Lab-1 4 Nov 2024

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$\boxed{01}$ $Y/\pi \sim \text{Binomial}(n=4, \pi)$

$P(\pi=0.4) = P(\pi=0.5) = P(\pi=0.6) = \frac{1}{3}$

Observed value of $Y=3$

Part a) Find the Likelihood of $Y=3$

$P(Y=3/\pi) = \binom{4}{3}\pi^3(1-\pi)^{4-3}$

1. $\pi=0.4$

$P(Y=3/\pi=0.4) = \binom{4}{3}(0.4)^3(1-0.4)^1$

$= 4 \times 0.064 \times 0.6$

$= 0.1536$

2. $\pi=0.5$

$P(Y=3/\pi=0.5) = \binom{4}{3}(0.5)^3(1-0.5)^1$

$= 4 \times (0.5)^3 (0.5)$

$= 0.25$

3. $\pi = 0.6$

$P(Y=3/\pi=0.6) = \binom{4}{3}(0.6)^3(1-0.6)$

$= 4 \times (0.6)^3 (0.4)$

$= 0.3456$

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b) Obtain the posterior probability of π given $Y=3$

① $$P(\pi / Y=3) = \frac{P(Y=3|\pi)P(\pi)}{\sum P(Y=3|\pi')P(\pi')}$$

$$P(\pi=0.4) = P(\pi=0.5) = P(\pi=0.6) = 1/3$$

$$\sum P(Y=3|\pi)P(\pi) = \frac{1}{3}(0.1536 + 0.25 + 0.3456)$$
$$= \frac{1}{3} \times 0.7492 = 0.24973$$

② $\pi = 0.4$

$$P(\pi=0.4 / Y=3) = \frac{0.1536 \times \frac{1}{3}}{0.24973} \approx 0.205$$

$\pi = 0.5$

$$P(\pi=0.5 / Y=3) = \frac{0.25 \times \frac{1}{3}}{0.24973} \approx 0.334$$

$\pi = 0.6$

$$P(\pi=0.6 / Y=3) = \frac{0.3456 \times \frac{1}{3}}{0.24973} \approx 0.461$$

c) $n=4$ trials & $Y=3$

$$P(Y=3|\pi) = \pi^3(1-\pi)$$

$\pi = 0.4$ $P(Y=3/\pi=0.4) = (0.4)^3 \times (1-0.4)$

002 $= 0.0384$

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$\pi = 0.5$

$$P(Y=3/\pi = 0.5) = (0.5)^3 \times (1-0.5)$$

$$= 0.125 \times 0.5 = 0.0625$$

$\pi = 0.6$

$$P(Y=3/\pi = 0.6) = (0.6)^3 \times (1-0.6) = 0.216 \times 0.4$$

$$= 0.0864$$

Posterior $\pi = 0.4$

$$P(\pi = 0.4 / Y=3) = \frac{0.0384 \times \frac{1}{3}}{0.06243} \simeq 0.205$$

$$P(\pi = 0.5 / Y=3) = \frac{0.0625 \times \frac{1}{3}}{0.06243} = 0.333$$

$$P(\pi = 0.6 / Y=3) = \frac{0.0864 \times \frac{1}{3}}{0.06243} = 0.462$$

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2) a) $X/\theta \sim \text{Poisson}(\theta)$

$$P(X = x/\theta) = \frac{\theta^{x} e^{-\theta}}{x!} \quad x = 0, 1, 2 \ldots$$

kernel $\theta^{x} e^{-\theta}$.

b) ~~X/θ~~ $Y/\theta, \beta \sim \text{Beta}(\beta\theta, \beta)$

$$f(y/\theta; \beta) = \frac{\Gamma(\beta\theta + \beta)}{\Gamma\beta\theta \, \Gamma\beta} y^{\beta\theta - 1} (1-y)^{\beta - 1}$$

kernel $y^{\beta\theta} (1-y)^{\beta - 1}$

c) $\theta / \alpha, \beta, x \sim \text{Gamma}(\alpha + x + 1, \, 1/(\beta - 3x))$

$$f(\theta/\alpha, \beta, x) = \frac{(\beta - 3x)^{\alpha + x + 1}}{\Gamma(\alpha + x + 1)}$$

$$\propto \theta^{\alpha + x} e^{-\theta(\beta - 3x)}$$

$\theta > 0$

kernel $\theta^{\alpha + x} e^{-\theta(\beta - 3x)}$

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d) $f(\phi | \mu, \eta, \tau) = \frac{1}{\sqrt{2\pi \eta^2 \tau^{-2}}}$

$\times \exp\left(-\frac{(\phi - (\tau\mu + (1-\tau)\eta))^2}{2\eta^2\tau^{-2}}\right)$

Kernel $\exp\left(-\frac{(\theta - \tau\mu + (1-\tau)\eta))^2}{2\eta^2\tau^{-2}}\right)$

Problem-3

a) $f(x) = 2e^{-2x} \quad x \geq 0$

$X \sim \text{exponential}(2)$

b) $f(x) \propto 1, \quad 0 \leq x < 1$

So $X \sim$ uniform $(0, 1)$

c) $f(y | \mu) \propto \frac{\mu^y}{y!} \quad y = 0, 1, 2 \ldots \quad \mu > 0$

$P(Y = y | \mu) = \frac{\mu^y e^{-\mu}}{y!}$

So $Y \sim$ poisson(μ)

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d) $f(y|\alpha,\beta) \propto y^{-\alpha-1} e^{-1/(2\beta y)}$, $y>0$

$\alpha>1$

$$f(y|\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma\alpha} y^{-\alpha-1} e^{-\beta/y}$$ $\beta>0$

$Y \sim$ Inverse Gamma(α,β) $y>0$

e) $f(x|m) = \text{constant}\ (1-x)^{m-1/2}$

So $x \sim$ Beta$(1, (m+1)/2)$ $0 \leq x \leq 1$

f) $f(z|\theta,\phi) \propto \exp\left(-\frac{\theta}{\phi} z^2 + (\phi+1)z\right)$

$\theta>0$ $\phi>0$

$z \sim$ Normal$\left(\frac{\phi(\phi+1)}{2\theta}, \frac{\phi}{2\theta}\right)$

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Problem - 4

a) $f(x, \theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{x_i - m)^2}{2\theta}\right.$

$L(\theta | x) = \frac{1}{(2\pi\theta)^{n/2}} \exp\left(-\frac{1}{2\theta}\sum_{i=1}^{n} x_i - m)^2\right.$

b) $P(X_i = x_i / \theta) = \frac{\theta^{x} e^{-\theta}}{x_i!}$

$L(\theta | x) = \prod_{i=1}^{18} P(X_i = x_i / \theta)$

$L(\theta | x) = \frac{\theta^{\sum_{i=1}^{18} x_i} e^{-18\theta}}{\prod_{i=1}^{18} x_i!}$

c) $P(x | \alpha, \beta) = \begin{cases} \alpha & x = 0 \\ (1-\alpha)\beta(1-\beta)^{x-1} & x = 1, 2 \end{cases}$

$L(\alpha, \beta / x) = \alpha(1-\alpha)^{n-n} \beta^{n-n}$

$(1-\beta)^{\sum (x-1)}$

Problem-5

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a) $\sum_{y=0}^{\infty} \frac{m^y}{y!} = e^m$

b) $B(\alpha, \beta)$

$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$

$= B(\alpha, \beta) = \frac{\Gamma\alpha \, \Gamma\beta}{\Gamma(\alpha+\beta)}$

c) $\int_0^1 \frac{1}{2}(\beta+3)(\beta+2)(\beta+1) x(1-x)^{\beta} dx$

$\int_0^1 \frac{1}{2}(\beta+3)(\beta+2)(\beta+1) x(1-x)^{\beta} dx$

$= \frac{(\beta+3)(\beta+2)(\beta+1)}{2(\beta+1)(\beta+2)(\beta+3)} = \frac{1}{2}$

d) $\int_{-\alpha}^{\alpha} 4m(\alpha-1)\left(\frac{\sigma^2+\tau^2}{\sigma\tau}\right)^{1/2}$

$\exp - \frac{\sigma^2+\tau^2}{\sigma\tau}(\theta-m)^2) \, d\theta$

Thus, $\int_{-\alpha}^{\alpha} 4m(\alpha-1)\sqrt{\frac{\sigma^2+\tau^2}{\sigma\tau}} \times \exp\left(-\frac{\sigma^2+\tau^2}{\sigma\tau}\right.$

$(\theta-m)^2 \, d\theta$

$= 4m(\alpha-1)\sqrt{\pi}$