

$$\frac{\partial \ell(\beta_0)}{\partial \beta_0} = \sum_{i=1}^n \left(\frac{y_i}{1 + e^{-\beta_0}} - \frac{1 - y_i}{1 + e^{-\beta_0}} \right) \times \left(\frac{e^{-\beta_0}}{(1 + e^{-\beta_0})^2} \right)$$

$$\frac{\partial \ell(\beta_0)}{\partial \beta_0} = \sum_{i=1}^n (y_i (1 + e^{\beta_0}) - (1 - y_i) e^{\beta_0})$$

Now,

$$\sum_{i=1}^n y_i (1 + e^{\hat{\beta}_0}) - (1 - y_i) e^{\hat{\beta}_0} = 0$$

$$e^{\hat{\beta}_0} = - \frac{\sum_{i=1}^n (2y_i - 1)}{\sum_{i=1}^n (2y_i - 1)}$$

$$e^{\hat{\beta}_0} = \frac{\sum_{i=1}^n (2y_i - 1)}{\sum_{i=1}^n (1 - 2y_i)}$$

$$\hat{\beta}_0 = \log \frac{\sum_{i=1}^n (2y_i - 1)}{\sum_{i=1}^n (1 - 2y_i)}$$