

# Quiz 1 - Probability Theory

7th February, 2025

## Instructions

Solve all questions.

You are not obliged to use any of the hints. Any correct solution will be accepted.

RELAX. Think.

## Problem 1

- Let  $\mathcal{F}$  be a  $\sigma$ -algebra. If  $|\mathcal{F}| < \infty$ , then prove that  $|\mathcal{F}| = 2^n$  for some natural number  $n \geq 1$ . (Hint: Let  $G$  be a finite group. Suppose  $g^2 = 1, \forall g \in G$ . Use the following fact: if prime  $p | \text{ord}(G)$  then  $\exists x \in G$  such that  $\text{ord}(x) = p$ .)

## Problem 2

- Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space such that  $\mathbb{P}(A) \in \{0, 1\}, \forall A \in \mathcal{F}$ . Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable. (Note:  $X$  need not be discrete.) Prove that there exists  $c \in \mathbb{R}$  such that  $\mathbb{P}(X = c) = 1$ .

## Problem 3

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space. Let  $\{A_n\}$  be a sequence of events in  $\mathcal{A}$ . We define:

$$\lim_{n \rightarrow \infty} \bigcup_{k=n}^{\infty} A_k = \limsup_{n \rightarrow \infty} A_n = \{\omega : \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$$

$$\lim_{n \rightarrow \infty} \bigcap_{k=n}^{\infty} A_k = \liminf_{n \rightarrow \infty} A_n = \{\omega : \exists m \in \mathbb{N} \text{ such that } \omega \in A_n, \forall n \geq m\}$$

- (Fatou's lemma) Prove that

$$\mathbb{P}(\liminf_{n \rightarrow \infty} A_n) \leq \liminf_{n \rightarrow \infty} \mathbb{P}(A_n) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(A_n) \leq \mathbb{P}(\limsup_{n \rightarrow \infty} A_n)$$

- (Borel-Cantelli lemma) Let  $\{B_n : n \in \mathbb{N}\}$  be a sequence of events such that  $\sum_{n \in \mathbb{N}} \mathbb{P}(B_n) < \infty$ . Prove that  $\mathbb{P}(\limsup_{n \rightarrow \infty} B_n) = 0$ .

Hint: Rewrite  $\limsup_{n \rightarrow \infty} A_n$  and  $\liminf_{n \rightarrow \infty} A_n$  using intersections and unions.

$$\begin{aligned} \mathbb{P}(\limsup_{n \rightarrow \infty} A_n) &= \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) \\ &\leq \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k=n}^{\infty} A_k\right) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) \\ &= 0 \end{aligned}$$

Handwritten notes and diagrams include Venn diagrams for unions and intersections, and various algebraic manipulations involving limits and probabilities.