A little book about waves

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"I would believe then that those who love to know the Causes of things and who are able to admire the marvels of Light, will find some satisfaction in these various speculations regarding it, and in the new explanation of its famous property which is the main foundation of the construction of our eyes and of those great inventions which extend so vastly the use of them. I hope also that there will be some who by following these beginnings will penetrate much further into this question than I have been able to do, since the subject must be far from being exhausted."

— Christiaan Huygens, *Treatise on Light*, 1690.

Lesson 5: Light (part I)

Light is a fascinating topic and our understanding of it has endless applications in the modern world[1]. It is one of the central themes in physics, not only because it involves many different phenomena, but also because its description has required the development of increasingly complex mathematical models.

[1] Microscopy, spectroscopy, thermography, lasers, wireless communication (e.g. radio, WiFi, Bluetooth, etc.), X-rays, MRI scans, resolving power, eye health, interferometry (e.g. detecting gravitational waves), detecting exoplanets and the expansion of the universe, fiber optics, etc.

While the story of light spans thousands of years due to its obvious presence in our lives, it was only after the scientific method was established in Europe around the 1600s that our knowledge about it increased significantly. An important breakthrough happened around the year 1800 when it was first experimentally verified that light has wave-like properties. A few decades later, in the early 1860s, James Clerk Maxwell's groundbreaking equations of electrodynamics showed us that *light is in fact an oscillation of an electric and magnetic field* – a so-called **electromagnetic wave**. We will talk more about the wave nature of light in the next lesson. Maxwell's achievement, however, was not the final word, and the development of relativity and quantum mechanics in the early 1900s led to an even more accurate understanding of light. We now know that light can behave both as a particle (a **photon**) and a wave! The lesson to learn from all this is just how difficult it is to properly understand reality in all its detail. As always, we probably still have more to learn and *the only way of knowing is to continue using the very reliable and effective scientific method of inquiry!*

Figure 1: A rainbow is one of the most magical everyday phenomenon related to light. The fact that white sunlight consists of all the different colors seen in a rainbow was first explained in a satisfactory manner by Isaac Newton in his work *Opticks* (1704).

Concept #1: Light as rays

If you've ever seen sunlight pass through a dusty (or foggy) room (see figure **??**), then it's not hard to accept the idea that light travels along straight lines. A thin beam of light that travels in this way is called a **light ray** and certain objects, typically very hot ones, emit visible light rays in all directions (see the hot metal in figure **??**). In most cases these rays "bounce" (or, to use a technical term that more correctly implies interaction: **scatter**) off other objects in all directions thus allowing us to see them. This type of reflection is sometimes referred to as **diffuse reflection** (see figure **??**) and most of what you see around you is light that reflects off other objects in

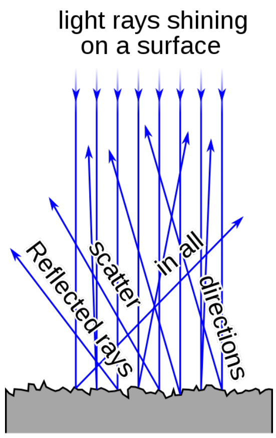


Figure 2: Diffuse reflection.

this way. For example, light rays falling into the limestone cave in figure **??** are scattering off small particles in the air, and the Moon is only visible because sunlight reflects off it (as is the case with most other objects in the Solar System).

Figure 3: Light is often seen in the form of light rays which suggests that light is some sort of small, fast-moving particle. Hot objects emit light rays in all directions.

Certain materials that are polished very smoothly (e.g. metals) reflect incoming light rays along only one direction and this type of reflection is called **specular reflection** (this is how mirrors work, more about this in a minute). Most materials combine these two types of reflection, e.g. a polished marble floor displays both diffuse and specular reflection.

1. Play around with light rays using an optics box or some other source of light rays (or use this simulation). If you use a laser pointer, then make sure not to point it at anyone's eyes. Shine a light ray through water with a bit of milk added to it to see a clear path taken by the light. Notice how light rays reflect off some surfaces and bend ("refract") through others as the pass from one material to another.

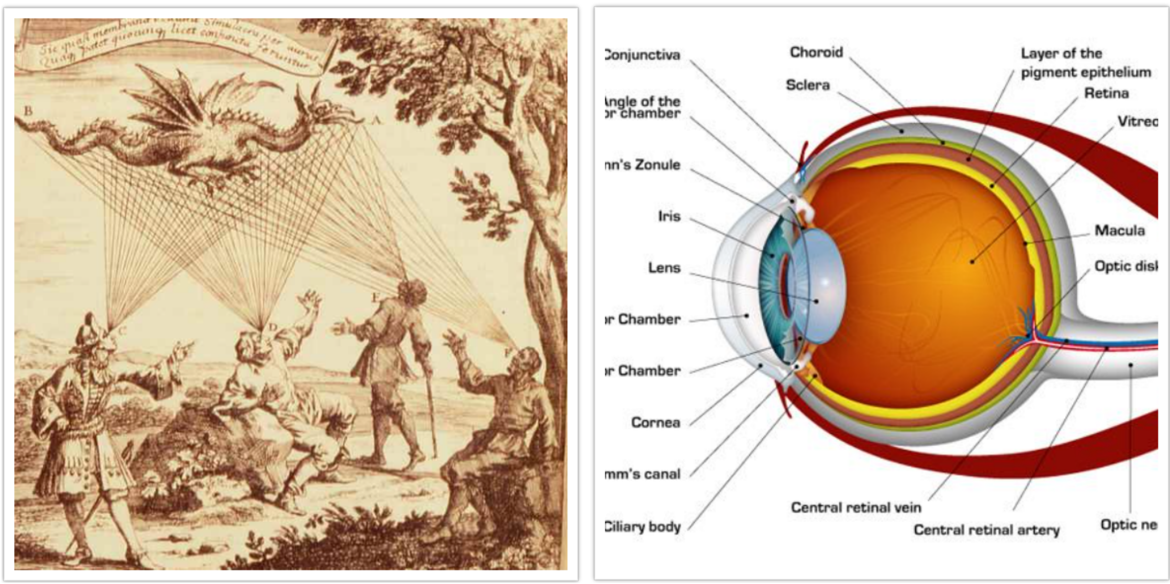


Figure 4: A medieval illustration of light emerging from the eye (source unknown, found here), and a diagram of the human eye (source unknown, found here).

Since the geometry of straight lines was well-known to the ancient Greek mathematicians, it's not surprising that they made some progress in describing light using this mathematical model. Euclid, Hero, and Ptolemy all wrote books on the subject and we use the phrase **geometrical optics** for this straight-line approach.[2] The Arab scientist al-Haytham also investigated light in this manner in his influential 11th century *Book of Optics*. Some early thinkers thought light originated inside the human eye and shone outwards like a flashlight, which is obviously incorrect since we can't see very well in darkness[3]. We now know that our eyes serve instead as **visible light detectors**, see figure **??**. Rays enter through the **pupil** (a hole in our **iris**) and hit light-sensitive cells (**rods** and **cones**) at the back of our eye (the **retina**). The existence of **color** is obvious to almost everyone and something we'll talk about a bit later (color blindness is due to the cone cells not working properly).[4]

[2] *Optics* is the general term we use for the branch of physics involving everything to do with light.

[3] The only place you can get away with thinking like this is in perspective drawing, an art form invented in Italy in the early 1400s.

[4] The human eye and its evolution is a fascinating topic to study.

Concept #2: The laws of reflection and refraction

Despite not knowing anything about the true nature of light, ancient cultures still invented the technology of mirrors and lenses. This is a good example of how *technology can be developed without having a full understanding of the science behind it*. Trial and error is a perfectly respectable approach to engineering! The earliest mirrors date back thousands of years and lenses that could bend light rays and serve as magnifying glasses were made from natural crystals as early as 750 BCE in the Middle East (see figure **??**). During the early Renaissance in Europe, wearable eyeglasses were developed (Italy, late 1200s) and a few centuries later, microscopes and telescopes were built (Netherlands, early 1600s), see figure **??**.

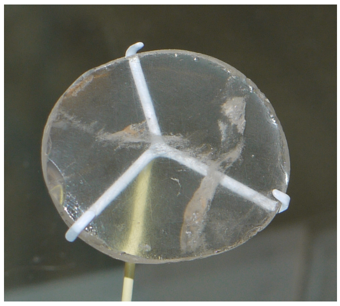


Figure 5: The Nimrud lens.

Figure 6: A page from Al-Haytham's work. Refraction of a light ray through glass. Early spectacles. An early telescope. Robert Hooke's influential book Micrographia with fantastic hand-drawn pictures of the microscopic world found using one of the very first microscopes. Robert Hooke was the person to coin the term *cell* for the small units of life that he discovered with this new technology.

All the above artifacts were based on the common experience that light **reflects** and **refracts** (= bends) in certain ways which can be summarised in the **law of reflection** and the **law of refraction**. At the end of the 1700s, these two laws represented our best understanding of optics. Let's look in more detail at these two early laws of physics.

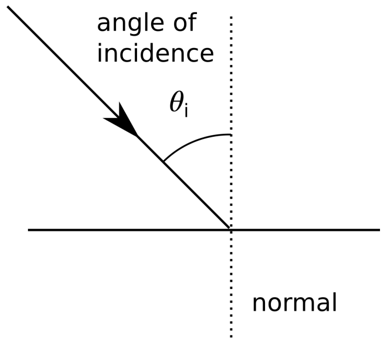


Figure 7: The angle of incidence and the normal direction.

First a bit of terminology: When a light ray is incident on a surface we define the **normal direction** as being the direction perpendicular to the surface. The **angle of incidence** is then defined as *the angle the incident ray makes with the normal*, see figure **??**. The **angle of reflection** is the angle the ray makes with the normal after it is reflected off the surface, and the law of reflection states that *it is equal to the angle of incidence*. This is illustrated in figure **??**. The law of reflection is very easy to verify experimentally and it has been fully accepted for thousands of years.[5]

[5] As simple as the law of reflection seems, it has a bunch of interesting applications. One is that of corner cubes. These are reflectors that return an incident light ray in the direct opposite direction (but shifted to the side a bit). Five arrays of corner cubes are currently on the Moon allowing us to measure the Earth-Moon distance with extreme precision and accuracy. Another application of the law of reflection is optical trickery, such as ghosts (!) and kaleidoscopes (= "observation of beautiful forms").

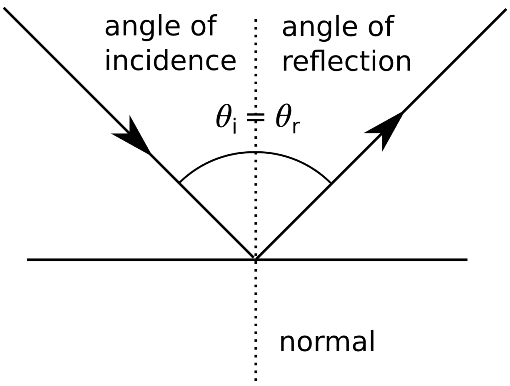


Figure 8: The law of reflection.

It was also known to many ancient cultures that when light passes from one transparent medium (e.g. air) to another (e.g. water or glass) it bends (= refracts). The angle that the refracted light ray makes with the normal direction is called the **angle of refraction**, see figure **??**. For a long time people thought that the angle of refraction was directly proportional to the angle of incidence, but that turned out to be incorrect. The correct formulation of the **law of refraction** is as follows: At the boundary between two different media, the incident angle, θ_1, and the refracted angle, θ_2, always satisfy the relationship

$$\frac{\sin\theta_1}{\sin\theta_2} = \text{constant}$$

where the constant has a value that depends on the properties of the two media.

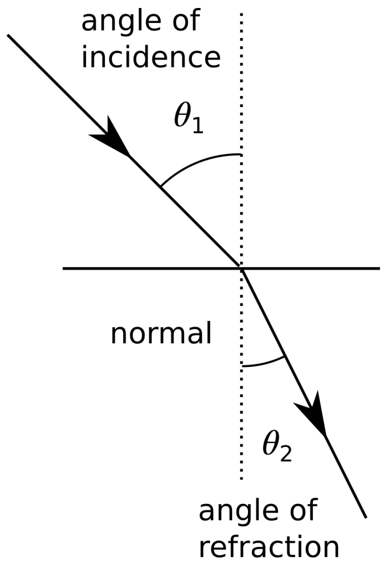


Figure 9: A light ray refracting (bending) as it passes from one medium into another.

1. Is the angle of refraction directly proportional to the angle of incidence for small angles?

2. Verify the law of refraction experimentally using an optics box (or use this simulation). Conduct a proper scientific investigation with the research question "*How does the angle of incidence affect the angle of refraction?*".

Concept #3: Scientific laws are verified, not "proven"

In this section, I'll spend a bit of time talking about **the nature (or 'philosophy') of science**. It's important to do that sometimes, because not many people actually take the time to understand the subtleties of science – even many well-educated people are not aware of the full implications of the scientific method of inquiry.[6]

One common mistake people make is to equate mathematical knowledge with scientific knowledge. Let's set that straight: Scientific laws are *discovered* by creating a model of reality, measuring variables associated with the model, and then empirically testing for causal relationships between those variables. When causality is established, the models have accurate predictive power which is the reason science has transformed our world so much over the past 450 years.[7] The models we create are typically mathematical in nature and we empirically verify the associated laws using mathematics as a language (data points, graphs, error bars, best-fit functions, statistics, etc.), *but it's important to understand that we are not directly proving the laws of science using the logic of mathematical deduction. A scientific law (or theory) is only considered 'true' as long as it is supported empirically by careful experiments. We don't prove a law in the same way that we prove a mathematical theorem. The 'logic' of science is the experiment – if carefully collected data does not support a claim within the precision of the measurements, then we are led to believe the claim is 'false' (= "wrong") and we have to look for a better understanding/explanation (a better model) of the phenomenon.*

Take, for example, the laws of reflection and refraction. These are laws associated with light being modelled as light rays and we can experimentally verify them to a high degree of accuracy. But this simple model does not explain *why* light behaves in this way. The mathematical model of light travelling along straight lines is rather superficial and it raises many new questions (which humanity had trouble answering for thousands of years): What are those light rays really made of? How fast do they travel? Why does light even reflect off a mirror? Why does light bend and why does the ratio of the sines stay constant when it bends? Despite not knowing any answers to these questions, the two laws can still be used to invent technology (telescopes and microscopes) that transform our understanding of nature in different areas (e.g. the microscopic world and the solar system).

But humans actually do want to know 'why', so scientists continue to make new, more advanced models that help explain new questions and phenomena that arise even after new technology has been invented. One example of this happened in 1662 when Pierre de Fermat tried to explain the law of refraction by making an assumption we now call **Fermat's principle**[8]. He decided to postulate that *light always travels between two points on the path which takes the least time*. In many ways, this principle/postulate raised more questions than answers. For example, who says this is even true? And

[6] I am the first data point in the data set that supports this claim: Earning a B.Sc. in Mathematics and Physics, a M.Sc. in Physics, and being accepted to a Ph.D. program in Mathematics, was not enough to make me fully understand the scientific method of inquiry! I only grasped the true importance of it after teaching science and mathematics for more than a decade to young curious minds. I honestly had to figure it out on my own because at no point in my education did anyone really teach this to me explicitly. I find that rather shocking now!

[7] For better or worse – mostly better (modern medicine, hard human labor replaced by machines, agriculture and technology), but in recent years maybe for worse (pollution, climate change, unchecked growth, unhealthy lifestyles, digital distractions and confusion).

[8] This principle was first used in a slightly different form by Hero to prove the law of reflection. Here's a good account of the history behind all this.

how does light "know" which path takes least time? But despite the mysterious nature of this principle, it did seem to nicely explain why the ratio of the sines was constant and it also allowed the speed of light to be included in the law of refraction[9]. The mathematical derivation leading to the law of reflection is given below:

Consider a light ray bending at point O as it passes a boundary from A to B (see figure **??**). Assume the light ray travels through medium 1 and medium 2 with speeds v_1 and v_2 respectively[10]. The total travel time from A to B is then given by the expression

$$t = \text{time taken to travel from } A \text{ to } B = \frac{AO}{v_1} + \frac{OB}{v_2}$$

[9] It also turned out to be equivalent to Huygen's wave theory of light (see next lesson), and it was the first example of a type of reasoning that goes under the name of the principle of least action which remains central in modern physics.

[10] It's an empirical fact that light travels at different speeds through different transparent substances.

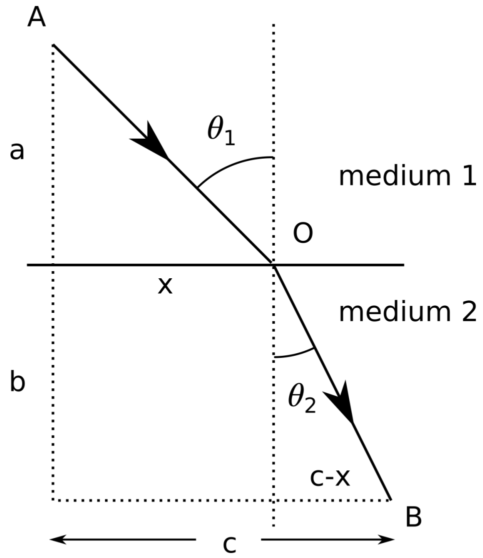


Figure 10: A diagram used to assist in deriving the law of refraction from Fermat's principle.

Applying Pythagoras' theorem to find expressions for AO and OB in terms of the quantities shown on the diagram, the above equation can be written as

$$t = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{(c-x)^2 + b^2}}{v_2} \tag{1}$$

where you should imagine x being a variable (by varying x, the position of point O changes), while the other quantities have fixed values. If, say $a = b = 2$, $c = 1$, $v_1 = 3$, $v_2 = 1$, then graphing this function as a function of x gives you the graph shown in figure **??** (you can explore the graph here):

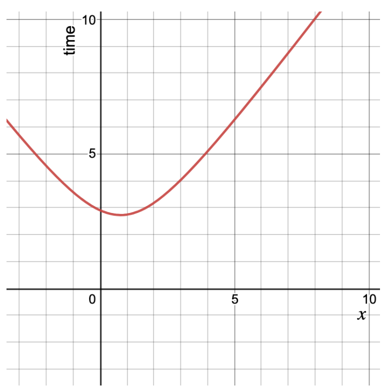


Figure 11: A graph of equation (**??**).

According to Fermat's principle, the light ray will take the path *that minimises this expression*, in other words, we need to find the value of x that gives us the least amount of time. As you can see on the graph, such a value does indeed exist. If you know **differential calculus**, then finding this minimum is easy: The minimum will satisfy the equation[11]

$$\frac{1}{v_1}\frac{x}{\sqrt{a^2+x^2}} - \frac{1}{v_2}\frac{(c-x)}{\sqrt{(c-x)^2+b^2}} = 0 \tag{2}$$

[11] This equation represents the fact that the slope of the graph is zero at the minimum point – we are setting the first derivative equal to zero, $f'(x) = 0$. If you haven't learnt this yet, then just take my word for it.

The above equation looks quite complicated, but it actually isn't. Since (look at figure **??** again)

$$\sin\theta_1 = \frac{x}{AO} = \frac{x}{\sqrt{a^2 + x^2}}$$

and

$$\sin\theta_2 = \frac{c-x}{OB} = \frac{(c-x)}{\sqrt{(c-x)^2 + b^2}}$$

then equation (**??**) simplifies to

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$$

which can be manipulated into

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

hence we arrive at the law of refraction and we can see that *the constant equals the ratio of the speed of light in the first medium to the speed of light in the second medium*. Quod erat demonstrandum![12]

Did we just "prove" the law of refraction? No, most certainly not! Just because a mathematical derivation based on a mathematical model leads to a particular equation, *it doesn't automatically mean that the equation holds true in reality!* **The real reason we believe in the law of refraction is because it can be verified experimentally.** All that the mathematical derivation does is give us an equation that we can test experimentally. **Scientific laws cannot be proven mathematically (as one proves theorems in math) and one should not mistake the language of mathematics with the nature of reality. Mathematical models do not actually exist, they are merely 'layers' that help us describe and understand the very real causal relationships that exist in nature.**

Here is another example that illustrates this subtle detail: Using Newton's laws of motion and his law of gravity we can mathematically derive the result that planets move in elliptical orbits. That fact alone does not make elliptical orbits true – *we only believe it is 'true' because we can actually measure those elliptical orbits.*[13] And because Newton's theory predicts this empirically verified (hence 'true') observation, we consider this observation to be a piece of evidence *in support of* Newton's theory. With millions of similar examples of empirical evidence, we firmly believe in Newton's theory of motion and gravity - despite the fact that we can never "prove" these theories using a purely deductive mathematical approach. Likewise, we believe in Darwin's theory of evolution and many other scientific theories because they are overwhelmingly supported by actual measurements – not because we can mathematically "prove" them.

1. It follows mathematically from Einstein's general theory of relativity that "time machines" exist inside rotating black holes (more specifically, closed time-like curves in 4D spacetime exist inside the event horizon of a Kerr black hole). Does this mean we can confidently claim that time travel is possible?

[12] Latin for "what was to be shown". Traditionally, the abbreviation Q.E.D. is placed at the end of mathematical proofs to indicate that the proof or the argument is complete.

[13] And this was difficult to measure! Initially astronomers measured circles, but as their measurements got more precise and didn't fit the model, they had to radically overthrow the 'geocentric model' of the solar system in favour of the 'heliocentric (sun-centred) model'.

Concept #4: The index of refraction – nothing new here

The law of refraction can be written in a different way by defining the **index of refraction** (or **refractive index**), n, which is simply a measure of *how much light slows down in a material*. It is precisely defined as *the ratio of the speed of light in empty space, c, to the speed of light in the material, v*

$$n \equiv \frac{c}{v}$$

Since the speed of light in empty space[14] is the fastest possible speed in the universe (one of the basic assumptions of Einstein's special theory of relativity), the index of refraction will always be a number greater than or equal to one, $n \geq 1$. Here are a few values of n:

$$\begin{aligned} \text{Air,} \quad & n = 1.0003 \approx 1 \\ \text{Water,} \quad & n = 1.33 \\ \text{Glass,} \quad & n = 1.6 \\ \text{Diamond,} \quad & n = 2.4 \end{aligned}$$

1. What is the speed of light in water as a percentage of the speed of light in empty space?
2. What is the speed of light in glass as a percentage of the speed of light in diamond?

The larger n is, the slower the speed of light[15], and we say the material is more **optically dense**. Using this new definition, the law of refraction can be written as

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1}$$

which reduces to the convenient[16]

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Concept #5: White light is full of colors

The final thing to mention in this lesson is the fact that 'white' light (e.g. sunlight or light from an incandescent light bulb) contains many different colours. Newton demonstrated this in 1672 by refracting sunlight through a glass prism, se figure **??**. It turns out the refractive index of a material, n, depends slightly on the colour of the light, a phenomenon known as **dispersion**. Hence different coloured light rays bend at slightly different angles and a rainbow is a lovely demonstration of this: White sunlight is refracted, dispersed and reflected in raindrops.[17] Newton wrote about all this in his influential book called 'Opticks' from 1704. Around this time in history, pretty much all our knowledge about optics followed from the laws of reflection and refraction, and even today, in the 21st century, most people's knowledge about light doesn't exceed this - they

[14] Recall, empty space is also called a *vacuum* and the speed of light in a vacuum is *defined to be exactly* $299,792,458$ m/s.

[15] Lene Hau, a Danish physicist at Harvard University, was the first person to slow light down to incredibly low speeds. In 2001 she actually brought light to a complete standstill.

[16] This version is typically referred to as **Snell's Law**, but whether or not he deserves credit for this is a bit complicated.

[17] See problem **??** and here is a great lecture by Walter Lewin about rainbows.

are still living in the 1700s in terms of their scientific understanding! We will eventually see that it was the wave-like properties of light that really advanced our understanding, but the mathematical description of waves is unfortunately much more complicated than straight-lines, which tends to be a common theme in physics: The mathematical language required to more accurately describe reality gets more and more complicated!

Figure 12: Newton doing science experiments instead of tidying up his room.

Time to go to the gym

People don't question the fact that in order to get a stronger body, you need to lift weights. Your muscles need to be pushed to (and beyond) their limit in order to grow back stronger. The same applies to physics, but instead of going to the gym, you need to sit down quietly and solve problems on a piece of paper! Making mistakes is the equivalent of breaking down your muscles. If you never exercise, your body suffers. If you never solve physics problems, your mind suffers.

If you want to learn physics, you have to work on solving problems. It's absolutely fine to make mistakes – it's actually the best way to learn! – but if you are completely stuck, then go back and read the relevant section again. If that doesn't help, then ask for help :)

The IB Data Booklet contains the equations shown below. Find all these equations in the text and pay attention to any differences.

Sub-topic 4.4 – Wave behaviour

$$\frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1} = \frac{v_2}{v_1}$$

Problems first

1. If you want to view your full height in a plane mirror, must the mirror be as tall as you are? Assume your body is completely flat and explain your answer using a diagram containing light rays (= a **ray diagram**).

2. In figure **??**, a beam of light reflects and refracts at point A on the interface between material 1 with index of refraction $n_1 = 1.33$ and material 2 with index of refraction $n_2 = 1.77$. The incoming beam makes an angle of 50° with the interface.

   (a) What is the angle of reflection at point A?

   (b) What is the angle of refraction at point A?

   (c) The light that enters material 2 at point A then reaches point B on the interface between material 2 and material 3, which is air, as shown in the figure below. The interface through B is parallel to that through A. At B, some of the light reflects and the rest enters the air. What is the angle of reflection? What is the angle of refraction into the air?

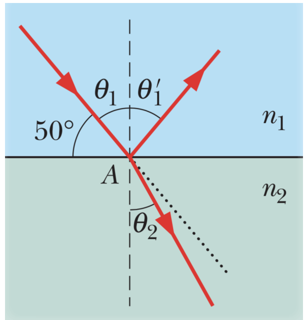


Figure 13: Problem **??** (a) and (b).

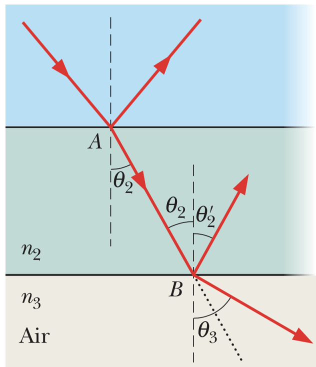
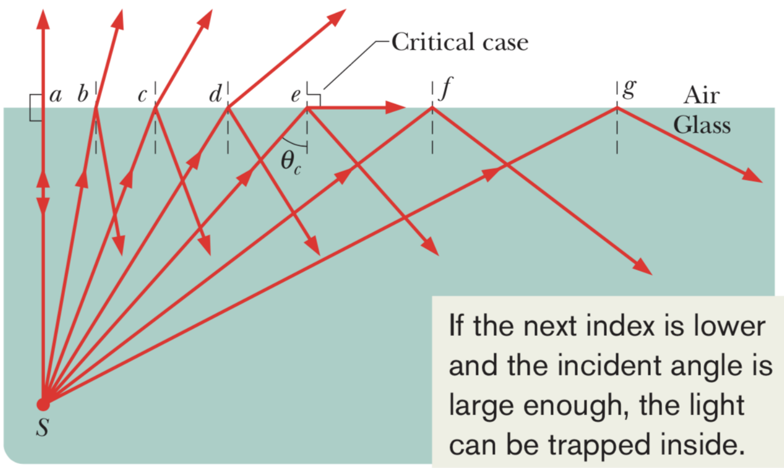


Figure 14: Problem **??** (c).

3. The figure below shows light rays from a point source S in glass incident on the interface between the glass and air.



For ray a, which is perpendicular to the interface, part of the light reflects at the interface and the rest travels through it with no change in direction. For rays b through e, which have progressively larger angles of incidence at the interface, there are also both reflection and refraction at the interface. As the angle of incidence increases, the angle of refraction increases; for ray e the angle of refraction is 90°, which means that the refracted ray points directly along the interface. The angle of incidence giving this situation is called the **critical angle**, θ_c. For angles of incidence larger than θ_c, rays f and g, there is no refracted ray and all the light is reflected; this effect is called **total internal reflection** because all the light remains inside the glass.

   (a) Show that

$$\theta_c = \sin^{-1}\left(\frac{n_{\text{air}}}{n_{\text{glass}}}\right)$$

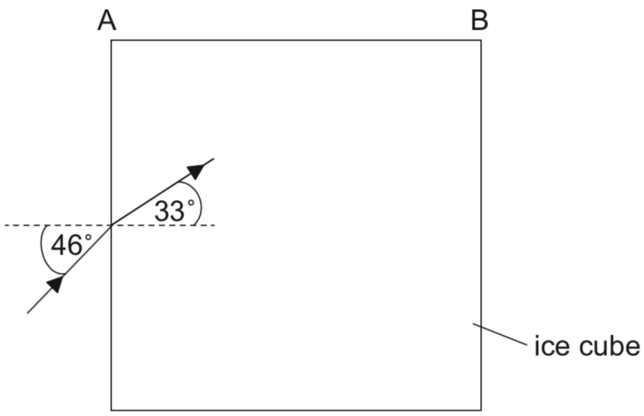
   (b) Calculate the critical angle when light goes from glass into air.

(c) Calculate the critical angle when light goes from glass into water.

(d) What is the condition and general formula for a light ray undergoing total internal reflection between two different media?

(e) Next time you take a bath, try this little demonstration of total internal reflection! Submerge yourself completely, lie horizontally and look at the water surface with your eyes rather close to the surface. Light rays coming from your feet should undergo total internal reflection in the water surface since they will be incident at an angle larger than the critical angle.

4. A large cube is formed from ice. A light ray is incident from a vacuum at an angle of 46° to the normal on one surface of the cube. The angle of refraction inside the cube is 33°.



(a) Calculate the speed of light inside the ice cube.

(b) Explain why no light emerges from side AB.

(c) Sketch, on the diagram, the path of the light ray as it leaves the ice cube again.

5. White light containing all colours is incident on a regular triangular glass prism at an angle of 45° to the normal. For red light $n_r = 1.7$ while for blue light $n_b = 1.8$. Make a careful drawing showing the red and blue light rays entering and leaving the prism.

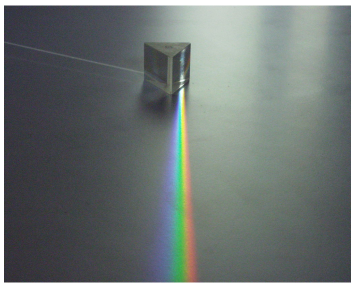


Figure 15: Problem **??**.

6. Rays of light are emitted upwards in all directions from a point at the bottom of a bowl containing liquid to a depth of 7.5 cm. The refractive index of the liquid is 1.25. Find the radius of the circle lying on the surface of the liquid with its centre vertically above the point, outside which all rays will be totally reflected.

7. A coin is viewed directly from above through a liquid 8.0 cm deep. If the coin appears to be brought 3.0 cm nearer to the

surface, calculate the refractive index of the liquid. (*Hint: Make a drawing, and assume the angles are small so* $\sin\theta \approx \tan\theta$.)

8. A red light beam is incident on a rectangular block of glass at an angle of 40°, as shown in figure **??**. The refractive index of red light in the glass is 1.45 and the thickness of the block is 4.00 cm.

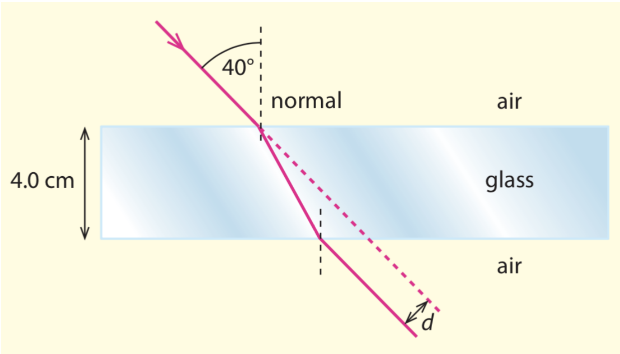


Figure 16: Problem **??**.

   (a) Calculate the amount d by which the ray shifts as it passes through the block.
   (b) How would d change if the block was thinner?
   (c) Blue light has refractive index 1.49 in the glass block. Assuming the red and blue light start on the same path, explain what happens as the two different colours pass through the block.

9. Try to explain how a mirage works, see figure **??**.

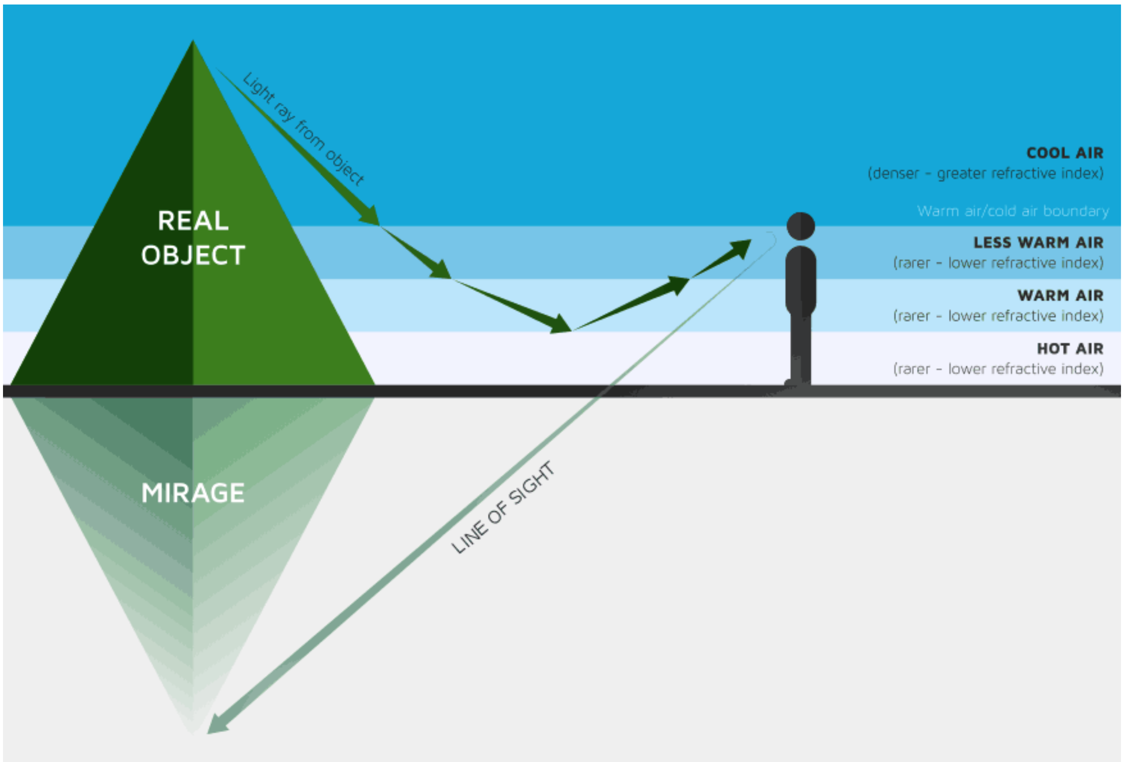


Figure 17: Problem **??**.

10. Rainbow problem

11. Check out this wonderful video from 3Blue1Brown that combines the brachistochrone problem with the law of refraction (and a great story about "the lion's claw"). And here's a nice example of how physics and mathematics go nicely together: A proof of the Basel problem (again from 3Blue1Brown).

12. The index of refraction of air depends on the temperature of the air. Hence it becomes possible to see hot air by sending light through it. This is called the Schlieren effect. Check out this cool video.

    Answers to all the problems.

Now take a quiz

Check your understanding of this lesson: Here is a quiz.