

Fast Divergence-Conforming Reduced Basis Methods for Steady Navier-Stokes Flow

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The stationary Navier-Stokes equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D, |\Gamma_D| > 0 \tag{3}$$

$$-p\mathbf{n} + \nu(\nabla \mathbf{u})\mathbf{n} = \mathbf{h} \quad \text{on } \Gamma_N. \tag{4}$$

Stationary NS: Weak formulation

Find (\mathbf{u}, p) so that for all (\mathbf{w}, q) it holds that

$$a(\mathbf{u}, \mathbf{w}) + c(\mathbf{u}, \mathbf{u}, \mathbf{w}) + b(p, \mathbf{w}) = d(\mathbf{w}), \quad (5)$$

$$b(q, \mathbf{u}) = 0, \quad (6)$$

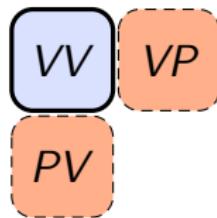
Where

$$a(\mathbf{u}, \mathbf{w}) = \nu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{w}, \quad b(p, \mathbf{w}) = - \int_{\Omega} p \nabla \cdot \mathbf{w},$$

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w}. \quad d(\mathbf{w}) = \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{w} + \int_{\Omega} \mathbf{f} \cdot \mathbf{w},$$

Anatomy of a reduced system

Given reduced bases for velocity and pressure (V and P), the system matrix looks like this (at any point in the nonlinear solver stage).

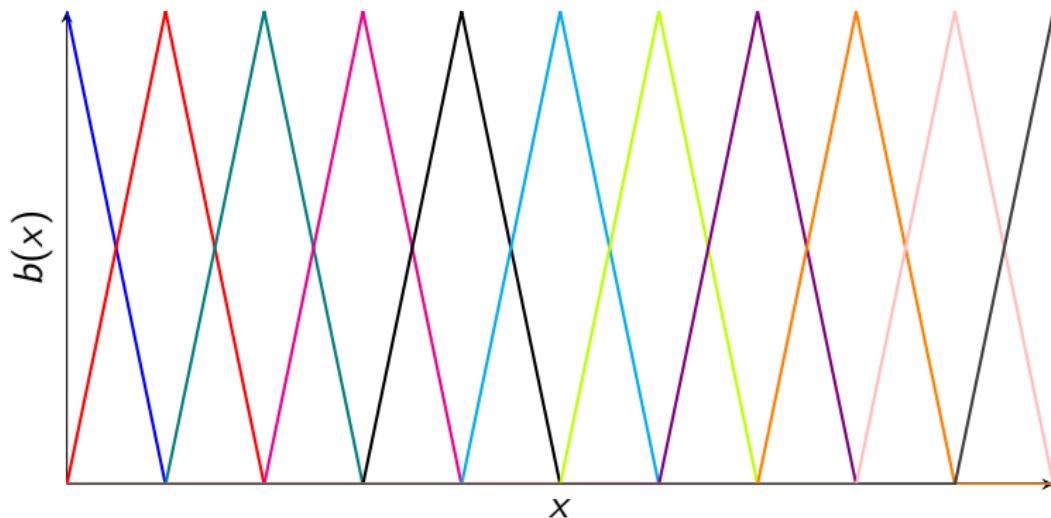


An idea

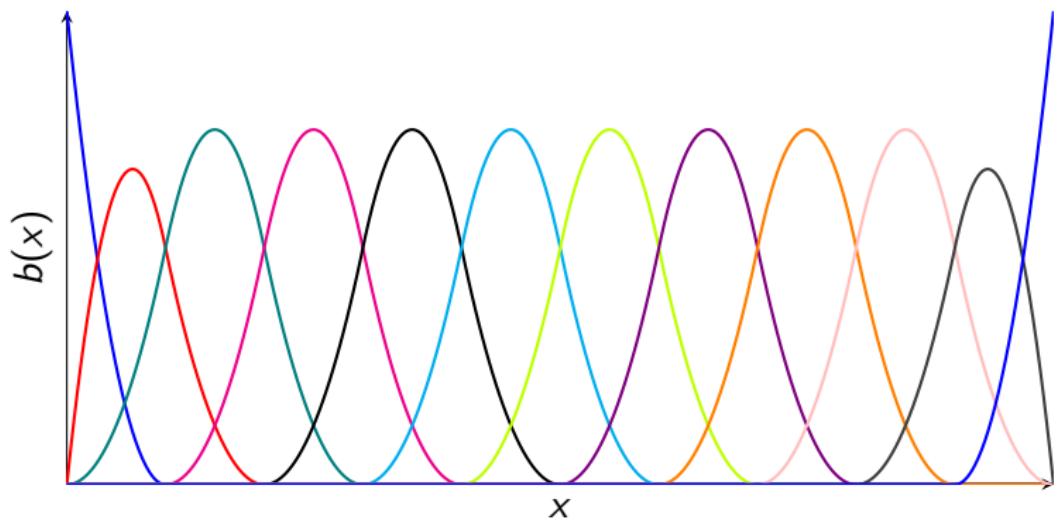
- In conventional solvers, pressure enters as a Lagrange multiplier to enforce the continuity equation.
- Incompatibilities between the velocity and pressure spaces are a frequent source of difficulties. (Also in reduced basis models; the VP matrix is often rank-deficient)
- If the velocity space were *a priori* divergence free, you wouldn't need a pressure field at all. You could solve directly for velocity and, if necessary, reconstruct the pressure in post.

Enter: Isogeometric Analysis (IGA).

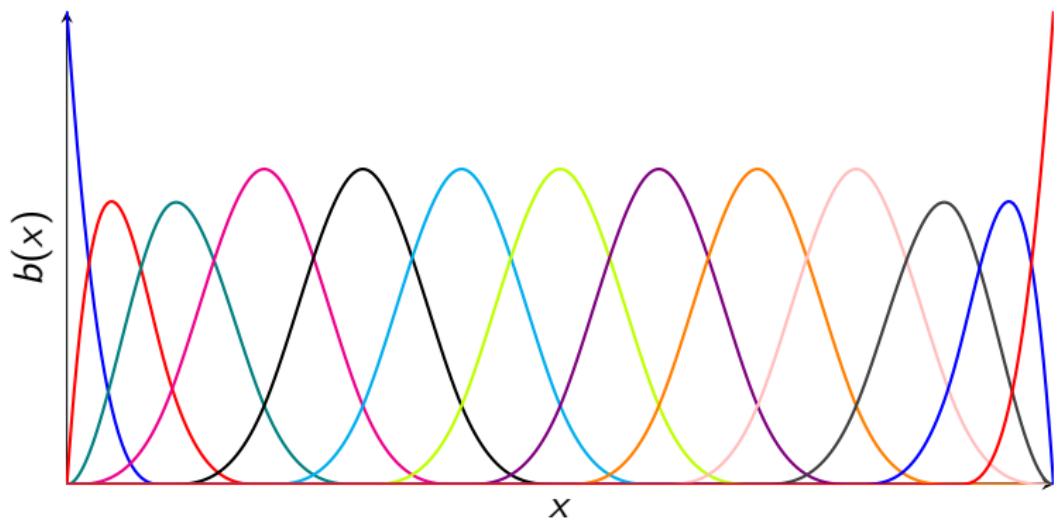
B-Spline basis functions.



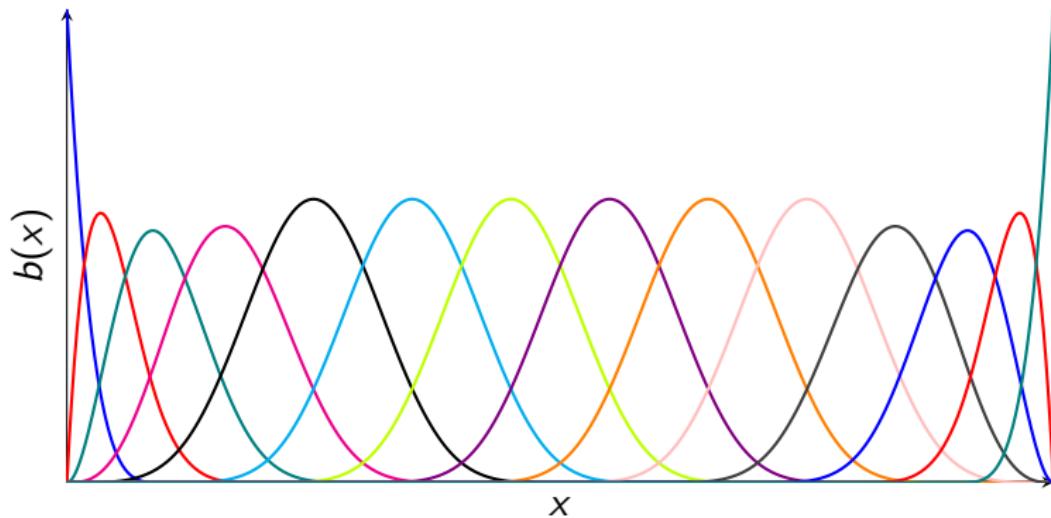
B-Spline basis functions.



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B-Spline basis functions.



Divergence-conforming methods

This way it's easy to create a divergence-conforming method on a rectilinear grid:

- v_1 : degree $(r+1, r)$ and continuity (C^r, C^{r-1})
- v_2 : degree $(r, r+1)$ and continuity (C^{r-1}, C^r)
- p : degree (r, r) and continuity (C^{r-1}, C^{r-1})

Then, $\nabla \cdot V = P$, and the inf-sup conditions are trivially satisfied.

Geometry mapping

We are usually not interested in rectangular domains, and the divergence-conforming property is not generally preserved using a pointwise velocity mapping:

$$\mathbf{v} = \hat{\mathbf{v}} \circ \pi^{-1}$$

where $\pi : \hat{\Omega}_{\text{param}} \rightarrow \hat{\Omega}$. Instead, the Piola mapping does the trick:

$$\mathbf{v} = \frac{\mathbf{J}}{|\mathbf{J}|} (\hat{\mathbf{v}} \circ \pi^{-1})$$

where \mathbf{J} is the Jacobian of π .

Geometry mapping (cont.)

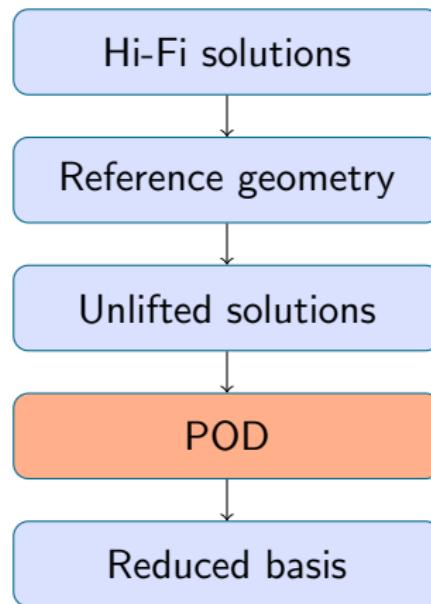
We use the same idea to map velocity fields from the reference domain $\hat{\Omega}$ to parametrized geometries $\Omega(\mu)$.

This generally has the effect of massively complicating affine representations,

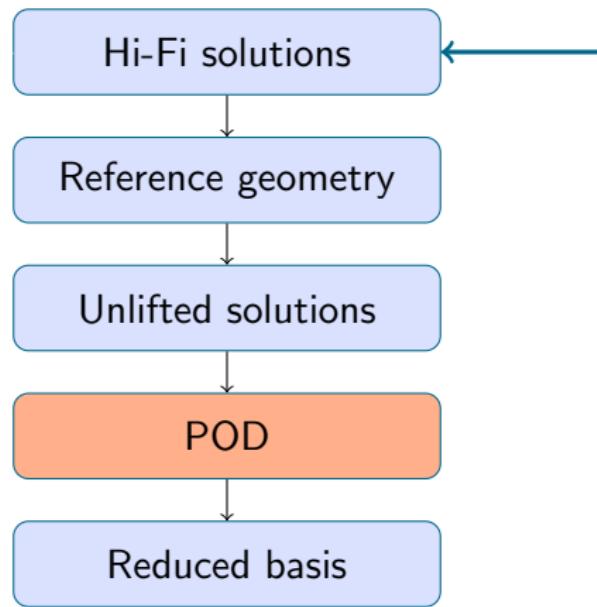
$$\mathbf{A}_h(\mu) = \sum_i \xi_i(\mu) \mathbf{A}_i, \quad \mathbf{f}_h(\mu) = \sum_i \chi_i(\mu) \mathbf{f}_i$$

however we intend to justify the *means* by the *end*.

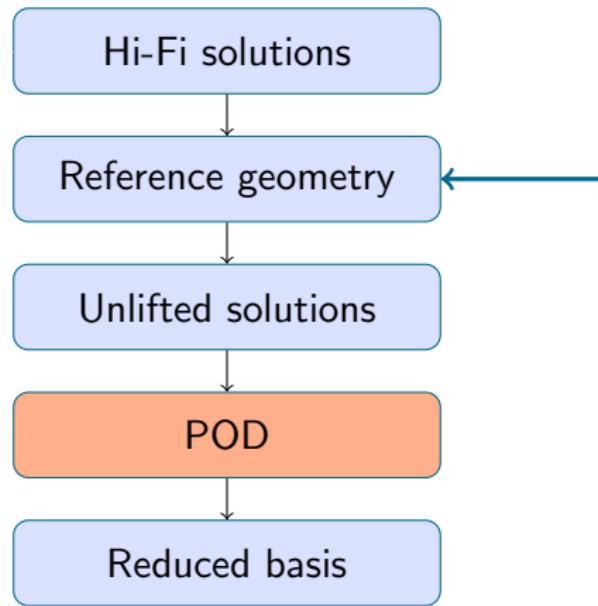
Divergence-free basis



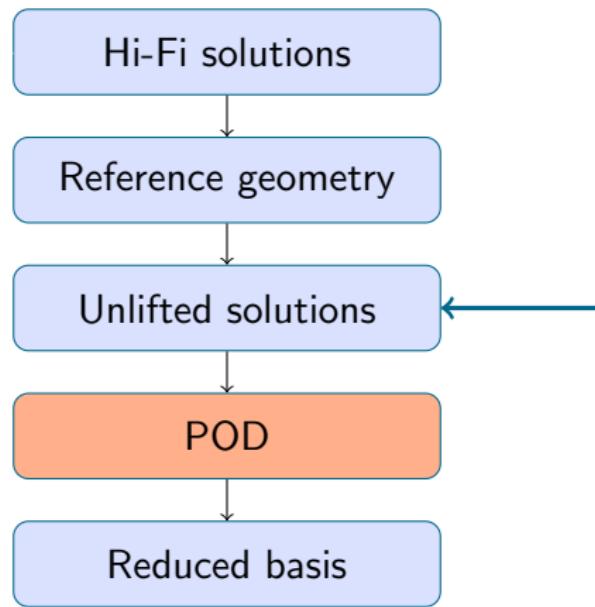
Divergence-free basis



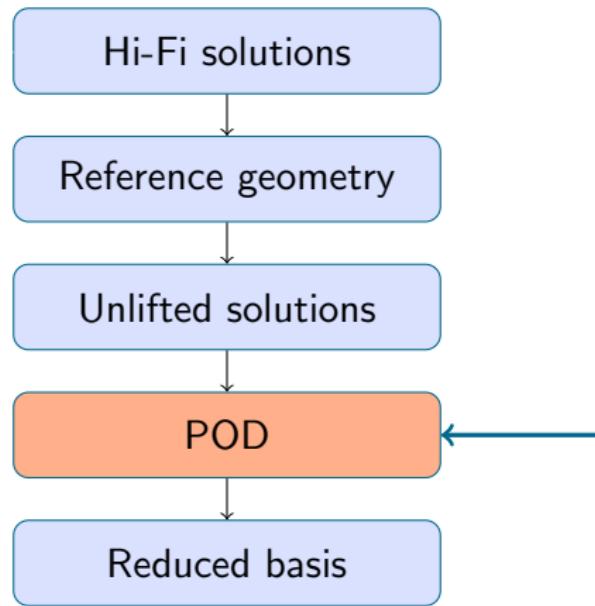
Divergence-free basis



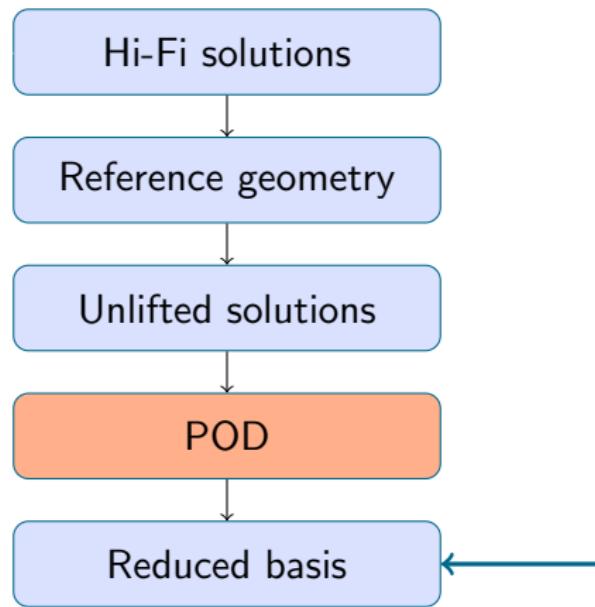
Divergence-free basis



Divergence-free basis



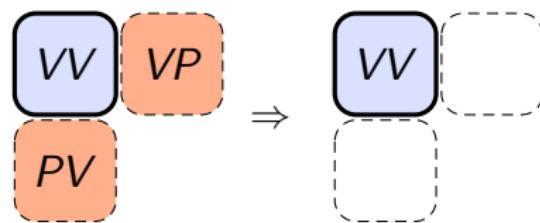
Divergence-free basis



Divergence-free basis

- The mapping from reference to physical geometry needs to preserve divergence-free functions (the Piola mapping).
- Lifting functions must also be divergence-free.
- That's it!

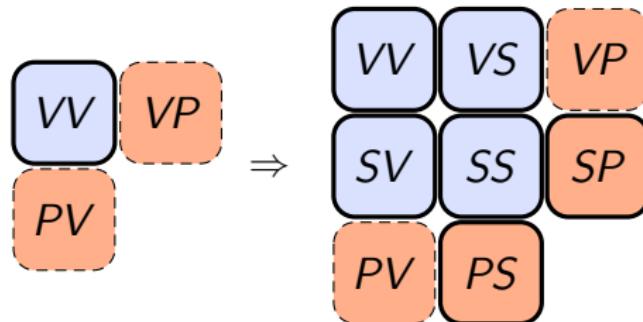
Anatomy of a divergence-free reduced system



A true velocity-only formulation is achieved.

Supremizers

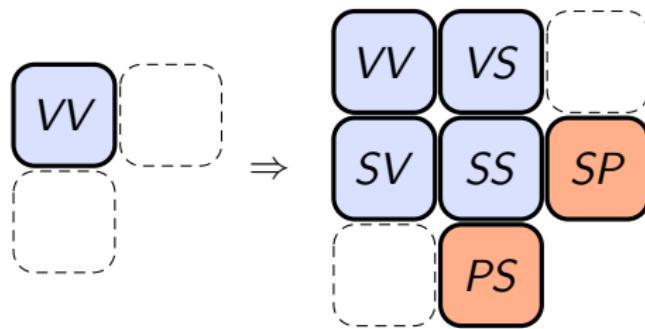
To solve for pressure, we leverage a technique from Ballarin et. al. (2015). There, the velocity space is enriched with *supremizers* to stabilize and control the LBB condition.



Supremizers are maximizers of the “sup” part of the inf-sup expression.

Supremizers (cont.)

For the same reason, supremizers are natural test functions when solving the momentum equation for pressure.



Note the block-triangular structure. (Also, the last equation is trivial.)

Block solver structure

$$B_{ps}\mathbf{s} = \mathbf{0} \quad \Rightarrow \quad \mathbf{s} = \mathbf{0}$$

So supremizers are truly test functions, have no influence on the solution.

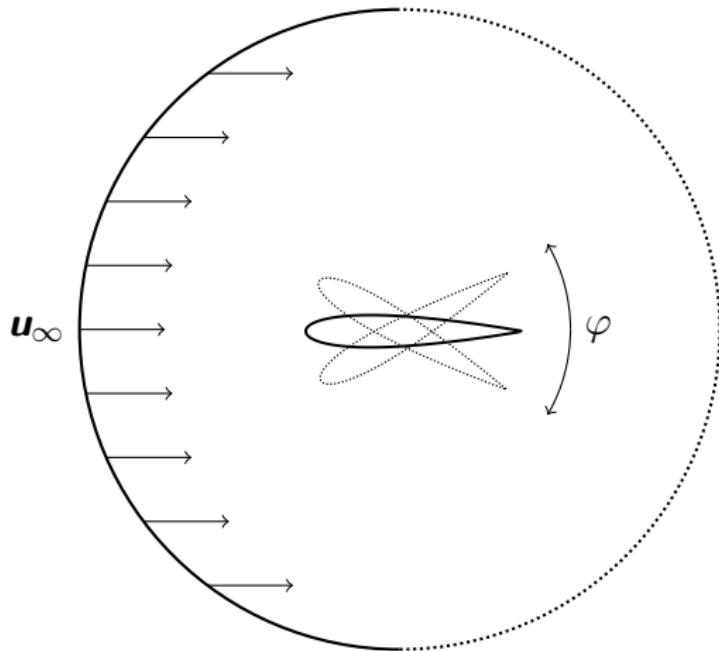
$$\mathbf{s} = \mathbf{0}$$

$$\mathbf{A}_{vv}\mathbf{v} = \mathbf{f}_v$$

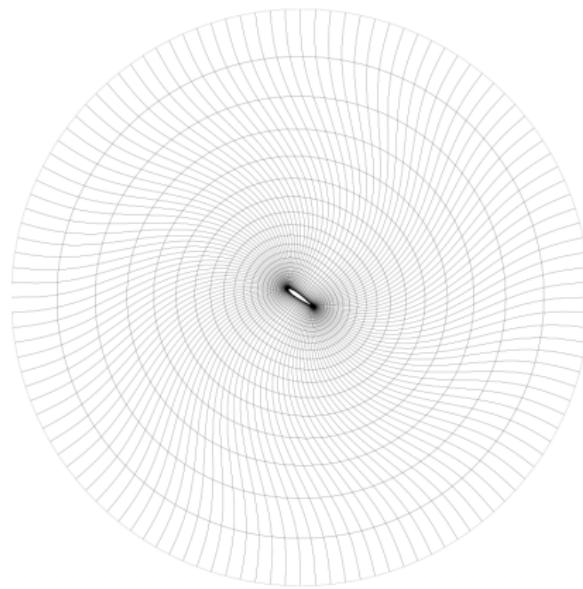
$$\mathbf{B}_{sp}\mathbf{p} = \mathbf{f}_p - \mathbf{A}_{sv}\mathbf{v}$$

Solve two systems of size M instead of one system of size $3M$.

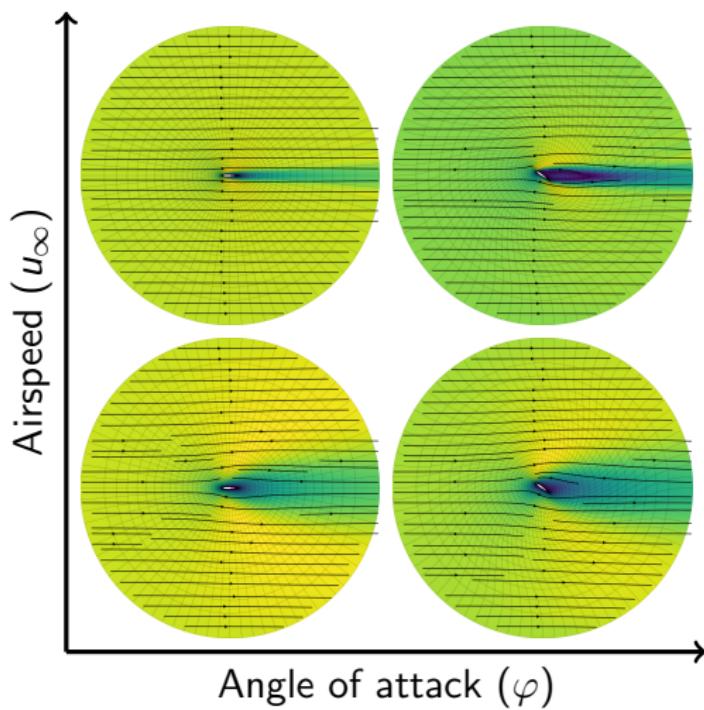
Flow around airfoil



Domain transformation



Parameter space



Affine representations

- We will try two high-fidelity methods: a TH (1,2)-method and an IGA (1,2) divergence-conforming method.
- Not possible to express the Navier Stokes problem as finite sums

$$\mathbf{A}_h(\boldsymbol{\mu}) = \sum_i \xi_i(\boldsymbol{\mu}) \mathbf{A}_i, \quad \mathbf{f}_h(\boldsymbol{\mu}) = \sum_i \chi_i(\boldsymbol{\mu}) \mathbf{f}_i$$

- Instead, we use truncated series expansions.
- For a maximal angle of attack of 35° we can expect about 10 digits of accuracy with a reasonable number of terms (~ 25 for TH, ~ 75 for DC).

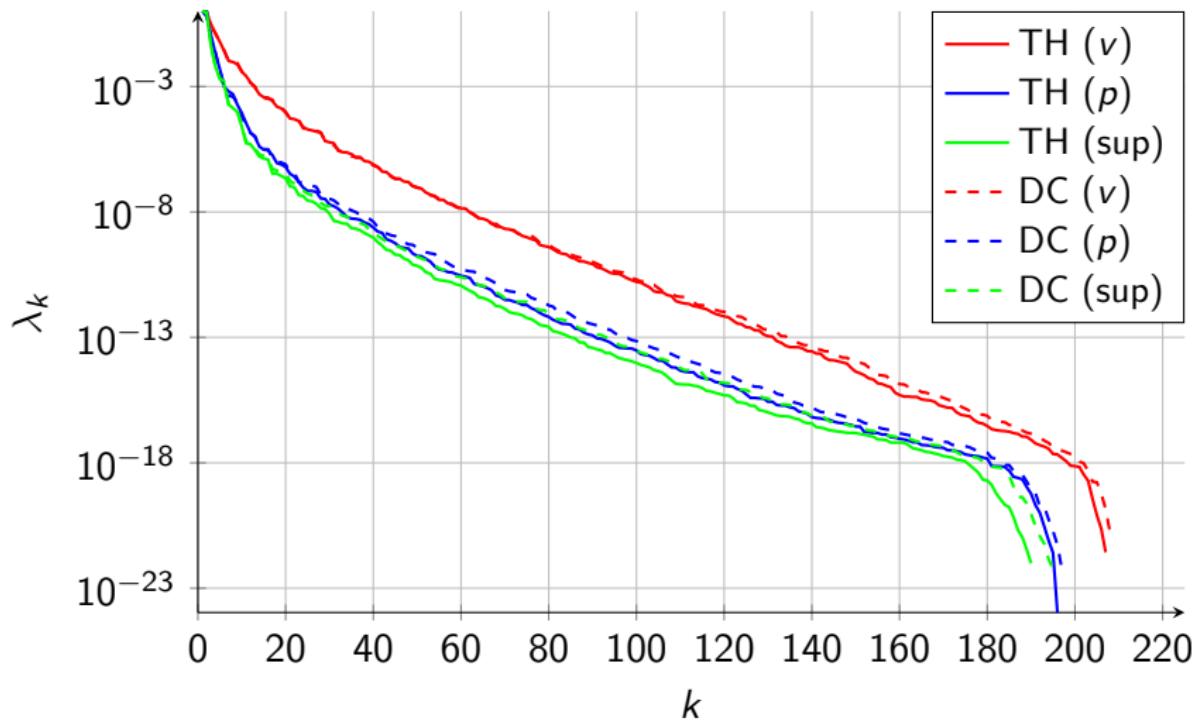
Ensembles

- We calculated 15×15 solutions at the Gauss quadrature nodes.
- The parameter domain was chosen as

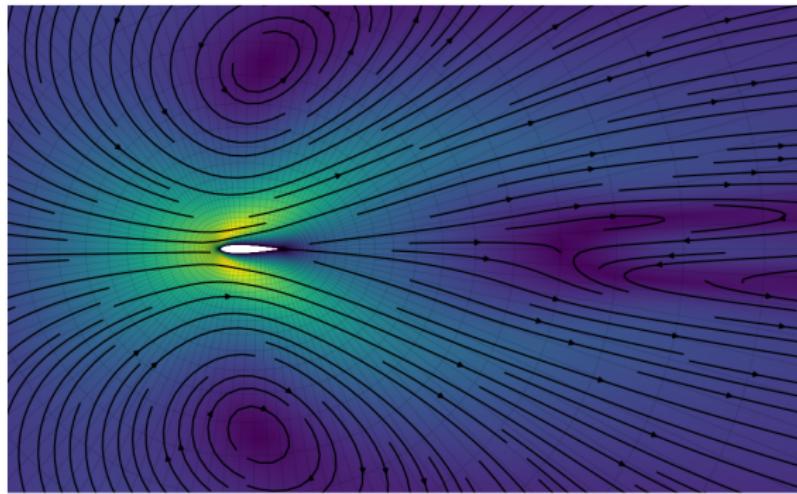
$$\mathcal{P} = [-35^\circ, +35^\circ] \times [1 \text{ m/s}, 20 \text{ m/s}].$$

- Only *stationary* Navier-Stokes, with $\nu = \frac{1}{6}$.

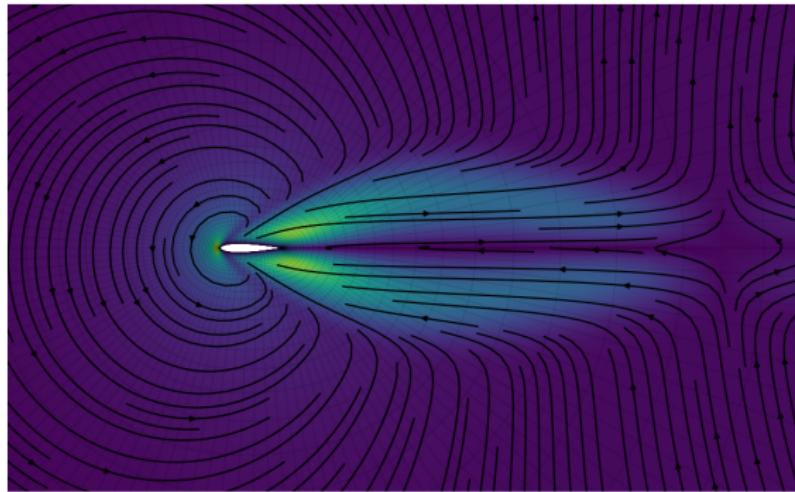
Spectrum



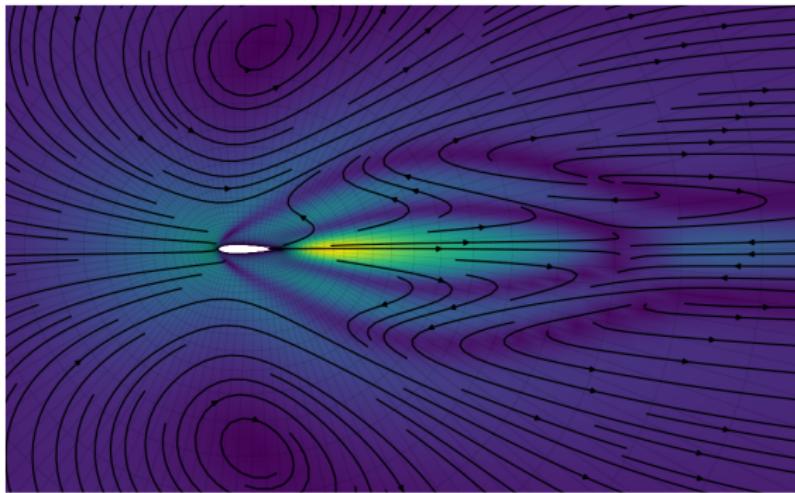
Basis functions (v , TH, 1)



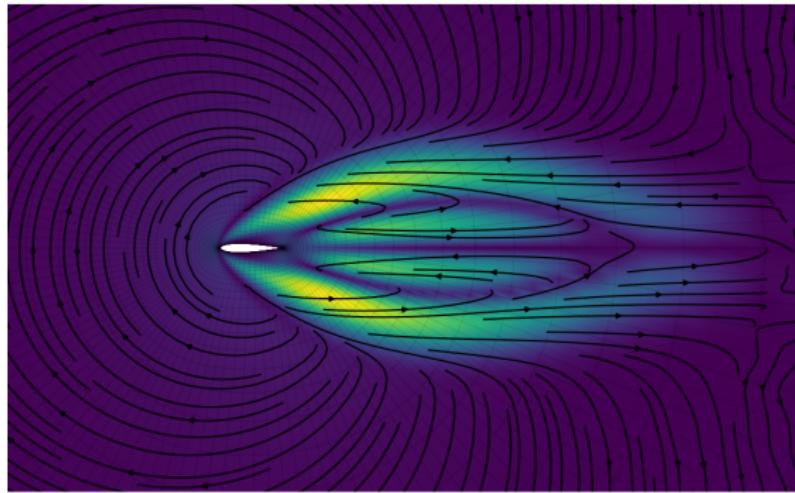
Basis functions (v , TH, 2)



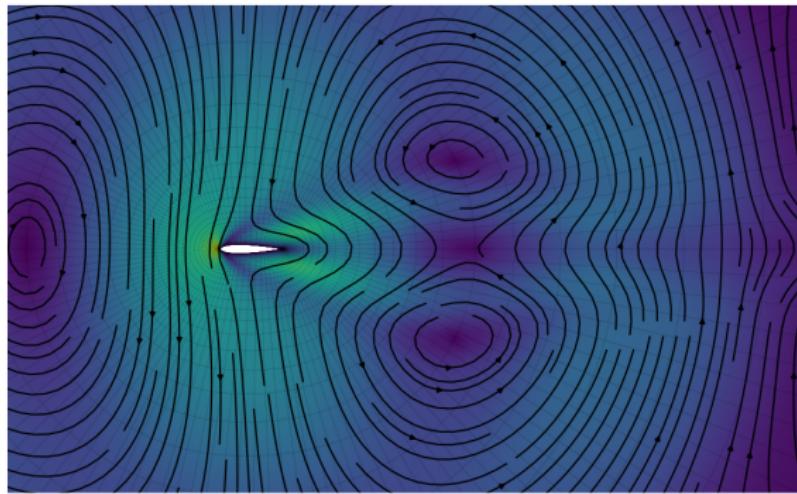
Basis functions (v , TH, 3)



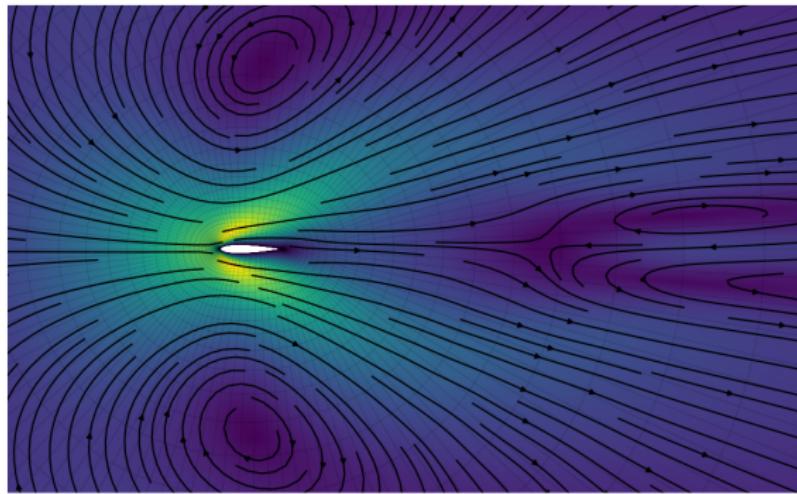
Basis functions (v , TH, 4)



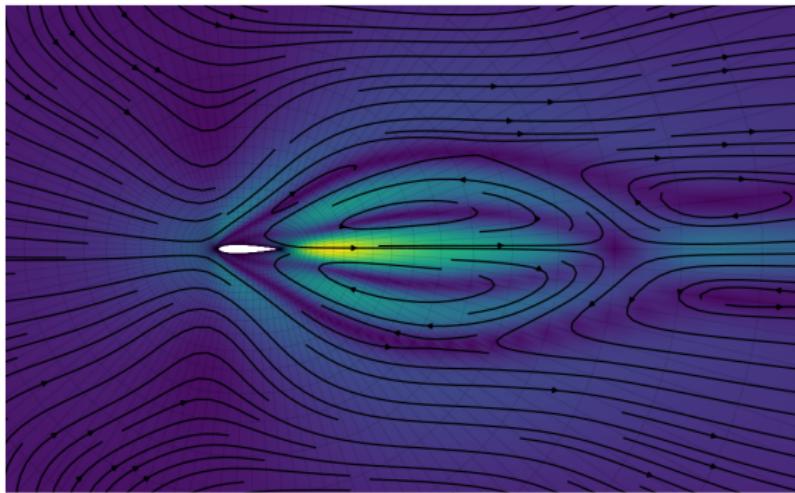
Basis functions (v , DC, 1)



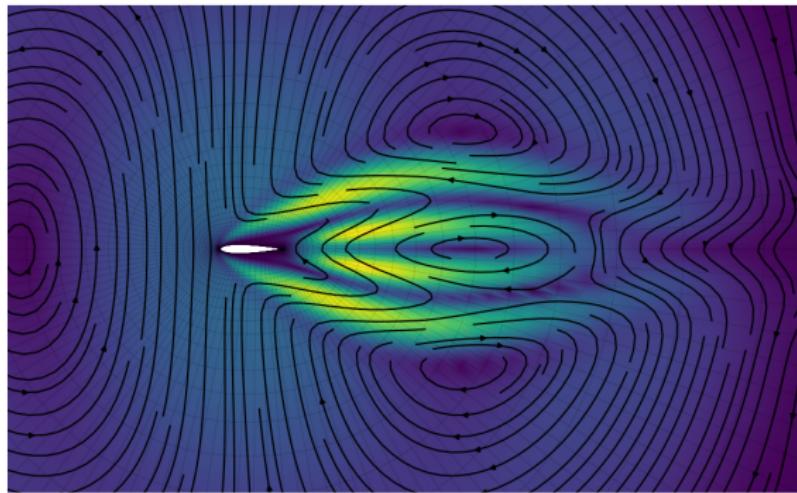
Basis functions (v , DC, 2)



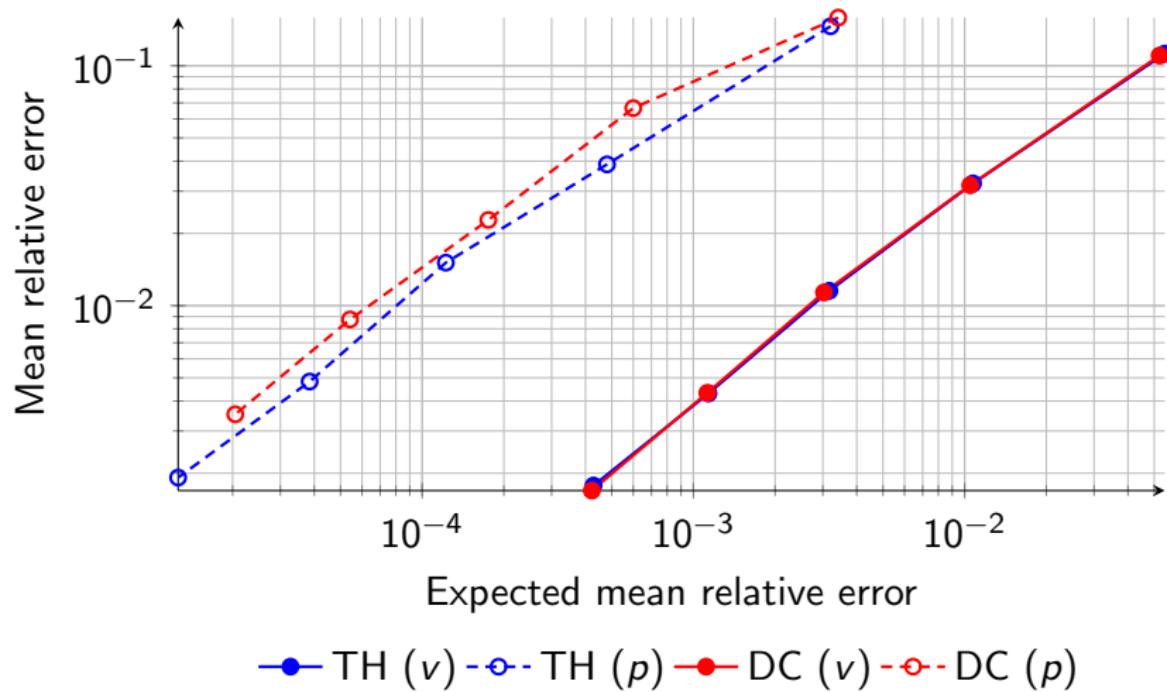
Basis functions (v , DC, 3)



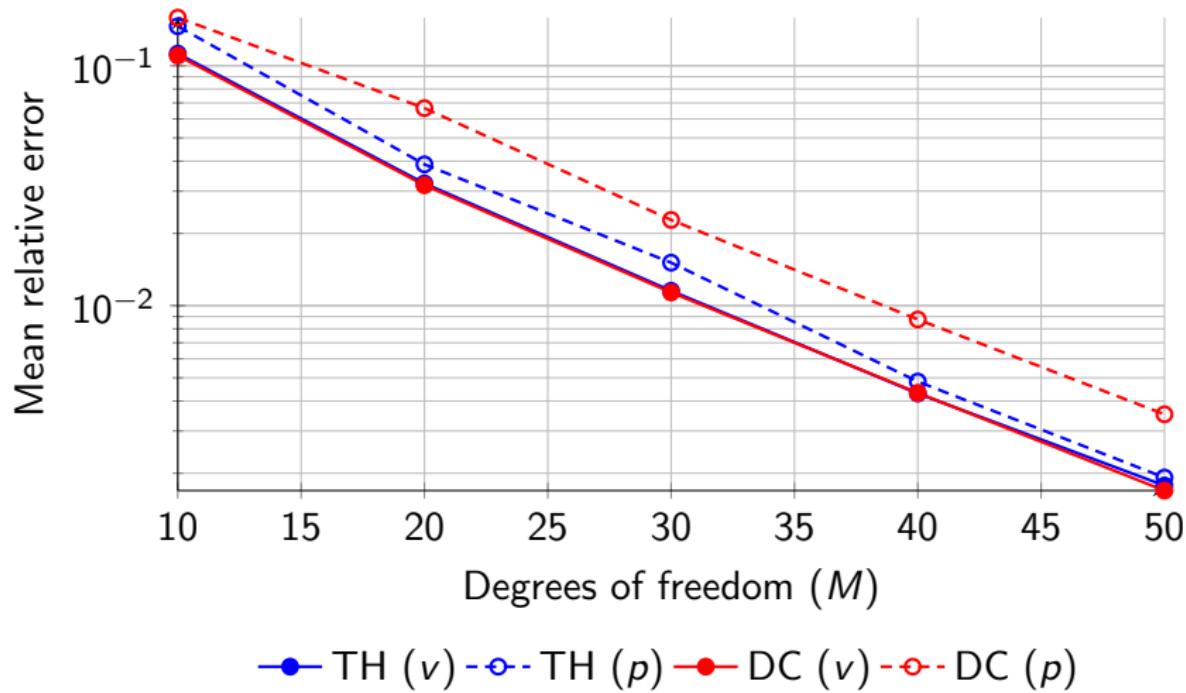
Basis functions (v , DC, 4)



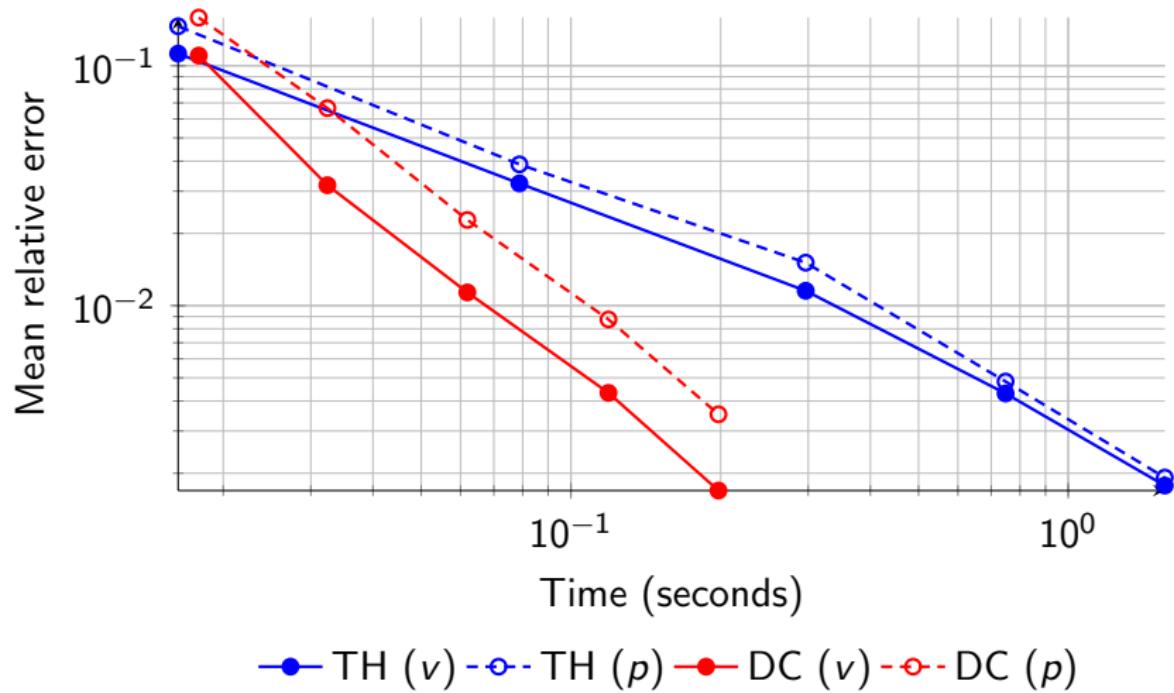
Convergence



Convergence



Convergence



Our guiding principle

All is fair in love, war and the offline stage.
— John Lyly (*Euphues*; 1579)

Summary

- Reduced order models can offer dramatic speed-ups for certain applications.
- They combine nicely with IGA and div-compatible spaces to form fully divergence-free function spaces without need for pressure fields.
- Divergence-free RBMs can be much faster than other RBMs, in spite of additional complexity in the offline stage (remember, all is fair there.)

Thanks!