Homework 4; Due THURSDAY, 09/28/2023
(MTH 317H, Honors Linear Algebra; Fall 2023)

1 Commentary

In this homework you will continue to understand the techniques related to linear dependence / independence, span, and basis. Additionally, you will start to use the notion of dimension.

1.1 required reading

"LADR" – Linear Algebra done right
"BOP" – Book of Proof

- LADR, sections 3A, 3B (and review 2A, 2B, 2C). For the moment, we will exclusively use the scalar field to be the real numbers, $\mathbb{R}$. That is to say, any time you encounter $\mathbf{F}$, or $\mathbf{F^n}$, you are free to assume $\mathbf{F} = \mathbb{R}$. (This will change later in the semester.)
- BOP, chp 5, chp 6.

2 Questions

Question 2.1. Let $\ell = v_1, \ldots, v_6$ and $s = w_1 \ldots, w_6$ be ordered lists of vectors in $\mathbb{R}^3$, where:

$$v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad v_2 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad v_3 = \begin{pmatrix}3\\4\\5\end{pmatrix} \quad v_4 = \begin{pmatrix}2\\1\\2\end{pmatrix} \quad v_5 = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad v_6 = \begin{pmatrix}1\\0\\3\end{pmatrix}$$

and

$$w_1 = \begin{pmatrix}1\\0\\3\end{pmatrix} \quad w_2 = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad w_3 = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad w_4 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad w_5 = \begin{pmatrix}3\\4\\5\end{pmatrix} \quad w_6 = \begin{pmatrix}2\\1\\2\end{pmatrix}.$$

Recall that the vectors in the canonical basis for $\mathbb{R}^3$ are

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(i) Prove that $e_1, e_2, e_3 \in \operatorname{span}(\ell)$.

(ii) Prove that $\operatorname{span}(\ell) = \mathbb{R}^3$.

(iii) Use the reduction algorithm in LADR Result 2.31 to reduce the list ℓ to a basis. What is the resulting list? For this you must show each step of the algorithm, including the calculation of whether or not a vector is in the span of the previous vectors. Then just state the resulting list. Give a one sentence explanation for why your answer for the span of the resulting list is what it is.

(iv) Use the reduction algorithm in LADR Result 2.31 to reduce the list s to a basis. What is the resulting list? Is it the same list as the algorithm applied to ℓ? Does order matter in this algorithm? Does the ordering of the original vectors affect the span of the reduced list?

Question 2.2. Define the list, $\ell = x_1, x_2, x_3$ of vectors in $\mathbb{R}^3$, where

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

You will prove that ℓ is a basis for $\mathbb{R}^3$ via two different methods.

(i) Prove, from the definitions of linear independence and span that ℓ is linearly independent and $\operatorname{span}(\ell) = \mathbb{R}^3$, hence that ℓ is a basis of $\mathbb{R}^3$. For the proof of the span, you must use the typical set equality proof, in which you will establish that $\operatorname{span}(\ell) \subseteq \mathbb{R}^3$ and $\mathbb{R}^3 \subseteq \operatorname{span}(\ell)$. Note, for your span proof, it may be easiest to first show that the canonical basis vectors satisfy $e_1, e_2, e_3 \in \operatorname{span}(\ell)$.

(ii) Use some combination of the results in LADR 2.C to prove that ℓ is a basis for $\mathbb{R}^3$.

(iii) Do you have a preference for either method?

Question 2.3. Define a list $\ell = p_1, p_2$ of vectors in $\mathcal{P}_3$, where

$$p_1(x) = 1 + x^2, \quad p_2(x) = 1 - x^2.$$

Add vectors to the list, ℓ, so that it becomes a basis for $\mathcal{P}_3$. Prove that your new list is both linearly independent and that it spans $\mathcal{P}_3$.

Question 2.4. Assume that the list $B = q_1, \ldots, q_m$ is a basis for $\mathcal{P}_4$.

(i) Is it possible that no polynomial in B has degree 4? Prove your answer.

(ii) Is it possible that no polynomial in B has degree 2? Prove your answer.

(iii) What must be the value of m (the length of the list, B)? Prove your answer.

Note: in the following questions the vectors, v_i, have nothing to do with Question 2.1. The vector space and vectors in the following questions are generic.

Question 2.5. Assume that v_1, v_2, v_3, v_4 is a linearly independent list of vectors in a vector space, V.

(i) Define the vectors, w_i as

$$w_1 = v_1 + v_2, \quad w_2 = v_2 + v_3, \quad w_3 = v_3 + v_4, \quad w_4 = v_4 + v_1.$$

Determine if the list of vectors w_1, w_2, w_3, w_4 is linearly dependent or is linearly independent. Justify your answer with a proof.

(ii) Define the vectors, u_i, as

$$u_1 = v_1 + v_2, \quad u_2 = v_2 + v_3, \quad u_3 = v_3 + v_4, \quad u_4 = v_4 - v_1.$$

Determine if the list of vectors u_1, u_2, u_3, u_4 is linearly dependent or is linearly independent. Justify your answer with a proof.

Question 2.6. Assume that V is a vector space, $v_1, v_2, v_3, v_4 \in V$, and

(a) v_1, v_2, v_3 is a linearly dependent list of vectors.

(b) v_2, v_3, w is a linearly independent list of vectors.

Prove that

(i) v_1 is a linear combination of v_2 and v_3.

(ii) w is not a linear combination of v_1, v_2, v_3.

Question 2.7. Prove that in any vector space, V, if the ordered list, $B = v_1, v_2, v_3, v_4$ is a basis for V, then for

$$C = w_1, w_2, w_3, w_4,$$

where

$$w_1 = v_1 + v_2, \quad w_2 = v_2 + v_3, \quad w_3 = v_2, \quad w_4 = v_2 + v_4,$$

C is also a basis for V.

Prove this by two different methods, as follows:

(i) Use the definitions of linear independence and span to show that C is a basis. (Hint: for span, try to show that $v_i \in \operatorname{span}(C)$.)

(ii) Use the results of LADR 2.C to show that C is a basis of V.