Stochastic optimization (short introduction)

Learning objectives

- Formulating a MILP
- Implement and solve MIP with Pulp
- Formulating a stochastic programs

Stochastic Programming

Many decisions are taken under uncertainties or limited knowledge (uncertainty)

- Demand from customers.
- Environmental uncertainties.
- Product/Process quality
- …

Mathematical programming offers a very generic framework for decision making

How to deal with uncertainties?

Find very good estimate of the value of the uncertain parameter?

Yes, if the unknown parameters don't have a serious impact on the decisions.

In many situation, the issue is not only about the quality of the estimate, but decision maker must hedge against the uncertainties.

Use Stochastic Programming!

Stochastic Programming

- Mathematical Programming (Optimization) is about decision making.
- Stochastic Programming is about decision making under uncertainty.
- Stochastic Programming can be seen as Mathematical Programming with random parameters.

Stochastic VS Robust optimization

Stochastic optimization:

- Fundamental assumption : rely on a probability distribution
- Computationally intensive
- Good performances on the considered distribution

Robust optimization:

- Rely on an uncertainty set.
- Fast computation

The Farmer's problem

(from Birge and Louveaux, 1997)

Consider a European farmer who specializes in raising wheat, corn, and sugar beets on his 500 acres of land. During the winter, he wants to decide how much land to devote to each crop. (We refer to the farmer as “he” for convenience and not to imply anything about the gender of European farmers.)

The farmer knows that at least 200 tons (T) of wheat and 240 T of corn are needed for cattle feed. These amounts can be raised on the farm or bought from a wholesaler. Any production in excess of the feeding requirement would be sold. Over the last decade, mean selling prices have been \$170 and \$150 per ton of wheat and corn, respectively. The purchase prices are 40% more than this due to the wholesaler’s margin and transportation costs.

Another profitable crop is sugar beet, which he expects to sell at \$36/T; however, the European Commission imposes a quota on sugar beet production. Any amount in excess of the quota can be sold only at \$10/T. The farmer’s quota for next year is 6000 T.

Based on past experience, the farmer knows that the mean yield on his land is roughly 2.5 T, 3 T, and 20 T per acre for wheat, corn, and sugar beets, respectively. Table 1 summarizes these data and the planting costs for these crops.

Data of the farmer's problem

	Wheat	Corn	Sugar Beets
Yield (T/acre)	2.5	3	20
Planting cost ($/acre)	150	230	260
Selling price ($/T)	170	150	36 under 6000 T 10 above 6000 T
Purchase price ($/T)	238	210	–
Minimum requirement (T)	200	240	–
Total available land: 500 acres			

Formulate this problem as a linear program.

Lab

Install Pycharm

(https://www.jetbrains.com/pycharm/download)

Setup Pulp

(https://pythonhosted.org/PuLP/)

Load the PuLP model for the News Vendor Problem available on Moodle

Optimal solution based on mean yield

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	120	80	300
Yield (T)	300	240	6000
Sales (T)	100	–	6000
Purchase (T)	–	–	–
Overall profit: $118,600			

After thinking about this solution, the farmer becomes worried. He has indeed experienced quite different yields for the same crop over different years mainly because of changing weather conditions. Most crops need rain during the few weeks after seeding or planting, then sunshine is welcome for the rest of the growing period.

Sunshine should, however, not turn into drought, which causes severe yield reductions. Dry weather is again beneficial during harvest. From all these factors, yields varying 20 to 25% above or below the mean yield are not unusual.

Lab

Modify the PuLP model to solve the News Vendor Problem with under above average (+20%) and under average (-20%) yield.

Optimal solution based on above average yield (+20%)

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	183.33	66.67	250
Yield (T)	550	240	6000
Sales (T)	350	–	6000
Purchase (T)	–	–	–
Overall profit: $167,667			

Optimal solution based on below average yield (-20%)

Culture	Wheat	Corn	Sugar Beets
Surface (acres)	100	25	375
Yield (T)	200	60	6000
Sales (T)	–	–	6000
Purchase (T)	–	180	–
Overall profit: $59,950			

Optimal solution based on below average yield (-20%)

Optimal solution is very **sensitive to yields**.

Main issue: sugar beets production:

- Large surface: Might have to sell some at the unfavorable price
- Small surface: Might miss the opportunity to sell the full quota at the favorable price

Stochastic Programming For the Farmer's problem

- There is no perfect decision (optimal in all circumstance).
- Assess the loss and profit in each scenario (s= 1, 2, 3):

Scenario 1: above average yield
Scenario 2: average yield
Scenario 3: below average yield

Stochastic Programming For the Farmer's problem

- Decision on land assignment (x_1, x_2, x_3) are made before to know the yield (**here and now**).

- The sales (w_1, w_2, w_3) and purchases (y_1, y_2) are decided once the yield is known (**recourse actions**).
    - Index these decisions per scenario
    - For example, W_{11} is the amount of wheat sold when the yield is below average

Lab

Implement the stochastic program with PuLP

Optimal solution based on stochastic model

		Wheat	Corn	Sugar Beets
First Stage	Area (acres)	170	80	250
$s = 1$ Above	Yield (T) Sales (T) Purchase (T)	510 310 –	288 48 –	6000 6000 (favor. price) –
$s = 2$ Average	Yield (T) Sales (T) Purchase (T)	425 225 –	240 – –	5000 5000 (favor. price) –
$s = 3$ Below	Yield (T) Sales (T) Purchase (T)	340 140 –	192 – 48	4000 4000 (favor. price) –
Overall profit: \$108,390				

Exercice 2: Farmers'problem extension

Consider the case where the farmer possesses four fields of sizes 185 , 145, 105 , and 65 acres, respectively. Observe that the total of 500 acres is unchanged. Now, the fields are unfortunately located in different parts of the village. For reasons of efficiency the farmer wants to raise only one type of crop on each field.

Formulate this model as a two-stage stochastic program with a first-stage program with binary variables.

Exercise 3: Assembly problem
(from Birge and Louveaux, 1997)

Consider a production or assembly problem. It consists of producing two products, say A and B. They are obtained by assembling two components, say $C1$ and $C2$, in fixed quantities. The following table shows the components usage for the two products:

Components usage	A	B
$C1$	6	10
$C2$	8	5

Exercise 3

Components are produced within the plant. Material (and / or operating) costs for $C1$ and $C2$ are 0.4 and 1.2, respectively. The level of production, or capacity, is related to the work-force and the equipment. Each unit of capacity costs 150 and 180 and can produce batches of 60 and 90 components, respectively for $C1$ and $C2$. Current capacity level is $(40,20)$ batches and cannot be decreased. The total number of batches must not exceed 120. An integer number of batches is not requested here.

In the deterministic case, the demands and unit selling prices are certain and are as follows:

	A	B
Demand	500	200
Unit selling price	50	60

Unmet demand results in lost sales. This does does not imply any additional penalty.

Exercise 3

(1) Model this problem as a Mixed integer linear program

Data:

Data	Nota.	Item A	Item B	Comp. C1	Comp. C2
Demand	d_i	500	200	-	-
Selling price	p_i	50	60	-	-
Capacity	c_i	-	-	40	20
Capacity cost	o_i	-	-	150	180
Production cost	v_i	-	-	0.4	1.2
Capcity consumption	k_i	-	-	1/60	1/90

Exercise 3

(2) Now consider a stochastic variant:

The selling prices are uncertain, and they are described as a stochastic parameter:

Scenario	Probability	Selling price A	Selling price B
1	0.3	54	56
2	0.4	50	60
3	0.3	46	64

Formulate a stochastic program with recourse for this case.

Exercise 3

(3) Consider the case where the selling price of A and B are independent, and:

- The selling price of A takes value 30, 50, 60 with probability 0.3, 0.4, 0.3, respectively
- The selling price of B takes value 50, 60, 70 with probability 0.3, 0.4, 0.3 , respectively

What are the scenarios to consider in this case?

Exercise 3

(4) Formulate a stochastic model with recourse for the case where the demand is uncertain. The demand (d_A, d_B) can take value (400, 100), (500, 200), (600, 300) with probability 0.3, 0.4, 0.3, respectively.