Chapter 2 Linear Equations

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1 Equations In Two Unknowns

Problem 7

Solve the following systems of equations for x and y.

$$\begin{aligned} 7x - y &= 2 \\ 2x + 2y &= 4 \end{aligned}$$

Solution 7

$$\begin{aligned} 7x - y = 2 &\leftrightarrow 14x - 2y = 4 \\ (14x - 2y) + (2x + 2y) &= 4 + 4 \\ 16x &= 8 \\ x &= \frac{8}{16} \\ x &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 7x - y &= 2 \\ 7(\frac{1}{2}) - y &= 2 \\ 7(\frac{1}{2}) - 2 &= y \\ (\frac{7}{2}) - \frac{4}{2} &= y \\ \frac{3}{2} &= y \end{aligned}$$

Problem 8

Solve the following systems of equations for x and y.

$$\begin{aligned} -4x - 7y &= 5 \\ 2x + y &= 6 \end{aligned}$$

Solution 8

$$\begin{aligned}
2(2x+y=6) &\leftrightarrow 4x+2y=12\\
(4x+2y)+(-4x-7y) &= 12+5\\
-5y &= 17\\
y &= \frac{-17}{5}
\end{aligned}$$

$$\begin{aligned}
2x+y &= 6\\
2x+\frac{-17}{5} &= 6\\
2x &= 6+\frac{17}{5}\\
(\frac{10}{5})x &= \frac{30}{5}+\frac{17}{5}\\
(\frac{10}{5})x &= \frac{47}{5}\\
x &= \frac{\frac{47}{5}}{(\frac{10}{5})}\\
x &= \frac{235}{50}\\
x &= \frac{47}{10}
\end{aligned}$$

Problem 9

Let a, b, c, d be numbers such that $ad-bc\neq 0$. Solve the following systems of equations for x and y in terms of a, b, c, d.

(a)

$$\begin{aligned}
ax+by &= 1\\
cx+dy &= 2
\end{aligned}$$

(b)

$$\begin{aligned}
ax+by &= 3\\
cx+dy &= -4
\end{aligned}$$

(c)

$$\begin{aligned}
ax+by &= -2\\
cx+dy &= 3
\end{aligned}$$

(d)

$$\begin{aligned}
ax+by &= 5\\
cx+dy &= 7
\end{aligned}$$

Solution 9 (a)
First multiply by d, $ax + by = 1 \leftrightarrow adx + bdy = d$.
Then multiply by b, $cx + dy = 2 \leftrightarrow bcx + bdy = 2b$.
Also multiply by c, $ax + by = 1 \leftrightarrow acx + bcy = c$.
And mutiply by a, $cx + dy = 2 \leftrightarrow acx + ady = 2a$.

$$\begin{aligned}(adx + bdy) - (bcx + bdy) &= d - 2b \\ adx - bcx &= d - 2b \\ x(ad - bc) &= d - 2b \\ x &= \frac{d - 2b}{ad - bc}\end{aligned}$$

$$\begin{aligned}(acx + ady) - (acx + bcy) &= 2a - c \\ ady - bcy &= 2a - c \\ y(ad - bc) &= 2a - c \\ y(ad - bc) &= \frac{2a - c}{ad - bc}\end{aligned}$$

Solution 9 (b)
First multiply by d, $ax + by = 3 \leftrightarrow adx + bdy = 3d$.
Then multiply by b, $cx + dy = -4 \leftrightarrow bcx + bdy = -4b$.
Also multiply by c, $ax + by = 3 \leftrightarrow acx + bcy = 3c$.
And mutiply by a, $cx + dy = -4 \leftrightarrow acx + ady = -4a$.

$$\begin{aligned}(adx + bdy) - (bcx + bdy) &= 3d + 4b \\ adx - bcx &= 3d + 4b \\ x(ad - bc) &= 3d + 4b \\ x &= \frac{3d + 4b}{ad - bc}\end{aligned}$$

$$\begin{aligned}(acx + ady) - (acx + bcy) &= -4a - 3c \\ ady - bcy &= -4a - 3c \\ y(ad - bc) &= -4a - 3c \\ y(ad - bc) &= \frac{-4a - 3c}{ad - bc}\end{aligned}$$

Solution 9 (c)
First multiply by d, $ax + by = -2 \leftrightarrow adx + bdy = -2d$.
Then multiply by b, $cx + dy = 3 \leftrightarrow bcx + bdy = 3b$.
Also multiply by c, $ax + by = -2 \leftrightarrow acx + bcy = -2c$.
And mutiply by a, $cx + dy = 3 \leftrightarrow acx + ady = 3a$.

$$\begin{aligned} (adx + bdy) - (bcx + bdy) &= -2d + 3b \\ adx - bcx &= -2d + 3b \\ x(ad - bc) &= -2d + 3b \\ x &= \frac{-2d + 3b}{ad - bc} \end{aligned}$$

$$\begin{aligned} (acx + ady) - (acx + bcy) &= 3a - c \\ ady - bcy &= 3a + 2c \\ y(ad - bc) &= 3a + 2c \\ y(ad - bc) &= \frac{3a + 2c}{ad - bc} \end{aligned}$$

Solution 9 (d)
First multiply by d, $ax + by = 5 \leftrightarrow adx + bdy = 5d$.
Then multiply by b, $cx + dy = 7 \leftrightarrow bcx + bdy = 7b$.
Also multiply by c, $ax + by = 5 \leftrightarrow acx + bcy = 5c$.
And mutiply by a, $cx + dy = 7 \leftrightarrow acx + ady = 7a$.

$$\begin{aligned} (adx + bdy) - (bcx + bdy) &= 5d - 7b \\ adx - bcx &= 5d - 7b \\ x(ad - bc) &= 5d - 7b \\ x &= \frac{5d - 7b}{ad - bc} \end{aligned}$$

$$\begin{aligned} (acx + ady) - (acx + bcy) &= 5a - 7c \\ ady - bcy &= 5a - 7c \\ y(ad - bc) &= 5a - 7c \\ y(ad - bc) &= \frac{5a - 7c}{ad - bc} \end{aligned}$$

Problem 10

Making the same assumptions as in Excersize 9, show that the solution of the system

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

must be $x = 0$ and $y = 0$.

Solution 10
First $ax + by = 0 \leftrightarrow adx + bdy = 0$.
Then $cx + dy = 0 \leftrightarrow bcx + bdy = 0$.
Also $ax + by = 0 \leftrightarrow acx + bcy = 0$.
And $cx + dy = 0 \leftrightarrow acx + ady = 0$.

$$\begin{aligned} (adx + bdy) - (bcx + bdy) &= 0 \\ adx - bcx &= 0 \\ x(ad - bc) &= 0 \\ x &= \frac{0}{ad - bc} \\ x &= 0 \end{aligned}$$

$$\begin{aligned} (acx + ady) - (acx + bcy) &= 0 \\ ady - bcy &= 0 \\ y(ad - bc) &= 0 \\ y &= \frac{0}{ad - bc} \\ y &= 0 \end{aligned}$$

Problem 11

Let a, b, c, d, u, v be numbers and assume that $ad - bc \neq 0$. Solve the following system of equations for x and y in terms of a, b, c, d, u, v

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned}$$

Verify that the answer you get is actually a solution.

Solution 9 (d)
First multiply first equation by d, $ax + by = u \leftrightarrow adx + bdy = ud$.
Then multiply second equation by b, $cx + dy = v \leftrightarrow bcx + bdy = vb$.
Also multiply first equation by c, $ax + by = u \leftrightarrow acx + bcy = uc$.
And multiply second equation by a, $cx + dy = v \leftrightarrow acx + ady = va$.

$$\begin{aligned} (adx + bdy) - (bcx + bdy) &= ud - vb \\ adx - bcx &= ud - vb \\ x(ad - bc) &= ud - vb \\ x &= \frac{ud - vb}{ad - bc} \end{aligned}$$

$$\begin{aligned} (acx + ady) - (acx + bcy) &= va - uc \\ ady - bcy &= va - uc \\ y(ad - bc) &= va - uc \\ y &= \frac{va - uc}{ad - bc} \end{aligned}$$

Veryfying the first equation.

$$\begin{aligned}
a\left(\frac{ud-vb}{ad-bc}\right)+b\left(\frac{va-uc}{ad-bc}\right) &= u \\
\frac{aud-avb+bva-buc}{ad-bc} &= u \\
\frac{v(-ab+ba)+u(ad-bc)}{ad-bc} &= u \\
\frac{u(ad-bc)}{ad-bc} &= u \\
u &= u
\end{aligned}$$

Verifying the second equation.

$$\begin{aligned}
c\left(\frac{ud-vb}{ad-bc}\right)+d\left(\frac{va-uc}{ad-bc}\right) &= v \\
\frac{cud-cvb+dva-duc}{ad-bc} &= v \\
\frac{u(cd-cd)+v(ad-bc)}{ad-bc} &= v \\
\frac{v(ad-bc)}{ad-bc} &= v \\
v &= v
\end{aligned}$$

2 Equations In Three Unknowns

Problem 7

Solve the following equations for x, y, z.
(1) $4x - 2y + 5z = 1$
(2) $x + y + z = 0$
(3) $-x + y - 2z = 2$

Solution 7

Summing (2) and (3).

$$\begin{aligned}(x+y+z)+(-x+y-2z) &= 0+2\\ (4)\quad 2y-z &= 2\end{aligned}$$

Summing (1) and (2) multiplied by -4.

$$\begin{aligned}(4x-2y+5z)+(-4x-4y-4z) &= 1+0\\ (5)\quad -6y+z &= 1\end{aligned}$$

Summing (4) and (5).

$$\begin{aligned}(2y-z)+(-6y+z) &= 2+1\\ -4y &= 3\\ y &= \frac{-3}{4}\end{aligned}$$

Summing (2) and (3). Setting $y = \frac{-3}{4}$.

$$\begin{aligned}(x+\frac{-3}{4}+z)+(-x+\frac{-3}{4}-2z) &= 0+2\\ 2\cdot\frac{-3}{4}-z &= 2\\ \frac{-3}{2}-2 &= z\\ \frac{-3}{2}-\frac{4}{2} &= z\\ \frac{-7}{2} &= z\end{aligned}$$

Using (2). Setting $y = \frac{-3}{4}$ and $z = \frac{-7}{2}$.

$$\begin{aligned}x+\frac{-3}{4}+\frac{-7}{2} &= 0\\ x+\frac{-3}{4}+\frac{-14}{4} &= 0\\ x+\frac{-17}{4} &= 0\\ x &= \frac{17}{4}\end{aligned}$$

$$\therefore x = \frac{17}{4}, y = \frac{-3}{4}, z = \frac{-7}{2}$$

Problem 8

Solve the following equations for x, y, z.
(1) $x + y + z = 0$
(2) $x - y - z = 1$
(3) $x + y - z = 1$

Solution 8

Summing (1) and (2).

$$\begin{aligned}(x + y + z) + (x - y - z) &= 0 + 1 \\ 2x &= 1 \\ x &= \frac{1}{2}\end{aligned}$$

Summing (2) and (3). Setting $x = \frac{1}{2}$.

$$\begin{aligned}(\frac{1}{2} - y - z) + (\frac{1}{2} + y - z) &= 1 + 1 \\ 1 - 2z &= 2 \\ -2z &= 1 \\ z &= \frac{-1}{2}\end{aligned}$$

Using (3). Setting $x = \frac{1}{2}$ and $z = \frac{-1}{2}$.

$$\begin{aligned}\frac{1}{2} + y - (\frac{-1}{2}) &= 1 \\ y + 1 &= 1 \\ y &= 0\end{aligned}$$

$$\therefore x = \frac{1}{2}, y = 0, z = \frac{-1}{2}$$

Problem 11

Solve the following equations for x, y, z.
(1) $\frac{1}{2}x + y - \frac{3}{4}z = 1$
(2) $x - \frac{1}{2}y + z = 0$
(3) $x + y - \frac{1}{3}z = 0$

Solution 11

Multiply (1) by -4.

$$\frac{1}{2}x + y - \frac{3}{4}z = 1$$
$$(4) \quad -2x - 4y + 3z = -4$$

Multiply (2) by 2.

$$x - \frac{1}{2}y + z = 0$$
$$(5) \quad 2x - y + 2z = 0$$

Multiply (3) by 3.

$$x + y - \frac{1}{3}z = 0$$
$$(6) \quad 3x + 3y - z = 0$$

Summing (4) and (5).

$$(-2x - 4y + 3z) + (2x - y + 2z) = -4 + 0$$
$$(7) \quad -5y + 5z = -4$$

Sum (5) times 3 and (6) times -2.

$$(6x - 3y + 6z) + (-6x - 6y + 2z) = 0$$
$$(8) \quad -9y + 8z = 0$$

Sum (7) times -9 and (8) times 5.

$$\begin{aligned}(45y - 45z) + (-45y + 40z) &= 36 \\ -5z &= 36 \\ z &= \frac{-36}{5}\end{aligned}$$

Using (7) and setting $z = \frac{-36}{5}$.

$$\begin{aligned}-9y + 8(\frac{-36}{5}) &= 0 \\ -9y - \frac{288}{5} &= 0 \\ -9y &= \frac{288}{5} \\ y &= \frac{-32}{5}\end{aligned}$$

Using (5) and setting $y = \frac{-32}{5}$, $z = \frac{-36}{5}$.

$$\begin{aligned}
2x - (\frac{-32}{5}) + 2(\frac{-36}{5}) &= 0 \\
2x + \frac{32}{5} - \frac{72}{5} &= 0 \\
2x - \frac{40}{5} &= 0 \\
x &= \frac{40}{10} \\
x &= 4
\end{aligned}$$

Problem 12

Solve the following equations for x, y, z.
(1) $\frac{1}{2}x - \frac{2}{3}y + z = 1$
(2) $x - \frac{1}{5}y + z = 0$
(3) $2x - \frac{1}{3}y + \frac{2}{5}z = 1$

Solution 12

Multiply (1) by 6.

$$\frac{1}{2}x - \frac{2}{3}y + z = 1$$
$$(4) \quad 3x - 4y + 6z = 6$$

Multiply (2) by -30.

$$x - \frac{1}{5}y + z = 0$$
$$(5) \quad -30x + 6y - 30z = 0$$

Multiply (3) by 15.

$$2x - \frac{1}{3}y + \frac{2}{5}z = 1$$
$$(6) \quad 30x - 5y + 6z = 15$$

Summing (5) and (6).

$$(-30x + 6y - 30z) + (30x - 5y + 6z) = 15$$
$$(7) \quad y - 24z = 15$$

Sum (4) times 10 and (5).

$$(30x - 40y + 60z) + (-30x + 6y - 30z) = 60$$
$$(8) \quad -34y + 30z = 60$$

Sum (7) times 34 and (8).

$$(34y - 816z) + (-34y + 30z) = (15 * 34) + 60$$
$$(34y - 816z) + (-34y + 30z) = 510 + 60$$
$$-786z = 570$$
$$z = \frac{-95}{131}$$

Using (7). Set $z = \frac{-95}{131}$.

$$y - 24(\frac{-95}{131}) = 15$$
$$y + \frac{2280}{131} = 15$$
$$y = \frac{1965}{131} - \frac{2280}{131}$$
$$y = \frac{-315}{131}$$

Using (7). Set $z = \frac{-95}{131}$ and $y = \frac{-315}{131}$.

$$
\begin{aligned}
x - \frac{1}{5} \cdot \frac{-315}{131} + \frac{-95}{131} &= 0 \\
x &= \frac{1}{5} \cdot \frac{-315}{131} - \frac{-95}{131} \\
x &= \frac{-315}{655} - \frac{-475}{655} \\
x &= \frac{160}{655} \\
x &= \frac{32}{131}
\end{aligned}
$$

$$\therefore x = \frac{32}{131}, y = \frac{-315}{131}, z = \frac{-95}{131}$$