Homework 3; Due THURSDAY, 09/21/2023
(MTH 317H, Honors Linear Algebra; Fall 2023)

1 Commentary

In this homework you will practice techniques related to span, linear combination, and linear independence.

1.1 required reading

"LADR" – Linear Algebra done right
"BOP" – Book of Proof

- LADR, sections 2A, 2B, 2C. For the moment, we will exclusively use the scalar field to be the real numbers, $\mathbb{R}$. That is to say, any time you encounter $\mathbf{F}$, or $\mathbf{F^n}$, you are free to assume $\mathbf{F} = \mathbb{R}$. (This will change later in the semester.)
- BOP, sections 2.8–2.12, chp 4 (and review sections 1.1 – 1.7, 2.1–2.7)

2 Questions

Question 2.1. Let v_1, v_2, v_3, v_4 be the following vectors in $\mathbb{R}^3$.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}.$$

(i) Prove that v_1, v_2, v_3, v_4 is a linearly dependent list.

(ii) Prove that $v_4 \in \operatorname{span}(v_1, v_2, v_3)$.

(iii) Prove that $v_1 \notin \operatorname{span}(v_2, v_3, v_4)$.

Question 2.2. Consider the same list of vectors v_1, v_2, v_3, v_4 as in Question 2.1.

(i) Prove that $\operatorname{span}(v_1, v_2, v_3, v_4) = \operatorname{span}(v_1, v_2, v_3)$.

(ii) Prove that $\operatorname{span}(v_1, v_2, v_3) = \mathbb{R}^3$.

Here, you must use the basic set equality proof structure, namely to show as sets that $A = B$, you must demonstrate $A \subseteq B$ and $B \subset A$.

(iii) Prove that $\operatorname{span}(v_1, v_2) \neq \mathbb{R}^3$.

Question 2.3. Let u_1, u_2, v_1, v_2 be vectors in $\mathbb{R}^3$, and let

$$U = \operatorname{span}(u_1, u_2) \quad \text{and} \quad V = \operatorname{span}(v_1, v_2).$$

Give an example of specific vectors u_1, u_2, v_1, v_2 in $\mathbb{R}^3$ so that

$$\{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset,$$

and yet, $U = V$. **No proof needed.**

Question 2.4. Assume that V is a vector space, $v_1, v_2, v_3, w \in V$, and

(a) v_1, v_2, v_3 is a linearly independent list of vectors.

(b) v_1, v_2, v_3, w is a linearly dependent list of vectors.

Prove that there is a unique choice of $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that

$$w = \sum_{i=1}^{3} \alpha_i v_i.$$

You must include the following two parts in your proof.

(i) Explain (prove) to your reader why there must exist at least one choice of $\alpha_1, \alpha_2, \alpha_3$.

(ii) Explain (prove) to your reader why this choice must be unique. As for right now, all uniqueness proofs of this type should have the following structure:

(1) Assume there exist two collections, $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ that give the desired construction of w.

(2) Prove that for $i = 1, 2, 3$, $\alpha_i = \beta_i$. (Hint: you have two equations that give w as a result. What happens if you subtract one of these equations from the other?)

Question 2.5. (i) Let V be a vector space, and let $A = \{v_1, \ldots, v_m\}$ and $B = \{w_1, \ldots, w_n\}$. Prove that

$$\operatorname{span}(A \cap B) \subseteq \operatorname{span}(A) \cap \operatorname{span}(B).$$

(ii) Give an example for $V = \mathbb{R}^3$ of two *non-empty* sets A and B as above, in which

$$\operatorname{span}(A) \cap \operatorname{span}(B) \not\subseteq \operatorname{span}(A \cap B).$$

Question 2.6. Assume that $u, v, w_1, \ldots, w_n$ are all distinct elements of a vector space, V. Define the lists of vectors, B and C, as

$$B = u, w_1, \ldots, w_n \quad \text{and} \quad C = v, w_1, \ldots, w_n.$$

Prove the following implication:

$$\text{if } B \text{ is a linearly independent list and } u \in \operatorname{span}(C), \quad \text{then} \quad \operatorname{span}(B) = \operatorname{span}(C).$$

Question 2.7. Consider the following vectors in $\mathbb{R}^3$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}.$$

Is the list of vectors v_1, v_2, v_3 a basis for $\mathbb{R}^3$? Justify your answer.

Question 2.8. Consider the polynomials

$$p_1(x) = 1 - x, \quad p_2(x) = x^2 + x, \quad p_3(x) = x^3 + x^2, \quad p_4(x) = x^3.$$

Prove that the list of polynomials p_1, p_2, p_3, p_4 is a basis for $\mathcal{P}_3$.