Algorithmic Game Theory

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based on course by Prajakta Nimbhorkar

1. Games and Equilibria : We are assuming games where everyone is acting selfishly. When everyone doesn't wish to switch strategies, we get an equilibrium.
    - We will see two-player games and multiplayer games.
    - Equilibrium and existence and computation.
2. Mechanism Design : Given multiple selfish agents, we need to make a mechanism that encourages (or discourages using hardness) the behavior we wish to encourage.
    - We will see auctions, voting etc.
3. Fair Division: This is an allocation problem where we want to allocate some resources in a 'fair' (or less unfair) way.
    - Distributing divisible and indivisible resources fairly among agents, given valuation.
    - EF, EQ, PROP and its relaxations.

Table of Contents

1. Games and Equilibria

We make the following assumptions:

1. Rationality/selfishness : Each agent attempts to maximize own payoff and believes others will do so as well as well has knowledge of same behavior being
2. Intelligence: Agents have enough computational resources to take into account the strategies and behaviors of others.
3. Common Knowledge: P, S_i, μ are known to everyone and everyone knows that everyone knows them and so on.

1.1. Braess's Paradox

Taking the example of traffic, let the network be:
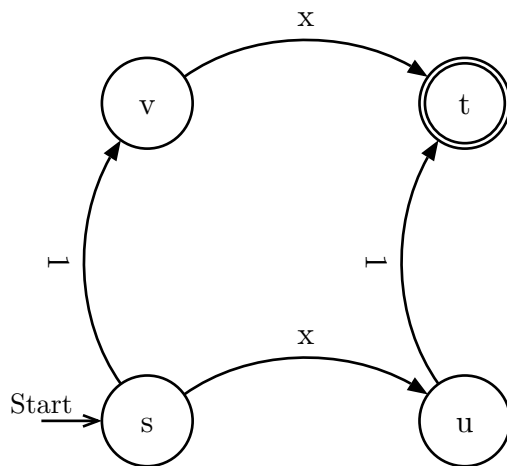
In this case, the equilibrium is 1.5 hours as traffic is divided in the two paths. Let's say the authorities build a bridge:
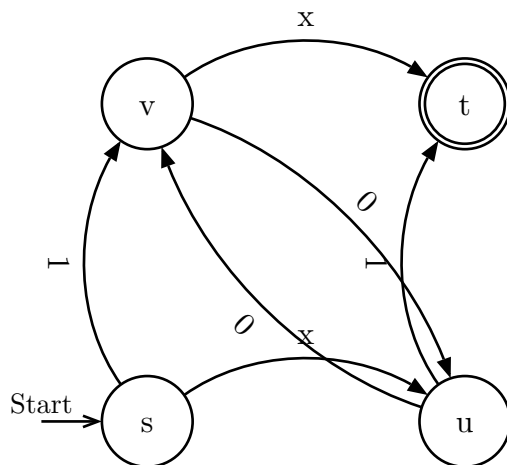
This will lead to a (weak) equilibrium of 2 hours as all agents take the route s-v-u-t.

This is called Braess's Paradox.

> **Side Note**
>
> We have actually seen this example occur almost verbatim in Seoul where the demolition of a bridge in 2005, solved a 2 decade old traffic problem.
>
> Furthermore, Steinberg and Zangwill, 1983; showed that in a random network, Braess's paradox is approximately 50% likely to occur in an random network.
>
> It is also a consideration in google maps where the navigation sometimes suggests a longer route, not to avoid a pre-existing traffic jam, but to avoid creating one. (This is why google map is not referentially transparent and may suggest two people different routes between same places at same time)
>
> **TODO.**

1.2. Prisoner's Dilemma

P2

	C	D
C	$4,4$	$5,1$
D	$1,5$	$2,2$

P1

Here both players have incentive to deviate from the mutually best case as confessing (strongly) dominates denial and hence, is a Nash equilibrium.

Let's define these words formally.

Definition: Game

To define a game we need to know the players and strategies for each player. Thus, a game $G = (P, S, \mu)$ where for i in P, S_i in S denotes the set of possible strategies for player i.

In a particular play of the game denoted as s, i plays strategy s_i forming the strategy profile or game vector $(s_1, s_2, ..., s_n)$.

We denote the payoffs or costs of the game for player i using a function $\mu_i : S_1 \times S_2 \times ... \times S_n \to \mathbb{R}$ which takes the game vector and tells the payoff for player i. $\mu : S_1 \times S_2 \times ... \times S_n \to \mathbb{R}^n; \mu(s) = (\mu_1(s), \mu_2(s), ..., \mu_n(s))$ is the payoff vector returning function.

Finally, for conciseness, we sometimes denote the choices for all players except i as s_{-i}. Thus, $\mu_i(s_i, s_{-i})$

Definition: Strongly Dominant Strategy

Player i's strategy $\hat{s}_i$ is a **Strongly Dominant Strategy** to the strategy s_{-i} of other players if:

$$\mu_i(\hat{s}_i, s_{-i}) > \mu_i(s'_i, s_{-i}) \forall s'_i \in S_i$$

Definition: Weakly Dominant Strategy

Player i's strategy $\hat{s}_i$ is a **Weakly Dominant Strategy** to the strategy s_{-i} of other players if:

$$\mu_i(\hat{s}_i, s_{-i}) \geq \mu_i(s'_i, s_{-i}) \forall s'_i \in S_i$$

and

$$\exists s'_i \in S_i \text{ s.t.} \mu_i(\hat{s}_i, s_{-i}) > \mu_i(s'_i, s_{-i})$$

> **Definition: Weakly Dominant Strategy / Best Response(BR)**
>
> Player i's strategy $\hat{s}_i$ is a **best response** to the strategy s_{-i} of other players if:
> $$\mu_i(\hat{s}_i, s_{-i}) \geq \mu_i(s'_i, s_{-i}) \forall s'_i \in S_i$$
> or if $\hat{s}_i$ solves $\max_{S_i} \mu_i(s_i, s_{-i})$.[1]

> **Definition: Strongly Dominated Strategy Equilibrium**
>
> Informally, this is when all players are playing dominant strategies.
>
> It is a strategy profile $(s_1^*, s_2^*, ..., s_N^*)$ is a SDSE if for each i, her choice s_i^* is strongly dominant to all .

> **Definition: Nash Equilibrium(NE)**
>
> Informally, this is when all players are playing BR to each other.
>
> Formally, it is a strategy profile $(s_1^*, s_2^*, ..., s_N^*)$ is a NE if for each i, her choice s_i^* is a best response to s_{-i}^*.

> **Remark**
>
> The reason Nash equilibrium is important is as it leads to no-regrets as no individual can do strictly better by deviating holding others fixed. Second, as it is a self-fulfilling prophecy (we will see what this means).

1.3. Pollution Control

Let there be n countries with the cost of pollution control being 3. If a country doesn't control pollution, it adds cost 1 to each country.

This leads to the dark case of no one controlling pollution. We can check that if only k countries control pollution, all have incentive to not control.

> **Definition**
>
> A dominant strategy is what is best for a player regardless of whatever strategy the rest of the players choose to adopt.

1.4. RPS

P2

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[1] Sometimes we can replace the actual strategies by others by p, which is i's belief of others' choices. This is often done in econ, but here, one of our assumptions (Common knowledge) allows us to not need it. The BR word is also from Econ.

P2

	R	P	S
R	$0,0$	$1,-1$	$-1,1$
P	$1,-1$	$0,0$	$-1,1$
S	$0,0$	$-1,1$	$1,-1$

P1 (row labels R, P, S)

There is no pure Nash equilibrium here. But allowing randomized strategy, we can get an equilibrium.

Definition: Mixed Strategy

A mixed strategy p_i is a randomization over i's pure strategies.
- $p_i(s_i)$ is the probability i assigns to the pure strategy s_i.
- $p_i(s_i)$ could be zero for some i.
- p_i could be one (in case of a pure strategy).

Definition: Mixed Strategy Profile

A mixed strategy profile is a tuple $(p_1, p_2, ..., p_n)$ where each p_i is a mixed strategy for player i. It represents the joint behavior of all players where each one plays a randomized strategy independently.

Definition: Expected Payoff

The expected payoff for player i is:

$$\sum_{(s_1,...,s_n) \in S_1 \times ... \times S_n} \mu_i(s_i, s_{-i}) \prod_{a=1}^{n} p_a(s_a)$$

Definition: Mixed Strategy Nash Equilibrium

A mixed strategy profile $(p_1^*, p_2^*, ..., p_n^*)$ is a mixed strategy NE (MSNE) if for each player i, p_i^* is a BR to p_{-i}^*. That is:

$$\mu_i(p_i^*, p_{-i}^*) \geq \mu_i(\sigma_i, \sigma_{-i}^*) \forall \sigma_i \in \text{probability distribution over } S_i$$

Nash's Theorem 1.1. *Every bimatrix game has a (pure or mixed) Nash equilibrium.*

Theorem 1.2. *For zero-sum games, it is polytime; otherwise, it is PPAD-hard.*

PPAD-hard $\Rightarrow$ NP-Hard $\iff$ NP = co-NP. This is open and unknown.

1.5. Sealed Bids Auctions

There is one seller and n buyers and one item that the seller wants to sell.

Buyers have non-negative valuations over this item v_i.

The rules of this auction are that each buyer submits a sealed bid b_i be the bid amounts. The highest bid gets the item (in case of ties, by the lower index).

If buyer i gets the item, and pays t_i, then their $\mu_i = v_i - t_i$ and payoff for other buyers is 0.

> **Definition: First Price Auctions**
>
> In first price auction, $t_i = b_i$.

> **Definition: Second Price Auction**
>
> Buyer who gets the items (say i) pays the second highest bid

> **Theorem 1.3.** *Second price auction is weakly revealing (that is it is a weakly dominant strategy to bid truthfully).*

Proof. We will analyze for v_i and then see that the same holds with small changes. ∎

1.6. Battle of the Sexes

> **Definition**
>
> We represent a situation where two agents must simultaneously take an action where each of them prefers one option over the other but prefer coordination over doing different things. And example occurrence would be[2]:
>
> **Wife**
>
> | | | Cricket | Music |
> |---|---|---|---|
> | | Cricket | 2,1 | 0,0 |
> | **Husband** | Music | 0,0 | 1,2 |

We can try to brute force this. Let's look at the payoffs for both the players

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[2]Which is extremely stereotypical and doesn't represent the author and hopefully the prof's views.

$$\mu_1(\sigma_1, \sigma_2) = \sigma_1(C)\sigma_2(C)\mu_i(C,C) +$$
$$\sigma_1(C)\sigma_2(D)\mu_i(C,D) +$$
$$\sigma_1(D)\sigma_2(C)\mu_i(D,C) +$$
$$\sigma_1(D)\sigma_2(D)\mu_i(D,D)$$
$$= 2\sigma_1(C)\sigma_2(C) + \sigma_1(D)\sigma_2(D)$$
$$= 2\sigma_1(C)\sigma_2(C) + (1 - \sigma_1(C))(1 - \sigma_2(C))$$
$$= 3\sigma_1(C)\sigma_2(C) - \sigma_1(C) - \sigma_2(C) + 1$$

And similarly

$$\mu_2 = 3\sigma_1(C)\sigma_2(C) - 2\sigma_1(C) - 2\sigma_2(C) + 2$$

Using $\mu_1(\sigma_1^*, \sigma_2^*) \geq \mu_1(\sigma_1, \sigma_2*) \forall \sigma_1 \in \Delta(S_1)$

$$\sigma_1^*(C)[3\sigma_2(C) - 1] \geq \sigma_1(1)[3\sigma_2(C) - 1] \forall \sigma_1(c) \in (0,1)$$

Similarly

$$\sigma_2^*(C)[3\sigma_1(C) - 2] \geq \sigma_2(1)[3\sigma_1(C) - 2] \forall \sigma_2(c) \in (0,1)$$

If $3\sigma_2^*(C) - 1 > 0 \Rightarrow \langle (1,0), (0,1) \rangle$.

If $3\sigma_2^*(C) - 1 < 0 \Rightarrow \langle (0,1), (1,0) \rangle$.

If $3\sigma_2^*(C) - 1 = 0 \Rightarrow \langle \left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right) \rangle$.

1.7. Properties of MSNE

Expected payoff for player i will be:

$$\mu_i(p_1, p_2, ..., p_n) = \sum_{(s_1,...,s_n) \in S_1 \times ... \times S_n} \mu_i(s_i, s_{-i}) \prod_{a=1}^{n} p_a(s_a)$$
$$= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} p_i(s_i) p_{-i}(s_{-i}) \mu_i(s_i, s_{-i})$$
$$= \sum_{s_i \in S_i} p_i(s_i) \sum_{s_{-i} \in S_{-i}} p_{-i}(s_{-i}) \mu_i(s_i, s_{-i})$$
$$= \sum_{s_i \in S_i} p_i(s_i) \mu_i(s_i, p_{-i})$$

This makes the utility of a player a convex combination of thier pure stratergy payoffs.

> **Definition: Convex Combination**
>
> A convex combination of $a_1, a_2, ..., a_n$ is $\sum_{i=1}^{n} \lambda_i a_i$ where $\lambda_i \in [0,1]$ and $\sum_{i=1}^{n} \lambda_i = 1$.

An obvious observation is that convex combination of $a_1, ..., a_n \leq \max\{a_i\}$.

Which implies that the payoff with mixed stratergy is less than equal to max payoff with a pure strategy s_i. This implies:

$$\max_{\sigma_i \in \Delta(S_i)} \mu_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} \mu_i(s_i, \sigma_{-i})$$

> **Theorem 1.4.** $(\sigma_1^*, ..., \sigma_n^*)$ *is a MSNE if and only if* $\forall i \in N$
> 1. $\mu_i(s_i, \sigma_{-i}^*)$ *is the same for all* $s_i \in$ *support of* σ_i^*.
> 2. $\mu_i(s_i, \sigma_{-i}^*) \geq \mu_i(s_i^*, \sigma_{-i}^*) \forall s_i' \notin$ *support of* σ_i^*, $s_i \in$ *support of* σ_i^*.

Remark: Implications

Each player gets the same payoff for any pure strategy in the support of the MSNE strategy.

1.8. Equilibrium Computation

1.8.1. Computing a SDSE

Example

P_2

		D	E	F
	A	4,3	5,1	6,2
P_1	B	2,1	8,4	3,6
	C	3,0	9,6	2,8

Notice, E is dominated by F as $\forall i, \mu_2(F, s_{-i}) > \mu_2(E, s_{-i})$. THis practically makes the game:

P_2

		D	F
	A	4,3	6,2
P_1	B	2,1	3,6
	C	3,0	2,8

Now A dominates B, C and after elimination, D will dominate E and we will be left with the nash equilibrium of $4, 3$.

Also note, (A, D) is a PSNE.

Note, this algorithm by no means gives all the nash equilibrium, if there are multiple. We don't run into such issues with SDSE as only one can exist by the definition of Strict dominance.

Example

$$P_2$$

	D	E	F
A	3,1	0,1	0,0
B	0,1	4,1	0,0
C	1,1	1,1	5,0

(P_1 labels the rows)

We can eliminate F as D, E are strictly better.

$$P_2$$

	D	E
A	3,1	0,1
B	0,1	4,1
C	1,1	1,1

(P_1 labels the rows)

Here, we can eliminate C as if P_2 plays D or E, we are better off playing A or B respectively.

Another way to argue the same is $\mu_1\left(\frac{A+B}{2}, s_{-i}\right) > \mu_1(C, s_{-i})$.

Side Note

Here, playing C is called the Bayesian Equilibrium as it is safe.

Example

$$P_2$$

	X	Y	Z
A	7,7	4,2	1,8
B	2,4	5,5	2,3
C	8,1	3,2	0,0

(P_1 labels the rows)

We can get the gurentee finding the nash equilibrium by identifying the highest entry per row and column wrt the respective players.

$$P_2$$

	X	Y	Z
A	$\underline{7}$,7	4,2	1,$\overline{8}$
B	2,4	$\underline{5}$,$\overline{5}$	2,3
C	$\underline{8}$,1	3,$\overline{2}$	0,0

(P_1 labels the rows)

> Making $(5, 5)$ the PSNE.

Algorithm 1.5. For the column player, mark the maximum entry for them.

For the row player, mark the maximum entry for them.

The intersections are PSNEs.

1.8.2. Two Player Zero Sum Games

Definition: Two Player Zero Sum Games

A game with:
- $N = \{1, 2\}$
- $S = \{S_1, S_2\}$
- $\mu = \{\mu_1, \mu_2\}$
- $\mu_1(s_1, s_2) = -\mu_2(s_1, s_2)$

In these games, we only need to specify the payoffs for one player. We, by convention, do this for the row player.

B

2	1	1
−1	1	2
1	0	1

A

For any stratergy, i of P_1 and P_2 will choose a stratergy such that,
- P_1 chooses $\max_i \min_j a_{ij}$ (maxmin)
- P_2 will choose $\min_j \max_i a_{ij}$. (minmax)

If maxmin and minmax are equal, we are done and the value is the nash equilibrium.

If a PSNE exists, we will get it by this process.

If there is no PSNE, this obviously doesn't work.

P2

	R	P	S
R	0	−1	1
P	1	0	−1
S	−1	1	0

P1

1.8.3. Saddle Point

Definition: Saddle Point

(i, j) is a saddle point of a matrix A if $a_{ij} \geq a_{kj} \forall k$ and $a_{ij} \leq a_{il} \forall l$

Theorem 1.6. *If i, j and k, h are saddle points, then (i, h) and k, j are also saddle points.*

Proof.

$$a_{ij} \geq a_{kj} \geq a_{kh} \geq a_{ih} \geq a_{ij}$$

And we are done by squeeze theorem. ∎

Definition: Mixed Stratergy in 2 player zero sum game

Let $|S_1| = m$ and $|S_2| = m$. Let $x = (x_1, x_2, ..., x_n)$ be a mixed strategy for P_1 and $y = (y_1, y_2, ..., y_n)$ be a mixed strategy for P_2.

The expected payoff is $\sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j a_{ij} = x^T A y$