Counting

Tutorial 10

The Product Rule

- If a procedure can be broken down into a sequence of m tasks, and
- there are $n_1, n_2, \ldots n_m$ ways to do the tasks,
- then there are $n_1 \cdot n_2 \cdot \cdots \cdot n_m$ ways to do the procedure.

The Sum Rule

- If a task can be done in one of n_1 ways,
- or in one of n_2 ways,
- $\ldots$
- or in one of n_m ways,
- where none of the set of n_i ways are the same as any of the n_j ways for all pairs i, j with $1 \le i < j \le m$,
- then there are $n_1 + n_2 + \cdots + n_m$ ways to do the task.

The Subtraction Rule

- If a task can be done in one of n_1 ways
- or in one of n_2 ways,
- and the set of n_1 ways and the set of n_2 ways have n ways in common,
- then there are $n_1 + n_2 - n$ ways to do the task.

The Division Rule

- If a task can be done using a procedure, which can be carried out in one of n ways,
- but for each way w, exactly d of n ways correspond to w,
- then there are n/d ways to do the task.

r-permutations

- If n is a non-negative integer and r is an integer with $0 \le r \le n$,
- then the number of r-permutations of a set with n **distinct** elements is:
- $P(n, r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$

1. How many binary strings (strings made up of 0s and 1s) of length 10 are there?

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   Solutions:

   - Select one character at a time: 2 choices for each character (0 or 1).
   - 10 characters total (10 tasks in our procedure).
   - Using the Product Rule: $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$

2. How many binary strings of length 10 start and end with 1s?

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Solutions:

- First and last characters are 1, so we are selecting 8 characters.
- Select one character at a time: 2 choices for each character (0 or 1).
- 8 characters total (8 tasks in our procedure).
- Using the Product Rule: 2^8.

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Solutions:

- We are counting 8-permutations of 8 distinct elements.
- $P(8, 8) = 8!$

4. How many permutations of the letters ABCDEFGH contain ED?

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Solutions:

- Distinct elements are: "A", "B", "C", "ED", "F", "G", "H".
- We are counting 7-permutations of 7 distinct elements.
- $P(7, 7) = 7!$

5. How many permutations of the letters ABCDEFGH contain CDE?

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Solutions:

- Distinct elements are: "A", "B", "CDE", "F", "G", "H".
- We are counting 6-permutations of 6 distinct elements.
- $P(6,6) = 6!$

6. How many permutations of the letters ABCDEFGH contain BA and FGH?

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Solutions:

- Distinct elements are: “BA”, “C”, “D”, “E”, “FGH”.
- We are counting 5-permutations of 5 distinct elements.
- $P(5,5) = 5!$

7. How many permutations of the letters ABCDEFGH contain BA or FGH?

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Solutions:

- How many contain BA? $P(7,7) = 7!$
- How many contain FGH? $P(6,6) = 6!$
- How many contain BA and FGH? $P(5,5) = 5!$
- How many contain BA or FGH? $7! + 6! - 5!$ by the Subtraction Rule.

8. How many 4 digit numbers are odd?

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Solutions:

- There are 9 choices for the first digit (1,2,...,9).
- There are 10 choices for the second digit.
- There are 10 choices for the third digit.
- There are 5 choices for the fourth (last) digit: 1,3,5,7,9.
- By the Product Rule, we have $9 \cdot 10 \cdot 10 \cdot 5 = 4500$.

9. We are playing a game with three dice – blue, red, and black. Each die has six sides, labelled with 1, 2, 3, 4, 5, or 6. We win if we roll the three dice and we get (at least) two values that are the same. How many ways are there to win?

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Solutions:

- Ways to win: XXY or XYX or YXX or XXX where X and Y are different.
- XXY: 6 ways to choose X and 5 ways to choose Y: $6 \cdot 5 = 30$ ways by the Product Rule
- XYX: 6 ways to choose X and 5 ways to choose Y: $6 \cdot 5 = 30$ ways by the Product Rule
- YXX: 6 ways to choose X and 5 ways to choose Y: $6 \cdot 5 = 30$ ways by the Product Rule
- XXX: 6 ways to choose X
- $30 + 30 + 30 + 6 = 96$ ways by the Sum Rule

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A different way to count the same thing:

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A different way to count the same thing:

- Either all values on the three dice are different OR at least two are the same.
- Number of possible rolls: $6 \cdot 6 \cdot 6 = 216$ by the Product Rule.
- Number of rolls where all values are different: $6 \cdot 5 \cdot 4 = 120$ by the Product Rule.
- Number of rolls where at least two value are the same: $216 - 120 = 96$.

10. How many positive integers less than 1,000 consist of distinct digits from $\{1, 3, 7, 9\}$?

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Solutions:

- 1-digit numbers: 4 ways
- 2-digit numbers: $4 \cdot 3 = 12$ ways (by the Product Rule)
- 3-digit numbers: $4 \cdot 3 \cdot 2 = 24$ ways (by the Product Rule)
- $4 + 12 + 24 = 40$ ways by the Sum Rule

11. We are making a tiny bracelet out of 6 beads, all different colours. We put the six beads on a string and then we join the ends of the string to make a bracelet. We are so good at this that nobody can tell where we made the join. How many possible bracelets can we make?

11. We are making a tiny bracelet out of 6 beads, all different colours. We put the six beads on a string and then we join the ends of the string to make a bracelet. We are so good at this that nobody can tell where we made the join. How many possible bracelets can we make?

Solutions:

- How many arrangements of 6 beads of all different colours on a string?
- We are counting 6-permutations of 6 distinct elements.
- $P(6, 6) = 6!$
- When we join the ends of the string, we realise that each choice of the first bead results in exactly the same set of arrangements.
- $6!/6 = 120$ bracelets by the Division Rule

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