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THE TOPOLOGICAL STRUCTURE OF AUSLANDER REITEN QUIVERS OF SPECIAL STRING ALGEBRAS

Christoph Thiele

Department of Mathematics
Yale University
New Haven, CT 06520, USA

INTRODUCTION

Let Λ be a finite dimensional, representation finite algebra over an algebraically closed field and denote by Γ_Λ its Auslander Reiten quiver. The translation quiver Γ_Λ carries the structure of a two dimensional simplicial complex, called the Auslander Reiten complex of Λ.

In the case that Λ is a special string algebra, we obtain in this way a compact Riemann surface with non-empty boundary. The aim of this paper is to prove:

Theorem 1 *Given any compact Riemann surface S with non-empty boundary, there is a special string algebra $\Lambda(S)$ whose Auslander Reiten complex is homeomorphic to the Riemann surface S.*

Recall that a compact Riemann surface S with boundary is uniquely determined by the three invariants orientability, number of boundary components, and reduced characteristic. The latter is the Euler characteristic less the number of boundary components, and takes a value in $\{-2, 0, 2, 4, \dots\}$ if S is orientable, and a value in $\{-1, 0, 1, 2, \dots\}$ if S is not orientable.

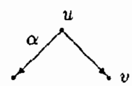
Hence theorem 2 below proves theorem 1 by giving an explicit construction of the string algebras needed in theorem 1.

1. CONSTRUCTION OF THE ALGEBRAS

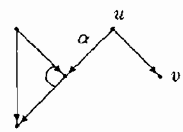
The algebras of theorem 2 are given by quivers with relations. A relation $\phi\psi$ is denoted by $\phi \frown \psi$.

We will call any of the following four quivers with relations a *moon*. For later reference an arrow and two points are named in each diagram.

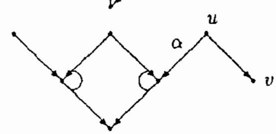
new moon:



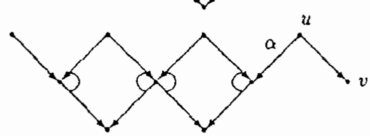
half moon:



full moon:



double moon:



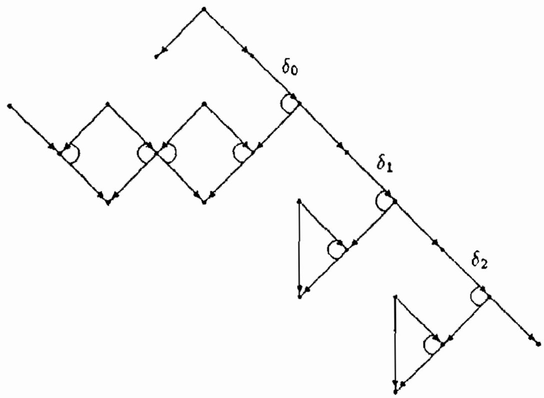
Let n be a non-negative integer and $\Lambda_0, \ldots, \Lambda_n$ a sequence of copies of moons. The following construction will be called an admissible gluing. Take the disjoint union of the quivers with relations $\Lambda_0, \ldots, \Lambda_n$, add for each successive pair Λ_i, Λ_{i+1} an arrow δ_i that starts in the point v of the copy Λ_i and ends in the point u of the copy Λ_{i+1}, and add for each new arrow δ_i the zero relation $\alpha\delta_i$ with the arrow α of the copy Λ_{i+1}.

An example of admissible gluing is shown below, with $n = 3$, Λ_0 isomorphic to the new moon, Λ_1 isomorphic to the double moon and Λ_2, Λ_3 isomorphic to the half moon.

Theorem 2 *Let $\Lambda_0, \ldots, \Lambda_n$ be a sequence of copies of moons containing h copies of the half moon, f copies of the full moon, and d copies of the double moon.*

Then admissible gluing yields a special string algebra, whose Auslander Reiten complex has $f + 1$ boundary components, reduced characteristic $h + 2d - 2$, and is orientable if and only if $h = 0$.

Note that the invariants of the Auslander Reiten complex occuring in the theorem do not depend on the number of copies of the new moon. But we need the new moon to get a non-empty sequence which does not contain a copy of the other moons.



2. CELL COMPLEXES

In the context of this paper, the suitable description of compact Riemann surfaces is by cell complexes. Of course the invariants of the cell complex have to be viewed as the invariants of the described Riemann surface. For the details of the connection between cell complexes and Riemann surfaces we refer to [1].

Definition 1 (Cell complex) *A cell complex system is given by a finite set of edges and a finite set of ordered triples of pairwise different edges. We call such an ordered triple a triangle.*

We demand that any edge is contained in exactly one or exactly two triangles. In the first case we call the edge outer edge, in the second case inner edge.

A cell complex is a non-empty, connected cell complex system. Connected means that the cell complex it is not the disjoint union of two non-empty cell complex systems.

It should be stressed that in this definition the triangles abc have a well defined first edge a, second edge b, and third edge c.

We fix a cell complex system.

If abc is a triangle, we call the ordered pairs of edges ab, bc and ca just edge pairs. There is a finest equivalence relation on the set of all edge pairs such that two edge pairs ab and $a'b'$ are equivalent whenever a is equal to a' or b is equal to b', and we call the equivalence classes with respect to this equivalence relation vertices of the cell complex system. If ab is an edge pair, we say that the edge a ends, and the edge b starts in the vertex $[ab]$. As usual we call an edge incident with a vertex if it starts or ends in the vertex.

We call the cell complex system orientable if we can simultanously assign to each triangle a value 0 or 1 such that each inner edge occurs in two triangles with different values, otherwise we call the cell complex system non-orientable.

A boundary component of the cell complex system consists of an indexed set $\{a_i\}_{i\in \mathbb{Z}/n\mathbb{Z}}$ of outer edges a_i, where $\mathbb{Z}/n\mathbb{Z}$ denotes the standard cyclic group with n elements. If n is equal to one, we demand that a_1 starts and ends in the same vertex. If n is equal to two, we demand that a_1 and a_2 are incident with two common vertices. Finally, if n is greater than two, we demand that the edges a_i and a_{i+2} are different and the edges a_i and a_{i+1} are incident with a common vertex for all i in the cyclic group. We do not distinguish between boundary components which consist of the same set of outer edges. If there is at least one boundary component, we say that the cell complex system has non-empty boundary.

If v, s and f denote the numbers of vertices, edges, and triangles of the cell complex, the Euler characteristic of the cell complex is $-v+s-f$.

3. SPECIAL STRING ALGEBRAS

Let k be an algebraically closed field and Q a finite quiver, consisting of points and arrows between these points. A relation in the path algebra kQ^* of the opposite quiver is given by a path of length at least two in the opposite quiver. Hence it is of the form $\beta_1\beta_2\ldots\beta_n$, where the β_i are arrows in Q, such that the starting point of β_i is the end point of β_{i+1} for all admissible indices i. Let P be a set of relations and denote by $\langle P\rangle$ the two sided ideal in kQ^* generated by P.

By abuse of language, we do not distinguish between the quiver with relations and the algebra $\Lambda = kQ^*/\langle P\rangle$.

Definition 2 (String algebra) *The algebra $\Lambda = kQ^*/\langle P\rangle$ is called a string algebra, provided the following three conditions are satisfied:*

1 *Any point of the quiver is starting point of at most two arrows, and it is end point of at most two arrows.*

2 *Given any arrow β of the quiver, there is at most one arrow γ such that $\beta\gamma$ is a path not in P, and there is at most one arrow γ' such that $\gamma'\beta$ is a path not in P.*

3 *There is a bound n such that any path of length n contains a subpath of P.*

The algebra Λ is called special string algebra, if in addition the following three conditions are satisfied.

4 *The algebra is representation finite.*

5 *The quiver is connected and contains at least one arrow.*

6 *If β,γ are arrows of the quiver and $\beta\gamma$ is a path, then there is an arrow α, not necessarily different from β and γ, such that $\alpha\gamma$ or $\beta\alpha$ is a path not in P.*

It is easy to verify that the four moons are examples of special string algebras.

As we are mainly interested in the structure of the Auslander Reiten quiver, rather than in any other aspect of representation theory, we use synonymously the term 'module' for the points of the Auslander Reiten quiver, and the term 'irreducible map' for the arrows of the Auslander Reiten quiver. For every pair X, Y of modules such that Y is the Auslander Reiten translate of X, we introduce an arrow, called translator, from X to Y. In diagrams the irreducible maps are visualized by solid arrows, and the translators by broken arrows.

The Auslander Reiten quiver of a special string algebra can be viewed as a cell complex, the Auslander Reiten complex. There the edges are the irreducible maps and the translators, and the triangles are the ordered triples fgt consisting of two irreducible maps f and g and a translator t, such that f starts in the end module of t, g starts in the end module of f, and t starts in the end module of g. The connectedness of the Auslander Reiten quiver in this case is a well known fact.

Note that for general string algebras we have to distinguish between the points of the Auslander Reiten quiver and the vertices of the Auslander Reiten complex, but we will show that, in the presence of condition 6. of the definition of special string algebras, they are in natural correspondence.

4. STRINGS

We fix a special string algebra.

Consider an alphabet whose letters are the arrows of the quiver and formal inverses of these arrows. Generally we use small greek letters for the arrows and write $\alpha^{-1}, \beta^{-1} \ldots$ for their inverses. It is understood that we get back an arrow, if we invert its inverse. A sequence of not necessarily distinct letters of this alphabet, written from left to right, constitutes a word of the length of the sequence. We compose words by starting one word at the end of the other, and the subwords of a word may be obtained by iterated decomposition of the word.

An inverse of a word is formed by inverting all letters and reversing the order of the inverted letters. We call a word direct if it contains only arrows as letters and antidirect if it contains only inverses of arrows as letters. Note that the paths of Q^* may be regarded as direct strings. For example for the words $A = \gamma\nu\omega^{-1}\theta\iota$ of length five and $B = \sigma\epsilon\alpha\upsilon\tau o^{-1}\nu$ of length seven we have the composition $AB = \gamma\nu\omega^{-1}\theta\iota\sigma\epsilon\alpha\upsilon\tau o^{-1}\nu$, the inverse $A^{-1} = \iota^{-1}\theta^{-1}\omega\nu^{-1}\gamma^{-1}$, and the direct subword $\alpha\upsilon$ of B.

We understand the starting point and the end point of an inverse of an arrow to be the end point and the starting point of the arrow. It should be pointed out that in our context one does not get into trouble with intuition if one imagines that the starting point of a letter is on the right side of the letter and the end point on the left side of the letter.

Definition 3 (Strings) *A proper string is a word which satisfies the following conditions.*

1. *The end point of the right letter of any subword of length two is the starting point of its left letter.*

2. *No subword of length two consists of two letters which are inverses of each other.*

3. *No subword is a path in P or the inverse of a path in P.*

For every letter we introduce a start string and an end string of the letter, both are given the length zero. Two such strings are defined to be equal, if they are equivalent with respect to the finest equivalence relation, such that for any proper string of length two the end string of its right letter is equivalent to the start string of its left letter. We call start and end strings just zero strings.

Clearly the map which assigns to start string (end string) of every letter the starting point (end point) of the letter, is well defined. We call the zero strings which are mapped to a given point the strings of this point. By going through the cases which are allowed by the conditions 1,2,5, and 6 in the definition of a special string algebra, we get the following lemma:

Lemma 4 *Each point of the quiver has two zero strings.*

Define the start string of a proper string to be the start string of the letter at the right end of the string, and the end string of a proper string the end string of the letter at the left end of the string. Start and end string of a zero string are the string itself.

If the start string of a string A is the end string of a string B, we define the composition AB as follows. If both strings are proper, we compose the words, and if one string has zero length, we define the composition to be the other string. There is no ambiguity in the case that both strings have zero length, because then the strings are equal. Note that the composition of two proper strings fails to be a string, if the composed word contains paths of P or their inverses. As in the case of words substrings are defined by iterated decomposition.

By symmetry of the definitions the inverse word of a proper string is again a string. We define the two zero strings of a point to be inverses of each other. Then any composed string AB is equal to the inverse of the string $B^{-1}A^{-1}$. We define zero strings to be both direct and antidirect.

Note that we have defined essentially the same strings as Butler and Ringel in [2]. Especially, we can reformulate the condition of representation finiteness of a special string algebra into the condition, that there are only finitely many strings.

We fix a zero string Z. In any string, which has end string Z, we replace every arrow by the latin letter a and every inverse of an arrow by the latin letter i, and we add at the right end of the word the latin letter e. Then the resulting words of the latin alphabet may be compared by the usual lexicographical order.

By the observation that, given a string, there is at most one arrow which might be added at the right end and at most one arrow whose inverse might be added at the right end, we get with the described order:

Lemma 5 *The strings ending with a fixed zero string are totally ordered.*

We say that a string C is greater than a string D, if both have the same end string and the latin word of C comes in lexicographical order before the latin word of D.

For every zero string Z there is a unique maximal string ending with Z. If a string A is not maximal, there is a unique lowest string that is greater than C, the successor of C. Analogously we have minimal strings and predecessors.

We have the following description of successors and maximal strings.

Lemma 6 *Let C be a string.*

1 *If there is an arrow β such that $C\beta$ is a string, then the successor of C is of the form $C\beta A$, where A is the minimal string ending in the start string of β.*

2 *If there is no arrow β such that $C\beta$ is a string, and if C contains an inverse of a string, then C is of the form $C_+\gamma^{-1}D$, where C_+ is the successor of C, γ an arrow, and D the maximal string ending in the start string of γ^{-1}.*

3 *If there is no arrow β such that $C\beta$ is a string, and if C contains no inverse of an arrow, then C is maximal.*

The analogous lemma holds for predecessors and minimal strings.

5. The AUSLANDER REITEN QUIVER IN TERMS OF STRINGS

The following description of the Auslander Reiten quiver of a special string algebra may be found for example in [2].

The points of the Auslander Reiten quiver, which we have called modules, may be viewed as the pairs of inverse strings. By the module of a string C we mean the pair C, C^{-1} .

Recall that by 'irreducible map' we mean an arrow of the Auslander Reiten quiver. For every string C which is not maximal, there is an irreducible map starting in the module of C and ending in the module of the successor of C. We get all irreducible maps in this way, and the parametrization is unique, which means that, if C and its inverse have successors, the modules of the two successors are different.

The module of a string C is not the end point of a translator if and only if the string C is the composition AD of two string A and D such that A^{-1} and D are maximal. Note that $C^{-1} = D^{-1}A^{-1}$ then has an analoguous decomposition. We call such modules and their strings injective.

The module of a string C is not starting point of a translator if and only if the string C is the composition DA of two strings D and A such that D^{-1} and A are minimal. We call such modules and their strings projective.

Any module is end module of at most one translator, hence we may parametrize the translators by their end modules, which are exactly the non-injective modules. Analogously, we might parametrize the translators by the non-projective modules.

Let the irreducible map f be parametrized by the non-injective string C. Then there is a unique triangle fgt which contains f as first edge. The irreducible map g in this triangle is parametrized by the inverse of the successor of C. Analogously, every irreducible map which ends in a non-projective module is the second edge of a unique triangle.

6. VERTICES AND MODULES

We show in this section that the abstractly defined vertices of the Auslander Reiten complex can be identified with the modules, such that start and end vertex of an irreducible map or translator are the start and end module. In the remaining sections, we will call the vertices of the Auslander Reiten complex synonymously modules.

We start with a vertex pair ab. Going through the definitions shows that the end module of a is equal to the start module of b. We denote this module by $M(ab)$. Obviously, if two edge pairs coincide in the first or the second edge, they are mapped by M to the same module, hence M may be viewed as a map from the set of vertices to the set of modules.

We have to show that M is a bijection.

Assume that M is not surjective. This means that there is a string C whose module is neither start nor end module of any irreducible map, because otherwise this irreducible map would be contained in a triangle, and this triangle would contain an edge pair that is a preimage of the module.

Hence C and its inverse are maximal and minimal, and by lemma 6 the string C does not contain any letter and there is no letter whose start or end string it is. But this is impossible for a string.

Hence M is surjective and it remains to show that it is injective.

We enumerate the steps of the proof.

0. It is enough to show that all edges which start in a given module start in the same vertex, because then all edge pairs, which are mapped to this module, belong to the same equivalence class.

1. We note that if C is not injective, a translator ends in the module of C, and any irreducible map starting in this module is the first edge of a triangle which contains this translator, hence the irreducible map starts in the end vertex of this translator.

2. Assume that C is projective, then no translator starts in the module of C, and by 1. we are done if C is not injective.

Hence we can assume that C is the composition AD of an antidirect string A and a maximal direct string D. As C is also projective, one of the two substrings must have length zero, and by replacing C by its inverse string if necessary, we can assume that it is equal to D. Then it is maximal and at most one irreducible map starts in its module. By this we have proved 0.

3. By 0., 1., 2., and the dual arguments, we may now assume that C is neither projective nor injective. We are going to show that there is an irreducible map, starting in the module of C, which starts in the same vertex as the translator starting in the module of C. This will complete the proof by 1.

4. In 5. and 6., we will find an irreducible map starting in the module of C which ends in a non-projective module. Then this map g is the second edge of a triangle. The first edge of this triangle is an irreducible map ending in the start vertex of g, and hence ending in the module of C. But by the dual argument of 1. this map also ends in the start vertex of the translator that starts in the module of C, and we are done by 3.

5. First assume that the string C is proper. By passing to its inverse if necessary, we find that it contains an inverse of an arrow, and consequently its successor is defined. If the successor was projective, it would be a composition DA of a direct string D and a minimal antidirect string A. By lemma 6 the string C would be a substring of the direct string D in contradiction to the fact that C contains an inverse of an arrow.

6. Now assume that C has length zero, and let it belong to the point u of the quiver Q. As C is not projective, we find an arrow β such that $C\beta^{-1}$ or βC is a string. Hence β starts in u . Dually we find an arrow γ that ends in u.

By condition **6** in the definition of a special string algebra we may assume that $\beta\gamma$ is not a relation in P, and hence a string. By replacing C by its inverse if necessary, we find that $C\gamma$ is defined. Then the successor of C has the form γA with a minimal string A, and is not projective because the composition $\beta\gamma A$ is a string.

7. ORIENTABILITY

By orientation of a letter, string, edge or triangle we mean a value 0 or 1, which is assigned to the letter, string, edge or triangle.

Lemma 7 (Orientability)
The Auslander Reiten complex of a special string algebra is orientable if and only if orientations of all letters exist simultanously, such that inverse letters have different orientations, and both letters of any string of length two have the same orientation.

Suppose first that orientations of the letters are given with the properties demanded in the lemma. Then given any proper string, all of its letters have the same orientation, because any two successive of its letters have the same orientation. By taking this value we get an orientation of the string.

Furthermore, we define the orientation of start string and end string of a letter to be the same as the orientation of the letter. As both letters of a string of length two have the same orientation, this gives the same orientation for equivalent start and end strings. Therefore the orientation is well defined on the set of zero strings.

Note that the successor of a string has the same orientation as the string itself, because one of these two strings is a substring of the other. Obviously inverse strings have different orientations.

Now we define for every irreducible map an orientation by taking the orientation of the string by which the irreducible map is parametrized. The second edge of a triangle is an irreducible map parametrized by the inverse of the succesor of the string, by which the first edge is parametrized, which is also an irreducible map. Hence the edges in the first and second position of a triangle have different orientations.

Let any triangle be oriented by the orientation of its first edge.

A triangle is determined by its first edge as well as by its second edge. Hence if an irreducible map is contained in two triangles, it occurs in one triangle as the first edge and in the other triangle as the second edge. Hence the two triangles have different values.

If a translator is contained in two triangles, the two different irreducible maps in the first positions of the triangles start both in the same module, and are parametrized by inverse strings. Consequently, the two irreducible maps have different orientations, and the two triangles have different orientations.

Hence these orientations of the triangles show that the Auslander Reiten complex is orientable.

Now suppose that all triangles are oriented such that each inner edge occurs in two triangles with different orientations.

We essentially reverse the steps of the proof above. First we define unambiguously orientations of the irreducible maps as follows. If an irreducible map occurs in the first position of a triangle, it has the same orientation as the triangle, if it occurs in the second position, it has the orientation different from that of the triangle.

Now we construct a section for the map which assigns to every string C its module. With this section every edge will have a start string and an end string defined as the image under this section of the start module and the end module.

Let f be the irreducible map which is parametrized by a string C. Our aim is that by the desired section the irreducible map, if it has the orientation 1, starts in C and ends in the successor of C, and if it has the orientation 0, starts in C^{-1} and ends in the inverse of the successor of C.

We start with an arbitrary edge pair ab. If a is an irreducible map, we assign to this edge pair the string in which a should end according to the list above, if b is an irreducible map, we assign to the edge pair the string in which b should start according to the list above. If both edges are irreducible maps, these two strings coincide.

We want to show that any two equivalent edge pairs are mapped to the same string. It is enough to show this for two edge pairs which have a common first or second edge. If this common edge is an irreducible map, the assertion is clear. Consider two different edge pairs which have a common translator as first or second edge. By duality, we may assume that this is the first edge. Then both second edges are irreducible maps and occur in the first position of the triangles which contain the edge pairs. The two different triangles have different orientations, because both contain the translator. Hence the two irreducible maps have different orientations.

As the two irreducible maps start in the same module, they are parametrized by inverse strings. Now it is clear that the strings to which the edge pairs are mapped are equal.

With that we have found our desired section. Let a string have the orientation 1, if it is in the image of the constructed section, and the orientation 0 otherwise.

The section is constructed in a way, that the successor of a string has the same orientation as the string. Hence all strings with a fixed end string have the same orientation. If two strings have the same orientation, their inverses have also the same orientation, hence by the previous statement all strings with a fixed start string have the same orientation. By these two statements clearly any substring of a string has the same orientation as the string.

Hence we have constructed orientations of the strings of length one, which are the letters, such that the conditions of the lemma are satisfied.

8. EULER CHARACTERISTIC

Recall that the number of arrows of a quiver less the number of points of the quiver is called the Euler characteristic of the quiver.

Lemma 8 (Euler characteristic)
The Euler characteristic of the Auslander Reiten complex of a special string algebra is equal to the Euler characteristic of the quiver of the algebra.

The number of modules of the cell complex is clearly half the number of strings, the latter we denote by S .

The irreducible maps are parametrized by the strings which are not maximal. To count the maximal strings note that every zero string has a unique maximal string ending in this zero string, and that we can of course step back from the maximal string by just taking the end string. As the number of zero strings is twice the number $|Q_0|$ of points of the quiver, we have exactly $S-2|Q_0|$ irreducible maps.

The translators are parametrized by the modules that are not injective. We count instead of injective modules the injective strings, which are clearly twice as many as the injective modules. Note that for every zero string we have the composition AD, where D is the maximal string that ends in the zero string, and A is the inverse of the maximal string that ends in the inverse of the zero string. The composition is clearly an injective string, especially there are no subwords which are paths or inverses of paths of P, because D is a direct string and A is an antidirect string. Again we can step back from the injective string to the zero string as follows. If the injective string contains an arrow, the zero string is the end string of the unique direct substring with maximal length of the injective string, otherwise the zero string is the start string of the injective string.

Hence we have exactly $S - 2|Q_0|$ strings which are not injective, and consequently $S/2 - |Q_0|$ translators.

The translators that are contained in only one triangle, are uniquely parametrized by the arrows of the quiver, as explained in the next section. All other translators are contained in two triangles, and as each triangle contains exactly one translator, the number of triangles is twice the number of translators less the number $|Q_1|$ of arrows.

Now the Euler characteristic

$$-(S/2) + (S - 2|Q_0| + S/2 - |Q_0|) - (S - 2|Q_0| - |Q_1|)$$

is easily simplified to $|Q_1| - |Q_0|$.

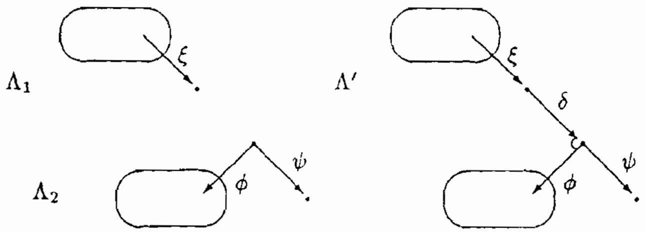
9. NUMBER OF BOUNDARY COMPONENTS

In the prove of our theorem we use the following lemma, which allows in very special cases to get the number of boundary components of the Auslander Reiten complex by induction.

Lemma 9 (Induction lemma)
Assume that a special string algebra Λ_1 contains an arrow ξ, such that no arrow starts, and no arrow except ξ ends in the end point of ξ. Assume that a further special string algebra Λ_2 contains an arrow ψ such that no arrow starts, and no arrow except ψ ends in the end point of ψ, and that no arrow ends in the starting point of ψ, but a second arrow ϕ starts in this point.

Connect the two algebras by an arrow δ, which starts in the end point of ξ and ends in the starting point of ψ, and add the relation $\phi\delta$. The result is again a special string algebra Λ', and its Auslander Reiten complex has one boundary component fewer than the union of the Auslander Reiten complexes of Λ_1 and Λ_2.

The following diagram visualizes the situation.



Note that there is no easy generalization of this lemma, because if two arbitrary special string algebras are connected with an arrow and appropriate relations, such that the result is again a special string algebra, the conclusion of the lemma is not necessarily true.

Proof of the induction lemma:

First of all, we check that Λ' is a special string algebra. Clearly the quiver of Λ' is non-empty and connected, because this holds for the quivers of the algebras Λ_1 and Λ_2. The conditions 1,2 and 6 in the definition of a special string algebra might only be violated for the starting point or end point of the new arrow δ and for arrows which start or end in these points, but there these conditions are obviously fulfilled.

To show that Λ' is representation finite, we have to show that it has only finitely many strings. Note that whenever the arrow δ occurs in a string of Λ', the substring to the left of the position where δ occurs has either length zero or is equal to ψ. Hence the new arrow δ occurs at most once in any string, and, by inverting the string, this is also true for δ^{-1}. Hence any string of Λ' is the composition of at most twice the letter δ or δ^{-1} and at most three further strings of Λ_1 or Λ_2. As these two algebras have only finitely many strings, the same is now true for the algebra Λ'.

Condition 3 follows in general from condition 4, because the paths that contain no subpath in the set of relations may be viewed as strings.

Hence we have proved that Λ' is a special string algebra.

Recall that we have assigned in the previous section to each zero string of a special string algebra a maximal and an injective string. Assign additionally to every zero string the minimal string which ends in this zero string and the projective string DA, where A is the minimal string of the given zero string and D is the inverse of the minimal string of C^{-1}, where C is the given zero string.

The following description of the outer edges may be found in [2]. There are three different kinds of outer edges of an Auslander Reiten complex, the irreducible maps which start in an injective module, the irreducible maps which end in a projective module and the translators which are contained in only one triangle. Each of these kinds is uniquely parametrized by the arrows of the string algebra. More precisely, we get for each arrow β three outer edges $(\beta, 1)$, $(\beta, 2)$, and $(\beta, 3)$ as follows, where D denotes the start string of β and A the start string of β^{-1}.

$(\beta, 1)$ starts in the module of the injective string of A, and ends in the module of the maximal string of D.

$(\beta, 2)$ starts in the module of the minimal string of A, and ends in the module of the projective string of D.

$(\beta, 3)$ starts in the module of the minimal string of D, and ends in the module of the maximal string of A.

We return to the special situation of the induction lemma.

To simplify notation, we introduce the disjoint union of the quivers with relations Λ_1 and Λ_2, which is again a string algebra Λ. Although this is not a special string algebra, we can apply all our definitions to get a cell complex system, which is the disjoint union of the Auslander Reiten complexes of Λ_1 and

Λ_2. In the following, objects of the cell complex system of Λ are for simplicity referred to as the objects of Λ.

Clearly every string of Λ may be viewed as a string of Λ', and it has the same inverse in Λ and Λ'. Hence we may identify the modules of Λ with modules of Λ'.

We introduce four critical modules N_1, N_2, N_3 and N_4, which are modules of both algebras and can be described as follows.

N_1 is the module of the maximal string of Λ that ends in the end string of ξ, (This string is also maximal w.r.t. Λ')

N_2 is the module of the end string of ξ,

N_3 is the module of the start string of ψ,

N_4 is the module of the string ψ.

The following lemma will reduce the proof of the induction lemma to the computation of a few outer edges and some simple combinatorics.

If β is an arrow of Λ, the symbol (β, i) may be interpreted as an edge of Λ or Λ'. In order to distinguish we write $(\beta, i)'$ in the second case.

Lemma 10 *Let ζ, θ be arrows of Λ and $i, j \in \{1,2,3\}$. Assume that both (ζ, i) and (θ, j) are incident with a non-critical module M.*

If M is the start module of (ζ, i), let M_1 be the start module of $(\zeta, i)'$, otherwise let M_1 be the end module of $(\zeta, i)'$. If M is the start module of (θ, j), let M_2 be the start module of $(\theta, j)'$, otherwise let M_2 be the end module of $(\theta, j)'$.

Then $M_1 = M_2$

Proof of lemma 10:

Note that any zero string of Λ, one time viewed as string in Λ and the other time viewed as string in Λ', gives two maximal, two minimal , two injective, and two projective strings. These strings may, but need not be different.

1. Assume that the two maximal strings of a zero string are different. Then the maximal string in Λ, denoted by C, cannot be maximal in Λ'. Hence its composition $C\delta$ with the new arrow must be a string in Λ'. Consequently C is either ψ or the start string of ψ, and the module of C is critical.

2. Assume that the two injective strings of a zero string are different. Then the injective string in Λ, denoted by C, cannot be injective in Λ', hence one of the compositions $C\delta$ or $\delta^{-1}C$ must be a string in Λ'. Again it follows that the module of C is critical.

3. Assume that the two projective strings of a zero string are different, and that the projective string in Λ , denoted by C, is minimal, but the projective string in Λ' not. Then the composition δC must be a string in Λ', and hence C ends with the end string of ξ. As this end string is already projective in Λ, it is equal to C and consequently the module of C is critical.

As a consequence of the classification of the outer edges, there is a zero string, namely the start string of ζ or ζ^{-1}, such that the modules M and M_1 are given by the two maximal, minimal, injective or projective strings of this zero string.

The same is true for the modules M and M_2.

If by our identification of the modules of Λ and Λ' the module M is equal to both M_1 and M_2, we are done. Hence without loss of generality, we may assume that M is not equal to M_1.

As M is not critical, we have by 1. and 2. that M and M_1 are not the modules of the two maximal or injective strings of the same zero string.

Assume first that M and M_1 are the modules of the two minimal strings of the same zero string C. Then the minimal string in Λ, which we denote by C, is not minimal in Λ', and the compositions $C\delta^{-1}$ and its inverse δC^{-1} are strings in Λ'. Then the end string of C^{-1} is the end string of ξ, and as the module M of C is not critical, we conclude that C^{-1} contains the substring ξ at its left end.

Then C^{-1} is not injective, because otherwise it would be the maximal string of the end string of ξ, and its module would be critical. By the same reason C^{-1} is not maximal, and neither is C maximal, because it contains ξ^{-1}.

We have seen that M is not the module of a maximal or injective string, hence it remains that M and M_2 may be the modules of the two minimal or projective strings of the same zero string. In the first case, as C^{-1} contains an arrow, the minimal string in Λ must again be C, and then both M_1 and M_2 are the modules of the minimal strings of the end string of C in Λ'. Assume the second case. Then clearly M and M_2 are also the modules of the projective strings of the inverse zero string, hence we may assume that the projective string in Λ is C, and hence it is the projective string of the end string of C. As C is minimal, it follows by 3. that the projective string in Λ' is also minimal, and again M_1 and M_2 are the modules of the minimal strings of the end string of C in Λ'. Hence $M_1 = M_2$.

Now assume that M and M_1 are the modules of the two projective strings of the same zero string. Let C denote the projective string in Λ. If C is either direct or antidirect, we may again pass to the inverse zero string if necessary and assume, that C is minimal. Then by 3. the projective string in Λ' is also minimal, and clearly both minimal strings have the same end string. But this is the previous case.

Hence we assume that C is neither direct nor antidirect. Clearly C and C^{-1} are neither maximal nor minimal nor injective, and hence M and M_2 are also the modules of the projective strings of the same zero string. Again, by inverting the strings if necessary, this zero string is the same as for M and M_1, hence we are done.

We have proved lemma 10 and return to the proof of the induction lemma.

Note the general fact, that every module, which is start or end module of an outer edge, is exactly twice the start or end module of an outer edge. Hence the following diagrams, which show incidences between modules and outer edges of Λ, give a complete overview of all outer edges of Λ, which start or end in a critical module.

The arrow χ in the first diagram is assumed to be an arrow such that χC^{-1} is a string, where C denotes the maximal string ending in the end string of ξ. If such an arrow exists, it is unique, if it does not exist, the edge $(\chi, 3)$ in the diagram has to be replaced by the edge $(\chi', 2)$ with the opposite direction, where χ' is the arrow which is contained in the proper string C as the letter at the right end. The prove of this second case goes analoguously to the following one.

1 $$\xleftarrow{(\chi,3)} N_1 \xrightarrow{(\xi,1)}$$

2 $$\xrightarrow{(\xi,3)} N_2 \xrightarrow{(\xi,2)}$$

3 $$\xrightarrow{(\phi,1)} N_3 \xleftarrow{(\psi,1)} N_4 \xrightarrow{(\phi,3)}$$

Recall that a boundary component is a cyclically indexed set of outer edges. We call a subset of successive outer edges of a boundary component an arc of the boundary component. Clearly every arc has a complementary arc, such that the disjoint union of both gives the whole boundary component.

Note that each outer edge occurs in exactly one boundary component and has two neighbours there. These neighbours are exactly the edges which are incident with a common module with this edge.

Hence with the list above, we have for Λ the arcs 1. $(\chi,3)(\xi,1)$, 2. $(\xi,3)(\xi,2)$, and 3. $(\phi,1)(\psi,1)(\phi,3)$.

All outer edges of Λ' are parametrized by pairs that contain an arrow of Λ, except the three outer edges $(\delta,i)'$ for i in $\{1,2,3\}$. These three outer edges occur in the following diagrams which show incidence relationships in Λ'.

The diagrams contain modules M_1, M_2 and M_3 of Λ', which can be described as follows.

M_1 is the module of the maximal string that ends in the end string of ψ,

M_2 is the module of the maximal string that ends in the end string of δ, and

M_3 is the module of the string $\psi\delta$.

4 $$\xleftarrow{(\chi,3)'} M_1 \xrightarrow{(\psi,1)'} M_2 \xrightarrow{(\delta,1)'} N_1 \xrightarrow{(\xi,1)'}$$

5 $$\xrightarrow{(\xi,3)'} N_2 \xrightarrow{(\delta,3)'} N_3 \xleftarrow{(\phi,1)'}$$

6 $$\xleftarrow{(\phi,3)'} N_4 \xrightarrow{(\delta,2)'} M_3 \xrightarrow{(\xi,2)'}$$

Again the diagrams may be read as arcs of Λ'.

Now we identify each outer edge (β,i) of Λ with the outer edge $(\beta,i)'$ of Λ'. Then an outer edge which does not occur in the diagrams above, is an outer edge of both Λ and Λ', and it has by lemma 10 the same neighbours in both cell complex systems, because it is not incident in Λ with a critical module.

The six outer edges which occur at the left and right ends of the diagrams also have by lemma 10 at their free ends in both cell complex systems the same neighbours.

It follows that the boundary components of Λ' can be constructed from the boundary components of Λ by replacing the arcs of the first three diagrams by the arcs of the second three diagrams, such that each edge that occurs at free ends of the arcs has in the boundary components of Λ and Λ' the same continuation at the free end.

Clearly the first arc is just replaced by the fourth arc, and this does not change the number of boundary components.

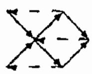
Recall that Λ is the disjoint union of Λ_1 and Λ_2. The the second arc belongs to Λ_1, and the third to Λ_2. Hence the second and the third arc have two independent complementary arcs, by which they are completed to two different boundary components. These two complementary arcs, one of them might have changed by the replacing of the first by the fourth arc, are in the cell complex of Λ' crosswise connected by the fifth and sixth arc. Hence the two boundary components become one boundary component.

Hence Λ' has one boundary component fewer than Λ, and we have proved the induction lemma.

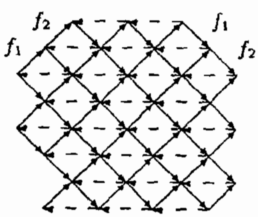
10. PROOF OF THE THEOREM

The Auslander Reiten quivers of the four moons are shown in the following diagrams, where the translators are given by broken arrows. In each diagram a few arrows have to be identified. These arrows are named.

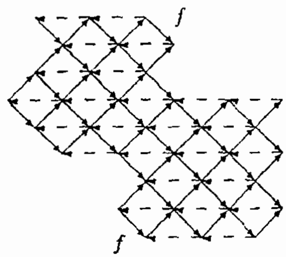
new moon:



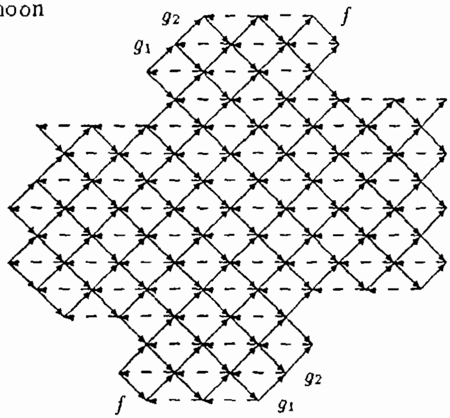
half moon:



full moon:



double moon



Clearly the four moons are special string algebras. We determine the invariants of their Auslander Reiten complexes.

In the defining diagrams of the new moon, full moon, and double moon, all arrows have either south east or south west direction. Let each arrow in south east direction have the orientation 1, each arrow in south west direction the orientation 0, and let their inverses have the opposite orientation. Then it is easy to check that this orientation of the letters satisfies in each case the conditions demanded in the orientability lemma 7. Hence the Auslander Reiten complexes of these moons are orientable.

To get a contradiction assume, that the Auslander Reiten complex of the half moon is orientable. Then there is an orientation of the letters of the half moon, satisfying the conditions of the orientability lemma 7. We consider the triangle of arrows in the diagram of the half moon. One of the two arrows of the triangle, whose composition is a zero relation, may be composed with α to a string, the other may be composed with α^{-1} to a string. Hence the two arrows have different orientations. But both may be composed with the inverse of the third arrow in the triangle to a string of length two, a contradiction.

The Euler characteristics of the Auslander Reiten complexes of the new moon, half moon, full moon, and double moon are in this order -1, 0, 0, and 1. As each of these complexes has at least one boundary component, by the possible values for the reduced characteristics, we see immediately that the complexes of the new moon, half moon, and full moon have in this order 1, 1, and 2 boundary components. For the Auslander Reiten complex of the double moon , we count explicitely in the diagram above one boundary component.

Now we come to the algebras of the theorem. If the sequence of algebras has length one, the constructed algebra is just one of the moons, and we are done with the results above. If the sequence $\Lambda_0, \ldots, \Lambda_n$ contains more than one copy, we can assume by induction, that the theorem is true for the algebra Λ constructed from the sequence $\Lambda_0, \ldots, \Lambda_{n-1}$. We see immediately that we are exactly in the situation of the induction lemma 9 if we connect the algebra Λ with the algebra Λ_n to the the algebra Λ' of the sequence $\Lambda_0, \ldots \Lambda_n$. Hence Λ' is a string algebra.

With the induction lemma we see that the number of boundary components of the Auslander Reiten complex of Λ' is equal to that of the Auslander Reiten complex of Λ, except if the Λ_n is a copy of the full moon. Then the Auslander Reiten complex of Λ' has one boundary component more than that one of Λ, which gives by induction the number of boundary components stated in the theorem.

To get the Euler characteristic of Λ', we have to add the Euler characteristics of Λ and Λ_n, and to increase the sum by one, the latter being caused from the connecting arrow. This gives easily the reduced characteristics stated in the theorem.

The statement about orientability in the theorem is proved with completely the same arguments as above for the special cases of the moons.

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