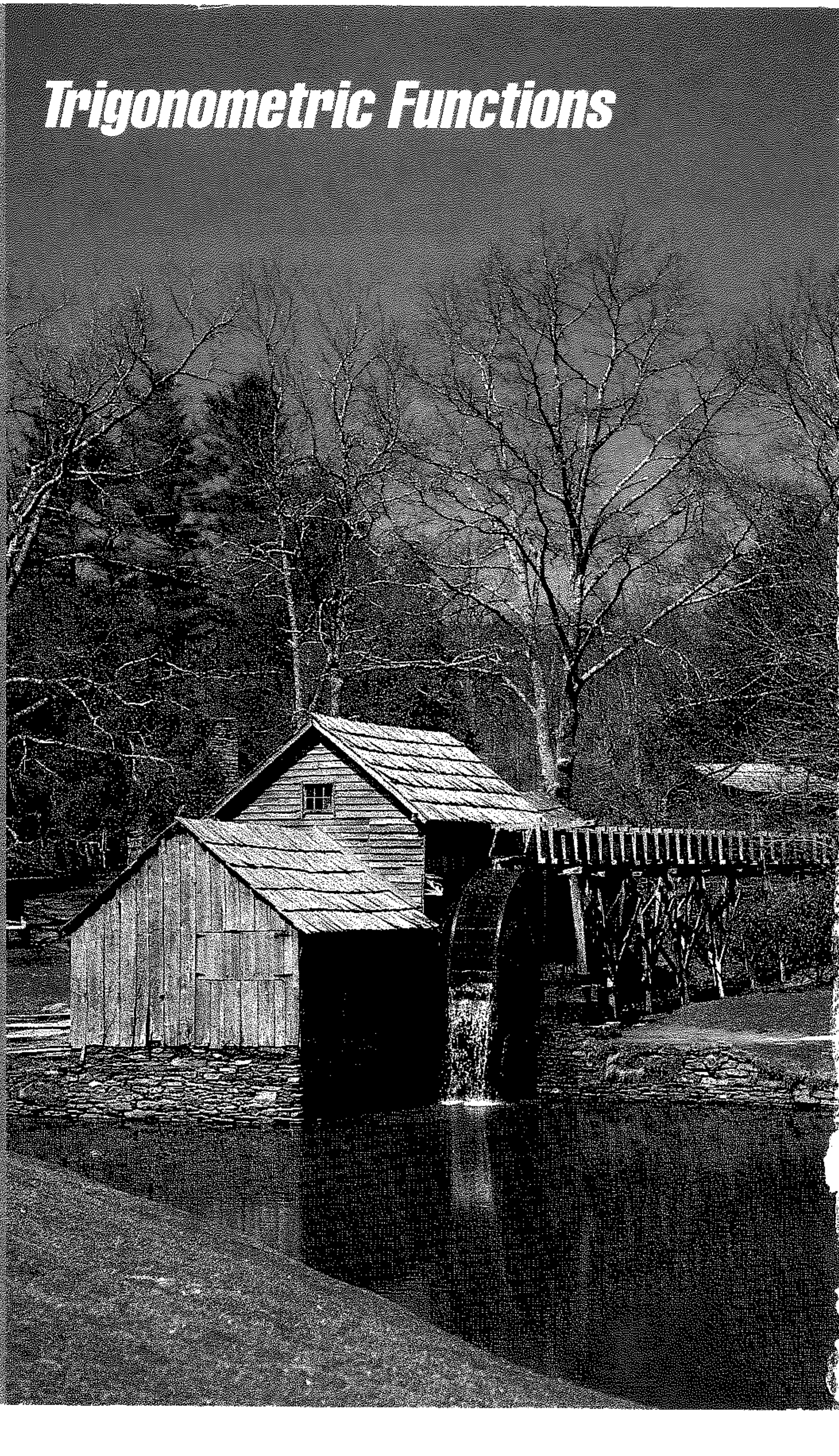
7 Trigonometric Functions

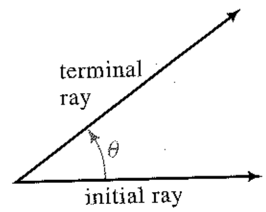


Angles, Arcs, and Sectors

7-1 Measurement of Angles

Objective *To find the measure of an angle in either degrees or radians and to find coterminal angles.*

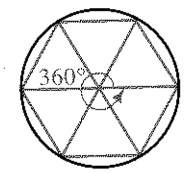
The word *trigonometry* comes from two Greek words, *trigonon* and *metron*, meaning "triangle measurement." The earliest use of trigonometry may have been for surveying land in ancient Egypt after the Nile River's annual flooding washed away property boundaries. In Chapter 9 we will discuss this use of trigonometry in greater detail. In Chapter 8 we will discuss more modern applications of trigonometry, such as the analysis of radio waves. The foundation for these applications is laid in this chapter, where we discuss the definitions and properties of the trigonometric functions.



In trigonometry, an angle often represents a rotation about a point. Thus, the angle θ shown is the result of rotating its *initial ray* to its *terminal ray*.

A common unit for measuring very large angles is the **revolution**, a complete circular motion. For example, when a car with wheels of radius 14 in. is driven at 35 mi/h, the wheels turn at an approximate rate of 420 revolutions per minute (abbreviated rpm).

A common unit for measuring smaller angles is the **degree**, of which there are 360 in one revolution. For example, when a door is opened, the doorknob is usually turned $\frac{1}{4}$ revolution, or 90 degrees.



The convention of having 360 degrees in 1 revolution can be traced to the fact that the Babylonian numeration system was based on the number 60. One theory suggests that Babylonian mathematicians subdivided the angles of an equilateral triangle into 60 equal parts (eventually called degrees). Since six equilateral triangles can be arranged within a circle, 1 revolution contained $6 \times 60 = 360$ degrees.

Angles can be measured more precisely by dividing 1 degree into 60 **minutes**, and by dividing 1 minute into 60 **seconds**. For example, an angle of 25 degrees, 20 minutes, and 6 seconds is written $25°20'6''$.

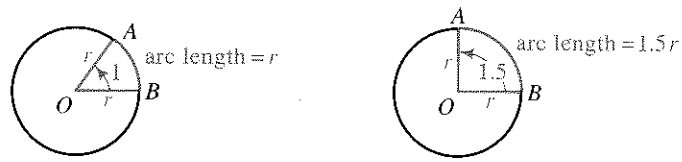
Angles can also be measured in decimal degrees. To convert between decimal degrees and degrees, minutes, and seconds, you can reason as follows:

$$12.3° = 12° + 0.3(60)' = 12°18'$$

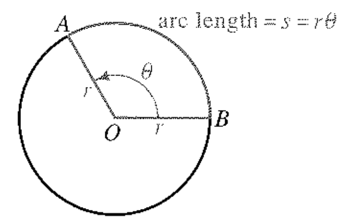
$$25°20'6'' = 25° + \left(\frac{20}{60}\right)^{\circ} + \left(\frac{6}{3600}\right)^{\circ} = 25.335°$$

Swiftly falling water propels this water wheel. Can you see how the radius of the wheel and the speed of the water determine the speed of revolution of the shaft of the water wheel?

Relatively recently in mathematical history, another unit of angle measurement, the *radian*, has come into widespread use. When an arc of a circle has the same length as the radius of the circle, as shown at the left below, the measure of the central angle, $\angle AOB$, is by definition 1 radian. Likewise, a central angle has a measure of 1.5 radians when the length of the intercepted arc is 1.5 times the radius, as shown at the right below.



In general, the **radian measure** of the central angle, $\angle AOB$, is the number of *radius* units in the length of arc AB. This accounts for the name radian. In the diagram at the right, the measure θ (Greek theta) of the central angle is:



$$\theta = \frac{s}{r}$$

Let us use this equation to see how many radians correspond to 1 revolution. Since the arc length of 1 revolution is the circumference of the circle, $2\pi r$, we have

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi.$$

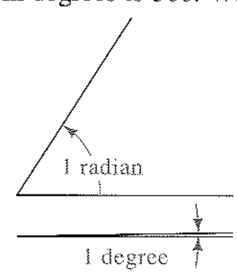
Thus, 1 revolution measured in radians is 2π and measured in degrees is 360. We therefore have 2π radians = 360 degrees, or

$$\pi \text{ radians} = 180 \text{ degrees}.$$

This gives us the following two conversion formulas:

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57.2958 \text{ degrees}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians} \approx 0.0174533 \text{ radians}$$



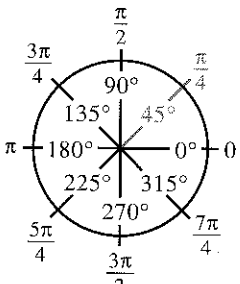
Example 1 **a.** Convert 196° to radians (to the nearest hundredth).
b. Convert 1.35 radians to decimal degrees (to the nearest tenth) and to degrees and minutes (to the nearest ten minutes).

Solution Use a calculator and the conversion formulas above. Note that some calculators have the conversion formulas already built in; consult the instruction manual for your calculator.

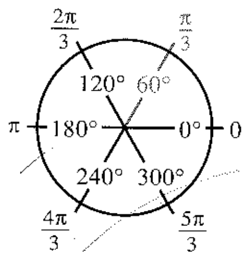
a. $196° = 196 \times \frac{\pi}{180} \approx 3.42$ radians

b. $1.35 \text{ radians} = 1.35 \times \frac{180}{\pi} \approx 77.3° \approx 77°20'$

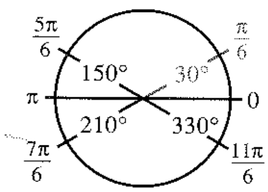
Angle measures that can be expressed evenly in degrees cannot be expressed evenly in radians, and vice versa. That is why angles measured in radians are frequently given as fractional multiples of π. Angles whose measures are multiples of $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{6}$ appear often in trigonometry. The diagrams below will help you keep the degree conversions for these special angles in mind. Note that a degree measure, such as 45°, is usually written with the degree symbol (°), while a radian measure, such as $\frac{\pi}{4}$, is usually written without any symbol.



Multiples of $\begin{cases} 45° \\ \frac{\pi}{4} \end{cases}$

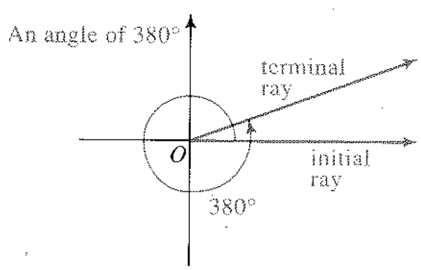


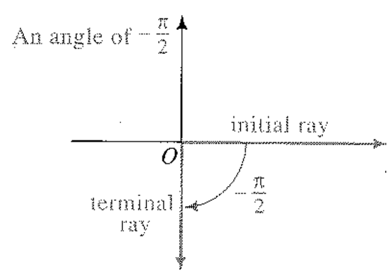
Multiples of $\begin{cases} 60° \\ \frac{\pi}{3} \end{cases}$



Other multiples of $\begin{cases} 30° \\ \frac{\pi}{6} \end{cases}$

When an angle is shown in a coordinate plane, it usually appears in **standard position**, with its vertex at the origin and its initial ray along the positive x-axis. Moreover, we consider a counterclockwise rotation to be positive and a clockwise rotation to be negative. The diagrams below give examples of positive and negative angles. (In this book we often do not distinguish between an angle and its measure. Thus, by "positive and negative angles" we mean angles with positive and negative measures.)





If the terminal ray of an angle in standard position lies in the first quadrant, as shown at the left above, the angle is said to be a first-quadrant angle. Second-, third-, and fourth-quadrant angles are similarly defined. If the terminal ray of an angle in standard position lies along an axis, as shown at the right above, the angle is called a **quadrantal angle**. The measure of a quadrantal angle is always a multiple of 90°, or $\frac{\pi}{2}$.

Two angles in standard position are called **coterminal** angles if they have the same terminal ray. For any given angle there are infinitely many coterminal angles.

Example 2 Find two angles, one positive and one negative, that are coterminal with the angle $\frac{\pi}{4}$. Sketch all three angles.

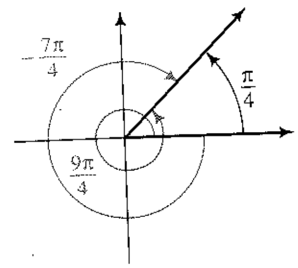
Solution A positive angle coterminal with $\frac{\pi}{4}$ is:

$$\frac{\pi}{4} + 2\pi = \frac{9\pi}{4}$$

A negative angle coterminal with $\frac{\pi}{4}$ is:

$$\frac{\pi}{4} - 2\pi = -\frac{7\pi}{4}$$

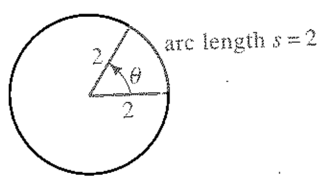
The three angles are shown at the right.



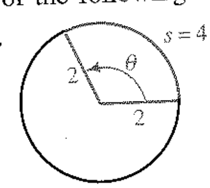
CLASS EXERCISES

1. Convert each degree measure to radians. Leave answers in terms of π.
   a. 180° **b.** 90° **c.** 315° **d.** 60°
   e. 120° **f.** 240° **g.** 30° **h.** 1°
2. Convert each radian measure to degrees.
   a. 2π **b.** π **c.** $\frac{\pi}{2}$ **d.** $\frac{\pi}{4}$
   e. $\frac{3\pi}{4}$ **f.** $\frac{5\pi}{3}$ **g.** $\frac{11\pi}{6}$ **h.** $\frac{5\pi}{6}$
3. Find two angles, one positive and one negative, that are coterminal with each given angle.
   a. 10° **b.** 100° **c.** −5° **d.** 400°
   e. π **f.** $\frac{\pi}{2}$ **g.** $-\frac{\pi}{3}$ **h.** 4π
4. **a.** The equation $\theta = (60 + 360n)°$, where n is an integer, represents all angles θ coterminal with an angle of __?__°.
   b. What would be the equivalent equation in radians?
5. Give the radian measure of θ in each of the following diagrams.

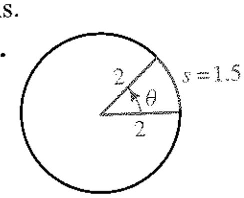
a.



b.



c.



6. Find the degree measure of an angle formed by each rotation described.
   a. $1\frac{2}{3}$ revolutions counterclockwise **b.** $2\frac{3}{4}$ revolutions clockwise

WRITTEN EXERCISES

Convert each degree measure to radians. Leave answers in terms of π.

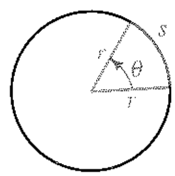
A

1. **a.** $315°$ **b.** $225°$ **c.** $15°$ **d.** $-45°$
2. **a.** $-90°$ **b.** $135°$ **c.** $-180°$ **d.** $-225°$
3. **a.** $-120°$ **b.** $-240°$ **c.** $300°$ **d.** $360°$
4. **a.** $210°$ **b.** $-135°$ **c.** $-210°$ **d.** $-315°$

Convert each radian measure to degrees.

5. **a.** $-\frac{\pi}{2}$ **b.** $\frac{4\pi}{3}$ **c.** $-\frac{3\pi}{4}$ **d.** $-\frac{\pi}{6}$
6. **a.** $-\frac{5\pi}{6}$ **b.** -2π **c.** $\frac{5\pi}{4}$ **d.** $-\frac{\pi}{3}$
7. **a.** π **b.** $-\frac{3\pi}{2}$ **c.** $\frac{2\pi}{3}$ **d.** $\frac{7\pi}{6}$
8. **a.** $-\frac{\pi}{4}$ **b.** $\frac{7\pi}{4}$ **c.** 4π **d.** $\frac{11\pi}{6}$

9. Give the radian measure of θ if:
   a. $r = 5$ and $s = 6$
   b. $r = 8$ and $s = 6$
10. Give the radian measure of θ if:
    a. $r = 4$ and $s = 5$
    b. $r = 6$ and $s = 15$



Exs. 9, 10

Convert each degree measure to radians. Give answers to the nearest hundredth of a radian.

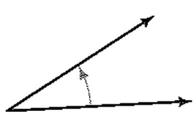
11. **a.** $95°$ **b.** $110°$ **c.** $95°10'$ **d.** $119.2°$
12. **a.** $212°$ **b.** $365°$ **c.** $200°40'$ **d.** $240.8°$

Convert each radian measure to degrees. Give answers to the nearest ten minutes or tenth of a degree.

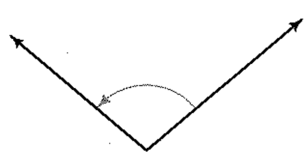
13. **a.** 1.6 **b.** 1.7 **c.** 1.21 **d.** 1.32
14. **a.** 2.2 **b.** 3.7 **c.** 2.82 **d.** 3.41

Visual Thinking **Estimate (by sight) the size in radians of each angle shown below. Then measure each angle with a protractor and convert from degrees to radians to find its actual size.**

15.



16.



Find two angles, one positive and one negative, that are coterminal with each given angle.

17. **a.** $500°$ **b.** $-60°$ **c.** $\frac{\pi}{4}$ **d.** $-\frac{2\pi}{3}$

18. **a.** $1000°$ **b.** $-100°$ **c.** $\frac{4\pi}{3}$ **d.** $-\frac{\pi}{6}$

B **19.** **a.** $28.5°$ **b.** $116.3°$ **c.** $-60.4°$ **d.** $-315.3°$

20. **a.** $38.4°$ **b.** $127.6°$ **c.** $-50.8°$ **d.** $-320.7°$

21. **a.** $360°30'$ **b.** $-90°40'$ **c.** $3°21'$ **d.** $115°15'$

22. **a.** $180°20'$ **b.** $-270°30'$ **c.** $11°44'$ **d.** $172°11'$

23. Give an expression in terms of the integer n for the measure of all angles that are coterminal with an angle of $29.7°$.

24. Give an expression in terms of the integer n for the measure of all angles that are coterminal with an angle of $-116°10'$.

Each of Exercises 25–30 gives the speed of a revolving gear. Find (a) the number of degrees per minute through which each gear turns and (b) the number of radians per minute. Give answers to the nearest hundredth.

25. 35 rpm **26.** 27 rpm **27.** 2.5 rpm

28. 6.5 rpm **29.** 14.6 rpm **30.** 19.8 rpm

31. ***Reading*** On page 257, you were told that when a car with wheels of radius 14 in. is driven at 35 mi/h, the wheels turn at an approximate rate of 420 rpm. Show how to obtain this rate of turn.

32. **Recreation** Suppose you can ride a bicycle a distance of 5 mi in 15 min. If you ride at a constant speed and if the bicycle's wheels have diameter 27 in., find the wheels' approximate rate of turn (in rpm).

33. ***Research*** Consult an encyclopedia or an atlas to see how points on a world map are located by using *latitude* and *longitude* coordinates given in degrees, minutes, and seconds.

a. If you travel south from a given point on Earth, about how many miles do you have to go to traverse an angle of $1°$?

b. Explain why your answer to part (a) might be different if you travel west instead of south.

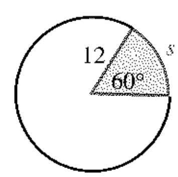
34. ***Research*** Consult a book of astronomy or a star atlas to see how stars on a celestial map are located by using angles of *right ascension* and *declination*. Describe how each of these angles is measured, and give examples.

7-2 Sectors of Circles

Objective *To find the arc length and area of a sector of a circle and to solve problems involving apparent size.*

A **sector** of a circle, shaded in red at the right below, is the region bounded by a central angle and the intercepted arc. Your geometrical intuition should tell you that the length s of the arc is some fraction of the circumference of the circle and that the area K of the sector is the same fraction of the area of the circle.

For example, suppose the central angle of a sector is 60° and the radius is 12. Then the arc length of the sector is $\frac{60}{360} = \frac{1}{6}$ of the whole circumference, or $\frac{1}{6}(2\pi r) = \frac{1}{6}(2\pi \cdot 12) = 4\pi$. Similarly, the area of the sector is $\frac{1}{6}$ of the area of the whole circle, or $\frac{1}{6}\pi r^2 = \frac{1}{6}\pi \cdot 12^2 = 24\pi$.



In general, we have the following formulas for the arc length s and area K of a sector with central angle θ.

If θ is in degrees, then: (1) $s = \frac{\theta}{360} \cdot 2\pi r$ (2) $K = \frac{\theta}{360} \cdot \pi r^2$

If θ is in radians, then: (1a) $s = r\theta$ (2a) $K = \frac{1}{2}r^2\theta$

Notice that formulas (1a) and (2a) are more straightforward than formulas (1) and (2). In fact, one reason for using radian measure is that many formulas in calculus are expressed more simply in radians than in degrees.

By combining formulas (1a) and (2a), we can obtain a third area formula:

$$K = \frac{1}{2}r^2\theta = \frac{1}{2}r(r\theta) = \frac{1}{2}rs$$

(2b) $K = \frac{1}{2}rs$

Example 1 A sector of a circle has arc length 6 cm and area 75 cm^2. Find its radius and the measure of its central angle.

Solution Using formula (2b), we have: $75 = \frac{1}{2}r(6)$

$$r = 25$$

Then, using formula (1a), we have: $6 = 25\theta$

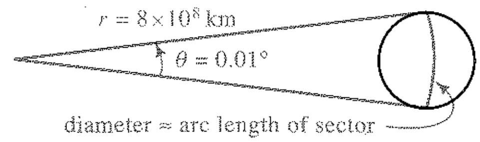
$$\theta = \frac{6}{25} = 0.24$$

Thus, the radius is 25 cm and the central angle is 0.24 radians $\approx 14°$.

Apparent Size

When there is nothing in our field of vision against which to judge the size of an object, we perceive the object to be smaller when it is farther away. For example, the sun is much larger than the moon, but we perceive the sun to be about the same size as the moon because the sun is so much farther from Earth. Thus, how big an object looks depends not only on its size but also on the angle that it subtends at our eyes. The measure of this angle is called the object's **apparent size**.

Example 2 Jupiter has an apparent size of $0.01°$ when it is 8×10^8 km from Earth. Find the approximate diameter of Jupiter.

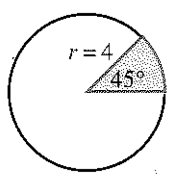


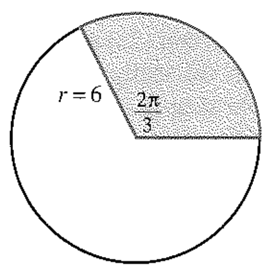
Solution As the exaggerated diagram above indicates, the diameter of Jupiter is approximately the same as the arc length of a sector with central angle $0.01°$ and radius 8×10^8 km. Using formula (1), we have:

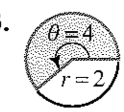
$$\text{diameter} \approx s \approx \frac{0.01}{360}(2\pi)(8 \times 10^8) \approx 140{,}000 \text{ km}$$

CLASS EXERCISES

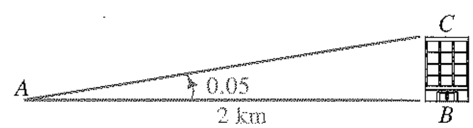
Find the arc length and area of each sector.

1. 

2. 

3. 

4. **a.** The apparent size of a tall building 2 km away is 0.05 radians. What is the building's approximate height?
   b. Explain why you can apply an arc length formula to $\triangle ABC$ and get a good approximation of BC.



WRITTEN EXERCISES

A

1. A sector of a circle has radius 6 cm and central angle 0.5 radians. Find its arc length and area.
2. A sector of a circle has radius 5 cm and central angle 3 radians. Find its arc length and area.

3. A sector of a circle has arc length 11 cm and central angle 2.2 radians. Find its radius and area.

4. A sector of a circle has arc length 2 cm and central angle 0.4 radians. Find its radius and area.

5. A sector of a circle has area 25 cm^2 and central angle 0.5 radians. Find its radius and arc length.

6. A sector of a circle has area 90 cm^2 and central angle 0.2 radians. Find its radius and arc length.

7. A sector of a circle has central angle 30° and arc length 3.5 cm. Find its area to the nearest square centimeter.

8. A sector of a circle has central angle 24°and arc length 8.4 cm. Find its area to the nearest square centimeter.

9. A sector of a circle has perimeter 7 cm and area 3 cm^2. Find all possible radii.

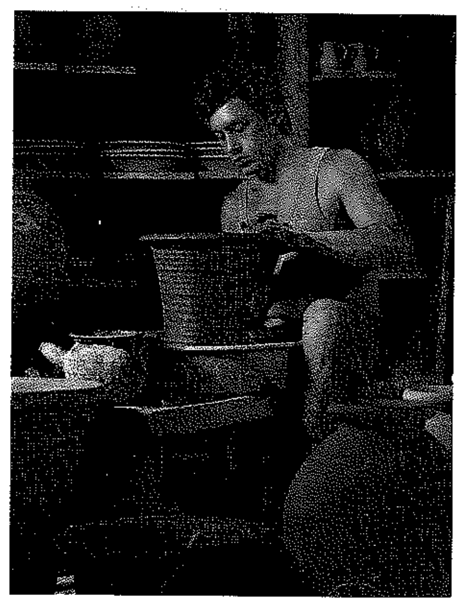
10. A sector of a circle has perimeter 12 cm and area 8 cm^2. Find all possible radii.

11. **Astronomy** The diameter of the moon is about 3500 km. Its apparent size is about 0.0087 radians. About how far is it from Earth?

12. **Astronomy** At its closest approach, Mars is about 5.6×10^7 km from Earth and its apparent size is about 0.00012 radians. What is the approximate diameter of Mars?

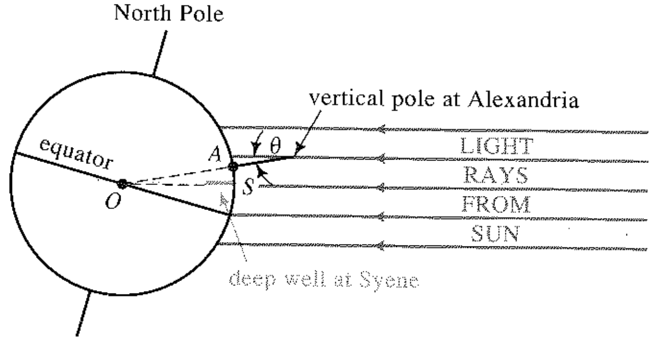
13. **Physics** A compact disc player uses a laser to read music from a disc. The player varies the rotational speed of the disc depending on the position of the laser. When the laser is at the outer edge of the disc, the player spins the disc at the slowest speed, 200 rpm.
    a. At the slowest speed, through how many degrees does the disc turn in a minute? Through how many radians does it turn in a minute?
    b. If the diameter of the disc is 11.9 cm, find the approximate distance that a point on the outer edge travels at the slowest speed in 1 min.
    c. Use part (b) to give the speed in cm/s.

14. **Physics** To make a clay vase, an artist uses a potter's wheel that has a diameter of 13 in. and spins at 120 rpm. Find the approximate distance traveled in 1 min by a point on the outer edge of the wheel.



B 15. **Astronomy** The moon and the sun have approximately the same apparent size for viewers on Earth. The distances from Earth to the moon and to the sun are about 4×10^5 km and 1.5×10^8 km, respectively. The diameter of the moon is about 3500 km. What is the approximate diameter of the sun?

16. Astronomy This exercise will show how the Greek mathematician and astronomer Eratosthenes (about 276 B.C.–194 B.C.) determined the circumference of Earth. It was reported to him that at noon on the first day of summer the sun was directly overhead in the city of Syene because there was no shadow in a deep well. Eratosthenes observed at this same time in the city of Alexandria that the sun's rays made an angle $\theta = 7.2°$ with a vertical pole.

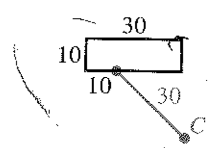


a. How did Eratosthenes conclude that the measure of $\angle AOS = \theta = 7.2°$?
b. If Alexandria was known to be 5000 stadia due north of Syene, show how Eratosthenes could conclude that the circumference of Earth was about 250,000 stadia. (1 stadium $\approx$ 0.168 km)
c. Given that

$$\text{percent difference} = \frac{\text{old value} - \text{modern value}}{\text{modern value}} \times 100,$$

what is the percent difference between Eratosthenes' value for the circumference of Earth and the modern value of 40,067 km?

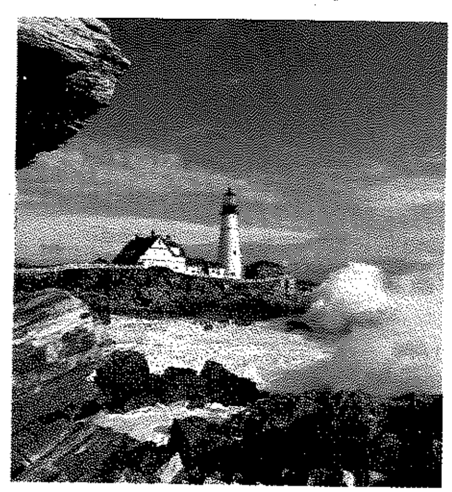
17. Farming A cow at C is tethered to a post alongside a barn 10 m wide and 30 m long. If the post is 10 m from a corner of the barn and if the rope is 30 m long, find the cow's total grazing area to the nearest square meter.



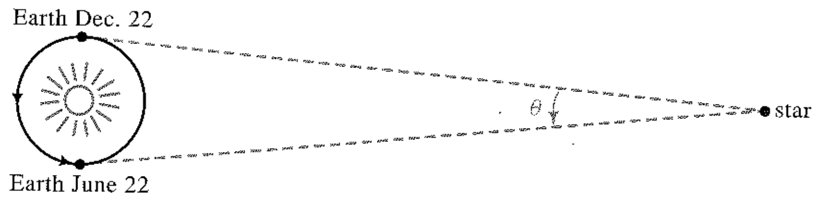
18. Optics What is the apparent size of an object 1 cm long held 80 cm from your eyes?

19. Optics You are traveling in a car toward a certain mountain at a speed of 80 km/h. The apparent size of the mountain is 0.5°. Fifteen minutes later the same mountain has an apparent size of 1°. About how tall is the mountain?

20. Optics A ship is approaching a lighthouse known to be 20 m high. The apparent size of the lighthouse is 0.005 radians. Ten minutes later the lighthouse has an apparent size of 0.010 radians. What is the approximate speed of the ship (in km/h)?



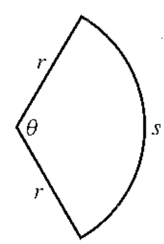
21. **Astronomy** Some stars are so far away that their positions appear fixed as Earth orbits the sun. Other stars, however, appear over time to shift their positions relative to the background of "fixed" stars. Suppose that the star shown below appears to shift through an arc of $\theta = 0°0'1.5''$ when viewed on the first day of winter and the first day of summer. If the distance from Earth to the sun is about 1.5×10^8 km, find the approximate distance from Earth to the star.



22. **Astronomy** Give the distance found in Exercise 21 in light years. (A *light year* is the distance light travels in one year. Use the fact that light travels 3.00×10^8 m/s.)

23. ***Writing*** A point traveling along a circle has both a *linear* speed, defined as the length of arc traversed per unit of time, and an *angular* speed, defined as the measure of angle moved through per unit of time. Write a paragraph in which you compare the linear and angular speeds of two points on a rotating line (for example, the tip of a clock's hand versus a point on the hand closer to the center of rotation). Then discuss the implication this has for ice skaters who form a rotating line by interlocking their arms and skating in a circle.

24. The sector shown at the right has perimeter 20 cm.

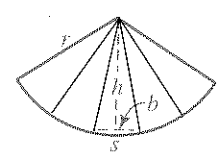


a. Show that $\theta = \frac{20}{r} - 2$ and that the area of the sector is $K = 10r - r^2$.

b. What value of r gives the maximum possible area of a sector of perimeter 20 cm? (*Hint:* K is a quadratic function of r.)

c. What is the measure of the central angle of the sector of maximum area?

25. The purpose of this exercise is to derive the formula for the area of a circle by first deriving the formula for the area of a sector. Consider the sector with radius r and arc length s shown in the diagram at the right. Inscribed in the sector are n congruent isosceles triangles, each with height h and base b.



a. Show that the total area of the inscribed triangles is $\frac{1}{2}nbh$.

b. As n increases, h gets closer and closer to __?__, and nb gets closer and closer to __?__.

c. Use parts (a) and (b) to derive the formula $K = \frac{1}{2}rs$.

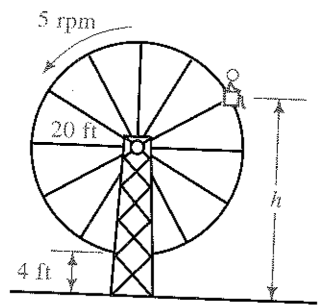
d. Derive the formula for the area of a circle from $K = \frac{1}{2}rs$.

The Trigonometric Functions

7-3 The Sine and Cosine Functions

Objective *To use the definitions of sine and cosine to find values of these functions and to solve simple trigonometric equations.*

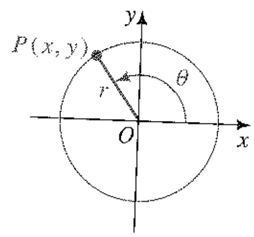
If you have ever ridden on a Ferris wheel, you may have wondered how to find your height above the ground at any given moment. Suppose a Ferris wheel has a radius of 20 ft and revolves at 5 rpm. If the bottom of the Ferris wheel sits 4 ft off the ground, then t seconds after the ride begins, a rider's height h above the ground is given in feet by:



$$h = 24 + 20\sin(30t - 90)°$$

The "sin" that appears in the equation above is an abbreviation of the *sine* function, one of the two trigonometric functions that we will discuss in this section.

Suppose $P(x, y)$ is a point on the circle $x^2 + y^2 = r^2$ and θ is an angle in standard position with terminal ray OP, as shown at the right. We define the **sine** of θ, denoted $\sin\theta$, by:



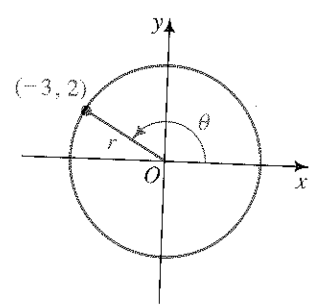
$$\sin\theta = \frac{y}{r}$$

and we define the **cosine** of θ, denoted $\cos\theta$, by:

$$\cos\theta = \frac{x}{r}$$

Example 1 If the terminal ray of an angle θ in standard position passes through $(-3, 2)$, find $\sin\theta$ and $\cos\theta$.

Solution Make a sketch as shown. To find the radius r of the circle, use the equation $x^2 + y^2 = r^2$ with $x = -3$ and $y = 2$:



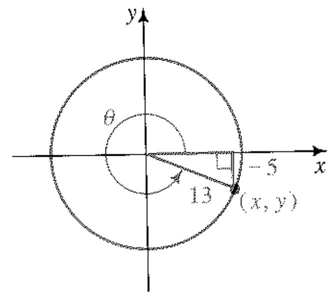
$$(-3)^2 + 2^2 = 13 = r^2$$
$$\sqrt{13} = r$$

Thus: $\sin\theta = \frac{y}{r} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$

and $\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$

Example 2 If θ is a fourth-quadrant angle and $\sin\theta = -\frac{5}{13}$, find $\cos\theta$.

Solution Make a sketch of a circle with radius 13 as shown. Since $\sin\theta = \frac{y}{r} = -\frac{5}{13}$ and r is always positive, $y = -5$. To find x, use the circle's equation, $x^2 + y^2 = r^2$:



$$x^2 + (-5)^2 = 13^2$$
$$x^2 + 25 = 169$$
$$x^2 = 144$$
$$x = \pm 12$$

Since θ is a fourth-quadrant angle, $x = 12$. Thus, $\cos\theta = \frac{x}{r} = \frac{12}{13}$.

Although the definitions of $\sin\theta$ and $\cos\theta$ involve the radius r of a circle, the values of $\sin\theta$ and $\cos\theta$ depend only on θ, as the following activity shows.

Activity

You will need graph paper, a ruler, a compass, a protractor, and a calculator.

a. Using graph paper, draw an xy-plane and an acute angle θ in standard position.

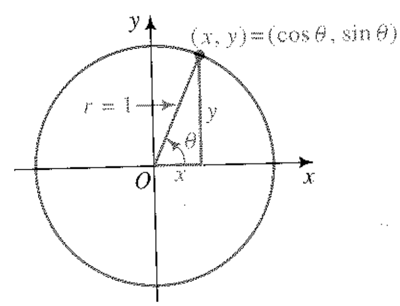
b. Draw three concentric circles centered at the origin, and mark the points P_1, P_2, and P_3 where the circles intersect the terminal ray of θ.

c. Carefully measure the radii r_1, r_2, and r_3 of the three circles as well as the vertical distances y_1, y_2, and y_3 between P_1, P_2, and P_3 and the x-axis.

d. Use a calculator to compute $\frac{y_1}{r_1}$, $\frac{y_2}{r_2}$, and $\frac{y_3}{r_3}$ to the nearest hundredth. Each ratio is an approximation of the sine of θ. What do you observe about the ratios?

e. Use your knowledge of geometry to support your observation from part (d).

The circle $x^2 + y^2 = 1$ has radius 1 and is therefore called the **unit circle**. This circle is the easiest one with which to work because, as the diagram shows, $\sin\theta$ and $\cos\theta$ are simply the y- and x-coordinates of the point where the terminal ray of θ intersects the circle.



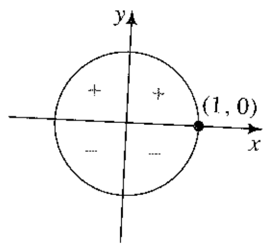
$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$$

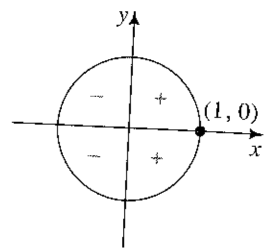
When a circle is used to define the trigonometric functions, they are sometimes called *circular functions*. (See Exercise 44 for another way to use the unit circle to define the trigonometric, or circular, functions.)

From the definitions and diagram at the bottom of the preceding page, we can see that the domain of both the sine and cosine functions is the set of all real numbers, since $\sin \theta$ and $\cos \theta$ are defined for any angle θ. Also, the range of both functions is the set of all real numbers between -1 and 1 inclusive, since $\sin \theta$ and $\cos \theta$ are the coordinates of points on the unit circle.

The diagrams below indicate where the sine and cosine functions have positive and negative values. For example, if θ is a second-quadrant angle, $\sin \theta$ is positive and $\cos \theta$ is negative.



$\sin \theta = y$

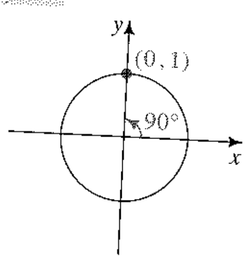


$\cos \theta = x$

Example 3 Find: **a.** $\sin 90°$ **b.** $\sin 450°$ **c.** $\cos(-\pi)$

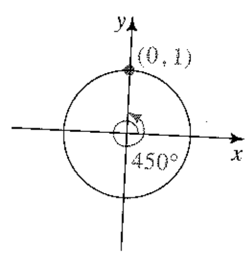
Solution

a.



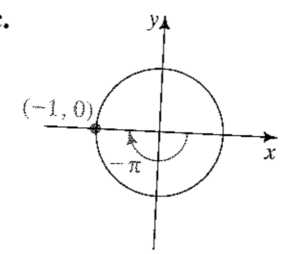
$\sin 90° = y\text{-coordinate} = 1$

b.



$\sin 450° = y\text{-coordinate} = 1$

c.



$\cos(-\pi) = x\text{-coordinate} = -1$

As figures (a) and (b) in Example 3 show, $\theta = 90°$ and $\theta = 450°$ are two solutions of the trigonometric equation $\sin \theta = 1$. The following example shows that there are infinitely many solutions of this equation.

Example 4 Solve $\sin \theta = 1$ for θ in degrees.

Solution You already know that $\theta = 90°$ is one solution of the equation $\sin \theta = 1$. Since any angle coterminal with 90° also has 1 as its sine value,

$$\theta = 90°,\ 90° \pm 360°,\ 90° \pm 2 \cdot 360°,\ 90° \pm 3 \cdot 360°,\ \ldots$$

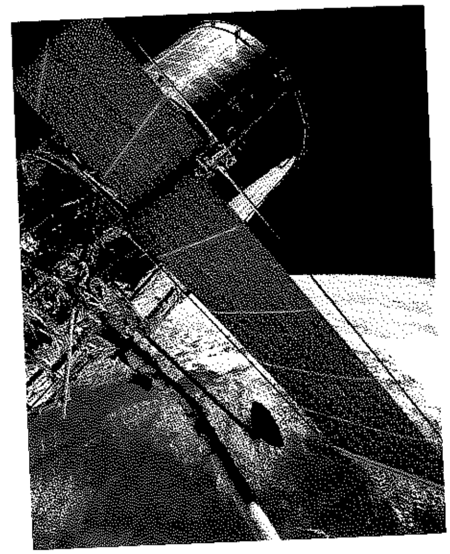
are all solutions of the equation. They can be written more conveniently as $\theta = 90° + n \cdot 360°$, where n is an integer. (In radians, the solutions would be written as $\theta = \frac{\pi}{2} + n \cdot 2\pi$ or $\theta = \frac{\pi}{2} + 2n\pi$.)

From Example 4 and the definitions of sin θ and cos θ, you can see that the sine and cosine functions repeat their values every 360° or 2π radians. Formally this means that for all θ:

$$\sin(\theta + 360°) = \sin\theta$$
$$\cos(\theta + 360°) = \cos\theta$$

$$\sin(\theta + 2\pi) = \sin\theta$$
$$\cos(\theta + 2\pi) = \cos\theta$$

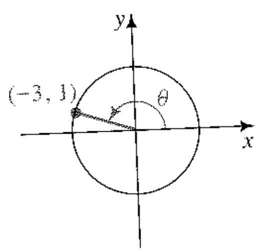
We summarize these facts by saying that the sine and cosine functions are *periodic* and that they have a *fundamental period* of 360°, or 2π radians. It is the periodic nature of these functions that makes them useful in describing many repetitive phenomena such as tides, sound waves, and the orbital paths of satellites.



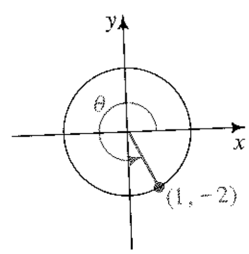
CLASS EXERCISES

Find sin θ and cos θ.

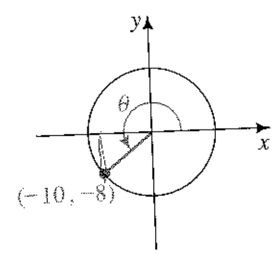
1.



2.



3.



4. State whether each expression is positive or negative.
   a. sin 165° **b.** sin 265° **c.** cos 210° **d.** cos 310°
   e. $\sin\frac{5\pi}{6}$ **f.** $\cos\frac{5\pi}{6}$ **g.** $\sin\frac{4\pi}{3}$ **h.** $\cos\frac{5\pi}{3}$
   i. sin 2 **j.** cos 2 **k.** sin 4 **l.** cos 4
5. Does cos θ increase or decrease as:
   a. θ increases from 0° to 90°? **b.** θ increases from 90° to 180°?
   c. θ increases from 180° to 270°? **d.** θ increases from 270° to 360°?
6. Answer Exercise 5 for sin θ.
7. Use the unit circle to justify the fact that for all θ:
$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$
8. There are infinitely many values of θ for which $\cos\theta = 0$. Name several.
9. **a.** Explain the meaning of $\theta = 45° + n \cdot 360°$, where n is an integer.
   b. What is the equivalent statement if θ is expressed in radians?

WRITTEN EXERCISES

Find the value of each expression without using a calculator or table.

A

1. a. $\sin 180°$ **b.** $\cos 180°$ **c.** $\sin 270°$ **d.** $\cos 270°$

2. a. $\sin(-90°)$ **b.** $\cos(-90°)$ **c.** $\sin 360°$ **d.** $\cos 360°$

3. a. $\sin(-\pi)$ **b.** $\cos \pi$ **c.** $\sin \frac{3\pi}{2}$ **d.** $\cos \frac{\pi}{2}$

4. a. $\cos 2\pi$ **b.** $\sin\left(-\frac{\pi}{2}\right)$ **c.** $\sin 3\pi$ **d.** $\cos\left(-\frac{3\pi}{2}\right)$

Name each quadrant described.

5. a. $\sin\theta > 0$ and $\cos\theta < 0$ **b.** $\sin\theta < 0$ and $\cos\theta < 0$

6. a. $\sin\theta < 0$ and $\cos\theta > 0$ **b.** $\sin\theta > 0$ and $\sin(90° + \theta) > 0$

Without using a calculator or table, solve each equation for *all* θ in radians.

7. a. $\sin\theta = 1$ **b.** $\cos\theta = -1$ **c.** $\sin\theta = 0$ **d.** $\sin\theta = 2$

8. a. $\cos\theta = 1$ **b.** $\sin\theta = -1$ **c.** $\cos\theta = 0$ **d.** $\cos\theta = -3$

Without using a calculator or table, state whether each expression is positive, negative, or zero.

9. a. $\sin 4\pi$ **b.** $\cos\frac{7\pi}{6}$ **c.** $\sin\left(-\frac{\pi}{4}\right)$ **d.** $\cos\frac{3\pi}{4}$

10. a. $\cos 3\pi$ **b.** $\sin\frac{2\pi}{3}$ **c.** $\sin\frac{11\pi}{6}$ **d.** $\cos\left(-\frac{\pi}{2}\right)$

11. a. $\sin 60°$ **b.** $\cos(-120°)$ **c.** $\cos 300°$ **d.** $\sin(-210°)$

12. a. $\cos 45°$ **b.** $\sin 135°$ **c.** $\cos(-225°)$ **d.** $\sin(-315°)$

13. a. $\sin\frac{7\pi}{4}$ **b.** $\sin\left(-\frac{\pi}{6}\right)$ **c.** $\cos\frac{3\pi}{2}$ **d.** $\cos\frac{\pi}{3}$

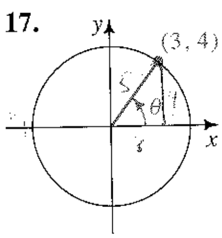
14. a. $\cos\left(-\frac{\pi}{3}\right)$ **b.** $\sin\frac{\pi}{6}$ **c.** $\sin\frac{5\pi}{4}$ **d.** $\cos\frac{7\pi}{4}$

15. a. $\cos 89°$ **b.** $\cos 91°$ **c.** $\sin 720°$ **d.** $\sin(-270°)$

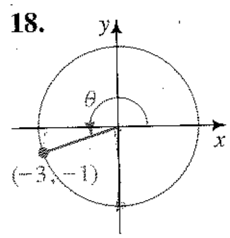
16. a. $\sin 1°$ **b.** $\sin(-1°)$ **c.** $\cos 90°$ **d.** $\cos 540°$

Find $\sin\theta$ and $\cos\theta$.

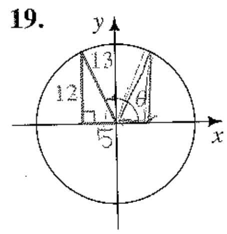
17.



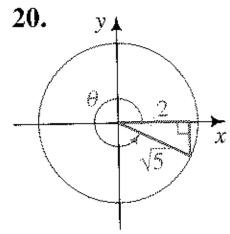
18.



19.



20.

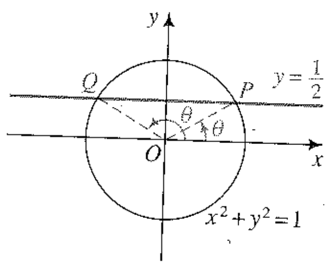


Complete the table. (A sketch like the one in Example 2 may be helpful.)

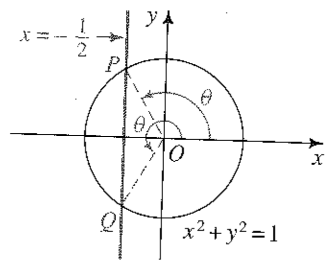
	21.	22.	23.	24.	25.	26.	27.	28.
Quadrant	I	II	III	IV	II	III	IV	II
$\sin\theta$	$\frac{3}{5}$	$\frac{5}{13}$	?	?	$\frac{1}{5}$	$-\frac{3}{7}$	?	$\frac{1}{9}$
$\cos\theta$	?	?	$-\frac{24}{25}$	$\frac{15}{17}$	?	?	$\frac{3}{4}$	?

B **29. a.** What are the coordinates of points P and Q where the line $y = \frac{1}{2}$ intersects the unit circle? (Refer to the diagram at the left below.)

b. Explain how part (a) shows that if $\sin\theta = \frac{1}{2}$, then $\cos\theta = \pm\frac{\sqrt{3}}{2}$.



Ex. 29



Ex. 30

30. a. What are the coordinates of points P and Q where the line $x = -\frac{1}{2}$ intersects the unit circle? (Refer to the diagram at the right above.)

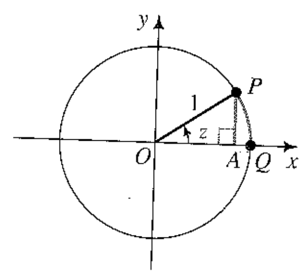
b. Explain how part (a) shows that if $\cos\theta = -\frac{1}{2}$, then $\sin\theta = \pm\frac{\sqrt{3}}{2}$.

31. ***Investigation*** In the diagram of the unit circle at the right, z is measured in radians.

a. Show that the length of arc PQ is z.

b. Show that the length of $\overline{PA}$ is $\sin z$.

c. What do parts (a) and (b) imply about the relationship between $\sin z$ and z for a small angle z? Confirm this relationship by using a calculator to compare $\sin z$ and z when z is a very small number of radians.



32. ***Investigation*** Refer to the diagram for Exercise 31.

a. Show that the length of $\overline{OA}$ is $\cos z$.

b. Use the results of part (c) of Exercise 31 and part (a) of this exercise to find an algebraic expression involving z (measure of the angle in radians) that approximates $\cos z$ for a small angle z.

c. Use a calculator to check the accuracy of the approximation in part (b) when z is a very small number of radians.

Without using a calculator or table, complete each statement with one of the symbols <, >, or =.

33. sin 40° _?_ sin 30°

34. cos 40° _?<_ cos 30°

35. sin 172° _?_ sin 8°

36. sin 310° _?_ sin 230°

37. sin 130° _?_ sin 50°

38. cos 50° _?_ cos (−50°)

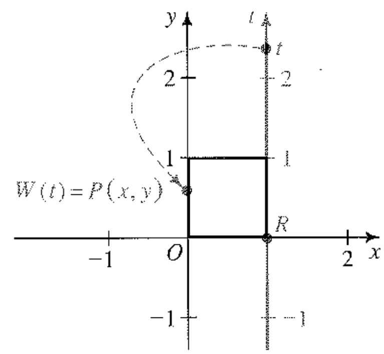
39. cos 214° _?_ cos 213°

40. sin 169° _?_ sin 168°

41. List in order of increasing size: sin 1, sin 2, sin 3, sin 4

42. List in order of increasing size: cos 1, cos 2, cos 3, cos 4

C 43. Consider a special type of function called a *wrapping function*. This function, denoted by W, wraps a vertical number line whose origin is at $R(1, 0)$ around a unit square, as shown at the right. With each real number t on the vertical number line, W associates a point $P(x, y)$ on the square. For example, $W(1) = (1, 1)$ and $W(-1) = (0, 0)$. From W we can define two simpler functions:



$$c(t) = x\text{-coordinate of } P,$$

and $$s(t) = y\text{-coordinate of } P.$$

a. Find $W(2)$, $W(3)$, $W(4)$, and $W(5)$.
b. Explain why W is a periodic function and give its fundamental period.
c. Explain how the periodicity of W guarantees the periodicity of c and s.
d. Sketch the graphs of $u = c(t)$ and $u = s(t)$ in separate tu-planes.

44. ***Writing*** Suppose the unit square in Exercise 43 is replaced with the unit circle. Write a paragraph in which you describe how the wrapping function can now be used to define the circular functions sine and cosine.

//// COMPUTER EXERCISES

1. Use a computer to obtain the approximate value (to five decimal places) of

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

when $x = 1$, $x = 2$, and $x = \frac{\pi}{2} \approx 1.5708$. Compare the results with the values of SIN(1), SIN(2), and SIN(1.5708) given directly by the computer.

2. Use a computer to obtain the approximate value (to five decimal places) of

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

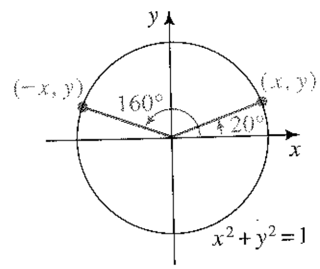
when $x = 1$, $x = 2$, and $x = \pi \approx 3.1416$. Compare the results with the values of COS(1), COS(2), and COS(3.1416) given directly by the computer.

7-4 Evaluating and Graphing Sine and Cosine

Objective *To use reference angles, calculators or tables, and special angles to find values of the sine and cosine functions and to sketch the graphs of these functions.*

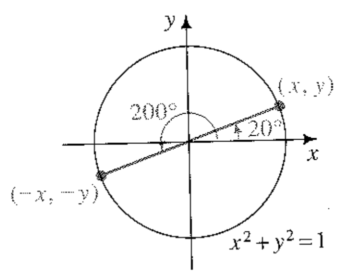
Reference Angles

Let α (Greek alpha) be an acute angle in standard position. Suppose, for example, that $\alpha = 20°$. Notice that the terminal ray of $\alpha = 20°$ and the terminal ray of $180° - \alpha = 160°$ are symmetric in the y-axis. If the sine and cosine of $\alpha = 20°$ are known, then the sine and cosine of $160°$ can be deduced, as shown in the diagram at the right.

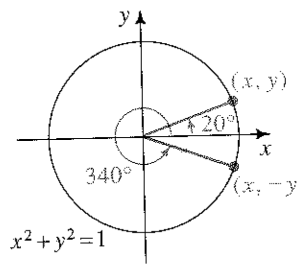


$\sin 160° = y = \sin 20°$
$\cos 160° = -x = -\cos 20°$

The angle $\alpha = 20°$ is called the *reference angle* for the $160°$ angle. It is also the reference angle for the $200°$ and $340°$ angles shown below.



$\sin 200° = -y = -\sin 20°$
$\cos 200° = -x = -\cos 20°$



$\sin 340° = -y = -\sin 20°$
$\cos 340° = x = \cos 20°$

In general, the acute angle α is the **reference angle** for the angles $180° - \alpha$, $180° + \alpha$, and $360° - \alpha$ as well as all coterminal angles. In other words, the reference angle for any angle θ is the acute positive angle α formed by the terminal ray of θ and the x-axis.

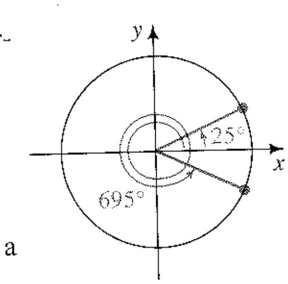
Example 1 Express $\sin 695°$ in terms of a reference angle.

Solution An angle between $0°$ and $360°$ that is coterminal with a $695°$ angle is:

$$695° - 360° = 335°$$

The reference angle for $335°$ is:

$$360° - 335° = 25°$$



(See the diagram at the right.) Since $695°$ is a fourth-quadrant angle, $\sin 695° < 0$. Thus:

$$\sin 695° = -\sin 25°$$

Using Calculators or Tables

The easiest way to find the sine or cosine of most angles is to use a calculator that has the sine and cosine functions. Always be sure to check whether the calculator is in degree or radian mode.

If you do not have access to a calculator, there are tables at the back of the book that evaluate $\sin\theta$ and $\cos\theta$ for first-quadrant values of θ. Instructions for using trigonometric tables are on page 800.

Example 2 Find the value of each expression to four decimal places.

a. $\sin 122°$ **b.** $\cos 237°$ **c.** $\cos 5$ **d.** $\sin(-2)$

Solution Note that in parts (a) and (b) the angles are given in degrees, while in parts (c) and (d) the angles are given in radians. Use your calculator and compare your results with those below.

a. $\sin 122° \approx 0.8480$ **b.** $\cos 237° \approx -0.5446$

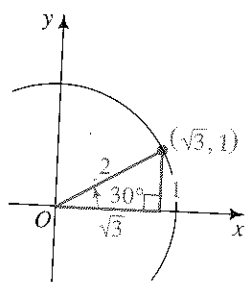
c. $\cos 5 \approx 0.2837$ **d.** $\sin(-2) \approx -0.9093$

Finding Sines and Cosines of Special Angles

Because angles that are multiples of 30° and 45° occur often in mathematics, it can be useful to know their sine and cosine values without resorting to a calculator or table. To do this, you need the following facts.

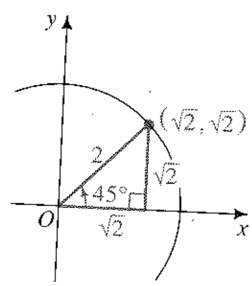
1. In a 30°-60°-90° triangle, the sides are in the ratio $1:\sqrt{3}:2$. (Note that in this ratio, 1 corresponds to the side opposite the 30° angle, $\sqrt{3}$ to the side opposite the 60° angle, and 2 to the side opposite the 90° angle.)
2. In a 45°-45°-90° triangle, the sides are in the ratio $1:1:\sqrt{2}$, or $\sqrt{2}:\sqrt{2}:2$.

These facts are used in the diagrams below to obtain the values of $\sin\theta$ and $\cos\theta$ for $\theta = 30°$, $\theta = 45°$, and $\theta = 60°$.



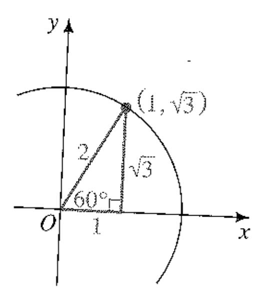
$\sin 30° = \frac{y}{r} = \frac{1}{2}$

$\cos 30° = \frac{x}{r} = \frac{\sqrt{3}}{2}$



$\sin 45° = \frac{y}{r} = \frac{\sqrt{2}}{2}$

$\cos 45° = \frac{x}{r} = \frac{\sqrt{2}}{2}$



$\sin 60° = \frac{y}{r} = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{x}{r} = \frac{1}{2}$

The information obtained from the diagrams above is summarized in the table at the top of the next page.

Although the table at the right only gives the sine and cosine values of special angles from 0° to 90°, reference angles can be used to find other multiples of 30° and 45°. For example:

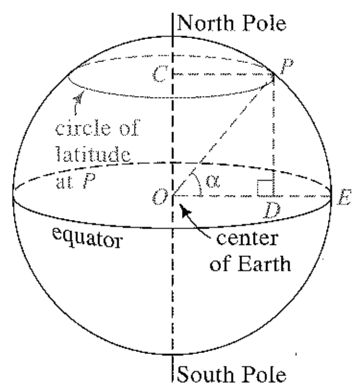
$$\sin 210° = -\sin 30° = -\frac{1}{2}$$

$$\cos 315° = \cos 45° = \frac{\sqrt{2}}{2}$$

θ (degrees)	θ (radians)	$\sin \theta$	$\cos \theta$
0°	0	0	1
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
90°	$\frac{\pi}{2}$	1	0

As the table suggests, $\sin \theta$ and $\cos \theta$ are both one-to-one functions for first-quadrant θ, a fact that we use in the next example to solve a geography problem.

Example 3 The *latitude* of a point on Earth is the degree measure of the shortest arc from that point to the equator. For example, the latitude of point P in the diagram equals the degree measure of arc PE. At what latitude is the circumference of the circle of latitude at P half the distance around the equator?



Solution Let R be the radius of Earth, and let r be the radius of the circle of latitude at P. Then:

$$\text{circumference of circle of latitude at } P = \frac{1}{2}(\text{circumference of Earth})$$

$$2\pi r = \frac{1}{2}(2\pi R)$$

$$r = \frac{1}{2}R$$

As shown in the diagram, point D is the intersection of the perpendicular from P to $\overline{OE}$. Since planes containing circles of latitude are perpendicular to the north-south axis of Earth, quadrilateral $OCPD$ is a rectangle, so that $OD = CP = r$ and

$$\cos \alpha = \frac{OD}{OP} = \frac{r}{R} = \frac{\frac{1}{2}R}{R} = \frac{1}{2}.$$

Since α is an acute angle, its measure must be 60° (see the table above). Thus, the latitude of point P is 60°N (north of the equator).

Graphs of Sine and Cosine

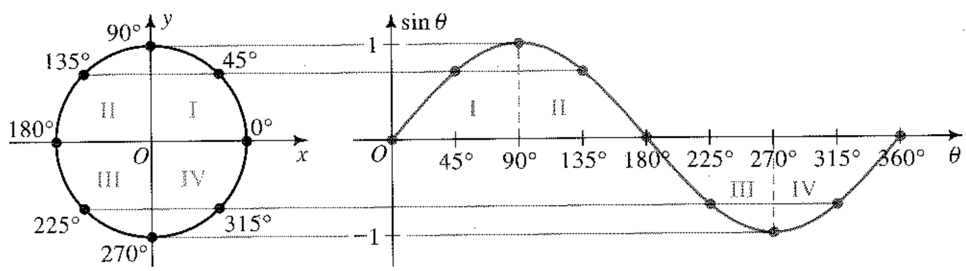
To graph the sine function, imagine a particle on the unit circle that starts at $(1, 0)$ and rotates counterclockwise around the origin. Every position (x, y) of the particle corresponds to an angle θ, where $\sin\theta = y$ by definition. As the particle rotates through the four quadrants, we get the four pieces of the sine graph shown below.

I From $0°$ to $90°$, the y-coordinate increases from 0 to 1.

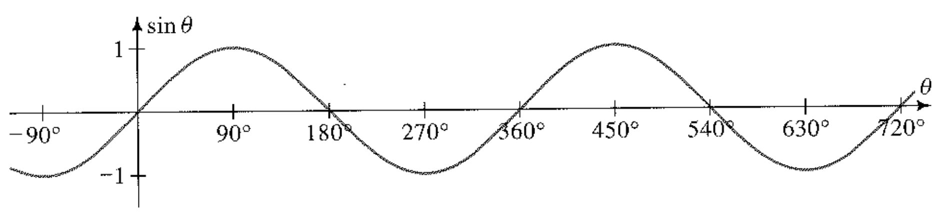
II From $90°$ to $180°$, the y-coordinate decreases from 1 to 0.

III From $180°$ to $270°$, the y-coordinate decreases from 0 to -1.

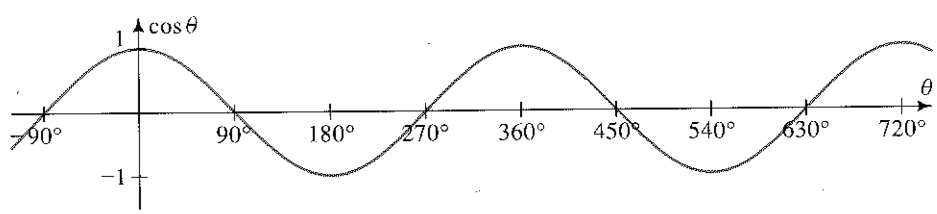
IV From $270°$ to $360°$, the y-coordinate increases from -1 to 0.



Since the sine function is periodic with a fundamental period of $360°$, the graph above can be extended left and right as shown below.



To graph the cosine function, we analyze the x-coordinate of the rotating particle in a similar manner. The cosine graph is shown below.



CLASS EXERCISES

1. Find the reference angle for θ.
   a. $\theta = 170°$ **b.** $\theta = 310°$ **c.** $\theta = 205.1°$ **d.** $\theta = 3$
2. Name another angle between $0°$ and $360°$ with the same sine as $70°$.

3. Name another angle between 0° and 360° with the same cosine as 40°.
4. Give each of the following in terms of the sine of a reference angle.
   a. sin 170° b. sin 330° c. sin (−15°) d. sin 400°
5. Give each of the following in terms of the cosine of a reference angle.
   a. cos 160° b. cos 182° c. cos (−100°) d. cos 365°

Use a calculator or table to find the value of each expression to four decimal places.

6. a. sin 188° b. sin (−110°) c. cos 350° d. cos (−230°)
7. a. cos 10.2° b. sin 28.6° c. sin (−54.7°) d. cos (−32.1°)
8. a. sin 3 b. cos 4 c. sin (−1) d. cos (−2)
9. a. cos 2.5 b. cos (−0.73) c. sin (−3.4) d. sin 0.39

Study the sine and cosine values of 30°, 45°, and 60°. Then give the exact value of each expression in simplest radical form.

10. a. sin 45° b. sin 135° c. sin 225° d. sin 315°
11. a. cos 60° b. cos 120° c. cos 240° d. cos 300°
12. a. sin 30° b. sin (−30°) c. cos 30° d. cos (−30°)
13. a. sin 330° b. cos 330° c. $\sin \frac{7\pi}{6}$ d. $\cos \frac{7\pi}{6}$
14. a. $\cos \frac{\pi}{4}$ b. $\sin \left(-\frac{\pi}{3}\right)$ c. $\cos \frac{5\pi}{6}$ d. sin (−300°)
15. For the graphs of the sine and cosine functions shown on the preceding page, the θ-axis is labeled in degrees. Redraw each graph, labeling the θ-axis in radians this time. (Your labels should be in terms of π.)
16. ***Visual Thinking*** Explain how translating the cosine graph can be used to justify the fact that for all θ:
$$\cos(\theta - 90°) = \sin \theta$$
17. a. What symmetry does the graph of the sine function have?
    b. What symmetry does the graph of the cosine function have?
18. Use Exercise 16 and part (b) of Exercise 17 to justify the fact that for all θ:
$$\cos(90° - \theta) = \sin \theta$$

WRITTEN EXERCISES

Express each of the following in terms of a reference angle.

A
1. a. sin 128° b. cos 128° c. sin (−37°) d. cos 500°
2. a. sin 310° b. cos 310° c. cos (−53°) d. sin 1000°
3. a. cos 224.5° b. cos 658° c. sin 145.7° d. sin (−201°)
4. a. cos 107.9° b. sin 271.3° c. sin 834° d. cos (−132°)

Use a calculator or table to find the value of each expression to four decimal places.

5. a. $\sin 28°$ **b.** $\sin 46°$ **c.** $\cos 65.3°$ **d.** $\sin 0.49$

6. a. $\sin 81°$ **b.** $\cos 0.3°$ **c.** $\cos 1.17$ **d.** $\sin 1.11$

7. a. $\sin 160°$ **b.** $\cos 205.3°$ **c.** $\cos 302.1°$ **d.** $\sin 207°18'$

8. a. $\cos 238°$ **b.** $\sin 403°$ **c.** $\sin 285.7°$ **d.** $\cos 132°6'$

9. a. $\sin 1.2$ **b.** $\sin 2.2$ **c.** $\cos 3.5$ **d.** $\cos 5$

10. a. $\sin 3.54$ **b.** $\cos 6$ **c.** $\sin(-2)$ **d.** $\cos 3$

Study the sine and cosine values of 30°, 45°, and 60°. Then give the exact value of each expression in simplest radical form.

11. a. $\cos 45°$ **b.** $\cos 135°$ **c.** $\sin 210°$ **d.** $\cos 150°$

12. a. $\sin(-45°)$ **b.** $\cos 315°$ **c.** $\cos 225°$ **d.** $\sin(-225°)$

13. a. $\sin 150°$ **b.** $\cos(-240°)$ **c.** $\sin(-135°)$ **d.** $\cos(-30°)$

14. a. $\cos 210°$ **b.** $\cos 90°$ **c.** $\sin(-120°)$ **d.** $\sin(-315°)$

15. a. $\cos \frac{\pi}{6}$ **b.** $\sin \frac{\pi}{3}$ **c.** $\cos \frac{2\pi}{3}$ **d.** $\sin \frac{3\pi}{4}$

16. a. $\cos \frac{\pi}{4}$ **b.** $\sin\left(-\frac{\pi}{4}\right)$ **c.** $\sin \frac{5\pi}{3}$ **d.** $\cos\left(-\frac{7\pi}{6}\right)$

17. a. $\cos 2\pi$ **b.** $\sin \frac{11\pi}{6}$ **c.** $\cos\left(-\frac{5\pi}{6}\right)$ **d.** $\cos \frac{3\pi}{4}$

18. a. $\cos\left(-\frac{\pi}{3}\right)$ **b.** $\sin \pi$ **c.** $\sin \frac{5\pi}{4}$ **d.** $\sin\left(-\frac{\pi}{6}\right)$

19. The "natural" way to graph $y = \sin x$ or $y = \cos x$ is to measure x in radians and use the same real-number scale on both axes. Use this method to sketch the graph of $y = \sin x$. (Note that π is a little more than 3 units on the x-axis.)

20. Sketch the graph of $y = \cos x$ using the method described in Exercise 19.

For part (b) of Exercises 21–23, you will need to use a computer or a graphing calculator. Be sure to use radian measure for x.

21. a. On a single set of axes, graph $y = \sin x$ and $y = 0.5x$. How many solutions does the equation $\sin x = 0.5x$ have?

b. Find each solution of the equation to the nearest hundredth.

22. a. On a single set of axes, graph $y = \cos x$ and $y = 0.1x$. How many solutions does the equation $\cos x = 0.1x$ have?

b. Find the smallest positive solution of the equation to the nearest tenth.

23. a. On a single set of axes, graph $y = |\cos x|$ and $y = x^2$. How many solutions does the equation $|\cos x| = x^2$ have?

b. Find each solution of the equation to the nearest hundredth.

24. Sketch the graph of $y = |\sin x|$. Then give the domain, range, and fundamental period of the functions $y = \sin x$ and $y = |\sin x|$.

25. ***Visual Thinking*** Imagine a particle starting at (1, 0) and making one counterclockwise revolution on the unit circle. Let θ be the angle in standard position that corresponds to the particle's position.

a. At how many points along the path of the particle are the x- and y-coordinates equal?

b. What values of θ correspond to the points in part (a)?

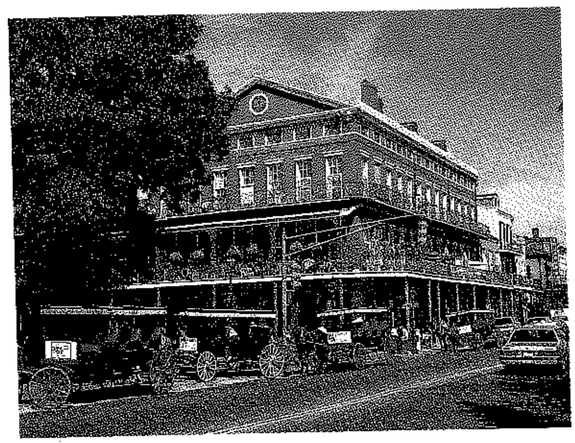
c. On a single set of axes, sketch the graphs of the sine and cosine functions for $0° \le \theta \le 360°$. Use the graphs to show where $\sin \theta = \cos \theta$ in order to verify your answers in part (b).

For Exercises 26–33, use 3963 mi for the radius of Earth.

B 26. **Geography** The latitude of Durham, North Carolina, is 36°N. About how far from Durham is the North Pole?

27. **Geography** The latitude of Lima, Peru, is 12°S. About how far from Lima is the South Pole?

28. **Geography** Beijing, China, is due north of Perth, Australia. The latitude of Beijing is 39°55′N and the latitude of Perth is 31°58′S. About how far apart are the two cities?



29. **Geography** Memphis, Tennessee, is due north of New Orleans, Louisiana. The latitude of Memphis is 35°6′N and the latitude of New Orleans is 30°N. About how far apart are the two cities?

30. **Physics** Earth's rotational speed at the equator is found by dividing the circumference of the equator by 24 hours:

$$24{,}902 \text{ mi} \div 24 \text{ h} \approx 1038 \text{ mi/h}$$

What is Earth's rotational speed at **(a)** Bangor, Maine (latitude 45°N) and **(b)** Esquina, Argentina (latitude 30°S)?

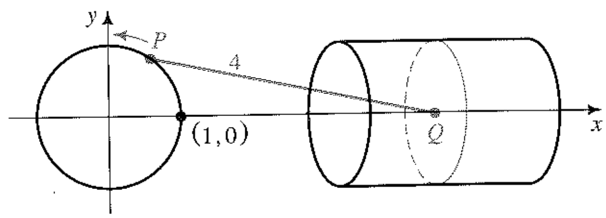
31. **Physics** What is Earth's rotational speed at **(a)** Anchorage, Alaska (latitude 61°N) and **(b)** Rio de Janeiro, Brazil (latitude 23°S)?

32. a. **Physics** Show that at latitude L, Earth's rotational speed in miles per hour is approximately equal to $1038 \cos L$.

b. **Physics** Find Earth's rotational speed at the North Pole.

33. **Physics** Rome, Italy, and Boston, Massachusetts, have approximately the same latitude (42°N). A plane flying from Rome due west to Boston is able to "stay with the sun," leaving Rome with the sun overhead and landing in Boston with the sun overhead. How fast is the plane flying? (*Hint*: See part (a) of Exercise 32.)

34. **Mechanics** A piston rod $\overline{PQ}$, 4 units long, is connected to the rim of a wheel at point P, and to a piston at point Q. As P moves counterclockwise around the wheel at 1 radian per second, Q slides left and right in the piston. What are the coordinates of P and Q in terms of time t in seconds? Assume that P is at $(1, 0)$ when $t = 0$.



7-5 The Other Trigonometric Functions

Objective *To find values of the tangent, cotangent, secant, and cosecant functions and to sketch the functions' graphs.*

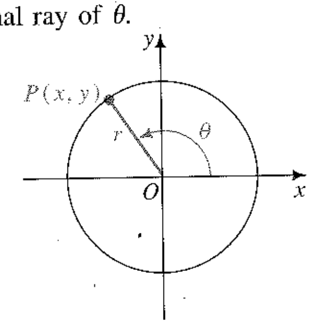
We can define four other trigonometric functions of an angle θ in terms of the x- and y-coordinates of a point on the terminal ray of θ.

tangent of θ: $\tan\theta = \frac{y}{x},\ x \neq 0$

cotangent of θ: $\cot\theta = \frac{x}{y},\ y \neq 0$

secant of θ: $\sec\theta = \frac{r}{x},\ x \neq 0$

cosecant of θ: $\csc\theta = \frac{r}{y},\ y \neq 0$



Since $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$, we can write these four new functions in terms of $\cos\theta$ and $\sin\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Notice that $\sec\theta$ and $\cos\theta$ are reciprocals, as are $\csc\theta$ and $\sin\theta$. This is also true of $\cot\theta$ and $\tan\theta$:

$$\cot\theta = \frac{1}{\tan\theta}$$

The signs of these functions in the various quadrants are summarized in the table at the right.

	I	II	III	IV
$\sin\theta$ and $\csc\theta$	+	+	−	−
$\cos\theta$ and $\sec\theta$	+	−	−	+
$\tan\theta$ and $\cot\theta$	+	−	+	−

Example 1 Find the value of each expression to four significant digits.

a. $\tan 203°$ **b.** $\cot 165°$ **c.** $\csc(-1)$ **d.** $\sec 11$

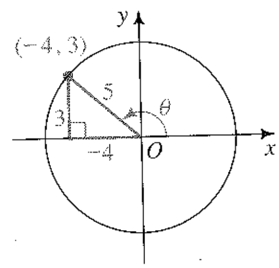
Solution Use a calculator to check each answer given below. If your calculator does not have the cotangent, secant, and cosecant as built-in functions, you must use the reciprocal relationships noted at the bottom of the preceding page. Be sure your calculator is in degree mode for parts (a) and (b), and in radian mode for parts (c) and (d).

a. $\tan 203° \approx 0.4245$ **b.** $\cot 165° = \frac{1}{\tan 165°} \approx -3.732$

c. $\csc(-1) = \frac{1}{\sin(-1)} \approx -1.188$ **d.** $\sec 11 = \frac{1}{\cos 11} \approx 226.0$

Example 2 If θ is a second-quadrant angle and $\tan\theta = -\frac{3}{4}$, find the values of the other five trigonometric functions.

Solution Since θ is a second-quadrant angle and $\tan\theta = \frac{y}{x} = -\frac{3}{4}$, we can draw a diagram as shown at the right. Then:



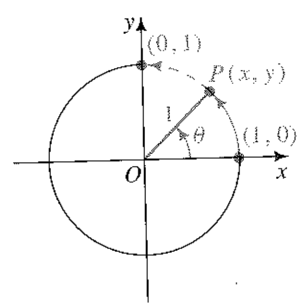
$\cos\theta = \frac{x}{r} = -\frac{4}{5}$ $\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{4}$

$\sin\theta = \frac{y}{r} = \frac{3}{5}$ $\csc\theta = \frac{1}{\sin\theta} = \frac{5}{3}$

$\cot\theta = \frac{1}{\tan\theta} = -\frac{4}{3}$

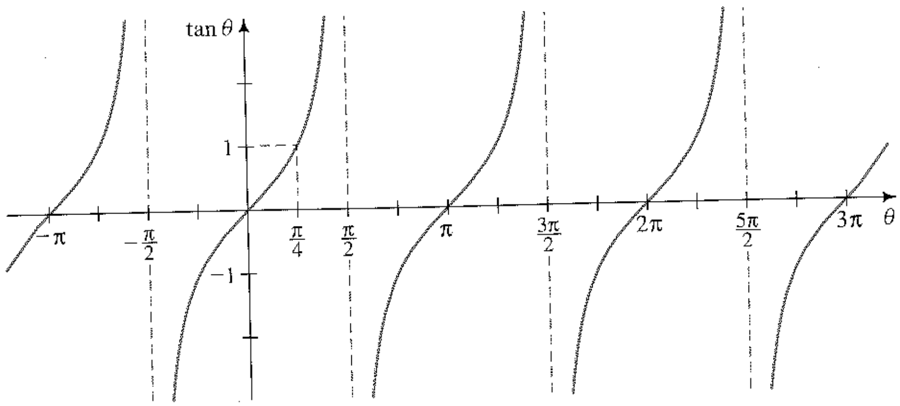
The Tangent Graph

Imagine on the unit circle a particle P that starts at (1, 0) and rotates counterclockwise around the origin. Every position (x, y) of the particle corresponds to an angle θ, where $\tan\theta = \frac{y}{x}$ by definition.



First consider what happens as the particle moves through the first quadrant. When P is at (1, 0), $\theta = 0$ and $\tan\theta = \frac{0}{1} = 0$. As P moves toward (0, 1), y increases and x decreases, so that $\tan\theta = \frac{y}{x}$ gets larger. When P reaches (0, 1), $\theta = \frac{\pi}{2}$. Although the definition of tangent would suggest that $\tan\frac{\pi}{2}$ is $\frac{1}{0}$, this expression is undefined, so we say that $\tan\frac{\pi}{2}$ is also undefined.

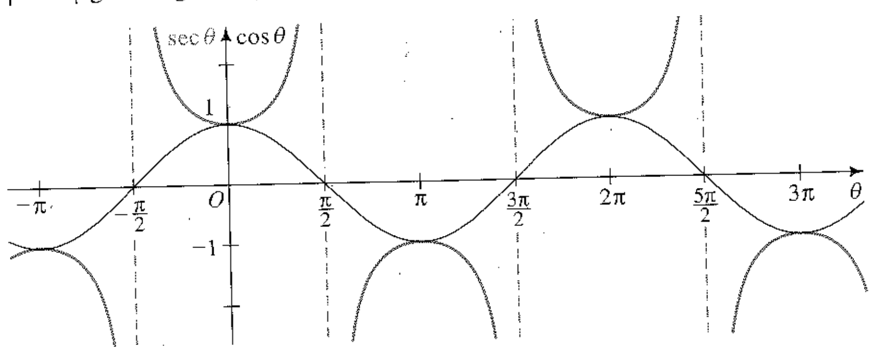
As the particle continues to move around the unit circle, we can analyze the other values of tan θ and obtain the tangent graph shown below. Notice that the graph has vertical asymptotes at odd multiples of $\frac{\pi}{2}$. Also notice that the tangent function is periodic with fundamental period π (or 180°). The graph of the cotangent function is similar and is left for you to sketch (see Written Exercise 9).



The Secant Graph

Since the secant function is the reciprocal of the cosine function, we can obtain the secant graph (shown in red below) using the cosine graph (black) and these facts:

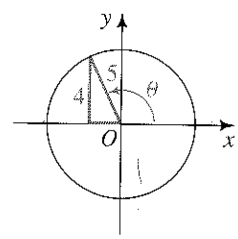
1. sec $\theta = 1$ when cos $\theta = 1$: at $\theta = 0, \pm 2\pi, \pm 4\pi, \ldots$
2. sec $\theta = -1$ when cos $\theta = -1$: at $\theta = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$
3. sec θ is undefined when cos $\theta = 0$: at $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$
4. $|\sec\theta|$ gets larger as $|\cos\theta|$ gets smaller.



Notice that the graph of the secant function has vertical asymptotes at odd multiples of $\frac{\pi}{2}$. Also notice that the secant function, like the cosine function, is periodic with fundamental period 2π (or 360°). The cosecant graph is like the secant graph and can be found by a similar analysis (see Written Exercise 10).

CLASS EXERCISES

1. Give the values of θ (in degrees) for which:
   a. $\cot\theta = 0$ **b.** $\csc\theta = 1$ **c.** $\cot\theta = -1$ **d.** $\csc\theta$ is undefined
2. If $\sec 15° \approx 1.035$, give the approximate value of:
   a. $\sec(-15°)$ **b.** $\sec 165°$ **c.** $\sec 195°$ **d.** $\sec 345°$
3. Find the value of each expression to four significant digits.
   a. $\tan 2$ **b.** $\cot 185°$ **c.** $\csc 3$ **d.** $\sec(-22°)$
4. In what quadrant is θ if $\csc\theta < 0$ and $\tan\theta > 0$?
5. The diagram shows an angle θ with $\sin\theta = \frac{4}{5}$. Find:
   a. $\cos\theta$ **b.** $\tan\theta$ **c.** $\cot\theta$ **d.** $\sec\theta$ **e.** $\csc\theta$



WRITTEN EXERCISES

Find the value of each expression to four significant digits.

A

1. **a.** $\tan 100°$ **b.** $\cot 276°$ **c.** $\csc 5$ **d.** $\sec 2.14$
2. **a.** $\sec(-11°)$ **b.** $\csc 233°$ **c.** $\tan 3$ **d.** $\cot 7.28$

Express each of the following in terms of a reference angle.

3. **a.** $\sin 195°$ **b.** $\sec 280°$ **c.** $\tan(-140°)$ **d.** $\sec 2$
4. **a.** $\cot 285°$ **b.** $\sec(-105°)$ **c.** $\csc 600°$ **d.** $\tan 3$
5. **a.** $\tan 820°$ **b.** $\sec 290°$ **c.** $\cot 185°$ **d.** $\csc 4$
6. **a.** $\tan 160°$ **b.** $\csc 115°$ **c.** $\sec 235°$ **d.** $\cot 5$
7. Give the values of x (in radians) for which $\csc x$ is:
   a. undefined **b.** 0 **c.** 1 **d.** -1
8. Give the values of x (in radians) for which $\cot x$ is:
   a. undefined **b.** 0 **c.** 1 **d.** -1
9. Sketch a graph of $\cot\theta$ versus θ for $-\pi \le \theta \le 3\pi$. Be sure to show where any vertical asymptotes occur.

You may find it helpful to have a computer or a graphing calculator to complete Exercise 10.

10. ***Writing*** On a single set of axes, graph $y = \sin x$ and $y = \csc x$, showing at least two full periods of each function. Using the reciprocal relationship between sine and cosecant, explain the features (x-intercepts, vertical asymptotes, periodicity, and so on) of the cosecant graph.
11. On a single set of axes, sketch the graphs of $y = \tan x$ and $y = 2x$. How many solutions does the equation $\tan x = 2x$ have?
12. On a single set of axes, sketch the graphs of $y = \sin x$ and $y = \sec x$. How many solutions does the equation $\sin x = \sec x$ have?

In Exercises 13–18, find the values of the other five trigonometric functions.

13. $\sin x = \frac{5}{13}, \frac{\pi}{2} < x < \pi$

14. $\cos x = \frac{24}{25}, -\frac{\pi}{2} < x < 0$

15. $\tan x = \frac{3}{4}, \pi < x < 2\pi$

16. $\cot x = -\frac{12}{5}, 0 < x < \pi$

17. $\sec x = -3, 0 < x < \pi$

18. $\csc x = -5, \frac{\pi}{2} < x < \frac{3\pi}{2}$

19. Explain why the cotangent graph has vertical asymptotes at multiples of π.

20. ***Reading*** Prepare a summary table for the six trigonometric functions introduced in this section and the preceding one. The table should give the definition, the domain, the range, the fundamental period, and a sketch of the graph of each function.

B **21. a.** Verify that $1 + \tan^2 \frac{\pi}{3} = \sec^2 \frac{\pi}{3}$. $\left[\textit{Note}\text{: } \tan^2 \frac{\pi}{3} \text{ means } \left(\tan \frac{\pi}{3}\right)^2.\right]$

b. Can you find any other values of x for which $1 + \tan^2 x = \sec^2 x$?

22. a. Evaluate $1 + \cot^2 x$ and $\csc^2 x$ for $x = \frac{\pi}{2}$, $x = \frac{3\pi}{4}$, and $x = \frac{7\pi}{6}$.

b. Make a conjecture about the relationship between $1 + \cot^2 x$ and $\csc^2 x$. Prove your conjecture using Exercise 7 on page 271 and the definitions of cotangent and cosecant.

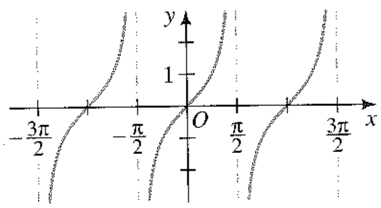
Find the exact value of each expression or state that the value is undefined.

23. a. $\cos 120°$	**b.** $\sec 120°$	**c.** $\sin 120°$	**d.** $\tan 120°$
24. a. $\sin 225°$	**b.** $\csc 225°$	**c.** $\tan 225°$	**d.** $\sec 225°$
25. a. $\csc 90°$	**b.** $\sec 180°$	**c.** $\tan 240°$	**d.** $\cot 0°$
26. a. $\csc 150°$	**b.** $\csc 0°$	**c.** $\tan 315°$	**d.** $\sec 315°$
27. a. $\csc \pi$	**b.** $\tan \frac{2\pi}{3}$	**c.** $\cot \frac{\pi}{2}$	**d.** $\sec \frac{5\pi}{6}$
28. a. $\tan \frac{\pi}{2}$	**b.** $\cot \frac{7\pi}{4}$	**c.** $\sec(-3\pi)$	**d.** $\csc \frac{7\pi}{6}$

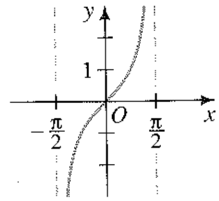
7-6 The Inverse Trigonometric Functions

Objective *To find values of the inverse trigonometric functions.*

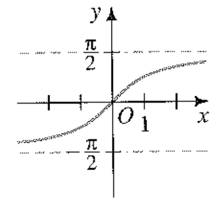
From the graph of $f(x) = \tan x$ shown on the left at the top of the next page, we can see that the tangent function is not one-to-one and thus has no inverse. However, if we restrict x to the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the restricted function, which we denote $F(x) = \text{Tan } x$, is one-to-one. Its inverse is denoted $\text{Tan}^{-1} x$ and is read "the inverse tangent of x." Notice that $\text{Tan}^{-1} x = y$ means that $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.



$f(x) = \tan x$ has no inverse.
Domain: $\{x \mid x \neq \frac{\pi}{2} + n\pi\}$
Range: Real numbers



$F(x) = \text{Tan}\, x$ has an inverse.
Domain: $\{x \mid -\frac{\pi}{2} < x < \frac{\pi}{2}\}$
Range: Real numbers



$F^{-1}(x) = \text{Tan}^{-1} x$
Domain: Real numbers
Range: $\{y \mid -\frac{\pi}{2} < y < \frac{\pi}{2}\}$

Example 1 Find $\text{Tan}^{-1}\, 2$ with a calculator.

Solution On most calculators, the inverse tangent function is symbolized by $\tan^{-1}$, INV tan, or ARC tan. Moreover, the calculator will give you answers in degrees or radians.

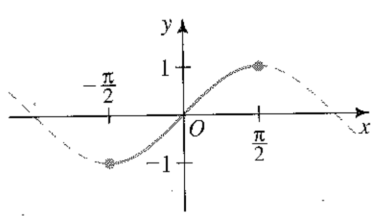
With the calculator in degree mode, $\text{Tan}^{-1}\, 2 \approx 63.4°$.
With the calculator in radian mode, $\text{Tan}^{-1}\, 2 \approx 1.11$.

Example 2 Find $\text{Tan}^{-1}(-1)$ without a calculator.

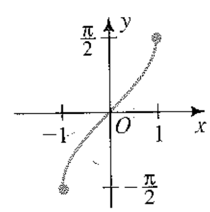
Solution $\text{Tan}^{-1}(-1) = x$ means that $\tan x = -1$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Thus, $\text{Tan}^{-1}(-1) = -\frac{\pi}{4}$.

By considering only the solid portion of the graph of $g(x) = \sin x$ shown at the left below, we obtain a new function $G(x) = \text{Sin}\, x$ with domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. This function has an inverse, $G^{-1}(x) = \text{Sin}^{-1}\, x$, whose graph is shown at the right below. Note that $\text{Sin}^{-1}\, x = y$ means that $\sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

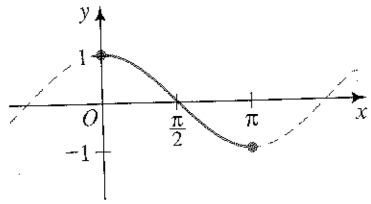


$G(x) = \text{Sin}\, x$ has an inverse.
Domain: $\{x \mid -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$
Range: $\{y \mid -1 \leq y \leq 1\}$



$G^{-1}(x) = \text{Sin}^{-1} x$
Domain: $\{x \mid -1 \leq x \leq 1\}$
Range: $\{y \mid -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\}$

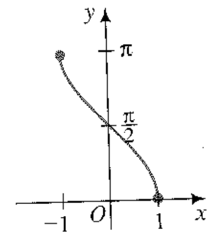
Similarly, by considering only the solid portion of the graph of $h(x) = \cos x$ at the left below, we obtain a new function $H(x) = \text{Cos}\, x$ with domain $0 \leq x \leq \pi$. This function has an inverse, $H^{-1}(x) = \text{Cos}^{-1} x$, whose graph is shown at the right below. Note that $\text{Cos}^{-1} x = y$ means that $\cos y = x$ and $0 \leq y \leq \pi$.



$H(x) = \text{Cos}\, x$ has an inverse.

Domain: $\{x \mid 0 \leq x \leq \pi\}$

Range: $\{y \mid -1 \leq y \leq 1\}$



$H^{-1}(x) = \text{Cos}^{-1} x$

Domain: $\{x \mid -1 \leq x \leq 1\}$

Range: $\{y \mid 0 \leq y \leq \pi\}$

Example 3 **a.** Find $\text{Sin}^{-1}(-0.8)$ with a calculator.
b. Find $\text{Cos}^{-1}(-0.5)$ without a calculator.

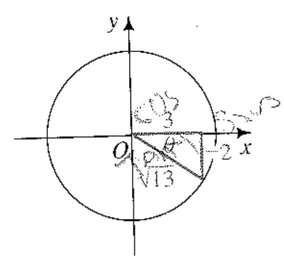
Solution **a.** With the calculator in radian mode, $\text{Sin}^{-1}(-0.8) \approx -0.93$.
b. $\text{Cos}^{-1}(-0.5) = x$ means that $\cos x = -0.5$ and $0 \leq x \leq \pi$.
Thus, $\text{Cos}^{-1}(-0.5) = \frac{2\pi}{3}$.

Example 4 **a.** Find $\cos\left(\text{Tan}^{-1}\left(-\frac{2}{3}\right)\right)$ with a calculator.
b. Find $\cos\left(\text{Tan}^{-1}\left(-\frac{2}{3}\right)\right)$ without a calculator.

Solution **a.** With the calculator in either degree or radian mode:

$$\cos\left(\text{Tan}^{-1}\left(-\frac{2}{3}\right)\right) \approx 0.83$$

b. Let $\theta = \text{Tan}^{-1}\left(-\frac{2}{3}\right)$, so that θ is a fourth-quadrant angle such that $\tan\theta = -\frac{2}{3}$. After sketching a right triangle as in the diagram at the right, we see:



$$\cos\left(\text{Tan}^{-1}\left(-\frac{2}{3}\right)\right) = \cos\theta$$

$$= \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

(Use a calculator to compare this result with part (a).)

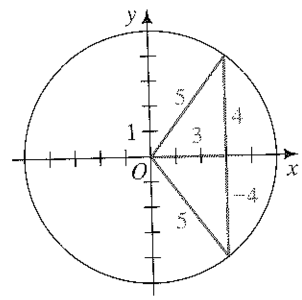
CLASS EXERCISES

Find the value of each expression to the nearest tenth of a degree.

1. $\text{Tan}^{-1} 1.2$
2. $\text{Sin}^{-1} (-0.3)$
3. $\text{Cos}^{-1} (-0.425)$

Find the value of each expression to the nearest hundredth of a radian.

4. $\text{Tan}^{-1} (-2.9)$
5. $\text{Sin}^{-1} 0.75$
6. $\text{Cos}^{-1} 0.058$
7. What happens when you try to evaluate $\text{Sin}^{-1} 1.7$ with a calculator? Explain why this happens.
8. Use the diagram at the right to find the value of each expression.



a. $\cos\left(\text{Sin}^{-1} \frac{4}{5}\right)$

b. $\tan\left(\text{Sin}^{-1} \frac{4}{5}\right)$

c. $\cot\left(\text{Sin}^{-1} \frac{4}{5}\right)$

WRITTEN EXERCISES

Use a calculator or table to find the value of each expression to the nearest tenth of a degree.

A

1. a. $\text{Sin}^{-1} 0.9$ b. $\text{Sin}^{-1} (-0.9)$ c. $\text{Cos}^{-1} \frac{3}{4}$ d. $\text{Cos}^{-1} \left(-\frac{3}{4}\right)$
2. a. $\text{Tan}^{-1} \frac{7}{3}$ b. $\text{Tan}^{-1} \left(-\frac{7}{3}\right)$ c. $\text{Cos}^{-1} 0.4$ d. $\text{Cos}^{-1} (-0.4)$

Use a calculator or table to find the value of each expression to the nearest hundredth of a radian.

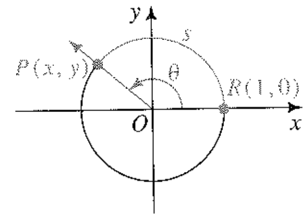
3. a. $\text{Tan}^{-1} 0.23$ b. $\text{Tan}^{-1} (-0.23)$ c. $\text{Cos}^{-1} 0.345$ d. $\text{Cos}^{-1} (-0.345)$
4. a. $\text{Sin}^{-1} \frac{3}{8}$ b. $\text{Sin}^{-1} \left(-\frac{3}{8}\right)$ c. $\text{Cos}^{-1} \frac{5}{6}$ d. $\text{Cos}^{-1} \left(-\frac{5}{6}\right)$

Without using a calculator or table, find the value of each expression in radians. Many answers can be given in terms of π.

5. a. $\text{Sin}^{-1} 0$ b. $\text{Cos}^{-1} 0$ c. $\text{Tan}^{-1} 1$ d. $\text{Tan}^{-1} (-1)$
6. a. $\text{Sin}^{-1} 1$ b. $\text{Sin}^{-1} (-1)$ c. $\text{Cos}^{-1} 1$ d. $\text{Cos}^{-1} (-1)$
7. a. $\text{Sin}^{-1} \frac{1}{2}$ b. $\text{Sin}^{-1} \left(-\frac{1}{2}\right)$ c. $\text{Cos}^{-1} \frac{1}{2}$ d. $\text{Cos}^{-1} \left(-\frac{1}{2}\right)$
8. a. $\text{Sin}^{-1} \frac{\sqrt{2}}{2}$ b. $\text{Sin}^{-1} \left(-\frac{\sqrt{2}}{2}\right)$ c. $\text{Cos}^{-1} \frac{\sqrt{2}}{2}$ d. $\text{Cos}^{-1} \left(-\frac{\sqrt{2}}{2}\right)$

9. What is wrong with the expression $\text{Cos}^{-1}\ 3$? What happens when you try to evaluate it with a calculator?

10. ***Writing*** Each of the trigonometric functions can be defined in terms of arc lengths instead of angle measures. In the diagram, angle θ (in standard position) intercepts arc PR on the unit circle. If arc PR has length s, then we can define the cosine of s as:



$$\cos s = \cos \theta = x\text{-coordinate of } P$$

Write a similar definition for the inverse cosine function and explain why the inverse cosine function is sometimes denoted Arccos (read "arc cosine"). (By similar reasoning, the inverse sine function is sometimes denoted Arcsin, the inverse tangent function is sometimes denoted Arctan, and so on.)

Find both the approximate value and the exact value of each expression.

11. **a.** $\tan\left(\text{Cos}^{-1}\frac{12}{13}\right)$ **b.** $\tan\left(\text{Cos}^{-1}\left(-\frac{12}{13}\right)\right)$

12. **a.** $\sin\left(\text{Cos}^{-1}\frac{1}{5}\right)$ **b.** $\sin\left(\text{Cos}^{-1}\left(-\frac{1}{5}\right)\right)$

13. **a.** $\csc\left(\text{Tan}^{-1}\ 1.05\right)$ **b.** $\sec\left(\text{Sin}^{-1}\ (-0.5)\right)$

14. **a.** $\cos\left(\text{Tan}^{-1}\ (-0.2)\right)$ **b.** $\sin\left(\text{Tan}^{-1}\ (-0.2)\right)$

You may find it helpful to have a computer or a graphing calculator to complete Exercises 15–18. Be sure to use radian measure for x.

B 15. **a.** Graph $y = \sin(\text{Sin}^{-1} x)$. Give the domain, range, and a simplified rule for the function $y = \sin(\text{Sin}^{-1} x)$.
b. Graph $y = \text{Sin}^{-1}(\sin x)$.
c. Does your graph from part (b) contradict the property of inverse functions which states that $f^{-1}(f(x)) = x$ for all x in the domain of f? Explain.

16. **a.** Graph $y = \text{Cos}^{-1}(\cos x)$. Give the coordinates of the high and low points of the graph.
b. For what values of x is $\text{Cos}^{-1}(\cos x) = x$?
c. Explain why $\text{Cos}^{-1}(\cos x) = x$ is not true for all values of x.

17. ***Writing*** In a paragraph, compare the graphs of $y = \tan(\text{Tan}^{-1} x)$ and $y = \text{Tan}^{-1}(\tan x)$.

18. Graph $f(x) = \text{Cos}^{-1} x + \text{Cos}^{-1}(-x)$. Give the domain and range of f.

Determine whether each equation is true for all x for which both sides of the equation are defined. If it is not true, give a counterexample.

19. **a.** $\tan(\text{Tan}^{-1} x) = x$ **b.** $\text{Tan}^{-1}(\tan x) = x$

20. **a.** $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1} x$ **b.** $\text{Cos}^{-1}(-x) = -\text{Cos}^{-1} x$

21. **a.** $\text{Cos}^{-1} x = \frac{\pi}{2} - \text{Sin}^{-1} x$ **b.** $\text{Tan}^{-1}\frac{1}{x} = \frac{\pi}{2} - \text{Tan}^{-1} x$

22. Let $y = \text{Cot } x$ be a function identical to the cotangent function except that its domain is $0 < x < \pi$. State the domain and range of $y = \text{Cot}^{-1} x$ and sketch its graph.

23. Let $y = \text{Sec } x$ be a function identical to the secant function except that its domain is $0 \leq x \leq \pi, x \neq \frac{\pi}{2}$. State the domain and range of $y = \text{Sec}^{-1} x$ and sketch its graph.

24. Let $y = \text{Csc } x$ be a function identical to the cosecant function except that its domain is $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0$. State the domain and range of $y = \text{Csc}^{-1} x$ and sketch its graph.

25. **a.** Compare the values of $\text{Sec}^{-1} 2$ and $\text{Cos}^{-1} \frac{1}{2}$.
  b. Define $\text{Sec}^{-1} x$ in terms of the inverse cosine function.

26. Define $\text{Csc}^{-1} x$ in terms of the inverse sine function.

In Exercises 27 and 28, the given expression always has the same value for all x between -1 and 1, inclusive. Find this value and explain why the expression has a constant value.

27. $\text{Cos}^{-1} x + \text{Cos}^{-1} (-x)$

28. $\text{Sin}^{-1} x + \text{Sin}^{-1} (-x)$

29. Using calculus, one can prove that

$$\text{Tan}^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \text{ for } |x| \leq 1.$$

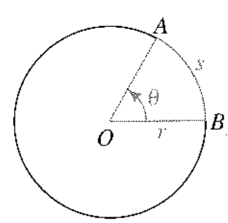
Use this relationship to show that $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$.

C 30. If $\text{Cos}^{-1} x + \text{Cos}^{-1} y = \frac{\pi}{2}$, show that $x^2 + y^2 = 1$. (*Hint*: Use Exercise 18 on page 279 and Exercise 7 on page 271.)

Chapter Summary

1. In trigonometry, an angle is often formed by rotating an *initial ray* to a *terminal ray*. When an angle's vertex is at the origin and its initial ray is along the positive x-axis, the angle is in *standard position*. Two angles in standard position are *coterminal angles* if they have the same terminal ray.

2. Angles can be measured in *revolutions*, *degrees*, or *radians*. Conversion formulas between degrees and radians are given on page 258.



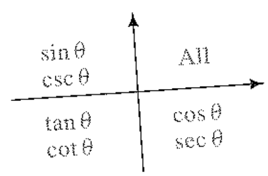
3. The radian measure of $\angle AOB$, shown at the right, is given by $\theta = \frac{s}{r}$. Thus, the length of arc AB is $s = r\theta$, and the area K of *sector AOB* is $K = \frac{1}{2}r^2\theta$ (θ in radians) or $K = \frac{1}{2}rs$.

4. If $P(x, y)$ is a point on the unit circle $x^2 + y^2 = 1$, and if θ is an angle in standard position with terminal ray OP, then the six *trigonometric functions* are defined as shown in the table below. Graphs of the sine, cosine, tangent, and secant functions are shown on pages 278 and 284.

Function	$\sin\theta = y$	$\cos\theta = x$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\csc\theta = \frac{1}{\sin\theta}$
Domain	all θ	all θ	$\theta \neq \frac{\pi}{2} + n\pi$	$\theta \neq n\pi$	$\theta \neq \frac{\pi}{2} + n\pi$	$\theta \neq n\pi$
Range	$\lvert\sin\theta\rvert \leq 1$	$\lvert\cos\theta\rvert \leq 1$	all reals	all reals	$\lvert\sec\theta\rvert \geq 1$	$\lvert\csc\theta\rvert \geq 1$
Period	2π	2π	π	π	2π	2π

5. Values of the sine and cosine functions for the angles 30°, 45°, and 60° are given in the table on page 277. To find the corresponding values of the other four trigonometric functions, use the definitions shown in the table above.

6. The acute angle α is the *reference angle* of $\theta = 180° \pm \alpha$, $\theta = 360° \pm \alpha$, and all coterminal angles. To find a trigonometric function of θ, first evaluate the function at α, the positive acute angle that θ makes with the x-axis. Use a plus or a minus sign depending on the quadrant involved. The diagram shows which functions are positive in each quadrant.



7. **a.** By restricting the domains of the trigonometric functions, we can define new one-to-one functions that have inverses. The domains and ranges of these *inverse trigonometric functions* are as follows:

Function	$y = \mathrm{Sin}^{-1}\, x$	$y = \mathrm{Cos}^{-1}\, x$	$y = \mathrm{Tan}^{-1}\, x$
Domain	$-1 \leq x \leq 1$	$-1 \leq x \leq 1$	all real x
Range	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$	$0 \leq y \leq \pi$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

b. The equation $y = \mathrm{Sin}^{-1} x$ is read "y is the inverse sine of x" and means that y is the value for which $\sin y = x$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

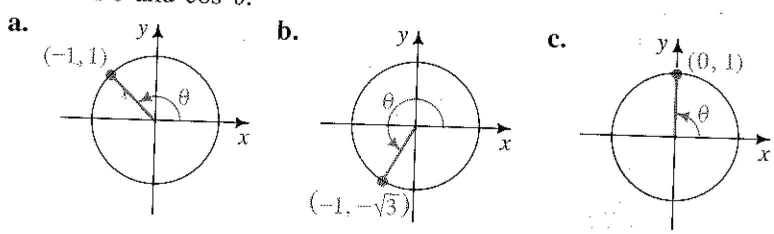
c. Graphs of three inverse trigonometric functions are on pages 287 and 288.

Key vocabulary and ideas

angle, initial ray, terminal ray (p. 257)
degree, minute, second (p. 257)
radian (p. 258)
standard position (p. 259)
coterminal angles (p. 260)
length of arc, area of a sector (p. 263)
sine and cosine functions (p. 268)
unit circle, circular function (p. 269)
reference angle (p. 275)
tangent and cotangent functions (p. 282)
secant and cosecant functions (p. 282)
Tan^{-1}, Sin^{-1}, Cos^{-1} (pp. 286–288)

Chapter Test

1. **a.** Convert 270° to radians. Give the answer in terms of π. 7-1
   b. Convert 192° to radians. Give the answer to the nearest hundredth.
2. **a.** Convert $\frac{5\pi}{3}$ radians to degrees.
   b. Convert 2.5 radians to degrees. Give the answer to the nearest ten minutes or tenth of a degree.
3. Find two angles, one positive and one negative, that are coterminal with each given angle.
   a. $-200°$ **b.** $313.2°$ **c.** $\frac{5\pi}{6}$ **d.** $142°10'$
4. A sector of a circle has radius 5 cm and central angle 48°. Find its approximate arc length and area. 7-2
5. The moon is about 4×10^5 km from Earth, and its apparent size is about 0.0087 radians. What is the moon's approximate diameter?
6. Find $\sin \theta$ and $\cos \theta$. 7-3



7. Complete each statement with one of the symbols <, >, or = .
   a. $\sin 60°$ _?_ $\sin 65°$ **b.** $\cos 60°$ _?_ $\cos 65°$
   c. $\sin 20°$ _?_ $\sin 160°$ **d.** $\cos 184°$ _?_ $\cos 185°$
8. Give the exact value of each expression in simplest radical form. 7-4
   a. $\sin \frac{5\pi}{4}$ **b.** $\cos(-90°)$ **c.** $\sin 150°$ **d.** $\cos \frac{11\pi}{6}$
9. Give the exact value of each expression or state that the value is undefined. 7-5
   a. $\tan 135°$ **b.** $\cot \frac{2\pi}{3}$ **c.** $\sec(-60°)$ **d.** $\csc(-\pi)$
10. If $\tan x = -\frac{1}{3}$ where $\frac{\pi}{2} < x < \pi$, find the values of the other five trigonometric functions.
11. Give the exact value of each expression. 7-6
    a. $\text{Tan}^{-1}\sqrt{3}$ **b.** $\cot\left(\text{Sin}^{-1}\frac{1}{2}\right)$ **c.** $\sec\left(\text{Cos}^{-1}\left(-\frac{3}{5}\right)\right)$
12. ***Writing*** To obtain the inverse sine function, we restrict the domain of $f(x) = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Write a short explanation in which you discuss why this restriction is necessary.