

# Algebraic Geometry by Thomas Garrity et. al.

Frosty

February 15, 2026

## Contents

1	Conics	1
1.1	Conics over the Reals	1
1.2	Changes of Coordinates	10
1.3	Conics over the Complex Numbers	21
1.4	The Complex Projective Plane $\mathbb{P}^2$	25
1.5	Projective Changes of Coordinates	32
1.6	The Complex Projective Line $\mathbb{P}^1$	33
1.7	Ellipses, Hyperbolas, and Parabolas as Spheres	36
1.8	Links to Number Theory	40
1.9	Degenerate Conics	43
1.10	Tangents and Singular Points	48

## 1 Conics

### 1.1 Conics over the Reals

#### Problem 1

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that for any  $(x, y) \in C$ , we also have

$$(-x, y) \in C.$$

Thus the curve is symmetric about the y-axis.

*Proof.* Let  $(x, y) \in C$ . Then  $P(x, y) = y - x^2 = 0$ . Let  $x' = -x$  and note that  $(-x)^2 = x^2$ . Thus

$$P(-x, y) = y - (-x)^2 = y - x^2 = 0.$$

Thus  $(-x, y) \in C$ . ■

#### Problem 2

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that if  $(x, y) \in C$ , then we have  $y \geq 0$ .

*Proof.* Suppose  $(x, y) \in C$ . Then

$$P(x, y) = y - x^2 = 0 \iff y = x^2 \geq 0.$$

Thus  $y \geq 0$ . ■

### Problem 3

$$P(x, y) = y - x^2, \quad C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}.$$

Show that for every  $y \geq 0$ , there is a point  $(x, y) \in C$  with this  $y$ -coordinate. Now, for points  $(x, y) \in C$ , show that if  $y$  goes to infinity, then one of the corresponding  $x$ -coordinates also approaches infinity while the other corresponding  $x$  coordinate must approach negative infinity.

*Proof.* Let  $y \in \mathbb{R}$  such that  $y \geq 0$ . Let  $x = \sqrt{y} \in \mathbb{R}$ . Then

$$y - x^2 = y - (\sqrt{y})^2 = y - y = 0.$$

Thus  $(x, y) = (\sqrt{y}, y) \in C$ .

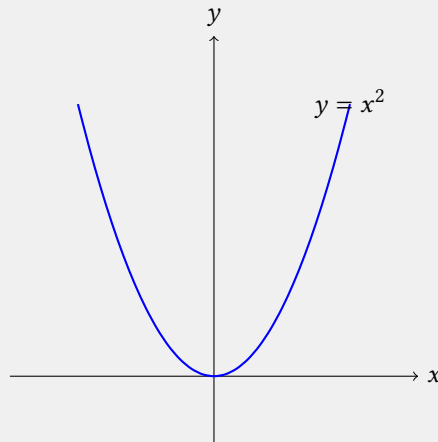
Now suppose  $y \rightarrow \infty$ . For points  $(x, y) \in C$ , we have

$$y - x^2 = 0 \iff x = \pm\sqrt{y}.$$

Since  $y \rightarrow \infty$ , we have  $\sqrt{y} \rightarrow \infty$  and  $-\sqrt{y} \rightarrow -\infty$ . Thus one corresponding  $x$ -coordinate approaches infinity, while the other approaches negative infinity. ■

### Problem 4

Sketch the curve  $C = \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}$ .



### Problem 5

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \right\}.$$

Show that if  $(x, y) \in C$ , then the three points  $(-x, y)$ ,  $(x, -y)$ ,  $(-x, -y)$  are also on  $C$ . Thus the curve  $C$  is symmetric about both the  $x$ - and  $y$ -axes.

*Proof.* Let  $(x, y) \in \mathbb{R}^2$ . Suppose  $\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0$ . Notice that  $x^2 = (-x)^2$  and  $y^2 = (-y)^2$ . Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{y^2}{9} - 1 = \frac{x^2}{4} + \frac{(-y)^2}{9} - 1 = \frac{(-x)^2}{4} + \frac{(-y)^2}{9} - 1 = 0.$$

Thus  $(-x, y), (x, -y), (-x, -y) \in C$ . ■

#### Problem 6

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \right\}.$$

Show that for every  $(x, y) \in C$ , we have  $|x| \leq 2$  and  $|y| \leq 3$ .

*Proof.* Let  $(x, y) \in C$ . Then

$$\frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \iff 9x^2 + 4y^2 - 36 = 0 \iff 9x^2 = -4y^2 + 36 \iff |x| = \sqrt{\frac{-4}{9}y^2 + 4} \leq \sqrt{4} = 2.$$

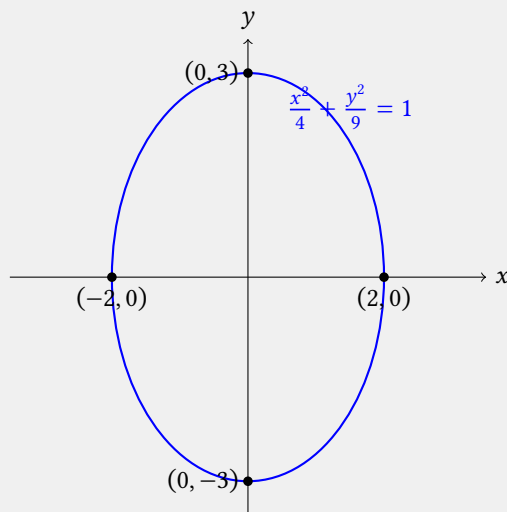
Similarly

$$9x^2 + 4y^2 - 36 = 0 \iff |y| = \sqrt{\frac{-9}{4}x^2 + 9} \leq \sqrt{9} = 3.$$
■

#### Problem 7

Sketch

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \right\}.$$



#### Problem 8

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\}.$$

Show that if  $(x, y) \in C$ , then the three points  $(-x, y)$ ,  $(x, -y)$ , and  $(-x, -y)$  are also on  $C$ . Thus the curve  $C$  is also symmetric about the  $x$ - and  $y$ -axes.

*Proof.* Let  $(x, y) \in \mathbb{R}^2$ . Suppose  $x^2 - y^2 - 4 = 0$ . Notice that  $x^2 = (-x)^2$  and  $y = (-y)^2$ . Then

$$x^2 - y^2 - 4 = (-x)^2 - y^2 = x^2 - (-y)^2 = (-x)^2 - (-y)^2 = 0.$$

Thus  $(-x, y), (x, -y), (-x, -y) \in C$ . ■

#### Problem 9

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\}.$$

Show that if  $(x, y) \in C$ , then we have  $|x| \geq 2$ .

*Proof.* Let  $(x, y) \in \mathbb{R}^2$ . Suppose  $x^2 - y^2 - 4 = 0$ . Then

$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff |x| = \sqrt{y^2 + 4} \geq \sqrt{4} = 2.$$

#### Problem 10

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\}.$$

Show that the curve  $C$  is unbounded in the positive and negative  $x$ -directions and also unbounded in the positive and negative  $y$ -directions.

*Proof.* First notice

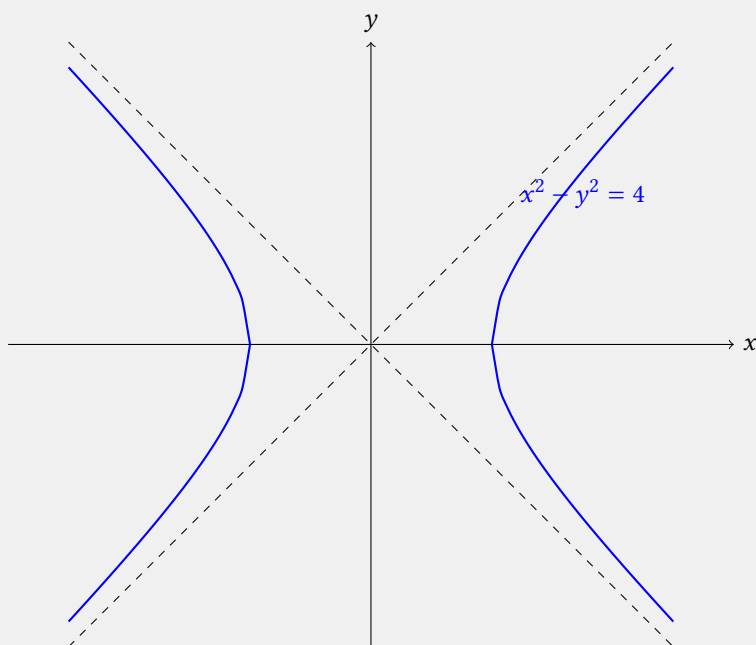
$$x^2 - y^2 - 4 = 0 \iff x^2 = y^2 + 4 \iff x = \pm\sqrt{y^2 + 4} \iff y = \pm\sqrt{x^2 - 4}.$$

As  $y \rightarrow \infty$ , we have  $x = \pm\sqrt{y^2 + 4} \rightarrow \infty$  and  $-\infty$ . Similarly, as  $x \rightarrow \infty$ , we have  $y = \pm\sqrt{x^2 - 4} \rightarrow \infty$  and  $-\infty$ . ■

#### Problem 11

Sketch

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 - 4 = 0\}.$$

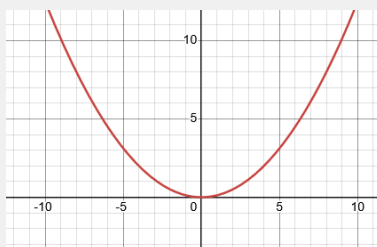


### Problem 12

Sketch the graph of each of the following conics in  $\mathbb{R}^2$ . Identify which are parabolas, ellipses, or Hyperbola.

1.  $V(x^2 - 8y)$ .
2.  $V(x^2 + 2x - y^2 - 3y - 1)$ .
3.  $V(4x^2 + y^2)$ .
4.  $V(3x^2 + 3y^2 - 75)$ .
5.  $V(x^2 - 9y^2)$ .
6.  $V(4x^2 + y^2 - 8)$ .
7.  $V(x^2 + 9y^2 - 36)$ .
8.  $V(x^2 - 4y^2 - 16)$ .
9.  $V(y^2 - x^2 - 9)$ .

**Solution (1):** Parabola.



**Solution (2):** Hyperbola.

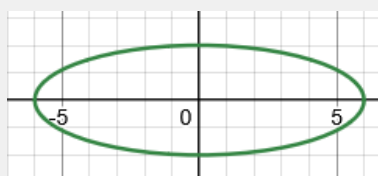
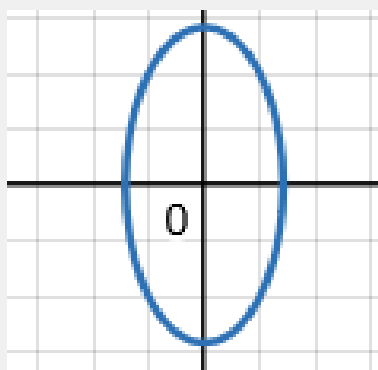
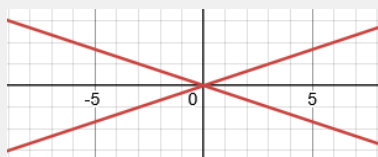
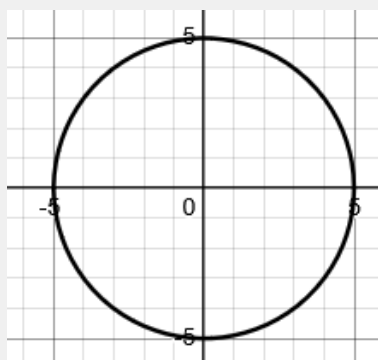
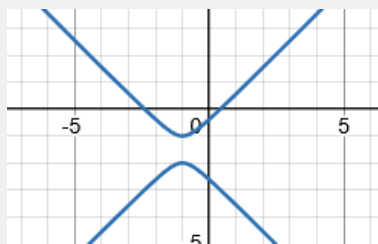
**Solution (3):** Point.

**Solution (4):** Ellipse.

**Solution (5):** Two lines.

**Solution (6):** Ellipse.

**Solution (7):** Ellipse.



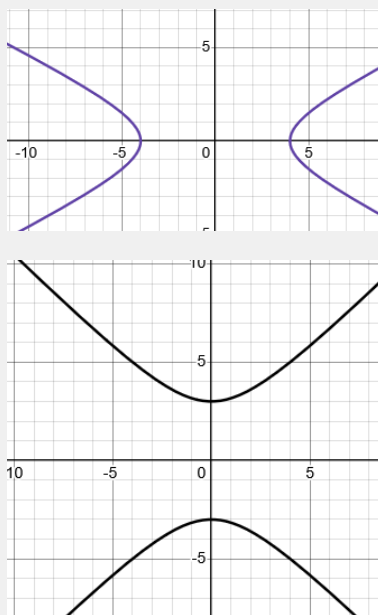
**Solution (8):** Hyperbola.

**Solution (9):** Hyperbola.

#### Problem 13

Express the polynomial  $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$  in the form

$$P(x, y) = Ax^2 + Bx + C,$$



where  $A, B$ , and  $C$  are polynomials in  $y$ . What are  $A, B$ , and  $C$ ?

*Proof.* Let  $A = a, B = by + d$ , and  $C = cy^2 + ey + h$ . Notice

$$ax^2 + bxy + cy^2 + dx + ey + h = ax^2 + bxy + dx + cy^2 + ey + h = ax^2 + (by + d)x + (cy^2 + ey + h) = Ax^2 + Bx + C.$$

#### Problem 14

Treating  $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$  as a polynomial in the variable  $x$ , show that the discriminant is

$$\Delta_x(y) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

*Proof.* From Problem 13 we have  $A = a, B = by + d$ , and  $C = cy^2 + ey + h$ . Then

$$\Delta_x(y) = B^2 - 4AC = (by + d)^2 - 4a(cy^2 + ey + h) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

#### Problem 15

1. Suppose  $\Delta_x(y_0) < 0$ . Explain why there is no point on  $V(P)$  whose  $y$ -coordinate is  $y_0$ .
2. Suppose  $\Delta_x(y_0) = 0$ . Explain why there is exactly one point on  $V(P)$  whose  $y$ -coordinate is  $y_0$ .
3. Suppose  $\Delta_x(y_0) > 0$ . Explain why there are exactly two points on  $V(P)$  whose  $y$ -coordinate is  $y_0$ .

**Solution (a):** In  $\mathbb{R}$ , the square root is undefined for values  $< 0$ .

**Solution (b):** If  $\Delta_x(y_0) = 0$ , then  $+\sqrt{B^2 - 4AC} = -\sqrt{B^2 - 4AC}$ , so there is exactly one point on  $V(P)$  whose  $y$ -coordinate is  $y_0$ .

**Solution (c):** If  $\Delta_x(y_0) > 0$ , then  $+\sqrt{B^2 - 4AC} \neq -\sqrt{B^2 - 4AC}$ , so there are exactly two points on  $V(P)$  whose  $y$ -coordinate is  $y_0$ .

### Problem 16

Suppose  $b^2 - 4ac = 0$ . Suppose further that  $2bd - 4ae > 0$ .

1. Show that  $\Delta_x(y) \geq 0$  if and only if  $y \geq \frac{4ah-d^2}{2bd-4ae}$ .
2. Conclude that if  $b^2 - 4ac = 0$  and  $2bd - 4ae > 0$ , then  $V(P)$  is a parabola.

*Proof.* Suppose  $\Delta_x(y) \geq 0$ . Then

$$\begin{aligned}\Delta_x(y) &= (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah) \\ &= 0y^2 + (2bd - 4ae)y + (d^2 - 4ah).\end{aligned}$$

Therefore,

$$(2bd - 4ae)y + (d^2 - 4ah) \geq 0.$$

Since  $2bd - 4ae > 0$ , we have

$$y \geq \frac{4ah - d^2}{2bd - 4ae}.$$

Conversely, suppose  $y \geq \frac{4ah-d^2}{2bd-4ae}$ . Then

$$\begin{aligned}\Delta_x(y) &= (2bd - 4ae)y + (d^2 - 4ah) \\ &\geq (2bd - 4ae) \left( \frac{4ah - d^2}{2bd - 4ae} \right) + (d^2 - 4ah) \\ &= 0.\end{aligned}$$

*Proof.* Suppose  $b^2 - 4ac = 0$  and  $2bd - 4ae > 0$ . Then  $\Delta_x(y) = (2bd - 4ae)y + (d^2 - 4ah)$ . Now,  $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ . It is clear that  $x$  is symmetrical, and since  $y \geq \frac{4ah-d^2}{2bd-4ae}$ ,  $V(P)$  is a parabola. ■

### Problem 17

Suppose  $b^2 - 4ac < 0$ .

1. Show that one of the following occurs:
  - (a)  $\{y \mid \Delta_x(y) \geq 0\} = \emptyset$ ,
  - (b)  $\{y \mid \Delta_x(y) \geq 0\} = \{y_0\}$ ,
  - (c) there exist real numbers  $\alpha$  and  $\beta$ ,  $\alpha < \beta$ , such that

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

2. Conclude that  $V(P)$  is either empty set, a point, or an ellipse.

*Proof.* Since  $b^2 - 4ac < 0$ , the graph of  $\Delta_x(y)$  is a downward opening parabola in  $y$ . There are three cases, depending on the number of real zeros of  $\Delta_x(y)$ .

1. If  $\Delta_x(y) < 0$  for all  $y$ , then

$$\{y \mid \Delta_x(y) \geq 0\} = \emptyset.$$

2. If  $\Delta_x(y)$  has exactly one real zero  $y_0$ , then

$$\{y \mid \Delta_x(y) \geq 0\} = \{y_0\}.$$



3. If  $\Delta_x(y)$  has two distinct real zeros  $\alpha < \beta$ , then

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

*Proof.* From part 1 the set of  $y$  values is either empty, a single point, or a bounded interval, it follows that  $V(P)$  is either empty, a point, or an ellipse. ■

#### Problem 18

Suppose  $b^2 - 4ac > 0$ .

1. Show that one of the following occurs:

- (a)  $\{y \mid \Delta_x(y) \geq 0\} = \mathbb{R}$  and  $\Delta_x(y) \neq 0$ ,
- (b)  $\{y \mid \Delta_x(y) = 0\} = \{y_0\}$  and  $\{y \mid \Delta_x(y) > 0\} = \{y \mid y \neq y_0\}$ ,
- (c) there exist real numbers  $\alpha$  and  $\beta$ ,  $\alpha < \beta$ , such that

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

- 2. If  $\{y \mid \Delta_x(y) \geq 0\} = \mathbb{R}$ , show that  $V(P)$  is a hyperbola opening left and right.
- 3. If  $\{y \mid \Delta_x(y) = 0\} = \{y_0\}$ , show that  $V(P)$  is two lines intersecting in a point.
- 4. If there are two real numbers  $\alpha$  and  $\beta$ ,  $\alpha < \beta$ , such that

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\},$$

show that  $V(P)$  is a hyperbola opening up and down.

*Proof.* Since  $b^2 - 4ac > 0$ , the graph of  $\Delta_x(y)$  is an upward opening parabola in  $y$ . There are three cases, depending on the number of real zeros of  $\Delta_x(y)$ .

1. If  $\Delta_x(y) > 0$  for all  $y$ , then

$$\{y \mid \Delta_x(y) \geq 0\} = \mathbb{R}.$$

2. If  $\Delta_x(y)$  has exactly one real zero  $y_0$ , then

$$\{y \mid \Delta_x(y) = 0\} = \{y_0\} \quad \text{and} \quad \{y \mid \Delta_x(y) > 0\} = \{y \mid y \neq y_0\}.$$

3. If  $\Delta_x(y)$  has two distinct real zeros  $\alpha < \beta$ , then

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

*Proof.* Since  $b^2 - 4ac > 0$ , the graph of  $\Delta_x(y)$  is an upward opening parabola in  $y$ . There are three cases, depending on the number of real zeros of  $\Delta_x(y)$ .

1. If  $\Delta_x(y) > 0$  for all  $y$ , then

$$\{y \mid \Delta_x(y) \geq 0\} = \mathbb{R}.$$

2. If  $\Delta_x(y)$  has exactly one real zero  $y_0$ , then

$$\{y \mid \Delta_x(y) = 0\} = \{y_0\} \quad \text{and} \quad \{y \mid \Delta_x(y) > 0\} = \{y \mid y \neq y_0\}.$$

3. If  $\Delta_x(y)$  has two distinct real zeros  $\alpha < \beta$ , then

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

*Proof.* Suppose  $\{y \mid \Delta_x(y) \geq 0\} = \mathbb{R}$ . Then for every  $y$  there exist two real solutions for  $x$ , and  $x$  is unbounded to the left and right. Since the equation is quadratic in  $x$ , the curve is symmetric in  $x$ . Thus  $V(P)$  is a hyperbola opening left and right.

*Proof.* Suppose  $\{y \mid \Delta_x(y) = 0\} = \{y_0\}$ . Then for  $y = y_0$  the equation has exactly one real solution for  $x$ , and for  $y \neq y_0$  it has two real solutions. Since the equation is quadratic in  $x$ ,  $V(P)$  consists of two lines intersecting at a point.

*Proof.* Suppose there exist real numbers  $\alpha$  and  $\beta$ ,  $\alpha < \beta$ , such that

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

For  $y \leq \alpha$  or  $y \geq \beta$ , the equation has two real solutions in  $x$ . If  $\alpha < y < \beta$  it has no real solutions. Thus  $x$  is bounded for each  $y$ , but  $y$  is unbounded above and below. Since the equation is quadratic in  $x$ , the curve is symmetric in  $x$ . Therefore  $V(P)$  is a hyperbola opening up and down.

#### Problem 19

Show that the discriminant of  $A'y^2 + B'y + C' = 0$  is

$$\Delta_y(x) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4ch).$$

*Proof.* Here  $A' = c$ ,  $B' = bx + e$ , and  $C' = ax^2 + dx + h$ . Then

$$\Delta_y(x) = (B')^2 - 4A'C' = (bx + e)^2 - 4c(ax^2 + dx + h) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4ch).$$

## 1.2 Changes of Coordinates

#### Problem 1

Show that the origin in the  $xy$ -coordinate system agrees with the origin in the  $uv$ -system if and only if  $e = f = 0$ . Thus the constants  $e$  and  $f$  describe translations of the origin.

*Proof.* Suppose the  $xy$ -coordinate system agrees with the origin of the  $uv$ -system. Then

$$u = 0 = a(0) + b(0) + e = e,$$

and

$$v = 0 = c(0) + d(0) + f = f.$$

Thus  $f = e = 0$ .

Conversely, suppose  $e = f = 0$ . Then

$$u = ax + by + e = ax + by + 0 = a(0) + b(0) = 0,$$

and

$$v = cx + dy + f = cx + dy + 0 = c(0) + d(0) = 0.$$

Thus the origin of the  $xy$ -coordinate system agrees with the origin of the  $uv$ -system.

### Problem 2

Show that if  $u = ax + by + e$  and  $v = cx + dy + f$  is a change of coordinates, then the inverse change of coordinates is

$$x = \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf).$$

$$y = \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af).$$

*Proof.* We need to solve the two equations  $u = ax + by + e$  and  $v = cx + dy + f$  in two unknowns  $x$  and  $y$ . Translating this to linear algebra, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u - e \\ v - f \end{bmatrix}.$$

Using Cramer's rule we see

$$x = \frac{\begin{vmatrix} u - e & b \\ v - f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d(u - e) - b(v - f)}{ad - bc},$$

$$y = \frac{\begin{vmatrix} a & u - e \\ c & v - f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c(u - e) + a(v - f)}{ad - bc}.$$

Therefore

$$x = \frac{du - bv - de + bf}{ad - bc}, \quad y = \frac{-cu + av + ce - af}{ad - bc}.$$

### Problem 3

Show that if

$$u = ax + by + e$$

$$v = cx + dy + f,$$

and

$$s = Au + Bv + E$$

$$t = Cu + Dv + F$$

are two real affine changes of coordinates from the  $xy$ -plane to the  $uv$ -plane and from the  $uv$ -plane to the  $st$ -plane, respectively, then the composition from the  $xy$ -plane to the  $st$ -plane is a real affine change of coordinates.

*Proof.* Suppose

$$u = ax + by + e$$

$$v = cx + dy + f,$$

and

$$s = Au + Bv + E$$

$$t = Cu + Dv + F$$

are two real affine changes of coordinates from the  $xy$ -plane to the  $uv$ -plane and from the  $uv$ -plane to the  $st$ -plane respectively. Substituting  $u, v$  into  $s, t$  we see

$$s = A(ax + by + e) + B(cx + dy + f) + E = (Aa + Bc)x + (Ab + Bd)y + (Ae + Bf + E),$$

and

$$t = C(ax + by + e) + D(cx + dy + f) + F = (Ca + Dc)x + (Cb + Dd)y + (Ce + Df + F).$$

Finally,

$$\det \begin{pmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{pmatrix} = (Aa + Bc)(Cb + Dd) - (Ab + Bd)(Ca + Dc) = (ad - bc)(AD - BC) \neq 0.$$

■

#### Problem 4

For each affine pair of ellipses, find a real affine change of coordinates that maps the ellipse in the  $xy$ -plane to the ellipse in the  $uv$ -plane.

1.  $V(x^2 + y^2 - 1), V(16u^2 + 9v^2 - 1)$ .
2.  $V((x - 1)^2 + y^2 - 1), V(16u^2 + 9(v + 2)^2 - 1)$ .
3.  $V(4x^2 + y^2 - 6y + 8), V(u^2 - 4u + v^2 - 2v + 4)$ .
4.  $V(13x^2 - 10xy + 13y^2 - 1), V(4u^2 + 9v^2 - 1)$ .

**Solution (1):** Let  $x = 4u$  and  $y = 3v$ . Then

$$x^2 + y^2 - 1 = (4u)^2 + (3v)^2 - 1 = 16u^2 + 9v^2 - 1 = 0.$$

**Solution (2):** Let  $x = 4u + 1$  and  $y = 3v + 6$ . Then

$$(x - 1)^2 + y^2 - 1 = (4u + 1 - 1)^2 + (3v + 6)^2 - 1 = 16u^2 + 9(v + 2)^2 - 1 = 0.$$

**Solution (3):** Let  $x = \frac{u}{2} - 1$  and  $y = v + 2$ . Then

$$\begin{aligned} 4x^2 + y^2 - 6y + 8 &= 4\left(\frac{u}{2} - 1\right)^2 + (v + 2)^2 - 6(v + 2) + 8 = \\ &= 4\left(\frac{u^2}{4} - 2\frac{u}{2} + 1\right) + v^2 + 4v + 4 - 6v - 12 + 8 = u^2 - 4u + 4 + v^2 - 2v = u^2 - 4u + v^2 - 2v + 4. \end{aligned}$$

**Solution (4):** Let  $x = \frac{u+v}{2}$  and  $y = \frac{u-v}{2}$ . Then

$$\begin{aligned} 13x^2 - 10xy + 13y^2 - 1 &= 13\left(\frac{u+v}{2}\right)^2 - 10\left(\frac{u+v}{2} \cdot \frac{u-v}{2}\right) + 13\left(\frac{u-v}{2}\right)^2 - 1 \\ &= 13\frac{(u+v)^2}{4} - 10\frac{u^2 - v^2}{4} + 13\frac{(u-v)^2}{4} - 1 \\ &= \frac{13}{4}(u^2 + 2uv + v^2) - \frac{10}{4}(u^2 - v^2) + \frac{13}{4}(u^2 - 2uv + v^2) - 1 \\ &= \frac{13 + 13 - 10}{4}u^2 + \frac{13 + 13 + 10}{4}v^2 + \frac{26 - 26}{4}uv - 1 \\ &= 4u^2 + 9v^2 - 1. \end{aligned}$$

### Problem 5

For each pair of hyperbolas, find a real affine change of coordinates that maps the hyperbola in the  $xy$ -plane to the hyperbola in the  $uv$ -plane.

1.  $V(xy - 1), V(u^2 - v^2 - 1)$ .
2.  $V(x^2 - y^2 - 1), V(16u^2 - 9v^2 - 1)$ .
3.  $V((x - 1)^2 - y^2 - 1), V(16u^2 - 9(v + 2)^2 - 1)$ .
4.  $V(x^2 - y^2 - 1), V(v^2 - u^2 - 1)$ .
5.  $V(8xy - 1), V(2u^2 - 2v^2 - 1)$ .

**Solution (1):** Let  $x = u - v$  and  $y = u + v$ . Then

$$xy - 1 = (u - v)(u + v) - 1 = u^2 - v^2 - 1.$$

**Solution (2):** Let  $x = 4u$  and  $y = 3v$ . Then

$$x^2 - y^2 - 1 = (4u)^2 - (3v)^2 - 1 = 16u^2 - 9v^2 - 1.$$

**Solution (3):** Let  $x = 4u + 1$  and  $y = 3v + 6$ . Then

$$(x - 1)^2 - y^2 - 1 = (4u + 1 - 1)^2 - (3v + 6)^2 = 16u^2 - 9(v + 2)^2 - 1.$$

**Solution (4):** Let  $x = v$  and  $y = u$ . Then

$$x^2 - y^2 - 1 = v^2 - u^2 - 1.$$

**Solution (5):** Let  $x = (u + v)/4$  and  $y = (u - v)/2$ . Then

$$8xy - 1 = 8((u + v)/4)((u - v)/2) - 1 = (u + v)(u - v) - 1 = u^2 - v^2 - 1.$$

### Problem 6

For each pair of parabolas, find a real affine change of coordinates that maps the parabola in the  $xy$ -plane to the parabola in the  $uv$ -plane.

1.  $V(x^2 - y), V(9v^2 - 4u)$ .
2.  $V((x - 1)^2 - y), V(u^2 - 9(v + 2))$ .
3.  $V(x^2 - y), V(u^2 + 2uv + v^2 - u + v - 2)$ .
4.  $V(x^2 - 4x + y + 4), V(4u^2 - (v + 1))$ .
5.  $V(4x^2 + 4xy + y^2 - y + 1), V(4u^2 + v)$ .

**Solution (1):** Let  $x = 3v$  and  $y = 4u$ . Then

$$x^2 - y = (3v)^2 - 4u = 9v^2 - 4u.$$

**Solution (2):** Let  $x = u + 1$  and  $y = 9v + 18$ . Then

$$(x - 1)^2 - y = (u + 1 - 1)^2 - (9v + 18) = u^2 - 9(v + 2).$$

**Solution (3):** Let  $x = (u + v)^2$  and  $y = u - v + 2$ . Then

$$x^2 - y = (u + v)^2 - (u - v + 2) = u^2 + 2uv + v^2 - u + v - 2.$$

**Solution (4):** Let  $x = 2u + 2$  and  $y = -(v + 1)$ . Then

$$x^2 - 4x + y + 4 = (2u + 2)^2 - 4(2u + 2) - (v + 1) + 4 = 4u^2 + 8u + 4 - 8u - 8 - (v + 1) + 4 = 4u^2 - (v + 1).$$

**Solution (5):** Let  $x = u - \frac{1}{2}v + \frac{1}{2}$  and  $y = v$ . Then

$$\begin{aligned}
 4x^2 + 4xy + y^2 - y + 1 &= 4\left(u - \frac{1}{2}v + \frac{1}{2}\right)^2 + 4\left(u - \frac{1}{2}v + \frac{1}{2}\right)v + v^2 - v + 1 \\
 &= 4\left(u^2 - uv + u + \frac{1}{4}v^2 - \frac{1}{2}v + \frac{1}{4}\right) + 4uv - 2v^2 + 2v + v^2 - v + 1 \\
 &= 4u^2 - 4uv + 4u + v^2 - 2v + 1 + 4uv - 2v^2 + 2v + v^2 - v + 1 \\
 &= 4u^2 + v.
 \end{aligned}$$

#### Problem 7

Explain why if  $b^2 - 4ac < 0$ , then  $ac > 0$ .

*Proof.* Suppose  $b^2 - 4ac < 0$ . Then  $0 \leq b^2 < 4ac \iff 0 \leq \frac{b^2}{4} < ac$ . Thus  $ac > 0$ . ■

#### Problem 8

Show that under the real affine transformation

$$x = \sqrt{\frac{c}{a}}u + v$$

$$y = u - \sqrt{\frac{a}{c}}v,$$

the ellipse  $V(ax^2 + bxy + cy^2 + dx + ey + h)$  in the  $xy$ -plane becomes an ellipse in the  $uv$ -plane whose defining equation is  $Au^2 + Cv^2 + Du + Ev + H = 0$ . Find  $A$  and  $C$  in terms of  $a, b, c$ . Show that if  $b^2 - 4ac < 0$ , then  $A \neq 0$  and  $C \neq 0$ .

*Proof.*

$$\begin{aligned}
 ax^2 + bxy + cy^2 + dx + ey + h &= a\left(\sqrt{\frac{c}{a}}u + v\right)^2 + b\left(\sqrt{\frac{c}{a}}u + v\right)\left(u - \sqrt{\frac{a}{c}}v\right) + c\left(u - \sqrt{\frac{a}{c}}v\right)^2 \\
 &\quad + d\left(\sqrt{\frac{c}{a}}u + v\right) + e\left(u - \sqrt{\frac{a}{c}}v\right) + h \\
 &= (cu^2 + 2\sqrt{ac}uv + av^2) + b\left(\sqrt{\frac{c}{a}}u^2 - \sqrt{\frac{a}{c}}v^2\right) + (cu^2 - 2\sqrt{ac}uv + av^2) \\
 &\quad + \left(d\sqrt{\frac{c}{a}} + e\right)u + \left(d - e\sqrt{\frac{a}{c}}\right)v + h \\
 &= (2c + b\sqrt{\frac{c}{a}})u^2 + (2a - b\sqrt{\frac{a}{c}})v^2 + \left(d\sqrt{\frac{c}{a}} + e\right)u + \left(d - e\sqrt{\frac{a}{c}}\right)v + h \\
 &= Au^2 + Cv^2 + Du + Ev + H.
 \end{aligned}$$
■

*Proof.* Suppose  $b^2 - 4ac < 0$ . Then

$$A = \sqrt{\frac{c}{a}}b + 2c, \quad C = -\sqrt{\frac{a}{c}}b + 2a.$$

Then

$$AC = (2c + b\sqrt{\frac{c}{a}})(2a - b\sqrt{\frac{a}{c}}) = 4ac - b^2.$$

Since  $b^2 - 4ac < 0$ ,

$$4ac - b^2 > 0 \implies AC > 0.$$

Therefore  $A \neq 0$  and  $C \neq 0$ . ■

### Problem 9

Show that there exists constants  $R, S$ , and  $T$  such that the equation

$$Au^2 + Cv^2 + Du + Ev + H = 0,$$

can be written in the form

$$A(u - R)^2 + C(v - S)^2 - T = 0.$$

Express  $R, S$ , and  $T$  in terms of  $A, C, D, E$ , and  $H$ .

*Proof.* Let  $R = -\frac{D}{2A}, S = -\frac{E}{2C}, T = \frac{D^2}{4A} + \frac{E^2}{4C} - H$ . Note  $A \neq 0$  and  $C \neq 0$  from problem 8. Notice

$$\begin{aligned} Au^2 + Cv^2 + Du + Ev + H &= A\left(u^2 + \frac{Du}{A}\right) + C\left(v^2 + \frac{Ev}{C}\right) + H \\ &= A\left(u^2 + \frac{Du}{A} + \left(\frac{D}{2A}\right)^2\right) - \frac{D^2}{4A} + C\left(v^2 + \frac{Ev}{C} + \left(\frac{E}{2C}\right)^2\right) - \frac{E^2}{4C} + H \\ &= A\left(u + \frac{D}{2A}\right)^2 + C\left(v + \frac{E}{2C}\right)^2 - \frac{D^2}{4A} - \frac{E^2}{4C} + H \\ &= A(u - R)^2 + C(v - S)^2 - T = 0. \end{aligned}$$

### Problem 10

Suppose  $A, C > 0$ . Find a real affine change of coordinates that maps the ellipse

$$V(A(x - R)^2 + C(y - S)^2 - T),$$

to the circle

$$V(u^2 + v^2 - 1).$$

*Proof.* Since  $A, C > 0$  we know  $T > 0$ . Notice

$$A(x - R)^2 + C(y - S)^2 = T \iff \frac{A(x - R)^2}{T} + \frac{C(y - S)^2}{T} = 1.$$

We set

$$u^2 = \frac{A(x - R)^2}{T}, \quad v^2 = \frac{C(y - S)^2}{T},$$

and solving for  $x, y$  shows

$$x = \sqrt{\frac{T}{A}} u + R, \quad y = \sqrt{\frac{T}{C}} v + S.$$

Substituting into the original equation, we find

$$\begin{aligned} A(x - R)^2 + C(y - S)^2 - T &= A\left(\sqrt{\frac{T}{A}} u\right)^2 + C\left(\sqrt{\frac{T}{C}} v\right)^2 - T \\ &= Tu^2 + Tv^2 - T \\ &= T(u^2 + v^2 - 1), \end{aligned}$$

### Problem 11

Consider the values  $A$  and  $C$  found in Exercise 1.2.8. Show that if  $b^2 - 4ac = 0$ , then either  $A = 0$  or  $C = 0$ , depending on the signs of  $a, b, c$ . [Hint: Recall,  $\sqrt{\alpha^2} = -\alpha$  if  $\alpha < 0$ .]

*Proof.* Suppose  $b^2 - 4ac = 0$ . From Exercise 1.2.8 we have

$$A = \sqrt{\frac{c}{a}}b + 2c, \quad C = -\sqrt{\frac{a}{c}}b + 2a.$$

We see that

$$AC = 4ac - b^2 = -(b^2 - 4ac) = -0 = 0.$$

Thus  $A = 0$  or  $C = 0$ . ■

### Problem 12

Show that there exists constants  $R$  and  $T$  such that the equation

$$Au^2 + Du + Ev + H = 0,$$

can be written as

$$A(u - R)^2 + E(v - T) = 0.$$

Express  $R$  and  $T$  in terms of  $A, D, E$ , and  $H$ .

*Proof.* First note  $A \neq 0$  therefore  $E \neq 0$ . Let

$$R = -\frac{D}{2A}, \quad T = -\left(\frac{H}{E} - \frac{D^2}{4AE}\right).$$

Then

$$\begin{aligned} Au^2 + Du + Ev + H &= A\left(u^2 + \frac{D}{A}u + \left(\frac{D}{2A}\right)^2\right) - \frac{D^2}{4A} + Ev + H \\ &= A\left(u + \frac{D}{2A}\right)^2 + E\left(v + \frac{H}{E} - \frac{D^2}{4AE}\right) \\ &= A(u - R)^2 + E(v - T) = 0. \end{aligned}$$
■

### Problem 13

Suppose  $A > 0$  and  $E \neq 0$ . Find a real affine change of coordinates that maps the parabola

$$V(A(x - R)^2 - E(y - T)),$$

to the parabola

$$V(u^2 - v).$$

*Proof.* We set  $A(x - R)^2 = u^2$  and  $-E(y - T) = -v$ . Then solving for  $x, y$  we have

$$x = \frac{u}{\sqrt{A}} + R, \quad y = \frac{v}{E} + T.$$



Then substituting into our original equation we have

$$A(x - R)^2 - E(y - T) = A\left(\frac{u}{\sqrt{A}} + R - R\right)^2 - E\left(\frac{v}{E} + T - T\right) = u^2 - v.$$

#### Problem 14

Suppose  $ac > 0$ . Use the real affine transformation in Exercise 1.2.8 to transform  $C$  to a conic in the  $uv$ -plane. Find the coefficients of  $u^2$  and  $v^2$  in the resulting equation and show that they have opposite signs.

*Proof.* Suppose  $ac > 0$ . From Exercise 1.2.8 we have

$$A = \sqrt{\frac{c}{a}}b + 2c, \quad C = -\sqrt{\frac{a}{c}}b + 2a.$$

We see that

$$AC = 4ac - b^2 = -(b^2 - 4ac) < 0.$$

Thus  $A$  and  $C$  have opposite signs.

#### Problem 15

Suppose  $ac < 0$  and  $b \neq 0$ . Use the real affine transformation

$$x = \sqrt{-\frac{c}{a}}u + v$$

$$y = u - \sqrt{-\frac{a}{c}}v,$$

to transform  $C$  to a conic in the  $uv$ -plane of the form

$$Au^2 + Cv^2 + Du + Ev + H = 0.$$

Find the coefficients of the resulting equation and show that they have opposite signs.

*Proof.*

$$\begin{aligned} ax^2 + bxy + cy^2 + dx + ey + h &= a\left(\sqrt{-\frac{c}{a}}u + v\right)^2 + b\left(\sqrt{-\frac{c}{a}}u + v\right)\left(u - \sqrt{-\frac{a}{c}}v\right) + c\left(u - \sqrt{-\frac{a}{c}}v\right)^2 \\ &\quad + d\left(\sqrt{-\frac{c}{a}}u + v\right) + e\left(u - \sqrt{-\frac{a}{c}}v\right) + h \\ &= (-cu^2 + 2\sqrt{-ac}uv - av^2) + b\left(\sqrt{-\frac{c}{a}}u^2 - \sqrt{-\frac{a}{c}}v^2\right) + (-cu^2 - 2\sqrt{-ac}uv - av^2) \\ &\quad + (d\sqrt{-\frac{c}{a}} + e)u + (d - e\sqrt{-\frac{a}{c}})v + h \\ &= (-2c + b\sqrt{-\frac{c}{a}})u^2 + (-2a - b\sqrt{-\frac{a}{c}})v^2 + (d\sqrt{-\frac{c}{a}} + e)u + (d - e\sqrt{-\frac{a}{c}})v + h \\ &= Au^2 + Cv^2 + Du + Ev + H. \end{aligned}$$

*Proof.* Since  $ac < 0$  and  $b \neq 0$ , we have

$$A = -2c + b\sqrt{-\frac{c}{a}}, \quad C = -2a - b\sqrt{-\frac{a}{c}}.$$

Then

$$AC = (-2c + b\sqrt{-\frac{c}{a}})(-2a - b\sqrt{-\frac{a}{c}}) = 4ac - b^2.$$

Since  $ac < 0$ ,

$$4ac - b^2 < 0 \implies AC < 0.$$

Therefore  $A$  and  $C$  have opposite signs. ■

#### Problem 16

Suppose  $ac = 0$  (so  $b \neq 0$ ). Since either  $a = 0$  or  $c = 0$ , we can assume  $c = 0$ . Use the real affine transformation

$$\begin{aligned} x &= u + v \\ y &= \left(\frac{1-a}{b}\right)u - \left(\frac{1+a}{b}\right)v, \end{aligned}$$

to transform  $V(ax^2 + bxy + dx + ey + h)$  to a conic in the  $uv$ -plane of the form

$$V(u^2 - v^2 + Du + Ev + H).$$

*Proof.*

$$\begin{aligned} ax^2 + bxy + dx + ey + h &= a(u+v)^2 + b(u+v)\left(\frac{1-a}{b}u - \frac{1+a}{b}v\right) \\ &\quad + d(u+v) + e\left(\frac{1-a}{b}u - \frac{1+a}{b}v\right) + h \\ &= a(u^2 + 2uv + v^2) + (u+v)((1-a)u - (1+a)v) \\ &\quad + d(u+v) + e\left(\frac{1-a}{b}u - \frac{1+a}{b}v\right) + h \\ &= (a+1-a)u^2 + (-(1+a)+a)v^2 + 2auv \\ &\quad + \left(d + e\frac{1-a}{b}\right)u + \left(d - e\frac{1+a}{b}\right)v + h \\ &= u^2 - v^2 + Du + Ev + H \end{aligned}$$

#### Problem 17

Show that there exists  $R, S$ , and  $T$  so that

$$Au^2 - Cv^2 + Du + Ev + H = A(u - R)^2 - C(v - S)^2 - T.$$

Express  $R, S$ , and  $T$  in terms of  $A, C, D, E$ , and  $H$ .

*Proof.* We set  $A(u - R)^2 = Au^2 + Du$  and  $-C(v - S)^2 = -Cv^2 + Ev$ . Then solving for  $R, S$  we have

$$R = -\frac{D}{2A}, \quad S = \frac{E}{2C}.$$

Then substituting into our original equation we have

$$\begin{aligned} Au^2 - Cv^2 + Du + Ev + H &= (A(u - R)^2 - AR^2) + (-C(v - S)^2 + CS^2) + H \\ &= A(u - R)^2 - C(v - S)^2 - (AR^2 - CS^2 - H) \\ &= A(u - R)^2 - C(v - S)^2 - T, \end{aligned}$$

where

$$T = AR^2 - CS^2 - H = \frac{D^2}{4A} - \frac{E^2}{4C} - H.$$

#### Problem 18

Suppose  $A, C, T > 0$ . Find a real affine change of coordinates that maps the hyperbola

$$V(A(x - R)^2 - C(y - S)^2 - T),$$

to the hyperbola

$$V(u^2 - v^2 - 1).$$

*Proof.* Notice

$$A(x - R)^2 - C(y - S)^2 - T = 0 \iff \frac{A(x - R)^2}{T} - \frac{C(y - S)^2}{T} = 1.$$

We set

$$u^2 = \frac{A(x - R)^2}{T}, \quad v^2 = \frac{C(y - S)^2}{T},$$

and solving for  $x, y$  shows

$$x = \sqrt{\frac{T}{A}}u + R, \quad y = \sqrt{\frac{T}{C}}v + S.$$

Substituting into the original equation, we find

$$\begin{aligned} A(x - R)^2 - C(y - S)^2 - T &= A\left(\sqrt{\frac{T}{A}}u\right)^2 - C\left(\sqrt{\frac{T}{C}}v\right)^2 - T \\ &= Tu^2 - Tv^2 - T \\ &= T(u^2 - v^2 - 1). \end{aligned}$$

#### Problem 19

Give an intuitive argument, based on the number of connected components, for the fact that no ellipse can be transformed into a hyperbola by a real affine change of coordinates.

**Solution:** A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve the number of connected components. Therefore, no real affine change can transform an ellipse into a hyperbola.

#### Problem 20

Show that there is no real affine change of coordinates

$$u = ax + by + e$$

$$v = cx + dy + f,$$

that transforms the ellipse  $V(x^2 + y^2 - 1)$  to the hyperbola  $V(u^2 - v^2 - 1)$ .

*Proof.* For contradiction, suppose such a real affine change exists.

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We must have

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the ellipse  $x^2 + y^2 = 1$ . Now substituting  $y^2 = 1 - x^2$  we see for this to vanish for all  $(x, y)$ , the coefficients of  $x^2$  and  $y^2$  must be

$$a^2 - c^2 = b^2 - d^2,$$

which would make the squared coefficients have the same sign, contradicting the requirement for a hyperbola that they have opposite signs. Thus there is no real affine transformation from an ellipse to a hyperbola. ■

#### Problem 21

Give an intuitive argument, based on boundedness, for the fact that no parabola can be transformed into an ellipse by a real affine change of coordinates.

**Solution:** A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve boundedness. Therefore, no real affine change can transform a parabola into an ellipse.

#### Problem 22

Show that there is no real affine change of coordinates that transforms the parabola  $V(x^2 - y)$  to the circle  $V(u^2 + v^2 - 1)$ .

*Proof.* For contradiction, suppose such a real affine change exists.

$$\begin{aligned} u^2 + v^2 &= (ax + by + e)^2 + (cx + dy + f)^2 \\ &= (a^2 + c^2)x^2 + (b^2 + d^2)y^2 + 2(ab + cd)xy + 2(ae + cf)x + 2(be + df)y + (e^2 + f^2). \end{aligned}$$

We must have

$$(a^2 + c^2)x^2 + (b^2 + d^2)y^2 + 2(ab + cd)xy + 2(ae + cf)x + 2(be + df)y + (e^2 + f^2) - 1 = 0$$

for all points on the parabola  $y = x^2$ . Now substituting  $y = x^2$ , we have

$$(b^2 + d^2)x^4 + 2(ab + cd)x^3 + (a^2 + c^2 + 2(be + df))x^2 + 2(ae + cf)x + (e^2 + f^2) - 1 = 0.$$

For this to vanish for all  $x$  all coefficients must be zero so

$$b^2 + d^2 = 0, \quad ab + cd = 0, \quad a^2 + c^2 + 2(be + df) = 0, \quad ae + cf = 0.$$

It follows that  $a = b = c = d = 0$  and therefore  $u^2 + v^2 = e^2 + f^2$  is constant, which cannot equal  $x^2 + y^2$  on the parabola. Thus there is no real affine transformation from the parabola to the circle. ■

### Problem 23

Give an intuitive argument, based on the number of connected components, for the fact that no parabola can be transformed into a hyperbola by a real affine change of coordinates.

**Solution:** A real affine change of coordinates can scale, rotate, shear, or translate a shape. These operations preserve the number of components. Therefore, no real affine change can transform a parabola into a hyperbola.

### Problem 24

Show that there is no real affine change of coordinates that transforms that parabola  $V(x^2 - y)$  to the hyperbola  $V(u^2 - v^2 - 1)$ .

*Proof.* For contradiction, suppose such a real affine change exists. Then

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We must have

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the parabola  $y = x^2$ . Substituting  $y = x^2$ , we get

$$(b^2 - d^2)x^4 + 2(ab - cd)x^3 + (a^2 - c^2)x^2 + 2(ae - cf)x + 2(be - df)x^2 + (e^2 - f^2) - 1 = 0.$$

For this to vanish for all  $x$ , the coefficient of  $x^4$  must be zero

$$b^2 - d^2 = 0 \implies b = \pm d.$$

Then, the  $x^3$  coefficient gives  $ab - cd = 0$ . Since  $b = \pm d$ , we have  $a = \pm c$ . Then, the  $x^2$  coefficient becomes  $a^2 - c^2 + 2(be - df)$ . Since  $a = \pm c$  and  $b = \pm d$ , this is zero if all coefficients vanish. Thus  $u^2 - v^2$  are constant, which cannot equal  $x^2 - y$  on the parabola. Therefore, there is no real affine transformation from the parabola to the hyperbola. ■

## 1.3 Conics over the Complex Numbers

### Problem 1

Show that the set

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 1 = 0\},$$

is empty but that the set

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 + 1 = 0\},$$

is not empty. In fact, show that given any complex number  $x$  there must exist  $y \in \mathbb{C}$  such that

$$(x, y) \in C.$$

Then show that if  $x \neq \pm i$ , then there are two distinct values  $y \in \mathbb{C}$  such that  $(x, y) \in C$ , while if  $x = \pm i$  there is only one such  $y$ .

*Proof.* Suppose  $(x, y) \in \mathbb{R}^2$  such that  $x^2 + y^2 + 1 = 0 \iff x^2 + y^2 = -1$ . Then  $x^2 \geq 0$  and  $y^2 \geq 0$  so  $x^2 + y^2 \geq 0$ , which is a contradiction. ■

*Proof.* Let  $(x, y) = (i, 0) \in \mathbb{C}^2$ . Then

$$x^2 + y^2 + 1 = -1 + 1 = 0.$$

Thus  $(x, y) \in C$ . ■

*Proof.* Let  $x$  be an arbitrary complex number. Furthermore, let  $y = \sqrt{-1 - x^2}$ . Then

$$x^2 + \left(\sqrt{-1 - x^2}\right)^2 + 1 = x^2 - 1 - x^2 + 1 = 0.$$

Thus  $(x, y) \in C$ . ■

*Proof.* Suppose  $x \neq \pm i$ . Then  $1 + x^2 \neq 0$ , so  $\sqrt{1 + x^2} \neq 0$ . Let

$$y = \pm i\sqrt{1 + x^2}.$$

These are two distinct values of  $y$ . Then

$$x^2 + y^2 + 1 = x^2 - (1 + x^2) + 1 = 0.$$

Now suppose  $x = \pm i$ . Then  $1 + x^2 = 0$ , so  $y^2 = 0$  and it follows that  $y = 0$ . Therefore, there is exactly one value of  $y$ . ■

#### Problem 2

Let

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + f,$$

with  $a \neq 0$ . Show that for any value  $y \in \mathbb{C}$ , there must be at least one  $x \in \mathbb{C}$ , but no more than two such  $x$ 's, such that

$$P(x, y) = 0.$$

[Hint: Write  $P(x, y) = Ax^2 + Bx + C$  as a function whose coefficients  $A$ ,  $B$ , and  $C$  are themselves functions of  $y$ , and use the quadratic formula. As mentioned before, this technique will be used frequently.]

*Proof.* Let  $A = a$ ,  $B = by + d$ , and  $C = cy^2 + ey + f$ . Notice

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = ax^2 + (by + d)x + (cy^2 + ey + f) = Ax^2 + Bx + C.$$

Applying the quadratic formula we find

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Since  $A = a \neq 0$  this is defined. Now if  $B^2 - 4AC = 0$  then we get one corresponding  $x$ . Otherwise, we get two corresponding  $x$ 's. ■

#### Problem 3

Let  $C = V\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) \subset \mathbb{C}^2$ . Show that  $C$  is unbounded in  $x$  and  $y$ .

*Proof.* We can solve for  $x$  in terms of  $y$

$$\frac{x^2}{4} = 1 - \frac{y^2}{9} \iff x = \pm 2\sqrt{1 - \frac{y^2}{9}}.$$

Since  $y \in \mathbb{C}$  is arbitrary and square roots always exist in  $\mathbb{C}$ , for any value of  $y$  there is a corresponding value of  $x$ . As  $|y|$  becomes arbitrarily large,  $1 - \frac{y^2}{9}$  becomes arbitrarily large, and thus the corresponding  $x$  is arbitrarily large. Thus  $C$  is unbounded in both  $x$  and  $y$ . ■

#### Problem 4

Let  $C = V(x^2 - y^2 - 1) \subset \mathbb{C}^2$ . Show that there is a continuous path on the curve  $C$  from the point  $(-1, 0)$  to the point  $(1, 0)$ , despite the fact that no such continuous path exists in  $\mathbb{R}^2$ .

*Proof.* Let  $x(t) = \cos(t)$  and  $y(t) = i \sin(t)$ . Then

$$x(t)^2 - y(t)^2 - 1 = \cos^2(t) - (i \sin(t))^2 - 1 = \cos^2(t) + \sin^2(t) - 1 = 0.$$

#### Problem 5

Show that if  $x = u$  and  $y = iv$ , then the circle  $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\}$  transforms into the hyperbola  $\{(u, v) \in \mathbb{C}^2 \mid u^2 - v^2 = 1\}$ .

*Proof.* Suppose  $x = u$  and  $y = iv$ . Then

$$x^2 + y^2 = u^2 + (iv)^2 = u^2 - v^2 = 1.$$

#### Problem 6

Show that if  $u = ax + by + e$  and  $v = cx + dy + f$  is a change of coordinates, then the inverse change of coordinates is

$$\begin{aligned} x &= \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf). \\ y &= \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af). \end{aligned}$$

*Proof.* We need to solve the two equations  $u = ax + by + e$  and  $v = cx + dy + f$  in two unknowns  $x$  and  $y$ . Translating this to linear algebra, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u - e \\ v - f \end{bmatrix}.$$

Using Cramer's rule we see

$$x = \frac{\begin{vmatrix} u - e & b \\ v - f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{d(u - e) - b(v - f)}{ad - bc},$$

$$y = \frac{\begin{vmatrix} a & u-e \\ c & v-f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{-c(u-e) + a(v-f)}{ad-bc}.$$

Therefore

$$x = \frac{du - bv - de + bf}{ad - bc}, \quad y = \frac{-cu + av + ce - af}{ad - bc}.$$

#### Problem 7

Use Theorem 1.2.25 together with the new result of Exercise 1.3.5 to conclude that all ellipses and hyperbolas are equivalent under complex affine changes of coordinates.

*Proof.* By Theorem 1.2.25, any ellipse can be transformed via an affine change of coordinates to a circle. Then, by Exercise 1.3.5 the circle can be transformed via a complex affine change to a hyperbola. ■

#### Problem 8

Show that the circle  $\{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 - 1 = 0\}$  is not equivalent under a complex affine change of coordinates to the parabola  $\{(u, v) \in \mathbb{C}^2 \mid u^2 - v^2 = 0\}$ .

*Proof.* For contradiction, suppose such a complex affine change exists

$$u = ax + by + e, \quad v = cx + dy + f.$$

Then

$$\begin{aligned} u^2 - v^2 &= (ax + by + e)^2 - (cx + dy + f)^2 \\ &= (a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2). \end{aligned}$$

We need

$$(a^2 - c^2)x^2 + (b^2 - d^2)y^2 + 2(ab - cd)xy + 2(ae - cf)x + 2(be - df)y + (e^2 - f^2) - 1 = 0$$

for all points on the circle. Substituting  $y = \sqrt{1 - x^2}$ , the lhs must vanish for all  $x$ . The highest-degree terms show  $b^2 - d^2 = 0 \implies b = \pm d$  and the other coefficients similarly give  $a = \pm c, e = \pm f$ . But then  $u^2 - v^2$  would be constant, which cannot equal  $x^2 + y^2 - 1$ . Therefore there is no complex affine transformation mapping the circle to the hyperbola  $u^2 - v^2 = 0$ . ■

#### Problem 9

Let

$$C = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}.$$

Give a bijection from

$$C \cap \{(x + iy, u + iv) \mid x, u \in \mathbb{R}, y = 0, v = 0\}.$$

to the real circle of the unit radius in  $\mathbb{R}^2$ .

**Solution:**

$$(x + iy, u + iv) \mapsto (x, u).$$



### Problem 10

Let

$$C = \{(z, w) \in \mathbb{C}^2 \mid z^2 + w^2 = 1\}.$$

Give a bijection from

$$C \cap \{(x + iy, u + iv) \in \mathbb{R}^4 \mid x, v \in \mathbb{R}, y = 0, u = 0\},$$

to the hyperbola  $V(x^2 - v^2 - 1)$  in  $\mathbb{R}^2$ .

**Solution:**

$$(x + 0i, 0 + iv) \mapsto (y, u).$$

## 1.4 The Complex Projective Plane $\mathbb{P}^2$

### Problem 1

1. Show that  $(2, 1 + i, 3i) \sim (2 - 2i, 2, 3 + 3i)$ .
2. Show that  $(1, 2, 3) \sim (2, 4, 6)$  and  $(-2, -4, -6) \sim (-i, -2i, -3i)$ .
3. Show that  $(2, 1 + i, 3i) \not\sim (4, 4i, 6i)$ .
4. Show that  $(1, 2, 3) \not\sim (3, 6, 8)$ .

*Proof.* Let  $\lambda = \frac{2}{2-2i} = \frac{1}{2} + \frac{1}{2}i$ . Then

$$\lambda(2 - 2i) = \frac{2}{2 - 2i}(2 - 2i) = 2,$$

$$\lambda \cdot 2 = \left(\frac{1}{2} + \frac{1}{2}i\right)2 = 1 + i,$$

$$\lambda(3 + 3i) = \left(\frac{1}{2} + \frac{1}{2}i\right)(3 + 3i) = 3i.$$

*Proof.* Let  $\lambda = \frac{1}{2}$ . Then

$$\lambda \cdot 2 = 1,$$

$$\lambda \cdot 4 = 2,$$

$$\lambda \cdot 6 = 3.$$

*Proof.* Let  $\lambda = 2i$ . Then

$$\lambda \cdot (-i) = -2,$$

$$\lambda \cdot (-2i) = -4,$$

$$\lambda \cdot (-3i) = -6.$$

*Proof.* Suppose there exists  $\lambda$  such that  $\lambda(4, 4i, 6i) = (2, 1 + i, 3i)$ . Then

$$\lambda \cdot 4 = 2 \implies \lambda = \frac{1}{2},$$

$$\lambda \cdot 4i = 2i \neq 1 + i.$$

Thus no such  $\lambda$  exists.

*Proof.* Suppose there exists  $\lambda$  such that  $\lambda(3, 6, 8) = (1, 2, 3)$ . Then

$$\lambda \cdot 3 = 1 \implies \lambda = \frac{1}{3},$$

$$\lambda \cdot 8 = \frac{8}{3} \neq 3.$$

Thus no such  $\lambda$  exists. ■

#### Problem 2

Show that  $\sim$  is an equivalence relation.

*Proof.* Suppose  $(x, y, z), (a, b, c), (d, e, f) \in \mathbb{C}^3 - \{(0, 0, 0)\}$ . Then  $\lambda = 1$  shows  $(x, y, z) \sim (x, y, z)$ . Thus  $\sim$  is reflexive.

Suppose  $(a, b, c) \sim (d, e, f)$ . Then there exists  $\lambda \in \mathbb{C} - \{0\}$  such that  $(a, b, c) = (\lambda d, \lambda e, \lambda f)$ . Therefore  $(\frac{1}{\lambda}a, \frac{1}{\lambda}b, \frac{1}{\lambda}c) = (d, e, f)$ . It follows that  $(d, e, f) \sim (a, b, c)$ . Thus  $\sim$  is symmetric.

Suppose  $(x, y, z) \sim (a, b, c)$  and  $(a, b, c) \sim (d, e, f)$ . Then there exist  $\lambda_1, \lambda_2 \in \mathbb{C} - \{0\}$  such that  $(x, y, z) = (\lambda_1 a, \lambda_1 b, \lambda_1 c)$  and  $(a, b, c) = (\lambda_2 d, \lambda_2 e, \lambda_2 f)$ . Then

$$(x, y, z) = (\lambda_1 a, \lambda_1 b, \lambda_1 c) = (\lambda_1 \lambda_2 d, \lambda_1 \lambda_2 e, \lambda_1 \lambda_2 f).$$

Thus  $(x, y, z) \sim (d, e, f)$ . Therefore  $\sim$  is transitive. ■

#### Problem 3

Suppose that  $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$  and that  $x_1 = x_2 \neq 0$ . Show that  $y_1 = y_2$  and  $z_1 = z_2$ .

*Proof.* Since  $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$  there exists  $\lambda \in \mathbb{C} - \{0\}$  such that  $(x_1, y_1, z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$ . Thus  $x_1 = \lambda x_2 = \lambda x_1$  therefore  $\lambda = \frac{x_1}{x_1} = 1$ . It follows that  $y_1 = y_2$  and  $z_1 = z_2$ . ■

#### Problem 4

Suppose that  $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$  with  $z_1 \neq 0$  and  $z_2 \neq 0$ . Show that

$$(x_1, y_1, z_1) \sim \left( \frac{x_1}{z_1}, \frac{y_1}{z_1}, 1 \right) = \left( \frac{x_2}{z_2}, \frac{y_2}{z_2}, 1 \right) \sim (x_2, y_2, z_2).$$

*Proof.* We see

$$(x_1, y_1, z_1) = \left( z_1 \cdot \frac{x_1}{z_1}, z_1 \cdot \frac{y_1}{z_1}, z_1 \cdot 1 \right).$$

Since  $z_1 \neq 0$  we see  $(x_1, y_1, z_1) \sim \left( \frac{x_1}{z_1}, \frac{y_1}{z_1}, 1 \right)$ . Now, since  $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$  there exists  $\lambda \in \mathbb{C} - \{0\}$  such that  $(x_1, y_1, z_1) = (\lambda x_2, \lambda y_2, \lambda z_2)$ . Since  $z_1 = \lambda z_2$  and  $z_1, z_2 \neq 0$  we see

$$\frac{x_1}{z_1} = \frac{\lambda x_2}{\lambda z_2} = \frac{x_2}{z_2} \quad \text{and} \quad \frac{y_1}{z_1} = \frac{\lambda y_2}{\lambda z_2} = \frac{y_2}{z_2}.$$

Thus

$$\left( \frac{x_1}{z_1}, \frac{y_1}{z_1}, 1 \right) = \left( \frac{x_2}{z_2}, \frac{y_2}{z_2}, 1 \right).$$

Since

$$(x_2, y_2, z_2) = \left( z_2 \cdot \frac{x_2}{z_2}, z_2 \cdot \frac{y_2}{z_2}, z_2 \cdot 1 \right),$$

and  $z_2 \neq 0$  we see

$$\left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2, y_2, z_2).$$

Therefore,

$$(x_1, y_1, z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2, y_2, z_2).$$

■

#### Problem 5

1. Find the equivalence class of  $(0, 0, 1)$ .
2. Find the equivalence class of  $(1, 2, 3)$ .

**Solution (1):**

$$\{(0, 0, c) \in \mathbb{C}^3 \mid c \neq 0\}.$$

**Solution (2):**

$$\{(\lambda, 2\lambda, 3\lambda) \in \mathbb{C}^3 \mid \lambda \neq 0\}.$$

#### Problem 6

Show that the equivalence class  $(1 : 2 : 3)$  and  $(2 : 4 : 6)$  are equal as sets.

*Proof.* Clearly, with  $\lambda = \frac{1}{2} \in \mathbb{C}$  we have  $(1, 2, 3) = (\lambda 2, \lambda 4, \lambda 6)$ . Thus  $(1, 2, 3) \sim (2, 4, 6)$  so  $(1 : 2 : 3) = (2 : 4 : 6)$ . ■

#### Problem 7

Explain why the elements of  $\mathbb{P}^2$  can intuitively be thought of as complex lines through the origin in  $\mathbb{C}^3$ .

**Solution:** Take a line passing through the origin in  $\mathbb{C}^3$  with direction vector  $(a, b, c) \neq (0, 0, 0)$ . This line consists of all points of the form  $(\lambda a, \lambda b, \lambda c)$  such that  $\lambda \in \mathbb{C}$ . If we require  $\lambda \neq 0$  we get the equivalence class  $(a : b : c) \in \mathbb{P}^2$ . Thus each element of  $\mathbb{P}^2$  represents a complex line through the origin in  $\mathbb{C}^3$ .

#### Problem 8

If  $c \neq 0$ , show, in  $\mathbb{C}^3$ , that the line  $x = \lambda a, y = \lambda b, z = \lambda c$  intersects the plane  $\{(x, y, z) \mid z = 1\}$  in exactly one point. Show that this point of intersection is  $\left(\frac{a}{c}, \frac{b}{c}, 1\right)$ .

*Proof.* Suppose  $c \neq 0$ . At the intersection we must have  $z = \lambda c = 1$  so  $\lambda = \frac{1}{c}$ . Thus

$$(\lambda a, \lambda b, \lambda c) = \left(\frac{a}{c}, \frac{b}{c}, 1\right).$$

■

#### Problem 9

Show that the map  $\psi : \mathbb{C}^2 \rightarrow \{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\}$  defined by  $\psi(x, y) = (x : y : 1)$  is a bijection.

*Proof.* Suppose  $(a, b), (x, y) \in \mathbb{C}^2$  such that  $\psi(x, y) = \psi(a, b)$ . Then

$$\psi(x, y) = \psi(a, b) \iff (x : y : 1) = (a : b : 1).$$

There exists  $\lambda \neq 0$  such that  $(x, y, 1) = (\lambda a, \lambda b, \lambda)$ . Therefore  $\lambda = 1$  thus  $x = a$  and  $y = b$ . Thus  $\psi$  is injective. Let  $(x : y : z)$  be an arbitrary element in  $\{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\}$ . Then

$$(x : y : z) = \left( \frac{x}{z} : \frac{y}{z} : 1 \right) = \psi \left( \frac{x}{z}, \frac{y}{z} \right).$$

Thus  $\psi$  is surjective. It follows that  $\psi$  is bijective. ■

#### Problem 10

Find a map from  $\{(x, y, z) \in \mathbb{P}^2 \mid z \neq 0\}$  to  $\mathbb{C}^2$  that is the inverse of the map  $\psi$  in Exercise 1.4.9.

**Solution:** Let

$$\phi : \{(x : y : z) \in \mathbb{P}^2 \mid z \neq 0\} \longrightarrow \mathbb{C}^2$$

be defined by

$$\phi(x : y : z) = \left( \frac{x}{z}, \frac{y}{z} \right).$$

#### Problem 11

Consider the line  $l = \{(x, y) \in \mathbb{C}^2 \mid ax + by + c = 0\}$  in  $\mathbb{C}^2$ . Assume  $a, b \neq 0$ . Explain why, as  $|x| \rightarrow \infty$ ,  $|y| \rightarrow \infty$ . (Hence,  $|x|$  is the modulus of  $x$ .)

*Proof.* We see  $y = \frac{-c-ax}{b}$  and  $x = \frac{-by-c}{a}$ . Since  $b$  and  $c$  are constants, as  $|y| \rightarrow \infty$  we have  $|x| \rightarrow \infty$ . ■

#### Problem 12

Consider again the line  $l$ . We know that  $a$  and  $b$  cannot both be 0, so we will assume without loss of generality that  $b \neq 0$ .

1. Show that the image of  $l$  in  $\mathbb{P}^2$  under  $\psi$  is the set

$$\{(bx : -ax - c : b) \mid x \in \mathbb{C}\}.$$

2. Show that this set equals the following union.

$$\{(bx : -ax - c : b) \mid x \in \mathbb{C}\} = \{(0 : -c : b)\} \cup \left\{ \left( 1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x} \right) \right\}.$$

3. Show that as  $|x| \rightarrow \infty$ , the second set in the above union becomes

$$\left\{ \left( 1 : -\frac{a}{b} : 0 \right) \right\}.$$

*Proof.* We start by solving explicitly for  $y$  and note  $b \neq 0$ .

$$ax + by + c = 0 \iff y = \frac{-ax - c}{b}.$$

Then

$$\psi \left( x, \frac{-ax - c}{b} \right) = \left( x : \frac{-ax - c}{b} : 1 \right) = (bx : -ax - c : b).$$
■

*Proof.* There are two cases. If  $x = 0$  then

$$(bx : -ax - c : b) = (0 : -c : b).$$

Otherwise, if  $x \neq 0$  then we can divide by  $bx \neq 0$  and see

$$(bx : -ax - c : b) = \left(1 : \frac{-ax - c}{bx} : \frac{1}{x}\right) = \left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right).$$

*Proof.* As  $|x| \rightarrow \infty$ , we have  $\frac{c}{bx} \rightarrow 0$  and  $\frac{1}{x} \rightarrow 0$ . Thus

$$\left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right) \rightarrow \left(1 : -\frac{a}{b} : 0\right).$$

### Problem 13

Explain why the following polynomials are homogeneous, and find each degree.

1.  $x^2 + y^2 - z^2$ .
2.  $xz - y^2$ .
3.  $x^3 + 3xy^2 + 4y^3$ .
4.  $x^4 + x^2y^2$ .

**Solution (1):** All monomials have total degree 2 thus it is homogeneous. The total degree is 2.

**Solution (2):** All monomials have total degree 2 thus it is homogeneous. The total degree is 2.

**Solution (3):** All monomials have total degree 3 thus it is homogeneous. The total degree is 3.

**Solution (4):** All monomials have total degree 4 thus it is homogeneous. The total degree is 4.

### Problem 14

Explain why the following polynomials are not homogeneous.

1.  $x^2 + y^2 - z$ .
2.  $xz - y$ .
3.  $x^2 + 3xy^2 + 4y^3 + 3$ .
4.  $x^3 + x^2y^2 + x^2$ .

**Solution (1):**  $z$  has total degree 1 and  $x^2$  has total degree 2 thus it is not homogeneous.

**Solution (2):**  $xz$  has total degree 2 and  $y$  has total degree 1 thus it is not homogeneous.

**Solution (3):** 3 has total degree 0 and  $x^2$  has total degree 2 thus it is not homogeneous.

**Solution (4):**  $x^3$  has total degree 3 and  $x^2$  has total degree 2 thus it is not homogeneous.

### Problem 15

Show that if the homogeneous equation  $Ax + By + Cz = 0$  holds for every point  $(x, y, z)$  in  $\mathbb{C}^3 - \{0, 0, 0\}$ , then it holds for every point in  $\mathbb{C}^3$  that belongs to the equivalence class  $(x : y : z)$  in  $\mathbb{C}^2$ .

*Proof.* Suppose the homogeneous equation  $Ax + By + Cz = 0$  holds for all  $(x, y, z) \in \mathbb{C}^3 - \{(0, 0, 0)\}$ . Consider the polynomial in  $x$

$$f(x) = Ax + (By + Cz).$$

Since  $f(x) = 0$  for all  $x \in \mathbb{C}$ ,  $A = 0$ . Similarly,  $B = 0$  and  $C = 0$ . Therefore  $Ax + By + Cz = 0$  for all  $(x, y, z) \in \mathbb{C}^3$ , which includes  $(x : y : z)$ .

**Problem 16**

Show that if the homogenous equation  $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$

*Proof.* Suppose the homogeneous equation  $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$ . Consider the polynomial in  $x$

$$f(x) = Ax^2 + Dxy + Exz + (By^2 + Cz^2 + Fyz).$$

Since  $f(x) = 0$  for all  $x \in \mathbb{C}$ ,  $A = 0$ . Similarly,  $B = C = D = E = F = 0$ . Therefore  $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$  for all  $(x, y, z) \in \mathbb{C}^3$ , which includes  $(x : y : z)$ . ■

**Problem 17**

State and prove the generalization of the previous two exercises for any degree  $n$  homogenous equation  $P(x, y, z) = 0$ .

*Proof.* Suppose the homogeneous equation  $P(x, y, z) = 0$  of degree  $n$ . Consider the polynomial in  $x$

$$f(x) = P(x, y, z).$$

Since  $f(x) = 0$  for all  $x \in \mathbb{C}$  coefficients of powers of  $x$  must vanish. Similarly, all other coefficients vanish. Therefore  $P(x, y, z) = 0$  for all  $(x, y, z) \in \mathbb{C}^3$ , which includes  $(x : y : z)$ . ■

**Problem 18**

Consider the non-homogeneous equation  $P(x, y, z) = x^2 + 2y + 2z = 0$ . Show that  $(2, -1, -1)$  satisfies the equation. Find a point of the equivalence class  $(2 : -1 : -1)$  that does not satisfy this equation.

**Solution:**

$$2^2 + 2(-1) + 2(-1) = 0$$

$$4^2 + 2(-2) + 2(-2) = 16 - 4 - 4 = 8 \neq 0$$

**Problem 19**

Homogenize the following equations. Then find point(s) where the curves intersect the line at infinity.

1.  $ax + by + c = 0$ .
2.  $x^2 + y^2 = 1$ .
3.  $y = x^2$ .
4.  $x^2 + 9y^2 = 1$ .
5.  $y^2 - x^2 = 1$ .

**Solution (1):**

$$z \cdot \left( a \frac{x}{z} + b \frac{y}{z} + c \right) = 0 \iff ax + by + cz = 0.$$

At  $z = 0$  we have

$$ax + by = 0.$$

A point at infinity is

$$(x : y : z) = (b : -a : 0).$$

**Solution (2):**

$$z^2 \cdot \left( \left( \frac{x}{z} \right)^2 + \left( \frac{y}{z} \right)^2 - 1 \right) = 0 \iff x^2 + y^2 - z^2 = 0.$$

At  $z = 0$  we have

$$x^2 + y^2 = 0.$$

Points at infinity are

$$(x : y : z) = (1 : i : 0), (1 : -i : 0).$$

**Solution (3):**

$$z^2 \cdot \left( \frac{y}{z} - \left( \frac{x}{z} \right)^2 \right) = 0 \iff x^2 - yz = 0.$$

At  $z = 0$  we have

$$x^2 = 0 \implies x = 0.$$

Point at infinity is

$$(x : y : z) = (0 : 1 : 0).$$

**Solution (4):**

$$z^2 \cdot \left( \left( \frac{x}{z} \right)^2 + 9 \left( \frac{y}{z} \right)^2 - 1 \right) = 0 \iff x^2 + 9y^2 - z^2 = 0.$$

At  $z = 0$  we have

$$x^2 + 9y^2 = 0.$$

Points at infinity are

$$(x : y : z) = (3i : 1 : 0), (-3i : 1 : 0).$$

**Solution (5):**

$$z^2 \cdot \left( \left( \frac{y}{z} \right)^2 - \left( \frac{x}{z} \right)^2 - 1 \right) = 0 \iff y^2 - x^2 - z^2 = 0.$$

At  $z = 0$  we have

$$y^2 - x^2 = 0 \implies (y - x)(y + x) = 0.$$

Points at infinity are

$$(x : y : z) = (1 : 1 : 0), (1 : -1 : 0).$$

#### Problem 20

Show that in  $\mathbb{P}^2$ , any two distinct lines will intersect in a point. Notice this implies that parallel lines in  $\mathbb{C}^2$ , when embeded in  $\mathbb{P}^2$ , intersect at the line at infinity.

*Proof.* Suppose we have the lines

$$l_1 : ax + by + cz = 0, \quad l_2 : dx + ey + fz = 0.$$

If  $l_1$  is not parallel to  $l_2$ , then set  $z = 1$  and solve the two equations for  $x$  and  $y$  to find the intersection point in  $\mathbb{C}^2$ . If  $l_1$  is parallel to  $l_2$ , then homogenizing shows that the intersection occurs at  $z = 0$ . ■

#### Problem 21

Once we have homogenized an equation, the original variables  $x$   $y$  are no more important than the variable  $z$ . Suppose we regard  $x$  and  $z$  as the original variables in our homogeneous equation. Then the image of the  $xz$ -plane in  $\mathbb{P}^2$  would be  $\{(x, y, z) \in \mathbb{P}^2 \mid y = 1\}$ .

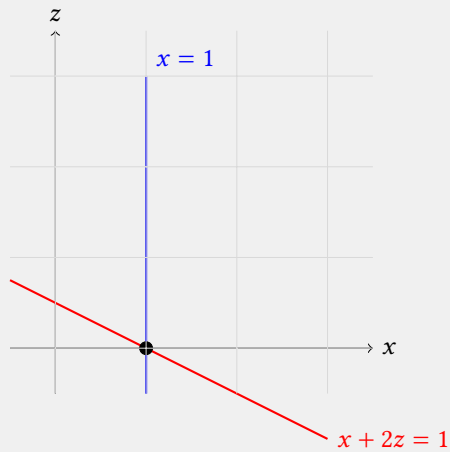
1. Homogenize the equations for the parallel lines  $y = x$  and  $y = x + 2$ .
2. Now regard  $x$  and  $z$  as the original variables, and set  $y = 1$  to sketch the image of the lines in the  $xz$ -plane.
3. Explain why the lines in part meet at the  $x$ -axis.

**Solution (1):** The lines are  $y = x$  and  $y = x + 2$ . Homogenizing with  $z$  gives:

$$\begin{aligned} y - x = 0 &\implies y - x = 0 \\ y - x - 2 = 0 &\implies y - x - 2z = 0 \end{aligned}$$

**Solution (2):** Regarding  $x$  and  $z$  as the original variables and setting  $y = 1$

$$\begin{aligned} 1 - x = 0 &\implies x = 1 \\ 1 - x - 2z = 0 &\implies x + 2z = 1 \end{aligned}$$



**Solution (3):** The lines intersect at the  $x$ -axis because in the  $xz$ -plane the  $x$ -axis is defined by  $z = 0$ .

## 1.5 Projective Changes of Coordinates

### Problem 1

For the complex affine change of coordinates

$$u = ax + by + e.$$

$$v = cx + dy + f,$$

where  $a, b, c, d \in \mathbb{C}$  and  $ad - bc \neq 0$ , show that

$$u = ax + by + ez,$$

$$v = cx + dy + fz,$$

$$w = z,$$

is the corresponding projective change of coordinates.

*Proof.* We have  $\psi(u, v) = \left( \frac{ax+by+e}{z} : \frac{cx+dy+f}{z} : 1 \right) = (ax + by + e : cx + dy + f : z)$ . ■

### Problem 2

Let  $C_1 = V(x^2 + y^2 - 1)$  be an ellipse in  $\mathbb{C}^2$  and let  $C_2 = V(u^2 - v)$  be a parabola in  $\mathbb{C}^2$ . Homogenize the defining polynomials for  $C_1$  and  $C_2$  and show that the projective change of coordinates

$$u = ix,$$



$$v = y + z,$$

$$w = y - z,$$

transforms the ellipse in  $\mathbb{P}^2$  into the parabola in  $\mathbb{P}^2$ .

*Proof.* We first homogenize  $C_1$  and  $C_2$  to obtain

$$x^2 + y^2 - z^2 = 0 \quad \text{and} \quad u^2 - vw = 0.$$

Then substituting our change of variables gives

$$u^2 - vw = (ix)^2 - (y + z)(y - z) = -x^2 - (y^2 - z^2) = -(x^2 + y^2 - z^2) = 0,$$

as required. ■

### Problem 3

Use the results of Section 1.3, together with the above problem, to show that, under a projective change of coordinates, all ellipses, hyperbolas, and parabolas are equivalent in  $\mathbb{P}^2$ .

*Proof.* By Section 1.3 ellipses and hyperbolas are equivalent under affine changes of coordinates, and by Problem 1 these extend to projective changes of coordinates in  $\mathbb{P}^2$ . By Problem 2 ellipses are equivalent to parabolas in  $\mathbb{P}^2$ . Therefore under a projective change of coordinates, all ellipses, hyperbolas, and parabolas are equivalent in  $\mathbb{P}^2$ . ■

## 1.6 The Complex Projective Line $\mathbb{P}^1$

### Problem 1

Suppose that  $(x_1, y_1) \sim (x_2, y_2)$  and that  $x_1 = x_2 \neq 0$ . Show that  $y_1 = y_2$ .

*Proof.* We know there exists  $\lambda$  such that  $x_1 = \lambda x_2$ . Dividing by  $x_2 \neq 0$  shows  $\lambda = 1$ . But then  $y_1 = \lambda y_2 = y_2$ . ■

### Problem 2

Suppose that  $(x_1, y_1) \sim (x_2, y_2)$  with  $y_1 \neq 0$  and  $y_2 \neq 0$ . Show that

$$(x_1, y_1) \sim \left( \frac{x_1}{y_1}, 1 \right) = \left( \frac{x_2}{y_2}, 1 \right) \sim (x_2, y_2).$$

*Proof.* Let  $\lambda = y_1$  to see that

$$\left( \lambda \frac{x_1}{y_1}, \lambda \cdot 1 \right) = (x_1, y_1).$$

Thus  $(x_1, y_1) \sim \left( \frac{x_1}{y_1}, 1 \right)$ . Similarly,  $(x_2, y_2) \sim \left( \frac{x_2}{y_2}, 1 \right)$ . Since  $(x_1, y_1) \sim (x_2, y_2)$  and  $\sim$  is an equivalence relation

$$\left( \frac{x_1}{y_1}, 1 \right) = \left( \frac{x_2}{y_2}, 1 \right).$$
■

### Problem 3

Explain why the elements of  $\mathbb{P}^1$  can intuitively be thought of as complex lines through the origin in  $\mathbb{C}^2$ .

**Solution:** Take a line passing through the origin in  $\mathbb{C}^2$  with direction vector  $(a, b) \neq (0, 0)$ . This line consists of all points of the form  $(\lambda a, \lambda b)$  such that  $\lambda \in \mathbb{C}$ . If we require  $\lambda \neq 0$  we get the equivalence class  $(a : b : c) \in \mathbb{P}^2$ . Thus each element of  $\mathbb{P}^1$  represents a complex line through the origin in  $\mathbb{C}^2$ .

### Problem 4

If  $b \neq 0$ , show that the line  $x = \lambda a, y = \lambda b$  will intersect the line  $\{(x, y) \mid y = 1\}$  in exactly one point. Show that the point of intersection is  $(\frac{a}{b}, 1)$ .

*Proof.* We have  $1 = y = \lambda b$  thus  $\lambda = \frac{1}{b}$ . Therefore  $(x, y) = (\lambda a, 1) = (\frac{a}{b}, 1)$ . ■

### Problem 5

Show that the map  $\psi : \mathbb{C} \rightarrow \{(x : y) \in \mathbb{P}^2 \mid y \neq 0\}$  defined by  $\psi(x) = (x : 1)$  is a bijection.

*Proof.* Suppose  $a, b \in \mathbb{C}$  such that  $\psi(a) = \psi(b)$ . Then

$$\psi(a) = \psi(b) \iff (a : 1) = (b : 1).$$

But then since  $1 = \lambda 1$  we have  $\lambda = 1$  so  $a = b$ . Let  $(a : y)$  be an arbitrary element in  $\{(x : y) \in \mathbb{P}^2 \mid y \neq 0\}$ . Then  $(a : y) = (\frac{a}{y} : 1) = \psi(\frac{a}{y})$ . Thus  $\psi$  is a bijection. ■

### Problem 6

Find a map  $\{(x : y) \in \mathbb{P}^1 \mid y \neq 0\}$  to  $\mathbb{C}$  that is the inverse of the map  $\psi$  in Exercise 1.6.5.

**Solution:**

$$\psi^{-1}(x : y) = \frac{x}{y}.$$

### Problem 7

Consider the map  $\psi : \mathbb{C} \rightarrow \mathbb{P}^2$  given by  $\psi(x) = (x : 1)$ . Show that as  $|x| \rightarrow \infty$ , we have  $\psi(x) \rightarrow (1 : 0)$ .

*Proof.* We have

$$(x : 1) = \left(1 : \frac{1}{x}\right).$$

As  $|x| \rightarrow \infty$ , we have  $\frac{1}{x} \rightarrow 0$  thus

$$\left(1 : \frac{1}{x}\right) \rightarrow (1 : 0).$$

### Problem 8

Let  $p$  denote the point  $(0, 0, 1) \in S^2$ , and let  $l$  denote the line through  $p$  and the point  $(x, y, 0)$  in the

$xy$ -plane, whose parametrization is given by

$$\rho(t) = (1-t)(0, 0, 1) + t(x, y, 0),$$

i.e.,

$$l = \{(tx, ty, 1-t) \mid t \in \mathbb{R}\}.$$

1.  $l$  clearly intersects  $S^2$  at the point  $p$ . Show that there is exactly one other point of intersection  $q$ .
2. Find the coordinates of  $q$ .
3. Define the map  $\psi : \mathbb{R}^2 \rightarrow S^2 - \{p\}$  to be the map that takes the point  $(x, y)$  to the point  $q$ . Show that  $\psi$  is a continuous bijection.
4. Show that as  $\sqrt{x^2 + y^2} \rightarrow \infty$ , we have  $\psi(x, y) \rightarrow p$ . Thus as we move away from the origin in  $\mathbb{R}^2$ ,  $\psi(x, y)$  moves toward the North Pole.

*Proof.* We substitute  $l$  into the unit sphere equation to find

$$x^2 + y^2 + z^2 - 1 = (tx)^2 + (ty)^2 + (1-t)^2 - 1 = t^2(x^2 + y^2 + 1) - 2t = t(t(x^2 + y^2 + 1) - 2) = 0.$$

Now  $t = 0$  corresponds to  $p$ . The other point corresponds to

$$t(x^2 + y^2 + 1) - 2 = 0 \implies t = \frac{2}{x^2 + y^2 + 1}.$$

Substituting back into the line gives

$$q = \left( \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

*Proof.* Each coordinate is a continuous fraction thus  $\psi$  is continuous. Suppose  $(x, y), (a, b) \in \mathbb{R}^2$  such that

$$\psi(x, y) = \psi(a, b) \iff \left( \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right) = \left( \frac{2a}{a^2 + b^2 + 1}, \frac{2b}{a^2 + b^2 + 1}, \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1} \right)$$

From the third coordinate we infer  $x^2 + y^2 + 1 = a^2 + b^2 + 1$ . Then from the first coordinates we see  $a = x$  and  $b = y$ . Thus  $\psi$  is injective. Suppose  $(X, Y, Z) \in S^2 - \{p\}$ . We can solve the following equations for  $x, y$

$$\frac{2x}{x^2 + y^2 + 1} = X, \quad \frac{2y}{x^2 + y^2 + 1} = Y, \quad \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} = Z.$$

Thus

$$x = \frac{X}{1-Z} \quad \text{and} \quad y = \frac{Y}{1-Z}.$$

Finally, we see

$$\psi(x, y) = (X, Y, Z).$$

Thus  $\psi$  is surjective and it follows that  $\psi$  is a bijection.

*Proof.* As  $\sqrt{x^2 + y^2} \rightarrow \infty$  we have  $\frac{2x}{x^2 + y^2 + 1} \rightarrow 0$ ,  $\frac{2y}{x^2 + y^2 + 1} \rightarrow 0$ , and  $\frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \rightarrow 1$ . Thus  $q \rightarrow (0, 0, 1) = p$ .

#### Problem 9

Determine which point(s) in  $\mathbb{P}^1$  do **not** have two representatives of the form  $(x : 1) = (1 : \frac{1}{x})$ .

**Solution:**

$$(0 : 1) \in \mathbb{P}^1.$$

### Problem 10

Map  $U \rightarrow \mathbb{P}^1$  via  $x \mapsto (x : 1)$  and map  $V \rightarrow \mathbb{P}^1$  via  $y \mapsto (1 : y)$ . Show that  $(x : 1) \mapsto (1 : \frac{1}{x})$  is a natural one-to-one map from  $U^*$  to  $V^*$ .

*Proof.* Let  $x, y \in U^*$  then under the map we have  $(1 : \frac{1}{x}) = (1 : \frac{1}{y}) \in V^*$ . Then since 1 is the first coordinate we have  $\lambda = 1$  thus  $x = y$ . Therefore the mapping is a natural one-to-one map from  $U^*$  to  $V^*$ . ■

### Problem 11

A sphere can be split into a neighborhood of its northern hemisphere and a neighborhood of its southern hemisphere. Show that a sphere can be obtained by correctly gluing together two copies of  $\mathbb{C}$ .

*Proof.* We take two spaces  $\mathbb{C}_1, \mathbb{C}_2$ . Then define maps  $\psi_1 : \mathbb{C}_1 \rightarrow \mathbb{R}^3 - \{0, 0, 1\}$ ,  $\psi_2 : \mathbb{C}_2 \rightarrow \mathbb{R}^3 - \{0, 0, -1\}$  by

$$\psi_1(x, y) = \left( \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right),$$

and

$$\psi_2(u, v) = \left( \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{1 - u^2 - v^2}{u^2 + v^2 + 1} \right).$$

Then

$$S^2 = \psi_1(\mathbb{C}_1) \cup \psi_2(\mathbb{C}_2).$$

■

### Problem 12

Put together the last two exercises to show that  $\mathbb{P}^1$  is topologically equivalent to a sphere.

*Proof.* From Problem 10, we can map the two spaces  $U^*$  and  $V^*$  to  $\mathbb{P}^1$ . We can then equate  $U^*$  and  $V^*$  to the two copies of  $\mathbb{C}$ . Then from Problem 11 there is a homeomorphism from the two spaces  $\mathbb{C}$  to  $S^2$ . Thus  $\mathbb{P}^1$  is homeomorphic to  $S^2$ . ■

## 1.7 Ellipses, Hyperbolas, and Parabolas as Spheres

### Problem 1

Find a bijective polynomial map from  $\mathbb{C}$  to the conic  $C = \{(x, y) \in \mathbb{C}^2 \mid x^2 - y = 0\}$

*Proof.* We first parametrize  $C$  as follows

$$x = t \text{ and } y = t^2.$$

Then, we define the mapping  $\psi : \mathbb{C} \rightarrow C$  as

$$\psi(\alpha) = (\alpha, \alpha^2).$$

Now, suppose  $\alpha_1, \alpha_2 \in \mathbb{C}$  and  $\psi(\alpha_1) = \psi(\alpha_2)$ . Then

$$\psi(\alpha_1) = \psi(\alpha_2) \iff (\alpha_1, \alpha_1^2) = (\alpha_2, \alpha_2^2) \iff \alpha_1 = \alpha_2.$$

Thus  $\psi$  is injective. Now, let  $(x, y) \in C$  and notice  $y = x^2$ . Then  $\psi(x) = (x, x^2) = (x, y)$ . Thus  $\psi$  is surjective. It follows that  $\psi$  is a bijection from  $\mathbb{C}$  to  $C$ . ■

### Problem 2

Let  $C = V(x^2 + y^2 - 1)$  be an ellipse in  $\mathbb{C}^2$ . Find a trigonometric parametrization of  $C$ . [Hint: Think high school trigonometry.]

*Proof.* Let  $x = \cos(t)$  and  $y = \sin(t)$  for  $t \in \mathbb{C}$ . ■

### Problem 3

Consider the ellipse  $C = V(x^2 + y^2 - 1) \subset \mathbb{C}^2$  and let  $p$  denote the point  $(0, 1) \in C$ .

1. Parametrize the line segment from  $p$  to the point  $(\lambda, 0)$  on the complex line  $y = 0$  as in Exercise 16.8.
2. This line segment clearly intersects  $C$  at the point  $p$ . Show that if  $\lambda \neq \pm i$ , then there is exactly one other point of intersection. Call this point  $q$ .
3. Find the coordinates of  $q \in C$ .
4. Show that if  $\lambda = \pm i$ , then the line segment intersects  $C$  only at  $p$ .

*Proof.* We have  $p + x(p - (\lambda, 0))$  for  $x \in \mathbb{C}$ . Then

$$p + x(p - (\lambda, 0)) = (0, 1) + x((0, 1) - (\lambda, 0)) = (0, 1) + x(-\lambda, 1) = (-\lambda x, 1 + x).$$

*Proof.* Suppose  $\lambda \neq \pm i$ . Then

$$(-\lambda x)^2 + (1 + x)^2 - 1 = \lambda^2 x^2 + (1 + 2x + x^2) - 1 = (\lambda^2 + 1)x^2 + 2x = x((\lambda^2 + 1)x + 2) = 0.$$

$x = 0$  corresponds to  $p$ .

$$x = -\frac{2}{\lambda^2 + 1}$$

corresponds to

$$q = \left( -\lambda \left( -\frac{2}{\lambda^2 + 1} \right), 1 - \frac{2}{\lambda^2 + 1} \right).$$

*Proof.* Suppose  $\lambda = \pm i$ . Then

$$(-\lambda x)^2 + (1 + x)^2 - 1 = (-\pm i x)^2 + (1 + 2x + x^2) - 1 = 2x = 0.$$

$x = 0$  corresponds to  $p$  and there are no other solutions. ■

### Problem 4

Define the map  $\tilde{\psi} : \mathbb{C} \rightarrow C \subset \mathbb{C}^2$  by

$$\tilde{\psi}(\lambda) = \left( \frac{2\lambda}{\lambda^2 + 1}, \frac{\lambda^2 - 1}{\lambda^2 + 1} \right).$$

Show that the above map can be extended to the map

$$\psi : \mathbb{P}^1 \rightarrow \{(x : y : z) \in \mathbb{P}^2 \mid x^2 + y^2 - z^2 = 0\}.$$

given by

$$\psi(\lambda : u) = (2\lambda u : \lambda^2 - u^2 : \lambda^2 + u^2).$$

*Proof.* Plugging in we find

$$\begin{aligned}(2\lambda u)^2 + (\lambda^2 - u^2)^2 - (\lambda^2 + u^2)^2 &= 4\lambda^2 u^2 + (\lambda^4 - 2\lambda^2 u^2 + u^4) - (\lambda^4 + 2\lambda^2 u^2 + u^4) \\ &= 4\lambda^2 u^2 + \lambda^4 - 2\lambda^2 u^2 + u^4 - \lambda^4 - 2\lambda^2 u^2 - u^4 \\ &= 0.\end{aligned}$$

#### Problem 5

1. Show that the map  $\psi$  is one-to-one.
2. Show that  $\psi$  is onto. [Hint: Consider the two cases:  $z \neq 0$  and  $z = 0$ . For  $z \neq 0$  follow the construction given above. For  $z = 0$ , find values of  $\lambda$  and  $u$  to show that these points are given by  $\psi$ . How does this relate to Part 4 of Exercise 1.7.3?]

*Proof.* Let  $(\lambda : u), (\lambda' : u') \in \mathbb{P}^1$ . Then

$$\psi(\lambda : u) = \psi(\lambda' : u') \iff (2\lambda u : \lambda^2 - u^2 : \lambda^2 + u^2) = (2\lambda' u' : \lambda'^2 - u'^2 : \lambda'^2 + u'^2).$$

There exists  $k \in \mathbb{C}$  such that

$$2\lambda u = k(2\lambda' u'), \quad \lambda^2 - u^2 = k(\lambda'^2 - u'^2) = k\lambda'^2 - ku'^2, \quad \lambda^2 + u^2 = k(\lambda'^2 + u'^2) = k\lambda'^2 + ku'^2.$$

Adding the second and third equations shows

$$\lambda^2 = k\lambda'^2.$$

Subtracting the second and third equations shows

$$-2u^2 = -2ku'^2 \implies u^2 = ku'^2.$$

Thus  $(\lambda : u) = (\lambda' : u')$  in  $\mathbb{P}^1$ . It follows that  $\psi$  is injective.

Suppose  $z = 0$ . We have

$$x^2 + y^2 - z^2 = 0 \iff -x^2 = y^2 \iff y = \pm ix.$$

Then

$$(x : \pm ix : z) = (1 : \pm i : 0).$$

We have

$$\lambda^2 + u^2 = 0 \iff \lambda^2 = -u^2 \iff \lambda = \pm iu.$$

Then

$$\psi(\pm iu : u) = (2(\pm iu)u : (\pm iu)^2 - u^2 : (\pm iu)^2 + u^2) = (\pm 2iu^2 : -2u^2 : 0) = (1 : \pm i : 0).$$

Now, suppose  $z \neq 0$ . Let  $(x : y : z)$  be an element in  $\mathbb{P}^2$  with  $z \neq 0$ . We require

$$2\lambda u = x, \quad \lambda^2 - u^2 = y, \quad \lambda^2 + u^2 = z.$$

Adding and subtracting the second and third equations we find

$$\lambda^2 = \frac{y+z}{2} \implies \lambda = \pm \sqrt{\frac{y+z}{2}}, \quad u^2 = \frac{z-y}{2} \implies u = \pm \sqrt{\frac{z-y}{2}}.$$

Then

$$\psi\left(\pm \sqrt{\frac{y+z}{2}}, \pm \sqrt{\frac{z-y}{2}}\right) = (x : y : z).$$

Thus  $\psi$  is surjective.

### Problem 6

For the following conics and the given point  $p$ , follow what we did for the conic  $x^2 + y^2 - 1 = 0$  to find a rational map from  $\mathbb{C}$  to the curve  $\mathbb{C}^2$  and then a one-to-one map from  $\mathbb{P}^1$  onto the conic in  $\mathbb{P}^2$ .

1.  $x^2 + 2x - y^2 - 4y - 4 = 0$  with  $p = (0, -2)$ .
2.  $3x^2 + 3y^2 - 75 = 0$  with  $p = (5, 0)$ .
3.  $4x^2 + y^2 - 8 = 0$  with  $p = (1, 2)$ .

**Solution (1):** Consider the line  $y = mx - 2$ . We have

$$\begin{aligned}
 x^2 + 2x - y^2 - 4y - 4 &= 0 \\
 \iff x^2 + 2x - (mx - 2)^2 - 4(mx - 2) - 4 &= 0 \\
 \iff x^2 + 2x - (m^2x^2 - 4mx + 4) - 4mx + 8 - 4 &= 0 \\
 \iff (1 - m^2)x^2 + 2x &= 0 \\
 \iff x[(1 - m^2)x + 2] &= 0.
 \end{aligned}$$

Then  $x = 0$  corresponds to  $p$ . So  $x = \frac{-2}{1-m^2}$ , thus  $y = \frac{-2m-2(1-m^2)}{1-m^2}$ . So we have the parametrization

$$x(m) = \frac{-2}{1-m^2}, \quad y(m) = \frac{2m^2 - 2m - 2}{1-m^2}.$$

Then define  $\psi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  as

$$\psi(m, u) = (-2 : 2m^2 - 2m - 2 : 1 - m^2).$$

**Solution (2):** Consider the line  $y = m(x - 5) = mx - 5m$ . We have

$$\begin{aligned}
 3x^2 + 3y^2 - 75 &= 0 \\
 \iff 3x^2 + 3(mx - 5m)^2 - 75 &= 0 \\
 \iff 3x^2 + 3(m^2x^2 - 10m^2x + 25m^2) - 75 &= 0 \\
 \iff 3x^2 + 3m^2x^2 - 30m^2x + 75m^2 - 75 &= 0 \\
 \iff 3(1 + m^2)x^2 - 30m^2x + 75(m^2 - 1) &= 0 \\
 \iff (1 + m^2)x^2 - 10m^2x + 25(m^2 - 1) &= 0 \\
 \iff [(1 + m^2)x - 5(m^2 + 1)][x - 5] &= 0.
 \end{aligned}$$

Then  $x = 5$  corresponds to  $p$ . So  $x = \frac{5(m^2+1)}{1+m^2} = 5$ , thus  $y = m(x - 5) = m(5 - 5) = 0$ . So we have the parametrization

$$x(m) = 5, \quad y(m) = 0.$$

Then define  $\psi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  as

$$\psi(m, u) = (5 : 0 : 1).$$

**Solution (3):** Consider the line  $y = m(x - 1) + 2 = mx - m + 2$ . We have

$$\begin{aligned}
 4x^2 + y^2 - 8 &= 0 \\
 \iff 4x^2 + (mx - m + 2)^2 - 8 &= 0 \\
 \iff 4x^2 + (m^2x^2 - 2m^2x + 4mx + m^2 - 4m + 4) - 8 &= 0 \\
 \iff (4 + m^2)x^2 + (-2m^2 + 4m)x + (m^2 - 4m - 4) &= 0 \\
 \iff (x - 1)((4 + m^2)x + (-4 + m^2 + 4m)) &= 0.
 \end{aligned}$$

Then  $x = 1$  corresponds to  $p$ . So  $x = \frac{4-m^2-4m}{4+m^2}$ , thus

$$y = m \left( \frac{4 - m^2 - 4m}{4 + m^2} - 1 \right) + 2.$$

So we have the parametrization

$$x(m) = \frac{4 - m^2 - 4m}{4 + m^2}, \quad y(m) = m \left( \frac{4 - m^2 - 4m}{4 + m^2} - 1 \right) + 2.$$

Then define  $\psi : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  as

$$\psi(m, u) = (4 - m^2 - 4m : m(4 - m^2 - 4m - (4 + m^2)) + 2(4 + m^2) : 4 + m^2).$$

## 1.8 Links to Number Theory

### Problem 1

Suppose  $(x_0, y_0, z_0)$  is a solution to  $x^2 + y^2 = z^2$ . Show that  $(mx_0, my_0, mz_0)$  is also a solution for any scalar  $m$ .

*Proof.* We have

$$x_0^2 + y_0^2 - z_0^2 = 0 \iff m^2(x_0^2 + y_0^2 - z_0^2) = 0 \iff m^2x_0^2 + m^2y_0^2 - m^2z_0^2 = 0 \iff (mx_0)^2 + (my_0)^2 - (mz_0)^2 = 0.$$

Thus  $(mx_0, my_0, mz_0)$  is also a solution for any scalar  $m$ . ■

### Problem 2

Let  $(a, b, c) \in \mathbb{Z}^3$  be a solution to  $x^2 + y^2 = z^2$ . Show that  $c = 0$  if and only if  $a = b = 0$ .

*Proof.* Suppose  $c = 0$  then clearly  $a = b = 0$ . Suppose  $a = b = 0$ . Then  $a^2 + b^2 = 0^2 + 0^2 = 0$ . Thus  $c = 0$ . ■

### Problem 3

Show that if  $(a, b, c)$  is a Pythagorean triple with  $c \neq 0$ , then the pair of rational number  $\left(\frac{a}{c}, \frac{b}{c}\right)$  is a solution to  $x^2 + y^2 = 1$ .

*Proof.* Suppose  $(a, b, c)$  is a Pythagorean triple with  $c \neq 0$ . Then

$$a^2 + b^2 = c^2 \iff \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \iff \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

### Problem 4

Let  $\left(\frac{a}{c_1}, \frac{b}{c_2}\right) \in \mathbb{Q}^2$  be a rational solution to  $x^2 + y^2 = 1$ . Find a corresponding Pythagorean triple.

*Proof.* First, write  $\frac{a}{c_1}, \frac{b}{c_2}$  in their lowest terms by dividing by common factors. Then

$$\left(\frac{a}{c_1}\right)^2 + \left(\frac{b}{c_2}\right)^2 = 1 \iff (ac_2)^2 + (bc_1)^2 = (c_1c_2)^2.$$

Thus  $(ac_2, bc_1, c_1c_2)$  is a Pythagorean triple. ■



### Problem 5

Let

$$C(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\}.$$

Define the map  $\tilde{\psi} : \mathbb{Q} \rightarrow \{(x, y) \in \mathbb{Q}^2 \mid x^2 + y^2 = 1\}$  as

$$\lambda \mapsto \left( \frac{2\lambda}{\lambda^2 + 1}, \frac{\lambda^2 - 1}{\lambda^2 + 1} \right).$$

Show that the above map  $\tilde{\psi}$  sends  $\mathbb{Q} \rightarrow C(\mathbb{Q})$ .

*Proof.* Let  $\lambda \in \mathbb{Q}$  Then  $\tilde{\lambda} = \left( \frac{2\lambda}{\lambda^2 + 1}, \frac{\lambda^2 - 1}{\lambda^2 + 1} \right)$ . Plugging in we find

$$\begin{aligned} \left( \frac{2\lambda}{\lambda^2 + 1} \right)^2 + \left( \frac{\lambda^2 - 1}{\lambda^2 + 1} \right)^2 &= \frac{4\lambda^2}{(\lambda^2 + 1)^2} + \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \\ &= \frac{4\lambda^2 + (\lambda^4 - 2\lambda^2 + 1)}{(\lambda^2 + 1)^2} \\ &= \frac{\lambda^4 + 2\lambda^2 + 1}{(\lambda^2 + 1)^2} \\ &= 1. \end{aligned}$$

### Problem 6

1. Show that  $\psi : \mathbb{P}^1(\mathbb{Q}) \rightarrow C(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{Q})$  is onto.
2. Show that every primitive triple is of the form  $(2\lambda u, \lambda^2 - u^2, \lambda^2 + u^2)$ .

*Proof.* This is the same as 1.7.5.  $\psi$  is a bijection thus every primitive triple is of that form.

### Problem 7

Find a rational point on the conic  $x^2 + y^2 - 2 = 0$ . Develop a parametrization and conclude that there are infinitely many rational points on this curve.

*Proof.* Consider  $p = (1, 1) \in \mathbb{Q}^2$ . Then  $x^2 + y^2 - 2 = 1 + 1 - 2 = 0$ . Consider the line  $y = m(x - 1) + 1 = mx - m + 1$ . Then

$$\begin{aligned} x^2 + (mx - m + 1)^2 - 2 &= x^2 + (mx - m + 1)^2 - 2 \\ &= x^2 + (m^2 x^2 - 2m^2 x + 2mx + m^2 - 2m + 1) - 2 \\ &= (1 + m^2)x^2 + 2m(1 - m)x + (m^2 - 2m - 1) \\ &= (x - 1)((1 + m^2)x - m^2 + 2m + 1) \end{aligned}$$

Now,  $x = 1$  corresponds to  $p$ . So we have  $x = \frac{m^2 - 2m - 1}{1 + m^2}$ . Then we have the parametrization

$$x(m) = \frac{m^2 - 2m - 1}{1 + m^2}, \quad y(m) = m \left( \frac{m^2 - 2m - 1}{1 + m^2} \right) - m + 1.$$

From this we can obtain infinitely many rational points on the curve.

### Problem 8

By mimicking the above, find four rational points on each of the following conics.

1.  $x^2 + 2x - y^2 - 4y - 4 = 0$  with  $p = (0, -2)$ .
2.  $3x^2 + 3y^2 - 75 = 0$  with  $p = (5, 0)$ .
3.  $4x^2 + y^2 - 8 = 0$  with  $p = (1, 2)$ .

*Proof.* (1) Consider  $p = (0, -2) \in \mathbb{Q}^2$ . Then  $x^2 + 2x - y^2 - 4y - 4 = 0$  at  $(0, -2)$ . Consider the line  $y = m(x - 0) - 2 = mx - 2$ . Then

$$\begin{aligned} x^2 + 2x - (mx - 2)^2 - 4(mx - 2) - 4 &= x^2 + 2x - (m^2x^2 - 4mx + 4) - (4mx - 8) - 4 \\ &= (1 - m^2)x^2 + (2 + 0)x + (-4 + 8 - 4) \\ &= (1 - m^2)x^2 + 2x + 0 \\ &= x((1 - m^2)x + 2) \end{aligned}$$

Now,  $x = 0$  corresponds to  $p$ . So we have  $x = \frac{-2}{1 - m^2}$ . Then we have the parametrization

$$x(m) = \frac{-2}{1 - m^2}, \quad y(m) = m\left(\frac{-2}{1 - m^2}\right) - 2.$$

From this we can obtain infinitely many rational points on the curve. ■

*Proof.* Consider  $p = (5, 0) \in \mathbb{Q}^2$ . Then  $3x^2 + 3y^2 - 75 = 0$  at  $(5, 0)$ . Consider the line  $y = m(x - 5) + 0 = m(x - 5)$ . Then

$$\begin{aligned} 3x^2 + 3(mx - 5m)^2 - 75 &= 3x^2 + 3(m^2x^2 - 10m^2x + 25m^2) - 75 \\ &= 3(1 + m^2)x^2 - 30m^2x + 75(m^2 - 1) \\ &= (x - 5)(3(1 + m^2)x - 30m^2 - 75) \end{aligned}$$

Now,  $x = 5$  corresponds to  $p$ . So the other intersection point satisfies

$$3(1 + m^2)x - 30m^2 - 75 = 0 \implies x = \frac{30m^2 + 75}{3(1 + m^2)} = \frac{10m^2 + 25}{1 + m^2}.$$

Then we have the parametrization

$$x(m) = \frac{10m^2 + 25}{1 + m^2}, \quad y(m) = m\left(\frac{10m^2 + 25}{1 + m^2} - 5\right) = m\left(\frac{10m^2 + 25 - 5 - 5m^2}{1 + m^2}\right) = m\left(\frac{5m^2 + 20}{1 + m^2}\right).$$

From this we can obtain infinitely many rational points on the curve. ■

*Proof.* Consider  $p = (1, 2) \in \mathbb{Q}^2$ . Then  $4x^2 + y^2 - 8 = 0$  at  $(1, 2)$ . Consider the line  $y = m(x - 1) + 2 = mx - m + 2$ . Then

$$\begin{aligned} 4x^2 + (mx - m + 2)^2 - 8 &= 4x^2 + (m^2x^2 - 2m^2x + 2mx + m^2 - 4m + 4) - 8 \\ &= (4 + m^2)x^2 + 2m(1 - m)x + (m^2 - 4m - 4) \\ &= (x - 1)((4 + m^2)x + 2m(1 - m) + (m^2 - 4m - 4)) \end{aligned}$$

Now,  $x = 1$  corresponds to  $p$ . So we have  $x = \frac{-2m^2 + 2m + 4}{4 + m^2}$ . Then we have the parametrization

$$x(m) = \frac{-2m^2 + 2m + 4}{4 + m^2}, \quad y(m) = m\left(\frac{-2m^2 + 2m + 4}{4 + m^2} - 1\right) + 2.$$

From this we can obtain infinitely many rational points on the curve. ■

### Problem 9

Show that the conic  $x^2 + y^2 = 3$  has no rational points.

*Proof.* Suppose  $x = \frac{a}{b}$  and  $y = \frac{c}{d}$  are in lowest terms with  $a, b, c, d \in \mathbb{Z}$ . Furthermore, suppose

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 3.$$

Then

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 3 \iff a^2d^2 + c^2b^2 = 3b^2d^2 \iff (ad)^2 = 3(bd)^2 - (bc)^2 = b^2(3d^2 - c^2).$$

Now we must have  $3d^2 - c^2 = k^2$  for some  $k \in \mathbb{Z}$ . We see  $3d^2 - c^2 \equiv 0, 2 \pmod{3}$  while  $k^2 \equiv 0, 1 \pmod{3}$ . But,  $\gcd(c, d) = 1$  thus  $3d^2 - c^2 \equiv 2 \pmod{3}$  while  $k^2 \equiv 0, 1 \pmod{3}$  which is a contradiction. ■

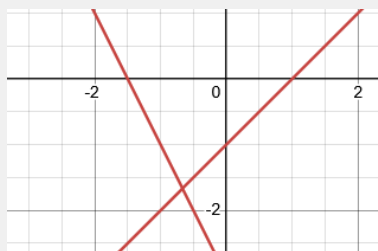
## 1.9 Degenerate Conics

### Problem 1

Dehomogenize  $f(x, y, z)$  by setting  $z = 1$ . Graph the curve

$$C(\mathbb{R}) = \{(x : y : z) \in \mathbb{P}^2 \mid f(x, y, 1) = 0\}.$$

in the real plane  $\mathbb{R}^2$ .



### Problem 2

Consider the two lines given by

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0,$$

and suppose

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0.$$

Show that the two lines intersect at a point where  $z \neq 0$ .

*Proof.* Suppose the two lines intersect at a point where  $z = 0$ . Thus

$$a_1x + b_1y = 0 \quad \text{and} \quad a_2x + b_2y = 0,$$

so

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus  $x = y = 0$  but  $(0, 0, 0) \notin \mathbb{P}^2$ . ■

### Problem 3

Dehomogenize the equation in the previous exercise by setting  $z = 1$ . Given an argument that, as lines in the complex plane  $\mathbb{C}^2$ , they have distinct slopes.

*Proof.* Dehomogenizing we find

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0.$$

Since they intersect at a point such that  $z \neq 0$  they are not parallel in  $\mathbb{C}^2$ . Thus either  $a_1 \neq a_2$  or  $b_1 \neq b_2$ . ■

#### Problem 4

Again consider the two lines

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0.$$

Suppose that

$$\det\left(A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right) = 0.$$

but that

$$\det\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \neq 0 \text{ or } \det\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \neq 0.$$

Show that the two lines still have one common point of intersection but that this point must have  $z = 0$ .

*Proof.* Suppose the two lines intersect at a point such that  $z \neq 0$ . Then dividing by  $z$  we find

$$\left(a_1 \frac{x}{z} + b_1 \frac{y}{z} + c_1\right)\left(a_2 \frac{x}{z} + b_2 \frac{y}{z} + c_2\right) = 0.$$

Let  $X = \frac{x}{z}, Y = \frac{y}{z}$ . We now have the following system

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \end{bmatrix}.$$

Since  $\det(A) = 0$  the system does not have a unique solution. It follows that the lines are linearly dependent. Now, wlog suppose

$$\det\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \neq 0.$$

Then  $a_1c_2 \neq a_2c_1$  and it follows that the lines are not equivalent. Therefore, the lines are parallel and meet at  $z = 0$ . ■

### Problem 5

Let

$$f(x, y, z) = (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z),$$

where at least one of  $a_1, b_1$ , or  $c_1$  is non-zero and at least one of the  $a_2, b_2$ , or  $c_2$  is non-zero. Show that the curve defined by  $f(x, y, z) = 0$  is a double line if and only if

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} = 0.$$

*Proof.* Suppose  $f(x, y, z) = 0$  is a double line. By Problem 4 we know

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} = 0.$$

Conversely, suppose

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = 0, \quad \det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} = 0.$$

Thus  $(a_2, b_2, c_2)$  is a scalar multiple of  $(a_1, b_1, c_1)$ . Therefore  $f(x, y, z) = 0$  is a double line. ■

### Problem 6

Consider

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0,$$

with

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \neq 0.$$

Find a projective change of coordinates from  $xyz$ -space to  $uvw$ -space so that the crossing lines become

$$uv = 0.$$

*Proof.* Consider

$$u = a_1x + b_1y + c_1z, \quad v = a_2x + b_2y + c_2z, \quad w = z.$$

Then,

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = uv.$$

■

### Problem 7

Consider the crossing lines  $(a_2x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0$ , with

$$\det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \neq 0.$$

Find a projective change of coordinates from  $xyz$ -space to  $uvw$ -space so that the crossing lines become

$$uv = 0.$$

*Proof.* Consider

$$u = a_1x + b_1y + c_1z, \quad v = a_2x + b_2y + c_2z, \quad w = z.$$

Then,

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = uv.$$

■

### Problem 8

Show that there is a projective change of coordinates from  $xyz$ -space to  $uvw$ -space so that the double lines  $(ax + by + cz)^2 = 0$  becomes the double line

$$u^2 = 0.$$

*Proof.* Consider

$$u = ax + by + cz, \quad v = ax + by + cz, \quad w = z.$$

Then,

$$(ax + by + cz)(ax + by + cz) = u^2.$$

■

### Problem 9

Argue that there are three distinct classes of conics in  $P^2$ .

*Proof.* There are three cases. If  $f(x, y, z)$  cannot be factored then it represents an ellipse, circle, parabola, or hyperbola, which are all equivalent in  $\mathbb{P}^2$ . If  $f(x, y, z)$  can be factored into two distinct linear factors then it represents two crossing lines. If  $f(x, y, z)$  can be factored into two identical linear factors then it represents a double line. ■

## 1.10 Tangents and Singular Points

### Problem 1

Explain why if both  $\frac{\partial f}{\partial x}(a, b) = 0$  and  $\frac{\partial f}{\partial y}(a, b) = 0$ , then the tangent line is not well-defined at  $(a, b)$ .

*Proof.* Suppose both  $\frac{\partial f}{\partial x}(a, b) = 0$  and  $\frac{\partial f}{\partial y}(a, b) = 0$ . The equation of the tangent line is

$$\frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) = 0.$$

Substituting we find

$$0(x - a) + 0(y - b) = 0,$$

which is true for all  $(x, y)$ . ■

### Problem 2

Show that the curve

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^2 + y^2 - 1 = 0\},$$

is smooth.

*Proof.* Let  $(a, b)$  be arbitrary points on  $C$ . Suppose

$$\frac{\partial f}{\partial x}(a, b) = 2a = 0,$$

and

$$\frac{\partial f}{\partial y}(a, b) = 2b = 0.$$

Then  $a = b = 0$  but  $x^2 + y^2 - 1 = -1 \neq 0$  which is a contradiction. Thus  $C$  is smooth. ■



### Problem 3

Show that the pair of crossing lines

$$C = \{(x, y) \in \mathbb{C}^2 \mid (x + y - 1)(x - y - 1) = 0\}$$

has exactly one singular point. Give a geometric interpretation of this singular point.

*Proof.* Expanding we find

$$(x + y - 1)(x - y - 1) = x^2 - 2x - y^2 + 1 = 0.$$

Suppose

$$\frac{\partial f}{\partial x}(a, b) = 2a - 2 = 0,$$

and

$$\frac{\partial f}{\partial y}(a, b) = -2b = 0.$$

Then  $a = 1$  and  $b = 0$ . Thus  $C$  is singular. This singular point is at the point of intersection. ■

### Problem 4

Show that every point on the double line

$$C = \{(x, y) \in \mathbb{C}^2 \mid (2x + 3y - 4)^2 = 0\},$$

is singular.

*Proof.* Expanding we find

$$(2x + 3y - 4)^2 = 4x^2 + 12xy - 16x + 9y^2 - 24y + 16 = 0.$$

Suppose

$$\frac{\partial f}{\partial x}(a, b) = 8a + 12b - 16 = 4(2a + 3b - 4) = 0,$$

and

$$\frac{\partial f}{\partial y}(a, b) = 12a + 18b - 24 = 6(2a + 3b - 4) = 0.$$

Thus all  $(a, b)$  vanish and therefore every point on the double line  $C$  is singular. ■

### Problem 5

Show that the curve

$$C = \{(x : y : z) \in \mathbb{P}^2 \mid x^2 + y^2 - z^2 = 0\}$$

is smooth.

*Proof.* Let  $(a : b : c) \in C$ . Suppose

$$\frac{\partial f}{\partial x}(a, b, c) = 2a = 0,$$

$$\frac{\partial f}{\partial y}(a, b, c) = 2b = 0,$$

and

$$\frac{\partial f}{\partial z}(a, b, c) = -2c = 0.$$

Then  $a = b = c = 0$  which is not possible in projective space. ■

### Problem 6

Show that the pair of crossing lines

$$C = \{(x : y : z) \in \mathbb{P}^2 \mid (x + y - z)(x - y - z) = 0\},$$

has exactly one point.

*Proof.* Let  $(a : b : c) \in C$ . Suppose

$$\frac{\partial f}{\partial x}(a, b, c) = 2a - 2c = 0,$$

$$\frac{\partial f}{\partial y}(a, b, c) = -2b = 0,$$

and

$$\frac{\partial f}{\partial z}(a, b, c) = -2a + 2c = 0.$$

Then  $b = 0$  and  $a = c$ . Thus  $(a : b : c) = (a : 0 : a) = (1 : 0 : 1)$ . ■

### Problem 7

Show that every point on the double line

$$C = \{(x : y : z) \in \mathbb{P}^2 \mid (2x + 3y - 4z)^2 = 0\},$$

is singular.

*Proof.* Let  $(a : b : c) \in C$ . Suppose

$$f(x, y, z) = (2x + 3y - 4z)^2.$$

Then

$$\frac{\partial f}{\partial x}(a, b, c) = 2(2a + 3b - 4c) \cdot 2 = 4(2a + 3b - 4c),$$

$$\frac{\partial f}{\partial y}(a, b, c) = 2(2a + 3b - 4c) \cdot 3 = 6(2a + 3b - 4c),$$

$$\frac{\partial f}{\partial z}(a, b, c) = 2(2a + 3b - 4c) \cdot (-4) = -8(2a + 3b - 4c).$$

But  $(a : b : c) \in C$  thus  $2a + 3b - 4c = 0$ . So all partial derivatives vanish on  $C$ . ■

### Problem 8

For

$$f(x, y, z) = x^2 + 3xy + 5xz + y^2 - 7yz + 8z^2,$$

show that

$$2f = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}.$$

*Proof.* For all  $(a, b, c)$  we have

$$2f(a, b, c) = 2(a^2 + 3ab + 5ac + b^2 - 7bc + 8c^2) = 2a^2 + 6ab + 10ac + 2b^2 - 14bc + 16c^2.$$

Then

$$\frac{\partial f}{\partial x}(a, b, c) = 2a + 3b + 5c,$$

$$\frac{\partial f}{\partial y}(a, b, c) = 3a + 2b - 7c,$$

$$\frac{\partial f}{\partial z}(a, b, c) = 5a - 7b + 16c.$$

Then

$$\begin{aligned} a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} + c \frac{\partial f}{\partial z} &= a(2a + 3b + 5c) + b(3a + 2b - 7c) + c(5a - 7b + 16c), \\ &= 2a^2 + 3ab + 5ac + 3ab + 2b^2 - 7bc + 5ac - 7bc + 16c^2 = 2a^2 + 6ab + 10ac + 2b^2 - 14bc + 16c^2. \end{aligned}$$
■

### Problem 9

For

$$f(x, y, z) = ax^2 + bxy + cxz + dy^2 + eyz + hz^2$$

show that

$$2f = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}.$$

*Proof.* Notice

$$\frac{\partial f}{\partial x} = 2ax + by + cz, \quad \frac{\partial f}{\partial y} = bx + 2dy + ez, \quad \frac{\partial f}{\partial z} = cx + ey + 2hz.$$

Then

$$\begin{aligned} x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} &= x(2ax + by + cz) + y(bx + 2dy + ez) + z(cx + ey + 2hz) \\ &= 2ax^2 + bxy + cxz + bxy + 2dy^2 + eyz + cxz + eyz + 2hz^2 \\ &= 2ax^2 + 2bxy + 2cxz + 2dy^2 + 2eyz + 2hz^2 \\ &= 2f(x, y, z), \end{aligned}$$

### Problem 10

Let  $f(x, y, z)$  be a homogeneous polynomial of degree  $n$ . Show that

$$nf = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}.$$

*Proof.* Let

$$f(x, y, z) = \sum_{i+j+k=n} c_{ijk} x^i y^j z^k$$

Then

$$\frac{\partial f}{\partial x} = \sum_{i+j+k=n} i c_{ijk} x^{i-1} y^j z^k, \quad \frac{\partial f}{\partial y} = \sum_{i+j+k=n} j c_{ijk} x^i y^{j-1} z^k, \quad \frac{\partial f}{\partial z} = \sum_{i+j+k=n} k c_{ijk} x^i y^j z^{k-1}.$$

Then

$$\begin{aligned} x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} &= \sum_{i+j+k=n} (i + j + k) c_{ijk} x^i y^j z^k \\ &= \sum_{i+j+k=n} n c_{ijk} x^i y^j z^k \\ &= nf(x, y, z). \end{aligned}$$

Problem 11

Use Exercise 1.10.10 to show that if  $p = (a : b : c)$  satisfies

$$\frac{\partial f}{\partial x}(a, b, c) = \frac{\partial f}{\partial y}(a, b, c) = \frac{\partial f}{\partial z}(a, b, c) = 0,$$

then  $p \in V(f)$ .

Problem 12

Problem 13

Problem 14

Problem 15

Problem 16

Problem 17

Problem 18

Problem 19

Problem 20