A little book about motion

Lesson 18: Uniform circular motion

Circular motion is an important example of two-dimensional motion. In this lesson we will learn how to describe the position, velocity, and acceleration of such motion. It's crucial to understand that circular motion is *accelerated* motion, hence there is always a net force acting on an object that is moving in a circle. We give this net force a new name, which is a bit unfortunate because many students make the mistake of thinking that it is an additional force to the ones already present. In this course we will only consider circular motion with a constant speed. More general circular motion is covered in more advanced courses on rotational motion.

When it comes to circles, you need to know that the **radian measure of an angle** $\Delta\theta$ subtending a circular arc of length s and radius r is defined as

$$\Delta\theta \equiv \frac{\text{arclength}}{\text{radius}} = \frac{s}{r}$$

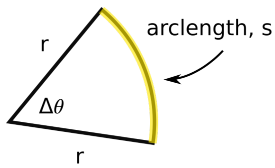


Figure 1: The definition of a radian.

This expression is often used in the form $s = r\Delta\theta$. Note that s and r are both lengths (measured in for example meters), so the ratio will not have a physical unit - it is a **dimensionless** quantity. So "radians" is not really a unit at all, just a label we use to specify that we are not working in degrees. When the angle is very small, we also often approximate the arc length to a straight line (zoom in on the circumference of a circle and it will start looking straight).

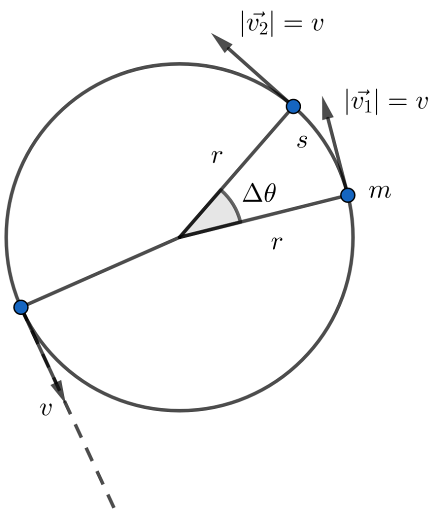


Figure 2: Uniform circular motion.

Figure 2 illustrates a mass m travelling in a circle with constant speed v. This is called **uniform circular motion** and here it is moving counterclockwise. The time it takes to complete one full circle is called the **period**, T, and since it travels a distance equal to one whole circumference in that time, we can calculate the speed as:

$$v = \frac{2\pi r}{T}$$

If it takes time Δt for the mass to sweep out the angle $\Delta\theta$, then the rate of change in angle,

$$\omega \equiv \frac{\Delta\theta}{\Delta t}$$

is called the **angular velocity**.[1] The unit of angular velocity is radians per second, but because the radian measure of an angle isn't a physical unit (as mentioned above), we often omit the 'rad':

$$\frac{\text{rad}}{\text{s}} = \text{rad/s} = \text{rad} \cdot \text{s}^{-1} = \text{s}^{-1}$$

Hence if you don't see a 'rad', it should still be clear from the context whether or not you are dealing with an angular velocity. As mentioned before, in *uniform* circular motion the angular velocity is always constant.

[1] In this course we will only look at circular motion with a constant angular velocity. When studying more general rotational motion, this expression is only the *average* angular velocity and calculus is needed to introduce the *instantaneous* angular velocity, $\omega = \mathrm{d}\theta/\mathrm{d}t$ and the *instantaneous* angular acceleration, $\alpha = \mathrm{d}\omega/\mathrm{d}t$

It's a common mistake to confuse the speed v with angular velocity ω. Here is the precise relationship between the two: When the time interval is sufficiently small, the arclength is very close to being a straight line with length $\Delta x = r\Delta\theta$, see figure 3. Since speed is defined as distance over time, we get

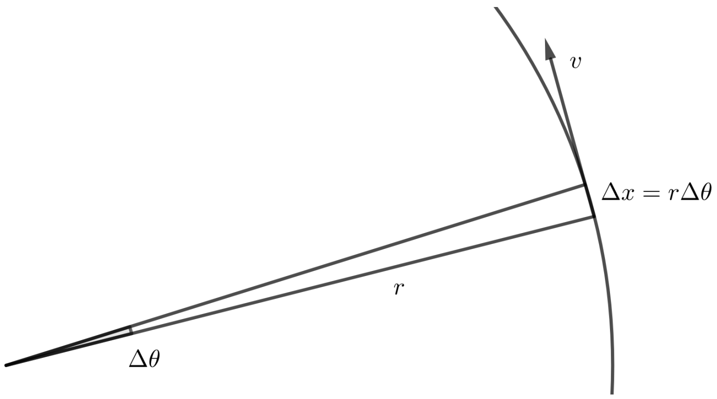


Figure 3: When the elapsed time, Δt, is very small, the arclength is approximately a straight line with length $\Delta x = r\Delta\theta$.

$$v \equiv \frac{\Delta x}{\Delta t} = \frac{r\Delta\theta}{\Delta t} = r\frac{\Delta\theta}{\Delta t} \equiv r\omega$$

which shows that the speed of the object moving around in a circle is proportional to the radius. Take for example a spinning disc. If the whole disc is spinning with a constant angular velocity (constant angle swept out per unit time), then the speed of various points on the disc depends on their distance to the center. A point close to the center has a relatively low speed, whereas a point further away from the center has a relatively high speed. This becomes obvious when you realize that the point farther away needs to cover a greater distance in the same amount of time (all points have the same period).

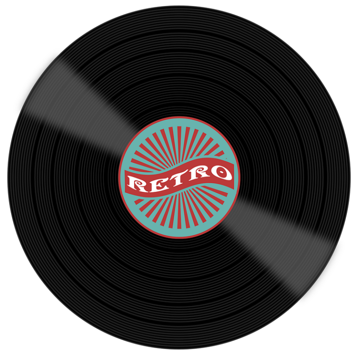


Figure 4: All points on a solid, spinning disc have the same angular velocity, ω, but the linear speed of a single point depends on its distance to the center $v = r\omega$.

Centripetal acceleration and force

As we know from Newton's first law, an object has a constant velocity (in magnitude *and* direction) unless acted upon by an unbalanced force. In uniform circular motion the magnitude of the velocity (speed) is constant but the direction is NOT constant, hence *it is accelerating and there must therefore be an unbalanced force acting on the object*. In analysing cases of circular motion, the main task is to identify this net force that is causing the acceleration.

Let's first investigate the acceleration in greater detail[2]: By considering figure 5, it's possible to realise that the triangle formed by the two velocity vectors is similar to the triangle containing angle $\Delta\theta$. The arc length is approximately $\Delta x = v\Delta t$ (and this becomes

[2] There are at least three other derivations of the following result. One uses vector calculus (probably the shortest), one uses a half-angle approach (similar to the one shown here) and finally, my favourite, uses regular polygons drawn inside the circle (I believe this was Newton's original approach). See the exercise section for more on this.

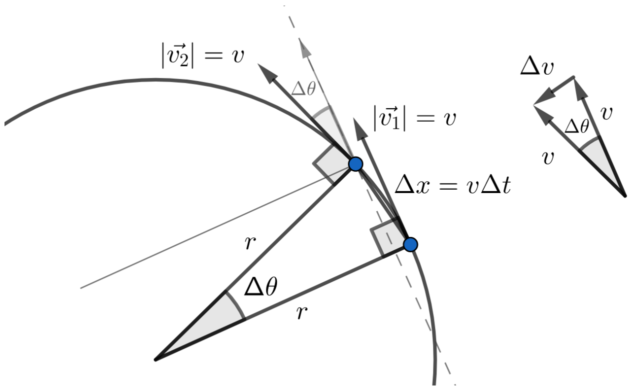


Figure 5: Similar triangles can be identified and used to derive the expression for the acceleration.

more accurate the smaller Δt gets), hence by SAS-similarity, we get:

$$\frac{\Delta v}{\Delta x} = \frac{v}{r} \quad \Rightarrow \quad \frac{\Delta v}{v\Delta t} = \frac{v}{r}$$

and rearranging we get the *magnitude* of the acceleration:

$$a \equiv \frac{\Delta v}{\Delta t} = \frac{v^2}{r} = r\omega^2$$

The *direction* of the acceleration, as seen in figure 5, is in the direction of Δv which is seen to be *towards the center of the circle* (and in the plane of the circle). We call this acceleration the **centripetal**[3] **acceleration**.

[3] 'Centripetal' means 'center-seeking'. It comes from the Latin 'centrum' = center and 'petere' = to seek.

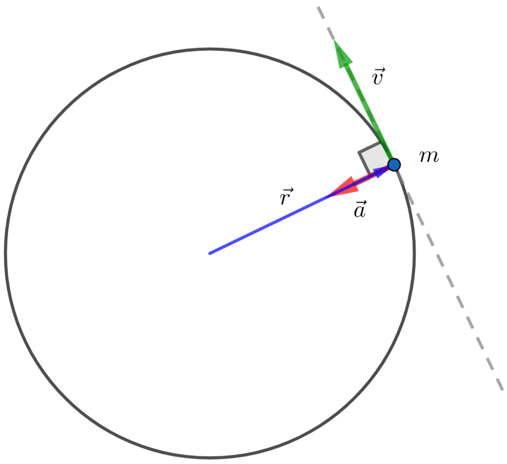


Figure 6: The position, velocity and acceleration vectors in uniform circular motion (the position vector is just a vector pointing from the center to the mass). All three vectors are constantly changing direction, but not magnitude. The position and acceleration vectors are parallel but opposite, the velocity vector is perpendicular to both position and acceleration. All vectors are in the plane of the circle.

To summarise: *When an object moves in a circle of radius r with constant angular velocity ω (i.e. constant speed v), then the acceleration* *always points towards the center and it has the constant magnitude*

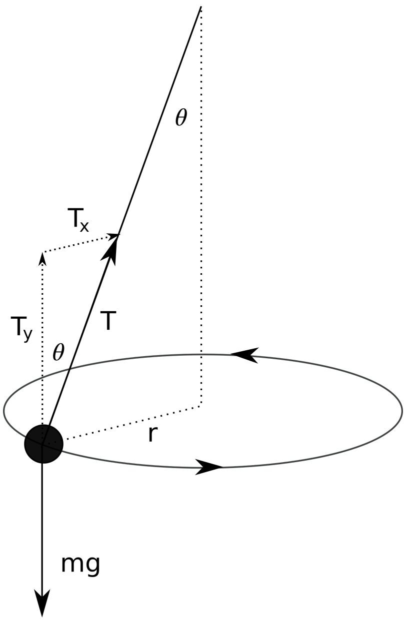
$$a = \frac{v^2}{r} = r\omega^2 \tag{1}$$

The centripetal acceleration is the net acceleration of an object moving uniformly in a circle. According to Newton's 2nd law of motion, this acceleration must of course be caused by a net force acting on the object, hence Newton's 2nd law takes the form

$$F_{\text{net}} = mr\omega^2 \quad \text{or} \quad F_{\text{net}} = m\frac{v^2}{r}$$

In this situation we often call the net force the **centripetal force**, but it's **very, very important** to know that we have just given *the net force a new name* – the centripetal force is NOT a "new force". It is simply the resultant force, the sum of all the actual forces acting on an object going around in a circle with constant speed.

For example, consider a **conical pendulum** (see figure below), which is a pendulum going around in a circle (the string sweeps out the surface of a cone, hence the name). Since the mass is going



around in a circle with constant speed *there has to be a net force acting on the mass which is pointing towards the center of the circle.* This net force is called the centripetal force but it is simply the sum of all the forces already acting on the mass. There are only two forces acting on the mass: The weight and the tension along the string. The vertical force components balance (since the mass is not moving vertically), which means the vertical component of tension must balance out the weight:

$$T_y = T\cos\theta = mg$$

In the horizontal direction, however, there is an unbalanced force which is the horizontal component of the tension $T_x = T\sin\theta$. *This* *is the centripetal force and we can therefore insert it in Newton's 2nd law, together with the expression for the centripetal acceleration:*

$$F_{\text{net}} = ma \quad \Rightarrow \quad T \sin\theta = m\frac{v^2}{r}$$

It's really important that you realise we are just applying Newton's 2nd law to the special case of uniform circular motion. Newton's 2nd law explains it all! One can eliminate T in the two equations above to get the following expression:

$$mg \tan\theta = m\frac{v^2}{r} \quad \Rightarrow \quad g \tan\theta = \frac{v^2}{r} = r\omega^2$$

which gives us an expression for the angle in terms of the other quantities:

$$\tan\theta = \frac{v^2}{gr} = \frac{r\omega^2}{g}$$

Planetary motion (approximately)

The Earth moves around the Sun in what is approximately uniform circular motion.[4] The centripetal force causing the Earth to move in this way is the gravitational pull due to the Sun. Hence we can write down Newton's 2nd law for uniform circular motion:

[4] The motion is more accurately elliptical. Newton's greatest achievement was to explain this perfectly using his laws of motion and gravity. To do this in its most general way is too advanced for this course.

$$ma = F_{\text{net}} \quad \Rightarrow \quad m\frac{v^2}{r} = G\frac{Mm}{r^2}$$

and simplifying this we arrive at the orbital speed expression:

$$v_{\text{orbit}} = \sqrt{\frac{GM}{r}}$$

From this we notice that the orbital speed is inversely proportional to the square root of the distance to the Sun, which means the farther away a planet is, the slower it moves. This is very different to the solid spinning disc discussed previously, so be aware of that!

Kepler's 3rd planetary law was an empirical law that Johannes Kepler discovered based on data collected by the Danish astronomer Tycho Brahe. He discovered – by only looking at the data – that the square of the period of a planet in the solar system is directly proportional to the cube of the radius of its orbit:

$$T^2 \propto r^3$$

One of Newton's great accomplishments was to show that this empirical law followed from his laws of motion and gravity, and you can now easily do that yourself for the case of circular orbits: Using the expression for the centripetal acceleration found in question 8 below, Newton's 2nd law becomes

$$F_{\text{net}} = ma \quad \Rightarrow \quad G\frac{Mm}{r^2} = m\frac{4\pi r}{T^2}$$

which can be manipulated into

$$T^2 = \frac{4\pi^2}{GM}r^3$$

This proves Kepler's 3rd law and even better than that, it gives us the constant of proportionality $4\pi^2/GM$. With this knowledge we can now weigh the Sun! (see exercise 14). Kepler's 3rd law is often used in this way to find the masses of astronomical objects (e.g. black holes).

Lesson 18: Exercises

1. What is the angle measure of a full circle in radians? And in degrees? The following expressions should help you remember the conversion factor between radians and degrees:

$$\frac{2\pi\,\text{rad}}{360\,^\circ} = \frac{\pi}{180}\,\text{rad}/^\circ \approx 0.017\,\text{rad}/^\circ \quad \text{or} \quad 57\,^\circ/\text{rad}$$

2. In a circle of radius 15 m, what is the length of an arc that subtends 0.28 radians?

3. An object is performing uniform circular motion with an angular velocity of 1.9 rad/s.

   (a) What angle does the object sweep out in 0.30 s? Express the angle in radians and degrees.

   (b) How long does it take for the object to complete 4 full revolutions?

4. Given an angle of 0.013 rad in a circle of radius 8.5 m, calculate

   (a) The length of the arc subtended by the angle

   (b) The length of the straight line segment approximating the arclength

   (c) Compare the two lengths - to how many significant digits are they equal?

5. For uniform circular motion, angular velocity ω and the period T are related in the following way (this is a very important formula which we will use a lot):

$$\omega = \frac{2\pi}{T}$$

   Use the definition of angular velocity to justify this.

6. The unit 'revolutions per minute' (rpm) is often used in describing rotational motion. Is this a linear speed or an angular velocity? For example, what does 500 rpm actually mean?

7. A bug sits in the middle of a spinning disc of radius 50 cm. The disc has an angular velocity of 4.0 rad/s. The bug starts moving outwards and stops to take a rest at distances 10, 20, 30 and 50 cm from the center. Calculate the speed of the bug at these rest points.

8. Show that the centripetal acceleration can also be expressed in terms of the period as
$$a = \frac{4\pi^2 r}{T^2}$$

9. (a) For a conical pendulum of length L derive a formula for the angular velocity in terms of only g, θ, and L.

   (b) A conical pendulum with length $L = 0.70\,\mathrm{m}$ makes an angle of $10°$ to the vertical.

      i. What is its angular velocity?

      ii. What is the linear speed of the mass?

      iii. If you want the pendulum to make an angle of $60°$ to the vertical, how fast does it need to spin?

      iv. If we were on Mars instead of Earth, how would your answer to iii) change?

      v. Derive a general formula for the linear speed of the mass in terms of only g, θ, and L. How does the speed vary with angle? Is it possible to reach an angle of $\pi/2$ rad?

10. A body of mass $1.00\,\mathrm{kg}$ is tied to a string and rotates on a horizontal, frictionless table.

   (a) If the length of the string is $40.0\,\mathrm{cm}$ and the speed of revolution is $2.0\,\mathrm{m/s}$, find the tension in the string.

   (b) If the string breaks when the tension exceeds $20.0\,\mathrm{N}$, what is the largest speed the mass can rotate at?

   (c) If the breaking tension is $20.0\,\mathrm{N}$ but you want the mass to rotate at $4.00\,\mathrm{m/s}$, what is the shortest length string that can be used?

11. A mass m is tied to a string and made to move in a vertical circle of radius R with constant speed v. Find the tension in the string at the lowest and highest points.

12. A mass of 2 kg is attached to the end of a stiff rod of length 1 m. The other end of the rod is fixed. The mass and rod moves around in a vertical uniform circle with angular velocity $\pi/2\,\mathrm{s}^{-1}$. What is the magnitude and direction of the contact force on the rod when the mass is at the lowest point? And the highest point?

13. *Activity: Whirling a bung on a string.* In class we will use a simple setup to verify the centripetal force expression.

14. *Activity: Weighing the Sun.* Go to a spreadsheet and let the first column be the distance to the Sun, r, in units of $10^9\,\mathrm{m}$ for all eight planets (here is a planetary fact sheet that you can use to look up the data - skip the Moon and Pluto). Let the second column be the orbital period, T, in days. Now use your amazing physics (and spreadsheet) skills to determine the mass of the Sun!

15. What would the length of a day be if the centripetal acceleration at the equator were equal to the acceleration due to gravity?

16. *Above and beyond:* What is the centripetal acceleration of a point on Earth at 50° latitude as a result of the Earth's rotation about its axis? Express the answer as a fraction of g, the acceleration due to gravity (and take g to be exactly $9.8\,\mathrm{ms}^{-2}$). What angle to the true vertical would a mass hanging at the end of a string make?

17. *Above and beyond:* One can derive the centripetal acceleration formula in a number of different ways. Here are some of them:

    (a) *Derivation using half-angle.* Use the following diagram to derive $a = v^2/r$:

    (b) *Derivation using polygons.* Check out this link and see if you can read in the code how to derive $a = v^2/r$.

    (c) *Derivation using vector calculus.* Express the 2D position vector for an object moving in a uniform circular motion in terms of cosine and sine. Use calculus to find the velocity and acceleration vectors. Find the magnitude of the acceleration vector. How does the acceleration vector compare to the position vector?

Lesson 18 Quiz

Check your understanding of this lesson: Here is a quiz.