3

Time Domain Circuits and Differential Equations

Introduction

Objectives

3.1 Math with Key Waveforms

All of the contributions that electronics make to our lives are carried on a small number of different wave shapes. Once you can write the time domain equation for each, its derivative, and its integral, you can determine the effect any circuit has on that waveform, or the circuit needed to shape what you have into what you need.

A more complex waveform can be assembled from the piece-wise linear combination of this small set of standard signals. And, derivatives and integrals can then be completed, in pieces.

DC

DC is just a constant. Mathematically, it is the simplest to manipulate. The time domain equation is a constant, since the voltage does not change. That also means that the derivative is 0 (no change with respect to time). The integral of a constant is that constant times the variable *t*. As time goes on, more and more is added to the summation (the integral). The area under the line increases as the width (t) increases.

Figure 3-1 DC waveform

$$v(t) = A$$

$$\frac{d}{dt} A = 0$$

$$\int_0^t A\,dt = At$$

Sinusoid

The sine wave is probably the second most common wave form. Commercial power is produced and distributed as sinusoids. It also is a valuable test signal since it does *not* change its shape as it is processed by *linear* components. Its amplitude and phase may be altered, but the fundamental shape passes through unchanged. Look at the derivative and integral below. This makes testing circuits and detecting nonlinearity and distortion simpler. Finally, with Fourier Analysis, *any* repetitive wave shape can be built by combining different frequency and amplitude sine waves.

$$v(t) = A sin(2\pi f t - \theta) \quad f = \frac{1}{T}$$

$$\frac{d}{dt} A sin(2\pi f t - \theta) = A \times 2\pi f \times cos(2\pi f t - \theta)$$

$$\int_0^t A sin(2\pi f t - \theta) dt = \frac{-A}{2\pi f}[\cos(2\pi f t - \theta) - \cos(\theta)]$$

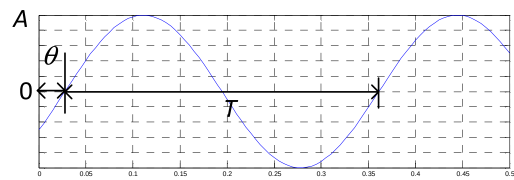


Figure 3-2 Sinusoid with amplitude (A), period (T), and phase (θ) defined

Basic calculus or fundamental tables give the derivative and the integral of sin (ϕ). But, what about all of the other variables? What do they do to the resulting function? The chain rule can get a little complicated to implement. The most direct way to evaluate this waveform's derivative and integral is with the TI-Nspire calculator.

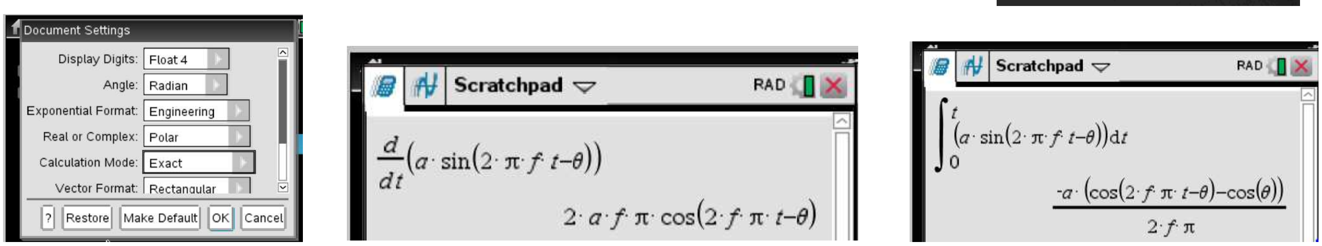


Figure 3-3 TI-Nspire derivative and integral of a general sine wave

The key screens are shown in Figure 3-3. First in the **Document settings**, select **Radians**, and **Exact**. Then, in the menu, select **4:Calculus**, then **1:Derivative**. The θ symbol is on the book key. The integral is found the same way, selecting **3:Integral** instead of **1:Derivative**.

Ramp

The ramp or triangle wave rises or falls at a constant rate, taking a set time to rise or fall one volt or one amp. The rising wave is shown in Figure 3-4. Often voltage is easier to sense than time. So many timing circuits have a ramp at their heart. Knowing how fast the ramp is rising, the circuit just waits until the voltage has risen to proscribed level. When voltage is an accurate measure of position, such as in a cathode ray tube or a robotic servo position control system, then a ramp can be used to move the controlled object at a constant speed.

$$v(t) = mt + b = \frac{A}{T}t + b$$

$$\frac{d}{dt}\left(\frac{A}{T}t + b\right) = \frac{A}{T}$$

$$\int_0^t \left(\frac{A}{T}t + b\right) dt = \frac{A}{2T}t^2 + bt$$

Figure 3-4 Ramp with amplitude (*A*), period (*T*) and offset (-*b*) defined

Figure 3-5 TI-Nspire derivative and integral of a ramp

The TI-Nspire calculations for the ramp are shown in Figure 3-5. This version of software does *not* recognize the difference between upper and lower case, *T* and *t*. So, the period has been represented by *p*.

Exponential Rise

Capacitance and inductance store energy in electrostatic and electromagnetic fields. As the stored energy increases, so does the voltage or current. But, the more voltage or current already present, the harder it is to force even more charge (and therefore even more voltage or current) onto the charging part. The part charges more slowly, and voltage or current rises more slowly. That is an *exponential* rise, *not* linear as in the preceding section. Look at Figure 3-6.

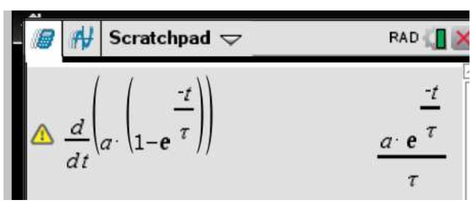
The time constant, τ, is defined as the length of time it takes the output to change by 63.2% (i.e. 1-e^{-1}) of its maximum possible change. In this example the output starts at 0 and will eventually reach *A*. So, its maximum possible change is *A**0.632.

$$v(t) = A\left(1 - e^{\frac{-t}{\tau}}\right)$$

$$\frac{d}{dt}\left[A\left(1 - e^{\frac{-t}{\tau}}\right)\right] = \frac{A}{\tau}e^{\frac{-t}{\tau}}$$

$$\int_0^t \left[A\left(1 - e^{\frac{-t}{\tau}}\right)\right]dt = A\left[t - \tau\left(1 - e^{\frac{-t}{\tau}}\right)\right]$$

Figure 3-6 Exponential rise with amplitude (*A*), and time constant (τ) defined



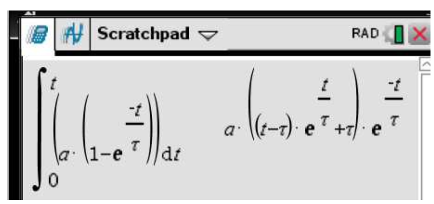


Figure 3-7 TI-Nspire derivative and integral of an exponential rise

When entering the exponential into the calculator, be sure to use the button especially for that function. Do *not* enter the letter *e* and then try to raise it to an exponent.

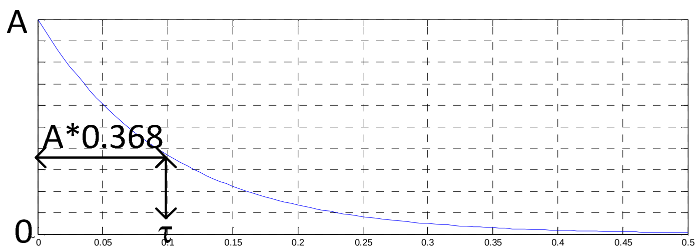
The integral returned by the TI-Nspire seems more complicated than the equation given beside Figure 3-6. But, a little algebra produces a match.

Exponential Spike or Fall

As capacitance or inductance *dis*charges, voltage or current also *falls* exponentially. This is shown in Figure 3-8.

Initially fully charged, energy flows rapidly off of the capacitance or inductance. Like charges oppose. So lots of charge forces charge off quickly. But as the charge falls, there is less charge left forcing charge away. So it leaves more slowly. The less there is, the more slowly it escapes.

The time constant is still the time it takes for the output to change 63.2% of its maximum change. Since it starts at *A*, a 63.2% change is a fall down to 36.8% of *A*.



$$v(t) = Ae^{\frac{-t}{\tau}}$$

$$\frac{d}{dt}\left(Ae^{\frac{-t}{\tau}}\right) = \frac{-A}{\tau}e^{\frac{-t}{\tau}}$$

$$\int_0^t \left(Ae^{\frac{-t}{\tau}}\right)dt = A\tau\left(1 - e^{\frac{-t}{\tau}}\right)$$

Figure 3-8 Exponential fall with amplitude (*A*), and time constant (τ) defined

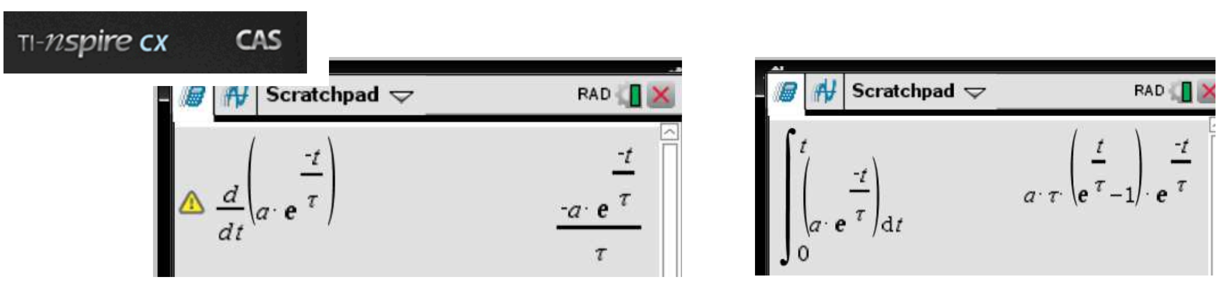


Figure 3-9 TI-Nspire derivative and integral of an exponential fall or spike

As with the exponential rise, it takes a little algebra to change the answer given by the TI-Nspire into a more convenient form.

3.2 Capacitance in the Time Domain

Capacitance exists anytime there are two parallel conductors separated by an insulator. Often, this is an intentional component, two plates separated by a film insulator and packaged with an external lead connecting to each plate. This is shown in Figure 3-10. But, any two associated conductors form a capacitance, such as the two traces on the top and bottom of a printed circuit board.

Each electron that arrives at the lower plate forces an electron to leave the upper plate, leaving an equal positive charge there. This charge, *Q*, creates an electrostatic field, shown by the arrows between the plates, storing energy and producing a resulting potential difference, *V*. How much charge and energy that can be stored depends on the size of the plates, how close together they are, and the properties of the insulation. These define the *capacitance*.

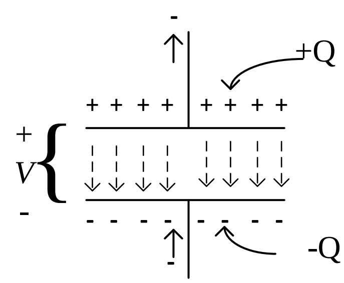


Figure 3-10 Capacitance, charge, electrostatic field, and voltage

$$C = \frac{Q}{V}$$

The more charge (and energy) a capacitor can hold (per volt), the larger its capacitance, (the more water a bucket can hold, the larger it must be).

$$V = \frac{Q}{C}$$

Taking the time derivative of both sides gives

$$\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt}$$

But, current is defined as

$$i = \frac{dQ}{dt}$$

$$\frac{dV}{dt} = \frac{1}{C} i$$

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$$i = C\frac{dV}{dt}$$

Current is the flow of charge onto and off of the capacitor as it charges and discharges. So, even though a charged capacitor looks like an open (to DC), *as* the capacitor charges or discharges, current flows onto and off of it. There is an insulator between the two plates, but *transient* current still flows (onto and off of => through). The more rapidly the voltage across the capacitor *changes* the more quickly charge flows on and off of the capacitor and the higher the resulting current.

Example 3-1

The voltage waveform shown in Figure 3-11 is impressed across a 5 μF capacitor. *Accurately* draw the current flowing into (and out of) the capacitor.

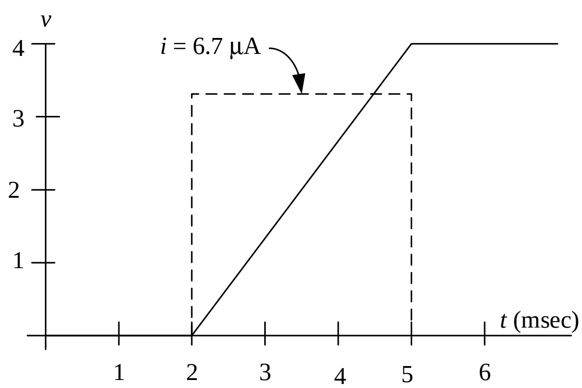


Figure 3-11 Ramp voltage resulting in a pulse of current

Solution

$$i = C\frac{dV}{dt}$$

From t = 0 to 2 msec, the voltage does not *change*.

$$i_{0 \text{ to } 2 \text{ msec}} = 0 \text{ A}$$

From $t = 2$ msec to 5msec, the voltage is a ramp, changing 4 V.

$$i = C\frac{dV}{dt} = 5\ \mu\text{F}\ \frac{4\text{ V}}{5\text{ sec} - 2\text{ sec}} = 6.7\ \mu\text{ A}$$

Since the voltage changes at a steady rate, the current is a constant 5 μA.

For $t > 6$ msec, the voltage does not *change*.

$$i_{t > 6\text{ msec}} = 0\text{ A}$$

Example 3-2
The voltage across the 5 μF capacitor is changed to an exponential rise

$$v = 4\text{ V}(1 - e^{-10t})$$

Calculate and plot the equation for the current.

Solution

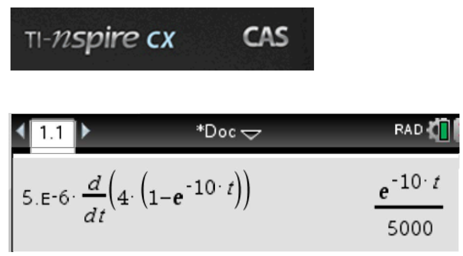
$$i = C\frac{dV}{dt} = 5\ \mu\text{F}\frac{d}{dt}\left(4\text{ V}(1 - e^{-10t})\right)$$

$$\frac{d}{dt}\left[A\left(1 - e^{\frac{-t}{\tau}}\right)\right] = \frac{A}{\tau}e^{\frac{-t}{\tau}}$$

$$A = 4\text{ V}$$
$$\tau = 0.1\text{ sec}$$

$$i = 5\ \mu\text{F}\frac{4\text{V}}{0.1\text{sec}}\text{e}^{-10\text{t}}$$

$$i = 200\ \mu\text{A e}^{-10\text{t}}$$



(a) TI-Nspire derivative

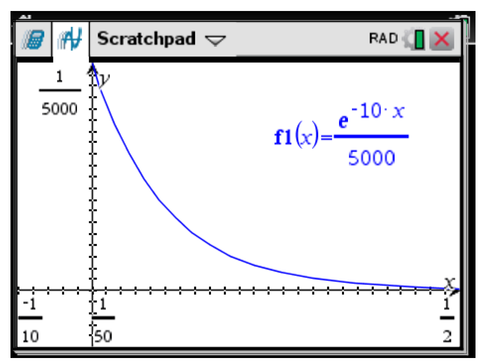


Figure 3-12 (b) Plot of current for Example 3-2

In Figure 3-12(a) the TI-Nspire completed the derivative. The answer was then copied, the calculator mode changed to Graph, and then the function pasted. The graph expects the horizontal axis to be *x*, not *t*, so that variable was changed. Finally the scaling was adjusted to show the full plot, *and* a little of the 0,0 origin.

Driving an exponentially increasing voltage across a capacitor produces a *spike* of current that falls to 0 when the voltage becomes constant at its maximum value.

Given the voltage, the current can be calculated.

$$i = C\frac{dV}{dt}$$

But, if current is driven into the capacitor, how is the voltage calculated?

$$i\,dt = C\,dv$$

$$dv = \frac{1}{C}i\,dt$$

$$v = \frac{1}{C}\int idt$$

To evaluate this integral requires limits.

$$v = \frac{1}{C}\int_{-\infty}^{t} idt = \frac{1}{C}\left(\int_{-\infty}^{0} idt + \int_{0}^{t} idt\right)$$

Integrating from the beginning of time (t = -∞) could be a problem. So, that part of the integral is separated. Whatever has happened before the beginning of the calculation (t = 0) is just the initial charge on the capacitor, V_o however it got there.

Given i_{cap}, find v_{cap}.

$$v = \frac{1}{C}\int_0^t i\,dt + V_o$$

Example 3-3

Given an exponential *spike* of current into a 5 μF capacitor with:

i_{max} = 1.5 A
τ = 100 μsec
V_o = -10 V

determine the equation for the voltage across the capacitor and plot it.

Solution

$$i = Ae^{-\frac{t}{\tau}} = 1.5\text{ A e}^{-\frac{t}{100\ \mu\text{sec}}}$$

$$v = \frac{1}{C}\int_0^t i\,dt + V_o = \frac{1}{5\ \mu F}\int_0^t 1.5\text{ A e}^{-\frac{t}{100\ \mu\text{sec}}}\text{dt} - 10\text{ V}$$

From the preceding section:

$$\int_0^t \left(Ae^{\frac{-t}{\tau}}\right)dt = A\tau\left(1 - e^{\frac{-t}{\tau}}\right)$$

$$v = \frac{1}{5\ \mu F}\int_0^t 1.5\text{ A e}^{-\frac{t}{100\ \mu\text{sec}}} - 10\ V = \frac{1.5\ A}{5\ \mu F} \times 100\ \mu sec\left(1 - e^{-\frac{t}{100\ \mu sec}}\right) - 10\ V$$

$$v = 30\ V\left(1 - e^{-\frac{t}{100\ \mu sec}}\right) - 10V$$

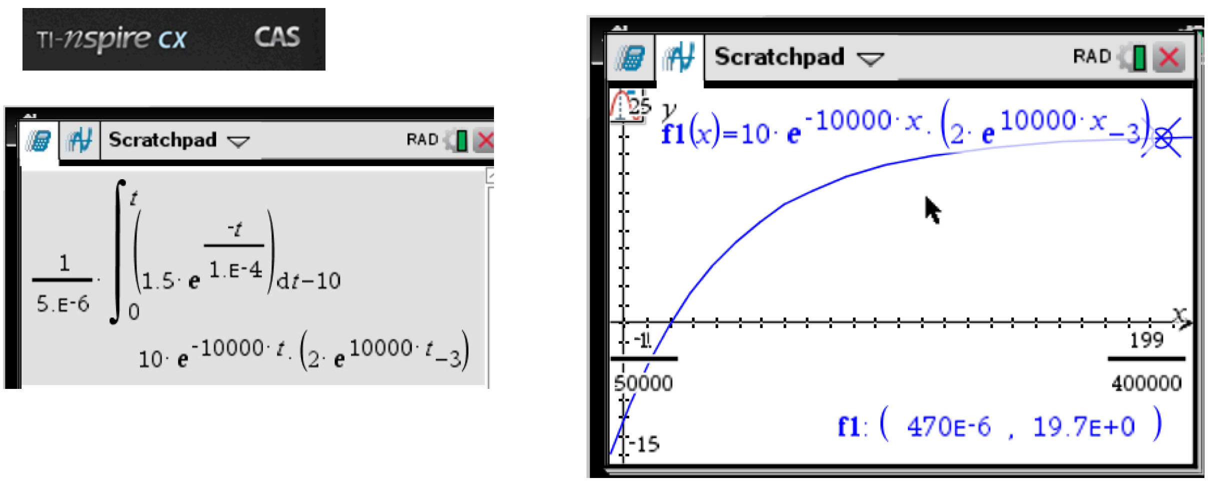


Figure 3-13 Calculator integration and plot

Although the calculator's answer to the integration seems different than the manual solution, a little algebra will show that the two results agree. The plot starts at V_o = -10 V, and in 5 time constants (500 µsec) rises exponentially to 20 V, a total rise of 30 V.

3.3 Inductance in the Time Domain

As current flows through a conductor a magnetic field is created that encircles the conductor. The stronger the current is, the further the field extends, circling further from the conductor.

The complement is also true. When a magnetic field intersects a conductor, current is induced to flow in the wire, producing a voltage in any connected circuit. It is the relative motion of the magnetic field moving across the wire that induces the current.

An inductor is a *coil* of wire, even though it may be so small that it is printed and come in a surface mount package. Current flowing through a segment of the coil creates a magnetic field. As that current increases (or decreases) the magnetic field increases or decreases. That expanding and contracting magnetic field cuts an adjacent segment of the coil inducing a current in it. All of these induced currents from all of the segments add producing a voltage across the two terminals of the inductor.

This induced voltage was created by the expanding and contracting magnetic fields from all of the adjacent wire segments. But, the magnetic fields expand and contract because the current through the coil is increasing and decreasing. The larger that current, or the more the resulting field moves, the more voltage is induces across the terminals of the inductor.

$$v \propto \frac{di}{dt}$$

The constant of proportionality is L, the inductance. How much voltage is induced depends on how well the magnetic fields cut adjacent wire segments and how big that magnetic field is, i.e. how the coil is wound, any magnetic core materials, the wire spacing and diameter, … .

$$v = L\frac{di}{dt}$$

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This is the complement of the relationship for a capacitor.

$$i = C\frac{dV}{dt}$$

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These relationships are summarized in Table 3-1.

Table 3-1 Voltage and current relationships

unknown	R	C	L
i	$\frac{v}{R}$	$C\frac{dv}{dt}$	$\frac{1}{L}\int_0^t v\,dt + I_o$
v	$i \times R$	$\frac{1}{C}\int_0^t i\,dt + V_o$	$L\frac{di}{dt}$

Example 3-4

The current waveform shown in Figure 3-14 is driven into a 5 mH inductor. *Accurately* draw the voltage across the inductor.

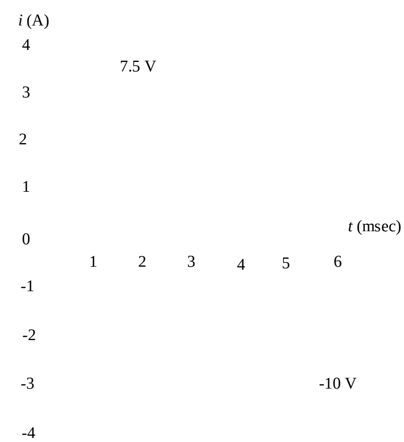


Figure 3-14 Waveforms for Example 3-4

Solution

$$v = L\frac{di}{dt}$$

For 0 msec < t < 2 msec, the current is *constant*. There is no change, so

$$v_{0\text{ msec to }2\text{ msec}} = 0\text{ V}$$

For 2 msec < t < 4 msec, the current ramps up, at a constant rate.

$$v = L\frac{di}{dt} = 5\text{ mH} \times \frac{4\text{ A} - 1\text{ A}}{4\text{ msec} - 2\text{ msec}} = 7.5\,V$$

For 4 msec < t < 5 msec, the current ramps down, at a constant rate.

$$v = L\frac{di}{dt} = 5\text{ mH} \times \frac{2\text{ A} - 4\text{ A}}{5\text{ msec} - 4\text{ msec}} = -10\,V$$

For 5 msec < t < 6 msec, the current is *constant*. There is no change, so

$$v_{5\text{ msec to }6\text{ msec}} = 0\text{ V}$$

Example 3-5

Given a current sine wave through a 5 mH inductor with:

i_{max} = 1.5 A
f= 100 Hz
θ = 45º

determine the equation for the voltage across it.

Solution

$$i = 1.5\ A\ \sin(2\ \pi \times 100\ \text{Hz} \times t + 45°)$$

$$v = L\frac{di}{dt}$$

$$\frac{d}{dt}\ Asin(2\pi ft - \theta) = A \times 2\pi f \times cos(2\pi ft - \theta)$$

$$v = 5\ \text{mH} \times 1.5\ A \times 2\pi \times 100\ \text{kHz} \times \cos(2\ \pi \times 100\text{Hz} \times t + 45°)$$

$$v = 4.71\ \text{V}\cos(2\ \pi \times 100\text{Hz} \times t + 45°)$$

or

$$v = 4.71\ \text{V}\sin(2\ \pi \times 100\text{Hz} \times t + 135°)$$

Using phasors

$$X = 2\pi \times 100\ \text{Hz} \times 5\ \text{mH} = 3.14\ \Omega$$

$$\bar{Z} = (3.14\ \Omega\ \angle 90°)$$

$$\bar{V} = \bar{I} \times \bar{Z} = \left(\frac{1.5\ A}{\sqrt{2}}\ \angle\ 45°\right) \times (3.14\ \Omega\ \angle 90°) = \left(\frac{4.71\ V}{\sqrt{2}} \angle\ 135°\right)$$

This matches the derivative calculation. Do not be confused by $\frac{1}{\sqrt{2}}$. Phasor calculations are done in V_{RMS} which is $\frac{V}{\sqrt{2}}$.

Example 3-6

Given a voltage ramp across an 80 mH inductor with:

I_o = 0 A (ramp starts at 0 and goes up)
v_{max} = 5 V
f = 100 Hz

determine the equation for the current through the inductor and plot it.

Solution

The equation for the ramp is

$$v(t) = mt + b = \frac{A}{T}t + b$$

where

$$A = 5\text{ V}$$

$$T = \frac{1}{100\text{ Hz}} = 10\text{ msec}$$

$$b = 0\text{ V}$$

$$v(t) = \frac{5\text{ V}}{10\text{ msec}}t = 500\,\frac{\text{V}}{\text{sec}}t$$

The current is

$$i = \frac{1}{L}\int_0^t v\,dt + I_o$$

$$i = \frac{1}{80\text{ mH}}\int_0^t 500\,\frac{\text{V}}{\text{sec}}\,t\,dt$$

The general integration is

$$\int_0^t \left(\frac{A}{T}t + b\right)dt = \frac{A}{2T}t^2 + bt$$

$$i = \frac{1}{80\text{ mH}}\int_0^t \left(500\,\frac{\text{V}}{\text{sec}}t\right)dt = \frac{1}{80\text{ mH}} \times \frac{500\frac{\text{V}}{\text{sec}}}{2}t^2$$

$$i = 3125\frac{A}{sec^2}t^2$$

This seems like an obscene amount of current. But that is multiplied by t^2. At the end of the ramp, 10 msec,

$$i = 3125\frac{\text{A}}{\text{sec}^2}\times(10\text{ msec})^2 = 312.5\text{ mA}$$

The calculator integration and plot (out to just beyond 10 msec) are shown in Figure 3-15.

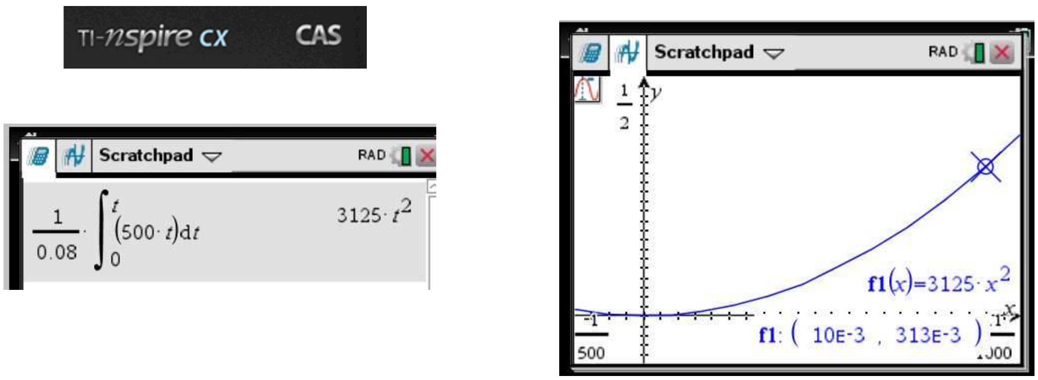


Figure 3-15 Calculator integration and plot for Example 3-6

The plot (and equation) are misleading. At t = 10 msec, at the end of the period, the wave does *not* continue to rise. It resets to 0 V and starts over. So, the output is valid only for 0 msec < t < 10 msec. Beyond that, different calculations are needed, since there now *is* an initial current, and time has been shifted.

Still, the math and the plot show that a linear increase in voltage produces a nonlinear rise in current through the inductor.

3.4 RC & RL Circuit Differential Equations

With derivatives and integrals, a capacitor's or inductor's current and voltage can be calculated. When those components are put into a circuit, then these derivatives and integrals must be incorporated with normal circuit analysis laws: Kirchhoff's Voltage and Current Laws, Thevenin's Theorem, Superposition, Impedance Combination, even Mesh Analysis.

Table 3-1 Voltage and current relationships

unknown	R	C	L
i	$\frac{v}{R}$	$C\frac{dv}{dt}$	$\frac{1}{L}\int_0^t v\,dt + I_o$
v	$i \times R$	$\frac{1}{C}\int_0^t i\,dt + V_o$	$L\frac{di}{dt}$

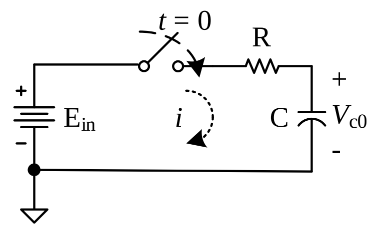


Figure 3-16 RC circuit with a step input

Figure 3-16 is a series circuit. Initially, the capacitor's charge (if any) is $V_{c0.}$ At t = 0, the switch is closed, applying a set voltage to the circuit, creating current i. The voltages and current change with time, as the capacitor changes its charge. The key voltage and current relationships are shown in Table 3-1 (again).

To begin, assume that the capacitor is initially uncharged.

$$V_{c0+} = 0 \text{ V}$$

Remember, a capacitor cannot change its voltage (charge) instantly. It takes time for the charge to flow from the source and to build up on the plates, creating an electrostatic field and voltage. So, just after the switch is closed $t = 0^+$, there is still no voltage across the capacitor. All of the source voltage must be across the resistor.

$$V_{R0+} = E_{in}$$

The resistor sets (limits) that initial current.

$$i_{0^+} = \frac{V_{R0^+}}{R} = \frac{E_{\text{in}}}{R}$$

By recognizing that a capacitor opposes a change in voltage, cannot change its voltage instantly, the *initial* ($t = 0^+$) voltages and current have been calculated.

As time goes on, performance becomes more involved. Begin by writing Kirchhoff's Voltage Law around the circuit.

$$E_{\text{in}} = v_{\text{R}} + v_{\text{C}}$$

To solve this single loop there can be only one variable. Both v_{R} and v_{C} can be written in terms of the circuit current, i, and know component values, E_{in}, R, and C. Look at Table 3-1.

$$E_{\text{in}} = i\text{R} + \frac{1}{\text{C}}\int_0^t i\,dt + V_o$$

Since the initial capacitor voltage is 0 V, the circuit equation becomes

$$E_{\text{in}} = i\text{R} + \frac{1}{\text{C}}\int_0^t i\,dt$$

There is a whole body of mathematics called Differential Equations. Those procedures can be used to solve this equation for i. But first, the integral must be dealt with. Differentiate both sides of the equation.

$$\frac{d}{dt}(E_{in}) = \frac{d}{dt}(i\text{R}) + \frac{d}{dt}\left(\frac{1}{\text{C}}\int_0^t i\,dt\right)$$

Since the voltage is a constant, its time derivative is 0. The derivative of an integral of a function is just the function.

$$0 = \text{R}\frac{di}{dt} + \frac{1}{\text{C}}i$$

This helps quite a bit. It is common to denote the time derivative of a variable as that variable followed by a prime, $\frac{di}{dt} = i'$

$$0 = \text{R}i' + \frac{1}{\text{C}}i$$

Finally, the equation is usually rearranged to put the highest derivative of the variable on the left of the equation, alone.

$$i' = -\frac{1}{\mathrm{RC}} i$$

This is the simplest form of the first order (one derivative) differential equation, and the simplest form of the equation that describes the performance of the series RC circuit to a step input.

The question now becomes, "What function can be differentiated and equal itself times a constant?" That is answered by *entire* mathematics courses on Differential Equations, way outside the content of this discussion of *electronics*. Fortunately, the TI-Nspire calculator can solve differential equations of electronics interest.

deSolve

Begin by adjusting the settings as shown in Figure 3-17.

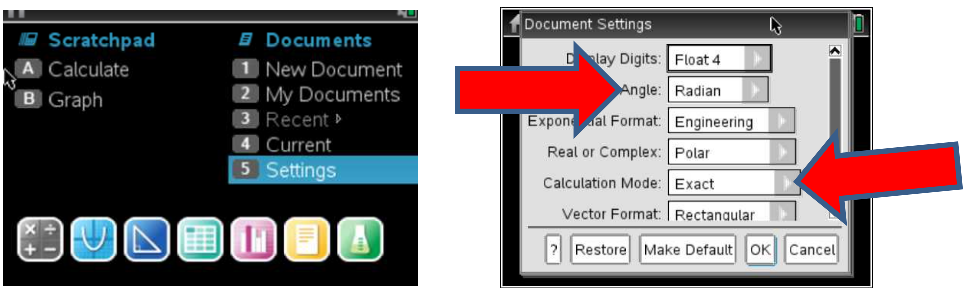


Figure 3-17 Calculator settings for differential equations solution

Then, from the Menu, select **4:Calculus** and **D:Differential Solver**

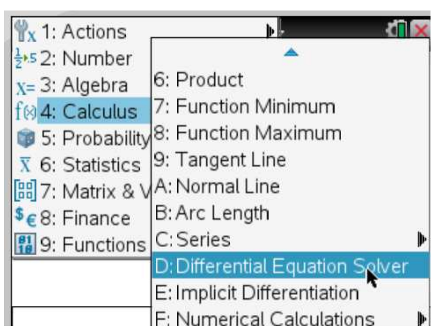


Figure 3-18 Menu selection to start the Different Solver

$$i' = -\frac{1}{\mathrm{RC}}i$$

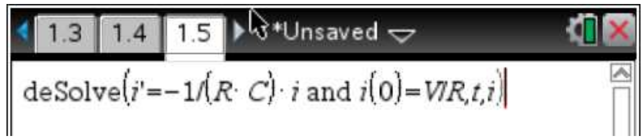


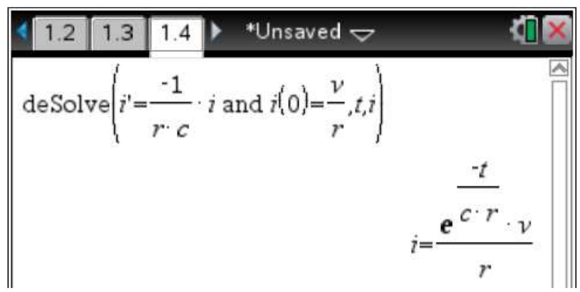
Figure 3-19 Entering the differential equation

$$i_{0^+} = \frac{V_{R0^+}}{R} = \frac{E_{\text{in}}}{R}$$

When the Differential Solver is selected, **deSolve()**
Appears. Enter the differential equation:

- The prime is on the π menu.
- Spaces before and after **and** are necessary.
- Then enter the initial conditions equation,
- Followed by a comma, the *x* variable, and the y variable.

$$i = \frac{E_{\text{in}}}{\mathrm{R}}\mathrm{e}^{-\frac{t}{RC}}$$



$$i(t) = Ae^{\frac{-t}{\tau}}$$
$$A = \frac{E_{\text{in}}}{R}$$
$$\tau = \frac{1}{RC}$$

Figure 3-20 Calculator solution of the differential equation

The solution, Figure 3-20, looks a little different from the *standard* forms, but a littl
are the same. The current spikes up to i_{0+}, and then falls, with a time constant of t
So, limit the current that the circuit must produce with R. Once R is chosen, the cap
mines how rapidly the circuit responds.

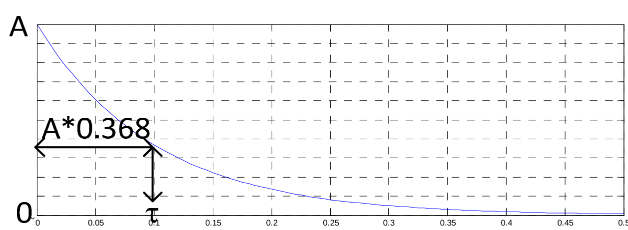


Figure 3-8 Exponential fall with amplitude (*A*), and
time constant (*τ*) defined (again)

Once the circuit current is known, the voltage across the resistor and capacitor can be calculated.

$$v_{\mathrm{R}} = i \times \mathrm{R}$$

$$i = \frac{E_{\mathrm{in}}}{\mathrm{R}} \mathrm{e}^{-\frac{t}{RC}}$$

$$v_R = i \times R = \left(\frac{E_{\mathrm{in}}}{\mathrm{R}} \mathrm{e}^{-\frac{t}{RC}}\right) \times R$$

$$v_{\mathrm{R}} = E_{\mathrm{in}}\, \mathrm{e}^{-\frac{t}{RC}}$$

$$v_{\mathrm{C}} = \frac{1}{C}\int_0^t i\, dt + V_o$$

$$v_{\mathrm{C}} = \frac{1}{\mathrm{C}}\int_0^t \left(\frac{E_{\mathrm{in}}}{\mathrm{R}} \mathrm{e}^{-\frac{t}{RC}}\right) dt$$

$$v_{\mathrm{C}} = \frac{E_{\mathrm{in}}}{\mathrm{RC}}\int_0^t \mathrm{e}^{-\frac{t}{RC}}\, dt \qquad \int_0^t \left(Ae^{\frac{-t}{\tau}}\right) dt = A\tau\left(1 - e^{\frac{-t}{\tau}}\right)$$

$$v_C = \frac{E_{\mathrm{in}}}{\mathrm{RC}} \times \mathrm{RC}\left(1 - e^{\frac{-t}{RC}}\right)$$

$$v_C = E_{\mathrm{in}}\left(1 - e^{\frac{-t}{RC}}\right)$$

If all these calculations are correct, then the original Kirchhoff's Voltage Law should work with these equations.

$$E_{\mathrm{in}} = v_{\mathrm{R}} + v_{\mathrm{C}}$$

$$E_{\mathrm{in}} \mathrel{?=} E_{\mathrm{in}}\, \mathrm{e}^{-\frac{t}{RC}} + E_{\mathrm{in}}\left(1 - e^{\frac{-t}{RC}}\right)$$

$$E_{\mathrm{in}} \mathrel{?=} E_{\mathrm{in}}\, \mathrm{e}^{-\frac{t}{RC}} + E_{\mathrm{in}} - E_{\mathrm{in}} e^{\frac{-t}{RC}}$$

$$E_{\mathrm{in}} \equiv E_{\mathrm{in}}$$

Example 3-7

Given:

E_{in} = 10 V
R = 1 kΩ
C = 10 nF

Calculate:

Initial current
Time constant
Final voltage across R
Final voltage across C

Plot v_R and v_C.

Solutions

$$i_{t=0} = \frac{E_{\text{in}}}{\text{R}}\text{e}^{-\frac{t}{RC}} = \frac{10\ V}{1\ \text{k}\Omega}\text{e}^{-\frac{0}{RC}}$$

$$i_{t=0} = 10\ \text{mA}$$

$$\tau = \text{RC} = 1\text{k}\Omega \times 10\text{nF} = 10\ \mu\text{sec}$$

$$v_{\text{R}} = E_{\text{in}}\ \text{e}^{-\frac{t}{RC}} = 10\ \text{V}\ \text{e}^{-\frac{\infty}{RC}}$$

$v_{\text{R}} = 0\ V$ Eventually there is *no* voltage across the resistor.

$$v_C = E_{\text{in}}\left(1 - e^{\frac{-t}{RC}}\right) = 10\ \text{V}\left(1 - e^{\frac{-\infty}{RC}}\right)$$

$$v_C = 10\ \text{V}(1-0) = 10\ \text{V}$$

Eventually *all* of the voltage is across the capacitor. It is fully charged.

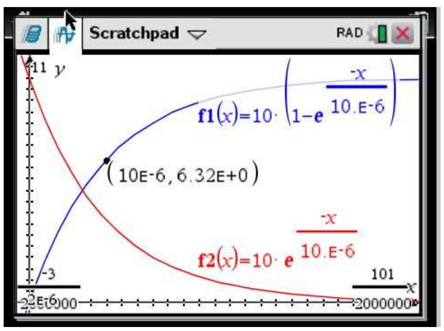


Figure 3-21 RC circuit voltage waveforms

The plots are shown in Figure 3-21. Initially, all of the voltage appears across the resistor [f2(x)], and the voltage across the capacitor is 0 V, [f1(x)]. As time goes on, the voltage across the resistor drops and the voltage across the capacitor rises, as it charges. Eventually, all of the input is across the capacitor, the capacitor is fully charged, there is no current, and so there is no voltage across the resistor.

After one time constant, 10 µsec, 63.2 % of the voltage is across the capacitor, leaving 36.8% of the voltage across the resistor.

The Time Constant, τ

For all *exponential* responses, the key timing parameter is the time constant, τ. For the rising waveforms, time t only appears in the exponent of the exponential e.

$$v(t) = A\left(1 - e^{\frac{-t}{\tau}}\right)$$

At $t = \tau$, the voltage has changed 63.2% of the maximum possible change.

$$v(t) = A(1 - e^{-1}) = 0.632 \times A$$

The rising wave starts at 0 V. So its maximum possible change is to increase to 63.2% of A. See Figure 3-22(a).

The interesting thing is that during the second time constant, the wave changes by 63.2% of the remaining maximum possible change. The rising wave starts at 0.632 * *A*. During the second time constant, the maximum remaining change is 0.368 * *A*. It changes 63.2% * 0.368 * *A* = 0.233 * *A*. Since it started the second time constant at 0.632 * *A* the voltage will arrive at 0.632 * *A* + 0.233 * *A* = 0.865 * *A*, Figure 3-22(b).

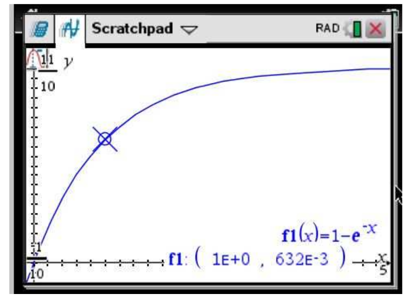
During the *third* time constant, there is only *A-0.865* * *A* = 0.135 * *A* for the voltage to rise. It will only rise 63.2% of this (63.2%*0.135 * *A* = 0.086 * *A*), arriving at 0.95 * *A*, Figure 3-22 (c).

Following this logic, as the voltage gets closer and closer to *A*, the function increases less and less. Taken to the extreme, the function can *never* reach *A*. But, after five time constants, the voltage rises to 0.993 * *A*, which is traditionally considered “close enough”.

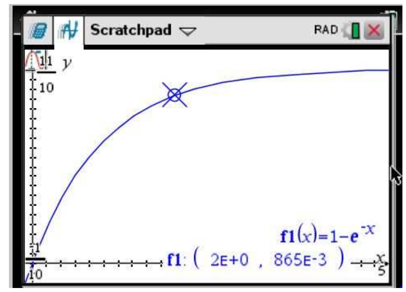
It takes 5τ for an RC circuit to fully (99.3%) respond.

A more useful measure is the *rise time*, t_{rise}. The rise time is the time it takes a wave to rise from 10% to 90%. This gets around any delays in getting started and the slow approach at the end. The plot in Figure 3-23 shows that the wave passes 10% (101E-3) at 0.1 τ (106E-3). The wave passes 90% (900E-3) at 2.3 τ. So, it takes 2.2 τ for the wave to rise from 10% to 90%.

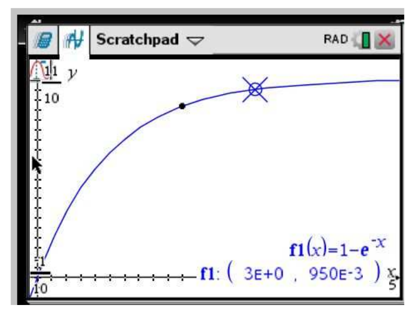
It takes 2.2τ for an RC circuit to complete its *rise time*.



(a) Rise after one time constant



(b) Rise after two time constants



(c) Rise after three time constants

Figure 3-22 Exponential rise

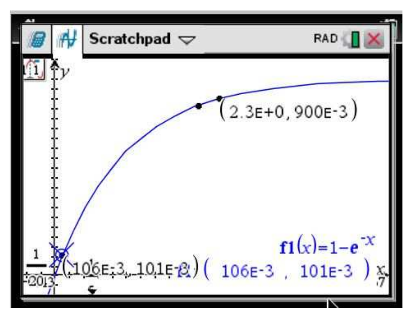


Figure 3-23 Exponential rise time

RL Differential Equation Circuit Analysis

Replacing the capacitor in Figure 3-16 with an inductor produces a complimentary response. The schematic is in Figure 3-24.

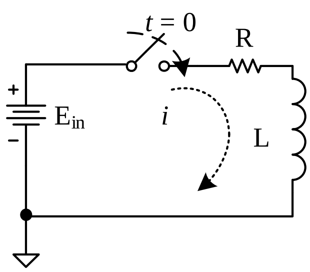


Figure 3-24 RL circuit with a step

Table 3-1 Voltage and current relationships

unknown	R	C	L
i	$\frac{v}{R}$	$C\frac{dv}{dt}$	$\frac{1}{L}\int_0^t v\,dt + I_o$
v	$i \times R$	$\frac{1}{C}\int_0^t i\,dt + V_o$	$L\frac{di}{dt}$

Magnetic fields building and cutting segments of the inductor induces current into those segments, establishing the current through the inductor. These magnetic fields cannot appear instantaneously. It takes time for them to build. So, the induced current cannot change instantly either.

Inductors oppose a change in current.

Starting with an open circuit the instant *before* the switch is closed gives $i_{t=0-} = 0$ A. The inductor opposes any change in current. So

$$i_{t=0+-} = 0 \text{ A}$$

the instant after the switch is closed, the initial current is also 0A.

Applying Kirchhoff's Voltage Law, just as with the RC circuit gives

$$E_{\text{in}} = v_{\text{R}} + v_L$$

Putting this equation in terms of current

$$E_{\text{in}} = i \times \mathrm{R} + \mathrm{L}\frac{di}{dt}$$

This is different from the RC circuit because there is no integral to get rid of, but now E_{in} remains part of the equation. A little algebra puts this circuit equation into the standard differential equations format,

$$E_{\text{in}} = i\mathrm{R} + \mathrm{L}i'$$

$$i' = -\frac{\mathrm{R}}{\mathrm{L}}i + \frac{E_{\text{in}}}{\mathrm{L}}$$

The TI-Nspire solution is shown in Figure 3-25.

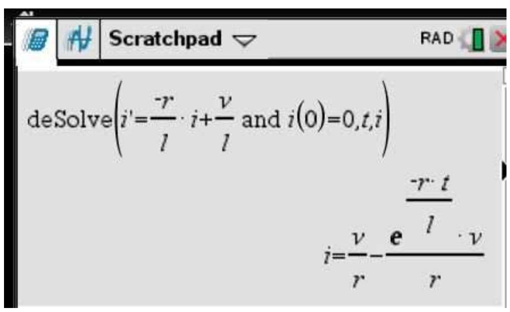


Figure 3-25 Calculator solution of the RL differential equation

This looks a clumsy. But, a little algebra cleans this up into one of the standard forms. Begin by factoring *v*/*r* out. Then clean up the exponent.

$$i = \frac{E_{\text{in}}}{\mathrm{R}}\left(1 - e^{-\frac{t}{\mathrm{L}/\mathrm{R}}}\right)$$

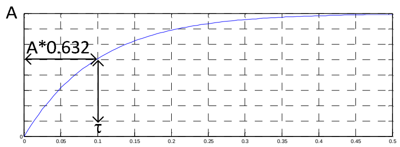
$$i(t) = A\left(1 - e^{\frac{-t}{\tau}}\right)$$

Standard form

$$A = \frac{E_{\text{in}}}{\mathrm{R}}$$

$$\tau = \frac{\mathrm{L}}{\mathrm{R}}$$

The voltage across the resistor is



$$v_{\mathrm{R}} = i \times \mathrm{R} = \left[\frac{E_{\mathrm{in}}}{\mathrm{R}}\left(1 - e^{-\frac{t}{L/R}}\right)\right] \times \mathrm{R}$$

$$v_{\mathrm{R}} = E_{\mathrm{in}}\left(1 - e^{-\frac{t}{L/R}}\right)$$

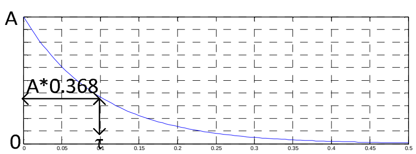
The voltage across the inductor is

$$v_{\mathrm{L}} = L\frac{di}{dt}$$

$$\frac{d}{dt}\left[A\left(1 - e^{\frac{-t}{\tau}}\right)\right] = \frac{A}{\tau}e^{\frac{-t}{\tau}}$$

$$v_{\mathrm{L}} = \mathrm{L}\frac{d}{dt}\left[\frac{E_{\mathrm{in}}}{\mathrm{R}}\left(1 - e^{-\frac{t}{L/R}}\right)\right]$$

$$v_{\mathrm{L}} = \frac{E_{in}\frac{\mathrm{L}}{\mathrm{R}}}{\frac{\mathrm{L}}{\mathrm{R}}}e^{-\frac{t}{L/R}}$$



$$v_{\mathrm{L}} = E_{in}e^{-\frac{t}{L/R}}$$

The TI-Nspire produces the same result, shown in Figre 2-26.

TI-nspire CX CAS

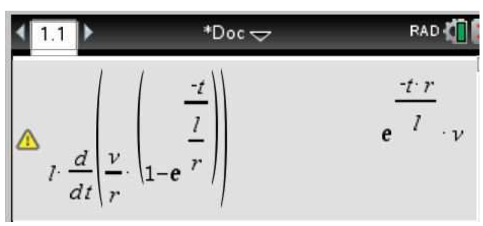
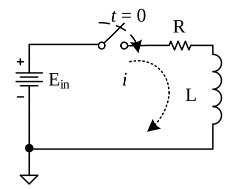


Figure 3-26 Calculator verification of the voltage across the inductor

$$i = \frac{E_{\text{in}}}{\text{R}}\left(1 - e^{-\frac{t}{L/R}}\right) \qquad v_{\text{R}} = E_{\text{in}}\left(1 - e^{-\frac{t}{L/R}}\right) \qquad v_{\text{L}} = E_{in}e^{-\frac{t}{L/R}}$$



Does all this math produce results that match what experience shows happens in the circuit?

The inductor opposes a change in current. So, the instant after the switch closes ($t = 0^+$), there will be *no* current in the circuit. Substituting $t = 0$ into the current equation produces $i = 0$.

With no current, the resistor can drop no voltage. Substituting $t = 0$ into the v_R equation gives $v_R = 0$.

With no voltage across the resistor, all of E_{in} must show up across the inductor. Substituting $t = 0$ in the the v_L equation gives $v_L = E_{in}$.

As time goes on and on, eventually the inductor becomes fully charged (i.e. the magnetic fields are fully established), and it looks like a piece of wire. At that time ($t = \infty$), the inductor (as a piece of wire) drops *no* voltage. Substituting $t = \infty$ into the v_L equation gives $v_L = 0$.

With no voltage across the inductor, all of E_{in} must appear across the resistor. Substituting $t = \infty$ into the v_R equation gives $v_R = E_{in}$.

Since there is E_{in} across the resistor, $i = E_{in}/\text{R}$. Substituting $t = \infty$ into the i equation gives that result.

3.5 Initial Conditions and Other Inputs

When the circuit does not start from rest, or when the input is something more complicated than the simple step used in the preceding section, the *algebra* becomes more involved. But the electronics and the procedures are the same.

Initial Conditions

For the RL circuit in Figure 3-24, how does the current behave differently, if at $t = 0^+$, $i = -5$ mA (the negative indicates it is flowing up, opposite to the indicated direction)?

The derivation of the circuit's differential equation is the same as originally done.

$$E_{\text{in}} = v_{\text{R}} + v_L$$

$$E_{\text{in}} = i \times \text{R} + \text{L}\frac{di}{dt}$$

$$E_{\text{in}} = i\text{R} + \text{L}i'$$

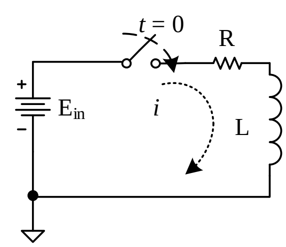


Figure 3-24 RL circuit with a step input

$$i' = -\frac{\mathrm{R}}{\mathrm{L}} i + \frac{E_{\text{in}}}{\mathrm{L}}$$

Entering this into the calculator for a solution, however requires that the initial current be included.

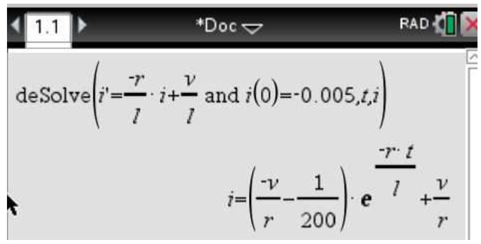


Figure 3-27 Calculator solution of RL circuit with initial current

Although pieces of this result have been seen before, as a whole this is not familiar. Algebra is needed. Begin by distributing the exponent.

$$i = -\frac{V}{R} e^{-\frac{t}{L/R}} - 5\ \text{mA}\ e^{-\frac{t}{L/R}} + \frac{V}{R}$$

Both the first and the last term have V/R, which is the eventual current. So, combine those.

$$i = \frac{V}{R} - \frac{V}{R} e^{-\frac{t}{L/R}} - 5\ \text{mA}\ e^{-\frac{t}{L/R}}$$

$$i = \frac{V}{R}\left(1 - e^{-\frac{t}{L/R}}\right) - 5\ \text{mA}\ e^{-\frac{t}{L/R}}$$

Now this looks like pieces seen before. The second term is just the exponential decay of the initial current. Then, the first term is the performance expected with no initial current.

It's just superposition, a voltage source and a current source. The voltage source produces a buildup in current, just as it did in the previous section (without the initial current). The (initial) current decays as its magnetic fields collapse.

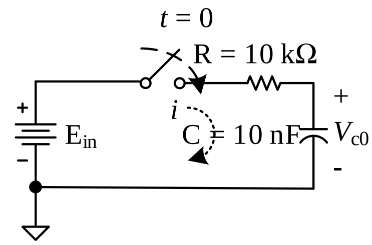


Figure 3-16 RC circuit with a step input

The RC circuit in Figure 3-16 is a little more involved to analyze when there is an initial condition. Let

$$E_{in} = 10\text{ V}$$

At $t = 0^+$ $$V_{c0} = 3\text{ V}$$

$$i_{0+} = \frac{10\text{ V} - 3\text{ V}}{1\text{ k}\Omega} = 7\text{ mA}$$

The circuit analysis with this initial condition now follows just as before.

$$E_{\text{in}} = v_{\text{R}} + v_{\text{C}}$$

$$E_{\text{in}} = i\text{R} + \frac{1}{\text{C}}\int_0^t i\,dt + V_o$$

$$E_{\text{in}} = i\text{R} + \frac{1}{\text{C}}\int_0^t i\,dt + V_0$$

$$\frac{d}{dt}(E_{in}) = \frac{d}{dt}(i\text{R}) + \frac{d}{dt}\left(\frac{1}{\text{C}}\int_0^t i\,dt\right) + \frac{d}{dt}V_0$$

$$0 = \text{R}\frac{di}{dt} + \frac{1}{\text{C}}i + 0$$

$$0 = \text{R}i' + \frac{1}{\text{C}}i$$

$$i' = -\frac{1}{\text{RC}}i$$

So, the circuit differential equation with an initial charge on the capacitor is the same as it is without the charge.

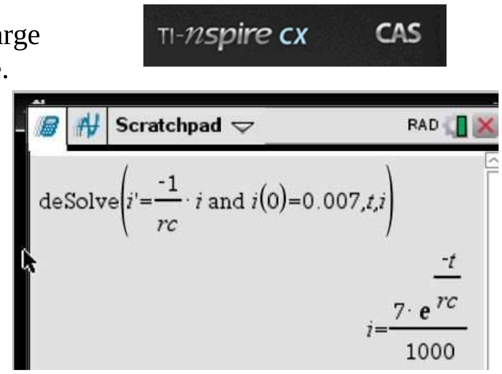


Figure 3-28 Solution of RC circuit with initial conditions

$$i = 7\ \text{mA}e^{-\frac{t}{RC}}$$

The initial charge on the capacitor has decreased the magnitude of the charging current.

The voltage across the capacitor involves an integration.

$$v_c = \frac{1}{\text{C}}\int_0^t i dt + V_0$$

$$v_c = \frac{1}{10\ \text{nF}}\int_0^t \left(7\text{mA}e^{-\frac{t}{1\,k\Omega\times 10\,nF}}\right) dt + 3\ \text{V}$$

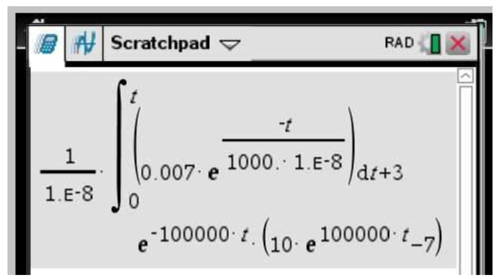


Figure 3-29 Capacitor voltage integral solution

Algebra is now needed to put this into a more standard form. Start with distributing the negative exponential

$$v_c = 10\ \text{V} - 7\ \text{V}e^{-\frac{t}{10\,\mu sec}}$$

$$v_c = 3\ \text{V} + 7\ \text{V} - 7\ \text{V}e^{-\frac{t}{10\,\mu sec}}$$

$$v_c = 3\ \text{V} + 7\ \text{V}\left(1 - e^{-\frac{t}{10\,\mu sec}}\right)$$

The voltage across the capacitor starts at 3 V and rises exponentially 7 V more, with a 10 μsec time constant (1 kΩ x 10 nF), eventually leaving the capacitor with a 10 V charge. The math properly aligns with what experience using RC charging circuits indicates should happen.

Ramp Input

The same procedure used so far, in setting up and solving a circuit's differential equation for a step input, is used with other shaped inputs. Figure 3-30 shows a CR series circuit driven by a ramp.

C

e_{in} i R

Figure 3-30 CR circuit with a ramp input

A = peak amplitude
T = period

$$e_{\text{in}} = \frac{A}{T}t$$

$$e_{in} = v_{\text{C}} + v_{\text{R}}$$

$$\frac{A}{T}t = \frac{1}{C}\int_0^t idt + V_0 + i \times \text{R}$$

Taking the derivative of both sides to get rid of the integral,

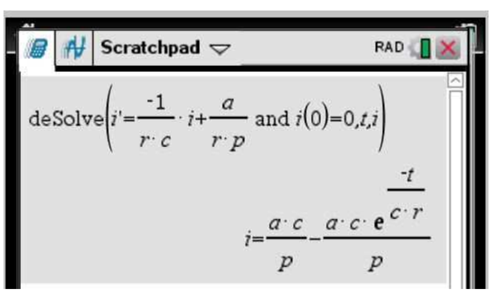
$$\frac{A}{T} = \frac{1}{C}i + i'\text{R}$$

Rearranging to put the equation into the standard form

$$i' = -\frac{1}{\text{RC}}i + \frac{\frac{\text{A}}{T}}{\text{R}}$$

$$i' = -\frac{1}{\text{RC}}\ i + \frac{A}{\text{R}T}$$

Since $e_{\text{in}\,t=0} = 0$, then $i_0 = 0$, the solution of this differential equation is



TI-nspire CX CAS

Figure 3-31 Solution of differential equation with a ramp input

RC Circuit with a step input

$$i' = -\frac{1}{\mathrm{RC}}i$$

$$i = \frac{E_{\mathrm{in}}}{\mathrm{R}}\mathrm{e}^{-\frac{t}{RC}}$$

$$i = \frac{AC}{T}\left(1 - e^{-\frac{t}{RC}}\right)$$

RC circuit driven by a ramp

This may be unexpected result. Driven by a ramp, the current *rises* exponentially. The results from the beginning two parts of section 3.4 is for the RC circuit driven with a step. It calculates the current *decreasing* exponentially. Look to the right.

Driven by a ramp, the voltage across the resistor is

$$v_R = i \times R = \frac{A\,RC}{T}\left(1 - e^{-\frac{t}{RC}}\right)$$

Figure 3-32 shows the results of a Multisim simulation of a ramp driving an RC Circuit. Sure enough, as the ramp increases linearly, the voltage across the resistor rises exponentially to a constant. The offset on v_R is caused by the charge left on the capacitor from previous cycles, i.e. the initial condition that was *assumed* to be zero.

MULTISIM

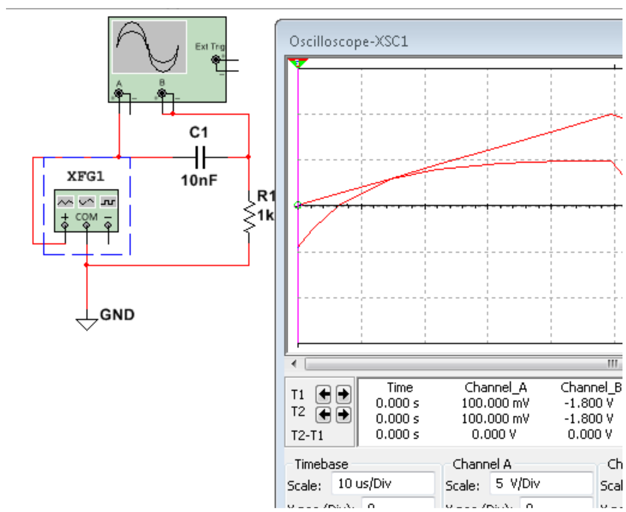


Figure 3-32 Simulation of RC circuit driven by a ramp

Exponential Spike Input

The circuit in Figure 3-33 is very similar to the RL circuit of Figure 3-24. However, the input has been changed to an exponential spike.

$$e_{\text{in}} = Ae^{-\frac{t}{\tau_{input}}}$$

A= peak amplitude

τ_{input} = input time constant

L

e_{in}

i R

Figure 3-33 RL circuit with an exponential spike input

As with analyzing other first order circuits, Kirchhoff's voltage law gives

$$e_{\text{in}} = v_{\text{L}} + v_{\text{R}}$$

$$Ae^{-\frac{t}{\tau_{input}}} = \text{L}\frac{di}{dt} + i\text{R}$$

$$Ae^{-\frac{t}{\tau_{input}}} = \text{L}i' + \text{R}i$$

$$\text{L}i' = Ae^{-\frac{t}{\tau_{input}}} - \text{R}i$$

$$i' = \frac{\text{A}}{\text{L}}e^{-\frac{t}{\tau_{input}}} - \frac{\text{R}}{\text{L}}i$$

$$i' = -\frac{\text{R}}{\text{L}}i + \frac{\text{A}}{\text{L}}e^{-\frac{t}{\tau_{input}}}$$

This is the circuit differential equation. It is ready to be solved in this *general* form. Usually getting a general solution, with the parameters represented by letters is a good idea. It shows how each parameter affects the outcome. The calculator is certainly capable of doing that.

However, in this case, the result is complex enough that it is hard to get a good idea of the circuit's response. So, instead, solve the differential equation with these realistic parameters, and use the variable *x* instead of *t*.

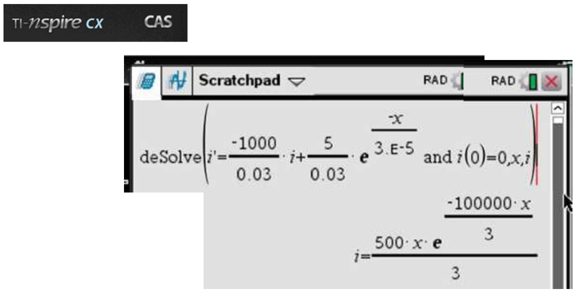
A = 5 V

τ_{input} = 30 μsec

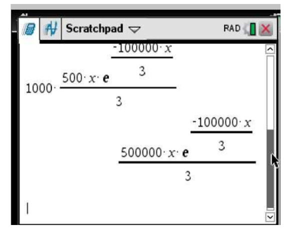
R = 1 kΩ

L = 30 mH

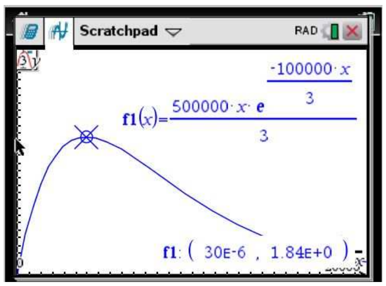
$i(0)$ = 0 A



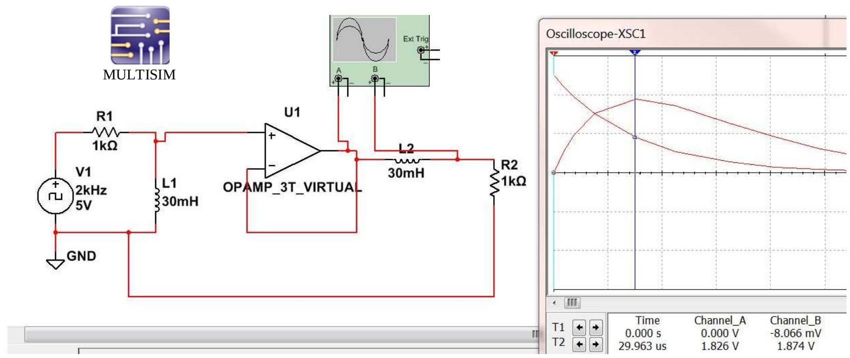
(a) Calculator solution of the differential equation



(b) $v_R = 1000\ \Omega * i$



(c) Equation copied into Graph, and then scaled with x_{max} = 150E-6 and y_{max} = 3.



(d) Multisim verification of calculation

Figure 3-34 RL circuit driven by an exponential pulse, differential equation solution

Figure 3-34a shows the solution of the differential equation for current.

The voltage across the resistor is calculate in Figure 3-34b, by copying the solution for *i* from *a*, pasting it and multiplying it by 1000 Ω.

That equation is copied and then pasted into the Graph portion of the calculator. For this to work, the horizontal variable must be *x*, not *t*. Finally, the graph is scaled using **menu/Window/1:Window Settings.**

The calculator shows a wave that starts up, but peaks after one input time constant and then falls, more-or-less exponentially to 0 V. That may not be intuitive. So, the Multisim simulation is shown in Figure 3-34d.

An RL circuit is used to produce the exponential spike for the input. That is followed by a voltage follower to prevent loading. So the voltage follower and everything to the left just create the input. The RL circuit of interest then follows. The oscilloscope shows the same result across the resistor as was calculated.

3.6 DC Motor

Fractional horse power motors are used extensively, from the small motor that moves the nose on a toy dog to the prime mover in an industrial robot. For a simple brushed DC motor to accurately complete its tasks, it must be driven with the correct voltage; the shaft begins to move; then it takes time for the motor to achieve its final response.

The motor converts electrical energy into mechanical (rotational) energy. But it takes time for the motor's speed to build up or coast down to a new level. That's just like the capacitor storing energy in its charge (electrostatic field) or the inductor storing energy in its magnetic field. Similar differential equations can be written and the motor's performance determined just as accurately. The derivation of that equation belongs in a study of rotating machines. The result is

http://www.leeson.com/Products/products/DCMotors/lowvoltage.html

Figure 3-35 DC motor

Speed

$$\omega' + \frac{1}{\tau}\omega = \frac{m}{\tau}v$$

ω = speed of the motor

$\frac{\text{revolutions}}{\text{minute}}$ (RPM)

τ = time constant
second

m = motor's gain

$\frac{\text{RPM}}{\text{V}}$

v = applied voltage

Motor manufactures' data sheets provide information about full speed RPM, maximum voltage, torque provided, and current requirements. Unfortunately, the two performance specifications needed to calculate the motor's dynamic response, m and τ are usually not specified. But, they can be *measured*.

Mount and load the motor as it will be used. Then apply a low voltage and record the speed once it has stabilized. Repeat this process across the range of the motor. An example tabulation is shown in Figure 3-36. Notice that it is typical for the motor not to start until there is 10% to 20% of the maximum voltage applied. This produces the nonlinearity and the offset shown in the scatter plot.

In Excel, right click on a data point in the scatter plot. In the menu provided, select **Trend Line**. Then pick **Linear** and **Display Equation on Chart.** This gives the equation of the best-fit straight line.

y = 205.7x - 231.64 i.e. y=mx+b

$$m = 205.7\,\frac{\text{RPM}}{\text{V}}$$

V	RPM
0	0
1	0
2	120
3	230
4	460
5	730
6	980
7	1210
8	1460
9	1750

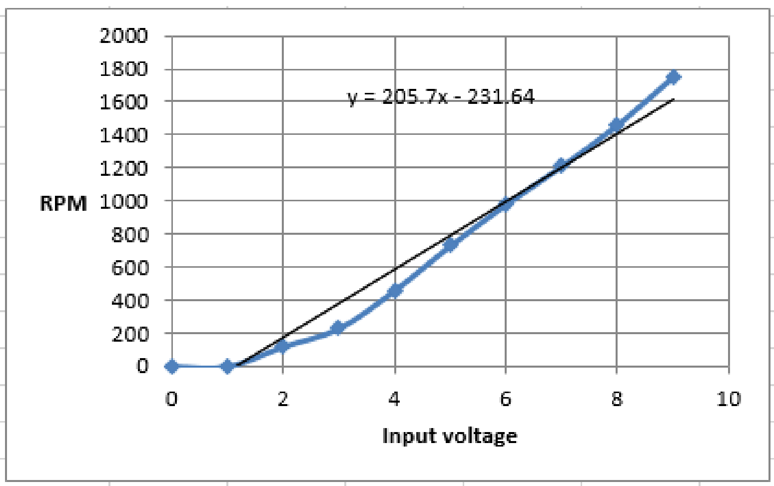


Figure 3-36 Typical DC motor static performance analysis

Measuring the time constant is a little more involved. The speed (RPM) must be captured as the motor changes speed. This can be done with an optical encoder. It is attached to the shaft and spins as the shaft spins. The output of the encoder is a frequency that is a direct measure of the speed. This frequency can be captured with a data acquisition card and displayed with Lab View. Or, the frequency can be converted to a voltage with a V-F IC, and the IC's output voltage captured with a digital oscilloscope. See Figure 3-37.

$$\omega' + \frac{1}{\tau}\omega = \frac{m}{\tau}v$$

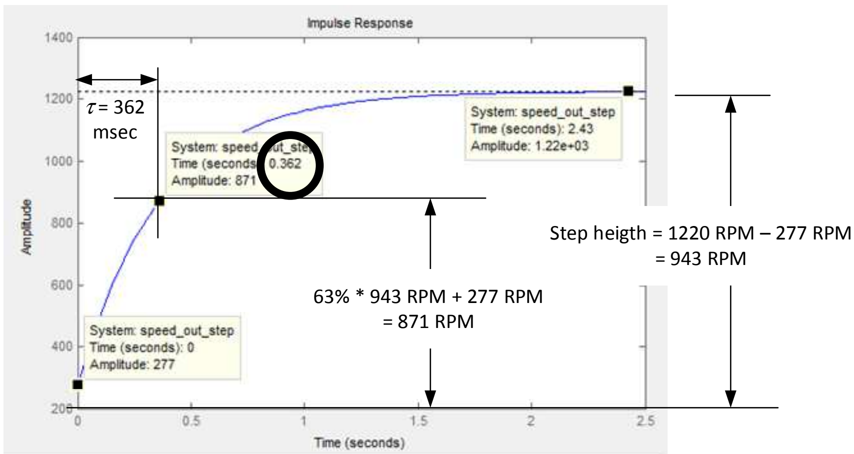


Figure 3-37 Motor time constant measurement

The test begins by applying a voltage to the motor that spins it at about 1/3 of its full speed. Once the motor is spinning *steadily*, trigger the data acquisition system or oscilloscope to begin grabbing speed data, and step the motor voltage to about 2/3 of full speed. Continue catching speed data until the motor settles at its new level. This process may take several iterations to capture a full set of data, starting just before the voltage step and ending a little after the change is complete.

Print the data. A table is also useful.

- Find the step height, 1220 RPM – 277 RPM = 943 RPM
- The time constant is the time it takes to change by 63% of the maximum change. Calculate the change in one time constant. 63%*943 RPM = 871 RPM
- The time from when the step occurred until when the speed changes by 871 RPM = τ = 362 msec

$$\omega' + \frac{1}{0.362\ \text{sec}}\omega = 568\,\frac{\text{RPM}}{\text{v sec}}\,v$$

Speed is in RPM. Time is in seconds

Example 3-8

a. Setup and solve the differential equation for the motor whose parameters were developed in Figure 3-36 and 3-37. Apply the same initial conditions and voltage step as shown in Figure 3-37.

b. Plot the result and compare these calculations to the experimental data of Figure 3-37.

Solution

From Figure 3-37, the initial speed is $\omega(0) = 277$ RPM

The other end point of the step is at 1200 RPM. From Figure 3-36, the trendline linear approximation starts at 0 RPM, 1 V and reaches 1200 RPM at 7 V. So the *step* height is

$$v = 7\,\mathrm{V} - 1\,\mathrm{V} = 6\,\mathrm{V}$$

$$\omega' + \frac{1}{0.362\ \mathrm{sec}}\omega = 568\,\frac{\mathrm{RPM}}{\mathrm{V\ sec}}\,v$$

$$\omega' + \frac{1}{0.362}\omega = 568\ \times 6$$

Figure 3-38 gives the solution of the differential equation. This looks awkward, but perhaps a little familiar. A little algebra is needed.

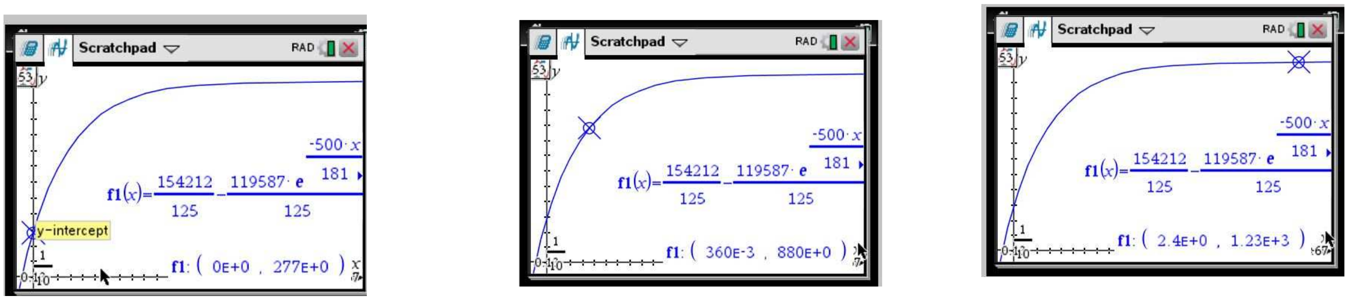
$$\omega = 1234 - 957e^{-\frac{t}{0.362}}$$

$$\omega = 277 + 957\left(1 - e^{-\frac{t}{0.362}}\right)$$

Figure 3-38 Motor differential equation solution

At $t = 0$ the second term falls out and the calculated speed is the measure initial speed.

The second term shows that the speed grows exponentially with a time constant of 0.362 seconds, exactly the results shown by the experimental step in Figure 3-37.



(a) Initial speed
$\omega(0) = 277$ RPM

(b) Time constant
$\tau \sim 0.360$ sec

(c) Final speed
$\omega(2.4 \text{ sec}) = 1230$ RPM

Figure 3-39 Calculated and plotted differential solution of the motor's response to a step

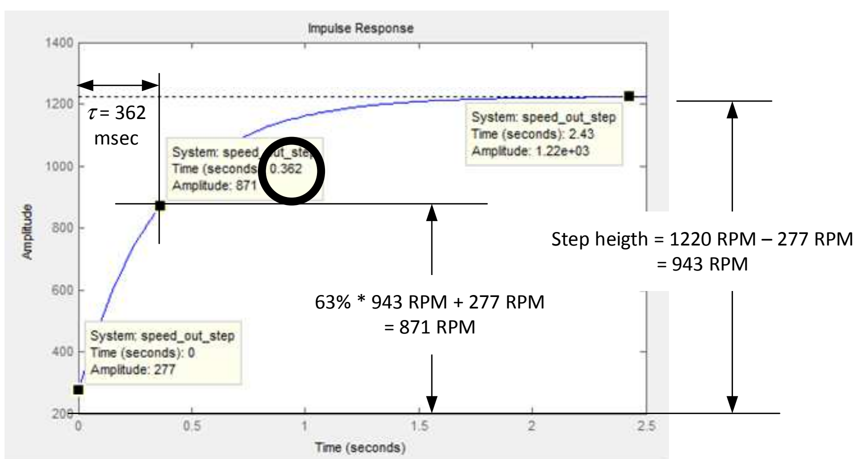


Figure 3-37 (again) Measured motor performance

The shape and the three parameters calculated all match well with the experimental data.

Position

Placing an element at a precise position is central to many manufacturing and nearly all robotics systems. Using the DC motor, with

$$\omega' + \frac{1}{\tau}\omega = \frac{m}{\tau}v$$

the relationship between shaft speed and its location is just

$$\omega = \frac{d\theta}{dt} = \theta'$$

where θ is the angular position.

$$\theta(t) = \int_0^t \omega \, dt + \theta_{t=0}$$

This looks straight forward, but remember, ω has been in RPM, i.e. revolutions per minute. Traditionally, time in an integration is in seconds and angle is in radians, though often degrees are preferred. The proper conversions must be included.

Example 3-9

A 1:100 reduction gear is added to the shaft of the motor in Example 3-8. Plot the angular position of the gear's output in response to that step. Assume that at t = 0 the shaft is passing through the origin 0º even though it is moving at 277 RPM.

Scale the vertical axis in degrees and determine

$$\theta_{t=1\,sec}$$

Solution

From Example 3-8 the speed of the motor's shaft is

$$\omega_{\text{motor}} = 277\frac{\text{rev}}{\text{min}} + 957\frac{\text{rev}}{\text{min}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

The gear reduces these speeds by 1/100.

$$\omega_{\text{gear}} = 277\frac{\text{rev}}{\text{min}} \times \frac{1}{100} + 957\frac{\text{rev}}{\text{min}} \times \frac{1}{100}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

$$\omega_{\text{gear}} = 2.77\frac{\text{rev}}{\text{min}} + 9.57\frac{\text{rev}}{\text{min}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

The minutes now must be converted into seconds.

$$\omega_{\text{gear}} = 2.77\frac{\text{rev}}{\text{min}} \times \frac{1\text{ min}}{60\text{ sec}} + 9.57\frac{\text{rev}}{\text{min}} \times \frac{1\text{ min}}{60\text{ sec}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

$$\omega_{\text{gear}} = 0.0462\frac{\text{rev}}{\text{sec}} + 0.160\frac{\text{rev}}{\text{sec}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

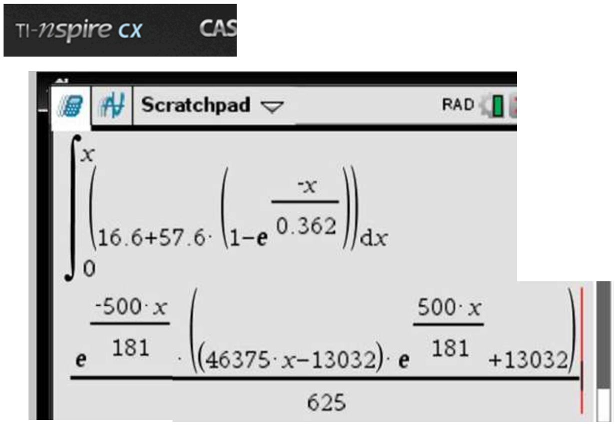
Now convert the revolutions to degrees.

$$\omega_{\text{gear}} = 0.0462\frac{\text{rev}}{\text{sec}} \times \frac{360\text{ deg}}{1\text{ rev}} + 0.160\frac{\text{rev}}{\text{sec}} \times \frac{360\text{ deg}}{1\text{ rev}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

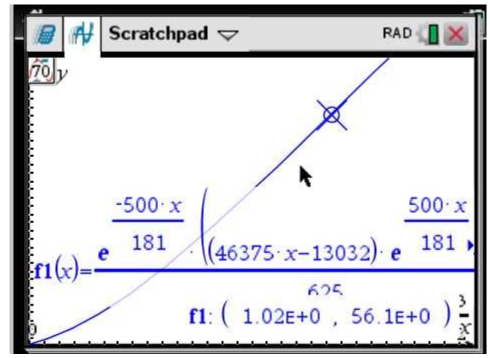
$$\omega_{\text{gear}} = 16.6\frac{\text{deg}}{\text{sec}} + 57.6\frac{\text{deg}}{\text{sec}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

$$\theta(t) = \int_0^t \omega\, dt + \theta_{t=0}$$

The integral solution is shown in Figure 3-40 a, and the plot with a marker at 1 second in Figure 3-40 b.



(a) Integral solution



(b) Graph with trace After 1 second the gear has moved 56 degrees

Figure 3-40 Example 3-9 position solution and plot

Since the problem asked for a plot and the angle at 1 second, the complicated form of the answer provided by the calculator's integration was just copied onto the Graph and then scaled. Initially, the gear's output moves nonlinearly, but eventually smoothes out to a steady rise. This makes physical sense. Eventually the shaft is turning at a new steady rate, sending the gear further and further along, at a steady rate. But, for the first several seconds, the shaft speed is rising exponentially. So position goes up, but slowly to start with and then faster and faster.

Simplifying the integral solution provides a position equation with a dominant first term that increases linearly with time, and a second term that goes to a constant.

$$\theta = 74.2\frac{\text{deg}}{\text{sec}}t - 20.9\text{ deg}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

This is just like

$$y = \text{m}x + \text{b}$$

Although this constant is subtracted from the first term, it is a steady *offset* (b), caused by the shaft starting slowly. It does not alter the slope (m), i.e. how quickly the position increases.