Lectures on the Hyperreals by Robert Goldblatt

Frosty

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1 Chapter Name

Problem 1

Prove using mathematical induction that for all positive integers n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof. Let $n = 1$ then $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$. Assume the formula is true for some integer $k = n - 1$, thus:

$$1 + 2 + 3 + \cdots + (n-1) = \frac{(n-1)((n-1)+1)}{2}$$

Thus:

$$\begin{aligned}
&1 + 2 + 3 + \cdots + (n-1) + n \\
&= \frac{(n-1)((n-1)+1)}{2} + n \\
&= \frac{(n-1)^2 + n - 1}{2} + \frac{2n}{2} \\
&= \frac{(n-1)^2 + 3n - 1}{2} \\
&= \frac{n^2 - 2n + 1 + 3n - 1}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

∎

Problem 3

You probably recall from your previous mathematical work the *triangle inequality:* for any real numbers x and y,

$$|x + y| \leq |x| + |y|$$

Accepts this as given (or see a calculus text to recall how it is proved). Generalize the triangle inequality, by proving that

$$|x_1 + x_2 + \cdots + x_n \leq |x_1| + |x_2| + \cdots |x_n|,$$

for any positive integer n.

Proof. For $n = 1$, trivially $|x_1| \leq |x_1|$. For $n = 2$, $|x_1 + x_2| \leq |x_1| + |x_2|$ by the triangle inequality. Now assume the formula holds for $k = n - 1$, thus:

$$|x_1 + x_2 + \cdots + x_{n-1}| \leq |x_1| + |x_2| + \cdots |x_{n-1}|$$

Thus:

$$\begin{aligned}
&|x_1 + x_2 + \cdots + x_{n-1} + x_n| \\
&\leq |(x_1 + x_2 + \cdots + x_{n-1}) + x_n| \\
&\leq |x_1 + x_2 + \cdots + x_{n-1}| + |x_n| && \text{triangle inequality} \\
&\leq |x_1| + |x_2| + \cdots + |x_n|
\end{aligned}$$

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Problem 4

Given a positive integer n, recall that $n! = 1 \cdot 2 \cdot 3 \cdots$ (this is read as n factorial). Provide an inductive definition for $n!$. (It is customary to actually start this defintion at $n = 0$, setting $0! = 1$)

Solution

We can define $n!$ as follows. If $n <= 1$, then $n! = 1$. If $n > 1$, then $n! = n(n-1)!$.

Problem 5

Prove that $2^n < n!$ for all $n \geq 4$.

Proof. Let $n = 4$, then $2^4 = 16 < 4! = 24$. Assume the inequality holds for $k = n - 1$, thus:

$$2^{n-1} < (n-1)!$$

Thus:

$$\begin{aligned}
&2^{n-1} \cdot 2 < (n-1)! \cdot n \quad \text{Note: } 2 < 4 \leq n \\
&2^n < n!
\end{aligned}$$

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Problem 7

Prove the familiar geometric progression formula. Namely, suppose that a and r are real numbers with $r \neq 1$. Then show that:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}$$

Proof. Let $n = 1$, then $a = \frac{a-ar^n}{1-r} = \frac{a-ar}{1-r} = \frac{a(1-r)}{1-r} = a$. Assume the formula holds for $k = n - 1$, thus:

$$a + ar + ar^2 + \cdots + ar^{n-2} = \frac{a - ar^{n-1}}{1 - r}$$

Thus

$$\begin{aligned}
&a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \\
&= \frac{a - ar^{n-1}}{1-r} + ar^{n-1} \\
&= \frac{a - ar^{n-1}}{1-r} + \frac{(1-r)ar^{n-1}}{1-r} \\
&= \frac{a - ar^{n-1} + (1-r)(ar^{n-1})}{1-r} \\
&= \frac{a - ar^{n-1} + ar^{n-1} - ar^n}{1-r} \\
&= \frac{a - ar^n}{1-r}
\end{aligned}$$

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Problem 12

Consider the sequence a_n defined inductively as follows:

$$a_1 = 5, a_2 = 7, a_{n+2} = 3a_{n+1} - 2a_n$$

Proof. Let $n = 1$, then $a_1 = 5 = 3 + 2^n = 3 + 2^1 = 5$. Let $n = 2$, then $a_2 = 7 = 3 + 2^n = 3 + 2^2 = 7$ Assume the formula holds for $k < n$ thus:

$$a_{n-1} = 3 + 2^{n-1}$$

and

$$a_{n-2} = 3 + 2^{n-2}$$

So $k = n$ is:

$$a_n = 3a_{n-1} - 2a_{n-2} = 3(3 + 2^{n-1}) - 2(3 + 2^{n-2})$$

Then:

$$\begin{aligned}
&3(3 + 2^{n-1}) - 2(3 + 2^{n-2}) \\
&= 9 + 3 \cdot 2^{n-1} - 6 - 2 \cdot 2^{n-2} \\
&= 3 + 3 \cdot 2^{n-1} - 2^{n-1} \\
&= 3 + 2 \cdot 2^{n-1} \\
&= 3 + 2^n
\end{aligned}$$

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Problem 14

In this problem you will prove some results about the binomial coefficients, using induction. Recall that:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

where n is a positive integer, and $0 \leq k \leq n$.
(a) Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$n \geq 2$ and $k < n$. Hint: You do not need induction to prove this. Bear in mind that $0! = 1$.

(b) Verify that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$. Use these facts, together with part a, to prove by induction on n that $\binom{n}{k}$ is an integer, for all k with $0 \leq k \leq n$. (Note: You may have encountered $\binom{n}{k}$ as the count of the number of k element subsets of a set of n objects; it follows that from this $\binom{n}{k}$ is an integer. What we are asking for here is an inductive proof based on algebra.)
(c) Use part a and induction to prove the Binomical Theorem: For non-negative n and variables x, y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof.

$$\begin{aligned}
&\binom{n-1}{k} + \binom{n-1}{k-1} \\
&= \frac{(n-1)!}{((n-1)-k)!k!} + \frac{(n-1)!}{((n-1)-(k-1))!(k-1)!} \\
&= (n-1)!\left(\frac{1}{((n-1)-k)!k!} + \frac{1}{((n-1)-(k-1))!(k-1)!}\right) \\
&= (n-1)!\left(\frac{1}{((n-1)-k)!k(k-1)!} + \frac{1}{((n-1)-(k-1))!(k-1)!}\right) \\
&= \frac{(n-1)!}{(k-1)!}\left(\frac{1}{((n-1)-k)!k} + \frac{1}{((n-1)-(k-1))!}\right) \\
&= \frac{(n-1)!}{(k-1)!}\left(\frac{1}{(n-k-1)!k} + \frac{1}{(n-k)!}\right) \\
&= \frac{(n-1)!}{(k-1)!}\left(\frac{1}{(n-k-1)!k} + \frac{1}{(n-k)(n-k-1)!}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{k} + \frac{1}{(n-k)}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{n-k}{k(n-k)} + \frac{k}{k(n-k)}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{n}{k(n-k)}\right) \\
&= \frac{n!}{k!(n-k)!}
\end{aligned}$$

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Proof. Let $k = 0$ then, $\binom{n}{0} = \frac{n!}{(n-0)!(0!)} = \frac{n!}{n!} = 1 \in \mathbb{Z}$. Let $k = n$ then, $\binom{n}{n} = \frac{n!}{(n-n)!(n!)} = \frac{n!}{n!} = 1 \in \mathbb{Z}$. Assume this holds for $n-1$, thus for all k where $0 \leq k \leq n-1$:

$$\binom{n-1}{k} \in \mathbb{Z}$$

Then:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Since each of these terms exist in $\mathbb{Z}$ their sum $\binom{n}{k}$ is in $\mathbb{Z}$ since the integers are closed over addition. ∎

Proof. Let $n = 0$. Then:

$$(x+y)^0 = 1 = \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k} = \binom{0}{0} x^0 y^0 = 1 \cdot 1 \cdot 1 = 1$$

Assume the formula holds for $n-1$, thus:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{(n-1)-k} = (x+y)^{n-1}$$

Then:

$$\begin{aligned}
(x+y)^n &= (x+y)^{n-1} \cdot (x+y)s \\
&= \left(\sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{(n-1)-k} \right) \cdot (x+y) \\
&= x \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{(n-1)-k} + y \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{(n-1)-k} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} x^k y^{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k} \\
&= \sum_{k=0}^{n} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) x^k y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\end{aligned}$$

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Problem 15

Criticize the following "proof" showing that all cows are the same color.
It suffices to show that any herd of n cows has the same color. If the herd has but one cow, then trivially all the cows in the herd have the same color. Now suppose that we have a herd of n cows and $n > 1$. Pick out a cow and remove it from the herd, leaving $n-1$ cows; by the induction hypothesis these cows all have the same color. Now put the cow back and remove another cow. (We can do so because $n > 1$.) The remaining $n-1$ again must all be the same color. Hence, the first cow selected and the second cow selected have the same color as those not selected, and so the entire herd of n cows has the same color.

Solution

The proof selects a different set of $n-1$ cows each time.

Problem 16

Prove the converse of Theorem 1.1; that is, prove that the Principle of Mathematical Induction implies the Well-ordering Princple. (This shows that these two principles are logically equivalent, and so from an axiomatic point of view it doesn't matter which we assume is an axiom for the natural numbers.)

Proof. Assume that the principle of mathematical induction holds. Let $G \subseteq \mathbb{N}$ be nonempty. For contradiction, suppose G has no least element. Define $P(n)$ to be the statement: "Nothing $\leq n$ is in G."

If $1 \in G$, then 1 would be the least element of G, a contradiction. So $1 \notin G$ and $P(1)$ is true.

Assume $P(n)$ holds meaning no element of G is $\leq n$. If $n+1 \in G$, then $n+1$ would be the least element of G, a contradiction. Therefore $n+1 \notin G$, and hence $P(n+1)$ holds.

By induction, $P(n)$ holds for all $n \in \mathbb{N}$. So no element of $\mathbb{N}$ is in G, so $G = \emptyset$, contradicting the assumption that G is nonempty. $\blacksquare$