A Friendly Introduction to Number Theory by Silverman

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January 13, 2026

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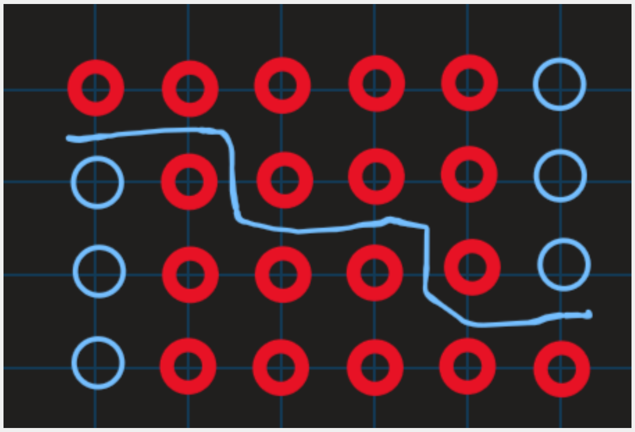
1 What is Number Theory

Problem 2

Try adding up the first few odd numbers and see if the numbers you get satisfy some sort of pattern. Once you find the pattern, express it as a formula. Give a geometric verification that your formula is correct.



Proof. We add up the first few odd numbers.

1. $1 = 1 = 1^2$
2. $1 + 3 = 4 = 2^2$
3. $1 + 3 + 5 = 9 = 3^2$
4. $1 + 3 + 5 + 7 = 16 = 4^2$

Please view the image above. The number of dots in the lower red triangle is what we are trying to discover a formula for. We first add k dots to the left of the triangle. Then we double the triangle and combine it with the first to make a rectangle with width $2k$ and height $k + 1$. Therefore the total number of dots is $2k^2 + 2k$. Now we doubled the triangle so we must remove dots to discover the number of dots in the original lower left triangle. We first halve the number of dots, $\frac{2k^2+2k}{2} = k^2 + k$. We also added k redundant blue dots, thus our total number of dots is $k^2 + k - k = k^2$. ∎

Problem 3

The consecutive odd numbers 3, 5, and 7 are all primes. Are there infinitely many such "prime triplets"? That is, are there infinitely many prime numbers p such that $p+2$ and $p+4$ are also primes?

Proof. Consider the sequence

$$p, p+1, p+2, p+3, p+4$$

There are three cases for the remainders of each term in the sequence when divided by 3. They are as follows

$$0, 1, 2, 0, 1$$

In which case $3 \mid p$.

$$1, 2, 0, 1, 2$$

In which case $3 \mid p+2$.

$$2, 0, 1, 2, 0$$

In which case $3 \mid p+4$. Thus there is only one set of prime triplets. ∎

Problem 4

It is generally believed that infinitely many primes have the form $N^2 + 1$, although no one knows for sure.

1. Do you think there are infinitely many primes of the form $N^2 - 1$?
2. How about of the form $N^2 - 3$? How about $N^2 - 4$.
3. Which values of a do you think give infinitely many primes of the form $N^2 - a$.

Solution (a): No, since $N^2 - 1 = (N+1)(N-1)$, which is composite for all $N > 2$.

Solution (b): For $N^2 - 3$ I think it does, and $N^2 - 4 = (N+2)(N-2)$, so there are finitely many primes.

Solution (c): Values of a such that $N^2 - a$ cannot be factored as a difference of squares $(N-b)(N+b)$ for some integer b.

2 Pythagorean Triples

Problem 1

1. We showed that in any primitive Pythagorean triple (a, b, c), either a or b is even. Use the same sort of argument to show that either a or b must be a multiple of 3.
2. By examining the above list of primitive triples, make a guess about when $a, b,$ or c is a multiple of 5. Try to show that your guess is correct.

Problem 2

Problem 6

Problem 7

Problem 9