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Derived representation theory and the algebraic K-theory of fields

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Abstract

In this paper, we prove a conjecture on the relationship of the algebraic K-theory of a field F, with abelian separable Galois group G_F, containing an algebraically closed subfield with the K-theory of the category of finite-dimensional continuous linear representations of G_F in an algebraically closed field. The connection is achieved through the use of a certain derived completion construction defined for commutative ring spectra. The paper proposes that the conjecture should hold for non-abelian separable Galois groups.

1. *Introduction*

Quillen's higher algebraic K-theory for fields F has been the object of intense study since its introduction in 1972 (see [**23**]). The main direction of research has been the construction of 'descent spectral sequences' whose E_2-term involved the cohomology of the separable Galois group G_F with coefficients in the algebraic K-theory of an algebraically closed field. The form of such a spectral sequence was conjectured in [**17**, **24**], but it was soon realized that it could not be expected to converge to algebraic K-theory exactly. However, it appeared likely that it could converge to algebraic K-groups in sufficiently high dimensions, that is, in dimensions greater than the cohomological dimension of G_F. This observation was formalized into the *Quillen–Lichtenbaum conjecture*. This conjecture has attracted a great deal of attention. Over the years, a number of special cases [**14**, **31**] have been treated, and partial progress has been made [**5**]. In [**33**], Voevodsky proved results that led to the verification of the conjecture at $p = 2$. More recently, fundamental work by Voevodsky and Rost and an important contribution by Weibel [**34**, **37**] have resolved this long-standing conjecture at odd primes as well. Moreover, it also resolves the Beilinson–Lichtenbaum conjecture about the form of a spectral sequence that converges exactly to the algebraic K-theory of F.

The existence of this spectral sequence does not, however, provide a homotopy-theoretic model for the algebraic K-theory spectrum of the field F. It is the goal of this paper to propose such a homotopy-theoretic model in the case of geometric fields, that is, fields containing separably closed subfields, and to verify that it is equivalent to the algebraic K-theory of F (completed at a prime l) in some cases. The model will depend only on the complex representation theory of G_F, without any other explicit arithmetic information about the field F, and is therefore contravariantly functorial in G_F. It is understood in algebraic topology that explicit space level models for spaces and spectra are generally preferable to strictly algebraic calculations of homotopy groups for the spaces. Knowledge of homotopy groups alone does not allow one to understand the behavior of various constructions and maps of spectra in an explicit way. In this case, there are specific reasons why one would find such a model desirable.

(1) It appears likely that one could begin to understand *finite* descent problems, as the behavior of restriction and induction maps for subgroups of finite index is relatively well understood. A conjectural program for carrying this out might look as follows. Suppose that we have a Galois extension $F \subseteq E$, with Galois group G. The conjecture would now relate $R[G_F]$ and $R[G_E]$ with the K-theory spectra of F and E, respectively, and suggests the possibility that one might be able to construct a spectral sequence using a derived endomorphism ring $\mathrm{End}_{R[G_F]}(R[G_E])$ to construct a spectral sequence, whose E_2-term would involve Ext-groups over this graded algebra of the K-theory of E.

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(2) It appears that the Milnor K-groups should be identified with the homotopy groups of the so-called *derived completion* of the complex representation ring of G_F. This would relate an explicitly arithmetic invariant (Milnor K-theory) with an explicitly group-theoretic invariant coming out of the representation theory. The Bloch–Kato conjecture provides one such connection, relating Milnor K-theory with Galois cohomology. These ideas provide another one, arising out of the representation theory of the group rather than from cohomology. There is a so-called *algebraic to geometric spectral sequence* for computing the homotopy groups of the derived completion, whose form in known cases appears similar to that of the Bloch–Lichtenbaum spectral sequence. The homotopy groups of the derived completion of the representation ring occur in this spectral sequence, and the right kind of comparison map might allow one to compare the Milnor K-groups with the homotopy groups of the derived completion.

The idea of the construction is as follows. Let F be a geometric field, with separable closure $\bar{F}$, and define the category of *descent data* for the extension $F \subset \bar{F}$ (denoted by $V^G(\bar{F})$) to have objects the finite-dimensional $\bar{F}$-vector spaces V with G_F-action satisfying $\gamma(fv) = f^\gamma\gamma(v)$ for all $f \in \bar{F}$, with the morphisms being the equivariant $\bar{F}$-linear isomorphisms. It is standard descent theory [28] that this category is equivalent to the category of finite-dimensional F-vector spaces. On the other hand, for any field E and profinite group G, let $\mathrm{Rep}_E[G]$ denote the category of finite-dimensional continuous E-linear representations of the profinite group G. There is a canonical homomorphism from $\mathrm{Rep}_F[G_F]$ to $V^{G_F}(\bar{F})$ obtained by applying $\bar{F} \otimes_F -$ and extending the action via the Galois action of G_F on $\bar{F}$. Let k denote an algebraically closed subfield of F. Also, let K denote the functor that takes a symmetric monoidal category to its K-theory spectrum. We shall permit ourselves to write KR or KX, when R is a ring or X is a scheme, for the K-theory of the corresponding categories of projective modules or vector bundles. We denote by $K^{G_F}(\bar{F})$ denotes the K-theory spectrum of the category $V^{G_F}(\bar{F})$. We then have the composite

$$\alpha_F : K\mathrm{Rep}_k[G_F] \longrightarrow K\mathrm{Rep}_F[G_F] \longrightarrow K^{G_F}(\bar{F}).$$

This map is of course very far from being an equivalence, as $\pi_0 K\mathrm{Rep}_k[G_F]$ is isomorphic to $R[G_F]$, a non-finitely generated abelian group, and $\pi_0 KF \cong \mathbb{Z}$. However, we observe that both spectra are *commutative ring spectra* and further that the map α_F is a homomorphism of commutative ring spectra. In [3], we constructed a derived version of completion, which is applicable to any homomorphism of commutative ring spectra. For such a homomorphism $f : A \to B$, we denote this derived completion by $A^\wedge_B$. We also let $\mathbb{H}_l$ denote the mod-l Eilenberg–Mac Lane spectrum. The spectrum $\mathbb{H}_l$ is also a commutative ring spectrum, and we have an evident commutative diagram of commutative ring spectra:

$$\begin{tikzcd} K\mathrm{Rep}_k[G_F] \arrow[r, "\alpha_F"] \arrow[d, "\varepsilon_l"'] & K^{G_F}(\bar{F}) \simeq KF \arrow[d, "\varepsilon_l"] \\ \mathbb{H}_l \arrow[r, "\mathrm{id}"] & \mathbb{H}_l. \end{tikzcd}$$

The vertical maps are constructed as follows. The ring $\mathbb{F}_l$ can be viewed as a symmetrical monoidal category with only identity morphisms, and with object set equal to the set $\mathbb{F}_l$. Applying an infinite loop space machine to it produces the mod-l Eilenberg–Mac Lane spectrum, and the multiplication makes it into a ring spectrum. There is an evident functor $\mathrm{Rep}_k[G_F] \to \mathbb{F}_l$, which carries any representation to its dimension mod-l, and the vertical maps are the induced maps on K-theory. The completion construction is natural for such commutative squares, and we obtain a map of spectra $A_F(l) : K\mathrm{Rep}_k[G_F]^\wedge_{\mathbb{H}_l} \longrightarrow KF^\wedge_{\mathbb{H}_l}$. We refer to $A_F(l)$ as the 'representational assembly (at l)'. In this paper, we make this construction precise, and then provide two results that act as 'proof of concept' that the representational assembly could be expected to be an equivalence for a general class of geometric fields. The results are as follows.

THEOREM 1.1. *The map of spectra $A_F(l)$ is an equivalence for the case of a geometric field whose separable Galois group is abelian.*

THEOREM 1.2. *The homomorphism $\pi_*(A_F(l))$ is surjective for all geometric fields of characteristic prime to l.*

We also pose the following.

Conjecture 1.3 (Rigidity). The map $A_F(l)$ is an equivalence for any geometric field of characteristic prime to l.

Conjecture 1.4. We conjecture that, for every prime l prime to the characteristic of F, there is a naturally defined isomorphism

$$\pi_*(\mathrm{Rep}_{\mathbb{C}}[G_F]^{\wedge}_{\mathbb{F}_l}) \longrightarrow K_*^{\mathrm{Mil}}(F)^{\wedge}_l,$$

for geometric fields, where $K_*^{\mathrm{Mil}}(F)^{\wedge}$ denotes the l-adic completion of Milnor K-theory, and where $\mathrm{Rep}_{\mathbb{C}}[G_F]^{\wedge}_{\mathbb{F}_l}$ denotes the derived completion of $\mathrm{Rep}_{\mathbb{C}}[G_F]$ at the homomorphism from $\mathrm{Rep}_{\mathbb{C}}[G_F]$ to $\mathbb{F}_l$ given by mod-l reduction of the augmentation map.

We remark that our results depend on the Beilinson–Lichtenbaum conjecture. It would of course be extremely interesting to find a proof independent of these conjectures. It might then be the case that a proof of the conjectures of Beilinson–Lichtenbaum and Bloch–Kato could be constructed by relating the representation theory of the absolute Galois group with its cohomology.

In future work, we plan to study various indirect methods for evaluation of the derived representation theory, including some which are particularly well suited for comparisons with the motivic methods. We hope to use these methods to develop methods of proof for Conjectures 1.3 and 1.4. We also hope to formulate a conjecture that applies to all fields, even fields of characteristic zero where the roots of unity are not present in the ground field. A step in this direction has been taken by Lyo [**18**] for fields of finite characteristic.

We describe the structure of the paper. Section 2 provides preliminaries on various categories of spectra we will use, including in some cases multiplicative properties, together with properties of various completion constructions we will use. Section 3 provides a description of equivariant spectra whose fixed point spectra are the K-theory spectra of fields, and concludes with the definition (in the level of generality of k-algebras with G-action, where k is a field) of the representational assembly map. Section 4 treats a key example of that map, namely, the case of a Laurent polynomial ring, and proves an equivalence result for the assembly map in that case, which will be used in the proof of our full result. Section 5 contains statements on the conjectures of Beilinson–Lichtenbaum and Bloch–Kato, as well as certain consequences that we find necessary in proving our result. Section 6 proves Theorem 1.1, and Section 7 proves Theorem 1.2.

The ultimate hope is that the clarification of the relationship between arithmetically defined descriptions of algebraic K-theory, such as the motivic spectral sequence, with descriptions that involve the Galois group and its representation theory directly, will shed more light on arithmetic and algebraic geometric questions.

2. *Preliminaries*

We assume the reader to be familiar with the category of spectra, as developed in [**6**, **7**] or [**15**]. We work in the category of symmetric spectra as developed in [**15**]. The category of symmetric spectra is shown to possess a coherently commutative and associative monoidal structure, called the smash product, which makes the category of spectra into a symmetric monoidal category. The presence of such a monoidal structure makes it possible to define *ring spectra* as monoid objects in $\underline{Spectra}$, as well as commutative ring spectra. It also makes it possible to define the notion of a module spectrum over a ring spectrum or a commutative ring spectrum. Appropriate model category structures on categories of ring spectra and module spectra were constructed in [**26**, **29**]. These references also introduce notions we will make use of, such as the universal coefficient and the Künneth spectral sequences. See [**3**, Section 2.1], for a discussion of the relevant notions as they will be used in the present paper. We point out that in order to make the constructions we study homotopy invariant, we will always compute them after suitable cofibrant replacement.

Recall that there are functors from the category of symmetric monoidal categories to spectra, sometimes referred to as 'infinite loop space machines'. The question of what kind of structures are required so that the spectrum associated to a symmetrical monoidal category is a ring spectrum, commutative ring spectrum, or module over a ring spectrum is addressed in [**8**]. Of course, our interest is in applications to K-theory. There are several models of the K-theory spectrum of a ring. One way to construct the algebraic K-theory spectrum of a ring is as the result of an infinite loop space machine applied to the symmetric monoidal category of finitely generated projective modules over the ring. Another is the so-called Waldhausen's $S_{.}$-construction [**36**] applied to the 'category with cofibrations and weak equivalences' consisting of finitely generated projective modules over the ring, or of finite chain complexes of such modules. The advantage of the first approach is that Elmendorf and Mandell [**8**] prove that, in that context, one can put a commutative ring spectrum structure on a variant of the K-theory spectrum on the category of projective modules over a commutative ring, and that there are criteria that specify module structures over the K-theory spectrum from combinatorial data. The second method, on the other hand, has available to it localization sequences and additional theorems that one finds useful in making comparisons between K-theory spectra, in particular for studying the homotopy fibers of maps of K-theory spectra. We shall use both, as well as some intermediate theories and comparisons between them. The fact that the two models are equivalent is proved in [**36**]. We refer the reader to [**23**, **36**] for results concerning the second model for K-theory spectra, notably the localization (Section V, Theorem 5), devissage (Section V, Theorem 4), and reduction by resolution (Section IV, Corollary 1) theorems.

We also remind the reader of the construction by Bousfield and Kan of the l-completion of a space (simplicial set) at a prime l, denoted by $X^{\wedge}_l$. Bousfield and Kan construct a functorial cosimplicial space $T^{\cdot}_l X$, and define the l-completion of X to be $\mathrm{Tot}(T^{\cdot}_l)$. This construction gives rise to a functorial notion of completion at a prime l. It extends in a straightforward way to spectra by applying completion at l levelwise. We record a technical property of the completion construction for which we shall find use later. For each positive integer n, let ζ_n denote the nth root of unity $e^{2\pi i/n}$. Let $S^1(n).$ denote the simplicial set whose set of vertices consists of the set of powers of ζ_n, whose 1-simplices consist of elements $\{\sigma_i\}_{i=0}^{n-1}$, with $d_0(\sigma_i) = \zeta_n^{i+1}$ and $d_1(\sigma_i) = \zeta_n^i$, and for which the powers of ζ_n and the simplices σ_i are the only non-degenerate simplices. We will denote the standard simplicial circle by $S^1(1)$, and there is an evident simplicial map $\lambda_n : S^1(n). \to S^1(1)$, which carries each vertex in $S^1(n).$ to the unique vertex in $S^1(1).$ and each non-degenerate 1-simplex in $S^1(n).$ to the unique non-degenerate 1-simplex in $S^1(1)$. For any simplicial set $X.$, we denote the cone on $X.$ by CX. We now define the mod-n Moore space (a simplicial set denoted by $M(n).$) to be the pushout of the diagram

$$\begin{tikzcd} S^1(n). \arrow[r, "\lambda_n"] \arrow[d] & S^1(1). \\ C(S^1(n)).. & \end{tikzcd}$$

Whenever $m|n$, there is a natural map $\theta : M(m). \to M(n).$ constructed out of the (m/n) th power map of the circle in the evident way. The vertex corresponding to the point 1 on the circle will always be denoted by $*$, and will act as a base point. There is a natural inclusion $S(1) \to M(l^j)$, and these maps are compatible in the sense that the diagrams

$$\begin{tikzcd} S(1) \arrow[r] \arrow[dr] & M(l^{j+1}) \arrow[d, "\theta"] \\ & M(l^j) \end{tikzcd}$$

commute. The cofibers $\mathrm{CF}(l^j)$ of the maps $S(1) \to M(l^j)$ are simplicial models of S^2, and there are induced maps $\mathrm{CF}(l^{j+1}) \to \mathrm{CF}(l^j)$, which we also denote by θ, and which are of degree l.

Proposition 2.1. *For any connective spectrum X, there is an inverse system of spectra*

$$\cdots \longrightarrow \Sigma^{-1}M(l^{i+1}).\wedge X \longrightarrow \Sigma^{-1}M(l^i).\wedge X \longrightarrow \Sigma^{-1}M(l^{i-1}).\wedge X \longrightarrow \cdots$$
$$\cdots \longrightarrow \Sigma^{-1}M(l).\wedge X \longrightarrow \Sigma^{-1}M(1).\wedge X,$$

whose homotopy inverse limit is equivalent to the Bousfield–Kan completion $X_l^\wedge$ of the spectrum X.

Proof. We construct three two-parameter inverse systems of spectra, $A_{..}$, $B_{..}$, and $C_{..}$ as follows. For any spectrum X we let $T^i(X)$ be $\mathrm{Tot}^i B^{\cdot}(X)$, where $B^{\cdot}(X)$ denotes the cosimplicial spectrum whose total spectrum is $X_l^\wedge$. We define B_{ij} to be $T^i(\Sigma^{-1}M(l^j)\wedge X)$, the maps in the i-direction to be the natural maps $T^{i+1}\to T^i$, and the maps in the j-direction to be

$$T(\Sigma^{-1}\theta\wedge \mathrm{id}_X) : T^i(\Sigma^{-1}M(l^{j+1})\wedge X) \longrightarrow T^i(\Sigma^{-1}M(l^j)\wedge X).$$

We define A_{ij} to be $T^i(\Sigma^{-1}S(1)\wedge X)$. The maps in the i-direction are again given by the natural transformations $T^{i+1}\to T^i$, and the maps in the j-direction are identities. Finally, C_{ij} is defined to be $T^i(\Sigma^{-1}\mathrm{CF}(l^j)\wedge X)$. We note also that we have a cofiber sequence

$$A_{..} \longrightarrow B_{..} \longrightarrow C_{..}$$

of two-parameter inverse systems of spectra, and therefore a cofiber sequence of homotopy limit spectra. We observe that $\mathrm{holim}_{i,j}\, A_{ij}\simeq X_l^\wedge$, as it is a two-parameter system that is constant in the j-direction, and in the i-direction, it is the inverse system defining the completion. Second, we observe that, fixing j, we have

$$\operatorname*{holim}_i B_{ij} \cong \Sigma^{-1}M(l^j)\wedge X.$$

This observation follows from the following facts.

(1) The spectrum $\Sigma^{-1}M(l^j)\wedge X$ is connective, consequently has a Postnikov tower whose layers are Eilenberg–Mac Lane spectra for groups of exponent l^j or less.

(2) The l-adic completion of an Eilenberg–Mac Lane spectrum Z for a group of exponent l^j or less is equivalent to Z itself.

(3) Here l-adic completion of spectra preserves fiber sequences up to homotopy.

(4) If we have a map $f: X\to Y$ of connective spectra that is N-connected, then the associated map of l-adic completions is also N-connected.

It is easy to check that the map on homotopy limits induced by the map $A_{..}\to B_{..}$ therefore yields a map $X_l^\wedge \to \mathrm{holim}_j\, \Sigma^{-1}M(l^j)\wedge X$. What is required to show that this map is an equivalence is that $\mathrm{holim}_{i,j}\, C_{ij}\simeq *$. But this follows directly from the observation that the system of abelian groups (for fixed i)

$$\{\pi_k T^i(\Sigma^{-1}\mathrm{CF}(l^j)\wedge X)\}_j$$

is pro-trivial, which in turn follows from the fact that all the maps in the system induce the zero map on mod-l homology. $\square$

The completion construction is extended further in [**3**] to the notion of completion of a module M over a commutative ring spectrum A at a commutative A-algebra B, denoted by $M_B^\wedge$. The Bousfield–Kan construction is the special case of this construction where A is the sphere spectrum and B is the mod-l Eilenberg–Mac Lane spectrum. A number of properties of this construction are established in [**3**]. We will use them extensively and record them here.

(1) The construction is defined as the total spectrum of a cosimplicial spectrum, which is given in codimension k by the spectrum

$$\underbrace{B\wedge_A\ldots\wedge_A B}_{k \text{ factors}}\wedge_A M.$$

(2) Given any diagram

$$A\longrightarrow B\longrightarrow C$$

of commutative ring spectra, there is an induced morphism of commutative ring spectra $A_C^\wedge \to B_C^\wedge$.

(3) Commutative rings may be viewed as commutative ring spectra via the Eilenberg–Mac Lane construction, and the construction agrees with ordinary completion in the case of finitely generated modules over a Noetherian ring. In particular, all higher homotopy groups of the completion vanish in that case. For non-Noetherian rings, however, it has the property that it creates higher derived groups of the completion constructions. These groups go far beyond the standard $\lim^1$ terms seen in studying the ordinary l-adic completion. There is a so-called *algebraic to geometric spectral sequence* [**3**, Theorem 7.1], whose E_2-term is computed solely from derived

completions over $\pi_0(A)$, and which converges to the homotopy groups of the spectrum level derived completion over A.

We now discuss how the construction works in some special cases. From the construction of the completion in [3], we have that the completion $M^{\wedge}_B$ is the total spectrum of a certain cosimplicial spectrum $\mathfrak{C}^{\cdot}(M;B)$. In [3], the dependence on A was not made explicit in the construction, but we will need to keep track of it, and therefore adopt the notation $\mathfrak{C}^{\cdot}_A(M;B)$. We also remark that although there is a functorial construction of $\mathfrak{C}^{\cdot}_A(M;B)$, it can be computed using various choices of cofibrant replacements for M and B, so it is useful to regard it as a homotopy type within the category of cosimplicial spectra, where the notion of homotopy equivalence is the levelwise equivalence of cosimplicial spectra. The homotopy groups of the total spectrum may be computed using the homotopy spectral sequence of a cosimplicial space $X^{\cdot}$ (see [2, Section 10.6]). This spectral sequence is based on the filtration obtained from a tower of fibrations

$$\cdots \longrightarrow \mathrm{Tot}^3 X^{\cdot} \longrightarrow \mathrm{Tot}^2 X^{\cdot} \longrightarrow \mathrm{Tot}^1 X^{\cdot} \longrightarrow \mathrm{Tot}^0 X^{\cdot},$$

which converges to $\mathrm{Tot}X^{\cdot}$ in the sense that there is a natural equivalence

$$\mathrm{Tot}X^{\cdot} \cong \operatorname*{holim}_i \mathrm{Tot}^i X^{\cdot}.$$

The tower of fibrations gives descending filtrations

$$F_i\pi_k\mathrm{Tot}X^{\cdot} = \mathrm{Ker}(\pi_k\mathrm{Tot}X^{\cdot} \longrightarrow \pi_k\mathrm{Tot}^i X^{\cdot}).$$

It is a second quadrant spectral sequence. The $E_1^{s,t}$-term of this spectral sequence is the group

$$\pi_t X^{-s} \cap \ \ker\, s^0 \cap \ \ker\, s^1 \cap \ldots \cap \ \ker\, s^{-s-1},$$

and the coboundary operator is the usual alternating sum of coface maps. The groups $E_\infty^{s,t}$ for $t+s=k$ form the associated graded groups to the filtration $F_i\pi_k\mathrm{Tot}X^{\cdot}$.

We describe the form of this spectral sequence, as well as the filtration arising from the E_∞-term, in some special cases.

Proposition 2.2. *Let $A=\mathbb{Z}$ and let $B=\mathbb{F}_l$. These are rings, which may be regarded as commutative ring spectra via the Eilenberg–Mac Lane construction. Then the E_1-term of the spectral sequence for $\mathfrak{C}^{\cdot}_A(A;B)$ is given by*

$$\begin{aligned} E_1^{s,t} &\cong \mathbb{F}_l \quad \text{for } s+t=0,\\ &\cong 0 \quad \text{otherwise.}\end{aligned}$$

The spectral sequence collapses at this point. It follows that $\pi_i A^{\wedge}_B = 0$ for $i \neq 0$, and that $\pi_0 A^{\wedge}_B \cong \mathbb{Z}_l$. The filtration on $\pi_0 A^{\wedge}_B$ is identified with the l-adic filtration on $\mathbb{Z}_l$.

Proof. The description of the spectral sequence is a straightforward computation. The description of $\pi_0\mathrm{Tot}\mathfrak{C}^{\cdot}_A(A;B) \cong \pi_0 A^{\wedge}_B$ is immediate from [3, Proposition 4.4]. The filtration statement is direct. □

We need to analyze the behavior of the completion construction for group rings of some abelian groups. We begin by recalling the Eilenberg–Moore spectral sequence, as described in [21, Section 8.2.3]. Let X be a based simplicial set. We form the cosimplicial space $\mathrm{EM}^{\cdot}(X)$, whose codimension k term is X^k, and whose coface and codegeneracies are given as follows:

$$\begin{aligned}
\delta^0(x_0,\ldots,x_{k-1}) &= (*,x_0,\ldots,x_{k-1}),\\
\delta^i(x_0,\ldots,x_{k-1}) &= (x_0,\ldots,x_{i-1},x_{i-1},\ldots,x_{k-1}) \quad \text{for } 1 \leqslant i \leqslant k,\\
\delta^{k+1}(x_0,\ldots,x_{k-1}) &= (x_0,\ldots,x_{k-1},*),\\
\sigma^i(x_0,\ldots,x_{k-1}) &= (x_0,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{k-1}).
\end{aligned}$$

Letting R be any commutative ring, we can then form the cosimplicial object $R[\mathrm{EM}^{\cdot}(X)]$ in the category of simplicial abelian groups. The Eilenberg–Moore spectral sequence is now the homotopy spectral sequence for the total space of this cosimplicial object. When X is simply connected, it converges to the homology of the loop space ΩX with coefficients in R. When X is not connected, however, the analysis of what it converges to is much more problematic. We consider the case $X=BG$, the classifying space of an abelian group G, and record the behavior of the Eilenberg–Moore spectral sequence for the case $G=\mathbb{Z}/l^\infty\mathbb{Z}$.

Lemma 2.3. *The E_2-term of the Eilenberg–Moore spectral sequence for $B\mathbb{Z}/l^\infty\mathbb{Z}$ and the ring $\mathbb{F}_l$ is isomorphic to an exterior algebra $\Lambda_{\mathbb{F}_l}(\xi)$, where ξ has bidegree $(-1, 2)$.*

Proof. The map $B\mathbb{Z}/l^\infty\mathbb{Z} \to BS^1 = \mathbb{C}P^\infty$ induces an isomorphism on homology with $\mathbb{F}_l$ coefficients, hence it clearly induces an isomorphism on E_1-terms of the Eilenberg–Moore spectral sequences. The E_2-term of the Eilenberg–Moore spectral sequence is known to have the form $\mathrm{CoTor}_{H_*(X,\mathbb{F}_l)}(\mathbb{F}_l, \mathbb{F}_l)$, whose vector space dual is $\mathrm{Ext}_{H^*(X,\mathbb{F}_l)}(\mathbb{F}_l, \mathbb{F}_l)$. As $H^*(\mathbb{C}P^\infty, \mathbb{F}_l)$ is a polynomial algebra on a two-dimensional generator, its Ext-algebra is exterior. Tracking down the dimensions gives the result. $\square$

Proposition 2.4. *There is a levelwise equivalence from the cosimplicial space obtained by applying the zeroth space functor to the cosimplicial spectrum*

$$\mathfrak{C}^{\cdot}_{R[G]}(R[G]; R)$$

with the cosimplicial space $R[\mathrm{EM}^{\cdot}(BG)]$. *Consequently*, *the homotopy spectral sequences for the two cosimplicial spaces are isomorphic.*

Proof. Recall that the completion must be computed using a cofibrant model for R as an $R[G]$-module. We select one as follows. Let $EG.$ denote the simplicial abelian group $k \to G^{k+1}$, with face maps given by projections and degeneracies given by diagonals. It is standard that this simplicial abelian group is contractible as a simplicial set, and there is a natural inclusion $G \hookrightarrow EG.$, where G is regarded as a constant simplicial group, and the inclusion in each level is a diagonal inclusion. The R-algebra $R[EG.]$ is now free as an $R[G]$-module, which is weakly equivalent to R, and so may be used to compute the derived completion. The homotopy type $\mathfrak{C}^{\cdot}_{R[G]}(R[G]; R)$ can now be computed using $R[EG.]$ as a cofibrant replacement for R. The resulting cosimplicial simplicial abelian group $\mathfrak{A}^{\cdot}$ has the form

$$k \longrightarrow \underbrace{R[EG.] \otimes_{R[G]} R[EG.] \otimes_{R[G]} \ldots \otimes_{R[G]} R[EG.]}_{k+1 \text{ factors}}.$$

Its coface and codegeneracy operators are given in [3]. On the other hand, the cosimplicial simplicial abelian group $\mathfrak{A}^{\cdot}$ is canonically isomorphic to the cosimplicial simplicial abelian group defined by

$$k \longrightarrow \underbrace{R[EG. \times_G EG. \times_G \ldots \times_G EG.]}_{k+1 \text{ factors}},$$

where $EG. \times_G EG. \times_G \ldots \times_G EG.$ denotes the orbit space of the action of G^k on $EG.^{k+1}$ defined by

$$(g_1, \ldots, g_k) \cdot (e_0, \ldots, e_k) = (e_0 g_1^{-1}, g_1 e_1 g_2^{-1}, \ldots, g_s e_s g_{s+1}^{-1}, \ldots, g_k e_k),$$

and where the coface and codegeneracy operators are defined by applying $R[-]$ to the coface and codegeneracy operators in the cosimplicial simplicial abelian group $\mathfrak{B}^{\cdot}$

$$k \longrightarrow \underbrace{EG. \times_G EG. \times_G \ldots \times_G EG.}_{k+1 \text{ factors}}$$

with

$$\delta^i(e_0, \ldots, e_k) = (e_0, \ldots, e_{i-1}, *, e_i, \ldots, e_k),$$
$$\sigma^i(e_0, \ldots, e_k) = (e_0, \ldots, e_{i-1}, e_i e_{i+1}, e_{i+2}, \ldots, e_k).$$

Let $\mathfrak{D}^{\cdot}$ denote the cosimplicial simplicial group

$$k \longrightarrow \underbrace{EG/G \times \ldots \times EG/G}_{k \text{ factors}}$$

with

$$\begin{aligned}\delta^0(e_0,\dots,e_k) &= (*,e_0,\dots,e_k),\\ \delta^i(e_0,\dots,e_k) &= (e_0,\dots,e_{i-1},e_i,e_i,e_{i+1},\dots,e_k),\\ \delta^{k+1}(e_0,\dots,e_k) &= (e_0,\dots,e_k,*),\\ \sigma^i(e_0,\dots,e_k) &= (e_0,\dots,e_{i-1},e_{i+1},\dots,e_k).\end{aligned}$$

We see that $\mathfrak{D}^{\cdot}$ is clearly isomorphic to the Eilenberg–Moore cosimplicial simplicial set

$$\mathrm{EM}^{\cdot}(EG/G) = \mathrm{EM}^{\cdot}(BG).$$

There is a natural map $\theta : \mathfrak{D}^{\cdot} \to \mathfrak{B}^{\cdot}$ given by

$$\theta(e_0,\dots,e_{k-1}) = (e_0^{-1}, e_0e_1^{-1},\dots,e_{k-2}e_{k-1}^{-1},e_{k-1}),$$

which is readily checked to be a levelwise homotopy equivalence. The result now follows by application of $R[-]$. □

Proposition 2.5. *Let G be an abelian group, and let A be the group ring $\mathbb{Z}[G]$. Let B be the ring $\mathbb{F}_l$, which becomes an A-algebra via the augmentation to $\mathbb{Z}$ followed by* mod-l *reduction. Then there is a model for $\mathfrak{C}^{\cdot}_A(A;B)$, which is a levelwise tensor product of a model for $\mathfrak{C}^{\cdot}_{\mathbb{Z}}(\mathbb{Z};\mathbb{F}_l)$ and a model for $\mathfrak{C}^{\cdot}_{\mathbb{F}_l}(\mathbb{F}_l[G];\mathbb{F}_l)$. The E_2-term of the spectral sequence for $\mathfrak{C}^{\cdot}_A(A,B)$ is therefore a tensor product*

$$\mathbb{F}_l[\beta]\otimes \mathrm{EM}^{*,*}(G),$$

where $\mathrm{EM}^{*,*}(G)$ denotes the E_2-term of the mod-l *Eilenberg–Moore spectral sequence for the loop space of BG.*

Proof. Let $\mathcal{F}$ denote any $\mathbb{Z}$-free simplicial commutative ring that is weakly equivalent to $\mathbb{F}_l$ and let $\mathcal{F}[EG_.]$ denote the tensor product $\mathcal{F}\otimes\mathbb{Z}[EG.]$. One model for the cosimplicial construction $\mathfrak{C}_A(A;B)$ is equivalent to the cosimplicial simplicial abelian group

$$k \longrightarrow \underbrace{\mathcal{F}[EG.]\otimes_{\mathbb{Z}[G]}\dots\otimes_{\mathbb{Z}[G]}\mathcal{F}[EG.]}_{k+1\text{ factors}}.$$

Given the tensor product decomposition of $\mathcal{F}[EG_.]$ and viewing $\mathbb{Z}[G]$ as $\mathbb{Z}\otimes\mathbb{Z}[G]$, we find that this cosimplicial simplicial abelian group is equivalent to

$$k \longrightarrow \underbrace{\mathcal{F}\otimes_{\mathbb{Z}}\dots\otimes_{\mathbb{Z}}\mathcal{F}}_{k+1\text{ factors}}\otimes_{\mathbb{Z}}\underbrace{\mathbb{Z}[EG.]\otimes_{\mathbb{Z}[G]}\dots\otimes_{\mathbb{Z}[G]}\mathbb{Z}[EG.]}_{k+1\text{ factors}}$$

with coface and codegeneracies being the tensor products of the corresponding operators for the factors. Proposition 2.4 now shows that this object is levelwise equivalent to

$$k \longrightarrow \underbrace{\mathcal{F}\otimes_{\mathbb{Z}}\dots\otimes_{\mathbb{Z}}\mathcal{F}}_{k+1\text{ factors}}\otimes_{\mathbb{Z}}\mathbb{Z}[\mathrm{EM}^{\cdot}(BG)],$$

which in turn is equivalent to the construction

$$k \longrightarrow \underbrace{\mathcal{F}\otimes_{\mathbb{Z}}\dots\otimes_{\mathbb{Z}}\mathcal{F}}_{k+1\text{ factors}}\otimes_{)}\mathcal{F}\mathcal{F}[\mathrm{EM}^{\cdot}(BG)].$$

This is the required result. □

Proposition 2.6. *We specialize Proposition 2.5 to the case where $G=\mathbb{Z}/l^{\infty}\mathbb{Z}$. In this case, the E_2-term of the spectral sequence for $\mathfrak{C}^{\cdot}_A(A;B)$ is isomorphic to a tensor product*

$$\mathbb{F}_l[\beta]\otimes\Lambda_{\mathbb{F}_l}(\xi),$$

where Λ denotes the Grassmann algebra, and the elements β and ξ have bidegree $(-1,1)$ and $(-1,2)$, respectively. The spectral sequence collapses, and the homotopy groups of the total spectrum are isomorphic to $\cong\mathbb{Z}_l$ for $i=0,1$ and vanish for other values of i. The filtration on π_0 is the l-adic filtration, and on π_1 it is given by $F_i\pi_1 = l^{i-1}\pi_1$. The pro-group $\{\pi_1(\mathrm{Tot}^i\mathfrak{C}^{\cdot}_A(A;B))\}_i$ is isomorphic to the pro-group $\{\mathbb{Z}/l^i\mathbb{Z}\}_i$.

Proof. The form of the spectral sequence follows immediately from Proposition 2.2 and Lemma 2.3. The computation of π_0 with its filtration follows directly by comparison with Proposition 2.2. Because of the ring structure on $\pi_* A_B^\wedge$, we have that $\pi_1 A_B^\wedge$ is a module over $\pi_0 A_B^\wedge$. Since its associated graded module is cyclic from the spectral sequence, it follows that it is itself cyclic, and that the filtration is as described. The pro-group statement is immediate. □

We wish to extend this result a bit. The l-adic completion is of an abelian group G defined to be the inverse limit $\varprojlim G/l^j G$. In [**2**], this construction is reformulated as $\mathrm{Ext}(\mathbb{Z}[1/l]/\mathbb{Z}, G)$. There is a natural homomorphism $\eta : G \to G_l^\wedge$, and an abelian group is said to be *l-complete* if η is an isomorphism. The following is proved in [**2**, VI.3.2].

Proposition 2.7. *If G is l-complete, then the group $G_l^\wedge$ is l-complete. In other words, the functor $(-)_l^\wedge$ is an idempotent functor from the category of abelian groups to itself.*

We also record the following facts, which will be useful later.

Proposition 2.8. *Let $f : G \to G'$ be a homomorphism of torsion-free abelian groups, with G' l-complete. Suppose that the composite $G \xrightarrow{f} G'/lG'$ is surjective. Then extension $\hat{f} : G_l^\wedge \to G'$ is surjective.*

Proof. An easy induction shows that the map $G/l^j G \to G'/l^j G'$ is surjective for all j. Since $G_l^\wedge$ and G' are given by the inverse limits of the systems

$$\cdots \longrightarrow G/l^3 G \longrightarrow G/l^2 G \longrightarrow G/lG$$

and

$$\cdots \longrightarrow G'/l^3 G' \longrightarrow G'/l^2 G' \longrightarrow G'/lG',$$

and the homomorphism $\hat{f}$ is obtained from the evident map of inverse systems, it follows that there is an exact sequence

$$G_l^\wedge \longrightarrow G' \longrightarrow \lim{}^1 K_j,$$

where K_j denotes the kernel of $G/l^j G \to G'/l^j G'$. We will prove that the inverse system

$$\cdots \longrightarrow K_3 \longrightarrow K_2 \longrightarrow K_1$$

is surjective, and therefore that its $\lim^1$ term vanishes. Consider now the diagram

$$\begin{tikzcd}
 & & l^j G/l^{j+1} G \arrow[r, "q"] \arrow[d] & l^j G'/l^{j+1} G' \arrow[d] & \\
0 \arrow[r] & K_{j+1} \arrow[r] \arrow[d] & G/l^{j+1} G \arrow[r] \arrow[d] & G'/l^{j+1} G' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_j \arrow[r] & G/l^j G \arrow[r] \arrow[d] & G'/l^j G' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. Observe that the torsion-freeness of the groups G and G' shows that $l^j G/l^{j+1} G \cong G/lG$ and $l^j G'/l^{j+1} G' \cong G'/lG'$. It now follows easily that the map q is surjective, and the five lemma now gives the result. □

Proposition 2.9. *Suppose that G and G' are both torsion-free l-complete groups, and that $f : G \to G'$ is a homomorphism. Assume further that the induced map $\bar{f} : G/lG \to G'/lG'$ is an isomorphism. Then f is an isomorphism.*

Proof. The surjectivity follows from Proposition 2.8. An inductive argument shows that the corresponding inverse systems $\{G/l^jG\}$ and $\{G'/l^jG'\}$ are mapped isomorphically under the map induced by f, which gives the result. □

The work of Harrison [**13**], as quoted in [**2**, Chapter VI, 4.5], gives the following.

Proposition 2.10. *Let $\mathfrak{D}_l$ and $\mathfrak{T}_l$ denote the categories of divisible l-torsion abelian groups and of torsion-free l-complete abelian groups, respectively. Then the functor $\mathcal{A}_l : \mathfrak{D}_l \to \mathfrak{T}_l$ defined by $\mathcal{A}_l(G) = \mathrm{Hom}(\mathbb{Z}/l^\infty\mathbb{Z}, G)$ is an equivalence of categories, with inverse functor $\mathcal{B}_l$ given by $\mathcal{B}_l(G) = \mathbb{Z}/l^\infty\mathbb{Z} \otimes G$. Moreover, every object in $\mathfrak{D}_l$ may be written as an arbitrary direct sum of copies of $\mathbb{Z}/l^\infty\mathbb{Z}$.*

We also define an l-completed version of the Grassmann algebra construction. Let A denote a commutative ring and let M be a module over A. We write $\Lambda_*(M)$ for the usual Grassmann algebra construction on M. Let $\mathfrak{p}$ denote any prime ideal of A, and denote by $\hat{\Lambda}^{\mathfrak{p}}_*(M)$ the direct sum $\bigoplus_s \lambda^s(M)^\wedge_{\mathfrak{p}}$. We see that $\hat{\Lambda}^{\mathfrak{p}}_*(M)$ is clearly a graded algebra over the completion $A_{\mathfrak{p}}$.

Proposition 2.11. *Let G denote an l-torsion divisible group, and let A denote the commutative ring $\mathbb{Z}[G]$. Let B denote $\mathbb{F}_l$ denote the finite field with l elements regarded as an A-algebra via the augmentation followed by* mod-l *reduction. Then the E_2-term of the spectral sequence for $\mathfrak{C}^{\cdot}_A(A;B)$ is isomorphic to a tensor product*

$$\mathbb{F}_l[\beta] \otimes \Lambda_{\mathbb{F}_l}(T),$$

where T denotes the l-torsion subgroup of G, regarded as an $\mathbb{F}_l$ vector space. The spectral sequence collapses at the E_2-term. We have

$$\pi_* \mathrm{Tot}(\mathfrak{C}^{\cdot}_A(A;B)) \cong \hat{\Lambda}^l_*(\mathcal{A}(G))$$

and the coskeletal filtration is given by $F_i\pi_s = l^{i-s}\pi_s$.

Proof. From Proposition 2.10 we know that G is a (possibly infinite) sum of copies of $\mathbb{Z}/l^\infty\mathbb{Z}$. It follows that the pro-group $\{\pi_1 \mathrm{Tot}^i \mathfrak{C}_A(A;B)\}_i$ is isomorphic to the corresponding direct sum of copies of the pro-group $\{\mathbb{Z}/l^i\mathbb{Z}\}_i$. This gives us the result for π_1. The higher groups now follow by multiplicativity of the spectral sequence. □

Corollary 2.12. *Let G, A, and B be as in Proposition 2.11. Then $\pi_1 A^\wedge_B$ is an l-complete group.*

Corollary 2.13. *Assume that G is given as a direct sum $\bigoplus_{\alpha\in A} \mathbb{Z}/l^\infty\mathbb{Z}$. Then the natural map*

$$\bigoplus_{\alpha\in A} \pi_1 \mathrm{Tot}(\mathfrak{C}^{\cdot}_{\mathbb{Z}[\mathbb{Z}/l^\infty\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}/l^\infty\mathbb{Z}];\mathbb{F}_l)) \longrightarrow \pi_1 \mathrm{Tot}(\mathfrak{C}^{\cdot}_{\mathbb{Z}[G]}(\mathbb{Z}[G];\mathbb{F}_l))$$

induces an isomorphism on l-completions.

Proof. Letting $\mathfrak{G}$ and $\hat{\mathfrak{G}}$ denote

$$\bigoplus_{\alpha\in A} \pi_1 \mathrm{Tot}(\mathfrak{C}^{\cdot}_{\mathbb{Z}[\mathbb{Z}/l^\infty\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}/l^\infty\mathbb{Z}];\mathbb{F}_l))$$

and

$$\pi_1 \mathrm{Tot}(\mathfrak{C}^{\cdot}_{\mathbb{Z}[G]}(\mathbb{Z}[G];\mathbb{F}_l)),$$

respectively, it is clear from the definitions that the map $\mathfrak{G}/l\mathfrak{G} \to \hat{\mathfrak{G}}/l\hat{\mathfrak{G}}$ is an isomorphism. The result now follows easily from Proposition 2.9. □

We also recall Suslin's theorem about the K-theory of an algebraically closed field [**30**].

THEOREM 2.14. *Let $k \to F$ be an inclusion of algebraically closed fields of characteristic $p \neq l$ (p may be 0). Then the natural map $Kk \to KF$ induces an equivalence $Kk_l^\wedge \to KF_l^\wedge$.*

REMARK 2.15. The completion functor $(-)_l^\wedge$ is obtained as the total spectrum of a cosimplicial spectrum, to which is associated a pro-spectrum whose individual terms are Tot_n applied to the cosimplicial spectrum. Applying homotopy groups now gives a pro-group. The proof of Suslin's theorem shows that, in fact, the map in question actually induces an isomorphism of pro-homotopy groups.

3. *Categories of descent data*

Let G be a profinite group and suppose that we are given an action of G on an abelian group N. We say that the action is *continuous* if the stabilizer of any element $n \in N$ is an open and closed subgroup of finite index in G. Let A be any commutative ring equipped with a group action by a profinite group G, which is continuous when A is regarded as an abelian group. For any category of A-modules $\mathcal{M}$, we say that the family is *G-invariant* if it is closed under the G-action in the sense that if M is in the category $\mathcal{M}$, then so are any of the module structures $*_g$ on M obtained by pulling back along the ring homomorphism $g : A \to A$, that is, so that $a *_g m = g(a) \cdot m$.

DEFINITION 3.1. By a *linear descent datum* for the pair (G, A), we mean a finitely generated projective A-module M, together with a continuous action of G on M (that is, the group action is continuous when M is regarded as an abelian group) so that $g(am) = a^g g(m)$ for all $g \in G$, $a \in A$, and $m \in M$. We define two categories of linear descent data, $V^G(A)$ and $V(G, A)$. The objects of $V^G(A)$ are all linear descent data for the pair (G, A), and the morphisms are all equivariant A-linear morphisms. The objects of $V(G, A)$ are also all linear descent data for (G, A), but the morphisms are all A-linear morphisms (without any equivariance requirements). The group G acts continuously on the category $V(G, A)$ by conjugation of maps (so the action is trivial on objects), and the fixed point subcategory is clearly $V^G(A)$. Note that both categories are symmetric monoidal categories under direct sum. More generally, when $\mathcal{M}$ is any G-invariant family of A-modules, we construct the corresponding categories $V^G(A; \mathcal{M})$ and $V(G, A; \mathcal{M})$ in identical fashion.

We note that $\otimes_A$ provides a coherently associative and commutative monoidal structure on $V(G, A)$ and $V^G(A)$.

DEFINITION 3.2. We define the spectra $K^G(A)$ and $K(G, A)$ to be the spectra obtained by applying an infinite loop space machine ([**19**] or [**27**]) to the symmetric monoidal categories of isomorphisms of $V^G(A)$ and $V(G, A)$, respectively. The spectrum $K(G, A)$ is equipped with a G-action, where fixed point spectrum is $K^G(A)$. The tensor product described above makes each of these spectra into commutative ring spectra using the results of [**8**], specifically the discussion following Definition 3.1.

REMARK 3.3. The spectrum $K(G, A)$ is here a 'naive' G-spectrum, that is, only a spectrum with an action of G on each of the deloopings, which are compatible under the bonding maps in the spectrum. In some cases, it will carry the structure of a fully equivariant ring spectrum, but we will not use this property.

There are various functors relating these categories (and therefore their K-theory spectra). We have the fixed point functor

$$(-)^G : V^G(A) \longrightarrow V^{\{e\}}(A^G) \simeq A^G\text{−mod}$$

defined on objects by $M \to M^G$. We also have the induction functor

$$A \otimes_{A^G} - : A^G\text{–mod} \simeq V^{\{e\}}(A^G) \longrightarrow V^G(A)$$

given on objects by $M \to A \otimes_{A^G} M$. The following is a standard result in descent theory.

PROPOSITION 3.4. *Let F be a field , and suppose that we are given a continuous action of a profinite group G on F, so that $F^G \hookrightarrow F$ is a Galois extension with Galois group G. Suppose further that G is the inverse limit of groups of order prime to the characteristic of F. Then both $(-)^G$ and $F \otimes_{F^G} -$ are equivalences of categories.*

Proof. The isomorphism classes of objects of $V^G(\bar{F})$ whose underlying $\bar{F}$-vector space has dimension n are in bijective correspondence with the non-abelian cohomology set $H^1(G, \mathrm{GL}_n(\bar{F}))$. This set consists of a single point (see [**28**, Chapter X, Proposition 3]), so every object in $V^G(\bar{F})$ is obtained up to isomorphism by tensoring up from an F-vector space with trivial action. The result now follows since the conjugation action by G on matrices over $\bar{F}$ has as fixed point set the corresponding matrices over F. □

We have an analog of this result when A is not a field.

PROPOSITION 3.5. *Suppose that A is a finite Galois extension of a commutative ring A_0, in the sense of [**4**]. The group Γ of automorphisms of A over A_0 is a finite group, and of course acts on A. The fixed point subring A^Γ is equal to A_0. Suppose further that A_0 contains a field k, and that no subgroup of Γ admits a continuous homomorphism to the group $\mathbb{Z}/p\mathbb{Z}$, where p is the characteristic of k. Then the functor $(-)^\Gamma : V^\Gamma(A) \to \mathrm{Proj}(A^\Gamma)$ is an equivalence of categories.*

Proof. The proof is Theorem 1.1, Chapter III of [**4**]. □

PROPOSITION 3.6. *Let G be any profinite group, and suppose that G acts continuously on a commutative ring A. Suppose further that, for any normal subgroup $N \subseteq G$ of finite index, the extension $A^G \subseteq A^N$ is a Galois extension of rings in the sense of [**4**], and further that A^G contains a subfield k and that no subgroup of G admits a homomorphism to $\mathbb{Z}/p\mathbb{Z}$, where p is the characteristic of k. Then the functor $(-)^G : V^G(A) \to \mathrm{Proj}(A^G)$ is an equivalence of categories.*

Proof. We will prove that the functor

$$\operatorname*{colim}_N V^{G/N}(A^N) \longrightarrow V^G(A)$$

is an equivalence of categories. The result will then follow from Proposition 3.5. Let P be an object of $V^G(A)$. As it is a projective A-module, it is a direct summand of a finitely generated free A-module F, say $F \cong P \oplus \bar{P}$. As P is equipped with a continuous G-action, we obtain a continuous G action on F by requiring that the action on $\bar{P}$ be the action by the identity. We equip F with a finite basis $\mathfrak{B}$. The direct summand P now corresponds to an idempotent matrix ϵ over A. Let N_0 denote the maximal normal subgroup that fixes all the entries of ϵ. From the continuity of the action, N_0 is a normal subgroup of finite index in G. Let N_1 denote the maximal normal subgroup of G that fixes the elements of $\mathfrak{B}$. It too is a normal subgroup of finite index due to the continuity condition on the G action. Let $N = N_0 \cap N_1$, which is also a normal subgroup of finite index. It is now clear that the A^N submodule of F generated by $\mathfrak{B}$ is invariant under the G action, and further that the matrix ϵ is of the form $A \otimes_{A^n} \epsilon^N$ for a matrix ϵ^N over A^N. It now follows immediately that the module P is of the form $A \otimes_{A^N} P^N$ for some object P^N of $V^{G/N}(A^N)$. Morphisms between objects of $V^G(A)$ can clearly be represented by finite matrices over A. Because of the finiteness of the matrix, it is clearly obtained by applying $A \otimes_{A^K} -$ for some K. The result follows. □

We also have the following proposition.

PROPOSITION 3.7. *Let F be a field, equipped with an action. The category $V(G, F)$ is canonically equivalent to the category $\mathrm{Vect}(F)$ of finitely generated vector spaces over F.*

Proof. There is a functor from $\mathrm{Vect}(F)$ into $V(G,F)$ that carries an F-vector space to itself, equipped with a trivial action. Similarly, there is a forgetful functor $V(G,F) \to \mathrm{Vect}(F)$, which simply ignores the action. One composite is the identity; the other is naturally isomorphic to the identity as the morphisms in $V(G,F)$ are computed without reference to the group action. □

REMARK 3.8. Note that when the group action is trivial, that is, G acts by the identity, and A is a field, then the category $V^G(F)$ is just the category of finite-dimensional continuous F-linear representations of G.

DEFINITION 3.9. In the case when the G-action on A is trivial, and A is a field F, we also write $\mathrm{Rep}_F[G]$ for $V^G(F)$.

PROPOSITION 3.10. *Let E be a field with continuous action by a profinite group G and let $F = E^G$ denote the fixed point subfield. Suppose further that G is the inverse limit of groups of order prime to the order of the characteristic of E. Then the functor*

$$\mathrm{Rep}_F[G] = V^G(F) \xrightarrow{E\otimes_F -} V^G(E) \simeq V^{\{e\}}(F) = \mathrm{Vect}(F)$$

respects the tensor product structure, and *$K(F)$ becomes an algebra over the ring spectrum $K\mathrm{Rep}_F[G]$.*

Proof. Let G be a group and suppose that G acts on a commutative ring A. We may form the skew group ring $A\langle G\rangle$ (see [**12**]). Given any left $A\langle G\rangle$-modules M, N, we may form the tensor product $M \otimes_A N$, and observe that it becomes an $A\langle G\rangle$-module by equipping it with the G-action given by $g(m \otimes n) = gm \otimes gn$. If the group G is finite, then this construction gives monoidal structures on the categories of finitely generated $A\langle G\rangle$-modules and on the full subcategory $P(A,G)$ of $A\langle G\rangle$-modules whose underlying A-modules are finitely generated projective. After rectifications identical to the ones performed in [**8**] after Definitions 3.1 and 3.7, $P(A,G)$ becomes a bipermutative category under this monoidal structure. Fixing G and supposing that we have two commutative rings A and B with actions by G together with an equivariant homomorphism of rings $f : A \to B$, we obtain a ring homomorphism $f\langle G\rangle : A\langle G\rangle \to B\langle G\rangle$ and an extension of scalars functor $B \otimes_A - : P(A,G) \to P(B,G)$. We also observe that there is a natural isomorphism of functors

$$(B \otimes_A M) \otimes_B (B \otimes_A N) \longrightarrow B \otimes_A (M \otimes_A N)$$

from which it follows that the functor $B \otimes_A -$ is a functor of bipermutative categories, and produces a homomorphism of commutative ring spectra $KP(G,A) \to KP(B,G)$. Let $G = \varprojlim G_i$, and $G_i = G/N_i$, where N_i is a normal subgroup of finite index. It is now easy to check that $V^G(F)$ is equivalent to the direct limit of bipermutative categories $P(F,G_i)$, and that similarly $V^G(E)$ is equivalent to the direct limit of the bipermutative categories $P(E^{N_i},G_i)$ under the evident inclusion maps. This gives the result. □

We now introduce some notation that we will use throughout the rest of the paper. Let k be an algebraically closed field and B be any k-algebra.

DEFINITION 3.11. Suppose that B is equipped with a continuous action by a profinite group G. Then by α_B^G we mean the natural map of k-theory spectra $K^G(k) \to K^G(B)$. Given a prime l, we let $A_B^G(l)$ denote the induced map on derived completions associated to the map

$$\rho_l : K^G(k) \longrightarrow \mathbb{H}_l,$$

where $\mathbb{H}_l$ denotes the mod-l Eilenberg–Mac Lane spectrum, and ρ is the reduction map that assigns to every finite-dimensional k-linear representation V the mod-l reduction of its dimension. When F is a field, we write α_F and $A_F(l)$ for the maps associated with the action of the separable Galois group G_F of F on the algebraic closure $\bar{F}$.

4. *An example*

Let k denote any algebraically closed field, and l be a prime distinct from the characteristic of k, and let L_j denote the ring $k[t^{\pm 1/l^j}]$ of Laurent polynomials over k, so $L_0 = k[t^{\pm 1}]$. We write L for L_0. We denote by L_∞ the union

$$\bigcup_j L_j.$$

Similarly, let $\mathcal{O}_j$ denote the ring $k[t^{1/l^j}]$ and $\mathcal{O}_\infty$ the union

$$\bigcup_j \mathcal{O}_j.$$

We choose a family of generators ζ_j for the groups of l^jth roots of unity in k, so that $\zeta_j^l = \zeta_{j-1}$ for each j. This family determines an action of the group $G = \mathbb{Z}_l$ on L_∞ by the requirement $T \cdot t^{1/l^j} = \zeta_j t^{1/l^j}$, where T denotes the topological generator 1 for $\mathbb{Z}_l$. The subring $\mathcal{O}_\infty \subseteq L_\infty$ is invariant under this action.

Remark 4.1. Throughout this section, all derived completions will be over the commutative ring spectrum $K^G(k)$, along the homomorphism of commutative ring spectra $K^G(k) \to \mathbb{H}_l$, where $\mathbb{H}_l$ denotes the mod-l Eilenberg–Mac Lane spectrum.

We require some further algebraic facts about the rings L_∞ and $\mathcal{O}_\infty$. Recall the following definition.

Definition 4.2. A commutative ring A is said to be *coherent* if every finitely generated ideal of A is finitely presented.

The theory of such rings is developed in [**11**]. In particular, we have the following result [**11**, Theorem 2.3.2(2)].

Proposition 4.3. *Let A be a commutative coherent ring. Then any finitely presented A-module M has the property that every finitely generated submodule of M is finitely presented.*

The following is a straightforward consequence.

Proposition 4.4. *Let A be a commutative coherent ring. Then the category of finitely presented A-modules is an abelian category.*

Proof. One needs only check that the kernel and cokernel of any homomorphism $f : M \to N$ of finitely presented A-modules are also finitely presented. This is obvious for the cokernel. For the kernel, one can use the fact that, for any commutative ring A and any exact sequence,

$$0 \longrightarrow K \longrightarrow L \longrightarrow H \longrightarrow 0$$

in which H is finitely presented and L is finitely generated, then K is finitely generated [**11**, Lemma 2.1.1]. For we have the exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow M \longrightarrow \mathrm{Im}(f) \longrightarrow 0.$$

Because N is finitely presented, Proposition 4.3 guarantees that $\mathrm{Im}(f)$ is finitely presented, and the quoted fact therefore guarantees that $\mathrm{Ker}(f)$ is finitely generated. Because it is a submodule of the finitely presented module M, it follows from coherence that it is finitely presented. $\square$

These results apply in our case because of the following fact.

Proposition 4.5. *Let R be a commutative ring, which is a union of subrings R_n. Suppose further that each ring R_n is Noetherian, and further that whenever $m \leqslant n$, R_n is flat as an R_m module. Then the ring R is coherent.*

Proof. Consider any finitely generated ideal $\mathfrak{I}$ in R_∞, generated say by the set $\{r_1, \ldots, r_k\}$. There is an integer N so that $r_i \in R_N$ for all i. As R_N is a Noetherian ring, it follows that the ideal $\mathfrak{I}_N$ of R_N generated by the set $\{r_1, \ldots, r_k\}$ is finitely presented, so we have an exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow \mathfrak{I}_N \longrightarrow 0,$$

where F_0 and F_1 are finitely generated free R_N-modules. The flatness of R over R_N now shows that we have an exact sequence

$$R \otimes_{R_N} F_1 \longrightarrow R \otimes_{R_N} F_0 \longrightarrow R \otimes_{R_N} \mathfrak{I} \to 0.$$

We claim that the natural map $R \otimes_{R_N} \mathfrak{I} \to \mathfrak{I}$ is an isomorphism. It is clearly surjective. To see that it is injective, we consider the exact sequence

$$F_0 \longrightarrow F_1 \longrightarrow R_N.$$

From the flatness of R over R_N, it follows that we obtain an exact sequence

$$R \otimes_{R_N} F_0 \longrightarrow R \otimes_{R_N} F_1 \longrightarrow R,$$

which demonstrates that the cokernel of $R \otimes_{R_N} F_0 \to R \otimes_{R_N} F_1$ injects into R, from which the result follows. $\square$

Corollary 4.6. *The rings L_∞ and $\mathcal{O}_\infty$ are coherent.*

We also record the following fact concerning L_∞ and $\mathcal{O}_\infty$.

Proposition 4.7. *Any finitely presented torsion-free module over L_∞ or $\mathcal{O}_\infty$ is finitely generated free.*

Proof. We do the case of L_∞, the case of $\mathcal{O}_\infty$ being identical. Let

$$F_1 \xrightarrow{\Lambda} F_0 \longrightarrow M \longrightarrow 0$$

be a presentation of M. The homomorphism Λ is represented by a matrix with entries in L_∞ once bases for F_0 and F_1 are chosen. As the entries form a finite set of elements of L_∞, there is a j so that all the entries lie in L_j. We then consider the L_j-module M_j defined by the presentation

$$G_1 \xrightarrow{\Lambda_j} G_0 \longrightarrow M_j \longrightarrow 0,$$

where G_0 and G_1 are, respectively, the L_j-submodules of F_0 and F_1 generated by the chosen bases, and Λ_j is the restriction of Λ to the submodule G_1. We claim that M_j is a torsion-free L_j-module. We clearly have that $M \cong L_\infty \otimes_{L_j} M_j$. For any $n \geqslant j$, L_n is a free L_j module, and the inclusion $L_j \hookrightarrow L_n$ is the inclusion on a direct summand of L_j-modules. If $m \in M_j \neq 0$, and $a \cdot m = 0$ in M_j, then it follows that the image of m in M is non-zero, and that it is an a-torsion element, contradicting the fact that M is torsion-free. As L_j is a principal ideal domain, it follows that any finitely generated torsion-free L_j-module is free, hence that M is a free L_∞ module. $\square$

We have observed in Proposition 3.10 that the spectrum KL, which is equivalent to $K^G(L_\infty)$, becomes an algebra spectrum over the ring spectrum $K\mathrm{Rep}_k[G] \simeq K^G k$. We wish to explore the nature of this algebra structure, and to use the derived completion to demonstrate that the algebraic K-theory of L can be constructed directly from the representation theory of G over an algebraically closed field, for example, $\mathbb{C}$.

We consider the map of ring spectra (defined in Definition 3.11)

$$\alpha^G_{L_\infty} : K^G(k) = K\mathrm{Rep}_k[G] \longrightarrow K^G(L_\infty) \simeq KL$$

induced by the functor

$$\mathrm{id}_k \otimes_k - : V^G(k) \longrightarrow V^G(L_\infty).$$

We note that as it stands this map does not seem to carry much structure, and is far from being an equivalence. It turns out, though, that we may obtain information from the derived completion as follows. As usual, let $\mathbb{H}_l$ denote the mod-l Eilenberg–Mac Lane spectrum. We have a commutative diagram of ring spectra

$$\begin{tikzcd} K^G(k) \arrow[r, "\alpha^G_{L_\infty}"] \arrow[d, "\varepsilon"'] & K^G L_\infty \arrow[d, "\varepsilon"] \\ \mathbb{H}_l \arrow[r, "\mathrm{id}"] & \mathbb{H}_l \end{tikzcd}$$

where ε in both cases denotes the augmentation map that sends any vector space or representation to its dimension mod-l. The naturality properties of the derived completion construction yield a map

$$A^G_{L_\infty}(l) : K^G(k)^\wedge_{\mathbb{H}_l} \longrightarrow K^G(L_\infty)^\wedge_{\mathbb{H}_l} \xrightarrow{\sim} KL^\wedge_{\mathbb{H}_l},$$

which we refer to as the *representational assembly map*. The rightmost equivalence is induced by the functor, which carries an object in $V^G(L_\infty)$ to its G-fixed point subspace, and the fact that it is an equivalence follows from Proposition 3.6. The goal of this section is to show that despite the fact that $\alpha^G_{L_\infty}$ carries little information about K_*L, $A^G_{L_\infty}(l)$ is an equivalence of spectra.

Proposition 4.8. *The spectrum $KL^\wedge_{\mathbb{H}_l}$ is equivalent to the l-adic completion of KL.*

Proof. We note that the $K\mathrm{Rep}_k[G]$-module structure on KL extends over the augmentation, that is, there is a commutative diagram

$$\begin{tikzcd} K\mathrm{Rep}_k[G] \wedge KL \arrow[r] \arrow[d, "\varepsilon\wedge\mathrm{id}"] & KL \arrow[d, "\mathrm{id}"'] \\ Kk \wedge KL \arrow[r] & KL \end{tikzcd}$$

where the horizontal maps are the structure maps for the module structures. Letting $B_0 = K\mathrm{Rep}[G]$ and $B_1 = Kk$, we write KL_{B_0} and KL_{B_1} for KL regarded as B_0 and B_1 module spectra, respectively. We obtain a natural map

$$(KL_{B_0})^\wedge_{\mathbb{H}_l} \longrightarrow (KL_{B_1})^\wedge_{\mathbb{H}_l},$$

which is an equivalence by Carlsson [**3**, Proposition 3.2(6)]. Next, let KL_S denote KL regarded as a module spectrum over the sphere spectrum S. Then we similarly obtain a natural map $(KL_S)^\wedge_{\mathbb{H}_l} \to (KL_{B_1})^\wedge_{\mathbb{H}_l}$, which is an equivalence by Carlsson [**3**, Theorem 6.10]. But [**3**, Example 8.2] shows that $(KL_S)^\wedge_{\mathbb{H}_l}$ is equivalent to the Bousfield–Kan l-adic completion of KL. □

We now proceed to prove that $A^G_{L_\infty}(l)$ is a weak equivalence. The overall strategy is as follows. We consider the map of $K^G(k)$-module spectra

$$\theta : K^G(\mathcal{O}_\infty) \longrightarrow K^G(L_\infty).$$

We will prove the following two facts about this situation.

(1) The map θ induces an equivalence $\theta^\wedge_{\mathbb{H}_l} : K^G(\mathcal{O}_\infty)^\wedge_{\mathbb{H}_l} \to K^G(L_\infty)^\wedge_{\mathbb{H}_l}$.

(2) The map

$$A^G_{\mathcal{O}_\infty}(l) : K^G(k)^\wedge_{\mathbb{H}_l} \longrightarrow K^G(\mathcal{O}_\infty)^\wedge_{\mathbb{H}_l}$$

is an equivalence.

It is clear that

$$A^G_{L_\infty}(l) = \theta^\wedge_{\mathbb{H}_l} \circ A^G_{\mathcal{O}_\infty}(l)$$

from which it follows that $A^G_{L_\infty}(l)$ is an equivalence if the facts (1) and (2) above hold.

We begin with fact (1).

Let Φ denote the homotopy fiber of the map θ. This fiber is a module spectrum over the commutative ring spectrum $K^G(k)$, and we may therefore form its derived completion $\Phi^{\wedge}_{\mathbb{H}_l}$.

PROPOSITION 4.9. *Suppose that $\Phi^{\wedge}_{\mathbb{H}_l} \simeq *$. Then the map $\theta^{\wedge}_{\mathbb{H}_l}$ is an equivalence of spectra.*

Proof. This is an immediate consequence of [**3**, Proposition 3.2(3)]. □

In order to prove $\Phi^{\wedge}_{\mathbb{H}_l} \simeq *$, we use the algebraic to geometric spectral sequence of [**3**, Theorem 7.1], which requires as input understanding of the $R[G] \cong \pi_0 K^G(k)$-module structure of $\pi_*\Phi$. Unfortunately, the model for K-theory spectra used in [**8**] does not have the right tools for analyzing homotopy fibers of maps of K-theory spectra. There is another model for K-theory spectra due to Waldhausen [**36**], which does have such tools. It takes as input a *category with cofibrations and weak equivalences* (see [**36**] for precise definitions). We summarize the properties of this model.

PROPOSITION 4.10. *Let A denote any ring, $P(A)$ the category of finitely generated projective A-modules, and* $\mathrm{Mod}(A)$ *the category of finitely generated modules. The category $P(A)$ is a category with cofibrations and weak equivalences in the sense of* [**36**], *where the cofibrations are inclusions of direct summands and the weak equivalences are isomorphisms. Any abelian category is a category with cofibrations and weak equivalences where the cofibrations are inclusions and the weak equivalences are the isomorphisms. Then we have the following.*

(1) *There is a natural equivalence $K(A) \to \mathcal{K}(P(A))$.*

(2) *Any abelian category can be viewed as a category with cofibrations and weak equivalences, where the cofibrations are inclusions and the weak equivalences are isomorphisms. Given any Serre subcategory T of an abelian category A, there is a homotopy fibration sequence*

$$\mathcal{K}T \longrightarrow \mathcal{K}A \longrightarrow \mathcal{K}A/T,$$

where A/T denotes the quotient as defined in [**32**].

(3) *Let A be a commutative ring, and let $\underline{B}$ denote an abelian category of A-modules containing $P(A)$. The inclusion $P(A) \hookrightarrow \underline{B}$ is an exact functor in the sense of* [**36**, Section I.1]. *Suppose further that every object of $\underline{B}$ has a finite projective resolution. Then the inclusion*

$$\mathcal{K}P(A) \longrightarrow \mathcal{K}\underline{B}$$

is a weak equivalence of spectra.

Proof. Assertion (1) is [**36**, Theorem 1.8.1]. Assertion (2) follows from the fibration [**36**, Theorem 1.6.4], or can be proved using the comparison result proved in [**36**, Appendix 1.9] together with Quillen's localization theorem [**23**, Theorem 5, Section 5]. Assertion (3) follows directly from the approximation theorem I.6.7 [**36**], or can be proved using the comparison result in [**36**, Appendix 1.9] together with the reduction by the resolution theorem [**23**, Theorem 3]. □

Let $\mathfrak{M}$ denote the category of finitely presented $\mathcal{O}_\infty$-modules. The category $\mathfrak{M}$ is a G-invariant category of $\mathcal{O}_\infty$-modules, and it is an abelian category by Propositions 4.4 and 4.6. Similarly, we let $\mathfrak{M}_L$ denote the category of finitely presented L_∞-modules. We let $\mathfrak{T}$ denote the full subcategory of $\mathfrak{M}$ consisting of the torsion modules, that is, the modules for which every element is annihilated by some power of the generator t. The following proposition gives a description of the $R[G]$-action on $\pi_*\Phi$.

PROPOSITION 4.11. *For any continuous k-linear representation ρ of G, we have the functor $\rho \otimes_k - : \mathfrak{T} \to \mathfrak{T}$. This family of functors gives an action of the semiring of isomorphism classes of representations on $K_*(V^G(\mathcal{O}_\infty, \mathfrak{T}))$, which evidently extends to an action of $R[G]$. There is an isomorphism of $R[G]$-modules from $\pi_*\Phi$ to $K_*(V^G(\mathcal{O}_\infty, \mathfrak{T}))$ equipped with this action.*

Proof. We have a diagram of spectra as follows:

$$
\begin{tikzcd}
\Phi \arrow[r] & K^G(\mathcal{O}_\infty) \arrow[r] & K^G(L_\infty) \\
\Psi_{\mathrm{sum}} \arrow[r] \arrow[u, "\alpha_\Phi"] \arrow[d, "\beta_\Phi"'] & \mathcal{K}_{\mathrm{sum}}(V^G(\mathcal{O}_\infty)) \arrow[r] \arrow[u, "\alpha_{\mathcal{O}}"] \arrow[d, "\beta_{\mathcal{O}}"] & \mathcal{K}_{\mathrm{sum}}(V^G(L_\infty)) \arrow[u, "\alpha_L"] \arrow[d, "\beta_L"] \\
\Psi \arrow[r] \arrow[d, "\gamma_\Phi"] & \mathcal{K}(V^G(\mathcal{O}_\infty)) \arrow[r] \arrow[d, "\gamma_{\mathcal{O}}"] & \mathcal{K}(V^G(L_\infty)) \arrow[d, "\gamma_L"] \\
\Psi' \arrow[r] & \mathcal{K}(V^G(\mathcal{O}_\infty, \mathfrak{M})) \arrow[r] & \mathcal{K}(V^G(L_\infty, \mathfrak{M}_L)) \\
\mathcal{K}(V^G(\mathcal{O}_\infty, \mathfrak{T})) \arrow[r] \arrow[u, "\delta_\Phi"] & \mathcal{K}(V^G(\mathcal{O}_\infty, \mathfrak{M})) \arrow[r] \arrow[u, "\delta_{\mathcal{O}}"] & \mathcal{K}(V^G(L_\infty, \mathfrak{M}_L)). \arrow[u, "\delta_L"]
\end{tikzcd}
$$

The top row is the fiber sequence associated to the map of Elmendorf–Mandell K-theory spectra for $\mathcal{O}_\infty$ and L_∞. The second row is the fiber sequence associated to the construction of spectra associated to the simplicial categories $wN_{\cdot}\mathcal{C}$ in [**36**, Section 1.8], applied to the functor of categories with sums $V^G(\mathcal{O}_\infty) \to V^G(L_\infty)$. Rows 3 and 4 are the fibration sequences associated to Waldhausen's $S_{\cdot}$-construction applied to the functors of categories with cofibrations and weak equivalences $V^G(\mathcal{O}_\infty) \to V^G(L_\infty)$ and $V^G(\mathcal{O}_\infty, \mathfrak{M}) \to V^G(L_\infty, \mathfrak{M}_L)$. The vertical α maps are composites of identifications of the spectra $\mathcal{K}_{\mathrm{sum}}$ with the standard Γ-space construction for permutative categories with the equivalence of the standard Γ-space model with the Elmendorf–Mandell multiplicative model introduced in [**8**]. We see that $\alpha_{\mathcal{O}}$ and α_L are equivalences by [**8**]. The maps $\beta_{\mathcal{O}}$ and β_L are the comparison maps from [**36**, Section 1.8], and are proved to be equivalences there. The maps $\gamma_{\mathcal{O}}$ and γ_L are induced by inclusions of categories with cofibrations and weak equivalences, and γ_L is induced by an isomorphism of categories, and so is an equivalence. The maps $\delta_{\mathcal{O}}$ and δ_L are identities, and δ_Φ is an equivalence obtained from the localization theorem [**36**, Theorem 1.6.4] using the identification of $V^G(L_\infty)$ with the quotient abelian category $V^G(\mathcal{O}_\infty, \mathfrak{M})/V^G(\mathcal{O}_\infty, \mathfrak{T})$ along the lines of Swan's proof in [**32**] of his Corollary 5.12. All spectra in this picture admit endofunctors $\rho \otimes_k$, and all the vertical and horizontal maps respect these actions on K-groups. The point is that the functors are both functors of permutative categories and categories with cofibrations and weak equivalences. If we can prove that $\gamma_{\mathcal{O}}$ is an equivalence, then we will have shown that the action of the endofunctors $\rho \otimes_k -$ on $\pi_*\Phi$ and $\pi_*\mathcal{T}$ described above is compatible with the equivalence, which gives the result. But $\gamma_{\mathcal{O}}$ is induced by the inclusion functor $P(\mathcal{O}) \hookrightarrow \mathfrak{M}$, and Proposition 4.7 shows that the hypothesis for part (3) of Proposition 4.10 is satisfied, which gives the result. □

In order to proceed with the analysis, we need some language. Suppose that a commutative ring R is described as the colimit of a sequence of inclusions of commutative rings

$$R_0 \hookrightarrow R_1 \hookrightarrow \cdots \hookrightarrow R_N \hookrightarrow \cdots.$$

Suppose that we are given a family of elements $\mathfrak{R} = \{r_i\}_{i=1}^{\infty}$, with $r_i \in R_i$. We may then form the new inductive system

$$R_0 \xrightarrow{\times r_1} R_1 \xrightarrow{\times r_2} R_2 \xrightarrow{\times r_3} \cdots$$

of abelian groups, whose colimit we denote by $M(\mathfrak{R})$. The colimit $M(\mathfrak{R})$ has the structure of an R-module, where the action is given as follows. For elements $r \in R$ and $m \in M(\mathfrak{R})$, select an integer i so that r and m can be represented as the images of $r_i \in R_i$ and $m_i \in R_i$, respectively. The product $r \cdot m$ is represented by the image of $r_i \cdot m_i \in R_i$ in $M(\mathfrak{R})$. It is readily checked that this produces a well-defined element in $M(\mathfrak{R})$, and that it defines an R-module structure on $M(\mathfrak{R})$.

PROPOSITION 4.12. *The module $M(\mathfrak{R})$ is isomorphic to the colimit $\mathfrak{M}$ of the system of R-modules*

$$R(0) \xrightarrow{\times r_1} R(1) \xrightarrow{\times r_2} R(2) \xrightarrow{\times r_3} \cdots,$$

where each $R(i)$ denotes a copy of the cyclic module R. The module $M(\mathfrak{R})$ is a flat R-module.

Proof. There is a diagram of inductive systems of abelian groups

$$\begin{tikzcd} R_0 \arrow[r, "\times r_1"] \arrow[d] & R_1 \arrow[r, "\times r_2"] \arrow[d] & R_2 \arrow[r, "\times r_3"] \arrow[d] & \cdots \\ R(0) \arrow[r, "\times r_1"] & R(1) \arrow[r, "\times r_2"] & R(2) \arrow[r, "\times r_3"] & \cdots, \end{tikzcd}$$

which induces a homomorphism of abelian groups $p : M(\mathfrak{R}) \to \mathfrak{M}$. Further, p is readily observed to be a homomorphism of R-modules. To prove that p is surjective, we take any element $m \in \mathfrak{M}$ and represent it as being in the image of an element $r \in R(i)$, for some i. The element r is in the image of R_j for some j. If $j \leqslant i$, then it is clear that m is in the image of p. If not, then we form the image of r in $R(j)$ under the transition maps in the inductive system defining $\mathfrak{M}$. With this new representative, we are in the earlier situation. Injectivity is immediate, as the map of inductive systems is injective, and colimits preserve injectivity. Flatness now follows as $\mathfrak{M}$ is a colimit of free R-modules. □

We now consider a specific case of this construction. Let G denote the profinite group $\mathbb{Z}_l$. Applying the representation ring functor gives an inductive system of rings

$$R[G_0] \hookrightarrow R[G_1] \hookrightarrow R[G_2] \hookrightarrow \cdots,$$

where $G_i = G/p^iG \cong \mathbb{Z}/p^i\mathbb{Z}$, whose colimit is $R[G]$. We recall that all representations of a cyclic group C over an algebraically closed field of characteristic zero are one-dimensional, and that the representation ring is isomorphic to the group ring of the character group $\chi(C)$. For any cyclic group C of prime power order l^i, we let $n_C \in R[C]$ denote the element

$$n_C = \sum_{\chi^l = e} \chi.$$

Let $\mathfrak{R}$ denote the family of elements $\{n_{G_i}\}_{i=1}^{\infty}$, and construct the $R[G]$-module $\mathfrak{M}(\mathfrak{R})$.

LEMMA 4.13. *The groups $\operatorname{Tor}_i^{R[G]}(\mathfrak{M}(\mathfrak{R}), \mathbb{F}_l)$ vanish for all i.*

Proof. For $i > 0$, this follows from the flatness of $\mathfrak{M}(\mathfrak{R})$. For $i = 0$, the group $\mathbb{F}_l \otimes_{R[G]} \mathfrak{M}(\mathfrak{R})$ is isomorphic to the colimit of the inductive system of abelian groups

$$\mathbb{F}_l \xrightarrow{\rho_1} \mathbb{F}_l \xrightarrow{\rho_2} \mathbb{F}_l \xrightarrow{\rho_3} \cdots,$$

where ρ_i is the mod-l reduction of the augmentation of the element n_{G_i}. As the augmentation of n_{G_i} is l, all the transition maps in the system vanish. □

We will use this construction to describe $\pi_*\mathcal{K}(V^G(\mathcal{O}_\infty, \mathfrak{T}))$. As a first step, for every i we consider the category $V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i))$, where G_i denotes the quotient group G/l^iG, $\mathcal{O}_i$ the fixed subring $\mathcal{O}_\infty^{l^iG}$, and $\mathfrak{T}(i)$ the category of finitely presented torsion $\mathcal{O}_i$-modules. We have inclusions

$$\cdots \hookrightarrow V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i)) \hookrightarrow V^{G_{i+1}}(\mathcal{O}_{i+1}, \mathfrak{T}(i+1)) \hookrightarrow \cdots \hookrightarrow V^G(\mathcal{O}_\infty, \mathfrak{T})$$

of categories, and consequently a functor

$$\mathrm{j} : \operatorname*{colim}_i V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i)) \longrightarrow V^G(\mathcal{O}_\infty, \mathfrak{T}).$$

LEMMA 4.14. *The functor* $\mathfrak{j}$ *is an equivalence of categories.*

Proof. A first observation is that every object in $V^G(\mathcal{O}_\infty, \mathfrak{T})$ is isomorphic to one in the image of $\mathfrak{j}$. To see this, let M be an object of $V^G(\mathcal{O}_\infty, \mathfrak{T})$, and let $\{m_1, \ldots, m_n\}$ be a generating set for the $\mathcal{O}_\infty$-module M. Let $\Gamma = \bigcap_s \mathrm{Stab}(m_s)$, the intersection of the stabilizer groups of the elements m_s. From the continuity of the G-action, Γ has finite index in G, and is therefore equal to G_i for some i. We obtain an equivariant surjection π from

$$k[G/\Gamma] \otimes_k \mathcal{O}_\infty^n = k[G/\Gamma] \otimes_k \left[\bigoplus_s \mathcal{O}_\infty(e_s)\right]$$

to M, defined by $\pi([g \otimes e_s]) = g \cdot m_s$, where the action on the domain is the diagonal action. Because of the coherence of $\mathcal{O}_\infty$ and the finite presentation of M, the kernel of π is a finitely generated submodule, say generated by $\{f_1, \ldots, f_t\}$. Let $\Gamma' \subseteq \Gamma$ be a subgroup of finite index that fixes the elements f_r for all r. It also exists by the continuity of the action of G on $\mathcal{O}_\infty$. We obtain an exact sequence

$$k[G/\Gamma'] \otimes_k \mathcal{O}_\infty^t \xrightarrow{f} k[G/\Gamma] \otimes_k \mathcal{O}_\infty^n \longrightarrow M \longrightarrow 0$$

of $\mathcal{O}_\infty$-modules with G-action. Let $j = \max(i, r)$. It is clear from the construction that the homomorphism f is of the form $f = \mathcal{O}_\infty \otimes_{\mathcal{O}_i} f'$ for a homomorphism f' of objects of $V^G(\mathcal{O}_j, \mathfrak{M}(j))$, so

$$M \cong \mathcal{O}_\infty \otimes_{\mathcal{O}_j} \mathrm{Coker}(f').$$

As M lies in $\mathfrak{T}$, it follows that $L_\infty \otimes_{\mathcal{O}_\infty} M = 0$. However,

$$L_\infty \otimes_{\mathcal{O}_\infty} M = \operatorname*{colim}_k L_k \otimes_{\mathcal{O}_j} \mathrm{Coker}(j'),$$

and it follows from the finite generation of $\mathrm{Coker}(j')$ that $L_k \otimes_{\mathcal{O}_j} \mathrm{Coker}(j') = 0$ for some k. Consequently, it is clear that $M_k = \mathcal{O}_k \otimes_{\mathcal{O}_j} \mathrm{Coker}(j')$ lies in $\mathfrak{T}(k)$, and that $M = \mathcal{O}_\infty \otimes_{\mathcal{O}_k} M_k$. This gives the result on objects. The lemma now follows from the fact that, given $M, M' \in V^G(\mathcal{O}_i, \mathfrak{M}(i))$, we have that $\mathrm{Hom}_{V^G(\mathcal{O}_\infty, \mathfrak{M})}(\mathcal{O}_\infty \otimes_{\mathcal{O}_i} M, \mathcal{O}_\infty \otimes_{\mathcal{O}_i} M')$ is isomorphic to the colimit

$$\operatorname*{colim}_j \mathrm{Hom}_{V^G(\mathcal{O}_j, \mathfrak{M})}(\mathcal{O}_j \otimes_{\mathcal{O}_i} M, \mathcal{O}_j \otimes_{\mathcal{O}_i} M').$$

This is a direct computation using matrix representations of liftings of homomorphisms to presentations of the modules in question. □

PROPOSITION 4.15. *We have an isomorphism* $\pi_* K^G(k) \cong R[G] \otimes K_* k$, *where* $R[G]$ *denotes the complex representation ring.* (*The complex representation ring of a profinite group is defined to be the direct limit of the representation rings of its finite quotients.*)

Proof. As k is algebraically closed of characteristic prime to l, the representation theory of G over k is identical to that over $\mathbb{C}$. This shows that

$$\pi_0 K^G(k) \cong \pi_0 K\mathrm{Rep}_k[G] \cong R[G].$$

In the category $\mathrm{Rep}_k[G]$, every object has a unique decomposition into irreducibles, each of which has k as its endomorphism ring. The result follows directly. □

PROPOSITION 4.16. *There is an isomorphism*

$$K_*(V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i))) \cong R[G_i] \otimes K_* k.$$

Proof. There is a natural inclusion $\mathrm{Rep}_k[G_i] \hookrightarrow V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i))$, with a representation ρ equipped with the trivial $\mathcal{O}_i$-action, that is, so that $\xi = t^{p^{-i}}$ acts identically by zero. Every object M in $V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i))$ admits a (finite) filtration

$$M \supseteq \xi M \supseteq \xi^2 M \supseteq \ldots \supseteq \{0\},$$

for which the subquotients are elements of the full subcategory of $V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i))$ consisting of the modules on which ξ acts trivially. The devissage theorem from [**23**, Section V, Theorem 4], together with the equivalence of Waldhausen's K-theory and Quillen's K-theory proved in [**36**, Appendix 1.9], now gives the result. □

It remains to express the effect of the inclusion

$$V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i)) \hookrightarrow V^{G_{i+1}}(\mathcal{O}_{i+1}, \mathfrak{T}(i+1))$$

on K-groups.

Proposition 4.17. *There is a commutative diagram*

$$\begin{tikzcd} K_*^{G_i}(V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i))) \arrow[r] \arrow[d] & R[G_i] \otimes K_*(k) \arrow[d, "\times n_{\chi(G_{i+1})} \otimes \mathrm{id}"] \\ K_*^{G_{i+1}}(V^{G_{i+1}}(\mathcal{O}_{i+1}, \mathfrak{T}(i+1))) \arrow[r] & R[G_{i+1}] \otimes K_*(k) \end{tikzcd}$$

where the right-hand vertical map denotes multiplication by the element

$$n_{\chi(G_{i+1})} \otimes 1 \in \mathbb{Z}[\chi(G_{i+1})] \otimes K_0(k) \cong R[G_{i+1}] \otimes K_0(k) \cong R[G_{i+1}].$$

Proof. As everything in the diagram is a module over $R[G_i] \otimes K_*(k)$, it suffices to determine where the trivial representation ρ_0 of G_i, interpreted as an object in $V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i))$, is taken by the inclusion

$$V^{G_i}(\mathcal{O}_i, \mathfrak{T}(i)) \hookrightarrow V^{G_{i+1}}(\mathcal{O}_{i+1}, \mathfrak{T}(i+1)).$$

It is clear that the image of ρ_0 in $V^{G_{i+1}}(\mathcal{O}_{i+1}, \mathfrak{T}(i+1))$ is the module

$$\mathcal{O}_{i+1}/t^{l^{-i}}\mathcal{O}_{i+1}$$

with G_{i+1}-action induced by the action of G_{i+1} given on $\mathcal{O}_{i+1}$. The subquotients in the filtration by powers of $t^{l^{-i-1}}$ on this module are the

$$\rho_0, \rho_1, \rho_2, \ldots, \text{ and } \rho_{l-1},$$

where ρ_i denotes the ith power of the standard representation of G_{i+1} in the lth power roots of unity. We have used an identification of the character group of G_{i+1} with the roots of unity in defining our action on $\mathcal{O}_{i+1}$. The additivity theorem [**36**, Theorem 1.4.2] now gives the result. □

Corollary 4.18. *There is an isomorphism of graded $R[G]$-modules*

$$\pi_*\Phi \cong \mathfrak{M}(\mathfrak{R}) \otimes K_*(k),$$

where, as in the earlier discussion, Φ is the homotopy fiber of the map $K^G(\mathcal{O}_\infty) \to K^G(\bar{F})$.

Proof. The proof is the immediate consequence of Proposition 4.17 together with Proposition 4.11. □

Theorem 4.19. *The map $\theta : K^G(\mathcal{O}_\infty) \to K^G(L_\infty)$ induces an equivalence $\theta^\wedge_{\mathbb{H}_l} : K^G(\mathcal{O}_\infty)^\wedge_{\mathbb{H}_l} \to K^G(L_\infty)^\wedge_{\mathbb{H}_l}$. Both $K^G(\mathcal{O}_\infty)$ and $K^G(L_\infty)$ are regarded as modules over the commutative ring spectrum $K^G(k)$.*

Proof. Corollary 4.18 shows that the groups $\mathrm{Tor}_i^{R[G]}(\pi_*\Phi, \mathbb{F}_l)$ all vanish. The Künneth spectral sequence for the derived smash product over a ring spectrum shows that it follows that

$$H_{\mathbb{H}_l} \bigwedge_{K\mathrm{Rep}_k[G]} \Phi$$

is contractible. Proposition 3.2(2) of [3] now shows that $\Phi^{\wedge}_{\mathbb{H}_l}$ is contractible. Proposition 4.9 yields the result. □

We next show that the natural map $K\mathrm{Rep}_k[G]^{\wedge}_{\mathbb{H}_l} \to K^G(\mathcal{O}_\infty)^{\wedge}_{\mathbb{H}_l}$ is an equivalence.

PROPOSITION 4.20. *The map $\mathcal{K}(V^{G_j}(k)) \to \mathcal{K}(V^{G_j}(\mathcal{O}_i))$ is an equivalence of spectra for any $j \geqslant i$.*

Proof. We first observe that the category $V^{G_j}(\mathcal{O}_i)$ is equivalent to the category of finitely generated modules over the skew group ring $\mathcal{O}_i * G_j$ (see [12]) associated to the action of G_j on $\mathcal{O}_i$. It is further clear that this ring satisfies the hypotheses of [23, Theorem 7, Section 6] (the homotopy property), from which the result follows. □

COROLLARY 4.21. *The natural map $K^G(k) \to K^G(\mathcal{O}_\infty)$ is an equivalence. Therefore, the map $K^G(k)^{\wedge}_{\mathbb{H}_l} \to K^G(\mathcal{O}_\infty)^{\wedge}_{\mathbb{H}_l}$ is also an equivalence.*

Proof. A straightforward direct limit argument similar to the one in the proof of Lemma 4.14 gives the result. □

COROLLARY 4.22. *The map $A^G_{L_\infty}(l)$ is an equivalence.*

5. *The Beilinson–Lichtenbaum conjecture*

This section contains the computational consequences of the conjectures of Bloch–Kato and Beilinson–Lichtenbaum, which we shall need in proving our results about the representational assembly map. Recall that, for any integer n, there is a mod-n version of the algebraic K-theory of any ring R, denoted by $K(R; \mathbb{Z}/n\mathbb{Z})$, and defined by

$$K(R; \mathbb{Z}/n\mathbb{Z}) = \Sigma^{-1} M(n). \wedge K(R),$$

where $M(n).$ was defined in Section 2. In Section 2, maps $\theta : M(m). \to M(n).$ were defined whenever m was divisible by n. Consequently, we obtain, for any prime l, an inverse system of spectra

$$\cdots \longrightarrow K(R; \mathbb{Z}/l^{j+1}\mathbb{Z}) \longrightarrow K(R; \mathbb{Z}/l^{j}\mathbb{Z}) \longrightarrow K(R; \mathbb{Z}/l^{j-1}\mathbb{Z}) \cdots \longrightarrow K(R; \mathbb{Z}/l\mathbb{Z}),$$

whose homotopy inverse limit is equivalent to the l-adic completion of $K(R)$ by Proposition 2.1. We next recall that there are motivic cohomology functors $H^p(X, \mathbb{Z}(q))$ for schemes over a field k, and that these groups applied to a field F containing k describe the E_2-term of a spectral sequence converging to the algebraic K-theory of F, as follows. For an integer l, prime to $\mathrm{char}(F)$, there are corresponding groups $H^p(X, \mathbb{Z}/l(q))$, and a corresponding spectral sequence converging to the mod-l version of algebraic K-theory of F. This spectral sequence was initially constructed in [1], and the interpretation as cohomology groups in a category of motives is described in [35, 20]. The multiplicative structure is proved in [9, Theorems 15.5 and 16.2].

THEOREM 5.1 (Bloch–Lichtenbaum spectral sequence). *For any field F there is a spectral sequence converging to the algebraic K-theory of F whose $E_2^{p,q}$-term is given by*

$$H^{p-q}(\mathrm{Spec}(F), \mathbb{Z}(-q)),$$

where the groups $E_\infty^{p,q}$ for $p+q=t$ form a composition series for $K_{-t}(F)$. The analogous result for the groups $H^{p-q}(\mathrm{Spec}(F), \mathbb{Z}/l(-q))$ and mod-l K-theory, where l is an integer, holds as well. Further, both in the absolute case and the mod-l case for l not congruent to 2 mod 4, the E_r-terms have the structure of a bigraded algebra, the d_r differential is a derivation, and the algebra structure on E_∞ is an associated graded version of the algebra structure on the corresponding K-theory algebras.

The Beilinson–Lichtenbaum conjecture asserts that the E_2-term in the mod-l case can be expressed entirely in terms of etale cohomology. It was proved in [**10**] that the Bloch–Kato conjecture implies the Beilinson–Lichtenbaum conjecture, and as the Bloch–Kato conjecture has been proved by Voevodsky and Rost, with an important contribution by Weibel [**34**, **37**], it is now a theorem.

THEOREM 5.2 (Beilinson–Lichtenbaum conjecture). *Let l be an integer prime to* $\mathrm{char}(F)$. *Then there are natural isomorphisms*

$$H^p(F, \mathbb{Z}/l(q)) \cong H^p_{\mathrm{et}}(F, \mu_l^{\otimes q}) \cong H^p(G_F, \mu_l^{\otimes q}),$$

where μ_l denotes the lth roots of unity when $p \leqslant q$, and G_F is the separable Galois group of F. When $p > q$, the groups $H^p(F, \mathbb{Z}/l(q))$ vanish. Further, the products in motivic cohomology coincide with those on etale cohomology.

Proof. The multiplicative statement is proved in [**10**, Appendix]. □

COROLLARY 5.3. *Let $E_2^{*,*}(F)$ denote the E_2-term of the Bloch–Lichtenbaum spectral sequence for F, regarded as a bigraded algebra. Suppose that F is a geometric field, that is, so that it contains an algebraically closed field. Then we have an isomorphism*

$$E_2^{*,*}(F) \cong \mathbb{Z}/l[\beta] \otimes \mathfrak{H}^{*,*},$$

where β is an element in bigrading $(-1, -1)$ and where $\mathfrak{H}^{*,*}$ is the algebra $\bigoplus_p H^p(G_F, \mu_l^{\otimes p})$, with the group $H^p(G_F, \mu_l^{\otimes p})$ occurring in the bigrading $(0, -p)$.

Proof. The fact that the field is geometric shows that the G_F-action on $\mu_l^{\otimes q}$ is trivial. The generator β is a generator for the group $H^0(G_F, \mu_l)$. The result follows directly from the Beilinson–Lichtenbaum conjecture. □

We also record the Bloch–Kato conjecture in a special case.

THEOREM 5.4 (Bloch–Kato conjecture). *Let F be a geometric field, and suppose again that l is an integer prime to the characteristic of F. Then the graded algebra $H^*(G_F, \mu_l^{\otimes *})$ is isomorphic to the graded algebra $K_*^{\mathrm{Mil}}(F)/l$, the* mod-l *Milnor K theory of F. The relations defining this algebra are given in* [**20**, Lecture 5]. *In particular, it is generated in dimension* 1, *and consequently the same is true of $H^*(G_F, \mu_l^{\otimes *})$.*

COROLLARY 5.5. *Let F be geometric, with l an integer prime to the characteristic of F. Then the Bloch–Lichtenbaum spectral sequence for $K_*(F, \mathbb{Z}/l)$ collapses at E_2.*

Proof. The generating classes in bidegree $(0, -1)$, corresponding to $H^1(G_F, \mu_l)$, are infinite cycles for dimensional reasons. Let $k \subseteq F$ denote an algebraically closed subfield. Then the class β_k in $E_2^{-1,-1}(k)$ is an infinite cycle, and the inclusion $k \hookrightarrow F$ carries it to a generator in $E_2^{-1,-1}(F)$, which must therefore also be an infinite cycle. The multiplicative structure of the spectral sequence now shows that it collapses. □

COROLLARY 5.6. *Let F be geometric, and suppose again that l is an integer prime to the characteristic of F. Then we have an isomorphism*

$$K_*(F, \mathbb{Z}/l\mathbb{Z}) \cong K_*^{\mathrm{Mil}}(F)/l \otimes \mathbb{Z}/l\mathbb{Z}[\beta],$$

where the degrees are as in Corollary 5.5.

Proof. Matsumoto's Theorem [**22**, Theorem 11.1] shows that there is a homomorphism from $K_*^{\mathrm{Mil}}(F)/l$ to $K_*(F, \mathbb{Z}/l\mathbb{Z})$. This gives a homomorphism of graded rings $K_*^{\mathrm{Mil}}(F)/l \otimes \mathbb{Z}/l\mathbb{Z}[\beta] \to K_*(F, \mathbb{Z}/l\mathbb{Z})$. The fact that it is an isomorphism is now a standard argument using the structure of the associated graded module in the Bloch–Lichtenbaum spectral sequence. □

Finally, this gives us the following.

Corollary 5.7. *Let F be a geometric field, containing an algebraically closed subfield k, and let l be a prime with $(l, \mathrm{char}(F)) = 1$. Then there is an isomorphism*

$$\pi_* K(F)^\wedge_l \cong K^{\mathrm{Mil}}_*(F)^\wedge_l \hat{\otimes} \mathbb{Z}_l[\beta],$$

where $\beta \in \pi_2 K(k)$ is a generator, and where $\hat{\otimes}$ denotes the completed tensor product over $\mathbb{Z}_l$. In particular, when the separable Galois group of F is abelian, we obtain that

$$\pi_* K(F)^\wedge_l \cong \hat{\Lambda}^l_*(F^*) \hat{\otimes} \mathbb{Z}_l[\beta].$$

Proof. We have now shown that the inverse system of graded abelian groups

$$\cdots \longrightarrow K_*(F; \mathbb{Z}/l^{n+1}\mathbb{Z}) \longrightarrow K_*(F; \mathbb{Z}/l^n\mathbb{Z}) \longrightarrow K_*(F; \mathbb{Z}/l^{n-1}\mathbb{Z}) \longrightarrow \cdots K_*(F; \mathbb{Z}/l\mathbb{Z})$$

is isomorphic to the surjective system

$$\begin{aligned} \cdots \longrightarrow K^{\mathrm{Mil}}_*(F)/l^{n+1} \hat{\otimes} \mathbb{Z}/l^{n+1}\mathbb{Z}[\beta] \longrightarrow K^{\mathrm{Mil}}_*(F)/l^n \otimes \mathbb{Z}/l^n\mathbb{Z}[\beta] \\ \longrightarrow K^{\mathrm{Mil}}_*(F)/l^{n-1} \hat{\otimes} \mathbb{Z}/l^{n-1}\mathbb{Z}[\beta] \longrightarrow \cdots. \end{aligned}$$

The Milnor exact sequence [**2**, Theorem 3.1, Chapter IX, Section 3] allows us to conclude that

$$\pi_* \operatorname*{holim}_n K(F; \mathbb{Z}/l^n\mathbb{Z}) \cong K^{\mathrm{Mil}}_*(F)^\wedge_l \hat{\otimes} \mathbb{Z}_l[\beta].$$

Proposition 2.1 now gives the result. □

6. *The case of an abelian separable Galois group*

We recall again the maps $\alpha^G_B : K^G(k) \to K^G(B)$ for any k-algebra B with continuous group action by a profinite group G, induced by the induction functors $V^G(k) \to V^G(B)$. We have the homomorphism of commutative ring spectra $K^G(k) \to \mathbb{H}_l$ for any prime l, where $\mathbb{H}_l$ denotes the mod-l Eilenberg–Mac Lane spectrum regarded as a commutative ring spectrum. We also have the associated induced maps

$$A^G_B(l) : K^G(k)^\wedge_{\mathbb{H}_l} \longrightarrow K^G(B)^\wedge_{\mathbb{H}_l}$$

on derived completions of $K^G(k)$-modules. In particular, if we consider a field F containing k, and let G_F denote the separable Galois group of F, then we obtain

$$A_F(l) (= A^{G_F}_F(l)) : K^G(k)^\wedge_{\mathbb{H}_l} \longrightarrow K^G(F)^\wedge_{\mathbb{H}_l} \cong K(F)^\wedge_l.$$

The following observation is clear.

Proposition 6.1. *Let $f : B \to C$ denote an equivariant homomorphism of k algebras with continuous actions by the profinite group G. Then we have the equality of maps $A^G_C(l) = K^G(f)^\wedge_{\mathbb{H}_l} \circ A^G_B(l)$.*

Next, suppose that we are given a field F (containing k) whose separable Galois group G is abelian and that l is prime to the characteristic of F. Our goal is to show that $A_F(l)$ is an equivalence of spectra. We recall some facts about the K-theory of F. We assume that l is fixed throughout the rest of this discussion, and let $\hat{K}(-)$ denote the functor $K(-)^\wedge_l$. Similarly, let $\hat{K}_*(-)$ denote the graded abelian group-valued functor $\pi_* \hat{K}(-)$.

Proposition 6.2. *The field F has the following properties.*
(1) *There is an isomorphism $\hat{K}_1(F) \cong (F^*)^\wedge_l$. In particular, $\hat{K}_1(F)$ is a l-complete group.*
(2) *The profinite group G is the product of its profinite Sylow subgroups.*
(3) *The l-Sylow subgroup G_l of G is of the form $\prod_\aleph \mathbb{Z}_p$, where $\aleph$ is a cardinal number.*
(4) *The group $(F^*)^\wedge_l$ is isomorphic to the group*

$$\mathrm{Hom}_c(G_l, \mathbb{Z}_l) \cong H^1_c(G_l, \mathbb{Z}_l),$$

where the subscript $_c$ indicates that we are referring to the continuous homomorphisms, where G_l is topologized using the subgroups of finite index as neighborhoods of the origin.

(5) *Let F_l be the fixed field of G_l under its action on $\bar{F}$, the separable closure of F. Then the inclusion $F \hookrightarrow F_l$ induces an equivalence $\hat{K}(F) \to \hat{K}(F_l)$.*

Proof. Part (1) is an immediate consequence of [**2**, Chapter VI, Proposition 5.1], together with the fact that $\mathrm{Hom}(\mathbb{Z}/l^\infty\mathbb{Z}, K_0(F)) = 0$. Parts (2) and (3) are proved in [**25**, Section 4.3]. Part (4) is a part of Kummer theory, which can be found in [**16**, Chapter VI, Section 8]. Kummer theory shows that

$$F^*/(F^*)^{l^j} \cong H^1(G_l, \mathbb{Z}/l^j\mathbb{Z}),$$

and further that there are commutative diagrams

$$\begin{tikzcd} H^1(G_l, \mathbb{Z}/l^j\mathbb{Z}) \arrow[r] \arrow[d] & F^*/(F^*)^{l^j} \arrow[d] \\ H^1(G_l, \mathbb{Z}/l^{j-1}\mathbb{Z}) \arrow[r] & F^*/(F^*)^{l^{j-1}} \end{tikzcd}$$

where the horizontal maps are the Kummer theory isomorphisms and the vertical maps are evident reductions. Consequently, we have an isomorphism

$$H^1_c(G_l, \mathbb{Z}_l) \cong \varprojlim H^1(G_l, \mathbb{Z}/l^j\mathbb{Z}) \cong \varprojlim F^*/(F^*)^{l^j} \cong (F^*)^\wedge_l.$$

This result gives part (4). Part (5) follows from a standard transfer argument applied to show that $K(F, \mathbb{Z}/l^j) \to K(F_l, \mathbb{Z}/l^j)$ is an isomorphism. The result for completions is now immediate. □

We also examine the representation ring and characters of G_l.

Proposition 6.3. *The representation ring $R[G_l]$ is isomorphic to the group ring $\mathbb{Z}[\chi_c(G_l)]$, where $\chi_c(-)$ denotes the group of continuous complex characters. $\chi_c(G_l)$ is an l-torsion divisible group, and $\mathcal{A}(\chi_c(G_l)) \cong \mathrm{Hom}_c(G_l, \mathbb{Z}_l)$.*

Proof. The first statement is generally true for abelian profinite groups, extending the corresponding result for the finite case. The second statement is proved using the following sequence of isomorphisms:

$$\begin{aligned} A(\chi_c(G_l)) &\cong \mathrm{Hom}(\mathbb{Z}/l^\infty\mathbb{Z}, \mathrm{Hom}(G_l, \mathbb{Z}/l^\infty\mathbb{Z})) \\ &\cong \varprojlim \mathrm{Hom}(G_l, \mathbb{Z}/l^\infty\mathbb{Z})^{l^j} \cong \varprojlim \mathrm{Hom}(G_l, \mathbb{Z}/l^{l^j}\mathbb{Z}) \cong \mathrm{Hom}_c(G_l, \mathbb{Z}_l), \end{aligned}$$

where, for an abelian group A, the notation A^n denotes the n-torsion subgroup of A. □

Proposition 6.4. *The map on completions $K^G(k)^\wedge_{\mathbb{H}_l} \to K^{G_l}(k)^\wedge_{\mathbb{H}_l}$ induced by the diagram*

$$\begin{tikzcd} K^G(k) \arrow[rr] \arrow[dr] & & K^{G_l}(k) \arrow[dl] \\ & \mathbb{H}_l & \end{tikzcd}$$

of commutative ring spectra is an equivalence. Note that $K^G(k)^\wedge_{\mathbb{H}_l}$ is a completion over the ring spectrum $K^G(k)$, and $K^{G_l}(k)^\wedge_{\mathbb{H}_l}$ is a completion over the ring spectrum $K^{G_l}(k)$.

Proof. This result follows from Proposition 2.5, because the map on mod-l Eilenberg–Moore spectral sequences induced by the map $B\chi(G) \to B\chi(G_l)$ is an isomorphism, as it is an isomorphism on homology with mod-l coefficients. □

Corollary 6.5. *In order to prove that $A_F(l)$ is an equivalence for all geometric fields with abelian separable Galois group, it suffices to prove the result for fields F whose separable Galois groups are abelian pro-l groups.*

Proof. The proof is direct from Proposition 6.2, part (5), and Proposition 6.4. $\square$

So, we now assume that the separable Galois group of our geometric field is an abelian pro-l group. We have identifications of the algebras $\pi_* \hat{K}(F)$ and $\pi_* K^G(k)^\wedge_{\mathbb{H}_l}$.

Proposition 6.6. *Both $\pi_* \hat{K}(F)$ and $\pi_* K^G(k)^\wedge_{\mathbb{H}_l}$ are isomorphic to the algebra $\hat{\Lambda}^l_*((F^*)^\wedge_l)\hat{\otimes}\mathbb{Z}_l[\beta]$, where $\hat{\otimes}$ denotes the completed tensor product over $\mathbb{Z}_l$.*

Proof. The proof follows from Propositions 6.2 and 6.3 in conjunction with Proposition 2.11 and the Beilinson–Lichtenbaum conjecture (Corollary 5.7). $\square$

With this identification, it clearly suffices to prove that $A_F(l)$ induces an isomorphism on π_1.

Theorem 6.7. *The map $\pi_1(A_F(l))$ is an isomorphism. Consequently, $A_F(l)$ is an equivalence of spectra for geometric fields with abelian absolute Galois group.*

Proof. We begin by selecting a basis $\{b_\alpha\}_{\alpha \in A}$ for the $\mathbb{F}_l$-vector space $F^*/(F^*)^l$. For each $\alpha \in A$ we select a lift of b_α to an element $f_\alpha \in F^*$. We let F_α denote the extension $\bigcup_n F(f_\alpha^{l^{-n}})$. The extension F_α is a Galois extension of F, and its Galois group is a quotient of G which we denote by G_α. The quotient G_α is abstractly isomorphic to $\mathbb{Z}_l$, and it is clear from the construction that the natural map $\prod_\alpha \pi_\alpha : G \to \prod_\alpha G_\alpha$ is an isomorphism of profinite groups. Consequently, we have the natural inclusions of spectra $i_\alpha : K\mathrm{Rep}_k[G_\alpha] \hookrightarrow K\mathrm{Rep}_k[G]$. As before, we let L and L_∞ denote the rings $k[t]$ and $\bigcup_n k[t^{\pm l^{-n}}]$, respectively. The ring L_∞ is equipped with an action by $\mathbb{Z}_l$ defined by a choice of l^nth roots of unity ζ_{l^n} for each n so that $\zeta_{l^n}^l = \zeta_{l^{n-1}}$ for all n. We select an isomorphism $\lambda_\alpha : G_\alpha \to \mathbb{Z}_l$. The choice of λ_α determines an extension of the homomorphism $j_\alpha : L \to F$, defined by $j_\alpha(t) = f_\alpha$, to an equivariant homomorphism of rings $J_\alpha : L_\infty \to \bar{F}$. We now have a commutative diagram

$$\begin{array}{ccc} V^{\mathbb{Z}_l}(k) & \longrightarrow & V^G(k) \\ \downarrow & & \downarrow \\ V^{\mathbb{Z}_l}(L_\infty) & \xrightarrow{V^G(J_\alpha)} & V^G(\bar{F}) \end{array}$$

of commutative ring spectra, and a corresponding diagram of derived completions

$$\begin{array}{ccc} K^{\mathbb{Z}_l}(k)^\wedge_{\mathbb{H}_l} & \longrightarrow & K^G(k)^\wedge_{\mathbb{H}_l} \\ {\scriptstyle A^{\mathbb{Z}_l}_{L_\infty}(l)}\downarrow & & \downarrow{\scriptstyle A_F(l)} \\ K^{\mathbb{Z}_l}(L_\infty)^\wedge_{\mathbb{H}_l} & \xrightarrow{K^G(J_\alpha)^\wedge_{\mathbb{H}_l}} & K^G(\bar{F})^\wedge_{\mathbb{H}_l} \\ \wr\downarrow & & \downarrow\wr \\ K(L)^\wedge_l & \xrightarrow{K(j_\alpha)^\wedge_l} & K(F)^\wedge_l. \end{array}$$

Note that the map $A^{\mathbb{Z}_l}_{L_\infty}(l)$ was proved to be an isomorphism in Corollary 4.22. This gives us a diagram of abelian groups

$$\begin{tikzcd}
\mathbb{Z}_l \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"'] & \pi_1(K^{\mathbb{Z}_l}(k)^\wedge_{\mathbb{H}_l}) \arrow[r] \arrow[d] & \pi_1(K^G(k)^\wedge_{\mathbb{H}_l}) \arrow[d] \\
\mathbb{Z}_l \arrow[r, "\sim"] & \pi_1(K(L)^\wedge_l) \arrow[r, "\pi_1(K(j_\alpha)^\wedge_l)"] & \pi_1(K(F)^\wedge_l).
\end{tikzcd}$$

We denote the upper horizontal composite by ϕ_α and the lower one by ψ_α. This means we may now assemble these maps together by summing over α, to get a diagram

$$\begin{tikzcd}
& \bigoplus_\alpha \mathbb{Z}_l \arrow[dl] \arrow[dr] & \\
\pi_1(K^G(k)^\wedge_{\mathbb{H}_l}) \arrow[rr, "\pi_1(A_F(l))"] & & \pi_1(K(F)^\wedge_l).
\end{tikzcd}$$

The algebraic to geometric spectral sequence [**3**, Theorem 7.1] shows that in this case $\pi_1(K^G(k)^\wedge_{\mathbb{H}_l}) \cong \pi_1 \mathrm{Tot}\mathfrak{C}^\cdot_{\mathbb{Z}[\chi(G)]}(\mathbb{Z}[\chi(G)], \mathbb{F}_l)$, and that the map Φ can be identified with the inclusion from Corollary 2.13, if $\chi(G)$ is given as a sum $\bigoplus_{\alpha\in A} \mathbb{Z}/l^\infty\mathbb{Z}$. It follows from Corollary 2.13 that Φ induces an isomorphism on l-completions. Similarly, it is immediate that Ψ induces an isomorphism on l-completions because it was chosen to induce an isomorphism on mod-l quotients. It follows that the homomorphism $\pi_1(A_F(l))$ is an isomorphism. The description of the homotopy groups given in Proposition 5.6 now shows that the homomorphism of graded rings $\pi_*(A_F(l))$ is an isomorphism, which is the desired result. □

7. *Surjectivity of* $\pi_* A_F(l)$

Let F be any field containing an algebraically closed subfield k, and suppose that l is prime to the characteristic of k. We claim that the map $A_F(l)$ induces a surjection on π_*.

Proposition 7.1. *For any field F we have*

$$\pi_1((KF)^\wedge_l) \cong (F^*)^\wedge_l,$$

where the right-hand term denotes the usual algebraic l-adic completion of the group F^*. In particular, $\pi_1((KF)^\wedge_l)$ is an l-complete group.

Proof. As $\pi_0(KF) \cong \mathbb{Z}$ is a finitely generated abelian group, this result follows from [**2**, Proposition 5.1, Chapter VI]. □

Proposition 7.2. *Let G be any torsion-free abelian pro-l group. Let A denote the representation ring $R[G]$ and B denote $\mathbb{F}_l$ regarded as an A-algebra via the composite of the augmentation with* mod-l *reduction. Then $\pi_1 A^\wedge_B$ is an l-complete group.*

Proof. The representation ring $R[G]$ can be identified with the group ring $\mathbb{Z}[\chi(G)]$, where $\chi(G)$ denotes the character group of G. Because of the torsion-freeness of G, $\chi(G)$ is divisible and is l-torsion. Corollary 2.12 now gives the result. □

Let G_F denote the separable Galois group of F, and G_l denote the maximal abelian pro-l quotient of G_F. The quotient G_l is a torsion-free abelian pro-l group by Kummer theory.

Functoriality of the assembly construction now gives us a density result.

Proposition 7.3. *The image of the composite*

$$\pi_1 K^{G_l}(k)^\wedge_{\mathbb{H}_l} \longrightarrow \pi_1 K^G(k)^\wedge_{\mathbb{H}_l} \xrightarrow{\pi_1(A_F(l))} \pi_1((KF)^\wedge_l) \cong (F^*)^\wedge_l$$

is dense in the l-adic topology on $(F^*)^\wedge_l$.

Proof. It clearly suffices to prove that, for any n and any element $x \in F^*/l^nF^*$, there is an element $\xi \in \pi_1(K\mathrm{Rep}_k[G]^\wedge_{\mathbb{H}_l})$ so that $\rho(\pi_1(A_F(l))(\xi)) = x$, where ρ is the projection $F^* \to F^*/l^nF^*$. In order to do this, we select $\bar{x} \in F^*$ so that $\rho(\bar{x}) = x$. We next consider the extension $\bigcup_n F(x^{1/l^n})$ of F. This extension is preserved under the action of the separable Galois group G, and therefore yields a particular topologically cyclic quotient of G, which we denote by Q. We note that Q is also a pro-l group, and is therefore isomorphic to $\mathbb{Z}_l$. Select a particular identification $\theta : Q \simeq \mathbb{Z}_l$. Next, define a ring homomorphism $L = k[t^{\pm 1}] \to F$ by sending the variable t to $\bar{x}$. Choosing a sequence of primitive l^nth roots of unity ζ_n as in the beginning of Section 6, we extend this map to a ring homomorphism

$$i : L_\infty \longrightarrow F,$$

via the requirement that $t^{\pm 1/l^n}$ be sent to the unique l^nth root of x on which the topological generator $\theta^{-1}(1)$ acts by multiplication with ζ_n. It is now clear from the construction that the diagram

$$\begin{tikzcd} K\mathrm{Rep}_k[\mathbb{Z}_l]^\wedge_{\mathbb{H}_l} \arrow[r, "A^Q_{L_\infty}(l)"] \arrow[dr, "A_F(l)"'] & (KL)^\wedge_l \arrow[d] \\ & KF^\wedge_l \end{tikzcd}$$

commutes, where the vertical arrow is induced by the ring homomorphism $L_\infty \to F$ defined above. It was proved in Section 6 that $A^Q_{L_\infty}(l)$ is an equivalence, and hence induces an isomorphism on π_1. The result now follows immediately. □

The actual surjectivity on π_1 now follows from Proposition 7.2 together with the following.

PROPOSITION 7.4. *Let $f : G_0 \to G_1$ denote a homomorphism between l-complete abelian groups, and suppose that the image of f is dense in the l-adic topology. Then f is surjective.*

Proof. The proof is immediate from the definition of the term l-complete. □

COROLLARY 7.5. *Let F be any geometric field, and l be a prime different from the characteristic of F. Let G_F denote the separable Galois group of F. Then the homomorphism $\pi_*A_F(l) : \pi_*K^{G_F}(k)^\wedge_{\mathbb{H}_l} \to \hat{K}_*(F)$ is surjective.*

Proof. Propositions 7.2 and 7.4 show that $\pi_1 A_F(l)$ is surjective. Corollary 5.7 now proves the surjectivity of $\pi_*A_F(l)$. □

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