Linear Algebra by Sterling K. Berberian

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1 Vector Spaces

1.1 Motivation (Vectors in 3-space)

Problem 1

Given the point $P = (2, -1, 3)$ and the vector $u = [-3, 4, 5]$, find the point Q such that $\overrightarrow{PQ} = u$.

Proof. We require $\vec{u} = [-3, 4, 5] = (q_1 - 2, q_2 - (-1), q_3 - 3)$. Solving componentwise, we see $(q_1, q_2, q_3) = (-1, 3, 8)$. Thus, $Q = (-1, 3, 8)$. ∎

Problem 2

Given the points $P(2, -1, 3)$, $Q(3, 4, 1)$, $R(4, -3, 4)$, $S(5, 2, 2)$. True or false (explain:) $PQSR$ is a parallelogram.

Proof. For $PQSR$ to be a parallelogram, we require $\vec{PQ} = \vec{SR}$.

$$\vec{PQ} = [3 - 2,\ 4 - (-1),\ 1 - 3] = [1, 5, -2]$$

$$\vec{SR} = [4 - 5,\ -3 - 2,\ 4 - 2] = [-1, -5, 2]$$

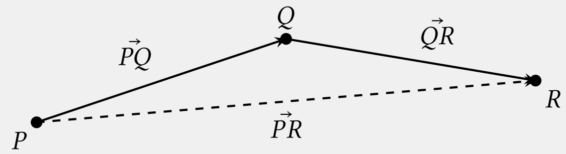
Since

$$[1, 5, -2] \neq [-1, -5, 2],$$

it follows that $PQSR$ is not a parallelogram. ∎

Problem 4

Show that for every triple points P, Q, R, $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$. [Method 1: Calculate components. Method 2: Draw a picture.]



1.2 $\mathbb{R}^n$ and $\mathbb{C}^n$

Problem 2

Consider the vectors $u = (2, 1)$, $v = (-5, 3)$, $w = (3, 4)$ in $\mathbb{R}^2$. Do there exists real numbers a, b such that $au + bv = w$? What if $v = (6, 3)$.

Proof. We require $au+bv = w \iff a(2, 1)+b(-5, 3) = (3, 4) \iff (2a, a)+(-5b, 3b) = (3, 4) \iff (2a-5b, a+3b) = (3, 4)$. Thus

$$\begin{cases} 2a - 5b = 3 \\ a + 3b = 4 \end{cases}$$

which has the solution $a = \frac{29}{11}$, $b = \frac{5}{11}$. If $v = (6, 3)$, we require

$$a(2, 1) + b(6, 3) = (3, 4) \iff (2a + 6b, a + 3b) = (3, 4).$$

Thus

$$\begin{cases} 2a + 6b = 3 \\ a + 3b = 4 \end{cases}$$

which has no solution. To see this has no solution, note

$$a + 3b = 4 \implies 6b = 8 - 2a.$$

Plugging into the first equation gives

$$2a + (8 - 2a) = 3 \implies 8 = 3,$$

which is a contradiction. ∎

Problem 3

Give a detailed proof of (8) of Theorem 1.2.3 (just checking!). [Write out all steps of the proof and give a reason for each step.]

Proof. Let $x = [x_1, x_2, \dots, x_n] \in F^n$ be a vector and let $c, d \in F$ where F is a field. Then

$$(cd)x = [(cd)x_1, (cd)x_2, \dots, (cd)x_n].$$

Since F is a field, scalar multiplication in F is associative thus

$$(cd)x_i = c(dx_i) \quad \text{for each } i = 1, 2, \dots, n.$$

Therefore,

$$(cd)x = [c(dx_1), c(dx_2), \dots, c(dx_n)] = c(dx),$$

∎

Problem 4

In Definition 1.2.1, $n = 1$ is not ruled out. What do the elements of $\mathbb{R}^1$ look like? Describe sums and scalar multiples in $\mathbb{R}^1$.

Proof. The elements of $\mathbb{R}^1$ look like $[x_1]$ where $x_1 \in \mathbb{R}$. Sums and scalar multiples behave identically to the operations in the field $\mathbb{R}$. ∎

1.3 Vectors Spaces: The Axioms some Examples

Problem 1

Let V be a vector space over a field F. Let T be a nonempty set and let $W = \mathscr{F}(T, V)$ be the set of all functions $x : T \longrightarrow V$. Show that W can be made into a vector space over F in a natural way. [Hint: Use the definitions in Example 1.3.4 as a guide.]

Proof. For $x, y \in W$, $x = y$ means that $x(t) = y(t)$ for all $t \in T$. If $x, y \in W$ and $c \in F$, define functions $x + y$ and cx by the formulas $(x + y)(t) = x(t) + y(t)$ and $(cx)(t) = cx(t)$ for all $t \in T$. Let θ be the function defined by $\theta(t) = 0$ for all $t \in T$, and for $x \in W$, let $-x$ be the function defined by $(-x)(t) = -x(t)$ for all $t \in T$. ∎

Problem 2

The definition of a vector space (1.3.1) can be formulated as follows. A vector space over F is a nonempty set V together with a pair of mappings $\sigma : V \longrightarrow V$ and $\mu : F \times V \longrightarrow V$. ($\sigma$ suggests 'sum' and μ suggests 'multiple') having the following properties $\sigma(x, y) = \sigma(y, x)$ for all $x, y \in V$; $\sigma(\sigma(x, y), z) = \sigma(x, \sigma(y, z))$ for all $x, y, z \in V$ etc. The exercise: Write out the 'etc.' in detail.

Proof. We list the 9 axioms below:

1. $x, y \in V \implies \sigma(x, y) \in V$
2. $x \in V, c \in F \implies \mu(c, x) \in V$
3. $x, y \in V \implies \sigma(x, y) = \sigma(y, x)$
4. $x, y, z \in V \implies \sigma(x, \sigma(y, z)) = \sigma(\sigma(x, y), z)$
5. $\exists 0 \in V$ such that $\sigma(0, v) = v = \sigma(v, 0)$
6. $x \in V \implies -x \in V$ such that $\sigma(x, -x) = 0 = \sigma(-x, x)$
7. $x, y \in V$ and $c \in F \implies \mu(c, \sigma(x, y)) = \sigma(\mu(c, x), \mu(c, y))$
8. $\exists 1 \in F$ such that $x \in V \implies \mu(1, x) = x$
9. $a, b \in F$ and $x \in V \implies \mu(a, \mu(b, x)) = \mu(ab, x)$

∎

Problem 3

Let V be a complex vector space (1.3.2). Show that V is also a real vector space (sums as usual, scalar multiplication restricted to real scalars). These two ways of looking at V may be indicated by writing V_c and V_r.

Proof. Clearly V_r is closed under scalar multiplication and vector addiction and is a vector space. ∎

Problem 4

Every real vector space V can be 'embedded' in a complex vector space W in the following way: let $W = V \times V$ be the real vector space constructed as in example 1.3.10 and define multiplication by complex scalars by the formula $(a+bi)(x,y) = (ax - by, bx + ay)$ for $a, b \in \mathbb{R}$ and $(x,y) \in W$. [In particular, $i(x,y) = (-y, x)$. Think of (x,y) as '$x + iy$'] Show that W satisfies the axioms for a complete vector space (W is called the *complexification* of V.)