Fast Divergence-Conforming Reduced Basis Methods for Steady Navier-Stokes Flow

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Outline

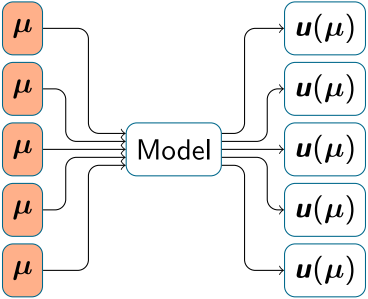
1. Motivation of RBMs
2. How RBMs are built, offline-online distinction
3. Affine representations (the catch)
4. Pretty pictures (backwards-facing step channel)
5. Where divergence-conformity enters into it
6. Pressure reconstruction
7. Flow around NACA0015 airfoil
8. Summary

Parameter-dependent models

$$\boxed{\boldsymbol{\mu}} \longrightarrow \boxed{\text{Model}} \longrightarrow \boxed{\boldsymbol{u}(\boldsymbol{\mu})}$$

- We are interested generating solutions $\boldsymbol{u}(\boldsymbol{\mu})$ to a physical model that depend on a set of pre-determined *parameters*, $\boldsymbol{\mu} \in \mathcal{P}$.
- viscosity, heat conductivity, varying boundary conditions, geometric variations, etc.

Parameter-dependent models



- Often we need repeated and quick sampling of the parameter space.
- Parameters may be unknown until shortly before a solution is needed.
- optimization, control, etc.

Dimensional reduction

- With conventional (read: FEM, FVM, FDM) methods, this may be impractical if not impossible.
- Too many DoFs N to finish in a realistic timeframe.
- Usually,

$$M = \dim\left(\{\boldsymbol{u}(\boldsymbol{\mu}) \,|\, \boldsymbol{\mu} \in \mathcal{P}\}\right) \ll N$$

- Idea: create a model with number of DoFs closely matching the physical dimension of the problem.
- Often, $M \sim 100$ or so!

Optimal reduced basis

If we knew enough about "typical" solutions, we could create an optimal basis in the following sense.

Let $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_M\}$ be a reduced basis, and let $\mathcal{I}_\mathcal{B} : u(\mathcal{P}) \to \operatorname{span} \mathcal{B}$ be the optimal-approximation projection in some norm $\| \cdot \|_a$. Let

$$\epsilon^2(\mathcal{B}) = \int_\mathcal{P} \|\boldsymbol{u}(\mu) - \mathcal{I}_\mathcal{B} \boldsymbol{u}(\boldsymbol{\mu})\|_a^2 \, \mathrm{d}\boldsymbol{\mu}.$$

Then $\mathcal{B}_{\text{opt}}$ is optimal if

$$\epsilon(\mathcal{B}) < \epsilon(\mathcal{B}_{\text{opt}}) \Longrightarrow \sharp\mathcal{B} > \sharp\mathcal{B}_{\text{opt}}.$$

In other words, no other basis of the same size can achieve a better ϵ.

Good-enough reduced basis

We can get close by employing *Proper Orthogonal Decomposition* (POD) on a suitably large *ensemble* of *snapshot* solutions.

The reduced basis is then optimal in the discrete quadrature sense of

$$\epsilon^2(\mathcal{B}) = \int_{\mathcal{P}} \|\boldsymbol{u}(\mu) - \mathcal{I}_{\mathcal{B}}\boldsymbol{u}(\boldsymbol{\mu})\|_a^2 \,\mathrm{d}\boldsymbol{\mu}.$$

In fact, the problem of choosing parameter values for our ensemble is equivalent to making a quadrature rule on $\mathcal{P}$.

The vision

Parametrized PDE

High-fidelity discr.

$$\boldsymbol{A}_h(\boldsymbol{\mu})\boldsymbol{u}_h(\boldsymbol{\mu}) = \boldsymbol{f}_h(\boldsymbol{\mu})$$

$$\boldsymbol{A}_h(\boldsymbol{\mu}) = \sum_i \xi_i(\boldsymbol{\mu})\boldsymbol{A}_i$$

$$\boldsymbol{f}_h(\boldsymbol{\mu}) = \sum_i \chi_i(\boldsymbol{\mu})\boldsymbol{f}_i$$

Generate a RB by applying POD to snapshot solutions, and construct projection matrix $\boldsymbol{V}$

Projection

$$\boldsymbol{A}_i^M = \boldsymbol{V}^\intercal \boldsymbol{A}_i \boldsymbol{V}$$

$$\boldsymbol{f}_i^M = \boldsymbol{V}^\intercal \boldsymbol{f}_i$$

$\boldsymbol{\mu}$

RB system assembly

$$\boldsymbol{A}_M = \sum_i \xi_q(\boldsymbol{\mu})\boldsymbol{A}_i^M$$

$$\boldsymbol{f}_M = \sum_i \chi_q(\boldsymbol{\mu})\boldsymbol{f}_i^M$$

RB system solution

$$\boldsymbol{A}_M(\boldsymbol{\mu})\boldsymbol{u}_M(\boldsymbol{\mu}) = \boldsymbol{f}_M(\boldsymbol{\mu})$$

Error estimate

$$\|\boldsymbol{u}_h(\boldsymbol{\mu}) - \boldsymbol{V}\boldsymbol{u}_M(\boldsymbol{\mu})\|$$

Recovery, visualization and post-processing

$$\boldsymbol{u}_M^h(\boldsymbol{\mu}) = \boldsymbol{V}\boldsymbol{u}_M(\boldsymbol{\mu})$$

Evaluating functionals of interest

Our guiding principle

Any* extra cost in the offline stage is worth paying,
no matter how much, if it makes the online stage faster.

Our guiding principle

All is fair in love, war and the offline stage.
— John Lyly (*Euphues*; 1579)

Assembly

- Why the insistence on forms like

$$\boldsymbol{A}_h(\boldsymbol{\mu}) = \sum_i \xi_i(\boldsymbol{\mu}) \boldsymbol{A}_i, \qquad \boldsymbol{f}_h(\boldsymbol{\mu}) = \sum_i \chi_i(\boldsymbol{\mu}) \boldsymbol{f}_i$$

- Because it makes *assembly* of reduced models fast.
- Each $\boldsymbol{A}_i$ and $\boldsymbol{f}_i$ can be projected independently onto a reduced basis and stored.
- This makes the online stage completely high-fidelity-agnostic.
- Deriving these *affine representations* is the core detail of RBM.

Example: The backwards-facing step channel

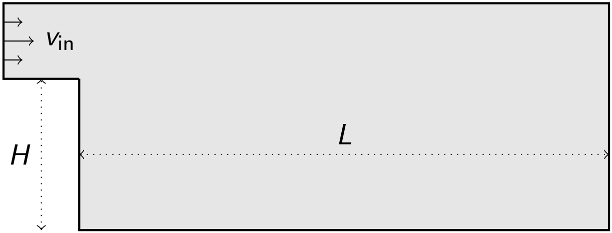
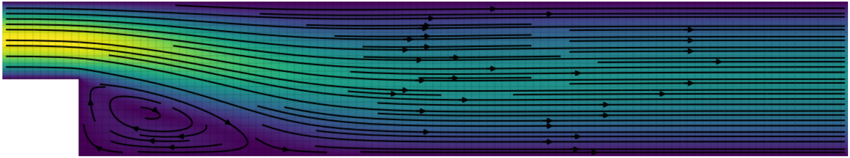


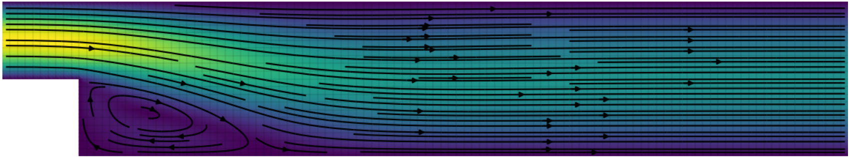
Figure: Problem domain with parameters. Additionally, the fluid viscosity ν is also a parameter. (See Quarteroni et. al. for more information.)

Example: The backwards-facing step channel



High-fidelity: 8 seconds.

Example: The backwards-facing step channel



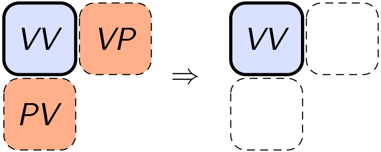
Reduced basis with 30 velocity DoFs: 20 milliseconds.

An idea

- In conventional solvers, pressure enters as a Lagrange multiplier to enforce the continuity equation.
- Incompatibilities between the velocity and pressure spaces are a frequent source of difficulties. (This is why div-conforming IGA formulations are useful.)
- If the velocity space were *a priori* divergence free, you wouldn't need a pressure field at all. You could solve directly for pressure and, if necessary, reconstruct the pressure in post.

An idea (cont.)

- This is usually not practical because divergence-free basis functions are difficult to make.
- Since reduced basis functions are derived from high-fidelity solutions (which are nearly or exactly divergence-free), it seems reasonable to expect them to be divergence-free too.



Divergence-free basis

Hi-Fi solutions

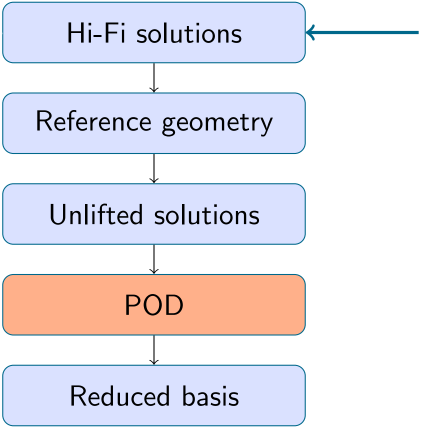
Reference geometry

Unlifted solutions

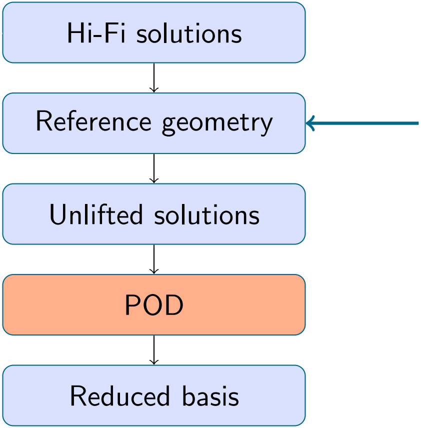
POD

Reduced basis

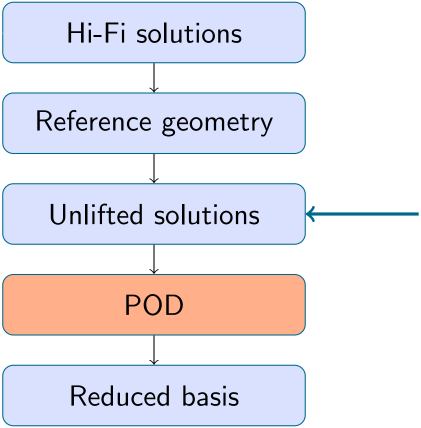
Divergence-free basis



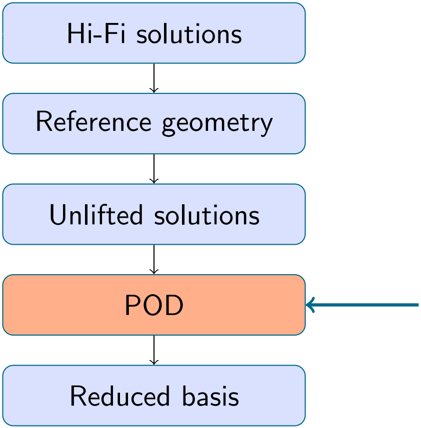
Divergence-free basis



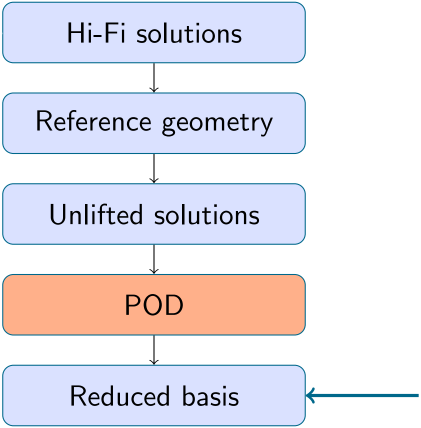
Divergence-free basis



Divergence-free basis



Divergence-free basis

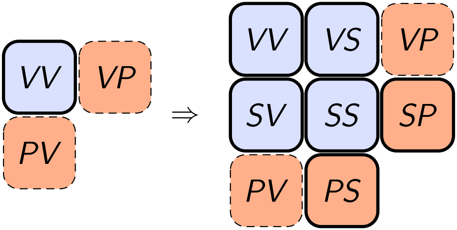


Divergence-free basis

- The mapping from reference to physical geometry needs to preserve divergence-free functions (the Piola mapping).
- Lifting functions must also be divergence-free.
- That's it!

Supremizers

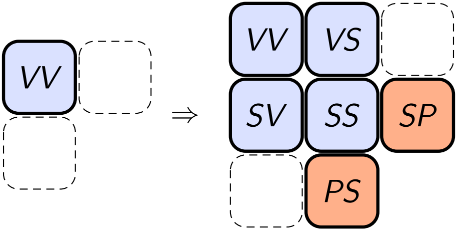
To solve for pressure, we leverage a technique from Ballarin et. al. (2015). There, the velocity space is enriched with *supremizers* to stabilize and control the LBB condition.



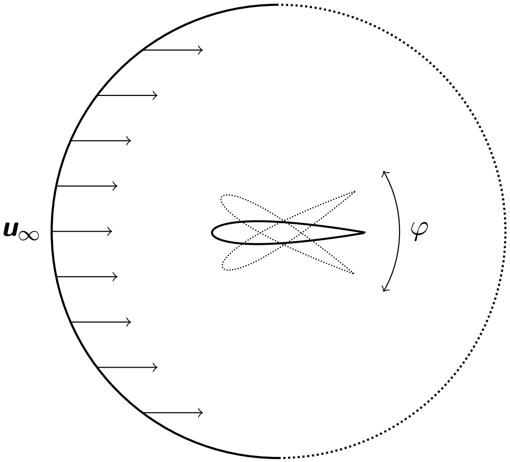
Supremizers are maximizers of the “sup” part of the inf-sup expression.

Supremizers (cont.)

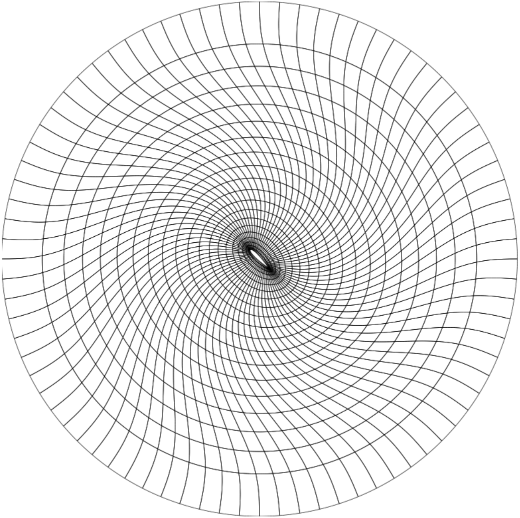
For the same reason, supremizers are natural test functions when solving the momentum equation for pressure.



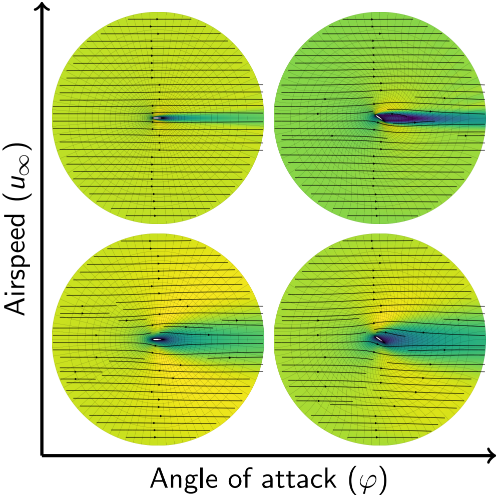
Flow around airfoil



Domain transformation



Parameter space



Affine representations

- We will try two high-fidelity methods: a TH (1,2)-method and an IGA (1,2) divergence-conforming method.
- Not possible to express the Navier Stokes problem as finite sums

$$\boldsymbol{A}_h(\boldsymbol{\mu}) = \textstyle\sum_i \xi_i(\boldsymbol{\mu})\boldsymbol{A}_i, \qquad \boldsymbol{f}_h(\boldsymbol{\mu}) = \textstyle\sum_i \chi_i(\boldsymbol{\mu})\boldsymbol{f}_i$$

- Instead, we use truncated series expansions.
- For a maximal angle of attack of 35° we can expect about 10 digits of accuracy with a reasonable number of terms ($\sim$ 25 for TH, $\sim$ 75 for DC).

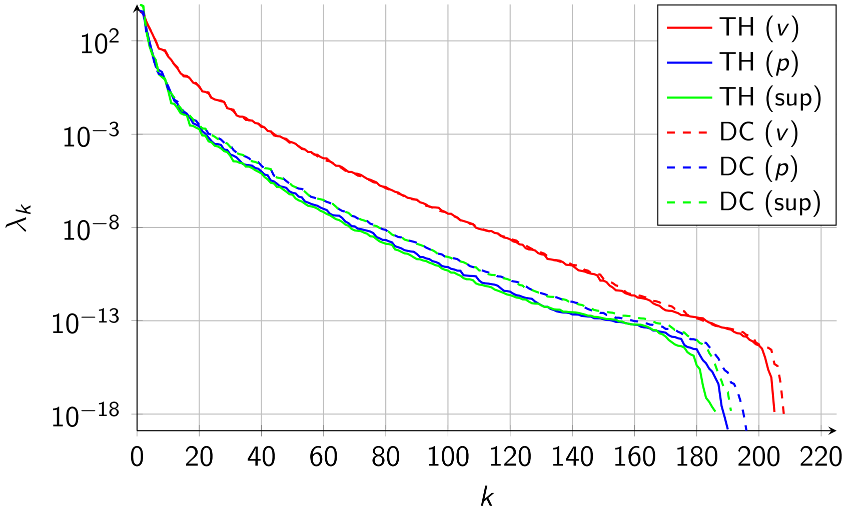
Ensembles

- We calculated 15×15 solutions at the Gauss quadrature nodes.
- The parameter domain was chosen as

$$\mathcal{P} = [-35^\circ, +35^\circ] \times [1\,\mathrm{m/s}, 20\,\mathrm{m/s}].$$

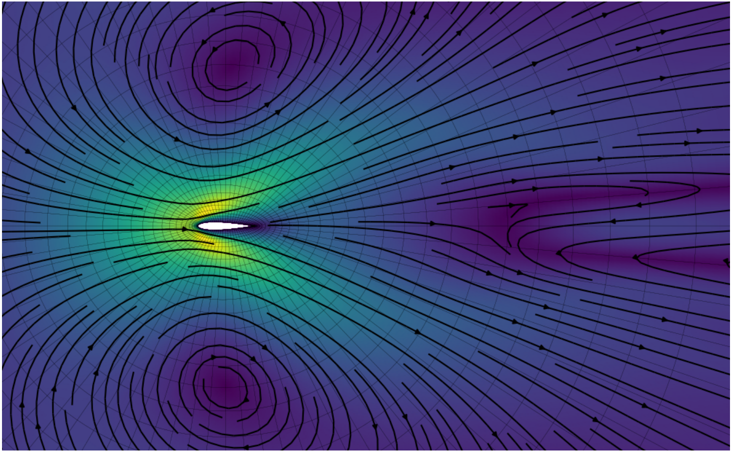
- Only *stationary* Navier-Stokes, with $\nu = \frac{1}{6}$.

Spectrum

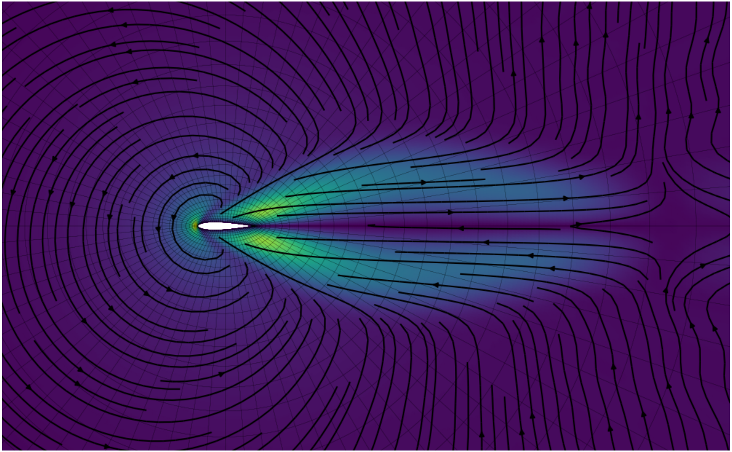


λ_k
10^{2}
10^{-3}
10^{-8}
10^{-13}
10^{-18}
0
20
40
60
80
100
120
140
160
180
200
220
k
TH (v)
TH (p)
TH (sup)
DC (v)
DC (p)
DC (sup)

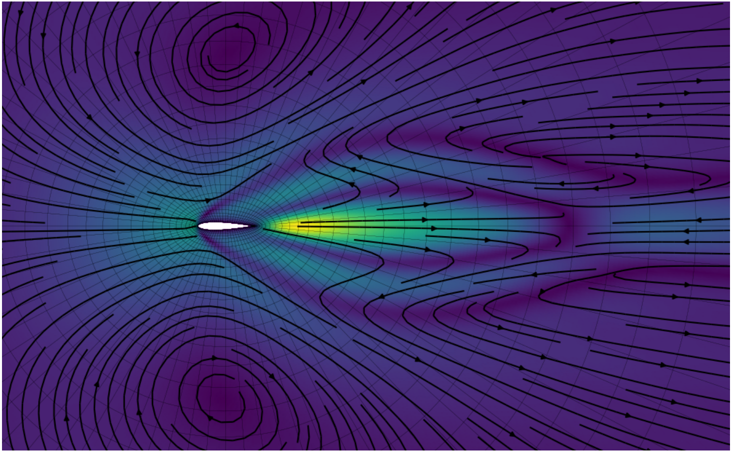
Basis functions (v, TH, 1)



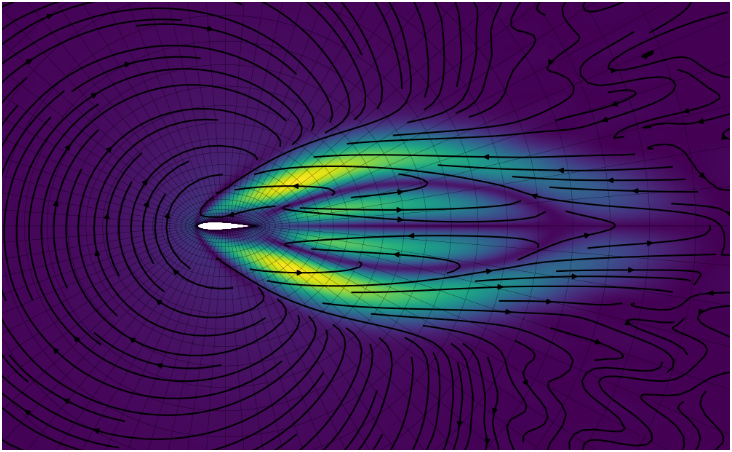
Basis functions (v, TH, 2)



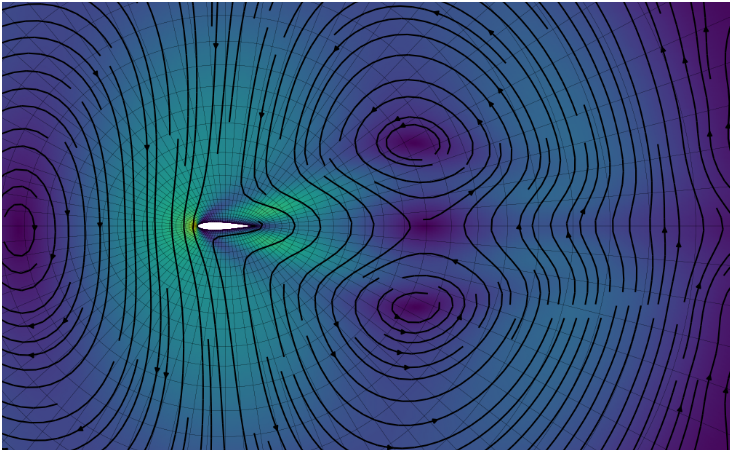
Basis functions (v, TH, 3)



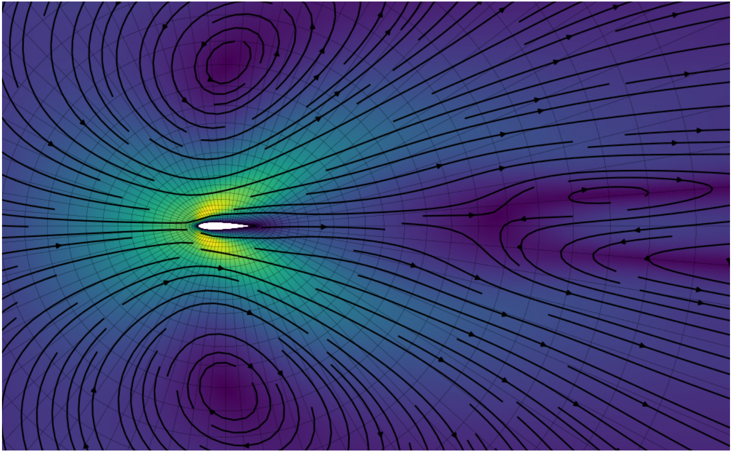
Basis functions (v, TH, 4)



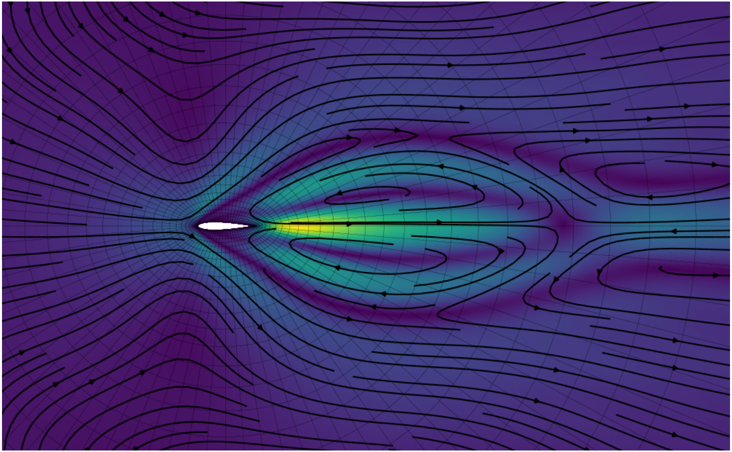
Basis functions (v, DC, 1)



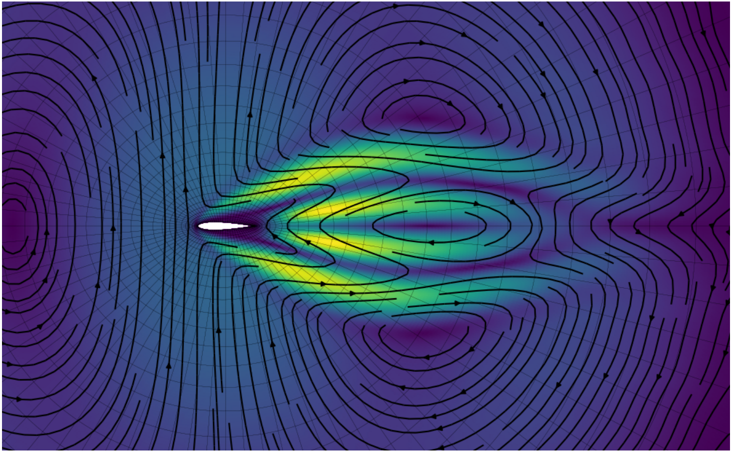
Basis functions (v, DC, 2)



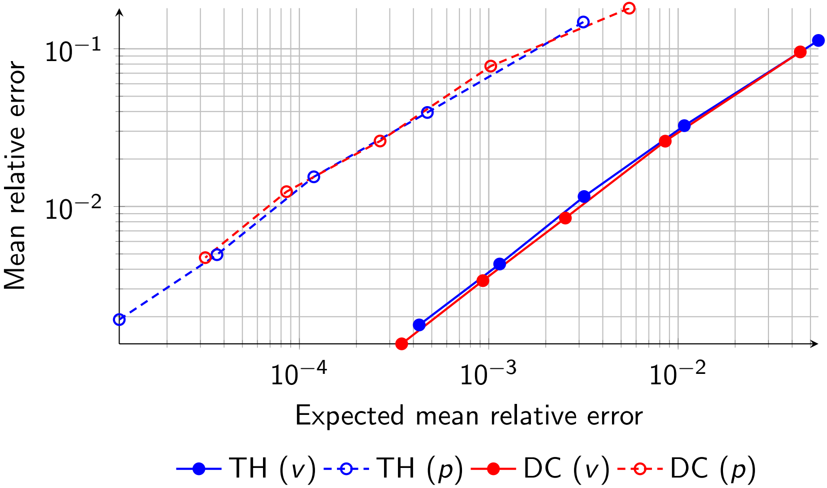
Basis functions (v, DC, 3)



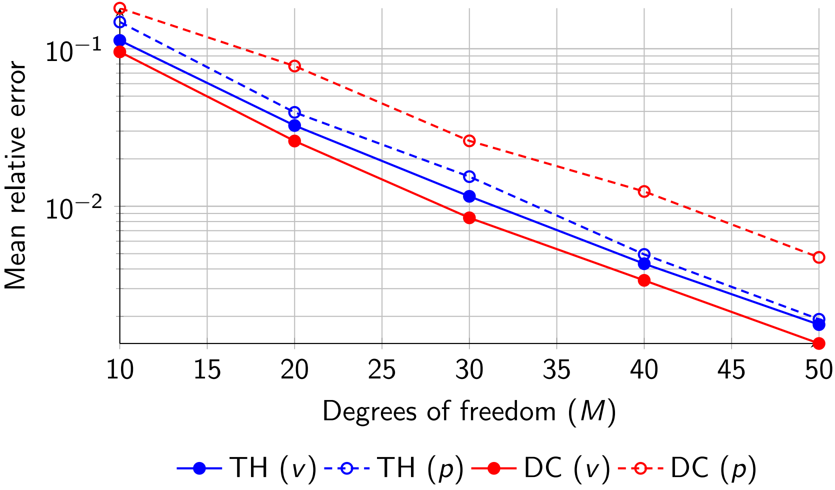
Basis functions (v, DC, 4)



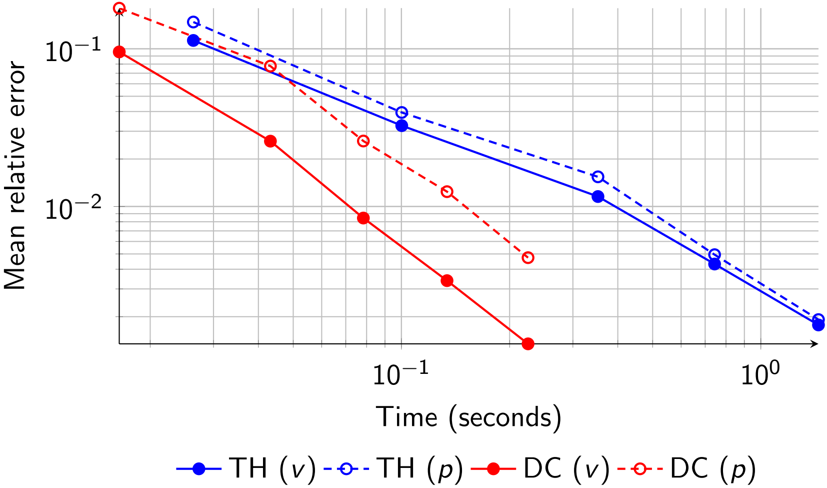
Convergence



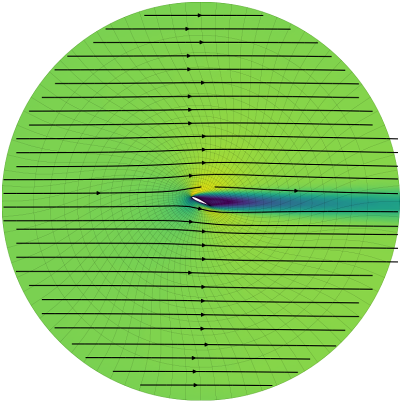
Convergence



Convergence

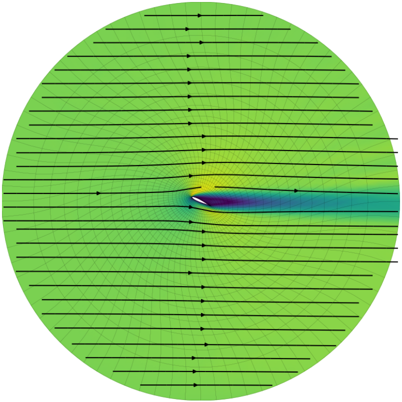


Solutions



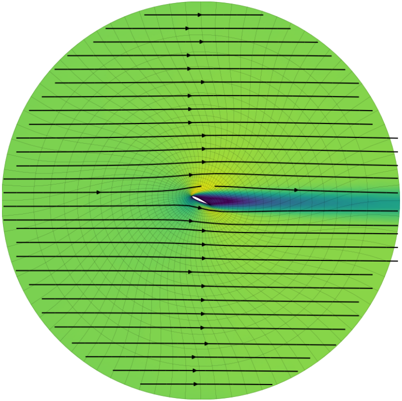
Full TH solution: 39 seconds.

Solutions



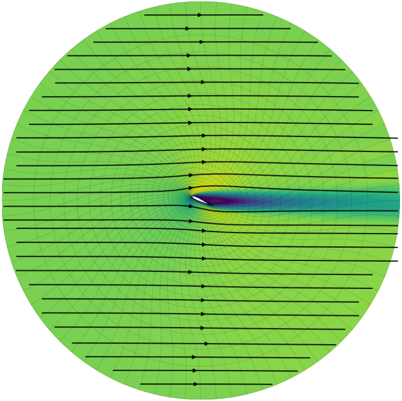
Full DC solution: 113 seconds.*

Solutions



Reduced TH solution (30 DoFs): 523 milliseconds.

Solutions



Reduced DC solution (30 DoFs): 96 milliseconds.

Summary

- Reduced order models can offer dramatic speed-ups for certain applications.
- They combine nicely with IGA and div-compatible spaces to form fully divergence-free function spaces without need for pressure fields.
- Divergence-free RBMs can be much faster than other RBMs, in spite of additional complexity in the offline stage (remember, all is fair there.)

Thanks!