Computational Techniques and Programming Languages

Course Code: PHY421

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February 8, 2024

The Logistic Map

- The logistic map has played a central role in the development of the theory of chaos.
- The logistic map has a quadratic nonlinearity and is represented by

$$x_{n+1} = ax_n(1 - x_n) = f(x_n), \quad n = 0, 1, \ldots$$

  where a is a parameter and we assume that $0 < x < 1$.
- This map is a discrete-time analog of the logistic equation for population growth.
- Now we wish to find what this model can tell us about the long time ($n \to \infty$) behaviour of the population fraction x.

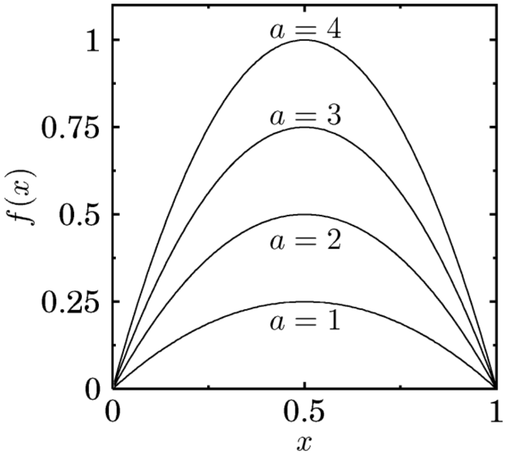

Figure: Graph of $f(x)$ for the logistic map

Equilibrium Points and Their Stability

- For an one-dimensional map the equilibrium points are obtained by writing
$$x_n = x_{n+1} = x^*.$$
- For the logistic map, this substitution leads to the following equation
$$x^* = ax^*\left(1 - x^*\right)$$
$$\implies x^*\left(1 - a + ax^*\right) = 0.$$
- Solving this equation, we obtain two equilibrium points, namely
$$x_1^* = 0, \quad x_2^* = \frac{(a-1)}{a}.$$

- What is the stability nature of the equilibrium points $x = x^*$?
- To answer this question, for a map $x_{n+1} = f(x_n)$, we consider the difference between the absolute value of x_n and x^* for n arbitrary.
- Let

$$|\delta_n| = |x_n - x^*|.$$

Then

$$\begin{aligned}
|\delta_{n+1}| &= |x_{n+1} - x^*| \\
&= |f(x_n) - x^*| \\
&= |f(x^* + \delta_n) - x^*| \\
&= |x^* + \mathrm{d}f/\mathrm{d}x_n|_{x_n = x^*}\, \delta_n + \ldots - x^*\,| \qquad \text{(Taylor expansion)} \\
&= |\mathrm{d}f/\mathrm{d}x_n|_{x_n = x^*}\,| \cdot |\delta_n\,|
\end{aligned}$$

- $|\delta_{n+1}| \to 0$ as $n \to \infty$.
- In other words, $|\delta_{n+1}| < |\delta_n|$.
- Consequently, for stability of the equilibrium point, we require that

$$|\delta_{n+1}| \,/\, |\delta_n| < 1$$

- Thus the condition for stability of x^* is

$$|\mathrm{d}f(x)/\mathrm{d}x|_{x=x^*} = \left|f'(x^*)\right| < 1.$$

- Applying the above criterion to the logistic map, the condition for stability becomes

$$\left|f'(x^*)\right| = |a\,(1 - 2x^*)| < 1.$$

- Thus we find that
  (1) the equilibrium point $x^* = 0$ is stable for $0 \leq a < 1$ and unstable for $a > 1$, since $|f'| = |a|$ is greater than 1 for $a > 1$.
  (2) The equilibrium point $x^* = (a-1)/a$ is stable for $1 < a < 3$, and unstable for all other values of a, as $|f'| = |2 - a|$ is less than 1 for $1 < a < 3$ and greater than 1 for a outside this range.
- That is, the map undergoes a transcritical bifurcation at $a = 1$.
- Transcritical bifurcation occurs when $f' = 1$.

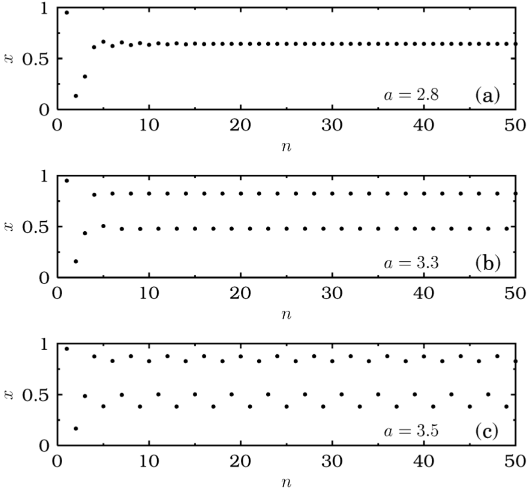

Figure: x_n versus n of the logistic map. (a) $a = 2.8$, the iterations asymptotically approach the stable equilibrium point $x^* \approx 0.643$. (b) $a = 3.3$, the long term behaviour is a period-2 cycle. (c) $a = 3.5$, the solution is a period-4 cycle

Stability When the First Derivative Equals to +1 or -1

- We have not stated its stability nature when $f' = \pm 1$.
- In this case the stability property depends upon the sign of second and third derivatives $f''(x^*)$ and $f'''(x^*)$.
- For $f' = 1$,
  (1) if $f'' < 0$ then x^* is (semi)stable for $x_0 > x^*$ and unstable for $x_0 < x^*$ and
  (2) if $f'' > 0$ then x^* is (semi)stable for $x_0 < x^*$ and unstable for $x_0 > x^*$.
  Further if $f'' = 0$, then
  (1) x^* is stable for $f''' < 0$, and
  (2) x^* is unstable for $f''' > 0$.

- In case $f''' = 0$ also, one can state the stability condition in term of $f^{(4)}(x^*) \neq 0$ and so on.
- For $f' = -1$, the stability is determined by the sign of the quantity

$$g''' = -2f''' - 3\left[f''\right]^2.$$

- Particularly, if $g''' < 0$ then x^* is stable, and if $g''' > 0$ then it is unstable.
- Now we apply the above criteria to the logistic map.

Periodic Solutions and Cycles

- One can find more general solutions other than the equilibrium points.
- One can find solutions which repeat after every two iterations, three iterations, ..., N iterations (N : arbitrary)
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$$\begin{aligned} 2- \ \text{cycle} &: x_{n+2} = x_n \\ 3- \ \text{cycle} &: x_{n+3} = x_n \\ \ldots \\ N- \ \text{cycle} &: x_{n+N} = x_n \end{aligned}$$

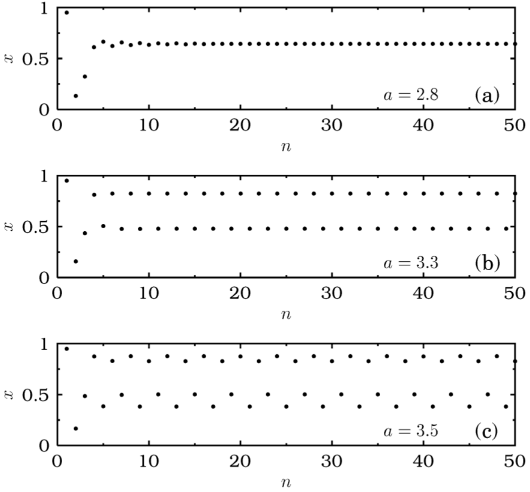


Figure: x_n versus n of the logistic map. (a) $a = 2.8$, the iterations asymptotically approach the stable equilibrium point $x^* \approx 0.643$. (b) $a = 3.3$, the long term behaviour is a period-2 cycle. (c) $a = 3.5$, the solution is a period-4 cycle

- A 2-cycle exists if and only if there are two points x_1^* and x_2^* for the given map $x_{n+1} = f(x_n)$ such that $f(x_1^*) = x_2^*$ and $f(x_2^*) = x_1^*$.
- Equivalently, such a solution must satisfy the relation $f(f(x_1^*)) = x_1^*$.
- For the logistic map, the period-2 solutions are obtained from the relations

$$x_2^* = ax_1^*(1 - x_1^*), \tag{1}$$
$$x_1^* = ax_2^*(1 - x_2^*). \tag{2}$$

- First we subtract (1) from (2) and obtain

$$\implies x_1^* + x_2^* = (1 + a)/a. \tag{3}$$

- We now multiply (1) by x_2^* and multiply (2) by x_1^* and subtract to obtain

$$x_1^* x_2^* = (x_1^* + x_2^*)/a = (1+a)/a^2. \tag{4}$$

- Eliminating x_2^* between (3) and (4), we have

$$a^2 x_1^{*2} - a(1+a)x_1^* + (1+a) = 0.$$

- Solving we find

$$x_1^* = \frac{(a+1) \pm \sqrt{(a+1)(a-3)}}{2a},$$
$$x_2^* = \frac{(a+1) \mp \sqrt{(a+1)(a-3)}}{2a}.$$

- In other words, we obtain the period- 2 equilibrium points as

$$x_{1,2}^* = \frac{(a+1) \pm \sqrt{(a+1)(a-3)}}{2a}. \tag{5}$$

- To obtain the stability condition for a period- 2 cycle, we define $|\delta_{n+1}| = |x_{n+1} - x_1^*|$ and $|\delta_{n+2}| = |x_{n+2} - x_2^*|$.

$$\begin{aligned} |\delta_{n+2}| &= |x_{n+2} - x_2^*| \\ &= |f(x_{n+1}) - x_2^*| \\ &= |f(x_1^* + \delta_{n+1}) - x_2^*| \\ &= \left| f(x_1^*) + f'(x_1^*)\,\delta_{n+1} - x_2^* \right| \\ &= \left| f'(x_1^*)\,\delta_{n+1} \right|, \end{aligned} \tag{6}$$

  where we have neglected higher orders in δ_{n+1} in the Taylor series expansion of $f(x_1^* + \delta_{n+1})$ about x_1^*.

- Similarly, expanding near x_2^* we obtain

$$|\delta_{n+1}| = \left| f'(x_2^*)\,\delta_n \right|. \tag{7}$$

- Substituting (7) into (6), we get

$$|\delta_{n+2}| = \left| f'(x_1^*)\, f'(x_2^*)\, \delta_n \right| .$$

- Now the stability condition $|\delta_{n+2}| / |\delta_n| < 1$ becomes

$$\left| f'(x_1^*)\, f'(x_2^*) \right| < 1.$$

- For the logistic map, the above stability condition for the period- 2 equilibrium point (5) becomes $\left|4 + 2a - a^2\right| < 1$.
- Suppose $s = 4 + 2a - a^2$ is positive.
- Then we must have $s < 1$ or $a^2 - 2a - 3 > 0$.
- The solutions of the equation $a^2 - 2a - 3 = 0$ are $a = -1, 3$.

- On the other hand, if s is negative then we must have $s > -1$ or $a^2 - 2a - 5 < 0$.
- The solutions of the equation $a^2 - 2a - 5 = 0$ are $a = 1 \pm \sqrt{6}$.
- It is clear that the condition $|s| < 1$ is satisfied for $a \in (1 - \sqrt{6}, -1)$ and $a \in (3, 1 + \sqrt{6})$.

Period Doubling Phenomenon

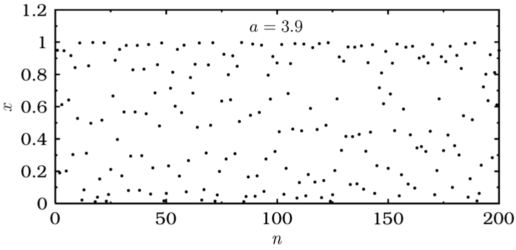
- We note that the period-1 solution $x^* = (a-1)/a$ is unstable for $a > 3$, because the stability determining slope crosses $f' = -1$ at $a = 3$.
- Similarly, the period- 2 solution born at this critical value becomes unstable at $a = 1 + \sqrt{6}$
- A stable period-4 solution again loses its stability when $f'(x_1^*)\, f'(x_2^*)\, f'(x_3^*)\, f'(x_4^*)$ becomes -1 .
- One can check that when the stability determining quantity of a period- k solution becomes -1 a bifurcation occurs giving birth to a stable period- $2k$ solution.
- We also note that to obtain the period-4 solution we have to solve a set of four coupled nonlinear algebraic equations.
- Similarly, to obtain the period- k solution, k-coupled equations have to be solved.

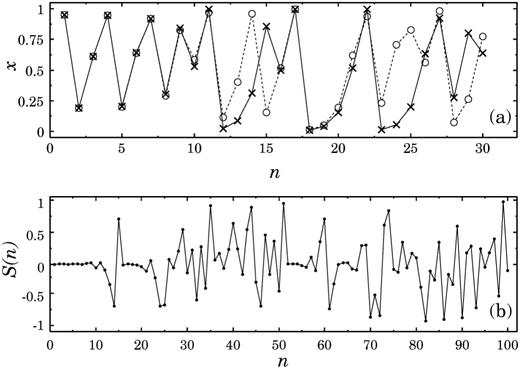
The combined analytical and numerical analysis gives the following picture.

- Period-1 solution exists in the range $0 < a < 3$, and it loses stability at the critical value of $a = a_1 = 3$, giving birth to (or bifurcating into) a period-2 cycle.
- Period-2 solution exists in the range $3 < a < 1 + \sqrt{6}(\approx 3.449)$, and it also loses stability at the critical value of $a = a_2 = 1 + \sqrt{6}$, giving birth to (or bifurcating into) a period $-4\left(= 2^2\right)$ cycle.
- Period- 2^2 cycle exists for $1 + \sqrt{6} < a < 3.544112$ and this solution bifurcates into a period $2^3(= 8)$ cycle solution at $a = a_3 = 3.544112$.
- This process proceeds further ad infinitum.

Period of the N-cycle	Bifurcation value of a
2^1	$a_1 = 3.000000$
2^2	$a_2 = 3.449489..$
2^3	$a_3 = 3.544112..$
2^4	$a_4 = 3.564445..$
2^5	$a_5 = 3.568809..$
2^6	$a_6 = 3.569745..$
2^∞	$a_\infty = 3.570000..$

Figure: First few bifurcation values of a


$a = 3.9$
x
n
0
0.2
0.4
0.6
0.8
1
1.2
0
50
100
150
200


x
n
(a)
$S(n)$
(b)
0
0.25
0.5
0.75
1
0
5
10
15
20
25
30
-1
-0.5
0
0.5
1
0
10
20
30
40
50
60
70
80
90
100

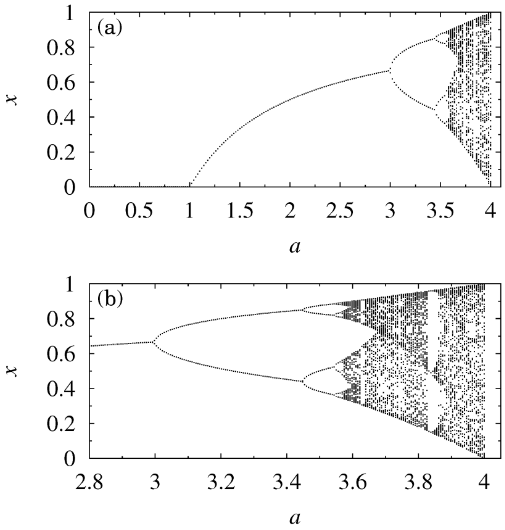


Figure: Bifurcation diagrams of the logistic map for (a) $a \in (0, 4)$, (b) $a \in (2.8, 4.0)$

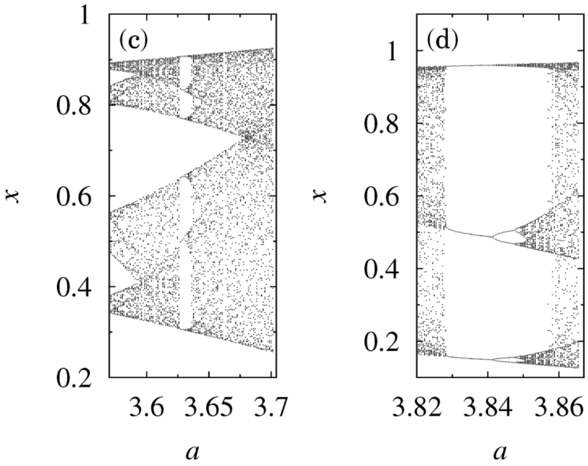


Figure: Bifurcation diagrams of the logistic map for (c) $a \in (3.57, 3.7)$, (d) $a \in (3.82, 3.87)$

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