Strong Induction

Tutorial 9

1 Strong Induction Review (do not submit)

Recall the Principle of Strong Induction we saw in class last week.

- What we prove: $\forall n \geq b, P(n)$
  - where $P(n)$ is a predicate (i.e., it has values True/False)
  - and universe of discourse is integers
- How we prove it:
  1. Prove $\forall k \geq b, (\forall b \leq i < k, P(i)) \rightarrow P(k)$:
     (a) Take arbitrary k and Suppose $k \geq b$.
     (b) Suppose $\forall b \leq i < k, P(i)$ (called **Induction Hypothesis (IH)**).
     (c) Prove $P(k)$.

Rule of Inference	**Name**
$\begin{array}{rl} & \forall k \geq b, (\forall b \leq i < k, P(i)) \rightarrow P(k) \\ \hline \therefore & \forall n \geq b, P(n) \end{array}$	Strong Induction

2 Strong Induction Exercises (to be submitted)

Please, aim to show solutions to the first two questions, and leave the rest as exercises.

2.1

The famous Fibonacci sequence is defined as follows.

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 1 \\
F_n &= F_{n-2} + F_{n-1}, \text{ for all } n \geq 2, n \in \mathbb{N}
\end{aligned}
$$

Perhaps surprisingly, the n^{th} Fibonacci number can be calculated for any n by using the following formula:

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}} \text{ where } \phi = \frac{1+\sqrt{5}}{2} \text{ and } \psi = \frac{1-\sqrt{5}}{2}$$

The number ϕ above is called the "golden ratio". Your task is to prove this result. Hint: First show that $\phi^2 = 1 + \phi$ and $\psi^2 = 1 + \psi$, and then use these results in your proof.

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The number ϕ above is called the "golden ratio". Your task is to prove this result. Hint: First show that $\phi^2 = 1 + \phi$ and $\psi^2 = 1 + \psi$, and then use these results in your proof.

Solutions:

Following the hint, we first show that $\phi^2 = 1 + \phi$ and $\psi^2 = 1 + \psi$:

$$\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1 + \phi$$

$$\psi^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2} = 1 + \psi \qquad (*)$$

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Let $P(n)$ be "$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$". To prove: $\forall n \geq 0, P(n)$.

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Let $P(n)$ be "$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$". To prove: $\forall n \geq 0, P(n)$.

To prove: $\forall k \geq 0, (\forall 0 \leq i < k, P(i)) \rightarrow P(k)$.

(1) Take arbitrary k

(2) Suppose $k \geq 0$

(3) Suppose $\forall 0 \leq i < k, P(i)$ Induction Hypothesis

(Rough work)

What are we trying to do?

$0 \leq k - 1 < k$ and $0 \leq k - 2 < k$

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$P(k-1)$ and $P(k-2)$ 3,11 U.M.P. (IH)

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$F_{k-1} = \dfrac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}}$ and $F_{k-2} = \dfrac{\phi^{k-2} - \psi^{k-2}}{\sqrt{5}}$ 12 def. of P

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$F_k = F_{k-1} + F_{k-2}$	def. of F

Spot the error!

$0 \leq k-1 < k$ and $0 \leq k-2 < k$

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$F_k = F_{k-1} + F_{k-2}$ def. of F

$F_0 = 0$
$F_1 = 1$
$F_n = F_{n-2} + F_{n-1}$, for all $n \geq 2, n \in \mathbb{N}$

Let $P(n)$ be "$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$".

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Suppose $k \geq 2$ case #3

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$F_k = F_{k-1} + F_{k-2}$ def. of F

Suppose $k = 0$ case #1

$$\frac{\phi^k - \psi^k}{\sqrt{5}} = \frac{\phi^0 - \psi^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} = 0 = F_0 = F_k$$ def. of F

$P(k)$ def. of P

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Suppose $k = 1$ case #2

$$\frac{\phi^k - \psi^k}{\sqrt{5}} = \frac{\phi^1 - \psi^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = 1 = F_1 = F_k \qquad \text{def. of } F$$

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$$\frac{\phi^k - \psi^k}{\sqrt{5}} = \frac{\phi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 = F_0 = F_k \qquad \text{def. of } F$$

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$P(k)$ def. of P

Suppose $k \geq 2$ case #3

$0 \leq k-1 < k$ and $0 \leq k-2 < k$

$P(k-1)$ and $P(k-2)$ U.M.P. (IH)

$$F_{k-1} = \frac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}} \text{ and } F_{k-2} = \frac{\phi^{k-2} - \psi^{k-2}}{\sqrt{5}} \qquad \text{def. of } P$$

$F_k = F_{k-1} + F_{k-2}$ def. of F

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(2)	Suppose $k \geq 0$	
(3)	Suppose $\forall 0 \leq i < k, P(i)$	Induction Hypothesis
(4)	Suppose $k = 0$	case #1
(5)	$\dfrac{\phi^k - \psi^k}{\sqrt{5}} = \dfrac{\phi^0 - \psi^0}{\sqrt{5}} = \dfrac{1-1}{\sqrt{5}} = 0 = F_0 = F_k$	4 def. of F
(6)	$P(k)$	5 def. of P
(7)	Suppose $k = 1$	case #2
(8)	$\dfrac{\phi^k - \psi^k}{\sqrt{5}} = \dfrac{\phi^1 - \psi^1}{\sqrt{5}} = \dfrac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = 1 = F_1 = F_k$	7 def. of F
(9)	$P(k)$	8 def. of P
(10)	Suppose $k \geq 2$	case #3
(11)	$0 \leq k-1 < k$ and $0 \leq k-2 < k$	
(12)	$P(k-1)$ and $P(k-2)$	3,11 U.M.P. (IH)
(13)	$F_{k-1} = \dfrac{\phi^{k-1} - \psi^{k-1}}{\sqrt{5}}$ and $F_{k-2} = \dfrac{\phi^{k-2} - \psi^{k-2}}{\sqrt{5}}$	12 def. of P
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	$= \dfrac{\phi^{k-1} + \phi^{k-2} - \psi^{k-1} - \psi^{k-2}}{\sqrt{5}}$	
	$= \dfrac{\phi^{k-2}(\phi + 1) - \psi^{k-2}(\psi + 1)}{\sqrt{5}}$	
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(16) $P(k)$ 4,6,7,9,10,15 cases

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(17) $(\forall 0 \leq i < k, P(i)) \rightarrow P(k)$ 3,16 implication

(18) $k \geq 0 \rightarrow ((\forall 0 \leq i < k, P(i)) \rightarrow P(k))$ 2,17 implication

(19) $\forall k \geq 0, (\forall 0 \leq i < k, P(i)) \rightarrow P(k)$ 1,18 U.G.

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(15)	$P(k)$	14 def. of P
(16)	$P(k)$	4,6,7,9,10,15 cases
(17)	$(\forall 0 \leq i < k, P(i)) \to P(k)$	3,16 implication
(18)	$k \geq 0 \to ((\forall 0 \leq i < k, P(i)) \to P(k))$	2,17 implication
(19)	$\forall k \geq 0, (\forall 0 \leq i < k, P(i)) \to P(k)$	1,18 U.G.

Therefore, $\forall n \geq 0, P(n)$ by Strong Induction.

2.2

Suppose x is a real number and $x \neq 0$ and $x + 1/x$ is an integer. Show that $x^n + 1/x^n$ is an integer for any $n \geq 0$. Hint: Begin by showing that, for an integer n:

$$x^n + \frac{1}{x^n} = \left(x + \frac{1}{x}\right)\left(x^{n-1} + \frac{1}{x^{n-1}}\right) - \left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

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Solutions:

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Solutions:

Following the hint, we first show that for any integer n and real $x \neq 0$:

$$\begin{aligned}
&\left(x + \frac{1}{x}\right)\left(x^{n-1} + \frac{1}{x^{n-1}}\right) - \left(x^{n-2} + \frac{1}{x^{n-2}}\right) \\
=&x^n + x^{n-2} + \frac{1}{x^{n-2}} + \frac{1}{x^n} - x^{n-2} - \frac{1}{x^{n-2}} \\
=&x^n + \frac{1}{x^n} \qquad (*)
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Let $P(n)$ be "$(x \in \mathbb{R}) \wedge (x \neq 0) \wedge (x + \frac{1}{x}$ is an integer$) \rightarrow (x^n + \frac{1}{x^n}$ is an integer$)$" or, more briefly, "$(x \in \mathbb{R}) \wedge (x \neq 0) \wedge (x + \frac{1}{x} \in \mathbb{Z}) \rightarrow (x^n + \frac{1}{x^n} \in \mathbb{Z})$". To prove: $\forall n \geq 0, P(n)$.

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Solutions:

Following the hint, we first show that for any integer n and real $x \neq 0$:

$$\begin{aligned}
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=&x^n + x^{n-2} + \frac{1}{x^{n-2}} + \frac{1}{x^n} - x^{n-2} - \frac{1}{x^{n-2}} \\
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To prove: $\forall k \geq 0, (\forall 0 \leq i < k, P(i)) \rightarrow P(k)$

(1) Take arbitrary k

(2) Suppose $k \geq 0$

(3) Suppose $\forall 0 \leq i < k, P(i)$ Induction Hypothesis

(1) Take arbitrary k

(2) Suppose $k \geq 0$

(3) Suppose $\forall 0 \leq i < k, P(i)$ Induction Hypothesis

(4) Suppose $(x \in \mathbb{R}) \wedge (x \neq 0) \wedge (x + \dfrac{1}{x} \in \mathbb{Z})$

(1) Take arbitrary k

(2) Suppose $k \geq 0$

(3) Suppose $\forall 0 \leq i < k, P(i)$ Induction Hypothesis

(4) Suppose $(x \in \mathbb{R}) \wedge (x \neq 0) \wedge (x + \frac{1}{x} \in \mathbb{Z})$

(17) $P(k)$ 16 def. of P

(18) $(\forall 0 \leq i < k, P(i)) \rightarrow P(k)$ 3,17 implication

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(20) $\forall k \geq 0, (\forall 0 \leq i < k, P(i)) \rightarrow P(k)$ 1,19 U.G.

Therefore, $\forall n \geq 0, P(n)$ by Strong Induction.

(10)	$x^k + \frac{1}{x^k} = \left(x + \frac{1}{x}\right)\left(x^{k-1} + \frac{1}{x^{k-1}}\right) - \left(x^{k-2} + \frac{1}{x^{k-2}}\right)$	4, (*)
(11)	$x + \frac{1}{x} \in \mathbb{Z}$	4
(12)	$x^{k-1} + \frac{1}{x^{k-1}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-1 < k)$
(13)	$x^{k-2} + \frac{1}{x^{k-2}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-2 < k)$

(9)	Suppose $k \geq 2$	case #3
(10)	$x^k + \frac{1}{x^k} = \left(x + \frac{1}{x}\right)\left(x^{k-1} + \frac{1}{x^{k-1}}\right) - \left(x^{k-2} + \frac{1}{x^{k-2}}\right)$	4, (*)
(11)	$x + \frac{1}{x} \in \mathbb{Z}$	4
(12)	$x^{k-1} + \frac{1}{x^{k-1}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-1 < k)$
(13)	$x^{k-2} + \frac{1}{x^{k-2}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-2 < k)$

(5)	Suppose $k = 0$	case #1
(6)	$x^k + \frac{1}{x^k} = x^0 + \frac{1}{x^0} = 2 \in \mathbb{Z}$	4,5

(9)	Suppose $k \geq 2$	case #3
(10)	$x^k + \frac{1}{x^k} = \left(x + \frac{1}{x}\right)\left(x^{k-1} + \frac{1}{x^{k-1}}\right) - \left(x^{k-2} + \frac{1}{x^{k-2}}\right)$	4, (*)
(11)	$x + \frac{1}{x} \in \mathbb{Z}$	4
(12)	$x^{k-1} + \frac{1}{x^{k-1}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-1 < k)$
(13)	$x^{k-2} + \frac{1}{x^{k-2}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-2 < k)$

(5)	Suppose $k=0$	case #1
(6)	$x^k+\dfrac{1}{x^k}=x^0+\dfrac{1}{x^0}=2\in\mathbb{Z}$	4,5
(7)	Suppose $k=1$	case #2
(8)	$x^k+\dfrac{1}{x^k}=x^1+\dfrac{1}{x^1}=x+\dfrac{1}{x}\in\mathbb{Z}$	4,7
(9)	Suppose $k\geq 2$	case #3
(10)	$x^k+\dfrac{1}{x^k}=\left(x+\dfrac{1}{x}\right)\left(x^{k-1}+\dfrac{1}{x^{k-1}}\right)-\left(x^{k-2}+\dfrac{1}{x^{k-2}}\right)$	4, (*)
(11)	$x+\dfrac{1}{x}\in\mathbb{Z}$	4
(12)	$x^{k-1}+\dfrac{1}{x^{k-1}}\in\mathbb{Z}$	3,9 U.M.P. (IH) $(0\leq k-1<k)$
(13)	$x^{k-2}+\dfrac{1}{x^{k-2}}\in\mathbb{Z}$	3,9 U.M.P. (IH) $(0\leq k-2<k)$

(5)	Suppose $k = 0$	case #1
(6)	$x^k + \frac{1}{x^k} = x^0 + \frac{1}{x^0} = 2 \in \mathbb{Z}$	4,5
(7)	Suppose $k = 1$	case #2
(8)	$x^k + \frac{1}{x^k} = x^1 + \frac{1}{x^1} = x + \frac{1}{x} \in \mathbb{Z}$	4,7
(9)	Suppose $k \geq 2$	case #3
(10)	$x^k + \frac{1}{x^k} = \left(x + \frac{1}{x}\right)\left(x^{k-1} + \frac{1}{x^{k-1}}\right) - \left(x^{k-2} + \frac{1}{x^{k-2}}\right)$	4, (*)
(11)	$x + \frac{1}{x} \in \mathbb{Z}$	4
(12)	$x^{k-1} + \frac{1}{x^{k-1}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-1 < k)$
(13)	$x^{k-2} + \frac{1}{x^{k-2}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-2 < k)$
(14)	$x^k + \frac{1}{x^k} \in \mathbb{Z}$	10-13 all integers
(15)	$x^k + \frac{1}{x^k} \in \mathbb{Z}$	5-9,14 cases
(16)	$(x \in \mathbb{R}) \wedge (x \neq 0) \wedge (x + \frac{1}{x} \in \mathbb{Z}) \to (x^k + \frac{1}{x^k} \in \mathbb{Z})$	4,15 implication

Let $P(n)$ be "$(x \in \mathbb{R}) \wedge (x \neq 0) \wedge (x + \frac{1}{x}$ is an integer$) \rightarrow (x^n + \frac{1}{x^n}$ is an integer$)$" or, more briefly, "$(x \in \mathbb{R}) \wedge (x \neq 0) \wedge (x + \frac{1}{x} \in \mathbb{Z}) \rightarrow (x^n + \frac{1}{x^n} \in \mathbb{Z})$". To prove: $\forall n \geq 0, P(n)$.

To prove: $\forall k \geq 0, (\forall 0 \leq i < k, P(i)) \rightarrow P(k)$

(1)	Take arbitrary k	
(2)	Suppose $k \geq 0$	
(3)	Suppose $\forall 0 \leq i < k, P(i)$	Induction Hypothesis
(4)	Suppose $(x \in \mathbb{R}) \wedge (x \neq 0) \wedge (x + \frac{1}{x} \in \mathbb{Z})$	
(5)	Suppose $k = 0$	case #1
(6)	$x^k + \frac{1}{x^k} = x^0 + \frac{1}{x^0} = 2 \in \mathbb{Z}$	4,5
(7)	Suppose $k = 1$	case #2
(8)	$x^k + \frac{1}{x^k} = x^1 + \frac{1}{x^1} = x + \frac{1}{x} \in \mathbb{Z}$	4,7
(9)	Suppose $k \geq 2$	case #3
(10)	$x^k + \frac{1}{x^k} = \left(x + \frac{1}{x}\right)\left(x^{k-1} + \frac{1}{x^{k-1}}\right) - \left(x^{k-2} + \frac{1}{x^{k-2}}\right)$	4, (*)
(11)	$x + \frac{1}{x} \in \mathbb{Z}$	4
(12)	$x^{k-1} + \frac{1}{x^{k-1}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-1 < k)$
(13)	$x^{k-2} + \frac{1}{x^{k-2}} \in \mathbb{Z}$	3,9 U.M.P. (IH) $(0 \leq k-2 < k)$
(14)	$x^k + \frac{1}{x^k} \in \mathbb{Z}$	10-13 all integers
(15)	$x^k + \frac{1}{x^k} \in \mathbb{Z}$	5-9,14 cases
(16)	$(x \in \mathbb{R}) \wedge (x \neq 0) \wedge (x + \frac{1}{x} \in \mathbb{Z}) \rightarrow (x^k + \frac{1}{x^k} \in \mathbb{Z})$	4,15 implication
(17)	$P(k)$	16 def. of P
(18)	$(\forall 0 \leq i < k, P(i)) \rightarrow P(k)$	3,17 implication
(19)	$k \geq 0 \rightarrow ((\forall 0 \leq i < k, P(i)) \rightarrow P(k))$	2,18 implication
(20)	$\forall k \geq 0, (\forall 0 \leq i < k, P(i)) \rightarrow P(k)$	1,19 U.G.

Therefore, $\forall n \geq 0, P(n)$ by Strong Induction.

Strong Induction

Tutorial 9