Mathematics from scratch

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The purpose of this document is to build some familiar mathematical structures from “scratch”, by which we mean the axioms of Zermelo-Fraenkel set theory (hereafter known as ZF). It is not our intention to be fully rigorous, we target here mostly interested laypeople and amateur mathematicians.

1 Zermelo-Fraenkel set theory

A *set* is to be understood as a collection of objects, in the sense that for any given set X and an object x, we can decide whether or not x belongs to X. In particular, sets do not count repetitions, merely membership (x cannot belong to X “more than once”).

The basic way to form a set is to merely list its elements, in the following way:

$$X = \{a, b, c, d\}.$$

Here, X is the set containing a, b, c and d, and nothing else.

The nature of the “objects” x is left unaddressed. For our purposes, all the mathematical objects we consider (sets, functions, ordered pairs, etc.) are ultimately defined in terms of sets, although we will generally not *think* of them as such. *Everything is a set!*

We write $x \in X$ to mean that x belongs to the set X, and $x \notin X$ if it does not. We also say that two sets X and Y are *disjoint* if they contain no common elements. A set Y is a *subset* of X if, whenever $z \in Y$, we have $z \in X$ also. This is denoted by $Y \subseteq X$.

The axioms of ZF describe what operations we are allowed to perform on sets without further prior justification. We will now consider them one by one.

Axiom 1 (Axiom of Existentionality)**.** *Let X and Y be sets. Assume for all sets z that $z \in X$ if and only if $z \in Y$. Then $X = Y$.*

This axiom tells us when two sets X and Y are equal, namely when they contain precisely the same elements. If so, then X can be substituted for Y in any formula.

Axiom 2 (Axiom of Regularity)**.** *Let X be a non-empty set. Then there exists a $z \in X$ so that z and X are disjoint.*

The Axiom of Regularity implies that no set can be a member of itself. Indeed, let X be a set and consider the set $\{X\}$. Since it is nonempty, it must have a member disjoint from itself, but X is its only member. Thus, X and $\{X\}$ have no common elements, and in particular $X \notin X$.

Axiom 3 (Axiom schema of Specification)**.** *Let X be a set, and ϕ any property that may or may not be satisfied by elements of X. Then there exists a set containing those elements of X that satisfy ϕ.*

The formal notation for this set is $\{x \in X \ : \ \phi\}$, to be read as "the set of all x in X satisfying ϕ". Rigorous treatment of set theory requires the formula ϕ to adhere to the language defined by ZF, but we will allow ourselves more natural language in this text.

If at least one set X exists, we can use Axiom 3 to define the empty set $\emptyset$ as

$$\emptyset := \{x \in X \ : \ x \in x\}$$

since by Axiom 2, no set x may be a member of itself. The empty set has the property that for all x, $x \notin \emptyset$.

Axiom 4 (Axiom of Pairing)**.** *Let x and y be sets. Then there exists a set Z such that $x \in Z$ and $y \in Z$.*

Note that Z may be larger than $\{x, y\}$. However, once we have the set Z from Axiom 4, we can use Axiom 3 to give the existence of $\{x, y\}$ as

$$\{x, y\} := \{z \in Z \ : \ z = x \vee z = y\}.$$

Axiom 5 (Axiom of Union)**.** *Let X be a set. Then there is a set Y that contains those sets which are members of some member of X:*

$$x \in z \wedge z \in X \implies z \in Y.$$

Note that, again, the set Y can be larger than *just* the members of members of X, but just like previously, we can use the axiom of specification to define *just* the union. We denote this set as the *union* of all members of X:

$$Y := \bigcup_{z \in X} z$$

In particular we have the following special notation for unions of a finite number of sets. If $X = \{z_1, z_2, \ldots, z_n\}$, then

$$\bigcup_{z \in X} z =: z_1 \cup z_2 \cup \cdots \cup z_n.$$

Axiom 6 (Axiom of Infinity). *Let the* successor *of a set x be given as $S(x) := x \cup \{x\}$. Then there exists a set X such that $\emptyset \in X$ and, whenever $x \in X$ we have $S(x) \in X$.*

Axiom 6 postulates the existence of an infinitely large set. In fact, as we will set later, this is just the set of natural numbers. Note also that this Axiom postulates the existence of the empty set $\emptyset$.

Axiom 7 (Axiom of Power Set). *Let X be a set. Then there is a set containing all the subsets of X.*

This set is called the *power set* of X, and is denoted by $\mathcal{P}(X)$[1].

These seven axioms will be largely sufficient for our purposes here. There are two omissions made:

- The axiom schema of replacement, which tells us that if f is a function on a domain A, then the range of f is a subset of some set B.
- The axiom of choice (or any of its equivalent notions).

These will be introduced if and when they are needed.

The reader will have noticed the pattern: the ZF axioms often guarantee the existence of sets that may be larger than necessary, while Axiom 3 is the tool that allows us to reduce sets to contain precisely what we need.

Let us now define intersections,

$$\bigcap_{z \in X} z := \left\{ x \in \bigcup_{z \in X} z \ : \ \forall z \in X, x \in z \right\},$$

finite intersections, for $X = \{z_1, z_2, \ldots, z_n\}$,

$$\bigcap_{z \in X} z =: z_1 \cap z_2 \cap \cdots \cap z_n,$$

and set differences,

$$X - Y := \{x \in X \ : \ x \notin Y\}.$$

In the context of a *universe* U, we may also define the *complement* of a set X as $X^c = U - X$.

2 Ordered tuples, relations and functions

Definition 8. Let a, b be sets. We define the ordered pair (a, b) as

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

[1]Like before, Axiom 7 does not by itself guarantee the existence of $\mathcal{P}(X)$, only of a set that is at least as large. We use Axiom 3 to "cut it down".

This definition provides a distinction between the pairs (a, b) and (b, a) if $a \neq b$. Since then $\{a\} \neq \{b\}$ (a and b are different) and $\{a\} \neq \{a, b\}$ (the two sets have different numbers of elements), we see that $\{\{a\}, \{a, b\}\} \neq \{\{b\}, \{b, a\}\}$, as $\{a\}$ is nowhere in the second set. This justifies the name *ordered* pair.

We may also produce tuples of any size by nesting pairs, e.g.

$$\begin{aligned}
(a, b, c, d, e) &:= (a, (b, (c, (d, e)))) \\
&= \{\{a\}, \{a, (b, (c, (d, e)))\}\} \\
&= \{\{a\}, \{a, \{\{b\}, \{b, (c, (d, e))\}\}\}\} \\
&= \{\{a\}, \{a, \{\{b\}, \{b, \{\{c\}, \{c, (d, e)\}\}\}\}\}\} \\
&= \{\{a\}, \{a, \{\{b\}, \{b, \{\{c\}, \{c, \{\{d\}, \{d, e\}\}\}\}\}\}\}\},
\end{aligned}$$

although we will provide a more suitable definition later.

We now wish to define the notion of the Cartesian product $X \times Y$ of two sets X and Y, namely the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. Note that $\{x\}$ and $\{x, y\}$ are both subsets of $X \cup Y$, thus $\{\{x\}, \{x, y\}$ is a subset of $\mathcal{P}(X \cup Y)$, or in other words, it is a member of $\mathcal{P}(\mathcal{P}(X \cup Y))$. We can then write

$$X \times Y = \{z \in \mathcal{P}(\mathcal{P}(X \cup Y)) \ : \ z = (x, y) \text{ for some } x \in X,\ y \in Y\}.$$

With this in hand, we can now define relations.

Definition 9. A set R is called a *relation* on a set X if it is a subset of $X \times X$.

Loosely speaking, a relation on X lists the pairs of elements of X that are "related". We often use the notation xRy to mean $(x, y) \in R$.

We observe, for example, that equality is a relation:

$$\{(x, y) \in X \times X \ : \ x = y\}.$$

In fact, equality belongs to a particularly important class of relations known as equivalence relations.

Definition 10. We say that a relation R is

- *reflexive* if xRx for all x (every element is related to itself),
- *symmetric* if xRy whenever yRx,
- *antisymmetric* if, whenever $x \neq y$, exactly one of xRy and yRx holds,
- *transitive* if, whenever xRy and yRz, we have xRz.

A relation R that is reflexive, symmetric and transitive is called an *equivalence relation*.