

A Radical Approach to Real Analysis by David M. Bressoud

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1 Crisis in Mathematics: Fourier's Series

2 Infinite Summations

2.1 The Archimedean Understanding

Definition 1 (Archimedean Understanding of an Infinite Series). *The Archimedean Understanding of an infinite series is that it is shorthand for the sequence of finite summations. The value of an infinite series, if it exists, is that number T such that given any $L < T$ and any $M > T$, all of the finite sums from some point on will be strictly contained in the interval between L and M . More precisely, given $L < T < M$, there is an integer n , whose value depends on the choice of L and M , such that every partial sum with at least n terms lies inside the interval (L, M) .*

2.2 Geometric Series

Definition 2 (Convergence of an Infinite Series). *An infinite series converges if there is a target value T such that for any $L < T$ and any $M > T$, all of the partial sums from some point on are strictly between L and M .*

2.3 Calculating π

Theorem 1 (Newton's Binomial Series). *For any real number a and any x such that $|x| < 1$, we have that*

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \dots.$$

2.4 Logarithms and Harmonic Series

Definition 3 (Divergence to Infinity). *When we write that an infinite series equals ∞ , we mean that no matter what number we pick, we can find an n so that the partial sums with at least n terms will exceed that number.*

Definition 4 (Euler's constant, γ). *Euler's constant is defined as the limit between the partial sum of the harmonic series and the natural logarithm,*

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n \right)$$

Definition 5 (Nested Interval Principle). *Given an increasing sequence, $x_1 \leq x_2 \leq x_3 \leq \dots$, and a decreasing sequence, $y_1 \geq y_2 \geq y_3 \geq \dots$, such that y_n is always larger than x_n but the difference between y_n and x_n can be made arbitrarily small by taking n sufficiently large, there is exactly one real number that is greater than or equal to every x_n and less than or equal to every y_n .*

2.5 Taylor Series

Definition 6 (Taylor Series). *If all of the derivatives of the function f exist at the point a , then the Taylor series f about a is the infinite series*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots.$$

This has a special case ($a = 0$):

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots.$$

Theorem 2 (Lagrange's Remainder Theorem). *Given a function f for which all derivatives exist at $x = a$, let $D_n(a, x)$ denote the difference between the n th partial sum of the Taylor series for f expanded about $x = a$ and the target value $f(x)$,*

$$D_n(a, x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} \right).$$

There is at least one real number c strictly between a and x for which

$$D_n(a, x) = \frac{f^{(n)}(c)}{n!}(x-a)^n.$$

Theorem 3 (Stirling's Formula). *The factorial function $n!$ is well approximated by the function $(n/e)^n \sqrt{2\pi n}$. Specifically, we have that*

$$\lim_{n \rightarrow \infty} \frac{n!}{(n/e)^n \sqrt{2\pi n}} = 1.$$

Theorem 4 (C^p and analytic functions). *Given an interval I , a function with a continuous first derivative in I is said to belong to the class C^1 . If the p th derivative exists and is continuous in I , the function belongs to the class C^p . If all derivatives exist, the function belongs to C^∞ and is called **analytic**.*

3 Differentiability and Continuity

3.1 Differentiability

Theorem 5 (Mean Value Theorem). *Given a function f that is differentiable at all points strictly between a and x and continuous at all points on the closed interval from a to x , there exists a real number c strictly between a and x such that*

$$\frac{f(x) - f(a)}{x - a} = f'(c).$$

Definition 7 (Archimedean Understanding of Limits). *When we write any limit statement such as*

$$\lim_{x \rightarrow a} f(x) = T.$$

what we actually mean is that if we take any number $M > T$, then we can force $f(x) < M$ by taking x to be sufficiently close to a . Similarly, if we take any $L < T$, then we can force $f(x) > L$ by taking x sufficiently close to (but not equal to) a .

Definition 8 (Derivative of f at $x = a$). *The derivative of f at a is that value, denoted $f'(a)$, such that for any $L < f'(a)$ and any $M > f'(a)$, we can force*

$$L < \frac{f(x) - f(a)}{x - a} < M,$$

by simply taking x sufficiently close to (but not equal to) a .

Definition 9 (Cauchy Definition of Derivative of f at $x = a$). *The derivative of f at a is that value, denoted $f'(a)$, such that for any $\epsilon > 0$, we have a response $\delta > 0$ so that if $0 < |x - a| < \delta$, then this forces*

$$E(x, a) = \left| f'(a) - \frac{f(x) - f(a)}{x - a} \right| < \epsilon.$$

3.2 Cauchy and the Mean Value Theorem

Definition 10 (Intermediate Value Property). *A function f is said to have the intermediate value property on the interval $[a, b]$ if given any two points $x_1, x_2 \in [a, b]$ and any number N satisfying*

$$f(x_1) < N < f(x_2),$$

then there is at least one value c between x_1 and x_2 for which $f(c) = N$.

Theorem 6 (Generalized Mean Value Theorem). *If f and F are both continuous at every point of $[a, b]$ and differentiable at every point on the open interval (a, b) and F' is never zero in this interval, then*

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(c)}{F'(c)},$$

for at least one point c , $a < c < b$.

3.3 Continuity

Definition 11 (Continuity). *We say that f is continuous at a if given any positive error bound ϵ , we can always reply with a tolerance δ such that if x is within δ of a , then $f(x)$ is within ϵ of $f(a)$:*

$$|x - a| = \delta \text{ implies that } |f(x) - f(a)| < \epsilon.$$

To say that f is continuous on an interval I means that it is continuous at every point a in the interval I .

Theorem 7 (Intermediate Value Theorem). *If f is continuous on the interval $[a, b]$, then f has the intermediate value property on this interval.*

Definition 12 (Monotonic). A function is **monotonic** on $[a, b]$ if it is **increasing** on this interval,

$$a \leq x_1 \leq x_2 < b \text{ implies that } f(x_1) \leq f(x_2),$$

or if it is **decreasing** on this interval,

$$a \leq x_1 \leq x_2 \leq b \text{ implies that } f(x_1) \geq f(x_2).$$

A function is **piecewise monotonic** on $[a, b]$ if we can find a partition of this interval into a finite number of subintervals

$$a = x_1 < x_2 < \dots < x_{n-1} < x_n = b,$$

for which the function is monotonic on each open subinterval (x_i, x_{i+1}) .

Theorem 8 (Modified Converse to IVT). If f is a piecewise monotonic and satisfies the intermediate value property on the interval $[a, b]$, then f is continuous at every point c in (a, b) .

Theorem 9 (Differentiable implies Continuous). If f is differentiable at $x = c$, then f is continuous at $x = c$.

Definition 13 (One-side Limits and Derivatives). The **limit from the right**, $\lim_{x \rightarrow a^+} f(x)$, is the target value T , with the property that for any $\epsilon > 0$, there is a response δ so that if $a < x < a + \delta$, then $|f(x) - T| < \epsilon$. The **one-sided derivatives** are defined by

$$f'_+(a) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}, \quad f'_-(a) = \lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a}.$$

3.4 Consequences of Continuity

Theorem 10 (Continuous implies Bounded). If f is continuous on the interval $[a, b]$, then there exists finite A and B such that

$$A \leq f(x) \leq B,$$

for all $x \in [a, b]$.

Definition 14 (Least Upper, Greatest Lower Bounds). Given a set S , the **least upper bound** or **supremum** of S , denoted $\sup S$, is the number G with the property that for any numbers $L < G$ and $M > G$, there is at least one element of S that is strictly larger than L and at least one upper bound for S that is strictly smaller than M . The **greatest lower bound** or **infimum** of S , denoted $\inf S$, is the negative of the least upper bound of $-S = \{-s \mid s \in S\}$.

Theorem 11 (Upperbound implies Least Upper Bound). In the real numbers, every set that has an upper bound also has a least upper bound and every set that has a lower bound also has a greatest lower bound.

Theorem 12 (Continuous implies Bounds Achieved). If f is continuous on $[a, b]$, then it achieves its greatest lower bound and its least upper bound. Equivalently, there exists $k_1, k_2 \in [a, b]$ such that

$$f(k_1) \leq f(x) \leq f(k_2),$$

for all $x \in [a, b]$.

Theorem 13 (Fermat's Theorem on Extrema). If f has an extremum at a point $c \in (a, b)$ [$f(c) \geq f(x)$ for all $x \in (a, b)$ or $f(c) \leq f(x)$ for all $x \in (a, b)$] and if f is differentiable at every point in (a, b) , then $f'(c) = 0$.

Theorem 14 (Rolle's Theorem). Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b) and for which $f(a) = f(b) = 0$. There exists at least one c , $a < c < b$, for $f'(c) = 0$.

3.5 Consequences of the Mean Value Theorem

Theorem 15 (Cauchy's Remainder Theorem). Given a function f for which all derivatives exist at $x = a$, let $D_n(a, x)$ denote the difference between the n th partial sum of the Taylor series for f expanded about $x = a$ and the target value $f(x)$,

$$D_n(a, x) = f(x) - \left(f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} \right).$$

There is at least one real number c strictly between a and x for which

$$D_n(a, x) = \frac{f^{(n)}(c)}{(n-1)!} (x-c)^{n-1} (x-a).$$

Definition 15 (Infinite Limit and Limit at Infinity). *The statement*

$$\lim_{x \rightarrow a} f(x) = \infty,$$

means that for any real number L , we can force $f(x) > L$ by restricting x to be sufficiently close to a . That is to say, there is a $\delta > 0$ so that $|x - a| < \delta$ implies $f(x) > L$. When we write

$$\lim_{x \rightarrow \infty} f(x) = T,$$

we mean that for any positive ϵ , we can force $f(x)$ to be within ϵ of T by taking x sufficiently large. In other words, there is an N so that $x > N$ implies that $|f(x) - T| < \epsilon$.

Theorem 16 (L'Hopital's Rule: 0/0). *If f and F are both differentiable inside an open interval that contains a , if*

$$\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} F(x),$$

if $F'(x) \neq 0$ for all x in this open interval, and if $\lim_{x \rightarrow a} f'(x)/F'(x)$ exists, then

$$\lim_{x \rightarrow a} \frac{f(x)}{F(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{F'(x)}.$$

Theorem 17 (L'Hopital's Rule: ∞/∞). *If f and F are both differentiable at every point except $x = a$ in an open interval that contains a , if*

$$\lim_{x \rightarrow a} |F(x)| = \infty,$$

if $F'(x) \neq 0$ for all x in this open interval, and if $\lim_{x \rightarrow a} f'(x)/F'(x)$ exists, then

$$\lim_{x \rightarrow a} \frac{f(x)}{F(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{F'(x)}.$$

Theorem 18 (Darboux's Theorem). *If f is differentiable on $[a, b]$, then f' has the intermediate value property on $[a, b]$.*

4 The Convergence of Infinite Series

4.1 The Basic Tests of Convergence

Definition 16. *An infinite series converges* if there is a value T with the property that for each $\epsilon > 0$ there is a response N so that all of the partial sums with at least N terms lie strictly within the open interval $(T - \epsilon, T + \epsilon)$.