Homework 9; Due date: 11/16/2023

(MTH 317H, Honors Linear Algebra; Fall 2023)

1 Commentary

This homework assignment explores properties and computations of the determinant.

1.1 required reading

- LADW (Linear Algebra Done Wrong) sections 3.2, 3.3, 3.4, 4.1.

2 Questions

Question 2.1. (i) Prove or give a counterexample: for $n \times n$ matrices A and B,

$$\det(A+B) = \det A + \det B.$$

(ii) If A is an $n \times n$ matrix, how are the determinants $\det A$ and $\det(5A)$ related? Justify your answer.

Question 2.2. Find the determinant of each of the following matrices, showing your work. You should think carefully about which approach to the determinant is best-suited for each matrix.

(i)

$$A = \begin{pmatrix} 1 & 6 & 3 & 2 & -3 & 1 \\ 0 & 2 & 9 & -1 & -9 & 3.4 \\ 0 & 0 & 1 & -3 & -7 & 2 \\ 0 & 0 & 0 & 3 & -1.1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii)

$$B = \begin{pmatrix} 1 & 2 & 3 & 2 & -3 & 1 \\ 0 & 0 & 9 & -1 & -9 & 3.4 \\ 0 & 0 & 0 & 2 & -7 & 2 \\ 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}$$

(iii)

$$C = \begin{pmatrix} 7 & 6 & 3 & 2 & -3 & 1 \\ \pi & 8 & 9 & -1 & -9 & 3.4 \\ 4 & 12 & 7 & -3 & -7 & 2 \\ 1 & 2.2 & 1.1 & 9 & -1.1 & 2 \\ 3 & 42 & 0 & 89 & 0 & 8 \\ 18 & 1 & -12 & 5 & 12 & 0 \end{pmatrix}$$

Question 2.3. For each of the following matrices:

(a) Compute the determinant of the matrix using cofactor expansion.
(b) Compute the determinant of the matrix again, this time using row operations to reduce the matrix to a triangular matrix.

(i)

$$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 3 & 1 \end{pmatrix}$$

(ii)

$$B = \begin{pmatrix} 1 & 0 & -2 & 7 \\ -3 & 1 & 1 & -4 \\ 0 & 4 & -1 & 11 \\ 2 & 3 & 0 & 8 \end{pmatrix}$$

Question 2.4. (i) A square matrix with entries in the real numbers Q is called *orthogonal* if $Q^TQ = I$. Here Q^T denotes the transpose of Q. Prove that if Q is an orthogonal matrix, then

$$\det(Q) = \pm 1.$$

(ii) A square matrix is called *nilpotent* if $A^k = 0$ for some positive integer k. Prove that for a nilpotent matrix A, $\det(A) = 0$.

Question 2.5. We say that a matrix A is *similar* to a matrix B if there exists a matrix Q such that
$$A = Q^{-1}BQ.$$

(i) Prove that if matrices A and B are similar, both A and B must be square matrices, and of the same size.

(ii) Prove that if A and B are similar matrices, then $\det(A) = \det(B)$.

Question 2.6. Recall that for a 2×2 matrix
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
the determinant is given by
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$
In this problem you will verify the following properties of the determinant in the 2×2 case, using this definition.

(i) *Anti-symmetry*: Show that
$$\det \begin{pmatrix} b & a \\ d & c \end{pmatrix} = -\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(ii) *Linearity*: Show that
$$\det \begin{pmatrix} \alpha a_1 + \beta a_2 & b \\ \alpha c_1 + \beta c_2 & d \end{pmatrix} = \alpha \det \begin{pmatrix} a_1 & b \\ c_1 & d \end{pmatrix} + \beta \det \begin{pmatrix} a_2 & b \\ c_2 & d \end{pmatrix}$$

(iii) *Normalization*: Show that
$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Question 2.7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear function defined as
$$T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} x_1 - x_2 \\ -x_1 + 2x_2 - 2x_3 \\ 2x_2 + x_3 \end{pmatrix}.$$

(i) Let A denote the matrix of the linear transformation T with respect to the standard basis of $\mathbb{R}^3$. Find the matrix A, and find $\det(A)$.

(ii) Consider the basis

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$$

of $\mathbb{R}^3$. Let B denote the matrix of the transformation T with respect to this basis for both the domain and the target. Find the matrix B, and find $\det(B)$.

(iii) Considering your answers to parts (i) and (ii) above, make a conjecture about how the determinant of matrix representing a linear transformation depends on the bases of the domain and target. You do not need to prove your conjecture.

Question 2.8. Let S be an invertible $l \times l$ matrix and D be an $l \times l$ matrix. Use mathematical induction to prove

$$\left(SDS^{-1}\right)^n = SD^nS^{-1}, \text{ for all } n \in \mathbb{N}.$$

Your inductive proof must include the following steps:

(i) identify and state what is the conditional statement, $P(n)$, for $n \in \mathbb{N}$.

(ii) state which natural number you will use as the base case, and prove that the conditional statement is true for the base case.

(iii) for a fixed $k \in \mathbb{N}$, state the inductive assumption.

(iv) prove the inductive step. That is to say, prove that $P(k+1)$ is also true.

(v) conclude.