Basic Mathematics by Lang

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Contents

1 Numbers

1.1 Rules For Addition

Leadup Instructions

Justify each step, using commutativity and associativity in proving the following identities.

Problem 1

$(a + b) + (c + d) = (a + d) + (b + c)$

Solution:

$$\begin{aligned} & (a+b)+(c+d) \\ &= ((a+b)+c)+d && \text{associative} \\ &= (a+(b+c))+d && \text{associative} \\ &= d+(a+(b+c)) && \text{commutative} \\ &= (d+a)+(b+c) && \text{associative} \\ &= (a+d)+(b+c) && \text{commutative} \end{aligned}$$

Problem 2

$(a + b) + (c + d) = (a + c) + (b + d)$

Solution:

$$\begin{aligned} (a+b)+(c+d) &= ((a+b)+c)+d && \text{associative} \\ &= (c+(a+b))+d && \text{associative} \\ &= ((c+a)+b)+d && \text{commutative} \\ &= (c+a)+(b+d) && \text{associative} \\ &= (a+c)+(b+d) && \text{commutative} \end{aligned}$$

Problem 3

$(a - b) + (c - d) = (a + c) + (-b - d)$

Solution:

$$\begin{aligned} (a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\ &= (c+(a-b))-d && \text{commutative} \\ &= ((c+a)-b)-d && \text{associative} \\ &= ((a+c)-b)-d && \text{commutative} \\ &= ((a+c)+(-b))+(-d) \\ &= (a+c)+(-b-d) && \text{associative} \end{aligned}$$

Problem 4

$(a - b) + (c - d) = (a + c) - (b + d)$

Solution:

$$
\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\
&= (c+(a-b))-d && \text{commutative} \\
&= ((c+a)+(-b))+(-d) \\
&= (c+a)+((-b)+(-d)) && \text{associative} \\
&= (c+a)-(b+d) \\
&= (a+c)-(b+d) && \text{commutative}
\end{aligned}
$$

Problem 5

$(a-b)+(c-d)=(a-d)+(c-b)$

Solution:

$$
\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\
&= ((a+(-b))+c)-d \\
&= (a+((-b)+c))-d && \text{associative} \\
&= (((-b)+c)+a)-d && \text{commutative} \\
&= ((-b)+c)+(a-d) && \text{associative} \\
&= (c+(-b))+(a-d) && \text{commutative} \\
&= (c-b)+(a-d) \\
&= (a-d)+(c-b) && \text{commutative}
\end{aligned}
$$

Problem 6

$(a-b)+(c-d)=-(b+d)+(a+c)$

Solution:

$$
\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d \quad \text{associative} \\
&= ((a+(-b))+c)+(-d) \\
&= (c+(a+(-b)))+(-d) \quad \text{commutative} \\
&= ((c+a)+(-b))+(-d) \quad \text{associative} \\
&= ((a+c)+(-b))+(-d) \quad \text{commutative} \\
&= (a+c)+((-b)+(-d)) \quad \text{associative} \\
&= ((-b)+(-d))+(a+c) \quad \text{commutative} \\
&= (-b-d)+(a+c) \\
&= -(b+d)+(a+c) \quad \text{distributive property}
\end{aligned}
$$

Problem 7

$(a-b)+(c-d)=-(b+d)-(-a-c)$

Solution:

$$\begin{aligned}
(a-b)+(c-d) &= (a+(-b))+(c+(-d)) && \\
&= ((a+(-b))+c)+(-d) && \text{associative} \\
&= (c+(a+(-b)))+(-d) && \text{commutative} \\
&= ((c+a)+(-b))+(-d) && \text{associative} \\
&= (c+a)+((-b)+(-d)) && \text{associative} \\
&= ((-b)+(-d))+(c+a) && \text{commutative} \\
&= -(b+d)+(c+a) && \\
&= -(b+d)+(-(-c)+-(-a)) && \\
&= -(b+d)-(-c-a) && \\
&= -(b+d)-(-a-c) && \text{commutative}
\end{aligned}$$

Problem 8

$((x+y)+z)+w = (x+z)+(y+w)$

Solution:

$$\begin{aligned}
((x+y)+z)+w &= (z+(x+y))+w && \text{commutative} \\
&= ((z+x)+y)+w && \text{associative} \\
&= (z+x)+(y+w) && \text{associative} \\
&= (x+z)+(y+w) && \text{commutative}
\end{aligned}$$

Problem 9

$(x-y)-(z-w) = (x+w)-y-z$

Solution:

$$\begin{aligned}
(x-y)-(z-w) &= (x+(-y))+((-z)+w) && \\
&= ((x+(-y))+(-z))+w && \text{associative} \\
&= (x+((-y)+(-z)))+w && \text{associative} \\
&= (((-y)+(-z))+x)+w && \text{commutative} \\
&= ((-y)+(-z))+(x+w) && \text{associative} \\
&= (x+w)+((-y)+(-z)) && \text{commutative} \\
&= (x+w)-y-z &&
\end{aligned}$$

Problem 10

$(x-y)-(z-w) = (x-z)+(w-y)$

Solution:

$$\begin{aligned}
(x-y)-(z-w) &= (x+(-y))+((-z)+w) && \text{distributive}\\
&= ((x+(-y))+(-z))+w && \text{associative}\\
&= (x+((-y)+(-z)))+w && \text{commutative}\\
&= (((-y)+(-z))+x)+w && \text{associative}\\
&= ((-y)+((-z)+x))+w && \text{associative}\\
&= w+((-y)+((-z)+x)) && \text{commutative}\\
&= (w+(-y))+((-z)+x) && \text{associative}\\
&= (w+(-y))+(x+(-z)) && \text{commutative}\\
&= (w-y)+(x-z)
\end{aligned}$$

Problem 11

Show that $-(a+b+c) = -a+(-b)+(-c)$.

Solution:

$$\begin{aligned}
-(a+b+c) &= -(a+(b+c))\\
&= (-a+-(b+c)) && \text{distributive}\\
&= (-a+(-b+(-c))) && \text{distributive}\\
&= -a+(-b)+(-c)
\end{aligned}$$

Problem 12

Show that $-(a-b-c) = -a+b+c$.

Solution:

$$\begin{aligned}
-(a-b-c) &= -(a+(-b)+(-c))\\
&= (-a-(-b)-(-c)) && \text{distributive}\\
&= (-a+b+c) && \text{double negation}\\
&= -a+b+c
\end{aligned}$$

Problem 13

Show that $-(a-b) = b-a$.

Solution:

$$\begin{aligned}
-(a-b) &= (-a)-(-b) && \text{distributive}\\
&= -a+b && \text{double negation}\\
&= b+(-a) && \text{commutative}\\
&= b-a
\end{aligned}$$

Solve for x in the following equations.

Problem 14

$-2 + x = 4$

Solution:

$$\begin{aligned} -2 + x &= 4 \\ = -2 + 2 + x &= 4 + 2 \\ = x &= 6 \end{aligned}$$

Problem 19

$-5 - x = -2$

Solution:

$$\begin{aligned}
-5 - x &= -2 \Leftrightarrow && \\
(-5 - x) + x &= -2 + x \Leftrightarrow && \\
-5 + ((-x) + x) &= -2 + x \Leftrightarrow && \text{associative} \\
-5 + 0 &= -2 + x \Leftrightarrow && \text{N2} \\
-5 &= -2 + x \Leftrightarrow && \text{N1} \\
-5 + 2 &= (-2 + x) + 2 \Leftrightarrow && \\
-5 + 2 &= 2 + (-2 + x) \Leftrightarrow && \text{commutative} \\
-5 + 2 &= (2 + (-2)) + x \Leftrightarrow && \text{associative} \\
-5 + 2 &= 0 + x \Leftrightarrow && \text{N2} \\
-3 &= x && \text{N1}
\end{aligned}$$

Problem 20

$-7 + x = -10$

Solution:

$$\begin{aligned}
-7 + x &= -10 \Leftrightarrow && \\
(-7 + x) + 7 &= -10 + 7 \Leftrightarrow && \\
7 + (-7 + x) &= -3 \Leftrightarrow && \text{commutative} \\
(7 + (-7)) + x &= -3 \Leftrightarrow && \text{associative} \\
0 + x &= -3 \Leftrightarrow && \text{N2} \\
x &= -3 &&
\end{aligned}$$

Problem 21

$-3 + x = 4$

Solution:

$$\begin{aligned}
-3 + x = 4 &\Leftrightarrow \\
(-3 + x) + 3 = 4 + 3 &\Leftrightarrow \\
3 + (-3 + x) = 7 &\Leftrightarrow && \text{commutative} \\
(3 + (-3)) + x = 7 &\Leftrightarrow && \text{associative} \\
0 + x = 7 &\Leftrightarrow && \text{N2} \\
x &= 7
\end{aligned}$$

22 Prove the cancellation law for addition

If $a + b = a + c$ then $b = c$.

Solution:

$$\begin{aligned}
a + b = a + c &\Leftrightarrow \\
(a + b) + (-a) = (a + c) + (-a) &\Leftrightarrow \\
-a + (a + b) = -a + (a + c) &\Leftrightarrow && \text{commutative} \\
(-a + a) + b = (-a + a) + c &\Leftrightarrow && \text{associative} \\
0 + b = 0 + c &\Leftrightarrow && \text{N2} \\
b &= c
\end{aligned}$$

23 Prove

If $a + b = a$, then $b = 0$.

Solution:

$$\begin{aligned}
a + b = a &\Leftrightarrow \\
(a + b) + (-a) = a + (-a) &\Leftrightarrow \\
(-a) + (a + b) &= a - a && \text{commutative} \\
(-a) + (a + b) &= 0 && \text{N2} \\
((-a) + a) + b = 0 &\Leftrightarrow && \text{associative} \\
0 + b &= 0 && \text{N2} \\
b &= 0
\end{aligned}$$

1.2 Rules For Multiplication

Express each of the following expressions in the form $2^m 3^n a^r b^s$, where m, n, r, s are positive integers.

Problem 1

(a) $8a^2b^3(27a^4)(2^5ab)$
(b) $16b^3a^2(6ab^4)(ab)^3$
(c) $3^2(2ab)^3(16a^2b^5)(24b^2a)$
(d) $24a^3(2ab^2)^3(3ab)^2$

(e) $(3ab)^2(27a^3b)(16ab^5)$
(f) $32a^4b^5a^3b^2(6ab^3)^4$

Solution: 1 (a)

$$\begin{aligned}
8a^2b^3(27a^4)(2^5ab) &= 8(27a^4)a^2b^3(2^5ab) && \text{commutative}\\
&= (8\cdot 27)a^4a^2b^3(2^5ab) && \text{associative}\\
&= (8\cdot 27)(2^5ab)a^4a^2b^3 && \text{commutative}\\
&= (8\cdot 27\cdot 2^5)aba^4a^2b^3 && \text{associative}\\
&= (8\cdot 27\cdot 2^5)aa^4a^2bb^3 && \text{commutative}\\
&= (2^3 3^3 2^5)a^7b^4 && \text{N11}\\
&= (2^3 2^5 3^3)a^7b^4 && \text{commutative}\\
&= (2^8 3^3)a^7b^4 && \text{N11}\\
&= 2^8 3^3 a^7 b^4
\end{aligned}$$

Solution: 1 (b)

$$\begin{aligned}
16b^3a^2(6ab^4)(ab)^3 &= b^3a^2(ab)^3(6ab^4)16 && \text{commutative}\\
&= b^3a^2(ab)^3 6(ab^4)16 && \text{associative}\\
&= b^3a^2(ab)^3(ab^4)16\cdot 6 && \text{commutative}\\
&= b^3a^2a^3b^3ab^4 16\cdot 6 && \text{N12}\\
&= a^2a^3ab^3b^3b^4 16\cdot 6 && \text{commutative}\\
&= a^2a^3ab^3b^3b^4 2^4 2\cdot 3\\
&= a^6b^{10}2^5 3 && \text{N11}\\
&= 2^5 3a^6b^{10} && \text{commutative}
\end{aligned}$$

Solution: 1 (c)

$$\begin{aligned}
3^2(2ab)^3(16a^2b^5)(24b^2a) &= 3^2 2^3 a^3 b^3 (16a^2b^5)(24b^2a) && \text{N12}\\
&= 2^3 24\cdot 3^2 16a^3a^2ab^3b^5b^2 && \text{commutative}\\
&= 2^3 3\cdot 2^3 3^2 2^4 a^3a^2ab^3b^5b^2\\
&= 2^3 2^3 2^4 3^2 3a^3a^2ab^3b^5b^2 && \text{associative}\\
&= 2^{10}3^3a^6b^{10} && \text{N11}
\end{aligned}$$

Solution: 1 (d)

$$\begin{aligned}
24a^3(2ab^2)^3(3ab)^2 &= 24a^3 2^3 a^3 (b^2)^3 3^2 a^2 b^2 && \text{N12}\\
&= 24a^3 2^3 a^3 b^6 3^2 a^2 b^2 && \text{N12}\\
&= 24 \cdot 2^3 3^2 a^3 a^3 a^2 b^6 b^2 && \text{commutative}\\
&= 2^3 3 \cdot 2^3 3^2 a^3 a^3 a^2 b^6 b^2 &&\\
&= 2^3 2^3 3^2 3 a^3 a^3 a^2 b^6 b^2 && \text{commutative}\\
&= 2^6 3^3 a^8 b^8 && \text{N11}
\end{aligned}$$

Solution: 1 (e)

$$\begin{aligned}
(3ab)^2(27a^3b)(16ab^5) &= 3^2 a^2 b^2 27 a^3 b 16 ab^5 && \text{N12}\\
&= 27 \cdot 16 \cdot 3^2 a^2 a^3 a b^2 b b^5 && \text{commutative}\\
&= 3^3 2^4 3^2 a^2 a^3 a b^2 b b^5 &&\\
&= 2^4 3^3 3^2 a^2 a^3 a b^2 b b^5 && \text{commutative}\\
&= 2^4 3^5 a^6 b^8 && \text{N11}
\end{aligned}$$

Solution: 1 (f)

$$\begin{aligned}
32a^4b^5a^3b^2(6ab^3)^4 &= 32a^4 b^5 a^3 b^2 6^4 a^4 (b^3)^4 && \text{N12}\\
&= 32a^4 b^5 a^3 b^2 6^4 a^4 b^{12} && \text{N12}\\
&= 6^4 32 a^3 a^4 a^4 b^5 b^2 b^{12} && \text{commutative}\\
&= (2 \cdot 3)^4 2^5 a^3 a^4 a^4 b^5 b^2 b^{12} &&\\
&= 2^4 3^4 2^5 a^3 a^4 a^4 b^5 b^2 b^{12} && \text{N12}\\
&= 2^4 2^5 3^4 a^3 a^4 a^4 b^5 b^2 b^{12} && \text{commutative}\\
&= 2^9 3^4 a^{11} b^{19} && \text{N11}
\end{aligned}$$

Problem 2

Prove
(a) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
(b) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Solution: 2 (a)

$$
\begin{aligned}
(a+b)^3 &= (a+b)(a+b)(a+b) \\
&= ((a+b)(a+b))(a+b) && \text{associative} \\
&= (a(a+b)+b(a+b))(a+b) && \text{distributive} \\
&= (a^2+ab+ba+b^2)(a+b) && \text{distributive} \\
&= (a^2+2ab+b^2)(a+b) \\
&= a^2(a+b)+2ab(a+b)+b^2(a+b) && \text{distributive} \\
&= a^2a+a^2b+2aba+2abb+b^2a+b^2b && \text{distributive} \\
&= a^3+a^2b+2a^2b+2ab^2+b^2a+b^3 && \text{N11} \\
&= a^3+3a^2b+3ab^2+b^3
\end{aligned}
$$

Solution: 2 (b)

$$
\begin{aligned}
(a-b)^3 &= (a-b)(a-b)(a-b) \\
&= ((a-b)(a-b))(a-b) && \text{associative} \\
&= (a(a-b)-b(a-b))(a-b) && \text{distributive} \\
&= (a^2-ab-ba+b^2)(a-b) && \text{distributive} \\
&= (a^2-2ab+b^2)(a-b) \\
&= a^2(a-b)-2ab(a-b)+b^2(a-b) && \text{distributive} \\
&= a^2a-a^2b-2aba+2abb+b^2a-b^2b && \text{distributive} \\
&= a^3-a^2b-2a^2b+2ab^2+b^2a-b^3 && \text{N11} \\
&= a^3-3a^2b+3ab^2-b^3
\end{aligned}
$$

Problem 3

Obtain expansions for $(a+b)^4$ and $(a-b)^4$.

Solution: 3

From 2: $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

$$
\begin{aligned}
(a+b)^3\cdot(a+b) &= (a+b)^4 && \text{N11} \\
&= ((a^3+3a^2b)+(3ab^2+b^3))\cdot(a+b) \\
&= (a^3+3a^2b)(a+b)+(3ab^2+b^3)(a+b) && \text{distributive} \\
&= (a^3(a+b)+3a^2b(a+b))+(3ab^2(a+b)+b^3(a+b)) && \text{distributive} \\
&= (aa^3+ba^3)+(a3a^2b+b3a^2b)+(a3ab^2+b3ab^2)+(ab^3+bb^3) && \text{distributive} \\
&= (a^4+a^3b)+(3a^3b+3a^2b^2)+(3a^2b^2+3ab^3)+(ab^3+b^4) && \text{N11} \\
&= a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{aligned}
$$

From prev: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

$$\begin{aligned}
(a-b)^4 &= (a+(-b))^4 \\
&= a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + (-b)^4 \\
&= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4
\end{aligned}$$

Problem 5
$(1-2x)^2$

Solution

$$\begin{aligned}
(1-2x)^2 &= (1-2x)\cdot(1-2x) \\
&= (1(1-2x) - 2x(1-2x)) && \text{distributive} \\
&= ((1-2x) - (2x - 2x2x)) && \text{distributive} \\
&= (1-2x) - (2x - 4x^2) && \text{N11} \\
&= ((1-2x) - 2x) - 4x^2 && \text{associative} \\
&= (1 + ((-2x) - 2x)) - 4x^2 && \text{associative} \\
&= (1 + (-4x)) - 4x^2 \\
&= 1 - 4x - 4x^2
\end{aligned}$$

Problem 7
$(x-1)^2$

Solution

$$\begin{aligned}
(x-1)^2 &= (x-1)\cdot(x-1) \\
&= x^2 - 2x + 1 \quad \text{perfect square}
\end{aligned}$$

Problem 11
$(1+x^3)(1-x^3)$

Solution

$$(1+x^3)(1-x^3) = (1-x^6) \quad \text{difference of squares}$$

Problem 13
$(x^2-1)^2$

Solution

$$(x^2-1)^2 = x^4 - 2x^2 + 1 \quad \text{perfect square}$$

Problem 17

$(x^3 - 4)(x^3 + 4)$

Solution

$$(x^3 - 4)(x^3 + 4) = x^6 - 16 \quad \text{difference of squares}$$

Problem 19

$(-2 + 3x)(-2 - 3x)$

Solution

$$(-2 + 3x)(-2 - 3x) = 4 - 9x^2 \quad \text{difference of squares}$$

Problem 23

$(-1 - x)(-2 + x)(1 - 2x)$

Solution

$$\begin{aligned}
(-1 - x)(-2 + x)(1 - 2x) &= (2 + x - x^2)(1 - 2x) && \text{distributive}\\
&= (2(1 - 2x) + x(1 - 2x) - x^2(1 - 2x)) && \text{distributive}\\
&= 2 - 4x + x - 2x^2 - x^2 + 2x^3 && \text{distributive}\\
&= 2 - 3x - 3x^2 + 2x^3
\end{aligned}$$

Problem 29

$(2x + 1)^2(2 - 3x)$

Solution

$$\begin{aligned}
(2x + 1)^2(2 - 3x) &= (4x^2 + 4x + 1)(2 - 3x) && \text{perfect square}\\
&= (4x^2(2 - 3x) + 4x(2 - 3x) + 1(2 - 3x)) && \text{distributive}\\
&= (8x^2 - 12x^3 + 8x - 12x^2 + 2 - 3x) && \text{distributive}\\
&= (-12x^3 - 4x^2 + 5x + 2)
\end{aligned}$$

Problem 30

The population of a city in 1910 was $50,000$, and it doubles every 10 years. What will it be (a) in 1970 (b) in 1990 (c) in $2,000$?

Solution

$$\begin{aligned}
&(a)\ 50000 \cdot 2^{((1970-1910)/10)} = 3200000\\
&(b)\ 50000 \cdot 2^{((1990-1910)/10)} = 12800000\\
&(c)\ 50000 \cdot 2^{((2000-1910)/10)} = 25600000
\end{aligned}$$

Problem 31

The population of a city in 1905 was 100, 000, and it doubles every 25 years. What will it be after (a) 50 years (b) 100 years (c) 150 years?

Solution

$$\begin{aligned}
&(a)\ 100000 \cdot 2^{(50/25)} = 400000\\
&(b)\ 100000 \cdot 2^{(100/25)} = 1600000\\
&(c)\ 100000 \cdot 2^{(150/25)} = 6400000
\end{aligned}$$

Problem 32

The population of a city was 200 thousand in 1915, and it triples every 50 years. What will be the population What will be the population
(a) in the year 2215?
(b) in the year 2165?

Solution

$$\begin{aligned}
&(a)\ 200000 \cdot 3^{((2215-1915)/50)} = 145800000\\
&(b)\ 200000 \cdot 3^{((2165-1915)/50)} = 48600000
\end{aligned}$$

Problem 33

The population of a city was 25, 000 in 1870, and it triples every 40 years. What will it be.
(a) in 1990?
(b) in 2030?

Solution

$$\begin{aligned}
&(a)\ 25000 \cdot 3^{((1990-1870)/40)} = 675000\\
&(b)\ 25000 \cdot 3^{((2030-1870)/40)} = 2025000
\end{aligned}$$

1.3 Even and Odd Integers; Divisibility

Problem 1

Give the proofs for the cases of theorem 1 which were not proved in the text.
(a) If a is even and b is even, then $a + b$ is even.
(b) If a is odd and b is even, then $a + b$ is odd.
(c) If a is odd and b is odd, then $a + b$ is even.

Solution (a)

Since a and b are even they can be written as $2n_1$ and $2n_2$ respectively, where n_1 and n_2 are integers. Let $x = n_1 + n_2$. Note x is an integer because the sum of two integers is an integer.

$$\begin{aligned} a + b &= 2n_1 + 2n_2 \\ &= 2(n_1 + n_2) \\ &= 2x \end{aligned}$$

Since $a + b$ can be written as $2x$ where x is an integer; $a + b$ is even.

Solution (b)

$$\begin{aligned} a + b &= 2n_1 + 1 + 2n_2 \\ &= 2n_1 + 2n_2 + 1 \\ &= 2(n_1 + n_2) + 1 && \text{let } x = n_1 + n_2 \\ &= 2x + 1 \end{aligned}$$

Solution (c)

$$\begin{aligned} a + b &= 2n_1 + 1 + 2n_2 + 1 \\ &= 2n_1 + 2n_2 + 2 \\ &= 2(n_1 + n_2 + 1) && \text{let } x = n_1 + n_2 + 1 \\ &= 2x \end{aligned}$$

Problem 2

If a is even and b is any positive integer then ab is even.

Proof. By def. of an even number a can be written as $2n$ where n is an integer.
Let $x = n \cdot b$. Note the product of two integers is an integer ig.
Something about multiplication being repeated addition and the sum of two integers being an integer.

$$\begin{aligned} a \cdot b &= 2n \cdot b \\ &= 2x \end{aligned}$$

Since ab can be written as $2x$ where x is an integer ab is even. ∎

Problem 3

If a is even, then a^3 is even.

Proof. By def. of an even number a can be written as $2n$ where n is an integer.
Let $x = 2^2n^3$. Note x is an integer.

$$\begin{aligned} a^3 &= (2n)^3 \\ &= 2^3n^3 && \text{N12} \\ &= 2 \cdot 2^2n^3 && \text{N11} \\ &= 2x \end{aligned}$$

Since a^3 can be written as $2x$ where x is an integer a^3 is even. ∎

Problem 4

If a is odd, then a^3 is odd.

Proof. By def. of an odd number a can be written as $2n + 1$ where n is an integer.
Let $x = 4n^3 + 6n^2 + 3n$. Note x is an integer.

$$\begin{aligned} a^3 &= (2n+1)^3 \\ &= 8n^3 + 12n^2 + 6n + 1 && \text{distributive} \\ &= 2(4n^3 + 6n^2 + 3n) + 1 && \text{distributive} \\ &= 2x + 1 \end{aligned}$$

Since a^3 can be written as $2x + 1$ where x is an integer a^3 is odd. ∎

Problem 5

If n is even, then $(-1)^n = 1$.

Proof. By def. of an even number n can be written as $2a$ where a is an integer.

$$\begin{aligned} (-1)^n &= (-1)^{2a} \\ &= ((-1)^2)^a && \text{N12} \\ &= 1^a \\ &= 1 \end{aligned}$$

∎

Problem 6

If n is odd, then $(-1)^n = -1$.

Proof. By def. of an odd number n can be written as $2a + 1$ where a is an integer.

$$\begin{aligned} (-1)^n &= (-1)^{2a+1} \\ &= (-1)^{2a} \cdot (-1)^1 && \text{N11} \\ &= 1 \cdot (-1) && \text{2a is even so by prob. 5} \\ &= -1 && \text{N7} \end{aligned}$$

∎

Problem 7

If m, n are odd, then the product mn is odd.

Proof. By def. of an odd number m and n can be written as $2n_1 + 1$ and $2n_2 + 1$ where n_1 and n_2 are integers. Let $x = 2n_1n_2 + n_1 + n_2$. Note x is an integer.

$$\begin{aligned} mn &= (2n_1 + 1)(2n_2 + 1) \\ &= 4n_1n_2 + 2n_1 + 2n_2 + 1 && \text{distributive} \\ &= 2(2n_1n_2 + n_1 + n_2) + 1 && \text{distributive} \\ &= 2x + 1 \end{aligned}$$

Since mn can be written as $2x + 1$ where x is an integer, therefore mn is odd. ∎

Problem 24

Let a, b be integers, Define $a \equiv b\ (mod\ 5)$, which we read "a is congruent to b modulo 5, to mean that $a - b$ is divisible by 5.
Prove if $a \equiv b\ (mod\ 5)$ and $x \equiv y\ (mod\ 5)$ then $a + x \equiv b + y\ (mod\ 5)$ and $ax = by\ (mod\ 5)$.

Proof. Need to show $(a + x) - (b + y) = 5n$ where n is an integer.
From $a \equiv b\ (mod\ 5)$, $a - b = 5n_1$ where n_1 is an integer.
From $x \equiv y\ (mod\ 5)$, $x - y = 5n_2$ where n_2 is an integer.
Let $t = n_1 + n_2$.

$$\begin{aligned} (a + x) - (b + y) &= (a - b) + (x - y) \\ &= 5n_1 + 5n_2 \\ &= 5(n_1 + n_2) \\ &= 5t \end{aligned}$$

Since $(a + x) - (b + y) = 5t$ where t is an integer, $a + x \equiv b + y\ (mod\ 5)$. ∎

Proof. Need to show $ax - by = 5n$ where n is an integer.
From $a \equiv b\ (mod\ 5)$, $a - b = 5n_1$ where n_1 is an integer.
From $x \equiv y\ (mod\ 5)$, $x - y = 5n_2$ where n_2 is an integer.
Let $t = bn_2 - yn_1 - 5n_1n_2$.

$$\begin{aligned} ax - by &= (b - 5n_1)(y + 5n_2) - by \\ &= by + 5bn_2 - 5yn_1 - 25n_2n_1 - by \\ &= 5bn_2 - 5yn_1 - 25n_2n_1 \\ &= 5(bn_2 - yn_1 - 5n_2n_1) \\ &= 5t \end{aligned}$$

Since $ax - by = 5t$ where t is an integer, $ax = by\ (mod\ 5)$. ∎

Problem 25

Let d be a positive integer. Let a, b be integers.
Define $a \equiv b \ (mod\ d)$ to mean that $a - b$ is divisible by d.
Prove that if $a \equiv b \ (mod\ d)$ and $x \equiv y \ (mod\ d)$, then $a + x \equiv b + y \ (mod\ d)$ and $ax = by \ (mod\ d)$.

Proof. Need to show $(a + x) - (b + y) = dn$ where n is an integer.
From $a \equiv b \ (mod\ d)$, $a - b = dn_1$ where n_1 is an integer.
From $x \equiv y \ (mod\ d)$, $x - y = dn_2$ where n_2 is an integer.

$$\begin{aligned}(a + x) - (b + y) &= (a - b) + (x - y) \\ &= dn_1 + dn_2 \\ &= d(n_1 + n_2) && \text{let } t = n_1 + n_2 \\ &= dt\end{aligned}$$

Since $(a + x) - (b + y)$ can be written as dt where t is an integer, $a + x \equiv b + y \ (mod\ d)$. ∎

Proof. Need to show $ax - by = dn$.
From $a \equiv b \ (mod\ d)$, $a - b = dn_1$ where n_1 is an integer.
From $x \equiv y \ (mod\ d)$, $x - y = dn_2$ where n_2 is an integer.

$$\begin{aligned}ax - by &= (b + dn_1)(y + dn_2) - by \\ &= by + bdn_2 + ydn_1 + dbn_1n_2 - by \\ &= bdn_2 + ydn_1 + dbn_1n_2 \\ &= d(bn_2 + yn_1 + bn_1n_2) && \text{Let } t = bn_2 + yn_1 + bn_1n_2 \\ &= dt\end{aligned}$$

Since $ax - by$ can be written as dt where t is an integer, $ax = by \ (mod\ d)$. ∎

Problem 26

Assume that every positive integer can be written in one of the forms $3k$, $3k + 1$, or $3k + 2$ for some integer k.
Show that if the square of a positive integer is divisible by 3, then so is the integer x.

Proof. From the assumptions x can either be written $3k$, $3k + 1$, or $3k + 2$.
Need to show that if $x^2 = 3n_1$, $x = 3n_2$ for some integers n_1 and n_2.
Case 1 ($x = 3k$):
Let $t_1 = 3k^2$

$$\begin{aligned}x^2 &= (3k)^2 \\ &= 3 \cdot 3k^2 \\ &= 3t_1\end{aligned}$$

Therefore in this case x is divisible by 3.
Case 2 ($x = 3k + 1$):

Let $t_2 = 2k^2 + 2k$

$$\begin{aligned} x^2 &= (3k+1)^2 \\ &= 6k^2 + 3k + 3k + 1 \\ &= 6k^2 + 6k + 1 \\ &= 3(2k^2 + 2k) + 1 \\ &= 3t_2 + 1 \end{aligned}$$

In this case x^2 is not divisible by 3 which contradicts our assumption, therefore $x \neq 3k + 1$.
Case 3 $(x = 3k + 2)$:
Let $t_3 = 3k^2 + 4k$

$$\begin{aligned} x^2 &= (3k+2)^2 \\ &= 9k^2 + 6k + 6k + 4 \\ &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k) + 4 \\ &= 3t_3 + 4 \end{aligned}$$

In this case x^2 is not divisible by 3 which contradicts our assumption, therefore $x \neq 3k + 2$.

Note there is no solution for $1 = 3m_1$ or $2 = 3m_2$ where m_1 and m_2 are integers.
Assume $3k_1 + 1 = 3m_1$ where k_1 and m_1 are integers.

$$\begin{aligned} 3k_1 + 1 &= 3m_1 \\ 1 &= 3m_1 - 3k_1 \\ 1 &= 3(m_1 - k_1) \end{aligned}$$

Therefore, $3k_1 + 1$ is not divisible by 3.

Assume $3k_2 + 2 = 3m_2$ where k_2 and m_2 are integers.

$$\begin{aligned} 3k_2 + 1 &= 3m_2 \\ 2 &= 3m_2 - 3k_2 \\ 2 &= 3(m_2 - k_2) \end{aligned}$$

Therefore, $3k_2 + 2$ is not divisible by 3.

∎

1.4 Rational Numbers

Problem 4

Let $a = \frac{m}{n}$ be a rational number expressed as a quotient of integers m, n with $m \neq 0$ and $n \neq 0$.
Show that there is a rational number b such that $ab = ba = 1$.

Proof. Let $b = \frac{n}{m}$. Since n and m are integers and $m \neq 0$, b is the ratio of two integers where the denomitor is not 0 making it a rational number by definition.

$$
\begin{aligned}
ab &= \frac{m}{n} \cdot \frac{n}{m} \\
&= \frac{mn}{nm} \\
&= \frac{nm}{nm} && \text{commutative} \\
&= 1 && \text{cancellation rule for fractions}
\end{aligned}
$$

$$
\begin{aligned}
ba &= \frac{n}{m} \cdot \frac{m}{n} \\
&= \frac{nm}{mn} \\
&= \frac{nm}{nm} && \text{commutative} \\
&= 1 && \text{cancellation rule for fractions}
\end{aligned}
$$

Therefore $ab = ba = 1$. ∎

Problem 6

Solve for x in the following equations.

Solution (d)

$$
\begin{aligned}
\frac{4x}{3} + \frac{3}{4} &= 2x - 5 \\
12\left(\frac{4x}{3} + \frac{3}{4}\right) &= 12(2x - 5) \\
16x + 9 &= 24x - 60 \\
9 + 60 &= 24x - 16x \\
69 &= 8x \\
x &= \frac{69}{8}
\end{aligned}
$$

Solution (e)

$$
\begin{aligned}
\frac{4(1 - 3x)}{7} &= 2x - 1 \\
4(1 - 3x) &= 7(2x - 1) \\
4 - 12x &= 14x - 7 \\
4 + 7 &= 14x + 12x \\
11 &= 26x \\
x &= \frac{11}{26}
\end{aligned}
$$

Solution (f)

$$\begin{aligned}\frac{2-x}{3} &= \frac{7}{8}x \\ 8(2-x) &= 3\cdot 7x \\ 16-8x &= 21x \\ 16 &= 29x \\ x &= \frac{16}{29}\end{aligned}$$

Problem 6

Let n be a positive integer. By n factorial, written $n!$, we mean the product:
$1\cdot 2\cdot 3\ldots n$
of the first n positive integers. For Instance
$2! = 2$
$3! = 2\cdot 3 = 6$
$4! = 2\cdot 3\cdot 4 = 24$
(a) Find the value $5!$, $6!$, $7!$, and $8!$.
(b) Define $0! = 1$. Define the binomial coefficient
$\binom{n}{m} = \frac{m!}{n!(m-n)!}$
for any natural numbers m, n such that n lies between 0 and m. Compute tons of binomial coefficients.
(c) Show that $\binom{m}{n} = \binom{m}{m-n}$.
(d) Show that if n is a positive integer at most equal to m, then
$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$.

Solution (a)

$$\begin{aligned}5! &= 2\cdot 3\cdot 4\cdot 5 && = 120 \\ 6! &= 2\cdot 3\cdot 4\cdot 5\cdot 6 && = 720 \\ 7! &= 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 && = 5040 \\ 8! &= 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8 && = 40320\end{aligned}$$

Solution (b)

$$\binom{3}{0} = \frac{3!}{0!(3-0)!} = 1$$
$$\binom{3}{1} = \frac{3!}{1!(3-1)!} = 3$$
$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$$
$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = 1$$
$$\binom{4}{0} = \frac{4!}{0!(4-0)!} = 1$$
$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = 4$$
$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$$
$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$$
$$\binom{4}{4} = \frac{4!}{4!(4-4)!} = 1$$
$$\binom{5}{0} = \frac{5!}{0!(5-0)!} = 1$$
$$\binom{5}{1} = \frac{5!}{1!(5-1)!} = 5$$
$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10$$
$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$$
$$\binom{5}{4} = \frac{5!}{4!(5-4)!} = 5$$
$$\binom{5}{5} = \frac{5!}{5!(5-5)!} = 1$$

Solution (c)

$$\binom{m}{n} = \binom{m}{m-n}$$
$$\frac{m!}{n!(m-n)!} = \frac{m!}{(m-n)!(m-(m-n))!}$$
$$\frac{m!}{n!(m-n)!} = \frac{m!}{(m-n)!(n)!}$$
$$\frac{m!}{n!(m-n)!} = \frac{m!}{(n)!(m-n)!}$$

Solution (d)
Need to show:

$$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$$

First note:

$$\binom{m+1}{n} = \frac{(m+1)!}{n!((m+1)-n)!}$$

Then:

$$\begin{aligned}
\binom{m}{n} + \binom{m}{n-1} &= \frac{m!}{n!(m-n)!} + \frac{m!}{(n-1)!(m-n+1)!} \\
&= \frac{m!(m-n+1) + m!n}{n!(m-n+1)!} \quad \text{(common denominator)} \\
&= \frac{m!(m-n+1+n)}{n!(m-n+1)!} \\
&= \frac{m!(m+1)}{n!(m-n+1)!} \\
&= \frac{(m+1)!}{n!((m+1)-n)!} \\
&= \binom{m+1}{n}
\end{aligned}$$

Problem 8

Prove that there is no positive rational number a such that $a^3 = 2$.

Proof. Let $a = \frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form. If $(\frac{m}{n})^3 = 2$.

$$\begin{aligned}
\frac{m^3}{n^3} &= 2 \\
m^3 &= 2n^3
\end{aligned}$$

If $m^3 = 2k$ for some integer k, then $m = 2a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned}
(2k)^3 &= 2n^3 \\
2^3k^3 &= 2n^3 \\
2^2k^3 &= n^3 \\
2 \cdot (2k^3) &= n^3
\end{aligned}$$

If $n^3 = 2k$ for some integer k, then $n = 2a$ for some integer a (shown in a previous problem). This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^3 = 2$. ∎

Problem 9

Prove that there is no positive rational number a such that $a^4 = 2$.

Proof. Suppose for contradiction $a^4 = 2$ where a is a rational number. Since a is rational, it can be expressed as $\frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form.

$$\frac{m^4}{n^4} = 2$$
$$m^4 = 2n^4$$

If $m^4 = 2k$ for some integer k, then $m = 2a$ for some integer a (shown in a previous problem). But:

$$(2k)^4 = 2n^4$$
$$2^4k^4 = 2n^4$$
$$2^3k^4 = n^4$$
$$2 \cdot (2^2k^3) = n^4$$

If $n^4 = 2k$ for some integer k, then $n = 2a$ for some integer a (shown in a previous problem). This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^4 = 2$. ∎

Problem 10

Prove that there is no positive rational number a such that $a^2 = 3$. You may assume that a positive integer can be written in one of the forms $3k$, $3k+1$, $3k+2$ for some integer k. Prove that if the square of a positive integer is divisible by 3 so is the integer. Then use a similar proof for $\sqrt{2}$.

Proof. Since a is rational, it can be expressed as $\frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form.

$$\frac{m^2}{n^2} = 3$$
$$m^2 = 3n^2$$

If $m^2 = 3k$ for some integer k, then $m = 3a$ for some integer a (shown in a previous problem). But:

$$(3a)^2 = 3n^2$$
$$3^2a^2 = 3n^2$$
$$3^3a^2 = n^2$$
$$3 \cdot (3^2a^2) = n^2$$

If $n^2 = 3k$ for some integer k, then $n = 3a$ for some integer a. This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^2 = 3$. ∎

Proof. Need to show that $a^2 = 2$ has no rational solution a. Since a is rational, it can be expressed as $\frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form.

$$\frac{m^2}{n^2} = 2$$
$$m^2 = 2n^2$$

If $m^2 = 2k$ for some integer k, then $m = 2a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned}(2a)^2 &= 2n^2\\ 2^2a^2 &= 2n^2\\ 2a^2 &= n^2\\ 2 \cdot a^2 &= n^2\end{aligned}$$

If $n^2 = 2k$ for some integer k, then $n = 2a$ for some integer a. This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^2 = 2$. ∎

Problem 16

A chemical substance decomposes in such a way that it halves every 3 min. If there are 6 grams (g) of the substance at present at the beginning, how much will will be left
(a) after 3 min?
(b) after 27 min?
(c) after 36 min?

Solution
(a) after 3 min? $6(\frac{1}{2})^{(\frac{3}{3})} = 3\text{g}$

(b) after 27 min? $6(\frac{1}{2})^{(\frac{27}{3})} = 0.01171875\text{g}$

(c) after 36 min? $6(\frac{1}{2})^{(\frac{36}{3})} = 0.001464843\text{g}$

Problem 18

A substance reacts in water in such a way that one-fourth of the undissolved parts disolves every 10 minutes. If you put 25g of a substance in water at a given time, how much will be left after:
(a) 10 min?
(b) 30 min?
(c) 50 min?

Solution
(a) after 10 min? $25(\frac{3}{4})^{(\frac{10}{10})} = 18.75\text{g}$

(b) after 30 min? $25(\frac{3}{4})^{(\frac{30}{10})} = 10.546875\text{g}$

(c) after 50 min? $25(\frac{3}{4})^{(\frac{50}{10})} = 5.933\text{g}$

Problem 20

A chemical pollutant is being emptied in a lake with 50, 000 fishes. Every month, one-thrid of the fish still alive die from this pollutant. How many fish will be alive after:
(a) 1 month?
(b) 2 month?
(c) 4 month?

Solution
(a) after 1 month? $50000(\frac{2}{3})^1 = 33333.33\text{fishes}$

(b) after 2 month? $50000(\frac{2}{3})^2 = 22222.22\text{fishes}$

(c) after 4 month? $50000(\frac{2}{3})^4 = 9876.54\text{fishes}$

Problem 21

Every 10 years the population of a city is five-fourths of what it was the 10 years before. How many years does it take
(a) before the population doubles
(b) before it triples

Formula: $p \cdot \frac{5}{4}^{\frac{y}{10}} = 2p$
Solution (a)

$$\begin{aligned} p \cdot \frac{5}{4}^{\frac{y}{10}} &= 2p \\ = y &\approx 31 \text{ years} \end{aligned}$$

Solution (b)

$$\begin{aligned} p \cdot \frac{5}{4}^{\frac{y}{10}} &= 3p \\ = y &\approx 49 \text{ years} \end{aligned}$$

1.5 Multiplicative Inverse

Problem 2

Prove the following relations. It is assumed that all values of x and y which occur are such that the denominators in the indicated fractions are not 0.
(a) $\frac{1}{x+y} - \frac{1}{x-y} = \frac{-2y}{x^2-y^2}$
(b) $\frac{x^3-1}{x-1} = 1 + x + x^2$
(c) $\frac{x^4-1}{x-1} = 1 + x + x^2 + x^3$

Proof.

$$\begin{aligned} \frac{1}{x+y} - \frac{1}{x-y} &= \frac{-2y}{x^2-y^2} \\ (x+y)[\frac{1}{x+y} - \frac{1}{x-y}] &= \frac{-2y}{x^2-y^2}(x+y) \\ 1 - \frac{x+y}{x-y} &= \frac{-2y}{x-y}(x+y) \\ (x-y)(1 - \frac{x+y}{x-y}) &= \frac{-2y}{x-y}(x-y) \\ (x-y)-(x+y) &= -2y \\ -2y &= -2y \end{aligned}$$

∎

Proof.

$$\begin{aligned} &\frac{x^3-1}{x-1} = 1 + x + x^2 \\ &x^3 - 1 = (x-1)(1+x+x^2) \\ &x^3 - 1 = x + x^2 + x^3 - 1 - x - x^2 \\ &x^3 - 1 = x^3 - 1 \end{aligned}$$

Proof.

$$\begin{aligned}
\frac{x^4-1}{x-1} &= 1+x+x^2+x^3 \\
x^4-1 &= (x-1)(1+x+x^2+x^3) \\
x^4-1 &= x+x^2+x^3+x^4-1-x-x^2-x^3 \\
x^4-1 &= x+x^2+x^3+x^4-1-x-x^2-x^3 \\
x^4-1 &= x^4-1
\end{aligned}$$

∎

Proof.

$$\begin{aligned}
\frac{x^n-1}{x-1} &= x^{n-1}+x^{n-2}+\cdots+x+1 \\
x^n-1 &= (x-1)(x^{n-1}+x^{n-2}+\cdots+x+1) \\
x^n-1 &= x^n+x^{n-1}+\cdots+x^2+x-x^{n-1}-x^{n-2}-\cdots-x-1 \\
x^n-1 &= x^n-1
\end{aligned}$$

∎

Problem 3

Prove the following relations.

(a) $\frac{1}{2x+y}+\frac{1}{2x-y}=\frac{4x}{4x^2-y^2}$

(b) $\frac{2x}{x+5}+\frac{3x+1}{2x+1}=\frac{x^2-14x-5}{2x^2+11x+5}$

(c) $\frac{1}{x+3y}+\frac{1}{x-3y}=\frac{2x}{x^2-9y^2}$

(c) $\frac{1}{3x-2y}+\frac{x}{x+y}=\frac{x+y+3x^2-2xy}{3x^2+xy-2y^2}$

Proof.

$$\begin{aligned}
\frac{1}{2x+y}+\frac{1}{2x-y} &= \frac{4x}{4x^2-y^2} \\
\leftrightarrow ((2x+y)(2x-y))\left(\frac{1}{2x+y}+\frac{1}{2x-y}\right) &= ((2x+y)(2x-y))\left(\frac{4x}{4x^2-y^2}\right) \\
\leftrightarrow (2x-y)+(2x+y) &= 4x \\
\leftrightarrow 4x &= 4x
\end{aligned}$$

∎

Proof.

$$\begin{aligned}
\frac{2x}{x+5} + \frac{3x+1}{2x+1} &= \frac{x^2-14x-5}{2x^2+11x+5} \\
\leftrightarrow ((x+5)(2x+1))\left(\frac{2x}{x+5} + \frac{3x+1}{2x+1}\right) &= ((x+5)(2x+1))\left(\frac{x^2-14x-5}{2x^2+11x+5}\right) \\
\leftrightarrow 2x(2x+1) - (3x+1)(x+5) &= x^2-14x-5 \\
\leftrightarrow (4x^2+2x) - (3x^2+15x+5) &= x^2-14x-5 \\
\leftrightarrow x^2-14x-5 &= x^2-14x-5
\end{aligned}$$

∎

Proof.

$$\begin{aligned}
\frac{1}{x+3y} + \frac{1}{x-3y} &= \frac{2x}{x^2-9y^2} \\
\leftrightarrow ((x+3y)(x-3y))\left(\frac{1}{x+3y} + \frac{1}{x-3y}\right) &= ((x+3y)(x-3y))\left(\frac{2x}{x^2-9y^2}\right) \\
\leftrightarrow (x-3y) + (x+3y) &= 2x \\
\leftrightarrow 2x &= 2x
\end{aligned}$$

∎

Proof.

$$\begin{aligned}
\frac{1}{3x-2y} + \frac{x}{x+y} &= \frac{x+y+3x^2-2xy}{3x^2+xy-2y^2} \\
\leftrightarrow ((3x-2y)(x+y))\left(\frac{1}{3x-2y} + \frac{x}{x+y}\right) &= ((3x-2y)(x+y))\left(\frac{x+y+3x^2-2xy}{3x^2+xy-2y^2}\right) \\
\leftrightarrow (x+y) + x(3x-2y) &= x+y+3x^2-2xy \\
\leftrightarrow (x+y) + (3x^2-2xy) &= x+y+3x^2-2xy \\
\leftrightarrow x+y+3x^2-2xy &= x+y+3x^2-2xy
\end{aligned}$$

∎

Problem 4

Prove the following relations.

(a) $\frac{x^3-y^3}{x-y} = x^2+xy+y^2$

(b) $\frac{x^4-y^4}{x-y} = x^3+x^2y+xy^2+y^3$

Let

$x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$

Show that $x^2+y^2 = 1$

Proof.

$$\begin{aligned}\frac{x^3-y^3}{x-y} &= x^2+xy+y^2\\ &\leftrightarrow x^3-y^3=(x-y)(x^2+xy+y^2)\\ &\leftrightarrow x^3-y^3=x^3+x^2y+xy^2-x^2y-xy^2-y^3\\ &\leftrightarrow x^3-y^3=x^3-y^3\end{aligned}$$

∎

Proof.

$$\begin{aligned}&\frac{x^4-y^4}{x-y} = x^3+x^2y+xy^2+y^3\\ &\leftrightarrow x^4-y^4=(x-y)(x^3+x^2y+xy^2+y^3)\\ &\leftrightarrow x^4-y^4=x^4+x^3y+x^2y^2+xy^3-x^3y-x^3y-x^2y^2-xy^3-y^4\\ &\leftrightarrow x^4-y^4=x^4-y^4\end{aligned}$$

∎

Proof.

$$\begin{aligned}x^2+y^2&=1\\ \leftrightarrow \left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2&=1\\ \leftrightarrow \frac{\left(1-t^2\right)^2+\left(2t\right)^2}{\left(1+t^2\right)^2}&=1\\ \leftrightarrow \frac{t^4-2t^2+1+4t^2}{\left(1+t^2\right)^2}&=1\\ \leftrightarrow \frac{t^4+2t^2+1}{\left(1+t^2\right)^2}&=1\\ \leftrightarrow \frac{\left(1+t^2\right)^2}{\left(1+t^2\right)^2}&=1\\ \leftrightarrow 1&=1\end{aligned}$$

∎

Problem 5

Prove the following relations.
(a) $\frac{x^3+1}{x+1}=x^2-x+1$
(b) $\frac{x^5+1}{x+1}=x^4-x^3+x^2-x+1$
(c) If n is an odd integerl, prove that
$\frac{x^n+1}{x+1}=x^{(n-1)}-x^{(n-2)}+x^{(n-3)}-\cdots-x+1$

Proof.

$$\begin{aligned}
&\frac{x^3+1}{x+1} = x^2 - x + 1 \\
\leftrightarrow\ & x^3 + 1 = (x+1)(x^2 - x + 1) \\
\leftrightarrow\ & x^3 + 1 = x^3 - x^2 + x + x^2 - x + 1 \\
\leftrightarrow\ & x^3 + 1 = x^3 + 1
\end{aligned}$$

∎

Proof.

$$\begin{aligned}
&\frac{x^5+1}{x+1} = x^4 - x^3 + x^2 - x + 1 \\
\leftrightarrow\ & x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1) \\
\leftrightarrow\ & x^5 + 1 = x^5 - x^4 + x^3 - x^2 + x + x^4 - x^3 + x^2 - x + 1 \\
\leftrightarrow\ & x^5 + 1 = x^5 + 1
\end{aligned}$$

∎

Proof.

$$\begin{aligned}
&\frac{x^n+1}{x+1} = x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1 \\
\leftrightarrow\ & x^n + 1 = (x+1)(x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1) \\
\leftrightarrow\ & x^n + 1 = x^n - x^{(n-1)} + x^{(n-2)} - \cdots - x^2 + x + x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1 \\
\leftrightarrow\ & x^n + 1 = x^n + 1
\end{aligned}$$

∎

Problem 7

If a solid has a uniform density d, occpies a volume v, and has a mass m, then we have the formula $m = vd$

Find the density if:

(a) $m = \frac{3}{10}$lb and $v = \frac{2}{3}$in^3

(a) $m = 6$lb and $v = \frac{4}{3}$in^3

Find the volume if the mass is 15 lb and the density is $\frac{2}{3}$ lb/in^3.

Solution (a)

$$\begin{aligned}
\frac{3}{10} &= \frac{2}{3}d \\
d &= \frac{9}{20}
\end{aligned}$$

Solution (b)

$$\begin{aligned}
6 &= \frac{4}{3}d \\
d &= \frac{18}{4}
\end{aligned}$$

Solution (**c**)

$$\begin{aligned}15 &= v\frac{2}{3}\\ v &= \frac{45}{2}\end{aligned}$$

Problem 13

Tickets for a performance sell $\$5.00$ and $\$2.00$. The total amount collected was $\$4,100$, and there are $1,300$ tickets in all. How many tickets of each price were sold.

Solution Let x = be the number of tickets sold at $\$2.00$.

$$\begin{aligned}2x + 5(1300 - x) &= 4100\\ 2x + 6500 - 5x &= 4100\\ -3x &= -2400\\ x &= 800\end{aligned}$$

800 tickets sold at $\$2.00$ and 500 sold at $\$5.00$.

Problem 16

A boat travels a distance of 500mi, along two rivers, for 50hr. The current goes in the same direction as the boat along one river, and then the boat averages 20mph. The current goes in the opposite dirction along the other river, and then the boat averages 8mph. During how many hours was the boat on the first river.

Solution Let x be the time spent on the first river.

$$\begin{aligned}20x + 8(50 - x) &= 500\\ 20x + 400 - 8x &= 500\\ 12x &= 100\\ x &= \frac{100}{12}\end{aligned}$$

$x = \frac{100}{12}$ hours on first river.

Problem 18

The radiator of a car can contain 10kg of liquid. If it is half full with a mixture having 60% antifreeze and 40% water, how much more water must be added so that the resulting mixture has only.

Solution (**a**) 40% **antifreeze**
40% antifreeze means 60% water.

$$\begin{aligned}\frac{4 + x}{10 + x} &= 0.6\\ 4 + x &= 0.6(10 + x)\\ 4 + x &= 6 + 0.6x\\ x &= 5\end{aligned}$$

15 kg will fit in the radiator.
Solution (**b**) 10% **antifreeze**

10% antifreeze means 90% water.

$$\begin{aligned}\frac{4+x}{10+x} &= 0.9\\ 4+x &= 0.9(10+x)\\ 4+x &= 9+0.9x\\ x &= 50\end{aligned}$$

55 kg will not fit in the radiator.

2 Linear Equations

2.1 Equations in Two Unknowns

Problem 7

Solve the following systems of equations for x and y.

$$\begin{aligned}7x-y&=2\\ 2x+2y&=4\end{aligned}$$

Solution 7

$$\begin{aligned}7x-y=2 \leftrightarrow 14x-2y&=4\\ (14x-2y)+(2x+2y)&=4+4\\ 16x&=8\\ x&=\frac{8}{16}\\ x&=\frac{1}{2}\end{aligned}$$

$$\begin{aligned}7x-y&=2\\ 7(\frac{1}{2})-y&=2\\ 7(\frac{1}{2})-2&=y\\ (\frac{7}{2})-\frac{4}{2}&=y\\ \frac{3}{2}&=y\end{aligned}$$

Problem 8

Solve the following systems of equations for x and y.

$$\begin{aligned}-4x-7y&=5\\ 2x+y&=6\end{aligned}$$

Solution 8

$$\begin{aligned} 2(2x + y = 6) \leftrightarrow 4x + 2y &= 12 \\ (4x + 2y) + (-4x - 7y) &= 12 + 5 \\ -5y &= 17 \\ y &= \frac{-17}{5} \end{aligned}$$

$$\begin{aligned} 2x + y &= 6 \\ 2x + \frac{-17}{5} &= 6 \\ 2x &= 6 + \frac{17}{5} \\ (\frac{10}{5})x &= \frac{30}{5} + \frac{17}{5} \\ (\frac{10}{5})x &= \frac{47}{5} \\ x &= \frac{\frac{47}{5}}{(\frac{10}{5})} \\ x &= \frac{235}{50} \\ x &= \frac{47}{10} \end{aligned}$$

Problem 9

Let a, b, c, d be numbers such that $ad - bc \neq 0$. Solve the following systems of equations for x and y in terms of a, b, c, d.

(a)

$$\begin{aligned} ax + by &= 1 \\ cx + dy &= 2 \end{aligned}$$

(b)

$$\begin{aligned} ax + by &= 3 \\ cx + dy &= -4 \end{aligned}$$

(c)

$$\begin{aligned} ax + by &= -2 \\ cx + dy &= 3 \end{aligned}$$

(d)

$$\begin{aligned} ax + by &= 5 \\ cx + dy &= 7 \end{aligned}$$

Solution 9 (a)
First multiply by d, $ax + by = 1 \leftrightarrow adx + bdy = d$.
Then multiply by b, $cx + dy = 2 \leftrightarrow bcx + bdy = 2b$.

Also multiply by c, $ax + by = 1 \leftrightarrow acx + bcy = c$.
And mutiply by a, $cx + dy = 2 \leftrightarrow acx + ady = 2a$.

$$\begin{aligned} (adx + bdy) - (bcx + bdy) &= d - 2b \\ adx - bcx &= d - 2b \\ x(ad - bc) &= d - 2b \\ x &= \frac{d - 2b}{ad - bc} \end{aligned}$$

$$\begin{aligned} (acx + ady) - (acx + bcy) &= 2a - c \\ ady - bcy &= 2a - c \\ y(ad - bc) &= 2a - c \\ y(ad - bc) &= \frac{2a - c}{ad - bc} \end{aligned}$$

Solution 9 (b)
First multiply by d, $ax + by = 3 \leftrightarrow adx + bdy = 3d$.
Then multiply by b, $cx + dy = -4 \leftrightarrow bcx + bdy = -4b$.
Also multiply by c, $ax + by = 3 \leftrightarrow acx + bcy = 3c$.
And mutiply by a, $cx + dy = -4 \leftrightarrow acx + ady = -4a$.

$$\begin{aligned} (adx + bdy) - (bcx + bdy) &= 3d + 4b \\ adx - bcx &= 3d + 4b \\ x(ad - bc) &= 3d + 4b \\ x &= \frac{3d + 4b}{ad - bc} \end{aligned}$$

$$\begin{aligned} (acx + ady) - (acx + bcy) &= -4a - 3c \\ ady - bcy &= -4a - 3c \\ y(ad - bc) &= -4a - 3c \\ y(ad - bc) &= \frac{-4a - 3c}{ad - bc} \end{aligned}$$

Solution 9 (c)
First multiply by d, $ax + by = -2 \leftrightarrow adx + bdy = -2d$.
Then multiply by b, $cx + dy = 3 \leftrightarrow bcx + bdy = 3b$.
Also multiply by c, $ax + by = -2 \leftrightarrow acx + bcy = -2c$.
And mutiply by a, $cx + dy = 3 \leftrightarrow acx + ady = 3a$.

$$\begin{aligned} (adx + bdy) - (bcx + bdy) &= -2d + 3b \\ adx - bcx &= -2d + 3b \\ x(ad - bc) &= -2d + 3b \\ x &= \frac{-2d + 3b}{ad - bc} \end{aligned}$$

$$\begin{aligned} (acx + ady) - (acx + bcy) &= 3a - c \\ ady - bcy &= 3a + 2c \\ y(ad - bc) &= 3a + 2c \\ y(ad - bc) &= \frac{3a + 2c}{ad - bc} \end{aligned}$$

Solution 9 (d)
First multiply by d, $ax + by = 5 \leftrightarrow adx + bdy = 5d$.
Then multiply by b, $cx + dy = 7 \leftrightarrow bcx + bdy = 7b$.
Also multiply by c, $ax + by = 5 \leftrightarrow acx + bcy = 5c$.
And mutiply by a, $cx + dy = 7 \leftrightarrow acx + ady = 7a$.

$$\begin{aligned}(adx + bdy) - (bcx + bdy) &= 5d - 7b\\ adx - bcx &= 5d - 7b\\ x(ad - bc) &= 5d - 7b\\ x &= \frac{5d - 7b}{ad - bc}\end{aligned}$$

$$\begin{aligned}(acx + ady) - (acx + bcy) &= 5a - 7c\\ ady - bcy &= 5a - 7c\\ y(ad - bc) &= 5a - 7c\\ y(ad - bc) &= \frac{5a - 7c}{ad - bc}\end{aligned}$$

Problem 10

Making the same assumptions as in Excersize 9, show that the solution of the system

$$\begin{aligned}ax + by &= 0\\ cx + dy &= 0\end{aligned}$$

must be $x = 0$ and $y = 0$.

Solution 10
First $ax + by = 0 \leftrightarrow adx + bdy = 0$.
Then $cx + dy = 0 \leftrightarrow bcx + bdy = 0$.
Also $ax + by = 0 \leftrightarrow acx + bcy = 0$.
And $cx + dy = 0 \leftrightarrow acx + ady = 0$.

$$\begin{aligned}(adx + bdy) - (bcx + bdy) &= 0\\ adx - bcx &= 0\\ x(ad - bc) &= 0\\ x &= \frac{0}{ad - bc}\\ x &= 0\end{aligned}$$

$$\begin{aligned}(acx + ady) - (acx + bcy) &= 0\\ ady - bcy &= 0\\ y(ad - bc) &= 0\\ y &= \frac{0}{ad - bc}\\ y &= 0\end{aligned}$$

Problem 11

Let a, b, c, d, u, v be numbers and assume that $ad - bc \neq 0$. Solve the following system of equations for x and y in terms of a, b, c, d, u, v

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned}$$

Verify that the answer you get is actually a solution.

Solution 9 (d)
First multiply first equation by d, $ax + by = u \leftrightarrow adx + bdy = ud$.
Then multiply second equation by b, $cx + dy = v \leftrightarrow bcx + bdy = vb$.
Also multiply first equation by c, $ax + by = u \leftrightarrow acx + bcy = uc$.
And multiply second equation by a, $cx + dy = v \leftrightarrow acx + ady = va$.

$$\begin{aligned} (adx + bdy) - (bcx + bdy) &= ud - vb \\ adx - bcx &= ud - vb \\ x(ad - bc) &= ud - vb \\ x &= \frac{ud - vb}{ad - bc} \end{aligned}$$

$$\begin{aligned} (acx + ady) - (acx + bcy) &= va - uc \\ ady - bcy &= va - uc \\ y(ad - bc) &= va - uc \\ y &= \frac{va - uc}{ad - bc} \end{aligned}$$

Veryfying the first equation.

$$\begin{aligned} a\left(\frac{ud - vb}{ad - bc}\right) + b\left(\frac{va - uc}{ad - bc}\right) &= u \\ \frac{aud - avb + bva - buc}{ad - bc} &= u \\ \frac{v(-ab + ba) + u(ad - bc)}{ad - bc} &= u \\ \frac{u(ad - bc)}{ad - bc} &= u \\ u &= u \end{aligned}$$

Verifying the second equation.

$$\begin{aligned} c\left(\frac{ud - vb}{ad - bc}\right) + d\left(\frac{va - uc}{ad - bc}\right) &= v \\ \frac{cud - cvb + dva - duc}{ad - bc} &= v \\ \frac{u(cd - cd) + v(ad - bc)}{ad - bc} &= v \\ \frac{v(ad - bc)}{ad - bc} &= v \\ v &= v \end{aligned}$$

2.2 Equations In Three Unknowns

Problem 7

Solve the following equations for x, y, z.
(1) $4x - 2y + 5z = 1$
(2) $x + y + z = 0$
(3) $-x + y - 2z = 2$

Solution 7

Summing (2) and (3).

$$(x + y + z) + (-x + y - 2z) = 0 + 2$$
$$(4) \quad 2y - z = 2$$

Summing (1) and (2) multiplied by -4.

$$(4x - 2y + 5z) + (-4x - 4y - 4z) = 1 + 0$$
$$(5) \quad -6y + z = 1$$

Summing (4) and (5).

$$(2y - z) + (-6y + z) = 2 + 1$$
$$-4y = 3$$
$$y = \frac{-3}{4}$$

Summing (2) and (3). Setting $y = \frac{-3}{4}$.

$$(x + \frac{-3}{4} + z) + (-x + \frac{-3}{4} - 2z) = 0 + 2$$
$$2 \cdot \frac{-3}{4} - z = 2$$
$$\frac{-3}{2} - 2 = z$$
$$\frac{-3}{2} - \frac{4}{2} = z$$
$$\frac{-7}{2} = z$$

Using (2). Setting $y = \frac{-3}{4}$ and $z = \frac{-7}{2}$.

$$x + \frac{-3}{4} + \frac{-7}{2} = 0$$
$$x + \frac{-3}{4} + \frac{-14}{4} = 0$$
$$x + \frac{-17}{4} = 0$$
$$x = \frac{17}{4}$$

$$\therefore x = \frac{17}{4}, y = \frac{-3}{4}, z = \frac{-7}{2}$$

Problem 8

Solve the following equations for x, y, z.
(1) $x + y + z = 0$
(2) $x - y - z = 1$
(3) $x + y - z = 1$

Solution 8

Summing (1) and (2).

$$\begin{aligned}(x + y + z) + (x - y - z) &= 0 + 1\\ 2x &= 1\\ x &= \frac{1}{2}\end{aligned}$$

Summing (2) and (3). Setting $x = \frac{1}{2}$.

$$\begin{aligned}(\frac{1}{2} - y - z) + (\frac{1}{2} + y - z) &= 1 + 1\\ 1 - 2z &= 2\\ -2z &= 1\\ z &= \frac{-1}{2}\end{aligned}$$

Using (3). Setting $x = \frac{1}{2}$ and $z = \frac{-1}{2}$.

$$\begin{aligned}\frac{1}{2} + y - (\frac{-1}{2}) &= 1\\ y + 1 &= 1\\ y &= 0\end{aligned}$$

$$\therefore x = \frac{1}{2}, y = 0, z = \frac{-1}{2}$$

Problem 11

Solve the following equations for x, y, z.
(1) $\frac{1}{2}x + y - \frac{3}{4}z = 1$
(2) $x - \frac{1}{2}y + z = 0$
(3) $x + y - \frac{1}{3}z = 0$

Solution 11

Multiply (1) by -4.

$$\begin{aligned}\frac{1}{2}x + y - \frac{3}{4}z &= 1\\ (4) \quad -2x - 4y + 3z &= -4\end{aligned}$$

Multiply (2) by 2.

$$\begin{aligned}x - \frac{1}{2}y + z &= 0\\ (5) \quad 2x - y + 2z &= 0\end{aligned}$$

Multiply (3) by 3.

$$x + y - \frac{1}{3}z = 0$$
$$(6) \quad 3x + 3y - z = 0$$

Summing (4) and (5).

$$(-2x - 4y + 3z) + (2x - y + 2z) = -4 + 0$$
$$(7) \quad -5y + 5z = -4$$

Sum (5) times 3 and (6) times -2.

$$(6x - 3y + 6z) + (-6x - 6y + 2z) = 0$$
$$(8) \quad -9y + 8z = 0$$

Sum (7) times -9 and (8) times 5.

$$\begin{aligned}(45y - 45z) + (-45y + 40z) &= 36\\ -5z &= 36\\ z &= \frac{-36}{5}\end{aligned}$$

Using (7) and setting $z = \frac{-36}{5}$.

$$\begin{aligned}-9y + 8(\frac{-36}{5}) &= 0\\ -9y - \frac{288}{5} &= 0\\ -9y &= \frac{288}{5}\\ y &= \frac{-32}{5}\end{aligned}$$

Using (5) and setting $y = \frac{-32}{5}$, $z = \frac{-36}{5}$.

$$\begin{aligned}2x - (\frac{-32}{5}) + 2(\frac{-36}{5}) &= 0\\ 2x + \frac{32}{5} - \frac{72}{5} &= 0\\ 2x - \frac{40}{5} &= 0\\ x &= \frac{40}{10}\\ x &= 4\end{aligned}$$

Problem 12

Solve the following equations for x, y, z.
(1) $\frac{1}{2}x - \frac{2}{3}y + z = 1$
(2) $x - \frac{1}{5}y + z = 0$
(3) $2x - \frac{1}{3}y + \frac{2}{5}z = 1$

Solution 12

Multiply (1) by 6.

$$\frac{1}{2}x - \frac{2}{3}y + z = 1$$
$$(4) \quad 3x - 4y + 6z = 6$$

Multiply (2) by -30.

$$x - \frac{1}{5}y + z = 0$$
$$(5) \quad -30x + 6y - 30z = 0$$

Multiply (3) by 15.

$$2x - \frac{1}{3}y + \frac{2}{5}z = 1$$
$$(6) \quad 30x - 5y + 6z = 15$$

Summing (5) and (6).

$$(-30x + 6y - 30z) + (30x - 5y + 6z) = 15$$
$$(7) \quad y - 24z = 15$$

Sum (4) times 10 and (5).

$$(30x - 40y + 60z) + (-30x + 6y - 30z) = 60$$
$$(8) \quad -34y + 30z = 60$$

Sum (7) times 34 and (8).

$$\begin{aligned}(34y - 816z) + (-34y + 30z) &= (15 \cdot 34) + 60 \\ (34y - 816z) + (-34y + 30z) &= 510 + 60 \\ -786z &= 570 \\ z &= \frac{-95}{131}\end{aligned}$$

Using (7). Set $z = \frac{-95}{131}$.

$$\begin{aligned}y - 24(\frac{-95}{131}) &= 15 \\ y + \frac{2280}{131} &= 15 \\ y &= \frac{1965}{131} - \frac{2280}{131} \\ y &= \frac{-315}{131}\end{aligned}$$

Using (7). Set $z = \frac{-95}{131}$ and $y = \frac{-315}{131}$.

$$\begin{aligned}x - \frac{1}{5} \cdot \frac{-315}{131} + \frac{-95}{131} &= 0 \\ x &= \frac{1}{5} \cdot \frac{-315}{131} - \frac{-95}{131} \\ x &= \frac{-315}{655} - \frac{-475}{655} \\ x &= \frac{160}{655} \\ x &= \frac{32}{131}\end{aligned}$$

$$\therefore x = \frac{32}{131}, y = \frac{-315}{131}, z = \frac{-95}{131}$$

3 Real Numbers

3.1 Addition and Multiplication

Problem 1

Let E be an abbreviation for even, and let I be an abbreviation for odd. We know that:
$E + E = E,$
$E + I = I + E = I,$
$I + I = E,$
$EE = E,$
$II = I$
$IE = EI = E.$
(a) Show that addition for E and I is associative and commutative. Show that E plays the role of a zero element for addition. What is the additive inverse of E? What is the additive inverse of I?
(b) Show that multiplication for E and I is commutative and associative. Which of E or I behaves like 1? Which behaves like 0 for multiplication? Show that multiplication is distributive with respect to addition.

Solution 1 (a)

Associative over Addition: We check that $(A + B) + C = A + (B + C)$ for all $A, B, C \in \{E, I\}$ by verifying all 8 cases:

- $(E + E) + E = E + E = E$, and $E + (E + E) = E + E = E$

- $(E + E) + I = E + I = I$, and $E + (E + I) = E + I = I$

- $(E + I) + E = I + E = I$, and $E + (I + E) = E + I = I$

- $(E + I) + I = I + I = E$, and $E + (I + I) = E + E = E$

- $(I + E) + E = I + E = I$, and $I + (E + E) = I + E = I$

- $(I + E) + I = I + I = E$, and $I + (E + I) = I + I = E$

- $(I + I) + E = E + E = E$, and $I + (I + E) = I + I = E$

- $(I + I) + I = E + I = I$, and $I + (I + I) = I + E = I$

Commutative over Addition: We check that $A + B = B + A$ for all $A, B \in \{E, I\}$.

- $E + E = E = E + E$

- $E + I = I = I + E$

- $I + I = E = I + I$

Zero Element: E plays the role of additive identity (zero element), since:

- $E + E = E$

- $I + E = I$

- $E + I = I$

Additive Inverse of E: E, since $E + E = E$.

Additive Inverse of I: I, since $I + I = E$.

Solution 1 (b)

Associative over Multiplication: We check that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ for all $A, B, C \in \{E, I\}$:

$$
\begin{aligned}
(E\cdot E)\cdot E &= E\cdot E = E, & E\cdot(E\cdot E) &= E\cdot E = E\\
(E\cdot E)\cdot I &= E\cdot I = E, & E\cdot(E\cdot I) &= E\cdot E = E\\
(E\cdot I)\cdot E &= E\cdot E = E, & E\cdot(I\cdot E) &= E\cdot E = E\\
(E\cdot I)\cdot I &= E\cdot I = E, & E\cdot(I\cdot I) &= E\cdot I = E\\
(I\cdot E)\cdot E &= E\cdot E = E, & I\cdot(E\cdot E) &= I\cdot E = E\\
(I\cdot E)\cdot I &= E\cdot I = E, & I\cdot(E\cdot I) &= I\cdot E = E\\
(I\cdot I)\cdot E &= I\cdot E = E, & I\cdot(I\cdot E) &= I\cdot E = E\\
(I\cdot I)\cdot I &= I\cdot I = I, & I\cdot(I\cdot I) &= I\cdot I = I
\end{aligned}
$$

Commutative over Multiplication: We check that $AB = BA$ for all $A, B \in \{E, I\}$.

$$
\begin{aligned}
E\cdot I = I\cdot E = E\\
I\cdot I = I\cdot I = I\\
E\cdot E = E\cdot E = E
\end{aligned}
$$

Multiplicative Identity: I behaves like 1 over multiplication.

- $II = I$

- $EI = E$

Multiplicative Zero: E behaves like 0 over multiplication.

- $IE = E$

- $EE = E$

Distributive Over Addition: We check that $A\cdot(B+C) = A\cdot B + A\cdot C$ for all $A, B, C \in \{E, I\}$. For example:

- $E(I+E) = E(I) = E = EI + EE = E + E = E$

- $I(I+E) = I(E) = E = II + IE = E + E = E$

- $E(E+I) = E(I) = E = EE + EI = E + E = E$

- $I(E+I) = I(I) = I = IE + II = E + I = I$

- $E(E+E) = E(E) = E = EE + EE = E + E = E$

- $I(E+E) = I(E) = E = IE + IE = E + E = E$

- $E(I+I) = E(E) = E = EI + EI = E + E = E$

- $I(I+I) = I(E) = E = II + II = I + I = E$

3.2 Real Numbers: Positivity

Problem 1

Prove:
(a) If a is a real number, then a^2 is positive.
(b) If a is positive and b is negative, then ab is negative.
(c) If a is negative and b is negative, then ab is positive.

Proof. By POS 2 either $a = 0$, $a > 0$, or $a < 0$.

Case 1 $(a = 0)$

If $a = 0$ then $a^2 = a \cdot a = 0 \cdot 0 = 0 \geq 0$.

Case 2 $(a > 0)$

If $a > 0$, then by POS 1, $a \cdot a = a^2 \geq 0$.

Case 3 $(a < 0)$

Since $a < 0$, by POS 2, $-a > 0$. Then by POS 1, $(-a) \cdot (-a) = a^2 > 0$.

Therefore, $a^2 \geq 0$. $\blacksquare$

Proof. Assume for contradiction, $ab > 0$. By POS 2, $-ab < 0$. Since $b < 0$ then, by POS 2, $-b > 0$. Then by POS 1, $a \cdot -b > 0$ so $-ab > 0$ which is a contradiction. Therefore, if a is positive and b is negative, then ab is negative. $\blacksquare$

Proof. Assume for contradiction, $ab < 0$. By POS 2, $-ab > 0$. Since $b < 0$, $a < 0$ then, by POS 2, $-b > 0$, $-a > 0$. Then by POS 1, $-a \cdot -b > 0$ so $ab > 0$ which is a contradiction. Therefore, if a is negative and b is negative, then ab is positive. $\blacksquare$

Problem 2

Prove: If a is positive, then a^{-1} is positive.

Proof. Suppose $a > 0$ and assume for contradiction $a^{-1} = \frac{1}{a} < 0$. By Excersize 1 part c, $a \cdot \frac{1}{a} < 0$. But $a \cdot \frac{1}{a} = \frac{a}{a} = 1 > 0$. Therefore, if a is positive, then a^{-1} is positive. $\blacksquare$

Problem 3

Prove: If a is negative, then a^{-1} is negative.

Proof. Suppose $a < 0$ and assume for contradiction $\frac{1}{a} > 0$. Since $a < 0$, by POS 2, $0 < -a$. Then by POS 1, $-a \cdot \frac{1}{a} > 0$. But $-a \cdot \frac{1}{a} = \frac{-a}{a} = -1 < 0$ which is a contradiction. Therefore, if a is negative, then a^{-1} is negative. $\blacksquare$

Problem 4

Prove: If a, b are positive numbers, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Proof.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \iff \sqrt{\frac{a}{b}}^2 = \left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 \iff \sqrt{\frac{a}{b}}^2 = \frac{\sqrt{a}^2}{\sqrt{b}^2} \iff \frac{a}{b} = \frac{a}{b}$$

$\blacksquare$

Problem 5

Prove that

$$\frac{1}{1-\sqrt{2}} = -(1+\sqrt{2})$$

Proof.

$$\begin{aligned}\frac{1}{1-\sqrt{2}} &= \frac{1+\sqrt{2}}{1+\sqrt{2}} \cdot \frac{1}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{1-2} = \frac{1+\sqrt{2}}{-1} \\ &= \frac{-1}{-1} \cdot \frac{1+\sqrt{2}}{-1} = \frac{-(1+\sqrt{2})}{1} = -(1+\sqrt{2})\end{aligned}$$

∎

Problem 8

Let a, b be rational numbers. Prove that the multiplicative inverse of $a + b\sqrt{2}$ can be expressed in the form $c + d\sqrt{2}$, where c, d are rational numbers.

Proof. First note since $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ therefore $a^2 - 2b^2 \in \mathbb{Q}$. In addition $a + b\sqrt{2} \neq 0$ (otherwise the inverse operation is undefined). If $b = 0$ then $a^2 \neq 0$ so $a^2 - 2b^2 \in \mathbb{Q}$ is defined. Now suppose $b \neq 0$.

$$a^2 = 2b^2 \iff \frac{a^2}{b^2} = 2 \iff \frac{a}{b} = \pm\sqrt{2}$$

But $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ so their quotient is rational. This is impossible since $\sqrt{2}$ is irrational, so $a^2 - 2b^2 \neq 0$. Futhermore since $a^2 - 2b^2 \in \mathbb{Q}$ and $a^2 - 2b^2 \neq 0$, $\frac{a}{a^2-2b^2} \in \mathbb{Q}$ and $\frac{-b}{a^2-2b^2} \in \mathbb{Q}$. Now, let $c = \frac{a}{a^2-2b^2}$ and $d = \frac{-b}{a^2-2b^2}$. Then

$$\begin{aligned}&(a+b\sqrt{2}) \cdot (c + d\sqrt{2}) \\ =&(a+b\sqrt{2}) \cdot \left(\frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2} \cdot \sqrt{2}\right) \\ =&(a+b\sqrt{2}) \cdot \left(\frac{a}{a^2-2b^2} + \frac{-b\sqrt{2}}{a^2-2b^2}\right) \\ =&(a+b\sqrt{2}) \cdot \left(\frac{a}{a^2-2b^2} - \frac{b\sqrt{2}}{a^2-2b^2}\right) \\ =&\left(\frac{a(a+b\sqrt{2})}{a^2-2b^2} - \frac{b\sqrt{2}(a+b\sqrt{2})}{a^2-2b^2}\right) \\ =&\frac{(a^2+ab\sqrt{2}) - (ab\sqrt{2} + 2b^2)}{a^2-2b^2} \\ =&\frac{a^2+ab\sqrt{2} - ab\sqrt{2} - 2b^2}{a^2-2b^2} \\ =&\frac{a^2-2b^2}{a^2-2b^2} \\ =&1\end{aligned}$$

∎

Problem 11

Generalize Excersize 10, replacing $\sqrt{5}$ by $\sqrt{a}$ for any positive integer a.

Proof. First note since $d \in \mathbb{Q}$ and $b \in \mathbb{Q}$ therefore $d^2 - ab^2 \in \mathbb{Q}$. In addition $d + b\sqrt{a} \neq 0$ (otherwise the inverse operation is undefined).

If $b = 0$ then $d^2 \neq 0$ so $d^2 - ab^2 \in \mathbb{Q}$ is defined.

Now suppose $b \neq 0$ and $\sqrt{a} \notin \mathbb{Q}$.

$$d^2 = ab^2 \iff \frac{d^2}{b^2} = a \iff \frac{d}{b} = \pm\sqrt{a}$$

But $d \in \mathbb{Q}$ and $b \in \mathbb{Q}$ so their quotient is rational. This is impossible if $\sqrt{a} \notin \mathbb{Q}$, so $d^2 - ab^2 \neq 0$.

Now suppose $b \neq 0$ and $\sqrt{a} \in \mathbb{Q}$.

$$d = b\sqrt{a} \iff d^2 = b^2 a \iff d^2 - ab^2 = 0$$

Since, $d \neq b\sqrt{a}$, $d^2 - ab^2 \neq 0$.

Furthermore since $d^2 - ab^2 \in \mathbb{Q}$ and $d^2 - ab^2 \neq 0$, $\frac{d}{d^2-ab^2} \in \mathbb{Q}$ and $\frac{-b}{d^2-ab^2} \in \mathbb{Q}$. Now let $c = \frac{d}{d^2-ab^2}$ and $e = \frac{-b}{d^2-ab^2}$. Then

$$\begin{aligned}
&(d + b\sqrt{a}) \cdot (c + e\sqrt{a}) \\
=&(d + b\sqrt{a}) \cdot \left(\frac{d}{d^2 - ab^2} + \frac{-b}{d^2 - ab^2} \cdot \sqrt{a}\right) \\
=&(d + b\sqrt{a}) \cdot \left(\frac{d}{d^2 - ab^2} + \frac{-b\sqrt{a}}{d^2 - ab^2}\right) \\
=&(d + b\sqrt{a}) \cdot \left(\frac{d}{d^2 - ab^2} - \frac{b\sqrt{a}}{d^2 - ab^2}\right) \\
=&\left(\frac{d(d + b\sqrt{a})}{d^2 - ab^2} - \frac{b\sqrt{a}(d + b\sqrt{a})}{d^2 - ab^2}\right) \\
=&\frac{(d^2 + db\sqrt{a}) - (db\sqrt{a} + ab^2)}{d^2 - ab^2} \\
=&\frac{d^2 + db\sqrt{a} - db\sqrt{a} - ab^2}{d^2 - ab^2} \\
=&\frac{d^2 - ab^2}{d^2 - ab^2} \\
=&1
\end{aligned}$$

∎

Problem 14

Find all possible numbers x such that
(a) $|2x - 1| = 3$
(b) $|3x + 1| = 2$
(c) $|2x + 1| = 4$
(d) $|3x - 1| = 1$
(e) $|4x - 5| = 6$

Solution 14 (a)

$x = 2$ or $x = -1$

Solution 14 (b)

$x = \frac{1}{3}$ or $x = -1$

Solution 14 (c)

$x = \frac{3}{2}$ or $x = \frac{-5}{2}$

Solution 14 (d)

$x = \frac{2}{3}$ or $x = 0$

Solution 14 (e)

$x = \frac{11}{4}$ or $x = \frac{-1}{4}$

Problem 15

Rationalize the numerator in the following expressions.

(a) $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

(b) $\frac{\sqrt{x+y}-\sqrt{y}}{\sqrt{x+y}+\sqrt{y}}$

(c) $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$

(d) $\frac{\sqrt{x-3}+\sqrt{x}}{\sqrt{x-3}-\sqrt{x}}$

(e) $\frac{\sqrt{x+y}-1}{3+\sqrt{x+y}}$

(f) $\frac{\sqrt{x+y}+x}{\sqrt{x+y}}$

Solution 15 (a)

$$\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \frac{x-y}{x-2\sqrt{xy}+y}$$

Solution 15 (b)

$$\frac{\sqrt{x+y}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} \cdot \frac{\sqrt{x+y}+\sqrt{y}}{\sqrt{x+y}+\sqrt{y}} = \frac{x}{\sqrt{x(x+y)}+\sqrt{xy}+\sqrt{y(x+y)}+y}$$

Solution 15 (c)

$$\begin{aligned}\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} \cdot \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} &= \frac{2}{(\sqrt{x+1}-\sqrt{x-1})(\sqrt{x+1}-\sqrt{x-1})} \\ &= \frac{2}{(\sqrt{x+1}-\sqrt{x-1})^2}\end{aligned}$$

Solution 15 (d)

$$\frac{\sqrt{x-3}+\sqrt{x}}{\sqrt{x-3}-\sqrt{x}} \cdot \frac{\sqrt{x-3}-\sqrt{x}}{\sqrt{x-3}-\sqrt{x}} = \frac{(x-3)+x}{(\sqrt{x-3}-\sqrt{x})^2}$$
$$= \frac{-3}{(\sqrt{x-3}-\sqrt{x})^2}$$

Solution 15 (e)

$$\frac{\sqrt{x+y}-1}{3+\sqrt{x+y}} \cdot \frac{\sqrt{x+y}+1}{\sqrt{x+y}+1} = \frac{x+y-1}{(3+\sqrt{x+y})(\sqrt{x+y}+1)}$$

Solution 15 (f)

$$\frac{\sqrt{x+y}+x}{\sqrt{x+y}} \cdot \frac{\sqrt{x+y}-x}{\sqrt{x+y}-x} = \frac{x+y-x^2}{\sqrt{x+y}(\sqrt{x+y}-x)}$$

Problem 17

Prove that there is no real number x such that

$$\sqrt{x-1} = 3 + \sqrt{x}$$

[*Hint*: Start by squaring both sides.]

Proof. Assume for contradiction there does exist a real number x such that $\sqrt{x-1} = 3+\sqrt{x}$. Then

$$\begin{aligned} &\sqrt{x-1} = 3+\sqrt{x} \\ &\leftrightarrow x-1 = 9+6\sqrt{x}+x \\ &\leftrightarrow -1 = 9+6\sqrt{x} \\ &\leftrightarrow -10 = 6\sqrt{x} \\ &\leftrightarrow \frac{-10}{6} = \sqrt{x} \end{aligned}$$

Which is a contradiction. Therefore, there is no real number x such that $\sqrt{x-1} = 3+\sqrt{x}$. ∎

Problem 20

If a, b are two numbers, prove that $|a-b| = |b-a|$.

Proof. Let $c = b-a$. By POS 2 there are three cases.

Case 1 ($c = 0$) If $b-a = 0$ then $b = a$ therefore $a-b = 0$.

$$\begin{aligned} &|b-a| = |a-b| \\ &\leftrightarrow |0| = |0| \\ &\leftrightarrow 0 = 0 \end{aligned}$$

Case 2 ($c > 0$) If $c > 0$ then $|c| = c$. Also $-c < 0$ so $|-c| = -(-c) = c$. Then

$$\begin{aligned} |b-a| &= |a-b| \\ \leftrightarrow |c| &= |-c| \\ \leftrightarrow c &= c \end{aligned}$$

Case 3 ($c < 0$) If $c < 0$ then $|c| = -c$. Also $-c > 0$ so $|-c| = -c$. Then

$$\begin{aligned} |b-a| &= |a-b| \\ \leftrightarrow |c| &= |-c| \\ \leftrightarrow -c &= -c \end{aligned}$$

Therefore $|a-b| = |b-a|$. ∎

3.3 Powers and Roots

Extra Problem

Suppose a is a nonzero rational number and b is an irrational real number. Show that ab is irrational.

Proof. A number is rational if it can be written as $\frac{x}{y}$ with $x, y \in \mathbb{Z}$ and $y \neq 0$. Assume for contradiction that $a \cdot b$ is rational, where $a \neq 0$ is rational and b is irrational. Since $a \neq 0$, we can divide both sides by a:

$$b = \frac{a \cdot b}{a}.$$

But the right-hand side is rational (a rational divided by a nonzero rational is rational), so b would be rational. This contradicts the assumption that b is irrational. Therefore, $a \cdot b$ must be irrational. ∎

Problem 1

Express each of the following in the form $2^k 2^m a^r b^s$ where k, m, r,s are integers.
(a) $\frac{1}{8}a^3b^{-4}2^5a^{-2}$
(b) $3^{-4}2^5a^3b^6 \cdot \frac{1}{2^3} \cdot \frac{1}{a^4} \cdot b^{-1} \cdot \frac{1}{9}$
(c) $\frac{3a^3b^4}{2a^5b^6}$
(d) $\frac{16a^{-3}b^{-5}}{9b^4a^72^{-3}}$

Solution (a):

$$\frac{1}{8}a^3b^{-4}2^5a^{-2} = \frac{2^5}{8}a^3a^{-2}b^{-4} = \frac{2^5}{8}a^1b^{-4} = \frac{2^5}{2^3}a^1b^{-4} = 2^23^0a^1b^{-4}$$

Solution (b):

$$3^{-4}2^5a^3b^6 \cdot \frac{1}{2^3} \cdot \frac{1}{a^4} \cdot b^{-1} \cdot \frac{1}{9} = \frac{2^5}{2^3}\frac{3^{-4}}{9}\frac{a^3}{a^4}\frac{b^6}{b} = 2^2\frac{3^{-4}}{3^2}\frac{a^3}{a^4}\frac{b^6}{b} = 2^23^{-6}a^{-1}b^5$$

Solution (c):

$$\frac{3a^3b^4}{2a^5b^6} = 2^{-1}3^1a^{-2}b^{-2}$$

Solution (d):

$$\frac{16a^{-3}b^{-5}}{9b^4a^72^{-3}} = \frac{2^4a^{-10}b^{-5}}{3^22^{-3}} = 2^73^{-2}a^{-10}b^{-9}$$

Problem 2

What integer is $81^{\frac{1}{4}}$ equal to?

Solution:

$$81^{\frac{1}{4}} = (81^{\frac{1}{2}})^{\frac{1}{2}} = 9^{\frac{1}{2}} = 3$$

Problem 3

What integer is $(\sqrt{2})^6$ equal to?

Solution:

$$(\sqrt{2})^6 = (\sqrt{2})^2(\sqrt{2})^2(\sqrt{2})^2 = 2 \cdot 2 \cdot 2 = 8$$

Problem 4

Is $(\sqrt{2})^5$ an integer?

Solution:

$$(\sqrt{2})^5 = (\sqrt{2})^2(\sqrt{2})^2(\sqrt{2}) = 2 \cdot 2 \cdot \sqrt{2} = 4\sqrt{2}$$

It is not an integer see extra problem proof.

Problem 5

Is $(\sqrt{2})^{-5}$ a rational number? Is $(\sqrt{2})^5$ a rational number?

Solution part 1:

$$(\sqrt{2})^{-5} = \frac{1}{(\sqrt{2})^5} = \frac{1}{4\sqrt{2}} = \frac{1}{4\sqrt{2}}\frac{4\sqrt{2}}{4\sqrt{2}} = \frac{4\sqrt{2}}{16 \cdot 2} = \frac{4\sqrt{2}}{32} = \frac{4}{32}\sqrt{2}$$

By the extra problem this is not a rational number.

Solution part 2: Same reason as problem 4.

Problem 6

In each case, the expression is equal to an integer. Which one?

(a) $16^{\frac{1}{4}}$
(b) $8^{\frac{1}{3}}$
(c) $9^{\frac{3}{2}}$
(d) $1^{\frac{5}{4}}$
(e) $8^{\frac{4}{3}}$
(f) $64^{\frac{2}{4}}$
(g) $25^{\frac{3}{2}}$

Solution:

(a) $16^{\frac{1}{4}} = (16^{\frac{1}{2}})^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2$

(b) $8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2$

(c) $9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27$

(d) $1^{\frac{5}{4}} = 1$

(e) $8^{\frac{4}{3}} = (8^{\frac{1}{3}})^4 = 2^4 = 16$

(f) $64^{\frac{2}{4}} = 64^{\frac{1}{2}} = 8$

(g) $25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3 = 5^3 = 125$

Problem 7

Express each of the following expressions as a simple decimal.
(a) $(0.09)^{\frac{1}{2}}$
(b) $(0.027)^{\frac{1}{3}}$
(c) $(0.125)^{\frac{2}{3}}$
(d) $(1.21)^{\frac{1}{2}}$

Solution:

(a) $(0.9)^{\frac{1}{2}} \approx 0.3$

(b) $(0.027)^{\frac{1}{3}} = 0.3$

(c) $(0.125)^{\frac{2}{3}} = ((0.125)^{\frac{1}{3}})^2 = 0.5^2 = 0.25$

(d) $(1.21)^{\frac{1}{2}} = 1.1$

Problem 8

Express each of the following expressions as a quotient $\frac{m}{n}$, where m, n are integers > 0.
(a) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$
(b) $\left(\frac{4}{9}\right)^{\frac{1}{2}}$
(c) $\left(\frac{25}{16}\right)^{\frac{3}{2}}$
(d) $\left(\frac{49}{4}\right)^{\frac{3}{2}}$

Solution:

$$\text{(a)}\quad \left(\frac{8}{27}\right)^{\frac{2}{3}} = \frac{8^{2/3}}{27^{2/3}} = \frac{4}{9}$$
$$\text{(b)}\quad \left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{2}{3}$$
$$\text{(c)}\quad \left(\frac{25}{16}\right)^{\frac{3}{2}} = \frac{(25^{1/2})^3}{(16^{1/2})^3} = \frac{125}{64}$$
$$\text{(d)}\quad \left(\frac{49}{4}\right)^{\frac{3}{2}} = \frac{(49^{1/2})^3}{(4^{1/2})^3} = \frac{343}{8}$$

Problem 9

Solve each of the following equations for x.
(a) $(x-2)^3 = 5$
(b) $(x+3)^2 = 4$
(c) $(x-5)^{-2} = 9$
(d) $(x+3)^3 = 27$
(e) $(2x-1)^{-3} = 27$
(f) $(3x+5)^{-4} = 64$

Solution:

$$\text{(a)}\quad x-2 = \sqrt[3]{5} \iff x = 2+\sqrt[3]{5}$$
$$\text{(b)}\quad x+3 = \pm 2 \iff x = -1 \text{ or } x = -5$$
$$\text{(c)}\quad \frac{1}{(x-5)^2} = 9 \iff (x-5)^2 = \frac{1}{9} \iff x = 5 \pm \frac{1}{3}$$
$$\text{(d)}\quad x+3 = 3 \iff x = 0$$
$$\text{(e)}\quad \frac{1}{(2x-1)^3} = 27 \iff (2x-1)^3 = \frac{1}{27} \iff 2x-1 = \frac{1}{3} \iff x = \frac{2}{3}$$
$$\text{(f)}\quad \frac{1}{(3x+5)^4} = 64 \iff (3x+5)^4 = \frac{1}{64} \iff 3x+5 = \frac{1}{2} \iff x = -\frac{3}{2}$$

3.4 Inequalities

Problem 1

Prove **IN 3**.

IN 3 If $a > b$ and $b > c$ then $a > c$.

Proof. Suppose $a > b$ and $b > c$. Since $a > b$, $a - b > 0$. Also, since $b > c$, $b - c > 0$. So $(a-b)+(b-c) > 0 \iff a-c > 0$. Therefore $a > c$. ∎

Problem 2

Prove: If $0 < a < b$, if $c < d$, and $c > 0$ then
$$ac < bd$$

Proof. Suppose $0 < a < b$, $c < d$, and $c > 0$. Since $a < b$ and $c > 0$ it follows that $ac < bc$ (**IN 2**). Since $c < d$ and $b > 0$ it follows that $bc < bd$ (**IN 2**). Since $ac < bc < bd$ it follows that $ac < bd$ (Problem 1). ∎

Problem 3

Prove: If $a < b < 0$, if $c < d < 0$ then

$$ac > bd$$

Proof. Suppose $a < b < 0$ and $c < d < 0$. Since $a < b$ it follows that $b - a > 0$. Since $b - a > 0$ and $c < 0$ it follows that $bc - ac < 0$ so $bc < ac$ (**IN 3**). Since $c < d$ it follows that $d - c > 0$. Since $d - c > 0$ and $b < 0$ it follows that $bd - bc < 0$ so $bd < bc$ (**IN 3**). So $bd < bc < ac$ and therefore $bd < ac$ (Problem 1). ∎

Problem 4

(a) If $x < y$ and $x > 0$, prove that $\frac{1}{y} < \frac{1}{x}$.
(b) Prove a rule of cross-multiplication of inequalities: If a, b, c, d are numbers and $b > 0$, $d > 0$, and if

$$\frac{a}{b} < \frac{c}{d}$$

prove that

$$ad < bc$$

Also prove the converse, that if $ad < bc$, then $\frac{a}{b} < \frac{c}{d}$.

Proof. Suppose $x > 0$. For contradiction, suppose $\frac{1}{x} < 0$. But $x > 0$ and $\frac{1}{x} \cdot x = 1 > 0$ which contradicts the fact that the product of a positive and a negative number is negative. Now suppose $y > 0$. Then, since $\frac{1}{x} > 0$ and $\frac{1}{y} > 0$ it follows that $\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy} > 0$. Thus, if $x, y > 0$ then $\frac{1}{xy} > 0$. ∎

Proof. Suppose $x < y$ and $x > 0$. Since $x < y$ it follows that $y - x > 0$. Since $y > x > 0$ it follows that $\frac{1}{xy} > 0$. Then $\frac{1}{xy}(y - x) > 0 \iff \frac{1}{x} - \frac{1}{y} > 0$ therefore $\frac{1}{x} > \frac{1}{y}$. ∎

Proof. Suppose a, b, c, and d are numbers such that $b > 0$ and $d > 0$. Suppose $\frac{a}{b} < \frac{c}{d}$. It follows that $\frac{c}{d} - \frac{a}{b} > 0$. Since $b > 0$ and $d > 0$ it follows that $bd > 0$. Then $bd(\frac{c}{d} - \frac{a}{b}) > 0 \iff cb - ad > 0 \iff ad < bc$. ∎

Proof. Suppose a, b, c, and d are numbers such that $b > 0$ and $d > 0$. Suppose $\frac{a}{b} > \frac{c}{d}$. So $\frac{a}{b} > \frac{c}{d} \iff \frac{c}{d} < \frac{a}{b}$. Since $\frac{c}{d} < \frac{a}{b}$ then $bc < ad$ (Previous Proof). ∎

Problem 5

Prove: If $a < b$ and c is any real number, then

$$a + c < b + c$$

Also,

$$a - c < b - c$$

Thus a number may be subtracted from each side of an inequality without changing the validity of the inequality.

Proof. Suppose $a < b$ and c is a real number. Since $a < b$ it follows that $b - a > 0$. Then $b - a > 0 \iff b - a + c - c > 0 \iff b + c - a - c > 0 \iff b + c - (a + c) > 0 \iff b + c > a + c$. ∎

Proof. Suppose $a < b$ and t is a real number. Apply previous proof with $-t$ in place of c. Therefore $a + (-t) < b + (-t) \iff a - t < b - t$ ∎

Problem 6

Prove: If $a < b$ and $a > 0$ that

$$a^2 < b^2$$

More generally, prove successively that

$$a^3 < b^3$$

$$a^4 < b^4$$

$$a^5 < b^5$$

Proceeding stepwise, we conclude that

$$a^n < b^n$$

for every positive integer n. To make this stepwise argument formal, one must state explicitly a property of integers which is called induction, and is discussed later in the book.

Proof. Suppose $a < b$ and $a > 0$. We proceed using induction on n considering a^n and b^n.

(**Base Case**) Clearly the inequality holds when $n = 1$. We now show the inequality holds when $n = 2$. Since $a < b$ it follows that $b - a > 0$. Since $a > 0$ it follows that $ab - a^2 > 0$ so $ab > a^2$. Also, since $b > 0$ it follows that $b^2 - ab > 0$ so $b^2 > ab$. Therefore $a^2 < ab < b^2$ so $a^2 < b^2$.

(**Induction Step**) Now, assume the inequality holds for $n-1$ and $n-2$. It follows that $a^{n-1} < b^{n-1}$ so $b^{n-1} - a^{n-1} > 0$. Since $a > 0$ and $b > 0$ it follows that $a + b > 0$. Then $(a+b)(b^{n-1} - a^{n-1}) > 0$ so $b^n - a^n + ab^{n-1} - a^{n-1}b > 0$. Now $ab^{n-1} - a^{n-1}b = ab(b^{n-2} - a^{n-2})$. Notice ab is clearly greater than 0 and by our hypothesis $b^{n-2} - a^{n-2} > 0$ so $ab^{n-1} - a^{n-1}b = ab(b^{n-2} - a^{n-2}) > 0$.

Then

$$\begin{aligned} & b^n - a^n + ab^{n-1} - a^{n-1}b > 0 \\ \iff & b^n - a^n + ab^{n-1} > a^{n-1}b \\ \iff & b^n - a^n > a^{n-1}b - ab^{n-1} > 0 \end{aligned}$$

It then follows that $b^n - a^n > 0$. Therefore $b^n > a^n$. ∎

Problem 7

Prove: If $0 < a < b$, then $a^{\frac{1}{n}} < b^{\frac{1}{n}}$. [Hint: Use Excersize 6.]

Proof. Suppose $0 < a < b$. Note $a = a^1 = a^{\frac{n}{n}} = \left(a^{\frac{1}{n}}\right)^n$. Similarly $b = \left(b^{\frac{1}{n}}\right)^n$. Then

$$\begin{aligned} & a < b \\ \iff & \left(a^{\frac{1}{n}}\right)^n < \left(b^{\frac{1}{n}}\right)^n \\ \iff & a^{\frac{1}{n}} < b^{\frac{1}{n}} \quad \text{Problem 6} \end{aligned}$$

∎

Problem 8

Let a, b, c, d be numbers and assume $b > 0$ and $d > 0$. Assume that

$$\frac{a}{b} < \frac{c}{d}$$

(a) Prove that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

(There are two inequalities to be proved here, the one on the left and the one on the right.)
(b) Let r be a number > 0. Prove that

$$\frac{a}{b} < \frac{a+rc}{b+rd} < \frac{c}{d}$$

(c) If $0 < r < s$, prove that

$$\frac{a+rc}{b+rd} = \frac{a+sc}{b+sd}$$

Proof. Since $\frac{a}{b} < \frac{c}{d}$ (Problem 6), it follows that $ad < bc$. Then

$$\begin{aligned}\frac{a+c}{b+d} - \frac{a}{b} &= \frac{b(a+c) - a(b+d)}{b(b+d)} \\ &= \frac{bc - ad}{b(b+d)}.\end{aligned}$$

Since $bc - ad > 0$ and $b(b+d) > 0$, it follows that $\frac{a}{b} < \frac{a+c}{b+d}$. Then

$$\begin{aligned}\frac{c}{d} - \frac{a+c}{b+d} &= \frac{c(b+d) - d(a+c)}{d(b+d)} \\ &= \frac{bc - ad}{d(b+d)}.\end{aligned}$$

Since $bc - ad > 0$ and $d(b+d) > 0$, it follows that $\frac{a+c}{b+d} < \frac{c}{d}$. Therefore

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

∎

Proof. Since $\frac{a}{b} < \frac{c}{d}$ (Problem 6), it follows that $ad < bc$. Then:

$$\begin{aligned}\frac{a+rc}{b+rd} - \frac{a}{b} &= \frac{b(a+rc) - a(b+rd)}{b(b+rd)} \\ &= \frac{r(bc - ad)}{b(b+rd)}\end{aligned}$$

Since $bc - ad > 0$ and $r, b, d > 0$, the numerator and denominator are positive. Therefore

$$\frac{a}{b} < \frac{a+rc}{b+rd}.$$

Also,

$$\begin{aligned}\frac{c}{d} - \frac{a+rc}{b+rd} &= \frac{c(b+rd) - d(a+rc)}{d(b+rd)} \\ &= \frac{bc - ad}{d(b+rd)}\end{aligned}$$

Since $bc - ad > 0$ and $d(b + rd) > 0$, it follows that

$$\frac{a + rc}{b + rd} < \frac{c}{d}.$$

$\blacksquare$

Proof. By part b we know that $\frac{a}{b} < \frac{a+rc}{b+rd} < \frac{c}{d}$. Also, $\frac{a}{b} < \frac{a+sc}{b+sd} < \frac{c}{d}$. Then:

$$\begin{aligned}
\frac{a + sc}{b + sd} - \frac{a + rc}{b + rd} &= \frac{ard - rbc + bsc - asd}{(b + sd)(b + rd)} \\
&= \frac{r(ad - bc) + s(bc - ad)}{(b + sd)(b + rd)} \\
&= \frac{(s - r)(bc - ad)}{(b + sd)(b + rd)}
\end{aligned}$$

Since $s > r$ and $bc - ad > 0$, the numerator is positive. Also, $b, d, s, r > 0$, so the denominator is positive. Therefore

$$\frac{a + rc}{b + rd} < \frac{a + sc}{b + sd}.$$

Then it follows that

$$\frac{a}{b} < \frac{a + rc}{b + rd} < \frac{a + sc}{b + sd} < \frac{c}{d}.$$

$\blacksquare$

4 Quadratics Equations

Problem 1

$$x^2 + 3x - 2 = 0$$

Solution:

$$a = 1, b = 3, c = -2$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-3 \pm \sqrt{3^2 - 4(1)(-2)}}{2(1)} \\
&= \frac{-3 \pm \sqrt{17}}{2}
\end{aligned}$$

So $x = \frac{-3+\sqrt{17}}{2}$ or $x = \frac{-3-\sqrt{17}}{2}$.

Problem 11

$$x^2 - \sqrt{2}x + 1 = 0$$

$$a = 1, b = -\sqrt{2}, c = 1$$

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-\sqrt{2}) \pm \sqrt{(-\sqrt{2})^2 - 4(1)(1)}}{2(1)} \\
&= \frac{\sqrt{2} \pm \sqrt{-2}}{2}
\end{aligned}
$$

There are no real solutions. In $\mathbb{C}$, $x = \frac{\sqrt{2}+\sqrt{-2}}{2}$ or $x = \frac{\sqrt{2}-\sqrt{-2}}{2}$.

5 Distance and Angles

5.1 Angles

Problem 2

Assume that the area of a disc of radius 1 is equal to the number π (approximately equal to $3.14159...$) and that the area of a disc of radius r is πr^2.

(a) What is the area of a sector in the disc of radius r lying between angles of θ_1 and θ_2 degrees, as shown in Fig. $5-20(a)$?

(b) What is the area of the band lying between two circles of radii r_1 and r_2 as shown in Fig. $5-20(b)$?

(c) What is the area in the region bounded by angles of θ_1 and θ_2 degrees and lying between circles of radii r_1 and r_2 as shown in Fig. $5-20(c)$?
Give your answers in terms of $\pi, \theta_1, \theta_2, r_2, r_1$.

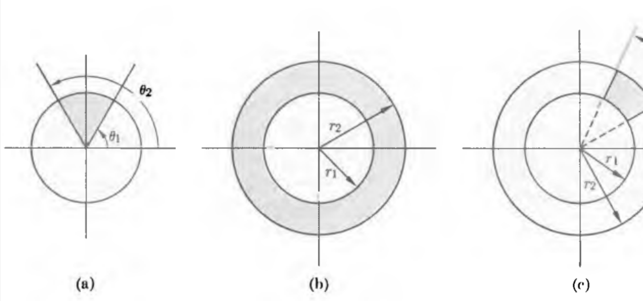


Fig. 5–20

Solution 2(a):

Area A of the sector is

$$A = \frac{|\theta_2 - \theta_1|}{360} \cdot \pi r_1^2$$

Solution 2(b):

Area A of the band is

$$A = \pi \cdot |r_2^2 - r_1^2|$$

Solution 2(c):

Area A of the region is

$$A = \frac{|\theta_2 - \theta_1|}{360} \cdot \pi \cdot |r_1^2 - r_2^2|$$

5.2 The Pythagoras Theorem

Problem 5

What is the length of the diagnol of a rectangle solid whose sides have lengths a, b, c? What if the sides have lengths ra, rb, rc.

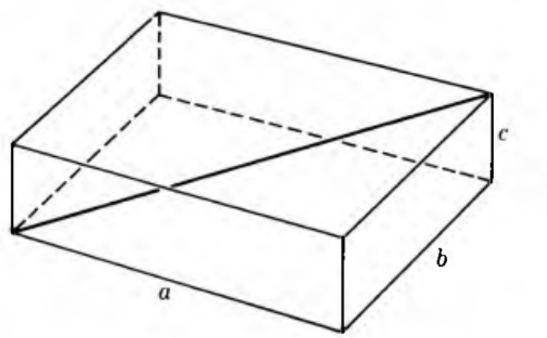


Fig. 5-30

Solution:

Let d be the length of the diagnol. Then

$$d = \sqrt{a^2 + b^2 + c^2}$$

If the sides have lengths ra, rb, rc then

$$d = \sqrt{(ra)^2 + (rb)^2 + (rc)^2} = \sqrt{r^2(a^2 + b^2 + c^2)} = r\sqrt{a^2 + b^2 + c^2}$$

Problem 9

Write down in detail the "similar steps" left to the reader in the proof of the corollary to the Pythagoras theorem.

Previous proof

Proof. Assume first that $d(P, M) = d(Q, M)$. By Pythagoras, we have

$$\begin{aligned} d(P, O)^2 + d(O, M)^2 &= d(P, M)^2 \\ &= d(Q, M)^2 \\ &= d(Q, O)^2 + d(O, M)^2 \end{aligned}$$

It follows that $d(P, O)^2 = d(Q, O)^2$ whence $d(P, O) = d(Q, O)$. ∎

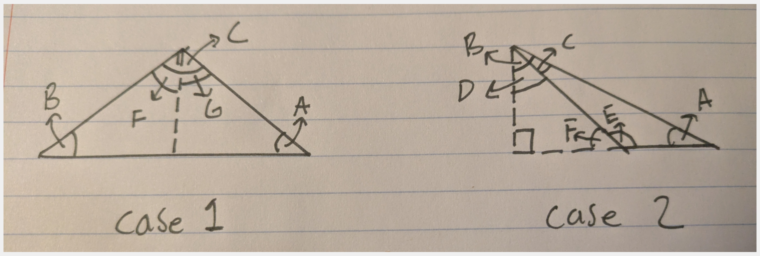
Proof. Suppose $d(P, O) = d(Q, O)$. By Pythagoras, we have $d(P, O)^2 + d(O, M)^2 = d(P, M)^2$. So $d(P, M)^2 - d(O, M)^2 = d(P, O)^2 = d(Q, O)^2 = d(Q, M)^2 - d(O, M)^2$. It follows that $d(P, M)^2 = d(Q, M)^2$ so $d(P, M) = d(Q, M)$. ∎

Problem 10

Prove that if A, B, C are the angles of an arbitrary triangle, then

$$m(A) + m(B) + m(C) = 180^\circ$$

by the following method: From any vertex draw the perpendicular to the line of the opposite side. Then use the result already known for right triangles.



Proof. Consider case 1. From the figure $m(C) = m(G)+m(F)$. The two subtriangles have sums $90^\circ+m(A)+m(G)$ and $90^\circ+m(B)+m(F)$. Now by Theorem 1, $m(A)+m(G) = 90^\circ$ and $m(B)+m(F) = 90^\circ$. So $90^\circ+m(A)+m(G) = 180^\circ$ then $m(A) = 90^\circ - m(G)$. Similarly, since $90^\circ + m(B) + m(F)$ then $m(B) = 90^\circ - m(F)$. Then the sum of the angles of the entire triangle is

$$\begin{aligned}
m(A) + m(B) + m(C) &= 90^\circ - m(G) + 90^\circ - m(F) + m(C) \\
&= 180^\circ - (m(G) + m(F)) + m(C) \\
&= 180^\circ - m(C) + m(C) \\
&= 180^\circ
\end{aligned}$$

Consider case 2. From the figure $m(D) + m(C) = m(B)$ so $m(C) = m(B) - m(D)$. Also $m(F) + m(E) = 180^\circ$ so $m(E) = 180^\circ - m(F)$. By Theorem 1, $m(D) + m(F) = 90^\circ$ and $m(B) + m(A) = 90^\circ$. Then the sum of the angles of the right-most inner triangle is

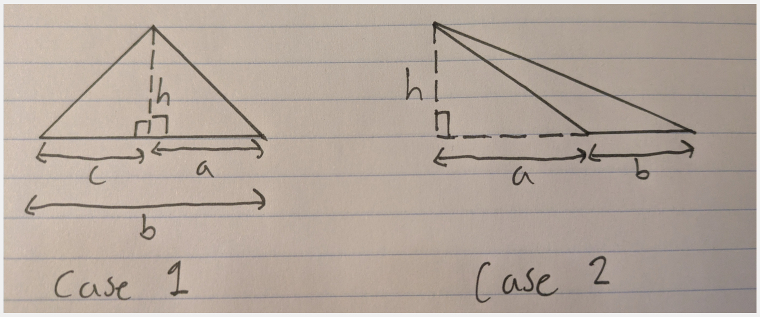
$$\begin{aligned}
m(C) + m(E) + 90^\circ &= m(B) - m(D) + 180^\circ - m(F) + m(A) \\
&= m(B) + m(A) + 180^\circ - (m(D) + m(F)) \\
&= m(B) + m(A) + 180^\circ - 90^\circ \\
&= m(B) + m(A) + 90^\circ \\
&= 90^\circ + 90^\circ \\
&= 180^\circ
\end{aligned}$$

∎

Problem 11

Show that the area of an arbitrary triangle of heigh h whose base has length b is $bh/2$. [*Hint:* Decompose the triangle into two right triangles. Distinguish between the two pictures in Fig. $5-31$. In one case the area of the triangle is the difference of the area of the two right triangles, and in the other case, it is the sum.]

Proof. Consider case 1. First note that $b = a+c$. By Theorem 2 (and for the rest of the proof) the area enclosed by the triangle on the right is $\frac{1}{2}ah$. The area enclosed by the triangle on the left is $\frac{1}{2}ch$. The area of the outer triangle is the sum of the inner triangles which is $\frac{1}{2}ch + \frac{1}{2}ah = \frac{1}{2}h(a+c) = \frac{1}{2}bh$.



Consider case 2. The area enclosed by the outer triangle is $\frac{1}{2}h(a+b)$. Then, the area enclosed by the right inner triangle is the area enclosed by the outer triangle minus the left inner triangle which is $\frac{1}{2}h(a+b) - \frac{1}{2}ha = \frac{1}{2}h(a+b-a) = \frac{1}{2}bh$. ∎

Problem 12

(a) Show that the length of the hypotenuse of a right triangle is $\geq$ the length of a leg.

(b) Let P be a point and L a line. Show that the smallest value for the distances $d(P,M)$ between P and points M on the line is the distance $d(P,Q)$, where Q is the point of the intersection between L and the line through P, perpendicular to L.

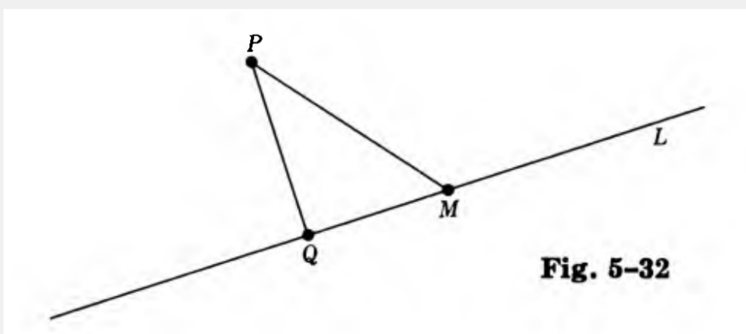


Fig. 5-32

Proof. Let a,b be the legs of a right triangle and c be the hypotenuse. By Pythagoras's Theorem we know that $c^2 = a^2 + b^2$. Since $b^2 \geq 0$ it follows that $c^2 \geq a^2$. Since $c, a \geq 0$ it follows that $c \geq a$. Similarly, $c^2 \geq b^2$ implies $c \geq b$. Therefore the hypotenuse c is greater than or equal to each leg a, b. ∎

Proof. Consider the right triangle formed by points P, Q, and M, where Q is the point on the line L such that $\overline{PQ}$ is perpendicular to L, and M is any other point on L. The hypotenuse of this triangle is $d(P,M)$, and we want to choose M to minimize this distance. By the Pythagorean Theorem,

$$d(P,M)^2 = d(P,Q)^2 + d(Q,M)^2.$$

The smallest value for $d(Q,M)^2$ is 0, which occurs if and only if $Q = M$. So when $Q = M$, we have $d(P,M) = d(P,Q)$. Therefore, the smallest distance from P to a point on the line L is $d(P,Q)$. ∎

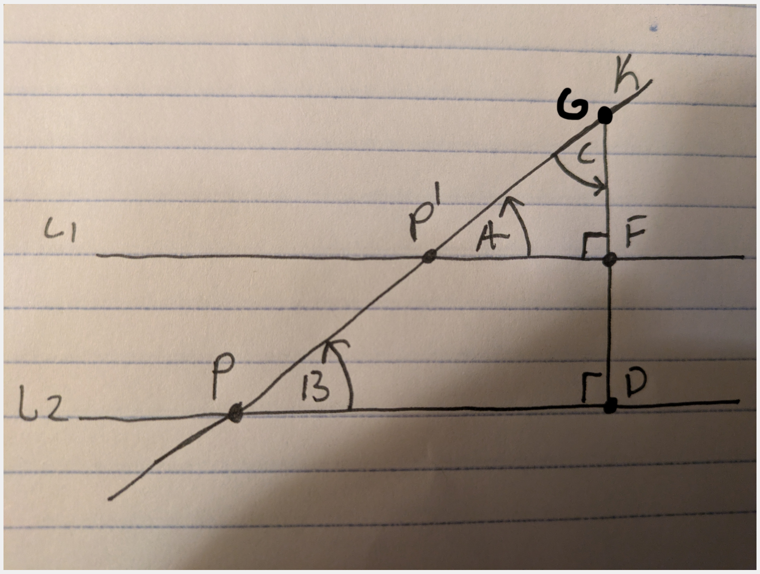
Problem 13

This exercize asks you to derive some standard properties of angles from elementary geometry. They are used very commonly. We refer to the following figures.

(a) In Fig. $5-33$(a), you are given two parallel lines L_1, L_2 and a line K which intersects them at points P and P' as shown. Let A and B then be angles which K makes with L_1 and L_2 respectively, as shown. Prove that $m(A) = m(B)$. [*Hint*: Draw a line from a point of K above L_1 and L_2. Then use the fact that the sum of the angles of a right triangle has $180°$.]

(b) In Fig. $5-33$(b), you are given L_1, L_2 and K again. Let B and B' be the alternate angles formed by K and L_1, L_2 respectively, as shown. Prove that $m(B) = m(B')$. (Actually, all you need to do here is refer to the appropriate portion of the text. Which is it?)

(c) Let K, L be two lines as shown on Fig. $5-33$(c). Prove that the opposite angles A and A' as shown have equal measure.



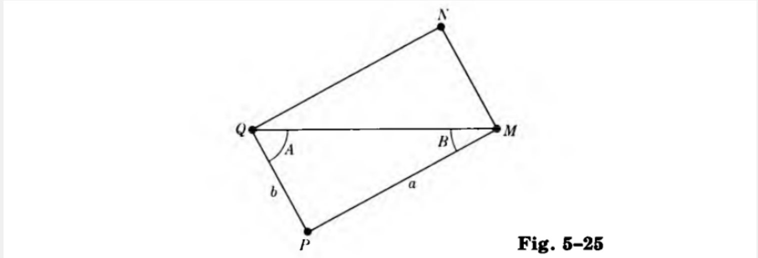
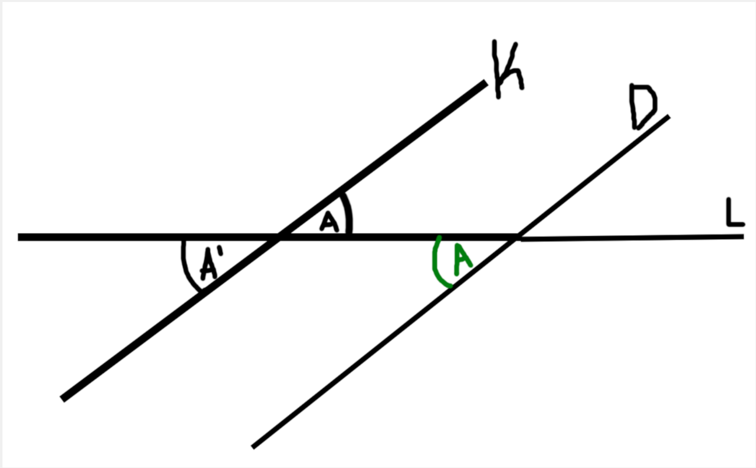


Fig. 5–25

Let A, B be the angles of the right triangle, other than the right angle. as shown in Fig. 5–25. It follows from RT that $\angle NQM$ has the same measure as B. Since $\angle NQP$ is a right angle, and since A and $\angle NQM$ together form

Proof. Refer to the figure. Consider the areas enclosed by the right triangles $\triangle P'FG$ and $\triangle PDG$. By Problem 10, the degrees of $\triangle P'FG$ is $90° + m(A) + m(C) = 180°$. Similarly, the degress of $\triangle PDG$ is $90° + m(B) + m(C) = 180°$. Then $90° + m(A) + m(C) = 90° + m(B) + m(C)$ and it follows that $m(A) = m(B)$. ∎

Proof. Refer to the text. Then let the parallel segments formed by points Q, P and N, M be the parallel lines L_1, L_2. Also let B be the B in our problem. Finally, let $\angle NQM$ be $m(B')$. It then follows from the text that $m(B) = m(B')$. ∎



Proof. Refer to the figure. Line D is parallel to line K. The measure of the green angle is equal to $m(A)$ by Problem 13(b). Then $m(A') = m(A)$ by Problem 13(a). ∎

Problem 14

Let $\triangle PQM$ be a triangle. Let L_1 be the perpendicular bisector of $\overline{PQ}$ and let L_2 be the perpendicular bisector of $\overline{QM}$. Let O be the point of intersection of L_1 and L_2. Show that $d(P,O) = d(M,O)$, and hence that O lies on the perpendicular bisector of $\overline{PM}$. Thus the perpendicular bisectors of the sides of the triangle meet in a point.

Proof. Since O lies on the perpendicular bisector of $\overline{PQ}$ it follows that $d(P,O) = d(Q,O)$. Similarly, $d(M,O) = d(Q,O)$. Then $d(P,O) = d(Q,O) = d(M,O)$. ∎

6 Isometries

6.1 Some Standard Mappings of the Plane

Problem 1

Let F be a mapping of the plane into itself. We define a **fixed point** for F to be a point P such that $F(P) = P$.

Describe the fixed points of the following mappings.
(a) The identity.
(b) Reflection through a given point O.
(c) Reflection through a line.
(d) A rotation not equal to the identity, with respect to a given point O.
(e) A translation not equal to the identity.
(f) Dilation by a number $r > 0$, relative to a given point O.

Solution (a):

All points in the plane are fixed points.

Solution (b):

The point O is a fixed point.

Solution (c):

All points on the line are fixed points.

Solution (d):

The point O is a fixed point.

Solution (e):

There are no fixed points on the plane.

Solution (f):

If $r = 1$, then all points are fixed points. If $r \neq 1$, then only O is a fixed point.

6.2 Isometries

Problem 2

For which values of r is dilation by r an isometry?

When $r = 1$ or $r = -1$.

Problem 4

Let L, K be two parallel lines, and let F be an isometry. Prove that $F(L)$ and $F(K)$ are parallel.

Proof. If $L = K$ then trivially $F(L)$ and $F(K)$ are parallel as they are equal since F is an isometry.

Suppose $L \neq K$. For contradiction, suppose $F(L)$ and $F(K)$ are not parallel. Then there is a point where $F(L)$ and $F(K)$ intersect. Since F is an isometry, this would imply that L and K also intersect, contradicting the fact that L and K are distinct parallel lines. ∎

Problem 5

Let K, L be perpendicular lines, and let F be an isometry. Prove that $F(K)$ and $F(L)$ are perpendicular. [Hint: Use the corollary of the Pythagoras theorem.]

Proof. Let I be the intersection of the lines K and L. Let P and Q be two points lying on the line L which are equal distances from I. Let O be a point on K such that $O \neq I$. By the corollary to the Pythagorean Theorem, $d(P, O) = d(Q, O)$. Now $F(P), F(Q)$ determine a line and $F(O), F(I)$ determine a line (corollary in the text). Then, since F is an isometry, $d(F(P), F(O)) = d(F(Q), F(O))$. It then follows that the line formed by $F(O), F(I)$ is the perpendicular bisector of $F(P), F(Q)$. ∎

Problem 6

Visualize 3-dimensional space. We also have the notion of distance in space, satisfying the same basic properties as in a plane. We can therefore define an isometry of 3-space in the same way that we defined an isometry of the plane. It is a mapping of 3-space into itself which is distance preserving. Are Theorems 1 and 2 valid in 3-space? How would you formulate Theorem 3? (Consider the plane in which the three points lie.) Now formulate a theorem in 3-space about an isometry being the identity provided that it leaves enough points fixed. Describe a proof for such a theorem, similar to the proof of Theorem 3. Make a list of what you need to assume to make such a proof go through. Write all of this up as if you were writing a book. Aside from learning mathematical substance, you will also learn how to think more clearly, and how to write mathematics in the process

Theorem 1. *Let F be an isometry. Let P, Q, M, S be four distinct points which do not lie on the same plane. Assume that P, Q, M, S are fixed points of F; that is*

$$F(P) = P, F(Q) = Q, F(M) = M, F(S) = S$$

Then F is the identity.

The proof could be layed out as follows.

1. First take the plane formed by P, Q, M and by Theorem 3 all points on that plane are fixed points.
2. Then note that point S does not lie on this plane.
3. Then since F is an isometry the $d(P, S) = d(F(P), F(S))$, $d(Q, S) = d(F(Q), F(S))$, and $d(M, S) = d(F(M), F(S))$.
4. Determine that $F(S) = S$, and hence that all points of space are fixed; therefore F is the identity.

6.3 Composition of Isometries

Problem 1

Let F be a reflection through a line L. What is the smallest positive integer n such that $F^n = I$.

Solution:

$n = 2$.

Problem 4

Give an example of two isometries F_1, F_2 such that

$$F_1 \circ F_2 \neq F_2 \circ F_1$$

Solution:

Let G_x = rotation by $x°$about the origin. Let T_1 = translation of 1 to the right. Then let $F_1 = G_{45}$ and $F_2 = T_1$. These two isometries do not commute

$$F_1 \circ F_2 \neq F_2 \circ F_1.$$

6.4 Inverse of Isometries

Problem 1

(a) Let F be an isometry which has an inverse F^{-1}. Let S be a circle of radius r, and center P. Show that the image of S under F is a circle. [Hint: Let S' be the circle of center $F(P)$ and radius r. Show that $F(S)$ is contained in S' and that every point of S' is the image under F of a point in S.]
(b) Let F be an isometry which has an inverse F^{-1}. Let D be a disc of radius r and center P. Show that the image of D under F is a disc.

Proof. Let S' be the circle of center $F(P)$ and radius r. We need to show that $F(S) \subseteq S'$ and $S' \subseteq F(S)$.

We first show $F(S) \subseteq S'$. Let T be a point such that $d(P, T) = r$. Since F is an isometry $d(F(P), F(T)) = r$. Since T was arbitrary all points T such that $d(P, T) = r$ are contained within the circle centered at $F(P)$ with radius r which is exactly S'.

We now show $S' \subseteq F(S)$. Let T be a point at distance r from the center $F(P)$ of the circle S'. Now let $Y = F^{-1}(T)$. It follows that $F(Y) = T$. We know that $d(F(Y), F(P)) = r$. Since F is an isometry $d(Y, P) = r$. ∎

Proof. Let D' be the disc of center $F(P)$ and radius r. We need to show that $F(D) \subseteq D'$ and $D' \subseteq F(D)$.

We first show $F(D) \subseteq D'$. Let T be a point such that $d(P,T) \leq r$. Since F is an isometry, $d(F(P), F(T)) \leq r$. Since T was arbitrary, all points T with $d(P,T) \leq r$ are contained within the disc centered at $F(P)$ with radius r, which is exactly D'.

We now show $D' \subseteq F(D)$. Let T be a point at distance $\leq r$ from the center $F(P)$ of the disc D'. Now let $Y = F^{-1}(T)$. It follows that $F(Y) = T$. We know that $d(F(Y), F(P)) \leq r$. Since F is an isometry, $d(Y, P) \leq r$. ∎

Problem 2

Let P, Q, P', Q' be points such that

$$d(P,Q) = d(P',Q')$$

Prove that there exists an isometry F such that $F(P) = P'$ and $F(Q) = Q'$. You may assume the statements we have assumed in this section.

Proof. First, perform a rotation G of Q about P such that the line formed by P and Q is parallel to the line formed by P' and Q'.

Next, let T be the translation along the ray from Q to Q' of length $d(Q,Q')$. Applying T moves Q exactly to Q'. Since the line through P and Q is parallel to the line through P' and Q', the same translation moves P to P'.

Since rotations and translations are isometries, the composition $T \circ G$ is an isometry. ∎

Problem 3

Let F, G, H be isomemetries and assume that F has an inverse. If

$$F \circ G = F \circ H$$

prove that $G = H$ (**cancellation law** for isomemetries).

Proof. Applying F^{-1} to both sides yields

$$\begin{aligned} & F^{-1} \circ (F \circ G) = F^{-1} \circ (F \circ H) \\ \iff & (F^{-1} \circ F) \circ G = (F^{-1} \circ F) \circ H \\ \iff & I \circ G = I \circ H \\ \iff & G = H \end{aligned}$$

∎

Problem 4

(a) Let F be an isometry such that $F^2 = I$ and $F^3 = I$. Prove that $F = I$.

(b) Let F be an isometry such that $F^4 = I$ and $F^7 = I$. Prove that $F = I$.

(c) Let F be an isometry such that $F^5 = I$ and $F^8 = I$. Prove that $F = I$.

Proof. Since $F^2 = I$ it follows that $F^3 = F \circ I$. Then $F = F \circ I = F^3 = I$. ∎

Proof. Since $F^4 = I$ it follows that $I = F^{-4} \circ I = F^{-4}$. Also since $F^4 = I$ it follows that $F = F^{-3} \circ I = F^{-3}$. Then $F = F^{-3} = F^7 \circ F^{-4} = I \circ I = I$. ∎

Proof. Since $F^5 = I$ it follows that $F = F^{-4} \circ I = F^{-4}$. Also, since $F^8 = I$ it follows that $F^3 = F^{-5} \circ I = F^{-5}$. Finally, since $F^8 = I$ it follows that $I = F^{-8} \circ I = F^{-8}$. Then

$$F = F^{-4} \circ I = F^{-4} \circ F^{-8} = F^{-12} = F^{-5} \circ F^{-8} = F^3 \circ F^{-8} = F^3 \circ F^{-4} \circ F^{-4} = F^3 \circ F \circ F = F^5 = I$$

∎

Problem 5

Write out the proof of the corollary of Theorem 3. (Consider $F^{-1} \circ G$.)

Corollary of Theorem 3. *Let P, Q, M be three distinct points which do not lie on the same line. Let F, G be isometries such that*

$$F(P) = G(P), \qquad F(Q) = G(Q), \qquad F(M) = G(M).$$

Assume that F^{-1} exists. Then $F = G$.

Proof. The proof is very easy and will be left as an exercise.

Proof. Since $F(P) = G(P)$, $F(Q) = G(Q)$, and $F(M) = G(M)$ it follows that $P = (F^{-1} \circ G)(P)$, $Q = (F^{-1} \circ G)(Q)$, and $M = (F^{-1} \circ G)(M)$. Now this is three fixed points so $F^{-1} \circ G$ is the identity mapping. Since $F^{-1} \circ G = I$ it follows that G is the inverse of F^{-1} which is F. ∎

Problem 6

Let $F \circ G \circ H$ be the composite of three isometries. Assume that F^{-1}, G^{-1}, H^{-1} exist. Prove that $(F \circ G \circ H)^{-1}$ exists, and express this inverse in terms of the inverses for F, G, H.

Proof. Let P be an arbitrary point in the plane such that $(F \circ G \circ H)(P) = X$. Then

$$\begin{aligned}
&(F \circ G \circ H)(P) = X \\
\iff &(H^{-1} \circ (F \circ G \circ H))(P) = H^{-1}(X) \\
\iff &((G^{-1} \circ H^{-1}) \circ (F \circ G \circ H))(P) = (G^{-1} \circ H^{-1})(X) \\
\iff &((F^{-1} \circ G^{-1} \circ H^{-1}) \circ (F \circ G \circ H))(P) = (F^{-1} \circ G^{-1} \circ H^{-1})(X) \\
\iff &((F \circ G \circ H)^{-1} \circ (F \circ G \circ H))(P) = (F^{-1} \circ G^{-1} \circ H^{-1})(X) \\
\iff &I(P) = (F^{-1} \circ G^{-1} \circ H^{-1})(X) \\
\iff &P = (F^{-1} \circ G^{-1} \circ H^{-1})(X)
\end{aligned}$$

Therefore, $(F \circ G \circ H)^{-1} = H^{-1} \circ G^{-1} \circ F^{-1}$. ∎

Problem 7

Let F be an isometry such that $F^7 = I$. Express F^{-1} as a positive power F.

Proof. Since $F^7 = I$, we have

$$F^{-1} = I \circ F^{-1} = F^7 \circ F^{-1} = F^6.$$

∎

Problem 8

Let n be a positive integer and let F be an isometry such that $F^n = I$. Express F^{-1} as a positive power of F.

Proof. Since $F^n = I$, we have

$$F^{-1} = I \circ I \circ F^{-1} = F^n \circ F^n \circ F^{-1} = F^{2n-1}.$$

Since $n \geq 1$ it follows that $2n \geq 2$ so $2n > 1$ and $2n - 1 > 0$. ∎

Problem 9

Consider the corners of a square centered at the origin. For convenience of notation, number these corners $1, 2, 3, 4$ as in Fig. $6 - 26$.
Write the image of each one of these corners under the isometries. $H, V, H \circ V, V \circ H$. Just to show you an easy notation to do this, we write down the images of these corners under rotation by $90°$in the following form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

This notation means that if G is a rotation by $90°$, then $G(1) = 2$, $G(2) = 3$, $G(3) = 4$, and $G(4) = 1$.
H, V are the reflections along the horizontal line and vertical line respectively.

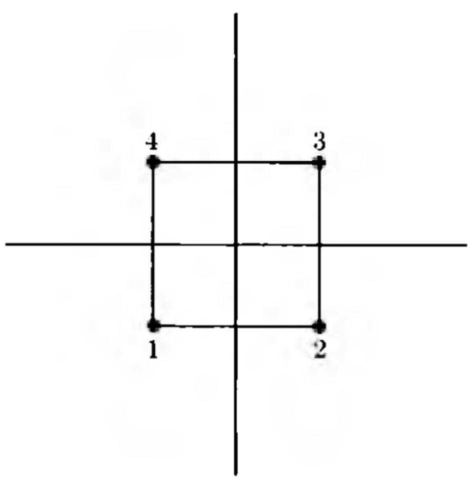


Fig. 6-26

Solution:

Image under H.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Image under V.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Image under $H \circ V$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Image under $V \circ H$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Problem 10

Let G be a rotation by $90°$so that $G^4 = I$. Express $H \circ G \circ H$ as a power of G. For what positive integer n do we have

$$H \circ G = G^n \circ H$$

Write down the images of the corner of the square as in the preceeding exercize, under the maps $I, G, G^2, G^3, H, H \circ G, H \circ G^2, H \circ G^3, G \circ H, G^2 \circ H, G^3 \circ H$.

Solution:

$$H \circ G \circ H = G^3$$

The following equation holds when $n = 3$.

$$H \circ G = G^n \circ H$$

Image under I.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Image under G.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

Image under G^2.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Image under G^3.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

Image under H.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Image under $H \circ G$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{bmatrix}$$

Image under $H \circ G^2$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{bmatrix}$$

Image under $H \circ G^3$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Image under $G \circ H$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

Image under $G^2 \circ H$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Image under $G^3 \circ H$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Problem 13

Consider a triangle whose three sides have equal length and whose three angles have the same measure, $60°$, as in Fig. $6-27$.
The vertices of the triangle are numbered $1, 2, 3$. Let G be a rotation by $120°$and let V, as usual, be reflection through the vertical axis.
(a) Give the effect of the six isometries I, G, G^2, V, VG, VG^2 on the vertices, using the same notation as exercize 9.
(b) Make up the multiplication table for these six isometries.

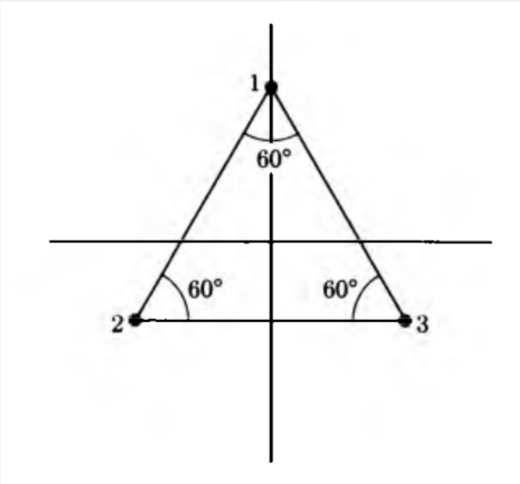


Fig. 6–27

Solution:

Image under I.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Image under G.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

Image under G^2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Image under V.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Image under VG.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Image under VG^2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

$\circ$	I	G	G^2	V	VG	VG^2
I	I	G	G^2	V	VG	VG^2
G	G	G^2	I	VG^2	V	VG
G^2	G^2	I	G	VG	VG^2	V
V	V	VG	VG^2	I	G^2	G
VG	VG	VG^2	V	G	I	G^2
VG^2	VG^2	V	VG	G^2	G	I

6.5 Characterization of Isometries

Problem 1

Prove that every isometry has an inverse.

Proof. Let F be an arbitrary isometry. There are four cases depending on the number of fixed points under F.

(≥ 3 **fixed points**) By Theorem 3, $F = I$ which clearly has an inverse: $F^{-1} = I$.

(2 **fixed points**) Let P and Q be the two fixed points under F. By Theorem 4, either F is the identity, or F is a reflection through the line L_{PQ} passing through P and Q. In the identity case, $F^{-1} = I$. If F is a reflection, the inverse is also F: $F^{-1} = F$.

(1 **fixed point**) Let P be the single fixed point under F. By Theorem 5, either F is a rotation about P, or F is a rotation composed with a reflection through a line through P. In the rotation case, F^{-1} is the rotation by the opposite angle about P. In the rotation-reflection case, F^{-1} is the reflection composed with the rotation by the opposite angle.

(**no fixed points**) By Theorem 6, either F is a translation, a composite of a translation and a rotation, or a composite of a translation, a rotation, and a reflection through a line. In the translation case, F^{-1} is the translation by the opposite vector. In the translation-rotation case, F^{-1} is the rotation by the opposite angle followed by translation by the opposite vector. In the translation-rotation-reflection case, F^{-1} is the reflection followed by rotation by the opposite angle and translation by the opposite vector.

Since these cases are exhaustive, every isometry has an inverse. ∎

Problem 2

If P is a fixed point for an isometry F, prove that P is also a fixed point for F^{-1}.

Proof. Since P is a fixed point under F is follows that $F(P) = P$. By Problem 1 we know that F^{-1} exists. Applying F^{-1} to both sides yields $P = F^{-1}(P)$. Therefore, P is a fixed point of F^{-1}. ∎

6.6 Congruences

Problem 1

Prove that two discs of the same radius are congruent.

Proof. Let the first disc be $D(r, O)$, of radius r, centered at O, and let the other disc be $C(r, O')$, centered at O'. Let T be the translation which maps O on O'. We know that T preserves distances. Hence if P is at distance $\leq r$ from O, then $T(P)$ is at distance $\leq r$ from $T(O) = O'$. Hence the image of the disc $D(r, O)$ is contained in the circle $D(r, O')$. We must still show that every point on $D(r, O')$ is the image of a point on $D(r, O)$ under T. Let Q be a point at distance $\leq r$ from O'. Note that the point

$$P = T^{-1}(Q)$$

is at distance $\leq r$ from O, and that $T(P) = T(T^{-1}(Q)) = Q$. ∎

Problem 2

Let S, S', S'' be sets in the plane. Prove that if S is congruent to S', and S' is congruent to S'', then S is congruent to S''. Prove that if S is congruent to S', then S' is congruent to S.

Proof. Suppose that S is congruent to S' and S' is congruent to S''. Let T be the isometry such that $T(S) = S'$. Let F be the isometry such that $F(S') = S''$. Then $(F \circ T)(S) = F(T(S)) = F(S') = S''$. Note that a composition of isometries is an isometry. Thus $F \circ T$ is an isometry mapping S to S'' so S is congruent to S''. ∎

Proof. Suppose that S is congruent to S'. Let F be the isometry such that $F(S) = S'$. Since F is an isometry it has an inverse F^{-1}. Then applying this to both sides yields $S = F^{-1}(S')$. Thus S' is congruent to S. ∎

Problem 3

Prove that two squares whose sides have the same length are congruent.

Proof. Let S and S' be two squares whose sides have the same length. Let A, B, C, D be the vertices of S in cyclic order. Let O be the intersection of the diagonals AC and BD. Since the diagonals of a square are equal in length and bisect each other at right angles, their intersection O is equidistant from all vertices. Thus O is the center of S.

Similarly, let A', B', C', D' be the vertices of S' in cyclic order, and let O' be the intersection of its diagonals (the center of S').

Let T be the translation mapping O to O'. Let G be the rotation with respect to O' of S' such that $(G \circ T)(A) = A'$. Because G is a rotation about the center O', it must also send the point on the opposite end of that diagonal, namely $(G \circ T)(C)$, to C', and $(G \circ T)(D)$ to D'. If $(G \circ T)(B) \neq B'$, then perform a reflection R across the line connecting $(G \circ T)(A)$ and $(G \circ T)(C)$. Otherwise, let $R = I$ such that I is the identity mapping. Let $F = R \circ G \circ T$.

Thus $F(A) = A', F(B) = B', F(C) = C', F(D) = D'$, and therefore $F(S) \subseteq S'$. Since F is an isometry, it preserves distances and lines, so every point of S maps to points of S' and vice versa, thus $F(S) = S'$. ∎

Problem 4

Prove that any two lines are congruent.

Proof. Let L and L' be two lines. Pick points $P, Q \in L$ and $P', Q' \in L'$. Using Section 6.4 Problem 2, since $d(P, Q) = d(P', Q')$, there exists an isometry F such that $F(P) = P'$ and $F(Q) = Q'$.

Since F preserves distances and maps two points on L to two points on L', it maps the entire line L onto L'. Therefore, the lines L and L' are congruent. ∎

Problem 5

Let $\triangle ABC$ be a triangle whose three angles all have $60°$. Prove that the sides have equal length. [Hint: From any vertex draw the perpendicular to the other side, and reflect through this perpendicular.]

Proof. Let A, B, C be the vertices of a triangle whose three angles all have $60°$. Let L be the perpendicular bisector of $\overline{AB}$, intersecting $\overline{AB}$ at its midpoint O. There are now two right triangles, namely $\triangle AOC$ and $\triangle BOC$, with right angles at O. These triangles share the side $\overline{OC}$. Since L is the perpendicular bisector of $\overline{AB}$, we have $d(A, O) = d(B, O)$. Then

$$d(A, C)^2 = d(A, O)^2 + d(O, C)^2 = d(B, O)^2 + d(O, C)^2 = d(B, C)^2.$$

It follows that $d(A, C) = d(B, C)$.

Apply similar steps but let L be the perpendicular bisector of $\overline{BC}$, thus showing that $d(A, B) = d(A, C)$.

Thus $d(A, B) = d(A, C) = d(B, C)$. ∎

Proof. Let A, B, C be the vertices of a triangle with all angles $60°$. Consider the perpendicular bisector L of side $\overline{AB}$. Reflect the vertex C across L to a point C'.

Since the triangle has equal angles, this reflection maps the triangle to itself, so $C' = C$. Reflection across the perpendicular bisector preserves distances, so

$$AC = BC.$$

Similarly, reflecting across the perpendicular bisector of $\overline{BC}$ gives

$$AB = AC.$$

Therefore, all three sides are equal:

$$AB = BC = AC.$$

∎

Problem 6

Prove Theorem 9. At first you are not allowed to use Theorem 10. If you were allowed to use Theorem 10, how would you deduce Theorem 9 from it.

Theorem 2. *Let $\triangle PQM$ and $\triangle P'Q'M'$ be right triangles whose right anlges are at Q and Q' respectively. Assume that the corresponding legs have the same lengths, that is:*

$$d(P, Q) = d(P', Q')$$

and

$$d(Q, M) = d(Q', M')$$

Then the triangles are congruent.

Proof. Let T be the translation such that $T(Q) = Q'$. Let G be the ration such that $(G \circ T)(P) = P'$. Finally, if $(G \circ T)(M) \neq M'$ apply a reflection on the line $\overline{P'Q'}$ Let $F = G \circ T$.

Thus, $G(Q) = Q'$, $G(P) = P'$, and $G(M) = M'$.

Now let O be any point on the segment $\overline{PQ}$. It follows that $d(P, Q) = d(F(P), F(Q)) = d(P, O) + d(Q, O) = d(F(P), F(O)) + d(F(Q), F(O))$. Then since $d(F(P), F(Q)) = d(F(P), F(O)) + d(F(Q), F(O))$, $F(O)$ lies on the segment $\overline{P'Q'}$.

Now let K be any point on the segment $\overline{P'Q'}$. Since F is an isometry, it has an inverse; let $U = F^{-1}(K)$. Then $d(P,Q) = d(F^{-1}(P'), F^{-1}(Q')) = d(P,U) + d(U,Q)$ so U lies on the segment $\overline{PQ}$. It follow that $F(U) = F(F^{-1}(K)) = K$.

Simlar arguments apply to $\overline{QM}$ and $\overline{PM}$. Thus the triangles are congruent. ∎

Proof. Since the legs have the same length, by the Pythagorean Theorem their hypotenuses are of equal length. Thus the lengths of the sides of the triangles are equal. One can then easily apply Theorem 10 to show the triangles are congruent. ∎

Problem 7

Let $\triangle PQM$ and $\triangle P'Q'M'$ be triangles having one corresponding angle of the same measure, say $\angle PQM$ and $\angle P'Q'M'$ have the same measure, and having adjacent sides of the same length, i.e.

$$d(P,Q) = d(P',Q') \text{ and } d(Q,M) = d(Q',M')$$

Prove that the triangles are congruent.

Proof. Using Section 6.4 Problem 2, since $d(P,Q) = d(P',Q')$, there exists an isometry T such that $T(P) = P'$ and $T(Q) = Q'$. From the figure, if $T(M) \neq M'$ then perform a reflection R through the line $\overline{P'Q'}$. Otherwise, let $R = I$ where I is the identity mapping. Let $F = R \circ T$.

If $R \neq I$, then reflecting about the line $\overline{P'Q'}$ ensures $F(M)$ points in the same direction as M'. Since F is an isometry, it preserves distances, so $d(Q', F(M)) = d(Q,M) = d(Q',M')$.

It follows that $F(P) = P', F(Q) = Q', F(M) = M'$. Thus the triangle's sides are the same length and applying Theorem 10 shows they are congruent. ∎

Problem 8

Prove that two rectangles having corresponding sides of equal lengths are congruent.

Proof. Let $ABCD$ and $A'B'C'D'$ be the corners encountered cyclicly of two rectangles such that $d(A,B) = d(A',B')$ and $d(B,C) = d(B',C')$.

Let T be the translation such that $T(A) = A'$. Let G be the rotation about A' such that $(G \circ T)(C)$ lies on the line through A' and C'. If necessary, let R be a reflection across the line $\overline{A'C'}$ to match B and B'. Otherwise, let $R = I$ where I is the identity mapping.

Thus $F = R \circ G \circ T$.

Draw the diagonal $\overline{AC}$ in both rectangles. This divides each rectangle into two right triangles: $\triangle ABC$ and $\triangle ADC$ in the first rectangle, and $\triangle A'B'C'$ and $\triangle A'D'C'$ in the second.

Now let O be any point on the segment $\overline{AC}$ of the first rectangle. It follows that $d(A,C) = d(A,O) + d(O,C) = d(A', F(O)) + d(F(O), C')$. Then, by the Pythagorean Theorem, $F(O)$ lies on the segment $\overline{A'C'}$.

Now let K be any point on the segment $\overline{A'C'}$. Since F is an isometry, it has an inverse; let $U = F^{-1}(K)$. Then $d(A,C) = d(F^{-1}(A'), F^{-1}(C')) = d(A,U) + d(U,C)$ so U lies on the segment $\overline{AC}$. By definition of U, $F(U) = F(F^{-1}(K)) = K$.

Hence the corresponding right triangles along the diagonals are congruent, and therefore the rectangles $ABCD$ and $A'B'C'D'$ are congruent. ∎

Problem 9

Give a definition of the region bounded by a square in terms of line segments. Same thing for a rectangle.

Definition 1. *Let S be a square with distinct parallel line segments $\overline{AB}$ and $\overline{CD}$. We can define the area of S to be all points on all line segments formed by points X, Y where X lies on $\overline{AB}$ and Y lies on $\overline{CD}$. The area for a rectangle can be defined in precisely the same way with exception to letting S be a rectangle.*

Problem 11

Let $\triangle PQM$ and $\triangle P'Q'M'$ be triangles whose corresponding angles have the same measures (i.e. the angle with vertex P has the same measure as the angle with vertex at P', and similarly for the angles with vertices at Q, Q' and M, M'). Assume that $d(P,Q) = d(P',Q')$. Prove that the triangles are congruent.

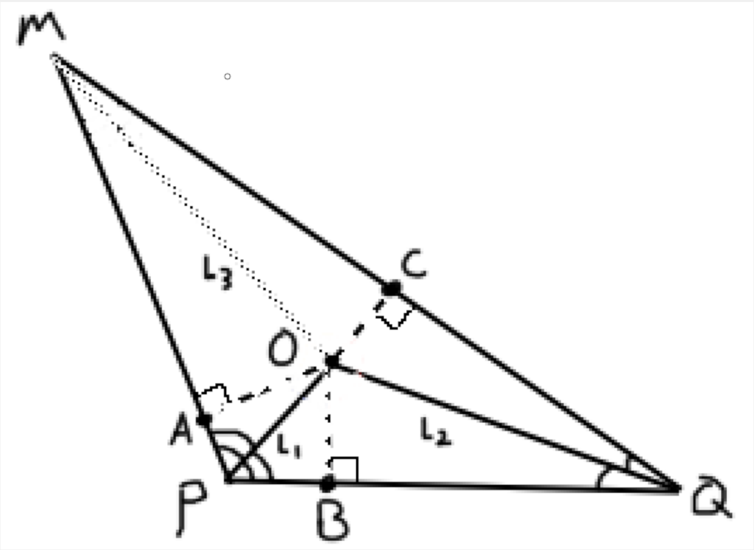
Proof. Using Section 6.4 Problem 2, since $d(P,Q) = d(P',Q')$, there exists an isometry T such that $T(P) = P'$ and $T(Q) = Q'$. From the figure, if $T(M) \neq M'$, then perform a reflection R through the line $\overline{P'Q'}$. Otherwise, let $R = I$, where I is the identity mapping. Let $F = R \circ T$.

If $R \neq I$, then reflecting about the line $\overline{P'Q'}$ ensures $F(M)$ lies on the same side of $\overline{P'Q'}$ as M'. Hence $F(P) = P'$ and $F(Q) = Q'$. Since corresponding angles are equal, the ray from $F(P)$ through $F(M)$ coincides with the ray from P' through M', and the ray from $F(Q)$ through $F(M)$ coincides with the ray from Q' through M'. Therefore $F(M) = M'$, and F maps all vertices of $\triangle PQM$ to the corresponding vertices of $\triangle P'Q'M'$.

Thus the triangles are congruent. ∎

Problem 12

Let $\triangle PQM$ be a triangle. Let L_1, L_2, L_3 be the three lines which bisect the three angles of the triangle, respectively. Let O be the point of intersection of L_1 and L_2. Prove that O lies on L_3. [Hint: From O, draw the perpendicular segments to the corresponding sides. Prove that their lengths are equal.]



Proof. Let A, B, and C be the feet of the perpendiculars from O to the sides $\overline{PM}$, $\overline{PQ}$, and $\overline{MQ}$ respectively.

We first show that $\angle AOP$ and $\angle BOP$ are equivalent. The sum of the angles of $\triangle AOP$ and $\triangle BOP$ can be expressed as $90 + \angle APO + \angle AOP$ and $90 + \angle BPO + \angle BOP$, respectively. These sums must both equal 180, so

$$90 + \angle APO + \angle AOP = 90 + \angle BPO + \angle BOP.$$

But $\angle APO = \angle BPO$, since L_1 bisects $\angle P$. Therefore, $\angle AOP = \angle BOP$.

Thus $\triangle AOP$ is a reflection of $\triangle BOP$ across L_1. Therefore their corresponding sides are equal, so the perpendicular distances from O to $\overline{PM}$ and $\overline{PQ}$ are equal, and thus $OA = OB$.

By the same reasoning, since O also lies on L_2, which bisects $\angle Q$, the right triangles $\triangle OBQ$ and $\triangle OCQ$ are reflections across L_2, giving $OB = OC$.

Therefore $OA = OB = OC$. A point equidistant from all three sides of a triangle lies on the bisector of the remaining angle. Hence, O lies on L_3. ∎

7 Area and Applications

7.1 Area of a Disc of Radius r

Problem 3

(a) Suppose that the sides of a rectangles S have lengths r and s. What are the lengths of the sides of the rectangle $F_{a,b}(S)$, i.e. of the rectangle obtained by the mixed dilation $F_{a,b}$?
(b) What is the area of $F_{a,b}$?
(c) If S is a bounded region in the plane with area A, what is the area of $F_{a,b}(S)$?

Solution (a):

$$F_{a,b}(S) \text{ has } ar \text{ width and } bs \text{ height.}$$

Solution (b):

$$\text{area} = ar \cdot bs = ab(rs)$$

Solution (c):

$$F_{a,b}(S) = ab(A)$$

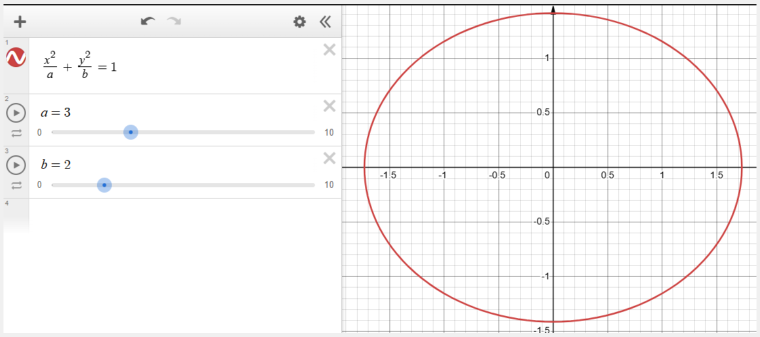
Problem 4

(a) Show that the set of points (u, v) satisfying the equation

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

is the image of the circle of radius 1 centered at O under the map $F_{a,b}$.
(b) Let $a = 3$ and $b = 2$. Sketch this set, which is called an **ellipse**.
(c) Can you guess and motivate your guess as to what the area of the region bounded by the ellipse in (a) should be.

Proof. Let C be the set of points making up the circle with radius $r = 1$ and center O. Let P be an arbitrary point in C. Since $r = 1$, $d(O, P) = 1$. Let L_1 and L_2 be the vertical and horizontal lines through O, respectively. Drop perpendiculars from P to L_1 and L_2, and let Q and R be the feet of these perpendiculars on L_1 and L_2, respectively. Then $\triangle OQR$ is a right triangle with hypotenuse $\overline{OP}$, and legs $\overline{OQ}$ and $\overline{OR}$ lying along L_1 and L_2. Let $u = d(OQ), v = d(OR)$ and by Pythagoras' Theorem $u^2 + v^2 = 1$. By applying the mapping $F_{a,b}$ to $\overline{OQ}, \overline{OR}$ lengths u, v are scaled by $\frac{1}{a}, \frac{1}{b}$ respectively we get $\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$. ∎



Solution (b):

Solution (c):

Start with a circle of radius $r = 1$ and suppose its area $A = \pi$. We can subdivide this region into small squares with width x and height y. By Problem 3 applying a $F_{a,b}$ to x, y gives ax, by resulting in an area of $ab(xy)$. Applying these dilations to the entire unit circle gives $ab\pi$.

Problem 7

Write up a discussion of how to give coordinates (x, y, z) to a point in 3−space. In terms of these coordinates, what would the effect of dilation by r?

Solution:

Let L_1, L_2, L_3 be perpendicular lines intersecting at a point O. Let P be an arbitrary point in space. Let the distances of the segments formed by dropping perpendiculars from P to L_1, L_2, L_3 on one side be (x, y, z). On the opposite side, they are $(-x, -y, -z)$.

Let V be the volume of a region in 3-space. The effect of a dilation by r would be to multiply the volume by r^3, giving r^3V.

Problem 8

Generalize the discussion of this section to the 3-dimensional case. Specifically:
(a) Under dilation by r, how does the volume of a cube change?
(b) How does the volume of a rectangle box with sides a, b, c change? Draw a picture, say $r = \frac{1}{2}, r = 2, r = 3$, arbitrary r.
(c) How would the volume of a 3−dimensional solid change under dilation by r?
(d) The volume of the solid ball of radius 1 in 3−space is equal to $\frac{4}{3}\pi$. What is the volume of the ball of radius r in 3−space?

Solution (a):

Under dilation r the new volume $V' = r^3V$ where V is the volume of the original cube.

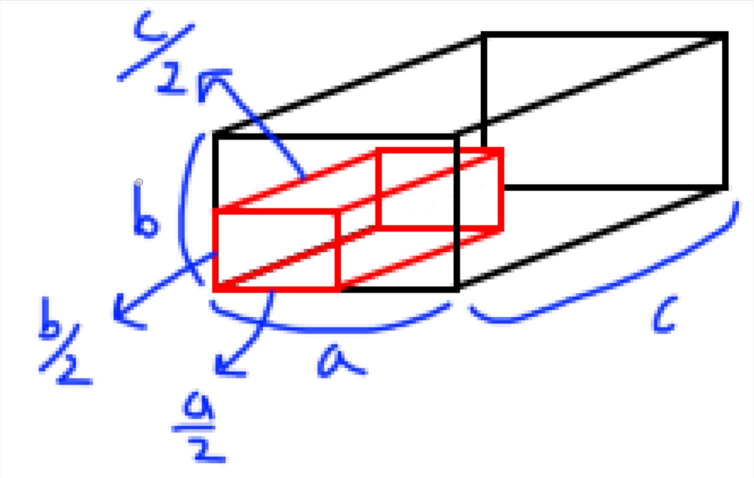
Solution (b):

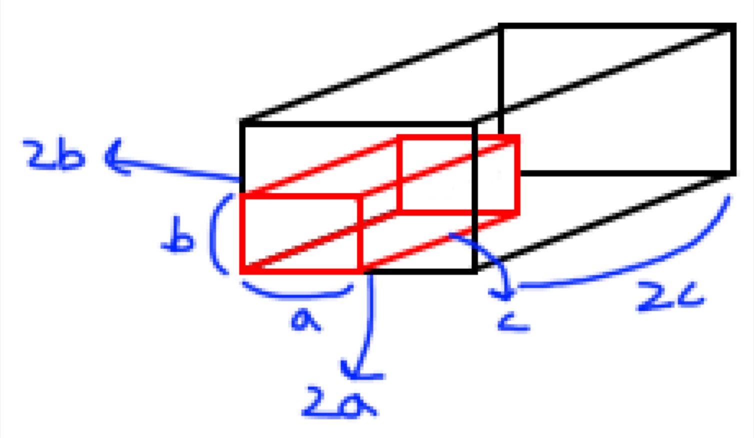
There are three cases for an arbitrary r shown below depending on if $r < 1, r = 1, r > 1$. If $r = 1$ the rectangle doesn't change that image isn't shown. The first image shows if $r < 1$ and the second shows if $r > 1$.

Solution (c):

Let V be the volume of a solid. Under dilation r the volume of the new solid is

$$r^3 \cdot V$$





Solution (d):

$$V = r^3 \cdot \frac{4}{3}\pi$$

Problem 9

Write down the equation of a sphere of radius r centered at the origin in $3-$space.

Solution:

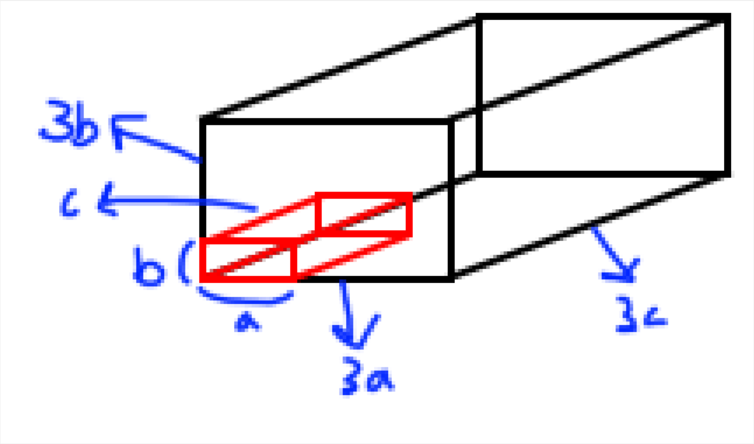
$$a^2 + b^2 + c^2 = r^2$$

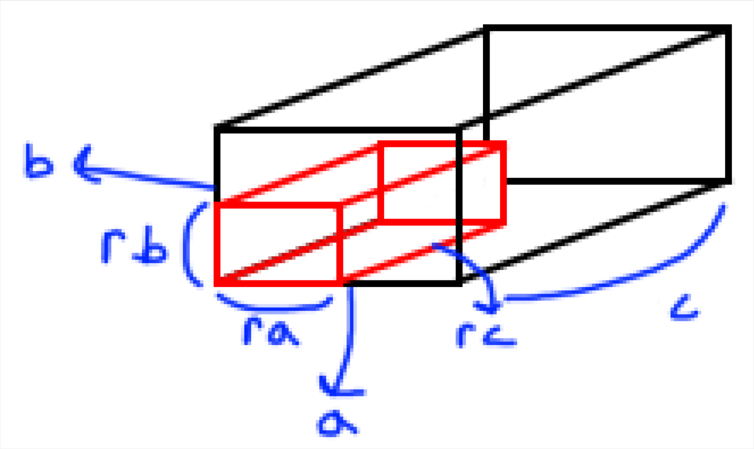
Problem 10

How would you define the volume of a rectangle solid whose sides have lengths a, b, c.

Definition 2. *Let the volume V of a rectangle R be defined as*

$$V = a \cdot b \cdot c$$





Problem 11

Let a, b, c be positive numbers. Let $\mathbb{R}^3$ be 3−space, that is, the set of all triples of numbers (x, y, z). Let

$$F_{a,b,c} : \mathbb{R}^3 \to \mathbb{R}^3$$

be the mapping

$$(x, y, z) \to (ax, by, cz)$$

Thus $F_{a,b,c}$ is a generalization to a 3−space of our mixed dilation $F_{a,b}$.
(a) What is the image of a cube whose sides have length 1 under $F_{a,b,c}$?
(b) A rectangular box S has sides of length r, s, t respectively. What are the lengths of the sides of the image $F_{a,b}(S)$? What is the volume of $F_{a,b,c}(S)$?
(c) Let S be a solid in 3−space, and let V be its volume. In terms of V, a, b, c what is the volume of the image S under $F_{a,b,c}$?

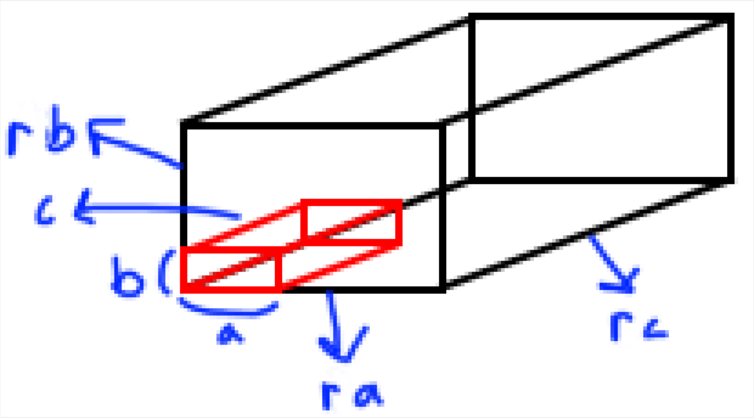
Solution (a):

The image of a cube whose sides have length 1 under $F_{a,b,c}$ is a rectangular prism of side lengths a, b, c.

Solution (b):

The lengths of the sides of the image $F_{a,b}(S)$ is ar, bs, t. The volume of $F_{a,b,c}(S)$ is $V = abc(rst)$.

Solution (c):



Let V' be the volume of the image.

$$V' = abcV$$

Problem 12

What is the volume of the solid in 3−space consisting of all points (x, y, z) satisfying the inequality

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{7}\right)^2 \leq 1$$

Solution:

$$V = 3 \cdot 2 \cdot 7 \cdot \frac{4}{3}\pi$$

Problem 14

Let a, b, c be numbers > 0. What is the volume of the solid in 3−space consisting of all points (x, y, z) satisfying the inequality

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1$$

Solution:

$$V = a \cdot b \cdot c \cdot \frac{4}{3}\pi$$

Problem 15

What about 4−space? n−space for arbitrary n?

Let V' be volume of 4−D sphere. The volume of the solid in 4−space consisting of all points (x, y, z, d) satisfying the inequality

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 + \left(\frac{p}{d}\right)^2 \leq 1$$

has volume

$$V = a \cdot b \cdot c \cdot d \cdot V'$$

Let V' be volume of n−D sphere. The volume of the solid in n−space consisting of all points $(x_1, x_2, x_3, \ldots, x_n)$ satisfying the inequality

$$\left(\frac{y_1}{x_1}\right)^2 + \left(\frac{y_2}{x_2}\right)^2 + \left(\frac{y_3}{x_3}\right)^2 + \ldots + \left(\frac{y_n}{x_n}\right)^2 \leq 1$$

has volume

$$V = x_1 \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_n \cdot V'$$

8 Coordinates and Geometry

8.1 Coordinate Systems

Problem 3

Let (x, y) be the coordinates of a point in the second quadrant. Is x positive or negative? Is y positive or negative?

Solution:

The x is negative. The y is positive.

Problem 4

Let (x, y) be the coordinates of a point in the third quadrant. Is x positive or negative. Is y positive or negative.

The x is negative. The y is negative.

8.2 Distance Between Points

Problem 11

Prove that if $d(P, Q) = 0$, then $P = Q$. Thus we have now proved two of the basic properties of distance.

Proof. Suppose $P \neq Q$. Let $(x_1, x_2) = P$ and $(y_1, y_2) = Q$. Either $x_1 \neq y_1$ or $x_2 \neq y_2$.

Suppose $x_1 \neq y_1$. It follows that $x_1 - y_1 \neq 0$ and therefore, since $(x_2 - y_2)^2 \geq 0$, $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \neq 0$. Thus $d(P, Q) \neq 0$.

Suppose $x_2 \neq y_2$. It follows similarly that $d(P, Q) \neq 0$.

Therefore, if $d(P, Q) = 0$, then $P = Q$. $\blacksquare$

Problem 12

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Let r be a positive number. Write down the formula for $d(A, B)$. Define the **dilation** rA be

$$rA = (ra_1, ra_2)$$

For instance, if $A = (-3, 5)$ and $r = 7$, then $rA = (-21, 35)$. Prove in general that

$$d(rA, rB) = r \cdot d(A, B)$$

Solution:

$$d(A, B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

Proof.

$$\begin{aligned}
d(rA, rB) &= \sqrt{(ra_1 - rb_1)^2 + (ra_2 - rb_2)^2} \\
&= \sqrt{(r^2a_1^2 - r^2a_1b_1 - r^2a_1b_1 + r^2b_1^2) + (r^2a_2^2 - r^2a_2b_2 - r^2a_2b_2 + r^2b_2^2)} \\
&= \sqrt{r^2(a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2)} \\
&= r\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \\
&= rd(A, B)
\end{aligned}$$

Therefore, $d(rA, rB) = rd(A, B)$. ∎

8.3 Equations of a Circle

Problem 19

(a) Write down the equation for a sphere of radius 1 centered at the origin in 3−space, in terms of coordinates (x, y, z).
(b) Same question for a sphere of radius 3.
(c) Same question for a sphere of radius r.

Solution (a):

$$x^2 + y^2 + z^2 = 1$$

Solution (b):

$$x^2 + y^2 + z^2 = 9$$

Solution (c):

$$x^2 + y^2 + z^2 = r^2$$

8.4 Rational Points on a Circle

Problem 2

Prove that if s, t are real numbers such that $0 \leq s < t$, then

$$\frac{1 - s^2}{1 + s^2} > \frac{1 - t^2}{1 + t^2}$$

[Hint: Prove appropriate inequalities for the numerators and denominators, before taking the quotient.] This proves that different values for $t > 0$ already give different values for x.

Proof. Suppose s, t are real numbers such that $0 \leq s < t$.

Since $s < t$ it follows that $s^2 < t^2$ and $s^2 - t^2 < 0$. Then

$$\begin{aligned}
(1 - s^2) - (1 - t^2) &= 1 - s^2 - 1 + t^2 \\
&= -s^2 + t^2 \\
&= -(s^2 - t^2)
\end{aligned}$$

Since $s^2 - t^2 < 0$, it follows that $-(s^2 - t^2) > 0$. Thus $1 - s^2 > 1 - t^2$.

Also

$$\begin{aligned}
(1 + s^2) - (1 + t^2) &= 1 + s^2 - 1 - t^2 \\
&= s^2 - t^2 < 0
\end{aligned}$$

Thus $1 + s^2 < 1 + t^2$.

Since $1 - s^2 > 1 - t^2$ and $1 + s^2 < 1 + t^2$ it follows that $(1 - s^2)(1 + t^2) > (1 - t^2)(1 + s^2)$. Therefore $\frac{1-s^2}{1+s^2} > \frac{1-t^2}{1+t^2}$ ∎

Problem 4

When t becomes very large positive, what happens to

$$\frac{1 - t^2}{1 + t^2}$$

When t becomes very large negative, what happens to

$$\frac{1 - t^2}{1 + t^2}$$

Substitute large values of t, like $10,000$ or $-10,000$, to get a feeling for what happens.

Solution:

Let $t = 10000$ then

$$\frac{1 - t^2}{1 + t^2} = \frac{1 - (10000)^2}{1 + (10000)^2} = -0.9999998$$

As t becomes very large positive $\frac{1-t^2}{1+t^2}$ approaches -1.

At $t = -10000$ it also approaches -1 since t is squared it doesn't make a difference.

Problem 5

Analyze what happens to

$$\frac{2t}{1 + t^2}$$

when $t \leq 0$ and when t becomes very large negative. Next analyze what happens when $t \geq 0$ and t becomes very large positive.

Let $t = -10000$ then

$$\frac{2t}{1 + t^2} = \frac{2(-10000)}{1 + (-10000)^2} = -0.0002$$

As t becomes very large negative $\frac{2t}{1+t^2}$ approaches 0.

Let $t = 10000$ then

$$\frac{2t}{1 + t^2} = \frac{2(10000)}{1 + (10000)^2} = 0.0002$$

As t becomes very large positive $\frac{2t}{1+t^2}$ approaches 0.

9 Operations on Points

9.1 Dilations and Reflections

Problem 2

Let A be a point, $A \neq 0$. If b, c are numbers such that $bA = cA$, prove that $b = c$.

Proof. Let $A = (a_1, a_2)$. Then $bA = (ba_1, ba_2)$ and $cA = (ca_1, ca_2)$. Suppose $bA = cA$. Then $ba_1 = ca_1$ and $ba_2 = ca_2$. Since $A \neq 0$ either $a_1 \neq 0$ or $a_2 \neq 0$. Suppose $a_1 \neq 0$ then since $ba_1 = ca_1$ dividing by a_1 shows $b = c$. Suppose $a_2 \neq 0$ then since $ba_2 = ca_2$ dividing by a_2 shows $b = c$.

Therefore $b = c$. ∎

Problem 3

Prove that reflection through O preserves distances. In other words, prove that

$$d(A, B) = d(-A, -B)$$

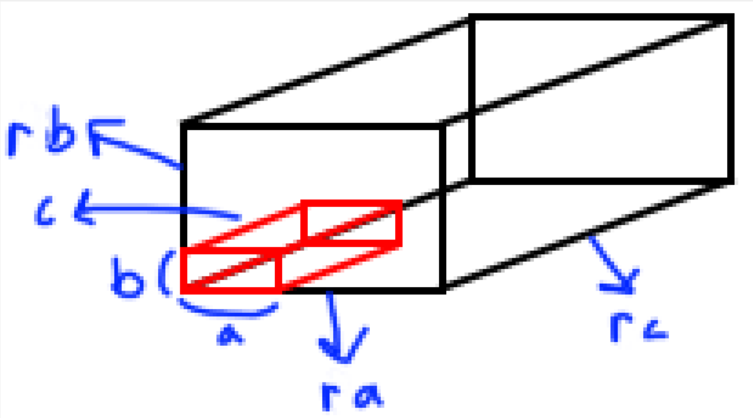
Proof. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Then

$$d(-A, -B) = \sqrt{((-a_1) - (-b_1))^2 + ((-a_2) - (-b_2))^2} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} = d(A, B).$$

∎

Problem 4

The $3-$ **dimensional case**
(a) Define the multiplication (dilation) of a point $A = (a_1, a_2, a_3)$ by a number c. Write out interpretations for this similar to those we did in the plane. Draw pictures.
(b) Define reflection A through $O = (0, 0, 0)$.
(c) State and prove the analogs of Theorems 1 and 2.



Definition 3. *Let r be a real number. Let the dilation of a point $A = (a_1, a_2, a_3)$ by r be defined as follows*

$$rA = (ra_1, ra_2, ra_3)$$

Definition 4. *Let the reflection R of A through $O = (0, 0, 0)$ dilation with $r = -1$.*

Theorem 3. *Let r be a postiive number. If A, B are points, then*

$$d(A, B) = r \cdot d(A, B)$$

Proof. Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Then $rA = (ra_1, ra_2, ra_3)$ and $rB = (rb_1, rb_2, rb_3)$. Hence

$$\begin{aligned} d(rA, rB)^2 &= (rb_1 - ra_1)^2 + (rb_2 - ra_2)^2 + (rb_3 - ra_3)^2 \\ &= (r(b_1 - a_1))^2 + (r(b_2 - a_2))^2 + (r(b_3 - a_3))^2 \\ &= r^2(b_1 - a_1)^2 + r^2(b_2 - a_2)^2 + r^2(b_3 - a_3)^2 \\ &= r^2 \cdot d(A, B)^2 \end{aligned}$$

Taking the square root proves our theorem. ∎

Theorem 4. *Let c be a number. Then*

$$d(cA, cB) = |c| \cdot d(A, B)$$

Proof. Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Then $cA = (ca_1, ca_2, ca_3)$ and $cB = (cb_1, cb_2, cb_3)$. Hence

$$\begin{aligned} d(cA, cB)^2 &= (cb_1 - ca_1)^2 + (cb_2 - ca_2)^2 + (cb_3 - ca_3)^2 \\ &= (c(b_1 - a_1))^2 + (c(b_2 - a_2))^2 + (c(b_3 - a_3))^2 \\ &= c^2(b_1 - a_1)^2 + c^2(b_2 - a_2)^2 + c^2(b_3 - a_3)^2 \\ &= c^2 \cdot d(A, B)^2 \end{aligned}$$

Taking the square root proves our theorem since $\sqrt{c^2} = |c|$. ∎

9.2 Addition, Subtraction, and Parallelogram Law

Problem 11

Let T_A be a translation by A. Prove that it is an isometry, in other words, that for any pair of points P, Q we have

$$d(P, Q) = d(T_A(P), T_A(Q))$$

Proof. Let $P = (p_1, p_2), Q = (q_1, q_2), A = (a_1, a_2)$. Now $d(P, Q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$. Then

$$\begin{aligned} d(T_A(P), T_A(Q))^2 &= ((p_1 + a_1) - (q_1 + a_1))^2 + ((p_2 + a_2) - (q_2 + a_2))^2 \\ &= (p_1 - q_1)^2 + (p_2 - q_2)^2. \end{aligned}$$

Taking the square root of both sides gives

$$d(T_A(P), T_A(Q)) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} = d(P, Q).$$

∎

Problem 12

Let $D(r, A)$ denote the disc of radius r centered at A. Show that $D(r, A)$ is the translation of A of the disc $D(r, O)$ of radius r centered at O.

Proof. Let X be a point in the disc of radius r centered at O. This means that $|X| < r$. The translation of X by A which is $X + A$, satisfies the condition $|X + A - A| < r$. Thus we see that $X + A$ is at distance $< r$ from A, and hence lies in the disc of radius r centered at A. Conversely, given a point Y in this disc, so that $|Y - A| < r$, let $X = Y - A$. Then $Y = X + A$ is the translation of X by A, and $|X| < r$. Therefore every point in the disc of radius r, centered at A, is the image T_A of a point in the disc of radius r centered at O. This proves our theorem ∎

Problem 13

Let $S(r, P)$ denote the circle of radius r centered at P.
(a) Show that the reflection of this circle through O is a again a circle. What is the center of the reflected circle.
(b) Show that the reflection of the disc $D(r, P)$ through O is a disc. What is the center of this reflected disc?

Proof. Let X be a point on the circle of radius r centered at P. This means that $|X - P| = r$. The reflection of X through the origin, which is $-X$, satisfies $|-X - (-P)| = |-(X - P)| = |X - P| = r$. Thus we see that $-X$ is at distance r from $-P$, and hence lies in the circle of radius r centered at $-P$. Conversely, given a point Y in this circle, so that $|Y - (-P)| = r$, let $X = -Y$. Then $Y = -X$ is the reflection of X through the origin, and $|X - P| = |-Y - P| = |Y - (-P)| = r$. Therefore every point on the circle of radius r, centered at $-P$, is the reflection of a point in the circle of radius r centered at P. This proves our theorem. ∎

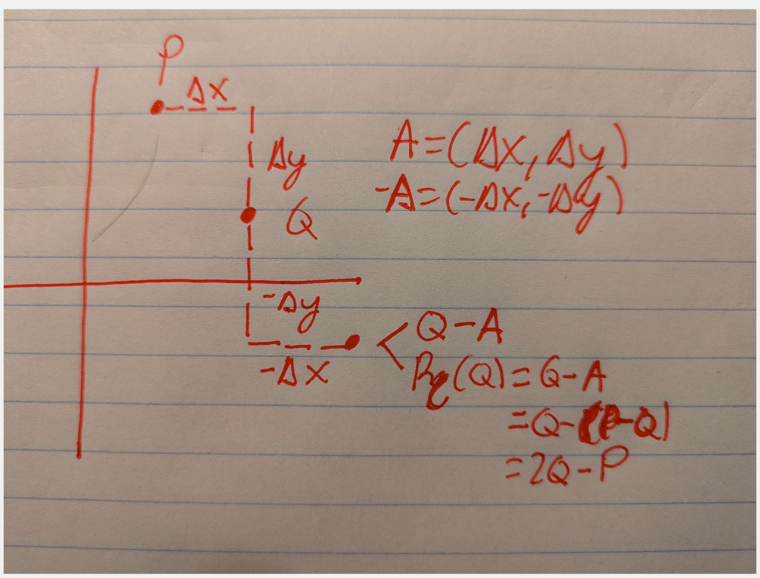
Solution: Let $(x, y) = P$. The image of P under a reflection through the origin is $(-x, -y) = -P$.

Proof. Let X be a point in the disc of radius r centered at P. This means that $|X - P| < r$. The reflection of X through the origin, which is $-X$, satisfies $|-X - (-P)| = |-(X - P)| = |X - P| < r$. Thus we see that $-X$ is at distance $< r$ from $-P$, and hence lies in the disc of radius r centered at $-P$. Conversely, given a point Y in this disc, so that $|Y - (-P)| < r$, let $X = -Y$. Then $Y = -X$ is the reflection of X through the origin, and $|X - P| = |-Y - P| = |Y - (-P)| < r$. Therefore every point in the disc of radius r, centered at $-P$, is the reflection of a point in the disc of radius r centered at P. This proves our theorem. ∎

Solution: Same as the center of a circle reflected through the origin.

Problem 14

Let P, Q be points. Write $P = Q + A$, where $A = P - Q$. Define the **reflection of** P **through** Q to be the point $Q - A$. If R_Q denotes reflection through Q, then we have $R_Q(P) = 2Q - P$. (Why?) Draw the picture, showing P, Q, A and $Q - A$ to convince yourself that this definition corresponds to our geometric intuition.



Solution: From the figure we see $Q - A$ is the reflection of P through Q. Since $A = P - Q$, $R_Q(P) = Q - (P - Q) = Q - A$.

Problem 15

(a) Prove that reflection through a point Q can be expressed in terms of reflection through O, followed by a translation.
(b) Let T_A be translation by A, and R_O reflection with respect to the origin. Prove that the composite $T_A \circ R_O$ is equal to R_Q for some point Q. Which one?

Proof. Let P be an arbitrary point. Let $P = (x, y)$. Reflecting through O gives $R_O(P) = -P = (-x, -y)$. Then translating by $2Q$ gives $-P + 2Q = 2Q - P = R_Q(P)$. ∎

Proof. Let P be an arbitrary point. Then

$$(T_A \circ R_O)(P) = T_A(R_O(P)) = T_A(2O - P) = 2O - P + A = A - P$$

Also $R_Q(P) = 2Q - P$. Then

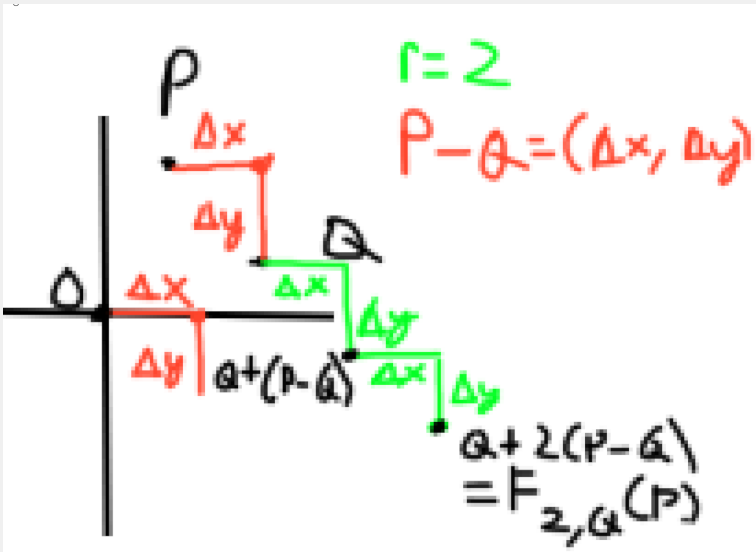
$$R_Q(P) = (T_A \circ R_O)(P) \iff 2Q - P = A - P \iff 2Q = A \iff Q = A/2$$

∎

Problem 16

(a) Let r be a positive number. Give an analytic definition of **dilation by** r **with respect to a point** Q, and denote this dilation by $F_{r,Q}$. To give this definition, look at Exercise 14. You may also want to look at the discussion about line segments in 4. If P is a point, draw the picture with $O, P, Q, P-Q$, and $F_{r,Q}(P)$.
(b) From your definition, it should be clear that $F_{r,Q}$ can be obtained as a composite of dilation with respect to O, and a translation. Translation by what point?

Definition 5. *Define the* ***dilation by*** r ***with respect to a point*** Q ***of*** P *be the point* $Q + r(P - Q)$.



Solution:

$$F_{r,Q}(P) = Q + r(P - Q) = rP - rQ + Q = rP - (r-1)Q = F_{r,O}(P) - (r-1)Q$$

So a translation by $-(r-1)Q$.

Problem 17

Let $S(r, A)$ be the circle of radius r and center A. Show that the reflection of this circle through a point Q is a circle. What is the center of this reflected circle? What is its radius? Draw a picture.

Proof. Let X be a point on the circle of radius r centered at P. This means that $|X - P| = r$. The reflection of P through a point Q is $2Q - P$, and the reflection of X through Q is $2Q - X$. Then

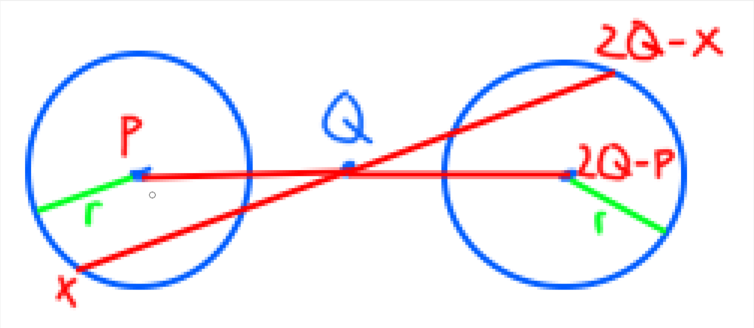
$$|(2Q - X) - (2Q - P)| = |-(X - P)| = |X - P| = r.$$

Thus $2Q - X$ is at distance r from $2Q - P$, and hence lies on the circle of radius r centered at $2Q - P$. Conversely, given a point Y on the circle of radius r centered at $2Q - P$, let $X = 2Q - Y$. Then

$$|X - P| = |(2Q - Y) - P| = |(2Q - P) - Y| = |Y - (2Q - P)| = r$$

So X is a point on the original circle centered at P, and Y is its reflection through Q. Therefore, the reflection of the circle through Q is again a circle with center $2Q - P$ and radius r. ∎

Solution: The center of the reflected circle is $2Q - P$, and its radius is r.



Problem 18

The inverse of the translation T_A is also a translation. By what? Prove your assertion.

Proof. The inverse of T_A is $T_{(-A)}$. Let P be an arbitrary point. Then $(T_A \circ T_{(-A)})(P) = T_A(P - A) = (P - A) + A = P$. Also $(T_{(-A)} \circ T_A)(P) = T_{(-A)}(P + A) = (P + A) - A = P$. Thus $T_A^{-1} = T_{(-A)}$. ∎

Problem 19

Let F_r be dilation by a postive number r, with respect to O, and let T_A be a translation by A.
(a) Show that F_r^{-1} is also a dilation. By what number?
(c) Show that $F_r \circ T_A \circ F_r^{-1}$ is a translation.

Proof. Consider $F_{(-r)}$. Let P be an arbitrary point. Then $(F_r \circ F_{1/r})(P) = F_r(P/r) = r \cdot P/r = P$. Also $(F_{1/r} \circ F_r)(P) = F_{1/r}(rP) = 1/r \cdot rP = P$. Thus $F_r^{-1} = F_{1/r}$. ∎

Proof. Let P be an arbitrary point. Then

$$(F_r \circ T_A \circ F_r^{-1})(P) = (F_r \circ T_A)(P/r) = F_r(P/r + A) = r(P/r + A) = P + Ar = T_{rA}(P)$$

∎

Problem 20

Show that the composite of two translations is a translation. $T_A \circ T_B = T_C$, how would you express C in terms of A and B?

Proof. Let P be an arbitrary point. Then

$$(T_A \circ T_B)(P) = T_A(P + B) = P + B + A = T_{B+A}(P) = T_C(P)$$

So $C = B + A$. ∎

Problem 21

Let R be a reflection through the origin.
(a) Show that R^{-1} exists.
(b) Show that $R \circ T_A \circ R^{-1}$ is a translation. By what?

Proof. Let P be an arbitrary point. Then $(R \circ R)(P) = R(-P) = -(-P) = P$. Thus $R^{-1} = R$. ∎

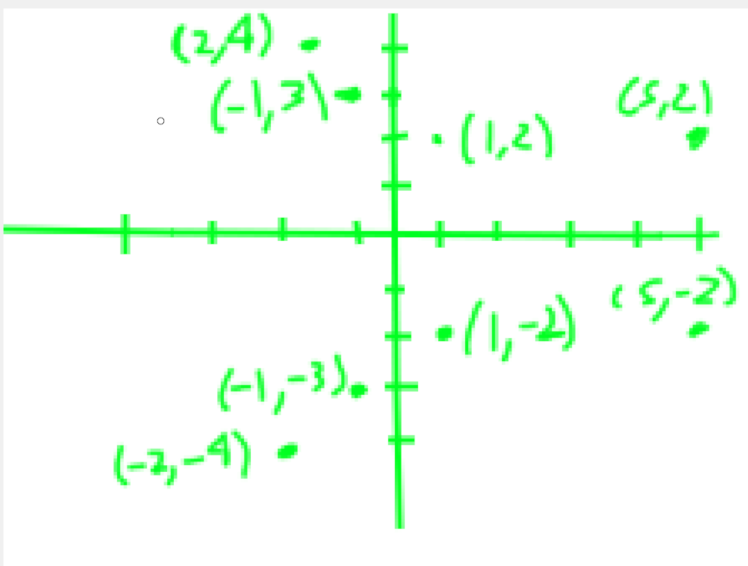
Proof. Let P be an arbitrary point. Then

$$(R \circ T_A \circ R^{-1})(P) = (R \circ T_A \circ R)(P) = (R \circ T_A)(-P) = R(-P + A) = P - A = T_{(-A)}(P)$$

∎

Problem 22

Let $A = (a_1, a_2)$ be a point. Define its **reflection through the x-axis** to be the point $(a_1, -a_2)$. Draw A and its reflection through the x-axis in the following cases.
(a) $A = (1, 2)$
(b) $A = (-1, 3)$
(c) $A = (-2, -4)$
(d) $A = (5, -2)$



Problem 23

Prove that reflection through the x-axis is an isometry.

Proof. Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be arbitrary points. Then

$$d(P, Q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} = d_{PQ}.$$

Reflecting both points through the x-axis gives

$$P' = (p_1, -p_2) \quad \text{and} \quad Q' = (q_1, -q_2).$$

Then
$$d(P',Q') = \sqrt{(p_1-q_1)^2+((-p_2)-(-q_2))^2} = \sqrt{(p_1-q_1)^2+(p_2-q_2)^2} = d_{PQ}.$$

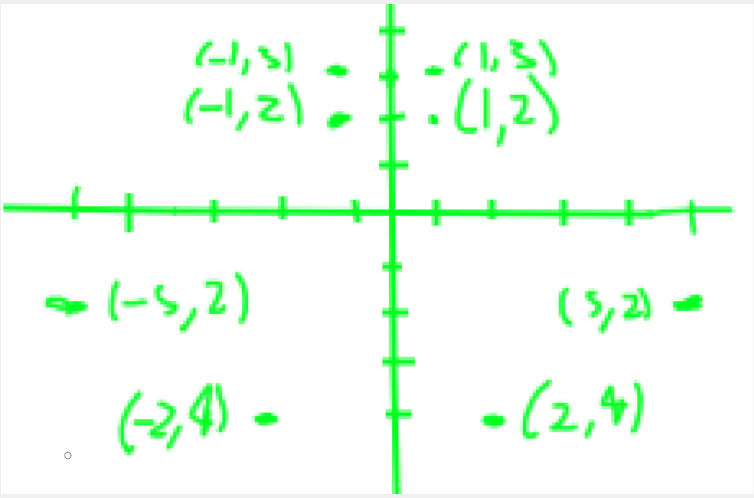
∎

Problem 24

Define the reflection through the y-axis in a similar way, and prove that it is an isometry. Draw the points of Exercise 22, and their reflections through the y-axis.

Proof. Let $P=(p_1,p_2)$ and $Q=(q_1,q_2)$ be arbitrary points. Then
$$d(P,Q) = \sqrt{(p_1-q_1)^2+(p_2-q_2)^2} = d_{PQ}.$$
Reflecting both points through the y-axis gives
$$P' = (-p_1,p_2) \quad \text{and} \quad Q' = (-q_1,q_2).$$
Then
$$d(P',Q') = \sqrt{((-p_1)-(-q_1))^2+(p_2-q_2)^2} = \sqrt{(p_1-q_1)^2+(p_2-q_2)^2} = d_{PQ}.$$

∎



Problem 25

Recall that a **fixed point** of a mapping F is a point P such that $F(P)=P$. Using the coordinate definition, determine the fixed points of
(a) translation
(b) reflection through O
(c) reflection through an arbitrary point P
(d) reflection through the x-axis and through the y-axis

Proof. Let P be an arbitrary point. If T_0, then all points are fixed points. Consider $T_0(P)=P+0=P$. Suppose T_A with $A\neq 0$. Then $T_A(P)=P+A\neq P$. ∎

Proof. The point O is a fixed point. Consider $F_O(O)=2O-O=O$. ∎

Proof. The point P is a fixed point. Consider $F_P(P) = 2P - P = P$. ∎

Proof. All points on the x-axis are fixed points with respect to x-axis reflection. Let $X = (x_1, x_2)$ be an arbitrary point on the x-axis. Then $X = (x_1, 0) = (x_1, -x_2)$.

All points on the y-axis are fixed points with respect to y-axis reflection. Let $Y = (y_1, y_2)$ be an arbitrary point on the y-axis. Then $Y = (0, y_2) = (-y_1, y_2)$. ∎

Problem 26

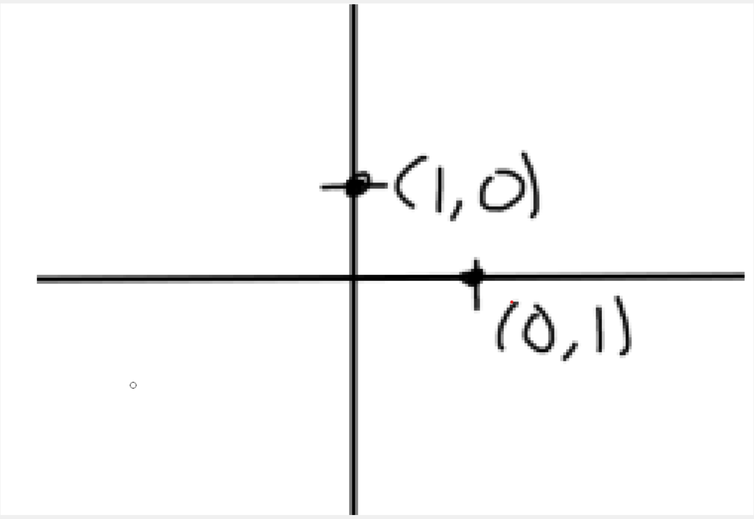
(a) Let

$$E_1 = (1, 0) \text{ and } E_2 = (0, 1)$$

We call E_1 and E_2 the **basic units points** of the plane. Plot these points. If $A = (a_1, a_2)$, prove that

$$A = a_1 E_1 + a_2 E_2$$

(b) If c is a number, what are the coordinates of cE_1, cE_2?



Proof. Suppose $A = (a_1, a_2)$. Consider

$$a_1 E_1 + a_2 E_2 = (a_1 \cdot 1, a_1 \cdot 0) + (a_2 \cdot 0, a_2 \cdot 1) = (a_1, 0) + (0, a_2) = (a_1, a_2) = A$$

∎

Solution (b):

$$cE_1 = (c \cdot 1, c \cdot 0) = (c, 0) \text{ and } cE_2 = (c \cdot 0, c \cdot 1) = (0, c)$$

Problem 29

Given a number $r > 0$ and a point A, we can define the corners of a square, having sides of length r parallel to the axes, and A as its lower left-hand corner, to be the points

$$A, A + rE_1, A + rE_2, A + rE_1 + rE_2$$

Let s be a postitive number. Show that if these four points are dilated by multiplication with s, they again form the corners of a square. What are the corners of this dilated square.

Proof. The dilation by multiplication with s sends each point X to sX. Applying this to the four corners of the square gives

$$A \mapsto sA, \qquad A + rE_1 \mapsto sA + srE_1, \qquad A + rE_2 \mapsto sA + srE_2, \qquad A + rE_1 + rE_2 \mapsto sA + srE_1 + srE_2.$$

These four points again form a square, since the sides remain parallel to the coordinate axes and each side has length sr. Thus the corners of the dilated square are

$$sA,\ sA + srE_1,\ sA + srE_2,\ sA + srE_1 + srE_2$$

∎

Problem 30

Let the notation be as in Exercise 29. What is the area of the dilated square? How does it compare with the area of the original square?

Solution: The original square has side length r and area r^2. The dilated square has side length sr and area s^2r^2. Thus the area of the square is multiplied by s^2 under the dilation.

Problem 31

Let A be a point and r, s positive numbers. How would you define the corners of a rectangle whose sides are parallel to the axes, with A as the lower left-hand corner, and such that the vertical side has length r and the horizontal side has length s.

Solution: A rectangle with lower left-hand corner A, vertical side of length r, horizontal side of length s, and sides parallel to the coordinate axes has corners

$$A, \quad A + sE_1, \quad A + rE_2, \quad A + sE_1 + rE_2.$$

Problem 32

Let t be a positive number. What is the effect of dilation by t on the sides and area of the rectangle in Exercise 31.

Solution: Dilation by t multiplies each side by t. Thus the horizontal side of length s becomes ts, and the vertical side of length r becomes tr. The area, originally rs, becomes

$$(tr)(ts) = t^2rs.$$

Thus the area is multiplied by t^2.

10 Segments, Rays, and Lines

10.1 Rays

Problem 1

Let P, Q be the indicated points. Give the coordinates of the point
(a) halfway
(b) one-third of the way
(c) two-thirds of the way
between P and Q.
$P = (1, 5)$, $Q = (3, -1)$

Solution: We first compute the segment (as a vector based at O) between P and Q:

$$Q - P = (3 - 1,\ -1 - 5) = (2,\ -6).$$

Then we scale this vector by $1/2$, $1/3$, and $2/3$:

$$\frac{1}{2}(2,-6) = (1,-3), \qquad \frac{1}{3}(2,-6) = \left(\frac{2}{3},-2\right), \qquad \frac{2}{3}(2,-6) = \left(\frac{4}{3},-4\right).$$

Finally, we add each of these to P:

$$P + (1,-3) = (2,2), \qquad P + \left(\tfrac{2}{3},-2\right) = \left(\tfrac{5}{3},3\right), \qquad P + \left(\tfrac{4}{3},-4\right) = \left(\tfrac{7}{3},1\right).$$

Problem 5

Prove that the image of a line segment $\overline{PQ}$ under translation T_A is also a line segment. What are the end points of this image.

Proof. A line segment from P to Q can be written as

$$\overline{PQ} = P + t(Q - P), \quad 0 \le t \le 1.$$

Then

$$T_A(P + t(Q - P)) = (P + t(Q - P)) + A = (P + A) + t((Q + A) - (P + A)).$$

This shows that the image is the line segment $\overline{(P + A)(Q + A)}$ with endpoints $P + A$ and $Q + A$. ∎

Problem 6

Let P, Q, M be the indicated points. In Exercise 6, find the point N such that $\overrightarrow{PQ}$ has the same direction as $\overrightarrow{MN}$ and such that the length of $\overrightarrow{MN}$ is
(a) 3 times the length of $\overrightarrow{PQ}$
(b) one-third the length of $\overrightarrow{PQ}$
$P = (1,4)$, $Q = (1,-5)$, $M = (-2,3)$

Solution (a): We require $N - M = 3(P - Q)$. So $N = 3(P - Q) + M \iff N = 3((1,4) - (1,-5)) + (-2,3) \iff N = 3(0,9) + (-2,3) \iff N = (0,27) + (-2,3) \iff N = (-2,30)$.

Solution (b): We require $N - M = (1/3)(P - Q)$. So $N = (1/3)(P - Q) + M \iff N = (1/3)((1,4) - (1,-5)) + (-2,3) \iff N = (1/3)(0,9) + (-2,3) \iff N = (0,3) + (-2,3) \iff N = (-2,6)$.

Problem 10

Let F be
(a) translation T_A
(b) reflection through O
(c) reflection through the x-axis
(d) reflection through the y-axis
(e) dilation by a number $r > 0$
In each one of these cases, prove that the image under F of (i) is a segment, (ii) a ray, is again (i) a segment, (ii) a ray, respectively. Thus you really have 10 cases to consider $10 = 5 \times 2$, but they are all easy.

Proof. We first show the image of a segment under $F = T_A$ is a segment. Let $\overline{PQ}$ be a segment. Consider

$$F(\overline{PQ}) = T_A(\overline{PQ}) = T_A(P + t(Q - P))$$

Where $0 \leq t \leq 1$. Then

$$P + t(Q - P) + A = (P + A) + t((Q + A) - (P + A)) = \overline{P + A, Q + A}$$

We now show the image of a ray under $F = T_A$ is a ray. Let $\overrightarrow{PQ}$ be a ray. Consider

$$F(\overrightarrow{PQ}) = T_A(\overrightarrow{PQ}) = T_A(P + t(Q - P))$$

Where $t \geq 0$. Then

$$P + t(Q - P) + A = (P + A) + t((Q + A) - (P + A)) = \overrightarrow{P + A, Q + A}$$

∎

Proof. We first show the image of a segment under reflection through O is a segment. Let $\overline{PQ}$ be a segment. Consider

$$F(\overline{PQ}) = -\overline{PQ} = -(P + t(Q - P))$$

where $0 \leq t \leq 1$. Then

$$-P - t(Q - P) = (-P) + t((-Q) - (-P)) = \overline{-P, -Q}.$$

We now show the image of a ray under reflection through O is a ray. Let $\overrightarrow{PQ}$ be a ray. Consider

$$F(\overrightarrow{PQ}) = -(P + t(Q - P))$$

where $t \geq 0$. Then

$$-P - t(Q - P) = (-P) + t((-Q) - (-P)) = \overrightarrow{-P, -Q}.$$

∎

Proof. We first show the image of a segment under reflection through the x-axis is a segment. Let $\overline{PQ}$ be a segment with $P = (p_x, p_y)$, $Q = (q_x, q_y)$. Consider

$$F(\overline{PQ}) = (p_x, -p_y) + t((q_x, -q_y) - (p_x, -p_y)), \quad 0 \leq t \leq 1$$

which equals

$$\overline{(p_x, -p_y), (q_x, -q_y)}.$$

We now show the image of a ray under reflection through the x-axis is a ray. Let $\overrightarrow{PQ}$ be a ray. Then

$$F(\overrightarrow{PQ}) = (p_x, -p_y) + t((q_x, -q_y) - (p_x, -p_y)), \quad t \geq 0$$

which equals

$$\overrightarrow{(p_x, -p_y), (q_x, -q_y)}.$$

∎

Proof. We first show the image of a segment under reflection through the y-axis is a segment. Let $\overline{PQ}$ be a segment with $P = (p_x, p_y)$, $Q = (q_x, q_y)$. Consider

$$F(\overline{PQ}) = (-p_x, p_y) + t((-q_x, q_y) - (-p_x, p_y)), \quad 0 \leq t \leq 1$$

which equals

$$\overline{(-p_x, p_y), (-q_x, q_y)}.$$

We now show the image of a ray under reflection through the y-axis is a ray. Let $\overrightarrow{PQ}$ be a ray. Then

$$F(\overrightarrow{PQ}) = (-p_x, p_y) + t((-q_x, q_y) - (-p_x, p_y)), \quad t \geq 0$$

which equals

$$\overrightarrow{(-p_x, p_y), (-q_x, q_y)}.$$

∎

Proof. We first show the image of a segment under dilation by a number $r > 0$ is a segment. Let $\overline{PQ}$ be a segment. Consider

$$F(\overline{PQ}) = r(P + t(Q - P)), \quad 0 \leq t \leq 1.$$

Then

$$rP + t\,r(Q - P) = (rP) + t((rQ) - (rP)) = \overline{rP, rQ}.$$

We now show the image of a ray under dilation by a number $r > 0$ is a ray. Let $\overrightarrow{PQ}$ be a ray. Consider

$$F(\overrightarrow{PQ}) = r(P + t(Q - P)), \quad t \geq 0.$$

Then

$$rP + t\,r(Q - P) = (rP) + t((rQ) - (rP)) = \overrightarrow{rP, rQ}.$$

∎

Problem 11

After you have read the definition of a straight line in the next section, prove that the image under F of a straight line is again a straight line. [Here F is any one of the mappings of Exercise 10.]

Proof. We show the image of a line under $F = T_A$ is a line. Let $\overline{PQ}$ be a line. Consider

$$F(\overline{PQ}) = T_A(\overline{PQ}) = T_A(P + t(Q - P))$$

Where $t \in \mathbb{R}$. Then

$$P + t(Q - P) + A = (P + A) + t((Q + A) - (P + A)) = \overline{P + A, Q + A}$$

∎

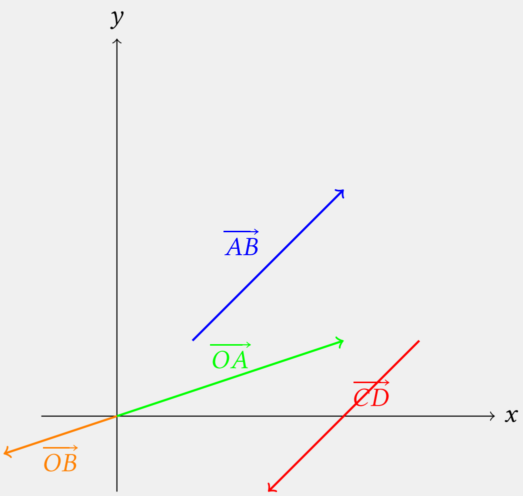
Problem 12

Give a definition for two located vectors to have **opposite direction**. Similary if $A \neq O$ and $B \neq O$, give a definition for A and B to have opposite direction. Draw the corresponding pictures.

Definition 6. *Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two vectors. We say that $\overrightarrow{AB}$ has **opposite direction** to $\overrightarrow{CD}$ if and only if there exists a positive scalar $r > 0$ such that*

$$\overrightarrow{AB} + r\,\overrightarrow{CD} = \vec{0}.$$

Definition 7. *Let A and B be points such that $A \neq O$ and $B \neq O$. We say that A and B have **opposite direction** if and only if $\overrightarrow{OA}$ has opposite direction to $\overrightarrow{OB}$ in the sense above.*



10.2 Lines

Problem 1

For Exercise 1: (a) write down the parametric representations of the lines passing through the indicated points P and Q, (b) find the point of intersection of the line and the x-axis, (c) find the point of intersection of the line and the y-axis.

$$P = (1, -1), Q = (3, 5)$$

Solution (a):

$$P + t(Q - P) = (1, -1) + t((3, 5) - (1, -1)) = (1, -1) + t(2, 6)$$

Thus

$$x = 1 + 2t, \quad y = -1 + 6t$$

Solution (b): We require $y = 0$, thus $0 = -1 + 6t \Rightarrow t = \frac{1}{6}$. Then plugging in we see $(x, y) = (1 + 2(\frac{1}{6}), 0) = (\frac{4}{3}, 0)$

Solution (c): We require $x = 0$, thus $0 = 1 + 2t \Rightarrow t = -\frac{1}{2}$. Then plugging in we see $(x, y) = (0, -1 + 6(-\frac{1}{2})) = (0, -4)$

Problem 12

Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Assume $A \neq O$ and $B \neq O$. Prove that A is parallel to B if and only if

$$a_1 b_2 - a_2 b_1 = 0$$

Proof. ($\longrightarrow$) Suppose A is parallel to B. Thus $tA = c(tB)$ thus $t(A - cB) = 0 \implies A - cB = 0$. Thus $(a_1, a_2) = c(b_1, b_2)$ and it follows that $a_1 = cb_1$ and $a_2 = cb_2$. Then $a_1 b_2 - a_2 b_1 = cb_1 b_2 - cb_2 b_1 = 0$.

($\longleftarrow$) Since $B \neq O$ either $b_1 \neq 0$ or $b_2 \neq 0$.

(**Case 1:** $b_1 \neq 0$) Let $c = \frac{a_1}{b_1}$. Then $A = c(B) \iff (a_1, a_2) = \frac{a_1}{b_1}(b_1, b_2)$ Thus

$$a_1 = \frac{a_1}{b_1} b_1 = a_1 \text{ and } a_2 = \frac{a_1}{b_1} b_2$$

Now $a_2 = \frac{a_1}{b_1} b_2 \iff b_1 a_2 = a_1 b_2$ which holds since $a_1 b_2 - a_2 b_1 = 0$.

(**Case 2:** $b_2 \neq 0$) Let $c = \frac{a_2}{b_2}$. Then $A = cB \iff (a_1, a_2) = \frac{a_2}{b_2}(b_1, b_2)$. Thus

$$a_2 = \frac{a_2}{b_2} b_2 = a_2 \text{ and } a_1 = \frac{a_2}{b_2} b_1$$

Now $a_1 = \frac{a_2}{b_2} b_1 \iff b_2 a_1 = a_2 b_1$, which holds since $a_1 b_2 - a_2 b_1 = 0$. ∎

Problem 13

Prove: If two lines are not parallel, then they have exactly one point in common. [Hint: Let the two lines by represented parametrically by

$$\{P + tA\}_{t \in R} = \{(p_1, p_2) + t(a_1, a_2)_{t \in R}\}$$

$$\{Q + sB\}_{s \in R} = \{(q_1, q_2) + s(b_1, b_2)_{s \in R}\}$$

Write down the general system of two equations for s and t and show that it can be solved.]

Proof. Suppose the two lines are not parallel. There are two cases. Either $a_1 = 0$ or $a_1 \neq 0$.

(**Case 1:** $a_1 = 0$) Since the lines are not parallel, we must have $b_1 \neq 0$. Then the lines intersect if and only if

$$① \; p_1 = q_1 + sb_1 \quad \text{and} \quad ② \; p_2 + ta_2 = q_2 + sb_2$$

Solving for s in ① shows $s = \frac{p_1 - q_1}{b_1}$. Substituting into ② shows

$$p_2 + ta_2 = q_2 + \left(\frac{p_1 - q_1}{b_1}\right) b_2$$

Solving for t shows

$$t = \frac{q_2 + \frac{b_2(p_1 - q_1)}{b_1} - p_2}{a_2}$$

Since $A \neq O$, $a_2 \neq 0$, therefore t is defined.

(**Case 2:** $a_1 \neq 0$) The lines intersect if and only if

$$① \; p_1 + ta_1 = q_1 + sb_1 \quad \text{and} \quad ② \; p_2 + ta_2 = q_2 + sb_2$$

Solving for t in ① shows $t = \frac{q_1 + sb_1 - p_1}{a_1}$. Substituting into ② shows

$$p_2 + \left(\frac{q_1 + sb_1 - p_1}{a_1}\right) a_2 = q_2 + sb_2$$

Multiplying through by a_1 shows

$$a_1 p_2 + (q_1 + sb_1 - p_1) a_2 = a_1 q_2 + a_1 s b_2$$

Solving for s shows

$$s = \frac{a_1(q_2 - p_2) - a_2(q_1 - p_1)}{a_2 b_1 - a_1 b_2}$$

Since the lines are not parallel, by Problem 13, $a_2 b_1 - a_1 b_2 \neq 0$. Thus s is defined. Substituting into ① shows

$$p_1 + ta_1 = q_1 + \left(\frac{a_1(q_2 - p_2) - a_2(q_1 - p_1)}{a_2 b_1 - a_1 b_2}\right) b_1$$

Then solving for t shows

$$t = \frac{q_1 + \left(\frac{a_1(q_2-p_2)-a_2(q_1-p_1)}{a_2b_1-a_1b_2}\right)b_1 - p_1}{a_1}$$

Since $a_1 \neq 0$, t is defined.

Therefore, in either case, the lines intersect at a unique point, as required. ∎

Problem 18

(Slightly harder.) Let S be the circle of radius $r > 0$ centered at the origin. Let $P = (p, q)$ be a point such that

$$p^2 + q^2 = r$$

In other words, P is a point in the disc of radius r centered at O. Show that any line passing through P must intersect the circle, and find the points of intersection. [Hint: Write the line in the form

$$P + tA$$

where $A = (a, b)$, substitute in the equation of the circle, and find the coordinates of the points of intersection in terms of p, q, a, b. Show that the quantity you get under the square root sign is ≥ 0.]

Proof. Let A be an arbitrary point such that $A \neq P$. Consider the line $P + tA$ for $t \in \mathbb{R}$ and let $A = (a, b)$. Then the parametric form is

$$x = p + ta, \quad y = q + tb.$$

These points must satisfy the circle equation $x^2 + y^2 = r^2$. Substituting shows

$$\begin{aligned} & (p+ta)^2 + (q+tb)^2 = r^2 \\ \iff & p^2 + 2pta + t^2a^2 + q^2 + 2qtb + t^2b^2 = r^2 \\ \iff & r^2 + 2pta + t^2a^2 + 2qtb + t^2b^2 = r^2 && \text{since } p^2 + q^2 = r^2 \\ \iff & 2t(pa + qb) + t^2(a^2 + b^2) = 0 \end{aligned}$$

Solving the quadratic equation with $a = a^2 + b^2, b = 2(pa + qb), c = 0$ shows

$$t = \frac{-2(pa+qb) \pm \sqrt{(2(pa+qb))^2}}{2(a^2+b^2)}.$$

Expanding $(2(pa + qb))^2$ shows

$$(2(pa+qb))^2 = 4(pa+qb)^2$$

Clearly $(pa + qb)^2 \geq 0$. ∎

10.3 Ordinary Equation for a Line

Problem 1

Find the ordinary equation of the line $\{P + tA\}_{t\in\mathbb{R}}$ of the following.

$$P = (3, 1), A = (7, 2)$$

Solution: The parametric equations for x, y are

$$x = 3 + 7t \text{ and } y = 1 + 2t$$

Multiplying x by -2 and y by 7 shows

$$-2x + 7y = -2(3 + 7t) + 7(1 + 2t) = -6 - 14t + 7 + 14t = 1$$

Thus $-2x + 7y = 1$.

Problem 7

Find the ordinary equation of the line $\{P + tA\}_{t\in\mathbb{R}}$ of the following.

$$P = (1, 1), A = (1, 1)$$

The parametric equations for x, y are

$$x = 1 + t \text{ and } y = 1 + t$$

Multiplying y by -1 shows

$$x + -y = 1 + t - (1 + t) = 0$$

Thus $x - y = 0$.

11 Trigonometry

11.1 Radian Measure

Problem 2

Give the following values in radians, as a fractional multiple of π.
(a) $20°$
(b) $40°$
(c) $140°$
(d) $310°$

Solution (a):

$$\frac{20}{360} \cdot 2\pi$$

Solution (b):

$$\frac{40}{360} \cdot 2\pi$$

Solution (c):

$$\frac{140}{360} \cdot 2\pi$$

Solution (d):

$$\frac{310}{360} \cdot 2\pi$$

11.2 Sine and Cosine

Problem 3

When the angle A has its defining ray in the fourth quadrant, determine whether the sine is positive or negative. Repeat for the cosine.

Solution: Sine is negative and cosine is positive.

Problem 4

A boat B starts from a point P and moves along a straight river. An observer O stands at a distance of 600 ft from P, on the perpendicular to the river passing through P. Find the distance from the boat to the observer when the angle θ formed by B, O, and P has
(a) $\pi/6$ radians
(b) $\pi/4$ radians
(c) $\pi/3$ radians

Solution (a): $\cos(\pi/6) = \frac{600}{c} \iff c = \frac{600}{\cos(\pi/6)}$ ft

Solution (b): $\cos(\pi/4) = \frac{600}{c} \iff c = \frac{600}{\cos(\pi/4)}$ ft

Solution (c): $\cos(\pi/3) = \frac{600}{c} \iff c = \frac{600}{\cos(\pi/3)}$ ft

Problem 5

A balloon B starts from a point P on earth and goes straight up. A man M is at a distance of $\frac{1}{2}$ mi from P. After 2 min, the angle θ formed by P, M, B has a cosine equal to
(a) 0.3
(b) 0.4
(c) 0.2
Find the distance between the man and the baloon at that time.

Solution (a): $\cos(\theta) = \frac{1/2}{c} = 0.3 \iff c = \frac{1/2}{0.3}$ mi

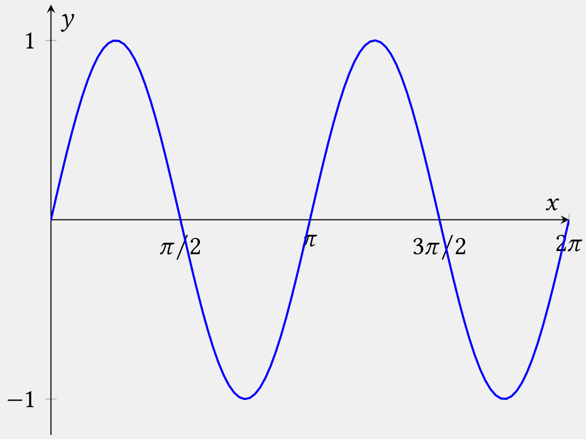
Solution (b): $\cos(\theta) = \frac{1/2}{c} = 0.4 \iff c = \frac{1/2}{0.4}$ mi

Solution (c): $\cos(\theta) = \frac{1/2}{c} = 0.2 \iff c = \frac{1/2}{0.2}$ mi

11.3 The Graphs

Problem 2

Sketch the graph of $\sin(2x)$, i.e. all points whose coordinates are $(x, \sin(2x))$.



11.4 The Tangent

Problem 2

Discuss how the tangent is increasing or decreasing for $-\pi/2 < x \leq 0$. Also for $\pi/2 < x < 3\pi/2$ and $-3\pi/2 < x < -\pi/2$.

Solution: Consider $-\pi/2 < x \leq 0$. As x goes from $-\pi/2$ to 0, $\sin(x)$ increases and $\cos(x)$ increases.

x	$\tan(x)$
$-\pi/4$	-1
$-\pi/6$	-0.577

Thus the ratio $\tan(x) = \frac{\sin(x)}{\cos(x)}$ increases.

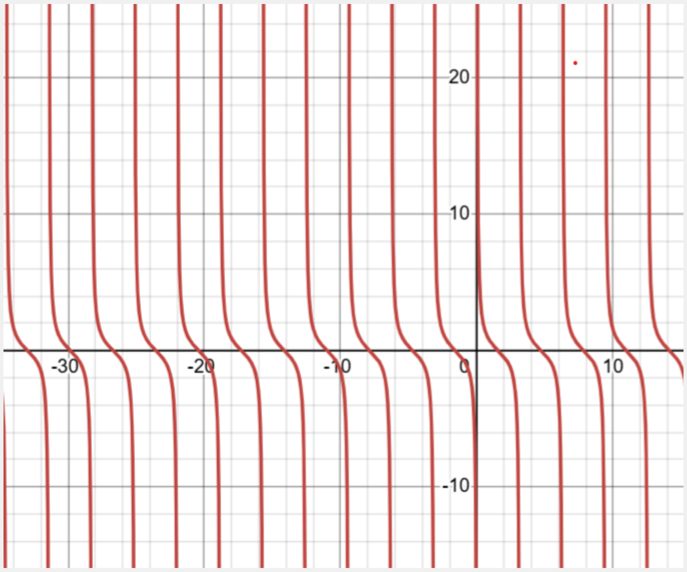
Consider $\pi/2 < x < 3\pi/2$. As x goes from $\pi/2$ to $3\pi/2$, $\sin(x)$ decreases and $\cos(x)$ decreases.

x	$\tan(x)$
$2\pi/3$	-1.732
$5\pi/6$	-0.577

Thus the ratio $\tan(x) = \frac{\sin(x)}{\cos(x)}$ increases.

Problem 3

Define the **cotangent** $\cot(x) = 1/\tan(x)$. Draw an approximate graph for hte cotangent, i.e. for the points $(x, \cot(x))$.

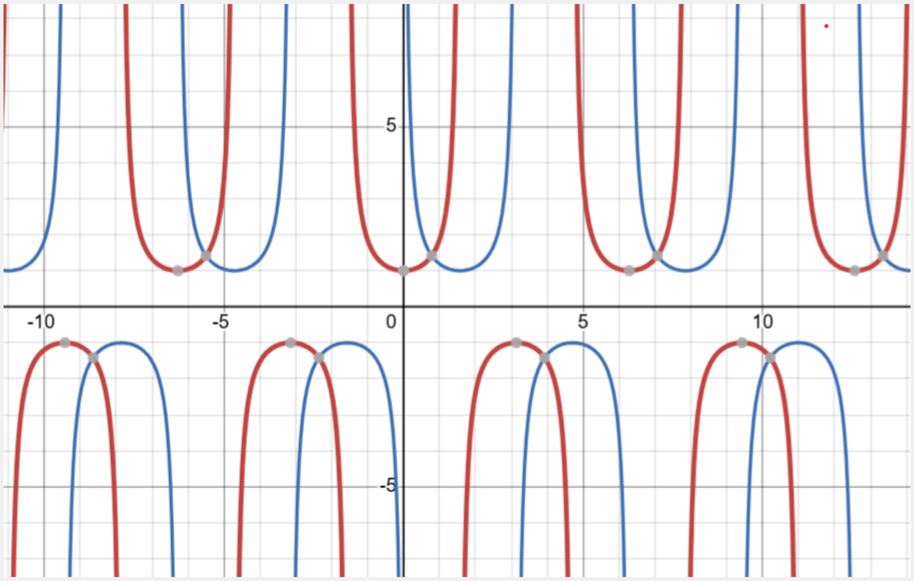


Problem 4

Define the secant and cosecant by

$$\sec(x) = \frac{1}{\cos(x)} \text{ and } cosec(x) = \frac{1}{\sin(x)}$$

for values of x where $\cos(x) \neq 0$ and $\sin(x) \neq 0$, respectively. Find enough values of the secant and cosecant until you feel that you have the hang of things. Draw their graphs.



Problem 5

Prove that $1 + \tan^2 x = \sec^2 x$.

Proof.

$$\begin{aligned} & 1 + \tan^2 x = \sec^2 x \\ \iff & 1 + \tan^2 x = \frac{1}{\cos^2 x} \\ \iff & \cos^2 x + \cos^2 x \tan^2 x = 1 \\ \iff & \cos^2 x + \cos^2 x \frac{\sin^2 x}{\cos^2 x} = 1 \\ \iff & \cos^2 x + \sin^2 x = 1 \end{aligned}$$

∎

Problem 6

State and prove a similar forumla relating the cotangent and cosecant.

Proof.

$$\begin{aligned} & 1 + \cot^2 x = \csc^2 x \\ \iff & 1 + \cot^2 x = \frac{1}{\sin^2 x} \\ \iff & \sin^2 x + \sin^2 x \cot^2 x = 1 \\ \iff & \sin^2 x + \sin^2 x \frac{\cos^2 x}{\sin^2 x} = 1 \\ \iff & \sin^2 x + \cos^2 x = 1 \end{aligned}$$

∎

Problem 9

You are looking at a tall building from a distance of 500 ft. The angle formed by the base of the building, your eyes, and the top of the building has
(a) $\pi/4$ radians
(b) $\pi/3$ radians
(c) $\pi/6$ radians
Find the height of the buildling.

Solution (a): $\tan(\pi/4) = \frac{h}{500} \iff h = 500 \cdot \tan(\pi/4)$ ft

Solution (b): $\tan(\pi/3) = \frac{h}{500} \iff h = 500 \cdot \tan(\pi/3)$ ft

Solution (c): $\tan(\pi/6) = \frac{h}{500} \iff h = 500 \cdot \tan(\pi/6)$ ft

11.5 Addition Formulas

Problem 1

Find $\sin(7\pi/12)$. [Hint: Write $7\pi/12 = 4\pi/12 + 3\pi/12$]

Solution:

$$\sin(7\pi/12) = \sin(4\pi/12 + 3\pi/12) = \sin(\pi/3)\cos(\pi/4) + \cos(\pi/3)\sin(\pi/4) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

Problem 4

Prove the following formulas. They should be memorizied.
(a) $\sin 2x = 2\sin x \cos x$
(b) $\cos 2x = \cos^2 x - \sin^2 x$
(c) $\cos^2 x = \frac{1+\cos 2x}{2}$
(d) $\sin^2 x = \frac{1-\cos 2x}{2}$

Proof.
$$\sin 2x = \sin(x + x) = \sin x \cos x + \cos x \sin x = \cos x(\sin x + \sin x) = 2\sin x \cos x$$
∎

Proof.
$$\cos 2x = \cos(x + x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$
∎

Proof.
$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1 \implies \cos^2 x = \frac{1 + \cos 2x}{2}$$
∎

Proof.
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x \implies \sin^2 x = \frac{1 - \cos 2x}{2}$$
∎

Problem 12

(a) A person throws a heavy ball at an angle θ from the ground. Let d be the distance from the person to the point where the ball strikes the ground. Then d is given by

$$d = \frac{2v^2}{2} \sin\theta\cos\theta$$

where v, g are constants. For what value of θ is the distance maximum? [Hint: Give another expression for $2\sin\theta\cos\theta$]
(b) You are watering the lawn, and point the watering hose at an angle of θ degrees from the ground. The distance from the nozzle at which the water strikes the ground is giving by

$$d = 2c\sin\theta\cos\theta$$

where c is a constant. For what value of θ is the distance at a maximum?

Solution (a): We have

$$d = \frac{2v^2}{2}\cos\theta\sin\theta = \frac{v^2}{2}(\sin\theta\cos\theta + \sin\theta\cos\theta) = \frac{v^2}{2}(\sin(2\theta))$$

Now $\sin(2\theta)$ is maximial at 1 thus we require $\sin(2\theta) = 1$ and it follows that $\theta = \pi/4$.

Solution (b): Similar to part (a).

$$d = c(\sin\theta\cos\theta + \sin\theta\cos\theta) = c\sin(2\theta)$$

and it follows that $\theta = \pi/4$.

Problem 13

Prove the following formulas for any integer m, n:

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$$

$$\frac{1}{2}[\cos(m-n)x - \cos(m+n)x] = \frac{1}{2}[(\cos mx\cos nx + \sin mx\sin nx) - (\cos mx\cos nx - \sin mx\sin nx)] = \sin mx\sin nx$$

$$\frac{1}{2}[\sin(m+n)x + \sin(m-n)x] = \frac{1}{2}[(\sin mx\cos nx + \cos mx\sin nx) + (\sin mx\cos nx - \cos mx\sin nx)] = \sin mx\cos nx$$

$$\frac{1}{2}[\cos(m+n)x + \cos(m-n)x] = \frac{1}{2}[(\cos mx\cos nx - \sin mx\sin nx) + (\cos mx\cos nx + \sin mx\sin nx)] = \cos mx\cos nx$$

11.6 Rotations

Problem 2

In the following case, write the matrix associated with the rotation of G_φ, when φ has the indicated value. Write explicitly the coordinates (x', y') of $G_\varphi(P)$ if P has coordinates (x, y).
$\varphi = \pi/4$.

$$\begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$
$$x' = (\cos\pi/4)x - (\sin\pi/4)y, y' = (\sin\pi/4)x + (\cos\pi/4)y$$

Problem 21

Associate a matrix with a dilation by r. Interpret dilation by r in terms of the matrix muliplication. Do the same for mixed dilations of type which we have written $F_{a,b}$.

Dilation:

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

Mixed Dilation:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Problem 22

Write down the matrices for the rotations $G_\varphi, G_\psi, G_{\varphi+\psi}$

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$
$$\begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$
$$\begin{bmatrix} \cos\varphi+\psi & -\sin\varphi+\psi \\ \sin\varphi+\psi & \cos\varphi+\psi \end{bmatrix}$$

12 Some Analytic Geometry

12.1 The Straight Line Again

Problem 28

For our purposes here, define two straight lines to be parallel if they have the same slope. Let

$$y = ax + b \text{ and } y = cx + d$$

be the equations of two lines with $b \neq d$.
(a) If they are parallel, show they they have no point in common.
(b) If they are not parallel, show theat they have exactly one point in common.

Proof. Suppose the lines are parallel and do share a comon point. Thus $ax + b = cx + d$, but $a = c$ thus $b = d$ which is a contradiction. ∎

Proof. Suppose the lines are not parallel. Thus

$$ax + b = cx + d \iff x(a - c) = d - b \iff x = \frac{d - b}{a - c}.$$

Note since the lines are not parallel $a \neq c$, thus $a - c \neq 0$. Plugging this back into either equation gives our point of intersection. ∎

Problem 30

If a straight line is expressed in parametric form,

$$\{P + tA\}_{t\in\mathbb{R}}$$

and $A = (a_1, a_2)$, what is the slope of the line in terms of the coordinates A? Does this slope depend on the coordinates of P.

Solution: Suppose $P = (p_1, p_2)$. Then

$$x = p_1 + ta_1, \quad y = p_2 + ta_2.$$

The slope of the line is

$$\frac{y - p_2}{x - p_1} = \frac{a_2 t}{a_1 t} = \frac{a_2}{a_1}.$$

Thus the slope does snot depend on the coordinates of P.

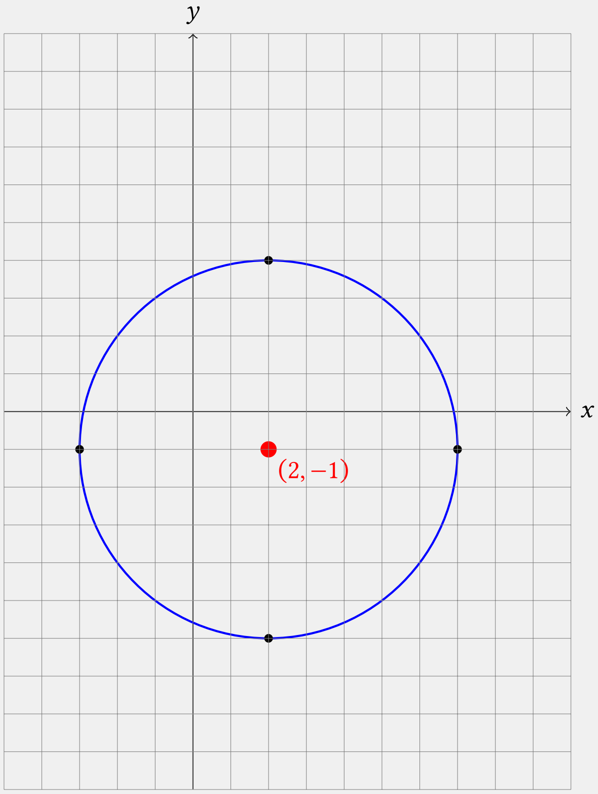
12.2 The Parabola

Problem 5

Sketch the graph $x^2 + y^2 - 4x + 2y - 20 = 0$.

Solution: $x^2 + y^2 - 4x + 2y - 20 = (x^2 - 4x) + (y^2 + 2y) - 20$. Completing the square on x gives $x^2 - 4x = (x-2)^2 - 4$, and completing the square on y gives $y^2 + 2y = (y + 1)^2 - 1$. Thus

$$(x - 2)^2 - 4 + (y + 1)^2 - 1 - 20 = 0 \implies (x - 2)^2 + (y + 1)^2 = 25$$



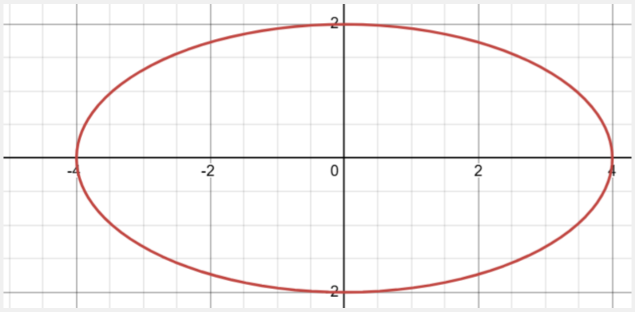
12.3 The Ellipse

Problem 1

Sketch the graph of the following equations. In each case, indicate the center of the ellipse, and its extremities.

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

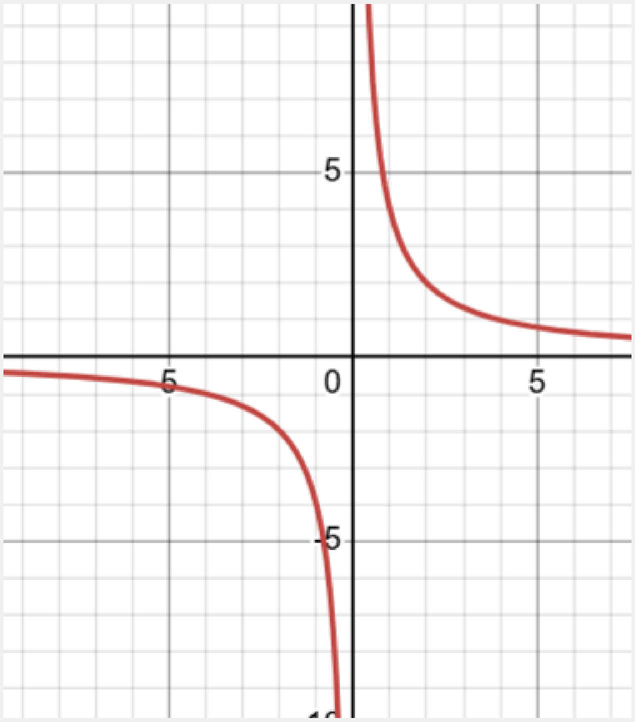
Solution: Center is $(0, 0)$. Extremities are $(\pm 4, 0)$ and $(0, \pm 2)$.



12.4 The Hyperbola

Problem 8

Sketch the graphs of the following curves, defined by the given equations.
$xy = 4$.



12.5 Rotation of Hyperbolas

Problem 2

Rotate the hyperbola H defined by the equation $xy = 1$ by $-\pi/4$. What is the equation satisfied by the image of H.

$$v^2 - u^2 = 2$$

Problem 8

Prove the statement made in the text: If G is rotation and F_r is a dilation by r, $G \circ F = F \circ G$.

Proof. Let $P = (x, y)$ be an arbitrary point. A dilation by r is

$$F_r(P) = r \begin{bmatrix} x \\ y \end{bmatrix},$$

and a rotation by angle θ is

$$G(P) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then

$$G(F_r(P)) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left(r \begin{bmatrix} x \\ y \end{bmatrix} \right) = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Similarly,

$$F_r(G(P)) = r \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

∎

13 Functions

13.1 Definition of a Function

Problem 2

For what numbers could you define a function f by the formula

$$f(x) = \frac{1}{x^2 - 2}$$

What is the value of the function for $x = 5$.

Solution: f is defined at all real numbers other than $\pm\sqrt{2}$.

$$f(5) = \frac{1}{5^2 - 2} = \frac{1}{23}$$

Problem 8

A function (defined for all numbers) is said to be an **even** function if $f(x) = f(-x)$ for all numbers x. It is said to be an **odd** function if $f(x) = -(f(-x))$ for all x. Determine which of the following functions are even or odd.
(a) $f(x) = x$
(b) $f(x) = x^2$
(c) $f(x) = x^3$
(d) $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$

Solution (a): odd

Solution (b): even

Solution (c): odd

Solution (d): odd

Problem 9

Show that any function defined for all numbers can be written as a sum of an even and an odd function. [Hint: The term

$$\frac{f(x) + f(-x)}{2}$$

will be an even function.]

Proof. Let any function f be defined for all numbers.

$$g(x) = \frac{f(x) + f(-x)}{2}, \quad l(x) = \frac{f(x) - f(-x)}{2}.$$

Then g is even and l is odd, and
$$f(x) = g(x) + l(x).$$
∎

Problem 11

(a) Show that the sum of odd functions is odd.
(b) Show that the sum of even functions is even.

Proof. Let f, g be odd functions. Then

$$(f+g)(x) = f(x) + g(x) = -f(-x) - g(-x) = -(f(-x) + g(-x)) = -(f+g)(-x),$$

so the sum is odd. Let f, g be even functions. Then

$$(f+g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f+g)(-x),$$

so the sum is even. ∎

Problem 12

Determine whether the product of the following types of functions is odd, even, or neither. Prove your assertions.
(a) Product of odd function with odd function
(b) Product of even function with odd function
(c) Product of even function with even function

Proof. If f and g are odd:

$$(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x)$$

The product is even.

If f is even and g is odd:

$$(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x) = -(fg)(x)$$

The product is odd.

If f and g are even:
$$(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$$

The product is even. ∎

13.2 Polynomial Functions

Problem 1

What is the degree of the following polynomials.
(a) $3x^2 - 4x + 5$
(b) $-5x^5 + x$
(c) $-38x^4 + x^3 - x - 1$
(d) $(3x^2 - 4x + 5)(-5x^5 + x)$
(e) $(-5x^5 + x)(-7x + 3)$
(f) $(-4x^2 + 5x - 4)(3x^3 + x - 1)$

(g) $(6x^7 - x^3 + 5)(7x^4 - 3x^2 + x - 1)$
(h) Let f, g be polynomials which are not the zero polynomials. Show that $deg(fg) = deg(f) + deg(g)$.

Solution (a): 2

Solution (b): 5

Solution (c): 4

Solution (d): 7

Solution (e): 6

Solution (f): 5

Solution (g): 11

Proof. Let f, g be arbitrary polynomials. Let

$$f = a_n x^n + \text{(other terms)}, \quad g = b_k x^k + \text{(other terms)}$$

where a_n, b_k are the coefficients of the largest-degree terms of f and g respectively, with $a_n \neq 0$ and $b_k \neq 0$. The term of largest degree in the product of fg is produced by multiplying the highest-degree terms

$$(a_n x^n)(b_k x^k) = (a_n b_k)x^{n+k}$$

This term as the degree $n + k$, and since $a_n b_k \neq 0$, the degree of fg is $n + k$. ∎

Problem 3

Let f be a polynomial of degree 3. If there exists polynomials g, h of degree ≥ 1 such that $f = gh$, show that f has a root.

Proof. By Problem 1 either $deg(g) = 1$ or $deg(h) = 1$. Suppose wlog $deg(g) = 1$. Let $g = ax + b$ where a, b are constants and $a \neq 0$. A root is found when $ax + b = 0$ thus $x = \frac{-b}{a}$. Then $g\left(\frac{-b}{a}\right)h\left(\frac{-b}{a}\right) = f\left(\frac{-b}{a}\right) = 0$ as required. ∎

Problem 4

Give an example of polynomials of degree 2, 4 which have no roots and degree 3 that has one root in the real numbers.

Solution:

$$x^2 + 2 = 0, (x^2 + 2)(x - 4) = 0, x^4 + 2 = 0$$

13.3 Graphs of Functions

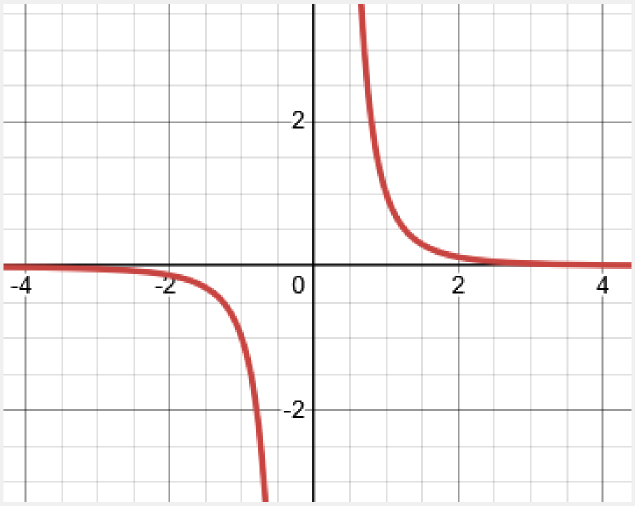
Problem 1

Sketch the graph of the following functions. Make a small table of values $f(x) = \frac{1}{x^3}$.

Solution:

$$x = -2, -1, -0.5, 0, 0.5, 1, 2$$

$$f(x) = -0.125, -1, -8, \text{undefined}, 8, 1, 0.125$$



13.4 Exponential Function

Problem 7

The function $f(t) = 4 \cdot 16^t$ describes the growth of bacteria.
(a) How many bacteria are present at the beggining when $t = 0$?
(b) After $\frac{1}{2}$ hr, how many bacteria are there?
(c) After $\frac{1}{4}$ hr, how many bacteria are there?
(d) After 1 hr, how many bacteria are there?

Solution (a): $4 \cdot 16^0$

Solution (b): $4 \cdot 16^{\frac{1}{2}}$

Solution (c): $4 \cdot 16^{\frac{1}{4}}$

Solution (d): $4 \cdot 16^1$

13.5 Logarithms

Problem 3

Let e be a fixed number > 1 and abbreviate $\log_e$ by log. If a is > 1 and x is an arbitrary number, prove that

$$\log(a^x) = x\log(a)$$

Proof.

$$\begin{aligned}
&\log a = \log a \\
&\iff \log(e^{\log a}) = \log a \\
&\iff e^{\log a} = a \\
&\iff (e^{\log a})^x = a^x \\
&\iff e^{x\log a} = a^x \\
&\iff \log a^x = \log(e^{x\log a}) = x\log a
\end{aligned}$$

Problem 6

Bacteria increase according to the formula

$$B(t) = Ce^{kt}$$

where C and k are constants, and $B(t)$ gives the number of bacteria as a function of t in min. At time $t = 0$, there are 10^6 bacteria. How long will it take before they increase to 10^7 if it takes 12 min to increase by 2×10^6.

Solution: At $t = 0$ there are 10^6 bacteria. Thus $C = 10^6$. Then plugging in 12 min we have the formula

$$3 \cdot 10^6 = 10^6 \cdot e^{12k}$$

Solving for k shows

$$k = \frac{\ln 3}{12}$$

Now plugging in

$$10^7 = 10^6 e^{kt} \implies 10 = e^{kt}$$
$$t = \frac{\ln 10}{k} = \frac{12 \ln 10}{\ln 3}$$

14 Mappings

14.1 Definition

Problem 1

A particle starts from the point $(0, 6)$ in the plant. It is attracted by a magnet below the x-axis, and repelled by a magnet along the y-axis in such a way that its coordinates are given as a function of t by

$$x(t) = 2t, y(t) = 6 - 15t^3$$

(a) Find the time at which it hits the x-axis.
(b) Give a simple equation in terms of x and y such that the coordinates $(x(t), y(t))$ of the particle satisfy this equation. Sketch the graph of this equation.
(c) Find the distance of the point at which the particle hits the x-axis from teh origin.
(d) Find the time at which the particle is at distance 2 units from the x-axis, below the x-axis.
(e) Find the time at which the particle is at distance 5 units from the x-axis, below the x-axis.
(f) Find the time at which the particle is at distance 7 units from the x-axis, below the x-axis.

Solution (a): We required $0 = 6 - 15t^3$. Thus $t = \sqrt[3]{\frac{6}{15}}$.

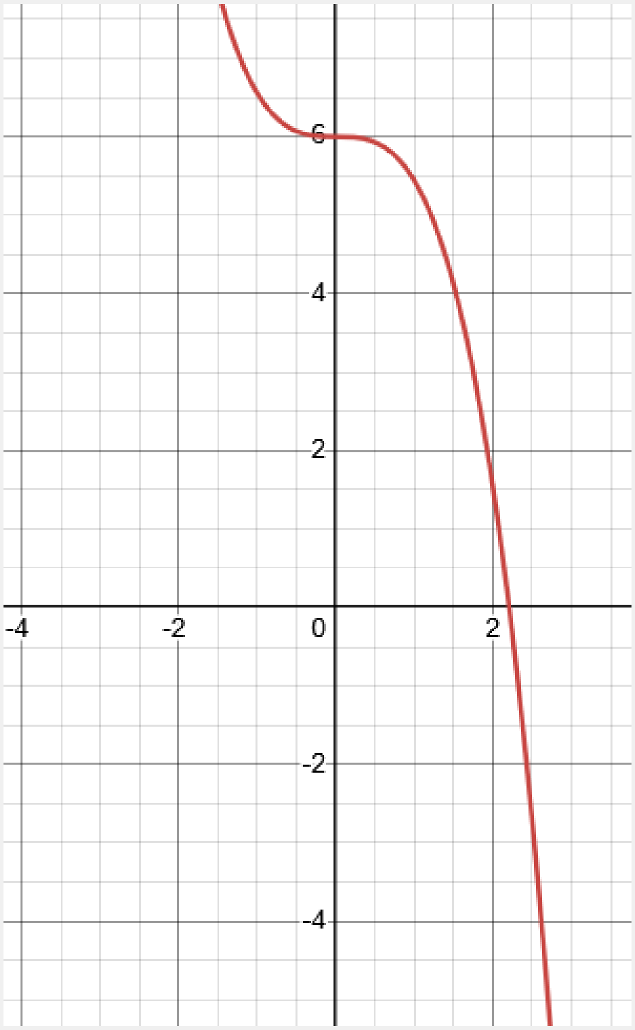
Solution (b):

$$y = 6 - 15\left(\frac{x}{3}\right)^3$$

Solution (c): The distance is $\sqrt{(3 \cdot \sqrt[3]{\frac{6}{15}})^2}$ units.

Solution (d): We require $-2 = 6 - 15t^3$. Thus $t = \sqrt[3]{\frac{8}{15}}$

Solution (e): We require $-5 = 6 - 15t^3$. Thus $t = \sqrt[3]{\frac{11}{15}}$



Solution (f): We require $-7 = 6 - 15t^3$. Thus $t = \sqrt[3]{\frac{13}{15}}$

14.2 Formalism of Mappings

Problem 1

Let $f : S \longrightarrow T$ and $g : S \longrightarrow T$ be mappings. Let

$$h = T \longrightarrow U$$

be a mapping have an inverse mapping denoted by

$$h^{-1} : U \longrightarrow T$$

If

$$h \circ f = h \circ g$$

Prove that $f = g$. This is the **cancellation law** for mappings.

Proof. Composing with h^{-1} shows $f = g$. ∎

Problem 2

Let $f : S \longrightarrow T$ be a mapping having an inverse mapping. Prove the following statements.
(a) If x, y are elements of S and $f(x) = f(y)$, then $x = y$.
(c) If z is an element of T, then there exists an element x of S such that $f(x) = z$.

Proof. Suppose x, y are elements of S, and $f(x) = f(y)$. Composing with f^{-1} shows $x = y$. ∎

Proof. Suppose z is an element of T. There exists $c \in S$ such that $f^{-1}(z) = c$. Composing with f shows $f(c) = z$. ∎

Problem 3

Let a, b be non-zero numbers. Let $F_{a,b} = \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the mapping such that

$$F_{a,b}(x, y) = (ax, by)$$

Show that $F_{a,b}$ has an inverse mapping.

Proof. Let $G_{a,b}(x, y) = \left(\frac{x}{a}, \frac{y}{b}\right)$. Consider

$$(F_{a,b} \circ G_{a,b})(x, y) = F_{a,b}(G_{a,b}(x, y)) = F_{a,b}\left(\frac{x}{a}, \frac{y}{b}\right) = (x, y)$$

Also

$$(G_{a,b} \circ F_{a,b})(x, y) = G_{a,b}(F_{a,b}(x, y)) = G_{a,b}(ax, by) = (x, y)$$

Therefore $G_{a,b} = F_{a,b}^{-1}$. ∎

Problem 6

Let $f : S \longrightarrow S$ be a mapping which has an inverse mapping.
(a) If $f^3 = I$ and $f^5 = I$, show that $f = I$.
(b) If $f^2 = I$ and $f^7 = I$, show that $f = I$.
(c) If $f^4 = I$ and $f^1 1 = I$, show that $f = I$.

Proof. Since $f^3 = I$, $f^2 = f^{-1}$. Then

$$f = f^5 \circ f^{-4} = f^5 \circ f^{-1} \circ f^{-1} \circ f^{-1} \circ f^{-1}$$

$$= f^5 \circ f^8 = f^{13} = f^3 \circ f^{10} = f^3 \circ f^5 \circ f^5 = I$$

∎

Proof. Since $f^2 = I$, $f = f^3$ and $f = f^{-1}$. Then

$$f = f^2 \circ f^{-1} = f^2 \circ f = f^2 \circ f^3 = f^4 \circ f = f^4 \circ f^3 = f^7 = I$$

∎

Proof. Since $f^4 = I$, $f = f^5$ and $f = f^{-3}$. Then

$$f = f^{11} \circ f^{-10} = f^{11} \circ (f^{-4})^2 \circ f^{-2} = f^{11} \circ I^2 \circ f^{-2} = f^{11} \circ f^{-2}.$$

Then $f^{11} = I$, so

$$f = I \circ f^{-2} = f^{-2}.$$

Composing both sides with f^2 gives

$$f \circ f^2 = f^{-2} \circ f^2 \implies f^3 = I.$$

Now $f^3 = I$ and $f^4 = I$, so

$$f = f^4 \circ f^{-3} = I \circ I = I.$$

∎

Problem 7

Let f, g be mappings of a set S into itself, and assume that they have inverse mappings. Assume also that $f \circ g = g \circ f$. Express each one of the following in the form $f^m \circ g^n$ where m, n are integers.
(a) $f^3 \circ g^2 \circ f^5 \circ g^{-5}$
(b) $f^7 \circ g \circ g^4 \circ f^{-6} \circ g^3$
(c) $f^4 \circ g^5 \circ f^{-5} \circ g^{-7} \circ g^2 \circ f^2$
(d) $f^4 \circ f^{-8} \circ g^2 \circ f^3 \circ g^3 \circ f^{-2}$

Solution (a):

$$f^3 \circ g^2 \circ f^5 \circ g^{-5} = f^3 \circ f^5 \circ g^2 \circ g^{-5} = f^8 \circ g^{-3}$$

Solution (b):

$$f^7 \circ g \circ g^4 \circ f^{-6} \circ g^3 = f^7 \circ f^{-6} \circ g \circ g^4 \circ g^3 = f \circ g^8$$

Solution (c):

$$f^4 \circ g^5 \circ f^{-5} \circ g^{-7} \circ g^2 \circ f^2 = f^4 \circ f^{-5} \circ f^2 \circ g^5 \circ g^{-7} \circ g^2 = f \circ g^0 = f$$

Solution (d):

$$f^4 \circ f^{-8} \circ g^2 \circ f^3 \circ g^3 \circ f^{-2} = f^4 \circ f^{-8} \circ f^3 \circ f^{-2} \circ g^2 \circ g^3 = f^{-3} \circ g^5$$

Problem 8

(a) Let f, g be mappings of a set S into itself, and assume that they have inverse mappings. Prove that $f \circ g$ has an inverse mapping, and express it in terms of $f^{-1} \circ g^{-1}$.
(b) Let $f_1, \ldots, f_m$ be maps of S into itself, and assume that each f_i has an inverse mapping. Show that $f_1 \circ f_2 \circ \ldots \circ f_m$ has an inverse mapping, and express this mapping in terms of the maps f_i^{-1}.

Proof. Consider

$$(f \circ g) \circ (g^{-1} \circ f^{-1}) = f \circ (g \circ g^{-1}) \circ f^{-1} = f \circ I \circ f^{-1} = f \circ f^{-1} = I$$

Also

$$(g^{-1} \circ f^{-1}) \circ (f \circ g) = g^{-1} \circ (f^{-1} \circ f) \circ g = g^{-1} \circ I \circ g = g^{-1} \circ g = I$$

Therefore, $g^{-1} \circ f^{-1}$ is the inverse of $f \circ g$. ∎

Proof. Consider

$$(f_1 \circ f_2 \circ \cdots \circ f_m) \circ (f_m^{-1} \circ f_{m-1}^{-1} \circ \cdots \circ f_1^{-1})$$

Then

$$f_1 \circ f_2 \circ \cdots \circ f_m \circ f_m^{-1} \circ \cdots \circ f_1^{-1} = f_1 \circ f_2 \circ \cdots \circ f_{m-1} \circ I \circ f_{m-1}^{-1} \circ \cdots \circ f_1^{-1} = \cdots = I.$$

Similarly,

$$(f_m^{-1} \circ \cdots \circ f_1^{-1}) \circ (f_1 \circ f_2 \circ \cdots \circ f_m) = I$$

Therefore,

$$(f_1 \circ f_2 \circ \cdots \circ f_m)^{-1} = f_m^{-1} \circ f_{m-1}^{-1} \circ \cdots \circ f_1^{-1}$$

∎

Problem 9

Let f be a mapping of a set S into itself, and assume that f has an inverse mapping.
(a) If $f^5 = I$, express f^{-1} as a positive power of I.
(b) In general, if $f^n = I$, for some positive power of f, express f^{-1} as a positive power of f.

Proof.

$$f^{-1} = f^{-1} \circ I = f^{-1} \circ f^5 = f^4$$

∎

Proof.

$$f^{-1} = f^{-1} \circ I = f^{-1} \circ f^n = f^{n-1}$$

∎

Problem 10

Let f, g be mappings of a set S into itself. Assume that $f^2 = g^2 = I$ and that $f \circ g = g \circ f$. Prove that $(f \circ g)^2 = I$. Prove that $(f \circ g)^2 = I$. Prove that $(f \circ g)^3 = I$. What about $(f \circ g)^n$ for any positive integer n? What about $(f \circ g)^n$ where n is a negative integer?

Proof. Notice

$$(f \circ g)^2 = (f \circ g) \circ (f \circ g) = f \circ (g \circ f) \circ g = f \circ (f \circ g) \circ g = (f \circ f) \circ (g \circ g) = I \circ I = I.$$

Also

$$(f \circ g)^3 = (f \circ g)^2 \circ (f \circ g) = I \circ (f \circ g) = f \circ g \neq I$$

Let n be an arbitrary positive integer. Then

$$(f \circ g)^n = \begin{cases} I & \text{if } n \text{ is even} \\ f \circ g & \text{if } n \text{ is odd.} \end{cases}$$

For negative integers the pattern is the same. ∎

14.3 Permutations

Problem 10

Prove that the number of odd permutations of J_n for $n \geq 2$ is equal to the number of even permutations.

Proof. Let $\sigma_1, \ldots, \sigma_m$ be all the distinct even permutations. Let γ be a transposition. Suppose $\sigma_i = \gamma_1\gamma_2 \ldots \gamma_{2k}$ for some $k \in \mathbb{Z}$. Composing with γ we see $\gamma\sigma_i = \gamma\gamma_1\gamma_2 \ldots \gamma_{2k}$ is a product of $2k + 1$ transpositions, thus an odd permuation. Suppose $i \neq j$, $\gamma\sigma_i = \gamma\sigma_j$. Composing with γ shows $\sigma_i = \sigma_j$, which is a contradiction. Thus $\gamma\sigma_1, \ldots, \gamma\sigma_m$ are distinct. Now suppose there exists an odd permuation not generated through this process. Let π be this permutation. Then, π can be written as a product of $2k + 1$ transpositions. Thus $\gamma\pi$ is a product of $2k$ transpositions, thus an even permutation, which must be in out list $\sigma_1, \ldots, \sigma_m$. Thus π is of the form $\gamma\sigma_i$ for some i, and no odd permutation is as required. ∎

Problem 11

Prove that the number of permuations of J_n is equal to $n!$.

Proof. (**Base Case**) Let $n = 1$. There is 1 way to permute 1 element, thus $1 = 1!$ as required.

(**Induction Step**) Suppose for some $n \in \mathbb{N}$ the theorem holds. Thus there are $n!$ ways to permute n elements. Consider a permuation of $n + 1$ elements. By the induction hypothesis there are $n!$ ways to permute the first n elements. For each of these permutations, there are $n + 1$ positions to place the $(n + 1)$-th element, including placing it in its original position. Thus we have a total of $n!(n + 1) = (n + 1)!$ permuations as required. ∎

15 Complex Numbers

15.1 The Complex Plane

Problem 3

Let z be a complex number $\neq 0$. What is the absolute value of $z\sqrt{z}$?

Solution: Let $z = x + yi$.

$$|z\sqrt{z}| = |z|\,|\sqrt{z}| = (x^2 + y^2)^{1/2} \cdot (x^2 + y^2)^{1/4} = (x^2 + y^2)^{3/4} = (z\overline{z})^{3/4} = (|z|^2)^{3/4} = |z|^{3/2}$$

Problem 4

Prove the statements of Theorem 1.

1. $\overline{zw} = \overline{zw}$
2. $\overline{z + w} = \overline{z} + \overline{w}$
3. $\overline{\overline{z}} = z$

Proof. Let $z = x + yi$ and $w = u + vi$. Then

$$zw = (x + yi)(u + vi) = (xu - yv) + (xv + yu)i, \quad \overline{zw} = (xu - yv) - (xv + yu)i$$

and

$$\overline{z}\,\overline{w} = (x - yi)(u - vi) = (xu - yv) - (xv + yu)i.$$

Therefore $\overline{zw} = \overline{z}\,\overline{w}$.

$$z + w = (x + u) + (y + v)i, \quad \overline{z + w} = (x + u) - (y + v)i = (x - yi) + (u - vi) = \overline{z} + \overline{w}.$$

$$\overline{\overline{z}} = \overline{x - yi} = x + yi = z.$$

∎

Problem 5

Show that for any complex number $z = x + iy$, with x, y real, we have

$$Im(z) \leq |Im(z)| \leq |z|$$

Proof. Notice

$$\mathrm{Im}(z) = y.$$

By definition of absolute value

$$y \leq |y| = |\operatorname{Im}(z)|.$$

Then

$$|\operatorname{Im}(z)| = |y| \leq \sqrt{x^2 + y^2} = |z|.$$

Combining these inequalities shows

$$\mathrm{Im}(z) \leq |\operatorname{Im}(z)| \leq |z|.$$

∎

15.2 Polar Form

Problem 3

Let z be a complex number $\neq 0$. Show that there are precisely two distinct complex numbers whose square is z.

Proof. We first show existence of two complex numbers whose square are z. Let $z = re^{\theta i}$ where $\theta, r \in \mathbb{R}$ and $0 \leq \theta < 2\pi$. Consider $\sqrt{r}e^{\theta/2i}$. Notice

$$\sqrt{r}e^{\theta/2i} \cdot \sqrt{r}e^{\theta/2i} = re^{\theta i}$$

Now consider $-\sqrt{r}e^{\theta/2i}$. Since $z \neq 0$, $\sqrt{r} \neq -\sqrt{r}$. Then

$$-\sqrt{r}e^{\theta/2i} \cdot -\sqrt{r}e^{\theta/2i} = re^{\theta i}$$

We now show these are the only two complex numbers whose square are z. Suppose w is a complex number such that $w^2 = z = re^{\theta i}$. Let $w = \rho e^{xi}$. Then

$$w^2 = \rho^2 e^{2xi} = re^{\theta i}$$

Matching moduli shows $\rho = \sqrt{r}$. Matching arguments shows $2x = \theta + 2\pi k$ for some $k \in \mathbb{Z}$. Thus $x = \theta/2 + k\pi$. Since angles differing by 2π give the same complex number, only $k = 0, 1$ produce distinct values. These are exactly $\sqrt{r}e^{\theta i/2}$ and $-\sqrt{r}e^{\theta i/2}$. ∎

Problem 4

Let z be a complex number $\neq 0$. Let n be a positive integer. Show that there are n distinct complex numbers w such that $w^n = z$. Write these complex numbers in polar form. The proof given that a polynomial of degree $\leq n$ has at most n roots applies to the complex case, and thus we see that there are no other complex numbers w such that $w^n = z$ other than those you have presumably written down.

Proof. We first show existence of n complex numbers whose n-th power is z. Let $z = re^{\theta i}$ where $r, \theta \in \mathbb{R}$ and $0 \le \theta < 2\pi$. Consider

$$w = r^{1/n} e^{(\theta + 2\pi k)i/n}, \quad k = 0, 1, \ldots, n-1.$$

Notice that

$$w^n = \left(r^{1/n} e^{(\theta+2\pi k)i/n}\right)^n = re^{\theta i} = z.$$

Since k ranges from 0 to $n-1$, these are n distinct numbers.

We now show that these are the only complex numbers whose n-th power is z. Suppose w is a complex number such that $w^n = z = re^{\theta i}$. Let $w = \rho e^{\phi i}$. Then

$$w^n = \rho^n e^{n\phi i} = re^{\theta i}.$$

Matching moduli gives $\rho = r^{1/n}$. Matching arguments gives $n\phi = \theta + 2\pi k$ for some $k \in \mathbb{Z}$, so

$$\phi = \frac{\theta + 2\pi k}{n}.$$

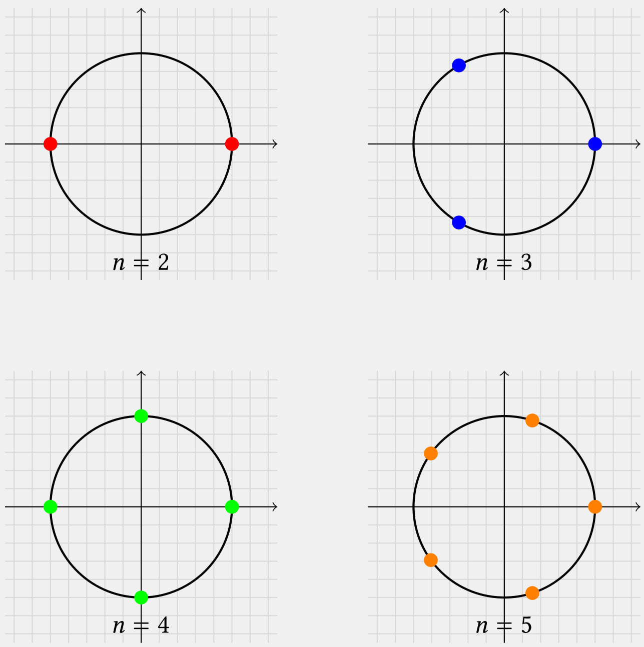
Angles differing by 2π give the same complex number, so only $k = 0, 1, \ldots, n-1$ produce distinct values. These are exactly

$$w = r^{1/n} e^{(\theta + 2\pi k)i/n}, \quad k = 0, 1, \ldots, n-1.$$

∎

Problem 5

Write in polar form the n complex numbers w such that $w^n = 1$. Plot all of these as points in the plane $n = 2, 3, 4, 5$.



Problem 6

If θ is real, show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Proof.

$$\begin{aligned}\frac{e^{i\theta}+e^{-i\theta}}{2} &= \frac{(\cos\theta + i\sin\theta) + (\cos(-\theta) + i\sin(-\theta))}{2} \\ &= \frac{(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)}{2} \\ &= \frac{2\cos\theta}{2} \\ &= \cos\theta\end{aligned}$$

∎

Proof.

$$\begin{aligned}\frac{e^{i\theta}-e^{-i\theta}}{2i} &= \frac{(\cos\theta + i\sin\theta) - (\cos(-\theta) + i\sin(-\theta))}{2i} \\ &= \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{2i} \\ &= \frac{2i\sin\theta}{2i} \\ &= \sin\theta\end{aligned}$$

∎

16 Induction and Summations

16.1 Induction

Problem 1

Prove that, for all integers $n \geq 1$, we have

$$1 + 3 + 5 + \ldots + (2n-1) = n^2$$

Proof. (**Base Case**) Let $n = 1$. Then $1 = n^2 = 1^2 = 1$ as required.

(**Induction Step**) Suppose the equation holds for some $n \in \mathbb{N}$. Thus $1 + 3 + 5 + \ldots + (2n-1) = n^2$. Consider $1 + 3 + 5 + \ldots + (2n-1) + (2n+1) = n^2 + (2n+1)$ by our hypothesis. Then $n^2 + 2n + 1 = (n+1)^2$ as required. ∎

Problem 4

Prove that $n(n^2+5)$ is divisible by 6 for all integers $n \geq 1$.

Proof. (**Base Case**) Let $n = 1$. Then $n(n^2+5) = 1(1^2+5) = 6$. Clearly $6 \mid 6$ as required.

(**Induction Step**) Suppose the equation holds for some $n \in \mathbb{N}$. Thus $n(n^2+5) = 6k$ for some $k \in \mathbb{Z}$. Consider $(n+1)((n+1)^2+5) = n^3+3n^2+8n+6 = n(n^2+5)+3n^2+3n+6$. Then, by our hypothesis, $n(n^2+5)+3n^2+3n+6 = 6k + 3n^2 + 3n + 6 = 6k + 3(n(n+1)+2)$. Now $n(n+1)$ is even thus $n(n+1) = 2j$ for some $j \in \mathbb{Z}$. Then, $6k + 3(n(n+1)+2) = 6k + 3(2j+2) = 6k + 6j + 6 = 6(k+j+1)$ as required. ∎

Problem 5

Prove that, for $x \neq 1$, we have

$$(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) = \frac{1-x^{2^{n+1}}}{1-x}$$

Proof. (**Base Case**) Let $n=0$. Then $1+x = \frac{1-x^{2^{0+1}}}{1-x} = \frac{1-x^2}{1-x} = \frac{(1+x)(1-x)}{1-x} = 1+x$ as required.

(**Induction Step**) Suppose the equation holds for some $n \in \mathbb{N}$. Thus

$$(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) = \frac{1-x^{2^{n+1}}}{1-x}$$

Then

$$(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})(1+x^{2^{n+1}}) = \frac{1-x^{2^{n+1}}}{1-x}(1+x^{2^{n+1}}) = \frac{1-x^{2^{n+2}}}{1-x}$$

∎

Problem 6

Let f be a function defined for all real numbers such that $f(xy) = f(x) + f(y)$. for all real numbers x, y. Show that $f(x^n) = nf(x)$ for all x.

Problem 7

Let f be a function defined for all numbers such that $f(xy) = f(x)f(y)$ for all real numbers x, y. Show that $f(x^n) = f(x)^n$ for all positive integers n and all real numbers x.

Problem 9

Binomial Coefficients. Let

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where n, k are integers ≥ 0, $0 \leq k \leq n$, and $0!$ is defined to be 1. Prove that
(a) $\binom{n}{k} = \binom{n}{n-k}$
(b) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ (for $k > 0$)
(c) Prove by induction that for all numbers x, y we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof.

$$\begin{aligned}
\binom{m}{n} &= \binom{m}{m-n} \\
\frac{m!}{n!(m-n)!} &= \frac{m!}{(m-n)!(m-(m-n))!} \\
\frac{m!}{n!(m-n)!} &= \frac{m!}{(m-n)!(n)!} \\
\frac{m!}{n!(m-n)!} &= \frac{m!}{(n)!(m-n)!}
\end{aligned}$$

∎

Proof.

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(k-1)!(n-k+1)!} \cdot \frac{k}{k} + \frac{n!}{k!(n-k)!} \cdot \frac{n-k+1}{n-k+1} \\
&= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!\big(k+(n-k+1)\big)}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!} \\
&= \binom{n+1}{k}
\end{aligned}$$

∎

Proof. Let $n = 0$. Then

$$(x+y)^0 = 1 = \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k} = \binom{0}{0} x^0 y^0 = 1 \cdot 1 \cdot 1 = 1$$

Assume the formula holds for $n-1$, thus

$$\sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{(n-1)-k} = (x+y)^{n-1}$$

Then

$$
\begin{aligned}
(x+y)^n &= (x+y)^{n-1} \cdot (x+y) \\
&= \left(\sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{(n-1)-k} \right) \cdot (x+y) \\
&= x \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{(n-1)-k} + y \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{(n-1)-k} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} x^k y^{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k} \\
&= \sum_{k=0}^{n} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) x^k y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\end{aligned}
$$

∎

Problem 12

Theorem 5. *All billiard balls have the same color.*

Proof. By induction, on the number of n billiard balls. Our theorem is certainly true for $n = 1$, i.e. for one billiard ball. Assume it for n billiard balls. We prove it for $n + 1$. Look at the first n billiard balls among those $n + 1$. By induction, they have the same color. Now look at the last n among those $n + 1$. They have the same color. Hence all $n + 1$ have the same color. ∎

Solution: When there are 2 balls there is no overlap in the first n and last n balls. Therefore, they do not necessarily have the same color.

Problem 13

Let E be the set with n elements, and let F be a set with m elements Show that the total number of mappings from E to F is m^n.

Proof. There are m choices in F for each of the n elements in E. The total number of mappings from E to F is therefore

$$\underbrace{m \cdot m \cdot \cdots \cdot m}_{n \text{ terms}} = m^n$$

∎

16.2 Summations

Problem 1

Get the forumula for the volume of a cone by approximating an arbitrary cone with cylinders.

Proof. Let C be an arbitrary cone with height h and base radius r. In the plane, the edge of the cone is the line $y = \frac{r}{h}x$. Suppose we split the x-axis into n segments of width h/n. Rotating these segments about the x-axis produces n cylinders, each with volume

$$\pi\left(\frac{k}{n}r\right)^2\frac{h}{n}, \quad 1 \le k \le n$$

Then taking the sum of the volumes of the n cylinders gives

$$\sum_{k=1}^{n}\pi\left(\frac{kr}{n}\right)^2\frac{h}{n} = \frac{\pi r^2 h}{n^3}\sum_{k=1}^{n}k^2 = \frac{\pi r^2 h}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

Simplifying

$$\frac{\pi r^2 h}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{\pi r^2 h}{6}\cdot\frac{2n^3+3n^2+n}{n^3} = \frac{\pi r^2 h}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$$

Taking n to be arbitrary large shows the volume is $\frac{1}{3}\pi r^2 h$. ∎

Problem 2

Rotate the curve $y = 3x$ about the x-axis. What is the volume of the solid obtained?
(a) when $0 \le x \le 2$?
(b) when $0 \le x \le 5$?
(c) when $0 \le x \le c$ with an arbitrary positive number c.

Solution: The solid is a cone.

Solution (a): Let $h = 2$ and $r = 3(2) = 6$. Then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\cdot 6^2\cdot 2 = 24\pi$$

Solution (b): Let $h = 5$ and $r = 3(5) = 15$. Then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\cdot 15^2\cdot 5 = 375\pi$$

Solution (c): Let $h = c$ and $r = 3c$. Then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\cdot (3c)^2\cdot c = 3\pi c^3$$

Problem 3

Rotate the curve $y = \sqrt{x}$ about the x-axis. What is the volume of the solid obtained when $0 \le x \le h$ and h has the value:
(a) $h = 1$?
(b) $h = 2$?
(c) $h = 3$?
(d) arbitrary h?

Proof. Let S be the solid obtained by rotating the curve $y = \sqrt{x}$ about the x-axis from $x = 0$ to $x = h$. In the plane, the edge of the solid is $y = \sqrt{x}$. Suppose we split the x-axis into n segments of width h/n. Rotating these segments about the x-axis produces n cylinders, each with volume

$$\pi\left(\sqrt{\frac{kh}{n}}\right)^2\frac{h}{n} = \pi\frac{kh}{n}\cdot\frac{h}{n} = \pi\frac{kh^2}{n^2}, \quad 1 \le k \le n$$

Then taking the sum of the volumes of the n cylinders gives

$$\sum_{k=1}^{n} \pi \frac{kh^2}{n^2} = \frac{\pi h^2}{n^2} \sum_{k=1}^{n} k = \frac{\pi h^2}{n^2} \cdot \frac{n(n+1)}{2}.$$

Simplifying

$$\frac{\pi h^2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{\pi h^2}{2} \cdot \frac{n+1}{n} = \frac{\pi h^2}{2} \left(1 + \frac{1}{n}\right).$$

Taking n to be arbitrarily large shows the volume is

$$V = \frac{\pi h^2}{2}.$$

∎

Solution (a): Let $h = 1$. Then

$$V = \frac{\pi h^2}{2} = \frac{\pi \cdot 1^2}{2} = \frac{\pi}{2}$$

Solution (b): Let $h = 2$. Then

$$V = \frac{\pi h^2}{2} = \frac{\pi \cdot 2^2}{2} = 2\pi$$

Solution (c): Let $h = 3$. Then

$$V = \frac{\pi h^2}{2} = \frac{\pi \cdot 3^2}{2} = \frac{9\pi}{2}$$

Solution (d): Let h be arbitrary. Then

$$V = \frac{\pi h^2}{2}.$$

Problem 4

Rotate the curve $y = \sqrt{r^2 - x^2}$ about the x-axis. What is the volume of the solid obtained when $0 \leq x \leq r$ has the value:
(a) $r = 1$,
(b) $r = 3$,
(c) $r = 2$,
(d) r is arbitrary?

Proof. Let S be the solid obtained by rotating the curve $y = \sqrt{r^2 - x^2}$ about the x-axis from $x = 0$ to $x = r$. In the plane, the edge of the solid is $y = \sqrt{r^2 - x^2}$. Suppose we split the x-axis into n segments of width r/n. Rotating these segments about the x-axis produces n cylinders, each with volume

$$\pi \left(\sqrt{r^2 - \left(\frac{kr}{n}\right)^2}\right)^2 \frac{r}{n} = \pi \left(r^2 - \frac{k^2 r^2}{n^2}\right) \cdot \frac{r}{n} = \pi \frac{r^3}{n} \left(1 - \frac{k^2}{n^2}\right), \quad 1 \leq k \leq n$$

Then taking the sum of the volumes of the n cylinders gives

$$\sum_{k=1}^{n} \pi \frac{r^3}{n} \left(1 - \frac{k^2}{n^2}\right) = \pi \frac{r^3}{n} \left(\sum_{k=1}^{n} 1 - \sum_{k=1}^{n} \frac{k^2}{n^2}\right) = \pi r^3 \left(1 - \frac{1}{n^3} \sum_{k=1}^{n} k^2\right)$$

Then

$$\pi r^3 \left(1 - \frac{n(n+1)(2n+1)}{6n^3}\right) = \pi r^3 \left(1 - \frac{(n+1)(2n+1)}{6n^2}\right) = \pi r^3 \left(\frac{6n^2 - (2n^2 + 3n + 1)}{6n^2}\right) = \pi r^3 \left(\frac{4n^2 - 3n - 1}{6n^2}\right)$$

Simplifying

$$\pi r^3 \left(\frac{4n^2 - 3n - 1}{6n^2}\right) = \pi r^3 \left(\frac{4n^2}{6n^2} - \frac{3n}{6n^2} - \frac{1}{6n^2}\right) = \pi r^3 \left(\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}\right)$$

Taking n to be arbitrarily large shows the volume is $\frac{2}{3}\pi r^3$. ∎

Solution (a): Let $r = 1$. Then

$$V = \frac{2}{3}\pi r^3 = \frac{2}{3}\pi \cdot 1^3 = \frac{2\pi}{3}$$

Solution (b): Let $r = 3$. Then

$$V = \frac{2}{3}\pi r^3 = \frac{2}{3}\pi \cdot 3^3 = 18\pi$$

Solution (c): Let $r = 2$. Then

$$V = \frac{2}{3}\pi r^3 = \frac{2}{3}\pi \cdot 2^3 = \frac{16\pi}{3}$$

Solution (d): Let r be arbitrary. Then

$$V = \frac{2}{3}\pi r^3$$

Problem 5

Look again at Exercise 4. What is the solid obtained? You should now be able to see that the volume of a spherical ball of radius r is equal to

$$\frac{4}{3}\pi r^3$$

Solution: The solid is a hemisphere. Yep I see it.

16.3 Geometric Series

Problem 2

Argue in a manner similar to that of the text to give a value to the geometric series when $-1 < c \leq 0$. Give the general formula, and also give the specific numerical values of the geometric series for the following numbers c.

(a) $\frac{-1}{3}$
(b) $\frac{-1}{4}$
(c) $\frac{-1}{5}$
(d) $\frac{-1}{6}$
(e) $\frac{-3}{4}$
(f) $\frac{-2}{3}$
(g) $\frac{-2}{5}$
(h) $\frac{-3}{5}$

Proof. Let c be a number $\neq 1$. Consider the sum $\sum_{k=0}^{n} c^k = 1+c+c^2+\cdots+c^n$. Observe that $(1+c+c^2+\cdots+c^n)(1-c) = 1 - c^{n+1}$. Thus $\sum_{k=0}^{n} c^k = \frac{1}{1-c} - \frac{c^{n+1}}{1-c}$. Now, suppose $-1 < c \leq 0$. Suppose $c = \frac{1}{2}$ and observe that $c^2 = \frac{1}{4}$, $c^3 = \frac{-1}{8}$, and $c^4 = \frac{-1}{16}$. We see that the denominator of $|0 - c|$ is increasing by powers of 2. Therefore the fraction c^{n+1} approaches 0. Hence, $\frac{c^{n+1}}{1-c}$ approaches 0 as n becomes arbitrarily large. Thus, $1 + c + c^2 + \cdots + c^n = \frac{1}{1-c}$. ∎

Solution (a):

$$\sum_{k=0}^{n}\left(-\frac{1}{3}\right)^k = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{3}{3+1} = \frac{3}{4}$$

Solution (b):

$$\sum_{k=0}^{n}\left(-\frac{1}{4}\right)^k = \frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{4}{4+1} = \frac{4}{5}$$

Solution (c):

$$\sum_{k=0}^{n}\left(-\frac{1}{5}\right)^k = \frac{1}{1-\left(-\frac{1}{5}\right)} = \frac{5}{5+1} = \frac{5}{6}$$

Solution (d):

$$\sum_{k=0}^{n}\left(-\frac{1}{6}\right)^k = \frac{1}{1-\left(-\frac{1}{6}\right)} = \frac{6}{6+1} = \frac{6}{7}$$

Solution (e):

$$\sum_{k=0}^{n}\left(-\frac{3}{4}\right)^k = \frac{1}{1-\left(-\frac{3}{4}\right)} = \frac{4}{4+3} = \frac{4}{7}$$

Solution (f):

$$\sum_{k=0}^{n}\left(-\frac{2}{3}\right)^k = \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{3}{3+2} = \frac{3}{5}$$

Solution (g):

$$\sum_{k=0}^{n}\left(-\frac{2}{5}\right)^k = \frac{1}{1-\left(-\frac{2}{5}\right)} = \frac{5}{5+2} = \frac{5}{7}$$

Solution (h):

$$\sum_{k=0}^{n}\left(-\frac{3}{5}\right)^k = \frac{1}{1-\left(-\frac{3}{5}\right)} = \frac{5}{5+3} = \frac{5}{8}$$

Problem 3

Let c be a complex number such that $0 \leq |c| < 1$. Again argue in a similar way to give a value to the geometric series.

Proof. Identical to the real case. ∎

Problem 4

What is the value of the sum

$$\sum_{k=0}^{n} c^k$$

where c is equal to $re^{i\theta}$ and $0 \leq r < 1$? Express your answer in the same form as that used to discuss the geometric series. Similarly, give the value of the series

$$\sum_{k=0}^{\infty} c^k$$

where $c = re^{i\theta}$ and $0 \leq r < 1$.

Solution:

$$\sum_{k=0}^{n} c^k = \frac{1-c^{n+1}}{1-c} = \frac{1-r^{n+1}e^{i(n+1)\theta}}{1-re^{i\theta}}.$$

$$\sum_{k=0}^{\infty} c^k = \frac{1}{1-c} = \frac{1}{1-re^{i\theta}}.$$

Problem 5

Let $c = e^{2\pi i/n}$ for some positive integer n. What is the value of

$$1 + c + c^2 + \cdots + c^{n-1}$$

Solution:

$$1 + c + \cdots + c^{n-1} = \frac{1-c^n}{1-c} = \frac{1-1}{1-c} = 0.$$

Problem 6

Let c be a complex number $\neq 1$, such that $c^n = 1$ for some positive integer n. What is the value of

$$1 + c + c^2 + \cdots + c^{n-1}$$

Solution

$$1 + c + \cdots + c^{n-1} = \frac{1-c^n}{1-c} = 0.$$

Problem 7

Consider the sums

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Show that these sums can be made to have arbitrarily large values, for sufficiently large n.

Proof. We can group the terms $1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$. Then for each group $2^m + 1, \ldots, 2^{m+1}$ there are 2^m terms, each at least $\dfrac{1}{2^{m+1}}$, so the block sum is at least $\dfrac{2^m}{2^{m+1}} = \frac{1}{2}$. Thus after k blocks the sum is at least $1 + \frac{1}{2} + k \cdot \frac{1}{2}$. which can be made arbitrarily large by choosing k large. ∎

17 Determinants

17.1 Matrices

In each of the following cases, write down the second row and first column of the indicated matrix. Also write down its transpose.

Problem 1

$$\begin{bmatrix} 2 & -5 \\ -3 & -7 \end{bmatrix}$$

Solution: Second row:

$$(-3, -7)$$

First column:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Transpose:

$$\begin{bmatrix} 2 & -3 \\ -5 & -7 \end{bmatrix}$$

Problem 4

$$\begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 3 \\ 7 & 3 & -2 \end{bmatrix}$$

Solution: Second row:

$$(-1, 2, 3)$$

First column:

$$\begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}$$

Transpose:

$$\begin{bmatrix} 3 & -1 & 7 \\ 5 & 2 & 3 \\ 6 & 3 & -2 \end{bmatrix}$$

Problem 7

Find the sum of the first two columns in the matrix in Exercise 4.

Solution:

$$\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 9 \end{bmatrix}$$

17.2 Determinants of Order 2

Problem 2

Compute the determinant of

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Solution:

$$D\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) = \cos\theta\cos\theta - (-\sin\theta)\sin\theta = \cos^2\theta + \sin^2\theta = 1$$

Problem 3

Compute the determinant

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(a) $\theta = \pi$,
(b) $\theta = \pi/2$
(c) $\theta = \pi/3$

(d) $\theta = \pi/4$

General solution:

$$D\left(\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) = \cos\theta\cos\theta - \sin\theta\sin\theta = cos^2\theta - sin^2\theta$$

Solution (a): $cos^2\pi - sin^2\pi$

Solution (b): $cos^2\pi/2 - sin^2\pi/2$

Solution (c): $cos^2\pi/3 - sin^2\pi/3$

Solution (d): $cos^2\pi/4 - sin^2\pi/4$

17.3 Properties of 2×2 Determinants

Problem 1

Prove the other half of **D1**, i.e. distributivity on the other side other than that given in the text.

Proof.

$$\begin{aligned} D(C, B' + B'') &= D\left(\begin{bmatrix} b_1' + b_1'' & c_1 \\ b_2' + b_2'' & c_2 \end{bmatrix}\right) \\ &= c_2(b_1' + b_1'') - c_1(b_2' + b_2'') \\ &= c_2 b_1' + c_2 b_1' - c_1 b_2' - c_1 b_2'' \\ &= D(C, B') + D(C, B'') \end{aligned}$$

∎

Problem 2

Prove the other half of **D2**.

Proof. We have

$$D(B, xC) = D\left(\begin{bmatrix} b_1 & xc_1 \\ b_2 & xc_2 \end{bmatrix}\right) = xb_1c_2 - xb_2c_1 = x(b_1c_2 - b_2c_1) = xD(B, C)$$

∎

Problem 3

Prove the other half of **D5**.

Proof. Using **D1, D2, D4** we find

$$D(B, C + xB) = D(B, C) + D(B, xB) = D(B, C) + xD(B, B) = D(B, C)$$

∎

Problem 4

Using the same method as at the end of the section, find the value for y.

Proof. Suppose x, y are solutions to the system of equations. We have

$$D(C, A) = D(xA + yB, A) = D(xA, A) + D(yB, A) = xD(A, A) + yD(B, A) = yD(B, A)$$

Thus $y = \frac{D(C,A)}{D(B,A)}$. ∎

Problem 6

Let c be a number, and let A be a 2×2 matrix. Define cA to be the matrix obtained by multiplying all components of A by c. How does $D(cA)$ differ from $D(A)$.

Solution: By **D2** it scales the value of the determinant by c^2.

17.4 Determinants of Order 3

Problem 2 (a)

Compute the following determinants by expandingaccording to the second row, and also according to the third column.
(a)

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{bmatrix}$$

Solution: By second row

$$3(3-8) - (-1)(3-4) = -15 - 1 = -16$$

By third column

$$2(0-12) - (-1)(3-4) + 1(9-0) = -24 - 1 + 9 = -16$$

Problem 4

Let a, b, c be numbers. In terms of a, b, c what is the value of the determinant.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Solution:

$$a(bc) = abc$$

Problem 6

In terms of the components of the matrix, what is the value of the determinant:

$$(a) \begin{bmatrix} a_1 1 & a_1 2 & a_1 3 \\ 0 & a_2 2 & a_2 3 \\ 0 & 0 & a_3 3 \end{bmatrix}, (b) \begin{bmatrix} a_1 1 & 0 & 0 \\ a_2 1 & a_2 2 & 0 \\ a_3 1 & a_3 2 & a_3 3 \end{bmatrix}$$

Solution (a):

$$a_{11}(a_{22}a_{33}) = a_{11}a_{22}a_{33}$$

Solution (b):

$$a_{11}a_{22}a_{33}$$

17.5 Properties of 3×3 Determinants

Problem 1

Write out in full and prove property **D1** with respect to the second and third column.

Theorem 6. *Suppose that the second column can be written as a sum,*

$$A^2 = B + C$$

that is,

$$\begin{bmatrix} a_1 2 \\ a_2 2 \\ a_3 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Then

$$D(A^1, B + C, A^3) = D(A^1, B, A^3) + D(A^1, C, A^3)$$

Proof. We use the definition of the determinant, namely the expansion according to the second column. Each term splits into a sum of two terms corresponding to B and C. For instance,

$$a_{21} \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} = b_1 \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + c_1 \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

$$a_{22} \begin{bmatrix} b_2 + c_2 & a_{13} \\ b_3 + c_3 & a_{33} \end{bmatrix} = a_{22} \begin{bmatrix} b_2 & a_{13} \\ b_3 & a_{33} \end{bmatrix} + a_{22} \begin{bmatrix} c_2 & a_{13} \\ c_3 & a_{33} \end{bmatrix}$$

$$a_{23} \begin{bmatrix} b_2 + c_2 & a_{12} \\ b_3 + c_3 & a_{32} \end{bmatrix} = a_{23} \begin{bmatrix} b_2 & a_{12} \\ b_3 & a_{32} \end{bmatrix} + a_{23} \begin{bmatrix} c_2 & a_{12} \\ c_3 & a_{32} \end{bmatrix}$$

Summing with the appropriate signs yields the desired relation. ∎

Theorem 7. *Suppose that the third column can be written as a sum,*

$$A^3 = B + C$$

that is,

$$\begin{bmatrix} a_1 3 \\ a_2 3 \\ a_3 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Then

$$D(A^1, A^2, B + C) = D(A^1, A^2, B) + D(A^1, A^2, C)$$

Proof. We use the definition of the determinant, namely the expansion according to the third column. Each term splits into a sum of two terms corresponding to B and C. For instance,

$$a_{31} \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} = b_1 \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} + c_1 \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

$$a_{32} \begin{bmatrix} b_2 + c_2 & a_{13} \\ b_3 + c_3 & a_{23} \end{bmatrix} = a_{32} \begin{bmatrix} b_2 & a_{13} \\ b_3 & a_{23} \end{bmatrix} + a_{32} \begin{bmatrix} c_2 & a_{13} \\ c_3 & a_{23} \end{bmatrix}$$

$$a_{33} \begin{bmatrix} b_2 + c_2 & a_{12} \\ b_3 + c_3 & a_{22} \end{bmatrix} = a_{33} \begin{bmatrix} b_2 & a_{12} \\ b_3 & a_{22} \end{bmatrix} + a_{33} \begin{bmatrix} c_2 & a_{12} \\ c_3 & a_{22} \end{bmatrix}$$

Summing with the appropriate signs yields the desired relation. ∎

Problem 2

Same thing for property **D2**.

Theorem 8. *If x is a number, then*
$$D(A^1, xA^2, A^3) = x \cdot D(A^1, A^2, A^3)$$

Proof. We have:
$$D(A^1, xA^2, A^3) = -a_{11}\begin{vmatrix} xa_{22} & a_{23} \\ xa_{32} & a_{33} \end{vmatrix} + a_{21}\begin{vmatrix} xa_{12} & a_{13} \\ xa_{32} & a_{33} \end{vmatrix} - a_{31}\begin{vmatrix} xa_{12} & a_{13} \\ xa_{22} & a_{23} \end{vmatrix} = x \cdot D(A^1, A^2, A^3).$$

∎

Theorem 9. *If x is a number, then*
$$D(A^1, A^2, xA^3) = x \cdot D(A^1, A^2, A^3)$$

Proof. We have:
$$D(A^1, A^2, xA^3) = a_{11}\begin{vmatrix} a_{22} & xa_{23} \\ a_{32} & xa_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & xa_{13} \\ a_{32} & xa_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & xa_{13} \\ a_{22} & xa_{23} \end{vmatrix} = x \cdot D(A^1, A^2, A^3).$$

∎

Problem 3

Prove the two cases not treated in the text for property **D4**.

Proof. Suppose that $A_2 = A_3$, and look at the expansion of the determinant according to the first row. Then $a_{13} = a_{12}$, and the first two terms cancel. The third term is equal to 0 because it involves a 2×2 determinant whose two columns are equal. ∎

Proof. Suppose that $A_1 = A_3$, and look at the expansion of the determinant according to the first row. Then $a_{13} = a_{11}$, and the first two terms cancel. The third term is equal to 0 because it involves a 2×2 determinant whose two columns are equal. ∎

Problem 4

Prove **D5**
(a) you add a multiple of the third column to the first;
(b) you add a multiple of the second column to the first;
(c) you add a third column to the second.

Proof. We have
$$D(A_1 + xA_3, A_2, A_3) = D(A_1, A_2, A_3) + D(xA_3, A_2, A_3) \text{ (by D1)}$$
$$= D(A_1, A_2, A_3) + x \cdot D(A_3, A_2, A_3) \text{ (by D2)} = D(A_1, A_2, A_3) \text{ (by D4)}.$$
We have
$$D(A_1 + xA_2, A_2, A_3) = D(A_1, A_2, A_3) + D(xA_2, A_2, A_3) \text{ (by D1)}$$
$$= D(A_1, A_2, A_3) + x \cdot D(A_2, A_2, A_3) \text{ (by D2)} = D(A_1, A_2, A_3) \text{ (by D4)}.$$
We have
$$D(A_1, A_2 + A_3, A_3) = D(A_1, A_2, A_3) + D(A_1, A_3, A_3) \text{ (by D1)} = D(A_1, A_2, A_3) \text{ (by D4)}.$$

∎

Problem 5

Prove **D6** in the second case.

Proof.

$$\begin{aligned} 0 &= D(A_1, A_2 + A_3, A_2 + A_3) = D(A_1, A_2, A_2 + A_3) + D(A_1, A_3, A_2 + A_3) \\ &= D(A_1, A_2, A_2) + D(A_1, A_2, A_3) + D(A_1, A_3, A_2) + D(A_1, A_3, A_3) = D(A_1, A_2, A_3) + D(A_1, A_3, A_2) \end{aligned}$$

Thus $D(A_1, A_2, A_3) = -D(A_1, A_3, A_2)$. ∎

Problem 6

If you interchange the first and third columns of the given matrix, how does its determinant change? What about interchanging the first and third row?

Solution: The determinant changes by a sign.

Problem 7

State **D5** and **D6** for rows.

Theorem 10 (D5 for rows)**.** *If we add a multiple of one row to another, then the value of the determinant does not change. In other words, let x be a number. Then, for instance,*

$$D(R_1 + xR_2, R_2, R_3) = D(R_1, R_2, R_3),$$

and similarly in all other cases.

Theorem 11 (D6 for rows)**.** *If two adjacent rows are interchanged, then the determinant changes by a sign. In other words, we have*

$$D(R_2, R_1, R_3) = -D(R_1, R_2, R_3),$$

and similarly in the other cases.

Problem 9

Let c be a number and multiply each componenet a_{ij} of a 3×3 matrix A by c, thus obtaining a new matrix which we denote by cA. How does $D(A)$ differ from $D(cA)$.

Solution: The determinant is scaled by c^3.

Problem 10

Let x_1, x_2, x_3 be numbers. Show that

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1)$$

Proof. Then

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = 1 \cdot (x_2 x_3^2 - x_3 x_2^2) - x_1 \cdot (1 \cdot x_3^2 - 1 \cdot x_2^2) + x_1^2 \cdot (1 \cdot x_3 - 1 \cdot x_2) = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1).$$

∎

Problem 13

State the analogous property to that of Exercise 12 with respect to the second column. Then with respect to the third column.

Solution (a): Let
$$A^2 = \sum_{j=1}^{n} x_j C^j$$
where C^j are column vectors and x_j are numbers. Then
$$D(A^1, A^2, A^3, \ldots, A^n) = \sum_{j=1}^{n} x_j D(A^1, C^j, A^3, \ldots, A^n).$$

Solution (b): Let
$$A^3 = \sum_{j=1}^{n} x_j C^j$$
where C^j are column vectors and x_j are numbers. Then
$$D(A^1, A^2, A^3, \ldots, A^n) = \sum_{j=1}^{n} x_j D(A^1, A^2, C^j, \ldots, A^n).$$

17.6 Cramer's Rule

Problem 1

Fill in the missing steps in the proof of Cramer's rule. Cf. Exercises 11 and 12 of the proceeding section.

Proof.
$$\begin{aligned}
D(B, A^2, A^3) &= D(x_1A^1 + x_2A^2 + x_3A^3, A^2, A^3) \\
&= D(x_1A^1 + x_2A^2, A^2, A^3) + D(x_3A^3, A^2, A^3) \\
&= D(x_1A^1, A^2, A^3) + D(x^2A^2, A^2, A^3) + D(x_3A^3, A^2, A^3) \\
&= x_1D(A^1, A^2, A^3) + x_2D(A^2, A^2, A^3) + x_3D(A^3, A^2, A^3) \\
&= x_1D(A^1, A^2, A^3)
\end{aligned}$$

Thus $x_1 = \frac{D(B,A^2,A^3)}{D(A^1,A^2,A^3)}$. ∎

Problem 2

Write out in full the proof of Cramer's rule for x_2 and x_3. It is very similar to the proof for x_1 in the text.

Proof.

$$\begin{aligned} D(B, A^1, A^3) &= D(x_1 A^1 + x_2 A^2 + x_3 A^3, A^1, A^3) \\ &= D(x_1 A^1, A^1, A^3) + D(x_2 A^2 + x_3 A^3, A^1, A^3) \\ &= D(x_1 A^1, A^1, A^3) + D(x_2 A^2, A^1, A^3) + D(x_3 A^3, A^1, A^3) \\ &= x_1 D(A^1, A^1, A^3) + x_2 D(A^2, A^1, A^3) + x_3 D(A^3, A^1, A^3) \\ &= x_2 D(A^2, A^1, A^3) \end{aligned}$$

Thus $x_2 = \frac{D(B,A^1,A^3)}{D(A^2,A^1,A^3)}$. ∎

Proof.

$$\begin{aligned} D(B, A^1, A^2) &= D(x_1 A^1 + x_2 A^2 + x_3 A^3, A^1, A^2) \\ &= D(x_1 A^1 + x_2 A^2, A^1, A^2) + D(x_3 A^3, A^1, A^2) \\ &= D(x_1 A^1, A^1, A^2) + D(x_2 A^2, A^1, A^2) + D(x_3 A^3, A^1, A^2) \\ &= x_1 D(A^1, A^1, A^2) + x_2 D(A^2, A^1, A^2) + x_3 D(A^3, A^1, A^2) \\ &= x_3 D(A^3, A^1, A^2) \end{aligned}$$

Thus $x_2 = \frac{D(B,A^1,A^2)}{D(A^3,A^1,A^2)}$. ∎

Problem 3

Let A^1, A^2, A^3 be columns of a 3×3 matrix A, and assume that there are numbers x_1, x_2, x_3 not all 0 such that

$$x_1 A^1 + x_2 A^2 + x_3 A^3 = 0$$

Prove that $D(A) = 0$.

Proof. Suppose wlog that $x_1 \neq 0$. Then

$$A^1 = -\frac{x_2}{x_1} A^2 - \frac{x_3}{x_1} A^3$$

Clearly, A^1 is a linear combination of A^2 and A^3, so $D(A) = 0$. ∎