# Learning Portfolios
(MTH 317H, Honors Linear Algebra; Sec 001; Fall 2023)

## 1 Instructions

This document contains 6 separate learning portfolio questions, 2.1 – 2.6.

Each learning portfolio will be graded out of 20 points. You will have 2 chances for each portfolio question to be graded, and your goal is to incorporate feedback from the first round of grading to obtain as high of a score as possible on the second round (ideally your second round submission will be perfect). Please note the differing due dates for first round submissions.

Turn in each portfolio question as its own document, in hard copy, separated from your other homework and learning portfolio submissions. Second round submissions must contain the graded first attempt.

## 2 Questions

**Notation:** In what follows, we will make an abuse of notation for the use of the canonical basis vectors in $\mathbb{R}^k$ for a generic $k$. When we say that $e_i$ is the $i$-th canonical vector in $\mathbb{R}^k$, we mean the vector $e_i \in \mathbb{R}^k$ with all entries 0, except for the $i$-th entry, which is 1. This is an abuse in the sense that we are using that same name for vectors that live in different spaces!!!! That is to say, within the context of specific questions, if the abuse is apparent, it is possible that we will sometimes say that

$$e_2 \in \mathbb{R}^3, \quad \text{with} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and on the other hand,} \quad e_2 \in \mathbb{R}^4, \quad \text{with} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Of course, these are not the same vector! But we will embrace the abuse of notation in order to make the writing of mathematics more bearable, at the expense of some possible logical confusion.

**Learning Portfolio 2.1** (first submission due by 10/27; second submission due by 12/1)**.** Prove, using induction, that for all $n \in \mathbb{N}$ the following equality holds.

$$\sum_{i=1}^{n} \frac{2}{(i+1)(i+2)} = \frac{n}{(n+2)}$$

Your proof must contain the following steps in a clearly presented fashion.

1. Write down a conditional statement, $P(n)$, so that the original statement can be written as

$$\forall\, n \in \mathbb{N},\ P(n).$$

2. Verify that the base case is true.

3. State the inductive assumption.

4. Use the inductive assumption to establish the inductive step.

**Learning Portfolio 2.2** (first submission due by 10/27; second submission due by 12/1)**.** Define the linear function $T : \mathbb{R}^3 \to \mathbb{R}^4$ via the formula

$$T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} x_1 + x_2 + x_3 \\ 0 \\ x_3 - 2x_2 \\ 8x_2 - 4x_3 \end{pmatrix}.$$

(i) Give a basis for null($T$) and range($T$). (no proof required)

(ii) Prove that your choices are indeed a basis for each respective subspace.

**Learning Portfolio 2.3** (first submission due by 10/27; second submission due by 12/1)**.** Let $L : \mathcal{P}_4 \to \mathcal{P}_4$ be a linear function that has the following properties:

(a) For $p_0, p_1, p_2 \in \mathcal{P}_4$ given by

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2,$$

$L$ has the values:

$$[L(p_0)](x) = x^4, \quad [L(p_1)](x) = 3x^3, \quad [L(p_2)](x) = x.$$

(b) For $p_3(x) = x^3$ and $p_4(x) = x^4$, $L(p_3 + p_4) =$ zero polynomial.

(c) $\dim(\text{null}(L)) = 1$.

(i) Characterize all possible linear functions, $L : \mathcal{P}_4 \to \mathcal{P}_4$, that satisfy the above properties. That is to say, since $\{p_0, \ldots, p_4\}$ is a basis for $\mathcal{P}_4$, explain all possible values of

$$L(p_3) \quad \text{and} \quad L(p_4).$$

**Note:** your answer to this should not contain a proof. Your answer is simply to give a description of all possible $b_0, \ldots, b_4$ and $c_0, \ldots, c_4$ so that if

$$[L(p_3)](x) = b_0 + b_1 x + \cdots + b_4 x^4 \quad \text{and} \quad [L(p_4)](x) = c_0 + c_1 x + \cdots + c_4 x^4,$$

these choices give all possibilities for an $L$ that satisfies (a), (b), (c).

(ii) Prove that your characterization is correct. That is to say, take a generic choice of one of the functions you gave in part (i) and prove that it does indeed satisfy conditions (a), (b), (c). Furthermore, take a generic choice of $L$ that satisfies (a), (b), (c), and show that it must also satisfy the conditions you have given in part (i).

(iii) Give a basis for $\text{null}(L)$ and a basis for $\text{range}(L)$.

Here are some hints: For $p_0, \ldots, p_4$, above, and $p \in \mathcal{P}_4$ a generic polynomial, you know that you have for some choice of $a_0, \ldots, a_4 \in \mathbb{R}$,

$$p(x) = a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) + a_4 p_4(x).$$

By linearity, you already know what happens to the first 3 terms when evaluating $L(p)$. So you are curious about the remaining two terms. Since you already know that value of $L(p_3 + p_4)$, you can obtain a relationship between $L(p_3)$ and $L(p_4)$. This should mean for

$$[L(p_3)](x) = b_0 + b_1 x + \cdots + b_4 x^4,$$

you need to find conditions on $b_0, \ldots, b_4$. These conditions will be a result of the requirement that any choice of the $b_i$ must keep property (c) intact, i.e. $\dim(\text{null}(L)) = 1$. You could use the rank nullity theorem (LADR fundamental theorem of linear functions) plus dimension counting and the fact that by a homework question,

$$\{L(p_0), L(p_1), L(p_2), L(p_3)\}$$

should be a basis for $\text{range}(L)$.

**Learning Portfolio 2.4** (first submission due by 11/10; second submission due by 12/8**).** Let $L : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear function defined by

$$L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_2 + x_3 - 5x_4 \\ x_1 + x_4 \end{pmatrix} \tag{1}$$

(i) In the *canonical basis* for both $\mathbb{R}^4$ and $\mathbb{R}^2$ (i.e. $\mathbb{R}^4 = \text{span}(\{e_1, e_2, e_3, e_4\})$, and, with an abuse of notation, $\mathbb{R}^2 = \text{span}(\{e_1, e_2\})$), give the unique matrix, $A$, so that

$$\text{whenever } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 \ \textbf{ in the canonical basis},$$
$$L(x) = Ax, \quad \text{defined via matrix multiplication.}$$

(ii) Without doing any computation involving $L$, give a lower bound for $\dim(\text{null}(L))$. That is to say find and integer, $k$, so that $\dim(\text{null}(L)) \geq k$, and $k > 0$.

(iii) Give a basis for null$(L)$. Prove that your choice of basis is correct.

(iv) Explain why, without calculating what is range$(L)$, that it must satisfy range$(L) = \mathbb{R}^2$.

(v) Start with the list, $\ell = \{L(e_1), L(e_2), L(e_3), L(e_4)\}$, and use the algorithm of LADR result 2.31 to reduce $\ell$ to a basis of range$(L)$. You must show each step of the algorithm and justify each choice. Prove that your resulting list is indeed a basis of range$(L)$.

(vi) Find some $x_0 \in \mathbb{R}^4$ so that

$$L(x_0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Check that indeed your choice of $x_0$ gives the desired output.

(vii) List all possible solutions, $v \in \mathbb{R}^4$, of the equation

$$L(v) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

**Learning Portfolio 2.5** (first submission due by 11/10; second submission due by 12/8**.** This is a direct continuation of the previous LP. Assume that $L$ is the same function defined in (1).

(i) For $v_1$, $v_2$, $v_3$, $v_4$ defined below, prove that $\ell = \{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{R}^4$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -5 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

(ii) For $i = 1, 2, 3, 4$, compute the value of $L(v_i)$.

(iii) (you do nothing for this part. it is just notation.) Let us define a notation to account for the fact that in this context, we have two different ways to represent any $x \in \mathbb{R}^4$:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_{\{e_i\}} \quad \text{is defined as} \quad x = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4,$$

where $\{e_i\}$ are the canonical basis vectors as above; and

$$x = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}_{\{v_i\}} \quad \text{is defined as} \quad x = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

(iv) For $x = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$, compute $L(x)$. Give your answer as a canonical vector in $\mathbb{R}^2$, i.e. find the values of $y_1$ and $y_2$ so that

$$L(x) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

(v) Find $a_1, \ldots, a_4$, so that when

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{\{e_i\}}, \quad \text{you also have} \quad x = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4.$$

(vi) Given $x_1, \ldots, x_4$ fixed, find $a_1, \ldots, a_4$, depending only on $x_i$ so that

$$x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4.$$

Check that your answer is correct. That is to say that you need to find that

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

or in our new notation that

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}_{\{v_i\}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}_{\{e_i\}}.$$

(vii) Write down a matrix, $B$, so that

$$\text{if} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} w_1 \\ w_1 \\ w_3 \\ w_4 \end{pmatrix} = Bx \quad \text{as matrix multiplication,}$$

$$\text{then} \quad x = w_1v_1 + w_2v_2 + w_3v_3 + w_4v_4.$$

Hint: look at the previous part. (Such a matrix, $B$, is called a change of basis matrix. It changed from the canonical basis to the basis $\{v_1, \ldots, v_4\}$.)

(viii) Write down a matrix, $\tilde{A}$, so that

$$\text{if} \quad x \in \mathbb{R}^4 \quad \text{and} \quad x = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}_{\{v_i\}}, \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \tilde{A} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \quad \text{as matrix multiplication,}$$

$$\text{then} \quad L(x) = y_1e_1 + y_2e_2, \quad \text{i.e.} \quad L(x) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{\{e_i\}}.$$

(ix) (Here is commentary, you do not do anything here.) Hopefully you have seen that the matrix representation of $L$ in terms of the domain in the basis $\{v_1, \ldots, v_4\}$ and the co-domain in the basis $\{e_1, e_2\}$, which is $\tilde{A}$ in the previous part, has a more intuitive and simple structure than the matrix that represents $L$ in the domain basis $\{e_1, \ldots e_4\}$ and co-domain basis $\{e_1, e_2\}$, which is $A$ from question 2.4. Notice that the function $L$ is the same function, defined on $\mathbb{R}^4$ regardless of which basis is used. However, a basis is a book-keeping mechanism which adds some more structure to the analysis of $L$. This should show that if you create a basis of the domain that takes into account which vectors span the nullspace and which vectors map to a basis of the range space, then the structure of $L$ becomes more apparent. But this is just an opinion, and so maybe you disagree! There is a conservation of difficulty here, in terms of whether or not you invest the extra energy to first convert $x$ from canonical coordinates to the coordinates given by $\{v_i\}$. This example has small dimension, and so you may not see the payoff, but in situations with much larger dimension of domain and co-domain, this can be very useful.

**Learning Portfolio 2.6** (first submission due by 11/10; second submission due by 12/8)**.** This LP assumes that you have been introduced to and have worked with the dot product on $\mathbb{R}^2$, and that you are familiar with the definition of length of vectors in $\mathbb{R}^2$. Here are some relevant definitions.

**Definition 2.7.** *If $u, v \in \mathbb{R}^2$ with $u = (u_1, u_2)$ and $v = (v_1, v_2)$, the dot product (or inner product) of $u$ and $v$ is deonted as "$u \cdot v$", and is defined as*

$$u \cdot v = u_1 v_1 + u_2 v_2.$$

**Definition 2.8.** *The length of a vector $v \in \mathbb{R}^2$, $v = (v_1, v_2)$, is given by*

$$|v| = \sqrt{(v_1)^2 + (v_2)^2}.$$

*Note that*

$$|v|^2 = v \cdot v.$$

We will use, without proof, the following fact about the dot product:

**Lemma 2.9.** *If $u, v \in \mathbb{R}^2$, then for $\theta \in [0, \pi]$ as the unique angle between $u$ and $v$,*

$$u \cdot v = |u|\,|v| \cos(\theta).$$

**Definition 2.10.** *If $A$ is the matrix,*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

*then the* determinant *of $A$ is defined as*

$$\det(A) = ad - cb. \tag{3}$$

In many results that follow, you will use the same linear function, $L : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$\text{whenever } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 e_1 + x_2 e_2 \ \ \textbf{in the canonical basis},$$
$$L(x) = Ax, \quad \text{defined via matrix multiplication.} \tag{4}$$

**Part (1).** Let $E$ be the cube in $\mathbb{R}^2$ that is the parallelogram whose two defining edges are $e_1$ and $e_2$, and whose diagonal is $e_1 + e_1$. (I.e. the parallelogram whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.) Let $F$ be the parallelogram with defining edges $L(e_1)$ and $L(e_2)$, and whose diagonal is $L(e_1) + L(e_2)$. (I.e. the parallelogram whose vertices are $(0,0)$, $L(e_1)$, $L(e_1) + L(e_2)$, $L(e_2)$.)

Prove that

$$\text{area of } F = |\det(A)|\,.$$

In this case, you are free to assume that $a > 0$, $b > 0$, $c > 0$, and $d > 0$. It is not necessary, but if it helps you, then that is fine.

Here are some steps you may want to consider. Let us call

$$v_1 = L(e_1) \ \text{ and } \ v_2 = L(e_2).$$

Assume that $v_1 \neq 0$ and $v_2 \neq 0$.

(a) Draw a picture!!!!

(b) You should know that the area of a parallelogram is Area $=$ (base)$\times$(height). If $\theta \in [0, \pi]$ is the unique angle between $L(e_1)$ and $L(e_2)$, then you should be able to obtain the area using: $|v_1|$, $|v_2|$, and $\sin(\theta)$.

(c) You can relate $\sin(\theta)$ to $v_1 \cdot v_2$ via

$$\cos(\theta) = \frac{v_1 \cdot v_2}{|v_1|\,|v_2|}.$$

(d) You should be able to obtain

$$(\text{base}) \times (\text{height}) = \sqrt{|v_1|^2\,|v_2|^2 - (v_1 \cdot v_2)^2}.$$

(e) Then just crunch through each expression as they involve $a, b, c, d$.

**Part (2).** Use the previous part to prove that for $L$ defined in (4),

$$\det(A) = 0 \quad \Longleftrightarrow \quad L \text{ is not injective.}$$

Here are some ideas to consider.

(a) Draw a picture!!!!

(b) For $F$ the parallelogram in the previous question, what are the only situations in which area$(F) = 0$? Think about this in terms of the angle, $\theta$.

(c) If $v_1$ and $v_2$ are co-linear, then you should be able to create a non-zero vector in null$(L)$.