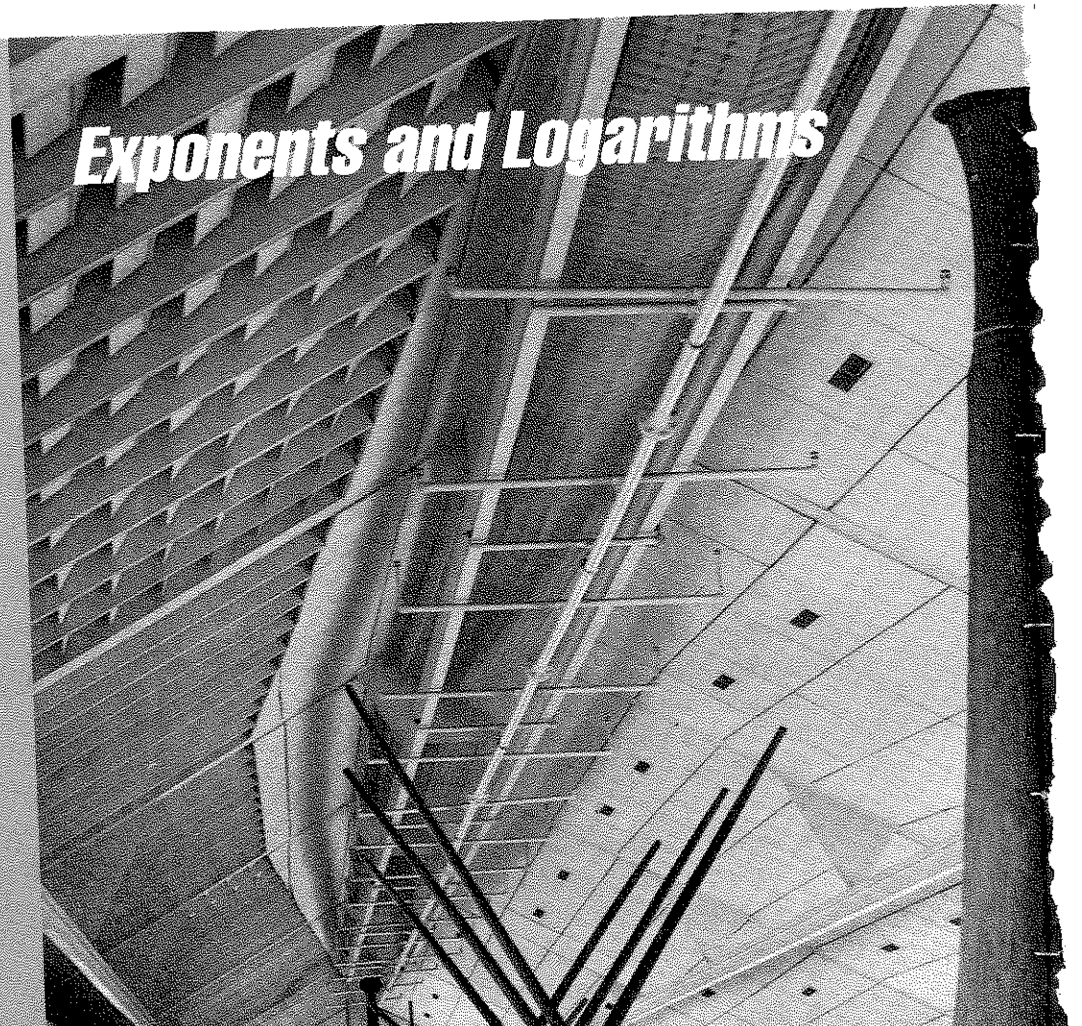
5 Exponents and Logarithms



Exponents

5-1 Growth and Decay: Integral Exponents

Objective *To define and apply integral exponents.*

Suppose that the cost of a hamburger has been increasing at the rate of 9% per year. Then, each year the cost is 1.09 times the cost in the previous year. Suppose that the cost now is \$4. Some projected future costs are given in the table below.

Time (years from now)	0	1	2	3	t
Cost (dollars)	4	$4(1.09)$	$4(1.09)^2$	$4(1.09)^3$	$4(1.09)^t$

$\times 1.09$ $\times 1.09$ $\times 1.09$ $\times 1.09$

The table suggests that the cost is a function of time t. Since the variable t occurs as an exponent, the cost is said to be an exponential function of time:

$$C(t) = 4(1.09)^t$$

When $t > 0$ the function gives future costs, and when $t < 0$ it gives costs in the past. Example 1 illustrates this.

Example 1 Use the cost function, $C(t) = 4(1.09)^t$, to find the cost of a hamburger **(a)** 5 years from now and **(b)** 5 years ago.

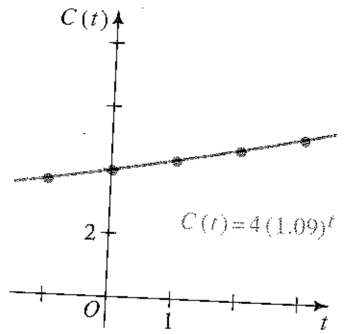
Solution Values of $C(t)$ are most easily found with a scientific calculator. (Use the exponent key.)

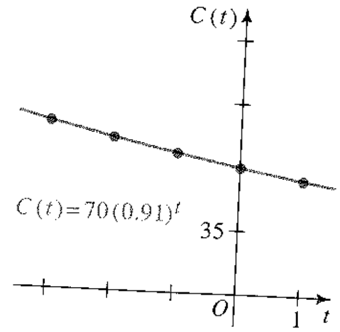
a. $C(5) = 4(1.09)^5 \approx 6.15$. The cost will be about \$6.15.
b. $C(-5) = 4(1.09)^{-5} \approx 2.60$. The cost was about \$2.60.

We will soon give exact meaning to expressions that involve negative exponents. However, first let us contrast the cost of a hamburger that has been increasing at 9% per year with the cost of a graphing calculator that has been *decreasing* at 9% per year. Each year the calculator cost is 100% − 9%, or 91%, of what it was the year before.

Hamburger cost: \$4 now Cost increasing at 9% per year $C(t) = 4(1.09)^t$	Calculator cost: \$70 now Cost decreasing at 9% per year $C(t) = 70(0.91)^t$

Ancient Egyptian astronomers used geometry to predict the future position of stars and planets. Current archeologists use mathematics to explore the Egyptian past. *Exponential functions,* for instance, are used in the carbon dating of ancient artifacts, such as this ceremonial boat dating from the reign of King Cheops, around 2400 B.C.





The graph at the left above shows exponential growth. The one at the right above shows exponential decay. Growth and decay can be modeled by:

$$A(t) = A_0(1 + r)^t$$

where A_0 is the initial amount, the amount at time $t = 0$, and r is the growth rate. If $r > 0$, then the initial amount grows exponentially. If $-1 < r < 0$, then the initial amount decays exponentially.

Example 2 Suppose that a radioactive isotope decays so that the radioactivity present decreases by 15% per day. If 40 kg are present now, find the amount present **(a)** 6 days from now and **(b)** 6 days ago.

Solution $A(t) = A_0(1 + r)^t = 40(1 - 0.15)^t = 40(0.85)^t$

a. $A(6) = 40(0.85)^6 \approx 15.1$
There will be about 15.1 kg.

b. $A(-6) = 40(0.85)^{-6} \approx 106.1$
There was about 106.1 kg.

Although we have used a calculator to evaluate expressions involving negative exponents, we must still define them. To do this, we will first review the laws of exponents for positive integers.

Laws of Exponents

Same bases

1. $b^x \cdot b^y = b^{x+y}$

2. $\dfrac{b^x}{b^y} = b^{x-y}\ (b \neq 0)$

3. If $b \neq 0$, 1, or -1, then $b^x = b^y$ if and only if $x = y$.

Same exponents

4. $(ab)^x = a^x b^x$

5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}\ (b \neq 0)$

6. If $x \neq 0$, $a > 0$, and $b > 0$, then $a^x = b^x$ if and only if $a = b$.

Power of a power

7. $(b^x)^y = b^{xy}$

If the laws on the preceding page are to make sense for the zero exponent and negative integral exponents, then such exponents must be defined as follows.

Definition of b^0: If law 1 is to hold for $y = 0$, then we have

$$b^x \cdot b^0 = b^{x+0} = b^x.$$

Since b^0 behaves like the number 1, we define it to be 1:

$$b^0 = 1 \; (b \neq 0)$$

Definition of b^{-x}: If law 1 is to hold for $y = -x$ and $b \neq 0$, then we have

$$b^x \cdot b^{-x} = b^{x+(-x)} = b^0 = 1.$$

Since b^x and b^{-x} have a product of 1, they are reciprocals of each other. Therefore, we define:

$$b^{-x} = \frac{1}{b^x} \; (x > 0 \text{ and } b \neq 0)$$

An expression is simplified when it contains neither negative exponents nor powers of powers. This is illustrated in the examples below.

Example 3 Simplify $\left(\frac{b^2}{a}\right)^{-2}\left(\frac{a^2}{b}\right)^{-3}$, where $a \neq 0$ and $b \neq 0$.

Solution $\left(\frac{b^2}{a}\right)^{-2}\left(\frac{a^2}{b}\right)^{-3} = \frac{(b^2)^{-2}}{a^{-2}} \cdot \frac{(a^2)^{-3}}{b^{-3}} = \frac{b^{-4}a^{-6}}{a^{-2}b^{-3}} = \frac{1}{a^4 b}$

Example 4 Simplify $(a^{-2} + b^{-2})^{-1}$, where $a \neq 0$ and $b \neq 0$.

Solution $(a^{-2} + b^{-2})^{-1} = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{-1} = \left(\frac{b^2 + a^2}{a^2 b^2}\right)^{-1} = \frac{a^2 b^2}{b^2 + a^2}$

As Example 4 shows, you do not simplify a power of a sum or difference by distributing the exponent over the individual terms. Laws 4 and 5 apply only to a power of a product or quotient:

$$(a^{-2}b^{-2})^{-1} = a^2 b^2 \quad \text{but } (a^{-2} + b^{-2})^{-1} \neq a^2 + b^2$$

Example 5 Simplify **(a)** $\frac{x^5 + x^{-2}}{x^{-3}}$ and **(b)** $\frac{x^5 \cdot x^{-2}}{x^{-3}}$, where $x \neq 0$.

Solution **a.** $\frac{x^5 + x^{-2}}{x^{-3}} = \frac{x^5}{x^{-3}} + \frac{x^{-2}}{x^{-3}}$
$= x^8 + x$

b. $\frac{x^5 \cdot x^{-2}}{x^{-3}} = \frac{x^3}{x^{-3}}$
$= x^6$

We shall assume throughout the rest of this book that variables are restricted so that there are no denominators of zero. In this section only, we assume that variables appearing as exponents represent integers.

CLASS EXERCISES

Copy and complete each table.

1.

a.	If A_0 increases by	3%	15%	4.6%	120%	?
b.	then multiply A_0 by	?	?	?	?	1.105

2.

a.	If A_0 decreases by	12%	7.5%	80%	?	100%
b.	then multiply A_0 by	?	?	?	0.68	?

3.

	Item	Annual rate of increase	Cost now	Cost in t years
a.	Bike	5%	$200.00	?
b.	Jeans	8%	$20.00	?
c.	Loaf of bread	?	$1.25	$1.25(1.06)^t$

4.

	Item	Annual rate of decrease	Value now	Value in t years
a.	Car	20%	$9800	?
b.	Boat	15%	$2200	?
c.	Skates	?	$100	$100(0.75)^t$

Simplify.

5. a. 8^{-1} **b.** 8^{-2}

6. a. $\left(\frac{2}{3}\right)^{-1}$ **b.** $\left(\frac{2}{3}\right)^{-2}$

7. a. $4 \cdot 3^{-2}$ **b.** $(4 \cdot 3)^{-2}$

8. a. $(2^{-1} \cdot 4^{-1})^{-1}$ **b.** $(2^{-1} + 4^{-1})^{-1}$

9. $\frac{12^3}{6^3}$

10. $\frac{8^n \cdot 3^n}{4^n}$

11. $\frac{(-2n)^2}{-2n^2}$

12. $x^{-3}(x^5 + x^3)$

13. $\frac{3a^3 \cdot 6a^6}{a^{-1}}$

14. $\frac{3a^3 - 6a^6}{a^{-1}}$

15. $\frac{5b^3 + 10b^6}{b^{-2}}$

16. $\frac{5a^3 \cdot 10b^6}{b^{-2}}$

WRITTEN EXERCISES

Simplify each expression.

A

1. a. $(-4)^{-2}$ **b.** -4^{-2} **2. a.** $(-3)^{-4}$ **b.** -3^{-4}

3. a. $5 \cdot 2^{-3}$ **b.** $(5 \cdot 2)^{-3}$ **4. a.** $2 \div 4^{-3}$ **b.** $(2 \div 4)^{-3}$

5. a. $(x^{-2})^{-1}$ **b.** $((x^{-2})^{-1})^{0}$ **6. a.** $(2x^{-3})^{-4}$ **b.** $2x^{-3} \cdot x^{-4}$

7. a. $(3a^{-1})^{-1}$ **b.** $(3 + a^{-1})^{-1}$ **8. a.** $(5x^{2} \cdot x^{-2})^{2}$ **b.** $(5x^{2} + x^{-2})^{2}$

9. a. $(2^{-2} + 2^{-3})^{-1}$ **b.** $(2^{-2} \cdot 2^{-3})^{-1}$ **10. a.** $(4^{-1} - 2^{-1})^{2}$ **b.** $(4^{-1} \div 2^{-1})^{2}$

11. a. $(a^{-1} - b^{-1})^{-1}$ **b.** $(a^{-1} \cdot b^{-1})^{-1}$ **12. a.** $(2 + x^{-2})^{-2}$ **b.** $(2 \cdot x^{-2})^{-2}$

Copy and complete the table. The cost of each item grows exponentially.

	Item	Annual rate of increase	Cost now	Cost in 10 years	Cost in 20 years
13.	Airplane ticket	15%	$300	?	?
14.	Swim suit	8%	$35	?	?
15.	Jar of mustard	7%	$1	?	?
16.	College tuition	10%	$12,000	?	?

Copy and complete the table. The value of each item decays exponentially.

	Item	Annual rate of decrease	Value now	Value in 3 years	Value in 6 years
17.	Farm tractor	25%	$65,000	?	?
18.	Industrial equipment	10%	$200,000	?	?
19.	Value of the dollar	6%	$1	?	?
20.	Value of the dollar	8%	$1	?	?

Simplify each expression.

21. $(3a^{-2})^{3} \cdot 3a^{5}$ **22.** $(-4x^{3})^{2} \cdot 3x^{-2}$ **23.** $(3n^{2})^{-1}\ (3n^{2})^{7}$

24. $(2r^{-1})^{4}(4r^{2})^{-2}$ **25.** $\dfrac{(2a^{-1})^{2}}{(2a^{-1})^{-2}}$ **26.** $\dfrac{(-3n^{-3})^{2}}{-9n^{-4}}$

27. $\left(\dfrac{a}{b^{2}}\right)^{-2}\left(\dfrac{a}{b^{2}}\right)^{-3}$ **28.** $\dfrac{(-2r)^{4}}{(-2r)^{-2}}$ **29.** $2x^{-3}(x^{5} - 2x^{3})$

30. $xy^{-2}(xy^{2} - 3y^{3})$ **31.** $\dfrac{6a^{-2} + 9a^{2}}{3a^{-2}}$ **32.** $\dfrac{8n^{4} - 4n^{-2}}{2n^{-2}}$

33. **Business** The value of a new car decreases 20% each year. Complete the table. The value $V(t)$ of the car is in dollars and its age t is in years. Give each value to the nearest hundred. Using the values in your table, make a graph to show the relationship between $V(t)$ and t.

t	0	1	2	3	4	5
$V(t)$	10,000	?	?	?	?	?

34. **Business** The value in dollars of a car t years from now is $V(t) = 12{,}500(0.85)^t$. **(a)** What is the annual rate of depreciation, the rate at which the car loses value? **(b)** In how many years will the value of the car be approximately half what it is now?

35. **Consumer Economics** If grocery prices increase 1% per month for a whole year, how much would groceries that cost \$100 at the beginning of the year cost at the end of the year?

36. **Consumer Economics** The cost of goods and services in an urban area increased 1.5% last month. At this rate, what will be the annual rate of increase?

Simplify by using powers of the same base.

37. a. $\dfrac{3^5 \cdot 9^4}{27^4}$ b. $\dfrac{125^{-3} \cdot 25}{5^{-8}}$ c. $\sqrt{\dfrac{8^n \cdot 2^7}{4^{-n}}}$

38. a. $\dfrac{4^9 \cdot 8^{-4}}{16^3}$ b. $\dfrac{3^7 \cdot 9^5}{\sqrt{27^{12}}}$ c. $\sqrt[3]{\dfrac{125^n \cdot 5^{4n}}{25^{-n}}}$

Write as a power of *b*.

39. a. $\dfrac{(b^n)^3}{b^n \cdot b^n}$ b. $\dfrac{(b^n)^2}{b^n \cdot b^{n+2}}$ c. $\dfrac{b \cdot b^n}{(b^3)^n}$

40. a. $\sqrt{\dfrac{b^{2n}}{b^{-2n}}}$ b. $\dfrac{(b \cdot b^n)^2}{(b^2)^n}$ c. $\sqrt{\dfrac{b^{1-n}}{b^{n-1}}}$

Simplify. (*Hint:* In Exercise 41(a), multiply the numerator and the denominator by 2^3.)

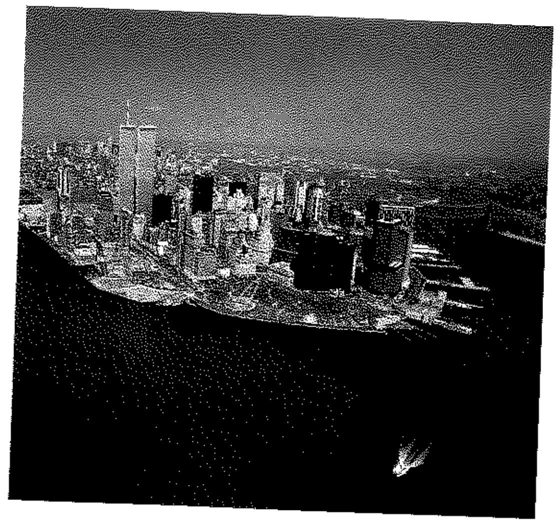
B 41. a. $\dfrac{2^{-1}}{2^{-2} + 2^{-3}}$ b. $\dfrac{4^{-5}}{4^{-2} + 4^{-3}}$

42. a. $\dfrac{3^{-2}}{3^{-3} + 3^{-2}}$ b. $\dfrac{2^{-1} - 2^{-2}}{2^{-1} + 2^{-2}}$

43. a. $\dfrac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}}$ b. $\dfrac{1 - y^{-1}}{y - y^{-1}}$

44. a. $\dfrac{x^{-1}}{x - x^{-1}}$ b. $\dfrac{4 - x^{-4}}{2 - x^{-2}}$

45. ***Discussion*** According to legend Manhattan Island was purchased in 1626 for trinkets worth about \$24. If the \$24 had been invested instead at a rate of 6% interest per year, what would be the value of the money in 1996? Compare this with a recent total of \$34,000,000,000 in assessed values for Manhattan Island.



C 46. Solve $2^x + 8 \cdot 2^{-x} = 9$.

47. Solve $2^x + 2^{-x} = \frac{5}{2}$.

48. Solve $2^{2x} - 3 \cdot 2^{x+1} + 8 = 0$.

49. Solve $3^{2x+1} - 10 \cdot 3^x + 3 = 0$.

5-2 Growth and Decay: Rational Exponents

Objective *To define and apply rational exponents.*

In Section 5-1, we considered a 9% annual growth in the cost of a hamburger. We saw that if a hamburger costs \$4 now (time $t = 0$), then its cost t years from now will be $C(t) = 4(1.09)^t$. To find the cost one-half year from now, we must evaluate an expression with a fractional exponent:

$$C\left(\frac{1}{2}\right) = 4(1.09)^{1/2}$$

Although a scientific calculator easily gives an approximate value of \$4.18, there remains the question of what a fractional exponent means.

The definition of a rational exponent given below is made in such a way that the laws for integral exponents on page 170 will continue to hold. However, the base b must now be a positive real number other than 1.

Definition of $b^{1/2}$: If exponent law 7 is to hold for rational exponents, then

$$(b^{1/2})^2 = b^{(1/2)(2)} = b^1.$$

Since we know $(\sqrt{b})^2 = b$, we define $b^{1/2}$ to be $\sqrt{b}$.

Definition of $b^{3/2}$: $b^{3/2} = (b^{1/2})^3 = (\sqrt{b})^3$ and $b^{3/2} = (b^3)^{1/2} = \sqrt{b^3}$

Either $(\sqrt{b})^3$ or $\sqrt{b^3}$ can be used as a definition of $b^{3/2}$.

Definition of $b^{p/q}$: Using reasoning similar to that above, we make these definitions:

$$b^{1/q} = \sqrt[q]{b} \quad \text{and} \quad b^{p/q} = (\sqrt[q]{b})^p \text{ or } \sqrt[q]{b^p}$$

Example 1 Simplify: **a.** $16^{1/4}$ **b.** $16^{-1/4}$ **c.** $8^{2/3}$ **d.** $8^{-2/3}$

Solution **a.** $16^{1/4} = \sqrt[4]{16} = 2$ **b.** $16^{-1/4} = (16^{1/4})^{-1} = 2^{-1} = \frac{1}{2}$

c. $8^{2/3} = (\sqrt[3]{8})^2 = 4$ **d.** $8^{-2/3} = (8^{2/3})^{-1} = 4^{-1} = \frac{1}{4}$

You can use a calculator to evaluate expressions that contain rational powers. For example, $5^{2/3} \approx 2.9240$.

Example 2 Suppose a car presently worth $8200 depreciates 20% per year. About how much will it be worth 2 years and 3 months from now?

Solution Express 2 years and 3 months as 2.25 years. Then:

$$C(t) = 8200(1 - 0.20)^t = 8200(0.80)^t$$
$$C(2.25) = 8200(0.80)^{2.25} \approx 4963$$

It will be worth about $4963.

Examples 3 and 4 show how the laws of exponents can be used to solve two kinds of equations containing exponents.

Example 3 Solve **(a)** $2^x = \frac{1}{8}$ and **(b)** $9^{x+1} = \sqrt{27}$.

Solution Express both sides of each equation as powers of the same base. Then apply law 3.

a.
$$2^x = \frac{1}{8}$$
$$2^x = 2^{-3}$$
$$x = -3$$

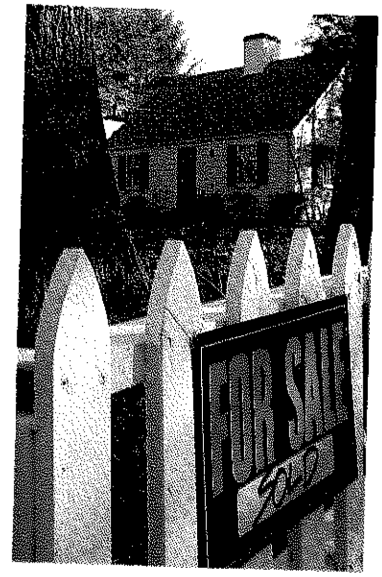
b.
$$9^{x+1} = \sqrt{27}$$
$$(3^2)^{x+1} = \sqrt{3^3}$$
$$3^{2x+2} = 3^{3/2}$$
$$2x + 2 = \frac{3}{2}$$
$$x = -\frac{1}{4}$$

Example 4 Solve **(a)** $4x^{3/2} = 32$ and **(b)** $(x - 1)^{-1/4} - 2 = 0$.

Solution **a.**
$$4x^{3/2} = 32$$
$$x^{3/2} = 8$$
$$(x^{3/2})^{2/3} = 8^{2/3}$$
$$x = 4$$

b.
$$(x - 1)^{-1/4} - 2 = 0$$
$$(x - 1)^{-1/4} = 2$$
$$((x - 1)^{-1/4})^{-4} = 2^{-4}$$
$$x - 1 = \frac{1}{16}$$
$$x = \frac{17}{16}$$

Example 5 A house bought five years ago for \$100,000 was just sold for \$135,000. To the nearest tenth of a percent what was the annual growth rate?



Solution

$$A(t) = A_0(1 + r)^t$$

Since $A_0 = 100{,}000$
and $A(5) = 135{,}000$,

$$135{,}000 = 100{,}000(1 + r)^5.$$

Therefore:

$$1.35 = (1 + r)^5$$
$$(1.35)^{1/5} = (1 + r)$$
$$r \approx 0.0619$$

To the nearest tenth of a percent, the growth rate was 6.2%.

CLASS EXERCISES

Simplify each expression.

1. a. $4^{1/2}$ **b.** $4^{-1/2}$ **2. a.** $4^{3/2}$ **b.** $4^{-3/2}$

3. a. $-9^{1/2}$ **b.** $-9^{-1/2}$ **4. a.** $(3^{1/2} \cdot 5^{1/2})^2$ **b.** $(3^{1/2} + 5^{1/2})^2$

5. $\left(\frac{49}{25}\right)^{-1/2}$ **6.** $\left(\frac{4}{9}\right)^{3/2}$ **7.** $(8^{-1/6})^{-2}$ **8.** $8^{3/2} \cdot 2^{3/2}$

9. $(2x^{-1/3})^3$ **10.** $\left(\frac{125}{x^6}\right)^{1/3}$ **11.** $\frac{x^{1/3}}{2x^{-2/3}}$ **12.** $2x^{3/2} \cdot 4x^{-1/2}$

13. **Consumer Economics** The cost in dollars of a new pair of running shoes t years from now is $C(t) = 62(1.05)^t$.

a. What is the cost now?

b. To find the cost in 2.5 years, use $t =$ __?__.

c. To find the cost 9 months ago, use $t =$ __?__.

Solve.

14. $3^{2x} = 3^{12}$ **15.** $9^x = 3^5$

16. $x^{2/3} = 9$ **17.** $x^{-1/2} = 4$

18. ***Discussion*** If the exponent is an integer n, then the laws of exponents are true for both positive and negative bases. When the definition of exponent is extended to rational numbers, the base must be restricted to positive numbers. Discuss why this restriction must be made.

WRITTEN EXERCISES

Write each expression using a radical sign and no negative exponents.

A **1. a.** $x^{2/3}$ **b.** $x^{3/2}$ **c.** $5^{1/2} \cdot x^{-1/2}$ **d.** $6^{1/3} \cdot x^{2/3}$

2. a. $3y^{2/5}$ **b.** $(3y)^{2/5}$ **c.** $a^{4/7} \cdot b^{-4/7}$ **d.** $a^{1/10} \cdot b^{-1/5}$

Write each expression using positive rational exponents.

3. a. $\sqrt{x^5}$ **b.** $\sqrt[3]{y^2}$ **c.** $(\sqrt[6]{2a})^5$ **d.** $\sqrt{x} \cdot \sqrt[3]{x} \cdot \sqrt[6]{x}$

4. a. $\sqrt[3]{8x^7}$ **b.** $(\sqrt[4]{16x})^3$ **c.** $\sqrt[3]{27x^{-6}y^2}$ **d.** $\sqrt[4]{x} \cdot \sqrt[3]{x} \div \sqrt[6]{x}$

Simplify.

5. a. $\left(\frac{9}{25}\right)^{1/2}$ **b.** $\left(\frac{9}{25}\right)^{-1/2}$ **c.** $\left(\frac{9}{25}\right)^{5/2}$ **d.** $\left(\frac{9}{25}\right)^{-1.5}$

6. a. $\left(\frac{27}{8}\right)^{1/3}$ **b.** $\left(\frac{27}{8}\right)^{2/3}$ **c.** $\left(\frac{27}{8}\right)^{-2/3}$ **d.** $\left(\frac{27}{8}\right)^{0}$

7. $(16^{-3/5})^{5/4}$ **8.** $(25^{-1/3})^{-3/2}$ **9.** $(81^{1/2} - 9^{1/2})^2$ **10.** $(3^{-2} + 4^{-2})^{-1/2}$

11. $(8a^{-6})^{-2/3}$ **12.** $(9n^{-5})^{-3/2}$ **13.** $(4x^{-3})^{-1/2} \cdot 4x^{1/2}$ **14.** $(4a^3)^{1/3} \div (4a^3)^{-2/3}$

15. **Consumer Economics** The cost of a certain brand of camera has been increasing at 8% per year. If a camera now costs $150, find the cost:
a. 2 years and 6 months from now **b.** 4 years and 3 months ago

16. **Business** The value of a computer depreciates at the rate of 25% per year. If a computer is now worth $2400, find its approximate value:
a. 3 years and 6 months from now **b.** 20 months ago

Simplify.

17. $a^{1/2}(a^{3/2} - 2a^{1/2})$ **18.** $2n^{1/3}(n^{2/3} + n^{-1/3})$ **19.** $x^{-1/2}(x^{5/2} - 2x^{3/2})$

20. $2n^{-2/3}(n^{8/3} - 3n^{5/3})$ **21.** $\dfrac{x^{1/2} - 2x^{-1/2}}{x^{-1/2}}$ **22.** $\dfrac{y^{-1/3} - 3y^{2/3}}{y^{-4/3}}$

23. $\dfrac{2n^{1/3} - 4n^{-2/3}}{2n^{-2/3}}$ **24.** $\dfrac{x^{-1/2}(2x^{1/2} - x^{-1/2})}{x^{-1}}$ **25.** $\dfrac{2n^{1/3}(3n^{1/3} - 4n^{4/3})}{2n^{-1/3}}$

26. $\dfrac{4ab^{-1/2} - 2ab^{1/2}}{(a^2b)^{-1/2}}$ **27.** $\dfrac{(\sqrt[3]{4a})^2}{\sqrt[6]{4a}}$ **28.** $\dfrac{(\sqrt{2x})^5}{(\sqrt{2x})^9}$

Solve.

29. $8^x = 2^6$ **30.** $9^{4x} = 81$ **31.** $8^{x-1} = 2^{x+1}$

32. $9^x = 3^{10}$ **33.** $8^x = 2^7 \cdot 4^9$ **34.** $27^{1-x} = \left(\frac{1}{9}\right)^{2-x}$

35. a. $(8x)^{-3} = 64$ **b.** $8x^{-3} = 64$ **c.** $(8 + x)^{-3} = 64$

36. a. $(2x)^{-2} = 16$ **b.** $2x^{-2} = 16$ **c.** $4(x - 2)^{-2} = 16$

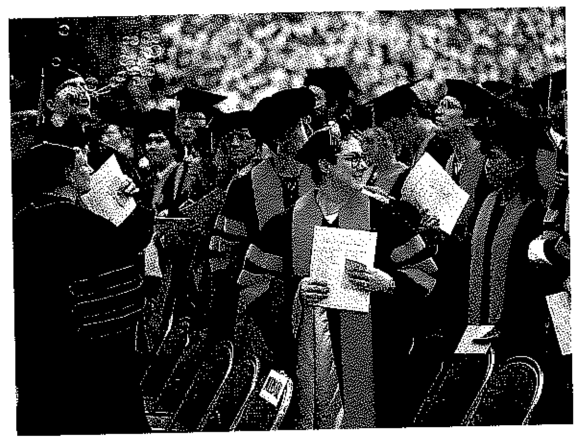
B 37. **Finance** A house bought for \$50,000 in 1980 was sold for \$150,000 in 1990. To the nearest percent, what was the annual rate of appreciation (increase) in the value of the house?

38. **Consumer Economics** The price of firewood four years ago was \$140 per cord. Today, a cord of wood costs \$182. To the nearest percent, what has been the annual rate of increase in the cost?

39. **Economics** The *consumer price index* (CPI) is a measure of the average cost of goods and services. The United States government set the index at 100 for the period 1982-1984. In 1988, the index was 118.3. What was the average annual rate of increase (to the nearest tenth of a percent) from 1984 to 1988?

40. ***Research*** Look in an almanac to find the current consumer price index. Then determine the average annual rate of increase (to the nearest tenth of a percent) for this index since 1984. (See Exercise 39.)

41. **Education** Yearly expenses at a state university have increased from \$14,000 to \$18,500 in the last 4 years. What has been the average annual growth rate in expenses? If this growth rate continues, what will the expenses be 4 years from now?



42. ***Research*** Find the present cost of something that interests you and its cost several years ago. Find the average annual growth rate.

Solve.

43. **a.** $(4x)^3 = 9^6$ **b.** $3^{4x} = 9^6$

44. **a.** $(4 - x)^{1/2} = 8$ **b.** $\left(\frac{1}{2}\right)^{4-x} = 8$

45. $\dfrac{2^{x^2}}{2^x} = 64$

46. $\dfrac{5^{x^2}}{(5^x)^2} = 125$

47. $\sqrt{\dfrac{9^{x+3}}{27^x}} = 81$

48. $\sqrt[3]{\dfrac{8^{x+1}}{16^x}} = 32$

Factor. (In Exercise 49(a), factor out $a^{1/2}b^{1/2}$.)

49. **a.** $a^{3/2}b^{1/2} - a^{1/2}b^{3/2}$ **b.** $a^{1/2}b^{-1/2} - a^{3/2}b^{1/2}$

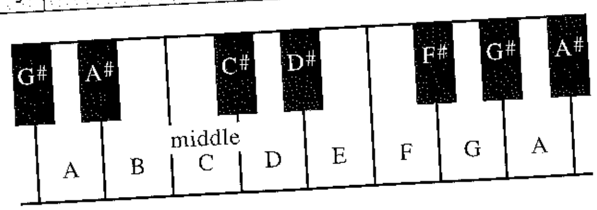
50. **a.** $(x - 1)^{1/2} - x(x - 1)^{-1/2}$ **b.** $(x + 1)^{3/2} - 4(x + 1)^{1/2}$

51. **a.** $(x^2 + 1)^{3/2} - x^2(x^2 + 1)^{1/2}$ **b.** $(x^2 + 2)^{1/2} - x^2(x^2 + 2)^{-1/2}$

52. **a.** $(2x + 1)^{2/3} - 4(2x + 1)^{-1/3}$ **b.** $(1 + x^2)^{-3/2} - (1 + x^2)^{-1/2}$

If you play the A below middle C on a piano, the piano string vibrates with a frequency of 220 vibrations per second, or 220 hertz (Hz). Frequencies for some notes above A are as follows:

Note	A# (A sharp)	B	Middle C	C#
Frequency	$220 \cdot 2^{1/12}$	$220 \cdot 2^{2/12}$	$220 \cdot 2^{3/12}$	$220 \cdot 2^{4/12}$



53. **Music** Give the frequencies of the notes D through A above middle C.

54. **a.** **Music** The frequency of G# below middle C is $220 \cdot 2^{-1/12}$ Hz and the frequency of G# above middle C is $220 \cdot 2^{11/12}$ Hz. Find the ratio of these frequencies. (*Note*: The interval between these two notes is called an "octave.")

    b. If a song is played in the key of C, the interval from C to the G above is called a "fifth." Show that the ratio of these two frequencies is approximately 3 to 2.

Solve for x by rewriting the equation in quadratic form.

55. $x^{2/3} - 7x^{1/3} + 12 = 0$ (*Hint:* $x^{2/3} = (x^{1/3})^2$)
56. $x^{4/3} - 6x^{2/3} + 8 = 0$
57. $9^{2x} - 2 \cdot 9^x - 3 = 0$
58. $4^{2x} - 10 \cdot 4^x + 16 = 0$

5-3 Exponential Functions

Objective *To define and use exponential functions.*

We have defined rational exponents, but not irrational ones. For example, how is $3^{\sqrt{2}}$ defined? To find out, complete the following activity.

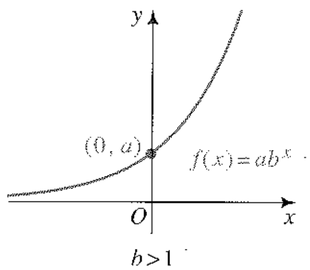
Activity

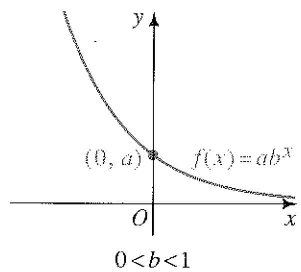
The first row in the table shows a sequence of decimal approximations for $\sqrt{2}$. Copy the table. Complete it by using the power key on a calculator. (Write all the digits displayed.) What do you notice about 3^x?

x	1.4	1.41	1.414	1.4142	1.41421
3^x	?	?	?	?	?

As you discovered, the sequence $3^{1.4}, 3^{1.41}, 3^{1.414}, \ldots$ seems to approach a fixed number. It can be proved that if any sequence has $\sqrt{2}$ as its limit, then the corresponding sequence of powers of 3 also has a limit, defined to be $3^{\sqrt{2}}$.

Any function of the form $f(x) = ab^x$, where $a > 0$, $b > 0$, and $b \neq 1$ is called an **exponential function with base b**. Its domain is the set of all real numbers. The range is the set of positive real numbers. See the graphs below.





Example 1 If f is an exponential function, $f(0) = 3$, and $f(2) = 12$, find $f(-2)$.

Solution Since f is an exponential function, $f(x) = ab^x$ for some constants a and b. We use the given values of f to find a and b.

Since $f(0) = 3$, then $ab^0 = 3$. So $a = 3$.

Since $a = 3$ and $f(2) = 12$, then $3b^2 = 12$. So $b = 2$.

Therefore, $f(x) = 3 \cdot 2^x$. Thus, $f(-2) = 3 \cdot 2^{-2} = \frac{3}{4}$.

When exponential functions are used to describe exponential growth and decay, the variable t is often used to represent time. Although these functions can always be written as $f(t) = ab^t$, they are often written in other forms such as the two listed below.

(1) $A(t) = A_0(1 + r)^t$ $\quad A_0$ = amount at time $t = 0$ and r = growth rate

(2) $A(t) = A_0 b^{t/k}$ $\quad k$ = time needed to multiply A_0 by b

Example 2 A bank advertises that if you open a savings account, you can double your money in 12 years. Express $A(t)$, the amount of money after t years, in each of the two forms listed above.

Solution Since 12 years is the time needed to multiply A_0 by 2, form (2) gives:

$$A(t) = A_0 \cdot 2^{t/12}$$

Notice that $A(12) = A_0 \cdot 2^{12/12} = 2A_0$.

To express $A(t)$ in form (1), reason as follows:

$$A(t) = A_0 \cdot 2^{t/12} = A_0 \cdot (2^{1/12})^t$$
$$\approx A_0(1.059)^t = A_0(1 + 0.059)^t$$

The so-called rule of 72 stated below provides an estimate of the time it takes for a quantity to double.

The Rule of 72

If a quantity is growing at $r\%$ per year (or month), then the **doubling time** is approximately $(72 \div r)$ years (or months).

For example, if a quantity grows at 8% per month, then its doubling time will be about $72 \div 8 = 9$ months. If a population grows exponentially at 2% per year, then it will double in about $72 \div 2 = 36$ years.

Example 3 A radioactive isotope has a **half-life** of 5 days. This means that half the substance decays in 5 days. At what rate does the substance decay each day?

Solution We use the form: $A(t) = A_0 b^{t/k} = A_0\left(\frac{1}{2}\right)^{t/5}$

(Notice that $A(5) = A_0 \cdot \frac{1}{2}$, which agrees with the half-life being 5 days.)

To find the daily decay rate, we rewrite $A(t)$ as follows.

$$A(t) = A_0\left(\left(\frac{1}{2}\right)^{1/5}\right)^t \approx A_0(0.87)^t = A_0(1 - 0.13)^t$$

Thus, the daily decay rate is approximately 13%.

CLASS EXERCISES

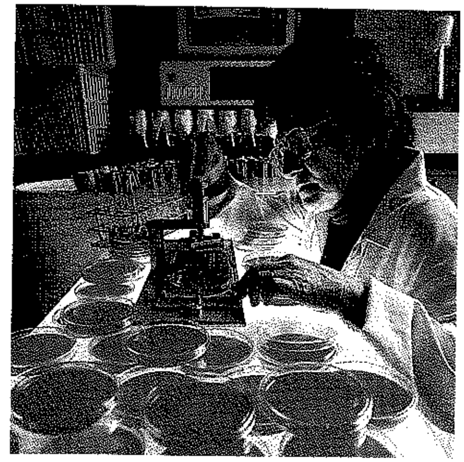
1. The graph of $y = ab^x$ has y-intercept 7. Find the value of a.
2. If $h(x) = ab^x$, $h(0) = 5$, and $h(1) = 15$, find the values of a and b.
3. For what values of b is $g(x) = b^x$ an increasing function? a decreasing function? (See the definitions of increasing and decreasing on page 138.)
4. State the domain and range of $f(x) = 3 \cdot 4^x$.
5. Give an integer estimate for 5^π. Find 5^π on a calculator. What is the difference between your estimate and the calculator value?
6. Finance According to the rule of 72, how long does it take to double an investment that grows at the rate of 9% per year? 6% per month?
7. Finance Suppose that today you invest some money that grows to the amount $A(t) = 1000 \cdot 2^{t/10}$ in t years.
   a. How much money did you invest?
   b. How long does it take to double your money?
8. Biology Suppose that t hours from now the population of a bacteria colony is given by $P(t) = 90(100)^{t/8}$.
   a. What is the population when $t = 0$?
   b. How long does it take for the population to be multiplied by 100?

9. **Biology** A colony of bacteria decays so that the population t days from now is given by $A(t) = 1000\left(\frac{1}{2}\right)^{t/4}$.

   a. What is the amount present when $t = 0$?

   b. How much will be present in 4 days?

   c. What is the half-life?



10. ***Discussion*** Can the data below be described as exponential growth?

x	0	1	2	3	4
y	4	4.4	4.8	5.3	5.8

Justify your answer by using a graph, an equation, or a logical argument. Can you think of a situation that the data might describe?

WRITTEN EXERCISES

Evaluate each expression with a calculator.

A 1. 6^{π} and π^{6}

2. $3.6^{\sqrt{2}}$ and $\sqrt{2}^{3.6}$

Find an exponential function having the given values.

3. $f(0) = 3, f(1) = 15$
4. $f(0) = 5, f(3) = 40$
5. $f(0) = 64, f(2) = 4$
6. $f(0) = 80, f(4) = 5$
7. **Physics** The half-life of a radioactive isotope is 4 days. If 3.2 kg are present now, how much will be present after:

   a. 4 days? **b.** 8 days? **c.** 20 days? **d.** t days?

8. **Physics** The half-life of radium is about 1600 years. If 1 kg is present now, how much will be present after:

   a. 3200 years? **b.** 16,000 years? **c.** 800 years? **d.** t years?

9. **Physics** The table shows the amount $A(t)$ in grams of a radioactive element present after t days. Suppose that $A(t)$ decays exponentially.

t(days)	0	2	4	6	8	10
$A(t)$	320	226	160	115	80	57

   a. What is the half-life of the element?

   b. About how much will be present after 16 days?

   c. Find an equation for $A(t)$.

10. **Geography** The table shows the population $P(t)$ (in thousands) for a small mythical nation at various times.

t (year)	1825	1850	1875	1900	1925	1950	1975
$P(t)$	200	252	318	401	504	635	800

a. Does it appear that this population is growing exponentially?
b. About how long does it take for the population to double?
c. Find an equation for $P(t)$. (*Hint:* The exponent contains $t - 1825$.)

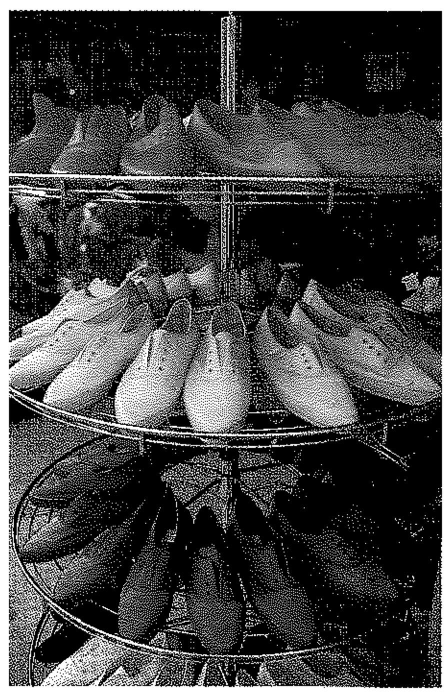
11. **Business** The value of a car t years from now is given by $V(t) = 4000(0.85)^t$.
a. What is the annual rate of depreciation?
b. In how many years will the value of the car be about half what it is now?

12. **a.** **Geography** Suppose the population of a nation grows at 3% per year. If the population was 30,000,000 people in 1990, what will be the population, to the nearest million, in the year 2000?
b. According to the rule of 72, how long does it take for the population to double?

13. **a.** **Finance** If \$1000 is invested so that it grows at the rate of 10% per year, what will the investment be worth in 20 years?
b. According to the rule of 72, in approximately how many years will the investment double in value?

14. **Biology** A bacteria colony triples every 4 days. The population is P_0 bacteria. What will the population $P(t)$ be t days later?

15. **Consumer Economics** If the price of sneakers increases 6% per year, about how long will it take for the price to double?



16. **Medicine** When a certain medicine enters the blood stream, it gradually dilutes, decreasing exponentially with a half-life of 3 days. The initial amount of the medicine in the blood stream is A_0 milliliters. What will the amount be 30 days later?

17. **Medicine** An amount A_0 of radioactive iodine has a half-life of 8.1 days. In terms of A_0, how much is present after 5 days? (Radioactive iodine is used to evaluate the health of the thyroid gland.)

18. **a.** Let $f(x) = 2^x$. Complete the table.

x	-2	-1	0	1	2
$f(x)$	?	?	?	?	?

b. Graph the function by plotting points.

c. Explain why f has an inverse.
d. Graph f^{-1} on the same set of axes as f.
e. State the domain and range of f and of f^{-1}.

You may find it helpful to have a graphing calculator to complete Exercises 19–21. You need a graphing calculator for Exercise 22.

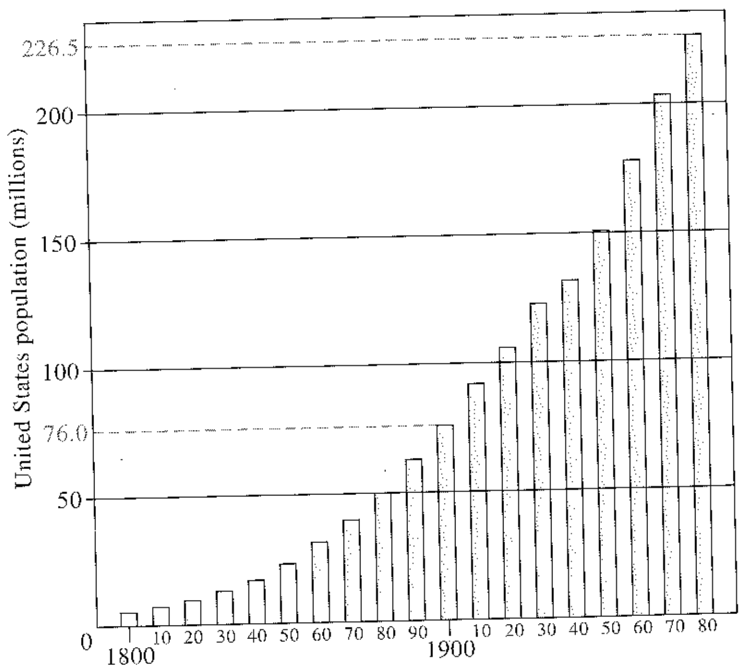
B 19. a. Graph the functions $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ on a single set of axes.
b. How are the graphs related to each other?

20. a. Graph the functions $y = 4^x$ and $y = 4^{x/2}$ on a single set of axes.
b. How are the graphs related to each other?

21. a. Graph the functions $y = 2^x$ and $y = 2^{x-1}$ on a single set of axes.
b. How are these graphs related to each other?

22. a. Graph the functions $y = x^2$ and $y = 2^x$ on a single set of axes.
b. Use a graphing calculator to solve $x^2 = 2^x$.

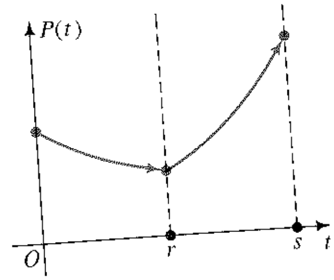
23. a. **Geography** The bar graph below gives the population of the United States for each census from 1800 to 1980. The tops of the bars lie approximately on the curve $y = ab^x$, where x is the number of years since 1800. Find the values of a and b to the nearest thousandth.
b. Use part (a) to predict the population in the years 2000 and 2050.



24. a. If $f(x) = 2^x$, show that $f(x + 1) = 2f(x)$.
b. Translating the graph of f 1 unit (left or right?) is equivalent to (vertically or horizontally?) stretching the graph of f by _?_ units.

25. *Investigation* Go to a bank in your area.
   a. What kinds of savings accounts does the bank offer? List the characteristics of each type of account. Write down the interest rate, any conditions placed on opening the account, any penalties placed on withdrawals, and so forth.
   b. Suppose that you have $1000 to invest. Which of the plans available would you choose?
   c. Make a table showing how the money in the account would grow.
   d. Discuss your investigation with the class.

26. *Visual Thinking* In the diagram, the red and blue curves are exponential in behavior.



   a. Describe the behavior of $P(t)$ on the interval $0 \leq t \leq r$.
   b. Describe the behavior of $P(t)$ on the interval $r < t \leq s$.
   c. Describe a real-world situation for which the diagram could be a model.

27. If $f(x) = 10^{2x+1}$ and x is a nonnegative integer, show that $f(x + 2) - f(x)$ is divisible by 99.

28. Over the past 60 years, the population of a country increased from 100 million people to 200 million people.
   a. Find an equation that gives the population t years from now.
   b. Use a calculator and trial and error to find approximately when the population will reach 300 million people.

C 29. Suppose that $f(x)$ is positive for all real x and
$$f(x + y) = f(x) \cdot f(y)$$
for all x and y. Prove that $f(0) = 1$, $f(2x) = [f(x)]^2$, and $f(3x) = [f(x)]^3$. Give an equation for a function that has the property that $f(x + y) = f(x) \cdot f(y)$ for all real x and y.

5-4 The Number e and the Function e^x

Objective *To define and apply the natural exponential function.*

You have already seen many exponential functions. In advanced mathematics, the most important base is the irrational number e, defined as
$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

This is read "the limit of $\left(1 + \frac{1}{n}\right)^n$ as n approaches infinity." Although you will not study limits until Chapter 13, you can get an idea of the value of e by completing the Activity on the next page.

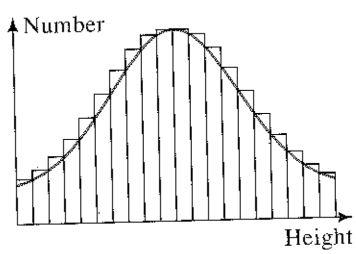
Activity

1. Copy and complete the table by using a calculator. What do you notice about the successive values of $\left(1 + \frac{1}{n}\right)^n$?

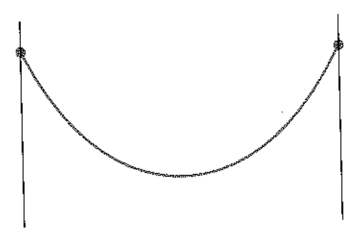
2. Using a graphing calculator, graph $y = \left(1 + \frac{1}{x}\right)^x$. What happens to y when x becomes large?

n	$\left(1 + \frac{1}{n}\right)^n$
10	?
100	?
1000	?
10,000	?
100,000	?

You can see that as n increases, $\left(1 + \frac{1}{n}\right)^n$ appears to get closer and closer to 2.718 . . . . The Swiss mathematician Leonhard Euler (1707–1783) proved this, and the limit is called e in his honor. Values of e^x can be obtained using a calculator or table (page 821). The function e^x is called the **natural exponential function**. In Exercise 13, you will be asked to draw its graph and the graph of its inverse. The number e is extremely important in advanced mathematics. It appears in unexpected places including statistics and physics.



Statistics: Distribution of heights of 18-year-olds



Physics: Hanging rope

The equations of these two graphs contain e^x.

Compound Interest and the Number *e*

Suppose you invest P dollars (the *principal*) at 12% interest compounded semiannually. Each half year, your money grows by 6%. At the end of the year, you have $P(1.06)^2 = 1.1236P$ dollars. If the interest had been compounded quarterly (4 times per year), then it would grow by 3% each quarter. At the end of the year, you would have $P(1.03)^4 = 1.1255P$ dollars. See the table at the top of the next page. The table also shows the results when the 12% annual interest rate is compounded monthly and daily. For simplicity, the investment is \$1.

Interest period	% growth each period	Growth factor during period	Amount
Annually	12%	$1 + \frac{0.12}{1}$	$1.12^1 = 1.12$
Semiannually	6%	$1 + \frac{0.12}{2}$	$1.06^2 = 1.1236$
Quarterly	3%	$1 + \frac{0.12}{4}$	$1.03^4 \approx 1.1255$
Monthly	1%	$1 + \frac{0.12}{12}$	$1.01^{12} \approx 1.1268$
Daily (360 days)	$\frac{12}{360}\%$	$1 + \frac{0.12}{360}$	$\left(1 + \frac{0.12}{360}\right)^{360} \approx 1.1275$
k times per year	$\frac{12}{k}\%$	$1 + \frac{0.12}{k}$	$\left(1 + \frac{0.12}{k}\right)^k$

The numbers in the right-hand column of the table suggest that the amount approaches a fixed value. To find this number, note that

$$\left(1 + \frac{0.12}{k}\right)^k = \left(1 + \frac{1}{\frac{k}{0.12}}\right)^k = \left[\left(1 + \frac{1}{\frac{k}{0.12}}\right)^{k/0.12}\right]^{0.12} = \left[\left(1 + \frac{1}{n}\right)^n\right]^{0.12}$$

where $n = \frac{k}{0.12}$. Since the limit of $\left(1 + \frac{1}{n}\right)^n$ as n increases is e, the limit of the expression above is $e^{0.12}$, approximately 1.1275. Since \$1 grows to \$1.1275 in one year, a bank that compounds daily will advertise that its 12% annual interest is equivalent to a 12.75% *effective annual yield.*

In general, if you invest P dollars at an annual rate r (expressed as a decimal) compounded continuously, then t years later your investment will be worth Pe^{rt} dollars. (See Written Exercises 17 and 18.)

The same principle applies to any quantity, such as population, where compounding takes place "all the time." If P_0 is the initial amount, then the amount at any future time t is $P(t) = P_0e^{rt}$.

CLASS EXERCISES

Use a calculator to evaluate the following.

1. e^2 **2.** $e^{3.2}$ **3.** e^{-4} **4.** $e^{\sqrt{2}}$ **5.** e^1

6. Which is larger, e^π or π^e? Try to estimate before using your calculator.

7. a. ***Reading*** From your reading, how is e defined?

b. Estimate the value of $\left(1 + \frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$.

8. Finance If money is invested at 8% compounded semiannually, then each year the investment is multiplied by 1.04^2. What is the investment multiplied by if interest is compounded:
   a. quarterly? b. 12 times a year? c. continuously?

9. Finance A bank advertises that its 5% annual interest rate compounded daily is equivalent to a 5.13% effective annual yield. What does this mean?

WRITTEN EXERCISES

A 1. a. Evaluate $\left(1 + \frac{1}{n}\right)^n$ when $n = 5000$, and $n = 5{,}000{,}000$.
   b. Compare your answers in part (a) with an approximation for e.

2. a. Evaluate $\left(1 - \frac{1}{n}\right)^n$ for $n = 100$, $n = 10{,}000$, and $n = 1{,}000{,}000$.
   b. Compare your answers in part (a) with an approximation for e^{-1}.
   c. What appears to be $\lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$?

3. Which is larger, $e^{\sqrt{2}}$ or $\sqrt{2}^e$? Try to estimate before using your calculator.

4. Evaluate: a. $e^{0.08}$ b. $e^{-0.08}$ c. $e^{4/3}$

5. Finance Suppose you invest \$1.00 at 6% annual interest. Calculate the amount that you would have after one year if the interest is compounded (a) quarterly, (b) monthly, (c) continuously.

6. Finance Repeat Exercise 5 if the annual rate is 8%.

7. Finance One hundred dollars deposited in a bank that compounds interest quarterly yields \$107.50 over 1 year. Find the effective annual yield.

8. Finance After a year during which interest is compounded quarterly, an investment of \$800 is worth \$851. What is the effective annual yield?

9. With which plan would an investor earn more, Plan A or B?
   Plan A: A 6% annual rate compounded annually over a 10-year period
   Plan B: A 5.5% annual rate compounded quarterly over a 10-year period

10. With which plan would an investor earn more, Plan A or B?
   Plan A: An 8% annual rate compounded quarterly for 5 years
   Plan B: A 7.5% annual rate compounded daily for 5 years

11. Finance Suppose that \$1000 is invested at 7% interest compounded continuously. How much money would be in the bank after 5 years?

12. Biology A population of ladybugs rapidly multiplies so that the population t days from now is given by $A(t) = 3000e^{0.01t}$.
   a. How many ladybugs are present now?
   b. How many will there be after a week?

You might find it helpful to have a graphing calculator to complete Exercises 13–16.

13. a. Use a calculator to sketch the graph of $f(x) = e^x$.
b. Explain why this function has an inverse.
c. Without using a calculator, graph the inverse.

14. On a single set of axes, graph $y = e^x$ and $y = e^{-x}$. How are the graphs related?

15. Graph $y = \frac{e^x + e^{-x}}{2}$. (See the hanging rope on page 187.)

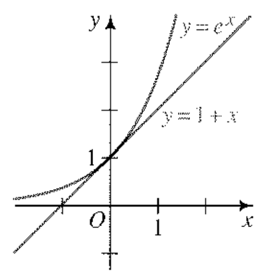
16. Sketch the graph of $y = e^{-x^2}$. (See the "bell-shaped curve" on page 187.)

B **17. a.** Finance Suppose that P dollars is invested at an annual interest rate r (r a decimal) with interest compounded k times a year. Explain why the value of the investment at the end of the year is $P\left(1 + \frac{r}{k}\right)^k$.
b. Show that the expression in part (a) is approximately Pe^r if k is large.

18. a. Finance Suppose that \$10,000 is invested at an annual rate of 9% and that interest is compounded every second for 365 days. Find the value of this investment at the end of one year by using the expression in part (a) of Exercise 17. (First find k, the number of seconds in one year.)
b. Compare your answer in part (a) with the value of $10{,}000e^{0.09}$.

C **19.** It can be proved that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$.
Approximate e by using the first five terms. (*Note:* $n!$, read "n factorial," denotes $n(n-1)(n-2)\cdots 2 \cdot 1$. For example, $3! = 3 \cdot 2 \cdot 1 = 6$.)

20. The points $A(0, f(0))$ and $B(h, f(h))$ are on the graph of $f(x) = e^x$. Find the slope of $\overline{AB}$ if h is **(a)** 1, **(b)** 0.1, and **(c)** 0.01.



21. It can be proved that the line $y = 1 + x$ is tangent to the graph of $y = e^x$ at (0, 1). Thus, $e^x \approx 1 + x$ when $|x|$ is small. Show that $(1 + x)^{1/x}$ is approximately e.

22. Show that $\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^n = \sqrt{e}$ by using the definition of e.

23. Determine the value of $\lim_{n\to\infty}\left(1 + \frac{2}{n}\right)^n$.
(See Exercise 22.)

//// COMPUTER EXERCISE

Write a program that prints a table giving the value of a \$1000 investment compounded quarterly at 4.5%, 5%, 5.5%, 6%, and 6.5% annual interest rates after each of ten successive years.

Logarithms

5-5 Logarithmic Functions

Objective *To define and apply logarithms.*

Now we will explore *logarithms*, numbers used to measure the severity of earthquakes, the loudness of sounds, and the brightness of stars.

We define the *common logarithm* of an integral power of 10 to be its exponent, as shown in the chart below.

$\times 10 \quad \times 10 \quad \times 10 \quad \times 10$

Number	10^0	10^1	10^2	10^3	10^4	$\cdots$	10^k
Common logarithm	0	1	2	3	4	$\cdots$	k

$+1 \quad +1 \quad +1 \quad +1$

We denote the common logarithm of x by $\log x$. For example,

$$\log 10^3 = 3,\ \log 10^4 = 4,\ \text{and } \log 10^k = k.$$

In general, the **common logarithm** of any positive real number x is defined to be the exponent you get when you write x as a power of 10.

$$\log x = a \text{ if and only if } 10^a = x.$$

For example:

$$\log 6.3 \approx 0.8 \text{ because } 6.3 \approx 10^{0.8}$$

You can verify this by evaluating $10^{0.8}$ on a calculator.

You can also use a scientific calculator to find common logarithms directly. On most calculators, you enter the number whose common logarithm you wish to find and then press the "log" key.

The table above illustrates that common logarithms increase arithmetically when numbers increase exponentially. This fact enables us to make a linear scale to measure quantities that increase exponentially.

Common logarithms are useful in applications involving the perception of sound. Every sound has an intensity level due to the power of the sound wave. In the table on the next page, I_0 represents the intensity of a sound barely audible. The intensity level I of any other sound is measured in terms of I_0. (See column 2 of the table on the next page.) The human ear perceives a sound as soft or loud. The unit for measuring the loudness of a sound is the decibel (dB) and it is related to the intensity of a sound by:

$$\text{decibel level of } I = 10 \log \frac{I}{I_0}$$

(See column 3 of the table below.) For example, a loud stereo set has an intensity level of $10^8 I_0$ and is perceived to have a decibel level of:

$$10 \log \frac{10^8 I_0}{I_0} = 10 \log 10^8$$
$$= 10 \times 8 = 80$$

Sound	I	Decibel level
Barely audible	I_0	0
Whisper	$10I_0$	10
Leaves in a breeze	$10^2 I_0$	20
Soft recorded music	$10^4 I_0$	40
Two-person conversation	$10^6 I_0$	60
Loud stereo set	$10^8 I_0$	80
Subway train	$10^{10} I_0$	100
Jet at takeoff	$10^{12} I_0$	120
Pain in eardrum	$10^{13} I_0$	130

Many people think that when the intensity of a sound is doubled, the decibel level is doubled also. The following example shows that this is not so.

Example 1 Two loud stereos are playing the same music simultaneously at 80 dB each. What is the decibel level of the combined sound? By how many decibels is the decibel level of the two stereos greater than the decibel level of one stereo?

Solution Since one stereo at 80 dB has an intensity $10^8 I_0$, two stereos will have an intensity that is twice that amount.

$$2(10^8 I_0)$$

With a calculator, we find that the decibel level corresponding to the two stereos is

$$10 \log\left(\frac{2 \times 10^8 I_0}{I_0}\right) \approx 83.$$

Since the decibel level of one stereo is 80 dB, there is only about a 3 dB increase in the decibel level when the two stereos are played at the same time.

The decibel scale is an example of a *logarithmic scale*. Such a scale is also used to measure acidity and brightness. (See Exercises 45 and 46.)

Common logarithms have base 10. Logarithms to other bases are sometimes used. The *logarithm to base b* of a positive number x, denoted by $\log_b x$, is defined to be the exponent a that you get when you write x as a power of b. (*Note*: $b > 0$, $b \neq 1$)

$$\log_b x = a \text{ if and only if } x = b^a.$$

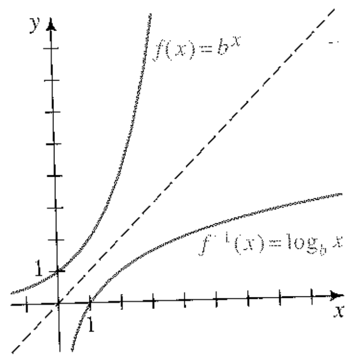
Example 2

$\log_5 25 = 2$	because	$5^2 = 25.$
$\log_5 125 = 3$	because	$5^3 = 125.$
$\log_2 \frac{1}{8} = -3$	because	$2^{-3} = \frac{1}{8}.$

The base b logarithmic function, whose graph is shown in blue below, is the inverse of the base b exponential function, whose graph is shown in red. Notice the domain and range of each function.

Base b exponential function:
$f(x) = b^x$
Domain: All reals
Range: Positive reals

Base b logarithmic function:
$f^{-1}(x) = \log_b x$
Domain: Positive reals
Range: All reals



The most important logarithmic function in advanced mathematics and statistics has the number e as its base. This function is called the *natural logarithm function*. The natural logarithm of x is usually denoted $\ln x$ although sometimes it is written $\log_e x$. The value of $\ln x$ can be found by using a scientific calculator.

$$\ln x = k \quad \text{if and only if} \quad e^k = x.$$

For example,

$$\ln 5 \approx 1.6 \text{ because } e^{1.6} \approx 5.$$

Example 3 Find the value of x to the nearest hundredth.
a. $10^x = 75$ **b.** $e^x = 75$

Solution
a. By definition, x is the common logarithm of 75. Thus, $x = \log 75 \approx 1.88$.
b. By definition, x is the natural logarithm of 75. Thus, $x = \ln 75 \approx 4.32$.

CLASS EXERCISES

1. Use a calculator to approximate each logarithm to four decimal places.
   a. $\log 30$ b. $\log 587$ c. $\log 6800$ d. $\log 0.3$ e. $\log 0.0003$
2. Try to find $\log(-7)$ on a calculator. What happens? Why?
3. Suppose you are told that $\log 8 \approx 0.9031$. Find each of the following without using a calculator. Then check your answers with a calculator.
   a. $\log 80$ b. $\log 800$ c. $\log 80{,}000$ d. $\log 0.8$ e. $\log 0.08$
4. a. The statement $\log_2 16 = 4$ means that $16 = \underline{\ ?\ }$.
   b. The statement $\log 31 \approx 1.49$ means that $31 \approx \underline{\ ?\ }$.
   c. The statement $\log 61 \approx 1.79$ means that $61 \approx \underline{\ ?\ }$.
5. Find each logarithm without using a calculator.
   a. $\log_7 49$ b. $\log_2 16$ c. $\log_2 \frac{1}{8}$ d. $\log_5 \frac{1}{5}$ e. $\log_5 \sqrt{5}$
6. Use a calculator to approximate each logarithm to four decimal places.
   a. $\ln 2$ b. $\ln 3$ c. $\ln 2.7$ d. $\ln 2.8$ e. $\ln e$
7. Find the value of x to the nearest hundredth: **(a)** $10^x = 50$ **(b)** $e^x = 50$

Solve each equation. (Do not use a calculator.)

8. a. $\log_5 x = 2$ b. $\log_6 x = 2$ c. $\log x = 2$ d. $\ln x = 2$
9. a. $\log_x 121 = 2$ b. $\log_x 64 = 3$ c. $\log_x \left(\frac{1}{2}\right) = -1$ d. $\log_x \sqrt{6} = \frac{1}{2}$

WRITTEN EXERCISES

Write each equation in exponential form.

A

1. $\log_4 16 = 2$ 2. $\log_4 64 = 3$ 3. $\log_6 \left(\frac{1}{36}\right) = -2$ 4. $\log_4 8 = 1.5$
5. $\log 1000 = 3$ 6. $\log 40 \approx 1.6$ 7. $\ln 8 \approx 2.1$ 8. $\ln 0.2 \approx -1.6$
9. a. What does it mean to say that x is the common logarithm of N?
   b. Solve **(1)** $10^x = 7$ and **(2)** $10^x = 0.562$ for x to the nearest hundredth.
   c. What does it mean to say that x is the natural logarithm of N?
   d. Solve **(1)** $e^x = 12$ and **(2)** $e^x = 0.06$ for x to the nearest hundredth.
10. Find the value of x to the nearest hundredth: **(a)** $10^x = 170$ **(b)** $e^x = 500$

Find each logarithm. (Do not use a calculator.)

11. a. $\log 100$ b. $\log 10{,}000$ c. $\log 0.01$ d. $\log 0.0001$
12. a. $\log_2 4$ b. $\log_2 32$ c. $\log_2 64$ d. $\log_2 2^{10}$
13. a. $\log_3 9$ b. $\log_3 27$ c. $\log_3 243$ d. $\log_3 3^8$
14. a. $\log_5 0.2$ b. $\log_5 \frac{1}{125}$ c. $\log_5 \sqrt[3]{5}$ d. $\log_5 1$

15. **a.** $\log_4 64$ **b.** $\log_4 \frac{1}{64}$ **c.** $\log_4 \sqrt[4]{4}$ **d.** $\log_4 1$

16. **a.** $\log_6 36$ **b.** $\log_{36} 6$ **c.** $\log_6 6\sqrt{6}$ **d.** $\log_6 \sqrt[3]{\frac{1}{6}}$

17. **a.** $\ln e$ **b.** $\ln e^2$ **c.** $\ln \frac{1}{e}$ **d.** $\ln \sqrt{e}$

18. **a.** $\log 10^8$ **b.** $\log_2 2^8$ **c.** $\log_5 5^8$ **d.** $\ln e^8$

19. Given $\log 4.17 \approx 0.6201$, find: **a.** $\log 417$ **b.** $\log 0.417$ **c.** $\log 0.0417$

20. Given $\log 6.92 \approx 0.8401$, find: **a.** $\log 692$ **b.** $\log 0.692$ **c.** $\log 0.00692$

21. Given $\ln 10 \approx 2.3026$, find: **a.** $\ln 0.1$ **b.** $\ln 0.01$ **c.** $\ln 100$

22. Given $\ln 5 \approx 1.6094$, find: **a.** $\ln 0.2$ **b.** $\ln 25$ **c.** $\ln 0.04$

B 23. **Physics** Find the decibel level for each sound with the given intensity I.
a. Average car at 70 km/h, $I = 10^{6.8}I_0$ **b.** Whisper, $I = 10^{1.5}I_0$

24. **Physics** Find the decibel level for each sound with the given intensity I.
a. Softly played flute, $I = 10^{4.1}I_0$ **b.** Vacuum cleaner, $I = 10^{7.5}I_0$

25. **a.** **Physics** Find the decibel level of two stereos, playing the same music simultaneously at 62 dB.
b. Find the decibel level if three stereos play instead of two.

26. **Physics** The decibel level of one car accelerating from rest to 50 km/h is 80 dB. Find the decibel level of four similar cars accelerating at once.

27. Graph $f(x) = 2^x$ and $f^{-1}(x) = \log_2 x$ on a single set of axes. Give the domain and range of each function.

28. **a.** Find $2^{\log_2 8}$, $5^{\log_5 25}$, and $3^{\log_3 x}$.
b. If $f(x) = 3^x$, what is $f^{-1}(x)$? What is $(f \circ f^{-1})(x)$?

Part (b) of Exercises 29 and 30 requires the use of a computer or a graphing calculator. You may also use a computer or a graphing calculator to confirm your answers to part (b) of Exercises 31 and 32.

29. **a.** Consider the functions $f(x) = e^x$ and $g(x) = \ln x$. Find rules for $(f \circ g)(x)$ and $(g \circ f)(x)$ and give the domain of each composite function.
b. To confirm the domains in part (a), graph the composite functions using a computer or a graphing calculator. Enter the equations of the composite functions in their *unsimplified* form. (See Example 2 on page 127.)

30. Repeat Exercise 29 for the functions $f(x) = e^{-x}$ and $g(x) = -\ln x$.

31. **a.** On separate sets of axes, graph $y = \log |x|$ and $y = |\log x|$.
b. Give the domain and range of each function.

32. Repeat Exercise 31 for the functions $y = -\ln x$ and $y = \ln (-x)$.

33. Give the domain, range, and zeros of $y = \log x + 3$ and $y = \log (x + 3)$.

34. Give the domain, range, and zeros of $y = \log_2 (x - 2)$ and $y = \log_2 x - 2$.

Solve for x without using a calculator. You may leave answers in terms of e if necessary.

35. **a.** $\log x = 3$ **b.** $\log |x| = 3$ **c.** $\log |x - 1| = 3$

36. **a.** $\log_6 x = 2$ **b.** $\ln x = 2$ **c.** $\ln |x| = 2$

37. **a.** $\log_4 x = 1.5$ **b.** $\ln x = 1.5$ **c.** $\ln x = 0$

38. **a.** $\log (x^2 - 1) = 2$ **b.** $\ln (x^2 - 1) = 2$ **c.** $\ln |x| = 1$

39. **a.** $\log_5 (\log_3 x) = 0$ **b.** $\log (\log x) = 1$ **c.** $\ln (\ln x) = 1$

40. **a.** $\log_6 (\log_2 x) = 1$ **b.** $\ln (x - 2) = 1$ **c.** $(\log x)^2 = 4$

Solve for x using a calculator. Give answers to the nearest hundredth.

41. **a.** $\log x = 0.7$ **b.** $\log x = 3.7$ **c.** $\log x = -0.3$

42. **a.** $\log x = 1.4$ **b.** $\log x = 0.4$ **c.** $\log x = -0.6$

43. **a.** $\ln x = 4.2$ **b.** $\ln x = -1.5$ **c.** $e^x = 5$

44. **a.** $\ln x = 1.73$ **b.** $\ln x = -0.52$ **c.** $e^x = 16$

45. **Chemistry** The pH of a solution is a measure of how acidic or alkaline the solution is. The pH of a solution is given by:

$$\text{pH} = -\log_{10} (\text{hydrogen ion concentration})$$

Pure water, which has a pH of 7, is considered neutral. A solution with pH less than 7 is acidic; a solution with pH greater than 7 is alkaline. Find the pH of the following solutions and classify them as acidic, neutral, or alkaline.

Solution	Hydrogen ion concentration (moles per liter)
Human gastric juices	10^{-2}
Acid rain	3×10^{-5}
Pure water	10^{-7}
Good soil for vegetables	5×10^{-7}
Sea water	10^{-8}

46. **Astronomy** The observed brightness of stars is classified by magnitude. Two stars can be compared by giving their magnitude difference d or their brightness ratio r. The numbers d and r are related by the equation $d = 2.5 \log r$. Comparing a first magnitude star with a sixth magnitude star, we have that $d = 6 - 1 = 5$. Find the value of r. What can you say about the relative brightness of the two stars?

47. a. Compare $\log_4 16$ and $\log_{16} 4$.
b. Compare $\log_9 27$ and $\log_{27} 9$.
c. State and prove a generalization based on parts (a) and (b).

48. a. Show that $\log_2 4 + \log_2 8 = \log_2 32$ by finding the three logarithms.
b. Verify that $\log_9 3 + \log_9 27 = \log_9 81$.
c. State and prove a generalization based on parts (a) and (b).

49. a. If $\log y = 1.5x - 2$, show that $y \approx 0.01(31.6)^x$.
b. If $\log y = 0.5x + 1$, express y in terms of x.

50. a. If $\ln y = 4x + 2$, show that $y \approx 7.4(54.6)^x$.
b. If $\ln y = 1 - 0.1x$, express y in terms of x.

C 51. In advanced mathematics, it can be proved that the number of prime numbers less than a positive integer n is approximately $\frac{n}{\ln n}$.
a. About how many primes are less than **(1)** 1000 and **(2)** 1,000,000?
b. There are four prime numbers less than 10. We say that the density of the primes in the interval from 1 to 10 is $\frac{4}{10}$. Use your answers to part (a) to find the approximate density of primes in the intervals from 1 to 1000 and from 1 to 1,000,000.
c. What happens to the density of primes less than n as n increases?

52. Prove that $\log 2$ is irrational. (*Hint:* Assume that $\log 2 = \frac{p}{q}$, where p and q are integers and $\frac{p}{q}$ is in lowest terms.)

5-6 Laws of Logarithms

Objective *To prove and apply laws of logarithms.*

Since the logarithmic function $y = \log_b x$ is the inverse of the exponential function $y = b^x$, it is not surprising that the laws of logarithms are very closely related to the laws of exponents on page 170.

Laws of Logarithms

If M and N are positive real numbers and b is a positive number other than 1, then:

1. $\log_b MN = \log_b M + \log_b N$
2. $\log_b \frac{M}{N} = \log_b M - \log_b N$
3. $\log_b M = \log_b N$ if and only if $M = N$
4. $\log_b M^k = k \log_b M$, for any real number k

To prove law 1, let $\log_b M = x$ and $\log_b N = y$. Then $M = b^x$ and $N = b^y$.

$$MN = b^x \cdot b^y = b^{x+y}$$

Therefore:

$$\log_b MN = x + y$$
$$\log_b MN = \log_b M + \log_b N$$

Laws 2 and 4 are proved in a similar fashion. Law 3 is a restatement, in terms of logarithms, of the third law of exponents on page 170.

If you know the logarithms of M and N, then you can use the laws of logarithms to find the logarithm of a more complicated expression in M and N.

Example 1 Express $\log_b MN^2$ in terms of $\log_b M$ and $\log_b N$.

Solution

$$\begin{aligned} \log_b MN^2 &= \log_b M + \log_b N^2 && \text{(law 1)} \\ &= \log_b M + 2\log_b N && \text{(law 4)} \end{aligned}$$

Example 2 Express $\log_b \sqrt{\frac{M^3}{N}}$ in terms of $\log_b M$ and $\log_b N$.

Solution

$$\begin{aligned} \log_b \sqrt{\frac{M^3}{N}} = \log_b \left(\frac{M^3}{N}\right)^{1/2} &= \frac{1}{2}\log_b \left(\frac{M^3}{N}\right) && \text{(law 4)} \\ &= \frac{1}{2}(\log_b M^3 - \log_b N) && \text{(law 2)} \\ &= \frac{1}{2}(3\log_b M - \log_b N) && \text{(law 4)} \end{aligned}$$

In Examples 1 and 2, the logarithm of an expression was written in terms of separate logarithms. In Examples 3 and 4, separate logarithms are combined into a single logarithm.

Example 3 Simplify $\log 45 - 2\log 3$.

Solution

$$\begin{aligned} \log 45 - 2\log 3 &= \log 45 - \log 3^2 && \text{(law 4)} \\ &= \log \frac{45}{3^2} && \text{(law 2)} \\ &= \log 5 \end{aligned}$$

Example 4 Express y in terms of x if $\ln y = \frac{1}{3}\ln x + \ln 4$.

Solution

$$\begin{aligned} \ln y = \frac{1}{3}\ln x + \ln 4 &= \ln x^{1/3} + \ln 4 && \text{(law 4)} \\ \ln y &= \ln 4x^{1/3} && \text{(law 1)} \\ y &= 4x^{1/3} && \text{(law 3)} \end{aligned}$$

Our final example shows how properties of logarithms can be used to solve certain equations.

Example 5 Solve $\log_2 x + \log_2 (x-2) = 3$.

Solution

$$\log_2 x + \log_2 (x-2) = 3$$
$$\log_2 x(x-2) = 3$$
$$x(x-2) = 8 \longleftarrow 2^3 = 8$$
$$x^2 - 2x - 8 = 0$$
$$(x-4)(x+2) = 0$$
$$x = 4 \text{ or } -2$$

Since $\log x$ is not defined for negative x, -2 is *not* a solution. The only solution is $x = 4$.

CLASS EXERCISES

Express the common logarithm of each of the following in terms of log M and log N.

1. M^2N
2. $\frac{M^2}{N}$
3. $\sqrt{\frac{M}{N}}$
4. $\sqrt[3]{MN}$
5. $M\sqrt{N}$
6. $\frac{M^2}{N^3}$

Use the laws of logarithms to express each of the following as a single logarithm.

7. $\log_5 2 + \log_5 3$
8. $\log_3 5 + \log_3 4$
9. $\log 12 - \log 3$
10. $\log 3 + \log 6 - \log 2$
11. $\ln 4 + 2 \ln 3$
12. $\frac{1}{2}\ln 25 - \ln 2$
13. $\log M + 2 \log N$
14. $2 \log P - \log Q$
15. $\log_b M + \log_b N + \log_b P$
16. $\log_b M + \log_b N - 3 \log_b P$
17. $\frac{1}{2}\ln a - \frac{1}{2}\ln b$
18. $\ln c + \frac{1}{3}\ln d$

In Exercises 19 and 20, give an example to show that in general each statement is false.

19. $\log_b (M + N) = \log_b M + \log_b N$
20. $\log_b \left(\frac{M}{N}\right) = \frac{\log_b M}{\log_b N}$
21. ***Discussion*** What is wrong with the following argument?
Since $\frac{1}{8} < \frac{1}{4}$, $\log \frac{1}{8} < \log \frac{1}{4}$. Therefore, $\log \left(\frac{1}{2}\right)^3 < \log \left(\frac{1}{2}\right)^2$. This means that $3 \log \left(\frac{1}{2}\right) < 2 \log \left(\frac{1}{2}\right)$. Thus, $3 < 2$.

WRITTEN EXERCISES

In Exercises 1–6, write each expression in terms of log M and log N.

A **1.** $\log (MN)^2$ **2.** $\log \frac{M}{N^2}$ **3.** $\log \sqrt[3]{\frac{M}{N}}$ **4.** $\log M\sqrt[4]{N}$ **5.** $\log M^2\sqrt{N}$ **6.** $\log \frac{1}{M}$

Write each expression as a rational number or as a single logarithm.

7. $\log 2 + \log 3 + \log 4$

8. $\log 8 + \log 5 - \log 4$

9. $\frac{1}{2}\log_6 9 + \log_6 5$

10. $\log_2 48 - \frac{1}{3}\log_2 27$

11. $2 \ln 6 - \ln 3$

12. $\frac{1}{2}\ln 5 + 3 \ln 2$

13. $\log M - 3 \log N$

14. $4 \log M + \frac{1}{2}\log N$

15. $\log A + 2 \log B - 3 \log C$

16. $\frac{1}{2}(\log_b M + \log_b N - \log_b P)$

17. $\frac{1}{3}(2 \log_b M - \log_b N - \log_b P)$

18. $5(\log_b A + \log_b B) - 2 \log_b C$

19. $\log \pi + 2 \log r$

20. $\log 4 - \log 3 + \log \pi + 3 \log r$

21. $\ln 2 + \ln 6 - \frac{1}{2}\ln 9$

22. $\ln 10 - \ln 5 - \frac{1}{3}\ln 8$

Simplify each expression.

23. a. $\ln e^2$ **b.** $\ln e^3$ **c.** $\ln \frac{1}{e}$ **d.** $\ln \sqrt{e}$

24. a. $\ln e^4$ **b.** $\ln \frac{1}{e^3}$ **c.** $\ln \sqrt[3]{e}$ **d.** $\ln 1$

25. a. $\ln e^x$ **b.** $e^{\ln x}$ **c.** $e^{2 \ln x}$ **d.** $e^{-\ln x}$

26. a. $\ln e^{3x}$ **b.** $e^{3 \ln x}$ **c.** $e^{\ln \sqrt{x}}$ **d.** $e^{(-1/2) \ln x}$

27. a. $10^{\log 6}$ **b.** $10^{2 \log 6}$ **c.** $10^{3 + \log 4}$ **d.** $e^{3 + \ln 4}$

28. a. $10^{3 \log 5}$ **b.** $e^{3 \ln 5}$ **c.** $10^{1 + 2 \log x}$ **d.** $e^{1 + 2 \ln x}$

Express y in terms of x.

29. a. $\log y = 2 \log x$ **b.** $\log y = 3 \log x + \log 5$

30. a. $\ln y - \ln x = 2 \ln 7$ **b.** $\ln y = 2 \ln x - \ln 4$

31. a. $\log y = -\log x$ **b.** $\log y = 2 \log x + \log 2$

32. a. $\log y + \frac{1}{2}\log x = \log 3$ **b.** $\ln y = \frac{1}{3}(\ln 4 + \ln x)$

33. a. $\log y = 1.2x - 1$ **b.** $\ln y = 1.2x - 1$

34. a. $\log y = 3 - 0.5x$ **b.** $\ln y = 3 - 0.5x$

You may find it helpful to have a graphing calculator to complete Exercises 35 and 36.

35. a. Predict how the graph of $y = \log_b \left(\frac{1}{x}\right)$ is related to the graph of $y = \log_b x$.

b. Check your prediction by graphing $y = \log x$ and $y = \log \left(\frac{1}{x}\right)$ on a single set of axes. .

36. a. Predict how the graph of $y = \log_b x^2$ is related to the graph of $y = \log_b x$.

b. Check your prediction by graphing $y = \log x$ and $y = \log x^2$ on a single set of axes.

37. a. If $f(x) = \log_2 x$, show that $f(2x) = f(x) + 1$.

b. Part (a) shows that horizontally shrinking the graph of f by a factor of 2 is equivalent to __?__.

38. a. If $f(x) = \log_3 x$, show that $f\left(\frac{x}{3}\right) = f(x) - 1$.

b. Make a statement similar to that in part (b) of Exercise 37.

39. Suppose that $\log y = ax + b$, where a and b are real numbers and a is nonzero. Express y in terms of x. What kind of expression defines $\log y$ in terms of x? What kind of expression defines y in terms of x?

40. **Conservation** The expiration time T of a natural resource is the time remaining until it is all used. If one assumes that the current growth rate of consumption remains constant, then the expiration time in years is given by

$$T = \frac{1}{r} \ln \left(\frac{rR}{C} + 1\right),$$

where C = current consumption, r = current growth rate of consumption, and R = resource size. Suppose that the world's consumption of oil is growing at the rate of 7% per year ($r = 0.07$) and current consumption is approximately 17×10^9 barrels per year. Find the expiration time for the following estimates of R.

a. $R \approx 1691 \times 10^9$ barrels (estimate of remaining crude oil)

b. $R \approx 1881 \times 10^9$ barrels (estimate of remaining crude plus shale oil)

41. If $\log_8 3 = r$ and $\log_8 5 = s$, express each logarithm in terms of r and s.

a. $\log_8 75$ b. $\log_8 225$ c. $\log_8 0.12$ d. $\log_8 \frac{3}{64}$

42. If $\log_9 5 = x$ and $\log_9 4 = y$, express each logarithm in terms of x and y.

a. $\log_9 100$ b. $\log_9 36$ c. $\log_9 \left(6\frac{1}{4}\right)$ d. $\log_9 3.2$

In Exercises 43–46, solve the given equation.

43. a. $\log_2 (x + 2) + \log_2 5 = 4$ b. $\log_4 (2x + 1) - \log_4 (x - 2) = 1$

44. a. $\log_6 (x + 1) + \log_6 x = 1$ b. $\log_3 x + \log_3 (x - 2) = 1$

45. a. $\log_4 (x - 4) + \log_4 x = \log_4 5$ b. $\log_2 (x^2 + 8) = \log_2 x + \log_2 6$

46. a. $\ln (x^2) = 16$ b. $\ln \frac{1}{x} = -5$

47. a. For what values of M is $\log_2 M < 0$?

b. Use your answer to part (a) to solve the inequality $\log_2 \left(\frac{x-1}{2}\right) < 0$.

c. For what values of M is $\log_3 M > 2$?

d. Use your answer to part (c) to solve the inequality $\log_3 (x^2 + 5) > 2$.

e. Use the fact that $\log_b M > \log_b N$ only if $M > N$ to solve the inequality $\log_6 5x > 2 \log_6 x$.

In Exercises 48–50, solve the given inequality.

48. a. $\ln (x - 4) + \ln 3 \le 0$ b. $\log (5 - x) - \log 7 > 0$

49. a. $2 \log x < \log (2x - 1)$ b. $\ln (x + 1) - \ln 2 > 3$

50. a. $\log_2 (x + 5) + \log_2 (x - 2) \ge 3$ b. $\log_4 (x - 1) + \log_4 (x + 1) < \log_4 6$

The *Richter scale* is a system for rating the severity of an earthquake. The severity can be measured either by the amplitude of the seismic wave or by the energy released by the earthquake. A one-point increase in the Richter scale number corresponds to a ten-fold increase in the *amplitude* of the seismic wave and to a thirty-one-fold increase in the *energy* released. Exercises 51 and 52 give some details.

51. **Geology** Richter scale numbers for some earthquakes that have occurred in this century are given in the table at the right. To find the ratio of the seismic wave amplitudes for the quakes in Japan and Alaska, we write:

$$\frac{10^{8.9}}{10^{8.4}} = 10^{0.5} \approx 3.16$$

a. Find the ratio of wave amplitudes for the 1906 and 1989 California earthquakes.

b. Find the ratio of wave amplitudes for the earthquakes in (1) Iran (1968) and Yugoslavia, and (2) Iran (1990) and Yugoslavia.

Year	Place	Richter scale
1906	California	8.3
1933	Japan	8.9
1963	Yugoslavia	6.0
1964	Alaska	8.4
1968	Iran	7.4
1989	California	7.1
1990	Iran	7.7

C 52. **Geology** The Richter scale was proposed in 1935 by Charles Richter. It was refined in 1979. The Richter magnitude, R, of an earthquake is given by,

$$R = 0.67 \log (0.37E) + 1.46,$$

where E is the energy in kW • h released by the earthquake.

a. Show that $E = 2.7 \cdot 10^{(R - 1.46)/0.67}$.
b. Show that if R increases by 1 unit, E increases by a factor of about 31.

The San Andreas fault

53. If $9^{(9^9)}$ is multiplied out and is typed on a strip of paper, 3 digits per centimeter, about how many kilometers long would the paper be? (*Hint*: The common logarithm of a number can tell you how many digits the number has.)

54. Suppose that all you know about a function f is that $f(ab) = f(a) + f(b)$ for all positive numbers a and b.
a. Find $f(1)$.
b. Prove that $f(a^2) = 2f(a)$ and $f(a^3) = 3f(a)$. What generalization does this suggest?
c. Prove that $f(\sqrt{a}) = \frac{1}{2}f(a)$ and $f(\sqrt[3]{a}) = \frac{1}{3}f(a)$. What generalization does this suggest?
d. Prove that $f\left(\frac{1}{b}\right) = -f(b)$.
e. Prove that $f\left(\frac{a}{b}\right) = f(a) - f(b)$.
f. Try to find a function f that satisfies the original equation.
g. If $f(10) = 1$, find the values of x for which $f(x) = 2$ and $f(x) = 3$.

5-7 Exponential Equations; Changing Bases

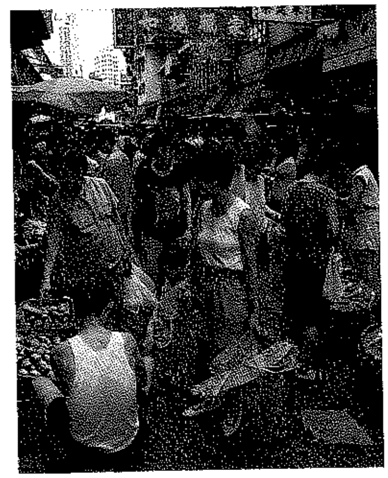
Objective *To solve exponential equations and to change logarithms from one base to another.*

An **exponential equation** is an equation that contains a variable in the exponent. Here are exponential equations you can solve from Section 5-2.

$$2^{t-3} = 8 \qquad 9^{2t} = 3\sqrt[3]{3}$$

These exponential equations are special because both sides of each equation can easily be expressed as powers of the same number. Usually exponential equations cannot be solved this way. In this section, you will see how to use logarithms to solve exponential equations involving a variable such as time, t.

Example 1 In 1990, there were about 5.4 billion people in the world. If the population has been growing at 1.95% per year, estimate the year when the population will be 8 billion people.



Solution

$$A_0\,(1 + r)^t = A(t)$$
$$5.4(1 + 0.0195)^t = 8$$
$$1.0195^t = 1.4815$$
$$\log\,(1.0195)^t = \log 1.4815$$
$$t \log\,(1.0195) = \log 1.4815$$

Thus: $t = \dfrac{\log 1.4815}{\log 1.0195} \approx 20.35$

The population will reach 8 billion people around the year 2010.

Example 2 Suppose you invest P dollars at an annual rate of 6% compounded daily. How long does it take **(a)** to increase your investment by 50%? **(b)** to double your money?

Solution

a. Since the interest is compounded daily, we can reliably use the formula for continuous compounding, $P(t) = Pe^{rt}$. We need to find the value of t for which $P(t) = P + 50\%P = 1.5P$.

$$P(t) = Pe^{rt} = 1.5P$$
$$Pe^{0.06t} = 1.5P$$
$$e^{0.06t} = 1.5$$

The most convenient logarithm to use is the natural logarithm.

$$\ln e^{0.06t} = \ln 1.5$$
$$0.06t \cdot \ln e = \ln 1.5$$
$$(0.06t)1 \approx 0.4055$$

Thus: $t \approx 6.76$

It will take about 6.76 years, or 6 years and 9 months.

b. Use the technique shown in part (a).

$$Pe^{0.06t} = 2P$$
$$e^{0.06t} = 2$$
$$0.06t = \ln 2$$

Thus: $t = \dfrac{\ln 2}{0.06} \approx \dfrac{0.693}{0.06} = 11.55$

It will take about 11.55 years, or 11 years and 6 months. (Note that the rule of 72 gives 12 years as an estimate.)

The **change-of-base formula** enables us to find the logarithm of a number in one base if logarithms in another base are known. The formula is stated below, and you are asked to derive it in Exercise 21.

$$\log_b c = \frac{\log_a c}{\log_a b}$$

For example, $\log_5 8 = \frac{\log 8}{\log 5} \approx \frac{0.903}{0.700} = 1.2900.$

CLASS EXERCISES

Solve each equation. Leave answers in radical form when possible.

1. $x^3 = 81$ **2.** $3^x = 81$ **3.** $4^x = 81$ **4.** $x^4 = 8$

Use a calculator to find the value of x to the nearest hundredth.

5. $10^x = 3$ **6.** $10^x = 8.1$ **7.** $10^x = 256$ **8.** $100^x = 302$

You may find it helpful to have a graphing calculator to complete Exercise 9.

9. ***Discussion*** Explain how the change-of-base formula enables you to use a graphing calculator or computer to graph $y = \log_2 x$.

WRITTEN EXERCISES

Use a calculator to find the value of x to the nearest hundredth.

A

1. $3^x = 12$ **2.** $2^x = 100$ **3.** $(1.06)^x = 3$ **4.** $(0.98)^x = 0.5$

5. $e^x = 18$ **6.** $e^{-x} = 0.01$ **7.** $\sqrt{e^x} = 50$ **8.** $(e^x)^3 = 200$

For each pair of equations, solve one of them by using powers of the same number. To the nearest hundredth, solve the other by using logarithms.

9. a. $4^x = 16\sqrt{2}$ **b.** $4^x = 20$ **10. a.** $9^x = \frac{3}{3^x}$ **b.** $9^x = 4$

11. a. $25^x = \sqrt[5]{5^x}$ **b.** $25^x = 2$ **12. a.** $8^x = \sqrt[3]{\frac{2}{4^x}}$ **b.** $8^x = \sqrt[3]{5}$

13. Geography The population of Kenya reached 25,000,000 people in 1990. When will it reach 50,000,000 people? Assume an annual rate of increase of 4.1%.

14. Finance An investment is made at 7% annual interest compounded daily. How long does it take to triple the investment?

15. **Finance** A $10,000 certificate of deposit at a certain bank will double in value in 9 years.
   a. Give a formula for the accumulated amount t years after the investment is made.
   b. How long does it take for the money to triple in value?

B 16. **Finance** According to the rule of 72, an investment at $r\%$ interest compounded continuously will double in approximately $\frac{72}{r}$ years. Show that a more accurate doubling time is $\frac{69.3}{r}$ years. (Note that the usual formula, $P(t) = Pe^{rt}$, where r is a decimal, must be rewritten as $P(t) = Pe^{0.01rt}$ where r is a percent.)

17. **Finance** Prove that the time needed to triple an investment at $r\%$ interest compounded continuously is approximately $\frac{110}{r}$.

18. **Finance** Tell how long it takes for $100 to become $1000 if it is invested at 8% interest compounded:
   a. annually   b. quarterly   c. daily

19. **Physics** A radioactive isotope has a half-life of 9.6 h.
   a. If there is 1 kg of the isotope now, how much will there be in 24 h?
   b. How long does it take for the isotope to decay to 1 g?

20. a. Find $\log_6 88$ by using the change-of-base formula.
   b. Find $\log_6 88$ by solving $6^x = 88$.

21. Derive the change-of-base formula.

22. Find the value of x to the nearest tenth.
   a. $x^5 = 98$ b. $5^x = 98$

You may find it helpful to have a graphing calculator to complete Exercises 23 and 24.

23. ***Discussion*** Suppose that you wished to solve $x = 2^x$.
   a. What happens when you try to solve $x = 2^x$ by taking the logarithm of each side of the equation?
   b. Give another approach to solve $x = 2^x$. How many solutions exist?
   c. Discuss the approach you would take to solve $x + 1 = 2^x$ and to solve $x + 2 = 2^x$. How many solutions are there in each case?
   d. Based on your answers to parts (a)–(c), how many solutions are there to $x + c = 2^x$, where c is a whole number?

24. Given: $x^x = \pi$. Find the value of x to the nearest hundredth.

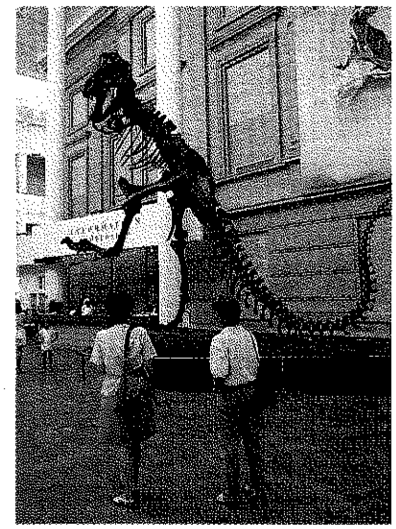
Solve. Express x as a logarithm if necessary.

25. $2^{2x} - 2^x - 6 = 0$ (*Hint:* $2^{2x} = (2^x)^2$)
26. $3^{2x} - 5 \cdot 3^x + 4 = 0$
27. $e^{2x} - 5e^x + 6 = 0$
28. $e^{2x} - e^x - 6 = 0$
29. $3^{2x+1} - 7 \cdot 3^x + 2 = 0$
30. $e^x + e^{-x} = 4$

31. **a.** If $b^m > b^n$ and $b > 1$, what can you say about m and n?
b. If $b^m > b^n$ and $0 < b < 1$, what can you say about m and n?

32. Solve: **a.** $8^x > 8^{7-x}$ **b.** $0.6^{5x} > 0.6^{x/2}$ **c.** $e^{3x} < e^x$ **d.** $\left(\frac{1}{2}\right)^x > 2^{6-x}$

33. **Archaeology** All living organisms contain a small amount of carbon 14, denoted C^{14}, a radioactive isotope. When an organism dies, the amount of C^{14} present decays exponentially. By measuring the radioactivity $N(t)$ of, say, an ancient skeleton of an animal and by comparing that radioactivity with the radioactivity N_0 of living animals, archaeologists can tell approximately when the animal died.
a. Given that the half-life of C^{14} is about 5700 years, write an equation relating $N(t)$, N_0, and the time t since the animal's death.
b. Suppose it is found that, for a certain animal, $N(t) = \frac{1}{10}N_0$. To the nearest 100 years, how long ago did the animal die?



34. **Archaeology** An archaeologist unearths a piece of wood that may have come from the Hanging Gardens of Babylon, about 600 B.C. The amount of radioactive C^{14} in the wood is $N(t) = 0.8N_0$. Is it possible that the wood could be from the Hanging Gardens? (The half-life of C^{14} is about 5700 years.)

35. Prove: $\log_b c = \frac{1}{\log_c b}$

36. Prove: $(\log_a b)(\log_b c) = \log_a c$

Evaluate each expression. (Use the results of Exercises 35 and 36.)

37. $\log_3 2 \cdot \log_2 27$ 38. $\log_{25} 8 \cdot \log_8 5$ 39. $\frac{1}{\log_2 6} + \frac{1}{\log_3 6}$ 40. $\frac{1}{\log_4 6} + \frac{1}{\log_9 6}$

For Exercises 41 and 42, use a computer or a graphing calculator to solve each inequality. Give answers to the nearest hundredth.

41. On a single set of axes, graph $y = \log_2 (x - 1)$ and $y = \log_3 x$. (See Class Exercise 9 on page 205.) Use your graph to solve $\log_2 (x - 1) > \log_3 x$.

42. Solve each inequality using the method suggested in Exercise 41.
a. $e^x < \ln (x + 5)$ **b.** $2^x \le \log_5 x$ **c.** $\log 20x > 2^{-x}$ **d.** $\log x \ge \log_4 x^2$

43. **Oceanography** After passing through a material t centimeters thick, the intensity $I(t)$ of a light beam is given by $I(t) = (4^{-ct})I_0$, where I_0 is the initial intensity and c is a constant called the absorption factor. Ocean water absorbs light with an absorption factor of $c = 0.0101$. At what depth will a beam of light be reduced to 50% of its initial intensity? 2% of its initial intensity?

C 44. Prove: $a^{\log b} = b^{\log a}$

45. Prove: $\frac{1}{\log_a ab} + \frac{1}{\log_b ab} = 1$

Chapter Summary

1. An *exponential function* has the form $f(x) = ab^x$, where $a > 0$, $b > 0$, and $b \neq 1$. Such functions are commonly used for calculating such things as the value of an investment deposited at a certain rate of compound interest, how much of a substance remains after a period of radioactive decay, or the size of a population growing at a certain rate.
2. The *laws of exponents* are given on page 170. They are used to define the zero exponent and negative integral exponents:

$$b^0 = 1$$

$$b^{-x} = \frac{1}{b^x}$$

   Integral exponents are used to define rational exponents:

$$b^{p/q} = (\sqrt[q]{b})^p = \sqrt[q]{b^p},$$

   where $b > 0$, p and q are integers, and $q \neq 0$.
3. The *rule of 72* provides an approximation of the doubling time for exponential growth. If a quantity is growing at $r\%$ per year, then

$$\text{doubling time} \approx 72 \div r.$$

4. The number e is defined as

$$\underset{n \to \infty}{\text{limit}}\left(1 + \frac{1}{n}\right)^n.$$

   The number e is the base for the *natural exponential function* and its inverse, the *natural logarithm function*.
5. The *logarithm* of x to the base b ($b > 0, b \neq 1$) is the exponent a such that $x = b^a$. Thus,

$$\log_b x = a \text{ if and only if } x = b^a.$$

   Common logarithms, logarithms to base 10, are usually denoted $\log x$, while natural logarithms, those to base e, are usually denoted $\ln x$.
6. Since logarithms are exponents, the laws of logarithms, found on page 197, are closely related to the laws of exponents stated on page 170.
7. *Exponential equations* can be solved by writing both sides of the equation in terms of a common base (Section 5-2) or taking logarithms of both sides (Section 5-7).
8. The *change-of-base formula* enables you to write logarithms in any given base in terms of logarithms in any other base.

$$\log_b c = \frac{\log_a c}{\log_a b}.$$

Key vocabulary and ideas

exponential function (p. 169, p. 181)
exponential growth and decay (p. 170)
zero exponent (p. 171)
negative exponent (p. 171)
rational exponent (p. 175)
the rule of 72 (p. 182)
the number e (p. 186)
natural exponential function (p. 187)
common logarithm (p. 191)
logarithm to base b (p. 193)
logarithmic function (p. 193)
natural logarithm function (p. 193)
exponential equation (p. 203)
change-of-base formula (p. 205)

Chapter Test

1. Give a general expression for the value of a piece of property t years from now if its current value is \$150,000 and property values are increasing at the rate of 9% per year. 5-1
2. Evaluate the following.
   a. $\frac{2^5 \cdot 2^{-4}}{2^{-2}}$ b. $(5^{-2} + 5^0)^{-1}$ c. $\sqrt{\frac{4^6}{2^{-4}}}$ d. $\frac{3^{-3} + 9^{-2}}{3^{-3}}$
3. Solve each equation: a. $2^{6-x} = 4^{2+x}$ b. $3\sqrt{27} = 9^{2x}$ 5-2
4. A gallon of milk cost \$1.99 two years ago. Now it costs \$2.19. To the nearest percent, what has been the annual rate of increase in the cost?
5. Graph $y = 2^x$ and $y = 2^{-x}$ on a single set of axes. How are the graphs related? 5-3
6. ***Writing*** Explain how $P(t) = P_0(1 + r)^t$ can be used to model exponential growth and decay. Give an example of each.
7. Suppose that \$1500 is invested at an interest rate of 8.5%. How much is the investment worth after 18 months if interest is compounded (a) quarterly? (b) continuously? 5-4
8. Given that $\log 25 \approx 1.3979$, find the value of: 5-5
   a. $\log 2.5$ b. $\log 2500$ c. $\log 0.04$
9. Find the exact value of: a. $\log_2 8$ b. $\log_8 2$ c. $2^{\log_4 64}$
10. Express y as a function of x. 5-6
    a. $\log_2 y = 2 \log_2 (2x)$ b. $\log_2 y = 2 + \log_2 x$
11. Express each of the following in terms of $\log_b M$ and $\log_b N$.
    a. $\log_b \sqrt[3]{\frac{M^2}{N}}$ b. $\log_b M^2N^3$
12. Solve $\log_5 x + \log_5 (x - 4) = 1$.
13. a. Between what two consecutive integers must $\log_5 21$ lie? 5-7
    b. Use the change-of-base formula to express $\log_5 21$ in terms of common logarithms and then evaluate it to the nearest thousandth.
14. Solve $5^x = 8$ for x to the nearest hundredth.

PROJECT

Newton's Law of Cooling

The best way to get a real feeling for mathematics is to see it in action. You've already seen functions that describe various processes. Did you ever wonder how you might develop a function to describe a process yourself?

Since this might be your first attempt, let's consider a relatively simple occurrence, such as cooling. Intuitively, you know that a cup of hot water set in a cool room will cool to room temperature. There must be some relationship between the temperature of the water over time and the temperature of the room.

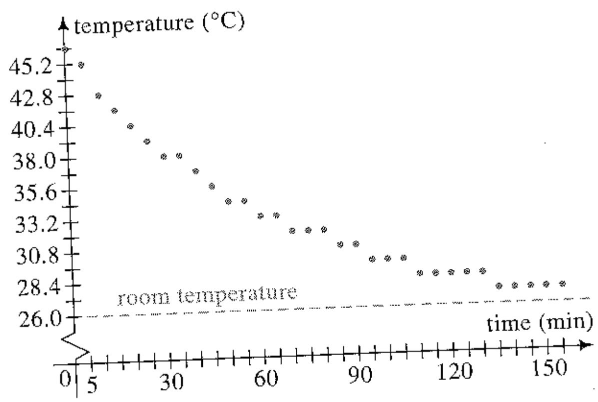
In this project, you will see how to gather data for a cooling experiment, model the data, report your findings, and explore the experiment further.

Materials:

a standard lab thermometer
a cup of hot water
a graphing calculator or software
a refrigerator

Gather the Data.

- Record the temperature, T_r, of the room.
- Next, fill a cup with some hot tap water. Place the thermometer in the cup. (Make certain that the water is not too hot for the thermometer scale.)
- Record the initial water temperature. Every five minutes thereafter, record the elapsed time t and the water temperature $T(t)$ until the temperature stops changing.
- Using a software package or a programmable graphing calculator, graph the ordered pairs $(t,\ T(t))$. Typically, your plot should resemble the graph below.



The graph probably confirms your expectations. The water temperature decreases, and the water cools more rapidly at first and then more slowly.

Model the Data.

In this part of the project, you will look for a function that reasonably represents the table of ordered pairs and the graph obtained. Which of the types of functions that you have studied so far might be a good model?

- Let's consider the exponential function $T(t) = Ce^{-kt}$, where C and k are positive constants. Why would there be a negative sign in front of k? (For a hint, see Exercise 14 in Section 5-4.)

  As t increases, what does Ce^{-kt} approach? (Use your calculator.) Now think about this. After a long period of time, what would you expect the temperature of the water to approach? If the temperature of the water approaches room temperature, then our exponential should approach T_r, or 26.0, instead of 0. Let's adjust our formula for $T(t)$ accordingly and write:

  $$T(t) = Ce^{-kt} + T_r$$

  Notice that our function $T(t)$ is the exponential $Ce^{-k}t$ translated T_r units vertically (Section 4-4).
- Next, find the value of $T(0)$. What does this tell us about the value of C? More specifically, then:

  $$T(t) = (T(0) - T_r)e^{-kt} + T_r$$
- How can we find a value for k? If the function fits all of the data, then any one of the ordered pairs should satisfy the function. For instance, we can use the ordered pair (30, 38.0) from the given data to find that $k \approx 0.02$. Then:

  $$T(t) = 20.4e^{-0.02t} + 26.0$$

  Try it! Find a value of k and the corresponding function by using one of the other ordered pairs.
- How well do you think the function above will fit the ordered pairs? To find out how well, try graphing the data points and the function on a single set of axes. Because of the variation in measurements and conditions, chances are high that there will be places where the data points do not lie exactly on the graph of the function. The value of k, after all, is just an estimate. A more accurate method of fitting an exponential function to a set of data involves all of the data points. You will see the method in Chapter 18.

Report Your Results.

A good written report should be logically organized and clearly written. It should enable someone to reproduce the experiment successfully. Be sure to state conditions, such as room temperature and initial water temperature, a listing of your ordered pairs of data, and your graph. Specify your calculations for finding the values of C and k, and your resulting function for modeling the data. In your report, analyze how well your function seems to fit your data. Explain why you do or do not think that cooling can be described by an exponential function.

Extend the Project.

Let's go one step further. What do you think would happen if instead of letting the cup of hot water cool to room temperature, you placed it in the refrigerator? To get an accurate comparison, heat the water to the same initial temperature and record your data over the same time intervals. Graph both your new and old ordered pairs on a single set of axes. How has the colder environment affected the cooling process?