Homework 8; due Thursday, 11/09/2023
(MTH 317H, Honors Linear Algebra; Fall 2023)

1 Commentary

This assignment covers row reduction, solving equations, inverse functions and matrices, and some basic computations involving 2x2 and 3x3 determinants.

1.1 required reading

- LADW (Linear Algebra Done Wrong) sections: 2.7, 2.8, 3.1, 3.5

2 Definitions

Here is some notation we will use in this homework regarding the bookkeeping for coordinates in a basis. This is a slightly more precise variation on the definition given in LADR 3.62, as we specifically note the basis in the notation.

Definition 2.1 (variation on LADR 3.62). *Let $\mathcal{B} = [v_1, \dots v_m]$ be a basis for a vector space, V. We say that $x \in V$ has the coordinates $a_1, \dots, a_m$ in the basis, $\mathcal{B}$, if*

$$x = a_1 v_1 + \cdots + a_m v_m.$$

We use the following shorthand notation for this as

$$x = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}_{\mathcal{B}} \quad \text{or} \quad x = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}_{[v_i]} \qquad \text{is defined as} \quad x = a_1 v_1 + \cdots + a_m v_m. \tag{1}$$

Note, when working on $\mathbb{R}^m$ with the *canonical* basis, $[e_1, \dots, e_m]$, we will often suppress the specific indication of the basis. That is to say, it is implied that when no basis is noted, it means the canonical one is used,

$$x = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}_{[e_i]}.$$

3 Questions

Question 3.1. Use the algorithm of LADW Chp 2, Sec 4 to compute the inverse of the following matrix A. Show your steps and confirm that your answer is correct. To confirm, this means that if you claim B is the inverse of A, then you must show that $AB = BA = \mathrm{Id}$ (where Id is the identity matrix).

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Question 3.2. Let A be the matrix,

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 3 & 3 & 2 \end{pmatrix}$$

(i) Compute the determinant of the matrix, A. Show your work.

(ii) Is it possible to solve, for all $b \in \mathbb{R}^3$, the equation

$$Ax = b?$$

If you answer is "yes", then prove that there is a solution for each b, and if your answer is "no", then you must give an example of a particular b so that there is no vector x that solves the equation, and prove that there is no solution.

Question 3.3. Let A be the 2×2 matrix given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let B_i be the matrices given by

$$B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

For each of the B_i matrices, above, complete the following:

(i) compute $B_i A$;

(ii) draw a picture of $\mathbb{R}^2$, showing e_1, e_2 and $B_i e_1, B_i e_2$, and describe what has changed geometrically from the list $[e_1, e_2]$ to $[B_i e_1, B_i e_2]$;

(iii) compute $\det(B_i A)$ in terms of the value of $\det(A)$. (You are meant to do this calculation directly, from the definition of 2×2 determinant. Do not invoke the properties of determinant, such as LADW Theorem 3.5.)

Question 3.4. Let the matrices A and B be given as

$$A = \begin{pmatrix} 0 & 1 & 5 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 5 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 5 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -5 & 1 \end{pmatrix}$$

(i) Show that the matrix A can be row-reduced to the matrix B, using row operations. Show your steps.

(ii) Write down an invertible matrix, T, so that $TA = B$. Do not supply a proof, just write down a matrix. (Hint, keep track of the elementary matrices– LADW p. 41, 42– that correspond to your steps in part (i))

(iii) Prove that T is invertible. (There are actually many ways one could argue this.)

Question 3.5. Let A be the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 3 & 1 \end{pmatrix}.$$

(i) Let R be the matrix that is the reduced echelon form of A. Write down R. You do not need to show your steps.

(ii) Define the linear function, $L_A : \mathbb{R}^4 \to \mathbb{R}^3$, given by

$$L_A(x) = Ax \quad \text{as matrix multiplication in the canonical bases.}$$

Use the reduction algorithm of LADR 2.31, starting with the ordered list,

$$[L_A(e_1), L_A(e_2), L_A(e_3), L_A(e_4)],$$

to give a basis for $\text{range}(L_A)$. You **do not** need to show your steps in the algorithm, just state the resulting basis.

(iii) Let $b \in \mathbb{R}^3$ be given as

$$b = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix}.$$

Represent b uniquely using the basis vectors for range(L_A) from part (ii). Use this to solve for x, in the equation

$$L_A(x) = b.$$

(iv) For the same b as in part (iii), use the row reduction algorithm in LADW section 2.2 to solve for one solution, x, in the equation

$$Ax = b.$$

(Because null(L_A) $\neq \{0\}$, there are infinitely many solutions, but you just need to give one here.)

(v) For the matrix, R, from part (i), define the linear function, $L_R : \mathbb{R}^4 \to \mathbb{R}^3$, given by

$$L_R(x) = Rx \quad \text{as matrix multiplication in the canonical bases.}$$

Why will solving for x in the equation

$$L_R(x) = b$$

not give the correct solution to $L_A(x) = b$? Give a new vector, z, so that

$$L_A(x) = b \quad \Longleftrightarrow \quad L_R(x) = z.$$

Question 3.6. Define the vector space V as

$$V = \{p \in \mathcal{P}_4 \ : \ p(0) = 0\}.$$

Define the function, $L : V \to \mathcal{P}_3$ as

$$L(f) = f' + f''.$$

(i) Briefly observe / justify that

$$V = \{f \in \mathcal{P}_4, \ : \ f(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4, \ \text{for some}, \ a_1, \ldots, a_4 \in \mathbb{R}\}.$$

(ii) Compute null(L). Prove that your answer is correct.

(iii) Prove that L is a bijection.

(iv) By a direct calculation, write down a formula for L^{-1}. That is to say, given a generic $q \in \mathcal{P}_3$, with

$$q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3,$$

if $L^{-1}(q) = p$, with

$$p(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4,$$

you need to specify the coefficients $a_1, \ldots, a_4$ in terms of the given $b_0, \ldots, b_3$.

(Hint: you know that $L^{-1}(q) = p$ if and only if $q = L(p)$. This should allow you to set up 4 equations in 4 unknowns– the a_i– and you should be able to solve directly for a_i in terms of b_i.)

(v) Confirm that your answer is correct. That is to say, you must demonstrate that $L(L^{-1}(q)) = q$ and $L^{-1}(L(p)) = p$.

Question 3.7. Assume that V and L are as in Question 3.6. Let $p_0, \ldots, p_3$ be the canonical basis polynomials for $\mathcal{P}_3$

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2, \quad p_3(x) = x^3.$$

Let the following $q_1, \ldots, q_4$ be a choice of basis for V:

$$q_1(x) = x, \quad q_2(x) = x^2, \quad q_3(x) = x^3, \quad q_4(x) = x^4.$$

(i) Write down a matrix, A, so that in the bases for V and $\mathcal{P}_3$ given above,

$$\text{if } \; p = a_1 q_1 + \cdots + a_4 q_4, \quad \text{i.e. } \; p = \begin{pmatrix} a_1 \\ \vdots \\ a_4 \end{pmatrix}_{[q_i]},$$

$$\text{and } \; L(p) = b_0 p_0 + \cdots + b_3 p_3 = \begin{pmatrix} b_0 \\ \vdots \\ b_3 \end{pmatrix}_{[p_i]},$$

$$\text{then } \begin{pmatrix} b_0 \\ \vdots \\ b_3 \end{pmatrix} = A \begin{pmatrix} a_1 \\ \vdots \\ a_4 \end{pmatrix} \quad \text{as matrix multiplication}$$

(ii) Using the algorithm of LADW Chp 2, Sec 4, compute the inverse matrix, A^{-1}. Confirm by matrix multiplication that your answer is correct. No need to show your steps, as you have done a similar computation in Question 3.1. Just write the matrix and confirm.

(iii) For the sake of comparing methods, do this part pretending you have not done Question 3.6 yet.

Using the matrix A^{-1}, write down the formula for L^{-1}. Given that a linear function is uniquely determined by its action on a basis, it is OK to specify L^{-1} by specifying $L^{-1}(p_i)$. Or, you can just write down a formula for $L^{-1}(q)$ for a generic q.