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1) a) $n = 50$

Yes response $\cancel{6}$ (y) $= 37$

μ prior $= 0.5$

σ^2(prior) $= 1/16$

μ prior $= \frac{a}{a+b}$, σ^2 prior $= \frac{ab}{(a+b)^2(a+b+1)}$

From the prior mean

$$0.5 = \frac{a}{a+b} \Rightarrow a \Rightarrow b$$

$$0.0625 = \frac{a^2}{(2a^2)(2a+1)} = \frac{1}{4a+2}$$

$$4a+2 = \frac{1}{0.0625} = 16$$

$$4a = 14 \Rightarrow a = 3.5 \quad b = 3.5$$

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b) $P(\theta < 0.6) = F(0.6, a = 3.5, b = 3.5)$

R pbeta(0.6, 3.5, 3.5)

0.6970499

c) The posterior distn is a Beta distn,

$\theta \sim Beta(a_{post}, b_{post})$

$a_{post} = a + x$, $b_{post} = b + n - x$

$= 3.5 + 37 =$

$b_{post} = 3.5 + 50 - 37 = 16.5$

$\theta \sim Beta(40.5, 16.5)$

d) $\mu_{post} = \dfrac{a_{post}}{a_{post} + b_{post}}$

$$\sigma^2_{post} = \frac{a_{post} b_{post}}{(a_{post} + b_{post})^2 (a_{post} + b_{post} + 1)}$$

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$$m_{post} = \frac{40.5}{40.5 + 16.5} = \frac{40.5}{57} \approx 0.7105$$

$$\sigma_{post} \approx \sqrt{0.00355} \approx 0.0592$$

Posterior mean $m_{post} \approx 0.7105$

$\sigma_{post} \approx 0.0592$

e)

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f) $P(\theta < 0.6) = F(0.6, \alpha_{post} = 40.5$

pbeta$(0.6, 40.5, 16.5)$ $b_{post} = 16.5)$

0.0382

#2) a)

$\mu \mid \text{data} \sim N(m_n, \tau_n^2)$

$\tau_n^2 = \left(\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}\right)^{-1}$

$m_n = \tau_n^2\left(\frac{m_0}{\tau_0^2} + \frac{n\bar{x}}{\sigma^2}\right)$

$n = 5, \ \bar{x} = \frac{78+80+95+87+72}{5}$

$= 82.2$

$\sigma^2 = 81, \quad m_0 = 70, \ \tau_0^2 = 24.01$

$\tau_n^2 = \left(\frac{1}{24.01} + \frac{5}{81}\right)^{-1}$

≈ 9.09

0.09

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$$m_n = 9.04\left(\frac{70}{29.01} + \frac{5\times 82.2}{82}\right)$$

$$= 72.2$$

μ | data $\sim N(72.2,\ 9.04)$

The 95% highest posterior density interval is

$$m_n \pm 1.96\sqrt{\tau_n^2}$$

$$72.2 \pm 1.96\sqrt{9.04}$$

$$72.2 \pm 5.88$$

$$HPDI = (66.32,\ 78.08)$$

c)

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d) $X_{new} \sim N(m_n, \sigma^2 + \tau_n^2)$

$m_n = 72.2 \quad \tau_n^2 = 9.09 \quad \sigma^2 = 87$

$V(X_{new}) = \tau_n^2 + \tau_n^2$

$= 9.09 + 87 = \cancel{90}.09$

$X_{new} \sim N(72.2, \sqrt{\cancel{9}\ldots})$

Predictive interval $(53.61, \cancel{90}.8)$

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~~#~~ 3) a)

The conjugate prior for λ is the Gamma distⁿ

$$\lambda \sim \text{Gamma}(\alpha, \beta)$$

mean, $\alpha/\beta = 5$

variance, $\alpha/\beta^2 = (0.25)^2 = 0.0625$

$$\alpha = \frac{m^2}{\sigma^2}, \quad \beta = \frac{m}{\sigma^2}$$

$$\alpha = \frac{5^2}{0.0625} = 400$$

$$\beta = \frac{5}{0.0625} = 80$$

$$\lambda \sim \text{Gamma}(400, 80)$$

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3) b)

$$P(\lambda > 7) = 1 - F(7, \alpha = 400, \beta = 80)$$

$$P(\lambda > 7) \simeq 0.00195$$

3) c)

The likelihood function for a poisson ~~distrib~~ is

$$L(\lambda) \propto \lambda^{49} e^{-6\lambda}$$

$$L(\lambda) \propto \lambda^{49} e^{-6\lambda}$$

$$\alpha_{post} = \alpha + \Sigma x = 400 + 49 = 449$$

$$\beta_{post} = \beta + n = 80 + 6 = 86$$

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#37 d) $\mu_{prior} = \frac{\alpha}{\beta} = \frac{400}{80} = 5$

$\sigma^2_{prior} = \frac{\alpha}{\beta} = \frac{400}{80^2}$

$= 0.0625$

Posterior mean and variance

$\mu_{post} = \frac{\alpha_{Post}}{\beta_{post}} = \frac{449}{86}$

≈ 5.221

$\sigma^2_{Post} = \frac{\alpha_{Post}}{\beta^2_{Post}} = \frac{449}{86^2}$

≈ 0.0606

e) Shift in Mean

- The posterior mean $\mu_{post} \approx 5.22$ is slightly higher than prior mean (5) reflecting the influence of observed data (average count $49/6 \approx 8.1670$) on the prior belief.

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- Reduced variance

The posterior variance is slightly smaller than full prior variance indicating greater certainty about λ after incorporating data.

- Likelihood Dominance

The posterior distn combines the prior with the likelihood centering closer to the observed data while still reflecting prior information

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4

a)

$\theta \sim \text{Gamma}(\alpha, \beta)$ $\quad \alpha = 1$
$\beta = 1$

$$L(\theta) \propto \theta^{n} e^{-\theta \sum x_i}$$

$$\theta \mid x \sim \text{Gamma}(\alpha + n, \beta + \sum x_i)$$

$n = 12$

$\sum x_i = 3 + 30 + \dots + 80 = 759$

Posterior $\alpha_{post} = \alpha + n$
$= 1 + 12 = 13$

$\beta_{post} = \beta + \sum x_i$
$= 1 + 759 = 760$

Posterior $\theta \mid x \sim \text{Gamma}(13, 760)$

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#4) c)

Prior Distribution: The prior Gamma(4,1) is highly dispersed, representing initial uncertainty about θ.

Posterior distribution

The posterior Gamma(13, 760) is more concentrated, reflecting reduced uncertainty after incorporating the data.

Central tendency

The posterior mean is closer to the observed data and more reliable due to the added weight of the sample information.

0/2

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4 d) $\sum x_i = 26 + \cdots + 15 = 3203$

$n = 12 + 12 = 24$

$\sum x_i = 789 + 3203 = 3692$

$\alpha_{post} = \alpha + n = 1 + 24 = 25$

$\beta_{post} = \beta + \sum x_i = 1 + 3962$

$= 3963$

$\theta \mid x \sim \text{Gamma}(25, 3963)$