Essay

Introduction to coordinate transformation systems and how they are applied in the detection of stellar wakes

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1 Introduction of coordinate systems

The need to point to a target differing from your own position has lasted as long as there has been an observer and an observed. The ancient Greeks, for instance, in their effort to map the night sky, used a sort of spherical coordinate system to most accurately represent the sky above them. However, it was much later that a French mathematician, going by the name René Descartes, published an article on coordinate systems in 1637. The legend surrounding the creation of such system says that Descartes, while in sick bed, was observing a fly crawling around on the ceiling. Trying to describe the fly's position, he came up with a system where he could take the corner of the room as the origin and use the ceiling tiles to measure the fly's position using a set of two real number coordinates.
The purpose of this essay is to introduce the basic Cartesian and polar systems in order to understand the cylindrical, spherical, galactic, and galactocentric systems.

1.1 Different systems

Cartesian: The Cartesian coordinate system is perhaps the most widely recognized system in the world. Maps, computer graphics, geometric grids, and pretty much anyone who wishes to display points in space use the Cartesian system because of its simple-to-understand design. The system is a space where every point is described by a set of real numbers called coordinates. The coordinates mark the distances from two or more perpendicular-oriented lines called the axes. Assuming a 3D space, the first coordinate marks the distance in units on the x-axis, the second coordinate on the y-axis, and the third on the z-axis, however the Cartesian space can describe any n-dimensional Euclidian space with a set of n coordinates. While Cartesian space also allows for more generalized cases where the axes do not need to be perpendicular to each other, these cases will not be covered here. [1]

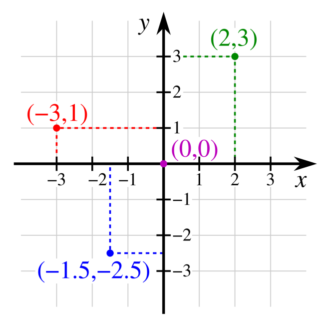


Figure 1: Example of a 2D Cartesian coordinate plane and a set of coordinates.
Source: Wikipedia

Polar: The polar coordinate system specifies a point on a plane using distance and angle as it's coordinates. The center is referred to as the pole, the distance component is called the radial coordinate, and the angle is called the angular coordinate. It is mainly used when it is important to specify the direction of an event. The polar coordinate system can be extended into the third dimension in two different ways: the cylindrical coordinate system, and the spherical coordinate system. The former case introduces a height component to create a cylindrical space. There, the components are (r, θ, z), where ρ and θ represent the radius and angle, and z represents the height from the origin plane. The spherical system creates a spherical space, where the added component, aside from ρ and θ is the angle measured between the pole and the radial component φ.
Radar systems for instance are a good example of a real-world application for the polar

coordinate system, as the observer's surroundings are shown in a way that can accurately portray where an event is happening relative to the observer. Even if distance is not known or is not accurate, the direction may provide enough information to locate an object or event.

Quick guide to celestial reference systems: There is a number of celestial reference systems, each solving a fundamental problem in astronomy - how to accurately refer to an object in a way that everyone can understand, and more importantly, find it. When leaving the confines of our planet, rather quickly a very noticeable problem emerges with the way we refer to objects not bound by the gravitational pull of Earth. On the planet, one thing has been decided for us, and that is the direction of *down*. A simple concept which becomes more ambiguous the higher you go from the surface. To illustrate this point, at a height of around $400km$ (around 6.3% the radius of the Earth) , the International Space Station is in a low-earth orbit (LEO) around the planet, constantly falling towards the people underneath and yet never reaching them. Due to this, the astronauts on board experience a weightless state, where the problem of differentiating from *up* and *down* starts to appear. To combat this dizzying confusion, a number of reference systems have been proposed to help the average hitchhiker reliably orient themselves in space. Among these, the most commonly used systems are horizontal (alt-az), equatorial, ecliptic, galactic, and supergalactic. [2, 3]

Horizontal - Also known as **alt-az** (short for "altitude-azimuth", which are the two angular components in a spherical coordinate system), is perhaps the most widely used system for Earth-based observations. This system divides the sky into two spheres, the upper sphere with everything above the horizon, and the lower sphere where things stay below the horizon. It a very observer-centric system, as this system differs from user to user based on their location on Earth. [4]

Equatorial - The equatorial coordinate system is a system in spherical coordinates. As opposed to the horizontal system, the equatorial coordinate system is a geocentric system where the origin of the coordinates is the center of the Earth. The sphere is divided into two hemispheres by a projection of the equator to the celestial sphere. The coordinate system consists of two components - declination and right ascension. Declination is measured from the equator to the pole using angles, the right ascension measures the angle between the primary direction and the observable spot. This system remains fixed against the background stars. [5]

Ecliptic - The ecliptic coordinate system is a spherical coordinate system. It is similar to the equatorial system in many ways, but the main difference is that the dividing line between the north and south hemispheres is tilted with respect to the equator. The ecliptic line is the orbital plane of Earth around the Sun, or in other words, the ecliptic is the apparent path the Sun takes in the sky in the course of 365 days. The center of this system can be either the Sun or the Earth. The primary direction is the March equinox, and has a right-hand convention. Using the geocentric version, the coordinates in this system are (λ, β, Δ), where the first two represent the angle measured along the ecliptic plane from the primary direction and angle from the ecliptic to the pole. The last one represents distance. The heliocentric system uses a different set of symbols (l, b, r), but they have the same meaning as the geocentric system. [6]

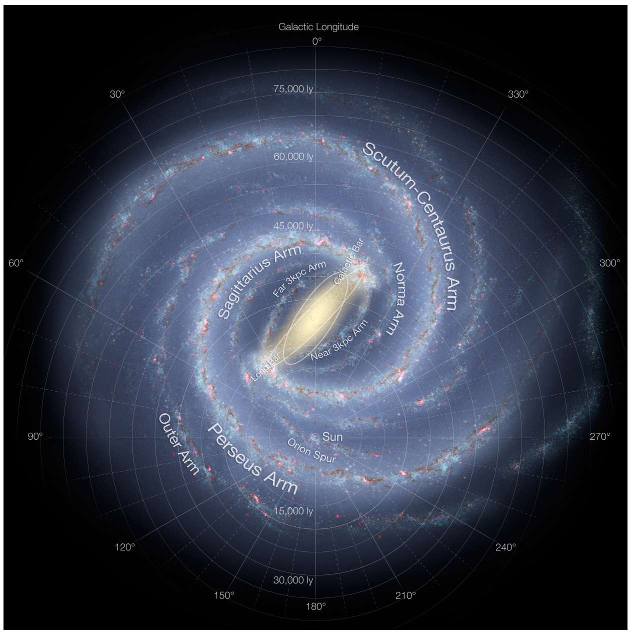


Figure 2: Example of galactic coordinates. Source: Wikipedia

<u>Galactic -</u> This is a celestial coordinate system in spherical coordinates. This system has the Sun in its origin and the primary path in the direction of the center of the Milky Way. The fundamental plane runs parallel to the approximation of the galactic plane with a slight offset. As with most systems, this uses the right-handed convention as well. This system uses (l, b, r) as its coordinates (longitude, latitude, and radius respectively). The longitude is measured eastward along the galactic equator from the Galactic Center. Latitude measures the angle of an object northward of the galactic equator as viewed from Earth. [7]

It may initially seem that some of these coordinate systems are haphazardly thrown together by a madman throwing darts in a dark room against a selection of keywords and just releasing them into the wild. One may as well wonder why such complexities are necessary, why not just pick one system and stick to it? The reality of the matter is that all of these systems have evolved to serve a specific purpose. But that isn't to say that they aren't connected in any way. It is possible to reference an object in one system and have it transformed into another using coordinate transformation systems.

2 Transformation between coordinate systems

In this section I will dive into the basics of coordinate transformation as well as bring up different ways to transform from one system to another.

Coordinate transformation in its most basic form is taking the axes of a coordinate system and manipulating it in any desired way to get a new coordinate system. In certain cases the vectors or points within the system don't change their position when transforming the coordinate systems, but instead the manipulation of the axes gives the points and vectors new coordinate references based on the new system. In other cases, the points within a system experience a translation or mirroring, in which case the axes remain as they were originally, but the points have now changed position. A simple example of the former would be the transformation from cartesian to polar coordinates. On a 2D Cartesian plane, the point $P = (2, 2)$ would be represented by a vector pointing from the origin to P. The vector pointing to P has a magnitude of $2\sqrt{2}$ units. After redefining the reference system to a polar coordinate system, the vector's position would be then described as $(2\sqrt{2}, \frac{\pi}{4})$ (transformation steps shown later). P has now been described using two different reference frames, through nothing about P itself has changed.

2.1 Transformations

To understand the different types of conversions from one coordinate system to another, we must first take a look at the different types of transformation systems which describe what happens to the poor coordinates caught in the crossfires of mathematicians. Each system describes a way to deform or move things in a space while preserving its original structure. The main types of transformations are *translation, rotation, reflection, and dilation*, where all but the first are some sort of linear transformations. [8, 9]
Linear transformations are usually presented by matrices and a transformation is denoted as $\mathbf{T}$.
Linear transformation - A linear transformation (also called as a linear map or vector space homomorphism) is a mapping between two vector spaces $\mathbf{V} \to \mathbf{W}$ that preserves the operations of vector addition and scalar multiplication. Mapping from $\mathbf{V} \to \mathbf{W}$ always maps the origin of $\mathbf{V}$ to the origin of $\mathbf{W}$. In linear algebra, a linear map is marked as $\mathbf{T}(\mathbf{s}) = \mathbf{A}\mathbf{s}$, where $\mathbf{A}$ is a matrix, and s is a vector being multiplied by the matrix $\mathbf{A}$. Linear maps can be presented as matrices.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where x and y are the transformed coordinates, a, b, c, d, or simply the matrix $\mathbf{A}$, describe the vectors of the new system in reference to the old system, and u and v are the coordinates in the old system.
In equation form this can be written as

$$x = au + bv, \quad y = cu + dv$$

In short, a linear transformation allows you to present a set of coordinates in a different reference system.
Translation - A translation is in essence a way to move a coordinate or every point of an object to another position while maintaining its original shape. This can be interpreted as shifting the origin of the coordinate system. Translation is given as $\mathbf{T_s}(\mathbf{p}) = \mathbf{p} + \mathbf{v}$, where $\mathbf{p}$ is the coordinate of a point and s is a translation vector.
Rotation - A rotation transformation allows you to rotate an object counterclockwise through an angle θ. This is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where θ is the angle you wish to rotate the system by. The rotation happens around the origin $\mathbf{O}$, so that when a set of points have their center of mass exactly in $\mathbf{O}$, the set of points will seemingly rotate around the origin. If the center of mass lies outside the origin, the object's movement during the translation can be described as "orbiting" the origin.
Reflection - A reflection is a mapping from a Euclidian space to itself. The process reflects the original object or set of points along an imaginary axis or plane of reflection. The reflection transformation preserves the distance from the axis of reflection (*isometry*) and has fixed point mapping in dimension $n-1$ (*hyperplane*), where n is the dimension you're working in, meaning that the reference system stays the same, and instead the object itself is moved through the space, along with everything which exists in that space. The end result is an object or set of points in mirror image of the original. When a reflection transformation is applied twice, then the object or set of points returns to its original place (*involution*)
For a 2D space, the reflection across an arbitrary line through the origin can be described by

$$Ref_l(v) = 2\frac{v \cdot l}{l \cdot l} l - v$$

where v is the vector being reflected and l is the vector of the line across which the reflection will be made. $v \cdot l$ is the dot product of v and l.
For a reflection through any n-dimensional hyperplane in Euclidian space through the origin, you use the following formula

$$Ref_a(v) = v - 2\frac{v \cdot a}{a \cdot a} a$$

where a is the orthogonal vector form the reflection plane.
Dilation - A dilation transformation changes the size of the object but not the shape. It is a function from a metric space $\mathbf{M}$ into itself that satisfies the identity

$$d(f(x), f(y)) = rd(x, y)$$

for all points $x, y \in \mathbf{M}$, where $d(x, y)$ is the distance from x to y and r is the scale factor, by which the distances between x and y are scaled.

2.2 Common coordinate conversions

Now that the transformation systems have been reviewed, the we can go into how exactly does a transformation from one coordinate system to another take place. For the most part, the systems are a variation of Cartesian and Polar coordinates, so we will go over the transformations between those systems. [3]

Cartesian $\rightarrow$ Cartesian: Transformations between two different Cartesian spaces follow the previously mentioned transformation methods. Most of the necessary work can be done using one or a combination of these transformations.

Cartesian → Polar (→ Cartesian again): A transformation between these two coordinate systems can be done using the following formulas

$$r = \sqrt{x^2 + y^2} \quad \textit{and} \quad \varphi = \arctan 2(y, x)$$

where

$$\arctan 2(y,x) = \begin{cases} \arctan(\frac{y}{x}), & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi, & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(\frac{y}{x}) - \pi, & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2}, & \text{if } x = 0 \text{ and } y > 0 \\ \frac{\pi}{2}, & \text{if } x = 0 \text{ and } y < 0 \\ undefined, & \text{if } x = 0 \text{ and } y = 0 \end{cases} \tag{1}$$

Here, x and y are the respective coordinates in Cartesian space. Using this formula, it's possible to show where the results from the previous example came from.
If we take point $P = (2, 2)$, the conversion would look like this

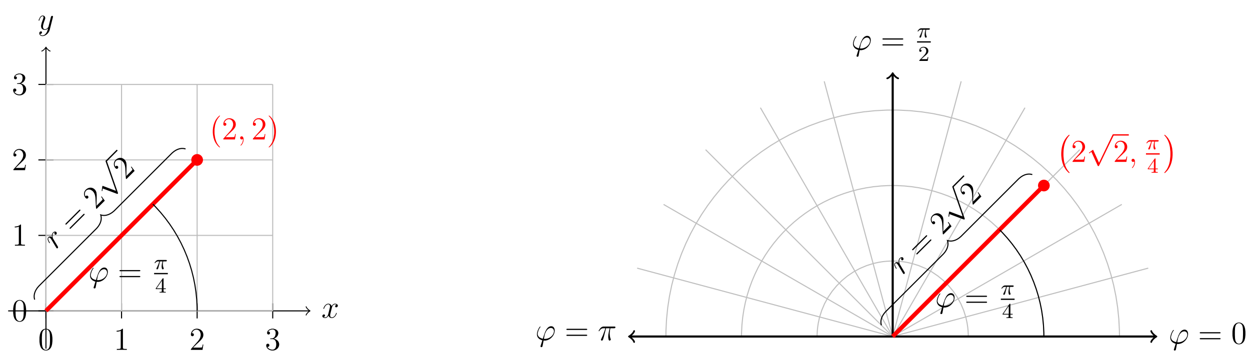
$$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \approx 2.83 \quad \textit{units}$$

similarly, the angle φ is found

$$\varphi = \arctan 2(y, x) = [\text{since } x = 2 \text{ and } y = 2\text{, this matches the case where } \arctan 2(y,x) = \arctan\left(\frac{y}{x}\right)] =$$

$$= \arctan\left(\frac{2}{2}\right) = \arctan(1) = \frac{\pi}{4}$$

Now we can see, that in fact the coordinates provided by the transformation does indeed correspond to the same vector in different reference systems.



(a) A line from the origin to $(2, 2)$ on a Cartesian grid.

(b) A line from the pole to $(2\sqrt{2}, \frac{\pi}{4})$ on a Polar coordinate grid.

Figure 3: Side-by-side comparison in Cartesian vs. Polar.

A reverse transformation from *Polar* → *Cartesian* would look like this:

$$\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi. \end{aligned} \tag{2}$$

This can also be presented as

$$\frac{\partial(x,y)}{\partial(r,\varphi} = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$$

This brings out an interesting point. To transform from Polar to Cartesian, a map is applied to the Polar coordinates

$$F : (r,\varphi) \rightarrow (x,y)$$

The Jacobian matrix of F is the 2×2 matrix of all first-order partial derivatives: [10]

$$J_F(r,\varphi) = \frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$$

The determinant of this matrix gives us the value r and basically tells us how the area elements scale under the change of variables. This transformation is only possible from Polar to Cartesian, as a reverse process, or an inverse Jacobian, does not give an accurate enough representation of how the system behaves as it only provides us with a local linear map and does not scale well to larger areas.

Spherical coordinates → 3D Cartesian (→ Spherical again): In a 3D space, the transformation follows a similar pattern. To transform from a spherical coordinate system to a 3D Cartesian, the formula is as follows

$$\begin{aligned} x &= \rho\sin\theta cos\varphi, \\ y &= \rho\sin\theta sin\varphi, \\ z &= \rho\cos\theta \end{aligned} \tag{3}$$

where ρ is the radial component from the origin, and θ, φ are the angular components in the spherical coordinates. In matrix form this would look like

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{bmatrix} \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\theta & -\rho\sin\theta & 0 \end{bmatrix} \tag{4}$$

and the volume element is revealed as

$$dxdydz = det\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}d\rho d\theta d\varphi = \rho^2\sin\theta d\rho d\theta d\varphi.$$

This basically shows how volume changes when tiny changes are made in the coordinates. Now, to transform all of this back to Spherical coordinates

$$\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \varphi &= \arctan\left(\frac{y}{x}\right) = \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \arcsin\left(\frac{Y}{\sqrt{x^2+y^2}}\right) \end{aligned} \tag{5}$$

and

$$\frac{\partial(\rho,\theta,\varphi)}{\partial(x,y,z)} = \begin{bmatrix} \frac{x}{\rho} & \frac{y}{\rho} & \frac{z}{\rho} \\ \frac{xz}{\rho^2\sqrt{x^2+y^2}} & \frac{yz}{\rho^2\sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{\rho^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{bmatrix} \tag{6}$$

Once again, the matrix defines how the target coordinate system changes when coordinates are changed. This is the best approximation of how a space behaves during a transformation.

Cylindrical coordinates → 3D Cartesian (→ Cylindrical again):

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z \end{aligned} \tag{7}$$

and

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

To transform back to Cylindrical coordinates

$$\begin{aligned} r &= \sqrt{x^2+y^2}, \\ \theta &= \arctan(\tfrac{y}{x}), \\ z &= z \end{aligned} \tag{9}$$

and

$$\frac{\partial(\rho,\theta,z)}{\partial(x,y,z)} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

Spherical coordinates → Cylindrical (→ Spherical again): Spherical coordinates → Cylindrical

$$\begin{aligned} r &= \rho\sin\varphi, \\ h &= \rho\cos\varphi, \\ \theta &= \theta \end{aligned} \tag{11}$$

and

$$\frac{\partial(r,h,\theta)}{\partial(\rho,\varphi,\theta)} = \begin{bmatrix} sin\varphi & \rho\cos\varphi & 0 \\ \cos\varphi & -\rho sin\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

where

$$det\frac{\partial(r,h,\theta)}{\partial(\rho,\varphi,\theta)} = -\rho$$

Cylindrical $\rightarrow$ Spherical

$$\begin{aligned} \rho &= \sqrt{r^2 + h^2}, \\ \theta &= \arctan(\tfrac{r}{h}), \\ \varphi &= \varphi \end{aligned} \tag{13}$$

Matrix form:

$$\frac{\partial(\rho, \theta, \varphi)}{\partial(r, h, \theta)} = \begin{bmatrix} \frac{r}{\sqrt{r^2+h^2}} & \frac{h}{\sqrt{r^2+h^2}} & 0 \\ \frac{h}{r^2+h^2} & \frac{-r}{r^2+h^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

where

$$det \frac{\partial(\rho, \theta, \varphi)}{\partial(r, h, \theta)} = -\frac{1}{\sqrt{r^2 + h^2}}$$

3 Importance of coordinate transformation systems in the search for dark matter

3.1 Background info

Now that we have covered the basics of coordinate transformation systems, it's time to dive into how is this all related to the search for dark matter. An essay on coordinate transformation systems may be enough to excite a room full of mathematicians, but to the layperson, this doesn't tell much in terms of who uses this outside of lecture halls and exhausted students presenting last second projects.

> "The story so far:
> In the beginning the Universe was created. This has made a lot of people very angry and been widely regarded as a bad move."
>
> *Douglas Adams:*
> *The Restaurant At The End of The Universe*

After the Big Bang the average temperature in the Universe dropped enough for the particles within to start forming matter. As time progressed, matter started gathering into larger clumps. Then there's a bit of a mystery about what happened next, but the result of it all was that the Universe started to be occupied by galaxies. Unimaginable collections of gas and dust orbiting a center of mass (usually a super massive black hole). Somewhere down the timeline, our own planet was created within the Milky Way galaxy, and even more recently, at some point, humans appeared. While many may argue, that this was also a mistake, the general consensus surrounding this is that there's not much to be done about it at this point. Be that as it may, one thing is for certain: the universe now has some observers to observe it (at least in our neck of the cosmic woods). And what we've found out about the Universe has been troubling scientist for nearly a century now.

In the 1900s[1] scientists began to notice a certain discrepancy in the predicted mass of cluster and galaxies, and the observed velocities of object in orbit around them. In the 1930s, a Swiss astronomer named Fritz Zwicky was observing the Coma Cluster, when he also reached the conclusion that the observable mass and the gravitational pull from the cluster on the orbiting galaxies did not match up the calculations. An estimation was made that the cluster was in fact 400 times more massive than was visually observed. He thought that because we cannot observe the missing mass, some of it has to be hidden from view. Though now it has been shown that Zwickys estimation for the mass was off, the idea itself wasn't. Further investigations into galaxy rotation curves showed that even galaxies were hiding some extra mass somewhere, making the velocities of stars in orbit higher than was expected and did not meet the expectations of Keplerian orbits. Today we've come to know this phenomenon as "dark-matter", for it's apparent ability to be visibly not so apparent. Today, countless observations have now confirmed, that *something* is boosting the mass of cosmic bodies. Generally, the effects of dark matter are apparent on larger scales. What we haven't yet determined, is what it actually is. There are many theories as to what it may be - WIMPs, axions, primordial black holes, etc. Unfortunately, due to it having almost no directly observable properties, this all remains theory.

3.2 Detection of stellar wakes within the Milky Way

Now that I've summarized the history of one of the most exciting mysteries our Universe holds into a short intro, let's return to the original question of "what do coordinate transformation systems have anything to do with dark matter?" In short: coordinate transformation systems are important tools to help us detect dark matter in small scale environments. It is understood how dark matter behaves in large scale structures such as the intergalactic medium and the best prediction we have to date is that dark matter behaves like CDM (Cold Dark Matter) [11]. Additionally we can observe its effects on matter by measuring the velocities of stars in other galaxies. To test our hypotheses, however, it is also important to see how dark matter behaves in small scale structures like galactic and sub-galactic environments. And one way to search for dark matter within these environments would be to look for stellar wakes within the Milky Way galaxy. CDM predicts, that lumps of dark matter in galactic or sub-galactic environments would interact with stars within the galaxy and alter their orbits forming wake like structures. Modeling these interactions based on what we know on how dark matter behaves, we can generate a simulation to test our theories. The best test for this would be to see how the *Large Magellanic Cloud* (LMC) moves through the halo of the Milky Way [12], and what the resulting interaction looks like. Now this is the part where coordinate transformations join the hunt for the invisible. Running idealized dark matter simulations and through applying a series of coordinate translations and rotations, it is possible to generate mock observations of such processes.

As the simulations are computationally very demanding, the easiest way to go about this problem is to run your simulation in a regular Cartesian reference frame, to avoid any unnecessary additional computations. Within this frame, you would place a point mass object

[1] Actually this question arose a bit earlier, nearing the end of the 19th century

and then run a wind tunnel simulation on this box, but instead of wind, the particles would represent environment in the Milky Way halo. The simulation would mimic the LMC moving through the galactic medium, and simulate the change of trajectories of stars within the galaxy's halo.

One of the most important parts in this process is data visualization. This is what gives us the final result of what the simulation looks like after it has run its course. The main thing which is checked for in these simulations is the overdensity distribution of matter within the the test, as this reveals how mass has repositioned itself after the perturbation from the dark matter source. While a plot of the simulation results itself may give an overview of what happens at the heart of the process, a coordinate transformation is needed to visualize how the results would look like if they were observed from the perspective of Earth.

The simulation box exists in its own reference frame and is independent from other frames of reference. A crucial step in visualizing the results is to apply these coordinate transformations to match up the simulation to reality. The simulation box itself does not provide sufficient information as to how the results would look like if we were to try and observe them. So, to get an accurate visual representation, a rotational transformation is needed to match up the velocity vectors from the simulation to the LMC's velocity vectors. This assures that the simulation's and the galaxy's orientations match up. Then, a translation would set the simulation box into the same space where the current day LMC would be. As the orbit in the simulation is a straight line, an additional piece of information is needed on the orbit of the LMC around the Milky Way. It's possible to take a function of time which describes its orbit, and fit the straight orbit from the simulation to match the curvature of the real orbit. The target celestial reference system for this transformation is ideally the galactic coordinate system. As this system has it's origin in the Solar system, such a transformation would give us a picture on how we would observe this simulation, if it were to take place in our universe. A final step in this process would be to convert the result into a Mollweide projection to show a 3D sphere on a 2D surface.

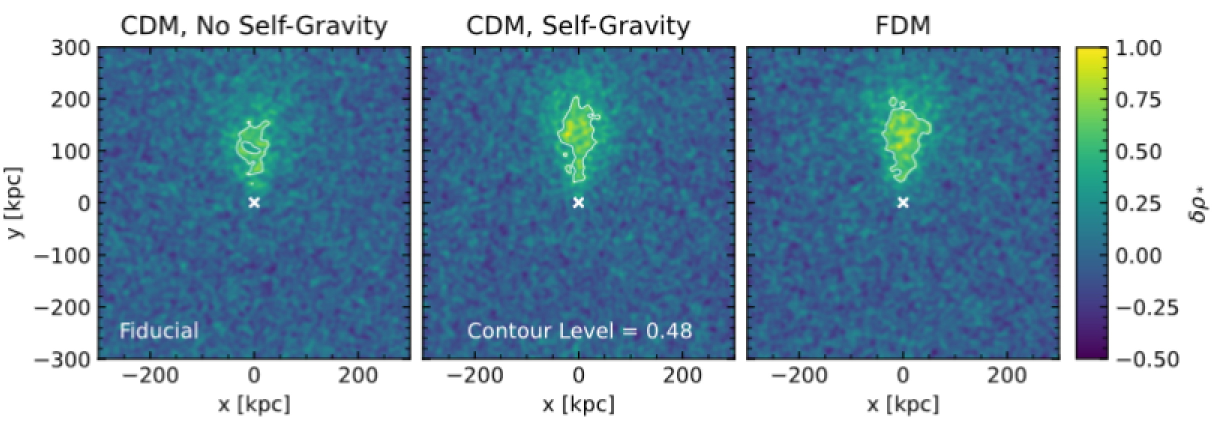


Figure 4: Example of a dark matter simulations with different dark matter models.
Source:
Foote et al 2023

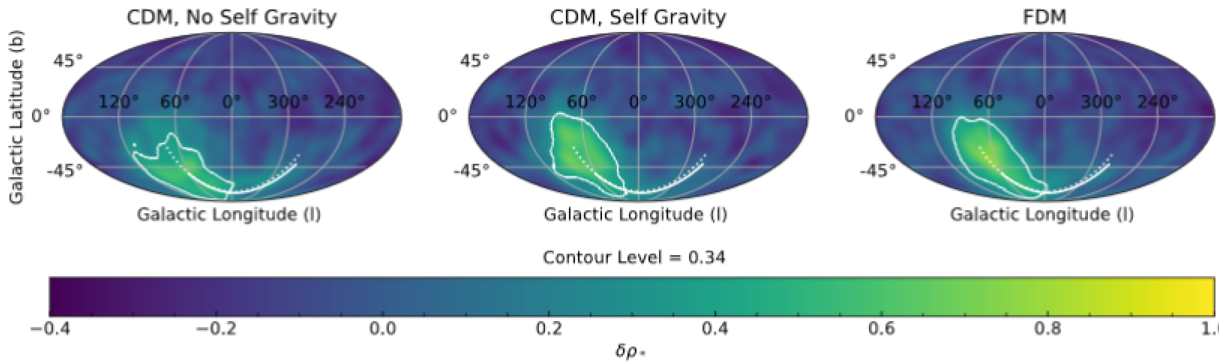


Figure 5: Dark matter simulation results projected into a Mollweide projection in a galactic reference frame. Plots include orbits of the LMC as well.
Source:

Foote et al 2023

4 Conclusion

This is a good example on how a purely theoretical construct in linear algebra can be put into practice. As we've seen, coordinate transformation systems are a useful tool when it comes to visualizing cosmological simulations. In the search for dark matter, they have become crucial to testing our theories, as it helps to visualize the results, which in turn helps us test the theories on dark matter. And even if we're not tackling the biggest mysteries in our universe, transformations of reference frames have helped shape the quality of life in every day processes, be it rotating images on your computer, steering a drone or converting GPS coordinates into a map on your phone.

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