3. (a) In case of generalized linear model $g(\mu_i) = \beta_0 + \beta_1 x_i$, the maximum likelihood estimates for the parameters β_0 and β_1 are $\hat{\beta}_0 = 1$ and $\hat{\beta}_1 = 0.5$. At the value $x_i = 5$, calculate the maximum likelihood estimate of μ_i, when the model is

i. $Y_i \sim Poi(\mu_i)$ and $\log(\mu_i) = \beta_0 + \beta_1 x_i$,
Solution:

$$\hat{\mu}_i = \exp(\hat{\beta}_0 + \hat{\beta}_1 x_i) = \exp(1 + 0.5 * 5) = \exp(3.5) = 33.11545.$$

ii. $Y_i \sim Poi(\mu_i)$ and $\sqrt{\mu_i} = \beta_0 + \beta_1 x_i$,
Solution:

$$\hat{\mu}_i = (\hat{\beta}_0 + \hat{\beta}_1 x_i)^2 = (1 + 0.5 * 5)^2 = (3.5)^2 = 12.25.$$

iii. $Y_i \sim Poi(\mu_i)$ and $\log\left(\frac{\mu_i}{t_i}\right) = \beta_0 + \beta_1 x_i$, where $t_i = 10$.
Solution:

$$\begin{aligned}\hat{\mu}_i &= \exp(\log(t_i) + \hat{\beta}_0 + \hat{\beta}_1 x_i) = \exp(\log(10) + 1 + 0.5 * 5) \\ &= \exp(5.802585) = 331.1545.\end{aligned}$$

(2 points)

(b) In generalized linear models, the likelihood equations can written in form

$$\frac{\partial l(\boldsymbol{\beta}, \phi)}{\partial \beta_j} = \sum_{i=1}^{n} \frac{y_i - \mu_i}{\mathrm{Var}(Y_i)} x_{ij} \left(\frac{\partial \mu_i}{\partial \eta_i}\right) = 0, \qquad j = 0, 1, 2 \ldots p.$$

Consider now the most simplest Poisson model with the identity link function

$$\begin{aligned} &Y_i \sim Poi(\mu_i), \\ &\mu_i = \eta_i = \beta_0. \end{aligned}$$

What kind of more simplified form the likelihood equations have in this case? That is, what form $\frac{\partial l(\beta_0)}{\partial \beta_0}$ has in the simplest Poisson model? By using the likelihood equations, find the maximum likelihood estimator $\hat{\beta}_0$.

(2 points)

Solution:

Since $\mathrm{E}(Y_i) = \mu_i, \mathrm{Var}(Y_i) = \mu_i, x_{i0} = 1$ and also $\frac{\partial \mu_i}{\partial \eta_i} = 1$, we have

$$\begin{aligned}\frac{\partial l(\boldsymbol{\beta}, \phi)}{\partial \beta_0} &= \sum_{i=1}^{n} \frac{y_i - \mu_i}{\mathrm{Var}(Y_i)} x_{i0} \left(\frac{\partial \mu_i}{\partial \eta_i}\right) \\ &= \sum_{i=1}^{n} \frac{y_i - \mu_i}{\mu_i} \cdot 1 \cdot (1) = \sum_{i=1}^{n} \frac{(y_i - \beta_0)}{\beta_0}\end{aligned}$$

Now $\frac{\partial l(\boldsymbol{\beta},\phi)}{\partial \beta_0} = 0$ only if $\sum_{i=1}^n (y_i - \beta_0) = 0$, i.e., only if

$$\sum_{i=1}^n (y_i - \beta_0) = \left(\sum_{i=1}^n y_i\right) - n\beta_0 = 0.$$

Hence it should hold for the solution $\hat{\beta}_0$ as

$$\begin{aligned} \left(\sum_{i=1}^n y_i\right) - n\hat{\beta}_0 &= 0, \\ -n\hat{\beta}_0 &= -\left(\sum_{i=1}^n y_i\right), \\ \hat{\beta}_0 &= \frac{\left(\sum_{i=1}^n y_i\right)}{n} = \bar{y}. \end{aligned}$$

(c) In generalized linear models, the likelihood equations can written in form

$$\frac{\partial l(\boldsymbol{\beta},\phi)}{\partial \beta_j} = \sum_{i=1}^n \frac{y_i - \mu_i}{\mathrm{Var}(Y_i)} x_{ij} \left(\frac{\partial \mu_i}{\partial \eta_i}\right) = 0, \qquad j = 0, 1, 2 \dots p.$$

Consider now the simple logit model with

$$\begin{aligned} Y_i &\sim Ber(\mu_i), \\ \mathrm{logit}(\mu_i) &= \eta_i = \beta_0. \end{aligned}$$

What kind of more simplified form the likelihood equations have in this case? That is, what form $\frac{\partial l(\beta_0)}{\partial \beta_0}$ has in the simple logit model? By using the likelihood equations, find the maximum likelihood estimator $\hat{\beta}_0$.

(2 points)

Solution:

Since $\mathrm{E}(Y_i) = \mu_i = \frac{e^{\eta_i}}{1+e^{\eta_i}} = \frac{e^{\beta_0}}{1+e^{\beta_0}}$, $\mathrm{Var}(Y_i) = \mu_i(1-\mu_i), x_{i0} = 1$ and

$$\frac{\partial \mu_i}{\partial \eta_i} = \frac{e^{\eta_i}\left(1+e^{\eta_i}\right) - e^{\eta_i}e^{\eta_i}}{\left(1+e^{\eta_i}\right)^2} = \frac{e^{\eta_i}}{1+e^{\eta_i}} \cdot \frac{1}{1+e^{\eta_i}} = \mu_i(1-\mu_i)$$

we have

$$\begin{aligned} \frac{\partial l(\boldsymbol{\beta},\phi)}{\partial \beta_0} &= \sum_{i=1}^n \frac{y_i - \mu_i}{\mathrm{Var}(Y_i)} x_{i0} \left(\frac{\partial \mu_i}{\partial \eta_i}\right) \\ &= \sum_{i=1}^n \frac{y_i - \mu_i}{\mu_i(1-\mu_i)} \cdot 1 \cdot (\mu_i(1-\mu_i)) = \sum_{i=1}^n (y_i - \mu_i) \\ &= \sum_{i=1}^n \left(y_i - \frac{e^{\beta_0}}{1+e^{\beta_0}}\right) = \sum_{i=1}^n (y_i) - n \cdot \frac{e^{\beta_0}}{1+e^{\beta_0}}. \end{aligned}$$

Now $\frac{\partial l(\boldsymbol{\beta},\phi)}{\partial \beta_0} = 0$ only if $\frac{e^{\beta_0}}{1+e^{\beta_0}} = \frac{\sum_{i=1}^n y_i}{n} = \bar{y}$. Hence it should hold for the solution $\hat{\beta}_0$ as

$$\begin{aligned}
\frac{e^{\hat{\beta}_0}}{1+e^{\hat{\beta}_0}} &= \bar{y}, \\
e^{\hat{\beta}_0} &= \frac{\bar{y}}{1-\bar{y}}, \\
\hat{\beta}_0 &= \log\left(\frac{\bar{y}}{1-\bar{y}}\right).
\end{aligned}$$