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Group representations without groups

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If k is an algebraically closed field of characteristic $p>0$ and G a finite group, we know by Dade [3], Janusz [7] and Kupisch [8] that blocks of the group-algebra $k[G]$ with cyclic defect groups are Morita-equivalent to algebras arising from Brauer trees [7]. Here we show that the latters coincide with the algebras, which are stably equivalent to symmetric Nakayama-algebras (as suggested by M. Auslander we call an algebra Nakayama if it is generalized uniserial in the sense of Nakayama, i.e. if it has finite dimension and every indecomposable module has only one composition series; for stable equivalence we refer to [1] or to 1.2 below).

1. The main results

1.1 We first recall that a *quiver* consists in vertices and in arrows connecting these vertices together. In Fig. 1 and Fig. 2 we give two concrete examples. The first of these quivers is the so-called *cycle* Z_e with e vertices.

If k is a field, we get a *k-representation* V of a quiver Q by attaching a k-vector space $V(i)$ to each vertex i and a k-linear map $V(\alpha): V(i) \to V(j)$ to

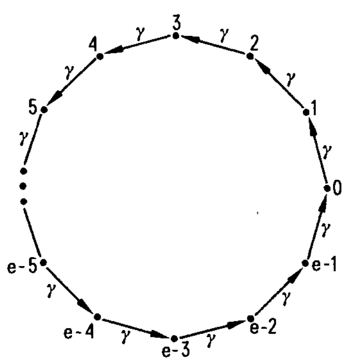


Figure 1

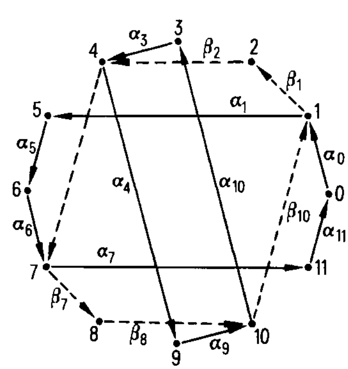


Figure 2

each arrow $\alpha: i \to j$ of Q. The dimension of V is by definition $\dim V = \sum_i \dim V(i)$. In practice, we shall have to restrict our representations by some constraints, which we describe now: A *composed arrow* from i to j is a sequence of vertices and arrows $(j \mid \alpha_n, \ldots, \alpha_1 \mid i)$ with $n \geq 0$, $i = \text{domain}\ (\alpha_1)$, $\text{range}\ (\alpha_m) = \text{domain}\ (\alpha_{m+1})$ for $1 \leq m < n$ and $\text{range}\ (\alpha_n) = j$; in case $n = 0$ we further require that $i = j$. The composed arrows from i to j freely generate a vector space $k(i, j)$, and the formal composition law given by $(m \mid \beta_p, \ldots, \beta_1 \mid j) \cdot (j \mid \alpha_n, \ldots, \alpha_1 \mid i) = (m \mid \beta_p, \ldots, \beta_1, \alpha_n, \ldots, \alpha_1 \mid i)$ clearly induces bilinear composition maps $k(i, j) \times k(j, m) \to k(i, m)$, $(\alpha, \delta) \mapsto \delta\alpha$. An *ideal* I of Q consists in subspaces $I(i, j) \subset k(i, j)$ such that $\beta I(i, j) \subset I(i, m)$ for any $\beta: j \to m$ and $I(i, j)\gamma \subset I(h, j)$ for any $\gamma: h \to i$.

The pair (Q, I) is called a *bounden quiver*. A *k-representation* of (Q, I) is a k-representation V of Q subjected to the supplementary condition $V(\alpha) = 0$, $\forall \alpha \in I(i, j)$ (where $V(\alpha)$ is defined in the obvious way, when α is a linear combination of composed arrows!). Of course, it suffices to submit V to the relations $V(\alpha_s) = 0$, where α_s runs through a family of generators of I. Therefore, if $(\alpha_s)_{s \in S}$ is such a family, we simple say that (Q, I) is the *bounden quiver defined by Q and the relations $\alpha_s = 0$, $s \in S$*. The category of its k-representations is denoted by $\text{Mod}_k\ (Q, I)$.

For instance, we denote by Z_e^h the *cycle of height* $\leq h+1$ with e vertices, i.e. the bounden quiver defined by Z_e and the e possible relations $\gamma^{h+1} = 0$. A k-representation of Z_e^h is called an *h-wreath of e vector spaces*. Besides Z_e^h we also consider the bounden quiver (Q, I) defined by the quiver Q of example 2 and the following relations:

$$
\begin{aligned}
0 &= \alpha_0\alpha_{11}\alpha_7\alpha_6\alpha_5\alpha_1\alpha_0\alpha_{11}\alpha_7\alpha_6 = \alpha_5\alpha_1\alpha_0\alpha_{11}\alpha_7\alpha_6\alpha_5\alpha_1\alpha_0\alpha_{11} \\
&= \alpha_1\beta_{10} = \beta_1\alpha_0 = \beta_7\beta_4\beta_2\beta_1 = \beta_2\beta_1\beta_{10}\beta_8 = \alpha_3\alpha_{10}\alpha_9 = \alpha_4\alpha_3\alpha_{10} \\
&= \alpha_4\beta_2 = \beta_4\alpha_3 = \alpha_7\alpha_6\alpha_5\alpha_1\alpha_0\alpha_{11}\alpha_7\alpha_6\alpha_5\alpha_1 = \alpha_6\alpha_5\alpha_1\alpha_0\alpha_{11}\alpha_7\alpha_6\alpha_5\alpha_1\alpha_0 \\
&= \alpha_{11}\alpha_7\alpha_6\alpha_5\alpha_1\alpha_0\alpha_{11}\alpha_7\alpha_6\alpha_5 = \alpha_7\beta_4 = \beta_7\alpha_6 = \beta_1\beta_{10}\beta_8\beta_7 = \beta_8\beta_7\beta_4\beta_2 \\
&= \alpha_9\alpha_4\alpha_3 = \alpha_{10}\alpha_9\alpha_4 = \alpha_{10}\beta_8 = \beta_{10}\alpha_9 = \alpha_1\alpha_0\alpha_{11}\alpha_7\alpha_6\alpha_5\alpha_1\alpha_0\alpha_{11}\alpha_7 \\
&= \alpha_6\alpha_5\alpha_1\alpha_0\alpha_{11}\alpha_7\alpha_6\alpha_5\alpha_1 + \beta_4\beta_2\beta_1 = \alpha_9\alpha_4 + \beta_8\beta_7\beta_4 \\
&= \alpha_0\alpha_{11}\alpha_7\alpha_6\alpha_5\alpha_1\alpha_0\alpha_{11}\alpha_7 + \beta_{10}\beta_8\beta_7 = \alpha_3\alpha_{10} + \beta_2\beta_1\beta_{10} = 0.
\end{aligned}
$$

1.2. One of our main theorems, stated in the particular case of the bounden quiver just defined, will be that $\text{Mod}_k\ (Q, I)$ is *stably equivalent* to $\text{Mod}_k Z_{12}^{18}$. Stable equivalence is defined as follows: first remember that the *stable category* $\bar{\mathscr{C}}$ attached to an abelian category $\mathscr{C}$ has the same objects as $\mathscr{C}$, and that the set of morphisms $\overline{\text{Hom}}\,(M, N)$ from M to N in $\bar{\mathscr{C}}$ consists in equivalence classes of morphisms from M to N in $\mathscr{C}$. More precisely, two morphisms $f, g : M \to N$ in $\mathscr{C}$ are considered as equivalent, if $g - f$ is factorized through a projective object. A *stable equivalence* is an equivalence between the stable categories.

If M is an object of $\mathscr{C}$, it will be convenient to write $\bar{M}$ for M considered as an object of $\bar{\mathscr{C}}$. If M is non-projective and has a local ring of endomorphisms, the quotient $\overline{\text{Hom}}\,(\bar{M}, \bar{M})$ of $\text{Hom}\,(M, M)$ is also local, hence $\bar{M}$ is indecomposable in $\bar{\mathscr{C}}$. In the cases we consider, the stable category $\bar{\mathscr{C}}$ therefore inherits from $\mathscr{C}$ the property that each object is a finite direct sum of indecomposable summands with local rings of endomorphisms. In particular, the map $M \mapsto \bar{M}$ induces a bijection between the *types* (= isomorphism classes) of non-projective indecomposable objects of $\mathscr{C}$ and the types of indecomposables of $\bar{\mathscr{C}}$.

The indecomposable objects of $\text{Mod}_k Z_e^h$ are easy to describe: Denote by $\varepsilon_0, \ldots, \varepsilon_h$ the natural basis of k^{h+1}. For any natural numbers s, i, m such that $0 \le s,\ i \le e-1$ and $1 \le m \le h+1$ set $V_{s,m}(i) = \bigoplus_j k\varepsilon_j$, where j is subjected to the conditions $0 \le j < m$ and $s + j \equiv i \mod e$. Connect the spaces $V_{s,m}(0), \ldots, V_{s,m}(e-1)$ by linear maps $\gamma = V(\gamma)$ such that $\gamma\varepsilon_j = \varepsilon_{j+1}$ if $j \le m-2$ and $\gamma\varepsilon_{m-1} = 0$. Thus we get an *indecomposable* wreath of length m, and it is well known that every h-wreath of e vector spaces is a direct sum of such representations $V_{s,m}$. Moreover $V_{s,m}$ is projective (and injective) in $\text{Mod}_k Z_e^h$ iff $m = h+1$.

In the example considered above we see, as a consequence of the existence of a stable equivalence between $\text{Mod}_k Z_{12}^{18}$ and $\text{Mod}_k\,(Q, I)$, that (Q, I) admits $12 \cdot 18$ types of non-projective indecomposable k-representations.

1.3. Let Q denote now a general quiver with finitely many vertices. We define the *quiver-algebra* of Q as

$$k[Q] = \bigoplus_{i,j} k(i, j),$$

the multiplication being defined in such a way, that

$$(m \mid \beta_p, \ldots, \beta_1 \mid h)(j \mid \alpha_n, \ldots, \alpha_1 \mid i) \quad \text{equals} \quad 0 \quad \text{if} \quad h \neq j$$

and $(m \mid \beta_p, \ldots, \beta_1, \alpha_n, \ldots, \alpha_1 \mid i)$ if $h = j$. The unit element is $\sum_i (i \| i)$. If V is a k-representation of Q, $\bigoplus_i V(i)$ bears an obvious (left-) module structure over $k[Q]$, and the functor $V \mapsto \bigoplus_i V(i)$ is a k-linear equivalence between $\text{Mod}_k Q$ and $\text{Mod}\, k[Q]$ (we denote by $\text{Mod}\, A$ the category of left modules over a given k-algebra A). If I is an ideal of Q, the "restricted" representations of the bounden quiver (Q, I) correspond under this equivalence to the modules annihilated by $\oplus_{i,j} I(i, j)$, i.e. to the modules over the *bounden quiver-algebra*

$$k[Q, I] := k[Q] / \bigoplus_{i,j} I(i, j).$$

In case $(Q, I) = Z_e^h$ we thus get algebras $k[Z_e^h]$, which are known to be representatives for all selfinjective split basic Nakayama-algebras over k (a finite dimensional k-algebra A with radical J is called *split basic* if $A/J \tilde{\to} k \times \cdots \times k$). The vector space wreaths furnish a suitable interpretation for the modules over these algebras.

1.4. Our purpose is to classify the finite-dimensional k-algebras A for which there exists a k-linear stable equivalence between Mod A and $\mathrm{Mod}_k Z_e^h \tilde{\to} \mathrm{Mod}\, k[Z_e^h]$ for some e, h. In this classification we need quivers of the following special kind:

We say that a quiver Q is a *Brauer-quiver* iff it is finite and connected and has the following properties:

a) Q is the union of the cycles which are contained in Q;

b) every vertex belongs to exactly two cycles;

c) any two cycles meet in one vertex at most. For the sake of illustration we give two examples (Fig. 3 and Fig. 4). Readers wishing to make themselves familiar with the notion may verify that there are 1 Brauer-quiver-type with 1 vertex, 1 with 2 vertices, 2 with 3, 3 with 4, 6 with 5, 14 with 6, 33 with 7 . . . ! With every Brauer-quiver Q we associate a tree T: its vertices correspond bijectively to the cycles of Q, and two vertices of T are matched together by a (non-oriented) edge iff the corresponding cycles of Q meet. Hence the edges of T

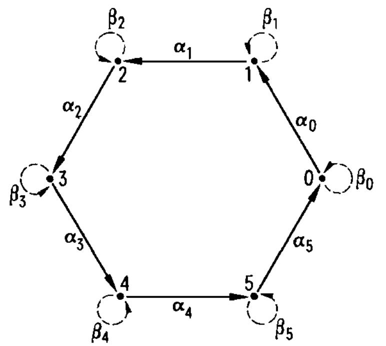


Figure 3

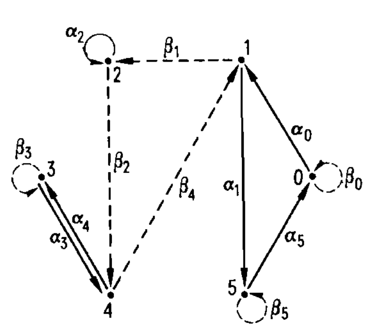


Figure 4

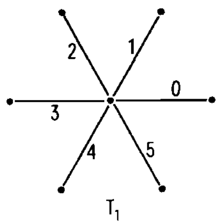


Figure 5

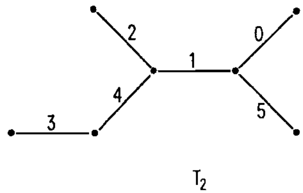


Figure 6

correspond to the vertices of Q. Moreover the edges of T converging at some common point correspond to the distinct vertices of some cycle of Q. They are therefore endowed with a cycle order, which we put in concrete form by drawing T in a plane in such a way that the edges converging at any vertex have the anticlockwise cyclic order. A tree endowed with such cyclic orderings is a *Brauer-tree.* Clearly, Q determines T and reversely. We draw explicitely the trees T_1 and T_2 attached to Q_1 and Q_2.

1.5. The cycles of a Brauer-quiver may be divided into two camps, an α- and a β-camp, in such a way that neighbouring cycles belong to different camps. We implicitely suppose in the sequel that one among the two possible camps has been baptized α, the other β; we say that Q is *oriented.*

For any vertex i we denote by αi and βi the terminal points of the α- and β-arrows starting at i. Thus we get two permutations $\alpha : i \mapsto \alpha i$ and $\beta : i \rightarrow \beta i$ of the vertices of Q.

LEMMA. *$\gamma = \alpha\beta$ is a cyclic permutation of the vertices of Q.*

Proof. If β is the identity, $\gamma = \alpha$ acts transitively, since Q is connected (example Q_1). Similarly, if α is the identity, $\gamma = \beta$ is transitive. In the other cases it is easily seen that there is a vertex s such that $\alpha s \neq s \neq \beta s$. Take for instance the

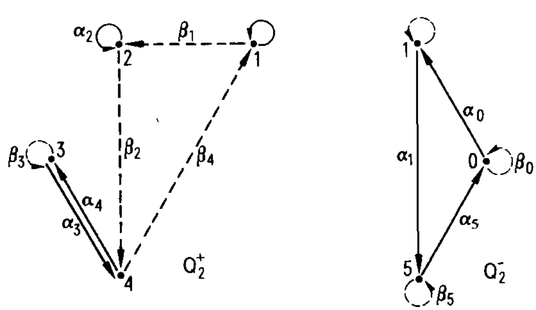


Figure 7

vertex 1 in Q_2 (1.4). Then s may be considered as the connecting vertex of two quivers Q^+ and Q^-, which we make more explicit by simply drawing them in case of Q_2.

We may now use induction on the number of vertices and get γ by "matching together" the cyclic permutations γ^+ and γ^- of Q^+ and Q^-. OK.

Let h be a natural number ≥ 1 and set $\exp(2i\pi x/h) = e_h(x)$ (or simply $e(x)$ if there is no danger of confusion). In case $s = e(x)$ and $t = e(y)$ with $0 < y - x < h$, we agree that $[s, t] = e([x, y])$ and similarly $[s, t[= e([x, y[) \ldots$ In the sequel, if Q is an oriented Brauer-quiver with h vertices, we shall identify the set of vertices with $\{e(0), e(1), \ldots, e(h-1)\} \subset \mathbf{S}_1 \subset \mathbf{C}$ in such a way that $\gamma e(i) = e(i+1)$ for each i. The inductive method used in the proof of the previous lemma leads us then to the following capital observation: if $s \neq \alpha s$ for some vertex s, the α-orbit of any $t \in]s, \alpha s[$ is contained in $]s, \alpha s[$; similarly, the β-orbit of any $t \in [s, \alpha s[$ is contained in $[s, \alpha s[$. An analogous result is obtained by permuting α and β: if $s \neq \beta s$, the β-orbit of any $t \in]s, \beta s[$ is contained in $]s, \beta s[$, whereas the α-orbit of any $t \in]s, \beta s]$ is contained in $]s, \beta s]$.

In the identification we have chosen, the vertices of Q delimit a regular polygon. We represent the α-arrows by full lines, the β-arrows by dotted lines. The observations made above then imply that the α-arrows joining the vertices of an α-orbit are the edges of the convex hull of this α-orbit. Moreover, the convex hulls of distinct α-orbits do not intersect. As a matter of fact, *the datum of an oriented Brauer quiver with h vertices is essentially equivalent to the datum of an equivalence relation on* $\{e(0), \ldots, e(h-1)\}$ *such that the convex hulls of two distinct equivalence classes do not intersect* (given such a relation, define αs as the first point equivalent to s coming after s in the anticlockwise orientation of the circle; then set $\beta e(i) = \alpha^{-1} e(i+1)$).

1.6. The universal covering $\tilde{Q}$ of an oriented Brauer-quiver Q is a quiver having $\mathbf{Z}$ as set of vertices. As for Q, the arrows of $\tilde{Q}$ are associated with permutations of the vertices, which we still call α and β and which are characterized as follows: $1 \leq \alpha i - i \leq h$, $e_h(\alpha i) = \alpha e_h(i)$, $1 \leq \beta i - i \leq h$ and $e_h(\beta i) = \beta e_h(i)$ for all $i \in \mathbf{Z}$. The permutations α and β of $\mathbf{Z}$ determine arrows $\alpha_i : i \to \alpha i$ and $\beta_i : i \to \beta i$ respectively. We illustrate our definitions by drawing *some* of the arrows in the cases of Q_1 and Q_2 (1.4):

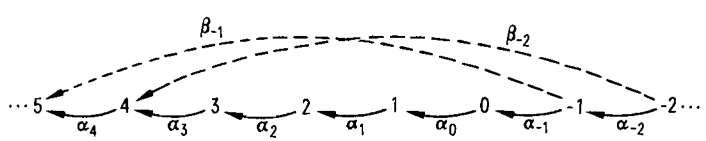


Figure 8

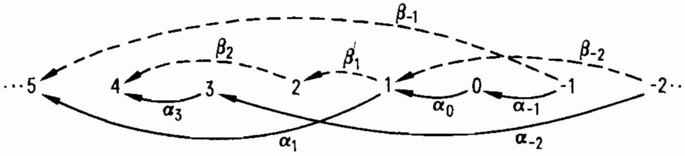


Figure 9

The properties already proved for oriented Brauer quivers infer the following relations among the permutations α and β of $\mathbf{Z}: \alpha\beta = \gamma^{h+1}$, $\alpha\gamma^h = \gamma^h\alpha$ and $\beta\gamma^h = \gamma^h\beta$, where $\gamma(i) = i+1$. Moreover, if we denote by ai and bi the cardinalities of the α- and β-orbits of $e(i)$ in Q, we have $\alpha^{ai}(i) = i+h = \beta^{bi}(i)$.

We endow the universal covering $\tilde{Q}$ with the relations

$$\beta_{\alpha i}\alpha_i = 0 \quad \text{and} \quad \alpha_{\beta i}\beta_i = 0$$

for every $i \in \mathbf{Z}$ and with

$$\alpha^{ai} + \beta^{bi} = 0,$$

where α^{ai} stands for the composition of ai arrows of type α, the first of which starts at $i \in \mathbf{Z}$; a similar definition holds for β^{bi}. In this way we get an infinite bounden quiver $(\tilde{Q}, \tilde{I})$.

In case $Q = Q_1$, the relations $\alpha^{ai} + \beta^{bi} = 0$ in $(\tilde{Q}_1, \tilde{I}_1)$ reduce to $\alpha^6 + \beta = 0$, or more explicitely to

$$\alpha_{i+5}\alpha_{i+4}\alpha_{i+3}\alpha_{i+2}\alpha_{i+1}\alpha_i + \beta_i = 0.$$

In a k-representation V of $(\tilde{Q}_1, \tilde{I}_1)$ the maps $V(\beta_i): V(i) \to V(i+6)$ are therefore uniquely determined by the maps $V(\alpha_i)$; moreover, the relations $V(\beta_{\alpha i})V(\alpha_i) = 0$, $V(\alpha_{\beta i})V(\beta_i) = 0$ may be reinterpreted in terms of the $V(\alpha_i)$ as follows

$$0 = V(\alpha_{i+6})V(\alpha_{i+5})V(\alpha_{i+4})V(\alpha_{i+3})V(\alpha_{i+2})V(\alpha_{i+1})V(\alpha_i), \qquad \forall i \in \mathbf{Z}.$$

This furnishes an isomorphism of $\mathrm{Mod}_k\,(\tilde{Q}_1, \tilde{I}_1)$ with $\mathrm{Mod}_k\,\tilde{Z}_6$, where $\tilde{Z}_6$ is the bounden quiver defined by

$$\cdots 6 \xleftarrow{\gamma} 5 \xleftarrow{\gamma} 4 \xleftarrow{\gamma} 3 \xleftarrow{\gamma} 2 \xleftarrow{\gamma} 1 \xleftarrow{\gamma} 0 \xleftarrow{\gamma} -1 \cdots$$

and the relations $\gamma^7 = 0$. Replacing 6 by any $h \in \mathbf{N}$ and the relations $\gamma^7 = 0$ by

$\gamma^{h+1}=0$ we get a bounden quiver denoted by $\tilde{Z}_h$, which generalizes $\tilde{Z}_6$ in a very obvious way.

1.7. Let Q be an oriented Brauer-quiver with h vertices. We construct a functor $R:\mathrm{Mod}_k \tilde{Z}_h \to \mathrm{Mod}_k(\tilde{Q},\bar{I})$ as follows. If V is a k-representation of $\tilde{Z}_h$, we set

$$(RV)(t)=V(t)\oplus V(\beta t)\oplus\cdots\oplus V(\beta^{bt-1}t)=\bigoplus_{0\le i<bt} V(\beta^i t).$$

Moreover, for any vertex $t\in\mathbf{Z}$ of $\tilde{Q}$, the k-linear map

$$(RV)(\beta)\colon (RV)(t)=\bigoplus_{0\le i<bt} V(\beta^i t)\to \bigoplus_{0<j\le bt} V(\beta^j t)=(RV)(\beta t)$$

is given by the matrix

$$B=\left[\begin{array}{cccc} 0 & \mathbf{1} & 0 & 0\cdots \\ 0 & 0 & \mathbf{1} & 0\cdots \\ 0 & 0 & 0 & \mathbf{1}\cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\gamma^{h} & -\gamma^{h-\beta t+t} & -\gamma^{h-\beta^2 t+t} & -\gamma^{h-\beta^3 t+t}\cdots \end{array}\right]$$

where we simply write γ instead of $V(\gamma)$. Similarly, the k-linear map

$$(RV)(\alpha)\colon (RV)(t)=\bigoplus_{0\le i<bt} V(\beta^i t)\to \bigoplus_{0\le j<b\alpha t} V(\beta^j \alpha t)$$

is given by the matrix

$$A=\left[\begin{array}{ccc} \gamma^{\alpha t-t} & \gamma^{\alpha t-\beta t} & \gamma^{\alpha t-\beta^2 t}\cdots \\ 0 & 0 & 0\cdots \\ 0 & 0 & 0\cdots \\ \cdots & \cdots & \cdots \end{array}\right]$$

THEOREM. *If the maps $V(\gamma)$ satisfy the relations $V(\gamma)^{h+1}=0$, then $(RV)(\beta)$ and $(RV)(\alpha)$ satisfy the relations of the bounden quiver $(\tilde{Q},\bar{I})$. The functor $R:\mathrm{Mod}_k \tilde{Z}_h\to\mathrm{Mod}_k(\tilde{Q},\bar{I})$, which is thus defined, maps projectives into projectives* *and induces by passing to the residual categories a stable equivalence*

$$\bar{R} : \overline{\mathrm{Mod}_k}\, \tilde{Z}_h \xrightarrow{\sim} \overline{\mathrm{Mod}_k}\, (\tilde{Q}, \tilde{I}).$$

This theorem will be proved in §4.

1.8. Let Q be an oriented Brauer-quiver with the h vertices

$$\mathrm{e}_h(0),\ \mathrm{e}_h(1), \ldots, \mathrm{e}_h(h-1) \in \mathbf{C}.$$

A natural number $e \geq 1$ is called a *period* of Q if we have

$$\alpha \mathrm{e}_h(i+e) = (\alpha \mathrm{e}_h(i))\mathrm{e}_h(e) \quad \text{and} \quad \beta \mathrm{e}_h(i+e) = (\beta \mathrm{e}_h(i))\mathrm{e}_h(e), \forall_i.$$

It is equivalent to say that the induced permutations of the universal covering $\tilde{Q}$ satisfy

$$\alpha(i+e) = \alpha(i) + e \quad \text{and} \quad \beta(i+e) = \beta(i) + e$$

for any $i \in \mathbf{Z}$. Clearly, h and any multiple of h is a period of Q. But there may be other periods: in the following example (Fig. 10), where $h = 18$, the periods are the multiples of 6.

Given a period e of Q, we define an *e-periodic k-representation* of $(\tilde{Q}, \tilde{I})$ as a

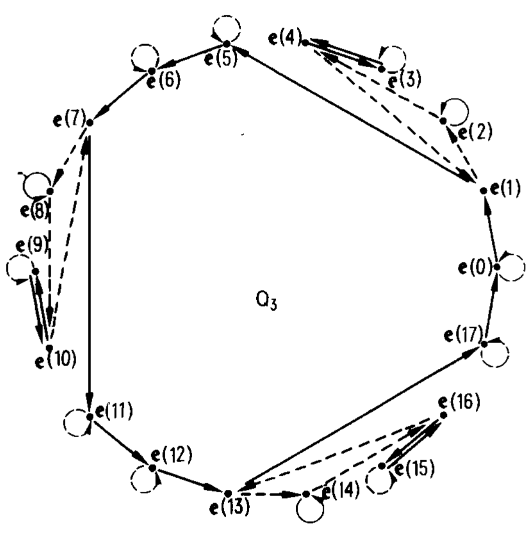


Figure 10

k-representation W of $(\tilde{Q}, \tilde{I})$ such that

$$W(i) = W(i+e),\ W(\alpha_i) = W(\alpha_{i+e}) \quad \text{and} \quad W(\beta_i) = W(\beta_{i+e})$$

for any $i \in \mathbf{Z}$. Similarly, an *e-periodic morphism* $f: W \to W'$ between two e-periodic representations is a morphism of $\mathrm{Mod}_k\,(\tilde{Q}, \tilde{I})$ such that $f(i) = f(i+e)$ for all $i \in \mathbf{Z}$. The e-periodic k-representations and the e-periodic morphisms form a subcategory of $\mathrm{Mod}_k\,(\tilde{Q}, \tilde{I})$, which we denote by $\mathrm{Mod}_k^e\,(\tilde{Q}, \tilde{I})$.

In an analogous way we may define e-periodic k-representations of the bounden quiver $\tilde{Z}_h$. The subcategory $\mathrm{Mod}_k^e\,\tilde{Z}_h$ of $\mathrm{Mod}_k\,\tilde{Z}_h$ formed by the e-periodic representations and morphisms is clearly identified with $\mathrm{Mod}_k\,Z_e^h$ (1.1).

Our main purpose in this paper is to prove the two following statements.

THEOREM 1. *The functor* $R: \mathrm{Mod}_k\,\tilde{Z}_h \to \mathrm{Mod}_k\,(\tilde{Q}, \tilde{I})$ *maps* $\mathrm{Mod}_k^e\,\tilde{Z}_h \xrightarrow{\sim}$ $\mathrm{Mod}\,k[Z_e^h]$ *into* $\mathrm{Mod}_k^e\,(\tilde{Q}, \tilde{I})$ *and induces a k-linear stable equivalence*

$$\overline{\mathrm{Mod}}\,k[Z_e^h] \xrightarrow{\sim} \overline{\mathrm{Mod}}_k^e\,(\tilde{Q}, \tilde{I}).$$

THEOREM 2 (Ch. Riedtmann). *Let A be a connected finite-dimensional algebra over k, for which there is a k-linear stable equivalence* $\overline{\mathrm{Mod}}\,A \xrightarrow{\sim} \overline{\mathrm{Mod}}\,k[Z_e^h]$ *with* $h \geq 2$. *Then* $\mathrm{Mod}\,A$ *is equivalent to* $\mathrm{Mod}_k^e\,(\tilde{Q}, \tilde{I})$ *for some Brauer-quiver Q with h vertices.*

Remember that an algebra A is called *connected*, if A does not admit any decomposition $A = A_1 \times A_2$ with $A_1 \neq 0 \neq A_2$. Theorem 2 will be proved in §3, Theorem 1 in §4.

If $h = 1$, every indecomposable module of length 2 over $k[Z_e^1]$ is projective. Hence we have $\overline{\mathrm{Hom}}\,(\bar{M}, \bar{N}) = 0$ in $\overline{\mathrm{Mod}}\,k[Z_e^1]$ whenever $\bar{M}$ and $\bar{N}$ are indecomposable and not isomorphic. Consequently, if $\mathrm{Mod}\,A$ is stably equivalent to $\mathrm{Mod}\,k[Z_e^1]$, A is a Nakayama-algebra of height ($=$Loewy-length) 2.

1.9. For any h we can interprete $\mathrm{Mod}_k^e\,(\tilde{Q}, \tilde{I})$ as the category of modules over some finite-dimensional k-algebra. We first define a finite bounden quiver (K, J) having as vertices the points

$$\mathrm{e}_e(0), \mathrm{e}_e(1), \ldots, \mathrm{e}_e(e-1).$$

The permutations α and β of $\mathbf{Z}$ induce permutations α and β of $\mathrm{e}_e(\mathbf{Z})$, which in turn determine arrows $\alpha_j : j = \mathrm{e}_e(i) \to \alpha j = \mathrm{e}_e(\alpha i)$ and $\beta_j : j = \mathrm{e}_e(i) \to \beta j = \mathrm{e}_e(\beta i)$. The vertices $j = \mathrm{e}_e(i)$ and the arrows α_j, β_j determine a quiver K which is subjected to the following relations: Any composition of an α-arrow and a β-arrow is 0; moreover, $\alpha^{aj}+\beta^{bj}$ equals 0 for any $j=e(i)$; here, as in 1.6, α^{aj} stands for the composition of $aj:=ai$ arrows of type α, the first of which starts at j; a similar definition holds for β^{bj}.

The k-representations of the bounden quiver (K, J) obtained by this construction are clearly identified with the e-periodic representations of $(\tilde{Q}, \tilde{I})$. Accordingly, $\mathrm{Mod}_k^e(\tilde{Q}, \tilde{I})$ is equivalent to the category of modules over the bounden quiver-algebra $k[K, J]$, which is finite-dimensional over k.

For instance, if Q is the oriented Brauer-quiver Q_3 of 1.8, and if $e=12$, $k[K, J]$ is isomorphic to the bounden quiver-algebra associated with example 2 of 1.1. In effect, this example is obtained from (K, J) by deleting the arrows of K corresponding to loops of Q_3. These deleted arrows arise in a linear way in the relations generating J. Therefore they may be eliminated.

1.10. LEMMA. *The periods f and the associated permutations γ^f of a Brauer-quiver Q are independent of the chosen orientation.*

Proof. For a given orientation of Q we have defined γ as $\alpha\beta$. Now set $\delta=\beta\alpha$. We have to prove that $\gamma^f=\delta^f$ for every period f. As a matter of fact we have $\gamma^f\beta=\beta\gamma^f=\beta(\alpha\beta\alpha\beta\cdots\alpha\beta)=(\beta\alpha\beta\alpha\cdots\beta\alpha)\beta=\delta^f\beta$, hence $\gamma^f=\delta^f$. OK.

Clearly, the periods of Q are multiples of some smallest period, which divides the number h of vertices of Q. Periods dividing h are therefore of particular interest. They are examined in the following proposition.

PROPOSITION. *Let Q be an oriented Brauer-quiver with h vertices and let $d\neq h$ be a period of Q dividing h. Then there is exactly one exceptional orbit in Q,* i.e. *either an α-orbit or a β-orbit which is stable under $\gamma^d=(\alpha\beta)^d$. Every non-exceptional α- or β-orbit has $m=h/d$ transforms under the action of γ^d.*

Proof. Identify the points of Q with the vertices of a regular polygon as explained in 1.5. Clearly, an orbit is stable iff its convex hull contains the center 0 of $D=\{z\in\mathbf{C}:|z|\leq 1\}$. Accordingly, there is at most one stable orbit. Suppose now that no β-orbit is stable under the "rotation" γ^d. Call Δ the union of the disjoint convex hulls of the different β-orbits. By assumption we have $0\notin\Delta$. Let Γ be the connected component of 0 in D-Δ. The intersection of Γ with the unit circle S_1 is the disjoint union of some open arcs

$$]e(i_1-1), e(i_1)[, \ldots,]e(i_a-1), e(i_a)[$$

with $i_1 < i_2 < \cdots < i_a < i_1 + h$. Together with the relation $\gamma = \alpha\beta$ this implies $\alpha e(i_1) = e(i_2)$, $\alpha e(i_2) = e(i_3)$, $\ldots$, $\alpha e(i_a) = e(i_1)$. Since 0 is fixed under the rotation γ^d, the component Γ of 0 in D-Γ is stable under γ^d. Consequently, $\Gamma \cap \mathbf{S}_1$ and the α-orbit $\{e(i_1), \ldots, e(i_a)\}$ are stable.

Clearly, the stability of the exceptional α-orbit implies for geometrical reasons that $1 \le \alpha i - i \le d$ whenever $e(i)$ lies on the exceptional orbit. If $e(i)$ does not lie on the exceptional orbit, we have $e(i) \in]e(j), \alpha e(j)[$ for some $e(j)$ belonging to the exceptional orbit. By 1.5 we know that the complete α-orbit Γ of $e(i)$ is then contained in $]e(j), \alpha e(j)[$. Therefore, the transforms Γ, $\gamma^d(\Gamma)$, $\gamma^{2d}(\Gamma), \ldots, \gamma^{(m-1)d}(\Gamma)$ must be disjoint. A similar argument holds for arbitrary β-orbits. OK.

Remark. Let us keep in mind that $1 \le \alpha i - i \le d$ whenever $e(i)$ lies as the exceptional (α-)orbit. If this is not the case, we know that, with the notations above, both $e(i)$ and $\alpha e(i)$ belong to $]e(j), \alpha e(j)[$; accordingly, we either have $1 \le \alpha i - i < d$ or $h - d < \alpha i - i \le h$ for obvious geometrical reasons. Analogously, for any i, we have either $1 \le \beta i - i < d$ or $h - d < \beta i - i \le h$.

1.11. In the situation described in Proposition 1.10 it will be convenient to consider besides Q the quotient $\bar{Q}$ of Q under the action of γ^d. As a matter of fact, the permutations α and β of Q and $\bar{Q}$ induce permutations of $\bar{Q} := e_d(\mathbf{Z})$, which we still call α and β and which in turn determine arrows $\alpha_j : j = e_d(i) \to \alpha j = e_d(\alpha i)$ and $\beta_j : j \to \beta j = e_d(\beta i)$. We denote by $p : Q \to \bar{Q}$ the map $e_h(i) \mapsto e_d(i) = e_h(mi)$ and call *exceptional* the image under p of the exceptional orbit of Q.

PROPOSITION. *The quiver $\bar{Q}$ consisting in $e_d(\mathbf{Z})$ and the arrows α_j, β_j is a Brauer-quiver. The exceptional orbit of Q is an m-fold covering of the exceptional orbit of $\bar{Q}$, whereas each non-exceptional orbit of $\bar{Q}$ is covered by m disjoint orbits of Q.*

Proof. Let us assume for instance that the exceptional orbit is an α-orbit. We first have to prove that the convex hulls of the α-orbits of $e_d(\mathbf{Z})$ are disjoint, or equivalently that, for any $i \in \mathbf{Z}$ and any $j \in]e_d(i), \alpha e_d(i)[$, the whole α-orbit of j lies in $]e_d(i), \alpha e_d(i)[$. Now we know by 1.10 that we have either $1 \le \alpha i - i \le d$ or $h - d + 1 \le \alpha i - i \le h$. In the first case, we choose a representative l of j in $]i, \alpha i[$; since the α-orbit Γ of $e_h(l)$ is contained in $]e_h(i), \alpha e_h(i)[$, the α-orbit $p(\Gamma)$ of $j = p(e_h(l))$ *is contained in* $p(]e_h(i), \alpha e_h(i)[) =]e_d(i), \alpha e_d(i)[$. In the second case, let i_0 be the greatest number smaller than $\alpha i - h$ such that $e_h(i_0)$ belongs to the exceptional orbit. Let further l be the representative of j in $[i_0, i_0 + d[$. Clearly, $e_h(l) \notin [\alpha e_h(i), e_h(i)]$. Therefore, if j does not belong to the exceptional orbit, the

α-orbit Γ of $e_h(l)$ is contained in $[e_h(i_0), e_h(i_0+d)[-[\alpha e_h(i), e_h(i)]$; accordingly $p(\Gamma)$ is contained in $p[e_h(i_0), e_h(i_0+h)[-p[\alpha e_h(i), e_h(i)]$, i.e. in $]e_d(i), \alpha e_d(i)[$, since p maps $[e_h(i_0), e_h(i_0+d)[$bijectively onto Q. Finally, if Γ is exceptional, we set $\Gamma_1 = \Gamma \cap [e_h(i_0), e_h(i_0+d)[$. Then Γ_1 is contained in $[e_h(i_0), e_h(i_0+d)[-[\alpha e_h(i), e_h(i)]$; accordingly, $p(\Gamma_1) = p(\Gamma)$ is contained in $]e_d(i), \alpha e_d(i)[= p[e_h(i_0), e_h(i_0+h)[$ $), e_h( )]$.

The last statements of our proposition follow directly from 1.10. OK.

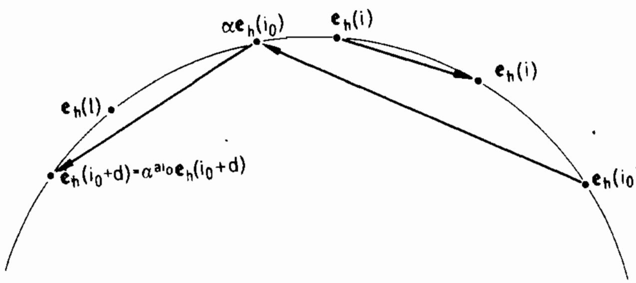


Figure 11

It is worth noticing that Q can be recovered from $\bar{Q}$, provided one knows the exceptional orbit of $\bar{Q}$ and the *multiplicity* $m = h/d$. Assume for instance that the exceptional orbit Δ of $\bar{Q}$ is an α-orbit. Then, if $\alpha e_d(i) \neq e_d(i)$ and if $]e_d(i), \alpha e_d(i)[$ does not contain Δ, we define $\alpha e_h(i)$ as $e_h(j)$, where j satisfies $e_d(j) = \alpha e_d(i)$ and $i+1 \leq j \leq i+d$. Otherwise, we set $\alpha e_h(i) = e_h(l)$, where l is determined by $e_d(l) = \alpha e_d(i)$ and $i+h-d+1 \leq l \leq i+h$. This shows that *the datum of a Brauer-quiver Q together with a period d dividing the number $h \neq d$ of vertices is equivalent to the datum of a Brauer-quiver $\bar{Q}$ having d vertices together with some exceptional cycle and some multiplicity $m = h/e > 1$.*

The relation between Q and $\bar{Q}$ is illustrated by setting $Q = Q_3$ (1.8), $d = 6$, $m = 3$ and $\bar{Q} = Q_2$ (1.4). The exceptional cycle of $\bar{Q}$ is $\{0, 1, 5\}$.

1.12. The results of 1.11 can be applied in the general case of an arbitrary period e by setting $d = (h, e) =$ greatest common divisor of h and e. In this case, the quiver K described in 1.9 is a (e/d)-fold covering of $\bar{Q}$.

As a matter of fact, we are mostly interested in the case where $e = d$. In this case, K is identified with $\bar{Q}$, and the relations generating the binding ideal J of 1.9 may be described directly: Denote by $A_j : j \to j$ and $B_j : j \to j$ the formal compositions of the arrows of the α- and β-cycle through a vertex j of $K = \bar{Q}$. Assuming that the exceptional cycle is an α-cycle, we get the following constraints

a) $A_j^m + B_j = 0$ if the α-cycle through j is exceptional ($m =$ fixed multiplicity),

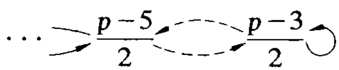
b) $A_j + B_j = 0$ if the α-cycle through j is not exceptional,

c) $\alpha_{\beta j}\beta_j = 0$ and $\beta_{\alpha j}\alpha_j = 0$ for any j.

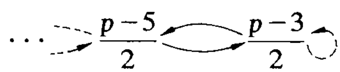
A k-representation of K satisfying these relations is called a Dade–Janusz–Kupisch-*representation* of K. The associated bounden quiver-algebra is called a DJK-*algebra*. Attaching to every DJK-representation V of $K = \bar{Q}$ the representation W of $\tilde{Q}$ such that $W(j) = V(e_e(j))$, we clearly get an isomorphism

$$\text{Mod } k[K, J] \xrightarrow{\sim} \text{Mod}_k^e(\tilde{Q}, \tilde{I}).$$

1.13. The preceding constructions are justified by the fact that DJK-algebras do occur in representation theory of finite groups. For instance, for any field k of characteristic $p > 2$, the group-algebra $k[SL_2(\mathbf{F}_p)]$ is *Morita-equivalent* to $k \times A \times A$, where A is the DJK-algebra attached to the Brauer-quiver Q below with $m = 2$ having the loop α_0 as exceptional cycle (the right end of the quiver is

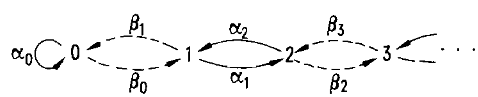


or



according as $p \equiv 1$ or 3 mod 4).

Q



More generally, let G be a finite group with cyclic p-Sylow-subgroups. Let P be a minimal p-subgroup of G and N its normalizer in G. By Michler [10], if k is an algebraically closed field of characteristic $p > 0$, $k[N]$ is a product of algebras each of which is Morita-equivalent to some $k[Z_e^{em}]$. Moreover, the scalar-extension-functor induces a stable equivalence between mod $k[N]$ and mod $k[G]$ (Thompson–Feit–Green).

We have to keep this relation to group theory in mind when proving Theorem 1 and 2 (1.8). In effect, by adapting methods of Green [6] and Peacock, it is rather easy to show that a "block" A (= connected finite dimensional algebra) is Morita-equivalent to a DJK-algebra, if some exact functor $R: \text{mod } k[Z_e^{em}] \to \text{mod } A$ induces a stable equivalence. The simplifying fact in this case, which is the group theory case, is that R necessarily induces isomorphisms $\text{Ext}^1(V, V') \xrightarrow{\sim} \text{Ext}_A^1(RV, RV')$ (see [11]). Now Ch. Riedtmann has shown in her thesis how to get rid of the existence of an exact functor. We reproduce her proof below.

2. The lattice-strip of irreducible morphisms

In order to prove our main theorem, we first have to scrutinize the morphisms of the category Mod $k[Z_e^h]$ and to give them a geometric description.

2.1. We represent the indecomposable $k[Z_e^h]$-module $V_{s,m}$ described in 1.2 by means of the pair $(s+\mathbf{Z}e, m)\in \mathbf{Z}/e\mathbf{Z}\times\mathbf{Z}$. Since $\mathbf{Z}/e\mathbf{Z}$ has as elements the subsets $\{s+re\colon r\in\mathbf{Z}\}$ of $\mathbf{Z}$, $V_{s,m}$ has infinitely many *representatives* $(s+re, m)$ in $\mathbf{Z}\times\mathbf{Z}$, which we identify with the points of the plane having $s+re$ and m as coordinates in the skew coordinate system below ($e=3$ and $h=8$ in the particular case of Fig. 12).

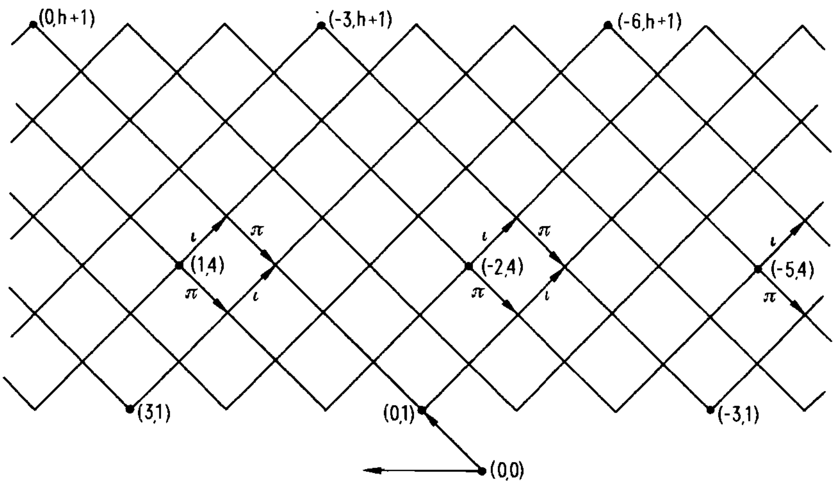


Figure 12

Just as we represent an indecomposable module by a series of points, we may represent a linear map by a series of arrows. In our picture the series of arrows $(s+re, m)\to(s+re, m-1)$, $r\in\mathbf{Z}$, stands for the projection $\pi: V_{s,m}\to V_{s,m-1}$ such that $\pi(\varepsilon_0)=\varepsilon_0$. Similarly, the injection $\iota: V_{s,m}\to V_{s-1,m+1}$ such that $\iota(\varepsilon_0)=\varepsilon_1$ is represented by a series of arrows $(s+re, m)\to(s-1+re, m+1)$. Thus we get a lattice-strip formed by vertices and arrows, where the series of meshes

$$\begin{tikzcd}[column sep=small] & (s-1+re, m+1) \arrow[dr] & \\ (s+re, m) \arrow[ur] \arrow[dr] & & (s-1+re, m), \; r\in\mathbf{Z} \\ & (s+re, m-1) \arrow[ur] & \end{tikzcd}$$

represents the commutative diagram

$$\begin{tikzcd} & V_{s-1,m+1} \arrow[dr, "\pi"] & \\ V_{s,m} \arrow[ur, "\iota"] \arrow[dr, "\pi"'] & & V_{s-1,m}, \quad 1<m\le h \\ & V_{s,m-1} \arrow[ur, "\iota"'] & \end{tikzcd}$$

($V_{-1,m}$ stands for $V_{e-1,m}$; notice that $\pi \circ \iota = 0$ if $m=1$).

2.2. Given two indecomposable wreaths $V_{s,m}$ and $V_{t,q}$, the morphisms between them are described as follows. Start with any representative vertex $(s+re, m)$ of $V_{s,m}$ and hatch the convex polygon generated by the vertices (i,j) such that $i \le s+re$ and $s+re+1 \le i+j \le s+re+m$. Let $(t+xe, q)$, $(t+(x-1)e, q), \ldots,$ $(t+(x-l)e, q)$ be the representatives of $V_{t,q}$ within the hatched polygon. For any such a representative $(t+(x-f)e, q)$ all possible compositions of π- and ι-arrows starting at $(s+re, m)$ and ending at $(t+(x-f)e, q)$ represent the same morphism $\mu_f : V_{s,m} \to V_{t,q}$. Moreover, the morphisms $\mu_0, \ldots, \mu_l$ form a basis of $\operatorname{Hom}(V_{s,m}, V_{t,q})$ over k.

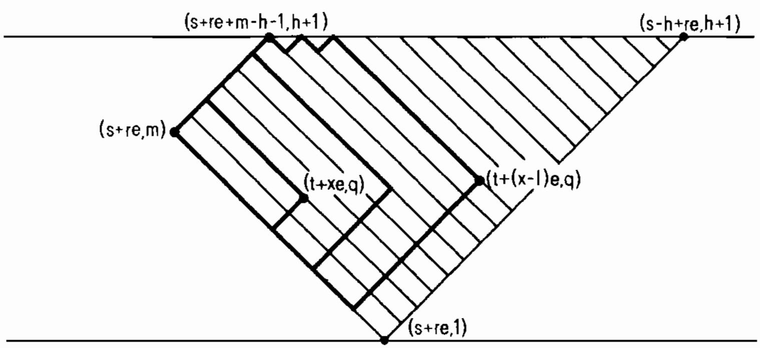


Figure 13

In order to describe $\operatorname{Hom}(V_{s,m}, V_{t,q})$ we might equally well start with some representative $(t+xe, q)$ of $V_{t,q}$ and hatch the polygon formed by the vertices (i,j) such that $t+xe \le i \le t+xe+q-1$ and $t+xe+q \le i+j$. The morphisms $\mu_0, \ldots, \mu_l$ are then represented by the compositions of π- and ι-arrows ending at $(t+xe, q)$ and starting at the vertices $(s+re, m)$, $(s+(r+1)e, m), \ldots, (s+(r+l)e, m)$, which lie within the hatched polygon and represent $V_{s,m}$ (see Figure 14).

2.3. Our geometric description of the morphisms suits also with the stable category $\underline{\operatorname{Mod}}_k\, Z_e^h$. In this residual category the projective indecomposables $V_{s,h+1}$

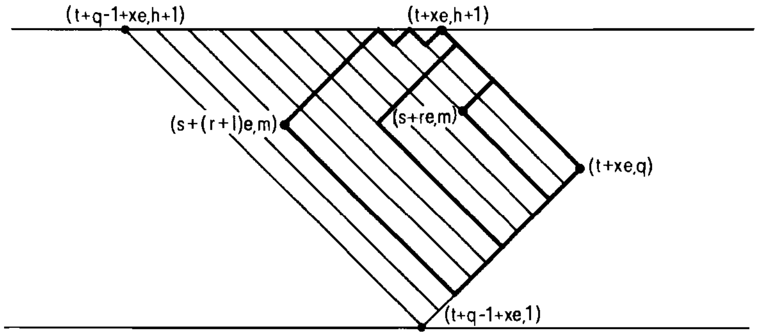


Figure 14

of $\mathrm{Mod}_k Z_e^h$ vanish. Hence we have to delete the vertices $(i, h+1)$ in the lattice-strip of 2.1. The types (=isomorphism classes) of indecomposables of $\overline{\mathrm{Mod}}_k Z_e^h$ correspond in a one-to-one onto way to the types of non-projective indecomposables of $\mathrm{Mod}_k Z_e^h$, hence to the sets $(s+\mathbf{Z}e, m)$ with $1 \leq m \leq h$.

The meshes

$$\begin{tikzcd} & V_{i-1,h+1} \arrow[dr, "\pi"] & \\ V_{i,h} \arrow[ur, "\iota"] \arrow[dr, "\pi"'] & & V_{i-1,h} \\ & V_{i,h-1} \arrow[ur, "\iota"'] & \end{tikzcd}$$

standing at the top of the lattice-strip of 2.1 give rise to the relations $0 = \bar{\iota} \circ \bar{\pi} = \bar{\pi} \circ \bar{\iota} : \bar{V}_{i,h} \to \bar{V}_{i-1,h}$ in $\overline{\mathrm{Mod}}_k Z_e^h$ ($\bar{\pi}$ and $\bar{\iota}$ denote the residue classes of π and ι). From these relations we deduce the following description of the spaces

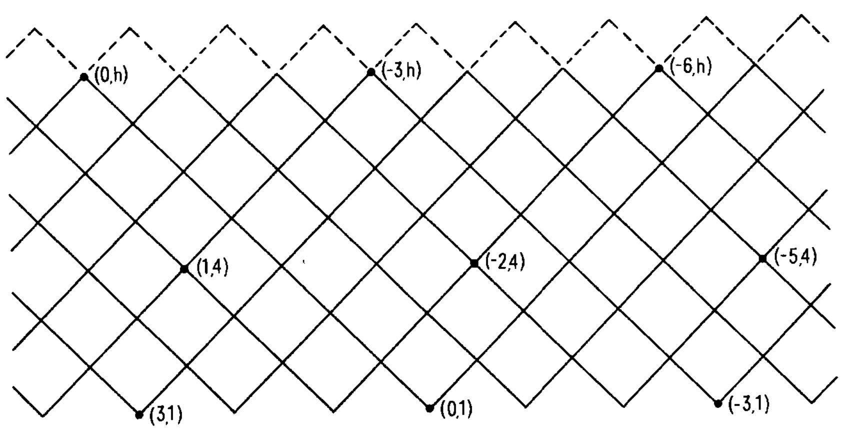


Figure 15

$\overline{\mathrm{Hom}}\,(\bar{V}_{s,m}, \bar{V}_{t,q})$ of morphisms in $\overline{\mathrm{Mod}_k\, Z_e^h}$. Start with any representative vertex $(s+re,\ m)$ of $\bar{V}_{s,m}$ and hatch the polygon determined by the points (i, j) such that $s+re \geq i \geq s+re+m-h$ and $s+re+m \geq i+j \geq s+re+1$. Let $(t+xe, q)$, $(t+(x-1)e, q), \ldots, (t+(x-g)e, q)$ be the representatives of $\bar{V}_{t,q}$ within the hatched rectangle. Clearly we have $g \leq l$ with the notations of 2.2. The residue classes $\bar{\mu}_{g+1}, \ldots, \bar{\mu}_l$ are 0, whereas $\bar{\mu}_0, \ldots, \bar{\mu}_g$ form a basis of $\overline{\mathrm{Hom}}\,(\bar{V}_{s,m}, \bar{V}_{t,q})$.

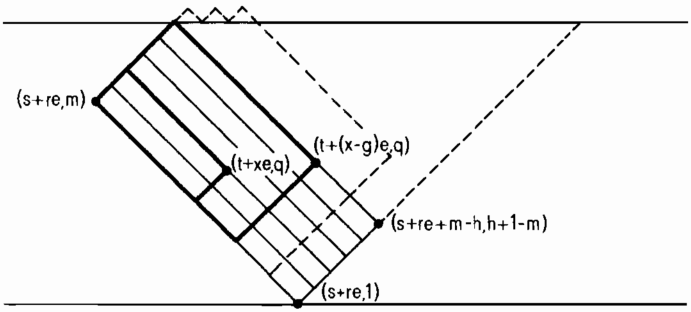


Figure 16

Similarly, if we start with some representative $(t+xe, q)$ of $\bar{V}_{t,q}$, $\bar{V}_{s,m}$ has representatives of the form $(s+re, m), \ldots,\ (s+(r+g)e, m)$ in the rectangle formed by the points (i, j) satisfying the inequalities $t+xe+q-1 \geq i \geq t+xe$ and $t+xe+h \geq i+j \geq t+xe+q$. The representatives correspond to the basis elements $\bar{\mu}_0, \ldots, \bar{\mu}_g$ of $\overline{\mathrm{Hom}}\,(\bar{V}_{s,m}, \bar{V}_{t,q})$(see Figure 17).

2.4. *Remarks.* a) The lattice obtained from the lattice-strip of 2.1 by deleting the points $(s, h+1)$ will be called the *stable lattice*. The rectangle formed by the points (i, j) satisfying the inequalities $s+re \geq i \geq s+re+m-h$ and $s+re+m \geq i+j \geq s+re+1$ will be called the *rectangle starting at* $(s+re, m)$ *and ending at* $(s+re+m-h, h+1-m)$.

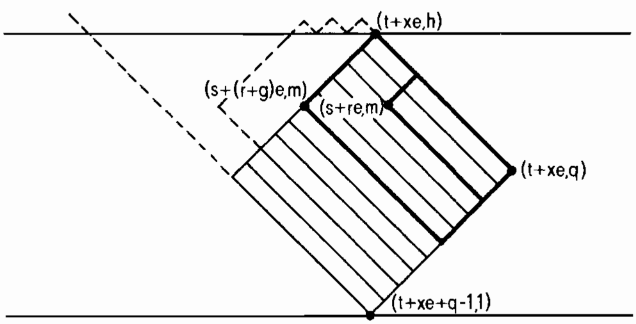


Figure 17

b) It should be clear how the morphisms $\bar{\mu}_n$ defined above are composed: Let $(t+(x-n)e, q)$ be the representative of $\bar{V}_{t,q}$ corresponding to $\bar{\mu}_n$ in the rectangle starting at $(s+re, m)$. Let further $(u+ye, p)$, $(u+(y-1)e, p), \ldots, (u+(y-d)e, p)$ be the representatives of some $\bar{V}_{u,p}$ within the rectangle starting at $(t+(x-n)e, q)$, and let $\bar{\mu}'_0, \ldots, \bar{\mu}'_d$ be the associated basis of $\overline{\mathrm{Hom}}\,(\bar{V}_{t,q}, \bar{V}_{u,p})$. Then $\bar{\mu}'_i \circ \bar{\mu}_n$ is $\neq 0$ iff $(u+(y-i)e, p)$ lies in the rectangle starting at $(s+re, m)$. If this is so, $\bar{\mu}'_i \circ \bar{\mu}_n$ is one of our basis elements in $\overline{\mathrm{Hom}}\,(\bar{V}_{s,m}, \bar{V}_{u,p})$.

c) Two morphisms $\mu: M \to P$ and $\nu: N \to Q$ in a category $\mathscr{C}$ are said to be *isomorphic* iff there are isomorphisms $\alpha: M \xrightarrow{\sim} N$ and $\beta: P \xrightarrow{\sim} Q$ such that $\nu\alpha = \beta\mu$. In the case of the category $\overline{\mathrm{Mod}_k}\, Z_e^h$ it is easily seen that every non-zero morphism $\mu: \bar{V}_{s,m} \to \bar{V}_{t,q}$ is isomorphic to one of the morphisms $\bar{\mu}_0, \ldots, \bar{\mu}_g$ described in 2.3.

2.5. In order to apply the stable lattice to categories, which are stably equivalent to $\mathrm{Mod}_k\, Z_e^h$, we need an internal characterization of the isomorphism classes of $\bar{\pi}$ and $\bar{\iota}$ in terms of the category $\overline{\mathrm{Mod}_k}\, Z_e^h$. For this sake we use the following notion introduced by Auslander and Reiten [1]. For the convenience of our readers we prove the needed results on irreducible morphisms directly.

DEFINITION. A morphism $f: M \to N$ between indecomposables of an additive category $\mathscr{C}$ is called *irreducible* iff it is not invertible and, given any factorization $M \xrightarrow{g} P \xrightarrow{h} N$ of f in $\mathscr{C}$, either g is a *section* (i.e. $\exists r$ such that $r \circ g = \mathbf{1}_M$), or h is a *retraction* (i.e. $\exists s$ such that $h \circ s = \mathbf{1}_N$). Notice that we do not require P to be indecomposable (cf. Remark 2.6).

LEMMA. *A morphism between indecomposables of* $\mathrm{Mod}_k\, Z_e^h$ *is irreducible iff it is isomorphic to some* $\pi: V_{s,m} \to V_{s,m-1}$ *or some* $\iota: V_{s,m} \to V_{s-1,m+1}$.

Proof. π is irreducible: Indeed, suppose that π admits the factorization

$$V_{s,m} \xrightarrow{(f_i)} \bigoplus_i W_i \xrightarrow{(g_i)} V_{s,m-1},$$

where each W_i is indecomposable, and where no f_i, g_i is invertible. Then, for each i, $g_i f_i$ factorizes through the radical ($\xrightarrow{\sim} V_{s+1,m-2}$ if $m \geq 3$, $\xrightarrow{\sim} 0$ if $m=2$) of $V_{s,m-1}$: Otherwise we would have $\mathrm{Coker}\, g_i f_i = 0$ and $1 = \lambda\,(\mathrm{Ker}\, g_i f_i)$: $=$ length of $\mathrm{Ker}\, g_i f_i$. In the canonical exact sequence

$$0 \to \mathrm{Ker}\, f_i \to \mathrm{Ker}\,(g_i f_i) \to \mathrm{Ker}\, g_i \xrightarrow{\partial} \mathrm{Coker}\, f_i \to \mathrm{Coker}\,(g_i f_i) \to \mathrm{Coker}\, g_i \to 0$$

the last two terms would be zero. Moreover, as W_i as uniserial (the subobjects are totally ordered by inclusion) and as $g_if_i \neq 0$, we would have $\operatorname{Im} f_i \supset \operatorname{Ker} g_i$; hence ∂ and $\operatorname{Coker} f_i$ would be zero. The equality $1 = \lambda\,(\operatorname{Ker} g_if_i) = \lambda\,(\operatorname{Ker} f_i) + \lambda\,(\operatorname{Ker} g_i)$ would imply $\lambda\,(\operatorname{Ker} f_i) = 0$ or $\lambda\,(\operatorname{Ker} g_i) = 0$. Hence f_i or g_i would be invertible. Therefore g_if_i and $\sum g_if_i$ factorize through the radical of $V_{s,m-1}$, a contradiction to the surjectivity of π.

The dual proof holds for ι.

Reversely, suppose that $\mu : V_{s,m} \to V_{t,q}$ is irreducible. In the canonical factorization $V_{s,m} \xrightarrow{\rho} \operatorname{Im} \mu \xrightarrow{\sigma} V_{t,q}$ of μ, either ρ or σ have to be isomorphisms. In the second case for instance, μ is surjective and admits a factorization $V_{s,m} \xrightarrow{\pi} V_{s,m-1} \xrightarrow{\tau} V_{t,q}$. By the irreducibility of μ, τ must be a reaction, hence an isomorphism, and μ is isomorphic to π. Similarly, if ρ was invertible, μ would be isomorphic to $\iota : V_{s,m} \to V_{s-1,m+1}$. OK.

2.6. LEMMA. *Let M,N be non-projective objects of an abelian category $\mathscr{C}$ and have local rings of endomorphisms. A morphism $f \in \operatorname{Hom}(M, N)$ is irreducible in $\mathscr{C}$ iff its residue class $\bar f \in \overline{\operatorname{Hom}}\,(\bar M, \bar N)$ is irreducible in the stable category $\bar{\mathscr{C}}$.*

Proof. Suppose that f is irreducible, and let $\bar M \xrightarrow{\bar g} \bar Q \xrightarrow{\bar h} \bar N$ be a factorization of $\bar f$. Then f-hg admits some factorization $M \xrightarrow{g_1} P \xrightarrow{h_1} N$ through a projective P, and f has the factorization $M \xrightarrow{(g,g_1)} Q \oplus P \xrightarrow{(h,h_1)} N$. The irreducibility of f then implies that either (g, g_1) or (h, h_1) must split. In the first case we know by Krull–Remak–Schmidt–Azumaya that either g or g_1 is a section. Hence g is a section, as M is non-projective. Similarly, if (h, h_1) splits, h is a retraction. In both cases, we see that either $\bar g$ or $\bar h$ splits.

Conversely, if $\bar f$ is irreducible, let $f = h \circ g$ be a factorization of f. Then either $\bar g$ or $\bar h$ splits. In the first case, there is some r such that $\bar r \bar g = \mathbf{1}_{\bar M}$. Then rg is an automorphism of M and g is a section. In the second case, the dual argument shows that h is a retraction.

Remark. From 2.5 and 2.6 it follows that the irreducible morphisms of $\overline{\operatorname{Mod}_k}\, Z_e^h$ coincide with the morphisms which are isomorphic either to some $\bar\pi$ or to some $\bar\iota$. From this simple description of the irreducible morphisms we may deduce that we would get the same notion of irreducibility both for $\overline{\operatorname{Mod}_k}\, Z_e^h$ and for $\operatorname{Mod}_k Z_e^h$, if we restricted the definition 2.5 to factorizations $M \xrightarrow{g} P \xrightarrow{h} N$ in which P is indecomposable. In the general case, however, such a restriction would furnish too many irreducible morphisms. Consider for instance the following morphism f between DJK-representations:

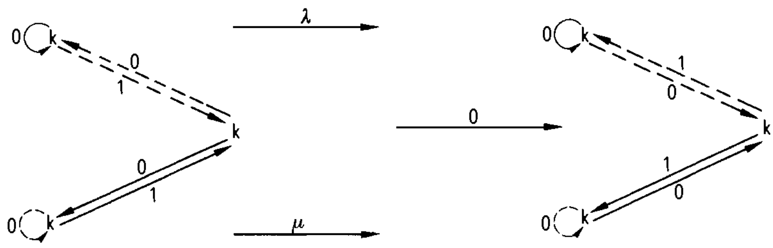


Figure 18

We suppose here that $h = d = e = 3$, so that there is no need for distinguishing some exceptional orbit. Clearly, Im f is semi-simple of length 2, so that f is not irreducible. If $\lambda = \mu$, f is factorized through the indecomposable projective DJK-representation

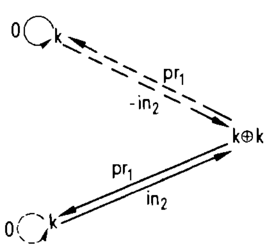


Figure 19

In case $\lambda \neq \mu$, $\lambda \neq 0$ and $\mu \neq 0$, however, f would be irreducible if we restricted the definition 2.5 to factorizations $M \xrightarrow{g} P \xrightarrow{h} N$, where P would have to run through the existing 12 types of indecomposable DJK-representations.

2.7. LEMMA. *Let A be a selfinjective* ($=$*quasifrobenius*) *artinian ring and P an indecomposable projective A-module of length ≥ 2.*

a) *An A-linear map with range P is irreducible iff it is isomorphic to the inclusion of* rad P ($=$radical of P) *into P.*

b) *An A-linear map with domain* rad P *is irreducible iff it is isomorphic either to the inclusion* rad $P \to P$ *or to the projection of* rad P *onto an indecomposable direct factor of* rad P/soc P (soc P = socle of P).

Proof. Clearly, an irreducible map is either surjective or injective. If the irreducible map $f: M \to P$ was surjective, it would admit a section (P is projective) and should therefore be invertible (M is indecomposable!). Hence f has to be a proper injection and is factorized through rad P. By the irreducibility of f, the factorization $M \to \operatorname{rad} P$ is a section; it is invertible, since rad P is indecomposable.

Conversely, let $\operatorname{rad} P \xrightarrow{g} Q \xrightarrow{h} P$ be a factorization of the inclusion $i: \operatorname{rad} P \to P$. If h is not a retraction, it cannot be surjective. Hence we get $h(Q) = \operatorname{rad} P$, and g is a section. This proves that i is irreducible.

b) Let $f: \operatorname{rad} P \to N$ be irreducible. If f is injective, the inclusion $i: \operatorname{rad} P \to P$ admits some factorization $\operatorname{rad} P \xrightarrow{f} N \xrightarrow{g} P$ (P is injective!). By the irreducibility of i, we see that g is a retraction, hence an isomorphism. On the other hand, if f is surjective, f admits a factorization $\operatorname{rad} P \xrightarrow{p} \operatorname{rad} P/\operatorname{soc} P \xrightarrow{g} N$, where p is the canonical projection. By the irreducibility of f, g has to be a retraction.

Conversely, let $\pi: \operatorname{rad} P/\operatorname{soc} P \to Q$ be the projection onto an indecomposable direct factor. Let us prove that πp is irreducible: consider any factorization $\operatorname{rad} P \xrightarrow{f} M \xrightarrow{g} Q$ of πp. Decompose $M = I \oplus N$ in such a way, that I is a direct sum of copies of P, whereas N does not contain any further copy of P. Suppose that f is not a section and denote by f_1, f_2 and g_1, g_2 the components of f and g relative to the decomposition $M = I \oplus N$. Then f_2 cannot be injective. Otherwise, there would be some $m: N \to P$ such that $mf_2 =$ inclusion. As f_2 is not a section, Im m would be distinct from rad P and hence equal to P. But then N would contain some copy of P.

Therefore f_2 equals some composition $\operatorname{rad} P \xrightarrow{h} \operatorname{rad} P/\operatorname{soc} P \xrightarrow{s} N$, and we get $(g_2 s - \pi)p = g_2 f_2 - \pi p = g_2 f_2 - (g_1 f_1 + g_2 f_2) = -g_1 f_1$. On the other hand $f_1: \operatorname{rad} P \to I$ clearly factorizes through rad I, hence $g_1 f_1$ and $h = g_2 s - \pi$ factorize through rad Q. This implies for the inclusion $\sigma: Q \to \operatorname{rad} P/\operatorname{soc} P$ that

$$g_2 s\sigma = \pi\sigma + h\sigma = \mathbf{1}_Q + h\sigma,$$

where $h\sigma$ maps Q into rad Q. Therefore $\mathbf{1}_Q + h\sigma$ is invertible, and g_2 is a retraction. OK.

3. The structure of wreath-like algebras

We call an algebra A over a field k *wreath-like* if A is finite-dimensional, connected, and if there is a k-linear stable equivalence between Mod A and

$\mathrm{Mod}_k Z_e^h$ for some e and $h \geq 2$. For the sequel of §3 we fix once for all a k-linear equivalence $\bar{L}: \overline{\mathrm{Mod}_k}\, Z_e^h \xrightarrow{\sim} \overline{\mathrm{Mod}}\, A$. Clearly, $\overline{\mathrm{Mod}}\, A$ inherits from $\mathrm{Mod}_k Z_e^h$ the property that every object is a (possibly infinite) direct sum of objects of type $\bar{L}\bar{V}_{s,m}$. This implies in particular that A has only finitely many types of indecomposables. These indecomposables are finitely generated, and every A-module is a direct sum of indecomposables [14].

Denote by mod A the category of *finitely generated* A-modules, by $\mathrm{mod}_k Z_e^h$ the category of finite-dimensional representations of Z_e^h. Then $\overline{\mathrm{mod}}\, A$ is a subcategory of $\overline{\mathrm{Mod}}\, A$, whose objects may be characterized up to isomorphisms within $\overline{\mathrm{Mod}}\, A$ as the finite direct sums of indecomposables. Consequently, we may assume without loss of generality that $\bar{L}$ maps $\overline{\mathrm{mod}_k}\, Z_e^h$ into $\overline{\mathrm{mod}}\, A$ and induces an equivalence between these categories. As I. Reiten has shown, [12], this implies that A is *self-injective.*

The equivalence $\bar{L}$ induces a bijection between the types of non-projective indecomposables of $\mathrm{mod}_k Z_e^h$ and of mod A. We shall say that an indecomposable A-module M is of *$\bar{L}$-type* (s, m) if $\bar{M} \xrightarrow{\sim} \bar{L}\bar{V}_{s,m}$; equivalently, we then say that each vertex $(s + re, m)$ is a *$\bar{L}$-representative* of M. The number m is the *$\bar{L}$-height* of M.

3.1. $\bar{L}$-representatives of simple modules: A splits over k and admits e types of simple modules. Furthermore there is one $\bar{L}$-representative of a simple A-module in each going down diagonal and one in each going up diagonal of the stable lattice. The $\bar{L}$-height l of a simple A-module has to satisfy at least one of the two following conditions

$$1 \leq l \leq e \quad \textit{and} \quad h - e + 1 \leq l \leq h.$$

Proof. Consider a going down diagonal

$$(s, h) \to (s, h-1) \to \cdots \to (s, 1)$$

in the stable lattice. If M is any indecomposable non-projective A-module, we know by 2.3 that $\overline{\mathrm{Hom}}\,(\bar{L}\bar{V}_{s,h}, \bar{M}) \neq 0$ iff M has some $\bar{L}$-representative on the given diagonal. In particular, if M is a simple A-module occurring in the *top* ($= N/\mathrm{rad}\, N$) of some indecomposable A-module N of $\bar{L}$-type (s, h), we infer that M is of $\bar{L}$-type (s, l) for some l. On the other hand, as we have $\overline{\mathrm{Hom}}\,(\bar{V}_{s,q}, \bar{V}_{s,r}) \neq 0$ for $q \geq r$, there cannot be more than one $\bar{L}$-represention of a simple A-module in the given diagonal. A similar argument applies to going up diagonals.

Suppose now that (s, l) $\bar{L}$-represents some indecomposable A-module S.

With notations from 2.3 we get $\overline{\mathrm{Hom}}_A(\bar{S},\bar{S})=\overline{\mathrm{Hom}}\ (\bar{V}_{s,l},\bar{V}_{s,l})=k\bar{\mu}_0\oplus\cdots\oplus k\bar{\mu}_g$, where $\bar{\mu}_i\bar{\mu}_j=\bar{\mu}_{i+j}$ or 0 according as $i+j\leq g$ or not. If S is simple, $\mathrm{Hom}_A(S,S)=\overline{\mathrm{Hom}}_A(\bar{S},\bar{S})$ is a field, and this implies $g=0$. On the other hand, we know that $g+1$ equals the number of $\bar{L}$-representatives $(s+re,l)$ within the rectangle starting at (s,l). This number is 1 iff $(s-e,l)$ does not belong to that rectangle, i.e. iff $h-l<e$ or $l-1<e$.

Finally, A splits, since we have $k \tilde{\to} \mathrm{Hom}_A(S,S)$ for any simple module S. OK.

Remarks a) We get more precise information on the $\bar{L}$-representatives of simple A-modules S, T by writing that $\overline{\mathrm{Hom}}(\bar{S},\bar{T})=0=\overline{\mathrm{Hom}}(\bar{T},\bar{S})$. This means that, if (s,l) is a $\bar{L}$-representative of a simple module, no other vertex within the two rectangles of Figure 20 below can be a $\bar{L}$-representative of a simple module. We shall make capital out of this information later on.

b) If a simple A-module S is of $\bar{L}$-type (s,l), the breadth of the rectangle starting at (s,l) is $<e$. Therefore, no indecomposable A-module can have two $\bar{L}$-representatives in this rectangle. By 2.3 this implies that the multiplicity of S in the socle of any indecomposable A-module is ≤ 1. The dual argument shows that the multiplicity of S in the top of any indecomposable A-module is ≤ 1.

Projective meshes

3.2. PROPOSITION. *Let T be a simple A-module of $\bar{L}$-type (t,l) with projective cover P. Then* rad *P is of $\bar{L}$-type $(t+l,h+1-l)$ and $P/$*soc *P of $\bar{L}$-type $(t+l-1,h+1-l)$.*

a) *If $l=1$,* rad *$P/$*soc *P is indecomposable of $\bar{L}$-type $(t+1,h-1)$.*

b) *If $l=h$,* rad *$P/$*soc *P is indecomposable of $\bar{L}$-type $(t+h-1,2)$.*

c) *If $1<l<h$,* rad *$P/$*soc *$P=P_\alpha\oplus P_\beta$ is the direct sum of two indecomposables P_α and P_β of $\bar{L}$-types respectively $(t+l,h-l)$ and $(t+l-1,h+2-l)$.*

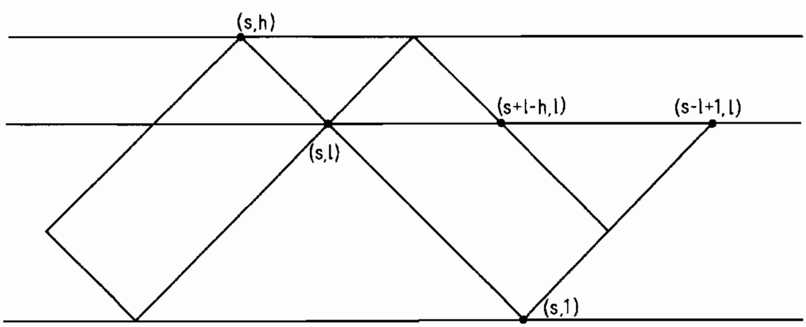


Figure 20

Proof. Let M be an indecomposable A-module of $\bar{L}$-type $(t+l-1, h+1-l)$. By 2.3 and 2.4 we know that the k-vector space $\overline{\mathrm{Hom}}\,(\bar{M}, \bar{T})$ is generated by some single $\bar{f}$ and that any morphism $\bar{g}: \bar{N} \to \bar{M}$, such that N is indecomposable and $\bar{f}\bar{g} \neq 0$, must be invertible. On the other hand, if $P \xrightarrow{p} M \xrightarrow{f} T$ is a factorization of the canonical projection $P \to T$, p cannot be injective; hence it is factorized through some $g: P/\mathrm{soc}\, P \to M$, which is invertible since $\bar{f}\bar{g} \neq 0$.

Let us now examine rad $P/\mathrm{soc}\, P$. By 3.1b) the top of rad P cannot contain two copies of the same simple module. Distinct summands in a direct sum decomposition of rad $P/\mathrm{soc}\, P$ are therefore not isomorphic. On the other hand we clearly have $(t+l-1, h+1-l) \not\equiv (t, l)$ modulo $e\mathbf{Z} \times 0$. Therefore we have $P/\mathrm{soc}\, P \not\twoheadrightarrow T$ and $\lambda(P) \geq 2$, so that the dual of Lemma 2.7 applies: since there are two arrows terminating at $(t+l-1, h+1-l)$ in case $1 < l < h$, there are two types of inclusion maps of direct summands of rad $P/\mathrm{soc}\, P$ into $P/\mathrm{soc}\, P$. Assertion c) follows immediately, and similar arguments hold in case $l = 1$, where P_β "vanishes," and in case $l = h$, where P_α "vanishes."

Finally we know by 2.7 that there are irreducible morphisms rad $P \to P_\alpha$ and rad $P \to P_\beta$. Hence rad P must be of $\bar{L}$-type $(t+l, h+1-l)$. (See Fig. 21–23.) OK.

Remark. Our proposition shows that, although a projective indecomposable A-module P has no $\bar{L}$-representative, the stable lattice bears some trace of P, namely the set of $\bar{L}$-representatives of rad P, P_α, P_β and $P/\mathrm{soc}\, P$. This set is called a *projective mesh* of P. Of course, the position of the projective meshes within the stable lattice depend on A and $\bar{L}$.

3.3. Proposition 3.2 admits a *dual statement*: Let S be a simple A-module of $\bar{L}$-type (s, i) with injective hull Q. Then Q is projective and indecomposable. A projective mesh of Q is formed by the vertices $(s+i-h, h+1-i)$, $(s+i-h, h-i)$, $(s+i-h-1, h+2-i)$ and $(s+i-h-1, h+1-i)$ $\bar{L}$-representing respectively rad Q, Q_α, Q_β and $Q/\mathrm{soc}\, Q$. Here again Q_α or Q_β may vanish (if $i = h$ or 1).

In the particular case where S *is the socle of the projective cover of* T, Q coincides with P. Writing that the projective meshes of Q and P coincide, we get $(s, i) = (t+h, l)$ or $s = t+h$ and $i = l$ for a convenient choice of the $\bar{L}$-representative (s, i) of S. This means that the *Nakayama permutation* $T \mapsto S$ of the types of simple A-modules is associated with the *translation* π of the stable lattice defined by $\pi(t, l) = (t+h, l)$. The set of $\bar{L}$-representatives of simple A-modules is stable under π, and projectives meshes are mapped into projective meshes.

On the other hand, if (r, g) $\bar{L}$-represents some indecomposable A-module M, we know that $(r+e, g)$ also $\bar{L}$-represents M. Therefore $\bar{L}$-representatives of simple A-modules and projective meshes are also stable under the translation ε

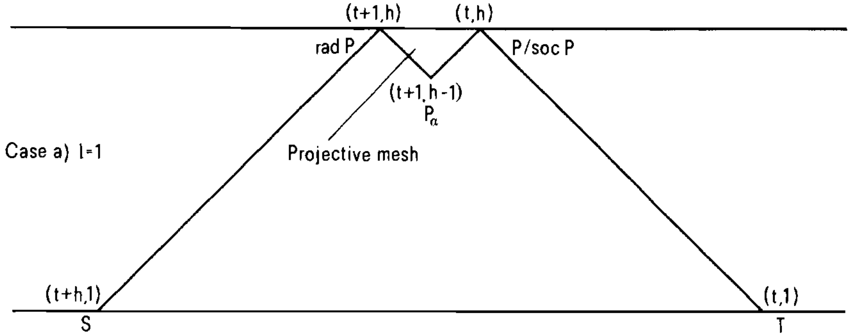


Figure 21

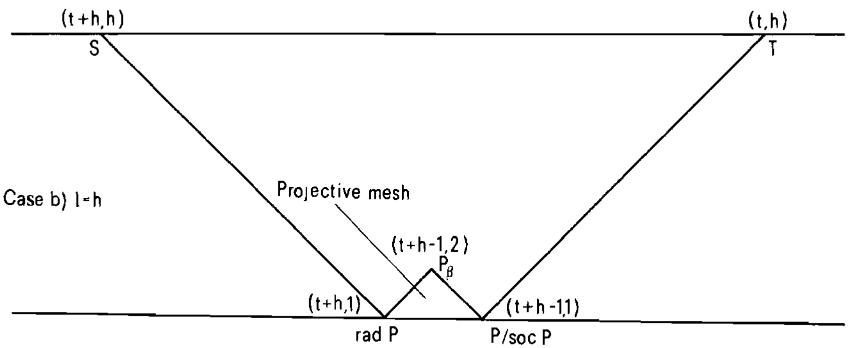


Figure 22

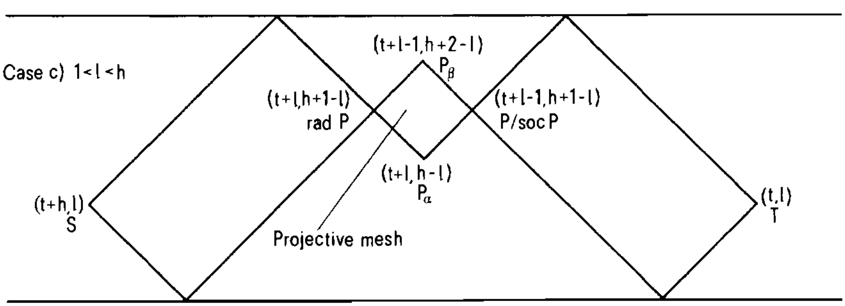


Figure 23

given by $\varepsilon(t, l) = (t+e, l)$. In other words, both h and e are "*periods*" of the configuration formed by the stable lattice and the $\bar{L}$-representatives of simple modules. Clearly, the greatest common divisor $d = (h, e)$ is a period too.

The structure of uniserial A-modules

3.4. For each $t \in \mathbf{Z}$ we denote by $(t, \alpha t - t)$ the vertex lying in the going down diagonal of $(t, 1)$ and $\bar{L}$-representing a simple A-module, say T. The vertex $(\alpha t, h+1-\alpha t+t)$ then $\bar{L}$-represents rad P, if P is the projective cover of T. In other words, α is the map induced on the first coordinates by the correspondence $T \mapsto \operatorname{rad} P$. Notice that α satisfies by construction the relations $t+1 \leq \alpha t \leq t+h$ and $\alpha(t+h) = \alpha(t)+h$ (confer 3.3).

LEMMA. *If $\alpha t \neq t+h$, the top of P_α is simple of $\bar{L}$-type $(\alpha t, \alpha^2 t - \alpha t)$, and we have $\alpha^2 t \leq t+h$.*

Proof. By 3.2 P_α is of $\bar{L}$-type $(\alpha t, h - \alpha t + t)$. By Remark (a) of 3.1 we know that besides $(t, \alpha t - t)$ no vertex of the rectangle ending at $(t, \alpha t - t)$ can $\bar{L}$-represent a simple A-module. By 2.3 every simple constituent of the top of P_α must therefore admit a $\bar{L}$-representative of the form $(\alpha t, l)$ with $l = \alpha^2 t - \alpha t \leq h - \alpha t + t$. The simplicity of the top of P_α follows from the fact that the going down diagonal of $(\alpha t, h - \alpha t + t)$ contains only one $\bar{L}$-representative of a simple module. (See Figure 24.)

3.5. LEMMA. *α is a permutation of* $\mathbf{Z}$.

Proof. Suppose that $\alpha t_1 = \alpha t_2$ with $t_1 > t_2$, and let T_1, T_2 be simple A-modules of $\bar{L}$-types $(t_1, \alpha t_1 - t_1)$ and $(t_2, \alpha t_2 - t_2)$. As $(t_1, \alpha t_1 - t_1)$ and $(t_2, \alpha t_2 - t_2)$ lie on the same going up diagonal of the stable lattice, we have $\overline{\operatorname{Hom}}(\bar{T}_1, \bar{T}_2) \neq 0$ and $T_1 \not\twoheadrightarrow T_2$, a contradiction!

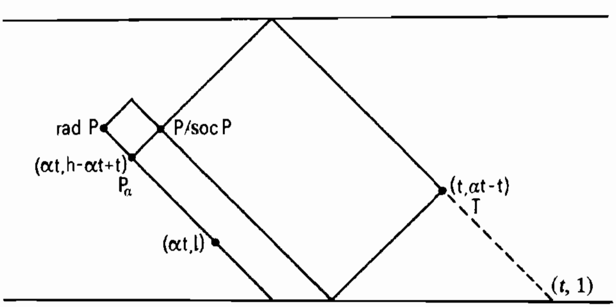


Figure 24

Now we claim that the injectivity of α together with the formula $\alpha(t+h) = \alpha(t)+h$ imply the surjectivity of α. Indeed, denote by $\bar{\alpha}: \mathbf{Z}/h\mathbf{Z} \to \mathbf{Z}/h\mathbf{Z}$ the map induced by α. It is easily verified, that the injectivity of α and our formula imply the injectivity of $\bar{\alpha}$, hence also the surjectivity of $\bar{\alpha}$, since $\mathbf{Z}/h\mathbf{Z}$ is finite. The surjectivity of $\bar{\alpha}$ and our formula in turn imply the surjectivity of α. OK.

3.6. DEFINITION. *Let M be a non-simple non-projective indecomposable A-module of $\bar{L}$-type (m, l) which has a simple top and a simple socle. If $l > \alpha m - m$, we say that M is of class α; if $l < \alpha m - m$, M is said to be of class β.*

PROPOSITION. *Let M be an indecomposable A-module of $\bar{L}$-type (m, l) and class α. Let $\alpha m, \alpha^2 m, \ldots, \alpha^{\lambda-1} m$ be the integers x lying in the α-orbit of m and such that $m < x < m+l$. Then M has a unique Jordan-Hölder series*

$$0 = M_\lambda \subset M_{\lambda-1} \subset \cdots \subset M_1 \subset M_0 = M,$$

and the uniserial subquotient M_i/M_{j+1} is of $\bar{L}$-type $(\alpha^i m, \alpha^{j+1} m - \alpha^i m)$. In particular, if $i = 0$ and $j = \lambda - 1$, we have $l = \alpha^\lambda m - m$.

Proof. We use induction on $\lambda(M) =$ length of M. By hypothesis the unique $\bar{L}$-representative of a simple A-module on the going down diagonal through (m, l) lies below (m, l). As M has simple top T, $(m, \alpha m - m)$ is the only $\bar{L}$-representative of a simple module within the rectangle starting at (m, l). In particular, the unique $\bar{L}$-representative $(m+l-g, g)$ of a simple module on the going up diagonal through (m, l) cannot lie higher than (m, l). On the other hand, as M has a simple socle S, $(m+l-g, g)$ necessarily $\bar{L}$-represents S.

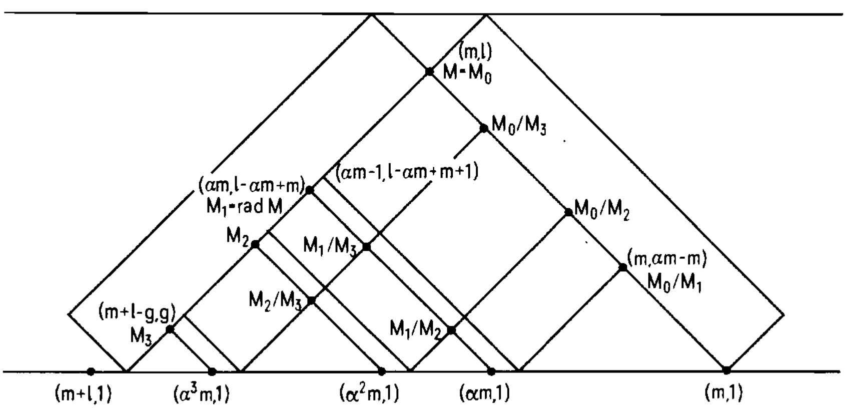


Figure 25

The lower "going down" edge of the rectangle ending at $(m, \alpha m-m)$ cuts the going up diagonal through (m, l) at $(\alpha m-1, l-\alpha m+m+1)$. As the projection $S \to M/\mathrm{rad}\, M$ is zero, we have $g \leq l-\alpha m+m$. We want to show that rad M is of $\bar{L}$-type $(\alpha m, l-\alpha m+m)$: For this sake we first consider any submodule $N \neq 0$ of M. The socle S of N being simple, N is indecomposable and has some $\bar{L}$-representative in the rectangle starting at $(m+l-g, g)$. On the other hand, the composed inclusion $S \to N \to M$ is not zero in the stable category. The unique $\bar{L}$-representative of N in the rectangle starting at $(m+l-g, g)$ must therefore lie on the going up diagonal between $(m+l-g, g)$ and (m, l). This holds in particular for rad M. Moreover, as the projection rad $M \to M/\mathrm{rad}\, M$ is zero, rad M must be of $\bar{L}$-type $(m+l-f, f)$ with $g \leq f \leq l-\alpha m+m$.

It remains to show that $f = l-\alpha m+m$: Indeed, let R be of $\bar{L}$-type $(\alpha m, l-\alpha m+m)$ and denote by $\mu : R \to M$ a map associated with the composed arrow $(\alpha m, l-\alpha m+m) \to (m, l)$. Since the composition $R \xrightarrow{\mu} M \to T$ is zero, μ factorizes through rad M. On the other hand, the inclusion rad $M \to M$ is isomorphic to some composition rad $M \to R \xrightarrow{\mu} M$. Hence μ is a retraction of R onto rad M; this implies $R \xrightarrow{\sim} \mathrm{rad}\, M$, since R is indecomposable.

This proves our proposition when $\lambda(M) = 2$. In the general case, the rectangle starting at $(\alpha m, l-\alpha m+m)$ is contained in the union of the rectangle ending at $(m, \alpha m-m)$ with the rectangle starting at (m, l) and the going down diagonal through $(\alpha m, l-\alpha m+m)$. As both rectangles contain no $\bar{L}$-representative of a simple module besides $(m, \alpha m-m)$, we infer that the top of rad M is simple of $\bar{L}$-type $(\alpha m, i)$ with $i \leq l-\alpha m+m$. By our induction hypothesis, rad M is uniserial; hence $M/\mathrm{soc}\, M$ is uniserial and our proposition follows by applying the induction hypothesis both to rad M and $M/\mathrm{soc}\, M$. OK.

Remark. Our proposition applies in particular if M is of $\bar{L}$-type (m, h). If T is the top of M and P its projective cover, M is then isomorphic to $(P/\mathrm{soc}\, P)/P_\beta$, which is an extension of T by P_α (set $P_\beta = 0$ if $\alpha m = m+1$ and $P_\alpha = 0$ if $\alpha m = m+h$). Clearly, M is a *maximal A-module of class α*. This gives an intrinsic characterization of the A-modules which are $\bar{L}$-represented by vertices on the upper border of the stable lattice.

If M is a maximal of class α, a simple glance at Figure 26 below shows that $\alpha^\lambda m = m+h$ ($\lambda = 4$ in the particular case of the picture). Therefore M has as length the number of points in the intersection of $[m, m+h[$ with the α-orbit of m.

Moreover, we see that *the top of P_α is of $\bar{L}$-type $(\alpha m, \alpha^2 m-\alpha m)$, whereas its socle is of $\bar{L}$-type $(s, \alpha s-s)$ with $s = \alpha^{\lambda-1} m = \alpha^{-1} m+h$.*

3.7. In the sequel we shall also need the dual statements of the preceding

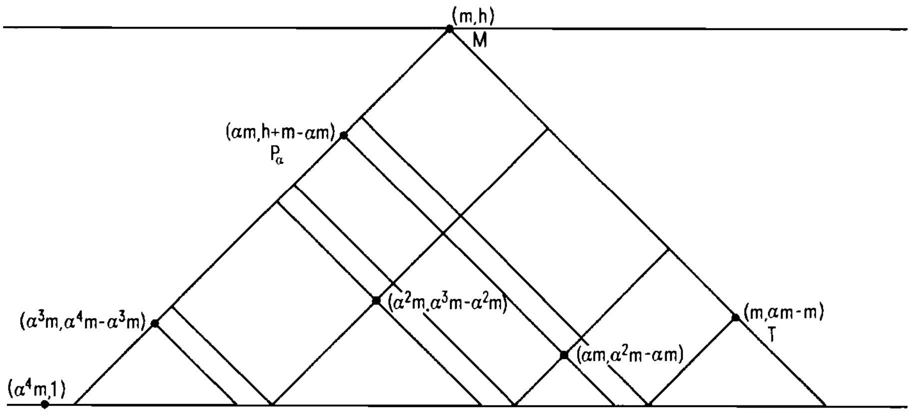


Figure 26

propositions. We produce them here without proofs:

a) Let I be the injective hull of a simple A-module S of $\bar{L}$-type $(s, \alpha s - s)$. If $\alpha s \neq s+1$, the socle of I_β is simple of $\bar{L}$-type $(\alpha s - h - 1, \alpha(\alpha s - 1) - \alpha s + 1)$. Moreover $\alpha(\alpha s - h - 1) \geq s + 1$.

b) The map $\mathbf{Z} \to \mathbf{Z}$, $s \mapsto \alpha s - h - 1$ is bijective. If we denote the inverse map by β, we get $\beta(\alpha s - h - 1) = s$, hence $\alpha\beta(\alpha s - h - 1) = \alpha s$. In other words, we get

$$\alpha\beta x = x + h + 1 = \pi(x) + 1$$

for any $x \in \mathbf{Z}$.

c) Let N be an indecomposable A-module of $\bar{L}$-type (n, g) and class β. Let $\beta^{-1}n, \beta^{-2}n, \ldots, \beta^{-\nu+1}n$ be the integers x lying in the β-orbit of n and such that $n > x > n + g - h - 1$. Then N has a unique Jordan–Hölder series

$$0 = N_\nu \subset N_{\nu-1} \subset \cdots \subset N_1 \subset N_0 = N,$$

and the uniserial subquotient N_i/N_{i+1} is of $\bar{L}$-type $(\beta^{-\nu+i+1}n, \alpha\beta^{-\nu+i+1}n - \beta^{-\nu+i+1}n)$. In particular, $g = \alpha\beta^{-\nu+1}n - n$. (See Figure 27).

d) The statement (c) applies in particular if N is of $\bar{L}$-type $(n, 1)$. If S is the socle of N and I its injective hull, N is then isomorphic to the inverse image of $I_\beta \subset I/S$ in I. The vertices lying on the lower border of the stable lattice *$\bar{L}$-represent the maximal A-modules of class β.*

e) If the module N of (c) is maximal of class β, we have $\beta^{-\nu}n = n - h$. The maximal A-module of class β and $\bar{L}$-type $(n, 1)$ has as length the number of points in the intersection of $]n-h, n]$ with the β-orbit of n.

f) Let I be the injective hull of a simple A-module of $\bar{L}$-type $(s, \alpha s - s)$. The

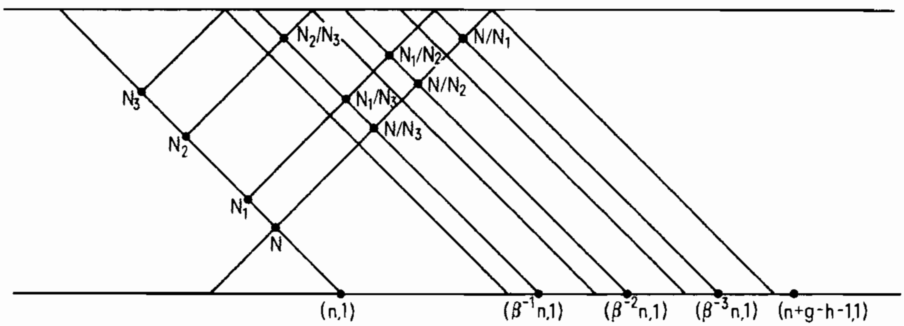


Figure 27

socle of I_β is of $\bar{L}$-type $(\beta^{-1}s, \alpha\beta^{-1}s-\beta^{-1}s)$; the top of I_β is of $\bar{L}$-type $(t, \alpha t-t)$ with $t=\beta s-h=\beta(s-h)$.

3.8. *The Brauer-quiver of a wreath-like algebra*

We know by 3.4 and 3.7(b) that the permutations α and β of $\mathbf{Z}$ commute with the translation $\pi: x \mapsto x+h$. Therefore they induce permutations $\bar{\alpha}$ and $\bar{\beta}$ of $e_h(\mathbf{Z})=\{\exp(2i\pi(x/h)): x \in \mathbf{Z}\}$ such that $\bar{\alpha}e_h(x)=e_h(\alpha x)$ and $\bar{\beta}e_h(x)=e_h(\beta x)$, $\forall x \in \mathbf{Z}$. We define a quiver Q by taking $e_h(\mathbf{Z})$ as set of vertices and by endowing this set with arrows $\bar{\alpha}_\rho: \rho \to \bar{\alpha}\rho$ and $\bar{\beta}_\rho: \rho \to \bar{\beta}\rho$.

PROPOSITION. *Q is an oriented Brauer quiver with cyclic permutation $\bar{\gamma}: \rho \mapsto \exp(2i\pi/h)\rho$. It has as universal covering $\tilde{Q}$ the set $\mathbf{Z}$ endowed with arrows $\alpha_x: x \to \alpha x$ and $\beta_x: x \to \beta x$.*

Proof. The relation $\alpha\beta x = x+h+1 = \pi x+1$ clearly implies $\bar{\gamma}=\bar{\alpha}\bar{\beta}$. It is therefore enough to prove that the convex hulls of distinct $\bar{\alpha}$-orbits do not intersect. This follows from the following property of the permutation α: if $x<y<\alpha x$, we have either $y<\alpha y<\alpha x$ or $x+h<\alpha y \leq y+h$. We illustrate the proof of this property simply by pictures representing the two possible cases:

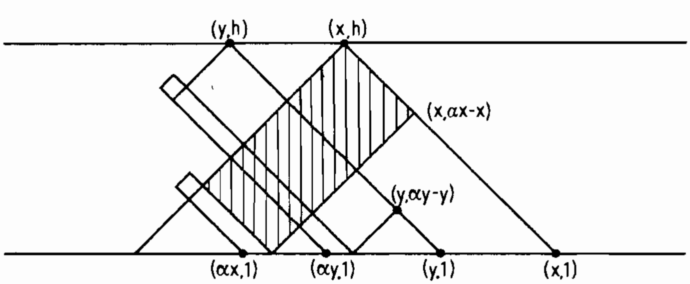


Figure 28a

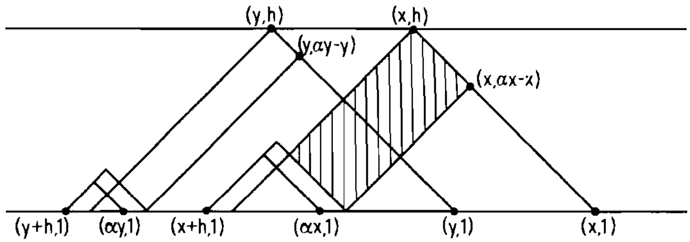


Figure 28b

The quiver representation attached to an A-module

3.9. Let J stand for the radical of A. We may assume that A is *basic*, i.e. that A/J is a product of fields. As we have $k \xrightarrow{\sim} \mathrm{Hom}_A(S, S)$ for any simple module S (3.1), all the simple factors of A/J are in fact isomorphic to k.

Let $1_A = \eta_1 + \cdots + \eta_e$ be a partition of unity into orthogonal primitive idempotents: $\eta_i\eta_j = \delta_{ij}\eta_i$. The primitivity means that, in the direct decomposition

$$A = A\eta_1 \oplus \cdots \oplus A\eta_e,$$

the non-isomorphic projective summands $A\eta_i$ are indecomposable. It is convenient to define η_n for every $n \in \mathbf{Z}$ by setting $\eta_n := \eta_{\bar{n}}$, where $\bar{n} \in \{1, 2, \ldots, e\}$ is congruent to n modulo e.

The associated decomposition of the top $\bar{A} := A/J$ is

$$\bar{A} = k\bar{\eta}_1 \oplus \cdots \oplus k\bar{\eta}_e$$

with $\bar{\eta}_i = \eta_i + J$. Clearly, the summands $k\bar{\eta}_i$ furnish a complete list of simple A-modules. We choose the numeration in such a way that $k\bar{\eta}_i$ is $\bar{L}$-represented by the vertex $(i, \alpha i - i)$.

We know by 3.4 and 3.7(f) that the top of the radical $J\eta_i$ of $A\eta_i$ has the form

$$J\eta_i/J^2\eta_i = (J/J^2)\eta_i = \bar{\eta}_{\alpha i}(J/J^2)\bar{\eta}_i \oplus \bar{\eta}_{\beta i}(J/J^2)\bar{\eta}_i$$

with

$$\bar{\eta}_{\alpha i}(J/J^2)\bar{\eta}_i \xrightarrow{\sim} \begin{cases} k\bar{\eta}_{\alpha i} & \text{if} \quad \alpha i \neq i+h \\ 0 & \text{if} \quad \alpha i = i+h \end{cases}$$

and

$$\bar{\eta}_{\beta i}(J/J^2)\bar{\eta}_i \xrightarrow{\sim} \begin{cases} k\bar{\eta}_{\beta i} & \text{if} \quad \alpha i \neq i+1 \\ 0 & \text{if} \quad \alpha i = i+1. \end{cases}$$

If $\alpha i \neq i+h$ we choose an $\alpha_i \in \eta_{\alpha i} J \eta_i - J^2$; its residue class $\bar{\alpha}_i = \alpha_i + J^2$ is then a basis of $\bar{\eta}_{\alpha i}(J/J^2)\bar{\eta}_i$. Similarly, if $\alpha i \neq i+1$, we choose an element $\beta_i \in \eta_{\beta i} J_{\eta i} - J^2$; its residue class $\bar{\beta}_i = \beta_i + J^2$ is a basis of $\bar{\eta}_{\beta i}(J/J^2)\bar{\eta}_i$. (As we shall see so, the elements α_i and β_i of A are related to the arrows $i \to \alpha i$ and $i \to \beta i$, which we already denoted by α_i and β_i. We hope, that this coincidence of notations will not lead to misunderstandings.)

THEOREM. *A is identified with the k-algebra with unity defined by the generators $\eta_i, \alpha_i, \beta_i$ subjected to the relations* a)–e) *below, where $1 \leq i, j \leq e$:*

a) $1_A = \eta_1 + \cdots + \eta_e$ *and* $\eta_i \eta_j = \delta_{ij}\eta_i$.

b) $\alpha_i = \eta_{\alpha i}\alpha_i\eta_i$ *and* $\beta_j = \eta_{\beta j}\beta_j\eta_j$, *if* $\alpha i \neq i+h$ *and* $\alpha j \neq j+1$.

c) $\beta_{\alpha i}\alpha_i = 0 = \alpha_{\beta j}\beta_j$ *if* $\alpha i \neq i+h$, $\alpha^2 i \neq \alpha i + 1$, $\alpha j \neq j+1$ *and* $\alpha\beta j \neq \beta j + h$.

d) $\alpha_{\alpha^{ai-1}i} \cdots \alpha_{\alpha i}\alpha_i + \lambda_i \beta_{\beta^{bi-1}i} \cdots \beta_{\beta i}\beta_i = 0$ *for some scalar* $\lambda_i \neq 0$ *if* $\alpha i \neq i+1$ *and* $\alpha i \neq i+h$.

e) $\alpha_{\alpha^{ai}i}\alpha_{\alpha^{ai-1}i} \cdots \alpha_{\alpha i}\alpha_i = 0$ *if* $\alpha i = i+1$, *and* $\beta_{\beta^{bj}j}\beta_{\beta^{bi-1}j} \cdots \beta_{\beta j}\beta_j = 0$ *if* $\alpha i = i+h$.

In the statement of the theorem ai and bj are such that $\alpha^{ai} i = i+h$ and $\beta^{bj} j = j+h$ (confer 1.6 and 3.8). Furthermore, we agree that $\alpha_n := \alpha_{\bar{n}}$ if $\alpha n \neq n+h$, and that $\beta_n := \beta_{\bar{n}}$ if $\alpha n \neq n+1$, where $\bar{n} \in \{1, \ldots, e\}$ is again assumed to be congruent to n modulo e.

3.10. *Proof of* a), b) *and* c). The relations a) and b) follow directly from our choices. Let us assume that for some i we have $\alpha i \neq i+h$, $\alpha^2 i \neq \alpha i + 1$ and $\beta_{\alpha i}\alpha_i \neq 0$. Consider a submodule N of $A\eta_i$ not containing $\beta_{\alpha i}\alpha_i$ and maximal for this condition, and set $M := A\eta_i/N$. Clearly, M is generated by $m := \eta_i + N$ and has simple top and simple socle. Moreover, the series

$$0 \subsetneqq A\beta_{\alpha i}\alpha_i m \subset J\alpha_i m \subsetneqq A\alpha_i m \subset Jm \subsetneqq Am = M$$

shows that M has at least the following three Jordan–Hölder factors:

$$\text{top}(M) = Am/Jm \xrightarrow{\sim} k\bar{\eta}_i,\ A\alpha_i m/J\alpha_i m \xrightarrow{\sim} k\bar{\eta}_{\alpha i} \quad \text{and}$$

$$\text{soc}(M) = A\beta_{\alpha i}\alpha_i m \xrightarrow{\sim} k\bar{\eta}_{\beta\alpha i}.$$

By Definition 3.6 M is either projective, of class α or of class β. Let us show that each of these possibilities leads to a contradiction.

First suppose that M is projective. Then $k\bar{\eta}_{\beta\alpha i} \xrightarrow{\sim} \text{soc}\, M \xrightarrow{\sim} k\bar{\eta}_{i+h}$ (3.3), hence $\beta\alpha i \equiv i+h \bmod e$ and $\mathrm{e}_d(\beta\alpha i) = \mathrm{e}_d(i)$, where $d = (h, e) =$ greatest common divisor of h and e. Accordingly, the β-orbit of $\mathrm{e}_d(\alpha i)$ meets the α-orbit of $\mathrm{e}_d(\alpha i)$ in $\mathrm{e}_d(\alpha i)$ and in $\mathrm{e}_d(i)$. Since $\bar{Q} = \mathrm{e}_d(\mathbf{Z})$ is a Brauer-quiver (1.11), we infer that

$e_d(\beta\alpha i) = e_d(i) = e_d(\alpha i)$. But the assumptions $\alpha i \neq i + h$ and $\alpha^2 i \neq \alpha i + 1$ mean that $e_h(i) \neq e_h(\alpha i) \neq e_h(\beta\alpha i)$. Since the projection $p : e_h(\mathbf{Z}) = Q \to e_d(\mathbf{Z}) = \bar{Q}$ acts bijectively on non-exceptional orbits (1.11), we infer that both the α-orbit and the β-orbit of $e_d(\alpha i)$ are exceptional: contradiction.

Now suppose that M is uniserial of class α. Then the Jordan–Hölder factors of M are $k\bar{\eta}_i, k\bar{\eta}_{\alpha i}, \ldots, k\bar{\eta}_{\alpha^a i}$ with $2 \leq a < ai$ (3.6). This implies $\beta\alpha i \equiv \alpha^a i \bmod e$, hence $e_d(\beta\alpha i) = e_d(\alpha^a i)$ and $e_d(\alpha i) = e_d(\beta\alpha i) = e_d(\alpha^a i)$ as in the first case. Accordingly, the α-orbit and the β-orbit of $e_d(\alpha i)$ are exceptional: contradiction.

Finally we assume that M is uniserial of class β. Then the Jordan–Hölder factors of M are $k\bar{\eta}_i, k\bar{\eta}_{\beta i}, \ldots, \beta\bar{\eta}_{\beta^b i}$ with $2 \leq b < bi$ (3.7). This implies $\alpha i \equiv \beta^c i \bmod e$ for some c, $1 \leq c < b$, hence $e_d(\alpha i) = e_d(\beta^c i)$ and further $e_d(\alpha i) = e_d(\beta^c i) = e_d(i)$. Here again we get the contradiction that both the α-orbit and the β-orbit of $e_d(i)$ are exceptional.

The relation $\alpha_{\beta j}\beta_j = 0$ is proved in a similar way.

3.11. *Proof of* d) *and* e). Set $P = A\eta_i$, and denote by P'_α and P'_β the counterimages of P_α and $P_\beta \subset P/\operatorname{soc} P$ in P. We have $P'_\alpha = A\alpha_i$ by construction of α_i and the length of P'_α satisfies the relation $\lambda(P'_\alpha) = \lambda((P/\operatorname{soc} P)/P_\beta) = \lambda(P/P'_\beta) = ai$ (3.6, Remark). On the other hand, the radical of P'_α is given by the equations

$$JP'_\alpha = J\alpha_i = J\eta_{\alpha i}\alpha_i = (A\alpha_{\alpha i} + A\beta_{\alpha i})\alpha_i = A\alpha_{\alpha i}\alpha_i,$$

which implies by induction

$$J^2 P'_\alpha = A\alpha_{\alpha^2 i}\alpha_{\alpha i}\alpha_i, \ldots, \operatorname{soc} P = J^{ai-1}P'_\alpha = A\alpha_{\alpha^{ai-1}i} \cdots \alpha_{\alpha i}\alpha_i.$$

Replacing α by β we get in a similar way

$$\operatorname{soc} P = A\beta_{\beta^{bi-1}i} \cdots \beta_{\beta i}\beta_i,$$

and hence $k\alpha_{\alpha^{ai-1}i} \cdots \alpha_{\alpha i}\alpha_i = k\beta_{\beta^{bi-1}i} \cdots \beta_{\beta i}\beta_i$, which is equivalent to relation d).

Let us now prove e): Again set $P = A\eta_i$. The relation $\alpha i = i + 1$ then implies $P_\beta = 0$, hence $\operatorname{rad} P = A\alpha_i$ and $\operatorname{soc} P = J^{ai-1}\alpha_i = A\alpha_{\alpha^{ai-1}i} \cdots \alpha_{\alpha i}\alpha_i$ as in the former case. The first relation of e) follows from the fact that $\alpha_{\alpha^{ai}i} \operatorname{soc} P \subset J \operatorname{soc} P = 0$. Similar arguments hold for the last relation.

3.12. *End of the proof of Theorem 1.9.* Let B be the algebra generated by the elements $\eta_i, \alpha_i, \beta_i$ and the relations a)–e). Let $\varphi : B \to A$ be the homomorphism which is the identity on the generators. Since the η_i form a basis of A modulo J, and since the α_i, β_i form a basis of J modulo J^2, it is easily seen that $\eta_i, \alpha_i, \beta_i$

generate the algebra A. Hence φ is surjective. Now it follows immediately from the relations a)–e) that the elements η_i, $\alpha_{\alpha^a i} \cdots \alpha_{\alpha i}\alpha_i$ and $\beta_{\beta^b i} \cdots \beta_{\beta i}\beta_i$ with $a < ai$ and $b < bi$ generate B as a vector space. Therefore we have $\dim_k B \leq \sum_i (ai + bi)$. On the other hand, the knowledge of the length of the maximal uniserial modules (3.6 and 3.7) implies immediately that $\dim_k A\eta_i = a_i + b_i$. This implies $\dim_k A = \sum_i (ai + bi) \geq \dim_k B$. Hence φ is injective. OK.

3.13. LEMMA. *The elements α_i and β_i may be chosen in such a way that we have*

$$\alpha_{\alpha^{ai-1}i} \cdots \alpha_{\alpha i}\alpha_i + \beta_{\beta^{bi-1}i} \cdots \beta_{\beta i}\beta_i = 0$$

whenever $\alpha i \neq i+1$ and $\alpha i \neq i+h$.

Proof. It is clear that, for a fixed i, the scalar λ_i appearing in relation d) of 1.9 may be replaced by 1 if we replace either α_i or β_i by some scalar multiple. The point is that an improvement at i might produce a deterioriation at some j. In order to proceed to our modifications in a coherent way, we first consider the Brauer-quiver $\bar{Q} = e_d(\mathbf{Z})$ introduced in 1.11, where $d = (e, h)$. If $d \neq h$, $\bar{Q}$ contains an exceptional orbit; if $d = h$, we choose an arbitrary orbit and call it exceptional. Since $\bar{Q}$ is associated with some Brauer tree (1.4), it is clearly possible to provide the cycles of $\bar{Q}$ with a total order satisfying the following conditions: a) The exceptional cycle is the smallest cycle. b) For any non-exceptional cycle Γ of $\bar{Q}$ there is exactly one cycle $\Delta < \Gamma$ having with Γ one common vertex.

The choice of such a total order determines our modification process. We proceed by induction: Assume that for some α-cycle Γ all the λ_i, such that $e_d(i)$ belongs to a cycle $E < \Gamma$, have already been replaced by 1 in our procedure. Let U be the set of elements $u \in \{1, \ldots, e\}$ such that $\lambda_u \neq 1$ and $e_d(u) \in \Gamma$. Since the β-orbit of any such $e_d(u)$ is non-exceptional, we know by 1.11 that the vertices $e_d(\beta^n u)$, where $u \in U$ and $0 < n < bu$, are all distinct, and they do not belong to any cycle $E \leq \Gamma$. In particular, all the elements $\beta_u \in A$ are distinct. Replacing then by scalar multiples, we can convert λ_u into 1 at once for all the elements $u \in U$. Moreover, our modification does not affect the coefficients $\lambda_i = 1$, such that $e_d(i) \in E < \Gamma$.

We proceed in a similar way when the first cycle Γ, which is not yet "clean," is a β-cycle. OK.

3.14. We can now simplify the description of A by generators and relations with the following *convention*: if $\alpha i = i + h$ we define an element $\alpha_i \in A$ by means of the equality $\alpha_i := -\beta_{\beta^{bi-1}i} \cdots \beta_{\beta i}\beta_i$; similarly, if $\alpha i = i + 1$ we set $\beta_i := -\alpha_{\alpha^{ai-1}i} \cdots \alpha_{\alpha i}\alpha_i$. The relations *e*) are then reduced to $\alpha_{\beta i}\beta_i = 0$ if $\alpha i = i + 1$, and to

$\beta_{\alpha j}\alpha_j = 0$ if $\alpha j = j + h$. Therefore, theorem 3.9 can be restated by saying that A is identified with the k-algebra defined by generators $\eta_i, \alpha_i, \beta_i$ $(i = 1, \ldots, e)$ submitted to the relations:

a) $1_A = \eta_1 + \cdots + \eta_e$ and $\eta_i\eta_j = \delta_{ij}\eta_i$
b) $\alpha_i = \eta_{\alpha i}\alpha_i\eta_i$ and $\beta_j = \eta_{\beta j}\beta_j\eta_j$
c) $\beta_{\alpha i}\alpha_i = 0 = \alpha_{\beta j}\beta_j$
d) $\alpha_{\alpha^{ai-1}} \cdots \alpha_{\alpha i}\alpha_i + \beta_{\beta^{bi-1}i} \cdots \beta_{\beta i}\beta_i = 0$
for any $i, j \in \{1, \ldots, e\}$.

Let $\tilde{Q}$ be the universal covering of the oriented Brauer-quiver associated with A (3.8). It is now easy to establish an equivalence between $\operatorname{Mod} A$ and the category of e-periodic representations of $\tilde{Q}$: we may attach to any A-module M an e-periodic representation V of $\tilde{Q}$ by setting $V(i) = \eta_i M$, $V(\alpha_i)(x) = \alpha_i x$ and $V(\beta_i)(x) = \beta_i x$ (Notice that on the left hand side α_i and β_i denote arrows, whereas on the right hand side they stand for elements of A!).

Reversely, if V is an e-periodic representation of $\tilde{Q}$, we set $M = V(1) \oplus \cdots \oplus V(e)$ and define the A-module structure on M as follows:

$$\eta_i x = x,\ \alpha_i x = V(\alpha_i)(x) \in V(\alpha i) \subset M$$

and

$$\beta_i x = V(\beta_i)(x) \in V(\beta i) \subset M$$

if $x \in V(i) \subset M$. If $x \in V(j)$, $j \neq i$, we set

$$0 = \eta_i x = \alpha_i x = \beta_i x.$$

This finishes the proof of Theorem 2 of § 1.8.

4. Bounden Brauer-quiver algebras are wreath-like

Our purpose in §4 is to prove Theorem 1 of §1.8. By Q we denote a fixed oriented Brauer-quiver with h vertices.

4.1. With any representation W of the universal covering $\tilde{Q}$ of Q (1.6) we associate vector spaces $(LW)(s)$, $s \in \mathbf{Z}$, and transition maps $(LW)(\gamma)$: $(LW)(s-1) \to (LW)(s)$ which are described as follows:

$$(LW)(s) = \bigoplus_{t \le s < \alpha t} W(t).$$

Moreover $\gamma := (LW)(\gamma)$ maps a family (w_t) onto the family (w'_t) such that

$$w'_t = w_t \quad \text{if} \quad t \neq s$$

and

$$w'_s = \alpha(w_{\alpha^{-1}s}) - \sum_{1 \leq i < bs} \beta^i(w_{\beta^{-i}s}),$$

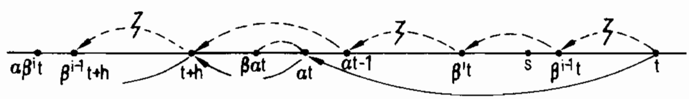
where α and β^i simply stand for $W(\alpha_{\alpha^{-1}s})$ and $W(\beta_{\beta^{-1}s})W(\beta_{\beta^{-2}s}) \cdots W(\beta_{\beta^{-i}s})$.

LEMMA. *If W is a representation of the bounden quiver $(\tilde{Q}, \tilde{I})$ (1.6), the transition maps $\gamma = (LW)(\gamma)$ satisfy the relations $\gamma^{h+1} = 0$, which means that LW is a representation of the bounden quiver $\tilde{Z}_h$.*

Proof. Denote by (t, w) the canonical image of $w \in W(t)$ in

$$(LW)(s) = \bigoplus_{t \leq s < \alpha t} W(t).$$

We have to prove that $\gamma^{h+1}(t, w) = 0$. With this aim in mind, we first consider the case $s \neq \alpha t - 1$ and have a look at the following figure where $\beta^{i-1}t \leq s < \beta^i t$, $i \geq 1$ (confer 1.5 and 1.6). The following relations hold (notice that $\alpha t - 1 = \beta^{bt-1}t$, $t + h = \alpha^{at}t$ and $\alpha\beta^i t = \beta^{i-1}t + h + 1$):



$$\gamma^{\beta^i t - s}(t, w) = (t, w) - (\beta^i t, \beta^i w)$$

$$\gamma^{\beta^{i+1}t - s}(t, w) = [(t, w) - (\beta^{i+1}t, \beta^{i+1}w)] - [(\beta^i t, \beta^i w) - (\beta^{i+1}t, \beta^{i+1}w)]$$

$$= (t, w) - (\beta^i t, \beta^i w)$$

$$\gamma^{\alpha t - 1 - s}(t, w) = (t, w) - (\beta^i t, \beta^i w)$$

$$\gamma^{\alpha t - s}(t, w) = (\alpha t, \alpha w) - (\beta^i t, \beta^i w) \quad \text{in case} \quad \alpha t \neq t + h$$

$$\gamma^{\beta\alpha t - s}(t, w) = [(\alpha t, \alpha w) - (\beta\alpha t, \beta\alpha w)] - (\beta^i t, \beta^i w) = (\alpha t, \alpha w) - (\beta^i t, \beta^i w)$$

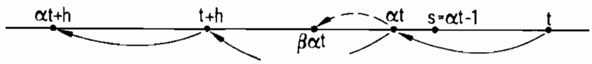
$$\gamma^{t+h-s}(t, w) = (t + h, \alpha^{at}w) - [(\beta^i t, \beta^i w) - (t + h, \beta^{bt}w)]$$

$$= (t + h, \alpha^{at}w + \beta^{bt}w) - (\beta^i t, \beta^i w) = -(\beta^i t, \beta^i w) \text{ even in case } \alpha t = t + h$$

$$\gamma^{\beta^{i-1}t+h-s}(t, w) = -(\beta^i t, \beta^i w)$$
$$\gamma^{\alpha\beta^i t-s}(t, w) = -(\alpha\beta^i t, \alpha\beta^i w) = 0.$$

Therefore we have $\gamma^{h+1}(t, w) = \gamma^{s-\beta^{i-1}t}\gamma^{\alpha\beta^i t-s}(t, w) = 0$.
In case $s = \alpha t - 1$, we get in a similar way:

$$\gamma(t, w) = (\alpha t, \alpha w)$$
$$\gamma^{\beta\alpha t-s}(t, w) = (\alpha t, \alpha w) - (\beta\alpha t, \beta\alpha w) = (\alpha t, \alpha w)$$
$$\gamma^{t+h-s}(t, w) = (t+h, \alpha^{at} w)$$
$$\gamma^{h+1}(t, w) = (\alpha t + h, \alpha\alpha^{at} w) = 0. \quad \text{OK.}$$



4.2. Define a representation $\tilde{V}_{s,m}$ of $\tilde{Z}_h$ by setting $\tilde{V}_{s,m}(r) = k\gamma^{r-s}\zeta_s$ if $s \le r < s+m$ and $\tilde{V}_{s,m}(r) = 0$ otherwise, where $\zeta_s \in \tilde{V}_{s,m}(s)$ denotes some symbol playing the part of a generator of $\tilde{V}_{s,m} (1 \le m \le h+1)$. Clearly, $\tilde{V}_{s,m}$ is indecomposable, and every representation of $\tilde{Z}_h$ is a direct sum of indecomposables of this type (compare with 1.2). Moreover, $\tilde{V}_{s,m}$ is projective in $\mathrm{Mod}_k\, \tilde{Z}_h$ iff $m = h+1$. If this is so, the map $f \mapsto f(\zeta_s)$ furnishes a bijection $\mathrm{Hom}\,(\tilde{V}_{s,h+1}, V) \tilde{\to} V(s)$ for any $V \in \mathrm{Mod}_k\, \tilde{Z}_h$.

On the other hand, consider the representation P_t of $(\tilde{Q}, \tilde{I})$ which is defined as follows: start with some "free generator" $\pi_t \in P_t(t)$, $t \in \mathbf{Z}$, and set

$$P_t(r) = \begin{cases} k\alpha^i\pi_t & \text{if} \quad t+h \ge \alpha^i t = r \ge t \\ k\beta^i\pi_t & \text{if} \quad t+h > \beta^i t = r > t \\ 0 & \quad\text{otherwise} \end{cases}$$

Clearly, P_t is indecomposable and projective (the map $\mathrm{Hom}\,(P_t, W) \to W(t)$, $f \mapsto f(\pi_t)$ is bijective for each $W \in \mathrm{Mod}_k\, (\tilde{Q}, \tilde{I})$). Using classical Nakayama-type arguments we see furthermore that any projective representation of $(\tilde{Q}, \tilde{I})$ is a direct sum of copies of P_t for various t.

LEMMA. *For any t, LP_t is freely generated by the elements*

$$(\beta^i t, \beta^i \pi_t) \in (LP_t)(\beta^i t), \qquad i = 0, 1, \ldots, bt-1.$$

In other words, the morphism

$$\bigoplus_{0\le i<bt} \tilde{V}_{\beta^i t, h+1} \xrightarrow{\mu_t} LP_t$$

sending $\zeta_{\beta^i t}$ *onto* $(\beta^i t, \beta^i \pi_t)$ *is an isomorphism.*

Proof. The constituents $(LP_t)(s)$ of LP_t are as follows:

a) $(LP_t)(s) = \bigoplus_{j=0}^{j=i} k(\beta^i t, \beta^i \pi_t)$ if $\beta^i t \le s < \inf(\beta^{i+1} t, \alpha t)$,

where $i = 0, 1, 2, \ldots, bt-1$.

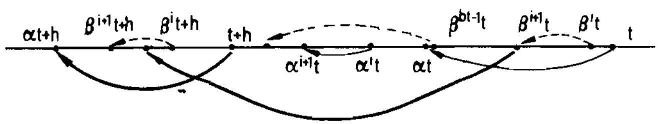
b) $(LP_t)(s) = k(\alpha^l t, \alpha^l \pi_t) \oplus \left(\bigoplus_{j=1}^{bt-1} k(\beta^i t, \beta^i \pi_t) \right)$

if $\alpha^l t \le s < \inf(\alpha^{l+1} t, t+h+1)$, where $l = 1, 2, \ldots,$ at.

c) $(LP_t)(s) = \bigoplus_{j=i+2}^{j=bt} k(\beta^j t, \beta^j \pi_t)$ if $\beta^i t + h < s \le \beta^{i+1} t + h$,

where $i = 0, 1, \ldots, bt-2$.

d) $(LP_t)(s) = 0$ in any other case.



For $i = 0, 1, \ldots, bt-1$, we have

$$\gamma(LP_t)(\beta^i t - 1) = \bigoplus_{j=0}^{j=i-1} k((\beta^i t, \beta^j \pi_t) - (\beta^i t, \beta^i \pi t)),$$

hence $(LP_t)(\beta^i t) = \gamma(LP_t)(\beta^i t - 1) \oplus k(\beta^i t, \beta^i \pi_t)$. For any others it is easily verified that $(LP_t)(s) = \gamma(LP_t)(s-1)$. By Nakayama's lemma this implies that μ_t is an epimorphism. In order to prove that μ_t is invertible, it remains to verify the equality $\sum_s \dim\ (LP_t)(s) = (h+1)(bt) = \sum_{i,s} \dim \tilde{V}_{\beta^i t, h+1}(s)$. This is pure routine. OK.

4.3. Since the functor L commutes with direct limits, it admits a right adjoint functor $R: \mathrm{Mod}_k\, \tilde{Z}_h \to \mathrm{Mod}_k\, (\tilde{Q}, \tilde{I})$, which homological algebra tells us to be the following: denote by $\bar{\alpha}: P_{\alpha t} \to P_t$ and $\bar{\beta}: P_{\beta t} \to P_t$ the morphisms sending $\pi_{\alpha t}$ onto $\alpha\pi_t$ and $\pi_{\beta t}$ onto $\beta\pi_t$ respectively. For any $V \in \mathrm{Mod}_k \tilde{Z}_h$ and any $t \in \mathbf{Z}$ we then have:

$$(RV)(t) = \mathrm{Hom}\,(LP_t, V), \qquad (RV)(\alpha) = \mathrm{Hom}\,(L\bar{\alpha}, V),$$
$$(RV)(\beta) = \mathrm{Hom}\,(L\bar{\beta}, V).$$

Moreover, the adjunction-bijection

$$u: \mathrm{Hom}\,(LW, V) \tilde{\to} \mathrm{Hom}\,(W, RV)$$

associates with a morphism $f: LW \to V$ the morphisms $u(f): W \to RV$, which maps $w \in W(t)$ onto the composition $LP_t \xrightarrow{Lw'} LW \xrightarrow{f} V$. Here w' denotes the morphism $P_t \to W$ such that $w'(\pi_t) = w$.

General rules need specification. In fact, the isomorphism μ_t of 4.2 allows the following identifications:

$$(RV)(t) = \mathrm{Hom}\,(LP_t, V) \tilde{\to} \mathrm{Hom}\,(\oplus \tilde{V}_{\beta^i t, h+1}, V) \tilde{\to} \bigoplus_{i=0}^{i=bt-1} V(\beta^i t).$$

A morphism $f: LP_t \to V$ is identified with the sequence $u_t(f) = (f(\beta^i t, \beta^i \pi_t)) \in \oplus_i V(\beta^i t)$.

LEMMA. $(RV)(\beta)$: $(RV)(t) \to (RV)(\beta t)$ *is identified with the map*

$$\bigoplus_{0 \le i < bt} V(\beta^i t) \to \bigoplus_{0 \le i < bt} V(\beta^{i+1} t)$$

given by the matrix

$$B = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & \cdots \\ 0 & 0 & \mathbf{1} & 0 & \cdots \\ 0 & 0 & 0 & \mathbf{1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\gamma^h & -\gamma^{h+t-\beta t} & -\gamma^{h+t-\beta^2 t} & -\gamma^{h+t-\beta^3 t} & \cdots \end{bmatrix}$$

Proof. Consider the square (where $\beta^* = \mathrm{Hom}(L\bar{\beta}_1 V)$)

$$\begin{array}{ccc} f \in \mathrm{Hom}\,(LP_t, V) & \xrightarrow{\beta^*} & \mathrm{Hom}\,(LP_{\beta t}, V) \\ {\scriptstyle u_t}\downarrow{\scriptstyle f} & & {\scriptstyle f}\downarrow{\scriptstyle u_{\beta t}} \\ \oplus V(\beta^i t) & \xrightarrow{B} & \oplus V(\beta^{i+1} t) \end{array}$$

By definition we have

$$u_{\beta t}(\beta^* f) = ((\beta^* f)(\beta^i \beta t, \beta^i \pi_{\beta t})) = (f(L\bar{\beta}(\beta^{i+1} t, \beta^i \pi_{\beta t})))$$
$$= (f(\beta^{i+1} t, \beta^{i+1} \pi_t)).$$

The last component of this sequence is $f(\beta^{bt}t, \beta^{bt}\pi t)$, where $\beta^{bt}t = t+h$ and $(\beta^{bt}t, \beta^{bt}\pi_t) \in (LP_t)(t+h)$. Let us express $(\beta^{bt}t, \beta^{bt}\pi_t)$ in terms of the "basis" $((\beta^i t, \beta^i \pi_t))_{0\le i<bt}$ of LP_t. The following relations hold in $(LP_t)(t+h)$ (they are reduced to the first equation in case $bt = 1$):

$$\gamma^h(t, \pi_t) = -(\beta t, \beta\pi_t)$$
$$\gamma^{h+t-\beta t}(\beta t, \beta\pi_t) = (\beta t, \beta\pi_t) - (\beta^2 t, \beta^2\pi_t)$$
$$\gamma^{h+t-\beta^2 t}(\beta^2 t, \beta^2\pi_t) = (\beta^2 t, \beta^2\pi_t) - (\beta^3 t, \beta^3\pi_t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\gamma^{h+t-\beta^{bt-1}t}(\beta^{bt-1}t, \beta^{bt-1}\pi_t) = (\beta^{bt-1}t, \beta^{bt-1}\pi_t) - (\beta^{bt}t, \beta^{bt}\pi_t).$$

Hence we get by addition

$$(\beta^{bt}t, \beta^{bt}\pi_t) = -\gamma^h(t, \pi_t) - \gamma^{h+t-\beta t}(\beta t, \beta\pi_t) - \cdots,$$

and the last component of $u_{\beta t}(\beta^* f)$ equals

$$f(\beta^{bt}t, \beta^{bt}\pi_t) = [-\gamma^h - \gamma^{h+t-\beta t} \cdots - \gamma^{h+t-\beta^{bt-1}t}] \begin{bmatrix} f(t, \pi_t) \\ f(\beta t, \beta\pi_t) \\ \vdots \end{bmatrix}$$

In this product the left factor is the last row of B, whereas the right factor is $u_t(f)$. Therefore $f(\beta^{bt}t, \beta^{bt}\pi_t)$ is also the last component of $Bu_t(f)$. Since the first components of $Bu_t(f)$ and $u_{\beta t}(\beta^* f)$ coincide trivially, we see that $Bu_t = u_{\beta t}\beta^*$. OK.

4.4. LEMMA. $(RV)(\alpha)$: $(RV)(t) \to (RV)(\alpha t)$ *is identified with the map*

$$\bigoplus_{0\le i<bt} V(\beta^i t) \to \bigoplus_{0\le j<b\alpha t} V(\beta^j \alpha t)$$

given by the matrix

$$A = \begin{bmatrix} \gamma^{\alpha t-t} & \gamma^{\alpha t-\beta t} & \gamma^{\alpha t-\beta^2 t} \cdots & \gamma^{\alpha t-\beta^{bt-1}t} \\ 0 & 0 & 0 \quad \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof. Consider the square

$$\begin{array}{ccc} f\in \operatorname{Hom}(LP_t, V) & \xrightarrow{\alpha^*=\operatorname{Hom}(L\bar{\alpha},V)} & \operatorname{Hom}(LP_{\alpha t}, V) \\ {\scriptstyle u_t}\downarrow\wr & & \wr\downarrow{\scriptstyle u_{\alpha t}} \\ \bigoplus V(\beta^i t) & \xrightarrow{\quad A \quad} & \bigoplus V(\beta^j \alpha t) \end{array}$$

By definition we have

$$\begin{aligned} u_{\alpha t}(\alpha^* f) = ((\alpha^* f)(\beta^j \alpha t, \beta^j \pi_{\alpha t})) &= (f(L\bar{\alpha}(\beta^j\alpha t, \beta^j \pi_{\alpha t}))) \\ &= (f(\beta^j \alpha t, \beta^j \alpha \pi_t)) \\ &= (f(\alpha t, \alpha\pi_t), 0, 0, \ldots), \end{aligned}$$

where $(\alpha t, \alpha\pi_t)\in(LP_t)(\alpha t)$. We express $(\alpha t, \alpha\pi_t)$ in terms of the basis $((\beta^i t, \beta^i\pi_t))$ of LP_t by using the following relations, which hold in $(LP_t)(\alpha t)$ if $bt\neq 1\neq$ at:

$$\begin{aligned} &\gamma^{\alpha t-t}(t,\pi_t) = (\alpha t, \alpha\pi_t) - (\beta t, \beta\pi_t) && (1)\\ &\gamma^{\alpha t-\beta t}(\beta t, \beta\pi_t) = (\beta t, \beta\pi_t) - (\beta^2 t, \beta^2\pi_t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\gamma^{\alpha t-\beta^{bt-2}t}(\beta^{bt-2}t, \beta^{bt-2}\pi_t) = (\beta^{bt-2}t, \beta^{bt-2}\pi_t) - (\beta^{bt-1}t, \beta^{bt-1}\pi_t) \\ &\gamma^{\alpha t-\beta^{bt-1}t}(\beta^{bt-1}t, \beta^{bt-1}\pi_t) = (\beta^{bt-1}t, \beta^{bt-1}\pi_t) && (2) \end{aligned}$$

By addition we get:

$$(\alpha t, \alpha\pi_t) = \gamma^{\alpha t-t}(t,\pi_t) + \gamma^{\alpha t-\beta t}(\beta t, \beta\pi_t) + \cdots + \gamma^{\alpha t-\beta^{bt-1}t}(\beta^{bt-1}t, \beta^{bt-1}\pi_t). \qquad (3)$$

If $bt=1$ this equation is reduced to $(\alpha t, \alpha\pi_t)=\gamma(t,\pi_t)$ and it is true by the very definition of γ. Finally, if at $=1$, we have to replace equation (1) by $\gamma^{\alpha t-t}(t,\pi_t) = -(\beta t, \beta\pi_t)$ and equation (2) by

$$\gamma^{\alpha t-\beta^{bt-1}t}(\beta^{bt-1}t, \beta^{bt-1}\pi_t) = (\beta^{bt-1}t, \beta^{bt-1}\pi_t) - (\beta^{bt}t, \beta^{bt}\pi_t).$$

Addition again furnishes equation (3).

In all the cases we get

$$f(\alpha t, \alpha\pi_t) = [\gamma^{\alpha t-t}\gamma^{\alpha t-\beta t}\cdots]\begin{bmatrix} f(t,\pi_t) \\ f(\beta t, \beta\pi_t) \\ \vdots \end{bmatrix}$$

and therefore $u_{\alpha t}(\alpha^* f) = Au_t(f)$. OK.

4.4. We come back now to the adjunction-bijection

$$\mathrm{Hom}\,(LW, V) \xrightarrow{\sim} \mathrm{Hom}\,(W, RV)$$

of 4.3. In case $V = LW$ we denote by $\Psi W : W \to RLW$ the morphism associated with $\mathbf{1}_{LW}$. Similarly, if $W = RV$, we write $\Phi V : LRV \to V$ for the inverse image of $\mathbf{1}_{RV}$.

LEMMA. *Let S_s be the simple representation of $(\tilde{Q}, \tilde{I})$ such that $S_s(t) = 0$ if $t \neq s$ and $S_s(s) = k$. Then $\Psi S_s : S_s \to RLS_s$ is a section and*

$$\mathrm{Coker}\,\Psi S_s \xrightarrow{\sim} \bigoplus_t P_t,$$

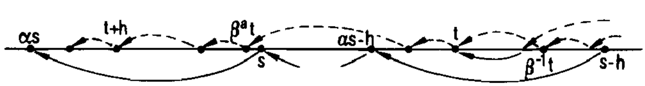
where t is subjected to the relations $\alpha s > \alpha t > s > t$.

Proof. Set $V := LS_s$ and denote by ζ a non-zero element in $V(s)$. We get $V(t) = k\gamma^{t-s}\zeta$ if $\alpha s > t \geq s$ and $V(t) = 0$ otherwise (in fact $V \xrightarrow{\sim} \tilde{V}_{s,\alpha s - s}$). Let us set further $W := RV = RLS_s$ and $W'(t) := \alpha W(\alpha^{-1}t) + \beta W(\beta^{-1}t) \subset W(t)$ for any $t \in \mathbf{Z}$. By construction we have $W(t) = \bigoplus_i k\gamma^{\beta^i t - s}\zeta$, where i is subjected to the conditions $0 \leq i < bt$ and $s \leq \beta^i t < \alpha s$. We want to examine $W'(t)$ for various t:

a) *Suppose that $\alpha s - h > t > \beta^{-1}t \geq s - h$.* By the geometric properties of Brauer-quivers examined in 1.5 this condition is equivalent to $\alpha s > \alpha t > s > t$. If $a > 0$ is the first integer such that $\beta^a t \geq s$, we get $W(t) = \bigoplus_i k\gamma^{\beta^i t - s}\zeta$ with $bt > i \geq a$, $W(\alpha^{-1}t) = 0$ and $W'(t) = \beta W(\beta^{-1}t) = \bigoplus_j k(\gamma^{\beta^j t - s}\zeta - \gamma^{\beta^{bt-1}t - s}\zeta)$ with $bt - 1 > j \geq a$. Hence we have $W(t) = W'(t) \oplus k\gamma^{\beta^a t - s}\zeta$.

Notice that $\alpha s > \alpha t = \beta^{-1}t + h + 1 > s$. This implies that $\alpha(\gamma^{\beta^a t - s}\zeta) = \gamma^{\alpha t - s}\zeta \neq 0$, and further that $0 \neq \gamma^{t+h-s}\zeta = \alpha^{at}(\gamma^{\beta^a t - s}\zeta) \in W(t+h)$. In other words, the morphism $\varphi_t : P_t \to RV$, $\pi_t \mapsto \gamma^{\beta^a t - s}\zeta$ maps the generator $\alpha^{at}\pi_t$ of the socle of P_t onto $\gamma^{t+h-s}\zeta \neq 0$. Accordingly, φ_t is a monomorphism. (See Figure below.)

b) *Suppose that $t = s$.* As in case a) we have $W(s) = \bigoplus_i k\gamma^{\beta^i s - s}\zeta$ with $bt > i \geq 0$, $W(\alpha^{-1}t) = 0$, $W'(s) = \beta W(\beta^{-1}s) = \bigoplus_j k(\gamma^{\beta^j s - s}\zeta - \gamma^{\beta^{bs-1}s - s}\zeta)$ with $bs - 1 > j \geq 0$, and finally $W(s) = W'(s) \oplus k\zeta$.



The important point now is that $\alpha\zeta = 0 = \beta\zeta$. Therefore ζ determines a monomorphism $\varphi_s : S_s \to W$, which maps $1 \in k = S_s(s)$ onto ζ.

c) *Suppose that* $\alpha s > t > s$. If $b \geq 0$ is the largest integer such that $\alpha s > \beta^b t$, we have $W(t) = \bigoplus_i k\gamma^{\beta^i t - s}\zeta$ with $0 \leq i \leq b$. Furthermore we have $t-1 = \beta^{b\alpha^{-1}t-1}(\alpha^{-1}t) \geq s$ and therefore $\gamma^{t-s-1}\zeta \in W(\alpha^{-1}t)$ and $\alpha W(\alpha^{-1}t) = k\gamma^{t-s}\zeta$. On the other hand, we have $\beta W(\beta^{-1}t) = \bigoplus_j k(\gamma^{\beta^j t - s}\zeta - \gamma^{\beta^{bt-1}t-s}\zeta)$ with $0 \leq j < bt-1$ or $\beta W(\beta^{-1}t) = \bigoplus_j k(\gamma^{\beta^j t - s}\zeta)$ with $0 \leq j \leq b$ according as $b = bt-1$ or $b < bt-1$. In both cases we get $W'(t) = W(t)$.

d) When t does not satisfy the conditions a), b) or c), we have $W'(t) = W(t) = 0$.

Our results concerning $W'(t)$ may now be exploited as follows: Look at the map

$$\varphi : S_s \oplus \Big(\bigoplus_t P_t\Big) \to W = RLS_s$$

with components φ_s and φ_t, where t is subjected to $\alpha s > \alpha t > s > t$. For each n we have $W(n) = W'(n) + \operatorname{Im}\varphi(n)$. Therefore φ is an epimorphism (Nakayama). On the other hand, the socle of $S_s \oplus (\bigoplus_t P_t)$ is generated by the elements $1 \in k = S_s(s)$ and $\alpha^{at}(\pi_t) \in P_t(t+h)$. These elements all have distinct degrees, and we know by a) and b) that they are mapped onto non-zero elements. Therefore φ is a monomorphism, hence an isomorphism.

It remains to show that ΨS_s maps S_s isomorphically onto $\varphi_s(S_s)$. In fact, we know that $\Psi S_s \neq 0$, because $L\Psi S_s$ admits the retraction ΦLS_s (general nonsense!) and $LS_s \neq 0$. On the other hand, the only copy of S_s contained in RLS_s is $\varphi_s(S_s)$, so that there is no alternative. OK.

4.6. LEMMA. *If* $V = \tilde{V}_{s,1}$ *the morphism* $\Phi V : LRV \to V$ *is a retraction, and*

$$\operatorname{Ker} \Phi V \xrightarrow{\sim} \bigoplus_i \tilde{V}_{\beta^{-i}s,h+1}$$

where $0 < i < bs$.

Proof. By 4.3 and 4.4 RV admits the following description:

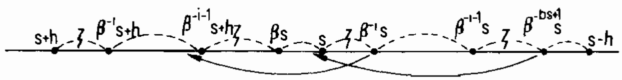
$$(RV)(t) = \begin{cases} k\varepsilon_i & \text{if} \quad t = \beta^{-i}s \quad \text{with} \quad 0 \leq i < bs \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta\varepsilon_0 = 0$, $\beta\varepsilon_i = \varepsilon_{i-1}$ for $i > 0$ and $\alpha\varepsilon_i = 0$ for any i. Therefore, if $U := LRV$, we get by the construction of L that

$$U(t) = (LRV)(t) = \bigoplus_i k(\beta^{-i}s, \varepsilon_i),$$

where i is subjected to the conditions $0 \le i < bs$ and $\alpha\beta^{-i}s - 1 = \beta^{-i-1}s + h \ge t \ge \beta^{-i}s$. The transition operator γ acts according to the formulae

$$\gamma(\beta^{-i}s, \varepsilon_i) = \begin{cases} (\beta^{-i}s, \varepsilon_i) - (\beta^{-j}s, \varepsilon_j) & \text{if} \quad t+1 = \beta^{-j}s \le s \\ 0 & \text{if} \quad t = \alpha\beta^{-i}s - 1 = \beta^{-i-1} + h \\ (\beta^{-i}s, \varepsilon_i) & \text{otherwise} \end{cases}$$



The element $(\beta^{-i}s, \varepsilon_i) \in U(\beta^{-i}s)$ satisfies the relation

$$\gamma^h(\beta^{-i}s, \varepsilon_i) = -(\beta^{-i+1}s, \varepsilon_{i-1}) \in U(\beta^{-i}s + h) \quad \text{if} \quad i > 0.$$

On the other hand, the element $\eta := (\beta^{-bs+1}s, \varepsilon_{bs-1}) \in U(s)$ satisfies $\gamma\eta = 0$. Accordingly, the morphism

$$\psi : \tilde{V}_{s,1} \oplus \Big(\bigoplus_{0<i<bs} \tilde{V}_{\beta^{-i}s,h+1} \Big) \to U = LRV,$$

which maps the canonical generator $\zeta_s \in \tilde{V}_{s,1}(s)$ onto $\eta \in U(s)$ and $\zeta_{\beta^{-i}s} \in \tilde{V}_{\beta^{-i}s,h+1}(\beta^{-i}s)$ onto $(\beta^{-i}s, \varepsilon_i) \in U(\beta^{-i}s)$, is injective on the socles. Therefore, ψ is a monomorphism. It is also an epimorphism by a Nakayama-type argument, since $\bigoplus_t U(t)/\gamma U(t-1)$ has the residue classes of the elements η and $(\beta^{-i}s, \varepsilon_i)$ as a basis.

Finally, ΦV has $\psi(\bigoplus_{0<i<bs} \tilde{V}_{\beta^{-i}s,h+1})$ as kernel, because this is true for any non-zero morphism from $\tilde{V}_{s,1} \oplus (\bigoplus \tilde{V}_{\beta^{-i}s,h+1})$ to $\tilde{V}_{s,1}$ (notice again that $\Phi V \ne 0$, since $R\Phi V$ admits the section ΨRV, and since $RV \ne 0$). OK.

4.7. *Proof of Theorem* 1.7. We already know that R is a functor from $\mathrm{Mod}_k \tilde{Z}_h$ to $\mathrm{Mod}_k(\tilde{Q}, \tilde{I})$. Accordingly, the maps $(RV)(\alpha)$ and $(RV)(\beta)$ satisfy the relations of the bounden quiver $(\tilde{Q}, \tilde{I})$. Furthermore, as L maps projectives onto projectives by 4.2, L induces a functor $\bar{L}:\overline{\mathrm{Mod}_k}(\tilde{Q}, \tilde{I}) \to \overline{\mathrm{Mod}_k}\,\tilde{Z}_h$. Since R is right adjoint to the *exact* functor L, R maps injectives onto injectives. But injectives coincide obviously with projectives in both categories. Therefore R induces also a functor $\bar{R}:\overline{\mathrm{Mod}_k}\,\tilde{Z}_h \to \overline{\mathrm{Mod}_k}(\tilde{Q}, \tilde{I})$.

By Lemma 4.5 we know that $\Psi W: W \to RLW$ is mono and has a projective cokernel if W is simple. Clearly, this is also true if W is semi-simple. Since RL is an exact functor, our assertion remains valid, if W is a extension of two semi-simple modules, and even more generally if W admits a finite increasing sequence of subrepresentations with semi-simple factors. As $\mathrm{Mod}_k(\tilde{Q}, \tilde{I})$ has height ($=$Loewy-length)$\leq h+1$, we infer that ΨW is mono and has projective cokernel for every W. Accordingly, $\bar{\Psi}:\mathbf{1} \xrightarrow{\sim} \bar{R}\bar{L}$ is an isomorphism. Using 4.6 instead of 4.5 we prove similarly that $\bar{\Phi}:\bar{L}\bar{R} \xrightarrow{\sim} \mathbf{1}$ is an isomorphism. OK.

4.8. Proof of Theorem 1 of 1.8. Let us use for $R:\mathrm{Mod}_k\,\tilde{Z}_h \to \mathrm{Mod}_k(\tilde{Q}, \tilde{I})$ the explicit description given by the formulae

$$(RV)(s) = \bigoplus_{0\leq i<bs} V(\beta^i s).$$

Clearly, the functor defined by these formulae maps e-periodic representations onto e-periodic representations and induces therefore a functor $R^e:\mathrm{Mod}_k^e\,\tilde{Z}_h \to \mathrm{Mod}_k^e(\tilde{Q}, \tilde{I})$. Moreover, if $V \in \mathrm{Mod}_k\,\tilde{Z}_h$ is projective and e-periodic, RV is projective by 4.7 and e-periodic. By the lemma below, R^e therefore maps projectives of $\mathrm{Mod}_k^e\,\tilde{Z}_h$ onto projectives of $\mathrm{Mod}_k^e(\tilde{Q}, \tilde{I})$ and induces a functor

$$\bar{R}^e:\overline{\mathrm{Mod}_k^e}\,\tilde{Z}_h \to \overline{\mathrm{Mod}_k^e}(\tilde{Q}, \tilde{I})$$

between the stable categories. Similarly, $L:\mathrm{Mod}_k(\tilde{Q}, \tilde{I}) \to \mathrm{Mod}_k\,\tilde{Z}_h$ induces functors $L^e:\mathrm{Mod}_k^e(\tilde{Q}, \tilde{I}) \to \mathrm{Mod}_k^e\,\tilde{Z}_h$ and $\bar{L}^e:\overline{\mathrm{Mod}_k^e}(\tilde{Q}, \tilde{I}) \to \overline{\mathrm{Mod}_k^e}\,\tilde{Z}_h$.

Now consider again the adjunction-bijection

$$u:\mathrm{Hom}(LW, V) \xrightarrow{\sim} \mathrm{Hom}(W, RV)$$

of 4.3. A morphism $f:LW \to V$ consists in maps

$$f(s): \bigoplus_{t\leq s<\alpha t} W(t) \to V(s),$$

the components of which will be denoted by $f_{s,t}$. Similarly, a morphism $g: W \to RV$ consists in maps $g(s): W(s) \to \oplus V(\beta^i s)$ with components $g_{s,i}$, $0 \leq i < bs$. In case $g=u(f)$ the description of u given in 4.3 furnishes the relation

$$(uf)_{s,i}=f_{\beta^i s,\beta^i s}\circ W(\beta^i).$$

This formula shows in particular that $u(Tf)=Su(f)$ if W and V are both e-periodic. Here $T(f)$ and $S(g)$ are defined by $T(f)(s)=f(s-e)$ and $S(g)=g(s-e)$ for any $f\in \operatorname{Hom}(LW, V)$ and any $g\in \operatorname{Hom}(W, RV)$. Consequently, $u(f)$ is e-periodic (i.e. we have $Su(f)=u(f)$) iff f is e-periodic (i.e. $Tf=f$). This shows that L^e and R^e are adjoint functors, and that the morphisms

$$\Psi W: W\to RLW \quad\text{and}\quad \Phi V: LRV\to V$$

are e-periodic if V and W are so. If this is the case, Coker ΨW and Ker ΦV are projective and e-periodic. Therefore they are projective in $\operatorname{Mod}_k^e(\tilde{Q},\tilde{I})$ and $\operatorname{Mod}_k^e \tilde{Z}_h$ respectively. Accordingly we have $\mathbf{1}\xrightarrow{\sim}\bar{R}^e\bar{L}^e$ and $\bar{L}^e\bar{R}^e\xrightarrow{\sim}\mathbf{1}$ as in 4.7. OK.

LEMMA. *An e-periodic representation W of $(\tilde{Q},\tilde{I})$* (resp. *V of $\tilde{Z}_h$*) *is projective in* $\operatorname{Mod}_k^e(\tilde{Q},\tilde{I})$ (resp. *in* $\operatorname{Mod}_k^e\tilde{Z}_h$) *iff it is projective in* $\operatorname{Mod}_k(\tilde{Q},\tilde{I})$ (resp. *in* $\operatorname{Mod}_k\tilde{Z}_h$).

Proof. We use the following characterization of the projective representations of $(\tilde{Q},\tilde{I})$: Start with any representation W and set $W'=\operatorname{rad} W\subset W$. For every $t\in\mathbf{Z}$, $W'(t)$ is defined by $W'(t)=\alpha W(\alpha^{-1}t)+\beta W(\beta^{-1}t)\subset W(t)$. Now choose for each t a k-subspace $W_1(t)$ of $W(t)$ such that $W(t)=W'(t)\oplus W_1(t)$. Then W is projective in $\operatorname{Mod}_k(\tilde{Q},\tilde{I})$ iff, for any $U\in\operatorname{Mod}_k(\tilde{Q},\tilde{I})$, every family of k-linear maps $h(t): W_1(t)\to U(t)$, $t\in\mathbf{Z}$, can be extended uniquely to a morphism $W\to U$.

When W is projective in $\operatorname{Mod}_k(\tilde{Q},\tilde{I})$ and e-periodic, we choose the supplementary subspaces $W_1(t)$ in such a way that $W_1(t-e)=W_1(t)$ for every $t\in\mathbf{Z}$. For any e-periodic representation U, every sequence of k-linear maps $h(t): W_1(t)\to U(t)$, $t=1,2,\ldots,e$, can then be extended uniquely to an e-periodic family $h(t): W_1(t)\to U(t)$, $t\in\mathbf{Z}$, hence to an e-periodic morphism $W\to U$. In other words, the map

$$\operatorname{Hom}^e(W,U)\to\bigoplus_{t=1}^{t=e}\operatorname{Hom}_k(W_1(t),U(t)),\ f\mapsto (f(t)\mid W_1(t))_{1\le t\le e}$$

is bijective (here $\operatorname{Hom}^e(W,U)$ stands for the space of e-periodic morphisms). Accordingly, the functor $\operatorname{Hom}^e(W,?)$ is exact, and W is projective in $\operatorname{Mod}_k^e(\tilde{Q},\tilde{I})$.

On the other hand, the inclusion-functor $\operatorname{Mod}_k^e(\tilde{Q},\tilde{I})\to\operatorname{Mod}_k(\tilde{Q},\tilde{I})$ is left adjoint to the exact functor Π defined by

$$(\Pi W)(t) = \prod_{n \in \mathbf{Z}} W(t+ne).$$

Therefore it maps projectives onto projectives.

The case $\tilde{Z}_h$ is proved in a similar way. OK.

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