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Interpolated models for non-intrusive affinization of reduced basis methods

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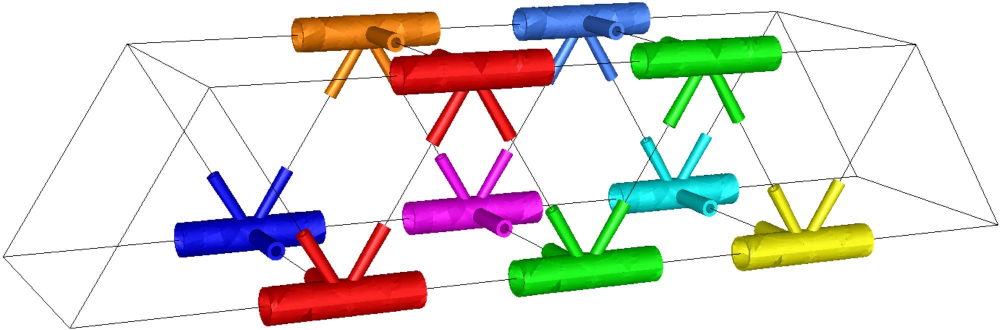
Jacket joints

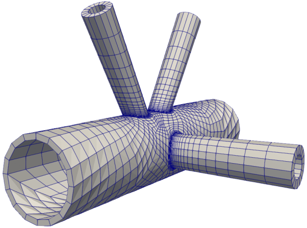
- Jacket structures used in marine operations are subject to rigorous compliance standards
- Much time is spent solving linear elasticity problems on such structures
- Beams are modeled cheaply using beam elements but the joints are not
- Want a joint ROM parametrized on material properties and geometry (plate thickness, radius, beam angles etc)

Example



Static condensation

- Static condensation ‘compresses’ each component to a $6n \times 6n$ superelement that acts as coupling terms between otherwise disconnected beam elements.
- Works well if each component is *identical*.
- Still requires the solution of a nontrivial system.
- Not parametrized over geometry, material, etc.



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Static condensation

$$\begin{pmatrix} \mathbf{A}_{BB} & \mathbf{A}_{BI} \\ \mathbf{A}_{IB} & \mathbf{A}_{II} \end{pmatrix} \begin{pmatrix} \mathbf{u}_B \\ \mathbf{u}_I \end{pmatrix} = \begin{pmatrix} \mathbf{f}_B \\ \mathbf{f}_I \end{pmatrix}$$

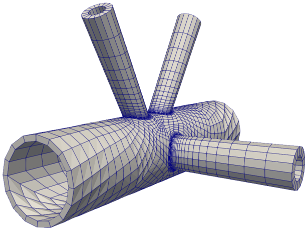
$$\mathbf{A}_{II}\mathbf{u}_I = \mathbf{f}_I - \mathbf{A}_{IB}\mathbf{u}_B$$

$$(\mathbf{A}_{BB} - \mathbf{A}_{BI}\mathbf{A}_{II}^{-1}\mathbf{A}_{IB})\mathbf{u}_B = \mathbf{f}_B - \mathbf{A}_{BI}\mathbf{A}_{II}^{-1}\mathbf{f}_I$$

Port condensation

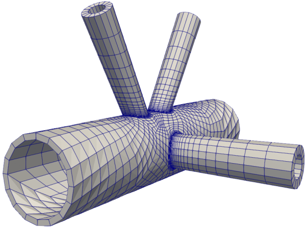
The technique follows in brief Eftang and Patera (2013).

- For each *port*, six ROMs are constructed - each evaluating the response on the structure from one degree of freedom on the boundary (three translations and three rotations)
- In addition, one ROM is constructed with *homogeneous* boundary conditions, evaluating the response from other sources, e.g. gravity



Port condensation

- In the online stage, each of the $6n$ ROMs are queried and the solutions form the internal basis functions for a $6n \times 6n$ superelement matrix
- The solution to the homogenous ROM enters the right hand side as a load term
- *Parametrized* and *faster* static condensation



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Port condensation

Let

- $\ell^{(i)}$ be the lifting function associated with port i
- $\left\{b_k^{(i)}\right\}_k$ be reduced basis functions associated with port i

Then the response due to excitation of port i is

$$\Psi^{(i)} = \ell^{(i)} + \sum_k c_k b_k^{(i)}$$

$$a(\Psi^{(i)}, v) = 0 \quad \forall v$$

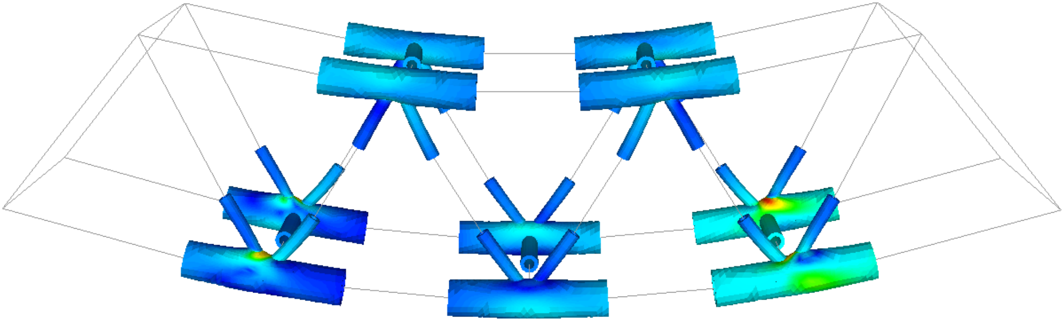
Port condensation

And the contribution to the (Galerkin) superelement is

$$
\begin{aligned}
a(\Psi^{(i)}, \Psi^{(j)}) &= a\left(\Psi^{(i)}, \ell^{(j)} + \sum_l c_l b_l^{(j)}\right) \\
&\approx a\left(\Psi^{(i)}, \ell^{(j)}\right) \\
&= a\left(\ell^{(i)}, \ell^{(j)}\right) + \sum_k c_k a\left(b_k^{(i)}, \ell^{(j)}\right)
\end{aligned}
$$

where we can discard the expensive terms due to (approximate) Galerkin orthogonality.

Example



Intrusiveness

- Many decisions made in the following are motivated by non-intrusiveness
- RBMs are typically quite intrusive: to reduce a FOM requires expert knowledge of it
- We aim to deliver a generalized RBM software package (AROMA) that is as non-intrusive as possible
  - snapshots as fully-featured solution vectors
  - stiffness matrix and load vector in restricted sense (i.e. with fixed DoFs removed and lift applied) — avaliable as debug output in most FOM software
  - knowledge of DoF classification (fixed/free) and lift — fairly mundane
- Aim is to view the FOM as a nearly opaque black box

Reduced basis methods

Substitute the solution of a large system

$$\mathbf{A}\,\mathbf{u} = \mathbf{f}$$

with a smaller one

$$\underbrace{\mathbf{V}^\top \mathbf{A}\, \mathbf{V}}_{\mathbf{A}^{\mathrm{r}}}\,\underbrace{\square}_{\mathbf{u}^{\mathrm{r}}} = \underbrace{\mathbf{V}^\top \mathbf{f}}_{\mathbf{f}^{\mathrm{r}}}$$

Solution and assembly

- Extremely fast *solutions* in the online stage: $\mathbf{A}^{\mathrm{r}}(\mu)$ frequently on the order of $\sim 10 \times 10$ or so.
- Assembly of $\mathbf{A}^{\mathrm{r}}(\mu)$ remains an open issue. On the face of it,

$$\mathbf{A}^{\mathrm{r}}(\mu) = \mathbf{V}^{\top}\mathbf{A}(\mu)\mathbf{V}$$

  requires the construction of the full-order matrix $\mathbf{A}(\mu)$ in the online stage.
- Many solutions have been proposed, most of which relates to...

The assumption of affine parametric dependence

If $\mathbf{A}(\mu)$ takes the form

$$\mathbf{A}(\mu) = \sum_i \xi_i(\mu) \mathbf{A}_i$$

then $\mathbf{A}^\mathrm{r}(\mu)$ takes the form

$$\mathbf{A}^\mathrm{r}(\mu) = \mathbf{V}^\top \mathbf{A}(\mu) \mathbf{V} = \sum_i \xi_i(\mu) \mathbf{V}^\top \mathbf{A}_i \mathbf{V} = \sum_i \xi_i(\mu) \mathbf{A}_i^\mathrm{r}$$

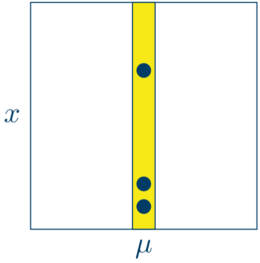
Each $\mathbf{A}_i^\mathrm{r}$ is small and easily storeable. Assuming the ξ_i are not pathological and the sum not too long, assembly in this form is competitive with time spent solving the system.

The assumption of affine parametric dependence

- Affinity tends to hold well with
  - explicit material parameters: elastic properties, viscosities, thermal conductivities
  - load and boundary data
- But not with geometric variability
  - even simple variation manifests in the integrands (for FEM) or fluxes (for FVM) in nontrivial ways (Fonn et. al., 2018)
- Approximate affine parametric dependence can be *forced* using various interpolation methods

Empirical Interpolation

- EIM requires the online stage to evaluate a function $g(x, \mu)$ at certain points x in the physical domain, or a vector-valued function $g_i(\mu)$ at certain entries i.
- Here, g is the Finite Element variational form or similar for FVM
- In the black-box FOM context we operate in, such restricted evaluation of the integrand is not possible (too intrusive)
- Instead, we resort to Matrix Least Squares



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Matrix Least Squares

Given reasonable guesses for $\xi_i(\mu)$, compute corresponding coefficient matrices $\mathbf{A}_i$ by minimizing

$$\int_\mu \left\| \mathbf{A}(\mu) - \sum_i \xi_i(\mu)\mathbf{A}_i \right\|_2^2 \mathrm{d}\mu.$$

In practice, this is implemented the same was as conventional L^2-fit on a sampling over μ:

$$\begin{aligned} \mathbf{B}_{ji} &= \xi_i(\mu_j) \\ \mathbf{A}_i &= \sum_{jk} (\mathbf{B}^\top \mathbf{B})^{-1}_{ik} \mathbf{B}_{jk} \mathbf{A}(\mu_j) \end{aligned}$$

Matrix Least Squares

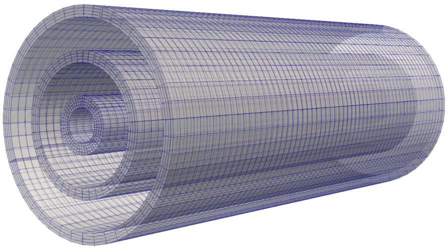
- In practice, sampling $\mathbf{A}, \mathbf{f}$ is done at the same time, and in the same parameter values as the snapshots.
- If the sampling strategy is known in advance, the contribution from $\mathbf{A}(\mu_j)$ can be immediately applied and the matrix then discarded.
- Much leeway is admitted in the choice of ξ_i
  - known affinities may be exploited, e.g.

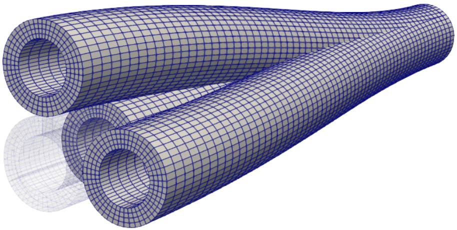
$$\xi_1(E,\nu) = \frac{E}{2(1+\nu)}, \qquad \xi_2(E,\nu) = \frac{E}{(1-2\nu)(1+\nu)}$$

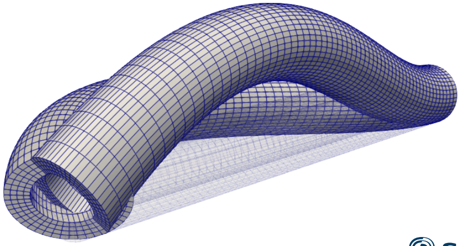
  - orthogonal polynomial families as a last resort, $\xi_i(r) = P_i(r)$
  - a combination, $\xi_i(E,\nu) = EP_i(\nu)$

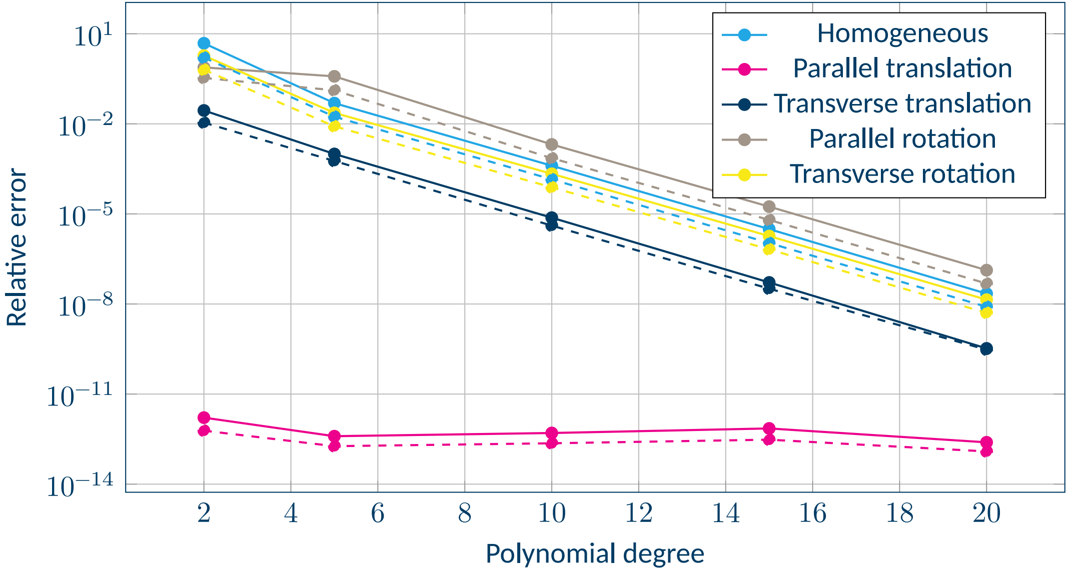
Example: two-port cylinder

Minimal example with variable radius








Homogeneous
Parallel translation
Transverse translation
Parallel rotation
Transverse rotation
10^{1}
10^{-2}
10^{-5}
10^{-8}
10^{-11}
10^{-14}
Relative error
2
4
6
8
10
12
14
16
18
20
Polynomial degree

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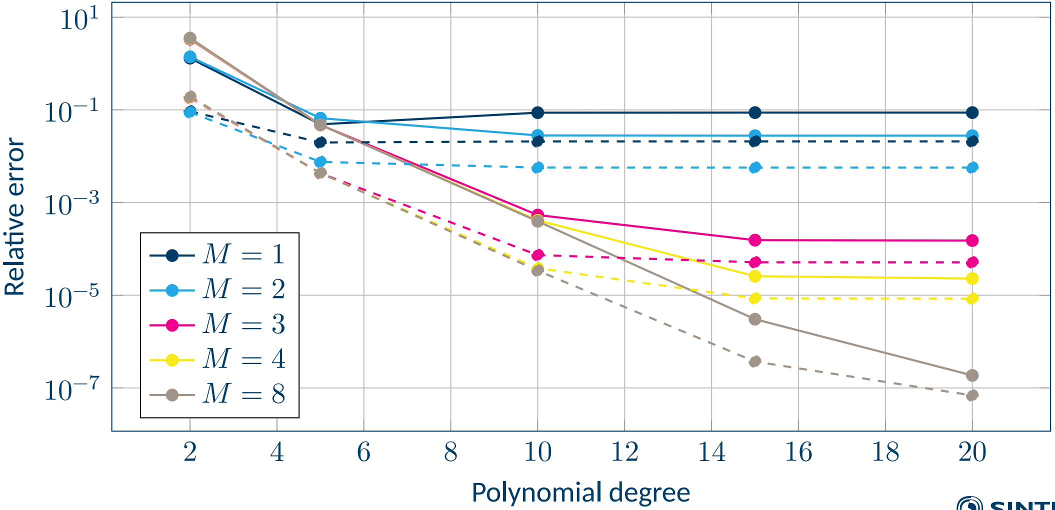
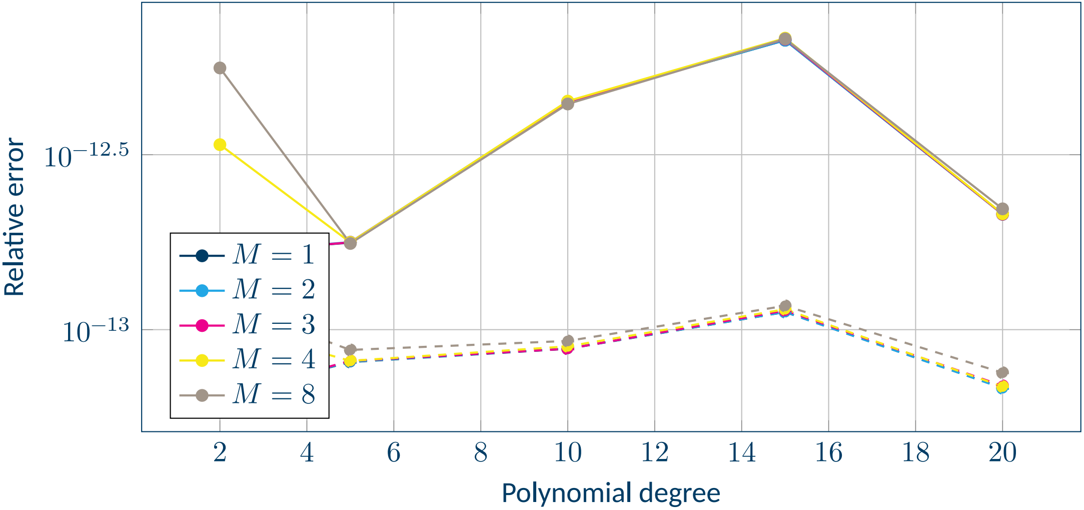
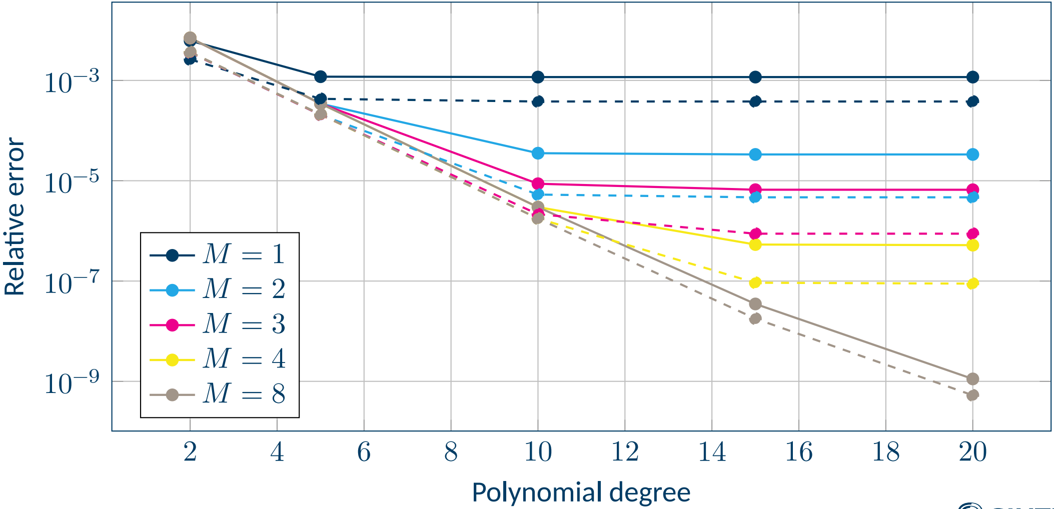
Homogenous case



Parallel translation case



Transverse translation case



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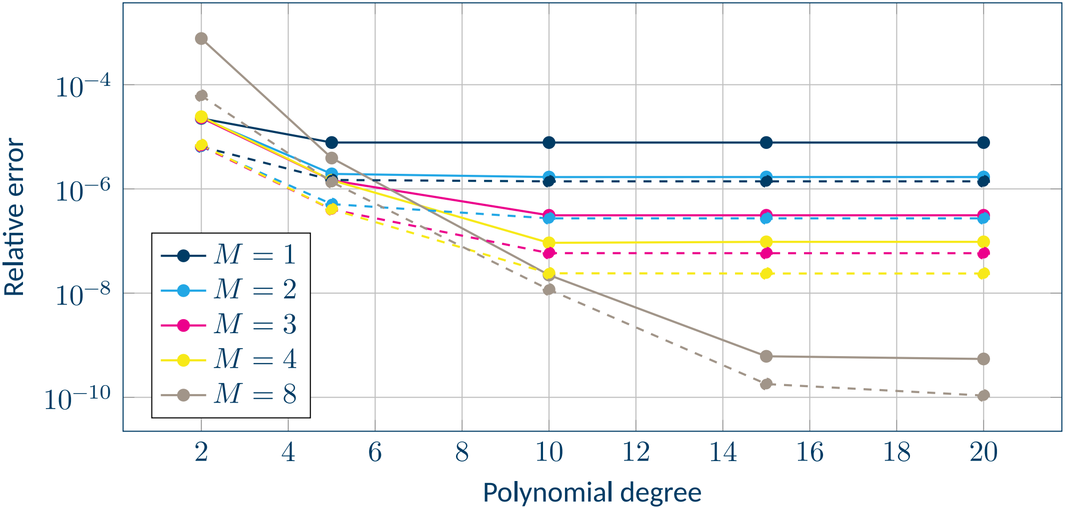
Parallel rotation case



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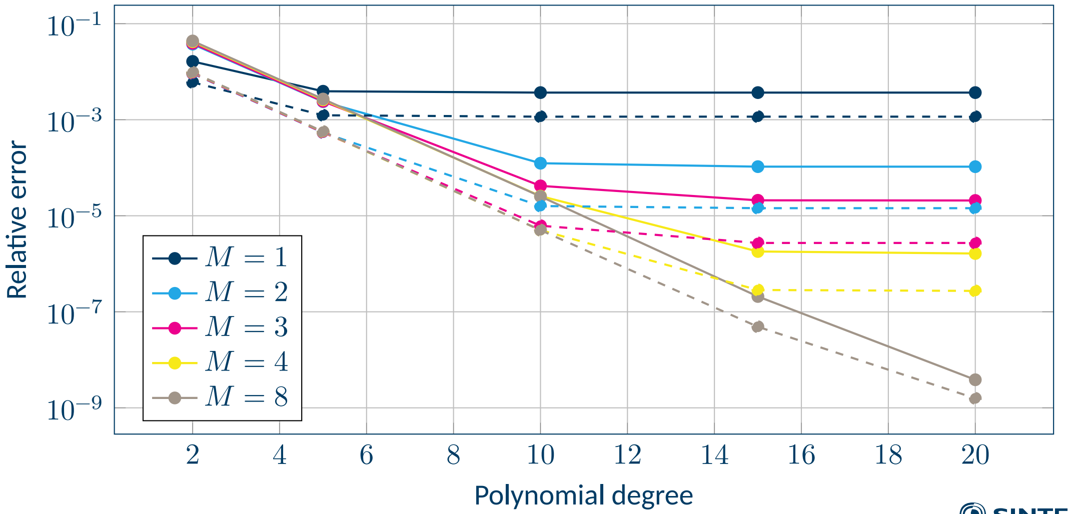
Transverse rotation case



Wish list

- Ability to both
  - not plan sampling scheme in advance, *and*
  - be able use and discard matrices on-the-fly
- Discover suitable ξ automatically
- Maintain black-box treatment of FOM
- Some more complicated case studies (meshing challenges)

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