

Reduced Order Models for Divergence-Conforming Isogeometric Flow Simulations

T. Kvamsdal^{1,2} E. Fonn² E. H. van Brummelen³ A. Rasheed²

¹Department of Mathematical Sciences, NTNU

²Applied Mathematics and Cybernetics, SINTEF Digital

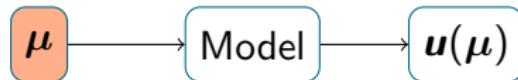
³Department of Mechanical Engineering, TU/e



Outline

- ① Basics of Reduced Basis Methods
- ② Stationary Navier-Stokes flow
- ③ Pressure recovery
- ④ Numerical examples
- ⑤ Results
- ⑥ Concluding remarks

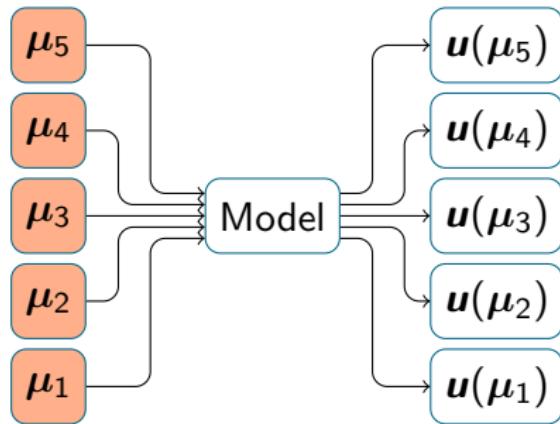
Parameter-dependent models



- We are interested generating solutions $u(\mu)$ to a physical model that depend on a set of pre-determined *parameters*, $\mu \in \mathcal{P}$.
- Parameters can be: viscosity, heat conductivity, varying boundary conditions, geometry changes, etc.

Parameter-dependent models

Motivation: *many-query* applications. E.g. control systems, optimization, inverse problems and real-time responsiveness.



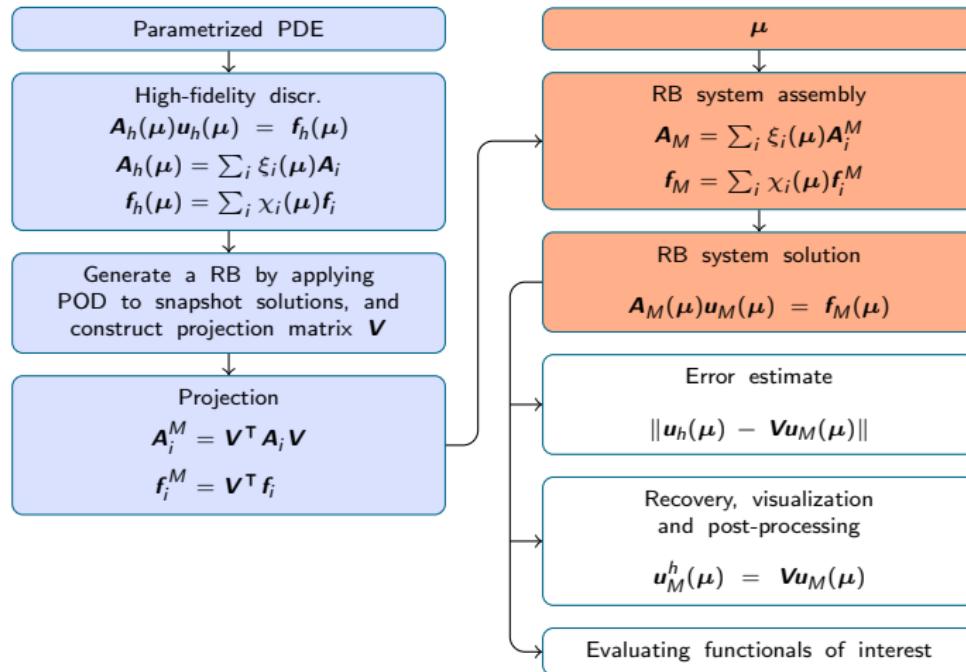
Dimensional reduction

- With conventional (read: FEM, FVM, FDM, and yes, even IGA) methods, this may be impractical if not impossible.
- Too many DoFs N to finish in a realistic timeframe.
- Usually,

$$M = \dim \text{"span"} (\{\boldsymbol{u}(\boldsymbol{\mu}) \mid \boldsymbol{\mu} \in \mathcal{P}\}) \ll N$$

- Idea: create a model with number of DoFs closely matching the physical dimension of the problem.
- Often, $M \sim 100$ or so!

The vision¹



¹See Quarteroni, Manzoni, and Negri 2016.

Our guiding principle

“**Any**” extra cost in the offline stage is worth paying,
no matter how much, if it makes the online stage faster.

Our guiding principle

All is fair in love, war *and the offline stage.*
— John Lyly (*Euphues*; 1579)

Assembly

- Why the insistence on forms like

$$\mathbf{A}_h(\boldsymbol{\mu}) = \sum_i \xi_i(\boldsymbol{\mu}) \mathbf{A}_i, \quad \mathbf{f}_h(\boldsymbol{\mu}) = \sum_i \chi_i(\boldsymbol{\mu}) \mathbf{f}_i$$

- Because it makes *assembly* of reduced models fast.
- Each \mathbf{A}_i and \mathbf{f}_i can be projected independently onto a reduced basis and stored.
- This makes the online stage completely high-fidelity-agnostic.
- Deriving these *affine representations* is the core detail of RBM.

Stationary Navier-Stokes flow

Navier-Stokes equations

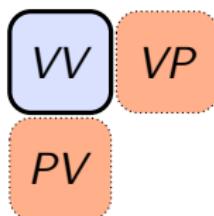
$$\begin{aligned} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma_D, |\Gamma_D| > 0 \\ -p \mathbf{n} + \nu (\nabla \mathbf{u}) \mathbf{n} &= \mathbf{h} && \text{on } \Gamma_N. \end{aligned}$$

Divergence-conforming methods

- Balance between velocity and pressure space is delicate.
- Too many velocity DoFs: poor enforcement of the continuity equation. Too many pressure DoFs: pressure instability.
- A pair of spaces (V, P) is divergence-conforming if $\operatorname{div} V = P$. Ensures the strong (pointwise) form of the continuity equation, and no pressure instabilities.
- IGA unlocks easy construction of such spaces.²

²Buffa, Sangalli, and Vázquez 2010; Evans and Hughes 2013.

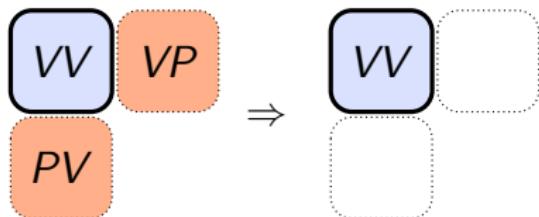
Anatomy of a reduced system



- Usually, the reduced method does not inherit the stability properties of the high-fidelity method.
- Rank-deficient velocity-pressure blocks are common.
- Carefully selecting number of velocity modes and pressure modes seems to help?

$$M_V \approx dM_P$$

Anatomy of a reduced system (cont.)

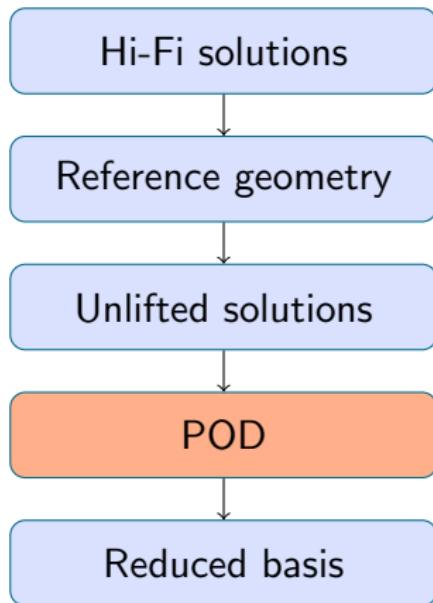


- A divergence-free reduced basis eliminates the coupling, leading to a fully stable velocity-only formulation.
- IGA³ enables divergence-free high-fidelity solutions,⁴ therefore also divergence-free reduced bases.

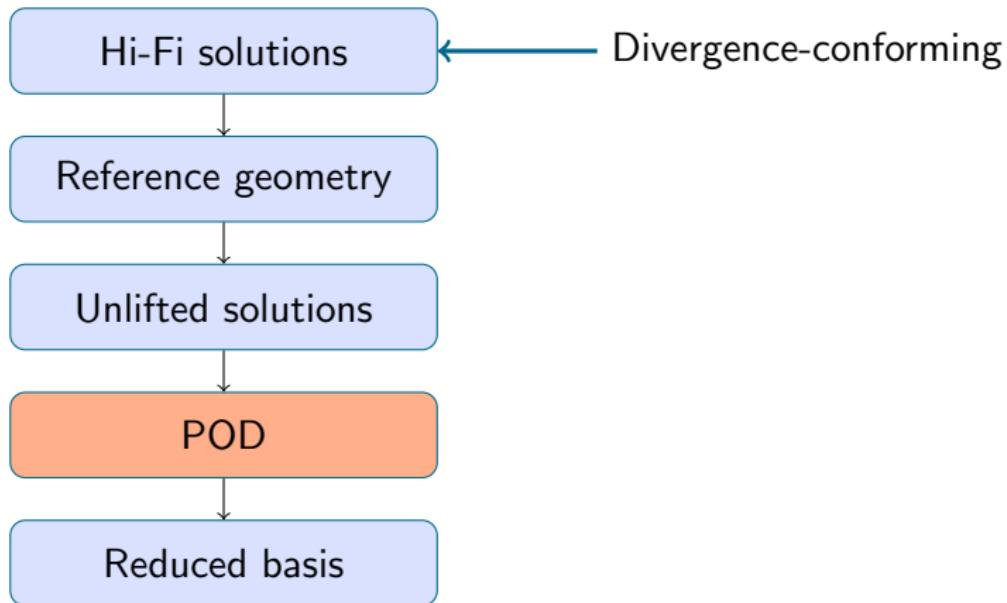
³Hughes, Cottrell, and Bazilevs 2005; Cottrell, Hughes, and Bazilevs 2009.

⁴Evans and Hughes 2013.

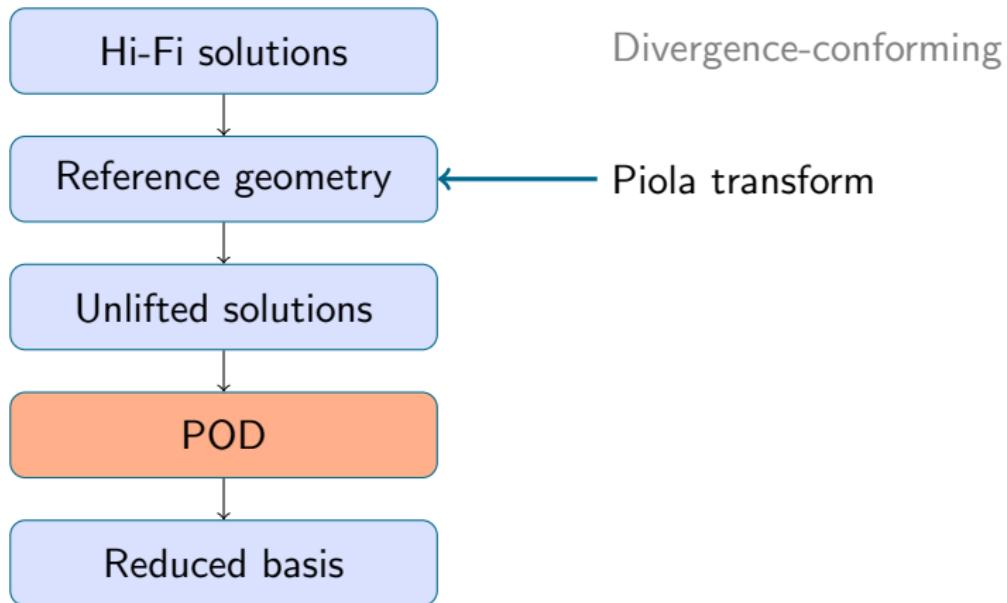
Divergence-free basis



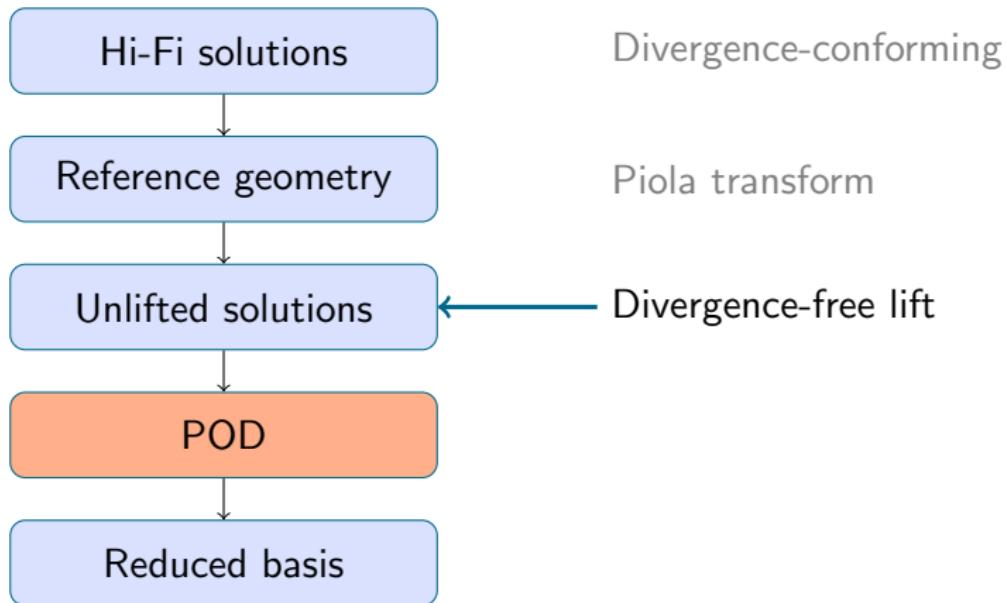
Divergence-free basis



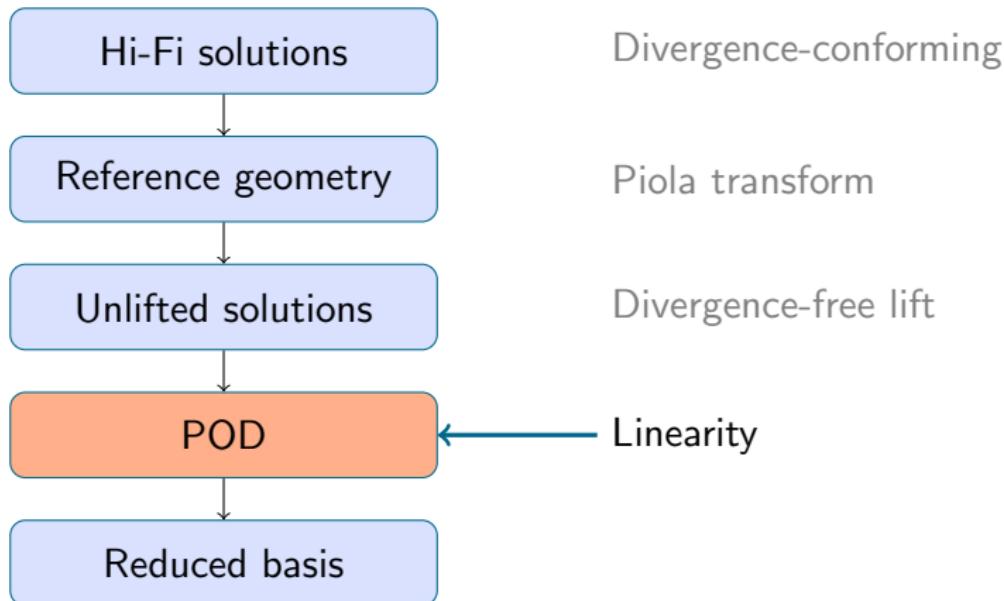
Divergence-free basis



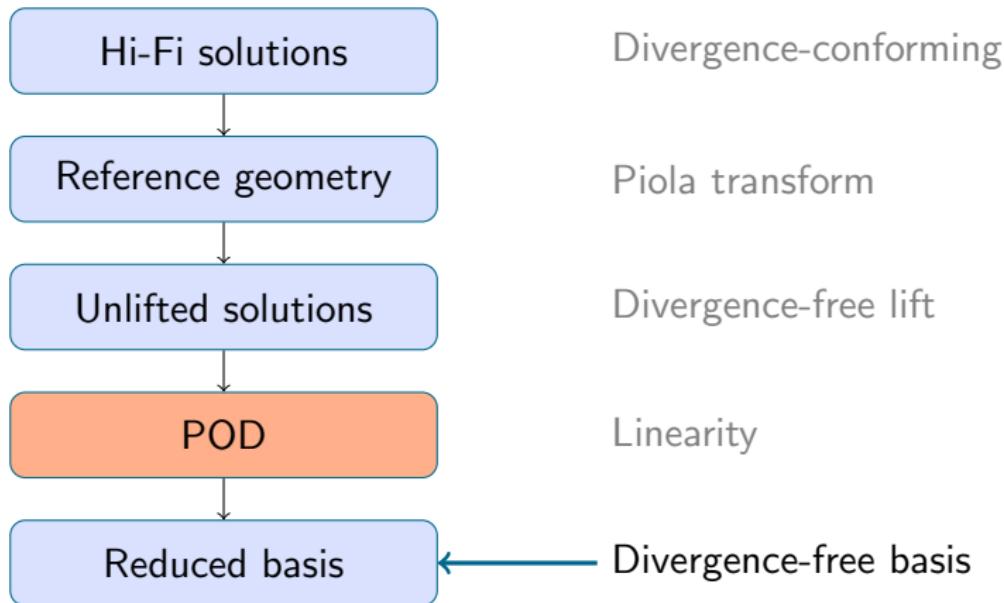
Divergence-free basis



Divergence-free basis



Divergence-free basis



Pressure recovery

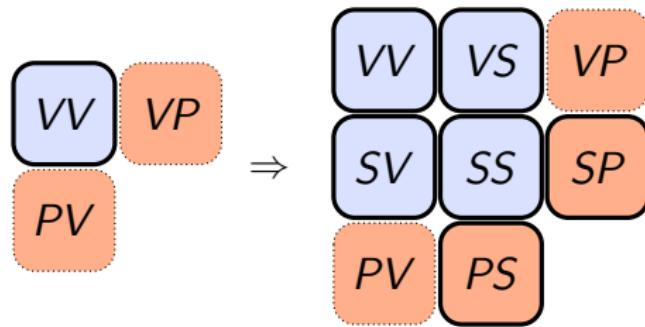
- The simplest approach is to attach pressure data to divergence-free velocity basis functions.
- Pressure recovery becomes “free” (no linear system to solve, use coefficients from RB velocity solution).
- However, this implies a linear velocity-pressure relationship, in violation of e.g. Navier-Stokes.
- Could work depending on problem.

Pressure recovery

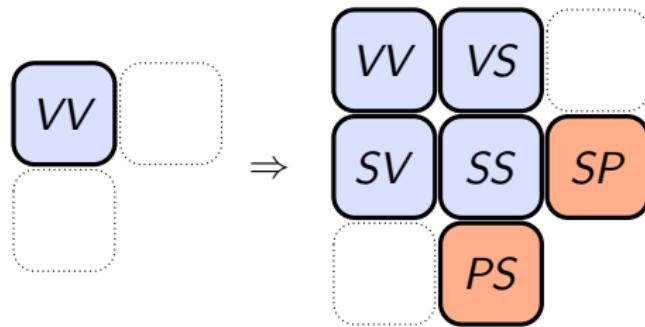
- Another approach is to solve the momentum equation with a different test space.
- The “optimal” test functions are *supremizers*, maximizers of the “sup” part of the inf-sup condition, for any specific pressure solution.
- Supremizers form a reduced space just like velocity and pressure do.
- Supremizers are commonly used to enrich non-divergence-free reduced velocity spaces for stability.⁵

⁵Ballarin et al. 2015.

Anatomy of a reduced system (cont.)



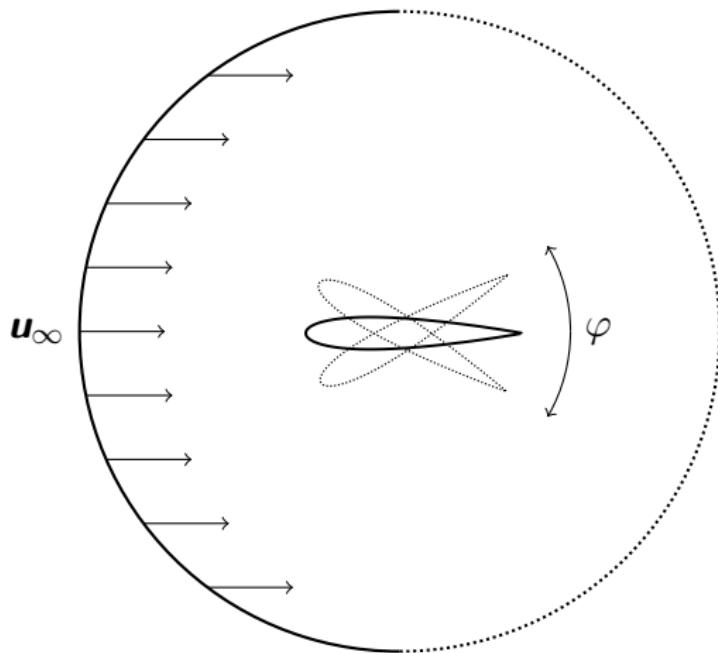
Anatomy of a reduced system (cont.)



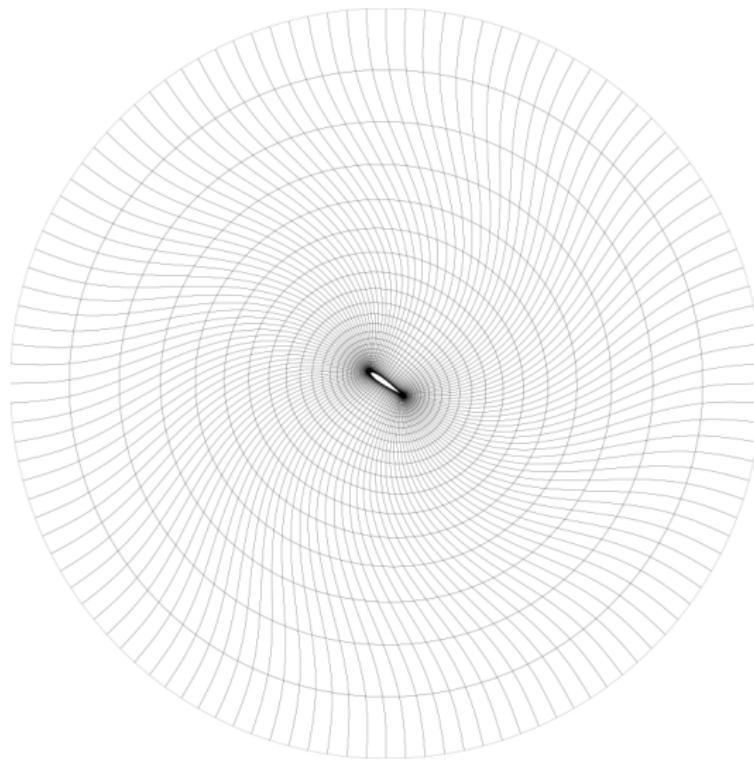
Note the block triangular structure.

Numerical examples

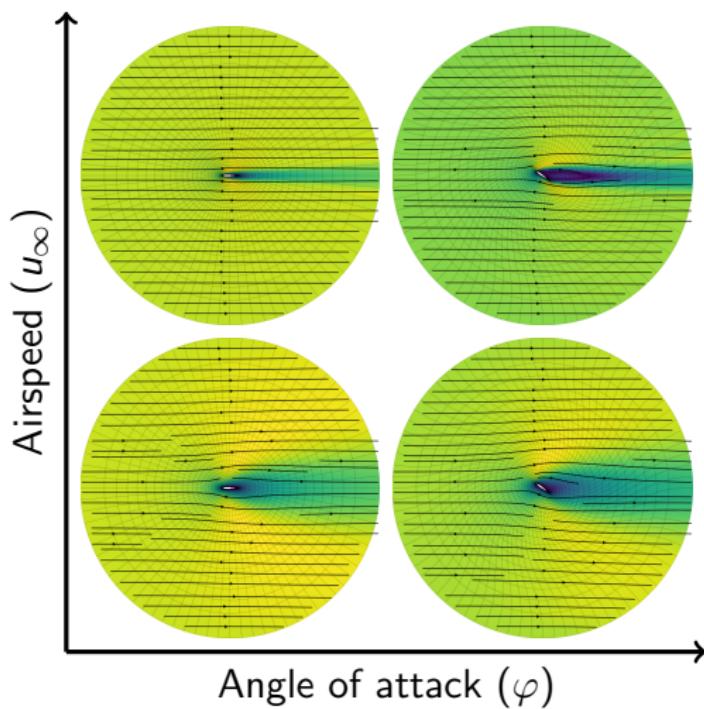
Flow around airfoil



Domain transformation



Parameter space



Problem specification

- We will try two high-fidelity methods: a Taylor-Hood (1,2)-method and an IGA (1,2) divergence-conforming method, with both approaches to pressure recovery.
- The parameter domain was chosen as

$$\mathcal{P} = [-35^\circ, +35^\circ] \times [1 \text{ m/s}, 20 \text{ m/s}].$$

- Only *stationary* Navier-Stokes, with $\nu = \frac{1}{6}$.
- We chose equal number of modes in all spaces: $N_V = N_S = N_P = M$.

Affine representations

- Not possible to express the Navier-Stokes problem as finite sums

$$\mathbf{A}_h(\boldsymbol{\mu}) = \sum_i \xi_i(\boldsymbol{\mu}) \mathbf{A}_i, \quad \mathbf{f}_h(\boldsymbol{\mu}) = \sum_i \chi_i(\boldsymbol{\mu}) \mathbf{f}_i$$

- Instead, we use truncated polynomial series in φ .⁶
- We can expect about 10 digits of accuracy with a reasonable number of terms (~ 25 for TH, ~ 75 for DC).
- Recall: the intention is to encode *all* parameters explicitly in the representation of the bi- or trilinear forms.

$$\mathbf{J} = \sum_i \varphi^i \mathbf{B}_i^{(+)} \quad \mathbf{J}^{-\top} = \sum_i \varphi_j \mathbf{B}_j^{(-)}$$

$$\mathbf{J}^{-1} \mathbf{J}^{-\top} = \mathbf{I} + \varphi \mathbf{D}_1 - \varphi^2 \mathbf{D}_2$$

⁶Fonn et al. 2018.

Affine representations (TH)

$$(\pi_{\mu}^* a)(\hat{\mathbf{u}}, \hat{\mathbf{w}}; \varphi) = \nu \int_{\hat{\Omega}} \nabla \hat{\mathbf{u}} : \nabla \hat{\mathbf{w}} + \nu \varphi \int_{\hat{\Omega}} \nabla \hat{\mathbf{u}} : (\mathbf{D}_1 \nabla) \hat{\mathbf{w}}$$

$$- \nu \varphi^2 \int_{\hat{\Omega}} \nabla \hat{\mathbf{u}} : (\mathbf{D}_2 \nabla) \hat{\mathbf{w}}$$

$$(\pi_{\mu}^* b)(\hat{p}, \hat{\mathbf{w}}; \varphi) \approx \sum_{i=0}^{2n} \varphi^i \int_{\hat{\Omega}} \hat{p} \mathbf{B}_i^{(-)} : \nabla \hat{\mathbf{w}}$$

$$(\pi_{\mu}^* c)(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}; \varphi) \approx \sum_{i=0}^{2n} \varphi^i \int_{\hat{\Omega}} (\hat{\mathbf{u}} \cdot \mathbf{B}_i^{(-)} \nabla) \hat{\mathbf{v}} \cdot \hat{\mathbf{w}}$$

Affine representations (IGA)

$$(\pi_{\mu}^* a)(\hat{\mathbf{u}}, \hat{\mathbf{w}}; \varphi) = \nu \sum_{i,j=0}^{2n} \varphi^{i+j} \int_{\hat{\Omega}} \nabla(\mathbf{B}_i^{(+)} \hat{\mathbf{u}}) : \nabla(\mathbf{B}_j^{(+)} \hat{\mathbf{w}})$$

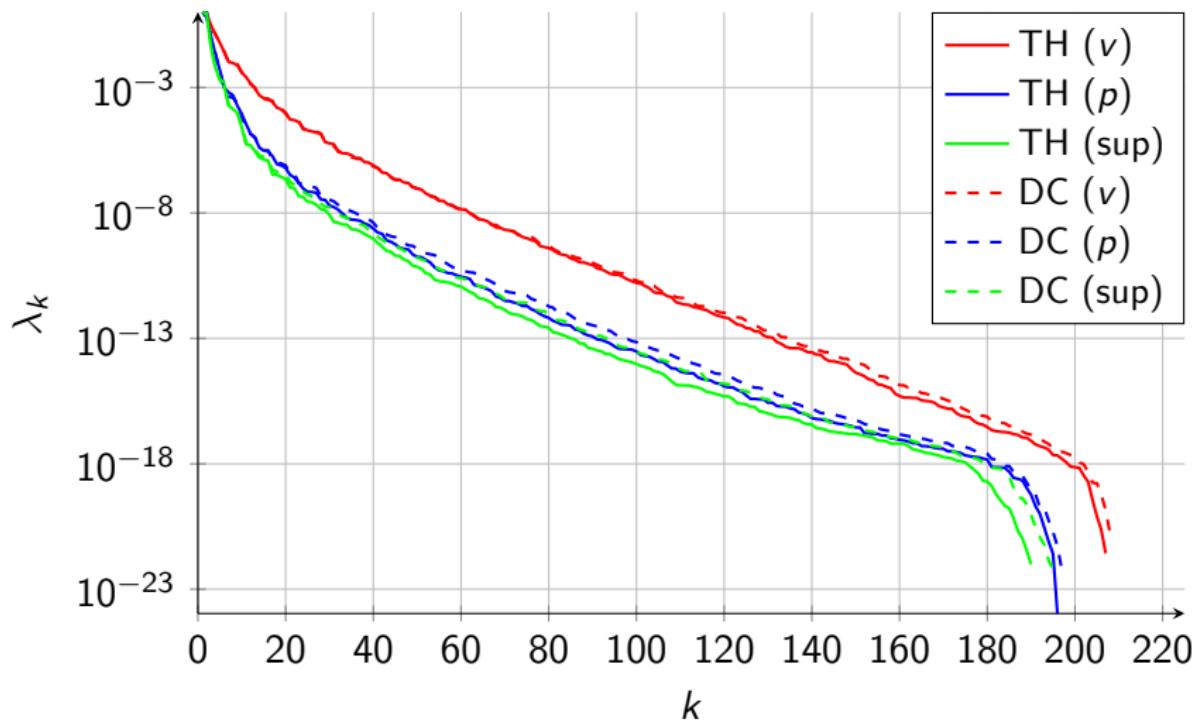
$$+ \nu \sum_{i,j=0}^{2n} \varphi^{i+j+1} \int_{\hat{\Omega}} \nabla(\mathbf{B}_i^{(+)} \hat{\mathbf{u}}) : (\mathbf{D}_1 \nabla)(\mathbf{B}_j^{(+)} \hat{\mathbf{w}})$$

$$- \sum_{i,j=0}^{2n} \varphi^{i+j+2} \int_{\hat{\Omega}} \nabla(\mathbf{B}_i^{(+)} \hat{\mathbf{u}}) : (\mathbf{D}_2 \nabla)(\mathbf{B}_j^{(+)} \hat{\mathbf{w}})$$

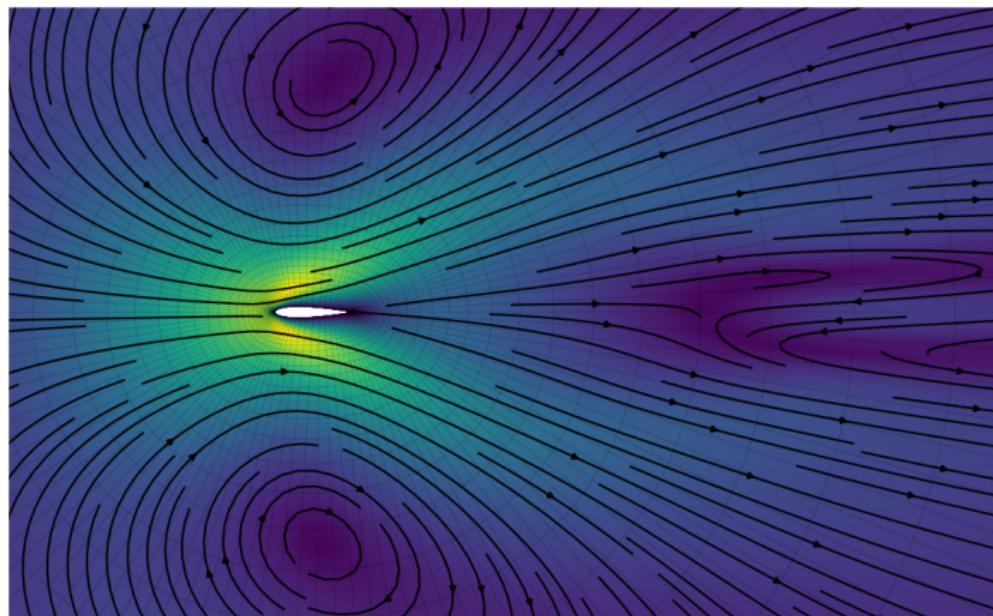
$$(\pi_{\mu}^* b)(\hat{p}, \hat{\mathbf{w}}; \varphi) = \sum_{i,j=0}^{2n} \varphi^{i+j} \int_{\hat{\Omega}} \hat{p} \mathbf{B}_i^{(-)} : \nabla(\mathbf{B}_j^{(+)} \hat{\mathbf{w}})$$

$$(\pi_{\mu}^* c)(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}; \varphi) = \sum_{i,j=0}^{2n} \varphi^{i+j} \int_{\hat{\Omega}} (\hat{\mathbf{u}} \cdot \nabla) \mathbf{B}_i^{(+)} \hat{\mathbf{v}} \cdot \mathbf{B}_j^{(+)} \hat{\mathbf{w}}.$$

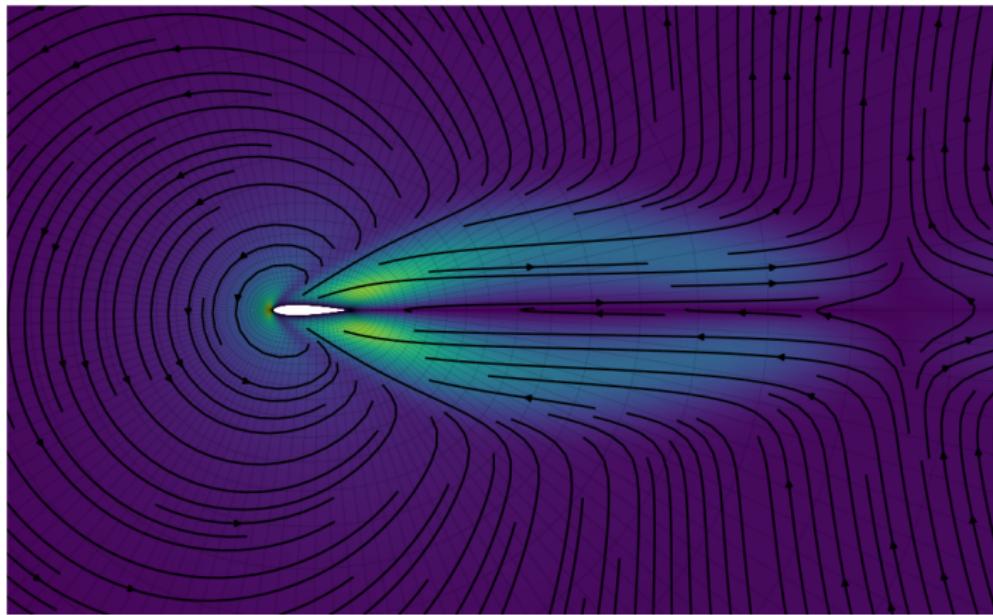
Spectrum



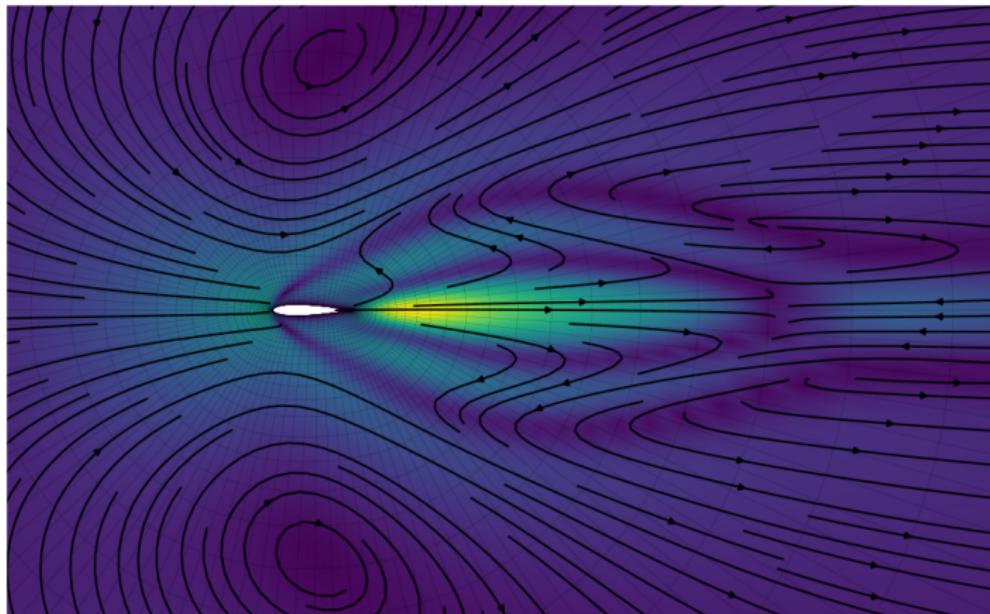
Basis functions (v , TH, 1)



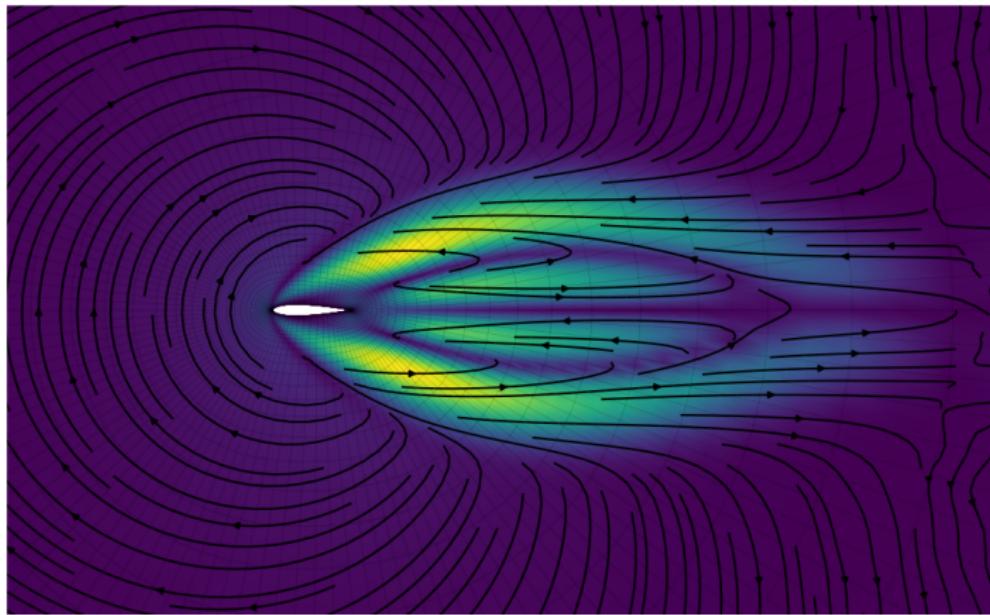
Basis functions (v , TH, 2)



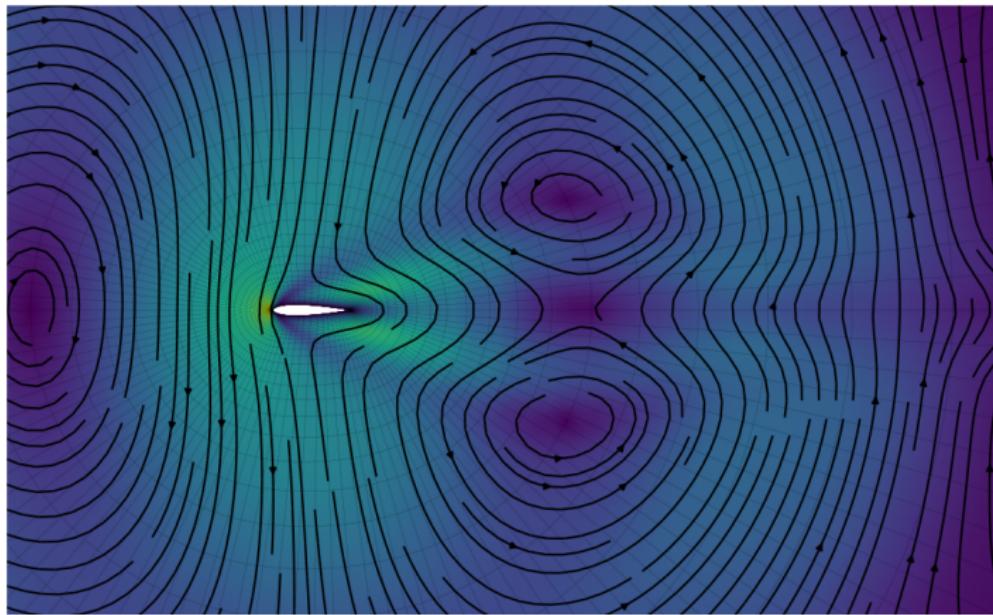
Basis functions (v , TH, 3)



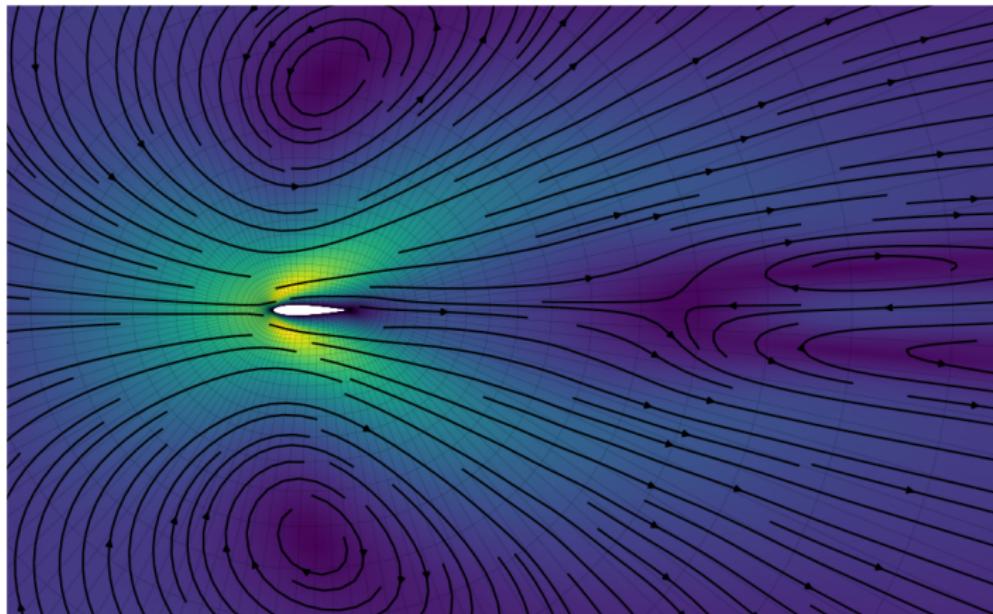
Basis functions (v , TH, 4)



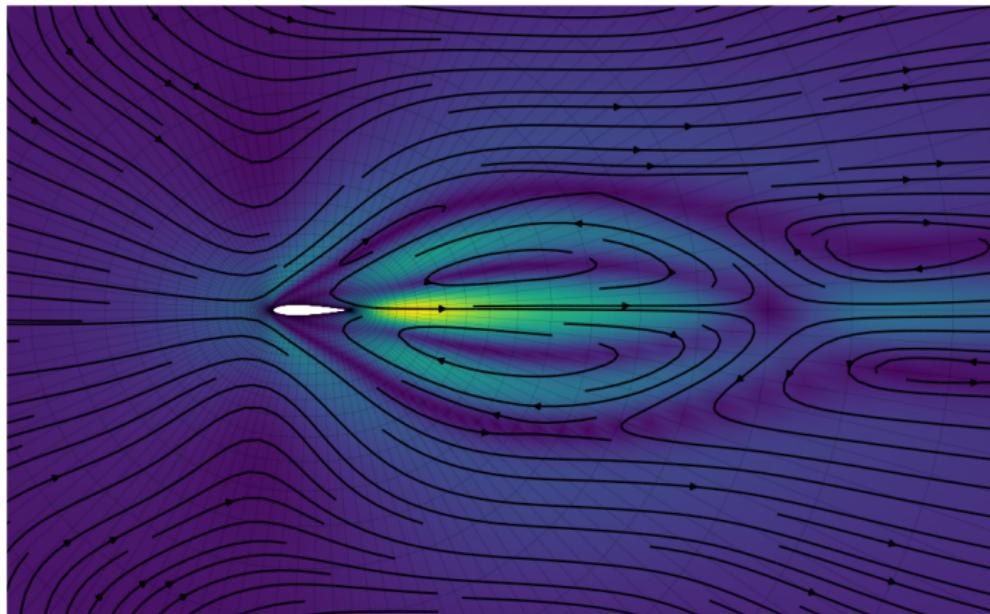
Basis functions (v , DC, 1)



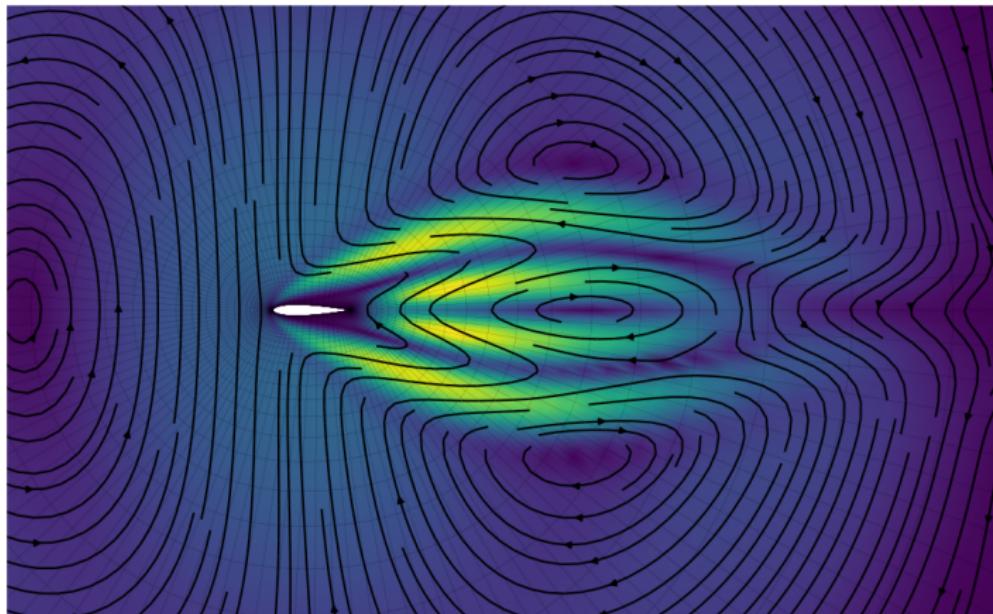
Basis functions (v , DC, 2)



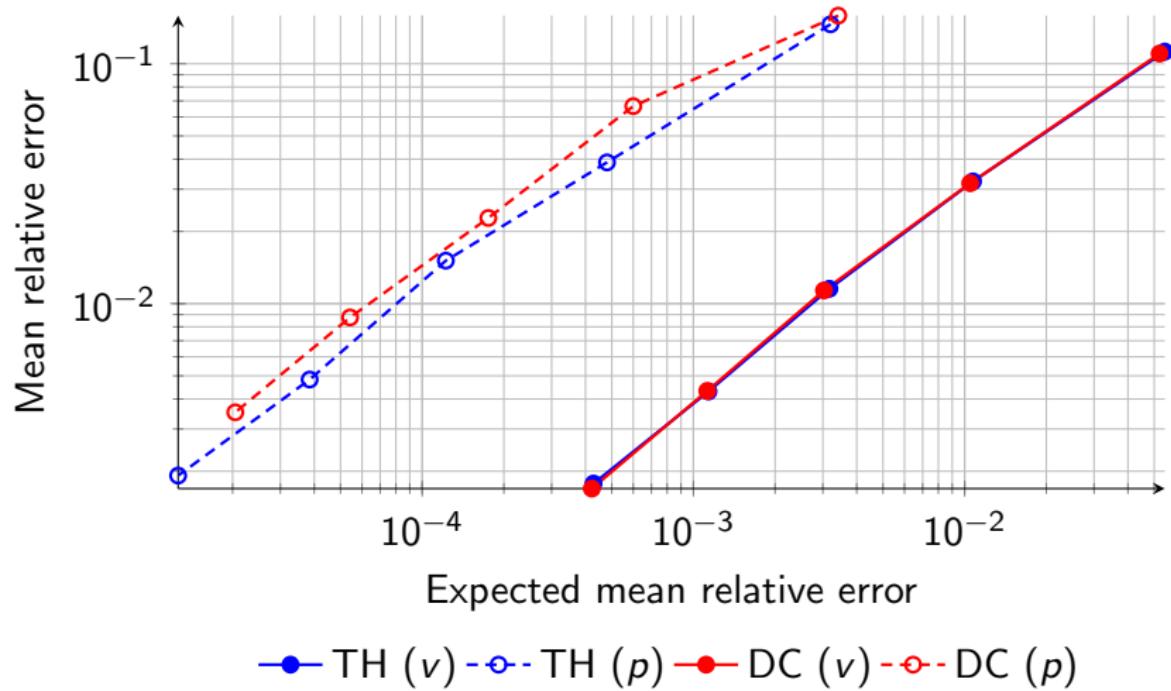
Basis functions (v , DC, 3)



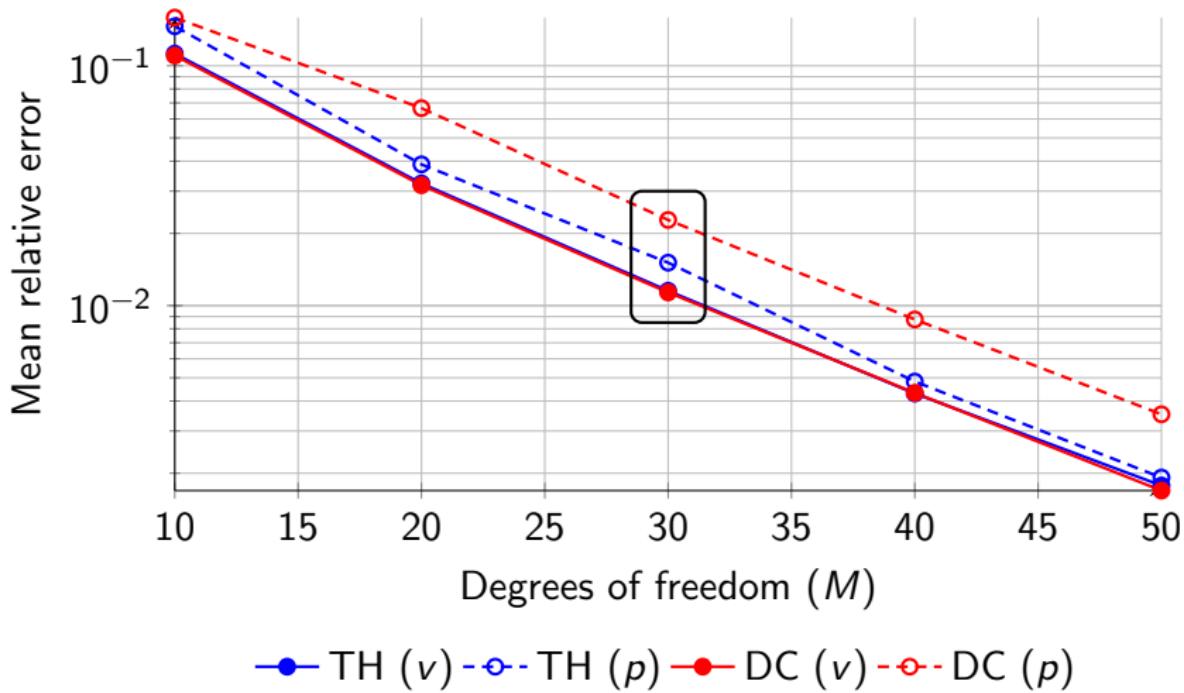
Basis functions (v , DC, 4)



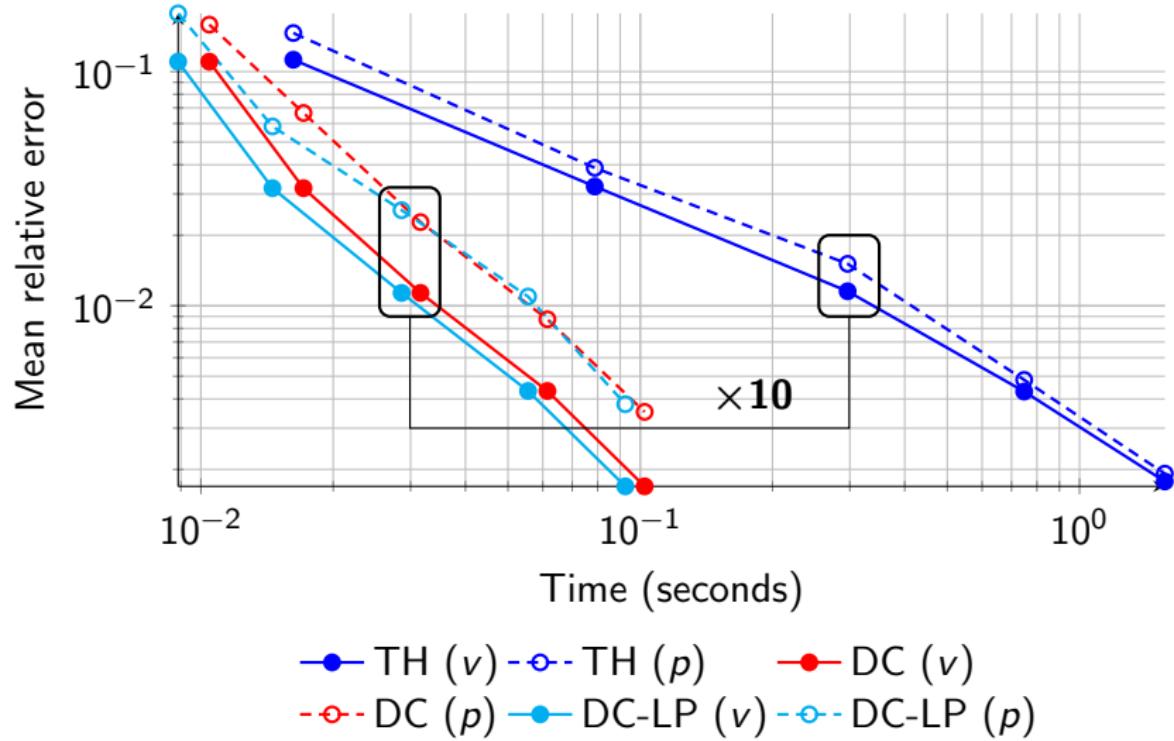
Convergence



Convergence



Convergence



Speedup factors

# DoFs (M)	Taylor-Hood	Conforming
10	1483	6890
20	390	2616
30	111	1441
40	52	843
50	27	502

Summary

- Reduced order models offer dramatic speed-ups for certain applications.
- They combine nicely with IGA and div-compatible spaces to form fully divergence-free function spaces without need for pressure fields.
- Divergence-free RBMs can be much faster than other RBMs, in spite of additional complexity in the offline stage (remember, all is fair there.)

Thank you!

References

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