The Dedekind-MacNeille completion

1 Ordered sets

1.1 Definition. A partially ordered set or poset is a set X equipped with an order relation $\leq$ satisfying the following axioms:

1. Reflexivity: $x \leq x$,
2. Anti-symmetry: If $x \leq y$ and $y \leq x$, the $x = y$.
3. Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$

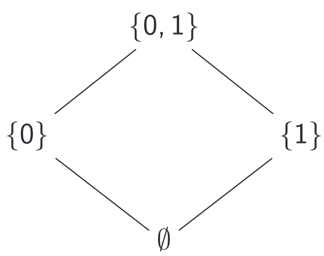
Furthermore, we say that X is totally ordered if it satisfies the trichotomy condition. That is, for each $x, y \in X$, we either have $x \leq y$ or $y \leq x$ (or both, in which case $x = y$).
Furthermore, we say that $x < y$ if $x \leq y$ and $x \neq y$.

A poset can often be represented by what is called a Hasse diagram. Indeed, if $(X, \leq)$ is a poset, then we can say that $x \prec y$ if $x < y$ and if there exists no z such that $x < z < y$. We say that $\prec$ is the covering relation.

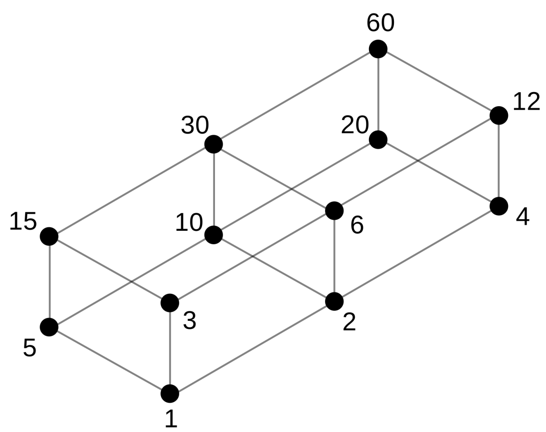
A Hasse diagram is represented by a collection of dots. We connect those dots for which the covering relation holds.

For example, consider $\{1, 2, 3, 4\}$ with the usual ordering. Then $1 \leq 3$, but not $1 \prec 3$ because there is a number between 1 and 3. On the other hand, we do have $1 \prec 2$, we have $2 \prec 3$ and $3 \prec 4$. We can represent this set by the following Hasse diagram:

Consider the set $A = \{0, 1\}$. Then consider $X = \mathcal{P}(A)$. This consists of 4 elements. Then X is ordered by inclusion. Thus $\{1\} \subseteq \{0, 1\}$, for example. We can represent this by the following Hasse diagram:



Another example of an ordered set is the set of divisors of 60. This set can be ordered using the division relation. This has the following Hasse diagram



Exercise: Show that the divisors of 30, the power set of $\{1, 2, 3\}$ all have a cube as Hasse Diagram:

Exercise: Find the Hasse diagrams of all ordered sets of 1, 2, 3 and 4 elements.

Exercise: What numbers have divisor-diagram consisting of cubes.

1.2 Definition. Let $(X, \leq)$ be a poset. Let $A \subseteq X$.

- We say that x is an upper bound of A if for each $a \in A$, we have $a \leq x$.
- We say that x is a lower bound of A if for each $a \in A$, we have $x \leq a$.
- We say that x is the supremum of A if x is the smallest upper bound.
- We say that x is the infimum of A if x is the largest lower bound.

A special type of ordered set is a lattice:

1.3 Definition. Let $(X, \leq)$ be an ordered set. We say that this is a lattice if every non-empty finite set has a supremum and an infimum. We denote the supremum of $\{x, y\}$ with $x \vee y$, and its infimum by $x \wedge y$.
If every set has a supremum and an infimum, then we say that X is a complete lattice. We denote the supremum of a set A by $\bigvee A$ and its infimum by $\bigwedge A$.

Exercise: Show that each power set is a complete lattice.

Exercise: Let $X = \mathbb{N}$ and take as relation that $n \leq m$ if n divides m. Show that this is a lattice.

Exercise: Find all lattices with 1 element, with 2 elements, with 3 elements, with 4 elements and with 5 elements.

Exercise: show that each totally ordered set is a lattice. Is it also a complete lattice?

2 The Dedekind-MacNeille completion

In every ordered set, we can add elements to the ordered set to make it a complete lattice. This is called the Dedekind-MacNeille completion.

2.1 Definition. Let $(X, \leq)$ be an ordered set. Let $A \subseteq X$. We define A^u the set of all upper bounds of A, and we set A^ℓ the set of all lower bounds of A. We let $A^{u\ell} = (A^u)^\ell$.

Exercise: Let $a \in X$, prove that $\{a\}^{u\ell} = \{x \in X \mid x \leq a\}$.

Exercise: Let $A \subseteq \mathbb{Q}$. Show that the complement of $A^{u\ell}$ is either $\emptyset$, entire $\mathbb{Q}$ or a Dedekind cut.

Exercise: conversely, show that every Dedekind cut is the complement of a set of the form $A^{u\ell}$.

We can do this in general:

2.2 Definition. Let $(X, \leq)$ be an ordered set. We let $\mathrm{DM}(X)$ be the set

$$\{A^{u\ell} \mid A \subseteq X\}$$

Then $\mathrm{DM}(X)$ is an ordered set under the inclusion. This is called the Dedekind-MacNeille completion of X.

2.3 Lemma. *For every poset $(X, \leq)$, we have the following:*

1. *$A \subseteq A^{u\ell}$ and $A \subseteq A^{\ell u}$.*
2. *If $A \subseteq B$, then $B^u \subseteq A^u$ and $B^\ell \subseteq A^\ell$.*
3. *$A^{\ell u \ell} = A^\ell$ and $A^{u \ell u} = A^u$.*
4. *If $A \in \mathrm{DM}(X)$, then $A^{u\ell} = A$.*

Proof. Exercise. □

Now we can prove our main results:

2.4 Theorem. *For any poset $(X, \leq)$, we have that $\mathrm{DM}(X)$ is a complete lattice.*

Proof. Exercise □

Now we can find an injection $i : X \to \mathrm{DM}(X)$ by sending $x \in X$ to $\{x\}^{u\ell} = \{y \in X \mid y \leq x\}$.

2.5 Theorem.

1. *i is an injection.*
2. *We have that $x \leq y$ if and only if $i(x) \leq i(y)$.*
3. *The image $i(X)$ is meet-dense in $\mathrm{DM}(X)$. This means that for any $A \in \mathrm{DM}(X)$, there is some $\mathcal{B} \subseteq i(X)$ such that $\bigcap \mathcal{B} = A$.*
4. *The image $i(X)$ is join-dense in $\mathrm{DM}(X)$. This means that for any $A \in \mathrm{DM}(X)$, there is some $\mathcal{B} \subseteq i(X)$ such that $\bigvee \mathcal{B} = A$.*
5. *If $B \subseteq X$ and if $\bigvee B$ exists, then $i\left(\bigvee B\right) = \bigvee i(B)$.*
6. *If $B \subseteq X$ and if $\bigwedge B$ exists, then $i\left(\bigwedge B\right) = \bigcap i(B)$.*
7. *If X is a complete lattice, then i is bijective.*

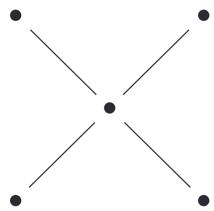
Proof. Exercise. □

Exercise: show that the Dedekind Mac-Neille completion of $\mathbb{Q}$ is $\mathbb{R} \cup \{-\infty, +\infty\}$.

Exercise: Calculate the Dedekind MacNeille completion of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{R} \cup \{-\infty, +\infty\}$.

Exercise: Let X be an antichain. That is: let the relation on X be defined as $x \leq x$, but no other two elements are related. Calculate $\mathrm{DM}(X)$.

Exercise: Consider the ordered set:



Compute the Dedekind-MacNeille completion of this.

Exercise: What would it mean for the Dedekind-MacNeille completion to be the smallest possible completion? Rigorize and prove it.

Exercise: Let X and Y be posets with their Dedekind MacNeille completions $i : X \to \mathrm{DM}(X)$ and $j : Y \to \mathrm{DM}(Y)$. Let $f : X \to Y$ be an order-preserving map, meaning that if $x \leq x'$ then $f(x) \leq f(x')$.

1. Show there is some order-preserving map $F : \mathrm{DM}(X) \to \mathrm{DM}(Y)$ such that $F \circ i = j \circ f$.
2. Give an example to show that F is not unique with this property.
3. Show that if $X = Y = \mathbb{Q}$, and f is bijective, then $F|_{\mathbb{R}}$ is unique.
4. There is an axiom system of the real numbers due to Tarski that does not mention multiplication of real numbers at all. Checking only Tarski's axioms would make the construction of the reals in Bloch a lot easier! Look it up, it is delightfully simple. But how would multiplication be defined? Given that $\mathbb{R}$ is defined as the Dedekind-MacNeille completion of $\mathbb{Q}$, how could you use F of this exercise to extend addition and multiplication from $\mathbb{Q}$ to $\mathbb{R}$? Checking that this multiplication and addition satisfy the required axioms would be standard but tedious and is not asked here.