Calculus by Elliott Mendelson

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1 Functions

Problem 1

If $f(x) = x^2 - 4x + 6$. Find
(a) $f(0)$.
(b) $f(3)$.
(c) $f(-2)$.
Show that $f(\frac{1}{2}) \neq f(\frac{7}{2})$ and $f(2-h) = f(2+h)$.

Solution (a):

$$f(0) = 0^2 - 4(0) + 6 = 6$$

Solution (a):

$$f(3) = 3^2 - 4(3) + 6 = 9 - 12 + 6 = 3$$

Solution (a):

$$f(-2) = (-2)^2 - 4(-2) + 6 = 18$$

Solution:

$$f(\frac{1}{2}) = (\frac{1}{2})^2 + 4(\frac{1}{2}) + 6 = \frac{1}{4} + 2 + 6 = \frac{1}{4} + 8 = \frac{33}{4}$$

$$f(\frac{7}{2}) = (\frac{7}{2})^2 + 4(\frac{7}{2}) + 6 = \frac{49}{4} + (2 * 7) + 6 = \frac{49}{4} + 14 + 6 = \frac{49}{4} + 20 = \frac{129}{4}$$

Clearly $f(\frac{1}{2}) \neq f(\frac{7}{2})$.

Solution:

$$f(2-h) = (2-h)^2 - 4(2-h) + 6 = 4 - 4h + h^2 - 8 + 4h + 6 = h^2 + 2$$

$$f(2=h) = (2+h)^2 - 4(2+h) + 6 = 4 + 4h + h^2 - 8 - 4h + 6 = h^2 + 2$$

Thus $f(2-h) = f(2+h)$.

Problem 16

Determine the domain of each of the following functions:
(a) $y = x^2 + 4$.
(b) $y = \sqrt{x^2 + 4}$.
(c) $y = \sqrt{x^2 - 4}$.
(d) $y = \frac{x}{x+3}$.

(e) $y = \frac{2x}{(x-2)(x+1)}$.
(f) $y = \frac{1}{\sqrt{9-x^2}}$.
(g) $y = \frac{x^2-1}{x^2+1}$.
(h) $y = \sqrt{\frac{x}{2-x}}$

Solution (a):

$$x \in \mathbb{R}$$

Solution (b):

$$x \in \mathbb{R}$$

Solution (c):

$$x \in \mathbb{R} \setminus \{-2, 2\}$$

Solution (d):

$$x \in \mathbb{R} \setminus \{-3\}$$

Solution (e):

$$x \in \mathbb{R} \setminus \{2, -1\}$$

Solution (f):

$$x \in \mathbb{R} \setminus \{-3, 3\}$$

Solution (g):

$$x \in \mathbb{R}$$

Solution (h):

$$x \in \mathbb{R} \setminus \{x \mid x < 0 \text{ or } x \geq 2\}$$

Problem 17

Compute $\frac{f(a+h)-f(a)}{h}$ in the following cases:
(a) $f(x) = \frac{1}{x-2}$.
(b) $f(x) = \sqrt{x-4}$ when $a \neq 2$ and $a + h \geq 4$.
(c) $f(x) = \frac{x}{x+1}$ when $a \neq -1$ and $a + h \neq -1$.

Solution (a):

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\frac{1}{(a+h)-2} - \frac{1}{a-2}}{h} \\
&= \frac{\frac{a-2}{a+h-2} - \frac{a+h-2}{(a+h-2)(a-2)}}{h} \\
&= \frac{\frac{-h}{(a+h-2)(a-2)}}{h} \\
&= \frac{-h}{h(a+h-2)(a-2)} \\
&= \frac{-1}{(a+h-2)(a-2)}
\end{aligned}$$

Solution (b):

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\sqrt{a+h-4}-\sqrt{a-4}}{h} \\
&= \frac{\sqrt{a+h-4}-\sqrt{a-4}}{h}\cdot\frac{\sqrt{a+h-4}+\sqrt{a-4}}{\sqrt{a+h-4}+\sqrt{a-4}} \\
&= \frac{a+h-(a-4)}{h(\sqrt{a+h-4}+\sqrt{a-4})} \\
&= \frac{1}{\sqrt{a+h-4}+\sqrt{a-4}}
\end{aligned}$$

Solution (c):

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\frac{a+h}{a+h+1}-\frac{a}{a+1}}{h} \\
&= \frac{\frac{(a+h)(a+1)}{(a+h+1)(a+1)}-\frac{a(a+h+1)}{(a+1)(a+h+1)}}{h} \\
&= \frac{\frac{a^2+a+ah+h-(a^2+ah+a)}{(a+h+1)(a+1)}}{h} \\
&= \frac{\frac{h}{(a+h+1)(a+1)}}{h} \\
&= \frac{1}{(a+h+1)(a+1)}
\end{aligned}$$

Problem 20

Evaluate the expression $\frac{f(x+h)-f(x)}{h}$ for the following functions f:
(a) $f(x) = 2x - x^2$.
(b) $f(x) = \sqrt{x-4}$ when $a \neq 2$ and $a + h \geq 4$.
(c) $f(x) = \frac{x}{x+1}$ when $a \neq -1$ and $a + h \neq -1$.
(d) $f(x) = \frac{x}{x+1}$ when $a \neq -1$ and $a + h \neq -1$.

Solution (a):

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{3(x+h)-(x+h)^2-(3x-x^2)}{h} \\
&= \frac{3x+3h-(x^2+2xh+h^2)-(3x-x^2)}{h} \\
&= \frac{3x+3h-x^2-2xh-h^2-3x+x^2}{h} \\
&= \frac{2h-2xh-h^2}{h} \\
&= \frac{h(3-2x-h)}{h} \\
&= 3-2x-h
\end{aligned}$$

Solution (b):

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{2(x+h)}-\sqrt{2x}}{h} \\
&= \frac{\sqrt{2(x+h)}-\sqrt{2x}}{h} \cdot \frac{\sqrt{2(x+h)}+\sqrt{2x}}{\sqrt{2(x+h)}+\sqrt{2x}} \\
&= \frac{2(x+h)-2x}{h(\sqrt{2(x+h)}+\sqrt{2x})} \\
&= \frac{2x+2h-2x}{h(\sqrt{2(x+h)}+\sqrt{2x})} \\
&= \frac{2h}{h(\sqrt{2(x+h)}+\sqrt{2x})} \\
&= \frac{2}{(\sqrt{2(x+h)}+\sqrt{2x})}
\end{aligned}$$

Solution (c):

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(3(x+h)-5)-(3x-5)}{h} \\
&= \frac{(3x+3h-5)-(3x-5)}{h} \\
&= \frac{3h}{h} \\
&= 3
\end{aligned}$$

Solution (d):

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{((x+h)^3-2)-(x^3-2)}{h} \\
&= \frac{(x^3+3x^2h+3xh^2+h^3-2)-(x^3-2)}{h} \\
&= \frac{h(3x^2+3xh+h^2)}{h} \\
&= 3x^2+3xh+h^2
\end{aligned}$$

Problem 21

Find a formula for the function f whose graph consists of all points satisfying each of the following equations. (In plain language, solve each equation for y.)
(a) $x^5y+4x-2=0$.
(b) $x=\frac{2+y}{2-y}$.
(c) $4x^2-4xy+y^2=0$.

Solution (a): Suppose $x \neq= 0$ then

$$x^5y+4x-2=0 \iff y=\frac{-4x+2}{x^5}$$

Thus

$$f(x)=\frac{-4x+2}{x^5}$$

Solution (b): Suppose $y \neq 2$ then

$$x = \frac{2+y}{2-y} \iff x(2-y)-2+y = 0 \iff 2x-xy-2-y = 0 \iff 2x-2 = xy+y \iff 2x-2 = x(y+1) \iff \frac{2x-2}{x+1} = y$$

Thus

$$f(x) = \frac{2x-2}{x+1}$$

Solution (c):

$$4x^2 - 4xy + y^2 = 0 \iff y^2 - 4xy + 4x^2 = 0 \iff (y-2x)(y-2x) = 0$$

Thus

$$f(x) = 2x$$

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