For $k, l, m \in \mathbb{R}^d$, A is a $3d$-dimensional tensor with

$$A(k,l,m) = \begin{cases} \beta(l,m) - \beta(m,m), & l+m=k, \\ 0, & \text{otherwise.} \end{cases}$$

There are two choices for β under consideration, here marked with superscripts M and P where appropriate. They are

$$\beta_d^P(l,m) = \int_{\mathcal{B}_{2R}} \int_{S^{d-1}} B(\|g\|, \cos\theta) \exp\left[-ig \cdot \frac{l+m}{2} - i\|g\|\sigma \cdot \frac{m-l}{2}\right] \mathrm{d}\sigma \, \mathrm{d}g \quad (1)$$

$$\beta_d^M(l,m) = \int_{\mathcal{B}_{\sqrt{2}R}} \int_{\mathcal{B}_{\sqrt{2}R}} \tilde{B}(x,y)\delta(x \cdot y) e^{il\cdot x} e^{im \cdot y} \, \mathrm{d}x \, \mathrm{d}y. \quad (2)$$

Here, R is a discretization parameter, and $\mathcal{B}_r$ is a ball of radius r around the origin. B and $\tilde{B}$ represent the Boltzmann collision kernel, in the latter case with a change of variables:

$$\tilde{B}(x,y) = 2^{d-1} \frac{1}{\|x+y\|^{d-2}} B\left(\|x+y\|, \frac{\|x\|}{\|x+y\|}\right).$$

The precise form of the collision kernel depends on the physical model, and a reasonable choice is the VHS (Variable Hard Spheres) model, where $B(\|g\|, \cos\theta) = C_\alpha \|g\|^\alpha$, with $\alpha \le 1$. Negative α give “soft potential” collisions, and positive α give “hard potential” collisions, with $\alpha = 1$ known as the typical hard sphere case (think billiard balls) and $\alpha = 0$ called “Maxwellian molecules”.

It can be shown that for the VHS model,

$$\beta_2^P(l,m) = C(R,\alpha) \int_0^1 r^{1+\alpha} J_0(\|l+m\|Rr) J_0(\|l-m\|Rr) \, \mathrm{d}r, \quad (3)$$

$$\begin{aligned} \beta_2^M(l,m) = C(R,\alpha) \int_0^1 \int_0^1 (r^2 + r'^2)^{\alpha/2} r r' \quad (4) \\ \left(J_0\left(\sqrt{2}R\|rl + r'm^\perp\|\right) + J_0\left(\sqrt{2}M\|rl - r'm^\perp\|\right)\right) \mathrm{d}r \, \mathrm{d}r', \end{aligned}$$

$$\beta_3^P(l,m) = C(R,\alpha) \int_0^1 r^{2+\alpha} \operatorname{sinc}(\|l+m\|Rr) \operatorname{sinc}(\|l-m\|Rr) \, \mathrm{d}r. \quad (5)$$

The constants $C(R,\alpha)$ are *not* the same! For (5) there are closed form expressions for integral α.

In the above, J_0 is the zero-order Bessel function of the first kind:

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\tau) \, \mathrm{d}\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \sin\tau} \, \mathrm{d}\tau.$$

So far we have only been working in $d = 2$ and with the P-model. The numbers I sent you are for $\alpha = 0$.