

# The Real Numbers and Real Analysis

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## 1 Construction of the Real Numbers

### 1.1 Axioms for the Natural Numbers

#### Problem 1

Fill in the missing details in the proof of Theorem 1.2.6.

*Proof.* We must show the uniqueness of the binary operation  $\cdot : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  that satisfies the following two properties for all  $n, m \in \mathbb{N}$ .

a.  $n \cdot 1 = n$ .

b.  $n \cdot s(m) = (n \cdot m) + n$ .

Suppose there are two binary operations  $\cdot$  and  $\times$  on  $\mathbb{N}$  that satisfy the two properties for all  $n, m \in \mathbb{N}$ . Let

$$G = \{x \in \mathbb{N} \mid n \cdot x = n \times x \text{ for all } n \in \mathbb{N}\}$$

We will prove that  $G = \mathbb{N}$ , which will imply that  $\cdot$  and  $\times$  are the same binary operation. It is clear that  $G \subseteq \mathbb{N}$ . By Part (a) applied to each of  $\cdot$  and  $\times$  we see that  $n \cdot 1 = n = n \times 1$  for all  $n \in \mathbb{N}$  and hence  $1 \in G$ . Now let  $q \in G$ . Let  $n \in \mathbb{N}$ . Then  $n \cdot q = n \times q$  by hypothesis on  $q$ . It then follows from Part (b) that  $n \cdot s(q) = (n \cdot q) + n = (n \times q) + n = n \times s(q)$ . Hence  $s(q) \in G$ . By Part (c) of the Peano Postulates we conclude that  $G = \mathbb{N}$ . ■

*Proof.* We must show the two properties hold. Now,  $n \cdot 1 = g_n(1) = n$ , which is Part (a), and  $n \cdot s(m) = g_n(s(m)) = (g_n \circ s)(m) = (h_n \circ g_n)(m) = g_n(m) + n = (n \cdot m) + n$ , which is Part (b). ■

#### Problem 2

Prove Theorem 1.2.7 (2) (3) (4) (7) (8) (9) (10) (11) (13).

*Proof.* Let  $a, b, c \in \mathbb{N}$ . We must show  $(a + b) + c = a + (b + c)$ . ■

*Proof.* Let  $a, b, c \in \mathbb{N}$ . We must show  $1 + a = s(a) = a + 1$ . ■

*Proof.* Let  $a, b, c \in \mathbb{N}$ . We must show  $a + b = b + a$ .

■

*Proof.* Let  $a, b, c \in \mathbb{N}$ . We must show  $a \cdot 1 = a = 1 \cdot a$ .

■

*Proof.* Let  $a, b, c \in \mathbb{N}$ . We must show  $(a + b)c = ac + bc$ .

■

*Proof.* Let  $a, b, c \in \mathbb{N}$ . We must show  $ab = ba$ .

■

*Proof.* Let  $a, b, c \in \mathbb{N}$ . We must show  $c(a + b) = ca + cb$ .

■

*Proof.* Let  $a, b, c \in \mathbb{N}$ . We must show  $(ab)c = a(bc)$ .

■

*Proof.* Let  $a, b, c \in \mathbb{N}$ . We must show  $ab = 1$  if and only if  $a = 1 = b$ .

■

Problem 3

Problem 4

Problem 5

Problem 6

Problem 7

Problem 8