Intrudction to Lattices and Order by Davey

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1 Ordered Sets

Problem 5

Prove that the ordered set $\sum^{**}$ of all binary strings is a **tree** (that is, an ordered set P with $\perp$ such that $\downarrow x$ is a chain for each $x \in P$). For each $u \in \sum^{**}$ describe the set of elements covering u.

Proof. Consider the empty string e. Now, $e \in \sum^{**}$, and for all $t \in \sum^{**}$, $e \leq t$. Thus $e = \perp$.

Let x be an arbitrary element in $\sum^{**}$. Note that $\downarrow x \neq \emptyset$ since $e \leq x$. Now, let s_1, s_2 be arbitrary elements in $\downarrow x$. Clearly, either $s_1 \leq s_2$ or $s_2 \leq s_1$, since both are prefixes of the same string x. Thus $\downarrow x$ is a chain as required.

Let s be an arbitrary element in $\sum^{**}$. Now, if s is infinite then there is no element covering s. Therefore, suppose s is finite. There are two elements $s', s'' \in \sum^{**}$ covering s. These are $s' = s0$ and $s'' = s1$, where concatenation appends the symbol to the end of the string. ∎

Problem 7

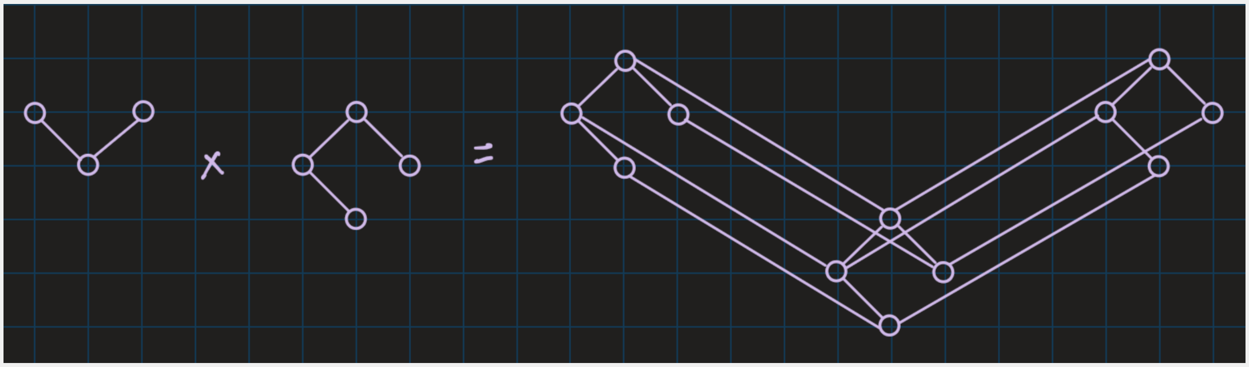
Let P and Q be ordered sets. Prove that $(a_1, b_1) \prec (a_2, b_2)$ in $P \times Q$ if and only if $(a_1 = a_2 \,\&\, b_1 \prec b_2)$ or $(a_1 \prec a_2 \,\&\, b_1 = b_2)$.

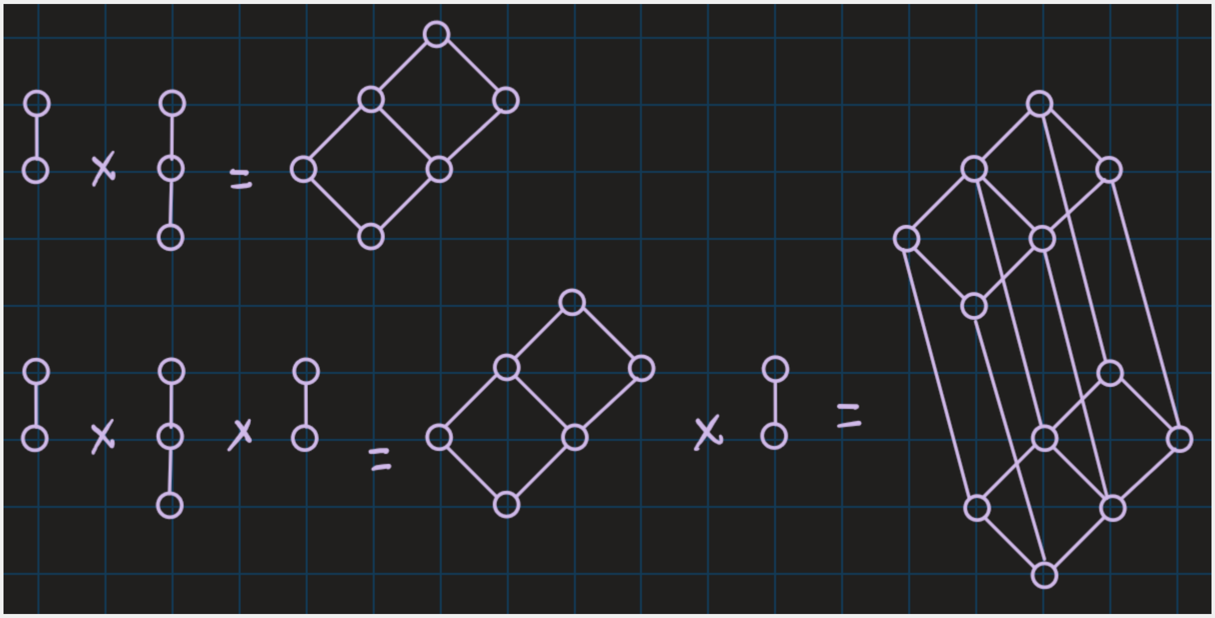
Proof. $(\longrightarrow)$ Suppose $(a_1, b_1) \prec (a_2, b_2)$ in $P \times Q$. Furthermore, suppose $a_1 < a_2$ and $b_1 < b_2$. Considering the pairs (a_2, b_1) and (a_1, b_2), we find the following two relations: $(a_1, b_1) < (a_2, b_1) < (a_2, b_2)$ and $(a_1, b_1) < (a_1, b_2) < (a_2, b_2)$. Both of which contradict the $\prec$ relation. Therefore, either $a_1 = a_2$ or $b_1 = b_2$. Suppose $a_1 = a_2$. Then $b_1 < b_2$. If $b_1 \not\prec b_2$ then there exists b' such that $b_1 < b' < b_2$. But then $(a_1, b_1) < (a_1, b') < (a_2, b_2)$. Thus $b_1 \prec b_2$. A similar argument shows $a_1 \prec a_2$ if $b_1 = b_2$.

$(\longleftarrow)$ Suppose $(a_1 = a_2 \,\&\, b_1 \prec b_2)$ or $(a_1 \prec a_2 \,\&\, b_1 = b_2)$. Now, suppose $(a_1 = a_2 \,\&\, b_1 \prec b_2)$. Futhermore, suppose there exists (a', b') such that $(a_1, b_1) < (a', b') < (a_2, b_2)$. Then $a_1 \leq a' \leq a_2$ and $b_1 \leq b' \leq b_2$. But $a_1 = a_2$ and $b_1 \prec b_2$, thus $a' = a_1$ and $b' = b_1$. It follows that $(a_1, b_1) = (a', b')$. Thus $(a_1, b_1) \prec (a_2, b_2)$. A similar argument shows $(a_1, b_1) \prec (a_2, b_2)$ if $b_1 = b_2$. ∎

Problem 8

Draw the diagrams of the products shown in Figure 1.12.





Problem 13

Draw and label a diagram for $\mathcal{O}(P)$ for each of the ordered sets P of Figure 1.13.

Problem 14

Let P be a finite ordered set.
(i) Show that $Q = \downarrow MaxQ$ for all $Q \in \mathcal{O}(P)$.
(ii) Establish a one-to-one correspondence between the elements of $\mathcal{O}(P)$ and antichains in P.
(iii) Hence show that, for all $x \in P$, $|\mathcal{O}(P)| = |\mathcal{O}(P \setminus \{x\}) + |\mathcal{O}(P \setminus (\downarrow x \cup \uparrow x))|$.

Problem 17

Problem 24

Problem 25

Problem 26

Problem 27