Basic Mathematics by Lang

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Contents

1 Numbers

1.1 Rules For Addition

Leadup Instructions

Justify each step, using commutativity and associativity in proving the following identities.

Problem 1

$(a+b)+(c+d)=(a+d)+(b+c)$

Solution:

$$\begin{aligned}
&(a+b)+(c+d) & \\
&=((a+b)+c)+d & \text{associative}\\
&=(a+(b+c))+d & \text{associative}\\
&=d+(a+(b+c)) & \text{commutative}\\
&=(d+a)+(b+c) & \text{associative}\\
&=(a+d)+(b+c) & \text{commutative}
\end{aligned}$$

Problem 2

$(a+b)+(c+d)=(a+c)+(b+d)$

Solution:

$$\begin{aligned}
(a+b)+(c+d) &=((a+b)+c)+d & \text{associative}\\
&=(c+(a+b))+d & \text{associative}\\
&=((c+a)+b)+d & \text{commutative}\\
&=(c+a)+(b+d) & \text{associative}\\
&=(a+c)+(b+d) & \text{commutative}
\end{aligned}$$

Problem 3

$(a-b)+(c-d)=(a+c)+(-b-d)$

Solution:

$$\begin{aligned}
(a-b)+(c-d) &=((a-b)+c)-d & \text{associative}\\
&=(c+(a-b))-d & \text{commutative}\\
&=((c+a)-b)-d & \text{associative}\\
&=((a+c)-b)-d & \text{commutative}\\
&=((a+c)+(-b))+(-d) & \\
&=(a+c)+(-b-d) & \text{associative}
\end{aligned}$$

Problem 4

$(a-b)+(c-d)=(a+c)-(b+d)$

Solution:

$$
\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\
&= (c+(a-b))-d && \text{commutative} \\
&= ((c+a)+(-b))+(-d) \\
&= (c+a)+((-b)+(-d)) && \text{associative} \\
&= (c+a)-(b+d) \\
&= (a+c)-(b+d) && \text{commutative}
\end{aligned}
$$

Problem 5

$(a-b)+(c-d)=(a-d)+(c-b)$

Solution:

$$
\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\
&= ((a+(-b))+c)-d \\
&= (a+((-b)+c))-d && \text{associative} \\
&= (((-b)+c)+a)-d && \text{commutative} \\
&= ((-b)+c)+(a-d) && \text{associative} \\
&= (c+(-b))+(a-d) && \text{commutative} \\
&= (c-b)+(a-d) \\
&= (a-d)+(c-b) && \text{commutative}
\end{aligned}
$$

Problem 6

$(a-b)+(c-d)=-(b+d)+(a+c)$

Solution:

$$
\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\
&= ((a+(-b))+c)+(-d) \\
&= (c+(a+(-b)))+(-d) && \text{commutative} \\
&= ((c+a)+(-b))+(-d) && \text{associative} \\
&= ((a+c)+(-b))+(-d) && \text{commutative} \\
&= (a+c)+((-b)+(-d)) && \text{associative} \\
&= ((-b)+(-d))+(a+c) && \text{commutative} \\
&= (-b-d)+(a+c) \\
&= -(b+d)+(a+c) && \text{distributive property}
\end{aligned}
$$

Problem 7

$(a-b)+(c-d)=-(b+d)-(-a-c)$

Solution:

$$
\begin{aligned}
(a-b)+(c-d) &= (a+(-b))+(c+(-d)) & \\
&= ((a+(-b))+c)+(-d) & \text{associative} \\
&= (c+(a+(-b)))+(-d) & \text{commutative} \\
&= ((c+a)+(-b))+(-d) & \text{associative} \\
&= (c+a)+((-b)+(-d)) & \text{associative} \\
&= ((-b)+(-d))+(c+a) & \text{commutative} \\
&= -(b+d)+(c+a) & \\
&= -(b+d)+(-(-c)+-(-a)) & \\
&= -(b+d)-(-c-a) & \\
&= -(b+d)-(-a-c) & \text{commutative}
\end{aligned}
$$

Problem 8

$((x+y)+z)+w = (x+z)+(y+w)$

Solution:

$$
\begin{aligned}
((x+y)+z)+w &= (z+(x+y))+w & \text{commutative} \\
&= ((z+x)+y)+w & \text{associative} \\
&= (z+x)+(y+w) & \text{associative} \\
&= (x+z)+(y+w) & \text{commutative}
\end{aligned}
$$

Problem 9

$(x-y)-(z-w) = (x+w)-y-z$

Solution:

$$
\begin{aligned}
(x-y)-(z-w) &= (x+(-y))+((-z)+w) & \\
&= ((x+(-y))+(-z))+w & \text{associative} \\
&= (x+((-y)+(-z)))+w & \text{associative} \\
&= (((-y)+(-z))+x)+w & \text{commutative} \\
&= ((-y)+(-z))+(x+w) & \text{associative} \\
&= (x+w)+((-y)+(-z)) & \text{commutative} \\
&= (x+w)-y-z &
\end{aligned}
$$

Problem 10

$(x-y)-(z-w) = (x-z)+(w-y)$

Solution:

$$\begin{aligned}
(x-y)-(z-w) &= (x+(-y))+((-z)+w) && \text{distributive}\\
&= ((x+(-y))+(-z))+w && \text{associative}\\
&= (x+((-y)+(-z)))+w && \text{commutative}\\
&= (((-y)+(-z))+x)+w && \text{associative}\\
&= ((-y)+((-z)+x))+w && \text{associative}\\
&= w+((-y)+((-z)+x)) && \text{commutative}\\
&= (w+(-y))+((-z)+x) && \text{associative}\\
&= (w+(-y))+(x+(-z)) && \text{commutative}\\
&= (w-y)+(x-z)
\end{aligned}$$

Problem 11

Show that $-(a+b+c) = -a+(-b)+(-c)$.

Solution:

$$\begin{aligned}
-(a+b+c) &= -(a+(b+c))\\
&= (-a+-(b+c)) && \text{distributive}\\
&= (-a+(-b+(-c))) && \text{distributive}\\
&= -a+(-b)+(-c)
\end{aligned}$$

Problem 12

Show that $-(a-b-c) = -a+b+c$.

Solution:

$$\begin{aligned}
-(a-b-c) &= -(a+(-b)+(-c))\\
&= (-a-(-b)-(-c)) && \text{distributive}\\
&= (-a+b+c) && \text{double negation}\\
&= -a+b+c
\end{aligned}$$

Problem 13

Show that $-(a-b) = b-a$.

Solution:

$$\begin{aligned}
-(a-b) &= (-a)-(-b) && \text{distributive}\\
&= -a+b && \text{double negation}\\
&= b+(-a) && \text{commutative}\\
&= b-a
\end{aligned}$$

Solve for x in the following equations.

Problem 14

$-2 + x = 4$

Solution:

$$
\begin{aligned}
-2 + x &= 4 \\
= -2 + 2 + x &= 4 + 2 \\
= x &= 6
\end{aligned}
$$

Problem 19

$-5 - x = -2$

Solution:

$$
\begin{aligned}
-5 - x &= -2 \Leftrightarrow && \\
(-5 - x) + x &= -2 + x \Leftrightarrow && \\
-5 + ((-x) + x) &= -2 + x \Leftrightarrow && \text{associative} \\
-5 + 0 &= -2 + x \Leftrightarrow && \text{N2} \\
-5 &= -2 + x \Leftrightarrow && \text{N1} \\
-5 + 2 &= (-2 + x) + 2 \Leftrightarrow && \\
-5 + 2 &= 2 + (-2 + x) \Leftrightarrow && \text{commutative} \\
-5 + 2 &= (2 + (-2)) + x \Leftrightarrow && \text{associative} \\
-5 + 2 &= 0 + x \Leftrightarrow && \text{N2} \\
-3 &= x && \text{N1}
\end{aligned}
$$

Problem 20

$-7 + x = -10$

Solution:

$$
\begin{aligned}
-7 + x &= -10 \Leftrightarrow && \\
(-7 + x) + 7 &= -10 + 7 \Leftrightarrow && \\
7 + (-7 + x) &= -3 \Leftrightarrow && \text{commutative} \\
(7 + (-7)) + x &= -3 \Leftrightarrow && \text{associative} \\
0 + x &= -3 \Leftrightarrow && \text{N2} \\
x &= -3 &&
\end{aligned}
$$

Problem 21

$-3 + x = 4$

Solution:

$$\begin{aligned}
-3 + x = 4 &\Leftrightarrow && \\
(-3 + x) + 3 = 4 + 3 &\Leftrightarrow && \\
3 + (-3 + x) = 7 &\Leftrightarrow && \text{commutative} \\
(3 + (-3)) + x = 7 &\Leftrightarrow && \text{associative} \\
0 + x = 7 &\Leftrightarrow && \text{N2} \\
x = 7 & &&
\end{aligned}$$

22 Prove the cancellation law for addition

If $a + b = a + c$ then $b = c$.

Solution:

$$\begin{aligned}
a + b = a + c &\Leftrightarrow && \\
(a + b) + (-a) = (a + c) + (-a) &\Leftrightarrow && \\
-a + (a + b) = -a + (a + c) &\Leftrightarrow && \text{commutative} \\
(-a + a) + b = (-a + a) + c &\Leftrightarrow && \text{associative} \\
0 + b = 0 + c &\Leftrightarrow && \text{N2} \\
b = c & &&
\end{aligned}$$

23 Prove

If $a + b = a$, then $b = 0$.

Solution:

$$\begin{aligned}
a + b = a &\Leftrightarrow && \\
(a + b) + (-a) = a + (-a) &\Leftrightarrow && \\
(-a) + (a + b) = a - a & && \text{commutative} \\
(-a) + (a + b) = 0 & && \text{N2} \\
((-a) + a) + b = 0 &\Leftrightarrow && \text{associative} \\
0 + b = 0 & && \text{N2} \\
b = 0 & &&
\end{aligned}$$

1.2 Rules For Multiplication

Express each of the following expressions in the form $2^m 3^n a^r b^s$, where m, n, r, s are positive integers.

Problem 1

(a) $8a^2b^3(27a^4)(2^5ab)$
(b) $16b^3a^2(6ab^4)(ab)^3$
(c) $3^2(2ab)^3(16a^2b^5)(24b^2a)$
(d) $24a^3(2ab^2)^3(3ab)^2$

(e) $(3ab)^2(27a^3b)(16ab^5)$

(f) $32a^4b^5a^3b^2(6ab^3)^4$

Solution: 1 (a)

$$\begin{aligned}
8a^2b^3(27a^4)(2^5ab) &= 8(27a^4)a^2b^3(2^5ab) && \text{commutative}\\
&= (8\cdot 27)a^4a^2b^3(2^5ab) && \text{associative}\\
&= (8\cdot 27)(2^5ab)a^4a^2b^3 && \text{commutative}\\
&= (8\cdot 27\cdot 2^5)aba^4a^2b^3 && \text{associative}\\
&= (8\cdot 27\cdot 2^5)aa^4a^2bb^3 && \text{commutative}\\
&= (2^3 3^3 2^5)a^7b^4 && \text{N11}\\
&= (2^3 2^5 3^3)a^7b^4 && \text{commutative}\\
&= (2^8 3^3)a^7b^4 && \text{N11}\\
&= 2^8 3^3 a^7 b^4
\end{aligned}$$

Solution: 1 (b)

$$\begin{aligned}
16b^3a^2(6ab^4)(ab)^3 &= b^3a^2(ab)^3(6ab^4)16 && \text{commutative}\\
&= b^3a^2(ab)^3 6(ab^4)16 && \text{associative}\\
&= b^3a^2(ab)^3(ab^4)16\cdot 6 && \text{commutative}\\
&= b^3a^2a^3b^3ab^4 16\cdot 6 && \text{N12}\\
&= a^2a^3ab^3b^3b^4 16\cdot 6 && \text{commutative}\\
&= a^2a^3ab^3b^3b^4 2^4 2\cdot 3\\
&= a^6b^{10}2^5 3 && \text{N11}\\
&= 2^5 3a^6b^{10} && \text{commutative}
\end{aligned}$$

Solution: 1 (c)

$$\begin{aligned}
3^2(2ab)^3(16a^2b^5)(24b^2a) &= 3^2 2^3a^3b^3(16a^2b^5)(24b^2a) && \text{N12}\\
&= 2^3 24\cdot 3^2 16a^3a^2ab^3b^5b^2 && \text{commutative}\\
&= 2^3 3\cdot 2^3 3^2 2^4a^3a^2ab^3b^5b^2\\
&= 2^3 2^3 2^4 3^2 3a^3a^2ab^3b^5b^2 && \text{associative}\\
&= 2^{10}3^3a^6b^{10} && \text{N11}
\end{aligned}$$

Solution: 1 (d)

$$\begin{aligned}
24a^3(2ab^2)^3(3ab)^2 &= 24a^3 2^3 a^3 (b^2)^3 3^2 a^2 b^2 && \text{N12}\\
&= 24a^3 2^3 a^3 b^6 3^2 a^2 b^2 && \text{N12}\\
&= 24 \cdot 2^3 3^2 a^3 a^3 a^2 b^6 b^2 && \text{commutative}\\
&= 2^3 3 \cdot 2^3 3^2 a^3 a^3 a^2 b^6 b^2 &&\\
&= 2^3 2^3 3^2 3 a^3 a^3 a^2 b^6 b^2 && \text{commutative}\\
&= 2^6 3^3 a^8 b^8 && \text{N11}
\end{aligned}$$

Solution: 1 (e)

$$\begin{aligned}
(3ab)^2(27a^3b)(16ab^5) &= 3^2 a^2 b^2 27 a^3 b 16 a b^5 && \text{N12}\\
&= 27 \cdot 16 \cdot 3^2 a^2 a^3 a b^2 b b^5 && \text{commutative}\\
&= 3^3 2^4 3^2 a^2 a^3 a b^2 b b^5 &&\\
&= 2^4 3^3 3^2 a^2 a^3 a b^2 b b^5 && \text{commutative}\\
&= 2^4 3^5 a^6 b^8 && \text{N11}
\end{aligned}$$

Solution: 1 (f)

$$\begin{aligned}
32a^4b^5a^3b^2(6ab^3)^4 &= 32a^4b^5a^3b^2 6^4 a^4 (b^3)^4 && \text{N12}\\
&= 32a^4b^5a^3b^2 6^4 a^4 b^{12} && \text{N12}\\
&= 6^4 32 a^3 a^4 a^4 b^5 b^2 b^{12} && \text{commutative}\\
&= (2 \cdot 3)^4 2^5 a^3 a^4 a^4 b^5 b^2 b^{12} &&\\
&= 2^4 3^4 2^5 a^3 a^4 a^4 b^5 b^2 b^{12} && \text{N12}\\
&= 2^4 2^5 3^4 a^3 a^4 a^4 b^5 b^2 b^{12} && \text{commutative}\\
&= 2^9 3^4 a^{11} b^{19} && \text{N11}
\end{aligned}$$

Problem 2

Prove
(a) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
(b) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Solution: 2 (a)

$$
\begin{aligned}
(a+b)^3 &= (a+b)(a+b)(a+b) \\
&= ((a+b)(a+b))(a+b) && \text{associative} \\
&= (a(a+b)+b(a+b))(a+b) && \text{distributive} \\
&= (a^2+ab+ba+b^2)(a+b) && \text{distributive} \\
&= (a^2+2ab+b^2)(a+b) \\
&= a^2(a+b)+2ab(a+b)+b^2(a+b) && \text{distributive} \\
&= a^2a+a^2b+2aba+2abb+b^2a+b^2b && \text{distributive} \\
&= a^3+a^2b+2a^2b+2ab^2+b^2a+b^3 && \text{N11} \\
&= a^3+3a^2b+3ab^2+b^3
\end{aligned}
$$

Solution: 2 (b)

$$
\begin{aligned}
(a-b)^3 &= (a-b)(a-b)(a-b) \\
&= ((a-b)(a-b))(a-b) && \text{associative} \\
&= (a(a-b)-b(a-b))(a-b) && \text{distributive} \\
&= (a^2-ab-ba+b^2)(a-b) && \text{distributive} \\
&= (a^2-2ab+b^2)(a-b) \\
&= a^2(a-b)-2ab(a-b)+b^2(a-b) && \text{distributive} \\
&= a^2a-a^2b-2aba+2abb+b^2a-b^2b && \text{distributive} \\
&= a^3-a^2b-2a^2b+2ab^2+b^2a-b^3 && \text{N11} \\
&= a^3-3a^2b+3ab^2-b^3
\end{aligned}
$$

Problem 3

Obtain expansions for $(a+b)^4$ and $(a-b)^4$.

Solution: 3

From 2: $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

$$
\begin{aligned}
(a+b)^3 * (a+b) &= (a+b)^4 && \text{N11} \\
&= ((a^3+3a^2b)+(3ab^2+b^3)) * (a+b) \\
&= (a^3+3a^2b)(a+b)+(3ab^2+b^3)(a+b) && \text{distributive} \\
&= (a^3(a+b)+3a^2b(a+b))+(3ab^2(a+b)+b^3(a+b)) && \text{distributive} \\
&= (aa^3+ba^3)+(a3a^2b+b3a^2b)+(a3ab^2+b3ab^2)+(ab^3+bb^3) && \text{distributive} \\
&= (a^4+a^3b)+(3a^3b+3a^2b^2)+(3a^2b^2+3ab^3)+(ab^3+b^4) && \text{N11} \\
&= a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{aligned}
$$

From prev: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

$$\begin{aligned}(a-b)^4 &= (a+(-b))^4 \\ &= a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + (-b)^4 \\ &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4\end{aligned}$$

Problem 5

$(1-2x)^2$

Solution

$$\begin{aligned}(1-2x)^2 &= (1-2x) * (1-2x) \\ &= (1(1-2x) - 2x(1-2x)) && \text{distributive} \\ &= ((1-2x) - (2x - 2x2x)) && \text{distributive} \\ &= (1-2x) - (2x - 4x^2) && \text{N11} \\ &= ((1-2x) - 2x) - 4x^2 && \text{associative} \\ &= (1 + ((-2x) - 2x)) - 4x^2 && \text{associative} \\ &= (1 + (-4x)) - 4x^2 \\ &= 1 - 4x - 4x^2\end{aligned}$$

Problem 7

$(x-1)^2$

Solution

$$\begin{aligned}(x-1)^2 &= (x-1)\cdot(x-1) \\ &= x^2 - 2x + 1 \quad \text{perfect square}\end{aligned}$$

Problem 11

$(1+x^3)(1-x^3)$

Solution

$$(1+x^3)(1-x^3) = (1-x^6) \quad \text{difference of squares}$$

Problem 13

$(x^2-1)^2$

Solution

$$(x^2-1)^2 = x^4 - 2x^2 + 1 \quad \text{perfect square}$$

Problem 17

$(x^3-4)(x^3+4)$

Solution

$$(x^3-4)(x^3+4) = x^6 - 16 \quad \text{difference of squares}$$

Problem 19

$(-2+3x)(-2-3x)$

Solution

$$(-2+3x)(-2-3x) = 4 - 9x^2 \quad \text{difference of squares}$$

Problem 23

$(-1-x)(-2+x)(1-2x)$

Solution

$$\begin{aligned}
(-1-x)(-2+x)(1-2x) &= (2+x-x^2)(1-2x) && \text{distributive}\\
&= (2(1-2x)+x(1-2x)-x^2(1-2x)) && \text{distributive}\\
&= 2-4x+x-2x^2-x^2+2x^3 && \text{distributive}\\
&= 2-3x-3x^2+2x^3
\end{aligned}$$

Problem 29

$(2x+1)^2(2-3x)$

Solution

$$\begin{aligned}
(2x+1)^2(2-3x) &= (4x^2+4x+1)(2-3x) && \text{perfect square}\\
&= (4x^2(2-3x)+4x(2-3x)+1(2-3x)) && \text{distributive}\\
&= (8x^2-12x^3+8x-12x^2+2-3x) && \text{distributive}\\
&= (-12x^3-4x^2+5x+2)
\end{aligned}$$

Problem 30

The population of a city in 1910 was $50,000$, and it doubles every 10 years. What will it be (a) in 1970 (b) in 1990 (c) in $2,000$?

Solution

$$(a)\ 50000 * 2^{((1970-1910)/10)} = 3200000$$
$$(b)\ 50000 * 2^{((1990-1910)/10)} = 12800000$$
$$(c)\ 50000 * 2^{((2000-1910)/10)} = 25600000$$

Problem 31

The population of a city in 1905 was $100,000$, and it doubles every 25 years. What will it be after (a) 50 years (b) 100 years (c) 150 years?

Solution

$$(a)\ 100000 * 2^{(50/25)} = 400000$$
$$(b)\ 100000 * 2^{(100/25)} = 1600000$$
$$(c)\ 100000 * 2^{(150/25)} = 6400000$$

Problem 32

The population of a city was 200 thousand in 1915, and it triples every 50 years. What will be the population What will be the population
(a) in the year 2215?
(b) in the year 2165?

Solution

$$(a)\ 200000 * 3^{((2215-1915)/50)} = 145800000$$
$$(b)\ 200000 * 3^{((2165-1915)/50)} = 48600000$$

Problem 33

The population of a city was $25,000$ in 1870, and it triples every 40 years. What will it be.
(a) in 1990?
(b) in 2030?

Solution

$$(a)\ 25000 * 3^{((1990-1870)/40)} = 675000$$
$$(b)\ 25000 * 3^{((2030-1870)/40)} = 2025000$$

1.3 Even and Odd Integers; Divisibility

Problem 1

Give the proofs for the cases of theorem 1 which were not proved in the text.
(a) If a is even and b is even, then $a + b$ is even.
(b) If a is odd and b is even, then $a + b$ is odd.
(c) If a is odd and b is odd, then $a + b$ is even.

Solution (a)

Since a and b are even they can be written as $2n_1$ and $2n_2$ respectively, where n_1 and n_2 are integers.
Let $x = n_1 + n_2$. Note x is an integer because the sum of two integers is an integer.

$$\begin{aligned} a + b &= 2n_1 + 2n_2 \\ &= 2(n_1 + n_2) \\ &= 2x \end{aligned}$$

Since $a + b$ can be written as $2x$ where x is an integer; $a + b$ is even.

Solution (b)

$$\begin{aligned} a + b &= 2n_1 + 1 + 2n_2 \\ &= 2n_1 + 2n_2 + 1 \\ &= 2(n_1 + n_2) + 1 && \text{let } x = n_1 + n_2 \\ &= 2x + 1 \end{aligned}$$

Solution (c)

$$\begin{aligned} a + b &= 2n_1 + 1 + 2n_2 + 1 \\ &= 2n_1 + 2n_2 + 2 \\ &= 2(n_1 + n_2 + 1) && \text{let } x = n_1 + n_2 + 1 \\ &= 2x \end{aligned}$$

Problem 2

If a is even and b is any positive integer then ab is even.

Proof. By def. of an even number a can be written as $2n$ where n is an integer.
Let $x = n \cdot b$. Note the product of two integers is an integer ig.
Something about multiplication being repeated addition and the sum of two integers being an integer.

$$\begin{aligned} a \cdot b &= 2n \cdot b \\ &= 2x \end{aligned}$$

Since ab can be written as $2x$ where x is an integer ab is even. ∎

Problem 3

If a is even, then a^3 is even.

Proof. By def. of an even number a can be written as $2n$ where n is an integer.
Let $x = 2^2n^3$. Note x is an integer.

$$\begin{aligned} a^3 &= (2n)^3 \\ &= 2^3n^3 && \text{N12} \\ &= 2 \cdot 2^2n^3 && \text{N11} \\ &= 2x \end{aligned}$$

Since a^3 can be written as $2x$ where x is an integer a^3 is even. ∎

Problem 4

If a is odd, then a^3 is odd.

Proof. By def. of an odd number a can be written as $2n+1$ where n is an integer.
Let $x = 4n^3 + 6n^2 + 3n$. Note x is an integer.

$$\begin{aligned} a^3 &= (2n+1)^3 \\ &= 8n^3 + 12n^2 + 6n + 1 && \text{distributive} \\ &= 2(4n^3 + 6n^2 + 3n) + 1 && \text{distributive} \\ &= 2x + 1 \end{aligned}$$

Since a^3 can be written as $2x+1$ where x is an integer a^3 is odd. ∎

Problem 5

If n is even, then $(-1)^n = 1$.

Proof. By def. of an even number n can be written as $2a$ where a is an integer.

$$\begin{aligned} (-1)^n &= (-1)^{2a} \\ &= ((-1)^2)^a && \text{N12} \\ &= 1^a \\ &= 1 \end{aligned}$$

∎

Problem 6

If n is odd, then $(-1)^n = -1$.

Proof. By def. of an odd number n can be written as $2a+1$ where a is an integer.

$$\begin{aligned} (-1)^n &= (-1)^{2a+1} \\ &= (-1)^{2a} \cdot (-1)^1 && \text{N11} \\ &= 1 \cdot (-1) && \text{2a is even so by prob. 5} \\ &= -1 && \text{N7} \end{aligned}$$

∎

Problem 7

If m, n are odd, then the product mn is odd.

Proof. By def. of an odd number m and n can be written as $2n_1 + 1$ and $2n_2 + 1$ where n_1 and n_2 are integers. Let $x = 2n_1n_2 + n_1 + n_2$. Note x is an integer.

$$\begin{aligned} mn &= (2n_1 + 1)(2n_2 + 1) & \\ &= 4n_1n_2 + 2n_1 + 2n_2 + 1 & \text{distributive} \\ &= 2(2n_1n_2 + n_1 + n_2) + 1 & \text{distributive} \\ &= 2x + 1 & \end{aligned}$$

Since mn can be written as $2x + 1$ where x is an integer, therefore mn is odd. ∎

Problem 24

Let a, b be integers, Define $a \equiv b \ (mod\ 5)$, which we read "a is congruent to b modulo 5, to mean that $a - b$ is divisible by 5.
Prove if $a \equiv b \ (mod\ 5)$ and $x \equiv y \ (mod\ 5)$ then $a + x \equiv b + y \ (mod\ 5)$ and $ax = by \ (mod\ 5)$.

Proof. Need to show $(a + x) - (b + y) = 5n$ where n is an integer.
From $a \equiv b \ (mod\ 5)$, $a - b = 5n_1$ where n_1 is an integer.
From $x \equiv y \ (mod\ 5)$, $x - y = 5n_2$ where n_2 is an integer.
Let $t = n_1 + n_2$.

$$\begin{aligned} (a + x) - (b + y) &= (a - b) + (x - y) \\ &= 5n_1 + 5n_2 \\ &= 5(n_1 + n_2) \\ &= 5t \end{aligned}$$

Since $(a + x) - (b + y) = 5t$ where t is an integer, $a + x \equiv b + y \ (mod\ 5)$. ∎

Proof. Need to show $ax - by = 5n$ where n is an integer.
From $a \equiv b \ (mod\ 5)$, $a - b = 5n_1$ where n_1 is an integer.
From $x \equiv y \ (mod\ 5)$, $x - y = 5n_2$ where n_2 is an integer.
Let $t = bn_2 - yn_1 - 5n_1n_2$.

$$\begin{aligned} ax - by &= (b - 5n_1)(y + 5n_2) - by \\ &= by + 5bn_2 - 5yn_1 - 25n_2n_1 - by \\ &= 5bn_2 - 5yn_1 - 25n_2n_1 \\ &= 5(bn_2 - yn_1 - 5n_2n_1) \\ &= 5t \end{aligned}$$

Since $ax - by = 5t$ where t is an integer, $ax = by \ (mod\ 5)$. ∎

Problem 25

Let d be a positive integer. Let a, b be integers.
Define $a \equiv b \ (mod\ d)$ to mean that $a - b$ is divisible by d.
Prove that if $a \equiv b \ (mod\ d)$ and $x \equiv y \ (mod\ d)$, then $a + x \equiv b + y \ (mod\ d)$ and $ax = by \ (mod\ d)$.

Proof. Need to show $(a + x) - (b + y) = dn$ where n is an integer.
From $a \equiv b \ (mod\ d)$, $a - b = dn_1$ where n_1 is an integer.
From $x \equiv y \ (mod\ d)$, $x - y = dn_2$ where n_2 is an integer.

$$\begin{aligned}(a + x) - (b + y) &= (a - b) + (x - y) \\ &= dn_1 + dn_2 \\ &= d(n_1 + n_2) && \text{let } t = n_1 + n_2 \\ &= dt\end{aligned}$$

Since $(a + x) - (b + y)$ can be written as dt where t is an integer, $a + x \equiv b + y \ (mod\ d)$. ∎

Proof. Need to show $ax - by = dn$.
From $a \equiv b \ (mod\ d)$, $a - b = dn_1$ where n_1 is an integer.
From $x \equiv y \ (mod\ d)$, $x - y = dn_2$ where n_2 is an integer.

$$\begin{aligned}ax - by &= (b + dn_1)(y + dn_2) - by \\ &= by + bdn_2 + ydn_1 + dbn_1n_2 - by \\ &= bdn_2 + ydn_1 + dbn_1n_2 \\ &= d(bn_2 + yn_1 + bn_1n_2) && \text{Let } t = bn_2 + yn_1 + bn_1n_2 \\ &= dt\end{aligned}$$

Since $ax - by$ can be written as dt where t is an integer, $ax = by \ (mod\ d)$. ∎

Problem 26

Assume that every positive integer can be written in one of the forms $3k$, $3k + 1$, or $3k + 2$ for some integer k.
Show that if the square of a positive integer is divisible by 3, then so is the integer x.

Proof. From the assumptions x can either be written $3k$, $3k + 1$, or $3k + 2$.
Need to show that if $x^2 = 3n_1$, $x = 3n_2$ for some integers n_1 and n_2.
Case 1 ($x = 3k$):
Let $t_1 = 3k^2$

$$\begin{aligned}x^2 &= (3k)^2 \\ &= 3 \cdot 3k^2 \\ &= 3t_1\end{aligned}$$

Therefore in this case x is divisible by 3.
Case 2 ($x = 3k + 1$):

Let $t_2 = 2k^2 + 2k$

$$\begin{aligned}
x^2 &= (3k+1)^2 \\
&= 6k^2 + 3k + 3k + 1 \\
&= 6k^2 + 6k + 1 \\
&= 3(2k^2 + 2k) + 1 \\
&= 3t_2 + 1
\end{aligned}$$

In this case x^2 is not divisible by 3 which contradicts our assumption, therefore $x \neq 3k + 1$.
Case 3 ($x = 3k + 2$):
Let $t_3 = 3k^2 + 4k$

$$\begin{aligned}
x^2 &= (3k+2)^2 \\
&= 9k^2 + 6k + 6k + 4 \\
&= 9k^2 + 12k + 4 \\
&= 3(3k^2 + 4k) + 4 \\
&= 3t_3 + 4
\end{aligned}$$

In this case x^2 is not divisible by 3 which contradicts our assumption, therefore $x \neq 3k + 2$.

Note there is no solution for $1 = 3m_1$ or $2 = 3m_2$ where m_1 and m_2 are integers.
Assume $3k_1 + 1 = 3m_1$ where k_1 and m_1 are integers.

$$\begin{aligned}
3k_1 + 1 &= 3m_1 \\
1 &= 3m_1 - 3k_1 \\
1 &= 3(m_1 - k_1)
\end{aligned}$$

Therefore, $3k_1 + 1$ is not divisible by 3.

Assume $3k_2 + 2 = 3m_2$ where k_2 and m_2 are integers.

$$\begin{aligned}
3k_2 + 1 &= 3m_2 \\
2 &= 3m_2 - 3k_2 \\
2 &= 3(m_2 - k_2)
\end{aligned}$$

Therefore, $3k_2 + 2$ is not divisible by 3.

∎

1.4 Rational Numbers

Problem 4

Let $a = \frac{m}{n}$ be a rational number expressed as a quotient of integers m, n with $m \neq 0$ and $n \neq 0$.
Show that there is a rational number b such that $ab = ba = 1$.

Proof. Let $b = \frac{n}{m}$. Since n and m are integers and $m \neq 0$, b is the ratio of two integers where the denomitor is not 0 making it a rational number by definition.

$$\begin{aligned}
ab &= \frac{m}{n} \cdot \frac{n}{m} \\
&= \frac{mn}{nm} \\
&= \frac{nm}{nm} && \text{commutative} \\
&= 1 && \text{cancellation rule for fractions}
\end{aligned}$$

$$\begin{aligned}
ba &= \frac{n}{m} \cdot \frac{m}{n} \\
&= \frac{nm}{mn} \\
&= \frac{nm}{nm} && \text{commutative} \\
&= 1 && \text{cancellation rule for fractions}
\end{aligned}$$

Therefore $ab = ba = 1$. ∎

Problem 6

Solve for x in the following equations.

Solution (d)

$$\begin{aligned}
\frac{4x}{3} + \frac{3}{4} &= 2x - 5 \\
12\left(\frac{4x}{3} + \frac{3}{4}\right) &= 12(2x - 5) \\
16x + 9 &= 24x - 60 \\
9 + 60 &= 24x - 16x \\
69 &= 8x \\
x &= \frac{69}{8}
\end{aligned}$$

Solution (e)

$$\begin{aligned}
\frac{4(1 - 3x)}{7} &= 2x - 1 \\
4(1 - 3x) &= 7(2x - 1) \\
4 - 12x &= 14x - 7 \\
4 + 7 &= 14x + 12x \\
11 &= 26x \\
x &= \frac{11}{26}
\end{aligned}$$

Solution (f)

$$\begin{aligned}
\frac{2-x}{3} &= \frac{7}{8}x \\
8(2-x) &= 3 \cdot 7x \\
16 - 8x &= 21x \\
16 &= 29x \\
x &= \frac{16}{29}
\end{aligned}$$

Problem 6

Let n be a positive integer. By n factorial, written $n!$, we mean the product:
$1 \cdot 2 \cdot 3 \ldots n$
of the first n positive integers. For Instance
$2! = 2$
$3! = 2 \cdot 3 = 6$
$4! = 2 \cdot 3 \cdot 4 = 24$
(a) Find the value $5!$, $6!$, $7!$, and $8!$.
(b) Define $0! = 1$. Define the binomial coefficient
$\binom{n}{m} = \frac{m!}{n!(m-n)!}$
for any natural numbers m, n such that n lies between 0 and m. Compute tons of binomial coefficients.
(c) Show that $\binom{m}{n} = \binom{m}{m-n}$.
(d) Show that if n is a positive integer at most equal to m, then
$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$.

Solution (a)

$$\begin{aligned}
5! &= 2 \cdot 3 \cdot 4 \cdot 5 && = 120 \\
6! &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 && = 720 \\
7! &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 && = 5040 \\
8! &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 && = 40320
\end{aligned}$$

Solution (b)

$$\binom{3}{0} = \frac{3!}{0!(3-0)!} = 1$$
$$\binom{3}{1} = \frac{3!}{1!(3-1)!} = 3$$
$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$$
$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = 1$$
$$\binom{4}{0} = \frac{4!}{0!(4-0)!} = 1$$
$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = 4$$
$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$$
$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$$
$$\binom{4}{4} = \frac{4!}{4!(4-4)!} = 1$$
$$\binom{5}{0} = \frac{5!}{0!(5-0)!} = 1$$
$$\binom{5}{1} = \frac{5!}{1!(5-1)!} = 5$$
$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10$$
$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$$
$$\binom{5}{4} = \frac{5!}{4!(5-4)!} = 5$$
$$\binom{5}{5} = \frac{5!}{5!(5-5)!} = 1$$

Solution (c)

$$\binom{m}{n} = \binom{m}{m-n}$$
$$\frac{m!}{n!(m-n)!} = \frac{m!}{(m-n)!(m-(m-n))!}$$
$$\frac{m!}{n!(m-n)!} = \frac{m!}{(m-n)!(n)!}$$
$$\frac{m!}{n!(m-n)!} = \frac{m!}{(n)!(m-n)!}$$

Solution (d)
Need to show:

$$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$$

First note:

$$\binom{m+1}{n} = \frac{(m+1)!}{n!((m+1)-n)!}$$

Then:

$$\begin{aligned}
\binom{m}{n} + \binom{m}{n-1} &= \frac{m!}{n!(m-n)!} + \frac{m!}{(n-1)!(m-n+1)!} \\
&= \frac{m!(m-n+1) + m!n}{n!(m-n+1)!} \quad \text{(common denominator)} \\
&= \frac{m!(m-n+1+n)}{n!(m-n+1)!} \\
&= \frac{m!(m+1)}{n!(m-n+1)!} \\
&= \frac{(m+1)!}{n!((m+1)-n)!} \\
&= \binom{m+1}{n}
\end{aligned}$$

Problem 8

Prove that there is no positive rational number a such that $a^3 = 2$.

Proof. Let $a = \frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form. If $(\frac{m}{n})^3 = 2$.

$$\begin{aligned}
\frac{m^3}{n^3} &= 2 \\
m^3 &= 2n^3
\end{aligned}$$

If $m^3 = 2k$ for some integer k, then $m = 2a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned}
(2k)^3 &= 2n^3 \\
2^3k^3 &= 2n^3 \\
2^2k^3 &= n^3 \\
2 \cdot (2k^3) &= n^3
\end{aligned}$$

If $n^3 = 2k$ for some integer k, then $n = 2a$ for some integer a (shown in a previous problem). This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^3 = 2$. ∎

Problem 9

Prove that there is no positive rational number a such that $a^4 = 2$.

Proof. Suppose for contradiction $a^4 = 2$ where a is a rational number. Since a is rational, it can be expressed as $\frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form.

$$\begin{aligned} \frac{m^4}{n^4} &= 2 \\ m^4 &= 2n^4 \end{aligned}$$

If $m^4 = 2k$ for some integer k, then $m = 2a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned} (2k)^4 &= 2n^4 \\ 2^4k^4 &= 2n^4 \\ 2^3k^4 &= n^4 \\ 2 \cdot (2^2k^3) &= n^4 \end{aligned}$$

If $n^4 = 2k$ for some integer k, then $n = 2a$ for some integer a (shown in a previous problem). This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^4 = 2$. ∎

Problem 10

Prove that there is no positive rational number a such that $a^2 = 3$. You may assume that a positive integer can be written in one of the forms $3k$, $3k+1$, $3k+2$ for some integer k. Prove that if the square of a positive integer is divisible by 3 so is the integer. Then use a similar proof for $\sqrt{2}$.

Proof. Since a is rational, it can be expressed as $\frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form.

$$\begin{aligned} \frac{m^2}{n^2} &= 3 \\ m^2 &= 3n^2 \end{aligned}$$

If $m^2 = 3k$ for some integer k, then $m = 3a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned} (3a)^2 &= 3n^2 \\ 3^2a^2 &= 3n^2 \\ 3^3a^2 &= n^2 \\ 3 \cdot (3^2a^2) &= n^2 \end{aligned}$$

If $n^2 = 3k$ for some integer k, then $n = 3a$ for some integer a. This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^2 = 3$. ∎

Proof. Need to show that $a^2 = 2$ has no rational solution a. Since a is rational, it can be expressed as $\frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form.

$$\begin{aligned} \frac{m^2}{n^2} &= 2 \\ m^2 &= 2n^2 \end{aligned}$$

If $m^2 = 2k$ for some integer k, then $m = 2a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned}(2a)^2 &= 2n^2\\ 2^2a^2 &= 2n^2\\ 2a^2 &= n^2\\ 2\cdot a^2 &= n^2\end{aligned}$$

If $n^2 = 2k$ for some integer k, then $n = 2a$ for some integer a. This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^2 = 2$. ∎

Problem 16

A chemical substance decomposes in such a way that it halves every 3 min. If there are 6 grams (g) of the substance at present at the beginning, how much will will be left
(a) after 3 min?
(b) after 27 min?
(c) after 36 min?

Solution
(a) after 3 min? $6(\frac{1}{2})^{(\frac{3}{3})} = 3g$

(b) after 27 min? $6(\frac{1}{2})^{(\frac{27}{3})} = 0.01171875g$

(c) after 36 min? $6(\frac{1}{2})^{(\frac{36}{3})} = 0.001464843g$

Problem 18

A substance reacts in water in such a way that one-fourth of the undissolved parts disolves every 10 minutes. If you put 25g of a substance in water at a given time, how much will be left after:
(a) 10 min?
(b) 30 min?
(c) 50 min?

Solution
(a) after 10 min? $25(\frac{3}{4})^{(\frac{10}{10})} = 18.75g$

(b) after 30 min? $25(\frac{3}{4})^{(\frac{30}{10})} = 10.546875g$

(c) after 50 min? $25(\frac{3}{4})^{(\frac{50}{10})} = 5.933g$

Problem 20

A chemical pollutant is being emptied in a lake with 50, 000 fishes. Every month, one-thrid of the fish still alive die from this pollutant. How many fish will be alive after:
(a) 1 month?
(b) 2 month?
(c) 4 month?

Solution
(a) after 1 month? $50000(\frac{2}{3})^1 = 33333.33$fishes

(b) after 2 month? $50000(\frac{2}{3})^2 = 22222.22$fishes

(c) after 4 month? $50000(\frac{2}{3})^4 = 9876.54$fishes

Problem 21

Every 10 years the population of a city is five-fourths of what it was the 10 years before. How many years does it take
(a) before the population doubles
(b) before it triples

Formula: $p \cdot \frac{5}{4}^{\frac{y}{10}} = 2p$
Solution (a)

$$\begin{aligned} p \cdot \frac{5}{4}^{\frac{y}{10}} &= 2p \\ = y &\approx 31 \text{ years} \end{aligned}$$

Solution (b)

$$\begin{aligned} p \cdot \frac{5}{4}^{\frac{y}{10}} &= 3p \\ = y &\approx 49 \text{ years} \end{aligned}$$

1.5 Multiplicative Inverse

Problem 2

Prove the following relations. It is assumed that all values of x and y which occur are such that the denominators in the indicated fractions are not 0.
(a) $\frac{1}{x+y} - \frac{1}{x-y} = \frac{-2y}{x^2-y^2}$
(b) $\frac{x^3-1}{x-1} = 1 + x + x^2$
(c) $\frac{x^4-1}{x-1} = 1 + x + x^2 + x^3$

Proof.

$$\begin{aligned} \frac{1}{x+y} - \frac{1}{x-y} &= \frac{-2y}{x^2-y^2} \\ (x+y)[\frac{1}{x+y} - \frac{1}{x-y}] &= \frac{-2y}{x^2-y^2}(x+y) \\ 1 - \frac{x+y}{x-y} &= \frac{-2y}{x-y}(x+y) \\ (x-y)(1 - \frac{x+y}{x-y}) &= \frac{-2y}{x-y}(x-y) \\ (x-y) - (x+y) &= -2y \\ -2y &= -2y \end{aligned}$$

∎

Proof.

$$\begin{aligned} &\frac{x^3-1}{x-1} = 1 + x + x^2 \\ &x^3 - 1 = (x-1)(1+x+x^2) \\ &x^3 - 1 = x + x^2 + x^3 - 1 - x - x^2 \\ &x^3 - 1 = x^3 - 1 \end{aligned}$$

Proof.

$$\begin{aligned}\frac{x^4-1}{x-1} &= 1+x+x^2+x^3\\ x^4-1 &= (x-1)(1+x+x^2+x^3)\\ x^4-1 &= x+x^2+x^3+x^4-1-x-x^2-x^3\\ x^4-1 &= x+x^2+x^3+x^4-1-x-x^2-x^3\\ x^4-1 &= x^4-1\end{aligned}$$

∎

Proof.

$$\begin{aligned}\frac{x^n-1}{x-1} &= x^{n-1}+x^{n-2}+\cdots+x+1\\ x^n-1 &= (x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)\\ x^n-1 &= x^n+x^{n-1}+\cdots+x^2+x-x^{n-1}-x^{n-2}-\cdots-x-1\\ x^n-1 &= x^n-1\end{aligned}$$

∎

Problem 3

Prove the following relations.

(a) $\frac{1}{2x+y}+\frac{1}{2x-y}=\frac{4x}{4x^2-y^2}$

(b) $\frac{2x}{x+5}+\frac{3x+1}{2x+1}=\frac{x^2-14x-5}{2x^2+11x+5}$

(c) $\frac{1}{x+3y}+\frac{1}{x-3y}=\frac{2x}{x^2-9y^2}$

(c) $\frac{1}{3x-2y}+\frac{x}{x+y}=\frac{x+y+3x^2-2xy}{3x^2+xy-2y^2}$

Proof.

$$\begin{aligned}\frac{1}{2x+y}+\frac{1}{2x-y} &= \frac{4x}{4x^2-y^2}\\ \leftrightarrow ((2x+y)(2x-y))\left(\frac{1}{2x+y}+\frac{1}{2x-y}\right) &= ((2x+y)(2x-y))\left(\frac{4x}{4x^2-y^2}\right)\\ \leftrightarrow (2x-y)+(2x+y) &= 4x\\ \leftrightarrow 4x &= 4x\end{aligned}$$

∎

Proof.

$$
\begin{aligned}
&\frac{2x}{x+5} + \frac{3x+1}{2x+1} = \frac{x^2-14x-5}{2x^2+11x+5} \\
\leftrightarrow\ &((x+5)(2x+1))\left(\frac{2x}{x+5} + \frac{3x+1}{2x+1}\right) = ((x+5)(2x+1))\left(\frac{x^2-14x-5}{2x^2+11x+5}\right) \\
\leftrightarrow\ &2x(2x+1) - (3x+1)(x+5) = x^2 - 14x - 5 \\
\leftrightarrow\ &(4x^2+2x) - (3x^2+15x+5) = x^2 - 14x - 5 \\
\leftrightarrow\ &x^2 - 14x - 5 = x^2 - 14x - 5
\end{aligned}
$$

∎

Proof.

$$
\begin{aligned}
&\frac{1}{x+3y} + \frac{1}{x-3y} = \frac{2x}{x^2-9y^2} \\
\leftrightarrow\ &((x+3y)(x-3y))\left(\frac{1}{x+3y} + \frac{1}{x-3y}\right) = ((x+3y)(x-3y))\left(\frac{2x}{x^2-9y^2}\right) \\
\leftrightarrow\ &(x-3y) + (x+3y) = 2x \\
\leftrightarrow\ &2x = 2x
\end{aligned}
$$

∎

Proof.

$$
\begin{aligned}
&\frac{1}{3x-2y} + \frac{x}{x+y} = \frac{x+y+3x^2-2xy}{3x^2+xy-2y^2} \\
\leftrightarrow\ &((3x-2y)(x+y))\left(\frac{1}{3x-2y} + \frac{x}{x+y}\right) = ((3x-2y)(x+y))\left(\frac{x+y+3x^2-2xy}{3x^2+xy-2y^2}\right) \\
\leftrightarrow\ &(x+y) + x(3x-2y) = x+y+3x^2-2xy \\
\leftrightarrow\ &(x+y) + (3x^2-2xy) = x+y+3x^2-2xy \\
\leftrightarrow\ &x+y+3x^2-2xy = x+y+3x^2-2xy
\end{aligned}
$$

∎

Problem 4

Prove the following relations.

(a) $\frac{x^3-y^3}{x-y} = x^2+xy+y^2$

(b) $\frac{x^4-y^4}{x-y} = x^3+x^2y+xy^2+y^3$

Let

$x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$

Show that $x^2+y^2=1$

Proof.

$$\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2$$
$$\leftrightarrow x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$
$$\leftrightarrow x^3 - y^3 = x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3$$
$$\leftrightarrow x^3 - y^3 = x^3 - y^3$$

∎

Proof.

$$\frac{x^4 - y^4}{x - y} = x^3 + x^2y + xy^2 + y^3$$
$$\leftrightarrow x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3)$$
$$\leftrightarrow x^4 - y^4 = x^4 + x^3y + x^2y^2 + xy^3 - x^3y - x^3y - x^2y^2 - xy^3 - y^4$$
$$\leftrightarrow x^4 - y^4 = x^4 - y^4$$

∎

Proof.

$$
\begin{aligned}
x^2 + y^2 &= 1 \\
\leftrightarrow \left(\frac{1 - t^2}{1 + t^2}\right)^2 + \left(\frac{2t}{1 + t^2}\right)^2 &= 1 \\
\leftrightarrow \frac{(1 - t^2)^2 + (2t)^2}{(1 + t^2)^2} &= 1 \\
\leftrightarrow \frac{t^4 - 2t^2 + 1 + 4t^2}{(1 + t^2)^2} &= 1 \\
\leftrightarrow \frac{t^4 + 2t^2 + 1}{(1 + t^2)^2} &= 1 \\
\leftrightarrow \frac{(1 + t^2)^2}{(1 + t^2)^2} &= 1 \\
\leftrightarrow 1 &= 1
\end{aligned}
$$

∎

Problem 5

Prove the following relations.

(a) $\frac{x^3+1}{x+1} = x^2 - x + 1$

(b) $\frac{x^5+1}{x+1} = x^4 - x^3 + x^2 - x + 1$

(c) If n is an odd integerl, prove that

$\frac{x^n+1}{x+1} = x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1$

Proof.

$$\begin{aligned} &\frac{x^3+1}{x+1} = x^2 - x + 1 \\ \leftrightarrow\ & x^3 + 1 = (x+1)(x^2 - x + 1) \\ \leftrightarrow\ & x^3 + 1 = x^3 - x^2 + x + x^2 - x + 1 \\ \leftrightarrow\ & x^3 + 1 = x^3 + 1 \end{aligned}$$

∎

Proof.

$$\begin{aligned} &\frac{x^5+1}{x+1} = x^4 - x^3 + x^2 - x + 1 \\ \leftrightarrow\ & x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1) \\ \leftrightarrow\ & x^5 + 1 = x^5 - x^4 + x^3 - x^2 + x + x^4 - x^3 + x^2 - x + 1 \\ \leftrightarrow\ & x^5 + 1 = x^5 + 1 \end{aligned}$$

∎

Proof.

$$\begin{aligned} &\frac{x^n+1}{x+1} = x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1 \\ \leftrightarrow\ & x^n + 1 = (x+1)(x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1) \\ \leftrightarrow\ & x^n + 1 = x^n - x^{(n-1)} + x^{(n-2)} - \cdots - x^2 + x + x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1 \\ \leftrightarrow\ & x^n + 1 = x^n + 1 \end{aligned}$$

∎

Problem 7

If a solid has a uniform density d, occpies a volume v, and has a mass m, then we have the formula $m = vd$
Find the density if:
(a) $m = \frac{3}{10}$lb and $v = \frac{2}{3}$in^3
(a) $m = 6$lb and $v = \frac{4}{3}$in^3
Find the volume if the mass is 15 lb and the density is $\frac{2}{3}$ lb/in^3.

Solution (a)

$$\begin{aligned} \frac{3}{10} &= \frac{2}{3}d \\ d &= \frac{9}{20} \end{aligned}$$

Solution (b)

$$\begin{aligned} 6 &= \frac{4}{3}d \\ d &= \frac{18}{4} \end{aligned}$$

Solution (c)

$$\begin{aligned} 15 &= v\frac{2}{3} \\ v &= \frac{45}{2} \end{aligned}$$

Problem 13

Tickets for a performance sell $5.00 and $2.00. The total amount collected was $4,100, and there are 1,300 tickets in all. How many tickets of each price were sold.

Solution Let x = be the number of tickets sold at $2.00.

$$\begin{aligned} 2x + 5(1300 - x) &= 4100 \\ 2x + 6500 - 5x &= 4100 \\ -3x &= -2400 \\ x &= 800 \end{aligned}$$

800 tickets sold at $2.00 and 500 sold at $5.00.

Problem 16

A boat travels a distance of 500mi, along two rivers, for 50hr. The current goes in the same direction as the boat along one river, and then the boat averages 20mph. The current goes in the opposite dirction along the other river, and then the boat averages 8mph. During how many hours was the boat on the first river.

Solution Let x be the time spent on the first river.

$$\begin{aligned} 20x + 8(50 - x) &= 500 \\ 20x + 400 - 8x &= 500 \\ 12x &= 100 \\ x &= \frac{100}{12} \end{aligned}$$

$x = \frac{100}{12}$ hours on first river.

Problem 18

The radiator of a car can contain 10kg of liquid. If it is half full with a mixture having 60% antifreeze and 40% water, how much more water must be added so that the resulting mixture has only.

Solution (a) 40% antifreeze
40% antifreeze means 60% water.

$$\begin{aligned} \frac{4+x}{10+x} &= 0.6 \\ 4 + x &= 0.6(10 + x) \\ 4 + x &= 6 + 0.6x \\ x &= 5 \end{aligned}$$

15 kg will fit in the radiator.
Solution (b) 10% antifreeze

10% antifreeze means 90% water.

$$\begin{aligned}
\frac{4+x}{10+x} &= 0.9 \\
4+x &= 0.9(10+x) \\
4+x &= 9+0.9x \\
x &= 50
\end{aligned}$$

55 kg will not fit in the radiator.

2 Linear Equations

2.1 Equations in Two Unknowns

Problem 7

Solve the following systems of equations for x and y.

$$\begin{aligned}
7x - y &= 2 \\
2x + 2y &= 4
\end{aligned}$$

Solution 7

$$\begin{aligned}
7x - y = 2 \leftrightarrow 14x - 2y &= 4 \\
(14x - 2y) + (2x + 2y) &= 4 + 4 \\
16x &= 8 \\
x &= \frac{8}{16} \\
x &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
7x - y &= 2 \\
7(\frac{1}{2}) - y &= 2 \\
7(\frac{1}{2}) - 2 &= y \\
(\frac{7}{2}) - \frac{4}{2} &= y \\
\frac{3}{2} &= y
\end{aligned}$$

Problem 8

Solve the following systems of equations for x and y.

$$\begin{aligned}
-4x - 7y &= 5 \\
2x + y &= 6
\end{aligned}$$

Solution 8

$$\begin{aligned}
2(2x+y=6) \leftrightarrow 4x+2y &= 12 \\
(4x+2y)+(-4x-7y) &= 12+5 \\
-5y &= 17 \\
y &= \frac{-17}{5}
\end{aligned}$$

$$\begin{aligned}
2x+y &= 6 \\
2x+\frac{-17}{5} &= 6 \\
2x &= 6+\frac{17}{5} \\
(\frac{10}{5})x &= \frac{30}{5}+\frac{17}{5} \\
(\frac{10}{5})x &= \frac{47}{5} \\
x &= \frac{\frac{47}{5}}{(\frac{10}{5})} \\
x &= \frac{235}{50} \\
x &= \frac{47}{10}
\end{aligned}$$

Problem 9

Let a, b, c, d be numbers such that $ad - bc \neq 0$. Solve the following systems of equations for x and y in terms of a, b, c, d.

(a)

$$\begin{aligned} ax + by &= 1 \\ cx + dy &= 2 \end{aligned}$$

(b)

$$\begin{aligned} ax + by &= 3 \\ cx + dy &= -4 \end{aligned}$$

(c)

$$\begin{aligned} ax + by &= -2 \\ cx + dy &= 3 \end{aligned}$$

(d)

$$\begin{aligned} ax + by &= 5 \\ cx + dy &= 7 \end{aligned}$$

Solution 9 (a)
First multiply by d, $ax + by = 1 \leftrightarrow adx + bdy = d$.
Then multiply by b, $cx + dy = 2 \leftrightarrow bcx + bdy = 2b$.

Also multiply by c, $ax + by = 1 \leftrightarrow acx + bcy = c$.
And mutiply by a, $cx + dy = 2 \leftrightarrow acx + ady = 2a$.

$$\begin{aligned}(adx + bdy) - (bcx + bdy) &= d - 2b \\ adx - bcx &= d - 2b \\ x(ad - bc) &= d - 2b \\ x &= \frac{d - 2b}{ad - bc}\end{aligned}$$

$$\begin{aligned}(acx + ady) - (acx + bcy) &= 2a - c \\ ady - bcy &= 2a - c \\ y(ad - bc) &= 2a - c \\ y(ad - bc) &= \frac{2a - c}{ad - bc}\end{aligned}$$

Solution 9 (b)
First multiply by d, $ax + by = 3 \leftrightarrow adx + bdy = 3d$.
Then multiply by b, $cx + dy = -4 \leftrightarrow bcx + bdy = -4b$.
Also multiply by c, $ax + by = 3 \leftrightarrow acx + bcy = 3c$.
And mutiply by a, $cx + dy = -4 \leftrightarrow acx + ady = -4a$.

$$\begin{aligned}(adx + bdy) - (bcx + bdy) &= 3d + 4b \\ adx - bcx &= 3d + 4b \\ x(ad - bc) &= 3d + 4b \\ x &= \frac{3d + 4b}{ad - bc}\end{aligned}$$

$$\begin{aligned}(acx + ady) - (acx + bcy) &= -4a - 3c \\ ady - bcy &= -4a - 3c \\ y(ad - bc) &= -4a - 3c \\ y(ad - bc) &= \frac{-4a - 3c}{ad - bc}\end{aligned}$$

Solution 9 (c)
First multiply by d, $ax + by = -2 \leftrightarrow adx + bdy = -2d$.
Then multiply by b, $cx + dy = 3 \leftrightarrow bcx + bdy = 3b$.
Also multiply by c, $ax + by = -2 \leftrightarrow acx + bcy = -2c$.
And mutiply by a, $cx + dy = 3 \leftrightarrow acx + ady = 3a$.

$$\begin{aligned}(adx + bdy) - (bcx + bdy) &= -2d + 3b \\ adx - bcx &= -2d + 3b \\ x(ad - bc) &= -2d + 3b \\ x &= \frac{-2d + 3b}{ad - bc}\end{aligned}$$

$$\begin{aligned}(acx + ady) - (acx + bcy) &= 3a - c \\ ady - bcy &= 3a + 2c \\ y(ad - bc) &= 3a + 2c \\ y(ad - bc) &= \frac{3a + 2c}{ad - bc}\end{aligned}$$

Solution 9 (d)
First multiply by d, $ax + by = 5 \leftrightarrow adx + bdy = 5d$.
Then multiply by b, $cx + dy = 7 \leftrightarrow bcx + bdy = 7b$.
Also multiply by c, $ax + by = 5 \leftrightarrow acx + bcy = 5c$.
And mutiply by a, $cx + dy = 7 \leftrightarrow acx + ady = 7a$.

$$\begin{aligned}(adx + bdy) - (bcx + bdy) &= 5d - 7b \\ adx - bcx &= 5d - 7b \\ x(ad - bc) &= 5d - 7b \\ x &= \frac{5d - 7b}{ad - bc}\end{aligned}$$

$$\begin{aligned}(acx + ady) - (acx + bcy) &= 5a - 7c \\ ady - bcy &= 5a - 7c \\ y(ad - bc) &= 5a - 7c \\ y(ad - bc) &= \frac{5a - 7c}{ad - bc}\end{aligned}$$

Problem 10

Making the same assumptions as in Excersize 9, show that the solution of the system

$$\begin{aligned}ax + by &= 0 \\ cx + dy &= 0\end{aligned}$$

must be $x = 0$ and $y = 0$.

Solution 10
First $ax + by = 0 \leftrightarrow adx + bdy = 0$.
Then $cx + dy = 0 \leftrightarrow bcx + bdy = 0$.
Also $ax + by = 0 \leftrightarrow acx + bcy = 0$.
And $cx + dy = 0 \leftrightarrow acx + ady = 0$.

$$\begin{aligned}(adx + bdy) - (bcx + bdy) &= 0 \\ adx - bcx &= 0 \\ x(ad - bc) &= 0 \\ x &= \frac{0}{ad - bc} \\ x &= 0\end{aligned}$$

$$\begin{aligned}(acx + ady) - (acx + bcy) &= 0 \\ ady - bcy &= 0 \\ y(ad - bc) &= 0 \\ y &= \frac{0}{ad - bc} \\ y &= 0\end{aligned}$$

Problem 11

Let a, b, c, d, u, v be numbers and assume that $ad - bc \neq 0$. Solve the following system of equations for x and y in terms of a, b, c, d, u, v

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned}$$

Verify that the answer you get is actually a solution.

Solution 9 (d)
First multiply first equation by d, $ax + by = u \leftrightarrow adx + bdy = ud$.
Then multiply second equation by b, $cx + dy = v \leftrightarrow bcx + bdy = vb$.
Also multiply first equation by c, $ax + by = u \leftrightarrow acx + bcy = uc$.
And multiply second equation by a, $cx + dy = v \leftrightarrow acx + ady = va$.

$$\begin{aligned} (adx + bdy) - (bcx + bdy) &= ud - vb \\ adx - bcx &= ud - vb \\ x(ad - bc) &= ud - vb \\ x &= \frac{ud - vb}{ad - bc} \end{aligned}$$

$$\begin{aligned} (acx + ady) - (acx + bcy) &= va - uc \\ ady - bcy &= va - uc \\ y(ad - bc) &= va - uc \\ y &= \frac{va - uc}{ad - bc} \end{aligned}$$

Veryfying the first equation.

$$\begin{aligned} a\left(\frac{ud - vb}{ad - bc}\right) + b\left(\frac{va - uc}{ad - bc}\right) &= u \\ \frac{aud - avb + bva - buc}{ad - bc} &= u \\ \frac{v(-ab + ba) + u(ad - bc)}{ad - bc} &= u \\ \frac{u(ad - bc)}{ad - bc} &= u \\ u &= u \end{aligned}$$

Verifying the second equation.

$$\begin{aligned} c\left(\frac{ud - vb}{ad - bc}\right) + d\left(\frac{va - uc}{ad - bc}\right) &= v \\ \frac{cud - cvb + dva - duc}{ad - bc} &= v \\ \frac{u(cd - cd) + v(ad - bc)}{ad - bc} &= v \\ \frac{v(ad - bc)}{ad - bc} &= v \\ v &= v \end{aligned}$$

2.2 Equations In Three Unknowns

Problem 7

Solve the following equations for x, y, z.
(1) $4x - 2y + 5z = 1$
(2) $x + y + z = 0$
(3) $-x + y - 2z = 2$

Solution 7

Summing (2) and (3).

$$\begin{aligned}(x + y + z) + (-x + y - 2z) &= 0 + 2\\ (4) \quad 2y - z &= 2\end{aligned}$$

Summing (1) and (2) multiplied by -4.

$$\begin{aligned}(4x - 2y + 5z) + (-4x - 4y - 4z) &= 1 + 0\\ (5) \quad -6y + z &= 1\end{aligned}$$

Summing (4) and (5).

$$\begin{aligned}(2y - z) + (-6y + z) &= 2 + 1\\ -4y &= 3\\ y &= \frac{-3}{4}\end{aligned}$$

Summing (2) and (3). Setting $y = \frac{-3}{4}$.

$$\begin{aligned}(x + \frac{-3}{4} + z) + (-x + \frac{-3}{4} - 2z) &= 0 + 2\\ 2 \cdot \frac{-3}{4} - z &= 2\\ \frac{-3}{2} - 2 &= z\\ \frac{-3}{2} - \frac{4}{2} &= z\\ \frac{-7}{2} &= z\end{aligned}$$

Using (2). Setting $y = \frac{-3}{4}$ and $z = \frac{-7}{2}$.

$$\begin{aligned}x + \frac{-3}{4} + \frac{-7}{2} &= 0\\ x + \frac{-3}{4} + \frac{-14}{4} &= 0\\ x + \frac{-17}{4} &= 0\\ x &= \frac{17}{4}\end{aligned}$$

$$\therefore x = \frac{17}{4}, y = \frac{-3}{4}, z = \frac{-7}{2}$$

Problem 8

Solve the following equations for x, y, z.
(1) $x + y + z = 0$
(2) $x - y - z = 1$
(3) $x + y - z = 1$

Solution 8

Summing (1) and (2).

$$\begin{aligned} (x+y+z) + (x-y-z) &= 0 + 1 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

Summing (2) and (3). Setting $x = \frac{1}{2}$.

$$\begin{aligned} (\frac{1}{2} - y - z) + (\frac{1}{2} + y - z) &= 1 + 1 \\ 1 - 2z &= 2 \\ -2z &= 1 \\ z &= \frac{-1}{2} \end{aligned}$$

Using (3). Setting $x = \frac{1}{2}$ and $z = \frac{-1}{2}$.

$$\begin{aligned} \frac{1}{2} + y - (\frac{-1}{2}) &= 1 \\ y + 1 &= 1 \\ y &= 0 \end{aligned}$$

$$\therefore x = \frac{1}{2}, y = 0, z = \frac{-1}{2}$$

Problem 11

Solve the following equations for x, y, z.
(1) $\frac{1}{2}x + y - \frac{3}{4}z = 1$
(2) $x - \frac{1}{2}y + z = 0$
(3) $x + y - \frac{1}{3}z = 0$

Solution 11

Multiply (1) by -4.

$$\begin{aligned} \frac{1}{2}x + y - \frac{3}{4}z &= 1 \\ (4) \quad -2x - 4y + 3z &= -4 \end{aligned}$$

Multiply (2) by 2.

$$\begin{aligned} x - \frac{1}{2}y + z &= 0 \\ (5) \quad 2x - y + 2z &= 0 \end{aligned}$$

Multiply (3) by 3.

$$x + y - \frac{1}{3}z = 0$$
$$(6) \quad 3x + 3y - z = 0$$

Summing (4) and (5).

$$(-2x - 4y + 3z) + (2x - y + 2z) = -4 + 0$$
$$(7) \quad -5y + 5z = -4$$

Sum (5) times 3 and (6) times -2.

$$(6x - 3y + 6z) + (-6x - 6y + 2z) = 0$$
$$(8) \quad -9y + 8z = 0$$

Sum (7) times -9 and (8) times 5.

$$\begin{aligned}(45y - 45z) + (-45y + 40z) &= 36\\ -5z &= 36\\ z &= \frac{-36}{5}\end{aligned}$$

Using (7) and setting $z = \frac{-36}{5}$.

$$\begin{aligned}-9y + 8(\frac{-36}{5}) &= 0\\ -9y - \frac{288}{5} &= 0\\ -9y &= \frac{288}{5}\\ y &= \frac{-32}{5}\end{aligned}$$

Using (5) and setting $y = \frac{-32}{5}$, $z = \frac{-36}{5}$.

$$\begin{aligned}2x - (\frac{-32}{5}) + 2(\frac{-36}{5}) &= 0\\ 2x + \frac{32}{5} - \frac{72}{5} &= 0\\ 2x - \frac{40}{5} &= 0\\ x &= \frac{40}{10}\\ x &= 4\end{aligned}$$

Problem 12

Solve the following equations for x, y, z.
(1) $\frac{1}{2}x - \frac{2}{3}y + z = 1$
(2) $x - \frac{1}{5}y + z = 0$
(3) $2x - \frac{1}{3}y + \frac{2}{5}z = 1$

Solution 12

Multiply (1) by 6.

$$\frac{1}{2}x - \frac{2}{3}y + z = 1$$
$$(4) \quad 3x - 4y + 6z = 6$$

Multiply (2) by -30.

$$x - \frac{1}{5}y + z = 0$$
$$(5) \quad -30x + 6y - 30z = 0$$

Multiply (3) by 15.

$$2x - \frac{1}{3}y + \frac{2}{5}z = 1$$
$$(6) \quad 30x - 5y + 6z = 15$$

Summing (5) and (6).

$$(-30x + 6y - 30z) + (30x - 5y + 6z) = 15$$
$$(7) \quad y - 24z = 15$$

Sum (4) times 10 and (5).

$$(30x - 40y + 60z) + (-30x + 6y - 30z) = 60$$
$$(8) \quad -34y + 30z = 60$$

Sum (7) times 34 and (8).

$$(34y - 816z) + (-34y + 30z) = (15 * 34) + 60$$
$$(34y - 816z) + (-34y + 30z) = 510 + 60$$
$$-786z = 570$$
$$z = \frac{-95}{131}$$

Using (7). Set $z = \frac{-95}{131}$.

$$y - 24(\frac{-95}{131}) = 15$$
$$y + \frac{2280}{131} = 15$$
$$y = \frac{1965}{131} - \frac{2280}{131}$$
$$y = \frac{-315}{131}$$

Using (7). Set $z = \frac{-95}{131}$ and $y = \frac{-315}{131}$.

$$x - \frac{1}{5} \cdot \frac{-315}{131} + \frac{-95}{131} = 0$$
$$x = \frac{1}{5} \cdot \frac{-315}{131} - \frac{-95}{131}$$
$$x = \frac{-315}{655} - \frac{-475}{655}$$
$$x = \frac{160}{655}$$
$$x = \frac{32}{131}$$

$$\therefore x = \frac{32}{131}, y = \frac{-315}{131}, z = \frac{-95}{131}$$

3 Real Numbers

3.1 Addition and Multiplication

Problem 1

Let E be an abbreviation for even, and let I be an abbreviation for odd. We know that:
$E + E = E$,
$E + I = I + E = I$,
$I + I = E$,
$EE = E$,
$II = I$
$IE = EI = E$.
(a) Show that addition for E and I is associative and commutative. Show that E plays the role of a zero element for addition. What is the additive inverse of E? What is the additive inverse of I?
(b) Show that multiplication for E and I is commutative and associative. Which of E or I behaves like 1? Which behaves like 0 for multiplication? Show that multiplication is distributive with respect to addition.

Solution 1 (a)

Associative over Addition: We check that $(A + B) + C = A + (B + C)$ for all $A, B, C \in \{E, I\}$ by verifying all 8 cases:

- $(E + E) + E = E + E = E$, and $E + (E + E) = E + E = E$

- $(E + E) + I = E + I = I$, and $E + (E + I) = E + I = I$

- $(E + I) + E = I + E = I$, and $E + (I + E) = E + I = I$

- $(E + I) + I = I + I = E$, and $E + (I + I) = E + E = E$

- $(I + E) + E = I + E = I$, and $I + (E + E) = I + E = I$

- $(I + E) + I = I + I = E$, and $I + (E + I) = I + I = E$

- $(I + I) + E = E + E = E$, and $I + (I + E) = I + I = E$

- $(I + I) + I = E + I = I$, and $I + (I + I) = I + E = I$

Commutative over Addition: We check that $A + B = B + A$ for all $A, B \in \{E, I\}$.

- $E + E = E = E + E$

- $E + I = I = I + E$

- $I + I = E = I + I$

Zero Element: E plays the role of additive identity (zero element), since:

- $E + E = E$

- $I + E = I$

- $E + I = I$

Additive Inverse of E: E, since $E + E = E$.

Additive Inverse of I: I, since $I + I = E$.

Solution 1 (b)

Associative over Multiplication: We check that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ for all $A, B, C \in \{E, I\}$:

$$
\begin{aligned}
(E \cdot E) \cdot E &= E \cdot E = E, & E \cdot (E \cdot E) &= E \cdot E = E \\
(E \cdot E) \cdot I &= E \cdot I = E, & E \cdot (E \cdot I) &= E \cdot E = E \\
(E \cdot I) \cdot E &= E \cdot E = E, & E \cdot (I \cdot E) &= E \cdot E = E \\
(E \cdot I) \cdot I &= E \cdot I = E, & E \cdot (I \cdot I) &= E \cdot I = E \\
(I \cdot E) \cdot E &= E \cdot E = E, & I \cdot (E \cdot E) &= I \cdot E = E \\
(I \cdot E) \cdot I &= E \cdot I = E, & I \cdot (E \cdot I) &= I \cdot E = E \\
(I \cdot I) \cdot E &= I \cdot E = E, & I \cdot (I \cdot E) &= I \cdot E = E \\
(I \cdot I) \cdot I &= I \cdot I = I, & I \cdot (I \cdot I) &= I \cdot I = I
\end{aligned}
$$

Commutative over Multiplication: We check that $AB = BA$ for all $A, B \in \{E, I\}$.

$$
\begin{aligned}
E \cdot I &= I \cdot E = E \\
I \cdot I &= I \cdot I = I \\
E \cdot E &= E \cdot E = E
\end{aligned}
$$

Multiplicative Identity: I behaves like 1 over multiplication.

- $II = I$

- $EI = E$

Multiplicative Zero: E behaves like 0 over multiplication.

- $IE = E$

- $EE = E$

Distributive Over Addition: We check that $A \cdot (B + C) = A \cdot B + A \cdot C$ for all $A, B, C \in \{E, I\}$. For example:

- $E(I + E) = E(I) = E = EI + EE = E + E = E$

- $I(I + E) = I(E) = E = II + IE = E + E = E$

- $E(E + I) = E(I) = E = EE + EI = E + E = E$

- $I(E + I) = I(I) = I = IE + II = E + I = I$

- $E(E + E) = E(E) = E = EE + EE = E + E = E$

- $I(E + E) = I(E) = E = IE + IE = E + E = E$

- $E(I + I) = E(E) = E = EI + EI = E + E = E$

- $I(I + I) = I(E) = E = II + II = I + I = E$

3.2 Real Numbers: Positivity

Problem 1

Prove:
(a) If a is a real number, then a^2 is positive.
(b) If a is positive and b is negative, then ab is negative.
(c) If a is negative and b is negative, then ab is positive.

Proof. By POS 2 either $a = 0$, $a > 0$, or $a < 0$.

Case 1 ($a = 0$)

If $a = 0$ then $a^2 = a \cdot a = 0 \cdot 0 = 0 \geq 0$.

Case 2 ($a > 0$)

If $a > 0$, then by POS 1, $a \cdot a = a^2 \geq 0$.

Case 3 ($a < 0$)

Since $a < 0$, by POS 2, $-a > 0$. Then by POS 1, $(-a) \cdot (-a) = a^2 > 0$.

Therefore, $a^2 \geq 0$. ∎

Proof. Assume for contradiction, $ab > 0$. By POS 2, $-ab < 0$. Since $b < 0$ then, by POS 2, $-b > 0$. Then by POS 1, $a \cdot -b > 0$ so $-ab > 0$ which is a contradiction. Therefore, if a is positive and b is negative, then ab is negative. ∎

Proof. Assume for contradiction, $ab < 0$. By POS 2, $-ab > 0$. Since $b < 0$, $a < 0$ then, by POS 2, $-b > 0$, $-a > 0$. Then by POS 1, $-a \cdot -b > 0$ so $ab > 0$ which is a contradiction. Therefore, if a is negative and b is negative, then ab is positive. ∎

Problem 2

Prove: If a is positive, then a^{-1} is positive.

Proof. Suppose $a > 0$ and assume for contradiction $a^{-1} = \frac{1}{a} < 0$. By Excersize 1 part c, $a \cdot \frac{1}{a} < 0$. But $a \cdot \frac{1}{a} = \frac{a}{a} = 1 > 0$. Therefore, if a is positive, then a^{-1} is positive. ∎

Problem 3

Prove: If a is negative, then a^{-1} is negative.

Proof. Suppose $a < 0$ and assume for contradiction $\frac{1}{a} > 0$. Since $a < 0$, by POS 2, $0 < -a$. Then by POS 1, $-a \cdot \frac{1}{a} > 0$. But $-a \cdot \frac{1}{a} = \frac{-a}{a} = -1 < 0$ which is a contradiction. Therefore, if a is negative, then a^{-1} is negative. ∎

Problem 4

Prove: If a, b are positive numbers, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Proof.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \iff \sqrt{\frac{a}{b}}^2 = \left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 \iff \sqrt{\frac{a}{b}}^2 = \frac{\sqrt{a}^2}{\sqrt{b}^2} \iff \frac{a}{b} = \frac{a}{b}$$

∎

Problem 5

Prove that

$$\frac{1}{1-\sqrt{2}} = -(1+\sqrt{2})$$

Proof.

$$\begin{aligned}\frac{1}{1-\sqrt{2}} &= \frac{1+\sqrt{2}}{1+\sqrt{2}} \cdot \frac{1}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{1-2} = \frac{1+\sqrt{2}}{-1} \\ &= \frac{-1}{-1} \cdot \frac{1+\sqrt{2}}{-1} = \frac{-(1+\sqrt{2})}{1} = -(1+\sqrt{2})\end{aligned}$$

∎

Problem 8

Let a, b be rational numbers. Prove that the multiplicative inverse of $a + b\sqrt{2}$ can be expressed in the form $c + d\sqrt{2}$, where c, d are rational numbers.

Proof. First note since $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ therefore $a^2 - 2b^2 \in \mathbb{Q}$. In addition $a + b\sqrt{2} \neq 0$ (otherwise the inverse operation is undefined). If $b = 0$ then $a^2 \neq 0$ so $a^2 - 2b^2 \in \mathbb{Q}$ is defined. Now suppose $b \neq 0$.

$$a^2 = 2b^2 \iff \frac{a^2}{b^2} = 2 \iff \frac{a}{b} = \pm\sqrt{2}$$

But $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ so their quotient is rational. This is impossible since $\sqrt{2}$ is irrational, so $a^2 - 2b^2 \neq 0$. Futhermore since $a^2 - 2b^2 \in \mathbb{Q}$ and $a^2 - 2b^2 \neq 0$, $\frac{a}{a^2-2b^2} \in \mathbb{Q}$ and $\frac{-b}{a^2-2b^2} \in \mathbb{Q}$. Now, let $c = \frac{a}{a^2-2b^2}$ and $d = \frac{-b}{a^2-2b^2}$. Then

$$\begin{aligned}&(a+b\sqrt{2}) \cdot (c + d\sqrt{2}) \\ =&(a+b\sqrt{2}) \cdot \left(\frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2} \cdot \sqrt{2}\right) \\ =&(a+b\sqrt{2}) \cdot \left(\frac{a}{a^2-2b^2} + \frac{-b\sqrt{2}}{a^2-2b^2}\right) \\ =&(a+b\sqrt{2}) \cdot \left(\frac{a}{a^2-2b^2} - \frac{b\sqrt{2}}{a^2-2b^2}\right) \\ =&\left(\frac{a(a+b\sqrt{2})}{a^2-2b^2} - \frac{b\sqrt{2}(a+b\sqrt{2})}{a^2-2b^2}\right) \\ =&\frac{(a^2+ab\sqrt{2}) - (ab\sqrt{2}+2b^2)}{a^2-2b^2} \\ =&\frac{a^2+ab\sqrt{2}-ab\sqrt{2}-2b^2}{a^2-2b^2} \\ =&\frac{a^2-2b^2}{a^2-2b^2} \\ =&1\end{aligned}$$

∎

Problem 11

Generalize Excersize 10, replacing $\sqrt{5}$ by $\sqrt{a}$ for any positive integer a.

Proof. First note since $d \in \mathbb{Q}$ and $b \in \mathbb{Q}$ therefore $d^2 - ab^2 \in \mathbb{Q}$. In addition $d + b\sqrt{a} \neq 0$ (otherwise the inverse operation is undefined).

If $b = 0$ then $d^2 \neq 0$ so $d^2 - ab^2 \in \mathbb{Q}$ is defined.

Now suppose $b \neq 0$ and $\sqrt{a} \notin \mathbb{Q}$.

$$d^2 = ab^2 \iff \frac{d^2}{b^2} = a \iff \frac{d}{b} = \pm\sqrt{a}$$

But $d \in \mathbb{Q}$ and $b \in \mathbb{Q}$ so their quotient is rational. This is impossible if $\sqrt{a} \notin \mathbb{Q}$, so $d^2 - ab^2 \neq 0$.

Now suppose $b \neq 0$ and $\sqrt{a} \in \mathbb{Q}$.

$$d = b\sqrt{a} \iff d^2 = b^2 a \iff d^2 - ab^2 = 0$$

Since, $d \neq b\sqrt{a}$, $d^2 - ab^2 \neq 0$.

Furthermore since $d^2 - ab^2 \in \mathbb{Q}$ and $d^2 - ab^2 \neq 0$, $\frac{d}{d^2-ab^2} \in \mathbb{Q}$ and $\frac{-b}{d^2-ab^2} \in \mathbb{Q}$. Now let $c = \frac{d}{d^2-ab^2}$ and $e = \frac{-b}{d^2-ab^2}$. Then

$$\begin{aligned}
&(d + b\sqrt{a}) \cdot (c + e\sqrt{a}) \\
=&(d + b\sqrt{a}) \cdot \left(\frac{d}{d^2 - ab^2} + \frac{-b}{d^2 - ab^2} \cdot \sqrt{a}\right) \\
=&(d + b\sqrt{a}) \cdot \left(\frac{d}{d^2 - ab^2} + \frac{-b\sqrt{a}}{d^2 - ab^2}\right) \\
=&(d + b\sqrt{a}) \cdot \left(\frac{d}{d^2 - ab^2} - \frac{b\sqrt{a}}{d^2 - ab^2}\right) \\
=&\left(\frac{d(d + b\sqrt{a})}{d^2 - ab^2} - \frac{b\sqrt{a}(d + b\sqrt{a})}{d^2 - ab^2}\right) \\
=&\frac{(d^2 + db\sqrt{a}) - (db\sqrt{a} + ab^2)}{d^2 - ab^2} \\
=&\frac{d^2 + db\sqrt{a} - db\sqrt{a} - ab^2}{d^2 - ab^2} \\
=&\frac{d^2 - ab^2}{d^2 - ab^2} \\
=&1
\end{aligned}$$

∎

Problem 14

Find all possible numbers x such that
(a) $|2x - 1| = 3$
(b) $|3x + 1| = 2$
(c) $|2x + 1| = 4$
(d) $|3x - 1| = 1$
(e) $|4x - 5| = 6$

Solution 14 (a)

$x = 2$ or $x = -1$

Solution 14 (b)

$x = \frac{1}{3}$ or $x = -1$

Solution 14 (c)

$x = \frac{3}{2}$ or $x = \frac{-5}{2}$

Solution 14 (d)

$x = \frac{2}{3}$ or $x = 0$

Solution 14 (e)

$x = \frac{11}{4}$ or $x = \frac{-1}{4}$

Problem 15

Rationalize the numerator in the following expressions.

(a) $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

(b) $\frac{\sqrt{x+y}-\sqrt{y}}{\sqrt{x+y}+\sqrt{y}}$

(c) $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$

(d) $\frac{\sqrt{x-3}+\sqrt{x}}{\sqrt{x-3}-\sqrt{x}}$

(e) $\frac{\sqrt{x+y}-1}{3+\sqrt{x+y}}$

(f) $\frac{\sqrt{x+y}+x}{\sqrt{x+y}}$

Solution 15 (a)

$$\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \frac{x-y}{x-2\sqrt{xy}+y}$$

Solution 15 (b)

$$\frac{\sqrt{x+y}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} \cdot \frac{\sqrt{x+y}+\sqrt{y}}{\sqrt{x+y}+\sqrt{y}} = \frac{x}{\sqrt{x(x+y)}+\sqrt{xy}+\sqrt{y(x+y)}+y}$$

Solution 15 (c)

$$\begin{aligned} \frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} \cdot \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} &= \frac{2}{(\sqrt{x+1}-\sqrt{x-1})(\sqrt{x+1}-\sqrt{x-1})} \\ &= \frac{2}{(\sqrt{x+1}-\sqrt{x-1})^2} \end{aligned}$$

Solution 15 (d)

$$\frac{\sqrt{x-3}+\sqrt{x}}{\sqrt{x-3}-\sqrt{x}} \cdot \frac{\sqrt{x-3}-\sqrt{x}}{\sqrt{x-3}-\sqrt{x}} = \frac{(x-3)+x}{(\sqrt{x-3}-\sqrt{x})^2}$$
$$= \frac{-3}{(\sqrt{x-3}-\sqrt{x})^2}$$

Solution 15 (e)

$$\frac{\sqrt{x+y}-1}{3+\sqrt{x+y}} \cdot \frac{\sqrt{x+y}+1}{\sqrt{x+y}+1} = \frac{x+y-1}{(3+\sqrt{x+y})(\sqrt{x+y}+1)}$$

Solution 15 (f)

$$\frac{\sqrt{x+y}+x}{\sqrt{x+y}} \cdot \frac{\sqrt{x+y}-x}{\sqrt{x+y}-x} = \frac{x+y-x^2}{\sqrt{x+y}(\sqrt{x+y}-x)}$$

Problem 17

Prove that there is no real number x such that

$$\sqrt{x-1} = 3 + \sqrt{x}$$

[*Hint*: Start by squaring both sides.]

Proof. Assume for contradiction there does exist a real number x such that $\sqrt{x-1} = 3+\sqrt{x}$. Then

$$\begin{aligned}
&\sqrt{x-1} = 3+\sqrt{x} \\
&\leftrightarrow x-1 = 9+6\sqrt{x}+x \\
&\leftrightarrow -1 = 9+6\sqrt{x} \\
&\leftrightarrow -10 = 6\sqrt{x} \\
&\leftrightarrow \frac{-10}{6} = \sqrt{x}
\end{aligned}$$

Which is a contradiction. Therefore, there is no real number x such that $\sqrt{x-1} = 3+\sqrt{x}$. ∎

Problem 20

If a, b are two numbers, prove that $|a-b| = |b-a|$.

Proof. Let $c = b - a$. By POS 2 there are three cases.

Case 1 ($c = 0$) If $b - a = 0$ then $b = a$ therefore $a - b = 0$.

$$\begin{aligned}
&|b-a| = |a-b| \\
&\leftrightarrow |0| = |0| \\
&\leftrightarrow 0 = 0
\end{aligned}$$

Case 2 ($c > 0$) If $c > 0$ then $|c| = c$. Also $-c < 0$ so $|-c| = -(-c) = c$. Then

$$\begin{aligned} |b-a| &= |a-b| \\ \leftrightarrow |c| &= |-c| \\ \leftrightarrow c &= c \end{aligned}$$

Case 3 ($c < 0$) If $c < 0$ then $|c| = -c$. Also $-c > 0$ so $|-c| = -c$. Then

$$\begin{aligned} |b-a| &= |a-b| \\ \leftrightarrow |c| &= |-c| \\ \leftrightarrow -c &= -c \end{aligned}$$

Therefore $|a-b| = |b-a|$. ∎

3.3 Powers and Roots

Extra Problem

Suppose a is a nonzero rational number and b is an irrational real number. Show that ab is irrational.

Proof. A number is rational if it can be written as $\frac{x}{y}$ with $x, y \in \mathbb{Z}$ and $y \neq 0$. Assume for contradiction that $a \cdot b$ is rational, where $a \neq 0$ is rational and b is irrational. Since $a \neq 0$, we can divide both sides by a:

$$b = \frac{a \cdot b}{a}.$$

But the right-hand side is rational (a rational divided by a nonzero rational is rational), so b would be rational. This contradicts the assumption that b is irrational. Therefore, $a \cdot b$ must be irrational. ∎

Problem 1

Express each of the following in the form $2^k 2^m a^r b^s$ where k, m, r,s are integers.
(a) $\frac{1}{8}a^3b^{-4}2^5a^{-2}$
(b) $3^{-4}2^5a^3b^6 \cdot \frac{1}{2^3} \cdot \frac{1}{a^4} \cdot b^{-1} \cdot \frac{1}{9}$
(c) $\frac{3a^3b^4}{2a^5b^6}$
(d) $\frac{16a^{-3}b^{-5}}{9b^4a^7 2^{-3}}$

Solution (a):

$$\frac{1}{8}a^3b^{-4}2^5a^{-2} = \frac{2^5}{8}a^3a^{-2}b^{-4} = \frac{2^5}{8}a^1b^{-4} = \frac{2^5}{2^3}a^1b^{-4} = 2^2 3^0 a^1 b^{-4}$$

Solution (b):

$$3^{-4}2^5a^3b^6 \cdot \frac{1}{2^3} \cdot \frac{1}{a^4} \cdot b^{-1} \cdot \frac{1}{9} = \frac{2^5}{2^3}\frac{3^{-4}}{9}\frac{a^3}{a^4}\frac{b^6}{b} = 2^2\frac{3^{-4}}{3^2}\frac{a^3}{a^4}\frac{b^6}{b} = 2^2 3^{-6} a^{-1} b^5$$

Solution (c):

$$\frac{3a^3b^4}{2a^5b^6} = 2^{-1}3^1a^{-2}b^{-2}$$

Solution (d):

$$\frac{16a^{-3}b^{-5}}{9b^4a^7 2^{-3}} = \frac{2^4a^{-10}b^{-5}}{3^2 2^{-3}} = 2^7 3^{-2} a^{-10} b^{-9}$$

Problem 2

What integer is $81^{\frac{1}{4}}$ equal to?

Solution:

$$81^{\frac{1}{4}} = (81^{\frac{1}{2}})^{\frac{1}{2}} = 9^{\frac{1}{2}} = 3$$

Problem 3

What integer is $(\sqrt{2})^6$ equal to?

Solution:

$$(\sqrt{2})^6 = (\sqrt{2})^2(\sqrt{2})^2(\sqrt{2})^2 = 2 \cdot 2 \cdot 2 = 8$$

Problem 4

Is $(\sqrt{2})^5$ an integer?

Solution:

$$(\sqrt{2})^5 = (\sqrt{2})^2(\sqrt{2})^2(\sqrt{2}) = 2 \cdot 2 \cdot \sqrt{2} = 4\sqrt{2}$$

It is not an integer see extra problem proof.

Problem 5

Is $(\sqrt{2})^{-5}$ a rational number? Is $(\sqrt{2})^5$ a rational number?

Solution part 1:

$$(\sqrt{2})^{-5} = \frac{1}{(\sqrt{2})^5} = \frac{1}{4\sqrt{2}} = \frac{1}{4\sqrt{2}}\frac{4\sqrt{2}}{4\sqrt{2}} = \frac{4\sqrt{2}}{16 \cdot 2} = \frac{4\sqrt{2}}{32} = \frac{4}{32}\sqrt{2}$$

By the extra problem this is not a rational number.

Solution part 2: Same reason as problem 4.

Problem 6

In each case, the expression is equal to an integer. Which one?

(a) $16^{\frac{1}{4}}$
(b) $8^{\frac{1}{3}}$
(c) $9^{\frac{3}{2}}$
(d) $1^{\frac{5}{4}}$
(e) $8^{\frac{4}{3}}$
(f) $64^{\frac{2}{4}}$
(g) $25^{\frac{3}{2}}$

Solution:

$$
\begin{aligned}
&\text{(a)} \quad 16^{\frac{1}{4}} = (16^{\frac{1}{2}})^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2\\
&\text{(b)} \quad 8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2\\
&\text{(c)} \quad 9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27\\
&\text{(d)} \quad 1^{\frac{5}{4}} = 1\\
&\text{(e)} \quad 8^{\frac{4}{3}} = (8^{\frac{1}{3}})^4 = 2^4 = 16\\
&\text{(f)} \quad 64^{\frac{2}{4}} = 64^{\frac{1}{2}} = 8\\
&\text{(g)} \quad 25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3 = 5^3 = 125
\end{aligned}
$$

Problem 7

Express each of the following expressions as a simple decimal.
(a) $(0.09)^{\frac{1}{2}}$
(b) $(0.027)^{\frac{1}{3}}$
(c) $(0.125)^{\frac{2}{3}}$
(d) $(1.21)^{\frac{1}{2}}$

Solution:

$$
\begin{aligned}
&\text{(a)} \quad (0.9)^{\frac{1}{2}} \approx 0.3\\
&\text{(b)} \quad (0.027)^{\frac{1}{3}} = 0.3\\
&\text{(c)} \quad (0.125)^{\frac{2}{3}} = ((0.125)^{\frac{1}{3}})^2 = 0.5^2 = 0.25\\
&\text{(d)} \quad (1.21)^{\frac{1}{2}} = 1.1
\end{aligned}
$$

Problem 8

Express each of the following expressions as a quotient $\frac{m}{n}$, where m, n are integers > 0.
(a) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$
(b) $\left(\frac{4}{9}\right)^{\frac{1}{2}}$
(c) $\left(\frac{25}{16}\right)^{\frac{3}{2}}$
(d) $\left(\frac{49}{4}\right)^{\frac{3}{2}}$

Solution:

$$\text{(a)}\quad \left(\frac{8}{27}\right)^{\frac{2}{3}} = \frac{8^{2/3}}{27^{2/3}} = \frac{4}{9}$$
$$\text{(b)}\quad \left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{2}{3}$$
$$\text{(c)}\quad \left(\frac{25}{16}\right)^{\frac{3}{2}} = \frac{(25^{1/2})^3}{(16^{1/2})^3} = \frac{125}{64}$$
$$\text{(d)}\quad \left(\frac{49}{4}\right)^{\frac{3}{2}} = \frac{(49^{1/2})^3}{(4^{1/2})^3} = \frac{343}{8}$$

Problem 9

Solve each of the following equations for x.
(a) $(x-2)^3 = 5$
(b) $(x+3)^2 = 4$
(c) $(x-5)^{-2} = 9$
(d) $(x+3)^3 = 27$
(e) $(2x-1)^{-3} = 27$
(f) $(3x+5)^{-4} = 64$

Solution:

$$\text{(a)}\quad x-2 = \sqrt[3]{5} \iff x = 2+\sqrt[3]{5}$$
$$\text{(b)}\quad x+3 = \pm 2 \iff x = -1 \text{ or } x = -5$$
$$\text{(c)}\quad \frac{1}{(x-5)^2} = 9 \iff (x-5)^2 = \frac{1}{9} \iff x = 5 \pm \frac{1}{3}$$
$$\text{(d)}\quad x+3 = 3 \iff x = 0$$
$$\text{(e)}\quad \frac{1}{(2x-1)^3} = 27 \iff (2x-1)^3 = \frac{1}{27} \iff 2x-1 = \frac{1}{3} \iff x = \frac{2}{3}$$
$$\text{(f)}\quad \frac{1}{(3x+5)^4} = 64 \iff (3x+5)^4 = \frac{1}{64} \iff 3x+5 = \frac{1}{2} \iff x = -\frac{3}{2}$$

3.4 Inequalities

Problem 1

Prove **IN 3**.

IN 3 If $a > b$ and $b > c$ then $a > c$.

Proof. Suppose $a > b$ and $b > c$. Since $a > b$, $a - b > 0$. Also, since $b > c$, $b - c > 0$. So $(a-b)+(b-c) > 0 \iff a-c > 0$. Therefore $a > c$. ∎

Problem 2

Prove: If $0 < a < b$, if $c < d$, and $c > 0$ then
$$ac < bd$$

Proof. Suppose $0 < a < b$, $c < d$, and $c > 0$. Since $a < b$ and $c > 0$ it follows that $ac < bc$ (**IN 2**). Since $c < d$ and $b > 0$ it follows that $bc < bd$ (**IN 2**). Since $ac < bc < bd$ it follows that $ac < bd$ (Problem 1). ∎

Problem 3

Prove: If $a < b < 0$, if $c < d < 0$ then
$$ac > bd$$

Proof. Suppose $a < b < 0$ and $c < d < 0$. Since $a < b$ it follows that $b - a > 0$. Since $b - a > 0$ and $c < 0$ it follows that $bc - ac < 0$ so $bc < ac$ (**IN 3**). Since $c < d$ it follows that $d - c > 0$. Since $d - c > 0$ and $b < 0$ it follows that $bd - bc < 0$ so $bd < bc$ (**IN 3**). So $bd < bc < ac$ and therefore $bd < ac$ (Problem 1). ∎

Problem 4

(a) If $x < y$ and $x > 0$, prove that $\frac{1}{y} < \frac{1}{x}$.
(b) Prove a rule of cross-multiplication of inequalities: If a, b, c, d are numbers and $b > 0$, $d > 0$, and if
$$\frac{a}{b} < \frac{c}{d}$$
prove that
$$ad < bc$$
Also prove the converse, that if $ad < bc$, then $\frac{a}{b} < \frac{c}{d}$.

Proof. Suppose $x > 0$. For contradiction, suppose $\frac{1}{x} < 0$. But $x > 0$ and $\frac{1}{x} \cdot x = 1 > 0$ which contradicts the fact that the product of a positive and a negative number is negative. Now suppose $y > 0$. Then, since $\frac{1}{x} > 0$ and $\frac{1}{y} > 0$ it follows that $\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy} > 0$. Thus, if $x, y > 0$ then $\frac{1}{xy} > 0$. ∎

Proof. Suppose $x < y$ and $x > 0$. Since $x < y$ it follows that $y - x > 0$. Since $y > x > 0$ it follows that $\frac{1}{xy} > 0$. Then $\frac{1}{xy}(y - x) > 0 \iff \frac{1}{x} - \frac{1}{y} > 0$ therefore $\frac{1}{x} > \frac{1}{y}$. ∎

Proof. Suppose a, b, c, and d are numbers such that $b > 0$ and $d > 0$. Suppose $\frac{a}{b} < \frac{c}{d}$. It follows that $\frac{c}{d} - \frac{a}{b} > 0$. Since $b > 0$ and $d > 0$ it follows that $bd > 0$. Then $bd(\frac{c}{d} - \frac{a}{b}) > 0 \iff cb - ad > 0 \iff ad < bc$. ∎

Proof. Suppose a, b, c, and d are numbers such that $b > 0$ and $d > 0$. Suppose $\frac{a}{b} > \frac{c}{d}$. So $\frac{a}{b} > \frac{c}{d} \iff \frac{c}{d} < \frac{a}{b}$. Since $\frac{c}{d} < \frac{a}{b}$ then $bc < ad$ (Previous Proof). ∎

Problem 5

Prove: If $a < b$ and c is any real number, then
$$a + c < b + c$$
Also,
$$a - c < b - c$$
Thus a number may be subtracted from each side of an inequality without changing the validity of the inequality.

Proof. Suppose $a < b$ and c is a real number. Since $a < b$ it follows that $b - a > 0$. Then $b - a > 0 \iff b - a + c - c > 0 \iff b + c - a - c > 0 \iff b + c - (a + c) > 0 \iff b + c > a + c$. ∎

Proof. Suppose $a < b$ and t is a real number. Apply previous proof with $-t$ in place of c. Therefore $a + (-t) < b + (-t) \iff a - t < b - t$ ∎

Problem 6

Prove: If $a < b$ and $a > 0$ that

$$a^2 < b^2$$

More generally, prove successively that

$$a^3 < b^3$$

$$a^4 < b^4$$

$$a^5 < b^5$$

Proceeding stepwise, we conclude that

$$a^n < b^n$$

for every positive integer n. To make this stepwise argument formal, one must state explicitly a property of integers which is called induction, and is discussed later in the book.

Proof. Suppose $a < b$ and $a > 0$. We proceed using induction on n considering a^n and b^n.

(**Base Case**) Clearly the inequality holds when $n = 1$. We now show the inequality holds when $n = 2$. Since $a < b$ it follows that $b - a > 0$. Since $a > 0$ it follows that $ab - a^2 > 0$ so $ab > a^2$. Also, since $b > 0$ it follows that $b^2 - ab > 0$ so $b^2 > ab$. Therefore $a^2 < ab < b^2$ so $a^2 < b^2$.

(**Induction Step**) Now, assume the inequality holds for $n-1$ and $n-2$. It follows that $a^{n-1} < b^{n-1}$ so $b^{n-1} - a^{n-1} > 0$. Since $a > 0$ and $b > 0$ it follows that $a + b > 0$. Then $(a + b)(b^{n-1} - a^{n-1}) > 0$ so $b^n - a^n + ab^{n-1} - a^{n-1}b > 0$. Now $ab^{n-1} - a^{n-1}b = ab(b^{n-2} - a^{n-2})$. Notice ab is clearly greater than 0 and by our hypothesis $b^{n-2} - a^{n-2} > 0$ so $ab^{n-1} - a^{n-1}b = ab(b^{n-2} - a^{n-2}) > 0$.

Then

$$\begin{aligned} & b^n - a^n + ab^{n-1} - a^{n-1}b > 0 \\ \iff & b^n - a^n + ab^{n-1} > a^{n-1}b \\ \iff & b^n - a^n > a^{n-1}b - ab^{n-1} > 0 \end{aligned}$$

It then follows that $b^n - a^n > 0$. Therefore $b^n > a^n$. ∎

Problem 7

Prove: If $0 < a < b$, then $a^{\frac{1}{n}} < b^{\frac{1}{n}}$. [Hint: Use Excersize 6.]

Proof. Suppose $0 < a < b$. Note $a = a^1 = a^{\frac{n}{n}} = \left(a^{\frac{1}{n}}\right)^n$. Similarly $b = \left(b^{\frac{1}{n}}\right)^n$. Then

$$\begin{aligned} & a < b \\ \iff & \left(a^{\frac{1}{n}}\right)^n < \left(b^{\frac{1}{n}}\right)^n \\ \iff & a^{\frac{1}{n}} < b^{\frac{1}{n}} \quad \text{Problem 6} \end{aligned}$$

∎

Problem 8

Let a, b, c, d be numbers and assume $b > 0$ and $d > 0$. Assume that

$$\frac{a}{b} < \frac{c}{d}$$

(a) Prove that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

(There are two inequalities to be proved here, the one on the left and the one on the right.)
(b) Let r be a number > 0. Prove that

$$\frac{a}{b} < \frac{a+rc}{b+rd} < \frac{c}{d}$$

(c) If $0 < r < s$, prove that

$$\frac{a+rc}{b+rd} = \frac{a+sc}{b+sd}$$

Proof. Since $\frac{a}{b} < \frac{c}{d}$ (Problem 6), it follows that $ad < bc$. Then

$$\begin{aligned}\frac{a+c}{b+d} - \frac{a}{b} &= \frac{b(a+c) - a(b+d)}{b(b+d)} \\ &= \frac{bc - ad}{b(b+d)}.\end{aligned}$$

Since $bc - ad > 0$ and $b(b+d) > 0$, it follows that $\frac{a}{b} < \frac{a+c}{b+d}$. Then

$$\begin{aligned}\frac{c}{d} - \frac{a+c}{b+d} &= \frac{c(b+d) - d(a+c)}{d(b+d)} \\ &= \frac{bc - ad}{d(b+d)}.\end{aligned}$$

Since $bc - ad > 0$ and $d(b+d) > 0$, it follows that $\frac{a+c}{b+d} < \frac{c}{d}$. Therefore

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

∎

Proof. Since $\frac{a}{b} < \frac{c}{d}$ (Problem 6), it follows that $ad < bc$. Then:

$$\begin{aligned}\frac{a+rc}{b+rd} - \frac{a}{b} &= \frac{b(a+rc) - a(b+rd)}{b(b+rd)} \\ &= \frac{r(bc - ad)}{b(b+rd)}\end{aligned}$$

Since $bc - ad > 0$ and $r, b, d > 0$, the numerator and denominator are positive. Therefore

$$\frac{a}{b} < \frac{a+rc}{b+rd}.$$

Also,

$$\begin{aligned}\frac{c}{d} - \frac{a+rc}{b+rd} &= \frac{c(b+rd) - d(a+rc)}{d(b+rd)} \\ &= \frac{bc - ad}{d(b+rd)}\end{aligned}$$

Since $bc - ad > 0$ and $d(b + rd) > 0$, it follows that

$$\frac{a + rc}{b + rd} < \frac{c}{d}.$$

∎

Proof. By part b we know that $\frac{a}{b} < \frac{a+rc}{b+rd} < \frac{c}{d}$. Also, $\frac{a}{b} < \frac{a+sc}{b+sd} < \frac{c}{d}$. Then:

$$\begin{aligned}
\frac{a + sc}{b + sd} - \frac{a + rc}{b + rd} &= \frac{ard - rbc + bsc - asd}{(b + sd)(b + rd)} \\
&= \frac{r(ad - bc) + s(bc - ad)}{(b + sd)(b + rd)} \\
&= \frac{(s - r)(bc - ad)}{(b + sd)(b + rd)}
\end{aligned}$$

Since $s > r$ and $bc - ad > 0$, the numerator is positive. Also, $b, d, s, r > 0$, so the denominator is positive. Therefore

$$\frac{a + rc}{b + rd} < \frac{a + sc}{b + sd}.$$

Then it follows that

$$\frac{a}{b} < \frac{a + rc}{b + rd} < \frac{a + sc}{b + sd} < \frac{c}{d}.$$

∎

4 Quadratics Equations

Problem 1

$$x^2 + 3x - 2 = 0$$

Solution:

$$a = 1, b = 3, c = -2$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)} \\
&= \frac{-3 \pm \sqrt{17}}{2}
\end{aligned}$$

So $x = \frac{-3+\sqrt{17}}{2}$ or $x = \frac{-3-\sqrt{17}}{2}$.

Problem 11

$$x^2 - \sqrt{2}x + 1 = 0$$

$$a = 1, b = -\sqrt{2}, c = 1$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-\sqrt{2}) \pm \sqrt{(-\sqrt{2})^2 - 4(1)(1)}}{2(1)} \\ &= \frac{\sqrt{2} \pm \sqrt{-2}}{2} \end{aligned}$$

There are no real solutions. In $\mathbb{C}$, $x = \frac{\sqrt{2}+\sqrt{-2}}{2}$ or $x = \frac{\sqrt{2}-\sqrt{-2}}{2}$.

5 Distance and Angles

5.1 Angles

Problem 2

Assume that the area of a disc of radius 1 is equal to the number π (approximately equal to 3.14159...) and that the area of a disc of radius r is πr^2.

(a) What is the area of a sector in the disc of radius r lying between angles of θ_1 and θ_2 degrees, as shown in Fig. $5-20(a)$?

(b) What is the area of the band lying between two circles of radii r_1 and r_2 as shown in Fig. $5-20(b)$?

(c) What is the area in the region bounded by angles of θ_1 and θ_2 degrees and lying between circles of radii r_1 and r_2 as shown in Fig. $5-20(c)$?
Give your answers in terms of $\pi, \theta_1, \theta_2, r_2, r_1$.

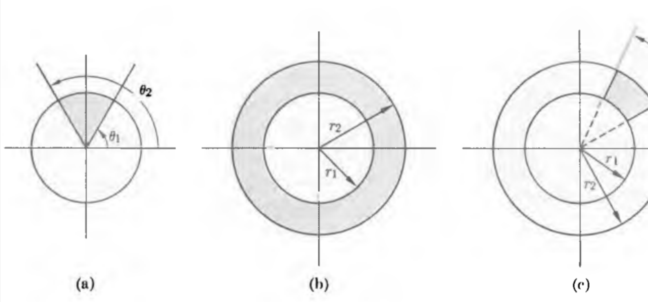


Fig. 5–20

Solution 2(a):

Area A of the sector is

$$A = \frac{|\theta_2 - \theta_1|}{360} \cdot \pi r_1^2$$

Solution 2(b):

Area A of the band is

$$A = \pi \cdot |r_2^2 - r_1^2|$$

Solution 2(c):

Area A of the region is

$$A = \frac{|\theta_2 - \theta_1|}{360} \cdot \pi \cdot |r_1^2 - r_2^2|$$

5.2 The Pythagoras Theorem

Problem 5

What is the length of the diagnol of a rectangle solid whose sides have lengths a, b, c? What if the sides have lengths ra, rb, rc.

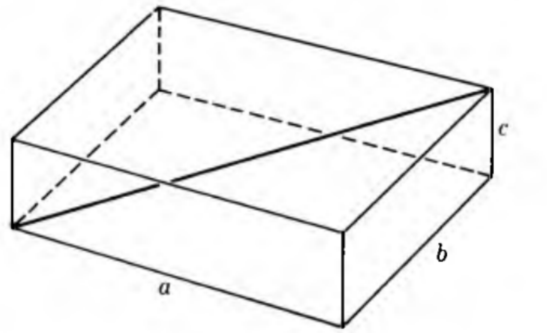


Fig. 5-30

Solution:

Let d be the length of the diagnol. Then

$$d = \sqrt{a^2 + b^2 + c^2}$$

If the sides have lengths ra, rb, rc then

$$d = \sqrt{(ra)^2 + (rb)^2 + (rc)^2} = \sqrt{r^2(a^2 + b^2 + c^2)} = r\sqrt{a^2 + b^2 + c^2}$$

Problem 9

Write down in detail the "similar steps" left to the reader in the proof of the corollary to the Pythagoras theorem.

Previous proof

Proof. Assume first that $d(P, M) = d(Q, M)$. By Pythagoras, we have

$$\begin{aligned} d(P, O)^2 + d(O, M)^2 &= d(P, M)^2 \\ &= d(Q, M)^2 \\ &= d(Q, O)^2 + d(O, M)^2 \end{aligned}$$

It follows that $d(P, O)^2 = d(Q, O)^2$ whence $d(P, O) = d(Q, O)$. ∎

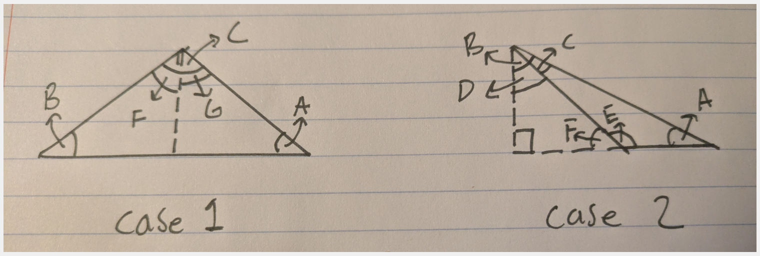
Proof. Suppose $d(P, O) = d(Q, O)$. By Pythagoras, we have $d(P, O)^2 + d(O, M)^2 = d(P, M)^2$. So $d(P, M)^2 - d(O, M)^2 = d(P, O)^2 = d(Q, O)^2 = d(Q, M)^2 - d(O, M)^2$. It follows that $d(P, M)^2 = d(Q, M)^2$ so $d(P, M) = d(Q, M)$. ∎

Problem 10

Prove that if A, B, C are the angles of an arbitrary triangle, then

$$m(A) + m(B) + m(C) = 180^\circ$$

by the following method: From any vertex draw the perpendicular to the line of the opposite side. Then use the result already known for right triangles.



Proof. Consider case 1. From the figure $m(C) = m(G) + m(F)$. The two subtriangles have sums $90^\circ + m(A) + m(G)$ and $90^\circ + m(B) + m(F)$. Now by Theorem 1, $m(A) + m(G) = 90^\circ$ and $m(B) + m(F) = 90^\circ$. So $90^\circ + m(A) + m(G) = 180^\circ$ then $m(A) = 90^\circ - m(G)$. Similarly, since $90^\circ + m(B) + m(F)$ then $m(B) = 90^\circ - m(F)$. Then the sum of the angles of the entire triangle is

$$\begin{aligned} m(A) + m(B) + m(C) &= 90^\circ - m(G) + 90^\circ - m(F) + m(C) \\ &= 180^\circ - (m(G) + m(F)) + m(C) \\ &= 180^\circ - m(C) + m(C) \\ &= 180^\circ \end{aligned}$$

Consider case 2. From the figure $m(D) + m(C) = m(B)$ so $m(C) = m(B) - m(D)$. Also $m(F) + m(E) = 180^\circ$ so $m(E) = 180^\circ - m(F)$. By Theorem 1, $m(D) + m(F) = 90^\circ$ and $m(B) + m(A) = 90^\circ$. Then the sum of the angles of the right-most inner triangle is

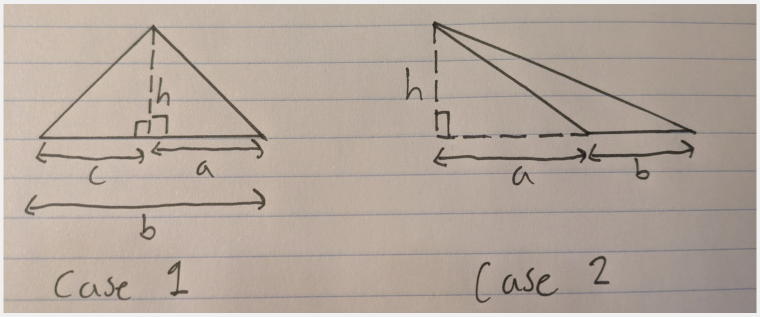
$$\begin{aligned} m(C) + m(E) + 90^\circ &= m(B) - m(D) + 180^\circ - m(F) + m(A) \\ &= m(B) + m(A) + 180^\circ - (m(D) + m(F)) \\ &= m(B) + m(A) + 180^\circ - 90^\circ \\ &= m(B) + m(A) + 90^\circ \\ &= 90^\circ + 90^\circ \\ &= 180^\circ \end{aligned}$$

∎

Problem 11

Show that the area of an arbitrary triangle of heigh h whose base has length b is $bh/2$. [*Hint:* Decompose the triangle into two right triangles. Distinguish between the two pictures in Fig. $5-31$. In one case the area of the triangle is the difference of the area of the two right triangles, and in the other case, it is the sum.]

Proof. Consider case 1. First note that $b = a + c$. By Theorem 2 (and for the rest of the proof) the area enclosed by the triangle on the right is $\frac{1}{2}ah$. The area enclosed by the triangle on the left is $\frac{1}{2}ch$. The area of the outer triangle is the sum of the inner triangles which is $\frac{1}{2}ch + \frac{1}{2}ah = \frac{1}{2}h(a + c) = \frac{1}{2}bh$.



Consider case 2. The area enclosed by the outer triangle is $\frac{1}{2}h(a+b)$. Then, the area enclosed by the right inner triangle is the area enclosed by the outer triangle minus the left inner triangle which is $\frac{1}{2}h(a+b)-\frac{1}{2}ha = \frac{1}{2}h(a+b-a) = \frac{1}{2}bh$. ∎

Problem 12

(a) Show that the length of the hypotenuse of a right triangle is ≥ the length of a leg.

(b) Let P be a point and L a line. Show that the smallest value for the distances $d(P,M)$ between P and points M on the line is the distance $d(P,Q)$, where Q is the point of the intersection between L and the line through P, perpendicular to L.

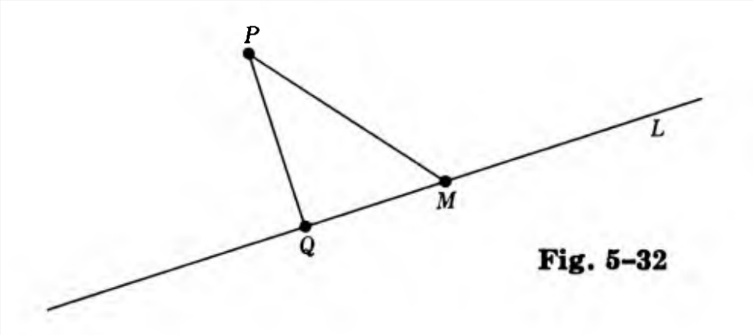


Fig. 5–32

Proof. Let a,b be the legs of a right triangle and c be the hypotenuse. By Pythagoras's Theorem we know that $c^2 = a^2 + b^2$. Since $b^2 \geq 0$ it follows that $c^2 \geq a^2$. Since $c, a \geq 0$ it follows that $c \geq a$. Similarly, $c^2 \geq b^2$ implies $c \geq b$. Therefore the hypotenuse c is greater than or equal to each leg a, b. ∎

Proof. Consider the right triangle formed by points P, Q, and M, where Q is the point on the line L such that $\overline{PQ}$ is perpendicular to L, and M is any other point on L. The hypotenuse of this triangle is $d(P,M)$, and we want to choose M to minimize this distance. By the Pythagorean Theorem,

$$d(P,M)^2 = d(P,Q)^2 + d(Q,M)^2.$$

The smallest value for $d(Q,M)^2$ is 0, which occurs if and only if $Q = M$. So when $Q = M$, we have $d(P,M) = d(P,Q)$. Therefore, the smallest distance from P to a point on the line L is $d(P,Q)$. ∎

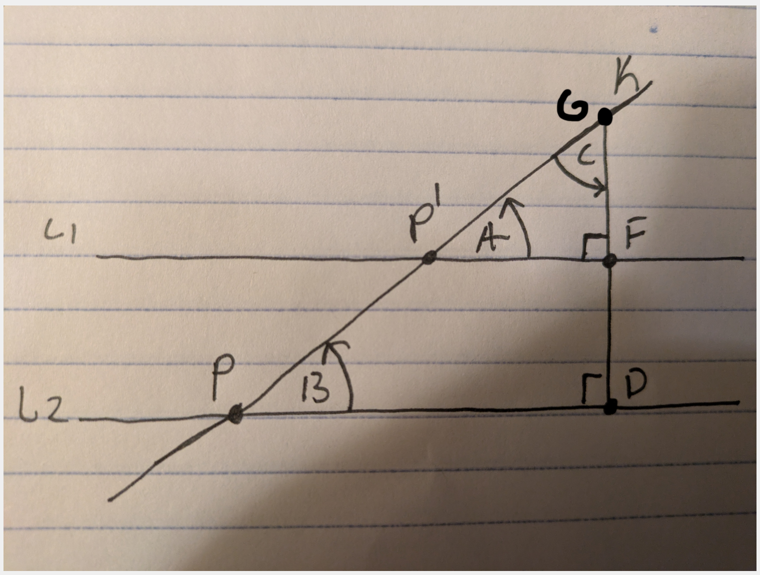
Problem 13

This exercize asks you to derive some standard properties of angles from elementary geometry. They are used very commonly. We refer to the following figures.

(a) In Fig. $5-33(a)$, you are given two parallel lines L_1, L_2 and a line K which intersects them at points P and P' as shown. Let A and B then be angles which K makes with L_1 and L_2 respectively, as shown. Prove that $m(A) = m(B)$. [*Hint*: Draw a line from a point of K above L_1 and L_2. Then use the fact that the sum of the angles of a right triangle has $180°$.]

(b) In Fig. $5-33(b)$, you are given L_1, L_2 and K again. Let B and B' be the alternate angles formed by K and L_1, L_2 respectively, as shown. Prove that $m(B) = m(B')$. (Actually, all you need to do here is refer to the appropriate portion of the text. Which is it?)

(c) Let K, L be two lines as shown on Fig. $5-33(c)$. Prove that the opposite angles A and A' as shown have equal measure.



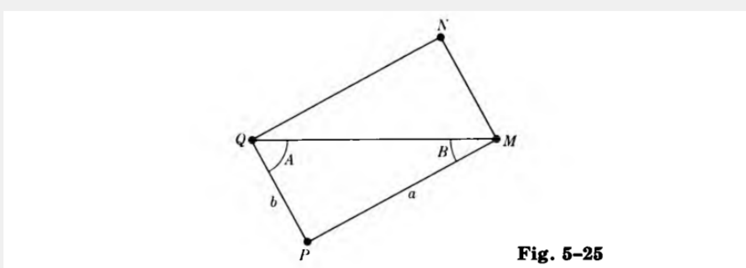
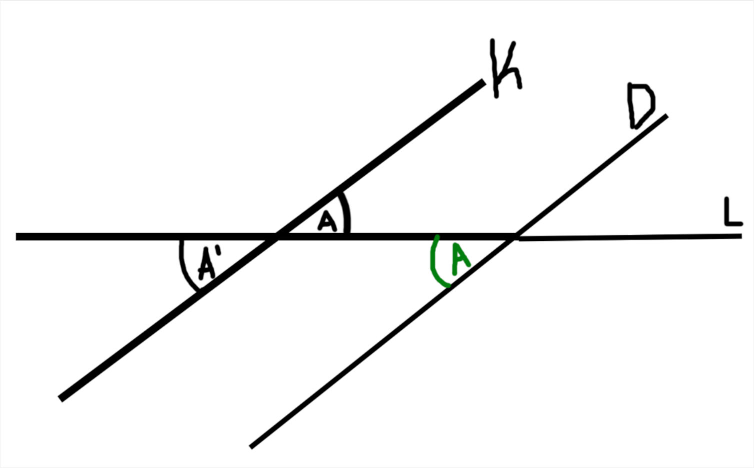


Fig. 5–25

Let A, B be the angles of the right triangle, other than the right angle as shown in Fig. 5–25. It follows from **RT** that $\angle NQM$ has the same measure as B. Since $\angle NQP$ is a right angle, and since A and $\angle NQM$ together form

Proof. Refer to the figure. Consider the areas enclosed by the right triangles $\triangle P'FG$ and $\triangle PDG$. By Problem 10, the degrees of $\triangle P'FG$ is $90° + m(A) + m(C) = 180°$. Similarly, the degress of $\triangle PDG$ is $90° + m(B) + m(C) = 180°$. Then $90° + m(A) + m(C) = 90° + m(B) + m(C)$ and it follows that $m(A) = m(B)$. ∎

Proof. Refer to the text. Then let the parallel segments formed by points Q, P and N, M be the parallel lines L_1, L_2. Also let B be the B in our problem. Finally, let $\angle NQM$ be $m(B')$. It then follows from the text that $m(B) = m(B')$. ∎



Proof. Refer to the figure. Line D is parallel to line K. The measure of the green angle is equal to $m(A)$ by Problem 13(b). Then $m(A') = m(A)$ by Problem 13(a). ∎

Problem 14

Let $\triangle PQM$ be a triangle. Let L_1 be the perpendicular bisector of $\overline{PQ}$ and let L_2 be the perpendicular bisector of $\overline{QM}$. Let O be the point of intersection of L_1 and L_2. Show that $d(P,O) = d(M,O)$, and hence that O lies on the perpendicular bisector of $\overline{PM}$. Thus the perpendicular bisectors of the sides of the triangle meet in a point.

Proof. Since O lies on the perpendicular bisector of $\overline{PQ}$ it follows that $d(P,O) = d(Q,O)$. Similarly, $d(M,O) = d(Q,O)$. Then $d(P,O) = d(Q,O) = d(M,O)$. ∎

6 Isometries

6.1 Some Standard Mappings of the Plane

Problem 1

Let F be a mapping of the plane into itself. We define a **fixed point** for F to be a point P such that $F(P) = P$.

Describe the fixed points of the following mappings.
(a) The identity.
(b) Reflection through a given point O.
(c) Reflection through a line.
(d) A rotation not equal to the identity, with respect to a given point O.
(e) A translation not equal to the identity.
(f) Dilation by a number $r > 0$, relative to a given point O.

Solution (a):

All points in the plane are fixed points.

Solution (b):

The point O is a fixed point.

Solution (c):

All points on the line are fixed points.

Solution (d):

The point O is a fixed point.

Solution (e):

There are no fixed points on the plane.

Solution (f):

If $r = 1$, then all points are fixed points. If $r \neq 1$, then only O is a fixed point.

6.2 Isometries

Problem 2

For which values of r is dilation by r an isometry?

When $r = 1$ or $r = -1$.

Problem 4

Let L, K be two parallel lines, and let F be an isometry. Prove that $F(L)$ and $F(K)$ are parallel.

Proof. If $L = K$ then trivially $F(L)$ and $F(K)$ are parallel as they are equal since F is an isometry.

Suppose $L \neq K$. For contradiction, suppose $F(L)$ and $F(K)$ are not parallel. Then there is a point where $F(L)$ and $F(K)$ intersect. Since F is an isometry, this would imply that L and K also intersect, contradicting the fact that L and K are distinct parallel lines. ∎

Problem 5

Let K, L be perpendicular lines, and let F be an isometry. Prove that $F(K)$ and $F(L)$ are perpendicular. [Hint: Use the corollary of the Pythagoras theorem.]

Proof. Let I be the intersection of the lines K and L. Let P and Q be two points lying on the line L which are equal distances from I. Let O be a point on K such that $O \neq I$. By the corollary to the Pythagorean Theorem, $d(P, O) = d(Q, O)$. Now $F(P), F(Q)$ determine a line and $F(O), F(I)$ determine a line (corollary in the text). Then, since F is an isometry, $d(F(P), F(O)) = d(F(Q), F(O))$. It then follows that the line formed by $F(O), F(I)$ is the perpendicular bisector of $F(P), F(Q)$. ∎

Problem 6

Visualize 3-dimensional space. We also have the notion of distance in space, satisfying the same basic properties as in a plane. We can therefore define an isometry of 3-space in the same way that we defined an isometry of the plane. It is a mapping of 3-space into itself which is distance preserving. Are Theorems 1 and 2 valid in 3-space? How would you formulate Theorem 3? (Consider the plane in which the three points lie.) Now formulate a theorem in 3-space about an isometry being the identity provided that it leaves enough points fixed. Describe a proof for such a theorem, similar to the proof of Theorem 3. Make a list of what you need to assume to make such a proof go through. Write all of this up as if you were writing a book. Aside from learning mathematical substance, you will also learn how to think more clearly, and how to write mathematics in the process

Theorem 1. *Let F be an isometry. Let P, Q, M, S be four distinct points which do not lie on the same plane. Assume that P, Q, M, S are fixed points of F; that is*

$$F(P) = P, F(Q) = Q, F(M) = M, F(S) = S$$

Then F is the identity.

The proof could be layed out as follows.

1. First take the plane formed by P, Q, M and by Theorem 3 all points on that plane are fixed points.
2. Then note that point S does not lie on this plane.
3. Then since F is an isometry the $d(P,S) = d(F(P), F(S))$, $d(Q,S) = d(F(Q), F(S))$, and $d(M,S) = d(F(M), F(S))$.
4. Determine that $F(S) = S$, and hence that all points of space are fixed; therefore F is the identity.

6.3 Composition of Isometries

Problem 1

Let F be a reflection through a line L. What is the smallest positive integer n such that $F^n = I$.

Solution:

$n = 2$.

Problem 4

Give an example of two isometries F_1, F_2 such that

$$F_1 \circ F_2 \neq F_2 \circ F_1$$

Solution:

Let G_x = rotation by $x°$about the origin. Let T_1 = translation of 1 to the right. Then let $F_1 = G_{45}$ and $F_2 = T_1$. These two isometries do not commute

$$F_1 \circ F_2 \neq F_2 \circ F_1.$$

6.4 Inverse of Isometries

Problem 1

(a) Let F be an isometry which has an inverse F^{-1}. Let S be a circle of radius r, and center P. Show that the image of S under F is a circle. [Hint: Let S' be the circle of center $F(P)$ and radius r. Show that $F(S)$ is contained in S' and that every point of S' is the image under F of a point in S.]
(b) Let F be an isometry which has an inverse F^{-1}. Let D be a disc of radius r and center P. Show that the image of D under F is a disc.

Proof. Let S' be the circle of center $F(P)$ and radius r. We need to show that $F(S) \subseteq S'$ and $S' \subseteq F(S)$.

We first show $F(S) \subseteq S'$. Let T be a point such that $d(P,T) = r$. Since F is an isometry $d(F(P), F(T)) = r$. Since T was arbitrary all points T such that $d(P,T) = r$ are contained within the circle centered at $F(P)$ with radius r which is exactly S'.

We now show $S' \subseteq F(S)$. Let T be a point at distance r from the center $F(P)$ of the circle S'. Now let $Y = F^{-1}(T)$. It follows that $F(Y) = T$. We know that $d(F(Y), F(P)) = r$. Since F is an isometry $d(Y,P) = r$. ∎

Proof. Let D' be the disc of center $F(P)$ and radius r. We need to show that $F(D) \subseteq D'$ and $D' \subseteq F(D)$.

We first show $F(D) \subseteq D'$. Let T be a point such that $d(P,T) \leq r$. Since F is an isometry, $d(F(P), F(T)) \leq r$. Since T was arbitrary, all points T with $d(P,T) \leq r$ are contained within the disc centered at $F(P)$ with radius r, which is exactly D'.

We now show $D' \subseteq F(D)$. Let T be a point at distance $\leq r$ from the center $F(P)$ of the disc D'. Now let $Y = F^{-1}(T)$. It follows that $F(Y) = T$. We know that $d(F(Y), F(P)) \leq r$. Since F is an isometry, $d(Y, P) \leq r$. ∎

Problem 2

Let P, Q, P', Q' be points such that

$$d(P,Q) = d(P',Q')$$

Prove that there exists an isometry F such that $F(P) = P'$ and $F(Q) = Q'$. You may assume the statements we have assumed in this section.

Proof. First, perform a rotation G of Q about P such that the line formed by P and Q is parallel to the line formed by P' and Q'.

Next, let T be the translation along the ray from Q to Q' of length $d(Q,Q')$. Applying T moves Q exactly to Q'. Since the line through P and Q is parallel to the line through P' and Q', the same translation moves P to P'.

Since rotations and translations are isometries, the composition $T \circ G$ is an isometry. ∎

Problem 3

Let F, G, H be isomemetries and assume that F has an inverse. If

$$F \circ G = F \circ H$$

prove that $G = H$ (**cancellation law** for isomemetries).

Proof. Applying F^{-1} to both sides yields

$$\begin{aligned}
& F^{-1} \circ (F \circ G) = F^{-1} \circ (F \circ H) \\
\iff & (F^{-1} \circ F) \circ G = (F^{-1} \circ F) \circ H \\
\iff & I \circ G = I \circ H \\
\iff & G = H
\end{aligned}$$

∎

Problem 4

(a) Let F be an isometry such that $F^2 = I$ and $F^3 = I$. Prove that $F = I$.

(b) Let F be an isometry such that $F^4 = I$ and $F^7 = I$. Prove that $F = I$.

(c) Let F be an isometry such that $F^5 = I$ and $F^8 = I$. Prove that $F = I$.

Proof. Since $F^2 = I$ it follows that $F^3 = F \circ I$. Then $F = F \circ I = F^3 = I$. ∎

Proof. Since $F^4 = I$ it follows that $I = F^{-4} \circ I = F^{-4}$. Also since $F^4 = I$ it follows that $F = F^{-3} \circ I = F^{-3}$. Then $F = F^{-3} = F^7 \circ F^{-4} = I \circ I = I$. ∎

Proof. Since $F^5 = I$ it follows that $F = F^{-4} \circ I = F^{-4}$. Also, since $F^8 = I$ it follows that $F^3 = F^{-5} \circ I = F^{-5}$. Finally, since $F^8 = I$ it follows that $I = F^{-8} \circ I = F^{-8}$. Then

$$F = F^{-4} \circ I = F^{-4} \circ F^{-8} = F^{-12} = F^{-5} \circ F^{-8} = F^3 \circ F^{-8} = F^3 \circ F^{-4} \circ F^{-4} = F^3 \circ F \circ F = F^5 = I$$

∎

Problem 5

Write out the proof of the corollary of Theorem 3. (Consider $F^{-1} \circ G$.)

> ***Corollary of Theorem 3.*** *Let P, Q, M be three distinct points which do not lie on the same line. Let F, G be isometries such that*
>
> $$F(P) = G(P), \qquad F(Q) = G(Q), \qquad F(M) = G(M).$$
>
> *Assume that F^{-1} exists. Then $F = G$.*
>
> ***Proof.*** The proof is very easy and will be left as an exercise.

Proof. Since $F(P) = G(P)$, $F(Q) = G(Q)$, and $F(M) = G(M)$ it follows that $P = (F^{-1} \circ G)(P)$, $Q = (F^{-1} \circ G)(Q)$, and $M = (F^{-1} \circ G)(M)$. Now this is three fixed points so $F^{-1} \circ G$ is the identity mapping. Since $F^{-1} \circ G = I$ it follows that G is the inverse of F^{-1} which is F. ∎

Problem 6

Let $F \circ G \circ H$ be the composite of three isometries. Assume that F^{-1}, G^{-1}, H^{-1} exist. Prove that $(F \circ G \circ H)^{-1}$ exists, and express this inverse in terms of the inverses for F, G, H.

Proof. Let P be an arbitrary point in the plane such that $(F \circ G \circ H)(P) = X$. Then

$$\begin{aligned}
&(F \circ G \circ H)(P) = X \\
\iff &(H^{-1} \circ (F \circ G \circ H))(P) = H^{-1}(X) \\
\iff &((G^{-1} \circ H^{-1}) \circ (F \circ G \circ H))(P) = (G^{-1} \circ H^{-1})(X) \\
\iff &((F^{-1} \circ G^{-1} \circ H^{-1}) \circ (F \circ G \circ H))(P) = (F^{-1} \circ G^{-1} \circ H^{-1})(X) \\
\iff &((F \circ G \circ H)^{-1} \circ (F \circ G \circ H))(P) = (F^{-1} \circ G^{-1} \circ H^{-1})(X) \\
\iff &I(P) = (F^{-1} \circ G^{-1} \circ H^{-1})(X) \\
\iff &P = (F^{-1} \circ G^{-1} \circ H^{-1})(X)
\end{aligned}$$

Therefore, $(F \circ G \circ H)^{-1} = H^{-1} \circ G^{-1} \circ F^{-1}$. ∎

Problem 7

Let F be an isometry such that $F^7 = I$. Express F^{-1} as a positive power F.

Proof. Since $F^7 = I$, we have

$$F^{-1} = I \circ F^{-1} = F^7 \circ F^{-1} = F^6.$$

∎

Problem 8

Let n be a positive integer and let F be an isometry such that $F^n = I$. Express F^{-1} as a positive power of F.

Proof. Since $F^n = I$, we have
$$F^{-1} = I \circ I \circ F^{-1} = F^n \circ F^n \circ F^{-1} = F^{2n-1}.$$
Since $n \geq 1$ it follows that $2n \geq 2$ so $2n > 1$ and $2n - 1 > 0$. ∎

Problem 9

Consider the corners of a square centered at the origin. For convenience of notation, number these corners $1, 2, 3, 4$ as in Fig. $6-26$.
Write the image of each one of these corners under the isometries. $H, V, H \circ V, V \circ H$. Just to show you an easy notation to do this, we write down the images of these corners under rotation by $90°$in the following form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

This notation means that if G is a rotation by $90°$, then $G(1) = 2$, $G(2) = 3$, $G(3) = 4$, and $G(4) = 1$.
H, V are the reflections along the horizontal line and vertical line respectively.

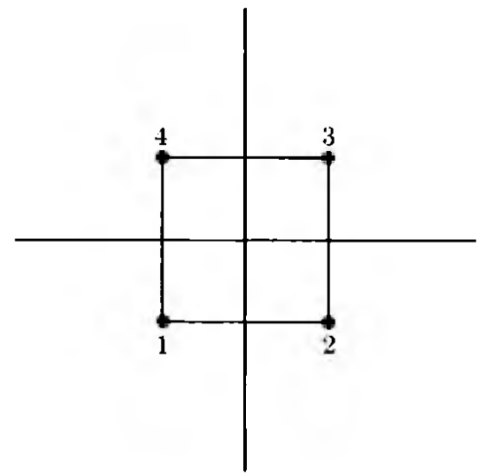


Fig. 6-26

Solution:

Image under H.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Image under V.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Image under $H \circ V$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Image under $V \circ H$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Problem 10

Let G be a rotation by $90°$so that $G^4 = I$. Express $H \circ G \circ H$ as a power of G. For what positive integer n do we have

$$H \circ G = G^n \circ H$$

Write down the images of the corner of the square as in the preceeding exercize, under the maps $I, G, G^2, G^3, H, H \circ G, H \circ G^2, H \circ G^3, G \circ H, G^2 \circ H, G^3 \circ H$.

Solution:

$$H \circ G \circ H = G^3$$

The following equation holds when $n = 3$.

$$H \circ G = G^n \circ H$$

Image under I.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Image under G.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

Image under G^2.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Image under G^3.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

Image under H.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Image under $H \circ G$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{bmatrix}$$

Image under $H \circ G^2$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{bmatrix}$$

Image under $H \circ G^3$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Image under $G \circ H$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Image under $G^2 \circ H$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Image under $G^3 \circ H$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Problem 13

Consider a triangle whose three sides have equal length and whose three angles have the same measure, $60°$, as in Fig. $6-27$.
The vertices of the triangle are numbered $1, 2, 3$. Let G be a rotation by $120°$and let V, as usual, be reflection through the vertical axis.
(a) Give the effect of the six isometries I, G, G^2, V, VG, VG^2 on the vertices, using the same notation as exercize 9.
(b) Make up the multiplication table for these six isometries.

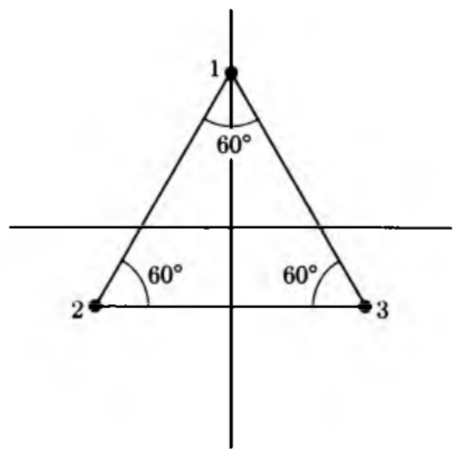


Fig. 6–27

Solution:

Image under I.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Image under G.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Image under G^2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Image under V.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Image under VG.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Image under VG^2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

$\circ$	I	G	G^2	V	VG	VG^2
I	I	G	G^2	V	VG	VG^2
G	G	G^2	I	VG^2	V	VG
G^2	G^2	I	G	VG	VG^2	V
V	V	VG	VG^2	I	G^2	G
VG	VG	VG^2	V	G	I	G^2
VG^2	VG^2	V	VG	G^2	G	I

6.5 Characterization of Isometries

Problem 1

Prove that every isometry has an inverse.

Proof. Let F be an arbitrary isometry. There are four cases depending on the number of fixed points under F.

(≥ 3 **fixed points**) By Theorem 3, $F = I$ which clearly has an inverse: $F^{-1} = I$.

(2 **fixed points**) Let P and Q be the two fixed points under F. By Theorem 4, either F is the identity, or F is a reflection through the line L_{PQ} passing through P and Q. In the identity case, $F^{-1} = I$. If F is a reflection, the inverse is also F: $F^{-1} = F$.

(1 **fixed point**) Let P be the single fixed point under F. By Theorem 5, either F is a rotation about P, or F is a rotation composed with a reflection through a line through P. In the rotation case, F^{-1} is the rotation by the opposite angle about P. In the rotation-reflection case, F^{-1} is the reflection composed with the rotation by the opposite angle.

(**no fixed points**) By Theorem 6, either F is a translation, a composite of a translation and a rotation, or a composite of a translation, a rotation, and a reflection through a line. In the translation case, F^{-1} is the translation by the opposite vector. In the translation-rotation case, F^{-1} is the rotation by the opposite angle followed by translation by the opposite vector. In the translation-rotation-reflection case, F^{-1} is the reflection followed by rotation by the opposite angle and translation by the opposite vector.

Since these cases are exhaustive, every isometry has an inverse. ∎

Problem 2

If P is a fixed point for an isometry F, prove that P is also a fixed point for F^{-1}.

Proof. Since P is a fixed point under F is follows that $F(P) = P$. By Problem 1 we know that F^{-1} exists. Applying F^{-1} to both sides yields $P = F^{-1}(P)$. Therefore, P is a fixed point of F^{-1}. ∎

6.6 Congruences

Problem 1

Prove that two discs of the same radius are congruent.

Proof. Let the first disc be $D(r, O)$, of radius r, centered at O, and let the other disc be $C(r, O')$, centered at O'. Let T be the translation which maps O on O'. We know that T preserves distances. Hence if P is at distance $\leq r$ from O, then $T(P)$ is at distance $\leq r$ from $T(O) = O'$. Hence the image of the disc $D(r, O)$ is contained in the circle $D(r, O')$. We must still show that every point on $D(r, O')$ is the image of a point on $D(r, O)$ under T. Let Q be a point at distance $\leq r$ from O'. Note that the point

$$P = T^{-1}(Q)$$

is at distance $\leq r$ from O, and that $T(P) = T(T^{-1}(Q)) = Q$. ∎

Problem 2

Let S, S', S'' be sets in the plane. Prove that if S is congruent to S', and S' is congruent to S'', then S is congruent to S''. Prove that if S is congruent to S', then S' is congruent to S.

Proof. Suppose that S is congruent to S' and S' is congruent to S''. Let T be the isometry such that $T(S) = S'$. Let F be the isometry such that $F(S') = S''$. Then $(F \circ T)(S) = F(T(S)) = F(S') = S''$. Note that a composition of isometries is an isometry. Thus $F \circ T$ is an isometry mapping S to S'' so S is congruent to S''. ∎

Proof. Suppose that S is congruent to S'. Let F be the isometry such that $F(S) = S'$. Since F is an isometry it has an inverse F^{-1}. Then applying this to both sides yields $S = F^{-1}(S')$. Thus S' is congruent to S. ∎

Problem 3

Prove that two squares whose sides have the same length are congruent.

Proof. Let S and S' be two squares whose sides have the same length. Let A, B, C, D be the vertices of S in cyclic order. Let O be the intersection of the diagonals AC and BD. Since the diagonals of a square are equal in length and bisect each other at right angles, their intersection O is equidistant from all vertices. Thus O is the center of S.

Similarly, let A', B', C', D' be the vertices of S' in cyclic order, and let O' be the intersection of its diagonals (the center of S').

Let T be the translation mapping O to O'. Let G be the rotation with respect to O' of S' such that $(G \circ T)(A) = A'$. Because G is a rotation about the center O', it must also send the point on the opposite end of that diagonal, namely $(G \circ T)(C)$, to C', and $(G \circ T)(D)$ to D'. If $(G \circ T)(B) \neq B'$, then perform a reflection R across the line connecting $(G \circ T)(A)$ and $(G \circ T)(C)$. Otherwise, let $R = I$ such that I is the identity mapping. Let $F = R \circ G \circ T$.

Thus $F(A) = A', F(B) = B', F(C) = C', F(D) = D'$, and therefore $F(S) \subseteq S'$. Since F is an isometry, it preserves distances and lines, so every point of S maps to points of S' and vice versa, thus $F(S) = S'$. ∎

Problem 4

Prove that any two lines are congruent.

Proof. Let L and L' be two lines. Pick points $P, Q \in L$ and $P', Q' \in L'$. Using Section 6.4 Problem 2, since $d(P, Q) = d(P', Q')$, there exists an isometry F such that $F(P) = P'$ and $F(Q) = Q'$.

Since F preserves distances and maps two points on L to two points on L', it maps the entire line L onto L'. Therefore, the lines L and L' are congruent. ∎

Problem 5

Let $\triangle ABC$ be a triangle whose three angles all have $60°$. Prove that the sides have equal length. [Hint: From any vertex draw the perpendicular to the other side, and reflect through this perpendicular.]

Proof. Let A, B, C be the vertices of a triangle whose three angles all have $60°$. Let L be the perpendicular bisector of $\overline{AB}$, intersecting $\overline{AB}$ at its midpoint O. There are now two right triangles, namely $\triangle AOC$ and $\triangle BOC$, with right angles at O. These triangles share the side $\overline{OC}$. Since L is the perpendicular bisector of $\overline{AB}$, we have $d(A, O) = d(B, O)$. Then

$$d(A,C)^2 = d(A,O)^2 + d(O,C)^2 = d(B,O)^2 + d(O,C)^2 = d(B,C)^2.$$

It follows that $d(A, C) = d(B, C)$.

Apply similar steps but let L be the perpendicular bisector of $\overline{BC}$, thus showing that $d(A, B) = d(A, C)$.

Thus $d(A, B) = d(A, C) = d(B, C)$. ∎

Proof. Let A, B, C be the vertices of a triangle with all angles $60°$. Consider the perpendicular bisector L of side $\overline{AB}$. Reflect the vertex C across L to a point C'.

Since the triangle has equal angles, this reflection maps the triangle to itself, so $C' = C$. Reflection across the perpendicular bisector preserves distances, so

$$AC = BC.$$

Similarly, reflecting across the perpendicular bisector of $\overline{BC}$ gives

$$AB = AC.$$

Therefore, all three sides are equal:

$$AB = BC = AC.$$

∎

Problem 6

Prove Theorem 9. At first you are not allowed to use Theorem 10. If you were allowed to use Theorem 10, how would you deduce Theorem 9 from it.

Theorem 2. *Let $\triangle PQM$ and $\triangle P'Q'M'$ be right triangles whose right anlges are at Q and Q' respectively. Assume that the corresponding legs have the same lengths, that is:*

$$d(P,Q) = d(P',Q')$$

and

$$d(Q,M) = d(Q',M')$$

Then the triangles are congruent.

Proof. Let T be the translation such that $T(Q) = Q'$. Let G be the ration such that $(G \circ T)(P) = P'$. Finally, if $(G \circ T)(M) \neq M'$ apply a reflection on the line $\overline{P'Q'}$ Let $F = G \circ T$.

Thus, $G(Q) = Q'$, $G(P) = P'$, and $G(M) = M'$.

Now let O be any point on the segment $\overline{PQ}$. It follows that $d(P,Q) = d(F(P),F(Q)) = d(P,O) + d(Q,O) = d(F(P),F(O)) + d(F(Q),F(O))$. Then since $d(F(P),F(Q)) = d(F(P),F(O)) + d(F(Q),F(O))$, $F(O)$ lies on the segment $\overline{P'Q'}$.

Now let K be any point on the segment $\overline{P'Q'}$. Since F is an isometry, it has an inverse; let $U = F^{-1}(K)$. Then $d(P,Q) = d(F^{-1}(P'),F^{-1}(Q')) = d(P,U) + d(U,Q)$ so U lies on the segment $\overline{PQ}$. It follow that $F(U) = F(F^{-1}(K)) = K$.

Simlar arguments apply to $\overline{QM}$ and $\overline{PM}$. Thus the triangles are congruent. ∎

Proof. Since the legs have the same length, by the Pythagorean Theorem their hypotenuses are of equal length. Thus the lengths of the sides of the triangles are equal. One can then easily apply Theorem 10 to show the triangles are congruent. ∎

Problem 7

Let $\triangle PQM$ and $\triangle P'Q'M'$ be triangles having one corresponding angle of the same measure, say $\angle PQM$ and $\angle P'Q'M'$ have the same measure, and having adjacent sides of the same length, i.e.

$$d(P,Q) = d(P',Q') \text{ and } d(Q,M) = d(Q',M')$$

Prove that the triangles are congruent.

Proof. Let $\triangle PQM$ and $\triangle P'Q'M'$ have a corresponding angle of the same measure, say $\angle PQM = \angle P'Q'M'$, and let the adjacent sides satisfy $d(P,Q) = d(P',Q') \quad \text{and} \quad d(Q,M) = d(Q',M')$.

Draw the altitude from Q to the side $\overline{PM}$ in $\triangle PQM$, and let H be the foot of this perpendicular. Similarly, draw the altitude from Q' to $\overline{P'M'}$ in $\triangle P'Q'M'$, and let H' be the foot.

Now consider the right triangles $\triangle QPH$ and $\triangle QMH$ in the first triangle, and $\triangle Q'P'H'$ and $\triangle Q'M'H'$ in the second. By construction and the given distances, the corresponding legs of these right triangles are equal $d(Q,P) = d(Q',P'), \quad d(Q,H) = d(Q',H'), \quad d(Q,M) = d(Q',M'), \quad d(Q,H) = d(Q',H')$

By Theorem 10, triangles with all three sides equal are congruent, so

$$\triangle QPH \cong \triangle Q'P'H' \quad \text{and} \quad \triangle QMH \cong \triangle Q'M'H'.$$

Since both pairs of right triangles are congruent, all points P, Q, M correspond to P', Q', M', and therefore ∎

Problem 8

Prove that two rectangles having corresponding sides of equal lengths are congruent.

Proof. Let $ABCD$ and $A'B'C'D'$ be the corners encountered cyclicly of two rectangles such that $d(A,B) = d(A',B')$ and $d(B,C) = d(B',C')$.

Let T be the translation such that $T(A) = A'$. Let G be the rotation about A' such that $(G \circ T)(C)$ lies on the line through A' and C'. If necessary, let R be a reflection across the line $\overline{A'C'}$ to match B and B'. Otherwise, let $R = I$ where I is the identity mapping.

Thus $F = R \circ G \circ T$.

Draw the diagonal $\overline{AC}$ in both rectangles. This divides each rectangle into two right triangles: $\triangle ABC$ and $\triangle ADC$ in the first rectangle, and $\triangle A'B'C'$ and $\triangle A'D'C'$ in the second.

Now let O be any point on the segment $\overline{AC}$ of the first rectangle. It follows that $d(A,C) = d(A,O) + d(O,C) = d(A', F(O)) + d(F(O), C')$. Then, by the Pythagorean Theorem, $F(O)$ lies on the segment $\overline{A'C'}$.

Now let K be any point on the segment $\overline{A'C'}$. Since F is an isometry, it has an inverse; let $U = F^{-1}(K)$. Then $d(A,C) = d(F^{-1}(A'), F^{-1}(C')) = d(A,U) + d(U,C)$ so U lies on the segment $\overline{AC}$. By definition of U, $F(U) = F(F^{-1}(K)) = K$.

Hence the corresponding right triangles along the diagonals are congruent, and therefore the rectangles $ABCD$ and $A'B'C'D'$ are congruent. ∎

Problem 9

Give a definition of the region bounded by a square in terms of line segments. Same thing for a rectangle.

Definition 1. *Let S be a square with distinct parallel line segments $\overline{AB}$ and $\overline{CD}$. We can define the area of S to be all points on all line segments formed by points X, Y where X lies on $\overline{AB}$ and Y lies on $\overline{CD}$. The area for a rectangle can be defined in precisely the same way with exception to letting S be a rectangle.*

Problem 11

Let $\triangle PQM$ and $\triangle P'Q'M'$ be triangles whose corresponding angles have the same measures (i.e. the angle with vertex P has the same measure as the angle with vertex at P', and similarly for the angles with vertices at Q, Q' and M, M'). Assume that $d(P,Q) = d(P',Q')$. Prove that the triangles are congruent.

Problem 12

Let $\triangle PQM$ be a triangle. Let L_1, L_2, L_3 be the three lines which bisect the three angles of the triangle, respectively. Let O be the point of intersection of L_1 and L_2. Prove that O lies on L_3. [Hint: From O, draw the perpendicular segments to the corresponding sides. Prove that their lengths are equal.]