Reduced Order Models for Divergence-Conforming Isogeometric Flow Simulations

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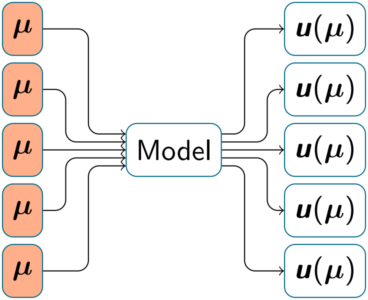
Parameter-dependent models

$$\boxed{\boldsymbol{\mu}} \longrightarrow \boxed{\text{Model}} \longrightarrow \boxed{\boldsymbol{u}(\boldsymbol{\mu})}$$

- We are interested generating solutions $\boldsymbol{u}(\boldsymbol{\mu})$ to a physical model that depend on a set of pre-determined *parameters*, $\boldsymbol{\mu} \in \mathcal{P}$.
- Parameters can be: viscosity, heat conductivity, varying boundary conditions, geometry changes, etc.

Parameter-dependent models

Motivation: *many-query* applications. E.g. control systems, optimization, inverse problems and real-time responsiveness.



Dimensional reduction

- With conventional (read: FEM, FVM, FDM, and yes, even IGA) methods, this may be impractical if not impossible.
- Too many DoFs N to finish in a realistic timeframe.
- Usually,

$$M = \dim\left(\{\boldsymbol{u}(\boldsymbol{\mu}) \,|\, \boldsymbol{\mu} \in \mathcal{P}\}\right) \ll N$$

- Idea: create a model with number of DoFs closely matching the physical dimension of the problem.
- Often, $M \sim 100$ or so!

The vision[1]

Parametrized PDE

High-fidelity discr.

$$A_h(\mu)u_h(\mu) = f_h(\mu)$$
$$A_h(\mu) = \sum_i \xi_i(\mu)A_i$$
$$f_h(\mu) = \sum_i \chi_i(\mu)f_i$$

Generate a RB by applying POD to snapshot solutions, and construct projection matrix $\boldsymbol{V}$

Projection

$$A_i^M = V^\mathsf{T} A_i V$$
$$f_i^M = V^\mathsf{T} f_i$$

$\boldsymbol{\mu}$

RB system assembly

$$A_M = \sum_i \xi_i(\mu)A_i^M$$
$$f_M = \sum_i \chi_i(\mu)f_i^M$$

RB system solution

$$A_M(\mu)u_M(\mu) = f_M(\mu)$$

Error estimate

$$\|u_h(\mu) - Vu_M(\mu)\|$$

Recovery, visualization and post-processing

$$u_M^h(\mu) = Vu_M(\mu)$$

Evaluating functionals of interest

[1] See Quarteroni, Manzoni, and Negri 2016.

Our guiding principle

Any* extra cost in the offline stage is worth paying,
no matter how much, if it makes the online stage faster.

Our guiding principle

All is fair in love, war and the offline stage.
— John Lyly (*Euphues*; 1579)

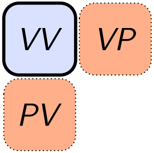
Assembly

- Why the insistence on forms like

$$\boldsymbol{A}_h(\boldsymbol{\mu}) = \sum_i \xi_i(\boldsymbol{\mu})\boldsymbol{A}_i, \qquad \boldsymbol{f}_h(\boldsymbol{\mu}) = \sum_i \chi_i(\boldsymbol{\mu})\boldsymbol{f}_i$$

- Because it makes *assembly* of reduced models fast.
- Each $\boldsymbol{A}_i$ and $\boldsymbol{f}_i$ can be projected independently onto a reduced basis and stored.
- This makes the online stage completely high-fidelity-agnostic.
- Deriving these *affine representations* is the core detail of RBM.

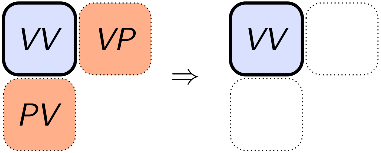
Anatomy of a reduced system



- Usually, the reduced method does not inherit the stability properties of the high-fidelity method.
- Rank-deficient velocity-pressure blocks are common.
- Carefully selecting number of velocity modes and pressure modes seems to help?

$$M_{\mathrm{V}} \approx d M_{\mathrm{P}}$$

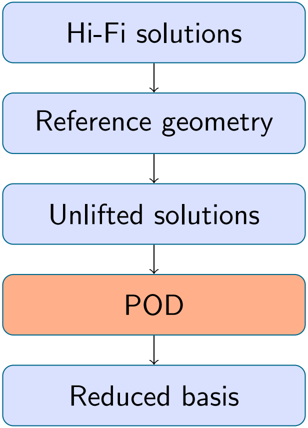
Anatomy of a reduced system (cont.)



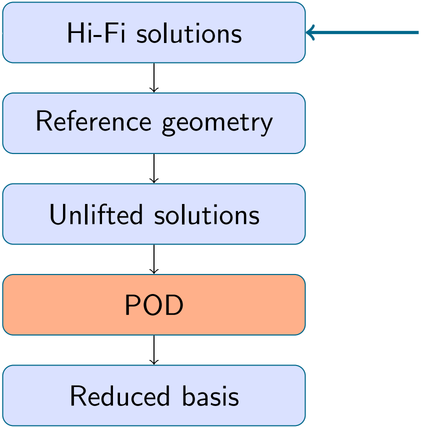
- A divergence-free reduced basis eliminates the coupling, leading to a fully stable velocity-only formulation.
- IGA[2] enables divergence-free high-fidelity solutions,[3] therefore also divergence-free reduced bases.

[2] Hughes, Cottrell, and Bazilevs 2005; Cottrell, Hughes, and Bazilevs 2009.
[3] Evans and Hughes 2013.

Divergence-free basis



Divergence-free basis



Divergence-free basis

Hi-Fi solutions

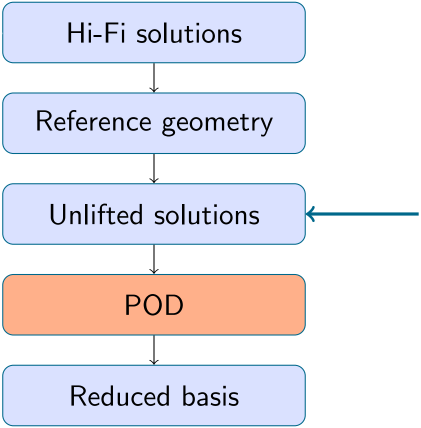
Reference geometry

Unlifted solutions

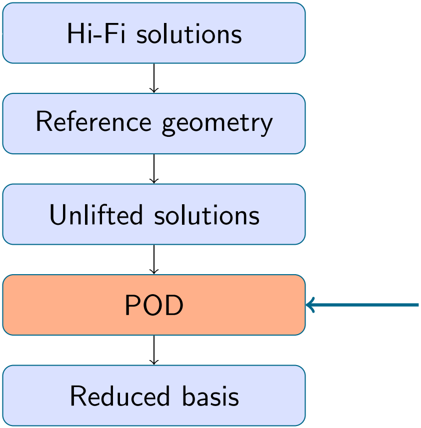
POD

Reduced basis

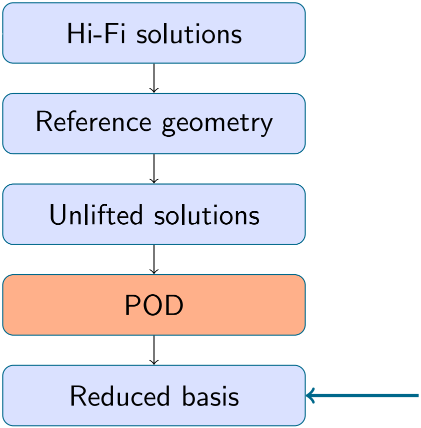
Divergence-free basis



Divergence-free basis



Divergence-free basis



Pressure recovery

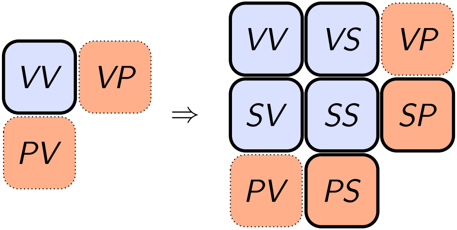
- The simplest approach is to attach pressure data to divergence-free velocity basis functions.
- Pressure recovery becomes "free" (no linear system to solve, use coefficients from RB velocity solution).
- However, this implies a linear velocity-pressure relationship, in violation of e.g. Navier-Stokes.
- Could work depending on problem.

Pressure recovery

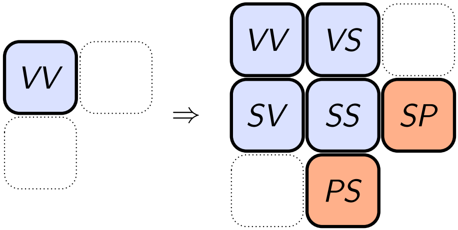
- Another approach is to solve the momentum equation with a different test space.
- The "optimal" test functions are *supremizers*, maximizers of the "sup" part of the inf-sup condition, for any specific pressure solution.
- Supremizers form a reduced space just like velocity and pressure do.
- Supremizers are commonly used to enrich non-divergence-free reduced velocity spaces for stability.[4]

[4] Ballarin et al. 2015.

Anatomy of a reduced system (cont.)

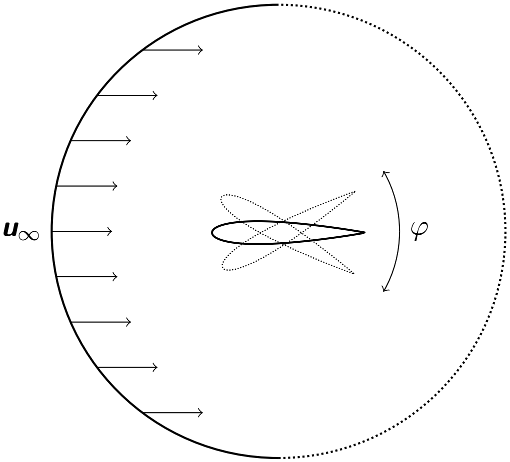


Anatomy of a reduced system (cont.)

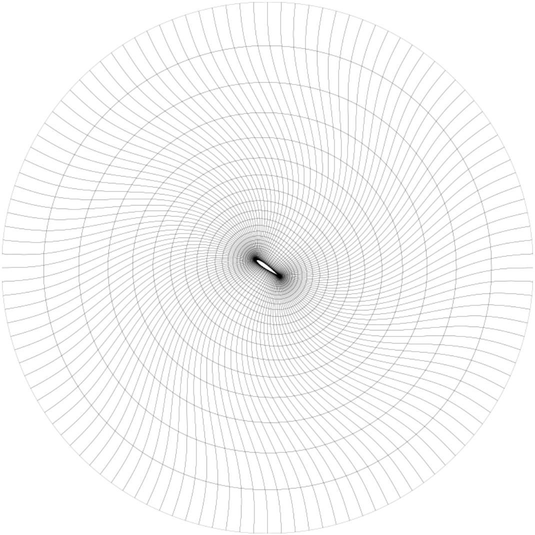


Note the block triangular structure.

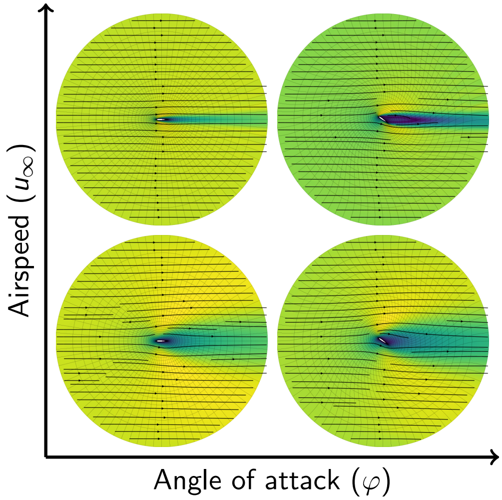
Flow around airfoil



Domain transformation



Parameter space



Problem specification

- We will try two high-fidelity methods: a Taylor-Hood (1,2)-method and an IGA (1,2) divergence-conforming method, with both approaches to pressure recovery.
- The parameter domain was chosen as

$$\mathcal{P} = [-35^\circ, +35^\circ] \times [1\,\mathrm{m/s}, 20\,\mathrm{m/s}].$$

- We can expect about 10 digits of accuracy with a reasonable number of terms (∼25 for TH, ∼75 for DC).
- Only *stationary* Navier-Stokes, with $\nu = \frac{1}{6}$.
- We chose equal number of modes in all spaces: $N_{\mathrm{V}} = N_{\mathrm{S}} = N_{\mathrm{P}} = M$.

Affine representations

- Not possible to express the Navier-Stokes problem as finite sums

$$\boldsymbol{A}_h(\boldsymbol{\mu}) = \sum_i \xi_i(\boldsymbol{\mu})\boldsymbol{A}_i, \qquad \boldsymbol{f}_h(\boldsymbol{\mu}) = \sum_i \chi_i(\boldsymbol{\mu})\boldsymbol{f}_i$$

- Instead, we use truncated polynomial series in φ.[5]
- Recall: the intention is to encode *all* parameters explicitly in the representation of the bi- or trilinear forms.

[5] Fonn et al. 2018.

Affine representations (TH)

$$(\pi_{\mu}^{*}a)(\hat{\boldsymbol{u}}, \hat{\boldsymbol{w}}; \varphi) = \nu \int_{\hat{\Omega}} \nabla \hat{\boldsymbol{u}} : \nabla \hat{\boldsymbol{w}} + \nu\varphi \int_{\hat{\Omega}} \nabla \hat{\boldsymbol{u}} : (\boldsymbol{D}_1 \nabla)\hat{\boldsymbol{w}} - \nu\varphi^2 \int_{\hat{\Omega}} \nabla \hat{\boldsymbol{u}} : (\boldsymbol{D}_2 \nabla)\hat{\boldsymbol{w}}$$

$$(\pi_{\mu}^{*}b)(\hat{p}, \hat{\boldsymbol{w}}; \varphi) \approx \sum_{i=0}^{2n} \varphi^i \int_{\hat{\Omega}} \hat{p}\boldsymbol{B}_i^{(-)} : \nabla \hat{\boldsymbol{w}}$$

$$(\pi_{\mu}^{*}c)(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}; \varphi) \approx \sum_{i=0}^{2n} \varphi^i \int_{\hat{\Omega}} (\hat{\boldsymbol{u}} \cdot \boldsymbol{B}_i^{(-)} \nabla)\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{w}}$$

Affine representations (IGA)

$$(\pi_{\mu}^{*}a)(\hat{\boldsymbol{u}},\hat{\boldsymbol{w}};\varphi)=\sum_{i,j=0}^{2n}\varphi^{i+j}\int_{\hat{\Omega}}\nabla(\boldsymbol{B}_i^{(+)}\hat{\boldsymbol{u}}):\nabla(\boldsymbol{B}_j^{(+)}\hat{\boldsymbol{w}})$$

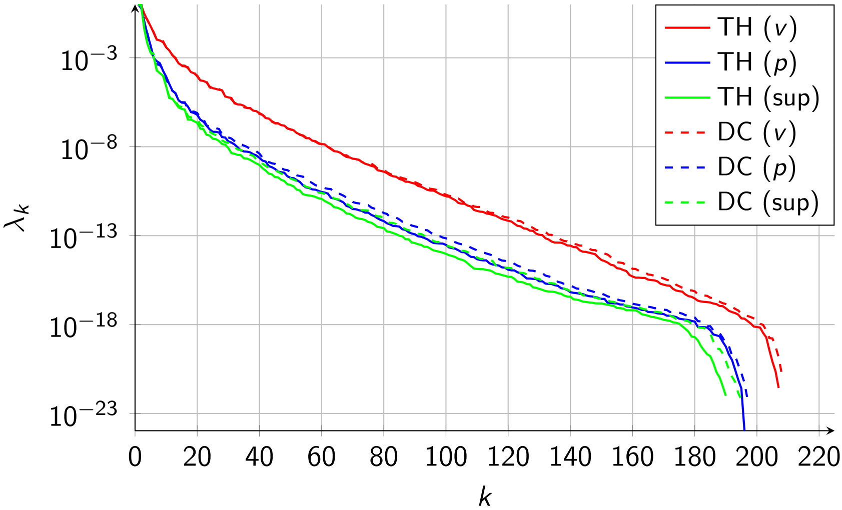
$$+\sum_{i,j=0}^{2n}\varphi^{i+j+1}\int_{\hat{\Omega}}\nabla(\boldsymbol{B}_i^{(+)}\hat{\boldsymbol{u}}):(\boldsymbol{D}_1\nabla)(\boldsymbol{B}_j^{(+)}\hat{\boldsymbol{w}})$$

$$-\sum_{i,j=0}^{2n}\varphi^{i+j+2}\int_{\hat{\Omega}}\nabla(\boldsymbol{B}_i^{(+)}\hat{\boldsymbol{u}}):(\boldsymbol{D}_2\nabla)(\boldsymbol{B}_j^{(+)}\hat{\boldsymbol{w}})$$

$$(\pi_{\mu}^{*}b)(\hat{p},\hat{\boldsymbol{w}};\varphi)=\sum_{i,j=0}^{2n}\varphi^{i+j}\int_{\hat{\Omega}}\hat{p}\boldsymbol{B}_i^{(-)}:\nabla\left(\boldsymbol{B}_j^{(+)}\hat{\boldsymbol{w}}\right)$$

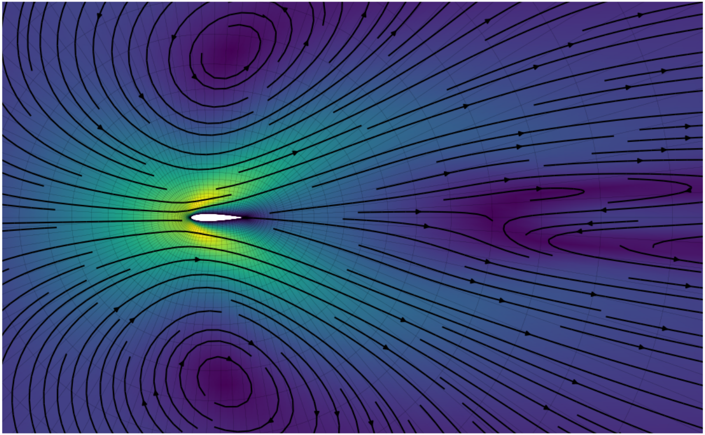
$$(\pi_{\mu}^{*}c)(\hat{\boldsymbol{u}},\hat{\boldsymbol{v}},\hat{\boldsymbol{w}};\varphi)=\sum_{i,j=0}^{2n}\varphi^{i+j}\int_{\hat{\Omega}}(\hat{\boldsymbol{u}}\cdot\nabla)\boldsymbol{B}_i^{(+)}\hat{\boldsymbol{v}}\cdot\boldsymbol{B}_j^{(+)}\hat{\boldsymbol{w}}.$$

Spectrum

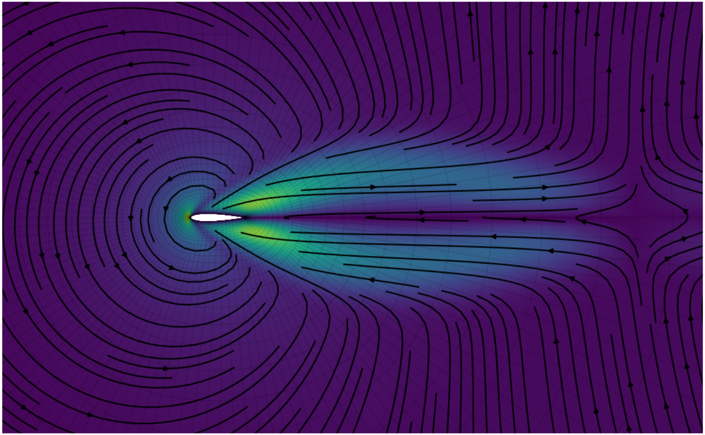


TH (v)
TH (p)
TH (sup)
DC (v)
DC (p)
DC (sup)
λ_k
10^{-3}
10^{-8}
10^{-13}
10^{-18}
10^{-23}
0 20 40 60 80 100 120 140 160 180 200 220
k

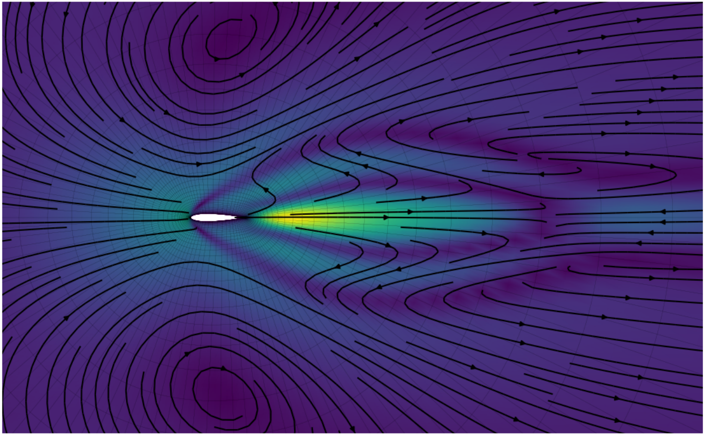
Basis functions (v, TH, 1)



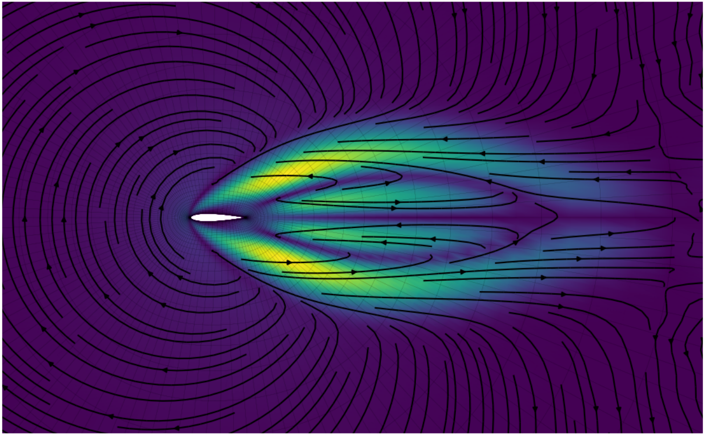
Basis functions (v, TH, 2)



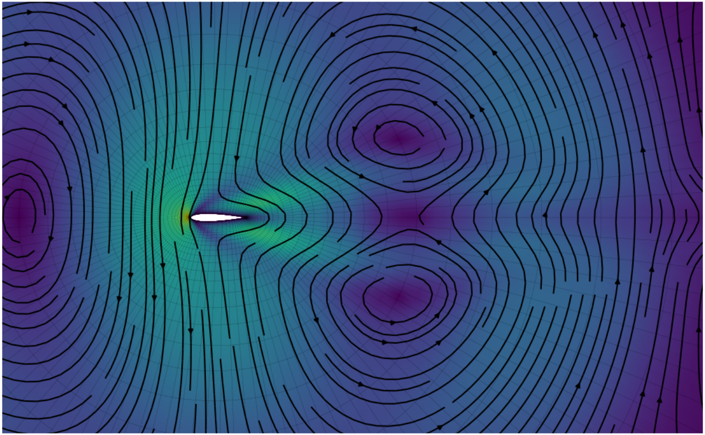
Basis functions (v, TH, 3)



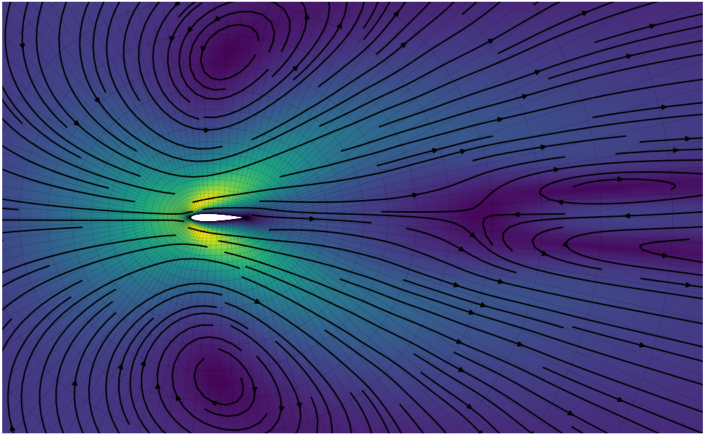
Basis functions (v, TH, 4)



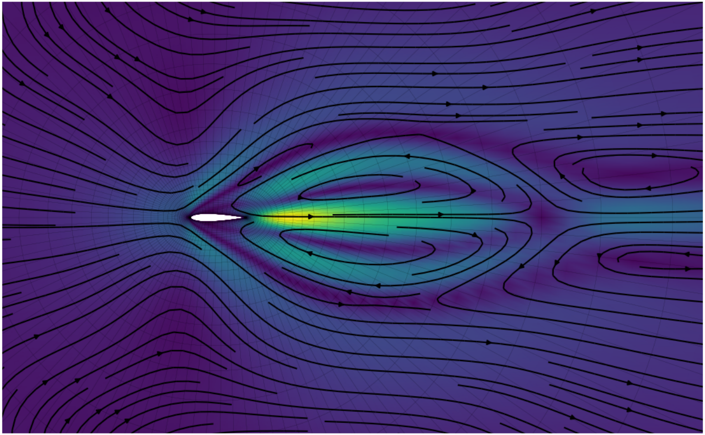
Basis functions (v, DC, 1)



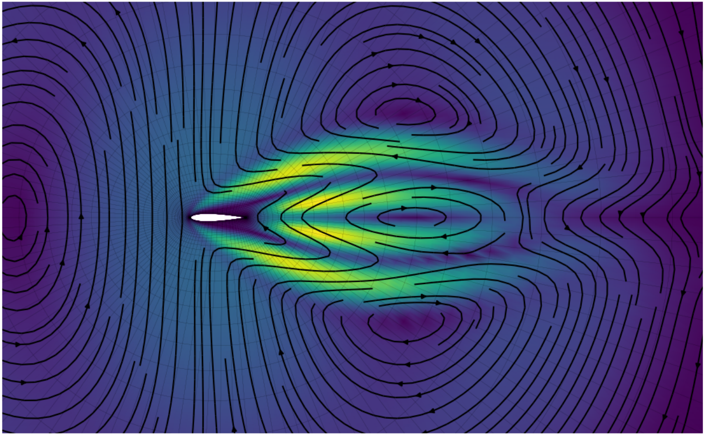
Basis functions (v, DC, 2)



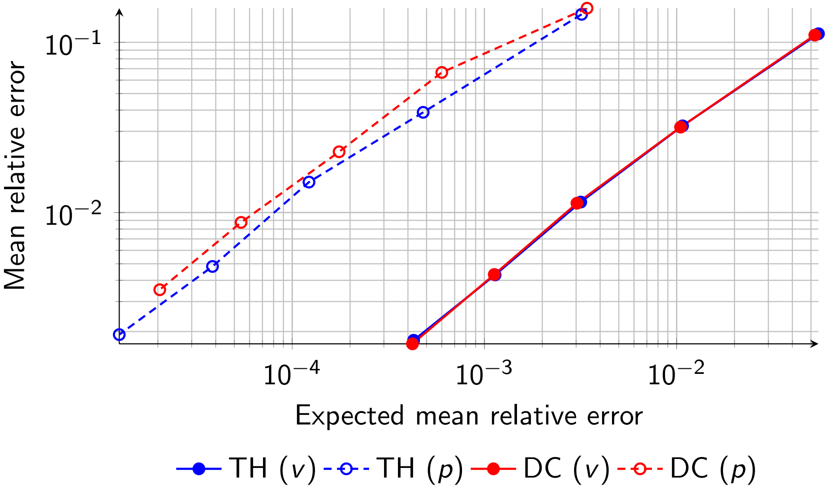
Basis functions (v, DC, 3)



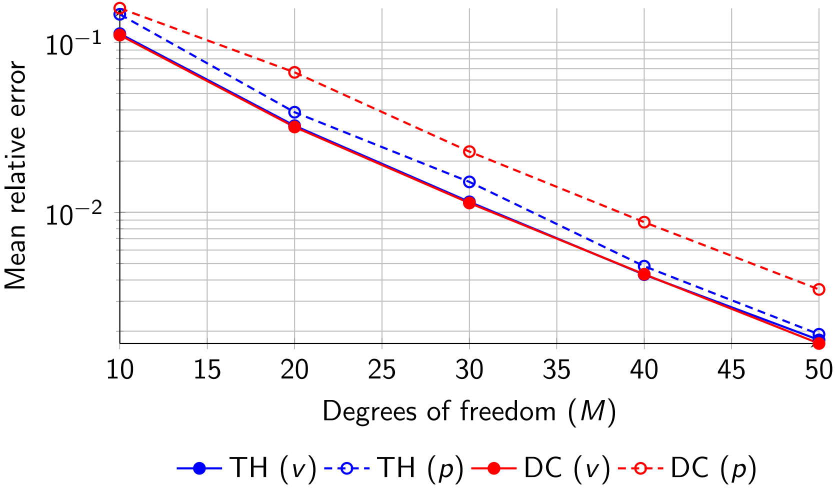
Basis functions (v, DC, 4)



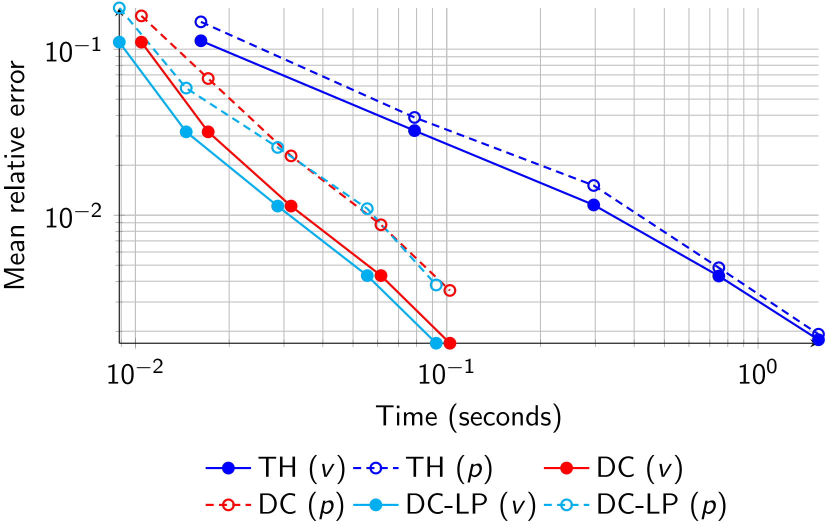
Convergence



Convergence



Convergence



Summary

- Reduced order models can offer dramatic speed-ups for certain applications.
- They combine nicely with IGA and div-compatible spaces to form fully divergence-free function spaces without need for pressure fields.
- Divergence-free RBMs can be much faster than other RBMs, in spite of additional complexity in the offline stage (remember, all is fair there.)

Thank you!

References

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