Lab - 09

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1 Let θ denote the probability of a head : $\theta \in (0.25, 0.5, 0.75)$.

Toss a randomly selected coin once, observing a head.

$X | \theta \sim$ Bernoulli (θ) and the observed data, $x = 1$.

likelihood : $L(\theta) = \theta^x (1-\theta)^{1-x}$

Coin θ	$P(\theta)$ prior	$P(x=1/\theta)$ lkhd	$P(x=1/\theta)$ lkhd×prior	Posterior
1 0.25	1/3	1/4	1/12	1/6
2 0.5	1/3	1/2	1/6	1/3
3 0.75	1/3	3/4	3/12	1/2
Sum	1		P(x): 6/12	1

Probability to choose coin 3 given one head observed $P(\theta = 0.75 / x = 1) = 0.5$

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b) Predictive distn for y given that one head was observed.

$$P(Y=y \mid X=1) = \sum_{\theta=\frac{1}{4},\frac{1}{2},\frac{3}{4}} P(Y=y \mid \theta)\, P(\theta \mid X=1)$$

$$= \left(\frac{1}{4}\right)^{y}\left(\frac{3}{4}\right)^{1-y} \times \frac{1}{6} + \left(\frac{1}{2}\right)^{y}\left(\frac{1}{2}\right)^{1-y} \times \left(\frac{1}{3}\right) + \left(\frac{3}{4}\right)^{y}\left(\frac{1}{4}\right)^{1-y} \times \frac{1}{2}$$

Predictive prob that next toss is head.

$$P(Y=1/X=1) = \left(\frac{1}{4}\right) \times \frac{1}{6} + \left(\frac{1}{2}\right) \times \left(\frac{1}{3}\right) + \left(\frac{3}{4}\right)^{y}\left(\frac{1}{4}\right)^{1-y} \times \frac{1}{2}$$

Predictive probability that next toss is a head

$$P(Y=1 \mid X=1) = \left(\frac{1}{4}\right) \times \left(\frac{1}{6}\right) + \left(\frac{1}{2}\right) \times \frac{1}{3} + \left(\frac{3}{4}\right) \times \frac{1}{2} = \frac{7}{12}$$

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c) Now, suppose that we observe x total response out of n tossings:

$x \mid \theta \sim \text{Binom}(n, \theta)$

Posterior predictive probability of the next toss being a head

$Y \mid \theta \sim \text{Bernoulli}(\theta)$

$$P(Y=1 \mid x) = \int_0^1 P(Y=1 \mid \theta)\, p(\theta \mid x)\, d\theta$$

$$= \int_0^1 \theta \times \frac{\Gamma(n+2)}{\Gamma(x+1)\,\Gamma(n-x+1)} \times \underbrace{\theta^{x}(1-\theta)^{n-x}\, d\theta}_{\text{density of Beta}(x+1,}$$

$$= \frac{\Gamma(n+2)}{\Gamma(x+1)\,\Gamma(n-x+1)} \int_0^1 \theta^{x+1} (1-\theta)^{n-x}\, d\theta$$

$$= \frac{\Gamma(n+2)}{\Gamma(x+1)\,\Gamma(n-x+1)} \cdot \frac{\Gamma(x+2)\,\Gamma(n-x+1)}{\Gamma(n+3)}$$

$$= \frac{x+1}{n+2}$$

as $\Gamma(n+3) = (n+2)\,\Gamma(n+2)$

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2 Five machines are put on tests for at most one hundred hours, three of them fail at 65, 89 and 95 hours

The remaining machines are still working at one hundred hours.

Assuming the lifetime of the machines has an exponential distn with mean $1/\theta$, the derived likelihood is $\propto \theta^3 \exp(-449\theta)$

The likelihood is found by

$$p(x/\theta) = \prod_{i=1}^{3} p(x_i/\theta) \times P(x_4 > 100) \times P(x_5 > 100)$$

$$P(x_j > 100) = \int_{100}^{\infty} \theta \exp(-x\theta)\, dx$$

$$= \exp(-100\theta)$$

Thus the likelihood is

$$p(x/\theta) = \prod_{i=1}^{3} [\theta \exp(-x\theta)] \times \exp(-100\theta) \times \exp(-100\theta)$$

$$= \theta^3 \exp(-(65 + 89 + 95)\theta) \exp(-200\theta)$$

$$= \theta^3 \exp(-449\theta)$$

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a) The prior distribution for θ is a
Gamma $(2, 1/180)$ distribution

Prior: $p(\theta) \propto \theta^{2} \exp(-180\theta)$

Posterior: $p(\theta|x) \propto p(x|\theta)\, p(\theta)$

$$\propto \theta^{3} \exp(-449\theta)$$
$$\times\, \theta^{2-1} \exp(-180\theta)$$
$$= \theta^{5-1} \exp(-629\theta)$$
$$\Rightarrow x|\theta \sim \text{Gamma}(5, 1/629)$$

b) ~~Find the~~ predictive dist^n of the
life time of another similar machine

$$p(y|x) = \int_0^\infty p(y|\theta)\, p(\theta|x)\, d\theta$$

$$= \int_0^\infty \theta e^{-y\theta} \times \frac{629^5}{\Gamma 5} \theta^4 e^{-629\theta}\, d\theta$$

$$= \frac{629^5}{\Gamma 5} \int_0^\infty \underbrace{\theta^5 e^{-\theta(y+629)}}_{\text{Kernel of Gamma}}$$

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$$= \frac{629^5}{\Gamma 5} \cdot \frac{\Gamma 6}{(y+629)^6}$$

$$= \frac{5 \times 629^5}{(y+629)^6}, \quad y > 0$$

c) What is the (predictive) probability that it works for more than 100 hours?

$$P[y > 100] = \int_{100}^{\infty} \frac{5 \times 629^5}{(y+629)^6}\, dy$$

$$= 5 \times 629^5 \left[\frac{-1}{5 \times (y+629)^5} \right]_{100}^{\infty}$$

$$= 5 \times 629^5 \frac{1}{5 \times 729^5} = \left(\frac{629}{729}\right)^5$$

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3. Let $X_1, \ldots X_n$ (independent given) be ~~to~~ from a poisson (λ) suppose that $\lambda \sim$ Gamma(α, $1/\beta$)

Let Z be a future observation.

Likelihood $X = (x_1, x_2 \ldots x_n) \mid \lambda \sim$ Poisson(λ)

prior $\lambda \sim$ Gamma(α, $1/\beta$)

Posterior: $\lambda / X \sim$ Gamma$\left(n\bar{x} + \alpha, \frac{1}{n+\beta}\right)$

$$E(Z) = E[E(Z/\lambda)] = E(\lambda)$$

$$\Rightarrow E(Z/X) = E(\lambda/X)$$

$$= (n\bar{x} + \alpha) \frac{1}{n+\beta}$$

$$Var(Z) = E[Var(Z/\lambda)] + Var[E(Z/\lambda)] = E(\lambda) + Var(\lambda)$$

$$Var[Z/X] = E[\lambda/X] + V[\lambda/X]$$

$$= \frac{n\bar{x} + \alpha}{n+\beta} + \frac{n\bar{x} + \alpha}{(n+\beta)^2}$$

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$$= \frac{(n\bar{x}+\alpha)(n+\beta+1)}{(n+\beta)^2}$$

~~we have grown in the~~

#4 i) $p_1(\theta) \propto \theta^2(1-\theta)^8$

$\Rightarrow p_1(\theta/x) \propto \theta^2(1-\theta)^8 \theta^5 (1-\theta)^{17}$

$= \theta^7(1-\theta)^{0.25}$; Beta $(8, 26)$

$p_2(\theta) \propto \theta^8(1-\theta)^2 \Rightarrow p_2(\theta/x) \propto$

$\theta^8(1-\theta)^2\theta^5(1-\theta)^{17}$

$= \theta^{13}(1-\theta)^{19}$; Beta $(14, 20)$

ii) Normalizing constants for each combination $C_j = \int p_j(\theta) p(x/\theta) d\theta$

$C_1 = \int p_1(\theta) p(x/\theta) d\theta$

$$= \int_0^1 \frac{\Gamma 12}{\Gamma 3 \, \Gamma 9} \theta^2 (1-\theta)^8 \times \theta^5 (1-\theta)^{17} d\theta$$

$$= \frac{\Gamma 12 \, \Gamma 8 \, \Gamma 26}{\Gamma 3 \, \Gamma 9 \, \Gamma 34}$$

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$$c_2 = \frac{\Gamma(12)\,\Gamma(14)\,\Gamma(20)}{\Gamma(8)\,\Gamma(3)\,\Gamma(34)}$$

iii)

$$a_1 = \frac{\Gamma(8)\,\Gamma(26)\times 0.5}{\Gamma(8)\,\Gamma 26 \times 0.5 + \Gamma(14)\,\Gamma 20 \times 0.5}$$

$$= \frac{7!\,28!}{7!\,25! + 13!\,19!} \simeq 0.9904$$

$$a_2 = 1 - a_1 = 0.0096$$

or $c_1 = \frac{11}{2!\,8!} = \frac{7!\,25!}{33!}$

$= 4.4565 \times 10^{-6}$

$$c_2 = \frac{11!}{8!\,2!}\,\frac{13!\,19!}{8!\,2!\,33!}$$

$= 4.3181 \times 10^{-8}$

$$a_1 = \frac{4.4565\times10^{-6}\times 0.5}{4.4565\times10^{-6}\times 0.5 + 4.3181\times10^{-8}\times 0.5}$$

$\simeq 0.9904$ $a_2 = 0.0096$

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8.

a) The likelihood is

$$p(y|\mu,\tau)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{(y-\mu)^2}{2\tau}\right)$$

The joint posterior of (μ,τ) is

$$p(\mu,\tau|y)\propto p(y|\mu,\tau)\,p(\mu,\tau)$$

$$\propto \tau^{-n/2}\exp\left(-\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{2\tau}\right)$$

$$\times\exp\left(-\frac{1}{2c}(\mu-m)^2\right)(\tau)^{-a-1}e^{-b/\tau}$$

$$p(\mu|\tau,y)\propto\exp\left(-\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{2\tau}\right)$$

$$\exp\left(-\frac{1}{2c}(\mu-m)^2\right)$$

$$=\exp\left(-\frac{1}{2\tau}\sum_{i=1}^{n}\left[y_i^2-2\mu y_i+\mu^2\right]\right)$$

$$\exp\left(-\frac{1}{2c}(\mu^2-2m\mu+m^2)\right)$$

$$\propto\exp\left(-\frac{1}{2}\left[\left(\frac{n}{\tau}+\frac{1}{c}\right)\mu^2\right.\right.$$

$$\left.\left.-2\left(\frac{n\bar{y}}{\tau}+\frac{1}{c}m\right)\mu\right]\right)$$

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Thus

$m \mid \tau, y \sim \text{Normal}(m^*, C^*)$

with $C^* = \left(\frac{n}{\tau} + \frac{1}{C}\right)^{-1}$

$m^* = C^*\left(\frac{n}{\tau}\bar{y} + \frac{1}{C} m_0\right)$

b) The conditional posterior for τ, $p(\tau \mid m, y)$ can be found by fixing m in $p(m, \tau \mid y)$.

$$p(m \mid \tau, y) \propto \tau^{-n/2} \exp\left(-\sum_{i=1}^{n} \frac{(y_i - m)^2}{2\tau}\right)$$

$$\propto (\tau)^{-a-1} e^{-b/\tau}$$

$$\propto \tau^{-n/2 - a - 1} \exp\left(-\frac{1}{\tau}\left(\sum \tfrac{1}{2}(y_i - m)^2 + b\right)\right)$$

Thus $\tau \mid m, y \sim IG\left(a + n/2, \dfrac{1}{\sum \frac{1}{2}(y_i - m)^2 + b}\right)$