Algorithmic Game Theory

Notes by Arjun Maneesh Agarwal

based on course by Prajakta Nimbhorkar

1. Games and Equilibria : We are assuming games where everyone is acting selfishly. When everyone doesn't wish to switch strategies, we get an equilibrium.
   - We will see two-player games and multiplayer games.
   - Equilibrium and existence and computation.
2. Mechanism Design : Given multiple selfish agents, we need to make a mechanism that encourages (or discourages using hardness) the behavior we wish to encourage.
   - We will see auctions, voting etc.
3. Fair Division: This is an allocation problem where we want to allocate some resources in a 'fair' (or less unfair) way.
   - Distributing divisible and indivisible resources fairly among agents, given valuation.
   - EF, EQ, PROP and its relaxations.

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1. Games and Equilibria

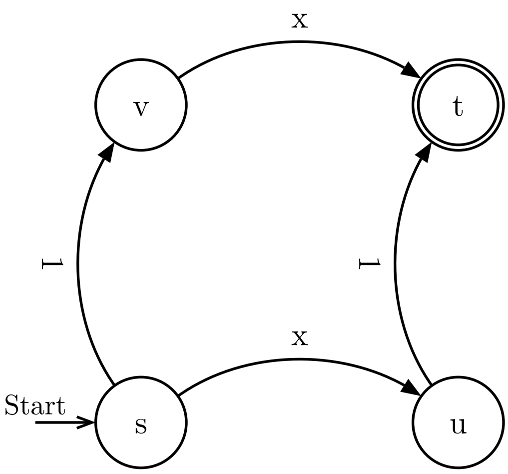
We make the following assumptions:

1. Rationality/selfishness : Each agent attempts to maximize own payoff and believes others will do so as well as well has knowledge of same behavior being
2. Intelligence: Agents have enough computational resources to take into account the strategies and behaviors of others.

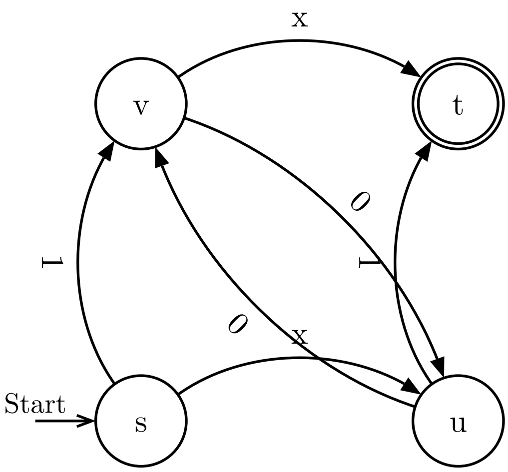
3. Common Knowledge: P, S_i, μ are known to everyone and everyone knows that everyone knows them and so on.

1.1. Braess's Paradox

Taking the example of traffic, let the network be:



In this case, the equilibrium is 1.5 hours as traffic is divided in the two paths. Let's say the authorities build a bridge:



This will lead to a (weak) equilibrium of 2 hours as all agents take the route s-v-u-t.

This is called Braess's Paradox.

Side Note

We have actually seen this example occur almost verbatim in Seoul where the demolition of a bridge in 2005, solved a 2 decade old traffic problem.

Furthermore, Steinberg and Zangwill, 1983; showed that in a random network, Braess's paradox is approximately 50% likely to occur in an random network.

It is also a consideration in google maps where the navigation sometimes suggests a longer route, not to avoid a pre-existing traffic jam, but to avoid creating one. (This is why google map is not referentially transparent and may suggest two people different routes between same places at same time)

TODO.

1.2. Prisoner's Dilemma

P1 \ P2	C	D
C	$4, 4$	$5, 1$
D	$1, 5$	$2, 2$

Here both players have incentive to deviate from the mutually best case as confessing (strongly) dominates denial and hence, is a Nash equilibrium.

Let's define these words formally.

Definition: Game

To define a game we need to know the players and strategies for each player. Thus, a game $G = (P, S, \mu)$ where for i in P, S_i in S denotes the set of possible strategies for player i.

In a particular play of the game denoted as s, i plays strategy s_i forming the strategy profile or game vector $(s_1, s_2, ..., s_n)$.

We denote the payoffs or costs of the game for player i using a function $\mu_i : S_1 \times S_2 \times ... \times S_n \to \mathbb{R}$ which takes the game vector and tells the payoff for player i. $\mu : S_1 \times S_2 \times ... \times S_n \to \mathbb{R}^n; \mu(s) = (\mu_1(s), \mu_2(s), ..., \mu_n(s))$ is the payoff vector returning function.

Finally, for conciseness, we sometimes denote the choices for all players except i as s_{-i}. Thus, $\mu_i(s_i, s_{-i})$

Definition: Strongly Dominant Strategy

Player i's strategy $\hat{s_i}$ is a **Strongly Dominant Strategy** to the strategy s_{-i} of other players if:

$$\mu_i(\hat{s_i}, s_{-i}) > \mu_i(s_i', s_{-i}) \forall s_i' \in S_i$$

Definition: Weakly Dominant Strategy

Player i's strategy $\hat{s_i}$ is a **Weakly Dominant Strategy** to the strategy s_{-i} of other players if:

$$\mu_i(\hat{s}_i, s_{-i}) \geq \mu_i(s'_i, s_{-i}) \forall s'_i \in S_i$$

and

$$\exists s'_i \in S_i \text{ s.t.} \mu_i(\hat{s}_i, s_{-i}) > \mu_i(s'_i, s_{-i})$$

Definition: Weakly Dominant Strategy / Best Response(BR)

Player i's strategy $\hat{s}_i$ is a **best response** to the strategy s_{-i} of other players if:

$$\mu_i(\hat{s}_i, s_{-i}) \geq \mu_i(s'_i, s_{-i}) \forall s'_i \in S_i$$

or if $\hat{s}_i$ solves $\max_{S_i} \mu_i(s_i, s_{-i})$.[1]

Definition: Strongly Dominated Strategy Equilibrium

Informally, this is when all players are playing dominant strategies.

It is a strategy profile $(s^*_1, s^*_2, ..., s^*_N)$ is a SDSE if for each i, her choice s^*_i is strongly dominant to all .

Definition: Nash Equilibrium(NE)

Informally, this is when all players are playing BR to each other.

Formally, it is a strategy profile $(s^*_1, s^*_2, ..., s^*_N)$ is a NE if for each i, her choice s^*_i is a best response to s^*_{-i}.

Remark

The reason Nash equilibrium is important is as it leads to no-regrets as no individual can do strictly better by deviating holding others fixed. Second, as it is a self-fulfilling prophecy (we will see what this means).

1.3. Pollution Control

Let there be n countries with the cost of pollution control being 3. If a country doesn't control pollution, it adds cost 1 to each country.

This leads to the dark case of no one controlling pollution. We can check that if only k countries control pollution, all have incentive to not control.

[1] Sometimes we can replace the actual strategies by others by p, which is i's belief of others' choices. This is often done in econ, but here, one of our assumptions (Common knowledge) allows us to not need it. The BR word is also from Econ.

Definition

A dominant strategy is what is best for a player regardless of whatever strategy the rest of the players choose to adopt.

1.4. RPS

		P2		
		R	P	S
	R	$0, 0$	$1, -1$	$-1, 1$
P1	P	$1, -1$	$0, 0$	$-1, 1$
	S	$0, 0$	$-1, 1$	$1, -1$

There is no pure Nash equilibrium here. But allowing randomized strategy, we can get an equilibrium.

Definition: Mixed Strategy

A mixed strategy p_i is a randomization over i's pure strategies.

- $p_i(s_i)$ is the probability i assigns to the pure strategy s_i.
- $p_i(s_i)$ could be zero for some i.
- p_i could be one (in case of a pure strategy).

Definition: Mixed Strategy Profile

A mixed strategy profile is a tuple $(p_1, p_2, ..., p_n)$ where each p_i is a mixed strategy for player i. It represents the joint behavior of all players where each one plays a randomized strategy independently.

Definition: Expected Payoff

The expected payoff for player i is:

$$\sum_{(s_1,...,s_n) \in S_1 \times ... \times S_n} \mu_i(s_i, s_{-i}) \prod_{a=1}^{n} p_a(s_a)$$

Definition: Mixed Strategy Nash Equilibrium

A mixed strategy profile $(p_1^*, p_2^*, ..., p_n^*)$ is a mixed strategy NE (MSNE) if for each player i, p_i^* is a BR to p_{-i}^*. That is:

$$\mu_i(p_i^*, p_{-i}^*) \geq \mu_i(\sigma_i, \sigma_{-i}^*) \forall \sigma_i \in \text{probability distribution over } S_i$$

Nash's Theorem 1.1. *Every bimatrix game has a (pure or mixed) Nash equilibrium.*

Theorem 1.2. *For zero-sum games, it is polytime; otherwise, it is PPAD-hard.*

PPAD-hard $\Rightarrow$ NP-Hard $\Longleftrightarrow$ NP = co-NP. This is open and unknown.

1.5. Sealed Bids Auctions

There is one seller and n buyers and one item that the seller wants to sell.

Buyers have non-negative valuations over this item v_i.

The rules of this auction are that each buyer submits a sealed bid b_i be the bid amounts. The highest bid gets the item (in case of ties, by the lower index).

If buyer i gets the item, and pays t_i, then their $\mu_i = v_i - t_i$ and payoff for other buyers is 0.

Definition: First Price Auctions

In first price auction, $t_i = b_i$.

Definition: Second Price Auction

Buyer who gets the items (say i) pays the second highest bid

Theorem 1.3. *Second price auction is weakly revealing (that is it is a weakly dominant strategy to bid truthfully).*

Proof. We will analyze for v_i and then see that the same holds with small changes. ∎

1.6. Battle of the Sexes

Definition

We represent a situation where two agents must simultaneously take an action where each of them prefers one option over the other but prefer coordination over doing different things. And example occurrence would be[2]:

		Wife: Cricket	Wife: Music
Husband	Cricket	2,1	0,0
	Music	0,0	1,2

We can try to brute force this. Let's look at the payoffs for both the players

[2]Which is extremely stereotypical and doesn't represent the author and hopefully the prof's views.

$$\begin{aligned}\mu_1(\sigma_1, \sigma_2) = \sigma_1(C)\sigma_2(C)\mu_i(C, C) +\\ \sigma_1(C)\sigma_2(D)\mu_i(C, D) +\\ \sigma_1(D)\sigma_2(C)\mu_i(D, C) +\\ \sigma_1(D)\sigma_2(D)\mu_i(D, D)\\ = 2\sigma_1(C)\sigma_2(C) + \sigma_1(D)\sigma_2(D)\\ = 2\sigma_1(C)\sigma_2(C) + (1 - \sigma_1(C))(1 - \sigma_2(C))\\ = 3\sigma_1(C)\sigma_2(C) - \sigma_1(C) - \sigma_2(C) + 1\end{aligned}$$

And similarly

$$\mu_2 = 3\sigma_1(C)\sigma_2(C) - 2\sigma_1(C) - 2\sigma_2(C) + 2$$

Using $\mu_1(\sigma_1^*, \sigma_2^*) \geq \mu_1(\sigma_1, \sigma_2 *)\forall\sigma_1 \in \Delta(S_1)$

$$\sigma_1^*(C)[3\sigma_2(C) - 1] \geq \sigma_1(1)[3\sigma_2(C) - 1]\forall\sigma_1(c) \in (0, 1)$$

Similarly

$$\sigma_2^*(C)[3\sigma_1(C) - 2] \geq \sigma_2(1)[3\sigma_1(C) - 2]\forall\sigma_2(c) \in (0, 1)$$

If $3\sigma_2^*(C) - 1 > 0 \Rightarrow \langle(1, 0), (0, 1)\rangle$.

If $3\sigma_2^*(C) - 1 < 0 \Rightarrow \langle(0, 1), (1, 0)\rangle$.

If $3\sigma_2^*(C) - 1 = 0 \Rightarrow \langle\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\rangle$.

1.7. Properties of MSNE

Expected payoff for player i will be:

$$\begin{aligned}\mu_i(p_1, p_2, ..., p_n) &= \sum_{(s_1,...,s_n)\in S_1\times...\times S_n} \mu_i(s_i, s_{-i}) \prod_{a=1}^{n} p_a(s_a)\\ &= \sum_{s_i\in S_i} \sum_{s_{-i}\in S_{-i}} p_i(s_i)p_{-i}(s_{-i})\mu_i(s_i, s_{-i})\\ &= \sum_{s_i\in S_i} p_i(s_i) \sum_{s_{-i}\in S_{-i}} p_{-i}(s_{-i})\mu_i(s_i, s_{-i})\\ &= \sum_{s_i\in S_i} p_i(s_i)\mu_i(s_i, p_{-i})\end{aligned}$$

This makes the utility of a player a convex combination of thier pure stratergy payoffs.

Definition: Convex Combination

A convex combination of $a_1, a_2, ..., a_n$ is $\sum_{i=1}^{n} \lambda_i a_i$ where $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$.

An obvious observation is that convex combination of $a_1, ..., a_n \leq \max\{a_i\}$.

Which implies that the payoff with mixed stratergy is less than equal to max payoff with a pure strategy s_i. This implies:

$$\max_{\sigma_i \in \Delta(S_i)} \mu_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} \mu_i(s_i, \sigma_{-i})$$

Theorem 1.4. *$(\sigma_1^*, ..., \sigma_n^*)$ is a MSNE if and only if $\forall i \in N$*
1. $\mu_i(s_i, \sigma_{-i}^*)$ is the same for all $s_i \in$ support of σ_i^*.
2. $\mu_i(s_i, \sigma_{-i}^*) \geq \mu_i(s_i^*, \sigma_{-i}^*) \forall s_i' \notin$ support of $\sigma_i^*, s_i \in$ support of σ_i^*.

Remark: Implications

Each player gets the same payoff for any pure strategy in the support of the MSNE strategy.

2. Equilibrium Computation

2.1. Computing a SDSE

Example

P_2

P_1		D	E	F
	A	4,3	5,1	6,2
	B	2,1	8,4	3,6
	C	3,0	9,6	2,8

Notice, E is dominated by F as $\forall i, \mu_2(F, s_{-i}) > \mu_2(E, s_{-i})$. THis practically makes the game:

P_2

P_1		D	F
	A	4,3	6,2
	B	2,1	3,6
	C	3,0	2,8

Now A dominates B, C and after elimination, D will dominate E and we will be left with the nash equilibrium of $4, 3$.

Also note, (A, D) is a PSNE.

Note, this algorithm by no means gives all the nash equilibrium, if there are multiple. We don't run into such issues with SDSE as only one can exist by the definition of Strict dominance.

Example

P_2

P_1		D	E	F
	A	3,1	0,1	0,0
	B	0,1	4,1	0,0
	C	1,1	1,1	5,0

We can eliminate F as D, E are strictly better.

P_2

P_1		D	E
	A	3,1	0,1
	B	0,1	4,1
	C	1,1	1,1

Here, we can eliminate C as if P_2 plays D or E, we are better off playing A or B respectively.

Another way to argue the same is $\mu_1\left(\frac{A+B}{2}, s_{-i}\right) > \mu_1(C, s_{-i})$.

Side Note

Here, playing C is called the Bayesian Equilibrium as it is safe.

Example

P_2

P_1		X	Y	Z
	A	7,7	4,2	1,8
	B	2,4	5,5	2,3
	C	8,1	3,2	0,0

We can get the gurentee finding the nash equilibrium by identifying the highest entry per row and column wrt the respective players.

P_2

P_1		X	Y	Z
	A	$\underline{7}$,7	4,2	1,$\overline{8}$
	B	2,4	$\underline{5}, \overline{5}$	2,3
	C	$\underline{8}$,1	3,$\overline{2}$	0,0

Making $(5,5)$ the PSNE.

Algorithm 2.1. For the column player, mark the maximum entry for them.

For the row player, mark the maximum entry for them.

The intersections are PSNEs.

2.2. Two Player Zero Sum Games

Definition: Two Player Zero Sum Games

A game with:
- $N = \{1, 2\}$
- $S = \{S_1, S_2\}$
- $\mu = \{\mu_1, \mu_2\}$
- $\mu_1(s_1, s_2) = -\mu_2(s_1, s_2)$

In these games, we only need to specify the payoffs for one player. We, by convention, do this for the row player.

$$\begin{array}{cc} & B \\ A & \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline -1 & 1 & 2 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \end{array}$$

For any stratergy, i of P_1 and P_2 will choose a stratergy such that,
- P_1 chooses $\max_i \min_j a_{ij}$ (maxmin)
- P_2 will choose $\min_j \max_i a_{ij}$. (minmax)

If maxmin and minmax are equal, we are done and the value is the nash equilibrium.

Definition: Maxmin-Minmax

The maxmin value refers to $\max_{i \in S_1} \min_{j \in S_2} a_{ij}$.

The minmax value refers to $\min_{j \in S_2} \max_{i \in S_1} a_{ij}$.

If a PSNE exists, we will get it by this process.

If there is no PSNE, this obviously doesn't work.

P2

R	P	S

		P2		
	R	0	-1	1
P1	P	1	0	-1
	S	-1	1	0

We can also discuss Saddle points.

Definition: Saddle Point

(i, j) is a saddle point of a matrix A if $a_{ij} \geq a_{kj} \forall k$ and $a_{ij} \leq a_{il} \forall l$

Theorem 2.2. *If i, j and k, h are saddle points, then (i, h) and k, j are also saddle points.*

Proof.

$$a_{ij} \geq a_{kj} \geq a_{kh} \geq a_{ih} \geq a_{ij}$$

And we are done by squeeze theorem. ■

Definition: Mixed Stratergy in 2 player zero sum game

Let $|S_1| = m$ and $|S_2| = m$. Let $x = (x_1, x_2, ..., x_n)$ be a mixed strategy for P_1 and $y = (y_1, y_2, ..., y_n)$ be a mixed strategy for P_2.

The expected payoff is $\sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j a_{ij} = x^T A y$

We can define maxmin and minmax values here as:

Definition: Maxmin-minmax

maxmin value $= \max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} x^T A y$

minmax value $= \min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} x^T A y$

Lemma 2.3.

$$\min_{y \in \Delta(S_2)} x^T A y = \min_{j \in S_2} \sum_{i=1}^{m} a_{ij} x_i$$

Proof. It is obvious that

$$\min_{y \in \Delta(S_2)} x^T A y \leq \min_{j \in S_2} \sum_{i=1}^{m} a_{ij} x_i$$

as $S_2 \subseteq \Delta(S_2)$.

To do the other way round,

$$x^T A y = \sum_{j=1}^{n} y_j \sum_{i=1}^{m} x_j a_{ij}$$
$$\geq \sum_{i=1}^{n} y_j \min_{k \in S_2} \sum_{i=1}^{m} x_i a_{ik}$$
$$= \min_{k \in S_2} \sum_{i=1}^{m} x_I a_{ik} \cdot \sum_{i=1}^{n} y_j$$
$$= \min_{k \in S_2} \sum_{i=1}^{m} x_I a_{ik}$$

Thus, by squeeze theorem, we are done. ■

2.3. Linear Programming for the the win

We make an LP for the row player as:

The row player has to to maximize $\min_{j \in S_2} \sum_{i=1}^{m} a_{ij} x_i$ with the constrains $\sum_{i=1}^{m} x_i = 1, x_i \geq 0 \forall i \in \{1, 2, ..., m\}$.

Equivalently, maximize z with the constrains $z \leq \sum_{i=1}^{m} a_{ij} x_i \forall j \in \{1, 2, ..., n\}$, $\sum_{i=1}^{n} x_i = 1$, $x_i \geq 0 \forall i \in [m]$.

Exercise : Column Player's LP

Define the column players LP.

Solution. minimize w such that $w \geq \sum_{j=1}^{n} a_{ij} y_j \forall i \in [m]$, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$ forall $j \in [n]$. ■

2.3.1. Duel of LP

Example : Toy LP

max $3x_1 + 2x_2 + x_3$ with constraints
1. $x_1 + x_2 + x_3 \leq 2$
2. $3x_1 + x_3 \leq 4$
3. $x_1 + x_2 + x_3 \leq 5$
4. $x_1, x_2, x_3 \geq 0$

We can get an upper bound by $3 *$ eq $3 \Rightarrow 3x_1 + 3x_2 + 3x_3 \leq 15$.

We can get a better upperbound by eq $1 +$ eq $2 \Rightarrow 3x_1 + 3x_2 + 3x_3 \leq 9$.

The idea is that as atleast one coefficient is bigger, the sum is bigger and is an upperbound.

But doing this for all combinations is going to be painfully time consuming. Let's try to do this algebraically and let the coefficients of equation 1,2,3 be y_1, y_2, y_3 in the linear combination.

$$x_1(y_1 + 3y_2 + y_3) + (y_1 + y_3)x_2 + (y_2 + y_3)x_3 \leq 2y_1 + 4y_2 + 5y_3$$

This leads to a LP

Minimize $2y_1 + 4y_2 + 5y_3$ with the constraints:
- $y_1 + 3y_2 + y_3 \geq 3$
- $y_1 + y_3 \geq 2$
- $y_2 + y_3 \geq 1$

Theorem 2.4. *The Row player LP and Column player LP are duel pairs.*

Proof.

TODO. *Computation*

■

2.4. Strong Duality

Optimum value of f is the optimal value of its primal duel.

Ler $x^* = (x_1^*, x_2^*, x_3^*, ..., x_m^*), z^*$ be the optimum values for Row player.

z^* must attain equality at some $j^* \in [n]$ that is $z^* = \sum_i a_{ij^*} x_i^*$ and $z^* \leq \sum_i a_{ij} x_i^* \forall j \in [n]$.

By lemma, $z^* = \min_{j \in S_2} \sum_{i=1}^n x_i^* = \min_{y \in \Delta(S_2)} {x^*}^T Ay$ and similarly, $w^* = \max \sum_{j=1}^n a_{ij^*} y_i = \max_{x \in \Delta(S_2)} x^T A y^*$.

Thus,

$$\min_{y \in \Delta(S_2)} {x^*}^T Ay = \max_{y \in \Delta(S_2)} x^T A y^*$$

This is a nash equilibrium as no player can unilaterally increase payoff by moving.

2.5. Existence of Nash Equilibrium (mixed) in finite strategic form games

Ler $(N, < s_i >, < \mu_i >)$ be the game where $N = \{1, 2, ..., n\}$, S_i is stratergy set for player i, with $|S_i| = m \forall i$ and and $\mu_i : S_i \times S_{-i} \to \mathbb{R}$ is the payoff function for player i.

Mixed strategy σ_i is a probability distribution over S_i.

Δ_i of $\Delta(S_i)$ denotes the set of all mixed stratergies for player i. $\Delta = \Delta_1 \times \Delta_2 \times ... \times \Delta_m$.

Expected payoff from δ_i for player i is

$$\mu_i(\sigma_i) \triangleq \sum_{(s_1,\ldots,s_n)\in S_1\times S_2\ldots S_n} \left(\prod_{j=1}^{n} \sigma_j(s_j)\right) \mu_i(s_1,\ldots,s_n)$$

Definition

σ^* is a Nash Equilibrium if for every player i,

$$\mu_i(\sigma^*) \geq \mu_i(s_i, \sigma^*_{-i}) \forall s_i \in S_i$$

Brouwer's Fixed Point Theorem 2.5. *Let B be a closed, bounded convex set. Let $f :: B \to B$ be a continuous function, then $\exists x \in B$ s.t. $f(x) = x$*

We want to define B, f such that the Nash Equilibriium is the fixed point of f. Define $B = \Delta$.

Define f such that $\sigma \in \Delta$ is not a NE then $f(\sigma) \neq \sigma$ NS IF σ^* is a NE then $f(\sigma^*) = \sigma^*$.

Attempt 1: Define $f(\sigma) = \rho$ where $\rho \in \Delta, \rho = (\rho_1, \rho_2, \ldots, \rho_n)$ such that

$$\rho_i = \arg\max_{\delta_i \in \Delta_i} \mu_i(\sigma_{-i}, \delta_i)$$

Recall, this is called the best response.

The problem is, this is not a function and if we apply a tie breaking rule, it is not continuous. We could use the powerset as the domain, but that would require Katakumi's fixed point which is more analytical than we wish to be.

Example : Matching Pennies

		P_2	
		H	T
P_1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

Let row player choose the mixed stratergy $(p, 1-p)$ and column player choose $(q, 1-q)$. The problem is

$$p > \frac{1}{2} \Rightarrow q = 0$$

$$p < \frac{1}{2} \Rightarrow q = 1$$

$$p = \frac{1}{2} \Rightarrow 0 \leq q \leq 1$$

This is an cointer example to the continuity of f.

Modifying f: We first define a gain function of player i for deviatoon to s_{ij} from σ_i.

$$g_{ij} = \max\{\mu_i(\sigma_{-i}, s_{ij}) - \mu_i(\sigma), 0\}$$

We now define

$$f_{ij}\sigma = \frac{\sigma_{ij} + g_{ij}(\sigma)}{\sum_{k=1}^{m}(\sigma_{ik} + g_{ik}(\sigma))} = \frac{\sigma_{ij} + g_{ij}(\sigma)}{1 + \sum_{k=1}^{m} g_{ik}(\sigma)}$$

We want

$$\sum_{k=1}^{m} f_{ik}(\sigma) = 1$$

We will define

$$\begin{aligned} f(\sigma) &=:= \rho \\ f(\sigma) &= f(\sigma_{11}, ..., \sigma_{1m}, ..., \sigma_{n1}, ...\sigma_{nm}) \\ &= (\rho_1, ..., \rho_{1m}, ..., \rho_{n1}, ..., \rho_{nm}). \end{aligned}$$

Let σ^* be a fixed point of f.

$$\begin{aligned} f_{ij} &= \sigma^*_{ij} \\ \sigma^*_{ij} &= \frac{\sigma_{ij} + g_{ij}(\sigma)}{1 + \sum_{k=1}^{m} g_{ik}(\sigma)} \\ &\iff g_{ij}(\sigma^*) = 0 \forall j \end{aligned}$$

TODO. *Will complete from Solan...*

Thus, by Brower's, we are done.