**Bayesian Analysis I**, PRACTICE problems

1. A major earthquake (exceeding 6.7 on the Richter scale) occurs immediately after one of three geological activities: a degassing burst (DB), an episode of intense air ionisation (IAO), or a continuous wave of magnetic pulsations (MP), all of which can be considered disjoint events.

   The table below shows the number of recent major earthquakes in San Francisco that have occurred following each of these three geological activities:

   | Geological activity | Number of earthquakes |
   |---|---|
   | DB | 6 |
   | IAO | 3 |
   | MP | 1 |

   Past studies show that a major earthquake in San Francisco will occur with probability 0.5, 0.3 and 0.2 if DB, IAO and MP are observed (respectively).

   a) Find the probability that a major earthquake will occur in San Francisco.

   b) Suppose an earthquake of magnitude 7.1 occurs in San Francisco. Find the posterior probability distribution for the geological activities. Express your answers in the form $l/41, m/41$ and $n/41$, where $l, m$ and $n$ are to be found.

2. Suppose that we have a random sample $X_i, i = 1, 2, ..., 50$ from a distribution with the following probability density function:

   $$f(x|\theta) = \frac{\theta}{x^2} e^{-\theta/x} \quad x, \theta > 0$$

   a) Find a conjugate prior distribution for $\theta$.

   b) Identify its resulting posterior distribution for $\theta$.

   c) Find the posterior mean given the sample.

3. Let $Y$ have binomial distribution with parameters $n = 3$ and $\theta$, and the unknown parameter $\theta$ is assumed to follow a prior distribution

   $$p(\theta) = \frac{27 - 75(\theta - 0.4)^2}{22} \quad 0 \le \theta \le 1$$

   Calculate the posterior distribution exactly when $Y = 2$.

4. We have $n = 9$ observations with sample mean $\overline{Y} = 12$ and are willing to assume that the observations are independent with $Y_i|\mu \sim \text{Normal}(\mu, 1/\psi)$, where $\psi$ is a known precision. The distribution for $\mu$ is assumed to be $\text{Normal}(\theta, \sigma^2/m)$ a priori.

   a) Derive the posterior distribution for $\mu$ given $Y_1, ..., Y_n$.

   b) Let $\psi = 1$. Assuming an improper flat prior by setting $\sigma^2 = 1, m = 0$, give a posterior 95% equal-tailed credible interval for $\mu$.

   c) How does a Bayesian credible interval for $\mu$ differ from a frequentist 95% confidence interval for $\mu$?

5. Suppose that we have a random sample $X_i, i = 1, ..., n$ from a distribution with the probability density function

$$f(x|\theta) = \left(\frac{\theta}{2\pi}\right)^{1/2} \exp\left(\frac{-\theta}{2}x^2\right) \quad \theta > 0.$$

Taking a Gamma$(a/2, 2/b)$ as a prior distribution, find the density of the predictive distribution of another independent observation $y$ given $X_1, ..., X_n$.