Fast Divergence-Conforming Reduced Basis Methods for Steady Navier-Stokes Flow

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The stationary Navier-Stokes equations

$$
\begin{aligned}
-\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, && (1)\\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega, && (2)\\
\boldsymbol{u} &= \boldsymbol{g} && \text{on } \Gamma_{\mathrm{D}}, |\Gamma_D| > 0 && (3)\\
-p\boldsymbol{n} + \nu(\nabla\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{h} && \text{on } \Gamma_{\mathrm{N}}. && (4)
\end{aligned}
$$

Stationary NS: Weak formulation

Find $(\boldsymbol{u}, p)$ so that for all $(\boldsymbol{w}, q)$ it holds that

$$a(\boldsymbol{u}, \boldsymbol{w}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) + b(p, \boldsymbol{w}) = d(\boldsymbol{w}), \tag{5}$$

$$b(q, \boldsymbol{u}) = 0, \tag{6}$$

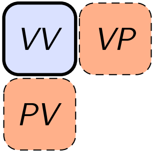
Where

$$a(\boldsymbol{u}, \boldsymbol{w}) = \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{w}, \qquad b(p, \boldsymbol{w}) = -\int_\Omega p \nabla \cdot \boldsymbol{w},$$

$$c(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u} \cdot \nabla) \boldsymbol{v} \cdot \boldsymbol{w}. \qquad d(\boldsymbol{w}) = \int_{\Gamma_\mathrm{N}} \boldsymbol{h} \cdot \boldsymbol{w} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w},$$

Anatomy of a reduced system

Given reduced bases for velocity and pressure (V and P), the system matrix looks like this (at any point in the nonlinear solver stage).

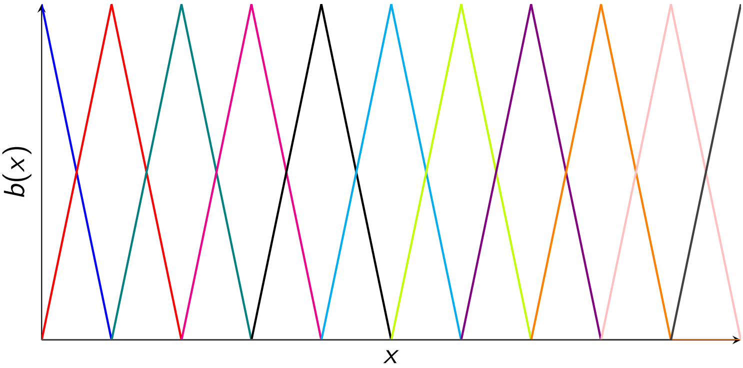


An idea

- In conventional solvers, pressure enters as a Lagrange multiplier to enforce the continuity equation.
- Incompatibilities between the velocity and pressure spaces are a frequent source of difficulties. (Also in reduced basis models; the VP matrix is often rank-deficient)
- If the velocity space were *a priori* divergence free, you wouldn't need a pressure field at all. You could solve directly for velocity and, if necessary, reconstruct the pressure in post.

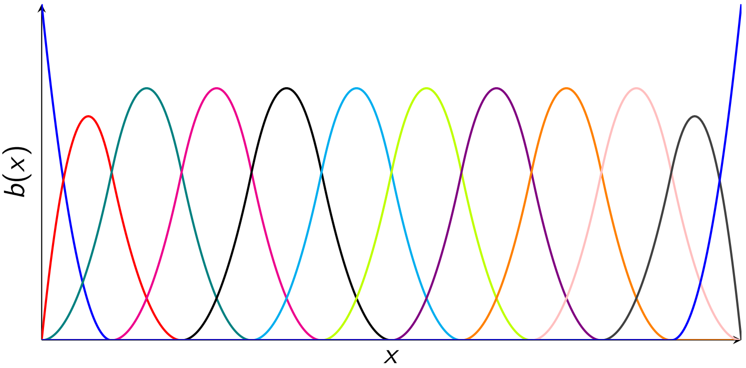
Enter: Isogeometric Analysis (IGA).

B-Spline basis functions.



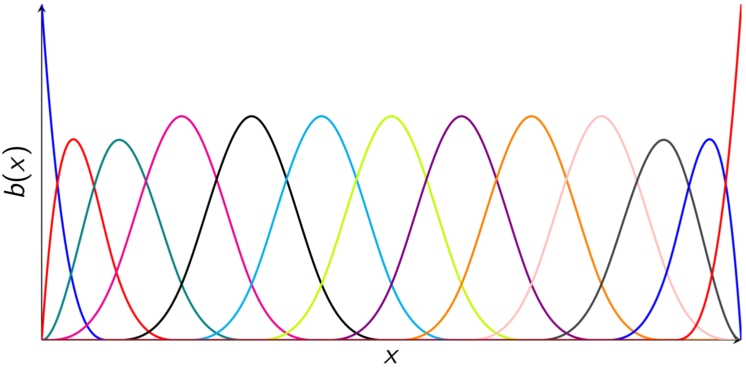
IGA

B-Spline basis functions.



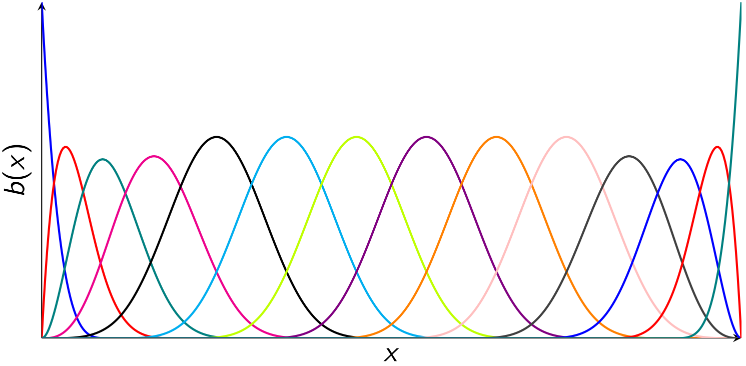
IGA

B-Spline basis functions.



IGA

B-Spline basis functions.



Divergence-conforming methods

This way it's easy to create a divergence-conforming method on a rectilinear grid:

- v_1: degree $(r+1, r)$ and continuity (C^r, C^{r-1})
- v_2: degree $(r, r+1)$ and continuity (C^{r-1}, C^r)
- p: degree (r, r) and continuity (C^{r-1}, C^{r-1})

Then, $\nabla \cdot V = P$, and the inf-sup conditions are trivially satisfied.

Geometry mapping

We are usually not interested in rectangular domains, and the divergence-conforming property is not generally preserved using a pointwise velocity mapping:

$$\boldsymbol{v} = \hat{\boldsymbol{v}} \circ \pi^{-1}$$

where $\pi : \hat{\Omega}_{\text{param}} \to \hat{\Omega}$. Instead, the Piola mapping does the trick:

$$\boldsymbol{v} = \frac{\boldsymbol{J}}{|\boldsymbol{J}|}(\hat{\boldsymbol{v}} \circ \pi^{-1})$$

where $\boldsymbol{J}$ is the Jacobian of π.

Geometry mapping (cont.)

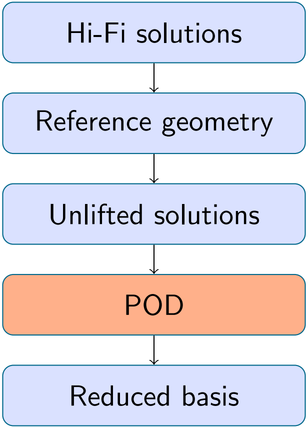
We use the same idea to map velocity fields from the reference domain $\hat{\Omega}$ to parametrized geometries $\Omega(\boldsymbol{\mu})$.

This generally has the effect of massively complicating affine representations,

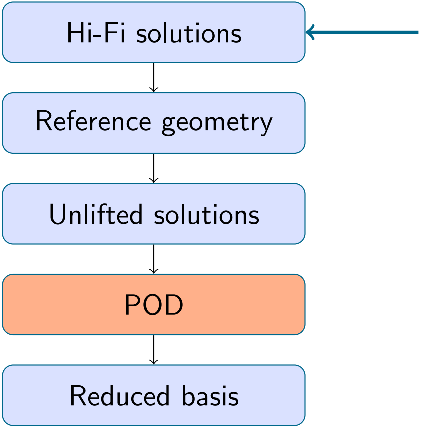
$$\boldsymbol{A}_h(\boldsymbol{\mu}) = \sum_i \xi_i(\boldsymbol{\mu})\boldsymbol{A}_i, \qquad \boldsymbol{f}_h(\boldsymbol{\mu}) = \sum_i \chi_i(\boldsymbol{\mu})\boldsymbol{f}_i$$

however we intend to justify the *means* by the *end*.

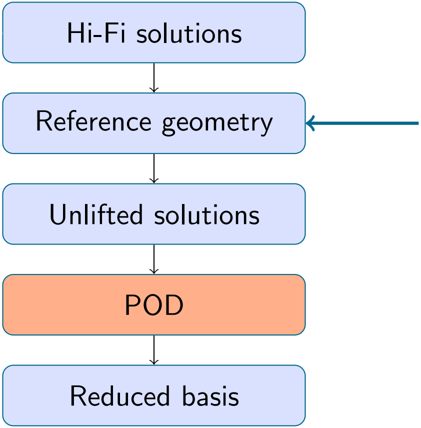
Divergence-free basis



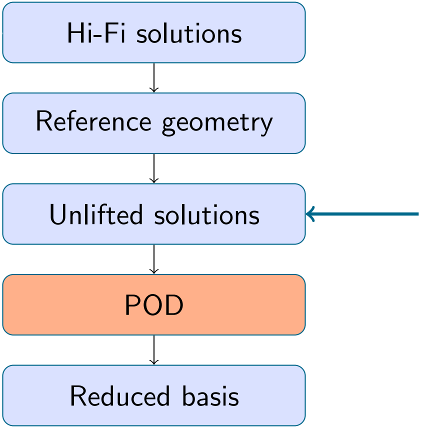
Divergence-free basis



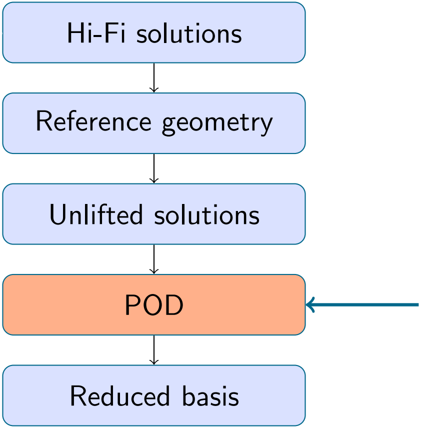
Divergence-free basis



Divergence-free basis



Divergence-free basis



Divergence-free basis

Hi-Fi solutions

↓

Reference geometry

↓

Unlifted solutions

↓

POD

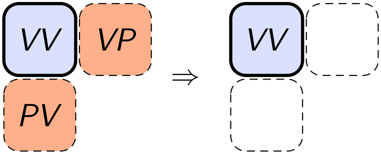
↓

Reduced basis ←

Divergence-free basis

- The mapping from reference to physical geometry needs to preserve divergence-free functions (the Piola mapping).
- Lifting functions must also be divergence-free.
- That's it!

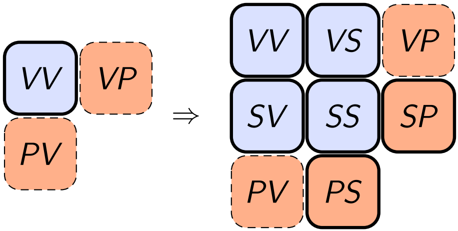
Anatomy of a divergence-free reduced system



A true velocity-only formulation is achieved.

Supremizers

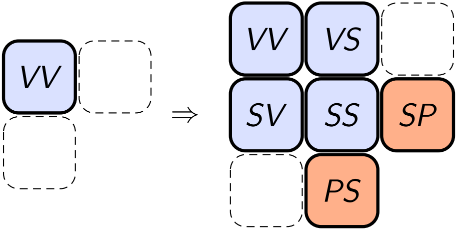
To solve for pressure, we leverage a technique from Ballarin et. al. (2015). There, the velocity space is enriched with *supremizers* to stabilize and control the LBB condition.



Supremizers are maximizers of the “sup” part of the inf-sup expression.

Supremizers (cont.)

For the same reason, supremizers are natural test functions when solving the momentum equation for pressure.



Note the block-triangular structure. (Also, the last equation is trivial.)

Block solver structure

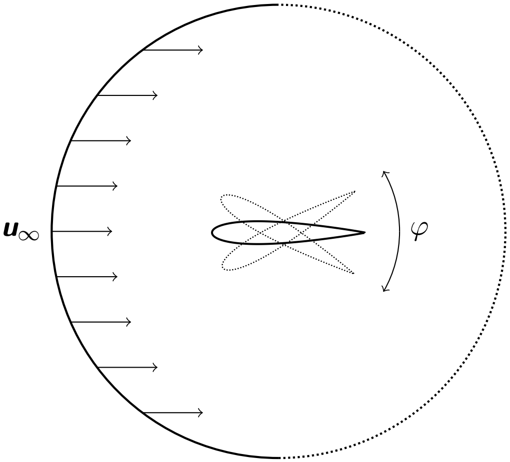
$$B_{ps}\boldsymbol{s} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{s} = \boldsymbol{0}$$

So supremizers are truly test functions, have no influence on the solution.

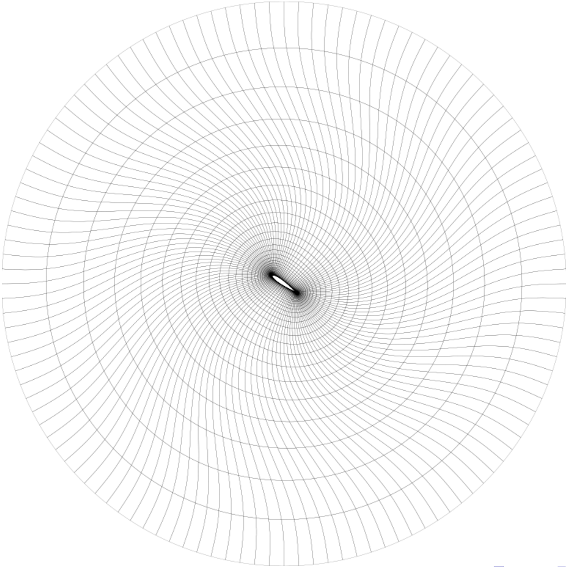
$$\begin{aligned} \boldsymbol{s} &= \boldsymbol{0} \\ \boldsymbol{A}_{vv}\boldsymbol{v} &= \boldsymbol{f}_v \\ \boldsymbol{B}_{sp}\boldsymbol{p} &= \boldsymbol{f}_p - A_{sv}\boldsymbol{v} \end{aligned}$$

Solve two systems of size M instead of one system of size $3M$.

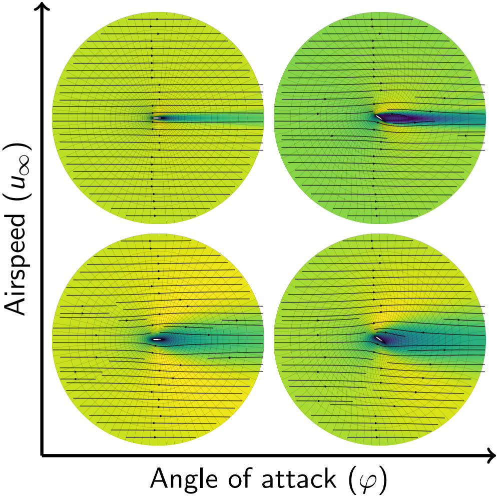
Flow around airfoil



Domain transformation



Parameter space



Affine representations

- We will try two high-fidelity methods: a TH (1,2)-method and an IGA (1,2) divergence-conforming method.
- Not possible to express the Navier Stokes problem as finite sums

$$\boldsymbol{A}_h(\boldsymbol{\mu}) = \textstyle\sum_i \xi_i(\boldsymbol{\mu})\boldsymbol{A}_i, \qquad \boldsymbol{f}_h(\boldsymbol{\mu}) = \textstyle\sum_i \chi_i(\boldsymbol{\mu})\boldsymbol{f}_i$$

- Instead, we use truncated series expansions.
- For a maximal angle of attack of 35° we can expect about 10 digits of accuracy with a reasonable number of terms
($\sim$ 25 for TH, $\sim$ 75 for DC).

Affine representations (TH)

$$a(\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}; \varphi) \approx \nu \int_{\hat{\Omega}} \nabla \hat{\boldsymbol{u}} : \left[(1 + \varphi \boldsymbol{D}_1 - \varphi^2 \boldsymbol{D}_2) \nabla \right] \hat{\boldsymbol{w}}$$

$$b(\hat{p}, \hat{\boldsymbol{w}}; \varphi) \approx \sum_{i=0}^{2n} \varphi^i \int_{\hat{\Omega}} \hat{p} \boldsymbol{B}_i^{(-)} : \nabla \hat{\boldsymbol{w}}$$

$$c(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}; \varphi) \approx \sum_{i=0}^{2n} \varphi^i \int_{\hat{\Omega}} (\hat{\boldsymbol{u}} \cdot \boldsymbol{B}_i^{(-)} \nabla) \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{w}}$$

Affine representations (TH)

$$a(\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}; \varphi) \approx \sum_{i,j=0}^{2n} \varphi^{i+j} \int_{\hat{\Omega}} \nabla(\boldsymbol{B}_i^{(+)} \hat{\boldsymbol{u}}) : \nabla(\boldsymbol{B}_j^{(+)} \hat{\boldsymbol{w}})$$

$$+ \sum_{i,j=0}^{2n} \varphi^{i+j+1} \int_{\hat{\Omega}} \nabla(\boldsymbol{B}_i^{(+)} \hat{\boldsymbol{u}}) : (\boldsymbol{D}_1 \nabla)(\boldsymbol{B}_j^{(+)} \hat{\boldsymbol{w}})$$

$$- \sum_{i,j=0}^{2n} \varphi^{i+j+2} \int_{\hat{\Omega}} \nabla(\boldsymbol{B}_i^{(+)} \hat{\boldsymbol{u}}) : (\boldsymbol{D}_2 \nabla)(\boldsymbol{B}_j^{(+)} \hat{\boldsymbol{w}})$$

$$b(\hat{p}, \hat{\boldsymbol{w}}; \varphi) \approx \sum_{i,j=0}^{2n} \varphi^{i+j} \int_{\hat{\Omega}} \hat{p} \boldsymbol{B}_i^{(-)} : \nabla \left(\boldsymbol{B}_j^{(+)} \hat{\boldsymbol{w}} \right)$$

$$c(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}; \varphi) \approx \sum_{i,j=0}^{2n} \varphi^{i+j} \int_{\hat{\Omega}} (\hat{\boldsymbol{u}} \cdot \nabla) \boldsymbol{B}_i^{(+)} \hat{\boldsymbol{v}} \cdot \boldsymbol{B}_j^{(+)} \hat{\boldsymbol{w}}$$

Our guiding principle

All is fair in love, war and the offline stage.
— John Lyly (*Euphues*; 1579)

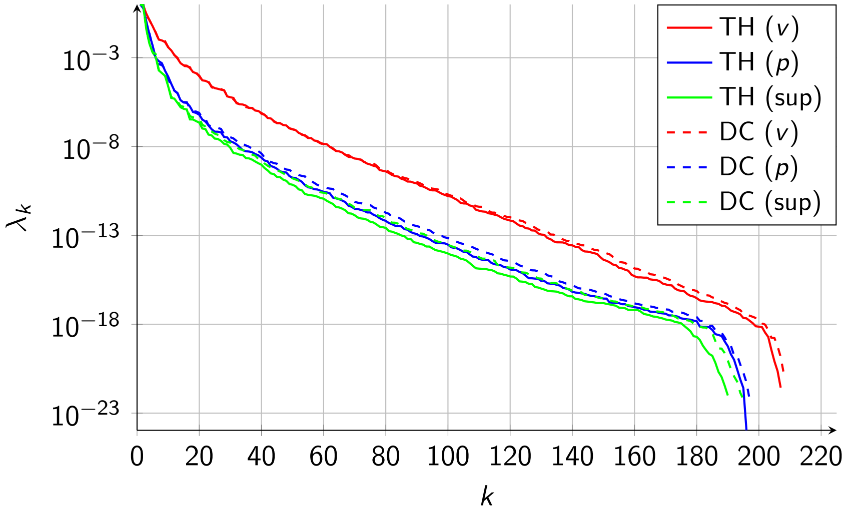
Ensembles

- We calculated 15×15 solutions at the Gauss quadrature nodes.
- The parameter domain was chosen as

$$\mathcal{P} = [-35^\circ, +35^\circ] \times [1\,\mathrm{m/s}, 20\,\mathrm{m/s}].$$

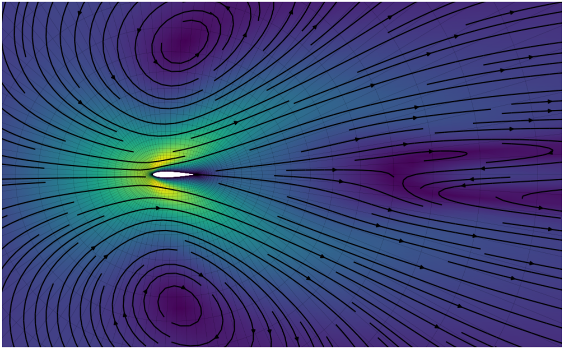
- Only *stationary* Navier-Stokes, with $\nu = \frac{1}{6}$.

Spectrum

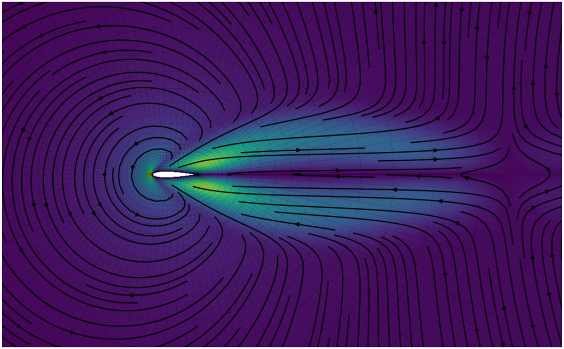


TH (v)
TH (p)
TH (sup)
DC (v)
DC (p)
DC (sup)
λ_k
k
10^{-3}
10^{-8}
10^{-13}
10^{-18}
10^{-23}
0 20 40 60 80 100 120 140 160 180 200 220

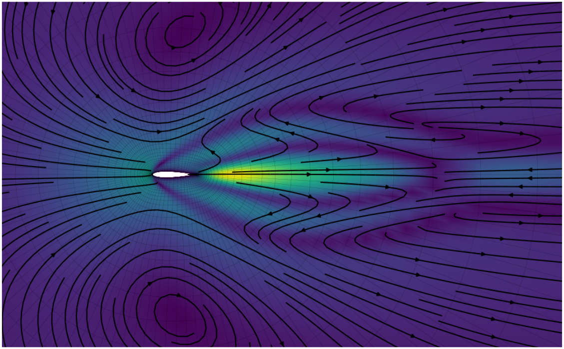
Basis functions (v, TH, 1)



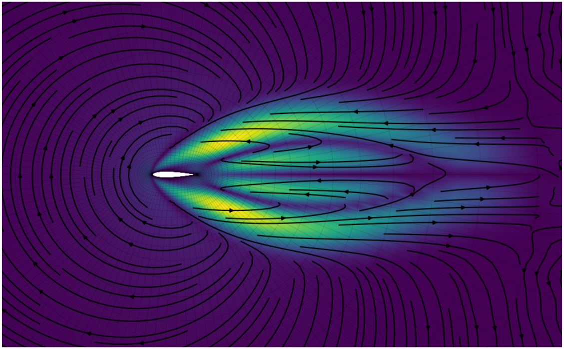
Basis functions (v, TH, 2)



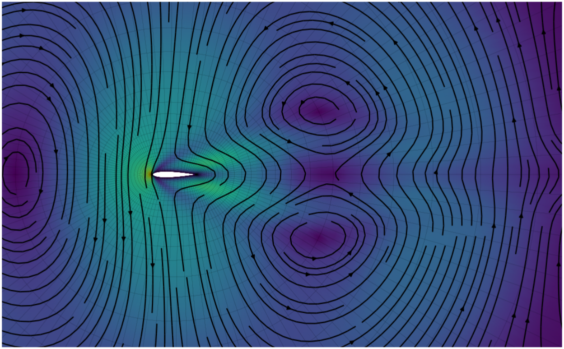
Basis functions (v, TH, 3)



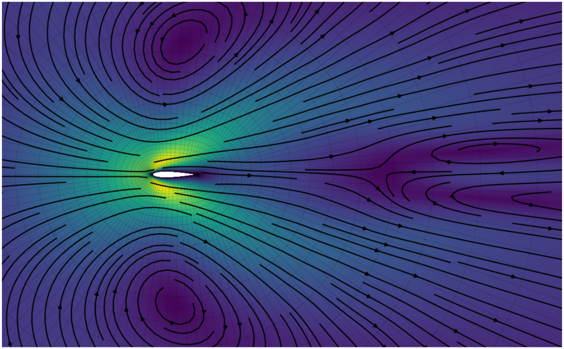
Basis functions (v, TH, 4)



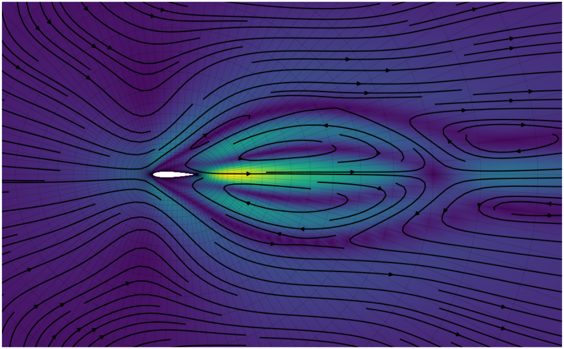
Basis functions (v, DC, 1)



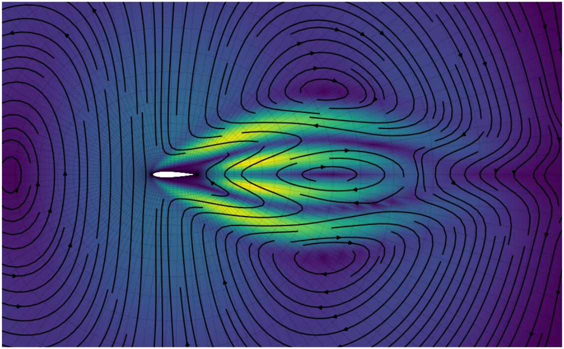
Basis functions (v, DC, 2)



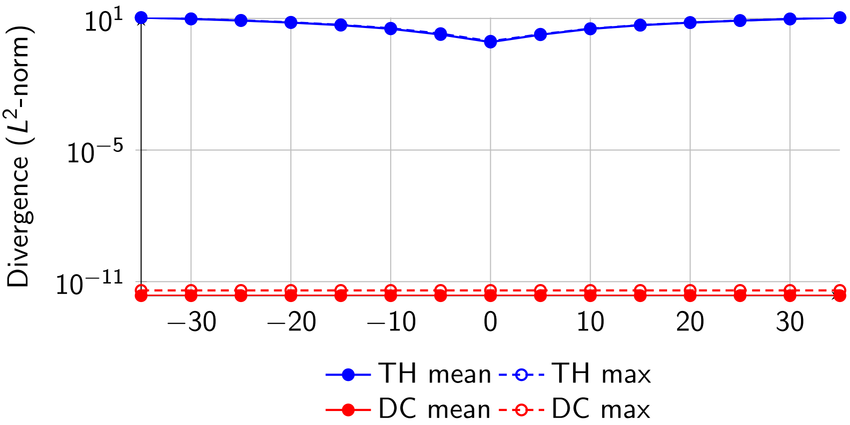
Basis functions (v, DC, 3)



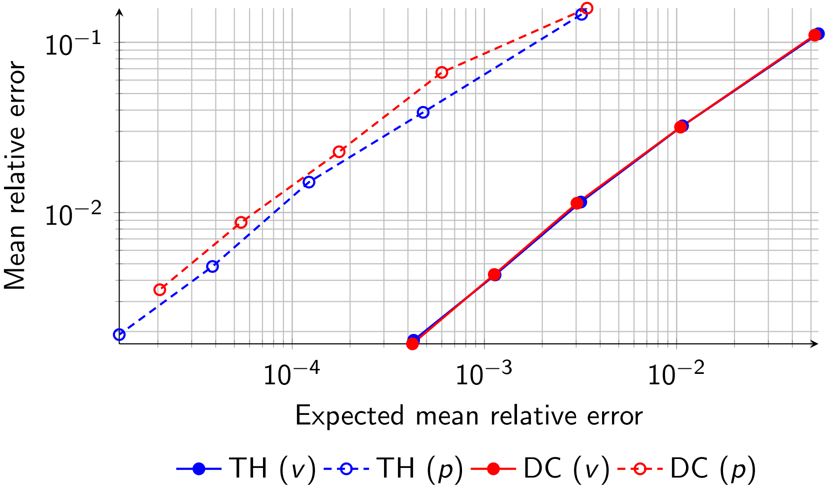
Basis functions (v, DC, 4)



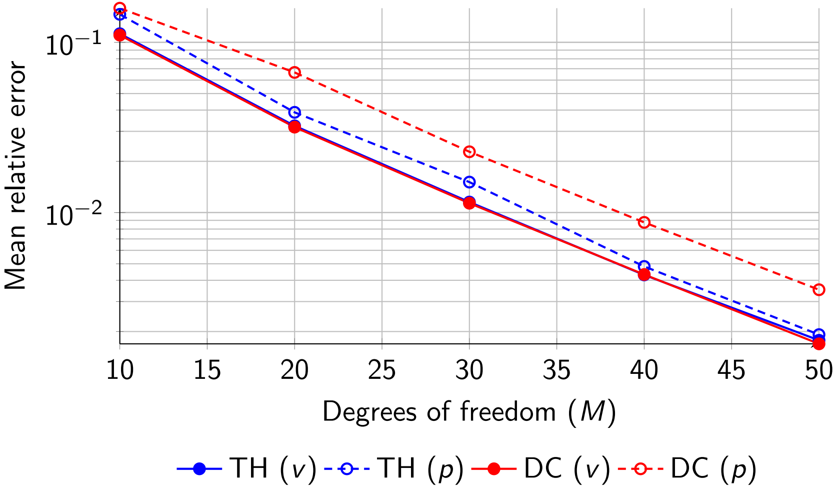
Basis divergence



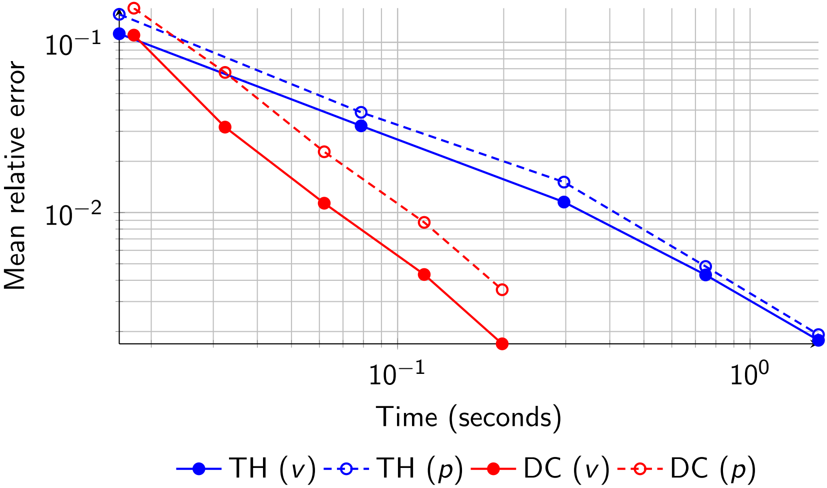
Convergence



Convergence



Convergence



Summary

- Reduced order models can offer dramatic speed-ups for certain applications.
- They combine nicely with IGA and div-compatible spaces to form fully divergence-free function spaces without need for pressure fields.
- Divergence-free RBMs can be much faster than other RBMs, in spite of additional complexity in the offline stage (remember, all is fair there.)

Thanks!