3. (a) In case of generalized linear model $g(\mu_i) = \beta_0 + \beta_1 x_i$, find the inverse function $g^{-1}$ (that is, solve what form the expected value $\mu_i$ has), when the link function $g$ is

i. $\sqrt{\mu_i} = \beta_0 + \beta_1 x_i$,
**Solution:**

$$\sqrt{\mu_i} = \beta_0 + \beta_1 x_i \Longrightarrow \mu_i = (\beta_0 + \beta_1 x_i)^2 .$$

ii. $\frac{1}{\mu_i^2} = \beta_0 + \beta_1 x_i$,
**Solution:**

$$\frac{1}{\mu_i^2} = \beta_0 + \beta_1 x_i \Longrightarrow \mu_i^2 = \frac{1}{\beta_0 + \beta_1 x_i} \Longrightarrow \mu_i = \frac{1}{\sqrt{\beta_0 + \beta_1 x_i}}.$$

iii. $\log\left(\frac{\mu_i}{1-\mu_i}\right) = \beta_0 + \beta_1 x_i$.
**Solution:**

$$\begin{aligned}\log\left(\frac{\mu_i}{1-\mu_i}\right) = \beta_0 + \beta_1 x_i &\Longrightarrow \frac{\mu_i}{1-\mu_i} = \exp(\beta_0 + \beta_1 x_i) \\ &\Longrightarrow \mu_i + \mu_i \exp(\beta_0 + \beta_1 x_i) = \exp(\beta_0 + \beta_1 x_i) \\ &\Longrightarrow \mu_i = \frac{\exp(\beta_0 + \beta_1 x_i)}{1 + \exp(\beta_0 + \beta_1 x_i)}\end{aligned}$$

(2 points)

(b) Let us assume $Y_i \sim IG(\mu_i, \phi)$. Consider the model

$$\log(\mu_i) = \beta_0 + \beta_1 \log(x_i).$$

Let the estimates of the parameters $\beta_0, \beta_1, \phi$ be as $\hat{\beta}_0 = 1, \hat{\beta}_1 = 0.5, \tilde{\phi} = 0.05$.

i. Calculate the maximum likelihood estimate for the expected value $\mu_i$ when $x_i = 5$.
**Solution:**
The model $\log(\mu_i) = \beta_0 + \beta_1 \log(x_i)$ means that the expected value has the form

$$\mu_i = e^{[\beta_0 + \beta_1 \log(x_i)]} = e^{\beta_0} x_i^{\beta_1}.$$

Thus the maximum likelihood estimate of $\mu_i$ is

$$\hat{\mu}_i = e^{\hat{\beta}_0} x_i^{\hat{\beta}_1} = e^1 \cdot 5^{0.5} = 6.078263.$$

ii. Calculate the Pearson residual

$$o_i = \frac{y_i - \hat{\mu}_i}{\sqrt{\widehat{\mathrm{Var}}(Y_i)}},$$

when $x_i = 5$ and the observed value is $y_i = 12$.

**Solution:**

Under Inverse Gaussian distribution $\mathrm{Var}(Y_i) = \phi\mu_i^3$. Hence

$$\widehat{\mathrm{Var}}(Y_i) = \tilde{\phi}\hat{\mu}_i^3 = 0.05 \cdot 6.078263^3 = 11.22816,$$

and

$$o_i = \frac{y_i - \hat{\mu}_i}{\sqrt{\widehat{\mathrm{Var}}(Y_i)}} = \frac{12 - 6.078263}{\sqrt{11.22816}} = 1.767237.$$

(2 points)

(c) In generalized linear models, the likelihood equations can written in form

$$\frac{\partial l(\boldsymbol{\beta}, \phi)}{\partial \beta_j} = \sum_{i=1}^{n} \frac{(y_i - \mu_i)}{\mathrm{Var}(Y_i)} \cdot x_{ij} \cdot \left(\frac{\partial \mu_i}{\partial \eta_i}\right) = 0, \qquad j = 0, 1, 2 \ldots p.$$

Consider now the simple Gamma model with

$$\begin{aligned} Y_i &\sim \mathit{Gamma}(\mu_i, \phi), \\ \mu_i &= \eta_i = \beta_0. \end{aligned}$$

What kind of more simplified form the likelihood equations have in this case? That is, what form $\frac{\partial l(\beta_0)}{\partial \beta_0}$ has in the simple Gamma model? By using the likelihood equations, find the maximum likelihood estimator $\hat{\beta}_0$.

(2 points)

**Solution:**

Since $\mathrm{E}(Y_i) = \mu_i = \beta_0, \mathrm{Var}(Y_i) = \phi\mu_i^2 = \phi\beta_0^2, x_{i0} = 1$ and also $\frac{\partial \mu_i}{\partial \eta_i} = 1$, we have

$$\begin{aligned} \frac{\partial l(\boldsymbol{\beta}, \phi)}{\partial \beta_0} &= \sum_{i=1}^{n} \frac{(y_i - \mu_i)}{\mathrm{Var}(Y_i)} x_{i0} \left(\frac{\partial \mu_i}{\partial \eta_i}\right) \\ &= \sum_{i=1}^{n} \frac{(y_i - \mu_i)}{\phi\mu_i^2} \cdot 1 \cdot 1 = \sum_{i=1}^{n} \frac{(y_i - \beta_0)}{\phi\beta_0^2} = \frac{1}{\phi\beta_0^2} \sum_{i=1}^{n} (y_i - \beta_0). \end{aligned}$$

Because $Y_i > 0$, also $\mu_i = \beta_0 > 0$ and hence $\frac{\partial l(\boldsymbol{\beta}, \phi)}{\partial \beta_0} = 0$ only if $\sum_{i=1}^{n} (y_i - \beta_0) = 0$, i.e., only if

$$\sum_{i=1}^{n} (y_i - \beta_0) = \left(\sum_{i=1}^{n} y_i\right) - n\beta_0 = 0.$$

Hence it should hold for the solution $\hat{\beta}_0$ as

$$\begin{aligned} \left(\sum_{i=1}^{n} y_i\right) - n\hat{\beta}_0 &= 0, \\ -n\hat{\beta}_0 &= -\left(\sum_{i=1}^{n} y_i\right), \\ \hat{\beta}_0 &= \frac{\left(\sum_{i=1}^{n} y_i\right)}{n} = \bar{y}. \end{aligned}$$