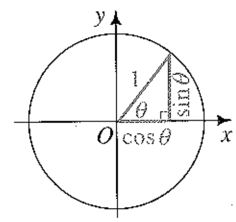
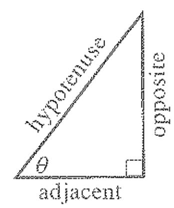
9 Triangle Trigonometry

9-1 Solving Right Triangles

Objective *To use trigonometry to find unknown sides or angles of a right triangle.*

In Chapter 7, we defined the trigonometric functions in terms of coordinates of points on a circle. In this chapter, our emphasis shifts from circles to triangles. When certain parts (sides and angles) of a triangle are known, you will see that trigonometric relationships can be used to find the unknown parts. This is called **solving a triangle**. For example, if you know the lengths of the sides of a triangle, then you can find the measures of its angles. In this section, we will consider how trigonometry can be applied to right triangles.





The right triangles shown in the diagrams at the right both have an acute angle of measure θ and are, therefore, similar. Consequently, the lengths of corresponding sides are proportional, and we have the equations shown below:

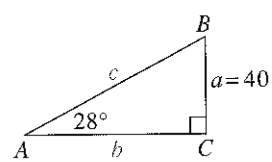
$$\frac{\sin\theta}{1} = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \frac{\cos\theta}{1} = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{adjacent}}$$

By the reciprocal relationships (see page 282), we also have:

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Applications of these equations are given in the following examples. In Example 1, notice the convention of using a capital letter to denote an angle and the corresponding lower-case letter to denote the length of the side opposite that angle.

Example 1 For the right triangle ABC shown at the right, find the value of b to three significant digits.



Solution To find the value of b, use either $\tan 28°$ or $\cot 28°$.

Using tan 28°:

$$\tan 28° = \frac{\text{opposite}}{\text{adjacent}} = \frac{40}{b}$$

$$b = \frac{40}{\tan 28°} \approx 75.2$$

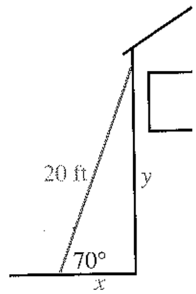
Using cot 28°:

$$\cot 28° = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{40}$$

$$b = 40\cot 28° \approx 75.2$$

◀ Triangle trigonometry is especially useful in navigation. The pilot of the ship shown can use it to find the height of the mountains around the ship, to measure the width of the channel ahead, and to plot the safest course along that channel.

Example 2 The safety instructions for a 20 ft ladder indicate that the ladder should not be inclined at more than a 70° angle with the ground. Suppose the ladder is leaned against a house at this angle, as shown. Find **(a)** the distance x from the base of the house to the foot of the ladder and **(b)** the height y reached by the ladder.



Solution

a. $\cos 70° = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{x}{20}$

$x = 20 \cos 70° \approx 6.84$

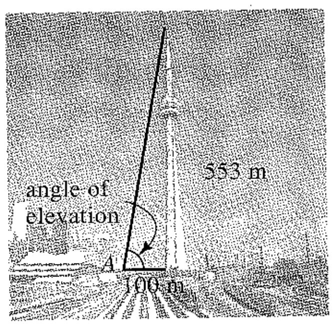
The foot of the ladder is about 6.84 ft from the base of the house.

b. $\sin 70° = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{y}{20}$

$y = 20 \sin 70° \approx 18.8$

The ladder reaches about 18.8 ft above the ground.

Example 3 The highest tower in the world is in Toronto, Canada, and is 553 m high. An observer at point A, 100 m from the center of the tower's base, sights the top of the tower. The *angle of elevation* is $\angle A$. Find the measure of this angle to the nearest tenth of a degree.



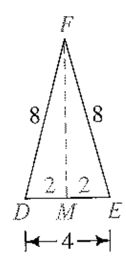
Solution

$\tan A = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{553}{100} = 5.53$ $\left\{ \begin{array}{l}\tan A \text{ is an abbreviation} \\ \text{for the tangent of } \angle A.\end{array}\right.$

$\angle A = \text{Tan}^{-1}\ 5.53 \approx 79.7°$

Because we can divide an isosceles triangle into two congruent right triangles, we can apply trigonometry to isosceles triangles, as shown in the following example.

Example 4 A triangle has sides of lengths 8, 8, and 4. Find the measures of the angles of the triangle to the nearest tenth of a degree.



Solution By drawing the altitude to the base of isosceles triangle DEF, we get two congruent right triangles. In $\triangle DMF$, we have:

$$\cos D = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{8} = 0.25$$

$$\angle D = \text{Cos}^{-1}\ 0.25 \approx 75.5°$$

Thus, $\angle E = \angle D \approx 75.5°$, and $\angle F \approx 180° - 2(75.5°) = 29.0°$.

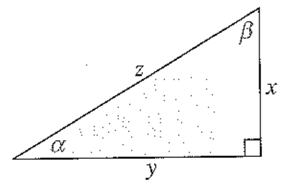
When calculators or trigonometric tables are used, only approximate results are possible in most cases. Throughout this chapter, therefore, you should round angle measures to the nearest tenth of a degree and give lengths accurate to three significant digits.

CLASS EXERCISES

1. Match each element of row A with an element in row B.

A:	sine	cosine	tangent	cotangent	secant	cosecant
B:	$\frac{\text{opposite}}{\text{adjacent}}$	$\frac{\text{opposite}}{\text{hypotenuse}}$	$\frac{\text{adjacent}}{\text{opposite}}$	$\frac{\text{hypotenuse}}{\text{opposite}}$	$\frac{\text{adjacent}}{\text{hypotenuse}}$	$\frac{\text{hypotenuse}}{\text{adjacent}}$

2. Refer to the diagram at the right.



a. Express the sine, cosine, and tangent of α in terms of x, y, and z.

b. Express the sine, cosine, and tangent of β in terms of x, y, and z.

c. Tell whether each of the following statements is true or false.

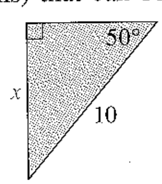
(1) $\sin \alpha = \cos \beta$ (2) $\tan \alpha = \cot \beta$ (3) $\sec \alpha = \csc \beta$

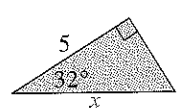
3. ***Reading*** In the solution of Example 1, suppose a calculator is not available. Would you prefer to find the value of b using $\tan 28°$ or $\cot 28°$? Explain.

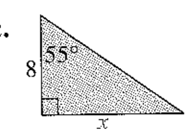
4. a. For right triangle ABC in Example 1, use trigonometry to find the value of c to three significant digits.

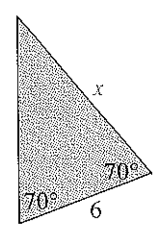
b. Knowing the values of a and b in Example 1, how could you find the value of c without using trigonometry?

5. For each triangle, state two equations (using reciprocal trigonometric functions) that can be used to find the value of x.

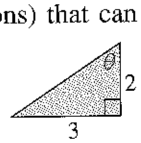
a.


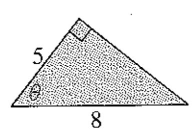
b.


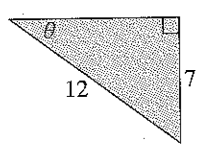
c.


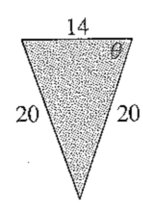
d.


6. For each triangle, state two equations (using reciprocal trigonometric functions) that can be used to find the measure of θ.

a.


b.


c.


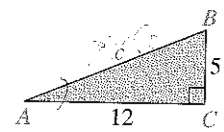
d.


WRITTEN EXERCISES

Throughout the exercises, give angle measures to the nearest tenth of a degree and lengths to three significant digits.

A

1. In $\triangle ABC$, $\angle A = 90°$, $\angle B = 25°$, and $a = 18$. Find b and c.
2. In $\triangle PQR$, $\angle P = 90°$, $\angle Q = 64°$, and $p = 27$. Find q and r.
3. In $\triangle DEF$, $\angle D = 90°$, $\angle E = 12°$, and $e = 9$. Find d and f.
4. In $\triangle XYZ$, $\angle X = 90°$, $\angle Y = 37°$, and $z = 25$. Find x and y.
5. Use the diagram at the right to find:
   a. $\sin A$ **b.** $\cos B$ **c.** $\tan A$
   d. $\cot B$ **e.** $\sec A$ **f.** $\csc B$

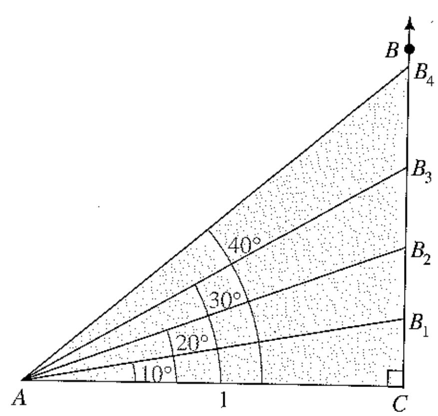


Ex. 5

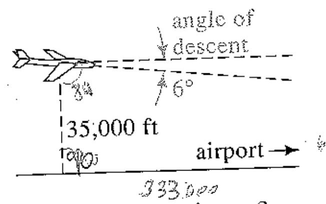
6. Sketch $\triangle ABC$ with $\angle C = 90°$. What is the relationship between:
   a. $\sin A$ and $\cos B$? **b.** $\tan A$ and $\cot B$? **c.** $\sec A$ and $\csc B$?
7. Find the measures of the acute angles of a 3-4-5 right triangle.
8. Find the measures of the acute angles of a right triangle whose legs are 9 cm and 16 cm long.
9. The legs of an isosceles right triangle are 1 unit long.
   a. Find the length of the hypotenuse in simplest radical form.
   b. Use part (a) to find the exact value of each of the following.
   (1) $\tan 45°$ (2) $\sin 45°$ (3) $\cos 45°$
   c. Convert the answers in part (b) to decimal form. Compare these with the values of $\tan 45°$, $\sin 45°$, and $\cos 45°$ obtained from the calculator.
10. The hypotenuse of a 30°-60°-90° triangle is 2 units long.
    a. Find the lengths of the legs in simplest radical form.
    b. Use part (a) to find the exact value of each of the following.
    (1) $\sin 30°$ (2) $\sin 60°$ (3) $\tan 30°$ (4) $\tan 60°$
    c. Convert the answers in part (b) to decimal form. Compare these with the values of $\sin 30°$, $\sin 60°$, $\tan 30°$, and $\tan 60°$ obtained from the calculator.

For Exercise 11, you may wish to use a computer with software that draws and measures geometric figures.

11. Draw a horizontal segment AC of length 1 unit and a vertical ray CB, as shown. Then draw segments that make angles of 10°, 20°, . . . , 80° with $\overline{AC}$. Measure the lengths CB_1, CB_2, . . . , CB_8. What is the significance of these lengths?
12. In the diagram that you drew for Exercise 11, note that $\angle CAB_2$ is twice as large as $\angle CAB_1$. Is $\overline{CB_2}$ twice as long as $\overline{CB_1}$? Is $\overline{CB_4}$ four times as long as $\overline{CB_1}$?



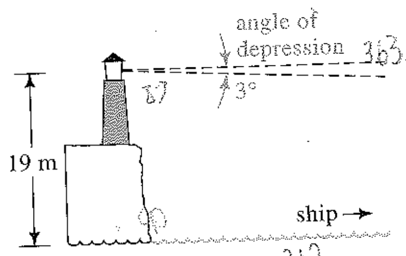
13. Aviation An airplane is at an elevation of 35,000 ft when it begins its approach to an airport. Its *angle of descent* is 6°.



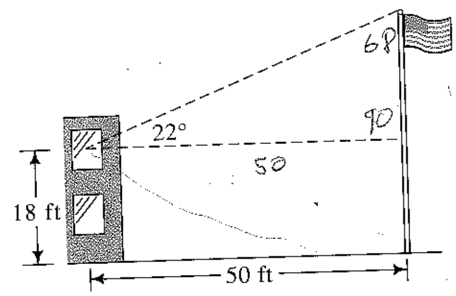
a. What is the distance between the airport and the point on the ground directly below the airplane?

b. What is the approximate air distance between the plane and the airport? What assumptions did you make in finding this distance?

14. Navigation A lighthouse keeper observes that there is a 3° *angle of depression* between the horizontal and the line of sight to a ship. If the keeper is 19 m above the water, how far is the ship from shore?



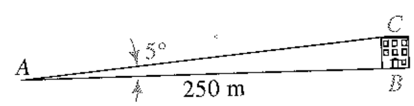
Ex.14



Ex. 15

15. A student looks out of a second-story school window and sees the top of the school flagpole at an angle of elevation of 22°. The student is 18 ft above the ground and 50 ft from the flagpole. Find the height of the flagpole.

16. For an observer at point A, 250 m from a building, the angle of elevation of the top of the building is 5°. In Chapter 7, we said that $\triangle ABC$ is about the same as a sector with central angle A.



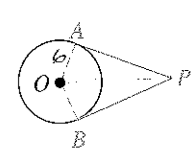
a. Use the arc length formula $s = r\theta$ to approximate BC. (Remember to express θ in radians.)

b. Use right-triangle trigonometry to find BC more accurately. Compare your answers.

17. Find the measures of the angles of an isosceles triangle whose sides are 6, 6, and 8. Also find the area of the triangle.

18. The legs of an isosceles triangle are each 21 cm long and the angle between them has measure 52°. What is the length of the third side?

B 19. In the figure at the right, $\overline{PA}$ and $\overline{PB}$ are tangents to a circle with radius $OA = 6$. If the measure of $\angle APB$ is 42°, find PA and PB.

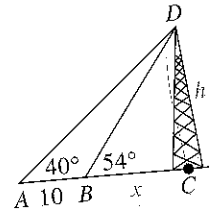


Ex. 19

20. Sketch the circle $(x - 6)^2 + (y - 8)^2 = 9$ and the two tangents to the circle from the origin. Find the measure of the angle between the tangents.

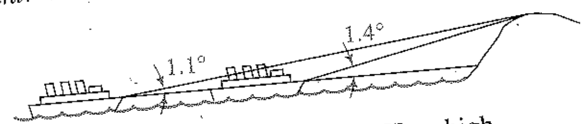
21. An isosceles trapezoid has sides whose lengths are in the ratio $5:8:5:14$. Find the measure of the angle between one of the legs and the shorter base. (*Hint*: Draw an altitude.)

22. A rectangle is 14 cm wide and 48 cm long. Find the measure of the acute angle between its diagonals.

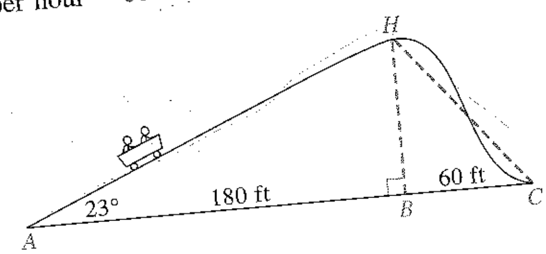


Ex. 23

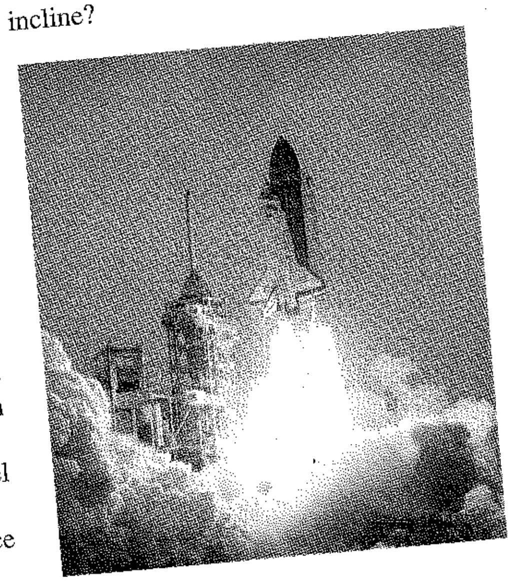
23. From points A and B, 10 m apart, the angles of elevation of the top of a tower are 40° and 54°, respectively, as shown at the right. Find the tower's height. (*Hint:* Write two equations in the unknowns x and h.)



24. **Navigation** From a ship offshore, the angle of elevation of a hill is 1.1°. After the ship moves inland at 4.5 knots for 20 min, the angle of elevation is 1.4°. How high is the hill? (1 knot = 1 nautical mile per hour ≈ 6080 feet per hour)



25. **Physics** The roller coaster car shown in the diagram takes 23.5 s to go up the 23° incline $\overline{AH}$ and only 2.8 s to go down the drop from H to C. The car covers horizontal distances of 180 ft on the incline and 60 ft on the drop.
   a. How high is the roller coaster above point B?
   b. Find the distances AH and HC.
   c. How fast (in ft/s) does the car go up the incline?
   d. What is the approximate average speed of the car as it goes down the drop? (Assume that the car travels along $\overline{HC}$. Since the actual path is longer than $\overline{HC}$, is your approximate answer too big or too small?)

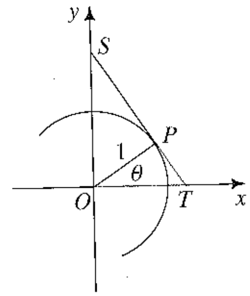


26. **Space Science** From a point 1.5 mi from a launch pad at Cape Canaveral, an observer sights a space shuttle at an angle of elevation of 10° moments after it is launched. After 10 s, the angle of elevation is 50°.
   a. How far does the space shuttle travel vertically during the 10 s interval?
   b. What is the average speed of the space shuttle during that time interval?

27. a. Draw $\triangle ABC$ with $\angle C = 90°$ and $\sin A = \frac{5}{13}$. Without using a calculator or table, find $\sec A$ and $\csc A$.
   b. Given that $\sin A = x$, find $\sec A$ in terms of x.

28. **a.** Draw $\triangle PQR$ with $\angle R = 90^\circ$ and $\tan P = \frac{1}{4}$. Without using a calculator or table, find $\sin P$ and $\cos Q$.

b. Given that $\tan P = x$, find $\sin P$ in terms of x.



29. A line tangent to the circle $x^2 + y^2 = 1$ at P intersects the axes at T and S, as shown in the diagram.

a. Show that $PT = \tan\theta$ and $OT = \sec\theta$. (This may explain the use of the names *tangent* and *secant*. Another interpretation is given in Exercise 37 on page 322.)

b. Show that $\sec^2\theta = 1 + \tan^2\theta$.

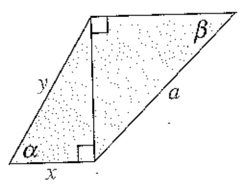
c. ***Visual Thinking*** Describe what happens to point T as θ increases from 0° to 90°, from 90° to 180°, from 180° to 270°, and from 270° to 360°. Your answers should agree with the graphs of $y = \tan\theta$ and $y = \sec\theta$.

30. **a.** Use the diagram in Exercise 29 to show that $PS = \cot\theta$ and $OS = \csc\theta$.

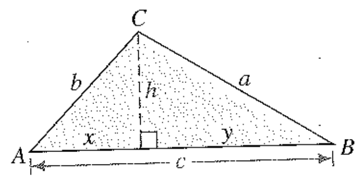
b. Show that $\csc^2\theta = 1 + \cot^2\theta$.

c. ***Visual Thinking*** Describe what happens to point S as θ increases from 0° to 90°, from 90° to 180°, from 180° to 270°, and from 270° to 360°. Your answers should agree with the graphs of $y = \cot\theta$ and $y = \csc\theta$.

31. Use the diagram at the left below. Express x and y in terms of α, β, and a.



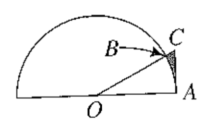
Ex. 31



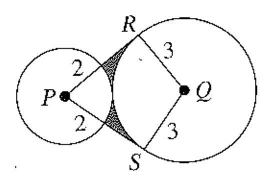
Ex. 32

32. Use the diagram at the right above. Show that:

a. $h = a\sin B = b\sin A$ **b.** area of $\triangle ABC = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$



33. In the diagram at the right, $\overline{AC}$ is tangent to a semicircle with radius $OA = 12$ cm. If the measure of $\angle AOB$ is 28°, find to the nearest square centimeter the area of the region that is inside $\triangle AOC$ and outside the circle.



34. As shown, two circles with radii 2 and 3 and centers P and Q, respectively, are externally tangent. From P, tangents $\overline{PR}$ and $\overline{PS}$ are drawn to the larger circle.

a. Find the measures of $\angle RPS$ and $\angle RQS$ to the nearest hundredth of a radian.

b. Find the area of the shaded region to the nearest hundredth of a square unit.

35. Each leg of an isosceles triangle is 8 cm long. Express the measure of the triangle's vertex angle as a function of the length of the triangle's base. Give an appropriate domain and range for this function.

36. The diagonals of a rectangle are 20 cm long. Express the measure of the acute angle formed by the diagonals as a function of the length of the rectangle's shorter sides. Give an appropriate domain and range for this function.

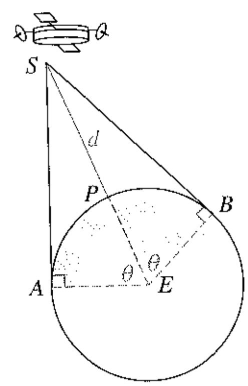
37. **Aviation** An airplane at an elevation of 30,000 ft begins to descend toward the runway on which it will land. Let d be the horizontal distance between the plane and the runway. Let θ be the plane's angle of descent (see Exercise 13).

a. Express θ as a function of d.

b. Suppose that the horizontal distance between the airplane and the runway is 60 mi. Find to the nearest tenth the angle at which the airplane must descend in order to land on the runway. (Recall that 1 mi = 5280 ft.)

You may wish to use a computer or a graphing calculator to complete part (c) of Exercise 38.

38. **Space Science** A telecommunications satellite S orbits above the earth. Suppose the shaded region, determined by tangents such as $\overline{SA}$ and $\overline{SB}$, is the portion of the earth's surface that can send and receive signals via the satellite. Assume point E to be the center of the earth and the radius of the earth to be 4000 mi.



a. Arc APB is an arc of a great circle (see Exercise 46 on page 301). The length of arc APB is the maximum distance a between two cities that can communicate via the satellite. Express a in terms of the central angle θ, and express θ in terms of the satellite's distance d above the earth's surface.

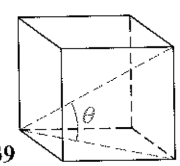
b. Use your answers in part (a) to express a as a function of d.

c. Give an appropriate domain and range for your function in part (b) and sketch its graph.

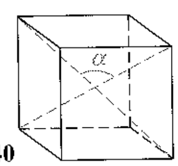
d. Many satellites are in *geostationary* orbits about 22,300 mi above the earth's surface. At this distance, the satellite's speed matches that of the earth's rotation, causing the satellite to appear stationary to an observer on the earth. If satellite S is in a geostationary orbit, can it transmit signals from Lima, Peru, to Cairo, Egypt, which is about 7726 mi away?

C 39. Find the measure of the angle θ formed by a diagonal of a cube and a diagonal of one of the faces of the cube.

40. Find the measure of the angle α formed by two diagonals of a cube.



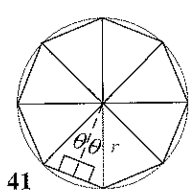
Ex. 39



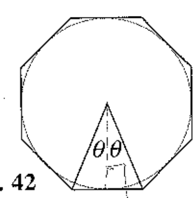
Ex. 40

41. Derive a formula for the area A of a regular polygon of n sides inscribed in a circle of radius r.

42. Derive a formula for the area K of a regular polygon of n sides circumscribed about a circle of radius r.



Ex. 41



Ex. 42

//// COMPUTER EXERCISE

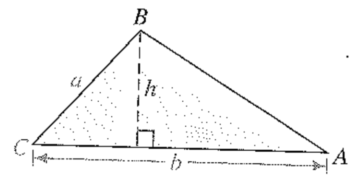
Use the answers from Exercises 41 and 42 to write a program that prints the values of A, K, and $K - A$ for $r = 1$ and $n = 10, 20, 30, \ldots, 100$. Interpret the results.

9-2 The Area of a Triangle

Objective *To find the area of a triangle given the lengths of two sides and the measure of the included angle.*

When the lengths of two sides of a triangle and the measure of the included angle are known, the triangle is uniquely determined. This fact is a consequence of the side-angle-side (SAS) condition for congruence. We now consider the problem of expressing the area of the triangle in terms of these measurements.

Suppose that we are given the lengths a and b and the measure of $\angle C$ in $\triangle ABC$. If the length of the altitude from B is h, then the area of the triangle is given by



$$K = \frac{1}{2}bh.$$

By right-triangle trigonometry, we know that $\sin C = \frac{h}{a}$, or

$$h = a \sin C.$$

Substituting $a \sin C$ for h, we find that

$$K = \frac{1}{2}ab \sin C.$$

If some other pair of sides and the included angle of $\triangle ABC$ were known, we could repeat the procedure for finding the area and thereby obtain two other area formulas. All three formulas are stated below.

The Area of a Triangle

The area K of $\triangle ABC$ is given by:

$$K = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B$$

Notice that the area formulas all have the basic pattern:

$$K = \frac{1}{2} \cdot (\text{one side}) \cdot (\text{another side}) \cdot (\text{sine of included angle})$$

Example 1 Two sides of a triangle have lengths 7 cm and 4 cm. The angle between the sides measures 73°. Find the area of the triangle.

Solution $K = \frac{1}{2} \cdot 7 \cdot 4 \cdot \sin 73° \approx 13.4$

Thus, the area is about 13.4 cm^2.

Example 2 The area of $\triangle PQR$ is 15. If $p = 5$ and $q = 10$, find all possible measures of $\angle R$.

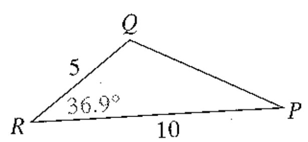
Solution Since $K = \frac{1}{2}pq \sin R$,

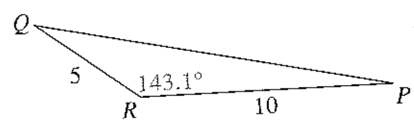
$$15 = \frac{1}{2} \cdot 5 \cdot 10 \cdot \sin R = 25 \sin R.$$

Thus, $\sin R = \frac{15}{25} = 0.6$. Since $\angle R$ could be acute *or* obtuse,

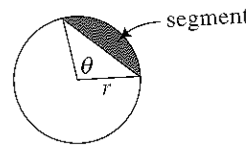
$$\angle R = \text{Sin}^{-1}\, 0.6 \approx 36.9^\circ \quad \text{or} \quad \angle R \approx 180^\circ - 36.9^\circ = 143.1^\circ$$

The diagrams below show the two possibilities for $\triangle PQR$.



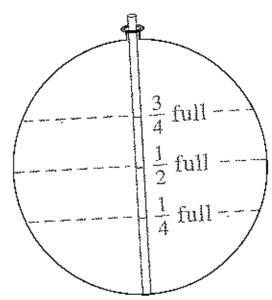


As shown at the right, a **segment** of a circle is the region bounded by an arc of the circle and the chord connecting the endpoints of the arc.

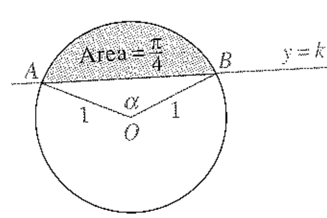


Next we use a computer or graphing calculator to solve an equation based on the area of a segment.

Example 3 The diagram shows an end view of a cylindrical oil tank with radius 1 m. Through a hole in the top, a vertical rod is lowered to touch the bottom of the tank. When the rod is removed, the oil level in the tank can be read from the oil mark on the rod. Where on the rod should marks be put to show that the tank is $\frac{3}{4}$ full, $\frac{1}{2}$ full, and $\frac{1}{4}$ full?



Solution Since the radius of the tank is $r = 1$ m, we can use the unit circle $x^2 + y^2 = 1$, shown at the right. To find where to put the "$\frac{3}{4}$ full" mark, we need to find the horizontal line $y = k$ that cuts off a segment having $\frac{1}{4}$ of the circle's area, or



$$\frac{1}{4}\pi r^2 = \frac{1}{4}(\pi \cdot 1^2) = \frac{\pi}{4}.$$

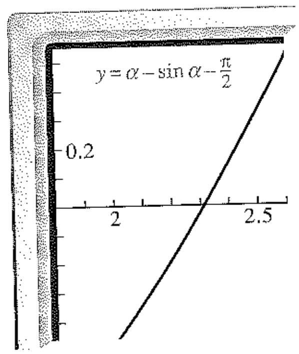
Area of segment = area of sector AOB − area of $\triangle AOB$

$$\frac{\pi}{4} = \frac{1}{2} \cdot 1^2 \cdot \alpha - \frac{1}{2} \cdot 1^2 \cdot \sin \alpha$$

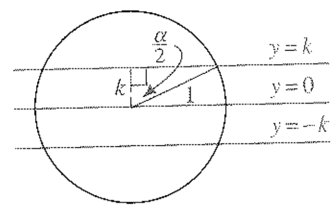
$$\frac{\pi}{4} = \frac{1}{2}(\alpha - \sin \alpha)$$

$$\frac{\pi}{2} = \alpha - \sin \alpha$$

$$0 = \alpha - \sin \alpha - \frac{\pi}{2}$$



To solve the last equation, we graph $y = \alpha - \sin \alpha - \frac{\pi}{2}$ using a computer or graphing calculator. From the graph at the right, we see that $y = 0$ when $\alpha \approx 2.3$. Using this value of α and the diagram at the right, we have:



$$k = \cos \frac{\alpha}{2}$$

$$\approx \cos \frac{2.3}{2} \approx 0.4$$

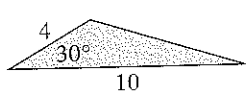
From the symmetry of the circle we deduce that the horizontal lines dividing the circle into four parts with equal areas have equations $y \approx 0.4$, $y = 0$, and $y \approx -0.4$. Thus, if one end of a vertical rod is at the bottom of the tank, then marks should be placed 0.6 m, 1 m, and 1.4 m from that end to indicate when the tank is $\frac{1}{4}$ full, $\frac{1}{2}$ full, and $\frac{3}{4}$ full, respectively.

CLASS EXERCISES

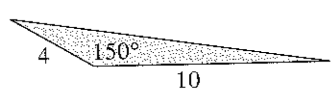
1. Two adjacent sides of a triangle have lengths 5 cm and 8 cm.
  a. If these sides form a 30° angle, what is the area of the triangle?
  b. If these sides form a 150° angle, what is the area of the triangle?

Find the area of each triangle.

2.



3.



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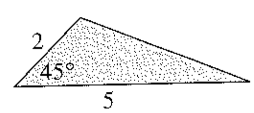


5. A triangle with area 5 cm^2 has two sides of lengths 4 cm and 5 cm.
  a. Find the sine of the angle included between these sides.
  b. Find the two possible measures of the included angle.

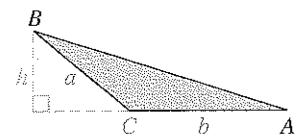
6. Find the area of a segment of a circle with radius 2 if the measure of the central angle of the segment is $\frac{\pi}{6}$.

WRITTEN EXERCISES

Throughout the exercises, give areas in radical form or to three significant digits. Give lengths to three significant digits and angle measures to the nearest tenth of a degree. In Exercises 1–4, find the area of each $\triangle ABC$.

A

1. **a.** $a = 4$, $b = 5$, $\angle C = 30°$
   b. $a = 4$, $b = 5$, $\angle C = 150°$
2. **a.** $b = 3$, $c = 8$, $\angle A = 120°$
   b. $b = 3$, $c = 8$, $\angle A = 60°$
3. **a.** $a = 6$, $c = 2$, $\angle B = 45°$
   b. $a = 6$, $c = 2$, $\angle B = 135°$
4. **a.** $a = 10$, $b = 20$, $\angle C = 70°$
   b. $a = 10$, $b = 20$, $\angle C = 110°$
5. What does the formula $K = \frac{1}{2}ab \sin C$ become when $\angle C$ is a right angle? Draw a sketch to illustrate.
6. As shown in the diagram at the right, $\angle C$ in $\triangle ABC$ is obtuse. Show that the formula $K = \frac{1}{2}ab \sin C$ gives the area K of $\triangle ABC$.



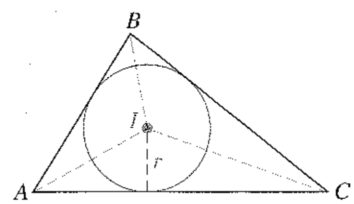
Ex. 6

7. Find the area of $\triangle XYZ$ if $x = 16$, $y = 25$, and $\angle Z = 52°$.
8. Find the area of $\triangle RST$ if $\angle S = 125°$, $r = 6$, and $t = 15$.
9. The area of $\triangle ABC$ is 15. If $a = 12$ and $b = 5$, find the measure(s) of $\angle C$.
10. The area of $\triangle PQR$ is 9. If $q = 4$ and $r = 9$, find the measure(s) of $\angle P$.
11. Find the area of a regular octagon inscribed in a circle of radius 40 cm.
12. Find the area of a regular 12-sided polygon inscribed in a circle of radius 8 cm.
13. Adjacent sides of a parallelogram have lengths 6 cm and 7 cm, and the measure of the included angle is 30°. Find the area of the parallelogram.
14. Sketch a parallelogram with sides of lengths a and b and with an acute angle θ. Express the area of the parallelogram in terms of a, b, and θ.
15. Suppose a triangle has two sides of lengths 3 cm and 4 cm and an included angle θ. Express the area of the triangle as a function of θ. State the domain and range of the function and sketch its graph.
16. Suppose a triangle has two sides of lengths a and b. If the angle between these sides varies, what is the maximum possible area that the triangle can attain? What can you say about the minimum possible area?

B

17. **a.** Given $\triangle ABC$ with an inscribed circle as shown at the right, show that the radius r of the circle is given by:

$$r = \frac{2(\text{area of } \triangle ABC)}{\text{perimeter of } \triangle ABC}$$



(*Hint*: If I is the center of the inscribed circle, then: area of $\triangle ABI$ + area of $\triangle ACI$ + area of $\triangle BCI$ = area of $\triangle ABC$.)

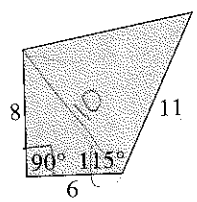
   b. Find the radius of the inscribed circle if $AB = AC = 10$ and $BC = 16$.

Part (b) of Exercise 18 requires the use of a computer or graphing calculator. Give your answer to the nearest tenth.

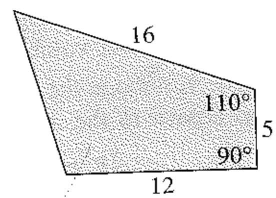
18. In isosceles triangle ABC, $AB = BC = 10$ and $AC = 2x$.
   a. Use Exercise 17 to show that the radius of the inscribed circle is:
   $$r = \frac{x\sqrt{100 - x^2}}{10 + x}$$
   b. Use a computer or graphing calculator to find the value of x that maximizes r. Interpret your answer.

Find the area of each quadrilateral to the nearest square unit.

19.



20.



21. Find the area of a segment of a circle of radius 5 if the measure of the central angle of the segment is 2 radians.
22. Find the area of the segment formed by a chord 24 cm long in a circle of radius 13 cm.

Exercises 23 and 24 require the use of a computer or graphing calculator. Give answers to the nearest tenth.

23. In a circle of radius 10, there is a segment with area 95. Use a computer or graphing calculator to find the measure of the central angle of the segment.
24. For the cylindrical oil tank described in Example 3 on pages 340 and 341, use a computer or graphing calculator to determine where on a measuring rod marks should be put to show that the tank is $\frac{1}{3}$ full and $\frac{2}{3}$ full.

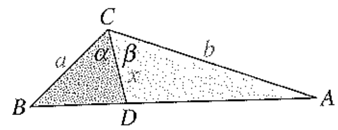
25. ***Visual Thinking*** In Example 3 on pages 340 and 341, suppose the cylindrical oil tank sits upright on one of its circular ends (with the opening for the measuring rod at the other end). Describe how the problem of measuring the amount of oil in the tank changes.

26. ***Research*** Find out and report on what the typical shape of a car's fuel tank is and how the amount of fuel in the tank is measured and indicated on the car's fuel gauge. Is the measuring instrument designed to give a truly accurate measurement of fuel? If not, why not?

Graph the region satisfying both inequalities and find its area.

27. $x^2 + y^2 \leq 36$, $y \geq 3$
28. $x^2 + y^2 \leq 9$, $x \geq 1$
29. $x^2 + y^2 \leq 9$, $x^2 + y^2 - 10x + 9 \leq 0$
30. $x^2 + y^2 - 8y \leq 0$, $x^2 + y^2 \leq 16$

31. In the diagram at the right, $\alpha = 60°$ and $\beta = 60°$.



a. Express the areas of $\triangle BCD$ and $\triangle ACD$ in terms of a, b, and x.

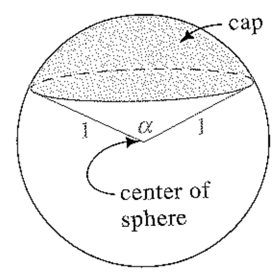
b. Express the area of $\triangle ABC$ in terms of a and b.

c. Show that $x = \dfrac{ab}{a + b}$.

d. Does part (c) agree with what you know from geometry if $a = b$? Explain.

Exercises 32–36 require the use of a computer or graphing calculator. Give answers to the nearest hundredth.

32. **Engineering** Suppose an oil tank is a sphere with radius 1 unit, as shown at the right.



a. Find the exact volume of a sphere with radius 1.

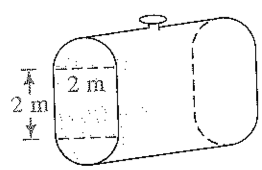
b. If the volume of the spherical cap formed by an angle α (as shown in the diagram) is given by

$$\frac{2\pi}{3} - \pi \cos \frac{\alpha}{2} + \frac{\pi}{3} \cos^3 \frac{\alpha}{2},$$

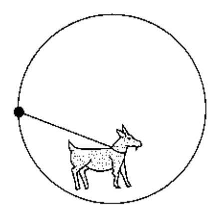
use a computer or graphing calculator to find the value of α for which the spherical tank is $\frac{3}{4}$ full.

c. Where on a measuring rod should marks be placed to show that the tank is $\frac{3}{4}$ full, $\frac{1}{2}$ full, and $\frac{1}{4}$ full?

33. **Engineering** A fuel tank has a cross section whose shape is a 2 m × 2 m square capped at the top and bottom by semicircles, as shown at the left below. Use a computer or graphing calculator to determine how to mark a measuring rod to show that the tank is only 10% full.



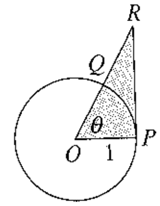
Ex. 33



Ex. 34

34. A goat is tethered to a stake at the edge of a circular field with radius 1 unit, as shown at the right above. Use a computer or graphing calculator to determine how long the rope should be so that the goat can graze over half the field.

35. $\overline{RP}$ is a tangent to a circle with radius 1, as shown at the right. If the area of the region shaded blue equals the area of the region shaded red, use a computer or graphing calculator to find θ in radians.

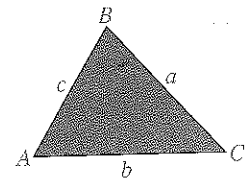


36. Use the diagram for Exercise 35 and a computer or graphing calculator to find the value of θ for which $\overline{QR}$ and arc QP have the same length.

Exs. 35, 36

9-3 The Law of Sines

Objective *To use the law of sines to find unknown parts of a triangle.*



When there are several methods for solving a problem, a comparison of the solutions can lead to new and useful results. From Section 9-2, we know three ways to find the area K of $\triangle ABC$, depending on which pair of sides is known.

$$K = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

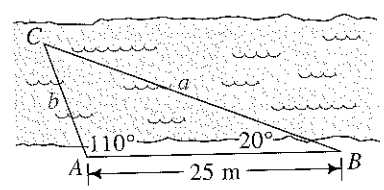
If each of these expressions is divided by $\frac{1}{2}abc$, we obtain the *law of sines.*

The Law of Sines

In $\triangle ABC$, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

If we know two angles and a side of a triangle, then we can use the law of sines to find the other sides, as shown in the following example.

Example 1 A civil engineer wants to determine the distances from points A and B to an inaccessible point C, as shown. From direct measurement the engineer knows that $AB = 25$ m, $\angle A = 110°$, and $\angle B = 20°$. Find AC and BC.



Solution First find the measure of $\angle C$:

$$\angle C = 180° - (110° + 20°) = 50°$$

Then, by the law of sines:

$$\frac{\sin 110°}{a} = \frac{\sin 20°}{b} = \frac{\sin 50°}{25}$$

Therefore:

$$a = \frac{25 \sin 110°}{\sin 50°} \approx 30.7 \quad \text{and} \quad b = \frac{25 \sin 20°}{\sin 50°} \approx 11.2$$

Thus, $BC \approx 30.7$ m and $AC \approx 11.2$ m.

In the activity at the top of the next page, you are given the lengths of two sides of a triangle and the measure of a nonincluded angle. You are then asked to investigate whether it is possible to construct the triangle and, if so, whether the triangle is unique.

Activity 1

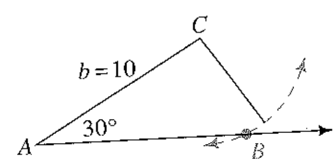
For this activity, use a ruler, compass, and protractor. Draw $\angle A$ with measure $30°$. Along one ray of $\angle A$, locate point C 10 cm from point A. For each of the following compass settings, draw a large arc. Then tell whether the arc crosses the other ray of $\angle A$ and, if so, in how many points.

a. Compass at C and opened to 4 cm
b. Compass at C and opened to 5 cm
c. Compass at C and opened to 6 cm

Activity 1 shows that when you are given the lengths of two sides of a triangle and the measure of a nonincluded angle (SSA), it may be possible to construct no triangle, one triangle, or two triangles. For this reason the SSA situation is called the *ambiguous case*.

Activity 2

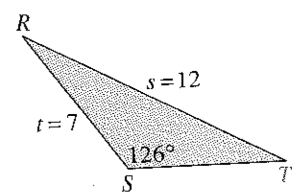
Show that your answers to Activity 1 agree with what the law of sines would give in each of the following SSA situations.



a. If $\angle A = 30°$, $b = 10$, and $a = 4$, find $\angle B$.
b. If $\angle A = 30°$, $b = 10$, and $a = 5$, find $\angle B$.
c. If $\angle A = 30°$, $b = 10$, and $a = 6$, find $\angle B$.

Example 2 In $\triangle RST$, $\angle S = 126°$, $s = 12$, and $t = 7$. Determine whether $\angle T$ exists. If so, find all possible measures of $\angle T$.

Solution Make a sketch, as shown. By the law of sines:



$$\frac{\sin T}{7} = \frac{\sin 126°}{12}$$

$$\sin T = \frac{7 \sin 126°}{12} \approx 0.4719$$

$$\angle T \approx 28.2° \quad \text{or} \quad \angle T \approx 151.8°$$

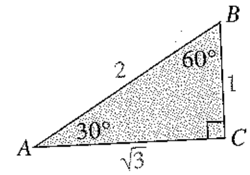
It seems that $\angle T$ exists and that there are two possible measures of $\angle T$. A triangle cannot have two obtuse angles, however, so we must reject $\angle T \approx 151.8°$. Thus, $\angle T \approx 28.2°$.

CLASS EXERCISES

1. ***Reading*** Rephrase the law of sines in words rather than symbols:

In any triangle, __?__ is constant.

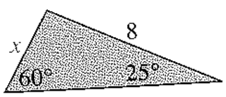
2. The diagram at the right shows a 30°-60°-90° triangle with a hypotenuse of length 2. Find the values of the ratios $\frac{\sin A}{a}$, $\frac{\sin B}{b}$, and $\frac{\sin C}{c}$. Are the ratios equal?

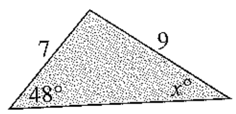


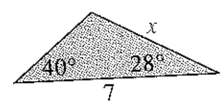
In Exercises 3–5, consider $\triangle ABC$.

3. If $\angle A \geq 90°$, what can you conclude about the measure of $\angle B$? Explain.
4. If $\angle B$ has a greater measure than $\angle C$, what must be true of b and c? Why?
5. If $a > b$, what must be true of $\angle A$ and $\angle B$? Why?

In Exercises 6–8, state an equation that you can use to solve for x.

6. 

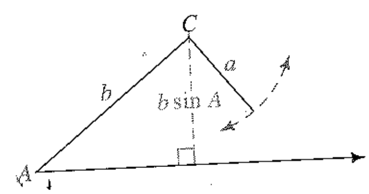
7. 

8. 

9. If $a = 8$ and $b = 6$, draw a diagram to show that $\triangle ABC$ is uniquely determined for each of the given measures of $\angle A$: **a.** 45° **b.** 90° **c.** 120°
10. Use the law of sines to show that $\triangle ABC$ in part (a) of Exercise 9 is unique.
11. Use the law of sines to show that there is no $\triangle XYZ$ with $\angle X = 30°$, $x = 3$, and $y = 8$.

WRITTEN EXERCISES

A 1. The purpose of this exercise is to determine whether $\triangle ABC$ can be constructed when two lengths a and b and the measure of an acute angle A are given. As shown at the right, first $\angle A$ and then the side of length b are constructed; finally a circular arc is drawn with center C and radius a. Given that the distance from C to the side opposite C is $b \sin A$, determine for each of the following conditions whether 0, 1, or 2 triangles can be formed.



a. $a < b \sin A$ **b.** $a = b \sin A$
c. $b \sin A < a < b$ **d.** $a \geq b$

2. Use Exercise 1 or the law of sines to determine whether there are 0, 1, or 2 triangles possible for each of the following sets of measurements.
a. $a = 2,\ b = 4,\ \angle A = 22°$ **b.** $b = 3,\ c = 6,\ \angle B = 30°$
c. $a = 7,\ c = 5,\ \angle A = 68°$ **d.** $b = 4,\ c = 3,\ \angle C = 76°$

In Exercises 3–14, solve each $\triangle ABC$. Give angle measures to the nearest tenth of a degree and lengths in simplest radical form or to three significant digits. Be alert to problems with no solution or two solutions.

3. $\angle A = 45°$, $\angle B = 60°$, $a = 14$
4. $\angle B = 30°$, $\angle C = 45°$, $b = 9$
5. $\angle B = 30°$, $\angle A = 135°$, $b = 4$
6. $\angle A = 60°$, $\angle B = 75°$, $c = 10$

7. $\angle C = 25°$, $b = 3$, $c = 2$

8. $\angle B = 36°$, $a = 10$, $b = 8$

9. $\angle A = 76°$, $a = 12$, $b = 4$

10. $\angle B = 130°$, $b = 15$, $c = 11$

11. $\angle C = 88°$, $b = 7$, $c = 7$

12. $\angle A = 95°$, $a = 13$, $c = 10$

13. $\angle B = 40°$, $a = 12$, $b = 6$

14. $\angle C = 112°$, $c = 5$, $a = 7$

15. In $\triangle RST$, $\angle R = 140°$ and $s = \frac{3}{4}r$. Find the measures of $\angle S$ and $\angle T$.

16. In $\triangle DEF$, $\angle F = 120°$ and $f = \frac{4}{3}e$. Find the measures of $\angle D$ and $\angle E$.

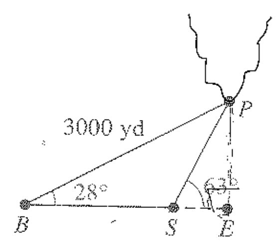
17. A fire tower at point A is 30 km north of a fire tower at point B. A fire at point F is observed from both towers. If $\angle FAB = 54°$ and $\angle ABF = 31°$, find AF.

18. From lighthouses P and Q, 16 km apart, a disabled ship S is sighted. If $\angle SPQ = 44°$ and $\angle SQP = 66°$, find the distance from S to the nearer lighthouse.

19. In $\triangle ABC$, $\tan A = \frac{3}{4}$, $\tan B = 1$, and $a = 10$. Find b in simplest radical form.

20. In $\triangle ABC$, $\cos A = \frac{1}{2}$, $\cos B = -\frac{1}{4}$, and $a = 6$. Find b in simplest radical form.

B 21. **Navigation** A ship passes by buoy B which is known to be 3000 yd from peninsula P. The ship is steaming east along line BE and $\angle PBE$ is measured as 28°. After 10 min, the ship is at S and $\angle PSE$ is measured as 63°.



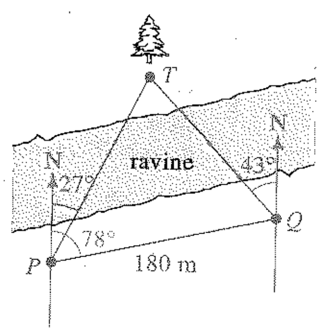
a. How far from the peninsula is the ship when it is at S?

b. If the ship continues east, what is the closest it will get to the peninsula?

c. How fast (in yd/min) is the ship traveling?

d. Ship speeds are often given in knots, where 1 knot = 1 nautical mile per hour ≈ 6080 feet per hour. Convert your answer in part (c) to knots.

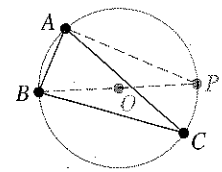
22. **Surveying** From points P and Q, 180 m apart, a tree at T is sighted on the opposite side of a deep ravine. From point P, a compass indicates that the angle between the north-south line and line of sight $\overline{PT}$ is 27° and that the angle between the north-south line and $\overline{PQ}$ is 78°. From point Q, the angle between the north-south line and $\overline{QT}$ is 43°.



a. How far from P is the tree?

b. How far from P is the point on $\overline{PQ}$ that is closest to the tree?

23. The purpose of this exercise is to show that in $\triangle ABC$ the three equal ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, and $\frac{c}{\sin C}$ are each equal to the diameter of the circumscribed circle of $\triangle ABC$.



a. Let the circle with center O be the circumscribed circle and let $\overline{BP}$ be the diameter through B. Show that $\angle P$ and $\angle C$ have the same measure.

b. Show that $\frac{AB}{BP} = \sin P$.

c. Use parts (a) and (b) to show that

$$\text{diameter} = \frac{c}{\sin C} = \frac{b}{\sin B} = \frac{a}{\sin A}.$$

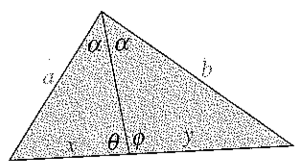
24. A triangle with angles 50°, 60°, and 70° has all three vertices on a circle of radius 8 cm. Find the lengths of the three sides. (*Hint*: See Exercise 23.)

25. a. Prove that the area of $\triangle ABC$ is given by $K = \frac{1}{2}\left(\frac{\sin B \sin C}{\sin A}\right)a^2$.

b. State two other formulas for K, one involving b and one involving c.

26. Use the results of Exercises 23 and 25 to show that the ratio of the area of $\triangle ABC$ to the area of its circumscribed circle is $\frac{2}{\pi} \sin A \sin B \sin C$.

27. The purpose of this exercise is to use the law of sines to prove that an angle bisector in a triangle divides the opposite side in the ratio of the two adjacent sides: that is, $\frac{x}{y} = \frac{a}{b}$ in the diagram at the right.



a. Show that $\sin \theta = \sin \phi$. (ϕ is the Greek letter *phi*.)

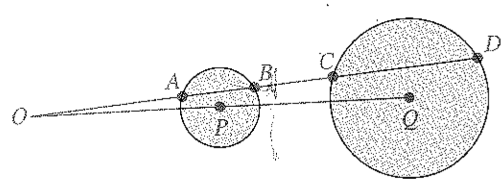
b. Show that $\frac{a}{x} = \frac{\sin \theta}{\sin \alpha}$ and $\frac{b}{y} = \frac{\sin \phi}{\sin \alpha}$.

c. Prove the stated theorem.

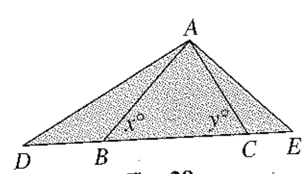
C 28. In the diagram at the left below, the two circles with centers P and Q have radii 1 and 2, respectively. P is the midpoint of $\overline{OQ}$. An arbitrary line through O intersects the circles at A, B, C, and D.

a. Use the law of sines to prove that A is the midpoint of $\overline{OC}$.

b. What special property do you think B has?



Ex. 28



Ex. 29

29. In the diagram at the right above, $\angle DAC = \angle BAE = 90°$. Prove that $DE = BC \tan x \tan y$.

9-4 The Law of Cosines

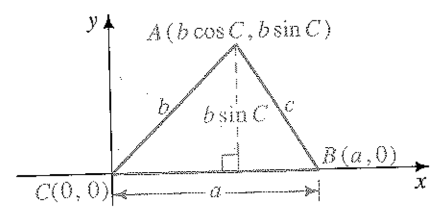
Objective *To use the law of cosines to find unknown parts of a triangle.*

In Section 9-2 we mentioned that by the SAS condition for congruence, a triangle is uniquely determined if the lengths of two sides and the measure of the included angle are known. By the side-side-side (SSS) condition for congruence, a triangle is also uniquely determined if the lengths of three sides are known. The *law of cosines* can be used to solve a triangle in either of these two cases.

The Law of Cosines

In $\triangle ABC$, $c^2 = a^2 + b^2 - 2ab \cos C$.

To derive this law, we introduce a coordinate system by placing the x-axis along $\overline{BC}$ of $\triangle ABC$ and the y-axis at C so that $C = (0, 0)$, $B = (a, 0)$, and $A = (b \cos C, b \sin C)$, as shown. Using the distance formula, we have:



$$\begin{aligned} c^2 = (AB)^2 &= (b \cos C - a)^2 + (b \sin C - 0)^2 \\ &= b^2 \cos^2 C - 2ab \cos C + a^2 + b^2 \sin^2 C \\ &= b^2(\cos^2 C + \sin^2 C) - 2ab \cos C + a^2 \\ &= b^2 \cdot 1 - 2ab \cos C + a^2 \\ &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

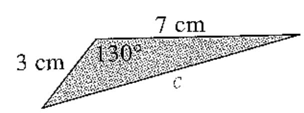
To help you remember the law of cosines, notice that it has the basic pattern:

$$\begin{pmatrix}\text{side}\\\text{opposite}\\\text{angle}\end{pmatrix}^2 = \begin{pmatrix}\text{side}\\\text{adjacent}\\\text{to angle}\end{pmatrix}^2 + \begin{pmatrix}\text{other side}\\\text{adjacent}\\\text{to angle}\end{pmatrix}^2 - 2\begin{pmatrix}\text{one}\\\text{adjacent}\\\text{side}\end{pmatrix}\begin{pmatrix}\text{other}\\\text{adjacent}\\\text{side}\end{pmatrix}\cos(\text{angle})$$

Note that when $\angle C = 90°$ the law of cosines reduces to $a^2 + b^2 = c^2$. Therefore the law of cosines includes the Pythagorean theorem as a special case and is, consequently, more flexible and useful than the Pythagorean theorem. When $\angle C$ is acute, c^2 is less than $a^2 + b^2$ by the amount $2ab \cos C$; when $\angle C$ is obtuse, $\cos C$ is negative and so c^2 is greater than $a^2 + b^2$.

Example 1 Suppose that two sides of a triangle have lengths 3 cm and 7 cm and that the angle between them measures 130°. Find the length of the third side.

Solution Make a sketch, as shown.

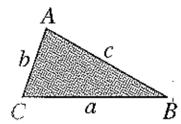


$$\begin{aligned} c^2 &= 3^2 + 7^2 - 2 \cdot 3 \cdot 7 \cdot \cos 130° \\ &\approx 85.0 \end{aligned}$$

Thus, the length of the third side is about $\sqrt{85.0}$ cm, or about 9.22 cm.

If we solve the law of cosines for cos C, we obtain:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

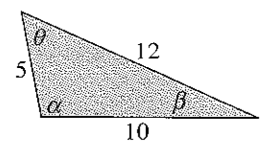


In this form, the law of cosines can be used to find the measures of the angles of a triangle when the lengths of the three sides are known. The basic pattern for this form of the law of cosines is:

$$\cos(\text{angle}) = \frac{(\text{adjacent})^2 + (\text{adjacent})^2 - (\text{opposite})^2}{2 \cdot (\text{adjacent}) \cdot (\text{adjacent})}$$

Example 2 The lengths of the sides of a triangle are 5, 10, and 12. Solve the triangle.

Solution Make a sketch of the triangle, as shown.



1. $\cos \alpha = \dfrac{5^2 + 10^2 - 12^2}{2 \cdot 5 \cdot 10} = -0.19$

   $\alpha \approx 101.0°$

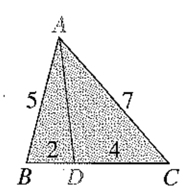
2. $\cos \beta = \dfrac{12^2 + 10^2 - 5^2}{2 \cdot 12 \cdot 10} = 0.9125$

   $\beta \approx 24.1°$

3. $\theta \approx 180° - (101.0° + 24.1°) = 54.9°$

Using the law of cosines, we can easily identify acute and obtuse angles. For instance, in Example 2, since cos α is negative and cos β is positive, we know that α is an obtuse angle and β is an acute angle. The law of sines does not distinguish between acute and obtuse angles, however, because both types of angle have positive sine values.

Example 3 In the diagram at the right, $AB = 5$, $BD = 2$, $DC = 4$, and $CA = 7$. Find AD.



Solution First we apply the law of cosines to $\triangle ABC$:

$$\cos B = \frac{5^2 + 6^2 - 7^2}{2 \cdot 5 \cdot 6} = 0.2$$

Then we apply the law of cosines to $\triangle ABD$:

$$(AD)^2 = 5^2 + 2^2 - 2 \cdot 5 \cdot 2 \cdot 0.2 = 25$$

Thus, $AD = \sqrt{25} = 5$.

In this section and the preceding section, we have seen various applications of the laws of sines and cosines. The situations in which these laws can be used are summarized at the top of the next page. The summary assumes that once the measures of two angles of a triangle are known, the measure of the third angle is found using the geometric fact that the sum of the measures of the angles is 180°.

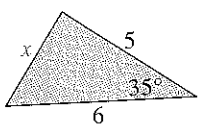
Given:	Use:	To find:
SAS	law of cosines	the third side and then one of the remaining angles.
SSS	law of cosines	any two angles.
ASA or AAS	law of sines	the remaining sides.
SSA	law of sines	an angle opposite a given side and then the third side. (Note that 0, 1, or 2 triangles are possible.)

When you use the law of sines, remember that every acute angle and its supplement have the same sine value. Class Exercises 4–7 show how you can tell which angle is correct for a given triangle.

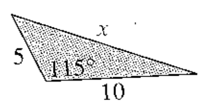
CLASS EXERCISES

In Exercises 1–3, state an equation that you can use to solve for x.

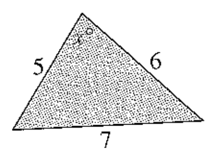
1.



2.



3.



In Exercises 4–7, consider $\triangle XYZ$, where $x = 4$, $y = 8$, and $\angle Z = 50°$.

4. Use the law of cosines to find z to the nearest hundredth.

5. Use the law of sines to find the measure of $\angle Y$ to the nearest tenth of a degree. Then find the measure of $\angle X$.

6. Since $x < z < y$, what can you say about the measures of $\angle X$, $\angle Y$, and $\angle Z$?

7. Do your answers to Exercises 5 and 6 agree? If not, find your error.

WRITTEN EXERCISES

Solve each triangle. Give lengths to three significant digits and angle measures to the nearest tenth of a degree.

A

1. $a = 8$, $b = 5$, $\angle C = 60°$

2. $t = 16$, $s = 14$, $\angle R = 120°$

3. $x = 9$, $y = 40$, $z = 41$

4. $a = 6$, $b = 10$, $c = 7$

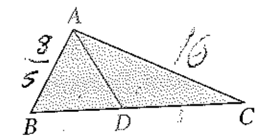
5. $p = 3$, $q = 8$, $\angle R = 50°$

6. $d = 5$, $e = 9$, $\angle F = 115°$

7. $a = 8$, $b = 7$, $c = 13$

8. $x = 10$, $y = 11$, $z = 12$

In Exercises 9 and 10, use the method of Example 3, page 351, to find AD in the diagram at the right.



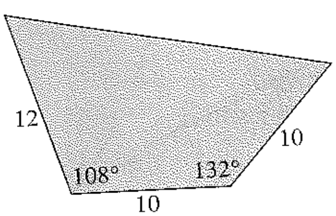
9. $AB = 8$, $BD = 7$, $DC = 5$, $AC = 10$
10. $AB = 5$, $BD = 5$, $DC = 3$, $AC = 7$

In Exercises 11 and 12, find the length of the median from A in the given $\triangle ABC$. Give your answers in simplest radical form.

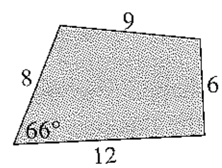
11. $a = 8$, $b = 4$, $c = 6$
12. $a = 12$, $b = 13$, $c = 5$
13. A parallelogram has a $70°$ angle and sides 6 cm and 10 cm long. How long are its diagonals?
14. An isosceles trapezoid has a height of 4 cm and bases 3 cm and 7 cm long. How long are its diagonals?

Find the area of each quadrilateral to the nearest square unit.

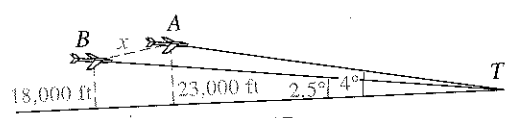
B 15.



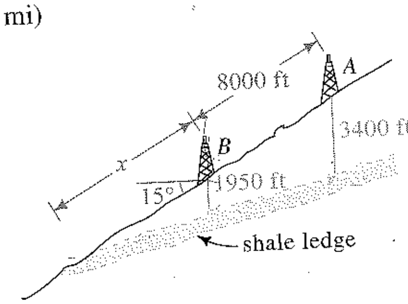
16.



17. **Aviation** Two airplanes, at points A and B in the diagram at the left below, have elevations of 23,000 ft and 18,000 ft, respectively. Both are flying east toward an airport control tower at T. From T, the angle of elevation of the airplane at A is $4°$, and the angle of elevation of the airplane at B is $2.5°$. How far apart (in mi) are the airplanes? (5280 ft = 1 mi)



Ex. 17



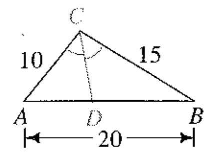
Ex. 18

18. **Geology** In the diagram at the right above, an oil well at A was drilled 3400 ft before it hit a ledge of shale. The same ledge was only 1950 ft deep when drilled from a well at B, which is 8000 ft directly downhill from A. The hill is inclined at $15°$ to the horizontal.
 a. If you assume that the ledge lies in a plane, how far down the hill from B would you expect the shale ledge to emerge?
 b. What is the angle between the ledge and the hill?

19. ***Visual Thinking*** Give a geometric interpretation of the law of cosines for the "triangle" with two sides a and b and "included angle" $C = 0°$.

20. ***Visual Thinking*** Give a geometric interpretation of the law of cosines for the "triangle" with two sides a and b and "included angle" $C = 180°$.

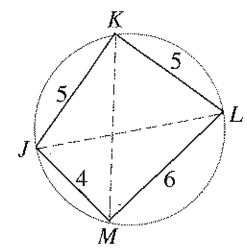
21. In $\triangle ABC$, $AB = 20$, $BC = 15$, $AC = 10$, and $\overline{CD}$ bisects $\angle ACB$, as shown at the right.



   a. Use the results of Exercise 27, page 349, to find AD and DB.

   b. Use the method of Example 3, page 351, to find CD.

22. Quadrilateral $JKLM$ is inscribed in a circle as shown. Find the length of each diagonal. (*Hint:* Opposite angles of an inscribed quadrilateral are supplementary. Express $\cos L$ in terms of $\cos J$ in order to find KM.)



Ex. 22

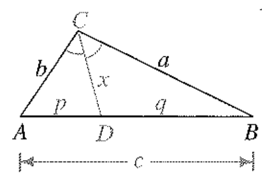
C 23. **a.** Prove that in $\triangle ABC$ the length x of the median from C is given by:

$$x = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

   b. What happens to this formula when $\angle C = 90°$?

   c. State the theorem from geometry that part (b) justifies.

24. In the diagram at the right, $\overline{CD}$ bisects $\angle ACB$.



   a. Use the method of part (a) of Exercise 21 to show:

$$p = \frac{bc}{a + b} \text{ and } q = \frac{ac}{a + b}$$

   b. Prove that $x^2 = ab - pq$.

   c. Use part (b) to find CD in Exercise 21.

25. A parallelogram has two adjacent sides of lengths a and b and diagonals of lengths x and y. Show that $x^2 + y^2 = 2a^2 + 2b^2$.

26. In this exercise, you are to derive *Hero's formula* for the area K of $\triangle ABC$:

$$K = \sqrt{s(s - a)(s - b)(s - c)} \text{ where } s = \frac{a + b + c}{2}$$

   a. In the partial derivation of Hero's formula shown below, use the law of cosines to show that the second expression in blue is equal to the first.

$$\begin{aligned}
\sin^2 C &= 1 - \cos^2 C \\
&= (1 + \cos C)(1 - \cos C) \\
&= \frac{(a + b)^2 - c^2}{2ab} \cdot \frac{c^2 - (a - b)^2}{2ab} \\
&= \frac{(a + b + c)(a + b - c)(c + a - b)(c - a + b)}{4a^2b^2} \\
&= \frac{(2s)(2s - 2c)(2s - 2b)(2s - 2a)}{4a^2b^2}
\end{aligned}$$

   b. Use the formula $K = \frac{1}{2}ab \sin C$ to complete the derivation in part (a).

//// COMPUTER EXERCISE

Write a program that accepts the input of the measures of any three parts of a triangle and then prints the measures of the remaining three parts. You might want to work in a group of five students and write a structured program that begins with a main routine like the one below (written in BASIC).

```
10 PRINT ''INDICATE THE PROBLEM TYPE.''
20 INPUT ''(ENTER SSS, SAS, ASA, AAS, OR SSA): ''; T$
30 IF T$ = ''SSS'' THEN GOSUB 100
40 IF T$ = ''SAS'' THEN GOSUB 200
50 IF T$ = ''ASA'' THEN GOSUB 300
60 IF T$ = ''AAS'' THEN GOSUB 400
70 IF T$ = ''SSA'' THEN GOSUB 500
80 END
```

Each member of the group can write one of the subroutines called by the main routine. Since many programming languages use radians instead of degrees, you may need to convert between the two types of angle measurement. Also, since many programming languages do not have inverse sine and inverse cosine as built-in functions but do have inverse tangent as a built-in function, you may need to use the following relationships, which are valid for $|x| < 1$:

$$\text{Sin}^{-1} x = \text{Tan}^{-1} \frac{x}{\sqrt{1 - x^2}} \quad \text{and} \quad \text{Cos}^{-1} x = \frac{\pi}{2} - \text{Tan}^{-1} \frac{x}{\sqrt{1 - x^2}}$$

MIXED TRIGONOMETRY EXERCISES

Where appropriate, give angle measures to the nearest tenth of a degree and lengths of sides in simplest radical form or to three significant digits.

A

1. Find the area of $\triangle PQR$ if $q = 6$, $r = 7$, and $\angle P = 50°$. Also find p.
2. Find the area of $\triangle PQR$ if $p = 7$, $q = 10$, and $\angle R = 130°$. Also find r.
3. Find the measure of the largest angle in a triangle with sides having lengths $3\sqrt{6}$, $6\sqrt{3}$, and $9\sqrt{2}$.
4. Find the measure of the smallest angle in a triangle with sides having lengths 7, 12, and 13.
5. In $\triangle RST$, $\angle R = 75°$, $\angle S = 45°$, and $t = 3$. Find r and s.
6. In $\triangle RST$, $\angle S = 100°$, $\angle T = 30°$, and $r = 8$. Find s and t.
7. Three measurements in $\triangle ABC$ are given as $\angle A = 60°$, $a = 4$, and $b = 5$. Show that at least one of the measurements is incorrect.
8. A regular polygon with 180 sides is inscribed in a circle with radius 1. Find its area. Compare your answer with π.
9. In $\triangle ABC$, $\cos A = -0.6$. Find $\sin A$ and $\tan A$.
10. In $\triangle ABC$, $a = 17$, $b = 10$, and $c = 21$. Find $\cos A$ and $\sin A$.

11. **Civil Engineering** A wheelchair ramp must rise 30 in. to meet the front door of a public library. If the ramp's angle of elevation is not to exceed 8°, what is the minimum horizontal length of the ramp?

12. **Navigation** A submarine dives at an angle of 16° with the horizontal. If it takes 4 min to dive from the surface to a depth of 300 ft, how fast does it move along its sloping path downward? Give your answer in feet per minute. Then convert it to nautical miles per hour. (*Note*: 1 nautical mile per hour $\approx$ 6080 feet per hour)

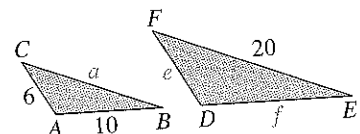
13. In $\triangle XYZ$, $\angle X = 21.1°$, $x = 6$, and $y = 9$. Find the measure(s) of $\angle Y$.

14. In parallelogram $ABCD$, $\angle A = 60°$, $AB = 5$, and $AD = 8$.
   a. Find the area of $ABCD$. **b.** Find the lengths of both diagonals.

15. A triangle has area 21 cm^2 and two of its sides are 9 cm and 14 cm long. Find the possible measures of the angle formed by these sides.

16. In $\triangle DEF$, $\angle D = 36°$, $\angle E = 64°$, and $f = 8$. Find d and e.

17. In the diagram at the right, $\triangle ABC$ is similar to $\triangle DEF$ and $\angle A = 120°$.
   a. Find the lengths a, e, and f.
   b. Find the ratio of the areas of the triangles.

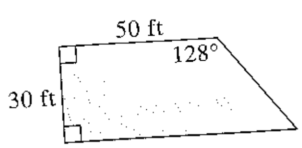


18. The diagonals of a parallelogram have lengths 8 and 14 and they meet at a 60° angle. Find the area and the perimeter of the parallelogram.

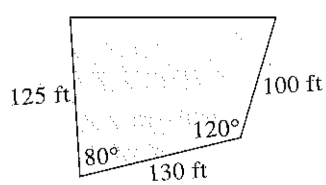
19. An obtuse triangle with area 12 has two sides of lengths 4 and 10. Find the length of the third side. (There are two answers.)

20. The perimeter of a regular decagon (10 sides) is 240. Find its area.

21. If fencing costs $2.50 per foot, how much will it cost to buy fencing to go around the plot of land shown at the left below?



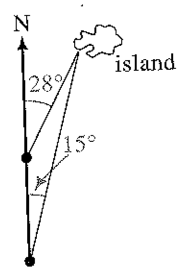
Ex. 21



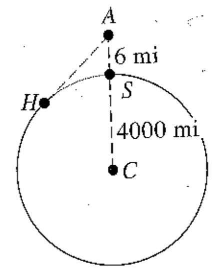
Ex. 22

22. In the township of Madison, rural undeveloped land is taxed at a rate of $115 per acre. Find the tax on the plot of land shown at the right above. (*Note*: 1 acre = 43,560 ft^2)

23. **Navigation** A ship is steaming north at 6 knots (6 nautical miles per hour) when the captain sights a small island at an angle of 15° to the east of the ship's course, as shown at the right. After 10 min, the angle is 28°. How far away is the island at this moment?



Ex. 23



Ex. 24

24. **Aviation** An airplane at A is flying at a height of 6 mi above Earth's surface at S, as shown at the far right.
    a. Find the distance to the nearest tenth of a mile from A to the horizon H. (The radius of Earth is about 4000 mi.)
    b. Find the curved distance to the nearest tenth of a mile from S along Earth's surface to H.

25. In $\triangle ABC$, $\tan A = 1$, $\tan B = \frac{3}{4}$, and $b = 18$. Find $\sin A$, $\sin B$, and a.

26. In $\triangle DEF$, $\sec F = -\sqrt{2}$. Find the measure of $\angle F$ and $\tan F$.

27. If $180° < x < 360°$ and $\tan x = -\frac{1}{5}$, find $\sin x$ and $\cos x$.

28. The area of $\triangle PQR$ is 84. If $r = 14$ and $q = 13$, find $\sin P$. Use a trigonometric identity to find two possible values of $\cos P$ and two possible values of p.

In geometry you can prove two triangles congruent by SSS. This means that when the lengths of the three sides of a triangle are given, its shape is completely determined. Exercises 29 and 30 below illustrate this principle. Exercises 31 and 32 illustrate the principle for SAS and ASA.

B 29. In $\triangle ABC$, $a = 5$, $b = 8$, and $c = 7$. (SSS)
    a. Solve $\triangle ABC$.
    b. Find the area of $\triangle ABC$. Use the formula $K = \frac{1}{2}ab \sin C$ or Hero's formula (Exercise 26, page 354).
    c. Find the length of the altitude to $\overline{AC}$.
    d. Find the length of the median from B to $\overline{AC}$ (Exercise 23, page 354).
    e. Find the length of the angle bisector from B to $\overline{AC}$ (Exercise 24, page 354).
    f. Find the radius R of the circumscribed circle (Exercise 23, page 349).
    g. Find the radius r of the inscribed circle (Exercise 17, page 342).

30. Repeat Exercise 29 if $a = 13$, $b = 14$, and $c = 15$. (SSS)

31. Repeat Exercise 29 if $a = 9$, $b = 10$, and $\cos C = -\frac{3}{5}$. (SAS)

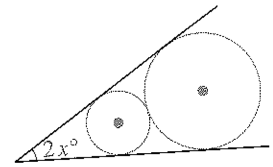
32. Repeat Exercise 29 if $\sin A = \frac{8}{17}$, $c = 21$, and $\sin B = \frac{\sqrt{2}}{2}$. (ASA) Assume that $\angle B$ is obtuse.

33. a. The consecutive sides of a quadrilateral inscribed in a circle have lengths 1, 4, 3, and 2. Find the length of each diagonal, using the fact that opposite angles are supplementary.
    b. Check your answer to part (a) by using Ptolemy's theorem: If $ABCD$ is inscribed in a circle, then $AC \cdot BD = AB \cdot DC + AD \cdot BC$.

34. $\triangle DEF$ is inscribed in a circle of radius 3. If $\angle D = 120°$ and $\angle E = 15°$, find the lengths d, e, and f. (*Hint:* Find the measures of arc DE and arc DF.)

35. Prove that the altitude from A in $\triangle ABC$ has length $h = \dfrac{a}{\cot B + \cot C}$.

C **36.** Two circles are externally tangent. Common tangents to the circles form an angle of measure $2x$. Prove that the ratio of the radii of the circles is $\dfrac{1 - \sin x}{1 + \sin x}$. (*Hint:* Express $\sin x$ in terms of the two radii.)

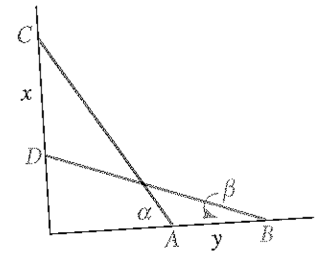


37. Prove that $\cot A + \cot B + \cot C = \dfrac{a^2 + b^2 + c^2}{4(\text{area of } \triangle ABC)}$ in any $\triangle ABC$.

38. Given $\triangle ABC$ with $c^2 = \dfrac{a^3 + b^3 + c^3}{a + b + c}$, find the measure of $\angle C$.

39. As shown at the right, a ladder, $\overline{AC}$, leans against the side of a house at an angle α with the ground. Suppose that the foot of the ladder slides y units from A to B, the top of the ladder slides x units from C to D, and the ladder makes an angle β with the ground. Prove:

$$x = \frac{y(\sin \alpha - \sin \beta)}{\cos \beta - \cos \alpha}$$



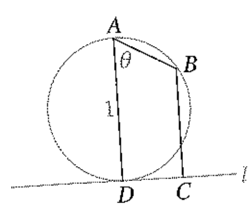
40. A sailor at the seashore watches a ship with a smokestack 30 m above water level as the ship steams out to sea. The sailor's eye level is 4 m above water level. About how far is the ship from shore when the stack disappears from the sailor's view? (The radius of Earth is about 6400 km.)

41. In $\triangle XYZ$, $\overline{YR}$ and $\overline{ZS}$ are altitudes. Prove that $RS = x \cos X$.

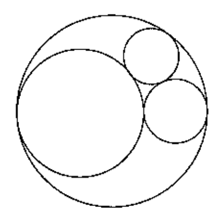
42. a. As shown in the diagram at the left below, $\overline{AD}$ is a diameter of the circle and is tangent to line l at D. If $AD = 1$ and $\overline{BC} \perp l$, show that:

$$AB + BC = 1 + \cos \theta - \cos^2 \theta$$

b. What value of θ makes the sum $AB + BC$ a maximum?



Ex. 42



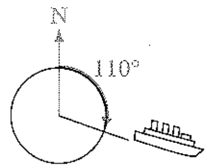
Ex. 43

43. Each circle in the diagram at the right above is tangent to the other three circles. The largest three circles have radii of 1, 2, and 3, respectively. Find the radius of the smallest circle.

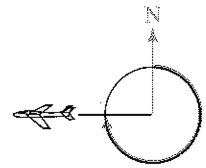
9-5 Applications of Trigonometry to Navigation and Surveying

Objective *To use trigonometry to solve navigation and surveying problems.*

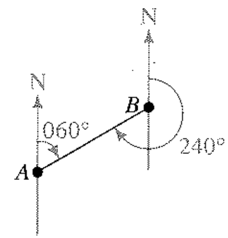
As shown below, the *course* of a ship or plane is the angle, measured clockwise, from the north direction to the direction of the ship or plane.



course of ship = 110°



course of plane = 270°

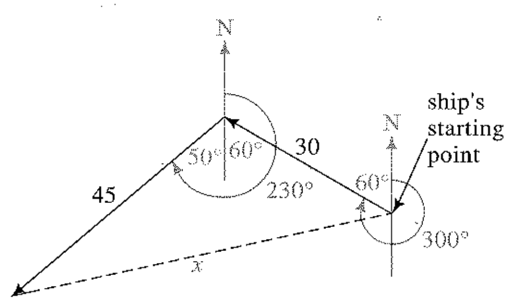


bearing of B from A = 060°
bearing of A from B = 240°

As shown at the right, the *compass bearing* of one location from another is measured in the same way. Note that compass bearings and courses are given with three digits, such as 060° rather than 60°.

Example 1 A ship proceeds on a course of 300° for 2 hours at a speed of 15 knots (1 knot = 1 nautical mile per hour). Then it changes course to 230°, continuing at 15 knots for 3 more hours. At that time, how far is the ship from its starting point?

Solution Make a diagram, as shown below.

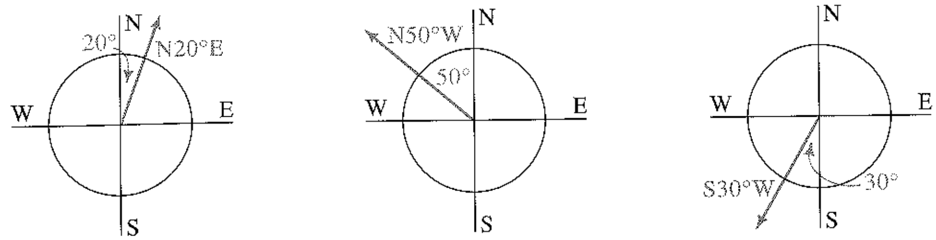


The ship travels first along a path of length $2 \cdot 15 = 30$ nautical miles and then along a path of length $3 \cdot 15 = 45$ nautical miles. The angle between the two paths is 110°. (You can find this angle by drawing north-south lines and using geometry.) To find x, the distance of the ship from its starting point, use the law of cosines:

$$x^2 = 30^2 + 45^2 - 2 \cdot 30 \cdot 45 \cdot \cos 110° \approx 3848$$

Thus, $x \approx \sqrt{3848} \approx 62.0$ nautical miles.

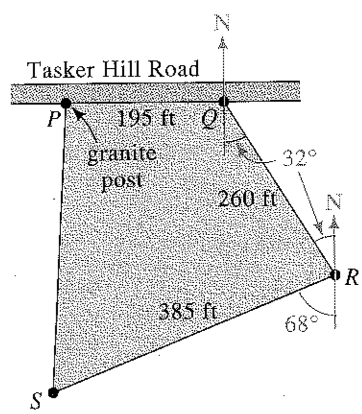
In surveying, a compass reading is usually given as an acute angle from the north-south line toward the east or west. A few examples are shown below.

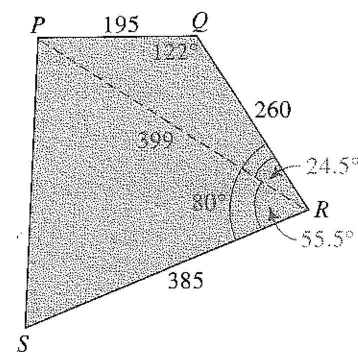


Example 2 Very often a plot of land is taxed according to its area. Sketch the plot of land described. Then find its area.

From a granite post, proceed 195 ft east along Tasker Hill Road, then along a bearing of S32°E for 260 ft, then along a bearing of S68°W for 385 ft, and finally along a line back to the granite post.

Solution We first sketch the plot of land, one side at a time and in the sequence described. (See the diagram at the left below.)





From the bearings given, we deduce that:

$$\angle PQR = 90° + 32° = 122°$$
$$\angle QRS = 180° - (32° + 68°) = 80°$$

To find the area of $PQRS$, we divide the quadrilateral into two triangles by introducing $\overline{PR}$. (See the diagram at the right above.) We can find the area of $\triangle PQR$ directly:

$$\text{Area of } \triangle PQR = \frac{1}{2} \cdot PQ \cdot QR \cdot \sin Q$$
$$= \frac{1}{2} \cdot 195 \cdot 260 \cdot \sin 122° \approx 21{,}500 \text{ ft}^2$$

To find the area of $\triangle PRS$, we must first find PR and $\angle PRS$.

To find PR, we use the law of cosines:

$$\begin{aligned} PR^2 &= 195^2 + 260^2 - 2 \cdot 195 \cdot 260 \cdot \cos 122° \\ &\approx 159{,}000 \end{aligned}$$

Therefore, $PR \approx \sqrt{159{,}000} \approx 399$ ft.

To find $\angle PRS$, we find $\angle PRQ$ by the law of sines:

$$\frac{\sin PRQ}{195} = \frac{\sin 122°}{399}$$

$$\sin PRQ = \frac{195 \sin 122°}{399} \approx 0.4145$$

$$\angle PRQ \approx 24.5°$$

Therefore, $\angle PRS = \angle QRS - \angle PRQ \approx 80° - 24.5° = 55.5°$.

Knowing that $PR \approx 399$ ft and $\angle PRS \approx 55.5°$, we have:

$$\begin{aligned} \text{Area of } \triangle PRS &= \tfrac{1}{2} \cdot PR \cdot RS \cdot \sin PRS \\ &\approx \tfrac{1}{2} \cdot 399 \cdot 385 \cdot \sin 55.5° \approx 63{,}300 \text{ ft}^2 \end{aligned}$$

Thus, we have:

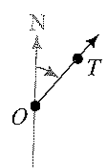
$$\begin{aligned} \text{Area of quadrilateral } PQRS &= \text{area of } \triangle PQR + \text{area of } \triangle PRS \\ &\approx 21{,}500 + 63{,}300 \\ &= 84{,}800 \text{ ft}^2 \end{aligned}$$

CLASS EXERCISES

1. ***Reading*** In the solution of Example 1, justify the fact that the angle opposite the side of length x in the diagram has a measure of 110°.
2. Suppose land is taxed at a rate of \$75 per acre. Determine the approximate tax on the land in Example 2. (1 acre = 43,560 ft^2)

Visual Thinking **In each diagram, a north-south line is given. If $\overrightarrow{OT}$ represents the path of a ship, estimate its course.**

3.



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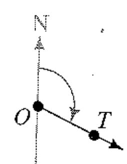
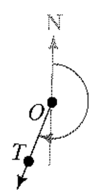
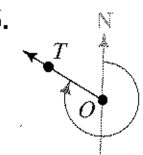


5.



6.



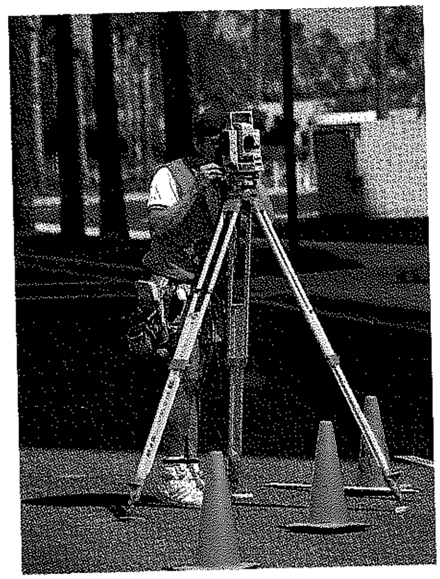
7. ***Visual Thinking*** Suppose point X is directly east of point Y.
   a. Give the bearing of X from Y.
   b. Give the bearing of Y from X.

For Exercises 8–11, make a sketch of each compass reading.

8. N70°E
9. N10°W
10. S15°E
11. S40°W

12. Match each compass direction with a course.

Direction	*Course*
northeast	135°
southeast	225°
northwest	315°
southwest	045°



13. A famous Alfred Hitchcock movie is *North by Northwest*. This direction is midway between north and northwest. What course corresponds to this direction?

14. From one corner of a triangular plot of land, a surveyor determines the directions to the other two corners to be N32°E and S76°E. What is the measure of the angle formed by the edges of the plot of land at the corner where the surveyor is?

WRITTEN EXERCISES

In Exercises 1–4, draw a diagram like those in Class Exercises 3–6 to show the path of a ship proceeding on each given course.

A 1. 070°
2. 150°
3. 340°
4. 225°

5. **Navigation** Ship A sights ship B on a compass bearing of 080°. Make a sketch and give the compass bearing of ship A from ship B.

6. **Navigation** Ship X sights ship Y on a bearing of 308°. What is the bearing of X from Y?

7. **Aviation** An airplane flies on a course of 110° at a speed of 1200 km/h. How far east of its starting point is it after 2 h?

8. A hunter walks east for 1 h and then north for $1\frac{1}{2}$ h. What course should the hunter take to return to his starting point? What assumptions do you make to answer the question?

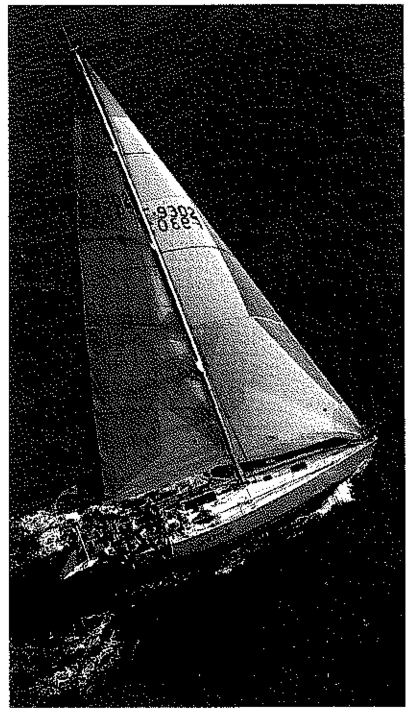
9. Point B is 10 km north of point A, and point C is 10 km from B on a bearing of 060° from B. Find the bearing and distance of C from A.

10. Point S is 4 km west of point R, and point T is 4 km southwest of S. Find the bearing and distance of R from T.

11. **Navigation** Traveling at a speed of 10 knots, a ship proceeds south from its port for $1\frac{1}{2}$ h and then changes course to 130° for $\frac{1}{2}$ h. At this time, how far from port is the ship?

12. **Navigation** A sailboat leaves its dock and proceeds east for 2 mi. It then changes course to 205° until it is due south of its dock. How far south is this?

13. **Navigation** Two ships, A and B, leave port at the same time. Ship A proceeds at 12 knots on a course of 040°, while ship B proceeds at 9 knots on a course of 115°. After 2 h, ship A loses power and radios for help. How far and on what course must ship B travel to reach ship A?



14. **Navigation** A ship leaves port and sails northwest for 1 h and then northeast for 2 h. If it does not change speed, find what course the ship should take to return directly to port. Also find how long this return will take.

In Exercises 15–18, sketch each plot of land described and find its area.

B

15. **Surveying** From an iron post, proceed 500 m northeast to the brook, then 300 m east along the brook to the old mill, then 200 m S15°E to a post on the edge of Wiggin's Road, and finally along Wiggin's Road back to the iron post.

16. **Surveying** From a cement marker, proceed 260 m southwest to the river, then 240 m south along the river to the bridge, then 280 m N40°E to a sign on the edge of Sycamore Lane, and finally along Sycamore Lane back to the cement marker.

17. **Surveying** From the southeast corner of the cemetery on Burnham Road, proceed S78°W for 250 m along the southern boundary of the cemetery until a granite post is reached, then S15°E for 180 m to Allard Road, then N78°E along Allard Road until it intersects Burnham Road, and finally N30°E along Burnham Road back to the starting point.

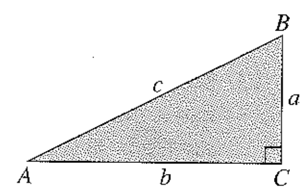
18. **Surveying** From the intersection of Simpson's Road and Mulberry Lane, proceed N32°W for 320 m along Simpson's Road, then S56°W for 280 m to the old oak tree, then S22°E until Mulberry Lane is reached, and finally N68°E along Mulberry Lane back to the starting point.

19. ***Research*** There is a difference between magnetic north, the direction in which a compass needle points, and true north. Research what this difference is and what the phrase "compass variation" means. In what parts of the world is the variation "east" and in what parts is it "west"? Also explain what is meant by the mariner's rhyme:

Error west, compass best.
Error east, compass least.

Chapter Summary

1. If you know the lengths of two sides of a right triangle, or the length of one side and the measure of one acute angle, then you can find the measures of the remaining sides and angles using the trigonometric functions, whose definitions are given below. (The definitions are based on the diagram at the right.)



$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} \qquad \csc A = \frac{1}{\sin A} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{a}$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} \qquad \sec A = \frac{1}{\cos A} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b}$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} \qquad \cot A = \frac{1}{\tan A} = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}$$

2. Knowing the lengths a and b of two sides of a triangle and the measure of the included angle C, you can obtain the area K of the triangle using the formula:

$$K = \frac{1}{2}ab \sin C$$

3. In any $\triangle ABC$, the following relationships hold:

Law of sines: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Law of cosines: $c^2 = a^2 + b^2 - 2ab \cos C$

Law of cosines (alternate form): $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

4. You can use the law of sines and the law of cosines to solve triangles, as summarized on page 352. It is important to remember that the SSA (side-side-nonincluded angle) case may result in 0, 1, or 2 triangles.

5. The application of trigonometry, particularly the laws of sines and cosines, to navigation and surveying is discussed in Section 9-5. In navigation, north represents 0° and the *course* or *compass bearing* of a ship or plane is measured by a clockwise rotation from north. In surveying, angles are given acute measures east or west of a north-south line.

Key vocabulary and ideas

right triangle definitions of the trigonometric functions (p. 331)
solving a triangle (p. 331) area of a triangle (p. 339) law of cosines (p. 350)
angle of elevation (p. 332) law of sines (p. 345) course, bearing (p. 359)

Chapter Test

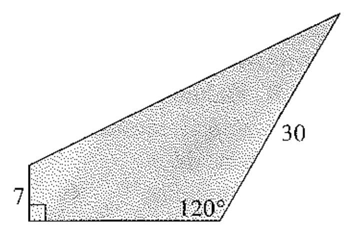
Where appropriate, give angle measures to the nearest tenth of a degree and lengths of sides in simplest radical form or to three significant digits.

1. The sides of an isosceles triangle have lengths 5, 10, and 10. What are the measures of its angles? 9-1

2. At a distance of 100 m, the angle of elevation to the top of a fir tree is 28°. About how tall is the tree?

3. A regular pentagon is inscribed in a circle of radius 4 in. Find the area of the pentagon. 9-2

4. Find the area of the quadrilateral shown at the right.

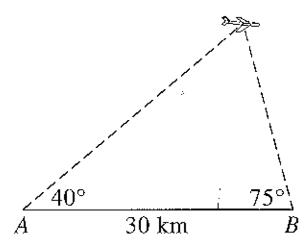


Ex. 4

5. How many different triangles PQR can be constructed using the given information? 9-3

a. $p = 5$, $q = 4$, $\angle Q = 74°$
b. $p = 9$, $q = 8$, $\angle P = 23°$

6. Observers at points A and B, 30 km apart, sight an airplane at angles of elevation of 40° and 75°, respectively, as shown in the diagram. How far is the plane from each observer?



Ex. 6

7. A triangle has sides of length 5, 8, and 10. What is the measure of its largest angle? 9-4

8. Two hikers follow a trail that splits into two forks. Each hiker takes a different fork. The forks diverge at an angle of 67° and both hikers walk at a speed of 3.5 mi/h. How far apart are the hikers after 1 h?

9. ***Writing*** When only three parts of a triangle are known, the law of sines and the law of cosines can be used to find the unknown parts. Discuss the specific circumstances for using each law.

10. After leaving an airport, a plane flies for 1.5 h at a speed of 200 km/h on a course of 200°. Then, on a course of 340°, the plane flies for 2 h at a speed of 250 km/h. At this time, how far from the airport is the plane? 9-5

Project

The People of Mathematics

Why do you think people choose to become mathematicians? What events in their lives steer them in the mathematical direction? How has mathematics changed over the centuries? Is mathematics discovered or created? Maybe you've thought about questions like these before. The answers, though, may actually surprise you. Here's a chance to do a little research in some areas of mathematics that perhaps you haven't yet formally explored.

Mathematics is such an extensive field, it's easy to get lost in all of the available information. You'll probably find it helpful, then, to narrow your study by choosing from the following list of suggested topics. Your goal will be to summarize your findings in a report of at least six pages in length.

Mathematical Essays and Autobiographies

- *Adventures of a Mathematician* by Stanislaw M. Ulam
- *I Want to Be a Mathematician* by Paul R. Halmos
- "Mathematical Creation" by Henri Poincaré
- "The Mathematician" by John von Neumann
- *A Mathematician's Apology* by G. H. Hardy
- "Olga Taussky-Todd: An Autobiographical Essay" by Olga Taussky-Todd

The Development of Numbers

- The mathematics of the Rhind papyrus
- The Mayan numeration system and the representation of zero
- The Babylonian (or Sumerian) numeration system
- The Hindu-Arabic numeration system
- Irrational numbers and the school of Pythagoras
- Calculators of π: Archimedes (Greece), Zǔ Chōngzhī and Zǔ Gěng (China), Aryabhata (India), al-Kashi (Persia), Viète (France), Kanada (Japan), Chudnovsky brothers (United States)

Famous Problems and Paradoxes

- The Chinese remainder theorem
- The Cantor set
- Goldbach's conjecture
- Russell's paradox
- The Möbius strip
- Chaotic motion and the Lorenzian waterwheel

A Few Masters of Discovery

- Benjamin Banneker
- János Bolyai
- Leonhard Euler
- Sophie Germain
- Sofia Kovalevskaia (also known as Sonya Kovalevsky)
- George Polya
- Srinivasa Ramanujan

People and Computational Machines

- The abacus, including the Chinese *suan pan* and the Japanese *soroban*
- Napier's rods
- Charles Babbage's difference engine
- Ada Byron Lovelace and the art of programming
- Alan Turing and the Turing machine
- Early computers: Mark I, ENIAC, EDSAC, and UNIVAC I

Writing Your Report

A report on any mathematical topic should be more than just a list of facts. It should also convey the elements of discovery and excitement. Imagine what it must have felt like to be among the first to experience the development of a new idea or theory. Try to reconstruct each episode. Include worked-out examples and diagrams or photographs. Investigate the social and historical events surrounding your subjects. Also investigate the economic or scientific needs that might have promoted each discovery. Finally, convince your readers that the mathematicians you mention are or were real people. Describe their personalities and traits as well as their accomplishments. Use quotations from their writings whenever possible.

Some Resources

Albers, Donald J., and G. L. Alexanderson, eds. *Mathematical People: Profiles and Interviews.* Boston: Birkhäuser, 1985.

Bell, E. T. *Men of Mathematics.* New York: Simon and Schuster, 1937.

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A Mayan Pyramid