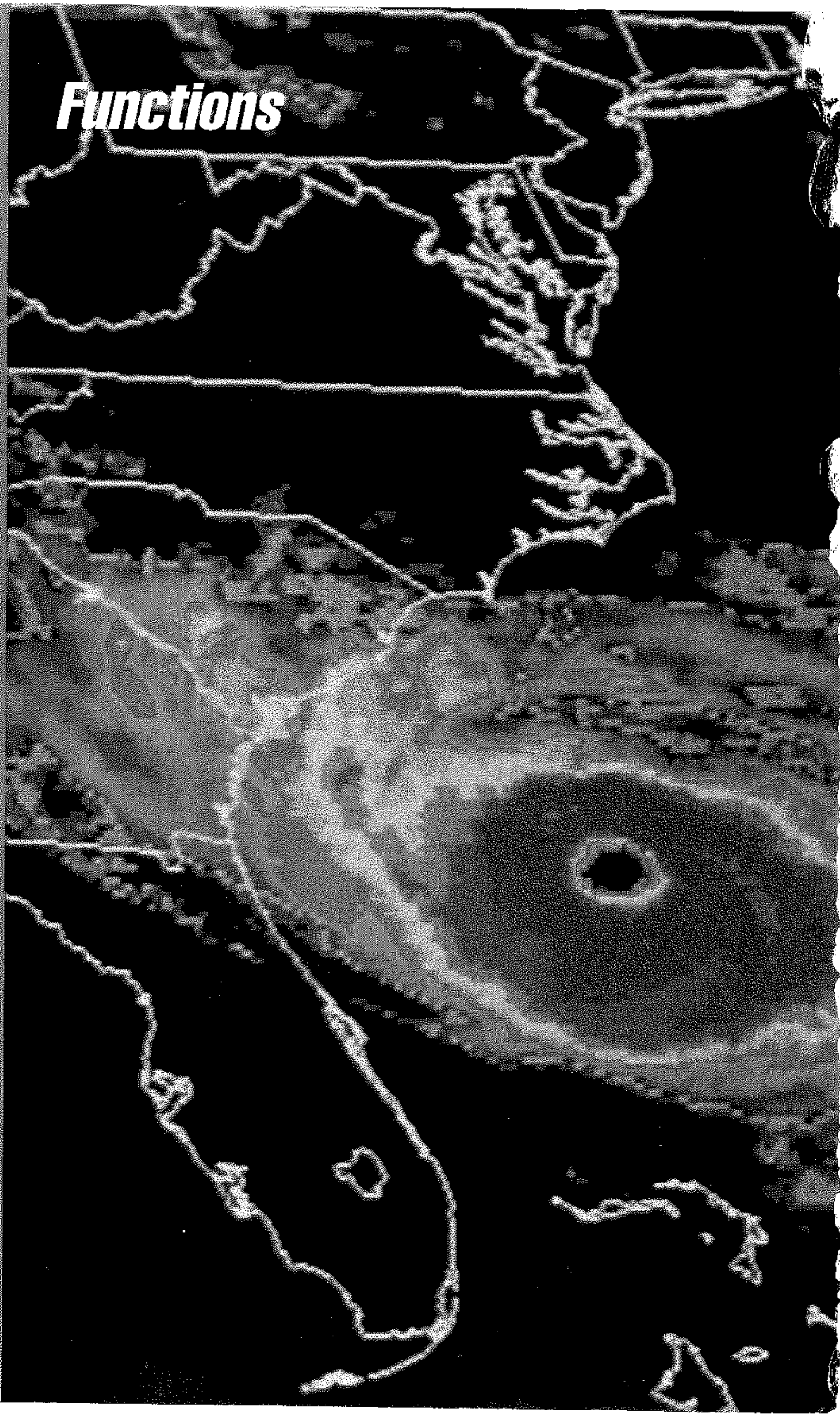
4 Functions



Properties of Functions

4-1 Functions

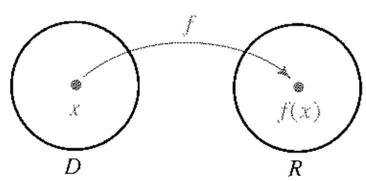
Objective *To identify a function, to determine the domain, range, and zeros of a function, and to graph a function.*

You have already studied linear functions, quadratic functions, and polynomial functions. The purpose of this chapter is to consider functions in general. You will see how functions can be combined to produce new functions and how simple changes in a function's rule will change its graph. These ideas will be used later in the book when you study exponential functions, logarithmic functions, and trigonometric functions.

Definition of a Function

A **function** is a correspondence or rule that assigns to every element in a set D exactly one element in a set R. The set D is called the **domain** of the function, and the set R is called the **range**.

The diagram at the right shows a function f mapping, or pairing, a domain element x to a range element $f(x)$, read "the value of f at x" or "f of x." Although f names the function and $f(x)$ gives its value at x, we sometimes refer to the function $f(x)$, thereby indicating both the function f and the variable x of its domain.



In this book the domain and range of a function are typically sets of numbers, but this need not be the case, as the following illustrations show.

1. Birthday function B maps a person to his or her birthday. For example,

   B(George Washington) = February 22

   and B(Martin Luther King, Jr.) = January 15.

2. Area function A maps a geometric figure to its area. For example, if $PQRS$ is a square with sides of length 5, then $A(PQRS) = 5^2$.

We can treat a function f as a set of ordered pairs (x, y) such that x is an element of the domain of f and y is the corresponding element of the range. This is written formally as $\{(x, y) \mid y = f(x)\}$ or more simply as

$$y = f(x).$$

Although the letters f, x, and y are commonly used in general discussions of functions, other letters can be used. For example, $v = g(u)$ is a function g that assigns a domain element u to a range element v.

◀ Weather maps often involve the visual representation of functions of more than one variable. Each colored band in the satellite photo at the left represents a particular range of wind velocity plotted as a function of latitude and longitude.

A function is frequently given in terms of a rule and a domain. If the domain of a function is not specified, then it is understood to consist of those real numbers for which the function produces real values.

Example 1 Give the domain of each function.

a. $g(x) = \dfrac{1}{x-7}$ **b.** $h(r) = \sqrt{1 - r^2}$

Solution **a.** The domain of g is the set of all real numbers except 7, since $\dfrac{1}{x-7}$ is not defined when the denominator is 0.

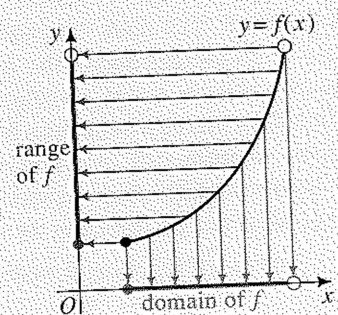
b. $\sqrt{1 - r^2}$ is a real number only if $1 - r^2 \geq 0$. Therefore, the domain of h is $\{r \mid -1 \leq r \leq 1\}$.

The Graph of a Function

The graph of a function $y = f(x)$ consists of all points $(x, f(x))$ in an xy-plane. We can obtain the domain, range, and zeros of a function from its graph, as indicated below.

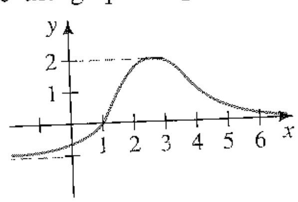
Analyzing the Graph of a Function

1. The domain of $y = f(x)$ is the set of x-coordinates of points on the graph of f. Projecting the graph of f on the x-axis gives a graph of the domain.
2. The range of $y = f(x)$ is the set of y-coordinates of points on the graph of f. Projecting the graph of f on the y-axis gives a graph of the range.
3. The zeros of $y = f(x)$ are the x-intercepts of the graph.

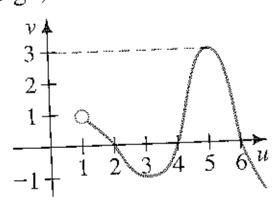


Example 2 Use the graph to give the domain, range, and zeros of each function.

a.



b.



Solution **a.** Domain = {all real numbers}
Range = $\{y \mid -1 < y \leq 2\}$
Zero: 1

b. Domain = $\{u \mid u > 1\}$
Range = $\{v \mid v \leq 3\}$
Zeros: 2, 4, and 6

In applied settings, we often work with equations relating two or more measured quantities. For example, if gasoline is priced at \$1.25 per gallon at a certain gas station, then a pump registers the cost C of g gallons of gas by using the equation

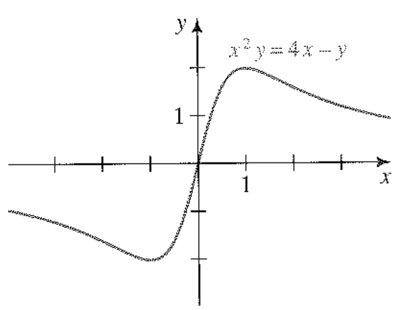
$$C = 1.25g.$$

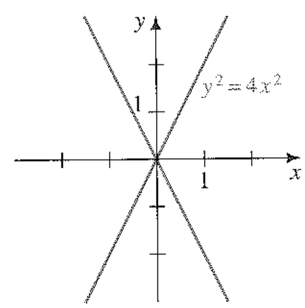
Alternatively, the equation

$$C(g) = 1.25g$$

emphasizes that C is a function of g. Because the cost depends on the number of gallons, C and g are called **dependent** and **independent variables**, respectively.

Functions are a subset of the more general class of correspondences called **relations**. A **relation** is *any* correspondence or rule that pairs the members of two sets (the domain and the range). In general, a relation expressed as an equation in two variables, say x and y, defines y as a function of x provided there is exactly one y-value for each x-value. For example, the relation described by the equation $x^2y = 4x - y$ defines y as a function of x, but the relation described by the equation $y^2 = 4x^2$ does not. One way to determine this is to solve each equation for y, if possible. The first equation becomes $y = \dfrac{4x}{x^2 + 1}$. Because there is only one y-value for each x-value, this relation is a function. The second equation becomes $y = \pm 2x$. Because there are two y-values for each nonzero x-value, this relation is not a function. The graphs of these two equations are shown below.





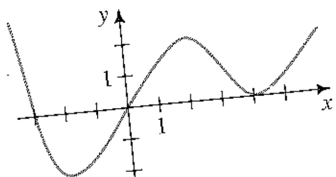
Another way to determine whether an equation in x and y defines y as a function of x is to apply the *vertical-line test* to the graph of the equation.

The Vertical-Line Test

If no vertical line intersects a given graph in more than one point, then the graph is the graph of a function.

CLASS EXERCISES

1. a. What are the domain and range of the cost-of-gas function described on the preceding page?
   b. Sketch the graph of the cost-of-gas function in a gC-plane.
2. Give the domain of each function.
   a. $f(x) = \dfrac{4}{x-2}$ b. $g(t) = \sqrt{t}$ c. $h(s) = \sqrt{s-4}$
3. a. Show that 2 is a zero of the function $f(x) = 3x^2 - 12$.
   b. What is another zero?
   c. Find the range of the function.
4. Give the domain, range, and zeros of the function whose graph is shown at the right.



Ex. 4

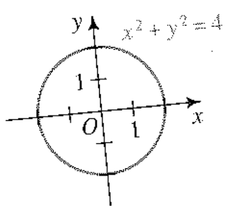
5. Apply the vertical-line test to the graphs of $x^2y = 4x - y$ and $y^2 = 4x^2$ shown on the preceding page.
6. Which of the relations described by the two equations $x^2 - y = 1$ and $y^2 - x = 1$ does *not* define y as a function of x? Explain.

Tell whether the relation described by each equation defines y as a function of x. If it does, sketch the graph.

7. $y = 2x + 5$ 8. $y = |x|$ 9. $x = |y|$ 10. $y^2 = x^2$

Tell whether the graph of each relation is the graph of a function. Explain your answer.

11.



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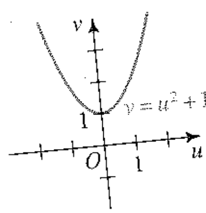

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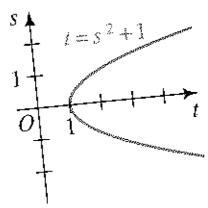
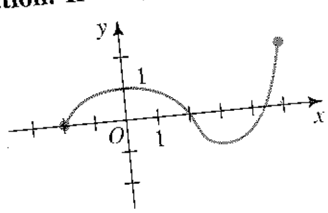


WRITTEN EXERCISES

In Exercises 1–6, tell whether the graph of each relation is the graph of a function. If it is, give the domain and range of the function.

A 1.



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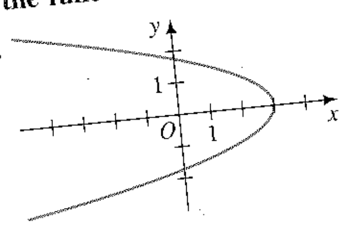

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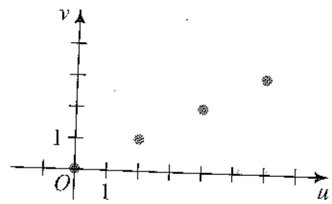

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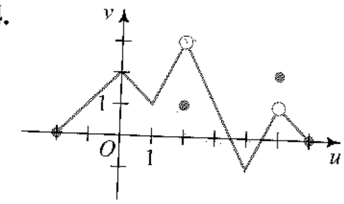

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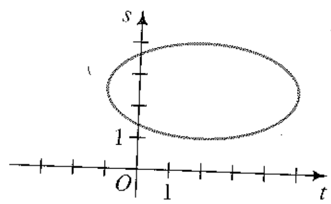

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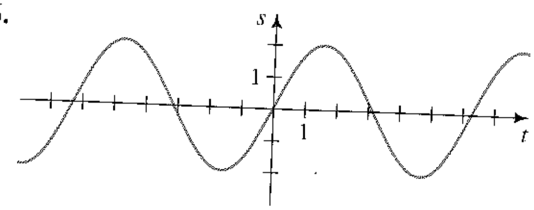


7. Explain why the equation $x^2 + y^2 = 1$ does not define y as a function of x.

8. Explain why the equation $x^3 + y^3 = 1$ defines y as a function of x.

Give the domain of each function.

9. **a.** $f(x) = \frac{1}{x}$ **b.** $g(x) = \frac{1}{x - 9}$ **c.** $h(x) = \frac{3x}{x^2 - 4}$

10. **a.** $f(t) = \frac{1}{t + 3}$ **b.** $g(t) = \frac{t + 2}{t^2 + 5t + 6}$ **c.** $h(t) = \frac{2t^2 - 3}{t^3 - 9t}$

Give the domain, range, and zeros of each function.

11. **a.** $f(x) = |x|$ **b.** $g(x) = |x - 2|$ **c.** $h(x) = |x| - 2$

12. **a.** $f(t) = \sqrt{t}$ **b.** $g(t) = \sqrt{9 - t}$ **c.** $h(t) = \sqrt{9 - t^2}$

In Exercises 13–18, sketch the graph of each function. Use the graph to find the range and zeros of the function.

13. $f(x) = x^2 - 6x + 8$

14. $g(x) = 4 - (x - 3)^2$

15. $f(t) = (t - 2)^3$

16. $g(t) = t^3 + 4t^2 - t - 4$

17. $h(u) = \begin{cases} u^2 & \text{if } -2 \le u < 1 \\ 2 - u & \text{if } 1 \le u < 4 \end{cases}$

18. $g(u) = \begin{cases} u - 1 & \text{if } u < 0 \\ u^2 - 2u - 3 & \text{if } 0 \le u \le 3 \\ 0 & \text{if } u > 3 \end{cases}$

19. **a.** Let V be the function that assigns to each solid its volume. If C is a cylinder with radius 3 and height 4, find $V(C)$.

b. Give the domain and range of V.

20. **a.** A formula from geometry states that $S = (n - 2)180°$. Give the meaning of this formula.

b. Is S a function of n? If so, give the domain and range of this function.

B **21.** The **greatest integer function** assigns to each number the greatest integer less than or equal to the number. If we denote the greatest integer in x by $\lfloor x \rfloor$, then we have $\lfloor 5.28 \rfloor = 5$, $\lfloor 5 \rfloor = 5$, $\lfloor \pi \rfloor = 3$, and $\lfloor -1.7 \rfloor = -2$.

a. Sketch the graph of $y = \lfloor x \rfloor$.

b. Give the domain and range of the greatest integer function.

22. The greatest integer function $f(x) = \lfloor x \rfloor$, described in Exercise 21, is sometimes called "the floor of x." By contrast, $c(x) = \lceil x \rceil$ is called "the ceiling of x" and is the least integer greater than or equal to x. Thus $\lceil 5.28 \rceil = 6$, $\lceil 5 \rceil = 5$, $\lceil \pi \rceil = 4$, and $\lceil -1.7 \rceil = -1$.

a. Sketch the graph of $y = \lceil x \rceil$.

b. The cost of parking a car in a municipal parking lot is \$3 for the first hour or any part thereof, plus \$2 for each additional hour or part thereof. Sketch the graph of this cost function and find a rule for the cost C as a function of time t. Your rule should use the ceiling function.

23. ***Writing*** Think about what it means for two functions to be *equal*. Would you say that the functions $f(x) = |x|$ and $g(x) = \sqrt{x^2}$ are equal? Are the functions $f(x) = |x|$ and $h(x) = (\sqrt{x})^2$ equal? Write a brief defense of your conclusions.

24. ***Research*** Use a mathematics dictionary to find the meaning of the phrase *implicit function*. Then determine what implicit functions, if any, are defined by each of the following equations.

a. $x = y^2$ **b.** $x^2 + y^2 = 1$ **c.** $x^2 - y^2 = 0$

25. For which of the following functions does $f(a + b) = f(a) + f(b)$?

a. $f(x) = x^2$ **b.** $f(x) = \frac{1}{x}$ **c.** $f(x) = 4x + 1$ **d.** $f(x) = 4x$

26. For which of the functions in Exercise 25 does $f(ab) = f(a) \cdot f(b)$?

27. If $f(a + b) = f(a) + f(b)$ for some function f, prove that $f(0) = 0$.

28. If $f(ab) = f(a) \cdot f(b)$ for some function f, prove that $f(1) = 1$.

4-2 Operations on Functions

Objective *To perform operations on functions and to determine the domains of the resulting functions.*

Suppose a company manufactures and sells a certain product. If the cost of manufacturing x items of the product is given by the function $C(x)$ and the revenue generated by the sale of the x items is given by the function $R(x)$, then the company's profit is given by the function $P(x)$ where

$$P(x) = R(x) - C(x).$$

That is, the profit function is the *difference* between the revenue and cost functions. As this example from economics suggests, it is possible to combine two given functions to produce a new function.

The Sum, Difference, Product, and Quotient of Functions

Each function listed below is defined for all x in the domains of both f and g.

1. **Sum** of f and g: $(f+g)(x) = f(x) + g(x)$
2. **Difference** of f and g: $(f-g)(x) = f(x) - g(x)$
3. **Product** of f and g: $(f \cdot g)(x) = f(x) \cdot g(x)$
4. **Quotient** of f and g: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$

Example 1 Let $f(x) = x + 1$ and $g(x) = x^2 - 1$. Find a rule for each of the following functions.

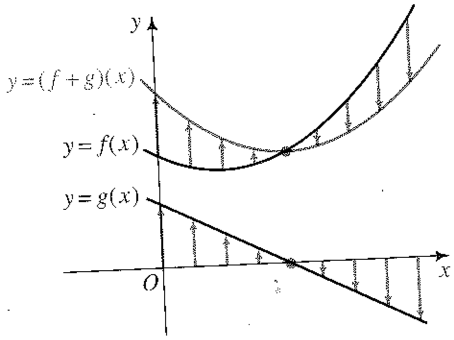
a. $(f+g)(x)$ **b.** $\left(\frac{f}{g}\right)(x)$

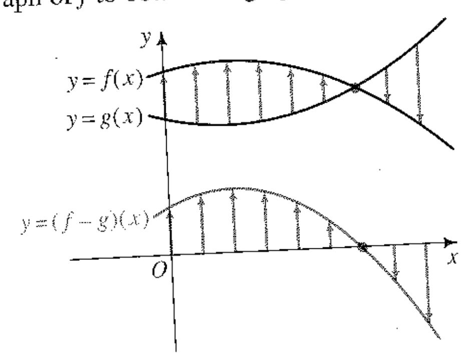
Solution

a. $(f+g)(x) = f(x) + g(x) = (x+1) + (x^2 - 1)$
$= x^2 + x$

b. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+1}{x^2-1}$
$= \frac{x+1}{(x+1)(x-1)}$
$= \frac{1}{x-1}$, provided $x \neq \pm 1$

The graph of the sum function, $f + g$, can be obtained directly from the graphs of f and g. As the diagram at the left below shows, vertical arrows from the x-axis to the graph of g are "added" to the graph of f to obtain the graph of $f + g$.





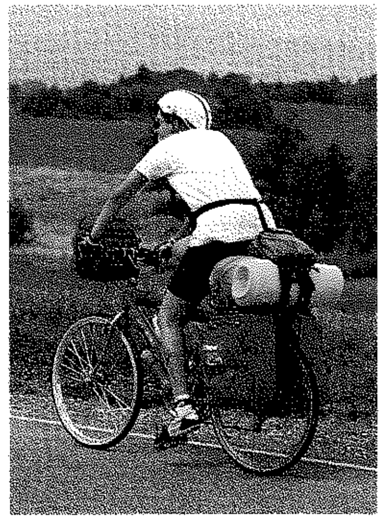
The diagram at the right above shows that the graph of the difference function, $f - g$, gives the vertical distance from the graph of g to the graph of f. Wherever the graph of f lies above the graph of g, $f - g$ is positive, and wherever the graph of f lies below the graph of g, $f - g$ is negative.

A fifth way of combining functions can be illustrated by the sport of cycling. When an 18-speed touring bicycle is in sixth gear, the gear ratio is 3 : 2, which means that the wheels of the bicycle revolve 3 times for every 2 revolutions of the pedals. This relationship can be expressed as

$$w = \frac{3}{2}p$$

where w and p represent wheel and pedal revolutions, respectively. Since the wheels of a touring bicycle have a diameter of 27 in., w revolutions of the wheels move the bicycle a distance d, in inches, given by:

$$d = 27\pi w$$



Notice that $d = 27\pi w$ gives distance as a function of wheel revolutions and that $w = \frac{3}{2}p$ gives wheel revolutions as a function of pedal revolutions. By substituting $\frac{3}{2}p$ for w in $d = 27\pi w$, we get

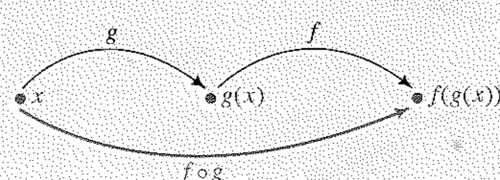
$$d = 27\pi\left(\frac{3}{2}p\right) = 40.5\pi p,$$

which gives distance as a function of pedal revolutions. This function $d(p)$ is said to be the *composite* of the functions $d(w)$ and $w(p)$.

The Composite of Functions

The **composite** of f and g, denoted $f \circ g$, is defined by two conditions:

1. $(f \circ g)(x) = f(g(x))$, which is read "f circle g of x equals f of g of x";
2. x is in the domain of g and $g(x)$ is in the domain of f.



The domain of $f \circ g$ is the set of x satisfying condition (2) above. The operation that combines f and g to produce their composite is called the **composition** of functions.

As Example 2 shows, using a computer or graphing calculator to graph a composite function helps you see that its domain may need to be restricted.

Example 2 Let $f(x) = x^4 - 3x^2$ and $g(x) = \sqrt{x-2}$. Find a rule for $(f \circ g)(x)$ and give the domain of the composite function. Confirm the domain by using a computer or graphing calculator to graph $y = (f \circ g)(x)$.

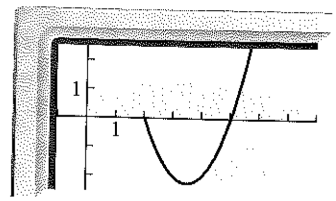
Solution

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(\sqrt{x-2}) \longleftarrow g(x) \text{ is defined for } x \geq 2.\\ &= (\sqrt{x-2})^4 - 3(\sqrt{x-2})^2 \longleftarrow f(g(x)) \text{ is defined for } x \geq 2.\\ &= x^2 - 7x + 10\end{aligned}$$

The domain of the composite function $f \circ g$ is $\{x \mid x \geq 2\}$ even though the expression $x^2 - 7x + 10$ is also defined for $x < 2$. To use a graphing calculator to confirm the domain, enter the equation of $f \circ g$ in its unsimplified form:

$$y = (\sqrt{x-2})^4 - 3(\sqrt{x-2})^2$$

Then the graph of $y = (f \circ g)(x)$ is the portion of the parabola $y = x^2 - 7x + 10$ for which $x \geq 2$.



Example 3 Let $f(x) = \frac{1}{x}$ and $g(x) = x + 1$. Find rules for $(f \circ g)(x)$ and $(g \circ f)(x)$ and give the domain of each composite function.

Solution

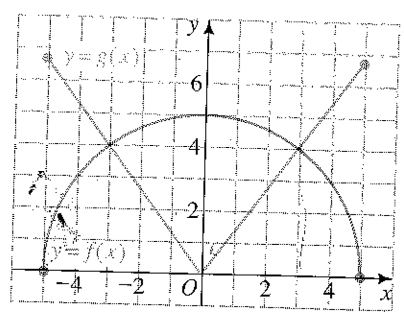
$(f \circ g)(x) = f(g(x)) = f(x+1) = \frac{1}{x+1}$ Domain $= \{x \mid x \neq -1\}$

$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 1$ Domain $= \{x \mid x \neq 0\}$

Notice in Example 3 that $(f \circ g)(x) \neq (g \circ f)(x)$. Since composition is not necessarily commutative, the *order* of two functions being composed is important.

CLASS EXERCISES

For Exercises 1–4, use the graphs of f and g shown at the right.



1. Find each of the following.
   a. $(f + g)(0)$ b. $(f + g)(3)$
2. For what values of x is:
   a. $(f - g)(x) = 0$?   b. $(f - g)(x) > 0$?
3. Sketch the graph of each of the following.
   a. $y = (f + g)(x)$ b. $y = (f - g)(x)$
4. Sketch the graph of each of the following.
   a. $y = f(x) + 1$ b. $y = g(x) - 1$

In Exercises 5–10, let $f(x) = x^2 + x$ and $g(x) = x + 1$. Find each of the following.

5. $(f + g)(x)$ **6.** $(f - g)(x)$ **7.** $(f \cdot g)(x)$ **8.** $\left(\frac{f}{g}\right)(x)$

9. a. $f(g(2))$ **b.** $(f \circ g)(x)$ **10. a.** $g(f(2))$ **b.** $(g \circ f)(x)$

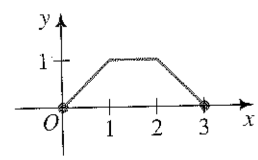
11. ***Visual Thinking*** On a single set of axes, sketch the graphs of $f(x) = \sqrt{x}$ and $g(x) = x + 1$. Demonstrate how the graphs of f and g can be used to find the value of **(a)** $f(g(3))$ and **(b)** $g(f(4))$.

12. If $F(X)$ is the father of X and $M(X)$ is the mother of X, what expression represents the maternal grandfather of X? the paternal grandmother of X?

13. If B is the birthday function defined on page 119 and if F is the father function defined in Exercise 12, which of the composite functions $B \circ F$ and $F \circ B$ is defined?

WRITTEN EXERCISES

A **1.** Copy the graph of $y = f(x)$ shown at the right. On a single set of axes, draw the graphs of $y = f(x) + 2$ and $y = f(x) - 3$.



Ex. 1

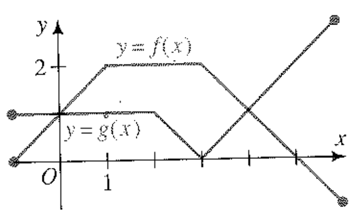
2. On a single set of axes, graph $y = |x|$, $y = |x| + 5$, and $y = |x| - 4$.

3. The graphs of $y = f(x)$ and $y = g(x)$ are shown at the left below.

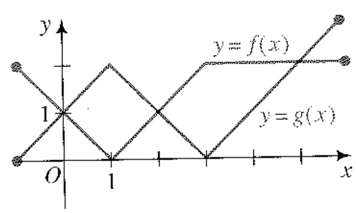
a. Find $f(1) - g(1)$.

b. For what values of x is $f(x) - g(x)$ positive? negative? zero?

c. What is the maximum value of $f(x) - g(x)$?



Ex. 3



Ex. 4

4. Repeat Exercise 3 using the graphs of $y = f(x)$ and $y = g(x)$ shown at the right above.

5–10. Let $f(x) = x^3 - 1$ and $g(x) = x - 1$. Evaluate the expressions in Class Exercises 5–10.

11. On a single set of axes, graph $f(x) = x$ in one color, graph $g(x) = |x|$ in a second color, and graph $f + g$ in a third color.

12. Using the graphs given in Exercise 4, graph $f + g$.

13. ***Visual Thinking*** Given two functions f and g, one way to obtain the real solutions of the equation $f(x) = g(x)$ is to graph the equations $y = f(x)$ and $y = g(x)$ in an xy-plane and then find the x-coordinates of any points of intersection. Describe another way to solve $f(x) = g(x)$ that also involves graphing in an xy-plane but that is based on the difference function $f - g$.

Use a computer or graphing calculator and one of the methods from Exercise 13 to find the real solutions of each of the following equations. Give answers to the nearest hundredth.

14. $x^3 = x + 1$ **15.** $\sqrt{x+1} = 2x$ **16.** $\sqrt{1 - x^2} = |x|$

B **17.** Let $f(x) = 2x - 3$, $g(x) = \frac{x+3}{2}$, and $h(x) = 3x + 2$.

a. Show that $f(g(x)) = g(f(x))$ for all x.
b. Show that $f(h(x)) \neq h(f(x))$ for any x.

18. Let $f(x) = x^3$, $g(x) = \sqrt[3]{x}$, and $h(x) = 3x$.

a. Show that $f(g(x)) = g(f(x))$ for all x.
b. Show that $f(h(x)) = h(f(x))$ for only one value of x.

Let $f(x) = \sqrt{x}$, $g(x) = 6x - 3$, and $h(x) = \frac{x}{3}$. Find each of the following.

19. a. $f(g(h(6)))$ **b.** $f(g(h(x)))$ **20. a.** $h(g(f(4)))$ **b.** $h(g(f(x)))$

21. a. $h\left(f\left(g\left(\frac{1}{2}\right)\right)\right)$ **b.** $h(f(g(x)))$ **22. a.** $g(h(f(9)))$ **b.** $g(h(f(x)))$

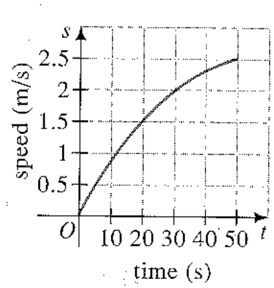
Let $f(x) = x^3$, $g(x) = \sqrt{x}$, $h(x) = x - 4$, and $j(x) = 2x$. Express each function k as a composite of three of these four functions.

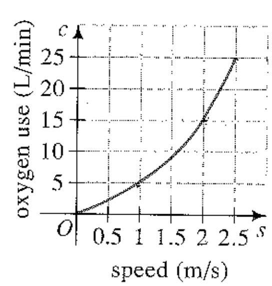
23. $k(x) = 2(x - 4)^3$ **24.** $k(x) = \sqrt{(x-4)^3}$

25. $k(x) = (2x - 8)^3$ **26.** $k(x) = \sqrt{x^3 - 4}$

27. Physiology The graph at the left below shows a swimmer's speed s as a function of time t. The graph at the right below shows the swimmer's oxygen consumption c as a function of s. Time is measured in seconds, speed in meters per second, and oxygen consumption in liters per minute.

a. What are the speed and oxygen consumption after 20 s of swimming?
b. How many seconds have elapsed if the swimmer's oxygen consumption is 15 L/min?

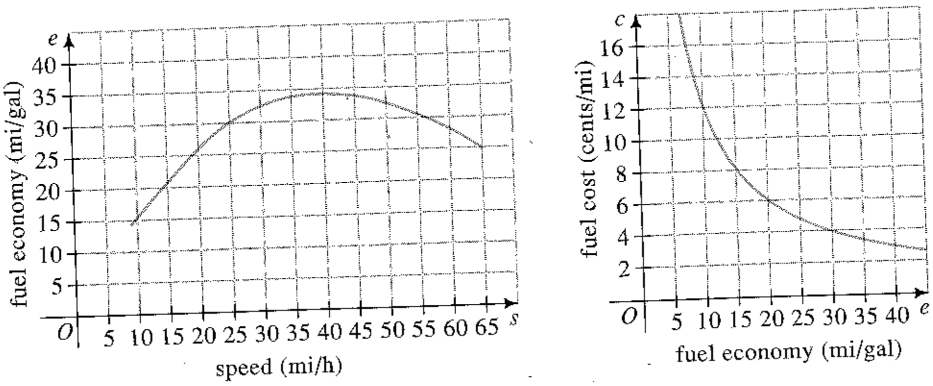




28. **Consumer Economics** The graph at the left below shows a car's fuel economy e as a function of the speed s at which the car is driven. The graph at the right below shows the per-mile fuel cost c as a function of e. Fuel economy is measured in miles per gallon, speed in miles per hour, and fuel cost in cents per mile.

a. If the car is driven at 55 mi/h, what is the fuel cost?

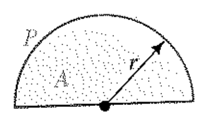
b. If the fuel cost is to be kept at or below 4 cents per mile, at what speeds should the car be driven?



29. **a.** Express the radius r of a circle as a function of the circumference C.

b. Express the area A of the circle as a function of C.

30. **a.** Express the area A and perimeter P of a semicircular region in terms of the radius r.



b. Express A as a function of P.

31. **Physics** The speed s of sound in air is given by the formula $s = 331 + 0.6C$ where s is measured in meters per second and C is the Celsius temperature. If $C = \frac{5}{9}(F - 32)$, express s as a function of F, the Fahrenheit temperature.

32. The surface area and volume of a sphere are given in terms of the radius by the following formulas: $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.

a. Express r as a function of A.

b. Express V as a function of A.

In Exercises 33–36, find rules for $(f \circ g)(x)$ and $(g \circ f)(x)$ and give the domain of each composite function. You may wish to use a computer or graphing calculator to confirm your answer for the domain.

33. $f(x) = 2x$, $g(x) = \sqrt{16 - x^2}$

34. $f(x) = \sqrt{x}$, $g(x) = \frac{1}{x - 4}$

35. $f(x) = x^2$, $g(x) = \sqrt{1 - x}$

36. $f(x) = x^2$, $g(x) = \sqrt{16 - x^2}$

37. If $g(x) = \frac{x + 3}{2}$, find $g(g(1))$, $g(g(g(1)))$, and $g(g(g(g(1))))$.

38. If $f(x) = 2x - 1$, show that $f(f(x)) = 4x - 3$. Find $f(f(f(x)))$.

C **39.** **Physics** The luminous intensity I, measured in candela (cd), of a 100 watt light bulb is 130 cd. The law of illumination states that $E = \frac{I}{d^2}$, where E is the illumination and d is the distance in meters to the light bulb. Suppose you hold a book 1 m away from a 100 watt bulb and begin walking away from the bulb at a rate of 1 m/s.

a. Express E in terms of the time t in seconds after you begin walking.

b. When will the illumination on the book be 1% of its original value?

Graphs and Inverses of Functions

4-3 Reflecting Graphs; Symmetry

Objective *To reflect graphs and to use symmetry to sketch graphs.*

In this section and the next we will see how the graph of an equation is transformed when the equation is altered. This will allow us to graph a simple equation and—by reflecting it, stretching or shrinking it, or sliding it—to obtain the graph of a related, more complicated equation.

We begin by considering the *reflection* of a graph in a line. The *line of reflection* acts like a mirror and is located halfway between a point and its reflection. (If the point being reflected is on the line, then the point is its own reflection.)

Activities

By completing the following activities, you will see how a change in an equation results in a reflection of its graph in some line. For parts (a) and (b) of each activity, graph each pair of equations on a single set of axes. You may wish to use a computer or graphing calculator to help you with your graphs.

1. a. Graph $y = x^2$ and $y = -x^2$.

b. Graph $y = x^3 + 2x^2$ and $y = -(x^3 + 2x^2)$.

c. In general, how are the graphs of $y = f(x)$ and $y = -f(x)$ related?

2. a. Graph $y = x^2 - 1$ and $y = |x^2 - 1|$.

b. Graph $y = x(x-1)(x-3)$ and $y = |x(x-1)(x-3)|$.

c. In general, how are the graphs of $y = f(x)$ and $y = |f(x)|$ related?

3. a. Graph $y = 2x - 1$ and $y = 2(-x) - 1$.

b. Graph $y = \sqrt{x}$ and $y = \sqrt{-x}$.

c. In general, how are the graphs of $y = f(x)$ and $y = f(-x)$ related?

4. a. Graph $y = 2x + 1$ and $x = 2y + 1$.

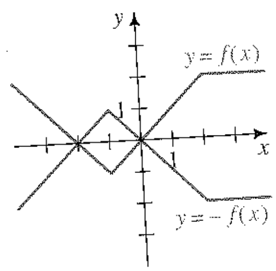
b. Graph $y = x^2$ and $x = y^2$.

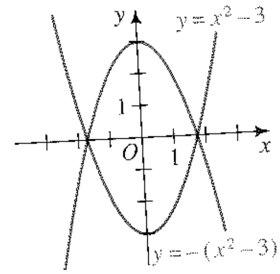
c. In general, how is the graph of an equation affected when you interchange the variables in the equation?

As the activities on the preceding page show, simple changes in an equation can produce reflections of its graph in the x-axis, the y-axis, and the line $y = x$.

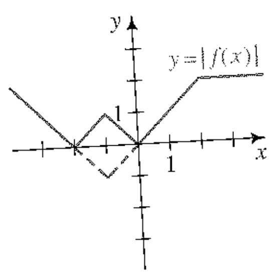
Reflection in the x-axis

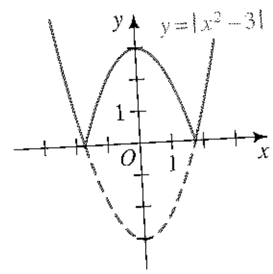
The graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis. In the graphs below, notice that each point (x, y) on the original (red) graph becomes the point $(x, -y)$ on the reflected (blue) graph.





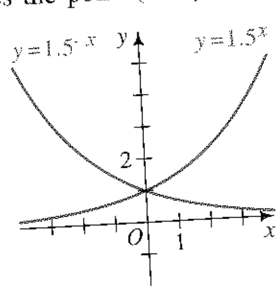
The graph of $y = |f(x)|$ is identical to the graph of $y = f(x)$ when $f(x) \geq 0$ and is identical to the graph of $y = -f(x)$ when $f(x) < 0$. This principle is applied to the graphs shown above to produce the following graphs. Notice that the graphs do not dip below the x-axis.

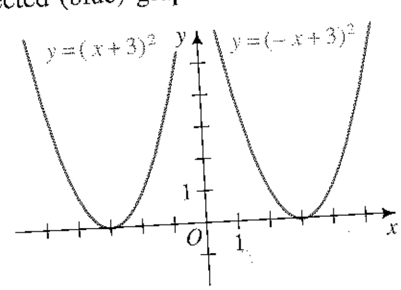




Reflection in the y-axis

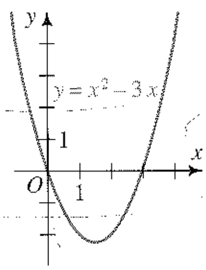
The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis. In the graphs below, notice that each point (x, y) on the original (red) graph becomes the point $(-x, y)$ on the reflected (blue) graph.



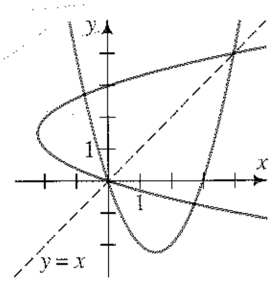


Reflection in the Line $y = x$

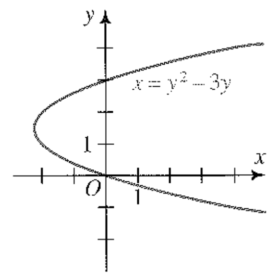
Reflecting the graph of an equation in the line $y = x$ is equivalent to interchanging x and y in the equation.



Original graph and equation



Reflection in $y = x$

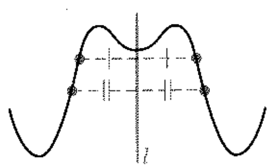


Reflected graph and altered equation

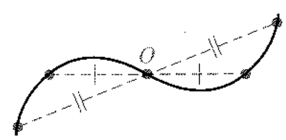
Symmetry

A line l is called an **axis of symmetry** of a graph if it is possible to pair the points of the graph in such a way that l is the perpendicular bisector of the segment joining each pair. (See the figure at the left below.)

A point O is called a **point of symmetry** of a graph if it is possible to pair the points of the graph in such a way that O is the midpoint of the segment joining each pair. (See the figure at the right below.)

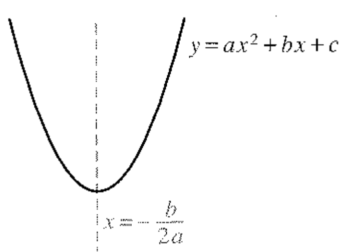


l = axis of symmetry

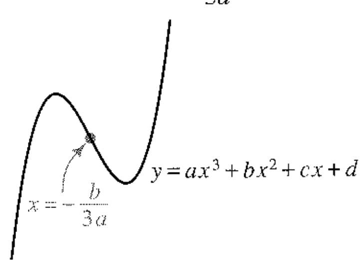


O = point of symmetry

As you know, the graph of $y = ax^2 + bx + c$ has an axis of symmetry with equation $x = -\frac{b}{2a}$. In Exercise 16 on page 145, we will show that the graph of $y = ax^3 + bx^2 + cx + d$ has a point of symmetry at $x = -\frac{b}{3a}$.

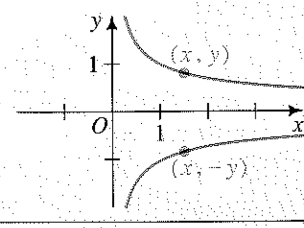
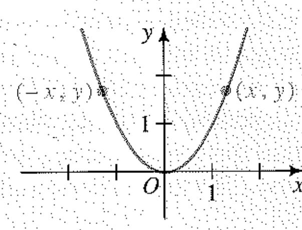
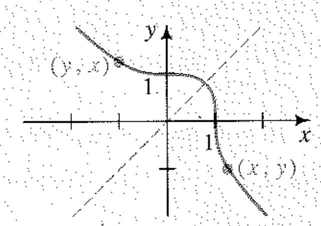
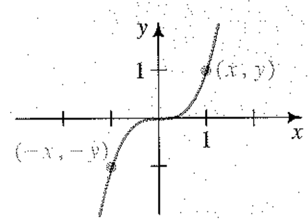


The graph of every quadratic function has a line of symmetry.



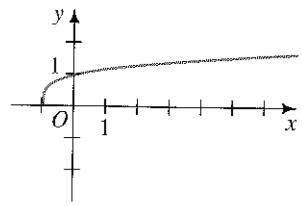
The graph of every cubic function has a point of symmetry.

Special Tests for the Symmetry of a Graph

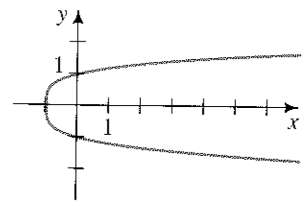
Type of Symmetry	Example
Symmetry in the x-axis *Meaning*: $(x, -y)$ is on the graph whenever (x, y) is. *Testing an equation of a graph*: In the equation, leave x alone and substitute $-y$ for y. Does an equivalent equation result?	$y^2x = 1$ $(-y)^2x = 1$ equivalent 
Symmetry in the y-axis *Meaning*: $(-x, y)$ is on the graph whenever (x, y) is. *Testing an equation of a graph*: In the equation, substitute $-x$ for x and leave y alone. Does an equivalent equation result?	$y = x^2$ $y = (-x)^2$ equivalent 
Symmetry in the line $y = x$ *Meaning*: (y, x) is on the graph whenever (x, y) is. *Testing an equation of a graph*: In the equation, interchange x and y. Does an equivalent equation result?	$x^3 + y^3 = 1$ $y^3 + x^3 = 1$ equivalent 
Symmetry in the origin *Meaning*: $(-x, -y)$ is on the graph whenever (x, y) is. *Testing an equation of a graph*: In the equation, substitute $-x$ for x and $-y$ for y. Does an equivalent equation result?	$y = x^3$ $-y = (-x)^3$ equivalent 

Example Use symmetry to sketch the graph of $y^4 = x + 1$.

Solution The equation $y^4 = x + 1$ can be rewritten as $y = \pm\sqrt[4]{x+1}$. You can see from either of these equations that the graph has symmetry in the x-axis. Therefore, you only need to graph $y = \sqrt[4]{x+1}$ (by plotting a few points or by using a computer or graphing calculator). You can then reflect the graph in the x-axis to obtain the graph of $y = -\sqrt[4]{x+1}$. The two pieces together comprise the complete graph of $y = \pm\sqrt[4]{x+1}$.



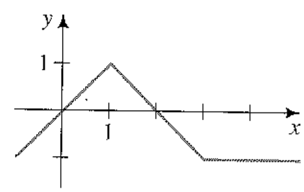
Graph of $y=\sqrt[4]{x+1}$



Graph of $y=\pm\sqrt[4]{x+1}$
or $y^4=x+1$

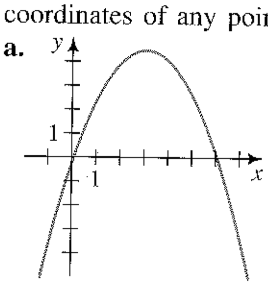
CLASS EXERCISES

1. The graph of $y = f(x)$ is shown at the right. Sketch the graph of each of the following equations.
   a. $y = -f(x)$
   b. $y = |f(x)|$
   c. $y = f(-x)$

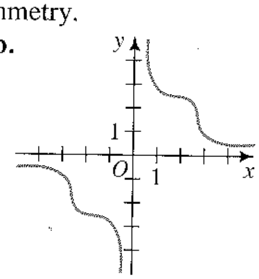


2. For each graph, give the equation(s) of any line(s) of symmetry and give the coordinates of any point of symmetry.

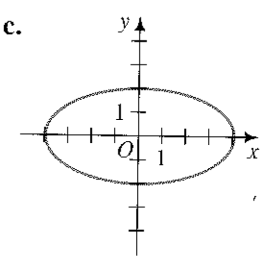
a.



b.



c.



3. Tell whether the graph of each equation has symmetry in:
   (i) the x-axis, (ii) the y-axis, (iii) the line $y = x$, and (iv) the origin.
   a. $x^4 + y^4 = 1$ **b.** $xy^3 = 1$ **c.** $x(x + y) = 1$
4. ***Visual Thinking*** If a graph has symmetry in both the x- and y-axes, what other symmetry must it have? Explain.

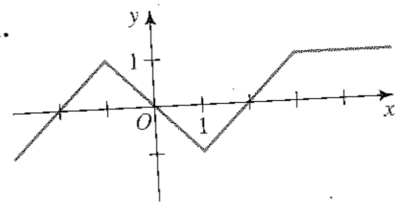
5. *Visual Thinking* Can a graph that has symmetry in the x-axis be the graph of a function? Explain.
6. *Visual Thinking* Describe how to obtain the graph of $y = \sqrt{|x|}$ from the graph of $y = \sqrt{x}$.
7. Give the equation of the axis of symmetry for the graph of each quadratic function.
   a. $f(x) = x^2 - 8x - 7$ **b.** $g(x) = 8x - 4x^2$ **c.** $h(x) = x^2 + 3$
8. Give the coordinates of the point of symmetry for the graph of each cubic function.
   a. $f(x) = x^3 - 6x^2 + 5x + 7$ **b.** $g(x) = 9x + 6x^2 + 2x^3$ **c.** $h(x) = 3x^3 - 3x + 7$

WRITTEN EXERCISES

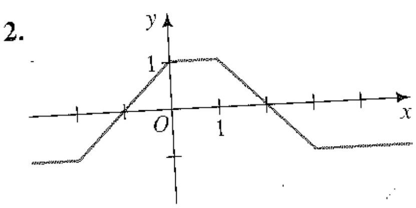
In Exercises 1–4, the graph of $y = f(x)$ is given. Sketch the graphs of:

a. $y = -f(x)$ **b.** $y = |f(x)|$ **c.** $y = f(-x)$

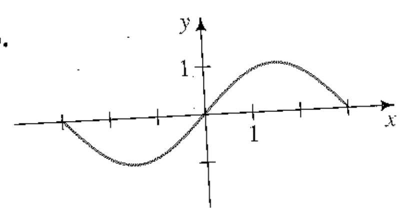
A **1.**



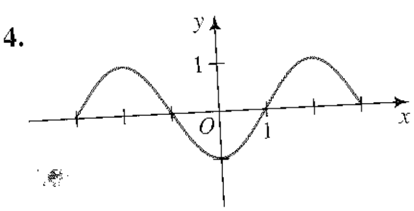
2.



3.



4.



5. Sketch the graphs of $y = x^2 - 9$, $y = 9 - x^2$, and $y = |9 - x^2|$ on a single set of axes.
6. Sketch the graphs of $y = |x| - 2$, $y = 2 - |x|$, and $y = |2 - |x||$ on a single set of axes.

In Exercises 7–14, sketch the graph of each equation and the reflection of the graph in the line $y = x$. Then give an equation of the reflected graph.

7. $y = 3x - 4$ **8.** $y = \frac{1}{2}x + 1$ **9.** $y = x^2 - 2x$ **10.** $y = x^2 + 3x$

11. $y = x^3$ **12.** $y = \sqrt{x}$ **13.** $y = |x| + 2$ **14.** $y = |x| - 3$

15. Test each equation to see if its graph has symmetry in:
(i) the x-axis, (ii) the y-axis, (iii) the line $y = x$, and (iv) the origin.
a. $y^2 - xy = 2$ **b.** $x^2 + y^2 = 1$ **c.** $y = x|x|$

16. Repeat Exercise 15 for each of the following equations.

a. $x^2 + xy = 4$ **b.** $|x| + |y| = 1$ **c.** $y = \frac{x}{|x|}$

Use symmetry to sketch the graph of each equation.

B 17. $|x| + |y| = 2$

18. $|x|^{1/2} + |y|^{1/2} = 2$

19. $x^2y = 1$

20. $|xy| = 12$

In Exercises 21–26, graph each parabola, showing the vertex with its coordinates and the axis of symmetry with its equation. The pairs of graphs in Exercises 21–23 should be done on a single set of axes.

21. **a.** $y = (x - 3)^2 + 5$
b. $x = (y - 3)^2 + 5$

22. **a.** $y = 2(x + 1)^2 + 3$
b. $x = 2(y + 1)^2 + 3$

23. **a.** $y = 3 - (x - 4)^2$
b. $x = 3 - (y - 4)^2$

24. $x = 2(y + 1)^2 + 4$

25. $x = y^2 + 6y + 8$

26. $x = y^2 + 2y - 3$

27. The graph of a cubic function has a local minimum at $(5, -3)$ and a point of symmetry at $(0, 4)$. At what point does a local maximum occur?

28. **a.** Find the point of symmetry of the graph of the cubic function $f(x) = -x^3 + 15x^2 - 48x + 45$.
b. The function has a local minimum at $(2, 1)$. At what point does a local maximum occur?

29. **a.** Graph $y = 3x^2 - x^3$. At what point does a local minimum occur?
b. Find the point of symmetry of the graph and then deduce the coordinates of the point where a local maximum occurs.

30. **a.** Graph $y = -x^3 - 6x^2 - 9x$. At what point does a local minimum occur?
b. Find the point of symmetry and then deduce the coordinates of the point where a local maximum occurs.

Use the following definitions to complete Exercises 31–36.

f is an **even** function if $f(-x) = f(x)$.
f is an **odd** function if $f(-x) = -f(x)$.

31. Classify each function as even, odd, or neither.

a. $f(x) = x^2$ **b.** $f(x) = x^3$ **c.** $f(x) = x^2 - x$
d. $f(x) = x^4 + 2x^2$ **e.** $f(x) = x^3 + 3x^2$ **f.** $f(x) = x^5 - 4x^3$

32. Use the results of Exercise 31 to guess the reasons for using the terms "even" and "odd" as they are applied to polynomial functions.

33. **a.** What kind of symmetry does the graph of an even function have?
b. What kind of symmetry does the graph of an odd function have?

34. Study the graphs shown in Exercises 3 and 4. Then tell whether each function graphed is even or odd.

35. If f and g are both odd functions and $h(x) = f(x) \cdot g(x)$, prove that h is even.

36. If f is an even function and g is an odd function, prove that $h(x) = f(x) \cdot g(x)$ is odd.

Use the following definition to complete Exercises 37–39. For Exercise 38, you will need to use a computer or graphing calculator.

Suppose x_1 and x_2 are any two domain elements of a function f. We say that f is **increasing** in its domain if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, and f is **decreasing** in its domain if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.

37. **a.** For what value(s) of m is the linear function $f(x) = mx$ an increasing function? a decreasing function? neither?

b. ***Visual Thinking*** Describe what the definitions of increasing and decreasing functions imply about the graphs of the functions.

38. Graph each function using a computer or a graphing calculator. Then use the graph to tell whether the function is increasing or decreasing in its domain.

a. $f(x) = x^3 + x - 1$ **b.** $f(x) = \sqrt[3]{1 - x^3}$ **c.** $f(x) = \dfrac{10}{1 + 2^x}$

39. ***Visual Thinking*** Suppose the graph of an increasing function is reflected in **(a)** the x-axis, **(b)** the y-axis, and **(c)** the line $y = x$. In each case, tell whether the reflected graph represents an increasing or a decreasing function.

40. On one set of axes, graph $y = |x - 2|$, $y = |x - 2| - 2$, and $y = ||x - 2| - 2|$.

C **41.** Use symmetry to sketch the graph of $x^{2/3} + y^{2/3} = 1$.

4-4 Periodic Functions; Stretching and Translating Graphs

Objective *To determine periodicity and amplitude from graphs, to stretch and shrink graphs both vertically and horizontally, and to translate graphs.*

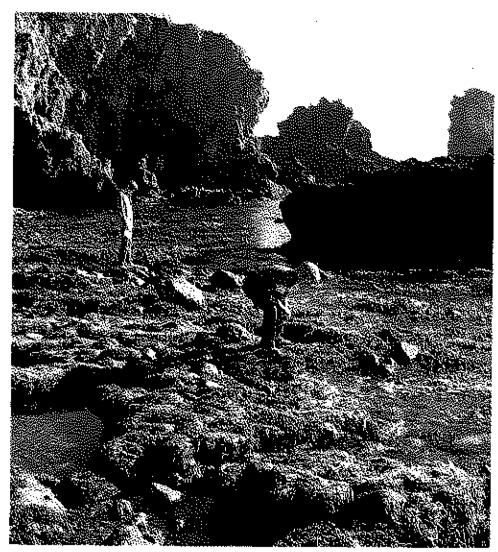
Periodic Functions

The world is full of periodic phenomena. The tides come in and go out again and again, each *cycle* or *period* lasting about 12.4 h. The amount of daylight increases and decreases with a period of one year. The functions that describe periodic behavior are called *periodic* functions.

A function f is **periodic** if there is a positive number p, called a **period** of f, such that

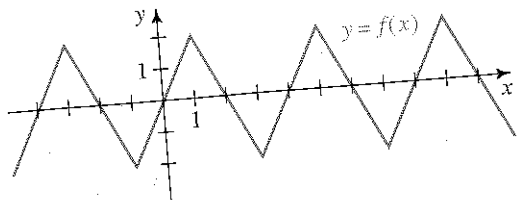
$$f(x + p) = f(x)$$

for all x in the domain of f. The smallest period of a periodic function is called the **fundamental period** of the function.



The definition of a periodic function implies that if f is a periodic function with period p, then $f(x) = f(x + mp)$ for all x and any integer m.

Example 1 The graph of a periodic function f is shown below. Find:
a. the fundamental period of f **b.** $f(99)$



Solution **a.** If we start at the origin and follow the graph to the right, the graph takes 4 units to complete one up-and-down cycle; another such cycle then begins. Thus, f is a periodic function with fundamental period 4.
b. Since the fundamental period of f is 4 and $99 \div 4 = 24$ with remainder 3, we have:

$$f(99) = f(99 - 24 \cdot 4) = f(3) = -2$$

If a periodic function has a maximum value M and a minimum value m, then the **amplitude** A of the function is given by:

$$A = \frac{M - m}{2}$$

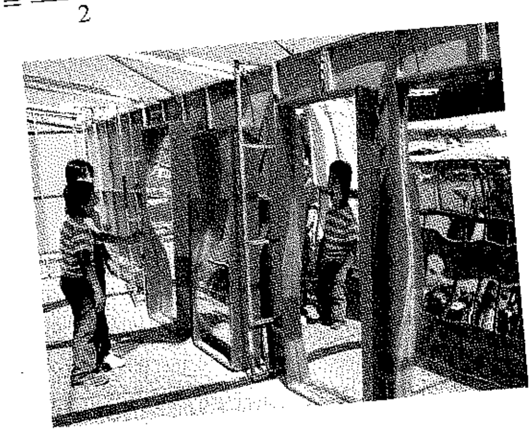
Example 2 Find the amplitude of the function f described in Example 1.

Solution The function's maximum value is $M = 2$; its minimum value is $m = -2$. The amplitude is half the difference between M and m:

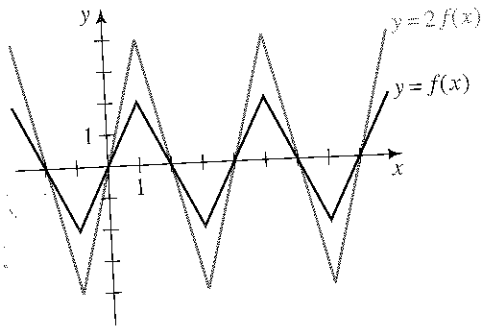
$$A = \frac{2 - (-2)}{2} = 2$$

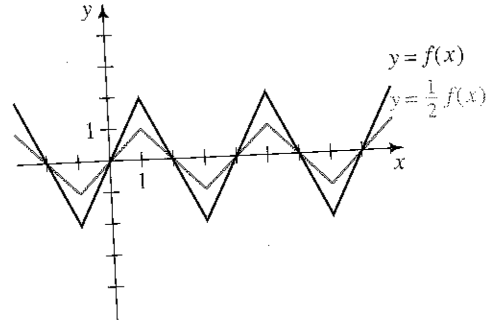
Stretching and Shrinking Graphs

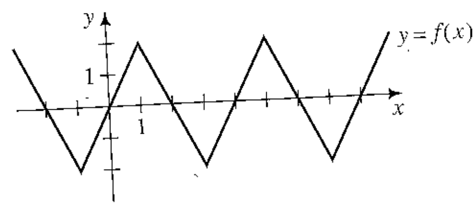
If you go into a house of mirrors at a circus or carnival, you may see your reflection distorted by some of the mirrors. For example, one mirror might make you look tall and thin, while another might make you look short and broad. Just as mirrors can stretch or shrink your reflection both vertically and horizontally, it is possible to stretch or shrink the graph of an equation both vertically and horizontally.



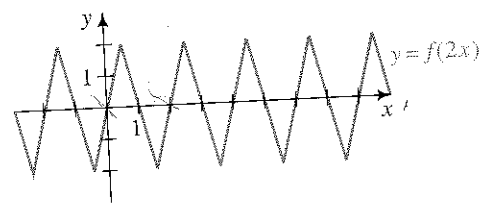
The graph of $y = cf(x)$ where c is positive (and not equal to 1) is obtained by *vertically* stretching or shrinking the graph of $y = f(x)$. For example, in the graphs below, notice that points on the x-axis remain fixed, while all other points move away from the x-axis for $c > 1$ (a vertical stretch) or toward the x-axis for $0 < c < 1$ (a vertical shrink).

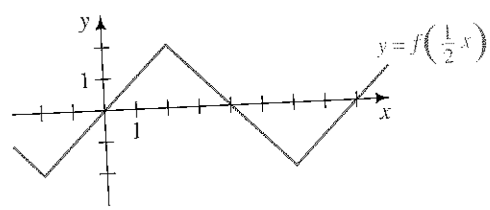






The graph of $y = f(cx)$ where c is positive (and not equal to 1) is obtained by *horizontally* stretching or shrinking the graph of $y = f(x)$. For example, in the graphs at the right, notice that points on the y-axis remain fixed, while all other points move toward the y-axis for $c > 1$ (a horizontal shrink) or away from the y-axis for $0 < c < 1$ (a horizontal stretch).





The graphs shown above are all based on a periodic function f with fundamental period 4 and amplitude 2. Notice that a vertical stretching or shrinking of the graph of f affects the amplitude but not the period, and a horizontal stretching or shrinking of the graph affects the period but not the amplitude. A more formal statement of these results is given at the top of the next page.

Changing the Period and Amplitude of a Periodic Function

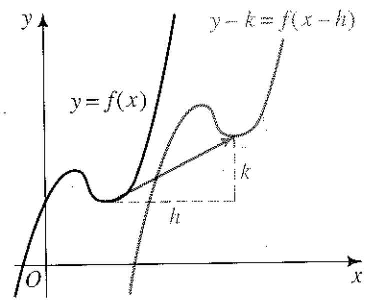
If a periodic function f has period p and amplitude A, then:

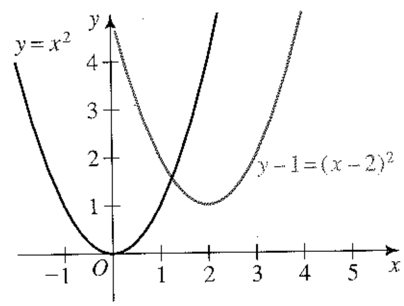
$y = cf(x)$ has period p and amplitude cA, and

$y = f(cx)$ has period $\frac{p}{c}$ and amplitude A.

Translating Graphs

The graph of $y - k = f(x - h)$ is obtained by translating the graph of $y = f(x)$ horizontally h units and vertically k units. For example, as shown at the right below, the graph of $y - 1 = (x - 2)^2$ is the graph of $y = x^2$ translated 2 units horizontally and 1 unit vertically.

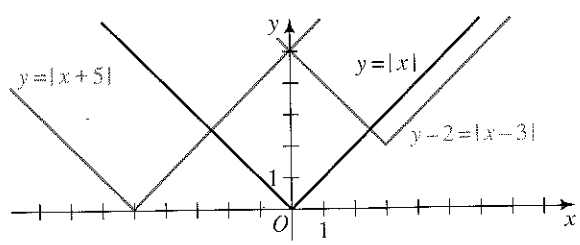




Example 3 Sketch the graphs of the following equations:

$$y = |x|,\ y - 2 = |x - 3|, \text{ and } y = |x + 5|$$

Solution Once we have graphed $y = |x|$, we can slide the graph 3 units horizontally and 2 units vertically to obtain the graph of $y - 2 = |x - 3|$. Since the equation $y = |x + 5|$ is equivalent to $y = |x - (-5)|$, we slide the original graph -5 units horizontally and 0 units vertically to obtain the graph of $y = |x + 5|$.

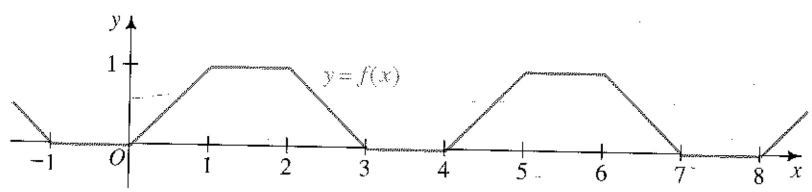


In this chapter we have seen how certain simple changes in the equation of a curve can stretch or shrink the curve, reflect it, or translate it. These results are summarized at the top of the next page.

If the equation $y = f(x)$ is changed to:	Then the graph of $y = f(x)$ is:
$y = -f(x)$	reflected in the x-axis.
$y = \|f(x)\|$	unchanged when $f(x) \geq 0$ and reflected in the x-axis when $f(x) < 0$.
$y = f(-x)$	reflected in the y-axis.
$x = f(y)$	reflected in the line $y = x$.
$y = cf(x)$, $c > 1$	stretched vertically.
$y = cf(x)$, $0 < c < 1$	shrunk vertically.
$y = f(cx)$, $c > 1$	shrunk horizontally.
$y = f(cx)$, $0 < c < 1$	stretched horizontally.
$y - k = f(x - h)$	translated h units horizontally and k units vertically.

CLASS EXERCISES

For Exercises 1–4, refer to the graph of a function $y = f(x)$ shown below.

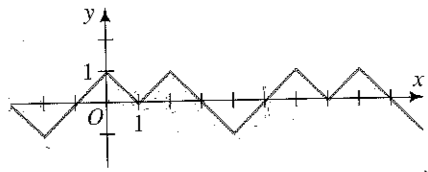


1. a. What is the fundamental period of f?
   b. What is the amplitude?
   c. Find $f(25)$ and $f(-25)$.
2. Sketch the graphs of $y = 2f(x)$ and $y = \frac{1}{2}f(x)$. Give the fundamental period and amplitude of each.
3. Sketch the graphs of $y = f(2x)$ and $y = f\left(\frac{1}{2}x\right)$. Give the fundamental period and amplitude of each.
4. Sketch the graphs of $y = f(x) - 2$ and $y = f(x - 2)$. Give the fundamental period and amplitude of each.
5. If a graph is vertically stretched or shrunk, explain why points on the x-axis remain fixed.
6. If a graph is horizontally stretched or shrunk, explain why points on the y-axis remain fixed.

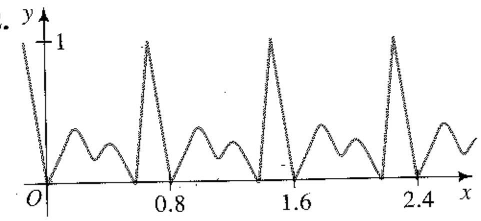
WRITTEN EXERCISES

In Exercises 1–4, the graph of a function f is given. Tell whether f appears to be periodic. If so, give its fundamental period and its amplitude, and then find $f(1000)$ and $f(-1000)$.

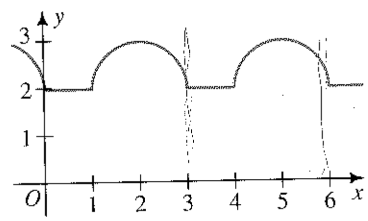
A 1.



2.



3.



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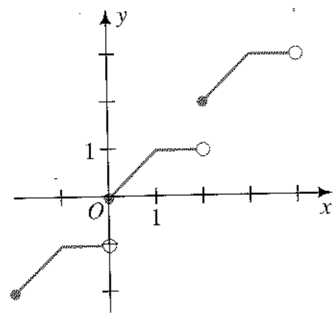
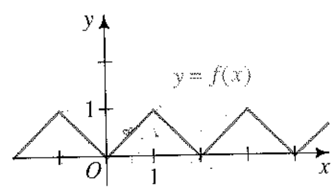


5. Use the graph of $y = f(x)$, shown at the right, to sketch the graph of each of the following.

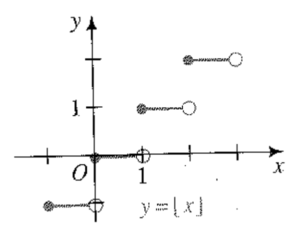


a. $y = 2f(x)$ **b.** $y = -\frac{1}{2}f(x)$

c. $y = f(-2x)$ **d.** $y = f\left(\frac{1}{2}x\right)$

e. $y = f\left(x - \frac{1}{2}\right)$ **f.** $y = f(-x) + 1$

6. The greatest integer function $y = \lfloor x \rfloor$ gives the greatest integer less than or equal to x. Thus, $\lfloor 2.1 \rfloor = 2$ and $\lfloor -3.1 \rfloor = -4$. Use the graph of this function, shown at the right, to sketch the graph of each of the following.



a. $y = \frac{1}{2}\lfloor x \rfloor$ **b.** $y = -2\lfloor x \rfloor$

c. $y = \left\lfloor -\frac{1}{2}x \right\rfloor$ **d.** $y = \lfloor 2x \rfloor$

e. $y = \lfloor x - 1 \rfloor$ **f.** $y = 2\lfloor x \rfloor + 1$

7. Sketch the graph of each of the following.

a. $y + 2 = |x|$ **b.** $y = |x - 3|$ **c.** $y - 4 = |x + 5|$

d. $y = 2|x + 1|$ **e.** $y + 1 = -|x|$ **f.** $y - 3 = |2x|$

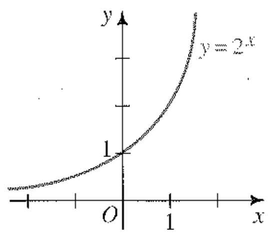
8. Sketch the graph of each of the following.

a. $y - 1 = \sqrt{x}$ **b.** $y = \sqrt{x + 4}$ **c.** $y + 2 = \sqrt{x - 5}$

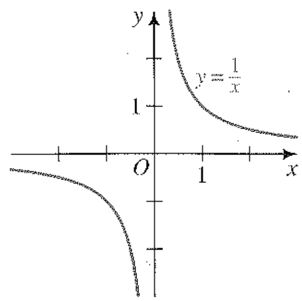
d. $y = 2\sqrt{x - 3}$ **e.** $y - 2 = \sqrt{-x}$ **f.** $y - 4 = \sqrt{4x}$

B **9.** Use the graph of $y = 2^x$, shown at the left below, to sketch the graph of each of the following.

a. $y = 2^{-x}$ **b.** $y = 2^{x-1}$ **c.** $y = 3 - 2^x$ **d.** $x = 2^y$



Ex. 9

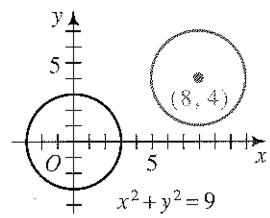


Ex. 10

10. Use the graph of $y = \frac{1}{x}$, shown at the right above, to sketch the graph of each of the following.

a. $y = -\frac{1}{x}$ **b.** $y = \frac{1}{x - 2}$ **c.** $y = 1 + \frac{1}{x}$ **d.** $x = \frac{1}{y}$

11. Given that the equation of a circle with radius 3 and center (0, 0) is $x^2 + y^2 = 9$ (graph at right), deduce the equation of the circle if it is translated so that its center is (8, 4).

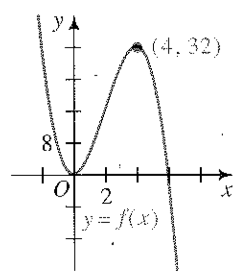


Exs. 11, 12

12. a. Refer to the circle with equation $x^2 + y^2 = 9$ in Exercise 11. Sketch the graph of $\left(\frac{x}{2}\right)^2 + y^2 = 9$.

b. In Exercise 11, the area of the circular region is $\pi r^2 = \pi \cdot 3^2 = 9\pi$. Make a conjecture about the area of the region enclosed by the graph in part (a).

13. The graph of $y = f(x)$, shown at the right, has x-intercepts at 0 and 6 and a local maximum at (4, 32).



a. Where do the x-intercepts and local maximum occur on the graph of $y = f(2x)$?

b. Where do the x-intercepts and local maximum occur on the graph of $y = 2f(x)$?

c. Where do the x-intercepts and local maximum occur on the graph of $y = f(x - 2)$?

d. Where do the x-intercepts and local maximum occur on the graph of $y = f(x + 2)$?

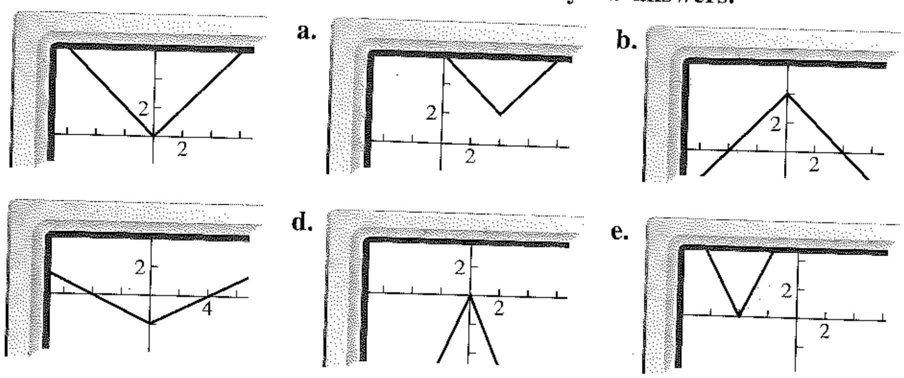
e. If f is a cubic polynomial, find a rule for $f(x)$.

14. **a.** Sketch the graph of $y = x^3 - 3x^2 + 2x$ and label all intercepts.

b. Sketch the graph of $y = \left(\frac{1}{2}x\right)^3 - 3\left(\frac{1}{2}x\right)^2 + 2\left(\frac{1}{2}x\right)$ by using the graph of part (a). Label all intercepts.

The first figure in Exercise 15 shows the calculator-drawn graph of $y = |x|$. Changes in the equation $y = |x|$ cause its graph to be reflected, stretched, shrunk, and/or translated to produce the graphs shown in parts (a)–(e). Give an equation for each graph. You may wish to use a computer or a graphing calculator to confirm your answers.

15.



16. In this exercise, you will show that the graph of

$$y = ax^3 + bx^2 + cx + d$$

has a point of symmetry at $x = -\frac{b}{3a}$.

always works
odd func.

a. If $y = f(x) = ax^3 + px$, show that $f(-x) = -f(x)$.

b. Explain how part (a) shows that the origin is a symmetry point of the graph of $y = ax^3 + px$.

c. Explain why $(0, q)$ is a symmetry point for the graph of $y = ax^3 + px + q$.

d. Explain why (h, q) is a symmetry point for the graph of $y = a(x - h)^3 + p(x - h) + q$.

moves graph h ... and q right

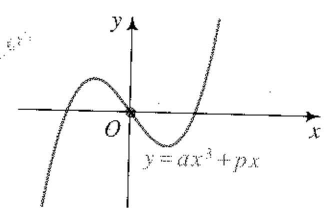
e. Suppose the equation

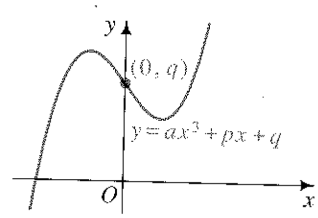
$$y = ax^3 + bx^2 + cx + d$$

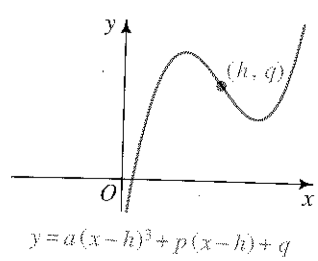
is rewritten in the equivalent form

$$y = a(x - h)^3 + p(x - h) + q.$$

By comparing the coefficients of the x^2 terms, show that $h = -\frac{b}{3a}$. Then use the results of part (d) to conclude that $y = ax^3 + bx^2 + cx + d$ has a point of symmetry at $x = -\frac{b}{3a}$.







17. If the periodic functions f and g both have fundamental period p and $h(x) = (f + g)(x)$, show that $h(x + p) = h(x)$ for all x in the domain of h.

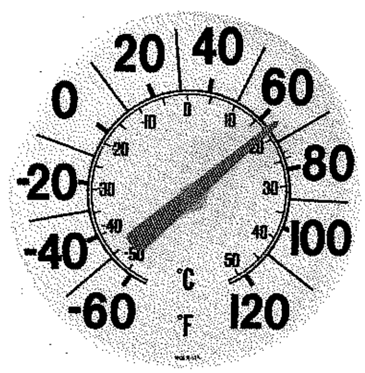
18. If f has fundamental period 2 and g has fundamental period 3, what is the fundamental period of $f + g$?

C 19. Is a constant function like $f(x) = 5$ periodic? If so, does it have a fundamental period? Explain.

20. A function f is defined for all real numbers as follows: $f(x) = 1$ if x is rational, and $f(x) = 0$ if x is irrational. Show that if p is a rational number, then $f(x + p) = f(x)$ for all x. Does f have a fundamental period? Explain.

4-5 Inverse Functions

Objective *To find the inverse of a function, if the inverse exists.*



Conversion formulas, such as those used in converting between U.S. customary and metric measurements, come in pairs. For example, consider the conversion formulas for the Fahrenheit and Celsius temperature scales:

$$F = \frac{9}{5}C + 32 \text{ and } C = \frac{5}{9}(F - 32)$$

The first formula gives a Fahrenheit temperature F as a function of a Celsius temperature C, while the second formula gives C as a function of F.

Notice that in the first formula, $F = 32$ when $C = 0$, and in the second formula, $C = 0$ when $F = 32$. Because each formula undoes what the other one does, the formulas are examples of *inverses*.

Two functions f and g are called **inverse functions** if the following statements are true:

1. $g(f(x)) = x$ for all x in the domain of f;
2. $f(g(x)) = x$ for all x in the domain of g.

Example 1 If $f(x) = \frac{x - 1}{2}$ and $g(x) = 2x + 1$, show that f and g are inverses of each other.

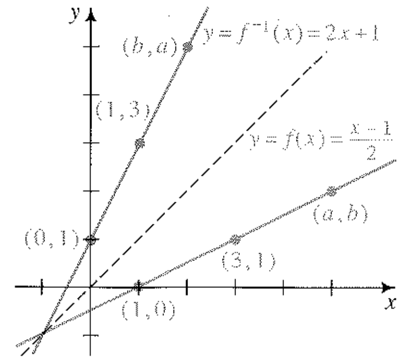
Solution We must check that conditions (1) and (2) stated in the definition above are satisfied:

1. $g(f(x)) = g\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x$ for all x

2. $f(g(x)) = f(2x + 1) = \frac{(2x + 1) - 1}{2} = x$ for all x

We denote the inverse of a function f by f^{-1}, read "f inverse." The symbol $f^{-1}(x)$, read "f inverse of x," is the value of f^{-1} at x. Note that $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$.

The graphs of f and f^{-1} in Example 1 are shown at the right. Notice that for every point (a, b) on the graph of $y = f(x)$, the point (b, a) is on the graph of $y = f^{-1}(x)$. This means that the graph of f^{-1} is the reflection of the graph of f in the line $y = x$.



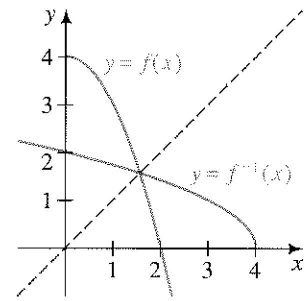
When a function f has an inverse, the graph of f^{-1} can be obtained from the graph of f by changing every point (x, y) on the graph to the point (y, x). Similarly, the rule for $f^{-1}(x)$ can be obtained by interchanging x and y in the equation $y = f(x)$, as shown in the following example.

Example 2 Let $f(x) = 4 - x^2$ for $x \geq 0$.

a. Sketch the graph of $y = f^{-1}(x)$.

b. Find a rule for $f^{-1}(x)$.

Solution **a.** Since the domain of $f(x) = 4 - x^2$ is $x \geq 0$, the graph of $y = f(x)$ is half of a parabola, as shown in red at the right. By reflecting the graph of $y = f(x)$ in the line $y = x$, we obtain the graph of $y = f^{-1}(x)$, as shown in blue at the right.



b. 1. Set $y = f(x)$. $\quad y = 4 - x^2,\ x \geq 0 \longleftarrow$ describes $y = f(x)$

2. Interchange x and y. $\quad x = 4 - y^2,\ y \geq 0 \longleftarrow$ describes $y = f^{-1}(x)$

3. Solve for y. $\quad y = \pm\sqrt{4 - x},\ y \geq 0$

$\quad y = \sqrt{4 - x}$

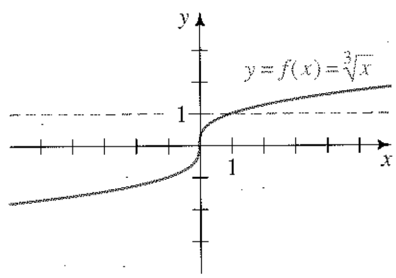
We therefore have the rule $f^{-1}(x) = \sqrt{4 - x}$. The graph shows that the domain of f^{-1} is $\{x \mid x \leq 4\}$.

Note that in Example 2, if the domain of $f(x) = 4 - x^2$ were the set of all real numbers instead of nonnegative real numbers, the reflection of the graph of $y = f(x)$ in the line $y = x$ would not be the graph of a function. Thus, not all functions have inverses.

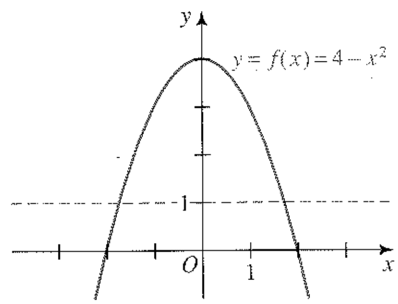
A function $y = f(x)$ that has an inverse is called a **one-to-one function**, because not only does each x-value correspond to exactly one y-value, but also each y-value corresponds to exactly one x-value. We can determine whether a function is one-to-one by applying the *horizontal-line test* to its graph.

The Horizontal-Line Test

If the graph of the function $y = f(x)$ is such that no horizontal line intersects the graph in more than one point, then f is one-to-one and has an inverse.



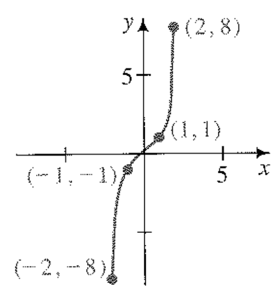
f is one-to-one and has an inverse.



f is not one-to-one and has no inverse. (Compare with f in Example 2.)

CLASS EXERCISES

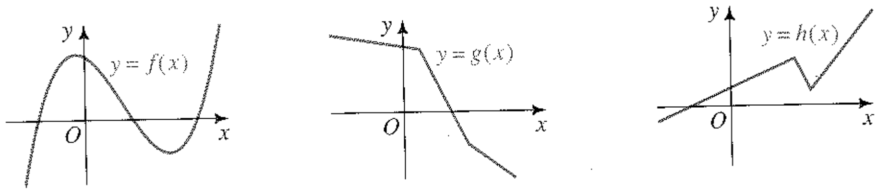
1. Consider the birthday function described on page 119. Is this function one-to-one? Does it have an inverse?
2. Suppose a function f has an inverse. If $f(2) = 3$, find each of the following.
   a. $f^{-1}(3)$ **b.** $f(f^{-1}(3))$ **c.** $f^{-1}(f(2))$
3. Find a rule for $g^{-1}(x)$ if:
   a. $g(x) = 4x$ **b.** $g(x) = 3x + 2$
   c. $g(x) = 2x - 1$ **d.** $g(x) = 4 - 5x$
4. The graph of $f(x) = x^3$ is shown at the right.
   a. Name several points on the graph of $y = f^{-1}(x)$ and then sketch the graph.
   b. Find a rule for $f^{-1}(x)$.



Ex. 4

5. **a.** Is the function $f(x) = x^3 - 1$ a one-to-one function? Explain.
   b. Is the function $g(x) = x^3 - x$ a one-to-one function? Explain.
6. Consider the two temperature conversion formulas given on page 146. Explain how one formula can be obtained from the other.

7. The graphs of f, g, and h are shown below. Which functions are one-to-one? Which functions have inverses?



8. Which of the following functions have inverses?
   a. $f(x) = |x|$ b. $f(x) = x^3$
   c. $f(x) = x^4$ d. $f(x) = x^4$, $x \leq 0$
9. On the dial or the buttons of a telephone, a telephone function T pairs letters of the alphabet with the digits 2–9. For example, $T(\text{A}) = 2$ and $T(\text{D}) = 3$. Does T have an inverse? Explain.
10. Explain how the vertical-line test (given on page 121) can be used to justify the horizontal-line test (given on page 148).
11. Explain why the domain of a one-to-one function f is the range of f^{-1} and why the domain of f^{-1} is the range of f.

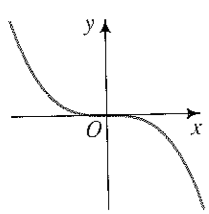
WRITTEN EXERCISES

A

1. Suppose a function f has an inverse. If $f(2) = 6$ and $f(3) = 7$, find:
   a. $f^{-1}(6)$ b. $f^{-1}(f(3))$ c. $f(f^{-1}(7))$
2. Suppose a function f has an inverse. If $f(0) = -1$ and $f(-1) = 2$, find:
   a. $f^{-1}(-1)$ b. $f^{-1}(f(0))$ c. $f(f^{-1}(2))$
3. If $g(3) = 5$ and $g(-1) = 5$, explain why g has no inverse.
4. Explain why $f(x) = x^3 + x^2$ has no inverse.
5. Let $h(x) = 4x - 3$.
   a. Sketch the graphs of h and h^{-1}. b. Find a rule for $h^{-1}(x)$.
6. Let $L(x) = \frac{1}{2}x - 4$.
   a. Sketch the graphs of L and L^{-1}. b. Find a rule for $L^{-1}(x)$.

In Exercises 7–10, the graph of a function is given. State whether the function has an inverse.

7.



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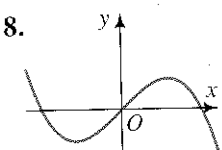
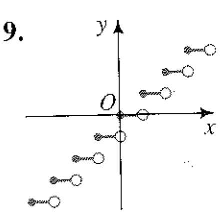
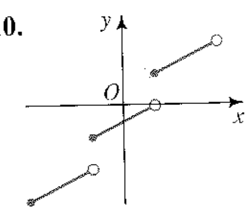


9.



10.



State whether the function f has an inverse. If f^{-1} exists, find a rule for $f^{-1}(x)$ and show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

11. $f(x) = 3x - 5$ **12.** $f(x) = |x| - 2$ **13.** $f(x) = \sqrt[4]{x}$

14. $f(x) = \frac{1}{x}$ **15.** $f(x) = \frac{1}{x^2}$ **16.** $f(x) = \sqrt{5 - x}$

17. $f(x) = \sqrt{4 - x^2}$ **18.** $f(x) = \sqrt{5 - x^2}$ **19.** $f(x) = \sqrt[3]{1 + x^3}$

Sketch the graphs of g and g^{-1}. Then find a rule for $g^{-1}(x)$.

B **20.** $g(x) = x^2 + 2, x \geq 0$ **21.** $g(x) = 9 - x^2, x \leq 0$

22. $g(x) = (x - 1)^2 + 1, x \leq 1$ **23.** $g(x) = (x - 4)^2 - 1, x \geq 4$

In Exercises 24–26, show that $h^{-1}(x) = h(x)$. Then sketch the graph of h. You may wish to use a computer or graphing calculator.

24. $h(x) = \sqrt[3]{1 - x^3}$ **25.** $h(x) = \frac{x}{x - 1}$ **26.** $h(x) = \sqrt{1 - x^2}, x \geq 0$

27. a. Using the results of Exercises 24–26, state how the graph of h is related to the line $y = x$ when $h^{-1}(x) = h(x)$.

b. Find a function h, different from those in Exercises 24–26, such that $h^{-1}(x) = h(x)$.

28. Refer to the definition of an increasing function given on page 138.

a. Explain why an increasing function must have an inverse.

b. Suppose f is an increasing function. Is f^{-1} also an increasing function? Explain your answer and support it with at least two examples.

C **29.** Which statement below is true? Prove it.

(i) $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$ (ii) $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

30. If f is a linear function such that $f(x + 2) - f(x) = 6$, find the value of $f^{-1}(x + 2) - f^{-1}(x)$.

31. Suppose a, b, and c are constants such that $a \neq 0$. Let $P(x) = ax^2 + bx + c$ for $x \leq -\frac{b}{2a}$. Find a rule for $P^{-1}(x)$.

CALCULATOR EXERCISES

In the next chapter we will define the functions $f(x) = e^x$ and $g(x) = \ln x$. You can use a calculator to learn something about these functions.

1. Enter any number. Press the $\boxed{e^x}$ and $\boxed{\ln x}$ keys alternately several times. What do you notice? Repeat this process for several other numbers. How would you describe the relationship between $f(x) = e^x$ and $g(x) = \ln x$?
2. By entering various numbers, determine whether $f(x) = e^x$ is defined for all real numbers.
3. By experimenting, determine the domain of $g(x) = \ln x$.

Applications of Functions

4-6 Functions of Two Variables

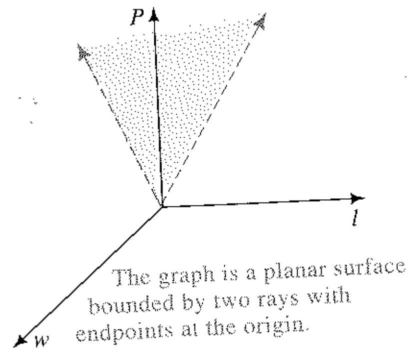
Objective *To graph functions of two variables in a two-dimensional coordinate system and to read such graphs.*

The formula for the perimeter of a rectangle, $P = 2l + 2w$, tells us that the perimeter P depends on both the length l and the width w of the rectangle. In other words, P is a function of two variables, l and w, and we write

$$P(l, w) = 2l + 2w$$

to emphasize the functional relationship.

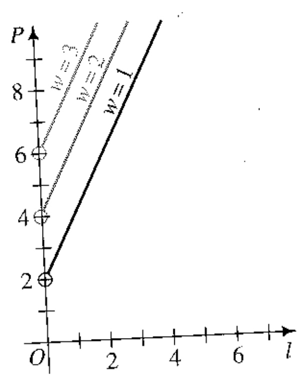
Graphing a function of two variables requires a three-dimensional coordinate system as shown at the right. Notice that there are three axes, each perpendicular to the other two. We will study such coordinate systems in Chapter 12, but for now let us consider two ways to describe P in a two-dimensional coordinate system.



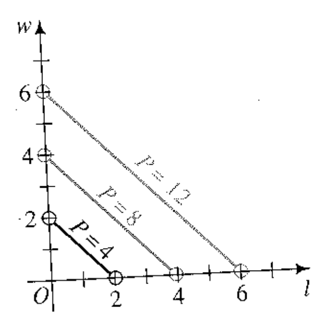
Graph of $P(l, w) = 2l + 2w$ (where $l > 0$ and $w > 0$) in a 3-dimensional coordinate system

One way is to draw curves for various constant values of either w or l. For example, by using constant values of w, we get such equations as $P(l, 1) = 2l + 2$, $P(l, 2) = 2l + 4$, and $P(l, 3) = 2l + 6$. The graphs of these equations, called curves of constant width, are shown in the lP-plane at the left below.

Another way to describe P in a two-dimensional plane is to draw curves along which the value of the perimeter function is constant. For example, $P(l, w) = 8$ when $(l, w) = (1, 3)$, $(2, 2)$, and $(3, 1)$. These points and others for which $P(l, w) = 8$ are shown on the red curve in the lw-plane at the right below. Other curves of constant perimeter are also shown.



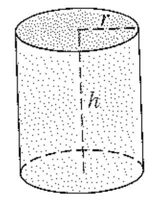
Curves of constant width



Curves of constant perimeter

Example

The volume of a cylinder is given by the formula $V = \pi r^2 h$.



a. Sketch the graphs of V versus r for $h = 1$, $h = 2$, and $h = 3$. (These graphs are called curves of constant height.)

b. Sketch the graphs of h versus r for $V = 36\pi$ and $V = 72\pi$. (These graphs are called curves of constant volume.)

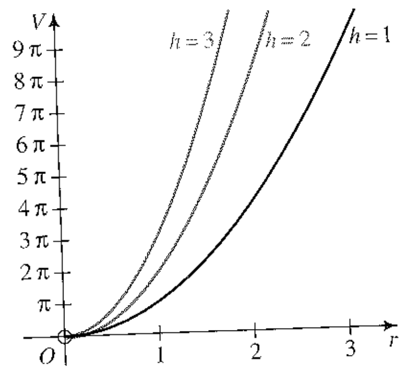
c. If h is held constant, what happens to V when r is doubled?

Solution

a. Since $V(r, h) = \pi r^2 h$, the three curves to be graphed have equations:

$$V(r, 1) = \pi r^2,$$
$$V(r, 2) = 2\pi r^2,$$
and
$$V(r, 3) = 3\pi r^2.$$

With both r and V being positive quantities, we use only the first quadrant when graphing the equations. Each curve is half of a parabola, as shown.



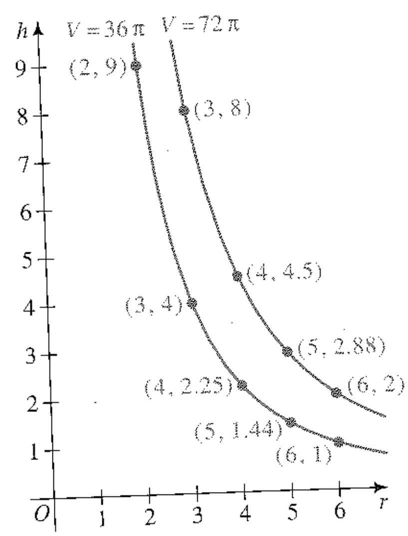
b. If $V = 36\pi$, we substitute $\pi r^2 h$ for V and solve for h:

$$\pi r^2 h = 36\pi$$
$$h = \frac{36}{r^2}$$

We use the last equation to make a table of values:

r	1	2	3	4	5	6
$h = \frac{36}{r^2}$	36	9	4	2.25	1.44	1

We then plot the pairs (r, h) to obtain the curve for $V = 36\pi$. In the same way, we sketch the curve for $V = 72\pi$. Both curves are shown at the right.

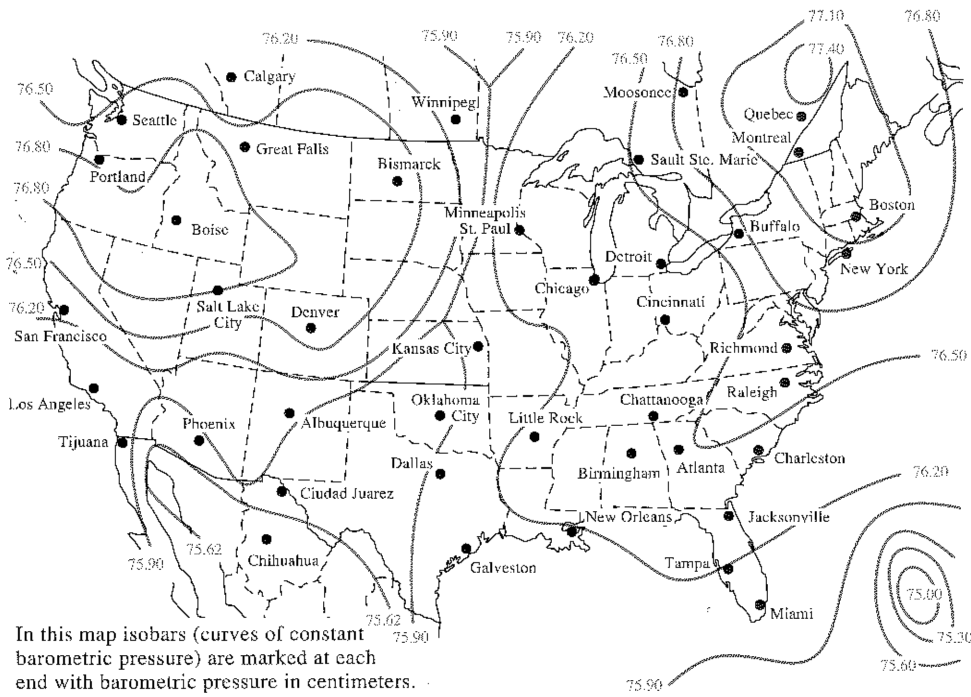


c. Holding h constant and doubling r, we have:

$$V(2r, h) = \pi(2r)^2 h = 4 \cdot \pi r^2 h = 4 \cdot V(r, h)$$

Thus, when h is held constant and r is doubled, V is quadrupled.

Real-world functions of more than one variable are often graphed in two dimensions. For example, meteorologists produce maps like the one below that show curves of constant barometric pressure.



In this map isobars (curves of constant barometric pressure) are marked at each end with barometric pressure in centimeters.

CLASS EXERCISES

Each of the following formulas from mathematics and science expresses some quantity as a function of two variables. Describe this functional relationship in words.

1. $A = lw$

2. $d = rt$

3. $V = \frac{1}{3}\pi r^2 h$

4. $F = ma$

5. $D = \frac{m}{V}$

6. $A = 2\pi rh + 2\pi r^2$

7. ***Visual Thinking*** Given the formula *density* = *mass* ÷ *volume* (see Exercise 5), what do curves of constant density look like in an mV-plane?

8. Refer to the barometric pressure graph shown above.

a. Name two other cities with approximately the same barometric pressure as New Orleans.

b. Bad weather usually accompanies low pressure. In which cities is the weather apt to be poorest?

c. Good weather usually accompanies high pressure. In which cities is the weather apt to be best?

9. ***Discussion*** The amount that a person pays for auto insurance is a function of many variables. Discuss what some of these variables are.

10. **a.** If $f(x, y) = \sqrt{x^2 + y^2}$, find $f(3, 4)$, $f(-4, 3)$, and $f(0, 5)$.
    b. Sketch some constant curves of the function f.

11. Name several pairs (a, b) not in the domain of $f(a, b) = \frac{a + b}{a - b}$.

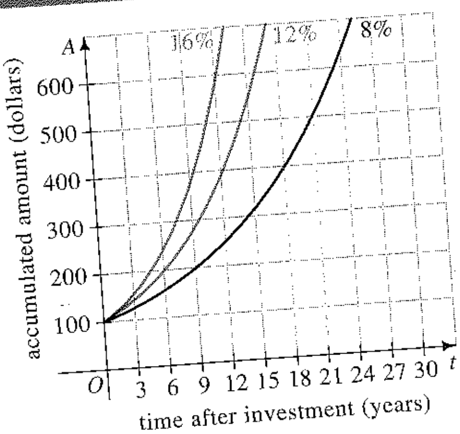
WRITTEN EXERCISES

A 1. **Consumer Economics** If \$100 is invested at interest rate r compounded annually, then the accumulated amount t years later is $A = 100(1 + r)^t$.

a. The formula shows that A is a function of __?__ and __?__.

b. The graphs of A versus t for several constant values of r are shown at the right. The equation of the rightmost curve is $A = 100(1.08)^t$. What are the equations of the other two curves?

c. About how many years does it take to double your money at 8% interest compounded annually? at 12% interest? at 16% interest?



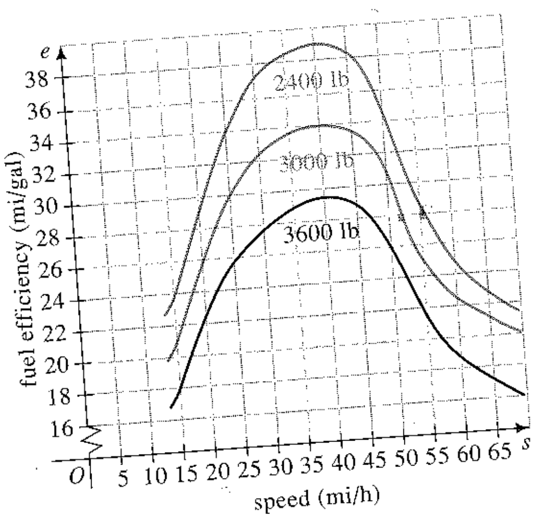
2. **Consumer Economics** The graph at the right below shows the fuel efficiency at various speeds for 2400 lb, 3000 lb, and 3600 lb cars. The fuel efficiency is given in miles per gallon (mi/gal).

a. Fuel efficiency is a function of __?__ and __?__.

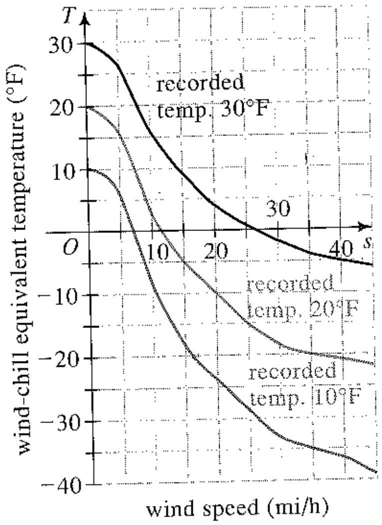
b. At what other speed will a 3000 lb car have approximately the same fuel efficiency as it has when traveling at 25 mi/h?

c. At what speeds does a 3600 lb car have approximately the same fuel efficiency as a 2400 lb car traveling at 55 mi/h?

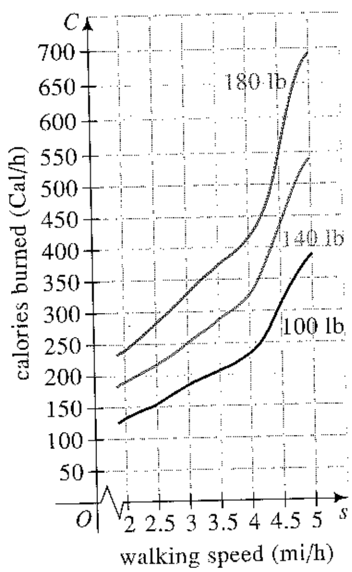
d. Regardless of what a car weighs, at approximately what speed is maximum fuel efficiency reached?



3. **Meteorology** The graph shows the wind-chill equivalent temperatures for recorded temperatures of 30°F, 20°F, and 10°F.
   a. Describe the functional relationship that the graph depicts.
   b. If the recorded temperature is 10°F, approximately what wind speed will produce a wind-chill equivalent temperature of -20°F?
   c. Would you rather be in a place where the recorded temperature is 30°F with a 30 mi/h wind or 20°F with a 10 mi/h wind? Why?
   d. Regardless of the recorded temperature, is there a greater change in the wind-chill equivalent temperature when the wind speed is between 5 mi/h and 10 mi/h or when it is between 35 mi/h and 40 mi/h?



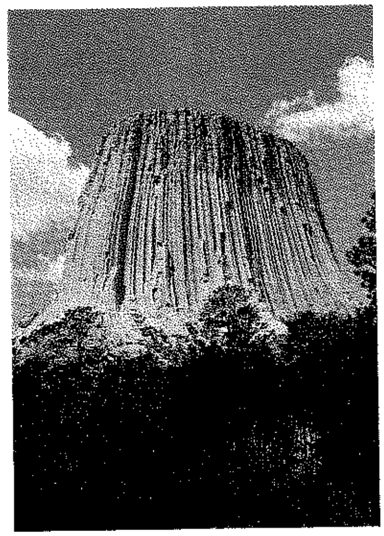
4. **Physiology** The graph shows the number of calories burned each hour when 100 lb, 140 lb, and 180 lb people walk at various speeds.
   a. Describe the functional relationship that the graph depicts.
   b. At approximately what speed must a 140 lb person walk in order to burn 450 Cal/h?
   c. At approximately what speed must a 100 lb person walk in order to burn as many calories per hour as a 180 lb person walking at 2 mi/h?
   d. Regardless of what a person weighs, is there a greater change in the number of calories burned per hour when the person increases his or her walking speed from 3 mi/h to 3.5 mi/h or from 4 mi/h to 4.5 mi/h?



5. a. If $A(b, h) = \frac{1}{2}bh$, find $A(8, 3)$ and $A(16, 6)$.
   b. Give a geometric interpretation of the function A.
   c. Give several pairs (b, h) for which $A(b, h) = 5$.

6. a. If $V(r, h) = \frac{1}{3}\pi r^2 h$, find $V(2, 6)$ and $V(4, 12)$.
   b. Give a geometric interpretation of the function V.
   c. Give several pairs (r, h) for which $V(r, h) = 75\pi$.

7. a. If $A(b, h) = \frac{1}{2}bh$, show that $A(3b, 3h) = 9 \cdot A(b, h)$.
   b. Give a verbal description of the equation $A(3b, 3h) = 9 \cdot A(b, h)$.
8. a. If $V(r, h) = \pi r^2 h$, show that $V(2r, 2h) = 8 \cdot V(r, h)$.
   b. Give a verbal description of the equation $V(2r, 2h) = 8 \cdot V(r, h)$.
9. a. If you travel at a constant rate r for t hours, the distance d that you travel is a function of r and t. Give a rule for this function.
   b. Sketch the constant distance curves $d(r, t) = 200$ and $d(r, t) = 400$ in an rt-plane.
10. a. The volume V of a square prism is a function of the length s of a side of the square base and of the height h of the prism. Give a rule for this function.
    b. Sketch two curves of constant volume in an sh-plane.



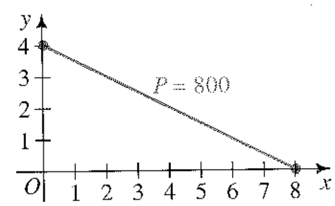
B 11. ***Research*** Look up "contour map" in a dictionary or encyclopedia. Draw a contour map of the butte shown at the right.

12. ***Research*** Most daily newspapers show a weather map on which curves of constant temperature are drawn. Obtain such a map and from it read the approximate forecast temperatures in Los Angeles, Seattle, Chicago, Dallas, New York City, and Miami.

13. Describe the domain of $f(x, y) = \dfrac{xy}{x^2 - y^2}$.

14. Describe the domain of $g(x, y) = \dfrac{4 + y^2}{4 - x^2}$.

15. **Retailing** The profit in selling x black-and-white television sets and y color television sets is $P(x, y) = 100x + 200y$ dollars, where $x \geq 0$ and $y \geq 0$. All points (x, y) on the line segment labeled $P = 800$ will give a profit of \$800. This line is a *constant profit line*. Copy this graph and draw the constant profit lines $P(x, y) = 1200$ and $P(x, y) = 1600$.



16. **Manufacturing** The cost of producing x wooden tennis rackets and y graphite tennis rackets is $C(x, y) = 24x + 60y$. Draw several lines of constant cost in an xy-plane.
17. a. If $A(l, w, h) = 2lw + 2wh + 2lh$, find $A(4, 3, 5)$ and $A(6, 4, 7)$.
    b. Give a geometric interpretation of the function A.
18. a. If $A(b_1, b_2, h) = \frac{1}{2}(b_1 + b_2)h$, find $A(3, 4, 2)$ and $A(5, 7, 3)$.
    b. Give a geometric interpretation of the function A.

For Exercises 19 and 20, refer to textbooks in the given fields to find several examples of functional relationships involving three or more variables. Be prepared to discuss these relationships in class.

19. ***Research*** Physics, chemistry, or biology

20. ***Research*** Medicine, psychology, or economics

C **21.** **Sports** For some international sailing competitions, the rating R of a yacht is a function of several variables:

$$R(A, L, V) = 0.9\left(\frac{L\sqrt{A}}{12\sqrt[3]{V}} + \frac{L + \sqrt{A}}{4}\right)$$

where A = surface area of the sails in square meters, L = length of the yacht in meters, and V = volume of water yacht displaces in cubic meters. To be in the R-5.5 class, a yacht must have a rating less than 5.5. Does a yacht that has 37 m^2 of sail, is 10 m long, and displaces 8.5 m^3 of water qualify?

4-7 Forming Functions from Verbal Descriptions

Objective *To form a function of one variable from a verbal description and, when appropriate, to determine the minimum or maximum value of the function.*

An important concern of mathematics is finding the minimum or maximum value of a function. You have already seen quadratic and cubic examples that involve minimizing costs and maximizing profits. Other such applications use mathematics to minimize the structural stress on a girder or to maximize the volume of a container made from a given amount of material.

Minimum and maximum values are often referred to as *extreme values*. Approximate extreme values of a function can be found using a computer or graphing calculator. Exact extreme values are most often found using calculus.

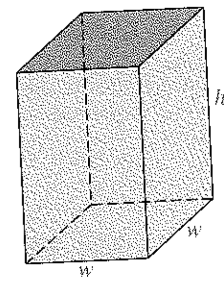
Whether technology or calculus is used, we almost always need to write a rule for the function to be minimized or maximized. If the rule depends on two or more variables, then we also need to find a relationship among the variables so that the function can be written in terms of only one variable. Developing these skills is the goal of this section.

Example 1 An open-top box with a square base is to be constructed from sheet metal in such a way that the completed box is made of 2 m^2 of sheet metal. Express the volume of the box as a function of the base width.

Solution

1. Sketch the box. Let h be the height of the box, and let w be the width. The volume of the box is a function of h and w:

$$V(w, h) = w^2h$$



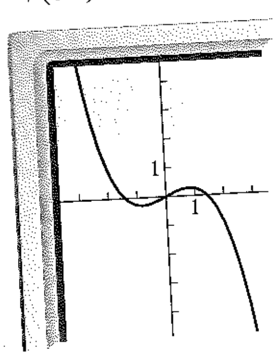
2. The equation above gives V in terms of w and h. To get V in terms of w alone, we must replace h in the above equation with a function of w. We use the fact that the box is made of 2 m^2 of sheet metal.

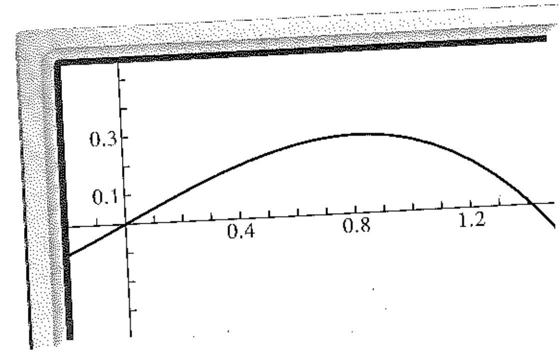
$$\begin{aligned} \text{Area of sheeting used} &= 2 \\ \text{Area of the base} + 4 \cdot (\text{Area of a side}) &= 2 \\ w^2 \quad + \quad 4wh &= 2 \\ h &= \frac{2 - w^2}{4w} \end{aligned}$$

3. To find V in terms of w alone, substitute $\frac{2 - w^2}{4w}$ for h.

$$V(w, h) = w^2h$$
$$V(w) = w^2\left(\frac{2 - w^2}{4w}\right) = \frac{2w - w^3}{4}$$

In Example 1, the volume of the box was expressed as a function of its width: $V(w) = \frac{2w - w^3}{4}$. To find the maximum volume, you can use a computer or graphing calculator to obtain the graph at the left below. Notice that this graph includes some values of w not in the domain of the function. By zooming in on the high point of the graph in the first quadrant, you can obtain the "blowup" shown at the right below. This shows that the maximum volume is approximately $V(0.8) = 0.3$ m^3.

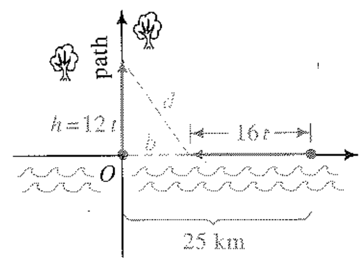




Example 2 A north-south bridle path intersects an east-west river at point O. At noon, a horse and rider leave O traveling north at 12 km/h. At the same time, a boat is 25 km east of O traveling west at 16 km/h. Express the distance d between the horse and the boat as a function of the time t in hours after noon.

Solution

1. Make a sketch showing the horse and boat at some time t. Let h be the horse's distance from O, and let b be the boat's distance east of O. Since the horse is traveling from O at 12 km/h,



$$h = 12t.$$

Since the boat is 25 km from O and traveling toward O at 16 km/h,

$$b = 25 - 16t.$$

2. By the Pythagorean theorem,

$$d(h, b) = \sqrt{h^2 + b^2}.$$

3. To find d in terms of t, substitute $12t$ for h and $25 - 16t$ for b.

$$\begin{aligned} d(t) &= \sqrt{(12t)^2 + (25 - 16t)^2} \\ &= \sqrt{144t^2 + 625 - 800t + 256t^2} \\ &= \sqrt{400t^2 - 800t + 625} \end{aligned}$$

In Example 2, it is possible to determine when the horse and boat are closest to each other (that is, when the distance between them is a minimum) and what the minimum distance is. Since $d(t)$ is the square root of the function

$$f(t) = 400t^2 - 800t + 625,$$

$d(t)$ will be a minimum when $f(t)$ is a minimum. Since the minimum or maximum value of $f(x) = ax^2 + bx + c$ occurs at $x = -\frac{b}{2a}$, the minimum value of $f(t) = 400t^2 - 800t + 625$ occurs at:

$$t = -\frac{-800}{2(400)} = 1$$

Therefore, the horse and boat are closest 1 h after noon. The minimum distance between them is $d(1)$.

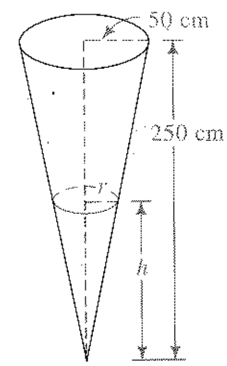
$$\begin{aligned} d(1) &= \sqrt{400 \cdot 1^2 - 800 \cdot 1 + 625} \\ &= \sqrt{225} = 15 \end{aligned}$$

Thus, the minimum distance between the horse and boat is 15 km.

Example 3 Water flows into a conical tank 100 cm wide and 250 cm deep at a rate of 40 cm^3/s. Find the volume V of the water in the tank as a function of the height h of the water. Represent h as a function of the time t that the water has been flowing into the empty tank.

Solution

1. Make a sketch. Let r be the radius of the water surface, and let h be the height of the water in the tank. The volume of water in the tank is a function of r and h:



$$V(r, h) = \frac{1}{3}\pi r^2 h$$

2. Find an expression for r in terms of h by using the similar triangles from a cross section of the tank.

$$\frac{r}{h} = \frac{50}{250}$$
$$r = \frac{h}{5}$$

3. To find V in terms of h alone, substitute $\frac{h}{5}$ for r.

$$V(r, h) = \frac{1}{3}\pi r^2 h$$
$$V(h) = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h = \frac{\pi}{75}h^3$$

4. To represent h as a function of the time t, note that the volume of water in the tank at t seconds is $V = 40t$. Therefore, substitute $40t$ for V and solve for h:

$$40t = \frac{\pi}{75}h^3$$
$$h = \sqrt[3]{\frac{3000t}{\pi}}$$

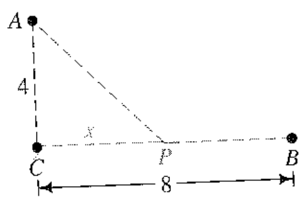
Thus, $h(t) = \sqrt[3]{\frac{3000t}{\pi}}$.

CLASS EXERCISES

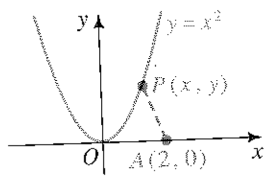
1. A square has a side of length s and a diagonal of length d, as shown.

   a. Express d as a function of s.
   b. Express s as a function of d.
   c. Express the area A of the square as a function of d.
2. a. Express the volume V of a cube as a function of the length e of an edge.
   b. The length of a diagonal of the cube is $d = e\sqrt{3}$. Express V as a function of d.

3. In the figure at the left below, point A is 4 km north of point C, point B is 8 km east of C, and P is a point on $\overline{BC}$ at a distance of x km from C.
   a. Express $AP + PB$ as a function of x.
   b. What is the domain of this function?



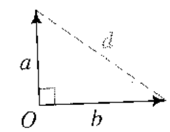
Ex. 3



Ex. 4

4. $P(x, y)$ is a point on the parabola $y = x^2$, as shown at the right above.
   a. Express the distance from P to $A(2, 0)$ as a function of x and y.
   b. Express the distance as a function of x alone.
   c. How can you find the minimum value of the function in part (b)?

5. A runner starts north from point O at 6 m/s. At the same time, a second runner sprints east from O at 8 m/s. (See the figure at the right.) Find the distance d between the runners t seconds later.



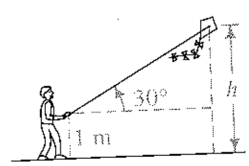
WRITTEN EXERCISES

A 1. Express the area A of a 30°-60°-90° triangle as a function of the length h of the hypotenuse.

2. Express the area A of an equilateral triangle as a function of the perimeter P.

3. A tourist walks n km at 4 km/h and then travels $2n$ km at 36 km/h by bus. Express the total traveling time t (in hours) as a function of n.

4. A student holds a ball of string attached to a kite, as shown at the right. The string is held 1 m above the ground and rises at a 30° angle to the horizontal. If the student lets the string out at a rate of 2 m/s, express the kite's height h (in meters) as a function of the time t (in seconds) after the kite begins to fly.



5. A store owner bought n dozen toy boats at a cost of \$3.00 per dozen, and sold them at \$.75 apiece. Express the profit P (in dollars) as a function of n.

6. The cost of renting a large boat is 30 dollars per hour plus a usage fee roughly equivalent to x^3 cents per hour when the boat is operated at a speed of x km/h. Express the cost C (in cents per kilometer) as a function of x.

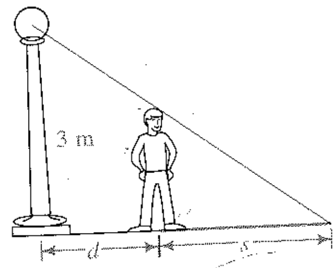
7. The height of a cylinder is twice the diameter. Express the total surface area A as a function of the height h.

8. A pile of sand is in the shape of a cone with a diameter that is twice the height. Express the volume V of sand as a function of the height h.

$d=2r$ $2r=2h \to r=h$

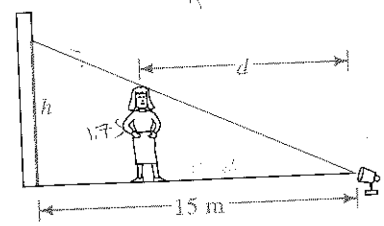
$V=\frac{1}{3}\pi r^2 h$

9. A light 3 m above the ground causes a boy 1.8 m tall to cast a shadow s meters long measured along the ground, as shown at the right. Express s as a function of d, the boy's distance in meters from the light.



Ex. 9

10. When a girl 1.75 m tall stands between a wall and a light on the ground 15 m away, she casts a shadow h meters high on the wall, as shown at the right. Express h as a function of d, the girl's distance in meters from the light.



Ex. 10

11. A box with a square base has a surface area (including the top) of 3 m^2. Express the volume V of the box as a function of the width w of the base.

12. A box with a square base and no top has a volume of 6 m^3. Express the total surface area A of the box as a function of the width w of the base.

13. A stone is thrown into a lake, and t seconds after the splash the diameter of the circle of ripples is t meters.
  a. Express the circumference C of this circle as a function of t.
  b. Express the area A of this circle as a function of t.

14. A balloon is inflated in such a way that its volume increases at a rate of 20 cm^3/s.
  a. If the volume of the balloon was 100 cm^3 when the process of inflation began, what will the volume be after t seconds of inflation?
  b. Assuming that the balloon is spherical while it is being inflated, express the radius r of the balloon as a function of t.

Part (b) of Exercises 15 and 16 requires the use of a computer or graphing calculator.

B 15. **Manufacturing** A box with a square base and no top has volume 8 m^3. The material for the base costs \$8 per square meter, and the material for the sides costs \$6 per square meter.
  a. Express the cost C of the materials used to make the box as a function of the width w of the base.
  b. Use a computer or graphing calculator to find the minimum cost.

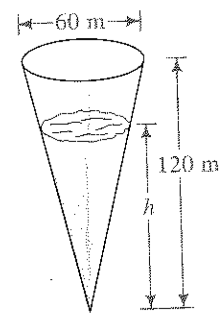
16. **Manufacturing** A cylindrical can has a volume of 400π cm^3. The material for the top and bottom costs 2¢ per square centimeter. The material for the vertical surface costs 1¢ per square centimeter.
  a. Express the cost C of the materials used to make the can as a function of the radius r.
  b. Use a computer or graphing calculator to find the minimum cost.

400π cm^3

17. At 2:00 P.M. bike A is 4 km north of point C and traveling south at 16 km/h. At the same time, bike B is 2 km east of C and traveling east at 12 km/h.
    a. Show that t hours after 2:00 P.M. the distance between the bikes is:
$$\sqrt{400t^2 - 80t + 20}$$
    b. At what time is the distance between the bikes the least?
    c. What is the distance between the bikes when they are closest?

26.25 = hd

18. A car leaves Oak Corners at 11:33 A.M. traveling south at 70 km/h. At the same time, another car is 65 km west of Oak Corners traveling east at 90 km/h.
    a. Express the distance d between the cars as a function of the time t after the first car left Oak Corners.
    b. Show that the cars are closest to each other at noon.

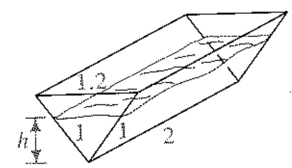


Ex. 19

$\left(\frac{26.25}{h}\right) = d$

19. **Engineering** Water is flowing at a rate of 5 m^3/s into the conical tank shown at the right.
    a. Find the volume V of the water as a function of the water level h.
    b. Find h as a function of the time t during which water has been flowing into the tank.

20. **Engineering** A trough is 2 m long, and its ends are triangles with sides of length 1 m, 1 m, and 1.2 m as shown at the right.
    a. Find the volume V of the water in the trough as a function of the water level h.
    b. If water is pumped into the empty trough at the rate of 6 L/min, find the water level h as a function of the time t after the pumping begins. ($1 \text{ m}^3 = 1000$ L)

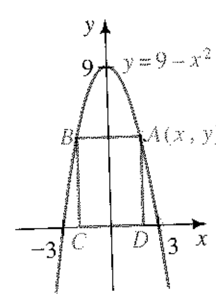


Ex. 20

21. $P(x, y)$ is an arbitrary point on the line $2x + y = 10$.
    a. Express the distance d from the origin to P as a function of the x-coordinate of P.
    b. What are the domain and range of this function?

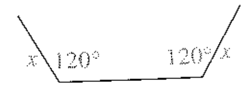
22. $P(x, y)$ is an arbitrary point on the parabola $y = x^2$.
    a. Express the distance d from P to the point $A(0, 1)$ as a function of the y-coordinate of P.
    b. What is the minimum distance d?

23. As shown at the right, rectangle $ABCD$ has vertices C and D on the x-axis and vertices A and B on the part of the parabola $y = 9 - x^2$ that is above the x-axis.
    a. Express the perimeter P of the rectangle as a function of the x-coordinate of A.
    b. What is the domain of the perimeter function?
    c. For what value of x is the perimeter a maximum?



Ex. 23

24. **Manufacturing** A sheet of metal is 60 cm wide and 10 m long. It is bent along its width to form a gutter with a cross section that is an isosceles trapezoid with 120° angles, as shown at the right.



Cross section of gutter

a. Express the volume V of the gutter as a function of x, the length in centimeters of one of the equal sides. (*Hint*: Volume = area of trapezoid × length of gutter)

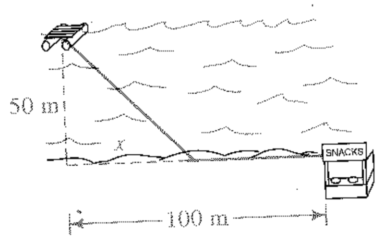
b. For what value of x is the volume of the gutter a maximum?

Part (b) of Exercises 25–29 requires the use of a computer or graphing calculator.

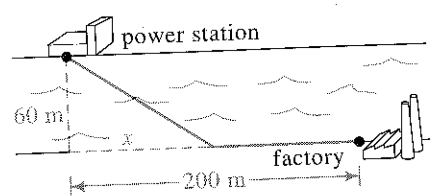
25. From a raft 50 m offshore, a lifeguard wants to swim to shore and run to a snack bar 100 m down the beach, as shown at the left below.

a. If the lifeguard swims at 1 m/s and runs at 3 m/s, express the total swimming and running time t as a function of the distance x shown in the diagram.

b. Use a computer or graphing calculator to find the minimum time.



Ex. 25



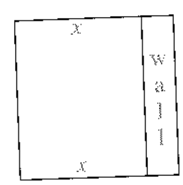
Ex. 26

26. **Engineering** A power station and a factory are on opposite sides of a river 60 m wide, as shown at the right above. A cable must be run from the power station to the factory. It costs \$25 per meter to run the cable in the river and \$20 per meter on land.

a. Express the total cost C as a function of x, the distance downstream from the power station to the point where the cable touches the land.

b. Use a computer or graphing calculator to find the minimum cost.

27. **Landscaping** A rectangular area of 60 m^2 has a wall as one of its sides, as shown. The sides perpendicular to the wall are made of fencing that costs \$6 per meter. The side parallel to the wall is made of decorative fencing that costs \$8 per meter.



a. Express the total cost C of the fencing as a function of the length x of a side perpendicular to the wall.

b. Use a computer or graphing calculator to find the minimum cost.

C **28.** A cylinder is inscribed in a sphere of radius 1.

a. Express the volume V of the cylinder as a function of the base radius r.

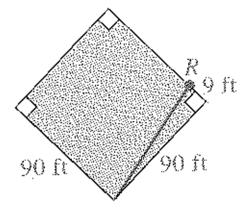
b. Use a computer or graphing calculator to find the radius and height of the cylinder having maximum volume.

29. A cone circumscribes a sphere with radius 1.

a. Express the volume V of the cone as a function of the base radius r.

b. Use a computer or graphing calculator to find the radius and height of the cone having minimum volume.

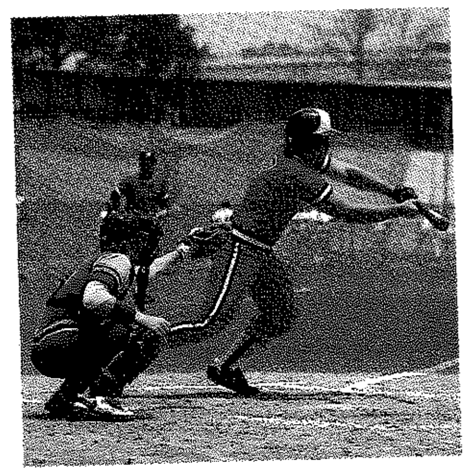
30. **Sports** A baseball diamond is a square with 90 ft sides. A runner (R in the diagram at the right) has taken a 9 ft lead from first base. At the moment the ball is pitched, the runner runs toward second base at 27 ft/s.



a. Express the runner's straight-line distance d from home plate as a function of the time t after the ball is thrown.

b. What are the domain and range of this function?

31. **Sports** A baseball player hits a ball into the farthest corner of the outfield and tries for an inside-the-park home run; that is, the player tries to run the bases and make it to home plate safely. Suppose that the player runs at 30 ft/s and stays strictly on the base lines, as shown in Exercise 30.



a. Express the player's straight-line distance d from home plate as a function of the time t after the ball was hit. (*Hint*: The function involves different rules for different intervals of time.)

b. Sketch the graph of the function in part (a).

Chapter Summary

1. A *function* consists of a set of real numbers, called the *domain* of the function, and a rule that assigns to each element in the domain exactly one real number. The set of real numbers assigned by the rule is called the *range* of the function. The *graph* of a function is the set of points corresponding to the ordered pairs that satisfy the functional rule.

2. Functions can be *added*, *subtracted*, *multiplied*, or *divided*:

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ provided } g(x) \neq 0$$

3. The *composite* of f and g, denoted $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$ such that x is in the domain of g and $g(x)$ is in the domain of f.

4. Simple changes in an equation of a graph can cause the graph to be reflected, translated, stretched, or shrunk. The chart on page 142 summarizes these changes.

5. Methods for testing an equation or its graph for symmetry in the x-axis, in the y-axis, in the origin, or in the line $y = x$ are summarized on page 134.
6. Properties of functions can be used to help graph and analyze functions:

   f is an *even* function if $f(-x) = f(x)$ for all x in the domain of f; the graph of an even function has symmetry in the y-axis.

   f is an *odd* function if $f(-x) = -f(x)$ for all x in the domain of f; the graph of an odd function has symmetry in the origin.
7. A function $y = f(x)$ is *periodic* if there is a positive number p, called a *period* of f, such that $f(x + p) = f(x)$ for all x in the domain of f. The smallest period of f is called the *fundamental period*. If f has maximum value M and minimum value m, then the *amplitude* of f is defined to be $\frac{M - m}{2}$.
8. If f is periodic with period p and amplitude A, then $y = cf(x)$ has period p and amplitude cA, and $y = f(cx)$ has period $\frac{p}{c}$ and amplitude A.
9. If $f(g(x)) = x$ for all x in the domain of g and $g(f(x)) = x$ for all x in the domain of f, then f and g are *inverses*. The inverse of f is denoted f^{-1} and its graph is the reflection of the graph of f in the line $y = x$. f^{-1} exists if f is *one-to-one*, that is, every horizontal line intersects the graph of f in at most one point.
10. In many situations, a problem is modeled by a function of more than one variable. By expressing one variable in terms of the other, a function of two variables can often be transformed into a function of one variable.

Key vocabulary and ideas

function, domain, range (p. 119)
relation (p. 121)
vertical-line test (p. 121)
sum and difference of f and g (p. 125)
product and quotient of f and g (p. 125)
composite function (p. 126)
reflection (pp. 131–133)
axis of symmetry (p. 133)
point of symmetry (p. 133)
periodic function, period (p. 138)
amplitude (p. 139)
vertical stretch or shrink (p. 140)
horizontal stretch or shrink (p. 140)
translation (p. 141)
inverse functions (p. 146)
one-to-one function (p. 148)
horizontal-line test (p. 148)
function of two variables (p. 151)

Chapter Test

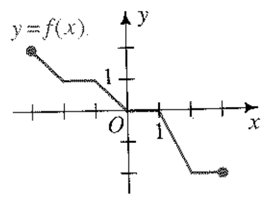
1. Give the domain, range, and zeros of the function $f(x) = \sqrt{9 - x^2}$. 4-1
2. Graph $g(x) = \begin{cases} x + 1 & \text{if } x < -1 \\ 1 - x^2 & \text{if } x \geq -1 \end{cases}$. Find the range and zeros of g.
3. Let $f(x) = x^2 + 2x$ and $g(x) = x + 2$. Find: 4-2

   a. $(f + g)(x)$ **b.** $(f - g)(x)$ **c.** $(f \cdot g)(x)$ **d.** $\left(\frac{f}{g}\right)(x)$

4. Using the functions f and g in Exercise 3, find:
   a. $(f \circ g)(x)$
   b. $(g \circ f)(x)$

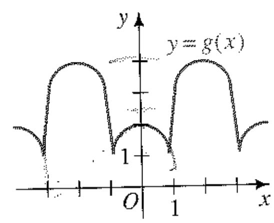
5. Determine whether the graph of $x^2 - xy = 4$ has symmetry in: (i) the x-axis, (ii) the y-axis, (iii) the line $y = x$, and (iv) the origin. 4-3

6. Given the graph of $y = f(x)$ shown at the right, sketch the graph of each of the following. 4-4
   a. $y = 2f(x)$
   b. $y = |f(x)|$
   c. $y = -f(x)$
   d. $y = f(x + 2)$



Ex. 6

7. The graph of a periodic function $y = g(x)$ is shown at the right.
   a. What is the fundamental period of g?
   b. What are the maximum and minimum values of g?
   c. What is the amplitude of g?



Ex. 7

8. ***Writing*** Describe and illustrate what happens when the graph of a periodic function with period p is horizontally translated p units. Then use the definition of a periodic function to explain why this happens.

9. a. Which one of the two functions $f(x) = \sqrt{3 + x^2}$ and $g(x) = 3 + x$ has an inverse? Find a rule for the inverse. 4-5
   b. Explain why the other function does not have an inverse.

10. The area A of a triangle is a function of the base b and height h. 4-6
    a. Express A as a function of b and h.
    b. Find $A(3, 4)$ and $A(6, 5)$.
    c. Draw the curve of constant area $A(b, h) = 3$ in a bh-plane.

11. A cylindrical tank 4 ft in diameter fills with water at the rate of $10 \text{ ft}^3/\text{s}$. Express the depth of the water in the tank as a function of the time t in seconds. Assume the tank is empty at time $t = 0$. 4-7

Part (c) of Exercise 12 requires the use of a computer or graphing calculator.

12. As shown at the right, triangle OAB is an isosceles triangle with vertex O at the origin and vertices A and B on the part of the parabola $y = 9 - x^2$ that is above the x-axis.
    a. Express the area of the triangle as a function of the x-coordinate of A.
    b. What is the domain of the area function?
    c. Use a computer or graphing calculator to find the maximum area.