Chapter 1 Numbers

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1 The Integers & Rules For Addition

Justify each step, using commutativity and associativity in proving the following identities.

Problem 1

$(a+b)+(c+d)=(a+d)+(b+c)$

Solution:

$$\begin{aligned}
&(a+b)+(c+d) \\
&=((a+b)+c)+d && \text{associative} \\
&=(a+(b+c))+d && \text{associative} \\
&=d+(a+(b+c)) && \text{commutative} \\
&=(d+a)+(b+c) && \text{associative} \\
&=(a+d)+(b+c) && \text{commutative}
\end{aligned}$$

Problem 2

$(a+b)+(c+d)=(a+c)+(b+d)$

Solution:

$$\begin{aligned}
(a+b)+(c+d) &=((a+b)+c)+d && \text{associative} \\
&=(c+(a+b))+d && \text{associative} \\
&=((c+a)+b)+d && \text{commutative} \\
&=(c+a)+(b+d) && \text{associative} \\
&=(a+c)+(b+d) && \text{commutative}
\end{aligned}$$

Problem 3

$(a-b)+(c-d)=(a+c)+(-b-d)$

Solution:

$$\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\
&= (c+(a-b))-d && \text{commutative} \\
&= ((c+a)-b)-d && \text{associative} \\
&= ((a+c)-b)-d && \text{commutative} \\
&= ((a+c)+(-b))+(-d) \\
&= (a+c)+(-b-d) && \text{associative}
\end{aligned}$$

Problem 4

$(a-b)+(c-d)=(a+c)-(b+d)$

Solution:

$$\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\
&= (c+(a-b))-d && \text{commutative} \\
&= ((c+a)+(-b))+(-d) \\
&= (c+a)+((-b)+(-d)) && \text{associative} \\
&= (c+a)-(b+d) \\
&= (a+c)-(b+d) && \text{commutative}
\end{aligned}$$

Problem 5

$(a-b)+(c-d)=(a-d)+(c-b)$

Solution:

$$\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\
&= ((a+(-b))+c)-d \\
&= (a+((-b)+c))-d && \text{associative} \\
&= (((-b)+c)+a)-d && \text{commutative} \\
&= ((-b)+c)+(a-d) && \text{associative} \\
&= (c+(-b))+(a-d) && \text{commutative} \\
&= (c-b)+(a-d) \\
&= (a-d)+(c-b) && \text{commutative}
\end{aligned}$$

Problem 6

$(a-b)+(c-d)=-(b+d)+(a+c)$

Solution:

$$\begin{aligned}
(a-b)+(c-d) &= ((a-b)+c)-d && \text{associative} \\
&= ((a+(-b))+c)+(-d) \\
&= (c+(a+(-b)))+(-d) && \text{commutative} \\
&= ((c+a)+(-b))+(-d) && \text{associative} \\
&= ((a+c)+(-b))+(-d) && \text{commutative} \\
&= (a+c)+((-b)+(-d)) && \text{associative} \\
&= ((-b)+(-d))+(a+c) && \text{commutative} \\
&= (-b-d)+(a+c) \\
&= -(b+d)+(a+c) && \text{distributive property}
\end{aligned}$$

Problem 7

$(a-b)+(c-d) = -(b+d)-(-a-c)$

Solution:

$$\begin{aligned}
(a-b)+(c-d) &= (a+(-b))+(c+(-d)) \\
&= ((a+(-b))+c)+(-d) && \text{associative} \\
&= (c+(a+(-b)))+(-d) && \text{commutative} \\
&= ((c+a)+(-b))+(-d) && \text{associative} \\
&= (c+a)+((-b)+(-d)) && \text{associative} \\
&= ((-b)+(-d))+(c+a) && \text{commutative} \\
&= -(b+d)+(c+a) \\
&= -(b+d)+(-(-c)+-(-a)) \\
&= -(b+d)-(-c-a) \\
&= -(b+d)-(-a-c) && \text{commutative}
\end{aligned}$$

Problem 8

$((x+y)+z)+w = (x+z)+(y+w)$

Solution:

$$\begin{aligned}
((x+y)+z)+w &= (z+(x+y))+w && \text{commutative} \\
&= ((z+x)+y)+w && \text{associative} \\
&= (z+x)+(y+w) && \text{associative} \\
&= (x+z)+(y+w) && \text{commutative}
\end{aligned}$$

Problem 9

$(x - y) - (z - w) = (x + w) - y - z$

Solution:

$$
\begin{aligned}
(x-y)-(z-w) &= (x+(-y))+((-z)+w) \\
&= ((x+(-y))+(-z))+w && \text{associative} \\
&= (x+((-y)+(-z)))+w && \text{associative} \\
&= (((-y)+(-z))+x)+w && \text{commutative} \\
&= ((-y)+(-z))+(x+w) && \text{associative} \\
&= (x+w)+((-y)+(-z)) && \text{commutative} \\
&= (x+w)-y-z
\end{aligned}
$$

Problem 10

$(x - y) - (z - w) = (x - z) + (w - y)$

Solution:

$$
\begin{aligned}
(x-y)-(z-w) &= (x+(-y))+((-z)+w) && \text{distributive} \\
&= ((x+(-y))+(-z))+w && \text{associative} \\
&= (x+((-y)+(-z)))+w && \text{commutative} \\
&= (((-y)+(-z))+x)+w && \text{associative} \\
&= ((-y)+((-z)+x))+w && \text{associative} \\
&= w+((-y)+((-z)+x)) && \text{commutative} \\
&= (w+(-y))+((-z)+x) && \text{associative} \\
&= (w+(-y))+(x+(-z)) && \text{commutative} \\
&= (w-y)+(x-z)
\end{aligned}
$$

Problem 11

Show that $-(a + b + c) = -a + (-b) + (-c)$.

Solution:

$$
\begin{aligned}
-(a+b+c) &= -(a+(b+c)) \\
&= (-a+-(b+c)) && \text{distributive} \\
&= (-a+(-b+(-c))) && \text{distributive} \\
&= -a+(-b)+(-c)
\end{aligned}
$$

Problem 12

Show that $-(a-b-c)=-a+b+c$.

Solution:

$$\begin{aligned} -(a-b-c) &= -(a+(-b)+(-c)) \\ &= (-a-(-b)-(-c)) && \text{distributive} \\ &= (-a+b+c) && \text{double negation} \\ &= -a+b+c \end{aligned}$$

Problem 13

Show that $-(a-b)=b-a$.

Solution:

$$\begin{aligned} -(a-b) &= (-a)-(-b) && \text{distributive} \\ &= -a+b && \text{double negation} \\ &= b+(-a) && \text{commutative} \\ &= b-a \end{aligned}$$

Solve for x in the following equations.

Problem 14

$-2+x=4$

Solution:

$$\begin{aligned} -2+x &= 4 \\ = -2+2+x &= 4+2 \\ = x &= 6 \end{aligned}$$

Problem 19

$-5-x=-2$

Solution:

$$\begin{aligned}
-5 - x = -2 &\Leftrightarrow && \\
(-5 - x) + x = -2 + x &\Leftrightarrow && \\
-5 + ((-x) + x) = -2 + x &\Leftrightarrow && \text{associative} \\
-5 + 0 = -2 + x &\Leftrightarrow && \text{N2} \\
-5 = -2 + x &\Leftrightarrow && \text{N1} \\
-5 + 2 = (-2 + x) + 2 &\Leftrightarrow && \\
-5 + 2 = 2 + (-2 + x) &\Leftrightarrow && \text{commutative} \\
-5 + 2 = (2 + (-2)) + x &\Leftrightarrow && \text{associative} \\
-5 + 2 = 0 + x &\Leftrightarrow && \text{N2} \\
-3 = x & && \text{N1}
\end{aligned}$$

Problem 20

$-7 + x = -10$

Solution:

$$\begin{aligned}
-7 + x = -10 &\Leftrightarrow && \\
(-7 + x) + 7 = -10 + 7 &\Leftrightarrow && \\
7 + (-7 + x) = -3 &\Leftrightarrow && \text{commutative} \\
(7 + (-7)) + x = -3 &\Leftrightarrow && \text{associative} \\
0 + x = -3 &\Leftrightarrow && \text{N2} \\
x = -3 & &&
\end{aligned}$$

Problem 21

$-3 + x = 4$

Solution:

$$\begin{aligned}
-3 + x = 4 &\Leftrightarrow && \\
(-3 + x) + 3 = 4 + 3 &\Leftrightarrow && \\
3 + (-3 + x) = 7 &\Leftrightarrow && \text{commutative} \\
(3 + (-3)) + x = 7 &\Leftrightarrow && \text{associative} \\
0 + x = 7 &\Leftrightarrow && \text{N2} \\
x = 7 & &&
\end{aligned}$$

22 Prove the cancellation law for addition

If $a + b = a + c$ then $b = c$.

Solution:

$$\begin{aligned}
a + b &= a + c \Leftrightarrow && \\
(a + b) + (-a) &= (a + c) + (-a) \Leftrightarrow && \\
-a + (a + b) &= -a + (a + c) \Leftrightarrow && \text{commutative} \\
(-a + a) + b &= (-a + a) + c \Leftrightarrow && \text{associative} \\
0 + b &= 0 + c \Leftrightarrow && \text{N2} \\
b &= c &&
\end{aligned}$$

23 Prove

If $a + b = a$, then $b = 0$.

Solution:

$$\begin{aligned}
a + b &= a \Leftrightarrow && \\
(a + b) + (-a) &= a + (-a) \Leftrightarrow && \\
(-a) + (a + b) &= a - a && \text{commutative} \\
(-a) + (a + b) &= 0 && \text{N2} \\
((-a) + a) + b &= 0 \Leftrightarrow && \text{associative} \\
0 + b &= 0 && \text{N2} \\
b &= 0 &&
\end{aligned}$$

2 Rules For Multiplication

Express each of the following expressions in the form $2^m 3^n a^r b^s$, where m, n, r, s are positive integers.

Problem 1

(a) $8a^2b^3(27a^4)(2^5ab)$
(b) $16b^3a^2(6ab^4)(ab)^3$
(c) $3^2(2ab)^3(16a^2b^5)(24b^2a)$
(d) $24a^3(2ab^2)^3(3ab)^2$
(e) $(3ab)^2(27a^3b)(16ab^5)$
(f) $32a^4b^5a^3b^2(6ab^3)^4$

Solution: 1 (a)

$$\begin{aligned}
8a^2b^3(27a^4)(2^5ab) &= 8(27a^4)a^2b^3(2^5ab) && \text{commutative}\\
&= (8\cdot 27)a^4a^2b^3(2^5ab) && \text{associative}\\
&= (8\cdot 27)(2^5ab)a^4a^2b^3 && \text{commutative}\\
&= (8\cdot 27\cdot 2^5)aba^4a^2b^3 && \text{associative}\\
&= (8\cdot 27\cdot 2^5)aa^4a^2bb^3 && \text{commutative}\\
&= (2^33^32^5)a^7b^4 && \text{N11}\\
&= (2^32^53^3)a^7b^4 && \text{commutative}\\
&= (2^83^3)a^7b^4 && \text{N11}\\
&= 2^83^3a^7b^4
\end{aligned}$$

Solution: 1 (b)

$$\begin{aligned}
16b^3a^2(6ab^4)(ab)^3 &= b^3a^2(ab)^3(6ab^4)16 && \text{commutative}\\
&= b^3a^2(ab)^3 6(ab^4)16 && \text{associative}\\
&= b^3a^2(ab)^3(ab^4)16\cdot 6 && \text{commutative}\\
&= b^3a^2a^3b^3ab^4 16\cdot 6 && \text{N12}\\
&= a^2a^3ab^3b^3b^4 16\cdot 6 && \text{commutative}\\
&= a^2a^3ab^3b^3b^4 2^4 2\cdot 3\\
&= a^6b^{10}2^53 && \text{N11}\\
&= 2^53a^6b^{10} && \text{commutative}
\end{aligned}$$

Solution: 1 (c)

$$\begin{aligned}
3^2(2ab)^3(16a^2b^5)(24b^2a) &= 3^22^3a^3b^3(16a^2b^5)(24b^2a) && \text{N12}\\
&= 2^324\cdot 3^216a^3a^2ab^3b^5b^2 && \text{commutative}\\
&= 2^33\cdot 2^33^22^4a^3a^2ab^3b^5b^2\\
&= 2^32^32^43^23a^3a^2ab^3b^5b^2 && \text{associative}\\
&= 2^{10}3^3a^6b^{10} && \text{N11}
\end{aligned}$$

Solution: 1 (d)

$$\begin{aligned}
24a^3(2ab^2)^3(3ab)^2 &= 24a^32^3a^3(b^2)^33^2a^2b^2 && \text{N12}\\
&= 24a^32^3a^3b^63^2a^2b^2 && \text{N12}\\
&= 24\cdot 2^33^2a^3a^3a^2b^6b^2 && \text{commutative}\\
&= 2^33\cdot 2^33^2a^3a^3a^2b^6b^2\\
&= 2^32^33^23a^3a^3a^2b^6b^2 && \text{commutative}\\
&= 2^63^3a^8b^8 && \text{N11}
\end{aligned}$$

Solution: 1 (e)

$$
\begin{aligned}
(3ab)^2(27a^3b)(16ab^5) &= 3^2a^2b^2 27a^3b16ab^5 && \text{N12}\\
&= 27\cdot 16\cdot 3^2a^2a^3ab^2bb^5 && \text{commutative}\\
&= 3^3 2^4 3^2a^2a^3ab^2bb^5 \\
&= 2^4 3^3 3^2a^2a^3ab^2bb^5 && \text{commutative}\\
&= 2^4 3^5 a^6 b^8 && \text{N11}
\end{aligned}
$$

Solution: 1 (f)

$$
\begin{aligned}
32a^4b^5a^3b^2(6ab^3)^4 &= 32a^4b^5a^3b^2 6^4a^4(b^3)^4 && \text{N12}\\
&= 32a^4b^5a^3b^2 6^4a^4b^{12} && \text{N12}\\
&= 6^4 32a^3a^4a^4b^5b^2b^{12} && \text{commutative}\\
&= (2\cdot 3)^4 2^5a^3a^4a^4b^5b^2b^{12} \\
&= 2^4 3^4 2^5a^3a^4a^4b^5b^2b^{12} && \text{N12}\\
&= 2^4 2^5 3^4a^3a^4a^4b^5b^2b^{12} && \text{commutative}\\
&= 2^9 3^4 a^{11} b^{19} && \text{N11}
\end{aligned}
$$

Problem 2

Prove
(a) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
(b) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Solution: 2 (a)

$$
\begin{aligned}
(a+b)^3 &= (a+b)(a+b)(a+b) \\
&= ((a+b)(a+b))(a+b) && \text{associative}\\
&= (a(a+b) + b(a+b))(a+b) && \text{distributive}\\
&= (a^2 + ab + ba + b^2)(a+b) && \text{distributive}\\
&= (a^2 + 2ab + b^2)(a+b) \\
&= a^2(a+b) + 2ab(a+b) + b^2(a+b) && \text{distributive}\\
&= a^2a + a^2b + 2aba + 2abb + b^2a + b^2b && \text{distributive}\\
&= a^3 + a^2b + 2a^2b + 2ab^2 + b^2a + b^3 && \text{N11}\\
&= a^3 + 3a^2b + 3ab^2 + b^3
\end{aligned}
$$

Solution: 2 (b)

$$\begin{aligned}
(a-b)^3 &= (a-b)(a-b)(a-b) \\
&= ((a-b)(a-b))(a-b) && \text{associative} \\
&= (a(a-b) - b(a-b))(a-b) && \text{distributive} \\
&= (a^2 - ab - ba + b^2)(a-b) && \text{distributive} \\
&= (a^2 - 2ab + b^2)(a-b) \\
&= a^2(a-b) - 2ab(a-b) + b^2(a-b) && \text{distributive} \\
&= a^2a - a^2b - 2aba + 2abb + b^2a - b^2b && \text{distributive} \\
&= a^3 - a^2b - 2a^2b + 2ab^2 + b^2a - b^3 && \text{N11} \\
&= a^3 - 3a^2b + 3ab^2 - b^3
\end{aligned}$$

Problem 3

Obtain expansions for $(a+b)^4$ and $(a-b)^4$.

Solution: 3

From 2: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

$$\begin{aligned}
(a+b)^3 * (a+b) &= (a+b)^4 && \text{N11} \\
&= ((a^3 + 3a^2b) + (3ab^2 + b^3)) * (a+b) \\
&= (a^3 + 3a^2b)(a+b) + (3ab^2 + b^3)(a+b) && \text{distributive} \\
&= (a^3(a+b) + 3a^2b(a+b)) + (3ab^2(a+b) + b^3(a+b)) && \text{distributive} \\
&= (aa^3 + ba^3) + (a3a^2b + b3a^2b) + (a3ab^2 + b3ab^2) + (ab^3 + bb^3) && \text{distributive} \\
&= (a^4 + a^3b) + (3a^3b + 3a^2b^2) + (3a^2b^2 + 3ab^3) + (ab^3 + b^4) && \text{N11} \\
&= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}$$

From prev: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

$$\begin{aligned}
(a-b)^4 &= (a + (-b))^4 \\
&= a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + (-b)^4 \\
&= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4
\end{aligned}$$

Problem 5

$(1-2x)^2$

Solution

$$\begin{aligned}
(1-2x)^2 &= (1-2x)*(1-2x) \\
&= (1(1-2x) - 2x(1-2x)) && \text{distributive} \\
&= ((1-2x) - (2x - 2x2x)) && \text{distributive} \\
&= (1-2x) - (2x - 4x^2) && \text{N11} \\
&= ((1-2x) - 2x) - 4x^2 && \text{associative} \\
&= (1 + ((-2x) - 2x)) - 4x^2 && \text{associative} \\
&= (1 + (-4x)) - 4x^2 \\
&= 1 - 4x - 4x^2
\end{aligned}$$

Problem 7

$(x-1)^2$

Solution

$$\begin{aligned}
(x-1)^2 &= (x-1)\cdot(x-1) \\
&= x^2 - 2x + 1 \quad \text{perfect square}
\end{aligned}$$

Problem 11

$(1+x^3)(1-x^3)$

Solution

$$(1+x^3)(1-x^3) = (1-x^6) \quad \text{difference of squares}$$

Problem 13

$(x^2-1)^2$

Solution

$$(x^2-1)^2 = x^4 - 2x^2 + 1 \quad \text{perfect square}$$

Problem 17

$(x^3-4)(x^3+4)$

Solution

$$(x^3-4)(x^3+4) = x^6 - 16 \quad \text{difference of squares}$$

Problem 19

$(-2+3x)(-2-3x)$

Solution

$$(-2+3x)(-2-3x) = 4 - 9x^2 \quad \text{difference of squares}$$

Problem 23

$(-1-x)(-2+x)(1-2x)$

Solution

$$\begin{aligned}
(-1-x)(-2+x)(1-2x) &= (2+x-x^2)(1-2x) && \text{distributive}\\
&= (2(1-2x)+x(1-2x)-x^2(1-2x)) && \text{distributive}\\
&= 2-4x+x-2x^2-x^2+2x^3 && \text{distributive}\\
&= 2-3x-3x^2+2x^3
\end{aligned}$$

Problem 29

$(2x+1)^2(2-3x)$

Solution

$$\begin{aligned}
(2x+1)^2(2-3x) &= (4x^2+4x+1)(2-3x) && \text{perfect square}\\
&= (4x^2(2-3x)+4x(2-3x)+1(2-3x)) && \text{distributive}\\
&= (8x^2-12x^3+8x-12x^2+2-3x) && \text{distributive}\\
&= (-12x^3-4x^2+5x+2)
\end{aligned}$$

Problem 30

The population of a city in 1910 was $50,000$, and it doubles every 10 years. What will it be (a) in 1970 (b) in 1990 (c) in $2,000$?

Solution

$$\begin{aligned}
&(a)\ 50000 * 2^{((1970-1910)/10)} = 3200000\\
&(b)\ 50000 * 2^{((1990-1910)/10)} = 12800000\\
&(c)\ 50000 * 2^{((2000-1910)/10)} = 25600000
\end{aligned}$$

Problem 31

The population of a city in 1905 was $100,000$, and it doubles every 25 years. What will it be after (a) 50 years (b) 100 years (c) 150 years?

Solution

$$\begin{aligned}
&(a)\ 100000 * 2^{(50/25)} = 400000\\
&(b)\ 100000 * 2^{(100/25)} = 1600000\\
&(c)\ 100000 * 2^{(150/25)} = 6400000
\end{aligned}$$

Problem 32

The population of a city was 200 thousand in 1915, and it triples every 50 years. What will be the population What will be the population
(a) in the year 2215?
(b) in the year 2165?

Solution

$$\begin{aligned}
&(a)\ 200000 * 3^{((2215-1915)/50)} = 145800000\\
&(b)\ 200000 * 3^{((2165-1915)/50)} = 48600000
\end{aligned}$$

Problem 33

The population of a city was $25,000$ in 1870, and it triples every 40 years. What will it be.
(a) in 1990?
(b) in 2030?

Solution

$$\begin{aligned}
&(a)\ 25000 * 3^{((1990-1870)/40)} = 675000\\
&(b)\ 25000 * 3^{((2030-1870)/40)} = 2025000
\end{aligned}$$

3 Even and Odd Integers; Divisibility

Problem 1

Give the proofs for the cases of theorem 1 which were not proved in the text.
(a) If a is even and b is even, then $a+b$ is even.
(b) If a is odd and b is even, then $a+b$ is odd.
(c) If a is odd and b is odd, then $a+b$ is even.

Solution (a)

Since a and b are even they can be written as $2n_1$ and $2n_2$ respectively, where n_1 and n_2 are integers.
Let $x = n_1 + n_2$. Note x is an integer because the sum of two integers is an integer.

$$\begin{aligned} a+b &= 2n_1 + 2n_2 \\ &= 2(n_1+n_2) \\ &= 2x \end{aligned}$$

Since $a+b$ can be written as $2x$ where x is an integer; $a+b$ is even.

Solution (b)

$$\begin{aligned} a+b &= 2n_1 + 1 + 2n_2 \\ &= 2n_1 + 2n_2 + 1 \\ &= 2(n_1+n_2) + 1 && \text{let } x = n_1 + n_2 \\ &= 2x + 1 \end{aligned}$$

Solution (c)

$$\begin{aligned} a+b &= 2n_1 + 1 + 2n_2 + 1 \\ &= 2n_1 + 2n_2 + 2 \\ &= 2(n_1+n_2+1) && \text{let } x = n_1 + n_2 + 1 \\ &= 2x \end{aligned}$$

Problem 2

If a is even and b is any positive integer then ab is even.

Proof. By def. of an even number a can be written as $2n$ where n is an integer.
Let $x = n \cdot b$. Note the product of two integers is an integer ig.
Something about multiplication being repeated addition and the sum of two integers being an integer.

$$\begin{aligned} a \cdot b &= 2n \cdot b \\ &= 2x \end{aligned}$$

Since ab can be written as $2x$ where x is an integer ab is even. □

Problem 3

If a is even, then a^3 is even.

Proof. By def. of an even number a can be written as $2n$ where n is an integer.
Let $x = 2^2 n^3$. Note x is an integer.

$$\begin{aligned} a^3 &= (2n)^3 & \\ &= 2^3 n^3 && \text{N12} \\ &= 2 \cdot 2^2 n^3 && \text{N11} \\ &= 2x \end{aligned}$$

Since a^3 can be written as $2x$ where x is an integer a^3 is even. $\square$

Problem 4

If a is odd, then a^3 is odd.

Proof. By def. of an odd number a can be written as $2n+1$ where n is an integer.
Let $x = 4n^3 + 6n^2 + 3n$. Note x is an integer.

$$\begin{aligned} a^3 &= (2n+1)^3 & \\ &= 8n^3 + 12n^2 + 6n + 1 && \text{distributive} \\ &= 2(4n^3 + 6n^2 + 3n) + 1 && \text{distributive} \\ &= 2x + 1 \end{aligned}$$

Since a^3 can be written as $2x+1$ where x is an integer a^3 is odd. $\square$

Problem 5

If n is even, then $(-1)^n = 1$.

Proof. By def. of an even number n can be written as $2a$ where a is an integer.

$$\begin{aligned} (-1)^n &= (-1)^{2a} & \\ &= ((-1)^2)^a && \text{N12} \\ &= 1^a \\ &= 1 \end{aligned}$$

$\square$

Problem 6

If n is odd, then $(-1)^n = -1$.

Proof. By def. of an odd number n can be written as $2a+1$ where a is an integer.

$$\begin{aligned} (-1)^n &= (-1)^{2a+1} \\ &= (-1)^{2a} \cdot (-1)^1 && \text{N11} \\ &= 1 \cdot (-1) && \text{2a is even so by prob. 5} \\ &= -1 && \text{N7} \end{aligned}$$

$\square$

Problem 7

If m, n are odd, then the product mn is odd.

Proof. By def. of an odd number m and n can be written as $2n_1+1$ and $2n_2+1$ where n_1 and n_2 are integers.
Let $x = 2n_1n_2 + n_1 + n_2$. Note x is an integer.

$$\begin{aligned} mn &= (2n_1+1)(2n_2+1) \\ &= 4n_1n_2 + 2n_1 + 2n_2 + 1 && \text{distributive} \\ &= 2(2n_1n_2 + n_1 + n_2) + 1 && \text{distributive} \\ &= 2x+1 \end{aligned}$$

Since mn can be written as $2x+1$ where x is an integer, therefore mn is odd. $\square$

Problem 24

Let a, b be integers, Define $a \equiv b \ (mod\ 5)$, which we read "a is congruent to b modulo 5, to mean that $a-b$ is divisible by 5.
Prove if $a \equiv b \ (mod\ 5)$ and $x \equiv y \ (mod\ 5)$ then $a+x \equiv b+y \ (mod\ 5)$ and $ax = by \ (mod\ 5)$.

Proof. Need to show $(a+x)-(b+y) = 5n$ where n is an integer.
From $a \equiv b \ (mod\ 5)$, $a-b = 5n_1$ where n_1 is an integer.
From $x \equiv y \ (mod\ 5)$, $x-y = 5n_2$ where n_2 is an integer.
Let $t = n_1 + n_2$.

$$\begin{aligned} (a+x)-(b+y) &= (a-b)+(x-y) \\ &= 5n_1 + 5n_2 \\ &= 5(n_1+n_2) \\ &= 5t \end{aligned}$$

Since $(a+x)-(b+y) = 5t$ where t is an integer, $a+x \equiv b+y \ (mod\ 5)$. $\square$

Proof. Need to show $ax - by = 5n$ where n is an integer.
From $a \equiv b \ (mod\ 5)$, $a-b = 5n_1$ where n_1 is an integer.

From $x \equiv y \ (mod\ 5)$, $x - y = 5n_2$ where n_2 is an integer.
Let $t = bn_2 - yn_1 - 5n_1n_2$.

$$\begin{aligned} ax - by &= (b - 5n_1)(y + 5n_2) - by \\ &= by + 5bn_2 - 5yn_1 - 25n_2n_1 - by \\ &= 5bn_2 - 5yn_1 - 25n_2n_1 \\ &= 5(bn_2 - yn_1 - 5n_2n_1) \\ &= 5t \end{aligned}$$

Since $ax - by = 5t$ where t is an integer, $ax = by \ (mod\ 5)$. □

Problem 25

Let d be a positive integer. Let a, b be integers.
Define $a \equiv b \ (mod\ d)$ to mean that $a - b$ is divisible by d.
Prove that if $a \equiv b \ (mod\ d)$ and $x \equiv y \ (mod\ d)$, then $a + x \equiv b + y \ (mod\ d)$ and $ax = by \ (mod\ d)$.

Proof. Need to show $(a + x) - (b + y) = dn$ where n is an integer.
From $a \equiv b \ (mod\ d)$, $a - b = dn_1$ where n_1 is an integer.
From $x \equiv y \ (mod\ d)$, $x - y = dn_2$ where n_2 is an integer.

$$\begin{aligned} (a + x) - (b + y) &= (a - b) + (x - y) \\ &= dn_1 + dn_2 \\ &= d(n_1 + n_2) && \text{let } t = n_1 + n_2 \\ &= dt \end{aligned}$$

Since $(a + x) - (b + y)$ can be written as dt where t is an integer, $a + x \equiv b + y \ (mod\ d)$. □

Proof. Need to show $ax - by = dn$.
From $a \equiv b \ (mod\ d)$, $a - b = dn_1$ where n_1 is an integer.
From $x \equiv y \ (mod\ d)$, $x - y = dn_2$ where n_2 is an integer.

$$\begin{aligned} ax - by &= (b + dn_1)(y + dn_2) - by \\ &= by + bdn_2 + ydn_1 + dbn_1n_2 - by \\ &= bdn_2 + ydn_1 + dbn_1n_2 \\ &= d(bn_2 + yn_1 + bn_1n_2) && \text{Let } t = bn_2 + yn_1 + bn_1n_2 \\ &= dt \end{aligned}$$

Since $ax - by$ can be written as dt where t is an integer, $ax = by \ (mod\ d)$. □

Problem 26

Assume that every positive integer can be written in one of the forms $3k$, $3k+1$, or $3k+2$ for some integer k.
Show that if the square of a positive integer is divisible by 3, then so is the integer x.

Proof. From the assumptions x can either be written $3k$, $3k+1$, or $3k+2$.
Need to show that if $x^2 = 3n_1$, $x = 3n_2$ for some integers n_1 and n_2.
Case 1 ($x = 3k$):
Let $t_1 = 3k^2$

$$\begin{aligned} x^2 &= (3k)^2 \\ &= 3 \cdot 3k^2 \\ &= 3t_1 \end{aligned}$$

Therefore in this case x is divisible by 3.
Case 2 ($x = 3k+1$):
Let $t_2 = 2k^2 + 2k$

$$\begin{aligned} x^2 &= (3k+1)^2 \\ &= 6k^2 + 3k + 3k + 1 \\ &= 6k^2 + 6k + 1 \\ &= 3(2k^2 + 2k) + 1 \\ &= 3t_2 + 1 \end{aligned}$$

In this case x^2 is not divisible by 3 which contradicts our assumption, therefore $x \neq 3k+1$.
Case 3 ($x = 3k+2$):
Let $t_3 = 3k^2 + 4k$

$$\begin{aligned} x^2 &= (3k+2)^2 \\ &= 9k^2 + 6k + 6k + 4 \\ &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k) + 4 \\ &= 3t_3 + 4 \end{aligned}$$

In this case x^2 is not divisible by 3 which contradicts our assumption, therefore $x \neq 3k+2$.

Note there is no solution for $1 = 3m_1$ or $2 = 3m_2$ where m_1 and m_2 are integers.
Assume $3k_1 + 1 = 3m_1$ where k_1 and m_1 are integers.

$$\begin{aligned} 3k_1 + 1 &= 3m_1 \\ 1 &= 3m_1 - 3k_1 \\ 1 &= 3(m_1 - k_1) \end{aligned}$$

Therefore, $3k_1 + 1$ is not divisible by 3.

Assume $3k_2 + 2 = 3m_2$ where k_2 and m_2 are integers.

$$\begin{aligned} 3k_2 + 1 &= 3m_2 \\ 2 &= 3m_2 - 3k_2 \\ 2 &= 3(m_2 - k_2) \end{aligned}$$

Therefore, $3k_2 + 2$ is not divisible by 3.

□

4 Rational Numbers

Problem 4

Let $a = \frac{m}{n}$ be a rational number expressed as a quotient of integers m, n with $m \neq 0$ and $n \neq 0$.
Show that there is a rational number b such that $ab = ba = 1$.

Proof. Let $b = \frac{n}{m}$. Since n and m are integers and $m \neq 0$, b is the ratio of two integers where the denomitor is not 0 making it a rational number by definition.

$$\begin{aligned} ab &= \frac{m}{n} \cdot \frac{n}{m} \\ &= \frac{mn}{nm} \\ &= \frac{nm}{nm} && \text{commutative} \\ &= 1 && \text{cancellation rule for fractions} \end{aligned}$$

$$\begin{aligned} ba &= \frac{n}{m} \cdot \frac{m}{n} \\ &= \frac{nm}{mn} \\ &= \frac{nm}{nm} && \text{commutative} \\ &= 1 && \text{cancellation rule for fractions} \end{aligned}$$

Therefore $ab = ba = 1$. □

Problem 6

Solve for x in the following equations.

Solution (d)

$$\begin{aligned}
\frac{4x}{3} + \frac{3}{4} &= 2x - 5 \\
12\left(\frac{4x}{3} + \frac{3}{4}\right) &= 12(2x - 5) \\
16x + 9 &= 24x - 60 \\
9 + 60 &= 24x - 16x \\
69 &= 8x \\
x &= \frac{69}{8}
\end{aligned}$$

Solution (e)

$$\begin{aligned}
\frac{4(1 - 3x)}{7} &= 2x - 1 \\
4(1 - 3x) &= 7(2x - 1) \\
4 - 12x &= 14x - 7 \\
4 + 7 &= 14x + 12x \\
11 &= 26x \\
x &= \frac{11}{26}
\end{aligned}$$

Solution (f)

$$\begin{aligned}
\frac{2 - x}{3} &= \frac{7}{8}x \\
8(2 - x) &= 3 \cdot 7x \\
16 - 8x &= 21x \\
16 &= 29x \\
x &= \frac{16}{29}
\end{aligned}$$

Problem 6

Let n be a positive integer. By n factorial, written $n!$, we mean the product:
$1 \cdot 2 \cdot 3 \ldots n$
of the first n positive integers. For Instance
$2! = 2$
$3! = 2 \cdot 3 = 6$
$4! = 2 \cdot 3 \cdot 4 = 24$
(a) Find the value 5!, 6!, 7!, and 8!.
(b) Define $0! = 1$. Define the binomial coefficient
$\binom{n}{m} = \frac{m!}{n!(m-n)!}$
for any natural numbers m, n such that n lies between 0 and m. Compute tons of binomial coefficients.

(c) Show that $\binom{m}{n} = \binom{m}{m-n}$.
(d) Show that if n is a positive integer at most equal to m, then
$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$.

Solution (a)

$$
\begin{aligned}
5! &= 2 \cdot 3 \cdot 4 \cdot 5 && = 120 \\
6! &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 && = 720 \\
7! &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 && = 5040 \\
8! &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 && = 40320
\end{aligned}
$$

Solution (b)

$$\binom{3}{0} = \frac{3!}{0!(3-0)!} = 1$$
$$\binom{3}{1} = \frac{3!}{1!(3-1)!} = 3$$
$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$$
$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = 1$$
$$\binom{4}{0} = \frac{4!}{0!(4-0)!} = 1$$
$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = 4$$
$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$$
$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$$
$$\binom{4}{4} = \frac{4!}{4!(4-4)!} = 1$$
$$\binom{5}{0} = \frac{5!}{0!(5-0)!} = 1$$
$$\binom{5}{1} = \frac{5!}{1!(5-1)!} = 5$$
$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10$$
$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$$
$$\binom{5}{4} = \frac{5!}{4!(5-4)!} = 5$$
$$\binom{5}{5} = \frac{5!}{5!(5-5)!} = 1$$

Solution (c)

$$\begin{aligned}
\binom{m}{n} &= \binom{m}{m-n} \\
\frac{m!}{n!(m-n)!} &= \frac{m!}{(m-n)!(m-(m-n))!} \\
\frac{m!}{n!(m-n)!} &= \frac{m!}{(m-n)!(n)!} \\
\frac{m!}{n!(m-n)!} &= \frac{m!}{(n)!(m-n)!}
\end{aligned}$$

Solution (d)
Need to show:

$$\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$$

First note:

$$\binom{m+1}{n} = \frac{(m+1)!}{n!((m+1)-n)!}$$

Then:

$$\begin{aligned}
\binom{m}{n} + \binom{m}{n-1} &= \frac{m!}{n!(m-n)!} + \frac{m!}{(n-1)!(m-n+1)!} \\
&= \frac{m!(m-n+1) + m!n}{n!(m-n+1)!} \quad \text{(common denominator)} \\
&= \frac{m!(m-n+1+n)}{n!(m-n+1)!} \\
&= \frac{m!(m+1)}{n!(m-n+1)!} \\
&= \frac{(m+1)!}{n!((m+1)-n)!} \\
&= \binom{m+1}{n}
\end{aligned}$$

Problem 8

Prove that there is no positive rational number a such that $a^3 = 2$.

Proof. Let $a = \frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form. If $(\frac{m}{n})^3 = 2$.

$$\begin{aligned}
\frac{m^3}{n^3} &= 2 \\
m^3 &= 2n^3
\end{aligned}$$

If $m^3 = 2k$ for some integer k, then $m = 2a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned}
(2k)^3 &= 2n^3 \\
2^3k^3 &= 2n^3 \\
2^2k^3 &= n^3 \\
2 \cdot (2k^3) &= n^3
\end{aligned}$$

If $n^3 = 2k$ for some integer k, then $n = 2a$ for some integer a (shown in a previous problem). This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^3 = 2$. □

Problem 9

Prove that there is no positive rational number a such that $a^4 = 2$.

Proof. Suppose for contradiction $a^4 = 2$ where a is a rational number. Since a is rational, it can be expressed as $\frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form.

$$\begin{aligned}
\frac{m^4}{n^4} &= 2 \\
m^4 &= 2n^4
\end{aligned}$$

If $m^4 = 2k$ for some integer k, then $m = 2a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned}
(2k)^4 &= 2n^4 \\
2^4k^4 &= 2n^4 \\
2^3k^4 &= n^4 \\
2 \cdot (2^2k^3) &= n^4
\end{aligned}$$

If $n^4 = 2k$ for some integer k, then $n = 2a$ for some integer a (shown in a previous problem). This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^4 = 2$. □

Problem 10

Prove that there is no positive rational number a such that $a^2 = 3$. You may assume that a positive integer can be written in one of the forms $3k$, $3k + 1$, $3k + 2$ for some integer k. Prove that if the square of a positive integer is divisible by 3 so is the integer. Then use a similar proof for $\sqrt{2}$.

Proof. Since a is rational, it can be expressed as $\frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form.

$$\begin{aligned}\frac{m^2}{n^2} &= 3\\ m^2 &= 3n^2\end{aligned}$$

If $m^2 = 3k$ for some integer k, then $m = 3a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned}(3a)^2 &= 3n^2\\ 3^2a^2 &= 3n^2\\ 3^3a^2 &= n^2\\ 3\cdot(3^2a^2) &= n^2\end{aligned}$$

If $n^2 = 3k$ for some integer k, then $n = 3a$ for some integer a. This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^2 = 3$. $\square$

Proof. Need to show that $a^2 = 2$ has no rational solution a. Since a is rational, it can be expressed as $\frac{m}{n}$ where m, n are integers, $n \neq 0$, and $\frac{m}{n}$ is in its lowest form.

$$\begin{aligned}\frac{m^2}{n^2} &= 2\\ m^2 &= 2n^2\end{aligned}$$

If $m^2 = 2k$ for some integer k, then $m = 2a$ for some integer a (shown in a previous problem). But:

$$\begin{aligned}(2a)^2 &= 2n^2\\ 2^2a^2 &= 2n^2\\ 2a^2 &= n^2\\ 2\cdot a^2 &= n^2\end{aligned}$$

If $n^2 = 2k$ for some integer k, then $n = 2a$ for some integer a. This contradicts our assumption that $\frac{m}{n}$ is in its lowest form, therefore there is no positive rational number a such that $a^2 = 2$. $\square$

Problem 16

A chemical substance decomposes in such a way that it halves every 3 min. If there are 6 grams (g) of the substance at present at the beginning, how much will will be left
(a) after 3 min?
(b) after 27 min?
(c) after 36 min?

Solution
(a) after 3 min? $6(\frac{1}{2})^{(\frac{3}{3})} = 3\text{g}$
(b) after 27 min? $6(\frac{1}{2})^{(\frac{27}{3})} = 0.01171875\text{g}$
(c) after 36 min? $6(\frac{1}{2})^{(\frac{36}{3})} = 0.001464843\text{g}$

Problem 18

A substance reacts in water in such a way that one-fourth of the undissolved parts disolves every 10 minutes. If you put 25g of a substance in water at a given time, how much will be left after:
(a) 10 min?
(b) 30 min?
(c) 50 min?

Solution
(a) after 10 min? $25(\frac{3}{4})^{(\frac{10}{10})} = 18.75\text{g}$
(b) after 30 min? $25(\frac{3}{4})^{(\frac{30}{10})} = 10.546875\text{g}$
(c) after 50 min? $25(\frac{3}{4})^{(\frac{50}{10})} = 5.933\text{g}$

Problem 20

A chemical pollutant is being emptied in a lake with $50,000$ fishes. Every month, one-thrid of the fish still alive die from this pollutant. How many fish will be alive after:
(a) 1 month?
(b) 2 month?
(c) 4 month?

Solution
(a) after 1 month? $50000(\frac{2}{3})^1 = 33333.33\text{fishes}$
(b) after 2 month? $50000(\frac{2}{3})^2 = 22222.22\text{fishes}$
(c) after 4 month? $50000(\frac{2}{3})^4 = 9876.54\text{fishes}$

Problem 21

Every 10 years the population of a city is five-fourths of what it was the 10 years before. How many years does it take
(a) before the population doubles
(b) before it triples

Formula: $p \cdot \frac{5}{4}^{\frac{y}{10}} = 2p$
Solution (a)

$$p \cdot \frac{5}{4}^{\frac{y}{10}} = 2p$$
$$= y \approx 31 \text{ years}$$

Solution (b)

$$p \cdot \frac{5}{4}^{\frac{y}{10}} = 3p$$
$$= y \approx 49 \text{ years}$$

5 Multiplicative Inverse

Problem 2

Prove the following relations. It is assumed that all values of x and y which occur are such that the denominators in the indicated fractions are not 0.
(a) $\frac{1}{x+y} - \frac{1}{x-y} = \frac{-2y}{x^2-y^2}$
(b) $\frac{x^3-1}{x-1} = 1 + x + x^2$
(c) $\frac{x^4-1}{x-1} = 1 + x + x^2 + x^3$

Proof.

$$\begin{aligned}
\frac{1}{x+y} - \frac{1}{x-y} &= \frac{-2y}{x^2-y^2} \\
(x+y)[\frac{1}{x+y} - \frac{1}{x-y}] &= \frac{-2y}{x^2-y^2}(x+y) \\
1 - \frac{x+y}{x-y} &= \frac{-2y}{x-y}(x+y) \\
(x-y)(1 - \frac{x+y}{x-y}) &= \frac{-2y}{x-y}(x-y) \\
(x-y) - (x+y) &= -2y \\
-2y &= -2y
\end{aligned}$$

□

Proof.

$$\begin{aligned}
&\frac{x^3-1}{x-1} = 1 + x + x^2 \\
&x^3 - 1 = (x-1)(1 + x + x^2) \\
&x^3 - 1 = x + x^2 + x^3 - 1 - x - x^2 \\
&x^3 - 1 = x^3 - 1
\end{aligned}$$

□

Proof.

$$\begin{aligned}
&\frac{x^4-1}{x-1} = 1 + x + x^2 + x^3 \\
&x^4 - 1 = (x-1)(1 + x + x^2 + x^3) \\
&x^4 - 1 = x + x^2 + x^3 + x^4 - 1 - x - x^2 - x^3 \\
&x^4 - 1 = x + x^2 + x^3 + x^4 - 1 - x - x^2 - x^3 \\
&x^4 - 1 = x^4 - 1
\end{aligned}$$

□

Proof.

$$\begin{aligned}
\frac{x^n-1}{x-1} &= x^{n-1}+x^{n-2}+\cdots+x+1 \\
x^n-1 &= (x-1)(x^{n-1}+x^{n-2}+\cdots+x+1) \\
x^n-1 &= x^n+x^{n-1}+\cdots+x^2+x-x^{n-1}-x^{n-2}-\cdots-x-1 \\
x^n-1 &= x^n-1
\end{aligned}$$

□

Problem 3

Prove the following relations.

(a) $\frac{1}{2x+y}+\frac{1}{2x-y}=\frac{4x}{4x^2-y^2}$

(b) $\frac{2x}{x+5}+\frac{3x+1}{2x+1}=\frac{x^2-14x-5}{2x^2+11x+5}$

(c) $\frac{1}{x+3y}+\frac{1}{x-3y}=\frac{2x}{x^2-9y^2}$

(c) $\frac{1}{3x-2y}+\frac{x}{x+y}=\frac{x+y+3x^2-2xy}{3x^2+xy-2y^2}$

Proof.

$$\begin{aligned}
\frac{1}{2x+y}+\frac{1}{2x-y} &= \frac{4x}{4x^2-y^2} \\
\leftrightarrow ((2x+y)(2x-y))\left(\frac{1}{2x+y}+\frac{1}{2x-y}\right) &= ((2x+y)(2x-y))\left(\frac{4x}{4x^2-y^2}\right) \\
\leftrightarrow (2x-y)+(2x+y) &= 4x \\
\leftrightarrow 4x &= 4x
\end{aligned}$$

□

Proof.

$$\begin{aligned}
\frac{2x}{x+5}+\frac{3x+1}{2x+1} &= \frac{x^2-14x-5}{2x^2+11x+5} \\
\leftrightarrow ((x+5)(2x+1))\left(\frac{2x}{x+5}+\frac{3x+1}{2x+1}\right) &= ((x+5)(2x+1))\left(\frac{x^2-14x-5}{2x^2+11x+5}\right) \\
\leftrightarrow 2x(2x+1)-(3x+1)(x+5) &= x^2-14x-5 \\
\leftrightarrow (4x^2+2x)-(3x^2+15x+5) &= x^2-14x-5 \\
\leftrightarrow x^2-14x-5 &= x^2-14x-5
\end{aligned}$$

□

Proof.

$$\begin{aligned}\frac{1}{x+3y}+\frac{1}{x-3y}&=\frac{2x}{x^2-9y^2}\\ \leftrightarrow ((x+3y)(x-3y))\left(\frac{1}{x+3y}+\frac{1}{x-3y}\right)&=((x+3y)(x-3y))\left(\frac{2x}{x^2-9y^2}\right)\\ \leftrightarrow (x-3y)+(x+3y)&=2x\\ \leftrightarrow 2x&=2x\end{aligned}$$

□

Proof.

$$\begin{aligned}\frac{1}{3x-2y}+\frac{x}{x+y}&=\frac{x+y+3x^2-2xy}{3x^2+xy-2y^2}\\ \leftrightarrow ((3x-2y)(x+y))\left(\frac{1}{3x-2y}+\frac{x}{x+y}\right)&=((3x-2y)(x+y))\left(\frac{x+y+3x^2-2xy}{3x^2+xy-2y^2}\right)\\ \leftrightarrow (x+y)+x(3x-2y)&=x+y+3x^2-2xy\\ \leftrightarrow (x+y)+(3x^2-2xy)&=x+y+3x^2-2xy\\ \leftrightarrow x+y+3x^2-2xy&=x+y+3x^2-2xy\end{aligned}$$

□

Problem 4

Prove the following relations.
(a) $\frac{x^3-y^3}{x-y}=x^2+xy+y^2$
(b) $\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3$
Let
$x=\frac{1-t^2}{1+t^2}$ and $y=\frac{2t}{1+t^2}$
Show that $x^2+y^2=1$

Proof.

$$\begin{aligned}\frac{x^3-y^3}{x-y}&=x^2+xy+y^2\\ &\leftrightarrow x^3-y^3=(x-y)(x^2+xy+y^2)\\ &\leftrightarrow x^3-y^3=x^3+x^2y+xy^2-x^2y-xy^2-y^3\\ &\leftrightarrow x^3-y^3=x^3-y^3\end{aligned}$$

□

Proof.

$$\begin{aligned}
&\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3\\
&\leftrightarrow x^4-y^4=(x-y)(x^3+x^2y+xy^2+y^3)\\
&\leftrightarrow x^4-y^4=x^4+x^3y+x^2y^2+xy^3-x^3y-x^3y-x^2y^2-xy^3-y^4\\
&\leftrightarrow x^4-y^4=x^4-y^4
\end{aligned}$$

□

Proof.

$$\begin{aligned}
x^2+y^2&=1\\
\leftrightarrow \left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2&=1\\
\leftrightarrow \frac{(1-t^2)^2+(2t)^2}{(1+t^2)^2}&=1\\
\leftrightarrow \frac{t^4-2t^2+1+4t^2}{(1+t^2)^2}&=1\\
\leftrightarrow \frac{t^4+2t^2+1}{(1+t^2)^2}&=1\\
\leftrightarrow \frac{(1+t^2)^2}{(1+t^2)^2}&=1\\
\leftrightarrow 1&=1
\end{aligned}$$

□

Problem 5

Prove the following relations.
(a) $\frac{x^3+1}{x+1}=x^2-x+1$
(b) $\frac{x^5+1}{x+1}=x^4-x^3+x^2-x+1$
(c) If n is an odd integerl, prove that
$\frac{x^n+1}{x+1}=x^{(n-1)}-x^{(n-2)}+x^{(n-3)}-\cdots-x+1$

Proof.

$$\begin{aligned}
&\frac{x^3+1}{x+1}=x^2-x+1\\
&\leftrightarrow x^3+1=(x+1)(x^2-x+1)\\
&\leftrightarrow x^3+1=x^3-x^2+x+x^2-x+1\\
&\leftrightarrow x^3+1=x^3+1
\end{aligned}$$

□

Proof.

$$\begin{aligned}
&\frac{x^5+1}{x+1} = x^4 - x^3 + x^2 - x + 1 \\
\leftrightarrow\ & x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1) \\
\leftrightarrow\ & x^5 + 1 = x^5 - x^4 + x^3 - x^2 + x + x^4 - x^3 + x^2 - x + 1 \\
\leftrightarrow\ & x^5 + 1 = x^5 + 1
\end{aligned}$$

□

Proof.

$$\begin{aligned}
&\frac{x^n+1}{x+1} = x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1 \\
\leftrightarrow\ & x^n + 1 = (x+1)(x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1) \\
\leftrightarrow\ & x^n + 1 = x^n - x^{(n-1)} + x^{(n-2)} - \cdots - x^2 + x + x^{(n-1)} - x^{(n-2)} + x^{(n-3)} - \cdots - x + 1 \\
\leftrightarrow\ & x^n + 1 = x^n + 1
\end{aligned}$$

□

Problem 7

If a solid has a uniform density d, occpies a volume v, and has a mass m, then we have the formula
$m = vd$
Find the density if:
(a) $m = \frac{3}{10}\text{lb}$ and $v = \frac{2}{3}\text{in}^3$
(a) $m = 6\text{lb}$ and $v = \frac{4}{3}\text{in}^3$
Find the volume if the mass is 15 lb and the density is $\frac{2}{3}$ lb/in^3.

Solution (a)

$$\begin{aligned}
\frac{3}{10} &= \frac{2}{3}d \\
d &= \frac{9}{20}
\end{aligned}$$

Solution (b)

$$\begin{aligned}
6 &= \frac{4}{3}d \\
d &= \frac{18}{4}
\end{aligned}$$

Solution (c)

$$\begin{aligned}
15 &= v\frac{2}{3} \\
v &= \frac{45}{2}
\end{aligned}$$

Problem 13

Tickets for a performance sell \$5.00 and \$2.00. The total amount collected was \$4,100, and there are 1,300 tickets in all. How many tickets of each price were sold.

Solution Let $x =$ be the number of tickets sold at \$2.00.

$$\begin{aligned} 2x + 5(1300 - x) &= 4100 \\ 2x + 6500 - 5x &= 4100 \\ -3x &= -2400 \\ x &= 800 \end{aligned}$$

800 tickets sold at \$2.00 and 500 sold at \$5.00.

Problem 16

A boat travels a distance of 500mi, along two rivers, for 50hr. The current goes in the same direction as the boat along one river, and then the boat averages 20mph. The current goes in the opposite dirction along the other river, and then the boat averages 8mph. During how many hours was the boat on the first river.

Solution Let x be the time spent on the first river.

$$\begin{aligned} 20x + 8(50 - x) &= 500 \\ 20x + 400 - 8x &= 500 \\ 12x &= 100 \\ x &= \frac{100}{12} \end{aligned}$$

$x = \frac{100}{12}$ hours on first river.

Problem 18

The radiator of a car can contain 10kg of liquid. If it is half full with a mixture having 60% antifreeze and 40% water, how much more water must be added so that the resulting mixture has only.

Solution (a) 40% antifreeze
40% antifreeze means 60% water.

$$\begin{aligned} &\frac{4 + x}{10 + x} = 0.6 \\ &4 + x = 0.6(10 + x) \\ &4 + x = 6 + 0.6x \\ &x = 5 \end{aligned}$$

15 kg will fit in the radiator.
Solution (b) 10% antifreeze

10% antifreeze means 90% water.

$$\begin{aligned}
\frac{4+x}{10+x} &= 0.9 \\
4+x &= 0.9(10+x) \\
4+x &= 9+0.9x \\
x &= 50
\end{aligned}$$

55 kg will not fit in the radiator.