Calculus II

Notes by Arjun Maneesh Agarwal

based on course by Krishna Hanumanthu

The course is mainly about multivarible integration.

Grades in the course will be based on the following weightage: 30% Homework, in-class quiz, 35% Midsem and 35% Final.

Do all probelms sent, as doing problems is doing mathematics, regardless if they are collected or not. These HW PSETs (regardless of if they are graded) can and will appear in Quizes, Midsems and Finals.

This course is based on Spivak's "Calculus on Manifolds" (ch 3, 4). We will also use other sources, like the notes on moodle (by TR Ramdas). Other books for some parts and excercises are: Rudin's "Principles of Mathematical Analysis", Apostal's "Mathematical Analysis".

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1. Hanumanthu's Wisdom

"Let us all agree that for calculations integrals are useless and hopefully, none of us will need to do them again."

2. 5 Auguest, 2025

2.1. Review of One Variable Integration

A typicall function we integrated looked like:

$$f : [a, b] \to \mathbb{R}$$

Assuming $a, b \in \mathbb{R}$ and $a < b$.

Definition: Upper and Lower sums

The upper sum is of the form

$$U(P,f) = \sum_{i=1} (t_i - t_{i-1}) \sup_{t_{i-1} \le x \le t_i} f(x)$$

And the lower sum is of the form:

$$L(P,f) = \sum_{i=1} (t_i - t_{i-1}) \inf_{t_{i-1} \le x \le t_i} f(x)$$

$$\int_{\underline{a}}^{b} f(x)\,\mathrm{d}x = \sup_{p(L(P,f))} \int_{a}^{\bar{b}} f(x)\,\mathrm{d}x = \inf_{p(U(P,f))}$$

If $\int_{\underline{a}}^{b} f(x)\,\mathrm{d}x = \int_{a}^{\bar{b}} f(x)\,\mathrm{d}x$, our function is integrable.

Note, this is always true when f is continous but it is not an if and only if statement.

We will now try to genralize this to two variables. The main jump is from 1 to 2 variables, beyond this, what is true for 2 is true for 150.

3. 7 Auguest, 2025

3.1. Review of One Variable Integration, cont.

$f : [a,b] \to \mathbb{R}$ is bounded.

$$\int_{\underline{a}}^{b} f \le \int_{a}^{\bar{b}} f$$

$$\Rightarrow \sup L(p,f) \le \inf U(p,f)$$

Definition: Riemaan Integrablity

f is Riemaan integrable if:

$$\int_{\underline{a}}^{b} f = \int_{a}^{\bar{b}} f$$

Proposition(Cauchy Critarion for Integrablity) 3.1. $f : I = [a,b] \to \mathbb{R}$ *bounded,*

$$f \text{ is integrable} \iff \text{given } \varepsilon > 0, \exists \text{ partition } R \text{ of } I \text{ s.t.}$$
$$U(R,f) - L(R,f) < \varepsilon$$

Proof. ($\Longrightarrow$) Assume f is integrable. Let $\varepsilon > 0$;

Choose P s.t.

$$\int_{\underline{a}}^{b} f - L(P,f) < \frac{\varepsilon}{2}$$

Choose Q s.t.

$$U(Q,f) - \int_{a}^{\bar{b}} f < \frac{\varepsilon}{2}$$

Adding,

$$U(Q,f) - P(P,f) < \varepsilon$$

Choose a refinement $R = P \cup Q$ and we are done.

($\Longleftarrow$) $\forall$ partitions R,

$$0 \leq \int_{a}^{\bar{b}} f - \int_{\underline{a}}^{b} f \leq U(R,f) - L(R,f)$$

■

Definition: Continuity

Given an ε and x, if there is a δ such that if y is δ-close to x, then $f(y)$ is ε-close to y.

Definition: Uniform Continuity

Given an ε, there is a δ such that if x and y are δ-close; then $f(x), f(y)$ are ε-close.

Definition: Compactness

TODO.

Theorem 3.2. *$f : [a,b] \to \mathbb{R}$ is continous $\implies$ f is integrable on $[a,b]$*

Proof.

Claim 3.3. f is uniformly continous on $[a,b]$.

TODO. *Proof of claim.*

Ler $\varepsilon > 0$. Thus, there exists a $\delta > 0$ s.t. $\forall x, y \in [a,b]$.

$$|x - y\ | < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Choose a partition $P = \bigcup_i I_i$ s.t. $I_{i+1}. - I_i < \delta$.

$$\begin{aligned}
&\Rightarrow \sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x) \\
&= \max_{x \in I_i} f(x) - \min_{x \in I_i} f(x) \\
&< \varepsilon \\
&\Rightarrow U(p, f) - L(p, f) < \varepsilon \sum_i I_{i+1} - I_i = \varepsilon(b - a)
\end{aligned}$$

Thus, by Cauchy Critarion, we are done. ■

We can take

$$V = \{f : [a, b] \to \mathbb{R} \mid f \text{ is Rieman integrable}\}$$

which is a vector space.

Integration is a linear map from $V \to \mathbb{R}$.

Theorem 3.4. *$f : [a, b] \to \mathbb{R}$ is integrable.*

Define $F : [a, b] \to \mathbb{R}; y \mapsto \int_a^y f(x) \,\mathrm{d}x$.

- *F is continous*
- *Further, if f is continous at $x_0 \in [a, b]$, then F is diffrentiable at x_0 and $F'(x_0) = f(x_0)$*

Theorem 3.5. *$F : I = [a, b] \to \mathbb{R}$ and diffrentiable, $F' = f$ is integrable on I, then:*

$$\int_a^b f(x) \,\mathrm{d}x = F(b) - F(a)$$

3.2. Integration in $\mathbb{R}^n$

The main set we will deal with is $R \coloneqq [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$.

We define $\mathrm{vol}(R) = \Pi_{i=1}^n (b_i - a_i)$

Most often, $n = 2$.

We can make partitions of $[a_1, b_1]; [a_2, b_2]; \ldots$ and just take the cartition products.

Definition: Partion index on R^2

Given a k_1 partition of $[a, b]$ and a k_2 partition on $[c, d]$.

$$\begin{aligned}
a = t_{1,0} < t_{1,1} < \ldots < t_{1,k_1} = b \\
c = t_{2,0} < t_{2,1} < \ldots < t_{2,k_2} = d
\end{aligned}$$

This gives us $R_{i,j} = \left[t_{1,i}, t_{1,i+1}\right] \times \left[t_{2,j}, t_{2,j+1}\right]$; where $0 \leq i \leq k_1 - 1; 0 \leq j \leq k_2 - 1$.

Example

Consider $R = [2,5] \times [3,7]$.

$2 < 4 < 5$, $t_{1,0} = 2 < t_{1,1} = 4 < t_{1,2} = 5$

$3 < 4 < 6 < 7$, $t_{2,0} = 3 < t_{2,1} = 4 < t_{2,2} = 6 < t_{2,3} = 7$

3.3. Exactly re-doing Riemaan integration!

Let $f : R \to \mathbb{R}$, $R \subseteq \mathbb{R}^2$ and be bounded.

$$L(p, f) = \sum_{\substack{0 \le i \le k_1 - 1 \\ 0 \le j \le k_2 - 1}} \left(\inf_{x \in R_{i,j}} f(x) \right) (\operatorname{vol}(R_{i,j}))$$

$$U(p, f) = \sum_{\substack{0 \le i \le k_1 - 1 \\ 0 \le j \le k_2 - 1}} \left(\sup_{x \in R_{i,j}} f(x) \right) (\operatorname{vol}(R_{i,j}))$$

If $\mathcal{P}$ is a refinement of $\mathcal{Q}$, then:

$$L(\mathcal{P}, f) \ge L(\mathcal{Q}, f)$$
$$U(\mathcal{P}, f) \le U(\mathcal{Q}, f)$$

Let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$ aka a common refinement.

$$\Rightarrow U(\mathcal{P}, f) \ge L(\mathcal{Q}, f)$$

Example : Common Refinement

Let $\mathcal{P} = (2 < 4 < 5) \times (3 < 4 < 6 < 7)$ and $\mathcal{Q} = (2 < 2.5 < 3 < 3.2 < 5) \times (3 < 5 < 7)$; then

$$\begin{aligned} \mathcal{R} &= \mathcal{P} \cup \mathcal{Q} \\ &= (2 < 2.5 < 3 < 3.2 < 4 < 5) \times (3 < 4 < 5 < 6 < 7) \end{aligned}$$

We will now define:

$$\underline{\int_R} f(x) \,\mathrm{d}x = \sup_p L(p, f)$$

$$\overline{\int_R} f(x) \,\mathrm{d}x = \inf_p U(p, f)$$

These will exist as by fixing a partion, we can get a upper bound for $L(p, f)$ and a lower bound for $U(p, f)$.

This gives the exact same definition for Riemaan integrablity in a multivarible case aswell.

4. 12 August, 2025

We had $R \in \mathbb{R}^n$ a rectangle with a function $f : R \to \mathbb{R}$ which is bounded.

We can partion R by P and define $L(P, f)$ and $U(P, f)$ similer to the one variable case.

We define

$$\int_R f := \inf U = \sup L$$

Example

$f : R \to \mathbb{R}$ is content function, that is $f(x_1, x_2, \ldots, x_n) = c$

Solution

Rather obviously

$$\int_R f = c\,\mathrm{vol}(R)$$

Example

$f : [0, 1] \times [0, 1] \to \mathbb{R}$ such that

$$f(x, y) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

Solution

This is just the Drinchlet function!

4.1. Integration on more general subsets of $\mathbb{R}^n$

Definition: Measure Zero

Let $A \subseteq \mathbb{R}^n$ be a subset. We say that A has measure 0 if $\forall \varepsilon > 0$ there is a cover $\{U_1, U_2, \ldots\}$ of A by closed rectangles U_i such that:

$$\sum_{i=1}^{\infty} \mathrm{vol}(U_i) < \varepsilon$$

Remark

1. A is finite $\implies A$ has measure 0
2. A is countable $\implies A$ has measure 0
3. $B \subset A$, A has measure 0 $\implies A$ is countable
4. $[0, 1]$ doesn't have measure 0.

Theorem 4.1. *Union of countable measure 0 sets is measure* 0.

Proof.

TODO. *From Spivak!*

■

Definition: Content Zero

$A \subseteq \mathbb{R}^n$ has content 0 if $\forall \varepsilon > 0$, there is a finite cover $\{U_1, ..., U_k\}$ of A by closed rectangles such that

$$\sum_{i=1}^{l} \text{vol}(U_i) < \varepsilon$$

Theorem 4.2. *A has content 0 $\implies$ A has measure 0; but the converse is not true.*

Theorem 4.3. *For a compact A, A has content 0 $\iff$ A has measure 0*

Theorem 4.4. *Let $R \subseteq \mathbb{R}^n$ be a closed rectangle, $f : R \to \mathbb{R}$ a bounded function.*

Then f is integrable $\iff$ $B := \{x \in R \mid f \text{ is not continous}\}$ has measure 0.

Proof. Let $\varepsilon > 0$,

$$B_\varepsilon := \{x \in R \mid o(f, x) \geq \varepsilon\} \subseteq B$$

This implies B_ε is measure 0 as B is measure 0.

B_ε is closed and bounded, thus, compact. That implies B_ε is content 0 as it is measure 0.

$$\implies \exists \text{ closed rectangles } U_1, ... \text{ s.t.}$$

$$B_\varepsilon \subseteq \bigcup_{i=1}$$

■

Example

$R \in \mathbb{R}^n$ be a closed rectangle. If $f, g : R \to \mathbb{R}$ are integrable, so is $f \cdot g$.

Recall that if $f : A \to \mathbb{R}$ is a bounded function and $A \in \mathbb{R}^n$.

Let $x \in A$, $\delta > 0$.

$$M(x, f, \delta) := \sup\{f(a) \mid a \in A, |x - a| < \delta\}$$
$$m(x, f, \delta) := \inf\{f(a) \mid a \in A, |x - a| < \delta\}$$

We define osciallation of f at x as:

$$o(f, x) := \lim_{\delta \to 0} M(x, f, \delta) - m(x, f, \delta)$$

Theorem 4.5. *If f is continous at $x \iff o(f, x) = 0$*

Theorem 4.6. *$A \in \mathbb{R}^n$ is closed and $f : A \to \mathbb{R}$ is bounded. Let $\varepsilon > 0$, then:*

$$\{x \in A \mid o(f, x) \geq \varepsilon\} \text{ is closed}$$