The Real Numbers and Real Analysis
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1 MacNeille

Problem 1

Show that the divisors of 30, the power set $\{1, 2, 3\}$ all have a cube Hasse Diagram.

Problem 2 breakable

Find the Hasse diagrams for all ordered sets of $1, 2, 3$ and 4 elements.

Problem 3

What numbers have divisor-diagrams consisting of cubes.

Solution: Any number n such that $n = abc$ where a, b, c are distinct primes. The nodes are thus

$$\{1, a, b, c, ab, ac, bc, n\}$$

And clearly

$$1 \prec a, b, c \text{ and } a \prec ab, ac \text{ and } b \prec ab, bc \text{ and } c \prec ac, bc \text{ and } ab, ac, bc \prec n$$

Thus giving us our 12 edges.

Problem 4

Show that each power set is a complete lattice.

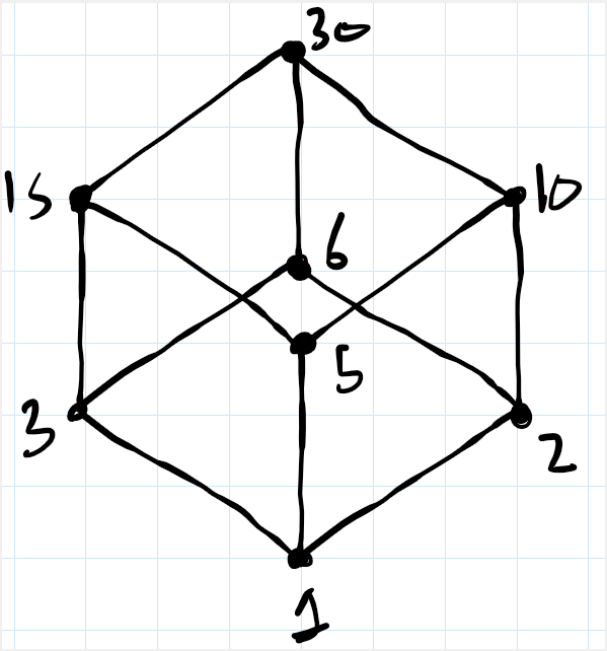


Figure 1: Exercise 1a

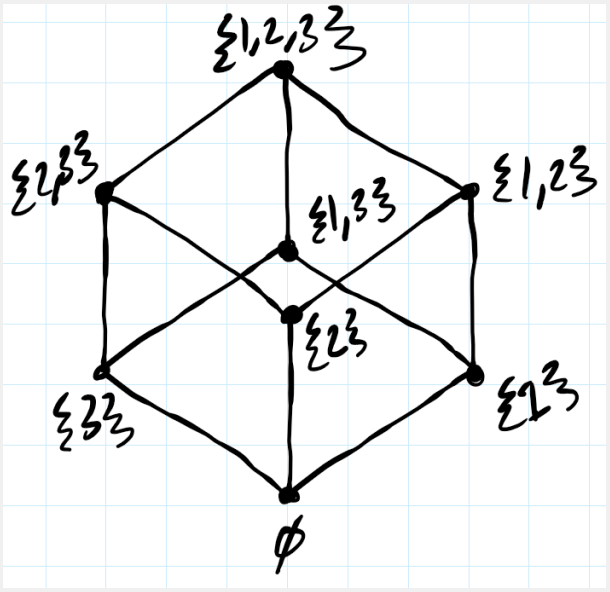
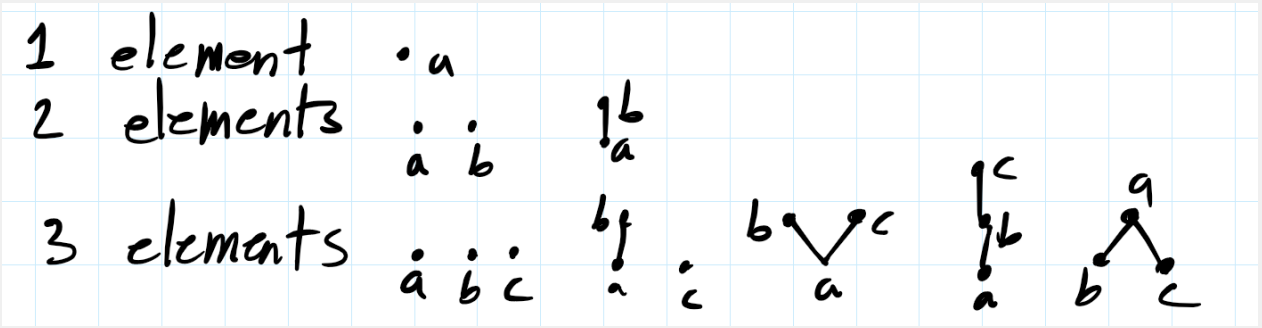


Figure 2: Exercise 1b



Proof. Let S be an arbitrary set. Let $A \in \mathscr{P}(\mathscr{P}(S))$.

We first show there exists a supremum of A. Let $C = \bigcup_{X \in A} X$. Let $X \in A$. Since $C = \bigcup_{X \in A} X$, $X \subseteq C$. Thus C is an upper bound of A. For contradiction, suppose there exists an upper bound $E \subset C$ of A. Let $x \in C \setminus E$. Then $x \in X$ for some $X \in A$. Since E is an upper bound, $X \subseteq E$, so $x \in E$, which is a contradiction. Thus $C = \bigvee A$.

We now show there exists an infimum of A. Let $D = \bigcap_{X \in A} X$. Let $X \in A$. Since $D = \bigcap_{X \in A} X$, $D \subseteq X$. Thus D is a lower bound of A. For contradiction, suppose there exists a lower bound $F \supset D$ of A. Let $x \in F \setminus D$. Then $x \notin X$ for some $X \in A$. Since F is a lower bound, $F \subseteq X$, so $x \in X$, which is a contradiction. Thus $D = \bigwedge A$.

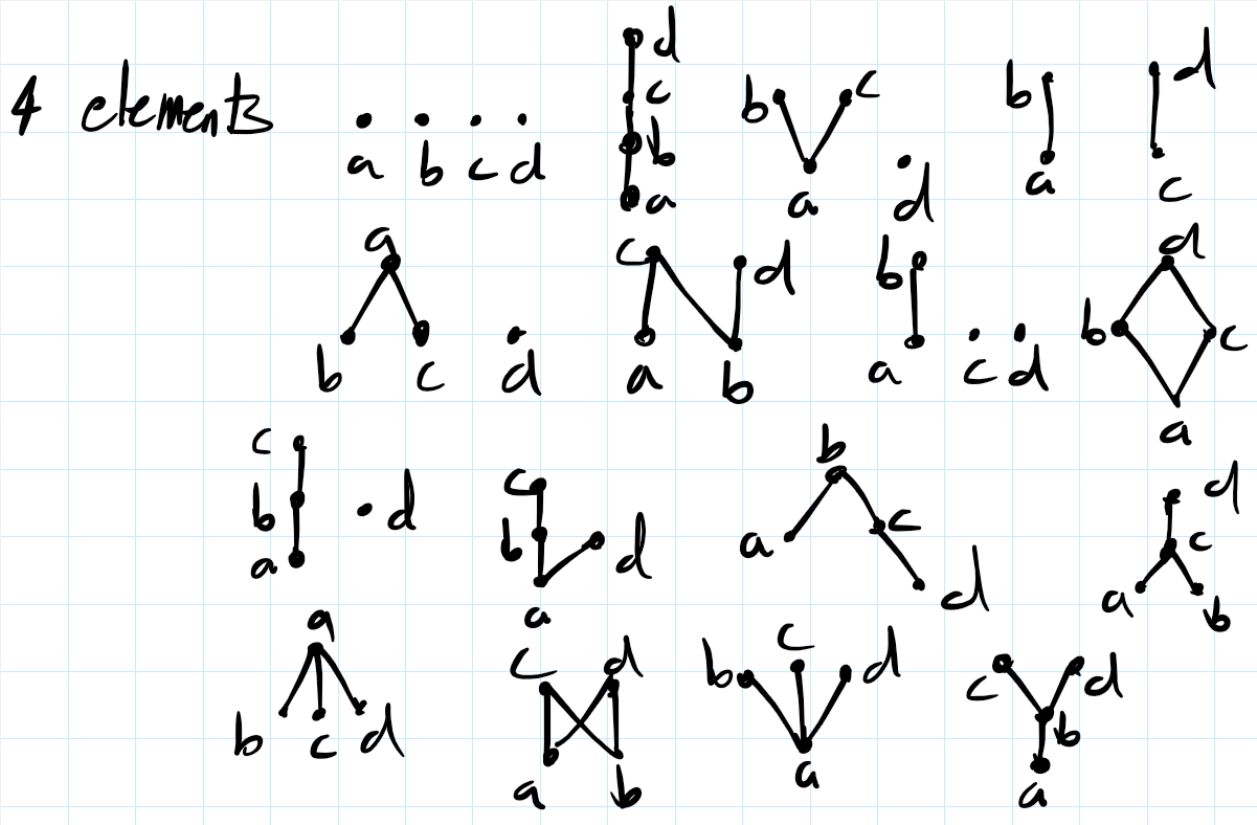
Therefore, $(\mathscr{P}(S), \subseteq)$ is a complete lattice. ∎

Problem 5

Let $X = \mathbb{N}$ and take as a relation $n \leq m$ if $n \mid m$. Show that this is a lattice.

Proof. Let $S \in \mathscr{P}(X)$ be a set with n elements, where $n \neq 0$.

We now show there exists an infimum of S. Let $C = \gcd(a_1, a_2, \ldots, a_n)$, where $a_i \in S$ for $1 \leq i \leq n$. Clearly $C \leq a_i$, so C is a lower bound of S. For contradiction suppose there exists a lower bound D of S with $D > C$. Then



$D \mid a_i$ for $1 \leq i \leq n$, so $D \mid C$, contradicting that C is the gcd. Thus $C = \bigwedge S$.

We now show there exists a supremum of S. Let $D = \operatorname{lcm}(a_1, a_2, \ldots, a_n)$, where $a_i \in S$ for $1 \leq i \leq n$. Clearly $a_i \leq D$, so D is an upper bound of S. For contradiction suppose there exists an upper bound E of S with $E < D$. Then $a_i \mid E$ for $1 \leq i \leq n$, so $D \mid E$, contradicting that D is the lcm. Thus $D = \bigvee S$.

Thus $(X, n \leq m \text{ if } n \mid m)$ is a lattice. ∎

Problem 6

Find all lattices with $1, 2, 3, 4, 5$ element(s).

Problem 7

Show that each totally ordered set is a lattice. Is it also a complete lattice?

Proof. Suppose S is a totally ordered set. We proceed using induction on the size of a finite subset of S.

(**Base Case**) Suppose $C \in \mathscr{P}(S)$ such that $C = \{a\}$. Clearly $a \leq a$ and $a \geq a$, thus $a = \bigvee C$ and $a = \bigwedge C$.

(**Induction Step**) Suppose the theorem holds for all finite subsets of size $n \in \mathbb{N}$. Consider $C \in \mathscr{P}(S)$ such that C has $n + 1$ elements. Let $C' = C \setminus \{a\}$ for some $a \in C$. By our induction hypothesis, C' has an infimum and supremum. Let $I = \bigwedge C'$ and $K = \bigvee C'$. Since S is totally ordered we can define the following

$$\bigwedge C = I \wedge a, \quad \bigvee C = K \vee a,$$

which gives the infimum and supremum of C. ∎

Solution (b): No.

Problem 8

Let $a \in X$, prove that $\{a\}^{ul} = \{x \in X \mid x \leq a\}$.

2 Construction of the Real Numbers

2.1 Axioms for the Natural Numbers

Problem 1

Fill in the missing details in the proof of Theorem 1.2.6.

Proof. We must show the uniquness of the binary operation $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that satisfies the following two properties for all $n, m \in \mathbb{N}$.

a. $n \cdot 1 = n$.

b. $n \cdot s(m) = (n \cdot m) + n$.

Suppose there are two binary operations $\cdot$ and $\times$ on $\mathbb{N}$ that satisfy the two properties for all $n, m \in \mathbb{N}$. Let

$$G = \{x \in \mathbb{N} \mid n \cdot x = n \times x \text{ for all } n \in \mathbb{N}\}$$

We will prove that $G = \mathbb{N}$, which will imply that $\cdot$ and $\times$ are the same binary operation. It is clear that $G \subseteq \mathbb{N}$. By part (a) applied to each of $\cdot$ and $\times$ we see that $n \cdot 1 = n = n \times 1$ for all $n \in \mathbb{N}$ and thus $1 \in G$. Now let $q \in G$. Let $n \in \mathbb{N}$. Then $n \cdot q = n \times q$ by hypothesis on q. It then follows from part (b) that $n \cdot s(q) = (n \cdot q) + n = (n \times q) + n = n \times s(q)$. thus $s(q) \in G$. By part (c) of the Peano Postulates we conclude that $G = \mathbb{N}$. ∎

Proof. We must show the two properties hold. Now, $n \cdot 1 = g_n(1) = n$, which is part (a), and $n \cdot s(m) = g_n(s(m)) = (g_n \circ s)(m) = (h_n \circ g_n)(m) = g_n(m) + n = (n \cdot m) + n$, which is part (b). ∎

Problem 2

Prove Theorem 1.2.7 (2) (3) (4) (7) (8) (9) (10) (11) (13).

Proof. Let $a, b, c \in \mathbb{N}$. We must show $(a + b) + c = a + (b + c)$. Consider the set

$$G = \{z \in \mathbb{N} \mid \text{if } x, y \in \mathbb{N} \text{ then } (x + y) + z = x + (y + z)\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. Suppose $z \in G$. Consider

$$(x + y) + 1 = s(x + y) = x + s(y) = x + (y + 1)$$

Thus $1 \in G$. Futher let $x, y, z \in \mathbb{N}$, and consider

$$(x + y) + s(z) = s((x + y) + z)$$

By our hypothesis on z, $(x + y) + z = x + (y + z)$ so

$$s((x + y) + z) = s(x + (y + z)) = x + s(y + z) = x + (y + s(z))$$

So $s(z) \in G$. Thus $G = \mathbb{N}$ by part (c) of the Peano Postulates. ∎

Proof. Let $a \in \mathbb{N}$. We must show $1 + a = s(a) = a + 1$. Consider the set

$$G = \{a \in \mathbb{N} \mid 1 + a = s(a) = a + 1\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. Let $a \in \mathbb{N}$ such that $a = 1$.

$$1 + a = s(a) = s(1) = 1 + 1 = a + 1$$

Thus $1 \in G$. Suppose $x \in \mathbb{N}$ and $x \in G$. By our hypothesis, $1 + x = x + 1$. Then

$$1 + s(x) = s(1 + x) = s(x + 1) = s(x) + 1$$

So $s(x) \in G$. Thus $G = \mathbb{N}$ by part (c) of the Peano Postulates. ∎

Proof. Let $a, b \in \mathbb{N}$. We must show $a + b = b + a$. Consider the set

$$G = \{x \in \mathbb{N} \mid \text{if } y \in \mathbb{N} \text{ then } x + y = y + x\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. Let $x \in \mathbb{N}$. By Theorem 1.2.7 part (3), $1 + x = x + 1$. Thus $1 \in G$. Now suppose $x \in G$. Let $y \in \mathbb{N}$. First note by Theorem 1.2.7 part (2), $1 + (x + y) = (1 + x) + y$. Consider

$$y + s(x) = s(y + x) = s(x + y) \text{ hypothesis on } x = 1 + (x + y) = (1 + x) + y = s(x) + y$$

So $s(x) \in G$. Thus $G = \mathbb{N}$ by part (c) of the Peano Postulates. ∎

Proof. Let $a \in \mathbb{N}$. We must show $a \cdot 1 = a = 1 \cdot a$. Consider the set

$$G = \{x \in \mathbb{N} \mid x \cdot 1 = x = 1 \cdot x\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. Consider

$$\begin{aligned} x \cdot 1 &= x && \text{Theorem 1.2.6 part (a)} \\ &= 1 \\ &= 1 \cdot 1 \\ &= x \cdot 1 \end{aligned}$$

Thus $1 \in G$. Consider

$$\begin{aligned} s(x) \cdot 1 &= s(x) && \text{Theorem 1.2.6 part (a)} \\ &= x + 1 && \text{Theorem 1.2.5 part (a)} \\ &= x \cdot 1 + 1 && \text{Theorem 1.2.6 part (a)} \\ &= 1 \cdot x + 1 && \text{Induction hypothesis} \\ &= 1 \cdot s(x) && \text{Theorem 1.2.6 part (b)} \end{aligned}$$

So $s(x) \in G$. Thus $G = \mathbb{N}$ by part (c) of the Peano Postulates. ∎

Proof. Let $a, b, c \in \mathbb{N}$. We must show $(a + b)c = ac + bc$. Consider the set

$$G = \{c \in \mathbb{N} \mid \text{if } a, b \in \mathbb{N} \text{ then } (a + b)c = ac + bc\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. Let $a, b \in \mathbb{N}$. Then

$$\begin{aligned} (a + b)1 &= a + b && \text{(Theorem 1.2.6 part (a))} \\ &= a \cdot 1 + b \cdot 1 && \text{(Theorem 1.2.6 part (a))} \end{aligned}$$

Suppose $a, b, c \in \mathbb{N}$ and $c \in G$. Then

$$\begin{aligned} (a + b) \cdot s(c) &= ((a + b)c) + (a + b) && \text{(Theorem 1.2.6 part (a))} \\ &= (ac + bc + a + b) && \text{(Induction Hypothesis)} \\ &= (ac + a + bc + b) && \text{(Theorem 1.2.7 part (4))} \\ &= a \cdot s(c) + b \cdot s(c) && \text{(Theorem 1.2.5 part (a))} \end{aligned}$$

So $s(c) \in G$. Thus $G = \mathbb{N}$ by part (c) of the Peano Postulates. ∎

Proof. Let $a, b \in \mathbb{N}$. We must show $ab = ba$. Consider the set

$$G = \{a \in \mathbb{N} \mid \text{if } b \in \mathbb{N} \text{ then } ab = ba\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. By Theorem 1.2.7 part (7), $a \cdot 1 = 1 \cdot a$. Thus $1 \in G$. Suppose $a, b \in \mathbb{N}$ and $a \in G$.

$$\begin{aligned}
s(a) \cdot b &= (a+1)b && \text{(Theorem 1.2.5 part (a))} \\
&= ab + 1b && \text{(Theorem 1.2.7 part (8))} \\
&= ab + b1 && \text{(Theorem 1.2.7 part (7))} \\
&= ab + b && \text{(Theorem 1.2.6 part (7))} \\
&= ba + b && \text{(Induction Hypothesis)} \\
&= b \cdot s(a) && \text{(Theorem 1.2.6 part (b))}
\end{aligned}$$

So $s(a) \in G$. Thus $G = \mathbb{N}$ by part (c) of the Peano Postulates. ∎

Proof. Let $a, b \in \mathbb{N}$. We must show $c(a+b) = ca + cb$. By Theorem 1.2.7 part (9), $c(a+b) = (a+b)c$. By Theorem 1.2.7 part (8), $(a+b)c = ac + bc$. By Theorem 1.2.7 part (9), $ac + bc = ca + cb$. ∎

Proof. Let $a, b, c \in \mathbb{N}$. We must show $(ab)c = a(bc)$. ∎

Proof. Let $a, b, c \in \mathbb{N}$. We must show $(ab)c = a(bc)$. Consider the set

$$G = \{c \in \mathbb{N} \mid \text{if } a, b \in \mathbb{N} \text{ then } (ab)c = a(bc)\}$$

We will show $G = \mathbb{N}$. Clearly $G \subseteq \mathbb{N}$. We first show $1 \in G$. Let $a, b \in \mathbb{N}$. Then

$$(ab)1 = ab \text{ (Theorem 1.2.7 part (7))} = a(b \cdot 1) \text{ (Theorem 1.2.6 part (a))}$$

Thus $1 \in G$. Suppose $a, b, c \in \mathbb{N}$ and $c \in G$. Then

$$\begin{aligned}
(ab) \cdot s(c) &= (ab)(c+1) && \text{(Theorem 1.2.5 part (a))} \\
&= (ab)c + (ab)1 && \text{(Theorem 1.2.7 part (10))} \\
&= a(bc) + (ab)1 && \text{(Induction Hypothesis)} \\
&= a(bc) + ab && \text{(Theorem 1.2.7 part (7))} \\
&= a(bc + b) && \text{(Theorem 1.2.7 part (8))} \\
&= a(bc + b \cdot 1) && \text{(Theorem 1.2.7 part (7))} \\
&= a(b(c+1)) && \text{(Theorem 1.2.7 part (8))} \\
&= a(b \cdot s(c)) && \text{(Theorem 1.2.5 part (a))}
\end{aligned}$$

So $s(c) \in G$. Thus $G = \mathbb{N}$ by part (c) of the Peano Postulates. ∎

Proof. Let $a, b \in \mathbb{N}$. We must show $ab = 1$ if and only if $a = 1 = b$.

Suppose $ab = 1$. For contradiction, suppose $a \neq 1$ or $b \neq 1$. Suppose $a \neq 1$. By Lemma 1.2.3 there exists $c \in \mathbb{N}$ such that $s(c) = a$. Then

$$ab = s(c)b = (c+1)b \text{ (Theorem 1.2.5 part (a))} = cb + b \text{ (Theorem 1.2.7 part (8))} = 1$$

Contradicting Theorem 1.2.7 part (5). Suppose $b \neq 1$. By Lemma 1.2.3 there exists $c \in \mathbb{N}$ such that $s(c) = b$. Then

$$ab = a \cdot s(c) = a(c+1) \text{ (Theorem 1.2.5 part (a))} = ac + a \text{ (Theorem 1.2.7 part (10))} = 1$$

Contradicting Theorem 1.2.7 part (5).

Suppose $a = 1 = b$. Then $ab = a \cdot 1 = a = 1$ by Theorem 1.2.6 part (a). ∎

Problem 3

Let $a, b \in \mathbb{N}$. Suppose $a < b$. Prove that there is a unique $p \in \mathbb{N}$ such that $a + p = b$

Proof. We first prove uniqueness. Let $a, b \in \mathbb{N}$ such that $a < b$. Suppose $x, y \in \mathbb{N}$ such that $a + x = b$ and $a + y = b$. Then $a + x = a + y$. By Theorem 1.2.7 part (4), $x + a = y + a$. Then by Theorem 1.2.7 part (1), $x = y$.

We now prove existence. Since $a < b$, by definition of $<$ there exists $p \in \mathbb{N}$ such that $a + p = b$. ∎

Problem 4

Prove Theorem 1.2.9 (1) (3) (4) (5) (11).

Proof. Let $a \in \mathbb{N}$. We must show $a \leq a$, and $a \not< a$, and $a < a + 1$.

To show $a \leq a$, suppose for contradiction $a = a$. Thus $a \leq a$. To show $a \not< a$, first, suppose $a < a$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = a$ contradicting Theorem 1.2.7 part (6). To show $a < a + 1$ consider $s(a) = a + 1 = a + 1$ thus $a < a + 1$. ∎

Proof. Let $a, b, c \in \mathbb{N}$. We must show if $a < b$ and $b < c$, then $a < c$; if $a \leq b$ and $b < c$, then $a < c$; if $a < b$ and $b \leq c$, then $a < c$; if $a \leq b$ and $b \leq c$, then $a \leq c$.

① Suppose $a < b$ and $b < c$. By definition of $<$, there exists $p_1, p_2 \in \mathbb{N}$ such that $a + p_1 = b$ and $b + p_2 = c$. Then $b + p_2 = (a + p_1) + p_2 = c$. By definition of $<$, $a < c$.

② Suppose $a \leq b$ and $b < c$. By definition of $\leq$, either $a = b$ or $a < b$. Suppose $a < b$. By ①, $a < c$. Suppose $a = b$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $b + p = c$. Then $b + p = a + p = c$. By definition of $<$, $a < c$.

③ Suppose $a < b$ and $b \leq c$. By definition of $\leq$, either $b = c$ or $b < c$. Suppose $b < c$. By ①, $a < c$. Suppose $b = c$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = b$. Then $b = a + p = c$ thus, by definition of $<$, $a < c$.

Suppose $a \leq b$ and $b \leq c$. There are four cases:

1. Suppose $a < b$ and $b < c$. By ①, $a < c$.
2. Suppose $a \leq b$ and $b < c$. By ②, $a < c$.
3. Suppose $a < b$ and $b \leq c$. By ③, $a < c$.
4. Suppose $a \leq b$ and $b \leq c$. There are four cases:
   (a) Suppose $a = b$ and $b < c$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $b + p = c$. Then $b + p = a + p = c$ so $a < c$.
   (b) Suppose $a < b$ and $b < c$. By ①, $a < c$.
   (c) Suppose $a = b$ and $b = c$. Clearly $a = b = c$ thus $a = c$.
   (d) Suppose $a < b$ and $b = c$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = b$. Then $a + p = b = c$ so $a < c$.

Thus either $a < c$ or $a = c$ thus, by definition of $\leq$, $a \leq c$. ∎

Proof. Let $a, b, c \in \mathbb{N}$. We must show if $a < b$ if and only if $a + c < b + c$.

Suppose $a < b$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = b$. By Theorem 1.2.7 part (1), $(a + p) + c = b + c$. By Theorem 1.2.7 part (2), $a + (p + c) = b + c$. By Theorem 1.2.7 part (4), $a + (c + p) = b + c$. By Theorem 1.2.7 part (2), $(a + c) + p = b + c$. Thus by definition of $<$, $a + c < b + c$.

Suppose $a+c < b+c$. There exists $p \in \mathbb{N}$ such that $(a+c)+p = b+c$. By Theorem 1.2.7 part (4), $p+(a+c) = b+c$. By Theorem 1.2.7 part (2), $(p+a)+c = b+c$. By Theorem 1.2.7 part (1), $p+a = b$ so, by Theorem 1.2.7 part (4), $a+p = b$. Thus by definition of $<$, $a < b$. ∎

Proof. Let $a, b, c \in \mathbb{N}$. We must show $a < b$ if and only if $ac < bc$.

Suppose $a < b$. For contradiction, suppose $ac \geq bc$. By definition of $\geq$, either $ac = bc$ or $ac > bc$.

Suppose $ac = bc$. By Theorem 1.2.7 part (12), $a = b$. But $a = b < b$ contradicting Theorem 1.2.9 part (1).

Suppose $ac > bc$. By definition of $<$, there exists $p_1, p_2 \in \mathbb{N}$ such that $a + p_1 = b$ and $bc + p_2 = ac$. Then $bc + p_2 = (a + p_1)c + p_2 = ac + p_1c + p_2$ (by Theorem 1.2.8 part (8) for distributivity) $= ac$. By definition of $<$, $ac < ac$ contradicting Theorem 1.2.9 part (1).

Suppose $ac < bc$. For contradiction, suppose $a \geq b$. By definition of $\geq$, either $a = b$ or $a > b$

Suppose $a = b$. Then $ac = bc < bc$ which contradicts Theorem 1.2.9 part (1).

Suppose $a > b$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $b + p = a$. Then, by Theorem 1.2.8 part (8), $ac = (b + p)c = bc + pc$. By definition of $<$, $bc < ac$. ∎

Proof. Let $a, b \in \mathbb{N}$. We must show $a < b$ if and only if $a + 1 \leq b$.

Suppose $a < b$. For contradiction, suppose $a+1 > b$. By definition of $<$, there exists $p_2 \in \mathbb{N}$ such that $a+p_2 = b$. Since $a + 1 > b$, there exists $p_1 \in \mathbb{N}$ such that $b + p_1 = a + 1$. Then $b + p_1 = (a + p_2) + p_1 = a + 1$. By Theorem 1.2.7 part (4), $p_1 + (a + p_2) = 1 + a$. By Theorem 1.2.7 part (4), $p_1 + (p_2 + a) = 1 + a$. By Theorem 1.2.7 part (2), $(p_1 + p_2) + a = 1 + a$. By Theorem 1.2.7 part (1), $p_1 + p_2 = 1$ contradicting Theorem 1.2.7 part (5).

Suppose $a + 1 \leq b$. By definition of $\leq$, either $a + 1 = b$ or $a + 1 < b$.

Suppose $a + 1 = b$. By definition of $<$, $a < b$.

Suppose $a + 1 < b$. For contradiction, suppose $a \geq b$. By definition of $\geq$, either $a = b$ or $a > b$. Suppose $a = b$, then $a + 1 = b + 1 > b$ contradicting Theorem 1.2.7 part (6). Suppose $a > b$. By definition of $<$, there exists $p_1, p_2 \in \mathbb{N}$ such that $(a + 1) + p_1 = b$ and $b + p_2 = a$. Then $(a + 1) + p_1 = ((b + p_2) + 1) + p_1 = b$. By definition of $<$, $b < b$ contradicting Theorem 1.2.9 part (1). ∎

Problem 5

Let $a, b \in \mathbb{N}$. Prove that if $a + a = b + b$, then $a = b$.

Proof. Suppose $a + a = b + b$. First, by Theorem 1.2.6 part (a), $a + a = a \cdot 1 + a \cdot 1$. Then, by Theorem 1.2.7 part (10), $a \cdot 1 + a \cdot 1 = a(1 + 1) = a \cdot 2$. Similarly $b + b = b \cdot 2$. Then, by Theorem 1.2.7 part (12), since $a \cdot 2 = b \cdot 2$, $a = b$. ∎

Problem 6

Let $b \in \mathbb{N}$. Prove that

$$\{n \in \mathbb{N} \mid 1 \leq n \leq b\} \cup \{n \in \mathbb{N} \mid b + 1 \leq n\} = \mathbb{N}$$

$$\{n \in \mathbb{N} \mid 1 \leq n \leq b\} \cap \{n \in \mathbb{N} \mid b + 1 \leq n\} = \emptyset$$

Proof. Let $A = \{n \in \mathbb{N} \mid 1 \leq n \leq b\}$ and $B = \{n \in \mathbb{N} \mid b+1 \leq n\}$. It is clear that $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$. Thus $A \cup B \subseteq \mathbb{N}$. Now let x be an arbitrary element in $\mathbb{N}$. By Theorem 1.2.9 part (6), either $x < b$, $x = b$, or $x > b$. Suppose $x < b$. Then $x \in A$, so $x \in A \cup B$. Suppose $x = b$. Then $x \in A$, so $x \in A \cup B$. Suppose $x > b$. Then $x \in B$, so $x \in A \cup B$. Therefore $\mathbb{N} \subseteq A \cup B$. It follows that $A \cup B = \mathbb{N}$.

Suppose $A \cap B \neq \emptyset$. Let $x \in A \cap B$. Then $1 \leq x \leq b$ and $b + 1 \leq x$. By Theorem 1.2.9 part (3), $b + 1 \leq x \leq b$ contradicting Theorem 1.2.9 part (9). ∎

Problem 7

Let $A \subseteq N$ be a set. The set A is **closed** if $a \in A$ implies $a + 1 \in A$. Suppose A is closed.

1. Prove that if $a \in A$ and $n \in \mathbb{N}$, then $a + n \in A$.
2. Prove that if $a \in A$, then $\{x \in \mathbb{N} \mid x \geq a\} \subseteq A$.

Proof. If $A = \emptyset$ then clearly the implication vacuously holds. Suppose $A \neq \emptyset$. Consider the set

$$G = \{x \in \mathbb{N} \mid a + x \in A\}.$$

We will show $G = \mathbb{N}$, proving our implication. Now, since $a \in A$ and A is closed, $a+1 \in A$, thus $1 \in G$. Suppose $x \in \mathbb{N}$ and $x \in G$. Then consider $a + s(x) = a + (x + 1)$. By Theorem 1.2.7 part (2), $a + (x + 1) = (a + x) + 1$. By our hypothesis, $a + x \in A$. But since A is closed, $(a + x) + 1 \in A$. Thus $s(x) \in G$. By the part (c) of the Peano Postulates, we conclude that $G = \mathbb{N}$. ∎

Proof. Suppose $a \in A$. Let $x \in \mathbb{N}$ such that $x \geq a$. Either $x = a$ or $a < x$. Suppose $x = a$, then trivially $x = a \in A$. Suppose $a < x$. By definition of $<$, there exists $p \in \mathbb{N}$ such that $a + p = x$. By the previous proof, $a + p = x \in A$. ∎

Problem 8

Suppose that the set $\mathbb{N}$ together with the element $1 \in \mathbb{N}$ and the function $s : \mathbb{N} \to \mathbb{N}$, and the set $\mathbb{N}'$ together with the element $1' \in \mathbb{N}$ and the function $s' : \mathbb{N}' \to \mathbb{N}'$, both satisfy the Peano Postulates. Prove that there is a bijective function $f : \mathbb{N} \to \mathbb{N}'$ such that $f(1) = 1'$ and $f \circ s = s' \circ f$. The existence of such a bijective function.

Proof. We can apply Theorem 1.2.4 to the set $\mathbb{N}'$, the element $1'$ and the function $s' : \mathbb{N}' \to \mathbb{N}'$, to deduce that there is a unique function $f : \mathbb{N} \to \mathbb{N}'$ such that $f \circ s = s' \circ f$ and $f(1) = 1'$.

We can apply Theorem 1.2.4 again, to the set $\mathbb{N}$, the element 1 and the function $s : \mathbb{N} \to \mathbb{N}$, to deduce that there is a unique function $f' : \mathbb{N}' \to \mathbb{N}$ such that $f' \circ s' = s \circ f'$ and $f'(1') = 1$.

Now we must show f' is the inverse of f.

Consider $f' \circ f$. Let $x \in \mathbb{N}$.

Base case: $x = 1$.

$$(f' \circ f)(x) = f'(f(1)) = f'(1') = 1 = x$$

Inductive step: Suppose $x > 1$. By Lemma 1.2.3 there exists $y \in \mathbb{N}$ such that $s(y) = x$. Suppose for $y \in \mathbb{N}$ such that $y < x$, $(f' \circ f)(y) = y$. Then

$$\begin{aligned}
(f' \circ f)(x) &= f'(f(s(y))) \\
&= f'(s'(f(y))) && (\text{by } f \circ s = s' \circ f) \\
&= s(f'(f(y))) && (\text{by } f' \circ s' = s \circ f') \\
&= s(y) && y < x \\
&= x
\end{aligned}$$

Consider $f \circ f'$. Let $x' \in \mathbb{N}'$.

Base case: $x' = 1'$.

$$(f \circ f')(x') = f(f'(1')) = f(1) = 1' = x'$$

Inductive step: Suppose $x' > 1'$. By Lemma 1.2.3 there exists $y' \in \mathbb{N}'$ such that $s'(y') = x'$. Suppose for $y' \in \mathbb{N}'$ such that $y' < x'$, $(f \circ f')(y') = y'$. Then

$$\begin{aligned}
(f \circ f')(x') &= f(f'(s'(y'))) \\
&= f(s(f'(y'))) && (\text{by } f' \circ s' = s \circ f') \\
&= s'(f(f'(y'))) && (\text{by } f \circ s = s' \circ f) \\
&= s'(y') && (\text{induction hypothesis}) \\
&= x'
\end{aligned}$$

Since $(f' \circ f)(x) = x$ and $(f \circ f')(x') = x'$, we conclude that f' is the inverse of f. Thus f is bijective. ∎

Extra Problem

Show the Peano axioms are independent. That is, for any two Peano axioms, find a structure that satisfies them but not the third. You may assume the regular math of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.

Axiom 1 (Peano Postulates)**.** *There exists a set* $\mathbb{N}$ *with an element* $1 \in \mathbb{N}$ *and a function* $s : \mathbb{N} \to \mathbb{N}$ *that satisfy the following three properties.*

- ***a.*** *There is no* $n \in \mathbb{N}$ *such that* $s(n) = 1$.
- ***b.*** *The function* s *is injective.*
- ***c.*** *Let* $G \subseteq \mathbb{N}$. *Suppose that* $1 \in G$, *and that if* $g \in G$ *then* $s(g) \in G$. *Then* $G = \mathbb{N}$.

Proof. (**a., b.**) Let $s : \mathbb{N} \to \mathbb{N}$ be defined by $s(x) = x + 2$. Let $G = \{x \mid \exists k \in \mathbb{Z}, x = 2k + 1\}$. Clearly s is injective, $1 \in G$, and $G \subseteq \mathbb{N}$. But $G \neq \mathbb{N}$, and if $g \in G$ then $s(g) = g + 2 \in G$. Clearly **a., b.** hold while **c.** does not hold.

(**a., c.**) Let $M = \{1, p\}$ and let $s : M \to M$ be defined by $s(1) = p$ and $s(p) = p$. Clearly **a., c.** hold while **b.** does not hold.

(**b., c.**) Let $M = \{1, p\}$ and let $s : M \to M$ be defined by $s(1) = p$ and $s(p) = 1$. Clearly **b., c** hold while **a.** does not hold. ∎

2.2 Constructing the Integers

Problem 2

Complete the proof of Lemma 1.3.2. That is, prove that the relation $\sim$ is transitive.

Proof. Let $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$. Assume $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. By definition of $\sim$, $a + d = b + c$ and $c + f = d + e$. Then taking sums shows $a + d + c + f = b + c + d + e$. Cancelling terms $a + f = b + e$. Thus, by definition of $\sim$, $(a, b) \sim (f, e)$. Since $\sim$ is symmetric, $(a, b) \sim (e, f)$. ∎

Problem 3

Test Complete the proof of Lemma 1.3.4. That is, prove that $\cdot$ and $-$ for $\mathbb{Z}$ are well-defined. The proof for $\cdot$ is a bit more complicated than might be expected. [Use Exercise 1.2.5.]

Proof. Let $(a, b), (c, d), (x, y), (z, w) \in \mathbb{N} \times \mathbb{N}$. Suppose $(a, b) \sim (c, d)$ and $(x, y) \sim (z, w)$. So $a + d = b + c$ and $x + w = y + z$. We compute the following equations.

1. $ax + aw = ay + az$. Multiply $x + w = y + z$ by a.

2. $by + bz = bx + bw$. Multiply $y + z = x + w$ by b.
3. $cx + cw = cy + cz$. Multiply $x + w = y + z$ by c.
4. $dy + dz = dx + dw$. Multiply $y + z = x + w$ by d.

Then taking sums.

$$ax + aw + by + bz + cx + cw + dy + dz = ay + az + bx + bw + cy + cz + dx + dw$$

Grouping terms.

$$ax + by + cw + dz + (aw + bz + cx + dy) = ay + bx + cz + dw + (az + bw + cy + dx)$$

We can complete the proof by ignoring bloch's hint because it doesn't help dumasses like me and cheating by showing $aw + bz + cx + dy = az + bw + cy + dx$.

$$\begin{aligned}
&\big([(aw+1, aw)] + [(bz+1, bz)] + [(cx+1, cx)] + [(dy+1, dy)]\big) \\
&\quad - \big([(az+1, az)] + [(bw+1, bw)] + [(cy+1, cy)] + [(dx+1, dx)]\big) \\
&= [(aw+1, aw)] + [(bz+1, bz)] + [(cx+1, cx)] + [(dy+1, dy)] \\
&\quad + [(az, az+1)] + [(bw, bw+1)] + [(cy, cy+1)] + [(dx, dx+1)] \\
&= [(aw + bz + cx + dy + az + bw + cy + dx + 4,\ aw + bz + cx + dy + az + bw + cy + dx + 4)] \\
&= [(1,1)] = 0
\end{aligned}$$

Thus $ax + by + cw + dz = ay + bx + cz + dw$. Then it follows that

$$(ax + by, ay + bx) \sim (cz + dw, cw + dz)$$

Then from the definition of $\cdot$

$$(a, b) \cdot (x, y) \sim (c, d) \cdot (z, w)$$

∎

Proof. Let $(a,b),(c,d),(x,y),(z,w) \in \mathbb{N} \times \mathbb{N}$. Suppose $(a,b) \sim (c,d)$ and $(x,y) \sim (z,w)$. So $a + d = b + c$ and $x + w = y + z$. Summing shows $a + y + d + z = b + x + c + w$. Which is to say $(a + y, b + x) \sim (c + w, d + z)$. Therefore $(a,b) + (y,x) \sim (c,d) + (w,z)$. It then follows that $(a,b) - (x,y) \sim (c,d) - (z,w)$. Thus $-$ is well defined. ∎

Problem 4

Let $a, b \in \mathbb{N}$.

1. Prove that $[(a,b)] = \hat{0}$ if and only if $a = b$.
2. Prove that $[(a,b)] = \hat{1}$ if and only if $a = b + 1$.
3. Prove that ① $[(a,b)] = [(n,1)]$ for some $n \in \mathbb{N}$ such that $n \neq 1$ if and only if ② $a > b$ if and only if ③ $[(a,b)] > \hat{0}$.
4. Prove that ① $[(a,b)] = [(1,m)]$ for some $m \in \mathbb{N}$ such that $m \neq 1$ if and only if ② $a < b$ if and only if ③ $[(a,b)] < \hat{0}$.

Proof. Suppose $[(a,b)] = \hat{0}$. Thus $(a,b) \sim (1,1)$. Therefore $a + 1 = b + 1$. It follows that $a = b$.

Suppose $a = b$. Then $a + 1 = b + 1$ Therefore $(a,b) \sim (1,1)$. It follows that $[(a,b)] = \hat{0}$. ∎

Proof. Suppose $[(a,b)] = \hat{1}$. Thus $(a,b) \sim (1+1,1)$. Therefore $a + 1 = b + (1 + 1)$. It follows that $a = b + 1$.

Suppose $a = b + 1$. Thus $a + 1 = b + (1 + 1)$. Thus $(a,b) \sim (1 + 1, 1)$. It follows that $[(a,b)] = \hat{1}$. ∎

Proof. (① →②) Suppose $[(a,b)] = [(n,1)]$ for some $n \in \mathbb{N}$ such that $n \neq 1$. Thus $a+1 = b+n$. Since $n \neq 1$, $n > 1$. There exists $p \in \mathbb{N}$ such that $s(p) = n$. Then $a + 1 = b + s(p) = b + p + 1$. It follows that $a = b + p$. Thus $b < a$.

(② →①) Suppose $a > b$. There exists $p \in \mathbb{N}$ such that $a = b + p$. Then $a + 1 = b + p + 1$. It follows that $a + 1 = b + s(p)$. Let $n = s(p)$. Therefore $[(a,b)] = [(n,1)]$ for some $n \in \mathbb{N}$ such that $n \neq 1$.

(② →③) Suppose $a > b$. There exists $p \in \mathbb{N}$ such that $a = b + p$. Then $a + 1 = b + 1 + p$. Therefore $[(a,b)] > \hat{0}$.

(③ →②) Suppose $[(a,b)] > \hat{0}$. It follows that $a + 1 > b + 1$. Thus there exists p such that $a + 1 = b + 1 + p$. Therefore $a = b + p$ and it follows that $a > b$. ∎

Proof. (① →②) Suppose $[(a,b)] = [(1,m)]$ for some $m \in \mathbb{N}$ such that $m \neq 1$. Then $a + m = b + 1$. Since $m \neq 1$, $m > 1$. There exists $p \in \mathbb{N}$ such that $s(p) = m$. Then $a + s(p) = b + 1 \implies a + p + 1 = b + 1$. It follows that $a = b - p$. Thus $a < b$.

(② →①) Suppose $a < b$. There exists $p \in \mathbb{N}$ such that $b = a + p$ with $p \neq 0$. Then $b + 1 = a + p + 1 = a + s(p)$. Let $m = s(p)$. Then $m \neq 1$. Therefore $[(a,b)] = [(1,m)]$ for some $m \in \mathbb{N}$ with $m \neq 1$.

(② →③) Suppose $a < b$. Then there exists $p \in \mathbb{N}$ such that $b = a + p$. Then $b + 1 = a + 1 + p$. Therefore $[(a,b)] < \hat{0}$.

(③ →②) Suppose $[(a,b)] < \hat{0}$. It follows that $b + 1 > a + 1$. Thus there exists $p \in \mathbb{N}$ such that $b + 1 = a + 1 + p$. Therefore $b = a + p$, so $a < b$. ∎

Problem 5

Prove Theorem 1.3.5 (1) (3) (4) (5) (6) (7) (8) (10) (11) (13) (14).

Proof. Let $x, y, z \in \mathbb{Z}$. We must show $(x + y) + z = z + (x + y)$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then

$$\begin{aligned}
(x + y) + z &= ([(x_1, x_2)] + [(y_1, y_2)]) + [(z_1, z_2)] \\
&= [(x_1 + y_1), (x_2 + y_2)] + [(z_1, z_2)] \\
&= [((x_1 + y_1) + z_1), ((x_2 + y_2) + z_2)] \\
&= [(x_1 + (y_1 + z_1)), (x_2 + (y_2 + z_2))] \\
&= [(x_1, x_2)] + [(y_1 + z_1), (y_2 + z_2)] \\
&= [(x_1, x_2)] + ([y_1, y_2] + [z_1, z_2]) \\
&= x + (y + z)
\end{aligned}$$

∎

Proof. We must show $x + \hat{0} = x$. Let $(x_1, x_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = [(x_1, x_2)]$. Then $x + \hat{0} = [(x_1, x_2)] + [(1,1)] = [(x_1 + 1, x_2 + 1)]$. Now $x_1 + x_2 + 1 = x_1 + x_2 + 1$ and rearranging shows $(x_1 + 1) + x_2 = (x_2 + 1) + x_1$. From which it follows $(x_1 + 1, x_2 + 1) \sim (x_1, x_2)$. Thus

$$[(x_1 + 1, x_2 + 1)] = [(x_1, x_2)] = x$$

∎

Proof. Let $x \in \mathbb{N}$ We must show $x + (-x) = \hat{0}$. Let $(x_1, x_2) \in \mathbb{N}$ such that $x = [(x_1, x_2)]$. Then

$$x + (-x) = [(x_1, x_2)] + (-[(x_1, x_2)]) = [(x_1, x_2)] + [(x_2, x_1)] = [(x_1 + x_2, x_2 + x_1)]$$

Now it is clearly $x_1 + x_2 + 1 = x_1 + x_2 + 1$ and rearranging shows $(x_1 + x_2) + 1 = (x_2 + x_1) + 1$. Thus $(x_1 + x_2, x_2 + x_1) \sim (1,1)$. Then

$$[(x_1 + x_2, x_2 + x_1)] = [(1,1)] = \hat{0}$$

∎

Proof. Let $x, y, z \in \mathbb{Z}$. We must show $(xy)z = x(yz)$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = [(x_1, x_2)], y = [(y_1, y_2)], z = [(z_1, z_2)]$. Then

$$\begin{aligned}
(xy)z &= ([(x_1, x_2)] \cdot [(y_1, y_2)]) \cdot [(z_1, z_2)] \\
&= [(x_1y_1 + x_2y_2, x_1y_2 + x_2y_1)] \cdot [(z_1, z_2)] \\
&= [((x_1y_1 + x_2y_2)z_1 + (x_1y_2 + x_2y_1)z_2, (x_1y_1 + x_2y_2)z_2 + (x_1y_2 + x_2y_1)z_1)] \\
&= [(x_1y_1z_1 + x_2y_2z_1 + x_1y_2z_2 + x_2y_1z_2, x_1y_1z_2 + x_2y_2z_2 + x_1y_2z_1 + x_2y_1z_1)] \\
&= [(x_1(y_1z_1 + y_2z_2) + x_2(y_2z_1 + y_1z_2), x_1(y_1z_2 + y_2z_1) + x_2(y_2z_2 + y_1z_1))] \\
&= [(x_1, x_2)] \cdot [(y_1z_1 + y_2z_2, y_1z_2 + y_2z_1)] \\
&= [(x_1, x_2)] \cdot ([(y_1, y_2)] \cdot [(z_1, z_2)]) \\
&= x \cdot (yz)
\end{aligned}$$

∎

Proof. Let $x, y \in \mathbb{N}$. We must show $xy = yx$. Let $(x_1, x_2), (y_1, y_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = [(x_1, x_2)], y = [(y_1, y_2)]$. Then

$$\begin{aligned}
xy &= [(x_1, x_2)] \cdot [(y_1, y_2)] \\
&= [(x_1y_1 + x_2y_2, x_1y_2 + x_2y_1)] \\
&= [(x_2y_2 + x_1y_1, x_2y_1 + x_1y_2)] \\
&= [(y_1, y_2)] \cdot [(x_1, x_2)] \\
&= yx
\end{aligned}$$

∎

Proof. Let $x \in \mathbb{Z}$. We must show $x \cdot \hat{1} = x$. Let $(x_1, x_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = [(x_1, x_2)]$. Then

$$x \cdot \hat{1} = [(x_1, x_2)] \cdot [(1 + 1, 1)] = [(x_1(1 + 1) + x_2 \cdot 1, x_1 \cdot 1 + x_2 \cdot 1)] = [(2x_1 + x_2, x_1 + x_2)]$$

Now $2x_1 + 2x_2 = 2x_1 + 2x_2$. It follows that $(2x_1 + x_2, x_1 + x_2) \sim (x_1, x_2)$. Therefore

$$[(2x_1 + x_2, x_1 + x_2)] = [(x_1, x_2)] = x$$

∎

Proof. Let $x, y, z \in \mathbb{Z}$. We must show $x(y + z) = xy + xz$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = [(x_1, x_2)], y = [(y_1, y_2)], z = [(z_1, z_2)]$.

$$\begin{aligned}
x(y + z) &= [(x_1, x_2)] \cdot ([(y_1, y_2)] + [(z_1, z_2)]) \\
&= [(x_1, x_2)] \cdot [(y_1 + z_1, y_2 + z_2)] \\
&= [(x_1(y_1 + z_1) + x_2(y_2 + z_2),\ x_1(y_2 + z_2) + x_2(y_1 + z_1))] \\
&= [(x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2,\ x_1y_2 + x_1z_2 + x_2y_1 + x_2z_1)] \\
&= [(x_1y_1 + x_2y_2,\ x_1y_2 + x_2y_1) + (x_1z_1 + x_2z_2,\ x_1z_2 + x_2z_1)] \\
&= xy + xz.
\end{aligned}$$

∎

Proof. Let $x, y \in \mathbb{Z}$. We must show precisely one of $x < y$, $x = y$, or $x > y$ holds. Let $(x_1, x_2), (y_1, y_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = [(x_1, x_2)], y = [(y_1, y_2)]$.

We first show no two hold simultaneously.

Suppose $x < y$ and $x > y$. Then $x_1 + y_2 < x_2 + y_1$ and $x_1 + y_2 > x_2 + y_1$, which is a contradiction.

Suppose $x < y$ and $x = y$. Then $x_1 + y_2 < x_2 + y_1$ and $x_1 + y_2 = x_2 + y_1$, which is a contradiction.

Suppose $x > y$ and $x = y$. Then $x_1 + y_2 > x_2 + y_1$ and $x_1 + y_2 = x_2 + y_1$, which is a contradiction.

Thus no two hold simultaneously.

We now show at least one holds. We know either $x_1 + y_2 < x_2 + y_1$, $x_1 + y_2 = x_2 + y_1$, or $x_1 + y_2 > x_2 + y_1$. Thus at least one of $x < y$, $x = y$, or $x > y$ holds. ∎

Proof. Let $x, y, z \in \mathbb{Z}$. We must show if $x < y$ then $x + z < y + z$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = [(x_1, x_2)], y = [(y_1, y_2)], z = [(z_1, z_2)]$. Suppose $x < y$. Then $x_1 + y_2 < x_2 + y_1$. There exists $p \in \mathbb{N}$ such that $x_1 + y_2 + p = x_2 + y_1$. It follows that $x_1 + y_2 + p + z_1 + z_2 = x_2 + y_1 + z_1 + z_2$. Rearranging terms $(x_1 + z_1) + (y_2 + z_2) + p = (x_2 + z_2) + (y_1 + z_1)$. Thus $(x_1 + z_1) + (y_2 + z_2) < (x_2 + z_2) + (y_1 + z_1)$. Then

$$[(x_1 + z_1, x_2 + z_2)] < [(y_1 + z_1, y_2 + z_2)] \iff [(x_1, x_2)] + [(z_1, z_2)] < [(y_1, y_2)]$$

Therefore $x + z < y + z$. ∎

Proof. Let $x, y, z \in \mathbb{Z}$. We must show if $x < y$ and $z > \hat{0}$, then $xz < yz$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = [(x_1, x_2)], y = [(y_1, y_2)], z = [(z_1, z_2)]$.

Suppose $x < y$ and $z > \hat{0}$. Then $x_1 + y_2 < x_2 + y_1$ and $z_1 > z_2$. Since $z_1 > z_2$, there exists $q \in \mathbb{N}$ such that $z_1 = z_2 + q$. From $x_1 + y_2 < x_2 + y_1$ there exists $p \in \mathbb{N}$ such that $x_1 + y_2 + p = x_2 + y_1$. From $x_1 + y_2 + p = x_2 + y_1$ multiply by z_1, $x_1z_1 + y_2z_1 + pz_1 = x_2z_1 + y_1z_1$. From $x_1 + y_2 + p = x_2 + y_1$ multiply by z_2, $x_1z_2 + y_2z_2 + pz_2 = x_2z_2 + y_1z_2$. Taking sums

$$(x_1z_1 + x_2z_2) + (y_1z_2 + y_2z_1) + pz_1 = (x_2z_1 + x_1z_2) + (y_2z_1 + y_1z_2)$$

Rearranging terms gives

$$(x_1z_1 + x_2z_2) + pz_1 < (x_2z_1 + x_1z_2)$$

Thus $(x_1z_1 + x_2z_2) < (x_2z_1 + x_1z_2)$. Then

$$[(x_1z_1 + x_2z_2, x_2z_1 + x_1z_2)] < [(y_1z_1 + y_2z_2, y_2z_1 + y_1z_2)] \iff [(x_1, x_2)] \cdot [(z_1, z_2)] < [(y_1, y_2)] \cdot [(z_1, z_2)]$$

Therefore $xz < yz$. ∎

Proof. We must show $\hat{0} \neq \hat{1}$. For contradiction suppose $\hat{0} = \hat{1}$. Then $(1, 1) \sim (1 + 1, 1)$ then $1 + 1 = 1 + 1 + 1$ Let $p \in \mathbb{N}$ such that $p = 1 + 1$. It follows that $p + 1 = p$ which is a contradiction. ∎

Problem 6

Prove Theorem 1.3.7 (1) (3) (4(b)) (4(c)).

Theorem 1. *Let $i : \mathbb{N} \to \mathbb{Z}$ be defined by $i(n) = [(n + 1), 1]$ for all $n \in \mathbb{N}$.*

1. *The function $i : \mathbb{N} \to \mathbb{Z}$ is injective.*
2. *$i(\mathbb{N}) = \{x \in \mathbb{Z} \mid x > \hat{0}\}$.*
3. *$i(1) = \hat{1}$.*
4. *Let $a, b \in \mathbb{N}$. Then*
   (a) *$i(a + b) = i(a) + i(b)$;*
   (b) *$i(ab) = i(a)i(b)$;*
   (c) *$a < b$ if and only if $i(a) < i(b)$.*

Proof. We must show $i : \mathbb{N} \to \mathbb{Z}$ is injective. Let $x_1, x_2 \in \mathbb{N}$ such that $i(x_1) = i(x_2)$. We must show $x_1 = x_2$. Now, $[(x_1 + 1, 1)] = [(x_2 + 1, 1)]$. Thus $(x_1 + 1) + 1 = 1 + (x_2 + 1)$ and cancelling terms shows that $x_1 = x_2$. ∎

Proof. We must show $i(1) = \hat{1}$. Now, $i(1) = [(1+1,1)] = \hat{1}$. ∎

Proof. We must show $i(ab) = i(a)i(b)$. Now $i(ab) = [(ab+1,1)]$. We know that $ab+a+b+3 = ab+a+b+3$ which is equivalent to $ab+1+a+1+b+1 = 1+(ab+a+b+1)+1$. Rearranging terms $(ab+1)+((a+1)+(b+1)) = 1+((a+1)(b+1)+1)$. Thus $(ab+1,1) \sim ((a+1)(b+1)+1,(a+1)+(b+1))$. Then $[(ab+1,1)] = [((a+1)(b+1)+1,(a+1)+(b+1))]$ and $[((a+1)(b+1)+1,(a+1)+(b+1))] = [(a+1,1)]\cdot[(b+1,1)]$. It follows that $[(a+1,1)]\cdot[(b+1,1)] = i(a)i(b)$. ∎

Proof. We must show $a < b$ if and only if $i(a) < i(b)$.

Suppose $a < b$. It follows that $(a+1)+1 < 1+(b+1)$. Thus $[(a+1,1)] < [(b+1,1)]$.

Suppose $i(a) < i(b)$. Then $[(a+1,1)] < [(b+1,1)]$. It follows that $(a+1)+1 < 1+(b+1)$. Cancelling terms shows $a < b$. ∎

Problem 7

Let $x, y, z \in \mathbb{Z}$
1. Prove that $x < y$ if and only if $-x > -y$.
2. Prove that if $z < 0$, then $x < y$ if and only if $xz > yz$.

Proof. Suppose $x < y$ then

$$\begin{aligned}
x < y &\iff x+((-x)+(-y)) < y+((-x)+(-y)) && \text{by Theorem 1.3.5 part (12)}\\
&\iff x+((-x)+(-y)) < y+((-y)+(-x)) && \text{by Theorem 1.3.5 part (2)}\\
&\iff (x+(-x))+(-y) < (y+(-y))+(-x) && \text{by Theorem 1.3.5 part (1)}\\
&\iff 0+(-y) < 0+(-x) && \text{by Theorem 1.3.5 part (4)}\\
&\iff (-y)+0 < (-x)+0 && \text{by Theorem 1.3.5 part (2)}\\
&\iff -y < -x && \text{by Theorem 1.3.5 (4)}
\end{aligned}$$

Suppose $-y < -x$ then

$$\begin{aligned}
-y < -x &\iff (-y)+(x+y) < (-x)+(x+y) && \text{by Theorem 1.3.5 part (12)}\\
&\iff (-y)+(y+x) < (-x)+(x+y) && \text{by Theorem 1.3.5 part (2)}\\
&\iff ((-y)+y)+x < ((-x)+x)+y && \text{by Theorem 1.3.5 part (1)}\\
&\iff 0+x < 0+y && \text{by Theorem 1.3.5 part (4)}\\
&\iff x+0 < y+0 && \text{by Theorem 1.3.5 part (2)}\\
&\iff x < y && \text{by Theorem 1.3.5 part (4)}
\end{aligned}$$

∎

Proof. Suppose $z < 0$. It follows that $-z > 0$.

Suppose $x < y$. By Theorem 1.3.5 part 13,2 it follows that $x(-z) < y(-z) \iff -zx < -zy$. By the previous problem, $zy > zx$. By Theorem 1.3.5 part 2, $xz > yz$,

Suppose $xz > yz$. By the previous problem, $-xz < -yz$. By Theorem 1.3.5 part 2, $x(-z) < y(-z)$. By Theorem 1.3.5 part 13, $x < y$. ∎

Problem 8

Let $x \in \mathbb{Z}$. Prove that if $x > 0$ then $x \geq 1$. Prove that if $x < 0$ then $x \leq -1$.

Proof. Suppose $x > 0$. For contradiction suppose $x < 1$. Then $0 < x < 1$ and it follows that $1 < x + 1 < 2$. Let i be the bijective function in Theorem 1.3.7. It follows that $i(1) < i(x + 1) < i(2) = i(1) + i(1)$. contradicting Theorem 1.2.9 part 9. ∎

Proof. Suppose $x < 0$. For contradiction suppose $x > -1$. Then $-1 < x < 0$ and it follows that $1 < x + 2 < 2$. Let i be the bijective function in Theorem 1.3.7. It follows that $i(1) < i(x + 2) < i(2) = i(1) + i(1)$, contradicting Theorem 1.2.9 part 9. ∎

Problem 9

1. Prove that $1 < 2$.
2. Let $x \in \mathbb{Z}$. Prove that $2x \neq 1$.

Proof. For contradiction suppose $1 \geq 2$. Either $1 = 2$ or $1 > 2$.

Suppose $1 = 2$. Let i be the bijective function in Theorem 1.3.7. Then $i(1) = i(2) = i(1) + i(1)$, which contradicts Theorem 1.2.7 part 6.

Suppose $1 > 2$. Then $i(1) > i(1) + i(1)$. There exists $p \in \mathbb{N}$ such that $i(1) = p + i(1) + i(1)$. This also contradicts Theorem 1.2.7 part 6.

It follows that $1 < 2$. ∎

Proof. For contradiction suppose $2x = 1$ Let $(x_1, x_2) \in \mathbb{N} \times \mathbb{N}$ such that $x = [(x_1, x_2)]$. Then $[(3, 1)] \cdot [(x_1, x_2)] = [(1 + 1, 1)] \iff [(3x_1 + x_2, 3x_2 + x_1)] = [(1 + 1, 1)]$ It follows that $3x_1 + x_2 + 1 = 3x_2 + x_1 + 1$. Cancelling terms shows $x_1 = x_2$. So $(x_1, x_2) \sim (1, 1)$ thus $2 \cdot \hat{0} = 0 \neq 1$. ∎

Problem 10

Prove that the Well-Order Principle (Theorem 1.2.10), which was stated for $\mathbb{N}$ in Section 1.2, still holds when we think of $\mathbb{N}$ as the set of positive integers. That is, let $G \subseteq \{x \in \mathbb{Z} \mid x > 0\}$ be a non-empty set. Prove that there is some $m \in G$ such that $m \leq g$ for all $g \in G$. Use Theorem 1.3.7.

Proof. Let $G \subseteq \{x \in \mathbb{Z} \mid x > 0\}$ such that $G \neq \emptyset$. Let i be the bijective function in Theorem 1.3.7. By Theorem 1.2.10, since $i^{-1}(G) \subseteq \mathbb{N}$ there exists $n \in i^{-1}(G)$ such that for all $x \in i^{-1}(G)$, $n \leq x$. It follows that for all $x \in G$, $i(n) \leq x$ ∎

Problem 11

Prove Theorem 1.3.8 (1) (3) (4) (5) (7) (10) (11).

Proof. We must show if $x + z = y + z$ then $x = y$. Suppose $x + z = y + z$. Then

$$
\begin{aligned}
& x + z = y + z \\
\iff & (x + z) + (-z) = (y + z) + (-z) \\
\iff & x + (z + (-z)) = y + (z + (-z)) && \text{by Theorem 1.3.5 part (1)} \\
\iff & x + 0 = y + z && \text{by Theorem 1.3.5 part (4)} \\
\iff & x = y && \text{by Theorem 1.3.5 part (3)}
\end{aligned}
$$

∎

Proof. We must show $-(x+y) = (-x) + (-y)$. Then

$$\begin{aligned}
& -(x+y) = (-x) + (-y) \\
\iff & -(x+y) + (x+y) = (-x) + (-y) + (x+y) \\
\iff & (x+y) + (-(x+y)) = (-x) + (x+y) + (-y) && \text{by Theorem 1.3.5 part (2)} \\
\iff & (x+y) + (-(x+y)) = (-x) + x + (y + (-y)) && \text{by Theorem 1.3.5 part (5)} \\
\iff & (x+y) + (-(x+y)) = x + (-x) + (y + (-y)) && \text{by Theorem 1.3.5 part (2)} \\
\iff & 0 = 0 + 0 && \text{by Theorem 1.3.5 part (4)} \\
\iff & 0 = 0 && \text{by Theorem 1.3.5 part (4)}
\end{aligned}$$

∎

Proof. We must show $x \cdot 0 = 0$.

$$\begin{aligned}
& (x \cdot 0) + (x \cdot 0) = x(0+0) && \text{by Theorem 1.3.5 part (8)} \\
\iff & (x \cdot 0) + (x \cdot 0) = x \cdot 0 && \text{by Theorem 1.3.5 part (3)} \\
\iff & (x \cdot 0) + (x \cdot 0) + (-(x \cdot 0)) = x \cdot 0 + (-(x \cdot 0)) \\
\iff & (x \cdot 0) + \big((x \cdot 0) + (-(x \cdot 0))\big) = x \cdot 0 + (-(x \cdot 0)) && \text{by Theorem 1.3.5 part (1)} \\
\iff & (x \cdot 0) + 0 = 0 && \text{by Theorem 1.3.5 part (4)} \\
\iff & x \cdot 0 = 0 && \text{by Theorem 1.3.5 part (3)}
\end{aligned}$$

∎

Proof. We must show that if $z \neq 0$ and $xz = yz$, then $x = y$. Suppose $z \neq 0$ and $xz = yz$. Then

$$\begin{aligned}
xz = yz & \iff xz - yz = 0 \\
& \iff (x-y)z = 0.
\end{aligned}$$

Since $z \neq 0$, it follows that $x + (-y) = 0$, so $x = y$. ∎

Proof. We must show $xy = 1$ if and only if $x = 1 = y$ or $x = -1 = y$.

($\rightarrow$) Suppose $xy = 1$. For contradiction, suppose $x \neq 1, y \neq 1$, and $x \neq -1, y \neq -1$.

To make things easier, we first show $x \neq 0, y \neq 0$. If $x = 0$ then $xy = 0y$ and from the 1.3.8 (4) it follows that $0y = 0$ contradicting that $xy = 1$. Similarly $y \neq 0$.

1. $x > 1, y > 1$.
2. $x < 1, y < 1$ so $x < 0, y < 0$.
3. $x > 1, y < 1$ so $y < 0$.
4. $x < 1, y > 1$ so $x < 0$.

Suppose $x > 1, y > 1$. Since $1 > 0$ it follows that $y > 0$ by Transitive Law. Since $1 < x$ and $y > 0$ it follows that $1 \cdot y < xy$. Then from Identity Law for Multiplication it follows that $y < xy$ showing $y < 1$ which is a contradiction.

Suppose $x < 0$ and $y < 0$. Since $-x > 0$ and $-y > 0$, we have $(-x)(-y) = xy$. But $xy = 1$, so $(-x)(-y) = 1$. For contradiction suppose $-x \neq 1$. Then either $-x > 1$ or $-x < 1$. Suppose $-x > 1$. Since $-y > 0$ it follows that $1 \cdot (-y) < (-x)(-y) = 1$. Then from Identity Law for Multiplication it follows that $-y < 1$. But $-y \in \mathbb{Z}$ and $-y > 0$ thus $-y \in \mathbb{N}$ contradicting that 1 is the lower bound of $\mathbb{N}$. Suppose $-x < 1$. So $-x \leq 0$. Suppose $-x = 0$. Thus $x = 0$ which is a contradiction. Thus $-x < 0$. Since $-y > 0$ it follows that $(-x)(-y) < 0 \cdot (-y)$. From which it follows that $1 < 0$ which is a contradiction.

Suppose $x > 1$ and $y < 0$. Since $1 < x$ and $y < 0$ it follows that $1 \cdot y > xy$. Then from Identity Law for Multiplication it follows that $y > xy$. But $xy = 1$ so $y > 1$ which contradicts $y < 1$.

Suppose $x < 0$ and $y > 1$. Since $1 < y$ and $x < 0$ it follows that $x \cdot 1 > xy$. Then from Identity Law for Multiplication it follows that $x > xy$. But $xy = 1$ so $x > 1$ which contradicts $x < 1$.

($\leftarrow$) Suppose $x = 1 = y$ or $x = -1 = y$. Suppose $x = 1 = y$. Then $xy = 1 \cdot 1 = 1$. Suppose $x = -1 = y$. Then $xy = (-1)(-1)$. Then by 1.3.8 (6), $(-1)(-1) = 1(-(-1))$. and by 1.3.8 (2), $1(-(-1)) = 1 \cdot 1 = 1$. Thus $xy = 1$. ∎

Proof. We must show if $x \leq y$ and $y \leq x$, then $x = y$. Suppose $x \leq y$ and $y \leq x$. For contradiction suppose $x \neq y$. Then either $x < y$ or $x > y$. Suppose $x < y$. This contradicts $y \leq x$. Suppose $x > y$. This contradicts $x \leq y$. Thus $x = y$. ∎

Proof. We must show that if $x > 0$ and $y > 0$, then $xy > 0$, and if $x > 0$ and $y < 0$, then $xy < 0$.

Suppose $x, y > 0$ and for contradiction $xy \leq 0$. Either $xy = 0$ or $xy < 0$. Suppose $xy = 0$ and it follows that $x = 0$ or $y = 0$ contradicting $x, y > 0$. Suppose $xy < 0$. It follows that $-xy > 0$. Thus $x(-y) > x \cdot 0$. Since $x > 0$ it follows that $-y > 0$ (Problem 7). Then $y < 0$ which is a contradiciton.

Suppose $x > 0$ and $y < 0$ and for contradiction $xy \geq 0$. Either $xy = 0$ or $xy > 0$. Suppose $xy = 0$ and it follows that $x = 0$ or $y = 0$ contradicting $x > 0, y < 0$. Suppose $xy > 0$. Since $-y > 0$ and $x > 0$ it follows that $x(-y) > 0$. Thus $-xy > 0$ and it follows that $xy < 0$. ∎

2.3 Axioms for the Integers

Problem 2

Let $n \in \mathbb{N}$. Prove that $n + 1 \in \mathbb{N}$.

Proof. Since $n \in \mathbb{N}$, $n \in \mathbb{Z}$ and $n > 0$. By Addition Law for Order, $n + 1 > 1$. By 1.4.5 (9), $n + 1 > 1 > 0$. By Transitive Law, $n + 1 > 0$. Since $n + 1 \in \mathbb{Z}$ and $n + 1 > 0$, by definition of $\mathbb{N}$, $n + 1 \in \mathbb{N}$. ∎

Problem 3

Let $x, y \in \mathbb{Z}$. Prove that $x \leq y$ if and only if $-x \geq -y$.

Proof. ($\rightarrow$) Suppose $x \leq y$. Then

$$
\begin{aligned}
x \leq y &\iff x + ((-x) + (-y)) \leq y + ((-x) + (-y)) \\
&\iff (x + (-x)) + (-y) \leq y + ((-x) + (-y)) && \text{1.4.1 (a)} \\
&\iff (x + (-x)) + (-y) \leq y + ((-y) + (-x)) && \text{1.4.1 (b)} \\
&\iff (x + (-x)) + (-y) \leq (y + (-y)) + (-x) && \text{1.4.1 (a)} \\
&\iff 0 + (-y) \leq 0 + (-x) && \text{1.4.1 (d)} \\
&\iff -y + 0 \leq -x + 0 && \text{1.4.1 (b)} \\
&\iff -y \leq -x && \text{1.4.1 (c)}
\end{aligned}
$$

($\leftarrow$) Suppose $-x \geq -y$. Then

$$\begin{aligned}
-x \geq -y &\iff -x + (x + y) \geq -y + (x + y) \\
&\iff (-x + x) + y \geq -y + (x + y) && 1.4.1\ (a) \\
&\iff (-x + x) + y \geq -y + (y + x) && 1.4.1\ (b) \\
&\iff (-x + x) + y \geq (-y + y) + x && 1.4.1\ (a) \\
&\iff (x + (-x)) + y \geq (y + (-y)) + x && 1.4.1\ (b) \\
&\iff 0 + y \geq 0 + x && 1.4.1\ (d) \\
&\iff y + 0 \geq x + 0 && 1.4.1\ (b) \\
&\iff y \geq x && 1.4.1\ (c)
\end{aligned}$$

∎

Problem 4

Prove that $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 1\}$.

Proof. Let $x \in \mathbb{N}$. By definition $x \in \mathbb{Z}$ and $x > 0$. For contradiction, suppose $x < 1$. Then $0 < x < 1$ contradicting 1.4.6. Thus $x \geq 1$. It follows that $x \in \{x \in \mathbb{Z} \mid x \geq 1\}$. Therefore $\mathbb{N} \subseteq \{x \in \mathbb{Z} \mid x \geq 1\}$.

Let $x \in \{x \in \mathbb{Z} \mid x \geq 1\}$. Either $x = 1$ or $x > 1$. In either case $x > 0$. Thus $x \in \mathbb{N}$. Therefore $\{x \in \mathbb{Z} \mid x \geq 1\} \subseteq \mathbb{N}$.

It follows that $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 1\}$. ∎

Problem 5

Let $a, b \in \mathbb{Z}$. Prove that if $a < b$ then $a + 1 \leq b$.

Proof. Suppose $a < b$. For contradiciton suppose $a + 1 > b$. Then $a + 1 > b > a$ contradicting 1.4.6. ∎

Problem 6

Let $n \in \mathbb{N}$. Suppose that $n \neq 1$. Prove that there is some $b \in \mathbb{N}$ such that $b + 1 = n$.

Proof. (**Base Case**) Suppose $n = 2$. Then $s(1) = 1 + 1 = 2$.

(**Induction Step**) Suppose the theorem holds for some $n \in \mathbb{N}$ such that $n \neq 1$. Consider $n + 1 = s(n)$. By our hypothesis there exists $b \in \mathbb{N}$ such that $s(b) = n$. Thus $n + 1 = s(s(b)) = s(b) + 1$. Thus proving our theorem. ∎

Problem 8

Let $a \in \mathbb{Z}$.

1. Let $G \subseteq \{x \in \mathbb{Z} \mid x \geq a\}$ be a set. Suppose that $a \in G$, and that if $g \in G$ then $g + 1 \in G$. Prove that $G = \{x \in \mathbb{Z} \mid x \geq a\}$.
2. Let $H \subseteq \{x \in \mathbb{Z} \mid x \leq a\}$ be a set. Suppose that $a \in H$, and that if $h \in H$ then $h + (-1) \in H$. Prove that $H = \{x \in \mathbb{Z} \mid x \leq a\}$.

Proof. We must show for a fixed $a \in \mathbb{Z}$, $\{x \in \mathbb{Z} \mid x \geq a\} \subseteq G$. Now, $x \geq a \iff x + (-a) \geq 0 \iff x + (-a) + 1 \geq 1$. So we need to show $\{x \in \mathbb{Z} \mid x + (-a) + 1 \geq 1\} \subseteq G$. Which is equivalent to $\{x \in \mathbb{Z} \mid x + (-a) + 1 \in \mathbb{N}\}$.

(**Base Case**) Let $x = a$. Then $x + (-a) + 1 = a + (-a) + 1 = 1 \in \mathbb{N}$. Thus $x = a \in G$.

(**Induction Step**) Suppose for some $x \in \mathbb{Z}$ that $x + (-a) + 1 \in \mathbb{N}$ and $x \in G$. Then consider $x + 1$. We have

$$(x + 1) + (-a) + 1 = (x + (-a) + 1) + 1 \in \mathbb{N}.$$

By the definition of G, since $x \in G$, we have $x + 1 \in G$.

Thus $G = \{x \in \mathbb{Z} \mid x \geq a\}$. ∎

Proof. We must show for a fixed $a \in \mathbb{Z}$, $\{x \in \mathbb{Z} \mid x \leq a\} \subseteq H$. Now, $x \leq a \iff a - x \geq 0 \iff a - x + 1 \geq 1$. So we need to show $\{x \in \mathbb{Z} \mid a - x + 1 \geq 1\} \subseteq H$. Which is equivalent to $\{x \in \mathbb{Z} \mid a - x + 1 \in \mathbb{N}\}$.

(**Base Case**) Let $x = a$. Then $a - x + 1 = a - a + 1 = 1 \in \mathbb{N}$. Thus $x = a \in H$.

(**Induction Step**) Suppose for some $x \in \mathbb{Z}$ that $a - x + 1 \in \mathbb{N}$ and $x \in H$. Then consider $x - 1$. We have

$$a - (x - 1) + 1 = (a - x + 1) + 1 \in \mathbb{N}.$$

By the definition of H, since $x \in H$, it follows that $x - 1 \in H$.

Thus $H = \{x \in \mathbb{Z} \mid x \leq a\}$. ∎

Extra Problem

There is a "unique" ordered integral domain that satisfies Axiom 1.4.4. Formulate this rigorously and prove it.

Theorem 2. *Let A and A' be ordered integral domains satisfying Axiom 1.4.4. Let $0, 1 \in A$ and $0', 1' \in A'$ such that $0 < 1$ and $0' < 1'$ and for all $x \in A$, $x + 0 = x$ and for all $x' \in B$, $x' + 0' = x'$. Then there exists a bijective function*

$$i : A \to B$$

such that $i(1) = 1'$ and for all $x, y \in A$ the following equations and relation holds.

1. $i(x + y) = i(x) + i(y)$.
2. $i(x - y) = i(x) - i(y)$.
3. $i(x \cdot y) = i(x) \cdot i(y)$.
4. $x < y \iff i(x) < i(y)$.

Proof. We can apply the recursive construction to the set B, the element $1' \in B$, and the function $s(x) = x + 1'$ on B, to deduce that there is a unique function $i : A \to B$ such that $i(x+1) = i(x)+1'$ for all $x \in A$ and $i(1) = 1'$.

Similarly, we can construct a function $i' : B \to A$ such that $i'(y + 1') = i'(y) + 1$ for all $y \in B$ and $i'(1') = 1$.

Now we show that i' is the inverse of i.

Consider $i' \circ i$. Let $x \in A$.

Base case: $x = 1$.

$$(i' \circ i)(1) = i'(i(1)) = i'(1') = 1 = x$$

Inductive step: Suppose $x > 1$. By the properties of the domain, there exists $y \in A$ such that $y + 1 = x$. Suppose for $y \in A$ with $y < x$ we have $(i' \circ i)(y) = y$. Then

$$\begin{aligned}
(i' \circ i)(x) &= i'(i(y + 1)) \\
&= i'(i(y) + 1') \\
&= i'(i(y)) + 1 \\
&= y + 1 && \text{(induction hypothesis)} \\
&= x
\end{aligned}$$

Now consider $i \circ i'$. Let $y \in B$.

Base case: $y = 1'$.

$$(i \circ i')(1') = i(i'(1')) = i(1) = 1' = y$$

Inductive step: Suppose $y > 1'$. By the properties of the domain, there exists $y_0 \in B$ such that $y_0 + 1' = y$. Suppose for $y_0 < y$ we have $(i \circ i')(y_0) = y_0$. Then

$$\begin{aligned}(i \circ i')(y) &= i(i'(y_0 + 1')) \\ &= i(i'(y_0) + 1) \\ &= i(i'(y_0)) + 1' \\ &= y_0 + 1' && \text{(induction hypothesis)} \\ &= y\end{aligned}$$

Since $(i' \circ i)(x) = x$ and $(i \circ i')(y) = y$, we conclude that i' is the inverse of i. Thus i is bijective.

Finally, we check that i preserves all operations:

- Addition: By construction, $i(x+1) = i(x) + 1'$; using induction on sums $x + y$, we get $i(x+y) = i(x) + i(y)$ for all $x, y \in A$.
- Subtraction: $i(x - y) = i(x) - i(y)$ follows from the additive inverse and induction.
- Multiplication: Using induction on y, $i(x \cdot 1) = i(x) \cdot 1'$ and $i(x \cdot (y+1)) = i(x \cdot y) + i(x)$, giving $i(x \cdot y) = i(x) \cdot i(y)$.
- Order: By definition, $x < y \iff \exists z \neq 0$ with $x + z = y$; using preservation of addition and $i(0) = 0'$, we have $i(x) < i(y) \iff x < y$.

∎

2.4 Constructing the Rational Numbers

Problem 1

Complete the proof of Lemma 1.5.2. That is, prove that the relation $\asymp$ is reflexive and symmetric.

Proof. Let $(a, b), (c, d) \in \mathbb{Q} \times \mathbb{Q}^*$.

We must show $(a, b) \asymp (a, b)$. But $ab = ab$ thus $(a, b) \asymp (a, b)$.

Suppose $(a, b) \asymp (c, d)$. We must show $(c, d) \asymp (a, b)$. But $ad = bc \iff cb = da$ thus $(c, d) \asymp (a, b)$. ∎

Redefining <

Let $[(a, b)] \in \mathbb{Q}$. Define $[(a, b)]$ to be in P iff both $a, b > 0$ or both $a, b < 0$.

1. If $[(a, b)] = [(c, d)]$ then $[(a, b)]$ in P iff $[(c, d)]$ in P.
2. Define $x < y$ if and only if $y - x$ in P.
3. Show $x < y$ if and only if Definition 1.5.3 is satisfied (this simplifies the proof of $<$ being well-defined).
4. Show that if x, y in P, then $x + y$ and xy in P.
5. Show that for any nonzero x in $\mathbb{Q}$, exactly one of $x, -x$ is in P.

Proof. Let $(x_1, x_2), (y_1, y_2) \in \mathbb{Q} \times \mathbb{Q}^*$ and let $x, y \in \mathbb{Q}$ such that $x = [(x_1, x_2)]$ and $y = [(y_1, y_2)]$. We first show $x < y$ if and only if Definition 1.5.3 is satisfied. Definition 1.5.3 states $<$ on $\mathbb{Q}$ is defined by

$$\begin{aligned}
[(x_1, x_2)] < [(y_1, y_2)] \iff & (x_2 > 0 \wedge y_2 > 0 \wedge x_1 y_2 < y_1 x_2) && ① \\
& \vee (x_2 < 0 \wedge y_2 < 0 \wedge x_1 y_2 < y_1 x_2) && ② \\
& \vee (x_2 > 0 \wedge y_2 < 0 \wedge x_1 y_2 > y_1 x_2) && ③ \\
& \vee (x_2 < 0 \wedge y_2 > 0 \wedge x_1 y_2 > y_1 x_2) && ④
\end{aligned}$$

($\longrightarrow$) Suppose $x < y$. It follows that $y - x \in P$. Then

$$[(y_1, y_2)] - [(x_1, x_2)] = [(y_1, y_2)] + [(-x_1, x_2)] = [(y_1 x_2 - x_1 y_2, y_2 x_2)] \in P$$

There are two cases. Either $(y_1 x_2 - x_1 y_2), (y_2 x_2) > 0$ or or $(y_1 x_2 - x_1 y_2), (y_2 x_2) < 0$.

(**Case 1**) Suppose $(y_1 x_2 - x_1 y_2), (y_2 x_2) < 0$. Since $y_2 x_2 > 0$ either $y_2, x_2 > 0$ or $y_2 x_2 < 0$. So, suppose $y_2, x_2 > 0$ we see ① holds. Similarly, suppose $y_2, x_2 < 0$ we see ② holds.

(**Case 2**) Suppose $(y_1 x_2 - x_1 y_2), (y_2 x_2) < 0$. Since $y_2 x_2 < 0$ either $y_2 > 0$ and $x_2 < 0$ or $y_2 < 0$ and $x_2 > 0$. So, suppose $y_2 > 0, x_2 < 0$ we see ③ holds. Similarly, suppose $y_2 < 0, x_2 > 0$ we see ④ holds.

So we have shown if $x < y$ then one case of Definition 1.5.3 holds.

($\longleftarrow$) Suppose $[(x_1, x_2)] < [(y_1, y_2)]$ then one of ①, ②, ③, ④ holds.

Case ①: $x_2 > 0$, $y_2 > 0$, and $x_1 y_2 < y_1 x_2$. Then $y_1 x_2 - x_1 y_2 > 0$ and $y_2 x_2 > 0$, so $[(y_1 x_2 - x_1 y_2, y_2 x_2)] \in P$. Thus $y - x \in P$ and therefore $x < y$.

Case ②: $x_2 < 0$, $y_2 < 0$, and $x_1 y_2 < y_1 x_2$. Then $y_1 x_2 - x_1 y_2 > 0$ and $y_2 x_2 > 0$, so $[(y_1 x_2 - x_1 y_2, y_2 x_2)] \in P$. Thus $y - x \in P$ and therefore $x < y$.

Case ③: $x_2 > 0$, $y_2 < 0$, and $x_1 y_2 > y_1 x_2$. Then $y_1 x_2 - x_1 y_2 < 0$ and $y_2 x_2 < 0$, so $[(y_1 x_2 - x_1 y_2, y_2 x_2)] \in P$. Thus $y - x \in P$ and therefore $x < y$.

Case ④: $x_2 < 0$, $y_2 > 0$, and $x_1 y_2 > y_1 x_2$. Then $y_1 x_2 - x_1 y_2 < 0$ and $y_2 x_2 < 0$, so $[(y_1 x_2 - x_1 y_2, y_2 x_2)] \in P$. Thus $y - x \in P$ and therefore $x < y$.

Thus proving our theorem. ∎

Proof. Let $x, y \in \mathbb{Z}$. Suppose $x, y \in P$. Let $[(x_1, x_2)] = x$ and $[(y_1, y_2)] = y$ such that $x_1, x_2, y_1, y_2 \in \mathbb{Z}$. By definition of $+$, $x + y = [(x_1 y_2 + y_1 x_2, x_2 y_2)]$. There are four cases.

Case $(x_1, x_2, y_1, y_2 > 0)$ Now, $y_1 > 0$ and since $x_2 > 0$, $y_1 x_2 > 0 \cdot x_2 \iff y_1 x_2 > 0$. Similarly $x_1 y_2 > 0$. Then $y_1 x_2 + x_1 y_2 > 0 + x_1 y_2 \iff y_1 x_2 + x_1 y_2 > x_1 y_2 > 0$. Thus $x_1 y_2 + y_1 x_2 > 0$. Now, $x_2 > 0$ and since $y_2 > 0$, $x_2 y_2 > 0 \cdot x_2 \iff x_2 y_2 > 0$. Since $x_1 y_2 + y_1 x_2 > 0$ and $x_2 y_2 > 0$, it follows that $x + y > 0$.

Case $(x_1, x_2, y_1, y_2 < 0)$ Now, $y_1 < 0$ and since $x_2 < 0$, $y_1 x_2 < 0 \cdot x_2 \iff y_1 x_2 > 0$. Similarly $x_1 y_2 < 0 \cdot y_2 \iff x_1 y_2 > 0$. Then $y_1 x_2 + x_1 y_2 > 0 + x_1 y_2 \iff y_1 x_2 + x_1 y_2 > x_1 y_2 > 0$. Now, $x_2 < 0$ and since $y_2 < 0$, $x_2 y_2 < 0 \cdot x_2 \iff x_2 y_2 > 0$. Since $x_1 y_2 + y_1 x_2 > 0$ and $x_2 y_2 > 0$, it follows that $x + y > 0$.

Case $(x_1, x_2 > 0, y_1, y_2 < 0)$ Now, $y_1 < 0$ and since $x_2 > 0$, $y_1 x_2 < 0 \cdot x_2 \iff y_1 x_2 < 0$. Similarly, $x_1 y_2 > 0 \cdot y_2 \iff x_1 y_2 < 0$. Then $y_1 x_2 + x_1 y_2 < 0 + x_1 y_2 \iff y_1 x_2 + x_1 y_2 < x_1 y_2 < 0$. Now, $x_2 > 0$ and $y_2 < 0$, so $x_2 y_2 > 0 \cdot y_2 \iff x_2 y_2 < 0$. Since $y_1 x_2 + x_1 y_2 < 0$ and $x_2 y_2 < 0$, it follows that $x + y > 0$.

Case $(x_1, x_2 < 0, y_1, y_2 > 0)$ Now, $y_1 > 0$ and $x_2 < 0$, so $y_1 x_2 < 0 \cdot x_2 \iff y_1 x_2 < 0$. Similarly, $x_1 y_2 < 0 \cdot y_2 \iff x_1 y_2 < 0$. Then $y_1 x_2 + x_1 y_2 < 0 + x_1 y_2 \iff y_1 x_2 + x_1 y_2 < x_1 y_2 < 0$. Now, $x_2 < 0$ and $y_2 > 0$, so $x_2 y_2 < 0 \cdot y_2 \iff x_2 y_2 < 0$. Since $y_1 x_2 + x_1 y_2 < 0$ and $x_2 y_2 < 0$, it follows that $x + y > 0$. ∎

Proof. Let $x, y \in \mathbb{Z}$. Suppose $x, y \in P$. Let $[(x_1, x_2)] = x$ and $[(y_1, y_2)] = y$ such that $x_1, x_2, y_1, y_2 \in \mathbb{Z}$. By definition of $\cdot$, $xy = [(x_1 y_1, x_2 y_2)]$. There are four cases.

Case $(x_1, x_2, y_1, y_2 > 0)$ Since $x_1 > 0$ and $y_1 > 0$ it follows that $x_1 y_1 > 0$. Since $x_2 > 0$ and $y_2 > 0$ it follows that $x_2 y_2 > 0$. Therefore $xy = [(x_1 y_1, x_2 y_2)] \in P$.

Case $(x_1, x_2, y_1, y_2 < 0)$ Since $x_1 < 0$ and $y_1 < 0$ it follows that $x_1 y_1 > 0$. Since $x_2 < 0$ and $y_2 < 0$ it follows that $x_2 y_2 > 0$. Therefore $xy = [(x_1 y_1, x_2 y_2)] \in P$.

Case $(x_1, x_2 > 0,\ y_1, y_2 < 0)$ Since $x_1 > 0$ and $y_1 < 0$ it follows that $x_1 y_1 < 0$. Since $x_2 > 0$ and $y_2 < 0$ it follows that $x_2 y_2 < 0$. Therefore $xy = [(x_1 y_1, x_2 y_2)] \in P$.

Case $(x_1, x_2 < 0,\ y_1, y_2 > 0)$ Since $x_1 < 0$ and $y_1 > 0$ it follows that $x_1 y_1 < 0$. Since $x_2 < 0$ and $y_2 > 0$ it follows that $x_2 y_2 < 0$. Therefore $xy = [(x_1 y_1, x_2 y_2)] \in P$. ∎

Proof. Let $x \in \mathbb{Q}$ such that $x \neq 0$. Let $x = [(x_1, x_2)]$ such that $x_1, x_2 \in \mathbb{Z}$. We want to show that exactly one of x or $-x$ is in P. Consider $-x = [(-x_1, x_2)]$. There are four cases.

Case $(x_1, x_2 > 0)$ Then $x \in P$ by definition. For $-x = [(-x_1, x_2)]$, so $-x_1 < 0$ and $x_2 > 0$, so $-x \notin P$. Thus exactly one of x or $-x$ is in P.

Case $(x_1, x_2 < 0)$ Then $x \in P$ by definition. For $-x = [(-x_1, x_2)]$, so $-x_1 > 0$ and $x_2 < 0$, so $-x \notin P$. Thus exactly one of x or $-x$ is in P.

Case $(x_1 > 0, x_2 < 0)$ Then $x \notin P$ by definition. For $-x = [(-x_1, x_2)]$, so $-x_1 < 0$ and $x_2 < 0$, so $-x \in P$. Thus exactly one of x or $-x$ is in P.

Case $(x_1 < 0, x_2 > 0)$ Then $x \notin P$ by definition. For $-x = [(-x_1, x_2)]$, so $-x_1 > 0$ and $x_2 > 0$, so $-x \in P$. Thus exactly one of x or $-x$ is in P. ∎

Problem 2

Complete the proof of Lemma 1.5.4. That is, prove that the binary relation $+$, the unary operation $^{-1}$ and the relation $<$, on all $\mathbb{Q}$, are well-defined.

Proof. Let $(a, b), (c, d), (x, y), (z, w) \in \mathbb{Q} \times \mathbb{Q}^*$. Suppose $(a, b) \asymp (c, d)$ and $(x, y) \asymp (z, w)$. Thus $ad = bc$ and $xw = zy$.

Then

$$\begin{aligned} & [(a,b)] + [(x,y)] = [(c,d)] + [(z,w)] \\ \iff & (ay + bx, by) \asymp (cw + dz, dw) \\ \iff & (ay + bx)dw = (cw + dz)by \\ \iff & adyw + bxdw = cbyw + dzby. \end{aligned}$$

Since $ad = bc$ and $xw = zy$ it follows that $bcyw + bzyd = bcyw + bdzy$ which holds. Thus $(ay + bx, by) \asymp (cw + dz, dw)$.

Suppose $a \neq 0$ and $c \neq 0$. $[(a, b)]^{-1} = [(b, a)]$ and $[(c, d)]^{-1} = [(d, c)]$. Then $(b, a) \asymp (d, c) \iff bc = da$ which holds. Thus $[(a, b)]^{-1} = [(b, a)] = [(d, c)] = [(c, d)]^{-1}$

Suppose $[(a, b)] < [(x, y)]$. Then $[(a, b)] - [(x, y)] \in P$. But $[(a, b)] = [(c, d)]$ and $[(x, y)] = [(z, w)]$ so $[(c, d)] - [(z, w)] \in P$. It follows that $[(c, d)] < [(z, w)]$. ∎

Problem 3

Let $x \in \mathbb{Z}$ and $y \in \mathbb{Z}^*$.

1. Prove that $[(x, y)] = \bar{0}$ if and only if $x = 0$.
2. Prove that $[(x, y)] = \bar{1}$ if and only if $x = y$.
3. Prove that $\bar{0} < [(x, y)]$ if and only if $0 < xy$.

Proof. Suppose $[(x,y)] = \bar{0}$. Then $[(x,y)] = [(0,1)]$. It follows that $x \cdot 1 = y \cdot 0$. Thus $x = 0$.

Suppose $x = 0$. Then $x \cdot 1 = y \cdot 0$ It follows that $[(x,y)] = [(0,1)]$. Thus $[(x,y)] = \bar{0}$. ∎

Proof. Suppose $[(x,y)] = \bar{1}$. Then $[(x,y)] = [(1,1)]$. It follows that $x \cdot 1 = y \cdot 1$. Thus $x = y$.

Suppose $x = y$. Then $x \cdot 1 = y \cdot 1$. It follows that $[(x,y)] = [(1,1)]$. Thus $[(x,y)] = \bar{1}$. ∎

Proof. Suppose $\bar{0} < [(x,y)]$. By definition, $[(x,y)] - \bar{0} \in P$. But $\bar{0} = [(0,1)]$, so

$$[(x,y)] - [(0,1)] = [(x \cdot 1 - 0 \cdot y, y \cdot 1)] = [(x,y)] \in P$$

Thus either $x, y > 0$ or $x, y < 0$, so in either case $xy > 0$.

Suppose $xy > 0$. Then either $x, y > 0$ or $x, y < 0$. Thus $[(x,y)] \in P$, so $\bar{0} < [(x,y)]$. ∎

Problem 4

Prove Theorem 1.5.5 (1) (2) (3) (5) (6) (8) (9) (11) (12) (14).

Proof. Let $r, s, t \in \mathbb{Q}$. We must show $(r+s)+t = r+(s+t)$. Let $(r_1, r_2), (s_1, s_2), (t_1, t_2) \in \mathbb{Z} \times \mathbb{Z}^*$. Then

$$\begin{aligned}
(r+s)+t &= ([(r_1,r_2)] + [(s_1,s_2)]) + [(t_1,t_2)] \\
&= [(r_1s_2 + r_2s_1, r_2s_2)] + [(t_1,t_2)] \\
&= [((r_1s_2 + r_2s_1)t_2 + t_1(r_2s_2), r_2s_2t_2)] \\
&= [(r_1s_2t_2 + r_2s_1t_2 + t_1r_2s_2, r_2s_2t_2)] \\
&= [(r_1s_2t_2 + (r_2s_1t_2 + t_1r_2s_2), r_2s_2t_2)] \\
&= [(r_1s_2t_2 + r_2(s_1t_2 + s_2t_1), r_2s_2t_2)] \\
&= [(r_1,r_2)] + [(s_1t_2 + s_2t_1, s_2t_2)] \\
&= [(r_1,r_2)] + ([(s_1,s_2)] + [(t_1,t_2)]) \\
&= r + (s+t)
\end{aligned}$$

∎

Proof. Let $r, s \in \mathbb{Q}$. We must show $r + s = s + r$. Let $(r_1, r_2), (s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}^*$. Then

$$\begin{aligned}
r+s &= [(r_1,r_2)] + [(s_1,s_2)] \\
&= [(r_1s_2 + s_1r_2, r_2s_2)] \\
&= [(s_1r_2 + r_1s_2, s_2r_2)] \\
&= [(s_1,s_2)] + [(r_1,r_2)] \\
&= s+r
\end{aligned}$$

∎

Proof. Let $r \in \mathbb{Q}$. We must show $r + \bar{0} = r$. Let $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z}^*$. Then

$$\begin{aligned}
r + \bar{0} = &= [(r_1,r_2)] + [(0,1)] \\
&= [(r_1 \cdot 1 + 0 \cdot r_2, r_2 \cdot 1)] \\
&= [(r_1,r_2)] \\
&= r
\end{aligned}$$

∎

Proof. Let $r, s, t \in \mathbb{Q}$. We must show $(rs)t = r(st)$. Let $(r_1, r_2), (s_1, s_2), (t_1, t_2) \in \mathbb{Z} \times \mathbb{Z}^*$. Then

$$\begin{aligned}(rs)t &= ([(r_1, r_2)] \cdot [(s_1, s_2)]) \cdot [(t_1, t_2)] \\ &= [(r_1 s_1, r_2 s_2)] \cdot [(t_1, t_2)] \\ &= [(r_1 s_1 t_1, r_2 s_2 t_2)] \\ &= [(r_1, r_2)] \cdot [(s_1 t_1, s_2 t_2)] \\ &= [(r_1, r_2)] \cdot ([(s_1, s_2)] \cdot [(t_1, t_2)]) \\ &= r(st)\end{aligned}$$

∎

Proof. Let $r, s \in \mathbb{Q}$. We must show $rs = sr$. Let $(r_1, r_2), (s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}^*$. Then

$$\begin{aligned}rs &= [(r_1, r_2)] \cdot [(s_1, s_2)] \\ &= [(r_1 s_1, r_2 s_2)] \\ &= [(s_1 r_1, s_2 r_2)] \\ &= [(s_1, s_2)] \cdot [(r_1, r_2)] \\ &= sr\end{aligned}$$

∎

Proof. Let $r \in \mathbb{Q}$. We must show if $r \neq \bar{0}$, then $r \cdot r^{-1} = \bar{1}$. Let $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z}^*$. Suppose $r \neq \bar{0}$. Then

$$\begin{aligned}r \cdot r^{-1} &= [(r_1, r_2)] \cdot [(r_1, r_2)]^{-1} \\ &= [(r_1, r_2)] \cdot [(r_2, r_1)] && r \neq \bar{0} \implies r_1 \neq 0 \implies r_1 \in \mathbb{Z}^* \\ &= [(r_1 r_2, r_2 r_1)] \\ &= \bar{1} && \text{Problem 3 b}\end{aligned}$$

∎

Proof. Let $r, s, t \in \mathbb{Q}$. We must show $r(s + t) = rs + rt$. Let $(r_1, r_2), (s_1, s_2), (t_1, t_2) \in \mathbb{Z} \times \mathbb{Z}^*$.

$$\begin{aligned}r(s + t) &= [(r_1, r_2)]([(s_1, s_2)] + [(t_1, t_2)]) \\ &= [(r_1, r_2)]([s_1 t_2 + t_1 s_2, s_2 t_2]) \\ &= [(r_1(s_1 t_2 + t_1 s_2), r_2(s_2 t_2))] \\ &= [(r_1 s_1 t_2 + r_1 t_1 s_2, r_2 s_2 t_2)] \\ &= [(r_1 s_1, r_2 s_2)] + [(r_1 t_1, r_2 t_2)] \\ &= rs + rt.\end{aligned}$$

∎

Proof. Let $r, s, t \in \mathbb{Q}$. We must show if $r < s$ and $s < t$, then $r < t$. Suppose $r < s$ and $s < t$. It follows that $s - r \in P$ and $t - s \in P$. It follows that $(s - r) + (t - s) = t - r \in P$. Thus $r < t$. ∎

Proof. Let $r, s, t \in \mathbb{Q}$. We must show if $r < s$ then $r + t < s + t$. Supose $r < s$. Then $s - r = s - r + 0 = s - r + t + (-t) = (s + t) - (r + t) \in P$. Thus $r + t < s + t$. ∎

Proof. We must show $\bar{0} \neq \bar{1}$. Suppose $\bar{0} = \bar{1}$. Then $[(0, 1)] = [(1, 1)] \iff 0 \cdot 1 = 1 \cdot 1 \iff 0 = 1$ which is a contradiction. Thus $\bar{0} \neq \bar{1}$. ∎

Problem 5

Prove Theorem 1.5.6 (1) (2) (3).

Theorem 3. *Let $i : \mathbb{Z} \longrightarrow \mathbb{Q}$ be defined by $i(x) = [(x, 1)]$ for all $x \in \mathbb{Z}$.*

1. *Then function $i : \mathbb{Z} \longrightarrow \mathbb{Q}$ is injective.*
2. *$i(0) = \bar{0}$ and $i(1) = \bar{1}$.*
3. *Let $x, y \in \mathbb{Z}$. Then*
   (a) *$i(x + y) = i(x) + i(y)$;*
   (b) *$i(-x) = -i(x)$;*
   (c) *$i(xy) = i(x)i(y)$;*
   (d) *$x < y$ if and only if $i(x) < i(y)$*
4. *For each $r \in \mathbb{Q}$ there are $x, y \in \mathbb{Z}$ such that $y \neq 0$ and $r = i(x)(i(y))^{-1}$.*

Proof. Let $x, y \in \mathbb{Z}$. Suppose $i(x) = i(y)$. Thus $[(x, 1)] = [(y, 1)]$ so $(x, 1) \asymp (y, 1)$. It follows that $x \cdot 1 = y \cdot 1$. From the Identity Law for Multiplication $x = y$. Thus i is injective. ∎

Proof. Notice $i(0) = [(0, 1)] = \bar{0}$ and $i(1) = [(1, 1)] = \bar{1}$. ∎

Proof. Let $x, y \in \mathbb{Z}$. Then

$$i(x + y) = [(x + y, 1)] = [(x \cdot 1 + y \cdot 1, 1 \cdot 1)] = [(x, 1)] + [(y, 1)] = i(x) + i(y)$$

Similarly

$$i(-x) = [(-x, 1)] = -[(x, 1)] = -i(x)$$

Similarly

$$i(xy) = [(xy, 1)] = [(xy, 1 \cdot 1)] = [(x, 1)] \cdot [(y, 1)] = i(x)i(y)$$

Finally suppose $x < y$. Then $i(y - x) = [(y - x, 1)] \in P$ since $y - x \in P$ and $1 \in P$. Thus

$$[(y - x, 1)] = [(y \cdot 1 - x \cdot 1, 1 \cdot 1)] = [(y, 1)] - [(x, 1)] = i(y) - i(x) \in P$$

It follows that $i(x) < i(y)$. ∎

Problem 6

Let $r, s, p, q \in \mathbb{Q}$.

1. Prove that $-1 < 0 < 1$.
2. Prove that if $r < s$ then $-s < -r$.
3. Prove that $r \cdot 0 = 0$.
4. Prove that if $r > 0$ and $s > 0$, then $r + s > 0$ and $rs > 0$.
5. Prove that if $r > 0$, then $\frac{1}{r} > 0$.
6. Prove that if $0 < r < s$, then $\frac{1}{s} < \frac{1}{r}$.
7. Prove that if $0 < r < p$ and $0 < s < q$, then $rs < pq$.

Proof. By Theorem 1.5.5 Part 14, $0 \neq 1$. By Theorem 1.5.5 Part 10, either $1 < 0$ or $0 < 1$. By our definition of $<$ either $-1 \in P$ or $1 \in P$. Suppose $-1 \in P$. Again, by our definition of $<$, $(-1) \cdot (-1) \in P$. But $(-1) \cdot (-1) = 1 \in P$, which is a contradiction. Thus $1 \in P$ and $-1 \notin P$. ∎

Proof. Suppose $r < s$. Then $s - r \in P$. But $(-r) - (-s) = s - r \in P$. Therefore $-s < -r$. ∎

Proof. Let $r = [(a,b)]$ such that $a, b \in \mathbb{Z}$, $b \neq 0$.

$$r \cdot 0 = [(a,b)] \cdot [(0,1)] = [(a \cdot 0, b \cdot 1)] = [(0,b)] = [(0,1)] = 0$$

∎

Proof. Suppose $r > 0$ and $s > 0$. Our definition of $<$ showed that $r + s > 0$ and $rs > 0$. ∎

Proof. Suppose $r > 0$ and for contradiction $\frac{1}{r} < 0$. Then $-\frac{1}{r} > 0$. Then $r \cdot -\frac{1}{r} = -\frac{r}{r} = -1 \notin P$ which is a contradiction. ∎

Proof. Suppose $0 < r < s$. Since $\frac{1}{r} > 0$ by the previous part, by Theorem 1.5.5 Part 13, $r \cdot \frac{1}{r} < s \cdot \frac{1}{r} \iff 1 < s \cdot \frac{1}{r}$. Similarly, multiplying both sides by $\frac{1}{s} > 0$ shows $\frac{1}{s} < \frac{1}{r}$. ∎

Proof. Suppose $0 < r < p$ and $0 < s < q$. Since $r < p$ and $0 < s$, by Theorem 1.5.5 Part 13, $rs < ps$. Similarly, since $s < q$ and $p > 0$, it follows that $ps < pq$. Then by transitive law, $rs < pq$ as required. ∎

Problem 7

1. Prove that $1 < 2$.
2. Let $s, t \in \mathbb{Q}$. Suppose $s < t$. Prove that $\frac{s+t}{2} \in \mathbb{Q}$, and that $s < \frac{s+t}{2} < t$.

Proof. By Problem 6 Part 1, $0 < 1$. By Addition Law for Order, $0 + 1 < 1 + 1 \iff 1 < 2$. ∎

Proof. Now $\frac{1}{2} \in \mathbb{Q}$. Since $\mathbb{Q}$ is closed under multiplication and addition $(s+t) \cdot \frac{1}{2} = (s+t)2^{-1} = \frac{s+t}{2} \in \mathbb{Q}$.

We now show $\frac{s+t}{2} < t$. First, notice $t - \frac{s+t}{2} = \frac{t}{1} + \frac{-(s+t)}{2} = \frac{2t+(-(s+t))1}{2 \cdot 1}$. Clearly $2 \in P$ and $2t + (-(s+t))1 = 2t - s - t = t - s$. Since $s < t$ it follows that $t - s \in P$. Thus $t - \frac{s+t}{2} \in P$. Therefore $\frac{s+t}{2} < t$.

We now show $\frac{s+t}{2} > s$. First, notice $\frac{s+t}{2} - s = \frac{s+t}{2} + \frac{-s}{1} = \frac{(s+t)1+(-2)s}{2 \cdot 1}$. Clearly $2 \in P$ and $(s+t)1 + (-2)s = t - s$. Since $s < t$ it follows that $t - s \in P$. Thus $\frac{s+t}{2} - s \in P$. Therefore $\frac{s+t}{2} > s$.

It follows that $s < \frac{s+t}{2} < t$. ∎

Problem 8

Let $r \in \mathbb{Q}$. Suppose that $r > 0$.

1. Prove that if $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ such that $b \neq 0$, then either $a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.
2. Prove that $r = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ such that $m > 0$ and $n > 0$.

Proof. Suppose $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ such that $b \neq 0$. Since $r > 0$ it follows that $a, b \in P$ or $a, b \notin P$. Suppose $a, b \in P$. Then $a = a - 0 \in P$ and $b = b - 0 \in P$. Thus $a > 0$ and $b > 0$. Suppose $a, b \notin P$. Then $a = a - 0 \notin P$ and $b = b - 0 \notin P$. Thus $-(a - 0) = -a + 0 = 0 - a \in P$ and $-(b - 0) = -b + 0 = 0 - b \in P$. Thus $0 > a$ and $0 > b$. ∎

Proof. Suppose $r = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. By part (1), either $a, b > 0$ or $a, b < 0$. If $a, b > 0$, let $m = a$ and $n = b$. Suppose $a, b < 0$. Then $-a > 0$ and $-b > 0$. Also, $(-a)(-b) = -(-ab) = ab$ so $(a, b) \asymp (-a, -b)$. Thus let $m = -a$ and $n = -b$. ∎

Problem 9

Let $r, s \in \mathbb{Q}$.

1. Suppose $r > 0$ and $s > 0$. Prove that there is some $n \in \mathbb{N}$ such that $s < nr$.
2. Suppose that $r > 0$. Prove that there is some $m \in \mathbb{N}$ such that $\frac{1}{m} < r$.
3. For each $x \in \mathbb{Q}$, let x^2 denote $x \cdot x$. Suppose that $r > 0$ and $s > 0$. Prove that if $r^2 < p$, then there is some $k \in \mathbb{N}$ such that $(r + \frac{1}{k})^2 < p$.

Proof. Since $r, s > 0$, let $r = \frac{a}{b}$ and $s = \frac{c}{d}$ such that $a, b, c, d \in \mathbb{N}$. Furthermore, $n = bc + 1$. Then $nr - s = n \cdot \frac{a}{b} - \frac{c}{d} = n \cdot \left(\frac{a}{b} \cdot \frac{d}{d}\right) - \left(\frac{c}{d} \cdot \frac{b}{b}\right) = \frac{adn}{bd} - \frac{bc}{bd} = \frac{adn-bc}{bd} = \frac{ad(bc+1)-bc}{bd} = \frac{ad(bc)+ad-bc}{bd} = \frac{bc(ad-1)+ad}{bd}$. Now $a, d \in \mathbb{N}$ thus $ad \geq 1$. Thus $bc(ad-1) + ad > 0$. It follows that $s < nr$ as required. ∎

Proof. Since $r > 0$, let $r = \frac{a}{b}$ such that $a, b \in \mathbb{N}$. Furthermore, let $m = b+1$. Then $r - \frac{1}{m} = \frac{a}{b} - \frac{1}{b+1} = \frac{a(b+1)}{b(b+1)} - \frac{b}{b(b+1)} = \frac{ab-b+a}{b(b+1)}$. Now, $a, b \in \mathbb{N}$ thus $ab > b$. Thus $ab - b + a > 0$. It follows that $\frac{1}{m} < r$ as required. ∎

Proof. Let k be an arbitrary natural number. Notice $(r+\frac{1}{k})^2 < p \iff r^2+\frac{2r}{k}+\frac{1}{k^2} < p$. Now $r^2+\frac{2r}{k}+\frac{1}{k^2} < r^2+\frac{2r}{k}+\frac{1}{k}$ thus it remains to show that $r^2 + \frac{2r}{k} + \frac{1}{k} < p$. Now

$$\begin{aligned} p - (r^2 + \frac{2r}{k} + \frac{1}{k}) &= \frac{pk}{k} - \frac{kr^2}{k} - \frac{2r}{k} + \frac{1}{k} \\ &= \frac{pk - kr^2 - 2r - 1}{k} \end{aligned}$$

Clearly $k > 0$ since $k \in \mathbb{N}$ so it remains to show that $pk - kr^2 - 2r - 1 > 0$ for some $k \in \mathbb{N}$. Now, since $r, p > 0$, let $r = \frac{a}{b}$ and $p = \frac{c}{d}$ such that $a, b, c, d \in \mathbb{N}$. Then

$$\begin{aligned} pk - kr^2 - 2r - 1 &= \frac{c}{d}k - k\frac{a^2}{b^2} - 2\frac{a}{b} - 1 \\ &= \frac{kcb^2 - ka^2d - 2abd - b^2d}{db^2} \end{aligned}$$

Now $d, b \in \mathbb{N}$ thus $db^2 > 0$. It remains to show that there is some $k \in \mathbb{N}$ such that $k(b^2c - a^2d) - 2abd - b^2d > 0$. Now since $r^2 < p \iff \frac{a^2}{b^2} < \frac{c}{d} \iff a^2d < b^2c$ it follows that $b^2c - a^2d \geq 1$. Then $k(b^2c - a^2d) - 2abd - b^2d \geq k - 2abd - b^2d$. Let $k = 2abd + b^2d + 1$. Then, clearly $k(b^2c - a^2d) - 2abd - b^2d > 0$ as required. ∎

2.5 Dedekind Cuts

Problem 1

Let $A, B \subseteq \mathbb{Q}$ be Dedekind cuts. Suppose that $A \subset B$. Prove that $B - A$ has more than one element. If you are familiar with the cardinality of sets, prove that $B - A$ is countably infinite.

Proof. Since $A \subset B$, $B - A \neq \emptyset$. Then, let $x \in B - A$ so $x \in B$ and $x \notin A$. By 1.6.1 Part (c) there exists $c < x$ such that $c \in B$. Now $x \in \mathbb{Q} - A$ and since $c < x$ by Lemma 1.6.5 Part (2), $c \in \mathbb{Q} - A$. Thus $c \in B$ and $c \notin A$ thus $c \in B - A$. ∎

Proof. The reader is not familiar with the cardinality of sets as required. ∎

Problem 2

Let T be the set defined in Equation 1.6.1.
1. Prove that T is Dedekind cut.
2. Prove that if $T = D_r$ for some $r \in \mathbb{Q}$, then $r^2 = 2$. [Use Exercise 1.5.7 and Exercise 1.5.9(3)]

Definition of T

$$T = \{x \mid x > 0 \text{ and } x^2 > 2\}$$

Proof. Clearly $1 \in \mathbb{Q}$, $1 > 0$, and $1^2 = 1 < 2$ thus $1 \notin T$. Therefore $T \neq \mathbb{Q}$. Also, $3 \in \mathbb{Q}$, $3 > 0$, and $3^2 = 9 > 2$ thus $3 \in T$. Therefore $T \neq \emptyset$. Satisyfing 1.6.1 Part (a).

Suppose $x \in A$ and $y \in \mathbb{Q}$ such that $y \geq x$. Clearly $y^2 \geq x^2 > 2$ and $y \geq x > 0$ thus $y \in T$. Satisyfing 1.6.1 Part (b).

Let $x \in T$ such that $x > 0$ and $x^2 > 2$. By 1.5.9 Part (2) there exists $k \in \mathbb{N}$ such that $\left(x - \frac{1}{k}\right)^2 > 2$. Clearly $x > x - \frac{1}{k}$ and $x - \frac{1}{k} \in T$. Satisyfing 1.6.1 Part (c). ∎

Proof. Suppose $T = D_r$ for some $r \in \mathbb{Q}$. Then $T = \{x \mid x > 0 \text{ and } x^2 > 2\} = \{x \mid x > r\}$ for some $r \in \mathbb{Q}$. Now, we know that $r \notin D_r = T$, thus $r^2 \leq 2$. For contradiction, suppose $r^2 < 2$. By 1.5.9 Part (2) there exists $k \in \mathbb{N}$ such that $(r^2 + \frac{1}{k})^2 < 2$. Thus $(r^2 + \frac{1}{k})^2 \notin T$ but $r < r + \frac{1}{k}$ thus $r + \frac{1}{k} \in D_r = T$ which is a contradicton. Therefore $r^2 = 2$ as required. ∎

Problem 3

Prove Lemma 1.6.8(3).

Proof. Let $A, B \subseteq \mathbb{Q}$ be a Dedekind cuts. Suppose that $0 \in \mathbb{Q} - A$ and $0 \in \mathbb{Q} - B$. Let

$$M = \{r \in \mathbb{Q} \mid r = ab \text{ for some } a \in A \text{ and } b \in B\}$$

(**a**) Now $0 \in M$ implies $ab = 0$ thus $a = 0 \in A$ or $b = 0 \in B$ which is a contradiction. Thus $M \neq \mathbb{Q}$. We know that $A, B \neq \emptyset$ and $A, B \neq \mathbb{Q}$. Let $a \in A$ and $b \in B$. Then $ab \in M$ thus $M \neq \emptyset$.

(**b**) Let $x \in M$. Suppose $y \in \mathbb{Q}$ such that $y \geq x$. Now $x = ab$ for some $a, b \in \mathbb{Q}$ and $a \neq 0, b \neq 0$. Then, since $y > x = ab$ it follows $\frac{y}{b} > a$. By the definition of Dedekind cuts Part (b), $\frac{y}{b} \in A$. Then $\frac{y}{b} \cdot b = y$ thus $y \in M$.

(**c**) Let $x \in M$. Now $x = ab$ for some $a \in A$ and $b \in B$. By the definition of Dedekind cuts Part (c), there exists $c < a$ such that $c \in A$. Then $cb < ab = x$ and since $c \in A$ and $b \in B$, $cb \in M$ as required. ∎

Problem 4

Let $A \subseteq \mathbb{Q}$ be a Dedekind cut, and let $r \in \mathbb{Q}$.
1. Prove that $A \subset D_r$ if and only if there is some $q \in \mathbb{Q} - A$ such that $r < q$.
2. Prove that ① $A \subseteq D_r$ if and only if ② $r \in \mathbb{Q} - A$ if and only if ③ $r < a$ for all $a \in A$.

Proof. ($\longrightarrow$) Suppose $A \subset D_r$. Let $x \in D_r - A$. Now, $x \notin A$ thus $x \in \mathbb{Q} - A$. Since $x \in D_r$, $x > r$ as required.

($\longleftarrow$) Suppose there is some $q \in \mathbb{Q} - A$ such that $r < q$. Let $x \in A$. Then $x > q > r$ thus $x > r$ and it follows that $x \in D_r$. Thus $A \subset D_r$. ∎

Proof. (① $\implies$ ②) Suppose $A \subseteq D_r$. Suppose $x \in A$ then $x \in D_r$ thus $x > r$. Suppose $r \in A$ then $r > r$ which is a contradiction. Thus $r \in \mathbb{Q} - A$ as required.

(② $\implies$ ③) Suppose $r \in \mathbb{Q} - A$. Let $a \in A$ and suppose $r > a$. By the definition of Dedekind cuts Part (b), $r \in A$ which is a contradicion. Thus for all $a \in A$, $r < a$ as required.

(③ $\implies$ ①) Suppose $r < a$ for all $a \in A$. Let $x \in A$ then $r < a$ by the definition of Dedekind cuts Part (c) $x \in D_r$. Thus $A \subseteq D_r$ as required. ∎

Problem 5

What we call a Dedekind cut is often called an "upper cut", to differentiate it from the analogous "lower cut". Both types of cuts are equally valid, and are mirror images of each other, though upper cuts are slightly simpler to use because the product of positive numbers is positive, whereas the product of negative numbers is not negative.

1. Write a precise definition of lower cuts, modeled upon Definition 1.6.1.
2. Let $A \subseteq Q$ be a Dedekind cut. Find an example to show $\mathbb{Q} - A$ is not necessarily a lower cut.
3. Let $A \subseteq \mathbb{Q}$ be a Dedekind cut. Prove that if $\mathbb{Q} - A$ is not a lower cut, then $m \in \mathbb{Q} - A$ such that $x \leq m$ for all $x \in \mathbb{Q} - A$.
4. Let $A \subseteq \mathbb{Q}$ be a Dedekind cut. Suppose that $\mathbb{Q} - A$ is not a lower cut. Prove that there is a unique element $k \in \mathbb{Q} - A$ such that $\mathbb{Q} - (A \cup \{k\})$ is a lower cut.
5. Let $A \subseteq Q$ be a Dedekind cut. Suppose that $\mathbb{Q} - A$ is not a lower cut. Let k be as in Part (4) of this lemma. Prove that $k \leq x$ for all $x \in A \cup \{k\}$.
6. Let $\mathscr{D}^u$ denote the set of of all Dedekind cuts of $\mathbb{Q}$, and let $\mathscr{D}^l$ denote the set of all lower cuts of $\mathbb{Q}$. Prove that there is a bijective function $\phi : \mathscr{D}^u \longrightarrow \mathscr{D}^l$ such that $A \subseteq B$ implies $\phi(A) \supseteq \phi(B)$ for all $A, B \in \mathscr{D}^u$. Because lower cuts are completely analogous to Dedekind cuts, you may assume that the analog of everything that has been previously proved about Dedekind cuts and lower cuts holds with the roles of Dedekind cuts and lower cuts reversed. [Use Exercise 1.6.1.]

Definition 1 (Lower Cuts)**.** *Let $A \subseteq \mathbb{Q}$ be a set. The set A is a **Dedikind cut** if the following three properties hold.*

1. *$A \neq \emptyset$ and $A \neq \mathbb{Q}$*
2. *Let $x \in A$. If $y \in \mathbb{Q}$ and $y \leq x$, then $y \in A$.*
3. *Let $x \in A$. Then there is some $y \in A$ such that $y > x$.*

Solution (2): Consider the upper cut $D_0 = \{x \in \mathbb{Q} \mid x > 0\}$. Now, $0 \notin D_0$, so $0 \in \mathbb{Q} - D_0$. For $\mathbb{Q} - D_0$ to be a lower cut, it must satisfy that for every $x \in \mathbb{Q} - D_0$ there exists $y \in \mathbb{Q} - D_0$ with $y > x$. But $0 \in \mathbb{Q} - D_0$ is the largest element, so there is no $y \in \mathbb{Q} - D_0$ with $y > 0$. Thus, $\mathbb{Q} - D_0$ is not a lower cut.

Proof. Suppose $\mathbb{Q} - A$ is not a lower cut. For contradiction, suppose there does not exist a maximal element in $\mathbb{Q} - A$.

(**a**) Now, clearly $\mathbb{Q} - A \neq \emptyset$ since $A \neq \mathbb{Q}$. Take $a \in A$ it follows that $a \notin \mathbb{Q} - A$ thus $L \neq \mathbb{Q}$.

(**b**) Let $x \in \mathbb{Q} - A$. Then $x - 1 < x$ and $x - 1 \in \mathbb{Q} - A$.

(**c**) Finally, suppose $x \in \mathbb{Q} - A$. There must exist $b \in \mathbb{Q} - A$ such that $x < b$. Otherwise, x would be the greatest element in $\mathbb{Q} - A$. Thus $\mathbb{Q} - A$ is a lower cut, which is a contradiction. ∎

Proof. Let k be the maximal element in $\mathbb{Q} - A$. We now show $L = \mathbb{Q} - (A \cup \{k\})$ is a lower cut.

(**a**) Now $k - 1 \in L$ thus $L \neq \emptyset$. Take $a \neq k \in A$ it follows that $a \notin L$ thus $L \neq \mathbb{Q}$.

(**b**) Let $x \in L$. Then $x - 1 < x$ and $x - 1 \in L$.

(**c**) Let $x \in L$. $x < \frac{x+k}{2} < k$ thus $\frac{x+k}{2} \in L$. ∎

Proof. Let $x \in A \cup \{k\}$. If $x = k$ then clearly $k \leq x$. If $x \neq k$ then $x > k$ since $k \in \mathbb{Q} - A$. Thus $k \leq x$ as required. ∎

Proof. Let $\phi : \mathscr{D}^u \longrightarrow \mathscr{D}^l$ be a mapping defined as $\phi(A) = \{-x : x \in A\}$.

We first show ϕ is a function. This is trivial since the inverse mapping $B = \{-x : x \in B\}$ exists, so ϕ is a bijection.

Let A be an arbitrary upper cut. We now show $\phi(A)$ is a lower cut. This is also trivial since the argument is identical to the upper cut but flipping the inequality.

Suppose $A, B \in \mathscr{D}^u$ and $A \subseteq B$. Let $x \in B$, if $x \notin A$ then $x \in B - A$, but all elements of A are in B thus $\{-x : x \in A\} \supseteq \{-x : x \in B\}$. Therefore $\phi(A) \supseteq \phi(B)$. ∎

Problem 6

In Definition 1.6.1, Dedekind cuts were defined as subsets of the set $\mathbb{Q}$. However, an examination of this definition reveals that it does not make use of the full features of $\mathbb{Q}$, but only the order relation $<$ on $\mathbb{Q}$. Thus, it is possible to define Dedekind cuts on sets equipped with only order relations, but not necessarily with binary operations such as addition and multiplication.
Let S be a non-empty set, and let $<$ be a relation on S. The relation $<$ is an **order relation** if it satisfies the Trichotemy Law and the Transitive Law, as stated, for example, in Theorem 1.5.5 (10) (11); the set S is an **ordered set** if $<$ is an order relation. For example, the natural numbers, the integers and the rational numbers are all ordered sets. Dedekind cuts can be defined for any ordered set exactly as in Definition 1.6.1.

1. Give an example of an ordered set for which the analog of Lemma 1.6.2 does not hold.
2. Find criteria on an ordered set that would guarantee that the analog of Lemma 1.6.2 holds. The criteria must be defined strictly in terms of the order relation.
3. Verify that the analog of Lemma 1.6.7 holds for arbitrary ordered sets.

Proof. Consider the set $\mathbb{Z}$ equipped with the standard order relation $<$. Let $a \in \mathbb{Z}$ and define

$$D_a = \{x \in \mathbb{Z} \mid x > a\}.$$

Now $a + 1 > a$, thus $a + 1 \in D_a$. Clearly, the definition of Dedekind cuts Part (3) cannot hold since there does not exist $k \in \mathbb{Z}$ such that $a < k < a + 1$. ∎

Lemma 1 (1.6.1)**.** *Let $A \subseteq \mathbb{Q}$ be a set. The set A is a **Dedekind cut** if the following three properties hold.*

1. *$A \neq \emptyset$ and $A \neq \mathbb{Q}$.*
2. *Let $x \in A$. If $y \in \mathbb{Q}$ and $y \geq x$, then $y \in A$.*
3. *Let $x \in A$. Then there is some $y \in A$ such that $y < x$.*

The following criteria are required of an ordered set for Lemma 1.6.2 to hold. Let A be an ordered set.

1. $A \neq \emptyset$.
2. Let $a, b \in A$. There must exist $k \in A$ such that $a < k < b$.
3. Let $a \in A$. There exists $b \in A$ such that $b < a$.

Solution (3): Clearly the proof provided only makes use of the ordering relation and not properties specific to $\mathbb{Q}$.

2.6 Construction of the Real Numbers

Problem 1

Let $r \in \mathbb{Q}$
1. Prove that $D_{-1} = -D_r$, using the Definition 1.6.4 and Definition 1.7.3.
2. Prove that $D_{r^{-1}} = [D_r]^{-1}$, using only Definition 1.7.5 and Definition 1.7.3.

Proof. According to Definition 1.7.3, $D_{-r} = \{x \in \mathbb{Q} \mid x > -r\}$. Futhermore, according to definition 1.7.3, $-D_r = \{x \in \mathbb{Q} \mid -x < c \text{ for some } c \in \mathbb{Q} - D_r\}$. Now suppose $x \in D_{-r}$. Then $x > -r$ and it follows that $-x < r$. Since $r \notin D_r$ it follows that $r \in \mathbb{Q} - D_r$. Thus $x \in -D_r$. Therefore $D_{-r} \subseteq -D_r$. Now suppose $x \in -D_r$. Then $-x < c$ for some $c \in \mathbb{Q} - D_r$. Now $c \leq r$ thus $x > -c > -r$. It follows that $x \in D_{-r}$. Therefore $-D_r \subseteq D_{-r}$. It follows that $D_{-r} = -D_r$. ∎

Proof. According to Definition 1.7.5, if $A > D_0$ then $A^{-1} = \{r \in \mathbb{Q} \mid r > 0 \text{ and } \frac{1}{r} < c \text{ for some } c \in \mathbb{Q} - A\}$; if $A < D_0$ then $A^{-1} = -(-A)^{-1}$. According to Definition 1.7.3 $-A = \{r \in \mathbb{Q} \mid -r < c \text{ for some } c \in \mathbb{Q} - A\}$. There are two cases. Suppose $D_r > D_0$. Now suppose $x \in D_{r^{-1}}$. Then $x > \frac{1}{r} > 0$. Since $r > 0$, we have $\frac{1}{x} < r$. Since $r \in \mathbb{Q} - D_r$, it follows that $x \in [D_r]^{-1}$. Now suppose $x \in [D_r]^{-1}$. Then $x > 0$ and $\frac{1}{x} < c \leq r$ for some $c \in \mathbb{Q} - D_r$. It follows that $x > \frac{1}{r}$, thus $x \in D_{r^{-1}}$. Thus if $D_r > D_0$ then $D_{r^{-1}} = [D_r]^{-1}$. Suppose $D_r < D_0$. Now $-D_r = D_{-r}$ by part 1. Then, since $-r > 0$, we can apply case one to find $(D_{-r})^{-1} = D_{(-r)^{-1}}$. Therefore $[D_r]^{-1} = -[D_{-r}]^{-1} = -D_{(-r)^{-1}} = D_{r^{-1}}$. Thus $D_{r^{-1}} = [D_r]^{-1}$. ∎

Problem 2

Let $A, B \in \mathbb{R}$. Suppose that $A > D_0$ and $B > D_0$. For this exercise, you may use results prior to Theorem 1.7.6.
1. Prove that $AB > D_0$.
2. Prove that $A^{-1} > D_0$.

Proof. Suppose $A > D_0$ and $B > D_0$. Thus $D_0 \supset A, B$. Let $x \in AB$. Then $x = ab$ for some $a \in A$ and $b \in B$. Since $D_0 \supset A, B$ it follows that $a, b \in D_0$ and therefore $a, b > 0$. Thus $x = ab > 0$ and it follows that $x \in D_0$. Therefore $D_0 \supset AB$ thus $AB > D_0$. ∎

Proof. Suppose $A > D_0$. Thus $D_0 \supset A$. Let $x \in A^{-1}$. Since $A > D_0$ it follows from the definition of inverse that $x > 0$ and $\frac{1}{x} < c$ for some $c \in \mathbb{Q} - A$. Since $x > 0$ it follows that $x \in D_0$. Thus $A^{-1} \subset D_0$ thus $A^{-1} > D_0$. ∎

Problem 3

Prove Theorem 1.7.6 (14). For this exercise you may use only results prior to Theorem 1.7.6. [Use Exercise 1.5.6 (1).]

Theorem 4 (1.7.6 Part (14))**.** $D_0 < D_1$ (Non-Triviality)

Proof. Clearly, $D_1 \neq D_0$ since $1 > \frac{1}{2} > 0$ thus $\frac{1}{2} \notin D_1$ and $\frac{1}{2} \in D_0$. Let $x \in D_1$ thus $x > 1 > 0$. Therefore $x > 0$ and it follows that $x \in D_0$. Thus $D_0 \supset D_1$ therefore $D_0 < D_1$. ∎

Problem 4

For this exercise, use only the properties of the real numers stated in Theorem 1.7.6 (1) (2) (3) (4) (10) (11) (12) (14); it is not necessary to use the definition of real numbers as Dedekind cuts. Let

$A, B \in \mathbb{R}$.

1. Prove that ① $A > D_0$ if and only if ② $-A < D_0$, and that ③ $A < D_0$ if and only if ④ $-A > D_0$.
2. Prove that $-(-A) = A$.
3. Prove that $-(A + B) = (-A) + (-B)$.
4. Prove that if $A > D_0$ and $B > D_0$, then $A + B > D_0$, and that if $A < D_0$ and $B < D_0$, then $A + B < D_0$.
5. Prove that $A = (-B) + (A + B) = B + [A + (-B)]$ and $-A = B + [-(B + A)]$.

Proof. (① ⟶ ②) Suppose $A > D_0$. Then

$$\begin{aligned} A > D_0 &\implies A + (-A) > D_0 + (-A) && (12)\\ &\iff D_0 > D_0 + (-A) && (4)\\ &\iff D_0 > (-A) + D_0 && (2)\\ &\iff D_0 > -A && (3) \end{aligned}$$

(② ⟶ ①) Suppose $-A < D_0$.

$$\begin{aligned} -A < D_0 &\implies -A + A < D_0 + A && (12)\\ &\iff A + (-A) < D_0 + A && (2)\\ &\iff D_0 < D_0 + A && (4)\\ &\iff D_0 < A && (3) \end{aligned}$$

(③ ⟶ ④) Suppose $A < D_0$.

$$\begin{aligned} A < D_0 &\implies A + (-A) < D_0 + (-A) && (12)\\ &\iff D_0 < D_0 + (-A) && (4)\\ &\iff D_0 < (-A) + D_0 && (2)\\ &\iff D_0 < -A && (3) \end{aligned}$$

(④ ⟶ ③)

$$\begin{aligned} -A > D_0 &\implies -A + A > D_0 + A && (12)\\ &\iff A + (-A) > D_0 + A && (2)\\ &\iff D_0 > D_0 + A && (4)\\ &\iff D_0 > A && (3) \end{aligned}$$

∎

Proof. By (4) we know $A + (-A) = D_0$ and $(-(-A)) + (-A) = D_0$. Thus $A + (-A) = (-(-A)) + (-A)$. Adding $-(-A)$ to both sides and applying (4) shows $A = -(-A)$. ∎

Proof. Notice

$$\begin{aligned} A + B + ((-A) + (-B)) &= A + ((-A) + (B + (-B))) && (2)\\ &= (A + (-A)) + (B + (-B)) && (2)\\ &= D_0 + D_0 && (4)\\ &= D_0 && (3) \end{aligned}$$

Thus $-(A + B) = (-A) + (-B)$. ∎

Proof. Suppose $A > D_0$ and $B > D_0$. Then, by (12),

$$\begin{aligned} A > D_0 &\implies A + (B + (-B)) > D_0 && (12)\\ &\implies (A + B) + (-B) > D_0 && (2) \end{aligned}$$

Adding B to both sides and cancelling shows $A + B > D_0 + B = B > D_0$. Thus $A + B > D_0$. ∎

Proof. Suppose $A < D_0$ and $B < D_0$. Then, by (12),

$$\begin{aligned} A < D_0 &\implies A + (B + (-B)) < D_0 && (12)\\ &\implies (A + B) + (-B) < D_0 && (2) \end{aligned}$$

Adding B to both sides and cancelling shows $A + B < D_0 + B = B < D_0$. Thus $A + B < D_0$. ∎

Proof. Notice

$$\begin{aligned} A &= A + D_0 && (3)\\ &= A + (B + (-B)) && (4)\\ &= (A + B) + (-B) && (1)\\ &= (-B) + (A + B) && (2)\\ &= ((-B) + A) + B && (1)\\ &= (A + (-B)) + B && (2)\\ &= B + (A + (-B)) && (2) \end{aligned}$$

Thus $A = (-B) + (A + B) = B + (A + (-B))$. Similarly

$$\begin{aligned} -A &= -A + D_0 && (3)\\ &= -A + (B + (-B)) && (4)\\ &= -A + ((-B) + B) && (2)\\ &= (-A + (-B)) + B && (1)\\ &= (-B + (-A)) + B && (2)\\ &= -(B + A) + B && (\text{Part } 3)\\ &= B + (-(B + A)) && (2) \end{aligned}$$

Thus $-A = B + (-(B + A))$. ∎

Problem 5

Prove Theorem 1.7.6 (5) (7). For this exercise, you may use only Parts (1) (2) (3) (4) (10) (11) (12) (14) of the theorem, and anything prior to the theorem. [Use Exercise 1.7.4]

Theorem 5 (1.7.4 Part (5))**.** *Let $A, B, C \in \mathbb{R}$.*

$$(AB)C = A(BC)$$

Theorem 6 (1.7.4 Part (7))**.** *Let $A \in \mathbb{R}$.*

$$A \cdot D_1 = A$$

Proof. Let $A, B, C \in \mathbb{R}$. There are 8 cases depending on whether each of A, B, C is $> D_0$ or $< D_0$.

(**Case 1:** $A, B > D_0, C > D_0$) Suppose $x \in (AB)C$. Then $x = dc$ for some $d \in AB$ and $c \in C$. Now $d = ab$ for some $a \in A$ and $b \in B$. Then $x = dc = (ab)c = a(bc)$. Since $b \in B$ and $c \in C$, $bc \in BC$. Since $a \in A$ and $bc \in BC$, it follows that $x \in A(BC)$.

Suppose $x \in A(BC)$. Then $x = ad$ for some $a \in A$ and $d \in BC$. Now $d = bc$ for some $b \in B$ and $c \in C$. Then $x = ad = a(bc) = (ab)c$. Since $a \in A$ and $b \in B$, $ab \in AB$. Since $ab \in AB$ and $c \in C$ it follows that $x \in (AB)C$.

Thus $(AB)C \subseteq A(BC)$.

(**Case 2:** $A < D_0, B > D_0, C > D_0$) Since $A < D_0$, $-A > D_0$. Now, part Part 1, $(-A)(BC) = ((-A)B)C$. Mulitplying by -1 shows $-[(-A)(BC)] = [-((-A)B)C]$. Then by problem 4, $A(BC) = (AB)C$.

(**Case 3:** $A > D_0, B < D_0, C > D_0$) Since $B < D_0$, $-B > D_0$. By Case 1, $A((-B)C) = (A(-B))C$. Multiplying by -1 shows $A(BC) = (AB)C$.

(**Case 4:** $A > D_0, B > D_0, C < D_0$) Since $C < D_0$, $-C > D_0$. By Case 1, $(AB)(-C) = A(B(-C))$. Multiplying by -1 shows $(AB)C = A(BC)$.

(**Case 5:** $A < D_0, B < D_0, C > D_0$) Since $A < D_0$ and $B < D_0$, $-A > D_0$ and $-B > D_0$. By Case 1, $((-A)(-B))C = (-A)((-B)C)$. Multiplying by -1 twice shows $A(BC) = (AB)C$.

(**Case 6:** $A < D_0, B > D_0, C < D_0$) Since $A < D_0$ and $C < D_0$, $-A > D_0$ and $-C > D_0$. By Case 1, $((-A)B)(-C) = (-A)(B(-C))$. Multiplying by -1 twice shows $A(BC) = (AB)C$.

(**Case 7:** $A > D_0, B < D_0, C < D_0$) Since $B < D_0$ and $C < D_0$, $-B > D_0$ and $-C > D_0$. By Case 1, $A((-B)(-C)) = (A(-B))(-C)$. Multiplying by -1 twice shows $A(BC) = (AB)C$.

(**Case 8:** $A < D_0, B < D_0, C < D_0$) Since $A < D_0$, $B < D_0$, $C < D_0$, we have $-A > D_0$, $-B > D_0$, $-C > D_0$. By Case 1, $((-A)(-B))(-C) = (-A)((-B)(-C))$. Multiplying by -1 three times shows $A(BC) = (AB)C$.

Thus, in all 8 cases, $(AB)C = A(BC)$. ∎

Proof. Let $A \in \mathbb{R}$.

(**Case 1:** $A > D_0$) Suppose $x \in AD_1$. Then $x = ad$ for some $a \in A$ and $d \in D_1$. Since $d > 1$, $ad > a$ thus $ad \in A$. Therefore $AD_1 \subseteq A$.

Suppose $x \in A$. Let $d \in D_1$. Then $d > 1$. Now $\frac{x}{d} < x$. Since $d > 0$ and $x > 0$, $\frac{x}{d} > 0$ thus $\frac{x}{d} \in A$. Then $x = \frac{x}{d} \cdot d \in AD_1$. Thus $A \subseteq AD_1$.

(**Case 2:** $A < D_0$) Now $-A > D_0$ and by **Case 1**, $-A = -AD_1$. Multiplying by -1 shows $A = AD_1$. ∎

Problem 6

Prove the remaining four cases in the proof of Theorem 1.7.6 (9). [Use Exercise 1.7.4]

Theorem 7 (1.7.6 Part 9)**.**

$$A(B+C) = AB + AC \quad (\textit{Distributive Law})$$

Proof. $(A < D_0, B < D_0, C < D_0)$

$(A < D_0, B \geq D_0, C \geq D_0)$

$(A < D_0, B < D_0, C \geq D_0)$

$(D < D_0, B \geq D_0, C < D_0)$ ∎

Problem 7

Prove Theorem 1.7.10. [Use Exercise 1.7.1]

Theorem 8 (1.7.10). *Let $i : \mathbb{Q} \longrightarrow \mathbb{R}$ be defined by $i(r) = D_r$, for all $r \in \mathbb{R}$.*

1. *The function $i : \mathbb{Q} \longrightarrow \mathbb{R}$ is injective.*
2. *$i(0) = D_0$ and $i(1) = D_1$.*
3. *Let $r, s \in \mathbb{Q}$ then*
   (a) *$i(r + s) = i(r) + i(s)$;*
   (b) *$i(-r) = -i(r)$;*
   (c) *$i(rs) = i(r)i(s)$;*
   (d) *if $r \neq 0$ then $i(r^{-1}) = [i(r)]^{-1}$;*
   (e) *$r < s$ if and only if $i(r) < i(s)$.*

Proof. Let x, y be arbitrary elements in $\mathbb{Q}$. Suppose $i(x) = i(y)$. Then $D_x = D_y$. Thus $x = y$. ∎

Proof. Applying the definition of i shows that $i(0) = D_0$ and $i(1) = D_1$. ∎

Proof. Let r, s be arbitrary elements in $\mathbb{Q}$. Then

$$i(r + s) = D_{r+s} = D_r + D_s = i(r) + i(s).$$

$$i(-r) = D_{-r} = -D_r = -i(r).$$

$$i(rs) = D_{rs} = D_r D_s = i(r)i(s).$$

Suppose $r \neq 0$. Then

$$i(r^{-1}) = D_{r^{-1}} = [D_r]^{-1} = [i(r)]^{-1}.$$

Suppose $r < s$. Then

$$D_r < D_s \implies i(r) < i(s).$$

Suppose $i(r) < i(s)$. Then

$$i(r) < i(s) \iff D_r < D_s \iff r < s.$$

∎

Problem 8

This exercise makes use of Exercise 1.6.6. Let S be a non-empty ordered set. The **Dedekind set** of S, denoted S^D, is defined by

$$S^D = \{A \subseteq S \mid A \text{ is a Dedekind cut}\}.$$

For example, we know by definition of $\mathbb{Q}^D = \mathbb{R}$. The order relation $<$ on S^D is defined analogously to Definition 1.7.2.

1. Find an example of an ordered set T for which $T^D = \emptyset$. It is sufficient to state informally the reasion why your example works.
2. Find an example of an ordered set U for which U^D has exactly one element. It is sufficient to state informally the reason why your example works.
3. Verify that S^D satisfies the Least Upper Bound Property.
4. What can you say about $\mathbb{R}^D$? It is sufficient to answer the question informally.

Solution (1): Let $T = \{a\}$ be a one-element ordered set. The only subsets of T are $\emptyset$ and T itself. The empty set is not a Dedekind cut, and T has a least element. Thus T has no Dedekind cuts, and therefore $T^D = \emptyset$.

Solution (2): Consider the ordered set $U = \{1, 2\}$ with $1 < 2$. The subset $\{2\}$ is an upper Dedekind cut: it is nonempty, proper, upward closed, and has no least element. No other subset of U satisfies the definition of an upper Dedekind cut. Thus U^D has exactly one element.

Solution (3): Let $\mathscr{C} \subseteq S^D$ be a nonempty set of Dedekind cuts that is bounded above. Define

$$A = \bigcap_{C \in \mathscr{C}} C.$$

Since each $C \in \mathscr{C}$ is an upper-cut, their intersection A is also an upper-cut. Also A has no least element and is a proper subset of S. Thus $A \in S^D$. We know that A is an upper bound for $\mathscr{C}$ and that it is the least upper bound. Thus S^D satisfies the Least Upper Bound Property.

Solution (4): $\mathbb{R}^D$ is order–isomorphic to $\mathbb{R}$.

3 Properties of the Real Numbers