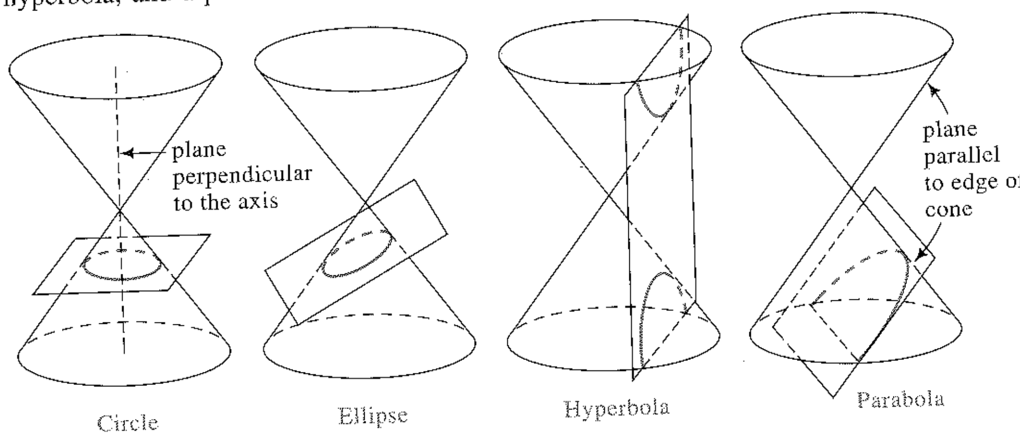
6 Analytic Geometry

Introduction

Analytic geometry is the study of geometric problems by means of analytic (or algebraic) methods. You saw some analytic geometry when you studied the distance formula, midpoint formula, and equations of lines in Sections 1-1 to 1-3. You saw more analytic geometry in Section 1-7 when you studied quadratic functions and their graphs, parabolas.

Imagine that the double cones shown below are extended indefinitely up and down. When these cones are sliced by a plane tilted at various angles, the resulting cross sections are called **conic sections**. As shown below, a circle, an ellipse, a hyperbola, and a parabola are conic sections.



It is also possible to slice the double cone to obtain a single point, a line, or a pair of lines. Do you see how? These extreme cases are called *degenerate* conic sections.

Using analytic geometry, we can find equations for these curves. Each equation is a special case of the **general second-degree equation** in two variables:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

For example,

$$x^2 + y^2 = 9$$
$$x^2 + 4y^2 = 9$$
$$x^2 - y^2 = 9$$
$$\frac{1}{4}x^2 - y = 0$$

are equations of a *circle*, an *ellipse*, a *hyperbola*, and a *parabola*, respectively.

Before we study the conic sections, we will show how to use analytic geometry to prove theorems.

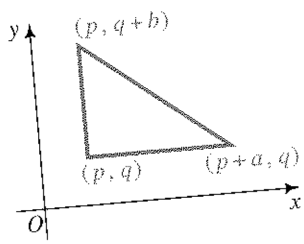
◀ In this Cassegrain reflecting telescope at Arizona's Kitt Peak Observatory, a large parabolic mirror reflects light to its focus, which is also one of the foci of a smaller hyperbolic mirror. The mirror reflects the light to the eyepiece, located at the second hyperbolic focus.

6-1 Coordinate Proofs

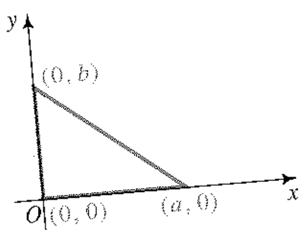
Objective *To prove theorems from geometry by using coordinates.*

In this section, you will see how to use analytic methods to prove theorems from geometry.

If a theorem is about a right triangle, there are several ways that we can place the coordinate axes on the triangle and then assign coordinates to the vertices of the triangle. For example, figures (1) and (2) below illustrate a right triangle with legs a and b units long. Most people prefer to work with figure (2). Because more of the coordinates are zero, the work is easier. Figure (3) shows that even if the right triangle is "tilted," we can choose the coordinate axes in such a way that the vertices of the triangle have several zero coordinates.



(1)



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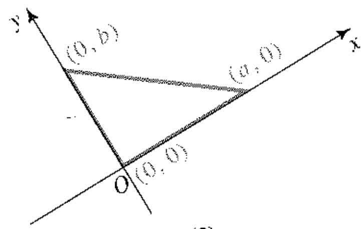
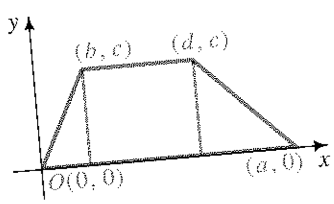
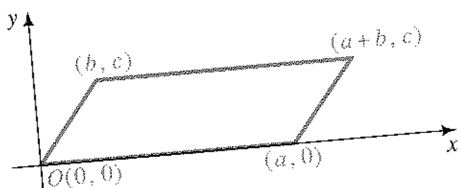


(3)

Similarly, if we wish to prove a theorem about a trapezoid or a parallelogram, we can always choose the axes in such a way that one of the vertices of the figure is at the origin and one of its parallel sides lies on the x-axis.



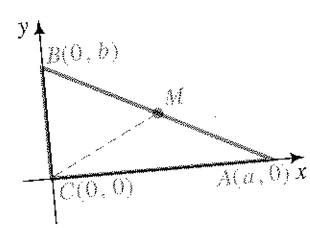
(4)



(5)

Example 1 Prove that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

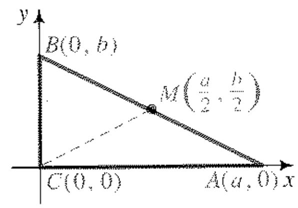
Solution **Step 1** *First, we make a coordinate diagram* of the triangle *and note what we are given and what we must prove.*



Given: $\angle C$ is a right angle.
M is the midpoint of $\overline{AB}$.

Prove: $MC = MA$
(We already know that $MB = MA$.)

Step 2 *Next, we use what is given to add information to the diagram or to express algebraically any given fact not shown in the original diagram.* In this example, we use the given fact that M is the midpoint of $\overline{AB}$ to find the coordinates of M.



Step 3 *Finally, we reword what we are trying to prove in algebraic terms.* To prove $MC = MA$:

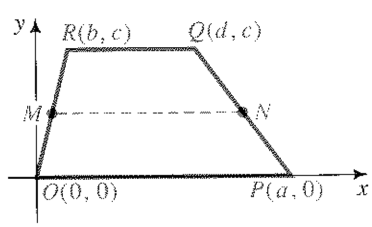
$$MC = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$
$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

$$MA = \sqrt{\left(\frac{a}{2} - a\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$$
$$= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}}$$

Therefore, $MC = MA$. Since $MA = MB$, we have that $MA = MB = MC$.

Example 2 Prove that the median of a trapezoid is parallel to the bases and has length equal to the average of the lengths of the bases.

Solution **Step 1** *We show a diagram, and the "Given" and "Prove."* Place the x-axis along the longer base of the trapezoid, with the origin at the endpoint of the longer base. Since the bases of a trapezoid are parallel and base $\overline{OP}$ has been chosen to be horizontal, base $\overline{RQ}$ is also horizontal. Thus R and Q have the same y-coordinate.

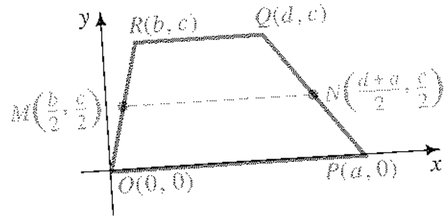


Given: Figure $OPQR$ is a trapezoid. Points M and N are midpoints of $\overline{OR}$ and $\overline{PQ}$, respectively.

Prove: (1) $\overline{MN} \parallel \overline{OP}$ and (2) $MN = \dfrac{OP + RQ}{2}$.

(Solution continues on the next page.)

Step 2 *Next, we use what is given to add information to the diagram.* In this example, we use the fact that M and N are midpoints to find their coordinates.



Step 3 *We reword what we are to prove in algebraic terms.*

(1) To prove $\overline{MN} \parallel \overline{OP}$, we must show that $\overline{MN}$ and $\overline{OP}$ have the same slope. A quick check shows that both slopes are zero, so this part of the proof is done.

(2) Lastly, we use algebra to show that $MN = \frac{1}{2}(OP + RQ)$.

$$MN = \frac{d + a}{2} - \frac{b}{2}$$

$$\frac{1}{2}(OP + RQ) = \frac{1}{2}(a + (d - b)) = \frac{d + a}{2} - \frac{b}{2}$$

Therefore, $MN = \frac{1}{2}(OP + RQ)$.

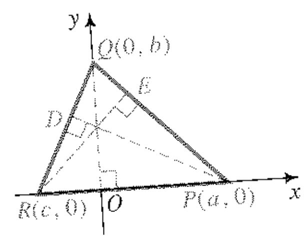
Example 3 Prove that the altitudes of a triangle meet in one point, that is, they are *concurrent*.

Solution **Step 1** *We show a diagram and the "Given" and "Prove."*

Given: $\triangle PQR$ with altitudes $\overline{PD}$, $\overline{QO}$, and $\overline{RE}$.

Prove: Lines PD, QO, and RE have a point in common.

Notice that the axes are placed in such a way that one of the altitudes lies on the y-axis.



Step 2 *We use the given information* to express algebraically the fact that $\overline{PD}$, $\overline{QO}$, and $\overline{RE}$ are altitudes.

a. To find the slope of line PD, we note that the slope of line QR is $\frac{b - 0}{0 - c} = -\frac{b}{c}$, so that the slope of line PD is $\frac{c}{b}$. Since line PD contains the point $(a, 0)$, its equation is

$$\frac{y - 0}{x - a} = \frac{c}{b}, \text{ or } cx - by = ca.$$

b. Likewise, an equation of line RE is

$$\frac{y - 0}{x - c} = \frac{a}{b}, \text{ or } ax - by = ca.$$

c. An equation of the vertical line QO is $x = 0$.

Step 3 *We reword what we are to prove in algebraic terms.* To prove that lines PD, QO, and RE have a point in common, we must show that their equations have a common solution. Using subtraction to solve

$$cx - by = ca$$
$$ax - by = ca,$$

we get
$$cx - ax = 0.$$

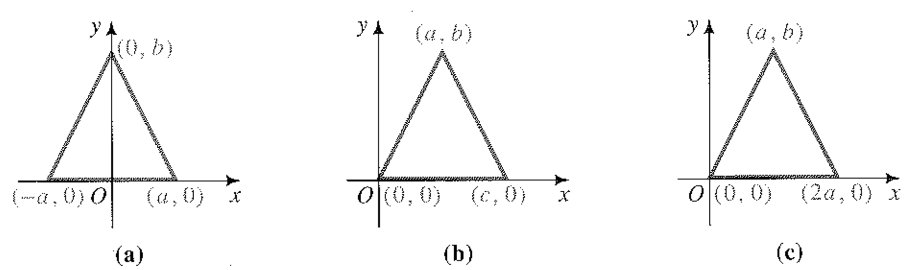
Therefore, $x(c - a) = 0$. Since $c \neq a$, $x = 0$. Substituting 0 for x in the equation $cx - by = ca$, we get $y = -\frac{ca}{b}$. Thus, the lines PD and RE intersect at $\left(0, -\frac{ca}{b}\right)$, a point on the y-axis, that is, on altitude QO, so we are done. (The point of concurrency of the altitudes is called the *orthocenter* of the triangle.)

Summary of Methods Commonly Used in Coordinate Proofs

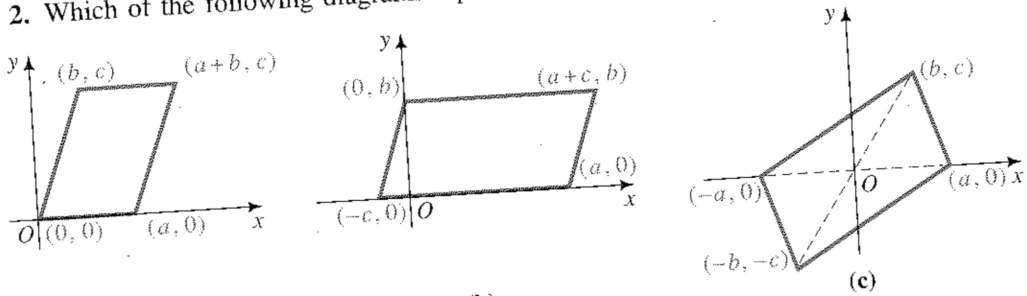
1. To prove line segments equal, use the distance formula to show that they have the same length.
2. To prove nonvertical lines parallel, show that they have the same slope.
3. To prove lines perpendicular, show that the product of their slopes is -1.
4. To prove that two line segments bisect each other, use the midpoint formula to show that each segment has the same midpoint.
5. To show that lines are concurrent, show that their equations have a common solution.

CLASS EXERCISES

1. Study the coordinates of the vertices in the following diagrams and tell which figures represent isosceles triangles.



2. Which of the following diagrams represent parallelograms?



(a) (b) (c)

WRITTEN EXERCISES

In these exercises, do not use any theorems from geometry other than the Pythagorean Theorem. You may use the formulas and theorems in Sections 1-1 and 1-2, and results proved in earlier exercises of this section.

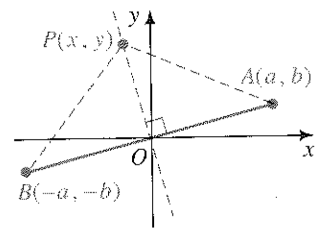
A

1. Use figure (a) or (c) of Class Exercise 1 to prove that the medians to the legs of an isosceles triangle are equal in length.
2. Prove that if a triangle has two congruent medians, then it is isosceles.
3. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and has length half that of the third side.
4. Use either figure (a) or (b) of Class Exercise 2 to prove that the diagonals of a parallelogram bisect each other.
5. Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. How does this result show that any parallelogram can be represented by figure (c) of Class Exercise 2?
6. Prove that the lengths of the diagonals of a rectangle are equal.
7. Prove that the lengths of the diagonals of an isosceles trapezoid are equal.
8. Prove that if the diagonals of a trapezoid are congruent, then the trapezoid is isosceles.
9. Prove that the line segments joining the midpoints of successive sides of any quadrilateral form a parallelogram.
10. Prove that the line segments joining the midpoints of successive sides of any rectangle form a rhombus.

B

11. Prove that if the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.
12. Prove that the diagonals of a rhombus are perpendicular.
13. Suppose that Q is any point in the plane of rectangle $RSTU$. Prove that $(QR)^2 + (QT)^2 = (QS)^2 + (QU)^2$.

14. Use the diagram at the right to prove that P is on the perpendicular bisector of $\overline{AB}$ if $PA = PB$.



Ex. 14

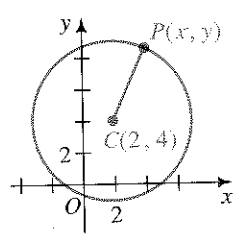
15. a. Use a figure similar to the one in Example 3 to prove that the medians of $\triangle PQR$ meet in a point G. (This point is called the *centroid* of the triangle.)
    b. Show that the x-coordinate of the centroid is the average of the x-coordinates of P, Q, and R, and that the y-coordinate of the centroid is the average of the y-coordinates of P, Q, and R.
16. Using Exercise 15, prove that the centroid G divides each median in a 2:1 ratio.
17. a. Use a figure similar to the one in Example 3 to prove that the perpendicular bisectors of the sides of $\triangle PQR$ meet in a point C. This point is called the *circumcenter* of the triangle.
    b. Prove that C is equidistant from P, Q, and R.

C 18. Let G be the intersection point of the medians of a triangle, let C be the intersection of the perpendicular bisectors of the sides, and let H be the intersection of the altitudes. Prove that G, C, and H are collinear and that $GH = 2GC$. (*Hint:* Use the results of Example 3 and Exercises 15 and 17.)

6-2 Equations of Circles

Objective *To find equations of circles and to find the coordinates of any points where circles and lines meet.*

The set of all points $P(x, y)$ in the plane that are 5 units from the point $C(2, 4)$ is a *circle*. To find an equation of this circle, we use the distance formula:



$$PC = 5$$
$$\sqrt{(x-2)^2 + (y-4)^2} = 5$$
$$(x-2)^2 + (y-4)^2 = 25$$

In general, if $P(x, y)$ is on the circle with center $C(h, k)$ and radius r, then:

$$PC = r$$
$$\sqrt{(x-h)^2 + (y-k)^2} = r$$
$$(x-h)^2 + (y-k)^2 = r^2$$

The steps leading to the last equation can be reversed to show that any point $P(x, y)$ satisfying the equation is on the circle with center (h, k) and radius r.

If the center of the circle is the origin, the equation becomes:

$$x^2 + y^2 = r^2$$

For brevity, we often refer to a circle with equation $x^2 + y^2 = r^2$, for example, as "the circle $x^2 + y^2 = r^2$."

Example 1 Find the center and radius of each circle.

a. $(x - 3)^2 + (y + 7)^2 = 19$

b. $x^2 + y^2 - 6x + 4y - 12 = 0$

Solution

a. Since $(x - 3)^2 + (y + 7)^2 = (x - 3)^2 + (y - (-7))^2$, the center of the circle is $(3, -7)$. The radius is $\sqrt{19}$.

b. We rewrite the equation in *center-radius form* by completing the squares in x and y.

Original equation: $x^2 + y^2 - 6x + 4y - 12 = 0$

$$(x^2 - 6x \quad\ ) + (y^2 + 4y \quad\ ) = 12$$

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4$$

Center-radius form: $(x - 3)^2 + (y + 2)^2 = 25$

Thus, the center of the circle is $(3, -2)$ and the radius is 5.

In Example 2, you will see how to use a graphing calculator (or computer) to graph an equation that represents a circle.

Example 2 Use a graphing calculator to graph the equation in Example 1(b).

Solution

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

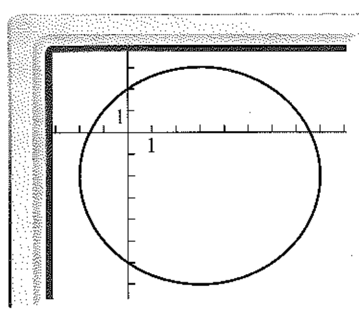
$$(x - 3)^2 + (y + 2)^2 = 25 \longleftarrow \text{center-radius form}$$

Solve for y in terms of x.

$$(y + 2)^2 = 25 - (x - 3)^2$$

$$y_1 = -2 + \sqrt{25 - (x - 3)^2} \quad (1)$$

$$y_2 = -2 - \sqrt{25 - (x - 3)^2} \quad (2)$$



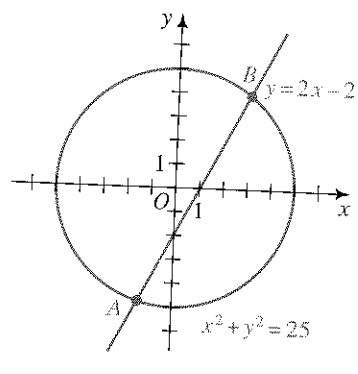
The figure at the right was obtained by graphing each of the *two* equations above on the same set of axes. Equation (1) represents the top half of the circle; equation (2) represents the bottom half.

Notice that the graph does not appear to be a perfect circle. You should consult your calculator manual to determine how to adjust the display to make the graph appear as a circle.

Example 3 Find the coordinates of the points where the line $y = 2x - 2$ and the circle $x^2 + y^2 = 25$ intersect.

Solution **Step 1** To find the coordinates of their intersection points, A and B, we solve these two equations simultaneously:

$$y = 2x - 2 \quad (1)$$
$$x^2 + y^2 = 25 \quad (2)$$



Step 2 Substituting for y in equation (2), we get:

$$x^2 + (2x - 2)^2 = 25$$
$$x^2 + 4x^2 - 8x + 4 = 25$$
$$5x^2 - 8x - 21 = 0$$
$$(5x + 7)(x - 3) = 0$$

Therefore, $x = -\frac{7}{5}$ or $x = 3$.

Step 3 Substituting these values for x in equation (1), we get:

$$y = 2\left(-\frac{7}{5}\right) - 2 = -\frac{24}{5} \quad \text{and} \quad y = 2(3) - 2 = 4$$

Thus, $A = \left(-\frac{7}{5}, -\frac{24}{5}\right)$ and $B = (3, 4)$. Check this result by substituting the coordinates of A and B in equations (1) and (2).

The solution to Example 3 involves an algebraic method, which is summarized below. You can also find the solution by simultaneously graphing the line and the circle with a graphing calculator or computer software.

To Find the Intersection of a Line and a Circle Algebraically:

1. Solve the linear equation for y in terms of x (or x in terms of y).
2. Substitute this expression for y (or x) in the equation of the circle. Then solve the resulting quadratic equation.
3. Substitute each real x-solution from Step 2 in the *linear* equation to get the corresponding value of y (or vice versa). Each point (x, y) is an intersection point.
4. You can check your result by substituting the coordinates of the intersection points in the two original equations.

When a line and a circle do not intersect, the quadratic equation in Step 2 will have a negative discriminant and the equation will have only imaginary roots. If the discriminant is zero, then there is only one real root. This indicates that the line intersects the circle in a single point and is tangent to the circle.

CLASS EXERCISES

Find the center and radius of each circle whose equation is given.

1. $x^2 + y^2 = 16$
2. $(x - 2)^2 + (y - 7)^2 = 36$
3. $(x - 4)^2 + (y + 7)^2 = 7$
4. $x^2 + y^2 + 12y = 0$
5. $x^2 - 2x + y^2 - 6y = 9$
6. $4x^2 + 4y^2 = 36$

In Exercises 7–9, find an equation of the circle described.

7. The circle with center $(7, 3)$ and radius 6.
8. The circle with center $(-5, 4)$ and radius $\sqrt{2}$.
9. The circle with center $(0, 0)$ that passes through $(-5, 12)$.
10. The graph of $x^2 + y^2 = 25$ consists of all points in the plane that are 5 units from the origin. Describe the graphs of:
    a. $x^2 + y^2 < 25$
    b. $x^2 + y^2 > 25$
11. ***Discussion*** Suppose that you wish to find where the line $3y + x = 6$ intersects the circle $x^2 + y^2 = 10$. Describe what you would do.
12. ***Discussion*** To find the intersection of the line $y = x + 8$ and the circle $x^2 + y^2 = 16$, Janice solved the equations simultaneously and found that $x = -4 \pm 2i\sqrt{2}$. What do the imaginary roots tell her?
13. ***Discussion*** Describe how to use a graphing calculator to graph $x^2 + y^2 = 9$ so that the circle does not appear distorted or flattened.

WRITTEN EXERCISES

In Exercises 1–12, write an equation of the circle described.

A

1. $C(4, 3)$, $r = 2$
2. $C(5, -6)$, $r = 7$
3. $C(-4, -9)$, $r = 3$
4. $C(a, b)$, $r = f$
5. $C(6, 0)$, $r = \sqrt{15}$
6. $C(-4, 2)$, $r = \sqrt{7}$
7. The center is $(2, 3)$; the circle passes through $(5, 6)$.
8. The points $(8, 0)$ and $(0, 6)$ are endpoints of a diameter.
9. The center is $(5, -4)$ and the circle is tangent to the x-axis.
10. The center is $(-3, 1)$ and the circle is tangent to the line $x = 4$.
11. The circle is tangent to the x-axis at $(4, 0)$ and has y-intercepts -2 and -8.
12. The circle contains $(-2, 16)$ and has x-intercepts -2 and -32.

Write each equation in center-radius form. Give the center and radius.

13. $x^2 + y^2 - 2x - 8y + 16 = 0$
14. $x^2 + y^2 - 4x + 6y + 4 = 0$
15. $x^2 + y^2 - 12y + 25 = 0$
16. $x^2 + y^2 + 14x = 0$
17. $2x^2 + 2y^2 - 10x - 18y = 1$
18. $2x^2 + 2y^2 - 5x + y = 0$

19. Show that the line $y = 2x + 8$ contains the center of the circle $x^2 + y^2 + 6x - 4y + 8 = 0$.

20. Determine whether the line $3x + 2y = 6$ contains the center of the circle $x^2 + y^2 + 4x - 12y + 24 = 0$.

In Exercises 21–23, use a graphing calculator or computer software to graph each equation.

21. $x^2 + y^2 = 50$ 22. $(x - 3)^2 + y^2 = 36$ 23. $x^2 + y^2 - 6y = 40$

In Exercises 24–26, on a single set of axes, sketch the graph of each semicircle whose equation is given.

24. a. $y = \sqrt{9 - x^2}$ b. $y = -\sqrt{9 - x^2}$

25. a. $x = \sqrt{9 - y^2}$ b. $x = -\sqrt{9 - y^2}$

26. a. $y = \sqrt{16 - (x - 5)^2}$ b. $x = 5 - \sqrt{16 - y^2}$

In Exercises 27–34, graph the equations. Solve the equations simultaneously to find the coordinates of any intersection points of their graphs. If the graphs are tangent or fail to intersect, say so.

27. $x + y = 23$, $x^2 + y^2 = 289$ 28. $9y - 8x = 10$, $x^2 + y^2 = 100$

29. $2x - y = 7$, $x^2 + y^2 = 7$ 30. $x + 2y = 10$, $x^2 + y^2 = 20$

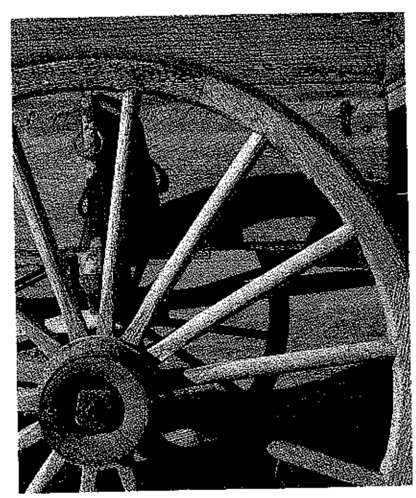
31. $5x + 2y = -1$, $x^2 + y^2 = 169$ 32. $y = 5$, $x^2 + y^2 - 4x - 6y = -9$

33. $y = \sqrt{3}x$, $x^2 + (y - 4)^2 = 16$ 34. $x - y = 3$, $x^2 + y^2 - 10x + 4y = -13$

35. ***Writing*** Write a description of the graphs of $x^2 + y^2 < 1$ and $x^2 + y^2 > 1$.

36. Sketch the graph of $(x - 3)^2 + (y - 4)^2 \leq 25$.

B 37. The line $x - 2y = 15$ intersects the circle $x^2 + y^2 = 50$ in points A and B. Show that the line joining the center of the circle to the midpoint of $\overline{AB}$ is perpendicular to $\overline{AB}$.



38. Find the length of a tangent line segment from $(10, 5)$ to the circle $x^2 + y^2 = 25$.

39. $P(2, 3)$ is on the circle with center $O(0, 0)$.
    a. Write an equation of the circle.
    b. Write an equation for the tangent l to the circle at P. (*Hint:* l is perpendicular to $\overline{OP}$.)

40. Show that $P(4, 2)$ is on the circle with equation $(x - 3)^2 + (y - 4)^2 = 5$. Find an equation of the tangent to the circle at P. (*Hint:* See the hint for Exercise 39.)

41. A circle with center $C(2, 4)$ has radius 13.
    a. Verify that $A(14, 9)$ and $B(7, 16)$ are points on this circle.
    b. If M is the midpoint of $\overline{AB}$, show that $\overline{CM} \perp \overline{AB}$.

42. A circle with center $C(-4, 0)$ has radius 15.

a. Verify that $A(8, 9)$ and $B(-13, 12)$ are points on this circle.

b. Write an equation of the perpendicular bisector of $\overline{AB}$ and show that the coordinates of point C satisfy the equation.

c. What theorem from geometry does this exercise illustrate?

43. A diameter of a circle has endpoints $A(13, 0)$ and $B(-13, 0)$.

a. Show that $P(-5, 12)$ is a point on this circle.

b. Show that $\overline{PA}$ and $\overline{PB}$ are perpendicular.

44. a. Find the coordinates of A and of B if $\overline{AB}$ is a horizontal diameter of the circle $x^2 + y^2 - 34x = 0$.

b. Show that $P(2, 8)$ is a point on this circle and that $\overline{PA} \perp \overline{PB}$.

45. Given $O(0, 0)$ and $N(12, 0)$, find an equation in terms of x and y for all points $P(x, y)$ such that $\overline{PO} \perp \overline{PN}$. Simplify this equation and show that P is on a circle. What are the center and radius of the circle?

46. ***Discussion*** Given $A(6, 8)$ and $B(-6, -8)$, write an equation in terms of x and y for all points $P(x, y)$ such that $\overline{PA} \perp \overline{PB}$. Simplify the equation and interpret your answer.

You may find it helpful to have a graphing calculator to complete Exercises 47–49.

47. A triangle is inscribed in a semicircle as shown at the right.

a. Find an equation for the semicircle.

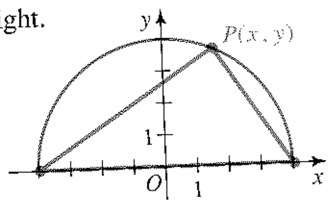
b. Write a function $A(x)$ for the area of the triangle.

c. What is the domain of the function from part (b)?

d. Graph $y = A(x)$.

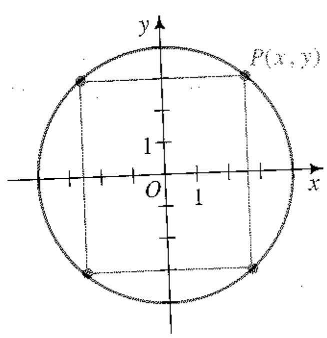
e. Use the graph from part (d) to find the value of x that maximizes $A(x)$.

f. What is the maximum value of $A(x)$?

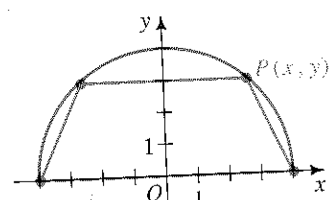


48. a. A rectangle is inscribed in a circle of radius 4 as shown at the left below. Write a function $A(x)$ for the area of the rectangle.

b. Graph $y = A(x)$. Use the graph from part (a) to find the value of x that maximizes $A(x)$. What is the maximum value of $A(x)$?



Ex. 48



Ex. 49

49. a. An isosceles trapezoid is inscribed in a semicircle of radius 4 as shown at the bottom of page 224. Write a function $A(x)$ for the area of the trapezoid.
b. Find the value of x that maximizes $A(x)$.

Find an equation of the circle that contains the given points.

50. $A(0, 0)$, $B(2, 0)$, and $C(2, 2)$
51. $P(0, 0)$, $Q(6, 0)$, and $R(0, 8)$
52. $L(8, 2)$, $M(1, 9)$, and $N(1, 1)$
53. $D(7, 5)$, $E(1, -7)$, and $F(9, -1)$

In Exercises 54–57, describe the set of points satisfying each equation.

54. $x^2 + y^2 + 2x + 2y + 2 = 0$
55. $x^2 + y^2 - 6x + 8y + 26 = 0$
C 56. $(x^2 + y^2 - 1)(x^2 + y^2 - 4) = 0$
57. $x^3y + xy^3 - xy = 0$

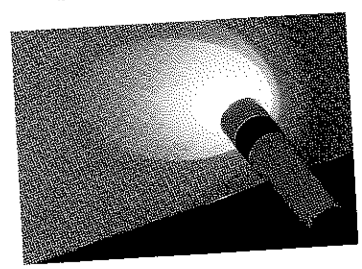
58. a. A point (x, y) lies inside the circle $x^2 + y^2 = 2$ and above the line $y = 1$. Give two inequalities that must be satisfied.
b. Sketch the region in which the point lies and find the area of the region.

59. a. Sketch the set of points that satisfies $|x| \geq 1$ and $x^2 + y^2 \leq 4$.
b. Find the area of the region.

60. Suppose that point $P(a, b)$ is any point on the circle with center $O(0, 0)$ and radius r. Suppose that line l is perpendicular to $\overline{OP}$ at P. Prove that l is tangent to the circle as follows:
a. Show that the equation of l can be written $ax + by = r^2$.
b. Solve the equations of l and the circle simultaneously. Show that there is only one solution.

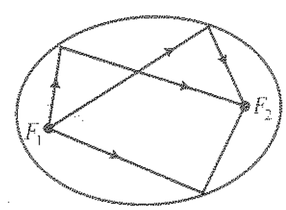
6-3 Ellipses

Objective *To find equations of ellipses and to graph them.*



If you stand a few feet away from a wall and shine a flashlight against it, you can make a lighted area in the form of an oval, or *ellipse*. The ellipse appears to be an elongated circle. Try it.

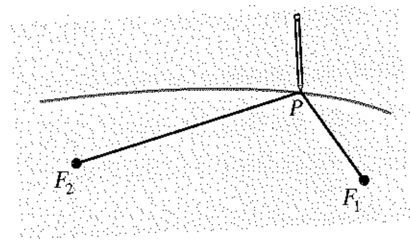
Ellipses are found in many applications. Light or sound waves from one *focus* F_1 are reflected from the ellipse to the other focus F_2. This reflection property is used to make a whispering chamber, where a person whispering at F_1 can be heard at F_2. The United States Capitol Building has such a chamber.



To sketch an ellipse, complete the Activity on the next page.

Activity

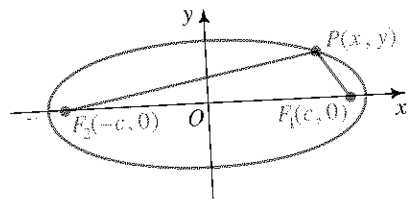
Fasten the ends of a string to a piece of cardboard with thumbtacks. Make sure the string has some slack. Keeping the string taut, draw a curve on the cardboard as shown. Describe the curve traced by the pencil point P. Repeat the experiment by moving the tacks farther apart or closer together.



If $F_1(c, 0)$ and $F_2(-c, 0)$ are two fixed points in the plane and a is a constant, $0 < c < a$, then the set of all points P in the plane such that

$$PF_1 + PF_2 = 2a$$

is an **ellipse**. This is the *geometric definition* of an ellipse. Points F_1 and F_2 are called the **foci** of the ellipse. (*Foci* is the plural of *focus*.) In the Activity above, the thumbtacks are the foci and the string has length $2a$.



To find an equation for the ellipse with foci $F_1(c, 0)$ and $F_2(-c, 0)$, we let $P(x, y)$ be any point on the ellipse, then express PF_1 and PF_2 in terms of x, y, and c. Since $PF_1 + PF_2 = 2a$, we have:

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x-(-c))^2 + y^2} = 2a$$
$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

Squaring: $(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + [(x-c)^2 + y^2]$

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

Simplifying: $4cx = 4a^2 - 4a\sqrt{(x-c)^2 + y^2}$

Dividing by 4 and rearranging terms:

$$cx - a^2 = -a\sqrt{(x-c)^2 + y^2}$$

Squaring again:
$$c^2x^2 - 2ca^2x + a^4 = a^2[(x-c)^2 + y^2]$$
$$= a^2x^2 - 2ca^2x + a^2c^2 + a^2y^2$$

Simplifying:
$$c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

Rearranging terms:
$$a^4 - a^2c^2 = a^2x^2 - c^2x^2 + a^2y^2$$
$$a^2(a^2 - c^2) = (a^2 - c^2)x^2 + a^2y^2$$

Since a must be greater than c, substitute $b^2 = a^2 - c^2$:

$$a^2b^2 = b^2x^2 + a^2y^2$$

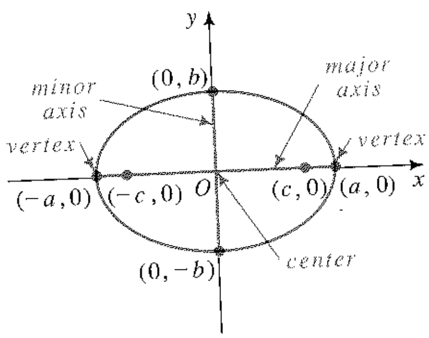
Dividing by a^2b^2, we have: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b^2 = a^2 - c^2$

The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an *algebraic definition* of an ellipse with center at the origin and foci on the x-axis. Sometimes it is convenient to consider an ellipse with foci $(0, c)$ and $(0, -c)$. In this case, we can derive an equation using similar reasoning and obtain:

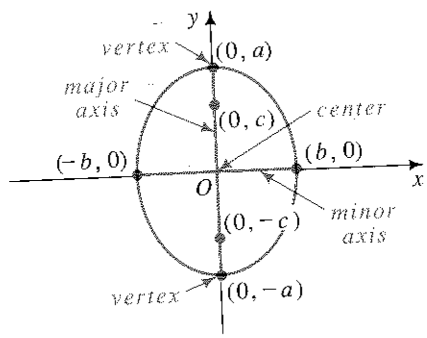
$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, \text{ where } b^2 = a^2 - c^2$$

The steps in the derivation of these equations can be reversed to show that any point satisfying the equations must also satisfy the condition $PF_1 + PF_2 = 2a$. Thus the algebraic and geometric definitions are equivalent.

All ellipses have two axes of symmetry. In the figures below, the axes of symmetry are the x- and y-axes. The portions of the axes of symmetry that lie on or within the ellipse are called the **major axis** and **minor axis** of the ellipse. The endpoints of the major axis are called the **vertices** of the ellipse. The midpoint of the major axis is the **center** of the ellipse.



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$

Example Find an equation of the ellipse with center at the origin, one vertex at $(0, 5)$, and one focus at $(0, 2)$. Sketch the ellipse, and label the vertices and the endpoints of the minor axis.

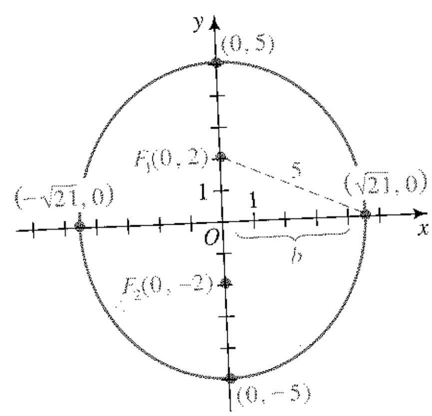
Solution Since one focus is $(0, 2)$, we have that $c = 2$. One vertex is at $(0, 5)$, so $a = 5$ and:

$$b^2 = a^2 - c^2 = 5^2 - 2^2 = 21$$

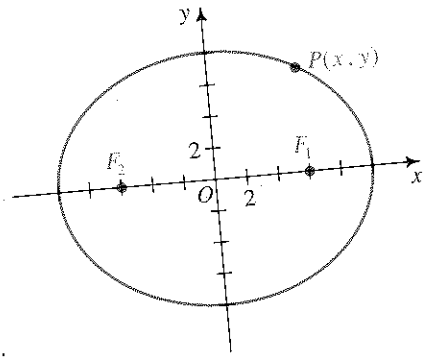
We substitute to write an equation of the ellipse:

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{25} + \frac{x^2}{21} = 1$$



CLASS EXERCISES



1. Study the ellipse shown and state:
   a. whether its major axis is horizontal or vertical.
   b. its vertices and foci.
   c. the constant value of $PF_1 + PF_2$. (*Hint:* Suppose P is at a vertex.)
   d. an equation of the ellipse.
2. An ellipse has equation $\frac{x^2}{25} + \frac{y^2}{169} = 1$.
   a. Is its major axis horizontal or vertical?
   b. Find the coordinates of its vertices and foci.
3. Describe the graphs of $\frac{x^2}{9} + y^2 < 1$ and $\frac{x^2}{9} + y^2 > 1$.
4. ***Discussion*** If F_1 and F_2 are fixed points in space, describe the set of points P such that **(a)** $PF_1 + PF_2 = 8$ and **(b)** $PF_1 + PF_2 < 8$.

WRITTEN EXERCISES

Sketch each ellipse. Find the coordinates of its vertices and foci.

A

1. $\frac{x^2}{36} + \frac{y^2}{16} = 1$ **2.** $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **3.** $\frac{x^2}{16} + \frac{y^2}{25} = 1$

4. $4x^2 + 25y^2 = 100$ **5.** $9x^2 + 25y^2 = 225$ **6.** $6.25x^2 + 4y^2 = 25$

7. Sketch the graph of each inequality: **(a)** $\frac{x^2}{25} + \frac{y^2}{9} \leq 1$ **(b)** $\frac{x^2}{4} + \frac{y^2}{16} \geq 1$

Find the domain and range of each function. Then graph the function. You may find a graphing calculator helpful.

8. $y = 3\sqrt{1 - x^2}$ **9.** $y = -\frac{1}{3}\sqrt{36 - x^2}$

10. a. On a single set of axes, graph $x^2 + y^2 = 1$ and $\left(\frac{x}{3}\right)^2 + y^2 = 1$.
b. Give the area of the circle and guess the area of the ellipse.

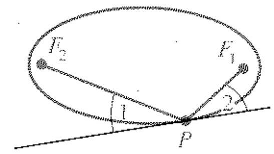
11. a. On a single set of axes, graph $x^2 + y^2 = 1$ and $x^2 + \left(\frac{y}{2}\right)^2 = 1$.
b. Give the area of the circle and guess the area of the ellipse.

Each ellipse has its center at the origin. Find an equation of the ellipse.

12. Vertex $(7, 0)$; minor axis 2 units long **13.** Vertex $(0, -9)$; minor axis 6 units long

14. Vertex $(0, -13)$; focus $(0, -5)$ **15.** Vertex $(17, 0)$; focus $(8, 0)$

16. a. *Investigation* Use the method described in the Activity to draw a large ellipse filling most of a full page of paper. Label the thumbtack holes F_2 and F_1. Then pick any point P on the ellipse. Draw $\overline{F_2P}$, $\overline{F_1P}$, and the tangent to the ellipse at P. Use a protractor to show that the angle between $\overline{F_2P}$ and the tangent is congruent to the angle between $\overline{F_1P}$ and the tangent. Repeat the experiment for another point P.



b. What property of the ellipse does part (a) illustrate?

17. *Investigation* Refer to the Activity on page 226. What happens to an ellipse as its foci F_1 and F_2 move toward each other? if F_1 and F_2 coincide?

Sketch the graphs of the given equations on a single set of axes. Then determine algebraically where the graphs intersect.

18. $9x^2 + 2y^2 = 18$
$3x + y = -3$

19. $x^2 + 4y^2 = 400$
$x - 2y = 28$

20. $2x^2 + y^2 = 9$
$y - 4x = 9$

21. $x^2 + 4y^2 = 16$
$|x| = 2$

22. a. What happens to the graph of an equation of an ellipse when x is replaced by $x - 5$?

b. Graph $\frac{x^2}{4} + y^2 = 1$ and $\frac{(x-5)^2}{4} + y^2 = 1$ on a single set of axes.

23. a. What happens to the graph of an equation of an ellipse when y is replaced by $y + 6$?

b. Graph $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and $\frac{x^2}{4} + \frac{(y+6)^2}{9} = 1$ on a single set of axes.

Sketch each ellipse. Label the center and vertices.

B 24. $\frac{(x-3)^2}{4} + \frac{(y-6)^2}{25} = 1$

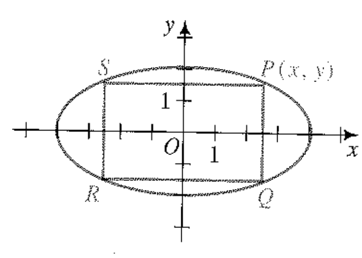
25. $\frac{(x+5)^2}{25} + \frac{(y-4)^2}{16} = 1$

26. $(x+7)^2 + \frac{(y-5)^2}{9} = 1$

27. $4(x+2)^2 + (y-5)^2 = 4$

You may find it helpful to have a graphing calculator to complete Exercise 28(b).

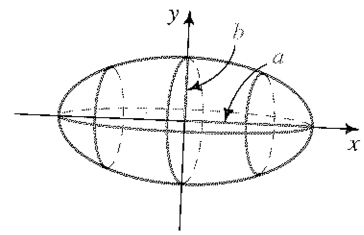
28. In the figure at the right, rectangle $PQRS$ is inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.



a. Show that the area of the rectangle is $A(x) = 4x\sqrt{4 - \frac{x^2}{4}}$.

b. Approximate the maximum area to the nearest tenth of a square unit.

29. When the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about either of its axes, an *ellipsoid* is formed.



a. The volume of the ellipsoid shown is $V = \frac{4}{3}\pi b^2 a$. Interpret this volume if the original ellipse is a circle.

b. Sketch the ellipsoid formed by rotating the given ellipse about its minor axis and guess its volume.

Sketch each ellipse. Find the coordinates of its vertices and foci.

30. $\frac{(x-5)^2}{25} + \frac{(y+3)^2}{9} = 1$

31. $\frac{(x+6)^2}{12} + \frac{(y-4)^2}{16} = 1$

32. $9(x-3)^2 + 4(y+5)^2 = 36$

33. $(x+1)^2 + 4(y+3)^2 = 9$

34. $x^2 + 25y^2 - 6x - 100y + 84 = 0$
(*Hint*: Complete the squares in x and in y. Begin by rewriting the equation in this form: $(x^2 - 6x + \quad) + 25(y^2 - 4y + \quad) = -84$.)

35. $9x^2 + y^2 + 18x - 6y + 9 = 0$

36. $9x^2 + 16y^2 - 18x - 64y - 71 = 0$

You may find a graphing calculator or graphing software helpful to complete Exercise 37.

37. A graphing calculator or computer software can be used to sketch the graph of an equation that represents a function.

a. Solve $\frac{x^2}{36} + \frac{y^2}{16} = 1$ for y. The result involves two equations.

b. Graph the ellipse in part (a). Does the result agree with the graph in Exercise 1?

In Exercises 38–41, find an equation of the ellipse described.

38. Center is (3, 7); one focus is (6, 7); one vertex is (8, 7).

39. Center is (4, −1); one vertex is (4, −5); one focus is (4, −3.5).

40. Vertices are (5, 9) and (5, 1); one focus is (5, 7).

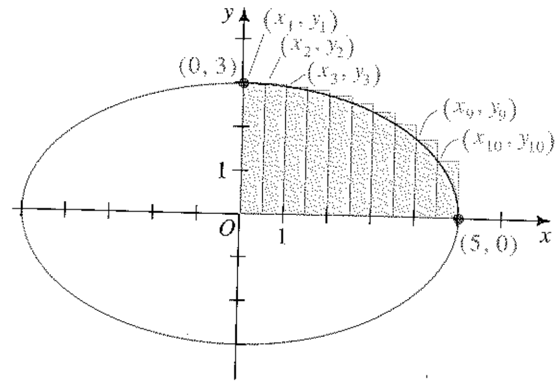
41. Center is (5, 6); the ellipse is tangent to both axes.

42. a. Suppose that $F_1 = (3, 0)$, $F_2 = (-3, 0)$, and $P = (x, y)$. Using the distance formula, write an equation that expresses the fact that $PF_1 + PF_2 = 10$.

b. Simplify your equation to one of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

//// COMPUTER EXERCISES

1. **a.** To approximate the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, shown at the right, you can add the areas of the ten rectangles in the first quadrant and multiply the sum by 4. The area of the leftmost rectangle is $\frac{1}{2} \cdot 3$; the area of the next is $\frac{1}{2}y_2$; and the area of the next is $\frac{1}{2}y_3$. How do you find y_2 and y_3?



   b. Write a program to print the approximate area of the ellipse.

2. Modify the program to print a more accurate approximation of the area.

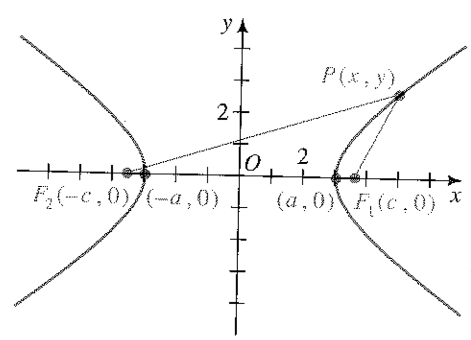
6-4 Hyperbolas

Objective *To find equations of hyperbolas and to graph them.*

The equation of a *hyperbola* with center at the origin and horizontal and vertical axes of symmetry is very similar to the equation of an ellipse. Moreover, the terminology for ellipses and hyperbolas is very similar.

Suppose that $F_1(c, 0)$ and $F_2(-c, 0)$ are fixed points in the plane, and that a is a constant, $a < c$. The *geometric definition* of a **hyperbola** states that a hyperbola is the set of all points $P(x, y)$ in the plane such that

$$|PF_1 - PF_2| = 2a.$$



To derive an equation of a hyperbola with foci $F_1(c, 0)$ and $F_2(-c, 0)$, we begin as follows:

$$|PF_1 - PF_2| = 2a$$
$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a$$

The rest of the derivation is similar to that for the ellipse on page 226, except that we let $b^2 = c^2 - a^2$ rather than $b^2 = a^2 - c^2$, since $c > a$. The equation that results is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Asymptotes

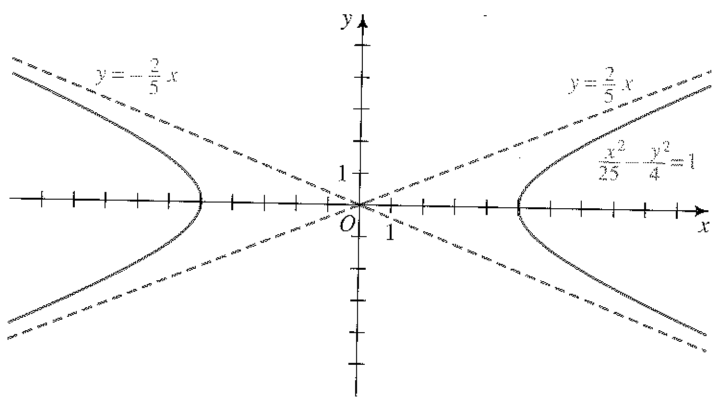
When we solve $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ for y, we obtain:

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

If $|x|$ is very large, $\sqrt{x^2 - a^2}$ is approximately the same as $\sqrt{x^2} = |x|$. Therefore, when $|x|$ is very large,

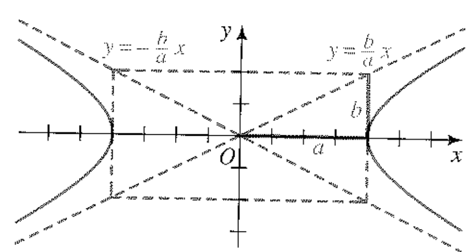
$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2} \approx \pm\frac{b}{a}x.$$

The lines $y = \pm\dfrac{b}{a}x$ are called the *asymptotes* of the hyperbola. An **asymptote** of a curve is a line that the curve approaches more and more closely, that is, the distance between the curve and its asymptote becomes less and less as $|x|$ becomes large. See the following figure and table.



x	10	20	100	1000
y (on asymptote) $y = \frac{2}{5}x$	4	8	40	400
y (on hyperbola) $y = \frac{2}{5}\sqrt{x^2 - 25}$	3.46	7.75	39.95	399.995
vertical distance (difference in y's)	0.54	0.25	0.05	0.005

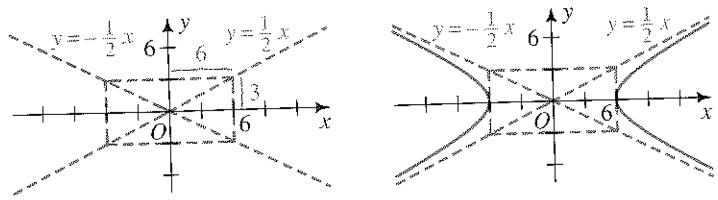
The asymptotes for $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ contain the diagonals of a rectangle with dimensions $2a$ and $2b$. The asymptotes and the rectangle are shown in blue. Drawing the rectangle and its diagonals is a good first step in graphing the equation.



Example 1 Graph the hyperbola $\frac{x^2}{36} - \frac{y^2}{9} = 1$. Find its foci.

Solution

1. Since the hyperbola has x-intercepts ± 6, the hyperbola contains $(6, 0)$ and $(-6, 0)$. Draw the rectangle centered at the origin having length $2 \cdot 6 = 12$ and width $2 \cdot 3 = 6$. Sketch the diagonals of the rectangle.



2. Sketch the curves through the vertices, $(\pm 6, 0)$, extending them towards the asymptotes as shown at the right above.
3. To find the foci, note that $c^2 = a^2 + b^2 = 36 + 9 = 45$. Thus, $c = \pm\sqrt{45} = \pm 3\sqrt{5}$. So the foci are $(\pm 3\sqrt{5}, 0)$.

The graph of

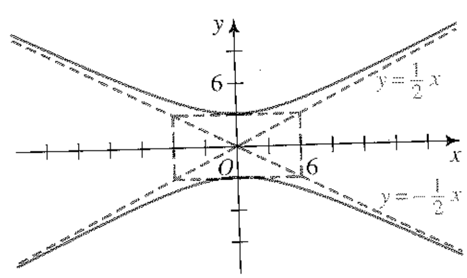
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

is also a hyperbola, but with a vertical major axis. Its asymptotes have equations $y = \pm\frac{a}{b}x$. Contrast the hyperbola in Example 2 with the one in Example 1.

Example 2 Graph the hyperbola $\frac{y^2}{9} - \frac{x^2}{36} = 1$. Find its foci.

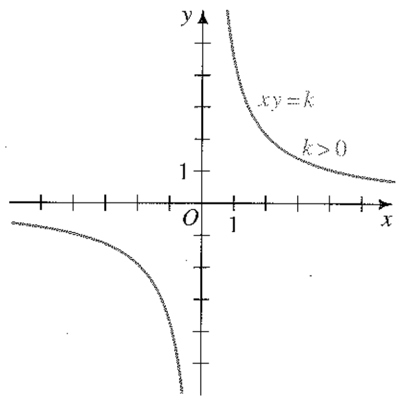
Solution

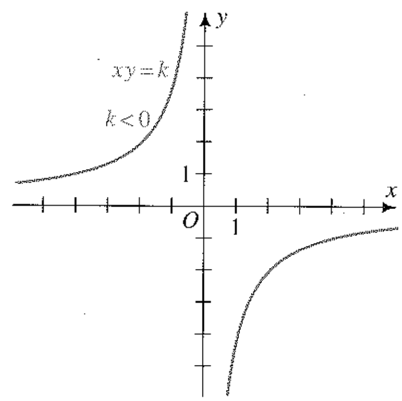
1. Since the hyperbola has y-intercepts ± 3, the hyperbola contains $(0, -3)$ and $(0, 3)$. Draw the rectangle centered at the origin having length $2 \cdot 6 = 12$ and width $2 \cdot 3 = 6$. Sketch the diagonals of the rectangle.



2. Sketch the curves through the vertices, $(0, \pm 3)$, extending them towards the asymptotes as shown above.
3. To find the foci, note that $c^2 = a^2 + b^2 = 9 + 36 = 45$ and $c = \pm 3\sqrt{5}$. So the foci are $(0, \pm 3\sqrt{5})$.

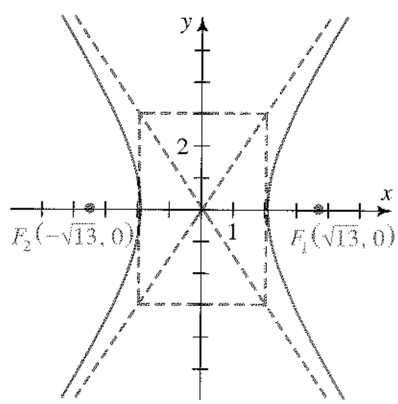
The diagrams below show hyperbolas with simple equations, that is, equations of the form $xy = k$ where k is a nonzero constant. The coordinate axes are the asymptotes for the hyperbolas.

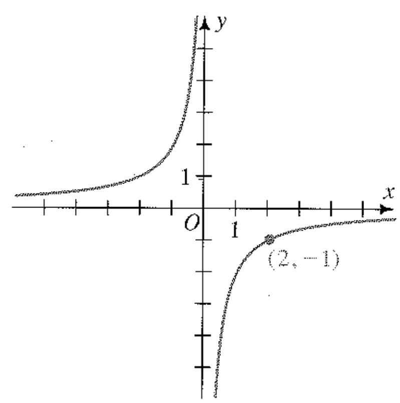




CLASS EXERCISES

1. **a.** Give the vertices and foci for the hyperbola at the left below.
   b. Is the major axis horizontal or vertical?
   c. Find an equation for the hyperbola and equations of its asymptotes.





2. **a.** Give an equation for the hyperbola at the right above.
   b. Rotate the hyperbola 90° about the origin. What is its new equation?

3. A hyperbola has equation $\dfrac{y^2}{25} - \dfrac{x^2}{1} = 1$.
   a. Is its major axis horizontal or vertical? Explain.
   b. What are its vertices and foci?
   c. What are the equations of its asymptotes?
   d. If the hyperbola were translated 6 units to the right and 5 units down, what would be its new equation?

4. Describe the graphs of $x^2 - y^2 < 4$ and $x^2 - y^2 > 4$.

WRITTEN EXERCISES

In Exercises 1–8, sketch each hyperbola and its asymptotes. Give equations for the asymptotes.

A

1. $\frac{x^2}{9} - \frac{y^2}{4} = 1$
2. $\frac{y^2}{16} - \frac{x^2}{4} = 1$
3. $x^2 - y^2 = 1$
4. $y^2 - x^2 = 1$
5. $4y^2 - x^2 = 4$
6. $25x^2 - 16y^2 = 400$
7. $xy = 4$
8. $xy = -4$

In Exercises 9 and 10, give the domain and the range of each function. Then graph the function. You may use a graphing calculator if you wish.

9. $y = \sqrt{x^2 - 4}$
10. $y = -\sqrt{9 + x^2}$

In Exercises 11 and 12, sketch each hyperbola and its asymptotes. Give equations for the asymptotes.

11. **a.** $xy = 12$ **b.** $xy = -12$ **c.** $(x + 3)(y + 4) = 12$
12. **a.** $xy = 8$ **b.** $xy = -8$ **c.** $(x - 2)(y + 4) = -8$
13. On a single set of axes, graph the hyperbolas with equations $x^2 - y^2 = k$, where $k = 4, 1, -1$, and -4.
14. On a single set of axes, graph the hyperbolas with equations $xy = k$, where $k = 12, 8, 4$, and 1.

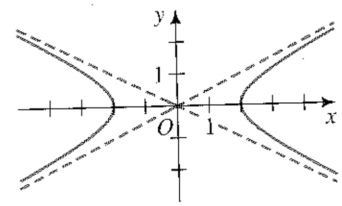
In Exercises 15–18, sketch the graphs of the given equations on a single set of axes. Then determine algebraically the coordinates of any points where the graphs intersect.

15. $x^2 - y^2 = 16$
    $x + y = 5$
16. $y^2 - x^2 = 94$
    $x - y = 3$
17. $x + 2y = 6$
    $xy = -20$
18. $2x + y = -14$
    $xy = 24$

In Exercises 19–22, find an equation of the hyperbola, with center at the origin, that satisfies the given conditions.

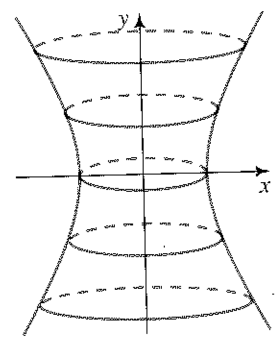
19. A vertex at $(6, 0)$ and a focus at $(10, 0)$
20. A vertex at $(0, -12)$ and a focus at $(0, -13)$
21. A vertex at $(0, -2)$ and an asymptote with equation $y = -x$
22. A vertex at $(8, 0)$ and an asymptote with equation $y = \frac{1}{2}x$

23. a. The diagram shows the graph of the hyperbola $\frac{x^2}{4} - y^2 = 1$. Show that this equation can be rewritten as $y = \pm\frac{1}{2}\sqrt{x^2 - 4}$.



b. Find the vertical distance between the hyperbola and its asymptote $y = \frac{1}{2}x$ when $x = 5$, $x = 10$, and $x = 100$.

24. Graph $y^2 - x^2 = 9$ and its asymptotes. Find a value of x for which the vertical distance between the hyperbola and an asymptote is less than 0.001.

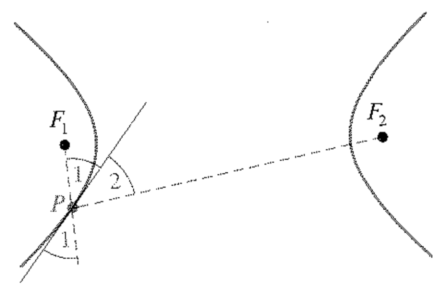


B **25.** ***Visual Thinking*** When the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is rotated about its vertical axis of symmetry, a *hyperboloid of one sheet* is formed. (See the figure at the right.) When the hyperbola is rotated about its horizontal axis of symmetry, a *hyperboloid of two sheets* is formed. Sketch this hyperboloid.

26. a. ***Visual Thinking*** The region enclosed by the hyperbola $y^2 - x^2 = 16$ and the lines $x = \pm 3$ is rotated about the x-axis. Sketch the solid formed.

b. Think of the solid as inscribed in a cylinder. Find the volume of the cylinder by using $V = \pi r^2 h$.

27. ***Investigation*** This exercise illustrates a reflection property of a hyperbola: The lines drawn from a point on a hyperbola to the foci form equal angles with the tangent at that point.



a. Carefully trace the hyperbola and the foci of the hyperbola at the right.

b. On one of the branches, choose a point P different from the one shown. Carefully sketch a line tangent to the hyperbola at P. Draw lines PF_1 and PF_2.

c. Measure $\angle 1$ and $\angle 2$. What is the relationship between these angles?

28. a. Suppose that $F_1 = (5, 0)$, $F_2 = (-5, 0)$, and $P = (x, y)$. Write an equation that expresses the fact that $|PF_1 - PF_2| = 8$.

b. Simplify your equation to the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

29. a. Suppose that $F_1 = (0, 10)$, $F_2 = (0, -10)$, and $P = (x, y)$. Write an equation that expresses the fact that $|PF_1 - PF_2| = 12$.

b. Simplify your equation to the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

On a single set of axes, sketch the graphs of the given equations.

30. $x^2 - y^2 = 1$
$(x-4)^2 - (y-3)^2 = 1$

31. $y^2 - 4x^2 = 4$
$(y+5)^2 - 4x^2 = 4$

32. Suppose the hyperbola in Example 1 is translated 8 units right and 6 units up. Give the equation of the new hyperbola and the equations of its asymptotes.

33. Suppose the hyperbola in Example 1 is rotated 90° about the origin. Give the equation of the new hyperbola and the equations of its asymptotes.

Sketch the graph of each inequality.

34. a. $xy \geq 8$ **b.** $xy \geq 1$ **c.** $xy \geq 0$

35. a. $xy \leq -4$ **b.** $xy \leq -1$ **c.** $xy \leq 0$

36. $x^2 - 4y^2 < 4$ **37.** $y^2 - x^2 > 1$ **38.** $(x-5)^2 - (y-4)^2 \geq 1$

Sketch the hyperbola whose equation is given. Find the coordinates of the vertices and the foci, and equations for the asymptotes.

39. $\dfrac{(x-6)^2}{36} - \dfrac{(y-8)^2}{64} = 1$

40. $\dfrac{(y+5)^2}{16} - \dfrac{x^2}{9} = 1$

41. $y^2 - x^2 - 2y + 4x - 4 = 0$

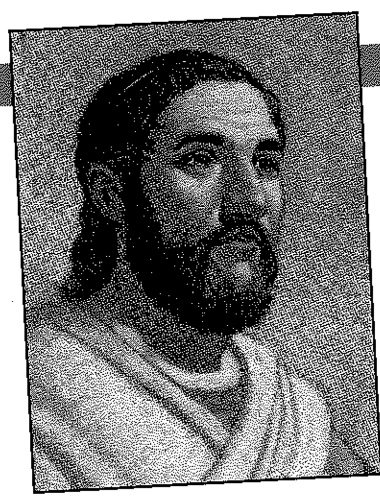
42. $x^2 - 4y^2 - 2x + 16y - 19 = 0$

Find an equation of the hyperbola described.

43. Center is $(5, 0)$; one vertex is $(9, 0)$; one focus is $(10, 0)$.

44. Vertices are $(4, 0)$ and $(4, 8)$; asymptotes have slopes ± 1.

C **45.** Find the values of a and b that make the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ congruent to the hyperbola $xy = 1$.



Apollonius (262–200 B.C.)

Apollonius of Perga was an astronomer and geometer noted for his eight-volume work entitled *On Conic Sections*. In it, he derived all the conic sections from a plane intersecting a right double cone. He was the first to use the terms *ellipse*, *parabola*, and *hyperbola* in reference to conic sections.

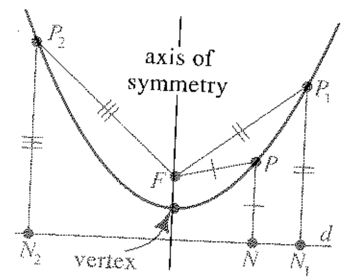
The problem of constructing a circle tangent to three given circles in the plane bears his name. It is called the problem of Apollonius.

6-5 Parabolas

Objective *To find equations of parabolas and to graph them.*

When we discussed quadratic functions in Section 1-7, we stated that their graphs are parabolas. In this section, we will state the geometric and algebraic definitions of a parabola and examine some of the properties of a parabola.

A **parabola** is the set of all points P in the plane that are equidistant from a fixed point F, the **focus**, and a fixed line d (not containing F), the **directrix**. That is, if F and d are fixed, then the set of all points P such that $PF = PN$ is a parabola.



The line through the focus F and perpendicular to the directrix d is the line of symmetry for the parabola. It intersects the parabola at its *vertex*.

If the line $y = -p$, $p \neq 0$, is the directrix of a parabola with focus $F(0, p)$, then by the geometric definition above:

$$PF = PN$$

$$\sqrt{(x-0)^2 + (y-p)^2} = |y + p|$$

Squaring and simplifying we get:

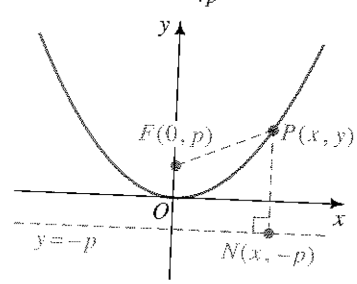
$$x^2 + (y-p)^2 = (y+p)^2$$

$$x^2 + y^2 - 2yp + p^2 = y^2 + 2yp + p^2$$

$$x^2 = 4py$$

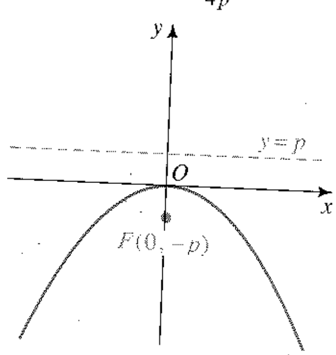
$$(1)\ y = \frac{1}{4p}x^2$$

(1) $y = \frac{1}{4p}x^2$

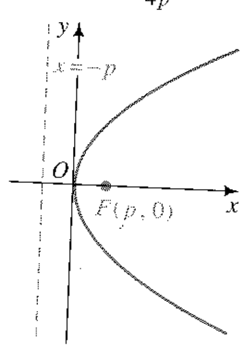


The figures below show other parabolas with vertex (0, 0). The derivation of each equation is analogous to that shown above.

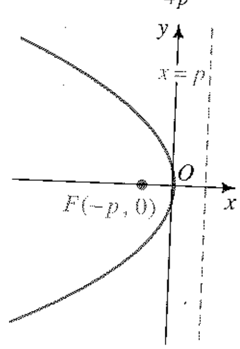
(2) $y = -\frac{1}{4p}x^2$



(3) $x = \frac{1}{4p}y^2$



(4) $x = -\frac{1}{4p}y^2$



Example 1 Find the focus and directrix of each parabola whose equation is given.

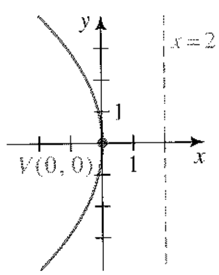
a. $y = 2x^2$ **b.** $x = \frac{1}{20}y^2$

Solution **a.** Comparing $y = 2x^2$ with $y = \frac{1}{4p}x^2$, we see that $p = \frac{1}{8}$. Thus, the focus is $F\left(0, \frac{1}{8}\right)$ and the directrix is $y = -\frac{1}{8}$.

b. Comparing $x = \frac{1}{20}y^2$ with $x = \frac{1}{4p}y^2$, we see that $p = 5$. Thus, the focus is $F(5, 0)$ and the directrix is $x = -5$.

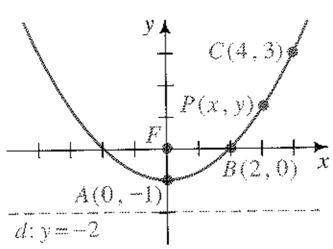
Example 2 Find an equation of the parabola with vertex $(0, 0)$ and directrix $x = 2$.

Solution A sketch shows that the parabola has an equation of type (4), $x = -\frac{1}{4p}y^2$. Since p is the distance from the vertex to the directrix, $p = 2$. Thus, $x = -\frac{1}{8}y^2$.

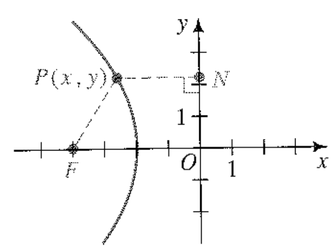


CLASS EXERCISES

1. Points A, B, C, and P are on the parabola $y = \frac{1}{4}x^2 - 1$, whose focus is $F(0, 0)$ and whose directrix d is the line $y = -2$. (See the figure at the left below.)
   a. Find the distances from A to F and A to d.
   b. Find the distances from B to F and B to d.
   c. Find the distances from C to F and C to d.
   d. Find the distances from P to F and P to d.



Ex. 1



Ex. 2

2. The point $P(x, y)$ is on the parabola whose focus F is $(-4, 0)$ and whose directrix is the y-axis. (See the figure at the right above.)
   a. Express (1) the distance PF and (2) the distance PN in terms of x and y.
   b. Explain how to *derive* an equation of the parabola.

3. ***Writing*** In a few sentences, describe how to tell whether a parabola opens up, down, left, or right when you are given an equation for the parabola.

Give the focus and the directrix of each parabola. Tell whether the parabola opens up, down, left, or right.

4. $y = \frac{1}{4}x^2$ **5.** $y = -\frac{1}{4}x^2$ **6.** $x = y^2$ **7.** $x = -y^2$

8. ***Discussion*** You know that a parabola can open up, down, left, or right. Is it possible for a parabola to open in some other direction? Explain.

9. ***Discussion*** Explain how to use the graph of the equation $y = x^2$ to graph the equation $y - k = (x - h)^2$.

WRITTEN EXERCISES

For each parabola give the coordinates of its vertex and focus and the equation of its directrix.

A **1. a.** $y = \frac{1}{8}x^2$ **b.** $x = \frac{1}{8}y^2$ **2. a.** $y = -\frac{1}{12}x^2$ **b.** $x = -\frac{1}{12}y^2$

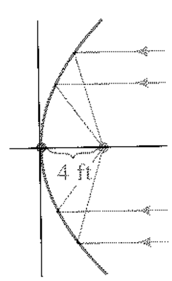
3. a. $y = -2x^2$ **b.** $x = -2y^2$ **4. a.** $y = x^2 + 1$ **b.** $x = y^2 + 1$

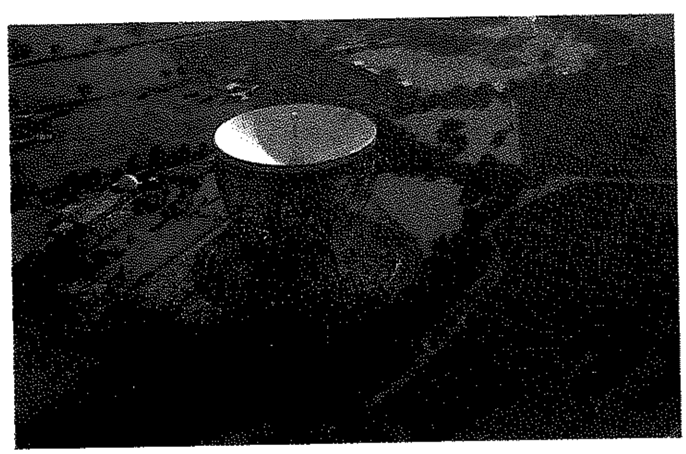
In Exercises 5–8, for each parabola give the coordinates of its vertex and focus and the equation of its directrix. (Use translations.)

5. $y - 1 = \frac{1}{4}(x - 2)^2$ **6.** $y + 3 = \frac{1}{8}(x - 5)^2$

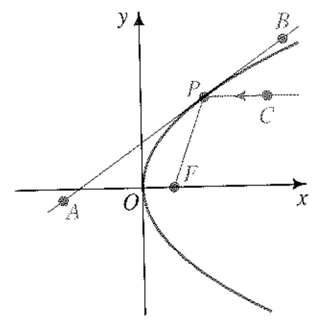
7. $x - 4 = (y - 7)^2$ **8.** $x + 2 = -2(y - 3)^2$

9. Telecommunications A satellite dish used to receive television signals has a parabolic cross-section as shown at the left below. Incoming signals that are parallel to the axis of the parabola are reflected to its focus. Thus, the incoming signal is magnified. Find an equation of the parabola.





10. ***Investigation*** On a sheet of graph paper, carefully draw a *large* graph of the equation $x = \frac{1}{16}y^2$. Draw a tangent line to the parabola at any point P. Mark two points A and B on the tangent line. Join P to the focus F (which you must first determine). Draw a ray through P parallel to the axis of the parabola. Label a point C on the ray. With a protractor, measure $\angle CPB$ and $\angle FPA$. What do you observe? Repeat the experiment for another point P on the parabola.



Find an equation of each parabola. Sketch its graph.

B 11. Focus, $(-1, 0)$; directrix, $x = 1$

12. Focus, $(0, 0.25)$; directrix, $y = -0.25$

13. Vertex, $(0, 0)$; focus, $(0, -0.25)$

14. Vertex, $(0, 0)$; focus, $(5, 0)$

15. Vertex, $(0, 0)$; directrix, $x = 4$

16. Vertex, $(0, 0)$; directrix, $y = -2$

17. Focus, $(0, 2)$; directrix, $y = 0$

18. Focus, $(3, 0)$; directrix, $x = -3$

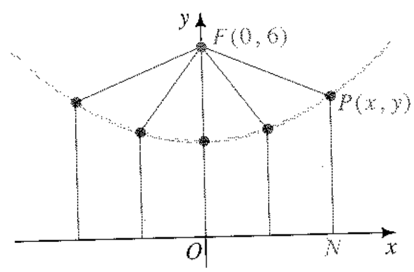
Sketch the graph of each "half-parabola." You may use a graphing calculator if you wish.

19. **a.** $y = \sqrt{x}$ **b.** $y = -\sqrt{x}$ **c.** $y = -\sqrt{x-3}$ **d.** $y = \sqrt{x+2} - 1$

20. **a.** $x = \sqrt{y}$ **b.** $x = -\sqrt{y}$ **c.** $x = \sqrt{y-4}$ **d.** $x = -\sqrt{y+3} + 1$

21. The diagram at the right shows many points $P(x, y)$ that are equidistant from $F(0, 6)$ and the x-axis.
    a. Write an equation that states that $PF = PN$.
    b. Rewrite the equation in the form

$$y - k = \frac{1}{4p}(x - h)^2.$$



The coordinates of F and an equation of line d are given. Write and simplify an equation that specifies the set of points $P(x, y)$ that are equidistant from F and d.

22. $F(0, 2)$
    d: $y = -2$

23. $F(0, -3)$
    d: $x = -2$

24. $F(-1, 3)$
    d: $x = 0$

25. $F(-7, -5)$
    d: $y = -7$

Find the vertex, focus, and directrix of each parabola. Graph the equation. Check your graph with a graphing calculator.

26. $y = 2x^2 - 8x + 3$

27. $y = 5 - 6x - 3x^2$

28. $6x - x^2 = 8y + 1$

29. $4y = x^2 - 8x + 12$

30. $x = y^2 - 2y - 5$

31. $y^2 - 6y + 16x + 25 = 0$

6-6 Systems of Second-Degree Equations

Objective *To solve systems of second-degree equations.*

In this section, we will investigate three methods of solving a system of second-degree equations in two variables.

Methods for Solving Systems of Second-Degree Equations

Method 1 Algebraic approaches, such as substitution or elimination
Method 2 A graphing calculator or a computer graphing utility
Method 3 A combination of algebraic and graphing methods

In the first solution to Example 1, you can see how to use technology to solve a system of second-degree equations.

Example 1 Solve the system: $xy = 6$ and $x^2 + 4y^2 = 64$

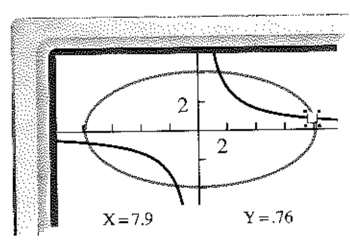
Solution 1 To solve the system graphically, you must first solve each equation for y.

$y = \frac{6}{x}$ hyperbola

$y = \sqrt{16 - \frac{x^2}{4}}$ upper half of ellipse

$y = -\sqrt{16 - \frac{x^2}{4}}$ lower half of ellipse

The figure shows a calculator or computer display of the graphs.



As you can see, there are four solutions. The figure suggests that the solutions in the third quadrant are reflections in the origin of the two solutions in the first quadrant. (You can verify this: If (a, b) satisfies each equation, show that $(-a, -b)$ satisfies each equation.) Using a calculator or computer, you can find that approximate solutions are $(1.5, 3.9)$, $(7.9, 0.76)$, $(-1.5, -3.9)$, and $(-7.9, -0.76)$.

Alternatively, the system in Example 1 can be solved algebraically by using substitution.

Solution 2 **Step 1** *Solve one equation for x or y.* Solve $xy = 6$ for x: $x = \frac{6}{y}$

Step 2 *Substitute in the other equation and solve.* Substitute $\frac{6}{y}$ for x in the second equation.

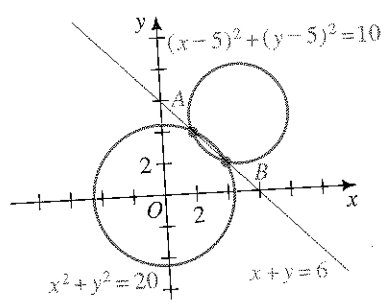
$$\left(\frac{6}{y}\right)^2 + 4y^2 = \frac{36}{y^2} + 4y^2 = 64$$
$$4y^4 - 64y^2 + 36 = 0$$
$$y^4 - 16y^2 + 9 = 0$$

Using the quadratic formula, we get:

$$y^2 = \frac{16 \pm \sqrt{16^2 - 4 \cdot 9}}{2} = 8 \pm \sqrt{55} \approx 15.4 \text{ or } 0.584$$
$$y \approx \pm 3.9 \text{ or } \pm 0.76$$

Step 3 *Find the corresponding values of x.* When $y = 3.9$, for example, $x = 6 \div 3.9 \approx 1.5$. The four approximate solutions are $(1.5, 3.9)$, $(7.9, 0.76)$, $(-1.5, -3.9)$, and $(-7.9, -0.76)$.

Example 2 The figure shows two circles with equations $x^2 + y^2 = 20$ and $(x - 5)^2 + (y - 5)^2 = 10$. Find the coordinates of their points of intersection, points A and B.



Solution Write both equations in expanded form and then subtract.

$$\begin{aligned} x^2 + y^2 &= 20 \quad (1) \\ x^2 + y^2 - 10x - 10y &= -40 \quad (2) \\ \hline 10x + 10y &= 60 \\ x + y &= 6 \quad (3) \end{aligned}$$

Equation (3) is the line containing A and B. From (3), $y = 6 - x$. Then substituting in (1) we get:

$$x^2 + (6 - x)^2 = 20$$
$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } 4$$

When $x = 2$, $y = 6 - x = 4$. When $x = 4$, $y = 6 - x = 2$. Thus, the intersection points are $A(2, 4)$ and $B(4, 2)$.

Example 3 Solve $2x^2 + y^2 = 4$ and $x^2 - 2y^2 = 12$ simultaneously.

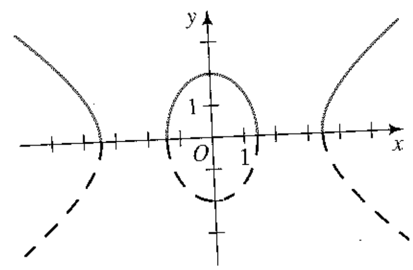
Solution 1 Multiply the first equation by 2. Then add.

$$\begin{aligned} 4x^2 + 2y^2 &= 8 \\ x^2 - 2y^2 &= 12 \\ \hline 5x^2 \qquad &= 20 \end{aligned}$$

Therefore, $x = \pm 2$. Substitute $x = \pm 2$ in either equation to obtain $y = \pm 2i$. Since there is no solution (x, y) where x and y are real numbers, $2x^2 + y^2 = 4$ and $x^2 - 2y^2 = 12$ have no common solution.

Alternatively, you can solve the system by graphing the upper half of the ellipse and the hyperbola and then using symmetry.

Solution 2 Solving $2x^2 + y^2 = 4$ and $x^2 - 2y^2 = 12$ for y, we get $y = \sqrt{4 - 2x^2}$ for the upper half of the ellipse and $y = \sqrt{\frac{x^2}{2} - 6}$ for the upper half of the hyperbola. By symmetry, we can see that there are no solutions.



CLASS EXERCISES

1. *Visual Thinking* Two quadratic equations in two variables may have 4, 3, 2, 1, or 0 real solutions. Illustrate each of these cases with a circle and a parabola.
2. Compare the graphs of $x^2 + y^2 = 9$ and $(x + y)^2 = 9$.
3. Compare the graphs of $x^2 - y^2 = 4$ and $(x - y)^2 = 4$.

WRITTEN EXERCISES

Sketch the graphs of each pair of circles to determine the number of points of intersection. If the circles are tangent or fail to intersect, say so. Then solve the system.

A
1. $x^2 + y^2 = 16$, $(x - 4)^2 + y^2 = 16$
2. $x^2 + y^2 = 4$, $x^2 + (y - 6)^2 = 25$
3. $x^2 + y^2 = 20$, $(x - 2)^2 + (y + 1)^2 = 13$
4. $x^2 + y^2 - 4y = 0$, $x^2 + y^2 - 2x = 4$
5. $x^2 + y^2 = 5$, $x^2 + y^2 - 12x + 6y = -25$
6. $x^2 + y^2 = 4$, $x^2 + y^2 - 6x - 6y = -14$

7. *Visual Thinking* Consider the ellipses $4x^2 + 5y^2 = 81$ and $5x^2 + 4y^2 = 81$.
   a. In how many points do they intersect? Give a convincing argument to justify your answer. Illustrate your answer graphically.
   b. Add the given equations. What is the graph of the resulting equation?
   c. Why do the intersection points of the ellipses lie on the graph from part (b)?

8. Two ellipses have equations $5x^2 + 3y^2 = 64$ and $3x^2 + 5y^2 = 64$. Show algebraically that their points of intersection are on the circle $x^2 + y^2 = 16$.

Solve each system algebraically or with a graphing calculator or computer software. (Note: For many of these systems, an algebraic approach may be more efficient.)

9. $x^2 + 4y^2 = 16$, $x^2 + y^2 = 4$

10. $4x^2 + y^2 = 16$, $x^2 - y^2 = -4$

11. $x^2 + 9y^2 = 36$, $x^2 - 2y = 4$

12. $4x^2 + 4y^2 = 25$, $2x + y^2 = 1$

13. $x^2 + y^2 = 25$, $xy = -12$

14. $y^2 - x^2 = 64$, $xy = 24$

15. $x^2 - y^2 = 1$, $y = -1 - x^2$

16. $9x^2 + 4y^2 = 36$, $x^2 + y^2 = 16$

17. $xy = 4$, $y = -4x^2$

18. $9x^2 - 16y^2 = 144$, $x + y^2 = -4$

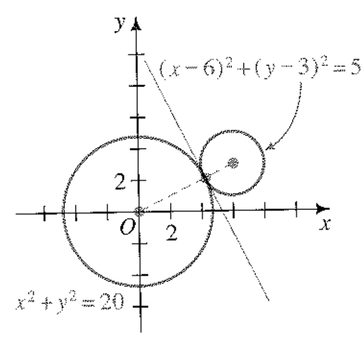
19. $(x + 3)^2 + y^2 = 1$, $x = -y^2$

20. $x^2 + 6y^2 = 9$, $5x^2 + y^2 = 16$

B 21. $x^2 + y^2 = 9$, $x^2 + y^2 + 8x + 7 = 0$

22. $4x^2 + y^2 - 4y - 32 = 0$, $x^2 - y - 7 = 0$

23. a. The circles $x^2 + y^2 = 20$ and $(x - 6)^2 + (y - 3)^2 = 5$ are tangent. Find the coordinates of the point of tangency and then find an equation of the common internal tangent shown. (*Hint*: How is this line related to the line containing the centers?)
   b. If you subtract the equations of the two circles from each other, you get a linear equation. Find this equation. Compare it with the equation found in part (a).



24. a. Find the radii and the distance between the centers of the circles with equations $x^2 + y^2 = 225$ and $(x - 6)^2 + (y - 8)^2 = 25$.
   b. Use your answers to part (a) to conclude that the circles must be internally tangent. Make a sketch.
   c. Find an equation of the common tangent of the two circles.

Graph the solution set of each inequality or system of inequalities.

25. $x^2 - 9y^2 \leq 0$

26. $y^2 - 2xy \geq 0$

27. $x^2 + 4y^2 - 10x + 24y + 61 \leq 0$

28. $x^2 - y^2 + 2x - 2y \geq 0$

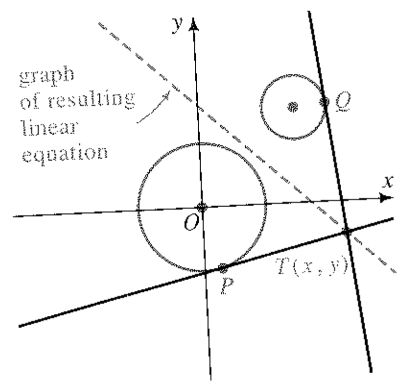
29. $9x^2 - y^2 \geq 0$
$x^2 + y^2 \leq 10$

30. $x^2 + 4y^2 \leq 16$
$x^2 - 2xy \geq 0$

C **31.** **Carpentry** A board 90 cm long just fits inside the bottom of a box 80 cm long and 60 cm wide. How wide is the board?

32. **a.** Given equations of two non-intersecting circles, show that the graph of the difference of the equations consists of those points $T(x, y)$ for which tangents $\overline{TP}$ and $\overline{TQ}$ are congruent.

b. Does the result in part (a) hold if the circles intersect?



COMMUNICATION: Reading

René Descartes was the first mathematician to use letters from the end of the alphabet to represent variables and letters from the beginning of the alphabet to represent constants, much as we do today. In the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, the letters A, B, C, D, E, and F represent constants, and the letters x and y represent variables.

The *American Heritage Dictionary* entry for the letter c is given below. Pick a letter of the alphabet. Write as many mathematical or scientific uses of that letter as you can without using a dictionary. Use a dictionary to expand your list.

c, C (sē) *n., pl.* **c's** or *rare* **cs. C's** or **Cs. 1.** The third letter of the modern English alphabet. See **alphabet**. **2.** Any of the speech sounds represented by this letter.
c, C, c., C. ***Note***: As an abbreviation or symbol, *c* may be a small or a capital letter, with or without a period. Established forms or those generally preferred precede the definition. When no form is given, all four forms are in general use in that sense. **1. c** *Physics.* candle. **2. C** *Electricity.* capacitance. **3. c., C.** capacity. **4. c., C.** cape. **5. c** carat. **6. C** The symbol for the element carbon. **7. c., C.** carton. **8. c., C.** case. **9. c., C.** *Baseball.* catcher. **10. C.** Catholic. **11. C** Celsius. **12. C.** Celtic. **13. c., C.** cent. **14. c** centi-. **15. C** centigrade. **16. c., C.** centime. **17. c., C.** century. **18. C.** chancellor. **19. c., C.** chapter. **20. C** *Physics.* charge conjugation. **21. C.** chief. **22. c., C.** church. **23. c., C.** circa. **24. C.** city. **25. c.** cloudy. **26. C.** companion. **27. c., C.** congius. **28. C.** Congress. **29. C.** Conservative. **30. c, C** *Mathematics.* constant. **31. c., C.** consul. **32. c., C.** copy. **33. c., C.** copyright. **34. c., C.** corps. **35. C** coulomb. **36. C.** court. **37. c** cubic. **38. c.** cup. **39. C** The Roman numeral for 100 (Latin *centum*). **40.** The third in a series. **41. C** The third highest in quality or rank: *a mark of C on a term paper.* **42. C** *Music.* **a.** The first tone in the scale of C major, or the third tone in the relative minor scale. **b.** The key or a scale in which C is the tonic. **c.** A written or printed note representing this tone. **d.** A string, key, or pipe tuned to the pitch of this tone.

6-7 A New Look at Conic Sections

Objective *To define conic sections in terms of eccentricity and to classify the graph of a second-degree equation by examining the coefficients in the equation.*

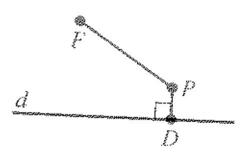
Although the various conic sections look quite different from one another, they do have some common properties.

1. The conic sections result from slicing a double cone with a plane. (See page 213.)
2. The conic sections can all be obtained from the single definition given below.
3. The conic sections all have second-degree equations. (See the discussion and the theorem on the next page.)

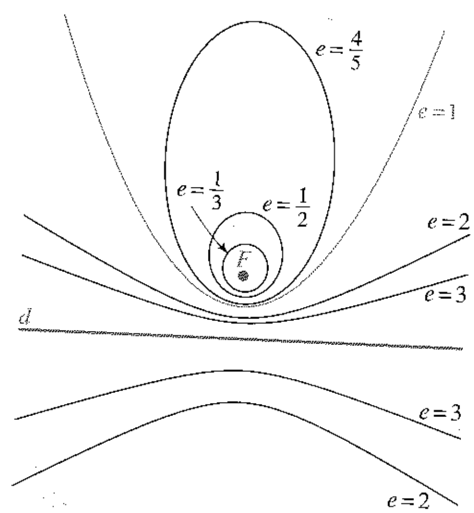
Common Definition of Conic Sections

Let F (the **focus**) be a fixed point not on a fixed line d (the **directrix**). Let P be a point in the plane of d and F, and let PD be the perpendicular distance from P to d. Consider the set of points for which the ratio $PF{:}PD$ is the constant e. The number e is the **eccentricity** of the conic section.

This set of points is:
(1) an ellipse if $0 < e < 1$;
(2) a parabola if $e = 1$;
(3) a hyperbola if $e > 1$.



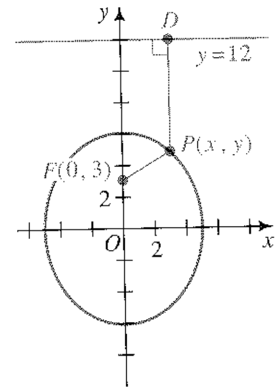
The diagram below suggests that once line d and point F are specified, the parabola (shown in blue) with directrix d and focus F separates the set of ellipses and the set of hyperbolas having this same directrix and focus. Notice that as the value of e approaches 0, the ellipse becomes more circular. The eccentricity of a circle is 0.



Example 1 Let the focus F be $(0, 3)$ and let the directrix d have equation $y = 12$. Find the equation of the set of points P for which $\frac{PF}{PD} = \frac{1}{2}$, and identify the graph.

Solution

$$2(PF) = PD$$
$$2\sqrt{(x-0)^2 + (y-3)^2} = |12 - y|$$
$$4[x^2 + (y-3)^2] = (12 - y)^2$$
$$4x^2 + 4y^2 - 24y + 36 = 144 - 24y + y^2$$
$$4x^2 + 3y^2 = 108$$
$$\frac{x^2}{27} + \frac{y^2}{36} = 1$$



The graph is an ellipse with vertices $(0, \pm 6)$ and foci $(0, \pm 3)$.

Second-Degree Equations in Two Variables

A conic section can be considered the graph of a *second-degree equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C are not all 0. For example, if $A = 4$, $B = 0$, $C = 3$, $D = 0$, $E = 0$, and $F = -108$, then we have $4x^2 + 3y^2 - 108 = 0$. In the example above, we saw that the graph of this equation is an ellipse.

So far we have considered primarily those second-degree equations in which $B = 0$. The one exception has been the hyperbola of the form $xy = k$. The theorem below enables us to classify a conic section by looking at its equation.

Classifying Second-Degree Equations

Consider the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. If A, B, and C are not all 0 and if the graph is not degenerate, then:
(1) the graph is a circle or an ellipse if $B^2 - 4AC < 0$; (In a circle, $B = 0$ and $A = C$.)
(2) the graph is a parabola if $B^2 - 4AC = 0$;
(3) the graph is a hyperbola if $B^2 - 4AC > 0$.

Example 2 Identify the graph of the equation $x^2 - 2xy + 3y^2 - 1 = 0$.

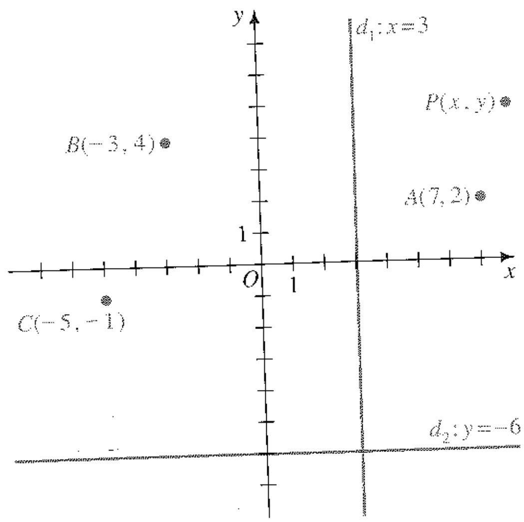
Solution

$$B^2 - 4AC = (-2)^2 - 4(1)(3)$$
$$= -8 < 0$$

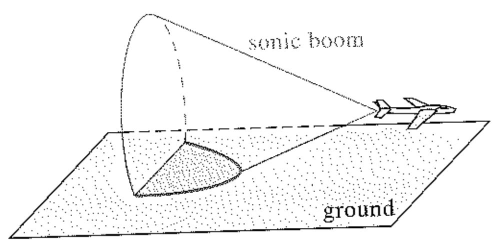
Since $B \neq 0$, the graph is an ellipse.

CLASS EXERCISES

Use the diagram for Exercises 1 and 2.



1. Find the distances from A, B, C, and P to:
   a. line d_1 with equation $x = 3$.
   b. line d_2 with equation $y = -6$.
2. State an equation that expresses the fact that:
   a. $PO = 2PA$
   b. $PO = \frac{1}{2}PB$
   c. the distance between P and C is equal to the distance between P and d_2.
   d. the distance between P and B is two times the distance between P and d_2.
   e. the distance between P and A is $\frac{1}{2}$ the distance between P and d_2.
3. **Aviation** The shock wave of a supersonic jet flying parallel to the ground is a cone. Points on the intersection of the cone and the ground receive a sonic boom at the same time. What conic section is the intersection?



4. Given that the following equations are not degenerate conics, tell whether the graph of each is a circle, an ellipse, a parabola, or a hyperbola.
   a. $x^2 - 3xy + 2y^2 + 2x - y + 6 = 0$
   b. $3x^2 - 4xy + 2y^2 - 3y = 0$
   c. $x^2 - 6xy + 9y^2 + x - y - 1 = 0$
   d. $144x^2 + 144y^2 - 216x + 96y - 47 = 0$

Describe the graph of each degenerate conic. If there is no graph, say so.

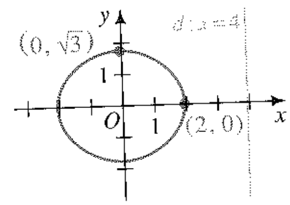
5. $x^2 + 9y^2 = 0$
6. $x^2 - 9y^2 = 0$
7. $x^2 + 9y^2 + 1 = 0$
8. ***Discussion*** Describe the graph of $Ax^2 + Cy^2 = 0$ if A and C are not both 0.
9. ***Investigation*** Shine a flashlight against a wall so that the edge of the lighted area is **(a)** a circle, **(b)** a parabola, and **(c)** a hyperbola. Relate the results of your investigation to the diagrams of the conic sections on page 213.

WRITTEN EXERCISES

A **1. a.** Find an equation for the ellipse.

b. Find the coordinates of F, the focus on the positive x-axis.

c. The distance from $P(x, y)$ to F is 0.5 times the distance from P to the line d: $x = 4$. Write and simplify an equation to show that P lies on the ellipse shown.



2. Given $F(0, 8)$ and d: $y = \frac{25}{2}$. The distance from $P(x, y)$ to F is $\frac{4}{5}$ the distance from P to d. Write an equation to show that P lies on an ellipse.

3. Given $F(-6, 0)$ and d: $x = -\frac{3}{2}$. The distance from $P(x, y)$ to F is twice the distance from P to d. Write an equation to show that P lies on a hyperbola.

The following are not degenerate conics. Identify the graph of each.

4. $x^2 - 2xy - y^2 = 4$

5. $x^2 - xy + 2y^2 = 2$

6. $y = x - \frac{1}{x}$

7. $x^2 - 3xy + y^2 = 5$

8. $x^2 + xy + y^2 - 4x\sqrt{2} - 4y\sqrt{2} = 0$

9. $x^2 + xy + y^2 - x\sqrt{2} - y\sqrt{2} = 0$

Describe the graph of each equation. If there is no graph, say so.

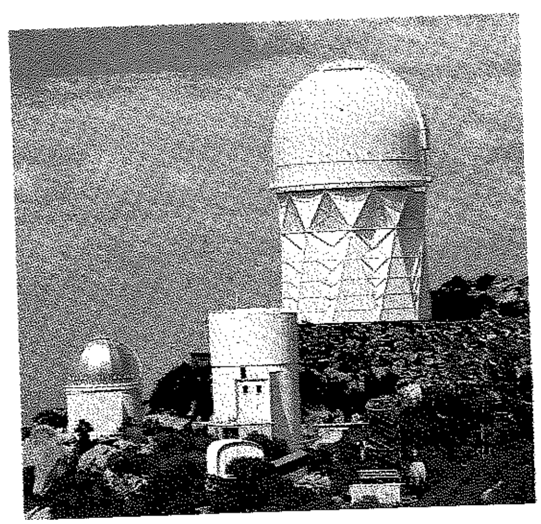
10. $4x^2 - y^2 = 0$

11. $4x^2 + y^2 = -1$

12. $y^2 - 3yx = 0$

When an equation of an ellipse is given in the form shown on page 226, it can be shown that $e = \frac{c}{a}$. Use this fact in Exercises 13 and 14.

B **13.** Astronomy Earth's orbit is an ellipse with the sun almost at one focus. During January, Earth is closest to the sun, a distance of 9.14×10^7 mi. During July, Earth is farthest from the sun, a distance of 9.44×10^7 mi. Find the eccentricity of Earth's orbit.



14. Astronomy Halley's comet has an elliptical orbit with the sun at a focus. When it is closest to the sun, it passes within 8.8×10^7 km of the sun. Its greatest distance from the sun is about 5.282×10^9 km. Sketch the orbit and give its eccentricity.

Solve.

15. $x^2 + y^2 = 9$
    $(x + y)^2 = 9$

16. $2x^2 + 3xy - 2y^2 = 0$
    $x^2 + 2y^2 = 6$

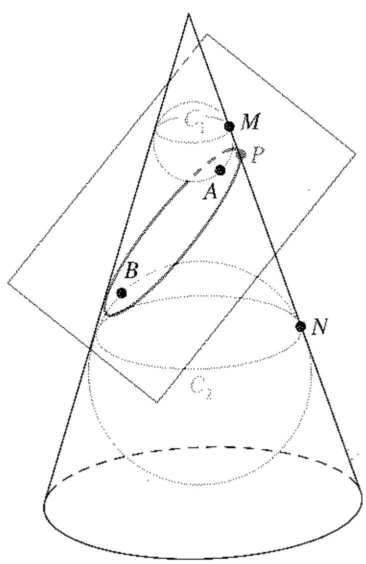
17. $x^2 + y^2 = 40$
    $x^2 + 2xy - 3y^2 = 0$

18. $3x^2 - 4xy - 4y^2 = 0$
    $x - 2y^2 + 4 = 0$

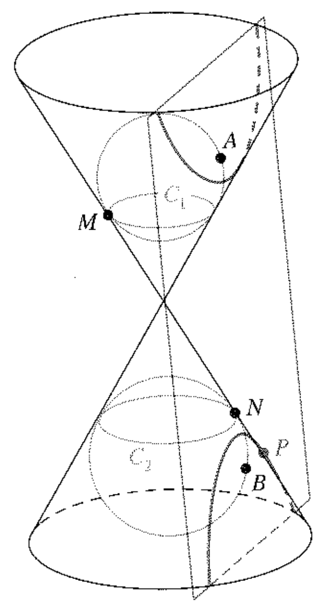
C 19. $x^2 + xy + y^2 = 3$
    $x^2 - y^2 = 3$

20. $x^2 + 3y^2 = 3$
    $3x^2 - xy = 6$

21. The diagram at the left below shows an ellipse formed by a plane cutting through a cone. It also shows two spheres tangent to the plane of the ellipse at A and B. They are also tangent to the cone, touching the cone in the parallel circles C_1 and C_2. Let P be any point on the ellipse. (For simplicity, P is shown where it is.)
    a. Explain why $PA = PM$ and $PB = PN$.
    b. Prove $PA + PB$ is a constant; thus proving that A and B are the foci of an ellipse. (The spheres are called **Dandelin spheres** after the Belgian mathematician Germinal Pierre Dandelin (1794–1847) who discovered them in 1822.)



Ex. 21



Ex. 22

22. The diagram at the right above shows a hyperbola formed by a plane cutting a double cone. Two spheres are tangent to the plane of the hyperbola at A and B. They are also tangent to the cone, touching the cone in the parallel circles C_1 and C_2. Let P be any point on the hyperbola. Show that $PA - PB$ is a constant, thus proving that A and B are the foci of a hyperbola.

Chapter Summary

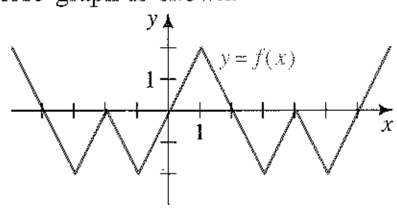
1. To prove a theorem from geometry by using coordinates, introduce a coordinate system in which the figure involved has as many zero coordinates as possible. Choose one or more of the methods listed on page 217.
2. A *circle* with radius r and center $C(h, k)$ has an equation of the form
$$(x-h)^2+(y-k)^2=r^2.$$
To find the intersection of a line and circle, follow the procedure on page 221.
3. If F_1 and F_2 are two fixed points (the *foci*) in the plane and a is a positive real number, then the set of all points $P(x, y)$ in the plane such that
$$PF_1+PF_2=2a$$
is an *ellipse*. An ellipse centered at the origin and having horizontal or vertical major axis has an equation of the form
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ or } \frac{y^2}{a^2}+\frac{x^2}{b^2}=1.$$
4. If F_1 and F_2 are two fixed points (the *foci*) in the plane and a is a positive real number, then the set of all points $P(x, y)$ in the plane such that
$$|PF_1-PF_2|=2a$$
is a *hyperbola*. A hyperbola centered at the origin and opening either horizontally or vertically has an equation of the form
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ or } \frac{y^2}{a^2}-\frac{x^2}{b^2}=1.$$
5. If F is a fixed point (the *focus*) in the plane and d (the *directrix*) is a fixed line not containing F, then the set of points equidistant from F and d is a *parabola*. A parabola whose vertex is at the origin and whose directrix is either horizontal or vertical has an equation of the form
$$y=\pm\frac{1}{4p}x^2 \text{ or } x=\pm\frac{1}{4p}y^2.$$
6. A system of two second-degree equations in two variables can be solved by algebraic methods, graphical methods, or a combination of algebraic and graphical methods.
7. Let F (the *focus*) be a fixed point not on a fixed line d (the *directrix*) and let e (the *eccentricity*) be positive. Let P be any point in the plane such that
$$\frac{PF}{PD}=e, \text{ where } PD \text{ is the distance from } P \text{ to } d.$$
If $0<e<1$, the set of points is an *ellipse*.
If $e=1$, the set of points is a *parabola*.
If $e>1$, the set of points is a *hyperbola*.
Conic sections are graphs of second-degree equations. A conic section can be identified by examining the coefficients of its equation.

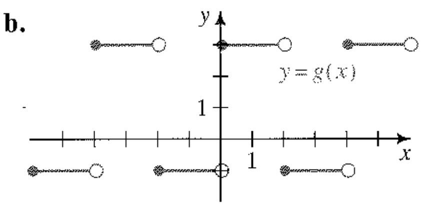
Key vocabulary and ideas

conic sections (p. 213, p. 247)
second-degree equation (p. 213, p. 248)
circle, center, radius (p. 219)
ellipse, foci of an ellipse (p. 226)
major axis, minor axis (p. 227)
vertices, center of an ellipse (p. 227)
hyperbola (p. 231)
asymptotes of a hyperbola (p. 232)
parabola, focus, directrix (p. 238)
eccentricity (p. 247)

Chapter Test

1. Prove that the diagonals of a square are perpendicular and congruent. 6-1
2. Find the coordinates of any points where the line $x - y = 2$ and the circle $x^2 + y^2 = 4$ intersect. 6-2
3. Find an equation of the circle having $(2, 5)$ and $(-2, -1)$ as endpoints of a diameter.
4. Find the coordinates of the vertices and the foci of the ellipse with equation $9x^2 + 5y^2 = 45$. Sketch the ellipse. 6-3
5. Find the coordinates of any points where the line $2x + 3y = 6$ and the ellipse $4x^2 + 9y^2 = 36$ intersect.
6. Find an equation for the ellipse that has $(0, -4)$ and $(0, 4)$ as vertices and $(-3, 0)$ and $(3, 0)$ as endpoints of its minor axis.
7. Find an equation for the hyperbola that satisfies the given conditions. 6-4
   a. Center at $(0, 0)$, a vertex at $(0, -3)$, and a focus at $(0, -\sqrt{13})$
   b. A vertex at $(2, 0)$ and asymptotes with equations $y = \pm 2x$
8. Graph the hyperbola $xy = -9$ and give equations for its asymptotes.
9. Find the coordinates of the vertex and focus, and the equation of the directrix, of the parabola $y = \frac{1}{6}x^2$. 6-5
10. Find an equation for the parabola whose directrix is $x = -3$ and whose focus is $(3, 0)$.
11. On a single set of axes, graph $x^2 + y^2 = 25$ and $x^2 + 10y^2 = 169$. Solve these equations simultaneously. 6-6
12. Given that the following equations are not degenerate conics, tell whether the graph is a circle, an ellipse, a hyperbola, or a parabola. 6-7
    a. $4x^2 + 4y^2 - 8x + 24y - 15 = 0$
    b. $-x^2 + 2xy - 3y^2 + 12 = 0$
    c. $2x^2 - 12xy + 18y^2 + 2x - 6y + 24 = 0$
13. Find and simplify an equation that expresses that $P(x, y)$ is equidistant from the point $F(1, 3)$ and the line $y = -4$.
14. ***Writing*** In a few sentences, describe the possibilities for the graph of the equation $Ax^2 + Bxy + Cy^2 = 0$, where $AC = 0$.

1. Sketch the graph of the function $f(x) = x^2 - 2x$. Then use the graph to find the range and zeros of f.
2. If $f(x) = x - 1$ and $g(x) = 1 - x$, find:
   a. $(f + g)(x)$ **b.** $(f - g)(x)$ **c.** $(f \cdot g)(x)$
   d. $\left(\frac{f}{g}\right)(x)$ **e.** $(f \circ g)(x)$ **f.** $(g \circ f)(x)$
3. Tell whether the graph of each equation has symmetry in (i) the x-axis, (ii) the y-axis, (iii) the line $y = x$, and (iv) the origin.
   a. $xy^2 + x^2y = 4$ **b.** $y = x^4 + 2x^2 + 1$ **c.** $y^2 = |x| + 1$
4. Give the fundamental period and amplitude of each periodic function whose graph is shown.
   a.


   b.


5. **a.** Using the graph of $y = f(x)$ in part (a) of Exercise 4, sketch the graph of $y = f(x + 1)$.
   b. Using the graph of $y = g(x)$ in part (b) of Exercise 4, sketch the graph of $y = g(-x) - 1$.
6. Tell whether each function f has an inverse. If f^{-1} exists, find a rule for $f^{-1}(x)$ and show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.
   a. $f(x) = 8 - 2x$ **b.** $f(x) = x^3 - x^2$ **c.** $f(x) = \dfrac{1}{x - 2}$
7. Tickets to a show cost \$6.00 for adults and \$4.00 for children. Let x and y represent the numbers of tickets sold to adults and children, respectively.
   a. Find a rule for the total sales function $S(x, y)$.
   b. Find $S(40, 25)$ and $S(32, 48)$.
   c. Sketch the graphs of the constant sales curves $S(x, y) = 300$ and $S(x, y) = 1200$ in an xy-plane.
8. Simplify.
   a. $(4^{-1} + 2^{-1})^2$ **b.** $(8y^4)(2y^{-3})^{-2}$ **c.** $\dfrac{(3x^2)^{-1}}{6x^{-3}}$
   d. $\left(\dfrac{81}{64}\right)^{-1/2}$ **e.** $(27^{-2})^{-1/3}$ **f.** $a^{3/4}(a^{-3/4} - a^{1/4})$
9. Solve.
   a. $3^{4x+1} = 81$ **b.** $\frac{1}{5}x^{2/3} = 20$ **c.** $(x - 1)^{-2} = 25$
10. Find an exponential function f such that $f(0) = \frac{1}{2}$ and $f(4) = \frac{9}{2}$.

11. The table shows the number $N(t)$ of bacteria in a colony at various times t (in hours).

t	0	1	2	3	4	5
$N(t)$	150	189	238	300	378	476

a. What is the doubling time for this colony of bacteria?
b. Find an equation for $N(t)$.
c. How many bacteria will there be after one full day?

12. Suppose you invest \$100 at 7% annual interest. Calculate the amount that you would have after one year if interest is compounded:
a. quarterly **b.** monthly **c.** continuously

13. Which plan yields a greater return on an investment?
Plan A: 8.1% annual rate compounded quarterly
Plan B: 8% annual rate compounded continuously

14. Simplify.
a. $\log \frac{1}{100}$ **b.** $\log_2 16$ **c.** $\log_{125} 5$
d. $\ln \frac{1}{e^5}$ **e.** $e^{2 \ln 3}$ **f.** $10^{1 + \log 5}$

15. Write each expression in terms of $\log M$ and $\log N$.
a. $\log M^2N^3$ **b.** $\log \sqrt{\frac{M^3}{N}}$ **c.** $\log 100M\sqrt{N}$

16. Express y in terms of x.
a. $\log y = \log x + 2$ **b.** $\ln y = \ln 2 - 3 \ln x$ **c.** $\log y = 2.5x + 1$

17. To the nearest hundredth, solve each equation.
a. $(e^x)^2 = 64$ **b.** $2^{x+1} = 50$ **c.** $(1.04)^x = 2$

18. Prove that the line segments joining the midpoints of successive sides of any rhombus form a rectangle.

19. Find the center and radius of the circle $x^2 + y^2 - 4x + 14y + 28 = 0$.

20. Sketch the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$. Find the coordinates of its center, vertices, and foci.

21. Find an equation of the hyperbola centered at the origin that has a focus at $(\sqrt{5}, 0)$ and a vertex at $(2, 0)$.

22. Sketch the parabola $x + 2 = -\frac{1}{4}(y - 1)^2$, and give its vertex, focus, and directrix.

23. Solve the system: $x^2 + y^2 = 25$
$2x^2 - 3y = 6$

24. Given that $x^2 - 2xy + 4y^2 = 4$ is not a degenerate conic, identify the graph of the equation.