Algebraic Geometry by Thomas Garrity et. al.

Frosty

February 14, 2026

Contents

1 Conics 1
- 1.1 Conics over the Reals . . . 1
- 1.2 Changes of Coordinates . . . 10
- 1.3 Conics over the Complex Numbers . . . 21
- 1.4 The Complex Projective Plane $\mathbb{P}^2$ . . . 25
- 1.5 Projective Changes of Coordinates . . . 32
- 1.6 The Complex Projective Line $\mathbb{P}^1$ . . . 33
- 1.7 Ellipses, Hyperbolas, and Parabolas as Spheres . . . 36
- 1.8 Links to Number Theory . . . 40
- 1.9 Degenerate Conics . . . 43

1 Conics

1.1 Conics over the Reals

Problem 1

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that for any $(x, y) \in C$, we also have

$$(-x, y) \in C.$$

Thus the curve is symmetric about the y-axis.

Proof. Let $(x, y) \in C$. Then $P(x, y) = y - x^2 = 0$. Let $x' = -x$ and note that $(-x)^2 = x^2$. Thus

$$P(-x, y) = y - (-x)^2 = y - x^2 = 0.$$

Thus $(-x, y) \in C$. ∎

Problem 2

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that if $(x, y) \in C$, then we have $y \geq 0$.

Proof. Suppose $(x, y) \in C$. Then

$$P(x, y) = y - x^2 = 0 \iff y = x^2 \geq 0.$$

Thus $y \geq 0$. ∎

Problem 3

$$P(x,y) = y - x^2, \quad C = \{(x,y) \in \mathbb{R}^2 \mid P(x,y) = 0\}.$$

Show that for every $y \geq 0$, there is a point $(x,y) \in C$ with this y-coordinate. Now, for points $(x,y) \in C$, show that if y goes to infinity, then one of the corresponding x-coordinates also approaches infinity while the other corresponding x coordinate must approach negative infinity.

Proof. Let $y \in \mathbb{R}$ such that $y \geq 0$. Let $x = \sqrt{y} \in \mathbb{R}$. Then

$$y - x^2 = y - (\sqrt{y})^2 = y - y = 0.$$

Thus $(x,y) = (\sqrt{y}, y) \in C$.

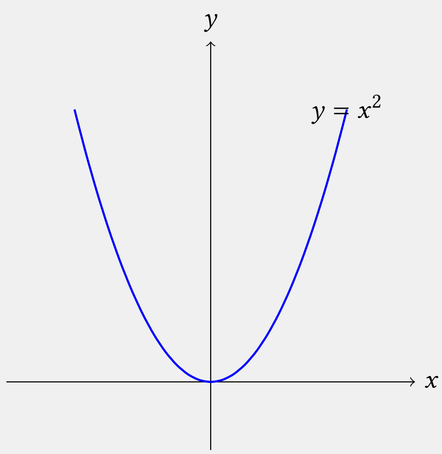
Now suppose $y \to \infty$. For points $(x,y) \in C$, we have

$$y - x^2 = 0 \iff x = \pm\sqrt{y}.$$

Since $y \to \infty$, we have $\sqrt{y} \to \infty$ and $-\sqrt{y} \to -\infty$. Thus one corresponding x-coordinate approaches infinity, while the other approaches negative infinity. ∎

Problem 4

Sketch the curve $C = \{(x,y) \in \mathbb{R}^2 \mid P(x,y) = 0\}$.



Problem 5

$$C = \left\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\right\}.$$

Show that if $(x,y) \in C$, then the three points $(-x,y), (x,-y), (-x,-y)$ are also on C. Thus the curve C is symmetric about both the x- and y-axes.

Proof. Let $(x,y) \in \mathbb{R}^2$. Suppose $\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0$. Notice that $x^2 = (-x)^2$ and $y = (-y)^2$. Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{y^2}{9} - 1 = \frac{x^2}{4} + \frac{(-y)^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{(-y)^2}{9} - 1 = 0.$$

Thus $(-x, y), (x, -y), (-x, -y) \in C$. ∎

Problem 6

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\right\}.$$

Show that for every $(x, y) \in C$, we have $|x| \leq 2$ and $|y| \leq 3$.

Proof. Let $(x, y) \in C$. Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \iff 9x^2 + 4y^2 - 36 = 0 \iff 9x^2 = -4y^2 + 36 \iff |x| = \sqrt{\frac{-4}{9}y^2 + 4} \leq \sqrt{4} = 2.$$

Similarly

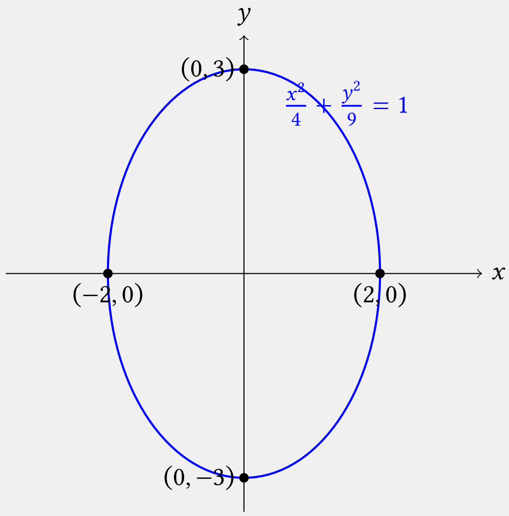
$$9x^2 + 4y^2 - 36 = 0 \iff |y| = \sqrt{\frac{-9}{4}x^2 + 9} \leq \sqrt{9} = 3.$$

∎

Problem 7

Sketch

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\right\}.$$



Problem 8

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

Show that if $(x, y) \in C$, then the three points $(-x, y), (x, -y)$, and $(-x, -y)$ are also on C. Thus the curve C is also symmetric about the $x-$ and y-axes.

Proof. Let $(x, y) \in \mathbb{R}^2$. Suppose $x^2 - y^2 - 4 = 0$. Notice that $x^2 = (-x)^2$ and $y = (-y)^2$. Then

$$x^2 - y^2 - 4 = (-x)^2 - y^2 = x^2 - (-y)^2 = (-x)^2 - (-y)^2 = 0.$$

Thus $(-x, y), (x, -y), (-x, -y) \in C$. ∎

Problem 9

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

Show that if $(x, y) \in C$, then we have $|x| \geq 2$.

Proof. Let $(x, y) \in \mathbb{R}^2$. Suppose $x^2 - y^2 - 4 = 0$. Then

$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff |x| = \sqrt{y^2 + 4} \geq \sqrt{4} = 2.$$

$\blacksquare$

Problem 10

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

Show that the curve C is unbounded in the positive and negative x-directions and also unbounded in the positive and negative y-directions.

Proof. First notice

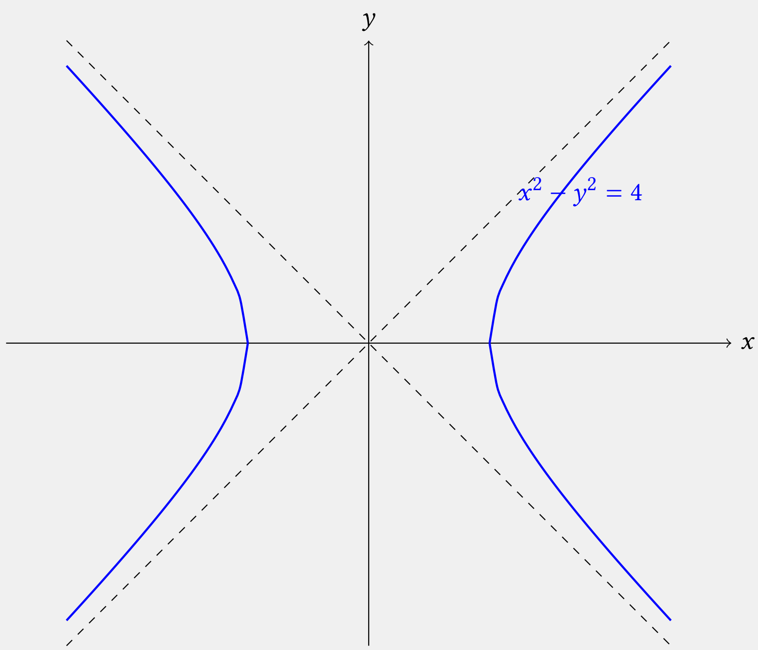
$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff x = \pm\sqrt{y^2 + 4} \iff y = \pm\sqrt{x^2 - 4}.$$

As $y \to \infty$, we have $x = \pm\sqrt{y^2 + 4} \to \infty$ and $-\infty$. Similarly, as $x \to \infty$, we have $y = \pm\sqrt{x^2 - 4} \to \infty$ and $-\infty$. $\blacksquare$

Problem 11

Sketch

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

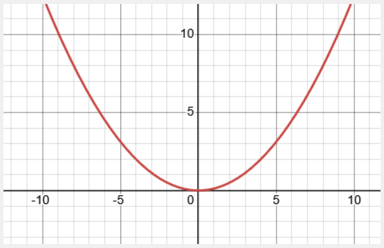


Problem 12

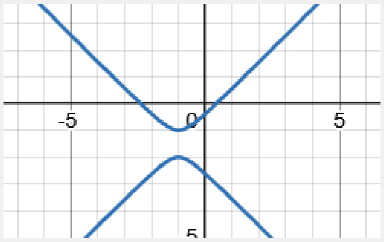
Sketch the graph of each of the following conics in $\mathbb{R}^2$. Identify which are parabolas, ellipses, or Hyperbola.

1. $V(x^2 - 8y)$.
2. $V(x^2 + 2x - y^2 - 3y - 1)$.
3. $V(4x^2 + y^2)$.
4. $V(3x^2 + 3y^2 - 75)$.
5. $V(x^2 - 9y^2)$.
6. $V(4x^2 + y^2 - 8)$.
7. $V(x^2 + 9y^2 - 36)$.
8. $V(x^2 - 4y^2 - 16)$.
9. $V(y^2 - x^2 - 9)$.

Solution (1): Parabola.

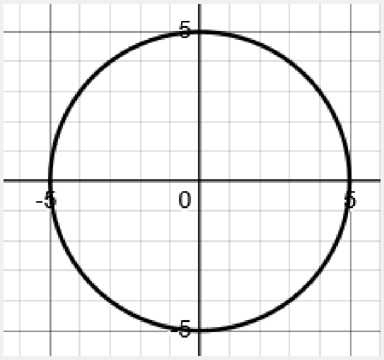


Solution (2): Hyperbola.



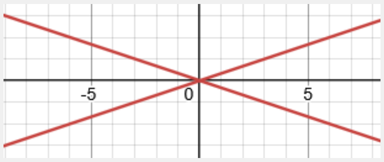
Solution (3): Point.

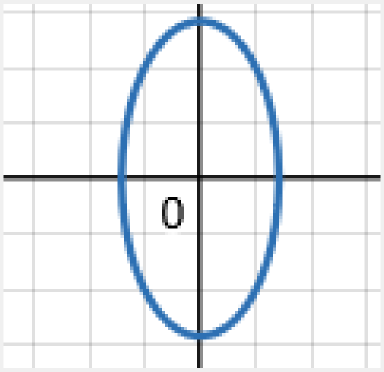
Solution (4): Ellipse.



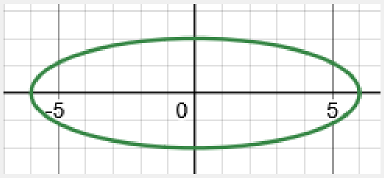
Solution (5): Two lines.

Solution (6): Ellipse.

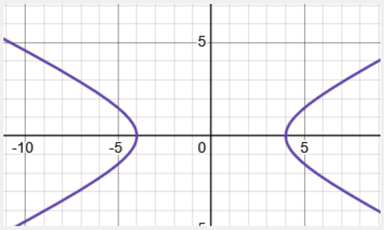




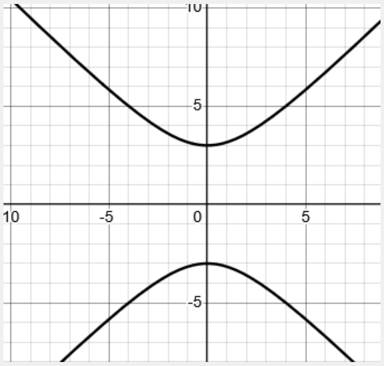
Solution (7): Ellipse.



Solution (8): Hyperbola.



Solution (9): Hyperbola.



Problem 13

Express the polynomial $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$ in the form

$$P(x, y) = Ax^2 + Bx + C,$$

where A, B, and C are polynomials in y. What are A, B, and C?

Proof. Let $A = a, B = by + d$, and $C = cy^2 + ey + h$. Notice

$$ax^2 + bxy + cy^2 + dx + ey + h = ax^2 + bxy + dx + cy^2 + ey + h = ax^2 + (by + d)x + (cy^2 + ey + h) = Ax^2 + Bx + C.$$

∎

Problem 14

Treating $P(x, y) = ax^2+bxy+cy^2+dx+cy+h$ as a polynomial in the variable x, show that the discriminant is

$$\triangle_x(y) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

Proof. From Problem 13 we have $A = a, B = by + d$, and $C = cy^2 + ey + h$. Then

$$\triangle_x(y) = B^2 - 4AC = (by + d)^2 - 4a(cy^2 + ey + h) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

∎

Problem 15

1. Suppose $\triangle_x(y_0) < 0$. Explain why there is no point on $V(p)$ whose y-coordinate is y_0.
2. Suppose $\triangle_x(y_0) = 0$. Explain why there is exactly one point on $V(P)$ whose y-coordinate is y_0.
3. Suppose $\triangle_x(y_0) > 0$. Explain why there are exactly two points on $V(P)$ whose y-coordinate is y_0.

Solution (a): In $\mathbb{R}$, the square root is undefined for values < 0.

Solution (b): If $\triangle_x(y_0) = 0$, then $+\sqrt{B^2 - 4AC} = -\sqrt{B^2 - 4AC}$, so there is exactly one point on $V(P)$ whose y-coordinate is y_0.

Solution (c): If $\triangle_x(y_0) > 0$, then $+\sqrt{B^2 - 4AC} \neq -\sqrt{B^2 - 4AC}$, so there are exactly two points on $V(P)$ whose y-coordinate is y_0.

Problem 16

Suppose $b^2 - 4ac = 0$. Suppose further that $2bd - 4ae > 0$.

1. Show that $\triangle_x(y) \geq 0$ if and only if $y \geq \frac{4ah-d^2}{2bd-4ae}$.
2. Conclude that if $b^2 - 4ac = 0$ and $2bd - 4ae > 0$, then $V(P)$ is a parabola.

Proof. Suppose $\triangle_x(y) \geq 0$. Then

$$\begin{aligned}\triangle_x(y) &= (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah)\\ &= 0y^2 + (2bd - 4ae)y + (d^2 - 4ah).\end{aligned}$$

Therefore,

$$(2bd - 4ae)y + (d^2 - 4ah) \geq 0.$$

Since $2bd - 4ae > 0$, we have

$$y \geq \frac{4ah - d^2}{2bd - 4ae}.$$

Conversely, suppose $y \geq \frac{4ah-d^2}{2bd-4ae}$. Then

$$\begin{aligned}
\triangle_x(y) &= (2bd - 4ae)y + (d^2 - 4ah) \\
&\geq (2bd - 4ae)\left(\frac{4ah - d^2}{2bd - 4ae}\right) + (d^2 - 4ah) \\
&= 0.
\end{aligned}$$

$\blacksquare$

Proof. Suppose $b^2 - 4ac = 0$ and $2bd - 4ae > 0$. Then $\triangle_x(y) = (2bd - 4ae)y + (d^2 - 4ah)$. Now, $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$. It is clear that x is symmetrical, and since $y \geq \frac{4ah-d^2}{2bd-4ae}$, $V(P)$ is a parabola. $\blacksquare$

Problem 17

Suppose $b^2 - 4ac < 0$.

1. Show that one of the following occurs:
   (a) $\{y \mid \triangle_x(y) \geq 0\} = \emptyset$,
   (b) $\{y \mid \triangle_x(y) \geq 0\} = \{y_o\}$,
   (c) there exist real numbers α and β, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

2. Conclude that $V(P)$ is either emptyset, a point, or an ellipse.

Proof. Since $b^2 - 4ac < 0$, the graph of $\triangle_x(y)$ is a downward opening parabola in y. There are three cases, depending on the number of real zeros of $\triangle_x(y)$.

1. If $\triangle_x(y) < 0$ for all y, then
$$\{y \mid \triangle_x(y) \geq 0\} = \emptyset.$$
2. If $\triangle_x(y)$ has exactly one real zero y_0, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y_0\}.$$
3. If $\triangle_x(y)$ has two distinct real zeros $\alpha < \beta$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

$\blacksquare$

Proof. From part 1 the set of y values is either empty, a single point, or a bounded interval, it follows that $V(P)$ is either empty, a point, or an ellipse. $\blacksquare$

Problem 18

Suppose $b^2 - 4ac > 0$.

1. Show that one of the following occurs:
   (a) $\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}$ and $\triangle_x(y) \neq 0$,

(b) $\{y \mid \triangle_x(y) = 0\} = \{y_0\}$ and $\{y \mid \triangle_x(y) > 0\} = \{y \mid y \neq y_0\}$,
(c) there exist real numbers α and β, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

2. If $\{y \mid \triangle_x(y)\} = \mathbb{R}$, show that $V(P)$ is a hyperbola opening left and right.
3. If $\{y \mid \triangle_x(y) = 0\} = \{y_0\}$, show that $V(P)$ is two lines intersecting in a point.
4. If there are two real numbers α and β, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\},$$

show that $V(P)$ is a hyperbola opening up and down.

Proof. Since $b^2 - 4ac > 0$, the graph of $\triangle_x(y)$ is an upward opening parabola in y. There are three cases, depending on the number of real zeros of $\triangle_x(y)$.

1. If $\triangle_x(y) > 0$ for all y, then
$$\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}.$$
2. If $\triangle_x(y)$ has exactly one real zero y_0, then
$$\{y \mid \triangle_x(y) = 0\} = \{y_0\} \quad \text{and} \quad \{y \mid \triangle_x(y) > 0\} = \{y \mid y \neq y_0\}.$$
3. If $\triangle_x(y)$ has two distinct real zeros $\alpha < \beta$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

∎

Proof. Since $b^2 - 4ac > 0$, the graph of $\triangle_x(y)$ is an upward opening parabola in y. There are three cases, depending on the number of real zeros of $\triangle_x(y)$.

1. If $\triangle_x(y) > 0$ for all y, then
$$\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}.$$
2. If $\triangle_x(y)$ has exactly one real zero y_0, then
$$\{y \mid \triangle_x(y) = 0\} = \{y_0\} \quad \text{and} \quad \{y \mid \triangle_x(y) > 0\} = \{y \mid y \neq y_0\}.$$
3. If $\triangle_x(y)$ has two distinct real zeros $\alpha < \beta$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

∎

Proof. Suppose $\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}$. Then for every y there exist two real solutions for x, and x is unbounded to the left and right. Since the equation is quadratic in x, the curve is symmetric in x. Thus $V(P)$ is a hyperbola opening left and right. ∎

Proof. Suppose $\{y \mid \triangle_x(y) = 0\} = \{y_0\}$. Then for $y = y_0$ the equation has exactly one real solution for x, and for $y \neq y_0$ it has two real solutions. Since the equation is quadratic in x, $V(P)$ consists of two lines intersecting at a point. ∎

Proof. Suppose there exist real numbers α and β, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

For $y \leq \alpha$ or $y \geq \beta$, the equation has two real solutions in x. If $\alpha < y < \beta$ it has no real solutions. Thus x is bounded for each y, but y is unbounded above and below. Since the equation is quadratic in x, the curve is symmetric in x. Therefore $V(P)$ is a hyperbola opening up and down. ∎

Problem 19

Show that the discriminant of $A'y^2 + B'y + C' = 0$ is

$$\triangle_y(x) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4ch).$$

Proof. Here $A' = c$, $B' = bx + e$, and $C' = ax^2 + dx + h$. Then

$$\triangle_y(x) = (B')^2 - 4A'C' = (bx + e)^2 - 4c(ax^2 + dx + h) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4ch).$$

∎

1.2 Changes of Coordinates

Problem 1

Show that the origin in the xy-coordinate system agrees with the origin in the uv-system if and only if $e = f = 0$. Thus the constants e and f describe translations of the origin.

Proof. Suppose the xy-coordinate system agrees with the origin of the uv-system. Then

$$u = 0 = a(0) + b(0) + e = e,$$

and

$$v = 0 = c(0) + d(0) + f = f.$$

Thus $f = e = 0$.

Conversely, suppose $e = f = 0$. Then

$$u = ax + by + e = ax + by + 0 = a(0) + b(0) = 0,$$

and

$$v = cx + dy + f = cx + dy + 0 = c(0) + d(0) = 0.$$

Thus the origin of the xy-coordinate system agrees with the origin of the uv-system. ∎

Problem 2

Show that if $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then the inverse change of coordinates is

$$x = \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf).$$

$$y = \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af).$$

Proof. We need to solve the two equations $u = ax + by + e$ and $v = cx + dy + f$ in two unknowns x and y. Translating this to linear algebra, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u-e \\ v-f \end{bmatrix}.$$

Using Cramer's rule we see

$$x = \frac{\begin{vmatrix} u-e & b \\ v-f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d(u-e) - b(v-f)}{ad-bc},$$

$$y = \frac{\begin{vmatrix} a & u-e \\ c & v-f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c(u-e) + a(v-f)}{ad-bc}.$$

Therefore

$$x = \frac{du - bv - de + bf}{ad-bc}, \quad y = \frac{-cu + av + ce - af}{ad-bc}.$$

∎

Problem 3

Show that if

$$u = ax + by + e$$
$$v = cx + dy + f,$$

and

$$s = Au + Bv + E$$
$$t = Cu + Dv + F$$

are two real affine changes of coordinates from the xy-plane to the uv-plane and from the uv-plane to the st-plane, respectively, then the composition from the xy-plane to the st-plane is a real affine change of coordinates.

Proof. Suppose

$$u = ax + by + e$$
$$v = cx + dy + f,$$

and

$$s = Au + Bv + E$$
$$t = Cu + Dv + F$$

are two real affine changes of coordinates from the xy-plane to the uv-plane and from the uv-plane to the st-plane respectively. Substituting u, v into s, t we see

$$s = A(ax + by + e) + B(cx + dy + f) + E = (Aa + Bc)x + (Ab + Bd)y + (Ae + Bf + E),$$

and

$$t = C(ax + by + e) + D(cx + dy + f) + F = (Ca + Dc)x + (Cb + Dd)y + (Ce + Df + F).$$

Finally,

$$\det\left(\begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix}\right) = (Aa + Bc)(Cb + Dd) - (Ab + Bd)(Ca + Dc) = (ad - bc)(AD - BC) \neq 0.$$

∎

Problem 4

For each affine pair of ellipses, find a real affine change of coordinates that maps the ellipse in the xy-plane to the ellipse in the uv-plane.

1. $V(x^2 + y^2 - 1), V(16u^2 + 9v^2 - 1)$.
2. $V((x-1)^2 + y^2 - 1), V(16u^2 + 9(v+2)^2 - 1)$.
3. $V(4x^2 + y^2 - 6y + 8), V(u^2 - 4u + v^2 - 2v + 4)$.
4. $V(13x^2 - 10xy + 13y^2 - 1), V(4u^2 + 9v^2 - 1)$.

Solution (1): Let $x = 4u$ and $y = 3v$. Then

$$x^2 + y^2 - 1 = (4u)^2 + (3v)^2 - 1 = 16u^2 + 9v^2 - 1 = 0.$$

Solution (2): Let $x = 4u + 1$ and $y = 3v + 6$. Then

$$(x-1)^2 + y^2 - 1 = (4u + 1 - 1)^2 + (3v + 6)^2 = 16u^2 + 9(v+2)^2 = 0.$$

Solution (3): Let $x = \frac{u}{2} - 1$ and $y = v + 2$. Then

$$4x^2 + y^2 - 6y + 8 = 4\left(\frac{u}{2} - 1\right)^2 + (v+2)^2 - 6(v+2) + 8 =$$

$$4\left(\frac{u^2}{4} - 2\frac{u}{2} + 1\right) + v^2 + 4v + 4 - 6v - 12 + 8 = u^2 - 4u + 4 + v^2 - 2v = u^2 - 4u + v^2 - 2v + 4.$$

Solution (4): Let $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Then

$$\begin{aligned}
13x^2 - 10xy + 13y^2 - 1 &= 13\left(\frac{u+v}{2}\right)^2 - 10\left(\frac{u+v}{2} \cdot \frac{u-v}{2}\right) + 13\left(\frac{u-v}{2}\right)^2 - 1 \\
&= 13\frac{(u+v)^2}{4} - 10\frac{u^2 - v^2}{4} + 13\frac{(u-v)^2}{4} - 1 \\
&= \frac{13}{4}(u^2 + 2uv + v^2) - \frac{10}{4}(u^2 - v^2) + \frac{13}{4}(u^2 - 2uv + v^2) - 1 \\
&= \frac{13 + 13 - 10}{4}u^2 + \frac{13 + 13 + 10}{4}v^2 + \frac{26 - 26}{4}uv - 1 \\
&= 4u^2 + 9v^2 - 1.
\end{aligned}$$

Problem 5

For each pair of hyperbolas, find a real affine change of coordinates that maps the hyperbola in the xy-plane to the hyperbola in the uv-plane.

1. $V(xy - 1), V(u^2 - v^2 - 1)$.
2. $V(x^2 - y^2 - 1), V(16u^2 - 9v^2 - 1)$.
3. $V((x-1)^2 - y^2 - 1), V(16u^2 - 9(v+2)^2 - 1)$.
4. $V(x^2 - y^2 - 1), V(v^2 - u^2 - 1)$.
5. $V(8xy - 1), V(2u^2 - 2v^2 - 1)$.

Solution (1): Let $x = u - v$ and $y = u + v$. Then

$$xy - 1 = (u - v)(u + v) - 1 = u^2 - v^2 - 1.$$

Solution (2): Let $x = 4u$ and $y = 3v$. Then

$$x^2 - y^2 - 1 = (4u)^2 - (3v)^2 - 1 = 16u^2 - 9v^2 - 1.$$

Solution (3): Let $x = 4u + 1$ and $y = 3v + 6$. Then

$$(x-1)^2 - y^2 - 1 = (4u+1-1)^2 - (3v+6)^2 = 16u^2 - 9(v+2)^2 - 1.$$

Solution (4): Let $x = v$ and $y = u$. Then

$$x^2 - y^2 - 1 = v^2 - u^2 - 1.$$

Solution (5): Let $x = (u+v)/4$ and $y = (u-v)/2$. Then

$$8xy - 1 = 8((u+v)/4)((u-v)/2) - 1 = (u+v)(u-v) - 1 = u^2 - v^2 - 1.$$

Problem 6

For each pair of parabolas, find a real affine change of coordinates that maps the parabola in the xy-plane to the parabola in the uv-plane.

1. $V(x^2 - y), V(9v^2 - 4u)$.
2. $V((x-1)^2 - y), V(u^2 - 9(v+2))$.
3. $V(x^2 - y), V(u^2 + 2uv + v^2 - u + v - 2)$.
4. $V(x^2 - 4x + y + 4), V(4u^2 - (v+1))$.
5. $V(4x^2 + 4xy + y^2 - y + 1), V(4u^2 + v)$.

Solution (1): Let $x = 3v$ and $y = 4u$. Then

$$x^2 - y = (3v)^2 - 4u = 9v^2 - 4u.$$

Solution (2): Let $x = u + 1$ and $y = 9v + 18$. Then

$$(x-1)^2 - y = (u+1-1)^2 - (9v+18) = u^2 - 9(v+2).$$

Solution (3): Let $x = (u+v)^2$ and $y = u - v + 2$. Then

$$x^2 - y = (u+v)^2 - (u - v + 2) = u^2 + 2uv + v^2 - u + v - 2.$$

Solution (4): Let $x = 2u + 2$ and $y = -(v+1)$. Then

$$x^2 - 4x + y + 4 = (2u+2)^2 - 4(2u+2) - (v+1) + 4 = 4u^2 + 8u + 4 - 8u - 8 - (v+1) + 4 = 4u^2 - (v+1).$$

Solution (5): Let $x = u - \frac{1}{2}v + \frac{1}{2}$ and $y = v$. Then

$$\begin{aligned}
4x^2 + 4xy + y^2 - y + 1 &= 4\left(u - \tfrac{1}{2}v + \tfrac{1}{2}\right)^2 + 4\left(u - \tfrac{1}{2}v + \tfrac{1}{2}\right)v + v^2 - v + 1 \\
&= 4\left(u^2 - uv + u + \tfrac{1}{4}v^2 - \tfrac{1}{2}v + \tfrac{1}{4}\right) + 4uv - 2v^2 + 2v + v^2 - v + 1 \\
&= 4u^2 - 4uv + 4u + v^2 - 2v + 1 + 4uv - 2v^2 + 2v + v^2 - v + 1 \\
&= 4u^2 + v.
\end{aligned}$$

Problem 7

Explain why if $b^2 - 4ac < 0$, then $ac > 0$.

Proof. Suppose $b^2 - 4ac < 0$. Then $0 \le b^2 < 4ac \iff 0 \le \frac{b^2}{4} < ac$. Thus $ac > 0$. ∎

Problem 8

Show that under the real affine transformation

$$x = \sqrt{\frac{c}{a}}u + v$$

$$y = u - \sqrt{\frac{a}{c}}v,$$

the ellipse $V(ax^2 + bxy + cy^2 + dx + ey + h)$ in the xy-plane becomes an ellipse in the uv-plane whose defining equation is $Au^2+Cv^2+Du+Ev+H = 0$. Find A and C in terms of a, b, c. Show that if $b^2-4ac < 0$, then $A \neq 0$ and $C \neq 0$.

Proof.

$$\begin{aligned} ax^2 + bxy + cy^2 + dx + ey + h &= a\big(\sqrt{\tfrac{c}{a}}\,u + v\big)^2 + b\big(\sqrt{\tfrac{c}{a}}\,u + v\big)\big(u - \sqrt{\tfrac{a}{c}}\,v\big) + c\big(u - \sqrt{\tfrac{a}{c}}\,v\big)^2 \\ &\quad + d\big(\sqrt{\tfrac{c}{a}}\,u + v\big) + e\big(u - \sqrt{\tfrac{a}{c}}\,v\big) + h \\ &= \big(cu^2 + 2\sqrt{ac}\,uv + av^2\big) + b\big(\sqrt{\tfrac{c}{a}}\,u^2 - \sqrt{\tfrac{a}{c}}\,v^2\big) + \big(cu^2 - 2\sqrt{ac}\,uv + av^2\big) \\ &\quad + \big(d\sqrt{\tfrac{c}{a}} + e\big)u + \big(d - e\sqrt{\tfrac{a}{c}}\big)v + h \\ &= \big(2c + b\sqrt{\tfrac{c}{a}}\big)u^2 + \big(2a - b\sqrt{\tfrac{a}{c}}\big)v^2 + \big(d\sqrt{\tfrac{c}{a}} + e\big)u + \big(d - e\sqrt{\tfrac{a}{c}}\big)v + h \\ &= Au^2 + Cv^2 + Du + Ev + H. \end{aligned}$$

∎

Proof. Suppose $b^2 - 4ac < 0$. Then

$$A = \sqrt{\frac{c}{a}}\,b + 2c, \qquad C = -\sqrt{\frac{a}{c}}\,b + 2a.$$

Then

$$AC = (2c + b\sqrt{\tfrac{c}{a}})(2a - b\sqrt{\tfrac{a}{c}}) = 4ac - b^2.$$

Since $b^2 - 4ac < 0$,

$$4ac - b^2 > 0 \implies AC > 0.$$

Therefore $A \neq 0$ and $C \neq 0$. ∎

Problem 9

Show that there exists constants R, S, and T such that the equation

$$Au^2 + Cv^2 + Du + Ev + H = 0,$$

can be written in the form

$$A(u - R)^2 + C(v - S)^2 - T = 0.$$

Express R, S, and T in terms of A, C, D, E, and H.

Proof. Let $R = -\frac{D}{2A}, S = -\frac{E}{2C}, T = \frac{D^2}{4A} + \frac{E^2}{4C} - H$. Note $A \neq 0$ and $C \neq 0$ from problem 8. Notice

$$\begin{aligned} Au^2 + Cv^2 + Du + Ev + H &= A\left(u^2 + \frac{Du}{A}\right) + C\left(v^2 + \frac{Ev}{C}\right) + H \\ &= A\left(u^2 + \frac{Du}{A} + \left(\frac{D}{2A}\right)^2\right) - \frac{D^2}{4A} + C\left(v^2 + \frac{Ev}{C} + \left(\frac{E}{2C}\right)^2\right) - \frac{E^2}{4C} + H \\ &= A\left(u + \frac{D}{2A}\right)^2 + C\left(v + \frac{E}{2C}\right)^2 - \frac{D^2}{4A} - \frac{E^2}{4C} + H \\ &= A(u-R)^2 + C(v-S)^2 - T = 0. \end{aligned}$$

∎

Problem 10

Suppose $A, C > 0$. Find a real affine change of coordinates that maps the ellipse

$$V(A(x-R)^2 + C(y-S)^2 - T),$$

to the circle

$$V(u^2 + v^2 - 1).$$

Proof. Since $A, C > 0$ we know $T > 0$. Notice

$$A(x-R)^2 + C(y-S)^2 = T \iff \frac{A(x-R)^2}{T} + \frac{C(y-S)^2}{T} = 1.$$

We set

$$u^2 = \frac{A(x-R)^2}{T}, \quad v^2 = \frac{C(y-S)^2}{T},$$

and solving for x, y shows

$$x = \sqrt{\frac{T}{A}}\, u + R, \quad y = \sqrt{\frac{T}{C}}\, v + S.$$

Substituting into the original equation, we find

$$\begin{aligned} A(x-R)^2 + C(y-S)^2 - T &= A\left(\sqrt{\frac{T}{A}}\, u\right)^2 + C\left(\sqrt{\frac{T}{C}}\, v\right)^2 - T \\ &= Tu^2 + Tv^2 - T \\ &= T(u^2 + v^2 - 1), \end{aligned}$$

∎

Problem 11

Consider the values A and C found in Exercise 1.2.8. Show that if $b^2 - 4ac = 0$, then either $A = 0$ or $C = 0$, depending on the signs of a, b, c. [Hint: Recall, $\sqrt{\alpha^2} = -\alpha$ if $\alpha < 0$.]

Proof. Suppose $b^2 - 4ac = 0$. From Exercise 1.2.8 we have

$$A = \sqrt{\frac{c}{a}}\, b + 2c, \qquad C = -\sqrt{\frac{a}{c}}\, b + 2a.$$

We see that

$$AC = 4ac - b^2 = -(b^2 - 4ac) = -0 = 0.$$

Thus $A = 0$ or $C = 0$. ∎

Problem 12

Show that there exists constants R and T such that the equation

$$Au^2 + Du + Ev + H = 0,$$

can be written as

$$A(u - R)^2 + E(v - T) = 0.$$

Express R and T in terms of $A, D, E,$ and H.

Proof. First note $A \neq 0$ therefore $E \neq 0$. Let

$$R = -\frac{D}{2A}, \qquad T = -\left(\frac{H}{E} - \frac{D^2}{4AE}\right).$$

Then

$$\begin{aligned} Au^2 + Du + Ev + H &= A\left(u^2 + \frac{D}{A}u + \left(\frac{D}{2A}\right)^2\right) - \frac{D^2}{4A} + Ev + H \\ &= A\left(u + \frac{D}{2A}\right)^2 + E\left(v + \frac{H}{E} - \frac{D^2}{4AE}\right) \\ &= A(u - R)^2 + E(v - T) = 0. \end{aligned}$$

∎

Problem 13

Suppose $A > 0$ and $E \neq 0$. Find a real affine change of coordinates that maps the parabola

$$V(A(x - R)^2 - E(y - T)),$$

to the parabola

$$V(u^2 - v).$$

Proof. We set $A(x - R)^2 = u^2$ and $-E(y - T) = -v$. Then solving for x, y we have

$$x = \frac{u}{\sqrt{A}} + R, \quad y = \frac{v}{E} + T.$$

Then substituting into our original equation we have

$$A(x - R)^2 - E(y - T) = A\left(\frac{u}{\sqrt{A}} + R - R\right)^2 - E\left(\frac{v}{E} + T - T\right) = u^2 - v.$$

∎

Problem 14

Suppose $ac > 0$. Use the real affine transformation in Exercise 1.2.8 to transform C to a conic in the uv-plane. Find the coefficients of u^2 and v^2 in the resulting equation and show that they have opposite signs.

Proof. Suppose $ac > 0$. From Exercise 1.2.8 we have

$$A = \sqrt{\frac{c}{a}}\,b + 2c, \qquad C = -\sqrt{\frac{a}{c}}\,b + 2a.$$

We see that

$$AC = 4ac - b^2 = -(b^2 - 4ac) < 0.$$

Thus A and C have opposite signs. ∎

Problem 15

Suppose $ac < 0$ and $b \neq 0$. Use the real affine transformation

$$x = \sqrt{-\frac{c}{a}}u + v$$

$$y = u - \sqrt{-\frac{a}{c}}v,$$

to transform C to a conic in the uv-plane of the form

$$Au^2 + Cv^2 + Du + Ev + H = 0.$$

Find the coefficients of the resulting equation and show that they have opposite signs.

Proof.

$$\begin{aligned}
ax^2 + bxy + cy^2 + dx + ey + h &= a\Big(\sqrt{-\tfrac{c}{a}}\,u + v\Big)^2 + b\Big(\sqrt{-\tfrac{c}{a}}\,u + v\Big)\Big(u - \sqrt{-\tfrac{a}{c}}\,v\Big) + c\Big(u - \sqrt{-\tfrac{a}{c}}\,v\Big)^2 \\
&\quad + d\Big(\sqrt{-\tfrac{c}{a}}\,u + v\Big) + e\Big(u - \sqrt{-\tfrac{a}{c}}\,v\Big) + h \\
&= \big(-cu^2 + 2\sqrt{-ac}\,uv - av^2\big) + b\Big(\sqrt{-\tfrac{c}{a}}\,u^2 - \sqrt{-\tfrac{a}{c}}\,v^2\Big) + \big(-cu^2 - 2\sqrt{-ac}\,uv - av^2\big) \\
&\quad + \Big(d\sqrt{-\tfrac{c}{a}} + e\Big)u + \Big(d - e\sqrt{-\tfrac{a}{c}}\Big)v + h \\
&= \Big(-2c + b\sqrt{-\tfrac{c}{a}}\Big)u^2 + \Big(-2a - b\sqrt{-\tfrac{a}{c}}\Big)v^2 + \Big(d\sqrt{-\tfrac{c}{a}} + e\Big)u + \Big(d - e\sqrt{-\tfrac{a}{c}}\Big)v + h \\
&= Au^2 + Cv^2 + Du + Ev + H.
\end{aligned}$$

∎

Proof. Since $ac < 0$ and $b \neq 0$, we have

$$A = -2c + b\sqrt{-\frac{c}{a}}, \qquad C = -2a - b\sqrt{-\frac{a}{c}}.$$

Then

$$AC = \Big(-2c + b\sqrt{-\tfrac{c}{a}}\Big)\Big(-2a - b\sqrt{-\tfrac{a}{c}}\Big) = 4ac - b^2.$$

Since $ac < 0$,

$$4ac - b^2 < 0 \implies AC < 0.$$

Therefore A and C have opposite signs. ∎

Problem 16

Suppose $ac = 0$ (so $b \neq 0$). Since either $a = 0$ or $c = 0$, we can assume $c = 0$. Use the real affine transformation

$$x = u + v$$
$$y = \left(\frac{1-a}{b}\right)u - \left(\frac{1+a}{b}\right)v,$$

to transform $V(ax^2 + bxy + dx + ey + h)$ to a conic in the uv-plane of the form

$$V(u^2 - v^2 + Du + Ev + H).$$

Proof.

$$\begin{aligned}
ax^2 + bxy + dx + ey + h &= a(u+v)^2 + b(u+v)\Big(\frac{1-a}{b}u - \frac{1+a}{b}v\Big) \\
&\quad + d(u+v) + e\Big(\frac{1-a}{b}u - \frac{1+a}{b}v\Big) + h \\
&= a(u^2 + 2uv + v^2) + (u+v)((1-a)u - (1+a)v) \\
&\quad + d(u+v) + e\Big(\frac{1-a}{b}u - \frac{1+a}{b}v\Big) + h \\
&= (a + 1 - a)u^2 + (-(1+a) + a)v^2 + 2auv \\
&\quad + \Big(d + e\frac{1-a}{b}\Big)u + \Big(d - e\frac{1+a}{b}\Big)v + h \\
&= u^2 - v^2 + Du + Ev + H
\end{aligned}$$

∎

Problem 17

Show that there exists R, S, and T so that

$$Au^2 - Cv^2 + Du + Ev + H = A(u-R)^2 - C(v-S)^2 - T.$$

Express R, S, and T in terms of A, C, D, E, and H.

Proof. We set $A(u-R)^2 = Au^2 + Du$ and $-C(v-S)^2 = -Cv^2 + Ev$. Then solving for R, S we have

$$R = -\frac{D}{2A}, \quad S = \frac{E}{2C}.$$

Then substituting into our original equation we have

$$\begin{aligned}
Au^2 - Cv^2 + Du + Ev + H &= \Big(A(u-R)^2 - AR^2\Big) + \Big(-C(v-S)^2 + CS^2\Big) + H \\
&= A(u-R)^2 - C(v-S)^2 - (AR^2 - CS^2 - H) \\
&= A(u-R)^2 - C(v-S)^2 - T,
\end{aligned}$$

where

$$T = AR^2 - CS^2 - H = \frac{D^2}{4A} - \frac{E^2}{4C} - H.$$

∎

Problem 18

Suppose $A, C, T > 0$. Find a real affine change of coordiantes that maps the hyperbola

$$V(A(x-R)^2 - C(y-S)^2 - T),$$

to the hyperbola

$$V(u^2 - v^2 - 1).$$

Proof. Notice

$$A(x-R)^2 - C(y-S)^2 - T = 0 \iff \frac{A(x-R)^2}{T} - \frac{C(y-S)^2}{T} = 1.$$

We set

$$u^2 = \frac{A(x-R)^2}{T}, \quad v^2 = \frac{C(y-S)^2}{T},$$

and solving for x, y shows

$$x = \sqrt{\frac{T}{A}}\,u + R, \quad y = \sqrt{\frac{T}{C}}\,v + S.$$

Substituting into the original equation, we find

$$\begin{aligned} A(x-R)^2 - C(y-S)^2 - T &= A\left(\sqrt{\frac{T}{A}}\,u\right)^2 - C\left(\sqrt{\frac{T}{C}}\,v\right)^2 - T \\ &= Tu^2 - Tv^2 - T \\ &= T(u^2 - v^2 - 1). \end{aligned}$$

∎

Problem 19

Give an intuitive argument, based on the number of connected components, for the fact that no ellpise can be transformed into a hyperbola by a real affine change of coordiantes.

Solution: A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve the number of connected components. Therefore, no real affine change can transform an ellipse into a hyperbola.

Problem 20

Show that there is no real affine change of coordinates

$$u = ax + by + e$$

$$v = cx + dy + f,$$

that transforms the ellipse $V(x^2 + y^2 - 1)$ to the hyperbola $V(u^2 - v^2 - 1)$.

Proof. For contradiction, suppose such a real affine change exists.

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We must have

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the ellipse $x^2 + y^2 = 1$. Now substituting $y^2 = 1 - x^2$ we see for this to vanish for all (x, y), the coefficients of x^2 and y^2 must be

$$a^2 - c^2 = b^2 - d^2,$$

which would make the squared coefficients have the same sign, contradicting the requirement for a hyperbola that they have opposite signs. Thus there is no real affine transformation from an ellipse to a hyperbola. ∎

Problem 21

Give an intuitive argument, based on boundedness, for the fact that no parabola can be transformed into an ellipse by a real affine change of coordinates.

Solution: A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve boundedness. Therefore, no real affine change can transform a parabola into an ellipse.

Problem 22

Show that there is no real affine change of coordinates that trasnforms the parabola $V(x^2 - y)$ to the circle $V(u^2 + v^2 - 1)$.

Proof. For contradiction, suppose such a real affine change exists.

$$\begin{aligned} u^2 + v^2 &= (ax + by + e)^2 + (cx + dy + f)^2 \\ &= (a^2 + c^2)x^2 + (b^2 + d^2)y^2 + 2(ab + cd)xy + 2(ae + cf)x + 2(be + df)y + (e^2 + f^2). \end{aligned}$$

We must have

$$(a^2 + c^2)x^2 + (b^2 + d^2)y^2 + 2(ab + cd)xy + 2(ae + cf)x + 2(be + df)y + (e^2 + f^2) - 1 = 0$$

for all points on the parabola $y = x^2$. Now substituting $y = x^2$, we have

$$(b^2 + d^2)x^4 + 2(ab + cd)x^3 + (a^2 + c^2 + 2(be + df))x^2 + 2(ae + cf)x + (e^2 + f^2) - 1 = 0.$$

For this to vanish for all x all coefficients must be zero so

$$b^2 + d^2 = 0, \quad ab + cd = 0, \quad a^2 + c^2 + 2(be + df) = 0, \quad ae + cf = 0.$$

It follows that $a = b = c = d = 0$ and therefore $u^2 + v^2 = e^2 + f^2$ is constant, which cannot equal $x^2 + y^2$ on the parabola. Thus there is no real affine transformation from the parabola to the circle. ∎

Problem 23

Give an intuitive argument, based on the number of connected components, for the fact that no parabola can be transformed into a hyperbola by a real affine change of coordinates.

Solution: A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve the number of components. Therefore, no real affine change can transform a parabola into a hyperbola.

Problem 24

Show that there is no real affine change of coordinates that transforms that parabola $V(x^2 - y)$ to the hyperbola $V(u^2 - v^2 - 1)$.

Proof. For contradiction, suppose such a real affine change exists. Then

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We must have

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the parabola $y = x^2$. Substituting $y = x^2$, we get

$$(b^2 - d^2)x^4 + 2(ab - cd)x^3 + (a^2 - c^2)x^2 + 2(ae - cf)x + 2(be - df)x^2 + (e^2 - f^2) - 1 = 0.$$

For this to vanish for all x, the coefficient of x^4 must be zero

$$b^2 - d^2 = 0 \implies b = \pm d.$$

Then, the x^3 coefficient gives $ab - cd = 0$. Since Since $b = \pm d$, we have $a = \pm c$. Then, the x^2 coefficient becomes $a^2 - c^2 + 2(be - df)$. Since $a = \pm c$ and $b = \pm d$, this is zero if all coefficients vanish. Thus $u^2 - v^2$ are constant, which cannot equal $x^2 - y$ on the parabola. Therefore, there is no real affine transformation from the parabola to the hyperbola. ∎

1.3 Conics over the Complex Numbers

Problem 1

Show that the set

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 1 = 0\},$$

is empty but that the set

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 + 1 = 0\},$$

is not empty. In fact, show that given any complex number x there must exist $y \in \mathbb{C}$ such that

$$(x, y) \in C.$$

Then show that if $x \neq \pm i$, then there are two distinct values $y \in \mathbb{C}$ such that $(x, y) \in C$, while if $x = \pm i$ there is only one such y.

Proof. Suppose $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 + 1 = 0 \iff x^2 + y^2 = -1$. Then $x^2 \geq 0$ and $y^2 \geq 0$ so $x^2 + y^2 \geq 0$, which is a contradiction. ∎

Proof. Let $(x, y) = (i, 0) \in \mathbb{C}^2$. Then

$$x^2 + y^2 + 1 = -1 + 1 = 0.$$

Thus $(x, y) \in C$. ∎

Proof. Let x be an arbitrary complex number. Furthermore, let $y = \sqrt{-1 - x^2}$. Then

$$x^2 + \left(\sqrt{-1 - x^2}\right)^2 + 1 = x^2 - 1 - x^2 + 1 = 0.$$

Thus $(x, y) \in C$. ∎

Proof. Suppose $x \neq \pm i$. Then $1 + x^2 \neq 0$, so $\sqrt{1 + x^2} \neq 0$. Let

$$y = \pm i\sqrt{1 + x^2}.$$

These are two distinct values of y. Then

$$x^2 + y^2 + 1 = x^2 - (1 + x^2) + 1 = 0.$$

Now suppose $x = \pm i$. Then $1 + x^2 = 0$, so $y^2 = 0$ and it follows that $y = 0$. Therefore, there is exactly one value of y. ∎

Problem 2

Let

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + f,$$

with $a \neq 0$. Show that for any value $y \in \mathbb{C}$, there must be at least one $x \in \mathbb{C}$, but no more than two such x's, such that

$$P(x, y) = 0.$$

[Hint: Write $P(x, y) = Ax^2 + Bx + C$ as a function whose coefficients A, B, and C are themselves functions of y, and use the quadratic formula. As mentioned before, this technique will be used frequently.]

Proof. Let $A = a$, $B = by + d$, and $C = cy^2 + ey + f$. Notice

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = ax^2 + (by + d)x + (cy^2 + ey + f) = Ax^2 + Bx + C.$$

Applying the quadratic formula we find

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Since $A = a \neq 0$ this is defined. Now if $B^2 - 4AC = 0$ then we get one corresponding x. Otherwise, we get two corresponding x's. ∎

Problem 3

Let $C = V\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) \subset \mathbb{C}^2$. Show that C is unbounded in x and y.

Proof. We can solve for x in terms of y

$$\frac{x^2}{4} = 1 - \frac{y^2}{9} \iff x = \pm 2\sqrt{1 - \frac{y^2}{9}}.$$

Since $y \in \mathbb{C}$ is arbitrary and square roots always exist in $\mathbb{C}$, for any value of y there is a corresponding value of x. As $|y|$ becomes arbitrarily large, $1 - \frac{y^2}{9}$ becomes arbitrarily large, and thus the corresponding x is arbitrarily large. Thus C is unbounded in both x and y. ∎

Problem 4

Let $C = V(x^2 - y^2 - 1) \subset \mathbb{C}^2$. Show that there is a continuous path on the curve C from the point $(-1, 0)$ to the point $(1, 0)$, despite the fact that no such continuous path exists in $\mathbb{R}^2$.

Proof. Let $x(t) = \cos(t)$ and $y(t) = i\sin(t)$. Then

$$x(t)^2 - y(t)^2 - 1 = \cos^2(t) - (i\sin(t))^2 - 1 = \cos^2(t) + \sin^2(t) - 1 = 0.$$

∎

Problem 5

Show that if $x = u$ and $y = iv$, then the circle $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\}$ transforms into the hyperbola $\{(u, v) \in \mathbb{C}^2 \mid u^2 - v^2 = 1\}$.

Proof. Suppose $x = u$ and $y = iv$. Then

$$x^2 + y^2 = u^2 + (iv)^2 = u^2 - v^2 = 1.$$

∎

Problem 6

Show that if $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then the inverse change of coordinates is

$$x = \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf).$$
$$y = \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af).$$

Proof. We need to solve the two equations $u = ax + by + e$ and $v = cx + dy + f$ in two unknowns x and y. Translating this to linear algebra, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u - e \\ v - f \end{bmatrix}.$$

Using Cramer's rule we see

$$x = \frac{\begin{vmatrix} u - e & b \\ v - f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d(u - e) - b(v - f)}{ad - bc},$$

$$y = \frac{\begin{vmatrix} a & u - e \\ c & v - f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c(u - e) + a(v - f)}{ad - bc}.$$

Therefore

$$x = \frac{du - bv - de + bf}{ad - bc}, \quad y = \frac{-cu + av + ce - af}{ad - bc}.$$

∎

Problem 7

Use Theorem 1.2.25 together with the new result of Exercise 1.3.5 to conclude that all ellipses and hyperbolas are equivalent under complex affine changes of coordinates.

Proof. By Theorem 1.2.25, any ellipse can be transformed via an affine change of coordinates to a circle. Then, by Exercise 1.3.5 the circle can be transformed via a complex affine change to a hyperbola. ∎

Problem 8

Show that the circle $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 - 1 = 0\}$ is not equivalent under a complex affine change of coordinates to the parabola $\{(u, v) \in \mathbb{C}^2 \mid u^2 - v^2 = 0\}$.

Proof. For contradiction, suppose such a complex affine change exists

$$u = ax + by + e, \quad v = cx + dy + f.$$

Then

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We need

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the circle. Substituting $y = \sqrt{1 - x^2}$, the lhs must vanish for all x. The highest-degree terms show $b^2 - d^2 = 0 \implies b = \pm d$ and the other coefficients similarly give $a = \pm c$, $e = \pm f$. But then $u^2 - v^2$ would be constant, which cannot equal $x^2 + y^2 - 1$. Therefore there is no complex affine transformation mapping the circle to the hyperbola $u^2 - v^2 = 0$. ∎

Problem 9

Let

$$C = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}.$$

Give a bijection from

$$C \cap \{(x + iy, u + iv) \mid x, u \in \mathbb{R}, y = 0, v = 0\}.$$

to the real circle of the unit radius in $\mathbb{R}^2$.

Solution:

$$(x + iy, u + iv) \mapsto (x, u).$$

Problem 10

Let

$$C = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}.$$

Give a bijection from

$$C \cap \{(x + iy, u + iv) \in \mathbb{R}^4 \mid x, v \in \mathbb{R}, y = 0, u = 0\},$$

to the hyperbola $V(x^2 - v^2 - 1)$ in $\mathbb{R}^2$.

Solution:

$$(x + 0i, 0 + iv) \mapsto (y, u).$$

1.4 The Complex Projective Plane $\mathbb{P}^2$

Problem 1

1. Show that $(2, 1+i, 3i) \sim (2-2i, 2, 3+3i)$.
2. Show that $(1, 2, 3) \sim (2, 4, 6)$ and $(-2, -4, -6) \sim (-i, -2i, -3i)$.
3. Show that $(2, 1+i, 3i) \nsim (4, 4i, 6i)$.
4. Show that $(1, 2, 3) \nsim (3, 6, 8)$.

Proof. Let $\lambda = \frac{2}{2-2i} = \frac{1}{2} + \frac{1}{2}i$. Then

$$\lambda(2-2i) = \frac{2}{2-2i}(2-2i) = 2,$$

$$\lambda \cdot 2 = \left(\frac{1}{2} + \frac{1}{2}i\right)2 = 1+i,$$

$$\lambda(3+3i) = \left(\frac{1}{2} + \frac{1}{2}i\right)(3+3i) = 3i.$$

∎

Proof. Let $\lambda = \frac{1}{2}$. Then

$$\lambda \cdot 2 = 1,$$

$$\lambda \cdot 4 = 2,$$

$$\lambda \cdot 6 = 3.$$

∎

Proof. Let $\lambda = 2i$. Then

$$\lambda \cdot (-i) = -2,$$

$$\lambda \cdot (-2i) = -4,$$

$$\lambda \cdot (-3i) = -6.$$

∎

Proof. Suppose there exists λ such that $\lambda(4, 4i, 6i) = (2, 1+i, 3i)$. Then

$$\lambda \cdot 4 = 2 \implies \lambda = \frac{1}{2},$$

$$\lambda \cdot 4i = 2i \neq 1+i.$$

Thus no such λ exists. ∎

Proof. Suppose there exists λ such that $\lambda(3, 6, 8) = (1, 2, 3)$. Then

$$\lambda \cdot 3 = 1 \implies \lambda = \frac{1}{3},$$

$$\lambda \cdot 8 = \frac{8}{3} \neq 3.$$

Thus no such λ exists. ∎

Problem 2

Show that $\sim$ is an equivalence relation.

Proof. Suppose $(x, y, z), (a, b, c), (d, e, f) \in \mathbb{C}^3 - \{(0, 0, 0)\}$. Then $\lambda = 1$ shows $(x, y, z) \sim (x, y, z)$. Thus $\sim$ is reflexive.

Suppose $(a, b, c) \sim (d, e, f)$. Then there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(a, b, c) = (\lambda d, \lambda e, \lambda f)$. Therefore $\left(\frac{1}{\lambda}a, \frac{1}{\lambda}b, \frac{1}{\lambda}c\right) = (d, e, f)$. It follows that $(d, e, f) \sim (a, b, c)$. Thus $\sim$ is symmetric.

Suppose $(x, y, z) \sim (a, b, c)$ and $(a, b, c) \sim (d, e, f)$. Then there exist $\lambda_1, \lambda_2 \in \mathbb{C} - \{0\}$ such that $(x, y, z) = (\lambda_1 a, \lambda_1 b, \lambda_1 c)$ and $(a, b, c) = (\lambda_2 d, \lambda_2 e, \lambda_2 f)$. Then

$$(x, y, z) = (\lambda_1 a, \lambda_1 b, \lambda_1 c) = (\lambda_1\lambda_2 d, \lambda_1\lambda_2 e, \lambda_1\lambda_2 f).$$

Thus $(x, y, z) \sim (d, e, f)$. Therefore $\sim$ is transitive. ∎

Problem 3

Suppose that $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ and that $x_1 = x_2 \neq 0$. Show that $y_1 = y_2$ and $z_2 = z_2$.

Proof. Since $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(x_1, y_1, z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$. Thus $x_1 = \lambda x_2 = \lambda x_1$ therefore $\lambda = \frac{x_1}{x_1} = 1$. It follows that $y_1 = y_2$ and $z_1 = z_2$. ∎

Problem 4

Suppose that $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ with $z_1 \neq 0$ and $z_2 \neq 0$. Show that

$$(x_1, y_1, z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2, y_2, z_2).$$

Proof. We see

$$(x_1, y_1, z_1) = \left(z_1 \cdot \frac{x_1}{z_1}, z_1 \cdot \frac{y_1}{z_1}, z_1 \cdot 1\right).$$

Since $z_1 \neq 0$ we see $(x_1, y_1, z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right)$. Now, since $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(x_1, y_1, z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$. Since $z_1 = \lambda z_2$ and $z_1, z_2 \neq 0$ we see

$$\frac{x_1}{z_1} = \frac{\lambda x_2}{\lambda z_2} = \frac{x_2}{z_2} \quad \text{and} \quad \frac{y_1}{z_1} = \frac{\lambda y_2}{\lambda z_2} = \frac{y_2}{z_2}.$$

Thus

$$\left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right).$$

Since

$$(x_2, y_2, z_2) = \left(z_2 \cdot \frac{x_2}{z_2}, z_2 \cdot \frac{y_2}{z_2}, z_2 \cdot 1\right),$$

and $z_2 \neq 0$ we see

$$\left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2, y_2, z_2).$$

Therefore,

$$(x_1, y_1, z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2, y_2, z_2).$$

∎

Problem 5

1. Find the equivalence class of $(0, 0, 1)$.
2. Find the equivalence class of $(1, 2, 3)$.

Solution (1):

$$\{(0, 0, c) \in \mathbb{C}^3 \mid c \neq 0\}.$$

Solution (2):

$$\{(\lambda, 2\lambda, 3\lambda) \in \mathbb{C}^3 \mid \lambda \neq 0\}.$$

Problem 6

Show that the equivalence class $(1 : 2 : 3)$ and $(2 : 4 : 6)$ are equal as sets.

Proof. Clearly, with $\lambda = \frac{1}{2} \in \mathbb{C}$ we have $(1, 2, 3) = (\lambda 2, \lambda 4, \lambda 6)$. Thus $(1, 2, 3) \sim (2, 4, 6)$ so $(1 : 2 : 3) = (2 : 4 : 6)$. ∎

Problem 7

Explain why the elements of $\mathbb{P}^2$ can intuitively be thought of as complex lines through the origin in $\mathbb{C}^3$.

Solution: Take a line passing through the origin in $\mathbb{C}^3$ with direction vector $(a, b, c) \neq (0, 0, 0)$. This line consists of all points of the form $(\lambda a, \lambda b, \lambda c)$ such that $\lambda \in \mathbb{C}$. If we require $\lambda \neq 0$ we get the equivalence class $(a : b : c) \in \mathbb{P}^2$. Thus each element of $\mathbb{P}^2$ represents a complex line through the origin in $\mathbb{C}^3$.

Problem 8

If $c \neq 0$, show, in $\mathbb{C}^3$, that the line $x = \lambda a$, $y = \lambda b$, $z = \lambda c$ intersects the plane $\{(x, y, z) \mid z = 1\}$ in exactly one point. Show that this point of intersection is $\left(\frac{a}{c}, \frac{b}{c}, 1\right)$.

Proof. Suppose $c \neq 0$. At the intersection we must have $z = \lambda c = 1$ so $\lambda = \frac{1}{c}$. Thus

$$(\lambda a, \lambda b, \lambda c) = \left(\frac{a}{c}, \frac{b}{c}, 1\right).$$

∎

Problem 9

Show that the map $\psi : \mathbb{C}^2 \to \{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\}$ defined by $\psi(x, y) = (x : y : 1)$ is a bijection.

Proof. Suppose $(a, b), (x, y) \in \mathbb{C}^2$ such that $\psi(x, y) = \psi(a, b)$. Then

$$\psi(x, y) = \psi(a, b) \iff (x : y : 1) = (a : b : 1).$$

There exists $\lambda \neq 0$ such that $(x, y, 1) = (\lambda a, \lambda b, \lambda)$. Therefore $\lambda = 1$ thus $x = a$ and $y = b$. Thus ψ is injective. Let $(x : y : z)$ be an arbitrary element in $\{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\}$. Then

$$(x : y : z) = \left(\frac{x}{z} : \frac{y}{z} : 1\right) = \psi\left(\frac{x}{z}, \frac{y}{z}\right).$$

Thus ψ is surjective. It follows that ψ is bijective. ∎

Problem 10

Find a map from $\{(x, y, z) \in \mathbb{P}^2 \mid z \neq 0\}$ to $\mathbb{C}^2$ that is the inverse of the map ψ in Exercise 1.4.9.

Solution: Let

$$\phi : \{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\} \longrightarrow \mathbb{C}^2$$

be defined by

$$\phi(x : y : z) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

Problem 11

Consider the line $l = \{(x, y) \in \mathbb{C}^2 \mid ax + by + c = 0\}$ in $\mathbb{C}^2$. Assume $a, b \neq 0$. Explain why, as $|x| \to \infty$, $|y| \to \infty$. (Hence, $|x|$ is the modulus of x.)

Proof. We see $y = \frac{-c-ax}{b}$ and $x = \frac{-by-c}{a}$. Since b and c are constants, as $|y| \to \infty$ we have $|x| \to \infty$. ∎

Problem 12

Consider again the line l. We know that a and b cannot both be 0, so we will assume without loss of generality that $b \neq 0$.

1. Show that the image of l in $\mathbb{P}^2$ under ψ is the set
$$\{(bx : -ax - c : b) \mid x \in \mathbb{C}\}.$$
2. Show that this set equals the following union.
$$\{(bx : -ax - c : b) \mid x \in \mathbb{C}\} = \{(0 : -c : b)\} \cup \left\{\left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right)\right\}.$$
3. Show that as $|x| \to \infty$, the second set in the above union becomes
$$\left\{\left(1 : -\frac{a}{b} : 0\right)\right\}.$$

Proof. We start by solving explicitly for y and note $b \neq 0$.

$$ax + by + c = 0 \iff y = \frac{-ax - c}{b}.$$

Then

$$\psi\left(x, \frac{-ax - c}{b}\right) = \left(x : \frac{-ax - c}{b} : 1\right) = (bx : -ax - c : b).$$

∎

Proof. There are two cases. If $x = 0$ then

$$(bx : -ax - c : b) = (0 : -c : b).$$

Otherwise, if $x \neq 0$ then we can divide by $bx \neq 0$ and see

$$(bx : -ax - c : b) = \left(1 : \frac{-ax - c}{bx} : \frac{1}{x}\right) = \left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right).$$

∎

Proof. As $|x| \to \infty$, we have $\frac{c}{bx} \to 0$ and $\frac{1}{x} \to 0$. Thus

$$\left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right) \to \left(1 : -\frac{a}{b} : 0\right).$$

∎

Problem 13

Explain why the following polynomials are homogeneous, and find each degree.

1. $x^2 + y^2 - z^2$.
2. $xz - y^2$.
3. $x^3 + 3xy^2 + 4y^3$.
4. $x^4 + x^2y^2$.

Solution (1): All monomials have total degree 2 thus it is homogeneous. The total degree is 2.

Solution (2): All monomials have total degree 2 thus it is homogeneous. The total degree is 2.

Solution (3): All monomials have total degree 3 thus it is homogeneous. The total degree is 3.

Solution (4): All monomials have total degree 4 thus it is homogeneous. The total degree is 4.

Problem 14

Explain why the following polynomials are not homogeneous.

1. $x^2 + y^2 - z$.
2. $xz - y$.
3. $x^2 + 3xy^2 + 4y^3 + 3$.
4. $x^3 + x^2y^2 + x^2$.

Solution (1): z has total degree 1 and x^2 has total degree 2 thus it is not homogeneous.

Solution (2): xz has total degree 2 and y has total degree 1 thus it is not homogeneous.

Solution (3): 3 has total degree 0 and x^2 has total degree 2 thus it is not homogeneous.

Solution (4): x^3 has total degree 3 and x^2 has total degree 2 thus it is not homogeneous.

Problem 15

Show that if the homogeneous equation $Ax + By + Cz = 0$ holds for every point (x, y, z) in $\mathbb{C}^3 - \{0, 0, 0\}$, then it holds for every point in $\mathbb{C}^3$ that belongs to the equivalence class $(x : y : z)$ in $\mathbb{C}^2$.

Proof. Suppose the homogeneous equation $Ax + By + Cz = 0$ holds for all $(x, y, z) \in \mathbb{C}^3 - \{(0, 0, 0)\}$. Consider the polynomial in x

$$f(x) = Ax + (By + Cz).$$

Since $f(x) = 0$ for all $x \in \mathbb{C}$, $A = 0$. Similarly, $B = 0$ and $C = 0$. Therefore $Ax + By + Cz = 0$ for all $(x, y, z) \in \mathbb{C}^3$, which includes $(x : y : z)$. ∎

Problem 16

Show that if the homogenous equation $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$

Proof. Suppose the homogeneous equation $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$. Consider the polynomial in x

$$f(x) = Ax^2 + Dxy + Exz + (By^2 + Cz^2 + Fyz).$$

Since $f(x) = 0$ for all $x \in \mathbb{C}$, $A = 0$. Similarly, $B = C = D = E = F = 0$. Therefore $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$ for all $(x, y, z) \in \mathbb{C}^3$, which includes $(x : y : z)$. ∎

Problem 17

State and prove the generalization of the previous two exercises for any degree n homogenous equation $P(x, y, z) = 0$.

Proof. Suppose the homogeneous equation $P(x, y, z) = 0$ of degree n. Consider the polynomial in x

$$f(x) = P(x, y, z).$$

Since $f(x) = 0$ for all $x \in \mathbb{C}$ coefficients of powers of x must vanish. Similarly, all other coefficients vanish. Therefore $P(x, y, z) = 0$ for all $(x, y, z) \in \mathbb{C}^3$, which includes $(x : y : z)$. ∎

Problem 18

Consider the non-homogeneous equation $P(x, y, z) = x^2 + 2y + 2z = 0$. Show that $(2, -1, -1)$ satisfies the equation. Find a point of the equivalence class $(2 : -1 : -1)$ that does not satisfy this equation.

Solution:

$$2^2 + 2(-1) + 2(-1) = 0$$

$$4^2 + 2(-2) + 2(-2) = 16 - 4 - 4 = 8 \neq 0$$

Problem 19

Homogenize the following equations. Then find point(s) where the curves intersect the line at infinity.

1. $ax + by + c = 0$.
2. $x^2 + y^2 = 1$.
3. $y = x^2$.
4. $x^2 + 9y^2 = 1$.
5. $y^2 - x^2 = 1$.

Solution (1):

$$z \cdot \left(a\frac{x}{z} + b\frac{y}{z} + c\right) = 0 \iff ax + by + cz = 0.$$

At $z = 0$ we have

$$ax + by = 0.$$

A point at infinity is

$$(x : y : z) = (b : -a : 0).$$

Solution (2):

$$z^2 \cdot \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right) = 0 \iff x^2 + y^2 - z^2 = 0.$$

At $z = 0$ we have

$$x^2 + y^2 = 0.$$

Points at infinity are

$$(x : y : z) = (1 : i : 0), (1 : -i : 0).$$

Solution (3):

$$z^2 \cdot \left(\frac{y}{z} - \left(\frac{x}{z} \right)^2 \right) = 0 \iff x^2 - yz = 0.$$

At $z = 0$ we have

$$x^2 = 0 \implies x = 0.$$

Point at infinity is

$$(x : y : z) = (0 : 1 : 0).$$

Solution (4):

$$z^2 \cdot \left(\left(\frac{x}{z} \right)^2 + 9 \left(\frac{y}{z} \right)^2 - 1 \right) = 0 \iff x^2 + 9y^2 - z^2 = 0.$$

At $z = 0$ we have

$$x^2 + 9y^2 = 0.$$

Points at infinity are

$$(x : y : z) = (3i : 1 : 0), (-3i : 1 : 0).$$

Solution (5):

$$z^2 \cdot \left(\left(\frac{y}{z} \right)^2 - \left(\frac{x}{z} \right)^2 - 1 \right) = 0 \iff y^2 - x^2 - z^2 = 0.$$

At $z = 0$ we have

$$y^2 - x^2 = 0 \implies (y - x)(y + x) = 0.$$

Points at infinity are

$$(x : y : z) = (1 : 1 : 0), (1 : -1 : 0).$$

Problem 20

Show that in $\mathbb{P}^2$, any two distinct lines will intersect in a point. Notice this implies that parallel lines in $\mathbb{C}^2$, when embeed in $\mathbb{P}^2$, intersect at the line at infinity.

Proof. Suppose we have the lines

$$l_1 : ax + by + cz = 0, \quad l_2 : dx + ey + fz = 0.$$

If l_1 is not parallel to l_2, then set $z = 1$ and solve the two equations for x and y to find the intersection point in $\mathbb{C}^2$. If l_1 is parallel to l_2, then homogenizing shows that the intersection occurs at $z = 0$. ∎

Problem 21

Once we have homogenized an equation, the original variables x y are no more important than the variable z. Suppose we regard x and z as the original variables in our homogeneous equation. Then the image of the xz-plane in $\mathbb{P}^2$ would be $\{(x, y, z) \in \mathbb{P}^2 \mid y = 1.\}$.

1. Homogenize the equations for the parallel lines $y = x$ and $y = x + 2$.
2. Now regard x and z as the original variables, and set $y = 1$ to sketch the image of the lines in the xz-plane.
3. Explain why the lines in part meet at the x-axis.

Solution (1): The lines are $y = x$ and $y = x + 2$. Homogenizing with z gives:

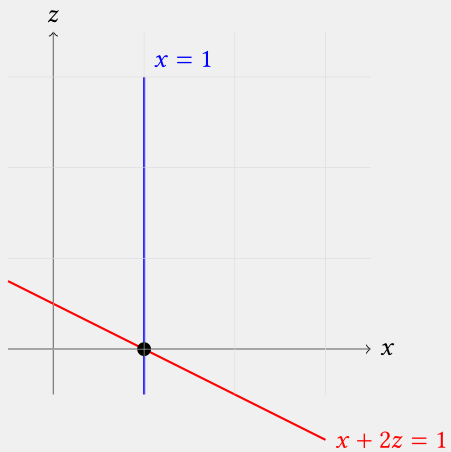
$$y - x = 0 \quad \implies \quad y - x = 0$$

$$y - x - 2 = 0 \quad \Longrightarrow \quad y - x - 2z = 0$$

Solution (2): Regarding x and z as the original variables and setting $y = 1$

$$1 - x = 0 \quad \Longrightarrow \quad x = 1$$

$$1 - x - 2z = 0 \quad \Longrightarrow \quad x + 2z = 1$$



Solution (3): The lines intersect at the x-axis because in the xz-plane the x-axis is defined by $z = 0$.

1.5 Projective Changes of Coordinates

Problem 1

For the complex affine change of coordinates

$$u = ax + by + e.$$

$$v = cx + dy + f,$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, show that

$$u = ax + by + ez,$$

$$v = cx + dy + fz,$$

$$w = z,$$

is the corresponding projective change of coordinates.

Proof. We have $\psi(u, v) = \left(\frac{ax+by+e}{z} : \frac{cx+dy+f}{z} : 1\right) = (ax + by + e : cx + dy + f : z)$. ∎

Problem 2

Let $C_1 = V(x^2 + y^2 - 1)$ be an ellipse in $\mathbb{C}^2$ and let $C_2 = V(u^2 - v)$ be a parabola in $\mathbb{C}^2$. Homogenize the defining polynomials for C_1 and C_2 and show that the projective change of coordinates

$$u = ix,$$

$$v = y + z,$$

$$w = y - z,$$

transforms the ellipse in $\mathbb{P}^2$ into the parabola in $\mathbb{P}^2$.

Proof. We first homogenize C_1 and C_2 to obtain

$$x^2 + y^2 - z^2 = 0 \quad \text{and} \quad u^2 - vw = 0.$$

Then substituting our change of variables gives

$$u^2 - vw = (ix)^2 - (y+z)(y-z) = -x^2 - (y^2 - z^2) = -(x^2 + y^2 - z^2) = 0,$$

as required. ∎

Problem 3

Use the results of Section 1.3, together with the above problem, to show that, under a projective change of coordinates, all ellipses, hyperbolas, and parabolas are equivalent in $\mathbb{P}^2$.

Proof. By Section 1.3 ellipses and hyperbolas are equivalent under affine changes of coordinates, and by Problem 1 these extend to projective changes of coordinates in $\mathbb{P}^2$. By Problem 2 ellipses are equivalent to parabolas in $\mathbb{P}^2$. Therefore under a projective change of coordinates, all ellipses, hyperbolas, and parabolas are equivalent in $\mathbb{P}^2$. ∎

1.6 The Complex Projective Line $\mathbb{P}^1$

Problem 1

Suppose that $(x_1, y_1) \sim (x_2, y_2)$ and that $x_1 = x_2 \neq 0$. Show that $y_1 = y_2$.

Proof. We know there exists λ such that $x_1 = \lambda x_2$. Dividing by $x_2 \neq 0$ shows $\lambda = 1$. But then $y_1 = \lambda y_2 = y_2$. ∎

Problem 2

Suppose that $(x_1, y_1) \sim (x_2, y_2)$ with $y_1 \neq 0$ and $y_2 \neq 0$. Show that

$$(x_1, y_1) \sim \left(\frac{x_1}{y_1}, 1\right) = \left(\frac{x_2}{y_2}, 1\right) \sim (x_2, y_2).$$

Proof. Let $\lambda = y_1$ to see that

$$\left(\lambda \frac{x_1}{y_1}, \lambda \cdot 1\right) = (x_1, y_1).$$

Thus $(x_1, y_1) \sim \left(\frac{x_1}{y_1}, 1\right)$. Similarly, $(x_2, y_2) \sim \left(\frac{x_2}{y_2}, 1\right)$. Since $(x_1, y_1) \sim (x_2, y_2)$ and $\sim$ is an equivalence relation

$$\left(\frac{x_1}{y_1}, 1\right) = \left(\frac{x_2}{y_2}, 1\right).$$

∎

Problem 3

Explain why the elements of $\mathbb{P}^1$ can intuitively be thought of as complex lines through the origin in $\mathbb{C}^2$.

Solution: Take a line passing through the origin in $\mathbb{C}^2$ with direction vector $(a, b) \neq (0, 0)$. This line consists of all points of the form $(\lambda a, \lambda b)$ such that $\lambda \in \mathbb{C}$. If we require $\lambda \neq 0$ we get the equivalence class $(a : b : c) \in \mathbb{P}^2$. Thus each element of $\mathbb{P}^1$ represents a complex line through the origin in $\mathbb{C}^2$.

Problem 4

If $b \neq 0$, show that the line $x = \lambda a$, $y = \lambda b$ will intersect the line $\{(x, y) \mid y = 1\}$ in exactly one point. Show that the point of intersection is $\left(\frac{a}{b}, 1\right)$.

Proof. We have $1 = y = \lambda b$ thus $\lambda = \frac{1}{b}$. Therefore $(x, y) = (\lambda a, 1) = \left(\frac{a}{b}, 1\right)$. ∎

Problem 5

Show that the map $\psi : \mathbb{C} \to \{(x : y) \in \mathbb{P}^2 \mid y \neq 0\}$ defined by $\psi(x) = (x : 1)$ is a bijection.

Proof. Suppose $a, b \in \mathbb{C}$ such that $\psi(a) = \psi(b)$. Then

$$\psi(a) = \psi(b) \iff (a : 1) = (b : 1).$$

But then since $1 = \lambda 1$ we have $\lambda = 1$ so $a = b$. Let $(a : y)$ be an arbitrary element in $\{(x : y) \in \mathbb{P}^2 \mid y \neq 0\}$. Then $(a : y) = \left(\frac{a}{y} : 1\right) = \psi\left(\frac{a}{b}\right)$. Thus ψ is a bijection. ∎

Problem 6

Find a map $\{(x : y) \in \mathbb{P}^1 \mid y \neq 0\}$ to $\mathbb{C}$ that is the inverse of the map ψ in Exercise 1.6.5.

Solution:

$$\psi^{-1}(x : y) = \frac{x}{y}.$$

Problem 7

Consider the map $\psi : \mathbb{C} \to \mathbb{P}^2$ given by $\psi(x) = (x : 1)$. Show that as $|x| \to \infty$, we have $\psi(x) \to (1 : 0)$.

Proof. We have

$$(x : 1) = \left(1 : \frac{1}{x}\right).$$

As $|x| \to \infty$, we have $\frac{1}{x} \to 0$ thus

$$\left(1 : \frac{1}{x}\right) \to (1 : 0).$$

∎

Problem 8

Let p denote the point $(0, 0, 1) \in S^2$, and let l denote the line through p and the point $(x, y, 0)$ in the

xy-plane, whose parametrization is given by

$$\rho(t) = (1-t)(0,0,1) + t(x,y,0),$$

i.e.,

$$l = \{(tx, ty, 1-t) \mid t \in \mathbb{R}\}.$$

1. l clearly intersects S^2 at the point p. Show that there is exactly one other point of intersection q.
2. Find the coordinates of q.
3. Define the map $\psi : \mathbb{R}^2 \to S^2 - \{p\}$ to be the map that takes the point (x,y) to the point q. Show that ψ is a continuous bijection.
4. Show that as $\sqrt{x^2+y^2} \to \infty$, we have $\psi(x,y) \to p$. Thus as we move away from the origin in $\mathbb{R}^2$, $\psi(x,y)$ moves toward the North Pole.

Proof. We substitute l into the unit sphere equation to find

$$x^2+y^2+z^2-1 = (tx)^2+(ty)^2+(1-t)^2-1 = t^2(x^2+y^2+1)-2t = t\big(t(x^2+y^2+1)-2\big) = 0.$$

Now $t=0$ corresponds to p. The other point corresponds to

$$t(x^2+y^2+1)-2 = 0 \qquad \Longrightarrow \qquad t = \frac{2}{x^2+y^2+1}.$$

Substituting back into the line gives

$$q = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

∎

Proof. Each coordinate is a continuous fraction thus ψ is continuous. Suppose $(x,y),(a,b) \in \mathbb{R}^2$ such that

$$\psi(x,y) = \psi(a,b) \iff \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right)$$

From the third coordinate we infer $x^2+y^2+1 = a^2+b^2+1$. Then from the first coordinates we see $a=x$ and $b=y$. Thus ψ is injective. Suppose $(X,Y,Z) \in S^2 - \{p\}$. We can solve the following equations for x, y

$$\frac{2x}{x^2+y^2+1} = X, \quad \frac{2y}{x^2+y^2+1} = Y, \quad \frac{x^2+y^2-1}{x^2+y^2+1} = Z.$$

Thus

$$x = \frac{X}{1-Z} \quad \text{and} \quad y = \frac{Y}{1-Z}.$$

Finally, we see

$$\psi(x,y) = (X,Y,Z).$$

Thus ψ is surjective and it follows that ψ is a bijection. ∎

Proof. As $\sqrt{x^2+y^2} \to \infty$ we have $\frac{2x}{x^2+y^2+1} \to 0$, $\frac{2y}{x^2+y^2+1} \to 0$, and $\frac{x^2+y^2-1}{x^2+y^2+1} \to 1$. Thus $q \to (0,0,1) = p$. ∎

Problem 9

Determine which point(s) in $\mathbb{P}^1$ do **not** have two representitives of the form $(x : 1) = (1 : \frac{1}{x})$.

Solution:

$$(0 : 1) \in \mathbb{P}^1.$$

Problem 10

Map $U \to \mathbb{P}^1$ via $x \mapsto (x : 1)$ and map $V \to \mathbb{P}^1$ via $y \mapsto (1 : y)$. Show that $(x : 1) \mapsto \left(1 : \frac{1}{x}\right)$ is a natural one-to-one map from U^* to V^*.

Proof. Let $x, y \in U^*$ then under the map we have $\left(1 : \frac{1}{x}\right) = \left(1 : \frac{1}{y}\right) \in V^*$. Then since 1 is the first coordinate we have $\lambda = 1$ thus $x = y$. Therefore the mapping is a natural one-to-one map from U^* to V^*. ∎

Problem 11

A sphere can be split into a neighborhood of its northern hemisphere and a neighborhood of its southern hemisphere. Show that a sphere can be obtained by correctly gluing together two copies of $\mathbb{C}$.

Proof. We take two spaces $\mathbb{C}_1, \mathbb{C}_2$. Then define maps $\psi_1 : \mathbb{C}_1 \to \mathbb{R}^3 - \{0,0,1\}$, $\psi_2 : \mathbb{C}_2 \to \mathbb{R}^3 - \{0,0,-1\}$ by

$$\psi_1(x, y) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right),$$

and

$$\psi_2(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{1 - u^2 - v^2}{u^2 + v^2 + 1}\right).$$

Then

$$S^2 = \psi_1(\mathbb{C}_1) \cup \psi_2(\mathbb{C}_2).$$

∎

Problem 12

Put together the last two exercises to show that $\mathbb{P}^1$ is topologically equivalent to a sphere.

Proof. From Problem 10, we can map the two spaces U^* and V^* to $\mathbb{P}^1$. We can then equate U^* and V^* to the two copies of $\mathbb{C}$. Then from Problem 11 there is a homeomorphism from the two spaces $\mathbb{C}$ to S^2. Thus $\mathbb{P}^1$ is homeomorphic to S^2. ∎

1.7 Ellipses, Hyperbolas, and Parabolas as Spheres

Problem 1

Find a bijective polynomial map from $\mathbb{C}$ to the conic $C = \{(x, y) \in \mathbb{C}^2 \mid x^2 - y = 0\}$

Proof. We first paramtrize C as follows

$$x = t \text{ and } y = t^2.$$

Then, we define the mapping $\psi : \mathbb{C} \to C$ as

$$\psi(\alpha) = (\alpha, \alpha^2).$$

Now, suppose $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\psi(\alpha_1) = \psi(\alpha_2)$. Then

$$\psi(\alpha_1) = \psi(\alpha_2) \iff (\alpha_1, \alpha_1^2) = (\alpha_2, \alpha_2^2) \iff \alpha_1 = \alpha_2.$$

Thus ψ is injective. Now, let $(x, y) \in C$ and notice $y = x^2$. Then $\psi(x) = (x, x^2) = (x, y)$. Thus ψ is surjective. It follows that ψ is a bijection from $\mathbb{C}$ to C. ∎

Problem 2

Let $C = V(x^2 + y^2 - 1)$ be an ellipse in $\mathbb{C}^2$. Find a trigonometric parametrization of C. [Hint: Think high school trigonometry.]

Proof. Let $x = \cos(t)$ and $y = \sin(t)$ for $t \in \mathbb{C}$. ∎

Problem 3

Consider the ellipse $C = V(x^2 + y^2 - 1) \subset \mathbb{C}^2$ and let p denote the point $(0,1) \in C$.

1. Parametrize the line segment from p to the point $(\lambda, 0)$ on the complex line $y = 0$ as in Exercise 16.8.
2. This line segment clearly intersects C at the point p. Show that if $\lambda \neq \pm i$, then there is exactly one other point of intersection. Call this point q.
3. Find the coordinates of $q \in C$.
4. Show that if $\lambda = \pm i$, then the line segment intersects C only at p.

Proof. We have $p + x(p - (\lambda, 0))$ for $x \in \mathbb{C}$. Then

$$p + x(p - (\lambda, 0)) = (0,1) + x((0,1) - (\lambda, 0)) = (0,1) + x(-\lambda, 1) = (-\lambda x, 1 + x).$$

∎

Proof. Suppose $\lambda \neq \pm i$. Then

$$(-\lambda x)^2 + (1+x)^2 - 1 = \lambda^2 x^2 + (1 + 2x + x^2) - 1 = (\lambda^2 + 1)x^2 + 2x = x\big((\lambda^2+1)x + 2\big) = 0.$$

$x = 0$ corresponds to p.

$$x = -\frac{2}{\lambda^2 + 1}$$

corresponds to

$$q = \left(-\lambda\left(-\frac{2}{\lambda^2+1}\right), 1 - \frac{2}{\lambda^2+1}\right).$$

∎

Proof. Suppose $\lambda = \pm i$. Then

$$(-\lambda x)^2 + (1+x)^2 - 1 = (-(\pm i)x)^2 + (1 + 2x + x^2) - 1 = 2x = 0.$$

$x = 0$ corresponds to p and there are no other solutions. ∎

Problem 4

Define the map $\tilde{\psi} : \mathbb{C} \to C \subset \mathbb{C}^2$ by

$$\tilde{\psi}(\lambda) = \left(\frac{2\lambda}{\lambda^2+1}, \frac{\lambda^2 - 1}{\lambda^2 + 1}\right).$$

Show that the above map can be extended to the map

$$\psi : \mathbb{P}^1 \to \{(x : y : z) \in \mathbb{P}^2 \mid x^2 + y^2 - z^2 = 0\}.$$

given by

$$\psi(\lambda : u) = (2\lambda u : \lambda^2 - u^2 : \lambda^2 + u^2).$$

Proof. Plugging in we find

$$\begin{aligned}(2\lambda u)^2 + (\lambda^2 - u^2)^2 - (\lambda^2 + u^2)^2 &= 4\lambda^2 u^2 + (\lambda^4 - 2\lambda^2 u^2 + u^4) - (\lambda^4 + 2\lambda^2 u^2 + u^4) \\ &= 4\lambda^2 u^2 + \lambda^4 - 2\lambda^2 u^2 + u^4 - \lambda^4 - 2\lambda^2 u^2 - u^4 \\ &= 0.\end{aligned}$$

∎

Problem 5

1. Show that the map ψ is one-to-one.
2. Show that ψ is onto. [Hint: Consider the two cases: $z \neq 0$ and $z = 0$. For $z \neq 0$ follow the contruction given above. For $z = 0$, find values of λ and u to show that these points are given by ψ. How does this relate to Part 4 of Exercise 1.7.3?]

Proof. Let $(\lambda : u), (\lambda' : u') \in \mathbb{P}^1$. Then

$$\psi(\lambda : u) = \psi(\lambda' : u') \iff (2\lambda u : \lambda^2 - u^2 : \lambda^2 + u^2) = (2\lambda' u' : \lambda'^2 - u'^2 : \lambda'^2 + u'^2).$$

There exists $k \in \mathbb{C}$ such that

$$2\lambda u = k(2\lambda' u'), \quad \lambda^2 - u^2 = k(\lambda'^2 - u'^2) = k\lambda'^2 - ku'^2, \quad \lambda^2 + u^2 = k(\lambda'^2 + u'^2) = k\lambda'^2 + ku'^2.$$

Adding the second and third equations shows

$$\lambda^2 = k\lambda'^2.$$

Subtracting the second and third equations shows

$$-2u^2 = -2ku'^2 \implies u^2 = ku'^2.$$

Thus $(\lambda : u) = (\lambda' : u')$ in $\mathbb{P}^1$. It follows that ψ is injective.

Suppose $z = 0$. We have

$$x^2 + y^2 - z^2 = 0 \iff -x^2 = y^2 \iff y = \pm ix.$$

Then

$$(x : \pm ix : z) = (1 : \pm i : 0).$$

We have

$$\lambda^2 + u^2 = 0 \iff \lambda^2 = -u^2 \iff \lambda = \pm iu.$$

Then

$$\psi(\pm iu : u) = (2(\pm iu)u : (\pm iu)^2 - u^2 : (\pm iu)^2 + u^2) = (\pm 2iu^2 : -2u^2 : 0) = (1 : \pm i : 0).$$

Now, suppose $z \neq 0$. Let $(x : y : z)$ be an element in $\mathbb{P}^2$ with $z \neq 0$. We require

$$2\lambda u = x, \quad \lambda^2 - u^2 = y, \quad \lambda^2 + u^2 = z.$$

Adding and subtracting the second and third equations we find

$$\lambda^2 = \frac{y+z}{2} \implies \lambda = \pm\sqrt{\frac{y+z}{2}}, \quad u^2 = \frac{z-y}{2} \implies u = \pm\sqrt{\frac{z-y}{2}}.$$

Then

$$\psi\left(\pm\sqrt{\frac{y+z}{2}}, \pm\sqrt{\frac{z-y}{2}}\right) = (x : y : z).$$

Thus ψ is surjective. ∎

Problem 6

For the following conics and the given point p, follow what we did for the conic $x^2+y^2-1=0$ to find a rational map from $\mathbb{C}$ to the curve $\mathbb{C}^2$ and then a one-to-one map from $\mathbb{P}^1$ onto the conic in $\mathbb{P}^2$.

1. $x^2+2x-y^2-4y-4=0$ with $p=(0,-2)$.
2. $3x^2+3y^2-75=0$ with $p=(5,0)$.
3. $4x^2+y^2-8=0$ with $p=(1,2)$.

Solution (1): Consider the line $y=mx-2$. We have

$$\begin{aligned} x^2+2x-y^2-4y-4=0 \\ &\iff x^2+2x-(mx-2)^2-4(mx-2)-4=0 \\ &\iff x^2+2x-(m^2x^2-4mx+4)-4mx+8-4=0 \\ &\iff (1-m^2)x^2+2x=0 \\ &\iff x[(1-m^2)x+2]=0. \end{aligned}$$

Then $x=0$ corresponds to p. So $x=\frac{-2}{1-m^2}$, thus $y=\frac{-2m-2(1-m^2)}{1-m^2}$. So we have the parametrization

$$x(m)=\frac{-2}{1-m^2}, \quad y(m)=\frac{2m^2-2m-2}{1-m^2}.$$

Then define $\psi : \mathbb{P}^1 \to \mathbb{P}^2$ as

$$\psi(m,u)=(-2 : 2m^2-2m-2 : 1-m^2).$$

Solution (2): Consider the line $y=m(x-5)=mx-5m$. We have

$$\begin{aligned} 3x^2+3y^2-75=0 \\ &\iff 3x^2+3(mx-5m)^2-75=0 \\ &\iff 3x^2+3(m^2x^2-10m^2x+25m^2)-75=0 \\ &\iff 3x^2+3m^2x^2-30m^2x+75m^2-75=0 \\ &\iff 3(1+m^2)x^2-30m^2x+75(m^2-1)=0 \\ &\iff (1+m^2)x^2-10m^2x+25(m^2-1)=0 \\ &\iff [(1+m^2)x-5(m^2+1)][x-5]=0. \end{aligned}$$

Then $x=5$ corresponds to p. So $x=\frac{5(m^2+1)}{1+m^2}=5$, thus $y=m(x-5)=m(5-5)=0$. So we have the parametrization

$$x(m)=5, \quad y(m)=0.$$

Then define $\psi : \mathbb{P}^1 \to \mathbb{P}^2$ as

$$\psi(m,u)=(5 : 0 : 1).$$

Solution (3): Consider the line $y=m(x-1)+2=mx-m+2$. We have

$$\begin{aligned} 4x^2+y^2-8=0 \\ &\iff 4x^2+(mx-m+2)^2-8=0 \\ &\iff 4x^2+(m^2x^2-2m^2x+4mx+m^2-4m+4)-8=0 \\ &\iff (4+m^2)x^2+(-2m^2+4m)x+(m^2-4m-4)=0 \\ &\iff (x-1)\big((4+m^2)x+(-4+m^2+4m)\big)=0. \end{aligned}$$

Then $x=1$ corresponds to p. So $x=\frac{4-m^2-4m}{4+m^2}$, thus

$$y=m\left(\frac{4-m^2-4m}{4+m^2}-1\right)+2.$$

So we have the parametrization

$$x(m) = \frac{4 - m^2 - 4m}{4 + m^2}, \quad y(m) = m\left(\frac{4 - m^2 - 4m}{4 + m^2} - 1\right) + 2.$$

Then define $\psi : \mathbb{P}^1 \to \mathbb{P}^2$ as

$$\psi(m, u) = (4 - m^2 - 4m : m(4 - m^2 - 4m - (4 + m^2)) + 2(4 + m^2) : 4 + m^2).$$

1.8 Links to Number Theory

Problem 1

Suppose (x_0, y_0, z_0) is a solution to $x^2 + y^2 = z^2$. Show that (mx_0, my_0, mz_0) is also a solution for any scalar m.

Proof. We have

$$x_0^2 + y_0^2 - z_0^2 = 0 \iff m^2(x_0^2 + y_0^2 - z_0^2) = 0 \iff m^2x_0^2 + m^2y_0^2 - m^2z_0^2 = 0 \iff (mx_0)^2 + (my_0)^2 - (mz_0)^2 = 0.$$

Thus (mx_0, my_0, mz_0) is also a solution for any scalar m. ∎

Problem 2

Let $(a, b, c) \in \mathbb{Z}^3$ be a solution to $x^2 + y^2 = z^2$. Show that $c = 0$ if and only if $a = b = 0$.

Proof. Suppose $c = 0$ then clearly $a = b = 0$. Suppose $a = b = 0$. Then $a^2 + b^2 = 0^2 + 0^2 = 0$. Thus $c = 0$. ∎

Problem 3

Show that if (a, b, c) is a Pythagorean triple with $c \neq 0$, then the pair of rational number $\left(\frac{a}{c}, \frac{b}{c}\right)$ is a solution to $x^2 + y^2 = 1$.

Proof. Suppose (a, b, c) is a Pythagorean triple with $c \neq 0$. Then

$$a^2 + b^2 = c^2 \iff \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \iff \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

∎

Problem 4

Let $\left(\frac{a}{c_1}, \frac{b}{c_2}\right) \in \mathbb{Q}^2$ be a rational solution to $x^2 + y^2 = 1$. Find a corresponding Pythagorean triple.

Proof. First, write $\frac{a}{c_1}, \frac{b}{c_2}$ in their lowest terms by dividing by common factors. Then

$$\left(\frac{a}{c_1}\right)^2 + \left(\frac{b}{c_2}\right)^2 = 1 \iff (ac_2)^2 + (bc_1)^2 = (c_1c_2)^2.$$

Thus (ac_2, bc_1, c_1c_2) is a Pythagorean triple. ∎

Problem 5

Let

$$C(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\}.$$

Define the map $\tilde{\psi} : \mathbb{Q} \to \{(x, y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\}$. as

$$\lambda \mapsto \left(\frac{2\lambda}{\lambda^2 + 1}, \frac{\lambda^2 - 1}{\lambda^2 + 1}\right).$$

Show that the above map $\tilde{\psi}$ sends $\mathbb{Q} \to C(\mathbb{Q})$.

Proof. Let $\lambda \in \mathbb{Q}$ Then $\tilde{\lambda} = \left(\frac{2\lambda}{\lambda^2+1}, \frac{\lambda^2-1}{\lambda^2+1}\right)$. Plugging in we find

$$\begin{aligned}
\left(\frac{2\lambda}{\lambda^2 + 1}\right)^2 + \left(\frac{\lambda^2 - 1}{\lambda^2 + 1}\right)^2 &= \frac{4\lambda^2}{(\lambda^2 + 1)^2} + \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \\
&= \frac{4\lambda^2 + (\lambda^4 - 2\lambda^2 + 1)}{(\lambda^2 + 1)^2} \\
&= \frac{\lambda^4 + 2\lambda^2 + 1}{(\lambda^2 + 1)^2} \\
&= 1.
\end{aligned}$$

∎

Problem 6

1. Show that $\psi : \mathbb{P}^1(\mathbb{Q}) \to C(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{Q})$ is onto.
2. Show that every primitive triple is of the form $(2\lambda u, \lambda^2 - u^2, \lambda^2 + u^2)$.

Proof. This is the same as 1.7.5. ψ is a bijection thus every primitive triple is of that form. ∎

Problem 7

Find a rational point on the conic $x^2 + y^2 - 2 = 0$. Develop a parametrization and conclude that there are infinitely many rational points on this curve.

Proof. Consider $p = (1, 1) \in \mathbb{Q}^2$. Then $x^2 + y^2 - 2 = 1 + 1 - 2 = 0$. Consider the line $y = m(x - 1) + 1 = mx - m + 1$. Then

$$\begin{aligned}
x^2 + (mx - m + 1)^2 - 2 &= x^2 + \big(mx - m + 1\big)^2 - 2 \\
&= x^2 + \big(m^2x^2 - 2m^2x + 2mx + m^2 - 2m + 1\big) - 2 \\
&= (1 + m^2)x^2 + 2m(1 - m)x + (m^2 - 2m - 1) \\
&= (x - 1)\big((1 + m^2)x - m^2 + 2m + 1\big)
\end{aligned}$$

Now, $x = 1$ corresponds to p. So we have $x = \frac{m^2-2m-1}{1+m^2}$. Then we have the paramtrization

$$x(m) = \frac{m^2 - 2m - 1}{1 + m^2}, \quad y(m) = m\left(\frac{m^2 - 2m - 1}{1 + m^2}\right) - m + 1.$$

From this we can obtain infinitely many rational points on the curve. ∎

Problem 8

By mimicking the above, find four rational points on each of the following conics.

1. $x^2 + 2x - y^2 - 4y - 4 = 0$ with $p = (0, -2)$.
2. $3x^+ 3y^2 - 75 = 0$ with $p = (5, 0)$.
3. $4x^2 + y^2 - 8 = 0$ with $p = (1, 2)$.

Proof. **(1)** Consider $p = (0, -2) \in \mathbb{Q}^2$. Then $x^2+2x-y^2-4y-4 = 0$ at $(0, -2)$. Consider the line $y = m(x-0)-2 = mx - 2$. Then

$$\begin{aligned} x^2 + 2x - (mx-2)^2 - 4(mx-2) - 4 &= x^2 + 2x - (m^2x^2 - 4mx + 4) - (4mx - 8) - 4 \\ &= (1-m^2)x^2 + (2+0)x + (-4+8-4) \\ &= (1-m^2)x^2 + 2x + 0 \\ &= x\big((1-m^2)x + 2\big) \end{aligned}$$

Now, $x = 0$ corresponds to p. So we have $x = \frac{-2}{1-m^2}$. Then we have the parametrization

$$x(m) = \frac{-2}{1-m^2}, \quad y(m) = m\left(\frac{-2}{1-m^2}\right) - 2.$$

From this we can obtain infinitely many rational points on the curve. ∎

Proof. Consider $p = (5, 0) \in \mathbb{Q}^2$. Then $3x^2 + 3y^2 - 75 = 0$ at $(5, 0)$. Consider the line $y = m(x-5) + 0 = m(x-5)$. Then

$$\begin{aligned} 3x^2 + 3(mx - 5m)^2 - 75 &= 3x^2 + 3(m^2x^2 - 10m^2x + 25m^2) - 75 \\ &= 3(1+m^2)x^2 - 30m^2x + 75(m^2 - 1) \\ &= (x-5)\big(3(1+m^2)x - 30m^2 - 75\big) \end{aligned}$$

Now, $x = 5$ corresponds to p. So the other intersection point satisfies

$$3(1+m^2)x - 30m^2 - 75 = 0 \implies x = \frac{30m^2 + 75}{3(1+m^2)} = \frac{10m^2 + 25}{1+m^2}.$$

Then we have the parametrization

$$x(m) = \frac{10m^2 + 25}{1+m^2}, \quad y(m) = m\left(\frac{10m^2+25}{1+m^2} - 5\right) = m\left(\frac{10m^2 + 25 - 5 - 5m^2}{1+m^2}\right) = m\left(\frac{5m^2+20}{1+m^2}\right).$$

From this we can obtain infinitely many rational points on the curve. ∎

Proof. Consider $p = (1, 2) \in \mathbb{Q}^2$. Then $4x^2 + y^2 - 8 = 0$ at $(1, 2)$. Consider the line $y = m(x-1) + 2 = mx - m + 2$. Then

$$\begin{aligned} 4x^2 + (mx - m + 2)^2 - 8 &= 4x^2 + (m^2x^2 - 2m^2x + 2mx + m^2 - 4m + 4) - 8 \\ &= (4+m^2)x^2 + 2m(1-m)x + (m^2 - 4m - 4) \\ &= (x-1)\big((4+m^2)x + 2m(1-m) + (m^2 - 4m - 4)\big) \end{aligned}$$

Now, $x = 1$ corresponds to p. So we have $x = \frac{-2m^2+2m+4}{4+m^2}$. Then we have the parametrization

$$x(m) = \frac{-2m^2 + 2m + 4}{4+m^2}, \quad y(m) = m\left(\frac{-2m^2+2m+4}{4+m^2} - 1\right) + 2.$$

From this we can obtain infinitely many rational points on the curve. ∎

Problem 9

Show that the conic $x^2 + y^2 = 3$ has no rational points.

Proof. Suppose $x = \frac{a}{b}$ and $y = \frac{c}{d}$ are in lowest terms with $a, b, c, d \in \mathbb{Z}$. Furthermore, suppose

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 3.$$

Then

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 3 \iff a^2d^2 + c^2b^2 = 3b^2d^2 \iff (ad)^2 = 3(bd)^2 - (bc)^2 = b^2(3d^2 - c^2).$$

Now we must have $3d^2 - c^2 = k^2$ for some $k \in \mathbb{Z}$. We see $3d^2 - c^2 \equiv 0, 2 \pmod 3$ while $k^2 \equiv 0, 1 \pmod 3$. But, $\gcd(c, d) = 1$ thus $3d^2 - c^2 \equiv 2 \pmod 3$ while $k^2 \equiv 0, 1 \pmod 3$ which is a contradiction. ∎

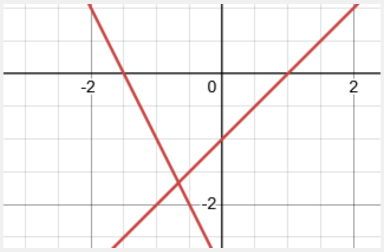
1.9 Degenerate Conics

Problem 1

Dehomogenize $f(x, y, z)$ by setting $z = 1$. Graph the curve

$$C(\mathbb{R}) = \{(x : y : z) \in \mathbb{P}^2 \mid f(x, y, 1) = 0\}.$$

in the real plane $\mathbb{R}^2$.



Problem 2

Consider the two lines given by

$$(a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z) = 0,$$

and suppose

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0.$$

Show that the two lines intersect at a point where $z \neq 0$.

Proof. Suppose the two lines intersect at a point where $z = 0$. Thus

$$a_1 x + b_1 y = 0 \quad \text{and} \quad a_2 x + b_2 y = 0,$$

so

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Then

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $x = y = 0$ but $(0, 0, 0) \notin \mathbb{P}^2$. ∎

Problem 3

Dehomogenize the equation in the previous exercise by setting $z = 1$. Given an argument that, as lines in the complex plane $\mathbb{C}^2$, they have distinct slopes.

Proof. Dehomogenizing we find

$$(a_1 x + b_1 y + c_1)(a_2 x + b_2 y + c_2) = 0.$$

Since they intersect at a point such that $z \neq 0$ they are not parallel in $\mathbb{C}^2$. Thus either $a_1 \neq a_2$ or $b_1 \neq b_2$. ∎

Problem 4

Again consider the two lines

$$(a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z) = 0.$$

Suppose that

$$\det\left(A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right) = 0.$$

but that

$$\det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \neq 0 \text{ or } \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \neq 0.$$

Show that the two lines still have one common point of intersection but that this point must have $z = 0$.

Proof. Suppose the two lines intersect at a point such that $z \neq 0$. Then dividing by z we find

$$\left(a_1 \frac{x}{z} + b_1 \frac{y}{z} + c_1\right)\left(a_2 \frac{x}{z} + b_2 \frac{y}{z} + c_2\right) = 0.$$

Let $X = \frac{x}{z}, Y = \frac{y}{z}$. We now have the following system

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}.$$

Since $\det(A) = 0$ the system does not have a unique solution. It follows that the lines are linearly dependent. Now, wlog suppose

$$\det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \neq 0.$$

Then $a_1 c_2 \neq a_2 c_1$ and it follows that the lines are not equivalent. Therefore, the lines are parallel and meet at $z = 0$. ∎

Problem 5

Let
$$f(x,y,z) = (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z),$$
where at least one of a_1, b_2, or c_1 is non-zero and at least one of the a_2, b_2, or c_2 is non-zero. Show that the curve defined by $f(x,y,z) = 0$ is a double line if and only if

$$\det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0, \quad \det\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = 0, \quad \det\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} = 0.$$

Proof. Suppose $f(x,y,z) = 0$ is a double line. By Problem 4 we know

$$\det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0, \quad \det\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = 0, \quad \det\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} = 0.$$

Conversely, suppose
$$\det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0, \quad \det\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = 0, \quad \det\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} = 0.$$

Thus (a_2, b_2, c_2) is a scalar multiple of (a_1, b_1, c_1). Therefore $f(x,y,z) = 0$ is a double line. ∎

Problem 6

Consider the crossing lines
$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0,$$
with
$$\det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right) \neq 0.$$

Find a projective change of coordinates from xyz-space to uvw-space so that the crossing lines become

$$uv = 0.$$

Problem 7

Consider the crossing lines $(a_2x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0$, with

$$\det\left(\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}\right) \neq 0.$$

Find a projective change of coordinates from xyz-space to uvw-space so that the crossing lines become

$$uv = 0.$$

Problem 8

Show that there is a projective change of coordinates from xyz-space to uvw-space so that the double lines $(ax + by + cz)^2 = 0$ becomes teh double line

$$u^2 = 0.$$

Problem 9

Argue that there are three distinct classes of conics in $\mathbb{P}^2$.