# Linear Algebra by Sterling K. Berberian

Frosty

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## Contents



## 1 Vector Spaces

### 1.1 Motivation (Vectors in 3-space)

**Problem 1**

Given the point $P = (2, -1, 3)$ and the vector $u = [-3, 4, 5]$, find the point $Q$ such that $\overrightarrow{PQ} = u$.

*Proof.* We require $\vec{u} = [-3, 4, 5] = (q_1 - 2, q_2 - (-1), q_3 - 3)$. Solving componentwise, we see $(q_1, q_2, q_3) = (-1, 3, 8)$. Thus, $Q = (-1, 3, 8)$. ∎

**Problem 2**

Given the points $P(2, -1, 3)$, $Q(3, 4, 1)$, $R(4, -3, 4)$, $S(5, 2, 2)$. True or false (explain:) $PQSR$ is a parallelogram.

*Proof.* For $PQSR$ to be a parallelogram, we require $\vec{PQ} = \vec{SR}$.

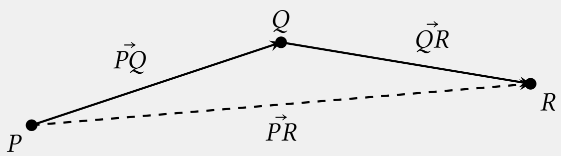
$$\vec{PQ} = [3 - 2,\ 4 - (-1),\ 1 - 3] = [1, 5, -2]$$

$$\vec{SR} = [4 - 5,\ -3 - 2,\ 4 - 2] = [-1, -5, 2]$$

Since
$$[1, 5, -2] \neq [-1, -5, 2],$$
it follows that $PQSR$ is not a parallelogram. ∎

**Problem 4**

Show that for every triple points $P, Q, R$, $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$. [Method 1: Calculate components. Method 2: Draw a picture.]




## 1.2 $\mathbb{R}^n$ and $\mathbb{C}^n$

**Problem 2**

Consider the vectors $u = (2, 1)$, $v = (-5, 3)$, $w = (3, 4)$ in $\mathbb{R}^2$. Do there exists real numbers $a, b$ such that $au + bv = w$? What if $v = (6, 3)$.

*Proof.* We require $au+bv = w \iff a(2, 1)+b(-5, 3) = (3, 4) \iff (2a, a)+(-5b, 3b) = (3, 4) \iff (2a-5b, a+3b) = (3, 4)$. Thus
$$\begin{cases} 2a - 5b = 3 \\ a + 3b = 4 \end{cases}$$
which has the solution $a = \frac{29}{11}$, $b = \frac{5}{11}$. If $v = (6, 3)$, we require
$$a(2, 1) + b(6, 3) = (3, 4) \iff (2a + 6b, a + 3b) = (3, 4).$$
Thus
$$\begin{cases} 2a + 6b = 3 \\ a + 3b = 4 \end{cases}$$
which has no solution. To see this has no solution, note
$$a + 3b = 4 \implies 6b = 8 - 2a.$$
Plugging into the first equation gives
$$2a + (8 - 2a) = 3 \implies 8 = 3,$$
which is a contradiction. ∎

**Problem 3**

Give a detailed proof of (8) of Theorem 1.2.3 (just checking!). [Write out all steps of the proof and give a reason for each step.]

*Proof.* Let $x = [x_1, x_2, \dots, x_n] \in F^n$ be a vector and let $c, d \in F$ where $F$ is a field. Then

$$(cd)x = [(cd)x_1, (cd)x_2, \dots, (cd)x_n].$$

Since $F$ is a field, scalar multiplication in $F$ is associative thus

$$(cd)x_i = c(dx_i) \quad \text{for each } i = 1, 2, \dots, n.$$

Therefore,

$$(cd)x = [c(dx_1), c(dx_2), \dots, c(dx_n)] = c(dx),$$

$\blacksquare$

**Problem 4**

In Definition 1.2.1, $n = 1$ is not ruled out. What do the elements of $\mathbb{R}^1$ look like? Describe sums and scalar multiples in $\mathbb{R}^1$.

*Proof.* The elements of $\mathbb{R}^1$ look like $[x_1]$ where $x_1 \in \mathbb{R}$. Sums and scalar multiples behave identically to the operations in the field $\mathbb{R}$. $\blacksquare$

## 1.3 Vectors Spaces: The Axioms some Examples

**Problem 1**

Let $V$ be a vector space over a field $F$. Let $T$ be a nonempty set and let $W = \mathscr{F}(T, V)$ be the set of all functions $x : T \longrightarrow V$. Show that $W$ can be made into a vector space over $F$ in a natural way. [Hint: Use the definitions in Example 1.3.4 as a guide.]

*Proof.* For $x, y \in W$, $x = y$ means that $x(t) = y(t)$ for all $t \in T$. If $x, y \in W$ and $c \in F$, define functions $x + y$ and $cx$ by the formulas $(x + y)(t) = x(t) + y(t)$ and $(cx)(t) = cx(t)$ for all $t \in T$. Let $\theta$ be the function defined by $\theta(t) = 0$ for all $t \in T$, and for $x \in W$, let $-x$ be the function defined by $(-x)(t) = -x(t)$ for all $t \in T$. $\blacksquare$

**Problem 2**

The definition of a vector space (1.3.1) can be formulated as follows. A vector space over $F$ is a nonempty set $V$ together with a pair of mappings $\sigma : V \longrightarrow V$ and $\mu : F \times V \longrightarrow V$. ($\sigma$ suggests 'sum' and $\mu$ suggests 'multiple') having the following properties $\sigma(x, y) = \sigma(y, x)$ for all $x, y \in V$; $\sigma(\sigma(x, y), z) = \sigma(x, \sigma(y, z))$ for all $x, y, z \in V$ etc. The exercise: Write out the 'etc.' in detail.

*Proof.* We list the $9$ axioms below:

1. $x, y \in V \implies \sigma(x, y) \in V$
2. $x \in V, c \in F \implies \mu(c, x) \in V$
3. $x, y \in V \implies \sigma(x, y) = \sigma(y, x)$
4. $x, y, z \in V \implies \sigma(x, \sigma(y, z)) = \sigma(\sigma(x, y), z)$
5. $\exists 0 \in V$ such that $\sigma(0, v) = v = \sigma(v, 0)$
6. $x \in V \implies -x \in V$ such that $\sigma(x, -x) = 0 = \sigma(-x, x)$
7. $x, y \in V$ and $c \in F \implies \mu(c, \sigma(x, y)) = \sigma(\mu(c, x), \mu(c, y))$
8. $\exists 1 \in F$ such that $x \in V \implies \mu(1, x) = x$

9. $a, b \in F$ and $x \in V \implies \mu(a, \mu(b, x)) = \mu(ab, x)$

∎

**Problem 3**

Let $V$ be a complex vector space (1.3.2). Show that $V$ is also a real vector space (sums as usual, scalar multiplication restricted to real scalars). These two ways of looking at $V$ may be indicated by writing $V_c$ and $V_r$.

*Proof.* Clearly $V_r$ is closed under scalar multiplication and vector addiction and is a vector space. ∎

**Problem 4**

Every real vector space $V$ can be 'embedded' in a complex vector space $W$ in the following way: let $W = V \times V$ be the real vector space constructed as in example 1.3.10 and define multiplication by complex scalars by the formula $(a + bi)(x, y) = (ax - by, bx + ay)$ for $a, b \in \mathbb{R}$ and $(x, y) \in W$. [In particular, $i(x, y) = (-y, x)$. Think of $(x, y)$ as '$x + iy$'] Show that $W$ satisfies the axioms for a complex vector space ($W$ is called the *complexification* of $V$.)

*Proof.* Below are the 9 vector space axioms.

1. Let $x, y \in W$. Then $x = (v_1, v_2), y = (w_1, w_2) \in V \times V$, and $(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2)$. Now, $v_1 + w_1 \in V$ and $v_2 + w_2 \in V$, thus $(v_1 + w_1, v_2 + w_2) \in V \times V$.
2. Let $x \in W$ and $c \in \mathbb{C}$. Then $x = (v_1, v_2) \in V \times V$, $c = a + bi \in \mathbb{C}$, and $(a + bi)(v_1, v_2) = (av_1 - bv_2, bv_1 + av_2)$. Clearly $av_1 - bv_2 \in V$ and $bv_1 + av_2 \in V$, thus $(av_1 - bv_2, bv_1 + av_2) \in V \times V$.
3. Let $x, y \in W$. Then $x = (v_1, v_2), y = (w_1, w_2) \in V \times V$. Then $x + y = (v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2) = (w_1 + v_1, w_2 + v_2) = (w_1, w_2) + (v_1, v_2) = y + x$.
4. Let $x, y, z \in W$. Then $x = (v_1, v_2), y = (w_1, w_2), z = (s_1, s_2) \in V \times V$. Then $(x + y) + z = [(v_1, v_2) + (w_1, w_2)] + (s_1, s_2) = (v_1 + w_1, v_2 + w_2) + (s_1, s_2) = ((v_1 + w_1) + s_1, (v_2 + w_2) + s_2) = (v_1 + (w_1 + s_1), v_2 + (w_2 + s_2)) = (v_1, v_2) + (w_1 + s_1, w_2 + s_2) = (v_1, v_2) + [(w_1, w_2) + (s_1, s_2)] = x + (y + z)$.
5. Let $x \in W$. Then $x = (v_1, v_2) \in V \times V$ and $x + 0 = (v_1, v_2) + (0, 0) = (v_1 + 0, v_2 + 0) = (v_1, v_2)$.
6. Let $x \in W$. Then $x = (v_1, v_2) \in V \times V$ and $x + (-x) = (v_1, v_2) + (-v_1, -v_2) = (v_1 - v_1, v_2 - v_2) = (0, 0) = 0 = (-v_1, -v_2) + (v_1, v_2) = (-x) + x$.
7. Let $a + bi \in \mathbb{C}$ and $x, y \in W$. Then $(a + bi)((v_1, v_2) + (w_1, w_2)) = (a + bi)(v_1 + w_1, v_2 + w_2) = (a(v_1 + w_1) - b(v_2 + w_2), b(v_1 + w_1) + a(v_2 + w_2)) = (av_1 - bv_2, bv_1 + av_2) + (aw_1 - bw_2, bw_1 + aw_2) = (a + bi)(v_1, v_2) + (a + bi)(w_1, w_2)$.
8. Let $a + bi, c + di \in \mathbb{C}$ and $x = (v_1, v_2) \in W$. Then $(a + bi)((c + di)(v_1, v_2)) = (a + bi)(cv_1 - dv_2, dv_1 + cv_2) = ((ac - bd)v_1 - (ad + bc)v_2, (bc + ad)v_1 + (bd + ac)v_2) = ((a + bi)(c + di))(v_1, v_2)$.
9. Let $x = (v_1, v_2) \in W$. Then $1 \cdot (v_1, v_2) = (1v_1 - 0v_2, 0v_1 + 1v_2) = (v_1, v_2)$.

∎

## 1.4 Vector Spaces: First Consequences of the Axioms

**Problem 1**

In a vector space, ① $x - y = \theta$ if and only if ② $x = y$; ③ $x + y = z$ if and only if ④ $x = z - y$.

*Proof.* (① ⟶ ②) Suppose $x - y = \theta \iff x + (-y) = \theta$. It follows that $-y = -x \iff y = x$.

(② ⟶ ①) Suppose $x = y \iff -x = -y$. Then $x + (-x) = x + (-y) = x - y = \theta$.

(③ ⟶ ④) Suppose $x + y = z \iff x + y + (-y) = z + (-y) \iff x = z - y$.

(④ ⟶ ③) Suppose $x = z - y \iff x + y = z - y + y \iff x + y = z$. ∎

**Problem 2**

Let $V$ be a vector space over a field $F$. If $x$ is a fixed nonzero vector, then the mapping $f : F \longrightarrow V$ defined by $f(c) = cx$ is injective. If $c$ is a fixed nonzero scalar, then the mapping $g : V \longrightarrow V$ defined by $g(x) = cx$ is bijective.

*Proof.* Suppose $x$ is a fixed nonzero vector. Let $c, c' \in F$ such that $f(c) = f(c') \iff cx = c'x \iff c = c'$. The last step follows from Corollary 1.4.7 (ii). Thus $f$ is injective. ∎

*Proof.* Suppose $c$ is a fixed nonzero scalar. The inverse function to $g(x) = cx$ is clearly $g^{-1}(x) = \frac{x}{c}$. Thus $g$ is bijective. ∎

**Problem 3**

If $V$ is a vector space and $y$ is a fixed vector, then the mapping $\tau : V \longrightarrow V$ defined by $\tau(x) = x + y$ is bijective (it is called *translation* by the vector $y$).

*Proof.* Suppose $V$ is a vector space and $y$ is a fixed vector. The inverse function to $\tau(x) = x + y$ is clearly $\tau^{-1}(x) = x - y$. Thus $\tau$ is bijective. ∎

**Problem 4**

If $a$ is a nonzero scalar and $b$ is vector in the space $V$, then the equation $ax + b = \theta$ has a unique solution $x$ in $V$.

*Proof.* Suppose $a$ is a nonzero scalar and $b$ is a vector in $V$. The equation $ax + b = \theta$ is equivalent to $ax = -b$. Multiplying both sides by $a^{-1}$ gives $a^{-1}(ax) = a^{-1}(-b) \iff (a^{-1}a)x = -a^{-1}b \iff x = -a^{-1}b$. ∎

**Problem 5**

If, in a vector space, $\theta'$ is a vector such that $0' + x = x$ for even a single vector $x$S, then $\theta' = \theta$. [Hint: Add $-x$ to both sides of the equation.]

*Proof.* Suppose in a vector space $\theta'$ is a vector such that $\theta' + x = \theta$. Adding $-x$ to both sides gives $\theta' + x + (-x) = x + (-x) \iff \theta' + \theta = \theta \iff \theta' = \theta$. ∎

## 1.5 Linear Combinations of Vectors

**Problem 1**

Express the function $f(t) = \cos(t - \pi/3)$ as a linear combination of the functions $\sin t$ and $\cos t$.

*Proof.* We use the standard trig identity to find $f(t) = \cos(t - \pi/3) = \cos(\pi/3)\cos(t) + \sin(\pi/3)\sin(t)$. ∎

**Problem 2**

Express the function $e^t$ as a linear combination of the hyperbolic functions $\cosh t$ and $\sinh t$.

*Proof.* We have the standard identity $e^t = \cosh(t) + \sinh(t)$. ∎

**Problem 3**

Convince yourself of the correctness of the following formulas pertaining to linear combinations in a vector space:

1. $c \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} c x_i$.
2. $\sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} (a_i + b_i) x_i$.
3. $\sum_{i=1}^{m} a_i x_I + \sum_{j=1}^{n} b_j y_j = \sum_{k=1}^{m+n} c_k z_k$. where $c_k = a_k$ and $z_k = x_k$ for $k = 1, 2, \dots, m$, while $c_k = b_{k-m}$ and $z_k = y_{k-m}$ for $k = m+1, m+2, \dots, m+n$.
4. $$\begin{aligned} \left(\sum_{i=1}^{m} c_i\right)\left(\sum_{j=1}^{n} x_j\right) &= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_i x_j\right) \\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{m} c_i x_j\right) \\ &= \sum_{i,j} c_i x_j \end{aligned}$$

The last expression signifying with sum, with $mn$ terms, of the vectors $c_i x_j$ for all possible combinations of $i$ and $j$.

*Proof.* I'm convinced. ∎

**Problem 5**

Show that in a vector space, $x - y$ is a linear combination of $x$ and $y$.

*Proof.* Let $\alpha = 1$ and $\beta = -1$ and it follows that $x - y = x + (-y) = \alpha x + \beta y$. ∎

**Problem 6**

True or false (explain): In $\mathbb{R}^3$,

1. the vector $x = (0, 6, 3)$ is a linear combination of $(2, 1, 0)$ and $(-3, 2, 0)$;
2. the vector $x = (2, 1, -2)$ is a linear combination of $(2, 1, 0)$ and $(-3, 2, 0)$.

*Proof.* We must have

$$(0, 6, 3) = \alpha(2, 1, 0) + \beta(-3, 2, 0) = (2\alpha - 3\beta, \alpha + 2\beta, 0)$$

for $\alpha, \beta \in \mathbb{R}$. We obtain

1. $0 = 2\alpha - 3\beta$
2. $6 = \alpha + 2\beta$
3. $3 = 0$

Since $3 \neq 0$, no solution exists. Now suppose

$$(2, 1, -2) = \alpha(2, 1, 0) + \beta(-3, 2, 0) = (2\alpha - 3\beta, \alpha + 2\beta, 0).$$

We obtain

1. $2 = 2\alpha - 3\beta$
2. $1 = \alpha + 2\beta$
3. $-2 = 0$

Since $-2 \neq 0$, no solution exists. ∎

## 1.6 Linear Subspaces

**Problem 1**

Let $M$ and $N$ be the subsets of $\mathbb{R}^2$ defined as follows:

$$M = \{(a, 0) \mid a \in \mathbb{R}\}, N = \{(0, b) \mid b \in \mathbb{R}\}$$

Show that $M$ and $N$ are linear subspaces of $\mathbb{R}^2$ and describe the linear subspaces $M \cap N$ and $M + N$.

*Proof.* First note $\theta = (0, 0) \in M, N$. Let $x, y \in M$ such that $x = (a_1, 0), y = (a_2, 0)$. Then $x + y = (a_1, 0) + (a_2, 0) = (a_1 + a_2, 0) \in M$. Let $c \in \mathbb{R}$. Then $cx = c(a_1, 0) = (ca_1, 0) \in M$. The argument for $N$ is the same. The linear subspace $M + N$ is all of $\mathbb{R}^2$, and the linear subspace $M \cap N$ is $\{(0, 0)\}$. ∎

**Problem 2**

Let $\mathscr{F} = \mathscr{R}$ as in Example 1.6.5, fix a nonempty subset $T$ of $\mathbb{R}$, and let

$$M = \{x \in \mathscr{F} \mid x(t) = 0 \text{ for all } t \in T\}$$

Show that $M$ is a linear subspace of $\mathscr{F}$.

*Proof.* Let $x, y \in M$. Then for all $t \in T$, $(x + y)(t) = x(t) + y(t) = 0 + 0 = 0$, so $x + y \in M$. Let $x \in M$ and $c \in \mathbb{R}$. Then for all $t \in T$, $(cx)(t) = c \cdot x(t) = c \cdot 0 = 0$, so $cx \in M$. Thus $M$ is a linear subspace of $\mathscr{F}$. ∎

**Problem 3**

With $\mathscr{F} = \mathscr{F}(\mathbb{R})$ as in 1.6.5, let

$$M = \{x \in \mathscr{F} \mid x = 0 \text{ on } (-\infty, 0]\}$$

$$N = \{x \in \mathscr{F} \mid x = 0 \text{ on } (0, +\infty)\}$$

Prove that $M + N = \mathscr{F}$ and $M \cap N = \{\theta\}$.

*Proof.* Let $f \in \mathscr{F}$. Define

$$x(t) = \begin{cases} f(t), & t > 0 \\ 0, & t \leq 0 \end{cases} \in M, \quad y(t) = \begin{cases} f(t), & t \leq 0 \\ 0, & t > 0 \end{cases} \in N.$$

Then $f = x + y$, so $M + N = \mathscr{F}$. Now, let $z \in M \cap N$. Then $z(t) = 0$ for $t \leq 0$ and $z(t) = 0$ for $t > 0$. Thus $z = \theta$ so $M \cap N = \{\theta\}$. ∎

**Problem 4**

Let $M$ and $N$ be linear subspaces of a vector space $V$, and let

$$M \cup N = \{x \in V \mid x \in M \text{ or } x \in N\}$$

be the union of $M$ and $N$. True or false (explain): $M \cup N$ is a linear subspace of $V$. [If true, give a proof. If false, give a counterexample (an example of $V$, $M$, $N$ for which $M \cup N$ is not a linear subspace of $V$).]

*Proof.* Let $M$ and $N$ be the subsets of $\mathbb{R}^2$ defined as follows:

$$M = \{(a, 0) \mid a \in \mathbb{R}\}, \quad N = \{(0, b) \mid b \in \mathbb{R}\}.$$

We showed in Problem 1 that $M$ and $N$ are linear subspaces of $\mathbb{R}^2$. But notice that $(1, 0) \in M$ and $(0, 1) \in N$, yet $(1, 0) + (0, 1) = (1, 1) \notin M \cup N$. $\blacksquare$

**Problem 6**

Let $V$ be a vector space and let $A$ be any subset of $V$. There exist linear subspaces of $V$ that contain $A$ (for instance, $V$ itself is such a subspace). Let $\mathscr{n} = \{N \mid N \text{ is a linear subspace of } V \text{ containing } A\}$. Show that $\mathscr{n}$ contains a smallest element. [Try $\cap\mathscr{n}$.] Thus, there exists a smallest linear subpsace of $V$ containing $A$.

*Proof.* Let $P$ be an arbitrary linear subspace in $\mathscr{n}$. Clearly, all elements in $\cap\mathscr{n}$ are in $P$, thus $\cap\mathscr{n} \subseteq P$. Now, since all linear subspaces in $\mathscr{n}$ contain $0$, we have $0 \in \cap\mathscr{n}$. Let $x, y \in \cap\mathscr{n}$ and $c \in F$, where $F$ is the field of scalars of $V$. Let $V$ be an arbitrary element in $\mathscr{n}$. Since $x, y \in \cap\mathscr{n}$, it follows that $x, y \in V$, thus $x + y \in V$. Therefore, $\cap\mathscr{n}$ is a linear subspace of $V$. $\blacksquare$

**Problem 7**

Let $V$ be a vector space and let $\mathscr{n}$ be any set of linear subspaces of $V$. Let $A$ be the *union* of the subspaces in $\mathscr{n}$, that is,

$$A = \cup\mathscr{n} = \{x \in V \mid x \in M \text{ for some } M \in \mathscr{n}\}$$

Apply Exercise 6 to conclude that there exists a linear subspace of $V$ that contains every $M \in \mathscr{n}$.

*Proof.* Clearly $\cup\mathscr{n} \subseteq V$. Let $A = \cup\mathscr{n}$. By Problem 6, there exists a smallest linear subspace of $V$ containing $A$. Since for every $M \in \mathscr{n}$, $M \subseteq A$, this linear subspace contains every $M \in \mathscr{n}$. $\blacksquare$

**Problem 8**

A linear subspace $M$ of a vector of a vector space $V$ is closed under subtraction: if $x, y \in M$ then $x - y \in M$.

*Proof.* Suppose $x, y \in M$, where $M$ is a linear subspace of a vector space $V$. Let $c = -1$. Then $cy = -1 \cdot y = -y \in M$. Thus $x + (-y) = x - y \in M$. $\blacksquare$

**Problem 9**

Let $V$ be a vector space and let $\mathscr{n}$ be a set of linear subspaces of $V$, having the following property: if $M, N \in \mathscr{n}$ then there exists $P \in \mathscr{n}$ such that $M \subset P$. (That is, any two elements of $\mathscr{n}$ are contained in a third.) Prove that $\cup\mathscr{n}$ is a linear subspace of $V$.

*Proof.* Let $x, y \in \cup\mathscr{n}$ and let $c$ be a scalar of the vector space $V$. Then $x \in M$ for some $M \in \mathscr{n}$. Since $M$ is a linear subspace, $cx \in M$, thus $cx \in \cup\mathscr{n}$. Now $y \in N$ for some $N \in \mathscr{n}$. There exists $P \in \mathscr{n}$ such that $M \subseteq P$ and $N \subseteq P$. Thus $x, y \in P$. Since $P$ is a linear subspace, $x + y \in P$, thus $x + y \in \cup\mathscr{n}$. It follows that $\cup\mathscr{n}$ is a linear subspace of $V$. ∎

### Problem 11

Let $M$ and $N$ be linear subspaces of $V$. Prove that the following conditions are equivalent:

1. $M \cap N = \{\theta\}$;
2. if $y \in M$, $z \in N$ and $y + z = \theta$, then $y = z = \theta$;
3. if $y + z = y' + z'$, where $y, y' \in M$ and $z, z' \in N$, then $y = y'$ and $z = z'$.

*Proof.* ($1 \longrightarrow 2$) Suppose $M \cap N = \{\theta\}$. Furthermore, suppose $y \in M$, $z \in N$ and $y + z = \theta$. Then $y = -z \in M$, so $z \in M \cap N$. Since $M \cap N = \{\theta\}$, it follows that $z = \theta$. Then $y = -z = -\theta = \theta$.

($2 \longrightarrow 3$) Suppose that if $y \in M$, $z \in N$ and $y + z = \theta$, then $y = z = \theta$. Suppose $y + z = y' + z'$, where $y, y' \in M$ and $z, z' \in N$. Then $(y - y') + (z - z') = \theta$. By 2 it follows that $y - y' = \theta$ and $z - z' = \theta$, so $y = y'$ and $z = z'$.

($3 \longrightarrow 1$) Suppose that if $y + z = y' + z'$, where $y, y' \in M$ and $z, z' \in N$, then $y = y'$ and $z = z'$. Let $x \in M \cap N$. Then $x \in M$ and $x \in N$. Consider $x + \theta = \theta + x$. By 3 it follows that$x = \theta$ and $\theta = x$. Therefore $M \cap N = \{\theta\}$. ∎

### Problem 12

Let $M$ and $N$ be linear subspaces of $V$. If $M + N = V$ and $M \cap N = \{\theta\}$ (cf. Exercise 11), $V$ is said to be the *direct sum* of $M$ and $N$, written $V = M \oplus N$. Prove that the following conditions are equivalent:

1. $V = M \oplus N$;
2. for each $x \in V$, there exist unique elements $y \in M$ and $z \in N$ such that $x = y + z$.

*Proof.* ($\longrightarrow$) Suppose $V = M \oplus N$. Let $x$ be an arbitrary element in $V$. Then there exists $m \in M$ and $n \in N$ such that $x = m + n$. Suppose there also exists $m' \in M$ and $n' \in N$ such that $x = m' + n'$. Then $m + n = m' + n'$. By Problem 11, $m = m'$ and $n = n'$, thus $m$ and $n$ are unique.

($\longleftarrow$) Suppose for each $x \in V$, there exist unique elements $y \in M$ and $z \in N$ such that $x = y + z$. Let $x$ be an arbitrary element in $V$. Clearly $x = y + z$ where $y \in M$ and $z \in N$, thus $x \in M + N$. Let $x$ be an arbitrary element in $M + N$. Thus $x = m + n$ for some $m \in M$ and $n \in N$. Therefore $V = M + N$.

Let $x \in M \cap N$. Then

$$x = x + \theta \quad \text{with } x \in M,\ \theta \in N,$$

and

$$x = \theta + x \quad \text{with } \theta \in M,\ x \in N.$$

Thus $x = \theta$. Therefore $M \cap N = \{\theta\}$. ∎

### Problem 13

Let $V$ be the real vector space of all functions $x : \mathbb{R} \longrightarrow \mathbb{R}$ (1.3.4). Call $y \in V$ *even* if $y(-t) = y(t)$ for all $t \in \mathbb{R}$, and call $z \in V$ *odd* if $z(-t) = -z(t)$ for all $t \in \mathbb{R}$. Let

$$M = \{y \in V \mid y \text{ is even}\}, \quad N = \{z \in V \mid z \text{ is odd}\}.$$

1. Prove that $M$ and $N$ are linear subspaces of $V$ and that $V = M \oplus N$ in the sense of Exercise 12. [Hint: If $x \in V$ consider the functions of $y$ and $z$ defined by $y(t) = \frac{1}{2}[x(t) + x(-t)]$, $z(t) =$

$\frac{1}{2}[x(t) - x(-t)]]$.

2. What does (i) say for $x(t) = e^t$?
3. What does (i) say for a polynomial function $x$?

*Proof.* Clearly the zero function is in both $M$ and $N$. Let $x, y \in M$ and let $c \in \mathbb{R}$. Then $x(-t) = x(t)$ and it follows that $cx(-t) = cx(t)$, thus $cx \in M$. Also,

$$(x + y)(-t) = x(-t) + y(-t) = x(t) + y(t) = (x + y)(t),$$

thus $x + y \in M$. Let $a, b \in N$ and let $c \in \mathbb{R}$. Then $a(-t) = -a(t)$ and it follows that $ca(-t) = -ca(t)$, thus $ca \in N$. Also,

$$(a + b)(-t) = a(-t) + b(-t) = -a(t) - b(t) = -(a + b)(t),$$

thus $a + b \in N$. It follows that $M$ and $N$ are linear subspaces of $V$. ∎

*Proof.* Let $x \in V$ be arbitrary. Define functions $y$ and $z$ by

$$y(t) = \frac{1}{2}\big[x(t) + x(-t)\big], \qquad z(t) = \frac{1}{2}\big[x(t) - x(-t)\big].$$

Notice

$$y(-t) = \frac{1}{2}\big[x(-t) + x(t)\big] = y(t),$$

so $y$ is even and thus $y \in M$. Also

$$z(-t) = \frac{1}{2}\big[x(-t) - x(t)\big] = -\frac{1}{2}\big[x(t) - x(-t)\big] = -z(t),$$

so $z$ is odd and thus $z \in N$. Now, for all $t \in \mathbb{R}$,

$$y(t) + z(t) = \frac{1}{2}\big[x(t) + x(-t)\big] + \frac{1}{2}\big[x(t) - x(-t)\big] = x(t).$$

Thus $x = y + z$ and therefore $V = M + N$. Suppose $x \in M \cap N$. Then $x$ is both even and odd and

$$x(t) = x(-t) \quad \text{and} \quad x(t) = -x(-t).$$

Thus $x(t) = -x(t)$ for all $t$, $x(t) = 0$ for all $t$. Thus $M \cap N = \{\theta\}$. Therefore $V = M \oplus N$. ∎

**Solution (2):** It says that $e^t$ can be written uniquely as the sum of an even function and an odd function.

**Solution (3):** It says that any polynomial function can be written uniquely as the sum of an even polynomial and an odd polynomial.

## Problem 16

Let $V$ and $W$ be vector spaces over the same field $F$ and let $V \times W$ be the product vector space (1.3.11). If $M$ is a linear subspace of $V$, and $N$ is a linear subspace of $W$, show that $M \times N$ is a linear subspace of $V \times W$.

*Proof.* Suppose $M$ is a linear subspace of $V$ and $N$ is a linear subspace of $W$. Let $x, y \in M \times N$ and let $c$ be a scalar from the field $F$. Then $x = (m_1, n_1)$ and $y = (m_2, n_2)$ are elements of $M \times N$, and $cx = c(m_1, n_1) = (cm_1, cn_1)$. Now, $cm_1 \in M$ and $cn_1 \in N$, thus $cx \in M \times N$. Also $x + y = (m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$. Since $m_1 + m_2 \in M$ and $n_1 + n_2 \in N$, it follows that $x + y \in M \times N$. Since $(\theta_V, \theta_W) \in M \times N$, it follows that $M \times N$ is a linear subspace of $V \times W$. ∎

Problem 17

If $V = V_1 \times V_2$ (1.3.11), $M_1 = \{(x_1, \theta) \mid x_1 \in V_1\}$ and $M_2 = \{(\theta, x_2) \mid x_2 \in V_2\}$, then $V = M_1 + M_2$ and $M_1 \cap M_2 = \{\theta\}$ (where $\theta$ stands for the zero vector of all vector spaces in sight). In the terminology of Exercise 12, $V$ is the direct sum of its subspaces in $M_1$ and $M_2$, that is, $V = M_1 \oplus M_2$. (Generalization?)

**Definition 1.** *If $V = V_1 \times V_2 \times \cdots \times V_n$, and*

1. $M_1 = \{(x_1, \theta_2, \dots, \theta_n) \mid x_1 \in V_1\}$
2. $M_2 = \{(\theta_1, x_2, \dots, \theta_n) \mid x_2 \in V_2\}$
3. ...
4. $M_n = \{(\theta_1, \theta_2, \dots, x_n) \mid x_n \in V_n\}$

*then $V$ is the direct sum of its subspaces $M_1, M_2, \dots, M_n$, or*

$$V = M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

Problem 18

Let $M$ and $N$ be linear subspaces of $V$ whose union $M \cup N$ is also a linear subspace. Prove that either $M \subset N$ or $N \subset M$. [Hint: Assume to the contrary that neither of $M, N$ is contained in the other; choose vectors $y \in M$, $z \in N$ with $y \notin N$, $z \notin M$ and try to find a home for $y + z$.]

*Proof.* Let $y \in M$ and $z \in N$ such that $y \notin N$ and $z \notin M$. Then $y \in M \cup N$ and $z \in M \cup N$. Since $M \cup N$ is a linear subspace, $y + z \in M \cup N$. Without loss of generality, suppose $y + z \in M$. Then $-y \in M$ by Problem 8. Since $M$ is closed under addition $y + z + (-y) = z \in M$ which is a contradiction. ∎

# 2 Linear Mappings

## 2.1 Linear Mappings

Problem 2

Let $T$ be a set, $V = \mathscr{F}(T, \mathbb{R})$ the vector space of all functions $x : T \longrightarrow \mathbb{R}$ (Example 1.3.4). Fix a point $t \in T$ and define $f : V \longrightarrow \mathbb{R}$ by the formula $f(x) = x(t)$. Then $f$ is a linear form on $V$.

*Proof.* Let $x, y \in V$ and $c \in \mathbb{R}$. Then $f(x + y) = (x + y)(t) = x(t) + y(t) = f(x) + f(y)$. and $f(cx) = (cx)(t) = c\,x(t) = cf(x)$. Thus $f$ is a linear mapping. Since it maps functions in $V$ to scalars in $\mathbb{R}$ it is a linear form. ∎

Problem 3

If $T : V \longrightarrow W$ is a linear mapping, then $T(x - y) = Tx - Ty$ for all $x, y \in V$.

*Proof.* Suppose $T : V \longrightarrow W$ is a linear mapping. Let $x, y$ be arbitrary elements in $V$. Then $T(x - y) = T(x + (-y)) = Tx + T(-y) = Tx + (-T(y)) = Tx - Ty$. ∎

Problem 4

If $V$ is a vector space, prove that the mapping $T : V \times V \longrightarrow V$ defined by $T(x, y) = x - y$ is linear. (It is understood that $V \times V$ has the product vector space structure described in Example 1.3.10.)

*Proof.* Suppse $V$ is a vector space. Let $(x, y), (a, b)$ be arbitrary elements in $V \times V$. Let $c \in \mathbb{R}$ be a scalar. Then $T((x, y)+(a, b)) = T(x+a, y+b) = (x+a)-(y+b) = (x-y)+(a-b) = T(x, y)+T(a, b)$. Also $T(c(x, y)) = T(cx, cy) = cx - cy = c(x - y) = c\,T(x, y)$. Thus $T$ is linear. ∎

**Problem 5**

With notations as in Theorem 2.1.3, an element $(a_1, \dots, a_n)$ of $F^n$ such that $T(a_1, \dots, a_n) = \theta$ is called a (linear) *relation* among the vectors $x_1, \dots, x_n$. For example, suppose that $V = F^2, n = 3$ and $x_1 = (2, -3), x_2 = (4, 1), x_3 = (8, 9)$.

1. Find a formula for the linear mapping $F^3 \longrightarrow F^2$ defined in Theorem 2.1.3.
2. Show that $(-2, 3, 1)$ is a relation among $x_1, x_2, x_3$.

**Solution (a):**

$$f(a_1, a_2, a_3) = a_1x_1 + a_2x_2 + a_3x_3$$

**Solution (b):**

$$f(-2, 3, -1) = -2x_1 + 3x_2 - x_3 = -2(2, -3) + 3(4, 1) - (8, 9) = (-4, 6) + (12, 3) - (8, 9) = (8, 9) - (8, 9) = (0, 0) = \theta$$

**Problem 6**

The proof of Theorem 2.1.2 uses only the additivity of the mapping $T$. Give a proof using only its homogeneity. [Hint: Corollaries 1.4.3, 1.4.5]

**Theorem 1.** *If $T : V \longrightarrow W$ is a linear mapping, then $T\theta = \theta$ and $T(-x) = -(Tx)$ for all $x \in V$.*

*Proof.* Suppose $T : V \longrightarrow W$ is a linear mapping. Then $T(\theta) = T(0 \cdot x) = 0 \cdot T(x)$. Then by Corollary 1.4.3, $0 \cdot T(x) = \theta$, hence $T(\theta) = \theta$. Also $T(-x) = T((-1) \cdot x) = (-1)T(x)$. Then by Corollary 1.4.5, $(-1)T(x) = -T(x)$. Thus $T(-x) = -Tx$. ∎

**Problem 7**

If $\mathscr{D}$ is the vector space of real polynomial functions (Example 1.3.5) and $f : \mathscr{D} \longrightarrow \mathbb{R}$ is the mapping defined by $f(p) = p'(1)$ (the value of the derivative of $p$ at 1), then $f$ is a linear form on $\mathscr{D}$. What is the geometric meaning of $f(p) = 0$.

*Proof.* Let $x, y \in \mathscr{D}$ and $c \in \mathbb{R}$. Then $f(x + y) = (x + y)'(1) = x'(1) + y'(1) = f(x) + f(y)$. and $f(cx) = (cx)'(1) = c\,x'(1) = cf(x)$. It follows that $f$ is a linear form on $\mathscr{D}$. ∎

**Solution:** $f(p) = 0$ is a critical point at the value $x = 1$ of the function $p$.

**Problem 8**

Let $S : \mathscr{D} \longrightarrow \mathscr{D}$ be the linear mapping of Example 2.1.5. Define $R : \mathscr{D} \longrightarrow \mathscr{D}$ by $Rp = Sp + p(5)1$, where $p(5)1$ is the constant function defined by the real number $p(5)$. Then $R$ is linear and $(Rp)' = p$. [So to speak, the constant of integration can be tailor-made for $p$ in a linear fashion.] More generally, look at the mapping $Rp = Sp + f(p)1$, where $f$ is any linear form on $\mathscr{D}$.

*Proof.* Let $x, y \in \mathscr{D}$ and $c \in \mathbb{R}$. Then $R(x + y) = S(x + y) + (x + y)(5)1 = S(x) + S(y) + x(5)1 + y(5)1 = R(x) + R(y)$. Also $R(cx) = S(cx) + (cx)(5)1 = cS(x) + cx(5)1 = cR(x)$. Thus $R$ is linear.

For the general form notice $R(x + y) = S(x + y) + f(x + y)1 = S(x) + S(y) + f(x)1 + f(y)1 = R(x) + R(y)$. Also $R(cx) = S(cx) + f(cx)1 = cS(x) + cf(x)1 = cR(x)$. ∎

## 2.2 Linear Mappings and Linear Subspaces: Kernel and Range

**Problem 2**

Let $V$ be a vector space over $F$, $f : V \longrightarrow F$ a linear form on $V$ (Definition 2.1.11). Assume that $f$ is not identically zero and choose a vector $y$ such that $f(y) \neq 0$. Let $N$ be the kernel of $f$. Prove that every vector $x$ in $V$ may be written $x = z + cy$ with $z \in N$ and $c \in F$, and that $z$ and $c$ are uniquely determined by $x$. [Hint: If $x \in V$, compute the value of $f$ at the vector $x - [f(x)/f(y)]y$.]

*Proof.* Suppose $x, y \in V$. Notice $f\left(x - \frac{f(x)}{f(y)}y\right) = f(x) - f\left(\frac{f(x)}{f(y)}y\right) = f(x) - \frac{f(x)}{f(y)}f(y) = f(x) - f(x) = 0$. We've just shown $x - \frac{f(x)}{f(y)}y \in \ker(f)$. Thus $x = \left(x - \frac{f(x)}{f(y)}y\right) + \frac{f(x)}{f(y)}y$. Let $z = x - \frac{f(x)}{f(y)}y$ and $c = \frac{f(x)}{f(y)}$, and we see $x = z + cy$ as required.

Suppose $x = z' + c'y$ with $z' \in \ker(f)$ and $c' \in F$. Then $z + cy = z' + c'y \implies z - z' = (c' - c)y$. Applying $f$ gives $0 = f(z - z') = (c' - c)f(y)$, so $c' - c = 0$ since $f(y) \neq 0$. Thus $c' = c$ and $z' = z$. ∎

**Problem 3**

Let $S : V \longrightarrow W$ and $T : V \longrightarrow W$ be a linear mappings and let $M = \{x \in V \mid Sx \in T(V)\}$. Prove that $M$ is a linear subspace of $V$.

*Proof.* Let $x, y \in M$ and let $c$ be a scalar of the vector space $V$. Since $x, y \in M$, we have $Sx \in T(V)$ and $Sy \in T(V)$. Since $T(V)$ is a linear subspace of $W$ it follows that $S(x + y) = Sx + Sy \in T(V)$. Thus $x + y \in M$. Similarly $S(cx) = cS(x) \in T(V)$ thus $cx \in M$. Therefore $M$ is a linear subspace of $V$. ∎

**Problem 4**

If $T : V \times V \longrightarrow V$ is the linear mapping $T(x, y) = x - y$ (2.1, Exercise 4), determine the kernel and range of $T$.

**Solution:** The kernel is $(x, x) \in V \times V$ and the range is $V$.

**Problem 5**

Let $V$ be a real vector space, $W$ its complexification (1.3, Exercise 4); write $W_R$ for $W$ regarded as a real vector space (1.3, Exercise 3). For $(x, y) \in W$, we have

$$(x, y) = (x, \theta) + (\theta, y) = (x, \theta) + i(y, \theta).$$

Prove that $x \mapsto (x, \theta)$ is an injective mapping $V \longrightarrow W_R$. [Identifying $x \in V$ with $(x, \theta) \in W$, one can suggestively write $W = V + iV$; so to speak, $V$ is the 'real part' of its complexification.]

*Proof.*

$$x(x_1) = x(x_2) \iff (x_1, 0) = (x_2, 0) \iff x_1 = x_2$$

∎

**Problem 6**

Let $\mathscr{D}$ be the vector space of real polynomial functions (1.3.5) and let $T : \mathscr{D} \longrightarrow \mathscr{D}$ be the linear mapping defined by $Tp = p - p'$, where $p'$ is the derivative of $p$. Prove that $T$ is injective.

*Proof.* Suppose $x, y \in \mathscr{D}$ such that $T(x) = T(y)$. Then $T(x) = T(y) \iff x - x' = y - y' \iff (x-y)-(x'-y') = 0$. Let $q = x - y$ then $q - q' = 0 \iff q = q'$. Thus $q = 0$ and it follows that $x = y$ and $x' = y'$. ∎

## 2.3 Spaces of Linear Mappings: $\mathscr{L}(V, W)$ and $\mathscr{L}(V)$

**Problem 1**

Complete the details in the proof of Theorem 2.3.11.

**Theorem 2.** *If $V$ and $W$ are vector spaces over $F$, then $\mathscr{L}(V, W)$ is also a vector space over $F$ for the operations defined in 2.3.9.*

*Proof.* Let $S, T \in \mathscr{L}(V, W)$ and let $a, b \in F$. For all $x \in V$, $(a(S+T))(x) = a((S+T)(x)) = a(S(x)+T(x)) = aS(x) + aT(x) = (aS + aT)(x)$. For all $x \in V$, $((a+b)T)(x) = (a+b)T(x) = aT(x) + bT(x) = (aT + bT)(x)$. For all $x \in V$, $((ab)T)(x) = ab\,T(x) = a(bT(x)) = (a(bT))(x)$. For all $x \in V$, $(1T)(x) = 1 \cdot T(x) = T(x)$. ∎

**Problem 2**

Let $V$ be a vector space. If $R, S, T \in \mathscr{L}(V)$ and $c$ is a scalar, the following equalities are true.

1. $(RS)T = R(ST)$;
2. $(R + S)T = RT + ST$;
3. $R(S + T) = RS + RT$;
4. $(cS)T = c(ST) = S(cT)$;
5. $TI = T = IT$; ($T$ is the identity mapping)

*Proof.* For all $x \in V$.

1. $(RS)T(x) = R(S(T(x))) = R(ST(x)) = (R(ST))(x)$.
2. $(R + S)T(x) = (R + S)(T(x)) = R(T(x)) + S(T(x)) = RT(x) + ST(x) = (RT + ST)(x)$.
3. $R(S + T)(x) = R(S(x) + T(x)) = R(S(x)) + R(T(x)) = RS(x) + RT(x) = (RS + RT)(x)$.
4. $(cS)T(x) = cS(T(x)) = c(ST(x)) = (c(ST))(x) = S(cT)(x)$.
5. $TI(x) = T(I(x)) = T(x)$ and $IT(x) = I(T(x)) = T(x)$.

∎

**Problem 4**

If $T : V \longrightarrow W$ is a linear mapping and $g$ is a linear form on $W$, show that the formula $f(x) = g(Tx)$ defines a linear form $f$ on $V$. [Shortcut: $f = gT = g \circ T$.] The formula $T'g = f$ defines a mapping $T' : \mathscr{L}(W, F) \longrightarrow \mathscr{L}(V, F)$, where $F$ is the field of scalars. Show that $T'$ is linear. [$T'$ is called the *transpose* (or 'adjoint' of $T$.)]

*Proof.* By Theorem 2.3.13 it directly follows that $f$ is linear.

Let $x, y \in \mathscr{L}(W, F)$ and $c$ be a scalar in $F$. Then for all $v \in V$, $T'(x + y)(v) = (x + y)(T(v)) = x(T(v)) + y(T(v)) = T'(x)(v) + T'(y)(v)$. Similarly, $T'(cx)(v) = (cx)(T(v)) = c\,x(T(v)) = c\,T'(x)(v)$. Therefore, $T'$ is linear. ∎

**Problem 5**

If $V$ is a vector space over $F$, the vector space $\mathscr{L}(V, F)$ of linear forms on $V$ is called the *dual space* of $V$ and is denoted $V'$. Thus, the correspondence $T \mapsto T'$ described in Exercise 4 defines a mapping

$\mathscr{L}(V,W) \longrightarrow \mathscr{L}(W',V')$. Show that this mapping is linear, that is, $(S+T)' = S'+T'$ and $(cT)' = cT'$ for all $S,T$ in $\mathscr{L}(V,W)$ and all scalars $c$.

*Proof.* Let $S,T \in \mathscr{L}(V,W)$ and let $c$ be a scalar. Let $v$ be a vector in $V$. Then $(S+T)'g(v) = g(S+T)(v) = g(S(v)+T(v)) = gS(v)+gT(v) = S'g(v)+T'g(v)$. Similarly $(cT)'g(v) = g(cT(v)) = cgT(v) = c(T'g)(v)$. Thus the correspondence $T \mapsto T'$ is linear. ∎

## Problem 6

With notations as in Exercise 4, prove that if $T$ is surjective then $T'$ is injective; in other words, show that if $T(V) = W$ then $T'g = 0 \implies g = 0$.

*Proof.* Suppose $T(V) = W$ and $T'g = 0$. Let $v \in V$ then $(T'g)(v) = g(T(v)) = 0$ for all $v \in V$. Thus $g = 0$ and $T'$ is injective. ∎

## Problem 7

As in Example 2.1.4, let $V = \mathscr{D}$ be the real vector space of all polynomial functions $p : \mathbb{R} \longrightarrow \mathbb{R}$, $D : V \longrightarrow V$ the linear mapping defined by $Dp = p'$ (the derivative of $p$); let $D' : V' \longrightarrow V'$ be the transpose of $D$ as defined in Exericise 4. (Caution: The prime is being used with three different meanings.)
For each $t \in \mathbb{R}$ let $\rho_i$ be the linear form on $V$ defined by $\rho_t(p) = p(t)$. Let $[a,b]$ be a closed interval on $\mathbb{R}$ and let $\rho$ be the linear form on $V$ defined by

$$\rho(p) = \int_a^b p(t)dt$$

Prove : $D'\rho = \rho_b - \rho_a$. [Hint: Fundamental Theorem of Calculus.]

*Proof.* Let $p \in V$. Notice $(D'\rho)(p) = \rho(Dp) = \int_a^b p'(t)dt = p(b) - p(a) = \rho_b(p) - \rho_a(p)$. ∎

## Problem 9

Let $U,V,W$ be vector spaces over $F$. Prove:

1. For fixed $T \in \mathscr{L}(U,V)$, $\psi = S \mapsto ST$ is a linear mapping $\mathscr{L}(V,W) \longrightarrow \mathscr{L}(U,W)$ (see Figure 8 2.3.12).
2. For fixed $S \in \mathscr{L}(V,W)$, $\phi = T \mapsto ST$ is a mapping $\mathscr{L}(U,V) \longrightarrow \mathscr{L}(U,W)$.

*Proof.* Let $x,y \in \mathscr{L}(V,W)$ and $v \in U$. Let $c$ be a scalar. Then $(\psi(x+y))(v) = ((x+y)T)(v) = (x+y)(T(v)) = xT(v)+yT(v) = \psi(x)(v)+\psi(y)(v)$. Also $(\psi(cx))(v) = ((cx)T)(v) = (cx)(T(v)) = c(xT(v)) = c\psi(x)(v)$. Thus $\psi$ is a linear mapping. ∎

*Proof.* Let $x,y \in \mathscr{L}(U,V)$ and $v \in U$. Let $c$ be a scalar. Then $(\phi(x+y))(v) = S(x+y)(v) = S(x(v)+y(v)) = S(x(v))+S(y(v)) = \phi(x)(v)+\phi(y)(v)$. Also $(\phi(cx))(v) = S(cx(v)) = c\,S(x(v)) = c\,\phi(x)(v)$. Thus $\phi$ is a linear mapping. ∎

## Problem 10

If $T : U \longrightarrow V$ and $S : V \longrightarrow W$ are linear mappings, prove that $Im(ST) \subset ImS$ and $Ker(ST) \supset KerT$.

*Proof.* Clearly $T$ can only restrict the domain of $S$ thus $Im(ST) \subset ImS$. Clearly any $x$ in $KerT$ will be in $Ker(ST)$ since $S(0) = 0$. ∎

**Problem 11**

Let $T : V \longrightarrow V$ be a linear mapping such that $ImT \subset Ker(T - I)$, where $I$ is the identity mapping (2.3.3). Prove that $T^2 = T$. [Recall $T^2 = TT$.]

*Proof.* First note that $Ker(T - I) = \{v \in V \mid (T - I)(v) = 0\}$. For any $v \in V$, $(T - I)(v) = 0 \iff T(v) - I(v) = 0 \iff T(v) - v = 0 \iff T(v) = v$. Let $v \in V$ be arbitrary. Then $T(v) \in Im\,T$. Since $Im\,T \subset Ker(T - I)$, it follows that $T(v) \in Ker(T - I)$, and thus $T(T(v)) = T(v)$. Thus $T^2 = T$. ∎

**Problem 12**

Supose $V = M \oplus N$ in the sense of 1.6, Exercise 12. For each $x \in V$, let $x = y + z$ be its unique decomposition with $y \in M, z \in N$ and define $Px = y$, $Qx = z$. Prove that $P, Q \in \mathscr{L}(V)$, $P^2 = P$, $Q^2 = Q$, $P + Q = I$ and $PQ = QP = 0$.

*Proof.* Let $a_1, a_2 \in M$ and $b_1, b_2 \in N$. Let $x = a_1 + b_1, y = a_2 + b_2 \in V$. Then $P(x + y) = P((a_1 + a_2) + (b_1 + b_2)) = a_1 + a_2 = Px + Py$. Also $P(cx) = P(c(a_1 + b_1)) = P(ca_1 + cb_1) = ca_1 = c(Px)$. Thus $P \in \mathscr{L}(V)$. The argument that $Q \in \mathscr{L}(V)$ is similar. ∎

*Proof.* Simply note that $ImP \subset Ker(P - I)$ thus $P^2 = P$. The argument that $Q^2 = Q$ is similar. ∎

*Proof.* Let $a \in M$ and $b \in N$. Let $x = a + b$. Then $(P + Q)(x) = P(x) + Q(x) = a + b = x$. Thus $P + Q = I$. ∎

*Proof.* Let $a \in M$ and $b \in N$. Let $x = a + b$. Then $(PQ)(x) = P(Q(x)) = P(Q(a + b)) = P(0 + Q(b)) = 0$. The argument for $QP$ is similar. ∎

**Problem 13**

Let $T : V \longrightarrow V$ be a linear mapping such that $T^2 = T$, and let $M = ImT$ and $N = KerT$. Prove that $M = \{x \in V \mid Tx = x\} = Ker(T - I)$ and that $V = M \oplus N$ in the sense of 1.6, Exercise 12.

*Proof.* Notice $x \in M \iff Tx = x \iff Tx - x = 0 \iff Tx - Ix = 0 \iff (T - I)(x) = 0 \iff x \in \ker(T - I)$.

Let $x \in V$. Let $y = Tx \in M$ and $z = x - Tx \in N$. Notice

$$T(z) = T(x - Tx) = Tx - T^2x = Tx - Tx = 0$$

Thus $z \in \ker T = N$. Therefore $x = y + z \in M + N$. Suppose $x = y_1 + z_1 = y_2 + z_2$ with $y_i \in M, z_i \in N$. Then

$$y_1 - y_2 = z_2 - z_1 \in M \cap N.$$

But then $M \cap N = \{0\}$ because suppose $v \in M \cap N$, then $Tv = v$ and $Tv = 0$, so $v = 0$. Thus $V = M \oplus N$. ∎

**Problem 14**

Find $S, T \in \mathscr{L}(\mathbb{R}^2)$ such that $ST \neq TS$.

*Proof.* Consider the linear maps $S, T \in \mathscr{L}(\mathbb{R}^2)$

$$S(x, y) = (y, 0), \quad T(x, y) = (0, x).$$

Then for any $(x, y) \in \mathbb{R}^2$

$$ST(x, y) = S(T(x, y)) = S(0, x) = (x, 0),$$
$$TS(x, y) = T(S(x, y)) = T(y, 0) = (0, y).$$

Thus $ST \neq TS$. ∎

**Problem 15**

Let $T : U \longrightarrow V$ and $S : V \longrightarrow W$ be linear mappings. Prove

1. If $S$ is injective, then $Ker(ST) = Ker(T)$.
2. If $T$ is surjective, then $Im(ST) = ImS$.

*Proof.* Suppose $S$ is injective. Let $x$ be an arbitrary element in $U$. Suppose $x \in Ker(T)$ thus $T(x) = 0$. Then $(ST)(x) = S(T(x)) = S(0) = 0$. Thus $x \in Ker(ST)$. Suppose $x \in Ker(ST)$ thus $(ST)(x) = 0$. Now $S(0) = 0$ and since $S(T(x)) = 0$ and $S$ is injective, $T(x) = 0$ thus $x \in KerT$. It follows that $Ker(ST) = Ker(T)$. ∎

*Proof.* Suppose $T$ is surjective. It follows that $T$ maps onto all elements in the domain of $S$. Therefore $Im(ST) = S$. ∎

**Problem 16**

Let $V$ be a real or complex vector space and let $t \in \mathscr{V}$ be such that $T^2 = I$. Define

$$M = \{x \in V \mid Tx = x\}, \quad N = \{x \in V \mid Tx = -x\}$$

Prove that $M$ and $N$ are linear subspaces of $V$ such that $V = M \oplus N$. [Hint: For every vector $x$, $x = \frac{1}{2}(x + Tx) + \frac{1}{2}(x - Tx)$.]

*Proof.* Let $x \in V$ and consider $x = \frac{1}{2}(x + Tx) + \frac{1}{2}(x - Tx)$. Let $m = \frac{1}{2}(x + Tx)$ and $n = \frac{1}{2}(x - Tx)$. Then $Tm = \frac{1}{2}(Tx + T^2x) = \frac{1}{2}(Tx + x) = m$ so $m \in M$, and $Tn = \frac{1}{2}(Tx - T^2x) = \frac{1}{2}(Tx - x) = -n$ so $n \in N$. Thus $x = m + n \in M + N$.

Suppose $v \in M \cap N$. Then $Tv = v$ and $Tv = -v$, which implies $v = 0$. Thus $M \cap N = \{0\}$. It follows that $V = M \oplus N$. ∎

**Problem 17**

Let $V = \mathscr{F}(\mathbb{R})$ be the real vector space of all functions $x : \mathbb{R} \longrightarrow \mathbb{R}$ (1.3.4). For $x \in V$ define $Tx \in V$ by the formula $(Tx)(t) = x(-t)$. Analyze $T$ in the light of Exercise 16. [Remember 1.6, Exercise 13]

**Solution:** The set of functions for which $x(t) = x(-t)$, i.e., the even functions, are in $M$. The functions for which $x(t) = -x(-t)$, i.e., the odd functions, are in $N$. Every function $x \in V$ can be expressed as $x = \frac{1}{2}(x + Tx) + \frac{1}{2}(x - Tx) \in M \oplus N$.

**Problem 18**

Let $T \in \mathscr{L}(U, V)$ and $S \in \mathscr{L}(V, W)$, prove that $(ST)' = T'S'$. (cf. Exercise 4).

*Proof.* Let $f \in W'$ be a linear form on $W$. Then $(ST)'(f) = f \circ (ST) = (f \circ S) \circ T = T'(S'(f))$. ∎

Problem 19

Let $S, T \in \mathscr{L}(V, W)$ and let $M = \{x \in V \mid Sx = Tx\}$. Prove that $M$ is a linear subspace of $V$. [Hint: Consider $Ker(S - T)$.]

*Proof.* Note that $M = \{x \in V \mid Sx - Tx = 0\} = \{x \in V \mid (S - T)x = 0\} = \ker(S - T)$. Since the kernel of a linear map is always a linear subspace, $M$ is a linear subspace of $V$. ∎

Problem 20

If $T : V \longrightarrow W, S : W \longrightarrow V$ are linear mappings such that $ST = I$, prove that $KerT = \{0\}$ and $ImS = V$.

*Proof.* Let $x \in KerT$ such that $Tx = 0$. Then $STx = Tx = 0$. Since $ST = I$, we have $x = 0$ thus $KernT = \{0\}$.

Next, let $v \in V$. Then $v = Iv = STv = S(Tv)$. Thus $v \in ImS$. Therefore $ImS = V$. ∎

Problem 21

If $V = \{\theta\}$ or $W = \{\theta\}$ then $\mathscr{L}(V, W) = \{0\}$. If $V \neq \{\theta\}$ then $I_v \neq 0$.

*Proof.* Suppose $V = \{\theta\}$ or $W = \{\theta\}$. Clearly there is only one linear mapping between $V$ and $W$ which is for all $x \in V$, $T(x) = 0 \in W$.

Suppose $V \neq \{\theta\}$. $V$ has at least two elements, one of which maps to $0$. The other must map to a nonzero element under $I_V$. ∎

Problem 22

A lightning proof of Theorem 2.3.11 can be based on an earlier exercise: Since $0 \in \mathscr{L}(V, W)$, Lemma 2.3.10 shows that $\mathscr{V}, \mathscr{W}$ is a linear subspace of the vector space $\mathscr{F}(V, W)$ defined as in 1.3, Exercise 1, therefore $\mathscr{L}(V, W)$ is also a vector space (1.6.2).

**Solution:** OK.

Problem 23

If $T : U \longrightarrow V$ and $S : V \longrightarrow W$ are linear mappings, prove that $Ker(ST) = T^{-1}(KerS)$.

*Proof.* Let $x \in U$. Then

$$x \in Ker(ST) \iff STx = 0 \iff S(Tx) = 0 \iff Tx \in KerS \iff x \in T^{-1}(KerS).$$

Therefore $Ker(ST) = T^{-1}(KerS)$. ∎

Problem 24

Let $V = \mathscr{D}$ be the vector space of real polynomial functions, $D : V \longrightarrow V$ the differentiation mapping $Dp = p'$ (2.1.4). Let $u$ be the monomial $u(t) = t$ and define another linear mapping $M : V \longrightarrow V$ by the formula $Mp = up$. (So to speak, $M$ is multiplication by $t$.) Prove that $DM - MD = I$ (the identity mapping). [Hint: The claim is that $(up)' - up' = p$. Remember the 'product rule'?]

*Proof.* By the product rule,

$$(up)' = u'p + up'.$$

Since $u(t) = t$ we know $u' = 1$. Thus

$$(up)' = p + up'.$$

∎

**Problem 25**

Let $D : \mathscr{D} \longrightarrow \mathscr{D}$ be the differentiating map of Exercise 24 and let $S : \mathscr{D} \longrightarrow \mathscr{D}$ be the linear mapping such that $(Sp)' = p$ and $(Sp)(0) = 0$ (2.1.5). Prove:

1. If $R : \mathscr{D} \longrightarrow \mathscr{D}$ is any linear mapping such that $(Rp)' = p$ for all $p \in \mathscr{D}$, then $DR = I$.
2. If $T : \mathscr{D} \longrightarrow \mathscr{D}$ is a linear mapping such that $DT = 0$, then there exists a linear form $f$ on $\mathscr{D}$ such that (in the notation of Exercise 2.3.4) $T = f \otimes 1$, where $1$ is the constant function $1$. [Hint: $ImT \subset KerD$.]
3. With $R$ as in (i), $R = S + f \otimes 1$ for a suitable linear form $f$ on $\mathscr{D}$. [Hint: Consider $T = R - S$.] Remember 2.1, Exercise 8?

*Proof.* Suppose $R : \mathscr{D} \longrightarrow \mathscr{D}$ is a linear mapping such that $(Rp)' = p$ for all $p \in \mathscr{D}$. Let $p \in \mathscr{D}$. Then $(DR)(p) = D(Rp) = (Rp)' = p$. Thus $DR = I$. ∎

*Proof.* If $T : \mathscr{D} \longrightarrow \mathscr{D}$ is a linear mapping such that $DT = 0$, then for all $p \in \mathscr{D}$, $Tp$ is a constant function. Thus $T = T \otimes 1$, where $1$ is the constant function $1$. ∎

*Proof.* Let $R : \mathscr{D} \longrightarrow \mathscr{D}$ and let $T = R - S$. Then for any $p \in \mathscr{D}$, $D(Tp) = D((R-S)p) = DR(p) - DS(p) = I(p) - I(p) = 0$. Thus $DT = 0$. By part (ii), there exists a linear form $f$ on $\mathscr{D}$ such that $T = f \otimes 1$. Therefore $R = S + T = S + f \otimes 1$. ∎

**Problem 26**

Let $S, T \in \mathscr{L}(V)$. If $ST - I$ is injective then so is $TS - I$. [Hint: $S(TS - I) = (ST - I)S$.]

*Proof.* Suppose $ST - I$ is injective. Thus $\ker(ST - I) = \{0\}$. Suppose $x \in V$ such that $(TS - I)(x) = 0$. Then applying $S$ we have $S(TS - I)(x) = 0$. By the hint we have $((ST - I)S)(x) = 0 \iff (ST - I)(S(x)) = 0$. Since $ST - I$ is injective we have $S(x) = 0$. Then $(TS - I)(x) = TS(x) - x = T0 - x = -x = 0$. Thus $x = 0$. ∎

## 2.4 Isomorphic Vector Spaces

**Problem 2**

Regard $\mathbb{C}^n$ as a real vector space in the natural way (sums as usual, multiplication only by real scalars); then $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as real vector spaces.

*Proof.* Let $r_1, r_2, \ldots, r_{2n-1}, r_{2n} \in \mathbb{R}$. Consider the mapping

$$(r_1, r_2, \ldots, r_{2n-1}, r_{2n}) \mapsto (r_1 + r_2 i, \ldots, r_{2n-1} + r_{2n} i).$$

We know that the map $(r_1, r_2) \mapsto r_1 + r_2 i$ is linear and bijective over $\mathbb{R}$. Our mapping is simply an extension of this map to $n$ components and is therefore linear and bijective. ∎

## Problem 3

Let $X$ be a set, $V$ a vector space, $f : X \longrightarrow V$ a bijection. There exists a unique vector space structure on $X$ for which $f$ is a linear mapping (hence a vector space isomorphism). [Hint: Define sums and scalar multiples in $X$ by the formulas $x + y = f^{-1}(f(x) + f(y)), cx = f^{-1}(cx)$. This trick is called 'transport of structure'.]

*Proof.* Clearly $X$ is a vector space, since after applying $f$ each vector space axiom becomes the corresponding axiom in $V$, and injectivity of $f$ implies the axioms hold in $X$. ∎

## Problem 4

Let $V$ be the set of n-ples $x = (x_1, \dots, x_n)$ of real numbers $> 0$, that is,

$$V = \{(x_1, \dots, x_n) \mid x_i \in (0, +\infty \text{ for } i = 1, \dots, n)\}.$$

For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ in $V$ and $c \in \mathbb{R}$, define

$$x \oplus y = (x_1 y_1, \dots, x_n y_n), c \cdot x = (x_1^c, \dots, x_n^c),$$

(here $x_i y_i$ and $c_i^c$ are the usual product and power), Define a mapping $T : V \longrightarrow \mathbb{R}^n$ by the formula $T(x_1, \dots, x_n) = (\log x_1, \dots, \log x_n)$.

1. Show that $T(x \oplus y) = Tx + Ty$ and $T(c \cdot x) = c(Tx)$ for all $x, y$ in $V$ and all $c \in \mathbb{R}$.
2. True or false (explain): $V$ is a real vector space for the operations $\cdot$.

*Proof.* Let $x, y \in V$ and $c$ be a scalar. Then

$$T(x \oplus y) = T((x_1 y_1, \dots, x_n y_n)) = (\log(x_1 y_1), \dots, \log(x_n y_n)) = (\log x_1 + \log y_1, \dots, \log x_n + \log y_n)$$

$$= (\log x_1, \dots, \log x_n) + (\log y_1, \dots, \log y_n) = Tx + Ty.$$

Also,

$$T(c \cdot x) = T((x_1^c, \dots, x_n^c)) = (\log(x_1^c), \dots, \log(x_n^c)) = (c \log x_1, \dots, c \log x_n) = c(Tx).$$

∎

**Solution:** False since scalar multiplication does not satisfy the vector space axioms.

## Problem 5

Let $V$ be a vector space, $T \in \mathscr{L}(V)$ bijective, and $c$ a nonzero scalar. Prove that $cT$ is bijective and that $(cT)^{-1} = c^{-1}T^{-1}$.

*Proof.* Let $v, v' \in V$ and suppose $cT(v) = cT(v')$. Then $cT(v) = cT(v') \iff cT(v) - cT(v') = 0 \iff c(T(v) - T(v')) = 0$. Since $c \neq 0$, we must have $T(v) = T(v')$. Since $T$ is injective, it follows that $v = v'$. Thus $cT$ is injective.

Let $v \in V$. Since $c \neq 0$, the scalar $\frac{1}{c}$ exists. Since $V$ is closed under scalar multiplication, $\frac{1}{c} \cdot v = \frac{v}{c} \in V$. Since $T$ is surjective, there exists $x \in V$ such that $T(x) = \frac{v}{c}$. Then $cT(x) = c\left(\frac{v}{c}\right) = v$. Thus $cT$ is surjective.

Let $v \in V$, and suppose $(cT)^{-1}(v) = x$. Then $cT(x) = v \implies T(x) = \frac{v}{c} \implies x = T^{-1}\left(\frac{v}{c}\right) = T^{-1}(c^{-1}v)$. Thus $(cT)^{-1} = c^{-1}T^{-1}$. ∎

**Problem 6**

Let $T : V \longrightarrow W$ be a bijective linear mapping. For each $S \in \mathscr{L}(V)$, define $\rho : \mathscr{L}(V) \longrightarrow TST^{-1}$ (note that the product is defined). Prove that $\rho : \mathscr{L}(V) \longrightarrow \mathscr{L}(W)$ is a bijective linear mapping such that $\rho(RS) = \rho(R)\rho(S)$ for all $R, S \in \mathscr{L}(V)$.

*Proof.* Since the composition of linear mappings is linear, $TST^{-1}$ is a linear mapping. Let $R, S \in \mathscr{L}(V)$. Then

$$\rho(RS) = T(RS)T^{-1} = (TRT^{-1})(TST^{-1}) = \rho(R)\rho(S).$$

Suppose $\rho(S)(v) = 0$ for some $v \in W$. Then $\rho(S)(v) = 0 \iff TST^{-1}(v) = 0$. Now $v = T(x)$ for some $x \in V$ thus $TST^{-1}(T(x)) = TS(x) = 0$. Then since $T$ is injective $S(x) = 0$ thus $\rho$ is injective. It is surjective because for any $U \in \mathscr{L}(W)$, setting $S = T^{-1}UT$ gives $\rho(S) = U$. ∎

**Problem 7**

Let $V$ be a vector space, $T \in \mathscr{L}(V)$. Prove:

1. If $T^2 = 0$ then $I - T$ is bijective.
2. If $T^n = 0$ for some positive integer $n$, then $I - T$ is bijective.

*Proof.* Suppose $T^2 = 0$. Let $v \in V$ and suppose $(I - T)(v) = 0 \implies T(v) - T^2(v) = 0 \implies T(v) = 0$. But $I(v) - T(v) = 0 \implies TI(v) - T^2(v) = T(0) \implies v - T(v) = 0 \implies v = T(v)$. Thus $v = 0$. It follows that $I - T$ is injective.

Let $y \in V$. We want to find $x \in V$ such that $(I-T)(x) = y$. Let $x = y + T(y)$. Then $(I-T)(x) = (I-T)(y+T(y)) = I(y + T(y)) - T(y + T(y)) = I(y) + IT(y) - T(y) - T^2(y) = y + T(y) - T(y) = y$. ∎

*Proof.* We proceed via induction on $n$. The previous proof shows the base case $n = 2$. Suppose for some $n-1$, $T^{n-1} = 0$ implies $I - T$ is bijective. Now suppose $T^n = 0$. Let $v \in V$ and suppose $(I - T)(v) = 0$. Composing with $T^{n-1}$, we get $T^{n-1}(v) - T^n(v) = T^{n-1}(v) = 0$. By the induction hypothesis, this implies $v = 0$, so $I - T$ is injective.

For surjectivity, let $y \in V$ and set $x = y + T(y) + \cdots + T^{n-1}(y)$. Then $(I - T)(x) = y$, so $I - T$ is bijective. ∎

**Problem 8**

Let $U, V, W$ be vector spaces over the same field. Form the product vector spaces $U \times V, V \times W$ (Definition 1.3.11), then the product vector spaces $(U \times V) \times W$ and $U \times (V \times W)$. Prove that $(U \times V) \times W \cong U \times (V \times W)$.

*Proof.* Consider the mapping

$$\rho : (U \times V) \times W \longrightarrow U \times (V \times W), \quad \rho((x_1, x_2), x_3) = (x_1, (x_2, x_3)).$$

Suppose $\rho((x_1, x_2), x_3) = \rho((y_1, y_2), y_3)$. Then $(x_1, (x_2, x_3)) = (y_1, (y_2, y_3))$, so $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$. Thus $((x_1, x_2), x_3) = ((y_1, y_2), y_3)$, and $\rho$ is injective.

Let $(u, (v, w)) \in U \times (V \times W)$. Then $\rho((u, v), w) = (u, (v, w))$. Thus $\rho$ is surjective. ∎

**Problem 9**

Prove that $\mathbb{R}^2 \times \mathbb{R}^3 \cong \mathbb{R}^5$.

*Proof.* Consider the mapping

$$\rho : \mathbb{R}^2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^5, \quad \rho(((x_1, x_2), (x_3, x_4, x_5))) = (x_1, x_2, x_3, x_4, x_5)$$

Suppose $\rho(((x_1, x_2), (x_3, x_4, x_5))) = \rho(((y_1, y_2), (y_3, y_4, y_5)))$. Then $(x_1, x_2) = (y_1, y_2)$ and $(x_3, x_4, x_5) = (y_3, y_4, y_5)$. Thus $((x_1, x_2), (x_3, x_4, x_5)) = ((y_1, y_2), (y_3, y_4, y_5))$, and $\rho$ is injective.

Let $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$. Then $\rho(((x_2, x_2), (x_3, x_4, x_5))) = (x_1, x_2, x_3, x_4, x_5)$. Thus $\rho$ is surjective. ∎

**Problem 10**

With notations as in Exercie 8, prove that $V \times W \cong W \times V$.

*Proof.* Consider the mapping

$$\rho : V \times W \longrightarrow W \times V, \quad \rho((v, w)) = (w, v).$$

Suppose $\rho((v_1, w_1)) = \rho((v_2, w_2))$. Then $(w_1, v_1) = (w_2, v_2)$, so $v_1 = v_2$ and $w_1 = w_2$. Thus $(v_1, w_1) = (v_2, w_2)$, and $\rho$ is injective.

Let $(w, v) \in W \times V$. Then $\rho((v, w)) = (w, v)$. Thus $\rho$ is surjective. ∎

**Problem 11**

With notations as in Example 2.4.2, prove that $(x_1, x_2) \mapsto (x_1, x_2 + x_2, 0)$ is an isomorphism $\mathbb{R}^2 \mapsto W$.

*Proof.* Consider the mapping

$$\rho : \mathbb{R}^2 \longrightarrow W, \quad \rho((x_1, x_2)) = (x_1, x_2, 0).$$

Suppose $\rho((x_1, x_2)) = \rho((y_1, y_2))$. Then $(x_1, x_2, 0) = (y_1, y_2, 0)$, so $x_1 = y_1$ and $x_2 = y_2$. Thus $(x_1, x_2) = (y_1, y_2)$, and $\rho$ is injective.

Let $(x_1, x_2, 0) \in W$. Then $\rho((x_1, x_2)) = (x_1, x_2, 0)$. Thus $\rho$ is surjective. ∎

## 2.5 Equivalence Relations and Quotient Sets

**Problem 2**

Let $X$ be the set of all *nonzero* vectors in $\mathbb{R}^n$, that is, $X = \mathbb{R}^n - \{\theta\}$. For $x, y$ in $X$, write $x \sim y$ if $x$ is proportional to $y$, that is, if there exists a scalar $c$ (necessarily nonzero) such that $x = cy$; this is an equivalence relation in $X$.

*Proof.* Notice that $x = 1x$, thus $x \sim x$.

Suppose $x \sim y$. Then $x = cy$ for some scalar $c$. Since $c \neq 0$, we have $\frac{1}{c}x = y$. Thus $y \sim x$.

Suppose $x \sim y$ and $y \sim z$. Then $x = c_1 y$ and $y = c_2 z$ for some scalars $c_1, c_2$. Then $x = c_1 y = c_1(c_2 z) = (c_1 c_2)z$. Thus $x \sim z$. ∎

Problem 4

If $\lambda$ is a nonempty set of vector spaces over the field $F$, then the relation of isomorphism (2.4.1) is an equivalence relation in $\lambda$.

*Proof.* Let $x, y, z \in \lambda$. Notice that $x \sim x$ through the identity mapping.

Suppose $x \sim y$. Then there exists an isomorphism $\rho$ such that $\rho(x) = y$. But then $x = \rho^{-1}(y)$, thus $y \sim x$.

Suppose $x \sim y$ and $y \sim z$. Then there exist $\rho_1, \rho_2$ such that $\rho_1(x) = y$ and $\rho_2(y) = z$. Then $z = \rho_2(\rho_1(x))$ Thus $x \sim z$. ∎

Problem 9

Let $f : X \longrightarrow Y$ be any mapping, $x \sim x'$ the equivalence relation defined by $f(x) = f(x')$ (2.5.1), $q : X \longrightarrow X/\sim$ the quotient mapping (2.5.17).
Define a mapping $g : X/\sim \longrightarrow f(X)$ as follows : if $u \in X/\sim$ define $g(u) = f(x)$, where $x$ is any element of $X$ such that $u = q(x)$. [If also $u = q(x')$ then $x \sim x'$, so $f(x) = f(x')$; thus $g(u)$ depends only on $u$ not on the particular $x$ chosen to represent the class $u$.] Prove (cf. Fig. 10):

1. $g$ is bijective;
2. $f = i \circ g \circ q$, where $i : f(X) \longrightarrow Y$ is the insertion mapping (Appendix A.3.8).

In particular, if $f : X \longrightarrow Y$ is surjective, then $g : X/\sim \longrightarrow Y$ is bijective.

*Proof.* Suppose $u, u' \in X/\sim$ such that $g(u) = g(u')$. By definition, $g(u) = f(x)$ and $g(u') = f(x')$ for some $x, x' \in X$ with $u = q(x)$ and $u' = q(x')$. Then $f(x) = f(x')$, so $x \sim x'$. Thus $q(x) = q(x')$, and therefore $u = u'$. Thus $g$ is injective.

Let $y$ be an arbitrary element of $f(X)$. Then $y = f(x)$ for some $x \in X$. Let $u = q(x) \in X/\sim$. By definition of $g$, we have $g(u) = f(x) = y$. Thus $g$ is surjective. ∎

*Proof.* Let $x \in X$. Then $q(x) \in X/\sim$ and $g(q(x)) = f(x)$ by definition of $g$. Since $i : f(X) \to Y$ is the insertion mapping, $i(f(x)) = f(x)$. Thus $(i \circ g \circ q)(x) = f(x)$ so $f = i \circ g \circ q$.

Suppose $f : X \longrightarrow Y$ is surjective. Then $f(X) = Y$, so $g : X/\sim \longrightarrow Y$.

Suppose $u, u' \in X/\sim$ such that $g(u) = g(u')$. Then $g(u) = f(x)$ and $g(u') = f(x')$ for some $x, x' \in X$. Thus $f(x) = f(x')$, so $x \sim x'$ and thus $u = u'$. Thus $g$ is injective.

Let $y \in Y$. Since $f$ is surjective there exists $x \in X$ such that $f(x) = y$. Let $u = q(x) \in X/\sim$. Then $g(u) = f(x) = y$. Thus $g$ is surjective. ∎

**Problem 10**

For $x, y$ in $\mathbb{R}$, write $x \sim y$ if $x \sim y$ is an integer (that is, $x - y \in \mathbb{Z}$); this is an equivalence relation in $\mathbb{R}$. The quotient set is denoted $\mathbb{R}/\mathbb{Z}$; the equivalence class $[x]$ of $x \in \mathbb{R}$ is the set $x + \mathbb{Z} = \{x + n \mid n \in \mathbb{Z}\}$.

1. There is a connection between functions defined on $\mathbb{R}/\mathbb{Z}$ and functions defined on $\mathbb{R}$ that are periodic of period 1; can you make it precise?
2. There exists a bijection $g : \mathbb{R}/\mathbb{Z} \longrightarrow U$, where $U = \{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\}$ is the 'unit circle' in $\mathbb{R}^2$. [Hint: Apply Exercise 9 to the function $f : \mathbb{R} \longrightarrow U$ defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$.]

**Solution:** Consider a periodic function of period 1, say $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$,

$$f(x + n) = f(x).$$

Thus $f$ takes the same value on all elements of the equivalence class

$$[x] = x + \mathbb{Z}.$$

Conversely, any function defined on $\mathbb{R}/\mathbb{Z}$ determines a function on $\mathbb{R}$ that is periodic of period 1 by composition with the quotient map.

*Proof.* Consider the function

$$f : \mathbb{R} \longrightarrow U, \qquad f(t) = (\cos 2\pi t, \sin 2\pi t).$$

Notice that $f(t + n) = f(t)$ for all $n \in \mathbb{Z}$, so $f$ is constant on equivalence classes of $\mathbb{R}/\mathbb{Z}$. By Exercise 9 there is a map

$$g : \mathbb{R}/\mathbb{Z} \longrightarrow U, \qquad g([t]) = f(t).$$

Suppose $g([t]) = g([s])$. Then $f(t) = f(s)$ so $t - s \in \mathbb{Z}$. Thus $[t] = [s]$. Suppose $(a, b) \in U$. There exists $t \in [0, 1)$ such that $(\cos 2\pi t, \sin 2\pi t) = (a, b)$, so $(a, b) = g([t])$. ∎

**Problem 11**

For $x, y$ in $\mathbb{R}$, write $x \sim y$ if $x - y$ is an integral multiple of $2\pi$ (that is, $x - y \in 2\pi\mathbb{Z}$). With an eye on Exercise 10, discuss periodic functions of period $2\pi$ and describe a bijection of $\mathbb{R}/2\pi\mathbb{Z}$ onto the unit circle $U$.

**Solution:** The equivalence class of $x \in \mathbb{R}$ in $\mathbb{R}/2\pi\mathbb{Z}$ is $[x] = \{x + 2\pi n \mid n \in \mathbb{Z}\}$. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is periodic of period $2\pi$ if $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. A bijection of $\mathbb{R}/2\pi\mathbb{Z}$ onto the unit circle $U = \{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\}$ can be defined as

$$g : \mathbb{R}/2\pi\mathbb{Z} \longrightarrow U, \qquad g([x]) = (\cos x, \sin x),$$

## 2.6 Quotient Vector Spaces

## Problem 1

Prove that if $V = M \oplus N$ (1.6, Exercise 12) then $V/M \cong N$. [Hint: Restrict the quotient mapping $V \longrightarrow V/M$ to $N$ (Appendix $A$.3.8, and calculate the kernel and range of the restricted mapping.)]

*Proof.* Suppose $V = M \oplus N$. Let $q : V \longrightarrow V/M$ be the quotient mapping. Then consider $(q \mid N) : N \longrightarrow V/M$, which is the restriction of $q$ to $N$. It was shown in the proof of Theorem 2.6.1 that $q$ is linear. Suppose $x \in N$ such that $(q \mid N)(x) = [0]$. By definition, this means $x \in M$. But $x \in N$ and $V = M \oplus N$ thus $x = 0$. Thus $(q \mid N)$ is injective. Let $[y] \in V/M$. Since $V = M \oplus N$ then $y = m + n$ with $m \in M$ and $n \in N$. Then $(q \mid N)(n) = [n] = [m+n] = [y]$. Thus $(q \mid N)$ is surjective. Therefore $q \mid N$ is bijective, and it follows that $V/M \cong N$. ∎

## Problem 2

Let $M$ and $N$ be linear subspaces of the vector spaces $V$ and $W$, respectively. Form the product vector space $V \times W$ (Definition 1.3.11). Then $M \times N$ is a linear subspace of $V \times W$ and

$$(V \times W)/(M \times N) \cong (V/M) \times (W/N).$$

Prove this by showing that

$$(x, y) + M \times N \longrightarrow (x + M, y + N),$$

defines a function $(V\times W)/(M\times N) \longrightarrow (V/M)\times(W/N)$ and that this function is linear and bijective. [The essential first step to show that if $u = (x, y)+M\times N = (x', y')+M\times N$, then $(x+M, y+N) = (x'+M, y'+N)$, that is, the proposed functional value at $u$ depends only on $u$ and not on the particular ordered pair $(x, y) \in u$ chosen to represent it (cf. the proof of Theorem 2.6.1).]

*Proof.* Let $q : (V\times W)/(M\times N) \cong (V/M)\times(W/N)$ be a linear mapping defined as $q(x+M, y+N) = (x'+M, y'+N)$. ∎

## Problem 3

## Problem 4

## Problem 5

## Problem 6

Let $X$ be a nonempty set. Let $R$ be a subset of the cartesian product $X \times X$, such that (i) $R$ contains the 'diagnol' $X \times X$, that is, $(x, x) \in R$ for all $x \in X$; (ii) $R$ is 'symmetric in the diagnol', that is, if $(x, y) \in R$ then $(y, x) \in R$; and (iii) if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$. Does this suggest a way of defining an equivalence relation $x \sim y$ in $X$.

## Problem 7

## Problem 8

**Solution:** Yes.

## 2.7 The First Isomorphism Theorem

Problem 1

Problem 2

Problem 3

Problem 4

Problem 5

Problem 6