Calculus by Spivak

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Contents

1 Basic Properties of Numbers

Problem 1

Prove the following:
(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.
(ii) $x^2 - y^2 = (x - y)(x + y)$.
(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.
(iv) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.
(v) $x^n - y^n = (x - y)(x^{(n-1)} + x^{(n-2)}y + \cdots + xy^{(n-2)} + y^{(n-1)})$
(vi) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. There is a particularly easy way to do this (iv), and it will show you how to find a factorization for $x^n + y^n$ whenever n is odd.

Proof.

$$\begin{aligned} & ax = a \\ \iff\ & a^{-1} \cdot ax = a^{-1} \cdot a \\ \iff\ & 1x = 1 && \text{(P7)} \\ \iff\ & x = 1 && \text{(P6)} \end{aligned}$$

∎

Proof.

$$\begin{aligned} x^2 - y^2 &= x^2 - xy + xy - y^2 && \text{(P2, P3)} \\ &= x(x - y) + y(x - y) && \text{(P9)} \\ &= (x - y)(x + y) && \text{(P9)} \end{aligned}$$

∎

Proof.

$$
\begin{aligned}
& x^2 = y^2 \\
\iff & x^2 - y^2 = y^2 - y^2 \\
\iff & x^2 - y^2 = 0 && \text{(P3)} \\
\iff & (x-y)(x+y) = 0 && \text{(1 ii)}
\end{aligned}
$$

It then follows that either $x = y$ or $x = -y$. ∎

Proof.

$$
\begin{aligned}
x^3 - y^3 &= x^3 - x^2y + x^2y - xy^2 + xy^2 - y^3 && \text{(P2, P3)} \\
&= x^2(x-y) + xy(x-y) + y^2(x-y) && \text{(P9)} \\
&= (x-y)(x^2 + xy + y^2) && \text{(P9)}
\end{aligned}
$$

∎

Proof.

$$
\begin{aligned}
&(x-y)(x^{(n-1)} + x^{(n-2)}y + \cdots + xy^{(n-2)} + y^{(n-1)}) \\
=&x^{(n-1)}(x-y) + x^{(n-2)}y(x-y) + \cdots + xy^{(n-2)}(x-y) + y^{(n-1)}(x-y) && \text{(P9)} \\
=&x^{(n-1)} \cdot x - x^{(n-1)} \cdot y + x^{(n-2)}y \cdot x - x^{(n-2)}y \cdot y + \\
&\cdots + xy^{(n-2)} \cdot x - xy^{(n-2)} \cdot y + y^{(n-1)} \cdot x - y^{(n-1)} \cdot y && \text{(P9)} \\
=&x^n - x^{n-1}y + x^{n-1}y - x^{n-2}y^2 + \cdots + x^2y^{n-2} - xy^{n-1} + xy^{n-1} - y^n \\
=&x^n - y^n && \text{(P3)}
\end{aligned}
$$

∎

Proof.

$$
\begin{aligned}
x^3 + y^3 &= x^3 - (-y)^3 \\
&= (x - (-y))(x^2 + x(-y) + (-y)^2) && \text{(1 iv)} \\
&= (x+y)(x^2 - xy + y^2)
\end{aligned}
$$

∎

Problem 3

Prove the following:

(i) $\frac{a}{b} = \frac{ac}{bc}$, if $b, c \neq 0$.

(ii) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, if $b, d \neq 0$.

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$. (To do this you must remember the defining property of $(ab)^{-1}$.)

(iv) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db}$, if $b, d \neq 0$.

(v) $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, if $b, c, d \neq 0$.

(vi) If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also determine when $\frac{a}{b} = \frac{b}{a}$.

Proof. Suppose $b, c \neq 0$. Then:

$$\begin{aligned}
& \frac{a}{b} = \frac{ac}{bc} & \\
\iff & ab^{-1} = ac(bc)^{-1} & \\
\iff & ab^{-1}(bc) = ac(bc)^{-1}bc & \\
\iff & ab^{-1}(bc) = ac \cdot 1 & \text{P7} \\
\iff & ab^{-1}(bc) = ac & \text{P6} \\
\iff & a(b^{-1}b)c = ac & \text{P5} \\
\iff & a \cdot 1 \cdot c = ac & \text{P7} \\
\iff & ac = ac & \text{P6}
\end{aligned}$$

∎

Proof. Suppose $b, d \neq 0$. Then:

$$\begin{aligned}
& \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} & \\
\iff & ab^{-1} + cd^{-1} = (ad + bc)(bd)^{-1} & \\
\iff & (bd)(ab^{-1} + cd^{-1}) = (ad + bc)(bd)^{-1}(bd) & \\
\iff & ab^{-1}(bd) + cd^{-1}(bd) = (ad + bc)(bd)^{-1}(bd) & \text{P9} \\
\iff & ab^{-1}(bd) + cd^{-1}(bd) = (ad + bc) \cdot 1 & \text{P7} \\
\iff & ab^{-1}(bd) + cd^{-1}(bd) = (ad + bc) & \text{P6} \\
\iff & a(b^{-1}b)d + cd^{-1}(bd) = (ad + bc) & \text{P5} \\
\iff & a(b^{-1}b)d + cd^{-1}(db) = (ad + bc) & \text{P8} \\
\iff & a(b^{-1}b)d + c(d^{-1}d)b = (ad + bc) & \text{P5} \\
\iff & a \cdot 1 \cdot d + c \cdot 1 \cdot b = (ad + bc) & \text{P7} \\
\iff & ad + cb = (ad + bc) & \text{P6} \\
\iff & ad + bc = ad + bc & \text{P8}
\end{aligned}$$

∎

Proof. Suppose $a, b \neq 0$. Then:

$$\begin{aligned}
& (ab)^{-1} = a^{-1}b^{-1} & \\
\iff & (ab)(ab)^{-1} = (ab)a^{-1}b^{-1} & \\
\iff & 1 = a(ba^{-1})b^{-1} & \text{P5} \\
\iff & 1 = a(a^{-1}b)b^{-1} & \text{P4} \\
\iff & 1 = (a \cdot a^{-1})b \cdot b^{-1} & \text{P5} \\
\iff & 1 = 1 \cdot b \cdot b^{-1} & \text{P7} \\
\iff & 1 = 1 \cdot 1 & \text{P7} \\
\iff & 1 = 1 & \text{P6}
\end{aligned}$$

∎

Proof. Suppose $b, d \neq 0$. Then:

$$
\begin{aligned}
& \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{db} \\
\iff & ab^{-1} \cdot cd^{-1} = ac(db)^{-1} \\
\iff & ab^{-1} \cdot cd^{-1} = ac(bd)^{-1} && \text{P8} \\
\iff & acb^{-1}d^{-1} = ac(bd)^{-1} && \text{P8} \\
\iff & acb^{-1}d^{-1}(bd) = ac(bd)^{-1}(bd) \\
\iff & acb^{-1}d^{-1}(bd) = ac \cdot 1 && \text{P7} \\
\iff & acb^{-1}d^{-1}(bd) = ac && \text{P6} \\
\iff & acb^{-1}d^{-1}(db) = ac && \text{P8} \\
\iff & acb^{-1}(d^{-1}d)b = ac && \text{P5} \\
\iff & acb^{-1} \cdot 1 \cdot b = ac && \text{P7} \\
\iff & acb^{-1}b = ac && \text{P6} \\
\iff & ac \cdot 1 = ac && \text{P7} \\
\iff & ac = ac && \text{P6}
\end{aligned}
$$

∎

Proof. Suppose $b, c, d \neq 0$. Then:

$$
\begin{aligned}
& \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} \\
\iff & \frac{a}{b}\left(\frac{c}{d}\right)^{-1} = \frac{ad}{bc} \\
\iff & \frac{a}{b}\left(\frac{c}{d}\right)^{-1} = \frac{a}{b} \cdot \frac{d}{c} && \text{Part (iv)} \\
\iff & \frac{a}{b}\left(\frac{c}{d}\right)^{-1} \cdot \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \cdot \frac{c}{d} \\
\iff & \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{d}{c} \cdot \frac{c}{d} && \text{P7} \\
\iff & \frac{a}{b} = \frac{a}{b} \cdot \frac{d}{c} \cdot \frac{c}{d} && \text{P6} \\
\iff & \frac{a}{b} = \frac{a}{b} \cdot \frac{dc}{cd} && \text{Part (iv)} \\
\iff & \frac{a}{b} = \frac{a}{b} \cdot \frac{dc}{dc} && \text{P8} \\
\iff & \frac{a}{b} = \frac{a}{b} \cdot dc(dc)^{-1} \\
\iff & \frac{a}{b} = \frac{a}{b} \cdot 1 && \text{P7} \\
\iff & \frac{a}{b} = \frac{a}{b} && \text{P6}
\end{aligned}
$$

∎

Proof. Suppose $b, d \neq 0$. Then:

$$\begin{aligned}
&\frac{a}{b} = \frac{c}{d} \\
&\iff ab^{-1} = cd^{-1} \\
&\iff ab^{-1}d = cd^{-1}d \\
&\iff ab^{-1}d = c \cdot 1 && \text{P7} \\
&\iff ab^{-1}d = c && \text{P6} \\
&\iff adb^{-1} = c && \text{P8} \\
&\iff adb^{-1}b = cb \\
&\iff ad \cdot 1 = cb && \text{P7} \\
&\iff ad = cb && \text{P7} \\
&\iff ad = bc && \text{P5}
\end{aligned}$$

Suppose $b, a \neq 0$ and $\frac{a}{b} = \frac{b}{a}$. Then:

$$\begin{aligned}
&\frac{a}{b} = \frac{b}{a} \\
&\iff a^2 = b^2 && \text{By previous answer} \\
&\iff |a| = |b|
\end{aligned}$$

Therefore $\frac{a}{b} = \frac{b}{a}$ iff $|a| = |b|$. ∎

Problem 5

Prove the following:
(i) If $a < b$ and $c < d$, then $a + c < b + d$.
(ii) If $a < b$, then $-b < -a$.
(iii) If $a < b$ and $c > d$, then $a - c < b - d$.
(iv) If $a < b$ and $c > 0$, then $ac < bc$.
(v) If $a < b$ and $c < 0$, then $ac > bc$.
(vi) If $a > 1$, then $a^2 > a$.
(vii) If $0 < a < 1$, then $a^2 < a$.
(viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
(ix) If $0 \leq a < b$, then $a^2 < b^2$. (Use (viii).)
(x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$. (Use (ix) backwards.)

Proof. Suppose $a < b$ and $c < d$. Then $b - a$ is in P and $d - c$ is in P. Therefore $(b - a) + (d - c)$ is in P. It follows that $(b - a) + (d - c) > 0$ and therefore $a + c < b + d$. ∎

Proof. Suppose $a < b$. Clearly $b - a$ is in P. It follows that $-(-b - (-a))$ is in P. Now $-b - (-a) < 0$ so $-b < -a$. ∎

Proof. Since $d < c$ by (ii) $-c < -d$. Since $-c < -d$ and $a < b$ by (i) $a + (-c) < b + (-d)$ therefore $a - c < b - d$. ∎

Proof. Suppose $a < b$ and $c > 0$. Since $a < b$ it follows $b - a$ is in P. Since $b - a$ and c are in P it follows that $c(b - a)$ is in P. Then $c(b - a) = bc - ac$ is in P so $ac < bc$. ∎

Proof. Suppose $a < b$ and $c < 0$ it follows that $-c > 0$. Then by (iv) $a(-c) < b(-c)$ so $-ac < -bc$. Then by (ii) it follows that $ac > bc$. ∎

Proof. Suppose $a > 1$. It follows that $a - 1 > 0$. Since $a > 1$ and $1 > 0$ it follows that $a > 0$. Since $0 < a - 1$ and $a > 0$ it follows that $0(a) < (a-1)a$ so $0 < a^2 - a$ and therefore $a^2 > a$. ∎

Proof. Suppose $0 < a < 1$. It follows that $a-1 < 0$. Since $a-1 < 0$ and $a > 0$ it follows by (iv) that $a(a-1) < 0(a)$. Therefore $a^2 - a < 0$ and $a^2 < a$. ∎

Proof. Suppose $0 \leq a < b$ and $0 \leq c < d$. If $a = 0$ or $c = 0$ then $ac = 0$. Now since $b > 0$ and $d > 0$ it follows that $bd > 0$ so $0 = ac < bd$. Suppose $a > 0$ and $c > 0$. Since $a < b$ and $d > 0$ it follows that $ad < bd$. Since $c < d$ and $a > 0$ it follows that $ac < ad$. Then $ac < ad < bd$ so $ac < bd$. ∎

Proof. Suppose $0 \leq a < b$. By part (viii) it follows that $a \cdot a < b \cdot b$ so $a^2 < b^2$. ∎

Proof. Suppose $a, b \geq 0$ and $a^2 < b^2$. Since $a^2 < b^2$ by (ix) it follows that $0 \leq a < b$ so $a < b$. ∎

Problem 7

Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b$$

Notice that the inequality $\sqrt{ab} \leq (a+b)/2$ holds for all $a, b \geq 0$. A generalization of this fact occurs in Problem $2-22$.

Proof. Suppose $0 < a < b$. Now let $x^2 = a$ and $y^2 = b$. By Problem 5 part (ix) since $x^2 < y^2$, $x < y$ so $\sqrt{a} < \sqrt{b}$. It then follows that $\sqrt{a} - \sqrt{b} < 0$. Since $\sqrt{a} > 0$ it follows that $\sqrt{a}(\sqrt{a} - \sqrt{b}) < 0$. Then $\sqrt{a}(\sqrt{a} - \sqrt{b}) < 0 \iff a - \sqrt{ab} < 0$ so $a < \sqrt{ab}$. Since $\sqrt{b} > 0$ it follows that $\sqrt{b}(\sqrt{a} - \sqrt{b}) < 0$. Then $\sqrt{b}(\sqrt{a} - \sqrt{b}) < 0 \iff \sqrt{ab} - b < 0$ so $\sqrt{ab} < b$. ∎

Problem 12

Prove the folowing:
(i) $|xy| = |x| \cdot |y|$
(ii) $|\frac{1}{x}| = \frac{1}{|x|}$, if $x \neq 0$. (The best way to do this is to remember what $|x|^{-1}$ is.)
(iii) $\frac{|x|}{|y|} = |\frac{x}{y}|$, if $y \neq 0$.
(iv) $|x - y| \leq |x| + |y|$. (Give a very short proof.)
(v) $|x| - |y| \leq |x - y|$. (A very short proof is possible, if you write things in the right way.)
(vi) $|(|x| - |y|)| \leq |x - y|$. (Why does this follow immediately from (v)?)
(vii) $|x + y + z| \leq |x| + |y| + |z|$. Indicate when equality holds, and prove your statement.

Proof. There are four cases to consider:

1. $x \leq 0$ and $y \leq 0$
2. $x \leq 0$ and $y \geq 0$
3. $x \geq 0$ and $y \leq 0$
4. $x \geq 0$ and $y \geq 0$

Suppose $x \leq 0$ and $y \leq 0$. Then $xy \geq 0$ so $|xy| = xy$. Now $|x| = -x$ and $|y| = -y$ so $|x| \cdot |y| = (-x)(-y) = xy = |xy|$.

Suppose $x \leq 0$ and $y \geq 0$. Then $xy \leq 0$ so $|xy| = -xy$. Now $|x| = -x$ and $|y| = y$ so $|x| \cdot |y| = -xy = |xy|$.

Suppose $x \geq 0$ and $y \leq 0$. Then $xy \leq 0$ so $|xy| = -xy$. Now $|x| = x$ and $|y| = -y$ so $|x| \cdot |y| = -xy = |xy|$.

Suppose $x \geq 0$ and $y \geq 0$. Then $xy \geq 0$ so $|xy| = xy$. Now $|x| = x$ and $|y| = y$ so $|x| \cdot |y| = xy = |xy|$.

Since these cases were exhaustive $|x||y| = |xy|$. $\blacksquare$

Proof. Suppose $x \neq 0$. So $|\frac{1}{x}||x| = |\frac{x}{x}| \quad \text{part (i)} = 1 = \frac{|x|}{|x|} = \frac{1}{|x|} \cdot |x|$. Then dividing by $|x| \neq 0$ it follows that $|\frac{1}{x}| = \frac{1}{|x|}$. $\blacksquare$

Proof. Suppose $y \neq 0$. So $|\frac{x}{y}||y| = |\frac{xy}{y}| \quad \text{part (i)} = |x| = \frac{|x||y|}{|y|}$. Then dividing by $|y| \neq 0$ it follows that $|\frac{x}{y}| = \frac{|x|}{|y|}$. $\blacksquare$

Proof. So $|x - y| = |x + (-y)| \leq |x| + |-y| \quad \text{triangle inequality} = |x| + |y|$. $\blacksquare$

Proof. So $|x| = |x + y - y| = |(x - y) + y| \leq |x - y| + |y| \quad \text{triangle inequality} = |x + y| + |y|$. Then subtracting $|y|$ on both sides gives $|x| - |y| \leq |x - y|$. $\blacksquare$

Proof. So $|(|x| - |y|)| \leq ||x - y|| \quad \text{part (v)} = |x - y|$. $\blacksquare$

Proof. So

$$\begin{aligned}
|x + y + z| &\leq |(x + y) + z| \\
&\leq |x + y| + |z| && \text{triangle inequality} \\
&\leq |x| + |y| + |z| && \text{triangle inequality.}
\end{aligned}$$

Now let us discover when $|x+y+z| = |x|+|y|+|z|$. Equality occurs when $|x+y| = |x|+|y|$ and $|x+y+z| = |x+y|+|z|$. Clearly $|x+y| = |x|+|y|$ when x, y are both non-positive or non-negative. We can take $|x+y|+|z| = |x|+|y|+|z|$ subtract $|z|$ from both sides and get $|x + y| = |x| + |y|$ which we already showed requires that $|x|$ and $|y|$ both be non-positive or non-negative. Now $|x + y| + |z| = |x| + |y| + |z|$ requires $x + y$ and z to be both non-positive or non-negative. If x and y have the same sign then $x + y$ also has this sign. Thus, $|x + y + z| = |x| + |y| + |z|$ if x, y, z are all non-positive or non-negative. $\blacksquare$

Problem 13

The maximum of two numbers x and y is denoted by $max(x, y)$. Thus $max(-1, 3) = max(3, 3) = 3$ and $max(-1, -4) = max(-4, -1) = -1$. The minimum of x and y is denoted by $min(x, y)$. Prove that

$$max(x, y) = \frac{x + y + |y - x|}{2}$$

$$min(x, y) = \frac{x + y - |y - x|}{2}$$

Derive a formula for $max(x, y, z)$ and $min(x, y, z)$, using, for example

$$max(x, y, z) = max(x, max(y, z))$$

Proof. Lets analyze $\frac{x+y}{2} + \frac{|y-x|}{2}$. Now if $y - x > 0$ then $y \geq x$ and $|y - x| = y - x$. Then $\frac{x+y}{2} + \frac{y-x}{2} = \frac{x+y+y-x}{2} = \frac{2y}{2} = y$ as expected. If $y - x < 0$ then $x > y$ and $|y - x| = -(y - x)$. Then $\frac{x+y}{2} + \frac{-(y-x)}{2} = \frac{x+y-y+x}{2} = \frac{2x}{2} = x$ as expected. The *min* equation simply negates $|y - x|$ and following similarly to our *max* computation would result in y if $y < x$ and x if $x \leq y$. $\blacksquare$

Formula for $max(x, y, z)$**:**

$$\begin{aligned} &max(x,y,z) = max(max(x,y),z) \\ =&max\left(\frac{x+y+|y-x|}{2}, z\right) \\ =&\frac{(x+y+|y-x|)+z+|z-(x+y+|y-x|)|}{2} \end{aligned}$$

Formula for $min(x, y, z)$**:**

$$\begin{aligned} &min(x,y,z) = min(min(x,y),z) \\ =&min\left(\frac{x+y-|y-x|}{2}, z\right) \\ =&\frac{(x+y-|y-x|)+z-|z-(x+y-|y-x|)|}{2} \end{aligned}$$

Problem 14

(a) Prove that $|a| = |-a|$. (The trick is not to become confused by too many cases. First prove the statement $a \geq 0$. Why is it then obvious for $a \leq 0$?)
(b) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular, it follows that $-|a| \leq a \leq |a|$.
(c) Use this fact to give a new proof that $|a + b| \leq |a| + |b|$.

Proof. Suppose $a \geq 0$ so $-a \leq 0$. So $|a| = a$ and $|-a| = -(-a)$. Then $|-a| = -(-a) = a = |a|$. Suppose $a < 0$ so $-a > 0$. So $|a| = -a$ and $|-a| = -a$. Then $|a| = -a = |-a|$. ∎

Proof. Suppose $-b \leq a \leq b$. Suppose $a \geq 0$ then $|a| = a$. So $-b \leq a \leq b \iff -b \leq |a| \leq b$. Suppose $a < 0$ then $|a| = -a$ So $-b \leq a \leq b \iff b \geq -a \geq -b \iff b \geq |a| \geq -b$. Therefore $|a| \leq b$.

Supose $|a| \leq b$. Suppose $a \geq 0$ then $|a| \leq b \iff a \leq b$. Suppose $a < 0$ then $|a| \leq b \iff -a \leq b \iff -b \leq a$. Since $-b \leq a$ and $a \leq b$ then $-b \leq a \leq b$.

Letting $b = a$ gives us $-|a| \leq a \leq |a|$. ∎

Proof. Trivially $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Taking the sum of these gives $-|a| + (-|b|) \leq a + b \leq |a| + |b| \iff -(|a| + |b|) \leq a + b \leq |a| + |b|$. Then by part (ii) we get $|a + b| \leq |a| + |b|$. ∎

Problem 16

(a) Show that

$$(x+y)^2 = x^2 + y^2 \quad \text{only when } x = 0 \text{ or } y = 0$$

$$(x+y)^3 = x^3 + y^3 \quad \text{only when } x = 0 \text{ or } y = 0 \text{ or } x = -y$$

(b) Using the fact that

$$x^2 + 2xy + y^2 = (x+y)^2 \geq 0$$

show that $4x^2 + 6xy + 4y^2 > 0$ unless x and y are both 0.
(c) Use part (b) to find out when $(x + y)^4 = x^4 + y^4$.
(d) Find out when $(x+y)^5 = x^5+y^5$. Hint: From the assumption $(x+y)^5 = x^5+y^5$ you should be able to derive the equation $x^3+2x^2+2xy^2+y^3 = 0$, if $xy \neq 0$. This implies that $(x+y)^3 = x^2y+xy^2 = xy(x+y)$. You should now be able to make a good guess as to when $(x + y)^n = x^n + y^n$; the proof is contained in Problem $11 - 57$.

Proof. First $(x+y)^2 = x^2 + 2xy + y^2$. Then $x^2 + 2xy + y^2 = x^2 + y^2 \iff 2xy = 0 \iff xy = 0$. Therefore $x = 0$ or $y = 0$.

Now $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$. Then $x^3+3x^2y+3xy^2+y^3 = x^3+y^3 \iff 3x^2y+3xy^2 = 0 \iff 3xy(x+y) = 0$. So either $3xy = 0$ in which case $x = 0$ or $y = 0$, or $x + y = 0$ in which case $x = -y$. ∎

Proof. Now $4x^2+2xy+4y^2 = 4(x^2+2xy+y^2)-6xy = 4(x+y)^2-6xy$. Then $(x+y)^2 \geq 0 \iff x^2+2xy+y^2 \geq 0 \iff x^2+y^2 \geq -2xy$. Similarly $(x-y)^2 \geq 0 \iff x^2 - 2xy + y^2 \geq 0 \iff x^2 + y^2 \geq 2xy$. Now since $x^2 + y^2 \geq -2xy$ it follows that $-(x^2+y^2) \leq 2xy$. Since $-(x^2+y^2) \leq 2xy$ and $x^2+y^2 \geq 2xy$, it follows that $-(x^2+y^2) \leq 2xy \leq x^2+y^2$. and therefore $|2xy| \leq x^2+y^2 \iff 2|xy| \leq x^2+y^2$. Now expanding, $4(x+y)^2-6xy > 0 \iff 4(x^2+2xy+y^2)-6xy > 0 \iff 4x^2+8xy+4y^2-6xy > 0 \iff 4x^2+4y^2+2xy > 0$. Now since $-(x^2+y^2) \leq 2xy \leq x^2+y^2$ it follows that $4x^2+4y^2+2xy > 4x^2+4y^2-(x^2+y^2) \iff 4(x^2+y^2)-(x^2+y^2) > 0 \iff 3(x^2+y^2) > 0$. Which is clearly true if $x, y \neq 0$, since $x^2 \geq 0$ and $y^2 \geq 0$ therefore $3(x^2+y^2) > 0$. Therefore $4x^2+2xy+4y^2 > 0$ if x, y are not both zero. ∎

Problem 18

(a) Suppose that $b^2 - 4c \geq 0$. Show that the numbers

$$\frac{-b+\sqrt{b^2-4c}}{2}, \quad \frac{-b-\sqrt{b^2-4c}}{2}$$

both satisfy the equation $x^2 + bx + c = 0$.
(b) Suppose that $b^2 - 4c < 0$. Show that there are not numbers x satisfying $x^2 + bx + c = 0$; in fact, $x^2 + bx + c > 0$ for all x. Hint: Complete the square.
(c) Use this fact to give another proof that if x and y are not both 0, then $x^2 + xy + y^2 > 0$.
(d) For which numbers α is it true that $x^2 + \alpha xy + y^2 > 0$ whenever x and y are not both 0?
(e) Find the smallest possible value of $x^2 + bx + c$ and of $ax^2 + bx + c$, for $a > 0$.

Proof.

$$\begin{aligned}
x^2+bx+c &= \left(\frac{-b+\sqrt{b^2-4c}}{2}\right)^2 + b\left(\frac{-b+\sqrt{b^2-4c}}{2}\right) + c \\
&= \frac{b^2-2b\sqrt{b^2-4c}+b^2-4c}{4} + \frac{-2b^2+2b\sqrt{b^2-4c}}{4} + \frac{4c}{4} \\
&= \frac{2b^2-2b^2+2b\sqrt{b^2-4c}-2b\sqrt{b^2-4c}+4c-4c}{4} \\
&= \frac{0}{4} = 0
\end{aligned}$$

∎

Proof.

$$\begin{aligned}
x^2+bx+c &= \left(\frac{-b-\sqrt{b^2-4c}}{2}\right)^2 + b\left(\frac{-b-\sqrt{b^2-4c}}{2}\right) + c \\
&= \frac{b^2+2b\sqrt{b^2-4c}+b^2-4c}{4} + \frac{-2b^2-2b\sqrt{b^2-4c}}{4} + \frac{4c}{4} \\
&= \frac{2b^2-2b^2+2b\sqrt{b^2-4c}-2b\sqrt{b^2-4c}+4c-4c}{4} \\
&= \frac{0}{4} = 0
\end{aligned}$$

$\blacksquare$

Proof. Suppose $b^2 - 4c < 0$. For contradiction, suppose $x^2 + bx + c = 0$ for some x. Then $x^2 + bx + c = 0 \iff x^2 + bx = -c \iff \left(x + \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c \iff \left(x + \frac{b}{2}\right)^2 = \frac{b^2-4c}{4}$. Now, $b^2 - 4c < 0$ thus $\frac{b^2-4c}{4} < 0$ contradicting the square of a non-zero number being > 0. $\blacksquare$

Proof. Suppose wlog that $x \neq 0$. Let $b = y$ and $c = y^2$. Notice $b^2 - 4c = y^2 - 4y^2$ which is certainly < 0. It directly follows from part (b) that $x^2 + xy + y^2 > 0$. $\blacksquare$

Proof. From part (b), for $x^2 + \alpha xy + y^2$ to be > 0 for all x, y not both 0, we require that $(\alpha y)^2 - 4y^2 < 0$. Then $(\alpha y)^2 - 4y^2 < 0 \iff (\alpha y)^2 < 4y^2 \iff \alpha^2 y^2 < 4y^2 \iff \alpha^2 < 4$. From which it follows that $|\alpha| < 2$. $\blacksquare$

Proof. We find the smallest value of $ax^2 + bx + c$ to derive the formula for the general solution then show $x^2 + bx + c$ as a specific case where $a = 1$. Notice

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}. \end{aligned}$$

It is clear that $x = -\frac{b}{2a}$ gives the minimum value. Plugging in shows $\min(ax^2 + bx + c) = c - \frac{b^2}{4a}$. Let $a = 1$, plugging in shows $\min(x^2 + bx + c) = c - \frac{b^2}{4}$. $\blacksquare$

Problem 19

The fact that $a^2 \geq 0$ for all numbers a, elementary as it may seem, is nevertheless a fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

(A more general form occurs in Problem $2-21$.) The three proofs of the Schwarz inequality outlined below have one thing in common - their reliance on the fact that $a^2 \geq 0$ for all a.

(a) Prove that if $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some number λ, then equality holds in the Schwarz inequality. Prove the same thing if $y_1 = y_2 = 0$. Now suppose that y_1 and y_2 are not both 0, and that there is no λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ Then

$$\begin{aligned} 0 &< (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 \\ &= \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2) \end{aligned}$$

Using Problem 18, complete the proof of the Schwarz inequality.

(b) Prove the Schwarz inequality by using $2xy \leq x^2 + y^2$ (how is this derived) with

$$x = \frac{x_i}{\sqrt{x_1^2 + x_2^2}}, \quad y = \frac{y_i}{\sqrt{y_1^2 + y_2^2}}$$

first show for $i = 1$ then for $i = 2$.

(c) Prove the Scwarz inequality by first proving that

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$$

(d) Deduce, from each of these three proofs, that equality holds only when $y_1 = y_2 = 0$ or when there is a number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.

Proof. Suppose $y_1 = y_2 = 0$. Then

$$0 = 0 + 0 = x_1 \cdot 0 + x_2 \cdot 0 = x_1 y_1 + x_2 y_2 = \sqrt{x_1^2 + x_2^2} \cdot 0 = \sqrt{x_1^2 + x_2^2}\sqrt{0} = \sqrt{x_1^2 + x_2^2}\sqrt{0^2 + 0^2} = \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

Thus $x_1 x_2 = \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$ as required. ∎

Problem 20

Prove that if

$$|x - x_0| < \frac{\epsilon}{2} \text{ and } |y - y_0| < \frac{\epsilon}{2}$$

then

$$|(x + y) - (x_0 + y_0)| < \epsilon$$

$$|(x - y) - (x_0 - y_0)| < \epsilon$$

Proof. Suppose $|x - x_0| < \frac{\epsilon}{2}$ and $|y - y_0| < \frac{\epsilon}{2}$. Then

$$|(x + y) - (x_0 + y_0)| = |(x - x_0) + (y - y_0)| \leq |x - x_0| + |y - y_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus $|(x + y) - (x_0 + y_0)| < \epsilon$. Similarly

$$|(x - y) - (x_0 - y_0)| = |(x - x_0) + (y_0 - y)| \leq |x - x_0| + |y_0 - y| = |x - x_0| + |y - y_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus $|(x - y) - (x_0 - y_0)| < \epsilon$. ∎

Problem 21

Prove that if

$$|x - x_0| < min\left(\frac{\epsilon}{2(|y_0| + 1)}, 1\right) \text{ and } |y - y_0| < \frac{\epsilon}{2(|x_0| + 1)}$$

then $|xy - x_0 y_0| < \epsilon$
(The notion "min" was defined in Problem 13, but the formula provided by that problem is irrevelant at the moment; the first inequality in the hypothesis just means that

$$|x - x_0| < \frac{\epsilon}{2(|y_0| + 1)} \text{ and } |x - x_0| < 1;$$

at one point in the argument you will need the first inequality, and at another point you will need the second. One more word of advice: since the hypotheses only provide information about $x - x_0$ and $y - y_0$, it is almost a forgone conclusion that the proof will depend up writing $xy - x_0 y_0$ in a way that involves $x - x_0$ and $y - y_0$.)

Problem 22

Prove that if $y_0 \neq 0$ and

$$|y - y_0| < min\left(\frac{|y_0|}{2}, \frac{\epsilon|y_0|^2}{2}\right)$$

then $y \neq 0$ and

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| < \epsilon$$

Proof. Suppose $y_0 \neq 0$. Then

$$|y| = |y - y_0 + y_0| = |y_0 + (y - y_0)| \geq ||y_0| - |y - y_0|| \geq |y_0| - |y - y_0| > |y_0| - \frac{|y_0|}{2} = \frac{|y_0|}{2}$$

Thus $y \neq 0$. Also

$$\left|\frac{1}{y} - \frac{1}{y_0}\right| = \left|\frac{y - y_0}{yy_0}\right| = \frac{|y - y_0|}{|y||y_0|} < \frac{\frac{\epsilon|y_0|^2}{2}}{\frac{|y_0|}{2}|y_0|} = \epsilon$$

∎

Problem 23

Replace the question marks in the following statement by expressions involving ϵ, x_0, and y_0 so that the conclusion will be true:
If $y_0 \neq 0$ and

$$|y - y_0| < ? \text{ and } |x - x_0| < ?$$

then $y \neq 0$ and

$$\frac{x}{y} - \frac{x_0}{y_0} < \epsilon$$

This problem is trivial in the sense that its solution follows from Problem 21 and 22 with almost now work at all (notice that $\frac{x}{y} = x \cdot \frac{1}{y}$). The crucial point is not to become confused; decide which of the two problems should be used first, and don't panic if your answer looks unlikely.

2 Numbers of Various Sorts

Problem 1

Prove the following formulas by induction.

1. $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
2. $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$

Proof. First notice

$$n(n+1)(2n+1) = 2n^3 + 3n^2 + n.$$

Also

$$(n+1)((n+1)+1)(2(n+1)+1) = 2n^3 + 9n^2 + 13n + 6.$$

Now, the base case is trivial. Suppose for some $n \in \mathbb{N}$ the equation holds. Then

$$1 + \cdots + n^2 + (n+1)^2 = \frac{2n^3 + 3n^2 + n}{6} + \frac{6(n^2 + 2n + 1)}{6} = \frac{2n^3 + 9n^2 + 13n + 6}{6}.$$

∎

Proof. The base case is trivial. Suppose for some $n \in \mathbb{N}$ the equation holds. Then

$$\begin{aligned}
(1 + \cdots + n + (n+1))^2 &= (1 + \cdots + n)^2 + [(n+1)(1 + \cdots + n) + (n+1)(1 + \cdots + n) + (n+1)(n+1)] \\
&= (1 + \cdots + n)^2 + (n+1)\big((1 + \cdots + n) + (1 + \cdots + n) + (n+1)\big) \\
&= (1 + \cdots + n)^2 + (n+1)(n^2 + 2n + 1) \\
&= (1 + \cdots + n)^2 + (n+1)^3 \\
&= 1^3 + \cdots + n^3 + (n+1)^3.
\end{aligned}$$

∎

Problem 2

Find a formula for

1. $\sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots+(2n-1)$
2. $\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+\cdots+(2n-1)$

Proof. Suppose n is even. We can view the sum as pairing terms as follows

$$\sum_{k=0}^{n}\big((n-k+1)+(2n-(2k+1))\big).$$

This gives the pattern

$$(1+(2n-1))+(3+(2n-3))+\cdots.$$

There are $\frac{n}{2}$ such pairs. Thus the sum is $\frac{n}{2}(2n)=n^2$. If n is odd, we simply add the middle unpaired term and find

$$\frac{n-1}{2}(2n)+(2n-2\cdot\tfrac{n-1}{2})=n^2.$$

$\blacksquare$

Proof. From Problem 1 part (i) we know

$$1^2+\cdots+(2n)^2=\frac{2n(2n+1)(2(2n)+1)}{6}=\frac{2n(2n+1)(4n+1)}{6}.$$

Now, we have overcounted the sum of the even squares between 1 and $2n$. The even squares are

$$2^2+4^2+\cdots+(2n)^2=4(1^2+2^2+\cdots+n^2)=4\cdot\frac{n(n+1)(2n+1)}{6}.$$

Thus

$$1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{2n(2n+1)(4n+1)}{6}-4\cdot\frac{n(n+1)(2n+1)}{6}=\frac{n(2n-1)(2n+1)}{3}.$$

$\blacksquare$