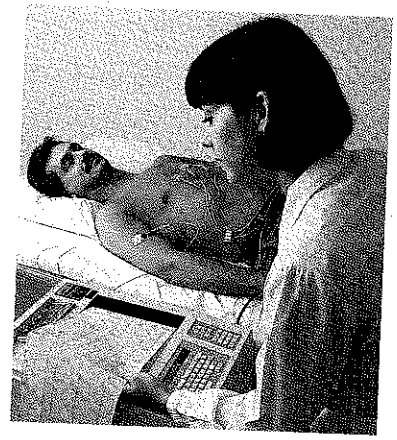
10 Trigonometric Addition Formulas

10-1 Formulas for cos $(\alpha \pm \beta)$ and sin $(\alpha \pm \beta)$

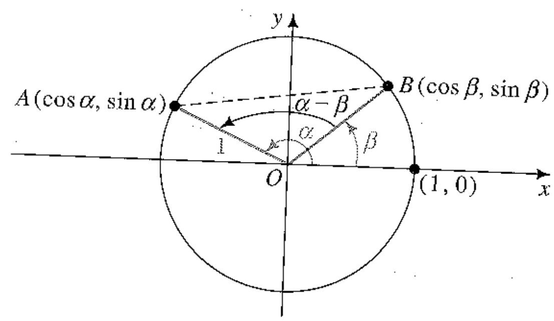
Objective *To derive and apply formulas for* $\cos(\alpha \pm \beta)$ *and for* $\sin(\alpha \pm \beta)$.

The photograph at the right shows a monitor (an electrocardiograph) recording the heartbeats of a healthy athlete. The monitor shows a wave that depicts the occurrence of each heartbeat over a period of time. The equation of this wave involves sines and cosines of α, 2α, 3α, and larger multiples of α. Our goal in this chapter is to gain experience working with expressions like $\sin 2\alpha$ and $\cos 3\alpha$. We will derive formulas showing, for example, how $\cos 2\alpha$ is related to the cosine of α and how $\cos(\alpha + \beta)$ is related to the sine and cosine of α and β.



Formulas for cos $(\alpha \pm \beta)$

To find a formula for $\cos(\alpha - \beta)$, let A and B be points on the unit circle with coordinates as shown in the diagram at the right. Then the measure of $\angle AOB$ is $\alpha - \beta$. The distance AB can be found by using either the law of cosines or the distance formula.



Using the law of cosines:

$$(AB)^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos(\alpha - \beta) = 2 - 2\cos(\alpha - \beta)$$

Using the distance formula:

$$\begin{aligned}(AB)^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2\\ &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta\\ &= (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\\ &= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\end{aligned}$$

Therefore,

$$2 - 2\cos(\alpha - \beta) = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta).$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad (1)$$

◀ The sound of every stringed instrument can be represented by the graph of a trigonometric function. The curve is determined by unique patterns of tone and overtone, which translate into sums of trigonometric functions.

To obtain a formula for $\cos(\alpha + \beta)$, we can use the formula for $\cos(\alpha - \beta)$ and replace β with $-\beta$. Recall that $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$.

$$\cos(\alpha - (-\beta)) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad (2)$$

Formulas for $\sin(\alpha \pm \beta)$

To find a formula for $\sin(\alpha + \beta)$, we use the cofunction relationship

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right).$$

Let $\theta = \alpha + \beta$. Then:

$$\begin{aligned}\sin(\alpha + \beta) &= \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right]\\ &= \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]\\ &= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta\end{aligned}$$

Since $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$ and $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad (3)$$

If we use the formula for $\sin(\alpha + \beta)$ and replace β with $-\beta$, we get:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad (4)$$

Sum and Difference Formulas for Cosine and Sine

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

There are two main purposes for the addition formulas: finding *exact* values of trigonometric expressions and simplifying expressions to obtain other identities. The following examples illustrate how to use these formulas.

Example 1 Find the exact value of $\sin 15°$.

Solution Use formula (4) to get the exact value.

$$\begin{aligned}\sin 15° &= \sin(45° - 30°)\\ &= \sin 45°\cos 30° - \cos 45°\sin 30°\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Example 2 Find the exact value of: **a.** $\cos 50° \cos 10° - \sin 50° \sin 10°$

b. $\sin \frac{5\pi}{12} \cos \frac{\pi}{12} + \cos \frac{5\pi}{12} \sin \frac{\pi}{12}$

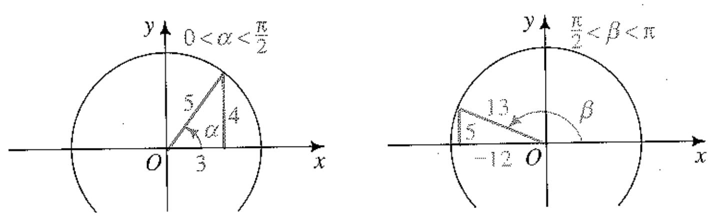
Solution The two given expressions have the patterns shown in formulas (2) and (3), respectively.

a.
$$\begin{aligned} \cos \alpha \cos \beta - \sin \alpha \sin \beta &= \cos(\alpha + \beta) \\ \cos 50° \cos 10° - \sin 50° \sin 10° &= \cos(50° + 10°) \\ &= \cos 60° \\ &= \frac{1}{2} \end{aligned}$$

b.
$$\begin{aligned} \sin \alpha \cos \beta + \cos \alpha \sin \beta &= \sin(\alpha + \beta) \\ \sin \frac{5\pi}{12} \cos \frac{\pi}{12} + \cos \frac{5\pi}{12} \sin \frac{\pi}{12} &= \sin\left(\frac{5\pi}{12} + \frac{\pi}{12}\right) \\ &= \sin \frac{\pi}{2} \\ &= 1 \end{aligned}$$

Example 3 Suppose that $\sin \alpha = \frac{4}{5}$ and $\sin \beta = \frac{5}{13}$, where $0 < \alpha < \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$. Find $\cos(\alpha + \beta)$.

Solution Sketch right-triangle diagrams to help find $\cos \alpha$ and $\cos \beta$. If $\sin \alpha = \frac{4}{5}$ and $0 < \alpha < \frac{\pi}{2}$, then $\cos \alpha = \frac{3}{5}$.



If $\sin \beta = \frac{5}{13}$ and $\frac{\pi}{2} < \beta < \pi$, then $\cos \beta = -\frac{12}{13}$.

Thus,

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$
$$\cos(\alpha + \beta) = \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) = -\frac{56}{65}$$

The sum or difference formulas can be used to verify many identities that we have seen, such as $\sin(90° - \theta) = \cos \theta$, and also to derive new identities.

Example 4 Show that $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$.

Solution By formula (4),

$$\begin{aligned}\sin\left(\frac{3\pi}{2} - x\right) &= \sin\left(\frac{3\pi}{2}\right)\cos x - \cos\left(\frac{3\pi}{2}\right)\sin x\\ &= (-1)\cos x - (0)\sin x\\ &= -\cos x\end{aligned}$$

Sometimes a problem involving a sum can be more easily solved if the sum can be expressed as a product. The formulas given below are derived from the addition formulas in Exercises 39–41.

Rewriting a Sum or Difference as a Product

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

CLASS EXERCISES

Find an example to show that, in general, each statement is true.

1. $\sin(x + y) \neq \sin x + \sin y$
2. $\cos(a - b) \neq \cos a - \cos b$

Are there any values of a and b for which each statement is true? If so, give an example.

3. $\sin(a - b) = \sin a - \sin b$
4. $\sin(a + b) = \sin a + \sin b$

Simplify each expression. Do not evaluate.

5. $\sin 1^\circ \cos 2^\circ + \cos 1^\circ \sin 2^\circ$
6. $\sin 20^\circ \cos 15^\circ - \cos 20^\circ \sin 15^\circ$
7. $\cos\frac{\pi}{4}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\sin\frac{\pi}{4}$
8. $\cos 75^\circ \cos 25^\circ + \sin 75^\circ \sin 25^\circ$
9. ***Discussion*** Explain how the identity $\sin(-\alpha) = -\sin\alpha$ is a special case of the difference formula for sine.

WRITTEN EXERCISES

Simplify the given expression.

A

1. $\sin 75° \cos 15° + \cos 75° \sin 15°$
2. $\cos 105° \cos 15° + \sin 105° \sin 15°$
3. $\cos \frac{5\pi}{12} \cos \frac{\pi}{12} - \sin \frac{5\pi}{12} \sin \frac{\pi}{12}$
4. $\sin \frac{4\pi}{3} \cos \frac{\pi}{3} - \cos \frac{4\pi}{3} \sin \frac{\pi}{3}$
5. $\sin 3x \cos 2x - \cos 3x \sin 2x$
6. $\cos 2x \cos x + \sin 2x \sin x$

In Exercises 7–10, prove that the given equation is an identity.

7. $\sin (x + \pi) = -\sin x$
8. $\cos (\pi + x) = -\cos x$
9. $\cos \left(x + \frac{\pi}{2}\right) = -\sin x$
10. $\cos \left(\frac{\pi}{2} - x\right) = \sin x$
11. ***Visual Thinking*** Translate the graph of $y = \sin x$ to the right by π units, and then reflect this curve about the x-axis. What graph results?
12. ***Visual Thinking*** Translate the graph of $y = \cos x$ to the left by $\frac{\pi}{2}$ units, and then reflect this curve about the x-axis. What graph results?

Find the exact value of each expression.

13. $\cos 75°$
14. $\cos 15°$
15. $\cos 105°$
16. $\sin 105°$
17. $\sin (-15°)$
18. $\cos (-165°)$
19. $\sin \frac{7\pi}{12}$
20. $\sin \frac{11\pi}{12}$

Simplify the given expression.

21. $\sin (30° + \theta) + \sin (30° - \theta)$
22. $\cos (30° + \theta) + \cos (30° - \theta)$
23. $\cos \left(\frac{\pi}{3} + x\right) + \cos \left(\frac{\pi}{3} - x\right)$
24. $\sin \left(\frac{\pi}{4} + x\right) + \sin \left(\frac{\pi}{4} - x\right)$
25. $\cos \left(\frac{3\pi}{2} + x\right) + \cos \left(\frac{3\pi}{2} - x\right)$
26. $\sin (\pi + x) + \sin (\pi - x)$

B

27. Suppose that $\sin \alpha = \frac{3}{5}$ and $\sin \beta = \frac{24}{25}$, where $0 < \alpha < \frac{\pi}{2} < \beta < \pi$. Find $\sin (\alpha + \beta)$.
28. Suppose that $\sin \alpha = \frac{4}{5}$ and $\sin \beta = \frac{1}{2}$, where $\frac{\pi}{2} < \beta < \alpha < \pi$. Find $\sin (\alpha - \beta)$.
29. Suppose that $\tan \alpha = \frac{4}{3}$ and $\tan \beta = \frac{12}{5}$, where $0 < \alpha < \beta < \frac{\pi}{2}$. Find $\cos (\alpha - \beta)$.
30. Suppose that $\sec \alpha = \frac{5}{4}$ and $\tan \beta = -1$, where $0 < \alpha < \frac{\pi}{2} < \beta < \pi$. Find $\cos (\alpha + \beta)$.

Use a graphing calculator set in radian mode to sketch each graph. Explain how the graph is related to a familiar graph and why.

31. $y = \sin x \cos 1 + \cos x \sin 1$

32. $y = \cos x \cos 2 + \sin x \sin 2$

Simplify the given expression.

33. $\dfrac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{\cos\alpha \cos\beta}$

34. $\dfrac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{\sin\alpha \sin\beta}$

35. $\cos x \cos y\ (\tan x + \tan y)$

36. $\sin x \sin y\ (\cot x \cot y - 1)$

37. $\sin(x + y) \sec x \sec y$

38. $\cos\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{6}\right)$

39. a. Derive the formula $\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$.
(*Hint*: Show that $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin\alpha \cos\beta$, and then substitute $\alpha = \frac{x + y}{2}$ and $\beta = \frac{x - y}{2}$.)

b. Use the formula in part (a) to show that $\sin 40° + \sin 20° = \cos 10°$.

c. Use the formula in part (a) to derive the formula for $\sin x - \sin y$.

40. a. Derive the formula $\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$. (Simplify $\cos(\alpha + \beta) + \cos(\alpha - \beta)$. Then substitute $\alpha = \frac{x + y}{2}$ and $\beta = \frac{x - y}{2}$, as was done in Exercise 39.)

b. Find the exact value of $\cos 105° + \cos 15°$.

41. a. Derive the formula $\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$. (*Hint*: See Exercise 39.)

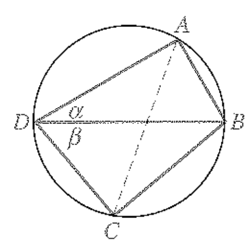
b. Find the exact value of $\cos 75° - \cos 15°$.

42. Evaluate $\sin\left(\mathrm{Tan}^{-1} \frac{1}{2} + \mathrm{Tan}^{-1} \frac{1}{3}\right)$ without using a calculator or tables.

43. Evaluate $\cos\left(\mathrm{Tan}^{-1} \frac{1}{2} + \mathrm{Tan}^{-1} 2\right)$ without using a calculator or tables.

44. Ptolemy's theorem states that if *ABCD* is inscribed in a circle, then

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$



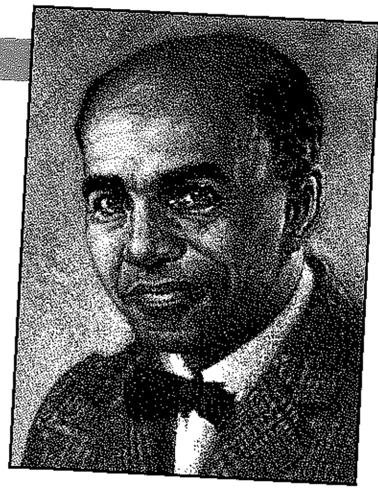
Consider a special case of this theorem in which $\overline{BD}$ is a diameter with length 1.

a. Show that $AB = \sin\alpha$ and $AD = \cos\alpha$.

b. Find *BC* and *CD* in terms of β.

c. Show that $AC = \sin(\alpha + \beta)$.

d. Use parts (a), (b), and (c), and Ptolemy's theorem to derive the formula for $\sin(\alpha + \beta)$.



David Blackwell (1919–)

David Blackwell expected to become a schoolteacher like his grandfather, but his interest in statistics led to a Ph.D. at the age of 22. After teaching at Howard University for ten years, he became a professor of statistics at the University of California at Berkeley in 1954.

Blackwell has made contributions to many fields of mathematics, including probability theory, game theory, set theory, and dynamic programming. In 1965, Blackwell was the first African American mathematician elected to the National Academy of Sciences.

10-2 Formulas for tan $(\alpha \pm \beta)$

Objective *To derive and apply formulas for tan $(\alpha \pm \beta)$.*

To derive a formula expressing $\tan(\alpha + \beta)$ in terms of $\tan \alpha$ and $\tan \beta$, we use the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}\\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\end{aligned}$$

Dividing the numerator and the denominator by $\cos\alpha\cos\beta$, we obtain:

$$\tan(\alpha + \beta) = \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}}$$

Therefore, we have the following formula.

Sum Formula for Tangent

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad (5)$$

This identity is valid for all values of α and β for which $\tan\alpha$, $\tan\beta$, and $\tan(\alpha + \beta)$ are defined.

To derive a formula for tan $(\alpha - \beta)$, simply replace β with $-\beta$ in formula (5) and use the fact that $\tan(-\beta) = -\tan \beta$. Thus, we have the following formula.

Difference Formula for Tangent

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \qquad (6)$$

Example 1 Suppose $\tan \alpha = \frac{1}{3}$ and $\tan \beta = \frac{1}{2}$.

a. Find $\tan(\alpha + \beta)$. **b.** Show that $\text{Tan}^{-1}\frac{1}{3} + \text{Tan}^{-1}\frac{1}{2} = \frac{\pi}{4}$.

Solution **a.** $\tan(\alpha + \beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \dfrac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{3} \cdot \frac{1}{2}} = \dfrac{\frac{5}{6}}{\frac{5}{6}} = 1$

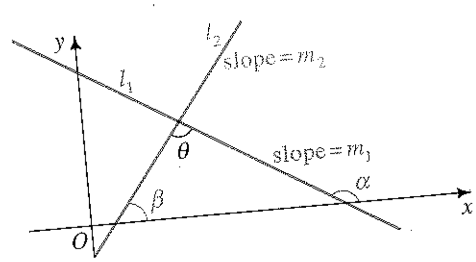
b. Let $\alpha = \text{Tan}^{-1}\frac{1}{3}$ and $\beta = \text{Tan}^{-1}\frac{1}{2}$. Since α and β are both between 0 and $\frac{\pi}{2}$, $\alpha + \beta$ is between 0 and π. From part (a), we know that $\tan(\alpha + \beta) = 1$. Thus, since $\frac{\pi}{4}$ is the only angle between 0 and π whose tangent is 1,

$$\alpha + \beta = \frac{\pi}{4}.$$

That is,

$$\text{Tan}^{-1}\frac{1}{3} + \text{Tan}^{-1}\frac{1}{2} = \frac{\pi}{4}.$$

Angles between Two Lines



The formula for $\tan(\alpha - \beta)$ can be used to find an angle between two intersecting lines. For example, consider the angle θ between lines l_1 and l_2 with slopes m_1 and m_2. Suppose the inclinations of these lines are α and β, respectively. Recall from Section 8-1 that the slope of a nonvertical line is the tangent of its inclination. Thus,

$$\tan \alpha = m_1 \text{ and } \tan \beta = m_2.$$

Since the measure of an exterior angle of a triangle equals the sum of the measures of the opposite interior angles, $\alpha = \beta + \theta$. Therefore,

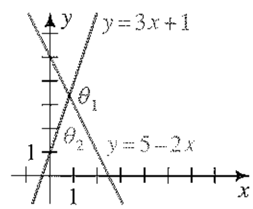
$$\theta = \alpha - \beta \quad \text{and} \quad \tan \theta = \tan(\alpha - \beta).$$

Substituting in $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$ gives the following formula:

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Example 2 Find an angle between the lines $y = 3x + 1$ and $y = 5 - 2x$.

Solution The lines have slopes 3 and -2. Thus, we can either let $m_1 = 3$ and $m_2 = -2$ or let $m_1 = -2$ and $m_2 = 3$. These two possibilities give us the two supplementary angles, θ_1 and θ_2, that are formed by the lines.



(1) The first case gives

$$\tan\theta_1 = \frac{3 - (-2)}{1 + 3(-2)} = -1. \text{ So, } \theta_1 = 135°.$$

(2) The second case gives

$$\tan\theta_2 = \frac{-2 - 3}{1 + (-2)3} = 1. \text{ So, } \theta_2 = 45°.$$

CLASS EXERCISES

1. Suppose $\tan\alpha = 2$ and $\tan\beta = 3$. Find **(a)** $\tan(\alpha + \beta)$ and **(b)** $\tan(\alpha - \beta)$.
2. Find the exact value: **a.** $\dfrac{\tan 15° + \tan 30°}{1 - \tan 15°\tan 30°}$ **b.** $\dfrac{\tan 85° - \tan 25°}{1 + \tan 85°\tan 25°}$
3. ***Discussion*** Interpret the formula for finding an angle between two lines when $1 + m_1 m_2 = 0$.

WRITTEN EXERCISES

In Exercises 1 and 2, find (a) $\tan(\alpha + \beta)$ and (b) $\tan(\alpha - \beta)$.

A 1. $\tan\alpha = \frac{2}{3}$ and $\tan\beta = \frac{1}{2}$

2. $\tan\alpha = 2$ and $\tan\beta = -\frac{1}{3}$

In Exercises 3–6, find the exact value of the given expression.

3. $\dfrac{\tan 75° - \tan 30°}{1 + \tan 75°\tan 30°}$

4. $\dfrac{\tan 100° + \tan 50°}{1 - \tan 100°\tan 50°}$

5. $\dfrac{\tan\frac{2\pi}{3} + \tan\frac{\pi}{12}}{1 - \tan\frac{2\pi}{3}\tan\frac{\pi}{12}}$

6. $\dfrac{\tan\frac{4\pi}{3} - \tan\frac{\pi}{12}}{1 + \tan\frac{4\pi}{3}\tan\frac{\pi}{12}}$

7. Evaluate $\tan\left(\frac{\pi}{4} + \theta\right)$ when $\tan \theta = \frac{1}{2}$.

8. Evaluate $\tan\left(\frac{3\pi}{4} - \theta\right)$ when $\tan \theta = \frac{1}{3}$.

9. Show that $\tan(-\alpha) = -\tan \alpha$.

10. ***Visual Thinking*** Simplify $\tan(x + \pi)$ and interpret your answer graphically. What does your answer illustrate about the period of the tangent function?

11. Evaluate $\tan 75°$ and $\tan 165°$ without using a calculator or tables.

12. Evaluate $\tan 15°$ and $\tan 105°$ without using a calculator or tables.

13. Find the two supplementary angles formed by the line $y = 3x - 5$ and the line $y = x + 4$.

14. Find the two supplementary angles formed by the line $3x + 2y = 5$ and the line $4x - 3y = 1$.

Use a graphing calculator set in radian mode to sketch each graph. Explain how the graph is related to the graph of $y = \tan x$ and why.

B 15. $y = \dfrac{\tan x + \tan 1}{1 - \tan x \tan 1}$

16. $y = \dfrac{\tan x - \tan \frac{\pi}{8}}{1 + \tan x \tan \frac{\pi}{8}}$

17. Suppose $\tan \alpha = \frac{1}{4}$ and $\tan \beta = \frac{3}{5}$, where $0 < \alpha < \beta < \frac{\pi}{2}$. Find $\tan(\alpha + \beta)$.

Then show that $\mathrm{Tan}^{-1} \frac{1}{4} + \mathrm{Tan}^{-1} \frac{3}{5} = \frac{\pi}{4}$.

18. Suppose $\tan \alpha = 3$ and $\tan \beta = \frac{1}{2}$, where $0 < \beta < \alpha < \frac{\pi}{2}$. Find $\tan(\alpha - \beta)$.

Then show that $\mathrm{Tan}^{-1} 3 - \mathrm{Tan}^{-1} \frac{1}{2} = \frac{\pi}{4}$.

19. Suppose $\alpha = \mathrm{Tan}^{-1} 2$ and $\beta = \mathrm{Tan}^{-1} 3$. Show that $\tan(\alpha + \beta) = -1$.

20. Suppose $\alpha = \mathrm{Tan}^{-1} 5$ and $\beta = \mathrm{Tan}^{-1} \frac{2}{3}$. Show that $\alpha - \beta = \frac{\pi}{4}$.

21. ***Investigation*** For what values of α and β does $\tan(\alpha + \beta) = \tan \alpha + \tan \beta$? For what values of α and β does $\tan(\alpha - \beta) = \tan \alpha - \tan \beta$?

22. Given $A(3, 1)$, $B(14, -1)$, and $C(5, 5)$, find the measure of $\angle BAC$ by using **(a)** the slopes of lines AB and AC and **(b)** the law of cosines.

23. Suppose $\cot \alpha = 2$ and $\cot \beta = \frac{2}{3}$, where $0 < \alpha < \beta < \frac{\pi}{2}$. Find $\cot(\alpha + \beta)$.

24. Suppose $\cot \alpha = \frac{3}{2}$ and $\cot \beta = \frac{1}{2}$, where $0 < \alpha < \beta < \frac{\pi}{2}$. Find $\cot(\alpha - \beta)$.

25. Suppose $\sin \alpha = \frac{3}{5}$ and $\cos \beta = \frac{5}{13}$, where $0 < \alpha < \beta < \frac{\pi}{2}$. Find:

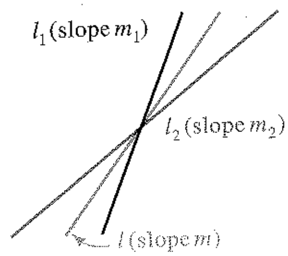
a. $\sin(\alpha + \beta)$ **b.** $\cos(\alpha + \beta)$ **c.** $\tan(\alpha + \beta)$

26. Suppose $\sin\alpha = \frac{4}{5}$ and $\tan\beta = -\frac{3}{4}$, where $\frac{\pi}{2} < \alpha < \beta < \pi$. Find:

a. $\sin(\alpha + \beta)$ **b.** $\cos(\alpha + \beta)$ **c.** $\tan(\alpha + \beta)$

C **27. a.** Line l bisects θ, the angle formed by lines l_1 and l_2. If the slopes of these three lines are m, m_1, and m_2, respectively, show that:

$$\frac{m_1 - m}{1 + m_1 m} = \frac{m - m_2}{1 + m_2 m}$$



b. If l_1 and l_2 have equations $y = 2x$ and $y = x$, find an equation of l.

28. Verify that in any $\triangle ABC$,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

29. Verify that $4\,\text{Tan}^{-1}\frac{1}{5} - \text{Tan}^{-1}\frac{1}{239} = \frac{\pi}{4}$. (*Hint*: Let $\alpha = \text{Tan}^{-1}\frac{1}{5}$ and let $\beta = \text{Tan}^{-1}\frac{1}{239}$. Then find $\tan(4\alpha - \beta)$.)

COMMUNICATION: Discussion

Small Group Discussion

Often when you are faced with a difficult problem, you can respond better if you have discussed it with others. Working cooperatively in a small group allows each of you to contribute your strengths and skills to solving the problem. Sometimes another person's perspective helps you better understand a problem. Likewise, your perspective may help someone else in the group gain insight. When working in a group, it is important that all members participate. This means that you must speak up when you have ideas and also listen carefully to the ideas of other persons in your group.

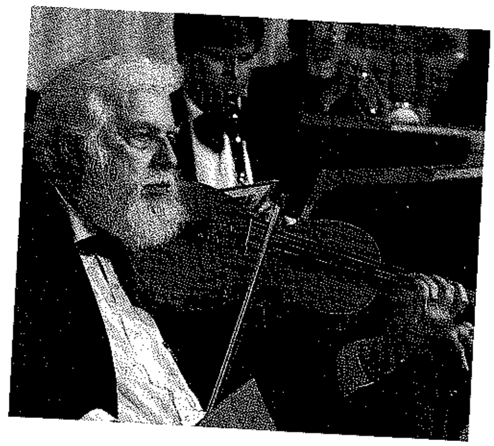
Working in groups of three or four, discuss one of the following questions. Then present your results to the entire class.

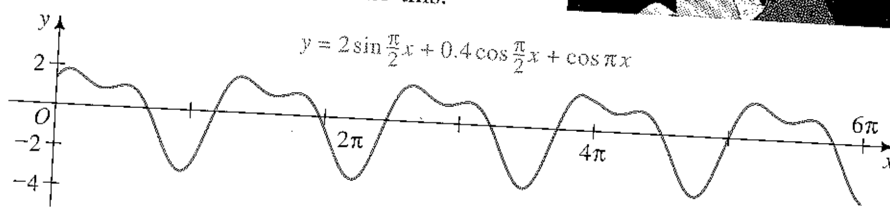
1. The decimal expansion of $\frac{1}{7}$, namely $0.\overline{142857}$, has a six-digit block of repeating digits. What is the maximum length of the block of repeating digits in the decimal expansion of $\frac{1}{n}$? Explain your answer.
2. Are there more rational numbers or irrational numbers?
3. Is the sum of two sine curves always another sine curve?
4. Why does it make sense that $b^0 = 1$ if $b \neq 0$? Why is 0^0 undefined?

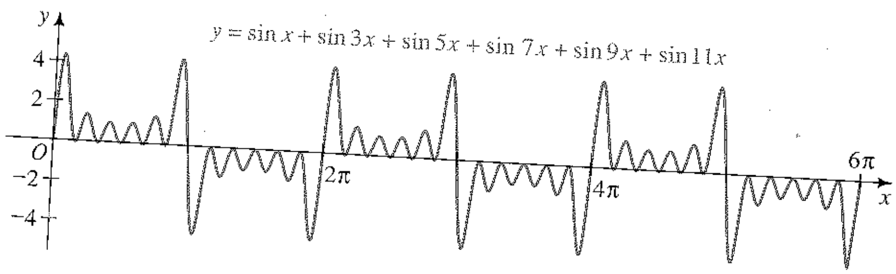
10-3 Double-Angle and Half-Angle Formulas

Objective *To derive and apply double-angle and half-angle formulas.*

Trigonometric functions are used in science and engineering to study light and sound waves. An important application is the wave pattern of a vibrating string. Consider the wave of a note sounded by a violin. Not only does the string vibrate as a whole, producing a fundamental tone, but it also vibrates in halves, thirds, and progressively smaller segments, producing overtones (called harmonics). An equation of this type of wave involves sums of sines and cosines of x, $2x$, $3x$, and greater multiples of x. The computer-generated graphs below illustrate this.







If you know the value of $\sin \alpha$, you do *not* double it to find $\sin 2\alpha$. Nor do you halve it to find $\sin \frac{1}{2}\alpha$. To see that this is true, complete the following activity.

Activity

1. On a single set of axes, graph $y = \sin 2x$ and $y = 2 \sin x$. Do the graphs coincide? What does this tell you about the values of $\sin 2x$ and $2 \sin x$?
2. On a single set of axes, graph $y = \sin \frac{1}{2}x$ and $y = \frac{1}{2}\sin x$. Do the graphs coincide? What does this tell you about the values of $\sin \frac{1}{2}x$ and $\frac{1}{2} \sin x$?

Double-Angle Formulas

The following double-angle formulas are special cases of the formulas for $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$, and $\tan(\alpha+\beta)$. If we let $\beta = \alpha$ in these formulas we obtain the following formulas.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad (7)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad (8a)$$

Using the fact that $\sin^2\alpha + \cos^2\alpha = 1$, we can obtain alternative formulas for $\cos 2\alpha$:

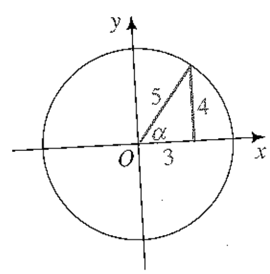
$$\cos 2\alpha = 1 - 2\sin^2\alpha \qquad (8b)$$
$$\cos 2\alpha = 2\cos^2\alpha - 1 \qquad (8c)$$

To express $\tan 2\alpha$ in terms of $\tan\alpha$, we again let $\beta = \alpha$:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\alpha+\alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \qquad (9)$$

Example 1 If $\sin\alpha = \frac{4}{5}$ and $0 < \alpha < \frac{\pi}{2}$, find $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$.

Solution From the given information and the diagram, we know that $\cos\alpha = \frac{3}{5}$ and $\tan\alpha = \frac{4}{3}$.



$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = -\frac{7}{25}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2} = -\frac{24}{7}$$

Example 2 Derive a formula for $\sin 4x$ in terms of functions of x.

Solution

$$\begin{aligned}\sin 4x = \sin[2(2x)] &= 2 \sin 2x \cos 2x \\ &= 2(2 \sin x \cos x)(\cos^2 x - \sin^2 x) \\ &= 4 \sin x \cos^3 x - 4 \sin^3 x \cos x\end{aligned}$$

Using another form for $\cos 2x$ will yield different but equivalent results.

Half-Angle Formulas

To obtain the sine and cosine half-angle formulas, we use formulas (8b) and (8c), replacing α with $\frac{x}{2}$.

$$\begin{aligned}\cos 2\alpha &= 1 - 2\sin^2 \alpha \\ \cos 2\left(\frac{x}{2}\right) &= 1 - 2\sin^2 \frac{x}{2} \\ \cos x &= 1 - 2\sin^2 \frac{x}{2} \\ 2\sin^2 \frac{x}{2} &= 1 - \cos x \\ \sin \frac{x}{2} &= \pm\sqrt{\frac{1 - \cos x}{2}} \qquad (10)\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= 2\cos^2 \alpha - 1 \\ \cos 2\left(\frac{x}{2}\right) &= 2\cos^2 \frac{x}{2} - 1 \\ \cos x &= 2\cos^2 \frac{x}{2} - 1 \\ 2\cos^2 \frac{x}{2} &= 1 + \cos x \\ \cos \frac{x}{2} &= \pm\sqrt{\frac{1 + \cos x}{2}} \qquad (11)\end{aligned}$$

When you use the half-angle formulas, choose + or − depending on the quadrant in which $\frac{x}{2}$ lies.

Example 3 Find the exact value of $\cos \frac{5\pi}{8}$.

Solution Because $\frac{5\pi}{8}$ is in the second quadrant, $\cos \frac{5\pi}{8}$ is negative. Since $\frac{5\pi}{8} = \frac{1}{2}\left(\frac{5\pi}{4}\right)$, we can let $x = \frac{5\pi}{4}$ in formula (11).

$$\begin{aligned}\cos \frac{5\pi}{8} &= -\sqrt{\frac{1 + \cos \frac{5\pi}{4}}{2}} = -\sqrt{\frac{1 + \left(-\frac{\sqrt{2}}{2}\right)}{2}} \\ &= -\sqrt{\frac{2 - \sqrt{2}}{4}} = -\frac{\sqrt{2 - \sqrt{2}}}{2}\end{aligned}$$

To derive a formula for $\tan \frac{x}{2}$, divide equation (10) by equation (11):

$$\tan \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} \qquad (12)$$

However, the following formulas, which can be derived by simplifying the radical expression in formula (12), may be more useful.

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x} \quad (12a) \qquad \tan\frac{x}{2} = \frac{1 - \cos x}{\sin x} \quad (12b)$$

Notice that these formulas don't need the ambiguous sign $\pm$. (Why?) You are asked to prove these formulas in Exercises 37 and 38.

The following table summarizes the double-angle and half-angle formulas.

Double-Angle and Half-Angle Formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} = \frac{\sin\alpha}{1 + \cos\alpha} = \frac{1 - \cos\alpha}{\sin\alpha}$$

CLASS EXERCISES

In Exercises 1–10, simplify the given expression.

1. $2\sin 10^\circ \cos 10^\circ$

2. $\cos^2 15^\circ - \sin^2 15^\circ$

3. $1 - 2\sin^2 35^\circ$

4. $2\cos^2 25^\circ - 1$

5. $\dfrac{2\tan 50^\circ}{1 - \tan^2 50^\circ}$

6. $\dfrac{2\tan 40^\circ}{1 - \tan^2 40^\circ}$

7. $1 - \sin^2 x$

8. $1 - 2\sin^2 x$

9. $2\sin 3\alpha \cos 3\alpha$

10. $\cos^2 5\theta - \sin^2 5\theta$

11. ***Discussion*** Given that $\cos 70^\circ \approx 0.3420$, explain how you can find $\cos 35^\circ$.

WRITTEN EXERCISES

In Exercises 1–12, simplify the given expression.

A

1. $2\cos^2 10^\circ - 1$

2. $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

3. $\dfrac{4\tan\beta}{1 - \tan^2\beta}$

4. $1 - 2\sin^2 20^\circ$

5. $2\sin 35^\circ \cos 35^\circ$

6. $\cos^2 4A - \sin^2 4A$

7. $\dfrac{2\tan 25^\circ}{1-\tan^2 25^\circ}$

8. $2\cos^2 3\alpha - 1$

9. $1 - 2\sin^2 \dfrac{x}{2}$

10. $\cos^2 40^\circ - \sin^2 40^\circ$

11. $\sqrt{\dfrac{1-\cos 80^\circ}{2}}$

12. $\sqrt{\dfrac{1+\cos 70^\circ}{2}}$

In Exercises 13–18, find the exact value of the given expression.

13. $2\cos^2 \dfrac{\pi}{8} - 1$

14. $\dfrac{2\tan \frac{\pi}{8}}{1-\tan^2 \frac{\pi}{8}}$

15. $\cos^2 \dfrac{\pi}{12} - \sin^2 \dfrac{\pi}{12}$

16. $1 - 2\sin^2 \dfrac{7\pi}{12}$

17. $\sin 15^\circ \cos 15^\circ$

18. $4\sin \dfrac{2\pi}{3} \cos \dfrac{2\pi}{3}$

In Exercises 19–24, $\angle A$ is acute.

19. If $\sin A = \frac{5}{13}$, find $\sin 2A$ and $\cos 2A$.

20. If $\tan A = \frac{1}{2}$, find $\cos 2A$ and $\tan 2A$.

21. If $\sin A = \frac{3}{5}$, find $\sin 2A$ and $\sin 4A$.

22. If $\cos A = \frac{1}{3}$, find $\cos 2A$ and $\cos 4A$.

23. If $\cos A = \frac{1}{5}$, find $\cos 2A$ and $\cos \frac{A}{2}$.

24. If $\cos A = \frac{1}{4}$, find $\sin 2A$ and $\sin \frac{A}{2}$.

25. Find $\cos 105^\circ$ using **(a)** an addition formula and **(b)** a half-angle formula.

26. Find $\sin 75^\circ$ using **(a)** an addition formula and **(b)** a half-angle formula.

Use a graphing calculator or computer to sketch the graph of each function. Then give the range and period of the function.

B 27. $y = \sin x + \cos 2x$

28. $y = \sin 3x + 2\cos x$

29. $y = 2\sin 2x + 4\cos 4x$

30. $y = 3\cos \dfrac{x}{2} - 2\cos 3x$

In Exercises 31–38, prove that the given equation is an identity.

31. $\dfrac{\sin 2A}{1-\cos 2A} = \cot A$

32. $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$

33. $\left(\sin \dfrac{x}{2} + \cos \dfrac{x}{2}\right)^2 = 1 + \sin x$

34. $\sin 4x = 4\sin x \cos x \cos 2x$

35. $\dfrac{1-\tan^2 x}{1+\tan^2 x} = \cos 2x$

36. $\dfrac{1+\sin A - \cos 2A}{\cos A + \sin 2A} = \tan A$

37. $\tan \dfrac{x}{2} = \dfrac{\sin x}{1+\cos x}$

38. $\tan \dfrac{x}{2} = \dfrac{1-\cos x}{\sin x}$

Simplify the given expression.

39. $\dfrac{1+\cos 2x}{\cot x}$

40. $\dfrac{(1+\tan^2 x)(1-\cos 2x)}{2}$

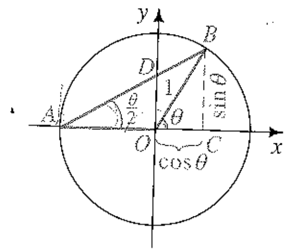
41. $(1 - \sin^2 x)(1 - \tan^2 x)$

42. $\sin x \tan x + \cos 2x \sec x$

43. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x}$

44. $\cos^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) - \sin^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)$

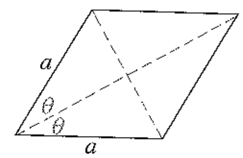
45. ***Visual Thinking*** The diagram at the right shows a unit circle, with the measure of $\angle BOC = \theta$.



a. Explain why the measure of $\angle BAO = \dfrac{\theta}{2}$.

b. Explain why $\tan \dfrac{\theta}{2} = \dfrac{\sin \theta}{1 + \cos \theta}$.

46. Use the rhombus at the right and the steps below to derive the formula for $\sin 2\theta$ for $0° < \theta < 90°$.



a. Show that the area of the rhombus is $a^2 \sin 2\theta$.

b. Use the fact that the diagonals of a rhombus are perpendicular to show that its area is

$$2a^2 \sin \theta \cos \theta.$$

c. Use your answers from parts (a) and (b) to obtain a formula for $\sin 2\theta$.

47. Find $\log_2 2 + \log_2 (\sin x) + \log_2 (\cos x)$ when $x = \dfrac{\pi}{12}$.

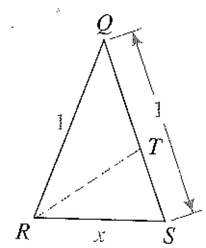
48. Find $\dfrac{4^{2\cos^2\theta}}{4}$ when $\theta = \dfrac{\pi}{3}$.

Prove that the given equation is an identity.

49. $\sin 3x = 3 \sin x - 4 \sin^3 x$

50. $\cos 3x = 4 \cos^3 x - 3 \cos x$

C 51. Given: $QR = QS = 1$, $\angle Q = 36°$, and $\overline{RT}$ bisects $\angle R$.



a. Prove that $\triangle QRS$ is similar to $\triangle RST$.

b. Use similar triangles to show that $\dfrac{x}{1-x} = \dfrac{1}{x}$.

c. Show that $x = \dfrac{\sqrt{5} - 1}{2}$.

d. Draw the bisector of $\angle Q$. Show that $\sin 18° = \dfrac{\sqrt{5} - 1}{4} = \cos 72°$.

52. In $\triangle ABC$, the measure of $\angle B$ is twice the measure of $\angle C$.

a. Use the law of sines to show that $b = 2c \cos C$.

b. Use the law of cosines to show that $b^2 = c(a + c)$.

53. The lengths of the sides of a triangle are consecutive integers n, $n + 1$, and $n + 2$, and the largest angle is twice the smallest angle θ.

a. Use the law of sines to show that $\cos \theta = \dfrac{n + 2}{2n}$.

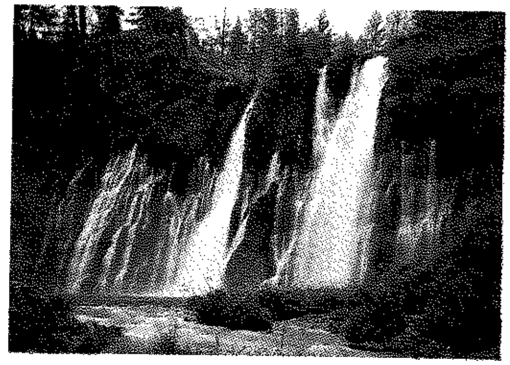
b. Use the law of cosines to show that $\cos \theta = \dfrac{n + 5}{2(n + 2)}$.

c. Use parts (a) and (b) to find n.

10-4 Solving Trigonometric Equations

Objective *To use identities to solve trigonometric equations.*

The acceleration of a body falling toward the Earth's surface is called acceleration due to gravity, often denoted by g. In theoretical physics, g is usually considered constant. However, g is not actually constant but varies slightly with latitude. A good approximation to the value of g can be found by using the following formula, which expresses g in terms of θ, the latitude in degrees.



Gravity acting on water creates this waterfall.

$$g \approx 9.78049(1 + 0.005288 \sin^2 \theta - 0.000006 \sin^2 2\theta)$$

For example, if you live in Chicago, which has a latitude of 42°N, $\sin \theta \approx 0.6691$ and $\sin 2\theta \approx 0.9945$. Therefore, $g \approx 9.8036$ m/s^2.

As you can see from this example, some problems involve trigonometric equations that have multiples of angles or numbers. The following suggests two methods that may be helpful in solving such equations. The first method gives a graphical method using a graphing calculator or computer, and the second method gives ways for solving the equation algebraically.

Methods for Solving the Trigonometric Equation $f(x) = g(x)$

Method 1

Use a graphing calculator or computer to graph $y = f(x)$ and $y = g(x)$ on the same set of axes. Use the zoom or trace feature to find the x-coordinates of any intersection points of the two graphs.

Method 2

Use the following guidelines.

a. It may be helpful to draw a quick sketch of $y = f(x)$ and $y = g(x)$ to see roughly where the solutions are.

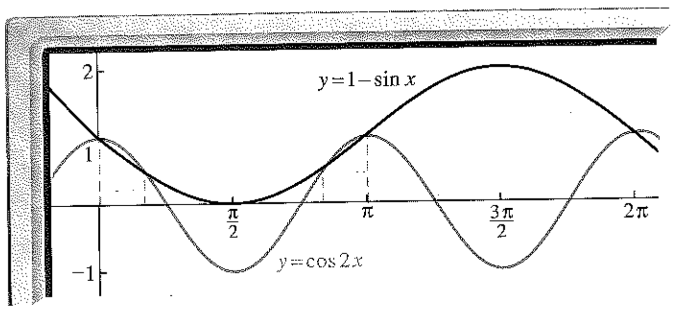
b. If the equation involves functions of $2x$ and x, transform the functions of $2x$ into functions of x by using identities.

c. If the equation involves functions of $2x$ only, it is usually better to solve for $2x$ directly and then solve for x.

d. Be careful not to lose roots when you divide both sides of an equation by a function of the variable. Review the discussion about losing roots on pages 32–33.

Example 1 Solve $\cos 2x = 1 - \sin x$ for $0 \le x < 2\pi$.

Solution Method 1 The diagram below shows the graphs of $y = \cos 2x$ and $y = 1 - \sin x$ on the same set of axes. There are four solutions in the interval $0 \le x < 2\pi$. From the diagram, you can see that 0 and π are solutions. Using a zoom feature, you can find that $0.52 \approx \frac{\pi}{6}$ and $2.62 \approx \frac{5\pi}{6}$ are also solutions.



Method 2

$$\cos 2x = 1 - \sin x$$
$$1 - 2\sin^2 x = 1 - \sin x$$
$$2\sin^2 x = \sin x$$

$\sin x = 0$	$2 \sin x = 1$
$x = 0, \pi$	$\sin x = \frac{1}{2}$
	$x = \frac{\pi}{6}, \frac{5\pi}{6}$

Graphs are useful for solving not only trigonometric equations but also trigonometric inequalities. For example, to solve the inequality $\cos 2x < 1 - \sin x$, look at the graph shown in Example 1. You can see that the graph of $y = \cos 2x$ is *below* the graph of $y = 1 - \sin x$ when $\frac{\pi}{6} < x < \frac{5\pi}{6}$ and when $\pi < x < 2\pi$.

Example 2 Solve $3 \cos 2x + \cos x = 2$ for $0 \le x < 2\pi$.

Solution

$$3\cos 2x + \cos x = 2$$
$$3(2\cos^2 x - 1) + \cos x = 2 \leftarrow \cos 2x = 2\cos^2 x - 1$$
$$6\cos^2 x + \cos x - 5 = 0$$
$$(6\cos x - 5)(\cos x + 1) = 0$$

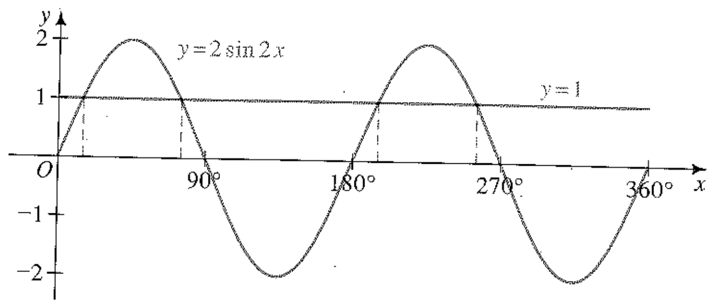
$\cos x = \frac{5}{6}$ or $\cos x = -1$

$x \approx 0.59, 5.70$ $\quad x = \pi$

$x \approx 3.14$

Example 3 Solve $2 \sin 2x = 1$ for $0° \leq x < 360°$.

Solution The graphs of $y = 2 \sin 2x$ and $y = 1$ over the interval $0° \leq x < 360°$ are shown below. You can see that there are four possible solutions.



The most efficient way to solve this equation algebraically is to solve directly for $2x$.

$$\sin 2x = \frac{1}{2}$$

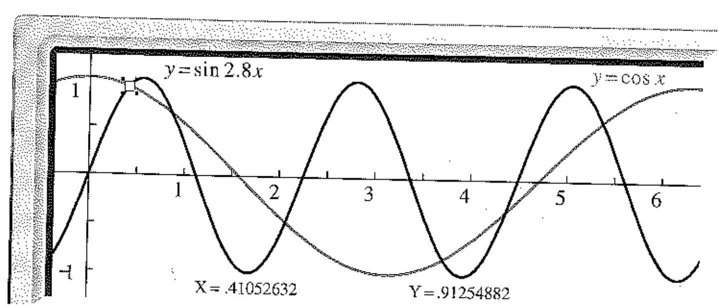
$$2x = 30°, 150°, 390°, 510° \quad \longleftarrow \; 0° \leq 2x < 720°$$

$$x = 15°, 75°, 195°, 255°$$

Examples 1 through 3 show that some trigonometric equations can be solved either algebraically or graphically. Example 4 shows that for some equations the best method is to use a graphing calculator or computer.

Example 4 Find the smallest positive root of $\sin 2.8x = \cos x$.

Solution You can make a sketch of the graphs of $y = \sin 2.8x$ and $y = \cos x$ on the same set of axes to see approximately where the solutions are, but then finding an algebraic solution would be very difficult. In this case, the most efficient method is to use a graphing calculator or a computer. The diagram below indicates that there are six roots between 0 and 2π. You are interested in the smallest one.



Using the zoom or trace feature, you will find that $x \approx 0.41$.

CLASS EXERCISES

Describe the method you would use to solve the following equations for $0° \leq x < 360°$. Some of these equations can be solved by methods discussed in Chapter 8.

1. $\cos 2x = \cos x$

2. $\sin^2 x = \sin x$

3. $\sin x = \cos x$

4. $\sin 2x = \cos 2x$

5. $\sin 3x = \cos 3x$

6. $\tan (x - 10°) = 1$

7. ***Discussion*** When solving $\sin 4x = \sin 2x$, would you begin by writing the expressions in terms of functions of x or $2x$? Why?

8. ***Discussion*** When solving $\cos 4x = 1 - 3 \cos 2x$, would you begin by writing the expressions in terms of functions of x or $2x$? Why?

9. a. On a single set of axes, sketch the graphs of $y = \sin x$ and $y = \cos x$ over the interval $0 \leq x \leq \frac{\pi}{2}$.

b. Find the x-coordinates of any points of intersection.

c. Over what interval(s) is $\sin x > \cos x$?

d. Over what interval(s) is $\sin x < \cos x$?

WRITTEN EXERCISES

A **1–8.** Solve the equations in Class Exercises 1–8 for $0° \leq x < 360°$.

9. a. On a single set of axes, sketch the graphs of $y = \sin 2x$ and $y = \tan x$ for $0 \leq x < 2\pi$.

b. Find the x-coordinates of the points where the graphs intersect by solving $\sin 2x = \tan x$.

c. Solve $\sin 2x > \tan x$ for $0 \leq x < 2\pi$.

10. a. On a single set of axes, sketch the graphs of $y = \cos 2x$ and $y = 3 \cos x$ for $0 \leq x < 2\pi$.

b. Find the x-coordinates of the points where the graphs intersect by solving $\cos 2x = 3 \cos x$.

c. Solve $\cos 2x < 3 \cos x$ for $0 \leq x < 2\pi$.

In Exercises 11–16, solve each inequality for $0 \leq x < 2\pi$. Give answers to the nearest hundredth. You may wish to use a graphing calculator or computer.

11. $\cos x > \frac{1}{2} \sin x$

12. $\tan x < 2 \sin x$

13. $\sin \left(3x - \frac{\pi}{2}\right) > 0$

14. $2 \cos x \leq \cos \left(x - \frac{\pi}{2}\right)$

15. $\cos x \leq \sin 2x$

16. $\cos 2x \geq 5 - \cos x$

In Exercises 17–22, solve each equation for $0° \leq x < 360°$ by using trigonometric identities. Give answers to the nearest tenth when necessary. You may wish to check your answers using a graphing calculator or computer.

B

17. $2 \cos (x + 45°) = 1$

18. $\cot (x - 20°) = 1$

19. $\sin (60° - x) = 2 \sin x$

20. $2 \sin (30° + x) = 3 \cos x$

21. $\sin x = \sin 2x$

22. $\tan^2 2x - 1 = 0$

In Exercises 23–32, solve each equation for $0 \leq x < 2\pi$ by using trigonometric identities. Give answers to the nearest hundredth when necessary. You may wish to check your answers using a graphing calculator or computer.

23. $\sin x \cos x = \frac{1}{2}$

24. $\cos (3x + \pi) = \frac{\sqrt{2}}{2}$

25. $\tan 2x = 3 \tan x$

26. $\tan 2x + \tan x = 0$

27. $\cos 2x = 5 \sin^2 x - \cos^2 x$

28. $\sin 2x \sec x + 2 \cos x = 0$

29. $3 \sin x = 1 + \cos 2x$

30. $\sin 2x = 5 \cos^2 x$

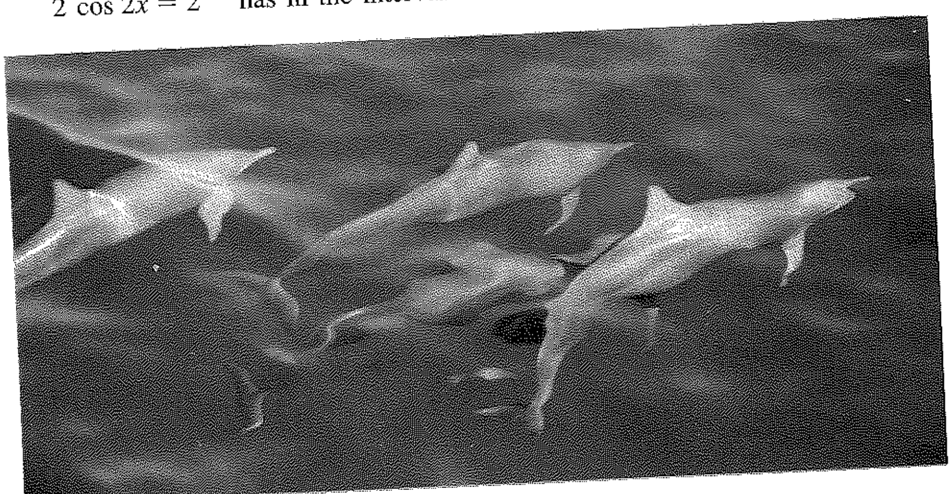
31. $\cos 2x = \sec x$

32. $\sin x \cos 2x = 1$

33. **a.** Is it possible to use trigonometric identities to solve $\sin 2x = x$?
b. On a single set of axes, sketch the graphs of $y = \sin 2x$ and $y = x$.
c. How many solutions are there to the equation $\sin 2x = x$?
d. Use a scientific calculator and trial and error to approximate the positive solution(s) to the nearest tenth.

34. ***Visual Thinking*** On the same set of axes, sketch the graphs of $y = \sin 2x$ and $y = \ln (x + 1)$ for $-2\pi < x < 2\pi$. Then tell *how many* solutions the equation $\sin 2x = \ln (x + 1)$ has in the interval $-2\pi < x < 2\pi$.

35. ***Visual Thinking*** On the same set of axes, sketch the graphs of $y = 2 \cos 2x$ and $y = 2^{-x}$ for $-2\pi < x < 2\pi$. Then tell *how many* solutions the equation $2 \cos 2x = 2^{-x}$ has in the interval $-2\pi < x < 2\pi$.



Dolphins locate objects by means of ultrasonic pulses.

Solve each equation graphically for $-2\pi \le x < 2\pi$ by using a graphing calculator or computer. Give answers to the nearest hundredth.

36. $\sin 1.5x = 2 \cos x$

37. $\tan 2x = x$

38. $\sin 2x = \ln (x + 1)$

39. $2 \cos 2x = 2^{-x}$

Solve each equation for $0 \le x < 2\pi$.

C **40.** $\sin 3x = \sin 5x + \sin x$ (*Hint:* Use one of the formulas given on page 372.)

41. $\cos 3x + \cos x = \cos 2x$

42. $\sin 3x - \sin x = 2 \cos 2x$

Solve.

43. $\text{Cos}^{-1} 2x = \text{Sin}^{-1} x$

44. $\text{Tan}^{-1} 2x = \text{Sin}^{-1} x$

//// COMPUTER EXERCISES

Write a computer program that will print a table of sines and cosines for 1°, 2°, . . . , 90° given only that $\sin 1° = 0.017452406$. *Hint:* Here is how you could begin:

1. Since $\sin^2 1° + \cos^2 1° = 1$,
$$\cos 1° = \sqrt{1 - \sin^2 1°} = \sqrt{1 - (0.017452406)^2},$$
which the computer may evaluate as 0.999847695.
2. $\sin 2° = \sin (1° + 1°) = \sin 1° \cos 1° + \cos 1° \sin 1° = 0.034899495.$
$\cos 2° = \sqrt{1 - \sin^2 2°} = 0.999390827.$
3. $\sin 3° = \sin (2° + 1°) = \sin 2° \cos 1° + \cos 2° \sin 1°$, and so on.

Chapter Summary

1. The *sum and difference formulas* for sine, cosine, and tangent are as follows:

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$
$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$
$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$
$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$
$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$
$$\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

2. The formulas for rewriting a trigonometric sum or difference as a product are as follows:

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

3. An angle θ formed by intersecting lines l_1 and l_2 with slopes m_1 and m_2 can be found by using the following formula:

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

4. The *double-angle formulas* shown below are derived from the formulas for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, and $\tan(\alpha + \beta)$.

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$
$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1$$
$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

5. The *half-angle formulas* shown below are derived from two of the double-angle formulas for cosine.

$$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{1 + \cos a}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}$$

6. Suggested methods for solving a trigonometric equation either graphically or algebraically are given on page 386.

Key vocabulary and ideas

sum and difference formulas for cosine and sine (p. 370)
formulas for rewriting a sum or difference as a product (p. 372)
sum and difference formulas for tangent (pp. 375–376)
angle formed by intersecting lines (pp. 376–377)
double-angle formulas (p. 381, p. 383)
half-angle formulas (pp. 382–383)

Chapter Test

10-1

1. Simplify the given expression.
   a. $\cos 75° \cos 15° + \sin 75° \sin 15°$
   b. $\sin 75° \cos 15° + \cos 75° \sin 15°$
   c. $\cos (30° + x) + \cos (30° - x)$
   d. $\sin (45° - x) - \sin (45° + x)$
2. Find the exact value of $\cos 15°$.

10-2

3. Find $\tan \left(\frac{5\pi}{4} - \theta\right)$ when $\tan \theta = -\frac{1}{3}$.
4. If $\tan \alpha = \frac{4}{3}$ and $\tan \beta = -\frac{1}{2}$, show that $\tan (\alpha + \beta) = \tan (\pi - \beta)$.
5. ***Writing*** Consider the lines $y = 2x + 1$ and $y = 4 - 3x$. Write a paragraph explaining how it is possible to find two different angles that are formed by the intersection of these lines. What is the relationship between these angles?

10-3

6. Suppose $\angle A$ is acute and $\cos A = \frac{4}{5}$. Find each of the following:
   a. $\sin A$
   b. $\cos 2A$
   c. $\sin 2A$
   d. $\sin 4A$
7. Simplify the given expression.
   a. $\dfrac{\sin 2x}{1 - \cos 2x}$
   b. $(1 + \tan^2 y)(\cos 2y - 1)$
   c. $\dfrac{\tan t}{\sec t + 1}$
   d. $\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$
8. Evaluate the given expression.
   a. $2 \cos^2 \frac{\pi}{12} - 1$
   b. $4 \sin \frac{\pi}{6} \cos \frac{\pi}{6}$
9. Prove that the given equation is an identity.
   a. $(1 + \cot^2 x)(1 - \cos 2x) = 2$
   b. $\dfrac{\sin \theta \sec \theta}{\tan \theta + \cot \theta} = \cos^2 \theta - \cos 2\theta$

10-4

10. a. On the same set of axes, sketch the graphs of $y = \cos 2\theta$ and $y = \sin \theta$ for $0° \le \theta < 360°$.
    b. Determine where the graphs intersect by solving $\cos 2\theta = \sin \theta$.
11. Solve the equation $\cos 2x = \cos x + 2$ either graphically or algebraically for $0 \le x < 2\pi$.