Lab 2

3. In a mass production process items are made to a nonnal weight of 1.0 kg.

$$1 = \int_0^{10} \frac{c}{100}\,d\theta + \int_{10}^{\infty} c\,\theta^{-2}\,d\theta$$

$$= \frac{c}{100}\,\theta\Big|_0^{10} + c\,\frac{\theta^{-2+1}}{2+1}\Big|_{10}^{\infty}$$

$$= \frac{c}{100}\times 10 + \frac{c}{10} =$$

$$= c\left[\frac{1}{10} + \frac{1}{10}\right] \Rightarrow c = 5$$

b) we observe ~~that~~ 10 items and

i) The density function for a random variable X with a uniform distn $(0, \theta)$ is $P(x/\theta) = \frac{1}{\theta}$ $0 \le x \le \theta$

ii) It follows that the likelihood is

$$p(x/\theta) = \frac{1}{\theta^{10}} \quad \text{for } \theta \geq \max x_i$$

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The range follows from the fact that

$\theta \geq x_1$, $\theta \geq x_2$ and so on.

Since θ is larger than each x_i.

It must be larger ~~that~~ than the largest of the x_i. for the data set given above $\max x_i = 4.9$

$$p(x/\theta) = \begin{cases} \frac{1}{\theta^{10}} & \text{for } \theta \geq 4.9 \\ & \text{for } \theta < 4.9 \end{cases}$$

$$\Rightarrow p(\theta/x) \propto \begin{cases} \frac{5}{100} \times \frac{1}{\theta^{10}} & \text{for } 4.9 \leq \theta \leq 10 \\ 5\theta^{-2} \times \frac{1}{\theta^{10}} & \text{for } \theta > 10 \end{cases}$$

III) The normalizing constant for the posterior density

$$1 = D\left[\int_{4.9}^{10} \frac{1}{20\,\theta^{10}}\,d\theta + \int_{10}^{\infty} \frac{5}{\theta^{12}}\,d\theta\right]$$

$$= D = \frac{1}{3.4106 \times 10^{-9}}$$

4) a) The pdf function

$\delta = 1$ is $P(x/\theta) = \theta x^{\theta-1}$ $\quad 0 < x < 1$

The likelihood is

$$P(x/\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

The prior is $P(\theta) = \frac{\beta^{\alpha}}{\Gamma\alpha} \theta^{\alpha-1} e^{-\theta\beta}$

By Bayes theorem the posterior dist^n is given by

$$P(\theta|x) \propto \theta^n \left(\prod_{i=1}^{n} x_i^{\theta-1} \right) \times \theta^{\alpha-1} e^{-\theta\beta}$$

$$= \theta^{n+\alpha-1} e^{-\theta\beta} \exp\left(\log \prod_{i=1}^{n} x_i^{\theta-1}\right)$$

$$= \theta^{n+\alpha-1} e^{-\theta\beta} \exp\left(\theta \sum_{i=1}^{n} \log x_i\right)$$

$$= \theta^{n+\alpha-1} e^{-\theta\beta} \exp\left(\sum_{i=1}^{n} (\theta-1) \log x_i\right)$$

$$\propto \theta^{n+\alpha-1} e^{-\theta\beta} \exp\left(\theta \sum_{i=1}^{n} \log x_i\right)$$

$$= \theta^{n+\alpha-1} \exp\left\{-\theta\left(\beta - \sum_{i=1}^{n} \log(x_i)\right)\right\}$$

$\therefore$ Kernal of Gamma$\left(n+\alpha, \frac{1}{\beta - \sum_{i=1}^{n} \log x_i}\right)$

Since the prior and posterior distn belong to the same Gamma family of distn Gamma$(\alpha, 1/\beta)$ is conjugate prior for θ.

The prior density function when $\theta = 2$ is $p(x/\delta) = 2\delta^2 x \quad 0 < x < \delta^{-1}$

$\Rightarrow$ The likelihood for $x_1, x_2, \dots x_n$ is

$$p(x/\delta) = \prod_{i=1}^{n} p(x_i/\delta)$$

$$= \prod_{i=1}^{n} 2\delta^2 x_i \propto \delta^{2n} \quad \max x_i < \delta^{-1}$$

equivalently $\min x_i^{-1} > \delta$

(b) The posterior is

$$p(\delta/x) \propto \delta^{-1} \delta^{2n} \quad \min x_i^{-1} > \delta$$

To find the normalizing constant let $\min x_i^{-1} = t$ then

$$1 = \int c\,\delta^{2n-1} d\delta = c \left. \frac{\delta^{2n}}{2n} \right|_0^t = c\,\frac{t^{2n}}{2n} \Rightarrow c = \frac{2n}{t^{2n}}$$

6. Data $n=12$ $\bar{x} = 16.35525$

$$p(\mu/x) \propto (\sqrt{2\pi})^{-n} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

sol

$$= (\sqrt{2\pi})^{-n} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i^2 - 2\mu x_i + \mu^2)\right]$$

$$\propto \exp\left(-\frac{1}{2}\, n\mu^2 - 2\mu \sum_{i=1}^{n} x_i\right)$$

$$= \exp\left(-\frac{1}{2}\left\{n(\mu^2 - 2\mu\bar{x})\right\}\right)$$

$$\propto \exp\left[-\frac{n}{2}(\mu-\bar{x})^2\right]$$: kernel of Normal $\left(\bar{x}, \frac{1}{n}\right)$

$\Rightarrow$ Posterior : Normal $\left(\bar{x}, \frac{1}{n}\right)$ = Normal $\left(16.35525, \frac{1}{12}\right)$

90% HPD for μ : $16.35525 \pm 1.65 \frac{1}{\sqrt{12}}$

$\approx (15.88)$