Bayesian Analysis I: Practice Problems

Question 1:

Geological Activity	Number of Earthquakes
DB	6
IAO	3
MP	1

Past studies show the following probabilities for a major earthquake occurring:

$$P(M|DB) = 0.6, \quad P(M|IAO) = 0.3, \quad P(M|MP) = 0.2$$

a) Find the Probability that a Major Earthquake Will Occur in San Francisco

The Law of Total Probability:

$$P(M) = P(M|DB)P(DB) + P(M|IAO)P(IAO) + P(M|MP)P(MP)$$

Where:

$$P(DB) = \frac{6}{10} = 0.6, \quad P(IAO) = \frac{3}{10} = 0.3, \quad P(MP) = \frac{1}{10} = 0.1$$

Substitute the values:

$$P(M) = (0.6)(0.5) + (0.3)(0.3) + (0.2)(0.2)$$
$$P(M) = 0.3 + 0.09 + 0.04 = 0.43$$

Thus, the probability that a major earthquake will occur is:

$$\boxed{P(M) = 0.43}$$

b) Find the Posterior Probability Distribution for Geological Activities

Using Bayes' Theorem, the posterior probabilities:

$$P(DB|M) = \frac{P(M|DB)P(DB)}{P(M)}, \quad P(IAO|M) = \frac{P(M|IAO)P(IAO)}{P(M)}, \quad P(MP|M) = \frac{P(M|MP)P(MP)}{P(M)}$$

Substitute the values:
1. **For DB:**

$$P(DB|M) = \frac{(0.6)(0.6)}{0.43} = \frac{0.36}{0.43} \approx 0.8372$$

2. **For IAO:**

$$P(IAO|M) = \frac{(0.3)(0.3)}{0.43} = \frac{0.09}{0.43} \approx 0.2093$$

3. **For MP:**

$$P(MP|M) = \frac{(0.2)(0.1)}{0.43} = \frac{0.02}{0.43} \approx 0.0465$$

Thus, the posterior probabilities are:

$$P(DB|M) \approx \frac{30}{41}, \quad P(IAO|M) \approx \frac{9}{41}, \quad P(MP|M) \approx \frac{2}{41}$$

Question 2

A random sample $X_i, i = 1, 2, \ldots, 50$ from a distribution

$$f(x|\theta) = \frac{\theta}{x^2} e^{-\frac{\theta}{x}} \quad \text{for} \quad x, \theta > 0$$

a) Find a Conjugate Prior Distribution for θ

The likelihood function is:

$$L(\theta|x) = \prod_{i=1}^{50} \frac{\theta}{x_i^2} e^{-\frac{\theta}{x_i}} = \theta^{50} \exp\left(-\theta \sum_{i=1}^{50} \frac{1}{x_i}\right) \prod_{i=1}^{50} \frac{1}{x_i^2}$$

For the conjugate prior, **Gamma distribution** is chosen:

$$p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}$$

Thus, the conjugate prior is:

$$\theta \sim \text{Gamma}(a, b)$$

b) Identify its Resulting Posterior Distribution for θ

Multiplying the likelihood by the prior:

$$p(\theta|x) \propto \theta^{50} \exp\left(-\theta \sum_{i=1}^{50} \frac{1}{x_i}\right) \theta^{a-1} e^{-b\theta}$$

This simplifies to:

$$p(\theta|x) \propto \theta^{50+a-1} \exp\left(-\theta \left(\sum_{i=1}^{50} \frac{1}{x_i} + b\right)\right)$$

Thus, the posterior distribution is:

$$\theta|x \sim \text{Gamma}(50 + a, \sum_{i=1}^{50} \frac{1}{x_i} + b)$$

c) Find the Posterior Mean Given the Sample

The mean of a Gamma distribution $\text{Gamma}(\alpha, \beta)$ is:

$$\mathbb{E}[\theta|x] = \frac{\alpha}{\beta}$$

For the posterior distribution $\text{Gamma}(50 + a, \sum_{i=1}^{50} \frac{1}{x_i} + b)$, the posterior mean is:

$$\mathbb{E}[\theta|x] = \frac{50 + a}{\sum_{i=1}^{50} \frac{1}{x_i} + b}$$

Thus, the posterior mean is:

$$\boxed{\mathbb{E}[\theta|x] = \frac{50 + a}{\sum_{i=1}^{50} \frac{1}{x_i} + b}}$$

Question 3

Given that Y follows a binomial distribution with parameters $n = 3$ and unknown parameter θ, which is assumed to have a prior distribution:

$$p(\theta) = \frac{27 - 75(\theta - 0.4)^2}{22} \quad \text{for} \quad 0 \leq \theta \leq 1$$

The posterior distribution when $Y = 2$.

Likelihood and Prior

The likelihood for a binomial distribution is:

$$p(y|\theta) = \binom{3}{y}\theta^y(1-\theta)^{3-y}$$

Substituting $y = 2$:

$$p(2|\theta) = \binom{3}{2}\theta^2(1-\theta) = 3\theta^2(1-\theta)$$

Now, using Bayes' Theorem, the posterior distribution is proportional to the likelihood multiplied by the prior:

$$p(\theta|2) \propto p(2|\theta)p(\theta)$$

Substituting the expressions for the likelihood and prior:

$$p(\theta|2) \propto 3\theta^2(1-\theta) \times \left(\frac{27 - 75(\theta - 0.4)^2}{22}\right)$$

Expanding the quadratic term:

$$(\theta - 0.4)^2 = \theta^2 - 0.8\theta + 0.16$$

Now, substituting this into the prior expression:

$$p(\theta|2) \propto 3\theta^2(1-\theta)\left(27 - 75(\theta^2 - 0.8\theta + 0.16)\right)$$

This simplifies to:

$$p(\theta|2) \propto 3\theta^2(1-\theta)\left(15 - 75\theta^2 + 60\theta\right)$$

Distribute $\theta^2(1-\theta)$ to get the final form of the posterior expression:

$$p(\theta|2) \propto 3\theta^2(1-\theta)\left(15 - 75\theta^2 + 60\theta\right)$$

Normalizing Constant

To normalize the posterior distribution,To compute the integral:

$$N.C. = \int_0^1 \left(15\theta^2 - 75\theta^4 + 60\theta^3 - 15\theta^3 + 75\theta^5\right) d\theta$$

After solving the integral

$$N.C. = \frac{15}{3} + \frac{45}{4} - 135 + 75 = \frac{4}{7}$$

Thus, the posterior distribution is:

$$p(\theta|2) = \frac{4}{7}\theta^2(1-\theta)\left(27 - 75(\theta - 0.4)^2\right)$$

Question 4

a) Posterior Distribution for μ

$n = 9$ observations with a sample mean $Y = 12$, and observations are independent with the distribution:

$$Y_i|\mu \sim \text{Normal}(\mu, \frac{1}{\psi})$$

where ψ is the known precision. The prior distribution for μ is assumed to be:

$$\mu \sim \text{Normal}(\theta, \frac{\sigma^2}{m})$$

The likelihood function for the sample mean is:

$$p(Y|\mu) \propto \exp\left(-\frac{n}{2\psi}(Y-\mu)^2\right)$$

The prior distribution for μ is:

$$p(\mu) \propto \exp\left(-\frac{1}{2}\frac{(\mu-\theta)^2}{\frac{\sigma^2}{m}}\right)$$

Now, applying Bayes' theorem:

$$p(\mu|Y) \propto p(Y|\mu)p(\mu)$$

Substituting the expressions for the likelihood and prior:

$$p(\mu|Y) \propto \exp\left(-\frac{n}{2\psi}(Y-\mu)^2\right)\exp\left(-\frac{1}{2}\frac{(\mu-\theta)^2}{\frac{\sigma^2}{m}}\right)$$

This gives the posterior distribution for μ as:

$$\mu|Y \sim \text{Normal}\left(\frac{nY + m\theta/\sigma^2}{n + m/\sigma^2}, \frac{\sigma^2}{n + m/\sigma^2}\right)$$

Thus, the posterior mean and variance are:
- Posterior mean: $\mathbb{E}[\mu|Y] = \frac{nY+m\theta/\sigma^2}{n+m/\sigma^2}$ - Posterior variance: $\text{Var}[\mu|Y] = \frac{\sigma^2}{n+m/\sigma^2}$

b) 95% Equal-Tailed Credible Interval for μ

Given that $\sigma^2 = 1$, $m = 0$, and $\psi = 1$, the posterior distribution for μ becomes:

$$\mu|Y \sim \text{Normal}\left(Y, \frac{1}{n}\right)$$

To find the 95% equal-tailed credible interval for μ, The fact that the 95% credible interval is given by:

$$\mu \in \left[Y \pm 1.96 \times \frac{1}{\sqrt{n}}\right]$$

Substituting the values:

$$\mu \in \left[12 \pm 1.96 \times \frac{1}{\sqrt{9}}\right] = \left[12 \pm 1.96 \times \frac{1}{3}\right] = [12 \pm 0.653]$$

Thus, the 95% credible interval for μ is:

$$\mu \in [11.347, 12.653]$$

c) Interpretation of the Credible Interval

The Bayesian credible interval has a straightforward interpretation: there is a 95% probability that μ lies within the interval. This means that, given the data and the prior, 95

The credible interval gives an estimate of the range in which the true value of μ lies with 95

Question 5

A random sample is given $X_i, i = 1, \ldots, n$ from a distribution with the probability density function:

$$f(x|\theta) = \frac{\theta^{1/2}}{\sqrt{2\pi}} \exp\left(-\frac{\theta x^2}{2}\right) \quad \theta > 0$$

Step 1: Likelihood Function

The likelihood function for the sample $X_1, \ldots, X_n$ is:

$$L(\theta|x) = \prod_{i=1}^{n} \frac{\theta^{1/2}}{\sqrt{2\pi}} \exp\left(-\frac{\theta x_i^2}{2}\right)$$

This simplifies to:

$$L(\theta|x) \propto \theta^{n/2} \exp\left(-\frac{\theta}{2} \sum_{i=1}^{n} x_i^2\right)$$

Step 2: Prior Distribution

A Gamma distribution for the prior of θ:

$$p(\theta) \propto \theta^{a/2-1} \exp\left(-\frac{b\theta}{2}\right)$$

Where a and b are the shape and rate parameters of the Gamma distribution.

Step 3: Posterior Distribution

The posterior distribution is proportional to the product of the likelihood and the prior:

$$p(\theta|x) \propto L(\theta|x)p(\theta)$$

Substitute the expressions for the likelihood and prior:

$$p(\theta|x) \propto \theta^{n/2} \exp\left(-\frac{\theta}{2} \sum_{i=1}^{n} x_i^2\right) \times \theta^{a/2-1} \exp\left(-\frac{b\theta}{2}\right)$$

This simplifies to:

$$p(\theta|x) \propto \theta^{(n+a)/2-1} \exp\left(-\frac{\theta}{2} \left(\sum_{i=1}^{n} x_i^2 + b\right)\right)$$

Thus, the posterior distribution for θ is:

$$\theta|x \sim \text{Gamma}\left(\frac{n+a}{2}, \frac{\sum_{i=1}^{n} x_i^2 + b}{2}\right)$$

Step 4: Predictive Distribution

To find the predictive distribution for a new observation y given $X_1, \ldots, X_n$,

$$p(y|x) = \int_0^\infty p(y|\theta)p(\theta|x)d\theta$$

Substitute the likelihood function for y and the posterior for θ:

$$p(y|x) = \int_0^\infty \frac{\theta^{1/2}}{\sqrt{2\pi}} \exp\left(-\frac{\theta y^2}{2}\right) \theta^{(n+a)/2-1} \exp\left(-\frac{\theta}{2}\left(\sum_{i=1}^n x_i^2 + b\right)\right) d\theta$$

This simplifies to:

$$p(y|x) \propto \int_0^\infty \theta^{(n+a)/2} \exp\left(-\frac{\theta}{2}\left(y^2 + \sum_{i=1}^n x_i^2 + b\right)\right) d\theta$$

The integral is in the form of the Gamma distribution, and after solving, Get the predictive distribution:

$$p(y|x) = \frac{\Gamma\left(\frac{n+a+1}{2}\right)}{\Gamma\left(\frac{n+a}{2}\right)} \left(\frac{1}{n+a}\right)^{1/2} \left(y^2 + \sum_{i=1}^n x_i^2 + b\right)^{(n+a)/2}$$

Thus, the predictive distribution for y is:

$$p(y|x) = \frac{\Gamma\left(\frac{n+a+1}{2}\right)}{\Gamma\left(\frac{n+a}{2}\right)} \left(\frac{1}{n+a}\right)^{1/2} \left(y^2 + \sum_{i=1}^n x_i^2 + b\right)^{(n+a)/2}$$