# Algebraic Geometry by Thomas Garrity et. al.

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## Contents



## 1 Conics

### 1.1 Conics over the Reals

**Problem 1**

$$P(x,y) = y - x^2, \quad C = \{(x,y) \in \mathbb{R}^2 \mid P(x,y) = 0\}.$$

Show that for any $(x,y) \in C$, we also have

$$(-x,y) \in C.$$

Thus the curve is symmetric about the y-axis.

*Proof.* Let $(x,y) \in C$. Then $P(x,y) = y - x^2 = 0$. Let $x' = -x$ and note that $(-x)^2 = x^2$. Thus

$$P(-x,y) = y - (-x)^2 = y - x^2 = 0.$$

Thus $(-x,y) \in C$. ∎

**Problem 2**

$$P(x,y) = y - x^2, \quad C = \{(x,y) \in \mathbb{R}^2 \mid P(x,y) = 0\}.$$

Show that if $(x,y) \in C$, then we have $y \geq 0$.

*Proof.* Suppose $(x,y) \in C$. Then

$$P(x,y) = y - x^2 = 0 \iff y = x^2 \geq 0.$$

Thus $y \geq 0$. ∎

## Problem 3

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that for every $y \geq 0$, there is a point $(x, y) \in C$ with this $y$-coordinate. Now, for points $(x, y) \in C$, show that if $y$ goes to infinity, then one of the corresponding $x$-coordinates also approaches infinity while the other corresponding $x$ coordinate must approach negative infinity.

*Proof.* Let $y \in \mathbb{R}$ such that $y \geq 0$. Let $x = \sqrt{y} \in \mathbb{R}$. Then

$$y - x^2 = y - (\sqrt{y})^2 = y - y = 0.$$

Thus $(x, y) = (\sqrt{y}, y) \in C$.

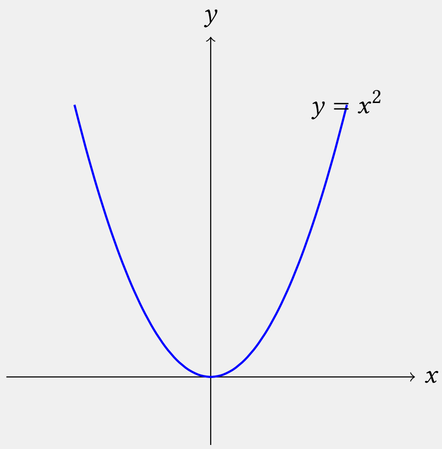
Now suppose $y \to \infty$. For points $(x, y) \in C$, we have

$$y - x^2 = 0 \iff x = \pm\sqrt{y}.$$

Since $y \to \infty$, we have $\sqrt{y} \to \infty$ and $-\sqrt{y} \to -\infty$. Thus one corresponding $x$-coordinate approaches infinity, while the other approaches negative infinity. ∎

## Problem 4

Sketch the curve $C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}$.




## Problem 5

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\right\}.$$

Show that if $(x, y) \in C$, then the three points $(-x, y), (x, -y), (-x, -y)$ are also on $C$. Thus the curve $C$ is symmetric about both the $x$- and $y$-axes.

*Proof.* Let $(x, y) \in \mathbb{R}^2$. Suppose $\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0$. Notice that $x^2 = (-x)^2$ and $y = (-y)^2$. Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{y^2}{9} - 1 = \frac{x^2}{4} + \frac{(-y)^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{(-y)^2}{9} - 1 = 0.$$

Thus $(-x, y), (x, -y), (-x, -y) \in C$. ∎

## Problem 6

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\right\}.$$

Show that for every $(x, y) \in C$, we have $|x| \leq 2$ and $|y| \leq 3$.

*Proof.* Let $(x, y) \in C$. Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \iff 9x^2 + 4y^2 - 36 = 0 \iff 9x^2 = -4y^2 + 36 \iff |x| = \sqrt{\frac{-4}{9}y^2 + 4} \leq \sqrt{4} = 2.$$

Similarly

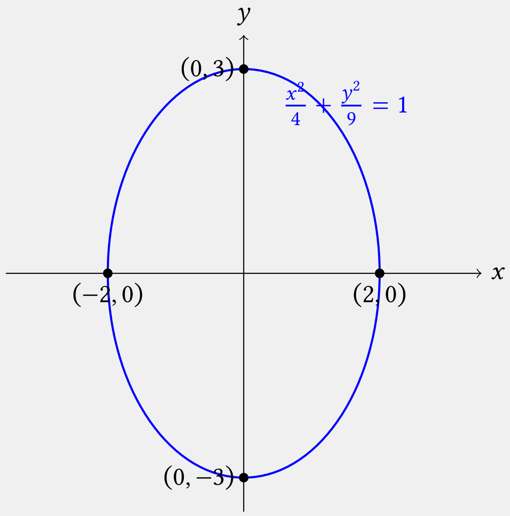
$$9x^2 + 4y^2 - 36 = 0 \iff |y| = \sqrt{\frac{-9}{4}x^2 + 9} \leq \sqrt{9} = 3.$$

∎

## Problem 7

Sketch

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\right\}.$$




## Problem 8

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

Show that if $(x, y) \in C$, then the three points $(-x, y), (x, -y)$, and $(-x, -y)$ are also on $C$. Thus the curve $C$ is also symmetric about the $x-$ and $y$-axes.

*Proof.* Let $(x, y) \in \mathbb{R}^2$. Suppose $x^2 - y^2 - 4 = 0$. Notice that $x^2 = (-x)^2$ and $y = (-y)^2$. Then

$$x^2 - y^2 - 4 = (-x)^2 - y^2 = x^2 - (-y)^2 = (-x)^2 - (-y)^2 = 0.$$

Thus $(-x, y), (x, -y), (-x, -y) \in C$. ∎

**Problem 9**

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

Show that if $(x, y) \in C$, then we have $|x| \geq 2$.

*Proof.* Let $(x, y) \in \mathbb{R}^2$. Suppose $x^2 - y^2 - 4 = 0$. Then

$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff |x| = \sqrt{y^2 + 4} \geq \sqrt{4} = 2.$$

$\blacksquare$

**Problem 10**

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

Show that the curve $C$ is unbounded in the positive and negative $x$-directions and also unbounded in the positive and negative $y$-directions.

*Proof.* First notice

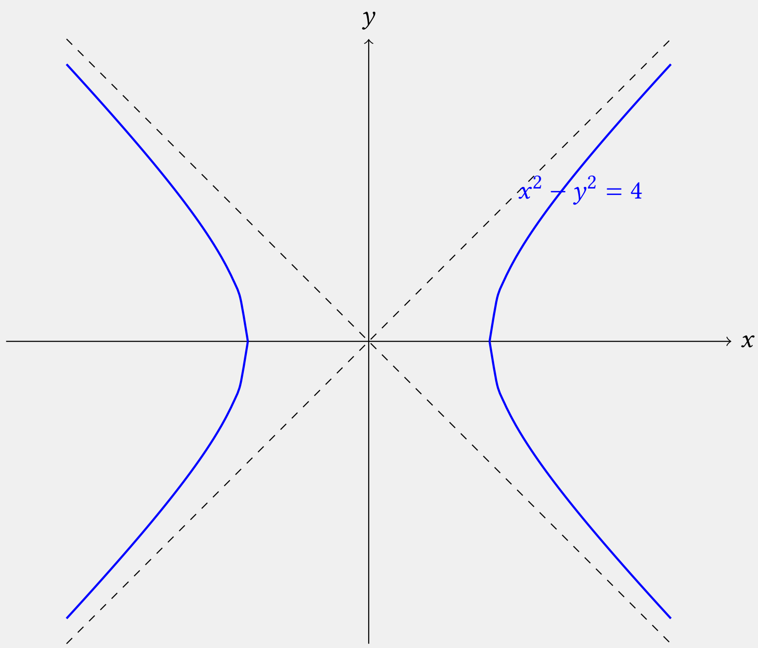
$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff x = \pm\sqrt{y^2 + 4} \iff y = \pm\sqrt{x^2 - 4}.$$

As $y \to \infty$, we have $x = \pm\sqrt{y^2 + 4} \to \infty$ and $-\infty$. Similarly, as $x \to \infty$, we have $y = \pm\sqrt{x^2 - 4} \to \infty$ and $-\infty$. $\blacksquare$

**Problem 11**

Sketch

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

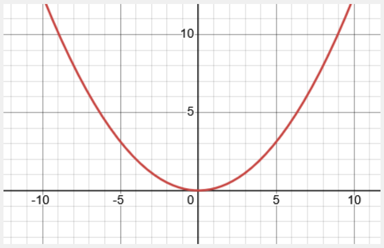


Problem 12

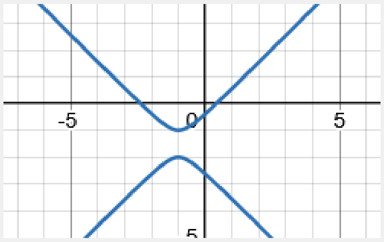
Sketch the graph of each of the following conics in $\mathbb{R}^2$. Identify which are parabolas, ellipses, or Hyperbola.

1. $V(x^2 - 8y)$.
2. $V(x^2 + 2x - y^2 - 3y - 1)$.
3. $V(4x^2 + y^2)$.
4. $V(3x^2 + 3y^2 - 75)$.
5. $V(x^2 - 9y^2)$.
6. $V(4x^2 + y^2 - 8)$.
7. $V(x^2 + 9y^2 - 36)$.
8. $V(x^2 - 4y^2 - 16)$.
9. $V(y^2 - x^2 - 9)$.

**Solution (1):** Parabola.

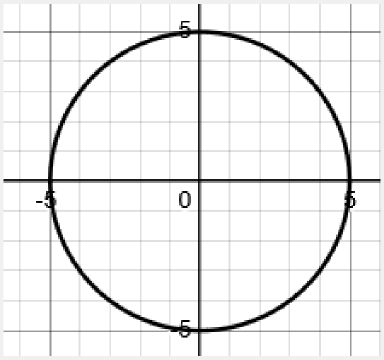



**Solution (2):** Hyperbola.



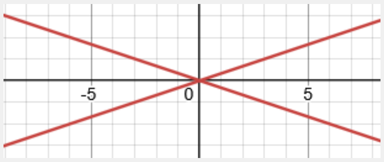

**Solution (3):** Point.

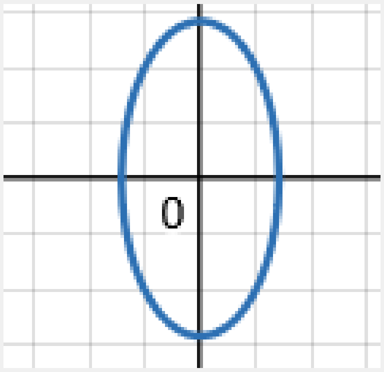
**Solution (4):** Ellipse.



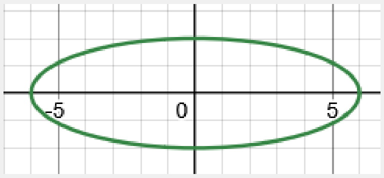

**Solution (5):** Two lines.

**Solution (6):** Ellipse.

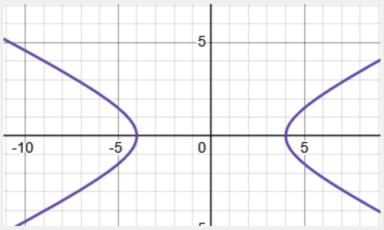


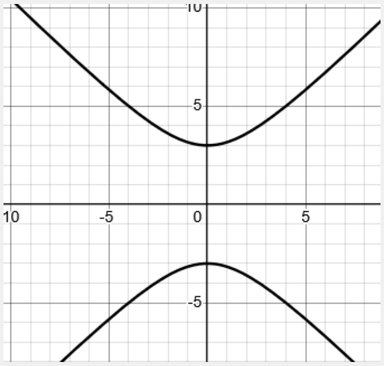

**Solution (7):** Ellipse.




**Solution (8):** Hyperbola.




**Solution (9):** Hyperbola.



**Problem 13**

Express the polynomial $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$ in the form

$$P(x, y) = Ax^2 + Bx + C,$$

where $A, B$, and $C$ are polynomials in $y$. What are $A, B$, and $C$?

*Proof.* Let $A = a, B = by + d$, and $C = cy^2 + ey + h$. Notice

$$ax^2 + bxy + cy^2 + dx + ey + h = ax^2 + bxy + dx + cy^2 + ey + h = ax^2 + (by + d)x + (cy^2 + ey + h) = Ax^2 + Bx + C.$$

∎

**Problem 14**

Treating $P(x, y) = ax^2+bxy+cy^2+dx+cy+h$ as a polynomial in the variable $x$, show that the discriminant is

$$\triangle_x(y) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

*Proof.* From Problem 13 we have $A = a, B = by + d$, and $C = cy^2 + ey + h$. Then

$$\triangle_x(y) = B^2 - 4AC = (by + d)^2 - 4a(cy^2 + ey + h) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

∎

**Problem 15**

1. Suppose $\triangle_x(y_0) < 0$. Explain why there is no point on $V(p)$ whose $y$-coordinate is $y_0$.
2. Suppose $\triangle_x(y_0) = 0$. Explain why there is exactly one point on $V(P)$ whose $y$-coordinate is $y_0$.
3. Suppose $\triangle_x(y_0) > 0$. Explain why there are exactly two points on $V(P)$ whose $y$-coordinate is $y_0$.

**Solution (a):** In $\mathbb{R}$, the square root is undefined for values $< 0$.

**Solution (b):** If $\triangle_x(y_0) = 0$, then $+\sqrt{B^2 - 4AC} = -\sqrt{B^2 - 4AC}$, so there is exactly one point on $V(P)$ whose $y$-coordinate is $y_0$.

**Solution (c):** If $\triangle_x(y_0) > 0$, then $+\sqrt{B^2 - 4AC} \neq -\sqrt{B^2 - 4AC}$, so there are exactly two points on $V(P)$ whose $y$-coordinate is $y_0$.

**Problem 16**

Suppose $b^2 - 4ac = 0$. Suppose further that $2bd - 4ae > 0$.

1. Show that $\triangle_x(y) \geq 0$ if and only if $y \geq \frac{4ah-d^2}{2bd-4ae}$.
2. Conclude that if $b^2 - 4ac = 0$ and $2bd - 4ae > 0$, then $V(P)$ is a parabola.

*Proof.* Suppose $\triangle_x(y) \geq 0$. Then

$$\begin{aligned}\triangle_x(y) &= (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah)\\ &= 0y^2 + (2bd - 4ae)y + (d^2 - 4ah).\end{aligned}$$

Therefore,

$$(2bd - 4ae)y + (d^2 - 4ah) \geq 0.$$

Since $2bd - 4ae > 0$, we have

$$y \geq \frac{4ah - d^2}{2bd - 4ae}.$$

Conversely, suppose $y \geq \frac{4ah-d^2}{2bd-4ae}$. Then

$$\begin{aligned}
\triangle_x(y) &= (2bd - 4ae)y + (d^2 - 4ah) \\
&\geq (2bd - 4ae)\left(\frac{4ah - d^2}{2bd - 4ae}\right) + (d^2 - 4ah) \\
&= 0.
\end{aligned}$$

∎

*Proof.* Suppose $b^2 - 4ac = 0$ and $2bd - 4ae > 0$. Then $\triangle_x(y) = (2bd - 4ae)y + (d^2 - 4ah)$. Now, $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$. It is clear that $x$ is symmetrical, and since $y \geq \frac{4ah-d^2}{2bd-4ae}$, $V(P)$ is a parabola. ∎

**Problem 17**

Suppose $b^2 - 4ac < 0$.

1. Show that one of the following occurs:
   (a) $\{y \mid \triangle_x(y) \geq 0\} = \emptyset$,
   (b) $\{y \mid \triangle_x(y) \geq 0\} = \{y_o\}$,
   (c) there exist real numbers $\alpha$ and $\beta$, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

2. Conclude that $V(P)$ is either emptyset, a point, or an ellipse.

*Proof.* Since $b^2 - 4ac < 0$, the graph of $\triangle_x(y)$ is a downward opening parabola in $y$. There are three cases, depending on the number of real zeros of $\triangle_x(y)$.

1. If $\triangle_x(y) < 0$ for all $y$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \emptyset.$$
2. If $\triangle_x(y)$ has exactly one real zero $y_0$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y_0\}.$$
3. If $\triangle_x(y)$ has two distinct real zeros $\alpha < \beta$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

∎

*Proof.* From part 1 the set of $y$ values is either empty, a single point, or a bounded interval, it follows that $V(P)$ is either empty, a point, or an ellipse. ∎

**Problem 18**

Suppose $b^2 - 4ac > 0$.

1. Show that one of the following occurs:
   (a) $\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}$ and $\triangle_x(y) \neq 0$,

(b) $\{y \mid \triangle_x(y) = 0\} = \{y_0\}$ and $\{y \mid \triangle_x(y) > 0\} = \{y \mid y \neq y_0\}$,
(c) there exist real numbers $\alpha$ and $\beta$, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

2. If $\{y \mid \triangle_x(y)\} = \mathbb{R}$, show that $V(P)$ is a hyperbola opening left and right.
3. If $\{y \mid \triangle_x(y) = 0\} = \{y_0\}$, show that $V(P)$ is two lines intersecting in a point.
4. If there are two real numbers $\alpha$ and $\beta$, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\},$$

show that $V(P)$ is a hyperbola opening up and down.

*Proof.* Since $b^2 - 4ac > 0$, the graph of $\triangle_x(y)$ is an upward opening parabola in $y$. There are three cases, depending on the number of real zeros of $\triangle_x(y)$.

1. If $\triangle_x(y) > 0$ for all $y$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}.$$
2. If $\triangle_x(y)$ has exactly one real zero $y_0$, then
$$\{y \mid \triangle_x(y) = 0\} = \{y_0\} \quad \text{and} \quad \{y \mid \triangle_x(y) > 0\} = \{y \mid y \neq y_0\}.$$
3. If $\triangle_x(y)$ has two distinct real zeros $\alpha < \beta$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

∎

*Proof.* Since $b^2 - 4ac > 0$, the graph of $\triangle_x(y)$ is an upward opening parabola in $y$. There are three cases, depending on the number of real zeros of $\triangle_x(y)$.

1. If $\triangle_x(y) > 0$ for all $y$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}.$$
2. If $\triangle_x(y)$ has exactly one real zero $y_0$, then
$$\{y \mid \triangle_x(y) = 0\} = \{y_0\} \quad \text{and} \quad \{y \mid \triangle_x(y) > 0\} = \{y \mid y \neq y_0\}.$$
3. If $\triangle_x(y)$ has two distinct real zeros $\alpha < \beta$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

∎

*Proof.* Suppose $\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}$. Then for every $y$ there exist two real solutions for $x$, and $x$ is unbounded to the left and right. Since the equation is quadratic in $x$, the curve is symmetric in $x$. Thus $V(P)$ is a hyperbola opening left and right. ∎

*Proof.* Suppose $\{y \mid \triangle_x(y) = 0\} = \{y_0\}$. Then for $y = y_0$ the equation has exactly one real solution for $x$, and for $y \neq y_0$ it has two real solutions. Since the equation is quadratic in $x$, $V(P)$ consists of two lines intersecting at a point. ∎

*Proof.* Suppose there exist real numbers $\alpha$ and $\beta$, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

For $y \leq \alpha$ or $y \geq \beta$, the equation has two real solutions in $x$. If $\alpha < y < \beta$ it has no real solutions. Thus $x$ is bounded for each $y$, but $y$ is unbounded above and below. Since the equation is quadratic in $x$, the curve is symmetric in $x$. Therefore $V(P)$ is a hyperbola opening up and down. ∎

**Problem 19**

Show that the discriminant of $A'y^2 + B'y + C' = 0$ is

$$\triangle_y(x) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4ch).$$

*Proof.* Here $A' = c$, $B' = bx + e$, and $C' = ax^2 + dx + h$. Then

$$\triangle_y(x) = (B')^2 - 4A'C' = (bx+e)^2 - 4c(ax^2 + dx + h) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4ch).$$

∎

## 1.2 Changes of Coordinates

**Problem 1**

Show that the origin in the $xy$-coordinate system agrees with the origin in the $uv$-system if and only if $e = f = 0$. Thus the constants $e$ and $f$ describe translations of the origin.

*Proof.* Suppose the $xy$-coordinate system agrees with the origin of the $uv$-system. Then

$$u = 0 = a(0) + b(0) + e = e,$$

and

$$v = 0 = c(0) + d(0) + f = f.$$

Thus $f = e = 0$.

Conversely, suppose $e = f = 0$. Then

$$u = ax + by + e = ax + by + 0 = a(0) + b(0) = 0,$$

and

$$v = cx + dy + f = cx + dy + 0 = c(0) + d(0) = 0.$$

Thus the origin of the $xy$-coordinate system agrees with the origin of the $uv$-system. ∎

**Problem 2**

Show that if $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then the inverse change of coordinates is

$$x = \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf).$$

$$y = \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af).$$

*Proof.* We need to solve the two equations $u = ax + by + e$ and $v = cx + dy + f$ in two unknowns $x$ and $y$. Translating this to linear algebra, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u - e \\ v - f \end{bmatrix}.$$

Using Cramer's rule we see

$$x = \frac{\begin{vmatrix} u-e & b \\ v-f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d(u-e) - b(v-f)}{ad - bc},$$

$$y = \frac{\begin{vmatrix} a & u-e \\ c & v-f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c(u-e) + a(v-f)}{ad - bc}.$$

Therefore

$$x = \frac{du - bv - de + bf}{ad - bc}, \quad y = \frac{-cu + av + ce - af}{ad - bc}.$$

∎

**Problem 3**

Show that if

$$u = ax + by + e$$
$$v = cx + dy + f,$$

and

$$s = Au + Bv + E$$
$$t = Cu + Dv + F$$

are two real affine changes of coordinates from the $xy$-plane to the $uv$-plane and from the $uv$-plane to the $st$-plane, respectively, then the composition from the $xy$-plane to the $st$-plane is a real affine change of coordinates.

*Proof.* Suppose

$$u = ax + by + e$$
$$v = cx + dy + f,$$

and

$$s = Au + Bv + E$$
$$t = Cu + Dv + F$$

are two real affine changes of coordinates from the $xy$-plane to the $uv$-plane and from the $uv$-plane to the $st$-plane respectively. Substituting $u, v$ into $s, t$ we see

$$s = A(ax + by + e) + B(cx + dy + f) + E = (Aa + Bc)x + (Ab + Bd)y + (Ae + Bf + E),$$

and

$$t = C(ax + by + e) + D(cx + dy + f) + F = (Ca + Dc)x + (Cb + Dd)y + (Ce + Df + F).$$

Finally,

$$\det\left(\begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix}\right) = (Aa + Bc)(Cb + Dd) - (Ab + Bd)(Ca + Dc) = (ad - bc)(AD - BC) \neq 0.$$

∎

Problem 4

For each affine pair of ellipses, find a real affine change of coordinates that maps the ellipse in the $xy$-plane to the ellipse in the $uv$-plane.

1. $V(x^2 + y^2 - 1), V(16u^2 + 9v^2 - 1)$.
2. $V((x-1)^2 + y^2 - 1), V(16u^2 + 9(v+2)^2 - 1)$.
3. $V(4x^2 + y^2 - 6y + 8), V(u^2 - 4u + v^2 - 2v + 4)$.
4. $V(13x^2 - 10xy + 13y^2 - 1), V(4u^2 + 9v^2 - 1)$.

**Solution (1):** Let $x = 4u$ and $y = 3v$. Then

$$x^2 + y^2 - 1 = (4u)^2 + (3v)^2 - 1 = 16u^2 + 9v^2 - 1 = 0.$$

**Solution (2):** Let $x = 4u + 1$ and $y = 3v + 6$. Then

$$(x-1)^2 + y^2 - 1 = (4u + 1 - 1)^2 + (3v + 6)^2 = 16u^2 + 9(v+2)^2 = 0.$$

**Solution (3):** Let $x = \frac{u}{2} - 1$ and $y = v + 2$. Then

$$4x^2 + y^2 - 6y + 8 = 4\left(\frac{u}{2} - 1\right)^2 + (v+2)^2 - 6(v+2) + 8 =$$

$$4\left(\frac{u^2}{4} - 2\frac{u}{2} + 1\right) + v^2 + 4v + 4 - 6v - 12 + 8 = u^2 - 4u + 4 + v^2 - 2v = u^2 - 4u + v^2 - 2v + 4.$$

**Solution (4):** Let $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Then

$$\begin{aligned}
13x^2 - 10xy + 13y^2 - 1 &= 13\left(\frac{u+v}{2}\right)^2 - 10\left(\frac{u+v}{2} \cdot \frac{u-v}{2}\right) + 13\left(\frac{u-v}{2}\right)^2 - 1 \\
&= 13\frac{(u+v)^2}{4} - 10\frac{u^2 - v^2}{4} + 13\frac{(u-v)^2}{4} - 1 \\
&= \frac{13}{4}(u^2 + 2uv + v^2) - \frac{10}{4}(u^2 - v^2) + \frac{13}{4}(u^2 - 2uv + v^2) - 1 \\
&= \frac{13 + 13 - 10}{4}u^2 + \frac{13 + 13 + 10}{4}v^2 + \frac{26 - 26}{4}uv - 1 \\
&= 4u^2 + 9v^2 - 1.
\end{aligned}$$

Problem 5

For each pair of hyperbolas, find a real affine change of coordinates that maps the hyperbola in the $xy$-plane to the hyperbola in the $uv$-plane.

1. $V(xy - 1), V(u^2 - v^2 - 1)$.
2. $V(x^2 - y^2 - 1), V(16u^2 - 9v^2 - 1)$.
3. $V((x-1)^2 - y^2 - 1), V(16u^2 - 9(v+2)^2 - 1)$.
4. $V(x^2 - y^2 - 1), V(v^2 - u^2 - 1)$.
5. $V(8xy - 1), V(2u^2 - 2v^2 - 1)$.

**Solution (1):** Let $x = u - v$ and $y = u + v$. Then

$$xy - 1 = (u - v)(u + v) - 1 = u^2 - v^2 - 1.$$

**Solution (2):** Let $x = 4u$ and $y = 3v$. Then

$$x^2 - y^2 - 1 = (4u)^2 - (3v)^2 - 1 = 16u^2 - 9v^2 - 1.$$

**Solution (3):** Let $x = 4u + 1$ and $y = 3v + 6$. Then

$$(x-1)^2 - y^2 - 1 = (4u+1-1)^2 - (3v+6)^2 = 16u^2 - 9(v+2)^2 - 1.$$

**Solution (4):** Let $x = v$ and $y = u$. Then

$$x^2 - y^2 - 1 = v^2 - u^2 - 1.$$

**Solution (5):** Let $x = (u+v)/4$ and $y = (u-v)/2$. Then

$$8xy - 1 = 8((u+v)/4)((u-v)/2) - 1 = (u+v)(u-v) - 1 = u^2 - v^2 - 1.$$

**Problem 6**

For each pair of parabolas, find a real affine change of coordinates that maps the parabola in the $xy$-plane to the parabola in the $uv$-plane.

1. $V(x^2 - y), V(9v^2 - 4u)$.
2. $V((x-1)^2 - y), V(u^2 - 9(v+2))$.
3. $V(x^2 - y), V(u^2 + 2uv + v^2 - u + v - 2)$.
4. $V(x^2 - 4x + y + 4), V(4u^2 - (v+1))$.
5. $V(4x^2 + 4xy + y^2 - y + 1), V(4u^2 + v)$.

**Solution (1):** Let $x = 3v$ and $y = 4u$. Then

$$x^2 - y = (3v)^2 - 4u = 9v^2 - 4u.$$

**Solution (2):** Let $x = u + 1$ and $y = 9v + 18$. Then

$$(x-1)^2 - y = (u+1-1)^2 - (9v+18) = u^2 - 9(v+2).$$

**Solution (3):** Let $x = (u+v)^2$ and $y = u - v + 2$. Then

$$x^2 - y = (u+v)^2 - (u - v + 2) = u^2 + 2uv + v^2 - u + v - 2.$$

**Solution (4):** Let $x = 2u + 2$ and $y = -(v+1)$. Then

$$x^2 - 4x + y + 4 = (2u+2)^2 - 4(2u+2) - (v+1) + 4 = 4u^2 + 8u + 4 - 8u - 8 - (v+1) + 4 = 4u^2 - (v+1).$$

**Solution (5):** Let $x = u - \frac{1}{2}v + \frac{1}{2}$ and $y = v$. Then

$$\begin{aligned}
4x^2 + 4xy + y^2 - y + 1 &= 4\left(u - \tfrac{1}{2}v + \tfrac{1}{2}\right)^2 + 4\left(u - \tfrac{1}{2}v + \tfrac{1}{2}\right)v + v^2 - v + 1 \\
&= 4\left(u^2 - uv + u + \tfrac{1}{4}v^2 - \tfrac{1}{2}v + \tfrac{1}{4}\right) + 4uv - 2v^2 + 2v + v^2 - v + 1 \\
&= 4u^2 - 4uv + 4u + v^2 - 2v + 1 + 4uv - 2v^2 + 2v + v^2 - v + 1 \\
&= 4u^2 + v.
\end{aligned}$$

**Problem 7**

Explain why if $b^2 - 4ac < 0$, then $ac > 0$.

*Proof.* Suppose $b^2 - 4ac < 0$. Then $0 \le b^2 < 4ac \iff 0 \le \frac{b^2}{4} < ac$. Thus $ac > 0$. ∎

Problem 8

Show that under the real affine transformation

$$x = \sqrt{\frac{c}{a}}u + v$$

$$y = u - \sqrt{\frac{a}{c}}v,$$

the ellipse $V(ax^2 + bxy + cy^2 + dx + ey + h)$ in the $xy$-plane becomes an ellipse in the $uv$-plane whose defining equation is $Au^2+Cv^2+Du+Ev+H = 0$. Find $A$ and $C$ in terms of $a, b, c$. Show that if $b^2-4ac < 0$, then $A \neq 0$ and $C \neq 0$.

*Proof.*

$$\begin{aligned}
ax^2 + bxy + cy^2 + dx + ey + h &= a\Big(\sqrt{\tfrac{c}{a}}\,u + v\Big)^2 + b\Big(\sqrt{\tfrac{c}{a}}\,u + v\Big)\Big(u - \sqrt{\tfrac{a}{c}}\,v\Big) + c\Big(u - \sqrt{\tfrac{a}{c}}\,v\Big)^2 \\
&\quad + d\Big(\sqrt{\tfrac{c}{a}}\,u + v\Big) + e\Big(u - \sqrt{\tfrac{a}{c}}\,v\Big) + h \\
&= \big(cu^2 + 2\sqrt{ac}\,uv + av^2\big) + b\Big(\sqrt{\tfrac{c}{a}}\,u^2 - \sqrt{\tfrac{a}{c}}\,v^2\Big) + \big(cu^2 - 2\sqrt{ac}\,uv + av^2\big) \\
&\quad + \Big(d\sqrt{\tfrac{c}{a}} + e\Big)u + \Big(d - e\sqrt{\tfrac{a}{c}}\Big)v + h \\
&= \Big(2c + b\sqrt{\tfrac{c}{a}}\Big)u^2 + \Big(2a - b\sqrt{\tfrac{a}{c}}\Big)v^2 + \Big(d\sqrt{\tfrac{c}{a}} + e\Big)u + \Big(d - e\sqrt{\tfrac{a}{c}}\Big)v + h \\
&= Au^2 + Cv^2 + Du + Ev + H.
\end{aligned}$$

∎

*Proof.* Suppose $b^2 - 4ac < 0$. Then

$$A = \sqrt{\frac{c}{a}}\,b + 2c, \qquad C = -\sqrt{\frac{a}{c}}\,b + 2a.$$

Then

$$AC = \Big(2c + b\sqrt{\tfrac{c}{a}}\Big)\Big(2a - b\sqrt{\tfrac{a}{c}}\Big) = 4ac - b^2.$$

Since $b^2 - 4ac < 0$,

$$4ac - b^2 > 0 \implies AC > 0.$$

Therefore $A \neq 0$ and $C \neq 0$. ∎

Problem 9

Show that there exists constants $R, S$, and $T$ such that the equation

$$Au^2 + Cv^2 + Du + Ev + H = 0,$$

can be written in the form

$$A(u - R)^2 + C(v - S)^2 - T = 0.$$

Express $R, S$, and $T$ in terms of $A, C, D, E$, and $H$.

*Proof.* Let $R = -\frac{D}{2A}, S = -\frac{E}{2C}, T = \frac{D^2}{4A} + \frac{E^2}{4C} - H$. Note $A \neq 0$ and $C \neq 0$ from problem 8. Notice

$$\begin{aligned}
Au^2 + Cv^2 + Du + Ev + H &= A\left(u^2 + \frac{Du}{A}\right) + C\left(v^2 + \frac{Ev}{C}\right) + H \\
&= A\left(u^2 + \frac{Du}{A} + \left(\frac{D}{2A}\right)^2\right) - \frac{D^2}{4A} + C\left(v^2 + \frac{Ev}{C} + \left(\frac{E}{2C}\right)^2\right) - \frac{E^2}{4C} + H \\
&= A\left(u + \frac{D}{2A}\right)^2 + C\left(v + \frac{E}{2C}\right)^2 - \frac{D^2}{4A} - \frac{E^2}{4C} + H \\
&= A(u-R)^2 + C(v-S)^2 - T = 0.
\end{aligned}$$

∎

**Problem 10**

Suppose $A, C > 0$. Find a real affine change of coordinates that maps the ellipse

$$V(A(x-R)^2 + C(y-S)^2 - T),$$

to the circle

$$V(u^2 + v^2 - 1).$$

*Proof.* Since $A, C > 0$ we know $T > 0$. Notice

$$A(x-R)^2 + C(y-S)^2 = T \iff \frac{A(x-R)^2}{T} + \frac{C(y-S)^2}{T} = 1.$$

We set

$$u^2 = \frac{A(x-R)^2}{T}, \quad v^2 = \frac{C(y-S)^2}{T},$$

and solving for $x, y$ shows

$$x = \sqrt{\frac{T}{A}}\,u + R, \quad y = \sqrt{\frac{T}{C}}\,v + S.$$

Substituting into the original equation, we find

$$\begin{aligned}
A(x-R)^2 + C(y-S)^2 - T &= A\left(\sqrt{\frac{T}{A}}\,u\right)^2 + C\left(\sqrt{\frac{T}{C}}\,v\right)^2 - T \\
&= Tu^2 + Tv^2 - T \\
&= T(u^2 + v^2 - 1),
\end{aligned}$$

∎

**Problem 11**

Consider the values $A$ and $C$ found in Exercise 1.2.8. Show that if $b^2 - 4ac = 0$, then either $A = 0$ or $C = 0$, depending on the signs of $a, b, c$. [Hint: Recall, $\sqrt{\alpha^2} = -\alpha$ if $\alpha < 0$.]

*Proof.* Suppose $b^2 - 4ac = 0$. From Exercise 1.2.8 we have

$$A = \sqrt{\frac{c}{a}}\,b + 2c, \qquad C = -\sqrt{\frac{a}{c}}\,b + 2a.$$

We see that

$$AC = 4ac - b^2 = -(b^2 - 4ac) = -0 = 0.$$

Thus $A = 0$ or $C = 0$. ∎

### Problem 12

Show that there exists constants $R$ and $T$ such that the equation

$$Au^2 + Du + Ev + H = 0,$$

can be written as

$$A(u-R)^2 + E(v-T) = 0.$$

Express $R$ and $T$ in terms of $A, D, E,$ and $H$.

*Proof.* First note $A \neq 0$ therefore $E \neq 0$. Let

$$R = -\frac{D}{2A}, \qquad T = -\left(\frac{H}{E} - \frac{D^2}{4AE}\right).$$

Then

$$\begin{aligned} Au^2 + Du + Ev + H &= A\left(u^2 + \frac{D}{A}u + \left(\frac{D}{2A}\right)^2\right) - \frac{D^2}{4A} + Ev + H \\ &= A\left(u + \frac{D}{2A}\right)^2 + E\left(v + \frac{H}{E} - \frac{D^2}{4AE}\right) \\ &= A(u-R)^2 + E(v-T) = 0. \end{aligned}$$

∎

### Problem 13

Suppose $A > 0$ and $E \neq 0$. Find a real affine change of coordinates that maps the parabola

$$V(A(x-R)^2 - E(y-T)),$$

to the parabola

$$V(u^2 - v).$$

*Proof.* We set $A(x-R)^2 = u^2$ and $-E(y-T) = -v$. Then solving for $x, y$ we have

$$x = \frac{u}{\sqrt{A}} + R, \quad y = \frac{v}{E} + T.$$

Then substituting into our original equation we have

$$A(x-R)^2 - E(y-T) = A\left(\frac{u}{\sqrt{A}} + R - R\right)^2 - E\left(\frac{v}{E} + T - T\right) = u^2 - v.$$

∎

### Problem 14

Suppose $ac > 0$. Use the real affine transformation in Exercise 1.2.8 to transform $C$ to a conic in the $uv$-plane. Find the coefficients of $u^2$ and $v^2$ in the resulting equation and show that they have opposite signs.

*Proof.* Suppose $ac > 0$. From Exercise 1.2.8 we have

$$A = \sqrt{\frac{c}{a}}\,b + 2c, \qquad C = -\sqrt{\frac{a}{c}}\,b + 2a.$$

We see that

$$AC = 4ac - b^2 = -(b^2 - 4ac) < 0.$$

Thus $A$ and $C$ have opposite signs. ∎

**Problem 15**

Suppose $ac < 0$ and $b \neq 0$. Use the real affine transformation

$$x = \sqrt{-\frac{c}{a}}u + v$$

$$y = u - \sqrt{-\frac{a}{c}}v,$$

to transform $C$ to a conic in the $uv$-plane of the form

$$Au^2 + Cv^2 + Du + Ev + H = 0.$$

Find the coefficients of the resulting equation and show that they have opposite signs.

*Proof.*

$$\begin{aligned}
ax^2 + bxy + cy^2 + dx + ey + h &= a\Big(\sqrt{-\tfrac{c}{a}}\,u + v\Big)^2 + b\Big(\sqrt{-\tfrac{c}{a}}\,u + v\Big)\Big(u - \sqrt{-\tfrac{a}{c}}\,v\Big) + c\Big(u - \sqrt{-\tfrac{a}{c}}\,v\Big)^2 \\
&\quad + d\Big(\sqrt{-\tfrac{c}{a}}\,u + v\Big) + e\Big(u - \sqrt{-\tfrac{a}{c}}\,v\Big) + h \\
&= \big(-cu^2 + 2\sqrt{-ac}\,uv - av^2\big) + b\Big(\sqrt{-\tfrac{c}{a}}\,u^2 - \sqrt{-\tfrac{a}{c}}\,v^2\Big) + \big(-cu^2 - 2\sqrt{-ac}\,uv - av^2\big) \\
&\quad + \Big(d\sqrt{-\tfrac{c}{a}} + e\Big)u + \Big(d - e\sqrt{-\tfrac{a}{c}}\Big)v + h \\
&= \Big(-2c + b\sqrt{-\tfrac{c}{a}}\Big)u^2 + \Big(-2a - b\sqrt{-\tfrac{a}{c}}\Big)v^2 + \Big(d\sqrt{-\tfrac{c}{a}} + e\Big)u + \Big(d - e\sqrt{-\tfrac{a}{c}}\Big)v + h \\
&= Au^2 + Cv^2 + Du + Ev + H.
\end{aligned}$$

∎

*Proof.* Since $ac < 0$ and $b \neq 0$, we have

$$A = -2c + b\sqrt{-\frac{c}{a}}, \qquad C = -2a - b\sqrt{-\frac{a}{c}}.$$

Then

$$AC = \Big(-2c + b\sqrt{-\tfrac{c}{a}}\Big)\Big(-2a - b\sqrt{-\tfrac{a}{c}}\Big) = 4ac - b^2.$$

Since $ac < 0$,

$$4ac - b^2 < 0 \implies AC < 0.$$

Therefore $A$ and $C$ have opposite signs. ∎

### Problem 16

Suppose $ac = 0$ (so $b \neq 0$). Since either $a = 0$ or $c = 0$, we can assume $c = 0$. Use the real affine transformation

$$\begin{aligned} x &= u + v \\ y &= \left(\frac{1-a}{b}\right)u - \left(\frac{1+a}{b}\right)v, \end{aligned}$$

to transform $V(ax^2 + bxy + dx + ey + h)$ to a conic in the $uv$-plane of the form

$$V(u^2 - v^2 + Du + Ev + H).$$

*Proof.*

$$\begin{aligned} ax^2 + bxy + dx + ey + h &= a(u+v)^2 + b(u+v)\Big(\frac{1-a}{b}u - \frac{1+a}{b}v\Big) \\ &\quad + d(u+v) + e\Big(\frac{1-a}{b}u - \frac{1+a}{b}v\Big) + h \\ &= a(u^2 + 2uv + v^2) + (u+v)((1-a)u - (1+a)v) \\ &\quad + d(u+v) + e\Big(\frac{1-a}{b}u - \frac{1+a}{b}v\Big) + h \\ &= (a + 1 - a)u^2 + (-(1+a) + a)v^2 + 2auv \\ &\quad + \Big(d + e\frac{1-a}{b}\Big)u + \Big(d - e\frac{1+a}{b}\Big)v + h \\ &= u^2 - v^2 + Du + Ev + H \end{aligned}$$

∎

### Problem 17

Show that there exists $R, S$, and $T$ so that

$$Au^2 - Cv^2 + Du + Ev + H = A(u-R)^2 - C(v-S)^2 - T.$$

Express $R, S$, and $T$ in terms of $A, C, D, E$, and $H$.

*Proof.* We set $A(u-R)^2 = Au^2 + Du$ and $-C(v-S)^2 = -Cv^2 + Ev$. Then solving for $R, S$ we have

$$R = -\frac{D}{2A}, \quad S = \frac{E}{2C}.$$

Then substituting into our original equation we have

$$\begin{aligned} Au^2 - Cv^2 + Du + Ev + H &= \Big(A(u-R)^2 - AR^2\Big) + \Big(-C(v-S)^2 + CS^2\Big) + H \\ &= A(u-R)^2 - C(v-S)^2 - (AR^2 - CS^2 - H) \\ &= A(u-R)^2 - C(v-S)^2 - T, \end{aligned}$$

where

$$T = AR^2 - CS^2 - H = \frac{D^2}{4A} - \frac{E^2}{4C} - H.$$

∎

**Problem 18**

Suppose $A, C, T > 0$. Find a real affine change of coordiantes that maps the hyperbola

$$V(A(x-R)^2 - C(y-S)^2 - T),$$

to the hyperbola

$$V(u^2 - v^2 - 1).$$

*Proof.* Notice

$$A(x-R)^2 - C(y-S)^2 - T = 0 \iff \frac{A(x-R)^2}{T} - \frac{C(y-S)^2}{T} = 1.$$

We set

$$u^2 = \frac{A(x-R)^2}{T}, \quad v^2 = \frac{C(y-S)^2}{T},$$

and solving for $x, y$ shows

$$x = \sqrt{\frac{T}{A}}\,u + R, \quad y = \sqrt{\frac{T}{C}}\,v + S.$$

Substituting into the original equation, we find

$$\begin{aligned} A(x-R)^2 - C(y-S)^2 - T &= A\left(\sqrt{\frac{T}{A}}\,u\right)^2 - C\left(\sqrt{\frac{T}{C}}\,v\right)^2 - T \\ &= Tu^2 - Tv^2 - T \\ &= T(u^2 - v^2 - 1). \end{aligned}$$

$\blacksquare$

**Problem 19**

Give an intuitive argument, based on the number of connected components, for the fact that no ellpise can be transformed into a hyperbola by a real affine change of coordiantes.

**Solution:** A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve the number of connected components. Therefore, no real affine change can transform an ellipse into a hyperbola.

**Problem 20**

Show that there is no real affine change of coordinates

$$u = ax + by + e$$

$$v = cx + dy + f,$$

that transforms the ellipse $V(x^2 + y^2 - 1)$ to the hyperbola $V(u^2 - v^2 - 1)$.

*Proof.* For contradiction, suppose such a real affine change exists.

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We must have

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the ellipse $x^2 + y^2 = 1$. Now substituting $y^2 = 1 - x^2$ we see for this to vanish for all $(x, y)$, the coefficients of $x^2$ and $y^2$ must be

$$a^2 - c^2 = b^2 - d^2,$$

which would make the squared coefficients have the same sign, contradicting the requirement for a hyperbola that they have opposite signs. Thus there is no real affine transformation from an ellipse to a hyperbola. ∎

**Problem 21**

Give an intuitive argument, based on boundedness, for the fact that no parabola can be transformed into an ellipse by a real affine change of coordinates.

**Solution:** A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve boundedness. Therefore, no real affine change can transform a parabola into an ellipse.

**Problem 22**

Show that there is no real affine change of coordinates that trasnforms the parabola $V(x^2 - y)$ to the circle $V(u^2 + v^2 - 1)$.

*Proof.* For contradiction, suppose such a real affine change exists.

$$\begin{aligned} u^2 + v^2 &= (ax + by + e)^2 + (cx + dy + f)^2 \\ &= (a^2 + c^2)x^2 + (b^2 + d^2)y^2 + 2(ab + cd)xy + 2(ae + cf)x + 2(be + df)y + (e^2 + f^2). \end{aligned}$$

We must have

$$(a^2 + c^2)x^2 + (b^2 + d^2)y^2 + 2(ab + cd)xy + 2(ae + cf)x + 2(be + df)y + (e^2 + f^2) - 1 = 0$$

for all points on the parabola $y = x^2$. Now substituting $y = x^2$, we have

$$(b^2 + d^2)x^4 + 2(ab + cd)x^3 + (a^2 + c^2 + 2(be + df))x^2 + 2(ae + cf)x + (e^2 + f^2) - 1 = 0.$$

For this to vanish for all $x$ all coefficients must be zero so

$$b^2 + d^2 = 0, \quad ab + cd = 0, \quad a^2 + c^2 + 2(be + df) = 0, \quad ae + cf = 0.$$

It follows that $a = b = c = d = 0$ and therefore $u^2 + v^2 = e^2 + f^2$ is constant, which cannot equal $x^2 + y^2$ on the parabola. Thus there is no real affine transformation from the parabola to the circle. ∎

**Problem 23**

Give an intuitive argument, based on the number of connected components, for the fact that no parabola can be transformed into a hyperbola by a real affine change of coordinates.

**Solution:** A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve the number of components. Therefore, no real affine change can transform a parabola into a hyperbola.

**Problem 24**

Show that there is no real affine change of coordinates that transforms that parabola $V(x^2 - y)$ to the hyperbola $V(u^2 - v^2 - 1)$.

*Proof.* For contradiction, suppose such a real affine change exists. Then

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We must have

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the parabola $y = x^2$. Substituting $y = x^2$, we get

$$(b^2 - d^2)x^4 + 2(ab - cd)x^3 + (a^2 - c^2)x^2 + 2(ae - cf)x + 2(be - df)x^2 + (e^2 - f^2) - 1 = 0.$$

For this to vanish for all $x$, the coefficient of $x^4$ must be zero

$$b^2 - d^2 = 0 \implies b = \pm d.$$

Then, the $x^3$ coefficient gives $ab - cd = 0$. Since Since $b = \pm d$, we have $a = \pm c$. Then, the $x^2$ coefficient becomes $a^2 - c^2 + 2(be - df)$. Since $a = \pm c$ and $b = \pm d$, this is zero if all coefficients vanish. Thus $u^2 - v^2$ are constant, which cannot equal $x^2 - y$ on the parabola. Therefore, there is no real affine transformation from the parabola to the hyperbola. ∎

## 1.3 Conics over the Complex Numbers

Problem 1

Show that the set

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 1 = 0\},$$

is empty but that the set

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 + 1 = 0\},$$

is not empty. In fact, show that given any complex number $x$ there must exist $y \in \mathbb{C}$ such that

$$(x, y) \in C.$$

Then show that if $x \neq \pm i$, then there are two distinct values $y \in \mathbb{C}$ such that $(x, y) \in C$, while if $x = \pm i$ there is only one such $y$.

*Proof.* Suppose $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 + 1 = 0 \iff x^2 + y^2 = -1$. Then $x^2 \geq 0$ and $y^2 \geq 0$ so $x^2 + y^2 \geq 0$, which is a contradiction. ∎

*Proof.* Let $(x, y) = (i, 0) \in \mathbb{C}^2$. Then

$$x^2 + y^2 + 1 = -1 + 1 = 0.$$

Thus $(x, y) \in C$. ∎

*Proof.* Let $x$ be an arbitrary complex number. Furthermore, let $y = \sqrt{-1 - x^2}$. Then

$$x^2 + \left(\sqrt{-1 - x^2}\right)^2 + 1 = x^2 - 1 - x^2 + 1 = 0.$$

Thus $(x, y) \in C$. ∎

*Proof.* Suppose $x \neq \pm i$. Then $1 + x^2 \neq 0$, so $\sqrt{1 + x^2} \neq 0$. Let

$$y = \pm i\sqrt{1 + x^2}.$$

These are two distinct values of $y$. Then

$$x^2 + y^2 + 1 = x^2 - (1 + x^2) + 1 = 0.$$

Now suppose $x = \pm i$. Then $1 + x^2 = 0$, so $y^2 = 0$ and it follows that $y = 0$. Therefore, there is exactly one value of $y$. ∎

## Problem 2

Let
$$P(x,y) = ax^2 + bxy + cy^2 + dx + ey + f,$$
with $a \neq 0$. Show that for any value $y \in \mathbb{C}$, there must be at least one $x \in \mathbb{C}$, but no more than two such $x$'s, such that
$$P(x,y) = 0.$$
[Hint: Write $P(x,y) = Ax^2+Bx+C$ as a function whose coefficients $A$, $B$, and $C$ are themselves functions of $y$, and use the quadratic formula. As mentioned before, this technique will be used frequently.]

*Proof.* Let $A = a$, $B = by + d$, and $C = cy^2 + ey + f$. Notice
$$P(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = ax^2 + (by+d)x + (cy^2 + ey + f) = Ax^2 + Bx + C.$$
Applying the quadratic formula we find
$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$
Since $A = a \neq 0$ this is defined. Now if $B^2 - 4AC = 0$ then we get one corresponding $x$. Otherwise, we get two corresponding $x$'s. ∎

## Problem 3

Let $C = V\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) \subset \mathbb{C}^2$. Show that $C$ is unbounded in $x$ and $y$.

*Proof.* We can solve for $x$ in terms of $y$
$$\frac{x^2}{4} = 1 - \frac{y^2}{9} \iff x = \pm 2\sqrt{1 - \frac{y^2}{9}}.$$
Since $y \in \mathbb{C}$ is arbitrary and square roots always exist in $\mathbb{C}$, for any value of $y$ there is a corresponding value of $x$. As $|y|$ becomes arbitrarily large, $1 - \frac{y^2}{9}$ becomes arbitrarily large, and thus the corresponding $x$ is arbitrarily large. Thus $C$ is unbounded in both $x$ and $y$. ∎

## Problem 4

Let $C = V(x^2 - y^2 - 1) \subset \mathbb{C}^2$. Show that there is a continuous path on the curve $C$ from the point $(-1,0)$ to the point $(1,0)$, despite the fact that no such continuous path exists in $\mathbb{R}^2$.

*Proof.* Let $x(t) = \cos(t)$ and $y(t) = i\sin(t)$. Then
$$x(t)^2 - y(t)^2 - 1 = \cos^2(t) - (i\sin(t))^2 - 1 = \cos^2(t) + \sin^2(t) - 1 = 0.$$
∎

### Problem 5

Show that if $x = u$ and $y = iv$, then the circle $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\}$ transforms into the hyperbola $\{(u, v) \in \mathbb{C}^2 \mid u^2 - v^2 = 1\}$.

*Proof.* Suppose $x = u$ and $y = iv$. Then

$$x^2 + y^2 = u^2 + (iv)^2 = u^2 - v^2 = 1.$$

∎

### Problem 6

Show that if $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then the inverse change of coordinates is

$$x = \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf).$$

$$y = \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af).$$

*Proof.* We need to solve the two equations $u = ax + by + e$ and $v = cx + dy + f$ in two unknowns $x$ and $y$. Translating this to linear algebra, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u - e \\ v - f \end{bmatrix}.$$

Using Cramer's rule we see

$$x = \frac{\begin{vmatrix} u - e & b \\ v - f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d(u - e) - b(v - f)}{ad - bc},$$

$$y = \frac{\begin{vmatrix} a & u - e \\ c & v - f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c(u - e) + a(v - f)}{ad - bc}.$$

Therefore

$$x = \frac{du - bv - de + bf}{ad - bc}, \quad y = \frac{-cu + av + ce - af}{ad - bc}.$$

∎

**Problem 7**

Use Theorem 1.2.25 together with the new result of Exercise 1.3.5 to conclude that all ellipses and hyperbolas are equivalent under complex affine changes of coordinates.

*Proof.* By Theorem 1.2.25, any ellipse can be transformed via an affine change of coordinates to a circle. Then, by Exercise 1.3.5 the circle can be transformed via a complex affine change to a hyperbola. ∎

**Problem 8**

Show that the circle $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 - 1 = 0\}$ is not equivalent under a complex affine change of coordinates to the parabola $\{(u, v) \in \mathbb{C}^2 \mid u^2 - v^2 = 0\}$.

*Proof.* For contradiction, suppose such a complex affine change exists

$$u = ax + by + e, \quad v = cx + dy + f.$$

Then

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We need

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the circle. Substituting $y = \sqrt{1 - x^2}$, the lhs must vanish for all $x$. The highest-degree terms show $b^2 - d^2 = 0 \implies b = \pm d$ and the other coefficients similarly give $a = \pm c$, $e = \pm f$. But then $u^2 - v^2$ would be constant, which cannot equal $x^2 + y^2 - 1$. Therefore there is no complex affine transformation mapping the circle to the hyperbola $u^2 - v^2 = 0$. ∎

**Problem 9**

Let

$$C = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}.$$

Give a bijection from

$$C \cap \{(x + iy, u + iv) \mid x, u \in \mathbb{R}, y = 0, v = 0\}.$$

to the real circle of the unit radius in $\mathbb{R}^2$.

**Solution:**

$$(x + iy, u + iv) \mapsto (x, u).$$

**Problem 10**

Let

$$C = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}.$$

Give a bijection from

$$C \cap \{(x + iy, u + iv) \in \mathbb{R}^4 \mid x, v \in \mathbb{R}, y = 0, u = 0\},$$

to the hyperbola $V(x^2 - v^2 - 1)$ in $\mathbb{R}^2$.

**Solution:**

$$(x + 0i, 0 + iv) \mapsto (y, u).$$

## 1.4 The Complex Projective Plane $\mathbb{P}^2$

**Problem 1**

1. Show that $(2, 1+i, 3i) \sim (2-2i, 2, 3+3i)$.
2. Show that $(1, 2, 3) \sim (2, 4, 6)$ and $(-2, -4, -6) \sim (-i, -2i, -3i)$.
3. Show that $(2, 1+i, 3i) \nsim (4, 4i, 6i)$.
4. Show that $(1, 2, 3) \nsim (3, 6, 8)$.

*Proof.* Let $\lambda = \frac{2}{2-2i} = \frac{1}{2} + \frac{1}{2}i$. Then

$$\lambda(2-2i) = \frac{2}{2-2i}(2-2i) = 2,$$

$$\lambda \cdot 2 = \left(\frac{1}{2} + \frac{1}{2}i\right) 2 = 1 + i,$$

$$\lambda(3+3i) = \left(\frac{1}{2} + \frac{1}{2}i\right)(3+3i) = 3i.$$

∎

*Proof.* Let $\lambda = \frac{1}{2}$. Then

$$\lambda \cdot 2 = 1,$$

$$\lambda \cdot 4 = 2,$$

$$\lambda \cdot 6 = 3.$$

∎

*Proof.* Let $\lambda = 2i$. Then

$$\lambda \cdot (-i) = -2,$$

$$\lambda \cdot (-2i) = -4,$$

$$\lambda \cdot (-3i) = -6.$$

∎

*Proof.* Suppose there exists $\lambda$ such that $\lambda(4, 4i, 6i) = (2, 1+i, 3i)$. Then

$$\lambda \cdot 4 = 2 \implies \lambda = \frac{1}{2},$$

$$\lambda \cdot 4i = 2i \neq 1 + i.$$

Thus no such $\lambda$ exists. ∎

*Proof.* Suppose there exists $\lambda$ such that $\lambda(3, 6, 8) = (1, 2, 3)$. Then

$$\lambda \cdot 3 = 1 \implies \lambda = \frac{1}{3},$$

$$\lambda \cdot 8 = \frac{8}{3} \neq 3.$$

Thus no such $\lambda$ exists. ∎

**Problem 2**

Show that $\sim$ is an equivalence relation.

*Proof.* Suppose $(x,y,z),(a,b,c),(d,e,f) \in \mathbb{C}^3-\{(0,0,0)\}$. Then $\lambda = 1$ shows $(x,y,z) \sim (x,y,z)$. Thus $\sim$ is reflexive.

Suppose $(a,b,c) \sim (d,e,f)$. Then there exists $\lambda \in \mathbb{C}-\{0\}$ such that $(a,b,c) = (\lambda d, \lambda e, \lambda f)$. Therefore $\left(\frac{1}{\lambda}a, \frac{1}{\lambda}b, \frac{1}{\lambda}c\right) = (d,e,f)$. It follows that $(d,e,f) \sim (a,b,c)$. Thus $\sim$ is symmetric.

Suppose $(x,y,z) \sim (a,b,c)$ and $(a,b,c) \sim (d,e,f)$. Then there exist $\lambda_1, \lambda_2 \in \mathbb{C}-\{0\}$ such that $(x,y,z) = (\lambda_1 a, \lambda_1 b, \lambda_1 c)$ and $(a,b,c) = (\lambda_2 d, \lambda_2 e, \lambda_2 f)$. Then

$$(x,y,z) = (\lambda_1 a, \lambda_1 b, \lambda_1 c) = (\lambda_1\lambda_2 d, \lambda_1\lambda_2 e, \lambda_1\lambda_2 f).$$

Thus $(x,y,z) \sim (d,e,f)$. Therefore $\sim$ is transitive. ∎

**Problem 3**

Suppose that $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$ and that $x_1 = x_2 \neq 0$. Show that $y_1 = y_2$ and $z_2 = z_2$.

*Proof.* Since $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$ there exists $\lambda \in \mathbb{C}-\{0\}$ such that $(x_1,y_1,z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$. Thus $x_1 = \lambda x_2 = \lambda x_1$ therefore $\lambda = \frac{x_1}{x_1} = 1$. It follows that $y_1 = y_2$ and $z_1 = z_2$. ∎

**Problem 4**

Suppose that $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$ with $z_1 \neq 0$ and $z_2 \neq 0$. Show that

$$(x_1,y_1,z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2,y_2,z_2).$$

*Proof.* We see

$$(x_1,y_1,z_1) = \left(z_1 \cdot \frac{x_1}{z_1}, z_1 \cdot \frac{y_1}{z_1}, z_1 \cdot 1\right).$$

Since $z_1 \neq 0$ we see $(x_1,y_1,z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right)$. Now, since $(x_1,y_1,z_1) \sim (x_2,y_2,z_2)$ there exists $\lambda \in \mathbb{C}-\{0\}$ such that $(x_1,y_1,z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$. Since $z_1 = \lambda z_2$ and $z_1, z_2 \neq 0$ we see

$$\frac{x_1}{z_1} = \frac{\lambda x_2}{\lambda z_2} = \frac{x_2}{z_2} \quad \text{and} \quad \frac{y_1}{z_1} = \frac{\lambda y_2}{\lambda z_2} = \frac{y_2}{z_2}.$$

Thus

$$\left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right).$$

Since

$$(x_2,y_2,z_2) = \left(z_2 \cdot \frac{x_2}{z_2}, z_2 \cdot \frac{y_2}{z_2}, z_2 \cdot 1\right),$$

and $z_2 \neq 0$ we see

$$\left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2,y_2,z_2).$$

Therefore,

$$(x_1,y_1,z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2,y_2,z_2).$$

∎

Problem 5

1. Find the equivalence class of $(0, 0, 1)$.
2. Find the equivalence class of $(1, 2, 3)$.

**Solution (1):**

$$\{(0, 0, c) \in \mathbb{C}^3 \mid c \neq 0\}.$$

**Solution (2):**

$$\{(\lambda, 2\lambda, 3\lambda) \in \mathbb{C}^3 \mid \lambda \neq 0\}.$$

Problem 6

Show that the equivalence class $(1 : 2 : 3)$ and $(2 : 4 : 6)$ are equal as sets.

*Proof.* Clearly, with $\lambda = \frac{1}{2} \in \mathbb{C}$ we have $(1, 2, 3) = (\lambda 2, \lambda 4, \lambda 6)$. Thus $(1, 2, 3) \sim (2, 4, 6)$ so $(1 : 2 : 3) = (2 : 4 : 6)$. ∎

Problem 7

Explain why the elements of $\mathbb{P}^2$ can intuitively be thought of as complex lines through the origin in $\mathbb{C}^3$.

**Solution:** Take a line passing through the origin in $\mathbb{C}^3$ with direction vector $(a, b, c) \neq (0, 0, 0)$. This line consists of all points of the form $(\lambda a, \lambda b, \lambda c)$ such that $\lambda \in \mathbb{C}$. If we require $\lambda \neq 0$ we get the equivalence class $(a : b : c) \in \mathbb{P}^2$. Thus each element of $\mathbb{P}^2$ represents a complex line through the origin in $\mathbb{C}^3$.

Problem 8

If $c \neq 0$, show, in $\mathbb{C}^3$, that the line $x = \lambda a$, $y = \lambda b$, $z = \lambda c$ intersects the plane $\{(x, y, z) \mid z = 1\}$ in exactly one point. Show that this point of intersection is $\left(\frac{a}{c}, \frac{b}{c}, 1\right)$.

*Proof.* Suppose $c \neq 0$. At the intersection we must have $z = \lambda c = 1$ so $\lambda = \frac{1}{c}$. Thus

$$(\lambda a, \lambda b, \lambda c) = \left(\frac{a}{c}, \frac{b}{c}, 1\right).$$

∎

### Problem 9

Show that the map $\psi : \mathbb{C}^2 \to \{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\}$ defined by $\psi(x, y) = (x : y : 1)$ is a bijection.

*Proof.* Suppose $(a, b), (x, y) \in \mathbb{C}^2$ such that $\psi(x, y) = \psi(a, b)$. Then

$$\psi(x, y) = \psi(a, b) \iff (x : y : 1) = (a : b : 1).$$

There exists $\lambda \neq 0$ such that $(x, y, 1) = (\lambda a, \lambda b, \lambda)$. Therefore $\lambda = 1$ thus $x = a$ and $y = b$. Thus $\psi$ is injective. Let $(x : y : z)$ be an arbitrary element in $\{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\}$. Then

$$(x : y : z) = \left(\frac{x}{z} : \frac{y}{z} : 1\right) = \psi\left(\frac{x}{z}, \frac{y}{z}\right).$$

Thus $\psi$ is surjective. It follows that $\psi$ is bijective. ∎

### Problem 10

Find a map from $\{(x, y, z) \in \mathbb{P}^2 \mid z \neq 0\}$ to $\mathbb{C}^2$ that is the inverse of the map $\psi$ in Exercise 1.4.9.

**Solution:** Let

$$\phi : \{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\} \longrightarrow \mathbb{C}^2$$

be defined by

$$\phi(x : y : z) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

### Problem 11

Consider the line $l = \{(x, y) \in \mathbb{C}^2 \mid ax + by + c = 0\}$ in $\mathbb{C}^2$. Assume $a, b \neq 0$. Explain why, as $|x| \to \infty$, $|y| \to \infty$. (Hence, $|x|$ is the modulus of $x$.)

*Proof.* We see $y = \frac{-c-ax}{b}$ and $x = \frac{-by-c}{a}$. Since $b$ and $c$ are constants, as $|y| \to \infty$ we have $|x| \to \infty$. ∎

## Problem 12

Consider again the line $l$. We know that $a$ and $b$ cannot both be $0$, so we will assume without loss of generality that $b \neq 0$.

1. Show that the image of $l$ in $\mathbb{P}^2$ under $\psi$ is the set

$$\{(bx : -ax - c : b) \mid x \in \mathbb{C}\}.$$

2. Show that this set equals the following union.

$$\{(bx : -ax - c : b) \mid x \in \mathbb{C}\} = \{(0 : -c : b)\} \cup \left\{\left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right)\right\}.$$

3. Show that as $|x| \to \infty$, the second set in the above union becomes

$$\left\{\left(1 : -\frac{a}{b} : 0\right)\right\}.$$

## Problem 13

## Problem 14

## Problem 15

## Problem 16

## Problem 17

## Problem 18

## Problem 19

## Problem 20

## Problem 21