

# Induction

Tutorial 8

## 1.1

In class we saw several proofs of statements of the form “For every integer  $n$ , such that  $n \geq 1, \dots$ ”. We used the proof format below:

- What we prove:  $\forall n \geq 1, P(n)$ 
  - where  $P(n)$  is a predicate (i.e., it has values True/False)
  - and universe of discourse is integers
- How we prove it:
  1. Prove  $P(1)$  (called **Base Case**).
  2. Prove  $\forall k \geq 1, P(k) \rightarrow P(k+1)$  (called **Induction Step**):
    - (a) Take arbitrary  $k$  and Suppose  $k \geq 1$ .
    - (b) Suppose  $P(k)$  (called **Induction Hypothesis (IH)**).
    - (c) Prove  $P(k+1)$ .

Rule of Inference	Name
$  \begin{array}{c}  P(1) \\  \forall k \geq 1, P(k) \rightarrow P(k+1) \\  \hline  \therefore \forall n \geq 1, P(n)  \end{array}  $	(Simple) Induction

## 1.2

Sometimes  $P(n)$  is true for all integers starting with some integer larger than 1. We use the same reasoning and the same format to prove “For every integer  $n$ , such that  $n \geq b$ , ...” :

- What we prove:  $\forall n \geq b, P(n)$ 
  - where  $P(n)$  is a predicate (i.e., it has values True/False)
  - and universe of discourse is integers
- How we prove it:
  1. Prove  $P(b)$  (called **Base Case**).
  2. Prove  $\forall k \geq b, P(k) \rightarrow P(k+1)$  (called **Induction Step**):
    - (a) Take arbitrary  $k$  and Suppose  $k \geq b$ .
    - (b) Suppose  $P(k)$  (called **Induction Hypothesis (IH)**).
    - (c) Prove  $P(k+1)$ .

Rule of Inference	Name
$  \begin{array}{c}  P(b) \text{ where } b \geq 1 \\  \forall k \geq b, P(k) \rightarrow P(k + 1) \\  \hline  \therefore \forall n \geq b, P(n)  \end{array}  $	(Simple) Induction

Note to self: write on board

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For every integer  $n \geq 1$ ,  $n^3 - n$  is divisible by 3.

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We notice that, with this definition of  $P$ , we have:

- $P(1)$  is “ $\exists m, 1^3 - 1 = 3m$ ”.
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To prove:  $\forall n \geq 1, P(n)$ .

Base Case:  $1^3 - 1 = 1 - 1 = 0 = 3 \cdot 0$ . Therefore,  $\exists m, 1^3 - 1 = 3m$ . Therefore,  $P(1)$ .

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- (1) Take arbitrary  $k$
- (2) Suppose  $k \geq 1$
- (3) Suppose  $P(k)$

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3 definition of  $P$

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4 E.I.

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|------|---|---------------------|
| (1)  | Take arbitrary $k$  |                     |
| (2)  | Suppose $k \geq 1$  |                     |
| (3)  | Suppose $P(k)$  |                     |
| (4)  | $\exists m, k^3 - k = 3m$                                     | 3 definition of $P$ |
| (5)  | $k^3 - k = 3a$  | 4 E.I.              |
|      |   |                     |
| (8)  | $\exists m, (k + 1)^3 - (k + 1) = 3m$                         | 6,7 E.G.            |
| (9)  | $P(k + 1)$  | 8 definition of $P$ |
| (10) | $P(k) \rightarrow P(k + 1)$                                   | 3,9 implication     |
| (11) | $k \geq 1 \rightarrow (P(k) \rightarrow P(k + 1))$            | 2,10 implication    |
| (12) | $\forall k, k \geq 1 \rightarrow (P(k) \rightarrow P(k + 1))$ | 1,11 U.G.           |

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- (2) Suppose  $k \geq 1$
- (3) Suppose  $P(k)$
- (4)  $\exists m, k^3 - k = 3m$  3 definition of  $P$
- (5)  $k^3 - k = 3a$  4 E.I.
- (6) 
$$\begin{aligned}(k+1)^3 - (k+1) &= (k+1)(k^2 + 2k + 1) - (k+1) \\&= k^3 + 2k^2 + k + k^2 + 2k + 1 - k - 1 \\&= k^3 + 3k^2 + 2k \\&= (k^3 - k) + 3k^2 + 3k \\&= 3a + 3k^2 + 3k && \text{5 I.H.} \\&= 3(a + k^2 + k)\end{aligned}$$
- (7)  $a + k^2 + k$  is an integer
- (8)  $\exists m, (k+1)^3 - (k+1) = 3m$  6,7 E.G.
- (9)  $P(k+1)$  8 definition of  $P$
- (10)  $P(k) \rightarrow P(k+1)$  3,9 implication
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Therefore,  $\forall n \geq 1, P(n)$  by (Simple) Induction.

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To prove:  $\forall n \geq 4, P(n)$ .

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To prove:  $\forall n \geq 4, P(n)$ .

Base Case:  $2^4 = 16 < 24 = 4!$ . Therefore,  $P(4)$ .

Induction Step:

- (1) Take arbitrary  $k$
- (2) Suppose  $k \geq 4$
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3,6 implication

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- (3) Suppose  $P(k)$
- (4)  $2^k < k!$
- (5)  $(k+1)! = (k+1)k!$   
 $> (k+1) \cdot 2^k$   
 $> 2 \cdot 2^k$   
 $= 2^{k+1}$
- (6)  $P(k+1)$
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