A little book about matter

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Lesson 7: Electrical circuits (part II)

In this lesson we will continue analysing circuits.

Series combinations

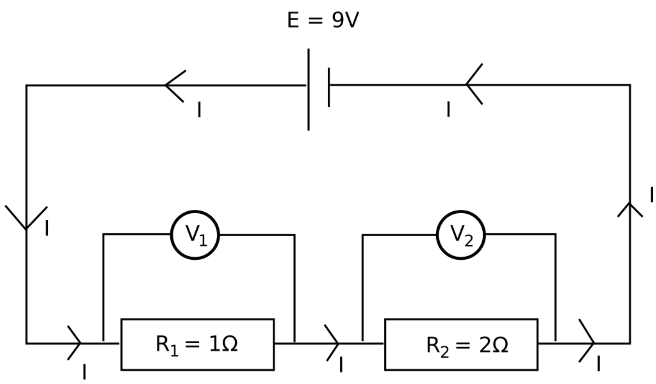


Figure 1: Resistors are sometimes drawn as rectangular boxes as shown here. In some education systems (especially the US) they are drawn as follows:

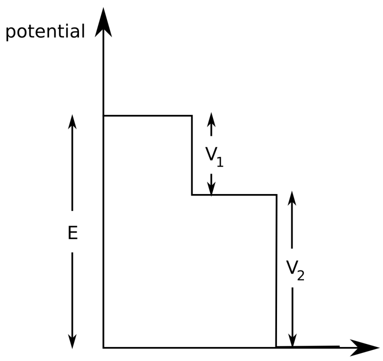


Figure 2: The two potential drops for the circuit shown in figure 1.

Consider two resistors, R_1 and R_2, in series and attached to an emf, $\mathcal{E}$, see figure 1. Let the pd across the two resistors be V_1 and V_2 respectively. The conservation of energy dictates that the two potential drops add up to the emf and you can visualise this as shown in figure 2. Since the resistors are in series, they have the same current flowing through them (otherwise charge would accumulate), hence we can write (from the definition of resistance):

$$I = \frac{V_1}{R_1} \quad \text{and} \quad I = \frac{V_2}{R_2} \quad \Rightarrow \quad \frac{V_1}{R_1} = \frac{V_2}{R_2}$$

This simplifies to the useful result that *in series the ratio of the potential drops is equal to the ratio of the resistances*.

$$\frac{V_1}{V_2} = \frac{R_1}{R_2}$$

For example, if $\mathcal{E} = 9\,\text{V}$, $R_1 = 1\,\Omega$, and $R_2 = 2\,\Omega$, then since the resistances are in a ratio $1 : 2$, the potential drops will also be in that ratio. Hence $V_1 = 3$ V and $V_2 = 6$ V. We can now also find the current in the circuit, since

$$I = \frac{V_1}{R_1} = \frac{V_2}{R_2} = \frac{3\,\text{V}}{1\,\Omega} = \frac{6\,\text{V}}{2\,\Omega} = 3\,\text{A}$$

We can also write

$$V_1 = R_1 I \qquad \text{and} \qquad V_2 = R_2 I$$

and from the conservation of energy we get

$$\mathcal{E} = V_1 + V_2 = (R_1 + R_2)I$$

If we define

$$R_{\text{total}} = R_1 + R_2$$

then we get

$$\mathcal{E} = R_{\text{total}} I$$

which shows that *two resistors in series behave exactly like one resistor with a resistance equal to the sum of the two resistances*. The resistances can be added up to get a total resistance of $R = 3\,\Omega$ and the main current can easily be calculated in this way too:

$$I = \frac{\mathcal{E}}{R} = \frac{9\,\text{V}}{3\,\Omega} = 3\,\text{A}$$

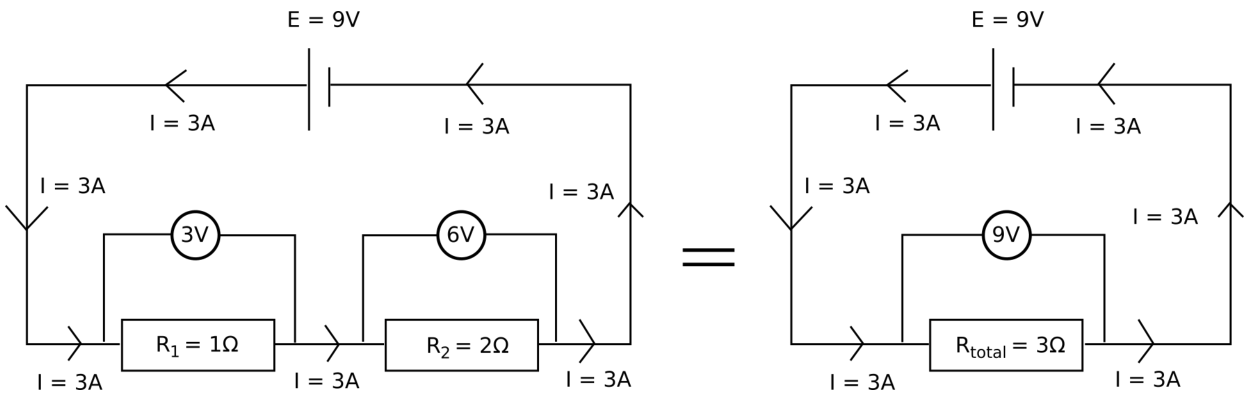


Figure 3: Two resistors in series behave exactly like one resistor with a resistance equal to the sum of the two resistances.

The internal resistance of a cell

Power supplies are designed to be a stable source of constant emf, but most individual cells do not behave like that. A setup to empirically verify that is shown in figure 4.

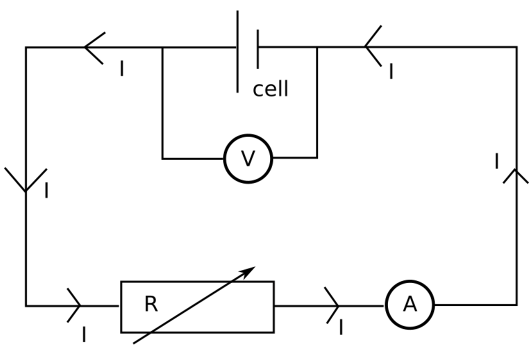


Figure 4: A simple setup to measure the internal resistance of a cell.

A **variable resistor** (the rectangle with an arrow across it) is used to change the resistance of the circuit which changes the current drawn from the cell. The corresponding **terminal voltage** (= the voltage across the terminals of the cell) is measured and the result is shown in figure 5

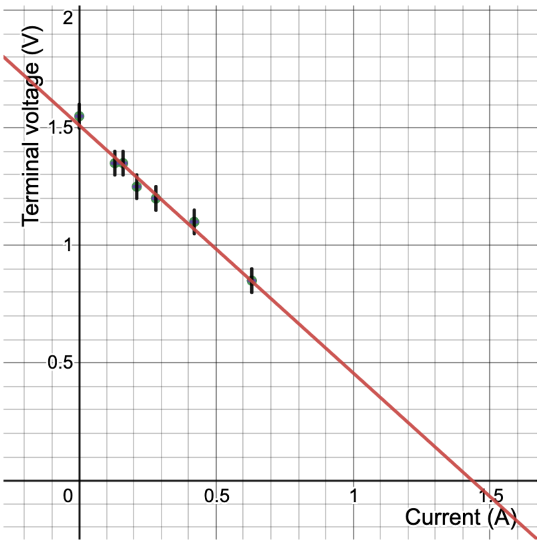


Figure 5: The voltage across the terminals of a cell depends on the amount of current drawn from the cell as shown here. This can be modelled by assuming the cell contains a constant amount of internal resistance. This internal resistance is due to the effect of the chemical reactions and as the cell gets older this resistance increases, sometimes to the point that the cell is not useful anymore and has to be discarded (if it's a primary cell) or recharged (if it's a secondary cell).

The graph shows a linear decrease of terminal voltage with current drawn from the cell. We can model this behaviour by assuming there is a certain amount of **internal resistance**, r, in the cell (due to the effect of the chemical reactions). Hence we think of a cell as made up of a *constant* source of emf, $\mathcal{E}$, in series with a resistor of resistance r, see figure 6. When we do this, the total resistance of the circuit in figure 4 is $R + r$ and we can write down the following expression for the circuit[1]:

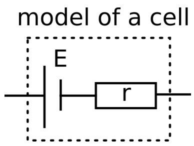


Figure 6: A model of a cell that agrees with empirical data.

$$\mathcal{E} = (R + r)I$$

Expanding this and expressing $V = RI$ as the terminal voltage[2], we get

$$\mathcal{E} = V + rI \quad \Rightarrow \quad V = \mathcal{E} - rI$$

which is exactly the equation of a straight line that fits out data. The y-intercept is $\mathcal{E}$ the constant emf (= the terminal voltage when no current is flowing) and the slope is $-r$ where r is the internal resistance. For the graph shown in figure 5 the values are $\mathcal{E} = 1.5\,\text{V}$ and the internal resistance is $r = 1.1\,\Omega$.

[1] I have seen this called Ohm's 2nd law by some people, but I don't find that particularly useful and the label "law" seems misplaced.

[2] This voltage is also the voltage applied across the variable resistor which makes up the entire 'external part' of the circuit. You can imagine the resistor symbolising any complicated arrangement of resistors. We sometimes use the term **load** for a resistor, so the terminal voltage is also the voltage across the load.

Parallel combinations

Consider two resistors, R_1 and R_2, in parallel and attached to an emf, $\mathcal{E}$, see figure 7. Since they are in parallel they have the same potential drop across them, namely the emf $\mathcal{E}$, so $V_1 = V_2 = \mathcal{E}$. But the main current, I, will split up when it comes to the junction, so let the current through R_1 be I_1 and the current in R_2 be I_2. Because the pds are the same, we can write

$$V_1 = V_2 \quad \Rightarrow \quad R_1 I_1 = R_2 I_2 \quad \Rightarrow \quad \frac{I_1}{I_2} = \frac{R_2}{R_1}$$

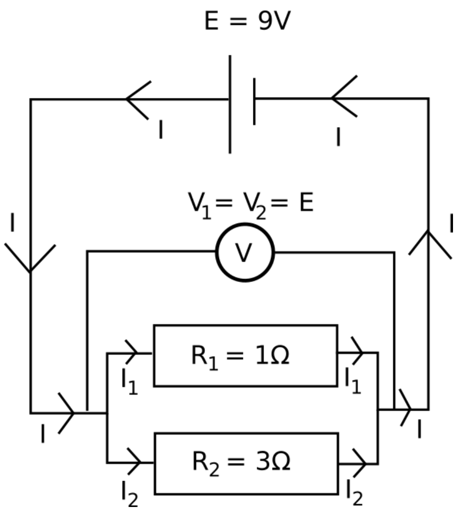


Figure 7: Two resistors in parallel have the same potential drop across them.

which shows that *in parallel the ratio of the currents is the* ***inverse*** *ratio of the resistances*. We can also write

$$I_1 = \frac{\mathcal{E}}{R_1} \qquad \text{and} \qquad I_2 = \frac{\mathcal{E}}{R_2}$$

and from the conservation of charge (the currents have to add up at the junction) we get

$$I = I_1 + I_2 = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\mathcal{E}$$

If we define

$$\frac{1}{R_{\text{total}}} = \frac{1}{R_1} + \frac{1}{R_2}$$

then we get

$$I = \frac{\mathcal{E}}{R_{\text{total}}}$$

which shows that *two resistors in parallel behave exactly like one resistor with a resistance equal to the reciprocal of the sum of the two reciprocals of the resistances*.

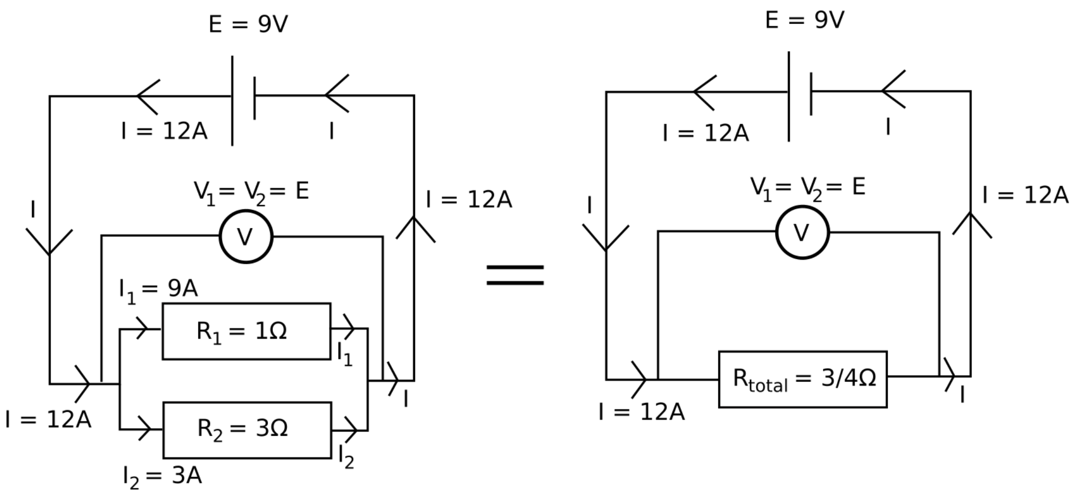


Figure 8: Two resistors in parallel behave exactly like one resistor with a total resistance equal to the reciprocal of the sum of the reciprocals of the two resistances.

Here's a simple example, see figure 8: Say $\mathcal{E} = 9\,\text{V}$, $R_1 = 1\,\Omega$, and $R_2 = 3\,\Omega$. We can find the total resistance by solving

$$\frac{1}{R_{\text{total}}} = \frac{1}{1\,\Omega} + \frac{1}{3\,\Omega} \quad \Rightarrow \quad R_{\text{total}} = \frac{3}{4}\,\Omega$$

and then it's easy to find the main current:

$$I = \frac{9\text{ V}}{3/4\,\Omega} = 12\,\text{A}$$

Since the resistances are in a ratio 1 : 3, the currents will be in that inverse ratio. Hence $I_1 = 3$ A and $I_2 = 9$ A (the largest current of course going through the smallest resistance).

Kirchhoff's circuit laws

When circuits get more complicated (which they obviously very quickly do), we need a general strategy to "solve them", that is, find all the relevant currents and potential drops. Luckily we have such a strategy and it is built around **Kirchhoff's circuit laws**. These two laws are general expressions for 1) *the conservation of energy along any closed path (= loop) contained in the whole circuit* and 2) *the conservation of charge at every junction where the current splits up*. We can express these as follows:

$$\sum_{\text{any loop}} V_i = 0 \qquad \text{and} \qquad \sum_{\text{any junction}} I_i = 0$$

Let's learn how to apply these laws by working through an example. Consider the circuit shown in figure 10. The little symbol in the lower right hand corner of the circuit is the symbol for 'ground' (a literal connection to the Earth) which can always be assumed to be at potential 0 V

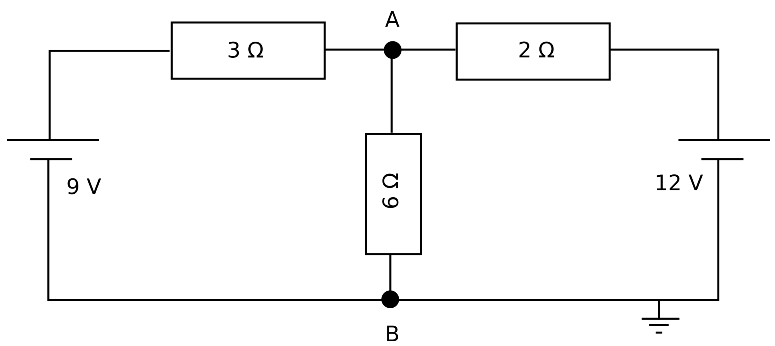


Figure 9: A more complex circuit that can be solved by applying Kirchhoff's laws.

Step #1: Label all currents

The current directions are often difficult to determine in a complicated circuit, and when applying Kirchhoff's laws it actually doesn't matter that you guess them correctly in the first step[3]. We do try, however, to predict the correct direction and in figure 10 it's not hard to imagine that the currents must flow upwards out of

[3] If you assign the incorrect direction, the current will come out as a negative quantity, hence you know the current is flowing opposite to the direction you initially assigned.

each cell and down through the $6\,\Omega$ resistor. Currents are different in different sections of the circuit, so label them carefully with different labels, I_1, I_2, and I_3, see figure **??**

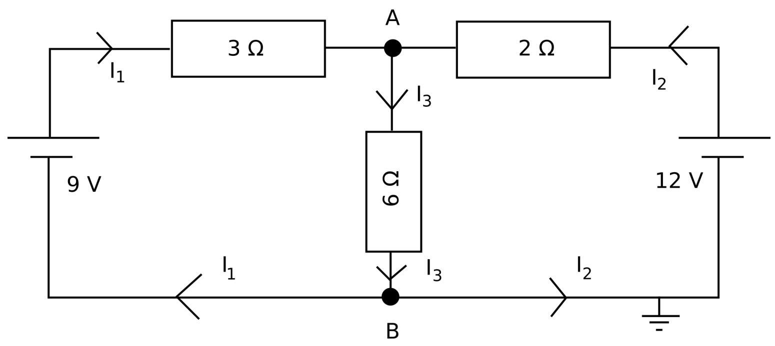


Figure 10: Currents have been labelled.

Step #2: Apply Kirchhoff's junction law

In order to reduce the number of unknowns, we apply the junction law to as many junctions as necessary. The sign convention is: *Currents approaching a junction are considered positive, while currents leaving a junction are considered negative*. Hence, for junction A we can write:

$$\sum_{\text{at A}} I_i = 0 \quad \Rightarrow \quad I_1 + I_2 - I_3 = 0$$

This is, of course, just the same as expressing 'current in' equals 'current out',

$$I_1 + I_2 = I_3,$$

which is simply a statement of the conservation of charge.

Step #3: Apply Kirchhoff's loop law

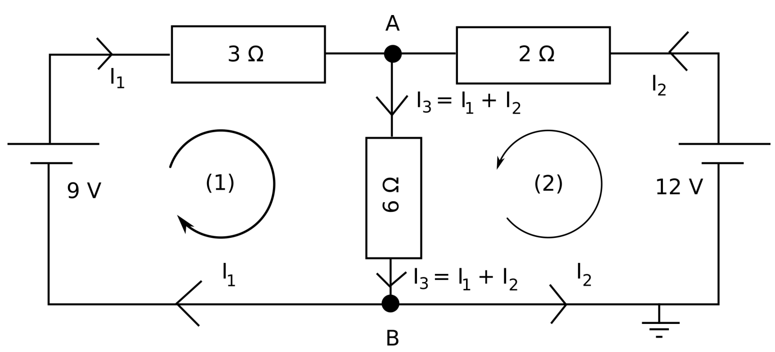


Figure 11: We have chosen two loops and their directions.

In order to get enough equations with enough unknowns, we now apply the loop law to as many loops as necessary. In figure 11 I've indicated two loops. It doesn't matter in which direction you go around the loop, but be consistent with the sign convention: *When moving across a resistor in the same direction as the current, the potential drop is negative, when moving across an emf from negative to* *positive in the same direction as the current, the potential drop is positive.* Hence, for loop (1) we write:

$$\sum_{\text{loop (1)}} V_i = 0 \quad \Rightarrow \quad 9 - 3I_1 - 6I_3 = 0$$

and for loop (2):

$$\sum_{\text{loop (2)}} V_i = 0 \quad \Rightarrow \quad 12 - 2I_2 - 6I_3 = 0$$

Step #4: Solve the equations

Inserting the expression for I_3 in terms of I_1 and I_2 that we got from the junction law in both equations,

$$\begin{aligned} 9 - 3I_1 - 6(I_1 + I_2) &= 0 \\ 12 - 2I_2 - 6(I_1 + I_2) &= 0 \end{aligned}$$

and then simplifying,

$$\begin{aligned} 9 - 9I_1 - 6I_2 &= 0 \\ 12 - 6I_1 - 8I_2 &= 0 \end{aligned}$$

and simplifying a bit more,

$$\begin{aligned} 3 - 3I_1 - 2I_2 &= 0 \\ 6 - 3I_1 - 4I_2 &= 0 \end{aligned}$$

gives us two linear equations that we proceed to solve simultaneously. Subtracting the first equation from the second, we get:

$$3 - 2I_2 = 0 \quad \Rightarrow \quad I_2 = +\frac{3}{2}\,\text{A}$$

and then it follows from the first equation that

$$3 - 3I_1 - 2\left(\frac{3}{2}\right) = 0 \quad \Rightarrow \quad -3I_1 = 0 \quad \Rightarrow \quad I_1 = 0$$

From the junction law we finally get $I_3 = +\frac{3}{2}\,\text{A}$.

Step #5: Reflect on your answer

We get the, perhaps slightly surprising, result that the current flowing out of the 9 V cell is zero. You can confirm this result by using the PhET circuit builder, see figure 12. This means there is only a constant current of 1.5 A flowing in loop (2). Since the current values are positive, the directions indicated on our diagram are correct directions for the conventional current.

Knowing all the currents and resistances, all the potential drops can be calculated, e.g. the potential drop across the 2 Ω resistor is

$$V = 2\,\Omega \cdot 1.5\,\text{A} = 3\,\text{V}$$

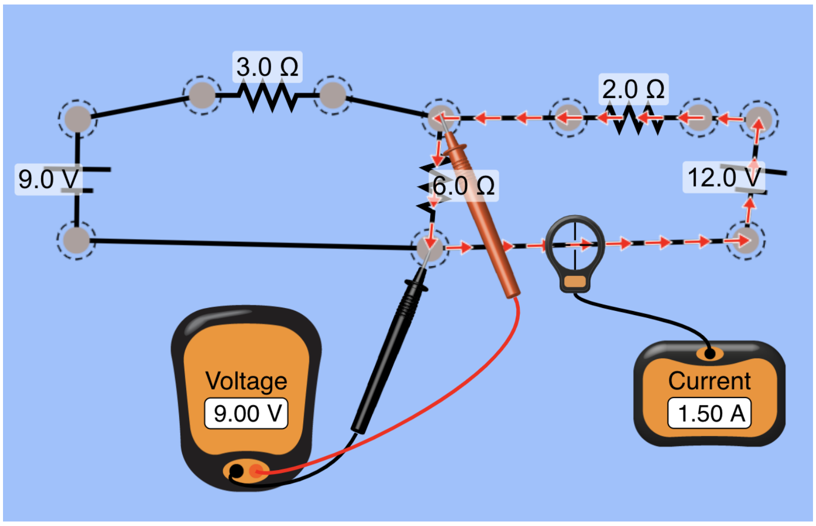


Figure 12: Our circuit example put together in the PhET circuit simulation.

and the potential drop across the $6\,\Omega$ resistor (and hence the potential drop between points A and B, see figure 11) is

$$V = 6\,\Omega \cdot 1.5\,\mathrm{A} = 9\,\mathrm{V}$$

which both add up to 12 V as they should.

As an additional check, we can see that the powers at which energy is created and consumed are equal, as they should be if energy is conserved:

$$\begin{aligned} P_{\text{generated}} &= IV = 1.5\,\mathrm{A} \cdot 12\,\mathrm{V} = 18\,\mathrm{W} \\ P_{\text{consumed}} &= (2\,\Omega)I_2^2 + (6\,\Omega)I_3^2 = 8\,\Omega \cdot 1.5\,\mathrm{A}^2 = 18\,\mathrm{W} \end{aligned}$$

In exercise 5 you will see what happens to the currents if the 12 V emf is replaced by a 10 V and a 14 V emf.

Lesson 7: Exercises

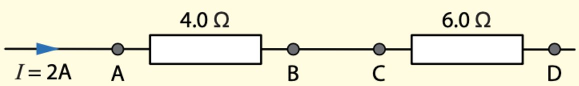
Make sure you have completed this series of small experiments before you attempt the exercises below.

1. Solve the following problem:

   The diagram shows two resistors with a current of 2.0 A flowing in the wire.

   a Calculate the potential difference across each resistor.

   b State the potential between points B and C.

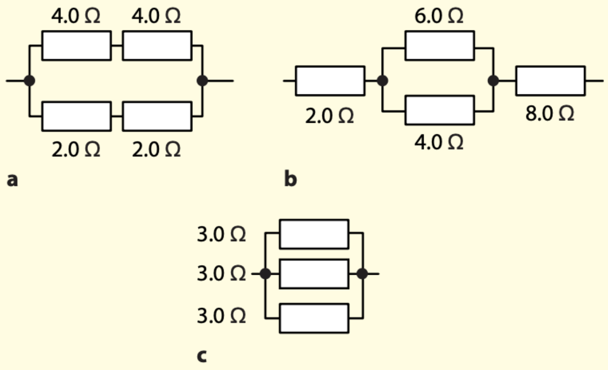
   

2. Solve the following problem:

> The filament of a lamp rated as 120 W at 220 V has resistivity $2.0 \times 10^{-6}\,\Omega\,\mathrm{m}$.
>
> **a** Calculate the resistance of the lamp when it is connected to a source of 220 V.
>
> **b** The radius of the filament is 0.030 mm. Determine its length.

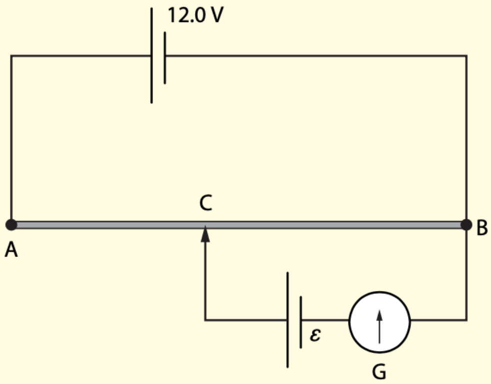
3. Solve the following problem:

> Determine the total resistance for each of the circuit parts in the diagram.



4. Solve the following problem:

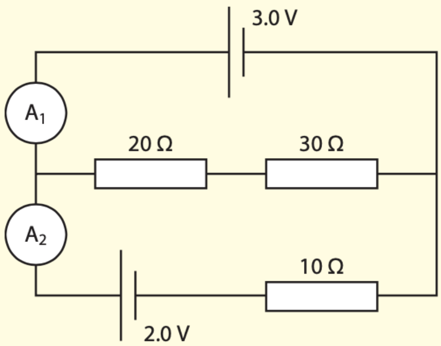
> In the potentiometer in the diagram, wire AB is uniform and has a length of 1.00 m. When contact is made at C with BC = 54.0 cm, the galvanometer G shows zero current. Determine the emf of the second cell.



5. Recall the example circuit from figure 10. Change the 12 V emf to 10 V and see how the currents change. Now change it to 14 V and see how the currents change. You could choose to do all this by using the PhET circuit construction kit.

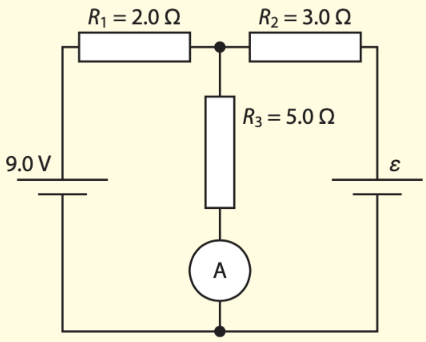
6. Solve the following problem:

In the circuit shown the top cell has emf 3.0V and the lower cell has emf 2.0V. Both cells have negligible internal resistance.
Calculate:
a the readings of the two ammeters
b the potential difference across each resistor.



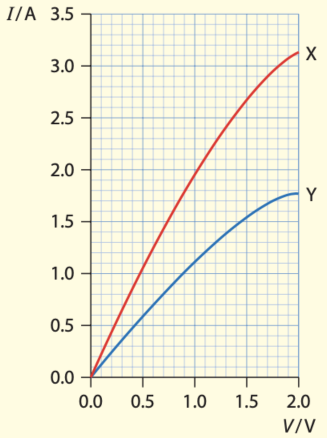
7. Solve the following problem:

In the circuit in the diagram the ammeter reads 7.0A. Determine the unknown emf ε.

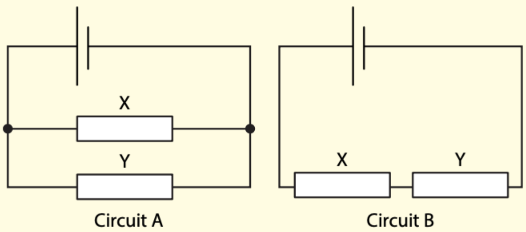


8. Solve the following problem:

Two resistors, X and Y, have I–V characteristics given by the graph.



a Circuit A shows the resistors X and Y connected in parallel to a cell of emf 1.5 V and negligible internal resistance. Calculate the total current leaving the cell.



b In circuit B the resistors X and Y are connected in series to the same cell. Estimate the total current leaving the cell in this circuit.

9. Solve the following problem:

A battery has emf = 10.0 V and internal resistance 2.0 Ω. The battery is connected in series to a resistance R. Make a table of the power dissipated in R for various values of R and then use your table to plot the power as a function of R. For what value of R is the power dissipated maximum?

10. Solve the following problem:

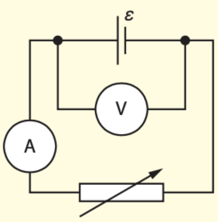
A battery of emf ε and internal resistance r sends a current I into a circuit.

a Sketch the potential difference across the battery as a function of the current.

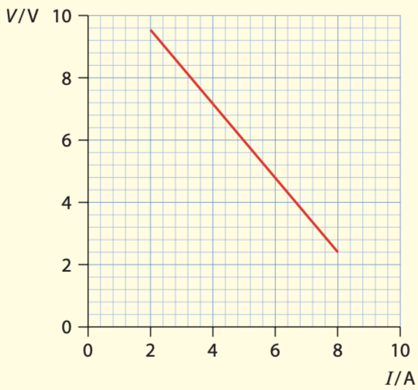
b What is the significance of **i** the slope and **ii** the vertical intercept of the graph?

11. Solve the following problem:

In an experiment, a voltmeter was connected across the terminals of a battery as shown in the diagram.



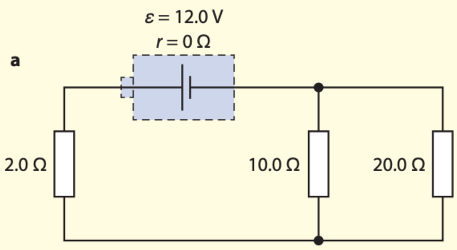
The current in the circuit is varied using the variable resistor. The graph shows the variation with current of the reading of the voltmeter.

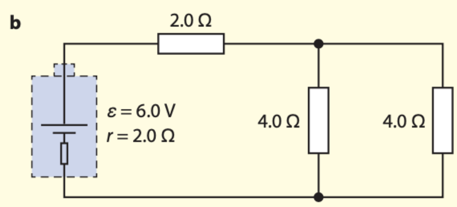


a Calculate the internal resistance of the battery.
b Calculate the emf of the battery.

12. Solve the following problem:

Calculate the current in, and potential difference across, each resistor in the circuits shown in the diagram.





13. Solve the following problem:

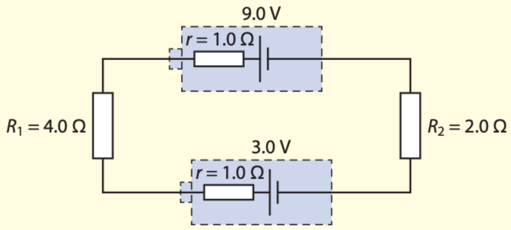
When two resistors, each of resistance $4.0\,\Omega$, are connected in parallel with a battery, the current leaving the battery is $3.0\,A$. When the same two resistors are connected in series with the battery, the total current in the circuit is $1.4\,A$. Calculate:

a the emf of the battery

b the internal resistance of the battery.

14. Solve the following problem:

In the circuit shown in the diagram each of the cells has an internal resistance of $1.0\,\Omega$.



a Determine the current in the circuit.

b Calculate the power dissipated in each cell.

c Comment on your answer to **b**.

Answers to all the exercises.

Lesson 7 Quiz

Check your understanding of this lesson: Here is a quiz.