Probability Theory

Math 4175

§3.2. Discrete Random Variable

A discrete random variable X consists of a finite set S (called the sample space) together with a probability distribution p defined on S. Here X varies over the elements of S.

Notation: The probability that the random variable X takes on the value x is denoted by $\Pr[X = x]$ or, for short, $\Pr[x]$ or $p[x]$.

Example 1: Suppose that a fair coin is tossed and let S be the set of all possible outcomes. Then $S = \{H, T\}$ where $p[H] = 0.5$ and $p[T] = 0.5$.

Example 2: Suppose that a fair coin is tossed twice and let S be the set of all possible outcomes. Then $S = \{HH, HT, TH, TT\}$ where $p[HH] = p[HT] = p[TH] = p[TT] = 0.25$.

In these two examples, S is said to have equally likely outcomes.

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Example 3: Suppose that a fair coin is tossed twice and let S be the number of possible heads. Then $S = \{0, 1, 2\}$ where $p[0] = p[2] = 0.25$ and $p[1] = 0.5$.

Example 4: Suppose that a fair red die and a fair blue die are rolled and S be the sum of the dots showing in both together.

Then $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ with following probability distribution:

X	2	3	4	5	6	7	8	9	10	11	12
p	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

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Roll of a red and a green dice:

R/B	1	2	3	4	5	6
1	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$	$(1, 6)$
2	$(2, 1)$	$(2, 2)$	$(2, 3)$	$(2, 4)$	$(2, 5)$	$(2, 6)$
3	$(3, 1)$	$(3, 2)$	$(3, 3)$	$(3, 4)$	$(3, 5)$	$(3, 6)$
4	$(4, 1)$	$(4, 2)$	$(4, 3)$	$(4, 4)$	$(4, 5)$	$(4, 6)$
5	$(5, 1)$	$(5, 2)$	$(5, 3)$	$(5, 4)$	$(5, 5)$	$(5, 6)$
6	$(6, 1)$	$(6, 2)$	$(6, 3)$	$(6, 4)$	$(6, 5)$	$(6, 6)$

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Notice that $0 \leq p[x] \leq 1$ and $\sum_{x \in S} p[x] = 1$.

Any subset E of S is called an event and $p[E] = \sum_{x \in E} p[x]$.

For example, let $E = \{2, 4, 6, 8, 10, 12\}$, that is all even sum. Then $p[E] = 0.5$.

§3.2. Joint Probability

Suppose that X is a random variable defined on S and Y is a random variable defined on T. The joint probability $p[x, y]$ is the probability that X takes on the value x and Y takes on the value y.

Warning: In general, $p[x, y] \neq p[x]p[y]$.

Example 5: As earlier, let us suppose that a fair red die and a fair blue die are rolled and S be the sum of the dots in both together. Then $S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Furthermore, let $T = \{E, O, N\}$ where Y takes the value E if both dice show even number of dots, Y takes the value O if both dice show odd, and Y takes the value N if one die shows even and other one shows odd.

Notice that $p[4, E] = \frac{1}{36}$ because $(2, 2)$ is the only outcome in this case. But on the other hand, $p[4]p[E] = \frac{3}{36}\frac{9}{36} = \frac{1}{48}$.

§3.2. Joint Probability

Example 6: Roll a die. Suppose we are interested in two things: whether the number of dots is odd **and** whether the number is at least four.
This can be modeled as follows:

- X: "observe whether face is even or odd"
  $S = \{e, o\}$; $p_X(e) = p_X(o) = 1/2$.
- Y: "observe whether face is high (4, 5, or 6) or low (1, 2, or 3)"
  $T = \{h, \ell\}$; $p_Y(h) = p_Y(\ell) = 1/2$.

Then $Z = (X, Y)$ gives us the result of both experiments together. The sample space for $Z = (X, Y)$ is $S \times T = \{(e, h); (e, \ell); (o, h); (o, \ell)\}$ with joint probability

- $p_Z[e, h] = 1/3$, $p_Z[e, \ell] = 1/6$.
- $p_Z[o, h] = 1/6$, $p_Z[o, \ell] = 1/3$.

Note that the probability that the number of dots is odd **and** greater than 3 is $p[Z = (o, h)] = 1/6$. This is not equal to

$$p[X = o] \cdot p[Y = h] = (1/2)(1/2) = 1/4.$$

§3.2. Conditional Probability

Suppose that X is a random variable defined on S and Y is a random variable defined on T. The conditional probability $p[x|y]$ is the probability that X takes on the value x given that (or known that) Y takes on the value y.

In the previous example 5, notice that $p[4|E] = \frac{1}{9}$ and $p[E|4] = \frac{1}{3}$. Now

$$p[4|E]p[E] = \frac{1}{9}\frac{9}{36} = \frac{1}{36} = p[4, E].$$

Formulas:

1. $p[x, y] = p[x|y]p[y]$
2. $p[x, y] = p[y|x]p[x]$

§3.2. Conditional Probability

Suppose that X is a random variable defined on S and Y is a random variable defined on T. Then X and Y are said to be independent random variables if $p[x|y] = p[x]$.

Notice that X and Y are independent random variables if and only if $p[x, y] = p[x]p[y]$ for all x and y.

We already noticed in our example that $p[4, E] \neq p[4]p[E]$.

So X and Y in our example are NOT independent random variables.

Example 7: Suppose that you toss a fair coin 4 times. Let X be the number of heads in the first two tosses and Y be the number of heads in the last two tosses. Then one can check that X and Y are independent random variable. In other words, $p[x|y] = p[x]$ for all x and y, or in other words, $p[x, y] = p[x]p[y]$ for all x and y.

§3.2. Conditional Probability

For example 6, we have,

$$p[h|e] = \frac{p_Z[e,h]}{p_X[e]} = \frac{1/3}{1/2} = \frac{2}{3}$$

$$p[h|o] = \frac{p_Z[o,h]}{p_X[o]} = \frac{1/6}{1/2} = \frac{1}{3}$$

So X and Y are not independent.

§3.2. Baye's Theorem

Baye's Theorem: If X and Y are two random variables and $p[y] > 0$, then

$$p[x|y] = \frac{p[y|x]p[x]}{p[y]}$$

Theorem follows from: $p[x|y]p[y] = p[x,y] = p[y|x]p[x]$.

In our example 5, notice that

$$p[4|E] = \frac{1}{9}, \;\; p[4] = \frac{3}{36}, \;\; p[E] = \frac{9}{36}$$

$$\frac{p[4|E]p[E]}{p[4]} = \frac{\frac{1}{9}\frac{9}{36}}{\frac{3}{36}} = \frac{1}{3} = p[E|4]$$