

Induction

Tutorial 8

1.1

In class we saw several proofs of statements of the form “For every integer n , such that $n \geq 1, \dots$ ”. We used the proof format below:

- What we prove: $\forall n \geq 1, P(n)$
 - where $P(n)$ is a predicate (i.e., it has values True/False)
 - and universe of discourse is integers
- How we prove it:
 1. Prove $P(1)$ (called **Base Case**).
 2. Prove $\forall k \geq 1, P(k) \rightarrow P(k + 1)$ (called **Induction Step**):
 - (a) Take arbitrary k and Suppose $k \geq 1$.
 - (b) Suppose $P(k)$ (called **Induction Hypothesis (IH)**).
 - (c) Prove $P(k + 1)$.

Rule of Inference	Name
$\frac{P(1) \quad \forall k \geq 1, P(k) \rightarrow P(k + 1)}{\therefore \forall n \geq 1, P(n)}$	(Simple) Induction

1.2

Sometimes $P(n)$ is true for all integers starting with some integer larger than 1. We use the same reasoning and the same format to prove “For every integer n , such that $n \geq b, \dots$ ” :

- What we prove: $\forall n \geq b, P(n)$
 - where $P(n)$ is a predicate (i.e., it has values True/False)
 - and universe of discourse is integers
- How we prove it:
 1. Prove $P(b)$ (called **Base Case**).
 2. Prove $\forall k \geq b, P(k) \rightarrow P(k + 1)$ (called **Induction Step**):
 - (a) Take arbitrary k and Suppose $k \geq b$.
 - (b) Suppose $P(k)$ (called **Induction Hypothesis (IH)**).
 - (c) Prove $P(k + 1)$.

Rule of Inference	Name
$P(b) \text{ where } b \geq 1$ $\frac{\forall k \geq b, P(k) \rightarrow P(k + 1)}{\therefore \forall n \geq b, P(n)}$	(Simple) Induction

Note to self: write on board

2.1

For every integer $n \geq 1$, $n^3 - n$ is divisible by 3.

2.1

For every integer $n \geq 1$, $n^3 - n$ is divisible by 3.

Solutions:

Let universe of discourse be integers.

2.1

For every integer $n \geq 1$, $n^3 - n$ is divisible by 3.

Solutions:

Let universe of discourse be integers. Let $P(n)$ be “ $n^3 - n$ is divisible by 3”, that is “ $\exists m, n^3 - n = 3m$ ”.

2.1

For every integer $n \geq 1$, $n^3 - n$ is divisible by 3.

Solutions:

Let universe of discourse be integers. Let $P(n)$ be “ $n^3 - n$ is divisible by 3”, that is “ $\exists m, n^3 - n = 3m$ ”.

We notice that, with this definition of P , we have:

- $P(1)$ is “ $\exists m, 1^3 - 1 = 3m$ ”.
- $P(k)$ is “ $\exists m, k^3 - k = 3m$ ”.
- $P(k + 1)$ is “ $\exists m, (k + 1)^3 - (k + 1) = 3m$ ”.

2.1

For every integer $n \geq 1$, $n^3 - n$ is divisible by 3.

Solutions:

Let universe of discourse be integers. Let $P(n)$ be “ $n^3 - n$ is divisible by 3”, that is “ $\exists m, n^3 - n = 3m$ ”.

We notice that, with this definition of P , we have:

- $P(1)$ is “ $\exists m, 1^3 - 1 = 3m$ ”.
- $P(k)$ is “ $\exists m, k^3 - k = 3m$ ”.
- $P(k + 1)$ is “ $\exists m, (k + 1)^3 - (k + 1) = 3m$ ”.

To prove: $\forall n \geq 1, P(n)$.

- $P(1)$ is “ $\exists m, 1^3 - 1 = 3m$ ”.
- $P(k)$ is “ $\exists m, k^3 - k = 3m$ ”.
- $P(k + 1)$ is “ $\exists m, (k + 1)^3 - (k + 1) = 3m$ ”.

To prove: $\forall n \geq 1, P(n)$.

Base Case: $1^3 - 1 = 1 - 1 = 0 = 3 \cdot 0$. Therefore, $\exists m, 1^3 - 1 = 3m$. Therefore, $P(1)$.

- $P(1)$ is “ $\exists m, 1^3 - 1 = 3m$ ”.
- $P(k)$ is “ $\exists m, k^3 - k = 3m$ ”.
- $P(k + 1)$ is “ $\exists m, (k + 1)^3 - (k + 1) = 3m$ ”.

To prove: $\forall n \geq 1, P(n)$.

Base Case: $1^3 - 1 = 1 - 1 = 0 = 3 \cdot 0$. Therefore, $\exists m, 1^3 - 1 = 3m$. Therefore, $P(1)$.

Induction Step:

- (1) Take arbitrary k
- (2) Suppose $k \geq 1$
- (3) Suppose $P(k)$

- $P(1)$ is “ $\exists m, 1^3 - 1 = 3m$ ”.
- $P(k)$ is “ $\exists m, k^3 - k = 3m$ ”.
- $P(k + 1)$ is “ $\exists m, (k + 1)^3 - (k + 1) = 3m$ ”.

To prove: $\forall n \geq 1, P(n)$.

Base Case: $1^3 - 1 = 1 - 1 = 0 = 3 \cdot 0$. Therefore, $\exists m, 1^3 - 1 = 3m$. Therefore, $P(1)$.

Induction Step:

(1) Take arbitrary k

(2) Suppose $k \geq 1$

(3) Suppose $P(k)$

(4) $\exists m, k^3 - k = 3m$

3 definition of P

- $P(1)$ is “ $\exists m, 1^3 - 1 = 3m$ ”.
- $P(k)$ is “ $\exists m, k^3 - k = 3m$ ”.
- $P(k + 1)$ is “ $\exists m, (k + 1)^3 - (k + 1) = 3m$ ”.

To prove: $\forall n \geq 1, P(n)$.

Base Case: $1^3 - 1 = 1 - 1 = 0 = 3 \cdot 0$. Therefore, $\exists m, 1^3 - 1 = 3m$. Therefore, $P(1)$.

Induction Step:

(1) Take arbitrary k

(2) Suppose $k \geq 1$

(3) Suppose $P(k)$

(4) $\exists m, k^3 - k = 3m$ 3 definition of P

(5) $k^3 - k = 3a$ 4 E.I.

- $P(1)$ is “ $\exists m, 1^3 - 1 = 3m$ ”.
- $P(k)$ is “ $\exists m, k^3 - k = 3m$ ”.
- $P(k + 1)$ is “ $\exists m, (k + 1)^3 - (k + 1) = 3m$ ”.

To prove: $\forall n \geq 1, P(n)$.

Base Case: $1^3 - 1 = 1 - 1 = 0 = 3 \cdot 0$. Therefore, $\exists m, 1^3 - 1 = 3m$. Therefore, $P(1)$.

Induction Step:

(1)	Take arbitrary k	
(2)	Suppose $k \geq 1$	
(3)	Suppose $P(k)$	
(4)	$\exists m, k^3 - k = 3m$	3 definition of P
(5)	$k^3 - k = 3a$	4 E.I.

(8)	$\exists m, (k + 1)^3 - (k + 1) = 3m$	6,7 E.G.
(9)	$P(k + 1)$	8 definition of P
(10)	$P(k) \rightarrow P(k + 1)$	3,9 implication
(11)	$k \geq 1 \rightarrow (P(k) \rightarrow P(k + 1))$	2,10 implication
(12)	$\forall k, k \geq 1 \rightarrow (P(k) \rightarrow P(k + 1))$	1,11 U.G.

Induction Step:

- (1) Take arbitrary k
- (2) Suppose $k \geq 1$
- (3) Suppose $P(k)$
- (4) $\exists m, k^3 - k = 3m$ 3 definition of P
- (5) $k^3 - k = 3a$ 4 E.I.
- (6)
$$\begin{aligned} (k+1)^3 - (k+1) &= (k+1)(k^2 + 2k + 1) - (k+1) \\ &= k^3 + 2k^2 + k + k^2 + 2k + 1 - k - 1 \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \\ &= 3a + 3k^2 + 3k \\ &= 3(a + k^2 + k) \end{aligned}$$
 5 I.H.
- (7) $a + k^2 + k$ is an integer
- (8) $\exists m, (k+1)^3 - (k+1) = 3m$ 6,7 E.G.
- (9) $P(k+1)$ 8 definition of P
- (10) $P(k) \rightarrow P(k+1)$ 3,9 implication
- (11) $k \geq 1 \rightarrow (P(k) \rightarrow P(k+1))$ 2,10 implication
- (12) $\forall k, k \geq 1 \rightarrow (P(k) \rightarrow P(k+1))$ 1,11 U.G.

Therefore, $\forall n \geq 1, P(n)$ by (Simple) Induction.

2.2

For every integer $n \geq 4$, $2^n < n!$.

Solutions:

2.2

For every integer $n \geq 4$, $2^n < n!$.

Solutions:

Let universe of discourse be integers.

2.2

For every integer $n \geq 4$, $2^n < n!$.

Solutions:

Let universe of discourse be integers. Let $P(n)$ be “ $2^n < n!$ ”.

2.2

For every integer $n \geq 4$, $2^n < n!$.

Solutions:

Let universe of discourse be integers. Let $P(n)$ be “ $2^n < n!$ ”.

We notice that, with this definition of P , we have:

- $P(4)$ is “ $2^4 < 4!$ ”.
- $P(k)$ is “ $2^k < k!$ ”.
- $P(k + 1)$ is “ $2^{k+1} < (k + 1)!$ ”.

2.2

For every integer $n \geq 4$, $2^n < n!$.

Solutions:

Let universe of discourse be integers. Let $P(n)$ be “ $2^n < n!$ ”.

We notice that, with this definition of P , we have:

- $P(4)$ is “ $2^4 < 4!$ ”.
- $P(k)$ is “ $2^k < k!$ ”.
- $P(k + 1)$ is “ $2^{k+1} < (k + 1)!$ ”.

To prove: $\forall n \geq 4, P(n)$.

2.2

For every integer $n \geq 4$, $2^n < n!$.

Solutions:

Let universe of discourse be integers. Let $P(n)$ be “ $2^n < n!$ ”.

We notice that, with this definition of P , we have:

- $P(4)$ is “ $2^4 < 4!$ ”.
- $P(k)$ is “ $2^k < k!$ ”.
- $P(k + 1)$ is “ $2^{k+1} < (k + 1)!$ ”.

To prove: $\forall n \geq 4, P(n)$.

Base Case: $2^4 = 16 < 24 = 4!$. Therefore, $P(4)$.

Induction Step:

- (1) Take arbitrary k
- (2) Suppose $k \geq 4$
- (3) Suppose $P(k)$

We notice that, with this definition of P , we have:

- $P(4)$ is “ $2^4 < 4!$ ”.
- $P(k)$ is “ $2^k < k!$ ”.
- $P(k + 1)$ is “ $2^{k+1} < (k + 1)!$ ”.

Induction Step:

- (1) Take arbitrary k
- (2) Suppose $k \geq 4$
- (3) Suppose $P(k)$
- (4) $2^k < k!$

We notice that, with this definition of P , we have:

- $P(4)$ is “ $2^4 < 4!$ ”.
- $P(k)$ is “ $2^k < k!$ ”.
- $P(k+1)$ is “ $2^{k+1} < (k+1)!$ ”.

3 definition of P

Induction Step:

- (1) Take arbitrary k
- (2) Suppose $k \geq 4$
- (3) Suppose $P(k)$
- (4) $2^k < k!$

We notice that, with this definition of P , we have:

- $P(4)$ is “ $2^4 < 4!$ ”.
- $P(k)$ is “ $2^k < k!$ ”.
- $P(k+1)$ is “ $2^{k+1} < (k+1)!$ ”.

3 definition of P

- (6) $P(k+1)$ 5 definition of P
- (7) $P(k) \rightarrow P(k+1)$ 3,6 implication
- (8) $k \geq 4 \rightarrow (P(k) \rightarrow P(k+1))$ 2,7 implication
- (9) $\forall k, k \geq 4 \rightarrow (P(k) \rightarrow P(k+1))$ 1,8 U.G.

Therefore, $\forall n \geq 4, P(n)$ by (Simple) Induction.

Induction Step:

(1) Take arbitrary k

(2) Suppose $k \geq 4$

(3) Suppose $P(k)$

(4) $2^k < k!$

(5) $(k+1)! = (k+1)k!$

$$> (k+1) \cdot 2^k$$

$$> 2 \cdot 2^k$$

$$= 2^{k+1}$$

We notice that, with this definition of P , we have:

- $P(4)$ is “ $2^4 < 4!$ ”.

- $P(k)$ is “ $2^k < k!$ ”.

- $P(k+1)$ is “ $2^{k+1} < (k+1)!$ ”.

3 definition of P

4 I.H.

from 2

5 definition of P

3,6 implication

2,7 implication

1,8 U.G.

Therefore, $\forall n \geq 4, P(n)$ by (Simple) Induction.