Lesson 3: The Doppler effect

Whenever you hear an ambulance or a police car approaching with high speed (hopefully not because you have done something wrong), then you will experience the **Doppler effect**: *A shift in frequency due to the relative motion of the source and the observer*[1]. Although mainly an effect associated with sound waves in our everyday lives (see for example this video if you don't know what I'm talking about), the Doppler effect is a general effect[2] that also applies to other types of waves (e.g. light).

[1] We can also refer to the 'observer' as the 'receiver'.

[2] Although a general effect, the derived equations depend on what type of wave you are dealing with. In this section we will assume the waves are sound (mechanical) waves, but if dealing with light waves, one has to follow the rules of special relativity and the derivation and result will be slightly different, see Concept #5.

Concept #1: A stationary source and a moving observer

In the first case, we will consider a source, S, at rest (stationary) relative to the medium and an observer, O, moving relative to the medium with speed u_O. Let us initially assume the observer is moving towards the source, see figure 1. Let the source emit a wave with frequency, f, and wave speed v. Recall that this wave speed only depends on the properties of the medium and it is a speed measured relative to the medium. The emitted wavelength is therefore

$$\lambda = \frac{v}{f} \tag{1}$$

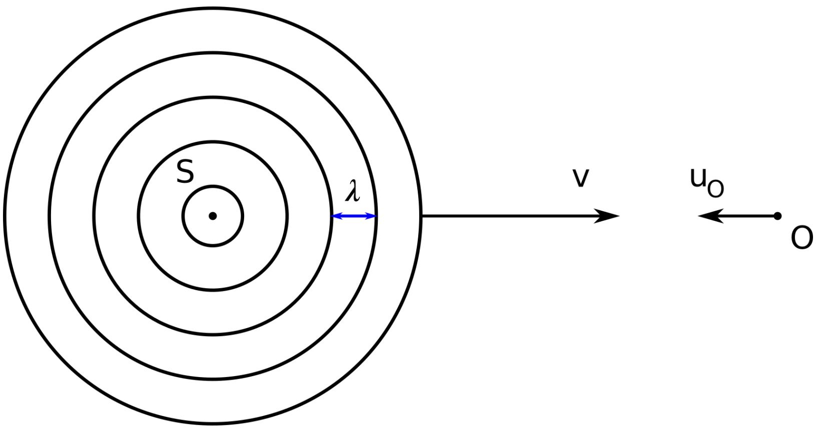


Figure 1: A stationary source and an observer moving towards the source.

Here is the key argument for this case: *Relative to O, the wavefronts will be approaching with a speed greater than v.* This larger relative speed[3] is $v + u_O$. Since the wavelength of the travelling wave is unchanged, the observer will receive the wavefronts at a *higher frequency* given by

$$f' = \frac{\text{wave speed}}{\text{wavelength}} = \frac{v + u_O}{\lambda}$$

[3] Think of yourself biking (with speed u_O) towards a car that is moving towards you (with speed v). Relative to you, the gap between you and the car will be decreasing at the faster rate $v + u_O$.

Inserting the expression for the wavelength, equation (1), we arrive at our first Doppler effect equation:

$$f' = \frac{v + u_O}{v/f} = f\left(\frac{v + u_O}{v}\right)$$

We can confirm that f' is greater than f, since the factor in parentheses is larger than one. If this was a sound wave, the observer would experience a higher pitch[4] than what a person at rest would experience.

[4] Recall that 'pitch' carries the same meaning as 'frequency'.

1. Now assume the observer is moving away from the stationary source. Show that the frequency measured by the observer is now the *lower frequency*

$$f' = f\left(\frac{v - u_O}{v}\right)$$

2. For a stationary source and a moving observer, we can summarize our results in one equation

$$f' = f\left(\frac{v \pm u_O}{v}\right)$$

Discuss how you would determine which sign to use.

Concept #2: A moving source and a stationary observer

In the second case, we will consider a source, S, moving relative to the medium with speed u_S, and an observer, O, at rest (stationary) relative to the medium. This models the situation of an ambulance approaching a person standing still. Let us initially assume the source is moving towards the observer, see figure 3. Let the source emit a wave with frequency, f, and wave speed $v > u_S$. Recall that this wave speed only depends on the properties of the medium and it is a speed measured relative to the medium.

Figure 2: A moving source of sound.

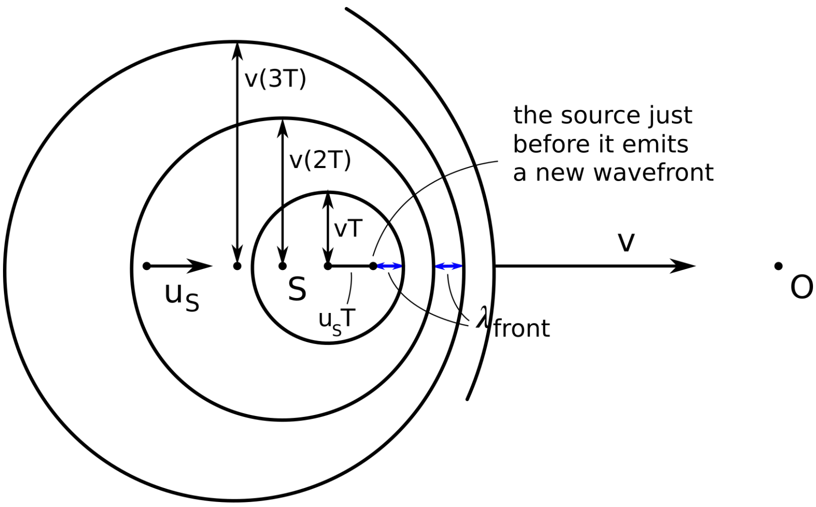


Figure 3: A moving source and an observer at rest. Note that we are assuming $v > u_S$.

Here is the key argument for this case: In one period T, the source moves a distance $u_S T$ while the wave travels[5] a longer distance vT. This implies that wavefronts bunch up in front of the moving source as shown in figure 3. The wavelength directly in front of the moving source[6] will therefore decrease to

$$\lambda_{\text{front}} = vT - u_S T = T(v - u_S)$$

Since the wave speed of the travelling wave is unchanged, this decrease in wavelength will result in the observer detecting wavefronts at a *higher frequency* given by

$$f' = \frac{\text{wave speed}}{\text{wavelength}} = \frac{v}{T(v - u_S)}$$

And inserting the expression $f = 1/T$ we arrive at our second Doppler effect equation:

$$f' = f\left(\frac{v}{v - u_S}\right)$$

We can confirm that f' is greater than f, since the factor in parentheses is larger than one. If this was a sound wave, the observer would experience a higher pitch compared to when the source was at rest,

1. Now assume the source is moving *away* from the stationary observer. Show that the frequency measured by the observer is now the *lower frequency*

$$f' = f\left(\frac{v}{v + u_S}\right)$$

2. For a stationary source and a moving observer, we can summarize our results in one equation

$$f' = f\left(\frac{v}{v \pm u_S}\right)$$

Discuss how you would determine which sign to use.

Concept #3: Combining cases

As shown in the previous sections, we get different equations for different cases of relative motion between a wave source and a receiver. It's important to realise that the two cases are not symmetrical[7] hence the frequency change is due to slightly different reasons: In the first case it is *the relative velocity of the wave* that changes, while in the second case it is *the wavelength of the travelling wave* that changes.

What happens if the source and the observer both move relative to the medium? Then we simply get a combined effect. E.g. if the source and observer both move towards each other, then the wavefronts bunch up (due to the motion of the source) and the relative

[5] Note again, that we are assuming the source speed is less than the wave speed, $u_S < v$, which is often the case. When the source speed exceeds the wave speed you get interesting effects such as the sonic boom (for sound) and Cherenkov radiation (for light), but this case is not part of the DP syllabus.

[6] The wavelength in other directions will be longer, reaching a maximum wavelength in the opposite direction to the source's motion. In which direction will the wavelength be unchanged?

[7] For *for light waves* however, the different cases do in fact turn out to be symmetrical. This is built into Einstein's theory of special relativity.

speed of the wavefronts increases (due to the motion of the observer). Hence we would expect the received frequency to be even higher than before:

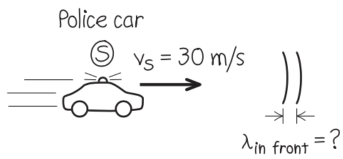
$$f' = f\left(\frac{v + u_O}{v - u_S}\right)$$

1. Although not part of the DP syllabus, here is one equation that covers all the cases considered in the previous sections:

$$f' = f\left(\frac{v \pm u_O}{v \mp u_S}\right)$$

   Discuss how you would determine which signs to use.

2. (a) A police car's siren emits a sinusoidal wave with frequency $f = 300$ Hz. The speed of sound is 340 m/s and the air is still.
      i. Find the wavelength of the waves if the siren is at rest.
      ii. Find the wavelengths of the waves in front of and behind the siren if it is moving at 30 m/s.



   (b) If *an observer is at rest and the siren is moving away* from the observer at 30 m/s, what frequency does the observer hear?
   (c) If *the siren is at rest and the observer is moving away* from it at 30 m/s, what frequency does the observer hear?
   (d) If the police car approaches a wall at 30 m/s, what frequency, f', will be received "by the wall"? The sound will reflect off the wall and the wall then becomes a stationary source of a wave with frequency f'. What frequency, f'', will the police officers in the moving car receive?
   (e) The police car moves away from an observer with a speed of 45 m/s relative to the air, and the observer moves toward the siren with a speed of 15 m/s relative to the air. What frequency does the observer hear?
   (f) Notice how the relative speed is constant 30 m/s in (b), (c) and (e), but in each case the change in frequency is different because *the Doppler effect also depends on how the source and observer are moving relative to the air/medium, not simply how they move relative to each other.*

Concept #4: Approximations are very useful

One of the defining aspects of science (if not *the* defining aspect) is *making useful approximations/models to the very complex reality that we live in*. In physics this often takes the form of *mathematical* approximations, but unfortunately, due to the way mathematics is taught in primary, middle and high school, students are often very uncomfortable with these. In my opinion, this is one of the main reasons science (physics in particular) is generally considered 'difficult to understand': Students never fully understand or appreciate that we are always developing approximate, but useful models. In this section, I'll show an example of what a common mathematical approximation looks like.

Let's begin by considering the infinite geometric series

$$1 + x + x^2 + x^3 + x^4 + \cdots$$

Most high school students learn that this sum converges[8] to $\frac{1}{1-x}$ provided $|x| < 1$,

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \quad \text{when } |x| < 1$$

For example if $x = 1/2$ then

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{1}{1-\frac{1}{2}} = 2$$

Now if x is much smaller than 1 (something we express as $x \ll 1$) then higher powers of x are vanishingly small compared to 1 and they can often be neglected. For example, if $x = \frac{1}{25} = 0.04 \ll 1$, then

$$x^5 = (0.04)^5 = 0.000\,000\,102\,4$$

which is indeed vanishingly small compared to 1. This allows us to confidently *discard the higher powers of* x and we can make the claim that[9]

$$1 + x + x^2 + x^3 + x^4 + \cdots \approx 1 + x$$

For example, if $x = \frac{1}{25}$ then we get

$$1 + \frac{1}{25} + \left(\frac{1}{25}\right)^2 + \left(\frac{1}{25}\right)^3 + \cdots = \frac{1}{1-\frac{1}{25}} \approx 1 + \frac{1}{25}$$

which you can evaluate to

$$1.041\,666\ldots \approx 1.04$$

These two numbers differ by less 0.2% which justifies that the right-hand-side is indeed a very good approximation to the left-hand side. The crucial point is that *the right-hand side* $(1 + x)$ *is much easier to calculate than the left-hand-side, so provided* x *is "small enough" we can use the simple calculation rather than the complicated calculation!* As long our measuring devices can't measure differences less than 0.2% then it doesn't matter which method we use and "the science will be the same".

Let's now apply this way of thinking to the Doppler equations. Consider an ambulance moving with speed u towards[10] a by-stander at rest. The frequency measured by the bystander will be

$$f' = f\left(\frac{v}{v-u}\right) = f\left(\frac{1}{1-\frac{u}{v}}\right)$$

where we have just rewritten the expression slightly in the last step (what did I do exactly?). Now *if the source speed is much smaller than*

[8] In other words, the two expression are exactly equal.

[9] Why not throw x away too? Well we could if x is small enough! And what about keeping x^2? Well we should if x is not small compared to 1! The exact details of the approximation often depends on *how we plan on using our result, i.e. how certain/accurate do we need to be?* In most cases we use approximations when x really is very small, but we still want to capture the variation that there is due to the change in x and the best approach is to choose the simplest type of variation which is to *linearise the expression* as $1 + x$. Linearising a more complicated expression is often called a **first order approximation**.

[10] I've omitted the subscript S in u_S for simplicity here.

the wave speed, $u \ll v$, then $\frac{u}{v} \ll 1$ and we can approximate the above expression to

$$f' \approx f\left(1 + \frac{u}{v}\right)$$

which can easily be manipulated into the expression

$$\frac{f' - f}{f} = \frac{\Delta f}{f} \approx \frac{u}{v}$$

where $\Delta f / f$ is the **relative/fractional change in frequency**. Hence we conclude that if $u \ll v$, then the ratio of the smaller speed to the larger speed is (approximately) equal to the relative change in frequency!

1. A loudspeaker on a bicycle is playing a very exciting song which consists of a single tone of frequency 300 Hz (check out the song here). The bicycle travels as a speed of $3\,\mathrm{m/s} \ll 340\,\mathrm{m/s}$ towards a bystander.
   (a) Would it be justified to use the approximate formula shown above?
   (b) Use the approximate formula to calculate the range of frequencies heard as the bicycle passes the bystander.
   (c) Now use the accurate formulas and compare with (b).
2. Show that

$$\frac{f' - f}{f} = \frac{\lambda - \lambda'}{\lambda'}$$

   and justify that *(Hint: Strictly speaking $\lambda' \neq \lambda$, but...)*

$$\frac{\Delta f}{f} \approx \frac{\Delta \lambda}{\lambda} \approx \frac{u}{v}$$

Concept #5: The Doppler effect for light waves

Although we haven't yet properly introduced the special case of electromagnetic waves (light waves), now is a good time to include something on that topic. It was mentioned earlier that the Doppler effect also applies to light waves but the derivation is a little bit different and you arrive at a slightly different formula for the frequency change. If we denote the wave speed as c (the speed of light in empty space) and the source velocity as v, the Doppler formula for light turns out to be[11]

$$f' = f\sqrt{\frac{1 + v/c}{1 - v/c}} \tag{2}$$

It turns out, however, that *this slightly different formula approximates in exactly the same way to the first order*[12], hence for light we also have:

$$\frac{\Delta f}{f} \approx \frac{\Delta \lambda}{\lambda} \approx \frac{v}{c}$$

Since the speed of a light source is very often much, much slower than lightspeed, this is the equation we always use for light waves.

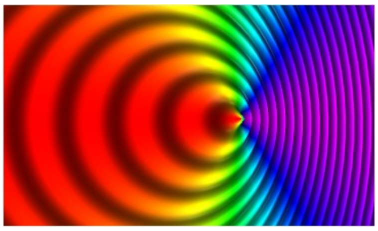


Figure 4: A light-emitting source moving to the right at a speed of $0.7c$ (70% the speed of light). In front of the source light is 'blue-shifted, at the back it is 'red-shifted'.

[11] You can find a derivation of this formula in *A Little Book About Special Relativity*.

[12] Which is not a coincidence because special relativity always reduces to our everyday Newtonian mechanics when the speed is much slower than the speed of light.

But if you are working with distant galaxies or subatomic particles moving at speeds close to the speed of light, then you have to use the more accurate equation (2).

1. You can investigate the approximation to these formulas in this desmos file.

A familiar application of the Doppler effect for light waves is the radar speed check device. Electromagnetic (radar) waves emitted by such a device is reflected from a moving car, which acts as a moving source, and the wave reflected back to the device is Doppler-shifted in frequency. The transmitted and reflected signals are then used to determine the speed of the car.[13] Similar techniques are used to measure wind velocities in the atmosphere. The Doppler effect is also used to track satellites and other space vehicles. In figure 5 a satellite emits a radio signal with constant frequency f_S. As the satellite orbits, it first approaches and then moves away from the receiver on Earth. The received frequency f_R changes from a value greater than f_S to a value less than f_S as the satellite passes overhead.

[13] The two signals are typically so close in frequency that they produce beats and the speed can be determined from the beat frequency.

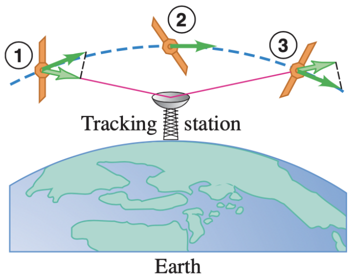


Figure 5: A satellite orbiting the Earth.

Lesson 3: Brain workout

If you want to learn physics, you must work on all the following problems! It's absolutely fine to make mistakes – it's actually preferable, because that's a great way to learn! If you are completely stuck, go back and read the relevant chapter again. If that didn't help at all, then ask me a question :)

1. The IB Data Booklet contains the equations shown in figure 6. Find all these equations in the text and notice if there are any differences.

Sub-topic 9.5 – Doppler effect
Moving source: $f' = f\left(\dfrac{v}{v \pm u_s}\right)$
Moving observer: $f' = f\left(\dfrac{v \pm u_o}{v}\right)$
$\dfrac{\Delta f}{f} = \dfrac{\Delta\lambda}{\lambda} \approx \dfrac{v}{c}$

Figure 6: Excerpt from the data booklet.

Lesson 3 Quiz

Check your understanding of this lesson: Here is a quiz.

Lesson 3 Problems

1. A source approaches (head on) a stationary observer at $40\,\mathrm{m\,s^{-1}}$ emitting sound of frequency 500 Hz.

(a) Determine the frequency the observer measures.

(b) Calculate the wavelength of the sound as measured by (i) the source and (ii) the observer.

2. A source is moving away from a stationary observer at $32\,\mathrm{ms}^{-1}$ emitting sound of frequency 480 Hz.

   (a) Determine the frequency the observer measures.

   (b) Calculate the wavelength of the sound as measured by (i) the source and (ii) the observer.

3. A sound wave of frequency 512 Hz is emitted by a stationary source toward an observer who is moving away at $12\,\mathrm{ms}^{-1}$.

   (a) Determine the frequency the observer measures.

   (b) Calculate the wavelength of the sound as measured by (i) the source and (ii) the observer.

4. A sound wave of frequency 628 Hz is emitted by a stationary source toward an observer who is approaching at $25\,\mathrm{ms}^{-1}$.

   (a) Determine the frequency the observer measures.

   (b) Calculate the wavelength of the sound as measured by (i) the source and (ii) the observer.

5. A sound wave of frequency 500 Hz is emitted by a stationary source toward a receding observer. The signal is reflected by the observer and received by the source, where the frequency is measured and found to be 480 Hz. Calculate the speed of the observer.

6. Ultrasound of frequency 5.000 MHz reflected from red blood cells moving in an artery is found to show a frequency shift of 2.4 kHz. The speed of ultrasound in blood is $v = 1500\,\mathrm{ms}^{-1}$.

   (a) Explain why the appropriate formula for the frequency shift is
   $$\frac{\Delta f}{f} = 2\frac{u}{v}$$
   where u is the speed of the blood cells

   (b) Estimate the speed of the blood cells.

   (c) In practice, a range of frequency shifts is observed. Explain this observation.

7. The concept of expressing a general function as an infinite sum of terms containing integer powers of the variable is quite important in physics and mathematics. Such a sum is called the **Taylor series** of the function and here is a list of some common ones.

   (a) Click on the above link and find the Taylor series that we used in this lesson.

   (b) Find the Taylor series for $f(x) = \sin(x)$.

i. Write down the first 3 terms of this infinite series.

ii. If $x \ll 1$ then what is $\sin(x)$ approximately equal to? Test this on your calculator. *(Hint: x is in radians!)*

iii. Imagine a right-angled triangle with hypotenuse r and one leg y much smaller than the other x. If θ (measured in radians) is the angle opposite the small leg y, show that

$$\theta \approx \frac{y}{r} \quad \Rightarrow \quad y \approx r\theta$$

and compare this with the arclength subtending angle θ in a circle of radius r.

iv. How large can x be in degrees for x and $\sin(x)$ to not deviate by more than 5%?

(c) If $x \ll 1$, then what is $\cos(x)$ and $\tan(x)$ approximately equal to?

8. Calculate the speed of a galaxy emitting light of wavelength 5.48×10^{-7} m which when received on Earth is measured to have a wavelength of 5.65×10^{-7} m.

9. The Sun rotates about its axis with a period that may be assumed to be constant 27 days. The radius of the Sun is 7.00×10^{8} m. Discuss the shifts in frequency of light emitted from the Sun's equator and received on Earth. Assume that the Sun emits monochromatic light of wavelength 5.00×10^{-7} m.

10. The gas cloud known as the Crab Nebula can be seen with even a small telescope. It is the remnant of a **supernova**, a cataclysmic explosion of a star. The explosion was seen on earth on July 4th, 1054 CE. The streamers glow with the characteristic red color of heated hydrogen gas. In a laboratory on the earth, heated hydrogen produces red light with frequency 4.568×10^{14} Hz. The red light received from streamers in the Crab Nebula pointed towards the earth has frequency 4.586×10^{14} Hz.

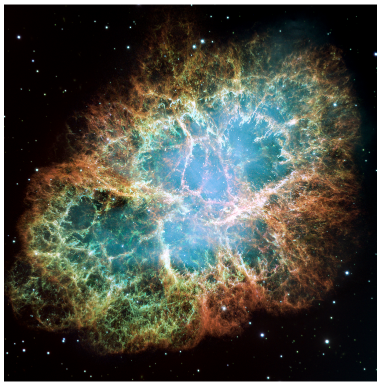


Figure 7: The Crab Nebula.

(a) Estimate the speed with which the outer edges of the Crab Nebula are expanding.

(b) Assuming that the expansion speed has been constant since the supernova explosion, estimate the diameter of the crab nebula. Give your answer in meters and light-years.

(c) The angular diameter of the Crab Nebula as seen from Earth is about 5 arc-minutes. Estimate (in light-years) the distance to the Crab Nebula, and estimate the year in which the supernova exploasion actually took place.

Answers to all the exercises.