How to Prove It by Velleman

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1 Sentential Logic

1.1 Operations On Sets

Problem 2

Let $A = \{\text{United States}, \text{Germany}, \text{China}, \text{Australia}\}$,
$B = \{\text{Germany}, \text{France}, \text{India}, \text{Brazil}\}$, and
$C = \{x \mid x \text{ is a country in Europe}\}$.

List the elements of the following sets. Are any of the sets below disjoint from any of the others? Are any of the sets below subsets of any others?

Problem 2 (a)

$A \cup B$

Solution:

$$A \cup B = \{\text{United States, Germany, China, Australia, France, India, Brazil}\}$$

Problem 2 (b)

$A \cap B \setminus C$

Solution:

$$\begin{aligned} A \cap B &= \{\text{Germany}\} \\ A \cap B \setminus C &= \varnothing \end{aligned}$$

Problem 2 (c)

$A \cap B \setminus A$

Solution:

$$\begin{aligned} A \cap B &= \{\text{Germany}\} \\ A \cap B \setminus A &= \varnothing \\ (A \cap B \setminus C) \text{ and } (A \cap B \setminus A) &\text{ are subsets of each other and } A \cup B \\ (A \cap B \setminus C) \text{ and } (A \cap B \setminus A) &\text{ are disjoint from each other and from } A \cup B \end{aligned}$$

Problem 5

Verify the identities in exercise 4 by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.
Excersize 4:
(a) $A \setminus (A \cap B) = A \setminus B$
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution a:

$$\text{Let } X \equiv (x \in A)$$
$$\text{Let } Y \equiv (x \in B)$$

$$\begin{aligned}
x \in A \setminus (A \cap B) &\equiv (x \in A) \land \neg(x \in A \land x \in B) \\
&\equiv (x \in A) \land (x \notin A \lor x \notin B) \\
&\equiv X \land (\neg X \lor \neg Y) \\
&\equiv (X \land \neg X) \lor (X \land \neg Y) \\
&\equiv X \land \neg Y \\
&\equiv x \in A \land x \notin B \\
&\equiv x \in A \setminus B
\end{aligned}$$

Solution b:

$$\text{Let } X \equiv (x \in A)$$
$$\text{Let } Y \equiv (x \in B)$$
$$\text{Let } Z \equiv (x \in C)$$

$$\begin{aligned}
x \in A \cup (B \cap C) &\equiv x \in A \lor x \in (B \cap C) && \text{def. of union} \\
&\equiv x \in A \lor x \in (x \in B \land x \in C) && \text{def. of intersection} \\
&\equiv X \lor (Y \land Z) && \text{def. of intersection} \\
&\equiv (X \lor Y) \land (X \lor Z) \\
&\equiv (x \in A \lor x \in B) \land (x \in A \lor x \in C) \\
&\equiv (x \in A \cup B) \land (x \in A \cup C) && \text{def. of union} \\
&\equiv x \in (A \cup B) \cap (A \cup C) && \text{def. of intersection}
\end{aligned}$$

Problem 7

Verify the identities in excersize 6 by writing out (using logical symbols) what it means for an object x to be an element of each set then using logical equivalences.
(a) $(A \cup B) \setminus C \equiv (A \setminus C) \cup (B \setminus C)$
(b) $A \cup (B \setminus C) \equiv (A \cup B) \setminus (C \setminus A)$

Solution a:

$$\begin{aligned}
&\text{Let } X \equiv (x \in A)\\
&\text{Let } Y \equiv (x \in B)\\
&\text{Let } Z \equiv (x \in C)
\end{aligned}$$

$$\begin{aligned}
x \in (A \cup B) \setminus C &\equiv x \in (A \cup B) \land x \notin C && \text{def. of set difference}\\
&\equiv (x \in A \lor x \in B) \land x \notin C && \text{def. of union}\\
&\equiv (X \lor Y) \land \neg Z\\
&\equiv (X \land \neg Z) \lor (Y \land \neg Z) && \text{distributive}\\
&\equiv (x \in A \land x \notin C) \lor (x \in B \land x \notin C)\\
&\equiv (x \in A \setminus C) \lor (x \in B \setminus C) && \text{def. of set difference}\\
&\equiv x \in (A \setminus C) \cup (B \setminus C) && \text{def. of union}
\end{aligned}$$

Solution b:

$$\begin{aligned}
&\text{Let } X \equiv (x \in A)\\
&\text{Let } Y \equiv (x \in B)\\
&\text{Let } Z \equiv (x \in C)
\end{aligned}$$

$$\begin{aligned}
x \in A \cup (B \setminus C) &\equiv x \in A \lor x \in (B \setminus C) && \text{def. of union}\\
&\equiv x \in A \lor (x \in B \land x \notin C) && \text{def. of set difference}\\
&\equiv X \lor (Y \land \neg Z)\\
&\equiv (X \lor Y) \land (X \lor \neg Z) && \text{distributive}\\
&\equiv (X \lor Y) \land \neg(\neg X \land Z) && \text{demorgans}\\
&\equiv (X \lor Y) \land \neg(Z \land \neg X) && \text{commutative}\\
&\equiv (x \in A \lor x \in B) \land \neg(x \in C \land x \notin A)\\
&\equiv x \in (A \cup B) \land \neg(x \in C \land x \notin A) && \text{def. of union}\\
&\equiv x \in (A \cup B) \land \neg(x \in (C \setminus A)) && \text{def. of set difference}\\
&\equiv x \in (A \cup B) \setminus (C \setminus A) && \text{def. of set difference}
\end{aligned}$$

Problem 9

For each of the following sets, write out (using logical symbols) what it means for an object x to be an element of the set. Then determine which of these sets must be equal to each other by determining which statements are equivalent.

(a) $(A \setminus B) \setminus C$
(b) $A \setminus B \setminus C$
(c) $(A \setminus B) \cup (A \cap C)$
(d) $(A \setminus B) \cap (A \setminus C)$
(e) $A \setminus (B \cup C)$

Solution

$$\begin{aligned}
\text{Let } X &\equiv (x \in A)\\
\text{Let } Y &\equiv (x \in B)\\
\text{Let } Z &\equiv (x \in C)
\end{aligned}$$

$$\begin{aligned}
\textbf{(a)}\ x \in (A \setminus B) \setminus C &\equiv x \in (A \setminus B) \wedge x \notin C\\
&\equiv (x \in A \wedge x \notin B) \wedge x \notin C\\
&\equiv (X \wedge \neg Y) \wedge \neg Z\\
&\equiv X \wedge \neg Y \wedge \neg Z\\
\textbf{(b)}\ x \in A \setminus (B \setminus C) &\equiv x \in A \wedge \neg(x \in B \setminus C)\\
&\equiv x \in A \wedge \neg(x \in B \wedge x \notin C)\\
&\equiv X \wedge \neg(Y \wedge \neg Z)\\
\textbf{(c)}\ x \in (A \setminus B) \cup (A \cap C) &\equiv (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C)\\
&\equiv (X \wedge \neg Y) \vee (X \wedge Z)\\
\textbf{(d)}\ x \in (A \setminus B) \cap (A \setminus C) &\equiv (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)\\
&\equiv (X \wedge \neg Y) \wedge (X \wedge \neg Z)\\
&\equiv X \wedge \neg Y \wedge \neg Z\\
\textbf{(e)}\ x \in A \setminus (B \cup C) &\equiv x \in A \wedge \neg(x \in B \vee x \in C)\\
&\equiv X \wedge \neg(Y \vee Z)\\
&\equiv X \wedge (\neg Y \wedge \neg Z)
\end{aligned}$$

(a) and (d) are equal
(b), (c), and (e) are equal

Problem 10

It was shown in this section that for any sets A and B, $(A \cup B) \subseteq A$.
(a) Give an example of two sets A and B for which $(A \cup B) \setminus B \neq A$.
(b) Show that for all sets A and B, $(A \cup B) \setminus B = A \setminus B$.

Problem 10 (a)

$$\begin{aligned}
A &= \{1, 2, 3\}\\
B &= \{4, 2, 5\}\\
A \cup B &= \{1, 2, 3, 4, 5\}\\
(A \cup B) \setminus B &= \{1, 3\}\\
\{1, 3\} &\neq A
\end{aligned}$$

Want to show $x \in (A \cup B) \setminus B$ is equivalent to $x \in A \setminus B$

Problem 10 (b)

$$\begin{aligned}
x \in (A \cup B) \setminus B &\equiv x \in (A \cup B) \land x \notin B && \text{(def. of set difference)} \\
&\equiv (x \in A \lor x \in B) \land x \notin B && \text{(def. of union)} \\
&\equiv (x \notin B \land x \in B) \lor (x \notin B \land x \in A) && \text{(distributive)} \\
&\equiv (x \notin B \land x \in A) && \text{(contradiction)} \\
&\equiv (x \in A \land x \notin B) && \text{(commutative)} \\
&\equiv x \in A \setminus B && \text{(def. of set difference)}
\end{aligned}$$

Problem 11

Suppose A and B are sets.
Is it necessarily true that $(A \setminus B) \cup B = A$?
If not, is one of these sets necessarily a subset of the other?
Is $(A \setminus B) \cup B$ always equal to either $A \setminus B$ or $A \cup B$.

Problem 11

No it is not necessarily true.

$$\begin{aligned}
A &= \{1, 2\} \\
B &= \{3\} \\
(A \setminus B) &= \{1, 2\} \\
(A \setminus B) \cup B &= \{1, 2, 3\} \\
A &\neq \{1, 2, 3\}
\end{aligned}$$

It is not always equal to $A \setminus B$ as shown above.

It is always equal to $A \cup B$. Proof:

$$\begin{aligned}
x \in (A \setminus B) \cup B &\equiv x \in (A \setminus B) \lor x \in B && \text{def. of union} \\
&\equiv (x \in A \land x \notin B) \lor x \in B && \text{def. of set difference} \\
&\equiv (x \in B \lor x \notin B) \land (x \in B \lor x \in A) && \text{distributive} \\
&\equiv x \in B \lor x \in A && \text{tautology} \\
&\equiv x \in A \lor x \in B && \text{commutative} \\
&\equiv x \in (A \cup B) && \text{def. of union}
\end{aligned}$$

A is always a subset of $(A \setminus B) \cup B$.

From above $(A \setminus B) \cup B \equiv A \cup B$.

From def. of union that means $x \in A$.

$(A \setminus B) \cup B$ is not always a subset of A.

From above $(A \setminus B) \cup B \equiv A \cup B$.

Therefore, x could be in B and not in A and still exist in $(A \setminus B) \cup B$.

Problem 16 (a)

Use any method you wish to verify the following identity. $(A \cup B) \triangle C = (A \triangle C) \triangle (B \setminus A)$

Proof. First note that commutativity holds for symmetric difference

$$\begin{aligned} A \triangle B &\equiv (A \cup B) \setminus (A \cap B) && \text{def. of symmetric difference} \\ &\equiv (B \cup A) \setminus (B \cap A) && \text{commutative} \\ &\equiv B \triangle A && \text{def. of symmetric difference} \end{aligned}$$

Also note $x \in A \cup B \equiv x \in A \triangle (B \setminus A)$. Let $X \equiv x \in A$, $Y \equiv x \in B$. Starting with the rhs

$$\begin{aligned} x \in A \triangle (B \setminus A) &\equiv x \in (A \cup (B \setminus A)) \setminus (A \cap (B \setminus A)) && \text{def. of symmetric difference} \\ &\equiv [X \vee (Y \wedge \neg X)] \wedge \neg [X \wedge (Y \wedge \neg X)] && \text{def. of union, intersection, set difference} \\ &\equiv [(X \vee \neg X) \wedge (X \vee Y)] \wedge \neg [(X \wedge \neg X) \wedge (X \wedge Y)] && \text{distributive} \\ &\equiv [X \vee Y] \wedge \neg [(X \wedge \neg X) \wedge (X \wedge Y)] && \text{tautology} \\ &\equiv [X \vee Y] && \text{contradiction} \\ &\equiv X \vee Y \\ &\equiv x \in X \cup Y && \text{def. of union} \end{aligned}$$

Now

$$\begin{aligned} (A \cup B) \triangle C &\equiv (A \triangle (B \setminus A)) \triangle C \\ &\equiv A \triangle ((B \setminus A) \triangle C) && \text{associative} \\ &\equiv A \triangle (C \triangle (B \setminus A)) && \text{commutativity} \\ &\equiv (A \triangle C) \triangle (B \setminus A) && \text{associative} \end{aligned}$$

Therefore, the identity $(A \cup B) \triangle C = (A \triangle C) \triangle (B \setminus A)$ holds. ∎

1.2 The Conditional and Biconditional Connectives

Problem 2

Analyze the logicla forms of the following statements:
(a) Mary will sell her house only if she can get a good price and find a nice apartment.
(b) Having both a good credit history and an adequate down payment is a necessary condition for getting a mortgage.
(c) John will drop out of school, unless someone stops him .(Hint: First try to rephrase this using the words *if* and *then* instead of *unless*).
(d) If x is divisible by either 4 or 6, then it isn't prime.

Problem (a)
Let G stand for the statement, "Mary can get a good price".
Let F stand for the statement, "Mary can find a nice aparment".
Let S stand for the statement, "Mary will sell her house".

$S \rightarrow G \wedge F$

Problem (b)
Let C stand for the statement, "Having a good credit score history".
Let A stand for the statement, "Having an adequate downpayment on the house."
Let M stand for the statement, "Getting a mortgage".

$M \to C \land A$

Problem (c)
Let D stand for the statement, ”John drops out of school”.
Let S stand for the statement, ”Someone stops him”.

$S \to \neg D$

Problem (d)
Let X stand for the statement, ”x is divisible by 4”.
Let Y stand for the statement, ”x is divisible by 6”.
Let Z stand for the statement, ”x is a prime”.

$X \lor Y \to \neg Z$.

Problem 3

Analyze the logical form of the following statement:
(a) If it is raining, then it is windy and the sun is not shining.
Now analyze the following statements. Also, for each statement determine whether the statement is equivalent to either statement (a) or its converse.
(b) It is windy and not sunny only if it is raining.
(c) Rain is a sufficient condition for wind with no sunshine.
(d) Rain is a necessary condition for wind with no sunshine.
(e) It’s not raining, if either the sun is shining or it’s not windy.
(f) Wind is a necessary condition for it to be rainy, and so is a lack of sunshine.
(g) Either it is windy only if it is rainy, or it is not sunny only if it is raining.

Solution (a)
Let R stand for the statement, ”It is raining”.
Let W stand for the statement, ”It is windy”.
Let S stand for the statement, ”The sun is shining”.

$R \to (W \land \neg S)$

Solution (b)

$(W \land \neg S) \to R$ Equivalent to (a) converse.

Solution (c)

$R \to (W \land \neg S)$ Equivalent to (a).

Solution (d)

$(W \land \neg S) \to R$ Equivalent to (a) converse.

Solution (e)

$(S \lor \neg W) \to \neg R$
$R \to (W \land \neg S)$ Equivalent to (a)

Solution (f)

$R \to (W \wedge \neg S)$ Equivalent to (a)

Solution (g)

$(W \wedge \neg S) \to R$ Equivalent to (a) converse

Problem 5

Use truth tables to determine whether or not the following arguments are valid:
(a) If Jones is convicted then he will go to prison. Jones will be convicted only if Smith testifies against him. Therefore, Jones won't go to prison unless Smith testifies against him.
(b) Either the Democrats or the Republicans will have a majority in the Senate, but not both. Having a Democratic majority is a necessary condition for the bill to pass. Therefore, if the Republicans have a majority in the Senate then the bill won't pass.

Problem 6 (a)

Show that $P \leftrightarrow Q$ is equivalent to $(P \wedge A) \vee (\neg P \wedge \neg Q)$.

Proof.

$$
\begin{aligned}
P \leftrightarrow Q &\equiv (P \to Q) \wedge (Q \to P) && \text{def. of iff} \\
&\equiv (\neg P \vee Q) \wedge (\neg Q \vee P) && \text{def. of imp.} \\
&\equiv [(\neg Q \vee P) \wedge \neg P] \vee [(\neg Q \vee P) \wedge Q] && \text{distributive} \\
&\equiv [(\neg Q \wedge \neg P) \vee (\neg P \wedge P)] \vee [(\neg Q \vee P) \wedge Q] && \text{distributive} \\
&\equiv (\neg Q \wedge \neg P) \vee [(\neg Q \vee P) \wedge Q] && \text{contradiction} \\
&\equiv (\neg Q \wedge \neg P) \vee [(Q \wedge \neg Q) \vee (Q \wedge P)] && \text{distributive} \\
&\equiv (\neg Q \wedge \neg P) \vee (Q \wedge P) && \text{contradiction} \\
&\equiv (Q \wedge P) \vee (\neg P \wedge \neg Q) && \text{commutativity}
\end{aligned}
$$

∎

Problem 6 (b)

Show that $(P \to Q) \vee (P \to R)$ is equivalent to $P \to (Q \vee R)$.

Proof.

$$
\begin{aligned}
P \to (Q \vee R) &\equiv \neg P \vee (Q \vee R) && \text{def. of imp.} \\
&\equiv (\neg P \vee Q) \vee (\neg P \vee R) && \text{distributive.} \\
&\equiv (P \to Q) \vee (P \to R) && \text{def. of imp.}
\end{aligned}
$$

∎

Problem 7 (a)

Show that $(P \to Q) \wedge (Q \to R)$ is equivalent to $(P \vee Q) \to R$.

Proof.

$$\begin{aligned}
(P \to Q) \land (Q \to R) &\equiv (\neg P \lor R) \land (\neg Q \lor R) && \text{def. of imp.}\\
&\equiv R \lor (\neg P \land \neg Q) && \text{distributive}\\
&\equiv R \lor \neg(P \lor Q) && \text{demorgans}\\
&\equiv \neg(P \lor Q) \lor R && \text{commutative}\\
&\equiv (P \lor Q) \to R && \text{def. of imp}
\end{aligned}$$

∎

Problem 7 (b)

Formulate and verify a similar equivalence involving $(P \to R) \lor (Q \to R)$.

Proof.

$$\begin{aligned}
(P \to R) \lor (Q \to R) &\equiv (\neg P \lor R) \lor (\neg Q \lor R) && \text{def. of imp.}\\
&\equiv R \lor (\neg P \lor \neg Q) && \text{distributive}\\
&\equiv R \lor \neg(P \land Q) && \text{demorgans}\\
&\equiv \neg(P \land Q) \lor R && \text{commutative}\\
&\equiv (P \land Q) \to R && \text{def. of imp}
\end{aligned}$$

∎

Problem 8 (a)

(a) Show that $(P \to Q) \land (Q \to R)$ is equivalent to $(P \to R) \land [(P \leftrightarrow Q) \lor (R \leftrightarrow Q)]$.
(b) Show that $(P \to Q) \lor (Q \to R)$ is a tautology.

Proof. First note that:

$$(P \to Q) \land (Q \to R) \equiv P \to R$$

which we label $N1$ (proof in image). Then:

$$\begin{aligned}
&(P \to R) \land [(P \leftrightarrow Q) \lor (R \leftrightarrow Q)]\\
&\equiv (P \to R) \land [((P \to Q) \land (Q \to P)) \lor ((R \to Q) \land (Q \to R))] && \text{(definition of } \leftrightarrow)\\
&\equiv [(P \to R) \land (P \to Q) \land (Q \to P)]\\
&\quad \lor [(P \to R) \land (R \to Q) \land (Q \to R)] && \text{(distributive law)}\\
&\equiv [(Q \to P) \land (P \to R) \land (P \to Q)]\\
&\quad \lor [(P \to R) \land (Q \to R) \land (R \to Q)] && \text{(associativity, commutativity)}\\
&\equiv [(P \to Q) \land (Q \to R)] \lor [(P \to Q) \land (Q \to R)] && \text{(by } N1)\\
&\equiv (P \to Q) \land (Q \to R) && \text{(idempotent law)}
\end{aligned}$$

∎

Proof.

$$\begin{aligned}
(P \to Q) \lor (Q \to R) &\equiv (\neg P \lor Q) \lor (\neg Q \lor R) && \text{def. of } \to\\
&\equiv \neg P \lor (Q \lor (\neg Q \lor R)) && \text{associative}\\
&\equiv \neg P \lor ((Q \lor \neg Q) \lor R) && \text{associative}\\
&\equiv \neg P \lor T \lor R && \text{tautology}\\
&\equiv T && \text{tautology}
\end{aligned}$$

Problem 9

Find a forumla using only connectives $\neg$ and $\to$ that is equivalent to $P \wedge Q$.

Solution

$$\begin{aligned}(P \to Q) \wedge P &\equiv (\neg P \vee Q) \wedge P && \text{def. of } \to\\ &\equiv (P \wedge \neg P) \vee (P \wedge Q) && \text{distributive}\\ &\equiv (P \wedge Q) && \text{idempotent}\end{aligned}$$

2 Qunatification Logic

2.1 Quantifiers

Problem 1

Analyize the logical forms of the following statements:
(a) Anyone who as forgiven at least one person is a saint.
(b) Nobody in the calculus class is smarter than everybody in the discrete math class.
(c) Everyone likes Mary, except Mary herself.
(d) Jane saw a police officer, and Roger saw one too.
(e) Jane saw a police officer, and Roger saw him too.

Solution 1 (a)
Let $P(x, y)$ mean "x has forgiven y".
Let $S(x)$ mean "x is a saint".

$$\forall x(\exists y P(x, y) \to S(x))$$

Solution 1 (b)
Let $S(x, y)$ mean "x is smarter than y".
Let $M(x)$ mean "x is in the calculus class".
Let $D(x)$ mean "x is in the discrete math class".

$$\forall x(M(x) \to (\exists y(D(y) \wedge S(y, x))))$$

Solution 1 (c)
Let $L(x)$ mean "x likes Mary".
Let $M(x)$ mean "x is Mary".

$$\forall x((\neg M(x) \to L(x)) \wedge (M(x) \to \neg L(x)))$$

Solution 1 (d)
Let $J(x)$ mean "x is Jane".
Let $R(x)$ mean "x is Roger".
Let $P(x)$ mean "x saw a police officer"

$$\forall x((J(x) \to P(x)) \wedge (R(x) \to P(x)))$$

Solution 1 (e)
Let $J(x)$ mean “x is Jane”.
Let $R(x)$ mean “x is Roger”.
Let $P(x, y)$ mean “x saw a police officer y”

$$\exists y(\forall x((J(x) \to P(x,y)) \wedge (R(x) \to P(x,y))))$$

Problem 3

Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?
(a) Every number that is larger than x is larger than y
(b) For every number a, the equation $ax^2 + 4x - 2 = 0$ has at least one solution iff $a >= -2$.
(c) All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.
(d) If there is a number x such that $x^2 + 5x = w$ and there is a number y such that $4 - y^2 = 2$ then w is strictly between -10 and 10.

Solution 3 (a)

$$\forall t((t > x) \to (t > y))$$

Free variables are x and y.
Solution 3 (b)

$$\forall t(\exists x(tx^2 + 4x - 2 = 0) \leftrightarrow t \geq -2)$$

No free variables.
Solution 3 (c)

$$\forall x(x^3 - 3x < 3 \to x < 10)$$

No free variables.
Solution 3 (d)

$$(\exists x(x^2 + 5x = w) \wedge \exists y(4 - y^2 = 2)) \to -10 < w < 10$$

Free variable is w.

Problem 5

Translate the following statements into idiomatic mathematical English.
(a) $\forall x[(P(x) \wedge \neg(x = 2)) \to O(x)]$, where $P(x)$ means “x is a prime number” and $O(x)$ means “x is odd”.
(b) $\exists x[P(x) \wedge \forall y(P(y) \to y \leq x)]$, where $P(x)$ means “x is a perfect number”.

Solution 5 (a)
For all x, if x is a prime number and x is not equal to 2 then x is odd.
Solution 5 (b)
There exists x such that x is a perfect number and for all y, if y is a perfect number then y is less than or equal to x.

Solution 5 (a)

Problem 8

Are these statements true or false? The universe of discourse is $\mathbb{N}$.
(a) $\forall x \exists y(2x - y = 0)$.
(b) $\exists y \forall x(2x - y = 0)$.
(c) $\forall x \exists y(x - 2y = 0)$.
(d) $\forall x(x < 10 \to \forall y(y < x \to y < 9))$.
(e) $\exists y \exists z(y + z = 100)$.
(f) $\forall x \exists y(y > x \land \exists z(y + z = 100))$

Solution 8 (a)
True.
Solution 8 (b)
False.
Solution 8 (c)
False.
Solution 8 (d)
True.
Solution 8 (e)
True.
Solution 8 (f)
False.

Problem 9

Same excersize as 8 but with $\mathbb{R}$ as the universe of discourse.

Solution 9 (a)
True.
Solution 9 (b)
False.
Solution 9 (c)
True.
Solution 9 (d)
False.
Solution 9 (e)
True.
Solution 9 (f)
True.

Problem 10

Same excersize as 8 but with $\mathbb{Z}$ as the universe of discourse.

Solution 9 (a)
True
Solution 9 (b)
False
Solution 9 (c)
False
Solution 9 (d)
True
Solution 9 (e)
True
Solution 9 (f)

True

2.2 Equivalences Involving Quantifiers

Problem 1

Negate these statements and then reexpress the results as equivalent positive statements. (See Example 2.2.1.)
(a) Everyone who is majoring in math has a friend who needs helps with his or her homework.
(b) Everyone has a roomate who dislikes everyone.
(c) $A \cup B \subseteq C \setminus D$
(d) $\exists x \forall x[y > x \to \exists z(z^2 + 5z = y)]$

Solution 1(a)
Let $M(x)$ mean "x is majoring in math".
Let $F(x, y)$ mean "x and y are friends".
Let $H(x)$ mean "x needs help with his or her homework".
Original statement means $\forall x(M(x) \to \exists y(F(x, y) \land H(y)))$.

$$\begin{aligned}
&\neg\forall x(M(x) \to \exists y(F(x, y) \land H(y)))\\
&\leftrightarrow \exists x\neg(M(x) \to \exists y(F(x, y) \land H(y)))\\
&\leftrightarrow \exists x\neg(\neg M(x) \lor \exists y(F(x, y) \land H(y)))\\
&\leftrightarrow \exists x(M(x) \land \neg\exists y(F(x, y) \land H(y)))\\
&\leftrightarrow \exists x(M(x) \land \forall y\neg(F(x, y) \land H(y)))\\
&\leftrightarrow \exists x(M(x) \land \forall y(\neg F(x, y) \lor \neg H(y)))\\
&\leftrightarrow \exists x(M(x) \land \forall y(F(x, y) \to \neg H(y)))
\end{aligned}$$

There is a math student for which all the students friends do not need help with their homework.
Solution 1(b)
Let $R(x, y)$ mean "x and y are roomates".
Let $D(x, y)$ mean "x dislikes y".
Original statement means $\forall x\exists y(R(x, y) \land \forall z D(y, z))$.

$$\begin{aligned}
&\neg\forall x\exists y(R(x, y) \land \forall z D(y, z))\\
&\leftrightarrow \exists x\neg\exists y(R(x, y) \land \forall z D(y, z))\\
&\leftrightarrow \exists x\forall y\neg(R(x, y) \land \forall z D(y, z))\\
&\leftrightarrow \exists x\forall y(\neg R(x, y) \lor \neg\forall z D(y, z))\\
&\leftrightarrow \exists x\forall y(\neg R(x, y) \lor \exists z\neg D(y, z))
\end{aligned}$$

There exists a student for which everyone is not his roomate or he has a roomate and that roomate likes someone.
Solution 1(c)
Original statement means $\forall x(x \in A \cup B \to C \setminus D)$.

$$\begin{aligned}
&\neg\forall x(x \in A \cup B \to x \in C \setminus D)\\
&\leftrightarrow \exists x\neg(\neg(x \in A \cup B) \lor x\ inC \setminus D)\\
&\leftrightarrow \exists x((x \in A \cup B) \land \neg(x \in C \setminus D))\\
&\leftrightarrow \exists x((x \in A \cup B) \land \neg(x \in C \land x \notin D))\\
&\leftrightarrow \exists x((x \in A \cup B) \land (x \notin C \lor x \in D))
\end{aligned}$$

There exists x in set A or B that is either not in C or is in D.
Solution 1(d)

$$
\begin{aligned}
&\neg\exists x\forall y[y > x \to \exists z(z^2 + 5z = y)]\\
&\leftrightarrow \forall x\neg\forall y[y > x \to \exists z(z^2 + 5z = y)]\\
&\leftrightarrow \forall x\exists y\neg[y > x \to \exists z(z^2 + 5z = y)]\\
&\leftrightarrow \forall x\exists y\neg[\neg(y > x) \vee \exists z(z^2 + 5z = y)]\\
&\leftrightarrow \forall x\exists y[y > x \wedge \neg\exists z(z^2 + 5z = y)]\\
&\leftrightarrow \forall x\exists y[y > x \wedge \exists z\neg(z^2 + 5z = y)]\\
&\leftrightarrow \forall x\exists y[y > x \wedge \exists z(z^2 + 5z \neq y)]
\end{aligned}
$$

For all x there exists y such that y is greater than x and there is a number z such that $z^2 + 5z$ is not equal to y.

Problem 2

Negate these statements and then reexpress the results as equivalent positive statements. (See Examples 2.2.1.)
(a) There is someone in the freshman class who doesn't have a room-mate.
(b) Everyone likes someone, but no ones likes everyone.
(c) $\forall a \in A \exists b \in B(a \in C \leftrightarrow b \in C)$.
(d) $\forall y > 0 \exists x(ax^2 + bx + c = y)$.

Solution 2(a)
Let $F(x)$ mean "x is a freshman".
Let $R(x)$ mean "x has a room-mate".
Original statement means $\exists x(F(x) \wedge \neg R(x))$.

$$
\begin{aligned}
&\neg\exists x(F(x) \wedge \neg R(x))\\
&\leftrightarrow \forall x\neg(F(x) \wedge \neg R(x))\\
&\leftrightarrow \forall x(\neg F(x) \vee R(x))
\end{aligned}
$$

For all people either they are not a freshmen or they have a roomate.
Solution 2(b)
Let $L(x, y)$ mean "x likes y".
Orignal statement means $\forall(x)(\exists y L(x, y) \wedge \exists z\neg L(x, z))$.

$$
\begin{aligned}
&\neg\forall x(\exists y L(x, y) \wedge \exists z\neg L(x, z))\\
&\leftrightarrow \exists x\neg(\exists y L(x, y) \wedge \exists z\neg L(x, z))\\
&\leftrightarrow \exists x(\neg\exists y L(x, y) \vee \neg\exists z\neg L(x, z))\\
&\leftrightarrow \exists x(\forall y\neg L(x, y) \vee \forall z L(x, z))
\end{aligned}
$$

There is someone who either everyone dislikes or everyone likes.
Solution 2(c)

$$
\begin{aligned}
&\neg\forall a \in A\exists b \in B(a \in C \leftrightarrow b \in C)\\
&\leftrightarrow \exists a \in A\neg\exists b \in B(a \in C \leftrightarrow b \in C)\\
&\leftrightarrow \exists a \in A\forall b \in B\neg(a \in C \leftrightarrow b \in C)\\
&\leftrightarrow \exists a \in A\forall b \in B\neg((a \in C \to b \in C) \wedge (b \in C \to a \in C))\\
&\leftrightarrow \exists a \in A\forall b \in B(\neg(a \in C \to b \in C) \vee \neg(b \in C \to a \in C))\\
&\leftrightarrow \exists a \in A\forall b \in B(\neg(\neg(a \in C) \vee b \in C) \vee \neg(\neg(b \in C) \vee a \in C))\\
&\leftrightarrow \exists a \in A\forall b \in B((a \in C) \wedge \neg(b \in C)) \vee ((b \in C) \wedge \neg(a \in C))\\
&\leftrightarrow \exists a \in A\forall b \in B(a \in C \wedge b \notin C) \vee (b \in C \wedge a \notin C)
\end{aligned}
$$

There exists a in set A for which all b in set B either a in set C and b is not in set C or a is not in C and b is in C.
Solution 2(d)

$$\begin{aligned}
&\neg\forall y > 0\exists x(ax^2 + bx + c = y)\\
&\leftrightarrow \exists y > 0\neg\exists x(ax^2 + bx + c = y)\\
&\leftrightarrow \exists y > 0\forall x\neg(ax^2 + bx + c = y)\\
&\leftrightarrow \exists y > 0\forall x(ax^2 + bx + c \neq y)
\end{aligned}$$

There exists a number y such that for all x, $ax^2 + bx + c$ is not equal to y.

Problem 3

Are these statements true or false? The universe of discourse is $\mathbb{N}$.
(a) $\forall x(x < 7 \rightarrow \exists a\exists b\exists c(a^2 + b^2 + c^2 = x))$.
(b) $\exists!x(x^2 + 3 = 4x)$.
(c) $\exists!x(x^2 = 4x + 5)$.
(d) $\exists x\exists y(x^2 = 4x + 5 \wedge y^2 = 4y + 5)$.

Solution 3(a)
True.
Solution 3(b)
False.
Solution 3(c)
True.
Solution 3(d)
True

Problem 4

Show that the second quantifier negation law, which says that $\neg\forall xP(x)$ is equivalent to $\exists x\neg P(x)$, can be derived from the first, which says that $\neg\exists xP(x)$ is equivalent to $\forall x\neg P(x)$. (Hint: Use the double negation law.)

Proof.

$$\begin{aligned}
\neg\forall xP(x) &\equiv \neg\forall x\neg(\neg P(x))\\
&\equiv \neg\neg\exists x(\neg P(x)) && \text{first quantifier negation law}\\
&\equiv \exists x\neg P(x)
\end{aligned}$$

∎

Problem 5

Show that $\neg\exists x \in AP(x)$ is equivalent to $\forall x \in A\neg P(x)$.

Proof.

$$\neg\exists x \in AP(x) = \forall x \in A\neg P(x) \qquad \text{second quantifier negation law}$$

∎

Problem 6

Show that the existential quantifier distributes over disjunction. In other words, show that $\exists x(P(x) \vee Q(x))$ is equivalent to $\exists x P(x) \vee \exists x Q(x)$. (Hint: Use the fact, discussed in this section, that the universal quantifier distributes over conjunction.)

Proof.

$$\begin{aligned}
\exists x(P(x) \vee Q(x)) &\equiv \neg\neg\exists x(P(x) \vee Q(x)) \\
&\equiv \neg\forall x \neg(P(x) \vee Q(x)) \\
&\equiv \neg\forall x(\neg P(x) \wedge \neg Q(x)) \\
&\equiv \neg(\forall x \neg P(x) \wedge \forall x \neg Q(x)) \\
&\equiv (\neg\forall x \neg P(x) \vee \neg\forall x \neg Q(x)) \\
&\equiv (\exists x \neg\neg P(x) \vee \exists x \neg\neg Q(x)) \\
&\equiv \exists x P(x) \vee \exists x Q(x)
\end{aligned}$$

∎

Problem 7

Show that $\exists x(P(x) \to Q(x))$ is equivalent to $\forall x P(x) \to \exists x Q(x)$.

Proof.

$$\begin{aligned}
\exists x(P(x) \to Q(x)) &\equiv \exists x(\neg P(x) \vee Q(x)) \\
&\equiv (\exists x \neg P(x) \vee \exists x Q(x)) && \text{from prob. 6} \\
&\equiv (\neg\forall x P(x) \vee \exists x Q(x)) \\
&\equiv \forall x P(x) \to \exists x Q(x)
\end{aligned}$$

∎

Problem 8

Show that $(\forall x \in A P(x)) \wedge (\forall x \in B P(x))$ is equivalent $\forall x \in (A \cup B) P(x)$. (Hint: Start by writing out the meanings of the bounded quantifiers in terms of unbounded quantifiers.)

Proof.

$$\begin{aligned}
(\forall x \in A P(x)) \wedge (\forall x \in B P(x)) &\equiv \forall x(x \in A \to P(x)) \wedge \forall x(x \in B \to P(x)) \\
&\equiv \forall x((x \in A \to P(x)) \wedge (x \in B \to P(x))) \\
&\equiv \forall x((\neg(x \in A) \vee P(x)) \wedge (\neg(x \in B) \vee P(x))) \\
&\equiv \forall x((\neg(x \in A) \wedge (\neg(x \in B)) \vee P(x))) \\
&\equiv \forall x(\neg(x \in A \vee x \in B) \vee P(x)) \\
&\equiv \forall x(x \in A \vee x \in B \to P(x)) \\
&\equiv \forall x(x \in A \vee B \to P(x)) \\
&\equiv \forall x(x \in A \vee B) P(x) \\
&\equiv \forall x \in (A \cup B) P(x)
\end{aligned}$$

∎

Problem 9

Is $\forall x(P(x) \vee Q(x))$ equivalent to $\forall x P(x) \vee \forall x Q(x)$? Explain. (Hint: Try assigning meanings to $P(x)$ and $Q(x)$.)

Solution No they are not equal. The first says that for every x, P or Q is true. In contrast the second statement means P is true for all x or Q is true for all x. If we let P mean "x goes to the store" and Q mean "x goes to the gym" then the first statement means for each person they go to the store or they go to the gym. The second statement means all people go the store or all people go to the gym.

Problem 10

(a) Show that $\exists x \in A P(x) \vee \exists x \in B P(x)$ is equivalent to $\exists x \in (A \cup B) P(x)$.
(b) Is $\exists x \in A P(x) \wedge \exists x \in B P(x)$ equivalent to $\exists x \in (A \cap B) P(x)$? Explain.

Proof.

$$\begin{aligned}
\exists x \in A P(x) \vee \exists x \in B P(x) &\equiv \exists x(x \in A P(x) \vee x \in B P(x)) && \text{prev prob} \\
&\equiv \exists x((x \in A \vee x \in B) \wedge P(x)) \\
&\equiv \exists x(x \in (A \cup B) \wedge P(x)) \\
&\equiv \exists x \in (A \cup B) P(x)
\end{aligned}$$

∎

Solution 10(b) False. The first statement means that there exists x in set A for which $P(x)$ is true and there exists another x possibly distinct from the former in set B for which $P(x)$ is true. The second statement says that there exists x in the set A and is also in the set B for which $P(x)$ is true.

Problem 11

Show that the statements $A \subseteq B$ and $A \setminus B = \varnothing$ are equivalent by writing each in logical symbols and then showing that the resulting formulas are equivalent.

Proof. Lhs is equavalent to $\forall x(x \in A \to x \in B)$
For the rhs:

$$\begin{aligned}
\neg \exists x(x \in A \wedge x \notin B) &\equiv \forall x \neg(x \in A \wedge x \notin B) \\
&\equiv \forall x(x \notin A \vee x \in B) \\
&\equiv \forall x(x \in A \to x \in B)
\end{aligned}$$

∎

Problem 12

Show that the statements $C \subseteq A \cup B$ and $C \setminus A \subseteq B$ are equivalent writing each in logical symbols and then showing that the resulting formulas are equivalent.

Proof. The lhs means $\forall x(x \in C \to x \in A \cup B)$.

For the rhs

$$\begin{aligned}
\forall x(x \in C \setminus A \to x \in B) &\equiv \forall x(\neg(x \in C \setminus A) \to x \in B)\\
&\equiv \forall x(\neg(x \in C \wedge x \notin A) \vee x \in B)\\
&\equiv \forall x((x \notin C \vee x \in A) \vee x \in B)\\
&\equiv \forall x(x \notin C \vee (x \in A \vee x \in B))\\
&\equiv \forall x(\neg(x \in C) \vee (x \in A \cup B))\\
&\equiv \forall x(x \in C \to x \in A \cup B)
\end{aligned}$$

∎

Problem 13

(a) Show that the statements $A \subseteq B$ and $A \cup B = B$ are equivalent by writing each in logical symbols and then showing that the resulting formulas are equivalent. (Hint: You may find excersize 11 from Section 1.5 useful.)

Proof. First note sec. 1 prob. 11 says $x \in P \vee x \in Q \leftrightarrow x \in Q \equiv x \in P \to x \in Q$.
The lhs means $\forall x(x \in A \to x \in B)$.
For the rhs

$$\forall x(x \in A \vee x \in B \leftrightarrow x \in B) \equiv \forall x(x \in A \to x \in B) \quad \text{from sec. 1, prob. 11}$$

∎

Problem 14

Show that the statements $A \cap B = \varnothing$ and $A \setminus B = A$ are equivalent.

Proof. The lhs means $\neg\exists x(x \in A \wedge x \in B)$
For the rhs

$$\begin{aligned}
\forall x(x \in (A \setminus B) \leftrightarrow x \in A) &\equiv \forall x((x \in (A \setminus B)) \to x \in A) \wedge (x \in A \to (x \in (A \setminus B)))\\
&\equiv \forall x(\neg(x \in (A \setminus B)) \vee x \in A) \wedge (\neg(x \in A) \vee (x \in (A \setminus B)))\\
&\equiv \forall x(\neg(x \in A \wedge x \notin B) \vee x \in A) \wedge (\neg(x \in A) \vee (x \in (A \setminus B)))\\
&\equiv \forall x(x \notin A \vee x \in B \vee x \in A) \wedge (\neg(x \in A) \vee (x \in (A \setminus B)))\\
&\equiv \forall x(\neg(x \in A) \vee (x \in (A \setminus B))) && \text{tautology}\\
&\equiv \forall x(x \notin A \vee (x \in (A \setminus B)))\\
&\equiv \forall x(x \notin A \vee (x \in A \wedge x \notin B))\\
&\equiv \forall x((x \notin A \vee x \in A) \wedge (x \notin A \vee x \notin B))\\
&\equiv \forall x(x \notin A \vee x \notin B) && \text{tautology}\\
&\equiv \forall x \neg(x \in A \wedge x \in B)\\
&\equiv \neg\exists x(x \in A \wedge x \in B)
\end{aligned}$$

∎

Problem 15

Let $T(x, y)$ mean "x is a teacher of y" What do the following statements mean? Under what circumstances would each one be true? Are any of them equivalent to each other?

(a) $\exists! y T(x,y)$.
(b) $\exists x \exists! y T(x,y)$.
(c) $\exists! x \exists y T(x,y)$.
(d) $\exists y \exists! x T(x,y)$.
(e) $\exists! x \exists! y T(x,y)$.
(f) $\exists x \exists y [T(x,y) \land \neg \exists u \exists v (T(u,v) \land (u \neq x \lor v \neq y))]$.

Solution 15 (a)
There is one student that x is a teacher of.
It is true when the teacher has only one student.
Solution 15 (b)
There is a teacher that has one student.
It is true if any teacher has one student.
Solution 15 (c)
There is one teacher that has a student.
It is true when there is a single teacher than has any students.
Solution 15 (d)
There is a student that has only one teacher.
It is true if any student has one teacher.
Solution 15 (e)
There is one student that has one teacher.
It is true if there is one student with one teacher.
Solution 15 (f)
There is a student and a teacher and there isn't another teacher with any other student other than the former.
It is true if there is a student and a teacher and there isn't another teacher with any other student other than the former.

None are equivalent.

2.3 More Operations On Sets

Problem 1

Analyze the logical forms of the following statements. You may use the symbols $\in$, $\notin$, $=$, $\neq$, $\land$, $\lor$, $\rightarrow$, $\leftrightarrow$, and $\exists$ in your answers, but not $\subseteq$, $\nsubseteq$, $\mathcal{P}$, $\cap$, $\cup$, $\setminus$, $\{$, $\}$, or $\neg$. (Thus you must write out the definitions of some theory notation, and you must use equivalences to get rid of any occurences of $\neg$).
(a) $\mathcal{F} \subseteq \mathcal{P}(A)$
(b) $A \subseteq \{2n+1 | n \in \mathbb{N}\}$
(c) $\{n^2+n+1 | n \in \mathbb{N}\} \subseteq \{2n+1 | n \in \mathbb{N}\}$
(d) $\mathcal{P}(\bigcup_{i \in I} A_i) \nsubseteq \bigcup_{i \in I} \mathcal{P}(A_i)$

Solution 1 (a)

$$\begin{aligned} \mathcal{F} \subseteq \mathcal{P}(A) &= \forall x (x \in \mathcal{F} \rightarrow x \in \mathcal{P}(A)) \\ &= \forall x (x \in \mathcal{F} \rightarrow \forall y (y \in x \rightarrow y \in A)) \end{aligned}$$

Solution 1 (b)

$$\begin{aligned} A \subseteq \{2n+1 | n \in \mathbb{N}\} &= \forall x (x \in A \rightarrow x \in \{2n+1 | n \in \mathbb{N}\}) \\ &= \forall x (x \in A \rightarrow \exists n \in \mathbb{N} (x = 2n+1)) \end{aligned}$$

Solution 1 (c)

$$\begin{aligned}\{n^2+n+1|n\in\mathbb{N}\}\subseteq\{2n+1|n\in\mathbb{N}\} &= \forall x(x\in\{n^2+n+1|n\in\mathbb{N}\}\to x\in\{2n+1|n\in\mathbb{N}\})\\ &= \forall x(\exists n\in\mathbb{N}(x=n^2+n+1)\to\exists n\in\mathbb{N}(x=2n+1))\end{aligned}$$

Solution 1 (d)

$$\begin{aligned}\mathcal{P}\left(\bigcup_{i\in I}A_i\right)\nsubseteq\bigcup_{i\in I}\mathcal{P}(A_i) &= \exists x\left(x\in\mathcal{P}\left(\bigcup_{i\in I}A_i\right)\wedge\neg\left(x\in\bigcup_{i\in I}\mathcal{P}(A_i)\right)\right)\\ &= \exists x\left(\forall y\left(y\in x\to y\in\mathcal{P}\left(\bigcup_{i\in I}A_i\right)\right)\wedge\neg\left(x\in\bigcup_{i\in I}\mathcal{P}(A_i)\right)\right)\\ &= \exists x\left((\forall y(y\in x\to\exists i\in I(y\in A_i)))\wedge\neg\left(\exists i\in I(x\in\mathcal{P}(A_i))\right)\right)\\ &= \exists x\left((\forall y(y\in x\to\exists i\in I(y\in A_i)))\wedge\neg\left(\exists i\in I(\forall y(y\in x\to y\in A_i))\right)\right)\\ &= \exists x\left((\forall y(y\in x\to\exists i\in I(y\in A_i)))\wedge\left(\forall i\in I(\exists y(y\in x\wedge y\notin A_i))\right)\right)\end{aligned}$$

Problem 2

Analyze the logical forms of the following statements. You may use the symbols $\in$, $\notin$, $=$, $\neq$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$, and $\exists$ in your answers, but not $\subseteq$, $\nsubseteq$, $\mathcal{P}$, $\cap$, $\cup$, $\setminus$, $\{$, $\}$, or $\neg$. (Thus you must write out the definitions of some theory notation, and you must use equivalences to get rid of any occurences of $\neg$).

(a) $x\in\bigcup\mathcal{F}\setminus\bigcup\mathcal{G}$

(b) $\{x\in B|x\notin C\}\in\mathcal{P}(A)$

(c) $x\in\bigcap_{i\in I}(A_i\cup B_i)$

(d) $x\in(\bigcap_{i\in I}A_i)\cup(\bigcap_{i\in I}B_i)$

Solution 2(a)

$$\begin{aligned}x\in\bigcup\mathcal{F}\setminus\bigcup\mathcal{G} &\equiv x\in\bigcup\mathcal{F}\wedge\neg\left(x\in\bigcup\mathcal{G}\right)\\ &\equiv\exists A(A\in F\wedge x\in A)\wedge\neg(\exists A(A\in G\wedge x\in A))\\ &\equiv\exists A(A\in F\wedge x\in A)\wedge\forall A\neg(A\in G\wedge x\in A)\\ &\equiv\exists A(A\in F\wedge x\in A)\wedge\forall A(A\notin G\vee x\notin A)\end{aligned}$$

Solution 2(b)

$$\begin{aligned}\{x\in\mathcal{B}|x\notin C\}\in\mathcal{P}(A) &\equiv\forall y(y\in\{x\in\mathcal{B}|x\notin C\}\to y\in A)\\ &\equiv\forall y((y\in\mathcal{B}\wedge\notin C)\to y\in A)\end{aligned}$$

Solution 2(c)

$$\begin{aligned}x\in\bigcap_{i\in I}(A_i\cup B_i) &\equiv\forall i\in I(x\in(A_i\cup B_i))\\ &\equiv\forall i\in I(x\in A_i\vee x\in B_i)\end{aligned}$$

Solution 2(d)

$$
\begin{aligned}
x \in \left(\bigcap_{i\in I} A_i\right) \cup \left(\bigcap_{i\in I} B_i\right) &\equiv x \in \left(\bigcap_{i\in I} A_i\right) \lor x \in \left(\bigcap_{i\in I} B_i\right) \\
&\equiv \forall i \in I (x \in A_i) \lor \forall i \in I (x \in B_i)
\end{aligned}
$$

Problem 3

We've seen that $\mathcal{P}(\varnothing) = \varnothing$, and $\{\varnothing\} \neq \varnothing$. What is $\mathcal{P}(\{\varnothing\})$?

Solution $\mathcal{P}(\{\varnothing\}) \equiv \{\varnothing, \{\varnothing\}\}$

Problem 6

Let $I = \{2, 3, 4, 5\}$, and for each $i \in I$ let $A_i = \{i, i+1, i-1, 2i\}$.
(a) List the elements of all the sets A_i, for each $i \in I$.
(b) Find $\bigcap_{i\in I} A_i$ and $\bigcup_{i\in I} A_i$.

Solution 6 (a)

$$
\begin{aligned}
A_2 &= \{2,\ 3,\ 1,\ 4\} \\
A_3 &= \{3,\ 4,\ 2,\ 6\} \\
A_4 &= \{4,\ 5,\ 3,\ 8\} \\
A_5 &= \{5,\ 6,\ 4,\ 10\}
\end{aligned}
$$

Solution 6 (b)

$$
\begin{aligned}
A_2 \cap A_3 &= \{2,\ 3,\ 4\} \\
A_4 \cap A_5 &= \{4,\ 5\} \\
\bigcap_{i\in I} A_i &= \{2,\ 3,\ 4\} \cap \{4,\ 5\} = \{4\}
\end{aligned}
$$

$$
\begin{aligned}
A_2 \cup A_3 &= \{1,\ 2,\ 3,\ 4,\ 6\} \\
A_4 \cup A_5 &= \{3,\ 4,\ 5,\ 6,\ 8,\ 10\} \\
\bigcup_{i\in I} A_i &= \{1,\ 2,\ 3,\ 4,\ 6\} \cup \{3,\ 4,\ 5,\ 6,\ 8,\ 10\} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10\}
\end{aligned}
$$

Problem 8

Let $I = \{2, 3\}$, and for each $i \in I$ let $A_i = \{i, 2i\}$ and $B_i = \{i, i+1\}$.
(a) List the elements of the sets A_i and B_i for $i \in I$.
(b) Find $\bigcap_{i\in I}(A_i \cup B_i)$ and $(\bigcap_{i\in I} A_i) \cup (\bigcap_{i\in I} B_i)$.
(c) In parts (c) and (d) of excersize 2 you analyzed the statements $x \in \bigcap_{i\in I}(A_i \cup B_i)$ and $x \in (\bigcap_{i\in I} A_i) \cup (\bigcap_{i\in I} B_i)$. What can you conclude from your answer to part (b) about whether or not these statements are equivalent?

Solution 8 (a)
$A_2 = \{2, 4\}$ and $A_3 = \{3, 6\}$
$B_2 = \{2, 3\}$ and $B_3 = \{3, 4\}$

Solution 8 (b)
Let $i = 2$, then $A_2 \cup B_2 = \{2,3,4\}$
Let $i = 3$, then $A_3 \cup B_3 = \{3,4,6\}$

$$\bigcap_{i \in I}(A_i \cup B_i) = \{2,3,4\} \cap \{3,4,6\} = \{3,4\}$$

$\bigcap_{i \in I} A_i = A_2 \cap A_3 = \{\}$
$\bigcap_{i \in I} B_i = B_2 \cap B_3 = \{3\}$

$$\left(\bigcap_{i \in I} A_i\right) \cup \left(\bigcap_{i \in I} B_i\right) = \{\} \cup \{3\} = \{3\}$$

Solution 8 (c)
They are not equivalent.

Problem 9

(a) Analyze the logical forms of the statements $x \in \bigcup_{i \in I}(A_i \setminus B_i)$, $x \in (\bigcup_{i \in I} A_i) \setminus (\bigcup_{i \in I} B_i)$, and $x \in (\bigcup_{i \in I} A_i) \setminus (\bigcap_{i \in I} B_i)$. Do you think that any of these statements are equivalent to each other?
(b) Let I, A_i, and B_i be defined as in excersize 8. Find $\bigcup_{i \in I}(A_i \setminus B_i)$, $(\bigcup_{i \in I} A_i) \setminus (\bigcup_{i \in I} B_i)$, and $(\bigcup_{i \in I} A_i) \setminus (\bigcap_{i \in I} B_i)$. Now do you think any of the statements in part (a) are equivalent?

Solution 9 (a)

$$\begin{aligned} x \in \left(\bigcup_{i \in I}(A_i \setminus B_i)\right) &\equiv \exists i \in I(x \in (A_i \setminus B_i)) \\ &\equiv \exists i \in I(x \in A_i \wedge x \notin B_i) \end{aligned}$$

$$\begin{aligned} x\ in \left(\bigcup_{i \in I} A_i\right) \setminus \left(\bigcup_{i \in I} B_i\right) &\equiv x \in \left(\bigcup_{i \in I} A_i\right) \wedge \neg\left(x \in \left(\bigcup_{i \in I} B_i\right)\right) \\ &\equiv \exists i \in I(x \in A_i) \wedge \neg(\exists i \in I(x \in B_i)) \\ &\equiv \exists i \in I(x \in A_i) \wedge \forall i \in I(x \notin B_i) \end{aligned}$$

$$\begin{aligned} x \in \left(\bigcup_{i \in I} A_i\right) \setminus \left(\bigcap_{i \in I} B_i\right) &\equiv x \in \left(\bigcup_{i \in I} A_i\right) \wedge \neg\left(x \in \left(\bigcap_{i \in I} B_i\right)\right) \\ &\equiv \exists i \in I(x \in A_i) \wedge \neg(\forall i \in I(x \in B_i)) \\ &\equiv \exists i \in I(x \in A_i) \wedge (\exists i \in I(x \notin B_i)) \end{aligned}$$

I do not think any of the statements are equivalent.

Solution 9 (b) From prob. 8
$I = \{2,3\}$
$A_2 = \{2,4\}$ and $A_3 = \{3,6\}$
$B_2 = \{2,3\}$ and $B_3 = \{3,4\}$

$$\bigcup_{i \in I}(A_i \setminus B_i) = \{4\} \cup \{6\} = \{4,6\}$$

$$\left(\bigcup_{i \in I} A_i\right) \setminus \left(\bigcup_{i \in I} B_i\right) = \{2,3,4,6\} \setminus \{2,3,4\} = \{6\}$$

$$\left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcap_{i\in I} B_i\right) = \{2,3,4,6\} \setminus \{3\} = \{2,4,6\}$$

Still not equal.

Problem 10

Give an example of an index set I and indexed families of sets $\{A_I | i \in I\}$ and $\{B_i | i \in I\}$ such that $\bigcup_{i\in I}(A_i \cap B_i) \neq (\bigcup_{i\in I} A_i) \cup (\bigcup_{i\in I} B_i)$.

Solution 10 $I = \{1,2\}$
$A_1 = \{1\}$
$A_2 = \{2\}$
$B_1 = \{1\}$
$B_2 = \varnothing$

$$\bigcup_{i\in I}(A_i \cap B_i) = \{1\} \cap \varnothing = \varnothing$$

$$(U_{i\in I} A_i) = \{1,2\}$$

$$(U_{i\in I} B_i) = \{1\}$$

$$(U_{i\in I} A_i) \cup (U_{i\in I} B_i) = \{1,2\}$$

$$\therefore \bigcup_{i\in I}(A_i \cap B_i) \neq (U_{i\in I} A_i) \cup (U_{i\in I} B_i)$$

Problem 11

Show that for any sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$, by showing that the statements $x \in \mathcal{P}(A \cap B)$ and $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$ are equivalent. (See Example 2.3.3)

Proof.

$$\begin{aligned} x \in \mathcal{P}(A \cap B) &\equiv \forall y(y \in x \rightarrow y \in A \cap B) \\ &\equiv \forall y(y \in x \rightarrow (y \in A \wedge y \in B)) \end{aligned}$$

$$\begin{aligned} x \in \mathcal{P}(A) \cap \mathcal{P}(B) &\equiv \forall y(y \in x \rightarrow y \in A) \wedge \forall y(y \in x \rightarrow y \in B) \\ &\equiv \forall y((\neg(y \in x) \vee y \in A) \wedge (\neg(y \in x) \vee y \in B)) \\ &\equiv \forall y(\neg(y \in x) \vee (y \in A \wedge y \in B)) \\ &\equiv \forall y(y \in x \rightarrow (y \in A \wedge y \in B)) \end{aligned}$$

$$\therefore \mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A) \cap \mathcal{P}(B)$$

∎

Problem 12

Give examples of sets A and B for which $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Solution 12 $A = \{1,2\}$
$B = \{1,3\}$
$A \cup B = \{1,2,3\}$

$$\mathcal{P}(A \cup B) = \{\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}, \varnothing\}$$

$$\mathcal{P}(A) = \{\{1,2\},\{1\},\{2\},\varnothing\}$$
$$\mathcal{P}(B) = \{\{1,3\},\{1\},\{3\},\varnothing\}$$
$$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\{1,2\},\{1,3\},\{2\},\{3\},\{1\},\varnothing\}$$
$$\therefore \mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$$

Problem 13

Verify the following identities by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.
(a) $\bigcup_{i\in I}(A_i \cup B_i) = (\bigcup_{i\in I} A_i) \cup (\bigcup_{i\in I} B_i)$
(b) $(\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G}) = \bigcap(\mathcal{F} \cup \mathcal{G})$
(c) $\bigcap_{i\in I}(A_i \setminus B_i) = (\bigcap_{i\in I} A_i) \setminus (\bigcup_{i\in I} B_i)$

Solution 13 (a)

$$\begin{aligned} x \in \bigcup_{i\in I}(A_i \cup B_i) &\equiv \exists i \in I(x \in (A_i \vee B_i)) \\ &\equiv \exists i \in I(x \in A_i \vee x \in B_i) \end{aligned}$$

$$\begin{aligned} x \in \left(\bigcup_{i\in I} A_i\right) \cup \left(\bigcup_{i\in I} B_i\right) &\equiv x \in \left(\bigcup_{i\in I} A_i\right) \vee x \in \left(\bigcup_{i\in I} B_i\right) \\ &\equiv \exists i \in I(x \in A_i) \vee \exists i \in I(x \in B_i) \\ &\equiv \exists i \in I(x \in A_i \vee x \in B_i) \end{aligned}$$

$$\therefore \bigcup_{i\in I}(A_i \cup B_i) = \left(\bigcup_{i\in I} A_i\right) \cup \left(\bigcup_{i\in I} B_i\right)$$

Solution 13 (b)

$$\begin{aligned} x \in \left(\bigcap \mathcal{F}\right) \cap \left(\bigcap \mathcal{G}\right) &\equiv x \in \left(\bigcap \mathcal{F}\right) \wedge x \in \left(\bigcap \mathcal{G}\right) \\ &\equiv \forall A \in F(x \in A) \wedge \forall A \in G(x \in G) \\ &\equiv \forall A \in (F \cup G)(x \in A) \quad \text{prob. 8 prev. sec.} \end{aligned}$$

$$x \in \bigcap(\mathcal{F} \cup \mathcal{G}) \equiv \forall A \in (F \cup G)(x \in A)$$
$$\therefore \left(\bigcap \mathcal{F}\right) \cap \left(\bigcap \mathcal{G}\right) = \bigcap(\mathcal{F} \cup \mathcal{G})$$

Solution 13 (c)

$$\begin{aligned} x \in \bigcap_{i\in I}(A_i \setminus B_i) &\equiv \forall i \in I(x \in (A_i \setminus B_i)) \\ &\equiv \forall i \in I(x \in A_i \wedge x \notin B_i) \end{aligned}$$

$$\begin{aligned} x \in \left(\bigcap_{i\in I} A_i\right) \setminus \left(\bigcup_{i\in I} B_i\right) &\equiv x \in \left(\bigcap_{i\in I} A_i\right) \wedge \neg(x \in \left(\bigcup_{i\in I} B_i\right)) \\ &\equiv \forall i \in I(x \in A_i) \wedge \neg(\exists i \in I(x \in B_i)) \\ &\equiv \forall i \in I(x \in A_i) \wedge \forall i \in I(x \notin B_i) \\ &\equiv \forall i \in I(x \in A_i \wedge x \notin B_i) \end{aligned}$$

$$\therefore \bigcap_{i\in I}(A_i \setminus B_i) = \left(\bigcap_{i\in I} A_i\right) \setminus \left(\bigcup_{i\in I} B_i\right)$$

Problem 14

Sometimes each set in an indexed family of sets has two indices. For this problem, use the following definitions: $I = \{1,2\}$, $J = \{3,4\}$. For each $i \in I$ and $j \in J$, let $A_{i,j} = \{i, j, i+j\}$. Thus, for example, $A_{2,3} = \{2,3,5\}$.
(a) For each $j \in J$, let $B_j = \bigcup_{i\in I} A_{i,j} = A_{1,j} \cup A_{2,j}$. Find B_3 and B_4.
(b) Find $\bigcap_{j\in J} B_j$. (Note that, replacing B_j with its definition, we could say that $\bigcap_{j\in J} B_j = \bigcap_{j\in J}(\bigcup_{i\in I} A_{i,j})$.)
(c) Find $\bigcup_{i\in I}(\bigcap_{j\in j} A_{i,j})$. (Hint: you may want to do this in two steps, coresponding to parts (a) and (b).) Are $\bigcap_{j\in J}(\bigcup_{i\in I} A_{i,j})$ and $\bigcup_{i\in I}(\bigcap_{j\in J})A_{i,j}$ equal?
(d) Analyze the logical forms of the statements $x \in \bigcap_{j\in J}(\bigcup_{i\in I} A_{i,j})$ and $x \in \bigcup_{i\in I}(\bigcap_{j\in J} A_{i,j})$. Are they equivalent?

Solution 14 (a)
$B_j = \bigcup_{i\in I} A_{i,j} = A_{1,j} \cup A_{2,j}$
$B_3 = A_{1,3} \cup A_{2,3}$
$A_{1,3} = \{1,3,4\}$ and $A_{2,3} = \{2,3,5\}$
$B_3 = \{1,2,3,4,5\}$

$B_3 = A_{1,4} \cup A_{2,4}$
$A_{1,4} = \{1,4,5\}$ and $A_{2,4} = \{2,4,6\}$
$B_4 = \{1,2,4,5,6\}$

Solution 14 (b)
$\bigcap_{j\in J} B_j = \bigcap_{j\in J}(\bigcup_{i\in I} A_{i,j})$
$U_{i\in I} = A_{1,j} \cup A_{2,j}$
$\bigcap_{j\in J}(A_{1,j} \cup A_{2,j}) = (A_{1,3} \cup A_{2,3}) \cap (A_{1,4} \cup A_{2,4})$
$B_3 \cap B_4 = \{1,2,4,5\}$

Solution 14 (c)
$\bigcup_{i\in I}(\bigcap_{j\in j} A_{i,j}) = \bigcup_{i\in I}(A_{i,3} \cap A_{i,4})$
$\bigcup_{i\in I}(A_{i,3} \cap A_{i,4}) = (A_{1,3} \cap A_{1,4}) \cup (A_{2,3} \cap A_{2,4})$
$(A_{1,3} \cap A_{1,4}) \cup (A_{2,3} \cap A_{2,4}) = \{1\} \cup \{2\} = \{1,2\}$

Solution 14 (d)

$$\begin{aligned} x \in \bigcap_{j\in J}\left(\bigcup_{i\in I} A_{i,j}\right) &\equiv \forall j \in J\left(x \in \left(\bigcup_{i\in I} A_{i,j}\right)\right) \\ &\equiv \forall j \in J(\exists i \in I(x \in A_{i,j})) \end{aligned}$$

$$\begin{aligned} x \in \bigcup_{i\in I}\left(\bigcap_{j\in J} A_{i,j}\right) &\equiv \exists i \in I\left(x \in \left(\bigcap_{j\in J} A_{i,j}\right)\right) \\ &\equiv \exists i \in I(\forall j \in J(x \in A_{i,j})) \end{aligned}$$

$$\bigcap_{j\in J}\left(\bigcup_{i\in I} A_{i,j}\right) \neq \bigcup_{i\in I}\left(\bigcap_{j\in J} A_{i,j}\right)$$

Problem 15

(a) Show that if $\mathcal{F} = \varnothing$, then the statement $x \in \bigcup \mathcal{F}$ will be false no matter what x is. It follows that $\bigcup \varnothing = \varnothing$.
(b) Show that if $\mathcal{F} = \varnothing$, then the statement $x \in \bigcap \mathcal{F}$ will be true no matter what x is. In a context in which it is clear what the universe of discourse U is, we might therefore want to say that $\bigcap \varnothing = U$. However, this has the unfortunate consequence that the notation $\bigcap \varnothing$ will mean different things in different contexts. Furthermore, when working with sets whose elements are sets, mathematicians often do not use a universe of discourse at all. (For more on this, see the next excersize). For these reasons, some mathematicians consider the notation $\bigcap \varnothing$ to be meaningless. We will avoid this problem in this book by using the notation $\bigcap \mathcal{F}$ only in contexts in which we can be sure that $\mathcal{F} \neq 0$.

Proof. Suppose for contradiction $x \in \bigcup F$ is true when $F = \varnothing$. Thus $\exists A \in F(x \in A)$, but for this to be true there must be at least one set in F but $F = \varnothing$.
$\therefore$ If $F = \varnothing$ then $x \in \bigcup F$ is always false. ∎

Proof. Suppose for contradiction $x \in \bigcap F$ is false when $F = \varnothing$. Thus $\forall A \in F(x \in A)$, but for this to be false there must be a case in which x is not in a set within F but $F = \varnothing$.
$\therefore$ If $F = \varnothing$ then $x \in \bigcap F$ is always true. ∎

Problem 16

In Section 2.3 we saw that a set can have other sets as elements. When discussing sets whose elements are sets, it might seem most natural to consider the universe of discourse to be the set of all sets. However, we will see in this problem, assuming there is such a set leads to contradictions.

Suppose U were the set of all sets. Note that in particular U is a set, so we should have $U \in U$. This is not yet a contradictions; although most sets are not elements of themselves, perhaps some sets are elements of themselves, But it suggests that the sets in the universe U could be split into two categories: the unusual sets that, like U itself, are elements of themselves, and the more typical sets that are not. Let R be the set of sets in the second category. In other words, $R = \{A \in U | A \notin A\}$. This means for that for any set A in the universe of U, A will be an element R iff $A \notin A$. In other words, we have $\forall A \in U(A \in R \leftrightarrow A \notin A)$.
(a) Show that applying this last fact to the set R itself (in other words plugging in R for A) leads to a contradiction. This contradiction was discovered by Bertrand Russell (1872 - 1970) in 1901, and is known as *Russel's paradox*.
(b) Think some more about the paradox in part (a). What do you think it tells us about sets?

Proof. Suppose for contradiction $\forall A \in U(A \in R \leftrightarrow A \notin A)$. Then setting R as A

$$\begin{aligned} R \in R \leftrightarrow R \notin R &\equiv (R \in R \rightarrow R \notin R) \land (R \notin R \rightarrow R \in R) \\ &\equiv (R \notin R \lor R \notin R) \land (R \in R \lor R \in R) \\ &\equiv (R \notin R) \land (R \in R) \end{aligned}$$

Thus we arrive at a contradiction. $\therefore$ You cannot construct $R = \{A \in U | A \notin A\}$. ∎

Solution 16 (b) With our current axioms not all set constuctions are valid. At the least not all self-referential set constructions are valid because as we have seen it may lead to a contradiction.

3 Proofs

3.1 Proof Strategies

Problem 2

Consider the following theorem. (The theorem is correct, but we will not ask you to prove it here.)

Theorem. Suppose that $b^2 > 4ac$. Then the quadratic equation $ax^2 + bx + c = 0$ has exactly two real solutions.

(a) Identify the hypothesis and conclusion of the theorem.
(b) To give an instance of the theorem, you must specify values for a, b, and c, but not x. Why?
(c) What can you conclude from the theorem in the case $a = 2$, $b = -5$, $c = 3$? Check directly that this conclusion is correct.
(d) What can you conclude from the theorem in the case $a = 2$, $b = 4$, $c = 3$? Check directly that this conclusion is correct.

Solution 1 (a)
Hypothesis: $b^2 > 4ac$ and a, b, c are real numbers.
Conclusion: $ax^2 + bx + c = 0$ has two real solutions.

Solution 1 (b)
a, b, and c are free and x is bound.

Solution 1 (c)
If $a = 2$, $b = -5$, and $c = 3$
Since $(-5)^2 > 4(2)(3) \leftrightarrow 25 > 24$ is true
$2x^2 + (-5)x + 3 = 0$ has exactly two real roots.

$$\begin{align} 2x^2 + (-5)x + 3 &= 0 \tag{1}\\ \leftrightarrow 2x^2 - 3x - 2x + 3 &= 0 \tag{2}\\ \leftrightarrow 2x(x-1) - 3(x-1) &= 0 \tag{3}\\ \leftrightarrow (2x-3)(x-1) &= 0 \tag{4}\end{align}$$

Therefore $x = 1$ and $x = \frac{3}{2}$.

Solution 1 (d)
If $a = 2$, $b = 4$, and $c = 3$
Since $4^2 > 4(2)(3) \leftrightarrow 16 > 24$ is false
$2x^2 + 4x + 3$ may or may not have exactly two real roots.

Problem 3

Consider the following theorem.

Theorem. Suppose n is a natural number larger than 2, and n is not a prime number. Then $2n + 13$ is not a prime number.

What are the hypotheses and conclusion of this theorem? Show that the theorem theorem is incorrect by finding a counterexample.

Solution Hypotheses

1. $n \in \mathbb{N}$
2. $n > 2$
3. n is not a prime number

Conclusion: $2n + 13$ is not a prime number.
Counterexample: Let $n = 9$, $2(9) + 13 = 31$ which is prime. Therefore the theorem is incorrect.

Problem 4

Complete the following alternative proof of the theorem in Example 3.1.2.

Proof. Suppose $0 < a < b$. Then $b - a > 0$.
(Fill in a proof of $b^2 - a^2 > 0$ here)
Since $b^2 - a^2 > 0$, it follows that $a^2 < b^2$. Therefore, if $0 < a < b$ then $a^2 < b^2$. ∎

Proof. Suppose $0 < a < b$. Then $b - a > 0$.
Since $b - a > 0$ we can multiply each side whithout flipping the inequality by $b + a$ which is a positive quantity because a and b are greater than 0. We then get $(b + a)(b - a) > (b + a)0$ and $b^2 - a^2 > 0$.
Since $b^2 - a^2 > 0$, it follows that $a^2 < b^2$. Therefore, if $0 < a < b$ then $a^2 < b^2$. ∎

Problem 5

Suppose a and b are real numbers. Prove that if $a < b < 0$ then $a^2 > b^2$.

Proof. Since $a < b$, then $a - b < 0$. Note that $a < 0$ and $b < 0$ so $a + b < 0$. Then $(a + b)(a - b) > (a + b)0$ and $a^2 - b^2 > 0$. It follows that $a^2 > b^2$. Therefore supposing a and b are real numbers, if $a < b < 0$ then $a^2 > b^2$. ∎

Problem 6

Suppose a and b are real numbers. Prove that if $a < b < 0$ then $\frac{1}{b} < \frac{1}{a}$.

Proof. Since $a < b$, then $a - b < 0$. Note that $a < 0$ and $b < 0$ therefore $ab > 0$. Then

$$\begin{aligned} &\frac{a-b}{ab} < \frac{0}{ab} \\ \leftrightarrow\ &\frac{a}{ab} - \frac{b}{ab} < 0 \\ \leftrightarrow\ &\frac{1}{b} - \frac{1}{a} < 0 \end{aligned}$$

It then follows that $\frac{1}{b} < \frac{1}{a}$. Therefore, supposing a and b are real numbers, if $a < b < 0$ then $\frac{1}{b} < \frac{1}{a}$. ∎

Problem 7

Suppose a is a real number. Prove that if $a^3 > a$ then $a^5 > a$.

Proof. Suppose $a^3 > a$. Then $a^3 - a > 0$. Note that $a^2 + 1 > 0$. Then

$$\begin{aligned} &a^3 - a > 0 \\ \leftrightarrow\ &a^3 - a(a^2 + 1) > (a^2 + 1)0 \\ \leftrightarrow\ &a^5 - a > 0 \end{aligned}$$

It then follows that $a^5 > a$. Therefore, supposing a is a real number if $a^3 > a$ then $a^5 > a$. ∎

Problem 8

Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

Proof. Suppose $x \notin D$. By contrapositive if $x \notin C$ or $x \notin D$ then $x \notin A$ or $x \in B$. Since $x \notin D$, either $x \notin A$ or $x \in B$, but $x \in A$ thus $x \in B$. Therefore supposing $A \setminus B \subseteq C \cap D$ and $x \in A$, if $x \notin D$ then $x \in B$. ∎

Problem 9

Suppose $A \cap C \subseteq C \setminus D$. Prove that if $x \in A$, then if $x \in D$ then $x \notin B$.

Proof. Suppose $x \in A$ and $x \in D$. By the contrapositive if $x \notin C$ or $x \in D$ then $x \notin A$ or $x \notin B$. Since $x \in D$, $x \notin A$ or $x \notin B$ but $x \in A$ thus $x \notin B$. Therefore, supposing $A \cap C \subseteq C \setminus D$, if $x \in A$, then if $x \in D$ then $x \notin B$. ∎

Problem 11

Suppose x is a real number and $x \neq 0$. Prove that if $(\sqrt[3]{(x)} + 5)/(x^2 + 6) = 1/x$ then $x \neq 8$.

Proof. By contrapositive if $x = 8$ then $(\sqrt[3]{(x)} + 5)/(x^2 + 6) \neq 1/x$. Suppose $x = 8$, $(\sqrt[3]{(8)} + 5)/(8^2 + 6) \neq 1/8$ if and only if $7/72 \neq 1/8$. Therefore, supposing x is a real number, if $(\sqrt[3]{(x)} + 5)/(x^2 + 6) = 1/x$ then $x \neq 8$. ∎

Problem 12

Suppose a, b, c, and d are real numbers $0 < a < b$, and $d > 0$. Prove that if $ac \geq bd$ then $c > d$.

Proof. Suppose $ac \geq bd$. Note that $d > 0$ and $a > 0$. Since $a < b$, $ad < bd$ and $ac \geq bd > ad$. It follows that $ac > ad$ and $c > d$. Therefore supposing a, b, c, and d are real numbers $0 < a < b$, and $d > 0$, if $ac \geq bd$ then $c > d$. ∎

Problem 14

Suppose that x and y are real numbers. Prove that if $x^2 + y = -3$ and $2x - y = 2$ then $x = -1$.

Proof. Assume $x^2 + y = -3$ and $2x - y = 2$. Then $y = 2x - 2$ and $x^2 + 2x - 2 = -3$ and finally $x^2 + 2x + 1 = 0$ which has two factors both of which are $x + 1 = 0$ therefore $x = -1$. Therefore supposing x and y are real numbers, if $x^2 + y = -3$ and $2x - y = 2$ then $x = -1$. ∎

Problem 15

Prove the first theorem in Example 3.1.1. (Hint: You might find it useful to apply the theorem from Example 3.1.2).

Theorem 3.1.1 Suppose $x > 3$ and $y < 2$. Then $x^2 - 2y > 5$.

Proof. Since $x > 3$ then $x^2 > 9$ by theorem 3.1.2. Also $y < 2$ then $2y < 4$ and $0 < 4 - 2y$. It follows that $x^2 > 9 > 4 > 2y$. Subtracting $2y$ and 5 gives $x^2 - 2y - 5 > 4 - 2y > -1 - 2y > -5$. Since $4 - 2y > 0$, $x^2 - 2y - 5 > 0$ and $x^2 - 2y > 5$. Therefore supposing $x > 3$ and $y < 2$. Then $x^2 - 2y > 5$. ∎

Problem 16

Consider the following theorem.

Theorem. Suppose x is a real number and $x \neq 4$. If $(2x-5)/(x-4) = 3$ then $x = 7$.

Proof. Suppose $x = 7$. Then $(2x-5)/(x-4) = (2(7)-5)/(7-4) = 9/3 = 3$. Therefore if $(2x-5)(x-4) = 3$ then $x = 7$. ∎

Solution 16 (a)
You cannot assume that the consequent of the conclusion is true.

Proof. Suppose $(2x-5)/(x-4) = 3$; it follows that $2x-5 = 3x-12$ and $-5 = x-12$. Thus $x = 7$. Therefore supposing x is a real number and $x \neq 4$. If $(2x-5)/(x-4) = 3$ then $x = 7$. ∎

Problem 17

Consider the following incorrect theorem:

Incorrect Theorem. Suppose that x and y are real numbers and $x \neq 3$. If $x^2y = 9y$ then $y = 0$.

(a) What's wrong with the following proof of the theorem.
Proof. Suppose that $x^2y = 9y$. Then $x^2 - y = 0$. Since $x \neq 3$, $x^2 \neq 9$, so $x^2 - 9 \neq 0$. Therefore we can divide both sides of the equation $(x^2-9)y = 0$ by $x^2 - 9$, which leads to the conclusion that $y = 0$. Thus, if $x^2y = 9y$ then $y = 0$.
(b) Show that the theorem is incorrect by finding a counterexample.

Solution 17 (a)
$x = -3$ is also a solution to $x^2 = 9$.

Solution 17 (b)
Suppose $x = -3$ and $y = 1$. Then $(-3)^2(1) = 9(1)$ which gives us $9 = 9$ which is true. So supposing x and y are real numbers if $x^2y = 9y$ then $y = 0$ is a false statement.

3.2 Proofs Involving Negations and Conditionals

Problem 1

This problem could be solved by using truth tables, but don't do it that way. Instead, use the methods for writing proofs discussed so far in this chapter. (See Example 3.2.4.)
(a) Suppose $P \to Q$ and $Q \to R$ are both true. Prove that $P \to R$ is true.
(b) Suppose $\neg R \to (P \to \neg Q)$ is true. Prove that $P \to (Q \to R)$ is true.

Proof. Suppose P therefore Q and since Q then R. Therefore, supposing $P \to Q$ and $Q \to R$ is true, $P \to R$ is true. Therefore, supposing $\neg R \to (P \to \neg Q)$ is true $P \to (Q \to R)$ is true. ∎

Proof. Suppose P and Q are true. By the contrapositive $\neg R \to (P \to \neg Q)$ is $\neg(P \to \neg Q) \to R$. This is equivalent to $(P \wedge Q) \to R$ and since $P \wedge Q$ therefore R. Therefore supposing $\neg R \to (P \to \neg Q)$ is true, $P \to (Q \to R)$ is true. ∎

Problem 2

This problem could be solved by using truth tables, but don't do it that way. Instead, use the methods for writing proofs discussed so far in this chapter. (See Example 3.2.4.)
(a) Suppose $P \to Q$ and $R \to \neg Q$ are both true. Prove that $P \to \neg R$ is true.
(b) Suppose that P is true. Prove that $Q \to \neg(Q \to \neg P)$ is true.

Proof. Suppose P then Q, and the contrapostive of $R \to \neg Q$ is $Q \to \neg R$. Since Q then $\neg R$. Therefore, supposing $P \to Q$ and $R \to \neg Q$ are both true $P \to \neg R$ is true. ∎

Proof. Suppose Q, we now need to show $\neg(Q \to \neg P)$ which is equivalent to $Q \land P$. Since Q is true and P is true, Q and P is true. Therefore, supposing that P is true $Q \to \neg(Q \to \neg P)$ is true. ∎

Problem 3

Suppose $A \subseteq C$, and B and C are disjoint. Prove that if $x \in A$ then $x \notin B$.

Proof. Suppose $x \in A$. Since A is a subset of C then $x \in C$. It follows from C being disjoint from B that since $x \in C$, $x \notin B$. Therefore, supposing $A \subseteq C$, and B and C are disjoint, if $x \in A$ then $x \notin B$. ∎

Problem 4

Suppose that $A \setminus B$ is disjoint from C and $x \in A$. Prove that if $x \in C$ then $x \in B$.

Proof. Suppose $x \in C$ and assume for contradiction $x \notin B$. Since $x \in A$ and $x \notin B$, $x \in A \setminus B$, but $x \in C$ and $A \setminus B$ is disjoint from C therefore x cannot be in both creating a contradiction. Therefore, supposing $A \setminus B$ is disjoint from C and $x \in A$ if $x \in C$ then $x \in B$. ∎

Problem 5

Prove that it cannot be the case that $x \in A \setminus B$ and $x \in B \setminus C$.

Proof. Assume for contradiction $x \in A \setminus B$. This means $x \in A$ and $x \notin B$. But x is also in B because $x \in B \setminus C$ which is a contradiction. Therefore, it cannot be the case that $x \in A \setminus B$ and $x \in B \setminus C$. ∎

Problem 6

Use the method of proof by contradiction to prove the theorem in Example 3.2.1.

Theorem 3.2.1 Suppose $A \cap C \subseteq B$ and $x \in C$. Prove that $x \notin A \setminus B$.

Proof. Assume for contradiction $x \in A \setminus B$. Therefore $x \in A$ and $x \notin B$, but since $x \in A$ and $x \in C$ and $A \cap C \subseteq B$, $x \in B$. This is a contradiction. Therefore supposing $A \cap C \subseteq B$ and $x \in C$, $x \notin A \setminus B$. ∎

Problem 7

Use the method of proof by contradiction to prove the theorem in Example 3.2.5.

Theorem 3.2.5 Suppose $A \subseteq B$, $x \in A$ and $x \notin B \setminus C$. Prove that $x \in C$.

Proof. Assume $x \notin C$. Since $x \notin B \setminus C$, $x \notin B$ or $x \notin C$, but $x \notin C$ therefore $x \notin B$. Since $x \in A$ and $A \subseteq B$, $x \in B$ which leads to a contradiction. Therefore, supposing $A \subseteq B$, $x \in A$ and $x \notin B \setminus C$, $x \in C$. ∎

Problem 8

Suppose that $y + x = 2y - x$, and x and y are not both zero. Prove that $y \neq 0$.

Proof. Assume $y = 0$ this means that $x \neq 0$. Then $y + x = 2y - x$ and setting $y = 0$ gives us $x = -x$. This is only possible if $x = 0$ which contradicts that $x \neq 0$ thus $y \neq 0$. Therefore, supposing $y + x = 2y - x$, and x and y are not both zero, $y \neq 0$. ∎

Problem 9

Suppose that a and b are nonzero real numbers. Prove that if $a < 1/a < b < 1/b$ then $a < -1$.

Proof. Suppose $a < 1/a < b < 1/b$ and assume for contradiction $a \geq -1$. There are two cases

Case 1 $a > 0$

Since $a > 0$ it follows that $a^2 > 0$. Since $0 < a < 1/a < b < 1/b$ we can multiply by a to get $0 < a^2 < 1 < ab < a/b$. Note that $b > 0$ and $1 < ab$. Dividing by b we get $1/b < a$ but we supposed that $a < 1/b$. Therefore $a \leq 0$.

Case 2 $a < 0$ **and** $a \geq -1$

Note that since $a < 0$ and $a \geq -1$, $a^2 \leq 1$. Since $a < 1/a$ and $a < 0$ we can multiply by a and get $a^2 > 1$. But $a^2 > 1$ and $a^2 \leq 1$ which is a contradiction. Therefore $a > 0$ or $a < -1$.

Since a being greater than or equal to -1 leads to a contradiction in all cases, a must be less than 1. Therefore, supposing that a and b are nonzero real numbers if $a < 1/a < b < 1/b$ then $a < -1$. ∎

Problem 11

Suppose that x and y are real numbers. Prove that if $x \neq 0$, then if $y = (3x^2 + 2y)/(x^2 + 2)$ then $y = 3$.

Proof. By the contrapositive $(y = (3x^2 + 2y)/(x^2 + 2) \wedge y \neq 3) \rightarrow x = 0$. Assume for contradiction $x \neq 0$ and suppose $y = (3x^2 + 2y)/(x^2 + 2)$ and $y \neq 3$.

$$\begin{aligned} y &= (3x^2 + 2y)/(x^2 + 2) \\ yx^2 + 2y &= 3x^2 + 2y \\ yx^2 &= 3x^2 \\ y &= 3 \end{aligned}$$

Since this is a contradiction $x = 0$. Therefore supposing x and y are real numbers if $x \neq 0$, then if $y = (3x^2 + 2y)/(x^2 + 2)$ then $y = 3$. ∎

Problem 12

Consider the following incorrect theorem: Incorrect Theorem. Suppose x and y are real numbers and $x + y = 10$. Then $x \neq 3$ and $y \neq 8$.
(a) What's wrong with the following proof of the theorem?

Proof. Suppose the conclusion of the theorem is false. Then $x = 3$ and $y = 8$. But then $x + y = 11$, which contradicts the given information that $x + y = 10$. Therefore the conclusion must be true. ∎

(b) Show that the theorem is incorrect by finding a counterexample.

Solution 12 (a) The negation of the conclusion is $x = 3$ or $y = 8$. It only rules out the case where $x = 3$ and $y = 8$.

Solution 12 (b) Let $x = 3$ and $y = 7$ then $x + y = 10$.

Problem 13

Consider the following incorrect theorem: Incorrect Theorem. Suppose that $A \subseteq C$, $B \subseteq C$, and $x \in A$. Then $x \in B$.
(a) What's wrong with the following proof of the theorem?

Proof. Suppose that $x \notin B$. Since $x \in A$ and $A \subseteq C$, $x \in C$. Since $x \notin B$ and $B \subseteq C$, $x \notin C$. But now we have proven both $x \in C$ and $x \notin C$, so we have reached a contradiction. Therefore $x \in B$. ∎

(b) Show that the theorem is incorrect by finding a counter example.

Solution 13 (a) This does not follow "since $x \notin B$ and $B \subseteq C$, $x \notin C$". B is a subset of C means if x is in B then it is in C. Since x is not in B it may or may not be in C.

Solution 13 (b)
Let $A = \{6\}$
Let $B = \{4, 5\}$
Let $C = \{4, 5, 6\}$
Now let $x = 6$.

Problem 18

Can the proof in Example 3.2.2 be modified to prove that if $x^2 + y = 13$ and $x \neq 3$ then $y \neq 4$ Explain.

Solution 18

No it cannot be modified to prove the new claim. Counterexample: $y = 4$ and $x = -3$ so $x \neq 3$ then $x^2 + y = (-3)^2 + 4 = 9 + 4 = 13$. So the premises are true and $y = 4$ so the implication is invalid.

3.3 Proofs Involving Quantifiers

Problem 1

In exercise 7 of Section 2.2 you used logical equivalences to show that $\exists x(P(x) \to Q(x))$ is equivalent to $\forall x P(x) \to \exists x Q(x)$. Now use the methods of this section to prove that if $\exists x(P(x) \to Q(x))$ is true, then $\forall x P(x) \to \exists x Q(x)$ is true. (Note: The other direction of the equivalence is quite a bit harder to prove. See exercise 30 of Section 3.5.)

Proof. Suppose x_0 exists such that $P(x_0)$ implies $Q(x_0)$ is true. Assume for contradiction for all z $P(z)$ is true, and for all y $Q(y)$ is false. This is a contradiction because P and Q are true for x_0. Therefore, if $\exists x(P(x) \to Q(x))$ is true, then $\forall x P(x) \to \exists x Q(x)$ is true. ∎

Problem 2

Prove that if A and $B \setminus C$ are disjoint, then $A \cap B \subseteq C$.

Proof. Suppose A and $B \setminus C$ are disjoint. Let x be an arbitrary element such that $x \in A \cap B$. Since $x \in A$ and $x \in B$, either $x \notin B$ or $x \in C$. Since $x \in B$ it follows that $x \in C$. Therefore if A and $B \setminus C$ are disjoint, then $A \cap B \subseteq C$. ∎

Problem 3

Prove that if $A \subseteq B \setminus C$ then A and C are disjoint.

Proof. Suppose $A \subseteq B \setminus C$. Let x be an arbitrary element in A. It follows that $x \in B$ and $x \notin C$. Therefore $x \in A$ and $x \notin C$. Since x was arbitrary then for all $x \in A$, $x \notin C$. ∎

Problem 4

Suppose $A \subseteq \mathcal{P}(A)$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$.

Proof. Suppose x, y are arbitrary and $x \in \mathcal{P}(A)$ and $y \in x$. Since $A \subseteq \mathcal{P}(A)$, $y \in x$, and $x \subseteq A$, $y \in \mathcal{P}(A)$. We know $y \in \mathcal{P}(A)$ for all $y \in x$ because y is arbitrary. So x is a set of elements that are all in $\mathcal{P}(A)$. This subset must be an element of $\mathcal{P}(\mathcal{P}(A))$. Since x was arbitrary, for all $x \in \mathcal{P}(A)$, $x \in \mathcal{P}(\mathcal{P}(A))$. Therefore $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$. ∎

Problem 5

The hypothesis of the theorem proven in exercise 4 is $A \subseteq P(A)$.
(a) Can you think of a set A for which this hypothesis is true?
(b) Can you think of another?

Solution 5(a)
Let $A = \varnothing$. Then $\mathcal{P}(A) = \varnothing$. In this case $A \subseteq \mathcal{P}(A)$.
Solution 5(b)
Let $A = \{\varnothing\}$. Then $\mathcal{P}(A) = \{\varnothing, \{\varnothing\}\}$. In this case $A \subseteq \mathcal{P}(A)$.

Problem 6

Suppose x is a real number.
(a) Prove that if $x \neq 1$ then there is a real number y such that $\frac{y+1}{y-2} = x$
(b) Prove that if there is a real number y such that $\frac{y+1}{y-2} = x$ then $x \neq 1$.

Proof. Suppose $x \neq 1$ and let $y = \frac{-1-2x}{1-x}$. Since $x \neq 1$ y is defined. Then

$$\begin{aligned}
\frac{y+1}{y-2} &= \frac{\frac{-1-2x}{1-x}+1}{\frac{-1-2x}{1-x}-2} \\
&= \frac{1-x}{1-x} \cdot \frac{\frac{-1-2x}{1-x}+1}{\frac{-1-2x}{1-x}-2} \\
&= \frac{1-2x+1-x}{-1-2x-2+2x} \\
&= \frac{1-2x+1-x}{-1-2x-2+2x} \\
&= \frac{-3x}{-3} \\
&= x
\end{aligned}$$

Since $\frac{y+1}{y-2} = x$ then there is a real number y such that $\frac{y+1}{y-2} = x$. ∎

Proof. Suppose there is a real number y such that $\frac{y+1}{y-2} = x$ and assume $x = 1$ for contradiction. Then $\frac{y+1}{y-2} = 1$ and multiplying by $y - 2$ on each sides gives $y + 1 = y - 2$. This is a contradiction, so $x \neq 1$. Therefore, if there is a real number y such that $\frac{y+1}{y-2} = x$ then $x \neq 1$. ∎

Problem 7

Prove that for every real number x, if $x > 2$ then there is a real number y such that $y + 1/y = x$.

Proof. Suppose $x > 2$. Then

$$\begin{aligned} y + \frac{1}{y} &= x \\ y^2 + 1 &= xy \\ y^2 - xy + 1 &= 0 \end{aligned}$$

We can then analyze the discriminant $b^2 - 4ac$ where $a = 1$, $b = -x$ and $c = 1$. So $(-x)^2 - 4(1)(1)$ and simplifying gives $(-x)^2 - 4$. Since $x > 2$ it follows that $(-x)^2 > 4$ and therefore $(-x)^2 - 4 > 0$. Since the discriminant is greater than 0, there are two distinct real roots y such that $y + 1/y = x$.

We need to now verify that $y \neq 0$ since $y + \frac{1}{y} = x$ where $y = 0$ is undefined. By the quadratic equation we need to show that $-b \pm \sqrt{b^2 - 4ac} \neq 0$. Note that $b \neq 0$ and $\sqrt{b^2 - 4ac} \neq 0$.

Suppose $-b + \sqrt{b^2 - 4ac} = 0$ where $a = 1$, $b = -x$ where $x > 2$, and $c = 1$. Then

$$\begin{aligned} -b &= \sqrt{b^2 - 4ac} \\ -b &= \sqrt{b^2 - 4(1)(1)} \\ -b &= \sqrt{b^2 - 4} \\ b^2 &= b^2 - 4 \\ 0 &= -4 \end{aligned}$$

This is a contradiction so $-b + \sqrt{b^2 - 4ac} \neq 0$.

Suppose $-b - \sqrt{b^2 - 4ac} = 0$ where $a = 1$, $b = -x$ where $x > 2$, and $c = 1$. Then

$$\begin{aligned} b &= \sqrt{b^2 - 4ac} \\ b &= \sqrt{b^2 - 4(1)(1)} \\ b &= \sqrt{b^2 - 4} \\ b^2 &= b^2 - 4 \\ 0 &= -4 \end{aligned}$$

This is a contradiction so $-b - \sqrt{b^2 - 4ac} \neq 0$.

Therefore, if $x > 2$ then there is a real number y such that $y + 1/y = x$. ∎

Problem 8

Prove that if F is a family of sets and $A \in F$, then $A \subseteq \bigcup F$.

Proof. Suppose $x \in A$. Since $x \in A$ and $A \in F$, $x \in \bigcup F$ by the definition of a union of a family of sets. Therefore, if F is a family of sets and $A \in F$, then $A \subseteq \bigcup F$. ∎

Problem 9

Prove that if F is a family of sets and $A \in F$, then $\bigcap F \subseteq A$.

Proof. Suppose $A \in F$, and $y \in \bigcap F$. For all sets B in F, $y \in B$. It follows that $y \in A$ since $A \in F$. Therefore, if F is a family of sets and $A \in F$, then $\bigcap F \subseteq A$. ∎

Problem 10

Suppose that F is a nonempty family of sets, B is a set, and $\forall A \in F(B \subseteq A)$. Prove that $B \subseteq \bigcap F$.

Proof. Suppose $x \in B$. Since $x \in B$ and for all sets A in F $B \subseteq A$, it must be the case that $x \in A$. Therefore for every set A in F $x \in A$. It follows by the definition of the intersection of a family of sets that $x \in \bigcap F$. ∎

Problem 11

Suppose that F is a family of sets. Prove that if $\varnothing \in F$ then $\bigcap F = \varnothing$.

Proof. Suppose $\varnothing \in F$ and assume for contradiction $\bigcap F \neq \varnothing$. Let x_0 be an element of $\bigcap F$. x_0 must be in all sets within F but $x_0 \notin \varnothing$ which is a conradiction. Therefore, if $\varnothing \in F$ then $\bigcap F = \varnothing$. ∎

Problem 12

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$.

Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$ and assume for contradiction $\bigcup \mathcal{F} \nsubseteq \bigcup \mathcal{G}$. This means there exists x_0 such that $x_0 \in \bigcup \mathcal{F}$ and $x_0 \notin \bigcup \mathcal{G}$. Since $x_0 \in \bigcup \mathcal{F}$, there is a set $A \in \mathcal{F}$ such that $x_0 \in A$. Note that $A \notin \mathcal{G}$ or x_0 would be in $\bigcup \mathcal{G}$. This means $A \in \mathcal{F}$ and $A \notin \mathcal{G}$ which is a contradiction since $\mathcal{F} \subseteq \mathcal{G}$. Therefore, if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$. ∎

Problem 13

Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$.

Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$ and assume for contradiction $\bigcap \mathcal{G} \nsubseteq \bigcap \mathcal{F}$. Let x_0 be the element such that $x_0 \in \bigcap \mathcal{G}$ and $x_0 \notin \bigcap \mathcal{F}$. This means x_0 is in every set in $\mathcal{G}$ and is not in at least one set in $\mathcal{F}$. Let A be the set such that $x_0 \notin A$ and $A \in \mathcal{F}$. But all sets in $\mathcal{G}$ contain x_0 so $A \notin \mathcal{G}$. This is contradiction since $\mathcal{F} \subseteq \mathcal{G}$. Therefore if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$. ∎

Problem 14

Suppose that $\{A_i | i \in I\}$ is an indexed family of sets. Prove that $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$ (Hint: First make sure you know what all the notation means!).

Proof. Assume for contradiction $\bigcup_{i \in I} \mathcal{P}(A_i) \nsubseteq \mathcal{P}(\bigcup_{i \in I} A_i)$. Let $A \in \bigcup_{i \in I} \mathcal{P}(A_i)$ and $A \notin \mathcal{P}(\bigcup_{i \in I} A_i)$. Since $A \in \bigcup_{i \in I} \mathcal{P}(A_i)$, $A \subseteq A_i$ for some $i \in I$. It follows that $A \subseteq \bigcup_{i \in I} A_i$ and therefore $A \in \mathcal{P}(\bigcup_{i \in I} A_i)$ which is a contradiction. Therefore $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$. ∎

Problem 15

Suppose $\{A_i | i \in I\}$ is an indexed family of sets and $I \neq \emptyset$. Prove that $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i)$.

Proof. First note that $\bigcap_{i \in I} \mathcal{P}(A_i)$ consists of the sets that are subsets of every A_i, where $i \in I$. $\bigcap_{i \in I} A_i$ is one such set since it is a subset of every A_i by definition of the intersection. Therefore, $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i)$. ∎

Problem 16

Prove the converse of the statement proven in Example 3.3.5. In other words, prove that if $\mathcal{F} \subseteq \mathcal{P}(B)$ then $\bigcup \mathcal{F} \subseteq B$.

Proof. Suppose $\mathcal{F} \subseteq \mathcal{P}(B)$. Let x_0 be an arbitrary element in $\bigcup \mathcal{F}$. x_0 must be in a set $A \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathcal{P}(B)$, $A \in \mathcal{P}(B)$. It then follows that $A \subseteq B$. Since $x_0 \in A$, $x_0 \in B$. Therefore, if $\mathcal{F} \subseteq \mathcal{P}(B)$ then $\bigcup \mathcal{F} \subseteq B$. ∎

Problem 17

Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets, and every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$. Prove that $\bigcup \mathcal{F} \subseteq \bigcap \mathcal{G}$.

Proof. Let x_0 be an arbitrary element such that $x_0 \in \bigcup \mathcal{F}$. There must exist $A \in \mathcal{F}$ such that $x_0 \in A$. Since A is a subset of all sets in $\mathcal{G}$, x_0 is in all sets in $\mathcal{G}$. It follows that $x_0 \in \bigcap \mathcal{G}$. Therefore, $\bigcup \mathcal{F} \subseteq \bigcap \mathcal{G}$. ∎

Problem 20

Consider the following theorem: Theorem. For every real number x, $x^2 \geq 0$. What's wrong with the following proof of the theorem?

Proof. Suppose not. Then for every real number x, $x^2 < 0$. In particular, plugging in $x = 3$ we would get $9 < 0$, which is clearly false. This contradiction shows that for every number x, $x^2 \geq 0$. ∎

Solution 20 The proof assumes an incorrect negation of for every real number x, $x^2 \geq 0$. The negation should be there exists x such that $x^2 < 0$.

Problem 21

Consider the following incorrect theorem: Incorrect Theorem. If $\forall x \in A(x \neq 0)$ and $A \subseteq B$ then $\forall x \in B(x \neq 0)$.
(a) What's wrong with the following proof of the theorem?

Proof. Suppose that $\forall x \in A(x \neq 0)$ and $A \subseteq B$. Let x be an arbitrary element of A. Since $\forall x \in A(x \neq 0)$, we can conclude that $x \neq 0$. Also, since $A \subseteq B$, $x \in B$. Since $x \in B$, $x \neq 0$, and x was arbitrary, we can conclude that $\forall x \in B(x \neq 0)$. ∎

(b) Find a counterexample to the theorem. In other words, find an example of sets A and B for which the hypotheses of the theorem are true but the conclusion is false.

Solution 21 (a)

The conclusion is incorrect. Just because $x \in B$ does not mean $0 \notin B$.

Solution 21 (b)

Let $A = \{1, 2, 3\}$.

Let $B = \{0, 1, 2, 3\}$.

The premises $\forall x \in A(x \neq 0)$ and $A \subseteq B$ are true yet the conclusion, $\forall x \in B(x \neq 0)$ is false.

Problem 22

Consider the following incorrect theorem: Incorrect Theorem. $\exists x \in R \forall y \in R(xy^2 = y - x)$. What's wrong with the following proof of the theorem?

Proof. Let $x = y/(y^2 + 1)$. Then

$$y - x = y - \frac{y}{y^2+1} = \frac{y^3}{y^2+1} = \frac{y}{y^2+1} \cdot y = xy^2$$

∎

Problem 22

It assumes that $y \neq 0$.

Problem 23

Consider the following incorrect theorem: Incorrect Theorem. Suppose F and G are families of sets. If $\bigcup F$ and $\bigcup G$ are disjoint, then so are F and G.
(a) What's wrong with the following proof of the theorem?

Proof. Suppose $\bigcup F$ and $\bigcup G$ are disjoint. Suppose F and G are not disjoint. Then we can choose some set A such that $A \in F$ and $A \in G$. Since $A \in F$, by exercise 8, $A \subseteq \bigcup F$, so every element of A is in $\bigcup F$. Similarly, since $A \in G$, every element of A is in $\bigcup G$. But then every element of A is in both $\bigcup F$ and $\bigcup G$, and this is impossible since $\bigcup F$ and $\bigcup G$ are disjoint. Thus, we have reached a contradiction, so F and G must be disjoint. ∎

(b) Find a counterexample to the theorem.

Solution 23 (a)

It is possible for every element in A to be in $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ if $A = \varnothing$.

Solution 23 (b)

Let $\mathcal{F} = \{\varnothing, \{1\}\}$

Let $\mathcal{G} = \{\varnothing, \{2\}\}$

The premise $\bigcup \mathcal{F} = \{1\}$ is disjoint $\bigcup \mathcal{G} = \{2\}$ is true, yet the conclusion $\mathcal{F}$ is disjoint from $\mathcal{G}$ is false since $\mathcal{F} \cap \mathcal{G} = \{\varnothing\}$.

Problem 24

Consider the following putative theorem:
Theorem. For all real numbers x and y, $x^2 + xy - 2y^2 = 0$.
(a) What's wrong with the following proof of the theorem?

Proof. Let x and y be equal to some arbitrary real number r. Then $x^2 + xy - 2y^2 = r2 + r \cdot r - 2r^2 = 0$. Since x and y were both arbitrary, this shows that for all real numbers x and y, $x^2 + xy - 2y^2 = 0$. ∎

(b) Is the theorem correct? Justify your answer with either a proof or a counterexample.

Solution 24 (a)

Assumes that x and y are both equal to r or in other words only considers cases where $x = y$.

Solution 24 (b)

Let $x = 1$ and $y = 2$ then

$$\begin{aligned} x^2 + xy - 2y^2 &= 0 \\ (1)^2 + (1)(2) - 2(2)^2 &= 0 \\ 1 + 2 - 8 &= 0 \\ -5 &= 0 \end{aligned}$$

This conclusion is false while the premise is true so the theorem is incorrect.

Problem 25

Prove that for every real number x there is a real number y such that for every real number z, $yz = (x+z)^2 - (x^2+z^2)$.

Proof. Let $y = 2x$. Then

$$\begin{aligned} 2x(z) &= (x+z)^2 - (x^2+z^2) \\ 2xz &= x^2 + 2xz + z^2 - x^2 - z^2 \\ 2xz &= 2xz \end{aligned}$$

Therefore, for every real number x there is a real number y such that for every real number z, $yz = (x+z)^2 - (x^2+z^2)$. ∎

3.4 Proofs Involving Conjunctions and Biconditionals

Problem 1

Use the methods of this chapter to prove that $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.

Proof. ($\rightarrow$) Assume for all x $P(x)$ and $Q(x)$ are true. Let x_0 be arbitrary, since for all x $P(x)$ and $Q(x)$ are true, $P(x_0)$ is true. Let y_0 be arbitrary, since for all x $P(x)$ and $Q(x)$ are true, $Q(y_0)$ is true. Since x_0 and y_0 were arbitrary, for all x $P(x)$ is true, and for all x $Q(x)$ is true.

($\leftarrow$) Assume for all x $P(x)$ is true, and for all x $Q(x)$ is true. Let x_0 be arbitrary. Since, for all x $P(x)$ is true, $P(x_0)$ is true. Also, since for all x $Q(x)$ is true, $Q(x_0)$ is true. It then follows that $P(x_0)$ and $Q(x_0)$ are true. Therefore, since x_0 was arbitrary, for all x $P(x)$ and $Q(x)$ are true. ∎

Problem 2

Prove that if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

Proof. Suppose $A \subseteq B$ and $A \subseteq C$. Let x_0 be an arbitrary element such that $x_0 \in A$. Since $A \subseteq B$, $x_0 \in B$. Also, since $A \subseteq C$, $x_0 \in C$. Since, $x_0 \in B$ and $x_0 \in C$, $x_0 \in B \cap C$. Therefore, if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$. ∎

Problem 3

Suppose $A \subseteq B$. Prove that for every set C, $C \setminus B \subseteq C \setminus A$.

Proof. Let x_0 be an arbitrary element such that $x_0 \in C \setminus B$. It follows that $x \in C$ and $x \notin B$. Since $A \subseteq B$ and $x \notin B$, $x \notin A$. Finally, since $x \in C$ and $x \notin A$, $x \in C \setminus A$. ∎

Problem 4

Prove that if $A \subseteq B$ and $A \nsubseteq C$ then $B \not\subset C$.

Proof. Suppose $A \subseteq B$ and $A \nsubseteq C$. Let x_0 be an element such that $x_0 \in A$. Since $A \subseteq B$, $x_0 \in B$. Also, since $A \nsubseteq C$, $x_0 \notin C$. Since $x \in B$ and $x \notin C$, $B \nsubseteq C$. ∎

Problem 5

Prove that if $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \nsubseteq C$.

Proof. Suppose $A \subseteq B \setminus C$ and $A \neq \varnothing$. Let x_0 be an element such that $x_0 \in A$. Since $A \subseteq B \setminus C$, $x_0 \in B$ and $x_0 \notin C$. It then follows that $B \nsubseteq C$. ∎

Problem 6

Prove that for any sets A, B, and C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$, by finding a string of equivalences starting with $x \in A \setminus (B \cap C)$ and ending with $x \in (A \setminus B) \cup A \setminus C$.

Proof. Let x be arbitrary. Then

$$\begin{aligned} x \in A \setminus (B \cap C) &\leftrightarrow x \in A \wedge x \notin B \cap C \\ &\leftrightarrow x \in A \wedge (x \notin B \vee x \notin C) \\ &\leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) \\ &\leftrightarrow (x \in A \setminus B) \vee (x \in A \setminus C) \\ &\leftrightarrow x \in (A \setminus B) \cup (A \setminus C) \end{aligned}$$

∎

Problem 7

Use the methods of this chapter to prove that for any sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Proof. Let C be arbitrary. Then

$$\begin{aligned} C \in \mathcal{P}(A \cap B) &\leftrightarrow C \subseteq A \cap B \\ &\leftrightarrow C \subseteq A \wedge C \subseteq B \\ &\leftrightarrow C \in \mathcal{P}(A) \wedge C \in \mathcal{P}(B) \\ &\leftrightarrow C \in \mathcal{P}(A) \cap \mathcal{P}(B) \end{aligned}$$

∎

Problem 8

Prove that $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof. ($\rightarrow$) Suppose $A \subseteq B$. Let y be an arbitrary set such that $y \in \mathcal{P}(A)$. Since $A \subseteq B$ for an arbitrary element $x \in A$, $x \in B$. So for all $x \in y$, $x \in B$ and therefore $y \subseteq B$. It follows that $y \in \mathcal{P}(B)$.

($\leftarrow$) Let x be an arbitrary element such that $x \in A$. Now let y be an arbitrary set such that $y \subseteq A$ and $x \in y$. Since $y \subseteq A$, $y \in \mathcal{P}(A)$. Then, since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $y \in \mathcal{P}(B)$. Since $y \in \mathcal{P}(B)$ it follows that $y \subseteq B$. Finally since $x \in y$ and $y \subseteq B$, $x \in B$.

Therefore $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. ∎

Problem 11

Prove that for every integer n, n^3 is even iff n is even.

Proof. ($\rightarrow$) Suppose n is even. It can be written in the form $2k$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^3 &= n \cdot n \cdot n \\ &= 2k \cdot 2k \cdot 2k \\ &= 2(2^2 \cdot k^3) \end{aligned}$$

($\leftarrow$) We will prove the contrapositive. Suppose n is odd and can therefore be written in the form $2k+1$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^3 &= n \cdot n \cdot n \\ &= (2k+1)(2k+1)(2k+1) \\ &= (4k^2+4k+1)(2k+1) \\ &= 8k^3+12k^2+6k+1 \\ &= 2(4k^3+6k^2+3k)+1 \end{aligned}$$

Therefore, for every integer n, n^3 is even iff n is even. ∎

Problem 12

Consider the following putative theorem:
Theorem? Suppose m is an even integer and n is an odd integer. Then $n^2 - m^2 = n + m$.
(a) What's wrong with the following proof of the theorem?

Proof. Since m is even, we can choose some integer k such that $m = 2k$. Similarly, since n is odd we have $n = 2k+1$. Therefore

$$\begin{aligned} n^2 - m^2 = (2k+1)^2 - (2k)^2 &= 4k^2 + 4k + 1 - 4k^2 = 4k + 1 \\ &= (2k+1) + (2k) = n + m \end{aligned}$$

∎

(b) Is the theorem correct? Justify your answer with either a proof or a counterexample.

Solution 12 (a)

The $2k$ in the equation for m is not necessarily equivalent to the $2k$ in the equation for n.

Solution 12 (b)

Let $m = 4$ and $n = 1$.

$$\begin{aligned} &n^2 - m^2 = n + m \\ &\leftrightarrow 1^2 - 4^2 = 1 + 4 \\ &\leftrightarrow 1 - 16 = 5 \\ &\leftrightarrow -15 = 5 \end{aligned}$$

This equation is false, and therefore, the theorem is false.

Problem 13

Prove that $\forall x \in \mathbb{R}[\exists y \in \mathbb{R}(x + y = xy) \leftrightarrow x \neq 1]$.

Proof. ($\rightarrow$) Suppose $x + y = xy$. Assume for contradiction $x = 1$. Then

$$x + y = xy \leftrightarrow 1 + y = (1)y \leftrightarrow 1 + y = y \leftrightarrow 1 = 0$$

Which is a contradiction. Therefore if $x + y = xy$ then $x \neq 1$.

($\leftarrow$) Suppose $x \neq 1$. Let $y = \frac{x}{x-1}$. Then

$$\begin{aligned} &x + y = xy \\ &\leftrightarrow x + \frac{x}{x-1} = x(\frac{x}{x-1}) \\ &\leftrightarrow (x-1)x + x = x^2 \\ &\leftrightarrow x^2 - x + x = x^2 \\ &\leftrightarrow x^2 = x^2 \end{aligned}$$

Therefore, $\forall x \in \mathbb{R}[\exists y \in \mathbb{R}(x + y = xy) \leftrightarrow x \neq 1]$. ∎

Problem 14

Prove that $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+[\exists y \in \mathbb{R}(y - x = y/x) \leftrightarrow x \neq z]$.

Proof. ($\rightarrow$) Suppose $y - x = y/x$. Let $z = 1$ and assume for contradiction $x = z = 1$. Then

$$\begin{aligned} y - x &= y/x \\ y - 1 &= y/1 \\ y - 1 &= y \\ -1 &= 0 \end{aligned}$$

Which is a contradiction.

($\leftarrow$) Suppose $x \neq z$. Let $z = 1$ and therefore $x \neq 1$. Let $y = \frac{x^2}{x-1}$. Then

$$\begin{aligned} y - x &= y/x \\ \frac{x^2}{x-1} - x &= \frac{x^2}{x-1}/x \\ (x-1) \cdot \frac{x^2}{x-1} - x(x-1) &= (x-1) \cdot \frac{(\frac{x^2}{x-1})}{x} \\ x^2 - x^2 + x &= x^2/x \\ x^3 - x^3 + x^2 &= x^3/x \\ x^2 &= x^2 \end{aligned}$$

Therefore, $\exists z \in \mathbb{R} \forall x \in \mathbb{R}^+[\exists y \in \mathbb{R}(y - x = y/x) \leftrightarrow x \neq z]$. ∎

Problem 15

Suppose B is a set and $\mathcal{F}$ is a family of sets. Prove that $\bigcup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \bigcup(\mathcal{F} \setminus \mathcal{P}(B))$.

Proof. Let x_0 be arbitrary such that $x_0 \in \bigcup\{A \setminus B \mid A \in \mathcal{F}\}$. It follows that $x_0 \in A$ where $A \in \mathcal{F}$ and $x_0 \notin B$. Since $A \in \mathcal{F}$ and since $x_0 \notin B$, it follows that $A \nsubseteq B$, so $A \notin \mathcal{P}(B)$. Therefore, $x_0 \in \bigcup(\mathcal{F} \setminus \mathcal{P}(B))$. Since x_0 was arbitrary, for all x, if $x \in \bigcup\{A \setminus B \mid A \in \mathcal{F}\}$ then $x \in \bigcup(\mathcal{F} \setminus \mathcal{P}(B))$. Therefore $\bigcup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \bigcup(\mathcal{F} \setminus \mathcal{P}(B))$. ∎

Problem 17

Prove that for any set A, $A = \bigcup \mathcal{P}(A)$.

Proof. Let x_0 be arbitrary such that $x_0 \in A$. Let $Y = \{x_0\} \subseteq A$. It follows that $Y \in \mathcal{P}(A)$. From the definition of the union of a family of sets $x_0 \in \bigcup \mathcal{P}(A)$. Therefore, since x_0 was arbitrary, for all $x \in A$, $x \in \bigcup \mathcal{P}(A)$.

Now let y_0 be arbitrary such that $y_0 \in \bigcup \mathcal{P}(A)$. By the definition of the union of a family of sets there is a set Z such that $y_0 \in Z$ and $Z \in \mathcal{P}(A)$. It follows that $Z \subseteq A$, and therefore, $y_0 \in A$. Therefore, since y_0 was arbitrary, for all $y \in \bigcup \mathcal{P}(A)$, $y \in A$.

Finally, since, for all $x \in A$, $x \in \bigcup \mathcal{P}(A)$, and, for all $y \in \bigcup \mathcal{P}(A)$, $y \in A$, $A = \bigcup \mathcal{P}(A)$. ∎

Problem 18

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.
(a) Prove $\bigcup(\mathcal{F} \cap \mathcal{G}) \subseteq (\mathcal{F}) \cap (\mathcal{G})$.
(b) What's wrong with the following proof that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$?

Proof. Suppose $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$. This means $x \in \bigcup \mathcal{F}$ and $x \in \bigcup \mathcal{G}$, so $\exists A \in \mathcal{F}(x \in A)$ and $\exists A \in \mathcal{G}(x \in A)$. Thus, we can choose a set A such that $A \in \mathcal{F}$, $A \in \mathcal{G}$, and $x \in A$. Since $A \in \mathcal{F}$ and $A \in \mathcal{G}$, $A \in \mathcal{F} \cap \mathcal{G}$. Therefore $\exists A \in \mathcal{F} \cap \mathcal{G}(x \in A)$, so $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$. Since x was arbitrary, we can conclude that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. ∎

(c) Find an example of famlies of sets $\mathcal{F}$ and $\mathcal{G}$ for which $\bigcup(\mathcal{F} \cap \mathcal{G}) \neq (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$.

Proof. Suppose $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$. There exists a set A such that $x \in A$ and $A \in (\mathcal{F} \cap \mathcal{G})$. Since $A \in (\mathcal{F} \cap \mathcal{G})$, $A \in \mathcal{F}$ and $A \in \mathcal{G}$. It follows that $x \in \bigcup \mathcal{F}$ and $x \in \bigcup \mathcal{G}$. ∎

Solution 18 (b)

The two sets $A \in \mathcal{F}$ and $A \in \mathcal{G}$ are not necessarily equivalent.

Solution 18 (c)

$\mathcal{F} = \{\{1\}\}$

$\mathcal{G} = \{\{1, 2\}\}$

$(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) = \{1\}$

$\bigcup(\mathcal{F} \cap \mathcal{G}) = \varnothing$

Problem 19

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$ iff $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G}))$.

Proof. ($\rightarrow$) Suppose $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. Let A, B be arbitrary sets such that $A \in \mathcal{F}$ and $B \in \mathcal{G}$. Let x_0 be an arbitrary element such that $x_0 \in A \cap B$. It follows that $x_0 \in A$ and $x_0 \in B$. Since $x_0 \in A$ and $A \in \mathcal{F}$, $x_0 \in (\bigcup \mathcal{F})$. Also, since $x_0 \in B$ and $B \in \mathcal{G}$, $x_0 \in (\bigcup \mathcal{G})$. Note that since $x_0 \in (\bigcup \mathcal{F})$ and $x_0 \in (\bigcup \mathcal{G})$, $x_0 \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$. Finally, since $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$, $x_0 \in \bigcup(\mathcal{F} \cap \mathcal{G})$.

($\leftarrow$) Suppose $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G}))$. Let x_0 be arbitrary such that $x_0 \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$. It follows that $x_0 \in (\bigcup \mathcal{F})$ and $x_0 \in (\bigcup \mathcal{G})$. Since $x_0 \in (\bigcup \mathcal{F})$, there exists a set A such that $x_0 \in A$ and $A \in \mathcal{F}$. Also, since $x_0 \in (\bigcup \mathcal{G})$, there exists a set B such that $x_0 \in B$ and $B \in \mathcal{G}$. Note that $x_0 \in A$ and $x_0 \in B$ so $x_0 \in A \cap B$. Finally, since $x_0 \in A$ and $x_0 \in B$ so $x_0 \in A \cap B$ and $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G}))$, $x_0 \in \bigcup(\mathcal{F} \cap \mathcal{G})$. ∎

Problem 20

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint iff for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint.

Proof. ($\rightarrow$) Suppose $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint. Assume for contradiction there exists A and B such that $A \in \mathcal{F}$ and $B \in \mathcal{G}$ and A and B are not disjoint. It follows there must exists an element x such that $x \in A$ and $x \in B$. Since $x \in A$ and $A \in \mathcal{F}$, $x \in \bigcup \mathcal{F}$. Also, since $x \in B$ and $B \in \mathcal{G}$, $x \in \bigcup \mathcal{G}$. But $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint which is a contradiction.

($\leftarrow$) Suppose for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint. Assume for contradiction $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are not disjoint. Since $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are not disjoint, there exists an element x such that $x \in \bigcup \mathcal{F}$ and $x \in \bigcup \mathcal{G}$. Furthermore, there exists a set A such that $x \in A$ and $A \in \mathcal{F}$ and there exists a set B such that $x \in B$ and $B \in \mathcal{G}$. The sets A and B are clearly not disjoint which is a contradiction.

Therefore, $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint iff for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint. ∎

Problem 21

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.
(a) Prove that $(\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \setminus \mathcal{G})$.
(b) What's wrong with the following proof that $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$.

Proof. Suppose $x \in \bigcup(\mathcal{F} \setminus \mathcal{G})$. Then we can choose some $A \in \mathcal{F} \setminus \mathcal{G}$ such that $x \in A$. Since $A \in \mathcal{F} \setminus \mathcal{G}$, $A \in \mathcal{F}$ and $A \notin \mathcal{G}$. Therefore $x \in (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$. ∎

(c) Prove that $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$ iff $\forall A \in (\mathcal{F} \setminus \mathcal{G}) \forall B \in \mathcal{G}(A \cap B = \varnothing)$.
(d) Find an example of families of sets $\mathcal{F}$ and $\mathcal{G}$ for which $\bigcup(\mathcal{F} \setminus \mathcal{G}) \neq (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$

Proof. Let x_0 be an arbitrary element such that $x_0 \in (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$. It follows that $x_0 \in (\bigcup \mathcal{F})$ and $x_0 \notin (\bigcup \mathcal{G})$. There exists a set A such that $x_0 \in A$ and $A \in \mathcal{F}$. There does not exist a set B such that $x_0 \in B$ and $B \in \mathcal{G}$. It follows that $A \in (\mathcal{F} \setminus \mathcal{G})$. By definition of union of a family of sets since $x_0 \in A$, $x_0 \in \bigcup(\mathcal{F} \setminus \mathcal{G})$. ∎

Solution 21 (b)

There could be a set B such that $x \in B$ and $B \in \mathcal{G}$ where $A \neq B$. It follows that $x \in \bigcup \mathcal{G}$.

Proof. ($\rightarrow$) Suppose $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$. Assume for contradiction there exists sets A, B such that $A \in \mathcal{F} \setminus \mathcal{G}$, $B \in \mathcal{G}$, and $A \cap B \neq \varnothing$. It follows that there exists x_0 such that $x_0 \in A$ and $x_0 \in B$. Since $x_0 \in A$ and $A \in \mathcal{F}$, $x_0 \in (\bigcup \mathcal{F})$. Since $x_0 \in B$, and $B \in \mathcal{G}$, $x_0 \in (\bigcup \mathcal{G})$. It follows that $x_0 \notin (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$. Since $x_0 \in A$ and $A \in \mathcal{F} \setminus \mathcal{G}$, it follows that $x_0 \in \bigcup(\mathcal{F} \setminus \mathcal{G})$. But $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$ which is a contradiction. So, if $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$ then $\forall A \in (\mathcal{F} \setminus \mathcal{G}) \forall B \in \mathcal{G}(A \cap B = \varnothing)$

($\leftarrow$) Suppose $\forall A \in (\mathcal{F} \setminus \mathcal{G})\forall B \in \mathcal{G}(A \cap B = \varnothing)$. Let x_0 be an arbitrary element such that $x_0 \in \bigcup(\mathcal{F} \setminus \mathcal{G})$. There must be a set A such that $x_0 \in A$ and $A \in \mathcal{F} \setminus \mathcal{G}$. It follows $A \in \mathcal{F}$ and $A \notin \mathcal{G}$. Since $x_0 \in A$ and $A \in \mathcal{F}$, $x_0 \in \bigcup \mathcal{F}$. Since $A \in \mathcal{F} \setminus \mathcal{G}$ and all sets in $\mathcal{G}$ are disjoint from A, $x_0 \notin (\bigcup \mathcal{G})$. Since $x_0 \in (\bigcup \mathcal{F})$ and $x_0 \notin (\bigcup \mathcal{G})$, $x_0 \in (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$.

Therefore, $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G})$ iff $\forall A \in (\mathcal{F} \setminus \mathcal{G})\forall B \in \mathcal{G}(A \cap B = \varnothing)$. ∎

Solution 21 (d)

$\mathcal{F} = \{\{1\}, \{2\}\}$

$\mathcal{G} = \{\{1, 2\}\}$

$\mathcal{F} \setminus \mathcal{G} = \{\{1\}, \{2\}\}$

$\bigcup(\mathcal{F} \setminus \mathcal{G}) = \{1, 2\}$

$(\bigcup \mathcal{F}) = \{1, 2\}$

$(\bigcup \mathcal{G}) = \{1, 2\}$

$(\bigcup \mathcal{F}) \setminus (\bigcup \mathcal{G}) = \varnothing$

Problem 23

Suppose B is a set, $\{A_i \mid i \in I\}$ is an indexed family of sets, and $I \neq \varnothing$.
(a) What proof strategies are used in the following proof of the equation $B \cap (\bigcup_{i \in I} A_i) = \bigcup_{i \in I}(B \cap A_i)$

Proof. Let x be arbitrary. Suppose $x \in B \cap (\bigcup_{i \in I} A_i)$. Then $x \in B$ and $x \in \bigcup_{i \in I} A_i$, so we can choose some $i_0 \in I$ such that $x \in A_{i_0}$. Since $x \in B$ and $x \in A_{i_0}$, $x \in B \cap A_{i_0}$. Therefore $x \in \bigcup_{i \in I}(B \cap A_i)$.
Now suppose $x \in \bigcup_{i \in I}(B \cap A_i)$. Then we can choose some $i_0 \in I$ such that $x \in B \cap A_{i_0}$. Therefore $x \in B$ and $x \in A_{i_0}$. Since $x \in A_{i_0}$, $x \in \bigcup_{i \in I} A_i$. Since $x \in B$ and $x \in \bigcup_{i \in I} A_i$, $x \in B \cap (\bigcup_{i \in I} A_i)$.
Since x was arbitrary, we have shown that $\forall x[x \in B \cap (\bigcup_{i \in I} A_i) \leftrightarrow x \in \bigcup_{i \in I}(B \cap A_i)]$, so $B \cap (\bigcup_{i \in I}) = \bigcup_{i \in I}(B \setminus A_i)$. ∎

(b) Prove that $B \setminus (\bigcup_{i \in I} A_i) = \bigcap_{i \in I}(B \setminus A_i)$.
(c) Can you discover and prove a similar theorem about $B \setminus (\bigcap_{i \in I} A_i)$? (Hint: Try to guess the theorem, and then try to prove it. If you can't finish the proof, it might be because your guess was wrong. Change your guess and try again.)

Solution 23(a)

The proof uses a direct proof strategy based on mutual inclusion: it shows that each side of the equation is a subset of the other. For any two sets A and B, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.

Solution 23(b)

Proof. ($\rightarrow$) Let x_0 be an arbitrary element such that $x_0 \in B \setminus (\bigcup_{i \in I} A_i)$. So $x_0 \in B$ and $x_0 \notin (\bigcup_{i \in I} A_i)$. So $x_0 \notin A_i$ for any $i \in I$. It follows that $x_0 \in \bigcap_{i \in I}(B \setminus A_i)$.

($\leftarrow$) Let x_0 be an arbitrary element such that $x_0 \in \bigcap_{i \in I}(B \setminus A_i)$. So $x_0 \in B$ and $x_0 \notin A_i$ for any $i \in I$. It follows that $x_0 \notin (\bigcup_{i \in I} A_i)$. So $x_0 \in B$ and $x_0 \notin (\bigcup_{i \in I} A_i)$, so $x_0 \in B \setminus (\bigcup_{i \in I} A_i)$.

Therefore, $B \setminus (\bigcup_{i \in I} A_i) = \bigcap_{i \in I}(B \setminus A_i)$. ∎

Solution 23(c)

$B \setminus (\bigcap_{i \in I} A_i) = \bigcup_{i \in I}(B \setminus A_i)$

Proof. ($\rightarrow$) Let x_0 be an arbitrary element such that $x_0 \in B \setminus (\bigcap_{i \in I} A_i)$. So $x_0 \in B$ and $x_0 \notin (\bigcap_{i \in I} A_i)$. It follows that since $x_0 \in B$ and $x_0 \notin A_i$ for some $i \in I$, $x_0 \in \bigcup_{i \in I}(B \setminus A_i)$.

($\leftarrow$) Let x_0 be an arbitrary element such that $x_0 \in \bigcup_{i\in I}(B \setminus A_i)$. So $x_0 \in B$ and $x_0 \notin A_i$ for some $i \in I$. It follows that $x_0 \notin (\bigcap_{i\in I} A_i)$ and, therefore, $x_0 \in B \setminus (\bigcap_{i\in I} A_i)$.

Therefore $B \setminus (\bigcap_{i\in I} A_i) = \bigcup_{i\in I}(B \setminus A_i)$. ∎

Problem 24

Suppose $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ are indexed family of sets and $I \neq \varnothing$.
(a) Prove that $\bigcup_{i\in I}(A_i \setminus B_i) \subseteq (\bigcup_{i\in I} A_i) \setminus (\bigcap_{i\in I} B_i)$.
(b) Find and example for which $\bigcup_{i\in I}(A_i \setminus B_i) \neq (\bigcup_{i\in I} A_i) \setminus (\bigcap_{i\in I} B_i)$.

Proof. Let x_0 be an arbitrary element such that $x_0 \in \bigcup_{i\in I}(A_i \setminus B_i)$. For some $i \in I$, $x_0 \in A_i$ and $x_0 \notin B_i$. Since $x_0 \in A_i$ for some $i \in I$, $x_0 \in \bigcup_{i\in I} A_i$. Since $x_0 \notin B_i$ for some $i \in I$, $x_0 \notin \bigcap_{i\in I} B_i$. It follows, since $x_0 \in \bigcup_{i\in I} A_i$ and $x_0 \notin \bigcap_{i\in I} B_i$, $x_0 \in \bigcup_{i\in I} A_i \setminus \bigcap_{i\in I} B_i$. Therefore, $\bigcup_{i\in I}(A_i \setminus B_i) \subseteq (\bigcup_{i\in I} A_i) \setminus (\bigcap_{i\in I} B_i)$. ∎

Solution 24(b)

$I = \{1, 2\}$

$A_1 = \{1, 2\}$, $A_2 = \{3\}$

$B_1 = \{1\}$, $B_2 = \{2\}$

$\bigcup_{i\in I}(A_i \setminus B_i) = \{2, 3\}$

$(\bigcup_{i\in I} A_i) \setminus (\bigcap_{i\in I} B_i) = \{1, 2, 3\}$

Problem 25

Suppose $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ are indexed family of sets and $I \neq \varnothing$.
(a) Prove that $\bigcup_{i\in I}(A_i \cap B_i) \subseteq (\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i)$.
(b) Find an example for which $\bigcup_{i\in I}(A_i \cap B_i) \neq (\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i)$.

Proof. Let x_0 be arbitrary such that $x_0 \in \bigcup_{i\in I}(A_i \cap B_i)$. For some $i \in I$, $x_0 \in A_i$ and $x_0 \in B_i$. Since x_0 in A_i for some $i \in I$, $x_0 \in (\bigcup_{i\in I} A_i)$. Since x_0 in B_i for some $i \in I$, $x_0 \in (\bigcup_{i\in I} B_i)$. So $x_0 \in (\bigcup_{i\in I} A_i)$ and $x_0 \in (\bigcup_{i\in I} B_i)$. It follows $x_0 \in (\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i)$. Therefore, $\bigcup_{i\in I}(A_i \cap B_i) \subseteq (\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i)$. ∎

Solution 25(b)

$I = \{1, 2\}$

$A_1 = \{1\}$, $A_2 = \{2\}$

$B_1 = \{2\}$, $B_2 = \{1\}$

$\bigcup_{i\in I}(A_i \cap B_i) = \varnothing$

$(\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i) = \{1, 2\}$

Problem 27

(a) Prove that for every integer n, $15 \mid n$ iff $3 \mid n$ and $5 \mid n$.
(b) Prove that it is not true that for every integer n, $60 \mid n$ iff $6 \mid n$ and $10 \mid n$.

Proof. ($\rightarrow$) Suppose $15 \mid n$. Then $n = 15k$ where $k \in \mathbb{Z}$. Then $n = 3(5k)$ and $n = 5(3k)$ showing $3 \mid n$ and $5 \mid n$.

($\leftarrow$) Suppose $3 \mid n$ and $5 \mid n$. So $n = 3k_1$ and $n = 5k_2$ for $k_1, k_2 \in \mathbb{Z}$.

$$\begin{aligned} 15(2k_2 - k_1) &= 30k_2 - 15k_1 \\ &= 6(5k_2) - 5(3k_1) \\ &= 6n - 5n = n \end{aligned}$$

So $15 \mid n$. Therefore, for every integer n, $15 \mid n$ iff $3 \mid n$ and $5 \mid n$. ∎

Solution 27 (b)

Counter example for backwards implication ($\leftarrow$).

Let $n = 30$. n is divisible by 6 and 10 but not by 60.

3.5 Proofs Involving Disjunctions

Problem 1

Suppose A, B, and C are sets. Prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof. Let x_0 be an arbitrary element such that $x_0 \in A \cap (B \cup C)$. So $x_0 \in A$ and $x_0 \in B \cup C$ and therefore, $x_0 \in B$ or $x_0 \in C$.

Case ($x_0 \in B$): Since $x_0 \in A$ and $x_0 \in B$, $x_0 \in A \cap B$. It follows that $x_0 \in (A \cap B) \cup C$.

Case ($x_0 \in C$): It immediately follows $x_0 \in (A \cap B) \cup C$.

Since these cases were exhaustive, $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. ∎

Problem 4

Suppose A, B, and C are sets. Prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.

Proof. We first show $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$ Let x_0 be an arbitrary element such that $x_0 \in A \setminus (B \setminus C)$. So $x_0 \in A$ and $x_0 \notin B \setminus C$. It follows that $x_0 \notin B$ or $x_0 \in C$.

Case ($x_0 \notin B$): Since $x_0 \in A$ and $x_0 \notin B$, $x_0 \in A \setminus B$. It follows that $x_0 \in (A \setminus B) \cup (A \cap C)$.

Case ($x_0 \in C$): Since $x_0 \in A$ and $x_0 \in C$, $x_0 \in A \cap C$. It follows that $x_0 \in (A \setminus B) \cup (A \cap C)$.

Since these cases were exhaustive, $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$.

We now show $(A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$. Let x_0 be an arbitrary element such that $x_0 \in (A \setminus B) \cup (A \cap C)$. So $x_0 \in A \setminus B$ or $x_0 \in A \cap C$.

Case ($x_0 \in A \cap C$): So $x_0 \in A$ and $x_0 \in C$. Since $x_0 \in C$ it follows $x_0 \notin B \setminus C$. Since $x_0 \in A$ and $x_0 \notin B \setminus C$, $x_0 \in A \setminus (B \setminus C)$.

Case ($x_0 \in A \setminus B$): So $x_0 \in A$ and $x_0 \notin B$. Since $x_0 \notin B$, $x_0 \notin B \setminus C$. Since $x_0 \in A$ and $x_0 \notin B \setminus C$, $x_0 \in A \setminus (B \setminus C)$.

Since these cases were exhaustive, $(A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$.

Since $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$ and $(A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$, $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$. ∎

Problem 6

Recall from Section 1.4 that the symmetric difference of two sets A and B is the set $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$. Prove that if $A \triangle B \subseteq A$ then $B \subseteq A$.

Proof. Suppose $A \triangle B \subseteq A$ and assume for contradiction $B \nsubseteq A$. Let x_0 be an element such that $x_0 \in B$ and $x_0 \notin A$. Since $A \triangle B \subseteq A$, $(A \setminus B) \cup (B \setminus A) \subseteq A$. Since $x_0 \in B$ and $x_0 \notin A$, $x_0 \in B \setminus A$ and therefore $x_0 \in (A \setminus B) \cup (B \setminus A)$. Since $x_0 \in (A \setminus B) \cup (B \setminus A)$ and $(A \setminus B) \cup (B \setminus A) \subseteq A$, $x_0 \in A$ which is a contradiction. Therefore, if $A \triangle B \subseteq A$ then $B \subseteq A$. ∎

Problem 8

Prove that for any sets A and B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof. Let x_0 be an arbitrary element such that $x_0 \in \mathcal{P}(A) \cup \mathcal{P}(B)$. So $x_0 \in \mathcal{P}(A)$ or $x_0 \in \mathcal{P}(B)$.

Case ($x_0 \in \mathcal{P}(A)$) So $x_0 \subseteq A$ and it follows that $x_0 \subseteq A \cup B$. Since $x_0 \subseteq A \cup B$, $x_0 \in \mathcal{P}(A \cup B)$.

Case ($x_0 \in \mathcal{P}(B)$) So $x_0 \subseteq B$ and it follows that $x_0 \subseteq A \cup B$. Since $x_0 \subseteq A \cup B$, $x_0 \in \mathcal{P}(A \cup B)$.

Since these cases were exhaustive, for any sets A and B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. ∎

Problem 9

Prove that for any sets A and B, if $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$.

Proof. Suppose $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$. Assume for contradiction $A \nsubseteq B$ and $B \nsubseteq A$. There exists elements x_0, y_0 such that $x_0 \in A \setminus B$ and $y_0 \in B \setminus A$. Now, $\{x_0, y_0\} \subseteq A \cup B$ and therefore $\{x_0, y_0\} \in \mathcal{P}(A \cup B)$. But $\{x_0, y_0\} \notin \mathcal{P}(A)$ and $\{x_0, y_0\} \notin \mathcal{P}(B)$ which is a contradiction. Therefore, for any sets A and B, if $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$. ∎

Problem 10

Suppose x and y are real numbers and $x \neq 0$. Prove that $y + 1/x = 1 + y/x$ iff either $x = 1$ or $y = 1$.

Proof. ($\rightarrow$) Suppose $y + 1/x = 1 + y/x$. Then:

$$\begin{aligned} y + 1/x &= 1 + y/x \\ xy + 1 &= x + y \\ x - 1 - y(x-1) &= 0 \\ (x-1)(1-y) &= 0 \end{aligned}$$

From this it follows that $x = 1$ or $y = 1$.

($\leftarrow$) Suppose $x = 1$ or $y = 1$. Suppose $x = 1$, then $y + 1/x = 1 + y/x$ and $y + 1 = y + 1$. Suppose $y = 1$, then $y + 1/x = 1 + y/x$ and $1 + 1/x = 1 + 1/x$.

Therefore, $y + 1/x = 1 + y/x$ iff either $x = 1$ or $y = 1$. ∎

Problem 13

(a) Prove that for all real numbers a and b, $|a| \leq b$ iff $-b \leq a \leq b$.
(b) Prove that for any real number x, $-|x| \leq x \leq |x|$. (Hint: Use part (a).)
(c) Prove that for all real numbers x and y, $|x + y| \leq |x| + |y|$. (This is called the triangle inequality. One way to do this is to combine parts (a) and (b), but you can also do it by considering a number of cases.)
(d) Prove that for all real numbers x and y, $|x + y| \geq |x| - |y|$. (Hint: Start with the equation $|x| = |(x + y) + (-y)|$ and then apply the triangle inequality to the right hand side.)

Proof. ($\rightarrow$) Suppose $|a| \leq b$. Either $a \geq 0$ or $a \leq 0$.

Case ($a \geq 0$). Since $a \geq 0$, $|a| = a$. Plugging into $|a| \leq b$ we get $a \leq b$. It follows that $a \geq -b$ and therefore $-b \leq a \leq b$.

Case ($a \leq 0$). Since $a \leq 0$, $|a| = -a$. Plugging into $|a| \leq b$ we get $-a \leq b$. Multiplying by -1 gives $a \geq -b$. It then follows that $a \leq b$ and therefore $-b \leq a \leq b$.

($\leftarrow$) Suppose $-b \leq a \leq b$. Either $a \geq 0$ or $a \leq 0$.

Case ($a \geq 0$). Since $a \geq 0$, $|a| = a$. Plugging into $a \leq b$ we get $|a| \leq b$.

Case ($a \leq 0$). Since $a \leq 0$, $|a| = -a$. From $a \geq -b$ we can multiply by -1 and get $-a \leq b$. Then plugging in we get $|a| \leq b$.

Therefore for all real numbers a and b, $|a| \leq b$ iff $-b \leq a \leq b$. ∎

Proof. Since $x \leq |x|$ using part i, $-|x| \leq x \leq |x|$. ∎

Proof. Since $x \leq |x|$ and $y \leq |y|$ we can can take their sum and get $x + y \leq |x| + |y|$. Now, since $-x \leq |x|$ and $-y \leq |y|$ we can take their sum and get $-(x + y) \leq |x| + |y|$. Since $x + y \leq |x| + |y|$ and $-(x + y) \leq |x| + |y|$, $|x + y| \leq |x| + |y|$. ∎

Proof. First:

$$
\begin{aligned}
|x| &= |x| \\
&= |x + 0| \\
&= |x + y - y| \\
&= |(x + y) + (-y)|
\end{aligned}
$$

From part iii (triangle inequality):

$$|x + y| + |-y| \geq |(x + y) + (-y)|$$

So:

$$
\begin{aligned}
|x + y| + |-y| &\geq |(x + y) + (-y)| \\
|x + y| + |y| &\geq |x| && \text{(since } |-y| = |y|\text{)} \\
|x + y| &\geq |x| - |y|
\end{aligned}
$$

∎

Problem 14

Prove that for every integer x, $x^2 + x$ is even.

Proof. Let $x \in \mathbb{Z}$. There are two cases. Either $x = 2k$ or $x = 2k + 1$ for some $k \in \mathbb{Z}$.

Case ($x = 2k$)

$$
\begin{aligned}
x + x^2 &= (2k) + (2k)^2 \\
&= 2k + 4k^2 \\
&= 2(k + 2k^2)
\end{aligned}
$$

Case $(x = 2k + 1)$

$$\begin{aligned} x + x^2 &= (2k+1) + (2k+1)^2 \\ &= 2k + 1 + 4k^2 + 4k + 1 \\ &= 4k^2 + 6k + 2 \\ &= 2(2k^2 + 3k + 1) \end{aligned}$$

Therefore, for every integer x, $x^2 + x$ is even. ∎

Problem 15

Prove that for every integer x, the remainder when x^4 is divided by 8 is either 0 or 1.

Proof. Either x is even or odd. Suppose x is even. Then $x = 2k$ where $k \in \mathbb{Z}$. Then $(2k)^4 = 16k^4 = 8(2k^4)$ which when divided by 8 has a remainder of 0.

Suppose x is odd. Then $x = 2k + 1$ where $k \in \mathbb{Z}$. Then:

$$\begin{aligned} (2k+1)^4 &= 16k^4 + 32k^3 + 24k^2 + 8k + 1 \\ &= 8(2k^4 + 4k^3 + 3k^2 + k) + 1 \end{aligned}$$

which when divided by 8 has a remainder of 1.

Since these cases were exhaustive, for every integer x, the remainder when x^4 is divided by 8 is either 0 or 1. ∎

Problem 16

Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets.
(a) Prove that $\bigcup(\mathcal{F} \cup \mathcal{G}) = (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$.
(b) Can you discover and prove a similar theorem about $\bigcap(\mathcal{F} \cup \mathcal{G})$.

Proof. First we show $\bigcup(\mathcal{F} \cup \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$. Let x_0 be an arbitrary element such that $x_0 \in \bigcup(\mathcal{F} \cup \mathcal{G})$. There exists a set A such that $x_0 \in A$ and either $A \in \mathcal{F}$ or $A \in \mathcal{G}$.

Case $A \in \mathcal{F}$: Then, since $x_0 \in A$, $x_0 \in (\bigcup \mathcal{F})$ and $x_0 \in (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$.

Case $A \in \mathcal{G}$: Then, since $x_0 \in A$, $x_0 \in (\bigcup \mathcal{G})$ and $x_0 \in (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$.

Since these cases are exhaustive, it follows that $\bigcup(\mathcal{F} \cup \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$.

We now show that $(\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cup \mathcal{G})$. Let x_0 be an arbitrary element such that $x_0 \in (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$. Either $x_0 \in (\bigcup \mathcal{F})$ or $x_0 \in (\bigcup \mathcal{G})$.

Case $x_0 \in (\bigcup \mathcal{F})$ There exists a set A such that $x_0 \in A$ and $A \in \mathcal{F}$. It follows that $A \in \mathcal{F} \cup \mathcal{G}$. Since $x_0 \in A$ and $A \in \mathcal{F} \cup \mathcal{G}$, $x_0 \in \bigcup(\mathcal{F} \cup \mathcal{G})$.

Case $x_0 \in (\bigcup \mathcal{G})$ There exists a set A such that $x_0 \in A$ and $A \in \mathcal{G}$. It follows that $A \in \mathcal{F} \cup \mathcal{G}$. Since $x_0 \in A$ and $A \in \mathcal{F} \cup \mathcal{G}$, $x_0 \in \bigcup(\mathcal{F} \cup \mathcal{G})$.

Since these cases are exhaustive, it follows that $(\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cup \mathcal{G})$.

Since $\bigcup(\mathcal{F} \cup \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$ and $(\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cup \mathcal{G})$, $\bigcup(\mathcal{F} \cup \mathcal{G}) = (\bigcup \mathcal{F}) \cup (\bigcup \mathcal{G})$. ∎

A similar theorem is $\bigcap(\mathcal{F} \cup \mathcal{G}) = (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$.

Proof. We first show $\bigcap(\mathcal{F} \cup \mathcal{G}) \subseteq (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$. Let x_0 be an arbitrary element such that $x_0 \in \bigcap(\mathcal{F} \cup \mathcal{G})$. For all sets $T \in \mathcal{F} \cup \mathcal{G}$, $x_0 \in T$. It follows that for all sets $A \in \mathcal{F}$, $x_0 \in A$; and for all sets $B \in \mathcal{G}$, $x_0 \in B$. Since $x_0 \in A$ for all $A \in \mathcal{F}$, it follows that $x_0 \in \bigcap \mathcal{F}$. Since $x_0 \in B$ for all $B \in \mathcal{G}$, it follows that $x_0 \in \bigcap \mathcal{G}$. Since $x_0 \in (\bigcap \mathcal{F})$ and $x_0 \in (\bigcap \mathcal{G})$, $x_0 \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$.

We now show $(\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G}) \subseteq \bigcap(\mathcal{F} \cup \mathcal{G})$. Let x_0 be an arbitrary element such that $x_0 \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$. It follows that $x_0 \in (\bigcap \mathcal{F})$ and $x_0 \in (\bigcap \mathcal{G})$. Since $x_0 \in A$ for all $A \in \mathcal{F}$ and $x_0 \in B$ for all $B \in \mathcal{G}$, it follows that $x_0 \in T$ for all $T \in \mathcal{F} \cup \mathcal{G}$. Therefore, $x_0 \in \bigcap(\mathcal{F} \cup \mathcal{G})$.

Therefore, $\bigcap(\mathcal{F} \cup \mathcal{G}) = (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$. ∎

Problem 17

Suppose $\mathcal{F}$ is a nonempty family of sets and B is a set.
(a) Prove that $B \cup (\bigcup \mathcal{F}) = \bigcup(\mathcal{F} \cup \{B\})$.
(b) Prove that $B \cup (\bigcap \mathcal{F}) = \bigcap_{A \in \mathcal{F}}(B \cup A)$.
(c) Can you discover and prove similar theorems about $B \cap (\bigcup \mathcal{F})$ and $B \cap (\bigcap \mathcal{F})$.

Proof. We first show $B \cup (\bigcup \mathcal{F}) \subseteq \bigcup(\mathcal{F} \cup \{B\})$. Let x_0 be an arbitrary element such that $x_0 \in B \cup (\bigcup \mathcal{F})$. Either $x_0 \in B$ or $x_0 \in (\bigcup \mathcal{F})$.

Suppose $x_0 \in B$ it immediately follows that $x_0 \in \bigcup(\mathcal{F} \cup \{B\})$.

Suppose $x_0 \in (\bigcup \mathcal{F})$. There exists a set A such that $x_0 \in A$ and $A \in \mathcal{F}$. It follows that $A \in \mathcal{F} \cup \{B\}$. Since $x_0 \in A$ then $x_0 \in \bigcup(\mathcal{F} \cup \{B\})$.

We now show $\bigcup(\mathcal{F} \cup \{B\}) \subseteq B \cup (\bigcup \mathcal{F})$. Let x_0 be an arbitrary element such that $x_0 \in \bigcup(\mathcal{F} \cup \{B\})$. There exists a set A such that $A \in \mathcal{F} \cup \{B\}$. So $A \in \mathcal{F}$ or $A \in \{B\}$.

Suppose $A \in \mathcal{F}$. Since $x_0 \in A$, $x_0 \in (\bigcup \mathcal{F})$. It follows that $x_0 \in B \cup (\bigcup \mathcal{F})$.

Suppose $A \in \{B\}$. Since $x_0 \in A$, $x_0 \in B$. It follows that $x_0 \in B \cup (\bigcup \mathcal{F})$.

Therefore, $B \cup (\bigcup \mathcal{F}) = \bigcup(\mathcal{F} \cup \{B\})$. ∎

Proof. We first show $B \cup (\bigcap \mathcal{F}) \subseteq \bigcap_{A \in \mathcal{F}}(B \cup A)$. Let x_0 be an arbitrary element such that $x_0 \in B \cup (\bigcap \mathcal{F})$. So $x_0 \in B$ or $x_0 \in (\bigcap \mathcal{F})$.

Suppose $x_0 \in B$. It follows that for all $A \in \mathcal{F}$ $x_0 \in B \cup A$. Therefore, $x_0 \in \bigcap_{A \in \mathcal{F}}(B \cup A)$.

Suppose $x_0 \in (\bigcap \mathcal{F})$. For all $A \in \mathcal{F}$, $x_0 \in A$ and, therefore, $x_0 \in B \cup A$. Therefore $x_0 \in \bigcap_{A \in \mathcal{F}}(B \cup A)$.

We now show $\bigcap_{A \in \mathcal{F}}(B \cup A) \subseteq B \cup (\bigcap \mathcal{F})$. Let x_0 be an arbitrary element such that $x_0 \in \bigcap_{A \in \mathcal{F}}(B \cup A)$. Either $x_0 \in B$ or for all sets $A \in \mathcal{F}$, $x_0 \in A$. Meaning $x_0 \in (\bigcap \mathcal{F})$.

Suppose $x_0 \in B$. It immediately follows that $x_0 \in B \cup (\bigcap \mathcal{F})$.

Suppose $x_0 \in (\bigcap \mathcal{F})$. It immediately follows that $x_0 \in B \cup (\bigcap \mathcal{F})$.

Therefore, $B \cup (\bigcap \mathcal{F}) = \bigcap_{A \in \mathcal{F}}(B \cup A)$. ∎

We now prove $B \cap (\bigcup \mathcal{F}) = \bigcup_{A \in \mathcal{F}}(B \cap A)$.

Proof. We first show $B \cap (\bigcup \mathcal{F}) \subseteq \bigcup_{A \in \mathcal{F}}(B \cap A)$. Let x_0 be an arbitrary element such that $x_0 \in B \cap (\bigcup \mathcal{F})$. So $x_0 \in B$ and $x_0 \in \bigcup \mathcal{F}$. Since $x_0 \in \bigcup \mathcal{F}$, there exists a set A such that $A \in \mathcal{F}$ and $x_0 \in A$. It follows that $x_0 \in B \cap A$ and therefore, $x_0 \in \bigcup_{A \in \mathcal{F}}(B \cap A)$.

We now show $\bigcup_{A \in \mathcal{F}}(B \cap A) \subseteq B \cap (\bigcup \mathcal{F})$. Let x_0 be an arbitrary element such that $x_0 \in \bigcup_{A \in \mathcal{F}}(B \cap A)$. Since $x_0 \in \bigcup_{A \in \mathcal{F}}(B \cap A)$, $x_0 \in B$ and $x_0 \in A$ for some $A \in \mathcal{F}$. Since $x_0 \in A$ for some $A \in \mathcal{F}$, $x_0 \in \bigcup(\mathcal{F})$. Since $x_0 \in B$ and $x_0 \in \bigcup(\mathcal{F})$, $x_0 \in B \cap \bigcup(\mathcal{F})$.

Therefore, $B \cap (\bigcup \mathcal{F}) = \bigcup_{A \in \mathcal{F}} (B \cap A)$. ∎

Problem 18

Suppose $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are nonempty families of sets and for every $A \in \mathcal{F}$ and every $B \in \mathcal{G}$, $A \cup B \in \mathcal{H}$. Prove that $\bigcap \mathcal{H} \subseteq (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$.

Proof. Assume for contradiction $\bigcap \mathcal{H} \nsubseteq (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$. Let x_0 be an element such that $x_0 \in \bigcap \mathcal{H}$ and $x_0 \notin (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$. There is a set $A \in \mathcal{F}$ such that $x_0 \notin A$ and there is a set $B \in \mathcal{G}$ such that $x_0 \notin B$. $A \cup B \in \mathcal{H}$, but $x_0 \notin A \cup B$ and $x_0 \in T$ for all $T \in \mathcal{H}$ which is a contradiction. Therefore, $\bigcap \mathcal{H} \subseteq (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$. ∎

Problem 26

Suppose A, B, and C are sets. Consider the sets $(A \setminus B) \triangle C$ and $(A \triangle C) \setminus (B \triangle C)$. Can you prove that either is a subset of the other? Justify your conclusions with either proofs or counterexamples.

We will prove that $(A \triangle C) \setminus (B \triangle C) \subseteq (A \setminus B) \triangle C$.

Proof. Let x_0 be an arbitrary element such that $x_0 \in (A \triangle C) \setminus (B \triangle C)$. So $x_0 \in (A \triangle C)$ and $x_0 \notin B \triangle C$. Since $x_0 \in A \triangle C$, either $x_0 \in A \setminus C$ or $x_0 \in C \setminus A$. Since $x_0 \notin B \triangle C$, it follows that either $x_0 \in B \cap C$ or $x_0 \notin B \cup C$. There are four cases.

Case ($x_0 \in A \setminus C$ and $x_0 \in B \cap C$) This isn't possible since it would require $x_0 \notin C$ since $x_0 \in A \setminus C$ and $x_0 \in C$ since $x_0 \in B \cap C$.

Case ($x_0 \in A \setminus C$ and $x_0 \notin B \cup C$) Since $x_0 \notin B \cup C$, $x_0 \notin C$. Since $x_0 \notin C$, $x_0 \notin (A \setminus B) \cap C$. Since $x_0 \in A \setminus C$, $x_0 \in A$. Since $x_0 \notin B \cup C$, $x_0 \notin B$. Since $x_0 \in A$ and $x_0 \notin B$, $x_0 \in A \setminus B$. It follows that $x_0 \in (A \setminus B) \cup C$. Since $x_0 \in (A \setminus B) \cup C$ and $x_0 \notin (A \setminus B) \cap C$, $x_0 \in (A \setminus B) \triangle C$.

Case ($x_0 \in C \setminus A$ and $x_0 \in B \cap C$) Since $x_0 \in B \cap C$, $x_0 \in C$. Since $x_0 \in C$, $x_0 \in (A \setminus B) \cup C$. Since $x_0 \in B \cap C$, $x_0 \in B$. Since $x_0 \in B$, $x_0 \notin A \setminus B$. Since $x_0 \notin A \setminus B$, $x_0 \notin (A \setminus B) \cap C$. Since $x_0 \in (A \setminus B) \cup C$ and $x_0 \notin (A \setminus B) \cap C$, $x_0 \in (A \setminus B) \triangle C$.

Case ($x_0 \in C \setminus A$ and $x_0 \notin B \cup C$) Since $x_0 \in C \setminus A$, $x_0 \in C$. Since $x_0 \notin B \cup C$, $x_0 \notin C$ which is a contradiction.

Therefore, $(A \triangle C) \setminus (B \triangle C) \subseteq (A \setminus B) \triangle C$. ∎

It is not true that $(A \setminus B) \triangle C \subseteq (A \triangle C) \setminus (B \triangle C)$. Counterexample:

$A = \varnothing$

$B = \varnothing$

$C = \{1\}$

$A \setminus B = \varnothing$

$(A \setminus B) \triangle C = \{1\}$

$A \triangle C = \{1\}$

$B \triangle C = \{1\}$

$(A \triangle C) \setminus (B \triangle C) = \varnothing$

Clearly $(A \setminus B) \triangle C = \{1\} \nsubseteq (A \triangle C) \setminus (B \triangle C) = \varnothing$.

Problem 27

Consider the following putative theorem.
Theorem? *For every real number x, if $|x-3|<3$ then $0<x<6$.*
Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Proof. Let x be an arbitrary real number, and suppose $|x-3|<3$. We consider two cases.
Case 1. $x-3 \geq 0$. Then $|x-3| = x-3$. Plugging this into the assumption that $|x-3|<3$, we get $x-3<3$, so clearly $x<6$.
Case 2. $x-3<0$. Then $|x-3| = 3-x$, so the assumption $|x-3|<3$ means that $3-x<3$. Therefore $3<3+x$, so $0<x$.
Since we have proven both $0<x$ and $x<6$, we can conclude that $0<x<6$. ∎

The theorem is correct. Proof is wrong it's missing the upper or lower bound of x.

Proof. Let x be an arbitrary real number, and suppose $|x-3|<3$. We consider two cases.

Case 1. $x-3 \geq 0$. Since $x-3 \geq 0$ it follows that $x \geq 3$ so $x \geq 0$. Then $|x-3| = x-3$. Plugging this into the assumption that $|x-3|<3$, we get $x-3<3$, so clearly $x<6$.

Case 2. $x-3<0$. Since $x-3<0$ it follows that $x<3$ so $x<6$. Then $|x-3| = 3-x$, so the assumption $|x-3|<3$ means that $3-x<3$. Therefore $3<3+x$, so $0<x$.

Since we have proven both $0<x$ and $x<6$, we can conclude that $0<x<6$. ∎

Problem 28

Consider the following putative theorem.
Theorem? *For any sets A, B, and C, if $A \setminus B \subseteq C$ and $A \nsubseteq C$ then $A \cap B \neq \varnothing$.*
Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Proof. Suppose $A \setminus B \subseteq C$ and $A \nsubseteq C$. Since $A \nsubseteq C$, so we can choose some x such that $x \in A$ and $x \notin C$. Since $x \notin C$ and $A \setminus B \subseteq C$, $x \notin A \setminus B$. Therefore either $x \notin A$ or $x \in B$. But we already know that $x \in A$, so it follows that $x \in B$. Since $x \in A$ and $x \in B$, $x \in A \cap B$. Therefore $A \cap B \neq \varnothing$. ∎

Solution 28

The proof is correct. It uses direct reasoning based on the definition of set difference. The theorem is correct.

Problem 29

Consider the following putative theorem.
Theorem? $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(xy^2 \neq y-x)$
Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Proof. Let x be an arbitrary real number.
Case 1. $x=0$. Let $y=1$. Then $xy^2=0$ and $y-x=1-0=1$, so $xy^2 \neq y-x$.
Case 2. $x \neq 0$. Let $y=0$. Then $xy^2=0$ and $y-x=-x \neq 0$, so $xy^2 \neq y-x$.
Since these cases are exhaustive, we have shown that $\exists y \in \mathbb{R}(xy^2 \neq y-x)$. Since x was arbitrary, this shows that $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(xy^2 \neq y-x)$. ∎

Solution 29

The proof is correct and it uses proof by cases. The theorem is correct.

Problem 31

Consider the following putative theorem.
Theorem? *Suppose A, B, and C are sets and $A \subseteq B \cup C$. Then either $A \subseteq B$ or $A \subseteq C$.*
Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Proof. Let x be an arbitrary element of A. Since $A \subseteq B \cup C$, it follows that either $x \in B$ or $x \in C$.
Case 1. $x \in B$. Since x was an arbitrary element of A, it follows that $\forall x \in A(x \in B)$, which means that $A \subseteq B$.
Case 2. $x \in C$. Similarly, since x was an arbitrary element of A, we can conclude that $A \subseteq C$.
Thus, either $A \subseteq B$ or $A \subseteq C$. ∎

Solution 31

The theorem is incorrect. It attempts to use proof by cases. From $A \subseteq B \cup C$, it does NOT follow that either $A \subseteq B$ or $A \subseteq C$. Proof is not fixable as the theorem is incorrect. In the first case, the proof shows that one particular $x \in A$ lies in B, but that does not imply all elements of A are in B. The same issue applies to Case 2.

Counterexample:

Let $B = \{1\}$.

Let $C = \{2\}$.

Let $A = \{1, 2\}$,

Obviously $A \subseteq B \cup C$ but $A \nsubseteq B$ and $A \nsubseteq C$.

Problem 33

Prove that $\exists x(P(x) \to \forall y P(y))$. (Note: Assume the universe of discourse is not the empty set.)

Proof. Since the universe of discourse is not empty there are two cases:

Case 1: $P(x)$ is false for some x. Then the implication $P(x) \to \forall y P(y)$ is true.

Case 2: $P(x)$ is true for all x. Then for all y, $P(y)$ is true, so the implication holds for every x.

Therefore, $\exists x(P(x) \to \forall y P(y))$. ∎

3.6 Existence and Uniqueness Proof

Problem 1

Prove that for every real number x there is a unique real number y such that $x^2 y = x - y$.

Proof. We first prove existence. Let $y = \frac{x}{x^2+1}$. Then:

$$\begin{aligned} x^2 y &= x - y \\ \leftrightarrow x^2 \cdot \frac{x}{x^2+1} &= x - \frac{x}{x^2+1} \\ \leftrightarrow x^2 \cdot x &= x(x^2+1) - x \\ \leftrightarrow x^2 \cdot x &= x^3 + x - x \\ \leftrightarrow x^3 &= x^3 \\ \leftrightarrow x &= x \end{aligned}$$

We now prove uniqueness. Suppose $x^2y + y = x$. Then:

$$\begin{aligned} x^2y &= x - y \\ \leftrightarrow x^2y + y &= x \\ \leftrightarrow y(x^2+1) &= x \\ \leftrightarrow y &= \frac{x}{x^2+1} \end{aligned}$$

∎

Problem 2

Prove that there is a unique real number x such that for every real number y, $xy + x - 4 = 4y$.

Proof. We first prove existence. Let $x = 4$. Then:

$$\begin{aligned} xy + x - 4 &= 4y \\ \leftrightarrow 4y + 4 - 4 &= 4y \\ \leftrightarrow 4y &= 4y \\ \leftrightarrow y &= y \end{aligned}$$

We now prove uniqueness.

$$\begin{aligned} xy + x - 4 - 4y &= 0 \\ \leftrightarrow x(y+1) - 4(y+1) &= 0 \\ \leftrightarrow (y+1)(x-4) &= 0 \end{aligned}$$

So there is exactly one value of x, namely $x = 4$, which is a solution for all y. ∎

Problem 3

Prove that for ever real number x, if $x \neq 0$ and $x \neq 1$ then there is a unique real number y such that $y/x = y - x$.

Proof. We first prove existence. Suppose $x \neq 0$ and $x \neq 1$. Let $y = \frac{-x^2}{1-x}$. Then:

$$\begin{aligned} \frac{y}{x} &= y - x \\ \leftrightarrow \frac{(\frac{-x^2}{1-x})}{x} &= \frac{-x^2}{1-x} - x \\ \leftrightarrow \frac{(\frac{-x^2}{1-x})}{x} \cdot (x)(1-x) &= \frac{-x^2}{1-x} \cdot (x)(1-x) - x \cdot (x)(1-x) && \text{since } x \neq 0 \text{ and } x \neq 1 \\ \leftrightarrow \frac{-x^2}{1-x} \cdot (1-x) &= -x^2 \cdot x - x \cdot (x)(1-x) \\ \leftrightarrow -x^2 &= -x^3 - (x^2)(1-x) \\ \leftrightarrow -x^2 &= -x^3 - x^2 + x^3 \\ \leftrightarrow -x^2 &= -x^2 \\ \leftrightarrow x &= x \end{aligned}$$

We now prove uniqueness.

$$\begin{aligned} \frac{y}{x} &= y - x \\ \leftrightarrow \frac{y}{x} &= y - x \\ \leftrightarrow y &= xy - x^2 && \text{Since } x \neq 0 \\ \leftrightarrow y - xy &= -x^2 \\ \leftrightarrow y(1-x) &= -x^2 \\ \leftrightarrow y &= \frac{-x^2}{1-x} \end{aligned}$$

$\blacksquare$

Problem 4

Prove that for every real number x, if $x \neq 0$ then there is a unique real number y such that for every real number z, $zy = z/x$.

Proof. We first prove existence. Suppose $x \neq 0$. Let $y = \frac{1}{x}$ which is defined since $x \neq 0$. Then:

$$\begin{aligned} zy &= \frac{z}{x} \\ \leftrightarrow z \cdot \frac{1}{x} &= \frac{z}{x} \\ \leftrightarrow \frac{z}{x} &= \frac{z}{x} \end{aligned}$$

We now prove uniqueness.

$$\begin{aligned} zy &= \frac{z}{x} \\ \leftrightarrow xzy &= z && \text{Since } x \neq 0 \\ \leftrightarrow xzy - z &= 0 \\ \leftrightarrow z(xy-1) &= 0 \end{aligned}$$

It then follows, since $xy - 1 = 0$ is a solution, that $y = \frac{1}{x}$. $\blacksquare$

Problem 6

Let U be any set.
(a) Prove that there is a unique $A \in \mathcal{P}(U)$ such that for every $B \in \mathcal{P}(U)$, $A \cup B = B$.
(b) Prove that there is a unique $A \in \mathcal{P}(U)$ such that for every $B \in \mathcal{P}(U)$, $A \cup B = A$.

Proof. We first show existence. Let $A = \varnothing$. Since $A \in \mathcal{P}(U)$, $A \subseteq U$. It trivially follows that $A \cup B = B$.

We now show uniqueness. Let C and D be arbitrary sets such that $C \cup B = B$, $C \in \mathcal{P}(U)$ and $D \cup B = B$, $D \in \mathcal{P}(U)$. Let $B = D$, then $C \cup D = D$. Let $B = C$, then $D \cup C = C$. Since $C \cup D = D \cup C$, $D = C$. $\blacksquare$

Problem 7

Let U be any set.
(a) Prove that there is a unique $A \in \mathcal{P}(U)$ such that for every $B \in \mathcal{P}(U)$, $A \cap B = B$.
(b) Prove that there is a unique $A \in \mathcal{P}(U)$ such that for every $B \in \mathcal{P}(U)$, $A \cap B = A$.

Proof. We first show existence. Let $A = U$. Clearly $A \subseteq U$, therefore $A \in \mathcal{P}(U)$. Let B be an arbitrary set such that $B \in \mathcal{P}(U)$. Since $B \in \mathcal{P}(U)$, $B \subseteq U$. It follows that for all $x \in B$, $x \in A$. Therefore, $A \cap B = B$.

We now show uniqueness. Let C and D be arbitrary sets such that for all $B \in \mathcal{P}(U)$, $C \cap B = B$ and $D \cap B = B$. Let $B = D$, then $C \cap D = D$. Let $B = C$, then $D \cap C = C$. Since $C \cap D = D \cap C$, $D = C$. ∎

Proof. We first show existence. Let $A = \varnothing$. Clearly $A \subseteq U$, therefore $A \in \mathcal{P}(U)$. Let B be an arbitrary set such that $B \in \mathcal{P}(U)$. Trivially $A \cap B = A$.

We now show uniqueness. Let C and D be arbitrary sets such that for all $B \in \mathcal{P}(U)$, $C \cap B = C$ and $D \cap B = D$. Let $B = D$, then $C \cap D = C$. Let $B = C$, then $D \cap C = D$. Since $C \cap D = D \cap C$, $C = D$. ∎

Problem 9

Recall that you showed in excersize 14 of Section 1.4 that symmetric difference is associative; in other words, for all sets A, B and $C, A \triangle (B \triangle C) = (A \triangle B) \triangle C$. You may also find it useful in this problem to note that the symmetric difference is clearly commutative; in other words, for all sets A and B, $A \triangle B = B \triangle A$.
(a) Prove that there is a unique identity element for symmetric difference. In other words, there is a unique set X such that for every set A, $A \triangle X = A$.
(b) Prove that every set has a unique inverse for the operation of symmetric difference. In other words, for every set A there is a unique set B such that $A \triangle B = X$, where X is the identity element from part (a).
(c) Prove that for any sets A and B there is a unique set C such that $A \triangle C = B$.
(d) Prove that for every set A there is a unique set $B \subseteq A$ such that for every set $C \subseteq A$, $B \triangle C = A \setminus C$.

Proof. We first show existence. Let $X = \varnothing$ and let A be an arbitrary set. By the definition of the symmetric difference $A \triangle X = (A \cup X) \setminus (A \cap X)$. Cleary $A \cup X = A \cup \varnothing = A$, and $A \cap X = A \cap \varnothing = \varnothing$. Therefore, $A \triangle X = (A \cup X) \setminus (A \cap X) = A \setminus \varnothing = A$.

We now show uniqueness. Let B and C be arbitrary sets such that for all A, $A \triangle B = A$ and $A \triangle C = A$. Let $A = B$, then $B \triangle C = B$. Let $A = C$, then $C \triangle B = C$. So, $B = B \triangle C = C \triangle B = C$ showing $B = C$. ∎

Proof. We first show existence. Let A be an arbitrary set. Let B be a set such that $B = A$. By the definition of the symmetric difference $A \triangle B = (A \cup B) \setminus (A \cap B)$. Clearly, $A \cup B = A \cup A = A$ and $A \cap B = A \cap A = A$. It follows that $A \triangle B = (A \cup B) \setminus (A \cap B) = A \setminus A = \varnothing$.

We now show uniqueness. Suppose there exists a set B such that $A \triangle B = \varnothing$ and $A \neq B$. Either $A \nsubseteq B$ or $B \nsubseteq A$.

Suppose $A \nsubseteq B$. Let x be an element such that $x \in A$ and $x \notin B$. Since $x \in A$, $x \in A \cup B$. Also, since $x \in A$ and $x \notin B$, $x \notin A \cap B$. Since $x \in A \cup B$ and $x \notin A \cap B$, $x \in (A \cup B) \setminus (A \cap B) = A \triangle B$. Contradicting $A \triangle B = \varnothing$.

Suppose $B \nsubseteq A$. Let x be an element such that $x \in B$ and $x \notin A$. Since $x \in B$, $x \in A \cup B$. Also, since $x \in B$ and $x \notin A$, $x \notin A \cap B$. Since $x \in A \cup B$ and $x \notin A \cap B$, $x \in (A \cup B) \setminus (A \cap B) = A \triangle B$. Contradicting $A \triangle B = \varnothing$. ∎

Proof. We first prove existence. Let A and B be arbitrary sets. Let $C = A \triangle B$. Now $A \triangle C = A \triangle (A \triangle B) = (A \triangle A) \triangle B$. By previous proof $A \triangle A = \varnothing$, therefore $(A \triangle A) \triangle B = \varnothing \triangle B$. Now, by the previous previous proof $\varnothing \triangle B = B$.

We now show uniqueness. Suppose X is another set such that $A \triangle X = B$. Then

$$X = (A \triangle A) \triangle X = A \triangle (A \triangle X) = A \triangle B = C.$$

So $X = C$. ∎

Proof. We first prove existence. Let $B = A$. Then for any $C \subseteq A$:

$$B \triangle C = A \triangle C = (A \setminus C) \cup (C \setminus A) = A \setminus C,$$

since $C \subseteq A$ implies $C \setminus A = \varnothing$.

We now prove uniqueness. Suppose X and Y are sets such that for every $C \subseteq A$, $X \triangle C = A \setminus C$ and $Y \triangle C = A \setminus C$. Take $C = \varnothing$, then

$$X \triangle \varnothing = A \setminus \varnothing = A \quad \text{and} \quad Y \triangle \varnothing = A \setminus \varnothing = A.$$

Therefore $X = Y = A$. ∎

Problem 10

Suppose A is a set, and for every family of sets $\mathcal{F}$, if $\bigcup \mathcal{F} = A$ then $A \in \mathcal{F}$. Prove that A has exactly one element.

Proof. Let A be an arbitrary set. For contradiction, assume for every family of sets $\mathcal{F}$, if $\bigcup \mathcal{F} = A$ then $A \in \mathcal{F}$ and A does not have exactly one element. Either $A = \varnothing$ or A has more than 1 element.

Suppose $A = \varnothing$. Let $\mathcal{F} = \varnothing$. Clearly $\bigcup \mathcal{F} = A$, but $A \notin \mathcal{F}$ so $A \neq \varnothing$.

Suppose A has more than one element. Let $\mathcal{T} = \{\{x\} \mid x \in A\}$. Clearly $\bigcup \mathcal{T} = A$, but $A \notin \mathcal{T}$ which is a contradiction. ∎

Problem 13

(a) Prove that there is a unique real number c such that there is a unique real number x such that $x^2 + 3x + c = 0$. (In other words, there is a unique real number c such that the equation $x^2 + 3x + c = 0$ has exactly one solution.)
(b) Show that it is not the case that there is a unique real number x such that there is a unique real number c such that $x^2 + 3x + c = 0$. (Hint: You should be able to prove that for every real number x there is a unique real number c such that $x^2 + 3x + c = 0$.)

Proof. We first show existence. Let $c = \frac{9}{4}$. Then: $x^2 + 3x + \frac{9}{4} = 0 \iff (x + \frac{3}{2})(x + \frac{3}{2}) = 0$. There is a single solution, namely $x = \frac{-3}{2}$.

We now show uniqueness. Given a polynomial of degree 2 there is a single solution if and only if the discriminant is zero. Letting $a = 1, b = 3$ we get $\sqrt{3^2 - 4c} = 0 \iff 3^2 - 4c = 0 \iff c = \frac{9}{4}$. ∎

Proof. We first show existence. Let x be an arbitrary real number. Let $c = -x^2 - 3x$. Then: $x^2 + 3x + c = 0 \iff x^2 + 3x - x^2 - 3x = 0 \iff 0 = 0$.

We now show uniqueness: $x^2 + 3x + c = 0 \iff x^2 + c = -3x \iff c = -x^2 - 3x$. ∎

3.7 More Examples of Proofs

Problem 2

Prove that there is a unique positive real number m that has the following two properties:
(a) For every positive real number x, $\frac{x}{x+1} < m$.
(b) If y is any positive real number with the property that for every positive real number x, $\frac{x}{x+1} < y$, then $m \leq y$.

Proof. We first show existence. Let $m = 1$. First note that $\frac{x}{x+1} < 1$. Trivially $\frac{x}{x+1} < 1 = m$. Suppose $y < 1$ and $y > \frac{x}{x+1}$ for all $x > 0$. Let $x = \frac{y}{1-y}$. Then:

$$\frac{x}{x+1} = \frac{\frac{y}{1-y}}{\frac{y}{1-y} + 1} = \frac{y}{y+1-y} = y$$

So $y \geq 1 = m$.

We now prove uniqueness. Suppose m_1, m_2 satisfy both properties. Then $m_1 \leq y$ and $m_2 \leq y$. Applying this to $y = m_1$ and $y = m_2$ gives $m_1 \leq m_2$ and $m_2 \leq m_1$, therefore, $m_1 = m_2$. ∎

Problem 6

Suppose $\mathcal{F}$ is a nonempty family of sets. Let $I = \bigcup \mathcal{F}$ and $J = \bigcap \mathcal{F}$. Suppose also that $J \neq \emptyset$, and notice that if follows that for every $X \in \mathcal{F}$, $X \neq \emptyset$, and also that $I \neq \emptyset$. Finally, suppose that $\{A_i \mid i \in I\}$ is an indexed family of sets.
(a) Prove that $\bigcup_{i \in I} A_i = \bigcup X \in \mathcal{F}(\bigcup_{i \in X} A_i)$.
(b) Prove that $\bigcap_{i \in I} A_i = \bigcup_{X \in \mathcal{F}}(\bigcap_{i \in X} A_i)$.
(c) Prove that $\bigcup_{i \in J} A_i \subseteq \bigcap_{X \in \mathcal{F}}(\bigcup_{i \in X} A_i)$. Is it always the case that $\bigcup_{i \in J} A_i = \bigcap_{X \in \mathcal{F}}(\bigcup_{i \in X} A_i)$? Give either a proof or a counterexample to justify your answer.
(d) Discover and prove a theorem relating $\bigcap_{i \in J} A_i$ and $\bigcup_{X \in \mathcal{F}}(\bigcap_{i \in X} A_i)$.

Proof. Suppose x is an arbitrary element such that $x \in \bigcup_{i \in I} A_i$. Let i be the index of the set x is in. ∎

Problem 10

Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct? (Note: The proof uses the fact that $\sqrt{2}$ is irrational, which we'll prove in Chapter 6 - see Theorem 6.4.5)

Proof. Either $\sqrt{2}^{\sqrt{2}}$ is ration or it's irrational.
Suppose $\sqrt{2}^{\sqrt{2}}$ is rational. Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then $a^b = \sqrt{2}^{\sqrt{2}}$, which is what we're assuming in this case is rational.
Suppose $\sqrt{2}^{\sqrt{2}}$ is irrational. Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. The a is irrational by assumption, and we know that b is irrational. Also,

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^{2} = 2$$

which is rational. ∎

Solution:

Yes the proof is correct. It uses proof by cases. Since the proof is valid the theorem is correct.

4 Relations

4.1 Ordered Pairs and Cartesian Products

Problem 5

Prove parts 2 and 3 of Theorem 4.13.

Proof. Show $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Let p be an arbitrary element in $A \times (B \cup C)$. Let $(x, y) = p$. Clearly $x \in A$ and $y \in (B \cup C)$. It follows that $y \in B$ or $y \in C$. Suppose $y \in B$. Then $x \in A$ and $y \in B$ so $p \in A \times B$. Suppose $y \in C$. Then $x \in A$ and $y \in C$ so $p \in A \times C$. So, $p \in A \times B$ or $p \in A \times C$. It follows that $p \in (A \times B) \cup (A \times C)$.

Let p be an arbitrary element in $(A \times B) \cup (A \times C)$. It follows that $p \in A \times B$ or $p \in A \times C$. Let $(x, y) = p$. Clearly $x \in A$ and y is in B or C. It follows that $p \in A \times (B \cup C)$. ∎

Proof. Show $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Let p be an arbitrary element in $(A \times B) \cap (C \times D)$. It follows that $p \in A \times B$ and $p \in C \times D$. Let $(x, y) = p$. Then, $(x, y) \in A \times B$ and $(x, y) \in C \times D$. So, $x \in A$ and $x \in C$. Therefore, $x \in A \cap C$. Also, $y \in B$ and $y \in D$. Therefore $y \in B \cap D$. Since, $x \in A \cap C$ and $y \in B \cap D$, $p \in (A \cap C) \times (B \cap D)$.

Let p be an arbitrary element in $(A \cap C) \times (B \cap D)$. Let $(x, y) = p$. Then, $x \in A \cap C$ and $y \in B \cap D$. Since $x \in A \cap C$, $x \in A$ and $x \in C$. Since $y \in B \cap D$, $y \in B$ and $y \in D$. Since $x \in A$ and $y \in B$, $p \in A \times B$. Since $x \in C$ and $y \in D$, $p \in C \times D$. S ince $p \in A \times B$ and $p \in C \times D$, $p \in (A \times B) \cap (C \times D)$. ∎

Problem 6

What's wrong with the following proof that for any sets A, B, C, and D, $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$? (Note that this is the reverse of the inclusion in part 4 of Theorem 4.1.3)

Proof. Suppose $(x, y) \in (A \cup C) \times (B \cup D)$.Then $x \in A \cup C$. and $y \in B \cup D$, so either $x \in A$ or $x \in C$, and either $y \in B$ or $y \in D$. We consider these cases seperately.
Case 1. $x \in A$ and $y \in B$. Then $(x, y) \in A \times B$.
Case 2. $x \in C$ and $y \in D$. Then $(x, y) \in C \times D$.
Thus, either $(x, y) \in A \times B$ or $(x, y) \in C \times D$, so $(x, y) \in (A \times B) \cup (C \times D)$. ∎

Solution

The proof doesn't account for all cases namely, $x \in A$ and $y \in D$ or $x \in C$ and $y \in B$.

Problem 7

If A has m elements and B has n elements, how many elements does $A \times B$ have?

Solution

$A \times B$ has $m \cdot n$ elements.

Problem 12

Prove that for any sets A, B, C, and D, if $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

Proof. For contradiction, suppose $A \times B$ and $C \times D$ are disjoint and A and C are not disjoint and B and D are not disjoint. Let x be an element of $A \cap C \neq \varnothing$. Let y be an element of $B \cap D \neq \varnothing$. Let $p = (x, y)$. Since

$x \in A$ and $y \in B$, $p \in A \times B$. Since $x \in C$ and $y \in D$, $p \in C \times D$. It follows that $p \in (A \times B) \cap (C \times D)$ contradicting the disjointness of $A \times B$ and $C \times D$. ∎

Problem 13

Suppose $I \neq \varnothing$. Prove that for any indexed family of sets $\{A_i \mid i \in I\}$ and any set B, $(\bigcap_{i \in I} A_i) \times B = \bigcap_{i \in I}(A_i \times B)$. Where in the proof does the assumption that $I \neq \varnothing$ get used?

For the intersection operation to be defined the set must not be emtpy (More operations on sets 15 b).

Proof. Let p be an arbitrary element in $(\bigcap_{i \in I} A_i) \times B$. Let $(x, y) = p$. Notice $x \in A_i$ for all $i \in I$ and $y \in B$. It follows that $p \in A_i \times B$ for all $i \in I$. So $p \in \bigcap_{i \in I}(A_i \times B)$.

Let p be an arbitrary element in $\bigcap_{i \in I}(A_i \times B)$. Notice $p \in A_i \times B$ for all $i \in I$. Let $(x, y) = p$. It follows that $x \in A_i$ for all $i \in I$. Therefore $x \in \bigcap_{i \in I} A_i$. Since $y \in B$, it follows that $p \in \bigcap_{i \in I}(A_i) \times B$. ∎

Problem 15

This problem was suggested by Professor Alan Taylor of Union College, NY. Consider the following putative theorem.
Theorem? *For any sets A, B, C, and D, if $A \times B \subseteq C \times D$ then $A \subseteq C$ and $B \subseteq D$.*
Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Proof. Suppose $A \times B \subseteq C \times D$. Let a be an arbitrary element of A and let b be an arbitrary element of B. Then $(a, b) \in A \times B$, so since $A \times B \subseteq C \times D$, $(a, b) \in C \times D$. Therefore $a \in C$ and $b \in D$. Since a and b were arbitrary elements of A and B, respectively, this shows $A \subseteq C$ and $B \subseteq D$. ∎

Solution:

The proof is incorrect. It implicitly assumes The sets A and B are not empty. It can be fixed by assuming A and B are not empty. The theorem is incorrect.

4.2 Relations

Problem 1

(a) $\{(p, q) \in P \times P \mid$ the person p is a parent of person $q\}$, where P is the set of all living people.
(b) $\{(x, y) \in \mathcal{R}^2 \mid y > x^2\}$

Solution (a):

Domain: People who have children that are alive.

Range: People whose parents are still living.

Solution (b):

Domain: All $x \in \mathcal{R}$.

Range: All $y > 0$.

Problem 5

Suppose that $A = \{1, 2, 3\}, B = \{4, 5, 6\}, R = \{(1, 4), (1, 5), (2, 5), (3, 6)\}$, and $S = \{(4, 5), (4, 6), (5, 4), (6, 6)\}$. Note that R is a relation from A to B and S is a relation from B to B. Find the following relations:
(a) $S \circ R$.

(b) $S \circ S^{-1}$.

Solution (a):

$S \circ R = \{(1,5),(1,6),(1,4),(2,4),(3,6)\}$

Solution (b):

$S \circ S^{-1} = \{(5,5),(5,6),(6,6),(6,5),(4,4)\}$

Problem 7

(a) Prove part 3 of Theorem 4.2.5 by imitating the proof of part 2 in the text.
(b) Give an alternative proof of part 3 of Theorem 4.2.5 by showing that it follows from part 1 and 2.
(c) Complete the proof of part 4 of Theorem 4.2.5.
(d) Prove part 5of Theorem 4.2.5.

(a) We want to prove that $Ran(R^{-1}) = Dom(R)$.

Proof.

$$\begin{aligned} a \in Ran(R^{-1}) &\iff \exists b \in B((b,a) \in R^{-1}) \\ &\iff \exists b \in B((a,b) \in R) \\ &\iff a \in Dom(R) \end{aligned}$$

∎

(b) We want to prove that $Ran(R^{-1}) = Dom(R)$ **using parts (1) and (2).**

Proof. Let a be an arbitrary element in $Ran(R^{-1})$. By part 2 $a \in Ran(R^{-1}) \iff a \in Dom((R^{-1})^{-1})$. By part 1 $a \in Dom((R^{-1})^{-1}) \iff a \in Dom(R)$. ∎

(c) We want to prove that $(T \circ S) \circ R = T \circ (S \circ R)$.

Proof. Let (a,d) be an arbitrary pair of elements in $(T \circ S) \circ R$. There exists c such that $(a,c) \in R$ and $(c,d) \in (T \circ S)$. There exists b such that $(c,b) \in S$ and $(b,d) \in T$. Since $(a,c) \in R$ and $(c,b) \in S$, $(a,b) \in S \circ R$. Since $(a,b) \in S \circ R$ and $(b,d) \in T$, $(a,d) \in T \circ (S \circ R)$. ∎

(d) We want to prove that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proof. Let (y,x) be an arbitrary pair of elements in $(S \circ R)^{-1}$. Then $(x,y) \in S \circ R$. There exists c such that $(x,c) \in R$ and $(c,y) \in S$. So $(c,x) \in R^{-1}$ and $(y,c) \in S^{-1}$. It follows that $(y,x) \in R^{-1} \circ S^{-1}$.

Let (y,x) be an arbitrary pair of elements in $R^{-1} \circ S^{-1}$. There exists c such that $(y,c) \in S^{-1}$ and $(c,x) \in R^{-1}$. Then $(c,y) \in S$ and $(x,c) \in R$. So $(x,y) \in S \circ R$ and $(y,x) \in (S \circ R)^{-1}$. ∎

Problem 8

Let $E = \{(p,q) \in P \times P \mid \text{ the person } p \text{ is an enemy of the person } q\}$, and $F = \{(p,q) \in P \times P \mid \text{ the person } p \text{ is a friend of the person } q\}$, where P is the set of all people. What does the saying "an enemy of one's enemy is one's friend" mean about the relations E and F.

$E \circ E \subseteq F$

Problem 9

Suppose R is a relation from A to B and S is a relation from B to C.
(a) Prove that $Dom(S \circ R) \subseteq Dom(R)$.
(b) Prove that if $Ran(R) \subseteq Dom(S)$ then $Dom(S \circ R) = Dom(R)$.
(c) Formulate and prove similar theorems about $Ran(S \circ R)$.

Proof. Let x be an arbitrary element in $Dom(S \circ R)$. There exists y such that $(x, y) \in S \circ R$. There exists c such that $(x, c) \in R$ and $(c, y) \in S$. Since $(x, c) \in R$, $x \in Dom(R)$. ∎

Proof. Suppose $Ran(R) \subseteq Dom(S)$. Let x be an arbitrary element such that $x \in Dom(S \circ R)$. There exists y, such that $(x, y) \in S \circ R$. There exists c, such that $(x, c) \in R$ and $(c, y) \in S$. Clearly $x \in Dom(R)$.

Let x be an arbitrary element such that $x \in Dom(R)$. There exists y such that $(x, y) \in R$. Clearly $y \in Ran(R)$. It follows that $y \in Dom(S)$. There exists c such that $(y, c) \in S$. Since $(x, y) \in R$ and $(y, c) \in S$, $(x, c) \in S \circ R$. Clearly $x \in Dom(S \circ R)$. ∎

Theorem: $Ran(S \circ R) \subseteq Ran(S)$.

Proof. Let y be an arbitrary element such that $y \in Ran(S \circ R)$. There exists x such that $(x, y) \in S \circ R$. There exists c such that $(x, c) \in R$ and $(c, y) \in S$. Clearly $y \in Ran(S)$. ∎

Theorem: If $Ran(R) = Dom(S)$ then $Ran(S \circ R) = Ran(S)$.

Proof. Suppose $Ran(R) = Dom(S)$.

Let y be an arbitrary element in $Ran(S \circ R)$. There exists x such that $(x, y) \in S \circ R$. There exists c such that $(x, c) \in R$ and $(c, y) \in S$. It follows that $y \in Ran(S)$.

Let y be an arbitrary element in $Ran(S)$. There exists b such that $(b, y) \in S$. It follows that $b \in Ran(R)$. There exists c such that $(c, b) \in R$. It follows that $(c, y) \in S \circ R$. Then $y \in Ran(S \circ R)$. ∎

Problem 10

Suppose R and S are relations from A to B. Must the following statements be true? Justify your answers with proofs or counterexamples.
(a) $R \subseteq Dom(R) \times Ran(R)$.
(b) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$.
(c) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.

Proof. Let (x, y) be arbitrary elements in R. Clearly $x \in Dom(R)$ and $y \in Ran(R)$. So $(x, y) \in Dom(R) \times Ran(R)$. ∎

Proof. Suppose $R \subseteq S$. Let (y, x) be arbitrary elements in R^{-1}. It follows that $(x, y) \in R$ therefore $(x, y) \in S$. Since $(x, y) \in S$, $(y, x) \in S^{-1}$. ∎

Proof. Let (y, x) be arbitrary elements in $(R \cup S)^{-1}$. Clearly $(x, y) \in R \cup S$.

Suppose $(x, y) \in R$, Then $(y, x) \in R^{-1}$ and $(y, x) \in R^{-1} \cup S^{-1}$.

Suppose $(x, y) \in S$, Then $(y, x) \in S^{-1}$ and $(y, x) \in S^{-1} \cup R^{-1}$.

Let (y, x) be arbitrary elements in $R^{-1} \cup S^{-1}$. Either $(y, x) \in R^{-1}$ or $(y, x) \in S^{-1}$.

Suppose $(y, x) \in R^{-1}$. Then $(x, y) \in R \subseteq R \cup S$. and $(y, x) \in (R \cup S)^{-1}$.

Suppose $(y,x) \in S^{-1}$. Then $(x,y) \in S \subseteq R \cup S$. and $(y,x) \in (R \cup S)^{-1}$. ∎

Problem 11

Suppose R is a relation from A to B and S is a relation from B to C. Prove that $S \circ R = \varnothing$ iff $Ran(R)$ and $Dom(S)$ are disjoint.

Proof. $(\rightarrow)$ Assume $S \circ R = \varnothing$ and for contradiction $Ran(R) \cap Dom(S) \neq \varnothing$. Let x be an element in $Ran(R) \cap Dom(S)$. Then $x \in Ran(R)$ and $x \in Dom(S)$. Since $x \in Ran(R)$ there exists c such that $(c,x) \in R$. Since $x \in Dom(S)$ there exists d such that $(x,d) \in S$. Since $(c,x) \in R$ and $(x,d) \in S$ then $(c,d) \in S \circ R$ contradicting $S \circ R = \varnothing$.

$(\leftarrow)$ Assume $Ran(R) \cap Dom(S) = \varnothing$ and for contradiction $S \circ R \neq \varnothing$. Let (x,y) be an arbitrary pair of elements such that $(x,y) \in S \circ R$. There exists c such that $(x,c) \in R$ and $(c,y) \in S$. Since $c \in Ran(R)$ and $c \in Dom(S)$, $c \in Ran(R) \cap Dom(S)$ contradicting $Ran(R) \cap Dom(S) = \varnothing$. ∎

Problem 12

Suppose R is a relation from A to B and S and T are relations from B to C.
(a) Prove that $(S \circ R) \setminus (T \circ R) \subseteq (S \setminus T) \circ R$.
(b) What's wrong with the following proof that $(S \setminus T) \circ R \subseteq (S \circ R) \setminus (T \circ R)$?

Proof. Suppose $(a,c) \in (S \setminus T) \circ R$. Then we can choose some $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S \setminus T$, so $(b,c) \in S$ and $(b,c) \notin T$. Since $(a,b) \in R$ and $(b,c) \notin T$, $(a,c) \notin T \circ R$. Therefore $(a,c) \in (S \circ R) \setminus (T \circ R)$. Since (a,c) was arbitrary, this shows that $(S \setminus T) \circ R \subseteq (S \circ R) \setminus (T \circ R)$. ∎

(c) Must it be true that $(S \setminus T) \circ R \subseteq (S \circ R) \setminus (T \circ R)$? Justify your answer with either a proof or a counterexample.

Proof. Let (x,y) be an arbitrary pair of elements in $(S \circ R) \setminus (T \circ R)$. There exists c such that $(x,c) \in R$ and $(c,y) \in S$. There does not exist d such that $(x,d) \in R$ and $(d,y) \in T$. Since $(x,c) \in R$, $(c,y) \notin T$ (Note: Let $c = d$ from previous statement). Since $(c,y) \in S$ and $(c,y) \notin T$, $(c,y) \in S \setminus T$. Since $(x,c) \in R$ and $(c,y) \in S \setminus T$, $(x,y) \in (S \setminus T) \circ R$. ∎

Solution:

It was never shown that $(a,c) \in S \circ R$.

Solution:

$$A = \{1\}, \quad B = \{2,3\}, \quad C = \{4\}$$
$$R = \{(1,2),(1,3)\}, \quad S = \{(2,4),(3,4)\}, \quad T = \{(3,4)\}$$
$$S \setminus T = \{(2,4)\}, \quad (S \setminus T) \circ R = \{(1,4)\}$$
$$S \circ R = \{(1,4)\}, \quad T \circ R = \{(1,4)\}, \quad (S \circ R) \setminus (T \circ R) = \varnothing$$
$$(S \setminus T) \circ R \nsubseteq (S \circ R) \setminus (T \circ R)$$

Problem 13

Suppose R is a relation from A to B, and S and T are relations from B to C. Must the following statements be true? Justify your answers with proofs or counterexamples.
(a) If R and S are disjoint, then so are R^{-1} and S^{-1}.
(b) If R and S are disjoint, then so are $T \circ R$ and $T \circ S$.
(c) If $T \circ R$ and $T \circ S$ are disjoint, then so are R and S.

Proof. Suppose R and S are disjoint and for contradiction R^{-1} and S^{-1} are not disjoint. Let $(y, x) \in R^{-1}$ and $(y, x) \in S^{-1}$. Then $(x, y) \in R$ and $(x, y) \in S$ which is a contradiction. ∎

Solution 13 (b):

$$R = \{(1,1)\} \quad S = \{(1,2)\} \quad T = \{(1,1),(2,1)\}$$
$$R \cap S = \varnothing$$
$$T \circ R = \{(1,1)\} \quad T \circ S = \{(1,1)\}$$
$$(T \circ R) \cap (T \circ S) = \{(1,1)\}$$

Solution 13 (c):

$$R = \{(1,1)\} \quad S = \{(1,1)\} \quad T = \varnothing$$
$$R \cap S = \{(1,1)\}$$
$$T \circ R = \varnothing \quad T \circ S = \varnothing$$
$$(T \circ R) \cap (T \circ S) = \varnothing$$

Problem 14

Suppose R is a relation from A to B, and S and T are relations from B to C. Must the following statements be true? Justify your answers with proofs or counterexamples.
(a) If $S \subseteq T$ then $S \circ R \subseteq T \circ R$.
(b) $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$.
(c) $(S \cap T) \circ R = (S \circ R) \cap (T \circ R)$.
(d) $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$.

Proof. Suppose $S \subseteq T$. Let (x, y) be arbitrary elements in $(S \circ R)$. There exists c such that $(x, c) \in R$ and $(c, y) \in S$. Since $(c, y) \in S$, $(c, y) \in T$. Since $(x, c) \in R$ and $(c, y) \in T$, $(x, y) \in T \circ R$. ∎

Proof. Let (x, y) be an arbitrary pair of elements such that $(x, y) \in (S \cap T) \circ R$. There exists c such that $(x, c) \in R$ and $(c, y) \in (S \cap T)$ So $(c, y) \in S$ and $(c, y) \in T$. Since $(x, c) \in R$ and $(c, y) \in S$, then $(x, y) \in S \circ R$. Since $(x, c) \in R$ and $(c, y) \in T$, then $(x, y) \in T \circ R$. Since $(x, y) \in S \circ R$ and $(x, y) \in T \circ R$, $(x, y) \in (S \circ R) \cap (T \circ R)$. ∎

Solution 14 (c):

$$A = \{1\}, \quad B = \{2,3\}, \quad C = \{4\}$$
$$R = \{(1,2),(1,3)\}, \quad S = \{(2,4)\}, \quad T = \{(3,4)\}$$
$$S \cap T = \varnothing$$
$$(S \cap T) \circ R = \varnothing$$
$$S \circ R = \{(1,4)\}, \quad T \circ R = \{(1,4)\}$$
$$(S \circ R) \cap (T \circ R) = \{(1,4)\}$$
$$(S \cap T) \circ R \neq (S \circ R) \cap (T \circ R)$$

Proof. Let (x, y) be an arbitrary pair of elements such that $(x, y) \in (S \cup T) \circ R$. There exists c such that $(x, c) \in R$ and $(c, y) \in S \cup T$. So either $(c, y) \in S$ or $(c, y) \in T$. If $(c, y) \in S$, then since $(x, c) \in R$, $(x, y) \in S \circ R$. If $(c, y) \in T$, then since $(x, c) \in R$, $(x, y) \in T \circ R$. In either case, $(x, y) \in (S \circ R) \cup (T \circ R)$. Therefore, $(S \cup T) \circ R \subseteq (S \circ R) \cup (T \circ R)$.

Let (x, y) be an arbitrary pair of elements such that $(x, y) \in (S \circ R) \cup (T \circ R)$. Then either $(x, y) \in S \circ R$ or $(x, y) \in T \circ R$. If $(x, y) \in S \circ R$, then there exists c such that $(x, c) \in R$ and $(c, y) \in S \subseteq S \cup T$. If $(x, y) \in T \circ R$, then there exists c such that $(x, c) \in R$ and $(c, y) \in T \subseteq S \cup T$. In either case, $(x, y) \in (S \cup T) \circ R$. Therefore, $(S \circ R) \cup (T \circ R) \subseteq (S \cup T) \circ R$. ∎

4.3 More About Relations

Problem 3

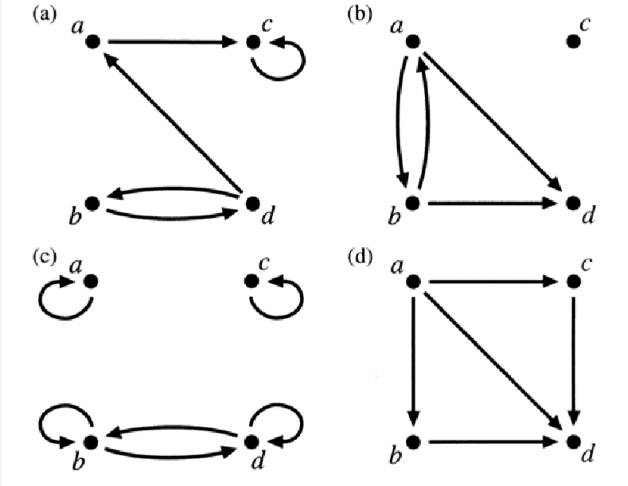
Let $A = \{1, 2, 3, 4\}$. Draw a directed graph i_A, the identity relation on A.

Solution:

$$\begin{array}{cccc} \{1\} < - - | & \{2\} < - - | & \{3\} < - - | & \{4\} < - - | \\ |- - - -| & |- - - -| & |- - - -| & |- - - -| \end{array}$$

Problem 4

List the ordered paris in the relations represented by the directed graphs in Figure 4.4. Determine whether each relation is reflexive, symmetric, or transitive.



Solution (a):

$$R = \{(a, c), (d, a), (d, b), (b, d)\}$$

Reflexive: NO, Symmetric: NO, Transitive: NO

Solution (b):

$$R = \{(a, d), (b, d), (b, a), (a, b)\}$$

Reflexive: NO, Symmetric: NO, Transitive: NO

Solution (c):

$$R = \{(a,a),(b,b),(c,c),(d,d),(b,d),(d,b)\}$$

Reflexive: YES, Symmetric: YES, Transitive: YES

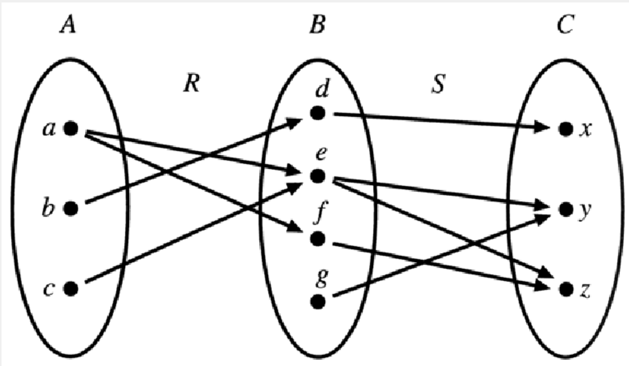
Solution (d):

$$R = \{(a,c),(a,b),(a,d),(b,d),(c,d)\}$$

Reflexive: NO, Symmetric: NO, Transitive: YES

Problem 5

Figure 4.5 shows two relations R and S. Find $S \circ R$.



Solution:

$S \circ R = \{(a,y),(a,z),(a,z),(b,x),(c,y),(c,z)\}$

Problem 7

Prove R is reflexive iff $i_A \subseteq R$, where i_A is the identity relation of A.

Proof. ($\rightarrow$) Suppose R is reflexive. It follows that for all $x \in A$, $(x,x) \in R$. So we can select two arbitrary elements $x, y \in A$ and if $x = y$ then $(x,y) = (x,x) \in R$. Therefore $i_A \subseteq R$.

($\leftarrow$) Suppose $i_A \subseteq R$. For all $x, y \in A$ where $x = y$, $(x,y) \in R$. So we can select two arbitrary elements $x, x \in A$ and since $x = x$, $(x,x) \in R$. Therefore, R is reflexive. ∎

Problem 8

Prove R is transitive iff $R \circ R \subset R$.

Proof. ($\rightarrow$) Suppose R is transitive. So for all $x, y, z \in A$, if $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$. Let (x,y) be an arbitrary pair of elements in $R \circ R$. There exists an element c such that $(x,c) \in R$ and $(c,y) \in R$. Since $(x,c) \in R$ and $(c,y) \in R$ it follows that $(x,y) \in R$.

($\leftarrow$) Suppose $R \circ R \subseteq R$. Let (x,y) be an arbitrary pair of elements in $R \circ R$. There exists an element c such that $(x,c) \in R$ and $(c,y) \in R$. It follows since $R \circ R \subseteq R$ that $(x,y) \in R$. Since (x,y) was arbitrary in $R \circ R$, this holds for all $x, y, z \in A$. So if $(x,z) \in R$ and $(z,y) \in R$, then $(x,y) \in R$. Therefore, R is transitive. ∎

Problem 9

Suppose A and B are sets.
(a) Show that for every relation R from A to B. $R \circ i_A = R$.
(b) Show that for every relation R from A to B. $i_B \circ R = R$.

Proof. We first show $R \circ i_A \subseteq R$. Let (x, y) be an arbitrary pair of elements in $R \circ i_A$. There exists an element c such that $(x, c) \in i_A$ and $(c, y) \in R$. But $(x, c) \in i_A$ so $c = x$. Therefore $(x, y) \in R$.

We now show $R \subseteq R \circ i_A$. Let (x, y) be an arbitrary pair of elements in R. We know $(x, x) \in i_A$. Since $(x, x) \in i_A$ and $(x, y) \in R$, $(x, y) \in R \circ i_A$.

Since $R \circ i_A \subseteq R$ and $R \subseteq R \circ i_A$ $R \circ i_A = R$. ∎

Proof. We first show $i_B \circ R \subseteq R$. Let (x, y) be an arbitrary pair of elements in $i_B \circ R$. There exists an element c such that $(x, c) \in R$ and $(c, y) \in i_B$. Since $(c, y) \in i_B$, $y = c$. Therefore $(x, y) \in R$.

We now show $R \subseteq i_B \circ R$. Let (x, y) be an arbitrary pair of elements in R. We know $(y, y) \in i_B$. Since $(x, y) \in R$ and $(y, y) \in i_B$, $(x, y) \in i_B \circ R$.

Since $i_B \circ R \subseteq R$ and $R \subseteq i_B \circ R$, $i_B \circ R = R$. ∎

Problem 10

Suppose S is a relation on A. Let $D = Dom(S)$ and $R = Ran(S)$.
Prove that $i_D \subseteq S^{-1} \circ S$ and $i_R \subseteq S \circ S^{-1}$.
Prove that $i_R \subseteq S \circ S^{-1}$.

Proof. Let (x, x) be an arbitrary pair in i_D. There exists y such that $(x, y) \in S$. Clearly $(y, x) \in S^{-1}$. Since $(x, y) \in S$ and $(y, x) \in S^{-1}$ then $(x, x) \in S \circ S^{-1}$. ∎

Proof. Let (y, y) be an arbitrary pair in i_R. There exists x such that $(x, y) \in S$. Clearly $(y, x) \in S^{-1}$. Since $(x, y) \in S$ and $(y, x) \in S^{-1}$ then $(y, y) \in S \circ S^{-1}$. ∎

Problem 11

Suppose R is a relation on A. Prove that if R is reflexive then $R \subseteq R \circ R$.

Proof. Suppose R is reflexive. Let (x, y) be an arbitrary pair in R. Since R is reflexive $(y, y) \in R$. Let $z = y$, then $(x, z) \in R$ and $(z, y) \in R$. Therefore $(x, y) \in R \circ R$. ∎

Problem 12

Suppose R is a relation on A.
(a) Prove that if R is reflexive, then so is R^{-1}
(b) Prove that if R is symmetric, then so is R^{-1}.
(c) Prove that if R is transitive, then so is R^{-1}.

Proof. Suppose R is reflexive. Let x be an arbitrary element in A. Since R is reflexive $(x, x) \in R$. Since $(x, x) \in R$, $(x, x) \in R^{-1}$. Since for all $x \in A$, $(x, x) \in R^{-1}$ it follows that R^{-1} is reflexive. ∎

Proof. Suppose R is symmetric. Let x, y be elements in A such that $(x, y) \in R$. Since R is symmetric $(y, x) \in R$. Since $(x, y) \in R$ then $(y, x) \in R^{-1}$. Since $(y, x) \in R$ then $(x, y) \in R^{-1}$. Since $(y, x) \in R^{-1}$ and $(x, y) \in R^{-1}$ it follows that R^{-1} is symmetric. ∎

Proof. Suppose R is transitive. Let x, y, z be arbitrary elements in A such that $(x, y) \in R$ and $(y, z) \in R$. Since R is transitive $(x, z) \in R$. Clearly, $(y, x) \in R^{-1}$, $(z, y) \in R^{-1}$, and $(z, x) \in R^{-1}$. Since $(z, x), (z, y), (y, x) \in R^{-1}$ it follows that R^{-1} is transitive. ∎

Problem 12

Suppose R is a relation on A.
(a) Prove that if R is reflexive, then so is R^{-1}.
(b) Prove that if R is symmetric, then so is R^{-1}.
(c) Prove that if R is transitive, then so is R^{-1}.

Proof. Suppose R is reflexive. Let x be an arbitrary element in A. Since R is reflexive $(x, x) \in R$. It then follows trivially that $(x, x) \in R^{-1}$. Therefore R^{-1} is reflexive. ∎

Proof. Suppose R is symmetric. Let (y, x) be an arbitrary element in R^{-1}. It follows that $(x, y) \in R$. Since R is symmetric $(y, x) \in R$. It follows that $(x, y) \in R^{-1}$. Therefore R^{-1} is symmetric. ∎

Proof. Suppose R is transitive. Let (x, y) and (y, z) be arbitrary elements in R^{-1}. It follows that $(y, x) \in R$ and $(z, y) \in R$. Since R is transitive $(z, x) \in R$. It follows that $(x, z) \in R^{-1}$. Therefore R^{-1} is transitive. ∎

Problem 13

Suppose R_1 and R_2 are relations on A. For each part give either a proof or a counterexample to justify your answer.
(a) If R_1 and R_2 are reflexive, must $R_1 \cup R_2$ be reflexive?
(b) If R_1 and R_2 are symmetric, must $R_1 \cup R_2$ be symmetric?
(c) If R_1 and R_2 are transitive, must $R_1 \cup R_2$ be transitive?

Proof. Suppose R_1 and R_2 are reflexive. Let x be an arbitrary element in A. Since R_1 and R_2 are reflexive it follows that $(x, x) \in R_1$ and $(x, x) \in R_2$. It follows that $(x, x) \in R_1 \cup R_2$. Therefore $R_1 \cup R_2$ is reflexive. ∎

Proof. Suppose R_1 and R_2 are symmetric. Let (x, y) be an arbitrary element in $R_1 \cup R_2$. It follows that $(x, y) \in R_1$ or $(x, y) \in R_2$. Suppose $(x, y) \in R_1$. Since R_1 is symmetric $(y, x) \in R_1$ and therefore $(y, x) \in R_1 \cup R_2$. Suppose $(x, y) \in R_2$. Since R_2 is symmetric $(y, x) \in R_2$ and therefore $(y, x) \in R_1 \cup R_2$. Therefore $R_1 \cup R_2$ is symmetric. ∎

Solution:

$$A = \{1, 2, 3\} \quad R_1 = \{(1, 2)\} \quad R_2 = \{(2, 3)\}$$

$$R_1 \cup R_2 = \{(1, 2), (2, 3)\}$$

Now clearly R_1 and R_2 are transitive. However, $R_1 \cup R_2$ is not transitive because $(1, 2) \in R_1 \cup R_2$ and $(2, 3) \in R_1 \cup R_2$ but $(1, 3) \notin R_1 \cup R_2$

Problem 22

Consider the following putative theorem:
Theorem 1. *Suppose R is a relation on A. If R is symmetric and transitive, then R is reflexive.*
Is the following proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

Proof. Let x be an arbitrary element of A. Let y be any element of A such that xRy. Since R is symmetric, it follows that yRx. But then by transitivity, since xRy and yRx we can conclude that xRx. Since x was arbitrary, we have shown that $\forall x \in A(xRx)$, so R is reflexive. ∎

Solution:

The proof is invalid since it assumes properties about $y \in A$ namely that $x, y \in R$. The theorem is incorrect so it cannot be fixed.

> Problem 24
>
> Let $R = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid |m - n| \leq 1\}$, which is a relation on $\mathbb{Z}$. This exercise will illustrate why, in part 1 of Definition 4.3.2, we define the phrase "R is reflexive on A", rather than simply "R is reflexive".
> (a) Is R reflexive on $\mathbb{N}$?
> (b) Is R reflexive on $\mathbb{Z}$?

Solution (a):

Yes R is reflexive on $\mathbb{N}$. Let x be an arbitrary element in $\mathbb{N}$. It follows that $|x - x| = 0 \leq 1$ and therefore $(x, x) \in R$.

Solution (b):

Yes R is reflexive on $\mathbb{Z}$. Let x be an arbitrary element in $\mathbb{Z}$. It follows that $|x - x| = 0 \leq 1$ and therefore $(x, x) \in R$.

4.4 Ordering Relations

> Problem 1
>
> In each case, say whether or not R is a partial order on A. If so, is it a total order?
> (a) $A = \{a, b, c\}, R = \{(a, a), (b, a), (b, b), (b, c), (c, c)\}$
> (b) $A = \mathbb{R}, R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x| \leq |y|\}$
> (c) $A = \mathbb{R}, R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x| < |y| \text{ or } x = y\}$

Solution (a):

R is a partial order on A. R is not a total order on A because aRc and cRa are false.

Solution (b):

R is not a partial order. Consider $(3, -3)$ so clearly $3R(-3)$ and $(-3)R3$ but $3 \neq -3$.

Solution (c):

R is a partial order on A. R is not a total order because $5R(-5)$ and $(-5)R5$ are false.

> Problem 2
>
> In each case, say whether or not R is a partial order on A. If so, is it a total order?
> (a) $A =$ the set of all English words. $R = \{(x, y) \in A \times A \mid$ where the word y occurs at least as late in alphabetical order as the word $x\}$.
> (b) $A =$ the set of all English words. $R = \{(x, y) \in A \times A \mid$ where the first letter of the word y occurs at least as late in the alphabet as the first letter of the word $x\}$.
> (c) $A =$ the set of all countries in the world. $R = \{(x, y) \in A \times A \mid$ where the population of country y is at least as large as the population of country $x\}$.

Solution (a):

R is a partial order on A and a total order on A.

Solution (b):

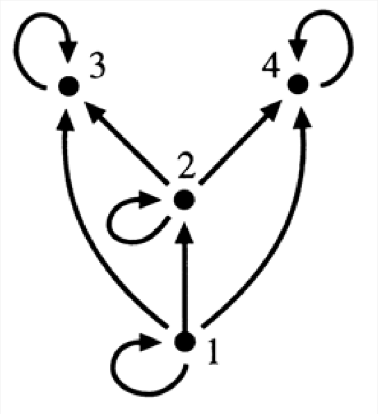
R is not a partial order on A. Consider "the" and "tip". Now "the"R"tip" and "tip"R"the" but "tip"$\neq$"the". So R is not antisymmetric.

Solution (c)

R is not a partial order on A. Consider $a =$ America population 100 and $m =$ Mexico population 100. Then aRm and mRa but $a \neq m$ so R is not antisymmetric.

Problem 3

In each case find all minimal and maximal elements of B. Also find, if they exist, the largest and smallest elements of B, and the least upper bound and greatest lower bound of B.
(a) $R =$ the relation shown in the directed graph in Figure 4.6, $B = \{2, 3, 4\}$.
(b) $R = \{(x, y \in \mathbb{R} \times \mathbb{R} \mid x \leq y)\}$, $B = \{(x \in \mathbb{R} \mid 1 \leq x < 2)\}$.
(c) $R = \{(x, y) \in \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \mid x \subseteq y\}$, $B = \{(x \in \mathcal{P}(\mathbb{N})) \mid x$ has at most 5 elements.$\}$



Solution (a):

Minimal elements: $\{2\}$, Maximal elements: $\{3, 4\}$

Smallest element: 2, Largest element: none

Greatest lower bound: 2, Least upper bound: none

Solution (b):

Minimal elements: $\{1\}$, Maximal elements: none

Smallest element: 1, Largest element: none

Greatest lower bound: 1, Least upper bound: 2

Solution (c):

Minimal elements: $\{\varnothing\}$, Maximal elements: Any 5 element subset of $\mathbb{N}$

Smallest element: $\varnothing$, Largest element: none

Greatest lower bound: $\varnothing$, Least upper bound: none

Problem 4

Suppose R is a relation on A. You might think R could not be both antisymmetric and symmetric, but this isn't true. Prove that R is both antisymmetric and symmetric iff $R \subseteq i_A$.

Proof. $(\rightarrow)$ Suppose R is both antisymmetric and symmetric. Let x, y be arbitrary elements in A such that xRy. Since R is symmetric and xRy it follows that yRx. Then, since R is antisymmetric and xRy and yRx, it follows that $x = y$. Therefore $(x, y) = (x, x) \in i_A$.

($\leftarrow$) Suppose $R \subseteq i_A$. Let x, y be arbitrary elements in A such that xRy. Since $R \subseteq i_A$ it follows that $(x, y) \in i_A$ and therefore $x = y$. So xRy and yRx implies $x = y$ so R is antisymmetric. Also, xRy implies yRx so R is symmetric.

Thus R is both antisymmetric and symmetric iff $R \subseteq i_A$. ∎

Problem 6

Suppose R_1 and R_2 are partial orders on A. For each part, give either a proof or a counterexample to justify your answer.
(a) Must $R_1 \cap R_2$ be a partial order on A?
(b) Must $R_1 \cup R_2$ be a partial order on A?

Proof. To prove that $R_1 \cap R_2$ is a partial order on A, we must show it is reflexive, antisymmetric, and transitive on A. First note that R_1, R_2 being partial orders on A implies that they are reflexive, antisymmetric, and transitive on A.

We first show $R_1 \cap R_2$ is reflexive. Let x be an arbitrary element in A. Since R_1, R_2 are reflexive on A it follows that xR_1x and xR_2x. It then follows that $x(R_1 \cap R_2)x$. Therefore $R_1 \cap R_2$ is reflexive.

We now show $R_1 \cap R_2$ is antisymmetric. Let x, y be arbitrary elements in A such that $x(R_1 \cap R_2)y$ and $y(R_1 \cap R_2)x$. This implies xR_1y and yR_1x. Since R_1 is antisymmetric it follows that $y = x$. A similar argument shows $y = x$ on R_2. Therefore $R_1 \cap R_2$ is antisymmetric.

Finally, we show $R_1 \cap R_2$ is transitive. Suppose x, y, z are arbitrary elements in A such that $x(R_1 \cap R_2)y$ and $y(R_1 \cap R_2)z$. It follows that xR_1y and yR_1z. Since R_1 is transitive it follows that xR_1z. A similar argument shows xR_2z. Therefore $R_1 \cap R_2$ is transitive.

Since $R_1 \cap R_2$ is reflexive, antisymmetric, and transitive, it is a partial order on A. ∎

Solution (b):

$$A = \{1, 2, 3\} \quad R_1 = \{(1,1), (2,2), (3,3), (1,2)\} \quad R_2 = \{(1,1), (2,2), (3,3)(2,3)\}$$
$$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$$

Notice $1(R_1 \cup R_2)2$ and $2(R_1 \cup R_2)3$ but not $1(R_1 \cup R_2)3$ therefore $R_1 \cup R_2$ is not transitive and not a partial order.

Problem 13

Suppose R is a partial order on A. Prove that R^{-1} is also a partial order on A. If R is a total order, will R^{-1} also be a total order?

Proof. Suppose R is a partial order on A.

Let x be an arbitrary element in A. Since R is a partial order it is reflexive therefore xRx. But xRx implies $xR^{-1}x$ so R^{-1} is reflexive.

Let x, y be arbitrary elements in A such that $xR^{-1}y$ and $yR^{-1}x$. Since $xR^{-1}y$, $yR^{-1}x$ it follows that yRx, xRy. Since R is a partial order it is antisymmetric therefore since yRx, xRy it follows that $x = y$. Thus $xR^{-1}y$ and $yR^{-1}x$ implies $x = y$ so R^{-1} is antisymmetric.

Let x, y, z be arbitrary elements in A such that $xR^{-1}y$ and $yR^{-1}z$. Since $xR^{-1}y$, $yR^{-1}z$ it follows that yRx, zRy. Since R is a partial order it is transitive therefore since zRy, yRx it follows that zRx and therefore $xR^{-1}z$. Thus $xR^{-1}y$ and $yR^{-1}z$ implies $xR^{-1}z$ so R^{-1} is transitive.

Since R^{-1} was reflexive, antisymmetric, and transitive it is a partial order. ∎

Solution (b):

By the previous proof if R is a total order than R^{-1} is at least a partial order. Now, R being a total order implies for all x, y either xRy or yRx. Clearly either $yR^{-1}x$ or $xR^{-1}y$ is true in that case therefore R^{-1} is also a total order.

Problem 14

Suppose R is a partial order on A, $B \subseteq A$, and $b \in B$. exercise 13 shows that R^{-1} is also a partial order on A.
(a) Prove that b is the R-largest element of B iff it is the R^{-1}-smallest element of B.
(b) Prove that b is an R-maximal element of B iff it is an R^{-1}-minimal element of B.

Proof. ($\rightarrow$) Suppose that b is an R-largest element of B. So for all $a \in B$, aRb. It then follows that $bR^{-1}a$. Since a was arbitrary, for all $a \in B$, $bR^{-1}a$. So b is the R^{-1}-smallest element of B.

($\leftarrow$) The argument is symmetrical.

Therefore b is the R-largest element of B iff it is the R^{-1}-smallest element of B. ∎

Proof. ($\rightarrow$) Suppose that b is an R-maximal element of B. There does not exist $a \in B$ such that bRa and $b \neq a$. So there does not exist $a \in B$ such that $aR^{-1}b$ and $a \neq b$. Therefore b is the R^{-1}-minimal element of B.

($\leftarrow$) The argument is symmetrical.

Therefore b is an R-maximal element of B iff it is an R^{-1}-minimal element of B. ∎

Problem 19

Consider the following putative theorem.

Theorem Suppose R is a total order on A and $B \subset A$. Then every element of B is either the smallest element of B or the largest element of B.

(a) What's wrong with the following proof of the theorem?

Proof. Suppose $b \in B$. Let x be an arbitrary element of B. Since R is a total order, either bRx or xRb.
Case 1. bRx. Since x was arbitrary, we can conclude that $\forall x \in B(bRx)$, so b is the smallest element of R.
Case 2. xRb. Since x was arbitrary, we can conclude that $\forall x \in B(xRb)$, so b is the largest element of R.
Thus, b is either the smallest element of B or the largest element of B. Since b was arbitrary, every element of B is either its smallest element or its largest element. ∎

(b) Is the theorem correct? Justify your answer with either a proof or a counterexample.

Solution (a):

Within each case x is not an arbitrary element of B. The issue is x has an additional property that bRx in case 1 and xRb in case 2.

Solution (b):

The theorem is incorrect. Consider the following counterexample:

$$A = \mathbb{N}, B = \{1, 2, 3\}, R = \{(x, y) \in A \times A \mid x \leq y\}$$

Clearly 2 is not the R-largest element since $(2,3) \in R$ and $2 \neq 3$. But 2 is also not the R-smallest element since $(1,2) \in R$ and $1 \neq 2$.

Problem 23

Prove Theorem 4.4.11.
Theorem 4.4.11 Suppose A is a set $\mathcal{F} \subseteq \mathcal{P}(A)$, and $\mathcal{F} \neq \varnothing$. Then the least upper bound of $\mathcal{F}$ (in the subset partial order) is $\bigcup \mathcal{F}$ and the greatest lower bound of $\mathcal{F}$ is $\bigcap \mathcal{F}$.

Proof. Let y be an arbitrary element in A such that $A \in \mathcal{F}$. For all $x \in B$ where $B \in \mathcal{F}$, $x \in \bigcup \mathcal{F}$. Since $y \in A$ where $A \in \mathcal{F}$, $y \in \bigcup \mathcal{F}$. Thus $\bigcup \mathcal{F}$ is an upper bound. Now, suppose there exists a smaller upper bound $\mathcal{G}$ such that $\mathcal{G} \subset \bigcup \mathcal{F}$. Let x be an element in $\bigcup \mathcal{F}$ that is not in $\mathcal{G}$. Then $x \in A$ for some $A \in \mathcal{F}$, but $x \notin \mathcal{G}$, so $A \nsubseteq \mathcal{G}$. So $\mathcal{G}$ is not an upper bound. Thus $\bigcup \mathcal{F}$ is the *l.u.b.* ∎

Proof. Let y be an arbitrary element in A such that $A \in \mathcal{F}$. For all $x \in \bigcap \mathcal{F}$, we have $x \in B$ for every $B \in \mathcal{F}$. Thus $\bigcap \mathcal{F} \subseteq A$ for all $A \in \mathcal{F}$, which shows that $\bigcap \mathcal{F}$ is a lower bound. Now, suppose there exists a greater lower bound $\mathcal{G}$ such that $\bigcap \mathcal{F} \subset \mathcal{G}$. Let x be an element in $\mathcal{G}$ that is not in $\bigcap \mathcal{F}$. Then $x \notin A$ for some $A \in \mathcal{F}$, so $\mathcal{G} \nsubseteq A$. So $\mathcal{G}$ is not a lower bound. Thus $\bigcap \mathcal{F}$ is the *g.l.b.* ∎

Problem 24

Suppose R is a relation on A. Let $S = R \cup R^{-1}$.
(a) Show that S is a symmetric relation on A and $R \subseteq S$.
(b) Show that if T is symmetric relation on A and $R \subseteq T$ then $S \subseteq T$.
Note that this exercise shows that S is the smallest element of the set $\mathcal{F} = \{T \subseteq A \times A \mid R \subseteq T \text{ and } T \textbf{ is symmetric}\}$; in other words, it is the smallest symmetric relation on A that contains R as a subset. This relation S is called the symmetric closure of R.

Proof. Suppose x, y are arbitrary elements in A and $(x,y) \in S$. So $(x,y) \in R \cup R^{-1}$. Thefore $(x,y) \in R$ or $(x,y) \in R^{-1}$. Suppose $(x,y) \in R$ then $(y,x) \in R^{-1}$ and it follows that $(y,x) \in R \cup R^{-1} = S$. Suppose $(x,y) \in R^{-1}$ then $(y,x) \in R$ and it follows that $(y,x) \in R \cup R^{-1} = S$. Therefore S is a symmetric relation on A. ∎

Proof. Let (x,y) be an arbitrary element in R. Cearly, $(x,y) \in R \cup R^{-1} = S$. Therefore $R \subseteq S$. ∎

Proof. Suppose (x,y) is an arbitrary pair of elements in S. It follows that $(x,y) \in R \cup R^{-1}$. So $(x,y) \in R$ or $(x,y) \in R^{-1}$. Suppose $(x,y) \in R$. Since $R \subseteq T$ then $(x,y) \in T$. Suppose $(x,y) \in R^{-1}$ so $(y,x) \in R$. Since $R \subseteq T$ then $(y,x) \in T$. Since T is symmetric it follows that $(x,y) \in T$. Therefore $S \subseteq T$. ∎

Problem 25

Suppose that R is a relation on A. Let $\mathcal{F} = \{T \subseteq A \times A \mid R \subseteq T \text{ and } T \text{ is transitive}\}$.
(a) Show that $\mathcal{F} \neq \varnothing$.
(b) Show that $\bigcap \mathcal{F}$ is a transitive relation on A and $R \subseteq \bigcap \mathcal{F}$.
(c) Show that $\bigcap \mathcal{F}$ is the smallest transitive relation on A that contains R as a subset. The relation $\bigcap \mathcal{F}$ is called the transitive closure of R.

Proof. Since $A \times A$ is a relation on A, $R \subseteq A \times A$, and $A \times A$ is transitive it follows that $A \times A \in \mathcal{F}$. Therefore $\mathcal{F} \neq \varnothing$. ∎

Proof. Suppose (x,y) and (y,z) are arbitrary elements in $\bigcap \mathcal{F}$. For all $\mathcal{G} \in \mathcal{F}$, $(x,y) \in \mathcal{G}$ and $(y,z) \in \mathcal{G}$. Since $\mathcal{G}$ is transitive then $(x,z) \in \mathcal{G}$. Since $\mathcal{G}$ was arbitrary it follows that $(x,z) \in \bigcap \mathcal{F}$. ∎

Proof. Suppose (x, y) is an arbitrary element in R, and $\mathcal{G}$ is an arbitrary set in $\mathcal{F}$. Since $R \subseteq \mathcal{G}$ it follows that $(x, y) \in \mathcal{G}$. Since $\mathcal{G}$ was arbitrary, $(x, y) \in \bigcap \mathcal{F}$. ∎

Proof. Suppose (x, y) is an arbitrary element in $\bigcap \mathcal{F}$, and T is a transitive relation on A such that $R \subseteq T$. Now for all $\mathcal{G} \in \mathcal{F}$, $(x, y) \in \mathcal{G}$. It then follows, since $T \in \mathcal{F}$ that $(x, y) \in T$. Therefore $\bigcap \mathcal{F} \subseteq T$. Showing that $\bigcap \mathcal{F}$ is the smallest transitive relation on A that contains R as a subset. ∎

Problem 26

Suppose R_1 and R_2 are relations on A, and let $R_1 \subseteq R_2$.
(a) Let S_1 and S_2 be the symmetric closures of R_1 and R_2, respectively. Prove that $S_1 \subseteq S_2$. (See exercise 24 for the definition of symmetric closure.)
(b) Let T_1 and T_2 be the transitive closures of R_1 and R_2, respectively. Prove that $T_1 \subseteq T_2$. (See exercise 25 for the definition of transitive closure.)

Proof. Let (x, y) be an arbitrary element in S_1. So $(x, y) \in R_1 \cup R_1^{-1}$ therefore $(x, y) \in R_1$ or $(x, y) \in R_1^{-1}$. Suppose $(x, y) \in R_1$. Since $R_1 \subseteq R_2$, $(x, y) \in R_2$. Then $(x, y) \in R_2 \subseteq R_2 \cup R_2^{-1} = S_2$. Suppose $(x, y) \in R_1^{-1}$ it follows that $(y, x) \in R_1$. Since $R_1 \subseteq R_2$, $(y, x) \in R_2$ so $(x, y) \in R_2^{-1}$. Then $(x, y) \in R_2 \cup R_2^{-1} = S_2$. Therefore $S_1 \subseteq S_2$. ∎

Proof. Suppose (x, y) is an arbitrary element in R_1. Since $R_1 \subseteq R_2$ it follows that $(x, y) \in R_2$. Thus $(x, y) \in T_2$ and $R_1 \subseteq T_2$. Suppose $(x, y), (y, z)$ are arbitrary pairs in R_1. Since $R_1 \subseteq R_2$ it follows that $(x, y), (y, z) \in R_2$. Thus $(x, y), (y, z), (x, z) \in T_2$. So T_2 is a transitive relation containing R_1. Since T_1 is the minimal transitive relation containing R_1, it follows that $T_1 \subseteq T_2$. ∎

Problem 27

Suppose R_1 and R_2 are relations on A, and let $R = R_1 \cup R_2$.
(a) Let S_1, S_2, and S be symmetric closures of R_1, R_2, and R, respectively. Prove that $S_1 \cup S_2 = S$. (See exercise 24 for the definition of symmetric closure.)
(b) Let T_1, T_2, and T be the transitive closures of R_1, R_2, and R, respectively. Prove that $T_1 \cup T_2 \subseteq T$, and give an example to show that it may happen that $T_1 \cup T_2 \neq T$. (See exercise 25 for the definition of transitive closure.)

Proof. We first show $S_1 \cup S_2 \subseteq S$. Suppose (x, y) is an arbitrary pair of elements in $S_1 \cup S_2$. So $(x, y) \in S_1$ or $(x, y) \in S_2$.

Suppose $(x, y) \in S_1$ So $(x, y) \in R_1 \cup R_1^{-1}$ and therefore $(x, y) \in R_1$ or $(x, y) \in R_1^{-1}$. Suppose $(x, y) \in R_1$ then it follows, since $R = R_1 \cup R_2$, $(x, y) \in R$ Then since $S = R \cup R^{-1}$ it follows that $(x, y) \in S$. A similar argument shows that supposing $(x, y) \in R_1^{-1}$ then $(x, y) \in S$. A similar argument shows that supposing $(x, y) \in S_2$ then $(x, y) \in S$. Therefore $S_1 \cup S_2 \subseteq S$.

We now show $S \subseteq S_1 \cup S_2$. Suppose (x, y) is an arbitrary pair of elements in S. So $(x, y) \in R$ or $(x, y) \in R^{-1}$. Suppose $(x, y) \in R$. Since $R = R_1 \cup R_2$ it follows that $(x, y) \in R_1 \cup R_2$. So $(x, y) \in R_1$ or $(x, y) \in R_2$. Suppose $(x, y) \in R_1$ then $(x, y) \in R_1 \cup R_1^{-1} = S_1 \subseteq S_1 \cup S_2$. A similar argument shows that supposing $(x, y) \in R_2$ then $(x, y) \in S_1 \cup S_2$. A similar argument shows that supposing $(x, y) \in R^{-1}$ then $(x, y) \in S_1 \cup S_2$. Therefore $S \subseteq S_1 \cup S_2$.

Since $S_1 \cup S_2 \subseteq S$ and $S \subseteq S_1 \cup S_2$ it follows that $S = S_1 \cup S_2$. ∎

Proof. Let (x, y) be an arbitrary pair in $T_1 \cup T_2$. Either $(x, y) \in T_1$ or $(x, y) \in T_2$.

Suppose $(x, y) \in R_1$. Since $R = R_1 \cup R_2$ it follows that $(x, y) \in R$. Then since $R \subseteq T$ it follows that $(x, y) \in T$. Thus $R_1 \subseteq T$.

Suppose $(x, y), (y, z) \in R_1$. Since $R = R_1 \cup R_2$ it follows that $(x, y), (y, z) \in R$. Then since T is the transitive closure of R, it follows that $(x, z) \in T$. Thus T is a transitive relation containing R_1. Since T_1 is the transitive closure of R_1 it follows that $T_1 \subseteq T$. Clearly $T_1 \cup T_2 \subseteq T$.

A similar argument shows that supposing $(x, y) \in R_2$ then $T_1 \cup T_2 \subseteq T$.

Therefore $T_1 \cup T_2 \subseteq T$. ∎

Solution:

$$R_1 = \{(3,4)\}, R_2 = \{(4,5)\}, R = \{(3,4), (4,5)\}$$

$$T_1 = \{(3,4)\}, T_2 = \{(4,5)\}, T_1 \cup T_2 = \{(3,4), (4,5)\}, T = \{(3,4), (4,5), (3,5)\}$$

Clearly $T_1 \cup T_2 \neq T$.

Problem 28

Suppose A is a set.
(a) Prove that if A has at least two elements then there is no largest antisymmetric relation on A. In other words, there is no relation R on A such that R is antisymmetric, and for every antisymmetric relation S on A, $S \subseteq R$.
(b) Suppose R is a total order on A. Prove that R is a maximal antisymmetric relation on A. In other words, there is no antisymmetric relation S on A such that $R \subseteq S$ and $R \neq S$.

Proof. Suppose A has at least two elements. Let R be an arbitrary antisymmetric relation on A. Let $a, b \in A$ such that $a \neq b$. If $\neg aRb$ and $\neg bRa$ then we can construct a relation S such that $S = R \cup \{(a, b)\}$ or $S = R \cup \{(b, a)\}$. In either case $S \not\subseteq R$. So either aRb or bRa. Suppose aRb. Then we can construct a relation S such that $S = (R \setminus \{(a, b)\}) \cup \{(b, a)\}$; then $S \not\subseteq R$. Suppose bRa. Then we can construct a relation S such that $S = (R \setminus \{(b, a)\}) \cup \{(a, b)\}$; then $S \not\subseteq R$. Therefore there is no largest antisymmetric relation on A. ∎

Proof. Suppose S is an antisymmetric relation on A such that $R \subseteq S$ and $R \neq S$. Let (a, b) be a pair of elements such that $(a, b) \in S$ and $(a, b) \notin R$. Since R is a total order and $(a, b) \notin R$, it follows that $a \neq b$ and $(b, a) \in R$. Since $(a, b) \in S$ and $a \neq b$ it follows that $(b, a) \notin S$. Finally, $(b, a) \in R$ and $(b, a) \notin S$ contradicts that $R \subseteq S$. Thus R is a maximal antisymmetric relation on A. ∎

Problem 30

Suppose R is a relation on A, and let T be the transitive closure of R. Prove that if R is symmetric, then so is T. (Hint: Assume that R is symmetric. Prove that $R \subseteq T^{-1}$ and T^{-1} is transitive. What can you conclude about T and T^{-1}? See exercise 25 for the definition of transitive closure.)

Proof. Suppose R is symmetric. Let (x, y) be an arbitrary element in R. Since R is symmetric it follows that $(y, x) \in R$. Then $(y, x) \in T$ so $(x, y) \in T^{-1}$. Therefore $R \subseteq T^{-1}$.

Suppose $(x, y), (y, z)$ are arbitrary pairs in T^{-1}. It follows that $(y, x), (z, y) \in T$. Then, since T is transitive, $(z, x) \in T$ so $(x, z) \in T^{-1}$. Therefore T^{-1} is transitive and contains R.

Now, since $R \subseteq T^{-1}$ and T^{-1} is transitive, T, being the minimal transitive closure of R, thus $T \subseteq T^{-1}$.

Let U be any transitive relation containing R. Then U^{-1} is transitive and contains R, so by minimality of T, $T \subseteq U^{-1}$. Taking inverses gives $T^{-1} \subseteq U$. Since U was arbitrary, T^{-1} is minimal. Thus $T^{-1} = T$, showing that T is symmetric. ∎

4.5 Equivalence Relations

Problem 1

Find all partitions of the set $A = \{1, 2, 3\}$

Solution:

$\mathcal{F}$ is a partition of A if it has the following properites:

1. $\bigcup \mathcal{F} = A$.
2. $\mathcal{F}$ is pairwise disjoint.
3. $\forall X \in \mathcal{F}(X \neq \varnothing)$.

Partitions of A:

1. $\{\{1\}, \{2\}, \{3\}\}$
2. $\{\{1, 2\}, \{3\}\}$
3. $\{\{1\}, \{2, 3\}\}$
4. $\{\{1, 3\}, \{2\}\}$
5. $\{\{1, 2, 3\}\}$

Problem 2

Find all equivalence relations on the set $A = \{1, 2, 3\}$.

Solution:

R is called an equivalence relation on A if it is reflexive, symmetric, and transitive.

All equivalence relations on A:

1. {(1, 1), (2, 2), (3, 3)}
2. {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}
3. {(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)}
4. {(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)}
5. {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)}

Problem 7

Let T be the set of all triangles, and let $S = \{(s, t) \in T \times T \mid \text{the triangles } s \text{ and } t \text{ are similar}\}$. (Recall that two triangles are similar if the angles of one triangle are equal to corresponding angles of the other.) Verify that S is an equivalent relation.

Proof. We need to show S is reflexive, symmetric, and transitive. Clearly a triangle is similar to itself so S is reflexive. Also, given two triangles A, B if A is similar to B then B is similar to A thus S is symmetric. Finally, given three triangles A, B, C if A is similar to B and B is similar to C then A is similar to C thus S is transitive. Therefore S is an equivalence relation on T. ∎

Problem 8

Complete the proof of Lemma 4.5.7.
We'll prove that R is reflexive and leave the rest for you to do in excersize 8.

Lemma 1. *Suppose A is a set and $\mathcal{F}$ is a partition of A. Let $R = \bigcup_{X\in\mathcal{F}}(X \times X)$. Then R is an equivalence relation on A. We will call R the equivalence relation determined by $\mathcal{F}$.*

Proof. We must show that R is symmetric and transitive.

To show that R is symmetric, let (x, y) be an arbitrary element of R. By the definition of R, $(x, y) \in \bigcup_{X\in\mathcal{F}}(X \times X)$. So there exists a set $X \in \mathcal{F}$ such that $(x, y) \in X \times X$. Since $x, y \in X$ it follows that $(y, x) \in X \times X$ and therefore $(y, x) \in \bigcup_{X\in\mathcal{F}}(X \times X) = R$. Thus R is symmetric.

To show that R is transitive, let $(x, y), (y, z)$ be arbitrary elements of R. By the definition of R, $(x, y), (y, z) \in \bigcup_{X\in\mathcal{F}}(X \times X)$. So there exists sets $X, Y \in \mathcal{F}$ such that $(x, y) \in X \times X$ and $(y, z) \in Y \times Y$. Since $y \in X$ and $y \in Y$ it follows that $X = Y$ since $\mathcal{F}$ is a partition of A. Since $X = Y$ it follows that $(x, y) \in X \times X$ and $(y, z) \in X \times X$. Therefore $(x, z) \in X \times X$ and $(x, z) \in \bigcup_{X\in\mathcal{F}}(X \times X) = R$. Thus R is transitive. ∎

Problem 11

Let $\equiv_m$ be the "congruence modulo m" relation defined in the text, for a postive integer m.
(a) Complete the proof of Theorem 4.5.10 by showing that $\equiv_m$ is reflexive and symmetric.
(b) Find all the equivalence classes for $\equiv_2$ and $\equiv_3$. How many equivalence classes are there in each case? In general how many equivalence classes do you think there are for $\equiv_m$.

Proof. We need to show that $\equiv_m$ is reflexive and symmetric.

Let x be an arbitrary integer. To show that $\equiv_m$ is reflexive notice $x - x = km$ when $k = 0$. Therefore $x \equiv_m x$. Thus $\equiv_m$ is reflexive.

We now show that $\equiv_m$ is symmetric. Let x, y be arbitrary integers such that $x \equiv_m y$. Then $m \mid x - y$ so $x - y = km$ for some integer k. Multiplying both sides by -1 shows that $y - x = (-k)m$. Therefore $y \equiv_m x$. Thus $\equiv_m$ is symmetric. ∎

Solution (b):

Two equivalent classes for $\equiv_2$:

$$[0]_2 = \{\dots, -4, -2, 0, 2, 4, \dots\}, \quad [1]_2 = \{\dots, -3, -1, 1, 3, 5, \dots\}.$$

Three equivalent classes for $\equiv_3$:

$$\begin{aligned} [0]_3 &= \{\dots, -6, -3, 0, 3, 6, \dots\}, \\ [1]_3 &= \{\dots, -5, -2, 1, 4, 7, \dots\}, \\ [2]_3 &= \{\dots, -4, -1, 2, 5, 8, \dots\}. \end{aligned}$$

Problem 12

Prove that for every integer n, either $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$.

Proof. Suppose n is an arbitrary integer. Either n is even or n is odd.

If n is even, it can be expressed as $2k$ where k is an integer. Now $n^2 = 4k^2$, so $4 \mid n^2$. Thus $n^2 \equiv 0 \pmod 4$.

If n is odd, it can be expressed as $2k + 1$ where k is an integer. Now $n^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. Since $4 \mid 4(k^2 + k)$, we have $n^2 \equiv 1 \pmod 4$. ∎

Problem 13

Suppose m is a positive integer. Prove that for all integers a, a', b and b', if $a' \equiv a \pmod{m}$ and $b' \equiv b \pmod{m}$ then $a' + b' = a + b \pmod{m}$ and $a'b' = ab \pmod{m}$.

Proof. Since $a' \equiv a \pmod{m}$ and $b' \equiv b \pmod{m}$, it follows that $a' - a = km$ and $b' - b = jm$ for some integers k, j. Then

$$\begin{aligned}(a' + b') - (a + b) &= km + jm \\ &= (k + j)m\end{aligned}$$

Thus $a' + b' \equiv a + b \pmod{m}$. ∎

Proof. Since $a' \equiv a \pmod{m}$ and $b' \equiv b \pmod{m}$, it follows that $a' = a + km$ and $b' = b + jm$ for some integers k, j. Then

$$\begin{aligned}a'b' - ab &= (a + km)(b + jm) - ab \\ &= ab + ajm + bkm + kjm^2 - ab \\ &= m(aj + bk + kjm)\end{aligned}$$

Thus $a'b' \equiv ab \pmod{m}$. ∎

Problem 14

Suppose that R is an equivalence relation on A and $B \subseteq A$. Let $S = R \cap (B \times B)$.
(a) Prove that S is an equivalence relation on B.
(b) Prove that for all $x \in B$, $[x]_S = [x]_R \cap B$.

Proof. We must show S is reflexive, symmetric, and transitive.

Let x be an arbitrary element in B. Since $x \in B$ it follows that $(x, x) \in B \times B$. We know $B \subseteq A$ so $x \in A$. Since R is an equivalence relation on A it follows that $(x, x) \in R$. Since $(x, x) \in R$ and $(x, x) \in B \times B$ it follows that $(x, x) \in R \cap B \times B$. Thus S is reflexive.

Let (x, y) be an arbitrary pair in S. It follows that $(x, y) \in R$ and $(x, y) \in B \times B$. Since R is symmetric it follows that $(y, x) \in R$. Since $(x, y) \in B \times B$ it follows that $(y, x) \in B \times B$. Since $(y, x) \in R$ and $(y, x) \in B \times B$ it follows that $(y, x) \in R \cap B \times B$. Thus S is symmetric.

Let $(x, y), (y, z)$ be arbitrary pairs in S. It follows that $(x, y), (y, z) \in R$ and $(x, y), (y, z) \in B \times B$. Since $(x, y), (y, z) \in R$ it follows that $(x, z) \in R$. Since $(x, y), (y, z) \in B \times B$ it follows that $(x, z) \in B \times B$. Since $(x, z) \in R$ and $(x, z) \in B \times B$ it follows that $(x, z) \in R \cap B \times B$. Thus S is transitive.

Since S is reflexive, symmetric, and transitive, it is an equivalence relation on B. ∎

Proof. Let x be an arbitrary element in B.

We first show $[x]_S \subseteq [x]_R \cap B$. Let y be an arbitrary element in $[x]_S$. It follows that $(y, x) \in S$. Thus $(y, x) \in R$ and $(y, x) \in B \times B$. It follows that $y \in [x]_R$ and $y \in B$. Therefore $y \in [x]_R \cap B$.

We now show $[x]_R \cap B \subseteq [x]_S$. Let y be an arbitrary element in $[x]_R \cap B$. It follows that $y \in [x]_R$ and $y \in B$. Since $x \in B$ it follows that $(y, x) \in B \times B$. Since $y \in [x]_R$ it follows that $(y, x) \in R$. Since $(y, x) \in R$ and $(y, x) \in B \times B$ it follows that $(y, x) \in S$. Thus $y \in [x]_S$. ∎

Problem 24

Supose R and S are relations on a set A, and S is an equivalence relation. We will say that R is compatible with S if for all x, y, x', and y' in A, if xSx' and ySy' then xRy iff $x'Ry'$.
(a) Prove that if R is compatible with S, then there is a unique relation T on A/S such that for all x and y in A, $[x]_S T [y]_S$ iff xRy.
(b) Suppose T is a relation on A/S and for all x and y in A, $[x]_S T [y]_S$ iff xRy. Prove that R is compatible with S.

Proof. Suppose R is compatible with S.

We first show existence. Let $T = \{([x]_S, [y]_S) \mid x, y \in A \text{ and } (x, y) \in R\}$. Let X, Y be two arbitrary sets in A/S. Let x, x' be arbitrary elements in X and y, y' be arbitrary elements in Y. We know $(x, x') \in S$ and $(y, y') \in S$. Since R is compatible with S it follows that $(x, y) \in R$ iff $(x', y') \in R$. Therefore $([x]_S, [y]_S) \in T$ iff $([x']_S, [y']_S) \in T$.

We now show uniqueness. Suppose there are two relations P and Q on A/S such that for all $x, y \in A$, $[x]_S P [y]_S \iff xRy$ and $[x]_S Q [y]_S \iff xRy$. Let x, y be arbitrary elements in A such that $([x]_S, [y]_S) \in P$. It then follows that $(x, y) \in R$. Thus $([x]_S, [y]_S) \in Q$ therefore $P \subseteq Q$. A similar argument shows $Q \subseteq P$. ∎

Proof. Let x, x', y, y' be arbitrary elements in A. We need to show if xSx' and ySy' then xRy iff $x'Ry'$. Suppose xSx' and ySy'. It follows that $([x]_S, [y]_S) \in T$ iff xRy and $([x']_S, [y']_S) \in T$ iff $x'Ry'$. Therefore xRy iff $x'Ry'$. ∎

Problem 25

Suppose R is a relation on A and R is reflexive and transitive. (Such a relation is called a *preorder* on A.) Let $S = R \cap R^{-1}$.
(a) Prove that S is an equivalence relation on A.
(b) Prove that there is a unique relation T on $A \setminus S$ such that for all x and y in A, $[x]_S T [y]_S$ iff xRy. (Hint: Use excersize 24.)
(c) Prove that T is a partial order on $A \setminus S$, where T is the relation from part (b).

Proof. We must show S is reflexive, symmetric, and transitive.

Let x be an arbitrary element in A. Since R is reflexive $(x, x) \in R$. It follows that $(x, x) \in R^{-1}$. Since $(x, x) \in R$ and $(x, x) \in R^{-1}$ it follows that $(x, x) \in S$. Thus S is reflexive.

Let (x, y) be an arbitrary element in S. It follows that $(x, y) \in R$ and $(x, y) \in R^{-1}$. Then $(y, x) \in R^{-1}$ and $(y, x) \in R$. Thus $(y, x) \in S$ therefore S is symmetric.

Let $(x, y), (y, z)$ be arbitrary elements in S. It follows that $(x, y), (y, z) \in R$ and $(x, y), (y, z) \in R^{-1}$. Then $(y, x), (z, y) \in R$. Since R is transitive it follows that $(x, z), (z, x) \in R$. Therefore $(x, z) \in R^{-1}$. Since $(x, z) \in R$ and $(x, z) \in R^{-1}$ it follows that $(x, z) \in S$. Thus S is transitive.

Since S is reflexive, symmetric, and transitive it is an equivalence relation on A. ∎

Proof. Follows directly from 24. ∎

Proof. We must show T is reflexive, transitive, and antisymmetric.

Let $[x]_S$ be an arbitrary element in A/S. Clearly $x \in [x]_S$ and since R is reflexive, xRx. Thus $([x]_S, [x]_S) \in T$. Therefore T is reflexive.

Let $([x]_S, [y]_S), ([y]_S, [z]_S) \in T$. It follows that xRy and yRz. Since R is transitive, xRz. Thus $([x]_S, [z]_S) \in T$. Therefore T is transitive.

Let $([x]_S, [y]_S), ([y]_S, [x]_S) \in T$. Then xRy and yRx, which implies $(x, y) \in S$. Thus $[x]_S = [y]_S$. Therefore T is antisymmetric. ∎

5 Functions

5.1 Functions

Problem 1

(a) Let $A = \{1, 2, 3\}, B = \{4\}$, and $f = \{(1, 4), (2, 4), (3, 4)\}$. Is f a function from A to B?
(b) Let $A = \{1\}, B = \{2, 3, 4\}$ and $f = \{(1, 2), (1, 3), (1, 4)\}$. Is f a function from A to B.
(c) Let C be the set of all cars registered in your state, and let S be the set of all finite sequences of letters and digits. Let $L = \{(c, s) \in C \times S \mid \text{the license plate number of the car } c \text{ is } s\}$. Is L a function from C to S.

Solution (a):

Yes.

Solution (b):

No.

Solution (c):

Yes.

Problem 2

(a) Let f be the relation represented by the graph in Figure 5.3. Is f a function from A to B.
(b) Let W be the set of all words of English, and let A be the set of all letters of the alphabet. Let

$$f = \{(w, a) \in W \times A \mid \text{the letter } a \text{ occurs in the word } w\}$$

and let

$$g = \{(w, a) \in W \times A \mid \text{the letter } a \text{ is the first letter of the word } w\}$$

Is f a function from W to A? How about g?
(c) John, Mary, Susan, and Fred go out to dinner and sit at a round table. Let

$$P = \{\text{John, Mary, Susan, Fred}\}$$

and let

$$R = \{(p, q) \in P \times P \mid \text{the person } p \text{ is sitting immediately to the right of person } q\}$$

Is R a function from P to P.

Solution (a):

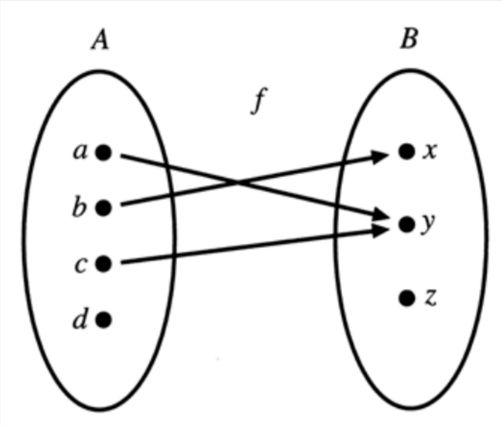
No.

Solution (b):

f: No. g: Yes.

Solution (c):

Yes.



Problem 7

Suppose $f : A \to B$ and $C \subseteq A$. The set $f \cap (C \times B)$, which is a relation from C to B, is called the *restriction* of f to C, and is sometimes denoted $f|C$. In other words.

$$f|C = f \cap (C \times B)$$

(a) Prove that $f|C$ is a function from C to B and that for all $c \in C$, $f(c) = (f|C)(c)$.
(b) Suppose $g : C \to B$. Prove that $g = f|C$ iff $g \subseteq f$.
(c) Let g and h be the functions defined in parts 2 and 3 of Example 5.1.3. Show that $g = h|\mathbb{Z}$.

$$g : \mathbb{Z} \to \mathbb{R} \text{ and } g(x) = 2x + 3$$

$$h : \mathbb{R} \to \mathbb{R} \text{ and } h(x) = 2x + 3$$

Proof. We must show that $f|C$ is a function. For all $x \in C$, there exists y such that $(x, y) \in f|C$, and there is exactly one $y \in B$ such that $(x, y) \in f|C$.

Let x be an arbitrary element in C. Since $C \subseteq A$, it follows that $x \in A$. Since f is a function from A to B, there exists $y \in B$ such that $(x, y) \in f$. Clearly, $(x, y) \in A \times B$. Since $(x, y) \in f$ and $(x, y) \in C \times B$, it follows that $(x, y) \in f \cap (C \times B) = f|C$. Thus $f|C$ maps all elements in C to B.

Notice that $f \cap (C \times B) \subseteq f$. Then, since $(x, y) \in f|C$, it follows that $(x, y) \in f$. It follows that, since f is a function, x maps to exactly one element in B, namely y. ∎

Proof. Suppose $g : C \to B$.

($\to$) Suppose $g = f|C$. Then $g = f|C = f \cap (C \times B) \subseteq f$.

($\leftarrow$) Suppose $g \subseteq f$. Let (x, y) be an arbitrary element in g. Since $g \subseteq f$, it follows that $(x, y) \in f$. Since $(x, y) \in g$, it follows that $(x, y) \in C \times B$. Thus $(x, y) \in f$ and $(x, y) \in C \times B$. Therefore $(x, y) \in f \cap (C \times B) = f|C$. It follows that $g \subseteq f|C$.

Let (x, y) be an arbitrary element in $f|C$. It follows that $(x, y) \in f$ and $(x, y) \in C \times B$. Since $g \subseteq f$ and $x \in C$, it follows that $(x, y) \in g$. Thus $f|C \subseteq g$. ∎

Proof. We need to show that $g = h|\mathbb{Z} = h \cap (\mathbb{Z} \times \mathbb{R})$.

Suppose (x, y) is an arbitrary element in g. We know that $(x, y) \in \mathbb{Z} \times \mathbb{R}$. It follows that $x \in \mathbb{Z}$ and $y \in \mathbb{R}$. Since $\mathbb{Z} \subseteq \mathbb{R}$, it follows that $x \in \mathbb{R}$. Therefore $(x, y) \in h$. Since $(x, y) \in h$ and $(x, y) \in \mathbb{Z} \times \mathbb{R}$, it follows that $(x, y) \in h|\mathbb{Z}$. Thus $g \subseteq h|\mathbb{Z}$.

Suppose (x, y) is an arbitrary element in $h|\mathbb{Z}$. It follows that $(x, y) \in h$ and $(x, y) \in \mathbb{Z} \times \mathbb{R}$. Since $x \in \mathbb{Z}$, it follows that $(x, y) \in g$. Thus $h|\mathbb{Z} \subseteq g$. ∎

Problem 8

Suppose $f : A \to B$ and $g \subseteq f$. Prove that there is a set $A' \subseteq A$ such that $g : A' \to B$.

Proof. Let $A' = dom(g)$. It is clear that $A' \subseteq A$, since $dom(g) \subseteq dom(f)$. Let $x \in A'$. Then there exists y such that $(x, y) \in g$. Since $g \subseteq f$, it follows that $(x, y) \in f$. There is only a single element that x maps to, since $g \subseteq f$ and f is a function. Thus $g : A' \to B$. ∎

Problem 9

Suppose $f : A \to B$, $B \neq \varnothing$, and $A \subseteq A'$. Prove that there is a function $g : A' \to B$ such that $f \subseteq g$.

Proof. Let $A'' = A' \setminus A$. Since $B \neq \varnothing$, let y be an element in B. Let $g = f \cup \{(x, y) \mid x \in A''\}$. Then for each $x \in A$ there is a unique y such that $(x, y) \in f$, and for each $x \in A''$ there is exactly one pair (x, y) in g. Thus g is a function from A' to B and $f \subseteq g$. ∎

Problem 11

Suppose A is a set. Show that i_A is the only relation on A that is both an equivalence relation and also a function from A to A.

Proof. For contradiction, suppose $R \neq i_A$ is an equivalence relation on A and $R : A \to A$. Then there exists a pair $(x, y) \in R$ such that $x \neq y$. Since R is reflexive, $(x, x) \in R$. Thus R contains both (x, x) and (x, y) with $y \neq x$, contradicting that R is a function. Therefore $R = i_A$. ∎

Problem 12

Suppose $f : A \to C$ and $g : B \to C$.
(a) Prove that if A and B are disjoint, then $f \cup g : A \cup B \to C$.
(b) Prove that $f \cup g : A \cup B \to C$ iff $f|(A \cap B) = g|(A \cap B)$. (See excersize 7 for the meaning of the notation used here.)

Proof. Suppose A and B are disjoint. Let x be an arbitrary element in $A \cup B$. Either $x \in A$ or $x \in B$. If $x \in A$, then under f there exists a unique $y \in C$ such that $(x, y) \in f$. If $x \in B$, then under g there exists a unique $y \in C$ such that $(x, y) \in g$. Since A and B are disjoint, $f \cup g$ assigns exactly one $y \in C$ to each $x \in A \cup B$. Therefore $f \cup g : A \cup B \to C$. ∎

Proof. $(\to)$ Suppose $f \cup g : A \cup B \to C$. For contradiction assume $f|(A \cap B) \neq g|(A \cap B)$. Suppose w.l.o.g. since $f|(A \cap B) \neq g|(A \cap B)$ there is a pair $(x, y) \in f|(A \cap B)$ such that $(x, y) \notin g|(A \cap B)$. Now clearly $(x, y) \in f$, but since g is a function on B and $x \in A \cap B$, there is a pair $(x, y') \in g$. Thus $f \cup g$ results in two mappings from x to y and y', which contradicts that $f \cup g$ is a function. Therefore, $f|(A \cap B) = g|(A \cap B)$.

$(\leftarrow)$ Suppose $f|(A \cap B) = g|(A \cap B)$. We must show that $f \cup g : A \cup B \to C$. Let (x, y) and (x, y') be elements of $f \cup g$. We need to show $y = y'$. If both pairs are in f or both in g, this follows since f and g are functions. If one pair is from f and the other from g, then $x \in A \cap B$ and hence $(x, y) \in f|(A \cap B)$ and $(x, y') \in g|(A \cap B)$. By assumption $f|(A \cap B) = g|(A \cap B)$, so $y = y'$. Thus $f \cup g$ is a function from $A \cup B$ to C. ∎

Problem 13

Suppose R is a relation from A to B, S is a relation from B to C, $Ran(R) = Dom(S) = B$, and $S \circ R : A \to C$.
(a) Prove that $S : B \to C$.
(b) Give an example to show that it need not be the case that $R : A \to B$.

Proof. Let b be an arbitrary element in B. Since $Ran(R) = Dom(S) = B$, there exists $a \in A$ such that $(a, b) \in R$. Because $S \circ R$ is a function, there is exactly one $c \in C$ such that $(a, c) \in S \circ R$. By the definition of composition, this means $(a, b) \in R$ and $(b, c) \in S$. Thus S maps b to a unique $c \in C$. Therefore, $S : B \to C$. ∎

Solution (b):

$$R = \{(1, 2), (1, 3)\}, S = \{(2, 4), (3, 4)\}, S \circ R = \{(1, 4)\}$$

Problem 17

Suppose A is a nonempty set and $f : A \to A$.
(a) Suppose there is some $a \in A$ such that $\forall x \in A(f(x) = a)$. (In this case, f is called a constant function.) Prove that for all $g : A \to A$, $f \circ g = f$.
(b) Suppose that for all $g : A \to A$, $f \circ g = f$. Prove that f is a constant function. (Hint: what happens if g is a constant function?)

Proof. Let (x, z) be an arbitrary pair in $f \circ g$. There exists y such that $(x, y) \in g$ and $(y, z) \in f$. It follows, since f is a constant function, that $(x, z) \in f$. Thus $f \circ g \subseteq f$.

Let (x, z) be an arbitrary pair in f. Now, since $g : A \to A$ is a function, for this $x \in A$ there exists a unique $y \in A$ such that $(x, y) \in g$. Since f is a constant function and $y \in A$, it follows that $(y, z) \in f$. Since $(x, y) \in g$ and $(y, z) \in f$, it follows that $(x, z) \in f \circ g$. Thus $f \subseteq f \circ g$.

Therefore $f \circ g = f$. ∎

Proof. Let $x_0 \in A$ and define $g : A \to A$ such that $g(x) = x_0$ for all $x \in A$. Then for every $x \in A$, $(f \circ g)(x) = f(g(x)) = f(x_0)$. By assumption, $f \circ g = f$, so for every $x \in A$, $f(x) = (f \circ g)(x) = f(x_0)$. Therefore, f maps every element of A to the same value. It follows that f is a constant function. ∎

Problem 19

Let $\mathcal{F} = \{f \mid f : \mathbb{Z}^+ \to \mathbb{R}\}$. For $g \in \mathcal{F}$, we define the set $O(g)$ as follows:

$$O(g) = \{f \in \mathcal{F} \mid \exists a \in \mathbb{Z}^+ \exists c \in \mathbb{R}^+ \forall x > a(|f(x)| \leq c|g(x)|)\}$$

(If $f \in O(g)$, then mathematicians say "f is big-oh of g".)
(a) Let $f : \mathbb{Z}^+ \to \mathbb{R}$ and $g : \mathbb{Z}^+ \to \mathbb{R}$ be defined by the formulas $f(x) = 7x + 3$ and $g(x) = x^2$. Prove that $f \in O(g)$, but $g \notin O(f)$.
(b) Let $S = \{(f, g) \in \mathcal{F} \times \mathcal{F} \mid f \in O(g)\}$. Prove that S is a preorder, but not a partial order. (See excersize 25 of Section 4.5 for the definition of *preorder*.)
(c) Suppose $f_1 \in O(g)$ and $f_2 \in O(g)$, and s and t are real numbers. Define a function $f : \mathbb{Z}^+ \to \mathbb{R}$ by the formula $f(x) = sf_1(x) + tf_2(x)$. Prove that $f \in O(g)$. (Hint: You may find the triangle inequality helpful. See excersize 13(c) of Section 3.5.)

Proof. Let $a = 500$ and $c = 1$. Then $|f(x)| \leq c|g(x)| \iff |7x + 3| \leq 1|x^2|$. With $x \geq a = 500$ we have $7x + 3 \leq x^2$. At the point $x = 500$, we have $7(500) + 3 = 3503 \leq 500^2 = 250000$. To show that $|f(x)| \leq c|g(x)|$ for all $x > a$, we can look at the derivatives: $f'(x) = 7$ and $g'(x) = 2x$, and for $x > 500$, $g'(x) > f'(x)$, so $g(x)$ grows faster than $f(x)$. Thus $f \in O(g)$. ∎

Proof. Suppose $g \in O(f)$ then it follows that there exists $c \in \mathbb{R}^+$ and $a \in \mathbb{Z}^+$ such that for all $x > a$, $|g(x)| \leq c|f(x)|$. Now plugging in gives $|x^2| \leq c|7x+3|$. Since x^2 grows faster than $7x+3$, for any fixed c there exists $x > a$ such that $x^2 > c(7x+3)$, which is a contradiction. Therefore, $g \notin O(f)$. ∎

Proof. We must show S is reflexive and transitive on $\mathcal{F}$. Suppose f is an arbitrary element in $\mathcal{F}$. Clearly if we let $a = 1, c = 1$ then $|f(x)| \leq |f(x)|$ thus $(f, f) \in S$.

Suppose $(f, g), (g, t)$ are arbitrary pairs in S. Now there exists $a_1 \in \mathbb{Z}^+$ and $c_1 \in \mathbb{R}^+$ such that for all $x > a_1$, $|f(x)| \leq c_1|g(x)|$. Similarly there exists $a_2 \in \mathbb{Z}^+$ and $c_2 \in \mathbb{R}^+$ such that for all $x > a_2$, $|g(x)| \leq c_2|t(x)|$. Let $a = \max(a_1, a_2)$. Then for all $x > a$, $|f(x)| \leq c_1|g(x)| \leq c_1c_2|t(x)|$. Thus $(f, t) \in S$. ∎

Proof. Since $f_1 \in O(g)$ and $f_2 \in O(g)$ it follows there exists $c_1, c_2 \in \mathbb{R}^+$ and $a_1, a_2 \in \mathbb{Z}^+$ such that for all $x > a_1$ and $x > a_2$, $|f_1(x)| \leq c_1|g(x)|$ and $|f_2(x)| \leq c_2|g(x)|$. Then since $|f_1(x)| \leq c_1|g(x)|$ it follows that $|sf_1(x)| \leq |s|c_1|g(x)|$. Similarly $|tf_2(x)| \leq |t|c_2|g(x)|$. Taking sums gives $|sf_1(x)| + |tf_2(x)| \leq (|s|c_1 + |t|c_2)|g(x)|$. By the triangle inequality $|sf_1(x) + tf_2(x)| \leq |sf_1(x)| + |tf_2(x)|$. Let $a = \max(a_1, a_2)$. Then for all $x > a$ it follows that $|sf_1(x) + tf_2(x)| \leq (|s|c_1 + |t|c_2)|g(x)|$. Let $c = |s|c_1 + |t|c_2$. Then $|f(x)| \leq c|g(x)|$ for all $x > a$, and therefore $f \in O(g)$. ∎

Problem 21

Suppose $f : A \to B$ and R is an equivalence relation on A. We will say that f is compatible with R if $\forall x \in A \forall y \in A(xRy \to f(x) = f(y))$.
(a) Suppose f is compatible with R prove that there is a unique function $h : A/R \to B$ such that for all $x \in A, h([x]_R) = f(x)$.
(b) Suppose $h : A/R \to B$ and for all $x \in A, h([x]_R) = f(x)$. Prove that f is compatible with R.

Proof. First note that since R is an equivalence relation, for all $x \in A$ there exists y such that xRy. This follows since R is reflexive, so xRx. Define a function $h : A/R \to B$ by setting $h([x]_R) = f(x)$ for each $x \in A$.

Suppose $x, y \in A$ such that $[x]_R = [y]_R$. Then xRy and it follows since f is compatible with R that $f(x) = f(y)$.

Uniqueness of h follows because such a function must satisfy $h([x]_R) = f(x)$ for all $x \in A$. ∎

Proof. Let x, y be arbitrary elements in A such that $(x, y) \in R$. Since $(x, y) \in R$ it follows that $f(x) = h([x]_R) = h([y]_R) = f(y)$. Thus $f(x) = f(y)$. Therefore f is compatible with R. ∎

Problem 22

Let $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \equiv y \pmod 5\}$. Note that by Theorem 4.5.10 and excersize 14 in Section 4.5, R is an equivalence relation on $\mathbb{N}$.
(a) Show that there is a unique function $h : \mathbb{N}/R \to \mathbb{N}/R$ such that for every natural number $x, h([x]_R) = [x^2]_R$. (Hint: Use excersize 21.)
(b) Show that there is no function $h : \mathbb{N}/R \to \mathbb{N}/R$ such that for every natural number $x, h([x]_r) = [2^x]_R$.

Proof. Suppose $x, y \in \mathbb{N}$ such that $[x]_R = [y]_R$. Then $x = 5k_1 + r$ and $y = 5k_2 + r$ for some $k_1, k_2 \in \mathbb{Z}$. Now $x^2 = 25k_1^2 + 10k_1r + r^2$ and $y^2 = 25k_2^2 + 10k_2r + r^2$. Taking differences gives $x^2 - y^2 = 25(k_1^2 - k_2^2) + 10r(k_1 - k_2)$. It follows that $x^2 - y^2$ is divisible by 5, so $[x^2]_R = [y^2]_R$. From problem 21, let $A = \mathbb{N}$ and $R' = R$, where R' refers to R from the previous problem. Then this follows immediately. ∎

Proof. For contradiction, suppose there exists a function $h : \mathbb{N}/R \to \mathbb{N}/R$ such that for every natural number $x, h([x]_R) = [2^x]_R$. Let $x, y \in \mathbb{N}$ such that $x \neq y$ and $y \in [x]_R$. It follows that $h([x]_R) = h([y]_R)$, so $[2^x]_R = [2^y]_R$. Now let $x = 1$ and $y = 6$. Then $y \in [x]_R$ because $6 \equiv 1 \pmod 5$. But $2^1 = 2 \equiv 2 \pmod 5$ and $2^6 = 64 \equiv 4 \pmod 5$, so $[2^1]_R \neq [2^6]_R$. This is a contradiction. ∎

5.2 One-to-One and Onto

Problem 1

Which of the functions in exercize 1 of Section 5.1 are one-to-one? Which are onto?
(a) Let $A = \{1, 2, 3\}, B = \{4\}$, and $f = \{(1, 4), (2, 4), (3, 4)\}$. Is f a function from A to B?
(b) Let $A = \{1\}, B = \{2, 3, 4\}$ and $f = \{(1, 2), (1, 3), (1, 4)\}$. Is f a function from A to B.
(c) Let C be the set of all cars registered in your state, and let S be the set of all finite sequences of letters and digits. Let $L = \{(c, s) \in C \times S \mid \text{the license plate number of the car } c \text{ is } s\}$. Is L a function from C to S.

Solution (a):

One-to-one: No, Onto: Yes.

Solution (b):

Was not a function.

Solution (c):

One-to-one: Yes, Onto: No (Some license plates haven't been assigned to vehicles.)

Problem 2

Which of the functions in exercize 2 of Section 5.1 are one-to-one? Which are onto?
(a) Let f be the relation represented by the graph in Figure 5.3. Is f a function from A to B.
(b) Let W be the set of all words of English, and let A be the set of all letters of the alphabet. Let

$$f = \{(w, a) \in W \times A \mid \text{the letter } a \text{ occurs in the word } w\}$$

and let

$$g = \{(w, a) \in W \times A \mid \text{the letter } a \text{ is the first letter of the word } w\}$$

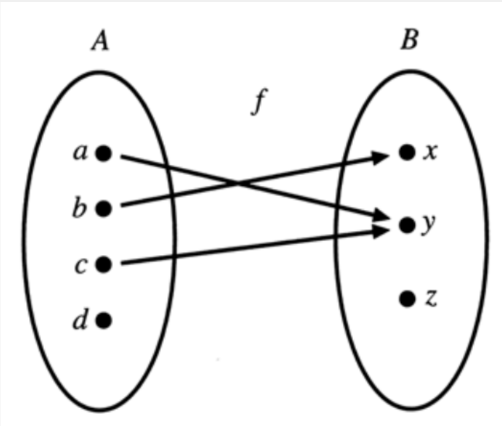
Is f a function from W to A? How about g?
(c) John, Mary, Susan, and Fred go out to dinner and sit at a round table. Let

$$P = \{\text{John, Mary, Susan, Fred}\}$$

and let

$$R = \{(p, q) \in P \times P \mid \text{the person } p \text{ is sitting immediately to the right of person } q\}$$

Is R a function from P to P.



Solution (a):

Was not a function.

Solution (b):

f was not a function. g One-to-one: no, Onto: yes.

Solution (c):

One-to-one: yes, Onto: yes.

Problem 3

Which of the functions in exercize 3 of Section 5.1 are one-to-one? Which are onto?
(a) Let $A = \{a, b, c\}, B = \{a, b\}$, and $f = \{(a, b), (b, b), (c, a)\}$.
(b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the formula $f(x) = x^2 - 2x$.
(c) Let $f = \{(x, n) \in \mathbb{R} \times \mathbb{Z} \mid n \leq x < n + 1\}$. Then $f : \mathbb{R} \to \mathbb{Z}$.

Solution (a):

One-to-one: no, Onto: yes.

Solution (b):

One-to-one: no, Onto: no.

Solution (c):

One-to-one: no, Onto: yes.

Problem 4

Which of the functions in exercize 4 of Section 5.1 are one-to-one? Which are onto?
(a) Let N be the set of all countries and C the set of all cities. Let $H : N \to C$ be the function defined by the rule that for every country n, $H(n) =$ the capital of the country n.
(b) Let $A = \{1, 2, 3\}$ and $B = \mathcal{P}(A)$. Let $F : B \to B$ be the function defined by the formula $F(X) = A \setminus X$.
(c) Let $f : \mathcal{R} \to \mathcal{R}$ be the function defined by the formula $f(x) = (x + 1, x - 1)$.

Solution (a):

One-to-one: no, Onto: yes.

Solution (b):

One-to-one: yes, Onto: yes.

Solution (c):

One-to-one: yes, Onto: no.

Problem 6

Suppose a and b are real numbers and $a \neq 0$. Define $f : \mathbb{R} \to \mathbb{R}$ by the formula $f(x) = ax + b$. Show that f is one-to-one and onto.

Proof. Let x_1, x_2 be arbitrary numbers in $\mathbb{R}$. Then

$$\begin{aligned} & f(x_1) = f(x_2) \\ \iff & a(x_1) + b = a(x_2) + b \\ \iff & a(x_1) = a(x_2) \\ \iff & x_1 = x_2 \quad \text{Note: } a \neq 0 \end{aligned}$$

Thus f is one-to-one. ∎

Proof. Let y be an arbitrary number in $\mathbb{R}$. Let $x = \frac{y-b}{a}$ which is defined since $a \neq 0$. Then

$$\begin{aligned} f\left(\frac{y-b}{a}\right) &= a\left(\frac{y-b}{a}\right) + b \\ &= y - b + b \\ &= y \end{aligned}$$

∎

Problem 8

Let $A = \mathcal{P}(\mathbb{R})$. Define $f : \mathbb{R} \to A$ by the formula $f(x) = \{y \in \mathbb{R} \mid y^2 < x\}$.
(a) Find $f(2)$.
(b) Is f one-to-one? Is it onto?

Solution (a):

$$f(2) = \{x \mid -\sqrt{2} < x < \sqrt{2}\}$$

Solution (b):

One-to-one: no, Onto: no.

Problem 10

Suppose $f : A \to B$ and $g : B \to C$.
(a) Prove that if $g \circ f$ is onto then g is onto.
(b) Prove that if $g \circ f$ is one-to-one then f is one-to-one.

Proof. Suppose $g \circ f$ is onto. Let y be an arbitrary element in C. Since $g \circ f$ is onto it follows that there exists x such that $(g \circ f)(x) = y$. Then there exists c such that $f(x) = c$ and $g(c) = y$. Thus g is onto. ∎

Proof. Suppose x_1, x_2 are arbitrary elements in A such that $f(x_1) = f(x_2)$. Then applying g to both sides gives $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then since $g \circ f$ is one-to-one it follows that $x_1 = x_2$. Thus f is one-to-one. ∎

Problem 11

Suppose $f : A \to B$ and $g : B \to C$.
(a) Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.
(b) Prove that if f is not onto and g is one-to-one, then $g \circ f$ is not onto.

Proof. Suppose f is onto and g is not one-to-one. Since g is not one-to-one there exist $x_1, x_2 \in B$ such that $x_1 \neq x_2$ and $g(x_1) = g(x_2)$. Now since f is onto there exist $c_1, c_2 \in A$ such that $f(c_1) = x_1$ and $f(c_2) = x_2$. If $c_1 = c_2$, then $x_1 = f(c_1) = f(c_2) = x_2$, a contradiction. Thus $c_1 \neq c_2$. Then $(g \circ f)(c_1) = g(f(c_1)) = g(x_1) = g(x_2) = g(f(c_2)) = (g \circ f)(c_2)$. Thus $g \circ f$ is not one-to-one. ∎

Proof. Since f is not onto there exists $y \in B$ such that for all $x \in A$, $f(x) \neq y$. Let $c = g(y) \in C$. Now if for some $x \in A$, $(g \circ f)(x) = g(y)$, then since g is one-to-one, $f(x) = y$, a contradiction. Thus $g \circ f$ is not onto. ∎

Problem 13

Suppose $f : A \to B$ and $C \subseteq A$. In exercize 7 of Section 5.1 we defined $f|C$ (the restriction of f to C), and you showed that $f|C : C \to B$
(a) Prove that if f is one-to-one, then so is $f|C$.
(b) Prove that if $f|C$ is onto, then so is f.
(c) Give examples to show that the converses of parts (a) and (b) are not always true.

Proof. Suppose f is one-to-one. Let x_1, x_2 be two arbitrary elements in C such that $(f|C)(x_1) = (f|C)(x_2)$. It follows that $(x_1, (f|C)(x_1)) \in f$ and $(x_2, (f|C)(x_2)) \in f$. Then since f is one-to-one it follows that $x_1 = x_2$. ∎

Proof. Suppose $f|C$ is onto. Let y be an arbitrary element in B. Since $f|C$ is onto there exists $x \in C$ such that $(f|C)(x) = y$. It follows that $(x, y) \in f$ thus $f(x) = y$. Therefore f is onto. ∎

Solution:

Counter example: If $f|C$ is one-to-one then f is one-to-one.

$$A = \{1, 2\}, B = \{1, 2\}, C = \{1\}, f = \{(1, 1), (2, 1)\}, f|C = \{(1, 1)\}$$

Counter example: If f is onto then $f|C$ is onto.

$$A = \{1, 2\}, B = \{1, 2\}, C = \{1\}, f = \{(1, 1), (2, 2)\}, f|C = \{(1, 1)\}$$

Problem 14

Suppose $f : A \to B$, and there is some $b \in B$ such that $\forall x \in A(f(x) = b)$. (Thus, f is a constant function.)
(a) Prove that if A has more than one element then f is not one-to-one.
(b) Prove that if B has more than one element then f is not onto.

Proof. Suppose that A has more than one element. Since A has more than one element there exists $x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = b$ and $f(x_2) = b$. Thus f is not one-to-one. ∎

Proof. Suppose that B has more than one element. Then there exists $y \in B$ such that $y \neq b$. Since $f(x) = b$ for all $x \in A$, there does not exist $x \in A$ such that $f(x) = y$. Therefore, f is not onto. ∎

Problem 15

Suppose $f : A \to C$, $g : B \to C$, and A and B are disjoint. In excersize $12(a)$ of Section 5.1 you proved that $f \cup g : A \cup B \to C$. Now suppose that f and g are both one-to-one. Prove that $f \cup g$ is one-to-one iff $Ran(f)$ and $Ran(g)$ are disjoint.

Proof. ($\to$) Suppose $f \cup g$ is one-to-one. For contradiction, suppose $Ran(f) \cap Ran(g) \neq \emptyset$. Let y be an element in $Ran(f) \cap Ran(g)$. It follows there exists $x_1 \in A$ and $x_2 \in B$ such that $f(x_1) = y$ and $g(x_2) = y$. Since A and B are disjoint, $x_1 \neq x_2$, contradicting that $f \cup g$ is one-to-one. Thus $Ran(f)$ and $Ran(g)$ are disjoint.

($\leftarrow$) Suppose $Ran(f)$ and $Ran(g)$ are disjoint. Suppose $x_1, x_2 \in A \cup B$ such that $(f \cup g)(x_1) = (f \cup g)(x_2)$. Since A and B are disjoint, either $x_1, x_2 \in A$ or $x_1, x_2 \in B$. Suppose w.l.o.g. $x_1 \in A$ and $x_2 \in B$ but then $(f \cup g)(x_1) \in Ran(f)$ and $(f \cup g)(x_2) \in Ran(g)$, which are disjoint. Suppose w.l.o.g. $x_1, x_2 \in A$. Since f is one-to-one, it follows that $x_1 = x_2$. Thus $f \cup g$ is one-to-one. ∎

Problem 16

Suppose R is a relation from A to B, S is a relation from B to C, $Ran(R) = Dom(S) = B$, and $S \circ R : A \to C$. In exercize $13(a)$ of Section 5.1 you proved that $S : B \to C$. Now prove that if S is one-to-one then $R : A \to B$.

Proof. Suppose S is one-to-one. Let x be an arbitrary element in A. Since $S \circ R$ is a function, there exists $y \in C$ such that $(S \circ R)(x) = y$. It follows that there exists $c \in B$ such that $R(x) = c$ and $S(c) = y$. Thus R maps each element in A to some element in B.

Suppose there exists $x \in A$ and $y_1, y_2 \in B$ such that $y_1 \neq y_2$ and $(x, y_1), (x, y_2) \in R$. Since $Ran(R) = Dom(S)$, there exist $z_1, z_2 \in C$ such that $(y_1, z_1) \in S$ and $(y_2, z_2) \in S$. Then $(x, z_1), (x, z_2) \in (S \circ R)$. Since S is one-to-one and $y_1 \neq y_2$, it follows that $z_1 \neq z_2$, contradicting that $S \circ R$ is a function. Thus for every $x \in A$ there exists a single $y \in B$ such that $(x, y) \in R$. ∎

Problem 18

Suppose R is an equivalence relation on A, and let $g : A \to A/R$ be defined by the formula $g(x) = [x]_R$, as in exersize $20(b)$ in Section 5.1.
(a) Show that g is onto.
(b) Show that g is one-to-one iff $R = i_A$.

Proof. Let T be an arbitrary element in A/R. Since T is not empty, there exists x such that $T = [x]_R$. It follows that $g(x) = [x]_R = T$. Thus g is onto. ∎

Proof. ($\to$) Suppose g is one-to-one. Let x be an arbitrary element in A. Since R is reflexive, $(x, x) \in R$, thus $x \in [x]_R$. Since g is one-to-one, no other element in A maps to $[x]_R$. It follows that each equivalence class contains exactly one element, so $R = i_A$.

($\leftarrow$) Suppose $R = i_A$. Let x_1, x_2 be arbitrary elements in A such that $g(x_1) = g(x_2)$. Then $[x_1]_R = [x_2]_R$. Since each equivalence class contains exactly one element, it follows that $x_1 = x_2$. Therefore, g is one-to-one. ∎

Problem 19

Suppose $f : A \to B$, R is an equivalence relation on A, and f is compatible with R. (See exersize 21 of Section 5.1 for the definition of *compatible*.) In exercize $21(a)$ of Section 5.1 you proved that there is a unique function $h : A/R \to B$ such that for all $x \in A$, $h([x]_R) = f(x)$. Now prove that h is one-to-one iff $\forall x \in A \forall y \in A(f(x) = f(y) \to xRy)$.

Proof. ($\rightarrow$) Suppose h is one-to-one. Let x, y be arbitrary elements in A. Furthermore, suppose $f(x) = f(y)$. It follows that $h([x]_R) = h([y]_R)$. Then, since h is one-to-one, $[x]_R = [y]_R$. It follows that $(x, y) \in R$.

($\leftarrow$) Suppose $\forall x, y \in A, (f(x) = f(y) \rightarrow xRy)$. Let x_1, x_2 be arbitrary elements in A such that $h([x_1]_R) = h([x_2]_R)$. Thus $f(x_1) = f(x_2)$, and it follows that $(x_1, x_2) \in R$. Therefore $[x_1]_R = [x_2]_R$. It follows that h is one-to-one. ∎

Problem 20

Suppose A, B, and C are sets and $f : A \rightarrow B$.
(a) Prove that if f is onto, $g : B \rightarrow C$, $h : B \rightarrow C$, and $g \circ f = h \circ f$, then $g = h$.
(b) Suppose that C has at least two elements, and for all functions g and h from B to C, if $g \circ f = h \circ f$ then $g = h$. Prove that f is onto.

Proof. Suppose f is onto, $g : B \rightarrow C$, $h : B \rightarrow C$, and $g \circ f = h \circ f$. Let b be an arbitrary element in B. Since f is onto, there exists $a \in A$ such that $f(a) = b$. Now, since $g \circ f = h \circ f$, it follows that $(g \circ f)(a) = (h \circ f)(a)$. Thus $g(f(a)) = h(f(a))$, and hence $g(b) = h(b)$. Since b was arbitrary, $g = h$. ∎

Proof. For contradiction, suppose f is not onto. There exists $y \in B$ such that for all $x \in A$, $f(x) \neq y$. Let $y_1, y_2 \in C$ with $y_1 \neq y_2$. For all $x \in B$ let $g(x) = y_1$, $h(x) = y_2$ if $x = y$; otherwise $g(x) = h(x) = y_1$. Clearly $g \neq h$. However, $g \circ f = h \circ f$ since for all $x \in A$ $f(x) \neq y$, which is a contradiction. Thus f is onto. ∎

Problem 21

Suppose A, B, and C are sets and $f : B \rightarrow C$.
(a) Prove that if f is one-to-one, $g : A \rightarrow B$, $h : A \rightarrow B$, and $f \circ g = f \circ h$, then $g = h$.
(b) Suppose that $A \neq \varnothing$, and for all functions g and h from A to B, if $f \circ g = f \circ h$ then $g = h$. Prove that f is one-to-one.

Proof. Suppose f is one-to-one, $g : A \rightarrow B$, $h : A \rightarrow B$, and $f \circ g = f \circ h$. Let x be an arbitrary element in A. There exists y_1, y_2 such that $h(x) = y_1$ and $g(x) = y_2$. Since $f \circ g = f \circ h$ it follows that $f(y_1) = f(y_2)$. Then since f is one-to-one it follows that $y_1 = y_2$. Therefore $g = h$. ∎

Proof. For contradiction, suppose f is not one-to-one. There exist $x_1, x_2 \in B$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$. For all $x \in A$ let $g(x) = x_1$ and $h(x) = x_2$. Clearly $g \neq h$. However, for all $x \in A$, $(f \circ g)(x) = (f \circ h)(x)$, which is a contradiction. Thus f is one-to-one. ∎

Problem 22

Let $\mathcal{F} = \{f \mid f : \mathbb{R} \rightarrow \mathbb{R}\}$, and define a relation R on $\mathcal{F}$ as follows:

$$R = \{(f, g) \in \mathcal{F} \times \mathcal{F} \mid \exists h \in \mathcal{F}(f = h \circ g)\}$$

(a) Let f, g, and h be the functions from $\mathbb{R}$ to $\mathbb{R}$ defined by the formulas $f(x) = x^2 + 1$, $g(x) = x^3 + 1$, and $h(x) = x^4 + 1$. Prove that hRf, but it is not the case gRf.
(b) Prove that R is a preorder. (See exercize 25 of Section 4.5 for the definition of *preorder*.)
(c) Prove that for all $f \in \mathcal{F}$, $fRi_{\mathbb{R}}$.
(d) Prove that for all $f \in \mathcal{F}$, $i_{\mathbb{R}}Rf$ iff f is one-to-one. (Hint for right-to-left direction: Suppose f is one-to-one. Let $A = Ran(f)$, and let $h = f^{-1} \cup ((\mathbb{R} \setminus A) \times \{0\})$. Now prove that $h : \mathbb{R} \rightarrow \mathbb{R}$ and $i_{\mathbb{R}} = h \circ f$.
(e) Suppose $g \in \mathcal{F}$ is a constant function; in other words, there is some real number c such that $\forall x \in \mathbb{R}(g(x) = c)$. Prove that for all $f \in \mathcal{F}$, gRf. (Hint: See exercize 17 of Section 5.1.)
(f) Suppose that $g \in \mathcal{F}$ is a constant function. Prove that for all $f \in \mathcal{F}$, fRg iff f is a constant function.

(g) As in exercize 25 of Section 4.5, if we let $S = R \cap R^{-1}$, then S is an equivalence relation on $\mathcal{F}$. Also, there is a unique relation T on $\mathcal{F}/S$ such that for all f and g in $\mathcal{F}$, $[f]_S T [g]_S$ iff fRg, and T is a partial order on $\mathcal{F}/S$. Prove that the set of all one-to-one functions from $\mathbb{R}$ to $\mathbb{R}$ is the largest element of $\mathcal{F}/S$ in the partial order T, and the set of all constant functions from $\mathbb{R}$ to $\mathbb{R}$ is the smallest element.

Solution (a):

Proof. Let $l(x) = x^2 - 2x + 2$. Then

$$\begin{aligned}(l \circ f)(x) &= l(f(x)) \\ &= (x^2+1)^2 - 2(x^2+1) + 2 \\ &= x^4 + 2x^2 + 1 - 2x^2 - 2 + 2 \\ &= x^4 + 1 \\ &= h\end{aligned}$$

Thus $h = l \circ f$, which shows that $(h, f) \in R$. ∎

Proof. For contradiction, suppose $(g, f) \in R$. There exists a function h such that $g = h \circ f$. Then

$$x^3 + 1 = h(x^2 + 1) \text{ for all } x \in \mathbb{R}.$$

Suppose $t_1, t_2 \in \mathbb{R}$ with $t_1 = -t_2 \neq 0$. Then $f(t_1) = t_1^2 + 1 = f(t_2)$, but $g(t_1) \neq g(t_2)$. Thus $g \neq h \circ f$ for any function h. ∎

Solution (b):

Proof. We must show R is reflexive and transitive. Suppose $f \in \mathcal{F}$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined be the formula $h(y) = y$. Suppose x is an arbitrary real number. Then $(h \circ f)(x) = h(f(x)) = f(x)$. Thus $(f, f) \in R$. It follows that R is reflexive. Suppose $(f, g), (g, l)$ are arbitrary elements in R. It follows that there exists h_1, h_2 such that $f = h_1 \circ g$ and $g = h_2 \circ l$. Then $f = h_1 \circ g = (h_1 \circ h_2) \circ l$. Thus $(f, l) \in R$. It follows that R is transitive. Therefore R is a preorder. ∎

Solution (c):

Proof. Let f be an arbitrary element in $\mathcal{F}$. Furthermore, let x be an arbitrary real number. Then $(i_{\mathbb{R}} \circ f)(x) = i_{\mathbb{R}}(f(x)) = f(x)$. Thus $(f, i_{\mathbb{R}}) \in R$. ∎

Solution (d):

Proof. (→) Let f be an arbitrary element in $\mathcal{F}$. Suppose $(i_{\mathbb{R}}, f) \in R$. There exists h such that $i_{\mathbb{R}} = h \circ f$. Let x_1, x_2 be arbitrary real numbers such that $f(x_1) = f(x_2)$. Applying h to both sides gives $(h \circ f)(x_1) = (h \circ f)(x_2)$. Then $x_1 = (h \circ f)(x_1) = (h \circ f)(x_2) = x_2$. Thus f is one-to-one.

(←) Suppose f is one-to-one. Let $A = \text{Ran}(f)$. Define

$$h = f^{-1} \cup ((\mathbb{R} \setminus A) \times \{0\}).$$

Then for each $x \in \mathbb{R}$, either $x \in A$ or $x \in \mathbb{R} \setminus A$. If $x \in A$, then $h(x) = f^{-1}(x)$. If $x \in \mathbb{R} \setminus A$, then $h(x) = 0$. Thus h is a function from $\mathbb{R}$ to $\mathbb{R}$. Furthermore, for all $x \in \mathbb{R}$,

$$(h \circ f)(x) = h(f(x)) = f^{-1}(f(x)) = x.$$

Thus $i_{\mathbb{R}} = h \circ f$, so $(i_{\mathbb{R}}, f) \in R$. ∎

Solution (e):

Proof. Let h be an arbitrary function in $\mathcal{F}$. By Section 5.1 Problem 17(a), $g \circ h = g$ Thus $(g, h) \in R$. ∎

Solution (f):

Proof. Let f be an arbitrary element in $\mathcal{F}$.

($\rightarrow$) Suppose $(f, g) \in R$. Then there exists h such that $f = h \circ g$. Since g is a constant function, there exists $y \in \mathbb{R}$ such that $g(x) = y$ for all $x \in \mathbb{R}$. Then for all $x \in \mathbb{R}$, $f(x) = (h \circ g)(x) = h(g(x)) = h(y)$, which shows that f is constant.

($\leftarrow$) Suppose f is a constant function. Let c be a real number such that $f(x) = c$ for all $x \in \mathbb{R}$. Furthermore, let h be a function from $\mathbb{R}$ to $\mathbb{R}$ such that $h(y) = c$ for all $y \in \mathbb{R}$. Then for all $x \in \mathbb{R}$, $(h \circ g)(x) = h(g(x)) = c = f(x)$. Thus $f = h \circ g$, so $(f, g) \in R$. ∎

Solution (g):

Proof. Let f be an arbitrary one-to-one function in $\mathcal{F}$. Let g be an arbitrary function in $\mathcal{F}$. Since f is one-to-one, f^{-1} exists on Ran(f). Let h be a function such that

$$h = g \circ f^{-1} \;\; \cup \;\; ((\mathbb{R} \setminus \mathrm{Ran}(f)) \times \{0\}).$$

Then for all $x \in \mathbb{R}$, $(h \circ f)(x) = h(f(x)) = g(x)$. Thus $(g, f) \in R$. ∎

Proof. Let f be an arbitrary constant function in $\mathcal{F}$. Let g be an arbitrary function in $\mathcal{F}$. From part (f) it follows that there exists h such that $f = h \circ g$. Thus $(f, g) \in R$. ∎

Problem 23

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined by the formula $f(n) = n$. Note that we could also say that $f : \mathbb{N} \rightarrow \mathbb{Z}$. This exersize will illustrate why, in Definition 5.2.1, we defined the phrase "f maps onto B," rather than simply "f is onto."
(a) Does f map onto $\mathbb{N}$.
(b) Does f map onto $\mathbb{Z}$.

Proof. Yes. Let y be an arbitrary natural number. Then let $x = y$. Clearly $f(x) = x = y$. Thus f maps onto $\mathbb{N}$. ∎

Proof. No. Suppose f maps onto $\mathbb{Z}$. Let y be an arbitrary integer such that $y < 0$. Since f is onto, exists a natural number x such that $f(x) = y$. Then $x = y$, but $y < 0$, contradicting that x is a natural number. Thus f does not map onto $\mathbb{Z}$. ∎

5.3 Inverses of Functions

Problem 1

Let R be the function defined in exercise 2(c) of Section 5.1. In exercise 2 of Section 5.2, you showed that R is one-to-one and onto, so $R^{-1} : P \rightarrow P$. If $p \in P$, what is $R^{-1}(p)$?

Problem 2

Let F be the function defined in exercise 4(b) of Section 5.1. In exercise 4 of Section 5.2, you showed that F is one-to-one and onto, so $F^{-1} : B \rightarrow B$. If $X \in B$, what is $F^{-1}(X)$.

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by the formula

$$f(x) = \frac{2x+5}{3}$$

Show that f is one-to-one and onto, and fine a formula for $f^{-1}(x)$. (You may want to imitate the method used in the example after Theorem 5.3.2,)

Problem 8

(a) Prove the second half of Theorem 5.3.2 by imitating the proof of the first half.
(b) Give and alternative proof of the second half of Theorem 5.1.2 by appying the first half to f^{-1}.

Problem 9

Prove part 2 of Theorem 5.3.3.

Problem 10

Use the following strategy to give an alternative proof of Theorem 5.3.5:
Let (b, a) be an arbitrary element of $B \times A$. Assume $(b, a) \in g$ and prove $(b, a) \in f^{-1}$. Then assume $(b, a) \in f^{-1}$ and prove $(b, a) \in g$.

Problem 11

Suppose $f : A \to B$ and $g : B \to A$.
(a) Prove that if f is one-to-one and $f \circ g = i_B$, then $g = f^{-1}$.
(b) Prove that if f is onto and $g \circ f = i_A$, then $g = f^{-1}$.
(c) Prove that if $f \circ g = i_B$ but $g \circ f \neq i_A$, then f is onto but one-to-one, and g is one-to-one but not onto.

Problem 12

Suppose $f : A \to B$ and f is one-to-one. Prove that there is some set $B' \subseteq B$ such that $f^{-1} : B' \to A$.

Problem 13

Suppose $f : A \to B$ and f is onto. Let $R = \{(x, y) \in A \times A \mid f(x) = f(y)\}$. By exercise $20(a)$ of Section 5.1, R is an equivalence relation on A.
(a) Prove that there is a function $h : A/R \to B$ such that for all $x \in A$, $h([x]_R) = f(x)$. (Hint: See exercise 21 of Section 5.1.)
(b) Prove that h is one-to-one and onto. (Hint: See exercise 19 of Section 5.2.)
(c) It follows from part (b) that $h^{-1} : B \to A/R$. Prove that for all $b \in B$, $h^{-1}(b) = \{x \in A \mid f(x) = b\}$.
(d) Suppose $g : B \to A$. Prove that $f \circ g = i_B$ iff $\forall b \in B(g(b) \in h^{-1}(b))$.

Problem 14

Suppose $f : A \to B$, $g : B \to A$, and $f \circ g = i_B$. Let $A' = Ran(g) \subseteq A$.
(a) Prove that for all $x \in A'$, $(g \circ f)(x) = x$.
(b) Prove that $f|A'$ is a one-to-one, onto function from A' to B and $g = (f|A')^{-1}$. (See exercise 7 of Section 5.1 for the meaning of the notation here.)

Problem 15

Let $B = \{x \in \mathbb{R} \mid x \geq 0\}$. Let $f : \mathbb{R} \to B$ and $g : B \to \mathbb{R}$ be defined by the formulas $f(x) = x^2$ and $g(x) = \sqrt{x}$. As we saw in part 2 of Example 5.3.6, $g \neq f^{-1}$. Show that $g = (f|B)^{-1}$. (Hint: See exercise 14.)

Problem 17

Suppose A is a set, and let $\mathcal{F} = \{f \mid f : A \to A\}$ and $\mathcal{P} = \{f \in \mathcal{F} \mid f$ is one-to-one and onto$\}$. Define a relation R on $\mathcal{F}$ as follows:

$$R = \{(f, g) \in \mathcal{F} \times \mathcal{F} \mid \exists h \in \mathcal{P}(f = h^{-1} \circ g \circ h)\}$$

(a) Prove that R is an equivalence relation.
(b) Prove that if fRg then $(f \circ f)R(g \circ g)$.
(c) For any $f \in \mathcal{F}$ and $a \in A$, if $f(a) = a$ then we say that a is a *fixed point* of f. Prove that if f has a fixed point and fRg, then g also has a fixed point.

Problem 18

Suppose $f : A \to C$, $g : B \to C$, and g is one-to-one and onto. Prove that there is a function $h : A \to B$ such that $g \circ h = f$.