Homework 5; due Thursday, 10/05/2023
(MTH 317H, Honors Linear Algebra; Fall 2023)

1 Commentary

In this homework you will begin working with linear functions, and hopefully continue to improve you understanding of span, independence, and basis.

1.1 required reading

In what follows below, you should read the section of LADR and the sections corresponding to BOP.
"LADR" – Linear Algebra done right
"BOP" – Book of Proof

- LADR, section 3C (and review 2A, 2B, 2C, 3A, 3B). For the moment, we will exclusively use the scalar field to be the real numbers, $\mathbb{R}$. That is to say, any time you encounter $\mathbf{F}$, or $\mathbf{F}^\mathbf{n}$, you are free to assume $\mathbf{F} = \mathbb{R}$. (This will change later in the semester.)
- BOP, chp 12, chp 10.

2 Definitions

We recall the following definitions, which you will need in this problem set:

Definition 2.1. *A function $T : V \to W$ is called* ***injective*** *if $u \neq v$ implies $T(u) \neq T(v)$.*

Equivalently, we can define injective functions in the following way, which is the contrapositive statement of the definition above:

Definition 2.2. *A function $T : V \to W$ is called* ***injective*** *if $T(u) = T(v)$ implies $u = v$.*

We will also be interested in surjective functions, which are defined as follows:

Definition 2.3. *A $T : V \to W$ is called* ***surjective*** *if for every $w \in W$, there is a $v \in V$ such that $T(v) = w$.*

3 Questions

Question 3.1. For each of the following functions, determine if it is linear or not, and justify your answer with a proof.

(i) Define $T : \mathcal{P}_4 \to \mathcal{P}_4$ by

$$(Tp)(x) = x^2 p''(x).$$

(Please, **do not** bother writing out the polynomial Tp in terms of its coefficients when you are writing your proof.)

(ii) Define $S : \mathcal{P}_4 \to \mathcal{P}_4$ by

$$(Sp)(x) = p''(x) + x^2.$$

(Please, **do not** bother writing out the polynomial Sp in terms of its coefficients when you are writing your proof.)

(iii) Define $H : \mathbb{R}^4 \to \mathbb{R}^3$, by

$$H(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}) = \begin{pmatrix} x_1 x_2 \\ x_1 \\ 5x_3 + x_4 \end{pmatrix}.$$

(iv) Define $I : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$I(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}) = \begin{pmatrix} x_1 - 3x_2 \\ x_1 + 7 \\ 5x_3 + x_4 \end{pmatrix}.$$

(v) Define $J : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$J(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}) = \begin{pmatrix} x_1 - 3x_2 \\ x_1 \\ 5x_3 + x_4 \end{pmatrix}.$$

Question 3.2. Assume V and W are vector spaces. Prove that if $L : V \to W$ is linear and injective and $v_1, \ldots, v_n$ is a linearly independent list of vectors, then $L(v_1), \ldots L(v_n)$ is a linearly independent list of vectors.
Note: You are free to use the following result that will be proved in class soon: for linear maps L, being injective is equivalent to the property that $L(v) = 0$ only when $v = 0$.

Question 3.3. ***NOTE***: *In this problem, you can use material from LADR up through Section 3.A, as well as the definitions of injective and surjective functions given in Section 2 of this assignment (above). You can NOT use material from LADR 3.B for these questions.*

Define the function, $L : \mathcal{P}_4 \to \mathcal{P}_2$ by the following rule:

$$\text{for } p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4, \quad (Lp)(x) = a_2 + a_3x + a_4x^2.$$

(i) Prove that $U = \{p \in \mathcal{P}_4 \ : \ Lp = \text{zero polynomial}\}$ is a subspace of $\mathcal{P}_4$.
Reminder: you cannot use results from LADR 3.B for these questions.

(ii) Give a basis for U. (No proof necessary.)

(iii) Using the definition of an injective function (see Section 2 above), prove that L is not injective.

(iv) Using the definition of a surjective function (See Section 2 above), prove that L is surjective.

Question 3.4. (i) Let $T : \mathbb{R}^2 \to \mathbb{R}$ be a *linear* function with the property that

$$T(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = 3 \quad \text{and} \quad T(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = 4.$$

Compute the value of $T(\begin{pmatrix} 3 \\ 1 \end{pmatrix})$.

(ii) Assume that $L : \mathbb{R}^2 \to \mathbb{R}$ is a *linear* function. Prove that there exist $a_1, a_2 \in \mathbb{R}$ so that

$$\forall\ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad L(x) = a_1x_1 + a_2x_2.$$

Note: Feel free to use the result from class that says a linear function is uniquely determined by its values on a basis.

NOTE: In the next two problems, we use the notation that if $X \subseteq V$, and $L : V \to W$, then

$$L(X) = \{L(x) \in W \ : \ x \in X\}.$$

Question 3.5. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear function defined as

$$L(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} x_1 - x_2 + 3x_3 \\ -3x_1 + 3x_2 - 9x_3 \end{pmatrix}.$$

Provide a vector, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ so that

$$L(\text{span}(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix})) = \text{span}(v).$$

Prove that your choice of v is correct.

Question 3.6. For two generic vector spaces V and W, let $T : V \to W$ be a linear function, and assume that $S \subset V$ is any finite list of vectors, say $S = v_1, \ldots, v_k$. Prove that

$$L(\text{span}(S)) = \text{span}(L(v_1), \ldots, L(v_k)).$$

Question 3.7. Let $L : \mathbb{R}^4 \to \mathbb{R}^4$ be given by the formula

$$L(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}) = \begin{pmatrix} x_1 - 2x_2 + 3x_3 - x_4 \\ 3x_1 - 2x_2 + 9x_3 + x_4 \\ -2x_2 - 2x_4 \\ x_1 + 3x_3 + x_4 \end{pmatrix}.$$

Give a basis for range(L). Prove your answer. You may use results from LADR 2.C, if you would like. You are strongly encouraged to invoke Question 3.6.