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~~K436765~~

Q1. Consider the linear model

Let us consider the estimation of β. Show or explain in details, one way or another, how the maximum likelihood estimator

$$\hat{\beta} = (X'X)^{-1} X'Y \text{ is obtained.}$$

Let us consider the estimation of unknown parameters of the linear model,

$$y = X\beta + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I) \quad (1)$$

Since $y \sim N(X\beta, \sigma^2 I)$ the likelihood function $L(\beta, \sigma^2/Y)$ & log likelihood function $l(\beta, \sigma^2/y)$

$$= \log(L(\beta, \sigma^2/y))$$

have the forms

$$L(\beta, \sigma^2/y) = \frac{1}{\sqrt{(2\pi)^n |\sigma^2 I|}} \times \exp\left(-\frac{(y-X\beta)'(y-X\beta)}{2\sigma^2}\right)$$

$$l(\hat{\beta}, \hat{\sigma}^2 / y) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{(y - X\beta)'(y - X\beta)}{2\sigma^2}$$

The maximum likelihood estimators $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, \ldots \hat{\beta}_p)$ of the parameters vector β & σ^2 of the scaler $\sigma^2 > 0$ can be obtained by finding the maximum of the log-likelihood function $l(\beta, \sigma^2 / y)$ with respect to β, σ^2 by the process of differentiation.

The maximum likelihood estimators $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, \ldots \hat{\beta}_p)$ of the parameters vector β & $\hat{\sigma}^2$ of the scaler $\sigma^2 > 0$ can be obtained by finding maximum of the log-likelihood function $l(\beta, \sigma^2 / y)$ with respect to β, σ^2 by the process of differentiation

$$\hat{\beta}\,\hat{\sigma}^2 = \underset{\beta, \sigma^2}{\arg\max}\, l(\beta, \sigma^2 / y) = \underset{\beta, \sigma^2}{\arg\max}\left(-\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right)$$

The local maximum $l(\beta, \sigma^2/y)$ with respect to β occurs at point, where the gradient vector (partial derivatives)

$$\frac{\partial l}{\partial \beta} = \begin{pmatrix} \frac{\partial l}{\partial \beta_0} \\ \frac{\partial l}{\partial \beta_p} \\ \vdots \\ \frac{\partial l}{\partial \beta_p} \end{pmatrix}$$

$$\frac{\partial l}{\partial \beta} = \frac{\partial\left(-\frac{1}{2\sigma^2}(y-X\beta)'(y-X\beta)\right)}{\partial \beta}$$

$$= \left(-\frac{1}{2\sigma^2}\right)\frac{\partial(-2y'X\beta)}{\partial \beta} + \left(-\frac{1}{2\sigma^2}\right) \times \left(\frac{\partial \beta' X' X\beta}{\partial \beta}\right)$$

$$= \left(-\frac{1}{2\sigma^2}\right)(-2X'y) + \left(-\frac{1}{2\sigma^2}\right) 2X'X\beta$$

$$= -\frac{1}{2\sigma^2}(X'X\beta - X'y) = 0$$

Since $\frac{\partial l}{\partial \beta} = 0$ for any, $\sigma^2 > 0$ if

& only if the normal equations

$$X'X\beta = X'y$$ holds for β

The maximum likelihood estimator $\hat{\beta}$ is

the solution to the normal equations

$$X'X\hat{\beta} = X'y$$

i.e $\hat{\beta} = (X'X)^{-1}X'y$.

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2. In generalized linear model, the likelihood equations can written ~~for~~ form

$$\frac{\partial l(\beta, \phi)}{\partial \beta_j} = \sum_{i=1}^{n} \frac{(y_i - \mu_i)}{var(y_i)} x_{ij} \left(\frac{\partial \mu_i}{\partial \eta_i}\right) = 0$$

$$j = 0, 1, 2 \dots n$$

Consider now the simple log-linear ~~Inverse~~ Inverse Gaussian model with

$$Y_i \sim IG(\mu_i, \phi)$$

$$\log(\mu_i) = \eta_i = \beta_0$$

What kind of more simplified form the likelihood equations have in this case, that is, what form $\frac{\partial l(\beta_0)}{\partial \beta_0}$ has in the simple Inverse Gaussian model? By using the likelihood equations, find ~~the~~ the maximum likelihood estimator $\hat{\beta_0}$

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Here, $\log(\mu_i) = \eta_i = \beta_0$

$$Y_i \sim IG(\mu_i, \phi)$$

$$\frac{\partial l(\beta_0)}{\partial \beta_0} = \sum_{i=1}^{n} \frac{(y_i-\mu_i)}{\phi\, \mu_i^2} \mu_i \cdot 1$$

$$= \frac{1}{\phi} \sum_{i=1}^{n} \left(\frac{y_i-\mu_i}{\mu_i}\right)$$

$\mu_i = e^{\beta_0}$

$$\frac{\partial l(\beta_0)}{\partial \beta_0} = \frac{1}{\phi} \sum_{i=1}^{n} \left(\frac{y_i - e^{\beta_0}}{e^{\beta_0}}\right)$$

For the MLE

$$\frac{1}{\phi} \sum_{i=1}^{n} \frac{(y_i - e^{\beta_0})}{e^{\beta_0}} = 0$$

$$\sum_{i=1}^{n} (y_i - e^{\beta_0}) = 0$$

$$\sum_{i=1}^{n} y_i - n e^{\beta_0} = 0$$

$$\sum_{i=1}^{n} y_i - n e^{\beta_0} = 0$$

$$\beta_0 = \log\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)$$

Thus

$$\hat{\beta_0} = \log(\bar{y})$$

This is the MLE for $\hat{\beta_0}$, for Simple non-linear Inverse Gaussian model.

Q3 Let us assume $Y_i \sim \text{Gamma}(\mu_i, \phi)$

Consider the model

$$\log(\mu_i) = \beta_0 + \beta_1 x_i$$

Let the estimate of the parameter β_0, β_1, ϕ be as $\hat{\beta_0} = 1$

$$\hat{\beta_1} = 0.5$$

$$\phi = 01$$

when

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$$X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 7 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \end{pmatrix}$$

Calculate the estimated covariance matrix

$$\widehat{\text{cov}}(\hat{\beta}) = (X' \hat{\Omega} X)^{-1}$$