A little book about matter

Lesson 3: The ideal gas law

Figure 1: When air heats up it expands and takes up more volume, hence the overall density decreases. That is the basic function of a hot air balloon.

The ideal gas law

$$pV = nRT$$

is an equation that models how gases behave and like all models it is based on certain assumptions and has certain limitations. The assumptions are as follows:

1. The gas consists of a very large number, N, identical particles (of same mass, etc.) with volumes negligible compared to the container volume (tiny dots moving around).
2. The particles in the gas are in constant, random motion and they only interact through elastic collisions with each other and the container walls (hence no other forces are involved).

Here is a simulation of an ideal gas, but note that this simulation doesn't have a very large number of particles and their volumes are not insignificant compared to the container. So although it isn't a great representation of an ideal gas, it gives you the rough idea. The ideal gas equation doesn't predict the behaviour of real gases perfectly, and here are the main limitations:

1. At low (absolute) temperatures the gases condense into liquids and hence the model breaks down.
2. At high particle densities the particles start exerting other forces on each other and their volumes and not insignificant relative to the container volume anymore.

Before developing the ideal gas model using Newtonian mechanics, let's discuss two important physical quantities in greater detail:

Temperature, T: All particles in a ideal gas are constantly jiggling around and changing speeds due to frequent collisions, but the total KE is constant since the collisions are assumed elastic. As we will see in a minute, it will make sense to *define temperature in terms of the average KE per particle*. The larger the average KE per particle, the higher the temperature we measure with a thermometer.

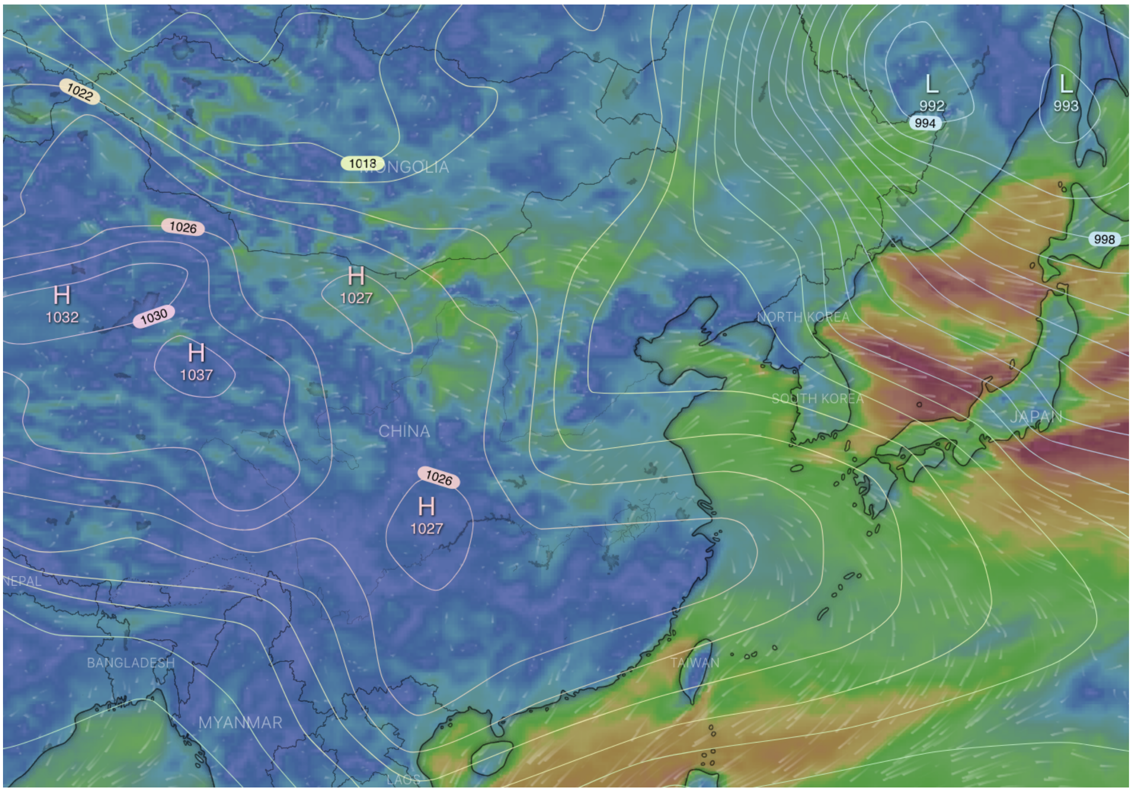


Figure 2: The weather in parts of Asia on November 24th 2021 as shown on the website windy.com. The lines (called isobars) are made of points which all have the same pressure. The unit is in hPa (hecto pascal = hundreds of pascal). Normal atmospheric pressure is around 1013 hPa. Where are the regions of low and high pressure?

Pressure, p: A gas continuously exerts an irregular force overall on whatever is constraining its volume due to frequent particle impacts. This is experienced as an average force and we define pressure as that average force per unit area:

$$p \equiv \frac{F}{A} \quad \left(\text{unit: Pa} \equiv \frac{\text{N}}{\text{m}^2}\right)$$

Many different units for pressure were developed over time (e.g. mmHg as we saw before), but we now mainly use the **pascal** (Pa) which is defined above as one newton per square meter. A unit called **standard atmosphere** (atm) is also commonly used and its definition is

$$1\,\text{atm} \equiv 101\,325\,\text{Pa} \approx 1013\,\text{hPa} \approx 10^5\,\text{Pa}$$

which is approximately equal to the average atmospheric pressure at sea level. See figure 2 to see an example of pressures given in hPa.

When pressure is measured in pascal, temperature in kelvin, quantity in moles and volume in cubic meters, then the gas constant takes on the value[1]

$$\frac{pV}{nT} \equiv R \approx 8.31\,\mathrm{JK^{-1}mol^{-1}}$$

Think of this as *the amount of energy required to heat up one mole of an ideal gas by one kelvin*.

The ideal gas law can be written in a slightly different way as follows: First express n (the number of moles) in terms of N (the number of particles) and Avogadro's constant N_A

$$n = \frac{N}{N_A}$$

Next, insert that expression into the ideal gas law:

$$pV = nRT = \left(\frac{N}{N_A}\right) RT$$

and by defining the **Boltzmann constant** k_B as

$$k_B \equiv \frac{R}{N_A} \approx 1.38 \times 10^{-23}\,\mathrm{JK^{-1}}$$

we get the ideal gas law in its other form:

$$pV = Nk_BT$$

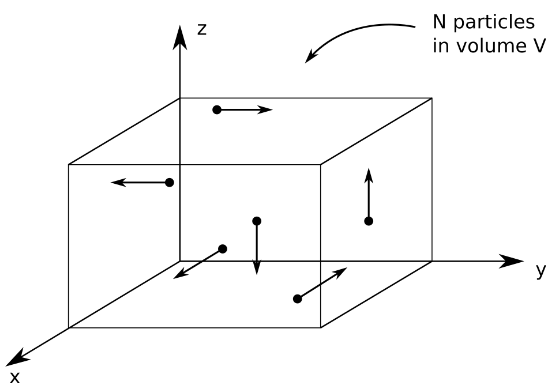
You can think of the Boltzmann constant as *the average energy per particle required to heat up an ideal gas by one kelvin*.

[1] From Wikipedia: "Some have suggested that it might be appropriate to name the symbol R the *Regnault constant* in honour of the French chemist Henri Victor Regnault, whose accurate experimental data were used to calculate the early value of the constant. However, the origin of the letter R to represent the constant is elusive." Due to the 2019 redefinition of the SI base units, the value of R is now the following exact number:

$$8.314\,462\,618\,153\,24\,\mathrm{J\,K^{-1}\,mol^{-1}}$$

A simple derivation

OK, enough chatter, let us now use Newtonian mechanics to arrive at a model of a gas that will allow us to define the concept of temperature in greater detail. As mentioned earlier we are not going to develop this model in the most rigorous way (because we lack the proper mathematical tools), but the derivation given here still contains the main conceptual ideas and therefore serves as a good example of how it is done.



Imagine a box of volume V and coordinate axes as shown above. We will assume there are N particles in the box and they are evenly spread out, so there is an average particle desnity of N/V. Now assume all particles only move with one of the following six velocities, distributed equally at any given time:

$$(\pm v, 0, 0), (0, \pm v, 0), (0, 0, \pm v)$$

This is of course a silly assumption to make, but this is how we can simplify the derivation considerably and in the end we get the same result.[2] Our assumption implies that at any given moment in time, $1/6$ of all the particles will be moving toward the bottom of the box. Consider an area A on the bottom of the box as shown in figure 3. All particles within a distance $v\Delta t$ from the area and

[2] In order to give a more rigorous derivation, we need to introduce the Maxwell-Boltzmann velocity distribution, which is outside the scope of this text.

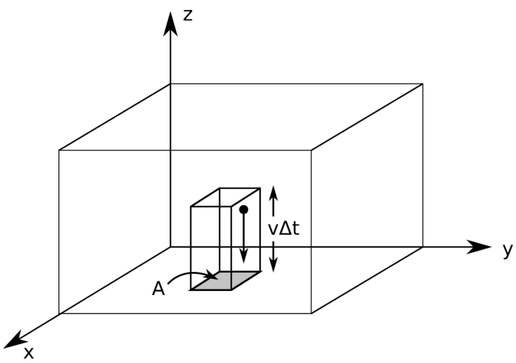


Figure 3: All particles within a distance $v\Delta t$ from the area and moving downwards, will hit the area during the time interval Δt

moving downwards, will hit the area during the time interval Δt. These particles are contained in the little volume that has cross-sectional area A and height $v\Delta t$, so the *number of particles hitting the area* in that time interval is given by the expression:

$$\frac{1}{6} \times \text{(density of particles)} \times \text{(little volume)} = \frac{1}{6}\left(\frac{N}{V}\right)(Av\Delta t)$$

When each particle hits the area A it will rebound elastically (= without changing speed). Hence a particle comes in with a magnitude of momentum mv (pointing down) and leaves with a magnitude of momentum mv (pointing up). The magnitude of the change in momentum is therefore $2mv$. This change in momentum is due to an average force F being exerted by the container wall on the particle. According to Newton's 3rd law, the particle also exerts this average force on the container. Hence, the total impulse that is transferred to the container wall during time Δt is

$$F\Delta t = 2mv\left(\frac{1}{6}\frac{N}{V}Av\Delta t\right)$$

This expression can be simplified and rearranged to the following:

$$\frac{F}{A} = \frac{1}{3}mv^2\frac{N}{V}$$

Here we notice that pressure appears on the left hand side ($p \equiv F/A$, force per unit area). If we also move V over to the left, we isolate pV on one side,

$$pV = \frac{2}{3}\left[N\left(\frac{1}{2}mv^2\right)\right] \tag{1}$$

It's starting to look like the ideal gas law! Note the expression for the kinetic energy on the right hand side. In the soft brackets you have the kinetic energy of a single particle, in the hard brackets you have the total kinetic energy of all the particles. Now, of course all particles don't have the same speed (recall our silly assumption) and when you do a more rigorous analysis, you find that the speed squared in equation (1) has to be replaced by *the average of all the squared speeds*. Let's briefly explore that concept in more detail: If you were to assign a velocity to each of the N particles, you get a long list of velocities

$$v_1, v_2, v_3, \ldots, v_N$$

Each particle would have a certain kinetic energy, so you also have a long list of kinetic energies:

$$\frac{1}{2}mv_1^2, \frac{1}{2}mv_2^2, \ldots, \frac{1}{2}mv_N^2$$

The average kinetic energy per particle, $\overline{KE}$, is then

$$\overline{KE} \equiv \frac{\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \cdots + \frac{1}{2}mv_N^2}{N} = \frac{1}{2}m\left(\frac{v_1^2 + v_2^2 + \cdots, v_N^2}{N}\right)$$

and what we have in the last bracket above is *the average of all the squared speeds* $\overline{v^2}$. The square root of this average is called the **root-mean-square**, v_{rms}, and it is a measure of the **average speed of the particles in the gas**:

$$v_{\text{rms}} \equiv \sqrt{\overline{v^2}} \quad \Rightarrow \quad v_{\text{rms}}^2 \equiv \overline{v^2}$$

So we can express the average kinetic energy per particle as (this is important)

$$\overline{KE} = \frac{1}{2}m\overline{v^2} = \frac{1}{2}mv_{\text{rms}}^2$$

and using this in expression (1) we get:

$$pV = \frac{2}{3}N\left(\frac{1}{2}mv_{\text{rms}}^2\right) = N\,\frac{2}{3}\,\overline{KE} \tag{2}$$

And now to wrap things up: Compare equation (2) with the empirically found ideal gas model in the form $pV = Nk_BT$ and notice that they are identical *if we associate the average kinetic energy per particle with absolute temperature in the following way*:

$$\frac{2}{3}\,\overline{KE} = k_BT \quad \Rightarrow \quad \overline{KE} = \frac{3}{2}k_BT$$

Hence we arrive at the understanding that *absolute temperature is a measure of the average kinetic energy of the particles* (they are directly proportional quantities) and the Boltzmann constant is really just a conversion factor from the energy unit to the kelvin unit.

Lesson 3: Exercises

1. Below is an excerpt from the IB DP Physics Data Booklet - make sure you know what the various equations mean:

Sub-topic 3.2 – Modelling a gas
$p = \dfrac{F}{A}$
$n = \dfrac{N}{N_A}$
$pV = nRT$
$\bar{E}_K = \dfrac{3}{2} k_B T = \dfrac{3}{2} \dfrac{R}{N_A} T$

   Notice that the equation $pV = Nk_BT$ isn't explicitly stated. How can you easily derive it from the information given?

2. (a) Calculate the number of molecules in 28 g of hydrogen gas. The molar mass is 2 g/mol (under normal conditions hydrogen is a **diatomic** gas, H_2, i.e. the particles consist of two hydrogen atoms bonded together).
   (b) Calculate the number of moles in 6.0 g of helium gas. Helium is a **monatomic** gas (i.e. the particles consist of a single atom).
   (c) Determine the number of moles in a sample of gas that contains 2.0×10^{24} molecules.
   (d) Determine the mass of carbon ($M = 12\,\text{g/mol}$) that contains as many molecules as 21 g of krypton ($M = 84\,\text{g/mol}$).

3. A sealed bottle contains air at 22.0 °C and 12.0×10^5 Pa. The temperature is raised to 120.0 °C. Calculate the new pressure.

4. A gas has pressure 8.2×10^6 Pa and volume $2.3 \times 10^{-3}\,\text{m}^3$. The pressure is reduced to 4.5×10^6 Pa at constant temperature. Calculate the new volume of the gas.

5. A mass of 12.0 kg of helium is required to fill a bottle of volume $5.00 \times 10^{-3}\,\text{m}^3$ at a temperature of 20.0 °C. Determine the pressure of the gas.

6. Determine the mass of carbon dioxide required to fill a tank of volume $12.0 \times 10^{-3}\,\text{m}^3$ at a temperature of 20.0 °C and a pressure of 4.00 atm.

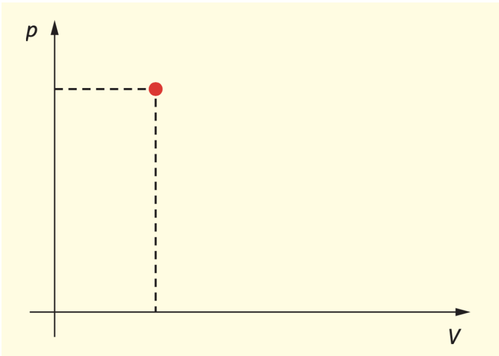
7. The point marked in the diagram represents the state of a fixed quantity of ideal gas in a container with moveable piston. The temperature of the gas in the state shown is 600 K. Copy the diagram into your notebook. Indicate on the diagram the point

representing the new state of the gas after the following separate changes:

(a) The volume doubles at constant temperature.

(b) The volume doubles at constant pressure.

(c) The pressure halves at constant volume.



8. The molar mass of a gas is 28 g/mol. A container holds 2.00 mol of this gas at 0.00 °C and a pressure of 1.00×10^5 Pa. Determine the mass and volume of the gas.

9. A balloon has a volume of 404 m^3 and is filled with helium of mass 70.0 kg. The temperature inside the balloon is 17.0 °C. Determine the pressure inside the balloon.

10. A flask has a volume of $5.0 \times 10^{-4}\,m^3$ and contains air at a temperature of 300 K and a pressure of 150 kPa.

    (a) Calculate the number of moles of air in the flask.

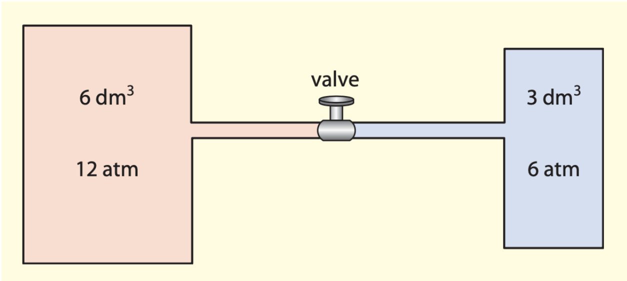
    (b) Determine the number of molecules in the flask.

    (c) Estimate the mass of air in the flask. You may take the molar mass of air to be 29 g/mol.

11. A flask of volume $300.0 \times 10^{-6}\,m^3$ contains air at a pressure of 5.0×10^5 Pa and a temperature of 27.0 °C. The flask loses molecules at a rate of 3.00×10^{19} per second. Estimate how long it will take for the pressure in the flask to fall to half its original value. Assume that the temperature remains constant during this time.

12. Two identical gases are kept at the same temperature in two containers separated by a valve, as shown in the diagram. Estimate the pressure when the valve is opened. The temperature stays the same. (Note: When two ideal gases mix, their separate pressures add to become a total pressure.)

13. The diagram shows a cylinder in a vacuum, which has a moveable, frictionless piston at the top. An ideal gas is kept in the cylinder. The piston is at a distance of 0.500 m from the bottom of the cylinder and the volume of the cylinder is 0.050 m^3. The

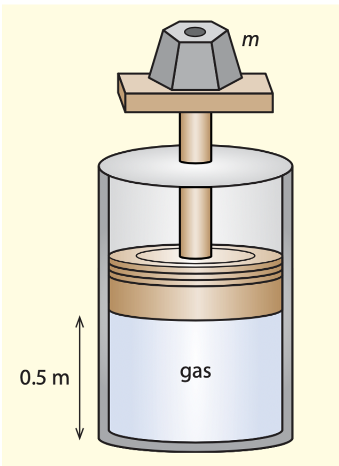


weight on top of the piston has a mass of 10.0 kg. The temperature of the gas is 19.0 °C.

(a) Calculate the pressure of the gas. (Hint: focus on the mass).

(b) Determine how many molecules there are in the gas.

(c) The temperature is increased to 152 °C. Calculate the new volume of the gas.



14. The molar mass of helium is 4.00 g/mol.

(a) Calculate the volume of 1.0 mol of helium at standard temperature and pressure (STP), i.e. at $T = 273\,\text{K}$ and $p = 1.0 \times 10^5\,\text{Pa}$.

(b) Determine the density of helium at STP.

(c) Estimate the density of oxygen gas at STP. The molar mass of oxygen is 32 g/mol.

15. The density of an ideal gas is $1.35\,\text{kg/m}^3$. The temperature in kelvin and the pressure are both doubled. Calculate the new density of the gas.

16. Calculate the average (root mean square) speed of helium atoms at a temperature of 850 K.

17. Show that the average (root mean square) speed of molecules of a gas of molar mass M (in kg/mol) kept at a temperature T is

given by

$$\sqrt{\frac{3RT}{M}}$$

18. (a) Calculate the average random kinetic energy per particle of a gas kept at a temperature of 300 K.

    (b) Determine the ratio of the average (rms) speeds of two ideal gases of molar mass 4.0 g/mol and 32 g/mol which are kept at the same temperature.

Answers to all the exercises.

Lesson 3 Quiz

Check your understanding of this lesson: Here is a quiz.