4.2 First Order Circuits in the Laplace Domain

First Order *RC* Circuit Analysis

The *transfer function* fully defines a linear circuit in terms of its Laplace domain components. With this simple equation, the circuit's response to *any* input can be calculated. For a circuit that is driven by a voltage source and outputs a voltage,

$$tranfer\ function = \frac{V_{\text{out}}(s)}{E_{\text{in}}(s)}$$

The circuit in Figure 4-3 is a simple RC circuit, the same as Figure 3-16. Instead of time domain component values, the Laplace domain impedances have been entered.

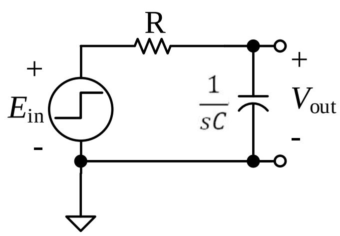

Figure 4-3 RC circuit in the Laplace domain

The transfer function comes directly from the voltage divider law.

$$\frac{V_{\text{out}}(s)}{E_{\text{in}}(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}}$$

This is a little awkward. So, multiply the numerator and denominator of the right side by *sC*.

$$\frac{V_{\text{out}}(s)}{E_{\text{in}}(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} \times \frac{sC}{sC} = \frac{1}{RCs + 1}$$

$$i = \frac{E_{\text{in}}}{\text{R}}\,\text{e}^{-\frac{t}{RC}}$$

$$v_R = i \times R = \left(\frac{E_{\text{in}}}{\text{R}}\,\text{e}^{-\frac{t}{RC}}\right) \times R$$

$$v_{\text{R}} = E_{\text{in}}\,\text{e}^{-\frac{t}{RC}}$$

$$v_{\text{C}} = \frac{1}{C}\int_0^t i\,dt + V_o$$

$$v_{\text{C}} = \frac{1}{\text{C}}\int_0^t \left(\frac{E_{\text{in}}}{\text{R}}\,\text{e}^{-\frac{t}{RC}}\right)\,dt$$

$$v_C = \frac{E_{\text{in}}}{\text{RC}} \times \text{RC}\left(1 - e^{\frac{-t}{RC}}\right)$$

$$v_C = E_{\text{in}}\left(1 - e^{\frac{-t}{RC}}\right)$$

From Chapter 3 $\qquad \tau = \text{RC}$

$$\frac{V_{\text{out}}(s)}{E_{\text{in}}(s)} = \frac{1}{\tau s + 1}$$

This *defines* the circuit.

To find the output in response to a given input

$$V_{\text{out}}(s) = \frac{E_{\text{in}}(s)}{\tau s + 1}$$

If the input is a step, V volts tall, substitute the Laplace transform for that step. Look at row 2 from Table 4-1

$$\mathcal{L}\{step\} = \frac{V}{s}$$

$$V_{\text{out}}(s) = \frac{\frac{V}{s}}{\tau s + 1} = \frac{V}{s(\tau s + 1)}$$

That is the output in the Laplace domain. To find the time domain equation, again look at Table 4-1. Row 17a gives

$$v_{\text{out}}(t) = \mathcal{L}^{-1}\left\{\frac{A}{s(\tau s + 1)}\right\} = A\left(1 - e^{-\frac{t}{\tau}}\right) = V\left(1 - e^{-\frac{t}{\tau}}\right)$$

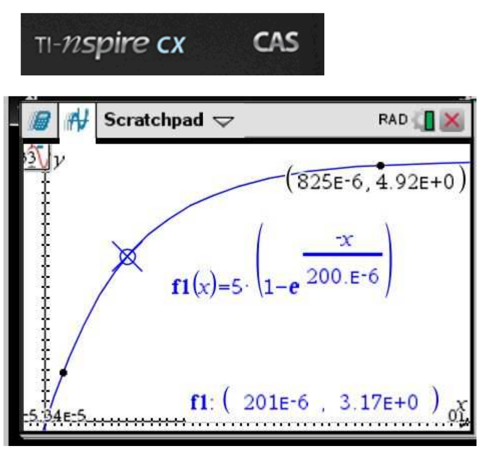

Figure 4-4 Capacitor output voltage

This is exactly the same result from the differential equation solution in Chapter 3. It required four steps in the Laplace domain. Solving the differential equations took a very powerful calculator and several *pages* of work. The *last* set of equations, *after* the differential equation has already been solved for $i(t)$, are shown above to the left.

Plotting this output waveform on the calculator, then, just requires values for the size of the input step, A, and the values of the resistor and the capacitor.

A = 5 V $\qquad$ R = 10 kΩ $\qquad$ C = 20 nF $\qquad$ τ = RC = 200 µsec

The capacitor voltage rises exponentially, settling at 5 V and rises 63% * 5 V = 3.16 V in one time constant.

Multisim has a Laplace block. It is located in the **Sources Group/Control_Functions Family**, and is called an **Arbitrary_Laplace_Function**. Figure 4-5 shows the simulation.

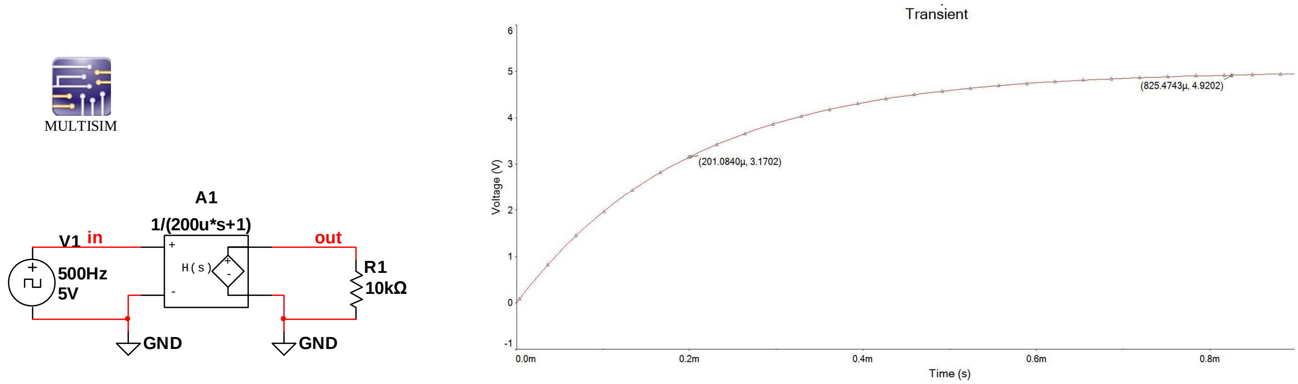

Figure 4-5 Multisim simulation using an Arbitrary_Laplace_Function and Transient Analysis

First Order *RL* Circuit Analysis

The Laplace domain circuit for a first order RL circuit is shown in Figure 4-6. It differs from the circuit in Chapter 3 *only* by replacing the time domain value of the R and L with their Laplace domain values *R* and *sL*

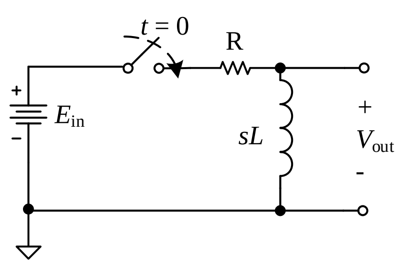

Figure 4-6 Laplace domain schematic of RL circuit

Using the Voltage Divider Law, the transfer function is

$$\frac{V_{\text{out}}(s)}{E_{\text{in}}(s)} = \frac{sL}{R + sL}$$

Dividing the numerator and denominator by *R* gives a more familiar form.

$$\frac{V_{\text{out}}(s)}{E_{\text{in}}(s)} = \frac{\frac{L}{R}s}{1 + s\frac{L}{R}}$$

Remember, from Chapter 3, for a RL circuit

$$\tau = \frac{L}{R}$$

$$\frac{V_{\text{out}}(s)}{E_{\text{in}}(s)} = \frac{\tau s}{\tau s + 1}$$

The denominator looks just like the RC transfer function. But, the numerator is different.

To find the output in response to a given input

$$V_{\text{out}}(s) = \frac{E_{\text{in}} \times \tau s}{\tau s + 1}$$

If the input is a step, *V* volts tall, substitute the Laplace transform for that step. Look at row 2 from Table 4-1

$$\mathcal{L}\{step\} = \frac{V}{s}$$

$$V_{\text{out}}(s) = \frac{\frac{V}{s} \times \tau s}{\tau s + 1} = \frac{V\tau}{(\tau s + 1)}$$

That is the output, in the Laplace domain. To find the time domain equation, again look at Table 4-1. Row 13a gives

$$v_{\text{out}}(t) = \mathcal{L}^{-1}\left\{\frac{A}{(\tau s + 1)}\right\} = \frac{A}{\tau}e^{-\frac{t}{\tau}}$$

Comparing the calculated function with the one provided by Table 4-1 shows that

$$V\tau = A$$

Substituting that into the v_{out} equation gives

$$v_{\text{out}}(t) = \frac{A}{\tau} e^{-\frac{t}{\tau}} = \frac{V\tau}{\tau} e^{-\frac{t}{\tau}}$$

$$v_{\text{out}}(t) = V e^{-\frac{t}{\tau}}$$

As with the RC circuit, working in the Laplace domain produces the same results in far fewer, simpler steps.

Matlab provides several powerful Laplace domain analysis and display tools. Look at Figure 4-7.

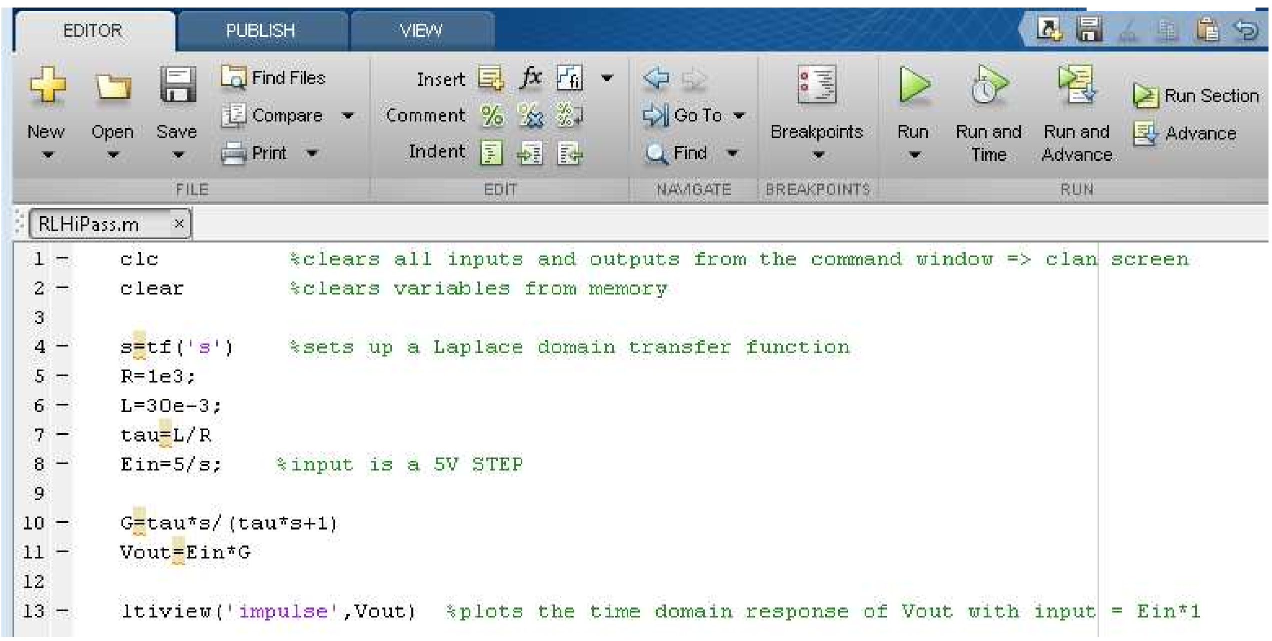

Figure 4-7 Matlab input file to plot the response of the voltage across a 30 mH inductor in series with a 1 kΩ resistor, with a 5 V step input.

This is a STEP response since `Ein=5/s`

The `Impuse Response` title is misleading.

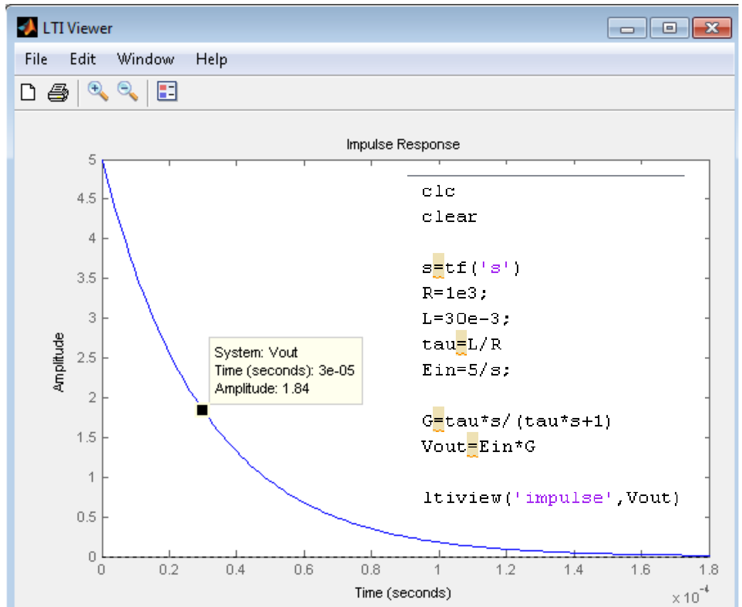

Figure 4-8 Matlab LTIView display of the voltage across a 30 mH inductor in series with a 1 kΩ resistor, with a 5 V step input.

Figure 4-8 shows the response as a 5V tall exponential drop with a time constant of 30 μsec = L/R = 30 mH/1 kΩ. In 1 τ, the V_{out} drops

$$\Delta V_{out} = 63.2\% * 5\text{ V} = 3.16\text{ V}$$

Starting at 5 V,

$$V_{out\,@\,1\,\tau} = 5\text{ V} - 3.16\text{ V} = 1.84\text{ V}$$

which is the value shown by the cursor in Figure 4-8.

The **`ltiview('impulse',Vout)`** statement causes the plot, automatically scaling it vertically and horizontally. These axis settings can be changed if you wish. Clicking on the line inserts the cursor. Additional cursors are placed by clicking and dragging on the line multiple times. That is far easier than plotting with the calculator.

The **`'impulse'`** part of the statement causes **`Vout`** to be multipled by 1, which is what you want. Omitting **`'impulse'`** would cause **`Vout`** to be multiplied by 1/s. But that has already been taken care of by **`Ein=5/s;`**

Section 3.5 explained how to handle initial conditions. With the TI Nspire deSolve, it was simply a matter of including them as the equation is entered. Line 10 in Table 4-1 handles the initial conditions when doing a Laplace transform.

$$sF(s) - f(0) \Leftrightarrow \frac{df(t)}{dt}$$

Be careful to notice that even when in the Laplace domain $\{F(s)\}$, the time domain value of the initial condition is used, *not* its Laplace transform.

The Laplace impedances of Table 4-2 do *not* include initial conditions. So, when they are present, go back to the time domain differential equation, section 3-5. For the RL circuit in Figure 3-24 the derivation of the circuit's differential equation is the same as originally done.

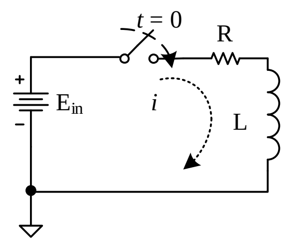

Figure 3-24 RL circuit with a step input

$$e_{\text{in}} = v_{\text{R}} + v_L$$

$$e_{\text{in}} = i \times \text{R} + \text{L}\frac{di}{dt}$$

$$e_{\text{in}} = i\text{R} + \text{L}i'$$

Now complete the Laplace transform of each term.

$$E_{\text{in}} = I\text{R} + \text{L}[(sI - i(0)]$$

Now a little algebra to get I alone on the left.

$$E_{\text{in}} = I\text{R} + \text{sL}I - \text{L} \times i(0)$$

$$\frac{E_{\text{in}}}{R} = I + \text{s}\frac{\text{L}}{\text{R}}I - \frac{\text{L}}{\text{R}} \times i(0)$$

$$\tau = \frac{\text{L}}{\text{R}}$$

$$\frac{E_{\text{in}}}{R} = I + \text{s}\tau I - \tau i(0)$$

$$I(\tau s + 1) = \frac{E_{\text{in}}}{R} + \tau i(0)$$

$$I = \frac{\frac{E_{\text{in}}}{R}}{(\tau s + 1)} + \frac{\tau i(0)}{(\tau s + 1)}$$

When E_{in} is a step $\quad E_{in} = \frac{V}{s}$

$$I = \frac{\frac{V}{R}}{s(\tau s + 1)} + \frac{\tau i(0)}{(\tau s + 1)}$$

17a. $\quad \dfrac{A}{s(\tau s + 1)} \quad A(1 - e^{-t/\tau})$

13a. $\quad \dfrac{A}{\tau s + 1} \quad \dfrac{A}{\tau} e^{-t/\tau}$

The first term can be returned to the time domain with Table 4-1, line 17a when $A = \frac{V}{R}$. The second term is transformed with line 13a, where $A = \tau i(0)$

$$i = \frac{V}{R}\left(1 - e^{-\frac{t}{\tau}}\right) + i(0)e^{-\frac{t}{\tau}}$$

This gives the same results as the calculator solution (and a little algebra from section 3.5, Figure 3-27.

TI-nspire CX CAS

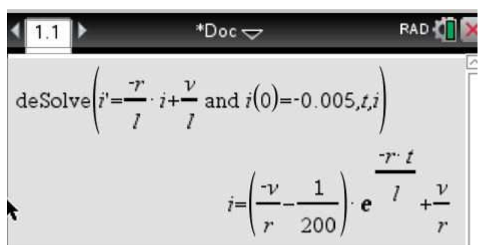

Figure 3-27 Calculator solution of RL circuit with initial current

$$i = -\frac{V}{R}e^{-\frac{t}{L/R}} - 5\text{ mA}\, e^{-\frac{t}{L/R}} + \frac{V}{R}$$

$$i = \frac{V}{R}\left(1 - e^{-\frac{t}{L/R}}\right) - 5\text{ mA}\, e^{-\frac{t}{L/R}}$$

.