Generating Functions

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Sourced from a Linus Hamilton handout.

Example 1. You flip one penny once. How many ways are there to get n cents worth of heads?

Example 2. You flip one nickel. How many ways are there to get n cents worth of heads?

Example 3. You flip one penny *and* one nickel. How many ways are there to get n cents worth of heads?

Example 4. You flip k pennies. How many ways are there to get n cents worth of heads?

Example 5. An *antipenny* is a coin worth -1 cents. You flip k pennies and k antipennies. How many ways are there to get n cents worth of heads?

Example 6. The currency of the Nomai has only pennies. How many ways are there for a Nomai to pay n cents?

Example 7. The Canterlot currency has only nickels. How many ways are there for a Nomai to pay n cents?

Example 8. The Revacholian currency has pennies *and* 3-cent coins. How many ways are there to pay n cents in Martinaise (a district of Revachol)?

Example 9. The Barovian currency has two types of 1-cent coins: pennies and blennies. How many ways are there to pay n cents in Barovia?

Example 10. The La-Mulana currency has 1-cent, 2-cent, 3-cent, 4-cent, ... coins: one type for each positive integer. How many ways are there to pay n cents in Nebur's shop (in La-Mulana)?

Example 11. 8 coins each have probability $\frac{1}{3}$ of flipping heads. What is the probability an even number of them lands heads?

Problems

Problem 1. Some coins have probabilities of landing heads $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\ldots$, $\frac{1}{2020}$. What's the probability of an even number of them landing heads?

Problem 2. If you roll two normal six-sided dice and add them together, you have a $\frac{1}{36}$ chance of rolling a 2, a $\frac{2}{36}$ chance of rolling a 3, $\ldots$, and a $\frac{1}{36}$ chance of rolling a 12.

Find a different way of labeling each die's sides with positive integers such that rolling both dice and adding them together gives the same distribution.

Problem 3. You flip n ordinary coins. What's the probability that the number of heads showing is a multiple of 4?

Problem 4. The Eg-Lana currency has a k-cent coin for every positive integer k. Show that in Eg-Lana, the number of ways to pay n cents using distinct-valued coins is the same as the number of ways to pay n cents using only odd-valued coins.

Problem 5. A *multiset* is a set but it can repeat elements, so $\{1, 1, 5\}$ is different from $\{1, 5\}$. Let S_k be the collection of k-element multisets of integers.

The function $f_n : S_n \to S_{\binom{n}{2}}$ maps S to the multisets of all sums of 2 not-the-same-element elements of S. For example, $f_3(\{1, 1, 5\}) = \{2, 6, 6\}$. Show that if n is not a power of 2, then f_n is injective.