A Radical Approach to Real Analysis by David M. Bressoud

Frosty

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Contents
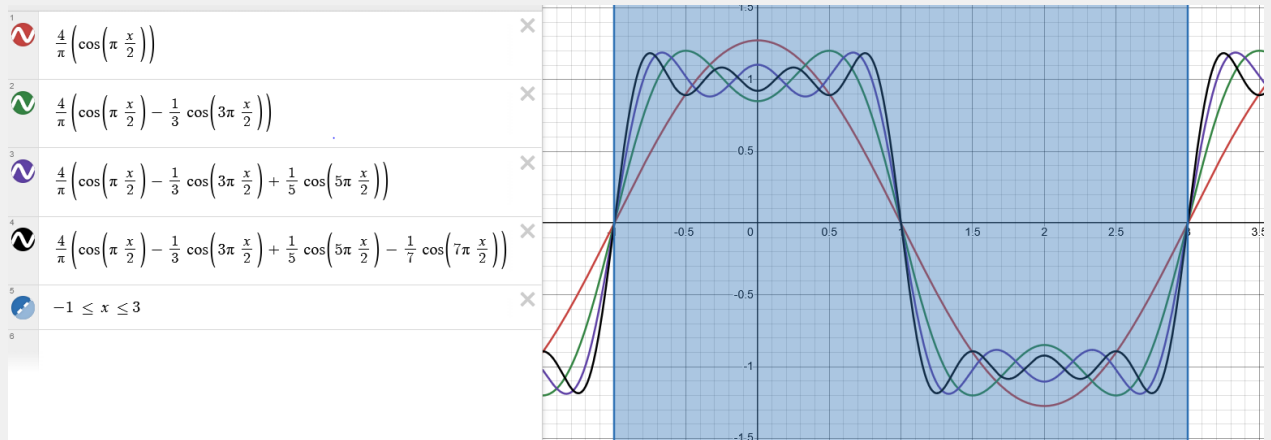
1 Crisis in Mathematics: Fourier's Series

Problem 1

Graph each of the following partial sums of Fourier's expansion over the inverval $-1 \le x \le 3$.
1. $\frac{4}{\pi} \cos(\pi x/2)$
2. $\frac{4}{\pi}(\cos(\pi x/2) - \frac{1}{3}\cos(3\pi x/2))$
3. $\frac{4}{\pi}(\cos(\pi x/2) - \frac{1}{3}\cos(3\pi x/2) + \frac{1}{5}\cos(5\pi x/2))$
4. $\frac{4}{\pi}(\cos(\pi x/2) - \frac{1}{3}\cos(3\pi x/2) + \frac{1}{5}\cos(5\pi x/2)) - \frac{1}{7}\cos(7\pi x/2))$

Problem 2

Let $F_n(x)$ denote the sum of the first n terms of the Fourier's series evaluated at x:

$$F_n(x) = \frac{4}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \cdots + \frac{(-1)^{n-1}}{2n-1}\cos\frac{(2n-1)\pi x}{2} \right)$$
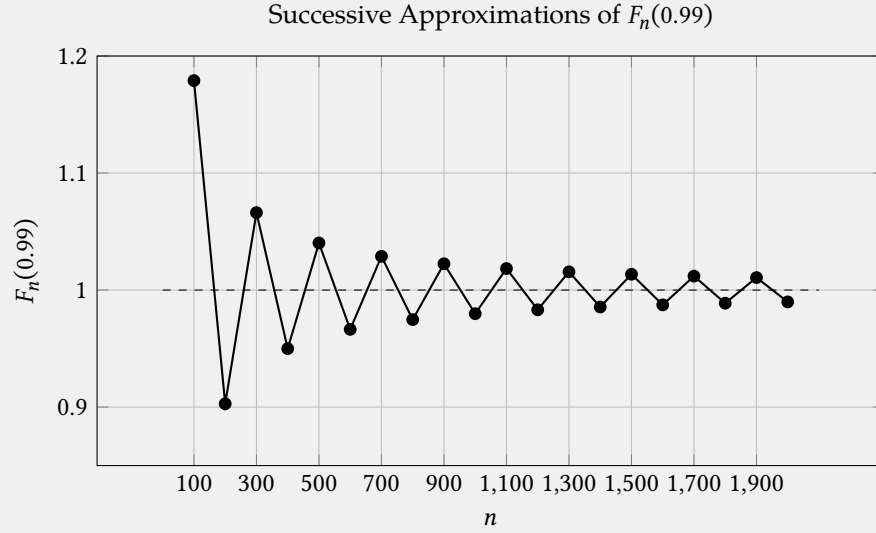
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1. Evaluate $F_{100}(x)$ at $x = 0, 0.5, 0.9, 0.99, 1.1$, and 2. Is this close to the expected value?
2. Evaluate $F_n(0.99)$ at $n = 100, 200, 300, \ldots, 2000$ and plot these successive approximations.
3. Evaluate $F_n(0.999)$ at $n = 100, 200, 300, \ldots, 2000$ and plot these successive approximations.
4. What is the value of this infinite series at $x = 1$?

Solution (a): I had no expectations.

1. $F_{100}(0) = 0.9968169807056898$.

2. $F_{100}(0.5) = 0.9954987558776579$.

3. $F_{100}(0.9) = 0.9796927699334861$.

4. $F_{100}(0.99) = 1.1789880778995547$

5. $F_{100}(1.1) = -0.9796927699334861$.

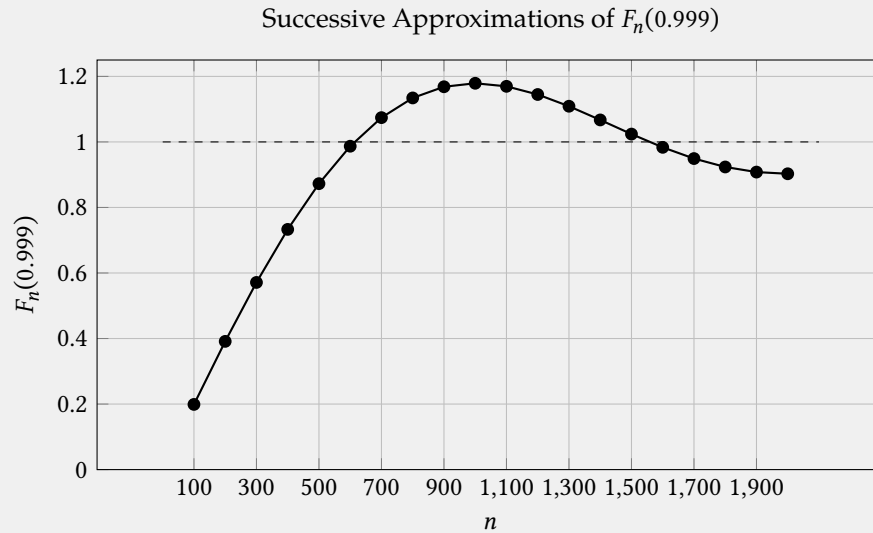6. $F_{100}(2) = -0.9968169807056898$.

Solution (b):

1. $F_{100}(0.99) = 1.1789880778995547$

2. $F_{200}(0.99) = 0.9028191668118976$

3. $F_{300}(0.99) = 1.0661892530888835$

4. $F_{400}(0.99) = 0.9499372563762823$

5. $F_{500}(0.99) = 1.0402159950960959$

6. $F_{600}(0.99) = 0.9664090460672992$

7. $F_{700}(0.99) = 1.0288331847566468$

8. $F_{800}(0.99) = 0.9747474084192721$

9. $F_{900}(0.99) = 1.0224612312263757$

10. $F_{1000}(0.99) = 0.9797755089505807$

11. $F_{1100}(0.99) = 1.0183922416138902$

12. $F_{1200}(0.99) = 0.9831360088728518$

13. $F_{1300}(0.99) = 1.0155699566691612$

14. $F_{1400}(0.99) = 0.9855398234944296$

15. $F_{1500}(0.99) = 1.0134979445155288$

16. $F_{1600}(0.99) = 0.9873443095081518$

17. $F_{1700}(0.99) = 1.0119123060445154$

18. $F_{1800}(0.99) = 0.9887486426910619$

19. $F_{1900}(0.99) = 1.010659859489416$

20. $F_{2000}(0.99) = 0.98987258246275$

Successive Approximations of $F_n(0.99)$

Solution (c):

1. $F_{100}(0.999) = 0.19890664596017577$

2. $F_{200}(0.999) = 0.39133027939704734$

3. $F_{300}(0.999) = 0.5711684699578463$

4. $F_{400}(0.999) = 0.733053348578498$

5. $F_{500}(0.999) = 0.8726544055639414$

6. $F_{600}(0.999) = 0.9869094176114769$

7. $F_{700}(0.999) = 1.074168253719476$

8. $F_{800}(0.999) = 1.134240867961708$

9. $F_{900}(0.999) = 1.1683479421219805$

10. $F_{1000}(0.999) = 1.1789798278055126$

11. $F_{1100}(0.999) = 1.1696760925989875$

12. $F_{1200}(0.999) = 1.1447435828791337$

13. $F_{1300}(0.999) = 1.10893505133833$

14. $F_{1400}(0.999) = 1.0671127580619808$

15. $F_{1500}(0.999) = 1.0239218847151361$

16. $F_{1600}(0.999) = 0.9834971044374297$

17. $F_{1700}(0.999) = 0.949222383157731$

18. $F_{1800}(0.999) = 0.9235593491634196$

19. $F_{1900}(0.999) = 0.9079537683550908$

20. $F_{2000}(0.999) = 0.9028232919136085$

Successive Approximations of $F_n(0.999)$

Solution (d): The value of the infinite series at $x = 1$ is 0.

Problem 6

Fourier series illustrate the dangers of trying to find limits by simply substituting the value that x approaches. Consider the Fourier's series:

$$f(x) = \frac{4}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \frac{1}{7}\cos\frac{7\pi x}{2} + \dots\right)$$

1. What value does this approach as x approaches 1 from the left?
2. What value does this approach as x approaches 1 from the right?
3. What is the value at $f(1)$?

These three answers are all different.

Solution (a):

$$\lim_{x\to 1^-} f(x) = 1.$$

Solution (b):

$$\lim_{x\to 1^+} f(x) = -1.$$

Solution (c):

$$f(1) = \frac{1 + (-1)}{2} = 0.$$

Problem 7

Consider the function that we get if we differentiate each summand of this function $f(x)$ defined in equation $g(x) = -2\left(\sin\frac{\pi x}{2} - \sin\frac{2\pi x}{2} + \sin\frac{5\pi x}{2} - \sin\frac{7\pi x}{2} + \dots\right)$.

1. For $-1 < x < 3$, graph the partial sums of this series consisting of the first $10, 20, 30, 40$, and 50 terms. Does it appear that these graphs are approaching a constant function 0.
2. Evaluate the partical sums up to at least 20 terms when $x = 0, 0.2, 0.3$, and 0.5. Does it appear that this series is approaching 0 at each of these values of $x/$
3. What *is* happening at $x = 0, 0.2, 0.3, 0.5$? What can you prove?
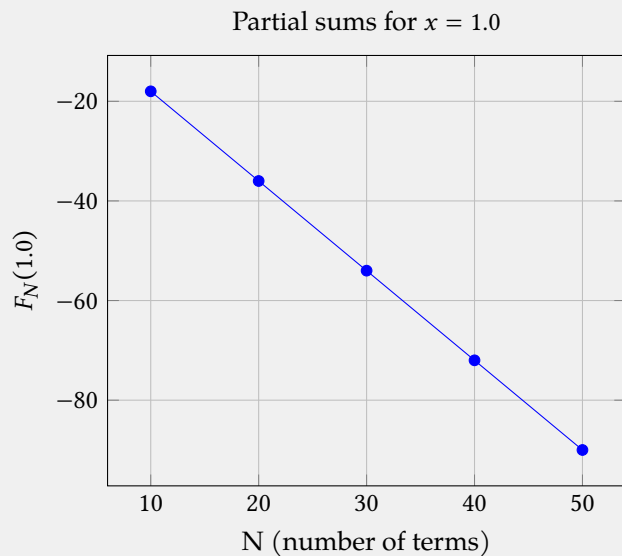

Solution (a):

1. $F_{10}(1.0) = -18.0$

4

2. $F_{20}(1.0) = -36.0$

3. $F_{30}(1.0) = -54.0$

4. $F_{40}(1.0) = -72.0$

5. $F_{50}(1.0) = -90.0$

Partial sums for $x = 1.0$

Solution (b):

1. $F_1(0) = -0.0$

2. $F_2(0) = -0.0$

3. $F_3(0) = -0.0$

4. $F_4(0) = -0.0$

5. $F_5(0) = -0.0$

6. $F_6(0) = -0.0$

7. $F_7(0) = -0.0$

8. $F_8(0) = -0.0$

9. $F_9(0) = -0.0$

10. $F_{10}(0) = -0.0$

11. $F_{11}(0) = -0.0$

12. $F_{12}(0) = -0.0$

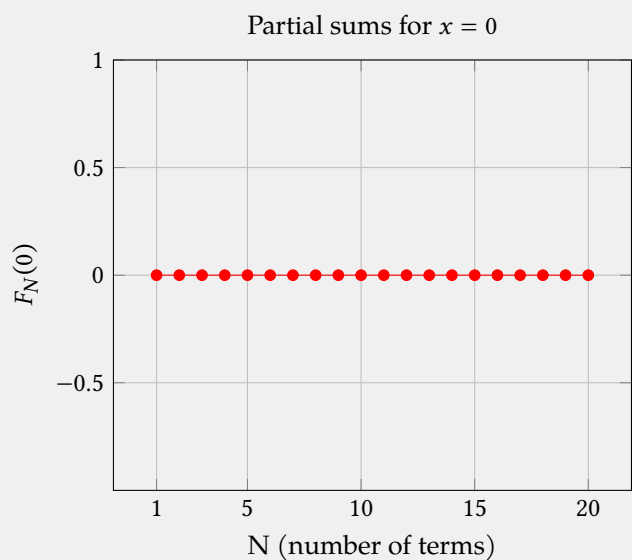13. $F_{13}(0) = -0.0$

14. $F_{14}(0) = -0.0$

15. $F_{15}(0) = -0.0$

16. $F_{16}(0) = -0.0$

17. $F_{17}(0) = -0.0$

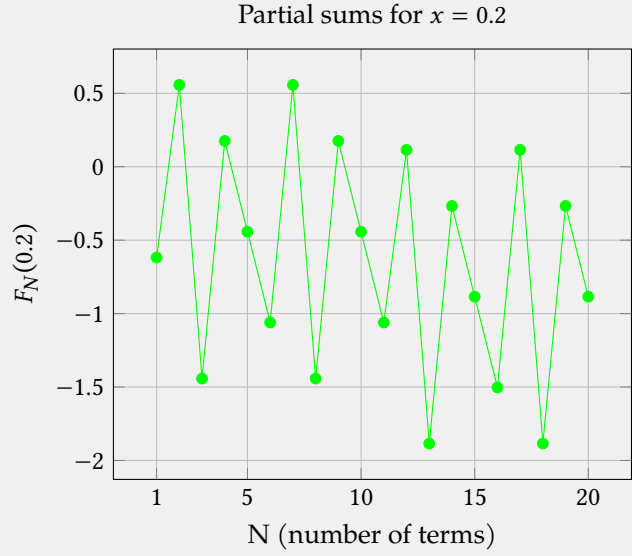18. $F_{18}(0) = -0.0$

19. $F_{19}(0) = -0.0$

20. $F_{20}(0) = -0.0$

Partial sums for $x = 0$
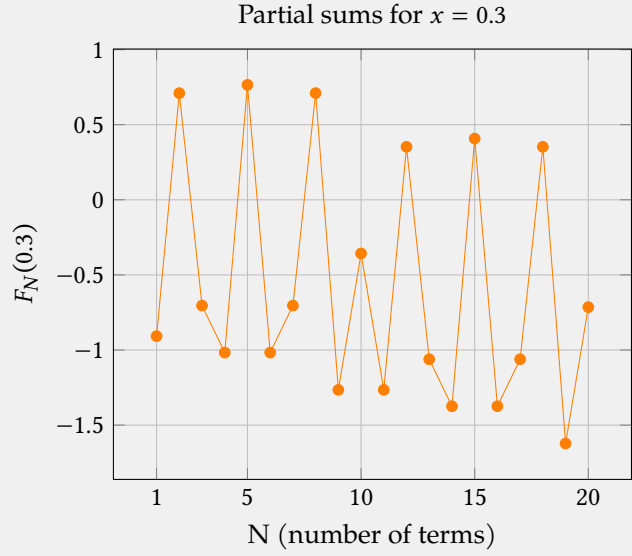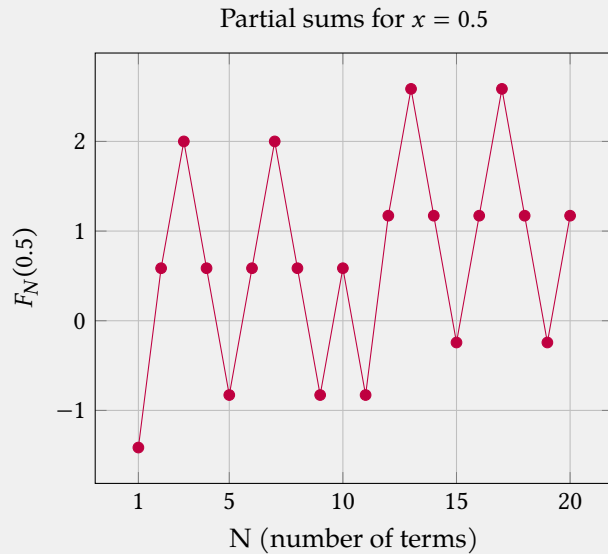

1. $F_1(0.2) = -0.6180339887498948$

2. $F_2(0.2) = 0.5575365158350515$

3. $F_3(0.2) = -1.4424634841649486$

4. $F_4(0.2) = 0.17557050458494627$

5. $F_5(0.2) = -0.44246348416494874$

6. $F_6(0.2) = -1.0604974729148433$

7. $F_7(0.2) = 0.5575365158350525$

8. $F_8(0.2) = -1.4424634841649475$

9. $F_9(0.2) = 0.1755705045849465$

10. $F_{10}(0.2) = -0.44246348416494874$

11. $F_{11}(0.2) = -1.0604974729148435$

12. $F_{12}(0.2) = 0.11507303167010274$

13. $F_{13}(0.2) = -1.8849269683298973$

14. $F_{14}(0.2) = -0.26689297958000235$

15. $F_{15}(0.2) = -0.8849269683298974$

16. $F_{16}(0.2) = -1.502960957079792$

17. $F_{17}(0.2) = 0.11507303167010385$

18. $F_{18}(0.2) = -1.8849269683298961$

19. $F_{19}(0.2) = -0.26689297958000213$

20. $F_{20}(0.2) = -0.8849269683298974$

Partial sums for $x = 0.2$

1. $F_1(0.3) = -0.9079809994790935$

2. $F_2(0.3) = 0.7100529892708014$

3. $F_3(0.3) = -0.7041605731022937$

4. $F_4(0.3) = -1.0170295031827552$

5. $F_5(0.3) = 0.7649835451939804$

6. $F_6(0.3) = -1.0170295031827563$

7. $F_7(0.3) = -0.7041605731022941$

8. $F_8(0.3) = 0.7100529892707993$

9. $F_9(0.3) = -1.265323691919476$

10. $F_{10}(0.3) = -0.3573426924403801$

11. $F_{11}(0.3) = -1.2653236919194737$

12. $F_{12}(0.3) = 0.3527102968304212$

13. $F_{13}(0.3) = -1.061503265542674$

14. $F_{14}(0.3) = -1.3743721956231354$

15. $F_{15}(0.3) = 0.4076408527536002$

16. $F_{16}(0.3) = -1.3743721956231365$

17. $F_{17}(0.3) = -1.0615032655426742$

18. $F_{18}(0.3) = 0.3527102968304192$

19. $F_{19}(0.3) = -1.622666384359856$

20. $F_{20}(0.3) = -0.7146853848807603$

Partial sums for $x = 0.3$

1. $F_1(0.5) = -1.414213562373095$

2. $F_2(0.5) = 0.5857864376269051$

3. $F_3(0.5) = 2.0$

4. $F_4(0.5) = 0.5857864376269046$

5. $F_5(0.5) = -0.8284271247461901$

6. $F_6(0.5) = 0.5857864376269066$

7. $F_7(0.5) = 2.0000000000000027$

8. $F_8(0.5) = 0.5857864376269057$

9. $F_9(0.5) = -0.8284271247461898$

10. $F_{10}(0.5) = 0.5857864376269073$

11. $F_{11}(0.5) = -0.8284271247461876$

12. $F_{12}(0.5) = 1.1715728752538124$

13. $F_{13}(0.5) = 2.5857864376269073$

14. $F_{14}(0.5) = 1.171572875253812$

15. $F_{15}(0.5) = -0.24264068711928277$

16. $F_{16}(0.5) = 1.171572875253814$

17. $F_{17}(0.5) = 2.5857864376269095$

18. $F_{18}(0.5) = 1.1715728752538126$

19. $F_{19}(0.5) = -0.242640687119283$

20. $F_{20}(0.5) = 1.1715728752538141$

Partial sums for $x = 0.5$

Solution (c): At $x = 0$, the series converges to 0. At $x = 0.2, 0.3, 0.5$, the partial sums oscillate and do not settle to 0 quickly.

2 Infinite Summations

2.1 The Archimedean Understanding

Problem 6

Consider the series
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} + \cdots.$$
Find the target value, T, of the partial sums. How do you know that for any M greater than your target value, all of the partial sums are strictly less than M? How many terms do you have to take in order to garuntee that all of the partial sums from that point on will be larger than $L = T - \frac{1}{10}$.

Proof. The target value T is equal to 2. Suppose M is a real number greater than 2. Then

$$2(1 - (1/2)^n) = M > 2 \implies 1 - (1/2)^n > 1 \implies -(1/2)^n > 0,$$

which is impossible since $(1/2)^n > 0$ for all n.

Notice $1.9 < 2(1 - (1/2)^n) \iff \frac{1.9}{2} < 1 - (1/2)^n \iff (1/2)^n < 1 - 0.95 = 0.05$. Taking logarithms shows $n > \log_{1/2}(0.05)$. ∎

Problem 9

What is the Archimedean understanding of the infinite series $1 - 1 + 1 - 1 + \cdots$? Explain why this series cannot have a value under this understanding.

Solution: The Archimedean understanding relies on the assumption that a sum is approximated by partial sums. But for even S_n the sum seems to go to 0, while for odd S_n the sum seems to go to 1. Since the partial sums do not approach a single number, the series cannot have a value with this understanding.

2.2 Geometric Series

Problem 1

Find the target value of the series

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^k} + \cdots$$

Find a value of n so that any partial sum with at least n terms is within 0.001 of the target value. Justify your answer.

Problem 5

It is tempting to differentiate each side of equation (2.11) with respect to x and to assert that

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}.$$

Following Cauchy's advice, we know we need to be careful. Differentiate each side of equation (2.10). What is the difference between $1 + 2x + 3x^2 + \cdots + nx^{n-1}$ and $(1-x)^2$? For which values of x will this difference approach 0 as n increases?