Homework 10; Due date: 11/30/2023
(MTH 317H, Honors Linear Algebra; Spring 2023)

1 Commentary

This homework assignment explores eigenvectors and eigenvalues.

1.1 required reading

- LADW (Linear Algebra Done Wrong) sections 4.1 and 4.2.

2 Questions

Question 2.1. Find the characteristic polynomials, eigenvalues, and eigenvectors of each of the following matrices. For each eigenvalue, find its algebraic multiplicity and geometric multiplicity.

(i)
$$A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$$

(ii)
$$B = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii)
$$C = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

Question 2.2. Determine whether each of the following statements is True or False. If it is true, provide a proof. If it is false, provide a counterexample.

(i) If a matrix has one eigenvector, it has infinitely many eigenvectors.

(ii) Similar matrices always have the same eigenvectors.

(iii) A non-zero sum of two eigenvectors of a matrix A is always an eigenvector.

(iv) A non-zero sum of two eigenvectors of a matrix A corresponding to the same eigenvalue λ is always an eigenvector.

Question 2.3. Recall that a matrix A is called *nilpotent* if $A^k = 0$ for some k.

(i) Show that the matrix

$$B = \begin{pmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{pmatrix}$$

is nilpotent.

(ii) Find the eigenvalues of the matrix B. You do not need to find the associated eigenvectors.

(iii) Prove that for any nilpotent matrix A, the only eigenvalue of A is 0.

Question 2.4. (i) Find the eigenvalues of the following matrix. You do not need to find the eigenvectors.

$$B = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 2 & 7 & -2 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(ii) Let A be an $n \times n$ triangular matrix. Prove that the eigenvalues of A (repeated with multiplicity) are the diagonal entries of A: $a_{1,1}, a_{2,2}, \ldots a_{n,n}$.

Question 2.5. Let A be an $n \times n$ matrix, and let $f(\lambda) = \det(A - \lambda I)$ be its characteristic polynomial. Prove that $f(\lambda)$ is a polynomial of degree n, and the coefficient of λ^n in $f(\lambda)$ is $(-1)^n$.
Hint: Use cofactor expansion and induction.

Question 2.6. In this problem you will prove that the determinant of an $n \times n$ matrix A is the product of its eigenvalues (repeated with multiplicity), i.e. $\det(A) = \lambda_1 \lambda_2 ... \lambda_n$. You will do this in two steps:

(i) First, show that

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda),$$

where $\lambda_1, \lambda_2, ...\lambda_n$ are the eigenvalues of A (repeated with multiplicity).

Hint: Use the fact that the eigenvalues are the roots of the characteristic polynomial, plus your result from Question 2.5 about the coefficient of λ^n in the characteristic polynomial.

(ii) Then, plug $\lambda = 0$ into your formula from part (i) to conclude that the determinant of A is the product of the eigenvalues.

Question 2.7. Let R be any polygon in the plane. Prove that it is possible to divide R into triangles, all of whose vertices are vertices of R. Your inductive proof must include the following steps:

(i) identify and state what is the conditional statement, $P(n)$, for $n \in \mathbb{N}, n \geq 3$.

(ii) state which natural number you will use as the base case, and prove that the conditional statement is true for the base case.

(iii) for a fixed $k \in \mathbb{N}$, state the inductive assumption.

(iv) prove the inductive step. That is to say, prove that $P(k + 1)$ is also true.

(v) conclude.