A Radical Approach to Real Analysis by David M. Bressoud

Frosty

February 2, 2026

Contents

1 Crisis in Mathematics: Fourier's Series

2 Infinite Summations

2.1 The Archimedean Understanding

Definition 1 (Archimedean Understanding of an Infinite Series). *The **Archimedean Understanding** of an infinite series is that it is shorthand for the sequence of finite summations. The **value** of an infinite series, if it exists, is that number T such that given any $L < T$ and any $M > T$, all of the finite sums from some point on will be strictly contained in the interval between L and M. More precisely, given $L < T < M$, there is an integer n, whose value depends on the choice of L and M, such that every partial sum with at least n terms lies inside the interval (L, M).*

2.2 Geometric Series

Definition 2 (Convergence of an Infinite Series). *An infinite series **converges** if there is a target value T such that for any $L < T$ and any $M > T$, all of the partial sums from some point on are strictly between L and M.*

2.3 Calculating π

Theorem 1 (Newton's Binomial Series). *For any real number a and any x such that $|x| < 1$, we have that*

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!} + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots.$$

2.4 Logarithms and Harmonic Series

Definition 3 (Divergence to Infinity). *When we write that an infinite series equals ∞, we mean that no matter what number we pick, we can find an n so that the partial sums with at least n terms will exceed that number.*

Definition 4 (Euler's constant, γ). *Euler's constant is defined as the limit between the partial sum of the harmonic series and the natural logarithm,*

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \ln n\right)$$

Definition 5 (Nested Interval Principle). *Given an increasing sequence, $x_1 \leq x_2 \leq x_3 \leq \cdots$, and a decreasing sequence, $y_1 \geq y_2 \geq y_3 \geq \cdots$, such that y_n is always larger than x_n but the difference between y_n and x_n can be made arbitrarily small by taking n sufficiently large, there is exactly one real number that is greater than or equal to every x_n and less than or equal to every y_n.*

2.5 Taylor Series

Definition 6 (Taylor Series). *If all of the derivatives of the function f exist at the point a, then the Taylor series f about a is the infinite series*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^3 + \cdots.$$

This has a special case ($a = 0$):

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots.$$

Theorem 2 (Lagrange's Remainder Theorem). *Given a function f for which all derivatives exist at $x = a$, let $D_n(a,x)$ denote the difference between the nth partial sum of the tayls series for f expanded about $x = a$ and the target value $f(x)$,*

$$D_n(a,x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}\right).$$

There is at least one real number c strictly between a and x for which

$$D_n(a,x) = \frac{f^{(n)}(c)}{n!}(x-a)^n.$$

Theorem 3 (Stirling's Formula). *The factorial function $n!$ is well approximated by the function $(n/e)^n\sqrt{2\pi n}$. Specifically, we have that*

$$\lim_{n\to\infty}\frac{n!}{(n/e)^n\sqrt{2\pi n}} = 1.$$

Theorem 4 (C^p and analytic functions). *Given an interval I, a function with a continuous first derivative in I is said to belong to the class C^1. If the pth derivative exists and is continuous in I, the function belongs to the class C^p. If all derivatives exist, the function belongs to C^∞ and is called **analytic**.*

3 Differentiability and Continuity

3.1 Differentiability

Theorem 5 (Mean Value Theorem). *Given a function f that is differentiable at all points strictly between a and x and continuous at all points on the closed interval from a to x, there exists a real number c strictly between a and x such that*

$$\frac{f(x) - f(a)}{x - a} = f'(c).$$

Definition 7 (Archimedian Understanding of Limits)**.** *When we write any limit statement such as*

$$\lim_{x \to a} f(x) = T.$$

what we actually mean is that if we take any number $M > T$, then we can force $f(x) < M$ by taking x to be sufficiently close to a. Similarly, if we take any $L < T$, then we can force $f(x) > L$ by taking x sufficiently close to $($but not equal to$)$ a.

Definition 8 (Derivative of f at $x = a$)**.** *The **derivative** of f at a is that value, denoted $f'(a)$, such that for any $L < f'(a)$ and any $M > f'(a)$, we can force*

$$L < \frac{f(x) - f(a)}{x - a} < M,$$

by simply taking x sufficiently close to $($but not equal to$)$ a.

Definition 9 (Cauchy Definition of Derivative of f at $x = a$)**.** *The **derivative** of f at a is that value, denoted $f'(a)$, such that for any $\epsilon > 0$, we have a response $\delta > 0$ so that if $0 < |x - a| < \delta$, then this forces*

$$E(x, a) = \left| f'(a) - \frac{f(x) - f(a)}{x - a} \right| < \epsilon.$$