*A little book about waves*



## Lesson 1: Simple harmonic motion (SHM)

An **oscillation** is the regular variation of a quantity above and below an equilibrium value. For example a pendulum *oscillates* when it swings back and forward (a regular variation of angle around a vertical equilibrium), a mass on a spring *oscillates* when it is set in motion (a regular variation of displacement around an equilibrium position), and alternating current is an *oscillation* (a regular variation of current around an equilibrium of no current).

Oscillations can take on many different forms, but one of the simplest to analyse is **simple harmonic motion (SHM)** which is *an oscillation that follows a sinusoidal function* ('sinusoidal' means the quantity varies as a sine function[1]). This type of oscillation models many types of behaviour, and it turns out to be the basic building block of more complex oscillations.[2]

[1] Or a cosine function. It doesn't matter which one you pick because they have the exact same shape and one can easily be transformed into the other.

[2] **Fourier analysis** shows that many types of periodic functions can be expressed as an infinite sum of sine and cosine functions.

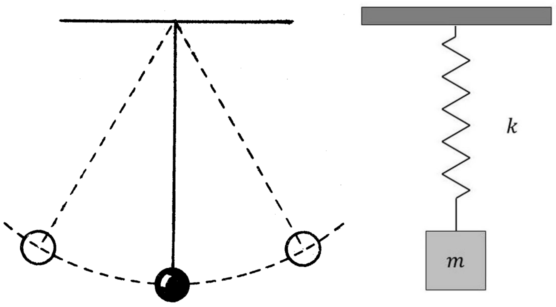


Figure 1: When set in motion, a pendulum and a mass on a spring are examples of oscillating systems.

### Concept #1: SHM defined as $\vec{a} \propto -\vec{x}$

Consider an oscillating mass on a spring, see figure 2. A mass is attached to a spring on a retort stand and when the mass is pull down slightly from its equilibrium position, it will oscillate up and down. A motion sensor can capture the oscillation in the form of a position vs. time motion graph as shown in figure 3. The time it takes to complete one full oscillation (for example, the time interval between two peaks) is called the **period**, $T$. One full oscillation is also called a **cycle**.

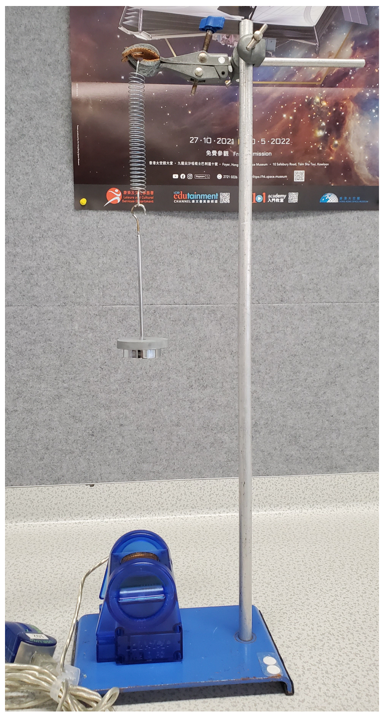

Figure 2: A setup to investigate oscillations.

1. Use a PASCO motion sensor to investigate how the mass affects the period of oscillation (if you don't have access to any equipment, then here is a very good simulation – set the damping to 0.00). Collect a good range of data (including small masses). Draw a conclusion and give a conceptual scientific explanation.

The **frequency**, $f$, of an oscillation is the number of oscillations per unit time:

$$f \equiv \frac{\text{number of oscillations}}{\text{time}}$$

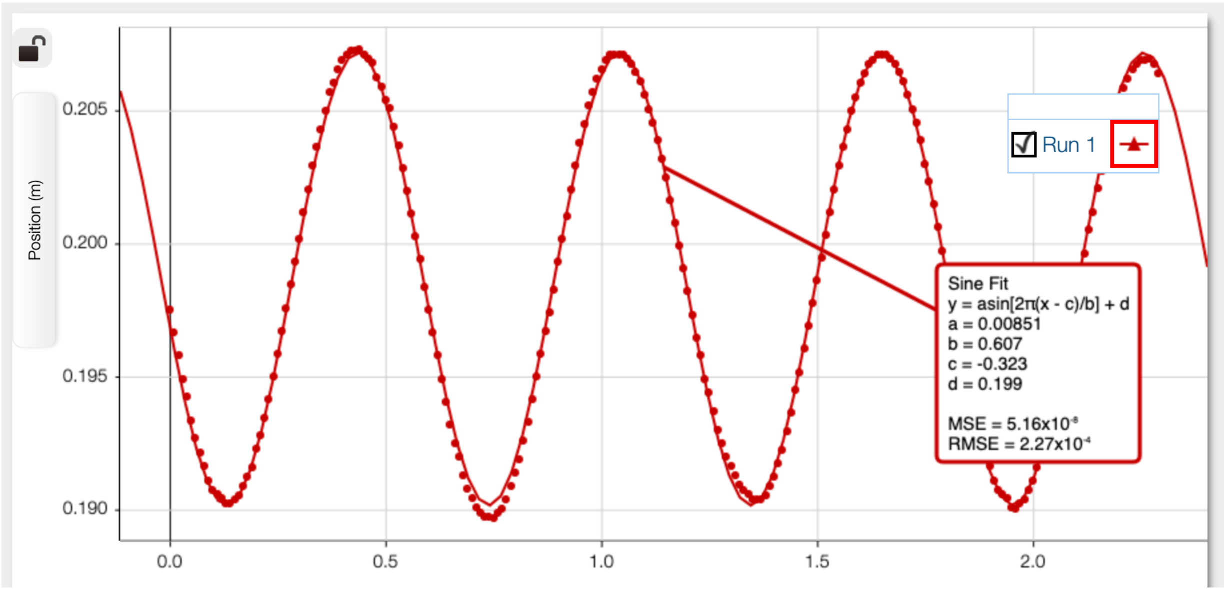


Figure 3: A sinusoidal graph.

For example, if a mass oscillates up and down 10 times in 2 seconds, then the frequency of that oscillation is

$$f \equiv \frac{10}{2\,\mathrm{s}} = 5\,\mathrm{s}^{-1} = 5\,\mathrm{Hz}$$

The last step above shows that we often use the unit **hertz (Hz)** instead of $\mathrm{s}^{-1}$.

2. Deduce that frequency is the reciprocal of the period: $f = \frac{1}{T}$.

3. For your data in exercise 1, use a spreadsheet to calculate the frequency of the oscillations. Notice that the higher the frequency, the shorter the period, etc. due to their inverse proportional relationship.

Figure 4 shows a mass on a spring when it is displaced from its equilibrium (we've changed to a horizontal spring here). The displacement in this case is a stretch. The elastic force on the mass after its displacement is also shown (assume all other forces are negligible or irrelevant).

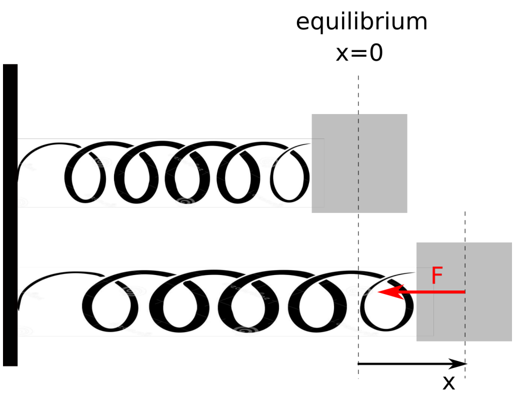


Figure 4: A spring is stretched and a restoring force appears. Hooke's law is the assumption that this restoring force is always proportional and opposite to the displacement. This assumption holds true for many small oscillations.

4. Write down Newton's 2nd law for the mass in its stretched position and isolate the acceleration. *(Hint: Recall Hooke's law $\vec{F}_{\text{elastic}} = -k\vec{x}$).*

5. Imagine the spring is compressed instead of stretched. What changes? Is the expression for the acceleration still the same as in exercise 4?

6. Does the *expression* for the acceleration depend on how much the spring has been stretched or compressed? Does the *magnitude* of the acceleration depend on how much the spring has been stretched or compressed? Can we use the suvat equations when describing this type of motion?

You should have seen above that the acceleration of the mass (at any given position of it's oscillation) can be expressed as

$$\vec{a} = -\frac{k}{m}\vec{x}$$

This shows that *the acceleration is always opposite (the minus sign) and proportional ($k/m$ is a constant) to the displacement from equilibrium.* Mathematically it can be proven that this relationship is equivalent to a sinusoidal variation[3], hence we can alternatively define **simple harmonic motion (SHM)** as follows: *Simple harmonic motion is an oscillation where the acceleration is always opposite and proportional to the displacement from equilibrium, $\vec{a} \propto -\vec{x}$.*

[3] To understand this, you need to learn how to solve 2nd order differential equations.

## *Concept #2: The sinusoidal nature of SHM*

Any SHM can be modelled by a general sine function with parameters $A$, $\omega$, and $\phi_0$ (assuming equilibrium is zero):

$$A\sin(\omega t + \phi_0) \tag{1}$$

The **amplitude** of the oscillation is defined as *the maximum distance from equilibrium*, and it is equal to[4] $|A|$ . The parameter $\omega$ is the **angular frequency** and it is the same quantity that we defined for uniform circular motion. It is defined as *the number of radians covered per unit time* and since $2\pi$ radians is covered in one full oscillation (or cycle) of period $T$, then it holds true that

[4] Sometimes the parameter $A$ in eq. (1) is negative, hence we need the absolute value because the amplitude is defined as a distance, a positive number.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

We'll see what SHM has to do with uniform circular motion in exercise 2 below. The parameter $\phi_0$ determines the starting point of the oscillation (when $t = 0$, the initial displacement is $A\sin(\phi_0)$), and we call it the **initial phase** of the oscillation.

1. Investigate whether the frequency (and hence period) depends on the amplitude. You should notice that for the mass-spring system *the frequency is independent of amplitude* (which is why tuning forks work). This is always true for SHM and this type

of oscillation is therefore called an **isochronous** oscillation. Pendulums swinging at small angles satisfy this which is why grandfather clocks can keep ticking at the same rate despite the amplitude of the swing decreasing over time.

2. Click on SHM (part 1) and get a feel for how the parameters change the SHM and what SHM has to do with circular motion.

We sometimes refer to the entire argument of the sine function (the expression "$\omega t + \phi_0$") as the **phase**. Two oscillators are said to be **in phase** when the difference between their phases (their 'phase difference') is an integer multiple of $2\pi$. *In this case, the oscillators have exactly the same displacements at equal times*. They are **out of phase** when that is not the case. **Completely out of phase** is when the phase difference is an odd integer multiple of $\pi$. *In this case, the oscillators have exactly opposite displacements at equal times*. **Coherence** is *when oscillators have a constant phase difference* and this will turn out to be an important assumption when dealing with more advanced wave phenomena. Understanding the concept of phase is absolutely crucial to many aspects of wave motion, but it can be a bit tricky to understand, so take your time!

3. Click on SHM (part 2) and explore the idea pf phase. Try to make the two oscillators oscillate completely out of phase.

A particle is often exposed to multiple forces, each of which makes it undergo SHM. For example, imagine two people holding the same particle and each wanting to make it oscillate sinusoidally by pushing and pulling it with the same varying force. If their pushes and pulls are synchronised (in phase), the particle will oscillate with a large amplitude (two pushes make it go further). On the other hand, if their pushes and pulls are exactly opposite (completely out of phase), then the particle won't move at all because when one person pulls, the other pushes! When multiple interactions act on an object, we simply "add the interactions" to get the "net effect". This fact of nature is called **the principle of superposition** and *it is one of the most important underlying principles in physics*.[5] For SHM, the principle of superposition means that we can add separate SHM displacements together to get the net SHM displacement of an oscillating particle.

4. Click on SHM (part 3) and explore the principle of superposition. What happens when the two oscillators oscillate in phase? What happens when the two oscillators are completely out of phase? (For the more mathematically inclined student: Work out an expression for the single sinusoidal function that describes the superposition of two SHMs).

5. Two oscillators of different angular frequencies are obviously incoherent *since the phase difference doesn't remain constant*. Click on SHM (part 4) to explore this.[6]

[5] Simple vector addition is an example of the principle of superposition that we have already been using: The net force is the vector sum of separate forces, and the net motion can be thought of as a sum of the individual motions.

[6] If the frequencies are close enough together, then it leads to a phenomenon called a **beat frequency**. You can see what the superposition looks like here.

If the function

$$x = A\sin(\omega t + \phi_0)$$

describes the position (displacement from zero) of a particle undergoing SHM, then the velocity is the instantaneous rate of change of this function and it can be shown[7] that you get

$$v = \omega A\cos(\omega t + \phi_0)$$

Since the acceleration is always proportional and opposite the displacement, we can deduce that

$$a \propto A\sin(\omega t + \phi_0)$$

and in the next section we'll see that in fact $a = -\omega^2 x$, hence[8]

$$a = -\omega^2 A\sin(\omega t + \phi_0)$$

6. Click on this link to see all three functions in desmos. What patterns do you notice? Can you visually check that $a \propto -x$ on these graphs?

7. Go to this very good simulation, set the damping to 0.00, and click on the 'Time Graph' tab. Investigate.

[7] For those who know calculus, this is of course just the derivative,

$$v = \mathrm{d}x/\mathrm{d}t$$

[8] We could also simply continue using differential calculus: We know that the instantaneous rate of change of the velocity function is the acceleration,

$$a = \mathrm{d}v/\mathrm{d}t$$

## *Concept #3: Newton's 2nd law and the angular frequency*

As mentioned earlier, simple harmonic motion is defined as being motion for which the acceleration is proportional and opposite to the displacement, $\vec{a} \propto -\vec{x}$. This can be expressed as the equation

$$a = -(\text{constant})x \tag{2}$$

where the constant of proportionality is shown (and I've omitted the vector notation for now). For example, the constant of proportionality in a mass-spring system equals $k/m$, because

$$a = -\frac{k}{m}x$$

It is possible to show mathematically[9], that any solution to the equation (2) can be written as

$$x = A\sin(\sqrt{\text{constant}} \cdot t + \phi_0)$$

where the constant of proportionality is under the square root sign. Comparing this with

$$x = A\sin(\omega t + \phi_0)$$

we can deduce that *the constant of proportionality is equal to the angular frequency squared*:

$$\omega = \sqrt{\text{constant}} \quad \Rightarrow \quad \omega^2 = \text{constant}.$$

So for the mass on a spring, we get

$$\omega^2 = \frac{k}{m} \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}}$$

where we used $T = 2\pi/\omega$ in the last step.

[9] To understand this, you need to learn how to solve 2nd order differential equations.

1. For the mass-spring system, how does the period depend on the mass according to the above equation? Compare this with your measurements from Concept #1, Exercise 1.

2. How does the period depend on the spring constant $k$ according to the above equation? Does that make sense?

We have arrived at the important realisation that the defining equation for SHM can always be written as

$$a = -\omega^2 x$$

where $\omega$ is the angular frequency of the oscillating system.

3. In the following examples, Newton's 2nd law has been applied to a particular system and the equation shows that it undergoes SHM. Verify the expression for the period and convince yourself that it makes sense.

   (a) A swinging pendulum of length $L$ only swinging at small angles[10] ($x$ is the horizontal displacement from the equilibrium):

   $$ma = -\frac{mg}{L}x \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{L}{g}}$$

   (b) A ball rolling in a spherical bowl with radius $R$:

   $$ma = -\frac{mg}{R}x \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{R}{g}}$$

   (c) A cylinder 'bobbing" up and down in the surface of a fluid with density $\rho$. The cylinder has cross-sectional area $A$:

   $$ma = -(\rho A g)x \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{\rho A g}}$$

   (d) A mass dropped into a tunnel drilled through the center of the Earth (!).

   $$ma = -\frac{GM_\oplus m}{R_\oplus^3}x \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{R_\oplus^3}{GM_\oplus}}$$

   Examples (e)-(g) below are very advanced and merely shown to illustrate that SHM is everywhere – you are not expected to understand the details at all.

   (e) A 'physical' pendulum, for example a swinging rod[11]. $I$ is the rotational inertia (the rotational 'mass' of the system), $\alpha$ is the angular acceleration, $\theta$ is the angular position, and $d$ is the distance to the center of mass:

   $$I\alpha = -(mgd)\theta \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{I}{mgd}}$$

   Notice that for the 'mathematical' pendulum, $I = mL^2$ and $d = L$, so it is just a special case of this 'physical pendulum'.

[10] Think of a small mass on a thin string. Such a pendulum is often called a 'mathematical' pendulum due to its simplicity.

[11] Here Newton's 2nd law for *rotational motion* has been used. Since the equation has the exact same form as before,

$$\alpha \equiv \frac{\mathrm{d}^2\theta}{\mathrm{d}t} \propto -\theta$$

we can apply the same understanding of SHM

(f) A mechanical watch with a balance wheel (with moment of inertia $I$) and a coil spring of torsion constant $\kappa$:

$$I\alpha = -\kappa\theta \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{I}{\kappa}}$$

(g) An $LC$ electrical circuit consisting of an inductor with inductance $L$ and a capacitor with capacitance $C$ connected together. The circuit can act as an 'electrical resonator', an electrical analogue of a tuning fork:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = -\frac{1}{LC} I \quad \Rightarrow \quad \omega = \frac{1}{\sqrt{LC}}$$

## *Concept #4: Energy in SHM*

Consider again a mass on a spring and imagine you pull it away from its equilibrium by a distance $A$. This will be the amplitude of the oscillation. As mentioned in the mechanics unit, the positive work that your pull force did on the system (in opposing the restoring force given by Hooke's law) is equal to

$$\frac{1}{2}kA^2$$

and this is now stored as **elastic potential energy (EPE)** in the system. This is also the **total amount of energy (TE)** in the system (provided you release the mass from rest and no other forces are involved). As the mass accelerates towards the equilibrium (due to the spring force), elastic potential energy is converted to **kinetic energy (KE)**. The mass will reach maximum kinetic energy when it passes the equilibrium point and after that, KE will convert back into EPE. Hence, the stored (total) energy 'sloshes' back and forward between elastic potential energy and kinetic energy (see this desmos simulation). At any position $-A \leq x \leq A$, the EPE is given by

$$EPE = \frac{1}{2}kx^2$$

so the kinetic energy at any position $-A \leq x \leq A$ is

$$KE = TE - EPE = \frac{1}{2}k(A^2 - x^2)$$

Since $KE = \frac{1}{2}mv^2$ we can insert that into the above expression and get a nice relationship[12] between velocity, $v$, and displacement, $x$:

$$\frac{1}{2}mv^2 = \frac{1}{2}k(A^2 - x^2) \quad \Rightarrow \quad v^2 = \frac{k}{m}(A^2 - x^2)$$

And from this expression we can find the maximum speed by setting $x = 0$:

$$v_{\text{max}}^2 = \frac{k}{m}A^2$$

[12] For those of you familiar with the equation of a circle and an ellipse, note that if you plot v vs. x you get an ellipse. Such a diagram represents the phase space of the system and such phase plots are useful in describing the behavior of complex dynamical systems.

1. The above energy equations all apply to the mass-spring system. Any SHM, however, has PE and KE that 'sloshes' back and forward. Write down the general energy equations for an arbitrary SHM in terms of the particle mass and angular frequency. *(Hint: If $a = -\omega^2 x$, then the force is given as $F = ma = -m\omega^2 x$ and the work done by this force is $\frac{1}{2}Fx = \frac{1}{2}m\omega^2 x^2$.)*

2. Write down the relationship between velocity $v$ and displacement $x$ for a general SHM.

## *Lesson 1: Brain workout*

***If you want to learn physics, you must work on all the following problems! It's absolutely fine to make mistakes – we have all been there – just try to learn from them. Ask questions if you are completely stuck.***

1. The IB Data Booklet contains the equations shown below. Check that we have covered all of them in this chapter and notice the small differences compared to the formulas shown in this lesson.

| **Sub-topic 4.1 – Oscillations** |
|---|
| $T = \frac{1}{f}$ |

| **Sub-topic 9.1 – Simple harmonic motion** |
|---|
| $\omega = \frac{2\pi}{T}$ |
| $a = -\omega^2 x$ |
| $x = x_0 \sin \omega t;\ x = x_0 \cos \omega t$ |
| $v = \omega x_0 \cos \omega t;\ v = -\omega x_0 \sin \omega t$ |
| $v = \pm\omega\sqrt{(x_0{}^2 - x^2)}$ |
| $E_K = \frac{1}{2}m\omega^2(x_0{}^2 - x^2)$ |
| $E_T = \frac{1}{2}m\omega^2 x_0{}^2$ |
| pendulum: $T = 2\pi\sqrt{\frac{l}{g}}$ |
| mass-spring: $T = 2\pi\sqrt{\frac{m}{k}}$ |

2. Here is a one-page overview of SHM for a mass on a spring. Read through the whole page and notice all the small details.

*Lesson 1 Quiz*

Check your understanding of this lesson: Here is a quiz.

*Lesson 1 Problems*

1. A pendulum has length $L = 1.0\,\mathrm{m}$. How many swings does it perform in one hour?

2. When a mass of $m = 1.0\,\mathrm{kg}$ is hung vertically from a certain spring, it extends the spring by $\Delta x = 0.10\,\mathrm{m}$. Find the period of oscillation of the mass-spring system.

3. (a) Write down an equation for the displacement of a particle undergoing SHM with an amplitude equal to $8.0\,\mathrm{cm}$ and a frequency of $14\,\mathrm{Hz}$, assuming that at $t = 0$ it is at equilibrium and has a positive velocity. (How would it change if the initial velocity had been negative?)

   (b) Determine the period of this motion.

   (c) Calculate the displacement, velocity, and acceleration of this particle at a time of $t = 0.025\,\mathrm{s}$.

4. A pendulum of length $L = 9.8\,\mathrm{m}$ hangs in equilibrium and is then given the velocity $0.20\,\mathrm{ms}^{-1}$ at its lowest point. What is the amplitude of the subsequent oscillation?

5. The displacement of a particle undergoing SHM is given by $x = 5.0\cos(2t)$, where $x$ is in millimeters and $t$ is in seconds. Calculate

   (a) The initial displacement of the particle.

   (b) The angular frequency, frequency, and period.

   (c) The displacement at $t = 1.2\,\mathrm{s}$.

   (d) The time at which the displacement first becomes $-2.0\,\mathrm{mm}$.

   (e) The displacement when the velocity of the particle is $6.0\,\mathrm{mm\,s}^{-1}$.

6. A 42.5 kg chair is attached to a spring and allowed to oscillate. When it is empty, the chair takes 1.30 s to make one complete vibration. But with an astronaut sitting in it, with her feet off the floor, the chair takes 2.54 s for one cycle. What is the mass of the astronaut? *(This procedure has actually been used to "weigh" astronauts in space, see here.)*

7. A point on a guitar string oscillates in SHM with an amplitude of $5.0\,\mathrm{mm}$ and a frequency of $460\,\mathrm{Hz}$. Determine the maximum velocity and acceleration of this point.

8. A block with mass $M$ rests on a frictionless surface and is connected to a horizontal spring of force constant $k$. The other end of the spring is attached to a wall (see figure 5). A second block with mass $m$ rests on top of the first block. The coefficient of static friction between the blocks is $\mu_s$. Find the maximum amplitude of oscillation such that the top block will not slip on the bottom block.

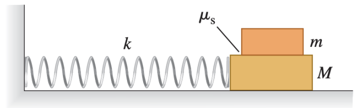


Figure 5: Problem 8.

9. Figure 6 shows the velocity $v$ of a particle executing SHM.

   (a) Using the graph, estimate the area between the curve and the time axis from 0.10 s to 0.30 s.

   (b) State what this area represents.

   (c) Hence write down an equation giving the displacement of the particle as a function of time.

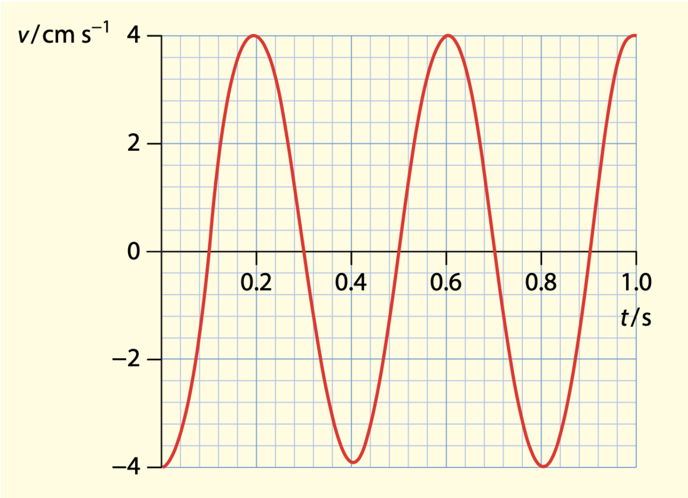


Figure 6: Problem 9.

10. Figure 7 shows the variation with displacement $x$ of the acceleration $a$ of a body of mass 0.150 kg.

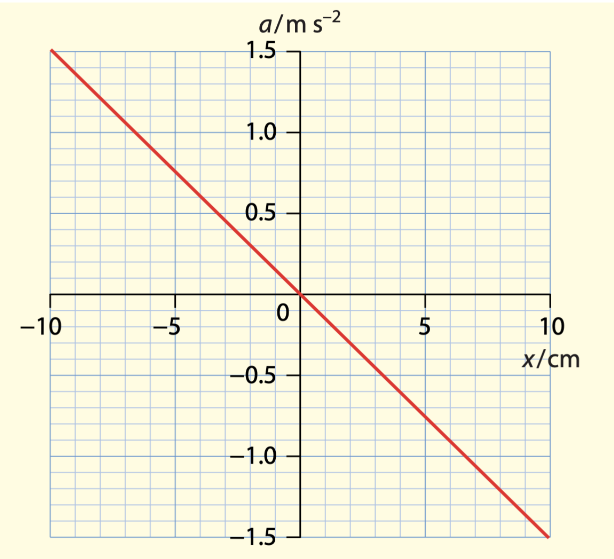


Figure 7: Problem 10.

(a) Use the graph to explain why the motion of the body is SHM.

(b) Determine the following:

i. the period of the motion

ii. the maximum velocity of the body during an oscillation

iii. the maximum net force exerted on the body

iv. the total energy of the body.

11. Figure 8 shows the displacement $x$ of a particle executing SHM. Draw a graph to show the variation with displacement $x$ of the acceleration $a$ of the particle (put numbers on the axes).

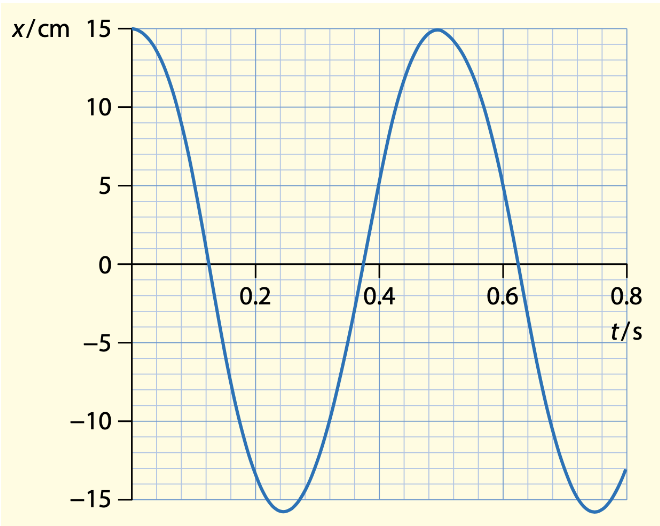


Figure 8: Problem 11.

12. A mass $m$ is attached to a vertical spring with spring constant $k$. The spring is stretched a distance $x_0$ when the mass is at rest.

(a) Draw a FBD for the mass when it is at rest and express $x_0$ in terms of $m$, $k$, and $g$.

(b) The mass is now pulled down further a distance $x$ and let go. Draw a FBD for $m$ at the moment it is released and write down Newton's 2nd law. Simplify the equation using (a) and show that

$$a = -\frac{k}{m}x$$

which means it will undergo SHM around $x_0$.

(c) Calculate the angular frequency of the SHM and compare it to the angular frequency of the oscillation when the spring is horizontal.

13. A spring of constant $k = 0.50\,\mathrm{N m^{-1}}$ and an attached mass $m$ oscillate on a smooth horizontal table. When the mass is at position $x_1 = 0.10\,\mathrm{m}$ its velocity is $v_1 = -1.0\,\mathrm{ms^{-1}}$, and $x_2 = -0.20\,\mathrm{m}$ it has velocity is $v_2 = 0.50\,\mathrm{ms^{-1}}$. Find $m$ and the amplitude $A$.

14. At what position in SHM is the KE equal to the PE? And at what time does this occur?

15. A guitar string, whose two ends are fixed oscillates as shown in figure 9. The vertical displacement of a point on the string a distance $x$ from the left end is given by

$$y = 6.0 \sin(\pi x) \cos(1040\pi t)$$

where $y$ is in millimeters, $x$ is in meters, and $t$ is in seconds. Use this expression to:

(a) Deduce that all points on the string execute SHM with a common frequency and common phase, and determine the common frequency.

(b) What does the string look like at time $t = \frac{1}{2080}$ s?

(c) Deduce that different points on the string have different amplitudes.

(d) Determine the maximum amplitude of the oscillation.

(e) Calculate the length $L$ of the string.

(f) Calculate the amplitude of oscillation of the point on the string where $x = \frac{3}{4}L$.

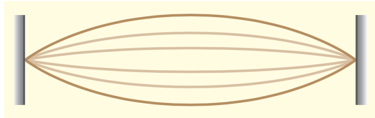

Figure 9: Problem 15.

16. A body of mass 0.120 kg is placed on a horizontal plate. The plate oscillates vertically in SHM making five oscillations per second.

(a) Determine the largest possible amplitude of oscillations such that the body never loses contact with the plate.

(b) Calculate the normal force on the body at the lowest point of the oscillation when the amplitude has the value found in (a).

17. Imagine you drill a hole through the center of the Earth and release a mass $m$ into the hole (yes, very unrealistic, but an entertaining thought experiment).

(a) When the mass falls through the Earth, the force of gravity on $m$ is only due to Earth's mass within the radius that $m$ is at. The mass 'above' $m$ does not contribute (this requires a rather technical mathematical proof, but just take that as a fact for now). Write down an expression for the force of gravity on $m$ when it is a distance $x$ away from the center.

(b) Write down Newton's 2nd law for $m$. Will it undergo SHM?

(c) What is the period of this motion? Compare it with the period of a satellite in orbit around the Earth with a circular orbit equal to the Earth's radius.

*Answers to all the problems.*