A Radical Approach to Real Analysis by David M. Bressoud

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February 5, 2026

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2.1 The Archimedean Understanding

Definition 1 (Archimedean Understanding of an Infinite Series). *The* ***Archimedean Understanding*** *of an infinite series is that it is shorthand for the sequence of finite summations. The* ***value*** *of an infinite series, if it exists, is that number T such that given any $L < T$ and any $M > T$, all of the finite sums from some point on will be strictly contained in the interval between L and M. More precisely, given $L < T < M$, there is an integer n, whose value depends on the choice of L and M, such that every partial sum with at least n terms lies inside the interval (L, M).*

2.2 Geometric Series

Definition 2 (Convergence of an Infinite Series). *An infinite series* ***converges*** *if there is a target value T such that for any $L < T$ and any $M > T$, all of the partial sums from some point on are strictly between L and M.*

2.3 Calculating π

Theorem 1 (Newton's Binomial Series). *For any real number a and any x such that $|x| < 1$, we have that*

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!} + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots .$$

2.4 Logarithms and Harmonic Series

Definition 3 (Divergence to Infinity). *When we write that an infinite series equals ∞, we mean that no matter what number we pick, we can find an n so that the partial sums with at least n terms will exceed that number.*

Definition 4 (Euler's constant, γ). *Euler's constant is defined as the limit between the partial sum of the harmonic series and the natural logarithm,*

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \ln n\right)$$

Definition 5 (Nested Interval Principle). *Given an increasing sequence, $x_1 \le x_2 \le x_3 \le \cdots$, and a decreasing sequence, $y_1 \ge y_2 \ge y_3 \ge \cdots$, such that y_n is always larger than x_n but the difference between y_n and x_n can be made arbitrarily small by taking n sufficiently large, there is exactly one real number that is greater than or equal to every x_n and less than or equal to every y_n.*

2.5 Taylor Series

Definition 6 (Taylor Series). *If all of the derivatives of the function f exist at the point a, then the Taylor series f about a is the infinite series*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^3 + \cdots.$$

This has a special case ($a = 0$):

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots.$$

Theorem 2 (Lagrange's Remainder Theorem). *Given a function f for which all derivatives exist at $x = a$, let $D_n(a,x)$ denote the difference between the nth partial sum of the tayls series for f expanded about $x = a$ and the target value $f(x)$,*

$$D_n(a,x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}\right).$$

There is at least one real number c strictly between a and x for which

$$D_n(a,x) = \frac{f^{(n)}(c)}{n!}(x-a)^n.$$

Theorem 3 (Stirling's Formula). *The factorial function $n!$ is well approximated by the function $(n/e)^n\sqrt{2\pi n}$. Specifically, we have that*

$$\lim_{n\to\infty}\frac{n!}{(n/e)^n\sqrt{2\pi n}} = 1.$$

Theorem 4 (C^p and analytic functions). *Given an interval I, a function with a continuous first derivative in I is said to belong to the class C^1. If the pth derivative exists and is continuous in I, the function belongs to the class C^p. If all derivatives exist, the function belongs to C^∞ and is called* ***analytic***.

3 Differentiability and Continuity

3.1 Differentiability

Theorem 5 (Mean Value Theorem). *Given a function f that is differentiable at all points strictly between a and x and continuous at all points on the closed interval from a to x, there exists a real number c strictly between a and x such that*

$$\frac{f(x)-f(a)}{x-a} = f'(c).$$

Definition 7 (Archimedian Understanding of Limits). *When we write any limit statement such as*

$$\lim_{x \to a} f(x) = T.$$

what we actually mean is that if we take any number $M > T$, then we can force $f(x) < M$ by taking x to be sufficiently close to a. Similarly, if we take any $L < T$, then we can force $f(x) > L$ by taking x sufficiently close to (but not equal to) a.

Definition 8 (Derivative of f at $x = a$). *The **derivative** of f at a is that value, denoted $f'(a)$, such that for any $L < f'(a)$ and any $M > f'(a)$, we can force*

$$L < \frac{f(x) - f(a)}{x - a} < M,$$

by simply taking x sufficiently close to (but not equal to) a.

Definition 9 (Cauchy Definition of Derivative of f at $x = a$). *The **derivative** of f at a is that value, denoted $f'(a)$, such that for any $\epsilon > 0$, we have a response $\delta > 0$ so that if $0 < |x - a| < \delta$, then this forces*

$$E(x, a) = \left| f'(a) - \frac{f(x) - f(a)}{x - a} \right| < \epsilon.$$

3.2 Cauchy and the Mean Value Theorem

Definition 10 (Intermediate Value Property). *A function f is said to have the intermediate value property on the interval $[a, b]$ if given any two points $x_1, x_2 \in [a, b]$ and any number N satisfying*

$$f(x_1) < N < f(x_2),$$

then there is at least one value c between x_1 and x_2 for which $f(c) = N$.

Theorem 6 (Generalized Mean Value Theorem). *If f and F are both continuous at every point of $[a, b]$ and differentiable at every point on the open interval (a, b) and F' is never zero in this interval, then*

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(c)}{F'(c)},$$

for at least one point c, $a < c < b$.

3.3 Continuity

Definition 11 (Continuity). *We say that f is **continuous at** a if given any positive error bound ϵ, we can always reply with a tolerance δ such that if x is within δ of a, then $f(x)$ is within ϵ of $f(a)$:*

$$|x - a| = \delta \text{ implies that } |f(x) - f(a)| < \epsilon.$$

*To say that f is **continuous on an interval** I means that it is continuous at every point a in the interval I.*

Theorem 7 (Intermediate Value Theorem). *If f is continuous on the interval $[a, b]$, then f has the intermediate value property on this interval.*

Definition 12 (Monotonic). *A function is **monotonic** on $[a, b]$ if it is **increasing** on this interval,*

$$a \le x_1 \le x_2 < b \text{ implies that } f(x_1) \le f(x_2),$$

*or if it is **decreasing** on this interval,*

$$a \le x_1 \le x_2 \le b \text{ implies that } f(x_1) \ge f(x_2).$$

*A function is **piecewise monotonic** on $[a, b]$ if we can find a partition of this interval into a finite number of subintervals*

$$a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

for which the function is monotonic on each open subinterval (x_i, x_{i+1}).

Theorem 8 (Modified Converse to IVT). *If f is a piecewise monotonic and satisfies the intermediate value property on the interval $[a, b]$, then f is continuous at every point c in (a, b).*

Theorem 9 (Differentiable implies Continuous). *If f is differentiable at $x = c$, then f is continuous at $x = c$.*

Definition 13 (One-side Limits and Derivatives). *The **limit from the right,** $\lim_{x \to a^+} f(x)$, is the target value T, with the property that for any $\epsilon > 0$, there is a reponse δ so that if $a < x < a + \delta$, then $|f(x) - T| < \epsilon$. The **one-sided derivatives** are defined by*

$$f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}, \quad f'_-(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}.$$

3.4 Consequences of Continuity

Theorem 10 (Continuous implies Bounded). *If f is continuous on the interval $[a, b]$, then there exists finite A and B such that*

$$A \leq f(x) \leq B,$$

for all $x \in [a, b]$.

Definition 14 (Least Upper, Greatest Lower Bounds). *Given a set S, the **least upper bound** or **supremum** of S, denoted $\sup S$, is the number G with the property that for any numbers $L < G$ and $M > G$, there is at least one elemement of S that is strictly larger than L and at least one upper bound for S that is strictly smaller than M. The **greatest lower bound** or **infimum** of S, denoted $\inf S$, is the negative of the least upper bound of $-S = \{-s \mid s \in S\}$.*

Theorem 11 (Upperbound implies Least Upper Bound). *In the real numbers, every set that has an upper bound also has a least upper bound and every set that has a lower bound also has a greatest lower bound.*

Theorem 12 (Continuous implies Bounds Achieved). *If f is continuous on $[a, b]$, then it achieves its greatest lower bound and its least upper bound. Equivalently, there exists $k_1, k_2 \in [a, b]$ such that*

$$f(k_1) \leq f(x) \leq f(k_2),$$

for all $x \in [a, b]$.

Theorem 13 (Fermat's Theorem on Extrema). *If f has an extremum at a point $c \in (a, b)[f(c) \geq f(x)$ for all $x \in (a, b)$ or $f(c) \leq f(x)$ for all $x \in (a, b)]$ and if f is differentiable at every point in (a, b), then $f'(c) = 0$.*