Limits and Derivatives

Angad Bisen

1 Basic Definition

Limits describe how a function behaves near a point, instead of at that point, they don't depend on the actual value of the function at the limit. They describe how the function behaves when it gets close to the limit.
Condition for existence of limits:

$$RHL = LHL$$

2 In determinant Forms

If a limit is given in either of these forms, we must first change the equation so that once we put the value given in the limit, neither of these forms is obtained. There are **7** total in-determinant forms.
$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0^{\infty}, 0 \times \infty, 1^{\infty}, \infty^{0}$

3 Fundamental rules of Limits

$$Assuming \lim_{x \to a} f(x) = l_1 and \lim_{x \to a} g(x) = l_2$$

1. $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = l_1 \pm l_2$
2. $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = l_1 \cdot l_2$
3. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l_1}{l_2}$
4. $\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x) = c \cdot l_1$

4 Basic formulae and important results

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$$

Proof. Assuming $x = a + h$
And $x \to a, h \to 0$

$$\lim_{h \to 0} \frac{(a+h)^n - a^n}{a + h - a}$$

$$\lim_{h\to 0} \frac{(a+h)^n - a^n}{h}$$

$$\lim_{h\to 0} \frac{a^n[(1+\frac{h}{a})^n - 1]}{h}$$

$$\boldsymbol{(1+x)^n = 1 = n \cdot x + \frac{n \cdot (n-1)}{2!} \cdot x^2 + \dots}$$

$$a^n \lim_{h\to 0} \frac{\left[(1 + n \cdot \frac{h}{a} + \frac{n\cdot(n-1)}{2!}(\frac{h}{a})^2 + \dots) - 1\right]}{h}$$

$$a^n \lim_{h\to 0} \left[(\frac{n}{a} + \frac{n \cdot (n-1)}{2!} \cdot \frac{h}{a^2} + \dots)\right]$$

$$\boxed{\boldsymbol{n \cdot a^{n-1}}}$$

$\square$

4.1 Important expansions

- $a^x = 1 + x \cdot \ln a + \frac{(x \cdot \ln a)^2}{2!} + \frac{(x \cdot \ln a)^3}{3!} + \dots$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

4.2 Direct results

1. $$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$$

2. $$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

3. $$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

4. $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

5. $$\lim_{x\to 0} \cos x = 1$$

6.
$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

7.
$$\lim_{x\to 0} \frac{\arcsin x}{x} = 1$$

8.
$$\lim_{x\to 0} \frac{\arctan x}{x} = 1$$

9.
$$\lim_{x\to a} \frac{\sin (x-a)}{(x-a)} = 1$$

10.
$$\lim_{x\to a} \frac{\tan (x-a)}{(x-a)} = 1$$

5 Derivatives

$$y = f(x)$$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Method of solving through the help of a question.

Q. Find the differential coefficient of x^n w.r.t x using the *first principle.*

A. *let* $y = x^n \to (i)$
When x is increased by Δx, y is also increased by Δy
$(y + \Delta y = (x+\Delta x)^n \to (ii)$
Subtracting (i) and (ii),
$(y + \Delta y - y = (x+\Delta x)^n - x^n$
Dividing by Δx,
$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$
Taking $\lim_{\Delta x\to 0}$ on LHS and RHS,
$\frac{dy}{dx} = \frac{\lim_{\Delta x\to 0}(x+\Delta x)^n - x^n}{\Delta x}$
$\frac{dy}{dx} = \frac{\lim_{x\to 0} x^n \cdot \left[(1+\frac{\Delta x}{x})^n - 1\right]}{\frac{\Delta x}{x}\times x}$
$\frac{dy}{dx} = \lim_{x\to 0} n \cdot \frac{x^n}{x}$
$\frac{dy}{dx} = \lim_{x\to 0} n \cdot x^{n-1}$
$$\boxed{\frac{dy}{dx} = \lim_{x\to 0} n \cdot x^{n-1}}$$

5.1 Derivative results to remember

1.
$$\frac{d(k)}{dx} = 0; k \in const$$

2.
$$\frac{d(x^n)}{dx} = n \cdot x^{n-1}$$

3.
$$\frac{d(\log x)}{dx} = \frac{1}{|x|}$$

4.
$$\frac{d(e^x)}{dx} = e^x$$

5.
$$\frac{d(a^x)}{dx} = a^x \cdot \log a$$

5.1.1 Trigonometric Results

6.
$$\frac{d(\sin x)}{dx} = \cos x$$

7.
$$\frac{d(\cos x)}{dx} = -\sin x$$

8.
$$\frac{d(\tan x)}{dx} = \sec^2 x$$

9.
$$\frac{d(\csc x)}{dx} = -\csc x \cdot \cot x$$

10.
$$\frac{d(\sec x)}{dx} = \sec x \cdot \tan x$$

11.
$$\frac{d(\cot x)}{dx} = -\csc^2 x$$

$$\boxed{\frac{d(\sin ax)}{dx} = a \cdot \cos (ax)}$$

$$\boxed{\frac{d(\sin ax \pm b)}{dx} = a \cdot \cos (ax + b)}$$

5.1.2 Inverse Trigonometric Results

12.
$$\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

13.
$$\frac{d(\arccos x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

14.
$$\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$$

15.
$$\frac{d(\operatorname{arccsc} x)}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

16.
$$\frac{d(\operatorname{arcsec} x)}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

17.
$$\frac{d(\operatorname{arccot} x)}{dx} = \frac{-1}{1+x^2}$$

5.2 Fundamental rules of differentiation

-
$$y = u \pm v$$

$\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

-
$$y = u \cdot v$$

$\frac{dy}{dx} = \frac{u \cdot dv}{dx} + \frac{v \cdot du}{dx}$

5.3 Complex derivative results

1.
$$\lim_{x \to 0} 1 + x^{\frac{1}{x}} = e$$

2.
$$\lim_{x \to \infty} 1 + \frac{a}{x}^{x} = e^a$$

3.
$$\lim_{x \to \infty} \frac{x+a}{x+b}^{x} = e^{a-b}$$

Newton Leibnitz's Formula

$$I = \int_{\psi_1}^{\psi_2(x)} f(t) \cdot dt$$

$$\frac{dI}{dx} = [f(\psi_2(x)) \cdot \psi_2'(x) - f(\psi_1(x)) \cdot \psi_1'(x)]$$

4.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(\frac{r}{n}) or \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} f(\frac{r}{n}) = \int_0^1 f(x) \cdot d(x)$$

6 *L'Hôpital's rule*

L'Hospital's rule is a general method of evaluating indeterminate forms such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$
L'Hospital's rule states that,

$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty, g'(x) \neq 0 \forall$ x in I with $x \neq c$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = \lim_{x\to c} \frac{f''(x)}{g''(x)}}$$

7 Important points to solve problems quicker

1. for $(\frac{0}{0})$ form, use *factorisation.*
2. for $(\frac{\infty}{\infty})$ form, take *highest power of variable common from numerator and denominator.*
3. for $(0 \times \infty)$ form, put $x = \frac{1}{t}$, or slight change (*rationalisation, LCM etc.*) will convert form into $(\frac{0}{0})$ or $(\frac{\infty}{\infty})$.
4. for $(\infty - \infty)$ form, *use substitution, or slight change* will convert form into $(\frac{0}{0})$ or $(\frac{\infty}{\infty})$.
5. LH rule can be used to evaluate $(\frac{0}{0})$ or $(\frac{\infty}{\infty})$ forms.
6. If square root is present, use *rationalisation.*
7. If $[x], \{x\}, |x|, sgn(x)$ and functions with different definitions are present, then *LHL and RHL must be calculated differently.*
8. $\sqrt{x^2} = |x|$
9. **Always check the form of the question before attempting.**