# Algebraic Geometry by Thomas Garrity et. al.

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## Contents



## 1 Conics

### 1.1 Conics over the Reals

**Problem 1**

$$P(x,y) = y - x^2, \quad C = \{(x,y) \in \mathbb{R}^2 \mid P(x,y) = 0\}.$$

Show that for any $(x,y) \in C$, we also have

$$(-x,y) \in C.$$

Thus the curve is symmetric about the y-axis.

*Proof.* Let $(x,y) \in C$. Then $P(x,y) = y - x^2 = 0$. Let $x' = -x$ and note that $(-x)^2 = x^2$. Thus

$$P(-x,y) = y - (-x)^2 = y - x^2 = 0.$$

Thus $(-x,y) \in C$. ∎

## Problem 2

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that if $(x, y) \in C$, then we have $y \geq 0$.

*Proof.* Suppose $(x, y) \in C$. Then

$$P(x, y) = y - x^2 = 0 \iff y = x^2 \geq 0.$$

Thus $y \geq 0$. ∎

## Problem 3

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that for every $y \geq 0$, there is a point $(x, y) \in C$ with this $y$-coordinate. Now, for points $(x, y) \in C$, show that if $y$ goes to infinity, then one of the corresponding $x$-coordinates also approaches infinity while the other corresponding $x$ coordinate must approach negative infinity.

*Proof.* Let $y \in \mathbb{R}$ such that $y \geq 0$. Let $x = \sqrt{y} \in \mathbb{R}$. Then

$$y - x^2 = y - (\sqrt{y})^2 = y - y = 0.$$

Thus $(x, y) = (\sqrt{y}, y) \in C$.

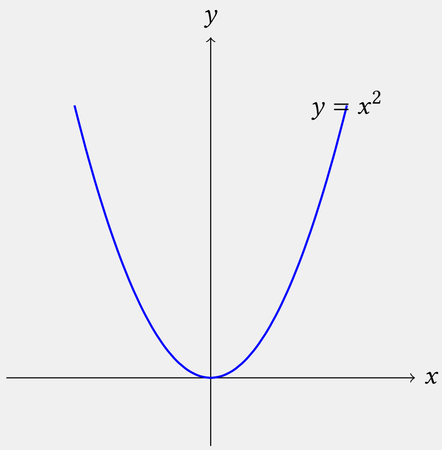
Now suppose $y \to \infty$. For points $(x, y) \in C$, we have

$$y - x^2 = 0 \iff x = \pm\sqrt{y}.$$

Since $y \to \infty$, we have $\sqrt{y} \to \infty$ and $-\sqrt{y} \to -\infty$. Thus one corresponding $x$-coordinate approaches infinity, while the other approaches negative infinity. ∎

## Problem 4

Sketch the curve $C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}$.



## Problem 5

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\right\}.$$

Show that if $(x, y) \in C$, then the three points $(-x, y), (x, -y), (-x, -y)$ are also on $C$. Thus the curve $C$ is symmetric about both the $x$- and $y$-axes.

*Proof.* Let $(x, y) \in \mathbb{R}^2$. Suppose $\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0$. Notice that $x^2 = (-x)^2$ and $y = (-y)^2$. Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{y^2}{9} - 1 = \frac{x^2}{4} + \frac{(-y)^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{(-y)^2}{9} - 1 = 0.$$

Thus $(-x, y), (x, -y), (-x, -y) \in C$. ∎

## Problem 6

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\right\}.$$

Show that for every $(x, y) \in C$, we have $|x| \leq 2$ and $|y| \leq 3$.

*Proof.* Let $(x, y) \in C$. Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \iff 9x^2 + 4y^2 - 36 = 0 \iff 9x^2 = -4y^2 + 36 \iff |x| = \sqrt{\frac{-4}{9}y^2 + 4} \leq \sqrt{4} = 2.$$

Similarly

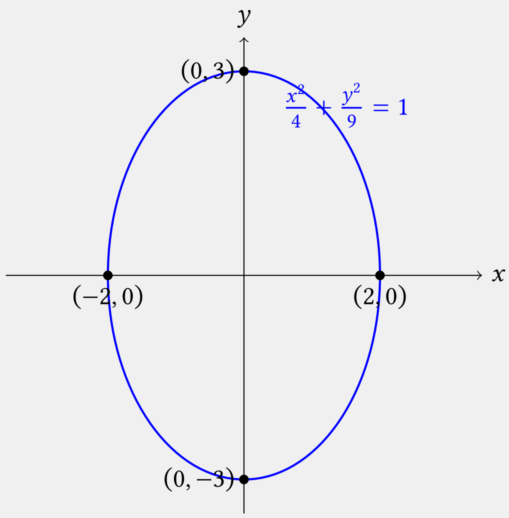
$$9x^2 + 4y^2 - 36 = 0 \iff |y| = \sqrt{\frac{-9}{4}x^2 + 9} \leq \sqrt{9} = 3.$$

∎

## Problem 7

Sketch

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0\right\}.$$



## Problem 8

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

Show that if $(x, y) \in C$, then the three points $(-x, y), (x, -y)$, and $(-x, -y)$ are also on $C$. Thus the curve $C$ is also symmetric about the $x-$ and $y$-axes.

*Proof.* Let $(x, y) \in \mathbb{R}^2$. Suppose $x^2 - y^2 - 4 = 0$. Notice that $x^2 = (-x)^2$ and $y = (-y)^2$. Then

$$x^2 - y^2 - 4 = (-x)^2 - y^2 = x^2 - (-y)^2 = (-x)^2 - (-y)^2 = 0.$$

Thus $(-x, y), (x, -y), (-x, -y) \in C$. ∎

## Problem 9

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

Show that if $(x, y) \in C$, then we have $|x| \geq 2$.

*Proof.* Let $(x, y) \in \mathbb{R}^2$. Suppose $x^2 - y^2 - 4 = 0$. Then

$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff |x| = \sqrt{y^2 + 4} \geq \sqrt{4} = 2.$$

∎

## Problem 10

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

Show that the curve $C$ is unbounded in the positive and negative $x$-directions and also unbounded in the positive and negative $y$-directions.

*Proof.* First notice

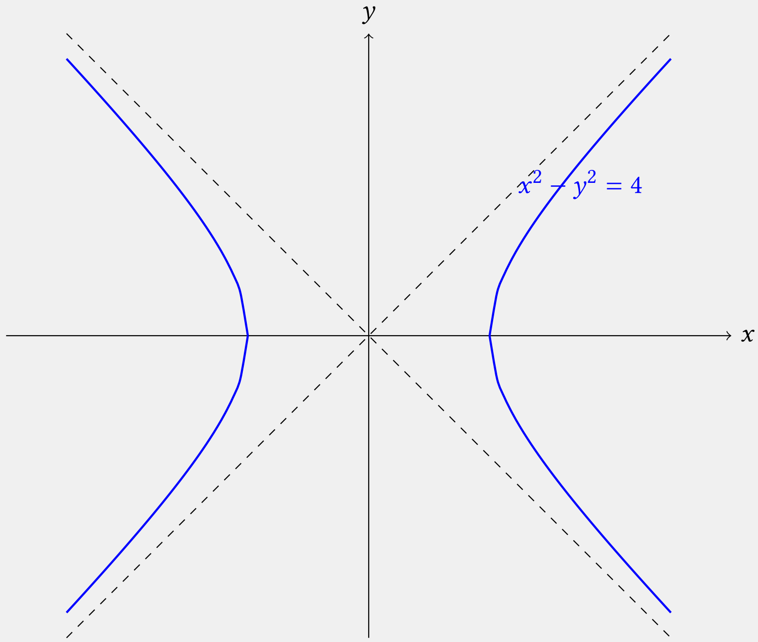
$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff x = \pm\sqrt{y^2 + 4} \iff y = \pm\sqrt{x^2 - 4}.$$

As $y \to \infty$, we have $x = \pm\sqrt{y^2 + 4} \to \infty$ and $-\infty$. Similarly, as $x \to \infty$, we have $y = \pm\sqrt{x^2 - 4} \to \infty$ and $-\infty$. ∎

## Problem 11

Sketch

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\right\}.$$

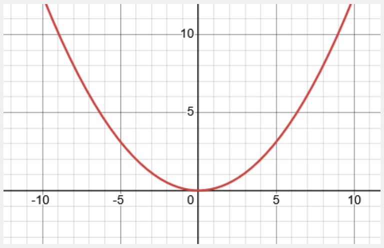



**Problem 12**

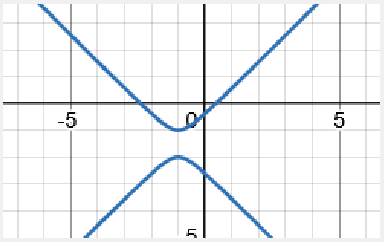
Sketch the graph of each of the following conics in $\mathbb{R}^2$. Identify which are parabolas, ellipses, or Hyperbola.

1. $V(x^2 - 8y)$.
2. $V(x^2 + 2x - y^2 - 3y - 1)$.
3. $V(4x^2 + y^2)$.
4. $V(3x^2 + 3y^2 - 75)$.
5. $V(x^2 - 9y^2)$.
6. $V(4x^2 + y^2 - 8)$.
7. $V(x^2 + 9y^2 - 36)$.
8. $V(x^2 - 4y^2 - 16)$.
9. $V(y^2 - x^2 - 9)$.

**Solution (1):** Parabola.

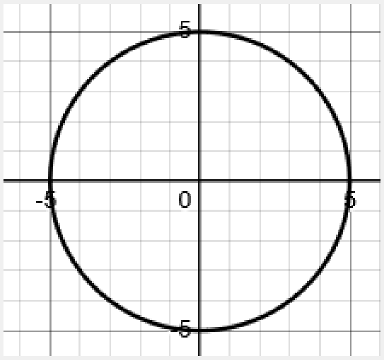



**Solution (2):** Hyperbola.

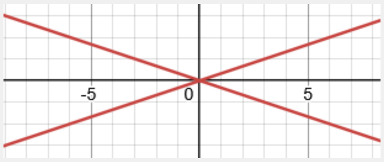



**Solution** (**3**)**:** Point.

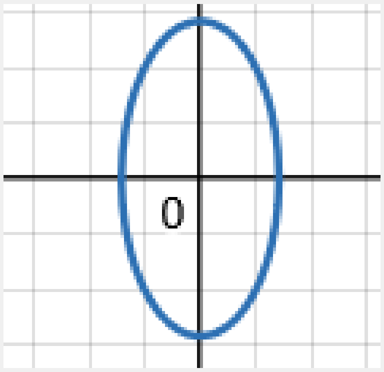
**Solution** (**4**)**:** Ellipse.



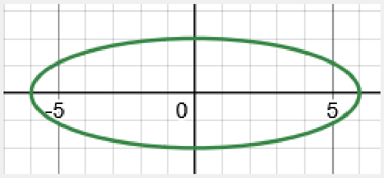

**Solution** (**5**)**:** Two lines.



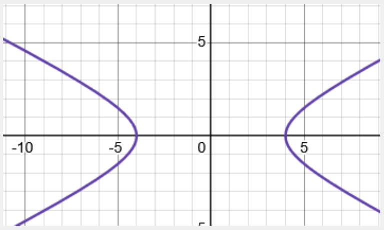

**Solution** (**6**)**:** Ellipse.



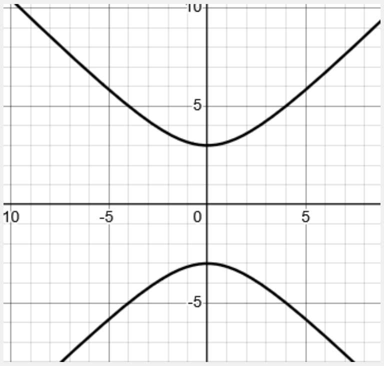

**Solution** (**7**)**:** Ellipse.




**Solution (8):** Hyperbola.




**Solution (9):** Hyperbola.




**Problem 13**

Express the polynomial $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$ in the form

$$P(x, y) = Ax^2 + Bx + C,$$

where $A, B,$ and $C$ are polynomials in $y$. What are $A, B,$ and $C$?

*Proof.* Let $A = a, B = by + d$, and $C = cy^2 + ey + h$. Notice

$$ax^2 + bxy + cy^2 + dx + ey + h = ax^2 + bxy + dx + cy^2 + ey + h = ax^2 + (by + d)x + (cy^2 + ey + h) = Ax^2 + Bx + C.$$

∎

**Problem 14**

Treating $P(x, y) = ax^2+bxy+cy^2+dx+cy+h$ as a polynomial in the variable $x$, show that the discriminant is

$$\triangle_x(y) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

*Proof.* From Problem 13 we have $A = a, B = by + d$, and $C = cy^2 + ey + h$. Then

$$\triangle_x(y) = B^2 - 4AC = (by + d)^2 - 4a(cy^2 + ey + h) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

∎

**Problem 15**

1. Suppose $\triangle_x(y_0) < 0$. Explain why there is no point on $V(p)$ whose $y$-coordinate is $y_0$.
2. Suppose $\triangle_x(y_0) = 0$. Explain why there is exactly one point on $V(P)$ whose $y$-coordinate is $y_0$.
3. Suppose $\triangle_x(y_0) > 0$. Explain why there are exactly two points on $V(P)$ whose $y$-coordinate is $y_0$.

**Solution (a):** In $\mathbb{R}$, the square root is undefined for values $< 0$.

**Solution (b):** If $\triangle_x(y_0) = 0$, then $+\sqrt{B^2 - 4AC} = -\sqrt{B^2 - 4AC}$, so there is exactly one point on $V(P)$ whose $y$-coordinate is $y_0$.

**Solution (c):** If $\triangle_x(y_0) > 0$, then $+\sqrt{B^2 - 4AC} \neq -\sqrt{B^2 - 4AC}$, so there are exactly two points on $V(P)$ whose $y$-coordinate is $y_0$.

**Problem 16**

Suppose $b^2 - 4ac = 0$. Suppose further that $2bd - 4ae > 0$.

1. Show that $\triangle_x(y) \geq 0$ if and only if $y \geq \frac{4ah-d^2}{2bd-4ae}$.
2. Conclude that if $b^2 - 4ac = 0$ and $2bd - 4ae > 0$, then $V(P)$ is a parabola.

*Proof.* Suppose $\triangle_x(y) \geq 0$. Then

$$\begin{aligned}\triangle_x(y) &= (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah)\\ &= 0y^2 + (2bd - 4ae)y + (d^2 - 4ah).\end{aligned}$$

Therefore,

$$(2bd - 4ae)y + (d^2 - 4ah) \geq 0.$$

Since $2bd - 4ae > 0$, we have

$$y \geq \frac{4ah - d^2}{2bd - 4ae}.$$

Conversely, suppose $y \geq \frac{4ah-d^2}{2bd-4ae}$. Then

$$\begin{aligned}\triangle_x(y) &= (2bd - 4ae)y + (d^2 - 4ah)\\ &\geq (2bd - 4ae)\left(\frac{4ah - d^2}{2bd - 4ae}\right) + (d^2 - 4ah)\\ &= 0.\end{aligned}$$

∎

*Proof.* Suppose $b^2 - 4ac = 0$ and $2bd - 4ae > 0$. Then $\triangle_x(y) = (2bd - 4ae)y + (d^2 - 4ah)$. Now, $x = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$. It is clear that $x$ is symmetrical, and since $y \geq \frac{4ah-d^2}{2bd-4ae}$, $V(P)$ is a parabola. ∎

Problem 17

Suppose $b^2 - 4ac < 0$.

1. Show that one of the following occurs:
   (a) $\{y \mid \triangle_x(y) \geq 0\} = \emptyset$,
   (b) $\{y \mid \triangle_x(y) \geq 0\} = \{y_o\}$,
   (c) there exist real numbers $\alpha$ and $\beta$, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

2. Conclude that $V(P)$ is either emptyset, a point, or an ellipse.

*Proof.* Since $b^2 - 4ac < 0$, the graph of $\triangle_x(y)$ is a downward opening parabola in $y$. There are three cases, depending on the number of real zeros of $\triangle_x(y)$.

1. If $\triangle_x(y) < 0$ for all $y$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \emptyset.$$
2. If $\triangle_x(y)$ has exactly one real zero $y_0$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y_0\}.$$
3. If $\triangle_x(y)$ has two distinct real zeros $\alpha < \beta$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

∎

*Proof.* From part 1 the set of $y$ values is either empty, a single point, or a bounded interval, it follows that $V(P)$ is either empty, a point, or an ellipse. ∎

Problem 18

Suppose $b^2 - 4ac > 0$.

1. Show that one of the following occurs:
   (a) $\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}$ and $\triangle_x(y) \neq 0$,
   (b) $\{y \mid \triangle_x(y) = 0\} = \{y_0\}$ and $\{y \mid \triangle_x(y) > 0\} = \{y \mid y \neq y_0\}$,
   (c) there exist real numbers $\alpha$ and $\beta$, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

2. If $\{y \mid \triangle_x(y)\} = \mathbb{R}$, show that $V(P)$ is a hyperbola opening left and right.
3. If $\{y \mid \triangle_x(y) = 0\} = \{y_0\}$, show that $V(P)$ is two lines intersecting in a point.
4. If there are two real numbers $\alpha$ and $\beta$, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\},$$

   show that $V(P)$ is a hyperbola opening up and down.

*Proof.* Since $b^2 - 4ac > 0$, the graph of $\triangle_x(y)$ is an upward opening parabola in $y$. There are three cases, depending on the number of real zeros of $\triangle_x(y)$.

1. If $\triangle_x(y) > 0$ for all $y$, then
$$\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}.$$

2. If $\triangle_x(y)$ has exactly one real zero $y_0$, then

$$\{y \mid \triangle_x(y) = 0\} = \{y_0\} \quad \text{and} \quad \{y \mid \triangle_x(y) > 0\} = \{y \mid y \neq y_0\}.$$

3. If $\triangle_x(y)$ has two distinct real zeros $\alpha < \beta$, then

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

∎

*Proof.* Since $b^2 - 4ac > 0$, the graph of $\triangle_x(y)$ is an upward opening parabola in $y$. There are three cases, depending on the number of real zeros of $\triangle_x(y)$.

1. If $\triangle_x(y) > 0$ for all $y$, then

$$\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}.$$

2. If $\triangle_x(y)$ has exactly one real zero $y_0$, then

$$\{y \mid \triangle_x(y) = 0\} = \{y_0\} \quad \text{and} \quad \{y \mid \triangle_x(y) > 0\} = \{y \mid y \neq y_0\}.$$

3. If $\triangle_x(y)$ has two distinct real zeros $\alpha < \beta$, then

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

∎

*Proof.* Suppose $\{y \mid \triangle_x(y) \geq 0\} = \mathbb{R}$. Then for every $y$ there exist two real solutions for $x$, and $x$ is unbounded to the left and right. Since the equation is quadratic in $x$, the curve is symmetric in $x$. Thus $V(P)$ is a hyperbola opening left and right. ∎

*Proof.* Suppose $\{y \mid \triangle_x(y) = 0\} = \{y_0\}$. Then for $y = y_0$ the equation has exactly one real solution for $x$, and for $y \neq y_0$ it has two real solutions. Since the equation is quadratic in $x$, $V(P)$ consists of two lines intersecting at a point. ∎

*Proof.* Suppose there exist real numbers $\alpha$ and $\beta$, $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

For $y \leq \alpha$ or $y \geq \beta$, the equation has two real solutions in $x$. If $\alpha < y < \beta$ it has no real solutions. Thus $x$ is bounded for each $y$, but $y$ is unbounded above and below. Since the equation is quadratic in $x$, the curve is symmetric in $x$. Therefore $V(P)$ is a hyperbola opening up and down. ∎

**Problem 19**

Show that the discriminant of $A'y^2 + B'y + C' = 0$ is

$$\triangle_y(x) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4ch).$$

*Proof.* Here $A' = c$, $B' = bx + e$, and $C' = ax^2 + dx + h$. Then

$$\triangle_y(x) = (B')^2 - 4A'C' = (bx+e)^2 - 4c(ax^2 + dx + h) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4ch).$$

∎

## 1.2 Changes of Coordinates

**Problem 1**

Show that the origin in the $xy$-coordinate system agrees with the origin in the $uv$-system if and only if $e = f = 0$. Thus the constants $e$ and $f$ describe translations of the origin.

*Proof.* Suppose the $xy$-coordinate system agrees with the origin of the $uv$-system. Then

$$u = 0 = a(0) + b(0) + e = e,$$

and

$$v = 0 = c(0) + d(0) + f = f.$$

Thus $f = e = 0$.

Conversely, suppose $e = f = 0$. Then

$$u = ax + by + e = ax + by + 0 = a(0) + b(0) = 0,$$

and

$$v = cx + dy + f = cx + dy + 0 = c(0) + d(0) = 0.$$

Thus the origin of the $xy$-coordinate system agrees with the origin of the $uv$-system. ∎

**Problem 2**

Show that if $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then the inverse change of coordinates is

$$x = \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf).$$

$$y = \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af).$$

*Proof.* We need to solve the two equations $u = ax + by + e$ and $v = cx + dy + f$ in two unknowns $x$ and $y$. Translating this to linear algebra, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u - e \\ v - f \end{bmatrix}.$$

Using Cramer's rule we see

$$x = \frac{\begin{vmatrix} u - e & b \\ v - f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d(u - e) - b(v - f)}{ad - bc},$$

$$y = \frac{\begin{vmatrix} a & u - e \\ c & v - f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c(u - e) + a(v - f)}{ad - bc}.$$

Therefore

$$x = \frac{du - bv - de + bf}{ad - bc}, \quad y = \frac{-cu + av + ce - af}{ad - bc}.$$

∎

**Problem 3**

Show that if

$$u = ax + by + e$$
$$v = cx + dy + f,$$

and

$$s = Au + Bv + E$$
$$t = Cu + Dv + F$$

are two real affine changes of coordinates from the $xy$-plane to the $uv$-plane and from the $uv$-plane to the $st$-plane, respectively, then the composition from the $xy$-plane to the $st$-plane is a real affine change of coordinates.

*Proof.* Suppose

$$u = ax + by + e$$
$$v = cx + dy + f,$$

and

$$s = Au + Bv + E$$
$$t = Cu + Dv + F$$

are two real affine changes of coordinates from the $xy$-plane to the $uv$-plane and from the $uv$-plane to the $st$-plane respectively. Substituting $u, v$ into $s, t$ we see

$$s = A(ax + by + e) + B(cx + dy + f) + E = (Aa + Bc)x + (Ab + Bd)y + (Ae + Bf + E),$$

and

$$t = C(ax + by + e) + D(cx + dy + f) + F = (Ca + Dc)x + (Cb + Dd)y + (Ce + Df + F).$$

Finally,

$$\det\left(\begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix}\right) = (Aa + Bc)(Cb + Dd) - (Ab + Bd)(Ca + Dc) = (ad - bc)(AD - BC) \neq 0.$$

∎

**Problem 4**

For each affine pair of ellipses, find a real affine change of coordinates that maps the ellipse in the $xy$-plane to the ellipse in the $uv$-plane.

1. $V(x^2 + y^2 - 1), V(16u^2 + 9v^2 - 1)$.
2. $V((x - 1)^2 + y^2 - 1), V(16u^2 + 9(v + 2)^2 - 1)$.
3. $V(4x^2 + y^2 - 6y + 8), V(u^2 - 4u + v^2 - 2v + 4)$.
4. $V(13x^2 - 10xy + 13y^2 - 1), V(4u^2 + 9v^2 - 1)$.

**Solution (1):** Let $x = 4u$ and $y = 3v$. Then

$$x^2 + y^2 - 1 = (4u)^2 + (3v)^2 - 1 = 16u^2 + 9v^2 - 1 = 0.$$

**Solution (2):** Let $x = 4u + 1$ and $y = 3v + 6$. Then

$$(x - 1)^2 + y^2 - 1 = (4u + 1 - 1)^2 + (3v + 6)^2 = 16u^2 + 9(v + 2)^2 = 0.$$

**Solution (3):** Let $x = \frac{u}{2} - 1$ and $y = v + 2$. Then

$$4x^2 + y^2 - 6y + 8 = 4\left(\frac{u}{2} - 1\right)^2 + (v + 2)^2 - 6(v + 2) + 8 =$$

$$4\left(\frac{u^2}{4}-2\frac{u}{2}+1\right)+v^2+4v+4-6v-12+8=u^2-4u+4+v^2-2v=u^2-4u+v^2-2v+4.$$

**Solution (4):** Let $x=\frac{u+v}{2}$ and $y=\frac{u-v}{2}$. Then

$$\begin{aligned}13x^2-10xy+13y^2-1&=13\left(\frac{u+v}{2}\right)^2-10\left(\frac{u+v}{2}\cdot\frac{u-v}{2}\right)+13\left(\frac{u-v}{2}\right)^2-1\\&=13\frac{(u+v)^2}{4}-10\frac{u^2-v^2}{4}+13\frac{(u-v)^2}{4}-1\\&=\frac{13}{4}(u^2+2uv+v^2)-\frac{10}{4}(u^2-v^2)+\frac{13}{4}(u^2-2uv+v^2)-1\\&=\frac{13+13-10}{4}u^2+\frac{13+13+10}{4}v^2+\frac{26-26}{4}uv-1\\&=4u^2+9v^2-1.\end{aligned}$$

**Problem 5**

For each pair of hyperbolas, find a real affine change of coordinates that maps the hyperbola in the $xy$-plane to the hyperbola in the $uv$-plane.

1. $V(xy-1), V(u^2-v^2-1)$.
2. $V(x^2-y^2-1), V(16u^2-9v^2-1)$.
3. $V((x-1)^2-y^2-1), V(16u^2-9(v+2)^2-1)$.
4. $V(x^2-y^2-1), V(v^2-u^2-1)$.
5. $V(8xy-1), V(2u^2-2v^2-1)$.

**Solution (1):** Let $x=u-v$ and $y=u+v$. Then

$$xy-1=(u-v)(u+v)-1=u^2-v^2-1.$$

**Solution (2):** Let $x=4u$ and $y=3v$. Then

$$x^2-y^2-1=(4u)^2-(3v)^2-1=16u^2-9v^2-1.$$

**Solution (3):** Let $x=4u+1$ and $y=3v+6$. Then

$$(x-1)^2-y^2-1=(4u+1-1)^2-(3v+6)^2=16u^2-9(v+2)^2-1.$$

**Solution (4):** Let $x=v$ and $y=u$. Then

$$x^2-y^2-1=v^2-u^2-1.$$

**Solution (5):** Let $x=(u+v)/4$ and $y=(u-v)/2$. Then

$$8xy-1=8((u+v)/4)((u-v)/2)-1=(u+v)(u-v)-1=u^2-v^2-1.$$

**Problem 6**

For each pair of parabolas, find a real affine change of coordinates that maps the parabola in the $xy$-plane to the parabola in the $uv$-plane.

1. $V(x^2-y), V(9v^2-4u)$.
2. $V((x-1)^2-y), V(u^2-9(v+2))$.
3. $V(x^2-y), V(u^2+2uv+v^2-u+v-2)$.
4. $V(x^2-4x+y+4), V(4u^2-(v+1))$.
5. $V(4x^2+4xy+y^2-y+1), V(4u^2+v)$.

**Solution (1):** Let $x = 3v$ and $y = 4u$. Then

$$x^2 - y = (3v)^2 - 4u = 9v^2 - 4u.$$

**Solution (2):** Let $x = u + 1$ and $y = 9v + 18$. Then

$$(x-1)^2 - y = (u+1-1)^2 - (9v+18) = u^2 - 9(v+2).$$

**Solution (3):** Let $x = (u+v)^2$ and $y = u - v + 2$. Then

$$x^2 - y = (u+v)^2 - (u - v + 2) = u^2 + 2uv + v^2 - u + v - 2.$$

**Solution (4):** Let $x = 2u + 2$ and $y = -(v+1)$. Then

$$x^2 - 4x + y + 4 = (2u+2)^2 - 4(2u+2) - (v+1) + 4 = 4u^2 + 8u + 4 - 8u - 8 - (v+1) + 4 = 4u^2 - (v+1).$$

**Solution (5):** Let $x = u - \frac{1}{2}v + \frac{1}{2}$ and $y = v$. Then

$$\begin{aligned}
4x^2 + 4xy + y^2 - y + 1 &= 4\left(u - \tfrac{1}{2}v + \tfrac{1}{2}\right)^2 + 4\left(u - \tfrac{1}{2}v + \tfrac{1}{2}\right)v + v^2 - v + 1 \\
&= 4\left(u^2 - uv + u + \tfrac{1}{4}v^2 - \tfrac{1}{2}v + \tfrac{1}{4}\right) + 4uv - 2v^2 + 2v + v^2 - v + 1 \\
&= 4u^2 - 4uv + 4u + v^2 - 2v + 1 + 4uv - 2v^2 + 2v + v^2 - v + 1 \\
&= 4u^2 + v.
\end{aligned}$$

**Problem 7**

Explain why if $b^2 - 4ac < 0$, then $ac > 0$.

*Proof.* Suppose $b^2 - 4ac < 0$. Then $0 \le b^2 < 4ac \iff 0 \le \frac{b^2}{4} < ac$. Thus $ac > 0$. ∎

**Problem 8**

Show that under the real affine transformation

$$x = \sqrt{\frac{c}{a}}u + v$$

$$y = u - \sqrt{\frac{a}{c}}v,$$

the ellipse $V(ax^2 + bxy + cy^2 + dx + ey + h)$ in the $xy$-plane becomes an ellipse in the $uv$-plane whose defining equation is $Au^2 + Cv^2 + Du + Ev + H = 0$. Find $A$ and $C$ in terms of $a, b, c$. Show that if $b^2 - 4ac < 0$, then $A \neq 0$ and $C \neq 0$.

*Proof.*

$$\begin{aligned}
ax^2 + bxy + cy^2 + dx + ey + h &= a\Big(\sqrt{\tfrac{c}{a}}\,u + v\Big)^2 + b\Big(\sqrt{\tfrac{c}{a}}\,u + v\Big)\Big(u - \sqrt{\tfrac{a}{c}}\,v\Big) + c\Big(u - \sqrt{\tfrac{a}{c}}\,v\Big)^2 \\
&\quad + d\Big(\sqrt{\tfrac{c}{a}}\,u + v\Big) + e\Big(u - \sqrt{\tfrac{a}{c}}\,v\Big) + h \\
&= \big(cu^2 + 2\sqrt{ac}\,uv + av^2\big) + b\Big(\sqrt{\tfrac{c}{a}}\,u^2 - \sqrt{\tfrac{a}{c}}\,v^2\Big) + \big(cu^2 - 2\sqrt{ac}\,uv + av^2\big) \\
&\quad + \Big(d\sqrt{\tfrac{c}{a}} + e\Big)u + \Big(d - e\sqrt{\tfrac{a}{c}}\Big)v + h \\
&= \Big(2c + b\sqrt{\tfrac{c}{a}}\Big)u^2 + \Big(2a - b\sqrt{\tfrac{a}{c}}\Big)v^2 + \Big(d\sqrt{\tfrac{c}{a}} + e\Big)u + \Big(d - e\sqrt{\tfrac{a}{c}}\Big)v + h \\
&= Au^2 + Cv^2 + Du + Ev + H.
\end{aligned}$$

∎

*Proof.* Suppose $b^2 - 4ac < 0$. Then

$$A = \sqrt{\frac{c}{a}}\,b + 2c, \qquad C = -\sqrt{\frac{a}{c}}\,b + 2a.$$

Then

$$AC = (2c + b\sqrt{\tfrac{c}{a}})(2a - b\sqrt{\tfrac{a}{c}}) = 4ac - b^2.$$

Since $b^2 - 4ac < 0$,

$$4ac - b^2 > 0 \implies AC > 0.$$

Therefore $A \neq 0$ and $C \neq 0$. ∎

**Problem 9**

Show that there exists constants $R, S$, and $T$ such that the equation

$$Au^2 + Cv^2 + Du + Ev + H = 0,$$

can be written in the form

$$A(u - R)^2 + C(v - S)^2 - T = 0.$$

Express $R, S$, and $T$ in terms of $A, C, D, E$, and $H$.

*Proof.* Let $R = -\frac{D}{2A}, S = -\frac{E}{2C}, T = \frac{D^2}{4A} + \frac{E^2}{4C} - H$. Note $A \neq 0$ and $C \neq 0$ from problem 8. Notice

$$\begin{aligned}
Au^2 + Cv^2 + Du + Ev + H &= A\left(u^2 + \frac{Du}{A}\right) + C\left(v^2 + \frac{Ev}{C}\right) + H \\
&= A\left(u^2 + \frac{Du}{A} + \left(\frac{D}{2A}\right)^2\right) - \frac{D^2}{4A} + C\left(v^2 + \frac{Ev}{C} + \left(\frac{E}{2C}\right)^2\right) - \frac{E^2}{4C} + H \\
&= A\left(u + \frac{D}{2A}\right)^2 + C\left(v + \frac{E}{2C}\right)^2 - \frac{D^2}{4A} - \frac{E^2}{4C} + H \\
&= A(u - R)^2 + C(v - S)^2 - T = 0.
\end{aligned}$$

∎

**Problem 10**

Suppose $A, C > 0$. Find a real affine change of coordinates that maps the ellipse

$$V(A(x - R)^2 + C(y - S)^2 - T),$$

to the circle

$$V(u^2 + v^2 - 1).$$

*Proof.* Since $A, C > 0$ we know $T > 0$. Notice

$$A(x - R)^2 + C(y - S)^2 = T \iff \frac{A(x - R)^2}{T} + \frac{C(y - S)^2}{T} = 1.$$

We set

$$u^2 = \frac{A(x - R)^2}{T}, \quad v^2 = \frac{C(y - S)^2}{T},$$

and solving for $x, y$ shows

$$x = \sqrt{\frac{T}{A}}\, u + R, \quad y = \sqrt{\frac{T}{C}}\, v + S.$$

Substituting into the original equation, we find

$$\begin{aligned} A(x-R)^2 + C(y-S)^2 - T &= A\Big(\sqrt{\frac{T}{A}}\, u\Big)^2 + C\Big(\sqrt{\frac{T}{C}}\, v\Big)^2 - T \\ &= Tu^2 + Tv^2 - T \\ &= T(u^2 + v^2 - 1), \end{aligned}$$

∎

**Problem 11**

Consider the values $A$ and $C$ found in Exercise 1.2.8. Show that if $b^2 - 4ac = 0$, then either $A = 0$ or $C = 0$, depending on the signs of $a, b, c$. [Hint: Recall, $\sqrt{\alpha^2} = -\alpha$ if $\alpha < 0$.]

*Proof.* Suppose $b^2 - 4ac = 0$. From Exercise 1.2.8 we have

$$A = \sqrt{\frac{c}{a}}\, b + 2c, \qquad C = -\sqrt{\frac{a}{c}}\, b + 2a.$$

We see that

$$AC = 4ac - b^2 = -(b^2 - 4ac) = -0 = 0.$$

Thus $A = 0$ or $C = 0$. ∎

**Problem 12**

Show that there exists constants $R$ and $T$ such that the equation

$$Au^2 + Du + Ev + H = 0,$$

can be written as

$$A(u-R)^2 + E(v-T) = 0.$$

Express $R$ and $T$ in terms of $A, D, E$, and $H$.

*Proof.* First note $A \neq 0$ therefore $E \neq 0$. Let

$$R = -\frac{D}{2A}, \qquad T = -\left(\frac{H}{E} - \frac{D^2}{4AE}\right).$$

Then

$$\begin{aligned} Au^2 + Du + Ev + H &= A\left(u^2 + \frac{D}{A}u + \left(\frac{D}{2A}\right)^2\right) - \frac{D^2}{4A} + Ev + H \\ &= A\left(u + \frac{D}{2A}\right)^2 + E\left(v + \frac{H}{E} - \frac{D^2}{4AE}\right) \\ &= A(u-R)^2 + E(v-T) = 0. \end{aligned}$$

∎

**Problem 13**

Suppose $A > 0$ and $E \neq 0$. Find a real affine change of coordinates that maps the parabola

$$V(A(x-R)^2 - E(y-T)),$$

to the parabola

$$V(u^2 - v).$$

*Proof.* We set $A(x-R)^2 = u^2$ and $-E(y-T) = -v$. Then solving for $x, y$ we have

$$x = \frac{u}{\sqrt{A}} + R, \quad y = \frac{v}{E} + T.$$

Then substituting into our original equation we have

$$A(x-R)^2 - E(y-T) = A\left(\frac{u}{\sqrt{A}} + R - R\right)^2 - E\left(\frac{v}{E} + T - T\right) = u^2 - v.$$

∎

**Problem 14**

Suppose $ac > 0$. Use the real affine transformation in Exercise 1.2.8 to transform $C$ to a conic in the $uv$-plane. Find the coefficients of $u^2$ and $v^2$ in the resulting equation and show that they have opposite signs.

*Proof.* Suppose $ac > 0$. From Exercise 1.2.8 we have

$$A = \sqrt{\frac{c}{a}}\, b + 2c, \qquad C = -\sqrt{\frac{a}{c}}\, b + 2a.$$

We see that

$$AC = 4ac - b^2 = -(b^2 - 4ac) < 0.$$

Thus $A$ and $C$ have opposite signs. ∎

**Problem 15**

Suppose $ac < 0$ and $b \neq 0$. Use the real affine transformation

$$x = \sqrt{-\frac{c}{a}}u + v$$

$$y = u - \sqrt{-\frac{a}{c}}v,$$

to transform $C$ to a conic in the $uv$-plane of the form

$$Au^2 + Cv^2 + Du + Ev + H = 0.$$

Find the coefficients of the resulting equation and show that they have opposite signs.

*Proof.*

$$
\begin{aligned}
ax^2 + bxy + cy^2 + dx + ey + h &= a\big(\sqrt{-\tfrac{c}{a}}\,u + v\big)^2 + b\big(\sqrt{-\tfrac{c}{a}}\,u + v\big)\big(u - \sqrt{-\tfrac{a}{c}}\,v\big) + c\big(u - \sqrt{-\tfrac{a}{c}}\,v\big)^2 \\
&\quad + d\big(\sqrt{-\tfrac{c}{a}}\,u + v\big) + e\big(u - \sqrt{-\tfrac{a}{c}}\,v\big) + h \\
&= \big(-cu^2 + 2\sqrt{-ac}\,uv - av^2\big) + b\big(\sqrt{-\tfrac{c}{a}}\,u^2 - \sqrt{-\tfrac{a}{c}}\,v^2\big) + \big(-cu^2 - 2\sqrt{-ac}\,uv - av^2\big) \\
&\quad + \big(d\sqrt{-\tfrac{c}{a}} + e\big)u + \big(d - e\sqrt{-\tfrac{a}{c}}\big)v + h \\
&= \big(-2c + b\sqrt{-\tfrac{c}{a}}\big)u^2 + \big(-2a - b\sqrt{-\tfrac{a}{c}}\big)v^2 + \big(d\sqrt{-\tfrac{c}{a}} + e\big)u + \big(d - e\sqrt{-\tfrac{a}{c}}\big)v + h \\
&= Au^2 + Cv^2 + Du + Ev + H.
\end{aligned}
$$

∎

*Proof.* Since $ac < 0$ and $b \neq 0$, we have

$$A = -2c + b\sqrt{-\frac{c}{a}}, \qquad C = -2a - b\sqrt{-\frac{a}{c}}.$$

Then

$$AC = \big(-2c + b\sqrt{-\tfrac{c}{a}}\big)\big(-2a - b\sqrt{-\tfrac{a}{c}}\big) = 4ac - b^2.$$

Since $ac < 0$,

$$4ac - b^2 < 0 \implies AC < 0.$$

Therefore $A$ and $C$ have opposite signs. ∎

**Problem 16**

Suppose $ac = 0$ (so $b \neq 0$). Since either $a = 0$ or $c = 0$, we can assume $c = 0$. Use the real affine transformation

$$
\begin{aligned}
x &= u + v \\
y &= \left(\frac{1-a}{b}\right)u - \left(\frac{1+a}{b}\right)v,
\end{aligned}
$$

to transform $V(ax^2 + bxy + dx + ey + h)$ to a conic in the $uv$-plane of the form

$$V(u^2 - v^2 + Du + Ev + H).$$

*Proof.*

$$
\begin{aligned}
ax^2 + bxy + dx + ey + h &= a(u+v)^2 + b(u+v)\Big(\frac{1-a}{b}u - \frac{1+a}{b}v\Big) \\
&\quad + d(u+v) + e\Big(\frac{1-a}{b}u - \frac{1+a}{b}v\Big) + h \\
&= a(u^2 + 2uv + v^2) + (u+v)((1-a)u - (1+a)v) \\
&\quad + d(u+v) + e\Big(\frac{1-a}{b}u - \frac{1+a}{b}v\Big) + h \\
&= (a + 1 - a)u^2 + (-(1+a) + a)v^2 + 2auv \\
&\quad + \Big(d + e\frac{1-a}{b}\Big)u + \Big(d - e\frac{1+a}{b}\Big)v + h \\
&= u^2 - v^2 + Du + Ev + H
\end{aligned}
$$

∎

## Problem 17

Show that there exists $R, S$, and $T$ so that

$$Au^2 - Cv^2 + Du + Ev + H = A(u-R)^2 - C(v-S)^2 - T.$$

Express $R, S$, and $T$ in terms of $A, C, D, E$, and $H$.

*Proof.* We set $A(u-R)^2 = Au^2 + Du$ and $-C(v-S)^2 = -Cv^2 + Ev$. Then solving for $R, S$ we have

$$R = -\frac{D}{2A}, \quad S = \frac{E}{2C}.$$

Then substituting into our original equation we have

$$\begin{aligned} Au^2 - Cv^2 + Du + Ev + H &= \left(A(u-R)^2 - AR^2\right) + \left(-C(v-S)^2 + CS^2\right) + H \\ &= A(u-R)^2 - C(v-S)^2 - (AR^2 - CS^2 - H) \\ &= A(u-R)^2 - C(v-S)^2 - T, \end{aligned}$$

where

$$T = AR^2 - CS^2 - H = \frac{D^2}{4A} - \frac{E^2}{4C} - H.$$

∎

## Problem 18

Suppose $A, C, T > 0$. Find a real affine change of coordiantes that maps the hyperbola

$$V(A(x-R)^2 - C(y-S)^2 - T),$$

to the hyperbola

$$V(u^2 - v^2 - 1).$$

*Proof.* Notice

$$A(x-R)^2 - C(y-S)^2 - T = 0 \iff \frac{A(x-R)^2}{T} - \frac{C(y-S)^2}{T} = 1.$$

We set

$$u^2 = \frac{A(x-R)^2}{T}, \quad v^2 = \frac{C(y-S)^2}{T},$$

and solving for $x, y$ shows

$$x = \sqrt{\frac{T}{A}}\,u + R, \quad y = \sqrt{\frac{T}{C}}\,v + S.$$

Substituting into the original equation, we find

$$\begin{aligned} A(x-R)^2 - C(y-S)^2 - T &= A\left(\sqrt{\frac{T}{A}}\,u\right)^2 - C\left(\sqrt{\frac{T}{C}}\,v\right)^2 - T \\ &= Tu^2 - Tv^2 - T \\ &= T(u^2 - v^2 - 1). \end{aligned}$$

∎

**Problem 19**

Give an intuitive argument, based on the number of connected components, for the fact that no ellpise can be transformed into a hyperbola by a real affine change of coordiantes.

**Solution:** A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve the number of connected components. Therefore, no real affine change can transform an ellipse into a hyperbola.

**Problem 20**

Show that there is no real affine change of coordinates

$$u = ax + by + e$$

$$v = cx + dy + f,$$

that transforms the ellipse $V(x^2 + y^2 - 1)$ to the hyperbola $V(u^2 - v^2 - 1)$.

*Proof.* For contradiction, suppose such a real affine change exists.

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We must have

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the ellipse $x^2 + y^2 = 1$. Now substituting $y^2 = 1 - x^2$ we see for this to vanish for all $(x, y)$, the coefficients of $x^2$ and $y^2$ must be

$$a^2 - c^2 = b^2 - d^2,$$

which would make the squared coefficients have the same sign, contradicting the requirement for a hyperbola that they have opposite signs. Thus there is no real affine transformation from an ellipse to a hyperbola. ∎

**Problem 21**

Give an intuitive argument, based on boundedness, for the fact that no parabola can be transformed into an ellipse by a real affine change of coordinates.

**Solution:** A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve boundedness. Therefore, no real affine change can transform a parabola into an ellipse.

**Problem 22**

Show that there is no real affine change of coordinates that trasnforms the parabola $V(x^2 - y)$ to the circle $V(u^2 + v^2 - 1)$.

*Proof.* For contradiction, suppose such a real affine change exists.

$$\begin{aligned} u^2 + v^2 &= (ax + by + e)^2 + (cx + dy + f)^2 \\ &= (a^2 + c^2)x^2 + (b^2 + d^2)y^2 + 2(ab + cd)xy + 2(ae + cf)x + 2(be + df)y + (e^2 + f^2). \end{aligned}$$

We must have

$$(a^2 + c^2)x^2 + (b^2 + d^2)y^2 + 2(ab + cd)xy + 2(ae + cf)x + 2(be + df)y + (e^2 + f^2) - 1 = 0$$

for all points on the parabola $y = x^2$. Now substituting $y = x^2$, we have

$$(b^2 + d^2)x^4 + 2(ab + cd)x^3 + (a^2 + c^2 + 2(be + df))x^2 + 2(ae + cf)x + (e^2 + f^2) - 1 = 0.$$

For this to vanish for all $x$ all coefficients must be zero so

$$b^2 + d^2 = 0, \quad ab + cd = 0, \quad a^2 + c^2 + 2(be + df) = 0, \quad ae + cf = 0.$$

It follows that $a = b = c = d = 0$ and therefore $u^2 + v^2 = e^2 + f^2$ is constant, which cannot equal $x^2 + y^2$ on the parabola. Thus there is no real affine transformation from the parabola to the circle. ∎

**Problem 23**

Give an intuitive argument, based on the number of connected components, for the fact that no parabola can be transformed into a hyperbola by a real affine change of coordinates.

**Solution:** A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve the number of components. Therefore, no real affine change can transform a parabola into a hyperbola.

**Problem 24**

Show that there is no real affine change of coordinates that transforms that parabola $V(x^2 - y)$ to the hyperbola $V(u^2 - v^2 - 1)$.

*Proof.* For contradiction, suppose such a real affine change exists. Then

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We must have

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the parabola $y = x^2$. Substituting $y = x^2$, we get

$$(b^2 - d^2)x^4 + 2(ab - cd)x^3 + (a^2 - c^2)x^2 + 2(ae - cf)x + 2(be - df)x^2 + (e^2 - f^2) - 1 = 0.$$

For this to vanish for all $x$, the coefficient of $x^4$ must be zero

$$b^2 - d^2 = 0 \implies b = \pm d.$$

Then, the $x^3$ coefficient gives $ab - cd = 0$. Since Since $b = \pm d$, we have $a = \pm c$. Then, the $x^2$ coefficient becomes $a^2 - c^2 + 2(be - df)$. Since $a = \pm c$ and $b = \pm d$, this is zero if all coefficients vanish. Thus $u^2 - v^2$ are constant, which cannot equal $x^2 - y$ on the parabola. Therefore, there is no real affine transformation from the parabola to the hyperbola. ∎

## 1.3 Conics over the Complex Numbers

**Problem 1**

Show that the set

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 1 = 0\},$$

is empty but that the set

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 + 1 = 0\},$$

is not empty. In fact, show that given any complex number $x$ there must exist $y \in \mathbb{C}$ such that

$$(x, y) \in C.$$

Then show that if $x \neq \pm i$, then there are two distinct values $y \in \mathbb{C}$ such that $(x, y) \in C$, while if $x = \pm i$ there is only one such $y$.

*Proof.* Suppose $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 + 1 = 0 \iff x^2 + y^2 = -1$. Then $x^2 \geq 0$ and $y^2 \geq 0$ so $x^2 + y^2 \geq 0$, which is a contradiction. ∎

*Proof.* Let $(x, y) = (i, 0) \in \mathbb{C}^2$. Then

$$x^2 + y^2 + 1 = -1 + 1 = 0.$$

Thus $(x, y) \in C$. ∎

*Proof.* Let $x$ be an arbitrary complex number. Furthermore, let $y = \sqrt{-1 - x^2}$. Then

$$x^2 + \left(\sqrt{-1 - x^2}\right)^2 + 1 = x^2 - 1 - x^2 + 1 = 0.$$

Thus $(x, y) \in C$. ∎

*Proof.* Suppose $x \neq \pm i$. Then $1 + x^2 \neq 0$, so $\sqrt{1 + x^2} \neq 0$. Let

$$y = \pm i\sqrt{1 + x^2}.$$

These are two distinct values of $y$. Then

$$x^2 + y^2 + 1 = x^2 - (1 + x^2) + 1 = 0.$$

Now suppose $x = \pm i$. Then $1 + x^2 = 0$, so $y^2 = 0$ and it follows that $y = 0$. Therefore, there is exactly one value of $y$. ∎

**Problem 2**

Let

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + f,$$

with $a \neq 0$. Show that for any value $y \in \mathbb{C}$, there must be at least one $x \in \mathbb{C}$, but no more than two such $x$'s, such that

$$P(x, y) = 0.$$

[Hint: Write $P(x, y) = Ax^2 + Bx + C$ as a function whose coefficients $A$, $B$, and $C$ are themselves functions of $y$, and use the quadratic formula. As mentioned before, this technique will be used frequently.]

*Proof.* Let $A = a$, $B = by + d$, and $C = cy^2 + ey + f$. Notice

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = ax^2 + (by + d)x + (cy^2 + ey + f) = Ax^2 + Bx + C.$$

Applying the quadratic formula we find

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Since $A = a \neq 0$ this is defined. Now if $B^2 - 4AC = 0$ then we get one corresponding $x$. Otherwise, we get two corresponding $x$'s. ∎

**Problem 3**

Let $C = V\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) \subset \mathbb{C}^2$. Show that $C$ is unbounded in $x$ and $y$.

*Proof.* We can solve for $x$ in terms of $y$

$$\frac{x^2}{4} = 1 - \frac{y^2}{9} \iff x = \pm 2\sqrt{1 - \frac{y^2}{9}}.$$

Since $y \in \mathbb{C}$ is arbitrary and square roots always exist in $\mathbb{C}$, for any value of $y$ there is a corresponding value of $x$. As $|y|$ becomes arbitrarily large, $1 - \frac{y^2}{9}$ becomes arbitrarily large, and thus the corresponding $x$ is arbitrarily large. Thus $C$ is unbounded in both $x$ and $y$. ∎

**Problem 4**

Let $C = V(x^2 - y^2 - 1) \subset \mathbb{C}^2$. Show that there is a continuous path on the curve $C$ from the point $(-1, 0)$ to the point $(1, 0)$, despite the fact that no such continuous path exists in $\mathbb{R}^2$.

*Proof.* Let $x(t) = \cos(t)$ and $y(t) = i\sin(t)$. Then

$$x(t)^2 - y(t)^2 - 1 = \cos^2(t) - (i\sin(t))^2 - 1 = \cos^2(t) + \sin^2(t) - 1 = 0.$$

∎

**Problem 5**

Show that if $x = u$ and $y = iv$, then the circle $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\}$ transforms into the hyperbola $\{(u, v) \in \mathbb{C}^2 \mid u^2 - v^2 = 1\}$.

*Proof.* Suppose $x = u$ and $y = iv$. Then

$$x^2 + y^2 = u^2 + (iv)^2 = u^2 - v^2 = 1.$$

∎

**Problem 6**

Show that if $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then the inverse change of coordinates is

$$x = \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf).$$

$$y = \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af).$$

*Proof.* We need to solve the two equations $u = ax + by + e$ and $v = cx + dy + f$ in two unknowns $x$ and $y$. Translating this to linear algebra, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u - e \\ v - f \end{bmatrix}.$$

Using Cramer's rule we see

$$x = \frac{\begin{vmatrix} u-e & b \\ v-f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d(u-e)-b(v-f)}{ad-bc},$$

$$y = \frac{\begin{vmatrix} a & u-e \\ c & v-f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c(u-e)+a(v-f)}{ad-bc}.$$

Therefore

$$x = \frac{du-bv-de+bf}{ad-bc}, \quad y = \frac{-cu+av+ce-af}{ad-bc}.$$

∎

**Problem 7**

Use Theorem 1.2.25 together with the new result of Exercise 1.3.5 to conclude that all ellipses and hyperbolas are equivalent under complex affine changes of coordinates.

*Proof.* By Theorem 1.2.25, any ellipse can be transformed via an affine change of coordinates to a circle. Then, by Exercise 1.3.5 the circle can be transformed via a complex affine change to a hyperbola. ∎

**Problem 8**

Show that the circle $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 - 1 = 0\}$ is not equivalent under a complex affine change of coordinates to the parabola $\{(u, v) \in \mathbb{C}^2 \mid u^2 - v^2 = 0\}$.

*Proof.* For contradiction, suppose such a complex affine change exists

$$u = ax + by + e, \quad v = cx + dy + f.$$

Then

$$\begin{aligned} u^2 - v^2 &= (ax+by+e)^2 - (cx+dy+f)^2 \\ &= (a^2-c^2)x^2 + (b^2-d^2)y^2 + 2(ab-cd)xy + 2(ae-cf)x + 2(be-df)y + (e^2-f^2). \end{aligned}$$

We need

$$(a^2-c^2)x^2 + (b^2-d^2)y^2 + 2(ab-cd)xy + 2(ae-cf)x + 2(be-df)y + (e^2-f^2) - 1 = 0$$

for all points on the circle. Substituting $y = \sqrt{1-x^2}$, the lhs must vanish for all $x$. The highest-degree terms show $b^2 - d^2 = 0 \implies b = \pm d$ and the other coefficients similarly give $a = \pm c$, $e = \pm f$. But then $u^2 - v^2$ would be constant, which cannot equal $x^2 + y^2 - 1$. Therefore there is no complex affine transformation mapping the circle to the hyperbola $u^2 - v^2 = 0$. ∎

**Problem 9**

Let

$$C = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}.$$

Give a bijection from

$$C \cap \{(x+iy, u+iv) \mid x, u \in \mathbb{R}, y = 0, v = 0\}.$$

to the real circle of the unit radius in $\mathbb{R}^2$.

**Solution:**

$$(x + iy, u + iv) \mapsto (x, u).$$

Problem 10

Let

$$C = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}.$$

Give a bijection from

$$C \cap \{(x + iy, u + iv) \in \mathbb{R}^4 \mid x, v \in \mathbb{R}, y = 0, u = 0\},$$

to the hyperbola $V(x^2 - v^2 - 1)$ in $\mathbb{R}^2$.

**Solution:**

$$(x + 0i, 0 + iv) \mapsto (y, u).$$

## 1.4 The Complex Projective Plane $\mathbb{P}^2$

Problem 1

1. Show that $(2, 1 + i, 3i) \sim (2 - 2i, 2, 3 + 3i)$.
2. Show that $(1, 2, 3) \sim (2, 4, 6)$ and $(-2, -4, -6) \sim (-i, -2i, -3i)$.
3. Show that $(2, 1 + i, 3i) \nsim (4, 4i, 6i)$.
4. Show that $(1, 2, 3) \nsim (3, 6, 8)$.

*Proof.* Let $\lambda = \frac{2}{2-2i} = \frac{1}{2} + \frac{1}{2}i$. Then

$$\lambda(2 - 2i) = \frac{2}{2 - 2i}(2 - 2i) = 2,$$

$$\lambda \cdot 2 = \left(\frac{1}{2} + \frac{1}{2}i\right)2 = 1 + i,$$

$$\lambda(3 + 3i) = \left(\frac{1}{2} + \frac{1}{2}i\right)(3 + 3i) = 3i.$$

∎

*Proof.* Let $\lambda = \frac{1}{2}$. Then

$$\lambda \cdot 2 = 1,$$

$$\lambda \cdot 4 = 2,$$

$$\lambda \cdot 6 = 3.$$

∎

*Proof.* Let $\lambda = 2i$. Then

$$\lambda \cdot (-i) = -2,$$

$$\lambda \cdot (-2i) = -4,$$

$$\lambda \cdot (-3i) = -6.$$

∎

*Proof.* Suppose there exists $\lambda$ such that $\lambda(4, 4i, 6i) = (2, 1 + i, 3i)$. Then

$$\lambda \cdot 4 = 2 \implies \lambda = \frac{1}{2},$$

$$\lambda \cdot 4i = 2i \neq 1 + i.$$

Thus no such $\lambda$ exists. ∎

*Proof.* Suppose there exists $\lambda$ such that $\lambda(3, 6, 8) = (1, 2, 3)$. Then

$$\lambda \cdot 3 = 1 \implies \lambda = \frac{1}{3},$$

$$\lambda \cdot 8 = \frac{8}{3} \neq 3.$$

Thus no such $\lambda$ exists. ∎

**Problem 2**

Show that $\sim$ is an equivalence relation.

*Proof.* Suppose $(x, y, z), (a, b, c), (d, e, f) \in \mathbb{C}^3 - \{(0, 0, 0)\}$. Then $\lambda = 1$ shows $(x, y, z) \sim (x, y, z)$. Thus $\sim$ is reflexive.

Suppose $(a, b, c) \sim (d, e, f)$. Then there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(a, b, c) = (\lambda d, \lambda e, \lambda f)$. Therefore $\left(\frac{1}{\lambda}a, \frac{1}{\lambda}b, \frac{1}{\lambda}c\right) = (d, e, f)$. It follows that $(d, e, f) \sim (a, b, c)$. Thus $\sim$ is symmetric.

Suppose $(x, y, z) \sim (a, b, c)$ and $(a, b, c) \sim (d, e, f)$. Then there exist $\lambda_1, \lambda_2 \in \mathbb{C} - \{0\}$ such that $(x, y, z) = (\lambda_1 a, \lambda_1 b, \lambda_1 c)$ and $(a, b, c) = (\lambda_2 d, \lambda_2 e, \lambda_2 f)$. Then

$$(x, y, z) = (\lambda_1 a, \lambda_1 b, \lambda_1 c) = (\lambda_1\lambda_2 d, \lambda_1\lambda_2 e, \lambda_1\lambda_2 f).$$

Thus $(x, y, z) \sim (d, e, f)$. Therefore $\sim$ is transitive. ∎

**Problem 3**

Suppose that $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ and that $x_1 = x_2 \neq 0$. Show that $y_1 = y_2$ and $z_2 = z_2$.

*Proof.* Since $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(x_1, y_1, z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$. Thus $x_1 = \lambda x_2 = \lambda x_1$ therefore $\lambda = \frac{x_1}{x_1} = 1$. It follows that $y_1 = y_2$ and $z_1 = z_2$. ∎

**Problem 4**

Suppose that $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ with $z_1 \neq 0$ and $z_2 \neq 0$. Show that

$$(x_1, y_1, z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2, y_2, z_2).$$

*Proof.* We see

$$(x_1, y_1, z_1) = \left(z_1 \cdot \frac{x_1}{z_1}, z_1 \cdot \frac{y_1}{z_1}, z_1 \cdot 1\right).$$

Since $z_1 \neq 0$ we see $(x_1, y_1, z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right)$. Now, since $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(x_1, y_1, z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$. Since $z_1 = \lambda z_2$ and $z_1, z_2 \neq 0$ we see

$$\frac{x_1}{z_1} = \frac{\lambda x_2}{\lambda z_2} = \frac{x_2}{z_2} \quad \text{and} \quad \frac{y_1}{z_1} = \frac{\lambda y_2}{\lambda z_2} = \frac{y_2}{z_2}.$$

Thus

$$\left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right).$$

Since

$$(x_2, y_2, z_2) = \left(z_2 \cdot \frac{x_2}{z_2}, z_2 \cdot \frac{y_2}{z_2}, z_2 \cdot 1\right),$$

and $z_2 \neq 0$ we see

$$\left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2, y_2, z_2).$$

Therefore,

$$(x_1, y_1, z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2, y_2, z_2).$$

∎

**Problem 5**

1. Find the equivalence class of $(0, 0, 1)$.
2. Find the equivalence class of $(1, 2, 3)$.

**Solution (1):**

$$\{(0, 0, c) \in \mathbb{C}^3 \mid c \neq 0\}.$$

**Solution (2):**

$$\{(\lambda, 2\lambda, 3\lambda) \in \mathbb{C}^3 \mid \lambda \neq 0\}.$$

**Problem 6**

Show that the equivalence class $(1 : 2 : 3)$ and $(2 : 4 : 6)$ are equal as sets.

*Proof.* Clearly, with $\lambda = \frac{1}{2} \in \mathbb{C}$ we have $(1, 2, 3) = (\lambda 2, \lambda 4, \lambda 6)$. Thus $(1, 2, 3) \sim (2, 4, 6)$ so $(1 : 2 : 3) = (2 : 4 : 6)$. ∎

**Problem 7**

Explain why the elements of $\mathbb{P}^2$ can intuitively be thought of as complex lines through the origin in $\mathbb{C}^3$.

**Solution:** Take a line passing through the origin in $\mathbb{C}^3$ with direction vector $(a, b, c) \neq (0, 0, 0)$. This line consists of all points of the form $(\lambda a, \lambda b, \lambda c)$ such that $\lambda \in \mathbb{C}$. If we require $\lambda \neq 0$ we get the equivalence class $(a : b : c) \in \mathbb{P}^2$. Thus each element of $\mathbb{P}^2$ represents a complex line through the origin in $\mathbb{C}^3$.

**Problem 8**

If $c \neq 0$, show, in $\mathbb{C}^3$, that the line $x = \lambda a$, $y = \lambda b$, $z = \lambda c$ intersects the plane $\{(x, y, z) \mid z = 1\}$ in exactly one point. Show that this point of intersection is $\left(\frac{a}{c}, \frac{b}{c}, 1\right)$.

*Proof.* Suppose $c \neq 0$. At the intersection we must have $z = \lambda c = 1$ so $\lambda = \frac{1}{c}$. Thus

$$(\lambda a, \lambda b, \lambda c) = \left(\frac{a}{c}, \frac{b}{c}, 1\right).$$

∎

## Problem 9

Show that the map $\psi : \mathbb{C}^2 \to \{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\}$ defined by $\psi(x, y) = (x : y : 1)$ is a bijection.

*Proof.* Suppose $(a, b), (x, y) \in \mathbb{C}^2$ such that $\psi(x, y) = \psi(a, b)$. Then

$$\psi(x, y) = \psi(a, b) \iff (x : y : 1) = (a : b : 1).$$

There exists $\lambda \neq 0$ such that $(x, y, 1) = (\lambda a, \lambda b, \lambda)$. Therefore $\lambda = 1$ thus $x = a$ and $y = b$. Thus $\psi$ is injective. Let $(x : y : z)$ be an arbitrary element in $\{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\}$. Then

$$(x : y : z) = \left(\frac{x}{z} : \frac{y}{z} : 1\right) = \psi\left(\frac{x}{z}, \frac{y}{z}\right).$$

Thus $\psi$ is surjective. It follows that $\psi$ is bijective. ∎

## Problem 10

Find a map from $\{(x, y, z) \in \mathbb{P}^2 \mid z \neq 0\}$ to $\mathbb{C}^2$ that is the inverse of the map $\psi$ in Exercise 1.4.9.

**Solution:** Let

$$\phi : \{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\} \longrightarrow \mathbb{C}^2$$

be defined by

$$\phi(x : y : z) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

## Problem 11

Consider the line $l = \{(x, y) \in \mathbb{C}^2 \mid ax + by + c = 0\}$ in $\mathbb{C}^2$. Assume $a, b \neq 0$. Explain why, as $|x| \to \infty$, $|y| \to \infty$. (Hence, $|x|$ is the modulus of $x$.)

*Proof.* We see $y = \frac{-c-ax}{b}$ and $x = \frac{-by-c}{a}$. Since $b$ and $c$ are constants, as $|y| \to \infty$ we have $|x| \to \infty$. ∎

## Problem 12

Consider again the line $l$. We know that $a$ and $b$ cannot both be $0$, so we will assume without loss of generality that $b \neq 0$.

1. Show that the image of $l$ in $\mathbb{P}^2$ under $\psi$ is the set

$$\{(bx : -ax - c : b) \mid x \in \mathbb{C}\}.$$

2. Show that this set equals the following union.

$$\{(bx : -ax - c : b) \mid x \in \mathbb{C}\} = \{(0 : -c : b)\} \cup \left\{\left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right)\right\}.$$

3. Show that as $|x| \to \infty$, the second set in the above union becomes

$$\left\{\left(1 : -\frac{a}{b} : 0\right)\right\}.$$

*Proof.* We start by solving explicitly for $y$ and note $b \neq 0$.

$$ax + by + c = 0 \iff y = \frac{-ax - c}{b}.$$

Then

$$\psi\left(x, \frac{-ax - c}{b}\right) = \left(x : \frac{-ax - c}{b} : 1\right) = (bx : -ax - c : b).$$

∎

*Proof.* There are two cases. If $x = 0$ then

$$(bx : -ax - c : b) = (0 : -c : b).$$

Otherwise, if $x \neq 0$ then we can divide by $bx \neq 0$ and see

$$(bx : -ax - c : b) = \left(1 : \frac{-ax - c}{bx} : \frac{1}{x}\right) = \left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right).$$

∎

*Proof.* As $|x| \to \infty$, we have $\frac{c}{bx} \to 0$ and $\frac{1}{x} \to 0$. Thus

$$\left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right) \to \left(1 : -\frac{a}{b} : 0\right).$$

∎

**Problem 13**

Explain why the following polynomials are homogeneous, and find each degree.

1. $x^2 + y^2 - z^2$.
2. $xz - y^2$.
3. $x^3 + 3xy^2 + 4y^3$.
4. $x^4 + x^2y^2$.

**Solution (1):** All monomials have total degree 2 thus it is homogeneous. The total degree is 2.

**Solution (2):** All monomials have total degree 2 thus it is homogeneous. The total degree is 2.

**Solution (3):** All monomials have total degree 3 thus it is homogeneous. The total degree is 3.

**Solution (4):** All monomials have total degree 4 thus it is homogeneous. The total degree is 4.

**Problem 14**

Explain why the following polynomials are not homogeneous.

1. $x^2 + y^2 - z$.
2. $xz - y$.
3. $x^2 + 3xy^2 + 4y^3 + 3$.
4. $x^3 + x^2y^2 + x^2$.

**Solution (1):** $z$ has total degree 1 and $x^2$ has total degree 2 thus it is not homogeneous.

**Solution (2):** $xz$ has total degree 2 and $y$ has total degree 1 thus it is not homogeneous.

**Solution (3):** 3 has total degree 0 and $x^2$ has total degree 2 thus it is not homogeneous.

**Solution (4):** $x^3$ has total degree 3 and $x^2$ has total degree 2 thus it is not homogeneous.

## Problem 15

Show that if the homogeneous equation $Ax + By + Cz = 0$ holds for every point $(x, y, z)$ in $\mathbb{C}^3 - \{0, 0, 0\}$, then it holds for every point in $\mathbb{C}^3$ that belongs to the equivalence class $(x : y : z)$ in $\mathbb{C}^2$.

*Proof.* Suppose the homogeneous equation $Ax + By + Cz = 0$ holds for all $(x, y, z) \in \mathbb{C}^3 - \{(0, 0, 0)\}$. Consider the polynomial in $x$

$$f(x) = Ax + (By + Cz).$$

Since $f(x) = 0$ for all $x \in \mathbb{C}$, $A = 0$. Similarly, $B = 0$ and $C = 0$. Therefore $Ax + By + Cz = 0$ for all $(x, y, z) \in \mathbb{C}^3$, which includes $(x : y : z)$. ∎

## Problem 16

Show that if the homogenous equation $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$

*Proof.* Suppose the homogeneous equation $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$. Consider the polynomial in $x$

$$f(x) = Ax^2 + Dxy + Exz + (By^2 + Cz^2 + Fyz).$$

Since $f(x) = 0$ for all $x \in \mathbb{C}$, $A = 0$. Similarly, $B = C = D = E = F = 0$. Therefore $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$ for all $(x, y, z) \in \mathbb{C}^3$, which includes $(x : y : z)$. ∎

## Problem 17

State and prove the generalization of the previous two exercises for any degree $n$ homogenous equation $P(x, y, z) = 0$.

*Proof.* Suppose the homogeneous equation $P(x, y, z) = 0$ of degree $n$. Consider the polynomial in $x$

$$f(x) = P(x, y, z).$$

Since $f(x) = 0$ for all $x \in \mathbb{C}$ coefficients of powers of $x$ must vanish. Similarly, all other coefficients vanish. Therefore $P(x, y, z) = 0$ for all $(x, y, z) \in \mathbb{C}^3$, which includes $(x : y : z)$. ∎

## Problem 18

Consider the non-homogeneous equation $P(x, y, z) = x^2 + 2y + 2z = 0$. Show that $(2, -1, -1)$ satisfies the equation. Find a point of the equivalence class $(2 : -1 : -1)$ that does not satisfy this equation.

**Solution:**

$$2^2 + 2(-1) + 2(-1) = 0$$

$$4^2 + 2(-2) + 2(-2) = 16 - 4 - 4 = 8 \neq 0$$

## Problem 19

Homogenize the following equations. Then find point(s) where the curves intersect the line at infinity.

1. $ax + by + c = 0$.
2. $x^2 + y^2 = 1$.
3. $y = x^2$.
4. $x^2 + 9y^2 = 1$.

5. $y^2 - x^2 = 1$.

**Solution (1):**

$$z \cdot \left(a\frac{x}{z} + b\frac{y}{z} + c\right) = 0 \iff ax + by + cz = 0.$$

At $z = 0$ we have

$$ax + by = 0.$$

A point at infinity is

$$(x : y : z) = (b : -a : 0).$$

**Solution (2):**

$$z^2 \cdot \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right) = 0 \iff x^2 + y^2 - z^2 = 0.$$

At $z = 0$ we have

$$x^2 + y^2 = 0.$$

Points at infinity are

$$(x : y : z) = (1 : i : 0), (1 : -i : 0).$$

**Solution (3):**

$$z^2 \cdot \left(\frac{y}{z} - \left(\frac{x}{z}\right)^2\right) = 0 \iff x^2 - yz = 0.$$

At $z = 0$ we have

$$x^2 = 0 \implies x = 0.$$

Point at infinity is

$$(x : y : z) = (0 : 1 : 0).$$

**Solution (4):**

$$z^2 \cdot \left(\left(\frac{x}{z}\right)^2 + 9\left(\frac{y}{z}\right)^2 - 1\right) = 0 \iff x^2 + 9y^2 - z^2 = 0.$$

At $z = 0$ we have

$$x^2 + 9y^2 = 0.$$

Points at infinity are

$$(x : y : z) = (3i : 1 : 0), (-3i : 1 : 0).$$

**Solution (5):**

$$z^2 \cdot \left(\left(\frac{y}{z}\right)^2 - \left(\frac{x}{z}\right)^2 - 1\right) = 0 \iff y^2 - x^2 - z^2 = 0.$$

At $z = 0$ we have

$$y^2 - x^2 = 0 \implies (y - x)(y + x) = 0.$$

Points at infinity are

$$(x : y : z) = (1 : 1 : 0), (1 : -1 : 0).$$

**Problem 20**

Show that in $\mathbb{P}^2$, any two distinct lines will intersect in a point. Notice this implies that parallel lines in $\mathbb{C}^2$, when embeed in $\mathbb{P}^2$, intersect at the line at infinity.

*Proof.* Suppose we have the lines

$$l_1 : ax + by + cz = 0, \quad l_2 : dx + ey + fz = 0.$$

If $l_1$ is not parallel to $l_2$, then set $z = 1$ and solve the two equations for $x$ and $y$ to find the intersection point in $\mathbb{C}^2$. If $l_1$ is parallel to $l_2$, then homogenizing shows that the intersection occurs at $z = 0$. ∎

**Problem 21**

Once we have homogenized an equation, the original variables $x$ $y$ are no more important than the variable $z$. Suppose we regard $x$ and $z$ as the original variables in our homogeneous equation. Then the image of the $xz$-plane in $\mathbb{P}^2$ would be $\{(x, y, z) \in \mathbb{P}^2 \mid y = 1.\}$.

1. Homogenize the equations for the parallel lines $y = x$ and $y = x + 2$.
2. Now regard $x$ and $z$ as the original variables, and set $y = 1$ to sketch the image of the lines in the $xz$-plane.
3. Explain why the lines in part meet at the $x$-axis.

**Solution (1):** The lines are $y = x$ and $y = x + 2$. Homogenizing with $z$ gives:

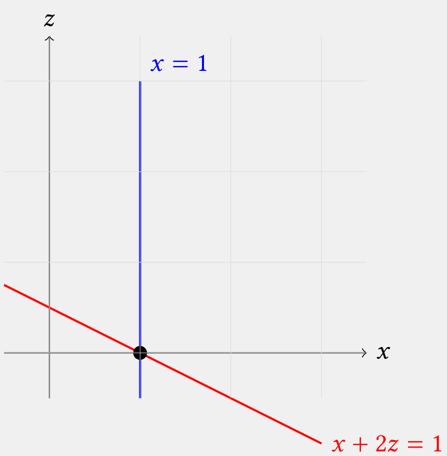
$$y - x = 0 \quad \Longrightarrow \quad y - x = 0$$

$$y - x - 2 = 0 \quad \Longrightarrow \quad y - x - 2z = 0$$

**Solution (2):** Regarding $x$ and $z$ as the original variables and setting $y = 1$

$$1 - x = 0 \quad \Longrightarrow \quad x = 1$$

$$1 - x - 2z = 0 \quad \Longrightarrow \quad x + 2z = 1$$




**Solution (3):** The lines intersect at the $x$-axis because in the $xz$-plane the $x$-axis is defined by $z = 0$.

## 1.5 Projective Changes of Coordinates

**Problem 1**

For the complex affine change of coordinates

$$u = ax + by + e.$$

$$v = cx + dy + f,$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, show that

$$u = ax + by + ez,$$

$$v = cx + dy + fz,$$

$$w = z,$$

is the corresponding projective change of coordinates.

*Proof.* We have $\psi(u, v) = \left(\frac{ax+by+e}{z} : \frac{cx+dy+f}{z} : 1\right) = (ax + by + e : cx + dy + f : z)$. ∎

**Problem 2**

Let $C_1 = V(x^2 + y^2 - 1)$ be an ellipse in $\mathbb{C}^2$ and let $C_2 = V(u^2 - v)$ be a parabola in $\mathbb{C}^2$. Homogenize the defining polynomials for $C_1$ and $C_2$ and show that the projective change of coordinates

$$u = ix,$$

$$v = y + z,$$

$$w = y - z,$$

transforms the ellipse in $\mathbb{P}^2$ into the parabola in $\mathbb{P}^2$.

*Proof.* We first homogenize $C_1$ and $C_2$ to obtain

$$x^2 + y^2 - z^2 = 0 \quad \text{and} \quad u^2 - vw = 0.$$

Then substituting our change of variables gives

$$u^2 - vw = (ix)^2 - (y + z)(y - z) = -x^2 - (y^2 - z^2) = -(x^2 + y^2 - z^2) = 0,$$

as required. ∎

**Problem 3**

Use the results of Section 1.3, together with the above problem, to show that, under a projective change of coordinates, all ellipses, hyperbolas, and parabolas are equivalent in $\mathbb{P}^2$.

*Proof.* By Section 1.3 ellipses and hyperbolas are equivalent under affine changes of coordinates, and by Problem 1 these extend to projective changes of coordinates in $\mathbb{P}^2$. By Problem 2 ellipses are equivalent to parabolas in $\mathbb{P}^2$. Therefore under a projective change of coordinates, all ellipses, hyperbolas, and parabolas are equivalent in $\mathbb{P}^2$. ∎

## 1.6 The Complex Projective Line $\mathbb{P}^1$

**Problem 1**

Suppose that $(x_1, y_1) \sim (x_2, y_2)$ and that $x_1 = x_2 \neq 0$. Show that $y_1 = y_2$.

*Proof.* We know there exists $\lambda$ such that $x_1 = \lambda x_2$. Dividing by $x_2 \neq 0$ shows $\lambda = 1$. But then $y_1 = \lambda y_2 = y_2$. ∎

## Problem 2

Suppose that $(x_1, y_1) \sim (x_2, y_2)$ with $y_1 \neq 0$ and $y_2 \neq 0$. Show that

$$(x_1, y_1) \sim \left(\frac{x_1}{y_1}, 1\right) = \left(\frac{x_2}{y_2}, 1\right) \sim (x_2, y_2).$$

*Proof.* Let $\lambda = y_1$ to see that

$$\left(\lambda\frac{x_1}{y_1}, \lambda \cdot 1\right) = (x_1, y_1).$$

Thus $(x_1, y_1) \sim \left(\frac{x_1}{y_1}, 1\right)$. Similarly, $(x_2, y_2) \sim \left(\frac{x_2}{y_2}, 1\right)$. Since $(x_1, y_1) \sim (x_2, y_2)$ and $\sim$ is an equivalence relation

$$\left(\frac{x_1}{y_1}, 1\right) = \left(\frac{x_2}{y_2}, 1\right).$$

∎

## Problem 3

Explain why the elements of $\mathbb{P}^1$ can intuitively be thought of as complex lines through the origin in $\mathbb{C}^2$.

**Solution:** Take a line passing through the origin in $\mathbb{C}^2$ with direction vector $(a, b) \neq (0, 0)$. This line consists of all points of the form $(\lambda a, \lambda b)$ such that $\lambda \in \mathbb{C}$. If we require $\lambda \neq 0$ we get the equivalence class $(a : b : c) \in \mathbb{P}^2$. Thus each element of $\mathbb{P}^1$ represents a complex line through the origin in $\mathbb{C}^2$.

## Problem 4

If $b \neq 0$, show that the line $x = \lambda a$, $y = \lambda b$ will intersect the line $\{(x, y) \mid y = 1\}$ in exactly one point. Show that the point of intersection is $\left(\frac{a}{b}, 1\right)$.

*Proof.* We have $1 = y = \lambda b$ thus $\lambda = \frac{1}{b}$. Therefore $(x, y) = (\lambda a, 1) = \left(\frac{a}{b}, 1\right)$. ∎

## Problem 5

Show that the map $\psi : \mathbb{C} \to \{(x : y) \in \mathbb{P}^2 \mid y \neq 0\}$ defined by $\psi(x) = (x : 1)$ is a bijection.

*Proof.* Suppose $a, b \in \mathbb{C}$ such that $\psi(a) = \psi(b)$. Then

$$\psi(a) = \psi(b) \iff (a : 1) = (b : 1).$$

But then since $1 = \lambda 1$ we have $\lambda = 1$ so $a = b$. Let $(a : y)$ be an arbitrary element in $\{(x : y) \in \mathbb{P}^2 \mid y \neq 0\}$. Then $(a : y) = \left(\frac{a}{y} : 1\right) = \psi\left(\frac{a}{b}\right)$. Thus $\psi$ is a bijection. ∎

## Problem 6

Find a map $\{(x : y) \in \mathbb{P}^1 \mid y \neq 0\}$ to $\mathbb{C}$ that is the inverse of the map $\psi$ in Exercise 1.6.5.

**Solution:**

$$\psi^{-1}(x : y) = \frac{x}{y}.$$

**Problem 7**

Consider the map $\psi : \mathbb{C} \to \mathbb{P}^2$ given by $\psi(x) = (x : 1)$. Show that as $|x| \to \infty$, we have $\psi(x) \to (1 : 0)$.

*Proof.* We have

$$(x : 1) = \left(1 : \frac{1}{x}\right).$$

As $|x| \to \infty$, we have $\frac{1}{x} \to 0$ thus

$$\left(1 : \frac{1}{x}\right) \to (1 : 0).$$

∎

**Problem 8**

Let $p$ denote the point $(0, 0, 1) \in S^2$, and let $l$ denote the line through $p$ and the point $(x, y, 0)$ in the $xy$-plane, whose parametrization is given by

$$\rho(t) = (1 - t)(0, 0, 1) + t(x, y, 0),$$

i.e.,

$$l = \{(tx, ty, 1 - t) \mid t \in \mathbb{R}\}.$$

1. $l$ clearly intersects $S^2$ at the point $p$. Show that there is exactly one other point of intersection $q$.
2. Find the coordinates of $q$.
3. Define the map $\psi : \mathbb{R}^2 \to S^2 - \{p\}$ to be the map that takes the point $(x, y)$ to the point $q$. Show that $\psi$ is a continuous bijection.
4. Show that as $\sqrt{x^2 + y^2} \to \infty$, we have $\psi(x, y) \to p$. Thus as we move away from the origin in $\mathbb{R}^2$, $\psi(x, y)$ moves toward the North Pole.

*Proof.* We substitute $l$ into the unit sphere equation to find

$$x^2 + y^2 + z^2 - 1 = (tx)^2 + (ty)^2 + (1 - t)^2 - 1 = t^2(x^2 + y^2 + 1) - 2t = t\big(t(x^2 + y^2 + 1) - 2\big) = 0.$$

Now $t = 0$ corresponds to $p$. The other point corresponds to

$$t(x^2 + y^2 + 1) - 2 = 0 \quad\implies\quad t = \frac{2}{x^2 + y^2 + 1}.$$

Substituting back into the line gives

$$q = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right).$$

∎

*Proof.* Each coordinate is a continuous fraction thus $\psi$ is continuous. Suppose $(x, y), (a, b) \in \mathbb{R}^2$ such that

$$\psi(x, y) = \psi(a, b) \iff \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right) = \left(\frac{2a}{a^2 + b^2 + 1}, \frac{2b}{a^2 + b^2 + 1}, \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1}\right)$$

From the third coordinate we infer $x^2 + y^2 + 1 = a^2 + b^2 + 1$. Then from the first coordinates we see $a = x$ and $b = y$. Thus $\psi$ is injective. Suppose $(X, Y, Z) \in S^2 - \{p\}$. We can solve the following equations for $x, y$

$$\frac{2x}{x^2 + y^2 + 1} = X, \quad \frac{2y}{x^2 + y^2 + 1} = Y, \quad \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} = Z.$$

Thus

$$x = \frac{X}{1-Z} \quad \text{and} \quad y = \frac{Y}{1-Z}.$$

Finally, we see

$$\psi(x, y) = (X, Y, Z).$$

Thus $\psi$ is surjective and it follows that $\psi$ is a bijection. ∎

*Proof.* As $\sqrt{x^2+y^2} \to \infty$ we have $\frac{2x}{x^2+y^2+1} \to 0$, $\frac{2y}{x^2+y^2+1} \to 0$, and $\frac{x^2+y^2-1}{x^2+y^2+1} \to 1$. Thus $q \to (0, 0, 1) = p$. ∎

**Problem 9**

Determine which point(s) in $\mathbb{P}^1$ do **not** have two representitives of the form $(x : 1) = (1 : \frac{1}{x})$.

**Solution:**

$$(0 : 1) \in \mathbb{P}^1.$$

**Problem 10**

Map $U \to \mathbb{P}^1$ via $x \mapsto (x : 1)$ and map $V \to \mathbb{P}^1$ via $y \mapsto (1 : y)$. Show that $(x : 1) \mapsto \left(1 : \frac{1}{x}\right)$ is a natural one-to-one map from $U^*$ to $V^*$.

*Proof.* Let $x, y \in U^*$ then under the map we have $\left(1 : \frac{1}{x}\right) = \left(1 : \frac{1}{y}\right) \in V^*$. Then since 1 is the first coordinate we have $\lambda = 1$ thus $x = y$. Therefore the mapping is a natural one-to-one map from $U^*$ to $V^*$. ∎

**Problem 11**

A sphere can be split into a neighborhood of its northern hemisphere and a neighborhood of its southern hemisphere. Show that a sphere can be obtained by correctly gluing together two copies of $\mathbb{C}$.

*Proof.* We take two spaces $\mathbb{C}_1, \mathbb{C}_2$. Then define maps $\psi_1 : \mathbb{C}_1 \to \mathbb{R}^3 - \{0, 0, 1\}$, $\psi_2 : \mathbb{C}_2 \to \mathbb{R}^3 - \{0, 0, -1\}$ by

$$\psi_1(x, y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right),$$

and

$$\psi_2(u, v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\right).$$

Then

$$S^2 = \psi_1(\mathbb{C}_1) \cup \psi_2(\mathbb{C}_2).$$

∎

**Problem 12**

Put together the last two exercises to show that $\mathbb{P}^1$ is topologically equivalent to a sphere.

*Proof.* From Problem 10, we can map the two spaces $U^*$ and $V^*$ to $\mathbb{P}^1$. We can then equate $U^*$ and $V^*$ to the two copies of $\mathbb{C}$. Then from Problem 11 there is a homeomorphism from the two spaces $\mathbb{C}$ to $S^2$. Thus $\mathbb{P}^1$ is homeomorphic to $S^2$. ∎

## 1.7 Ellipses, Hyperbolas, and Parabolas as Spheres

**Problem 1**

Find a bijective polynomial map from $\mathbb{C}$ to the conic $C = \{(x, y) \in \mathbb{C}^2 \mid x^2 - y = 0\}$

*Proof.* We first paramtrize $C$ as follows

$$x = t \text{ and } y = t^2.$$

Then, we define the mapping $\psi : \mathbb{C} \to C$ as

$$\psi(\alpha) = (\alpha, \alpha^2).$$

Now, suppose $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\psi(\alpha_1) = \psi(\alpha_2)$. Then

$$\psi(\alpha_1) = \psi(\alpha_2) \iff (\alpha_1, \alpha_1^2) = (\alpha_2, \alpha_2^2) \iff \alpha_1 = \alpha_2.$$

Thus $\psi$ is injective. Now, let $(x, y) \in C$ and notice $y = x^2$. Then $\psi(x) = (x, x^2) = (x, y)$. Thus $\psi$ is surjective. It follows that $\psi$ is a bijection from $\mathbb{C}$ to $C$. ∎

**Problem 2**

Let $C = V(x^2 + y^2 - 1)$ be an ellipse in $\mathbb{C}^2$. Find a trigonometric parametrization of $C$. [Hint: Think high school trigonometry.]

*Proof.* Let $x = \cos(t)$ and $y = \sin(t)$ for $t \in \mathbb{C}$. ∎

**Problem 3**

Consider the ellipse $C = V(x^2 + y^2 - 1) \subset \mathbb{C}^2$ and let $p$ denote the point $(0, 1) \in C$.

1. Parametrize the line segment from $p$ to the point $(\lambda, 0)$ on the complex line $y = 0$ as in Exercise 16.8.
2. This line segment clearly intersects $C$ at the point $p$. Show that if $\lambda \neq \pm i$, then there is exactly one other point of intersection. Call this point $q$.
3. Find the coordinates of $q \in C$.
4. Show that if $\lambda = \pm i$, then the line segment intersects $C$ only at $p$.

*Proof.* We have $p + x(p - (\lambda, 0))$ for $x \in \mathbb{C}$. Then

$$p + x(p - (\lambda, 0)) = (0, 1) + x((0, 1) - (\lambda, 0)) = (0, 1) + x(-\lambda, 1) = (-\lambda x, 1 + x).$$

∎

*Proof.* Suppose $\lambda \neq \pm i$. Then

$$(-\lambda x)^2 + (1 + x)^2 - 1 = \lambda^2 x^2 + (1 + 2x + x^2) - 1 = (\lambda^2 + 1)x^2 + 2x = x\big((\lambda^2 + 1)x + 2\big) = 0.$$

$x = 0$ corresponds to $p$.

$$x = -\frac{2}{\lambda^2 + 1}$$

corresponds to

$$q = \left(-\lambda\left(-\frac{2}{\lambda^2 + 1}\right), 1 - \frac{2}{\lambda^2 + 1}\right).$$

∎

*Proof.* Suppose $\lambda = \pm i$. Then

$$(-\lambda x)^2 + (1+x)^2 - 1 = (-(\pm i)x)^2 + (1 + 2x + x^2) - 1 = 2x = 0.$$

$x = 0$ corresponds to $p$ and there are no other solutions. ∎

**Problem 4**

Define the map $\tilde{\psi} : \mathbb{C} \to C \subset \mathbb{C}^2$ by

$$\tilde{\psi}(\lambda) = \left( \frac{2\lambda}{\lambda^2+1}, \frac{\lambda^2-1}{\lambda^2+1} \right).$$

Show that the above map can be extended to the map

$$\psi : \mathbb{P}^1 \to \{(x : y : z) \in \mathbb{P}^2 \mid x^2 + y^2 - z^2 = 0\}.$$

given by

$$\psi(\lambda : u) = (2\lambda u : \lambda^2 - u^2 : \lambda^2 + u^2).$$

*Proof.* Plugging in we find

$$\begin{aligned}(2\lambda u)^2 + (\lambda^2 - u^2)^2 - (\lambda^2 + u^2)^2 &= 4\lambda^2 u^2 + (\lambda^4 - 2\lambda^2 u^2 + u^4) - (\lambda^4 + 2\lambda^2 u^2 + u^4) \\ &= 4\lambda^2 u^2 + \lambda^4 - 2\lambda^2 u^2 + u^4 - \lambda^4 - 2\lambda^2 u^2 - u^4 \\ &= 0.\end{aligned}$$

∎

**Problem 5**

1. Show that the map $\psi$ is one-to-one.
2. Show that $\psi$ is onto. [Hint: Consider the two cases: $z \neq 0$ and $z = 0$. For $z \neq 0$ follow the contruction given above. For $z = 0$, find values of $\lambda$ and $u$ to show that these points are given by $\psi$. How does this relate to Part 4 of Exercise 1.7.3?]

*Proof.* Let $(\lambda : u), (\lambda' : u') \in \mathbb{P}^1$. Then

$$\psi(\lambda : u) = \psi(\lambda' : u') \iff (2\lambda u : \lambda^2 - u^2 : \lambda^2 + u^2) = (2\lambda' u' : \lambda'^2 - u'^2 : \lambda'^2 + u'^2).$$

There exists $k \in \mathbb{C}$ such that

$$2\lambda u = k(2\lambda' u'), \quad \lambda^2 - u^2 = k(\lambda'^2 - u'^2) = k\lambda'^2 - ku'^2, \quad \lambda^2 + u^2 = k(\lambda'^2 + u'^2) = k\lambda'^2 + ku'^2.$$

Adding the second and third equations shows

$$\lambda^2 = k\lambda'^2.$$

Subtracting the second and third equations shows

$$-2u^2 = -2ku'^2 \implies u^2 = ku'^2.$$

Thus $(\lambda : u) = (\lambda' : u')$ in $\mathbb{P}^1$. It follows that $\psi$ is injective.

Suppose $z = 0$. We have

$$x^2 + y^2 - z^2 = 0 \iff -x^2 = y^2 \iff y = \pm ix.$$

Then

$$(x : \pm ix : z) = (1 : \pm i : 0).$$

We have

$$\lambda^2 + u^2 = 0 \iff \lambda^2 = -u^2 \iff \lambda = \pm iu.$$

Then

$$\psi(\pm iu : u) = (2(\pm iu)u : (\pm iu)^2 - u^2 : (\pm iu)^2 + u^2) = (\pm 2iu^2 : -2u^2 : 0) = (1 : \pm i : 0).$$

Now, suppose $z \neq 0$. Let $(x : y : z)$ be an element in $\mathbb{P}^2$ with $z \neq 0$. We require

$$2\lambda u = x, \quad \lambda^2 - u^2 = y, \quad \lambda^2 + u^2 = z.$$

Adding and subtracting the second and third equations we find

$$\lambda^2 = \frac{y+z}{2} \implies \lambda = \pm\sqrt{\frac{y+z}{2}}, \quad u^2 = \frac{z-y}{2} \implies u = \pm\sqrt{\frac{z-y}{2}}.$$

Then

$$\psi\left(\pm\sqrt{\frac{y+z}{2}}, \pm\sqrt{\frac{z-y}{2}}\right) = (x : y : z).$$

Thus $\psi$ is surjective. ∎

**Problem 6**

For the following conics and the given point $p$, follow what we did for the conic $x^2 + y^2 - 1 = 0$ to find a rational map from $\mathbb{C}$ to the curve $\mathbb{C}^2$ and then a one-to-one map from $\mathbb{P}^1$ onto the conic in $\mathbb{P}^2$.

1. $x^2 + 2x - y^2 - 4y - 4 = 0$ with $p = (0, -2)$.
2. $3x^2 + 3y^2 - 75 = 0$ with $p = (5, 0)$.
3. $4x^2 + y^2 - 8 = 0$ with $p = (1, 2)$.

**Solution (1):** Consider the line $y = mx - 2$. We have

$$\begin{aligned} x^2 + 2x - y^2 - 4y - 4 = 0 & \\ &\iff x^2 + 2x - (mx-2)^2 - 4(mx-2) - 4 = 0 \\ &\iff x^2 + 2x - (m^2x^2 - 4mx + 4) - 4mx + 8 - 4 = 0 \\ &\iff (1-m^2)x^2 + 2x = 0 \\ &\iff x[(1-m^2)x + 2] = 0. \end{aligned}$$

Then $x = 0$ corresponds to $p$. So $x = \frac{-2}{1-m^2}$, thus $y = \frac{-2m-2(1-m^2)}{1-m^2}$. So we have the parametrization

$$x(m) = \frac{-2}{1-m^2}, \quad y(m) = \frac{2m^2 - 2m - 2}{1-m^2}.$$

Then define $\psi : \mathbb{P}^1 \to \mathbb{P}^2$ as

$$\psi(m, u) = (-2 : 2m^2 - 2m - 2 : 1 - m^2).$$

**Solution (2):** Consider the line $y = m(x-5) = mx - 5m$. We have

$$\begin{aligned} 3x^2 + 3y^2 - 75 = 0 & \\ &\iff 3x^2 + 3(mx - 5m)^2 - 75 = 0 \\ &\iff 3x^2 + 3(m^2x^2 - 10m^2x + 25m^2) - 75 = 0 \\ &\iff 3x^2 + 3m^2x^2 - 30m^2x + 75m^2 - 75 = 0 \\ &\iff 3(1+m^2)x^2 - 30m^2x + 75(m^2-1) = 0 \\ &\iff (1+m^2)x^2 - 10m^2x + 25(m^2-1) = 0 \\ &\iff [(1+m^2)x - 5(m^2+1)][x-5] = 0. \end{aligned}$$

Then $x = 5$ corresponds to $p$. So $x = \frac{5(m^2+1)}{1+m^2} = 5$, thus $y = m(x-5) = m(5-5) = 0$. So we have the parametrization

$$x(m) = 5, \quad y(m) = 0.$$

Then define $\psi : \mathbb{P}^1 \to \mathbb{P}^2$ as

$$\psi(m,u) = (5 : 0 : 1).$$

**Solution (3):** Consider the line $y = m(x-1) + 2 = mx - m + 2$. We have

$$\begin{aligned}
4x^2 + y^2 - 8 = 0 \\
&\iff 4x^2 + (mx - m + 2)^2 - 8 = 0 \\
&\iff 4x^2 + (m^2x^2 - 2m^2x + 4mx + m^2 - 4m + 4) - 8 = 0 \\
&\iff (4 + m^2)x^2 + (-2m^2 + 4m)x + (m^2 - 4m - 4) = 0 \\
&\iff (x-1)\big((4+m^2)x + (-4 + m^2 + 4m)\big) = 0.
\end{aligned}$$

Then $x = 1$ corresponds to $p$. So $x = \frac{4-m^2-4m}{4+m^2}$, thus

$$y = m\left(\frac{4 - m^2 - 4m}{4 + m^2} - 1\right) + 2.$$

So we have the parametrization

$$x(m) = \frac{4 - m^2 - 4m}{4 + m^2}, \quad y(m) = m\left(\frac{4 - m^2 - 4m}{4 + m^2} - 1\right) + 2.$$

Then define $\psi : \mathbb{P}^1 \to \mathbb{P}^2$ as

$$\psi(m,u) = (4 - m^2 - 4m : m(4 - m^2 - 4m - (4 + m^2)) + 2(4 + m^2) : 4 + m^2).$$

## 1.8 Links to Number Theory

**Problem 1**

Suppose $(x_0, y_0, z_0)$ is a solution to $x^2 + y^2 = z^2$. Show that $(mx_0, my_0, mz_0)$ is also a solution for any scalar $m$.

*Proof.* We have

$$x_0^2 + y_0^2 - z_0^2 = 0 \iff m^2(x_0^2 + y_0^2 - z_0^2) = 0 \iff m^2x_0^2 + m^2y_0^2 - m^2z_0^2 = 0 \iff (mx_0)^2 + (my_0)^2 - (mz_0)^2 = 0.$$

Thus $(mx_0, my_0, mz_0)$ is also a solution for any scalar $m$. ∎

**Problem 2**

Let $(a,b,c) \in \mathbb{Z}^3$ be a solution to $x^2 + y^2 = z^2$. Show that $c = 0$ if and only if $a = b = 0$.

*Proof.* Suppose $c = 0$ then clearly $a = b = 0$. Suppose $a = b = 0$. Then $a^2 + b^2 = 0^2 + 0^2 = 0$. Thus $c = 0$. ∎

**Problem 3**

Show that if $(a,b,c)$ is a Pythagorean triple with $c \neq 0$, then the pair of rational number $\left(\frac{a}{c}, \frac{b}{c}\right)$ is a solution to $x^2 + y^2 = 1$.

*Proof.* Suppose $(a, b, c)$ is a Pythagorean triple with $c \neq 0$. Then

$$a^2 + b^2 = c^2 \iff \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \iff \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

∎

**Problem 4**

Let $\left(\frac{a}{c_1}, \frac{b}{c_2}\right) \in \mathbb{Q}^2$ be a rational solution to $x^2 + y^2 = 1$. Find a corresponding Pythagorean triple.

*Proof.* First, write $\frac{a}{c_1}, \frac{b}{c_2}$ in their lowest terms by dividing by common factors. Then

$$\left(\frac{a}{c_1}\right)^2 + \left(\frac{b}{c_2}\right)^2 = 1 \iff (ac_2)^2 + (bc_1)^2 = (c_1c_2)^2.$$

Thus $(ac_2, bc_1, c_1c_2)$ is a Pythagorean triple. ∎

**Problem 5**

Let

$$C(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\}.$$

Define the map $\tilde{\psi} : \mathbb{Q} \to \{(x, y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\}.$ as

$$\lambda \mapsto \left(\frac{2\lambda}{\lambda^2 + 1}, \frac{\lambda^2 - 1}{\lambda^2 + 1}\right).$$

Show that the above map $\tilde{\psi}$ sends $\mathbb{Q} \to C(\mathbb{Q})$.

*Proof.* Let $\lambda \in \mathbb{Q}$ Then $\tilde{\lambda} = \left(\frac{2\lambda}{\lambda^2+1}, \frac{\lambda^2-1}{\lambda^2+1}\right)$. Plugging in we find

$$\begin{aligned}
\left(\frac{2\lambda}{\lambda^2 + 1}\right)^2 + \left(\frac{\lambda^2 - 1}{\lambda^2 + 1}\right)^2 &= \frac{4\lambda^2}{(\lambda^2 + 1)^2} + \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \\
&= \frac{4\lambda^2 + (\lambda^4 - 2\lambda^2 + 1)}{(\lambda^2 + 1)^2} \\
&= \frac{\lambda^4 + 2\lambda^2 + 1}{(\lambda^2 + 1)^2} \\
&= 1.
\end{aligned}$$

∎

**Problem 6**

1. Show that $\psi : \mathbb{P}^1(\mathbb{Q}) \to C(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{Q})$ is onto.
2. Show that every primitive triple is of the form $(2\lambda u, \lambda^2 - u^2, \lambda^2 + u^2)$.

*Proof.* This is the same as 1.7.5. $\psi$ is a bijection thus every primitive triple is of that form. ∎

Problem 7

Find a rational point on the conic $x^2 + y^2 - 2 = 0$. Develop a parametrization and conclude that there are infinitely many rational points on this curve.

*Proof.* Consider $p = (1,1) \in \mathbb{Q}^2$. Then $x^2+y^2-2 = 1+1-2 = 0$. Consider the line $y = m(x-1)+1 = mx-m+1$. Then

$$\begin{aligned} x^2 + (mx-m+1)^2 - 2 &= x^2 + \left(mx - m + 1\right)^2 - 2 \\ &= x^2 + \left(m^2x^2 - 2m^2x + 2mx + m^2 - 2m + 1\right) - 2 \\ &= (1+m^2)x^2 + 2m(1-m)x + (m^2 - 2m - 1) \\ &= (x-1)\big((1+m^2)x - m^2 + 2m + 1\big) \end{aligned}$$

Now, $x = 1$ corresponds to $p$. So we have $x = \frac{m^2-2m-1}{1+m^2}$. Then we have the paramtrization

$$x(m) = \frac{m^2-2m-1}{1+m^2}, \quad y(m) = m\left(\frac{m^2-2m-1}{1+m^2}\right) - m + 1.$$

From this we can obtain infinitely many rational points on the curve. ∎

Problem 8

By mimicking the above, find four rational points on each of the following conics.

1. $x^2 + 2x - y^2 - 4y - 4 = 0$ with $p = (0, -2)$.
2. $3x^+ 3y^2 - 75 = 0$ with $p = (5, 0)$.
3. $4x^2 + y^2 - 8 = 0$ with $p = (1, 2)$.

*Proof.* **(1)** Consider $p = (0,-2) \in \mathbb{Q}^2$. Then $x^2+2x-y^2-4y-4 = 0$ at $(0,-2)$. Consider the line $y = m(x-0)-2 = mx-2$. Then

$$\begin{aligned} x^2 + 2x - (mx-2)^2 - 4(mx-2) - 4 &= x^2 + 2x - (m^2x^2 - 4mx + 4) - (4mx - 8) - 4 \\ &= (1-m^2)x^2 + (2+0)x + (-4+8-4) \\ &= (1-m^2)x^2 + 2x + 0 \\ &= x\big((1-m^2)x + 2\big) \end{aligned}$$

Now, $x = 0$ corresponds to $p$. So we have $x = \frac{-2}{1-m^2}$. Then we have the parametrization

$$x(m) = \frac{-2}{1-m^2}, \quad y(m) = m\left(\frac{-2}{1-m^2}\right) - 2.$$

From this we can obtain infinitely many rational points on the curve. ∎

*Proof.* Consider $p = (5,0) \in \mathbb{Q}^2$. Then $3x^2 + 3y^2 - 75 = 0$ at $(5,0)$. Consider the line $y = m(x-5)+0 = m(x-5)$. Then

$$\begin{aligned} 3x^2 + 3(mx-5m)^2 - 75 &= 3x^2 + 3(m^2x^2 - 10m^2x + 25m^2) - 75 \\ &= 3(1+m^2)x^2 - 30m^2x + 75(m^2-1) \\ &= (x-5)\big(3(1+m^2)x - 30m^2 - 75\big) \end{aligned}$$

Now, $x = 5$ corresponds to $p$. So the other intersection point satisfies

$$3(1+m^2)x - 30m^2 - 75 = 0 \implies x = \frac{30m^2+75}{3(1+m^2)} = \frac{10m^2+25}{1+m^2}.$$

Then we have the parametrization

$$x(m) = \frac{10m^2 + 25}{1 + m^2}, \quad y(m) = m\left(\frac{10m^2 + 25}{1 + m^2} - 5\right) = m\left(\frac{10m^2 + 25 - 5 - 5m^2}{1 + m^2}\right) = m\left(\frac{5m^2 + 20}{1 + m^2}\right).$$

From this we can obtain infinitely many rational points on the curve. $\blacksquare$

*Proof.* Consider $p = (1, 2) \in \mathbb{Q}^2$. Then $4x^2 + y^2 - 8 = 0$ at $(1, 2)$. Consider the line $y = m(x - 1) + 2 = mx - m + 2$. Then

$$\begin{aligned} 4x^2 + (mx - m + 2)^2 - 8 &= 4x^2 + (m^2x^2 - 2m^2x + 2mx + m^2 - 4m + 4) - 8 \\ &= (4 + m^2)x^2 + 2m(1 - m)x + (m^2 - 4m - 4) \\ &= (x - 1)\big((4 + m^2)x + 2m(1 - m) + (m^2 - 4m - 4)\big) \end{aligned}$$

Now, $x = 1$ corresponds to $p$. So we have $x = \frac{-2m^2+2m+4}{4+m^2}$. Then we have the parametrization

$$x(m) = \frac{-2m^2 + 2m + 4}{4 + m^2}, \quad y(m) = m\left(\frac{-2m^2 + 2m + 4}{4 + m^2} - 1\right) + 2.$$

From this we can obtain infinitely many rational points on the curve. $\blacksquare$

**Problem 9**

Show that the conic $x^2 + y^2 = 3$ has no rational points.

*Proof.* Suppose $x = \frac{a}{b}$ and $y = \frac{c}{d}$ are in lowest terms with $a, b, c, d \in \mathbb{Z}$. Furthermore, suppose

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 3.$$

Then

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 3 \iff a^2d^2 + c^2b^2 = 3b^2d^2 \iff (ad)^2 = 3(bd)^2 - (bc)^2 = b^2(3d^2 - c^2).$$

Now we must have $3d^2 - c^2 = k^2$ for some $k \in \mathbb{Z}$. We see $3d^2 - c^2 \equiv 0, 2 \pmod 3$ while $k^2 \equiv 0, 1 \pmod 3$. But, $\gcd(c, d) = 1$ thus $3d^2 - c^2 \equiv 2 \pmod 3$ while $k^2 \equiv 0, 1 \pmod 3$ which is a contradiction. $\blacksquare$

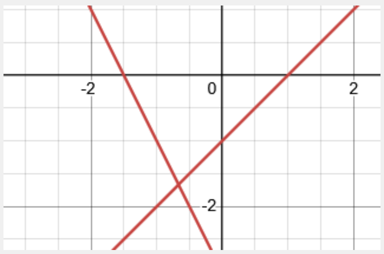
## 1.9 Degenerate Conics

**Problem 1**

Dehomogenize $f(x, y, z)$ by setting $z = 1$. Graph the curve

$$C(\mathbb{R}) = \{(x : y : z) \in \mathbb{P}^2 \mid f(x, y, 1) = 0\}.$$

in the real plane $\mathbb{R}^2$.



**Problem 2**

Consider the two lines given by

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0,$$

and suppose

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0.$$

Show that the two lines intersect at a point where $z \neq 0$.

*Proof.* Suppose the two lines intersect at a point where $z = 0$. Thus

$$a_1x + b_1y = 0 \quad \text{and} \quad a_2x + b_2y = 0,$$

so

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Then

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $x = y = 0$ but $(0, 0, 0) \notin \mathbb{P}^2$. ∎

**Problem 3**

Dehomogenize the equation in the previous exercise by setting $z = 1$. Given an argument that, as lines in the complex plane $\mathbb{C}^2$, they have distinct slopes.

*Proof.* Dehomogenizing we find

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0.$$

Since they intersect at a point such that $z \neq 0$ they are not parallel in $\mathbb{C}^2$. Thus either $a_1 \neq a_2$ or $b_1 \neq b_2$. ∎

**Problem 4**

Again consider the two lines

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0.$$

Suppose that

$$\det \left( A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \right) = 0.$$

but that

$$\det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \neq 0 \text{ or } \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \neq 0.$$

Show that the two lines still have one common point of intersection but that this point must have $z = 0$.

*Proof.* Suppose the two lines intersect at a point such that $z \neq 0$. Then dividing by $z$ we find

$$\left(a_1 \frac{x}{z} + b_1 \frac{y}{z} + c_1\right)\left(a_2 \frac{x}{z} + b_2 \frac{y}{z} + c_2\right) = 0.$$

Let $X = \frac{x}{z}, Y = \frac{y}{z}$. We now have the following system

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}.$$

Since $\det(A) = 0$ the system does not have a unique solution. It follows that the lines are linearly dependent. Now, wlog suppose

$$\det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \neq 0.$$

Then $a_1 c_2 \neq a_2 c_1$ and it follows that the lines are not equivalent. Therefore, the lines are parallel and meet at $z = 0$. ∎

### Problem 5

Let

$$f(x, y, z) = (a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z),$$

where at least one of $a_1, b_2$, or $c_1$ is non-zero and at least one of the $a_2, b_2$, or $c_2$ is non-zero. Show that the curve defined by $f(x, y, z) = 0$ is a double line if and only if

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} = 0.$$

*Proof.* Suppose $f(x, y, z) = 0$ is a double line. By Problem 4 we know

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} = 0.$$

Conversely, suppose

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} = 0.$$

Thus $(a_2, b_2, c_2)$ is a scalar multiple of $(a_1, b_1, c_1)$. Therefore $f(x, y, z) = 0$ is a double line. ∎

### Problem 6

Consider

$$(a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z) = 0,$$

with

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0.$$

Find a projective change of coordinates from $xyz$-space to $uvw$-space so that the crossing lines become

$$uv = 0.$$

*Proof.* Consider

$$u = a_1 x + b_1 y + c_1 z, \quad v = a_2 x + b_2 y + c_2 z, \quad w = z.$$

Then,

$$(a_1 x + b_1 y + c_1 z)(a_2 x + b_2 y + c_2 z) = uv.$$

∎

**Problem 7**

Consider the crossing lines $(a_2x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0$, with

$$\det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \neq 0.$$

Find a projective change of coordinates from $xyz$-space to $uvw$-space so that the crossing lines become

$$uv = 0.$$

*Proof.* Consider

$$u = a_1x + b_1y + c_1z, \quad v = a_2x + b_2y + c_2z, \quad w = z.$$

Then,

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = uv.$$

∎

**Problem 8**

Show that there is a projective change of coordinates from $xyz$-space to $uvw$-space so that the double lines $(ax + by + cz)^2 = 0$ becomes the double line

$$u^2 = 0.$$

*Proof.* Consider

$$u = ax + by + cz, \quad v = ax + by + cz, \quad w = z.$$

Then,

$$(ax + by + cz)(ax + by + cz) = u^2.$$

∎

**Problem 9**

Argue that there are three distinct classes of conics in $\mathbb{P}^2$.

*Proof.* There are three cases. If $f(x, y, z)$ cannot be factored then it represents an ellipse, circle, parabola, or hyperbola, which are all equivalent in $\mathbb{P}^2$. If $f(x, y, z)$ can be factored into two distinct linear factors then it represents two crossing lines. If $f(x, y, z)$ can be factored into two identical linear factors then it represents a double line. ∎

## 1.10 Tangents and Singular Points

**Problem 1**

Explain why if both $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$, then the tangent line is not well-defined at $(a, b)$.

*Proof.* Suppose both $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$. The equation of the tangent line is

$$\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) = 0.$$

Substituting we find

$$0(x-a)+0(y-b)=0,$$

which is true for all $(x,y)$. ∎

**Problem 2**

Show that the curve

$$C=\{(x,y)\in\mathbb{C}^2 \mid x^2+y^2-1=0\},$$

is smooth.

*Proof.* Let $(a,b)$ be arbitrary points on $C$. Suppose

$$\frac{\partial f}{\partial x}(a,b)=2a=0,$$

and

$$\frac{\partial f}{\partial y}(a,b)=2b=0.$$

Then $a=b=0$ but $x^2+y^2-1=-1\neq 0$ which is a contradiction. Thus $C$ is smooth. ∎

**Problem 3**

Show that the pair of crossing lines

$$C=\{(x,y)\in\mathbb{C}^2 \mid (x+y-1)(x-y-1)=0\}$$

has exactly one singular point. Give a geometric interpretation of this singular point.

*Proof.* Expanding we find

$$(x+y-1)(x-y-1)=x^2-2x-y^2+1=0.$$

Suppose

$$\frac{\partial f}{\partial x}(a,b)=2a-2=0,$$

and

$$\frac{\partial f}{\partial y}(a,b)=-2b=0.$$

Then $a=1$ and $b=0$. Thus $C$ is singular. This singular point is at the point of intersetion. ∎

**Problem 4**

Show that every point on the double line

$$C=\{(x,y)\in\mathbb{C}^2 \mid (2x+3y-4)^2=0\},$$

is singular.

*Proof.* Expanding we find

$$(2x+3y-4)^2=4x^2+12xy-16x+9y^2-24y+16=0.$$

Suppose

$$\frac{\partial f}{\partial x}(a,b) = 8a + 12b - 16 = 4(2a + 3b - 4) = 0,$$

and

$$\frac{\partial f}{\partial y}(a,b) = 12a + 18b - 24 = 6(2a + 3b - 4) = 0.$$

Thus all $(a,b)$ vanish and therefore every point on the double line $C$ is singular. ∎

**Problem 5**

Show that the curve

$$C = \{(x : y : z) \in \mathbb{P}^2 \mid x^2 + y^2 - z^2 = 0\}$$

is smooth.

*Proof.* Let $(a : b : c) \in C$. Suppose

$$\frac{\partial f}{\partial x}(a,b,c) = 2a = 0,$$

$$\frac{\partial f}{\partial y}(a,b,c) = 2b = 0,$$

and

$$\frac{\partial f}{\partial z}(a,b,c) = -2c = 0.$$

Then $a = b = c = 0$ which is not possible in projective space. ∎

**Problem 6**

Show that the pair of crossing lines

$$C = \{(x : y : z) \in \mathbb{P}^2 \mid (x + y - z)(x - y - z) = 0\},$$

has exactly one point.

*Proof.* Let $(a : b : c) \in C$. Suppose

$$\frac{\partial f}{\partial x}(a,b,c) = 2a - 2c = 0,$$

$$\frac{\partial f}{\partial y}(a,b,c) = -2b = 0,$$

and

$$\frac{\partial f}{\partial z}(a,b,c) = -2a + 2c = 0.$$

Then $b = 0$ and $a = c$. Thus $(a : b : c) = (a : 0 : a) = (1 : 0 : 1)$. ∎

**Problem 7**

Show that every point on the double line

$$C = \{(x : y : z) \in \mathbb{P}^2 \mid (2x + 3y - 4z)^2 = 0\},$$

is singular.

*Proof.* Let $(a : b : c) \in C$. Suppose
$$f(x, y, z) = (2x + 3y - 4z)^2.$$
Then
$$\frac{\partial f}{\partial x}(a, b, c) = 2(2a + 3b - 4c) \cdot 2 = 4(2a + 3b - 4c),$$
$$\frac{\partial f}{\partial y}(a, b, c) = 2(2a + 3b - 4c) \cdot 3 = 6(2a + 3b - 4c),$$
$$\frac{\partial f}{\partial z}(a, b, c) = 2(2a + 3b - 4c) \cdot (-4) = -8(2a + 3b - 4c).$$
But $(a : b : c) \in C$ thus $2a + 3b - 4c = 0$. So all partial derivatives vanish on $C$. ∎

**Problem 8**

For
$$f(x, y, z) = x^2 + 3xy + 5xz + y^2 - 7yz + 8z^2,$$
show that
$$2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}.$$

*Proof.* For all $(a, b, c)$ we have
$$2f(a, b, c) = 2(a^2 + 3ab + 5ac + b^2 - 7bc + 8c^2) = 2a^2 + 6ab + 10ac + 2b^2 - 14bc + 16c^2.$$
Then
$$\frac{\partial f}{\partial x}(a, b, c) = 2a + 3b + 5c,$$
$$\frac{\partial f}{\partial y}(a, b, c) = 3a + 2b - 7c,$$
$$\frac{\partial f}{\partial z}(a, b, c) = 5a - 7b + 16c.$$
Then
$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z} = a(2a + 3b + 5c) + b(3a + 2b - 7c) + c(5a - 7b + 16c),$$
$$= 2a^2 + 3ab + 5ac + 3ab + 2b^2 - 7bc + 5ac - 7bc + 16c^2 = 2a^2 + 6ab + 10ac + 2b^2 - 14bc + 16c^2.$$
∎

**Problem 9**

For
$$f(x, y, z) = ax^2 + bxy + cxz + dy^2 + eyz + hz^2$$
show that
$$2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}.$$

*Proof.* Notice
$$\frac{\partial f}{\partial x} = 2ax + by + cz, \quad \frac{\partial f}{\partial y} = bx + 2dy + ez, \quad \frac{\partial f}{\partial z} = cx + ey + 2hz.$$

Then

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} &= x(2ax + by + cz) + y(bx + 2dy + ez) + z(cx + ey + 2hz) \\ &= 2ax^2 + bxy + cxz + bxy + 2dy^2 + eyz + cxz + eyz + 2hz^2 \\ &= 2ax^2 + 2bxy + 2cxz + 2dy^2 + 2eyz + 2hz^2 \\ &= 2f(x, y, z), \end{aligned}$$

∎

**Problem 10**

Let $f(x, y, z)$ be a homogeneous polynomial of degree $n$. Show that

$$nf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}.$$

*Proof.* Let

$$f(x, y, z) = \sum_{i+j+k=n} c_{ijk} x^i y^j z^k$$

Then

$$\frac{\partial f}{\partial x} = \sum_{i+j+k=n} i\, c_{ijk} x^{i-1} y^j z^k, \quad \frac{\partial f}{\partial y} = \sum_{i+j+k=n} j\, c_{ijk} x^i y^{j-1} z^k, \quad \frac{\partial f}{\partial z} = \sum_{i+j+k=n} k\, c_{ijk} x^i y^j z^{k-1}.$$

Then

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} &= \sum_{i+j+k=n} (i + j + k) c_{ijk} x^i y^j z^k \\ &= \sum_{i+j+k=n} n\, c_{ijk} x^i y^j z^k \\ &= nf(x, y, z). \end{aligned}$$

∎

**Problem 11**

Use Exercise 1.10.10 to show that if $p = (a : b : c)$ satisfies

$$\frac{\partial f}{\partial x}(a, b, c) = \frac{\partial f}{\partial y}(a, b, c) = \frac{\partial f}{\partial z}(a, b, c) = 0,$$

then $p \in V(f)$.

*Proof.* Suppose $f$ has degree $n \neq 0$. Suppose $p = (a : b : c)$ such that

$$\frac{\partial f}{\partial x}(a, b, c) = \frac{\partial f}{\partial y}(a, b, c) = \frac{\partial f}{\partial z}(a, b, c) = 0.$$

Then

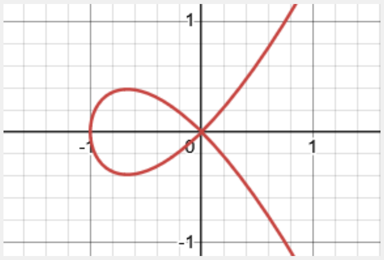
$$nf(a, b, c) = a\frac{\partial f}{\partial x}(a, b, c) + b\frac{\partial f}{\partial y}(a, b, c) + c\frac{\partial f}{\partial z}(a, b, c) = 0.$$

Thus $f(a, b, c) = 0$ so $(a : b : c) \in V(f)$. ∎

**Problem 12**

Graph the curve
$$f(x, y) = x^3 + x^2 - y^2 = 0,$$
in the real plane $\mathbb{R}^2$. What is happening at the origin $(0, 0)$? Find the singular points.

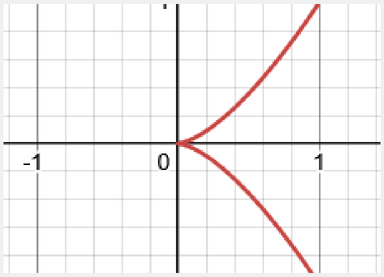
**Solution:** The curve intersects itself at the origin, which is a singular point.




**Problem 13**

Graph the curve
$$f(x, y) = x^3 - y^2 = 0,$$
in the real plane $\mathbb{R}^2$. What is happening at the origin $(0, 0)$? Find the singular points.

**Solution:** The curve has no derivative at the origin, which is a singular point.




**Problem 14**

Suppose that
$$f_1(a, b) = 0 \text{ and } f_2(a, b) = 0,$$
for a point $(a, b) \in \mathbb{C}^2$. Show that $(a, b)$ is a singular point on $V(f)$, where $f = f_1 f_2$.

*Proof.* Notice
$$\frac{\partial f}{\partial x} = f_1 \frac{\partial f_2}{\partial x} + f_2 \frac{\partial f_1}{\partial x}, \quad \frac{\partial f}{\partial y} = f_1 \frac{\partial f_2}{\partial y} + f_2 \frac{\partial f_1}{\partial y}.$$
Then
$$\frac{\partial f}{\partial x}(a, b) = f_1(a, b) \frac{\partial f_2}{\partial x}(a, b) + f_2(a, b) \frac{\partial f_1}{\partial x}(a, b) = 0,$$
$$\frac{\partial f}{\partial y}(a, b) = f_1(a, b) \frac{\partial f_2}{\partial y}(a, b) + f_2(a, b) \frac{\partial f_1}{\partial y}(a, b) = 0.$$
Therefore $(a, b)$ is a singular point of $V(f)$. ∎

**Problem 16**

Consider the curve
$$C = \{(u, v, w) \in \mathbb{P}^2 \mid u^2 - v^2 - w^2 = 0\}.$$
Suppose we have the projective change of coordinates given by
$$u = x + y,$$
$$v = x - y,$$
$$w = z.$$
Show that $C$ corresponds to the curve
$$\tilde{C} = \{(x, y, z) \in \mathbb{P}^2 \mid 4xy - z^2 = 0\}.$$
In other words, if $f(u, v, w) = u^2 - v^2 - w^2$, then $\tilde{f}(x, y, z) = 4xy - z^2$.

*Proof.* We have
$$\begin{aligned} u^2 - v^2 - w^2 &= (x + y)^2 - (x - y)^2 - z^2 \\ &= x^2 + 2xy + y^2 - (x^2 - 2xy + y^2) - z^2 \\ &= 4xy - z^2. \end{aligned}$$
∎

**Problem 17**

Suppose we have the projective change of coordinates given by
$$u = x + y,$$
$$v = x - y,$$
$$w = x + y + z.$$
If $f(u, v, w) = u^2 + uw + v^2 + vw$, find $\tilde{f}(x, y, z)$.

*Proof.* We have
$$\begin{aligned} u^2 + uw + v^2 + vw &= (x + y)^2 + (x + y)(x + y + z) + (x - y)^2 + (x - y)(x + y + z) \\ &= (x^2 + 2xy + y^2) + (x^2 + 2xy + y^2 + (x + y)z) + (x^2 - 2xy + y^2) + (x^2 - 2xy + y^2 + (x - y)z) \\ &= 4x^2 + 4y^2 + 2xz + 2yz \\ &= 2(2x^2 + 2y^2 + (x + y)z). \end{aligned}$$
∎

**Problem 18**

For a general projective change of coordinates given by
$$u = a_{11}x + a_{12}y + a_{13}z,$$
$$v = a_{21}x + a_{22}y + a_{23}z,$$
$$w = a_{31}x + a_{32}y + a_{33}z,$$

and a polynomial $f(u,v,w)$, describe how to find the corresponding $\tilde{f}(x,y,z)$.

**Solution:** Substitute $u, v, w$ which are written in terms of $x, y, z$ into $f(u,v,w)$. Thus,

$$\tilde{f}(x,y,z) = f\big(a_{11}x + a_{12}y + a_{13}z,\ a_{21}x + a_{22}y + a_{23}z,\ a_{31}x + a_{32}y + a_{33}z\big).$$

**Problem 19**

Let

$$u = a_{11}x + a_{12}y + a_{13}z,$$
$$v = a_{21}x + a_{22}y + a_{23}z,$$
$$w = a_{31}x + a_{32}y + a_{33}z,$$

be a projective change of coordinates. Show that $(u_0 : v_0 : w_0)$ is a singular point of the curve $C = \{(u : v : w) : f(u,v,w) : 0\}$ if and only if the corresponding curve $\tilde{C} = \{(x : y : z) \mid \tilde{f}(x,y,z) = 0\}$.

*Proof.* Suppose $(u_0 : v_0 : w_0)$ is a singular point of $C$. Then

$$f(u_0,v_0,w_0) = 0, \quad \frac{\partial f}{\partial u}(u_0,v_0,w_0) = 0, \quad \frac{\partial f}{\partial v}(u_0,v_0,w_0) = 0, \quad \frac{\partial f}{\partial w}(u_0,v_0,w_0) = 0.$$

Let $(x_0 : y_0 : z_0)$ be the point under the linear change. Then $\tilde{f}(x_0,y_0,z_0) = f(u_0,v_0,w_0) = 0$ so

$$\begin{aligned}
\frac{\partial \tilde{f}}{\partial x}(x_0,y_0,z_0) &= a_{11}\frac{\partial f}{\partial u}(u_0,v_0,w_0) + a_{21}\frac{\partial f}{\partial v}(u_0,v_0,w_0) + a_{31}\frac{\partial f}{\partial w}(u_0,v_0,w_0) = 0,\\
\frac{\partial \tilde{f}}{\partial y}(x_0,y_0,z_0) &= a_{12}\frac{\partial f}{\partial u}(u_0,v_0,w_0) + a_{22}\frac{\partial f}{\partial v}(u_0,v_0,w_0) + a_{32}\frac{\partial f}{\partial w}(u_0,v_0,w_0) = 0,\\
\frac{\partial \tilde{f}}{\partial z}(x_0,y_0,z_0) &= a_{13}\frac{\partial f}{\partial u}(u_0,v_0,w_0) + a_{23}\frac{\partial f}{\partial v}(u_0,v_0,w_0) + a_{33}\frac{\partial f}{\partial w}(u_0,v_0,w_0) = 0.
\end{aligned}$$

Thus $(x_0 : y_0 : z_0)$ is singular in $\tilde{C}$.

Conversely, if $(x_0 : y_0 : z_0)$ is singular on $\tilde{C}$ a symmetrical argument shows that $(u_0 : v_0 : w_0)$ is singular in $C$. ∎

**Problem 20**

Use the previous exercise to prove Theorem 1.10.15.

*Proof.* By Problem 19 if $(u_0 : v_0 : w_0)$ is a singular point of a curve

$$C = \{(u : v : w) \mid f(u,v,w) = 0\},$$

and $(x_0 : y_0 : z_0)$ is the corresponding point under a projective change of coordinates then $(x_0 : y_0 : z_0)$ is singular on the curve

$$\tilde{C} = \{(x : y : z) \mid \tilde{f}(x,y,z) = 0\}.$$

Conversely, any singular point of $\tilde{C}$ corresponds to a singular point of $C$. ∎

## 1.11 Conics via Linear Algebra

**Problem 1**

Write the following conics in the form

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

That is, find a symmetrix matrix $A$ for each quadratic equation.

1. $x^2 + y^2 + z^2 = 0$.
2. $x^2 + y^2 - z^2 = 0$.
3. $x^2 - y^2 = 0$.
4. $x^2 + 2xy + y^2 + 3xz + z^2 = 0$.

**Solution (1)**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution (2)**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

**Solution (3)**

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution (4)**

$$A = \begin{bmatrix} 1 & 1 & \frac{3}{2} \\ 1 & 1 & 0 \\ \frac{3}{2} & 0 & 1 \end{bmatrix}$$

**Problem 2**

Show that any conic

$$f(x, y, z) = ax^2 + bxy + cy^2 + dxz + eyz + hz^2,$$

can be written as

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

*Proof.*

$$A = \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & h \end{bmatrix}.$$

∎

**Problem 3**

Let $C$ be a $3 \times 3$ matrix and let $X$ be a $3 \times 1$ matrix. Show that $(CX)^T = X^T C^T$.

*Proof.* Let

$$C = (c_{ij}), \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then

$$\begin{aligned}(CX)^T &= \begin{bmatrix} \sum_{k=1}^3 c_{1k}x_k \\ \sum_{k=1}^3 c_{2k}x_k \\ \sum_{k=1}^3 c_{3k}x_k \end{bmatrix}^T \\ &= \begin{bmatrix} \sum_{k=1}^3 x_k c_{k1} & \sum_{k=1}^3 x_k c_{k2} & \sum_{k=1}^3 x_k c_{k3} \end{bmatrix} \\ &= X^T C^T.\end{aligned}$$

∎

**Problem 4**

Let $M$ be a projective change of coordinates

$$\begin{bmatrix} u \\ v \\ m \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and suppose

$$f(u,v,w) = \begin{bmatrix} u & v & w \end{bmatrix} A \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \tilde{f}(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} B \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Show that

$$B = M^T A M.$$

*Proof.* We have

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}^T = \left( M \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T M^T.$$

But then substituting for $u, v, w$ we see

$$f(u,v,w) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T M^T A M \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Since

$$\tilde{f}(x,y,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T B \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

it follows that $B = M^T A M$. ∎

**Problem 5**

Given a $3 \times 3$ matrix $A$, show that $A$ has exactly three eigenvalues, counting multiplicity.

*Proof.* We want to find non-zero vectors $v$ such that

$$Av = \lambda v.$$

Now let

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ such that } x, y, \text{ or } z \text{ is non-zero.}$$

Then

$$AV = \lambda V \iff (A - \lambda I)V = 0.$$

Since $V \neq 0$ we have

$$\det(A - \lambda I) = 0.$$

The roots of this third degree polynomial are the three eigenvalues, counting multiplicity. ∎

**Problem 6**

1. Let $A$ and $B$ be two symmetric matrices, neither of which has a zero eigenvalue. Show there is an invertible $3 \times 3$ matrix $C$ such that
$$A = C^T BC.$$
2. Let $A$ and $B$ be two symmetric matrices, each of which has exactly one zero eigenvalue (with the other two eigenvalues being non-zero). Show that there is an invertible $3 \times 3$ matric $C$ such that
$$A = C^T BC.$$
3. Now let $A$ and $B$ be two symmetric matrices, each of which has a zero eigenvalue with multiplicity two (and hence the remaining eigenvalues being non-zero). Show that there is an invertible $3 \times 3$ matrix such that
$$A = C^T BC.$$

**Problem 7**

1. Show that the $3\times3$ matrix associated to the ellipse $V(x^2+y^2-z^2)$ has three non-zero eigenvalues.
2. Show that the $3 \times 3$ matrix associated to the two crossing lines $V(xy)$ has one zero eigenvalue and two non-zero eigenvalues.
3. Finally, show that the $3\times3$ matrix associated to the double line $V((x-y)^2)$ has a zero eigenvalue of multiplicity two and a non-zero eigenvalue.

*Proof.* We must solve

$$\det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix} = 0,$$

for $\lambda$. Computing the determinant we find

$$(1-\lambda)(1-\lambda)(-1-\lambda) = (1-\lambda)^2(-1-\lambda).$$

Thus $\lambda = 1$ and $\lambda = -1$. Substituting in $\lambda$ we solve for $v$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix} v = 0.$$

For $\lambda = 1$, this gives

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \implies z = 0, \quad v = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

For $\lambda = -1$, this gives

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \implies x = 0,\ y = 0, \quad v = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}.$$

∎

*Proof.* We must solve

$$\det\begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

for $\lambda$. Computing the determinant we find

$$(-\lambda)(1-\lambda)(1-\lambda) = -\lambda(1-\lambda)^2.$$

Thus $\lambda = 0$ and $\lambda = 1$. Substituting in $\lambda$, we solve for $v$

$$\begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} v = 0.$$

For $\lambda = 0$, this gives

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \implies y = 0,\ z = 0, \quad v = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda = 1$, this gives

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \implies x = 0, \quad v = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}.$$

∎

*Proof.* We must solve

$$\det\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = 0$$

for $\lambda$. Computing the determinant we find

$$(-\lambda)((1-\lambda)^2 - (-1)^2) = (-\lambda)(\lambda^2 - 2\lambda) = -\lambda^2(\lambda - 2).$$

Thus $\lambda = 0$ and $\lambda = 2$. Substituting in $\lambda$, we solve for $v$

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} v = 0.$$

For $\lambda = 0$, this gives

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \implies x = y, \quad v = \begin{bmatrix} x \\ x \\ z \end{bmatrix}.$$

For $\lambda = 2$, this gives

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \implies x = -y,\ z = 0, \quad v = \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix}.$$

∎

Problem 8

Based on the material of this section, give another proof that under projective changes of coordinates all ellipses, hyperbolas, and parabolas are the same, all crossing lines are the same, and all double lines are the same.

*Proof.* Let $A$ be the $3 \times 3$ symmetric matrix for a conic or line. By Problem 6 we obtain a projective change of coordinates

$$A' = C^T AC.$$

where $A'$ represents the standard form for an ellipse, crossing line, or double line. By Problem 7 the eigenvalues are preserved up to scaling. Thus under projective changes of coordinates, ellipses, hyperbolas, and parabolas are equivalent, crossing lines are equivalent, and double lines are equivalent. ∎

Problem 9

Explain why we need to only consider only the second order terms.

**Solution:** In $\mathbb{P}^2$ all equations are homogeneous of degree 2. Thus linear and constant terms become second order terms involving $z$.

Problem 10

Find the discriminant of each of the following conics.

1. $9x^2 + 4y^2 = 1$.
2. $9x^2 - 4y^2 = 1$.
3. $9x^2 - y = 0$.

**Solution (1):**

$$\triangle = -4 \det \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} = -4 \det \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} = -4(36) = -144$$

**Solution (2):**

$$\triangle = -4 \det \begin{bmatrix} 9 & 0 \\ 0 & -4 \end{bmatrix} = -4(-36) = 144$$

**Solution (3):**

$$\triangle = -4 \det \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix} = -4(0) = 0$$

Problem 11

Based on the previous exercise, describe the conics obtained of $\triangle = 0$, $\triangle < 0$, and $\triangle > 0$. State what the general result ought to be.

Problem 12

Consider the equation $ax^2 + bxy + cy^2 = 0$, where all coefficients are real numbers. Dehomogenize the equation by setting $y = 1$. Solve the resulting quadratic equation for $x$. You should see a factor involving $\triangle$ in your solution. How does $\triangle$ relate to the discriminant used in the quadratic formula?

*Proof.* Let $y = 1$, then

$$ax^2 + bxy + cy^2 = ax^2 + bx + c = 0.$$

Notice

$$\triangle = -4\det\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = -4\left(ac - \frac{b^2}{4}\right) = b^2 - 4ac.$$

Using the quadratic formula we find

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{-\triangle}}{2a},$$

If $\triangle < 0$, there are two real roots. If $\triangle = 0$, there is one real root. If $\triangle > 0$, there are no real roots. ∎

**Problem 13**

The discriminant in the quadratic formula tells us how many (real) solutions a given quadratic equation in a single variable has. Classify a conic $V(f(x, y))$ based on the number of solutions to its dehomogenized quadratic equation.

*Proof.* If $\triangle = 0$ then the dehomogenized quadratic has one real solution so the conic is tangent to the line at infinity and is thus a parabola. If $\triangle > 0$ then the dehomogenized quadratic has two real solution, so the conic intersects the line at infinity twice and is thus a hyperbola. If $\triangle < 0$ then the dehomogenized quadratic has no real solutions so the conic does not intersect the line at infinity and is thus an ellipse. ∎

## 1.12 Duality

**Problem 1**

Show that the line associated to $a_1 = 1, b_1 = 2, c_1 = 3$ is the same line as that associated to $a_2 = -2, b_2 = -4, c_2 = -6$.

*Proof.* We have

$$L_1 = \{(x : y : z) \in \mathbb{P}^2 \mid x + 2y + 3z = 0\}$$

and

$$L_2 = \{(x : y : z) \in \mathbb{P}^2 \mid -2x - 4y - 6z = 0\}.$$

But

$$(-2, -4, -6) = \lambda(1, 2, 3) \quad (\lambda = -2),$$

Thus

$$L_1 = L_2.$$

∎

**Problem 2**

Show that the line associated to $a_1, b_1, c_1$ is the same as the line associated to $a_2, b_2, c_2$ if and only if there is a non-zero constant $\lambda \in \mathbb{C}$ such that $a_1 = \lambda a_2, b_1 = \lambda b_2, c_1 = \lambda c_2$.

*Proof.* Suppose the line associated to $a_1, b_1, c_1$ is the same as the line associated to $a_2, b_2, c_2$. Then

$$L_1 = \{(x, y, z) \in \mathbb{P}^2 \mid a_1 x + b_1 y + c_1 z = 0\}$$

and

$$L_2 = \{(x, y, z) \in \mathbb{P}^2 \mid a_2 x + b_2 y + c_2 z = 0\}.$$

There exists a nonzero constant $\lambda \in \mathbb{C}$ such that

$$a_1 = \lambda a_2, \quad b_1 = \lambda b_2, \quad c_1 = \lambda c_2.$$

Conversely suppose that there is a non-zero constant $\lambda \in \mathbb{C}$ such that $a_1 = \lambda a_2, b_1 = \lambda b_2, c_1 = \lambda c_2$. Then

$$a_1 x + b_1 y + c_1 z = \lambda(a_2 x + b_2 y + c_2 z).$$

Since $\lambda \neq 0$ the equations have the same zero set. Thus the lines are the same. ∎

**Problem 3**

Show that the set of all lines in $\mathbb{P}^2$ can be identified with $\mathbb{P}^2$ itself.

*Proof.* By Problem 2, each line can be associated with exactly one point in $\mathbb{P}^2$, and each point in $\mathbb{P}^2$ can be associated with exactly one line. Thus the set of all lines can be identified with the set of all points in $\mathbb{P}^2$. ∎

**Problem 4**

Explain in your own words why, given $(x_0 : y_0 : z_0) \in \mathbb{P}^2$, we can interpret $M_{(x_0:y_0:z_0)}$ as the set of all lines containing the point $(x_0 : y_0 : z_0)$.

**Solution:** $M_{(x_0:y_0:z_0)}$ consists of all lines for which $(x_0 : y_0 : z_0)$ is a solution, so we can interpret it as the set of all lines containing the point $(x_0 : y_0 : z_0)$.

**Problem 5**

Show that both maps $\pi_1$ and $\pi_2$ are onto.

*Proof.* Let $(a : b : c) \in \tilde{\mathbb{P}}^2$ and $(x_0 : y_0 : z_0) \in \mathbb{P}^2$. Then

$$\pi_1((a : b : c), (x_0, y_0, z_0)) = (a : b : c),$$

and

$$\pi_2((a : b : c), (x_0, y_0, z_0)) = (x_0 : y_0 : z_0).$$

Thus $\pi_1, \pi_2$ are onto. ∎

**Problem 6**

Given a point $(a : b : c) \in \tilde{\mathbb{P}}^2$, consider the set

$$\pi_1^{-1}(a : b : c) = \{((a : b : c), (x_0 : y_0 : c_0)) \in \textstyle\sum\}.$$

Show that the set $\pi_2(\pi_1^{-1}(a : b : c))$ is identical to a set in $\mathbb{P}^2$ that we defined near the beggining of this section.

*Proof.* Let $(a : b : c) \in \tilde{\mathbb{P}}^2$. By definition,

$$\pi_1^{-1}(a : b : c) = \{((a : b : c), (x_0 : y_0 : z_0)) \in \textstyle\sum\}.$$

Applying $\pi_2$

$$\pi_2(\pi_1^{-1}(a : b : c)) = \{(x_0 : y_0 : z_0) \in \mathbb{P}^2 \mid ((a : b : c), (x_0 : y_0 : z_0)) \in \textstyle\sum\}.$$

This is the set of all lines in $\mathbb{P}^2$ passing through $(a : b : c)$. ∎

**Problem 7**

Given a point $(x_0 : y_0 : z_0) \in \mathbb{P}^2$, consider the set

$$\pi_2^{-1}(x_0 : y_0 : z_0) = \{((a : b : c), (x_0 : y_0 : z_0)) \in \sum\}.$$

Show that the set $\pi_1(\pi_2^{-1}(x_0 : y_0 : z_0))$ is identical to a set in $\tilde{\mathbb{P}}^2$ that we defined near the beggining of this section.

*Proof.* Let $(x_0 : y_0 : z_0) \in \mathbb{P}^2$. By definition,

$$\pi_2^{-1}(x_0 : y_0 : z_0) = \{((a : b : c), (x_0 : y_0 : z_0)) \in \sum\}.$$

Applying $\pi_1$

$$\pi_1(\pi_2^{-1}(x_0 : y_0 : z_0)) = \{(a : b : c) \in \tilde{\mathbb{P}}^2 \mid ((a : b : c), (x_0 : y_0 : z_0)) \in \sum\}.$$

This is the set of all points in $\tilde{\mathbb{P}}^2$ that lie on the line $(x_0 : y_0 : z_0)$. ∎

**Problem 8**

Let $(1 : 2 : 3), (2 : 5 : 1) \in \tilde{\mathbb{P}}^2$. Find

$$\pi_2(\pi_1^{-1}(1 : 2 : 3)) \cap \pi_2(\pi_1^{-1}(2 : 5 : 1)).$$

Explain why this is just a fancy way for finding the point of intersection of the two lines

$$x + 2y + 3z = 0,$$

$$2x + 5y + z = 0.$$

*Proof.* We have

$$\pi_2(\pi_1^{-1}(1 : 2 : 3)) = \pi_2(\{((1 : 2 : 3), (x_0 : y_0 : z_0)) \in \sum\}),$$

and

$$\pi_2(\pi_1^{-1}(2 : 5 : 1)) = \pi_2(\{((2 : 5 : 1), (x_1 : y_1 : z_1)) \in \sum\}).$$

Then

$$\pi_2(\pi_1^{-1}(1 : 2 : 3)) \cap \pi_2(\pi_1^{-1}(2 : 5 : 1))$$

is the set of all lines passing through both $(1 : 2 : 3)$ and $(2 : 5 : 1)$. In projective space there is only one line through two distinct points so this intersection is a single line. This line corresponds to the point of intersection of the two lines

$$x + 2y + 3z = 0 \quad \text{and} \quad 2x + 5y + z = 0.$$

∎

**Problem 9**

Let $(1 : 2 : 3), (2 : 5 : 1) \in \mathbb{P}^2$. Find

$$\pi_1(\pi_2^{-1}(1 : 2 : 3)) \cap \pi_1(\pi_2^{-1}(2 : 5 : 1)).$$

Explain why this is just a fancy way for finding the unique line containing the points $(1 : 2 : 3), (2 : 5 : 1)$.

*Proof.* We have

$$\pi_2(\pi_1^{-1}(1:2:3)) = \pi_2(\{((1:2:3),(x_0:y_0:z_0)) \in \textstyle\sum\}),$$

and

$$\pi_2(\pi_1^{-1}(2:5:1)) = \pi_2(\{((2:5:1),(x_1:y_1:z_1)) \in \textstyle\sum\}).$$

Then

$$\pi_2(\pi_1^{-1}(1:2:3)) \cap \pi_2(\pi_1^{-1}(2:5:1))$$

is the set of all lines passing through both $(1:2:3)$ and $(2:5:1)$. In projective space there is only one line through two points. This line is the unique line passing through $(1:2:3)$ and $(2:5:1)$. ∎

**Problem 10**

Use the duality principle to find the corresponding theorem to :
**Theorem 1.** *Any two distinct points in $\mathbb{P}^2$ determine a unique line.*

**Theorem 2.** *Any two distinct lines in $\mathbb{P}^2$ determine a unique point.*

**Problem 12**

Given $(x_0, y_0, z_0, w_0), (x_1, y_1, z_1, w_1) \in \mathbb{C}^4 - \{(0,0,0,0)\}$, define

$$(x_0, y_0, z_0, w_0) \sim (x_1, y_1, z_1, w_1),$$

if there exists a non-zero $\lambda$ such that

$$x_0 = \lambda x_1, y_0 = \lambda y_1, z_0 = \lambda z_1, w_0 = \lambda w_1.$$

Define

$$\mathbb{P}^3 = \mathbb{C}^4 \setminus \{(0,0,0,0)\} \big/ \sim .$$

Show that the set of all planes in $\mathbb{P}^3$ can be identified with another copy of $\mathbb{P}^3$. Explain how the duality principle can be used to link the fact that three non-colinear points define a unique plane to the fact three planes with linearly dependent normal vectors intersect in a unique point.

*Proof.* A plane in $\mathbb{P}^3$ is defined by

$$ax + by + cz + dw = 0,$$

where $(a,b,c,d) \in \mathbb{C}^4 \setminus \{(0,0,0,0)\}$. If $(a,b,c,d) \sim (\lambda a, \lambda b, \lambda c, \lambda d)$, they define the same plane. Thus the coefficients $(a,b,c,d)$ determine exactly one point in $\mathbb{P}^3$. Conversely, each point $(a,b,c,d) \in \mathbb{C}^4 \setminus \{(0,0,0,0)\}$ determines the plane

$$ax + by + cz + dw = 0.$$

Therefore, the set of all planes in $\mathbb{P}^3$ can be identified with another copy of $\mathbb{P}^3$. ∎

**Solution:** Three non-colinear points determine a unique plane, while dually, three planes with linearly independent normal vectors intersect in a unique point.

**Problem 13**

Show for any $p = (x_0, y_0, z_0) \in C$, we have

$$T_p(C) = \left\{ (x:y:z) \in \mathbb{P}^2 \;\middle|\; (x - x_0)\frac{\partial f}{\partial x}(p) + (y - y_0)\frac{\partial f}{\partial y}(p) + (z - z_0)\frac{\partial f}{\partial z}(p) = 0 \right\}.$$

*Proof.* We have

$$\begin{aligned}(x-x_0)\frac{\partial f}{\partial x}(p)+(y-y_0)\frac{\partial f}{\partial y}(p)+(z-z_0)\frac{\partial f}{\partial z}(p) &= x\frac{\partial f}{\partial x}(p)-x_0\frac{\partial f}{\partial x}(p)\\ &\quad+y\frac{\partial f}{\partial y}(p)-y_0\frac{\partial f}{\partial y}(p)\\ &\quad+z\frac{\partial f}{\partial z}(p)-z_0\frac{\partial f}{\partial z}(p)\\ &= x\frac{\partial f}{\partial x}(p)+y\frac{\partial f}{\partial y}(p)+z\frac{\partial f}{\partial z}(p)\\ &\quad-\left(x_0\frac{\partial f}{\partial x}(p)+y_0\frac{\partial f}{\partial y}(p)+z_0\frac{\partial f}{\partial z}(p)\right)\end{aligned}$$

Now, we know

$$x_0\frac{\partial f}{\partial x}(p)+y_0\frac{\partial f}{\partial y}(p)+z_0\frac{\partial f}{\partial z}(p)=0.$$

So our equation reduces to

$$x\frac{\partial f}{\partial x}(p)+y\frac{\partial f}{\partial y}(p)+z\frac{\partial f}{\partial z}(p)=0,$$

which is the tangent line equation at $p$. ∎

**Problem 14**

For $f(x,y,z)=x^2+y^2-z^2$, let $C=V(f(x,y,z))$. Show for any $(x_0:y_0:z_0)\in C$ that

$$\mathscr{D}(x_0:y_0:z_0)=(2x_0:2y_0:-2z_0).$$

Show that in this case the dual curve $\tilde{C}$ is the same as the original $C$.

*Proof.* Let $(x_0:y_0:z_0)$ be an arbitrary point in $C$. Then

$$\frac{\partial f}{\partial x}(x_0,y_0,z_0)=2x_0,\quad \frac{\partial f}{\partial y}(x_0,y_0,z_0)=2y_0,\quad \frac{\partial f}{\partial z}(x_0,y_0,z_0)=-2z_0.$$

Then

$$\mathscr{D}(x_0:y_0:z_0)=\left(\frac{\partial f}{\partial x}(p):\frac{\partial f}{\partial y}(p):\frac{\partial f}{\partial z}(p)\right)=(2x_0:2y_0:-2z_0).$$

Then

$$x^2+y^2-z^2=(2x_0)^2+(2y_0)^2+(-2z_0)^2=4(x_0^2+y_0^2-z_0^2)=0.$$

∎

**Problem 15**

Consider $f(x,y,z)=x^2-yz=0$. Then for any $(x_0:y_0:z_0)\in C$, where $C=V(f)$, show that

$$\mathscr{D}(x_0,y_0,z_0)=(2x_0:-z_0:-y_0).$$

Show that the image is in $\mathbb{P}^2$ by showing that $(2x_0,-z_0,-y_0)\neq(0,0,0)$. Letting $(u:v:w)=(2x:-z:-y)$, show that $u^2-4vw=0$ defines the dual curve $\tilde{C}$. Note that here $\tilde{C}\neq C$.

*Proof.* Let $(x_0 : y_0 : z_0) \in C$. Then

$$\frac{\partial f}{\partial x}(p) = 2x_0, \quad \frac{\partial f}{\partial y}(p) = -z_0, \quad \frac{\partial f}{\partial z}(p) = -y_0.$$

Then

$$\mathscr{D}(x_0 : y_0 : z_0) = (2x_0 : -z_0 : -y_0).$$

since $(x_0 : y_0 : z_0) \in C$

$$x_0^2 - y_0 z_0 = 0,$$

so either $x_0, y_0, z_0$ is non-zero. Thus $(2x_0, -z_0, -y_0) \in \mathbb{P}^2$. Let $(u : v : w) = (2x : -z : -y)$. Then

$$u^2 - 4vw = (2x)^2 - 4(-z)(-y) = 4(x^2 - yz) = 0 \iff x^2 - yz = 0.$$

∎

## Problem 16

For $C = V(x^2 + 4y^2 - 9z^2)$, show that the dual curve is

$$\tilde{C} = \{(x, y, z) \in \mathbb{P}^2 \mid x^2 + \frac{1}{4}y^2 - \frac{1}{9}z^2 = 0\}.$$

*Proof.* Let $(x_0 : y_0 : z_0) \in C$. Then

$$\frac{\partial f}{\partial x}(p) = 2x_0, \quad \frac{\partial f}{\partial y}(p) = 8y_0, \quad \frac{\partial f}{\partial z}(p) = -18z_0.$$

Then

$$\mathscr{D}(x_0 : y_0 : z_0) = (2x_0 : 8y_0 : -18z_0).$$

Since $(x_0 : y_0 : z_0) \in C$ at least one coordinate is non-zero. Thus $(2x_0, 8y_0, -18z_0) \in \mathbb{P}^2$. Let $(u : v : w) = (2x : 8y : -18z)$. Then

$$u^2 + \frac{1}{4}v^2 - \frac{1}{9}w^2 = (2x)^2 + \frac{1}{4}(8y)^2 - \frac{1}{9}(-18z)^2 = 4x^2 + 16y^2 - 36z^2 = 4(x^2 + 4y^2 - 9z^2) = 0.$$

Thus

$$\tilde{C} = \{(x : y : z) \in \mathbb{P}^2 \mid x^2 + \frac{1}{4}y^2 - \frac{1}{9}z^2 = 0\}.$$

∎

## Problem 17

For $C = V(5x^2 + 2y^2 - 8z^2)$, find the dual curve.

*Proof.* Let $(x_0 : y_0 : z_0) \in C$. Then

$$\frac{\partial f}{\partial x}(p) = 10x_0, \quad \frac{\partial f}{\partial y}(p) = 4y_0, \quad \frac{\partial f}{\partial z}(p) = -16z_0.$$

Then

$$\mathscr{D}(x_0 : y_0 : z_0) = (10x_0 : 4y_0 : -16z_0).$$

Since $(x_0 : y_0 : z_0) \in C$, at least one coordinate is non-zero. Thus $(10x_0, 4y_0, -16z_0) \in \mathbb{P}^2$. Let $(u : v : w) = (10x : 4y : -16z)$. Then

$$u^2 + \frac{25}{100}v^2 - \frac{25}{256}w^2 = (10x)^2 + \frac{25}{100}(4y)^2 - \frac{25}{256}(-16z)^2 = 100x^2 + 4y^2 - 25z^2 = 25(4x^2 + \frac{4}{25}y^2 - z^2) = 25(4x^2 + \frac{4}{25}y^2 - z^2) = 0.$$

Thus

$$\tilde{C} = \{(x : y : z) \in \mathbb{P}^2 \mid 4x^2 + \tfrac{1}{25}y^2 - z^2 = 0\}.$$

∎

**Problem 18**

For a line $L = \{(x : y : z) \in \mathbb{P}^2 \mid ax + bx + cz\}$, find the dual curve. Explain why calling this set the "dual curve" might seem strange.

*Proof.* Let $(x_0 : y_0 : z_0) \in L$. Then

$$\frac{\partial f}{\partial x}(p) = a, \quad \frac{\partial f}{\partial y}(p) = b, \quad \frac{\partial f}{\partial z}(p) = c.$$

Then

$$\mathscr{D}(x_0 : y_0 : z_0) = (a : b : c).$$

Since $(a, b, c) \neq (0, 0, 0)$, we have $(a : b : c) \in \mathbb{P}^2$. Let $(u : v : w) = (a : b : c)$. Then

$$\tilde{L} = \{(u : v : w) \in \mathbb{P}^2 \mid (u : v : w) = (a : b : c)\}.$$

∎

**Solution:** It is strange because the "curve" is a point.

# 2 Cubic Curves and Elliptic Curves

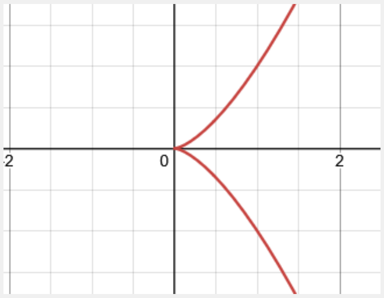
## 2.1 Cubics in $\mathbb{C}^2$

**Problem 1**

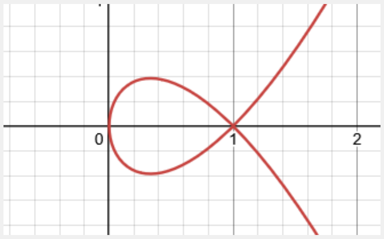
Sketch the following cubics in the real plane $\mathbb{R}^2$.

1. $y^2 = x^3$.
2. $y^2 = x(x-1)^2$.
3. $y^2 = x(x-1)(x-2)$.
4. $y^2 = x(x^2 + x + 1)$.

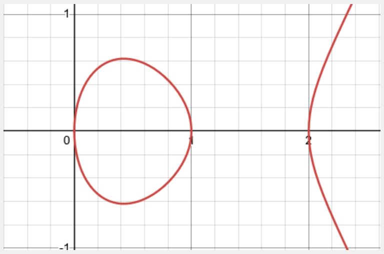
**Solution (1):**



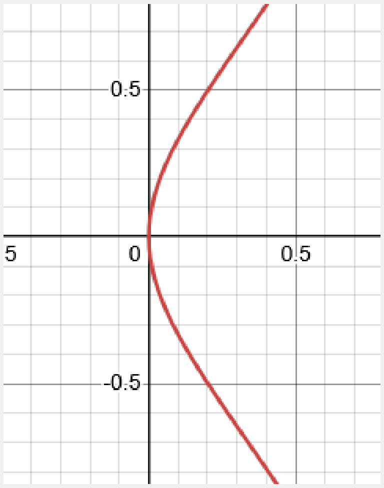

**Solution (2):**




**Solution (3):**




**Solution (4):**

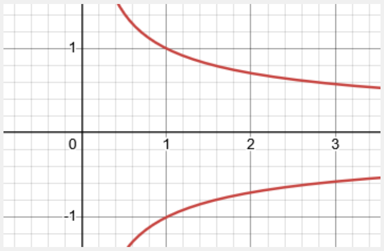



> **Problem 2**
>
> Consider the cubics in the above exercise.
> 1. Give the homogeneous form for each cubic, which extends each of the above cubics to the complex projective plane.
> 2. For each of the above cubics, dehomogenize by setting $x = 1$, and graph the resulting cubic in $\mathbb{R}^2$ with coordinates $y$ and $z$.

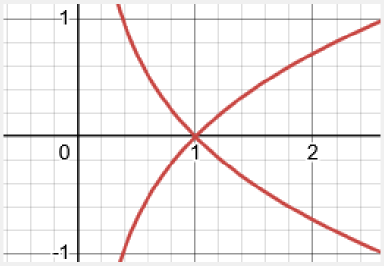
**Solution:**

1. $y^2 = x^3$ becomes $Y^2Z = X^3$.
2. $y^2 = x(x-1)^2$ becomes $Y^2Z = X(X-Z)^2$.
3. $y^2 = x(x-1)(x-2)$ becomes $Y^2Z = X(X-Z)(X-2Z)$.
4. $y^2 = x(x^2+x+1)$ becomes $Y^2Z = X(X^2+XZ+Z^2)$.

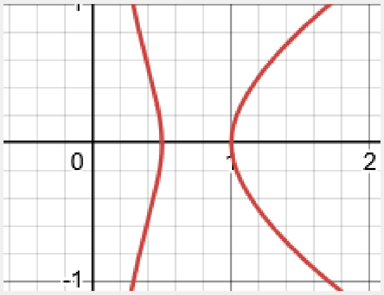
**Solution (1):**



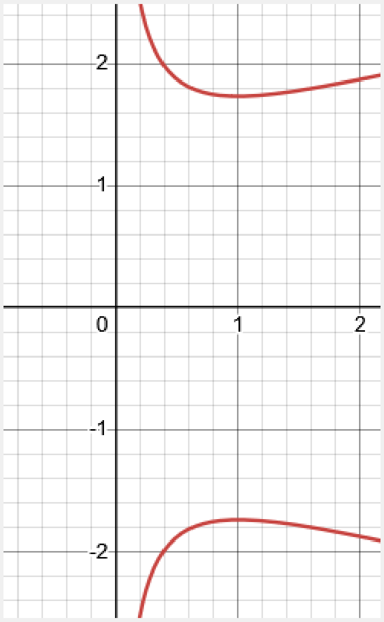

**Solution (2):**




**Solution (3):**




**Solution (4):**




> **Problem 3**
>
> Show that the following cubics are singular.
> 1. $V(xyz)$.
> 2. $V(x(x^2+y^2-z^2))$.
> 3. $V(x^3)$.

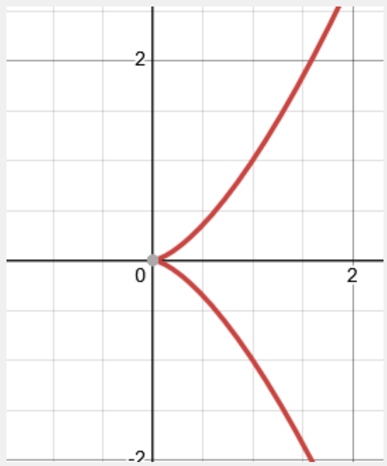
*Proof.* Here we list each function and calculate its gradient.

1. For $f(x,y,z) = xyz$
$$\nabla f = (yz, xz, xy).$$
$(1:0:0)$ is a singular point.
2. For $f(x,y,z) = x(x^2+y^2-z^2) = x^3 + xy^2 - xz^2$
$$\nabla f = \left(3x^2 + y^2 - z^2, 2xy, -2xz\right).$$
$(0:0:0)$ is a singular point.
3. For $f(x,y,z) = x^3$
$$\nabla f = (3x^2, 0, 0).$$
Any point with $x = 0$ is a singular point.

∎

> **Problem 4**
>
> Sketch the cubic $y^2 = x^3$ in the real plane $\mathbb{R}^2$. Show that the corresonding cubic $V(x^3 - y^2z)$ in $\mathbb{P}^2$ has a singular point at $(0:0:1)$. Show that this is the only singular point on this cubic.

**Solution**




*Proof.* We have

$$\frac{\partial f}{\partial x}(x^3 - y^2 z) = 3x^2, \quad \frac{\partial f}{\partial y}(x^3 - y^2 z) = -2yz, \quad \frac{\partial f}{\partial z}(x^3 - y^2 z) = -y^2.$$

Thus $x = 0$ and $y = 0$. Then $z$ is any element in $\mathbb{C} - \{0\}$. Therefore $(0 : 0 : 1) \in \mathbb{C} - \{(0, 0, 0)\}$ is a singularity. ∎

**Problem 5**

Show that the polynomial $P(x, y, z) = x^3 - y^2 z$ is irreducible, i.e., cannot be factored into two polynomials. (This is a fairly brute force high school algebra problem).

*Proof.* The only possible factorization is

$$(x + A)(x^2 + Bx + C) = x^3 + Bx^2 + Cx + Ax^2 + ABx + AC = x^3 + (A + B)x^2 + (AB + C)x + AC,$$

where $A, B, C \in P(y, z)$. Comparing to $P(x, y, z)$ we must have

$$A + B = 0, \quad AB + C = 0, \quad \text{and} \quad AC = -y^2 z.$$

Then $A = -B$ thus $AB + C = A(-A) + C = -A^2 + C = 0 \iff C = A^2$. Then

$$AC = A(A^2) = A^3 \text{ thus } A^3 = -y^2 z.$$

This is impossible since the cube of a polynomial must have all exponents divisible by 3, but in $y^2 z$ the exponents of $y$ and $z$ are not divisible by 3. Therefore $P(x, y, z)$ is irreducible. ∎

## 2.2 Inflection Points

**Problem 1**

If $(x - a)$ divides $P(x)$, show that $a$ is a root of $P(x)$.

*Proof.* We have $P(x) = (x - a)g(x)$ for some $g(x)$. Then $P(a) = (a - a)g(a) = 0 \cdot g(a) = 0$. ∎

**Problem 2**

If $a$ is a root of $P(x)$, show that $(x - a)$ divides $P(x)$.

*Proof.* Suppose $a$ is a root of $P(x)$. By the Division Algorithm we obtain $q(x)$ and $r(x)$ such that $P(x) = q(x)(x-a) + r(x)$. Then $P(a) = 0 = (a-a)q(a) + r(a) = r(a)$. Thus $r(a) = 0$. Since $r(x)$ has degree $< 1$ it is a constant so $r(x) = 0$. ∎

**Problem 3**

Suppose that $a$ is a root of multiplicity two for $P(x)$. Show that there is a polynomial $g(x)$, with $g(a) \neq 0$, such that
$$P(x) = (x-a)^2 g(x).$$

*Proof.* By definition of multiplicity two, we can write $P(x) = (x-a)^2 g(x)$ for some $g(x)$. Now suppose $g(a) = 0$ then $g(x) = (x-a)g'(x)$ for some $g'(x)$. But then $P(x) = (x-a)^2(x-a)g'(x) = (x-a)^3 g'(x)$ contradicting that $a$ has multiplicity two for $P(x)$. Therefore $g(a) \neq 0$. ∎

**Problem 4**

Suppose $a$ is a root of multiplicity two for $P(x)$. Show that $P(a) = 0$ and $P'(a) = 0$ but $P''(a) \neq 0$.

*Proof.* By definition of multiplicity two, we can write $P(x) = (x-a)^2 g(x)$ for some $g(x)$ with $g(a) \neq 0$. Clearly, $P(a) = (a-a)^2 g(a) = 0 \cdot g(a) = 0$. Then

$$P'(x) = 2(x-a)g(x) + (x-a)^2 g'(x) \text{ and } P''(x) = 2g(x) + 4(x-a)g'(x) + (x-a)^2 g''(x).$$

Then $P'(a) = 0$ and $P''(a) = 2g(a)$ where $g(a) \neq 0$ thus $P''(a) \neq 0$. ∎

**Problem 5**

Suppose that $a$ is a root of multiplicity $k$ for $P(x)$. Show there is a polynomial $g(x)$ such that

$$P(x) = (x-a)^k g(x)$$

with $g(a) \neq 0$.

*Proof.* By definition of multiplicity $k$, we can write $P(x) = (x-a)^k g(x)$ for some $g(x)$. Now suppose $g(a) = 0$ then $g(x) = (x-a)g'(x)$ for some $g'(x)$. But then $P(x) = (x-a)^k(x-a)g'(x) = (x-a)^{k+1} g'(x)$ contradicting that $a$ has multiplicity $k$ for $P(x)$. Therefore $g(a) \neq 0$. ∎

**Problem 6**

Suppose that $a$ is a root of multiplicity $k$ for $P(x)$. Show that $P(a) = P'(a) = \cdots = P^{(k-1)}(a) = 0$ but $P^k(a) \neq 0$.

*Proof.* Suppose $a$ is a root of multiplicity $k$. Then

$$P(x) = (x-a)^k g(x)$$

for some polynomial $g(x)$ with $g(a) \neq 0$. Taking the $k$-th derivative and applying the product rule repeatedly gives
$$P^{(k)}(x) = \underbrace{k!g(x)}_{\text{no factor of } (x-a)} + \underbrace{k!(x-a)g'(x)}_{\text{has factor } (x-a)} + \underbrace{\cdots}_{\text{has factor } (x-a)} + \underbrace{(x-a)^k g^{(k)}(x)}_{\text{has factor } (x-a)}.$$

Every term except the first contains a factor of $(x-a)$. Therefore, evaluating at $x=a$ gives

$$P^{(k)}(a) = \underbrace{k!g(a)}_{\neq 0} + \underbrace{0+\cdots+0}_{\text{all contain factor }(x-a)} \neq 0,$$

since $g(a) \neq 0$. Similarly, for any derivative of order $m < k$, every term contains a factor of $(x-a)$, so

$$P(a) = P'(a) = \cdots = P^{(k-1)}(a) = 0.$$

∎

**Problem 7**

Suppose that $(a:b)$ is a root of multiplicity two for $P(x,y)$. Show that

$$P(a,b) = \frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0,$$

but at least one of the second partials does not vanish at $(a:b)$.

*Proof.* Suppose $(a:b)$ is a root of multiplicity two for $P(x,y)$. Then

$$P(x,y) = (bx-ay)^2 g(x,y)$$

for some polynomial $g(x,y)$ such that $g(a,b) \neq 0$. Then

$$\frac{\partial P}{\partial x} = 2(bx-ay)\cdot b\, g(x,y) + (bx-ay)^2\frac{\partial g}{\partial x}, \quad \frac{\partial P}{\partial y} = 2(bx-ay)(-a)\, g(x,y) + (bx-ay)^2\frac{\partial g}{\partial y}.$$

Then

$$\frac{\partial P}{\partial x}(a,b) = \frac{\partial P}{\partial y}(a,b) = 0.$$

Then

$$\begin{aligned}
\frac{\partial^2 P}{\partial x^2} &= 2b^2 g(x,y) + 4b(bx-ay)\frac{\partial g}{\partial x} + (bx-ay)^2\frac{\partial^2 g}{\partial x^2},\\
\frac{\partial^2 P}{\partial x \partial y} &= -2ab g(x,y) + 2b(bx-ay)\frac{\partial g}{\partial y} - 2a(bx-ay)\frac{\partial g}{\partial x} + (bx-ay)^2\frac{\partial^2 g}{\partial x \partial y},\\
\frac{\partial^2 P}{\partial y^2} &= 2a^2 g(x,y) - 4a(bx-ay)\frac{\partial g}{\partial y} + (bx-ay)^2\frac{\partial^2 g}{\partial y^2}.
\end{aligned}$$

Plugging in $(x,y)=(a,b)$ shows

$$\frac{\partial^2 P}{\partial x^2}(a,b) = 2b^2 g(a,b), \quad \frac{\partial^2 P}{\partial x \partial y}(a,b) = -2ab g(a,b), \quad \frac{\partial^2 P}{\partial y^2}(a,b) = 2a^2 g(a,b).$$

Since $g(a,b) \neq 0$ and $(a,b) \neq (0,0)$, some second partial derivative is nonzero. ∎

**Problem 8**

Suppose $(a:b)$ is a root of multiplicity $k$ for $P(x,y)$. Show that

$$P(a,b) = \frac{\partial P}{\partial x}(a,b) = \frac{\partial P}{\partial y}(a,b) = \cdots = \frac{\partial^{k-1} P}{\partial x^i \partial y^j}(a,b) = 0,$$

where $i + j = k - 1$ but

$$\frac{\partial^k P}{\partial x^i \partial y^j}(a,b) \neq 0,$$

for at least one pair $i + j = k$.

*Proof.* Suppose $(a : b)$ is a root of multiplicity $k$. Then

$$P(x,y) = (bx - ay)^k g(x,y)$$

for some polynomial $g(x,y)$ with $g(a,b) \neq 0$.

Taking a partial derivative of total order $k$ and applying the product rule repeatedly gives

$$\frac{\partial^k P}{\partial x^i \partial y^j} = \underbrace{k!\, b^i (-a)^j\, g(x,y)}_{\text{no factor of } (bx-ay)} + \underbrace{(bx - ay)(\cdots)}_{\text{has factor } (bx-ay)} + \underbrace{\cdots}_{\text{has factor } (bx-ay)} + \underbrace{(bx - ay)^k \frac{\partial^k g}{\partial x^i \partial y^j}}_{\text{has factor } (bx-ay)}.$$

Every term except the first contains a factor of $(bx - ay)$. Therefore, evaluating at $(x,y) = (a,b)$ gives

$$\frac{\partial^k P}{\partial x^i \partial y^j}(a,b) = \underbrace{k!\, b^i (-a)^j\, g(a,b)}_{\neq 0} + \underbrace{0 + \cdots + 0}_{\text{all contain factor } (bx-ay)} \neq 0,$$

since $g(a,b) \neq 0$.

Similarly, any derivative of total order $m < k$ still contains a factor of $(bx - ay)$, so

$$P(a,b) = \frac{\partial P}{\partial x}(a,b) = \frac{\partial P}{\partial y}(a,b) = \cdots = \frac{\partial^m P}{\partial x^i \partial y^j}(a,b) = 0.$$

∎

**Problem 9**

Supppose

$$P(a,b) = \frac{\partial P}{\partial x}(a,b) = \frac{\partial P}{\partial y}(a,b) = \cdots = \frac{\partial^{k-1} P}{\partial x^i \partial y^j}(a,b) = 0,$$

where $i + j = k - 1$ and

$$\frac{\partial^k P}{\partial x^i \partial y^j}(a,b) \neq 0,$$

for at least one pair $i + j = k$. Show that $(a : b)$ is a root of multiplicity $k$ for $P(x,y)$.

*Proof.* Suppose for contradiction that $(a : b)$ is a root of multiplicity $m \neq k$. Now suppose $m < k$. Then

$$P(x,y) = (bx - ay)^m g(x,y)$$

where $g(a,b) \neq 0$. Taking a partial derivative of total order $m$ gives

$$\frac{\partial^m P}{\partial x^i \partial y^j}(a,b) = \underbrace{m! b^i (-a)^j g(a,b)}_{\neq 0} + \underbrace{0 + \cdots + 0}_{\text{all contain factor } (bx-ay)} \neq 0,$$

which contradicts the assumption that all derivatives of order less than $k$ vanish. Next suppose $m > k$. Then

$$P(x,y) = (bx - ay)^m g(x,y).$$

Taking any derivative of total order $k$ every term contains a factor of $(bx - ay)$ thus

$$\frac{\partial^k P}{\partial x^i \partial y^j}(a, b) = 0,$$

which contradicts the assumption that at least one derivative of order $k$ is nonzero. Therefore $m = k$, and $(a : b)$ is a root of multiplicity $k$. ∎

**Problem 10**

Let $(x_0 : y_0 : z_0) \in V(P) \cap V(l)$. Show that $(x_0 : z_0)$ is a root of the homogeneous two-variable polynomial $P(x, ax + cz, z)$ show that $y_0 = ax_0 + cz_0$.

*Proof.* Since $(x_0 : y_0 : z_0) \in V(l)$

$$ax_0 - y_0 + cz_0 = 0.$$

Solving for $y_0$ gives

$$y_0 = ax_0 + cz_0.$$

Then define

$$Q(x, z) := P(x, ax + cz, z).$$

Then

$$Q(x_0, z_0) = P(x_0, ax_0 + cz_0, z_0) = P(x_0, y_0, z_0) = 0,$$

since $(x_0 : y_0 : z_0) \in V(P)$. ∎

**Problem 11**

Let $P(x, y, z) = x^2 - yz$ and $l(x, y, z) = \lambda x - y$. Show that the intersection multiplicity of $V(P)$ and $V(l)$ at $(0 : 0 : 1)$ is one when $\lambda \neq 0$ and two when $\lambda = 0$.

*Proof.* We have $y = \lambda x$.

$$P(x, y, z) = P(x, \lambda x, 1) = x^2 - \lambda x \cdot 1 = x(x - \lambda) = 0.$$

If $\lambda = 0$ then $x = 0$ is the only solution. If $\lambda \neq 0$ then $x = 0$ and $x = \lambda$ are solutions. ∎

**Problem 12**

Let $(x_0 : y_0 : z_0) \in V(P) \cap V(l)$. Let $x = a_1 s + b_1 t, y = a_2 s + b_2 t, z = a_3 s + b_3 t$ and $x = c_1 u + d_1 v, y = c_2 u + d_2 v, z = c_3 u + d_3 v$ be two parametrizations of the line $V(l)$ such that $(x_0 : y_0 : z_0)$ corresponds to $(s_0 : t_0)$ and $(u_0 : v_0)$, respectively. Show that the multiplicity of the root $(s_0 : t_0)$ of $P(a_1 s + b_1 t, a_2 s + b_2 t, a_3 s + b_3 t)$ is equal to the multiplicityof the root $(u_0, v_0)$ of $P(c_1 u + d_1 v, c_2 u + d_2 v, c_3 u + d_3 v)$. Conclude that our definition of the intersection multiplicity of $V(P)$ and $V(l)$ is independent of the paramtrization used for the line $V(l)$.

*Proof.* Let

$$f(s, t) = P(a_1 s + b_1 t, a_2 s + b_2 t, a_3 s + b_3 t), \quad g(u, v) = P(c_1 u + d_1 v, c_2 u + d_2 v, c_3 u + d_3 v).$$

Since $(s_0 : t_0)$ and $(u_0 : v_0)$ correspond to the same point $(x_0 : y_0 : z_0)$, there exist constants $\alpha, \beta, \gamma, \delta$ such that

$$s = \alpha u + \beta v, \quad t = \gamma u + \delta v$$

and therefore $f(s, t) = g(u, v)$ for all points on the line. It follows that the parametrizations have the same multiplicity of roots. ∎

Problem 13

Let $P(x,y,z) = x^2 + 2xy - yz + z^2$. Show that the intersection and multiplicity of $V(P)$ and any line $l$ at a point of intersection is at most two.

*Proof.* Let $l = ax_0 + by_0 + cz_0$ be an arbitrary line with $a, b$, or $c$ being nonzero. Wlog suppose $b \neq 0$. We can let $b = -1$ and get

$$y_0 = ax_0 + cz_0.$$

Then substituting into $P$ we have

$$P(x_0, ax_0 + cz_0, z_0) = x_0^2 + 2x_0(ax_0 + cz_0) - (ax_0 + cz_0)z_0 + z_0^2.$$

Simplifying, we get

$$P(x_0, ax_0 + cz_0, z_0) = (1 + 2a)x_0^2 + (2c - a)x_0z_0 + (1 - c)z_0^2.$$

Therefore any root has multiplicity at most 2. ∎

Problem 14

Let $P(x,y,z)$ be an irreducible second degree homogeneous polynomial. Show that the intersection multiplicity of $V(P)$ and any line $l$ at a point of intersection is at most two.

*Proof.* If we substitute the equation of the line into the second degree homogeneous polynomial, we get another second degree polynomial in one or two variables. But if a root had multiplicity greater than 2, it would require a polynomial with total degree greater than 2. Thus the intersection multiplicity is at most two. ∎

Problem 15

Let $P(x,y,z) = x^2 + y^2 + 2xz - yz$.

1. Find the tangent line $l = V(l)$ to $V(P)$ at $(-2 : 1 : 1)$.
2. Show that the intersection multiplicity of $V(P)$ and $l$ at $(-2 : 1 : 1)$ is two.

*Proof.* Notice

$$\nabla P(x,y,z) = (2x + 2z,\ 2y - z,\ 2x - y).$$

Then, at $(-2, 1, 1)$ we have

$$\nabla P(-2,1,1) = (-2,\ 1,\ -5).$$

Thus the tangent plane is

$$-2(x+2) + 1(y-1) - 5(z-1) = 0 \quad \Longrightarrow \quad -2x + y - 5z = 0.$$

Parametrizing

$$(x,y,z) = (-2,1,1) + t(2,9,1) = (-2 + 2t,\ 1 + 9t,\ 1 + t).$$

Plugging this into $P(x,y,z)$ gives

$$\begin{aligned} P(-2+2t,\ 1+9t,\ 1+t) &= (-2+2t)^2 + (1+9t)^2 + 2(-2+2t)(1+t) - (1+9t)(1+t) \\ &= 4 - 8t + 4t^2 + 1 + 18t + 81t^2 + 2(-2+2t)(1+t) - (1+9t)(1+t) \\ &= 85t^2. \end{aligned}$$

Thus the polynomial vanishes with multiplicity 2 at $t = 0$. ∎

**Problem 16**

Let $P(x,y,z) = x^3 - y^2z + z^3$.

1. Find the tangent line to $V(P)$ at $(2 : 3 : 1)$ and show directly that the intersection multiplicity of $V(P)$ and its tangent at $(2 : 3 : 1)$ is two.
2. Find the tangent line to $V(P)$ at $(0 : 1 : 1)$ and show directly that the intersection multiplicity of $V(P)$ and its tangent at $(0 : 1 : 1)$ is three.

*Proof.* Notice

$$\nabla P(x,y,z) = (3x^2,\ -2yz,\ -y^2 + 3z^2).$$

Then, at $(2,3,1)$ we have

$$\nabla P(2,3,1) = (12, -6, -6).$$

Thus the tangent plane is

$$12(x-2) - 6(y-3) - 6(z-1) = 0 \quad \Longrightarrow \quad 2x - y - z = 0.$$

Parametrizing

$$(x,y,z) = (2,3,1) + t(1,2,0) = (2+t,\ 3+2t,\ 1).$$

Plugging this into $P(x,y,z)$ gives

$$\begin{aligned} P(2+t,\ 3+2t,\ 1) &= (2+t)^3 - (3+2t)^2 \cdot 1 + 1^3 \\ &= 8 + 12t + 6t^2 + t^3 - (9 + 12t + 4t^2) + 1 \\ &= 2t^2 + t^3. \end{aligned}$$

Let

$$f(t) = 2t^2 + t^3.$$

Then

$$f(0) = 0, \quad f'(t) = 4t + 3t^2, \quad f'(0) = 0,$$

and

$$f''(t) = 4 + 6t, \quad f''(0) \neq 0.$$

Thus by Exercise 2.2.9, the root has multiplicity 2 at $t = 0$.

Now, at $(0,1,1)$ we have

$$\nabla P(0,1,1) = (0, -2, 2).$$

Thus the tangent plane is

$$0(x-0) - 2(y-1) + 2(z-1) = 0 \quad \Longrightarrow \quad y = z.$$

Parametrizing

$$(x,y,z) = (0,1,1) + t(1,1,1) = (t,\ 1+t,\ 1+t).$$

Plugging this into $P(x,y,z)$ gives

$$\begin{aligned} P(t,\ 1+t,\ 1+t) &= t^3 - (1+t)^2(1+t) + (1+t)^3 \\ &= t^3. \end{aligned}$$

Let

$$g(t) = t^3.$$

Then

$$g(0) = 0, \quad g'(0) = 0, \quad g''(0) = 0,$$

but

$$g'''(0) \neq 0.$$

Thus by Exercise 2.2.9, the root has multiplicity 3 at $t = 0$. ∎

**Problem 17**

Redo the previous two exercises using Exercise 2.2.9.

*Proof.*

$$P(x, y, z) = x^2 + y^2 + 2xz - yz, P = (-2 : 1 : 1).$$

The tangent line computed previously is

$$2x - y + 2z - z = 0 \implies y = 2x + z.$$

Plugging into the curve

$$\begin{aligned} P(x, 2x + z, z) &= x^2 + (2x + z)^2 + 2xz - (2x + z)z \\ &= x^2 + 4x^2 + 4xz + z^2 + 2xz - 2xz - z^2 \\ &= 5x^2 + 4xz. \end{aligned}$$

Compute derivatives at $x = -2$

$$f(x) = 5x^2 + 4x \quad \Rightarrow \quad f(-2) = 0, \quad f'(-2) = 0, \quad f''(-2) \neq 0.$$

Thus by Exercise 2.2.9, the intersection multiplicity at $(-2 : 1 : 1)$ is 2. ∎

*Proof.*

$$P(x, y, z) = x^3 - y^2 z + z^3, P = (2 : 3 : 1).$$

The tangent line computed previously is

$$2x - y - z = 0 \implies y = 2x - z.$$

Plugging into the curve

$$\begin{aligned} P(x, 2x - z, z) &= x^3 - (2x - z)^2 z + z^3 \\ &= x(x - 2z)^2. \end{aligned}$$

Compute derivatives $x = 2$

$$f(2) = 0, \quad f'(2) = 0, \quad f''(2) \neq 0.$$

Thus by Exercise 2.2.9, the intersection multiplicity at $P$ is 2. ∎

*Proof.*

$$Q = (0 : 1 : 1)$$

The tangent line computed previously is

$$y - z = 0 \implies y = z.$$

Plugging into the curve

$$P(x, z, z) = x^3.$$

Compute derivatives at $x = 0$

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 0, \quad f'''(0) \neq 0.$$

Thus by Exercise 2.2.9, the intersection multiplicity at $Q$ is 3. ∎

Problem 18

Show for any nonsingular curve $V(P) \subset \mathbb{P}^2$, the intersection of multiplicity of $V(P)$ and its tangent line $l$ at the point of tangency is at least two.

*Proof.* Suppose $V(P) \subset \mathbb{P}^2$ is nonsingular at a point $(x_0 : y_0 : z_0)$. Then

$$\nabla P(x_0, y_0, z_0) \neq 0.$$

The tangent line $l$ at $(x_0 : y_0 : z_0)$ is

$$P_x(x_0, y_0, z_0)(x - x_0) + P_y(x_0, y_0, z_0)(y - y_0) + P_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Parametrize this line by $l(t)$ such that $l(0) = (x_0, y_0, z_0)$ thus

$$f(t) = P(l(t)).$$

Then

$$f(0) = P(x_0, y_0, z_0) = 0, \quad f'(0) = 0$$

since $l$ is tangent at the point. Since $V(P)$ is nonsingular we have

$$f''(0) \neq 0.$$

By Exercise 2.2.9 the intersection multiplicity of $V(P)$ and $l$ at $(x_0 : y_0 : z_0)$ is at least $2$. ∎

Problem 19

1. Let $P(x, y, z)$ be an irreducible degree three homogeneous polynomial. Show that the intersection multiplicity of $V(P)$ and any line $l$ at a point of intersection is at most three.
2. Let $P(x, y, z)$ be an irreducible homogeneous polynomial of degree $n$. Show that the intersection multiplicity of $V(P)$ and any line $l$ at a point of intersection is at most $n$.

*Proof.* Composing $P$ with $l$ gives

$$f(t) = P(l(t)),$$

which has degree at most $3$. ∎

*Proof.* Composing a degree $n$ polynomial $P$ with a line $l$ gives

$$f(t) = P(l(t)),$$

which has degree at most $n$. ∎

Problem 20

Let $P(x, y, z) = x^3 + yz^2$. Show that $(0 : 0 : 1)$ is an inflection point of $V(P)$.

*Proof.* We have

$$\nabla P(x, y, z) = (3x^2, z^2, 2yz).$$

At $(0 : 0 : 1)$ we have

$$\nabla P(0, 0, 1) = (0, 1, 0).$$

The tangent line at $(0 : 0 : 1)$ is

$$P_x(0, 0, 1)(x - 0) + P_y(0, 0, 1)(y - 0) + P_z(0, 0, 1)(z - 1) = 0,$$

which simplifies to

$$0 \cdot x + 1 \cdot y + 0 \cdot (z - 1) = y = 0.$$

Thus $y = 0$. Plugging into $P$ we have

$$P(x, 0, z) = x^3 + 0 \cdot z^2 = x^3.$$

Thus the tangent line intersects the curve with multiplicity $3$ at this point. Therefore, $(0 : 0 : 1)$ is an inflection point of $V(P)$. ∎

**Problem 21**

Let $P(x, y, z) = x^3 + y^3 + z^3$ (the Fermat curve). Show that $(1 : -2 : 0)$ is an inflection point of $V(P)$.

*Proof.* We have

$$\nabla P(x, y, z) = (3x^2, 3y^2, 3z^2).$$

At $(1 : -2 : 0)$ we have

$$\nabla P(1, -2, 0) = (3, 12, 0).$$

The tangent line at $(1 : -2 : 0)$ is

$$P_x(1, -2, 0)(x - 1) + P_y(1, -2, 0)(y + 2) + P_z(1, -2, 0)(z - 0) = 0,$$

which simplifies to

$$3(x - 1) + 12(y + 2) + 0 \cdot z = 3(x - 1) + 12(y + 2) = 0.$$

Dividing by $3$ we have

$$(x - 1) + 4(y + 2) = 0 \quad \Longrightarrow \quad x + 4y + 7 = 0.$$

Thus the tangent line is $x + 4y + 7 = 0$. Plugging into $P$ we have

$$P(-4y - 7, y, z) = (-4y - 7)^3 + y^3 + z^3.$$

Thus the tangent line intersects the curve with multiplicity $3$ at this point. Therefore, $(1 : -2 : 0)$ is an inflection point of $V(P)$. ∎

**Problem 22**

Compute $H(P)$ for the following cubic polynomials.

1. $P(x, y, z) = x^3 + yz^2$.
2. $P(x, y, z) = y^3 + z^3 + xy^2 - 3yz^2 + 3zy^2$.
3. $P(x, y, z) = x^3 + y^3 + z^3$.

**Solution (1):**

$$\det \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix} = \det \begin{bmatrix} 6x & 0 & 0 \\ 0 & 0 & 2z \\ 0 & 2z & 2y \end{bmatrix} = 6x \cdot (0 \cdot 2y - 2z \cdot 2z) = -24xz^2.$$

**Solution (2):**

$$\det \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix} = \det \begin{bmatrix} 0 & 2y & 0 \\ 2y & 6y + 2x + 6z & -6z + 6y \\ 0 & -6z + 6y & 6z - 6y \end{bmatrix}$$

$$\begin{aligned} &= 0 \cdot ((6y + 2x + 6z)(6z - 6y) - (-6z + 6y)(-6z + 6y)) \\ &\quad - 2y \cdot (2y(6z - 6y) - (-6z + 6y) \cdot 0) \\ &\quad + 0 \cdot (2y(-6z + 6y) - (6y + 2x + 6z) \cdot 0) \\ &= -2y \cdot (12y(z - y)) \\ &= -24y^2(z - y). \end{aligned}$$

**Solution (3):**

$$\det\begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix} = \det\begin{bmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{bmatrix} = 6x \cdot 6y \cdot 6z = 216xyz.$$

**Problem 23**

Let $P(x, y, z)$ be an irreducible homogeneous polynomial of degree three. Show that $H(P)$ is also a third degree homogeneous polynomial.

**Problem 24**

Let $P(x, y, z) = x^3 + y^3 + z^3$ (th Fermat curve). Show that $(1 : -1 : 0) \in V(P) \cap V(H(P))$.

*Proof.* Clearly

$$P(1, -1, 0) = 1^3 + (-1)^3 + 0^3 = 0.$$

Then

$$H(P) = \det\begin{bmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{bmatrix} = 6^3 xyz.$$

Then

$$H(P)(1, -1, 0) = 6^3 \cdot 1 \cdot (-1) \cdot 0 = 0.$$

Therefore, $(1 : -1 : 0) \in V(P) \cap V(H(P))$. ∎

**Problem 25**

Let $P(x, y, z) = y^3 + z^3 + xy^2 - 3yz^2 + 3zy^2$. Show that $(-2 : 1 : 1) \in V(P) \cap V(H(P))$.

*Proof.* Clearly

$$P(-2, 1, 1) = 1^3 + 1^3 + (-2)(1^2) - 3 \cdot 1 \cdot 1^2 + 3 \cdot 1 \cdot 1^2 = 1 + 1 - 2 - 3 + 3 = 0.$$

Then

$$H(P) = \det\begin{bmatrix} 0 & 2y & 0 \\ 2y & 6y + 2x + 6z & -6z + 6y \\ 0 & -6z + 6y & 6z - 6y \end{bmatrix} = -24y^2(z - y).$$

Then

$$H(P)(-2, 1, 1) = -24 \cdot 1^2 \cdot (1 - 1) = 0.$$

Therefore, $(-2 : 1 : 1) \in V(P) \cap V(H(P))$. ∎

**Problem 26**

Let $P(x, y, z) = x^3 + yz^2$. Show that $(0 : 0 : 1) = V(P) \cap V(H(P))$.

*Proof.* Clearly

$$P(0, 0, 1) = 0^3 + 0 \cdot 1^2 = 0.$$

Then

$$H(P) = \det\begin{bmatrix} 6x & 0 & 0 \\ 0 & 0 & 2z \\ 0 & 2z & 2y \end{bmatrix} = -4z^2 \cdot 6x = -24xz^2.$$

Then

$$H(P)(0,0,1) = -24 \cdot 0 \cdot 1^2 = 0.$$

Therefore, $(0 : 0 : 1) \in V(P) \cap V(H(P))$. ∎

**Problem 27**

Let $P(x,y,z) = x^3 + yz^2$.

1. Show that $(0 : 1 : 0) \in V(P) \cap V(H(P))$.
2. Explain why $(0 : 1 : 0)$ is not an inflection point of $V(P)$.

*Proof.* Clearly

$$P(0,1,0) = 0^3 + 1 \cdot 0^2 = 0.$$

Then

$$H(P) = \det \begin{bmatrix} 6x & 0 & 0 \\ 0 & 0 & 2z \\ 0 & 2z & 2y \end{bmatrix} = -24xz^2.$$

Then

$$H(P)(0,1,0) = -24 \cdot 0 \cdot 0^2 = 0.$$

Therefore, $(0 : 1 : 0) \in V(P) \cap V(H(P))$.

Now, note that $P_x = P_y = P_z = 0$. The tangent line of $P$ at $(0 : 1 : 0)$ is

$$P_x(x-0) + P_y(y-1) + P_z(z-0) = 0.$$

Thus $(0 : 1 : 0)$ is not an inflection point. ∎

Problem 29

Consider the following projective change of coordinates

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Suppose that under the projective transformation $A$ the polynomial $P(x, y, z)$ becomes the polynomial $P(u, v, w)$.

1. Show that the Hessian matrix of $P$ and $Q$ are related by

$$\begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix} = A^T \begin{bmatrix} Q_{uu} & Q_{uv} & Q_{uw} \\ Q_{vu} & Q_{vv} & Q_{vw} \\ Q_{wu} & Q_{wv} & Q_{ww} \end{bmatrix} A$$

2. Conclude that $H(P)(x, y, z) = 0$ if and only if $H(Q)(u, v, w) = 0$.

*Proof.* First notice

$$P(x, y, z) = Q(u(x, y, z), v(x, y, z), w(x, y, z)).$$

Then using the chain rule

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial Q}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial Q}{\partial w}\frac{\partial w}{\partial x} = Q_u \frac{\partial u}{\partial x} + Q_v \frac{\partial v}{\partial x} + Q_w \frac{\partial w}{\partial x}.$$

Now

$$\frac{\partial u}{\partial x} = a_{11}, \quad \frac{\partial v}{\partial x} = a_{21}, \quad \frac{\partial w}{\partial x} = a_{31},$$

so

$$\frac{\partial P}{\partial x} = Q_u a_{11} + Q_v a_{21} + Q_w a_{31}.$$

Then

$$\frac{\partial^2 P}{\partial x^2} = a_{11}\frac{\partial Q_u}{\partial x} + a_{21}\frac{\partial Q_v}{\partial x} + a_{31}\frac{\partial Q_w}{\partial x}.$$

Applying the chain rule again

$$\frac{\partial Q_u}{\partial x} = Q_{uu}a_{11} + Q_{uv}a_{21} + Q_{uw}a_{31}, \quad \frac{\partial Q_v}{\partial x} = Q_{vu}a_{11} + Q_{vv}a_{21} + Q_{vw}a_{31}, \quad \frac{\partial Q_w}{\partial x} = Q_{wu}a_{11} + Q_{wv}a_{21} + Q_{ww}a_{31}.$$

Plugging these in gives

$$\frac{\partial^2 P}{\partial x^2} = \sum_{k,l=1}^{3} a_{k1} H(Q)_{kl} a_{l1},$$

which is the $(1, 1)$ entry of $A^T H(Q)A$. Similarly every other entry of $H(P)$ can be computed this way thus

$$H(P) = A^T H(Q)A.$$

∎

*Proof.* Notice

$$\begin{aligned} H(P)(x,y,z)=0 &\iff A^T H(Q)(u,v,w)A = 0 \\ &\iff (A^T)^{-1}\,0\,A^{-1} = H(Q)(u,v,w) \\ &\iff H(Q)(u,v,w)=0. \end{aligned}$$

∎

**Problem 30**

Suppose $p$ is a point of inflection $V(P)$, and that under a projective change of cooridnates the polynomial $P$ becomes the polynomial $Q$ and $p \mapsto q$. Show that $q$ is a point of inflection of $Q$.

*Proof.* Let $l$ be the tangent line to $V(P)$ at $p$, and let $p, p', p''$ be the three points of intersection of $l$ with $V(P)$. Under a linear projective change of coordinates, $l \mapsto l'$, $V(P) \mapsto V(Q)$, $p \mapsto q$, $p' \mapsto q'$, and $p'' \mapsto q''$. Since the change of coordinates preserve intersection multiplicities, the tangent line $l'$ intersects $V(Q)$ at $q$ with multiplicity three. Thus $q$ is a point of inflection of $V(Q)$. ∎

**Problem 31**

Use Exercises 2.2.29 and 2.2.30 to explain why, in proving Theorem 2.2.28, it is enough to show that $p$ is a point of inflection if and only if $H(P)=0$ in the case where $p=(0:0:1)\in V(P)$ and the tangent line $l$ to $V(P)$ at $p$ is $y=0$, i.e. $l=V(y)$.

*Proof.* Suppose $p$ is a point of inflection if and only if $H(P)=0$ in the case where $p=(0:0:1)\in V(P)$ and the tangent line $l$ to $V(P)$ at $p$ is $y=0$, i.e., $l=V(y)$.

For a general point and tangent line we can perform a linear change of coordinates to move $p$ to $(0:0:1)$ and $l$ to $V(y)$. By Exercise 2.2.29 part 2, the Hessian vanishes at the transformed point if and only if it vanishes at the original point under the change of coordinates. By Exercise 2.2.30 part 1 the vanishing of the Hessian in the transformed coordinates implies the vanishing of the Hessian for the original curve. Thus it is enough to check the Hessian at the point $(0:0:1)$ with tangent $y=0$. ∎

**Problem 32**

Explain why in the affine patch $z=1$ the dehomogenized curve is

$$\lambda y + (ax^2+bxy+cy^2) + \text{ higher order terms},$$

where $\lambda \neq 0$. [Hint: We know that $p\in V(P)$ and $p$ is nonsingular.]

**Solution:** We can perform a change of coordinates to map $p$ to $(0:0:1)$ with tangent $y=0$. Then, since $P$ is nonsingular, either $\frac{\partial P}{\partial x}(p)\neq 0$ or $\frac{\partial P}{\partial y}(p)\neq 0$. If $\frac{\partial P}{\partial y}(p)=0$, we can perform another change of coordinates so that $\frac{\partial P}{\partial y}(p)\neq 0$ and $\frac{\partial P}{\partial x}(p)=0$. Thus $\lambda$ is the slope of the tangent with respect to this final curve, and we have the curve in the form

$$\lambda y + (ax^2+bxy+cy^2) + \text{ higher order terms},$$

with $\lambda \neq 0$.

Problem 33

Explain why the intersection of $V(P)$ with the tangent $V(y)$ at $p$ corresponds to the root $(0 : 1)$ of the equation

$$P(x, 0, z) = ax^2 z^{d-2} + \text{ higher order terms } = 0,$$

where $z = deg(P)$.

**Solution:** We write the point as $(x/z, 0, 1)$ which lies on the tangent line. Thus, plugging into $P$ gives

$$P(x/z, 0, 1) = P(x, 0, z) = 0,$$

so the intersection corresponds to the root $(0 : 1)$.

Problem 34

Show that $p$ is a point of inflection of $V(P)$ if and only if $a = 0$. [Hint: For $p$ to be an inflection point, what must the multiplicity $(0 : 1)$ be in the equation in Exercise 2.2.33?]

*Proof.* Suppose $p$ is a point of inflection of $V(P)$. The multiplicity of the tangent line substituted into the curve must be $3$ since it is an inflection point. Thus the root $(0 : 1)$ of $P(x, 0, z)$ has multiplicity $3$, so $x^3 \mid P(x, 0, z)$ and therefore the $x^2$ term must vanish, implying $a = 0$.

Conversely, suppose $a = 0$. Then the lowest degree term in $P(x, 0, z)$ is degree at least $3$ in $x$, so the root $(0 : 1)$ has multiplicity $3$. Thus the tangent line intersects the curve with multiplicity $3$, so $p$ is a point of inflection of $V(P)$. ∎

Problem 35

1. Show that
$$H(P)(p) = \det \begin{bmatrix} 2a & b & 0 \\ b & 2c & \lambda(d-1) \\ 0 & \lambda(d-1) & 0 \end{bmatrix}.$$
2. Conclude that $p \in V(H(P))$ if and only if $a = 0$.

*Proof.* Notice
$$H(P) = \det \begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix}.$$
From Exercise 2.2.33
$$P(x,y,z) = ax^2z^{d-2} + bxyz^{d-2} + cy^2z^{d-2} + \lambda yz^{d-1} + \text{higher order terms}.$$
Thus the partial derivatives are

1. $P_{xx} = 2az^{d-2} +$ (higher order terms in $x, y, z$)
2. $P_{xy} = bz^{d-2} +$ (higher order terms in $x, y, z$)
3. $P_{xz} = 2a(d-2)xz^{d-3} + b(d-2)yz^{d-3} +$ (higher order terms in $x, y, z$)
4. $P_{yy} = 2cz^{d-2} +$ (higher order terms in $x, y, z$)
5. $P_{yz} = b(d-2)xz^{d-3} + 2c(d-2)yz^{d-3} + \lambda(d-1)z^{d-2} +$ (higher order terms in $x, y, z$)
6. $P_{zz} = \lambda(d-1)(d-2)yz^{d-3} +$ (higher order terms in $x, y, z$)

Evaluating at the point $p = (0 : 1)$ gives
$$P_{xx}(p) = 2a, \quad P_{xy}(p) = b, \quad P_{xz}(p) = 0, \quad P_{yy}(p) = 2c, \quad P_{yz}(p) = \lambda(d-1), \quad P_{zz}(p) = 0.$$
Evaluating at the point $p$
$$H(P)(p) = \det \begin{bmatrix} 2a & b & 0 \\ b & 2c & \lambda(d-1) \\ 0 & \lambda(d-1) & 0 \end{bmatrix}.$$
Using the third column
$$H(P)(p) = -\lambda(d-1) \det \begin{bmatrix} 2a & b \\ 0 & \lambda(d-1) \end{bmatrix} = -2a\lambda^2(d-1)^2.$$
Since $\lambda \neq 0$ and $d \neq 1$ this equation is $0$ if and only if $a = 0$. Thus $p \in V(H(P))$ if and only if $a = 0$. ∎

**Problem 36**

Let $P(x, y, z)$ be an irreducible second degree homogeneous polynomial. Using the Hessian curve, show that $V(P)$ has no points of inflection.

*Proof.* Since $P$ is quadratic, each second partial derivative $P_{ij}$ is either constant or zero. Therefore the Hessian determinant $H(P)$ is a constant, independent of $x, y, z$. There are two cases.

Suppose $H(P) = 0$ for all $(x, y, z)$. Then $P$ must be degenerate meaning it is not irreducible.

Suppose $H(P) \neq 0$ for all $(x, y, z)$. Then there are no inflection points in $P$. ∎

**Problem 38**

Use Exercise 2.2.38 and Theorems 2.2.28 and 2.2.37 to show that if $V(P)$ is a smooth curve, then $V(P)$ has exactly nine inflection points.

*Proof.* Suppose $V(P)$ is a smooth cubic curve. By Exercise 2.2.38, the Hessian curve $H(P)$ also has degree 3. If $H(P)$ and $P$ have a common component then $H(P)$ is degenerate and therefore has a singular point. Thus $H(P)$ and $P$ have no common components. By Theorem 2.2.37, the curves $V(P)$ and $V(H(P))$ intersect in exactly 9 points. Finally, by Theorem 2.2.28, each of these 9 points of intersection is an inflection point of $V(P)$. Therefore $V(P)$ has exactly nine inflection points. ∎

Problem 39

Find all nine points of inflection of the Fermat curve,

$$P(x,y,z) = x^3 + y^3 + z^3.$$

*Proof.* Compute the second partial derivatives

$$P_{xx} = 6x, \quad P_{yy} = 6y, \quad P_{zz} = 6z,$$

$$P_{xy} = P_{xz} = P_{yz} = 0.$$

Thus

$$H(P) = \begin{bmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{bmatrix},$$

Therefore

$$H(P) = \det \begin{bmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{bmatrix} = 216\, xyz.$$

We require

$$x^3 + y^3 + z^3 = 0 \text{ and } \quad xyz = 0.$$

There are three cases $x = 0, y = 0, z = 0$. Suppose $x = 0$. $y^3 + z^3 = 0 \implies y^3 = -z^3$. We can let $y = 1$ then clearly $z = -1$ is one root and the roots of unity give the other two. The other two cases $y = z = 0$ follow similarly. ∎

## 2.3 Group Law

Problem 1

Explain why the chord-tangent composition law is commutative, i.e. $PQ = QP$ for all points $P, Q$ on $C$.

**Solution:** The third point of intersection of $V(p)$ and $V(l)$ is the same since the line between $P, Q$ is equivalent to the line between $Q, P$. Therefore, the resulting point is the same, and thus $PQ = QP$.

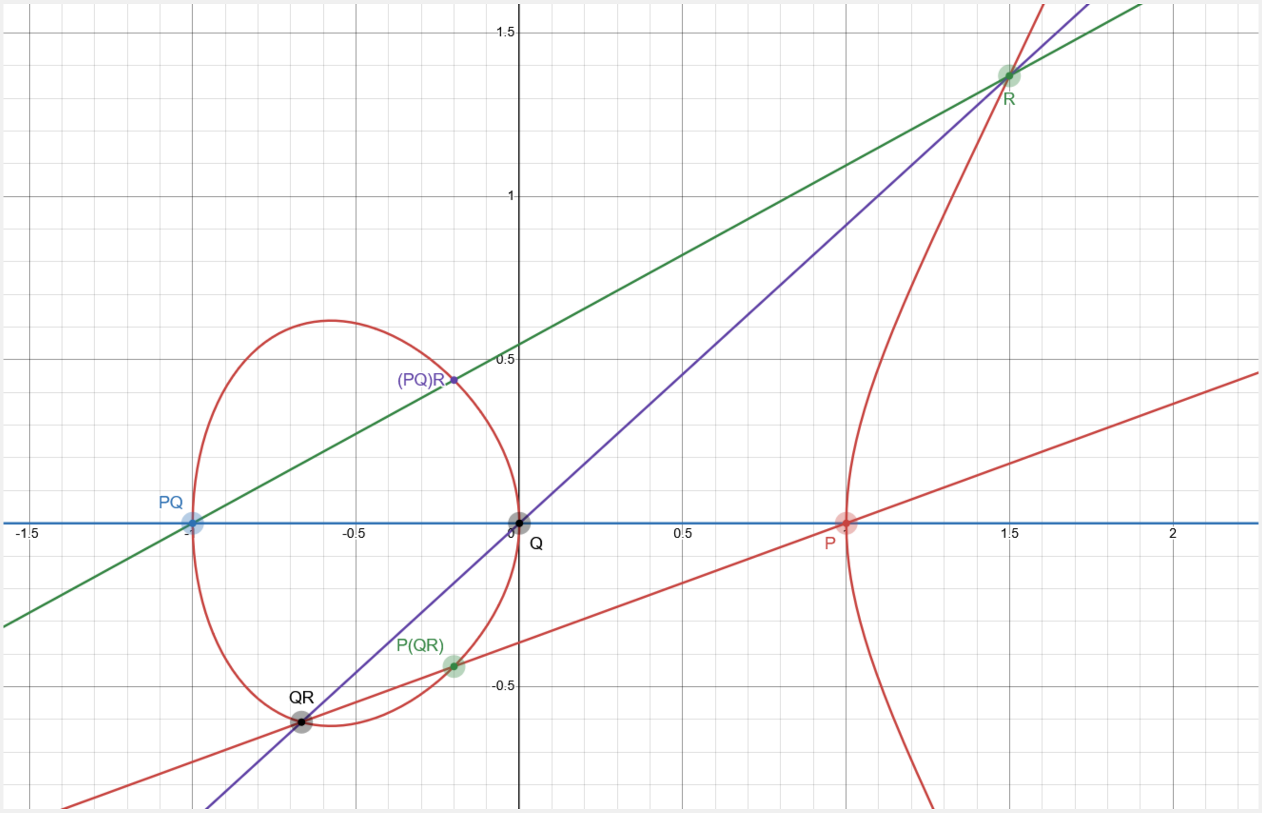
Problem 2

Consider the cubic curve

$$C = \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 - x\},$$

and the points $P, Q, R$ on $C$, as shown below. (Note that only the real part of $C$ is shown.) Using a straightedge, locate $PQ$ and then $(PQ)R$ on the curve $C$. Now locat ethe point $QR$ and the point $P(QR)$ on the curve $C$. Is it true that $P(QR) = (PQ)R$? That is, is the chord-tangent composition law associative for these points on $C$?

**Solution:** Clearly not from the figure below.




**Problem 3**

Consider the cubic curve

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^3 + y^3 = 1\},$$

and the points $P = (0, 1)$ and $Q = (1, 0)$ on $C$.

1. Using the equation of the cubic curve $C$ and its Hessian, verify that $P$ and $Q$ are inflection points of $C$.
2. Verify that $PP = P$. Conclude that if $C$ has an idententity element $e$, then $e = P$.
3. Verify that $QQ = Q$. Conclude that if $C$ has an idententity element $e$, then $e = Q$.
4. Conclude that $C$ does not have an idententity element for the chord-tangent composition law.

*Proof.* Clearly

$$0^3 + 1^3 - 1 = 0 \text{ and } 1^3 + 0^3 - 1 = 0.$$

Thus $P, Q \in V(C)$. Now

$$F_x = 3x^2, \quad F_y = 3y^2,$$

and

$$F_{xx} = 6x, \quad F_{yy} = 6y, \quad F_{xy} = 0, \quad F_{yx} = 0.$$

Finally

$$H(F) = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}.$$

Computing the determinant we have

$$\det(H(F)) = (6x)(6y) - 0 = 36xy.$$

Then

$$36(0)(1) = 0 \text{ and } 36(1)(0) = 0,$$

thus $P, Q \in V(H(C))$. It follows that $P, Q$ are inflection points. Since $PP = P$ it follows that if $C$ has an idententity element $e$ then $e = P$. Similarly, since $QQ = Q$ it follows that if $C$ has an idenity element $e$ then $e = Q$. The idenity is unique and since $P \neq Q$ the chord-tangent is not a group. ∎

**Problem 4**

Let $P_1 = (2 : 3 : 1), P_2 = (0 : 1 : 1), P_3 = (-1 : 0 : 1), P_4 = (0 : -1 : 1), P_5 = (2 : -3 : 1)$ and

$$C = V(x^3 - y^2 z + z^3) \subset \mathbb{P}^2.$$

Use the equations of the cubic curve $C$ and its Hessian to verify that $P_2$ and $P_4$ are inflection points of $C$.

*Proof.* Clearly

$$0^3 - 1^2 \cdot 1 + 1^3 = 0 \text{ and } 0^3 - (-1)^2 \cdot 1 + 1^3 = 0.$$

Thus $P_2, P_4 \in V(C)$. Now

$$F_x = 3x^2, \quad F_y = -2yz, \quad F_z = -y^2 + 3z^2,$$

and

$$F_{xx} = 6x, \quad F_{yy} = -2z, \quad F_{zz} = 6z, \quad F_{xy} = F_{yx} = 0, \quad F_{xz} = F_{zx} = 0, \quad F_{yz} = F_{zy} = -2y.$$

Finally

$$H(F) = \begin{bmatrix} 6x & 0 & 0 \\ 0 & -2z & -2y \\ 0 & -2y & 6z \end{bmatrix}.$$

Computing the determinant we have

$$\det(H(F)) = 6x((-2z)(6z) - (-2y)(-2y)) - 0 + 0 = 6x(-12z^2 - 4y^2) = -72xz^2 - 24xy^2.$$

Then

$$\det(H(F))(0,1,1) = -72 \cdot 0 \cdot 1^2 - 24 \cdot 0 \cdot 1^2 = 0 \quad \text{and} \quad \det(H(F))(0,-1,1) = -72 \cdot 0 \cdot 1^2 - 24 \cdot 0 \cdot (-1)^2 = 0,$$

thus $P_2, P_4 \in V(H(C))$. ∎

**Problem 5**

Let
$$C = V(x^3 - y^2z + z^3) \subset \mathbb{P}^2,$$
and
$$P_1 = (2 : 3 : 1), P_2 = (0 : 1 : 1), P_3 = (-1 : 0 : 1), P_4 = (0 : -1 : 1), P_5 = (2 : -3 : 1).$$
Let $O = P_2$ be the specified inflection point so that $+$ is defined relative to $P_2$, i.e., $Q + R = P_2(QR)$ for points $Q, R$ on $C$.

1. Compute $P_1 + P_2, P_2 + P_2, P_3 + P_2, P_4 + P_2$, and $P_5 + P_2$.
2. Explain why $P_2$ is the idententity element for $C$.
3. Find the inverses of $P_1, P_2, P_3, P_4$ and $P_5$ on $C$.
4. Verify that $P_1 + (P_3 + P_4) = (P_1 + P_3) + P_4$. In general, addition of points on $C$ is associative.

*Proof.* We use the affine patch $z = 1$ then compute the lines through $P_iP_2$ where $i \in \{1, 2, 3, 4, 5\}$.

1. $P_1P_2$ had $m = \frac{3-1}{2-0} = 1$
$$y - 1 = 1 \cdot (x - 0) \implies y = x + 1.$$
2. $P_2P_2$ is a trivial case.
3. $P_3P_2$ has $m = \frac{1-0}{0-(-1)} = 1$
$$y - 0 = 1 \cdot (x + 1) \implies y = x + 1.$$
4. $P_4P_2$ is vertical
$$x = 0.$$
5. $P_5P_2$ has $m = \frac{-3-1}{2-0} = -2$
$$y - 1 = -2 \cdot (x - 0) \implies y = -2x + 1.$$

Plugging each of these into our curve we have

1. $P_1P_2$: $(x + 1)^2 = x^3 + 1 \implies x^3 - x^2 - 2x = 0 \iff x = 0, 2, -1.$
2. $P_2P_2$: is trivally $P_2$.
3. $P_3P_2$: $(x + 1)^2 = x^3 + 1 \implies x^3 - x^2 - 2x = 0 \iff x = -1, 0, 2.$
4. $P_4P_2$: $x = 0 \implies y^2 = 0^3 + 1 = 1 \iff y = \pm 1.$
5. $P_5P_2$: $(-2x + 1)^2 = x^3 + 1 \implies 4x^2 - 4x + 1 = x^3 + 1 \implies x^3 - 4x^2 + 4x = 0 \iff x = 0, 1, 3.$

Then plugging solution points into our curve we have

1. $P_1 + P_2 = (2, 3) = P_1$
2. $P_2 + P_2 = (0, 1) = P_2$
3. $P_3 + P_2 = (-1, 0) = P_3$
4. $P_4 + P_2 = (0, -1) = P_4$
5. $P_5 + P_2 = (2, -3) = P_5$

∎

*Proof.* Since $P_i + P_2 = P_i$ for all $i \in \{1, 2, 3, 4, 5\}$, $P_2$ must be the idenity element. ∎

*Proof.* The inverses of $P_i$ are the points $P_i'$ such that $P_i + P_i' = P_2$, thus

1. $-P_1 = P_5 = (2, -3)$
2. $-P_2 = P_2 = (0, 1)$
3. $-P_3 = P_4 = (0, -1)$
4. $-P_4 = P_3 = (-1, 0)$
5. $-P_5 = P_1 = (2, 3)$

∎

*Proof.* Will go through this one casually... We have

$$P_1 + (P_3 + P_4) = P_1 + O(P_3P_4) = O(P_1O(P_3P_4)) = P_2(P_1P_2(P_3P_4)),$$

and

$$(P_1 + P_3) + P_4 = O(P_1P_3) + P_4 = O(O(P_1P_3)P_4) = P_2(P_2(P_1P_3)P_4).$$

Then $P_3P_4 = P_5$ and $P_1P_3 = P_2$. Thus

$$P_2(P_1P_2(P_3P_4)) = P_2(P_1P_2P_5) = P_2(P_4).$$

and

$$P_2(P_2(P_1P_3)P_4) = P_2(P_2(P_2)P_4) = P_2(P_4).$$

At $x = 0$ our curve is $z(z - y)(z + y)$ where $z = 0$ is our point of interest. Both have the line equation $x = 0$ and thus intersect at the same point at infinity. ∎

**Problem 6**

Now let $O = P_4$ be the specified inflection point so that $+$ is defined relative to $P_4$, i.e., $Q + R = P_4(QR)$ for points $Q, R$, on $C$.

1. Compute $P_1 + P_2, P_2 + P_2, P_3 + P_2, P_4 + P_2$, and $P_5 + P_2$.
2. Now compute $P_1 + P_4, P_2 + P_4, P_3 + P_4, P_4 + P_4$, and $P_5 + P_4$. Explain why $P_4$ is now the idenity element for $C$.
3. Using the fact that $P_4$ is now the indepentity element on $C$, find the inverses of $P_1, P_2, P_3, P_4$ and $P_5$ on $C$.

**Problem 7**

Explain why $P + Q = Q + P$ for all points $P, Q$ on $C$. This establishes that $+$ is a commutative binary operation.

*Proof.* Notice $PQ = QP$ thus $O(PQ) = O(QP) = P + Q = Q + P$. ∎

**Problem 8**

Let $C$ be a smooth cubic curve and let $O$ be one of its inflection points. Define + of points on $C$ relative to $O$. Show that $P + O = P$ for all points $P$ on $C$ and that there is no other point on $C$ with this property. Thus $O$ is the idenity element for + on $C$.

*Proof.* We have

$$P + O = O(PO).$$

First notice the third point of intersection of the line $l(PO) = l(OP)$ with the curve is some point $P'$. Then

$$O(PO) = O(P') = P.$$

Suppose $O$ and $O'$ both have this property. Then $O + O' = O$ and $O + O' = O'$, so $O = O'$. ∎

Problem 9

Let $C$ be a smooth cubic curve and let $O$ be one of its inflection points. Define addition + of points on $C$ relative to the idenity $O$.

1. Suppose that $P, Q, R$ are collinear points on $C$. Show that $P + (Q + R) = O$ and $(P + Q) + R = O$.
2. Let $P$ be any point on $C$. Assume that $P$ has an inverse element $P^{-1}$ on $C$. Prove that the points $P, P^{-1}$, and $O$ must be colinear.
3. Use the results of Part (1) and Part (2) to show that for any $P$ there is an element $P'$ on $C$ statisfying $P + P' = P' + P = O$, i.e., every element $P$ has an inverse $P^{-1}$. Then show this inverse is unique.

*Proof.* $P, Q, R$ must lie on a line of inflection of $C$. Thus $PQ = R, QR = P, PR = Q$. Let $O'$ be the third point of intersection of the line $OP$. Let $R'$ be the third point of intersection of the line $OR$. Then

$$P + (Q + R) = O(P(Q + R)) = O(PO(QR)) = O(PO(P)) = O(PO') = OO = O,$$

and

$$(P + Q) + R = O((P + Q)R) = O(O(PQ)R) = O(O(R)R) = O(R'R) = OO = O.$$

Thus

$$P + (Q + R) = (P + Q) + R = O.$$

∎

*Proof.* We have

$$P + P^{-1} = O \iff O(PP^{-1}) = O,$$

where $PP^{-1}$ is the third point of intersection of the line through $P$ and $P^{-1}$. Thus the line through $P$ and $P^{-1}$ passes through $O$, so $P, P^{-1}, O$ are collinear. ∎

*Proof.* Let $P$ be an arbitrary point on $C$. Let $P'$ be the third point of intersection, $\neq P, O$, of the line $l(P, O)$ with $C$. Then

$$P + P' = P' + P = O(PP') = OO = O.$$

Thus $P' = P^{-1}$.

Suppose $P'$ is another inverse of $P$. Then the line through $P$ and $P'$ must intersect $C$ at $O$ but so does the line between $P$ and $P^{-1}$. Since a line intersects a cubic in at most three points we have $P' = P^{-1}$. ∎

## Problem 10

Start with two cubic curves, $C = V(f)$ and $D = V(g)$. By Theorem 1.2.37, there are exactly nine points of intersection, counting multiplicities, of $C$ and $D$. Denote these points by $P_1, P_2, \dots, P_9$.

1. Let $\lambda, \mu \in \mathbb{C}$ be arbitrary constants. Show that $P_1, P_2, \dots, P_9$ are points on the cubic curve defined by $\lambda f + \mu g = 0$.
2. Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ be arbitrary constants. Show that $P_1, P_2, \dots, P_9$ are the nine points of intersection of the curbic curves $C_1 = V(\lambda_1 f + \mu_1 g)$ and $C_2 = V(\lambda_2 f + \mu_2 g)$.

*Proof.* Let $P_i$ where $i \in \{1, \dots, 9\}$. Then

$$(\lambda f + \mu g)(P_i) = \lambda f(P_i) + \mu g(P_i) = \lambda \cdot 0 + \mu \cdot 0 = 0,$$

so $P_i$ lies on the cubic curve $V(\lambda f + \mu g)$. Also

$$(\lambda_1 f + \mu_1 g)(P_i) = \lambda_1 f(P_i) + \mu_1 g(P_i) = 0,$$

$$(\lambda_2 f + \mu_2 g)(P_i) = \lambda_2 f(P_i) + \mu_2 g(P_i) = 0.$$

Thus $P_i \in C_1 \cap C_2$. ∎

## Problem 11

Consider eight distinct points $P_1, P_2, \dots, P_8$ in $\mathbb{P}^2$, such that no four are collinear and no seven are on a single conic. Let $F$ be a generic cubic polynomial with unknown coefficients $a_1, a_2, \dots, a_{10}$. The system of simultaneous equations

$$F(P_1) = F(P_2) = \cdots = F(P_{10}) = 0,$$

is a system of eight linear equations in the ten unknowns $a_1, a_2, \dots, a_{10}$. Prove that the vector space of solutions to this linear system as dimension equal to $2$ by considering each of the following cases.

1. The eight points are in *general position*, which means that no three are colinear and no six are on a conic.
2. Three of the points are collinear.
3. Six of the points are on a conic.

Problem 12

Show that there are two linearly independent cubics

$$P_1(x, y, z) \text{ and } F_2(x, y, z)$$

such that any cubic curve passing thorugh the eight points

$$P_1, P_2, \ldots, P_8,$$

has the form $\lambda F_1 + \mu F_2$. Conclude that for any collection of eight points with no four collinear and no seven on a conic, there is a *unique* ninth point $P_9$ such that *every* cubic curve passing through the eight given points must also pass through $P_9$.

Problem 13

Let $C$ be a smooth cubic curve in $\mathbb{P}^2$ and let $P, Q, R$ be three points on $C$. We will show that $P+(Q+R) = (P+Q)+R$.

1. Let $V(l_1) = l(P, Q)$ and $S_2 = PQ$, so $V(l_1) \cap C = \{P, Q, S_1\}$.
2. Let $V(l_2) = l(S_1, O)$ and $S_2 = OS_1 = P + Q$, so $V(l_2) \cap C = \{S_1, O, S_2\}$.
3. Let $V(l_3) = l(S_2, R)$ and $S_3 = (P + Q)R$ so $V(l_3) \cap C = \{S_2, R, S_3\}$.

Similarly:

1. Let $V(m_1) = l(Q, R)$ and $T_1 = QT$, so $V(m_1) \cap C = \{Q, R, T_1\}$.
2. Let $V(m_2) = l(T_1, O)$ and $T_2 = OT_1 = Q + R$, so $V(m_2) \cap C = \{T_1, O, T_2\}$.
3. Let $V(m_3) = l(T_2, P)$ and $T_3 = P(Q + R)$, so $V(m_3) \cap C = \{T_2, P, T_3\}$.

1. Notice that $C' = V(l_1 m_2 l_3)$ is a cubic. Find $C' \cap C$.
2. Likewise, $C'' = V(m_1 l_2 m_3)$ is a cubic. Find $C'' \cap C$.
3. Using Parts (1) and Part (2) together with Exercise 2.3.11, deduce that $(P + Q)R = P(Q + R)$.
4. Explain why $(P+Q)R = P(Q+R)$ implies that $(P+Q)+R = P+(Q+R)$. Conclude that the addition of points on cubics is associative.

Problem 14

Show that $2 \cdot P = O$ if and only if $l(O, P)$ is tangent to $C$ at $P$.

Problem 15

Show that if $P$ and $Q$ are two points on $C$ of order two, then $PQ$, the third point of intersection of $C$ with $l(P, Q)$ is also a point of order two on $C$.

Problem 16

Let $C$ be the cubic curve defined by $y^2 z = x^3 - xz^2$.

1. Show that $O = (0 : 1 : 0)$ is an inflection point.
2. Graph $G$ in the affine patch $z = 1$.
3. Show that lines through $(0 : 1 : 0)$ correspond to vertical lines in the affine patch $z = 1$.
4. Find three points of order two in the group $(C, O, +)$.

Problem 17

Let $P$ be any inflection point on $C$. Show that $3 \cdot P = O$.

Problem 18

Suppose $P$ is a point on $C$ and $3 \cdot P = 0$. Conclude that $PP = P$. From this, deduce that $P$ is a point of inflection on $C$.

## 2.4 Normal Forms of Cubics

Problem 1

Consider the smooth cubic curve defined by $x^3 + y^3 - z^3 = 0$.

1. Show that $O = (1 : 0 : 1)$ is an inflection point of $C$.
2. Show that $x - z = 0$ is the equation of the tangent line to $C$ at $O$.
3. Find a $3 \times 3$ matrix $M$ such that, under the change of variables
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M^{-1} \begin{bmatrix} x_1 \\ y_1 \\ z_2 \end{bmatrix},$$
we have $(1 : 0 : 1) \mapsto (0 : 1 : 0)$ and $l(x, y, z) = x - z$ becoming $l_1(x_1, y_1, z_1) = z_1$.
4. Find the equation $f_1(x_1, y_1, z_1) = 0$ for the curve $C_1$ that is associated to this projective change of coordinates.

*Proof.* We have
$$\nabla C(x, y, z) = (3x^2, 3y^2, -3z^2).$$
Then
$$\nabla C(1, 0, 1) = (3, 0, -3).$$
Thus the tangent line is
$$l(x, y, z) = 3(x - 1) - 3(z - 1) = 3x - 3z = 0 \iff x - z = 0.$$
Substituting $x = z$ into our curve we have
$$x^3 + y^3 - x^3 = y^3 = 0 \implies y = 0.$$
Thus we have $y = 0$ with multiplicity $3$, and $(1 : 0 : 1)$ is an inflection point of $C$. Let
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}.$$
Then
$$O = (1 : 0 : 1) \mapsto (0 : 1 : 0), \quad l(x, y, z) = x - z \mapsto l_1(x_1, y_1, z_1) = z_1.$$
∎

Problem 2

1. Explain why the homogeneous polynomial $f_1(x_1, y_1, z_1)$ can be expressed as

$$f_1(x_1, y_1, z_1) = \alpha x_1^3 + z_1 F(x_1, y_1, z_1),$$

where $\alpha \neq 0$ and $F(0, 1, 0) \neq 0$.
2. Explain why the highest power of $y_1$ in the homogeneous polynomial $f_1(x_1, y_1, z_1)$ is two.
3. Explain how by rescaling we can introduce new coordinates $(x_2 : y_2 : z_2)$ so that the coefficient of $x_2^3$ is 1 and the coefficient of $y_2^2 z_2$ is $-1$ in the new homogeneous polynomial $f_2(x_2, y_2, z_2) = 0$.

*Proof.* First, restrict the homogeneous polynomial to the line $z_1 = 0$ and we find

$$f_1(x_1, y_1, 0) = ax_1^3 + bx_1 y_1^2 + cx_1^2 y_1 + dy_1^3.$$

Then plugging in the point on the tangent line $(0 : 1 : 0)$ we find

$$f_1(0, 1, 0) = d = 0.$$

This shows why the highest power of $y_1$ in $f_1(x_1, y_1, z_1)$ is 2. We can then factor $f_1$ to get

$$f_1(x_1, y_1, 0) = x_1(ax_1^2 + by_1^2 + cx_1 y_1).$$

But since $(0 : 1 : 0)$ has multiplicity three we must have $b = c = 0$, thus $f_1(x_1, y_1, 0) = \alpha x_1^3$ where $\alpha \neq 0$. The rest of the polynomial must have a factor of $z_1$, thus $f_1(x_1, y_1, z_1) = \alpha x_1^3 + z_1 F(x_1, y_1, z_1)$. Suppose $F(0, 1, 0) = 0$, then all partial derivatives would be zero, contradicting that the curve is smooth. Therefore $f_1(x_1, y_1, z_1) = \alpha x_1^3 + z_1 F(x_1, y_1, z_1)$ such that $\alpha \neq 0$ and $F(0, 1, 0) \neq 0$.

Let $x_2 = \alpha^{-1/3} x_1$, $y_2 = \beta y_1$, and $z_2 = \gamma z_1$, where $\beta, \gamma \neq 0$ are constants chosen so that the coefficient of $y_2^2 z_2$ is $-1$. ∎

Problem 3

Use the Fermat curve defined in Exercise 2.4.1 for the following.

1. Show that

$$M^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

is the desired matrix that solves Part (3) of Exercise 2.4.1.

Problem 4

Problem 5

Problem 6

Problem 7

Problem 8

Problem 9

Problem 10

Problem 11

Problem 12

Problem 13

Problem 14

Problem 15

Problem 16

Problem 17

Problem 18

Problem 19

Problem 20

Problem 21

Problem 22

Problem 23

Problem 24

Problem 25

Problem 26

Problem 27

Problem 28

Problem 29

Problem 30

Problem 31

Problem 32

Problem 33