# The Real Numbers and Real Analysis
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## Contents



## 1 Construction of the Real Numbers

### 1.1 Axioms for the Natural Numbers

**Problem 1**

Fill in the missing details in the proof of Theorem 1.2.6.

*Proof.* We must show the uniquness of the binary operation $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that satisfies the following two properties for all $n, m \in \mathbb{N}$.

**a.** $n \cdot 1 = n$.

**b.** $n \cdot s(m) = (n \cdot m) + n$.

Suppose there are two binary operations $\cdot$ and $\times$ on $\mathbb{N}$ that satisfy the two properties for all $n, m \in \mathbb{N}$. Let

$$G = \{x \in \mathbb{N} \mid n \cdot x = n \times x \text{ for all } n \in \mathbb{N}\}$$

We will prove that $G = \mathbb{N}$, which will imply that $\cdot$ and $\times$ are the same binary operation. It is clear that $G \subseteq \mathbb{N}$. By part (a) applied to each of $\cdot$ and $\times$ we see that $n \cdot 1 = n = n \times 1$ for all $n \in \mathbb{N}$ and hence $1 \in G$. Now let $q \in G$. Let $n \in \mathbb{N}$. Then $n \cdot q = n \times q$ by hypothesis on $q$. It then follows from part (b) that $n \cdot s(q) = (n \cdot q) + n = (n \times q) + n = n \times s(q)$. Hence $s(q) \in G$. By part (c) of the Peano Postulates we conclude that $G = \mathbb{N}$. ∎

*Proof.* We must show the two properties hold. Now, $n \cdot 1 = g_n(1) = n$, which is part (a), and $n \cdot s(m) = g_n(s(m)) = (g_n \circ s)(m) = (h_n \circ g_n)(m) = g_n(m) + n = (n \cdot m) + n$, which is part (b). ∎

**Problem 2**

Prove Theorem 1.2.7 (2) (3) (4) (7) (8) (9) (10) (11) (13).

*Proof.* Let $a, b, c \in \mathbb{N}$. We must show $(a + b) + c = a + (b + c)$. Consider the set

$$G = \{z \in \mathbb{N} \mid \text{if } x, y \in \mathbb{N} \text{ then } (x + y) + z = x + (y + z)\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. Suppose $z \in G$. Consider

$$(x + y) + 1 = s(x + y) = x + s(y) = x + (y + 1)$$

Thus $1 \in G$. Futher let $x, y, z \in \mathbb{N}$, and consider

$$(x + y) + s(z) = s((x + y) + z)$$

By our hypothesis on $z$, $(x + y) + z = x + (y + z)$ so

$$s((x + y) + z) = s(x + (y + z)) = x + s(y + z) = x + (y + s(z))$$

So $s(z) \in G$. Thus $G = \mathbb{N}$ by part (c) of the Peano Postulates. ∎

*Proof.* Let $a \in \mathbb{N}$. We must show $1 + a = s(a) = a + 1$. Let $x \in \mathbb{N}$. Suppose $x = 1$. Then

$$1 + x = 1 + 1 = s(1) = x + 1$$

Suppose $x > 1$. By Lemma 1.2.3 there exists $y \in \mathbb{N}$ such that $s(y) = x$. First note $y + (1 + 1) = (y + 1) + 1$ by Theorem 1.2.7 part (2). Then

$$1 + s(y) = 1 + (y + 1) = s(y + 1) = y + s(1) = y + (1 + 1) = (y + 1) + 1 = s(y) + 1$$

Thus $1 + x = x + 1$. ∎

*Proof.* Let $a, b \in \mathbb{N}$. We must show $a + b = b + a$. Consider the set

$$G = \{x \in \mathbb{N} \mid \text{if } y \in \mathbb{N} \text{ then } x + y = y + x\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. Let $x \in \mathbb{N}$. By Theorem 1.2.7 part (3), $1 + x = x + 1$. Thus $1 \in G$. Now suppose $x \in G$. Let $y \in \mathbb{N}$. First note by Theorem 1.2.7 part (2), $1 + (x + y) = (1 + x) + y$. Consider

$$y + s(x) = s(y + x) = s(x + y) \text{ hypothesis on } x = 1 + (x + y) = (1 + x) + y = s(x) + y$$

So $s(x) \in G$. Thus $G = \mathbb{N}$ by part (c) of the Peano Postulates. ∎

*Proof.* Let $x, y \in \mathbb{N}$. We must show $a \cdot 1 = 1 \cdot a$. ∎

*Proof.* STILL NEED TO DO THIS!!! Let $a \in \mathbb{N}$. We must show $a \cdot 1 = a = 1 \cdot a$. Consider the set

$$G = \{x \in \mathbb{N} \mid x \cdot 1 = x = 1 \cdot x\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. Consider $x \cdot 1 = 1 \cdot 1 = x \cdot 1 = x$ by Theorem 1.2.6 part (a). Suppose $x \in \mathbb{N}$ and assume $x \in G$. ∎

*Proof.* Let $a, b, c \in \mathbb{N}$. We must show $(a + b)c = ac + bc$. ∎

*Proof.* Let $a, b, c \in \mathbb{N}$. We must show $ab = ba$. ∎

*Proof.* Let $a, b, c \in \mathbb{N}$. We must show $c(a + b) = ca + cb$. ∎

*Proof.* Let $a, b, c \in \mathbb{N}$. We must show $(ab)c = a(bc)$. ∎

*Proof.* Let $a, b, c \in \mathbb{N}$. We must show $ab = 1$ if and only if $a = 1 = b$. ∎

**Problem 3**

Let $a, b \in \mathbb{N}$. Suppose $a < b$. Prove that there is a unique $p \in \mathbb{N}$ such that $a + p = b$

*Proof.* We first prove uniqueness. Let $a, b \in \mathbb{N}$ such that $a < b$. Suppose $x, y \in \mathbb{N}$ such that $a + x = b$ and $a + y = b$. Then $a + x = a + y$. By Theorem 1.2.7 part (4), $x + a = y + a$. Then by Theorem 1.2.7 part (1), $x = y$.

We now prove existence. Since $a < b$ it follows that $a + 1 < b$ by Theorem 1.2.9 part (11). ∎

**Problem 4**

Prove Theorem 1.2.9 (1) (3) (4) (5) (11).

*Proof.* Let $a \in \mathbb{N}$. We must show $a \leq a$, and $a \not< a$, and $a < a + 1$.

To show $a \leq a$ consider $a = a$ thus $a \leq a$. To show $a \not< a$, first, suppose $a < a$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = a$ contradicting Theorem 1.2.7 part (6). To show $a < a + 1$ consider $s(a) = a + 1 = a + 1$ thus $a < a + 1$. ∎

*Proof.* Let $a, b, c \in \mathbb{N}$. We must show if $a < b$ and $b < c$, then $a < c$; if $a \leq b$ and $b < c$, then $a < c$; if $a < b$ and $b \leq c$, then $a < c$; if $a \leq b$ and $b \leq c$, then $a \leq c$.

① Suppose $a < b$ and $b < c$. By definition of $<$, there exists $p_1, p_2 \in \mathbb{N}$ such that $a + p_1 = b$ and $b + p_2 = c$. Then $b + p_2 = (a + p_1) + p_2 = c$. By definition of $<$, $a < c$.

② Suppose $a \leq b$ and $b < c$. By definition of $\leq$, either $a = b$ or $a < b$. Suppose $a < b$. By ①, $a < c$. Suppose $a = b$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $b + p = c$. Then $b + p = a + p = c$. By definition of $<$, $a < c$.

③ Suppose $a < b$ and $b \leq c$. By definition of $\leq$, either $b = c$ or $b < c$. Suppose $b < c$. By ①, $a < c$. Suppose $b = c$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = b$. Then $b = a + p = c$ thus, by definition of $<$, $a < c$.

Suppose $a \leq b$ and $b \leq c$. There are four cases:

1. Suppose $a < b$ and $b < c$. By ①, $a < c$.
2. Suppose $a \leq b$ and $b < c$. By ②, $a < c$.
3. Suppose $a < b$ and $b \leq c$. By ③, $a < c$.
4. Suppose $a \leq b$ and $b \leq c$. There are four cases:
   (a) Suppose $a = b$ and $b < c$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $b + p = c$. Then $b + p = a + p = c$ so $a < c$.
   (b) Suppose $a < b$ and $b < c$. By ①, $a < c$.
   (c) Suppose $a = b$ and $b = c$. Clearly $a = b = c$ thus $a = c$.
   (d) Suppose $a < b$ and $b = c$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = b$. Then $a + p = b = c$ so $a < c$.

Thus either $a < c$ or $a = c$ thus, by definition of $\leq$, $a \leq c$. ∎

*Proof.* Let $a, b, c \in \mathbb{N}$. We must show if $a < b$ if and only if $a + c < b + c$.

Suppose $a < b$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = b$. By Theorem 1.2.7 part (1), $(a + p) + c = b + c$. By Theorem 1.2.7 part (2), $a + (p + c) = b + c$. By Theorem 1.2.7 part (4), $a + (c + p) = b + c$. By Theorem 1.2.7 part (2), $(a + c) + p = b + c$. Thus by definition of $<$, $a + c < b + c$.

Suppose $a+c < b+c$. There exists $p \in \mathbb{N}$ such that $(a+c)+p = b+c$. By Theorem 1.2.7 part (4), $p+(a+c) = b+c$. By Theorem 1.2.7 part (2), $(p+a)+c = b+c$. By Theorem 1.2.7 part (1), $p+a = b$ so, by Theorem 1.2.7 part (4), $a+p = b$. Thus by definition of $<$, $a < b$. ∎

*Proof.* Let $a, b, c \in \mathbb{N}$. We must show $a < b$ if and only if $ac < bc$.

Suppose $a < b$. Now suppose $ac > bc$. By definition of $<$, there exists $p_1, p_2 \in \mathbb{N}$ such that $a + p_1 = b$ and $bc + p_2 = ac$. Then, by Theorem 1.2.8 part (8), $(a + p_1)c + p_2 = ac + p_1c + p_2 = ac$. By definition of $<$, $ac < ac$ contradicting Theorem 1.2.9 part (1).

Suppose $ac < bc$. Now suppose $a > b$. By definition of $<$, there exists $p$ such that $b + p = a$. Then, by Theorem 1.2.8 part (8), $(b + p)c = bc$. So $bc + pc = bc$. From this we deduce that $bc < bc$ contradicting Theorem 1.2.9 part (1). ∎

*Proof.* Let $a, b \in \mathbb{N}$. We must show $a < b$ if and only if $a + 1 \leq b$.

Suppose $a < b$. Now suppose $a + 1 > b$. By definition of $<$, there exists $p_1, p_2 \in \mathbb{N}$ such that $b + p_1 = a + 1$ and $a + p_2 = b$. Then $b + p_1 = (a + p_2) + p_1 = a + 1$. By Theorem 1.2.7 part (4), $p_1 + (a + p_2) = 1 + a$. By Theorem 1.2.7 part (4), $p_1 + (p_2 + a) = 1 + a$. By Theorem 1.2.7 part (2), $(p_1 + p_2) + a = 1 + a$. By Theorem 1.2.7 part (1), $p_1 + p_2 = 1$ contradicting Theorem 1.2.7 part (5).

Suppose $a + 1 \leq b$. By definition of $\leq$, either $a + 1 = b$ or $a + 1 < b$.

Suppose $a + 1 < b$. Now suppose $a > b$. By definition of $<$, there exists $p_1, p_2 \in \mathbb{N}$ such that $(a + 1) + p_1 = b$ and $b + p_2 = a$. Then $(a + 1) + p_1 = ((b + p_2) + 1) + p_1 = b$. By definition of $<$, $b < b$ contradicting Theorem 1.2.9 part (1).

Suppose $a + 1 = b$. Now suppose $a > b$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $b + p = a$. Then $a + 1 = b + p + 1 = b$. By definition of $<$, $b < b$ contradicting Theorem 1.2.8 part (6). ∎

## Problem 5

Let $a, b \in \mathbb{N}$. Prove that if $a + a = b + b$, then $a = b$.

*Proof.* Suppose $a + a = b + b$. First, by Theorem 1.2.6 part (a), $a + a = a \cdot 1 + a \cdot 1$. Then, by Theorem 1.2.7 part (10), $a \cdot 1 + a \cdot 1 = a(1 + 1) = a \cdot 2$. Similarly $b + b = b \cdot 2$. Then, by Theorem 1.2.7 part (12), since $a \cdot 2 = b \cdot 2$, $a = b$. ∎

## Problem 6

Let $b \in \mathbb{N}$. Prove that

$$\{n \in \mathbb{N} \mid 1 \leq n \leq b\} \cup \{n \in \mathbb{N} \mid b + 1 \leq n\} = \mathbb{N}$$

$$\{n \in \mathbb{N} \mid 1 \leq n \leq b\} \cap \{n \in \mathbb{N} \mid b + 1 \leq n\} = \emptyset$$

*Proof.* Let $A = \{n \in \mathbb{N} \mid 1 \leq n \leq b\}$ and $B = \{n \in \mathbb{N} \mid b+1 \leq n\}$. It is clear that $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$. Thus $A \cup B \subseteq \mathbb{N}$. Now let $x$ be an arbitrary element in $\mathbb{N}$. By Theorem 1.2.9 part (6), either $x < b$, $x = b$, or $x > b$. Suppose $x < b$. Then $x \in A$, so $x \in A \cup B$. Suppose $x = b$. Then $x \in A$, so $x \in A \cup B$. Suppose $x > b$. Then $x \in B$, so $x \in A \cup B$. Therefore $\mathbb{N} \subseteq A \cup B$. It follows that $A \cup B = \mathbb{N}$. ∎

*Proof.* Suppose $\{n \in \mathbb{N} \mid 1 \leq n \leq b\} \cap \{n \in \mathbb{N} \mid b + 1 \leq n\} \neq \emptyset$. Let $x \in \{n \in \mathbb{N} \mid 1 \leq n \leq b\} \cap \{n \in \mathbb{N} \mid b + 1 \leq n\}$. Then $1 \leq x \leq b$ and $b+1 \leq x$. By Theorem 1.2.9 part (3), $b+1 \leq x \leq b$ contradicting Theorem 1.2.9 part (9). ∎

## Problem 7

Let $A \subseteq N$ be a set. The set $A$ is **closed** if $a \in A$ implies $a+1 \in A$. Suppose $A$ is closed.

1. Prove that if $a \in A$ and $n \in \mathbb{N}$, then $a+n \in A$.
2. Prove that if $a \in A$, then $\{x \in \mathbb{N} \mid x \geq a\} \subseteq A$.

*Proof.* If $A = \emptyset$ then clearly the implication vacuously holds. Suppose $A \neq \emptyset$. Consider the set

$$G = \{x \in \mathbb{N} \mid a + x \in A\}.$$

We will show $G = \mathbb{N}$, proving our implication. Now, since $a \in A$ and $A$ is closed, $a+1 \in A$, thus $1 \in G$. Suppose $x \in \mathbb{N}$ and $x \in G$. Then consider $a + s(x) = a + (x+1)$. By Theorem 1.2.7 part (2), $a + (x+1) = (a+x)+1$. By our hypothesis, $a + x \in A$. But since $A$ is closed, $(a+x)+1 \in A$. Thus $s(x) \in G$. By the part (c) of the Peano Postulates, we conclude that $G = \mathbb{N}$. ∎

*Proof.* Suppose $a \in A$. Let $x \in \mathbb{N}$ such that $x \geq a$. Either $x = a$ or $a < x$. Suppose $x = a$, then trivially $x = a \in A$. Suppose $a < x$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = x$. By the previous proof, $a + p = x \in A$. ∎

## Problem 8

Suppose that the set $\mathbb{N}$ together with the element $1 \in mathbbN$ and the function $s : \mathbb{N} \to \mathbb{N}$, and the set $\mathbb{N}'$ together with the element $1' \in \mathbb{N}$ and the function $s' : \mathbb{N}' \to \mathbb{N}'$, both satisfy the Peano Postulates. Prove that there is a bijective function $f : \mathbb{N} \to \mathbb{N}'$ such that $f(1) = 1'$ and $f \circ s = s' \circ f$. The existence of such a bijective function

## Extra Problem

Show the Peano axioms are independent. That is, for any two Peano axioms, find a structure that satisfies them but not the third. You may assume the regular math of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.