Lectures on the Hyperreals by Robert Goldblatt

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Problem 1

If $\emptyset \neq A \subseteq I$, there is an ultrafilter $\mathscr{F}$ on I with $A \in \mathscr{F}$.

Proof. Consider the set $\{A\}$, which has the finite intersection property since $A \neq \emptyset$. Thus, by Theorem 2.6.1, there exists an ultrafilter $\mathscr{F}$ on I such that

$$\{A\} \subseteq \mathscr{F}.$$

Therefore, $A \in \mathscr{F}$. ∎

Problem 2

There exists a nonprincipal ultrafilter on $\mathbb{N}$ containing the set of even numbers, and another containing the set of odd numbers.

Proof. Let
$$\mathscr{E} = \{x \in \mathbb{N} \mid x = 2k \text{ for some } k \in \mathbb{N}\}$$
and
$$\mathscr{O} = \{x \in \mathbb{N} \mid x = 2k+1 \text{ for some } k \in \mathbb{N}\}.$$
Since $\mathscr{E}$ is infinite, by Corollary 2.6.2 there exists a nonprincipal ultrafilter $\mathscr{U}_E$ on $\mathscr{E}$. Let

$$\mathscr{F}_E = \{A \subseteq \mathbb{N} \mid A \cap \mathscr{E} \in \mathscr{U}_E\}.$$

We first show $\mathscr{F}_E$ is a filter. Since $\mathscr{E} \in \mathscr{U}_E$, we have $\mathbb{N} \cap \mathscr{E} = \mathscr{E} \in \mathscr{U}_E$, so $\mathbb{N} \in \mathscr{F}_E$. Also $\emptyset \notin \mathscr{F}_E$ since $\emptyset \cap \mathscr{E} = \emptyset \notin \mathscr{U}_E$. If $A, B \in \mathscr{F}_E$, then $A \cap \mathscr{E} \in \mathscr{U}_E$ and $B \cap \mathscr{E} \in \mathscr{U}_E$. Since $\mathscr{U}_E$ is a filter,

$$(A \cap \mathscr{E}) \cap (B \cap \mathscr{E}) = (A \cap B) \cap \mathscr{E} \in \mathscr{U}_E,$$

so $A \cap B \in \mathscr{F}_E$. If $A \in \mathscr{F}_E$ and $A \subseteq B \subseteq \mathbb{N}$, then $A \cap \mathscr{E} \subseteq B \cap \mathscr{E}$. Since $\mathscr{U}_E$ is upward closed, $B \cap \mathscr{E} \in \mathscr{U}_E$, so $B \in \mathscr{F}_E$. Thus $\mathscr{F}_E$ is a filter. Now, let $X \subseteq \mathbb{N}$. Since $\mathscr{U}_E$ is an ultrafilter on $\mathscr{E}$, either $X \cap \mathscr{E} \in \mathscr{U}_E$ or

$$\mathscr{E} \setminus (X \cap \mathscr{E}) = X^C \cap \mathscr{E} \in \mathscr{U}_E.$$

Thus either $X \in \mathscr{F}_E$ or $X^C \in \mathscr{F}_E$. Thus $\mathscr{F}_E$ is an ultrafilter on $\mathbb{N}$. Notice

$$\mathscr{E} \cap \mathscr{E} = \mathscr{E} \in \mathscr{U}_E,$$

so $\mathscr{E} \in \mathscr{F}_E$. Since $\mathscr{U}_E$ is nonprincipal, $\mathscr{F}_E$ is also nonprincipal. The same argument applied to $\mathscr{O}$ produces a nonprincipal ultrafilter on $\mathbb{N}$ containing $\mathscr{O}$. ∎

Problem 3

An ultrafilter on a finite set must be principle.

Proof. Suppose $\mathscr{F}$ is an ultrafilter on a finite set I. Since I is finite, there exists a smallest set $X \in \mathscr{F}$. If $|X| = 1$ then obviously $\mathscr{F}$ is principal. Therefore, suppose, $|X| \geq 2$. Take $p \in X$ and consider $X - \{p\}$. Since $\mathscr{F}$ is an ultrafilter
$$X - \{p\} \in \mathscr{F} \quad \text{or} \quad I - (X - \{p\}) \in \mathscr{F}.$$
But X has the smallest size among sets in $\mathscr{F}$, so $X - \{p\} \notin \mathscr{F}$. Therefore

$$I - (X - \{p\}) = \{p\} \cup (I - X) \in \mathscr{F}.$$

Then
$$X \cap (\{p\} \cup (I - X)) = \{p\} \in \mathscr{F}.$$

Thus $\mathscr{F}$ contains a singleton and is a principal ultrafilter generated by p. ∎

Problem 4

For $\mathscr{H} \subseteq \mathscr{P}(I)$, let $\mathscr{F}^{\mathscr{H}}$ be defined as in Example 2.4(3).

1. Show that $\mathscr{F}^{\mathscr{H}}$ is a filter that includes $\mathscr{H}$, i.e., $\mathscr{H} \subseteq \mathscr{F}^{\mathscr{H}}$.
2. Show that $\mathscr{F}^{\mathscr{H}}$ is included in any other filter that includes $\mathscr{H}$.