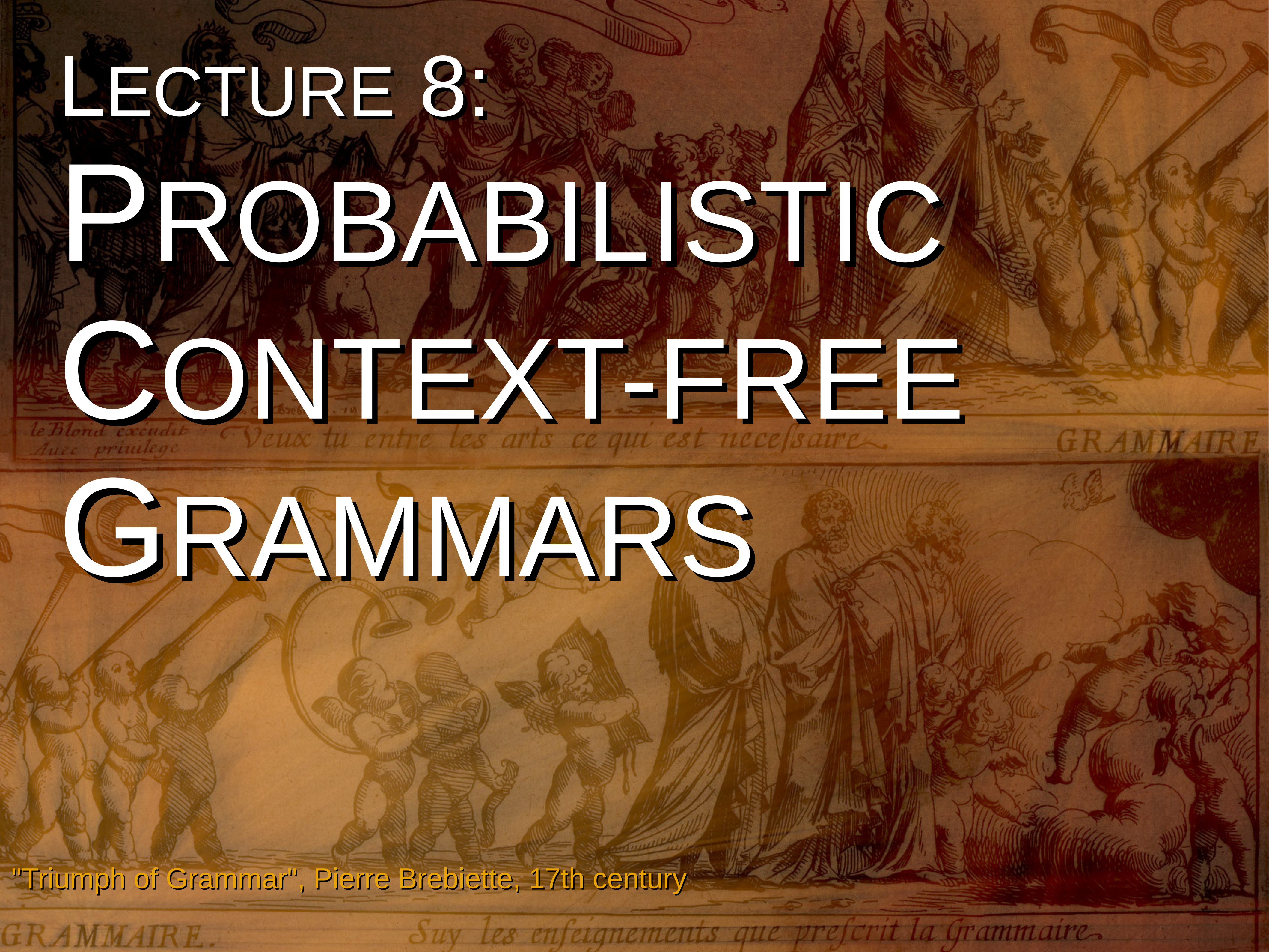
Lecture 8: Probabilistic Context-free Grammars



"Triumph of Grammar", Pierre Brebiette, 17th century

Different ways of speaking

- *"No more training do you require. Already know you, that which you need."*
  - Yoda, *Return of the Jedi*
- *"To a dark place, this line of thought will carry us. Hmmm. Great care, we must take."*
  - Yoda, *Revenge of the Sith*
- *This, he announced, was "a thing up with which I will not put".*
  - anecdote in *The West Sussex Gazette*, 1941, later misattributed to Winston Churchill

- The generative models of word sequences we have seen so far (N-grams, HMMs) assume sequences are generated beginning-to-end based on previous words/states.
- But what about more general ways of constructing sequences? For example, we know that question sentences tend to have particular structure - can we model how particular sentences arise from more general 'templates' of their parts? This can be done by **grammars**.

Probabilistic context-free grammar

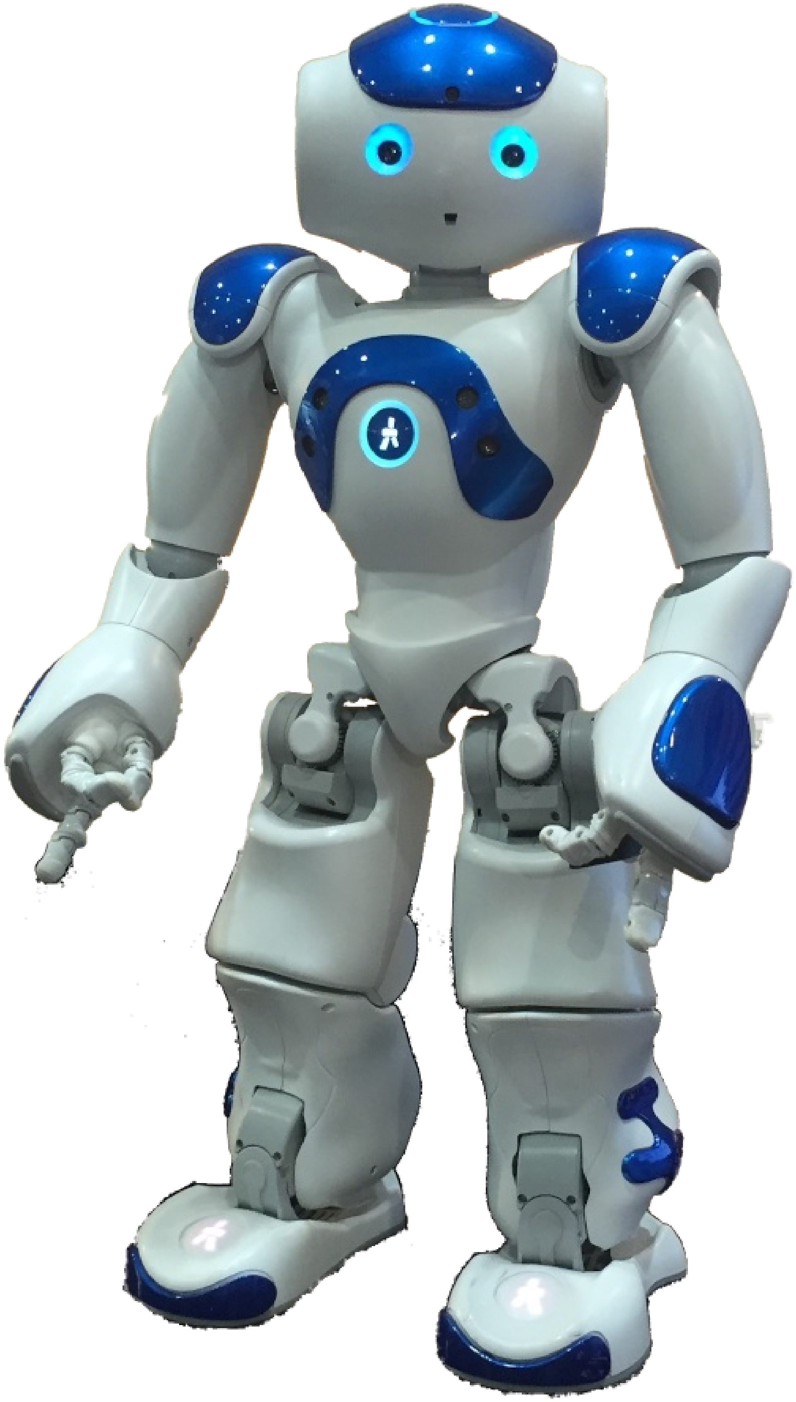
- A **probabilistic context-free grammar** (PCFG), also called a stochastic context-free grammar, is defined by:
  - A **start symbol** n_1
  - A set of **nonterminal symbols** $\{n_i\}, i=1,\ldots,N$
  - A set of **terminal symbols** (specific words, punctuation, phrases): $\{w_i\}, i=1,\ldots,V$
  - A set of **rules**, $\{n_{i_r} \rightarrow S_r\}, r=1,\ldots,R$ where each rule r transforms a particular nonterminal symbol n_{i_r} into a sequence S_r of nonterminal and terminal symbols
  - A set of **conditional probabilities** of the rules given their left-hand-side nonterminal symbol, so that for each nonterminal symbol the probabilities of its rules sum to 1

$$\{p_r = p(n_{i_r} \rightarrow S_r | n_{i_r})\}, r=1,\ldots,R, \qquad \forall i=1,\ldots,N \quad \sum_{r=1}^{R} \underbrace{\delta(i_r=i)}_{\text{1 if } i_r=i\text{, 0 otherwise}} p_r = 1$$

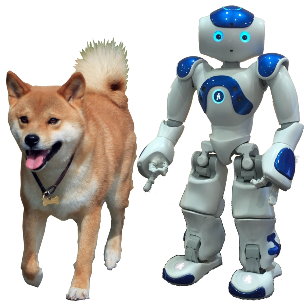
Probabilistic context-free grammar

- The PCFG is called a "**context-free**" grammar because each rule can be applied to a nonterminal symbol regardless of the context where the nonterminal symbol appears, i.e. regardless of the surrounding symbols
- A PCFG can be used to generate sentences, or to parse existing observed sentences.
- There can be several ways to generate the same sentence (through different sequences of applying the rules), and thus there can be several parses of an observed sentence.
- Because the rules in a PCFG have probabilities, the PCFG can be used to find **the most likely parse** of an observed sentence

Dogs versus Robots



Probabilistic context-free grammar

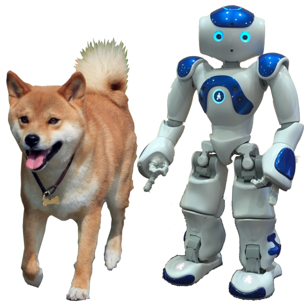


- Example grammar:
  - Start symbol: $n_1 = Sentence$
  - Set of nonterminal symbols: $\{Sentence, Noun, Article, Verb\}$
  - Terminal symbols: $\{a, the, robot, dog, bit, fed, walked, petted, .\}$
  - Rules and their probabilities:

$Sentence \rightarrow Article \;\; Noun \;\; Verb \;\; Article \;\; Noun \;\; .$	1
$Article \rightarrow a$	0.7
$Article \rightarrow the$	0.3
$Noun \rightarrow robot$	0.6
$Noun \rightarrow dog$	0.4
$Verb \rightarrow bit$	0.1
$Verb \rightarrow fed$	0.2
$Verb \rightarrow walked$	0.4
$Verb \rightarrow petted$	0.3

Probabilistic context-free grammar

- Example sentences and their probabilities:
  - The robot walked a dog. $1 \cdot 0.3 \cdot 0.6 \cdot 0.4 \cdot 0.7 \cdot 0.4 = 0.02016$



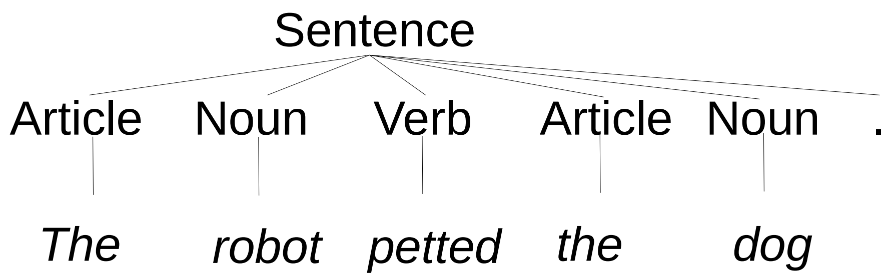
$$\begin{array}{c} \textit{Sentence} \\ \textit{Article Noun Verb Article Noun .} \\ \textit{The Noun Verb Article Noun .} \\ \textit{The robot Verb Article Noun .} \\ \textit{The robot walked Article Noun .} \\ \textit{The man walked a Noun .} \\ \textit{The robot walked a dog .} \end{array}$$

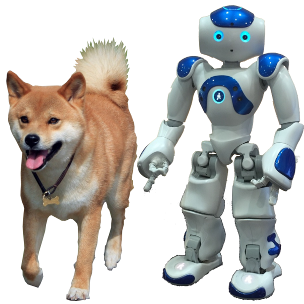
$\textit{Sentence} \rightarrow \textit{Article Noun Verb Article Noun .}$	1
$\textit{Article} \rightarrow \textit{the}$	0.3
$\textit{Noun} \rightarrow \textit{robot}$	0.6
$\textit{Verb} \rightarrow \textit{walked}$	0.4
$\textit{Article} \rightarrow \textit{a}$	0.7
$\textit{Noun} \rightarrow \textit{dog}$	0.4

  - A dog petted a dog . $1 \cdot 0.7 \cdot 0.4 \cdot 0.3 \cdot 0.7 \cdot 0.4 = 0.02352$
- The probability of the sentence is the probability that out of all valid sentences in the grammar, it will generate this one.
- All possible sentences (**for this example grammar** their probabilities sum to 1): A robot bit a robot. A robot bit a dog. A robot bit the robot. A robot bit the dog. A robot fed a robot. A robot fed a dog. A robot fed the robot. A robot fed the dog. A robot petted a robot. A robot petted a dog. A robot petted the robot. A robot petted the dog. A robot walked a robot. A robot walked a dog. A robot walked the robot. A robot walked the dog. A dog bit a robot. A dog bit a dog. A dog bit the robot. A dog bit the dog. A dog fed a robot. A dog fed a dog. A dog fed the robot. A dog fed the dog. A dog petted a robot. A dog petted a dog. A dog petted the robot. A dog petted the dog. A dog walked a dog. A dog walked a dog. A dog walked the robot. A dog walked the dog. The robot bit a robot. The robot bit a dog. The robot bit the robot. The robot bit the dog. The robot fed a robot. The robot fed a dog. The robot fed the robot. The robot fed the dog. The robot petted a robot. The robot petted a dog. The robot petted the robot. The robot petted the dog. The robot walked a robot. The robot walked a dog. The robot walked the robot. The robot walked the dog. The dog bit a robot. The dog bit a dog. The dog bit the robot. The dog bit the dog. The dog fed a robot. The dog fed a dog. The dog fed the robot. The dog fed the dog. The dog petted a robot. The dog petted a dog. The dog petted the robot. The dog petted the dog. The dog walked a dog. The dog walked a dog. The dog walked the robot. The dog walked the dog.

Probabilistic context-free grammar

- Parse tree of a sentence:





- (In this simple example grammar each valid sentence has only one possible **parse tree:** robot, dog can only come from Noun; a, the from Article, and so on. Generally there can be many possible parses for the same sentence.)
- PCFGs have useful independence properties:
  - **Place invariance**: probability of a subtree, which yields some substring of words in the text, is independent of where that subtree occurs in a longer text
  - **Context-free**: probability of a subtree is independent of any words that are not part of the subtree
  - **Ancestor-free**: probability of a subtree is independent of any nonterminal symbols (parse tree nodes) that are not part of the subtree

Probabilistic context-free grammar

- A PCFG may define a proper language model (proper distribution) but not always. Example:

  This kind of grammar is actually called a probabilistic regular grammar

  Sentence → *blah* 0.5
  Sentence → *blah Sentence* 0.5
- Possible sentences and their probabilities:
  blah 0.5, blah blah 0.25, blah blah blah 0.125, ...
- Sum of probabilities: 0.5 + 0.25 + 0.125 + ... = 1, **ok (proper distribution)**!
- But consider this alternative PCFG:
  Sentence → *blah* 0.3
  Sentence → *Sentence Sentence* 0.7
- Now:
  blah 0.3 1 way to generate = 0th Catalan number
  blah blah 0.7*0.3*0.3 1 way to generate = 1st Catalan number
  blah blah blah 2*(0.7*0.7*0.3*0.3*0.3) 2 ways S->SS->Sb->SSb->... , S->SS->bS->bSS->...
  blah blah blah blah 5*(0.7*0.7*0.7*0.3*0.3*0.3*0.3) 5 ways = 3rd Catalan number
  blah blah blah blah blah 14*(0.7*0.7*0.7*0.7*0.3*0.3*0.3*0.3*0.3) 14 ways = 4th C.n.
  blah blah blah blah blah blah 42*(0.7*0.7*0.7*0.7*0.7*0.3*0.3*0.3*0.3*0.3*0.3) 42 ways=5th C.n.
- Sum: about 0.43, **less than 1 - not a proper distribution!**
- Grammars whose parameters are learned from a parsed training corpus can be guaranteed to yield a proper distribution (Chi and German, 1998)

Chomsky normal form PCFG

- A PCFG in **Chomsky normal form** only has **unary and binary rules** of this form:

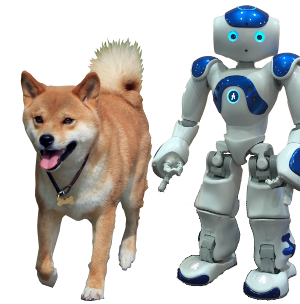
$$n_i \rightarrow n_j n_k \quad (Nonterminal_i \rightarrow Nonterminal_j \, Nonterminal_k)$$
$$n_i \rightarrow w_j \quad (Nonterminal_i \rightarrow word_j)$$

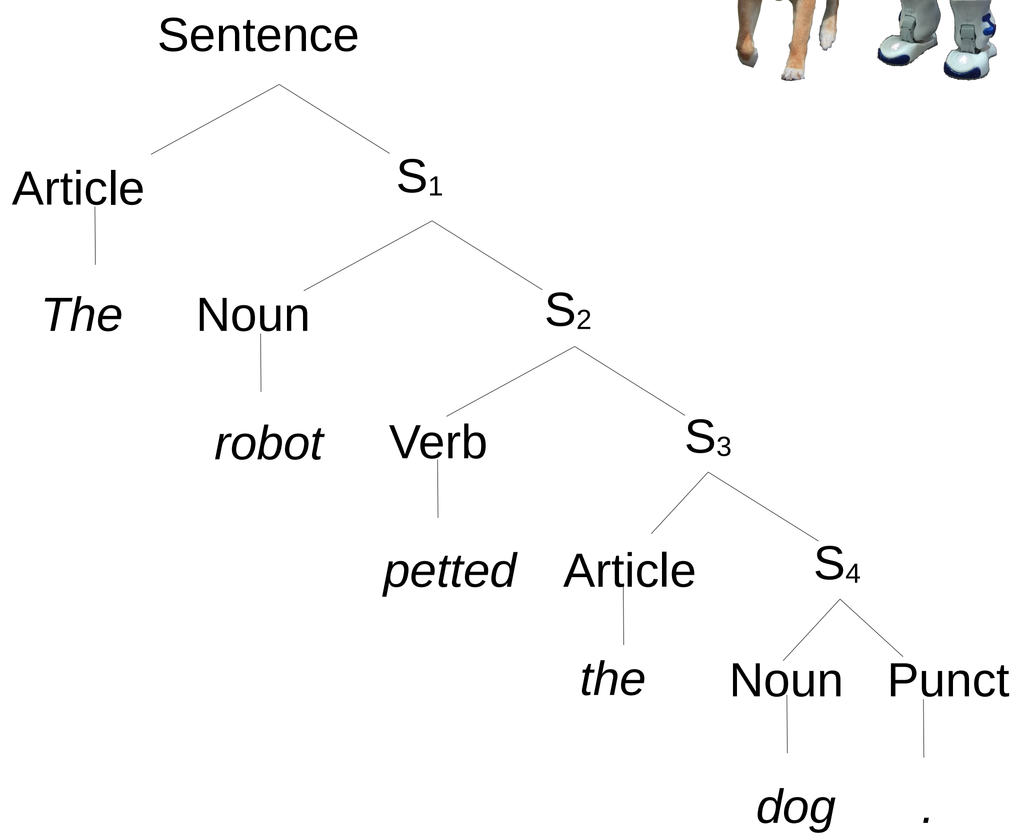
- Parameters: conditional probabilities of all N^3 binary rules (all combinations i,j,k) and all $N \cdot V$ unary rules (all combinations i,j)
- As before, the conditional probabilities of all rules with the same left-hand-side must sum to 1: $\forall i=1,\ldots,N \quad \sum_{j=1}^{N}\sum_{k=1}^{N} p(n_i \rightarrow n_j n_k | n_i) + \sum_{j=1}^{V} p(n_i \rightarrow w_j | n_i) = 1$
- Any context-free grammar (CFG) can be represented by a CFG in Chomsky normal form that is **weakly equivalent** to the original CFG:
  - Two grammars are said to be weakly equivalent if they generate the same language (same set of possible sentences).
  - Two grammars are **strongly equivalent** if they also give sentences the same tree structure.
  - In the probabilistic case also the probabilities of the sentences must be the same in the two grammars

Chomsky normal form PCFG

- Previous example in Chomsky normal form:

$Sentence$	→	$Article\ \ S_1$	1
S_1	→	$Noun\ \ S_2$	1
S_2	→	$Verb\ \ S_3$	1
S_3	→	$Article\ \ S_4$	1
S_4	→	$Noun\ \ Punct$	1
$Article$	→	a	0.7
$Article$	→	the	0.3
$Noun$	→	$robot$	0.6
$Noun$	→	dog	0.4
$Verb$	→	bit	0.1
$Verb$	→	fed	0.2
$Verb$	→	$walked$	0.4
$Verb$	→	$petted$	0.3
$Punct$	→	.	1





Probability of a string

- There can be exponentially many possible parse trees for a string of M words, probability of the string needs to be summed over them all **efficiently**.
- We define **inside probabilities** and **outside probabilities**.
- Inside probability: given a specific nonterminal n_i, probability that it generates the substring of words l - m in the text

$$p_i^{Inside}(l,m)=p(w_l,\ldots,w_m|n_i)$$

- Outside probability: given the starting symbol, probability to generate all words outside l-m and a nonterminal n_i responsible for the contents of l-m which we denote as $n_i(l,m)$

$$p_i^{Outside}(l,m)=p(w_1,\ldots,w_{l-1},n_i(l,m),w_{m+1},\ldots,w_M|n_1)$$

- The probability of a string can be computed by a dynamic programming algorithm: the **inside algorithm**

Inside algorithm

- Probability of a whole string is an inside probability:

$$p(w_1,\ldots,w_M)=p(w_1,\ldots,w_M|n_1)=p_1^{Inside}(1,M)$$

- We need to break down the inside probability into terms we know.
- If l = m, the string is a single word which must result from a unary rule $n_i \rightarrow w_m$ so

$$p_i^{Inside}(m,m)=p(n_i \rightarrow w_m|n_i)$$

- If l < m the string must result from some binary rule and its follow-up rules: $n_i \rightarrow n_j n_k \rightarrow \cdots \rightarrow w_l,\ldots,w_m$
- In such a rule, the nonterminals n_j and n_k must each be responsible for a substring: $w_l,\ldots,w_u$ and $w_{u+1},\ldots,w_m$ where u is the breakpoint

Inside algorithm

- The inside probability can then be computed inductively, as a sum over the possible intermediate nonterminals and breakpoints:

$$
\begin{aligned}
& p_i^{Inside}(l,m) \\
&= p(w_l,\ldots,w_m|n_i) \\
&= p(w_l,\ldots,w_u,w_{u+1},\ldots,w_m|n_i) \\
&= \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{u=l}^{m-1} p(w_l,\ldots,w_u,w_{u+1},\ldots,w_m,n_j(l,u),n_k(u+1,m)|n_i) \\
&= \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{u=l}^{m-1} p(w_l,\ldots,w_u|w_{u+1},\ldots,w_m,n_j(l,u),n_k(u+1,m),n_i) \\
&\cdot p(w_{u+1},\ldots,w_m|n_j(l,u),n_k(u+1,m),n_i)\cdot p(n_j(l,u),n_k(u+1,m)|n_i) \\
&= \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{u=l}^{m-1} p(w_l,\ldots,w_u|n_j)\cdot p(w_{u+1},\ldots,w_m|n_k)\cdot p(n_j,n_k|n_i) \\
&= \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{u=l}^{m-1} p_j^{Inside}(l,u)\, p_k^{Inside}(u+1,m)\, p(n_i \rightarrow n_j n_k|n_i)
\end{aligned}
$$

- Context-free: probability of the l-u subtree does not depend on words not in the l-u subtree
- Ancestor-free: probability of the l-u subtree does not depend on nodes not in the l-u subtree
- Same for the subtree of (u+1)-m

- Substrings l-u and (u+1)-m are shorter than l-m, so eventually they become single-word substrings whose probabilities come from the unary rules. Thus the above algorithm suffices to calculate the probability of a string of text.

Outside algorithm

- Probability of a whole string can also be computed using outside probabilities. For any breakpoint l:

$$
\begin{aligned}
& p(w_1,\ldots,w_M|n_1) \\
&= p(w_1,\ldots,w_{l-1},w_l,w_{l+1},\ldots,w_M|n_1) \\
&= \sum_{j=1}^{N} p(w_1,\ldots,w_{l-1},w_l,w_{l+1},\ldots,w_M,n_j(l,l)|n_1) \\
&= \sum_{j=1}^{N} p(w_l,w_1,\ldots,w_{l-1},n_j(l,l),w_{l+1},\ldots,w_M|n_1) \\
&= \sum_{j=1}^{N} p(w_l|w_1,\ldots,w_{l-1},n_j(l,l),w_{l+1},\ldots,w_M,n_1)\, p(w_1,\ldots,w_{l-1},n_j(l,l),w_{l+1},\ldots,w_M|n_1) \\
&= \sum_{j=1}^{N} p(w_l|n_j(l,l))\, p(w_1,\ldots,w_{l-1},n_j(l,l),w_{l+1},\ldots,w_M|n_1) \\
&= \sum_{j=1}^{N} p(n_j \rightarrow w_l|n_j)\, p_j^{Outside}(l,l)
\end{aligned}
$$

Outside algorithm

- We start top down from the top nonterminal:
  $$p_1^{Outside}(1,M) = p(n_1(1,M)|n_1) = 1$$
  $$\forall\, i \neq 1 \quad p_i^{Outside}(1,M) = p(n_i(1,M)|n_1) = 0$$
- Consider a nonterminal n_v that is not the top one and is responsible for words l-m.
  - The nonterminal must have resulted from a binary rule $n_i \rightarrow n_j n_k$, thus it must be the **left child** or **right child** of a parent nonterminal.
  - If n_v is the left child, there must be a right child n_k which is responsible for a substring (m+1)-q just after m, their parent n_i is then responsible for l-q.
  - If n_v is the right child, there must be a left child n_j which is responsible for a substring q-(l-1) just before l, their parent n_i is then responsible for q-m.

Outside algorithm

- We can then write:

$$p_v^{Outside}(l,m) = p(w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M|n_1)$$

$$= \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{q=m+1}^{M} p(w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_q,w_{q+1},\ldots,w_M,n_k(m+1,q),n_i(l,q)|n_1)$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{q=1}^{l-1} p(w_1,\ldots,w_{q-1},w_q,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M,n_j(q,l-1),n_i(q,m)|n_1)$$

$$= vIsLeftChild + vIsRightChild$$

- Note that in rules of the form $n_i \rightarrow n_v n_v$ the nonterminal n_v appears as both a left child and a right child in the same rule. We must not count such rules twice, so we exclude them from one of the sums (e.g. from $vIsLeftChild$)

Outside algorithm

- The case where v is the right child becomes:

$$
\begin{aligned}
& vIsRightChild \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{q=1}^{l-1} p(w_1,\ldots,w_{q-1},w_q,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M,n_j(q,l-1),n_i(q,m)|n_1) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{q=1}^{l-1} p(w_1,\ldots,w_{q-1},n_i(q,m),w_{m+1},\ldots,w_M,w_q,\ldots,w_{l-1},n_j(q,l-1),n_v(l,m)|n_1) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{q=1}^{l-1} p(w_q,\ldots,w_{l-1}|w_1,\ldots,w_{q-1},n_i(q,m),w_{m+1},\ldots,w_M,n_j(q,l-1),n_v(l,m),n_1) \\
&\cdot p(n_j(q,l-1),n_v(l,m)|w_1,\ldots,w_{q-1},n_i(q,m),w_{m+1},\ldots,w_M,n_1) \\
&\cdot p(w_1,\ldots,w_{q-1},n_i(q,m),w_{m+1},\ldots,w_M|n_1) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{q=1}^{l-1} p(w_q,\ldots,w_{l-1}|n_j(q,l-1)) \\
&\cdot p(n_i(q,m)\rightarrow n_j(q,l-1),n_v(l,m)|n_i(q,m)) \\
&\cdot p(w_1,\ldots,w_{q-1},n_i(q,m),w_{m+1},\ldots,w_M|n_1) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{q=1}^{l-1} p_j^{Inside}(q,l-1) \;\cdot\; p(n_i(q,m)\rightarrow n_j(q,l-1),n_v(l,m)|n_i(q,m)) \;\cdot\; p_i^{Outside}(q,m)
\end{aligned}
$$

Outside algorithm

- The case where v is the left child becomes (here $k \neq v$ to avoid counting rules $n_i \rightarrow n_v n_v$ twice):

$$vIsLeftChild$$

$$= \sum_{i=1}^{N} \sum_{k=1, k\neq v}^{N} \sum_{q=m+1}^{M} p\left(w_1, \ldots, w_{l-1}, n_v(l,m), w_{m+1}, \ldots, w_q, w_{q+1}, \ldots, w_M, n_k(m+1,q), n_i(l,q) \middle| n_1\right)$$

$$= \sum_{i=1}^{N} \sum_{k=1, k\neq v}^{N} \sum_{q=m+1}^{M} p\left(w_{m+1}, \ldots, w_q, n_v(l,m), n_k(m+1,q), w_1, \ldots, w_{l-1}, n_i(l,q), w_{q+1}, \ldots, w_M \middle| n_1\right)$$

$$= \sum_{i=1}^{N} \sum_{k=1, k\neq v}^{N} \sum_{q=m+1}^{M} p\left(w_{m+1}, \ldots, w_q \middle| n_v(l,m), n_k(m+1,q), w_1, \ldots, w_{l-1}, n_i(l,q), w_{q+1}, \ldots, w_M, n_1\right)$$
$$\cdot p\left(n_v(l,m), n_k(m+1,q) \middle| w_1, \ldots, w_{l-1}, n_i(l,q), w_{q+1}, \ldots, w_M, n_1\right)$$
$$\cdot p\left(w_1, \ldots, w_{l-1}, n_i(l,q), w_{q+1}, \ldots, w_M \middle| n_1\right)$$

$$= \sum_{i=1}^{N} \sum_{k=1, k\neq v}^{N} \sum_{q=m+1}^{M} p\left(w_{m+1}, \ldots, w_q \middle| n_k(m+1,q)\right)$$
$$\cdot p\left(n_v(l,m), n_k(m+1,q) \middle| n_i(l,q)\right)$$
$$\cdot p\left(w_1, \ldots, w_{l-1}, n_i(l,q), w_{q+1}, \ldots, w_M \middle| n_1\right)$$

$$= \sum_{i=1}^{N} \sum_{k=1, k\neq v}^{N} \sum_{q=m+1}^{M} p_k^{Inside}(m+1,q) \cdot p\left(n_i(l,q) \rightarrow n_v(l,m), n_k(m+1,q) \middle| n_i(l,q)\right) \cdot p_i^{Outside}(l,q)$$

- Because the outside probabilities are of ever larger substrings, at some point this reaches the top level whose probability we know.

Outside algorithm

- Using both the inside and outside probabilities we can write the probability that a particular nonterminal n_v will be generated and it will generate substring l-m:

$$
\begin{aligned}
&p(w_1,\ldots,w_{l-1},w_l,\ldots,w_m,w_{m+1},\ldots,w_M,n_v(l,m)|n_1)\\
&=p(w_l,\ldots,w_m|w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M,n_1)\cdot p(w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M|n_1)\text{¿}\\
&=p(w_l,\ldots,w_m|n_v(l,m))\cdot p(w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M|n_1)\\
&=p_v^{Inside}(l,m)\cdot p_v^{Outside}(l,m)
\end{aligned}
$$

- Or the probability that substring l-m arises from a single nonterminal (no matter which one):

$$
\begin{aligned}
&\sum_{v=1}^{N} p(w_1,\ldots,w_{l-1},w_l,\ldots,w_m,w_{m+1},\ldots,w_M,n_v(l,m)|n_1)\\
&=\sum_{v=1}^{N} p_v^{Inside}(l,m)\cdot p_v^{Outside}(l,m)
\end{aligned}
$$

Most likely parse of a string

- Remember that in the Inside algorithm we had

$$p_i^{Inside}(l,m)=p(w_l,\ldots,w_m|n_i)=\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{u=l}^{m-1} p_j^{Inside}(l,u)\, p_k^{Inside}(u+1,m)\, p(n_i\rightarrow n_j n_k|n_i)$$

$$=\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{u=l}^{m-1} p\left(n_i\rightarrow \left(n_j \rightarrow\cdots\rightarrow w_l,\ldots,w_u, n_k \rightarrow\cdots\rightarrow w_{u+1},\ldots,w_m\right)|n_i\right)$$

$$=\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{u=l}^{m-1} p\left(w_l,\ldots,w_m, n_j(l,u), n_k(u+1,m)|n_i\right)$$

- Each sum term is a probability of l-m and a rule $n_i\rightarrow n_j n_k$. Thus the maximum a posteriori parse of a string can be computed by dynamic programming.

Most likely parse of a string

- Computation phase: For each l-m and each n_i we can record what continuation rule (and breakpoint u) $r_i^{MaxInside}(l,m)$ will give the highest probability $p_i^{MaxInside}(l,m)$
  - Start from bottom up (unary rules): for each i and l, $p_i^{MaxInside}(l,l)=p(n_i \rightarrow w_l)$
  - Then for each i, l, m:

$$p_i^{MaxInside}(l,m)=max_{j=1,\ldots,N,\ k=1,\ldots,N,\ u=1,\ldots,m-1}\, p_j^{Inside}(l,u)\, p_k^{Inside}(u+1,m)\, p(n_i \rightarrow n_j n_k | n_i)$$

$$r_i^{MaxInside}(l,m)=arg\,max_{j=1,\ldots,N,\ k=1,\ldots,N,\ u=1,\ldots,m-1}\, p_j^{Inside}(l,u)\, p_k^{Inside}(u+1,m)\, p(n_i \rightarrow n_j n_k | n_i)$$

Most likely parse of a string

- The highest-probability parse of the whole string 1-M is then written starting from $r_1^{MaxInside}(1,M)$ corresponding to the probability $p_1^{MaxInside}(1,M)$:
  - Suppose $r_1^{MaxInside}(1,M)=(j,k,u)$, then the first rule of the parse is $n_1(1,M)\rightarrow n_j(1,u)n_k(u+1,M)$
  - Then find the next rules. For the left symbol $n_j(1,u)$, if $r_j^{MaxInside}(1,u)=(j',k',u')$ the rule is $n_j(1,u)\rightarrow n_{j'}(1,u')n_{k'}(u'+1,u)$

    For the right symbol $n_k(u+1,M)$, if $r_k^{MaxInside}(u+1,M)=(\tilde{j},\tilde{k},\tilde{u})$ the rule is $n_k(u+1,M)\rightarrow n_{\tilde{j}}(u+1,\tilde{u})n_{\tilde{k}}(\tilde{u}+1,M)$
  - Continue like this until the substrings are single words l: then $r_i^{MaxInside}(l,l)$ corresponds to the unary rule $n_i(l,l)\rightarrow w_l$ at the bottom of the parse

Properties of PCFGs

- PCFGs can compute the probabilities of different possible parses of a sentence
- PCFGs can be learned from positive data alone (examples of data generated from the grammar) without negative examples (data that does not follow the grammar)
- The probabilistic nature of PCFGs makes it somewhat robust to mistakes and errors in data
- Predictive power can be greater than of a hidden Markov model with the same number of parameters
- However, might still be a worse model than an N-gram which takes local lexical context (nearby words) into account.
- Bias: probability of a smaller parse tree tends to be larger than of a long tree: too much probability for small sentences, whereas real sentences tend to be middle-length.

Learning PCFG probabilities

- The parameters of a PCFG can be learned from data by maximum likelihood, to maximize the probability of observing a set of training data.
- If we had already-parsed sentences, we could get a maximum-likelihood estimate simply by counting from all sentences the fraction of how many times each rule is taken starting from each nonterminal:

$$\hat{p}(n_i \rightarrow S | n_i) = \frac{count(n_i \rightarrow S | n_i)}{\sum_{S_r} count(n_i \rightarrow S_r | n_i)} = \frac{count(n_i \rightarrow S, n_i)}{\sum_{S_r} count(n_i \rightarrow S_r, n_i)} = \frac{count(n_i \rightarrow S, n_i)}{count(n_i)}$$

general version

- When ready-made parses of the training-data sentences are not available, they are **latent variables** that must be estimated during the parameter optimization
- The algorithm that does this is a variant of expectation maximization, and is called the **Inside-Outside algorithm**

Learning PCFG probabilities

- In Chomsky normal form: first, choose the vocabulary, and the number of nonterminals. This defines the set of possible unary and binary rules. Then optimize their parameters (rule probabilities) by maximum likelihood.
- Start by setting the rule probabilities to some initial values
- Idea: we can compute **expected counts of times** that a particular nonterminal, or a particular rule following a nonterminal, is used. Then the maximum likelihood estimate is

$$\hat{p}(n_i \rightarrow n_j n_k | n_i) = \frac{E[count(n_i \rightarrow n_j n_k, n_i)]}{E[count(n_i)]}$$

Chomsky normal form version, binary rules

$$\hat{p}(n_i \rightarrow w_j | n_i) = \frac{E[count(n_i \rightarrow w_j, n_i)]}{E[count(n_i)]}$$

Chomsky normal form version, unary rules

- We need to compute these expected counts given the observed string of words.
- We will first consider just one string, then a data set of multiple strings.

Learning PCFG probabilities

- **Part 1: The denominator.** Recall that

$$p(w_1,\ldots,w_{l-1},w_l,\ldots,w_m,w_{m+1},\ldots,w_M,n_v(l,m)|n_1)=p_v^{Inside}(l,m)\cdot p_v^{Outside}(l,m)$$

$$p(w_1,\ldots,w_M|n_1)=p_1^{Inside}(1,M)$$

- Then the probability nonterminal n_v is used when it would lead to l-m is

$$p(n_v(l,m)|w_1,\ldots,w_{l-1},w_l,\ldots,w_m,w_{m+1},\ldots,w_M,n_1)$$

$$=\frac{p(w_1,\ldots,w_{l-1},w_l,\ldots,w_m,w_{m+1},\ldots,w_M,n_v(l,m)|n_1)}{p(w_1,\ldots,w_{l-1},w_l,\ldots,w_m,w_{m+1},\ldots,w_M|n_1)}=\frac{p_v^{Inside}(l,m)\cdot p_v^{Outside}(l,m)}{p_1^{Inside}(1,M)}$$

- and the expected count of the nonterminal n_v is the sum over all positions where it could appear (all substrings it could lead to):

$$E[count(n_v|w_{1,}\ldots,w_M,n_1)]=\sum_{l=1}^{M}\sum_{m=l}^{M}\frac{p_v^{Inside}(l,m)\cdot p_v^{Outside}(l,m)}{p_1^{Inside}(1,M)}=\frac{\sum_{l=1}^{M}\sum_{m=l}^{M}p_v^{Inside}(l,m)\cdot p_v^{Outside}(l,m)}{p_1^{Inside}(1,M)}$$

Learning PCFG probabilities

- **Part 2: Numerator for binary rules.** The probability of all the words, and that the nonterminal n_v is used in a binary rule leading to nonterminals n_j, n_k and those in turn to words l-u, u+1-m is:

$$p(w_1,\ldots,w_{l-1},w_l,\ldots,w_u,w_{u+1},\ldots,w_m,w_{m+1},\ldots,w_M,n_j(l,u),n_k(u+1,m),n_v(l,m)|n_1)$$

$$=p(w_l,\ldots,w_u,w_{u+1},\ldots,w_m,n_j(l,u),n_k(u+1,m),w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M|n_1)$$

$$\begin{aligned}=&p(w_l,\ldots,w_u|w_{u+1},\ldots,w_m,n_j(l,u),n_k(u+1,m),w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M,n_1)\\ &\cdot p(w_{u+1},\ldots,w_m|n_j(l,u),n_k(u+1,m),w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M,n_1)\\ &\cdot p(n_j(l,u),n_k(u+1,m)|w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M,n_1)\\ &\cdot p(w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M|n_1)\end{aligned}$$

$$\begin{aligned}=&p(w_l,\ldots,w_u|n_j(l,u))\\ &\cdot p(w_{u+1},\ldots,w_m|n_k(u+1,m))\\ &\cdot p(n_j(l,u),n_k(u+1,m)|n_v(l,m))\\ &\cdot p(w_1,\ldots,w_{l-1},n_v(l,m),w_{m+1},\ldots,w_M|n_1)\end{aligned}$$

$$=p_j^{Inside}(l,u)\cdot p_k^{Inside}(u+1,w_m)\cdot p(n_v(l,m)\rightarrow n_j(l,u),n_k(u+1,m)|n_v(l,m))\cdot p_v^{Outside}(l,m)$$

(treatment of binary rule continues on next slides)

Learning PCFG probabilities

- (repeat from previous slide) The probability of all words, and that nonterminal n_v is used in a binary rule leading to nonterminals n_j, n_k and those in turn to words l-u, u+1-m is:

$$p\left(w_1,\ldots,w_{l-1},w_l,\ldots,w_u,w_{u+1},\ldots,w_m,w_{m+1},\ldots,w_M,n_j(l,u),n_k(u+1,m),n_v(l,m)\middle|n_1\right)$$

$$=p_j^{Inside}(l,u)\cdot p_k^{Inside}(u+1,w_m)\cdot p\left(n_v(l,m)\rightarrow n_j(l,u),n_k(u+1,m)\middle|n_v(l,m)\right)\cdot p_v^{Outside}(l,m)$$

- Thus the probability n_v is used leading to l-m, with $n_v \rightarrow n_j n_k$ used as a rule, is a sum over possible breakpoints u:

$$p\left(w_1,\ldots,w_{l-1},w_l,\ldots,w_m,w_{m+1},\ldots,w_M,n_j,n_k,n_v(l,m)\middle|n_1\right)$$

$$=\sum_{u=l}^{m}p_j^{Inside}(l,u)\cdot p_k^{Inside}(u+1,w_m)\cdot p\left(n_v(l,m)\rightarrow n_j(l,u),n_k(u+1,m)\middle|n_v(l,m)\right)\cdot p_v^{Outside}(l,m)$$

(treatment of binary rule continues on next slides)

Learning PCFG probabilities

- Then the conditional probability, given the text, to use n_v leading to l-m with $n_v \rightarrow n_j n_k$ is:

$$p(n_v(l,m), n_j, n_k | w_1,\dots,w_{l-1}, w_l,\dots,w_m, w_{m+1},\dots,w_M, n_1)$$
$$= \frac{p(w_1,\dots,w_{l-1}, w_l,\dots,w_m, w_{m+1},\dots,w_M, n_v(l,m), n_j, n_k | n_1)}{p(w_1,\dots,w_{l-1}, w_l,\dots,w_m, w_{m+1},\dots,w_M | n_1)}$$
$$= \frac{\sum_{u=l}^{m} p_j^{Inside}(l,u)\cdot p_k^{Inside}(u+1, w_m)\cdot p(n_v(l,m)\rightarrow n_j(l,u), n_k(u+1,m) | n_v(l,m))\cdot p_v^{Outside}(l,m)}{p_1^{Inside}(1,M)}$$

- Then the expected number of times n_v is used with $n_v \rightarrow n_j n_k$ is a sum over possible positions:

$$E[count(n_v, n_j, n_k | w_1,\dots,w_M, n_1)]$$
$$= \frac{\sum_{l=1}^{M}\sum_{m=l+1}^{M}\sum_{u=l}^{m} p_j^{Inside}(l,u)\cdot p_k^{Inside}(u+1, w_m)\cdot p(n_v(l,m)\rightarrow n_j(l,u), n_k(u+1,m) | n_v(l,m))\cdot p_v^{Outside}(l,m)}{p_1^{Inside}(1,M)}$$

(treatment of binary rule continues on next slides)

Learning PCFG probabilities

- Now that we have the two kinds of expected counts, we can compute the maximum likelihood estimate for the probability of a binary rule:

$$\hat{p}(n_v \rightarrow n_j n_k | n_v) = \frac{E[count(n_v, n_j, n_k | w_1, \ldots, w_M, n_1)]}{E[count(n_v | w_1, \ldots, w_M, n_1)]}$$

$$= \frac{\sum_{l=1}^{M} \sum_{m=l+1}^{M} \sum_{u=l}^{m} p_j^{Inside}(l,u) \cdot p_k^{Inside}(u+1, w_m) \cdot p(n_v(l,m) \rightarrow n_j(l,u), n_k(u+1,m) | n_v(l,m)) \cdot p_v^{Outside}(l,m)}{\sum_{l=1}^{M} \sum_{m=l}^{M} p_v^{Inside}(l,m) \cdot p_v^{Outside}(l,m)}$$

- Next we will do a similar computation for unary rules.

(treatment of unary rule continues on next slides)

Learning PCFG probabilities

- **Part 3: numerator for unary rules.** The probability of all the words, and n_v is used in a unary rule leading to a word whose dictionary index is w, in position m, is:

$$p\left(w_1,\ldots,w_{m-1},w_m,w_{m+1},\ldots,w_M,n_v(m,m)\rightarrow w,n_v(m,m)\middle|n_1\right)$$

$$=p\left(w_m,n_v(m,m)\rightarrow w,w_1,\ldots,w_{m-1},n_v(m,m),w_{m+1},\ldots,w_M\middle|n_1\right)$$

$$\begin{aligned}&=p\left(w_m\middle|n_v(m,m)\rightarrow w,w_1,\ldots,w_{m-1},n_v(m,m),w_{m+1},\ldots,w_M,n_1\right)\\&\cdot p\left(n_v(m,m)\rightarrow w\middle|w_1,\ldots,w_{m-1},n_v(m,m),w_{m+1},\ldots,w_M,n_1\right)\\&\cdot p\left(w_1,\ldots,w_{m-1},n_v(m,m),w_{m+1},\ldots,w_M\middle|n_1\right)\end{aligned}$$

$$=p\left(w_m\middle|n_v(m,m)\rightarrow w\right)\cdot p\left(n_v(m,m)\rightarrow w\middle|n_v(m,m)\right)\cdot p\left(w_1,\ldots,w_{m-1},n_v(m,m),w_{m+1},\ldots,w_M\middle|n_1\right)$$

$$=\delta(w_m,w)\cdot p\left(n_v\rightarrow w\middle|n_v\right)\cdot p_v^{Outside}(m,m)$$

$$=\delta(w_m,w)\cdot p_v^{Inside}(m,m)\cdot p_v^{Outside}(m,m)$$

- The conditional probability is then:

$$p\left(n_v(m,m)\rightarrow w,n_v(m,m)\middle|w_1,\ldots,w_{m-1},w_m,w_{m+1},\ldots,w_M,n_1\right)$$

$$=\frac{\delta(w_m,w)\cdot p_v^{Inside}(m,m)\cdot p_v^{Outside}(m,m)}{p_1^{Inside}(1,M)}$$

(treatment of unary rule continues on next slides)

Learning PCFG probabilities

- The expected count is then a sum over positions m:

$$E[count(n_v \rightarrow w, n_v | w_1, \ldots, w_M, n_1)] = \frac{\sum_{m=1}^{M} \delta(w_m, w) \cdot p_v^{Inside}(m,m) \cdot p_v^{Outside}(m,m)}{p_1^{Inside}(1,M)}$$

- The probability estimate of the unary rule is then

$$\hat{p}(n_v \rightarrow w | n_v) = \frac{E[count(n_v \rightarrow w, n_v | w_1, \ldots, w_M, n_1)]}{E[count(n_v | w_1, \ldots, w_M, n_1)]} = \frac{\sum_{m=1}^{M} \delta(w_m, w) \cdot p_v^{Inside}(m,m) \cdot p_v^{Outside}(m,m)}{\sum_{l=1}^{M} \sum_{m=l}^{M} p_v^{Inside}(l,m) \cdot p_v^{Outside}(l,m)}$$

(treatment of unary rule continues on next slides)

Learning PCFG probabilities

- When we have multiple strings $W^t=[w_1^{(t)},\ldots,w_{M(t)}^{(t)}],\ \ t=1,\ldots,T$, and we assume they are independently generated, we can take expected counts from each:

$$E_v^{(t)}=E[count(n_v|w_1^{(t)},\ldots,w_{M(t)}^{(t)},n_1)]=\frac{\sum_{l=1}^{M(t)}\sum_{m=l}^{M(t)} p_v^{Inside(t)}(l,m)\cdot p_v^{Outside(t)}(l,m)}{p_1^{Inside(t)}(1,M)}$$

$$E_{v,j,k}^{(t)}=E[count(n_v,n_j,n_k|w_1^{(t)},\ldots,w_{M(t)}^{(t)},n_1)]$$

$$=\frac{\sum_{l=1}^{M(t)}\sum_{m=l+1}^{M(t)}\sum_{u=l}^{m} p_j^{Inside(t)}(l,u)\cdot p_k^{Inside(t)}(u+1,w_m)\cdot p(n_v(l,m)\rightarrow n_j(l,u),n_k(u+1,m)|n_v(l,m))\cdot p_v^{Outside(t)}(l,m)}{p_1^{Inside(t)}(1,M)}$$

$$E_{v,w}^{(t)}=E[count(n_v\rightarrow w,n_v|w_1^{(t)},\ldots,w_{M(t)}^{(t)},n_1)]$$

$$=\frac{\sum_{m=1}^{M(t)}\delta(w_m^{(t)},w)\cdot p_v^{Inside(t)}(m,m)\cdot p_v^{Outside(t)}(m,m)}{p_1^{Inside(t)}(1,M)}$$

where $p_v^{Inside(t)}(l,m)$ and $p_v^{Outside(t)}(l,m)$ are inside and outside probabilities computed for string t.

(treatment of multiple strings continues on next slides)

Learning PCFG probabilities

- Then the probability estimates for binary and unary rules are again ratios of expected counts, but the numerator and denominator are summed over all the strings:

$$\hat{p}(n_v \rightarrow w | n_v) = \frac{\sum_{t=1}^{T} E[count(n_v \rightarrow w, n_v | w_1^{(t)}, \ldots, w_{M(t)}^{(t)}, n_1)]}{\sum_{t=1}^{T} E[count(n_v | w_1^{(t)}, \ldots, w_{M(t)}^{(t)}, n_1)]} = \frac{\sum_{t=1}^{T} E_{v,w}^{(t)}}{\sum_{t=1}^{T} E_v^{(t)}}$$

$$\hat{p}(n_v \rightarrow n_j n_k | n_v) = \frac{\sum_{t=1}^{T} E[count(n_v, n_j, n_k | w_1^{(t)}, \ldots, w_{M(t)}^{(t)}, n_1)]}{\sum_{t=1}^{T} E[count(n_v | w_1^{(t)}, \ldots, w_{M(t)}^{(t)}, n_1)]} = \frac{\sum_{t=1}^{T} E_{v,j,k}^{(t)}}{\sum_{t=1}^{T} E_v^{(t)}}$$

- The right-hand side of these rules depend on previous estimates of the rule probabilities, to give a new estimate.
- The updates must be repeated iteratively many times (recomputing the expectations each time), until the change in the rule probabilities is small enough. This is the **inside-outside algorithm**.

Learning PCFG probabilities

- Properties of the inside-outside algorithm:
  - It is guaranteed that the algorithm **never decreases the total likelihood** (product of probabilities of the observed strings 1-T): each iteration either increases the likelihood or keeps it the same.
  - It is **not guaranteed to converge to a global maximum** of the likelihood, only to a local maximum which depends on the initialization of the probabilities.
    - There are some general approaches in machine learning, such as simulated annealing, that aim to more comprehensively search for a good local maximum.
  - It has a **high computational cost**: for each sentence t the cost of each iteration is $O\left(M^3(t)N^3\right)$ where M(t) is the length of the sentence.
  - It is not guaranteed that resulting nonterminals and rules resemble those typically used to describe the language (e.g. that there would be nonterminals corresponding to noun phrases, verb phrases etc.)
- Some observations:
  - Charniak, 1993: in an experiment with artificial data, each of 300 trials found a different local optimum (sensitive to initialization)
  - Lari and Young, 1990: good results may require more nonterminals than are theoretically needed to describe the language. In artificial data generated with N nonterminals they needed 3*N nonterminals to learn the grammar.