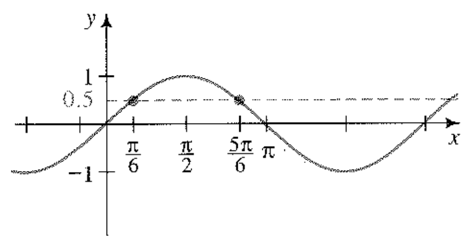
8 Trigonometric Equations and Applications

Equations and Applications of Sine Waves

8-1 Simple Trigonometric Equations

Objective *To solve simple trigonometric equations and to apply them.*

The sine graph at the right illustrates that there are many solutions to the trigonometric equation $\sin x = 0.5$. We know that $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ are two *particular* solutions. Since the period of $\sin x$ is 2π, we can add integral multiples of 2π to get the other solutions.



$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi,$$

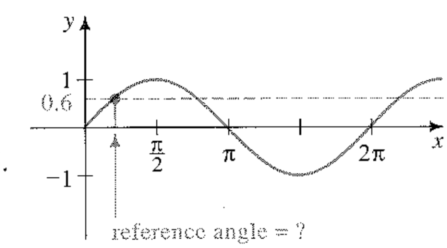
where n is any integer, are the *general solutions* of $\sin x = 0.5$.

When solving an equation such as $\sin x = 0.6$, we can use a calculator or a table of values.

Example 1 Find the values of x between 0 and 2π for which $\sin x = 0.6$.

Solution **Step 1** Set the calculator in radian mode and use the inverse sine key. Thus,

$x = \text{Sin}^{-1}\ 0.6 \approx 0.6435.$



Step 2 0.6435 is the reference angle for other solutions. Since $\sin x$ is positive, a Quadrant II angle also satisfies the equation.

$x = \pi - 0.6435 \approx 2.4981$. Therefore, $x \approx 0.6435, 2.4981$.

Note that if you had been asked to find *all* values of x for which $\sin x = 0.6$, then your answer would be

$x \approx 0.6435 + 2\pi n$ and $x \approx 2.4981 + 2\pi n$, for any integer n.

To solve an equation involving a single trigonometric function, we first transform the equation so that the function is alone on one side of the equals sign. Then we follow the same procedure used in Example 1.

The hour of sunset varies with the time of year; tides advance and recede in a pattern that is determined by the moon. Studied since ancient times, these natural cycles can be modeled by sine and cosine curves.

Example 2 To the nearest tenth of a degree, solve $3 \cos \theta + 9 = 7$ for $0° \le \theta < 360°$.

Solution

$$3 \cos \theta + 9 = 7$$
$$3 \cos \theta = -2$$
$$\cos \theta = -\frac{2}{3}$$

Since $\cos \theta < 0$, the solutions are between 90° and 180°, and 180° and 270°. (See graph.) One way to find these solutions is to first find the reference angle by ignoring the negative sign.

The reference angle is:

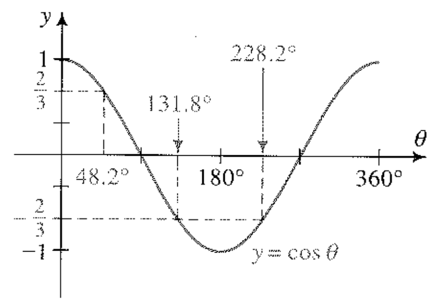
$\mathrm{Cos}^{-1}\left(\frac{2}{3}\right) \approx 48.2°$

The first solution is:

$\theta \approx 180° - 48.2° = 131.8°$

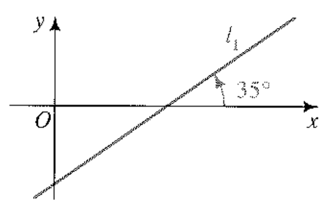
The second solution is:

$\theta \approx 180° + 48.2° = 228.2°$

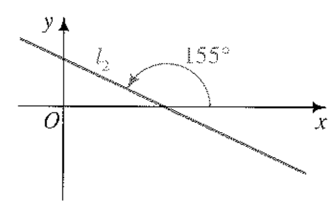


Inclination and Slope

The **inclination** of a line is the angle α, where $0° \le \alpha < 180°$, that is measured from the positive x-axis to the line. The line at the left below has inclination 35°. The line at the right below has inclination 155°. The theorem that follows states that the slope of a nonvertical line is the tangent of its inclination.



slope of $l_1 = \tan 35° \approx 0.7002$



slope of $l_2 = \tan 155° \approx -0.4663$

Theorem

For any line with slope m and inclination α,

$$m = \tan \alpha \qquad \text{if } \alpha \ne 90°.$$

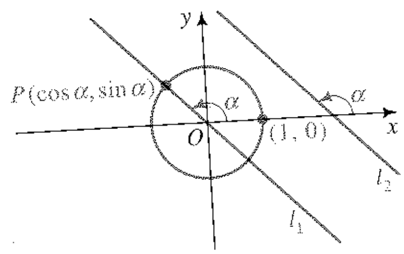
If $\alpha = 90°$, then the line has no slope. (The line is vertical.)

Proof: Refer to the diagram on the next page. Suppose that l_1 passes through the origin and has inclination α. Let P be the point where l_1 intersects the unit circle.

Then $P = (\cos \alpha, \sin \alpha)$. The slope of l_1 is

$$m = \frac{\sin \alpha - 0}{\cos \alpha - 0} = \tan \alpha.$$

If the line l_2 has inclination α and does not contain the origin, it is parallel to l_1. Therefore, the slope of l_2 also equals $\tan \alpha$.



Example 3 To the nearest degree, find the inclination of the line $2x + 5y = 15$.

Solution Rewrite the equation as $y = -\frac{2}{5}x + 3$.

$$\text{slope} = -\frac{2}{5} = \tan \alpha$$

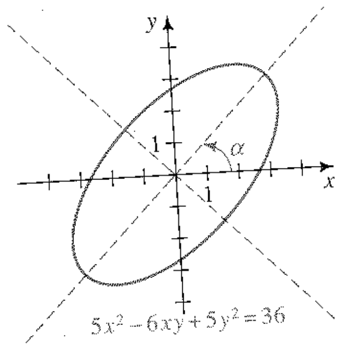
$$\alpha = \text{Tan}^{-1}\left(-\frac{2}{5}\right) \approx -21.8° \text{ (The reference angle is } 21.8°.)$$

Since $\tan \alpha$ is negative and α is a positive angle, $90° < \alpha < 180°$. The inclination is $180° - 21.8° \approx 158.2°$.

In Section 6-7, you learned to graph conic sections whose equations have no xy-term. That is, equations of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where $B = 0$. The graph at the right shows a conic section with center at the origin whose equation has an xy-term ($B \neq 0$). Conics like this have one of their two axes inclined at an angle α to the x-axis. To find this *direction angle* α, use the formula below.



$$\alpha = \frac{\pi}{4} \qquad \text{if } A = C$$

$$\tan 2\alpha = \frac{B}{A - C} \qquad \text{if } A \neq C, \text{ and } 0 < 2\alpha < \pi$$

The direction angle α is useful in finding the equations of the axes of these conic sections. This is shown in Method 1 of Example 4 on the next page.

Example 4 Identify the graph of the equation, find the direction angle α, and sketch the curve $x^2 - 2xy + 3y^2 = 1$.

Solution $A \neq C$ and $B^2 - 4AC = (-2)^2 - 4(1)(3) = -8 < 0$, so the graph is an ellipse.

$$\tan 2\alpha = \frac{B}{A - C} = \frac{-2}{1 - 3} = 1 \qquad \text{Thus, } 2\alpha = \frac{\pi}{4} \text{ and } \alpha = \frac{\pi}{8}.$$

(Solution continues on the next page.)

To sketch the conic, we can use a graphing calculator or a computer, as shown in Method 1, or we can use substitution and our knowledge of the particular conic, as shown in Method 2.

Method 1

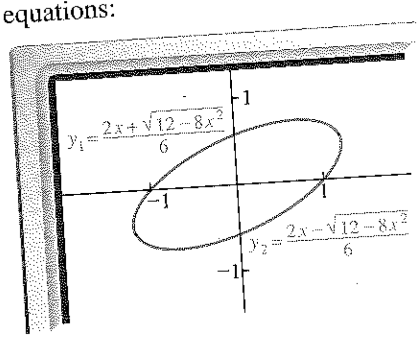
With a graphing calculator, it is often necessary to rewrite the equation so that it is quadratic in y. Then the equation can be solved for y by using the quadratic formula.

$$x^2 - 2xy + 3y^2 = 1$$
$$3y^2 + (-2x)y + (x^2 - 1) = 0$$
$$y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 3(x^2 - 1)}}{2 \cdot 3}$$
$$y = \frac{2x \pm \sqrt{12 - 8x^2}}{6}$$

Rewrite the equation above as two equations:

$$y_1 = \frac{2x + \sqrt{12 - 8x^2}}{6} \quad \text{and}$$

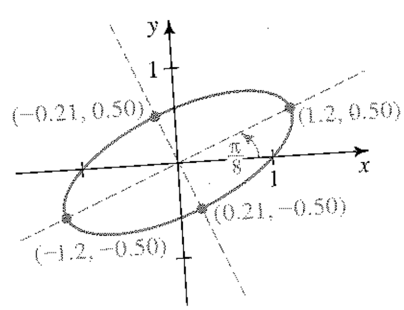
$$y_2 = \frac{2x - \sqrt{12 - 8x^2}}{6}.$$



Entering these two functions into a graphing calculator or computer gives the graph shown at the right.

Method 2

Use the direction angle found on page 297 to find equations of the axes of the ellipse. Since $\alpha = \frac{\pi}{8}$, the slope of one axis is $\tan \frac{\pi}{8} \approx 0.41$, and the slope of the other axis is $-\frac{1}{0.41} \approx -2.4$. Thus, equations of the two axes are $y = 0.41x$ and $y = -2.4x$. By substituting for y into $x^2 - 2xy + 3y^2 = 1$, we find that these axes intersect the ellipse at points $(1.2, 0.50)$, $(-1.2, -0.50)$, $(0.21, -0.50)$, and $(-0.21, 0.50)$. Also, by substituting $y = 0$ and $x = 0$, we find that the graph intersects the x-axis at $(\pm 1, 0)$ and intersects the y-axis at $\left(0, \pm\frac{\sqrt{3}}{3}\right)$. We can use these points to sketch the graph of the ellipse, as shown at the right.



CLASS EXERCISES

Solve for $0° \leq \theta < 360°$ without using tables or a calculator.

1. $\cos \theta = \frac{1}{2}$ **2.** $\sin \theta = -4$ **3.** $\csc \theta = 2$ **4.** $\cot \theta = -1$

Solve for $0 \leq x < 2\pi$ without using tables or a calculator.

5. $\cos x = -\frac{\sqrt{3}}{2}$ **6.** $\cot x = 1$ **7.** $\tan x = -\sqrt{3}$ **8.** $\sec x = \frac{1}{2}$

Solve for θ in degrees, giving *all* solutions.

9. $\cos \theta = -1$ **10.** $\sin \theta = -\frac{\sqrt{2}}{2}$ **11.** $\tan \theta = 1$

12. The inclination of a line is 140°. Find its slope.

13. The slope of a line is $\frac{3}{5}$. Find its inclination.

14. Find to the nearest degree a direction angle α for the conic whose equation is $5x^2 + 2xy - y^2 = 6$.

15. ***Discussion*** Comment on why you think it may be necessary to enter two equations when graphing a conic using a graphing calculator, as mentioned in Method 1 of Example 4.

WRITTEN EXERCISES

A **Solve for $0° \leq \theta < 360°$. Give answers to the nearest tenth of a degree.**

1. $\sin \theta = -0.7$ **2.** $\cos \theta = 0.42$ **3.** $\tan \theta = 1.2$

4. $\cot \theta = -0.3$ **5.** $\sec \theta = -5$ **6.** $\csc \theta = 14$

7. $3 \cos \theta = 1$ **8.** $4 \sin \theta = 3$ **9.** $5 \sec \theta + 6 = 0$

10. $2 \tan \theta + 1 = 0$ **11.** $6 \csc \theta - 9 = 0$ **12.** $4 \cot \theta - 5 = 0$

Solve for $0 \leq x < 2\pi$. Give answers to the nearest hundredth of a radian.

13. $\tan x = -1.5$ **14.** $\sec x = 2.5$ **15.** $\csc x = -1.4$

16. $\cos x = -0.8$ **17.** $\cot x = 6$ **18.** $3 \sin x + 2 = 4$

19. $8 = 9 \cos x + 2$ **20.** $\frac{5 \csc x}{3} = \frac{9}{4}$ **21.** $\frac{3 \cot x}{4} + 1 = 0$

Find the slope and equation of each line described. Sketch the line.

22. inclination = 45°
y-intercept = 4

23. inclination = 120°
contains (2, 3)

24. inclination = 158°
contains (−3, 5)

Find the inclination of each line. Give your answers to the nearest degree.

25. The line $3x + 5y = 8$

26. The line $x - 4y = 7$

27. The line joining $(-1, 2)$ and $(4, 1)$

28. The line joining $(-1, 1)$ and $(4, 2)$

29. A line perpendicular to $4x + 3y = 12$

30. A line parallel to $2x + 3y = -6$

Solve for $0 \le x < 2\pi$ without using tables or a calculator.

B **31.** $|\csc x| = 1$

32. $|\sec x| = \sqrt{2}$

33. $\log_2 (\sin x) = 0$

34. $\log_2 (\cos x) = -1$

35. $\log_3 (\tan x) = \frac{1}{2}$

36. $\log_{\sqrt{3}} (\cot x) = 1$

Identify the graph of each equation, find its direction angle α, and sketch the curve. You may wish to use a graphing calculator or computer.

37. $x^2 + xy + y^2 = 1$

38. $x^2 - 2xy - y^2 = 4$

39. $x^2 - xy + y^2 = 1$

40. $x^2 - xy + 2y^2 = 2$

41. $x^2 - 3xy + y^2 = -6$

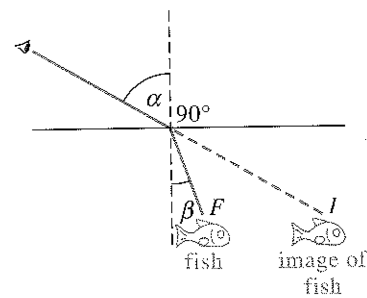
42. $y = x - \frac{1}{x}$

43. $x^2 - 2xy + y^2 - 4\sqrt{2}x - 4\sqrt{2}y = 0$

44. $x^2 - 2xy + y^2 - \sqrt{2}x - \sqrt{2}y = 0$

You may find it helpful to have a computer or a graphing calculator to complete part (b) of Exercise 45. Use degree measure for α and β.

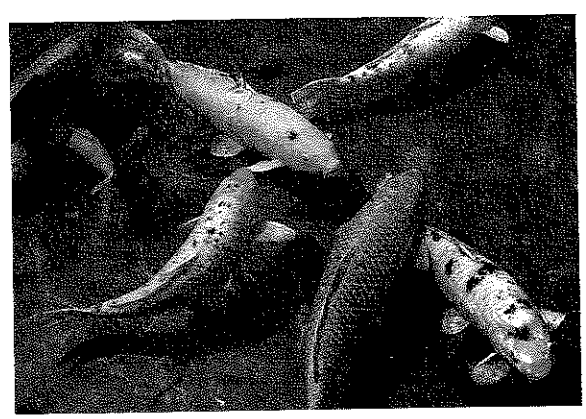
45. Optics The rays of light that are reflected from a fish in a pond to your eyes travel at different speeds through water and through air. Consequently, these rays bend at the water's surface. Your mind, which expects light to travel in a straight line, perceives the fish to be at location I when the fish is actually at location F. According to Snell's law,



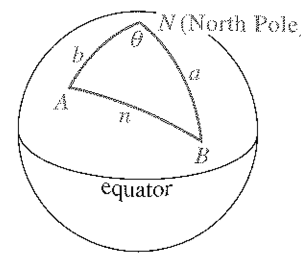
$$\frac{\text{speed of light in air}}{\text{speed of light in water}} = \frac{\sin \alpha}{\sin \beta}.$$

a. Using the fact that the speed of light in air is 3.00×10^8 m/s and the speed of light in water is 2.25×10^8 m/s, show that $\beta = \text{Sin}^{-1}(0.75 \sin \alpha)$. Give an appropriate domain and range for this function.

b. Use the graphs of $y = \beta$ and $y = 0.5\alpha$ to decide if there is a positive angle α for which $\beta = \frac{1}{2}\alpha$.



46. **Navigation** A *great circle* on Earth's surface is a circle whose center is at Earth's center. The shortest distance measured in degrees between points A and B on Earth's surface is along the arc of a great circle. To find this shortest distance, we can work with the spherical triangle ABN shown in the diagram, whose sides are arcs of great circles. The important measurements in this triangle are:



great circle arc $AB = n°$
great circle arc $AN = b° = 90° -$ latitude of A
great circle arc $BN = a° = 90° -$ latitude of B
$\theta =$ difference in longitudes of A and B

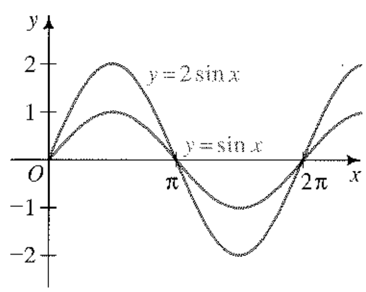
Use the formula $\cos n = \cos a \cos b + \sin a \sin b \cos \theta$ and the following data to find the great circle distance measured in degrees between Rome and Boston. Give your answer to the nearest tenth of a degree.

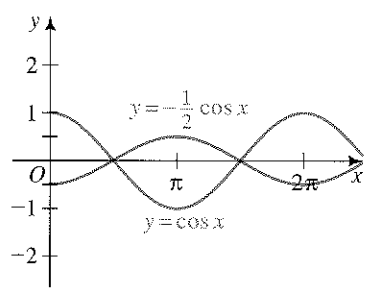
	Latitude	Longitude
Rome	41.53°N	12.30°E
Boston	42.20°N	71.05°W

8-2 Sine and Cosine Curves

Objective *To find equations of different sine and cosine curves and to apply these equations.*

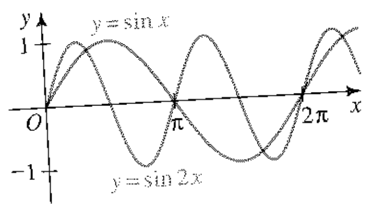
Recall from Section 4-4 that the graph of $y = cf(x)$ can be obtained by vertically stretching or shrinking the graph of $y = f(x)$. This is illustrated by the sine and cosine curves below.

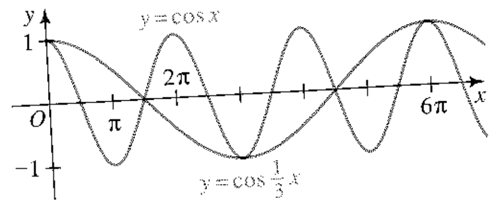




Notice that $y = -\frac{1}{2}\cos x$ is the graph of $y = \frac{1}{2}\cos x$ that has been reflected in the x-axis. You can see that the graph of $y = 2\sin x$ has amplitude 2 and that the graph of $y = -\frac{1}{2}\cos x$ has amplitude $\frac{1}{2}$.

Also in Section 4-4, you learned that the graph of $y = f(cx)$ is obtained by horizontally stretching or shrinking the graph of $y = f(x)$. This is illustrated below.





The period of $y = \sin x$ and $y = \cos x$ is 2π. (In this chapter, when we refer to the period p of a function, we mean the *fundamental period* of the function.) You can see from above that the period of $y = \sin 2x$ is $\frac{2\pi}{2}$, or π, and that the period of $y = \cos \frac{1}{3}x$ is $\frac{2\pi}{\frac{1}{3}}$, or 6π.

In general, we can determine useful information about the graphs of $y = A \sin Bx$ and $y = A \cos Bx$ by analyzing the factors A and B.

Period and Amplitude of Sine and Cosine Curves

For functions $y = A \sin Bx$ and $y = A \cos Bx$ ($A \neq 0$ and $B > 0$):

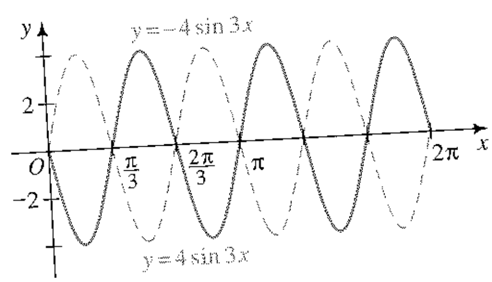
$$\text{amplitude} = |A| \qquad \text{period} = \frac{2\pi}{B}$$

Example 1 Give the amplitude and period of the function $y = -4 \sin 3x$. Then sketch at least one cycle of its graph.

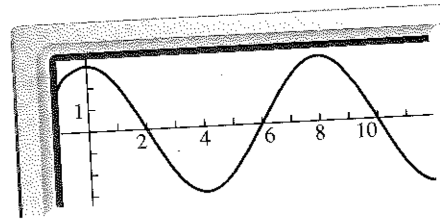
Solution amplitude $= |-4| = 4$

period $= \frac{2\pi}{B} = \frac{2\pi}{3}$

Reflect the graph of $y = 4 \sin 3x$ in the x-axis to get the graph of $y = -4 \sin 3x$.



Example 2 Give the amplitude, period, and an equation of the curve shown at the right.



Solution Use the cosine curve, $y = A \cos Bx$, with amplitude 3 and period 8. Thus, $8 = \frac{2\pi}{B}$, which gives $B = \frac{\pi}{4}$. The equation is

$$y = A \cos Bx = 3 \cos \frac{\pi}{4}x.$$

Example 3 shows two methods of solving a trigonometric equation. Method 1 is an algebraic solution, as developed in Section 8-1. Method 2 is a graphical solution in which a graphing calculator or a computer can be used.

Example 3 Solve the equation $6 \sin 2x = 5$ for $0 \le x < 2\pi$. Give answers to the nearest hundredth of a radian.

Solution **Method 1** Transform the equation as follows:

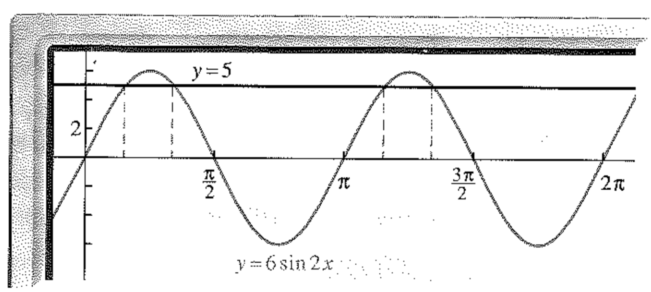
$$6 \sin 2x = 5$$

$$\sin 2x = \frac{5}{6}$$

$$2x \approx 0.99, 2.16, 7.27, 8.44 \longleftarrow \text{If } 0 \le x < 2\pi, \text{ then}$$

$$x \approx 0.49, 1.08, 3.63, 4.22 \qquad 0 \le 2x < 4\pi.$$

Method 2 Use a graphing calculator or computer to sketch the graphs of $y = 5$ and $y = 6 \sin 2x$ on the same set of axes. Then find the x-coordinates of the intersection points over the interval $0 \le x < 2\pi$.



Using the zoom feature to get more accuracy, you will be able to see that the line $y = 5$ intersects the curve $y = 6 \sin 2x$ at

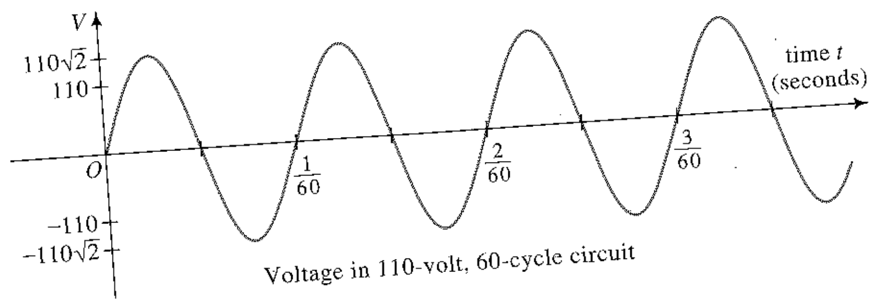
$$x \approx 0.49, 1.08, 3.63, \text{ and } 4.22.$$

Applications to Electricity

Most household circuits in the United States are 60-cycle alternating current (AC) circuits. This means that the voltage oscillates like the sine curve at a *frequency* of 60 cycles per second. In other words, 1 cycle is completed every $\frac{1}{60}$ of a second, which is the period. The period of a sine or cosine curve is always the reciprocal of the frequency.

In the following diagram, the frequency is 60, and the period is $\frac{1}{60}$. Thus, $\frac{1}{60} = \frac{2\pi}{B}$, and so $B = 120\pi$. Most household circuits are called 110-volt circuits because they deliver energy at the same rate as a direct current of 110 volts. In actuality, maximum voltage in a 110-volt circuit is $110\sqrt{2}$ volts, and so $A = 110\sqrt{2}$. The voltage equation is:

$$V = A \sin Bx$$
$$V = 110\sqrt{2} \sin 120\pi t \quad (t \text{ in seconds})$$



Voltage in 110-volt, 60-cycle circuit

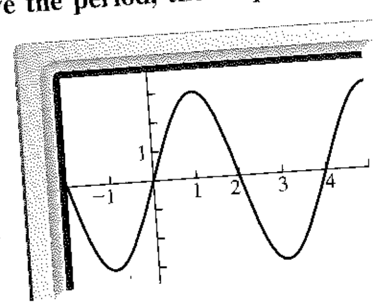
CLASS EXERCISES

Give the period and amplitude of each function.

1. $y = 4 \cos 2x$
2. $y = 3 \sin \frac{1}{2}x$
3. $y = 5 \sin \frac{2\pi}{7}x$
4. $y = 6 \cos \frac{2\pi}{3}t$

Give the period, the amplitude, and an equation for each curve.

5.



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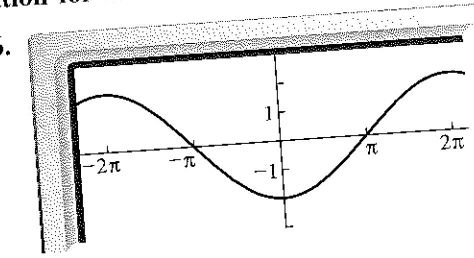


7. ***Visual Thinking*** Draw a quick sketch to tell how many solutions each equation has between 0 and 2π.

   a. $\sin x = 1$ **b.** $\sin 2x = \frac{1}{2}$ **c.** $\sin 3x = -1$

WRITTEN EXERCISES

A **For Exercises 1–4, sketch and label the graphs given on a single set of axes.**

1. $y = \cos x,\ y = 3 \cos x,\ y = \frac{1}{3} \cos x$

2. $y = \sin x,\ y = 4 \sin x,\ y = -4 \sin x$

3. $y = \sin x,\ y = \sin \frac{1}{2}x$

4. $y = \cos x,\ y = \cos 3x$

Give the amplitude and period of each function. Then sketch its graph.

5. $y = 2 \sin 3x$

6. $y = 4 \cos 2x$

7. $y = -2 \cos 2t$

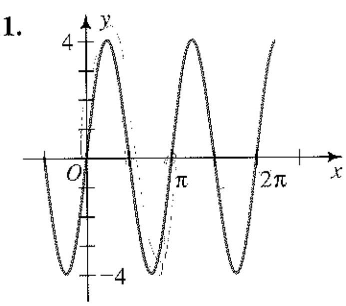
8. $y = -4 \sin \frac{x}{3}$

9. $y = \frac{1}{2} \cos 2\pi t$

10. $y = 1.5 \sin \frac{\pi}{2}x$

Give the amplitude, period, and an equation for each curve.

11.



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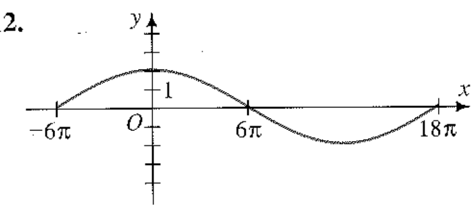

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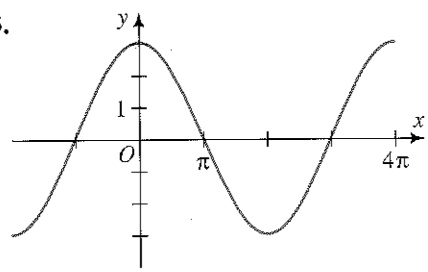
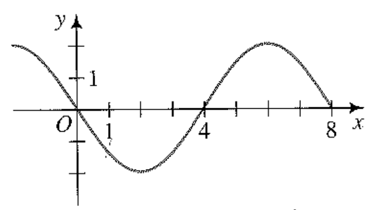


14.



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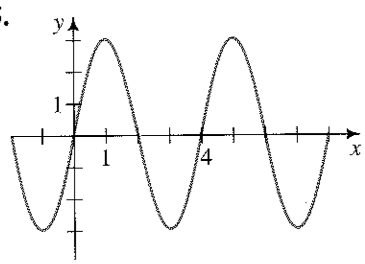

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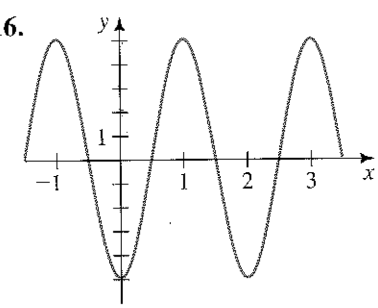


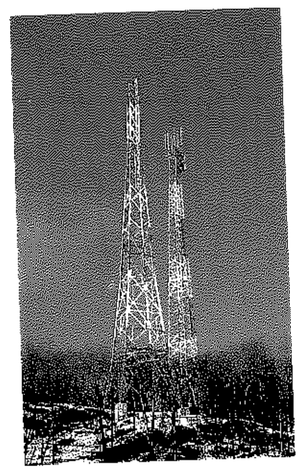
17. A sine curve varies between 4 and -4 with period 12. Find its equation.
18. A cosine curve varies between -9 and 9 with period 5. Find its equation.

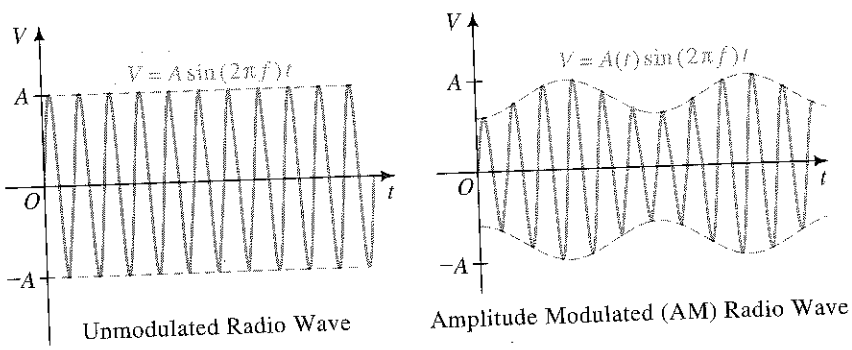
Solve each equation for $0 \le x < 2\pi$ either algebraically or graphically using a computer or graphing calculator. Give answers to the nearest hundredth of a radian.

19. **a.** $\cos x = -1$ **b.** $\cos 2x = -1$ **c.** $\cos 3x = -1$
20. **a.** $2 \sin x = 1$ **b.** $2 \sin 2x = 1$ **c.** $2 \sin \frac{x}{2} = 1$

B

21. $8 \cos 2x = 1$
22. $5 \sin 3x = -2$
23. $3 \sin \frac{x}{2} = -1$
24. $1.5 \cos \frac{x}{2} = \frac{1}{2}$
25. $4 \sin \frac{\pi}{2}x = 1$
26. $-3 \cos \frac{\pi}{4}x = 1$
27. Sketch the graphs of $y = \tan 2x$ and $y = \tan \frac{1}{2}x$.
28. Sketch the graphs of $y = \sec 2x$ and $y = \sec \frac{1}{2}x$.
29. ***Writing*** Write a paragraph to explain why it does not make sense to talk about the amplitude of the graph of $y = \tan x$ and of the graph of $y = \sec x$.
30. **Electronics** An AM radio wave means that the *amplitude* is *modulated.* Thus, instead of a constant amplitude A, there is a varying amplitude $A(t)$ as shown below. Each AM radio station in the United States is assigned a frequency between 540 kHz and 1700 kHz. (A hertz (Hz) is one cycle per second, so 1 kHz is 1000 cycles per second.)



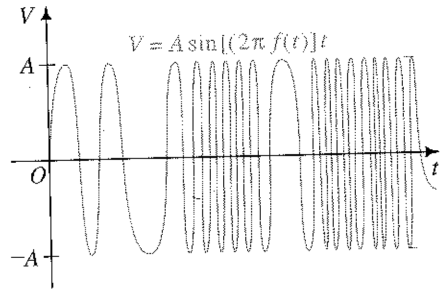


Unmodulated Radio Wave

Amplitude Modulated (AM) Radio Wave

a. Give the period of an AM radio wave with a frequency of 800 kHz.
b. An AM radio wave has equation $V(t) = A(t) \sin 1{,}850{,}000\pi t$. What is its frequency in kHz?

31. **Electronics** The initials FM stand for *frequency modulation*. In FM broadcasting, information is communicated by varying the frequency; that is, the constant frequency f is replaced by a variable frequency $f(t)$, which is a function of time. An exaggerated graph of this situation appears at the right. A complication of FM broadcasting is that $f(t)$ must remain near the radio station's assigned frequency. FM frequencies are given in MHz (millions of cycles per second).



Frequency Modulated (FM) Radio Wave

a. Suppose that an FM radio wave has an equation of the following form: $V(t) = A \sin [2\pi(200{,}000{,}000 + 10{,}000 \sin 500\pi t)]t$. Thus, the varying frequency is $f(t) = 200{,}000{,}000 + 10{,}000 \sin 500\pi t$. What is the assigned frequency in MHz?

b. Give the maximum and minimum frequencies of this FM radio wave over a long period of time.

32. ***Research*** In a reference book find information about two of the following:

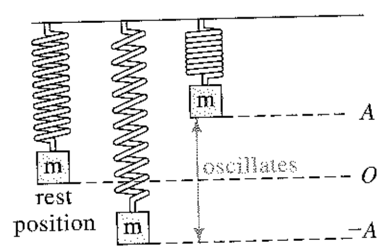
a. the range of assigned frequencies of the FM broadcasting channels in the United States

b. the frequency ranges for VHF (very high frequency) and UHF (ultra high frequency) television

c. the frequencies of ultraviolet light and infrared light

d. the frequency of the musical pitch called "middle C"

33. **Physics** Suppose a weight with mass m grams hangs on a spring. If you pull the weight A centimeters downward and let go of it, the weight will oscillate according to the formula below. Let d represent the displacement in centimeters t seconds after the initial displacement. Let k be a constant measuring the spring's stiffness.



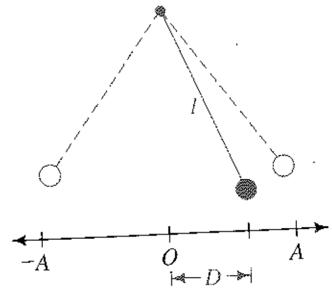
$$d = -A \cos \left[\left(\sqrt{\frac{k}{m}} \right) t \right]$$

a. Show that the period of the motion is $2\pi\sqrt{\frac{m}{k}}$.

b. Suppose you put a weight with known mass 100 g on the spring and you time its period to be $t = 1.1$ s. Find the value of the spring constant k.

c. Suppose you put a weight with unknown mass m on the spring and time its period to be $t = 1.4$ s. Find m. (Use the value of k found in part (b).)

34. **Physics** When a pendulum swings back and forth through a small arc, its horizontal displacement is given by the formula below. Let D be the horizontal displacement in centimeters of the pendulum t seconds after passing through O, let A represent the maximum displacement, and let l represent the length of the pendulum.



$$D \approx A \sin\left[\left(\sqrt{\frac{980}{l}}\right)t\right]$$

a. If the length l of the pendulum is 100 cm, find the earliest time t for which the displacement is maximum.

b. How long is a clock pendulum that has a period of 1 second?

C 35. On the same set of axes, sketch the graphs of $y = 2^{-x}$, $y = -2^{-x}$, and $y = 2^{-x} \sin x$ for $x > 0$.

36. **Music** When the note called *concert* A is sounded on a piano, the piano string vibrates with a frequency of 440 Hz. The equation that gives the displacement of a point on the vibrating piano string is

$$D = B(2^{-t}) \sin 880\pi t,$$

where t is the number of seconds after the string is struck and B is a constant that depends on how hard the string is hit and the point's position on the string. (This formula applies only for the first few seconds.)

a. What part of the equation models the gradual dying out of the vibrations?

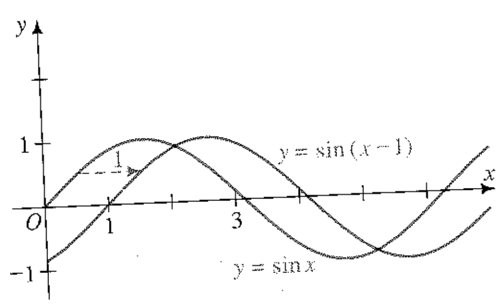
b. What is the earliest time t for which the displacement is maximum?

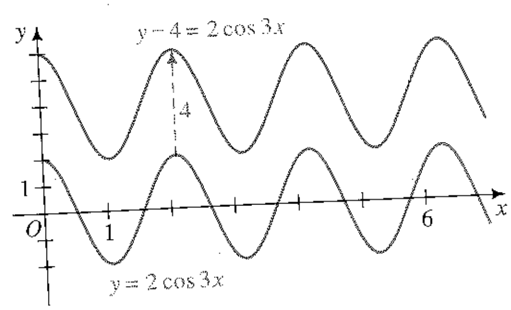
c. The piano string for the A an octave lower than concert A has a frequency of 220 Hz. Write an equation for its displacement.

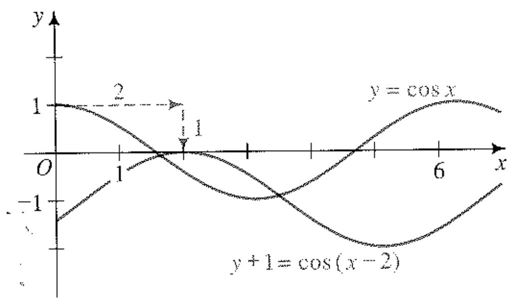
8-3 Modeling Periodic Behavior

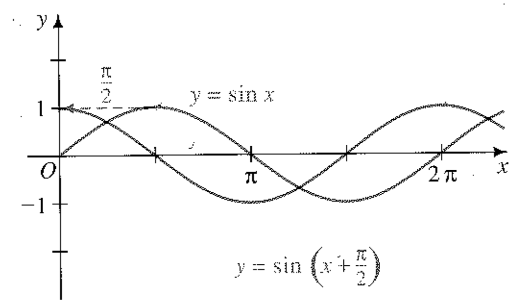
Objective *To use trigonometric functions to model periodic behavior.*

When a graph is translated h units horizontally and k units vertically, then x and y must be replaced by $(x - h)$ and $(y - k)$, respectively. This idea was introduced in Section 4-4 and is illustrated below and on the next page.









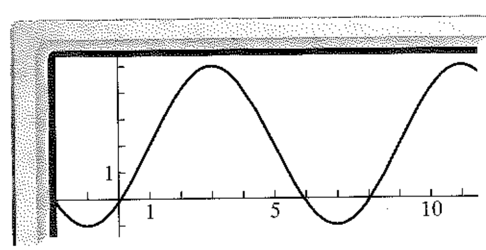
Notice that the last graph, whose equation is $y = \sin\left(x + \frac{\pi}{2}\right)$, is also the graph of $y = \cos x$. Thus, the cosine curve is the sine curve that has been shifted $\frac{\pi}{2}$ units to the left. Since the cosine curve is congruent to the sine curve, we refer to sine and cosine curves as *sine waves*.

General Sine Waves

If the graphs of $y = A \sin Bx$ and $y = A \cos Bx$ are translated horizontally h units and vertically k units, then the resulting graphs have equations

$$y - k = A \sin B(x - h) \quad \text{and} \quad y - k = A \cos B(x - h).$$

Example 1 Give an equation of the sine wave shown at the right.



Solution The graph suggests a translation of $y = A \sin Bx$ or $y = A \cos Bx$. To find A and B, reason as follows:

$$\text{amplitude:} \quad A = \frac{\text{Max} - \text{min}}{2} = \frac{5 - (-1)}{2} = 3$$

$$\text{period:} \quad p = \text{horizontal distance between successive maximums}$$
$$= 11 - 3 = 8$$

$$\text{Since } 8 = \frac{2\pi}{B},\ B = \frac{\pi}{4}.$$

The given sine wave is a translation of $y = 3 \sin \frac{\pi}{4}x$ or of $y = 3 \cos \frac{\pi}{4}x$.

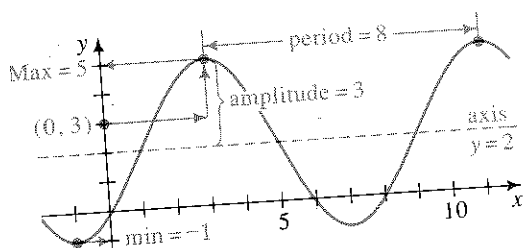
(Solution continues on the next page.)

To find the amounts of the translation, first find the *axis of the wave*, which is the horizontal line midway between the maximum and minimum points of the curve.

$$\text{axis of wave:} \quad y = \frac{\text{Max} + \text{min}}{2} = \frac{5 + (-1)}{2} = 2$$

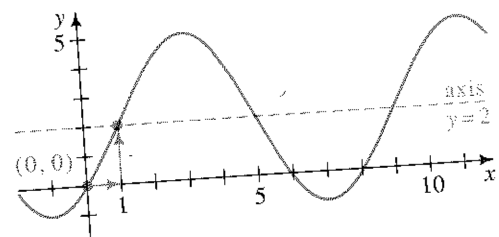
(1) If the equation is to be in terms of cosine, select a highest point on the curve. Determine the translation amounts in moving from the point $(0, A)$ to this point. See the diagram at the left below.

(2) If the equation is to be in terms of sine, select a point where the curve intersects its axis. Determine the translation amounts in moving from the point $(0, 0)$ to this point. See the diagram at the right below.



The graph of $y = 3 \cos \frac{\pi}{4}x$ is translated 3 units right and 2 units up.

$$y - 2 = 3 \cos \frac{\pi}{4}(x - 3)$$



The graph of $y = 3 \sin \frac{\pi}{4}x$ is translated 1 unit right and 2 units up.

$$y - 2 = 3 \sin \frac{\pi}{4}(x - 1)$$

Thus, an equation of the given graph is

$$y - 2 = 3 \cos \frac{\pi}{4}(x - 3) \quad \text{or} \quad y - 2 = 3 \sin \frac{\pi}{4}(x - 1).$$

Trigonometric Models

Trigonometric functions are useful in solving many problems that involve periodic behavior, such as the motion of the tide. In many applications, the variable might represent something other than angles. For example, an application might involve $\sin t$ or $\cos t$, where t represents time. In this case, you evaluate $\sin t$ or $\cos t$ as if t were in radians.

As the example on the next page shows, a sketch of a graph showing given information is very helpful in analyzing a problem in order to find an equation of the trigonometric model. Then you could use a graphing calculator or a computer to draw quickly the functions involved in the equation, and then zoom in on the intersection points to find the time values.

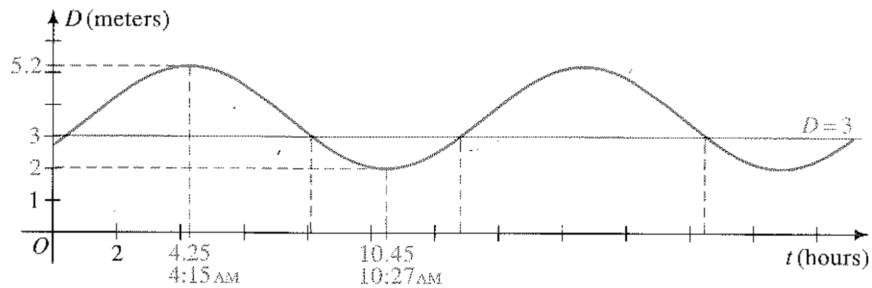
Example 2 The depth of water at the end of a pier varies with the tides throughout the day. Today the high tide occurs at 4:15 A.M. with a depth of 5.2 m. The low tide occurs at 10:27 A.M. with a depth of 2.0 m.

a. Find a trigonometric equation that models the depth of the water t hours after midnight.

b. Find the depth of the water at noon.

c. A large boat needs at least 3 m of water to moor at the end of the pier. During what time period after noon can it safely moor?

Solution **a.** First make a sketch, noting the given information.



The amplitude is $A = \frac{5.2 - 2.0}{2} = 1.6$

The period is $p = 2 \cdot (\text{time of low tide} - \text{time of high tide})$

$$= 2(10.45 - 4.25)$$
$$= 12.4$$

Since $12.4 = \frac{2\pi}{B}$, $B \approx 0.507$.

The axis of the wave has the equation $y = \frac{5.2 + 2.0}{2} = 3.6$.

Thus, an equation that models the depth D at time t hours after midnight is:

$$D - 3.6 = 1.6 \cos [0.507(t - 4.25)], \text{ or}$$
$$D = 1.6 \cos [0.507(t - 4.25)] + 3.6$$

b. Substitute $t = 12$ into the equation above to find the depth at noon.

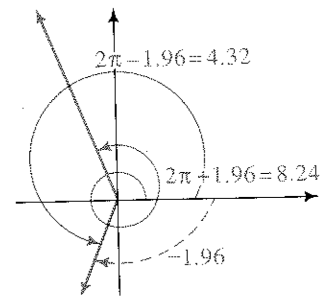
$$D = 1.6 \cos [0.507(12 - 4.25)] + 3.6$$
$$D \approx 2.47 \text{ m}$$

c. Using a graphing calculator or a computer, first enter the equation $D - 3.6 = 1.6 \cos\ [0.507(t - 4.25)]$ to get the curve shown above. Then graph the line $D = 3$ and find that it intersects the curve at $t = 8.1$, $t = 12.8$, and $t = 20.5$ (the times when the depth is 3 m).

Without a graphing calculator, you can find these values of t algebraically, as shown on the next page.

$$D = 1.6 \cos [0.507(t - 4.25)] + 3.6$$
$$3 = 1.6 \cos [0.507(t - 4.25)] + 3.6$$
$$-0.6 = 1.6 \cos [0.507(t - 4.25)]$$
$$-0.375 = \cos [0.507(t - 4.25)]$$
$$\text{Cos}^{-1} (-0.375) = 0.507(t - 4.25)$$

Use a scientific calculator to find that $\text{Cos}^{-1} (-0.375) \approx 1.96$. Any radian value coterminal with 1.96 or with -1.96 would also have -0.375 as its cosine value. The diagram shows that 4.32 and 8.24 are such possible radian values. Substituting these values gives:

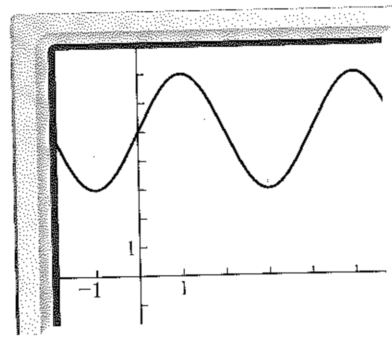


$1.96 = 0.507(t - 4.25)$	$4.32 = 0.507(t - 4.25)$	$8.24 = 0.507(t - 4.25)$
$t \approx 8.1$ h	$t \approx 12.8$ h	$t \approx 20.5$ h
$t = 8{:}06$ A.M.	$t = 12{:}48$ P.M.	$t = 8{:}30$ P.M.

You should locate these three times on the graph shown on page 311. Looking at the graph, you can see that the curve is above the line $D = 3$ when $12.8 < t < 20.5$. Therefore, the boat can safely moor between 12:48 P.M. and 8:30 P.M.

CLASS EXERCISES

For Exercises 1–5, refer to the sine wave shown.

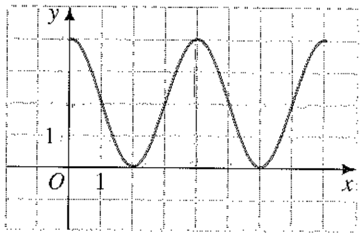


1. What is the amplitude?
2. What is the period?
3. What is the axis of the wave?
4. **a.** If the wave is considered to be the translation of the graph of $y = A \sin Bx$, what are the horizontal and vertical translation amounts?
   b. What is this equation?
5. **a.** If the wave is considered to be the translation of the graph of $y = A \cos Bx$, what are the translation amounts?
   b. What is this equation?
6. The equation of a sine wave is $y - 6 = 4 \sin 3(x + 5)$. What do the numbers 3, 4, 5, and 6 tell you about the wave?

WRITTEN EXERCISES

Give an equation for each trigonometric graph.

A **1.**



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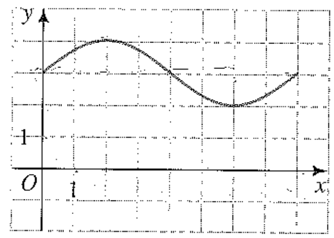

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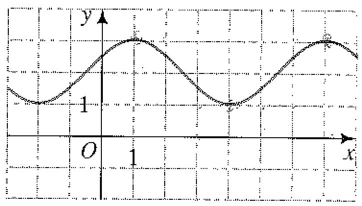

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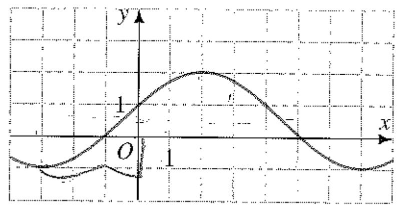

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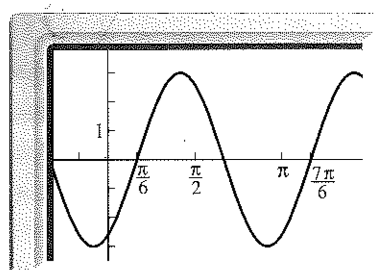

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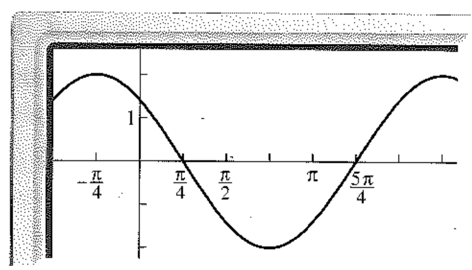

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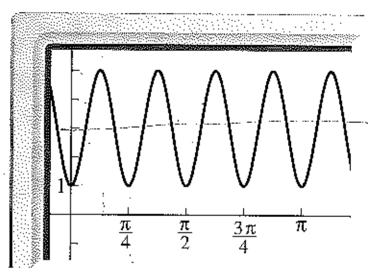
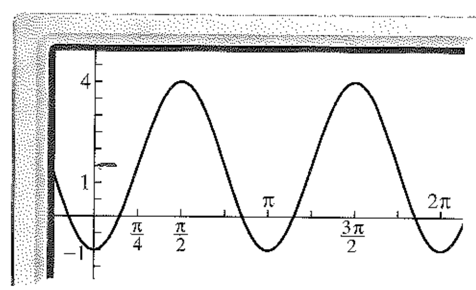


8.



Sketch the graph of each equation.

9. $y = 3 + 5 \sin 2x$

10. $y = -2 + 2 \cos \frac{1}{2}x$

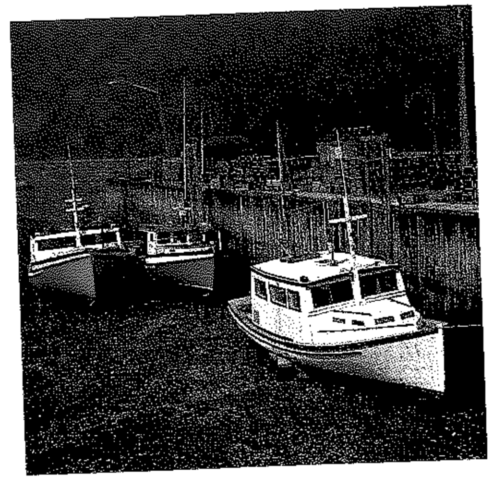
11. $y = -3 \sin\left(x - \frac{\pi}{6}\right)$

12. $y = 4 - 4 \cos 2(x - \pi)$

13. $y - 3 = 2 \cos \frac{\pi}{2}(x - 1)$

14. $y + 3 = 6 \sin \frac{\pi}{4}(x + 2)$

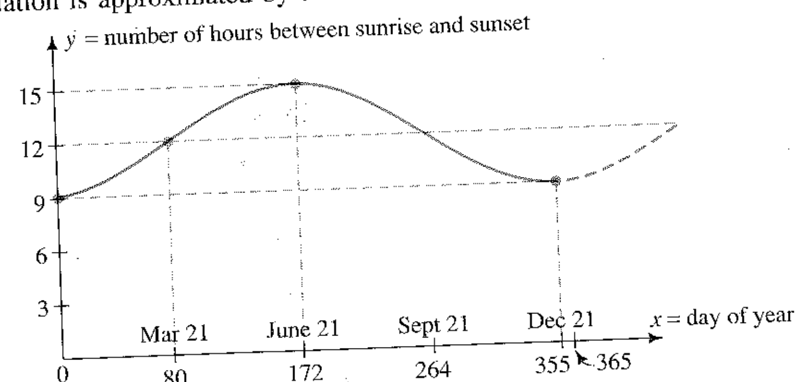
15. Oceanography The Bay of Fundy is an inlet of the Atlantic Ocean bounded by Maine and New Brunswick on the north and Nova Scotia on the south. It is famous for its high tides. At a dock there, the depth of water is 2 ft at low tide and 58 ft at high tide, which occurs 6 h 12 min after low tide. Draw a graph showing the depth of water at the dock as a function of the time since high tide occurs. Find an equation of your graph.



16. Oceanography Rework the tide problem in Example 2 assuming that the first high tide today occurs at 3:00 A.M. with a depth of 4.0 m, and the first low tide occurs at 9:24 A.M. with a depth of 1.8 m.

For some parts of the following exercises, you may wish to use a graphing calculator or a computer. Be sure to use radian measure.

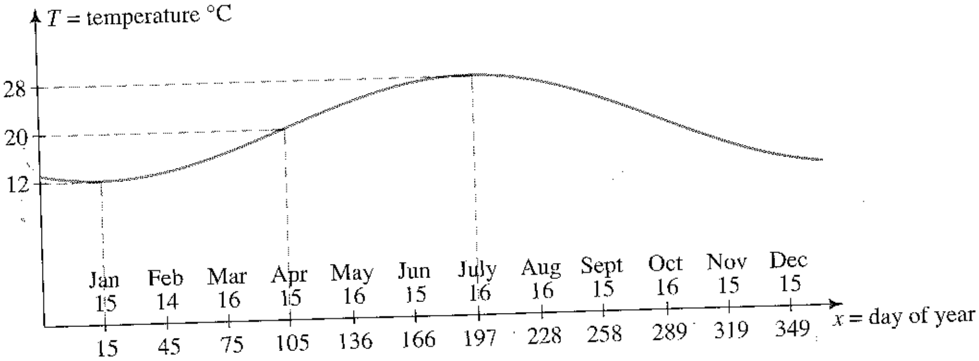
B 17. Astronomy The approximate number of hours between sunrise and sunset in Denver, Colorado, varies throughout the year as shown in the graph. This variation is approximated by the sine wave shown below.



a. Give the period, amplitude, and equation of the daylight-hours graph.
b. Find the amount of daylight in Denver on January 1 and on July 4.
c. Over the course of a year, during what period of time is the amount of daylight in Denver at least 14 hours?
d. If you were to draw a daylight-hours graph for Seattle, Washington, which is north of Denver, do you think its amplitude would be less than or greater than that for the Denver curve?

18. Astronomy The graph given in Exercise 17 applies to Denver and to all other locations at latitude 39°44′N. Modify the graph for a location at 39°44′S. Give an equation of your modified graph.

19. **Meteorology** Average monthly temperatures for New Orleans are plotted for the middle of each month. These points are connected to give a smooth curve, shown below, that gives an approximation of the average daily temperatures.



a. Find an equation of the average-daily-temperature graph.

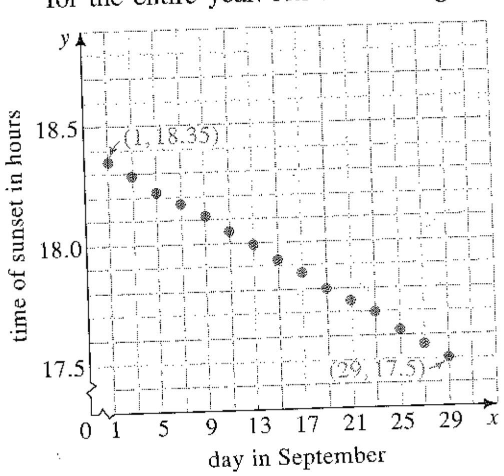
b. Over the course of a year, during what periods of time is the average daily temperature in New Orleans no more than 15°C?

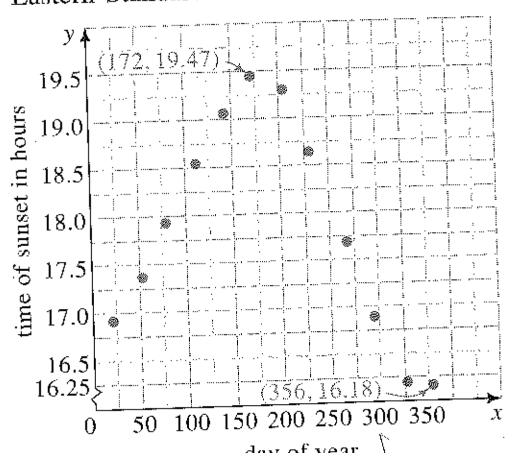
20. **Meteorology** The average maximum and minimum temperatures of two cities are given below. For each location, sketch a temperature sine wave like that in Exercise 19. Then give an equation of the curve.

a. Winnipeg, Canada: 26°C (July 16) and −14°C (January 15)

b. Rio de Janeiro, Brazil: 28°C (January 15) and 22°C (July 16)

21. **Astronomy** The graph at the left below shows the time of sunsets occurring every other day during September in Exeter, New Hampshire. The graph at the right shows the time of sunsets on either the 21st or 22nd day of each month for the entire year. All times are given in Eastern Standard Time.





a. Find a linear function that models the data in the graph at the left.

b. Find a trigonometric function that models the data in the graph at the right.

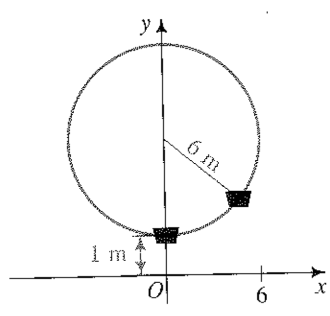
c. Use the function in part (b) to find the period of time, over the course of a year, during which the sun sets after 7:00 P.M. EST.

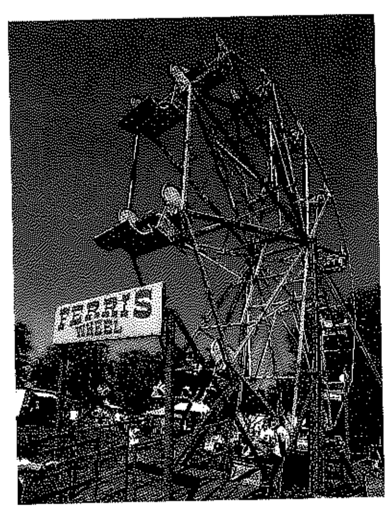
22. ***Research*** Refer to Exercise 20. Consult a local almanac or newspaper to find the average maximum and minimum temperatures for the area where you live. Sketch a temperature sine wave and give an equation of the curve.

23. ***Research*** Refer to Exercise 21. Consult a local almanac or newspaper to find the date and time of the latest sunrise and of the earliest sunrise in your area. (For simplicity, use standard time, not daylight-saving time.)
    a. Draw a graph that approximates the times of sunrises for the whole year and give its equation.
    b. What is the time of sunrise on your birthday?

24. **Thermodynamics** On a cold winter day, a house is heated until it is warm enough for the thermostat to turn off the heat. Then the house cools until it is cool enough for the thermostat to turn on the heat again. Assume that this periodic change in temperature can be modeled by a trigonometric function.
    a. Find an equation of this function given the following data:
        1. Thermostat turns heat on at 10:15 A.M. when house temperature is 18°C.
        2. Thermostat turns heat off at 10:30 A.M. when house temperature is 20°C.
    b. ***Writing*** Write a paragraph explaining why this trigonometric function might not be a good model. Think about how long it takes for the heating and cooling parts of the cycle. Think about the effects of temperature changes outdoors.

C 25. **Physics** Suppose that after you are loaded into a Ferris wheel car, the wheel begins turning at 4 rpm. The wheel has diameter 12 m and the bottom seat of the wheel is 1 m above the ground. (See the diagram below.) Express the height h of the seat above the ground as a function of time t seconds after it begins turning. (*Suggestion*: Begin your solution by sketching the graph of height versus time. Label the times at which you reach the lowest and highest points.)





26. **Physics** A reflector is fastened to the front wheel of a bicycle 20 cm from the center of the wheel. The diameter of the wheel and inflated tire is 70 cm. If the bike is traveling at 10 km/h, express the height of the reflector above the ground as a function of time. Assume that at time $t = 0$ seconds, the reflector is at its highest point. (*Hint:* The hardest part is determining the period.)

Identities and Equations

8-4 Relationships Among the Functions

Objective *To simplify trigonometric expressions and to prove trigonometric identities.*

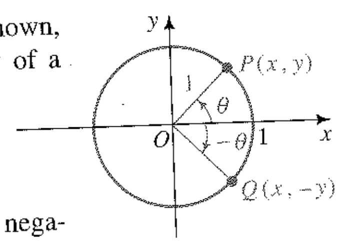
In this section, we will investigate some of the relationships among the trigonometric functions. Some of these relationships you have seen before. In Section 7-5, you saw that certain functions are reciprocals of others.

Reciprocal Relationships

$\csc\theta = \dfrac{1}{\sin\theta}$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\qquad$ $\cot\theta = \dfrac{1}{\tan\theta}$

Recall that if P is a point on the unit circle as shown, then $x = \cos\ \theta$ and $y = \sin\ \theta$. From the symmetry of a circle, the following relationships are true.

$$\sin(-\theta) = -y = -\sin\theta$$
$$\cos(-\theta) = x = \cos\theta$$



Using the reciprocal relationships, we can find negative relationships for the other functions.

Relationships with Negatives

$\sin(-\theta) = -\sin\theta$	and	$\cos(-\theta) = \cos\theta$
$\csc(-\theta) = -\csc\theta$	and	$\sec(-\theta) = \sec\theta$
$\tan(-\theta) = -\tan\theta$	and	$\cot(-\theta) = -\cot\theta$

We can find other relationships using the unit circle. Again, refer to the diagram above. Since $P(\cos\theta, \sin\theta)$ is on the unit circle,

$$(\sin\theta)^2 + (\cos\theta)^2 = x^2 + y^2 = 1.$$

This is the first of the three Pythagorean relationships that follow. You are asked to explore the other two relationships in Activity 2 and in Exercises 25 and 26.

Pythagorean Relationships

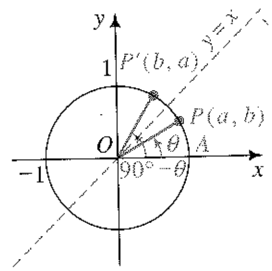
$\sin^2\theta + \cos^2\theta = 1$ $\qquad$ $1 + \tan^2\theta = \sec^2\theta$ $\qquad$ $1 + \cot^2\theta = \csc^2\theta$

The sine and cosine are called **cofunctions**, as are the tangent and cotangent, and the secant and cosecant. There is a special relationship between a function and its cofunction, as you will discover in the following activity.

Activity 1

1. Use a calculator to evaluate the following.
   - **a.** sin 50°, cos 40° **b.** sin 25°, cos 65°
   - **c.** cos 11°, sin 79° **d.** sin 83°, cos 7°
2. Complete each of the following.
   - **a.** sin 18° = cos (?°) **b.** cos 89° = sin (?°)
   - **c.** $\sin \theta = \cos$ (?) **d.** $\cos \theta = \sin$ (?)

The reason for the cofunction relationships can be seen from the diagram at the right. If the sum of the measures of $\angle POA$ and $\angle P'OA$ is 90°, then P and P' are symmetric with respect to the line $y = x$. Hence, if $P = (a, b)$, then $P' = (b, a)$. Consequently,



$$\begin{aligned} \sin \theta &= y\text{-coordinate of } P \\ &= x\text{-coordinate of } P' \\ &= \cos(90° - \theta) \end{aligned} \qquad \begin{aligned} \cos \theta &= x\text{-coordinate of } P \\ &= y\text{-coordinate of } P' \\ &= \sin(90° - \theta) \end{aligned}$$

You should convince yourself that this argument remains valid if the diagram is changed so that θ is not in Quadrant I. In general,

function of θ = *co*function of the *co*mplement of θ.

The cofunction relationships are summarized below.

Cofunction Relationships

$\sin \theta = \cos(90° - \theta)$	and	$\cos \theta = \sin(90° - \theta)$
$\tan \theta = \cot(90° - \theta)$	and	$\cot \theta = \tan(90° - \theta)$
$\sec \theta = \csc(90° - \theta)$	and	$\csc \theta = \sec(90° - \theta)$

Identities

Each of the trigonometric relationships given is true for all values of the variable for which *each side of the equation is defined*. Such relationships are called **trigonometric identities**, just as $(a + b)^2 = a^2 + 2ab + b^2$ is called an *algebraic identity*. The following activity will help you understand identities.

Activity 2

Use a graphing calculator or computer to answer each question. Be sure you use radian measure. If your calculator does not have a particular function, use its reciprocal function. For example, you can use $\frac{1}{\cos x}$ instead of $\sec x$.

1. a. On the same set of axes, graph $y = 1 + \tan^2 x$ and $y = \sec^2 x$.
   b. Compare the two graphs. Would you say that the equation $1 + \tan^2 x = \sec^2 x$ is an identity? Explain.
2. a. On the same set of axes, graph $y = \sin 2x$ and $y = 2 \sin x$.
   b. Compare the two graphs. Would you say that the equation $\sin 2x = 2 \sin x$ is an identity? Explain.
3. a. Graph $y = \sec x - \sin x \tan x$. What is the domain of the sine function? of the secant function? of the tangent function?
   b. On the same set of axes, graph $y = \cos x$. What is the domain of the cosine function?
   c. If you eliminate those values of x for which any of these functions are undefined, what can you say about the comparison of the two graphs for the remaining values of x? Does the screen display suggest an identity?

In the following examples, we will use trigonometric identities to simplify expressions and to prove other identities.

Example 1 Simplify $\sec x - \sin x \tan x$.

Solution Express each function in terms of $\sin x$ and $\cos x$.

$$\begin{aligned}\sec x - \sin x \tan x &= \frac{1}{\cos x} - \sin x \cdot \frac{\sin x}{\cos x}\\ &= \frac{1 - \sin^2 x}{\cos x}\\ &= \frac{\cos^2 x}{\cos x}\\ &= \cos x\end{aligned}$$

Example 2 Prove: $\frac{\cot A(1 + \tan^2 A)}{\tan A} = \csc^2 A$

Solution Use identities to simplify the expression on either the left or right side of the equals sign. Usually, it's a good idea to choose the more complicated expression. Here, start simplifying the left side by replacing the expression $1 + \tan^2 A$ with $\sec^2 A$, as shown on the next page.

$$\frac{\cot A\,(1+\tan^2 A)}{\tan A} = \frac{\cot A \cdot \sec^2 A}{\tan A} \quad \longleftarrow 1+\tan^2 A = \sec^2 A$$

$$= \cot^2 A \cdot \sec^2 A \quad \longleftarrow \frac{1}{\tan A} = \cot A$$

$$= \frac{\cos^2 A}{\sin^2 A} \cdot \frac{1}{\cos^2 A} \quad \longleftarrow \cot A = \frac{\cos A}{\sin A} \text{ and } \sec A = \frac{1}{\cos A}$$

$$= \frac{1}{\sin^2 A}$$

$$= \csc^2 A$$

Since the result gives the expression on the other side of the equals sign, the given equation is an identity.

Sometimes when simplifying an expression, you may not see which identity applies. If this happens, try expressing the functions involved in terms of sine and cosine only. Usually this method takes longer, but it can be effective if all else fails.

CLASS EXERCISES

1. Since $\sin^2\theta + \cos^2\theta = 1$: **a.** $1 - \sin^2\theta = \underline{\ ?\ }$ **b.** $1 - \cos^2\theta = \underline{\ ?\ }$
2. Since $1 + \tan^2\theta = \sec^2\theta$: **a.** $\sec^2\theta - 1 = \underline{\ ?\ }$ **b.** $\sec^2\theta - \tan^2\theta = \underline{\ ?\ }$
3. Since $1 + \cot^2\theta = \csc^2\theta$: **a.** $\csc^2\theta - 1 = \underline{\ ?\ }$ **b.** $\csc^2\theta - \cot^2\theta = \underline{\ ?\ }$

Simplify each expression.

4. **a.** $\tan\theta \cdot \cos\theta$ **b.** $\tan(90° - A)$ **c.** $\cos\left(\frac{\pi}{2} - x\right)$
5. **a.** $(1 - \sin x)(1 + \sin x)$ **b.** $\sin^2 x - 1$ **c.** $(\sec x - 1)(\sec x + 1)$
6. **a.** $\tan A \cdot \cot A$ **b.** $\cot y \cdot \sin y$ **c.** $\cot^2 x - \csc^2 x$
7. Evaluate each expression.
   a. $\sin^2\frac{5\pi}{6} + \cos^2\frac{5\pi}{6}$ **b.** $\sec^2\pi - \tan^2\pi$ **c.** $\csc^2 135° - \cot^2 135°$
8. ***Discussion*** Tell how you would simplify each complex fraction. The result of the first fraction given should help you simplify the second fraction.

   a. $\dfrac{t + \frac{1}{t}}{t}$ $\qquad \dfrac{\tan A + \frac{1}{\tan A}}{\tan A}$ **b.** $\dfrac{a}{\frac{1}{a}} - \dfrac{b}{\frac{1}{b}}$ $\qquad \dfrac{\sec\theta}{\cos\theta} - \dfrac{\tan\theta}{\cot\theta}$

   c. $\dfrac{\frac{y}{x} + \frac{x}{y}}{\frac{1}{xy}}$ $\qquad \dfrac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta\sin\theta}}$ **d.** $\dfrac{1}{x + \frac{y^2}{x}}$ $\qquad \dfrac{1}{\cos\theta + \frac{\sin^2\theta}{\cos\theta}}$

WRITTEN EXERCISES

Simplify.

A

1. a. $\cos^2\theta + \sin^2\theta$ b. $(1 - \cos\theta)(1 + \cos\theta)$ c. $(\sin\theta - 1)(\sin\theta + 1)$
2. a. $1 + \tan^2\theta$ b. $(\sec x - 1)(\sec x + 1)$ c. $\tan^2 x - \sec^2 x$
3. a. $1 + \cot^2 A$ b. $(\csc A - 1)(\csc A + 1)$ c. $\dfrac{1}{\sin^2 A} - \dfrac{1}{\tan^2 A}$
4. a. $\dfrac{1}{\cos(90^\circ - \theta)}$ b. $1 - \dfrac{\sin^2\theta}{\tan^2\theta}$ c. $\dfrac{1}{\cos^2\theta} - \dfrac{1}{\cot^2\theta}$
5. a. $\cos\theta\cot(90^\circ - \theta)$ b. $\csc^2 x(1 - \cos^2 x)$ c. $\cos\theta(\sec\theta - \cos\theta)$
6. a. $\cot A \sec A \sin A$ b. $\cos^2 A(\sec^2 A - 1)$ c. $\sin\theta(\csc\theta - \sin\theta)$
7. $\sin A \tan A + \sin(90^\circ - A)$
8. $\csc A - \cos A \cot A$
9. $(\sec B - \tan B)(\sec B + \tan B)$
10. $(1 - \cos B)(\csc B + \cot B)$
11. $(\csc x - \cot x)(\sec x + 1)$
12. $(1 - \cos x)(1 + \sec x)\cos x$

Simplify each expression.

13. $\dfrac{\sin x \cos x}{1 - \cos^2 x}$
14. $\dfrac{\tan x + \cot x}{\sec^2 x}$
15. $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$
16. $(\sec^2\theta - 1)(\csc^2\theta - 1)$
17. $\dfrac{\cot^2\theta}{1 + \csc\theta} + \sin\theta\csc\theta$
18. $\dfrac{\tan^2\theta}{\sec\theta + 1} + 1$
19. $\cos^3 y + \cos y \sin^2 y$
20. $\dfrac{\sec y + \csc y}{1 + \tan y}$

B

21. $\dfrac{\cos\theta}{1 + \sin\theta} + \dfrac{1 + \sin\theta}{\cos\theta}$
22. $\dfrac{\sin\theta\cot\theta + \cos\theta}{2\tan(90^\circ - \theta)}$
23. $\dfrac{\sin^4\theta - \cos^4\theta}{\sin^2\theta - \cos^2\theta}$
24. $\dfrac{\sin^2\theta}{1 + \cos\theta}$ (*Hint:* See Exercise 1(b).)
25. Use the equation $\sin^2\theta + \cos^2\theta = 1$ to prove that $\tan^2\theta + 1 = \sec^2\theta$.
26. Use the equation $\sin^2\theta + \cos^2\theta = 1$ to prove that $\cot^2\theta + 1 = \csc^2\theta$.

In Exercises 27 and 28, use a graphing calculator or a computer to graph the given functions.

27. Graph the function $y = \sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x$. What is the domain of this function? What other function could have this same graph? Use trigonometric relationships to verify the suggested identity.
28. Graph the function $y = (\sin x \div \cos x) \div \tan x$. What is the domain of this function? What other function could have this same graph? Use trigonometric relationships to verify the suggested identity.

In Exercises 29–36, prove the given identity.

29. $\cot^2 \theta + \cos^2 \theta + \sin^2 \theta = \csc^2 \theta$

30. $\dfrac{\cot \theta - \tan \theta}{\sin \theta \cos \theta} = \csc^2 \theta - \sec^2 \theta$

31. $\dfrac{\sin \theta}{\csc \theta} + \dfrac{\cos \theta}{\sec \theta} = \sin \theta \csc \theta$

32. $\dfrac{1 - \sin^2 \theta}{1 + \cot^2 \theta} = \sin^2 \theta \cos^2 \theta$

33. $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$

34. $\dfrac{\tan^2 x}{1 + \tan^2 x} = \sin^2 x$

35. $\dfrac{\sin \theta}{\sin \theta + \cos \theta} = \dfrac{\tan \theta}{1 + \tan \theta}$

36. $\dfrac{\tan x}{1 + \sec x} + \dfrac{1 + \sec x}{\tan x} = 2 \csc x$

37. $\overline{AB}$ is tangent to the unit circle at $B(1, 0)$.

a. Why is $\triangle OPQ \sim \triangle OAB$?

b. Use part (a) to explain why

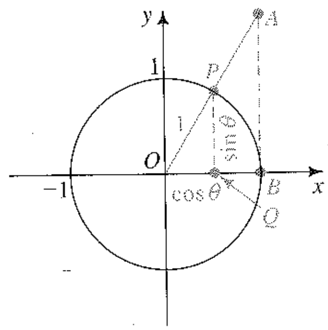
$$\frac{PQ}{OQ} = \frac{AB}{OB} \text{ and } \frac{OP}{OQ} = \frac{AO}{BO}.$$

c. Use part (b) to show that $AB = \tan \theta$ and $AO = \sec \theta$.

d. ***Visual Thinking*** Use the diagram to explain why the name *tangent* is given to the expression $\dfrac{\sin x}{\cos x}$ and the name *secant* is given to the expression $\dfrac{1}{\cos x}$.

e. Use right triangle AOB to prove that $\sec^2 \theta = 1 + \tan^2 \theta$.

f. Extend $\overline{AO}$ to intersect the circle at C. A theorem from geometry states that $(AB)^2 = AP \cdot AC$. Use this fact to prove $\tan^2 \theta = (\sec \theta - 1)(\sec \theta + 1)$.

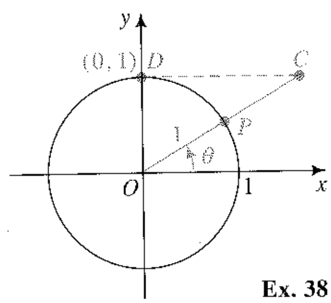


Ex. 37

38. $\overline{CD}$ is tangent to the unit circle at $D(0, 1)$. Show that

$$CD = \cot \theta \text{ and } CO = \csc \theta.$$

(*Hint:* See Exercise 37.)



Ex. 38

39. ***Writing*** Jon expected that the graph of $y = \sqrt{1 + \tan^2 x}$ would be the same as the graph of $y = \sec x$, but his graphing calculator showed that this was not the case. Write a paragraph explaining why this happened.

40. Express $\tan \theta$ in terms of $\cos \theta$ only.

41. Express $\sec \theta$ in terms of $\sin \theta$ only.

C **42.** Prove $\sqrt{\dfrac{1 - \sin x}{1 + \sin x}} = |\sec x - \tan x|$. For what values of x is this identity true?

//// COMPUTER EXERCISE

Imagine that your computer can calculate only the sine function. Write a program for which the input is any x, where $0 \le x \le \frac{\pi}{2}$, and the outputs are the six trigonometric functions of x. (*Hint:* See Exercise 41.)

Amalie Emmy Noether (1882–1935)

Emmy Noether was born and educated in Erlangen, Germany. She lectured at the University of Göttingen from 1915 until 1933 when she and other Jewish mathematicians were denied the right to teach. She emigrated to the United States, where she became a visiting professor at Bryn Mawr College and lectured at the Institute of Advanced Study at Princeton.

Noether's contributions centered on invariants and on noncommutative algebras. Her work on invariants culminated in a theorem, known to physicists as Noether's theorem, which is basic to the general theory of relativity. Her impact, however, extended far beyond her own work. Her insight, advice, and encouragement affected the research and publications of many colleagues and students.

8-5 Solving More Difficult Trigonometric Equations

Objective *To use trigonometric identities or technology to solve more difficult trigonometric equations.*

Many trigonometric equations can be solved in the same way that algebraic equations are solved.

Example 1 Solve $2\sin^2\theta - 1 = 0$ for $0° \le \theta < 360°$.

Solution First solve for $\sin^2\theta$.

$$2\sin^2\theta - 1 = 0$$
$$2\sin^2\theta = 1$$
$$\sin^2\theta = \frac{1}{2}$$
$$\sin\theta = \pm\frac{\sqrt{2}}{2} \longleftarrow \sin^2\theta = \frac{1}{2} \text{ implies that } \sin\theta = \pm\sqrt{\frac{1}{2}}.$$
$$\theta = 45°,\ 135°,\ 225°, \text{ and } 315°$$

Notice that there are four solutions because $\sin\theta$ can be either positive or negative, and so we need to look at all four quadrants.

Compare the solving of the quadratic equation $x^2 - 3x - 4 = 0$ with that of $\cos^2 \theta - 3 \cos \theta - 4 = 0$:

$$\begin{aligned} x^2 - 3x - 4 &= 0 \\ (x + 1)(x - 4) &= 0 \\ x + 1 = 0 \quad &\text{or} \quad x - 4 = 0 \\ x = -1 \quad &\text{or} \quad x = 4 \end{aligned}$$

$$\begin{aligned} \cos^2 \theta - 3 \cos \theta - 4 &= 0 \\ (\cos \theta + 1)(\cos \theta - 4) &= 0 \\ \cos \theta = -1 \quad &\text{or} \quad \cos \theta = 4 \\ \theta = 180° + n \cdot 360° \end{aligned}$$

(Since the range of the cosine function is all real numbers between -1 and 1, $\cos \theta = 4$ has no solution.)

Some trigonometric equations that are not quadratic can be transformed into equations that have the quadratic form.

Example 2 Solve $\sin^2 x - \sin x = \cos^2 x$ for $0 \leq x < 2\pi$.

Solution To get an equation involving only $\sin x$, substitute $1 - \sin^2 x$ for $\cos^2 x$.

$$\begin{aligned} \sin^2 x - \sin x &= \cos^2 x \\ \sin^2 x - \sin x &= 1 - \sin^2 x \\ 2 \sin^2 x - \sin x - 1 &= 0 \\ (2 \sin x + 1)(\sin x - 1) &= 0 \\ \sin x = -\frac{1}{2} \text{ or } \sin x &= 1 \\ x &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2} \end{aligned}$$

Example 3 Solve $\sin x \tan x = 3 \sin x$ for $0 \leq x < 2\pi$. Give answers to the nearest hundredth of a radian.

Solution

$$\begin{aligned} \sin x \tan x &= 3 \sin x \\ \sin x \tan x - 3 \sin x &= 0 \\ \sin x(\tan x - 3) &= 0 \end{aligned}$$

$$\sin x = 0 \quad \text{or} \quad \tan x = 3$$

$$x \approx 0, 3.14 \qquad x \approx 1.25, 4.39$$

The solutions are 0, 1.25, 3.14, and 4.39.

Notice that in Example 3 we did not divide both sides of the equation by the factor $\sin x$. Doing so would have caused us to lose a root. (See the discussion concerning losing roots on page 32.) In the next example, there is no common factor for both sides of the equation. In this case, there is no difficulty in dividing both sides by $\sin \theta$, as long as $\sin \theta \neq 0$, of course. We do not lose a root since values of θ for which $\sin \theta = 0$ are clearly not solutions to the original equation.

Example 4 Solve $2 \sin \theta = \cos \theta$ for $0° \leq \theta < 360°$.

Solution

$$2 \sin \theta = \cos \theta$$

$$2 = \frac{\cos \theta}{\sin \theta} \longleftarrow \text{Divide both sides by } \sin \theta.$$

$$2 = \cot \theta \longleftarrow \text{You can use } \tan \theta = \frac{1}{2} \text{ instead.}$$

$$\theta \approx 26.6°, 206.6°$$

The next example shows two strategies for solving a trigonometric equation. One method is to write a simpler equation by using identities and then solve the equation algebraically. Another method is to use a graphing calculator or computer to draw graphs and then zoom in on the intersection points.

Example 5 Solve $2 \sin \theta = \cos \theta + 1$ for $0° \leq \theta < 360°$.

Solution

Method 1

Try to rewrite the equation so that there is only one function. You can replace $\sin \theta$ by $\pm\sqrt{1 - \cos^2 \theta}$ since $\sin^2 \theta + \cos^2 \theta = 1$. Then the equation is in terms of $\cos \theta$ only.

$$2 \sin \theta = \cos \theta + 1$$

$$2(\pm\sqrt{1 - \cos^2 \theta}) = \cos \theta + 1$$

$$[2(\pm\sqrt{1 - \cos^2 \theta})]^2 = (\cos \theta + 1)^2$$

$$4(1 - \cos^2 \theta) = \cos^2 \theta + 2 \cos \theta + 1$$

$$5 \cos^2 \theta + 2 \cos \theta - 3 = 0$$

$$(5 \cos \theta - 3)(\cos \theta + 1) = 0$$

$$\cos \theta = 0.6 \quad \text{or} \quad \cos \theta = -1$$

$$\theta \approx 53.1°, 306.9° \qquad \theta = 180°$$

Since we squared the original equation, it is possible that we may have gained a root. Therefore, we must check each of these solutions in the *original equation.*

$$2 \sin 53.1° \stackrel{?}{=} \cos 53.1° + 1 \qquad 2 \sin 306.9° \stackrel{?}{=} \cos 306.9° + 1$$

$$1.6 = 1.6 \qquad -1.6 \neq 1.6$$

$$2 \sin 180° \stackrel{?}{=} \cos 180° + 1$$

$$0 = 0$$

Thus, 306.9° is an extraneous root and must be rejected.

The solutions are 53.1° and 180°.

(Solution continues on the next page.)

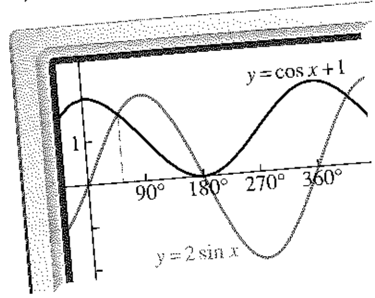
Method 2

Use a computer or graphing calculator to graph the equations

$$y = 2 \sin x \text{ and } y = \cos x + 1.$$

(If you can't use your computer in the degree mode, use the radian mode and then convert your answers to degrees.) For the graphs, choose the intervals

$0° \le x < 360°$ and $-2 \le y \le 2$.



The graphs are shown at the right. Notice that for values of x between $0°$ and $360°$, the graphs intersect in two points. By using a zoom (or trace) feature, you can find that the solutions are about $53.1°$ and $180°$.

WRITTEN EXERCISES

Solve for $0° \le \theta < 360°$. Give answers to the nearest tenth of a degree.

A

1. $\sec^2 \theta = 9$
2. $\tan^2 \theta = 1$
3. $1 - \csc^2 \theta = -3$
4. $8 \cos^2 \theta - 3 = 1$
5. $6 \sin^2 \theta - 7 \sin \theta + 2 = 0$
6. $2 \tan^2 \theta = 3 \tan \theta - 1$
7. $6 \sin^2 \theta = 7 - 5 \cos \theta$
8. $\cos^2 \theta - 3 \sin \theta = 3$

Solve for $0 \le x < 2\pi$. Give answers to the nearest hundredth of a radian when necessary.

9. $\cos x \tan x = \cos x$
10. $\sec x \sin x = 2 \sin x$
11. $\sin^2 x = \sin x$
12. $\tan^2 x = \tan x$
13. $2 \cos^2 x = \cos x$
14. $3 \sin x = \cos x$
15. $\sin x + \cos x = 0$
16. $\sec x = 2 \csc x$

Solve each equation for $0 \le x < 2\pi$ algebraically by using identities or graphically by using a graphing calculator or computer. Give answers to the nearest hundredth of a radian when necessary.

B

17. $\tan^2 x = 2 \tan x \sin x$
18. $2 \sin x \cos x = \tan x$
19. $2 \csc^2 x = 3 \cot^2 x - 1$
20. $2 \sec^2 x + \tan x = 5$
21. $\sin^2 x + \sin x - 1 = 0$
22. $\cos^2 x - 2 \cos x - 1 = 0$
23. $3 \cos x \cot x + 7 = 5 \csc x$
24. $2 \sin^3 x - \sin^2 x - 2 \sin x + 1 = 0$
25. $2 \cos^2 x - \cos x = 2 - \sec x$
26. $\csc^2 x - 2 \csc x = 2 - 4 \sin x$

27. $2 \cos x = 2 + \sin x$

28. $1 - \sin x = 3 \cos x$

Solve each equation for $0 \le x < 2\pi$ by using a graphing calculator or computer. Give answers to the nearest hundredth of a radian.

29. $\cos x = x$

30. $\tan x = x$

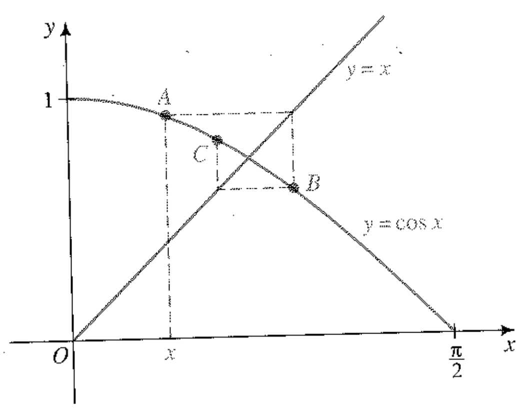
31. $\sin x = \frac{x}{2}$

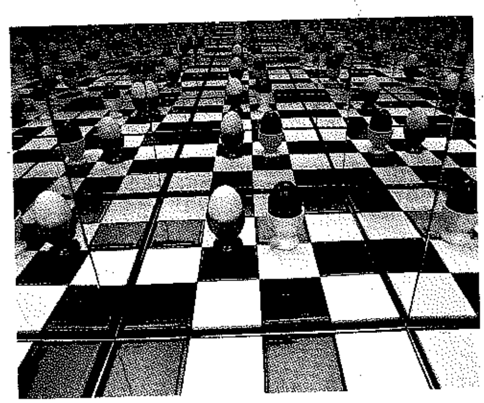
32. $\sin x = 2x$

33. $\cos x = 2x - 3$

34. $\sin x = x^2$

35. ***Investigation*** Consider the graphs of $y = \cos x$ and $y = x$ shown at the left below. Suppose you choose an arbitrary value of x between 0 and $\frac{\pi}{2}$. From this x-value move directly up to point A on the cosine curve. To get to point B, move horizontally to the line $y = x$ and from there move vertically back to the cosine curve. To get to point C, move horizontally and vertically again.





A set of mirrors iterates the basic image of two eggs to create a striking photograph.

a. The x-coordinate of point A is obviously x. Find the x-coordinates of points B and C in terms of x.

b. If you continue to move from point to point on the cosine curve as described above, what appears to happen?

c. Based on your answers to parts (a) and (b), what relationship exists between the x-coordinates of the points $A, B, C, \ldots$ and the solution of the equation $\cos x = x$?

d. Describe a method for approximating the solution of the equation $\cos x = x$ using a scientific calculator.

e. Implement your method of part (d) to find the solution of the equation $\cos x = x$ to the nearest thousandth of a radian.

f. Determine whether the initial choice of x needs to be limited to the interval from 0 to $\frac{\pi}{2}$.

Chapter Summary

1. Techniques for solving simple *trigonometric equations* are illustrated in Examples 1 and 2 of Section 8-1.
2. If a line has *inclination* α and *slope* m, then $m = \tan \alpha$.
3. If $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is the equation of a non-degenerate conic section, one of its axes has *direction angle* α, where α is an angle formed by the positive x-axis and this axis of the conic. Also:

$$\alpha = \frac{\pi}{4} \text{ (if } A = C\text{) and } \tan 2\alpha = \frac{B}{A - C} \text{ (if } A \neq C, 0 < 2\alpha < \pi\text{)}$$

   Example 4 on pages 297–298 shows how to graph such a conic.
4. The functions $y = A \sin Bx$ and $y = A \cos Bx$ have amplitude $|A|$ and period $\frac{2\pi}{B}$, where $A \neq 0$ and $B > 0$.
5. Sine curves and cosine curves are both called *sine waves*. If the graphs of $y = A \sin Bx$ and $y = A \cos Bx$ are translated horizontally h units and vertically k units, then the resulting graphs have equations

$$y - k = A \sin B(x - h) \text{ and } y - k = A \cos B(x - h).$$

   Information about how to find the translation amounts h and k for either a sine curve or a cosine curve is given on page 310.
6. The graphs of the sine waves have many applications, including the description of tides, radio waves, sunrises, musical tones, and the motion of springs.
7. A *trigonometric identity* is an equation that is true for all values of the variable for which both sides of the equation are defined. In proving such identities, the following relationships are helpful: the reciprocal relationships, the negative relationships, the Pythagorean relationships, and the cofunction relationships. These relationships are summarized on pages 317–318.
8. More advanced techniques for solving various trigonometric equations either algebraically or graphically are discussed in Examples 1–5 of Section 8-5.

Key vocabulary and ideas

inclination of a line, direction angle of a conic (pp. 296–297)
period and amplitude of sine and cosine curves (p. 302)
equations of general sine waves (p. 309)
trigonometric identity (p. 318)

Chapter Test

1. Find the inclination of the line $x - 3y = 9$ to the nearest degree. 8-1
2. Solve $5 \cos \theta = -1$ for $0° \leq \theta < 360°$. Give answers to the nearest tenth of a degree.

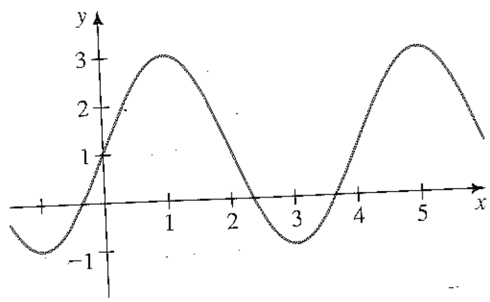
3. Solve $3 - \csc x = 7$ for $0 \leq x < 2\pi$. Give answers to the nearest hundredth of a radian.

8-2

4. Solve $2 \sin 3x = \sqrt{2}$ for $0 \leq x < 2\pi$.

5. Graph $y = 3 \sin \frac{x}{2}$, and give its period and amplitude.

6. Find an equation for the trigonometric function whose graph is shown below. 8-3



7. At Ocean Tide Dock the first high tide today occurs at 2:00 A.M. with depth 5 m, and the first low tide occurs at 8:30 A.M. with depth 2.2 m.
   a. Sketch and label a graph showing the depth of the water at the dock as a function of time after midnight.
   b. ***Writing*** Write a paragraph or two to describe how you would use the given information about the high and low tides to find a trigonometric equation of this function.

8-4

8. Simplify each expression:
   a. $\cot A\,(\sec A - \cos A)$
   b. $\dfrac{\cot \theta}{\sin (90^\circ - \theta)}$
   c. $(\sec x + \tan x)\,(1 - \sin x)$
   d. $\dfrac{\cot \alpha + \tan \alpha}{\csc^2 \alpha}$

9. Solve each equation for $0 \leq x < 2\pi$. Give answers to the nearest hundredth of a radian when necessary. 8-5
   a. $2 \cos x = \sin x$
   b. $\sin x = \csc x$

10. Solve each equation for $0^\circ \leq \theta < 360^\circ$. Give answers to the nearest tenth of a degree.
   a. $2 \cos^2 \theta + 3 \sin \theta - 3 = 0$
   b. $\cos \theta \cot \theta = 2 \cos \theta$