Contents

Haskell for CMI

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This is (still!) an incomplete draft.
Please send any corrections, comments etc. to feedback_host@mailthing.com
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To someone

Table of Contents

Basic Theory

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§1.1. Precise Communication

Haskell (as well as a lot of other programming languages) and Mathematics, both involve communicating an idea in a language that is precise enough for them to be understood without ambiguity.

The main difference between mathematics and haskell is **who** reads what we write.

When writing any form of mathematical expression, it is the expectation that it is meant to be read by humans, and convince them of some mathematical proposition.
On the other hand, haskell code is not *primarily* meant to be read by humans, but rather by machines. The computer reads haskell code, and interprets it into steps for manipulating some expression, or doing some action.

When writing mathematics, we can choose to be a bit sloppy and hand-wavy with our words, as we can rely to some degree on the imagination and pattern-sensing abilities of the reader to fill in the gaps.

However, in the context of Haskell, computers, being machines, are extremely unimaginative, and do not possess any inherent pattern-sensing abilities. Unless we spell out the details for them in excruciating detail, they are not going to understand what we want them to do.

Since in this course we are going to be writing for computers, we need to ensure that our writing is very precise, correct and generally **idiot-proof**. (Because, in short, computers are idiots)

In order to practice this more formal style of writing required for **haskell code**, the first step we can take is to know how to **write our familiar mathematics more formally**.

§1.2. The Building Blocks

The language of writing mathematics is fundamentally based on two things -

- **Symbols:** such as $0, 1, 2, 3, x, y, z, n, \alpha, \gamma, \delta, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \in, <, >, f, g, h, \Rightarrow, \forall, \exists$ etc. Along with;
- **Expressions:** which are sentences or phrases made by chaining together these symbols, such as
  - $x^3 \cdot x^5 + x^2 + 1$
  - $f(g(x, y), f(a, h(v), c), h(h(h(n))))$
  - $\forall \alpha \in \mathbb{R}\ \exists L \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \ | \, x - \alpha \, | < \delta \Rightarrow | \, f(x) - f(\alpha) \, | < \varepsilon$

  etc.

§1.3. Values

÷ **mathematical value**

A **mathematical value** is a single and specific well-defined mathematical object that is constant, i.e., does not change from scenario to scenario nor represents an arbitrary object.

The following examples should clarify further.

Examples include -

- The real number π
- The order $<$ on $\mathbb{N}$

- The function of squaring a real number : $\mathbb{R} \to \mathbb{R}$
- The number d , defined as the smallest number in the set
  $\{n \in \mathbb{N} \mid \exists \text{ infinitely many pairs } (p, q) \text{ of prime numbers with } |p - q| \leq n\}$

Therefore we can see that relations and functions can also be **values**, as long as they are specific and not scenario-dependent. For example, the order $<$ on $\mathbb{N}$ does not have different meanings or interpretations in different scenarios, but rather has a fixed meaning which is independent of whatever the context is.

In fact, as we see in the last example, we don't even currently know the exact value of d.
The famous "Twin Primes Conjecture" is just about whether $d == 2$ or not.

So, the moral of the story is that even if we don't know what the exact value is,
we can still know that it is **some** ≒ **mathematical value**,
as it does not change from scenario to scenario and remains constant, even though it is an unknown constant.

§1.4. Variables

≒ **mathematical variable**

A **mathematical variable** is a symbol or chain of symbols
meant to represent an arbitrary element from a set of ≒ **mathematical value**s,
usually as a way to show that whatever process follows is general enough so that the process can be carried out with any arbitrary value from that set.

The following examples should clarify further.

For example, consider the following function definition -

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) := 3x + x^2$$

Here, x is a ≒ **mathematical variable** as it isn't any one specific ≒ **mathematical value**,
but rather **represents an arbitrary** element from the set of real numbers.

Consider the following theorem -

Theorem Adding 1 to a natural number makes it bigger.

Proof Take n to be an arbitrary natural number.
We know that $1 > 0$.
Adding n to both sides of the preceding inequality yields

$$n + 1 > n$$

Hence Proved !! ■

Here, n is a ≒ **mathematical variable** as it isn't any one specific ≒ **mathematical value**,
but rather **represents an arbitrary** element from the set of natural numbers.

Here is another theorem -

Theorem For any $f : \mathbb{N} \to \mathbb{N}$, if f is a strictly increasing function, then f(0) < f(1)

Proof Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Thus

$$\forall n, m \in \mathbb{N}, n < m \Rightarrow f(n) < f(m)$$

Take n to be 0 and m to be 1. Thus we get

$$f(0) < f(1)$$

Hence Proved! ■

Here, f is a ≒ **mathematical variable** as it isn't any one specific ≒ **mathematical value**, but rather **represents an arbitrary** element from the set of all $\mathbb{N} \to \mathbb{N}$ strictly increasing functions.

It has been used to show a certain fact that holds for **any** natural number.

§1.5. Well-Formed Expressions

Consider the expression -

$$xyx \Longleftarrow \forall \Rightarrow f(\nleftarrow > \vec{v}$$

It is an expression as it **is** a bunch of symbols arranged one after the other, but the expression is obviously meaningless.

So what distinguishes a meaningless expression from a meaningful one? Wouldn't it be nice to have a systematic way to check whether an expression is meaningful or not?

Indeed, that is what the following definition tries to achieve - a systematic method to detect whether an expression is well-structured enough to possibly convey any meaning.

≒ **checking whether mathematical expression is well-formed**

It is difficult to give a direct definition of a **well-formed expression**.

So before giving the direct definition,
we define a ***formal procedure*** to check whether an expression is a **well-formed expression** or not.

The procedure is as follows -

Given an expression e,

- first check whether e is
  - a ≒ **mathematical value**, or
  - a ≒ **mathematical variable**

  in which cases e passes the check and is a **well-formed expression**.

Failing that,

- check whether e is of the form $f(e_1, e_2, e_3, ..., e_n)$, where
  - f is a function
  - which takes n inputs, and
  - $e_1, e_2, e_3, ..., e_n$ are all ***well-formed expressions*** which are ***valid inputs*** to f.

And only if e passes this check will it be a **well-formed expression**.

≒ **well-formed mathematical expression**

A ***mathematical expression*** is said to be a **well-formed mathematical expression** if and only if it passes the formal checking procedure defined in ≒ **checking whether mathematical expression is well-formed**.

Let us use ÷ **checking whether mathematical expression is well-formed** to check if $x^3 \cdot x^5 + x^2 + 1$ is a well-formed expression.
(We will skip the check of whether something is a valid input or not, as that notion is still not very well-defined for us.)

$x^3 \cdot x^5 + x^2 + 1$ is $+$ applied to the inputs $x^3 \cdot x^5$ and $x^2 + 1$.
Thus we need to check that $x^3 \cdot x^5$ and $x^2 + 1$ are well-formed expressions which are valid inputs to $+$.

$x^3 \cdot x^5$ is $\cdot$ applied to the inputs x^3 and x^5.
Thus we need to check that x^3 and x^5 are well-formed expressions.

x^3 is $(\ )^3$ applied to the input x.
Thus we need to check that x is a well-formed expression.

x is a well-formed expression, as it is a ÷ **mathematical variable**.

x^5 is $(\ )^5$ applied to the input x.
Thus we need to check that x is a well-formed expression.

x is a well-formed expression, as it is a ÷ **mathematical variable**.

$x^2 + 1$ is $+$ applied to the inputs x^2 and 1.
Thus we need to check that x^2 and 1 are well-formed expressions.

x^2 is $(\ )^2$ applied to the input x.
Thus we need to check that x is a well-formed expression.

x is a well-formed expression, as it is a ÷ **mathematical variable**.

1 is a well-formed expression, as it is a ÷ **mathematical value**.

Done!

X **checking whether expression is well-formed**

Suppose a, b, v, f, g are ÷ **mathematical value**s.

Suppose x, y, n, h are ÷ **mathematical variable**s.

Check whether the expression

$$f(g(x, y), f(a, h(v), c), h(h(h(n))))$$

is well-formed or not.

§1.6. Function Definitions

Functions are a very important tool in mathematics and they form the foundations of Haskell programming.
Thus, it is very helpful to have a deeper understanding of **how function definitions in mathematics work**.

§1.6.1. Using Expressions

In its simplest form, a function definition is made up of a left-hand side, ':=' in the middle[1], and a right-hand side.

A few examples -
- $f(x) := x^3 \cdot x^5 + x^2 + 1$
- $\text{second}(a, b) := b$
- $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$

On the left we write the name of the function followed by a number of variables which represent its inputs.

In the middle we write ':=', indicating that right-hand side is the definition of the left-hand side.

On the right, we write a ≒ **well-formed mathematical expression** using the variables of the left-hand side, describing to how to combine and manipulate the inputs to form the output of the function.

§1.6.2. Some Conveniences

Often in the complicated definitions of some functions, the right-hand side expression can get very convoluted, so there are some conveniences which we can use to reduce this mess.

§1.6.2.1. Where, Let

Consider the definition of the famous sine function -

$$\text{sine} : \mathbb{R} \to \mathbb{R}$$

Given an angle θ,
Let T be a right-angled triangle, one of whose angles is θ.
Let p be the length of the perpendicular of T.
Let h be the length of the hypotenuse of T.
Then

$$\text{sine}(\theta) := \frac{p}{h}$$

Here we use the variables p and h in the right-hand side of the definition, but to get their meanings one will have to look at how they are defined beforehand in the lines beginning with "let".

We can also do the exact same thing using "where" instead of "let".

$$\begin{aligned} &\text{sine} : \mathbb{R} \to \mathbb{R} \\ &\text{sine}(\theta) := \frac{p}{h} \\ &\quad \text{,where} \\ &\qquad T := \text{a right-angled triangle with one angle} == \theta \\ &\qquad p := \text{the length of the perpendicular of } T \\ &\qquad h := \text{the length of the hypotenuse} \quad \text{of } T \end{aligned}$$

Here we use the variables p and h in the right-hand side of the definition, but to get their meanings one will have to look at how they are defined after "where".

[1] In order to have a clear distinction between definition and equality, we use $A := B$ to mean "A is defined to be B", and we use $A == B$ to mean "A is equal to B".

§1.6.2.2. Anonymous Functions

A function definition such as

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) := x^3 \cdot x^5 + x^2 + 1$$

for convenience, can be rewritten as -

$$(x \mapsto x^3 \cdot x^5 + x^2 + 1) : \mathbb{R} \to \mathbb{R}$$

Notice that we did not use the symbol f, which is the name of the function, which is why this style of definition is called "anonymous".

Also, we used $\mapsto$ in place of $:=$

This style is particularly useful when we (for some reason) do not want name the function.

This notation can also be used when there are multiple inputs.

Consider -

$$\text{harmonicSum} : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$$
$$\text{harmonicSum}(x, y) := \frac{1}{x} + \frac{1}{y}$$

which, for convenience, can be rewritten as -

$$\left(x, y \mapsto \frac{1}{x} + \frac{1}{y}\right) : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$$

§1.6.2.3. Piecewise Functions

Sometimes, the expression on the right-hand side of the definition needs to depend upon some condition, and we denote that in the following way -

$$< \text{functionName} > (x) := \begin{cases} < \text{expression}_1 > \text{ ; if } < \text{condition}_1 > \\ < \text{expression}_2 > \text{ ; if } < \text{condition}_2 > \\ < \text{expression}_3 > \text{ ; if } < \text{condition}_3 > \\ . \\ . \\ . \\ < \text{expression}_n > \text{ ; if } < \text{condition}_n > \end{cases}$$

For example, consider the following definition -

$$\text{signum} : \mathbb{R} \to \mathbb{R}$$
$$\text{signum}(x) := \begin{cases} +1 \text{ ; if } x > 0 \\ 0 \text{ ; if } x == 0 \\ -1 \text{ ; if } x < 0 \end{cases}$$

The "signum" of a real number tells the "sign" of the real number ; whether the number is positive, zero, or negative.

§1.6.2.4. Pattern Matching

Pattern Matching is another way to write piecewise definitions which can work in certain situations.

For example, consider the last definition -

$$\text{signum}(x) := \begin{cases} +1 \ ; \text{ if } x \ > \ 0 \\ 0 \ ; \text{ if } x == 0 \\ -1 \ ; \text{ if } x \ < \ 0 \end{cases}$$

which can be rewritten as -

$$\text{signum}(0) := 0$$
$$\text{signum}(x) := \frac{x}{|x|}$$

This definition relies on checking the form of the input.

If the input is of the form "0", then the output is defined to be 0.
For any other number x, the output is defined to be $\frac{x}{|x|}$

However, there might remain some confusion -
If the input is "0", then why can't we take x to be 0, and apply the second line ($\text{signum}(x) := \frac{x}{|x|}$) of the definition ?

To avoid this confusion, we adopt the following convention -
Given any input, we start reading from the topmost line of the function definition to the bottom-most, and we apply the first applicable definition.

So here, the first line ($\text{signum}(0) := 0$) will be used as the definition when the input is 0.

§1.6.3. Recursion

A function definition is recursive when the name of the function being defined appears on the right-hand side as well.

For example, consider defining the famous fibonacci function -

$$F : \mathbb{N} \to \mathbb{N}$$
$$F(0) := 1$$
$$F(1) := 1$$
$$F(n) := F(n-1) + F(n-2)$$

§1.6.3.1. Termination

But it might happen that a recursive definition might not give a final output for a certain input.

For example, consider the following definition -

$$f(n) := f(n+1)$$

It is obvious that this definition does not define an actual output for, say, $f(4)$.

However, the previous definition of F obviously defines a specific output for $F(4)$ as follows -

$$\begin{aligned} F(4) &= F(3) + F(2) \\ &= (F(2) + F(1)) + F(2) \\ &= ((F(1) + F(0)) + F(1)) + F(2) \\ &= ((1 + F(0)) + F(1)) + F(2) \\ &= ((1 + 1) + F(1)) + F(2) \\ &= (2 + F(1)) + F(2) \\ &= (2 + 1) + F(2) \\ &= 3 + F(2) \\ &= 3 + (F(1) + F(0)) \\ &= 3 + (1 + F(0)) \\ &= 3 + (1 + 1) \\ &= 3 + 2 \\ &= 5 \end{aligned}$$

÷ **termination of recursive definition**

In general, a recursive definition is said to **terminate on an input** ***if and only if*** it eventually gives an ***actual specific output for that input***.

But what we cannot do this for every $F(n)$ one by one.

What we can do instead, is use a powerful tool known as the ÷ **principle of mathematical induction**.

§1.6.3.2. Induction

÷ **principle of mathematical induction**

Suppose we have an infinite sequence of statements $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \ldots$
and we can prove the following 2 statements -

- φ_0 is true
- For each $n > 0$, if φ_{n-1} is true, then φ_n is also true.

then all the statements $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \ldots$ in the sequence are true.

The above definition should be read as follows, given a sequence of formulas:

- The first one is true.
- Any formula being true, implies that the next one in the sequence is true.

Then all of the formulas in the sequence are true. Something like a chain of dominoes falling.

X **Exercise**

Show that n^2 is the same as the sum of first n odd numbers using induction.

The scenic way

(a) Prove the following theorem of Nicomachus by induction:

$$\begin{aligned} 1^3 &= 1 \\ 2^3 &= 3+5 \\ 3^3 &= 7+9+11 \\ 4^3 &= 13+15+17+19 \\ &\vdots \end{aligned}$$

(b) Use this result to prove the remarkable formula

$$1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2$$

There is enough information!

Given $a_0 = 100$ and $a_n = -a_{n-1} - a_{n-2}$, what is a_{2025}?

2-3 Color Theorem

A k-coloring is said to exist if the regions the plane is divided off in can be colored with three colors in such a way that no two regions sharing some length of border are the same color.

(a) A finite number of circles (possibly intersecting and touching) are drawn on a paper. Prove that a valid 2-coloring of the regions divided off by the circles exists.

(b) A circle and a chord of that circle are drawn in a plane. Then a second circle and chord of that circle are added. Repeating this process, until there are n circles with chords drawn, prove that a valid 3-coloring of the regions in the plane divided off by the circles and chords exists.

Square-full

Call an integer square-full if each of its prime factors occurs to a second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.

Hint: We recommend using induction. Given $(a, a+1)$ are square-full, can we generate another?

Same Height?

Here is a proof by induction that all people have the same height. We prove that for any positive integer n, any group of n people all have the same height. This is clearly true for $n = 1$. Now assume it for n, and suppose we have a group of $n+1$ persons, say $P_1, P_2, \cdots, P_{n+1}$. By the induction hypothesis, the n people $P_1, P_2, \cdots, P_n$ all have the same height. Similarly the n people $P_2, P_3, \cdots, P_{n+1}$ all have the same height. Both groups of people contain $P_2, P_3, \cdots, P_n$, so P_1 and P_{n+1} have the same height as $P_2, P_3, \cdots, P_n$. Thus all of $P_1, P_2, \cdots, P_{n+1}$ have the same height. Hence by induction, for any n any group of n people have the same height. Letting n be the total number of people in the world, we conclude that all people have the same height. Is there a flaw in this argument?

proving the principle of induction

We know that, any subset of the NN has a smallest element.

Using the above fact, prove the ≒ **principle of mathematical induction**

§1.6.3.3. Proving Termination using Induction

So let's see the ÷ **principle of mathematical induction** in action, and use it to prove that

Theorem The definition of the fibonacci function F terminates for any natural number n.

Proof For each natural number n, let φ_n be the statement

" The definition of F terminates for every natural number which is $\leq n$ "

To apply the ÷ **principle of mathematical induction**, we need only prove the 2 requirements and we'll be done. So let's do that -

- $\langle\langle$ **φ_0 is true** $\rangle\rangle$
  The only natural number which is ≤ 0 is 0, and $F(0) := 1$, so the definition terminates immediately.
- $\langle\langle$ **For each $n > 0$, if φ_{n-1} is true, then φ_n is also true.** $\rangle\rangle$
  Assume that φ_{n-1} is true.
  Let m be an arbitrary natural number which is $\leq n$.
  - $\langle\langle$ Case 1 $(m \leq 1)$ $\rangle\rangle$
    $F(m) := 1$, so the definition terminates immediately.
  - $\langle\langle$ Case 2 $(m > 1)$ $\rangle\rangle$
    $F(m) := F(m-1) + F(m-2)$,
    and since $m-1$ and $m-2$ are both $\leq n-1$,
    φ_{n-1} tells us that both $F(m-1)$ and $F(m-2)$ must terminate.
    Thus $F(m) := F(m-1) + F(m-2)$ must also terminate.

  Hence φ_n is proved!

Hence the theorem is proved!! ■

§1.7. Infix Binary Operators

Usually, the name of the function is written before the inputs given to it. For example, we can see that in the expression $f(x, y, z)$, the symbol f is written to the left of / before any of the inputs x, y or z.

However, it's not always like that. For example, take the expression

$$x + y$$

Here, the function name is $+$, and the inputs are x and y.

But $+$ has been written in-between x and y, not before!

Such a function is called an infix binary operator[2]

÷ **infix binary operator**

An **infix binary operator** is a ***function*** which takes exactly 2 inputs and whose function name is written between the 2 inputs rather than before them.

Examples include -
- $+$ (addition)

[2]
infix - because the function name is **in-between** the inputs
binary - because exactly **2** inputs, and binary refers to **2**
operator - another way of saying **function**

- − (subtraction)
- × or ∗ (multiplication)
- / (division)

§1.8. Trees

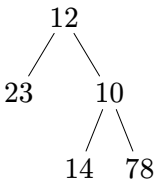
Trees are a way to structure a collection of objects.

Trees are a fundamental way to understand expressions and how haskell deals with them.

In fact, any object in Haskell is internally modelled as a tree-like structure.

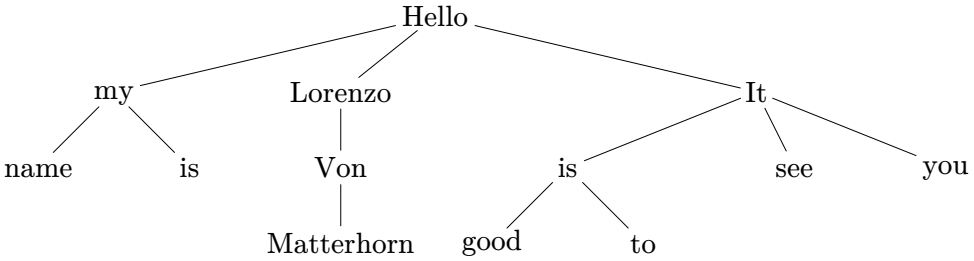
§1.8.1. Examples of Trees

Here we have a tree which defines a structure on a collection of natural numbers -



The line segments are what defines the structure.

The following tree defines a structure on a collection of words from the English language -

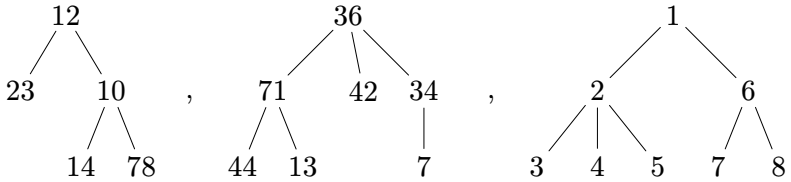


§1.8.2. Making Larger Trees from Smaller Trees

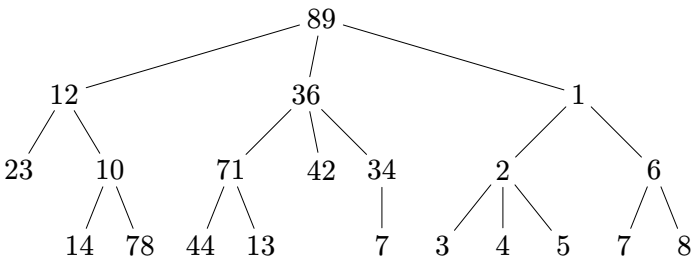
If we have an object -

89

and a few trees -



we can put them together into one large tree by connecting them with line segments, like so -



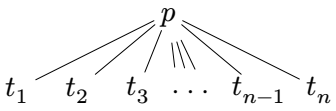
In general, if we have an object

$$p$$

and a bunch of trees

$$t_1, t_2, t_3, ..., t_{n-1}, t_n$$

, we can put them together in a larger tree, by connecting them with n line segments, like so -



We would like to define trees so that only those which are made in the above manner qualify as trees.

§1.8.3. Formal Definition of Trees

A **tree over a set S** defines a meaningful structure on a collection of elements from S.
The examples we've seen include trees over the set $\mathbb{N}$, as well as a tree over the set of English words.

We will adopt a similar approach to defining trees as we did with expressions, i.e., we will provide a formal procedure to check whether a mathematical object is a tree, rather than directly defining what a tree is.

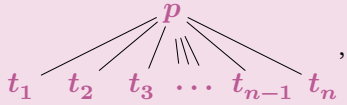
≒ checking whether object is tree

The formal procedure to determine whether an object is a **tree over a set S** is as follows -

Given a mathematical object t,

- first check whether $t \in S$, in which case t passes the check, and is a **tree over S**

Failing that,

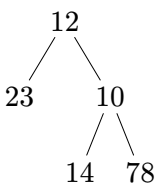
- check whether t is of the form 

, where
  - $p \in S$
  - and each of $t_1, t_2, t_3, ..., t_{n-1}, \text{and } t_n$ is a **tree over S**.

≒ tree

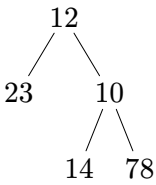
Given a set S, a ***mathematical object*** is said to be a **tree over S** if and only if it passes the formal checking procedure defined in **≒ checking whether object is tree**.

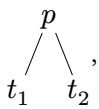
Let us use this definition to check whether

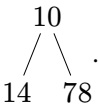


is a **tree over the natural numbers**.

Let's start -

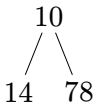


is of the form 

, where p is 12 , t_1 is 23 , and t_2 is 

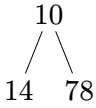
.

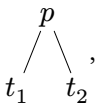
Of course, $12 \in \mathbb{N}$ and therefore $p \in S$.

So we are only left to check that 23 and 

are trees over the natural numbers.

$23 \in \mathbb{N}$, so 23 is a tree over $\mathbb{N}$ by the first check.



is of the form 

, where p is 10 , t_1 is 14, and t_2 is 78

Now, obviously $10 \in \mathbb{N}$, so $p \in S$.
Also, $14 \in \mathbb{N}$ and $78 \in \mathbb{N}$, so both pass by the first check.

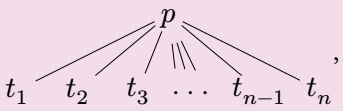
§1.8.4. Structural Induction

In order to prove things about trees, we have a version of the ≒ **principle of mathematical induction** for trees -

≒ **structural induction for trees**

Suppose for each tree t over a set S, we have a statement φ_t .

If we can prove the following two statements -

- For each $s \in S, \varphi_s$ is true
- For each tree T of the form 

,

  if φ_{t_1} , φ_{t_2} , φ_{t_3} , ... , $\varphi_{t_{n-1}}$ and φ_{t_n} are all true,

  then φ_T is also true.

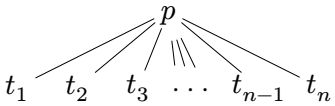
then φ_t is true for all trees t over S.

§1.8.5. Structural Recursion

We can also define functions on trees using a certain style of recursion.

From the definition of ≒ **tree**, we know that trees are

- either of the form $s \in S$

- or of the form 

So, to define any function (f : Trees over $S \to X$), we can divide taking the input into two cases, and define the outputs respectively.

÷ **tree size**

Let's use this principle to define the function

$$\text{size} : \text{Trees over } S \to \mathbb{N}$$

which is meant to give the number of times the elements of S appear in a tree over S.

$\text{size}(s) := 1$

$\text{size}\left(\begin{array}{c} p \\ t_1 \quad t_2 \quad t_3 \quad \cdots \quad t_{n-1} \quad t_n \end{array} \right) := 1 + \text{size}(t_1) + \text{size}(t_2) + \text{size}(t_3) + \ldots + \text{size}(t_{n-1}) + \text{size}(t_n)$

§1.8.6. Termination

Using ÷ **structural induction for trees**, let us prove that

Theorem The definition of the function "size" terminates on any tree.

Proof For each tree t, let φ_t be the statement

$$\text{“ The definition of size}(t) \text{ terminates “}$$

To apply ÷ **structural induction for trees**, we need only prove the 2 requirements and we'll be done. So let's do that -

- $\langle\langle\ \forall s \in S, \varphi_s$ **is true** $\rangle\rangle$
  $\text{size}(s) := 1$, so the definition terminates immediately.
- $\langle\langle$ **For each tree T of the form ... then** φ_T **is also true**$\rangle\rangle$
  Assume that each of $\varphi_{t_1},\ \varphi_{t_2},\ \varphi_{t_3}, \ldots,\ \varphi_{t_{n-1}},\ \varphi_{t_n}$ is true.
  That means that each of $\text{size}(t_1),\ \text{size}(t_2),\ \text{size}(t_3), \ldots,\ \text{size}(t_{n-1}),\ \text{size}(t_n)$ will terminate.

  Now, $\text{size}(T) := 1 + \text{size}(t_1) + \text{size}(t_2) + \text{size}(t_3) + \ldots + \text{size}(t_{n-1}) + \text{size}(t_n)$

  Thus, we can see that each term in the right-hand side terminates.
  Therefore, the left-hand side "size(T)",
  being defined as an addition of these terms,
  must also terminate.
  (since addition of finitely many terms always terminates)

  Hence φ_T is proved!

Hence the theorem is proved!! ■

tree depth

Fix a set S.

tree depth

depth : Trees over $S \to \mathbb{N}$

$\text{depth}(s) := 1$

$$\text{depth}\left(\begin{array}{c} p \\ t_1 \quad t_2 \quad t_3 \quad \cdots \quad t_{n-1} \quad t_n \end{array}\right) := 1 + \max_{1 \le i \le n} \{\text{depth}(t_i)\}$$

1. Prove that the definition of the function "depth" terminates on any tree over S.
2. Prove that for any tree t over the set S,
$$\text{depth}(t) \le \text{size}(t)$$
3. When is $\text{depth}(t) == \text{size}(t)$?

Exercise

This exercise is optional as it can be difficult, but it can be quite illuminating to understand the solution. So even if you don't solve it, you should ask for a solution from someone.

Using the **principle of mathematical induction**,

prove **structural induction for trees**.

§1.9. Why Trees?

But why care so much about trees anyway? Well, that is mainly due to the previously mentioned fact - "**In fact, any object in Haskell is internally modelled as a tree-like structure.**"

But why would Haskell choose to do that? There is a good reason, as we are going to see.

§1.9.1. The Problem

Suppose we are given that $x = 5$ and then asked to find out the value of the expression $x^3 \cdot x^5 + x^2 + 1$.

How can we do this?

Well, since we know that $x^3 \cdot x^5 + x^2 + 1$ is the function $+$ applied to the inputs $x^3 \cdot x^5$ and $x^2 + 1$, we can first find out the values of these inputs and then apply $+$ on them!

Similarly, as long as we can put an expression in the form $f(x_1, x_2, x_3, ..., x_{n-1}, x_n)$, we can find out its value by finding out the values of its inputs and then applying f on these values.

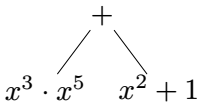
So, for dumb Haskell to do this (figure out the values of expressions, which is quite an important ability) , a vital requirement is to be able to easily put expressions in the form $f(x_1, x_2, x_3, ..., x_{n-1}, x_n)$.

But this can be quite difficult - In $x^3 \cdot x^5 + x^2 + 1$, it takes our human eyes and reasoning to figure it out fully, and for long, complicated expressions it will be even harder.

§1.9.2. The Solution

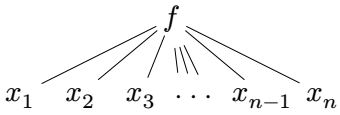
One way to make this easier to represent the expression in the form of a tree -

For example, if we represent $x^3 \cdot x^5 + x^2 + 1$ as

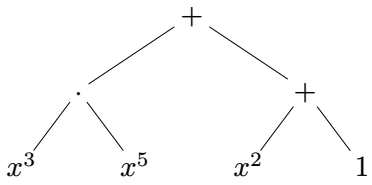


, it becomes obvious what the function is and what the inputs are to which it is applied.

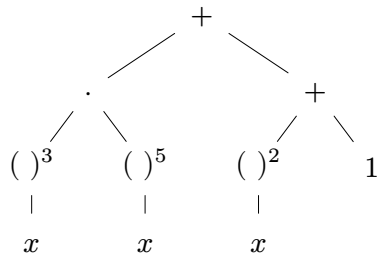
In general, we can represent the expression $f(x_1, x_2, x_3, ..., x_{n-1}, x_n)$ as



But why stop there, we can represent the sub-expressions (such as $x^3 \cdot x^5$ and $x^2 + 1$) as trees too -



and their sub-expressions can be represented as trees as well -



This is known as the as an Abstract Syntax Tree, and this is (approximately) how Haskell stores expressions, i.e., how it stores everything.

≒ abstract syntax tree

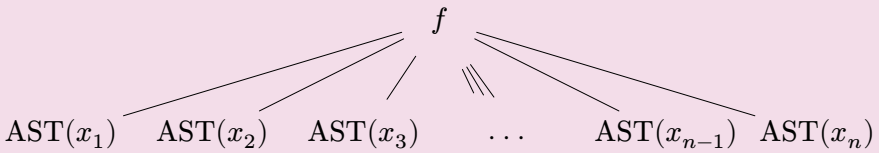
The **abstract syntax tree of a well-formed expression** is defined by applying the "function" **AST** to the expression.

The "function" **AST** is defined as follows -

$\text{AST} : \text{Expressions} \to \text{Trees over values and variables}$

$\text{AST}(v) := v, \text{if } v \text{ is a value or variable}$

$\text{AST}(f(x_1, x_2, x_3, ..., x_{n-1}, x_n)) :=$



§1.9.3. Exercises

All the following exercises are optional, as they are not the most relevant for concept-building. They are just a collection of problems we found interesting and arguably solvable with the theory of this chapter. Have fun!

X Turbo The Snail(IMO 2024, P5)

Turbo the snail is in the top row of a grid with $s \geq 4$ rows and $s - 1$ columns and wants to get to the bottom row. However, there are $s - 2$ hidden monsters, one in every row except the first and last, with no two monsters in the same column. Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an orthogonal neighbor. (He is allowed to return to a previously visited cell.) If Turbo reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move between attempts, and Turbo remembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and Turbo wins.

Find the smallest integer n such that Turbo has a strategy which guarantees being able to reach the bottom row in at most n attempts, regardless of how the monsters are placed.

X Points in Triangle

Inside a right triangle a finite set of points is given. Prove that these points can be connected by a broken line such that the sum of the squares of the lengths in the broken line is less than or equal to the square of the length of the hypotenuse of the given triangle.

X Joining Points(IOI 2006, 6)

A number of red points and blue points are drawn in a unit square with the following properties:

- The top-left and top-right corners are red points.
- The bottom-left and bottom-right corners are blue points.
- No three points are collinear.

Prove it is possible to draw red segments between red points and blue segments between blue points in such a way that: all the red points are connected to each other, all the blue points are connected to each other, and no two segments cross.

As a bonus, try to think of a recipie or a set of instructions one could follow to do so.

Hint: Try using the 'trick' you discovered in X **Points in Triangle**.

X Usmions(USA TST 2015, simplified)

A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to arrange the usamons in a line such that all the charged usamons are to the left of the uncharged usamons, regardless of the number of charged usamons. Is there any series of diode usage that makes this possible?

X Battery

(a) There are $2n+1 (n>2)$ batteries. We don't know which batteries are good and which are bad but we know that the number of good batteries is greater by 1 than the number of bad batteries. A lamp uses two batteries, and it works only if both of them are good. What is the least number of attempts sufficient to make the lamp work?

(b) The same problem but the total number of batteries is $2n (n>2)$ and the numbers of good and bad batteries are equal.

X Seven Tries (Russia 2000)

Tanya chose a natural number $X \leq 100$, and Sasha is trying to guess this number. He can select two natural numbers M and N less than 100 and ask about $\gcd(X+M, N)$. Show that Sasha can determine Tanya's number with at most seven questions.

Note: We know of atleast 5 ways to solve this. Some can be genralized to any number k other than 100, with $\lceil \log_2(k) \rceil$ many tries, other are a bit less general. We hope you can find atleast 2.

The best (trollest) codeforces question ever!

Let $s(k)$ be sum of digits in decimal representation of positive integer k. Given two integers $1 \leq m, n \leq 1129$ and n, find two integers $1 \leq a, b \leq 10^{2230}$ such that

- $s(a) \geq n$
- $s(b) \geq n$
- $s(a+b) \leq m$

For Example

Input1 : 6 5

Output1 : 6 7

Input2 : 8 16

Output2 : 35 53

Rope

Given a $r \times c$ grid with $0 \leq n \leq r * c$ painted cells, we have to arrange ropes to cover the grid. Here are the rules through example:

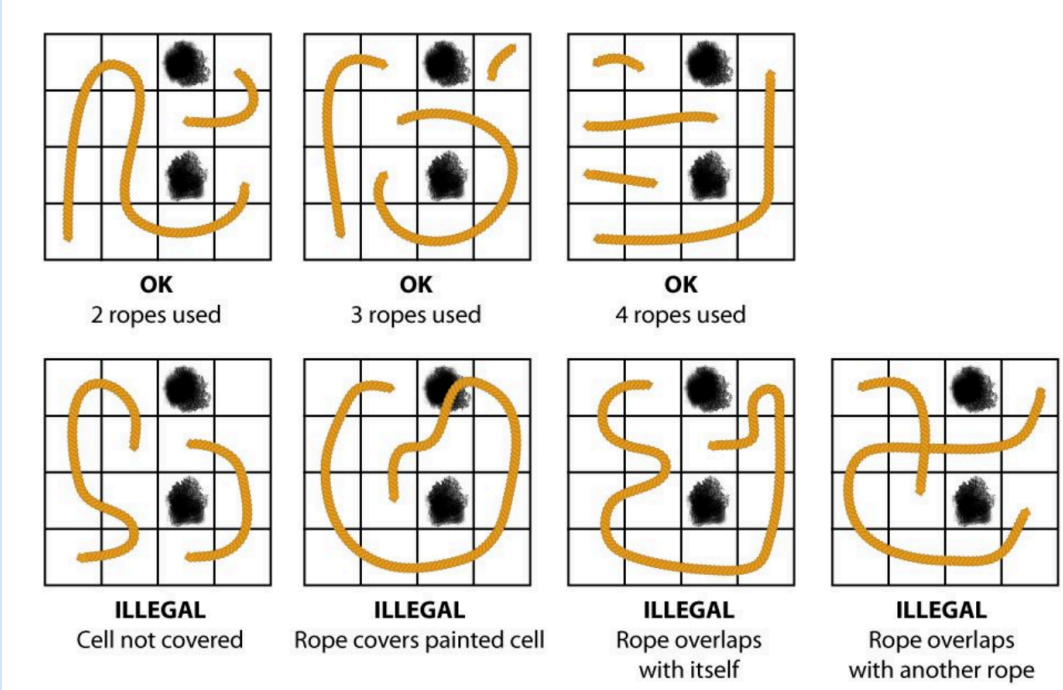


Figure out an algorithm/recipie to covering the grid using $n + 1$ ropes leagally.

Hint: Try to first do the $n = 0$ case. Then $r = 1$ case, with arbitrary n. Does this help?

n composite

Given N, find N consecutive integers that are all composite numbers.

Divided by 5^n

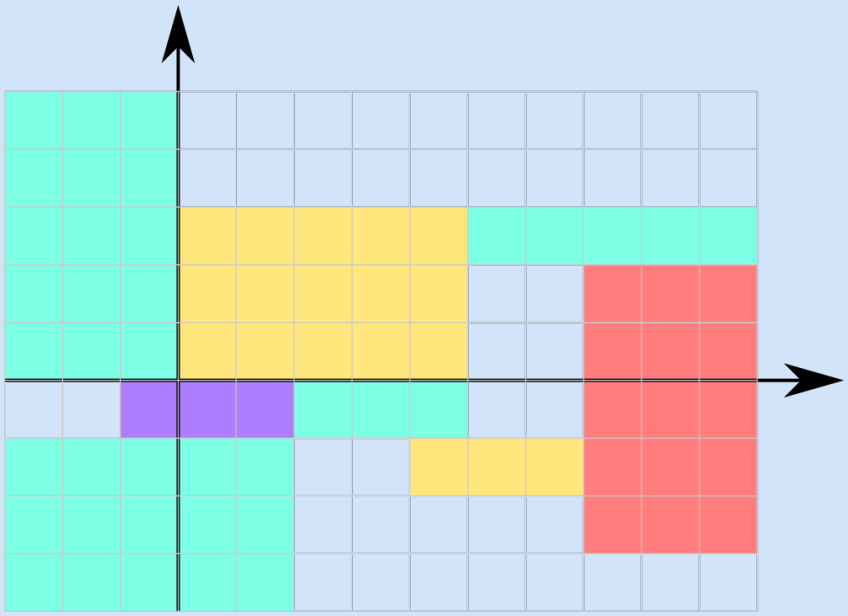
Prove that for every positive integer n, there exists an n-digit number divisible by 5^n, all of whose digits are odd.

X This was rated 2100? (Timofey's Colourbook Problem, Codeforces)

One of Timofey's birthday presents is a colourbook in the shape of an infinite plane. On the plane, there are n rectangles with sides parallel to the coordinate axes. All sides of the rectangles have odd lengths. The rectangles do not intersect, but they can touch each other.

Your task is, given the coordinates of the rectangles, to help Timofey color the rectangles using four different colors such that any two rectangles that **touch each other by a side** have **different colors**, or determine that it is impossible.

For example,



is a valid filling. Make an algorithm/recipe to fulfill this task.

PS: You will feel a little dumb once you solve it.

X Seating

Wupendra Wulkarni storms into the exam room. He glares at the students.

“Of course you all sat like this on purpose. Don’t act innocent. I know you planned to copy off each other. Do you all think I’m stupid? Hah! I’ve seen smarter chairs.

Well, guess what, darlings? I’m not letting that happen. Not on my watch.

Here’s your punishment - uh, I mean, assignment:

You’re all sitting in a nice little grid, let’s say n rows and m columns. I’ll number you from 1 to $n \cdot m$, row by row. That means the poor soul in row i, column j is student number $(i-1) \cdot m + j$. Got it?

Now, you better rearrange yourselves so that none of you little cheaters ends up next to the same neighbor again. Side-by-side, up-down—any adjacent loser you were plotting with in the original grid? Yeah, stay away from them.“

Your task is this: Find a new seating chart (in general an algorithm/recipie), using n rows and m columns, using every number from 1 to $n \cdot m$ such that no two students who were neighbors in the original grid are neighbors again.

And if you think it’s impossible, then prove it as Wupendra won’t satisfy for anything less.

X Yet some more Fibonnaci Identity

Fibonnaci sequence is defined as $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

(i) Prove that

$$\sum_{n=2}^{\infty} \arctan\left(\frac{(-1)^n}{F_2 n}\right) = \frac{1}{2}\arctan\left(\frac{1}{2}\right)$$

Hint : What is this problem doing on this list of problems?

(ii) Every natural number can be expressed uniquely as a sum of Fibonacci numbers where the Fibonacci numbers used in the sum are all distinct, and no two consecutive Fibonacci numbers appear.

(iii) Evaluate

$$\sum_{i=2}^{\infty} \frac{1}{F_{i-1}F_{i+1}}$$

X Round Robin

A group of n people play a round-robin chess tournament. Each match ends in either a win or a lost. Show that it is possible to label the players $P_1, P_2, P_3, \cdots, P_n$ in such a way that P_1 defeated P_2, P_2 defeated $P_3, \ldots, P_{n-1}$ defeated P_n.

Stamps

(i) The country of Philatelia is founded for the pure benefit of stamp-lovers. Each year the country introduces a new stamp, for a denomination (in cents) that cannot be achieved by any combination of older stamps. Show that at some point the country will be forced to introduce a 1-cent stamp, and the fun will have to end.

(ii) Two officers in Philatelia decide to play a game. They alternate in issuing stamps. The first officer to name 1 or a sum of some previous numbers (possibly with repetition) loses. Determine which player has the winning strategy.

Seven Dwarfs

The Seven Dwarfs are sitting around the breakfast table; Snow White has just poured them some milk. Before they drink, they perform a little ritual. First, Dwarf 1 distributes all the milk in his mug equally among his brothers' mugs (leaving none for himself). Then Dwarf 2 does the same, then Dwarf 3, 4, etc., finishing with Dwarf 7. At the end of the process, the amount of milk in each dwarf's mug is the same as at the beginning! What was the ratio of milt they started with?

Coin Flip Scores

A gambling graduate student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring exactly n points at some time in a sequence of n tosses is $\frac{2+\left(-\frac{1}{2}\right)^n}{3}$

Coins (IMO 2010 P5)

Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

(1) Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};

(2) Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

X Caves (IOI 2013, P4)

While lost on the long walk from the college to the UQ Centre, you have stumbled across the entrance to a secret cave system running deep under the university. The entrance is blocked by a security system consisting of N consecutive doors, each door behind the previous; and N switches, with each switch connected to a different door.

The doors are numbered $0, 1, \cdots, 4999$ in order, with door 0 being closest to you. The switches are also numbered $0, 1, \cdots, 4999$, though you do not know which switch is connected to which door.

The switches are all located at the entrance to the cave. Each switch can either be in an up or down position. Only one of these positions is correct for each switch. If a switch is in the correct position then the door it is connected to will be open, and if the switch is in the incorrect position then the door it is connected to will be closed. The correct position may be different for different switches, and you do not know which positions are the correct ones.

You would like to understand this security system. To do this, you can set the switches to any combination, and then walk into the cave to see which is the first closed door. Doors are not transparent: once you encounter the first closed door, you cannot see any of the doors behind it. You have time to try $70,000$ combinations of switches, but no more. Your task is to determine the correct position for each switch, and also which door each switch is connected to.

X Carnivel (CEIO 2014)

Each of Peter's N friends (numbered from 1 to N) bought exactly one carnival costume in order to wear it at this year's carnival parties. There are C different kinds of costumes,numbered from 1 to C. Some of Peter's friends, however, might have bought the same kind of costume. Peter would like to know which of his friends bought the same costume. For this purpose, he organizes some parties, to each of which he invites some of his friends.

Peter knows that on the morning after each party he will not be able to recall which costumes he will have seen the night before, but only how many different kinds of costumes he will have seen at the party. Peter wonders if he can nevertheless choose the guests of each party such that he will know in the end, which of his friends had the same kind of costume. Help Peter!

Peter has $N \leq 60$ friends and we can not have more than 365 parties(as we want to know the costumes by the end of the year).

Installing Haskell

Ryan Hota, Shubh Sharma, Arjun Maneesh Agarwal

§2.1. Installation

§2.1.1. General Instructions

1. This may take a while, so make sure that you have enough time on your hands.
2. Make sure that your device has enough charge to last you the entire installation process.
3. Make sure that you have a strong and stable internet connection.
4. Make sure that any antivirus(es) that you have on your device is fully turned off during the installation process. You can turn it back on immediately afterwards.
5. Make sure to follow the following instructions **IN ORDER**.
   Make sure to **COMPLETE EACH STEP** fully **BEFORE** moving on to the **NEXT STEP**.

§2.1.2. Choose your Operating System

§2.1.2.1. Linux

1. **Install Haskell**
   1. Read the general instructions very carefully, and ensure that you have complied with all the requirements properly.
   2. Close all open windows and running processes other than wherever you are reading this.
   3. Open the directory `haskellSupport/haskell/installation/Linux` in your text editor.
      (We have more support for Visual Studio Code, but any text editor should do)
   4. Open the terminal of your text editor and ensure that current directory is `Linux`.
   5. Type in `installHaskell` in the terminal.
   6. This may take a while.
   7. You will know installation is complete at the point when it says `Press any key to exit`.
   8. Restart (shut down and open again) your device.
2. **Install HaskellSupport**
   1. Read the general instructions very carefully, and ensure that you have complied with all the requirements properly.

2. Close all open windows and running processes other than wherever you are reading this.
3. Open the directory `haskellSupport/haskell/installation/Linux` in your text editor.
   (We have more support for Visual Studio Code, but any text editor should do)
4. Open the terminal of your text editor and ensure that current directory is `Linux`.
5. Type in `installHaskellSupport` in the terminal.
6. This may take a while.
7. You will know installation is complete at the point when it says `haskellSupport installation complete`.
8. Restart (shut down and open again) your device.

§2.1.2.2. MacOS

1. **Install Haskell**
   1. Read the general instructions very carefully, and ensure that you have complied with all the requirements properly.
   2. Close all open windows and running processes other than wherever you are reading this.
   3. Open the folder `haskellSupport` in Finder.
   4. Open the folder `haskell` in Finder .
   5. Open the folder `installation` in Finder.
   6. Right click on the folder `MacOS` in Finder, and select `Open in Terminal`.
   7. Type in `chmod +x installHaskell.command` in the terminal.
   8. Close the terminal window.
   9. Open the folder `MacOS` in Finder.
   10. Double-click on `installHaskell.command`.
   11. This may take a while.
   12. You will know installation is complete at the point when it says `Press any key to exit`.
   13. Restart (shut down and open again) your device.
2. **Install Visual Studio Code**
   Get it *here*.
3. **Install HaskellSupport**.
   1. Read the general instructions very carefully, and ensure that you have complied with all the requirements properly.
   2. Close all open windows and running processes other than wherever you are reading this.
   3. Open the folder `haskellSupport` in Finder.
   4. Open the folder `haskell` in Finder .
   5. Open the folder `installation` in Finder.
   6. Right click on the folder `MacOS` in Finder, and select `Open in Terminal`.

7. Type in `chmod +x installHaskellSupport.command` in the terminal.
8. Close the terminal window.
9. Open the folder `MacOS` in Finder.
10. Double-click on `installHaskellSupport.command`.
11. This may take a while.
12. You will know installation is complete if a new window pops up with `helloWorld` written in it.
13. Restart (shut down and open again) your device.

§2.1.2.3. Windows

1. **Install Haskell**.
    1. Read the general instructions very carefully, and ensure that you have complied with all the requirements properly.
    2. Close all open windows and running processes other than wherever you are reading this.
    3. Open the folder `haskellSupport` in File Explorer.
    4. Open the folder `haskell` in File Explorer .
    5. Open the folder `installation` in File Explorer.
    6. Open the folder `Windows` in File Explorer.
    7. Double-click on `installHaskell`.
    8. This may take a while.
    9. You will know installation is complete at the point when it says `Press any key to exit`.
    10. Restart (shut down and open again) your device.
2. **Install Visual Studio Code**
   Get it *here*.
3. **Install HaskellSupport**.
    1. Read the general instructions very carefully, and ensure that you have complied with all the requirements properly.
    2. Close all open windows and running processes other than wherever you are reading this.
    3. Open the folder `haskellSupport` in File Explorer.
    4. Open the folder `haskell` in File Explorer.
    5. Open the folder `installation` in File Explorer.
    6. Open the folder `Windows` in File Explorer.
    7. Double-click on `installHaskellSupport`.
    8. This may take a while.
    9. You will know installation is complete if a new window pops up with `helloWorld` written in it.

10. Restart (shut down and open again) your device.

Basic Syntax

Arjun Maneesh Agarwal

We will now gradually move to actually writing in Haskell. Programmers refer to this step as learning the "syntax" of a language.

To do this we will slowly translate the syntax of mathematics into the corresponding syntax of Haskell.

§3.1. The Building Blocks

Just like in math, the Haskell language relies on the symbols and expressions. The symbols include whatever characters can be typed by a keyboard, like `q`, `w`, `e`, `r`, `t`, `y`, `%`, `(`, `)`, `=`, `1`, `2`,etc.

§3.2. Values

Haskell has values just like in math.

value

A **value** is a single and specific well-defined object that is constant, i.e., does not change from scenario to scenario nor represents an arbitrary object.

Examples include -

- The number `pi` with the decimal expansion `3.141592653589793 ...`
- The order `<` on the `Integer`s
- The function of squaring an `Integer`
- the character `'a'` from the keyboard
- `True` and `False`

§3.3. Variables

Haskell also has its own variables.

variable

A **variable** is a symbol or chain of symbols,
meant to represent an arbitrary object of some ***type***,
usually as a way to show that whatever process follows is general enough so that the process can be carried out with ***any arbitrary value*** from that ***type***.

The following examples should clarify further.

We have previously seen how variables are used in function definitions and theorems.

Even though we can prove theorems about Haskell, the Haskell language itself supports only function definitions and not theorems.

So we can use variables in function definitions. For example -

λ double
```
double :: Integer → Integer
double x = x + x
```

This reads - "`double` is a function that takes an `Integer` as input and gives an `Integer` as output.
The `double` of an input `x` is the output `x + x`"

Note that `x` here is a variable.

Also, in mathematics we would write `double(x)`, but Haskell does not need those brackets.
So we can simply put some space between `double` and `x`, i.e.,
we write `double x`,
in order to indicate that `double` is the name of the function and `x` is its input.

Also, **Note** that the names of Haskell ÷ **variables** have to begin with an lowercase English letter.

§3.4. Types

Every ÷ **value** and ÷ **variable** in Haskell must have a "type".

For example,

- `'a'` has the type `Char`,
  indicating that it is a character from the keyboard.
- `5` can have the type `Integer`,
  indicating that it is an integer.
- `double` has the type `Integer → Integer`,
  indicating that is a function that takes an integer as input and gives an integer as output.
- In the definition of `double`, specifically "`double x = x + x`",
  the variable `x` has type `Integer`,
  indicating that it is an integer.

The type of an object is like a short description of the object's "nature".

Also, **Note** that the names of types usually have to begin with an uppercase English letter.

§3.4.1. Using GHCi to get Types

GHCi allows us to get the type of any value using the command `:type +d` followed by the value -

λ :type +d
```
>>> :type +d 'a'
'a' :: Char

>>> :type +d 5
5 :: Integer

>>> :type +d double
double :: Integer → Integer
```

`x :: T` is just Haskell's way of saying "`x` is of type `T`".

Note - The `+d` at the end of `:type +d` stands for "default", which means that its a more basic version of the more powerful command `:type`

For example -

```
>>> :type +d (+)
(+) :: Integer → Integer → Integer
```

This reads - "The function `+` takes in two `Integer`s as inputs and gives an `Integer` as output"

Or more generally -

λ `:type`

```
>>> :type (+)
(+) :: Num a ⇒ a → a → a
```

This reads - "The function `+` takes in two `Num`bers as inputs and gives a `Num`ber as output"

In summary, `:type +d` is specific, whereas `:type` is general.

For now, we will be assuming `:type +d` throughout, until we get to Chapter 6.

§3.4.2. Types of Functions

As we have seen before, `double` has type `Integer → Integer`. This function has a single input.

And the "basic" type of the ÷ **infix binary operator** `+` is `Integer → Integer → Integer`. This function has two inputs.

We can also define functions which takes a greater number of inputs -

λ `functions with many inputs`

```
sumOf2 :: Integer → Integer → Integer
sumOf2 x y = x + y
-- The above function has 2 inputs

sumOf3 :: Integer → Integer → Integer → Integer
sumOf3 x y z = x + y + z
-- The above function has 3 inputs

sumOf4 :: Integer → Integer → Integer → Integer → Integer
sumOf4 x y z w = x + y + z + w
-- The above function has 4 inputs

.
.
.
```

So we can deduce that in general,
if a function takes n inputs of types `T1`, `T2`, `T3`,..., `Tn` respectively,
and gives an output of type `T`,
then the function itself will have type `T1 → T2 → T3 → . . . → Tn → T` .

§3.5. Well-Formed Expressions

Of course, since we have ≒ **value**s and ≒ **variable**s, we can define "well-formed expressions" in a very similar manner to what we had before -

≒ **checking whether expression is well-formed**

It is difficult to give a direct definition of a **well-formed expression**.

So before giving the direct definition,
we define a ***formal procedure*** to check whether an expression is a **well-formed expression** or not.

The procedure is as follows -

Given an expression e,

- first check whether e is
  - a ≒ **value**, or
  - a ≒ **variable**

  in which cases e passes the check and is a **well-formed expression**.

Failing that,

- check whether e is of the form $f(e_1, e_2, e_3, ..., e_n)$, where
  - f is a function
  - which takes n inputs, and
  - $e_1, e_2, e_3, ..., e_n$ are all ***well-formed expressions*** which are ***valid inputs*** to f.

And only if e passes this check will it be a **well-formed expression**.

≒ **well-formed expression**

An ***expression*** is said to be a **well-formed expression** if and only if it passes the formal checking procedure defined in ≒ **checking whether expression is well-formed**.

Recall, that last time in Section §1.5., when we were formally checking that $x^3 \cdot x^5 + x^2 + 1$ is indeed a ≒ **well-formed expression**, we skipped the part about checking whether

$$\text{“}e_1, e_2, e_3, ..., e_n \text{ are ... } \textit{valid inputs} \text{ to } f\text{.”}$$

which is present in the very last part of the formal procedure
≒ **checking whether mathematical expression is well-formed**.

That is, we didn't have a very good way to check whether

$$\text{the input to a function} \in \text{the domain of the function}$$

, Thus we could potentially face mess-ups like

$$(1, 2) + 3$$

Here, the expression is not well-formed because $(1, 2)$ is not a valid input for $+$
(in other words $(1, 2) \notin$ the domain of $+$)
, but we had no way to prevent this before.

Now, with types, this problem is solved!

If a function has type `T1 → T2`,
and Haskell wants to check whether whatever input has been given to it is a valid input or not,
it need only check that this input is of type `T1`.

We can see this in action with `double` -

```
>>> double 12
24
```

`12` has type `Integer`, and therefore Haskell is quite happy to take it as input to the function `double` of type `Integer → Integer`.

However -

```
>>> double 'a'

<interactive>:1:8: error: [GHC-83865]
    * Couldn't match expected type `Integer' with actual type `Char'
    * In the first argument of `double', namely 'a'
      In the expression: double 'a'
      In an equation for `it': it = double 'a'
```

Since `double` has type `Integer → Integer`, Haskell tries to check whether the input `'a'` has type `Integer`, but discovers that it actually has a different type (`Char`), and therefore disallows it.

This is actually the point of types, and the consequences are very powerful.
Why? Recall that ÷ **well-formed expression**s are supposed to be only those expressions which are meaningful. Since Haskell has the power to check whether expressions are well-formed or not, it will never allow us to write a "meaningless" expression.

Other programming languages which don't have types allows one to write these "meaningless" expressions and that creates "bugs" a.k.a logical errors.

The very powerful consequence is that Haskell manages to **provably avoid any of these logical errors**!

§3.6. Infix Binary Operators

If we enclose **an** ÷ **infix binary operator** in **brackets**, we can use it just as we would a **function**

λ using infix operator as function

```
>>> 12 + 34 -- usage as infix binary operator
46
>>> (+) 12 34 -- usage as a normal Haskell function
46

>>> 12 - 34 -- as infix binary operator
-22
>>> (-) 12 34 -- usage as a normal Haskell function
-22

>>> 12 * 34 -- as infix binary operator
408
>>> (*) 12 34 -- usage as a normal Haskell function
408
```

Conversely, if we enclose a **function** in **backticks** (`), we can use it just like an ÷ **infix binary operator**.

λ using function as infix operator

```
>>> f x y = x*y + x + y -- function definition
>>> f 3 4 -- usage as a normal Haskell function
19
>>> 3 `f` 4 -- usage as an infix binary operator
19
```

§3.6.1. Precedence

÷ **infix binary operator**s sometimes introduce a small complication.

For example, when we write `a + b * c`,

do we mean `a + ( b * c )`

or do we mean `( a + b ) * c` ?

We know that the method to solve these problems are the BODMAS or PEMDAS conventions.

So Haskell assumes the first option due to BODMAS or PEMDAS conventions, whichever one takes your fancy.

This problem is called the problem of "precedence", i.e.,
"which operations in an expression are meant to be applied first (preceding) and which to be applied later?"

Haskell has a convention for handling all possible ÷ **infix binary operator**s that extends the PEMDAS convention.

(It assigns to each ÷ **infix binary operator** a number indicating the precedence, and those with greater value of precedence are evaluated first)

But there still remains an issue -

What about `a - b - c`?

Does it mean `( a - b ) - c` ,

or does it mean `a - ( b - c )`?

Observe that this issue is not solved by the BODMAS or PEMDAS convention.

Haskell chooses `( a - b ) - c`, because `-` is "left-associative".

÷ **left-associative**

If an ÷ **infix binary operator** `#` is **left-associative**, it means that the expression

```
x1 # x2 # x3 # ... # xn
```

is equivalent to

```
( x1 # x2 ) # x3 # ... # xn
```

which means that the leftmost `#` is evaluated first.

Therefore `a - b - c` is equivalent to `( a - b ) - c` and not `a - ( b - c )`.

But what about `a - b - c - d`?

```
a - b - c - d
-- take # as -, n as 4, x1 as a, x2 as b, x3 as c, x4 as d
== ( a - b ) - c - d
-- take # as -, n as 3, x1 as ( a - b ), x2 as c, x3 as d
== ( ( a - b ) - c ) - d
```

X **order of operations**

Find out the value of `7 - 8 - 4 - 15 - 65 - 42 - 34`

We also have the complementary notion of being "right-associative".

÷ **right-associative**

If an ÷ **infix binary operator** `#` is **right-associative**, it means that the expression

```
x1 # x2 # x3 # ... # xn-2 # xn-1 # xn
```

is equivalent to

```
x1 # x2 # x3 # ... # xn-2 # ( xn-1 # xn )
```

which means that the rightmost `#` is evaluated first.

§3.7. Logic

§3.7.1. Truth

The way to represent truth or falsity in Haskell is to use the value `True` or the value `False` respectively. Both values are of type `Bool`.

```
>>> :type True
True :: Bool

>>> :type False
False :: Bool
```

The `Bool` type means “true or false”.

The values `True` and `False` called `Bool`eans.

§3.7.2. Statements

Haskell can check the correctness of some very simple mathematical statements -

λ simplest logical statements

```
>>> 1 < 2
True

>>> 2 < 1
False

>>> 5 == 5
True

>>> 5 /= 5
False

>>> 4 == 5
False

>>> 4 /= 5
True
```

Note that `/=` is written as `/=`

Note that `≤` is written as `<=`

etc.

But the very nice fact is that Haskell does not require any new syntax or mechanism for these.

The way Haskell achieves this is an inbuilt ÷ **infix binary operator** named `<`,

which takes two inputs, `x` and `y`,

and outputs `True` if `x` is less than `y`, and otherwise outputs `False`.

So, in the statement `1 < 2`,

the `<` function is given the two inputs `1` and `2`,

and then GHCi evaluates this and outputs the correct value, `True`.

```
>>> 1 < 2
True
```

So let’s see if all this makes sense with respect to the type of `<` -

λ type of <

```
>>> :type (<)
(<) :: Ord a ⇒ a → a → Bool
```

Indeed we see that `<` takes two inputs of type `a`, and gives an output of type `Bool`.

§3.8. Conditions

So we can use these functions to define some "condition" on a ≒ **variable**.

For example -

λ condition on a variable

```
isLessThan5 :: Integer → Bool
isLessThan5 x = x < 5
```

This function encodes the "condition" that the input variable must be less than 5.

However, we would definitely like to express some more complicated conditions as well. For example, we might want to express the condition -

$$x \in (4, 10]$$

We know that $x \in (4, 10]$ if and only both $x > 4$ AND $x \leq 10$ hold true.

Using this fact, we can express the condition "$x \in (4, 10]$" as

```
( x > 4 ) && ( x ⩽ 10 )
```

in Haskell, since `&&` represents "AND" in Haskell.

Let's take `x` to be `7` and see what is happening here step by step -

```
   ( x > 4 ) && ( x ⩽ 10 )
== ( 7 > 4 ) && ( 7 ⩽ 10 )
==    True   && ( 7 ⩽ 10 )
==    True   &&    True
-- now applying the definition of && aka AND
== True
```

which is correct since " $7 \in (4, 10]$ " is indeed a true statement.

So the type of `&&` is -

```
>>> :type (&&)
(&&) :: Bool → Bool → Bool
```

It takes two `Bool`eans as inputs and outputs another `Bool`ean.

§3.8.1. Logical Operators

≒ **logical operator**

Functions like `&&`, which take in some `Bool`ean(s) as input(s), and give a single `Bool`ean as output are called **logical operators**.

You might have seen some logical operators before with names such AND, OR, NOT, NAND, NOR etc.

As we just saw, they are very useful in combining two conditions into one, more complicated condition.

For example -

- if we want to express the condition

$$x \in (-\infty, 6] \cup (15, \infty)$$

, we would re-express it as

"$x \leq 6$ OR $x > 15$"

, which could finally be expressed in Haskell as

```
( x ⩽ 6 ) || ( x > 15 )
```

, since `||` is Haskell's way of writing OR.

- if we want to express the condition

$$x \notin (-\infty, 4)$$

, we could re-express it as

$$\text{NOT } (x \in (-\infty, 4))$$

, which could be further re-written as

$$\text{NOT } (x < 4)$$

, which then can be expressed in Haskell as

```
not ( x < 4 )
```

We include the definition of `not` as it is quite simple -

λ not

```
not :: Bool → Bool
not True = False
not False = True
```

§3.8.1.1. Exclusive OR aka XOR

Finally, we define a logical operator called XOR.

÷ **XOR**

(boolean$_1$ XOR boolean$_2$) is defined to be true
if and only if
at least one of the 2 inputs is true, but not both,
and otherwise is defined to be false.

Suppose P and Q are two people running a race against each other.

Then at least one of them will win, but not both.

Therefore ((A wins) XOR (B wins)) would be true.

Also, (false XOR false) would be false, since at least one of the inputs need to be true.

Finally, 7(true XOR true) would be false, as both inputs are true.

§3.9. Function Definitions

Functions are a very important tool in mathematics and they form the foundations of Haskell programming.

Nearly everything in Haskell is done using functions, so there various ways of defining many kinds of functions.

§3.9.1. Using Expressions

In its simplest form, a function definition is made up of a `left-hand side` describing the function name and input(s), `=` in the middle and a `right-hand side` describing the output.

An example -

If we want write the following definition

$$f(x, y) := x^3 \cdot x^5 + y^3 \cdot x^2 + 14$$

Then we can write in Haskell -

λ basic function definition

```
f x y = x^3 * x^5 + y^3 * x^2 + 14
```

On the left we write the name of the function followed by a number of variables which represent its inputs.

In the middle we write `=`, indicating that right-hand side is the definition of the left-hand side.

On the right, we write a ÷ **well-formed expression** using the variables of the left-hand side, describing to how to combine and manipulate the inputs to form the output of the function.

Also, we know that $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

We can include this information in the definition -

λ function definition with explicit type

```
f :: Integer → Integer → Integer
f x y = x^3 * x^5 + y^3 * x^2 + 14
```

Even though it is not mandatory, it is **always** advised to follow the above style and **explicitly provide a particular type** for the function being defined.

Even if an explicit type is not provided, Haskell will assume the most general type the function could have, like what we observed in the `:type` command of GHCi.

Let's try to define ÷ **XOR** in Haskell -

λ xor

```
xor :: Bool → Bool → Bool
xor b1 b2 =
--    at least one of the inputs is True, but   not both
-- ⇒ b1 is True OR b2 is True           , but   not both
-- ⇒ ( b1 == True ) OR ( b2 == True )   , but   not both
-- ⇒ ( b1 == True ) OR ( b2 == True )   , but   not ( b1 AND b2 )
-- ⇒ ( b1 == True ) OR ( b2 == True )   , but ( not ( b1 AND b2 ) )
-- ⇒ ( b1 == True ) OR ( b2 == True )    AND  ( not ( b1 AND b2 ) )
    ( ( b1 == True ) || ( b2 == True ) ) &&   ( not ( b1 &&  b2 ) )
```

§3.9.2. Some Conveniences

§3.9.2.1. Piecewise Functions

If we have a function definition like

$$< \text{functionName} > (x) := \begin{cases} < \text{expression}_1 > \ ; \text{if} < \text{condition}_1 > \\ < \text{expression}_2 > \ ; \text{if} < \text{condition}_2 > \\ < \text{expression}_3 > \ ; \text{if} < \text{condition}_3 > \\ . \\ . \\ . \\ < \text{expression}_N > ; \text{if} < \text{condition}_N > \end{cases}$$

, it can be written in Haskell as

λ guards

```
functionName
    | condition1 = expression1
    | condition2 = expression2
    | condition3 = expression3
    .
    .
    .
    | conditionN = expressionN
```

For example,

$$\text{signum} : \mathbb{R} \to \mathbb{R}$$

$$\text{signum}(x) := \begin{cases} +1 \ ; \text{if } x \ > \ 0 \\ \ \ 0 \ ; \text{if } x == 0 \\ -1 \ ; \text{if } x \ < \ 0 \end{cases}$$

can written in Haskell as

λ basic usage of guards

```
signum :: Double → Double
signum x
    | x >  0 =  1
    | x == 0 =  0
    | x <  0 = -1
```

If a piecewise definition has a “catch-all” or “otherwise” clause at the end, as in

$$< \text{functionName} > (x) := \begin{cases} < \text{expression}_1 > & ; \text{if} < \text{condition}_1 > \\ < \text{expression}_2 > & ; \text{if} < \text{condition}_2 > \\ < \text{expression}_3 > & ; \text{if} < \text{condition}_3 > \\ . & \\ . & \\ . & \\ < \text{expression}_N > & ; \text{if} < \text{condition}_N > \\ < \text{expression}_{N+1} > & ; \text{otherwise} \end{cases}$$

, it can be written in Haskell as

λ guards

```
functionName
    | condition1 = expression1
    | condition2 = expression2
    | condition3 = expression3
    .
    .
    .
    | conditionN = expressionN
    | otherwise  = expression(N+1)
```

This | syntax symbol is called a “guard”.

For example -

λ otherwise

```
xor1 :: Bool → Bool → Bool
xor1 b1 b2
    | (not b1) && (not b2) = False -- when none of the inputs are True
    | b1 && b2             = False -- when both of the inputs are True
    | otherwise            = True  -- any other situation
```

If a piecewise definition has ony two parts

$$< \text{functionName} > (x) := \begin{cases} < \text{expression}_1 > & ; \text{if} < \text{condition} > \\ < \text{expression}_2 > & ; \text{otherwise} \end{cases}$$

then a lot programming languages have a simple construct called “if-else” to express this -

λ if-then-else

```
functionName = if condition then expression1 else expression2
```

For example -

λ if-then-else example

```
xor2 :: Bool → Bool → Bool
xor2 b1 b2 = if b1 == b2 then False else True
-- if both inputs to xor are the same, then output False, otherwise True
```

§3.9.2.2. Pattern Matching

We can write the map of every possible input one by one. This is called “exhaustive pattern matching”.

λ exhaustive pattern matching

```
xor3 :: Bool → Bool → Bool -- answer True iff at least one input is True,
but not both
xor3 False False = False -- at least one input should be True
xor3 True  True  = False -- since both inputs are True
xor3 False True  = True
xor3 True  False = True
```

We could be smarter and save some keystrokes.

λ pattern matching

```
xor4 :: Bool → Bool → Bool
xor4 False b = b
xor4 b False = b
xor4 b1 b2 = False
```

Another small pattern match equivalent to `xor1` -

λ unused variables in pattern match

```
xor5 :: Bool → Bool → Bool
xor5 False False = True
xor5 True True = True
xor5 b1 b2 = False
```

But since the variables `b1` and `b2` are not used in the right-hand side,

we can replace them with `_` (read as “wildcard”)

λ wildcard

```
xor6 :: Bool → Bool → Bool
xor6 False True = True
xor6 True False = True
xor6 _ _ = False
```

Wildcard (`_`) just means that any pattern will be accepted.

We can use other functions to help us as well -

λ using other functions in RHS

```
xor7 :: Bool → Bool → Bool
xor7 False b = b
xor7 True b = not b
```

We can also piecewise definitions in a pattern match -

λ pattern matches mixed with guards

```
xor8 :: Bool → Bool → Bool
xor8 False b2 = b2 -- Notice, we can have part of the definition unguarded
before entering the guards.
xor8 True b2
  | b2 == False = True
  | b2 == True  = False
```

Now we introduce the `case .. of ..` syntax. It is used to pattern-matching for any expression, not necessarily just the input variables, which are the only kinds of examples we’ve seen till now.

```
case <expression> of
  <pattern1> → <result1>
  <pattern2> → <result2>
  ...
```

The case syntax evaluates the `<expression>`, and matches it against each pattern in order. The first matching pattern's corresponding result is returned.

λ trivial case

```
xor9 :: Bool → Bool → Bool
xor9 b1 b2 = case b1 of
  False → b2
  True  → not b2
```

λ non-trivial case

```
xor10 :: Bool → Bool → Bool
xor10 b1 b2 = case ( b1 , b2 ) of
    ( False , False ) → False
    ( True  , True  ) → False
    _                 → True
```

§3.9.2.3. Where, Let

λ where

```
xor11 :: Bool → Bool → Bool
xor11 False b = b
xor11 True b = b'
         where b' = not b
```

λ let

```
xor12 :: Bool → Bool → Bool
xor12 False b = b
xor12 True b = let b' = not b in
               b'
```

§3.9.2.4. Without Inputs

Let us recall for a moment the definition for `xor2` (in λ **if-then-else example**)

λ if-then-else example

```
xor2 :: Bool → Bool → Bool
xor2 b1 b2 = if b1 == b2 then False else True
-- if both inputs to xor are the same, then output False, otherwise True
```

We can see that this just equivalent to

```
xor13 b1 b2 = not ( b1 == b2 )
```

which can be shortened even further

```
xor14 b1 b2 = b1 /= b2
```

, rewritten by λ **using infix operator as function**

```
xor14 b1 b2 = (/=) b1 b2
```

and thus can finally be shortened to the extreme

λ function definition without input variables

```
xor14 = (/=)
```

Notice the curious thing that the above function definition doesn't have any input variables. This ties into a fundamentally important concept called currying which we will explore later.

§3.9.2.5. Anonymous Functions

An anonymous function like

$$(x \mapsto x^3 \cdot x^5 + x^2 + 1) : \mathbb{R} \to \mathbb{R}$$

can written as

λ basic anonymous function

```
( \ x → x^3 * x^5 + x^2 + 1 ) :: Double → Double
```

Note that we used $\to$ in place of $\mapsto$,
and also added a `\` (pronounced "lambda") before the input variable.

For an example with multiple inputs, consider

$$\left(x, y \mapsto \frac{1}{x} + \frac{1}{y}\right)$$

which can be written as

λ multi-input anonymous function

```
( \ x y → 1/x + 1/y )
```

X only nand

It is a well know fact that one can define all logical operators using only `nand`. Well, let's do so.

Redefine `and`, `or`, `not` and `xor` using only `nand`.

§3.9.3. Recursion

A lot of mathematical functions are defined recursively. We have already seen a lot of them in chapter 1 and exercises. Factorial, binomials and fibonacci are common examples.

We can use the recurrence

$$n! = n \cdot (n-1)!$$

to define the factorial function.

λ factorial

```
factorial :: Integer → Integer
factorial 0 = 1
factorial n = n * factorial (n-1)
```

We can use the standard Pascal's recurrence

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

to define the binomial or "choose" function.

λ binomial
```
choose :: Integer → Integer → Integer
0 `choose` 0 = 1
0 `choose` _ = 0
n `choose` r = (n-1) `choose` r + (n-1) `choose` (r-1)
```

And we have already seen the recurrence relation for the fibonacci function in Section §1.6.3..

λ naive fibonacci definition
```
fib :: Int → Int
fib 0 = 1
fib 1 = 1
fib n = fib (n-1) + fib (n-2)
```

§3.10. Optimization

For fibonacci, note that in λ **naive fibonacci definition** is, well, naive.

This is because we keep recomputing the same values again and again. For example computing `fib 5` according to this scheme would look like:

λ computation of naive fibonacci
```
fib 5 = fib 4 + fib 3
      = (fib 3 + fib 2) + (fib 2 + fib 1)
      = ((fib 2 + fib 1) + (fib 1 + fib 0)) + ((fib 1 + fib 0) + 1)
      = (((fib 1 + fib 0) + 1) + (1 + 1)) + ((1 + 1) + 1)
      = (((1 + 1) + 1) + (1 + 1)) + ((1 + 1) + 1)
      = 8
```

If we can manage to avoid recomputing the same values over and over again, then the computation will take less time.

That is what the following definition achieves.

λ fibonacci by tail recursion
```
fibonacci :: Integer → Integer
fibonacci n = go n 1 1 where
  go 0 a _ = a
  go n a b = go (n - 1) b (a + b)
```

We can see that this is much more efficient. Tracing the computation of `fibonacci 5` now looks like:

λ computation of tail recursion fibonacci
```
fibonacci 5
= go 5 1 1
= go 4 1 2
= go 3 2 3
= go 2 3 5
= go 1 5 8
= go 0 8 13
= 8
```

This is called tail recursion as we carry the tail of the recursion to speed things up. It can be used to speed up naive recursion, although not always.

Another way to evaluate fibonacci will be seen in end of chapter exercises, where we will translate Binet's formula straight into Haskell. Why can't we do so directly? As we can't represent $\sqrt{5}$ exactly and the small errors in the approximation will accumulate due to the number of operations. This

exercise should allow you to end up with a blazingly fast algorithm which can compute the 12.5-th million fibonacci number in 1 sec. Our tail recursive formula takes more than 2 mins to reach there.

§3.11. Numerical Functions

≒ Integer and Int

`Int` and `Integer` are the types used to represent integers.

`Integer` can hold any number no matter how big, up to the limit of your machine's memory, while `Int` corresponds to the set of positive and negative integers that can be expressed in 32 or 64 bits(based on system) with the bounds changing depending on implementation (guaranteed at least -2^{29} to 2^{29}). Going outside this range may give weird results.

The reason for `Int` existing is historical. It was the only option at one point and continues to be available for backwards compatibility.

We will assume `Integer` wherever possible.

≒ Rational

`Rational` and `Double` are the types used to deal with non-integral numbers. The former is used for fractions or rationals while the latter for reals with varying amount of precision.

`Rational`s are declared using `%` as the viniculum(the dash between numerator and denominator). For example `1%3`, `2%5`, `97%31`, which respectively correspond to $\frac{1}{3}, \frac{2}{5}, \frac{97}{31}$.

≒ Double

`Double` or Double Precision Floating Point are high-precision approximations of real numbers. For example, consider the "square root" function -

```
>>> sqrt 2 :: Double
1.4142135623730951

>>> sqrt 99999 :: Double
316.226184874055

>>> sqrt 999999999 :: Double
31622.776585872405
```

A lot of numeric operators and functions come predefined in Haskell. Some natural ones are

```
>>> 7 + 3
10

>>> 3 + 8
11

>>> 97 + 32
129

>>> 3 - 7
-4

>>> 5 - (-6)
11

>>> 546 - 312
234

>>> 7 * 3
21

>>> 8*4
32

>>> 45 * 97
4365

>>> 45 * (-12)
-540

>>> (-12) * (-11)
132

>>> abs 10
10

>>> abs (-10)
10
```

The internal definition of addition and subtraction is discussed in the appendix while we talk about some multiplication algorithms in chapter 10. For now, assume that these functions work exactly as expected.

`Abs` is also implemented in a very simple fashion.

λ Implementation of abs function

```
abs :: Num a ⇒ a → a
abs a = if a ⩾ 0 then a else -a
```

§3.11.1. Division, A Trilogy

Now let's move to the more interesting operators and functions.

`recip` is a function which reciprocates a given number, but it has rather interesting type signature. It is only defined on types with the `Fractional` "type-class". This refers to a lot of things, but the

most common ones are `Rational, Float` and `Double`. `recip`, as the name suggests, returns the reciprocal of the number taken as input. The type signature is `recip :: Fractional a ⇒ a → a`

```
>>> recip 5
0.2
>>> k = 5 :: Int
>>> recip k
<interactive>:47:1: error: [GHC-39999] ...
```

It is clear that in the above case, 5 was treated as a `Float` or `Double` and the expected output provided. In the following case, we specified the type to be `Int` and it caused a horrible error. This is because for something to be a fractional type, we literally need to define how to reciprocate it. We will talk about how exactly it is defined in < some later chapter probably 8 >. For now, once we have `recip` defined, division can be easily defined as

```
(/) :: Fractional a ⇒ a → a → a
x / y = x * (recip y)
```

Again, notice the type signature of `(/)` is `Fractional a ⇒ a → a → a`.[3]

However, suppose that we want to do integer division and we want a quotient and remainder.

Say we want only the quotient, then we have `div` and `quot` functions.

These functions are often coupled with `mod` and `rem` are the respective remainder functions. We can get the quotient and remainder at the same time using `divMod` and `quotRem` functions. A simple example of usage is

```
>>> 100 `div` 7
14

>>> 100 `mod` 7
2

>>> 100 `divMod` 7
(14,2)

>>> 100 `quot` 7
14

>>> 100 `rem` 7
2

>>> 100 `quotRem` 7
(14,2)
```

One must wonder here that why would we have two functions doing the same thing? Well, they don't actually do the same thing.

[3]It is worth pointing out that one could define `recip` using `(/)` as well given 1 is defined. While this is not standard, if `(/)` is defined for a data type, Haskell does automatically infer the reciprocation. So technically, for a datatype to be a member of the type class `Fractional` it needs to have either reciprocation or division defined, the other is inferred.

X Div vs Quot

From the given example, what is the difference between `div` and `quot`?

```
>>> 8 `div` 3
2

>>> (-8) `div` 3
-3

>>> (-8) `div` (-3)
2

>>> 8 `div` (-3)
-3

>>> 8 `quot` 3
2

>>> (-8) `quot` 3
-2

>>> (-8) `quot` (-3)
2

>>> 8 `quot` (-3)
-2
```

X Mod vs Rem

From the given example, what is the difference between `mod` and `rem`?

```
>>> 8 `mod` 3
2

>>> (-8) `mod` 3
1

>>> (-8) `mod` (-3)
-2

>>> 8 `mod` (-3)
-1

>>> 8 `rem` 3
2

>>> (-8) `rem` 3
-2

>>> (-8) `rem` (-3)
-2

>>> 8 `rem` (-3)
2
```

While the functions work similarly when the divisor and dividend are of the same sign, they seem to diverge when the signs don't match.

The thing here is we always want our division algorithm to satisfy $d * q + r = n, |r| < |d|$ where d is the divisor, n the dividend, q the quotient and r the remainder.

The issue is for any $-d < r < 0 \Rightarrow 0 < r < d$. This means we need to choose the sign for the remainder.

In Haskell, `mod` takes the sign of the divisor (comes from floored division, same as Python's `%`), while `rem` takes the sign of the dividend (comes from truncated division, behaves the same way as Scheme's `remainder` or C's `%`.).

Basically, `div` returns the floor of the true division value (recall $\lfloor -3.56 \rfloor = -4$) while `quot` returns the truncated value of the true division (recall $\text{trunicate}(-3.56) = -3$ as we are just truncating the decimal point off). The reason we keep both of them in Haskell is to be comfortable for people who come from either of these languages.

Also, The `div` function is often the more natural one to use, whereas the `quot` function corresponds to the machine instruction on modern machines, so it's somewhat more efficient (although not much, I had to go upto 10^{100000} to even get millisecond difference in the two).

A simple exercise for us now would be implementing our very own integer division algorithm. We begin with a division algorithm for only positive integers.

λ A division algorithm on positive integers by repeated subtraction

```haskell
divide :: Integer → Integer → (Integer, Integer)
divide n d = go 0 n where
  go q r = if r ⩾ d then go (q+1) (r-d) else (q,r)
```

Now, how do we extend it to negatives by a little bit of case handling?

```haskell
divideComplete :: Integer → Integer → (Integer, Integer)
divideComplete _ 0 = error "DivisionByZero"
divideComplete n d

  | d < 0     = let (q, r) = divideComplete n (-d) in
                (-q, r)

  | n < 0     = let (q, r) = divideComplete (-n) d in
                if r == 0 then (-q, 0) else (-q - 1, d - r)

  | otherwise = divide n d

divide :: Integer → Integer → (Integer, Integer)
divide n d = go 0 n where
  go q r = if r ⩾ d then go (q+1) (r-d) else (q,r)
```

X Another Division

Figure out which kind of division have we implemented above, floored or truncated.

Now implement the other one yourself by modifying the above code appropriately.

§3.11.2. Exponentiation

Haskell defines for us three exponentiation operators, namely `(^^), (^), (**)`.

X Can you see the difference?

What can we say about the three exponentiation operators?

```
>>> a = 5 :: Int
>>> b = 0.5 :: Float
>>>
>>> a^a
3125
>>> a^^a
<interactive>:4:2: error: [GHC-39999]
>>> a**a
<interactive>:5:2: error: [GHC-39999]
>>>
>>> a^b
<interactive>:6:2: error: [GHC-39999]
>>> a^^b
<interactive>:7:2: error: [GHC-39999]
>>> a**b
<interactive>:8:4: error: [GHC-83865]
>>>
>>> b^a
3.125e-2
>>> b^^a
3.125e-2
>>> b**a
<interactive>:11:4: error: [GHC-83865]
>>>
>>> b^b
<interactive>:12:2: error: [GHC-39999]
>>> b^^b
<interactive>:13:2: error: [GHC-39999]
>>> b**b
0.70710677
>>>
>>> a^(-a)
*** Exception: Negative exponent
>>> a^^(-a)
<interactive>:16:2: error: [GHC-39999]
>>> a**(-a)
<interactive>:17:2: error: [GHC-39999]
>>>
>>> b^(-a)
*** Exception: Negative exponent
>>> b^^(-a)
32.0
>>> b**(-a)
<interactive>:20:6: error: [GHC-83865]
```

Unlike division, they have almost the same function. The difference here is in the type signature. While, inhering the exact type signature was not expected, we can notice:

- `^` is raising general numbers to positive integral powers. This means it makes no assumptions about if the base can be reciprocated and just produces an exception if the power is negative and error if the power is fractional.
- `^^` is raising fractional numbers to general integral powers. That is, it needs to be sure that the reciprocal of the base exists (negative powers) and doesn't throw an error if the power is negative.
- `**` is raising numbers with floating point to powers with floating point. This makes it the most general exponentiation.

The operators clearly get more and more general as we go down the list but they also get slower. However, they are also reducing in accuracy and may even output `Infinity` in some cases. The `...` means I am truncating the output for readability, GHCi did give the complete answer.

```
>>> 2^1000
10715086071862673209484250490600018105614048117055336074 ...

>>> 2 ^^ 1000
1.0715086071862673e301

>>> 2^10000
19950631168807583848837421626835850838 2 ...

>>> 2^^10000
Infinity

>>> 2 ** 10000
Infinity
```

The exact reasons for the inaccuracy comes from float conversions and approximation methods. We will talk very little about this specialist topic somewhat later.

However, something within our scope is implementing `(^)` ourselves.

λ A naive integer exponentiation algorithm

```
exponentiation :: (Num a, Integral b) ⇒ a → b → a
exponentiation a 0 = 1
exponentiation a b = if b < 0
  then error "no negative exponentiation"
  else a * (exponentiation a (b-1))
```

This algorithm, while the most naive way to do so, computes 2^{100000} in merely 0.56 seconds.

However, we could do a bit better here. Notice, to evaluate a^b, we are making b multiplications.

A fact, which we shall prove in chapter 10, is that multiplication of big numbers is faster when it is balanced, that is the numbers being multiplied have similar number of digits.

So to do better, we could simply compute $a^{\frac{b}{2}}$ and then square it, given b is even, or compute $a^{\frac{b-1}{2}}$ and then square it and multiply by a otherwise. This can be done recusrsivly till we have the solution.

```haskell
λ A better exponentiation algorithm using divide and conquer
exponentiation :: (Num a, Integral b) ⇒ a → b → a
exponentiation a 0 = 1
exponentiation a b
  | b < 0      = error "no negative exponentiation"
  | even b     = let half = exponentiation a (b `div` 2)
                 in half * half
  | otherwise = let half = exponentiation a (b `div` 2)
                 in a * half * half
```

The idea is simple: instead of doing b multiplications, we do far fewer by solving a smaller problem and reusing the result. While one might not notice it for smaller b's, once we get into the hundreds or thousands, this method is dramatically faster.

This algorithm brings the time to compute 2^{100000} down to 0.07 seconds.

The idea is that we are now making at most 3 multiplications at each step and there are at most $\lceil \log_2(b) \rceil$ steps. This brings us down from b multiplications to $3\log(b)$ multiplications. Furthermore, most of these multiplications are somewhat balanced and hence optimized.

This kind of a strategy is called divide and conquer. You take a big problem, slice it in half, solve the smaller version, and then stitch the results together. It's a method/technique that appears a lot in Computer Science (in sorting, in searching through data, in even solving differential equations and training AI models) and we will see it again shortly.

§3.11.3. `gcd` and `lcm`

A very common function for number theoretic use cases is `gcd` and `lcm`. They are pre-defined as

```haskell
>>> :t gcd
gcd :: Integral a ⇒ a → a → a

>>> :t lcm
lcm :: Integral a ⇒ a → a → a

>>> gcd 12 30
6

>>> lcm 12 30
60
```

We will now try to define these functions ourselves.

Let's say we want to find $g := \gcd(p, q)$ and $p > q$. That would imply $p = dq + r$ for some $r < q$. This means $g \mid p, q \Rightarrow g \mid q, r$ and by the maximality of g, $\gcd(p, q) = \gcd(q, r)$. This helps us out a lot as we could eventually reduce our problem to a case where the larger term is a multiple of the smaller one and we could return the smaller term then and there. This can be implemented as:

λ Fast GCD and LCM

```haskell
gcd :: Integer → Integer → Integer
gcd p 0 = p -- Using the fact that the moment we get q | p, we will reduce
to this case and output the answer.
gcd p q = gcd q (p `mod` q)


lcm :: Integer → Integer → Integer
lcm p q = (p * q) `div` (gcd p q)
```

We can see that this is much faster. The exact number of steps or time taken is a slightlty involved and not very related to what we cover. Intrested readers may find it and related citrations here.

This algorithm predates computers by approximatly 2300 years. If was first decribed by Euclid and hence is called the Euclidean Algorithm. While, faster algorithms do exist, the ease of implementation and the fact that the optimizations are not very dramatic in speeding it up make Euclid the most commonly used algorithm.

While we will see these class of algorithms, including checking if a number is prime or finding the prime factorization, these require some more weapons of attack we are yet to devlop.

§3.12. Mathematical Functions

We will now talk about mathematical functions like `log`, `sqrt`, `sin`, `asin` etc. We will also take this oppurtunity to talk about real exponentiation. To begin, Haskell has a lot of pre-defined functions.

```
>>> sqrt 81
9.0

>>> log (2.71818)
0.9999625387017254

>>> log 4
1.3862943611198906

>>> log 100
4.605170185988092

>>> logBase 10 100
2.0

>>> exp 1
2.718281828459045

>>> exp 10
22026.465794806718

>>> pi
3.141592653589793

>>> sin pi
1.2246467991473532e-16

>>> cos pi
-1.0

>>> tan pi
-1.2246467991473532e-16

>>> asin 1
1.5707963267948966

>>> asin 1/2
0.7853981633974483

>>> acos 1
0.0

>>> atan 1
0.7853981633974483
```

`pi` is a predefined variable inside haskell. It carries the value of π upto some decimal places based on what type it is forced in.

```
>>> a = pi :: Float
>>> a
3.1415927

>>> b = pi :: Double
>>> b
3.141592653589793
```

All the functions above have the type signature `Fractional a ⇒ a → a` or for our purposes `Float → Float`. Also, notice the functions are not giving exact answers in some cases and instead are giving approximations. These functions are quite unnatural for a computer, so we surely know that the computer isn't processing them. So what is happening under the hood?

§3.12.1. Taylor Series

We know that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$. For small $x, \ln(1+x) \approx x$. So if we can create a scheme to make x small enough, we could get the logarithm by simply multiplying. Well, $\ln(x^2) = 2\ln(|x|)$. So, we could simply keep taking square roots of a number till it is within some error range of 1 and then simply use the fact $\ln(1+x) \approx x$ for small x.

λ Log defined using Taylor Approximation

```
logTay :: Float → Float → Float
logTay tol n
  | n ⩽ 0                    = error "Negative log not defined"
  | abs(n - 1) ⩽ tol         = n - 1  -- using log(1 + x) ≈ x
  | otherwise                = 2 * logTay tol (sqrt n)
```

This is a very efficient algorithm for approximating `log`. Doing better requires the use of either pre-computed lookup tables(which would make the programme heavier) or use more sophisticated mathematical methods which while more accurate would slow the programme down. There is an exercise in the back, where you will implement a state of the art algorithm to compute `log` accurately upto 400-1000 decimal places.

Finally, now that we have `log = logTay 0.0001`, we can easily define some other functions.

```
logBase a b = log(b) / log(a)
exp n = if n == 1 then 2.71828 else (exp 1) ** n
(**) a b = exp (b * log(a))
```

We will use this same Taylor approximation scheme for `sin` and `cos`. The idea here is: $\sin(x) \approx x$ for small x and $\cos(x) = 1$ for small x. Furthermore, $\sin(x + 2\pi) = \sin(x), \cos(x + 2\pi) = \cos(x)$ and $\sin(2x) = 2\sin(x)\cos(x)$ as well as $\cos(2x) = \cos^2(x) - \sin^2(x)$.

This can be encoded as

λ Sin and Cos using Taylor Approximation

```
sinTay :: Float → Float → Float
sinTay tol x
  | abs(x) ⩽ tol        = x  -- Base case: sin(x) ≈ x when x is small
  | abs(x) ⩾ 2 * pi     = if x > 0
                          then sinTay tol (x - 2 * pi)
                          else sinTay tol (x + 2 * pi)  -- Reduce x to
[-2π, 2π]
  | otherwise           = 2 * (sinTay tol (x/2)) * (cosTay tol (x/2))  --
sin(x) = 2 sin(x/2) cos(x/2)

cosTay :: Float → Float → Float
cosTay tol x
  | abs(x) ⩽ tol        = 1.0  -- Base case: cos(x) ≈ 1 when x is small
  | abs(x) ⩾ 2 * pi     = if x > 0
                          then cosTay tol (x - 2 * pi)
                          else cosTay tol (x + 2 * pi)  -- Reduce x to
[-2π, 2π]
  | otherwise           = (cosTay tol (x/2))**2 - (sinTay tol (x/2))**2  --
cos(x) = cos²(x/2) - sin²(x/2)
```

As one might notice, this approximation is somewhat poorer in accuracy than `log`. This is due to the fact that the taylor approximation is much less truer on `sin` and `cos` in the neighbourhood of `0` than for `log`.

We will see a better approximation once we start using lists, using the power of the full Taylor expansion.

Finally, similer to our above things, we could simply set the tolerance and get a function that takes an input and gives an output, name it `sin` and `cos` and define `tan x = (sin x) / (cos x)`.

X Inverse Trig

Use taylor approximation and trignometric identites to define inverse sin(`asin`), inverse cos(`acos`) and inverse tan(`atan`).

§3.13. Exercises

X Collatz

Collatz conjucture states that for any $n \in \mathbb{N}$ exists a k such that $c^{k(n)} = 1$ where c is the Collatz function which is $\frac{n}{2}$ for even n and $3n + 1$ for odd n.

Write a functuon `col :: Integer → Integer` which, given a n, finds the smalltest k such that $c^{k(n)} = 1$, called the Collatz chain length of n.

Newton–Raphson method

Newton–Raphson method

Newton–Raphson method is a method to find the roots of a function via subsequent approximations.

Given $f(x)$, we let x_0 be an inital guess. Then we get subsequent guesses using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

As $n \to \infty$, $f(x_n) \to 0$.

The intution for why this works is: imagine standing on a curve and wanting to know where it hits the x-axis. You draw the tangent line at your current location and walk down it to where it intersects the x-axis. That's your next guess. Repeat. If the curve behaves nicely, you converge quickly to the root.

Limitations of Newton–Raphson method are

- Requires derivative: The method needs the function to be differentiable and requires evaluation of the derivative at each step.
- Initial guess matters: A poor starting point can lead to divergence or convergence to the wrong root.
- Fails near inflection points or flat slopes: If $f'(x)$ is zero or near zero, the method can behave erratically.
- Not guaranteed to converge: Particularly for functions with multiple roots or discontinuities.

Considering, $f(x) = x^2 - a$ and $f(x) = x^3 - a$ are well behaved for all a, implement `sqrtNR :: Float → Float → Float` and `cbrtNR :: Float → Float → Float` which finds the square root and cube root of a number upto a tolerance using the Newton–Raphson method.

Hint: The number we are trying to get the root of is a sufficiently good guess for numbers absolutly greater than 1. Otherwise, 1 or −1 is a good guess. We leave it to your mathematical intution to figure out when to use what.

Digital Root

The digital root of a number is the digit obtained by summing digits until you get a single digit.

For example `digitalRoot 9875 = digitalRoot (9+8+7+5) = digitalRoot 29 = digitalRoot (2+9)`.
`= digitalRoot 11 = digitalRoot (1+1) = 2`

Implement the function `digitalRoot :: Int → Int`.

X AGM Log

A rather uncommon mathematical function is AGM or arthmatic-geometric mean. For given two numbers,

$$\mathrm{AGM}(x,y) = \begin{cases} x & \text{if } x = y \\ \mathrm{AGM}\left(\frac{x+y}{2}, \sqrt{xy}\right) & \text{otherwise} \end{cases}$$

Write a function `agm :: (Float, Float) -> Float -> Float` which takes two floats and returns the AGM within some tolerance(as getting to the exact one recusrsivly takes, about infinite steps).

Using AGM, we can define

$$\ln(x) \approx \frac{\pi}{2\,\mathrm{AGM}\left(1, \frac{2^{2-m}}{x}\right)} - m\ln(2)$$

which is precise upto p bits where $x2^m > 2^{\frac{p}{2}}$.

Using the above defined `agm` function, define `logAGM :: Int -> Float -> Float -> Float` which takes the number of bits of precision, the tolerance for `agm` and a number greater than 1 and gives the natural logithrm of that number.

Hint: To simplify the question, we added the fact that the input will be greater than 1. This means a simplification is taking `m = p/2` directly. While geting a better `m` is not hard, this is just simpler.

X Multiplexer

A multiplexer is a hardware element which chooses the input stream from a variety of streams.

It is made up of $2^n + n$ components where the 2^n are the input streams and the n are the selectors.

(i) Implement a 2 stream multiplex `mux2 :: Bool -> Bool -> Bool -> Bool` where the first two booleans are the inputs of the streams and the third boolean is the selector. When the selector is `True`, take input from stream 1, otherwise from stream 2.

(ii) Implement a 2 stream multiplex using only boolean operations.

(iii) Implement a 4 stream multiplex. The type should be `mux4 :: Bool -> Bool -> Bool -> Bool -> Bool -> Bool -> Bool`. (There are 6 arguments to the function, 4 input streams and 2 selectors). We encourage you to do this in atleast 2 ways (a) Using boolean operations (b) Using only `mux2`.

Could you describe the general scheme to define `mux2^n` (a) using only boolean operations (b) using only `mux2^(n-1)` (c) using only `mux2` ?

X Moduler Exponation

Implement modular exponentiation $(a^b \bmod m)$ efficiently using the fast exponentiation method. The type signature should be `modExp :: Int -> Int -> Int -> Int`

X Bean Nim (Putnam 1995, B5)

A game starts with four heaps of beans containing a, b, c, and d beans. A move consists of taking either

(a) one bean from a heap, provided at least two beans are left behind in that heap, or

(b) a complete heap of two or three beans.

The player who takes the last heap wins. Do you want to go first or second?

Write a recursive function to solve this by brute force. Call it `naiveBeans :: Int → Int → Int → Int → Bool` which gives `True` if it is better to go first and `False` otherwise. Play around with this and make some observations.

Now write a much more efficient (should be one line and has no recursion) function `smartBeans :: Int → Int → Int → Int → Bool` which does the same.

X Squares and Rectangles on a chess grid

Write a function `squareCount :: Int → Int` to count number of squares on a $n \times n$ grid. For example, `squareCount 1 = 1` and `squareCount 2 = 5` as four 1x1 squares and one 2x2 square.

Furthermore, also make a function `rectCount :: Int → Int` to count the number of rectangles on a $n \times n$ grid.

Finally, make `genSquareCount :: (Int, Int) → Int` and `genRectCount :: (Int, Int) → Int` to count number of squares and rectangle in a $a \times b$ grid.

X Knitting Baltik (COMPFEST 13, Codeforces)

Mr. Chanek wants to knit a batik, a traditional cloth from Indonesia. The cloth forms a grid with size $m \times n$. There are k colors, and each cell in the grid can be one of the k colors.

Define a sub-rectangle as an pair of two cells $((x_1, y_1), (x_2, y_2))$, denoting the top-left cell and bottom-right cell (inclusively) of a sub-rectangle in the grid. Two sub-rectangles $((x_1, y_1), (x_2, y_2))$ and $((x_1, y_1), (x_2, y_2))$ have the same pattern if and only if the following holds:

(i) they have the same width ($x_2 - x_1 = x_4 - x_3$);

(ii) they have the same height ($y_2 - y_1 = y_4 - y_3$);

(iii) for every pair i, j such that $0 \leq i \leq x_2 - x_1$ and $0 \leq j \leq y_2 - y_1$, the color of cells $(x_1 + i, y_1 + j)$ and $(x_3 + i, y_3 + j)$ is the same.

Write a function `countBaltik` of type

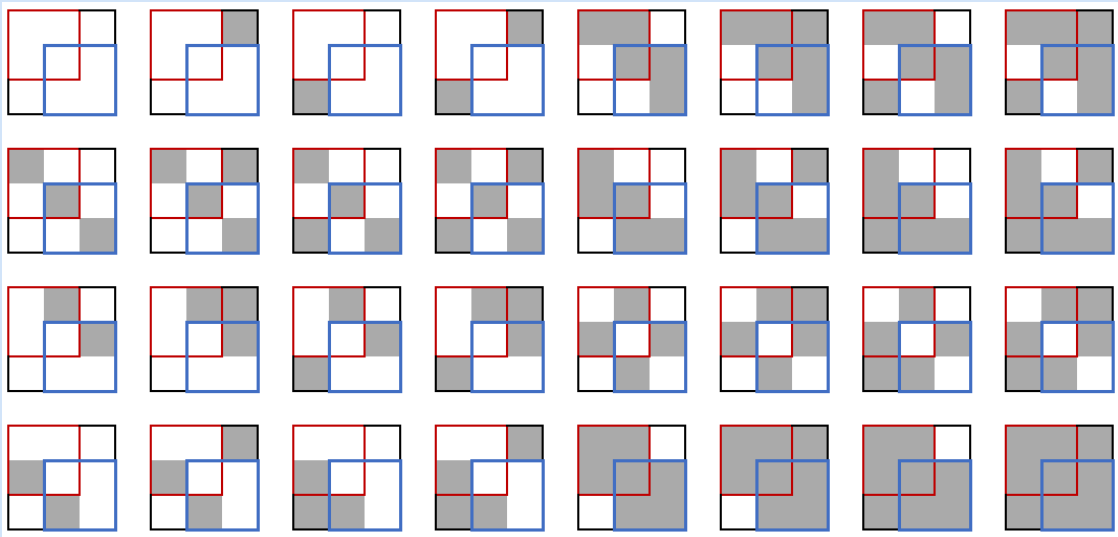
`(Int, Int) → Int → (Int, Int) → (Int, Int) → (Int, Int) → Integer` to count the number of possible batik color combinations, such that the subrectangles $\big((a_x, a_y), (a_x + r - 1, a_y + c - 1)\big)$ and $\big((b_x, b_y), (b_x + r - 1, b_y + c - 1)\big)$ have the same pattern.

Input `countBaltik` takes as input:

- The size of grid (m, n)
- Numer of colors k
- Size of sub-rectangle (r, c)
- (a_x, a_y)
- (b_x, b_y)

and should output an integer denoting the number of possible batik color combinations.

For example: `countBaltik (3,3) 2 (2,2) (1,1) (2,2) = 32`. The following are all 32 possible color combinations in the first example.



Modulo Inverse

Given a prime modulus $p > a$, according to Euclidean Division $p = ka + r$ where

$$\begin{aligned} ka + r &\equiv 0 \bmod p \\ &\Rightarrow ka \equiv -r \bmod p \\ &\Rightarrow -ra^{-1} \equiv k \bmod p \\ &\Rightarrow a^{-1} \equiv -kr^{-1} \bmod p \end{aligned}$$

Using this, implement `modInv :: Int → Int → Int` which takes in a and p and gives $a^{-1} \bmod p$.

Note that this reasoning does not hold if p is not prime, since the existence of a^{-1} does not imply the existence of r^{-1} in the general case.

New Bakery(Codeforces)

Bob decided to open a bakery. On the opening day, he baked n buns that he can sell. The usual price of a bun is a coins, but to attract customers, Bob organized the following promotion:

- Bob chooses some integer $k (0 \le k \le \min(n, b))$.
- Bob sells the first k buns at a modified price. In this case, the price of the i-th $(1 \le i \le k)$ sold bun is $(b - i + 1)$ coins.
- The remaining $(n - k)$ buns are sold at a coins each.

Note that k can be equal to 0. In this case, Bob will sell all the buns at a coins each.

Write a function `profit :: Int → Int → Int → Int` Help Bob determine the maximum profit he can obtain by selling all n buns with a being the normal price and b the price of first bun to be sold at a modified price.

Example

```
profit          4          4           5 = 17
profit          5          5           9 = 35
profit         10         10           5 = 100
profit 1000000000 1000000000 1000000000 = 1000000000000000000
profit 1000000000 1000000000          1 = 1000000000000000000
profit       1000          1       1000 = 500500
```

Note

In the first test case, it is optimal for Bob to choose $k = 1$. Then he will sell one bun for 5 coins, and three buns at the usual price for 4 coins each. Then the profit will be $5 + 4 + 4 + 4 = 17$ coins.

In the second test case, it is optimal for Bob to choose $k = 5$. Then he will sell all the buns at the modified price and obtain a profit of $9 + 8 + 7 + 6 + 5 = 35$ coins.

In the third test case, it is optimal for Bob to choose $k = 0$. Then he will sell all the buns at the usual price and obtain a profit of $10 \cdot 10 = 100$ coins.

Sumac

A Sumac sequence starts with two non-zero integers t_1 and t_2.

The next term, $t_3 = t_1 - t_2$

More generally, $t_n = t_{n-2} - t_{n-1}$

The sequence ends when $t_n \leq 0$. All values in the sequence must be positive.

Write a function `sumac :: Int → Int → Int` to compute the length of a Sumac sequence given the initial two terms, t_1 and t_2.

Examples(Sequence is included for clarification)

```
(t1,t2)   Sequence           n
-- ------------------------- -
(120,71)  [120,71,49,22,27] 5
(101,42)  [101,42,59]       3
(500,499) [500,499,1,498]   4
(387,1)   [387,1,386]       3
(3,-128)  [3]               1
(-2,3)    []                0
(3,2)     [3,2,1,1]         4
```

Binet Formula

Binet's formula is an explicit, closed form formula used to find the nth term of the Fibonacci sequence. It is so named because it was derived by mathematician Jacques Philippe Marie Binet, though it was already known by Abraham de Moivre.

Thr problem with this remarkable formula is that it is clutered with irrational numbers, specifically $\sqrt{5}$.

$$F_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$$

While computing using the binet formula would only take $2 * \log(n) + 2$ operations (exponentiation takes $\log(n)$ time), doing so directly is out of the question as we can't reprasent $\sqrt{5}$ exactly and the small errors in the approximation will accumalate due to the number of operations.

So an idea is to do all computations on a tuple (a, b) which reprasents $a + b\sqrt{5}$, We will need to define **addition, subtraction, multiplication and division** on these tuples as well as define a **fast exponentiation** here.

With that in hand, Write a function `fibMod :: Integer → Integer` which computes Fibonacci numbers using this method.

X A puzzle (UVA 10025)

A classic puzzle involves replacing each ? with one can set operators $+$ or $-$, in order to obtain a given k

$$?\,1?\,2?\ldots\ ?\,n = k$$

For example: to obtain $k = 12$, the expression to be used will be:

$$-1 + 2 + 3 + 4 + 5 + 6 - 7 = 12$$

with $n = 7$

Write function `puzzleCount :: Int → Int` which given a k tells us the smallest n such that the puzzle can be solved.

Examples

```
puzzleCount 12 = 7
puzzleCount -3646397 = 2701
```

X Rating Recalculation (Code Forces)

It is well known in the Chess Federation that the boundary for the Grandmaster title is carefully maintained just above the rating of International Master Wupendra Wulkarni. However, due to a recent miscalculation in the federation's new rating system, Wulkarni was mistakenly awarded the Grandmaster title.

To correct this issue, the federation has decided to revamp the division system, ensuring that Wupendra is placed into Division 2 (International Master), well below Grandmaster status.

A simple rule like `if rating ⩽ wupendraRating then div = max div 2` would be too obvious and controversial. Instead, the head of the system, Mike, proposes a more subtle and mathematically elegant solution.

First, Mike chooses the integer parameter $k \geq 0$.

Then, he calculates the value of the function $f(r-k, r)$, where r is the user's rating, and

$$f(n, x) := \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}}{e^x}$$

Finally, the user's division is defined by the formula

$$\mathrm{div}(r) = \left\lfloor \frac{1}{f(r-k, r)} \right\rfloor - 1$$

.

Write function `ratingCon :: Int → Int` to find the minimum k, given Wupendra's rating, so that the described algorithm assigns him a division strictly greater than 1 and GM Wulkarni doesn't become a reality.

Examples

```
ratingCon 5 = 2
ratingCon 100 = 5
ratingCon 200 = 7
ratingCon 2500 = 23
ratingCon 3000 = 25
ratingCon 3500 = 27
```

X Knuth's Arrow

Knuth introduced the following notation for a family of math notation:

$$3 \cdot 4 = 12$$

$$3 \uparrow 4 = 3 \cdot (3 \cdot (3 \cdot 3)) = 3^4 = 81$$

$$3 \uparrow\uparrow 4 = 3 \uparrow (3 \uparrow (3 \uparrow 3)) = 3^{3^{3^3}} = 3^{7625597484987}$$

$$\approx 1.2580142906274913178603906982032812155180467143165960151896 7 \times 10^{3638334640024}$$

$$3 \uparrow\uparrow\uparrow 4 = 3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3))$$

You can see the pattern as well as the extreme growth rate. Make a function `knuthArrow :: Integer → Int → Integer → Integer` which takes the first argument, number of arrows and second argument and provides the answer.

Types as Sets

Ryan Hota

§4.1. Sets

≑ **set**

A **set** is a ***well-defined collection of "things"***.
These "things" can be values, objects, or other sets.
For any given set, the "things" it contains are called its **elements**.

Some basic kinds of sets are -

- ≑ **empty set**

  The **empty set** is the ***set that contains no elements*** or equivalently, $\{\}$.

- ≑ **singleton set**

  A **singleton set** is a ***set that contains exactly one element***, such as $\{34\}$, $\{\triangle\}$, the set of natural numbers strictly between 1 and 3, etc.

We might have encountered some mathematical sets before, such as the set of real numbers $\mathbb{R}$ or the set of natural numbers $\mathbb{N}$, or even a set following the rules of vectors (a vector space).

We might have encountered sets as data structures acting as an unordered collection of objects or values, such as Python sets - `set([])`,`{1,2,3}`, etc.

Note that sets can be finite ($\{12, 1, \circ, \vec{x}\}$), as well as infinite ($\mathbb{N}$) .

A fundamental keyword on sets is "$\in$", or "belongs".

≑ **belongs**

Given a value x and a set S,
$x \in S$ is a ***claim*** that x ***is an element of*** S,

Other common operations include -

≑ **union**

$A \cup B$ is the ***set containing all those*** x ***such that either*** $x \in A$ ***or*** $x \in B$.

≑ **intersection**

$A \cap B$ is the ***set containing all those*** x ***such that*** $x \in A$ ***and*** $x \in B$.

≑ **cartesian product**

$A \times B$ is the ***set containing all ordered pairs*** (a, b) ***such that*** $a \in A$ ***and*** $b \in B$.

So,

$$X == \{x_1, x_2, x_3\} \text{ and } Y == \{y_1, y_2\}$$
$$\Rightarrow$$
$$X \times Y == \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2)\}$$

÷ set exponent

B^A is the *set of all functions with domain A and co-domain B*,
or equivalently , the *set of all functions f such that $f : A \to B$*,
or equivalently , the *set of all functions from A to B*.

X size of exponent set

If A has $|A|$ elements, and B has $|B|$ elements, then how many elements does B^A have?

§4.2. Types

We have encountered a few types in the previous chapter, such as `Bool`, `Integer` and `Char`. For our limited purposes, we can think about each such **type** as the **set of all values of that type**.

For example,

- `Bool` can be thought of as the **set of all boolean values**, which is {`False`, `True`}.
- `Integer` can be thought of as the **set of all integers**, which is {`0`, `1`, `-1`, `2`, `-2`, . . . }.
- `Char` can be thought of as the **set of all characters**, which is {`'\NUL'`, `'\SOH'`, `'\STX'`, . . . , `'a'`, `'b'`, `'c'`, . . . , `'A'`, `'B'`, `'C'`, . . . }

If this analogy were to extend further, we might expect to see analogues of the basic kinds of sets and the common set operations for types, which we can see in the following -

§4.2.1. `::` is analogous to $\in$ or ÷ belongs

Whenever we want to claim a value `x` is of type `T`, we can use the `::` keyword, in a similar fashion to $\in$, i.e., we can say `x :: T` in place of $x \in T$.

In programming terms, this is known as declaring the variable `x`.

For example,

- λ declaration of x

```
x :: Integer
x = 42
```

  This reads - "Let $x \in \mathbb{Z}$. Take the value of x to be 42."

- λ declaration of y

```
y :: Bool
y = xor True False
```

  This reads - "Let $y \in$ {False, True}. Take the value of y to be the $\oplus$ of True and False."

X declaring a variable

Declare a variable of type `Char`.

§4.2.2. `A → B` is analogous to B^A or ÷ set exponent

As B^A contains all functions from A to B,
so is each function `f` defined to take an input of type `A` and output of type `B` satisfy `f :: A→B`.

For example -

- λ function
```
succ :: Integer → Integer
succ x = x + 1
```

- λ another function
```
even :: Integer → Bool
even n = if n `mod` 2 == 0 then True else False
```

X basic function definition

Define a non-constant function of type `Bool → Integer`.

X difference between declaration and function definition

What are the differences between declaring a variable and defining a function?

§4.2.3. `( A , B )` is analogous to $A \times B$ or ÷ cartesian product

As $A \times B$ contains all pairs (a, b) such that $a \in A$ and $b \in B$,
so is every pair `(a,b)` of type `(A,B)` if `x` is of type `A` and `b` is of type `B`.

For example, if I ask GHCi to tell me the type of `(True, 'c')` (which I can do using the command `:t`), then it would tell me that the value's type is `(Bool, Char)` -

λ type of a pair
```
>>> :t (True, 'c')
(True, 'c') :: (Bool, Char)
```

This reads - "GHCi, what is the type of `(True, 'c')`?
Answer : the type of `(True, 'c')` is `(Bool, Char)`."

If we have a type `X` with elements `X1`, `X2`, and `X3`, and another type `Y` with elements `Y1` and `Y2`, we can use the author-defined function `listOfAllElements` to obtain a list of all elements of certain types -

λ elements of a product type
```
>>> listOfAllElements :: [X]
[X1,X2,X3]

>>> listOfAllElements :: [Y]
[Y1,Y2]

>>> listOfAllElements :: [(X,Y)]
[(X1,Y1),(X1,Y2),(X2,Y1),(X2,Y2),(X3,Y1),(X3,Y2)]

>>> listOfAllElements :: [(Char,Bool)]
[('\NUL',False),('\NUL',True),('\SOH',False),('\SOH',True), . . . ]
```

There are two fundamental inbuilt operations from a product type -

A function to get the first component of a pair -

λ first component of a pair
```
fst (a,b) = a
```

and a similar function to get the second component -

λ second component of a pair
```
snd (a,b) = b
```

We can define our own functions from a product type using these -

λ function from a product type
```
xorOnPair :: ( Bool , Bool ) → Bool
xorOnPair pair = ( fst pair ) ≠ ( snd pair )
```

or even by pattern matching the pair -

λ another function from a product type
```
xorOnPair' :: ( Bool , Bool ) → Bool
xorOnPair' ( a , b ) = a ≠ b
```

Also, we can define our functions to a product type -
For example, consider the useful inbuilt function `divMod`, which **divides a number by another**, and **returns** both the **quotient and the remainder as a pair**. Its definition is equivalent to the following -

λ function to a product type
```
divMod :: Integer → Integer → ( Integer , Integer )
divMod n m = ( n `div` m , n `mod` m )
```

X size of a product type

If a type `T` has n elements, and type `T'` has m elements, then how many elements does `(T,T')` have?

§4.2.4. `()` is analogous to ÷ singleton set

`()`, pronounced Unit, is a type that contains exactly one element.
That unique element is `()`.

So, it means that `():()`, which might appear a bit confusing.

The `()` on the left of `::` is just a simple value, like `1` or `'a'`.
The `()` on the right of `::` is a type, like `Integer` or `Char`.

This value `()` is the only value whose type is `()`.

On the other hand, other types might have multiple values of that type. (such as `Integer`, where both `1` and `2` have type `Integer`.)

We can even check this using `listOfAllElements` -

λ elements of unit type
```
>>> listOfAllElements :: [()]
[()]
```

This reads - "The list of all elements of the type `()` is a list containing exactly one value, which is the value `()`."

x function to unit

Define a function of type `Bool → ()`.

x function from unit

Define a function of type `() → Bool`.

§4.2.5. No ÷ intersection of Types

We now need to discuss an important distinction between sets and types. While two different sets can have elements in common, like how both $\mathbb{R}$ and $\mathbb{N}$ have the element 10 in common, on the other hand, two different types `T1` and `T2` cannot have any common elements.

For example, the types `Int` and `Integer` have no elements in common. We might think that they have the element `10` in common, however, the internal structures of `10 :: Int` and `10 :: Integer` are very different, and thus the two `10`s are quite different.

Thus, the intersection of two different types will always be empty and doesn't make much sense anyway.

Therefore, no intersection operation is defined for types.

§4.2.6. No ÷ union of Types

Suppose the type `T1 ∪ T2` were an actual type. It would have elements in common with the type `T1`. As discussed just previously, this is undesirable and thus disallowed.

But there is a promising alternative, for which we need to define the set-theoretic notion of **disjoint union**.

x subtype

Do you think that there can be an analogue of the *subset* relation $\subseteq$ for types?

§4.2.7. Disjoint Union of Sets

÷ disjoint union

$A \sqcup B$ is defined to be $(\{0\} \times A) \cup (\{1\} \times B)$, or equivalently, *the set of all pairs either of the form* $(0, a)$ *such that* $a \in A$*, or of the form* $(1, b)$ *such that* $b \in B$.

So,

$$X == \{x_1, x_2, x_3\} \text{ and } Y == \{y_1, y_2\}$$
$$\Rightarrow$$
$$X \sqcup Y == \{(0, x_1), (0, x_2), (0, x_3), (1, y_1), (1, y_2)\}$$

The main advantage that this construct offers us over the usual ÷ union is that given an element x from a disjoint union $A \sqcup B$, it is very easy to see whether x comes from A, or whether it comes from B.

For example, consider the statement - $(0, 10) \in \mathbb{R} \sqcup \mathbb{N}$.
It is obvious that this 10 comes from $\mathbb{R}$ and does not come from $\mathbb{N}$.
$(1, 10) \in \mathbb{R} \sqcup \mathbb{N}$ would indicate exactly the opposite, i.e, the 10 here comes from $\mathbb{N}$, not $\mathbb{R}$.

§4.2.8. `Either A B` is analogous to $A \sqcup B$ or ÷ disjoint union

The term "either" is motivated by its appearance in the definition of ÷ **disjoint union**.

Recall that in a ÷ **disjoint union** , each element has to be
- of the form $(0, a)$, where $a \in A$, and A is the set to the left of the $\sqcup$ symbol,
- or they can be of the form $(1, b)$, where $b \in B$, and B is the set to the right of the $\sqcup$ symbol.

Similarly, in `Either A B`, each element has to be
- of the form `Left a`, where `a :: A`
- or of the form `Right b`, where `b :: B`

If we have a type `X` with elements `X1`, `X2`, and `X3`, and another type `Y` with elements `Y1` and `Y2`, we can use the author-defined function `listOfAllElements` to obtain a list of all elements of certain types -

λ elements of an either type
```
>>> listOfAllElements :: [X]
[X1,X2,X3]

>>> listOfAllElements :: [Y]
[Y1,Y2]

>>> listOfAllElements :: [Either X Y]
[Left X1,Left X2,Left X3,Right Y1,Right Y2]

>>> listOfAllElements :: [Either Bool Char]
[Left False,Left True,Right '\NUL',Right '\SOH',Right '\STX', . . . ]
```

We can define functions to an `Either` type.
Consider the following problem : We have to make a function that provides feedback on a quiz. We are given the marks obtained by a student in the quiz marked out of 10 total marks. If the marks obtained are less than 3, return `'F'`, otherwise return the marks as a percentage -

λ function to an either type
```
feedback :: Integer → Either Char Integer
--                    Left ~ Char,Integer ~ Right
feedback n
  | n < 3     = Left  'F'
  | otherwise = Right ( 10 * n ) -- multiply by 10 to get percentage
```

This reads - "

Let `feedback` be a function that takes an `Integer` as input and returns `Either` a `Char` or an `Integer`.

As `Char` and `Integer` occurs on the left and right of each other in the expression `Either Char Integer`, thus `Char` and `Integer` will henceforth be referred to as `Left` and `Right` respectively.

Let the input to the function `feedback` be `n`.

If `n<3`, then we return `'F'`. To denote that `'F'` is a `Char`, we will tag `'F'` as `Left`. (remember that `Left` refers to `Char`!)

`otherwise`, we will multiply `n` by `10` to get the percentage out of 100 (as the actual quiz is marked out of 10). To denote that the output `10*n` is an `Integer`, we will tag it with the word `Right`. (remember that `Right` refers to `Integer`!)

“

We can also define a function from an `Either` type.
Consider the following problem : We are given a value that is either a boolean or a character. We then have to represent this value as a number.

```
top
import Data.Char(ord)
```

```
λ function from an either type
representAsNumber :: Either Bool Char → Int
--                   Left ~ Bool,Char ~ Right
representAsNumber ( Left  bool ) = if bool then 1 else 0
representAsNumber ( Right char ) = ord char
```

This reads - “

Let `representAsNumber` be a function that takes either a `Bool` or a `Char` as input and returns an `Int`.

As `Bool` and `Char` occurs on the left and right of each other in the expression `Either Bool Char`, thus `Bool` and `Char` will henceforth be referred to as `Left` and `Right` respectively.

If the input to `representAsNumber` is of the form `Left bool`, we know that `bool` must have type `Bool` (as `Left` refers to `Bool`). So if the `bool` is `True`, we will represent it as `1`, else if it is `False`, we will represent it as `0`.

If the input to `representAsNumber` is of the form `Right char`, we know that `char` must have type `Bool` (as `Right` refers to `Char`). So we will represent `char` as `ord char`.

“

We might make things clearer if we use a deeper level of pattern matching, like in the following function (which is equivalent to the last one).

```
λ another function from an either type
representAsNumber' :: Either Bool Char → Int
representAsNumber' ( Left  False ) = 0
representAsNumber' ( Left  True  ) = 1
representAsNumber' ( Right char  ) = ord char
```

X size of an either type

If a type `T` has n elements, and type `T'` has m elements, then how many elements does `Either T T'` have?

§4.2.9. The `Maybe` Type

Consider the following problem : We are asked make a function `reciprocal` that reciprocates a rational number, i.e., $(x \mapsto \frac{1}{x}) : \mathbb{Q} \to \mathbb{Q}$.

Sounds simple enough! Let's see -

λ naive reciprocal

```
reciprocal :: Rational → Rational
reciprocal x = 1/x
```

But there is a small issue! What about $\frac{1}{0}$?
What should be the output of `reciprocal 0`?

Unfortunately, it results in an error -

```
>>> reciprocal 0
*** Exception: Ratio has zero denominator
```

To fix this, we can do something like this - Let's add one *extra element* to the output type `Rational`, and then `reciprocal 0` can have this *extra element* as its output!

So the new output type would look something like this - ($\{$*extra element*$\} \sqcup$ `Rational`)

Notice that this $\{$*extra element*$\}$ is a ÷ **singleton set**.
Which means that if we take this *extra element* to be the value `()`,
and take $\{$*extra element*$\}$ to be the type `()`,
then we can obtain ($\{$*extra element*$\} \sqcup$ `Rational`) as the type `Either () Rational`.

Then we can finally rewrite λ **naive reciprocal** to handle the case of `reciprocal 0` -

λ reciprocal using either

```
reciprocal :: Rational → Either () Rational
reciprocal 0 = Left  ()
reciprocal x = Right (1/x)
```

There is already an inbuilt way to express this notion of `Either () Rational` in Haskell, which is the type `Maybe Rational`.

`Maybe Rational` just names it elements a bit differently compared to `Either () Rational` -

- where
  `Either () Rational` has `Left ()`,
  `Maybe Rational` instead has the value `Nothing`.
- where
  `Either () Rational` has `Right r` (where `r` is any `Rational`),
  `Maybe Rational` instead has the value `Just r`.

Which means that we can rewrite λ **reciprocal using either** using `Maybe` instead -

λ function to a maybe type

```
reciprocal :: Rational → Maybe Rational
reciprocal 0 = Nothing
reciprocal x = Just (1/x)
```

But we can also do this for any arbitrary type `T` in place of `Rational`. In that case -

There is already an inbuilt way to express the notion of `Either () T` in Haskell, which is the type `Maybe T`.

`Maybe T` just names it elements a bit differently compared to `Either () T` -

- where
  `Either () T` has `Left ()`,
  `Maybe T` instead has the value `Nothing`.
- where
  `Either () T` has `Right t` (where `t` is any value of type `T`),
  `Maybe T` instead has the value `Just t`.

If we have a type `X` with elements `X1`, `X2`, and `X3`, and another type `Y` with elements `Y1` and `Y2`, we can use the author-defined function `listOfAllElements` to obtain a list of all elements of certain types -

λ elements of a maybe type

```
>>> listOfAllElements :: [X]
[X1,X2,X3]

>>> listOfAllElements :: [Maybe X]
[Nothing,Just X1,Just X2,Just X3]

>>> listOfAllElements :: [Y]
[Y1,Y2]

>>> listOfAllElements :: [Maybe Y]
[Nothing,Just Y1,Just Y2]

>>> listOfAllElements :: [Maybe Bool]
[Nothing,Just False,Just True]

>>> listOfAllElements :: [Maybe Char]
[Nothing,Just '\NUL',Just '\SOH',Just '\STX',Just '\ETX', . . . ]
```

X size of a maybe type

If a type `T` has n elements, then how many elements does `Maybe T` have?

We can define functions to a `Maybe` type. For example consider the problem of making an inverse function of `reciprocal`, i.e., a function `inverseOfReciprocal` s.t.

$$\forall\ \texttt{x::Rational}\ ,\ \texttt{inverseOfReciprocal (reciprocal x) == x}$$

as follows -

λ function from a maybe type

```
inverseOfReciprocal :: Maybe Rational → Rational
inverseOfReciprocal Nothing  = 0
inverseOfReciprocal (Just x) = (1/x)
```

§4.2.10. `Void` is analogous to {} or ≒ empty set

The type `Void` has no elements at all.

This also means that no actual value has type `Void`.

Even though it is out-of-syllabus, an interesting exercise is to

Exercise

try to define a function of type `( Bool → Void ) → Void`.

Introduction to Lists

Ryan Hota

A list is an ordered collection of objects, possibly with repetitions, denoted by

$$[\ \text{object}_0\ ,\ \text{object}_1\ ,\ \text{object}_2\ ,\ldots\ ,\ \text{object}_{n-1}\ ,\ \text{object}_n\ ]$$

These objects are called the **elements of the list**.

In Haskell, the elements of a particular list all have to have the same type.

Thus, a list such as `[1,2,True,4]` is not allowed.

§5.1. Type of List

If the elements of a list each have type `T`, then the list is given the type `[T]`.

```
>>> :t +d [1,2,3]
[1,2,3] :: [Integer]

>>> :t +d ['a','Z','\STX']
['a','Z','\STX'] :: [Char]

>>> :t +d [True,False]
[True,False] :: [Bool]
```

§5.2. Creating Lists

There are several nice ways to create a list in Haskell.

§5.2.1. Empty List

The most basic approach is to create the empty list by writing `[]`.

§5.2.2. Arithmetic Progression

Haskell has some luxurious syntax for declaring lists containing arithmetic progressions -

```
>>> [1..6]
[1,2,3,4,5,6]

>>> [1,3..6]
[1,3,5]

>>> [1,-3.. -10]
[1,-3,-7]

>>> [0.5..4.9]
[0.5,1.5,2.5,3.5,4.5]
```

But, very usefully, it just doesn't work for numbers, but other types as well.

```
>>> [False ..True]
[False,True]

>>> ['a'..'z']
"abcdefghijklmnopqrstuvwxyz"
```

§5.3. Functions on Lists

Now that we know how to create a list, how do we manipulate them into the data that we would want?

§5.4. List Comprehension

Well, the way we achieve this in sets is through **set comprehension**.

When we want the set of squares of the even natural numbers $\leq n$, we write -

$$\{m^2 \mid m \in \{0,1,2,3,...,n-1,n\}, 2 \text{ divides } m\}$$

Haskell lets us do the same with lists -

```
>>> n = 10
>>> [ m*m | m ← [0..n] , m `mod` 2 == 0 ]
[0,4,16,36,64,100]
```

When we want the set of pairs of numbers $\leq n$ whose highest common factor is 1, we write -

$$\{(x,y) \mid x,y \in \{0,1,2,3,...,n-1,n\}, \text{HCF}(x,y) == 1\}$$

,which can be expressed in haskell as

```
>>> n = 10
>>> [ (x,y) | x ← [1..n] , y ← [1..n] , gcd x y == 1 ]
[(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),(1,10),(2,1),(2,3),
(2,5),(2,7),(2,9),(3,1),(3,2),(3,4),(3,5),(3,7),(3,8),(3,10),(4,1),(4,3),
(4,5),(4,7),(4,9),(5,1),(5,2),(5,3),(5,4),(5,6),(5,7),(5,8),(5,9),(6,1),
(6,5),(6,7),(7,1),(7,2),(7,3),(7,4),(7,5),(7,6),(7,8),(7,9),(7,10),(8,1),
(8,3),(8,5),(8,7),(8,9),(9,1),(9,2),(9,4),(9,5),(9,7),(9,8),(9,10),(10,1),
(10,3),(10,7),(10,9)]
```

§5.4.1. Cons or `(:)`

The operator `:` (read as "cons") can be used to add a single element to the the beginning of a list.

```
>>> 5 : [8,2,3,0]
[5,8,2,3,0]

>>> 1 : [2,3,4]
[1,2,3,4]

>>> 7 : [10,2,35,92]
[7,10,2,35,92]

>>> True : [False,True,True,False]
[True,False,True,True,False]
```

However, the `:` operator is much more special than it appears, since -

- It can be used to pattern match lists
- It is how lists are defined in the first place

So, how can we use it for pattern matching?

λ pattern matching lists

```
>>> (x:xs) = [5,8,3,2,0]
>>> x
5
>>> xs
[8,3,2,0]
```

When we use the pattern `(x:xs)` to refer to a list, `x` refers to the first element of the list, and `xs` refers to the list containing the rest of the elements.

§5.5. Length

One of the most basic questions we could ask about lists is the number of elements they contain. The `length` function gives us that answers, counting repetitions as separate.

```
>>> length [5,5,5,5,5,5]
6

>>> length [5,8,3,2,0]
5

>>> length [7,10,2,35,92]
5

>>> length [False,True,True,False]
4
```

Ans we can use pattern matching to define it -

λ length of list

```
length []     = 0
length (x:xs) = 1 + length xs
```

This reads - " If the list is empty, then `length` is `0`.

If the list has a first element `x`, then the `length` is `1 + length of the list of the rest of the elements`. "

§5.5.1. Concatenate or `(++)`

The `++` (read as "concatenate") operator can be used to join two lists together.

```
>>> [5,8,2,3,0] ++ [122,32,44]
[5,8,2,3,0,122,32,44]

>>>  [False,True,True,False] ++ [True,False,True]
[False,True,True,False,True,False,True]
```

Again, we can define it by using pattern matching

λ concatenation of lists

```
[]     ++ ys = ys
(x:xs) ++ ys = x : ( xs ++ ys )
```

This reads - " Suppose we are concatenating a list to the front of the list `ys`.

If the list is empty, then of course the answer is just `ys`.

If the list has a first element `x`, and the rest of the elements form a list `xs`, then we can first concatenate `xs` and `ys`, and then add `x` at the beginning of the resulting list. "

§5.5.2. Head and Tail

The `head` function gives the first element of a list.

```
>>> head [5,8,3,2,0]
5

>>> head [7,10,2,35,92]
7

>>> head [False,True,True,False]
False
```

And it can be defined using pattern-matching -

λ head of list

```
head (x:xs) = x
```

The `tail` function provides the rest of the list after the first element.

```
>>> tail [5,8,3,2,0]
[8,3,2,0]

>>> tail [7,10,2,35,92]
[10,2,35,92]

>>> tail [False,True,True,False]
[True,True,False]
```

And it can be defined using pattern-matching -

λ tail of list

```
tail (x:xs) = xs
```

But how are these functions supposed to work if there is no first element at all, such as in the case of `[]`? They produce errors when applied to the empty list! -

```
>>> head []
*** Exception: Prelude.head: empty list
CallStack (from HasCallStack):
  error, called at libraries\base\GHC\List.hs:1644:3 in base:GHC.List
  errorEmptyList, called at libraries\base\GHC\List.hs:87:11 in
base:GHC.List
  badHead, called at libraries\base\GHC\List.hs:83:28 in base:GHC.List
  head, called at <interactive>:6:1 in interactive:Ghci6
```

```
>>> tail []
*** Exception: Prelude.tail: empty list
CallStack (from HasCallStack):
  error, called at libraries\base\GHC\List.hs:1644:3 in base:GHC.List
  errorEmptyList, called at libraries\base\GHC\List.hs:130:28 in
base:GHC.List
  tail, called at <interactive>:7:1 in interactive:Ghci6
```

Note that, in our definitions, we have not handled the case of the input being `[]`!

So, it is advised to use the function `uncons` from `Data.List`, which adopts the philosophy we saw in λ **function to a maybe type**, which is

> if the function gives an error, output Nothing instead of the error

Thus, for non-empty `l`, `uncons l` returns `Just (head l, tail l)`,
and when `l` is empty, `uncons l` returns `Nothing`.

Let's test this in GHCi -

```
>>> import Data.List
>>> uncons [5,8,3,2,0]
Just (5,[8,3,2,0])
>>> uncons []
Nothing
```

And the definition -

λ **uncons of list**

```
uncons []     = Nothing
uncons (x:xs) = Just ( x , xs )
```

Also consider the functions `safeHead` and `safeTail` from `Distribution.Simple.Utils`.

§5.5.3. Take and Drop

There are some "generalized" functions corresponding to `head` and `tail`, namely `take` and `drop`,

`take n l` gives the first `n` elements of `l`.

```
>>> take 3 [5,8,3,2,0]
[5,8,3]

>>> take 4 [7,10,2,35,92]
[7,10,2,35]

>>> take 2 [False,True,True,False]
[False,True]
```

And the definition -

λ take from list

```
take 0   l     = []
take n (x:xs) = x : take (n-1) xs
take n []     = []
```

This reads - " If we `take` only `0` elements, the result will of course be the empty list `[]`.

If we want to take `n` elements, then we can take the first element and then the first `n-1` elements from the rest.

But why the last line of the definition? " The last line of the function may look strange, but -

Exercise

Explain why, without the last line of the definition, the function might give an unexpected error.

`drop n l` gives `l`, excluding the first `n` elements.

```
>>> drop 3 [5,8,3,2,0]
[2,0]

>>> drop 4 [7,10,2,35,92]
[92]

>>> drop 2 [False,True,True,False]
[True,False]
```

And the definition -

λ drop from list

```
drop 0   l     = l
drop n (x:xs) = drop (n-1) xs
drop n []     = []
```

Exercise

Prove that the above definition works as told in the description of the functionality of the `drop` function.

The `splitAt` function combines these two functionalities by returning both answers in a pair.

That is ; `splitAt n l == ( take n l , drop n l )`

```
>>> splitAt 3 [5,8,3,2,0]
([5,8,3],[2,0])
```

§5.5.4. Elem

The `elem` function takes a value and a list, and answers whether the value appears in the list or not, answering in either `True` or `False`.

```
>>> elem 5 [5,8,3,2,0]
True
>>> elem 8 [5,8,3,2,0]
True
>>> elem 3 [5,8,3,2,0]
True
>>> elem 2 [5,8,3,2,0]
True
>>> elem 0 [5,8,3,2,0]
True
```

```
>>> elem 7 [5,8,3,2,0]
False
>>> elem 6 [5,8,3,2,0]
False
>>> elem 4 [5,8,3,2,0]
False
```

And the definition -

```
elem x []     = False
elem x (y:ys) = x == y || elem x ys
```

This reads - " `x` does not appear in the empty list.
`x` appears in a list if and only if it is equal to the first element or it appears somewhere in the rest of the list. "

§5.5.5. (!!)

The `!!` (read as bang-bang) operator takes a list and a number `n :: Int`, and returns the n^{th} element of the list, counting from `0` onwards.

```
>>> [5,8,3,2,0] !! 0
5
>>> [5,8,3,2,0] !! 1
8
>>> [5,8,3,2,0] !! 2
3
>>> [5,8,3,2,0] !! 3
2
>>> [5,8,3,2,0] !! 4
0
```

But what happens if `n` is not between `0` and `length l`?

Error!

```
>>> [5,8,3,2,0] !! (-1)
*** Exception: Prelude.!!: negative index
CallStack (from HasCallStack):
  error, called at libraries\base\GHC\List.hs:1369:12 in base:GHC.List
  negIndex, called at libraries\base\GHC\List.hs:1373:17 in base:GHC.List
  !!, called at <interactive>:8:13 in interactive:Ghci6

>>> [5,8,3,2,0] !! 5
*** Exception: Prelude.!!: index too large
CallStack (from HasCallStack):
  error, called at libraries\base\GHC\List.hs:1366:14 in base:GHC.List
  tooLarge, called at libraries\base\GHC\List.hs:1376:50 in base:GHC.List
  !!, called at <interactive>:9:13 in interactive:Ghci6
```

So, again, it is advised to avoid using the `!!` operator.

X Exercise

Provide a definition for the `!!` operator.

§5.6. Strings

A string is how we represent text (like English sentences and words) in programming.

Like many modern programming languages, Haskell defines a string to be just a list of characters.

In fact, the type `String` is just a way to refer to the actual type `[Char]`.

So, if we want write the text "hello there!", we can write it in GHCi as `['h','e','l','l','o',' ','t','h','e','r','e','!']`.

Let's test it out -

```
>>> ['h','e','l','l','o',' ','t','h','e','r','e','!']
"hello there!"
```

But we see GHCi replies with something much simpler - `"hello there!"`

This simplified form is called syntactic sugar. It allows us to read and write strings in a simple form without having to write their actual verbose syntax each time.

So, we can write -

```
>>> "hello there!"
"hello there!"

>>> :t +d "hello there!"
"hello there!" :: String
```

The type `String` is just a way to refer to the actual type `[Char]`.

And since strings are just lists, all the list functions apply to strings as well.

```
>>> 'h' : "ello there!"
"hello there!"

>>> "hello " ++ "there!"
"hello there!"

>>> head "hello there!"
'h'

>>> tail "hello there!"
"ello there!"

>>> take 5 "hello there!"
"hello"

>>> drop 5 "hello there!"
" there!"

>>> elem 'e' "hello there!"
True

>>> elem 'w' "hello there!"
False

>>> "hello there!" !! 7
'h'

>>> "hello there!" !! 6
't'
```

But there are some special functions just for strings -

`words` breaks up a string into a list of the words in it.

```
>>> words "hello there!"
["hello","there!"]
```

And `unwords` combines the words back into a single string.

```
>>> unwords ["hello","there!"]
"hello there!"
```

`lines` breaks up a string into a list of the lines in it.

```
>>> lines "hello there!\nI am coding ... "
["hello there!","I am coding ... "]
```

Ans `unlines` combines the lines back into a single string.

```
>>> unlines ["hello there!","I am coding ... "]
"hello there!\nI am coding ... \n"
```

§5.7. Structural Induction for Lists

Suppose we wan prove some fact about lists.

We can use the following version of the ≒ **principle of mathematical induction** -

> ≒ **structural induction for lists**
>
> Suppose for each list `l` of type `[T]`, we have a statement $\varphi_{\texttt{l}}$. If we can pore the following two statements -
>
> - $\varphi_{\texttt{[]}}$
> - For each list of the form `(x:xs)`, if $\varphi_{\texttt{xs}}$ is true, then $\varphi_{\texttt{(x:xs)}}$ is also true.
>
> then $\varphi_{\texttt{l}}$ for all finite lists `l`.

Let use this principle to prove that

Theorem The definition of `length` terminates on all finite lists.

Proof Let $\varphi_{\texttt{l}}$ be the statement

The definition of `length l` terminates.

To use ≒ **structural induction for lists**, we need to prove -

- $\langle\langle\varphi_{\texttt{[]}}\rangle\rangle$
  The definition of `length []` directly gives `0`.
- $\langle\langle$ For each list `(x:xs)`, if $\varphi_{\texttt{xs}}$, then $\varphi_{\texttt{(x:xs)}}$ also. $\rangle\rangle$
  Assume $\varphi_{\texttt{xs}}$ is true.
  The definition for `length (x:xs)` is `1 + length xs`.
  By $\varphi_{\texttt{xs}}$,we know that `length xs` will finally give return some number `n`.
  Therefore `1 + length xs` reduces to `1 + n`.
  And `1 + n` obviously terminates. ■

§5.8. Optimization

Suppose we want to reverse the the order of elements in a list.

For example, transforming the list `[5,8,3,2,0]` into `[0,2,3,8,5]`.

So how do we define the function `reverse`?

An obvious definition is -

λ naive reverse

```
reverse []     = []
reverse (x:xs) = ( reverse xs ) ++ [x]
```

But this is not “optimal”?

What does this mean? Let’s see -

Let’s apply the definitions of `reverse` and `(++)` to see how `reverse [5,8,3]` is computed -

```
reverse [5,8,3,2] == ( reverse [8,3]  ) ++ [5]

                  == ( ( reverse [3]  ) ++ [8] ) ++ [5]

                  == ( ( ( reverse [] ) ++ [3] ) ++ [8] ) ++ [5]

                  == ( ( [] ++ [3] ) ++         [8]   ) ++         [5]

                  == (         [3]   ++         [8]   ) ++         [5]

                  == (          3    :  ( [] ++ [8] ) ) ++         [5]

                  == (          3    :          [8]   ) ++         [5]

                  == (          3    : (        [8]     ++         [5] )

                  ==            3    : (         8      :  ( [] ++ [5] ) )

                  ==            3    : (         8      :          [5]   )

                  -- which finally is
                       [3,8,5]
```

So we see that this takes 10 steps of computation.

Let us take an alternative definition of `reverse` -

λ optimized reverse

```
reverse l = help [] l where
    help xs (y:ys) = help (y:xs) ys
    help xs []     = xs
```

Let us how this one is computed step by step -

```
reverse [5,8,3] == help      [] [5,8,3]

                == help     [5] [8,3]

                == help   [8,5] [3]

                == help [3,8,5] []

                == [3,8,5]
```

So we see this computation takes only 5 steps, as compared to 10 from last time.

So, in some way, the second definition is better as it requires much less steps.

We can comment on something similar for `splitAt`

λ naive splitAt

```
splitAt n l = ( take n l , drop n l )
```

λ optimized splitAt

```
splitAt n []     = []
splitAt n (x:xs) = ( x:ys , zs ) where
    (ys,zs) = splitAt (n-1) xs
```

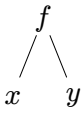
Exercise

(1) Prove that the two definitions are equivalent using ÷ structural induction for lists.

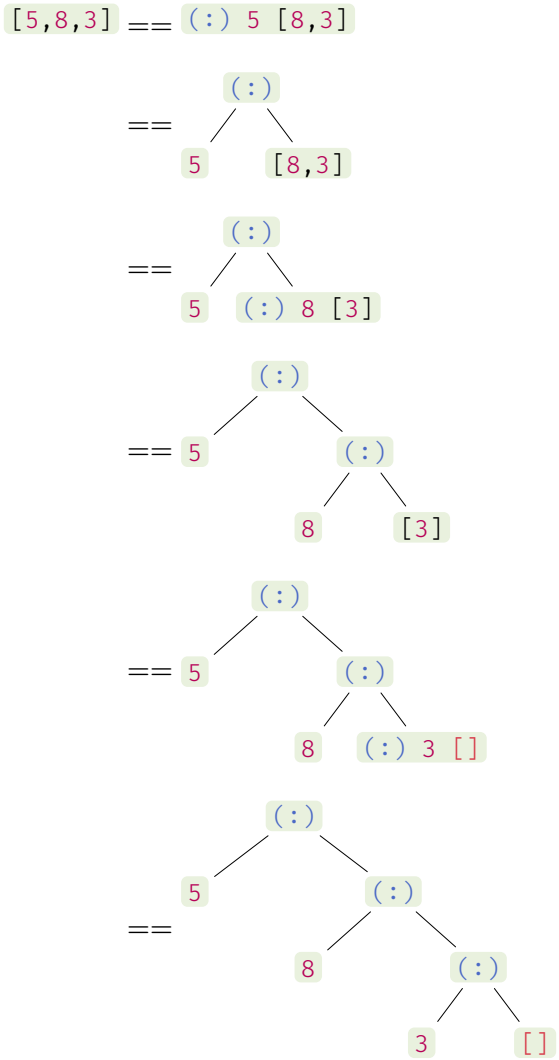
(2) See which definition takes more steps to compute `splitAt 2 [5,8,3]`

§5.9. Lists as Syntax Trees

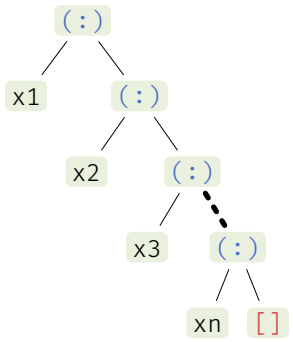
Recall ÷ abstract syntax tree.

Remember that we represent $f(x, y)$ as 

Using this rule, see whether the following steps make sense -



In fact any list `[x1,x2,x3, ... ,xn]` can be represented as



This is the representation that Haskell actually uses to store lists.

§5.10. Dark Magic

We can use our arithmetic progression notation to generate infinite arithmetic progressions.

```
>>> [0..]
[0,1,2,3,4,5,6,7,8,9, ...]

>>> [2,5..]
[2,5,8,11,14,17,20,23,26,29, ...]
```

We can define infinite lists like -

a list of infinitely many `0`s -

```
zeroes = 0 : zeroes
```

```
>>> zeroes
[0,0,0,0,0,0,0,0,0,0, ...]
```

the list of all natural numbers -

```
naturals = l 0 where l n = n : l (n+1)
```

```
>>> naturals
[0,1,2,3,4,5,6,7,8,9, ...]
```

and the list of all fibonacci numbers -

```
fibs = l 0 1 where l a b = a : l b (a+b)
```

```
>>> fibs
[0,1,1,2,3,5,8,13,21,34, ...]
```

Since we obviously cannot view the entirety of an infinite list, it is advisable to use `take` to view an initial section of the list, rather than the whole thing.

§5.10.1. Excercises

Ballons

In an ICPC contest, balloons are distributed as follows:

- Whenever a team solves a problem, that team gets a balloon.
- The first team to solve a problem gets an additional balloon.

A contest has 26 problems, labelled $A, B, ..., Z$. You are given the order of solved problems in the contest, denoted as a string s, where the i-th character indicates that the problem s_i has been solved by some team. No team will solve the same problem twice.

Write a function `balloons :: String → Int` to determine the total number of balloons used in the contest. Note that some problems may be solved by none of the teams.

Example :

```
balloons "ABA" = 5
balloons "A" = 2
balloons "ORZ" = 6
balloons "BAAAA" = 7
balloons "BAAAA" = 7
balloons "BKPT" = 8
balloons "BKPT" = 8
balloons "HASKELL" = 13
```

Neq Array (INOI 2025 P1)

Given a list A of length N, we call a list of integers B of length N such that:

- All elemeents of B are positive, ie $\forall 1 \leq i \leq N, B_i > 0$
- B is non-decreasing, ie $B_1 \leq B_2 \leq ... \leq B_N$
- $\forall 1 \leq i \leq N, B_i = A_i$

Let $\text{neq}(A)$ denote the minimum possible value of the last element of B for a valid array B.

Write a function `neq :: [Int] → Int` that takes a list A and returns the $\text{neq}(A)$.

Example :

```
neq [2,1] = 2
neq [1,2,3,4] = 5
neq [2,1,1,3,2,1] = 3
```

X Kratki (COCI 2014)

Given two integers N and K, write a function `krat :: Int → Int → Maybe [Int]` which constructs a permutation of numbers from 1 to N such that the length of its longest monotone subsequence (either ascending or descending) is exactly K or declare that the following is not possible.

A monotone subsequence is a subsequence where elements are either in non-decreasing order (ascending) or non-increasing order (descending).

Example:

```
krat 4 3 = Just [1,4,2,3]
krat 5 1 = Nothing
krat 5 5 = Just [1,2,3,4,5]
```

For example 1: The permutation (1, 4, 2, 3) has longest ascending subsequence (1, 2, 3) of length 3, and no longer monotone subsequence exists. For example 2: It's impossible to create a permutation of 5 distinct numbers with longest monotone subsequence of length 1. For example 3: The permutation (1, 2, 3, 4, 5) itself is the longest monotone subsequence of length 5.

X Putnik (COCI 2013

Chances are that you have probably already heard of the travelling salesman problem. If you have, then you are aware that it is an NP-hard problem because it lacks an efficient solution. Well, this task is an uncommon version of the famous problem! Its uncommonness derives from the fact that this version is, actually, solvable.

Our vacationing mathematician is on a mission to visit N cities, each exactly once. The cities are represented by numbers $1, 2, ..., N$. What we know is the direct flight duration between each pair of cities. The mathematician, being the efficient woaman that she is, wants to modify the city visiting sequence so that the total flight duration is the minimum possible.

Alas, all is not so simple. In addition, the mathematician has a peculiar condition regarding the sequence. For each city labeled K must apply: either all cities with labels smaller than K have been visited before the city labeled K or they will all be visited after the city labeled K. In other words, the situation when one of such cities is visited before, and the other after is not allowed.

Assist the vacationing mathematician in her ambitious mission and write a function `time :: [[Int]] → Int` to calculate the minimum total flight duration needed in order to travel to all the cities, starting from whichever and ending in whichever city, visiting every city exactly once, so that her peculiar request is fulfilled. Example :

```
time [
  [0,5,2],
  [5,0,4],
  [2,4,0]] = 7


time [
  [0,15,7,8],
  [15,0,16,9],
  [7,16,0,12],
  [8,9,12,0]] = 31
]
```

In the first example: the optimal sequence is 2, 1, 3 or 3, 1, 2. The sequence 1, 3, 2 is even more favourable, but it does not fulfill the condition. In the second example: the sequence is either 3, 1, 2, 4 or 4, 2, 1, 3.

X Look and Say

Look-and-say sequences are generated iteratively, using the previous value as input for the next step. For each step, take the previous value, and replace each run of digits (like 111) with the number of digits (3) followed by the digit itself (1).

For example:

- 1 becomes 11 (1 copy of digit 1).
- 11 becomes 21 (2 copies of digit 1).
- 21 becomes 1211 (one 2 followed by one 1).
- 1211 becomes 111221 (one 1, one 2, and two 1s).
- 111221 becomes 312211 (three 1s, two 2s, and one 1).

Write function `lookNsay :: Int → Int` which takes an number and generates the next number in its look and say sequence.

X Triangles (Codeforces)

Pavel has several sticks with lengths equal to powers of two. He has a_0 sticks of length $2^0 = 1$, a_1 sticks of length $2^1 = 2$, ..., a_n sticks of length 2^n.

Pavel wants to make the maximum possible number of triangles using these sticks. The triangles should have strictly positive area, each stick can be used in at most one triangle.

It is forbidden to break sticks, and each triangle should consist of exactly three sticks. Write a function `triangles :: [Int] → Int` to find the maximum possible number of triangles.

Examples

```
triangles [1,2,2,2,2] = 3
triangles [1,1,1] = 0
triangles [3,3,3] = 1
```

In the first example, Pavel can, for example, make this set of triangles (the lengths of the sides of the triangles are listed): $(2^0, 2^4, 2^4)$, $(2^1, 2^3, 2^3)$, $(2^1, 2^2, 2\hat{}2)$.

In the second example, Pavel cannot make a single triangle.

In the third example, Pavel can, for example, create this set of triangles (the lengths of the sides of the triangles are listed): $(2^0, 2^0, 2^0)$, $(2^1, 2^1, 2^1)$, $(2^2, 2^2, 2^2)$.

X Thanos Sort (Codeforces)

Thanos sort is a supervillain sorting algorithm, which works as follows: if the array is not sorted, snap your fingers* to remove the first or the second half of the items, and repeat the process.

Given an input list, what is the size of the longest sorted list you can obtain from it using Thanos sort? Write function `thanos :: Ord a ⇒ [a] → Int` to determine that.

* Infinity Gauntlet required.

Examples

```
thanos [1,2,2,4] = 4
thanos [11, 12, 1, 2, 13, 14, 3, 4] = 2
thanos [7,6,5,4] = 1
```

In the first example the list is already sorted, so no finger snaps are required.

In the second example the list actually has a subarray of 4 sorted elements, but you can not remove elements from different sides of the list in one finger snap. Each time you have to remove either the whole first half or the whole second half, so you'll have to snap your fingers twice to get to a 2-element sorted list.

In the third example the list is sorted in decreasing order, so you can only save one element from the ultimate destruction.

X Deadfish

Deadfish XKCD is a fun, unusual programming language. It only has one variable, called s, which starts at 0. You change s by using simple commands.

Your task is to write a Haskell program that reads Deadfish XKCD code from a file, runs it, and prints the output.

Deadfish XKCD has the following commands:

Command	What It Does
x	Add 1 to s.
k	print s as a number.
c	Square s,
d	Subtract 1 from s.
X	Start defining a function (the next character is the function name).
K	Print s as an ASCII character.
C	End the function definition or run a function.
D	Reset s back to 0.
{}	Everything inside curly braces is considered a comment.

Extra Rules:

- s must stay between 0 and 255.
- If s goes above 255 or below 0, reset it back to 0.
- Ignore spaces, newlines, and tabs in the code.
- Other characters (not commands) work differently depending on the subtask.

While you can do the whole exercise in one go, we reccomend doing the following subtasks in order.

1. Basic (x, k, c, d only)

```
run "xxcxkdk" = "54"
```

2. Extended (add K, D)

```
run "xxcxxxxcxxxxxxxxK" = "H"
```

3. Functions (all commands)

```
run "XUxkCxxCUCUCU" = "345"
```

4. Comments (ignore {})

Ignore content inside curly braces.

Hint: Don't be afraid to use tuples!

X Weakness and Poorness (Codeforces)

You are given a sequence of n integers $a_1, a_2, ..., a_n$.

Write a function `solve :: [Int] → Float` to determine a real number x such that the weakness of the sequence $a_1 - x, a_2 - x, ..., a_n - x$ is as small as possible.

The weakness of a sequence is defined as the maximum value of the poorness over all segments (contiguous subsequences) of a sequence.

The poorness of a segment is defined as the absolute value of sum of the elements of segment.

Examples

```
solve [1,2,3] = 2.0
solve [1,2,3.4] = 2.5
```

Note For the first case, the optimal value of x is 2 so the sequence becomes $-1, 0, 1$ and the max poorness occurs at the segment -1 or segment 1.

For the second sample the optimal value of x is 2.5 so the sequence becomes $-1.5, -0.5, 0.5, 1.5$ and the max poorness occurs on segment $-1.5, -0.5$ or $0.5, 1.5$.

Polymorphism and Higher Order Functions

Shubh Sharma

§6.1. Polymorphism

§6.1.1. Classification has always been about *shape* and *behvaiour* anyway

Functions are our way, to interact with the elements of a type, and one can define functions in one of the two following ways:

1. Define an output for every single element.
2. Consider the general property of elements, that is, how they look like, and the functions defined on them.

And we have seen how to define functions from a given type to another given type using the above ideas, for example:

`nand` is a function that accepts 2 `Bool` values, and checks if it at least one of them is `False`. We will show two ways to write this function.

The first is too look at the possible inputs and define the outputs directly:

```
nand :: Bool → Bool → Bool
nand False _     = True
nand True True   = False
nand True False = True
```

The other way is to define the function in terms of other functions and how the elements of the type `Bool` behave

```
nand :: Bool → Bool → Bool
nand a b = not (a && b)
```

The situation is something similar, for a lot of other types, like `Int`, `Char` and so on.

But with the addition of the List type from the previous chapter, we were able to add *shape* to the elements of a type, in the following sense:

Consider the type `[Integer]`, the elements of these types are lists of integers, the way one would interact with these would be to treat it as a collection of objects, in which each element is an integer.

- A function for lists would thus have 2 components, at least conceptually if not explicit in the code itself:
  - The first being that of a list, which can be interacted with using functions like `head`.
  - The second being that of `Integer`, So that functions on `Integer` can be applied to the elements of the list.

  consider the following example:

  λ squaring all elements of a list
  ```
  squareAll :: [Integer] → [Integer]
  squareAll []       = []
  squareAll (x : xs) = x * x : squareAll xs
  ```

  Here, in the definition when we match patterns, we figure out the shape of the list element, and if we can extract an integer from it, then we square it and put it back in the list.

Something similar can be done with the type `[Bool]`:

- Once again, to write a function, one needs to first look at the *shape* an element as a list, Then pick elements out of them and treat them as `Bool` elements.
- An example of this will be the `and` function, that takes in a collection of `Bool` and returns `True` if and only if all of them are `True`.

  λ and
  ```
  and :: [Bool] → Bool
  and []       = True  -- We call scenarios like this 'vacuously true'
  and (x : xs) = x && and xs
  ```

  Once again, the pattern matching handles the shape of an element as a list, and the definition handles each item of a list as a `Bool`.

Then we see functions like the following:

- `elem`, which checks in an element belong to a list.
- `(==)`, which checks if 2 elements are equal.
- `drop`, which takes a list and discards a specfied about of items in the list from the beginning.

These functions seem to note care about all of the properties (shape and behaviour together) of their inputs.

- The `elem` function wants its inputs to be list does not care about the internal type of list items as long as some notion of equality if defined.
- The `(==)` works on all types where some notion of equality is defined, this is the only behaviour it is interested in. (A counter example would be the type of functions: `Integer → Integer`, and we will discuss why this is the case soon.)
- The `drop` function just cares about the list structre of an element, and does not look at the behaviour of the list items at all.

To define any function in haskell, one needs to give them a type, haskell demands so, so lets look at the case of the `drop` function. One possible way to have it would be to define one for every single type, as shown below:

```
dropIntegers :: Integer → [Integer] → [Integer]
dropIntegers = ...
dropChars :: Integer → [Char] → [Char]
dropChars = ...
dropBools :: Integer → [Bool] → [Bool]
dropBools = ...
.
.
.
```

but that has 2 problems:

- The first is that the defintion of all of these functions is the exact same, so doing this would be a lot of manual work, and one would also need to have different name for different types, which is very inconvenient.
- The second, and arguably a more serious issue, is that it stops us from abstracting, abstraction is the process of looking at a scenario and removing information that is not relevant to the problem.
  - An example would be that the `drop` simply lets us treat elememts as lists, while we can ignore the type of items in the list.
  - All of Mathematics and Computer Science is done like this, in some sense it is just that.
    - Linear Algebra lets us treat any set where addition and scaling is defined as one *kind* of thing, without worrying about any other structure on the elements.
    - Metric Spaces let us talk about all sets where there is a notion of distance.
    - Differential Equations let us talk about "change" in many different scenarios.

    in all of these fields of study, say linear algebra, a theorem generally involes working with an object, whose exact details we don't assume, just that it satisfies the conditions required for it to be a vector space and seeing what can be done with just that much information.
  - And this is a powerful tool because solving a problem in the *abstract* version solves the problem in all *concretized* scenarios.

John Locke, An Essay Concerning Human Understanding (1690)

The acts of the mind, wherein it exerts its power over simple ideas, are chiefly these three:

1. Combining several simple ideas into one compound one, and thus all complex ideas are made.
2. The second is bringing two ideas, whether simple or complex, together, and setting them by one another so as to take a view of them at once, without uniting them into one, by which it gets all its ideas of relations.
3. The third is separating them from all other ideas that accompany them in their real existence: this is called **abstraction**, and thus all its general ideas are made.

One of the ways abstraction is handled in Haskell, and a lot of other programming languages is **Polymorphism**.

Polymorphism

A **polymorphic** function is one whose output type depends on the input type. Such a property of a function is called **polymorphism**, and the word itself is ancient greek for ***many forms***.

A polymorphic function differs from functions we have seen in the following ways:

- It can take input from multiple different input types (not necessarily all types, restrictions are allowed).
- Its output type can be different for different inputs types.

An example for such a function that we have seen in the previous section would be:

λ drop

```
drop :: Integer → [a] → [a]
drop _ []     = []
drop 0 ls     = ls
drop n (x:xs) = drop (n-1) xs
```

The polymorphism of this function is shown in the type `drop :: Integer → [a] → [a]` where we have used the variable `a` (usually called a type variable) instead of explicitly mentioning a type.

The goal of polymorphic functions is to let us **abstract** over a collection of types. That take a collection of types, based on some common property (either shape, or behaviour, maybe both) and treat that as a collection of elements. This lets us build functions that work on "all lists" or "all maybe types" and so on.

The example λ **drop** brings together all types of lists and only looks at the *shape* of the element, that of a list, and does not look at the bhevaiour at all. This is shown by using the type variable `a` in the definition, indicating that we don't care about the properties of the list items.

Datatypes of some list functions

A nice exercise would be to write the types of the following functions defined in the previous section: `head`, `tail`, `(!!)`, `take` and `splitAt`.

We have now given a type to one of the 3 functions discussed above, by giving a way to group together types by their common *shape*. This is not enough to give types of the other two functions (`(==)` and `elem`), to do so we define the following:

Behaviour

Given a type `T`, the **behaviour** of the elements in `T` is the set of definable functions whose type includes `T`.

We use this to define the two types of polymorphism, one of which we have already seen in this section, and we will look at the other one more deepy in the next.

2 Types of Polymorphism

- Polymorphism done by grouping types that with common ***shape*** is called **Parametric Polymorphism**.
- Polymorphism done by grouping types that with common ***behaviour*** is called **Ad-Hoc Polymorphism**.

We will come back to **parametric polymorphism** in the second half of the chapter, but for now we discuss **Ad-Hoc polymorphism**.

§6.1.2. A Taste of Type Classes

Consider the case of the `Integer` functions

```haskell
f :: Integer → Integer
f x = x^2 + 2*x + 1

g :: Integer → Integer
g x = (x + 1)^2
```

We know that both functions, do the same thing in the mathematical sense, given any input, both of then have the same output, so mathematicans call them the same, and write $f = g$ this is called **function extensionality**. But the does the following expression make sense in haskell?

λ Function Extensionality

```haskell
f == g
```

This definitely seems like a fair thing to ask, as we already have a definition for equality of mathematical functions, but we run into 2 issues:

- Is it really fair to say that? In computer science, we care about the way things are computed, that is where the subject gets its name from. A lot of times, one will be able to distinguish distinguish between functions, by simply looking at which one works faster or slower on big inputs, and that might be something people would want to factor into what they mean by "sameness". So maybe the assumption that 2 functions being equal pointwise imply the functions are equal is not wise.
- The second is that in general it is not possible, in this case we have a mathematical identity that lets us prove so, but given any 2 function, it might be that the only way to prove that they are equal would be to actually check on every single value, and since domains of functions can be infinite, this would simply not be possible to compute.

So we can't have the type of `(==)` to be `a → a → Bool`. In fact, if I try to write it, the haskell compiler will complain to me by saying

```
funext.hs:8:7: error: [GHC-39999]
    • No instance for ‘Eq (Integer → Integer)’ arising from a use of ‘==’
      ... more error
  |
8 | h = f == g
  |
```

To tackle the problem of giving a type for `(==)`, we define the following:

≒ **Typeclasses**

Typeclasses are a collection of types, characterized by the common *behaviour*.

The previous section talked about grouping types together by the common *shape* of the elements but λ **Function Extensionality** tells us that there are other properties shared by elements of different types, which we call their *behaviour*. By that we mean the functions that are defined for them.

Typeclasses are how one expresses in haskell, what a collection of types looks like, and the way to do so is by defining the common functions that work for all of them. Some examples are:

- `Eq`, which is the collection of all types for which the function `(==)` is defined.
- `Ord`, which is the collection of all types for which the function `(<)` is defined.
- `Show`, which is the collection of all types for which there is a function that converts them to `String` using the function `show`.

Note that in the above cases, defining one function lets you define some other functions, like `(/=)` for `Eq` and `(<=)`, `(>=)` and others for the `Ord` typeclass.

Now we come back to the `elem` function, the goal of this function is to check if a given element belongs to a list. And the following is a way to write it:

```
elem _ []       = False
elem e (x : xs) = e == x || elem e xs
```

Now lets try to give this a type.

First we see that the `e` must have the same types as the items in the list, but if we try to give it the type

```
elem :: a -> [a] -> Bool
```

we will encounter the same issue as we did in λ **Function Extensionality**, because of `(==)`. We need to find a way to say that `a` belongs to the collection `Eq`, and this leads to the correct type:

λ `elem`

```
elem :: Eq a => a -> [a] -> Bool
elem _ []       = False
elem e (x : xs) = e == x || elem e xs
```

X Checking if a list is sorted

Write the function `isSorted` which takes in a list as an argument, such that the elements of the list have a notion of ordering between them, and the output should be true if the list in an ascending order (equal elements are allowed to be next to each other), and false otherwise.

X Shape is behaviour?

The two types of polymorphism, that is parametric and ad-hoc, are not exlusive, there are plenty of function where both are seen together, an example would be `elem`.

These two happen to not be that different conceptually either, we give elements their *shape* using functions, try figuring out what the functions are for list types, maybe type, tuples and either type.

That being said, the syntax used to define parametric polymorphism sets us to set operations while defining the type of the function which is very powerful.

§6.2. Higher Order Functions

One of the most powerful features of functional programming languages is that it lets one pass in functions as argument to another function, and have functions return other functions as outputs, these kinds of functions are known as:

÷ Higher Order Functions

A **higher order function** is a function that does at least one of the following things:

- It takes one or more functions as its arguments.
- It returns a function as an argument.

This is again a way of generalization and is very handy, as we will see in the rest of the chapter.

§6.2.1. Currying

Perhaps the first place where we have encountered higher order functions is when we defined `(+) :: Int → Int → Int` way back in Section §3.4.. We have been suggesting to think of the type as `(+) :: (Int, Int) → Int`, because that is really what we want the function to do, but in haskell it would actually mean `(+) :: Int → (Int → Int)`, which says the function has 1 interger argument, and it returns a function of type `Int → Int`.

An example from mathematics would be finding the derivative of a differentiable function f at a point x. This is generally represented as $f'(x)$ and the process of computing the derivative can be given to have the type

$$(f, x) \mapsto f'(x) : \left((\mathbb{R} \to \mathbb{R})^d \times \mathbb{R}\right) \to \mathbb{R}$$

Here $(\mathbb{R} \to \mathbb{R})^d$ is the type of real differentiable functions.

But one can also think of the derivative operator, that takes a differentiable function f and produces the function f', which can be given the following type:

$$\frac{d}{dx} : (\mathbb{R} \to \mathbb{R})^d \to (\mathbb{R} \to \mathbb{R})$$

In general, we have the following theorem:

Theorem **Currying**: Given any sets A, B, C, there is a *bijection* called curry between the sets $C^{A\times B}$ and the set $\left(C^B\right)^A$ such that given any function $f : C^{A\times B}$ we have

$$(\text{curry } f)(a)(b) = f(a, b)$$

Category theorists call the above condition *naturality* (or say that the bijection is *natural*). The notation Y^X is the set of functions from X to Y.

Proof We prove the above by defining $\text{curry} : C^{A\times B} \to \left(C^B\right)^A$, and then defining its inverse.

$$\text{curry}(f) := x \mapsto (y \mapsto f(x, y))$$

The inverse of curry is called uncurry : $\left(C^B\right)^A \to C^{A\times B}$

$$\text{uncurry}(g) := (x, y) \mapsto g(x)(y)$$

To complete the proof we need to show that the above functions are inverses.

Exercise

Show that the uncurry is the inverse of curry, and that the *naturality* condition holds.

(Note that one needs to show that uncurry is the 2-way inverse of curry, that is, uncurry ∘ curry = id and curry ∘ uncurry = id, one direction is not enough.)

■

The above theorem, is a concretization of the very intuitive idea:

- Given a function f that takes in a pair of type $(A, B) \to C$, if one fixes the first argument, then we get a function $f(A, -)$ which would take an element of type B and then give an element of types C.
- But every different value of type A that we fix, we get a different function.

- Thus we can think of f as a function that takes in an element of type A and returns a function of type $B \to C$.

And the above theorem is also "implemented" in haskell using the following functions:

λ curry and uncurry
```haskell
curry :: ((a, b) → c) → a → b → c
curry f a b = f (a, b)

uncurry :: (a → b → c) → (a, b) → c
uncurry g (a, b) = g a b
```

Currying lets us take a function with with argument, and lets us apply the function to each of them one at a time, rather than applying it on the entire tuple at once. One very interesting result of that is called **partial application**.

Partial applicaion is precisely the process of fixing some arugments to get a function over the remaining, let us look at some examples

```haskell
suc :: Integer → Integer
suc = (+ 1) -- suc 5 = 6

-- | curry examples
neg :: Integer → Integer
neg = (-1 *) -- neg 5 = -5
```

We will find many more examples in the next section.

§6.2.2. Functions on Functions

We have already seen examples of a couple of functions whose arguments themselves are functions. The most recent ones being λ **curry and uncurry**, both of them take functions as inputs and return functions as outputs (note that our definition takes in functions and values, but we can always use partial application), these functions can be thought of as useful operations on functions.

Another very useful example, that a lot of us have seen is composition of functions, when we allow functions as inputs, composition can be treated like a function:

λ composition
```haskell
(.) :: (b → c) → (a → b) → (a → c)
g . f = \a → g (f a)

-- example
square :: Integer → Integer
square x = x * x

-- checks if a number is the same if written in reverse
is_palindrome :: Integer → Bool
is_palindrome x = (s == reverse s)
  where
    s = show x -- convert x to string

is_square_palindrome :: Integer → Bool
is_square_palindrome = is_palindrome . square
```

Breaking a complicated function into simpler parts, and being able to combime them is fairly standard problem solving strategy, in both Mathematics and Computer Science, and in fact in a lot more general scenarios too! Having a clean notation for a tool that used fairly frequently is always a good idea!

Higher order functions are where polymorphism shines it brightest, see how the composition function works on all pairs of functions that can be composed in the mathematical sense, this would have been significantly less impressive if say it was only composition between functions from `Integer → Integer` and `Integer → Bool`.

Another similar function that makes writing code in haskell much cleaner is the following:

λ function application function

```
($) :: (a → b) → a → b
f $ a = f a

(&) :: a → (a → b) → b
a & f = f a
```

These may seem like a fairly trivial function that really doesn't offer anything apart from an extra `$`, but the following 3 lines make them useful

λ operator precedence

```
-- The 'r' in infixr says a.b.c.d is interpreted by haskell as a.(b.(c.d))
infixr 9 .
infixr 0 $
infixl 1 &
```

These 2 lines are saying that, whenever there is an expression, which contains both `($)` and `(.)`, haskell will first evaluate `(.)`, using these 2 one can write a chain of function applications as follows:

```
-- old way
f (g (h (i x)))

-- new way
f . g . h . i $ x

-- also
x & f & g & h & i
```

which in my opinion is much simpler to read!

X Exercise

Write a function `apply_n_times` that takes a function `f` and an argument `a` along with a natural number `n` and applies the function `n` times on `a`, for exaxmple: `apply_n_times (+1) 5 3` would return `8`. Also figure out the type of the function.

§6.2.3. A Short Note on Type Inference

Haskell is a statically typed language. What that means is that it requires the types for the data that is being processed by the program, and it needs to do so for an analysis that happens before running called **type checking**.

It is not however required to give types to all functions (we do strongly recommend it though!), in fact one can simply not give any types at all. This is possible because the haskell compiler is smart enough

to figure all of it out on its own! It's so good that when you do write type annotations for functions, haskell ignores it, figures the types out on its own and can then check if you have given the types correctly. This is called **type inference**.

Haskell's type inference also gives the most general possible type for a function. To see that, one can open ghci, and use the `:t` command to ask haskell for types of any given expression.

```
>>> :t flip
flip :: (a → b → c) → b → a → c
>>> :t (\ x y → x == y)
(\ x y → x == y) :: Eq a ⇒ a → a → Bool
```

The reader should now be equipped with everything they need to understand how types can be read and can now use type inference like this to understand haskell programs better.

§6.2.4. Higher Order Functions on Maybe Type : A Case Study

The **Maybe Type**, as defined in Chapter 3 is another playground for higher order functions.

As a refresher on **Maybe Types**, given a type `a`, one can add an *extra element* to it by making it the type `Maybe a`. For example, given the type `Integer`, whose elements are all the integers, the type `Maybe Integer` will be the collection of integers along with an extra element, which we call `Nothing`.

Maybe Types are meant to capture failure, for example, the λ **function to a maybe type** defines the `reciprocal` function, which takes a rational number, and returns its reciprocal, except when the input is `0`, in which case it returns the *extra value* which is `Nothing`.

To state that elements belong to a **Maybe Type** they are decorated with `Just`. For example:

- The type of `5` is `Integer`
- The type of `Just 5` is `Maybe Integer`.

To see an example of some functions that use `Maybe` in their type definitions are:

- A safe version of `head` and `tail`:
  - `safeHead :: [a] → Maybe a`
  - `safeTail :: [a] → Maybe [a]`
- A safe way to index a list, that is a safe version of `(!!)`:
  - `safeIndex :: [a] → Int → Maybe a`

X Safety First

Define the functions `safeHead`, `safeTail` and `safeIndex`.

Something that should be noted is that so far in the book, `head`, `tail` and `(!!)` are the only functions for which we need safe versions. This is because these are the only functions that are not defined for all possible inputs and can hence give an error while the program executes (that would be like passing empty list to head, or idexing an element at a negative position). Every other function we have seen will always have a valid output, that is, it is literally impossible for functions to fail for not having a valid input if one only uses safe functions!

This may seem like a fairly trivial fact for those who are learning haskell as thier first programming language, but for those who has programmed in languages like Java, Python, C or so on, it is impossible to write a program that would lead to an error which is equivalent to the following:

- Nonetype does not have this attribute: Python
- Null Pointer Exception: Java
- Memory Access Violation or Segfault for derefencing a null pointer: C

If these erros have haunted you, you have our condolences, all of these would have been completely avoided if the langauge had some version of `Maybe`, or even some bare bones type system in case of python.

All of the safety provided by `Maybe` types has 1 potential drawback: When using `Maybe` types, one eventually runs into a problem that looks something like this:

- While solving a complicated problem, one would break it down into simpler parts, that would correspond to many tiny functions, that will come to gether to form the functions which solves the problem.
- Turns out that one the functions, maybe something in the very beginning returns a `Maybe Integer` instead of an `Integer`.
- This means that the next function along the chain, would have had to have its input type as `Maybe Integer` to account for the potentially case of `Nothing`.
- This also forces the output type to be a `Maybe` type, this makes sense, if the process fails in the beginning, one might not want to continue.
- The `Maybe` now propogates in this manner through a large section of your code, this means that a huge chunk of code needs to be rewritten to looks something like:

```
f :: a -> b
f inp = <some expression to produce output>

f' :: Maybe a -> Maybe b
f' (Just inp) = Just $ <some epression to produce output>
f' Nothing    = Nothing
```

Note that `$` here is making our code a little bit cleaner, otherwise we would have to put the enter expression in paranthesis.

This is still not a very elegant way to write things though, and its just a lot of repetitive work (all of it is just book keeping really, one isn't really adding much to the program by making these changes, except for safety, programmers usually like to call it boilerplate.)

Instead of going and modifying each function manually, we make a function modifier, which is precisely what a higher order function: Our goal, which is obvious from the problem: `(a -> b) -> (Maybe a -> Maybe b)` and we define it as follows:

λ maybeMap

```haskell
maybeMap :: (a → b) → Maybe a → Maybe b
maybeMap f (Just a) = Just . f $ a
maybeMap _ Nothing  = Nothing

(<$>) :: (a → b) → Maybe a → Maybe b -- symbol version
f <$> a = maybeMap f a

(<.>) :: (b → c) → (a → Maybe b) → a → Maybe c
g <.> f = \x → g <$> f x

infixr 1 <$>
infixr 8 <.>
```

Note: The symbol `<$>` is written as `<$>`.

So consider the following chain of functions:

```haskell
f . g . h . i . j $ x
```

where say `i` was the function that turned out to be the one with `Maybe` output, the only change we need to the code would be the following!

```haskell
f . g . h <.> i . j $ x
```

Higher order functions, along with polymorphism help our code be really expressive, so we can write very small amounds of code that looks easy to read, which also does a lot. In the next chapter we will see a lot more examples of such functions.

X Beyond map

The above shows how haskell can elegantly handle cases when we want to convert a function from type `a → b` to a function from type `Maybe a → Maybe b`. This can be thought of as some sort of a *change in context*, where our function is now aware that its inputs can contain a possible fail value, which is `Nothing`. The reason for needing such a *change in context* were function of type `f :: a → Maybe b`, that is ones which can fail. They add the possiblility of failure to the *context*.

But since we have the power to be able to change *contexts* whenever wanted easily, we have a responsibility to keep it consistent when it makes sense. That is, what if there are multiple function with type `f :: a → Maybe b` we then would just want to use `<.>` or `maybeMap` to get something like:

```
f :: a → Maybe b
g :: b → Maybe c

h x = g <$> f x :: a → Maybe (Maybe c)
```

This is most likely undesirable, the point of `Maybe` was to say that there is a possiblility of error, the point of `(<$>)` was to propogate that possible error then the type `Maybe (Maybe c)` seems to not have a place here.

To rectify this, we find a way to compose such functions together:

```
maybe_comp :: (a → Maybe b) → (b → Maybe c) → (a → Maybe c)
infixr 8 >=>
```

This cute looking function is called the **fish** operator. This will be our way to compose functions of the shape `a → Maybe b` together, but note that the order of inputs is reversed, so it not looks like a pipe through which the value is passed. The above function `h` is defined as follows:

```
h = f >=> g :: a → Maybe c
```

This function, takes a value of type `a`, first applies `f` to it, and then applies `g` to it in a way that the final output is of type `Maybe c`, and of course, we can use this to make longer chains!

```
func1 :: a → Maybe b
func2 :: b → Maybe c
func3 :: c → Maybe d
func4 :: d → Maybe e

final :: a → Maybe e
final = func1 >=> func2
    >=> func3 >=> func4
```

Define and `(>=>)` and see how both of then are used in programs, and compare then by how one would define `final` without these.

Note The symbol `(>=>)` is written as `(>=>)`.

Advanced List Operations

Arjun Maneesh Agarwal

§7.1. List Comprehensions

As we have talked about before, Haskell tries to make it's syntax look as similer as possible to math notation. This is reprasented in one of the most powerful syntactic sugers in Haskell, list comprehension.

If we want to talk about all pythagorian triplets using integers from $1 - n$, we could express it mathematically as

$$\{(x, y, z) \mid x, y, z \in \{1, 2, ..., n\}, x^2 + y^2 = z^2\}$$

which can be written in Haskell as

```
[(x,y,z) | x ← [1..n], y ← [1..n], z ← [1..n], x^2 + y^2 == z^2]
```

This allows us to define a lot of operations we have seen before, in ch 1, in rather concise manner.

For example, `map :: (a→ b) → [a] → [b]` which used to apply a function to a list of elements of a suitable input type and gave a list of the suitable output type. Basically, `map f [a1,a2,a3] = [f a1, f a2, f a3]`. We can define this in two ways:

λ Defining map using pattern matching and list comprehension

```
map _ [] = []
map f (x:xs) = (f x) : (map f xs)

-- and much more clearly and concisely as
map f ls = [f l | l ← ls]
```

Similerly, we had seen `filter :: (a → Bool) → [a] → [a]` which used to take a boolean function, some predicate to satisfy, and return the list of elements satisfying this predicate. We can define this as:

λ Defining filter using pattern matching and list comprehension

```
filter _ [] = []
filter p (x:xs) = let rest = p xs in
  if p x then x : rest else rest

-- and much more cleanly as
filter p ls = [l | l ← ls, p l]
```

Another operation we can consider, though not explictly defined in Haskell, is cartesian product. Hopefully, you can see where we are going with this right?

λ Defining cartesian product using pattern matching and list comprehension

```
cart :: [a] → [b] → [(a,b)]
cart xs ys = [(x,y) | x ← xs, y ← ys]

-- Trying to define this recursively is much more cumbersome.

cart [] _ = []
cart (x:xs) ys = (go x ys) ++ (cart xs ys) where
  go _ [] = []
  go l (m:ms) = (l,m) : (go l ms)
```

Finally, let's talk a bit more about our pythagorian triplets example at the start of this section.

λ A naive way to get pythagorian triplets

```
pythNaive :: Int → [(Int, Int, Int)]
pythNaive n = [(x,y,z) |
          x ← [1..n],
          y ← [1..n],
          z ← [1..n],
          x^2 + y^2 == z^2]
```

For `n = 1000`, we get the answer is some 13 minutes, which makes sense as our code is basically considering the 1000^3 triplets and then culling the ones which are not pythagorian. But could we do better?

A simple idea would be to not check for `z` as it is implied by the choice of `x` and `y` and instead set the condition as

λ A mid way to get pythagorian triplets

```
pythMid n = [(x, y, z) |
    x ← [1..n],
    y ← [1..n],
    let z2 = x^2 + y^2,
    let z = floor (sqrt (fromIntegral z2)),
    z * z == z2]
```

This is clearly better as we will be only considering some 1000^2 triplets. Continuing with our example, for `n = 1000`, we finish in 1.32 seconds. As we expected, that is already much, much better than the previous case.

Also notice that we can define variables inside the comprehension by using the `let` syntax.

However, there is one final optimization we can do. The idea is that $x > y$ or $x < y$ for pythagorian triplets as `sqrt 2` is irrational. So if we can somehow, only evaluate only the cases where $x < y$ and then just genrate (x, y, z) and (y, x, z); we almost half the number of cases we check. This means, our final optimized code would look like:

λ The optimal way to get pythagorian triplets

```
pythOpt n = [t|
    x ← [1..n],
    y ← [(x+1)..n],
    let z2 = x^2 + y^2,
    let z = floor (sqrt (fromIntegral z2)),
    z * z == z2,
    t ← [(x,y,z), (y,x,z)]
    ]
```

This should only make some $\frac{1000*999}{2}$ triplets and cull the list from there. This makes it about twice as fast, which we can see as for $n = 1000$, we finish in 0.68 seconds.

Notice, we can't return two things in a list comprehension. That is, `pythOpt n = [(x,y,z), (y,x,z) | <blah blah>]` will given an error. Intead, we have to use `pythOpt n = [ t | <blah blah>, t ← [(x,y,z), (y,x,z)]]`.

Another intresting thing we can do using list comprehension is sorting. While further sorting methods and their speed is discussed in chapter 10, we will focus on a two methods of sorting: Merge Sort and Quick Sort.

We have seen the idea of divide and conquor before. If we can divide the problem in smaller parts and combine them, without wasting too much time in the spliting or combining, we can solve the problem. Both these methods work on this idea.

Merge Sort divides the list in two parts, sorts them and then merges these sorted lists by comparing element to elemeent. We can do this recursion with peace of mind as once we reach 1 element lists, we just say they are sorted. That is `mergeSort [x] = [x]`.

Just to illustrate, the merging would work as follows: `merge [1,2,6] [3,4,5]` would take the smaller of the two heads till both lists are empty. This works as as both the lists are sorted. The complete evaluation is something like:

```
merge [1,2,6] [3,4,5]
= 1 : merge [2,6] [3,4,5]
= 1 : 2 : merge [6] [3,4,5]
= 1 : 2 : 3 : merge [6] [4,5]
= 1 : 2 : 3 : 4 : merge [6] [5]
= 1 : 2 : 3 : 4 : 5 : merge [6] []
= 1 : 2 : 3 : 4 : 5 : 6 : merge [] []
= 1 : 2 : 3 : 4 : 5 : 6 : []
= [1,2,3,4,5,6]
```

So we can implement `merge`, rather simply as

λ The merge function of mergesort

```
merge :: Ord a ⇒ [a] → [a] → [a]
merge [] [] = []
merge [] ys = ys
merge xs [] = xs
merge (x:xs) (y:ys) = if x < y
  then x : merge xs (y:ys)
  else y : merge (x:xs) ys
```

Note, we can only sort a list which has some definition of order on the elements. That is the elements must be of the typeclass `Ord`.

To implement merge sort, we now only need a way to split the list in half. This is rather easy, we have already seen `drop` and `take`. An inbuilt function in Haskell is `splitAt :: Int → [a] → ([a], [a])` which is basically equivalent to `splitAt n xs = (take n xs, drop n xs)`.

That means, we can now merge sort using the function

λ An implementation of mergesort

```
mergeSort :: Ord a ⇒ [a] → [a]
mergeSort [] = []
mergeSort [x] = [x]
mergeSort xs = merge (mergeSort left) (mergeSort right) where
  (left,right) = splitAt (length xs `div` 2) xs
```

X MergeSort Works?

Prove that merge sort indeed works. A road map is given

(i) Prove that `merge` defined by taking the smaller of the heads of the lists recursively, produces a sorted list given the two input lists were sorted. The idea is that the first element choosen has to be the smallest. Use induction of the sum of lengths of the lists.

(ii) Prove that `mergeSort` works using induction on the size of list to be sorted.

This is also a very efficient way to sort a list. If we define a function C that count the number of comparisons we make, $C(n) < 2 * C(\lceil \frac{n}{2} \rceil) + n$ where the n comes from the merge.

This implies

$$\begin{aligned}
C(n) &< n\lceil \log(n) \rceil C(1) + n + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{\lceil \frac{n}{2} \rceil}{2} \right\rceil + \ldots + 1 \\
&< n\lceil \log(n) \rceil + n + \frac{n+1}{2} + \left\lceil \frac{n+1}{4} \right\rceil + \ldots + 1 \\
&= n\lceil \log(n) \rceil + n + \frac{n}{2} + \frac{1}{2} + \frac{n}{4} + \frac{1}{2} + \ldots + 1 \\
&< n\lceil \log(n) \rceil + 2n + \frac{1}{2}\lceil \log(n) \rceil \\
&< n(\log(n) + 1) + 2n + \frac{1}{2}(\log(n) + 1) \\
&= n\log(n) + 3n + \frac{1}{2}\log(n) + \frac{1}{2}
\end{aligned}$$

Two things to note are that the above computation was a bit cumbersome. We will later see a way to make it a bit less cumbersome, albeit at the cost of some information.

The second, for sufficiently large n, $n\log(n)$ dominates the equation. That is

$$\exists m \, s.t. \, \forall n > m : n\log(n) > 3n > \frac{1}{2}\log(n) > \frac{1}{2}$$

This means that as n becomes large, we can sort of ignore the other terms. We will later prove, that given no more information other than the fact that the shape of the elemeents in the list is such that they can be compared, we can't do much better. The dominating term, in the number of comparisins, will be $n\log(n)$ times some constant. This later refers to chapter 10.

In practice, we waste some amount of operations dividing the list in 2. What if we take our chances and approximatly divide the list into two parts?

This is the idea of quick sort. If we take a random element in the list, we expect half the elements to be lesser than it and half to be greater. We can use this fact to define quickSort by splitting the list on the basis of the first element and keep going. This can be implemented as:

λ An implementation of Quick Sort

```
quickSort :: Ord a ⇒ [a] → [a]
quickSort [] = []
quickSort [x] = [x]
quickSort (x:xs) = quickSort [l | l ← xs, l > x] ++ [x] ++ quickSort [r | r
← xs, r ⩽ x]
```

X Quick Sort works?

Prove that Quick Sort does indeed works. The simplest way to do this is by induction on length.

Clearly, With n being the length of list, $C(n)$ is a random variable dependent on the permutation of the list.

Let l be the number of elements less than the first elements and $r = n - l - 1$. This means $C(n) = C(l) + C(r) + 2(n-1)$ where the $n-1$ comes from the list comprehension.

In the worst case scenario, our algoritm could keep spliting the list into a length 0 and a length $n-1$ list. This would screw us very badly.

As $C(n) = C(0) + C(n-1) + 2(n-1)$ where the $n-1$ comes from the list comprehension and the $(n-1)+1$ from the concatination. Using $C(0) = 0$ as we don't make any comparisons, This evaluates to

$$\begin{aligned} C(n) &= C(n-1) + 2(n-1) \\ &= 2(n-1) + 2(n-2) + \ldots + 2 \\ &= 2 * \frac{n(n-1)}{2} \\ &= n^2 - n \end{aligned}$$

Which is quite bad as it grows quadratically. Furthermore, the above case is also common enough. How common?

X A Strange Proof

Prove $2^{n-1} \leq n!$

Then why are we intrested in Quick Sort? and why is named quick?

Let's look at the average or expected number of comparison we would need to make!

Consider the list we are sorting a permutation of $[x_1, x_2, \ldots, x_n]$. Let $X_{i,j}$ be a random variable which is 1 if the x_i and x_j are compared and 0 otherwise. Let $p_{i,j}$ be the probability that x_i and x_j are compared. Then, $\mathbb{E}(X_{i,j}) = 1 * p + 0 * (1-p) = p$.

Using the linearity of expectation (remember $\mathbb{E}(\sum X) = \sum \mathbb{E}(x)$?), we can say $\mathbb{E}(C(n)) = \sum_{i,j} \mathbb{E}(X_{i,j}) = \sum_{i,j} p_{i,j}$.

Using the same idea we used to reduce the number of pythogoream triplets we need to check, we rewrite this summation as

$$\begin{aligned} \mathbb{E}(C(n)) &= \sum_{i,j} p_{i,j} \\ &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_{i,j} \end{aligned}$$

Despite a toothy appearence, this is rather easy and elegent way to actually compute $p_{i,j}$.

Notice that each element in the array (except the pivot) is compared only to the pivot at each level of the recurrence. To compute $p_{i,j}$, we shift our focus to the elemenents $[x_i, x_{i+1}, ..., x_j]$. If this is split into two parts, x_i and x_j can no longer be compared. Hence, x_i and x_j are compared only when from the first pivot from the range $[x_i, x_{i+1}, ..., x_j]$ is either x_i or x_j.

This clearly has probability $p_{i,j} = \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}$. Thus,

$$\begin{aligned}
\mathbb{E}(C(n)) &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n} 2\left(\frac{1}{2} + ... + \frac{1}{n-i+1}\right) \\
&= 2\sum_{i=1}^{n}\left(1 + \frac{1}{2} + ... + \frac{1}{n-i+1} - 1\right) \\
&\leq 2\sum_{i=1}^{n} \log(i) \\
&\leq 2\sum_{i=1}^{n} \log(n) \\
&\leq 2n\log(n)
\end{aligned}$$

Considering the number of cases where the comparisons with $n^2 - n$ operations is 2^{n-1}, Quick Sort's expected number of operations is still less than $2n\log(n)$ which, as we discussed, is optimal.

This implies that there are some lists where Quick Sort is extreamly efficient and as one might expect there are many such lists. This is why languages which can keep states (C++, C, Rust etc) etc use something called Introsort which uses Quick Sort till the depth of recursion reaches $\log(n)$ (at which point it is safe to say we are in one of the not nice cases); then we fallback to Merge Sort or a Heap/ Tree Sort(which we will see in chapter 11).

Haskell has an inbuilt `sort` function you can use by putting `import Data.List` at the top of your code. This used to use quickSort as the default but in 2002, Ian Lynagh changed it to Merge Sort. This was motivated by the fact that Merge Sort gurentees sorting in $n\log(n) + ...$ comparisons while Quick Sort will sometimes finish much quicker (pun not intended) and other times, just suffer.

As a dinal remark, our implementation of the Quick Sort is not the most optimal as we go through the list twice, but it is the most aesthetically pleasing and concise.

X Faster Quick Sort

A slight improvment can be made to the implementation by not using list comprehension and instead using a helper function, to traverse the list only once.

Try to figure out this implementation.

§7.2. Zip it up!

Have you ever suffered through a conversation with a very dry person with the goal of getting the contact information of a person you are actually intrested in? If you haven't well, that is what you will have to do now.

The boring zip

Haskell has an inbuilt function called `zip`. It's behaviour is as follows

```
>>> zip [1,2,3] [4,5,6]
[(1,4),(2,5),(3,6)]
>>> zip [1,2,3] [4,5,6,7]
[(1,4),(2,5),(3,6)]
>>> zip [0,1,2,3] [4,5,6]
[(0,4),(1,5),(2,6)]
>>> zip [0,1,2,3] [True, False, True, False]
[(0,True),(1,False),(2,True),(3,False)]
>>> zip [True, False, True, False] "abcd"
[(True,'a'),(False,'b'),(True,'c'),(False,'d')]
>>> zip [1,3..] [2,4..]
[(1,2),(3,4),(5,6),(7,8),(9,10),(11,12),(13,14),(15,16),(17,18),
(19,20)...]
```

What is the type signature of `zip`? How would one implement `zip`?

The solution to the above exercise is, rather simply:

λ Implementation of zip function

```
zip :: [a] → [b] → [(a,b)]
zip [] _ = []
zip _ [] = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

While one could think of some places where this is useful, all of the uses seem rather dry. But now that `zip` has opened up to us, we will ask them about `zipWith`. The function `zipWith` takes two lists, a binary function, and joins the lists using the function. The possible implementations are:

λ Implementation of zipWith function

```
zipWith :: (a → b → c) → [a] → [b] → [c]
zipWith _ [] _ = []
zipWith _ _ [] = []
zipWith f (x:xs) (y:ys) = (x `f` y) : zipWith f xs ys
```

Alternate definitions

While we have defined `zip` and `zipWith` independently here, can you: (i) Define `zip` using `zipWith`? (ii) Define `zipWith` using `zip`?

Now one might feel there is nothing special about `zipWith` as well, but they would be wrong. First, it saves us form defining a lot of things: `zipWith (+) [0,2,5] [1,3,3] = [1,5,8]` is a common enough use. And then, it leads to a lot of absolutly mindblowing pieces of code.

λ The zipWith fibonnaci

```
fibs = 0 : 1 : zipWith (+) fibs (tail fibs)
```

Belive it or not, this should output the fibonnaci sequence. The idea is that Haskell is lazy! This means lists are computed one element at a time, starting from the first. Tracing the computation of the elements of `fibs`:

1. Since by definition `fibs = 0: 1: (something)`, the first element is `0`.

2. This is again easy, since `fibs = 0 : 1 : (something)`, so the second element is `1`.

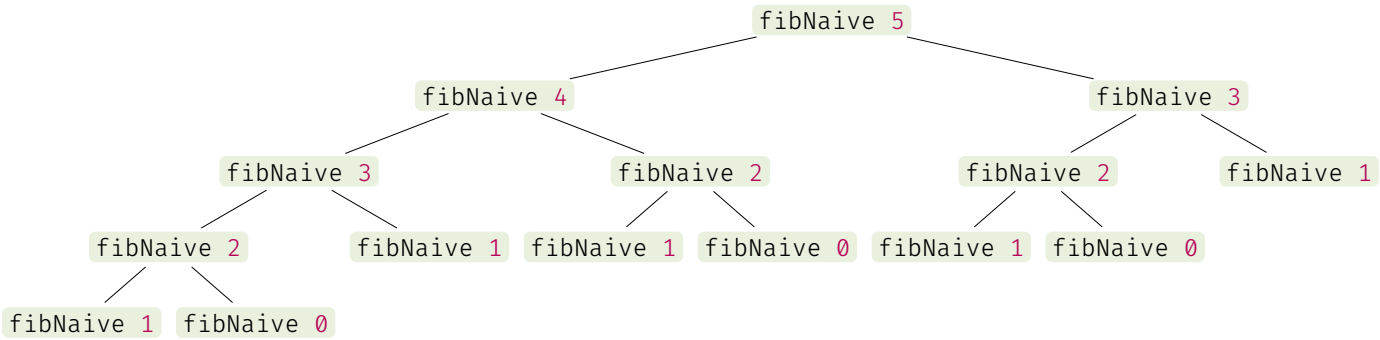
3. This is going to be the first element of `something`, i.e. the part that comes after the `0 : 1 :`. So, we need to compute the first element of `zipWith (+) fibs (tail fibs)`. How do we do this? We compute the first element of `fibs` and the first element of `tail fibs` and add them. We know already, that the first element of `fibs` is `0`. And we also know that the first element of `tail fibs` is the second element of `fibs`, which is `1` So, the first element of `zipWith (+) fibs (tail fibs)` is `0 + 1 = 1`.

4.It is going to be the fourth element of `fibs` and the `second of zipWith (+) fibs (tail fibs)`. Again, we do this by taking the second elements of `fibs` and `tail fibs` and adding them together. We know that the second element of `fibs` is `1`. The second element of `tail fibs` is the third element of `fibs`. But we just computed the third element of `fibs`, so we know it is `1`. Adding them together we get that the fourth element of fibs is `1 + 1 = 2`.

This goes on and one to generate the fibonnaci sequence. To recall, the naive

```
fibNaive 0 = 0
fibNaive 1 = 1
fibNaive n = fibNaive (n-1) + fibNaive (n-2)
```

is much slower. This is becuase the computation tree for say `fib 5` looks something like:



And one can easily see that we make a bunch of unneccesary recomputations, and thus a lot of unneccesary additions. On the other hand, using our `zipWith` method, only computes things once, and hence makes only as many additions as required.

Exercise

Try to trace the computation of `fib !! 5` and make a tree.

Let's now try using this trick to solve some harder problems.

Tromino's Pizza I

Tromino's sells slices of pizza in only boxes of 3 pieces or boxes of 5 pieces. You can only buy a whole number of such boxes. Therefore it is impossible to buy exactly 2 pieces, or exactly 4 pieces, etc. Create list `possiblePizza` such that if we can buy exactly `n` slices, `possiblePizza !! n` is `True` and `False` otherwise.

The solution revolves around the fact $f(x) = f(x-3) \vee f(x-5)$. A naive implementation could be

```
possiblePizzaGo :: Int → Bool
possiblePizzaGo x
  | x == 0    = True
  | x < 3     = False
  | x == 5    = True
  | otherwise = possiblePizzaGo (x - 3) || possiblePizzaGo (x - 5)

possiblePizza = [possiblePizzaGo x | x ← [0..]]
```

This is slow for the same reason as `fibNaive`. So what can we do? Well, use zipWith.

```
possiblePizza = True : False : False : True : False : zipWith (||)
(possiblePizza) (drop 3 possiblePizza)
```

Note, we need to define till the 5th place as otherwise the code has no way to know we can do 5 slices.

X Tromino's Pizza II

Tromino's has started to charge a box fees. So now given a number of slices, we want to know the minimum number of boxes we can achive the order in. Create a list `minBoxPizza` such that if we can buy exactly `n` slices, the list displays `Just` the minimum number of boxes the order can be achived in, and `Nothing` otherwise. The list is hence of type `[Maybe Int]`.

Hint : Create a helper function to use with the `zipWith` expression.

One more intresting thing we can talk about is higher dimensional `zip` and `zipWith`. One way to talk about them is `zip3 :: [a] → [b] → [c] → [(a,b,c)]` and `zipWith3 :: (a → b → c → d) → [a] → [b] → [c] → [d]`. These are defined exactly how might expect them to be.

```
zip3 :: [a] → [b] → [c] → [(a,b,c)]
zip3 [] _ _ = []
zip3 _ [] _ = []
zip3 _ _ [] = []
zip3 (x:xs) (y:ys) (z:zs) = (x,y,z) : zip3 xs ys zs


zipWith3 :: (a → b → c → d) → [a] → [b] → [c] → [d]
zipWith3 _ [] _ _ = []
zipWith3 _ _ [] _ = []
zipWith3 _ _ _ [] = []
zipWith3 f (x:xs) (y:ys) (z:zs) = (f x y z) : zipWith3 f xs ys zs
```

X Exercise

A slightly tiresome exercise, try to define `zip4` and `zipWith4` blind.

Haskell predefines till `zip7` and `zipWith7`. We are yet to see anything beyond `zipWith3` used in code, so this is more than enough. Also, if you truly need it, `zip8` and `zipWith8` are not that hard to define.

X Tromino's Pizza III

Tromino's has introduced a new box of size 7 slices. Now they sell $3, 5, 7$ slice boxes. They still charge the box fees. So now given a number of slices, we still want to know the minimum number of boxes we can achive the order in. Create a list `minBoxPizza` such that if we can buy exactly `n` slices, the list displays `Just` the minimum number of boxes the order can be achived in, and `Nothing` otherwise. The list is hence of type `[Maybe Int]`.

Another idea of dimension would be something that could join together two grids, something with type signature `zip2d :: [[a]] → [[b]] → [[(a,b)]]` and `zipWith2d :: (a → b → c) → [[a]] → [[b]] → [[c]]`.

```
zip2d :: [[a]] → [[b]] → [[(a,b)]]
zip2d = map zip

zipWith2d :: (a → b → c) → [[a]] → [[b]] → [[c]]
zipWith2d = zipWith . zipWith
```

The second definition should raise immediate alarms. It seems too good to be true. Let's formally check

```
zipWith . zipWith $ (a → b → c) [[a]]  [[b]]
= zipWith (zipWith (a → b → c))  [[a]] [[b]] -- Using the fact that
composition only allows one of the inputs to be pulled inside
= [
    zipWith (a → b → c) [a1] [b1],
    zipWith (a → b → c) [a2] [b2],
    ...
  ]
= [[c1], [c2], ...]
= [[c]]
```

This also implies `zip2d = zipWith.zipWith $ (\x y → (x,y))` is also a correct definition. Also surprisingly, `zipWith . zipWith . zipWith` has the type signature `(a → b → c) → [[[a]]] → [[[b]]] → [[[c]]]`. You can see where we are going with this...

X Composing zipWith's

What should the type signature and behaviour of `zipWith . zipWith . <n times> . zipWith` be? Prove it.

X Unzip

Haskell has an inbuilt function called `unzip :: [(a,b)] → ([a],[b])` which takes a list of pairs and provides a pair of list in the manner inverse of `zip`.

Try to figure out the implementation of `unzip`.

§7.3. Folding, Scaning and The Gate to True Powers

§7.3.1. Orgami of Code!

A lot of reccursion on lists has the following structure

```
g [] = v -- The vacuous case
g (x:xs) = x `f` (g xs)
```

That is, the function `g :: [a] → b` maps the empty list to a value `v`, of say type `b`, and for non-empty lists, the head of the list and the result of recursively processing the tail are combined using a function or operator `f :: a → b → b`.

Some commone examples from the inbuilt functions are:

```
sum :: [Int] → Int
sum [] = 0
sum (x:xs) = x + (sum xs)

product :: [Int] → Int
product [] = 1 -- The structure forces this choice as other wise, the
product of full lists may become incorrect.
product (x:xs) = x * (product xs)

or :: [Bool] → Bool
or [] = False -- As the structure of our implementation forces this to be
false, or otherwise, everything is true.
or (x:xs) = x || (or xs)

and :: [Bool] → Bool
and [] = True
and (x:xs) = x && (and xs)
```

We will also see a few more examples in a while, but one can notice that this is a common enough pattern. So what do we do? We abstract it.

λ **Definition of foldr**

```
foldr :: (a → b → b) → b → [a] → b
foldr _ v [] = v
foldr f v (x:xs) = x `f` (foldr f v xs)
```

This shortens our definitons to

```
sum = foldr (+) 0
product = foldr (*) 1
or = foldr (||) False
and = foldr (&&) True
```

Sometimes, we don't wish to define a base case or maybe the logic makes it so that doing so is not possible, then you use `foldr1 :: (a → b → b) → [a] → a` defined as

λ **Definition of foldr1**

```
foldr1 :: (a → a → a) → [a] → a
foldr1 _ [x] = x
foldr1 f (x:xs) = x `f` (foldr f xs)
```

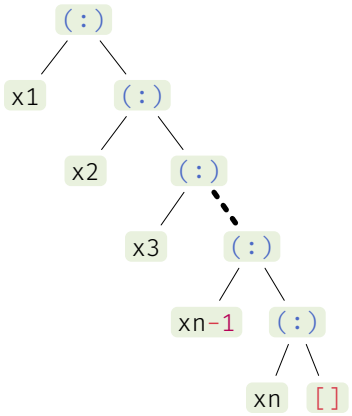
Like we could now define `product = foldr1 (*)` which is much more clean as we don't have to define a weird vacous case.

Let's now discuss the naming of the pattern. Recall, `[1,2,3,4]` is syntactic suger for `1 : 2 : 3 : 4 : []`. We are just allowed to write the former as it is more aesthetic and convinient. One could immidietly see that

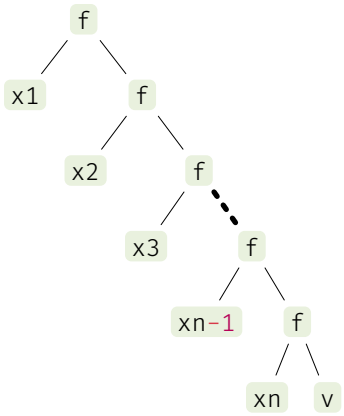
```
foldr v f [1,2,3,4] = 1 `f` (2 `f` (3 `f` (4 `f` v)))
-- and if f is right associative
= 1 `f` 2 `f` 3 `f` 4 `f` v
```

We have basically changed the cons (`:`) into the function and the empyy list (`[]`) into `v`. But notice the brackets, the evaluation is going from right to left.

Using trees, A list can be reprasented in the form



which is converted to:



However, what if our function is left associative? After all, if this was the only option, we would have called it `fold`, not `foldr` right?

The recursive pattern

```
g :: (b → a → b) → b → [a] → b
g _ v [] = v
g f v (x:xs) = g (f v x) xs
```

is abstracted to `foldl` and `foldl1` respectively.

λ Definition of foldl and foldl1

```
foldl :: (b → a → b) → b → [a] → b
foldl _ v [] = v
foldl f v (x:xs) = foldl (f v x) xs

foldl1 :: (a → a → a) → [a] → a
foldl1 f (x:xs) = foldl f x xs
```

And as the functions we saw were commutative, we could define them as

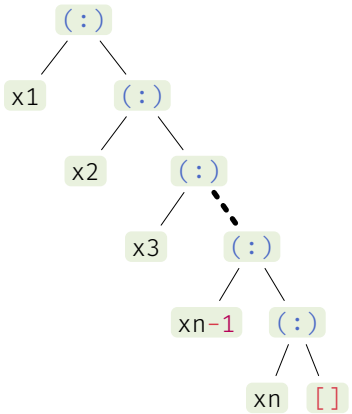
```
sum       = foldl (+) 0
product   = foldl (*) 1
or        = foldl (||) False
and       = foldl (&&) True
```

There is one another pair of function defined in the fold family called `foldl'` and `foldl1'` which are faster than `foldl` and `foldl1` and don't require a lot of working memory. This makes them the defualts used in most production code, but to understand them well, we need to discuss how haskell's lazy computation actually works and is there a way to bypass it. This is done in chapter 9. We will use `foldl` and `foldl1` till then.

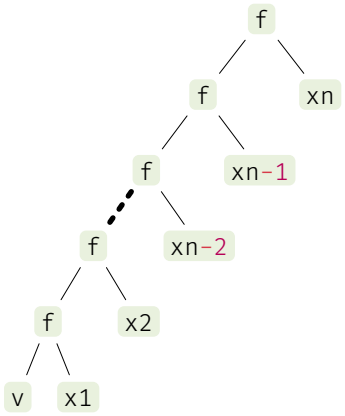
The computation of `foldl` proceeds like

```
foldl v f [1,2,3,4] = ((((v `f` 1) `f` 2) `f` 3) `f` 4)
--  and if f is left associative
= v `f` 1 `f` 2 `f` 3 `f` 4
```

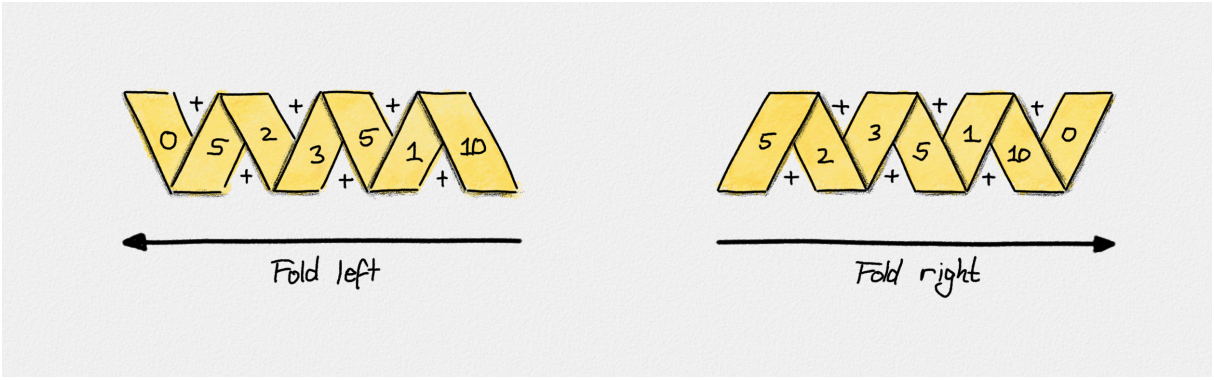
Or in tree form as



which is converted to:



Another very cute picture to summerize the diffrences is:



Similer to how `unzip` was for `zip`, could we define `unfoldr`, something that takes a generator function and a seed value and genrates a list out of it.

What could the type of such a function be? Well, like with every design problem; let's see what our requirements are:

- The list should not just be one element over and over. Thus, we need to be able to update the seed after every unfolding.
- There should be a way for the list to terminate.

So what could the type be? We can say that our function must spit out pairs: of the new seed value and the element to add to the list. But what about the second condition?

Well, what if we can spit out some seed which can never come otherwise and use that to signal it? The issue is, that would mean the definition of the function has to change from type to type.

Instead, can we use something we studied in ch-6? Maybe.[4]

λ Implementation of unfoldr

```
unfoldr :: (a → Maybe (a,b)) → a → [b]
unfoldr gen seed = go seed
  where
    go seed = case gen seed of
        Just (x, newSeed) → x : go newSeed
        Nothing           → []
```

For example, we could now define some library functions as:

[4]Pun intended.

```haskell
replicate :: Int → a → [a]
-- replicate's an value n times
replicate n x = unfoldr gen n x where
    rep 0 = Nothing
    rep m = Just (x, m - 1)

iterate :: (a → a) → a → [a]
-- given a function f and some starting value x
-- outputs the infinite list [x, f x, f f x, ...]
iterate f seed = unfold (\x → Just (x, f x)) seed
```

While `foldr` and `foldl` are some of the most common favorite function of haskell proggramers; `unfoldr` remains mostly ignored. It is so ignored that to get the inbuilt version, one has to `import Data.List`. We will soon see an eggregious case where Haskell's own website ignored it. One of the paper we referred was litrally titled "The Under-Appreciated Unfold".

X Some more inbuilt functions

Implement the following functions using fold and unfold.

(i) `concat :: [[a]] → [a]` concats a list of lists into a single list. For example: `concat [[1,2,3],[4,5,6],[7,8],[],[10]] = [1,2,3,4,5,6,7,8,9, 10]`

(ii) `cycle :: [a] → [a]` cycles through the list endlessly. For example: `cycle [2, 3, 6, 18] = [2, 3, 6, 18, 2, 3, ...]`

(iii) `filter :: (a → Bool) → [a] → [a]` takes a predicate and a list and filters out the elements satisfying that predicate.

(iv) `concatMap :: (a → [b]) → [a] → [b]` maps a function over all the elements of a list and concatenate the resulting lists. Do not use `map` in your definition.

(v) `length :: [a] → Int` gives the number of elements in the provided list. Use `foldr` or `foldl`.

X Base Conversion

(i) Comvert list of digits in base `k` to a number. That is `lis2num :: Int → [Int] → Int` with the usage `lis2num base [digits]`.

(ii) Given a number in base 10, convert to a list of digits in base `k`. `num2lis :: Int → Int → [Int]` with the usage `num2list base numberInBase10`

Let's go part by part. The idea of the first question is simply to understand that `[4,2,3]` in base `k` reprasents $4 * k^2 + 2 * k + 3 * k^0 = ((0 * k + 4) * k + 2) * k + 3$; doesn't this smell like `foldl`?

```haskell
lis2num :: [Int] → Int → Int
lis2num k = foldl (\x y → k * x + y) 0
```

For part two, the idea is that we can base convert using repeated division. That is,

```
423 `divMod` 10 = (42, 3)
42 `divMod` 10 = (4, 2)
4 `divMod` 10 = (0, 4)
```

It is clear that we terminate when the quotient reaces 0 and then just take the remainders. Does this sound like `unfoldr`?

```
num2lis :: Int → Int → [Int]
num2lis k = reverse . unfoldr gen where
  gen 0 = Nothing
  gen x = Just $ (x `mod` k, x `div` k)
```

X A list of Primes

This is the time when Haskell itself forgot that the `unfoldr` function exists. The website offers the following method to make a list of primes in Haskell as an advertisment for the language.

```
primes = filterPrime [2..] where
filterPrime (p:xs) =
  p : filterPrime [x | x ← xs, x `mod` p /= 0]
```

Understand this code (write a para explaining exactly what is happening!) and try to define a shorter (and more aesthetic) version using `unfoldr`.

The answer is litrally doing what one would do on paper. Like describing it would be a disservice to the code.

λ list of primes using unfoldr

```
sieve (x:xs) = Just (x, filter (\y → y `mod` x /= 0) xs)
primes = unfoldr sieve [2..]
```

X Subsequences

Write a function `subslists :: [a] → [[a]]` which takes a list and returns a list of sublists of the given list. For example: `sublists "abc" = "","a","b","ab","c","ac","bc","abc"` and `sublists [24, 24] = [[],[24],[24],[24,24]]`.

Try to use the fact that a sublist either contains an element or not. Second, the fact that sublists correspond nicely to binery numerals may also help.

You function must be compatable with infinite lists, that is `take 10 $ sublists [1..] = [[], [1], [2], [1,2], [3], [1,3], [2,3], [1,2,3], [4], [1,4]]` should work.

Please fill in the blanks below

A naive, non-infinite compatable definiton is:

```
sublists [] = ______
subslists (x:xs) = concatMap (\ys → ______) (sublists xs)
```

On an infinite list, this definition gets stuck in an non-productive loop because we must traverse the entire list before it returns anything.

Note that on finite cases, the first sublist returned is always `_____`. This means we can state this as `sublists xs = _____ : _____ (sublists xs)`. It is sensible to extend this equality to the infinite case, due to the analogy of `_____`.

By making this substitution, we produce the definition that can handle infinite lists, from which we can calculate a definition that's more aesthetically pleasing and slightly more efficient:

```
_____ -- Base case
sublists (x:xs)  = _____ : _____ : concatMap (\ys → _____) (tail .
sublists xs)
```

We can clean this definition up by calculating definitions for `tail.sublists x` and renaming it something like `nonEmpties`. We start by applying tail to both sides of the two cases. `nonEmpties [] = tail.sublists [] = _____` and `nonEmpties (x:xs) = tail.sublists (x:xs) = _____`

Substituting all thins through the definition.

λ Space to write the definition of sublists

This function can be called in Haskell through the `subsequences` function one gets on importing `Data.Lists`. Our definition is the most efficient and is what is used internally.

Finally, a question which would require you to use a lot of functions we just defined:

X The Recap Problem (Euler's Project 268)

It can be verified that there are 23 positive integers less than 1000 that are divisible by at least four distinct primes less than 100.

Find how many positive integers less than 10^{16} are divisible by at least four distinct primes less than 100.

Hint : Thik about PIE but not π.

Something we mentioned was that `foldr` and `unfoldr` are inverse (or more accutately duel) of each other. But their types seem so different. How do we reconcile this?

$$\begin{aligned}
\text{foldr} :: (a \to b \to b) \to b &\to [a] \to b \\
&\cong (a \times b \to b) \to b &\to [a] \to b \\
&\cong (a \times b \to b) \to (1 \to b) &\to [a] \to b \\
&\cong (a \times b \cup 1 \to b) &\to [a] \to b \\
&\cong (\text{Maybe}(a, b) \to b) &\to [a] \to b \\
\text{Notice, } \text{unfoldr} :: (b \to \text{Maybe}(a, b)) &\to b \to [a]
\end{aligned}$$

And now the duality emerges. $(\text{foldr } f)^{-1} = \text{unfoldr } f^{-1}$.

Some more ideas on the nature of fold can be found in the upcoming chapters on datatypes as well as in the appendix.

Another type of function we sometimes want to define are:

```
sumlength :: [Int] → (Int,Int)
sumlength xs = (sum xs, length xs)
```

This is bad as we traverse the list twice. We could do this twice as fast using

```
sumlength :: [Int] → (Int, Int)
sumlength = foldr (\x (a,b) → (a+x, b + 1)) (0,0)
```

This might seem simple enough, but this idea can be taken to a diffrent level rather immidietly.

X Ackerman Function

The Ackerman function is defined as follows:

```
ack :: [Int] → [Int] → [Int]
ack [] ys = 1 : ys
ack (x : xs) [] = ack xs [1]
ack (x : xs) (y : ys) = ack xs (ack (x : xs) ys)
```

Define this in one single line using `foldr`.

Let's say `foldr f v` ≅ `ack`.

```
⇒ ack [] = v
⇒ ack (x:xs) = f x (ack xs)
```

This means `v = (1:)`. unfortunatly, figuring out `f` seems out of reach. Luckily, we are yet to use all the information the function provides.

Let's say `foldr g w` ≅ `ack (x:xs)`.

```
⇒ ack (x:xs) [] = w
⇒ ack (x:xs) (y:ys) = g y (ack (x:xs) ys)
```

This means `w = ack xs [1]` and

```
ack (x:xs) (y:ys)
  = g y (ack (x:xs) ys) ⟺  ack xs (ack (x : xs) ys)
  (canceling on both sides)
  ⇒ g y = ack xs
  ⇒ g = (\y z → ack xs z)
```

Thus, `g = (\y z→ ack xs z)`.

And finally, now working towards `f`, we get

```
ack (x:xs)
  = f x (ack xs) ⟺ foldr (\y z→ ack xs z) (ack xs [1])
  (substitution of a = ack xs)
  ⇒ f x a ⟺ foldr (\y z → a z) (a [1])
  ⇒ f = (\x a → foldr (\y z → a z) (a [1]))
```

This gives us the definiton

```
ack :: [Int] → [Int] → [Int]
ack = foldr (\x a → foldr (\y z → a z) (a [1])) (1:)
```

This might seem like a rather messy definition, but from a theoretical point of view, even this has it's importence. The main thing is that folding is faster than recursion at runtime so if no additional overhead is there, folds will run faster.

It is possible, but out of the scope of our current undertaking, to prove that all primitive recursive functions can be written as folds. What does primitive recursive functions mean? Well, that is left for your curiosity.

X Removing duplicates

Haskell has inbuilt function `nub :: Eq a ⇒ [a] → [a]` which is used to remove duplicates in a list. Write a recursive definition of `nub` and then write a definition using folds.

Haskell also has an inbuilt function `nubBy :: (a → a → Bool) → [a] → [a]` which is used to remove elements who report true to some property. That is `nubBy (\x y → x + y == 4) [1,2,3,4,2, 0] = [1,2,4]` as `1+3 = 4, 2+2 = 4, 4 + 0 = 4`. Write a recursive definition of `nubBy` and then write a definition using folds.

X More droping and more taking

`dropWhile :: (a → Bool) → [a] → [a]`

and `takeWhile :: (a → Bool) → [a] → [a]` take a predicate and a list and drop all elements while the predicate is satisfied and take all objects while the predicate is satisfied respectively.

Implement them using recursion and then using folds.

§7.3.2. Numerical Integration

To quickly revise all the things we just learnt, we will try to write our first big-boy code.

Let's talk about numerical Integration. Numerical Integration refers to finding the value of integral of a function, given the limits. This is also a part of the mathematical computing we first studied in chapter 3. To get going, a very naive idea would be:

```
easyIntegrate :: (Float → Float) → Float → Float → Float
easyIntegrate f a b = (f b + f a) * (b-a) / 2
```

This is quite inaccurate unless `a` and `b` are close. We can be better by simply dividing the integral in two parts, ie $\int_a^b f(x)\partial x = \int_a^m f(x)\partial x + \int_m^b f(x)\partial x$ where $a < m < b$ and approximate these parts. Given the error term is smaller in these parts than that of the full integral, we would be done. We can make a sequence converging to the integral we are intrested in as:

λ Naive Integration

```
integrate :: (Float → Float) → Float → Float → [Float]
integrate f a b = (easyIntegrate f a b) : zipWith (+) (integrate f a m)
(integrate f m b) where m = (a+b)/2
```

If you are of the kind of person who likes to optimize, you can see a very simple inoptimality here. We are computing `f m` far too many times. Considering, `f` might be slow in itself, this seems like a bad idea. What do we do then? Well, ditch the aesthetic for speed and make the naive integrate as:

λ **Naive Integration without repeated computation**

```
integrate f a b = go f a b (f a) (f b)

integ f a b fa fb = ((fa + fb) * (b-a)/2) : zipWith (+) (integ f a m fa fm)
(integ f m b fm fb) where
  m = (a + b)/2
  fm = f m
```

This process is unfortunatly rather slow to converge for a lot of fucntions. Let's call in some backup from math then.

The elements of the sequence can be expressed as the correct answer plus some error term, ie $a_i = A + \mathrm{E}$. This error term is roughly propotional to some power of the seperation between the limits evaluated (ie $(b-a)$, $\frac{b-a}{2}$...) (the proof follows from Taylor exmapnsion of f. You are reccomended to prove the same). Thus,

$$a_i = A + B \times \left(\frac{b-a}{2^i}\right)^n$$

$$a_{i+1} = A + B \times \left(\frac{b-a}{2^{i+1}}\right)^n$$

$$\Rightarrow a_{i+1} - \frac{1}{2^n} a_i = A\left(1 - \frac{1}{2^n}\right)$$

$$\Rightarrow A = \frac{2^n \times a_{i+1} - a_i}{2^n - 1}$$

This means we can improve our sequence by eliminating the error

```
elimerror :: Int → [Float] → [Float]
elimerror n (x:y:xs) = (2^^n * y - x) / (2^^n - 1) : elimerror n (y:xs)
```

However, we have now found a new problem. How in the world do we get `n`?

$$
\begin{aligned}
a_i &= A + B \times \left(\frac{b-a}{2^i}\right)^n \\
a_{i+1} &= A + B \times \left(\frac{b-a}{2^{i+1}}\right)^n \\
a_{i+2} &= A + B \times \left(\frac{b-a}{2^{i+2}}\right)^n \\
\Rightarrow a_i - a_{i+1} &= B \times \left(\frac{b-a}{2^i}\right)^n \times \left(1 - \frac{1}{2^n}\right) \\
\Rightarrow a_{i+1} - a_{i+2} &= B \times \left(\frac{b-a}{2^i}\right)^n \times \left(\frac{1}{2^n} - \frac{1}{4^n}\right) \\
\Rightarrow \frac{a_i - a_{i+1}}{a_{i+1} - a_{i+2}} &= \frac{4^n - 2^n}{2^n - 1} = \frac{2^n(2^n-1)}{2^n - 1} = 2^n \\
\Rightarrow n &= \log_2\left(\frac{a_i - a_{i+1}}{a_{i+1} - a_{i+2}}\right)
\end{aligned}
$$

Thus, we can estimate `n` using the function `order`. We will be using the in-built function `round :: (RealFrac a, Integral b) → a → b` in doing so. In our case, `round :: Float → Int`.

```
order :: [Float] → Int
order (x:y:z:xs) = round $ logBase 2 $ (x-y)/(y-z)
```

This allows us to improve our sequence

```
improve :: [Float] → [Float]
improve xs = elimerror (order xs) xs
```

One could make a very fast converging sequence as say `improve $ improve $ improve $ integrate f a b`.

But based on the underlying function, the number of `improve` may differ.

So what do we do? We make an extreamly clever move to define a super sequence `super` as

```
super :: [Float] → [Float]
super xs = map (!! 2) (iterate improve xs) -- remeber iterate from the
excercises above?
```

I will re-instate, the implementation of `super` is extreamly clever. We are recursivly getting a sequence of more and more improved sequences of approximations and constructs a new sequence of approximations by taking the second term from each of the improved sequences. It turns out that the second one is the best one to take. It is more accurate than the first and doesn't require any extra work to compute. Anything further, requires more computations to compute.

Finally, to complete our job, we define a function to choose the term upto some error.

```haskell
within :: Float → [Float] → Float
within error (x:y:xs)
  | abs(x-y) < error = y
  | otherwise = within error (y:xs)
```

λ An optimalized function for numerical integration

```haskell
ans :: (Float → Float) → Float → Float → Float → Float
ans f a b error = within error $ super $ integrate f a b
```

With this we are done!

X Simpson's Rule

Here we have used the approximation $\int_a^b f(x)\,\mathrm{d}x = (f(a)+f(b))\frac{b-a}{2}$ and used divide and conquor. This is called the Trapazoidal Rule in Numerical Analysis.

A better approximation is called the Simpson's (First) Rule.

$$\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Modify the code to now use Simpson's Rule. Furthermore, show that this approximation makes sense (the idea is to find a quadratic polynomial which takes the same value as our function at a, $\frac{a+b}{2}$ and b and using its area).

§7.3.3. Time to Scan

We will now talk about folds lesser known cousing scans.

Scans

While fold takes a list and compresses it to a single value, scan takes a list and makes a list of the partial compressions. Basically,

```
scanr :: (a → b → b) → b → [a] → [b]
scanr f v [x1, x2, x3, x4]
  = [
      foldr f v [x1, x2, x3, x4],
      foldr f v [x2, x3, x4],
      foldr f v [x3, x4],
      foldr f v [x4],
      foldr f v []
    ]
  = [
      x1 `f` x2 `f` x3 `f` x4 `f` v,
      x2 `f` x3 `f` x4 `f` v,
      x3 `f` x4 `f` v,
      x4 `f` v,
      v
    ]
```

and very much similerly as

```
scanl :: (b → a → b) → b → [a] → [b]
scanl f v [x1, x2, x3, x4]
  = [
      foldl f v [],
      foldr f v [x1],
      foldr f v [x1, x2],
      foldr f v [x1, x2, x3],
      foldr f v [x1, x2, x3, x4]
    ]
  = [
      v,
      v `f` x1,
      v `f` x1 `f` x2,
      v `f` x1 `f` x2 `f` x3,
      v `f` x1 `f` x2 `f` x3 `f` x4,
    ]
```

There are also very much similer `scanr1` and `scanl1`.[5]

The reason the naming is as the internal implementation of these funtions look like similer to the definition of the fold they borrow their name from.

[5]Similer to our note in fold, there is a function pair scanl' and scanl1', which similer to foldl' and foldl1', and have the same set of benefits. This makes them the defualts, but similerly, to understand them well, we need to discuss how haskell's lazy computation actually works and the way to bypass it. This is done in chapter 9.

```haskell
scanr :: (a → b → b) → b → [a] → [b]
scanr _ v [] = [v]
scanr f v (x:xs) = x `f` (head part) : part where part = scanr f v xs

scanl :: (b → a → b) → b → [a] → [b]
scanl _ v [] = [v]
scanl f v (x:xs) = v : scanl f (v `f` x) xs
```

Scan as a fold

We can define `scanr` using `foldr`, try to figure out a way to do so.

Defining scanl1 and scanr1

Modify these definitions and define `scanl1` and `scanr1`.

This seems like a much more convoluted recursion pattern. So why have we decided to study it? Let's see by example

Not Quite Lisp (AOC 2015, 1)

Santa is trying to deliver presents in a large apartment building, but he can't find the right floor - the directions he got are a little confusing. He starts on the ground floor (floor 0) and then follows the instructions one character at a time.

An opening parenthesis, `(`, means he should go up one floor, and a closing parenthesis, `)`, means he should go down one floor.

The apartment building is very tall, and the basement is very deep; he will never find the top or bottom floors.

For example:

- `(())` and `()()` both result in floor 0.
- `(((` and `(()(()(` both result in floor 3.
- `))(((((` also results in floor 3.
- `())` and `))(` both result in floor −1 (the first basement level).
- `)))` and `)())())` both result in floor −3.

Write a function `parse :: String → Int` which takes the list of parenthesis as a sting in input and gives the correct integer as output.

This is quite simple using folds.

```haskell
parse :: String → Int
parse = foldl (\x y → if y == '(' then x+1 else x - 1) 0
```

But every AOC question always has a part 2!

Not Quite Lisp, 2

Now, given the same instructions, find the position of the first character that causes him to enter the basement (floor −1). The first character in the instructions has position 1, the second character has position 2, and so on.

For example:

- `)` causes him to enter the basement at character position 1.
- `()())` causes him to enter the basement at character position 5.

Make a function `ans` which takes the list of parenthesis as a sting in input and output the position(1 indexed) of the first character that causes Santa to enter the basement

If we had no idea of scans, this would be harder. In this case, it is just a simple replacement.

```
ans :: String → Int
ans = length.takeWhile (/= -1).scanl (\x y → if y == '(' then x+1 else x -
1) 0
```

The `takewhile` chooses all the floors we reach before −1. As 0th floor is counted (as it is the scan on empty list), we will have a list of `length` as much as the position of the character that caused us to enter −1.

Now, here is a conincidence we didn't expect. We were told that AOC 2015's first question was a good `foldl` to `scanl` example. What I was not preperaed to see was scanning showing up in AOC 2015's third question as well.

Perfectly Spherical Houses in a Vacuum (AOC 2015)

Santa is delivering presents to an infinite two-dimensional grid of houses.

He begins by delivering a present to the house at his starting location, and then an elf at the North Pole calls him via radio and tells him where to move next. Moves are always exactly one house to the north (^), south (v), east (>), or west (<). After each move, he delivers another present to the house at his new location.

However, the elf back at the north pole has had a little too much eggnog, and so his directions are a little off, and Santa ends up visiting some houses more than once. How many houses receive at least one present?

For example:

- `>` delivers presents to 2 houses: one at the starting location, and one to the east.
- `^>v<` delivers presents to 4 houses in a square, including twice to the house at his starting/ending location.
- `^v^v^v^v^v` delivers a bunch of presents to some very lucky children at only 2 houses.

Create function `solve1 :: String → Int` which takes the list of instructions as string in input and outputs the number of hourses visited.

Perfectly Spherical Houses in a Vacuum II

The next year, to speed up the process, Santa creates a robot version of himself, Robo-Santa, to deliver presents with him.

Santa and Robo-Santa start at the same location (delivering two presents to the same starting house), then take turns moving based on instructions from the elf, who is eggnoggedly reading from the same script as the previous year.

This year, how many houses receive at least one present?

For example:

- `^v` delivers presents to 3 houses, because Santa goes north, and then Robo-Santa goes south.
- `^>v<` now delivers presents to 3 houses, and Santa and Robo-Santa end up back where they started.
- `^v^v^v^v^v` now delivers presents to 11 houses, with Santa going one direction and Robo-Santa going the other.

Create function `solve2 :: String → Int` which takes the list of instructions as string in input and outputs the number of hourses visited.

We will also breifly talk about something called Segmented Scan.

Segmented Scan

A scan can be broken into segments with flags so that the scan starts again at each segment boundary. Each of these scans takes two vectors of values: a data list and a flag list. The segmented scan operations present a convenient way to execute a scan independently over many sets of values.

For example, a segmented looks like is:

```
1 2 3 4 5 6 -- Input
T F F T F T -- Flag
1 3 6 4 9 6 -- Result
```

We will name this function `segScan :: (a → a → b) → [Bool] → [a] → [b]`.

The implementation of function is as follows

```
segScan :: (a → a → b) → [Bool] → [a] → [b]
segScan f flag str = scanl (\r (x,y) → if x then y else r `f` y) (head str)
(tail (zip flag str))
```

This might seem complex but we are merely `zip`-ing the flags and input values, and defining a new function, say `g` which applies the function `f`, but resets to `y` (the new value) whenever `x` (the flag) is `True`. The `head` and `tail` are to ensure that the first element is the beginning of the first segment.

This will be the end of my discussion of this. The major use of segmented scan is in parallel computation algorithms. A rather complex quick sort parallel algorithm can be created using this as the base.

§7.4. Excercises

X Factors

(i) Write an optimized function `factors :: Int → [Int]` which takes in an integer and provides a list of all it's factors.

(ii) Write an optimized function `primeFactors :: Int → [Int]` which takes in an integer and provides a list of all it's prime factors, repeated wrt to multiplicity. That is `primeFactors 100 = [2,2,5,5]`.

X Trojke (COCI 2006, P3)

Mirko and Slavko are playing a new game, Trojke (Triplets). First they use a chalk to draw an $N \times N$ square grid on the road. Then they write letters into some of the squares. No letter is written more than once in the grid.

The game consists of trying to find three letters on a line as fast as possible. Three letters are considered to be on the same line if there is a line going through the centre of each of the three squares (horizontal, vertical and diagonal).

After a while, it gets harder to find new triplets. Mirko and Slavko need a program that counts all the triplets, so that they know if the game is over or they need to search further.

Write a function `trojke :: [String] → Int` which takes the contents of the lines and outputs the numper of lines.

Example:

```
trojke [
  "...D",
  "..C.",
  ".B..",
  "A..."
] = 4

trojke [
"..T..",
"A....",
".FE.R",
"....X",
"S...."
] = 3

trojke [
"....AB....",
"..C....D..",
".E......F.",
"...G..H...",
"I........J",
"K........L",
"...M..N...",
".O......P.",
"..Q....R..",
"....ST...."
] = 0
```

X Deathstar (COCI 2015, P3)

Young jedi Ivan has infiltrated in The Death Star and his task is to destroy it. In order to destroy The Death Star, he needs an array A of non-negative integers of length N that represents the code for initiating the self-destruction of The Death Star. Ivan doesn't have the array, but he has a piece of paper with requirements for that array, given to him by his good old friend Darth Vader.

On the paper, a square matrix of the size is written down. In that matrix, in the row i and column j there is a number that is equal to bitwise and between numbers a_i and a_j. Unfortunately, a lightsaber has destroyed all the fields on the matrix's main diagonal and Ivan cannot read what is on these fields. Help Ivan to reconstruct an array for the self-destruction of The Death Star that meets the requirements of the matrix.

The solution doesn't need to be unique, but will always exist. Your function `destroy :: [[Int]] → [Int]` only needs to find one of these sequences.

Example:

```
destroy [
  [0,1,1],
  [1,0,1],
  [1,1,0]] = [1,1,1]
destroy [
  [0,0,1,1,1]
  [0,0,2,0,2]
  [1,2,0,1,3]
  [1,0,1,0,1]
  [1,2,3,1,0]] = [1,2,3,1,11]
```

Extra Credit : There is a way to do this in one line.

Nucleria (CEOI 2015 P5

Long ago, the people of Nuclearia decided to build several nuclear plants. They prospered for many years, but then a terrible misfortune befell them. The land was hit by an extremely strong earthquake, which caused all the nuclear plants to explode, and radiation began to spread throughout the country. When the people had made necessary steps so that no more radiation would emanate, the Ministry of Environment started to find out how much individual regions were polluted by the radiation. Your task is to write a function `quary :: (Int, Int) → [(Int, Int, Int, Int)] → Float` that will find the average radiation in Nuclearia given data.

Nuclearia can be viewed as a rectangle consisting of $W \times H$ cells. Each nuclear plant occupies one cell and is parametrized by two positive integers: a, which is the amount of radiation caused to the cell where the plant was, and b, which describes how rapidly the caused radiation decreases as we go farther from the plant.

More precisely, the amount of radiation caused to cell $C = (x_C, y_C)$ by explosion of a plant in cell $P = (x_P, y_P)$ is $\max(0, a - b \cdot d(P, C))$, where $d(P, C)$ is the distance of the two cells, defined by $d(P, C) = \max(|x_P - x_C\ |, |y_P - y_C\ |)$ (i.e., the minimum number of moves a chess king would travel between them).

The total radiation in a cell is simply the sum of the amounts that individual explosions caused to it. As an example, consider a plant with $a = 7$ and $b = 3$. Its explosion causes 7 units of radiation to the cell it occupies, 4 units of radiation to the 8 adjacent cells, and 1 unit of radiation to the 16 cells whose distance is 2.

The Ministry of Environment wants to know the average radiation per cell. The input will be in the form `quary (W, H) [(x1,y1,a1,b1), (x2,y2,a2,b2)]` where we first give the size of Nucleria and then the position of plants and their paramenter.

Example:

```
quary (4,3) [(1,1,7,3),(3,2,4,2)] = 3.67
```

The radiation in Nuclearia after the two explosions is as follows:

$$\begin{matrix} 7632 \\ 4652 \\ 1332 \end{matrix}$$

X Garner's Algorithm

A consequence of the Chinese Remainder Theorem is, that we can represent big numbers using an array of small integers. For example, let p be the product of the first 1000 primes. p has around 3000 digits.

Any number a less than p can be represented as a list $a_1, ..., a_k$, where $a_i \equiv a \bmod(p_i)$. But to do this we obviously need to know how to get back the number a from its representation (which we will call the CRT form).

Another form for numbers is called the mixed radix form. We can represent a number a in the mixed radix representation as:$a = x_1 + x_2p_1 + x_3p_1p_2 + ... + x_kp_1...p_{k-1}$ with $x_i \in [0, p_i)$

(i) Make a list of first 1000 primes. Call it `primeThousand :: [Int]`.

(ii) Write function `encode :: Int → [Int]` which encodes a number into the CRT form.

(iii) You had constructed an extreamly fast way to compute modulo inverses in X **Modulo Inverse**. Use it to create `residue :: [[Int]]` such that r_{ij} denotes he inverse of p_i modulo p_j.

(iv) Garner's algorithm converts from the CRT from to the mixed radix form. We want you to implement `garner :: [Int] → [Int]`. The idea of the algorithm is as follows:

Substituting a from the mixed radix representation into the first congruence equation we obtain

$$a_1 \equiv x_1 \bmod(p_1).$$

Substituting into the second equation yields

$$a_2 \equiv x_1 + x_2p_1 \bmod(p_2),$$

which can be rewritten by subtracting x_1 and dividing by p_1 to get

$$\begin{aligned} a_2 - x_1 &\equiv x_2p_1 && \bmod(p_2) \\ (a_2 - x_1)r_{12} &\equiv x_2 && \bmod(p_2) \\ x_2 &\equiv (a_2 - x_1)r_{12} && \bmod(p_2) \end{aligned}$$

Similarly we get that

$$x_3 \equiv ((a_3 - x_1)r_{13} - x_2)r_{23} \bmod(p_3).$$

(v) Finally, now write a function `decodeMixed :: [Int] → Int` which decodes from the mixed radix form.

(vi) Finally, combine these functions and write a `decode :: [Int] → Int` which decodes from CRT form.

Note : We find it extreamly cool to know that so much of math goes on in simply reprasenting big integers in high accuracy systems. Like from airplane cockpits to rocker simulations to some games like Valorent, a very cool algorithm is keeping it up and running.

X Shanks Baby Steps-Giant Steps algorithm

A surprisingly hard problem is, given a, b, m computing x such that $a^x \equiv b \bmod m$ efficiently. This is called the discrete logirithm. We will try to walk through implementing an algorithm to do so efficiently. At the end, you are expected to make a function `dlog :: Int → Int → Int → Maybe Int` which takes in a, b and m and returns x such that $a^x \equiv b \bmod m$ if it exists.

Let $x = np - q$, where n is some pre-selected constant (by the end of the description, we want you to think of how to choose n).

We can also see that $p \in \{1, 2, ..., \lceil \frac{m}{n} \rceil\}$ and $q \in \{0, 1, ..., n-1\}$.

p is known as giant step, since increasing it by one increases x by n. Similarly, q is known as baby step. Try to find the bounds

Then, the equation becomes: $a^{np-q} \equiv b \bmod m$.

Using the fact that a and m are relatively prime, we obtain:

$a^{np} \equiv ba^q \bmod m$

So we now need to find the p and q which satisfies this. Well, that can be done quite easily, right?

Keeping in mind X **Moduler Exponation**, what n should we choose to be most optimal?

X A very cool DP (Codeforces)

Giant chess is quite common in Geraldion. We will not delve into the rules of the game, we'll just say that the game takes place on an h × w field, and it is painted in two colors, but not like in chess. Almost all cells of the field are white and only some of them are black. Currently Gerald is finishing a game of giant chess against his friend Pollard. Gerald has almost won, and the only thing he needs to win is to bring the pawn from the upper left corner of the board, where it is now standing, to the lower right corner. Gerald is so confident of victory that he became interested, in how many ways can he win?

The pawn, which Gerald has got left can go in two ways: one cell down or one cell to the right. In addition, it can not go to the black cells, otherwise the Gerald still loses. There are no other pawns or pieces left on the field, so that, according to the rules of giant chess Gerald moves his pawn until the game is over, and Pollard is just watching this process.

Write a function `wins :: (Int, Int) → [(Int,Int)] → Int` to compute the number of ways to win on a (w, h) grid with black squares at given coordinates. The pawn starts at $(1, 1)$ and we must go till (w, h).

Examples

```
wins (3,4) [(2,2),(2,3)] = 2
wins (100,100) [(15,16),(16,15),(99,98)] = 545732279
```

Hint : This is a very hard question to do optimally. The 'standard' way to do so would be making a function `ways :: (Int,Int) → Integer` and counting the ways to every square recursively and setting the black squares as 0.

This is not optimal if we have a huge grid. Here the idea is to sort the black squares lexiographically. Let this sorted list be $b_1, b_2, ..., b_n$. We add $b_{n+1} = (w, h)$. Let the paths (ignoring black squares) from $(1, 1)$ to b_i be d_i. Let the paths respecting black squares be c_i. Our goal is to find c_{n+1}.

Also try defining a function `paths :: (Int, Int) → (Int, Int) → Int` which counts the paths from (x_1, y_1) to x_2, y_2 without any black squares. This function should give us d_i's. Can we use these d_i and the `paths` function to get c_i's?

X Mars Rover (Codeforces)

If you felt bad that I gave a hint in the last problem, here is a similar problem for you to do all on your own.

Research rover finally reached the surface of Mars and is ready to complete its mission. Unfortunately, due to the mistake in the navigation system design, the rover is located in the wrong place.

The rover will operate on the grid consisting of n rows and m columns. We will define as (r, c) the cell located in the row r and column c. From each cell the rover is able to move to any cell that share a side with the current one.

The rover is currently located at cell $(1, 1)$ and has to move to the cell (n, m). It will randomly follow some shortest path between these two cells. Each possible way is chosen equiprobably.

The cargo section of the rover contains the battery required to conduct the research. Initially, the battery charge is equal to s units of energy.

Some of the cells contain anomaly. Each time the rover gets to the cell with anomaly, the battery looses half of its charge rounded down. Formally, if the charge was equal to x before the rover gets to the cell with anomaly, the charge will change to $\lceil (\frac{x}{2}) \rceil$.

While the rover picks a random shortest path to proceed, write function `charge :: (Int, Int) → [(Int, Int)] → Int → Float` to compute the expected value of the battery charge after it reaches cell (n, m), with the anomalies at some positions $[(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)]$ if we started with some c charge.

Note: If the cells $(1, 1)$ and (n, m) contain anomaly, they also affect the charge of the battery.

Examples

```
charge (3,3) [(2,1),(2,3)] 11 = 6.33333333333
```

X Vegetables (ZCO 2024)

You are a farmer, and you want to grow a wide variety of vegetables so that the people in your town can eat a balanced diet.

In order to remain healthy, a person must eat a diet that contains N essential vegetables, numbered from 1 to N . In total, your town requires A_i units of each vegetable i , for $1 \leq i \leq N$. In order to grow a single unit of vegetable i , you require B_i units of water.

However, you can use upgrades to improve the efficiency of your farm. In a single upgrade, you can do one of the following two actions:

1. You can improve the nutritional value of your produce so that your town requires one less unit of some vegetable i . Specifically, you can choose any one vegetable i such that $A_i \geq 1$, and reduce A_i by 1 .
2. You can improve the quality of your soil so that growing one unit of some vegetable i requires one less unit of water. Specifically, you can choose any one vegetable i such that $B_i \geq 1$, and reduce B_i by 1.

If you use at most X upgrades, what is the minimum possible number of units of water you will need to feed your town? Write a function `water :: [Int] → [Int] → Int → Int` to answer this where the first list is A, second is B and the integer is X.

X de Polignac Numbers (Rosetta Code)

Alphonse de Polignac, a French mathematician in the 1800s, conjectured that every positive odd integer could be formed from the sum of a power of 2 and a prime number.

He was subsequently proved incorrect.

The numbers that fail this condition are now known as de Polignac numbers.

Technically 1 is a de Polignac number, as there is no prime and power of 2 that sum to 1. De Polignac was aware but thought that 1 was a special case. However, 127 is also fails that condition, as there is no prime and power of 2 that sum to 127.

As it turns out, de Polignac numbers are not uncommon, in fact, there are an infinite number of them.

- Find and display the first fifty de Polignac numbers.
- Find and display the one thousandth de Polignac number.
- Find and display the ten thousandth de Polignac number.

X

The Bifid cipher is a polygraphic substitution cipher which was invented by Félix Delastelle in around 1901. It uses a 5 x 5 Polybius square combined with transposition and fractionation to encrypt a message. Any 5 x 5 Polybius square can be used but, as it only has 25 cells and there are 26 letters of the (English) alphabet, one cell needs to represent two letters - I and J being a common choice.

Operation Suppose we want to encrypt the message “ATTACKATDAWN”.

We use this archetypal Polybius square where I and J share the same position.

```
x/y 1 2 3 4 5
-- ──────────
1 │ A B C D E
2 │ F G H I K
3 │ L M N O P
4 │ Q R S T U
5 │ V W X Y Z
```

The message is first converted to its x, y coordinates, but they are written vertically beneath.

```
A T T A C K A T D A W N
1 4 4 1 1 2 1 4 1 1 5 3
1 4 4 1 3 5 1 4 4 1 2 3
```

They are then arranged in a row.

```
1 4 4 1 1 2 1 4 1 1 5 3 1 4 4 1 3 5 1 4 4 1 2 3
```

Finally, they are divided up into pairs which are used to look up the encrypted letters in the square.

```
14 41 12 14 11 53 14 41 35 14 41 23
D  Q  B  D  A  X  D  Q  P  D  Q  H
```

The encrypted message is therefore “DQBDAXDQPDQH”.

Decryption can be achieved by simply reversing these steps.

Write functions in haskell to encrypt and descrypt a message using the Bifid cipher.

Use them to verify (including subsequent decryption):

- The above example.
- The example in the Wikipedia article using the message and Polybius square therein.
- The above example but using the Polybius square in the Wikipedia article to illustrate that it doesn’t matter which square you use as long, of course, as the same one is used for both encryption and decryption.

In addition, encrypt and decrypt the message “The invasion will start on the first of January” using any Polybius square you like. Convert the message to upper case and ignore spaces.

X Colorful Numbers (Rosetta Code)

A colorful number is a non-negative base 10 integer where the product of every sub group of consecutive digits is unique.

For example: 24753 is a colorful number. $2, 4, 7, 5, 3, (2 \times 4)8, (4 \times 7)28, (7 \times 5)35, (5 \times 3)15, (2 \times 4 \times 7)56, (4 \times 7 \times 5)140, (7 \times 5 \times 3)105, (2 \times 4 \times 7 \times 5)280, (4 \times 7 \times 5 \times 3)420, (2 \times 4 \times 7 \times 5 \times 3)840$

Every product is unique.

2346 is not a colorful number. $2, 3, 4, 6, (2 \times 3)6, (3 \times 4)12, (4 \times 6)24, (2 \times 3 \times 4)48, (3 \times 4 \times 6)72, (2 \times 3 \times 4 \times 6)144$

The product 6 is repeated.

Single digit numbers are considered to be colorful. A colorful number larger than 9 cannot contain a repeated digit, the digit 0 or the digit 1. As a consequence, there is a firm upper limit for colorful numbers; no colorful number can have more than 8 digits.

- Write a function to test if a number is a colorful number or not.
- Use that function to find all of the colorful numbers less than 100.
- Use that function to find the largest possible colorful number.
- Find and display the count of colorful numbers in each order of magnitude.
- Find and show the total count of all colorful numbers.

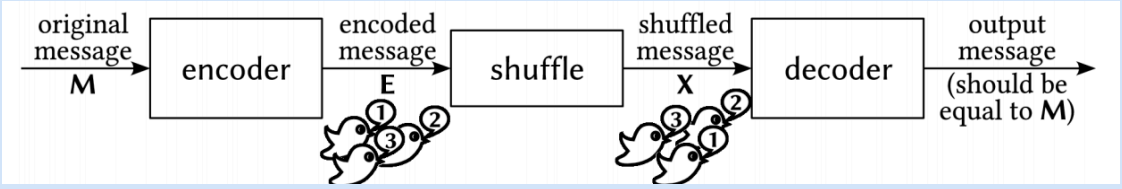
X Data Transfer Protocol on IOI? Parrots (IOI 2011, P6)

Yanee is a bird enthusiast. Since reading about IP over Avian Carriers (IPoAC), she has spent much of her time training a flock of intelligent parrots to carry messages over long distances.

Yanee's dream is to use her birds to send a message M to a land far far away. Her message M is asequence of N (not necessarily distinct) integers, each between 0 and 255, inclusive. Yanee keeps some K specially-trained parrots. All the parrots look the same; Yanee cannot tell them apart. Each bird can remember a single integer between 0 and R, inclusive.

Early on, she tried a simple scheme: to send a message, Yanee carefully let the birds out of the cage one by one. Before each bird soared into the air, she taught it a number from the message sequence in order. Unfortunately, this scheme did not work. Eventually, all the birds did arrive at the destination, but they did not necessarily arrive in the order in which they left. With this scheme, Yanee could recover all the numbers she sent, but she was unable to put them into the right order.

To realize her dream, Yanee will need a better scheme, and for that she needs your help.



- Try to design a function that receives the **Original Message** (of some length N with numbers between $0 - 255$) and encode the original messages to another message sequence, it's called **Encoded Message** with limit the size of message must not exceed K and using numbers from $0 - R$.
- The **encoded message** from will be shuffled.
- Receive the **Shuffled Message** and Decode back to **Original Message**.

You must implement the `encode :: [Int] → [Int]` and `decode :: [Int] → [Int]` process with you own method. We suggest the following roadmap,

Subtask 1 : $N = 8$, all elements of the orignal message are either 0 or 1, $R = 2^{16} - 1$, $K = 10 * N = 80$.

Subtask 2 : $1 \leq N \leq 16$, $R = 2^{16} - 1$, $K = 10 * N$

Subtask 3: $1 \leq N \leq 16$, $R = 255$, $K = 10 * N$

Subtask 4: $1 \leq N \leq 32$, $R = 255$, $K = 10 * N$

Subtask 4: $1 \leq N \leq 32$, $R = 255$, $K = 10 * N$

Subtask 5: We now want you to try to reduce the ratio between encoded message length and original message length upto

$$1 \leq N \leq 64$$

. The mathematical limit for the best ratio one can get is slightly above $\frac{261}{64} \approx 4.08$ (Derive the limit!). Anything below 7 is very good, although we (the authours) are very intrested if someone can find the optimal solution.

Note: When the problem came in IOI, no one found the optimal solution. Furthermore, most solutions which got 100, did so they were optimal on the test cases and not in the general case.

Broken Device (JOI 2016, Spring Training Camp)

Anna wants to send a 60-bit integer to Bruno. She has a device that can send a sequence of 150 numbers that are either 0 or 1. The twist is that L ($0 \leq L \leq 40$) of the positions of the device are broken and can only send 0. Bruno receives the sequence Anna sent, but the does not know the broken positions.

Anna knows the broken positions but Bruno doesn't. Write functions `encode :: Int → [Int] → [Int]` which given the integer and the broken position encodes the message and function `decode :: [Int] → Int` which decodes the message and recovers the integer sent.

Subtask 1: $K = 0$, This should be very simple as you are just converting to binery.

Subtask 2: $K = 1$, We will have one broken position. If we can somehow indicate the start of out 60 bit sequence, can we find a continous 61 bit sequence?

Subtask 3: $K = 15$, This is where one needs to be creative. Note $\frac{150}{2} = 75$ and $75 - 15 = 60$. Can you think of something now?

Subtask 4: $K = 40$, The last question had 2, now try with 3 but have some sequences encode more than 1 bit.

Coins (IOI 2017 Practice)

You have a number C ($0 \leq C < 63$) and an line of 64 coins that are either heads or tails.

As an secret agent, you want to communicate your number to your handler by flipping some of the coins. To avoid being caught, you want to use as few flips as needed. To make the handler aware that you are communicating, you wish to flip atleast one coin. (The handler doesn't know the initial sequence of coins).

In the encoding part, you may flip at least one coins and at most K coins of the line.

In the decoding part, you receive the coins already with the changes, in a line, and you must recover the number C.

Write functions `encode :: Int → [Bool] → [Bool]` which takes a number and a list of coins (`True` is heads and `False` is tails) and returns a list of bools with atleast 1 and atmost K of them flipped. Write a function `decode :: [Bool] → Int` to recover the number.

Subtask 1: $K = 64$, This is easy.

Subtask 2: $K = 6$, Using our friend binery.

Subtask 3: $K = 1$, Note that bitwise Xor $\hat{}$ has some very useful properties. One of these is the fact that it is extreamly easy to change and second is that for numbers between $1 - 64$, taking bitwise xor of some set of numbers will result in a number between $0 - 63$. How can we abuse it?

X Holes (Singapore 2007)

A group of scientists want to monitor a huge forest. They plan to airdrop small sensors to the forest. Due to many unpredictable conditions during airdropping, each sensor will land in a random location in the forest. After all sensors have landed, there will be square regions in the forest that do not contain any sensor. Let us call such a region a hole. It is desirable that all holes are small. This can be achieved by airdropping very large number of sensors. On the other hand, those sensors are expensive. Hence, the scientists want to conduct a computer simulation to determine how many sensors should be airdropped, so that the chances of having a large hole are small.

To conduct this simulation, a function `hole :: [Bool] → Int` is required that, given if a grid of booleans reprasenting the presence of the sensors, outputs the size of the largest hole. This function has to be very efficient since the simulation will be repeated many times with different parameters. You are tasked to write this function.

Example: We will use a matrix of one's and zero's to reprasent the input for convenience.

$$\text{hole} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 5$$

as we have a 5×5 grid of 0's, made bold.

X Restorers and Destroyers (Codeforces)

You have to restore a temple in Greece. While the roof is long gone, their are N pillars of marble slabs stil standing, the height of the i-th pillar is initially equal to h_i, the height is measured in number of marble slabs. After the restoration all the N pillars should have equal heights.

You are allowed the following operations:

- put a new slab on top of one pillar, the cost of this operation is A;
- remove a slab from the top of one non-empty pillar, the cost of this operation is R;
- move a slab from the top of one non-empty pillar to the top of another pillar, the cost of this operation is M.

As the name of the temple is based on the number of pillers, you cannot create additional pillars or ignore some of pre-existing pillars even if their height becomes 0.

What is the minimal total cost of restoration, in other words, what is the minimal total cost to make all the pillars of equal height?

Write a function `cost :: Int → Int → Int → [Int] → Int` which takes the costs A, R and M; and the list of height of pillers and returns the cost of restoration.

Examples

```
cost 1 100 100 [1,3,8] = 12
cost 100 1 100 [1,3,8] = 9
cost 1 2 4 [5,5,3,6,5] = 4
cost 1 2 2 [5,5,3,6,5] = 3
```

X Numerical Diffrentiation

Using the techniques decribed in numerical integration, create a numerical Diffrentiation function.

You will need to write the following functions. You will be able to reuse some of the definitions from Numerical integration.

```
easyDiff :: (Float → Float) → Float → Float → Float
diffrentiate :: (Float → Float) → Float → Float → [Float]
elimerror :: Int → [Float] → [Float]
order :: [Float] → Int
improve :: [Float] → [Float]
super :: [Float] → [Float]
within :: Float → [Float] → Float
ans :: (Float → Float) → Float → Float → Float → Float
```

X Cutting Grass

After attempting to program in Grass for the entire morning, you decide to go outside and mow some real grass. The grass can be viewed as a string consisting exclusively of the following characters: `wWv`. `w` denotes tall grass which takes 1 unit of energy to mow. `W` denotes extremely tall grass which takes 2 units of energy to mow. Lastly v denotes short grass which does not need to be mowed.

You decide to mow the grass from left to right (beginning to the end of the string). However, every time you encouter a v (short grass), you stop to take a break to replenish your energy, before carrying on with the mowing. Your task is to calculate the maximum amount of energy expended while mowing. In other words, write a function `energy :: String → Int` to find the maximum total energy of mowing a patch of grass, that of which does not contain v.

Examples

```
energy "WwwvWWWWvvwwwwwwwwvWwWw" = 8
energy "w" = 1
energy "W" = 2
energy "vwww" = 3
energy "vWWW" = 6
energy "v" =  0
energy "vvvvvvv" = 0
energy "vwvWvwvWv" = 2
energy "vWWWWWWWWWWvwwwwwwwwwwwwwwwwwwwwwv" = 21
energy "vWWWWWWWWWWvwwwwwwwwwwwwwwwwwwwwv" = 20
energy "vvWvv" = 2
```

X Hacking Cyper (Codeforces)

Polycarpus participates in a competition for hacking into a new secure messenger. He's almost won.

Having carefully studied the interaction protocol, Polycarpus came to the conclusion that the secret key can be obtained if he properly cuts the public key of the application into two parts. The public key is a long integer which may consist of even a million digits!

Polycarpus needs to find such a way to cut the public key into two nonempty parts, that the first (left) part is divisible by a as a separate number, and the second (right) part is divisible by b as a separate number. Both parts should be positive integers that have no leading zeros. Polycarpus knows values a and b.

Write function `crack :: Integer → Integer → Integer → Maybe (Integer, Integer)` to help Polycarpus and find any suitable method to cut the public key, given the key, a and b; if possible.

Examples

```
crack 116401024 97 1024 = Just (11640, 1024)
crack 28425458915392817191128181l000 1009 1000 = Just (2842545891539,
28171911281811000)
crack 120 12 1 = Nothing
```

Mohak has explictly approved this (Codeforces 1874D)

There are `n+1` cities with numbers from `0` to `n`, connected by `n` roads. The `i`-th road connects city `i-1` and city `i` bi-directionally.

After Mohak flew back to city 0, he found out that he had left her Miku fufu in city `n`.

Each road has a positive integer level of beauty. Denote the beauty of the `i`-th road as `a_i`.

Mohak is trying to find his fufu. As all Miku fans are obsessive indoor creatues and have no sense of direction and are basically useless without Miku fufu for encouragement, he doesn't know which way to go. Every day, he randomly chooses a road connected to the city she currently is in and traverses it.

Let `s` be the sum of the beauty of the roads connected to the current city. For each road connected to the current city, Mohak will traverse the road with a probability of `x/s`, where `x` is the beauty of the road, reaching the city on the other side of the road.

Mohak starts in cuty `0` and Fufu is in city `n`.

In order to help Mohak, you want to choose the beauty of the roads such that the expected number of days Mohak takes to find his fufu will be the minimum possible. However, due to limited funding, the sum of beauties of all roads must be less than or equal to `m`.

Write a function `miku :: Int → Int → Float` which takes the value of `m` and `n` and tells us the expected time for Mohak to find his fufu.

Examples

```
miku 3 8 = 5.2
miku 10 98 = 37.721155173329
```

Note In the first example, the optimal assignment of beauty is [?]=[1,2,5]. The expected number of days Mohak needs to get his fufu back is 5.2.

Hint: Let's say you are given the beauty list a. Now, how will you compute the expected number of days? Consider $f(i)$ to be expected number of days to reach the i-th city. Can you find these recursivly? What about $g(i) = f(i) - f(i-1)$? Can you find a form in terms of a, can you use `zipWith` to compute these?

Now, can you make a function to make lists that sum up to m and are of length n? You can save a lot of time by imposing one condition on the lists. The idea is that we want to keep moving forward.

Introduction to Datatypes

Shubh Sharma

§8.1. Datatypes (Once Again)

In Chapter 4 we saw how Haskell datatypes correspond to sets of values. Like `Integer` is the set of all integers and `String → Bool` is the set of all functions that take in a `String` as an argument and return a `Boolean` as their output. This was the first time we gave explicit attention to datatypes and learned the following:

÷ Types 1

A **Datatype**, in its simplest form, is the name of a set.

In Section §6.1., where we defined polymorphic functions, the *shape* and *behaviour* of an element were 2 properties that we built off of.

As a small recap, consider function `elem`, this is a function which checks if a given element belongs to a given list. The input requires to be a list of elements of a type, such that there is a notion of equality between types.

```
elem :: Eq a ⇒ a → [a] → Bool
elem _ []       = False
elem e (x : xs) = e == x || elem e xs
```

Our requirements for the function are very clearly mentioned in the type. We are starting with a type `a` which has a notion of equality defined on it, as depicted by `Eq a`, and our arguments are an element of the type `a` and a list of elements of the type, that is, `[a]`. Here we used datatypes to specify the properties of the elements that we use. So we extend the previous definition.

÷ Types 2

A **Datatype** is the name of a ***homogenous*** collection of object, where the common properties, like the shape of elements, is depicted in the name.

Some examples of datatypes we have already seen are:

- `[Integer]`, which is the collection of lists of integers.
- `Maybe Char`, which is the collection of characters along with the extra element `Nothing`.
- `Integer → String`, which is the collection of functions with their domain as the set of integers and range as the set of strings.

This definition suggests that datatypes can be used to *structure* the data we want to work with. And this is actually something we have seen before!

In Section §4.1., we saw operations on sets such as

- `(A, B)` being analogous to ÷ **cartesian product**.
- `Either A B` being analogous to ÷ **disjoint union**.

Here we will spend some time to see how we can define dataypes like these on our own.

Before getting to defining our own datatypes, its good to remember what the purpose of datatypes is: The point of datatype is to make thinking about programs simpler, for both the programmer and Haskell. This is done in the following ways:

- Types indicate the *shape* of elements and can add information about the functions, for example:
  - `Either [Integer] Bool` tells us that every element of the type is either a list of integers, or a boolean value.
  - `Eq a ⇒ a → [a] → Maybe Integer` tells us that `a` has a notion of equality defined on it, and the output should be an integer, but the function can potentially fail (that is return `Nothing`).
- Types tell the compiler information about domains and codomains of functions which lets it, to a great extent, check if functions are given inputs they are defined on, and that functions are defined on the values of the domain. (It is not always capable of doing so, for example trying to `sum` an infinite list, but this avoids runtime errors by a lot!)

We will now see how to define our own types.

§8.2. Finite Types

The first step we take in defining types is by creating values and put them together in a collection, this is done using the `data` keyword.

λ finite types

```
data Colour = Red | Green | Blue

data Bool = True | False       -- This is how Haskell defines Bool!
data Ordering = LT | EQ | GT   -- This is also a type defined by Haskell

data Coin = Heads | Tails
```

The last example is there to emphasize that the `data` keyword really creates new types. The `Coin` type is a 2-element type, but is not the same as `Bool` and Haskell will give a type error if its used in its place. Each element of a type defined like this is called a **constructor**, which is name that will get its justification by the end of the chapter.

To define a function out of a finite type, one needs to define the output all all constructors, for example:

```
isRed :: Colour → Bool
isRed Red   = True
isRed Blue  = False
isRed Green = False
```

Defining finite types is really helpful when one wants to have a finite number of variants in a type, for example, there are a finite number of chess pieces, in languages that do not have a syntax that lets us do something like this, one would make do with strings. The benifit of these finite types is that

now Haskell will make sure that the functions are only defined on intended values (unlike all possible strings), and will also give warnings if any function is not defined on all variants.

X Finite Types

Define the types `Month`, `Day` of the week and `DiceHead` as finite types.

§8.3. Product Types

These are what we get when take ≒ **cartesian product** of other, simpler types. The purpose of product types is to define data, that has multiple smaller components. For example:

- A `Point` on a 2D grid which has 2 integer components.
- A `Profile` representing a profile on a dating app, which would contain the person's name, their age, some images and more information about them. We will be using the first example to keep things simple.
- Complex Numbers can be thought of as having 2 components, real and imaginary.

The first way to create a product, which is something we have already seen before is a tuple. As our example we will consider the type `(Integer, Integer)` which we are supposed to interpret as the the set of points on a 2-dimensional lattice. where the two `Integer`s represent the x and y coordinates of a point on the lattice.

And we can extract components using `fst` and `snd` functions. (Note `Point` is just a synonym for `(Integer, Integer)`).

Another way to do so is to use the `data` keyword again in a much more powerful way!

```
data Point = Coord Integer Integer -- Constructors can take inputs!
                   --  X       Y

x_coord, y_coord :: Point → Integer
x_coord (Point x _) = x
y_coord (Point _ y) = y
-- we use underscores to state that we don't care about the value
```

Here we need to define our own functions to extract components as `Point` is differnet from `(Integer, Integer)`.

The second important thing to highlight here is that constructors are functions! They are called so because they "construct" and element of the type associated with them, like `Coord` constructs elements of type `Point`, infact Haskell will even give us a type for it. Constructors for finite types can be thought of as functions that take 0 arguments (so, they just behave as values).

```
>>> :t Coord
Coord :: Integer → Integer → Point
```

Since defining a product type, and then defining functions to extract the components is a fairly common practice, haskell has another way to define product types.

```haskell
data Point = Coord {
  x_coord :: Integer,
  y_coord :: Integer
}

-- This is how one can create an element!
origin :: Point
origin = Coord { x_coord = 0, y_coord = 0 }
```

These are called **Records** its a syntactic sugar, which means internally haskell treats it just like the previous way of defining product types, so `Coord 0 0` also works. But now we have the 2 functions `x_coord` and `y_coord` defined!

X Dating Profile

As described above, the profile of a dating app can be also thought of as a product type, one which is more complicated than a simple point in the 2d grid. Define the type `Profile` and try to see how elaborate you can make it. A fun rabbit hole do dive into would be to see how dating apps work.

X Complex Numbers

Define the datatype `Complex`, we will be looking at this again in later sections of the chapter.

§8.4. Parametric Types

We will once again extend the use of `data` keyword using ideas form Section §6.1..

We compared product types with tuples, we even treated `Point` as a special case `(Integer, Integer)` for a while. Turns out we can define our tuples, in its full generality as follows:

```haskell
Tuple A B = Pair A B

ex :: Tuple Integer String
ex = Pair 5 "Heyy!"
```

Here `Tuple` is called a **parametric type**, and this is similar to how haskell defines its tuples, it just adds an extra syntactic sugar so we can write it as `(a,b)`.

Some other **parametric types** that we have seen before, and we will be discussing in depth in the next section are:

- `Maybe a`
- `Either a b`
- `[a]`, The list type
- The function type `a → b`

§8.5. Sum Types

Sum types are what type theory people like to call ÷ **disjoint union**. And already have seen everything we need to construct sum types:

- Finite Types

- Viewing constructors as functions
- Parmetric Types

The purpose of having sum types is to have a collection of many possible *variants* in a type. This is similar to what we did with Finite types but we can have an entire collection as a variant with the help of Parametric types. Here are some examples:

```
IntOrString = I Integer | S String

Maybe a = Just a | Nothing    -- This is how Haskell defines Maybe types!
Either a b = Left a | Right b -- This is how Haskell defines Either types!

Shape = Circle { radius :: Double } -- Record syntax works!
      | Square { side :: Double }
      | Rectangle { len :: Double, width :: Double }
-- Remember, the records are just syntactic sugar, which are converted to
-- Shape = Circle Double | Square Double | Rectangle Double Double
```

Just like finite types, to define a function on a sum type, one needs to define it on all variants. This is also called Pattern Matching!

```
area :: Shape → Double
area (Circle r)      = pi * r * r
area (Square s)      = s * s
area (Rectangle l b) = l * b
-- If this doesn't make a lot of sense, read the comment below the
definition of the type Shape.
```

Better Dating Profile

Think of some interesting questions and possible answers for those questions / information bits for a dating profile and incorporate it into you `Profile` type.

§8.6. Inductive Types

Inductive types will be the final tool in the type construction toolbox that we will be looking at. While it can be considered as a slight modification of the previous methods of building types, we will give it some special attention.

§8.6.1. Inductive Types (as a Mathematician)

We defined a **set** as a well-defined collection of objects. So consider the following description:

$$B = \{\text{Set of squares reachable by a bishop (in 0 or more moves)} \\ \text{given that the bottom left square is in the set}\}$$

Here is a quick refresher on the relevant rules of chess:

- There is an 8x8 square grid and each piece lies inside a square.
- Bishop is one of the chess pieces that can only move diagonally in a line, but it is allowed to move as far as possible.

One can now create the set B by starting at the bottom left square and one by one adding sqaures, where the bishop can reach, to set. Here we say that the set B was generated by the bottom left under the bishop movement rules. There was one other important piece of information other than the "base case" and the "operation", the extra structure imposed by the chess board itself, specifically,

the operations $\langle$go top right$\rangle \to \langle$go top left$\rangle$ is the same as $\langle$go top left$\rangle \to \langle$go top right$\rangle$ and the board is restricted to an 8×8 grid. This is called **generating** a set from a value (bottom left square) using a function (bishop movement).

Now consider the following example: You are trying to braid your (or your long haired friend's hair), you do so by starting with 3 bunches of hair (ordered as left, middle and right) and you plan to put them together. This is done by swapping positions of the middle bunch, with either the left or the right bunch (middle one always goes from below), done in an alternating manner. Turns our you're new to this and did not know about the alternating part. And so you start braiding. An important observation you make as you do this every other day is that every different sequence of swaps gives you a different looking braids! (most of them will probably not look good, but you are (or your friend is) a math person, this makes you happy). If we consider B to be the set of all braids that you can create, we can define it using just 2 pieces of information:

- You started with the non-braid which is in the set.
- You either swapped the middle bunch with the left, or with the right bunch and kept doing this.

This is another example of a **generated** set, but here there are no extra rules other than the starting element and the operations (unlike the restrictions imposed by the geometry of the chessboard). In such cases we say that the set is **freely generated** (free as is no restrictions). A very useful fact about such sets is that each element can be identified with the sequence of operations used to create them, in fact a lot of the times this is how people talk about elements of freely generated sets.

÷ Freely Generated Sets

Given a collection of of **base values** $\mathcal{B} = \{b_1, b_2 ... b_n\}$ and a collection of **operationd** $\mathcal{F} = \{f_1, f_2, ... f_m\}$ we say that a set S is freely generated from $\mathcal{B}$ using $\mathcal{F}$ if S satisfies the following properties:

- $\mathcal{B} \subseteq S$, that is, all the base values are in the set.
- Given $s_1, s_2 ... s_n \in S$ and any $f \in \mathcal{F}$ we have that $f(s_1, s_2 ... s_n) \in S$.
- If there are 2 elements in S constructed as $s_1 = f_1(v_1, v_2 ... v_p)$ and $s_2 = f_2(w_1, w_2 ... w_q)$ such that $s_1 = s_2$, then we can say that
  - $f_1 = f_2$,
  - $p = q$ and
  - $v_1 = w_1, v_2 = w_2 ... \ v_p = w_q$
- S is the smallest such set satisfying these properties.

Now we see some more examples.

§8.6.1.1. Natural Numbers as Inductive Types

Let $\mathbb{N}$ be the set freely generated by the following:

- The element $0 :: \mathbb{N}$
- The operation $\text{succ} :: \mathbb{N} \to \mathbb{N}$

Here the names are suggestive, but if simply follow the rules for freely generated sets, we get the following:

- We know that 0 is in the set.
- That means succ 0 is in the set.
- Which means succ (succ 0) is in the set.
- and so on...

So we will end up with the set

$$\{\ 0,\quad \text{succ } 0,\quad \text{succ succ } 0,\quad \text{succ succ succ } 0,\quad \text{succ succ succ succ } 0\ ...\ \}$$

This way is very similar to how mathematicians usually formalize natural numbers.[6].

These "freely generated sets" are what programmers call Inductive types, and one can define the type of natural numbers in haskell as follows:

λ nat

```haskell
data Nat = Z | Succ Nat

three :: Nat
three = Succ (Succ (Succ Z))
```

And functions on a natural number would be usually given by a recursive function:

```haskell
add' :: Nat → Nat → Nat
add'  Z       n = n
add' (Succ m) n = Succ (add' m n)
```

We can also define functions to convert between Integers and Natural Numbers:

λ nat and integer

```haskell
natToInteger :: Nat → Integer
natToInteger Z = 0
natToInteger (Succ n) = natToInteger n + 1

integerToNat :: Integer → Maybe Nat
integertoNat n | n <  0 = Nothing
               | n == 0 = Just Z
               | n > 1  = Just $ Succ (integerToNat (n-1))
```

X Exercise

Natural numbers are the default way people count things. A lot of the haskell functions that involve counting, like `(!!)`, `takeWhile`, `drop` and so on are functions that can potentially fail because of an attept to access a negative index. redefine these functions our definition of natural numbers (λ nat).

X Functions on naturals

Define versions of functions `max`, `sum`, `prod`(product), `min` and `==` for natural numbers. Note that you would have to use new names.

§8.6.1.2. Lists as Inductive Types

Another interesting example of an inductive type is the set of lists over the type `A`.

Given a set/type `A` we can define the set of list over it using the following:

- $\square :: [A]$, the empty list.
- For each element $a :: A$, a function $(a :) :: [A] \to [A]$.

[6]The usualy way to define natural numbers was written by Guiseppe Peano and are called the 'Peano Axioms' which invole a bunch of rules, whose relevant part can be summarizes as :

- 0 is a natural number
- Natural numbers are closed under the succ operation
- For each number x that is not 0, there is a unique number y such that $x = \text{succ } y$
- 0 is not the successor of any number.

We leave it to the reader to show that this inductive type generates the set of all lists of elements of type `A`. We will justify for the example $[0, 1, 2, 3]$

- $\square$ belongs to the set $[\mathbb{Z}]$
- Then $3 : \square$ belongs to the set $[\mathbb{Z}]$
- Then $2 : 3 : \square$ belongs to the set $[\mathbb{Z}]$
- Then $1 : 2 : 3 : \square$ belongs to the set $[\mathbb{Z}]$
- Then $0 : 1 : 2 : 3 : \square$ belongs to the set $[\mathbb{Z}]$

And we treat $0 : 1 : 2 : 3 : \square$ as $[0, 1, 2, 3]$.

In haskell, we cannot write infinitely many constructors in the definition of a type, so we instead define it as follows:

λ list
```
data List A = Nil | Cons A (List A)
```

Haskell will not let us use `[]`, this is sytactic sugar given by the compiler a user (like us) cannot give our own definitions to, so we will us `Nil` and `Cons` instead (similar to the bracket and comma syntax for tuples).

Fixing as `A` as `Integer` for now `Nil` represents `[]` and `Cons` takes an integer `n` and gives the constructor `n:`. This is work around to not being able to write infintely many constructors. The above definition (apart from the syntactic sugar) is how Haskell internally defines lists.

This idea is very much inspired by the concept of Currying which was discussed in Section §6.2.1..

§8.6.2. (Not Quite) Inductive Types (as a Programmer)

As a programmer, we will be using these types as a *blueprint* for the shape of the element in the type. Specifically to indicate that an element is created by combining other elements of the type together, hence, these are also called **recrusive datatypes**.

In fact, the constructors defined are often used to describe a procedure to check if an element belong to the type, very similar to what we did in ≒ **tree** and in ≒ **well-formed mathematical expression**.

We will see how to extract such a procedure from the constructors of a type in Section §8.6.2.2..

But first we see a simple example of a recursive datatype, a type to represent arithmetic expressions.

§8.6.2.1. Calculator

For our purposes, we say that our calculator can compute:

- Addition
- Subtraction
- Multiplication
- Division
- Exponentiation

The plan will to have an inductive type `Expr` of expressions (because tiny expressions combine to give big expressions), which we define as follows:

λ expression

```
data Expr = Val Double
          | Add Expr Expr
          | Sub Expr Expr
          | Mul Expr Expr
          | Div Expr Expr
          | Exp Expr Expr
          | Neg Expr
```

Here the goal of the type was to specify the structure of the data (arithmetic expression) we want to working with, lets see a few examples!

The expression $3 + 5 * 10 + \frac{8^3}{2}$ corresponds to

λ expr example

```
ex :: Expr
ex = Add (Val 3.0)
         (Add (Mul (Val 5.0)
                   (Val 10.0))
              (Div (Exp (Val 8.0)
                        (Val 3.0))
                   (Val 2.0)))
```

The following is the procedure to check

X Evaluate and extend

Write a function `eval :: Expr → Double` that takes an expression and returns its value. The potential failure case here is division by 0. To deal with it, either add an failure value to the expression type, or make the function have a `Maybe` output.

Also try extending the Expression type to include more operations.

For those with keen eyes and good memory the shape of `ex` should remind you of the discussion in Why Trees? section in Section §1.9..

We will now justify the satement made there:

Ryan Hota, Haskell2025

In fact, any object in Haskell is internally modelled as a tree-like structure.

We will now see that Haskell verifies that the element `ex` (from λ **expr example**) is an `Expr` using a procedure that is very similar to what was defined in ≒ **tree**.

- We see that the value `ex` is created using the constructor `Add`, so it must produce an `Expr`, now we need to make sure that both of its arguments are also of type `Expr`.
  - The fisrt argument of the above is `Val 3.0` which defines a `Expr`.
  - The second argument of the above constructor is also `Add` so it must produce an `Expr` given that both its arguments are also of type `Expr`.
    - The first argument to this is produced using the constructor `Mul` hence it must produce an element of type `Expr` given the correct arguments.
      - Its first argument is `Val 5.0`

    - Its second argument is `Val 10.0`, both of which are `Expr`.
  - The second argument to the `Add` is constructed using `Div`, so it must be a tree given both its arguments are `Expr`.
    - Its first argument is constructed using `Exp`, so it must be a `Expr` given its arguments are also `Expr`
      - Its first argument is `Val 8.0`
      - Its second argument is `Val 3.0`, both of which are `Expr`.
    - Its second argument is `Val 2.0` which is a `Expr`.

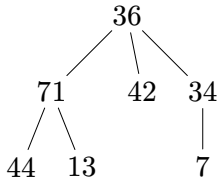
by simply checking that all constructors get inputs of the correct type, Haskell has gone through the procedure defined in ≒ **tree** to check that the element `ex` is well defined.

§8.6.2.2. Trees as Inductive Types

On the topic of trees, while working with such inductive one finds that all inductive dataypes follow a tree structure, this is a result of *free generation*. Trees happen to be a ubiquitous data-structure (way to structure data) in computer science and has applications everwhere. The following is a very tiny subset of those:

- Compilers (like both haskell and our calculator)
- File Systems
- Databases
- Data representation formats like JSON and XML
- Data Compression (huffman encoding)
- Space partitioning (oct-trees and quad-trees)

So we now define trees, recall ≒ **tree**, which defines a tree as a meaningful structure on data involving a main **root** node, and each node having 0 or more children, as shown in the following diagram.



Looking at the structure, we can define a tree as follows in haskell:

λ tree

```
data Tree a = Node { value :: a, children :: [Tree a] }
```

And the above tree can be represented as follows:

```
ex :: Tree Integer
ex = Node 36
        [ Node 71 [Node 44 [], Node 13 []]
        , Node 42 []
        , Node 37 [Node 7 []]]
```

X Tree Functions

Define the following functions for the tree datatype:

- `depth :: Tree a → Nat`, this defines the longest path one can take starting from the route, for example, the `depth ex` is 2.
- `size :: Tree a → Nat`, this defines the number of nodes in a tree, for example `size ex` is 7.

Also define versions of the `elem` and `sum` functions for the `Tree` datatype.

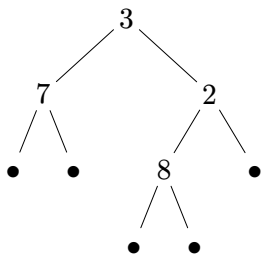
§8.6.2.3. Binary Trees

Binary Trees are a special case of trees, where each node has either exactly 2, or 0 children. Nodes with 0 children are called **leaves**.

Out of the uses cases mentioned for trees the following involve binary trees:

- Compilers (for functional languages)
- Databases
- Data compression (Huffman Encoding)

A binary tree over intergers looks like:



Here unlike the (not necessarily binary) tree, we don't allow leaves to hold values (represeted by •), these allow us to have nodes that behave like they have just 1 child (like 2 in the above example). This also allows are definition to look like the definition of lists:

λ Btree

```
data Btree a = BNode {left :: Btree a, Val :: a, right :: Btree a}
             | Leaf
```

and the example above can be written as:

λ Btree ex

```
bex:: Btree Integer
bex = BNode (BNode 7 Leaf Leaf)
            3
            (BNode (BNode 8 Leaf Leaf)
                   2
                   Leaf)
```

X Binary Tree Functions

Define all of the X Tree Functions for binary trees.

We will see how these datatypes are used in Section §11.1..

X All trees are Binary Trees

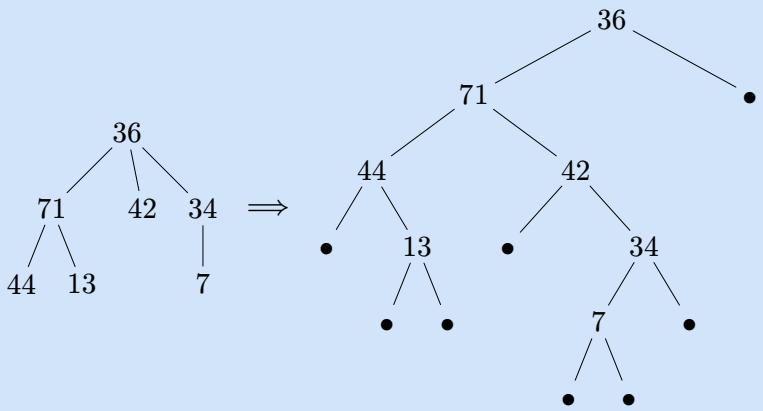
There is a way to convert a (not necessarily binary) tree into a binary tree without losing any information (that is, in a way that one can reconstruct the original tree back from the binary tree).

The idea here is that a binary tree has 2 types of relations between elements, that of a left child, and that of a right child. And we use those to capture the 2 types of relations in trees which are: Being the *first* child, being the *next* sibling. In fact, one can get to each element of a tree from the root by these 2 operations.

So given a tree, we construct its corresponding binary tree as follows:

1. We set the root as also the root of the binary tree.
2. If the node we are looking at in the tree is a leaf and does not have siblings to the right of it, we are done with the node.
3. If the node we are looking at is not a left, let `(x:xs)` be the list of children. Then we make an edge from the current node to `x` and we connect `x` to the next element in the list. That to the one after it and so on.
4. We repeat the previous 2 steps with all new vertices added in step 3 until there are no new nodes left to add.

As an example:



Write a function `treeToBtree :: Tree a → Btree a` to convert to a tree. Also write the function `btreeToTree :: Tree a → Maybe (Btree a)` and show why `Maybe` is required here.

§8.6.2.4. Addressing the "Not Quite"...

The title of Section §8.6.2. is **(Not Quite) Inductive Types (as a Programmer)**. Turns out that while we have been discussing inductive types this entire section of the chapter, because of the way haskell works, all of the types that we have defined like :

- `Nat`
- `Expr`
- `Tree`
- `Btree`
- `List`

aren't exactly inductive types, a consequence of this is that the type `Nat` isn't exactly the set $\mathbb{N}$ either.

The culprit here is laziness and its exactly what was discussed in Section §5.10., that is infinite elements, to understand that one of the important points in ≒ **Freely Generated Sets** is that that the set that is being generated must be the **smallest** set that satisfies the other properties given in the definition.

Consider the case of the type `Nat = Zero | Succ Nat`, this is supposed to represent the set $\mathbb{N}$ and it satisfies all the properties fo a freely generated set, aka an inductive type except for one: The one about $\mathbb{N}$ being the smallests set that satisfies the given properties because of the following extra elements:

```
infinity :: Nat
infinity = Succ infinity

-- so the element looks like
-- infinity = Succ( Succ( Succ( Succ( Succ( Succ ... )))))
```

So Haskell is letting the `infinity` sneak into our type `Nat`.

The same also holds for all of the other types where we can define

```
inflist :: a → [a]
inflist a = a : inflist a
-- which is simply
-- a : a : a : a : a ...
```

And even infinite expressions like

```
infexpr :: Double → Expr
infexpr x = Add x (infexpr x)
```

which is a funny term that is simply evalutates to

```
infexpr x = x + x + x + x + ...
```

One can make weirder terms that make even less sense than the above and are encouraged to do so, its fun.

There is a formal way to reason about such infinite data structures, and this feature is captured in programming languages like Lean and Agda and is called **coinduction**, but we will not be discussing it.

We still chose to put emphasis on **inductive datatypes** as we think that its an idea helpful in designing programs. A lot of the functions that we have discussed so far, like

- `eval` from X **Evaluate and extend**
- `depth` and `size` from X **Tree Functions**
- `natToInteger` from λ **nat and integer**

and many more from the list chapter and so on are designed with the idea of inductive types. The purpose of types, as discussed, was to be able to give information to the Haskell compiler about what the functions should expect as an argument, this is one of the places where the Haskell type system fails to express that.

Nonetheless, most of the time people do tend to asssume that the functions are going to get arugments that are finite, which is reasonable in most cases, for example if you are writing a full fledged calculator, using the type described in Section §8.6.2.1., it would involve something like

- Being able to take a string input

- Parse it into an `Expr` element
- Evaluate it and return the answer

In such cases it is very easy to make sure that expressions are going to be finite (I don't any user has enough free time to enter an infinite expression).

If you are disappointed by the fact that the Haskell compiler is letting a potential error pass through, not that the alternative would be, being able to prove that all programs in haskell would terminate. This problem is a version of the **Halting Problem** which is one of the most famous problems in computer science and, in some sense, is the problem that the field of modern computer science stems from. Alan Turing proposed this problem and prove that it is *unsolvable*, so it simply isn't possible for a language like haskell to check for infinite structures like this. Languages like Lean and Adga achieve this by also disallowing some programs that do terminate. These languages are not Turing Complete (its not possible to write every valid program in this language).

§8.7. Same Same but Different

This section is a bit differnt from the previous one, the goal here is not to create new types but to repurpose the old ones.

§8.7.1. Same Same

One of the two reasons we had given in Section §8.1. for creating types was to make programs make more sense to people who try to read them. Datatypes often indiciate the structure of the code and it doing so, they can express the intent of the programmer. One of the ways in which haskell makes this easy for us by letting us come up with aliases for types. This is done using the `type` keyword.

λ type aliases
```
type Point = (Integer, Integer)
type String = [Char] -- This is how Haskell defines String!

type Name = String
type Age = Integer

type Person = (Name, Age)
```

note that any type defined using the keyword `type` is simply an alias for another type and Haskell does not treat it any differently.

Nonetheless, this can be very helpful for interpreting the type for a human. For example the type `Person` which is an alias for `(String, Integer)`, when written as `(Name, Age)`, is very clearly meant to be a pair containg the *name* and *age* of a **person**.

§8.7.2. Different

Another thing one might want to do with an existing type is to use it as your own and define your own functions on it.

One reason you might want to do this would be to use the same datatype to for different pieces of data, and you might not want them to get mixed up. For example, say you're playing a game involving you going down a cave to find treasures. Your 'health' would be an `Integer`, if that goes to `0` your character dies. The 'depth' you're at would also be an `Integer` which would be used to decide how valuable the minerals you find would be. (For those interested, the game in mind is Minecraft).

We can use `data` to let us define separate types like:

```
data Health = Health Integer
data Depth = Depth Integer
```

Now you have 2 copies of `Integer` that haskell will treat differently, making sure that they never get mixed up.

One change that we can make here is to use the `newtype` keyword instead of `data` and get the following:

```
newtype Health = Health Integer
newtype Depth = Depth Integer
```

This just tells Haskell that your datatype has only 1 constructor with exacty 1 field which will allow Haskell to apply some optimizations. It otherwise behaves just like `data` except for the above mentioned restrictions.

Contents

Computation as Reduction

Shubh Sharma

Complexity

Arjun Maneesh Agarwal

§10.1. Introduction

< to do >

§10.2. Asymptotics

§10.2.1. Big El

Big Ell notation

The symbol $\mathcal{L}$ means absolutly atmost. For example, $\mathcal{L}(4)$ refers to a value whoose absolute value is less than or equal to 4. For example, $\pi = \mathcal{L}(4)$.

Note, the $=$ sign is not trasitive in this regard. The reason for this is because the notation came from math and unlike Computer scientists, mathematicains have a small vocabulary and couldn't think of another symbol. As a Mathematician once put in words(quite surprising),

De Bruijn

Mathematicians customarily use the $=$ sign as they use the word "is" in English: Aristotle is a man, but a man isn't necessarily Aristotle.

You can also see this by the story, let's say a textbook author's typewriter doesn't have $>$ sign. So they start using $5 = \mathcal{L}(3)$ to reprasent that 3 is lesser than 5. They can also write $\mathcal{L}(3) = \mathcal{L}(5)$ but not $\mathcal{L}(5) = \mathcal{L}(3)$

Comming back to mathematics, Big-A notation is very compatible with arithmatic.

Theorem

1. $\pi = 3.14 + \mathcal{L}(0.005)$
2. $10^{\mathcal{L}(2)} = \mathcal{L}(100)$

Proof We will just use the defination of $-k \leq A(k) \leq k$.

Thus, $\pi = 3.14 + \mathcal{L}(0.005) \iff \pi \in [3.14 - 0.05, 3.14 + 0.05]$ which is true as $3.140 < \pi = 3.1415926... < 3.142$.

Similerly, $10^{\mathcal{L}(2)} = \mathcal{L}(100) \iff [10^{-2}, 10^{2}] \subseteq [-100, 100]$ which is true. ■

Theorem $\mathcal{L}(x) \cdot \mathcal{L}(y) = \mathcal{L}(xy)$

Proof We can eliminate the $\mathcal{L}$ to get that the above statement is equivalent to, $[-xy, xy] \subseteq [-xy, xy]$ which is trivially true. ■

We can use the notation for variables as well,

Theorem

1. $\sin x = \mathcal{L}(1)$;
2. $\mathcal{L}(x) = x \cdot \mathcal{L}(1)$;
3. $\mathcal{L}(x) + \mathcal{L}(y) = \mathcal{L}(x + y)$, where$x, y \geq 0$;
4. $(1 + \mathcal{L}(t))^2 = 1 + 3\mathcal{L}(t)$, where $t = A(1)$.

Proof Left as an excercise to reader. ■

Big Ell Analysis

1. Simplify: $(5 + \mathcal{L}(0.2)) + (3 + \mathcal{L}(0.1))$
2. Simplify: $(5 + \mathcal{L}(0.2)) * (3 + \mathcal{L}(0.1))$
3. Express e^x for $|x| \leq 1$ in the form $1 + \mathcal{L}(u)$ for some $u = kx$. **Hint**: Use the fact that $e^x = 1 + x + \frac{x^2}{2!} + ...$
4. Define a sequence $x_0 =, x_n = 2x_{n-1} + A(\varepsilon_n)$ where $\varepsilon_n > -$. Then solve for x_n when
   1. $\varepsilon_n = \frac{1}{3^n}$
   2. $\varepsilon = \frac{1}{n^2}$

§10.2.2. The Hierarchy of Functions

Asymptotic Dominance

Asymptotic Dominance is a relation over functions, where $f(n) \prec g(n)$ when $g(n)$ approaches infinity faster than $f(n)$. We can formalize this by saying:

$$f(n) \prec g(n) \iff \limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 0$$

Furthermore, $f(n) \asymp g(n) \iff \limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} = k$ where $0 < k < \infty$ is constent.

Theorem Asymptotic dominence is transistive.

Proof Let $f(n) \prec g(n)$ and $g(n) \prec h(n)$. We wish to show $f(n) \prec h(n)$.

$$\limsup_{n\to\infty} \frac{|f(n)|}{|h(n)|} = \limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} \cdot \frac{|g(n)|}{|h(n)|} = 0 \cdot 0 = 0$$

■

Theorem If $\alpha < \beta$ then $x^\alpha \prec x^\beta$

Proof Left as an excercise to reader. ■

Theorem $f(n) \prec g(n) \iff \frac{1}{f(n)} \succ \frac{1}{g(n)}$

Proof Left as an excercise to reader. ■

Theorem $e^{f(n)} \prec e^{g(n)} \iff \limsup_{n\to\infty}(f(n) - g(n)) = -\infty$

Proof Using the definition, we will get:

$$
\begin{aligned}
& e^{f(n)} \prec e^{g(n)} \\
\iff & \limsup_{n\to\infty} \frac{e^{f(n)}}{e^{g(n)}} = 0 \\
\iff & \limsup_{n\to\infty} e^{\ln\left(\frac{e^{f(n)}}{e^{g(n)}}\right)} = 0 \\
\iff & \limsup_{n\to\infty} e^{f(n)-g(n)} = 0 \\
\iff & \limsup_{n\to\infty} f(n) - g(n) = -\infty
\end{aligned}
$$

■

Theorem Let $f(x) = k * g(x)$ for some real $k \neq 0$, then $f(x) \asymp g(x)$

Proof Left as an excercise to reader. ■

Theorem Let $f(x) \succ g(x)$ and $f(x) \asymp h(x)$ thebm $f(x) \asymp h(x) + g(x)$

Proof Left as an excercise to reader. ■

This previous last two theorems tells us that the equivalence induced by ≒ **Asymptotic Dominance** is invarient under scaling and translation by dominated functions. That is, in a sum of functions, we only need to consider the most dominent function when comparing asymptotic dominence. This allows us to talk about a group of functions using a single 'reprasentative' function.

≒ **Reprasentive Function wrt asymptotic dominance**

The representative function of a set of asymptotically equivalent functions is the simplest form of the equivalence class, defined as:

- A function with a leading coefficient of 1,
- No additive subdominant terms (i.e., no terms in addition that are asymptotically dominated by others in the class).

For example x^2 is the reprasentative function of the equivalence class of quadratic functions, $ax^2 + bx + c$.

Theorem If $f(x), g(x)$ are the reprasentative functions of the equivalence class F, G; then show that $f(x) \succ g(x) \iff \hat{f}(x) \succ \hat{g}(x)$ for all $\hat{f} \in F$ and $\hat{g} \in G$

Proof ($\implies$) Using the fact $f, \hat{f} \in F$, we get $f(x) \asymp \hat{f}(x) \iff \lim_{x\to\infty} \frac{f(x)}{\hat{f}(x)} = c_1$ where $c_1 \neq 0$ is a constent.

Similerly, $g(x) \asymp \hat{g}(x) \iff \lim_{x\to\infty} \frac{g(x)}{\hat{g}(x)} = c_2$ where $c_2 \neq 0$ is a constent.

Using $f(x) \succ g(x) \iff \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$

Thus,

$$
\begin{aligned}
& \limsup_{x\to\infty} \frac{|\hat{g}(x)|}{|\hat{f}(x)|} \\
= & \limsup_{x\to\infty} \frac{|\hat{g}(x)|}{|g(x)|} \cdot \frac{|g(x)|}{|f(x)|} \cdot \frac{|f(x)|}{|\hat{f}(x)|} \\
= & \frac{1}{c_2} \cdot 0 \cdot c_1 \\
= & 0 \\
\implies & \hat{f}(x) \succ \hat{g}(x)
\end{aligned}
$$

($\Longleftarrow$) $f \in F$ and $g \in g$ implies $f(x) \succ g(x)$ as this is true for all functions in the equivalence clases.[7] ■

An Hieraarchy of Common Functions

Prove that:

$$1 \prec \log(\log(n)) \prec \log(n) \prec n^{\varepsilon} \prec n^{c} \prec n^{\log(n)} \prec c^{n} \prec n! \prec n^{n} \prec c^{c^{n}}$$

§10.2.3. Big Oh notation

These topics seem disconected, right? Let's fix that.

Big Oh (from Big Ell)

$f(x) = o(g(x))(n \to \infty) \iff \exists k, n_0 \text{ s.t. } \forall n \geq n_0 f(n) = k\mathcal{L}(g(n))$

Big Oh (from hierarchy)

$f(n)o(g(n))(n \to \infty) \Leftrightarrow f(n) \preceq g(n)$

Theorem The above definations are equivalent.

Proof **Lemma:** Both the definitions are equivalent to $\limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty$.

From the first definiton, there exists an k, n_0 such that $\forall n_0 \leq n$:

$$\begin{aligned} & f(n) = k\mathcal{L}(g(n)) \\ \Leftrightarrow & |f(n)| \leq k\ |g(n)| \\ \Leftrightarrow & \frac{|f(n)|}{|g(n)|} \leq (k) \end{aligned}$$

As $n_0 < \infty$ and $k < \infty$; thus,

$$\limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty$$

The equivalence from the second defination follows from the definition of asymptotic dominance. ■

We also allow this lemma to also be a defination of o.

Big Oh (limit)

$f(n)o(g(n))(n \to \infty) \iff \limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} < \infty.$

We also give the classical defination.

Big Oh (Classical)

$f(n) = o(g(n))(n \to \infty) \iff \exists c, n_0 \text{ s.t. } \forall n \geq n_0, |f(n)| \leq c \cdot |g(n)|$

Theorem The above definations are equivalent.

Proof Left as excercise. ■

While we have used lim sup here, it is only to deal with the cases where the limit is not gurenteed to exist. We can use lim in place for all the definations till here in most cases we will be concerned with.

In the family of big oh, we also have some more notations one must be aware of. This is called the Bachmann–Landau family, with contributions from Hardy and Knuth.

[7] We could just say this part was trivial.

÷ Bachman-Landau Notation

Notation	Name	Classical Definition	Limit Definition
$f(n) = \Theta(g(n))$	Big Theta	$\exists k_1, k_2, n \; s.t. \; \forall n > n_0, k_1 g(n) \leq \lvert f(n) \rvert \leq k_2 g(n)$	$f(n) \asymp g(n)$
$f(n) = \Omega(g(n))$	Big Omega	$\exists k, n_0 \; s.t. \; \forall n \geq n_0, \lvert f(n) \rvert \geq k g(n)$	$\liminf_{n\to\infty} \frac{\lvert f(n) \rvert}{g(n)} = 0$
$f(n) \sim g(n)$	Total Asymptotic Equivalence (diffrent for asymptotic eqivalende defined using $\asymp$.)		$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$
$f(n) = o(g(n))$	Small Oh	$\exists k, n_0 \; s.t. \; \forall n \geq n_0, f(n) \leq k g(n)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
$f(n) = \omega(g(n))$	Small Omega	$\exists k, n_0 \; s.t. \; \forall n \geq n_0, f(n) > k g(n)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

Theorem If $f \in F$ evivalence class and $\hat{f}$ is the reprasentative of the class, then $f(x) = \Theta\left(\hat{f}(x)\right)$

Proof Follows trivially from defination. ■

Here is an example to illustrate the diffrence between the notations.

- $3n^2 - 100n + 6 = o(n^2)$, because I choose $c = 3, n_0 = \frac{6}{100}$;
- $3n^2 - 100n + 6 = o(n^3)$, because I choose $c = 1, n_0 = \frac{7}{100}$;

– $3n^2 - 100n + 6 \neq o(n)$, because for any c I can take $n > 100 + c$;

- $3n^2 - 100n + 6 = \Omega(n^2)$, because I choose $c = 2, n_0 = 100$;
- $3n^2 - 100n + 6 \neq \Omega(n^3)$, because for an c, I can take $n > \frac{100}{c}$, if $c < 1$ and $n > c$ otherwise;

– $3n^2 - 100n + 6 = \Omega(n)$, because for any c, I can take $n_0 = 100c$;

- $3n^2 - 100n + 6 = \Theta(n^2)$, because both o and Ω apply;
- $3n^2 - 100n + 6 \neq \Theta(n^3)$, because Ω fails;

– $3n^2 - 100n + 6 \neq \Theta(n)$, because o fails.

Also note, $3n^2 - 100n + 6 \sim 3n^2$ by taking the limit.

Finally, note that o, ω are equal to o, Ω here as our limits exist.

X Big Oh Arithematic

Prove the following:

1. $n^m = o\left(n^{m'}\right) \quad m \leq m'$
2. $o(f(n)) + o(g(n)) = o(\lvert f(n) \rvert + \lvert g(n) \rvert)$
3. $f(n) = o(f(n))$
4. $o(o(f(n))) = o(f(n))$
5. $co(f(n)) = o(f(n))$ where c is constent
6. $o(f(n))o(g(n)) = o(f(n)g(n)) = f(n)o(g(n)) = g(n)o(f(n))$
7. $o(f(n)^2) = o(f(n))^2$

§10.3. Asymptotic Mathematics

Notice that Hardy was involoved in devloping some of this notation. What has an analytical number theory got to do with a tool for algorithmic analysis?

We wrote $n \to \infty$ next to o in the definitions as we are consider the limits approaching ∞. We can also define similerly for $n \to 0$ and $n \to k$. These definitions are more common in math than CS, as mathematicains care about behavior of functions at a lot of places other than ∞. As this is a CS book, in absence of clarification, assume $n \to \infty$.

This allows us to write taylor series in big-oh notation, for example as $\sin(x) = x - \frac{x^3}{3!} + o(x^5)(x \to 0)$.

All this can be made useful in problems like:

Sum of Power of Numbers

Comment on the asymptotic behavior of

$$f_{k(n)} = \sum_{i=1}^{n} i^k$$

where $k, n \in \mathbb{N}$.

Let's start with some small cases.

$$f_0(n) = \sum_{i=1}^{n} i^0 = \sum_{i=1}^{n} 1 = n = o(n)$$

$$f_1(n) = \sum_{i=1}^{n} i^1 = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = o(n^2)$$

$$f_2(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = o(n^3)$$

This seems to indicate $f_{k(n)} = o(n^{k+1})$, but how do we prove this?

Derivative with big Oh

We can define the derivative of f through this equation[8]

$$f(x+h) = f(x) + hf'(x) + o(h^2) \quad (h \to 0)$$

Binoial Exapansion with Big Oh

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \ldots + \binom{n}{k}x^k + o(x^{k+1}) \quad (n \to 0)$$

Taylor Series with Big Oh

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \ldots + f^{(i)}(0)\frac{x^i}{i!} + \mathcal{O}(x^{i+1}) \quad (x \to 0)$$

This allows us to define $(x+h)^{k+1} = x^{k+1} + h(k+1)x^k + o(h^2)$, taking $h = 1$;

[8] This only works if f has a strong or strict derivative. For most functions we care about, this is true (and equals their usual derivative). This note is included to clarify in case you have doubts, despite the fact we do not wish to go into the further technicalities of this.

$$(x+1)^{k+1} = (x)^{k+1} + (k+1)x^k + o(1)$$
$$(x+1)^{k+1} - (x)^{k+1} = (k+1)x^k + o(1)$$
$$\sum_{x=0}^{n}[(x+1)^{k+1} - (x)^{k+1}] = \sum_{x=0}^{n}[(k+1)x^k + o(1)]$$
$$\sum_{i=1}^{n+1}[i^{k+1} - (i-1)^{k+1}] = (k+1)\sum_{x=1}^{n} x^k + o(n)$$
$$\frac{(n+1)^{k+1} - o(n)}{k+1} = \sum_{x=1}^{n} x^k$$
$$\sum_{x=1}^{n} x^k = \frac{n^{k+1} + (k+1)n^k + o(1) - o(n)}{k+1}$$
$$\sum_{x=1}^{n} x^k = o\left(n^{k+1}\right)$$

We can also use this notion to reprasent some mathematics results. For example:

Prime Number Theorem

Let $\pi(n)$ reprasent the number of primes less then n. Then:

$$\pi(n) \sim \frac{n}{\ln(n)}$$

Stirling Approximation

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + o\left(\frac{1}{n^2}\right)\right)$$

[9] Also note,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$
$$\Rightarrow n! = \Theta\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right)$$

Binomial Coefficient

When k is fixed,

$$\binom{n}{k} \sim \frac{n^k}{k!} = \Theta(n^k)$$

When n, k are both large, we can use the Stirling Approximation to get:

$$\binom{n}{k} \sim \sqrt{\frac{n}{2\pi k(n-k)}}\left(\frac{n^n}{k^k(n-k)^{n-k}}\right)$$

One case we have suspiciously left out is when k is small.

[9] More terms can be obtained by using the Bernolli numbers. We have gone for this form as the proof follows from probability theory and doesn't involve results from complex analysis.

Small k binomials

We will restrict ourselves to $k = o(n)$ as other cases get messy rather quickly.

Notice

$$\binom{n}{k} = A(n,k)\frac{n^k}{k!}$$

where

$$A(n,k) := \prod_{i=1}^{k-1}\left(1 - \frac{i}{n}\right)$$

Thus, to find the asymptotics, we just need to solve for A.

$$\ln(A(n,k)) = \sum_{i=1}^{k-1}\ln\left(1 - \frac{i}{n}\right)$$

Solve when

(i) $k = o(\sqrt{n})$

(ii) $k = c\sqrt{n}$ and c is known

(iii) $k = o\left(n^{\frac{2}{3}}\right)$

(iv) $k = o\left(n^{\frac{3}{4}}\right)$

(v) $k = o(n)$

We will solve the first 3 cases and leave the rest for you, the reader.

(i) Using ÷ Taylor Series with Big Oh on $\ln(1+x) = x + o(x^2)$, we will get:

$$\begin{aligned}
\ln(A(n,k)) &= \sum_{i=1}^{k-1}\ln\left(1 - \frac{i}{n}\right) \\
&= -\sum_{i=1}^{k-1}\left(\frac{i}{n} + \mathcal{O}\left(\frac{i}{n}\right)^2\right) && \left(\frac{i}{n} \to 0\right) \\
&\sim -\frac{k^2}{2n} + \mathcal{O}(k^3 n^{-2}) && \left(\frac{k}{n} \to 0\right) \quad \text{using the sum of consecutive terms} \\
&= o(1) && (k, n \to \infty) \quad \text{using } k = o(\sqrt{n})
\end{aligned}$$

This implies $A = e^{o(1)} = o(1) \Rightarrow \binom{n}{k} = o(1) \cdot \frac{n^k}{k!} = o\left(\frac{n^k}{k!}\right)$.

(ii) Here the c being specified gives us $A = e^{-\frac{c^2}{2}} \Rightarrow \binom{n}{k} = e^{-\frac{c^2}{2}}\frac{n^k}{k!}$.

Hint for (iii), (iv), (v): We were able to disolve the $\mathcal{O}(k^3 n^{-2})$ term as if $k = o(\sqrt{n})$, the term would be $\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$ and goes to zero, rapidly.

We can 'fix' this by having a slightly bigger $k = o\left(n^{\frac{2}{3}}\right)$. The same happens in **(iv)** as we take a more and more accurate Taylor exapansion for $\ln\left(1 - \frac{i}{n}\right)$. Can we get some sort of a limiting case and solve for $k = o(n)$?

X Bootstrap(Erdos, Greene)

Find the asymptotics of f satisfying:

$$f(t)e^{f(t)} = t$$

as $t \to \infty$

The idea here is to begin with a weak bound and progressivle make it stronger.

Here we start by simplyfying the equation:

$$\ln(f(t)) + f(t) = \ln(t) \quad (t \to \infty)$$

We know that $f(x) \succ \ln(f(x))$, thus

$$\lim_{t\to\infty} \frac{\ln(f(t)) + f(t)}{\ln(t)} = 1$$
$$\Rightarrow \lim_{t\to\infty} \frac{f(t)}{\ln(t)} = 1$$
$$\Rightarrow f(t) = \Theta(\ln(t))$$

Making this substitution gives us,

$$f(t) = \ln(t) - \Theta(\ln(\ln(t)))$$

Now, making this substitution gives us,

$$\ln(\ln(t) - \Theta(\ln(\ln(t)))) + f(t) = \ln(t)$$
$$\ln\left(\ln(t)\left(1 - \Theta\left(\frac{\ln(\ln(t))}{\ln(t)}\right)\right)\right) + f(t) = \ln(t)$$
$$\ln(\ln(t)) + \ln\left(1 - \Theta\left(\frac{\ln(\ln(t))}{\ln(t)}\right)\right) + f(t) = \ln(t)$$
$$f(t) = \ln(t) - \ln(\ln(t)) + \Theta\left(\frac{\ln(\ln(t))}{\ln(t)}\right)$$

And one can continue getting more and more terms by this process. We stop here as getting the logitherm now will be much harder.

X A Weird Condition

Prove that there are infinitely many integers n such that the sum of digits of n in base $p < n$ exceeds 2025 for every prime p.

This question is incredibly hard to answer through regular means of construction of contradiction.

We will try to set up an asymptotic bound on number of integers violating the condition and show that their density is less than 1, hence, showing that the condition is true for infinitly many numbers.

Let's say $x < N$ integrs violate these condition for some large N. Notice,

$$\forall \text{ primes } p \leq n, \exists k \in \mathbb{N}\, s.t.\, n^{\frac{1}{k+1}} < p \leq n^{\frac{1}{k}}$$

Thus, given a p, any number from $1..N$ has atmost $k+1$ digits in base p. Thus, a violation can only occcur if the sum of digits: $d_1, d_2, ..., d_{k+1} < p$ is less than 2025.

We can (grossly) overcount the number of violaters using the sum:

$$x < \sum_{k} \sum_{\substack{N^{\frac{1}{k+1}} < p \leq N^{\frac{1}{k}} \\ p \text{ is prime}}} \binom{k+2026}{2025}$$

To might be as good time as any to get the bounds on k.

$$\pi\left(N^{\frac{1}{k_m}}\right) - \pi\left(N^{\frac{1}{k_m+1}}\right) = 0$$

$$\frac{k_m N^{\frac{1}{k_m}}}{\ln(n)} - \frac{(k_m+1)N^{\frac{1}{k_m+1}}}{\ln(n)} = 0$$

$$k_m N^{\frac{1}{k_m}} = (k_m+1)N^{\frac{1}{k_m+1}}$$

$$N^{\frac{1}{k_m(k_m+1)}} = 1 + \frac{1}{k_m}$$

$$\frac{1}{k_m(k_m+1)}\ln(N) = \ln\left(1+\frac{1}{k_m}\right)$$

$$\frac{1}{k_m(k_m+1)}\ln(N) = \frac{1}{k_m} + O\left(\frac{1}{k_m^2}\right)$$

$$\frac{1}{k_m+1}\ln(N) = 1 + O\left(\frac{1}{k}\right)$$

$$\ln(N) = k + O(1) + O\left(\frac{1}{k_m}\right)$$

$$k_m = \Theta(\ln(N))$$

Notice, for $k = 1$,

$$\sum_{\substack{\sqrt{N} < p \leq N \\ p \text{ is prime}}} \binom{2027}{2025} = \mathcal{O}(\pi(N)) < \frac{N}{3}$$

And for $k > 1$,

$$\sum_{k>1}^{k_m} \sum_{\substack{N^{\frac{1}{k+1}} < p \leq N^{\frac{1}{k}} \\ p \text{ is prime}}} \binom{k+2026}{2025}$$

$$= \sum_{k>1}^{k_m} \sqrt{N}\Theta(k^{2025})$$

$$= \sqrt{N}\Theta(\ln(N)^{2025})\Theta(\ln(n))$$

$$= \sqrt{N}\Theta(\ln(N)^{2026})$$

$$< \frac{N}{3}$$

Thus, $x < \frac{2N}{3}$ for some n. Thus, atleast $\frac{N}{3}$ integers exist satisfying our property for some large N.

§10.4. Analysis of Algoritms

With the (damn scary) math out of the way, we will now move to algoritms and the real reason we are concerned with asymptotics. Take a deep breath, relax; the scary part is beyond us.

Before we get to designing and analyzing algorithms, let's pause and breifly question what 'algorithm' actually means. To quote Hannah Fry,

> **Hannah Fry**
>
> It's a term that, although used frequently, routinely fails to convey much actual information. This is partly because the word itself is quite vague. Officially, it is defined as follows:
>
> *algorithm (noun): A step-by-step procedure for solving a problem or accomplishing some end especially by a computer.*
>
> An algorithm is simply a series of logical instructions that show, from start to finish, how to accomplish a task. By this broad definition, a cake recipe counts as an algorithm. So does a list of directions you might give to a lost stranger. IKEA manuals, YouTube troubleshooting videos, even self-help books – in theory, any self-contained list of instructions for achieving a specific, defined objective could be described as an algorithm. But that's not quite how the term is used. Usually, algorithms refer to something a little more specific. They still boil down to a list of step-by-step instructions, but these algorithms are almost always mathematical objects. They take a sequence of mathematical operations – using equations, arithmetic, algebra, calculus, logic and probability – and translate them into computer code. They are fed with data from the real world, given an objective and set to work crunching through the calculations to achieve their aim. They are what makes computer science an actual science, and in the process have fuelled many of the most miraculous modern achievements made by machines.

We will take Prof. Fry's defination as the gospel as trying to go into more details will open up questions which are more philosophical than we wish to be here.

§10.4.1. Multiplication

The word Algorithm orignates from French where it was the mistranslated name of the 9th Century Arabic scholer Al-Khwarizmi, who was born in present day Uzbekistan, who studied and worked in Baghdad. His text on multiplying indo-arabic numerals travelled to Europe and his name was mis translated to "Algorisme" which later evolved into algorithm. While we will see other algorithms of the ancients, let's begin with the OG multiplication. We will consider the multiplication of two n digit numbers, given it takes a single operation to solve for $n = 1$ (base cases). We assume additions of sigle digit takes $\mathcal{O}(1)$ (constent) time, and hence, adding a m, n digit number takes $\mathcal{O}(\min(m, n))$ time. Similerly, if a function calls some other function, we say this call takes $\mathcal{O}(1)$ time.

The naive way to do so would be to define multiplication as repeated addition. Something of the sort `multiply 1 b = b` and the reccurence. `multiply a b = b + multiply (a-1) b`. For two `n` digit numbers, this will take $\mathcal{O}(10^n) \cdot \mathcal{O}(n) = \mathcal{O}(n10^n)$ operations as $b < 10^n$ and we need to make b function calls and as many additions with the smallest number of size n. That is very bad, we will see the quantitatatives in a moment.[10]

Notice, this is a departure from our usual method of counting every operation and we are instead taking the big-oh approximation. We will talk why this is a good idea most of the times in some while, but in this case, it is the only way to deal with the mix of operations we are using and reconciling number of operations.

An improvement in the multiplication algorithm, we are already familier with, is the one taught in school. This was also the algorithm Al-Khwarizmi found. The number of operations fort this algorithm

[10] We obviously know that the function definition is not complete. We need to deal with negitives and zero, but all of that doesn't really change the time complexity.

is $\mathcal{O}(n^2) + \mathcal{O}(n)\mathcal{O}(n-1) = \mathcal{O}(n)$ as we multiply all the digits in the latter number with the digits in the former number and then add the results suitably one by one. That is in $32 * 45$, we will compute $32 * 5$ and $32 * 4$ and add them.

Doing a lot better than this took about a millenia, with the improvement comming from Anatoly Karatsuba in 1962. The idea used is the usual divide and conquor.

We divide the fist number into $x = a * 10^{\frac{n}{2}} + b$ and the second number as $y = c * 10^{\frac{n}{2}} + d$. Thus,

$$\begin{aligned} xy &= \left(a * 10^{\frac{n}{2}} + b\right) * \left(c * 10^{\frac{n}{2}} + d\right) \\ &= ac * 10^n + bc * 10^{\frac{n}{2}} + ad * 10^{\frac{n}{2}} + bd \end{aligned}$$

Let's say it take $T(n)$ operations to multiply two n digit numbers. Thus, our problem of multiplying two n digit numbers can be reduced to multiplying two $\frac{n}{2}$ digit number 4 times.

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) + \mathcal{O}(1) \\ \Rightarrow T\left(\frac{n}{2}\right) &= 4T\left(\frac{n}{4}\right) + \mathcal{O}\left(\frac{n}{2}\right) + O(1) \\ \Rightarrow \ldots \Rightarrow T(n) &= 4^{\log(n)} + \mathcal{O}(n)1 * \log(n) \\ \Rightarrow T(n) &= n^2 + \mathcal{O}(\log(n)) = \mathcal{O}(n^2) \end{aligned}$$

But we didn't do any better! All the effort seems to be in vain. Well, this is where Karatsuba's brillience comes to play.

$$\begin{aligned} xy &= ac * 10^n + bc * 10^{\frac{n}{2}} + ad * 10^{\frac{n}{2}} + bd \\ xy &= ac * 10^n + (bc + ad) * 10^{\frac{n}{2}} + bd \end{aligned}$$

We only need three terms. If we could somehow figure out $bc + ad$ without needing to find them both individually. The genius idea was:

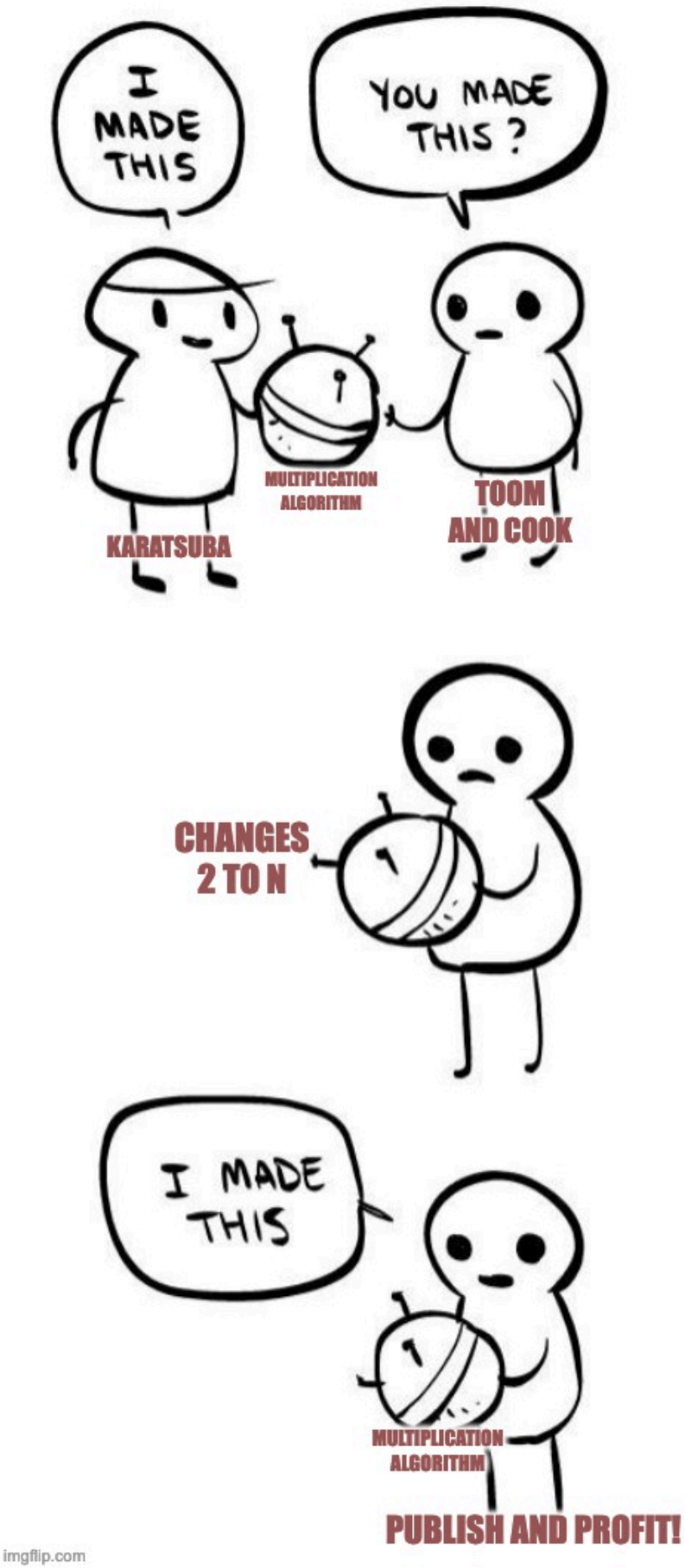
$$(a+b)(c+d) = ac + bd + (bc + ad)$$

and we already have computed ac and bd. If we subtract them, we will have the third term. Notice, $a + b, c + d$ are atmost an $\frac{n}{2} + 1$ digit numbers. Thus, we can now only make three smaller multiplications. We will claim $T(n+1) = T(n) + \mathcal{O}(n)$ as we can divide the given $n+1$ digit numbers into two parts, the most significent n digits and then the last digit and complete the computation.

This will give us,

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) + \mathcal{O}(1) \\ \Rightarrow T(n) &= 3^{\log(n)} + \mathcal{O}(n) + \mathcal{O}(1)\log(n) \\ \Rightarrow T(n) &= \mathcal{O}\left(n^{\log(3)}\right) \approx \mathcal{O}\left(n^{1.6}\right) \end{aligned}$$

This is a lot better. The next improvement came just an year later in 1963 by Tooom and Cook, making it $\mathcal{O}\left(n^{\log_3(5)}\right)$. Here is what we belive their research process looked like:


I MADE THIS
YOU MADE THIS?
KARATSUBA
MULTIPLICATION ALGORITHM
TOOM AND COOK
CHANGES 2 TO N
I MADE THIS
MULTIPLICATION ALGORITHM
PUBLISH AND PROFIT!
imgflip.com

If you are wondering, they showed that we can break the multiplication in five $\frac{n}{3}$ sized products. Actually, we can split in any number of parts we want. Karatsuba is Toom-2, the $\mathcal{O}\left(n^{\log_3(5)}\right)$ algorithm is Toom-3. When split in some k parts, the complexity is $\mathcal{O}(n^E)$ where $E = \log_{k(2k-1)}$).

This can in theory do $\mathcal{O}(1)$ multiplication. As we will see in the upcoming section on the dark secrets of big-oh, $\mathcal{O}$ sweeps the constents under the rug. This works when the rug is heavy, large and the constents tiny. But if, the constent is large: well then $\Theta\left(10^{10^{100}}n\right)$ is worse than $\Theta(n^2)$ for all values we could care about, irrespective of the fact the former is $\mathcal{O}(n)$ while the latter is $\mathcal{O}(n^2)$.

Doing an exact complexity analysis can allow us to compute the exact speed of growth of the constent of $\mathcal{O}(n^E)$ (hint: It is basically exponential).

This leads us to the $\mathcal{O}(n\log(n)\log(\log(n)))$ Schönhage–Strassen algorithm (1971) which uses the Discrete Fast Fouries Transform algorithm described in chapter 8. The exact implementation is left as excercise to the morbidly curious. In this paper, Arnold Schönhage and Volker Strassen also conjectured a lower bound of $\Omega(n\log(n))$.

This is about the end of multiplication algorithms I can hope to talk about with the material in this book. Also, the constents hidden by big-oh become so large that most implementations use Karatsuba or Toom-3 till some size and then switch to Schönhage–Strassen. So everything here onwards are just fun facts.

The next leap came in 2007, when Martin Fürer improved the bound to $\mathcal{O}\left(n\log(n)2^{\mathcal{O}(\log^*(n))}\right)$ where the $\log^*(n)$ denotes the number of times we must take $\log(n)$ before we go below 1. This leap was made possible due to half-DFT's, lots of ring theory and complex analysis and certain results about primes of form $p = 2^{2^k} + 1$ turining up true. This algorithm beats Schönhage–Strassen for integers with about 10^{19} digits.

The next improvement came by using number theory inplace of complex analysis and other changes courtasy Anindya De, Piyush P Kurur, Chandan Saha and Ramprasad Saptharishi[11] (2008) which beats this Fürer for numbers with about 10^{4796} digits.

In 2015, David Harvey, Joris van der Hoeven and Grégoire Lecerf gave a new algorithm which replaced the $\mathcal{O}(\log^*(n))$ with $3\log^*(n)$. The only issue is that this paper used certain unproven conjuctures on Mersenne primes.

In 2015-16, in a series of two papers, Svyatoslav Covanov and Emmanuel Thomé first made a new algorithm with same complexity as Fürer and then, using unproven conjuctures on Fermat Primes and genralizations, produced an algorithm where $\mathcal{O}(\log^*(n))$ is replaced with $2\log^*(n)$.

Not to be defeated so easily, Harvey and Hoevan snapped back in 2018 with an algorithm which acives the $2\log^*(n)$ complexity without any conjuctures. They use Minkowski's theorem (which funnily was proven in 1889).

And to end this on a win, Harvey and Hoeveen publish the first $\mathcal{O}(n\log(n))$ in March 2019.

> **David Harvey and Joris van der Hoeven**
>
> ...our work is expected to be the end of the road for this problem, although we don't know yet how to prove this rigorously."

So well, can we do better is still an open question.

Anyways, for this paper, they shared the de Bruijn medal in 2022. Well, we have come full circle I guess.

[11]

Advanced Data Structures

Arjun Maneesh Agarwal

§11.1. post-complexity data types (feel free to change it)

§11.1.1. Stacks and Queues

Type Classes

Ryan Hota

§12.1. typeclasses (feel free to change it)

Monads

Ryan Hota

§13.1. Monad (feel free to change it)

Appendix

Ryan Hota, Shubh Sharma, Arjun Maneesh Agarwal

Nothing past this point is for exam, obviously.

§14.1. Table of Contents

- return to the above heading

§14.2. Basic Theory

- return to the above heading

§14.3. Precise Communication

- return to the above heading

§14.4. The Building Blocks

- return to the above heading

§14.5. Values

- return to the above heading

§14.6. Variables

- return to the above heading

§14.7. Well-Formed Expressions

- return to the above heading

§14.8. Function Definitions

- return to the above heading

§14.8.1. Using Expressions

- return to the above heading

§14.8.2. Some Conveniences

- return to the above heading

§14.8.2.1. Where, Let

- return to the above heading

§14.8.2.2. Anonymous Functions

- return to the above heading

§14.8.2.3. Piecewise Functions

- return to the above heading

§14.8.2.4. Pattern Matching

- return to the above heading

§14.8.3. Recursion

- return to the above heading

§14.8.3.1. Termination

- return to the above heading

§14.8.3.2. Induction

- return to the above heading

§14.8.3.3. Proving Termination using Induction

- return to the above heading

§14.9. Infix Binary Operators

- return to the above heading

§14.10. Trees

- return to the above heading

§14.10.1. Examples of Trees

- return to the above heading

§14.10.2. Making Larger Trees from Smaller Trees

- return to the above heading

§14.10.3. Formal Definition of Trees

- return to the above heading

§14.10.4. Structural Induction

- return to the above heading

§14.10.5. Structural Recursion

- return to the above heading

§14.10.6. Termination

- return to the above heading

§14.11. Why Trees?

- return to the above heading

§14.11.1. The Problem

- return to the above heading

§14.11.2. The Solution

- return to the above heading

§14.11.3. Exercises

- return to the above heading

§14.12. Installing Haskell

- return to the above heading

§14.13. Installation

- return to the above heading

§14.13.1. General Instructions

- return to the above heading

§14.13.2. Choose your Operating System

- return to the above heading

§14.13.2.1. Linux

- return to the above heading

§14.13.2.2. MacOS

- return to the above heading

§14.13.2.3. Windows

- return to the above heading

§14.14. Basic Syntax

- return to the above heading

§14.15. The Building Blocks

- return to the above heading

§14.16. Values

- return to the above heading

§14.17. Variables

- return to the above heading

§14.18. Types

- return to the above heading

§14.18.1. Using GHCi to get Types

- return to the above heading

§14.18.2. Types of Functions

- return to the above heading

§14.19. Well-Formed Expressions

- return to the above heading

§14.20. Infix Binary Operators

- return to the above heading

§14.20.1. Precedence

- return to the above heading

§14.21. Logic

- return to the above heading

§14.21.1. Truth

- return to the above heading

§14.21.2. Statements

- return to the above heading

§14.22. Conditions

- return to the above heading

§14.22.1. Logical Operators

- return to the above heading

§14.22.1.1. Exclusive OR aka XOR

- return to the above heading

§14.23. Function Definitions

- return to the above heading

§14.23.1. Using Expressions

- return to the above heading

§14.23.2. Some Conveniences

- return to the above heading

§14.23.2.1. Piecewise Functions

- return to the above heading

§14.23.2.2. Pattern Matching

- return to the above heading

§14.23.2.3. Where, Let

- return to the above heading

§14.23.2.4. Without Inputs

- return to the above heading

§14.23.2.5. Anonymous Functions

- return to the above heading

§14.23.3. Recursion

- return to the above heading

§14.24. Optimization

- return to the above heading

§14.25. Numerical Functions

- return to the above heading

§14.25.1. Division, A Trilogy

- return to the above heading

§14.25.2. Exponentiation

- return to the above heading

§14.25.3. `gcd` and `lcm`

- return to the above heading

§14.26. Mathematical Functions

- return to the above heading

§14.26.1. Taylor Series

- return to the above heading

§14.27. Exercises

- return to the above heading

§14.28. Types as Sets

- return to the above heading

§14.29. Sets

- return to the above heading

§14.30. Types

- return to the above heading

§14.30.1. `::` is analogous to $\in$ or ≒ belongs

- return to the above heading

§14.30.2. `A → B` is analogous to B^A or ≒ set exponent

- return to the above heading

§14.30.3. `( A , B )` is analogous to $A \times B$ or ≒ cartesian product

- return to the above heading

§14.30.4. `()` is analogous to ≒ singleton set

- return to the above heading

§14.30.5. No ≒ intersection of Types

- return to the above heading

§14.30.6. No ≒ union of Types

- return to the above heading

§14.30.7. Disjoint Union of Sets

- return to the above heading

§14.30.8. `Either A B` is analogous to $A \sqcup B$ or ≒ disjoint union

- return to the above heading

§14.30.9. The `Maybe` Type

- return to the above heading

§14.30.10. `Void` is analogous to $\{\}$ or ≒ empty set

- return to the above heading

§14.31. Introduction to Lists

- return to the above heading

§14.32. Type of List

- return to the above heading

§14.33. Creating Lists

- return to the above heading

§14.33.1. Empty List

- return to the above heading

§14.33.2. Arithmetic Progression

- return to the above heading

§14.34. Functions on Lists

- return to the above heading

§14.35. List Comprehension

- return to the above heading

§14.35.1. Cons or `(:)`

- return to the above heading

§14.36. Length

- return to the above heading

§14.36.1. Concatenate or `(++)`

- return to the above heading

§14.36.2. Head and Tail

- return to the above heading

§14.36.3. Take and Drop

- return to the above heading

§14.36.4. Elem

- return to the above heading

§14.36.5. (!!)

- return to the above heading

§14.37. Strings

- return to the above heading

§14.38. Structural Induction for Lists

- return to the above heading

§14.39. Optimization

- return to the above heading

§14.40. Lists as Syntax Trees

- return to the above heading

§14.41. Dark Magic

- return to the above heading

§14.41.1. Excercises

- return to the above heading

§14.42. Polymorphism and Higher Order Functions

- return to the above heading

§14.43. Polymorphism

- return to the above heading

§14.43.1. Classification has always been about *shape* and *behvaiour* anyway

- return to the above heading

§14.43.2. A Taste of Type Classes

- return to the above heading

§14.44. Higher Order Functions

- return to the above heading

§14.44.1. Currying

- return to the above heading

§14.44.2. Functions on Functions

- return to the above heading

§14.44.3. A Short Note on Type Inference

- return to the above heading

§14.44.4. Higher Order Functions on Maybe Type : A Case Study

- return to the above heading

§14.45. Advanced List Operations

- return to the above heading

§14.46. List Comprehensions

- return to the above heading

§14.47. Zip it up!

- return to the above heading

§14.48. Folding, Scaning and The Gate to True Powers

- return to the above heading

§14.48.1. Orgami of Code!

- return to the above heading

§14.48.2. Numerical Integration

- return to the above heading

§14.48.3. Time to Scan

- return to the above heading

§14.49. Excercises

- return to the above heading

§14.50. Introduction to Datatypes

- return to the above heading

§14.51. Datatypes (Once Again)

- return to the above heading

§14.52. Finite Types

- return to the above heading

§14.53. Product Types

- return to the above heading

§14.54. Parametric Types

- return to the above heading

§14.55. Sum Types

- return to the above heading

§14.56. Inductive Types

- return to the above heading

§14.56.1. Inductive Types (as a Mathematician)

- return to the above heading

§14.56.1.1. Natural Numbers as Inductive Types

- return to the above heading

§14.56.1.2. Lists as Inductive Types

- return to the above heading

§14.56.2. (Not Quite) Inductive Types (as a Programmer)

- return to the above heading

§14.56.2.1. Calculator

- return to the above heading

§14.56.2.2. Trees as Inductive Types

- return to the above heading

§14.56.2.3. Binary Trees

- return to the above heading

§14.56.2.4. Addressing the “Not Quite”...

- return to the above heading

§14.57. Same Same but Different

- return to the above heading

§14.57.1. Same Same

- return to the above heading

§14.57.2. Different

- return to the above heading

§14.58. Computation as Reduction

- return to the above heading

§14.59. Complexity

- return to the above heading

§14.60. Introduction

- return to the above heading

§14.61. Asymptotics

- return to the above heading

§14.61.1. Big El

- return to the above heading

§14.61.2. The Hierarchy of Functions

- return to the above heading

§14.61.3. Big Oh notation

- return to the above heading

§14.62. Asymptotic Mathematics

- return to the above heading

§14.63. Analysis of Algoritms

- return to the above heading

§14.63.1. Multiplication

- return to the above heading

§14.64. Advanced Data Structures

- return to the above heading

§14.65. post-complexity data types (feel free to change it)

- return to the above heading

§14.65.1. Stacks and Queues

- return to the above heading

§14.66. Type Classes

- return to the above heading

§14.67. typeclasses (feel free to change it)

- return to the above heading

§14.68. Monads

- return to the above heading

§14.69. Monad (feel free to change it)

- return to the above heading

§14.70. Appendix

- return to the above heading