

Algebraic Geometry by Thomas Garrity et. al.

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1 Conics

1.1 Conics over the Reals

Problem 1

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that for any $(x, y) \in C$, we also have

$$(-x, y) \in C.$$

Thus the curve is symmetric about the y-axis.

Proof. Let $(x, y) \in C$. Then $P(x, y) = y - x^2 = 0$. Let $x' = -x$ and note that $(-x)^2 = x^2$. Thus

$$P(-x, y) = y - (-x)^2 = y - x^2 = 0.$$

Thus $(-x, y) \in C$.

Problem 2

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that if $(x, y) \in C$, then we have $y \geq 0$.

Proof. Suppose $(x, y) \in C$. Then

$$P(x, y) = y - x^2 = 0 \iff y = x^2 \geq 0.$$

Thus $y \geq 0$.

Problem 3

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that for every $y \geq 0$, there is a point $(x, y) \in C$ with this y -coordinate. Now, for points $(x, y) \in C$,

show that if y goes to infinity, then one of the corresponding x -coordinates also approaches infinity while the other corresponding x coordinate must approach negative infinity.

Proof. Let $y \in \mathbb{R}$ such that $y \geq 0$. Let $x = \sqrt{y} \in \mathbb{R}$. Then

$$y - x^2 = y - (\sqrt{y})^2 = y - y = 0.$$

Thus $(x, y) = (\sqrt{y}, y) \in C$.

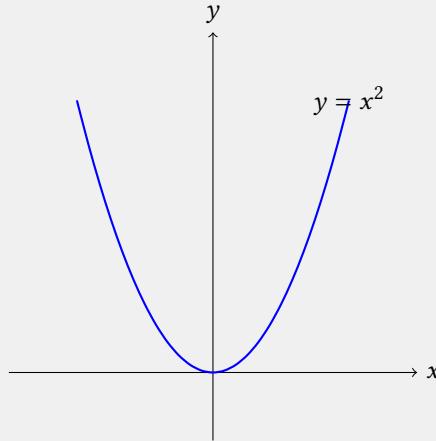
Now suppose $y \rightarrow \infty$. For points $(x, y) \in C$, we have

$$y - x^2 = 0 \iff x = \pm\sqrt{y}.$$

Since $y \rightarrow \infty$, we have $\sqrt{y} \rightarrow \infty$ and $-\sqrt{y} \rightarrow -\infty$. Thus one corresponding x -coordinate approaches infinity, while the other approaches negative infinity. \blacksquare

Problem 4

Sketch the curve $C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}$.



Problem 5

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \right\}.$$

Show that if $(x, y) \in C$, then the three points $(-x, y), (x, -y), (-x, -y)$ are also on C . Thus the curve C is symmetric about both the x - and y -axes.

Proof. Let $(x, y) \in \mathbb{R}^2$. Suppose $\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0$. Notice that $x^2 = (-x)^2$ and $y = (-y)^2$. Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{y^2}{9} - 1 = \frac{x^2}{4} + \frac{(-y)^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{(-y)^2}{9} - 1 = 0.$$

Thus $(-x, y), (x, -y), (-x, -y) \in C$. \blacksquare

Problem 6

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \right\}.$$

Show that for every $(x, y) \in C$, we have $|x| \leq 2$ and $|y| \leq 3$.

Proof. Let $(x, y) \in C$. Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \iff 9x^2 + 4y^2 - 36 = 0 \iff 9x^2 = -4y^2 + 36 \iff |x| = \sqrt{\frac{-4}{9}y^2 + 4} \leq \sqrt{4} = 2.$$

Similarly

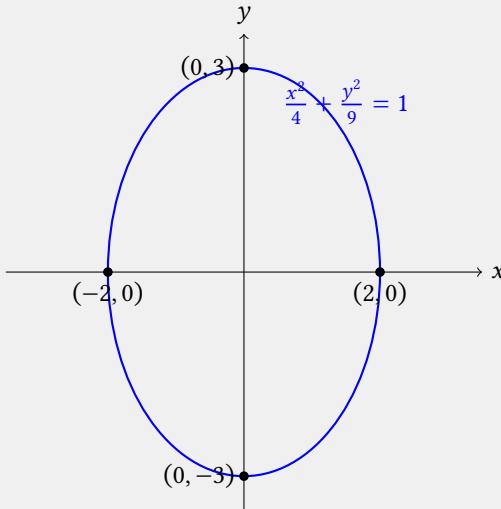
$$9x^2 + 4y^2 - 36 = 0 \iff |y| = \sqrt{\frac{-9}{4}x^2 + 9} \leq \sqrt{9} = 3.$$

■

Problem 7

Sketch

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \right\}.$$



Problem 8

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0 \right\}.$$

Show that if $(x, y) \in C$, then the three points $(-x, y)$, $(x, -y)$, and $(-x, -y)$ are also on C . Thus the curve C is also symmetric about the x - and y -axes.

Proof. Let $(x, y) \in \mathbb{R}^2$. Suppose $x^2 - y^2 - 4 = 0$. Notice that $x^2 = (-x)^2$ and $y^2 = (-y)^2$. Then

$$x^2 - y^2 - 4 = (-x)^2 - y^2 = x^2 - (-y)^2 = (-x)^2 - (-y)^2 = 0.$$

Thus $(-x, y), (x, -y), (-x, -y) \in C$.

■

Problem 9

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\}.$$

Show that if $(x, y) \in C$, then we have $|x| \geq 2$.

Proof. Let $(x, y) \in \mathbb{R}^2$. Suppose $x^2 - y^2 - 4 = 0$. Then

$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff |x| = \sqrt{y^2 + 4} \geq \sqrt{4} = 2.$$

■

Problem 10

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\}.$$

Show that the curve C is unbounded in the positive and negative x -directions and also unbounded in the positive and negative y -directions.

Proof. First notice

$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff x = \pm\sqrt{y^2 + 4} \iff y = \pm\sqrt{x^2 - 4}.$$

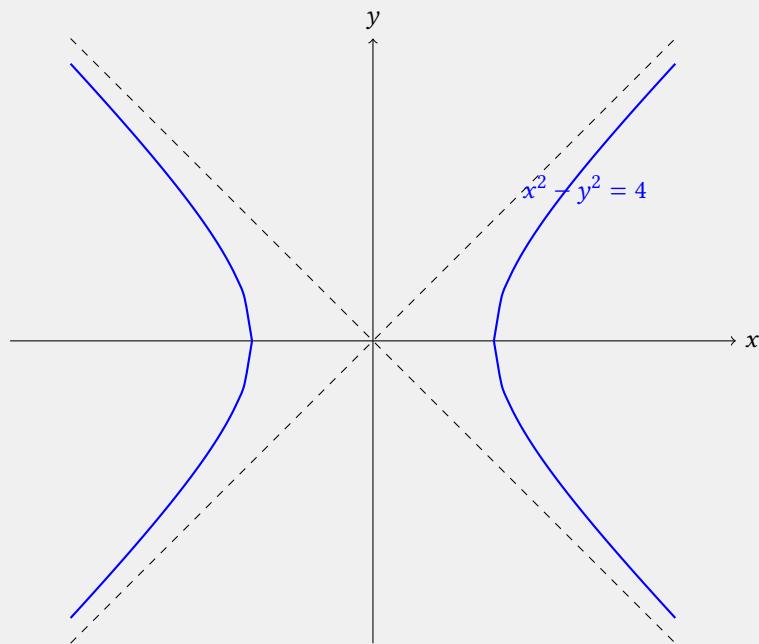
As $y \rightarrow \infty$, we have $x = \pm\sqrt{y^2 + 4} \rightarrow \infty$ and $-\infty$. Similarly, as $x \rightarrow \infty$, we have $y = \pm\sqrt{x^2 - 4} \rightarrow \infty$ and $-\infty$.

■

Problem 11

Sketch

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\}.$$

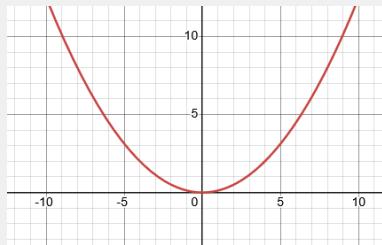


Problem 12

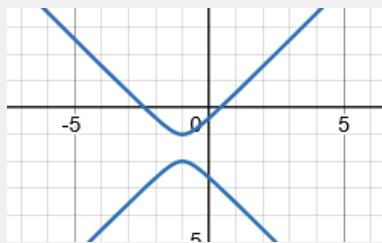
Sketch the graph of each of the following conics in \mathbb{R}^2 . Identify which are parabolas, ellipses, or Hyperbola.

1. $V(x^2 - 8y)$.
2. $V(x^2 + 2x - y^2 - 3y - 1)$.
3. $V(4x^2 + y^2)$.
4. $V(3x^2 + 3y^2 - 75)$.
5. $V(x^2 - 9y^2)$.
6. $V(4x^2 + y^2 - 8)$.
7. $V(x^2 + 9y^2 - 36)$.
8. $V(x^2 - 4y^2 - 16)$.
9. $V(y^2 - x^2 - 9)$.

Solution (1): Parabola.

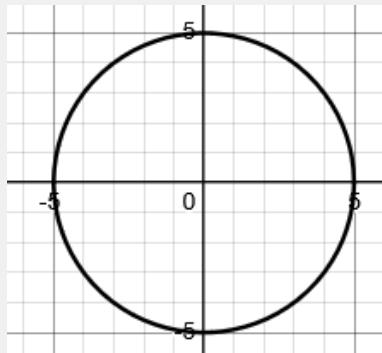


Solution (2): Hyperbola.



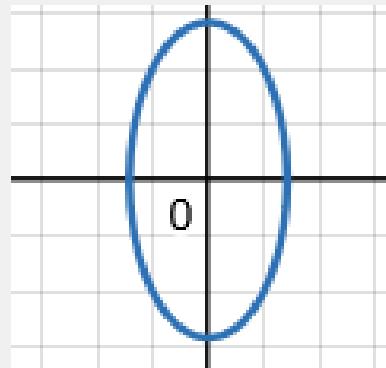
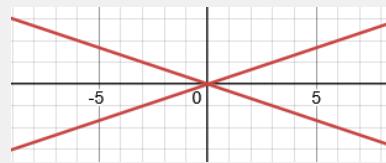
Solution (3): Point.

Solution (4): Ellipse.

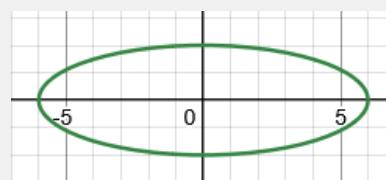


Solution (5): Two lines.

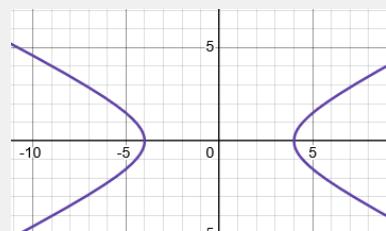
Solution (6): Ellipse.



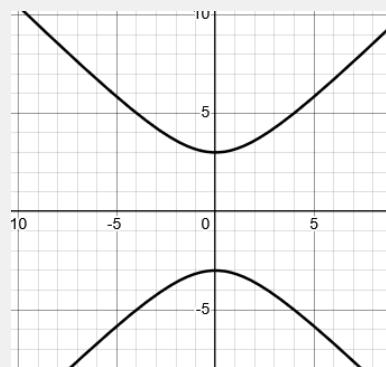
Solution (7): Ellipse.



Solution (8): Hyperbola.



Solution (9): Hyperbola.



Problem 13

Express the polynomial $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$ in the form

$$P(x, y) = Ax^2 + Bx + C,$$

where A, B , and C are polynomials in y . What are A, B , and C ?

Proof. Let $A = a$, $B = by + d$, and $C = cy^2 + ey + h$. Notice

$$ax^2 + bxy + cy^2 + dx + ey + h = ax^2 + bxy + dx + cy^2 + ey + h = ax^2 + (by + d)x + (cy^2 + ey + h) = Ax^2 + Bx + C.$$



Problem 14

Treating $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$ as a polynomial in the variable x , show that the discriminant is

$$\Delta_x(y) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

Proof. From Problem 13 we have $A = a$, $B = by + d$, and $C = cy^2 + ey + h$. Then

$$\Delta_x(y) = B^2 - 4AC = (by + d)^2 - 4a(cy^2 + ey + h) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$



Problem 15

1. Suppose $\Delta_x(y_0) < 0$. Explain why there is no point on $V(p)$ whose y -coordinate is y_0 .
2. Suppose $\Delta_x(y_0) = 0$. Explain why there is exactly one point on $V(P)$ whose y -coordinate is y_0 .
3. Suppose $\Delta_x(y_0) > 0$. Explain why there are exactly two points on $V(P)$ whose y -coordinate is y_0 .

Solution (a): In \mathbb{R} , the square root is undefined for values < 0 .

Solution (b): If $\Delta_x(y_0) = 0$, then $+\sqrt{B^2 - 4AC} = -\sqrt{B^2 - 4AC}$, so there is exactly one point on $V(P)$ whose y -coordinate is y_0 .

Solution (c): If $\Delta_x(y_0) > 0$, then $+\sqrt{B^2 - 4AC} \neq -\sqrt{B^2 - 4AC}$, so there are exactly two points on $V(P)$ whose y -coordinate is y_0 .

Problem 16

Suppose $b^2 - 4ac = 0$. Suppose further that $2bd - 4ae > 0$.

1. Show that $\triangle_x(y) \geq 0$ if and only if $y \geq \frac{4ah-d^2}{2bd-4ae}$.
2. Conclude that if $b^2 - 4ac = 0$ and $2bd - 4ae > 0$, then $V(P)$ is a parabola.

Proof. Suppose $\triangle_x(y) \geq 0$. Then

$$\begin{aligned}\triangle_x(y) &= (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah) \\ &= 0y^2 + (2bd - 4ae)y + (d^2 - 4ah).\end{aligned}$$

Therefore,

$$(2bd - 4ae)y + (d^2 - 4ah) \geq 0.$$

Since $2bd - 4ae > 0$, we have

$$y \geq \frac{4ah - d^2}{2bd - 4ae}.$$

Conversely, suppose $y \geq \frac{4ah - d^2}{2bd - 4ae}$. Then

$$\begin{aligned}\triangle_x(y) &= (2bd - 4ae)y + (d^2 - 4ah) \\ &\geq (2bd - 4ae)\left(\frac{4ah - d^2}{2bd - 4ae}\right) + (d^2 - 4ah) \\ &= 0.\end{aligned}$$

Proof. Suppose $b^2 - 4ac = 0$ and $2bd - 4ae > 0$. Then $\triangle_x(y) = (2bd - 4ae)y + (d^2 - 4ah)$. Now, $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$. It is clear that x is symmetrical, and since $y \geq \frac{4ah - d^2}{2bd - 4ae}$, $V(P)$ is a parabola. ■

Problem 17

Suppose $b^2 - 4ac < 0$.

1. Show that one of the following occurs:
 - (a) $\{y \mid \triangle_x(y) \geq 0\} = \emptyset$,
 - (b) $\{y \mid \triangle_x(y) \geq 0\} = \{y_0\}$,
 - (c) there exist real numbers α and β , $\alpha < \beta$, such that

$$\{y \mid \triangle_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

2. Conclude that $V(P)$ is either emptyset, a point, or an ellipse.

Proof. We can compare the quadratic term $(b^2 - 4ac)y^2 \leq 0$ with the linear term $(2bd - 4ae)y + (d^2 - 4ah)$ in $\triangle_x(y)$. ■

Problem 18

Problem 19