A Radical Approach to Real Analysis by David M. Bressoud

Frosty

January 13, 2026

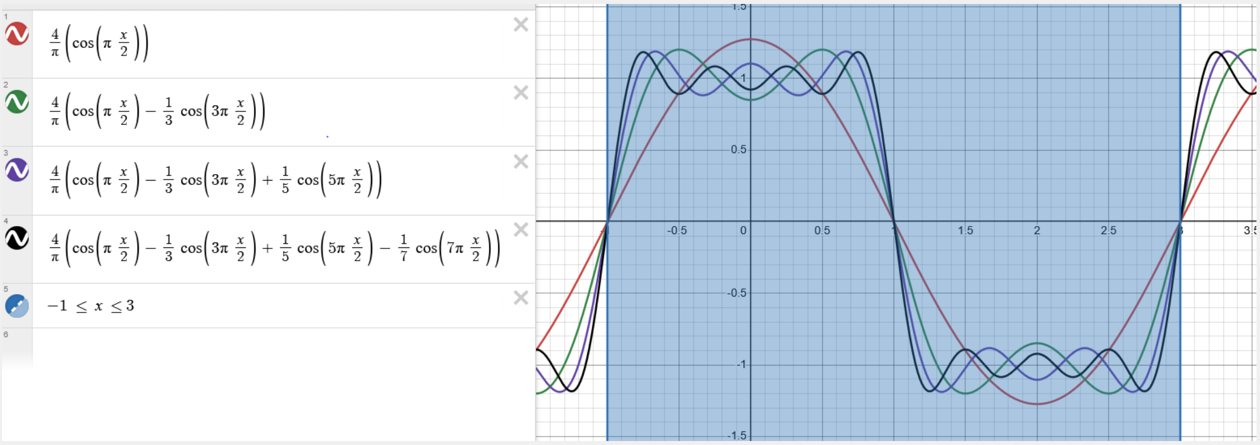
Contents

1 Crisis in Mathematics: Fourier's Series

Problem 1

Graph each of the following partial sums of Fourier's expansion over the inverval $-1 \leq x \leq 3$.

1. $\frac{4}{\pi}\cos(\pi x/2)$
2. $\frac{4}{\pi}(\cos(\pi x/2) - \frac{1}{3}\cos(3\pi x/2))$
3. $\frac{4}{\pi}(\cos(\pi x/2) - \frac{1}{3}\cos(3\pi x/2) + \frac{1}{5}\cos(5\pi x/2))$
4. $\frac{4}{\pi}(\cos(\pi x/2) - \frac{1}{3}\cos(3\pi x/2) + \frac{1}{5}\cos(5\pi x/2)) - \frac{1}{7}\cos(7\pi x/2))$



Problem 2

Let $F_n(x)$ denote the sum of the first n terms of the Fourier's series evaluated at x:

$$F_n(x) = \frac{4}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \cdots + \frac{(-1)^{n-1}}{2n-1}\cos\frac{(2n-1)\pi x}{2}\right)$$

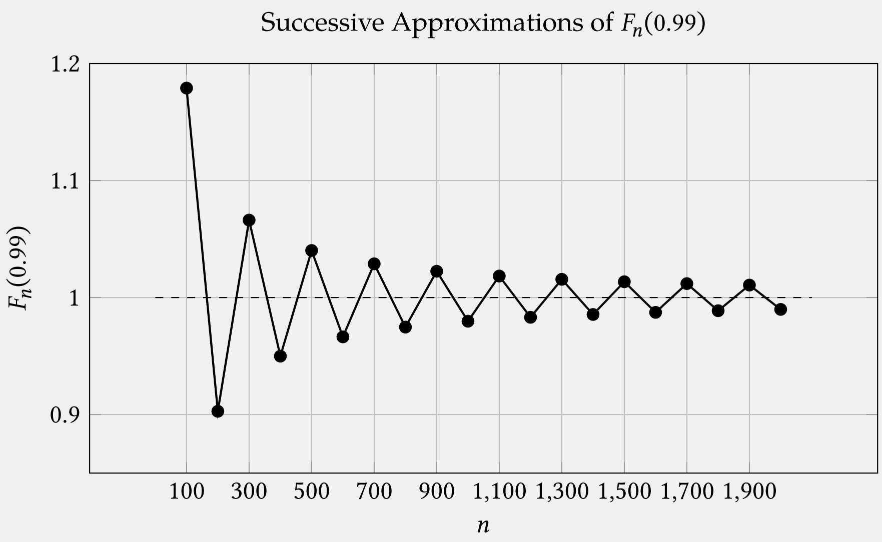
1. Evaluate $F_{100}(x)$ at $x = 0, 0.5, 0.9, 0.99, 1.1,$ and 2. Is this close to the expected value?
2. Evaluate $F_n(0.99)$ at $n = 100, 200, 300, \ldots, 2000$ and plot these successive approximations.
3. Evaluate $F_n(0.999)$ at $n = 100, 200, 300, \ldots, 2000$ and plot these successive approximations.
4. What is the value of this infinite series at $x = 1$?

Solution (a): I had no expectations.

1. $F_{100}(0) = 0.9968169807056898$.
2. $F_{100}(0.5) = 0.9954987558776579$.
3. $F_{100}(0.9) = 0.9796927699334861$.
4. $F_{100}(0.99) = 1.1789880778995547$
5. $F_{100}(1.1) = -0.9796927699334861$.
6. $F_{100}(2) = -0.9968169807056898$.

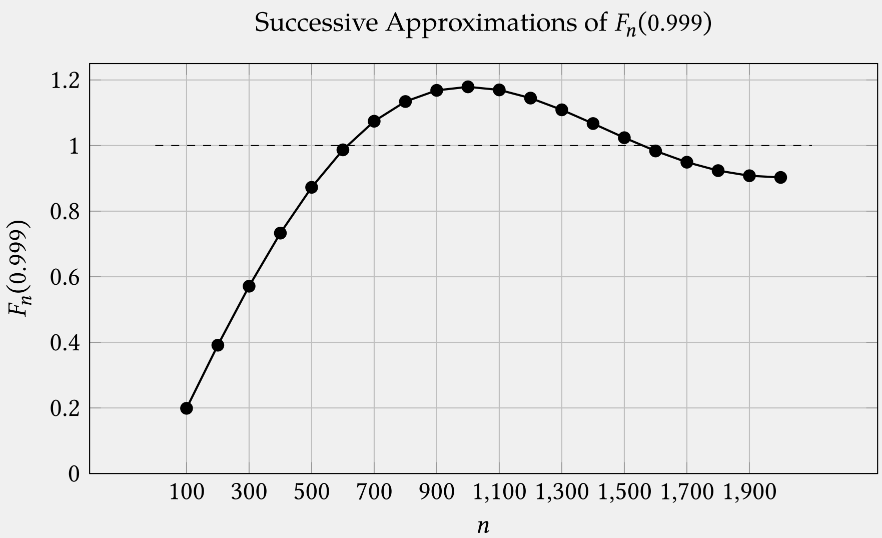
Solution (b):

1. $F_{100}(0.99) = 1.1789880778995547$
2. $F_{200}(0.99) = 0.9028191668118976$
3. $F_{300}(0.99) = 1.0661892530888835$
4. $F_{400}(0.99) = 0.9499372563762823$
5. $F_{500}(0.99) = 1.0402159950960959$
6. $F_{600}(0.99) = 0.9664090460672992$
7. $F_{700}(0.99) = 1.0288331847566468$
8. $F_{800}(0.99) = 0.9747474084192721$
9. $F_{900}(0.99) = 1.0224612312263757$
10. $F_{1000}(0.99) = 0.9797755089505807$
11. $F_{1100}(0.99) = 1.0183922416138902$
12. $F_{1200}(0.99) = 0.9831360088728518$
13. $F_{1300}(0.99) = 1.0155699566691612$
14. $F_{1400}(0.99) = 0.9855398234944296$
15. $F_{1500}(0.99) = 1.0134979445155288$
16. $F_{1600}(0.99) = 0.9873443095081518$
17. $F_{1700}(0.99) = 1.0119123060445154$
18. $F_{1800}(0.99) = 0.9887486426910619$
19. $F_{1900}(0.99) = 1.010659859489416$
20. $F_{2000}(0.99) = 0.98987258246275$



Solution (c):

1. $F_{100}(0.999) = 0.19890664596017577$
2. $F_{200}(0.999) = 0.39133027939704734$
3. $F_{300}(0.999) = 0.5711684699578463$
4. $F_{400}(0.999) = 0.733053348578498$
5. $F_{500}(0.999) = 0.8726544055639414$
6. $F_{600}(0.999) = 0.9869094176114769$
7. $F_{700}(0.999) = 1.074168253719476$
8. $F_{800}(0.999) = 1.134240867961708$
9. $F_{900}(0.999) = 1.1683479421219805$
10. $F_{1000}(0.999) = 1.1789798278055126$
11. $F_{1100}(0.999) = 1.1696760925989875$
12. $F_{1200}(0.999) = 1.1447435828791337$
13. $F_{1300}(0.999) = 1.10893505133833$
14. $F_{1400}(0.999) = 1.0671127580619808$
15. $F_{1500}(0.999) = 1.0239218847151361$
16. $F_{1600}(0.999) = 0.9834971044374297$
17. $F_{1700}(0.999) = 0.949222383157731$
18. $F_{1800}(0.999) = 0.9235593491634196$
19. $F_{1900}(0.999) = 0.9079537683550908$
20. $F_{2000}(0.999) = 0.9028232919136085$



Solution (d): The value of the infinite series at $x = 1$ is 0.

Problem 6

Fourier series illustrate the dangers of trying to find limits by simply substituting the value that x approaches. Consider the Fourier's series:

$$f(x) = \frac{4}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \frac{1}{7}\cos\frac{7\pi x}{2} + \ldots\right)$$

1. What value does this approach as x approaches 1 from the left?
2. What value does this approach as x approaches 1 from the right?
3. What is the value at $f(1)$?

These three answers are all different.

Solution (a):

$$\lim_{x\to 1^-} f(x) = 1.$$

Solution (b):

$$\lim_{x\to 1^+} f(x) = -1.$$

Solution (c):

$$f(1) = \frac{1 + (-1)}{2} = 0.$$

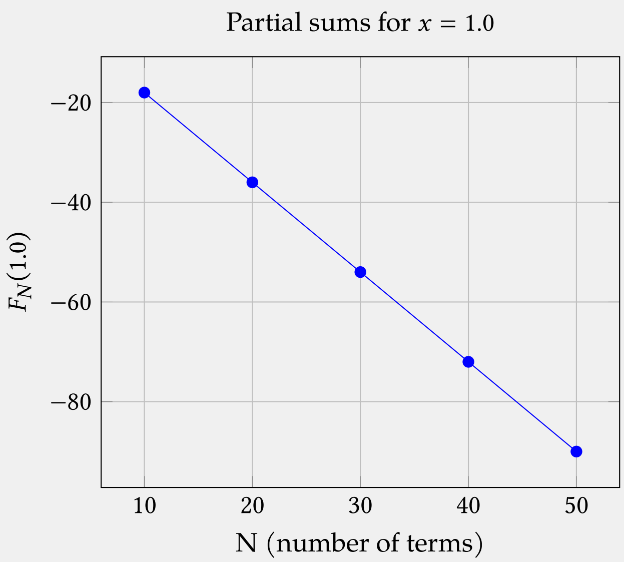
Problem 7

Consider the function that we get if we differentiate each summand of this function $f(x)$ defined in equation $g(x) = -2\left(\sin\frac{\pi x}{2} - \sin\frac{2\pi x}{2} + \sin\frac{5\pi x}{2} - \sin\frac{7\pi x}{2} + \cdots\right)$.

1. For $-1 < x < 3$, graph the partial sums of this series consisting of the first $10, 20, 30, 40$, and 50 terms. Does it appear that these graphs are approaching a constant function 0.
2. Evaluate the partical sums up to at least 20 terms when $x = 0, 0.2, 0.3$, and 0.5. Does it appear that this series is approaching 0 at each of these values of x/
3. What *is* happening at $x = 0, 0.2, 0.3, 0.5$? What can you prove?

Solution (a):

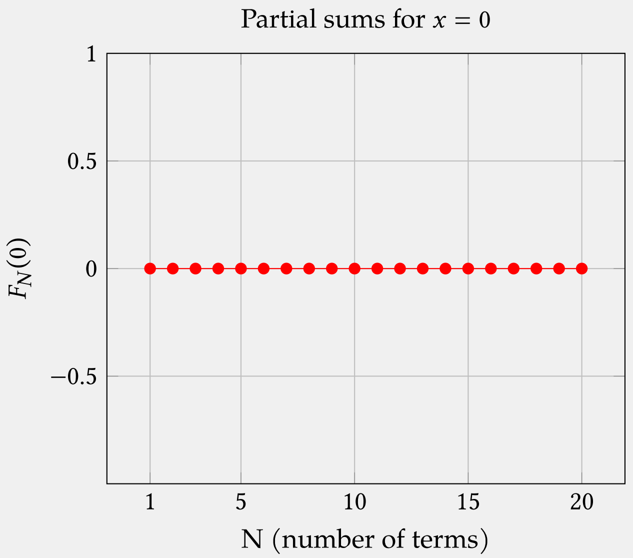
1. $F_{10}(1.0) = -18.0$
2. $F_{20}(1.0) = -36.0$
3. $F_{30}(1.0) = -54.0$
4. $F_{40}(1.0) = -72.0$
5. $F_{50}(1.0) = -90.0$



Solution (b):

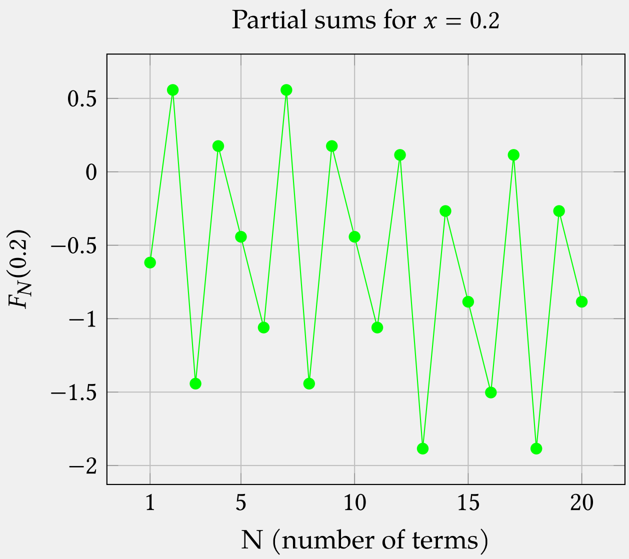
1. $F_1(0) = -0.0$
2. $F_2(0) = -0.0$
3. $F_3(0) = -0.0$
4. $F_4(0) = -0.0$
5. $F_5(0) = -0.0$
6. $F_6(0) = -0.0$
7. $F_7(0) = -0.0$
8. $F_8(0) = -0.0$
9. $F_9(0) = -0.0$
10. $F_{10}(0) = -0.0$

11. $F_{11}(0) = -0.0$
12. $F_{12}(0) = -0.0$
13. $F_{13}(0) = -0.0$
14. $F_{14}(0) = -0.0$
15. $F_{15}(0) = -0.0$
16. $F_{16}(0) = -0.0$
17. $F_{17}(0) = -0.0$
18. $F_{18}(0) = -0.0$
19. $F_{19}(0) = -0.0$
20. $F_{20}(0) = -0.0$



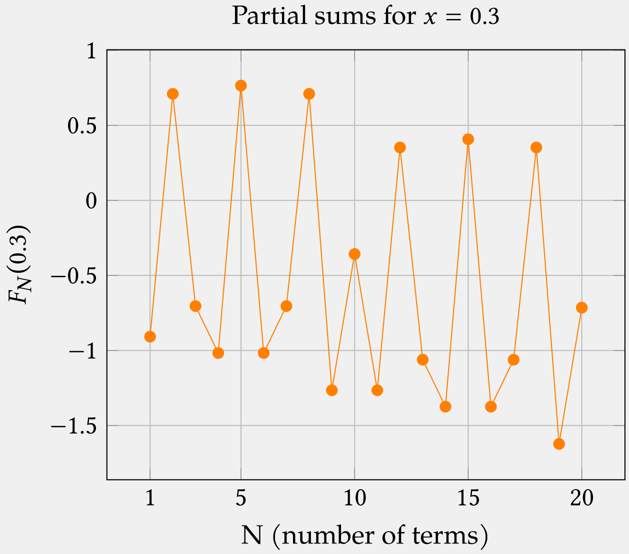
1. $F_1(0.2) = -0.6180339887498948$
2. $F_2(0.2) = 0.5575365158350515$
3. $F_3(0.2) = -1.4424634841649486$
4. $F_4(0.2) = 0.17557050458494627$
5. $F_5(0.2) = -0.44246348416494874$
6. $F_6(0.2) = -1.0604974729148433$
7. $F_7(0.2) = 0.5575365158350525$
8. $F_8(0.2) = -1.4424634841649475$
9. $F_9(0.2) = 0.1755705045849465$
10. $F_{10}(0.2) = -0.44246348416494874$
11. $F_{11}(0.2) = -1.0604974729148435$
12. $F_{12}(0.2) = 0.11507303167010274$
13. $F_{13}(0.2) = -1.8849269683298973$
14. $F_{14}(0.2) = -0.26689297958000235$

15. $F_{15}(0.2) = -0.8849269683298974$
16. $F_{16}(0.2) = -1.502960957079792$
17. $F_{17}(0.2) = 0.11507303167010385$
18. $F_{18}(0.2) = -1.8849269683298961$
19. $F_{19}(0.2) = -0.26689297958000213$
20. $F_{20}(0.2) = -0.8849269683298974$

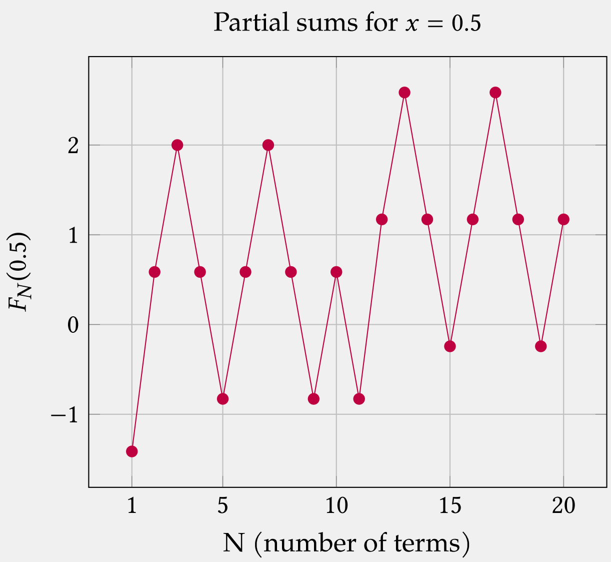


1. $F_1(0.3) = -0.9079809994790935$
2. $F_2(0.3) = 0.7100529892708014$
3. $F_3(0.3) = -0.7041605731022937$
4. $F_4(0.3) = -1.0170295031827552$
5. $F_5(0.3) = 0.7649835451939804$
6. $F_6(0.3) = -1.0170295031827563$
7. $F_7(0.3) = -0.7041605731022941$
8. $F_8(0.3) = 0.7100529892707993$
9. $F_9(0.3) = -1.265323691919476$
10. $F_{10}(0.3) = -0.3573426924403801$
11. $F_{11}(0.3) = -1.2653236919194737$
12. $F_{12}(0.3) = 0.3527102968304212$
13. $F_{13}(0.3) = -1.061503265542674$
14. $F_{14}(0.3) = -1.3743721956231354$
15. $F_{15}(0.3) = 0.4076408527536002$
16. $F_{16}(0.3) = -1.3743721956231365$
17. $F_{17}(0.3) = -1.0615032655426742$
18. $F_{18}(0.3) = 0.3527102968304192$

19. $F_{19}(0.3) = -1.622666384359856$
20. $F_{20}(0.3) = -0.7146853848807603$



1. $F_1(0.5) = -1.414213562373095$
2. $F_2(0.5) = 0.5857864376269051$
3. $F_3(0.5) = 2.0$
4. $F_4(0.5) = 0.5857864376269046$
5. $F_5(0.5) = -0.8284271247461901$
6. $F_6(0.5) = 0.5857864376269066$
7. $F_7(0.5) = 2.0000000000000027$
8. $F_8(0.5) = 0.5857864376269057$
9. $F_9(0.5) = -0.8284271247461898$
10. $F_{10}(0.5) = 0.5857864376269073$
11. $F_{11}(0.5) = -0.8284271247461876$
12. $F_{12}(0.5) = 1.1715728752538124$
13. $F_{13}(0.5) = 2.5857864376269073$
14. $F_{14}(0.5) = 1.171572875253812$
15. $F_{15}(0.5) = -0.24264068711928277$
16. $F_{16}(0.5) = 1.171572875253814$
17. $F_{17}(0.5) = 2.5857864376269095$
18. $F_{18}(0.5) = 1.1715728752538126$
19. $F_{19}(0.5) = -0.242640687119283$
20. $F_{20}(0.5) = 1.1715728752538141$



Solution (c): At $x = 0$, the series converges to 0. At $x = 0.2, 0.3, 0.5$, the partial sums oscillate and do not settle to 0 quickly.