On the cycle of solving and writing

Practice both solving and writing a lot and do so with continual self-reflection. If problem solving is hard, so is writing — a different skill requiring its own practice. Fortunately, it is easier to learn than learning to solve problems. Moreover, the two activities are closely intertwined, supporting each other and together enhancing overall learning. Thus the separation made below between writing tips and solving tips is somewhat artificial.

So far your mindset likely has been something like "first you solve, then you write, then you get your marks and you move on". Change this model into one where writing becomes an integral component of learning. Go back and forth as necessary between thinking and writing, *using each to improve the other.* This can propel your understanding and writing to become transparent enough for you or even your writing alone to be capable of *teaching* someone. That is a much more worthwhile goal than merely satisfying an evaluator.

Writing tips

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"Easy reading is damned hard writing." – Thomas Hood

1. Choose good notation. Clearly write down any notation before using it.
2. Write in complete sentences. Take good care of your quantifiers ($\forall, \exists$), negations, ANDs, ORs. Beware of the pitfalls of ordinary language. Should the "or" in the first sentence of exercise 0 be "and" instead?
3. The sentences should fit together to give a correct, complete and coherent argument proving all claims.
4. After you write your solution, edit it! As a beginning writer, you may need to *rewrite from scratch*, repeatedly. Do not resent this. You are *learning* when you are (re)writing, provided you do it mindfully.
5. Don't write a lot to try to be thorough or "just to be safe". It creates more room for errors. Even if not incorrect, unnecessary content can make the exposition poorer by distracting from the essence of the argument. Even if not badly written, longer writing can reduce the punch. Several writers have expressed regret for writing more because they "did not have the time to make it shorter". Think about the point they're making. (Is this paragraph a cautionary example?) Do write everything necessary to do the intended job, but no more. Weighing this issue is hard. Start by doing the best you can. There isn't a unique right answer (but many wrong ones). Your personal style will emerge with experience.

Further reading. There is a lot of material available on the internet. Two famous sources are *How to Write Mathematics* by Paul Halmos and *Mathematical Writing* by Donald Knuth et al, especially pages 1-6. These are geared towards writing papers, but much of the material is relevant for you. For a student-oriented checklist see appendix E.2 of *A ProblemText in Advanced Calculus* by John Erdman. Also read the last paragraph on the first page of the preface of this book and then go to the appendices as per your need.

Solving tips

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Some "obvious" but useful strategies that should become second nature

1. **Read the problem** carefully and understand its meaning. (Find the source of the next quote. Enjoy.)

   "Confusion hi confusion hai. Solution kuch pata nahin.
   Solution jo mila to saala question kya tha pata nahin."

2. **Clearly formulate what is to be proved.** Just by looking at the semantics of what is asked, one can get started by *writing down* some things for free, setting up useful notation along the way. See point 6 below for a basic illustration. Observe that in HW 1 I did this for you to some extent. The resulting linguistic clarity will at least bring into sharp relief what needs to be done and set the stage for "real work". But it can help even more by imposing some discipline, thereby *structuring* your thoughts, and hence *suggesting* further steps. Often this sets you already well on your way.

3. Can you **break the goal into parts**, e.g., into cases or a cascade of subclaims? Does reformulation help? All mathematical proofs consist “only” of repeated reformulations! Practically, one can try working *forward*: explore consequences of the hypotheses to try to works towards the desired conclusion. Or try working *backwards*: try to find something that implies the desired conclusion, a second something that implies the first something, and see if you can make a chain all the way back to the hypothesis. Often a combination of both methods is useful.

4. It may be easier to show the **equivalent contrapositive statement**, i.e., do proof by contradiction. Showing $A \Rightarrow B$ is equivalent to showing (not B) $\Rightarrow$ (not A). Everything in the previous item comes into play once again. *Caution*: The statement $A \Rightarrow B$ and its converse $B \Rightarrow A$ are very different. Do not confuse one for the other.

5. **The power of examples.** Working out small/extreme examples and proving special cases of the claim can guide your intuition, even if such work will not be part of the solution you write. Sometimes it may help to generalize. The result may become false and a counterexample can shed some light, or stay true and the reason may become more transparent.

6. **Some nuts and bolts**. Showing that two statements A and B are equivalent requires *two* arguments, each showing that one implies the other. Sometimes this can be done by a chain of statements connected at each step by $\Leftrightarrow$. But be careful not to introduce a one way implication in the chain. Similarly to prove equality of sets $S = T$ one shows that each is a subset of the other. Template: “Take $x \in S$ ... some logic ..., so $x \in T$. Take $y \in T$ ... some logic ..., so $y \in S$.” Showing equality of functions f and g means — after ensuring that they have the same domain and the same codomain — showing that $f(x) = g(x)$ for each x in the common domain.

7. **Psycholoigcal, social and practical aspects.** To be good at solving requires the ability to bring your full mental faculties to bear on a problem, hold it in the center of your attention with a laser focus and do this in a sustained manner for hours, possibly days and maybe (much) longer. This is a superpower. Unfortunately most of us are not super(wo)men. Attention can flag or wander, especially when it feels like one is beating one’s head against a wall. *What to do?*

   - First, do not give up too quickly. You *can* get used to feeling stuck and learn to be ok it.
   - After giving it a good try (and a cry?), take a break and come back to it, maybe the next day.
   - Discuss with a partner/study group. **This method is enormously helpful if used correctly.**
   - Get help but use it actively. E.g., ask for a minimal hint, not a solution. If you see a solution, cover it and uncover one line or even word at a time. The moment you see the smallest thought that you did not have or see as being relevant, STOP reading and try to get ahead on your own.
   - For harder/longer problems, multiple rounds of a combination of these steps may be needed.
   - If this looks like a long haul, it is, but that is ok. The rewards will be worth it. So *be patient with yourself* and play the long game. Try to enjoy the process. It *will* get easier with time.

8. **On harder problems.** Easier problems are often more important for learning than tricky ones in that the former help in drilling central concepts and results into your mind. One should definitely solve *some* harder problems to train one’s mental faculties and keep them sharp, but generally speaking it is not useful to make some particular problem a hill to die on, much less to insist on solving every problem in a section before moving ahead. Does knowing a city well mean visiting every single house?

Further reading. *How to solve it* by George Polya is famous. The wikipedia page has a useful summary and references to other material. *For any method to be of any help, one has to keep solving problems!*

Postscript about communicating mathematics. Mathematics is very much a social activity and as such relies heavily on communication, both written and oral. As in life, how to communicate well depends critically on the context, not just the intended content. *Who* is trying to communicate *what*, with *whom*, for *what purpose* and in *what setting*? Even just within mathematics, with so many scenarios possible, this is a large topic! I’ll just end by asking you to be cognizant of the problem, remain thoughtful about the answers in various contexts, and start by seeking answers in the context of your current roles as a student and a colleague.