# Homework 1

MTH 317H

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## Problem 2.1

Let $\mathcal{P}_3$ denote the set of polynomials of degree less than or equal to 3, with real coefficients. For the following two sets, $A$ and $B$, prove that $A = B$.

$$A = \{p \in \mathcal{P}_3 \ : \ p'' = 0\}$$

$$B = \{f(x) = a_0 + a_1 x \ : \ a_0, a_1 \in \mathbb{R}\}.$$

*Proof.* We will show that $A = B$ by proving that $A \subseteq B$ and $B \subseteq A$.

To show that $A \subseteq B$, let $p \in A$ be arbitrary. Since $p \in A \subseteq \mathcal{P}_3$, $p$ is a function of $x$ such that

$$p(x) = a_0 + a_1 x + a_2 x^2,$$

in which $a_0, a_1, a_2$ are coefficients in $\mathbb{R}$. Then,

$$p'(x) = a_1 + 2a_2 x$$

and

$$p''(x) = 2a_2$$

Solving for $p''(x) = 0$, it follows that $a_2 = 0$. Thus,

$$p(x) = a_0 + a_1 x,$$

which implies that $p \in B$. Therefore, we have proven that

$$A \subseteq B. \tag{1}$$

To show that $B \subseteq A$, let $p \in B$ be arbitrary. Then, there exist $a_0, a_1 \in \mathbb{R}$ such that

$$p(x) = a_0 + a_1 x$$

for any $x \in \mathbb{R}$. It is easy to see that $p(x)$ is twice differentiable, and

$$p'(x) = a_1$$
$$p''(x) = 0.$$

Thus, $p \in A$, or

$$B \subseteq A. \tag{2}$$

From (1) and (2), it follows that $A = B$. ■

## Problem 2.2

Let the following sets, $A$ and $B$, be defined as

$$A = \{p \in \mathcal{P}_3 : \ p''(1) = 0\}$$
$$B = \{q(x) = a_0 + a_3(1-x)^3 \ : \ a_0, a_3 \in \mathbb{R}\}.$$

(i) Prove that $B \subseteq A$.

(ii) Is it true that $A \subseteq B$? If your answer is "yes", then give the set containment proof outlined in Question 2.1. If your answer is "no", then give a concrete example of an element in $A$ that is not in $B$.

*Proof.* (i) Let $q = a_0 + a_3(1-x)^3 \in B$ for some $a_0, a_3 \in \mathbb{R}$. It is easy to see that $q$ is twice differentiable, and

$$q'(x) = -3a_3(1-x)^2$$
$$q''(x) = 6a_3(1-x)$$

Then, $q''(1) = 0$, or $q \in A$. Since $q$ is an arbitrary element of $B$, this implies that $B \subseteq A$. ■

(ii) Let $p = a_1 x$ for some $a_1 \in \mathbb{R}$. Then, $p \in \mathcal{P}_3$ and is twice differentiable, meaning

$$p'(x) = a_1$$
$$p''(x) = 0.$$

Thus, $p''(1) = 0$, meaning $p \in A$. However, it is not the case that there exist $a_0, a_3 \in \mathbb{R}$ such that $p(x) = a_0 + a_3(1-x)^3$ for any $x \in \mathbb{R}$. Therefore, $p \notin B$, or $A \not\subseteq B$. The answer to the question is "no". ■

## Problem 2.3

Let $\mathcal{P}_n$ denote the vector space of real polynomials of degree less than or equal to $n$. For this question, assume the usual operations of addition and scalar multiplication that are given on the vector spaces $\mathbb{R}^n$ and $\mathcal{P}_n$.

For each of the following sets, $X$, complete the following

(a) List 3 different and concrete elements of $X$.

(b) Verify whether or not the set, $X$, satisfies the following 3 properties with respect to the usual addition and scalar multiplication associated with $\mathbb{R}^n$ or $\mathcal{P}_n$. For each property, your answer will be "yes" or "no", and you must provide a justification. If your answer is "yes", you will demonstrate the property for generic inputs. If your answer is "no", you will choose specific concrete elements and/or scalars that show the property fails.

(1) for two generic elements of $X$, say $u, v \in X$, also $u + v \in X$.

(2) for a generic scalar, $\lambda \in \mathbb{R}$ and a generic element, $v \in X$, also $\lambda v \in X$.

(3) the zero vector, $\vec{0}$, is in $X$.

Here are the sets, $X$:

(i) $X = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} \; : \; x_1, x_3 \in \mathbb{R} \right\}$

(ii) $X = \left\{ \begin{pmatrix} x_1 \\ 1 \\ x_3 \end{pmatrix} \; : \; x_1, x_3 \in \mathbb{R} \right\}$

(iii) $X = \left\{ \begin{pmatrix} x \\ 3x \\ 3x^2 \end{pmatrix} \; : \; x \in \mathbb{R} \right\}$

(iv) $X = \left\{ a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \; : \; a, b \in \mathbb{R} \right\}$

(v) $X = \{p \in \mathcal{P}_3, \; : \; p(5) = 0\}$

(vi) $X = \{p \in \mathcal{P}_3, \; : \; \deg(p) = 3\}$

*Solution.* (a) Let $p_1(x) = a_0$, $p_2(x) = a_1x$, $p_3(x) = a_2x^2$, where $a_1, a_2, a_3 \in \mathbb{R} - \{0\}$ and $a_1$, $a_2$, $a_3$ are pairwise different. Then, for all $x \in \mathbb{R}$, $p_1(x)$, $p_2(x)$, $p_3(x)$ are pairwise different and $p_1(x), p_2(x), p_3(x) \in \mathbb{R}$. Hence, $p_1, p_2, p_3$ are three different and concrete elements of $X$.

(b) (i) The first set satisfies the three following properties:

(1) Let $x = \begin{pmatrix} x_1 \\ 0 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ 0 \\ y_2 \end{pmatrix}$ be elements of $X$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Then, $x + y = \begin{pmatrix} x_1 + y_1 \\ 0 \\ x_2 + y_2 \end{pmatrix} \in X$, since $x_1 + y_1 \in \mathbb{R}$ and $x_2 + y_2 \in \mathbb{R}$.

(2) Let $x = \begin{pmatrix} x_1 \\ 0 \\ x_2 \end{pmatrix}$ and $\lambda \in \mathbb{R}$. Then, $\lambda x = \begin{pmatrix} \lambda x_1 \\ 0 \\ \lambda x_2 \end{pmatrix} \in X$, since both $\lambda x_1$ and $\lambda x_2$ are in $X$.

(3) Let $x = \begin{pmatrix} x_1 \\ 0 \\ x_2 \end{pmatrix}$, where $x_1 = x_2 = 0$. Then, $x$ is the zero vector of $X$.

Therefore, $X$ satisfies the three following properties, or the answer to the question is "yes".

(ii) The set does not contain the zero vector since every $x \in X$ has the second component to be 1. Therefore, the answer to the question is "no".

(iii) Let $x = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$, $y = \begin{pmatrix} 2 \\ 6 \\ 12 \end{pmatrix}$. It is easy to see that $x, y \in X$. However, $x + y = \begin{pmatrix} 3 \\ 9 \\ 15 \end{pmatrix} \notin X$ since $15 \neq 3 \times 3^2$. Therefore, the set fails property (1), or the answer to the question is "no".

(iv) The fourth set satisfies the three following properties:

(1) Let $x = a_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b_1 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $y = a_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ be elements of $X$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then, $x + y = (a_1 + a_2) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (b_1 + b_2) \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \in X$ since $a_1 + a_2 \in \mathbb{R}$ and $b_1 + b_2 \in \mathbb{R}$.

(2) Let $x = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \in X$ and $\lambda \in \mathbb{R}$. Then, $\lambda x = \lambda a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda b \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \in X$ since both $\lambda a$ and $\lambda b$ are in $X$.

(3) Let $x = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \in X$, where $a = b = 0$. Then, $x$ is the zero vector of $X$.

Therefore, $X$ satisfies the three following properties, or the answer to the question is "yes".

(v) The fifth set satisfies the three following properties:

(1) Let $p, q \in X$. Then, $p(5) = q(5) = 0$ and $deg(p), deg(q) \leq 3$. This implies that $deg(p+q) \leq 3$ and $(p+q)(5) = p(5) + q(5) = 0$. Thus, $p + q \in X$.

(2) Let $p \in X$ and $\lambda \in \mathbb{R}$. Then, $(\lambda p)(5) = \lambda p(5) = 0$. Therefore, $\lambda p \in X$.

(3) Let $p = 0$. It is easy to see that $p \in X$ and $p$ is the zero vector.

Therefore, $X$ satisfies the three following properties, or the answer to the question is "yes".

(vi) The set does not contain the zero vector since every element of $X$ has a degree of 3, while the degree of the zero polynominal is not defined. Therefore, the answer to the question is "no".

## Problem 2.4

Determine for which $m \in \mathbb{R}$ and for which $b \in \mathbb{R}$ the following set, $X$, satisfies the 3 properties required in Question (2.3).

$$X = \left\{ \begin{pmatrix} x_1 \\ mx_1 + b \end{pmatrix} \ : \ x_1 \in \mathbb{R} \right\}.$$

For the $m$ and $b$ that work, prove they work– i.e. establish the 3 requirements. For the $m$ and $b$ that don't work, pick one of the 3 properties that fail, and then give specific examples of vectors and scalars that show the property fails.

*Solution.* It is easy to see that $b$ must be 0, since if $b \neq 0$ then the zero vector does not belong to $X$ (if the first component of the vector (i.e. $x_1$) is zero, the second component (i.e. $mx_1 + b$) of the vector is equal to $b \neq 0$).

$m$ can be any element of $\mathbb{R}$ since all three properties are satisfied for any $m$:

(1) Let $x = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ my_1 \end{pmatrix}$ be elements of $X$. Then, $x+y = \begin{pmatrix} x_1 + y_1 \\ m(x_1 + y_1) \end{pmatrix} \in X$.

(2) Let $x = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix}$ and $\lambda \in \mathbb{R}$. Then, $\lambda x = \begin{pmatrix} \lambda x_1 \\ m(\lambda x_1) \end{pmatrix} \in X$.

(3) Let $x = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} \in X$ such that $x_1 = 0$. Then, $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or the zero vector is in $X$.

Therefore, in order for the set $X$ to satisfy all three properties from Question (2.3), $b$ must be 0 but $m$ can be any element of $\mathbb{R}$.

## Problem 2.5

Consider the set, $\mathbb{R}^2$, with the usual addition operation for vectors in $\mathbb{R}^2$. Define an operation, $\odot$, so that

$$\odot : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,$$

$$\lambda \in \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \quad \lambda \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \lambda x_1 \\ x_2 \end{pmatrix}.$$

Does $\odot$ satisfy the requirements of a scalar multiplication in a vector space (i.e. does it satisfy the requirements as given in definitions 1.18 and 1.19 of LADR)? Your answer will be "yes" or "no", followed by a justification of your choice. If you answer yes, then you must verify that all requirements of scalar multiplication are true. If your answer is no, then you must identify one requirement that fails, and you will demonstrate the failure with a concrete choice of vectors and scalars.

*Solution.* Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}$ for some $x_1, x_2 \in \mathbb{R}$ such that $x_2 \neq 0$. Then,

$$3 \odot x = \begin{pmatrix} 3x_1 \\ x_2 \end{pmatrix},$$

$$\begin{aligned} (1+2) \odot x &= 1 \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 2 \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} \\ &= \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix} \end{aligned}$$

However, since $x_2 \neq 0$,

$$x_2 \neq 3x_2,$$

or $3 \odot x \neq (1+2) \odot x$. This implies that $(\mathbb{R}^2, \odot)$ does not satisfy the distributive property of scalar multiplication with respect to real addition, or $\odot$ does not satisfy the requirements of a scalar multiplication in a vector space. Therefore, the answer to the question is ”no”.

## Problem 2.6

Consider the set of 3-dimensional columns of real numbers

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

with addition and scalar multiplication defined by:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1 - v_1 \\ w_2 - v_2 \\ w_3 - v_3 \end{pmatrix}, \quad \text{and} \qquad \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix},$$

respectively. Does this define a vector space? If your answer is "yes," you need to verify all of the vector space axioms. If your answer is "no" you need to identify an axioms that fails, and demonstrate the failure with a concrete counterexample.

*Solution.* Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, where $v_1 \neq w_1$. Then, $v + w = \begin{pmatrix} v_1 - w_1 \\ v_2 - w_2 \\ v_3 - w_3 \end{pmatrix}$, while $w - v = \begin{pmatrix} w_1 - v_1 \\ w_2 - v_2 \\ w_3 - v_3 \end{pmatrix}$. However, since $v_1 \neq w_1$,

$$v_1 - w_1 \neq w_1 - v_1,$$

or $v + w \neq w + v$. Therefore, the following set equipped with addition and scalar multiplication does not form a vector space, or the answer to the question is "no".

## Problem 2.7

Assume that $V$ is a vector space. By the vector space axioms, for each $v \in V$, there exists an additive inverse, which we denote $(-v)$. This additive inverse $(-v)$ is unique (see LADR 1.26). Since $(-v)$ is in $V$, it too has an additive inverse, which we will write as $-(-v)$. Prove that

$$-(-v) = v.$$

*Solution.* Let $v \in V$ and $w$ be the additive inverse of $-v$. Then,

$$w + (-v) = \vec{0},$$

or $v + w + (-v) = v$. From the commutative and associative property of vector addition,

$$\begin{aligned} v &= v + w + (-v) \\ &= w + v + (-v) \\ &= w + (v + (-v)) \\ &= w + \vec{0} \\ &= w \end{aligned}$$

(since $-v$ is the additive inverse of $v$). Therefore, $-(-v) = w = v$ ∎

## Problem 2.8

Assume that $V$ is a vector space, $a \in \mathbb{R}$ and $v \in V$. Prove the following implication:

$$\text{if } av = \vec{0}, \text{ then } a = 0 \text{ or } v = \vec{0}.$$

*Solution.* For $a \in \mathbb{R}$ and $v \in V$, suppose $av = \vec{0}$.

If $a = 0$, the proof is complete.

If $a \neq 0$, there exists $a^{-1} \in \mathbb{R}$ such that

$$v = (a^{-1}a)v = a^{-1}(av) = a^{-1}\vec{0} = \vec{0}$$

Therefore, either $a = 0$ or $v = \vec{0}$. ■