A little book about motion

> "Any measurement that you make without any knowledge of the uncertainty is meaningless."
>
> — Walter Lewin, *8.01x - Lect 1 - Powers of 10, Units, Dimensions, Uncertainties, Scaling Arguments*

Lesson 2a: Error propagation

Assume you measure a quantity $x \pm \delta x$ and another quantity $y \pm \delta y$. What happens with the uncertainties if you perform calculations with these quantities? As any rational thinker can imagine, uncertainties somehow need to 'combine' all depending on what type of calculation you are making – they 'propagate' through the calculations. This is the topic we will address in this lesson.

I prefer writing absolute uncertainties as δx instead of Δx (for some reason I don't like our usual 'delta' notation getting mixed up with the concept of an absolute uncertainty – blame this awesome book for giving me that idea). But you should know that often people just write absolute uncertainties using the Δ-notation.

The sum and difference rule

Say you measure $x \pm \delta x$ and $y \pm \delta y$, and you need to calculate the difference between them. Let's denote the difference by z, so

$$z = x - y$$

The smallest value z could be is $z - \delta x - \delta y = z - (\delta x + \delta y)$ and the largest it could be is $z + (\delta x + \delta y)$. So we see that the absolute uncertainty of z is simply the sum of the absolute uncertainties:

$$\delta z = \delta x + \delta y$$

This also holds true when adding two measured quantities, in summary: If $x \pm \delta x$ and $y \pm \delta y$, then $z = x \pm y$ has the absolute uncertainty

$$\delta z = \delta x + \delta y$$

Here's an example: You need to calculate a time interval, $\Delta t = t_2 - t_1$, and both times are subject to a human reaction error of $0.3\,\mathrm{s}$. If $t_1 = 4.5\,\mathrm{s}$ and $t_2 = 16.9\,\mathrm{s}$, then we would end up with $\Delta t = 12.4 \pm 0.6\,\mathrm{s}$.

The constant multiple rule

Say you measure $x \pm \delta x$ and you need to multiply x by a constant k. In this case the absolute uncertainty of $z = kx$, is also simply multiplied by the constant, hence

$$\delta z = |k|\delta x$$

Here's an example: Say you measure a mass, $m = 0.45 \pm 0.01\,\mathrm{g}$ and you want to double the mass. Then we would end up with $2m = 0.90 \pm 0.02\,\mathrm{g}$. Notice how this doesn't change the percentage uncertainty (around 2% in both cases) which makes sense: Just by multiplying by a constant we are not inherently changing the precision of the measurement.

The product and quotient rule

Say you measure $x \pm \delta x$ and $y \pm \delta y$, and you need to calculate the product $z = xy$ or the ratio $z = x/y$. It turns out that in both cases *the relative uncertainty of z is the sum of the relative uncertainties*[1] (hence you just add the percentage uncertainties):

$$\frac{\delta z}{z} = \frac{\delta x}{x} + \frac{\delta y}{y}$$

This process is a bit more lengthy than the previous rules, so be careful! Especially because you often need to calculate the absolute uncertainties from the relative uncertainties.

Here's an example: You need to measure the density of an object. You measure the mass as $m = 21 \pm 3\,\mathrm{g}$ and the volume as $V = 78 \pm 6\,\mathrm{cm}^3$. Since density is mass divided by volume, $\rho = m/V$, the relative uncertainty of the density is

$$\frac{\delta\rho}{\rho} = \frac{\delta m}{m} + \frac{\delta V}{V} = \frac{3}{21} + \frac{6}{78} = 0.2197\ldots \approx 22\%$$

To get the absolute uncertainty, we need to multiply this value by the best estimate of the density:

$$\delta\rho = 0.2197\ldots \times \frac{21}{78} \approx 0.06$$

So finally we can write our calculated value as

$$\rho = 0.27 \pm 0.06\,\mathrm{g/cm^3}$$

Here's another example: When you take the reciprocal of a measured quantity, you can view that as a division where the numerator is a constant that has no uncertainty. So the end result is that *the reciprocal of a measured quantity has the same relative uncertainty as the measured quantity itself*. For example, if you have a measurement

$$0.00225 \pm 0.00004\,\mathrm{s\,cm^{-1}}$$

then the reciprocal of this would have units of speed, and if you follow the error propagation rule correctly you get

$$444 \pm 8\,\mathrm{cm\,s^{-1}}$$

Notice that the relative uncertainty in both cases is the same (around 2%).

[1] This follows by a bit of easy algebra, here it is for the product rule: $z_{\mathrm{max}} = (x + \delta x)(y + \delta y) \approx xy + x\delta y + y\delta x$ and $z_{\mathrm{min}} = (x - \delta x)(y - \delta y) \approx xy - x\delta y - y\delta x = xy - (x\delta y + y\delta x)$, hence $\delta z = x\delta y + y\delta x$ and by dividing this equation by z we get the answer.

The power rule

Say you measure $x \pm \delta x$ and you need to calculate $z = x^n$. In this case the relative uncertainty of z is the absolute value of n times the relative uncertainty:

$$\frac{\delta z}{z} = |n| \frac{\delta x}{x}$$

For integer values of n this follows from the product rule, but the proper proof follows from the next general result. A quick example: A length os measured as $L = 0.20 \pm 0.01\,\text{cm}$, and if we wish to calculate the square root of the length $z = \sqrt{L} = L^{\frac{1}{2}}$, then the relative uncertainty is

$$\frac{\delta z}{z} = \frac{1}{2}\frac{\delta L}{L} = \frac{1}{2} \cdot \frac{0.01}{0.20} = 0.025$$

with an absolute uncertainty of $\delta z = 0.025 \cdot z = 0.025 \cdot \sqrt{0.20} \approx 0.01\,\text{cm}^{\frac{1}{2}}$, so the final result should be given as

$$\sqrt{L} = 0.45 \pm 0.01\,\text{cm}^{\frac{1}{2}}$$

Functions

If z is some function of a measured quantity $x \pm \delta x$ then the simplest option is to take a max-min approach. E.g. if $\theta = 40 \pm 1^\circ$ and $z = \sin\theta$, then we could find the absolute uncertainty by considering the difference between the maximum and minimum value of z:

$$\delta z = \frac{z_{\text{max}} - z_{\text{min}}}{2} = \frac{\sin(41^\circ) - \sin(39^\circ)}{2} \approx 0.013$$

and the final result would be

$$\sin\theta = 0.643 \pm 0.013$$

If you know a bit of calculus, you can instead find the absolute uncertainty of z as follows ('a little change in z is the rate of change of z times a little change in x'):

$$\delta z = \left|\frac{\mathrm{d}z}{\mathrm{d}x}\right| \delta x$$

This rule can in fact be used to prove most of the rules above.

The very general case

If z is a function of more than one measured variable, (e.g. $z = f(x_1, \ldots, x_n)$), then we can again use calculus (this time *multivariable* calculus) to find the absolute uncertainty of z:

$$\delta z = \sqrt{\left(\frac{\partial z}{\partial x_1}\delta x_1\right)^2 + \cdots + \left(\frac{\partial z}{\partial x_n}\delta x_n\right)^2}$$

Lesson 2a: Questions and activities

1. Using a normal ruler and a digital scale, determine the density of all the metal blocks given to you by first measuring the length, height, width, and mass (= raw data) and then process the data (using a spreadsheet) to find the density with uncertainty. Your uncertainties will be different for each block so you need an extra column for that. Which block is the "imposter" block!?

2. Using a vernier caliper, measure the diameter of the large metal sphere given to you in class. Measure its mass and determine its density with uncertainty. Steel has a density of around $7.8\,\mathrm{g/cm^3}$ - what can you conclude?

   Answers to all the questions.

Lesson 2a Quiz

Check your understanding of this lesson: Here is a quiz.