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001

a) $\bar{Y}_1 = 12.42 \quad \bar{Y}_2 = 12.27$

$S.D = \sigma_1 = \sigma_2 = 0.12$

Difference in means $\theta \sim N(12.42 - 12.27,$
$0.1^2(\frac{1}{9} + \frac{1}{12}))$

90% HPI for θ

$0.15 \pm 1.64\sqrt{0.0019} = (0.077, 0.223)$

b) Prior $NIC(-2.0, m, \sigma - 1)$

$P^* = 19 \quad \sigma^* = 0.1965 \;\&$

Posterior $\theta \sim t_{19}(0.15, 0.00201)$

90% HPI for θ.

$0.15 \pm 1.729\sqrt{\frac{0.1965}{17}(\frac{1}{9} + \frac{1}{12})}$

$\Rightarrow (0.1068, 0.232)$

001

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Q2

Joint prior $P(\mu, T) = P(\mu|T)P(T)$

$\mu|T \sim N(\delta, T/s)$, $T \sim IG(a, b)$

1. Posterior ~~dist~~ derivation

$P(\mu, T|x) \propto P(x|\mu, \tau) P(\mu|T) P(T)$

Combining terms and simplifying

Posterior $\mu|T, x \sim N(\delta^*, T/s^*)$

$s^* = s+n$, $\delta^* = \dfrac{\delta s + n\bar{x}}{s+n}$

Posterior $\tau|x \sim IG(a^*, b^*)$

$a^* = a+n$, $b^* = b + s + \dfrac{sn}{s+n}(\bar{x}-\delta)^2$

Posterior

$\mu|T, x \sim N\left(\dfrac{n\bar{x}}{n+s}, \dfrac{T}{n+s}\right)$

$T|x \sim IG\left(a+n, b+s+\dfrac{ns}{n+s}(\bar{x}-\delta)^2\right)$

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003 — Given $y|\theta \sim N(A\theta, V)$ with known A & V.

Prior $P(\theta) \simeq$ constant

Posterior Derivation:

$$P(\theta|y) \propto P(y|\theta)P(\theta) \propto \exp\left(-\frac{1}{2}(y - A\theta)^T V^{-1}(y - A\theta)\right)$$

$$Q = (y - A\theta)^T V^{-1}(y - A\theta)$$

$$Q \propto \theta^T A^T V^{-1} A\theta - 2\theta^T A^T V^{-1} y.$$

Terms $P(\theta|y) \sim N(m^*, V^*)$

where

$$(V^*)^{-1} = A^T V^{-1} A \text{ or } V^* = (A^T V^{-1} A)^{-1}$$

$$m^* = V^* A^T V^{-1} y.$$

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004

a) $y_i \sim \theta$

$y_1|\theta, \gamma \sim N(\theta+\gamma, 1)$, $y_2|\theta, \gamma \sim N(\theta-\gamma, 1)$

$$\text{Prior}(\theta,\gamma) \sim N\left(\begin{bmatrix}0\\m\end{bmatrix}, \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\right)$$

Combine prior and likelihood

$$\theta, \gamma|Y \sim N(Bb, B)$$

where

$$B = (A_1^T C_1^{-1} A_1 + C_2^{-1})^{-1}$$

$$b = A_1^T C_1^{-1} y + C_2^{-1} A_2 \theta_2$$

Thus, $$(\theta, \gamma|Y) \sim N\left(\begin{Bmatrix}\frac{3}{7}(y_1+y_2)\\ \frac{3}{7}\left(y_1-y_2+\frac{m}{3}\right)\end{Bmatrix}, \begin{bmatrix}\frac{3}{7} & 0\\ 0 & \frac{3}{7}\end{bmatrix}\right)$$

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b) Three stage linear model

$Y/\theta_1 \sim N(A_1\theta_1, C_1)$

$\theta/\theta_2 \sim N(A_2\theta_2, C_2)$

$\theta_2 \sim N(\mu, C_3)$

~~Given~~ marginal distribution of θ_1

$$\theta_1 \sim N(A_2\mu, C_2 + A_2 C_2 A_2^T)$$

Posterior Distn of θ_1/Y

using Lindley - Smith theorem:

$$\theta_1/Y \sim N(Bb, B)$$

where $B^{-1} = [A_1^T C_1^{-1} A_1 + C_2 + A_2 C_3 A_2^T]^{-1}$

$b = [A_1^T C_1^{-1} Y + C_2 + A_2 C_3 A_2^T]^{-1}$

$A_2\mu$.

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~~Q#5~~

~~#~~#5

$$y = (y_1, y_2, y_3) = (666, 857, 114)$$

$n = 163 \quad \theta, (\theta_1, \theta_2, \theta_3)$

a) Posterior distribution

$$P(\theta|y) \propto \theta_1^{y_1} \theta_2^{y_2} \theta_3^{y_3}$$

$\sim$ Dirichlet $(y_1 + 1, y_2 + 1, y_3 + 1)$

Thus, $\theta|y \sim$ Dirichlet $(667, 852, 115)$

b) Posterior mean

$$E(\theta_1 - \theta_2 | y) = \frac{\alpha_1}{\alpha_0} - \frac{\alpha_2}{\alpha_0} = \frac{667 - 852}{1639}$$

where $\alpha_0 = 667 + 852 + 115 = 1639 \quad = -0.113$

variance $\theta_1 - \theta_2$

$$var(\theta_1 - \theta_2|y) = v[\theta_1|y] + var(\theta_2|y] - 2cov(\theta_1, \theta_2|y)$$

$$var(\theta_j] = \frac{\alpha_j(\alpha_0 - \alpha_j)}{\alpha_0^2(\alpha_0 + 1)}$$

$$cov(\theta_i, \theta_j] = \frac{-\alpha_i \alpha_j}{\alpha_0^2(\alpha_0 + 1)}$$

Simplifying

$$var[\theta_1 - \theta_2 | y] = \frac{852(667+115) - 2\times 667\times 852}{}$$

$$= \frac{852(667+115) + 667(852+115)}{1634^2 \times 1635}$$

$$= 5.607 \times 10^{-9}$$

c) 95% Posterior credible interval

$$E(\theta_1 - \theta_2 | y) \pm 1.96 \pm \sqrt{var(\theta_1 - \theta_2 | y)}$$

$$\Rightarrow (-0.160, -0.067)$$

6) #

a) Joint posterior $P(\theta, \alpha, \beta | y)$

Likelihood

$$P(y|\theta) \propto \prod_{j=1}^{J} \theta_j^{y_j} - e^{-\theta}$$

Prior $P(\theta_j | \alpha; \beta) \propto \theta_j^{\alpha-1} e^{-\theta_j \beta}$

Joint posterior

$$P(\theta, \alpha, \beta | y) \propto \frac{1}{\alpha\beta} \prod_{j=1}^{J} \theta_j^{y_j+\alpha-1} \times e^{-\theta_j(1+\beta)}$$

b) Marginal conditional density

$p(\theta | \alpha, \beta, Y)$;

Form de posterior

$p(\theta_j | \alpha, \beta, y_j)$

$\propto \theta_j^{\,y_j+\alpha-1} e^{-\theta_j(1+\beta)}$

$\theta_j | \alpha, \beta, Y \sim \text{Gamma}(y_j+\alpha, 1+\beta)$

c) $p(\alpha, \beta | Y) \propto \frac{1}{\alpha\beta} \prod_{j=1}^{J} \frac{\Gamma(y_j+\alpha)}{\Gamma\alpha} (1+\beta)^{-(y_j+\alpha)}$