[dr] & & & \\
{} \arrow[r] \arrow[ur] & \tau_A^2 I_2 \arrow[r] & \tau_A^2 I_3 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A I_2 \arrow[r] & \tau_A I_3 \arrow[ur] \arrow[r] \arrow[dr] & I_2 \arrow[r] & I_3 \arrow[dr] & & \\
\cdots & {} \arrow[ur] & {} \arrow[r] & \tau_A^2 I_4 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A^2 I_5 \arrow[r] & \tau_A I_4 \arrow[ur] \arrow[r] \arrow[dr] & \tau_A I_5 \arrow[r] & I_4 \arrow[r] \arrow[dr] & I_5 \\
& & {} \arrow[ur] & & \tau_A^2 I_6 \arrow[ur] & & \tau_A I_6 \arrow[ur] & & I_6
\end{tikzcd}$$

and the regular part $\mathcal{R}(A)$ contains a stable tube $\mathcal{T}_{\lambda_0}^{A}$ of rank 3 and two stable tubes $\mathcal{T}_{\lambda_1}^{A}$ and $\mathcal{T}_{\lambda_2}^{A}$ of rank 2. Moreover, the mouth of the stable tube $\mathcal{T}_{\lambda_0}^{A}$ is formed by the simple modules S_3 and S_4 at the vertices 3 and 4 of Q, and the module R with $[R] = (1, 1, 1, 1, 1, 1)$ in $K_0(A) = \mathbb{Z}^6$, and such that $\tau_A R = S_4$, $\tau_A S_4 = S_3$, $\tau_A S_3 = R$. We also observe that the simple modules S_1 and S_2 at the vertices 1 and 2 of Q are projective, and the simple modules S_5 and S_6 at the vertices 5 and 6 of Q are injective.

Consider the trivial extension algebra $T = \mathrm{T}(A) = A \ltimes D(A)$. We will describe the Auslander–Reiten quiver Γ_T of T. Corollaries 8.12 and 8.13 show that the connecting meshes in Γ_T are of the forms

$$\begin{tikzcd}[column sep=small]
 & \Omega_T(I_i) \arrow[dr] & & P_i & & & & & I_3 \arrow[dr] & \\
\mathfrak{M}_{P_i}\colon & & \Omega_T(I_3) \arrow[ur] & & , & {}_{I_i}\mathfrak{M}\colon & & I_i \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_i)\ , \\
 & & & & & & & & I_T(P_i) \arrow[ur] &
\end{tikzcd}$$

for $i \in \{1, 2\}$,

$$\begin{tikzcd}[column sep=small]
 & & P_1 \arrow[dr] & & & & \Omega_T^{-1}(P_1) \arrow[dr] & \\
\mathfrak{M}_{P_3}\colon & \Omega_T(I_3) \arrow[ur] \arrow[r] \arrow[dr] & P_2 \arrow[r] & P_3\ , & {}_{I_3}\mathfrak{M}\colon & I_3 \arrow[ur] \arrow[r] \arrow[dr] & \Omega_T^{-1}(P_2) \arrow[r] & \Omega_T^{-1}(P_3)\ , \\
 & & \Omega_T(I_4) \arrow[ur] & & & & I_4 \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & P_3 \arrow[dr] & & & & \Omega_T^{-1}(P_3) \arrow[dr] & \\
\mathfrak{M}_{P_4}\colon & \Omega_T(I_4) \arrow[ur] \arrow[r] \arrow[dr] & \Omega_T(I_5) \arrow[r] & P_4\ , & {}_{I_4}\mathfrak{M}\colon & I_4 \arrow[ur] \arrow[r] \arrow[dr] & I_5 \arrow[r] & \Omega_T^{-1}(P_4)\ , \\
 & & \Omega_T(I_6) \arrow[ur] & & & & I_6 \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & P_4 \arrow[dr] & & & & & & \\
\mathfrak{M}_{P_j}\colon & \Omega_T(I_j) \arrow[ur] \arrow[dr] & & P_j\ , & {}_{I_j}\mathfrak{M}\colon & & \Omega_T^{-1}(P_4) \arrow[dr] & & , \\
 & & P_T(I_j) \arrow[ur] & & & I_j \arrow[ur] & & \Omega_T^{-1}(P_j) &
\end{tikzcd}$$

for $j \in \{5, 6\}$. Then the acyclic component $\mathcal{X}(A)$ of Γ_T, obtained by gluing $\Omega_T(\mathcal{Q}(A))$ and $\mathcal{P}(A)$, is of the form

$$\begin{tikzcd}
\Omega_T(\tau_A I_1) \arrow[dr] & & \Omega_T(I_1) \arrow[dr] & & P_1 \arrow[dr] & & \tau_A^{-1} P_1 \arrow[dr] & & & \\
\Omega_T(\tau_A I_2) \arrow[r] & \Omega_T(\tau_A I_3) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T(I_2) \arrow[r] & \Omega_T(I_3) \arrow[r] \arrow[ur] \arrow[dr] & P_2 \arrow[r] & P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A^{-1} P_2 \arrow[r] & \tau_A^{-1} P_3 \arrow[ur, dash] \arrow[dr] & & \cdots \\
\cdots \arrow[ur] & & \Omega_T(\tau_A I_4) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T(\tau_A I_5) \arrow[r] & \Omega_T(I_4) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T(I_5) \arrow[r] \arrow[dr] & P_4 \arrow[r] \arrow[ur] \arrow[dr] & P_5 \arrow[r] & \tau_A^{-1} P_4 \arrow[r] \arrow[ur, dash] \arrow[dr] & \tau_A^{-1} P_5 \\
& {} \arrow[ur] & & \Omega_T(\tau_A I_6) \arrow[ur] & & \Omega_T(I_6) \arrow[ur] \arrow[dr] & P_T(I_5) \arrow[ur] & P_6 \arrow[ur] & & \tau_A^{-1} P_6 \\
& & & & & & P_T(I_6) \arrow[ur] & & &
\end{tikzcd}$$

with $P_T(I_5) = P_T(S_5)$ and $P_T(I_6) = P_T(S_6)$, and the acyclic component $\mathcal{Y}(T)$ of Γ_T, obtained by gluing $\mathcal{Q}(A)$ and $\Omega_T^{-1}(\mathcal{P}(A))$, is of the form

$$\begin{tikzcd}
& & & I_T(P_1) \arrow[dr] & & & & & & \\
\tau_A I_1 \arrow[dr] & & I_1 \arrow[ur] \arrow[dr] & I_T(P_2) \arrow[dr] & \Omega_T^{-1}(P_1) \arrow[dr] & & \Omega_T^{-1}(\tau_A^{-1} P_1) \arrow[dr] & & & \\
& \tau_A I_3 \arrow[r] \arrow[ur] \arrow[dr] & I_2 \arrow[r] \arrow[ur] & I_3 \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T^{-1}(P_2) \arrow[r] & \Omega_T^{-1}(P_3) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T^{-1}(\tau_A^{-1} P_2) \arrow[r] & \Omega_T^{-1}(\tau_A^{-1} P_3) \arrow[ur, dash] \arrow[dr] & & \cdots \\
\cdots \arrow[ur] & & \tau_A I_4 \arrow[r] \arrow[ur] \arrow[dr] & \tau_A I_5 \arrow[r] & I_4 \arrow[r] \arrow[ur] \arrow[dr] & I_5 \arrow[r] & \Omega_T^{-1}(P_4) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T^{-1}(P_5) \arrow[r] & \Omega_T^{-1}(\tau_A^{-1} P_4) \arrow[r] \arrow[ur, dash] \arrow[dr] & \Omega_T^{-1}(\tau_A^{-1} P_5) \\
& {} \arrow[ur] & & \tau_A I_6 \arrow[ur] & & I_6 \arrow[ur] & & \Omega_T^{-1}(P_6) \arrow[ur] & & \Omega_T^{-1}(\tau_A^{-1} P_6)
\end{tikzcd}$$

with $I_T(P_1) = P_T(S_1)$ and $I_T(P_2) = P_T(S_2)$.

Further, the family of components $\mathcal{R}(A)^+ = \mathcal{R}(A)^-$ consists of the stable tubes $\Omega_T(\mathcal{T}_\lambda^A)$, $\lambda \in \Lambda(A) \setminus \{\lambda_0\}$, and a quasi-tube obtained from the stable tube $\Omega_T(\mathcal{T}_{\lambda_0}^A)$ in Γ_T^{s} of rank 3 by adding the projective modules $P_T(S_3) = I_T(S_3)$ and $P_T(S_4) = I_T(S_4)$ as follows:

$$\begin{tikzcd}
 & & & P_T(S_3) \arrow[dr] & & P_T(S_4) \arrow[dr] & \\
\Omega_T(R) \arrow[dr] \arrow[dd, dash, dashed] & & \Omega_T(S_3) \arrow[ur] \arrow[dr] & & \Omega_T(S_4) \arrow[ur] \arrow[dr] & & \Omega_T(R) \arrow[dd, dash, dashed] \\
 & \Omega_T(R[2]) \arrow[ur] \arrow[dr] & & \Omega_T(S_3[2]) \arrow[ur] \arrow[dr] & & \Omega_T(S_4[2]) \arrow[ur] \arrow[dr] & \\
\Omega_T(S_4[3]) \arrow[ur] \arrow[dr, dash] \arrow[d, dash, dashed] & & \Omega_T(R[3]) \arrow[ur] \arrow[dr, dash] & & \Omega_T(S_3[3]) \arrow[ur] \arrow[dr, dash] & & \Omega_T(S_4[3]) \arrow[d, dash, dashed] \\
{} & \vdots \arrow[ur] & & \vdots \arrow[ur] & & \vdots \arrow[ur] & {}
\end{tikzcd}$$

Example 8.23. Let A be the following $\mathbb{R}$-subalgebra of the matrix algebra $M_2(\mathbb{H})$

$$\begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ x & b \end{bmatrix} \in M_2(\mathbb{H}) \;\middle|\; a \in \mathbb{R}, b \in \mathbb{C}, x \in \mathbb{H} \right\},$$

considered in Example VII.9.24. Then A is a 7-dimensional hereditary $\mathbb{R}$-algebra of wild type with the quiver Q_A of the form

$$2 \xrightarrow{(4,2)} 1\,.$$

We proved in Example VII.9.24 that the Auslander–Reiten quiver Γ_A of A has a disjoint union decomposition

$$\Gamma_A = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A),$$

where $\mathcal{P}(A)$ is the postprojective component of the form $(-\mathbb{N})Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
 & P_2 \arrow[dr, "{(4,2)}"] & & \tau_A^{-1} P_2 \arrow[dr, "{(4,2)}"] & & \tau_A^{-2} P_2 \arrow[dr, "{(4,2)}"] & & \tau_A^{-3} P_2 \arrow[dr, dash] & & \\
P_1 \arrow[ur, "{(2,4)}"] & & \tau_A^{-1} P_1 \arrow[ur, "{(2,4)}"] & & \tau_A^{-2} P_1 \arrow[ur, "{(2,4)}"] & & \tau_A^{-3} P_1 \arrow[ur, "{(2,4)}"] & & {} & \cdots
\end{tikzcd}$$

$\mathcal{Q}(A)$ is the preinjective component of the form $\mathbb{N}Q_A^{\mathrm{op}}$

$$\begin{tikzcd}
\cdots \arrow[dr] & & \tau_A^3 I_2 \arrow[dr, "(4,2)"] & & \tau_A^2 I_2 \arrow[dr, "(4,2)"] & & \tau_A I_2 \arrow[dr, "(4,2)"] & & I_2 \\
& \tau_A^3 I_1 \arrow[ur, "(2,4)"] & & \tau_A^2 I_1 \arrow[ur, "(2,4)"] & & \tau_A I_1 \arrow[ur, "(2,4)"] & & I_1 \arrow[ur, "(2,4)"] &
\end{tikzcd}$$

and $\mathcal{R}(A)$ is an infinite family of components of type $\mathbb{Z}\mathbb{A}_\infty$. We also note that $\mathcal{R}(A)$ does not contain a simple module.

Consider the trivial extension algebra $T = \mathrm{T}(A) = A \ltimes D(A)$. We will describe the Auslander–Reiten quiver Γ_T of T, using Corollary 8.20. Corollaries 8.12 and 8.13 show that the connecting valued meshes of Γ_T are of the forms

$$\begin{tikzcd}
& & \Omega_T(I_2) \arrow[dr, "(4,2)"] & \\
\mathfrak{M}_{P_1}: & \Omega_T(I_1) \arrow[ur, "(2,4)"] & & P_1 ,
\end{tikzcd}$$

$$\begin{tikzcd}
& & I_2 \arrow[dr, "(4,2)"] & \\
{}_{I_1}\mathfrak{M}: & I_1 \arrow[ur, "(2,4)"] \arrow[dr] & & \Omega_T^{-1}(P_1) , \\
& & I_T(P_1) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& & P_T(I_2) \arrow[dr] & \\
\mathfrak{M}_{P_2}: & \Omega_T(I_2) \arrow[ur] \arrow[dr, "(4,2)"'] & & P_2 , \\
& & P_1 \arrow[ur, "(2,4)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& I_2 \arrow[dr, "(4,2)"] & & \Omega_T^{-1}(P_2) \\
{}_{I_2}\mathfrak{M}: & & \Omega_T^{-1}(P_1) \arrow[ur, "(2,4)"] &
\end{tikzcd}$$

Then the acyclic component $\mathcal{X}(A)$ of Γ_T, obtained by gluing $\Omega_T(\mathcal{Q}(A))$ and $\mathcal{P}(A)$, is of the form

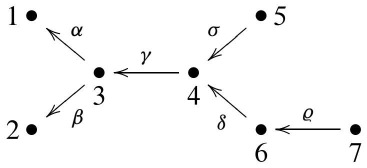
$$\begin{tikzcd}
& & & & & P_T(I_2) \arrow[dr] & & & & \\
\cdots \arrow[dr] & & \Omega_T(\tau_A I_2) \arrow[dr, "(4,2)"] & & \Omega_T(I_2) \arrow[ur] \arrow[dr, "(4,2)"] & & P_2 \arrow[dr, "(4,2)"] & & \tau_A^{-1} P_2 \arrow[dr, dash] & \cdots \\
& \Omega_T(\tau_A I_1) \arrow[ur, "(2,4)"] & & \Omega_T(I_1) \arrow[ur, "(2,4)"] & & P_1 \arrow[ur, "(2,4)"] & & \tau_A^{-1} P_1 \arrow[ur, "(2,4)"] & &
\end{tikzcd}$$

and the acyclic component $\mathcal{Y}(T)$ of Γ_T, obtained by gluing $\mathcal{Q}(A)$ and $\Omega_T^{-1}(\mathcal{P}(A))$, is of the form

$$\begin{tikzcd}
\cdots \arrow[dr] & & \tau_A I_2 \arrow[dr, "{(4,2)}"] & & I_2 \arrow[dr, "{(4,2)}"] & & \Omega_T^{-1}(P_2) \arrow[dr, "{(4,2)}"] & & \Omega_T^{-1}(\tau_A^{-1}P_2) \arrow[dr, dash] & & \cdots \\
& \tau_A I_1 \arrow[ur, "{(2,4)}"] & & I_1 \arrow[ur, "{(2,4)}"] \arrow[dr] & & \Omega_T^{-1}(P_1) \arrow[ur, "{(2,4)}"] & & \Omega_T^{-1}(\tau_A^{-1}P_1) \arrow[ur, "{(2,4)}"] & & {} & \\
& & & & I_T(P_1) \arrow[ur] & & & & & &
\end{tikzcd}$$

We note that $P_T(I_2) = P_T(S_2)$ and $I_T(P_1) = P_T(S_1)$. Moreover, since $\mathcal{R}(A)$ does not contain a simple module, we have $\Omega_T(\mathcal{R}(A)) = \mathcal{R}(A)^+ = \mathcal{R}(A)^- = \Omega_T^{-1}(\mathcal{R}(A))$, which is an infinite family of components of type $\mathbb{Z}\mathbb{A}_\infty$.

Example 8.24. Let K be a field, Δ the quiver



and $H = K\Delta$ the path algebra of Δ over K, considered in Example VII.9.26. Then H is a hereditary K-algebra of wild type, and Δ is the quiver Q_H of H. Moreover, the Auslander–Reiten quiver Γ_H of H has a disjoint union decomposition

$$\Gamma_H = \mathcal{P}(H) \cup \mathcal{R}(H) \cup \mathcal{Q}(H),$$

where $\mathcal{P}(H)$ is the postprojective component of the form $(-\mathbb{N})\Delta^{\mathrm{op}}$

$$\begin{tikzcd}
P_1 \arrow[dr] & & \tau_H^{-1}P_1 \arrow[dr] & & \tau_H^{-2}P_1 \arrow[dr] & & & & & \\
P_2 \arrow[r] & P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^{-1}P_2 \arrow[r] & \tau_H^{-1}P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^{-2}P_2 \arrow[r] & \tau_H^{-2}P_3 \arrow[ur, dash] \arrow[dr] & & & & \cdots \\
& & P_4 \arrow[r] \arrow[ur] \arrow[dr] & P_5 \arrow[r] & \tau_H^{-1}P_4 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^{-1}P_5 \arrow[r] & \tau_H^{-2}P_4 \arrow[r] \arrow[ur, dash] \arrow[dr] & \tau_H^{-2}P_5 & & \\
& & & P_6 \arrow[ur] \arrow[dr] & & \tau_H^{-1}P_6 \arrow[ur] \arrow[dr] & & \tau_H^{-2}P_6 \arrow[ur, dash] \arrow[dr] & & \\
& & & & P_7 \arrow[ur] & & \tau_H^{-1}P_7 \arrow[ur] & & \tau_H^{-2}P_7 \arrow[ur, dash] &
\end{tikzcd}$$

$\mathcal{Q}(A)$ is the preinjective component of the form $\mathbb{N}\Delta^{\mathrm{op}}$

$$\begin{tikzcd}
 & \tau_H^2 I_1 \arrow[dr] & & \tau_H I_1 \arrow[dr] & & I_1 \arrow[dr] & & & & \\
{} \arrow[r] \arrow[ur] & \tau_H^2 I_2 \arrow[r] & \tau_H^2 I_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H I_2 \arrow[r] & \tau_H I_3 \arrow[r] \arrow[ur] \arrow[dr] & I_2 \arrow[r] & I_3 \arrow[dr] & & & \\
 & {} \arrow[ur] & {} \arrow[r, tail] & \tau_H^2 I_4 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^2 I_5 \arrow[r] & \tau_H I_4 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H I_5 \arrow[r] & I_4 \arrow[r] \arrow[dr] & I_5 & \\
\cdots & & {} \arrow[ur] & & \tau_H^2 I_6 \arrow[ur] \arrow[dr] & & \tau_H I_6 \arrow[ur] \arrow[dr] & & I_6 \arrow[dr] & \\
 & & & {} \arrow[ur] & & \tau_H^2 I_7 \arrow[ur] & & \tau_H I_7 \arrow[ur] & & I_7
\end{tikzcd}$$

and $\mathcal{R}(A)$ is a family of components of type $\mathbb{Z}\mathbb{A}_\infty$, containing the component $\mathcal{C}$ of the form

$$\begin{tikzcd}
 & \tau_H S_3 \arrow[dr] & & S_3 \arrow[dr] & & S_4 \arrow[dr] & & R \arrow[dr] & & \tau_H^{-1} R \arrow[dr, dash] & \\
{} \arrow[ur] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr, dash] & & {} \\
 & {} \arrow[ur] & & \bullet \arrow[ur] \arrow[dr, dash] & & \bullet \arrow[ur] \arrow[dr, dash] & & \bullet \arrow[ur] \arrow[dr, dash] & & {} & \\
 & & {} \arrow[ur] & & {} \arrow[ur] & & {} \arrow[ur] & & {} & &
\end{tikzcd}$$

with the simple modules S_3 and S_4 being its quasi-simple regular modules. We also note that the simple modules S_1, S_2 are projective, the simple modules S_5, S_7 are injective, and the simple module $S_6 = \tau_H S_7$ is preinjective.

Consider the trivial extension algebra $T = \mathrm{T}(H) = H \ltimes D(H)$. We will describe the Auslander–Reiten quiver Γ_T of T, using Corollary 8.20. Corollaries 8.12 and 8.13 show that the connecting meshes in Γ_T are of the forms

$$\begin{tikzcd}
 & & & & & & & I_3 \arrow[dr] & \\
\mathfrak{M}_{P_i}\colon & \Omega_T(I_i) \arrow[dr] & & P_i & , & {}_{I_i}\mathfrak{M}\colon & I_i \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_i)\,, \\
 & & \Omega_T(I_3) \arrow[ur] & & & & & I_T(P_i) \arrow[ur] &
\end{tikzcd}$$

for $i \in \{1, 2\}$,

$$\begin{tikzcd}
 & & P_1 \arrow[dr] & \\
\mathfrak{M}_{P_3}\colon & \Omega_T(I_3) \arrow[ur] \arrow[r] \arrow[dr] & P_2 \arrow[r] & P_3\,,\\
 & & \Omega_T(I_4) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Omega_T^{-1}(P_1) \arrow[dr] & \\
{}_{I_3}\mathfrak{M}\colon & I_3 \arrow[ur] \arrow[r] \arrow[dr] & \Omega_T^{-1}(P_2) \arrow[r] & \Omega_T^{-1}(P_3)\,,\\
 & & I_4 \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P_3 \arrow[dr] & \\
\mathfrak{M}_{P_4}\colon & \Omega_T(I_4) \arrow[ur] \arrow[r] \arrow[dr] & \Omega_T(I_5) \arrow[r] & P_4\,,\\
 & & \Omega_T(I_6) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Omega_T^{-1}(P_3) \arrow[dr] & \\
{}_{I_4}\mathfrak{M}\colon & I_4 \arrow[ur] \arrow[r] \arrow[dr] & I_5 \arrow[r] & \Omega_T^{-1}(P_4)\,,\\
 & & I_6 \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P_4 \arrow[dr] & \\
\mathfrak{M}_{P_5}\colon & \Omega_T(I_5) \arrow[ur] \arrow[dr] & & P_5\,,\\
 & & P_T(I_5) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Omega_T^{-1}(P_4) \arrow[dr] & \\
{}_{I_5}\mathfrak{M}\colon & I_5 \arrow[ur] & & \Omega_T^{-1}(P_5)\,,
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P_4 \arrow[dr] & \\
\mathfrak{M}_{P_6}\colon & \Omega_T(I_6) \arrow[ur] \arrow[dr] & & P_6\,,\\
 & & \Omega_T(I_7) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Omega_T^{-1}(P_4) \arrow[dr] & \\
{}_{I_6}\mathfrak{M}\colon & I_6 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_6)\,,\\
 & & I_7 \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & P_6 \arrow[dr] & & & & \Omega_T^{-1}(P_6) \arrow[dr] & \\
\mathfrak{M}_{P_7}\colon & \Omega_T(I_7) \arrow[ur] \arrow[dr] & & P_7\,, & {}_{I_7}\mathfrak{M}\colon & & & \\
 & & P_T(I_7) \arrow[ur] & & & I_7 \arrow[uur] & & \Omega_T^{-1}(P_7)\,.
\end{tikzcd}$$

Then the acyclic component $\mathcal{X}(H)$ of Γ_T, obtained by gluing $\Omega_T(\mathcal{Q}(H))$ and $\mathcal{P}(H)$, is of the form

$$\begin{tikzcd}[column sep=tiny]
 & \Omega_T(\tau_H I_1) \arrow[dr] & & \Omega_T(I_1) \arrow[dr] & & P_1 \arrow[dr] & & \tau_H^{-1} P_1 \arrow[dr] & & & & & \\
{} \arrow[ur] & \Omega_T(\tau_H I_2) \arrow[r] & \Omega_T(\tau_H I_3) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T(I_2) \arrow[r] & \Omega_T(I_3) \arrow[r] \arrow[ur] \arrow[dr] & P_2 \arrow[r] & P_3 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H^{-1} P_2 \arrow[r] & \tau_H^{-1} P_3 \arrow[r, dash] \arrow[ur, dash] \arrow[dr] & {} & & \cdots & \\
 & {} \arrow[ur] & & \Omega_T(\tau_H I_4) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T(\tau_H I_5) \arrow[r] & \Omega_T(I_4) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T(I_5) \arrow[r] \arrow[dr] & P_4 \arrow[r] \arrow[ur] \arrow[dr] & P_5 \arrow[r] & \tau_H^{-1} P_4 \arrow[r] \arrow[ur, dash] \arrow[dr] & \tau_H^{-1} P_5 & & \\
 & \cdots & {} \arrow[ur] & & \Omega_T(\tau_H I_6) \arrow[ur] \arrow[dr] & & \Omega_T(I_6) \arrow[ur] \arrow[dr] & P_T(I_5) \arrow[ur] & P_6 \arrow[ur] \arrow[dr] & & \tau_H^{-1} P_6 \arrow[ur, dash] \arrow[dr] & & \\
 & & & {} \arrow[ur] & & \Omega_T(S_6) = \Omega_T(\tau_H I_7) \arrow[ur] \arrow[dr] & & \Omega_T(I_7) \arrow[ur] \arrow[dr] & & P_7 \arrow[ur] & & \tau_H^{-1} P_7 \arrow[ur, dash] & {} \\
 & & & & & & P_T(S_6) \arrow[ur] & & P_T(I_7) \arrow[ur] & & & &
\end{tikzcd}$$

with $P_T(I_5) = P_T(S_5)$ and $P_T(I_7) = P_T(S_7)$, and the acyclic component $\mathcal{Y}(H)$ of Γ_T, obtained by gluing $\mathcal{Q}(H)$ and $\Omega_T^{-1}(\mathcal{P}(H))$, is of the form

$$\begin{tikzcd}
& & & & I_T(P_1) \arrow[dr] & & & & & & & & \\
& \tau_H I_1 \arrow[dr] & & I_1 \arrow[ur] \arrow[dr] & I_T(P_2) \arrow[dr] & \Omega_T^{-1}(P_1) \arrow[dr] & & \Omega_T^{-1}(\tau_H^{-1} P_1) \arrow[dr] & & & & & \\
\arrow[r] \arrow[ur] & \tau_H I_2 \arrow[r] & \tau_H I_3 \arrow[r] \arrow[ur] \arrow[dr] & I_2 \arrow[r] \arrow[ur] & I_3 \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T^{-1}(P_2) \arrow[r] & \Omega_T^{-1}(P_3) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T^{-1}(\tau_H^{-1} P_2) \arrow[r] & \Omega_T^{-1}(\tau_H^{-1} P_3) \arrow[ur, no head] \arrow[dr] & & & \cdots & \\
& \arrow[ur] & & \tau_H I_4 \arrow[r] \arrow[ur] \arrow[dr] & \tau_H I_5 \arrow[r] & I_4 \arrow[r] \arrow[ur] \arrow[dr] & I_5 \arrow[r] & \Omega_T^{-1}(P_4) \arrow[r] \arrow[ur] \arrow[dr] & \Omega_T^{-1}(P_5) \arrow[r] & \Omega_T^{-1}(\tau_H^{-1} P_4) \arrow[r] \arrow[ur, no head] \arrow[dr] & \Omega_T^{-1}(\tau_H^{-1} P_5) & & \\
& \cdots & \arrow[ur] & & \tau_H I_6 \arrow[ur] \arrow[dr] & & I_6 \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(P_6) \arrow[ur] \arrow[dr] & & \Omega_T^{-1}(\tau_H^{-1} P_6) \arrow[ur, no head] \arrow[dr] & & \\
& & & \arrow[ur] & & \tau_H I_7 \arrow[ur] & & I_7 \arrow[ur] & & \Omega_T^{-1}(P_7) \arrow[ur] & & \Omega_T^{-1}(\tau_H^{-1} P_7) \arrow[ur, no head] &
\end{tikzcd}$$

with $I_T(P_1) = P_T(S_1)$ and $I_T(P_2) = P_T(S_2)$. Further, the family of components $\mathcal{R}(H)^+ = \mathcal{R}(H)^-$ consists of components of type $\mathbb{Z}\mathbb{A}_\infty$, and the component obtained from the component $\Omega_T(\mathcal{C}) = \Omega_T^{-1}(\mathcal{C})$ in Γ_T^{s} by adding the projective modules $P_T(S_3) = I_T(S_3)$ and $P_T(S_4) = I_T(S_4)$ as follows

$$\begin{tikzcd}
& & & & P_T(S_3) \arrow[dr] & & P_T(S_4) \arrow[dr] & & & & \\
& \Omega_T(\tau_H S_3) \arrow[dr] & & \Omega_T(S_3) \arrow[ur] \arrow[dr] & & \Omega_T(S_4) \arrow[ur] \arrow[dr] & & \Omega_T(R) \arrow[dr] & & \Omega_T(\tau_H^{-1} R) \arrow[dr, no head] & \\
\arrow[ur] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr, no head] & & \\
& \arrow[ur] & & \bullet \arrow[ur] \arrow[dr, no head] & & \bullet \arrow[ur] \arrow[dr, no head] & & \bullet \arrow[ur] \arrow[dr, no head] & & & \\
& & \arrow[ur] & & \arrow[ur] & & \arrow[ur] & & & &
\end{tikzcd}$$

9 Exercises

1. The aim of this exercise is to show that the equivalence of Hochschild extensions is finer than the isomorphism of Hochschild extensions.

Let $A = \mathbb{R}[X]/(X^2)$ and $\Lambda = \mathbb{R}[X]/(X^4)$ be the quotient $\mathbb{R}$-algebras of the polynomial algebra $\mathbb{R}[X]$ in one variable X over $\mathbb{R}$ by the ideals (X^2) and (X^4), respectively, and $M = (X^2)/(X^4)$ the ideal of Λ.

(a) Let $\varphi\colon \Lambda \to \Lambda$ be the map defined by

$$\varphi\big(f(X) + (X^4)\big) = f\left(\frac{1}{2}X\right) + (X^4)$$

for any polynomial $f(X)$ of $\mathbb{R}[X]$. Show that φ is an $\mathbb{R}$-algebra isomorphism.

(b) Show that M can be naturally regarded as an A-bimodule.

(c) Let $\mathbb{E}\colon 0 \to M \overset{\omega}{\to} \Lambda \overset{\varrho}{\to} A \to 0$ and $\mathbb{E}'\colon 0 \to M \overset{\omega'}{\to} \Lambda \overset{\varrho}{\to} A \to 0$ be the Hochschild extensions of A by the A-bimodule M such that ϱ is a canonical surjective algebra homomorphism, ω is an inclusion homomorphism, and ω' is a Λ-bimodule homomorphism with $\omega'(x) = \frac{1}{4}x$ for all $x \in M$.

(i) Show that the extensions $\mathbb{E}$ and $\mathbb{E}'$ are isomorphic, that is, there are $\mathbb{R}$-algebra isomorphisms $u\colon A \to A$ and $v\colon \Lambda \to \Lambda$, and a Λ-bimodule isomorphism $w\colon M \to M$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{E}\colon & 0 \arrow[r] & M \arrow[r, "\omega"] \arrow[d, "w"] & \Lambda \arrow[r, "\varrho"] \arrow[d, "v"] & A \arrow[r] \arrow[d, "u"] & 0 \\
\mathbb{E}'\colon & 0 \arrow[r] & M \arrow[r, "\omega'"] & \Lambda \arrow[r, "\varrho"] & A \arrow[r] & 0\,.
\end{tikzcd}$$

(ii) Show that the extensions $\mathbb{E}$ and $\mathbb{E}'$ are not equivalent, that is, it is not possible to replace u and w in (i) by the identities id_A and id_M, respectively.

2. Let L be a finite field extension of K and $\alpha\colon L \times L \to L$ be a 2-cocycle of the K-algebra L by the L-bimodule L.

Prove that the following equalities hold for $[\ ,\ ] = [\ ,\ ]_\alpha$.

(a) For polynomials $f(x), g(x) \in K[x]$ and $a, b \in K$,

$$[f(a), g(b)] = f'(a)g'(b)[a, b],$$

where $f' = \frac{df}{dx}$ and $g' = \frac{dg}{dx}$ are the derivative functions of f and g, respectively.

(b) For natural numbers $l, m \geq 0$ and $f(x) \in K[x]$, we have

$$\begin{aligned}\big[f(a)b^m, a\big] &= f(a)\big[b^m, a\big] = \Big(\frac{\partial}{\partial b}\big(f(a)b^m\big)\Big)[b,a],\\ \big[f(a)b^m, b\big] &= \Big(\frac{\partial}{\partial a}\big(f(a)b^m\big)\Big)[a,b],\end{aligned}$$

where $\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial b}$ denote the partial derivatives.

3. Let K, L be the fields and $\alpha \colon L \times L \to L$ the K-linear map considered in Example 5.11. Show that α is a 2-cocycle.

4. Let $K[X]$ be the polynomial algebra in one variable X over a field K. Let n be a positive integer and $A_n = K[X]/(X^n)$ be the quotient K-algebra of $K[X]$ by (X^n).

(a) Prove that A_n is a symmetric algebra.

(b) Find a bound quiver presentation of the trivial extension algebra $\mathrm{T}(A_n) = A_n \ltimes A_n$.

(c) Prove that A_n is isomorphic to the trivial extension algebra $\mathrm{T}(A_m)$, for a positive integer m, if and only if $n = 2$.

(d) Show that A_{2n} is isomorphic to a non-splittable Hochschild extension algebra of A_n by A_n.

(e) Show that the algebra A_{2n+1} is not isomorphic to any Hochschild extension algebra of an algebra by its duality module.

(f) Show that $H^2(A_n, A_n) \neq 0$ for any integer $n > 1$.

(g) Show that the algebra A_n, for $n > 1$, is not isomorphic to $B \ltimes D(B)_\sigma$ for any finite dimensional K-algebra B and a K-algebra automorphism σ of B.

5. Let $A = K[X, Y]/(X^2, Y^2)$ be the quotient algebra of the polynomial algebra $K[X, Y]$ in two variables X, Y over a field K by the ideal (X^2, Y^2). Let $x = X + (X^2, Y^2)$ and $y = Y + (X^2, Y^2)$ in A.

Show the following statements:

(a) A is a symmetric algebra with a K-basis $\{1, x, y, xy\}$.

(b) For an element $a \in A$, let $a = a_0 + a_1 x + a_2 y + a_3 xy$, where a_0, a_1, a_2, a_3 are elements of K. Then the mapping $\alpha \colon A \times A \to A$ defined by $\alpha(a, b) = a_1 b_2 xy$ for $a, b \in A$ is a 2-cocycle.

(c) The Hochschild extension algebra $\mathrm{T}_A(A, \alpha)$ is a local non-symmetric selfinjective K-algebra.

(d) $\mathrm{T}_A(A,\alpha)$ is isomorphic to the K-algebra $K[X,Y,Z]/I$, where $K[X,Y,Z]$ is the polynomial algebra in three variables X,Y,Z over K, and I is the ideal of $K[X,Y,Z]$ generated by

$$XY - YX - XYZ,\ ZX - XZ,\ YZ - ZY.$$

(e) Let C and J be the subalgebra and the ideal of $\mathrm{T}_A(A,\alpha)$ such that $C = K + Kx + Kz + Kxz$, $J = Ky + Kxy + Kyz + Kxyz$, where x,y,z are the classes of X,Y,Z in $\mathrm{T}_A(A,\alpha)$. We define the K-algebra automorphism σ of C by

$$\sigma(1) = 1,\ \sigma(x) = x - xz,\ \sigma(z) = z.$$

  (a) Show that $A \cong C$ as K-algebras.

  (b) Show that $\mathrm{T}_A(A,\alpha) \cong C \ltimes C_\alpha$ as K-algebras.

  (c) Let τ be the K-algebra automorphism of A such that

$$\tau(1) = 1,\ \tau(x) = x - xy,\ \tau(y) = y,$$

  where x,y are the classes of X,Y in A. Show that $\mathrm{T}_A(A,\alpha) \cong A \ltimes D(A)_\tau$ as K-algebras.

6. Let $A = K[X,Y,Z]/(X,Y,Z)^2$ be the quotient algebra of the polynomial algebra $K[X,Y,Z]$ in three variables X,Y,Z over a field K by the ideal $(X,Y,Z)^2$. Let $x = X + (X,Y,Z)^2$, $y = Y + (X,Y,Z)^2$ and $z = Z + (X,Y,Z)^2$ in A.

Show the following statements:

(a) A has a K-basis $\{1,x,y,z\}$.

(b) For an element $a \in A$, let $a = a_0 + a_1x + a_2y + a_3z$, for elements a_0,a_1,a_2,a_3 of K. Let $\alpha\colon A \times A \to D(A) = \mathrm{Hom}_K(A,K)$ be the map defined by

$$\alpha(a,b) = (a_1b_2 + a_2b_3 + a_3b_1)1^* - (a_2a_3x^* + a_3b_1y^* + a_1b_2z^*),$$

for $a,b \in A$, where $\{1^*,x^*,y^*,z^*\}$ is the dual K-basis of $\mathrm{Hom}_K(A,K)$. Then α is a 2-cocycle.

(c) Let $T = \mathrm{T}_{D(A)}(A,\alpha)$ be a Hochschild extension algebra of A by α. Then T is a local symmetric selfinjective K-algebra.

7. Let K be a field, $\lambda \in K \setminus \{0\}$, and

$$A_\lambda = K\langle X,Y\rangle/(X^2,Y^2,XY - \lambda YX),$$

where $K\langle X,Y\rangle$ be the polynomial algebra in two noncommuting variables X and Y over K (see Example IV.2.8). Let $B = K + Kx$ and $J = Ky + Kxy$.

(a) Show that B is a symmetric K-subalgebra of A and J is an ideal of A.

(b) Let σ be the K-algebra automorphism of B defined by

$$\sigma(1) = 1, \quad \sigma(x) = \lambda^{-1}x.$$

Then show that $B_\sigma \cong J$ as B-bimodules, and $A_\lambda \cong B \ltimes \operatorname{Hom}_K(B, K)_\sigma$ as K-algebras.

(c) Show that A_λ is a Hochschild extension algebra of a finite dimensional K-algebra C by $\operatorname{Hom}_K(C, K)$ if and only if $\lambda = 1$.

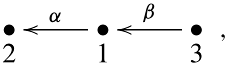
8. Let L be a field and let G be a finite group of automorphisms of L. Let $K = \{x \in L \mid \sigma(x) = x, \text{ for all } \sigma \in G\}$ be the fixed field. For $\sigma \in G$, let $T_\sigma = L \ltimes L_\sigma$ be the trivial extension K-algebra of L by the L-bimodule L_σ.

(a) Show that $T_\sigma \cong T_\tau$ as L-algebras if and only if $\sigma = \tau$.

(b) Show that T_σ weakly symmetric.

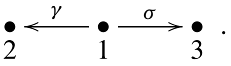
(c) Show that T_σ is symmetric if and only if $\sigma = 1_G$.

9. Let K be a field, Q the quiver



I the ideal in KQ generated by $\beta\alpha$, and $\Lambda = KQ/I$ the associated bound quiver algebra.

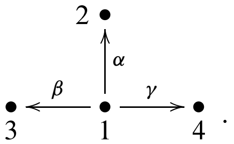
(a) Prove that the trivial extension algebra $\mathrm{T}(\Lambda)$ of Λ is isomorphic to the trivial extension algebra $\mathrm{T}(A)$ of the path algebra $K\Delta$ of the quiver Δ of the form



(b) Find bound quiver presentations of the K-algebras $\mathrm{T}(\Lambda)$ and $\mathrm{T}(A)$.

(c) Describe the Auslander–Reiten quiver $\Gamma_{\mathrm{T}(\Lambda)}$ of $\mathrm{T}(\Lambda)$.

10. Let K be a field and A be the path algebra $K\Delta$ of the quiver Δ of the form



(a) Let σ be the K-algebra automorphism of A such that $\sigma(\alpha) = \beta$, $\sigma(\beta) = \gamma$, $\sigma(\gamma) = \alpha$. Determine a Nakayama permutation of $A \ltimes D(A)_\sigma$.

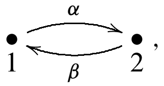
(b) Let σ' be the K-algebra automorphism of A such that $\sigma'(\alpha) = \alpha$, $\sigma'(\beta) = \gamma$, $\sigma'(\gamma) = \beta$. Determine a Nakayama permutation of $A \ltimes D(A)_{\sigma'}$.

(c) Prove that the K-algebras $A \ltimes D(A)$, $A \ltimes D(A)_{\sigma}$, $A \ltimes D(A)_{\sigma'}$ are pairwise nonisomorphic.

(d) Find bound quiver presentations of the algebras $A \ltimes D(A)$, $A \ltimes D(A)_{\sigma}$, $A \ltimes D(A)_{\sigma'}$.

(e) Describe the Auslander–Reiten quivers of the algebras $A \ltimes D(A)$, $A \ltimes D(A)_{\sigma}$, $A \ltimes D(A)_{\sigma'}$.

11. Let K be a field, Q the quiver



I the ideal of KQ generated by $\alpha\beta$ and $\beta\alpha$, and $A = KQ/I$ the associated bound quiver algebra.

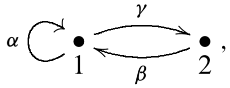
(a) For nonzero elements $\lambda, \mu \in K$, let σ be the automorphism of the K-algebra A defined by $\sigma(\alpha) = \lambda\alpha$, $\sigma(\beta) = \mu\beta$. When is $A \ltimes D(A)_{\sigma}$ isomorphic to $A \ltimes D(A)$ as a K-algebra?

(b) Let σ be the automorphism of the K-algebra A defined by

$$\sigma(e_1) = e_2,\ \sigma(e_2) = e_1,\ \sigma(\alpha) = \beta,\ \sigma(\beta) = \alpha.$$

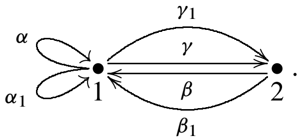
Prove that $A \ltimes D(A)_{\sigma}$ is isomorphic to the bound quiver algebra KQ/J, where J is the ideal of KQ generated by $(\alpha\beta)^2$ and $(\beta\alpha)^2$.

12. Let K be a field, Q the quiver



I the ideal in KQ generated by α^2, $\gamma\beta$, $\beta\alpha$, $\beta\gamma$, and $A = KQ/I$ the associated bound quiver algebra.

(a) Let Δ be the quiver



Find an ideal J in $K\Delta$ such that the K-algebra $K\Delta/J$ is isomorphic to the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

(b) Let I_1 be the ideal in KQ generated by $\alpha^2 - \gamma\beta$, $\beta\gamma$, I_2 the ideal in KQ generated by $\alpha^2 - \gamma\beta$, $\beta\gamma - \beta\alpha\gamma$, $\beta\alpha\gamma\beta$, and let $T_1 = KQ/I_1$, $T_2 = KQ/I_2$ be the associated bound quiver algebras (see Exercise IV.16.36).

(i) Show that there exist non-splittable Hochschild extensions

$$\mathbb{E}_i\colon\ 0 \longrightarrow D(A) \xrightarrow{\omega_i} T_i \xrightarrow{\rho_i} A \longrightarrow 0, \quad i \in \{1,2\}.$$

(ii) Find 2-cocycles $\varepsilon_i\colon A \times A \to D(A)$ such that T_i is isomorphic to $A \ltimes_{\varepsilon_i} D(A)$ as a K-algebra, $i \in \{1,2\}$.

(iii) Prove that the extensions $\mathbb{E}_1$ and $\mathbb{E}_2$ are equivalent if and only if K is of characteristic different from 2.

(iv) Describe the Auslander–Reiten quivers of T_1 and T_2.

13. Let H be the $\mathbb{R}$-subalgebra of the matrix algebra $M_2(\mathbb{C})$

$$\begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \in M_2(\mathbb{C}) \;\middle|\; a \in \mathbb{R}, b, c \in \mathbb{C} \right\},$$

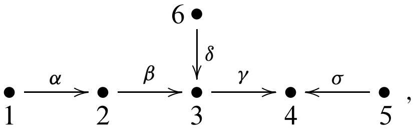
and $A = H \times H \times H$. Determine a basic indecomposable Hochschild extension $\mathbb{R}$-algebra T of A by a duality module Q and the Auslander–Reiten quiver Γ_T of T.

14. Let A be the $\mathbb{R}$-subalgebra of the matrix algebra $M_4(\mathbb{C})$

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{R} & 0 \\ 0 & 0 & \mathbb{R} & \mathbb{R} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ x & b & y & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & z & d \end{bmatrix} \in M_4(\mathbb{C}) \;\middle|\; \begin{matrix} a, b, x, y \in \mathbb{C} \\ c, d, z \in \mathbb{R} \end{matrix} \right\},$$

considered in Example VII.7.10. Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

15. Let K be a field, Q the Dynkin quiver



and $A = KQ$ the path algebra of Q over K (see Example VII.7.11). Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

16. Let K be a field, Q the Dynkin quiver

$$\begin{tikzcd}
\underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} \arrow[r, "\beta"] & \underset{3}{\bullet} & \underset{4}{\bullet} \arrow[l, "\gamma"'] \arrow[r, "\sigma"] & \underset{5}{\bullet} \arrow[r, "\delta"] & \underset{6}{\bullet}
\end{tikzcd},$$

and $A = KQ$ the path algebra of Q over K. Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

17. Let K be a field, Q the Dynkin quiver

$$\begin{tikzcd}
\underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} \arrow[r, "\beta"] & \underset{3}{\bullet} & \underset{4}{\bullet} \arrow[l, "\gamma"'] & \underset{5}{\bullet} \arrow[l, "\delta"']
\end{tikzcd},$$

and $A = KQ$ the path algebra of Q over K. Moreover, let σ be the K-algebra automorphism of A such that $\sigma(e_1) = e_5$, $\sigma(e_2) = e_4$, $\sigma(e_3) = e_3$, $\sigma(e_4) = e_2$, $\sigma(e_5) = e_1$, $\sigma(\alpha) = \delta$, $\sigma(\beta) = \gamma$, $\sigma(\gamma) = \beta$, $\sigma(\delta) = \alpha$, and $T = A \ltimes D(A)_\sigma$. Describe the Auslander–Reiten quiver of T.

18. Let K be a field, Q the Dynkin quiver

$$\begin{tikzcd}
 & & & \bullet\, 4 \arrow[dl, "\gamma"'] \\
\underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l, "\beta"'] & \\
 & & & \bullet\, 5 \arrow[ul, "\delta"']
\end{tikzcd},$$

and $A = KQ$ the path algebra of Q over K. Moreover, let σ be the K-algebra automorphism of A such that $\sigma(e_1) = e_1$, $\sigma(e_2) = e_2$, $\sigma(e_3) = e_3$, $\sigma(e_4) = e_5$, $\sigma(e_5) = e_4$, $\sigma(\alpha) = \alpha$, $\sigma(\beta) = \beta$, $\sigma(\gamma) = \delta$, $\sigma(\delta) = \gamma$, and $T = A \ltimes D(A)_\sigma$. Describe the Auslander–Reiten quiver of T.

19. Let K be a field, $\mathbb{M}$ a K-species with the quiver $Q_{\mathbb{M}}$ of the form

$$\begin{tikzcd}
\underset{1}{\bullet} \arrow[r, "{(1,2)}"] & \underset{2}{\bullet} \arrow[r] & \underset{3}{\bullet} & \underset{4}{\bullet} \arrow[l] \arrow[r] & \underset{5}{\bullet}
\end{tikzcd},$$

and $A = T(\mathbb{M})$ the tensor algebra of $\mathbb{M}$ (see Exercise VII.11.13). Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

20. Let K be a field, Δ the Kronecker quiver

$$\begin{tikzcd}
1\, \bullet & \bullet\, 2 \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\beta"]
\end{tikzcd},$$

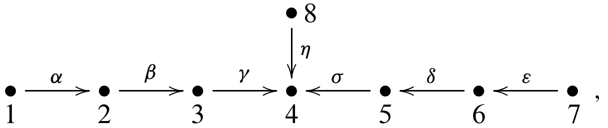
$H = K\Delta$ the path algebra of Δ over K, and $A = H \times H$. Determine a basic indecomposable Hochschild extension K-algebra T of A by a duality module Q and the Auslander–Reiten quiver Γ_T of T.

21. Let A be the $\mathbb{R}$-subalgebra of the matrix algebra $M_5(\mathbb{C})$

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 & 0 \\ 0 & 0 & \mathbb{R} & 0 & 0 \\ 0 & 0 & \mathbb{R} & \mathbb{R} & 0 \\ 0 & 0 & \mathbb{R} & \mathbb{R} & \mathbb{R} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ x & b & y & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & z & d & 0 \\ 0 & 0 & u & v & e \end{bmatrix} \in M_5(\mathbb{C}) \;\middle|\; \begin{matrix} a, b, x, y \in \mathbb{C} \\ c, d, e, z, u, v \in \mathbb{R} \end{matrix} \right\}$$

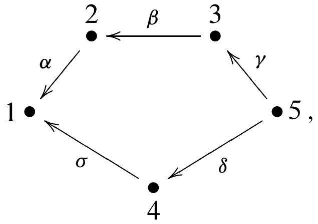
considered in Example VII.8.29. Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

22. Let K be a field, Q the Euclidean quiver



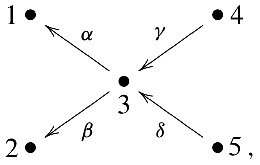
and $A = KQ$ the path algebra of Q over K. Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

23. Let K be a field, Q the Euclidean quiver



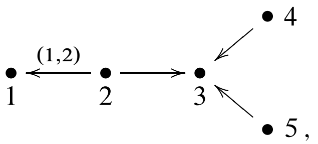
and $A = KQ$ the path algebra of Q over K. Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

24. Let K be a field, Q the Euclidean quiver



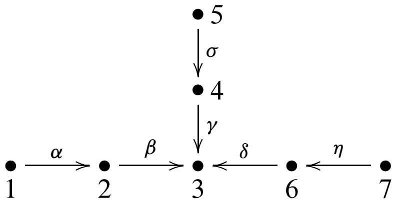
and $A = KQ$ the path algebra of Q over K. Moreover, let σ be the K-algebra automorphism of A such that $\sigma(e_1) = e_2$, $\sigma(e_2) = e_1$, $\sigma(e_3) = e_3$, $\sigma(e_4) = e_5$, $\sigma(e_5) = e_4$, $\sigma(\alpha) = \beta$, $\sigma(\beta) = \alpha$, $\sigma(\gamma) = \delta$, $\sigma(\delta) = \gamma$, and $T = A \ltimes D(A)_\sigma$. Describe the Auslander–Reiten quiver Γ_T of T.

25. Let $\mathbb{M}$ be a $\mathbb{C}$-species with the quiver $Q_{\mathbb{M}}$ of the form



and $A = T(\mathbb{M})$ the tensor algebra of $\mathbb{M}$ (see Exercise VII.11.19). Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

26. Let $A = \mathbb{R}Q$ be the path algebra of the quiver Q of the form



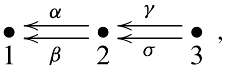
over the field $\mathbb{R}$ (see Exercise VII.11.16). Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

27. Let H be the $\mathbb{R}$-subalgebra of the matrix algebra $M_2(\mathbb{H})$

$$\begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ x & b \end{bmatrix} \in M_2(\mathbb{H}) \;\middle|\; a \in \mathbb{R}, b \in \mathbb{C}, x \in \mathbb{H} \right\},$$

and $A = H \times H \times H \times H$. Determine a basic indecomposable Hochschild extension $\mathbb{R}$-algebra T of A by a duality module Q and the Auslander–Reiten quiver Γ_T of T.

28. Let K be a field, Q the quiver



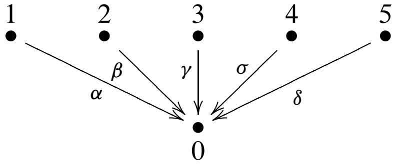
and $A = KQ$ the path algebra of Q over K (see Exercise VII.11.22). Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

29. Let A be the $\mathbb{R}$-subalgebra of the matrix algebra $M_3(\mathbb{H})$

$$\begin{bmatrix} \mathbb{R} & 0 & 0 \\ \mathbb{H} & \mathbb{H} & 0 \\ \mathbb{H} & \mathbb{H} & \mathbb{H} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{bmatrix} \in M_3(\mathbb{H}) \;\middle|\; \begin{matrix} a \in \mathbb{R}, \\ b, c, x, y, z \in \mathbb{H} \end{matrix} \right\}.$$

(see Exercise VII.11.20). Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

30. Let K be a field, Q the quiver



and $A = KQ$ the path algebra of Q over K (see Exercise VII.11.23). Describe the Auslander–Reiten quiver of the trivial extension algebra $\mathrm{T}(A) = A \ltimes D(A)$.

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Andrzej Skowroński
Kunio Yamagata

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