Chapter 3 Real Numbers

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1 Addition and Multiplication

Problem 1

Let E be an abbreviation for even, and let I be an abbreviation for odd. We know that:
$E + E = E$,
$E + I = I + E = I$,
$I + I = E$,
$EE = E$,
$II = I$
$IE = EI = E$.
(a) Show that addition for E and I is associative and commutative. Show that E plays the role of a zero element for addition. What is the additive inverse of E? What is the additive inverse of I?
(b) Show that multiplication for E and I is commutative and associative. Which of E or I behaves like 1? Which behaves like 0 for multiplication? Show that multiplication is distributive with respect to addition.

Solution 1 (a)

Associative over Addition: We check that $(A+B)+C = A+(B+C)$ for all $A, B, C \in \{E, I\}$ by verifying all 8 cases:

- $(E + E) + E = E + E = E$, and $E + (E + E) = E + E = E$

- $(E + E) + I = E + I = I$, and $E + (E + I) = E + I = I$

- $(E + I) + E = I + E = I$, and $E + (I + E) = E + I = I$

- $(E + I) + I = I + I = E$, and $E + (I + I) = E + E = E$

- $(I + E) + E = I + E = I$, and $I + (E + E) = I + E = I$

- $(I + E) + I = I + I = E$, and $I + (E + I) = I + I = E$

- $(I + I) + E = E + E = E$, and $I + (I + E) = I + I = E$

- $(I + I) + I = E + I = I$, and $I + (I + I) = I + E = I$

Commutative over Addition: We check that $A + B = B + A$ for all $A, B \in \{E, I\}$.

- $E + E = E = E + E$

- $E + I = I = I + E$

- $I + I = E = I + I$

Zero Element: E plays the role of additive identity (zero element), since:

- $E + E = E$

- $I + E = I$

- $E + I = I$

Additive Inverse of E: E, since $E + E = E$.

Additive Inverse of I: I, since $I + I = E$.

Solution 1 (b)

Associative over Multiplication: We check that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ for all $A, B, C \in \{E, I\}$:

$$
\begin{aligned}
(E \cdot E) \cdot E = E \cdot E = E, &\qquad E \cdot (E \cdot E) = E \cdot E = E \\
(E \cdot E) \cdot I = E \cdot I = E, &\qquad E \cdot (E \cdot I) = E \cdot E = E \\
(E \cdot I) \cdot E = E \cdot E = E, &\qquad E \cdot (I \cdot E) = E \cdot E = E \\
(E \cdot I) \cdot I = E \cdot I = E, &\qquad E \cdot (I \cdot I) = E \cdot I = E \\
(I \cdot E) \cdot E = E \cdot E = E, &\qquad I \cdot (E \cdot E) = I \cdot E = E \\
(I \cdot E) \cdot I = E \cdot I = E, &\qquad I \cdot (E \cdot I) = I \cdot E = E \\
(I \cdot I) \cdot E = I \cdot E = E, &\qquad I \cdot (I \cdot E) = I \cdot E = E \\
(I \cdot I) \cdot I = I \cdot I = I, &\qquad I \cdot (I \cdot I) = I \cdot I = I
\end{aligned}
$$

Commutative over Multiplication: We check that $AB = BA$ for all $A, B \in \{E, I\}$.

$$
\begin{gathered}
E \cdot I = I \cdot E = E \\
I \cdot I = I \cdot I = I \\
E \cdot E = E \cdot E = E
\end{gathered}
$$

Multiplicative Identity: I behaves like 1 over multiplication.

- $II = I$

- $EI = E$

Multiplicative Zero: E behaves like 0 over multiplication.

- $IE = E$

- $EE = E$

Distributive Over Addition: We check that $A \cdot (B + C) = A \cdot B + A \cdot C$ for all $A, B, C \in \{E, I\}$. For example:

- $E(I + E) = E(I) = E = EI + EE = E + E = E$

- $I(I+E) = I(E) = E = II + IE = E + E = E$

- $E(E+I) = E(I) = E = EE + EI = E + E = E$

- $I(E+I) = I(I) = I = IE + II = E + I = I$

- $E(E+E) = E(E) = E = EE + EE = E + E = E$

- $I(E+E) = I(E) = E = IE + IE = E + E = E$

- $E(I+I) = E(E) = E = EI + EI = E + E = E$

- $I(I+I) = I(E) = E = II + II = I + I = E$

Problem 1

Prove:
(a) If a is a real number, then a^2 is positive.
(b) If a is positive and b is negative, then ab is negative.
(c) If a is negative and b is negative, then ab is positive.

Proof. By POS 2 either $a = 0$, $a > 0$, or $a < 0$.

Case 1 $(a = 0)$

If $a = 0$ then $a^2 = a \cdot a = 0 \cdot 0 = 0 \geq 0$.

Case 2 $(a > 0)$

If $a > 0$, then by POS 1, $a \cdot a = a^2 \geq 0$.

Case 3 $(a < 0)$

Since $a < 0$, by POS 2, $-a > 0$. Then by POS 1, $(-a) \cdot (-a) = a^2 > 0$.

Therefore, $a^2 \geq 0$. □

Proof. Assume for contradiction, $ab > 0$. By POS 2, $-ab < 0$. Since $b < 0$ then, by POS 2, $-b > 0$. Then by POS 1, $a \cdot -b > 0$ so $-ab > 0$ which is a contradiction. Therefore, if a is positive and b is negative, then ab is negative. □

Proof. Assume for contradiction, $ab < 0$. By POS 2, $-ab > 0$. Since $b < 0$, $a < 0$ then, by POS 2, $-b > 0$, $-a > 0$. Then by POS 1, $-a \cdot -b > 0$ so $ab > 0$ which is a contradiction. Therefore, if a is negative and b is negative, then ab is positive. □

Problem 2

Prove: If a is positive, then a^{-1} is positive.

Proof. Suppose $a > 0$ and assume for contradiction $a^{-1} = \frac{1}{a} < 0$. By Excersize 1 part c, $a \cdot \frac{1}{a} < 0$. But $a \cdot \frac{1}{a} = \frac{a}{a} = 1 > 0$. Therefore, if a is positive, then a^{-1} is positive. □

Problem 3

Prove: If a is negative, then a^{-1} is negative.

Proof. Suppose $a < 0$ and assume for contradiction $\frac{1}{a} > 0$. Since $a < 0$, by POS 2, $0 < -a$. Then by POS 1, $-a \cdot \frac{1}{a} > 0$. But $-a \cdot \frac{1}{a} = \frac{-a}{a} = -1 < 0$ which is a contradiction. Therefore, if a is negative, then a^{-1} is negative. □

Problem 4

Prove: If a, b are positive numbers, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Proof.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \iff \sqrt{\frac{a}{b}}^2 = \left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 \iff \sqrt{\frac{a}{b}}^2 = \frac{\sqrt{a}^2}{\sqrt{b}^2} \iff \frac{a}{b} = \frac{a}{b}$$

□

Problem 5

Prove that

$$\frac{1}{1-\sqrt{2}} = -(1+\sqrt{2})$$

Proof.

$$\begin{aligned}\frac{1}{1-\sqrt{2}} &= \frac{1+\sqrt{2}}{1+\sqrt{2}} \cdot \frac{1}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{1-2} = \frac{1+\sqrt{2}}{-1} \\ &= \frac{-1}{-1} \cdot \frac{1+\sqrt{2}}{-1} = \frac{-(1+\sqrt{2})}{1} = -(1+\sqrt{2})\end{aligned}$$

□

Problem 8

Let a, b be rational numbers. Prove that the multiplicative inverse of $a + b\sqrt{2}$ can be expressed in the form $c + d\sqrt{2}$, where c, d are rational numbers.

Proof. First note since $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ therefore $a^2 - 2b^2 \in \mathbb{Q}$. In addition $a + b\sqrt{2} \neq 0$ (otherwise the inverse operation is undefined). If $b = 0$ then $a^2 \neq 0$ so $a^2 - 2b^2 \in \mathbb{Q}$ is defined. Now suppose $b \neq 0$.

$$a^2 = 2b^2 \iff \frac{a^2}{b^2} = 2 \iff \frac{a}{b} = \pm\sqrt{2}$$

But $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$ so their quotient is rational. This is impossible since $\sqrt{2}$ is irrational, so $a^2 - 2b^2 \neq 0$. Futhermore since $a^2 - 2b^2 \in \mathbb{Q}$ and $a^2 - 2b^2 \neq 0$, $\frac{a}{a^2-2b^2} \in \mathbb{Q}$ and $\frac{-b}{a^2-2b^2} \in \mathbb{Q}$.

Now, let $c = \frac{a}{a^2-2b^2}$ and $d = \frac{-b}{a^2-2b^2}$. Then

$$\begin{aligned}
&(a+b\sqrt{2})\cdot(c+d\sqrt{2})\\
=&(a+b\sqrt{2})\cdot\left(\frac{a}{a^2-2b^2}+\frac{-b}{a^2-2b^2}\cdot\sqrt{2}\right)\\
=&(a+b\sqrt{2})\cdot\left(\frac{a}{a^2-2b^2}+\frac{-b\sqrt{2}}{a^2-2b^2}\right)\\
=&(a+b\sqrt{2})\cdot\left(\frac{a}{a^2-2b^2}-\frac{b\sqrt{2}}{a^2-2b^2}\right)\\
=&\left(\frac{a(a+b\sqrt{2})}{a^2-2b^2}-\frac{b\sqrt{2}(a+b\sqrt{2})}{a^2-2b^2}\right)\\
=&\frac{(a^2+ab\sqrt{2})-(ab\sqrt{2}+2b^2)}{a^2-2b^2}\\
=&\frac{a^2+ab\sqrt{2}-ab\sqrt{2}-2b^2}{a^2-2b^2}\\
=&\frac{a^2-2b^2}{a^2-2b^2}\\
=&1
\end{aligned}$$

$\square$

Problem 11

Generalize Excersize 10, replacing $\sqrt{5}$ by $\sqrt{a}$ for any positive integer a.

Proof. First note since $d \in \mathbb{Q}$ and $b \in \mathbb{Q}$ therefore $d^2 - ab^2 \in \mathbb{Q}$. In addition $d + b\sqrt{a} \neq 0$ (otherwise the inverse operation is undefined).

If $b = 0$ then $d^2 \neq 0$ so $d^2 - ab^2 \in \mathbb{Q}$ is defined.

Now suppose $b \neq 0$ and $\sqrt{a} \notin \mathbb{Q}$.

$$d^2 = ab^2 \iff \frac{d^2}{b^2} = a \iff \frac{d}{b} = \pm\sqrt{a}$$

But $d \in \mathbb{Q}$ and $b \in \mathbb{Q}$ so their quotient is rational. This is impossible if $\sqrt{a} \notin \mathbb{Q}$, so $d^2 - ab^2 \neq 0$.

Now suppose $b \neq 0$ and $\sqrt{a} \in \mathbb{Q}$.

$$d = b\sqrt{a} \iff d^2 = b^2 a \iff d^2 - ab^2 = 0$$

Since, $d \neq b\sqrt{a}$, $d^2 - ab^2 \neq 0$.

Furthermore since $d^2 - ab^2 \in \mathbb{Q}$ and $d^2 - ab^2 \neq 0$, $\frac{d}{d^2-ab^2} \in \mathbb{Q}$ and $\frac{-b}{d^2-ab^2} \in \mathbb{Q}$. Now let $c = \frac{d}{d^2-ab^2}$ and $e = \frac{-b}{d^2-ab^2}$. Then

$$\begin{aligned}
&(d + b\sqrt{a}) \cdot (c + e\sqrt{a}) \\
=&(d + b\sqrt{a}) \cdot \left(\frac{d}{d^2 - ab^2} + \frac{-b}{d^2 - ab^2} \cdot \sqrt{a}\right) \\
=&(d + b\sqrt{a}) \cdot \left(\frac{d}{d^2 - ab^2} + \frac{-b\sqrt{a}}{d^2 - ab^2}\right) \\
=&(d + b\sqrt{a}) \cdot \left(\frac{d}{d^2 - ab^2} - \frac{b\sqrt{a}}{d^2 - ab^2}\right) \\
=&\left(\frac{d(d + b\sqrt{a})}{d^2 - ab^2} - \frac{b\sqrt{a}(d + b\sqrt{a})}{d^2 - ab^2}\right) \\
=&\frac{(d^2 + db\sqrt{a}) - (db\sqrt{a} + ab^2)}{d^2 - ab^2} \\
=&\frac{d^2 + db\sqrt{a} - db\sqrt{a} - ab^2}{d^2 - ab^2} \\
=&\frac{d^2 - ab^2}{d^2 - ab^2} \\
=&1
\end{aligned}$$

$\square$

Problem 14

Find all possible numbers x such that
(a) $|2x-1|=3$
(b) $|3x+1|=2$
(c) $|2x+1|=4$
(d) $|3x-1|=1$
(e) $|4x-5|=6$

Solution 14 (a)

$x=2$ or $x=-1$

Solution 14 (b)

$x=\frac{1}{3}$ or $x=-1$

Solution 14 (c)

$x=\frac{3}{2}$ or $x=\frac{-5}{2}$

Solution 14 (d)

$x=\frac{2}{3}$ or $x=0$

Solution 14 (e)

$x=\frac{11}{4}$ or $x=\frac{-1}{4}$

Problem 15

Rationalize the numerator in the following expressions.
(a) $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$
(b) $\frac{\sqrt{x+y}-\sqrt{y}}{\sqrt{x+y}+\sqrt{y}}$
(c) $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$
(d) $\frac{\sqrt{x-3}+\sqrt{x}}{\sqrt{x-3}-\sqrt{x}}$
(e) $\frac{\sqrt{x+y}-1}{3+\sqrt{x+y}}$
(f) $\frac{\sqrt{x+y}+x}{\sqrt{x+y}}$

Solution 15 (a)

$$\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}\cdot\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}}=\frac{x-y}{x-2\sqrt{xy}+y}$$

Solution 15 (b)

$$\frac{\sqrt{x+y}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\cdot\frac{\sqrt{x+y}+\sqrt{y}}{\sqrt{x+y}+\sqrt{y}}=\frac{x}{\sqrt{x(x+y)}+\sqrt{xy}+\sqrt{y(x+y)}+y}$$

Solution 15 (c)

$$\begin{aligned}\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\cdot\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} &= \frac{2}{(\sqrt{x+1}-\sqrt{x-1})(\sqrt{x+1}-\sqrt{x-1})}\\ &= \frac{2}{\left(\sqrt{x+1}-\sqrt{x-1}\right)^2}\end{aligned}$$

Solution 15 (d)

$$\begin{aligned}\frac{\sqrt{x-3}+\sqrt{x}}{\sqrt{x-3}-\sqrt{x}}\cdot\frac{\sqrt{x-3}-\sqrt{x}}{\sqrt{x-3}-\sqrt{x}} &= \frac{(x-3)+x}{\left(\sqrt{x-3}-\sqrt{x}\right)^2}\\ &= \frac{-3}{\left(\sqrt{x-3}-\sqrt{x}\right)^2}\end{aligned}$$

Solution 15 (e)

$$\frac{\sqrt{x+y}-1}{3+\sqrt{x+y}}\cdot\frac{\sqrt{x+y}+1}{\sqrt{x+y}+1} = \frac{x+y-1}{(3+\sqrt{x+y})(\sqrt{x+y}+1)}$$

Solution 15 (f)

$$\frac{\sqrt{x+y}+x}{\sqrt{x+y}}\cdot\frac{\sqrt{x+y}-x}{\sqrt{x+y}-x} = \frac{x+y-x^2}{\sqrt{x+y}(\sqrt{x+y}-x)}$$

Problem 17

Prove that there is no real number x such that

$$\sqrt{x-1} = 3+\sqrt{x}$$

[*Hint*: Start by squaring both sides.]

Proof. Assume for contradiction there does exist a real number x such that $\sqrt{x-1} = 3+\sqrt{x}$. Then

$$\begin{aligned}&\sqrt{x-1} = 3+\sqrt{x}\\ &\leftrightarrow x-1 = 9+6\sqrt{x}+x\\ &\leftrightarrow -1 = 9+6\sqrt{x}\\ &\leftrightarrow -10 = 6\sqrt{x}\\ &\leftrightarrow \frac{-10}{6} = \sqrt{x}\end{aligned}$$

Which is a contradiction. Therefore, there is no real number x such that $\sqrt{x-1} = 3+\sqrt{x}$. □

Problem 20

If a, b are two numbers, prove that $|a-b| = |b-a|$.

Proof. Let $c = b - a$. By POS 2 there are three cases.

Case 1 ($c = 0$) If $b - a = 0$ then $b = a$ therefore $a - b = 0$.

$$\begin{aligned}
|b-a| &= |a-b| \\
\leftrightarrow |0| &= |0| \\
\leftrightarrow 0 &= 0
\end{aligned}$$

Case 2 ($c > 0$) If $c > 0$ then $|c| = c$. Also $-c < 0$ so $|-c| = -(-c) = c$. Then

$$\begin{aligned}
|b-a| &= |a-b| \\
\leftrightarrow |c| &= |-c| \\
\leftrightarrow c &= c
\end{aligned}$$

Case 3 ($c < 0$) If $c < 0$ then $|c| = -c$. Also $-c > 0$ so $|-c| = -c$. Then

$$\begin{aligned}
|b-a| &= |a-b| \\
\leftrightarrow |c| &= |-c| \\
\leftrightarrow -c &= -c
\end{aligned}$$

Therefore $|a-b| = |b-a|$. □

2 Powers and Roots

Extra Problem

Suppose a is a nonzero rational number and b is an irrational real number. Show that ab is irrational.

Proof. A number is rational if it can be written as $\frac{x}{y}$ with $x, y \in \mathbb{Z}$ and $y \neq 0$. Assume for contradiction that $a \cdot b$ is rational, where $a \neq 0$ is rational and b is irrational. Since $a \neq 0$, we can divide both sides by a:

$$b = \frac{a \cdot b}{a}.$$

But the right-hand side is rational (a rational divided by a nonzero rational is rational), so b would be rational. This contradicts the assumption that b is irrational. Therefore, $a \cdot b$ must be irrational. □

Problem 1

Express each of the following in the form $2^k 2^m a^r b^s$ where k, m, r,s are integers.
(a) $\frac{1}{8}a^3b^{-4}2^5a^{-2}$
(b) $3^{-4}2^5a^3b^6 \cdot \frac{1}{2^3} \cdot \frac{1}{a^4} \cdot b^{-1} \cdot \frac{1}{9}$
(c) $\frac{3a^3b^4}{2a^5b^6}$
(d) $\frac{16a^{-3}b^{-5}}{9b^4a^72^{-3}}$

Solution (a):

$$\frac{1}{8}a^3b^{-4}2^5a^{-2} = \frac{2^5}{8}a^3a^{-2}b^{-4} = \frac{2^5}{8}a^1b^{-4} = \frac{2^5}{2^3}a^1b^{-4} = 2^23^0a^1b^{-4}$$

Solution (b):

$$3^{-4}2^5a^3b^6 \cdot \frac{1}{2^3} \cdot \frac{1}{a^4} \cdot b^{-1} \cdot \frac{1}{9} = \frac{2^5}{2^3}\frac{3^{-4}}{9}\frac{a^3}{a^4}\frac{b^6}{b} = 2^2\frac{3^{-4}}{3^2}\frac{a^3}{a^4}\frac{b^6}{b} = 2^23^{-6}a^{-1}b^5$$

Solution (c):

$$\frac{3a^3b^4}{2a^5b^6} = 2^{-1}3^1a^{-2}b^{-2}$$

Solution (d):

$$\frac{16a^{-3}b^{-5}}{9b^4a^72^{-3}} = \frac{2^4a^{-10}b^{-5}}{3^22^{-3}} = 2^73^{-2}a^{-10}b^{-9}$$

Problem 2

What integer is $81^{\frac{1}{4}}$ equal to?

Solution:

$$81^{\frac{1}{4}} = (81^{\frac{1}{2}})^{\frac{1}{2}} = 9^{\frac{1}{2}} = 3$$

Problem 3

What integer is $(\sqrt{2})^6$ equal to?

Solution:

$$(\sqrt{2})^6 = (\sqrt{2})^2(\sqrt{2})^2(\sqrt{2})^2 = 2 \cdot 2 \cdot 2 = 8$$

Problem 4

Is $(\sqrt{2})^5$ an integer?

Solution:

$$(\sqrt{2})^5 = (\sqrt{2})^2(\sqrt{2})^2(\sqrt{2}) = 2 \cdot 2 \cdot \sqrt{2} = 4\sqrt{2}$$

It is not an integer see extra problem proof.

Problem 5

Is $(\sqrt{2})^{-5}$ a rational number? Is $(\sqrt{2})^{5}$ a rational number?

Solution part 1:

$$(\sqrt{2})^{-5} = \frac{1}{(\sqrt{2})^5} = \frac{1}{4\sqrt{2}} = \frac{1}{4\sqrt{2}}\frac{4\sqrt{2}}{4\sqrt{2}} = \frac{4\sqrt{2}}{16\cdot 2} = \frac{4\sqrt{2}}{32} = \frac{4}{32}\sqrt{2}$$

By the extra problem this is not a rational number.

Solution part 2: Same reason as problem 4.

Problem 6

In each case, the expression is equal to an integer. Which one?
(a) $16^{\frac{1}{4}}$
(b) $8^{\frac{1}{3}}$
(c) $9^{\frac{3}{2}}$
(d) $1^{\frac{5}{4}}$
(e) $8^{\frac{4}{3}}$
(f) $64^{\frac{2}{4}}$
(g) $25^{\frac{3}{2}}$

Solution:

$$\begin{aligned}
&\text{(a)} \quad 16^{\frac{1}{4}} = (16^{\frac{1}{2}})^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2\\
&\text{(b)} \quad 8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2\\
&\text{(c)} \quad 9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27\\
&\text{(d)} \quad 1^{\frac{5}{4}} = 1\\
&\text{(e)} \quad 8^{\frac{4}{3}} = (8^{\frac{1}{3}})^4 = 2^4 = 16\\
&\text{(f)} \quad 64^{\frac{2}{4}} = 64^{\frac{1}{2}} = 8\\
&\text{(g)} \quad 25^{\frac{3}{2}} = (25^{\frac{1}{2}})^3 = 5^3 = 125
\end{aligned}$$

Problem 7

Express each of the following expressions as a simple decimal.
(a) $(0.09)^{\frac{1}{2}}$
(b) $(0.027)^{\frac{1}{3}}$
(c) $(0.125)^{\frac{2}{3}}$
(d) $(1.21)^{\frac{1}{2}}$

Solution:

$$\begin{aligned} &\text{(a)} \quad (0.9)^{\frac{1}{2}} \approx 0.3 \\ &\text{(b)} \quad (0.027)^{\frac{1}{3}} = 0.3 \\ &\text{(c)} \quad (0.125)^{\frac{2}{3}} = ((0.125)^{\frac{1}{3}})^2 = 0.5^2 = 0.25 \\ &\text{(d)} \quad (1.21)^{\frac{1}{2}} = 1.1 \end{aligned}$$

Problem 8

Express each of the following expressions as a quotient $\frac{m}{n}$, where m, n are integers > 0.
(a) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$
(b) $\left(\frac{4}{9}\right)^{\frac{1}{2}}$
(c) $\left(\frac{25}{16}\right)^{\frac{3}{2}}$
(d) $\left(\frac{49}{4}\right)^{\frac{3}{2}}$

Solution:

$$\begin{aligned} &\text{(a)} \quad \left(\frac{8}{27}\right)^{\frac{2}{3}} = \frac{8^{2/3}}{27^{2/3}} = \frac{4}{9} \\ &\text{(b)} \quad \left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{2}{3} \\ &\text{(c)} \quad \left(\frac{25}{16}\right)^{\frac{3}{2}} = \frac{(25^{1/2})^3}{(16^{1/2})^3} = \frac{125}{64} \\ &\text{(d)} \quad \left(\frac{49}{4}\right)^{\frac{3}{2}} = \frac{(49^{1/2})^3}{(4^{1/2})^3} = \frac{343}{8} \end{aligned}$$

Problem 9

Solve each of the following equations for x.
(a) $(x-2)^3 = 5$
(b) $(x+3)^2 = 4$
(c) $(x-5)^{-2} = 9$
(d) $(x+3)^3 = 27$
(e) $(2x-1)^{-3} = 27$
(f) $(3x+5)^{-4} = 64$

Solution:

$$\begin{array}{ll}
\text{(a)} & x-2=\sqrt[3]{5} \iff x=2+\sqrt[3]{5} \\
\text{(b)} & x+3=\pm 2 \iff x=-1 \text{ or } x=-5 \\
\text{(c)} & \dfrac{1}{(x-5)^2}=9 \iff (x-5)^2=\dfrac{1}{9} \iff x=5\pm\dfrac{1}{3} \\
\text{(d)} & x+3=3 \iff x=0 \\
\text{(e)} & \dfrac{1}{(2x-1)^3}=27 \iff (2x-1)^3=\dfrac{1}{27} \iff 2x-1=\dfrac{1}{3} \iff x=\dfrac{2}{3} \\
\text{(f)} & \dfrac{1}{(3x+5)^4}=64 \iff (3x+5)^4=\dfrac{1}{64} \iff 3x+5=\dfrac{1}{2} \iff x=-\dfrac{3}{2}
\end{array}$$

3 Inequalities

Problem 1

Prove **IN 3**.

IN 3 If $a > b$ and $b > c$ then $a > c$.

Proof. Suppose $a > b$ and $b > c$. Since $a > b$, $a - b > 0$. Also, since $b > c$, $b - c > 0$. So $(a-b)+(b-c) > 0 \iff a - c > 0$. Therefore $a > c$. □

Problem 2

Prove: If $0 < a < b$, if $c < d$, and $c > 0$ then

$$ac < bd$$

Proof. Suppose $0 < a < b$, $c < d$, and $c > 0$. Since $a < b$ and $c > 0$ it follows that $ac < bc$ (**IN 2**). Since $c < d$ and $b > 0$ it follows that $bc < bd$ (**IN 2**). Since $ac < bc < bd$ it follows that $ac < bd$ (Problem 1). □

Problem 3

Prove: If $a < b < 0$, if $c < d < 0$ then

$$ac > bd$$

Proof. Suppose $a < b < 0$ and $c < d < 0$. Since $a < b$ it follows that $b - a > 0$. Since $b - a > 0$ and $c < 0$ it follows that $bc - ac < 0$ so $bc < ac$ (**IN 3**). Since $c < d$ it follows that $d - c > 0$. Since $d - c > 0$ and $b < 0$ it follows that $bd - bc < 0$ so $bd < bc$ (**IN 3**). So $bd < bc < ac$ and therefore $bd < ac$ (Problem 1). $\square$

Problem 4

(a) If $x < y$ and $x > 0$, prove that $\frac{1}{y} < \frac{1}{x}$.
(b) Prove a rule of cross-multiplication of inequalities: If a, b, c, d are numbers and $b > 0$, $d > 0$, and if

$$\frac{a}{b} < \frac{c}{d}$$

prove that

$$ad < bc$$

Also prove the converse, that if $ad < bc$, then $\frac{a}{b} < \frac{c}{d}$.

Proof. Obviously we can assume $x \neq 0$ and $y \neq 0$. Suppose $x < y$ and $x > 0$. Since $x < y$ it follows that $y - x > 0$. Since $y > x > 0$ it follows that $\frac{1}{xy} > 0$. Then $\frac{1}{xy}(y-x) > 0 \iff \frac{1}{x} - \frac{1}{y} > 0$ therefore $\frac{1}{x} > \frac{1}{y}$. $\square$

Proof. Suppose a, b, c, and d are numbers such that $b > 0$ and $d > 0$. Suppose $\frac{a}{b} < \frac{c}{d}$. It follows that $\frac{c}{d} - \frac{a}{b} > 0$. Since $b > 0$ and $d > 0$ it follows that $bd > 0$. Then $bd(\frac{c}{d} - \frac{a}{b}) > 0 \iff cb - ad > 0 \iff ad < bc$. $\square$

Proof. Suppose a, b, c, and d are numbers such that $b > 0$ and $d > 0$. Suppose $\frac{a}{b} > \frac{c}{d}$. So $\frac{a}{b} > \frac{c}{d} \iff \frac{c}{d} < \frac{a}{b}$. Since $\frac{c}{d} < \frac{a}{b}$ then $bc < ad$ (Previous Proof). $\square$

Problem 5

Prove: If $a < b$ and c is any real number, then

$$a + c < b + c$$

Also,

$$a - c < b - c$$

Thus a number may be subtracted from each side of an inequality without changing the validity of the inequality.

Proof. Suppose $a < b$ and c is a real number. Since $a < b$ it follows that $b - a > 0$. Then $b-a > 0 \iff b-a+c-c > 0 \iff b+c-a-c > 0 \iff b+c-(a+c) > 0 \iff b+c > a+c$. $\square$

Proof. Suppose $a < b$ and t is a real number. Apply previous proof with $-t$ in place of c. Therefore $a + (-t) < b + (-t) \iff a - t < b - t$ $\square$

Problem 6

Prove: If $a < b$ and $a > 0$ that

$$a^2 < b^2$$

More generally, prove successively that

$$a^3 < b^3$$

$$a^4 < b^4$$

$$a^5 < b^5$$

Proceeding stepwise, we conclude that

$$a^n < b^n$$

for every positive integer n. To make this stepwise argument formal, one must state explicitly a property of integers which is called induction, and is discussed later in the book.

Proof. Suppose $a < b$ and $a > 0$. It follows that $b - a > 0$. Since $b > 0$ and $b - a > 0$ it follows that $b^2 - ab > 0$ so $b^2 > ab$. Also, since $a > 0$ and $b - a > 0$ it follows that $ab - a^2 > 0$ so $ab > a^2$. Since $a^2 < ab < b^2$ it follows that $a^2 < b^2$ $\square$

Proof. Suppose $a < b$ and $a > 0$. It follows that $a^2 < b^2$ (Previous Proof). Therefore $b^2 - a^2 > 0$. Since $b > 0$ and $b^2 - a^2 > 0$ it follows that $b^3 - a^2 b > 0$. Also, since $a > 0$ and $b^2 - a^2 > 0$ it follows that $ab^2 - a^3 > 0$. Since $b^3 - a^2 b > 0$ and $ab^2 - a^3 > 0$ it follows that $(b^3 - a^2 b) + (ab^2 - a^3) > 0 \iff (b^3 - a^3) + (ab^2 - a^2 b) > 0 \iff (b^3 - a^3) + (ab(b - a)) > 0$. Since $a, b > 0$ it follows that $ab > 0$. Since $ab > 0$ and $(b - a) > 0$ it follows that $ab(b - a) > 0$. Therefore $(b^3 - a^3) + (ab(b - a)) \geq b^3 - a^3 > 0$. It then follows that $b^3 > a^3$. $\square$

Proof. Suppose $a < b$ and $a > 0$. It follows that $a^2 < b^2$ (Previous Proof). Therefore $b^2 - a^2 > 0$. Since $b > 0$ and $b^2 - a^2 > 0$ it follows that $b^3 - a^2 b > 0$. Also, since $a > 0$ and $b^2 - a^2 > 0$ it follows that $ab^2 - a^3 > 0$. Since $b^3 - a^2 b > 0$ and $ab^2 - a^3 > 0$ it follows that $(b^3 - a^2 b) + (ab^2 - a^3) > 0 \iff (b^3 - a^3) + (ab^2 - a^2 b) > 0 \iff (b^3 - a^3) + (ab(b-a)) > 0$. Since $a, b > 0$ it follows that $ab > 0$. Since $ab > 0$ and $(b - a) > 0$ it follows that $ab(b - a) > 0$. Therefore $(b^3 - a^3) + (ab(b - a)) \geq b^3 - a^3 > 0$. It then follows that $b^3 > a^3$. □

Proof. Suppose $a < b$ and $a > 0$. It follows that $a^3 < b^3$ (Previous Proof). Therefore $b^3 - a^3 > 0$. Since $b > 0$ and $b^3 - a^3 > 0$ it follows that $b^4 - a^3 b > 0$. Also, since $a > 0$ and $b^3 - a^3 > 0$ it follows that $ab^3 - a^4 > 0$. Since $b^4 - a^2 b^2 > 0$ and $ab^3 - a^4 > 0$ it follows that $(b^4 - a^2 b^2) + (ab^3 - a^4) > 0 \iff (b^4 - a^4) + (ab^3 - a^2 b^2) > 0 \iff (b^4 - a^4) + (ab^2(b - a)) > 0$. Since $a, b > 0$ it follows that $ab^2 > 0$. Since $ab^2 > 0$ and $(b - a) > 0$ it follows that $ab^2(b - a) > 0$. Therefore $(b^4 - a^4) + (ab^2(b - a)) \geq b^4 - a^4 > 0$. It then follows that $b^4 > a^4$. □

Proof. Suppose $a < b$ and $a > 0$. It follows that $a^4 < b^4$ (Previous Proof). Therefore $b^4 - a^4 > 0$ (Previous Proof). Since $b > 0$ and $b^4 - a^4 > 0$ it follows that $b^5 - a^4 b > 0$. Also, since $a > 0$ and $b^4 - a^4 > 0$ it follows that $ab^4 - a^5 > 0$. Since $b^5 - a^3 b^2 > 0$ and $ab^4 - a^5 > 0$ it follows that $(b^5 - a^3 b^2) + (ab^4 - a^5) > 0 \iff (b^5 - a^5) + (ab^4 - a^3 b^2) > 0 \iff (b^5 - a^5) + (ab^2(b^2 - a^2)) > 0$. Since $a, b > 0$ it follows that $ab^2 > 0$. Since $ab^2 > 0$ and $(b^2 - a^2) > 0$ it follows that $ab^2(b^2 - a^2) > 0$. Therefore $(b^5 - a^5) + (ab^2(b^2 - a^2)) \geq b^5 - a^5 > 0$. It then follows that $b^5 > a^5$. □

Problem 7

Prove: If $0 < a < b$, then $a^{\frac{1}{n}} < b^{\frac{1}{n}}$. [Hint: Use Excersize 6.]

Problem 8

Let a, b, c, d be numbers and assume $b > 0$ and $d > 0$. Assume that

$$\frac{a}{b} < \frac{c}{d}$$

(a) Prove that

$$\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$$

(There are two inequalities to be proved here, the one on the left and the one on the right.)
(b) Let r be a number > 0. Prove that

$$\frac{a}{b} < \frac{a + rc}{b + rd} < \frac{c}{d}$$

(c) If $0 < r < s$, prove that

$$\frac{a + rc}{b + rd} = \frac{a + sc}{b + sd}$$