4.4 Series RLC Second Order Circuit

Transfer Function

An inductor introduces a derivative, and an *s*, into a circuit's transfer function. That is first order, one energy storage element. A capacitor introduces an integral, and another *s*, into the transfer function. That makes the resulting equation a function of s^2, *second order*. Both the circuit behavior and the math get much more interesting as the two parts interchange energy, alternately charging and discharging each other as well as sending energy to the load resistor. The schematic is shown in Figure 4-12.

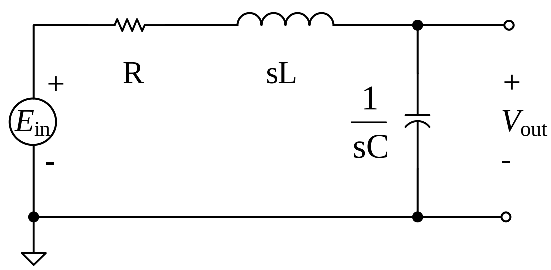


Figure 4-12 Series RLC Laplace schematic

Applying the voltage divider law,

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}}$$

Basic Voltage Divider Law

$$v_{Z3} = \frac{Z3}{Z1 + Z2 + Z3} \times e_{in}$$

Traditionally, one form of a second order transfer function is

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{A\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where *A*, ξ and ω_n have very specific effects on the circuit's behavior. The next step is to put the function of R, L, and C into this standard form.

Begin by clearing the 1/sC from the denominator. Multiply numerator and denominator by sC.

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} \times \frac{sC}{sC}$$

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{1}{RCs + s^2CL + 1}$$

Now, make the coefficient of s^2 unity. Divide numerator and denominator by LC

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{1}{RCs + s^2CL + 1} \times \frac{\frac{1}{LC}}{\frac{1}{LC}}$$

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{\frac{1}{LC}}{\frac{R}{L}s + s^2 + \frac{1}{LC}}$$

Then rearrange the denominator to align with the standard form

***V_{cap}* - RLC transfer function**

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Finally, equate coefficients of the RLC transfer function with those of the standard form.

Standard second order form

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{A\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

ω_n is called the natural or *resonant frequency*.

$$\omega_n^2 = \frac{1}{LC}$$

Resonant frequency

$$\omega_n = \frac{1}{\sqrt{LC}}$$

ξ is the damping. Equating coefficients of *s* gives

$$2\xi\omega_n = \frac{R}{L}$$

$$\xi = \frac{R}{2L\omega_n}$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

$$\xi = \frac{R}{2L\frac{1}{\sqrt{LC}}}$$

$$\xi = \frac{R\sqrt{LC}}{2L}$$

$$\xi = \frac{R\sqrt{LC}}{2\sqrt{L^2}}$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Damping

A is the gain. Equating the numerators gives

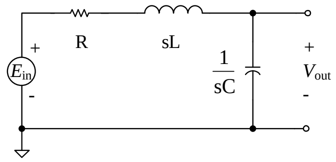
$$A\omega_n^2 = \frac{1}{LC}$$

And, $\omega_n^2 = \frac{1}{LC}$

This leaves $A = 1$

Gain

So, the gain is fixed at 1, the resonant frequency can be set by *L* and or *C*. Then the damping can be adjusted with *R*, without changing ω_n.



$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{A\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Roots

How does this circuit behave? The Laplace domain approach is to specify an input, then look in Table 4-1 for the transform. For a unit step input

$$E_{\text{in}} = \frac{1\,V}{s}$$

giving

$$V_{\text{out}} = \frac{\frac{1}{LC}}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

In Table 4-1 there are three distinct transforms, depending on how the denominator is *factored*.

20b. $\dfrac{A}{s(s+a)(s+b)} \quad \rightarrow \quad \dfrac{A}{ab}\left(1 + \dfrac{ae^{-bt} - be^{-at}}{b-a}\right)$

21b. $V_{\text{out}} = \dfrac{A}{s(s+a)^2} \quad \rightarrow \quad \dfrac{A}{a^2}\ [1 - (1+at)e^{-at}]$

22. $\dfrac{A\omega_n}{s(s^2+2\xi\omega_n s+\omega_n^2)} \rightarrow A\left[1 + \dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} sin\left(\omega_n\sqrt{1-\xi^2}t - \psi\right)\right]$ where $\psi = tan^{-1}\left(\dfrac{\sqrt{1-\xi^2}}{-\xi}\right)$ $(0 < \psi < \pi)$

Over **damped**

If the denominator can be factored into two different factors (called *roots)* a step input produces an output that contains *two* different exponential decays. Eventually, when they both disappear, the output is a steady level that depends on the *gain A*, and the two roots, *a* and *b*.

Critically **damped**

If the denominator can be factored into a *perfect* square, then the output is a simpler exponential decay. The final level also depends on the *gain* and the *root*, *a*.

Under **damped**

If the denominator cannot be factored into *real* roots, the output contains an exponentially decaying sine wave, oscillating up and down before finally settling to a level set by the *gain*.

Factoring the second order transfer function's denominator is central to determining the circuit's performance. That is done with the quadratic equation.

$$s^2 + 2\xi\omega_n s + {\omega_n}^2 = 0$$

$$Ax^2 + Bx + C = 0$$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

where

$$A = 1 \qquad B = 2\xi\omega_n \quad C = {\omega_n}^2$$

Damping

Overdamped

For the quadratic equation to produce two different, real solutions (roots) then the square root must be positive.

$$B^2 - 4AC > 0$$

$$B^2 > 4AC$$

$$B \quad > \sqrt{4AC}$$

20b. $\dfrac{A}{s(s+a)(s+b)}$

$$\rightarrow \frac{A}{ab}\left(1 + \frac{ae^{-bt} - be^{-at}}{b-a}\right)$$

In terms of circuit parameters this becomes

$$2\xi\omega_n > \sqrt{4{\omega_n}^2}$$

$$2\xi\omega_n > 2\omega_n$$

$$\xi > 1$$

In terms of circuit components

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}} > 1$$

Multiplying both sides by the same constant

$$\frac{R}{2}\sqrt{\frac{C}{L}} \times 2\sqrt{\frac{L}{C}} > 2\sqrt{\frac{L}{C}}$$

$$R > 2\sqrt{\frac{L}{C}}$$

Critically damped

For the quadratic equation to produce two equal, solutions (roots) then the square root must be zero.

$$B^2 - 4AC = 0$$

$$B^2 = 4AC$$

$$B = \sqrt{4AC}$$

In terms of circuit parameters this becomes

$$2\xi\omega_n = \sqrt{4\omega_n{}^2}$$

$$2\xi\omega_n = 2\omega_n$$

$$\xi = 1$$

In terms of circuit components

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}} = 1$$

Multiplying both sides by the same constant

$$\frac{R}{2}\sqrt{\frac{C}{L}} \times 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{L}{C}}$$

$$R = 2\sqrt{\frac{L}{C}}$$

21b. $\frac{A}{s(s+a)^2}$

$\frac{A}{a^2}\ [1 - (1 + at)e^{-at}]$

Under damped

The final option is that the quadratic equation produces complex roots. That is, the square root is taken of a negative value, which can only be *imagined*.

$$B^2 - 4AC < 0$$

$$B^2 < 4AC$$

$$B \quad < \sqrt{4AC}$$

In terms of circuit parameters this becomes

$$2\xi\omega_{\mathrm{n}} < \sqrt{4{\omega_n}^2}$$

$$2\xi\omega_{\mathrm{n}} < 2\omega_{\mathrm{n}}$$

$$\xi < 1$$

In terms of circuit components

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}} < 1$$

Multiplying both sides by the same constant

$$\frac{R}{2}\sqrt{\frac{C}{L}} \times 2\sqrt{\frac{L}{C}} < 2\sqrt{\frac{L}{C}}$$

$$R < 2\sqrt{\frac{L}{C}}$$

In summary, the larger ξ is, the more heavily damped and sluggish the response. As ξ falls below 1, the system becomes underdamped, and tends to oscillate. Eventually, as ξ approaches 0, the system tries to break into oscillations.

For the series RLC circuit, damping is directly proportional to resistance. The lower the resistance, the more responsive, and oscillatory, the system.

22. $\frac{A\omega_n}{s(s^2+2\xi\omega_n s+\omega_n^2)}$

$$A\left[1 + \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} sin\left(\omega_n\sqrt{1-\xi^2}t - \psi\right)\right]$$

$$\psi = tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{-\xi}\right) \quad (0 < \psi < \pi)$$

Example 4-3

Find the response of the series RLC circuit in Figure 4-12, with:

R = 3.3 kΩ
L = 33 mH
C = 47 nF

a. Calculate the three parameters, A , ω_n and ξ.

b. Given a 1 V input step, determine the time domain response.

c. Plot that response.

Solutions

a.

$$A = 1$$

$$V_{\text{out}} = \frac{\frac{1}{LC}}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{33\text{ mH} \times 47\text{ nF}}}$$

$$\omega_n = 25.39\,k\,\frac{\text{radians}}{\text{sec}}$$

$$V_{\text{out}} = \frac{\frac{1}{33\text{ mH} \times 47\text{ nF}}}{s\left(s^2 + \frac{3.3\,k}{33\,mH}s + \frac{1}{33\text{ mH} \times 47\text{ nF}}\right)}$$

$$\omega_n = 2\pi f_n$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{25.39\,\frac{\text{radians}}{\text{sec}}}{2\pi} = 4.04\text{ kHz}$$

$$V_{\text{out}} = \frac{644{,}7\text{ M}}{s(s^2 + 100\text{ k}\,s + 644.7\text{ M})}$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{3.3\text{ k}\Omega}{2}\sqrt{\frac{47\text{ nF}}{33\text{ mH}}}$$

$$\xi = 1.97$$

The circuit it *overdamped.*

b. The transform into the time domain requires that the Laplace domain equation be factored, to get a and b.

$$\frac{A}{s(s+a)(s+b)} \quad \rightarrow \quad \frac{A}{ab}\left(1 + \frac{ae^{-bt} - be^{-at}}{b-a}\right)$$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

where

$$\mathrm{A} = 1 \qquad \mathrm{B} = 2\xi\omega_{\mathrm{n}} \qquad C = {\omega_n}^2$$

$$\mathrm{B} = 2\xi\omega_{\mathrm{n}} = 2 \times 1.97 \times 25.39\ \mathrm{k} = 100\ \mathrm{k}$$

$$C = {\omega_n}^2 = (25.39\ k)^2 = 644.7\ \mathrm{M}$$

$$x = \frac{-100k \pm \sqrt{(100\mathrm{k})^2 - 4 \times 1 \times 644.7\ \mathrm{M}}}{2 \times 1}$$

$$x = \frac{-100k \pm \sqrt{(100\mathrm{k})^2 - 4 \times 1 \times 644.7\ \mathrm{M}}}{2 \times 1}$$

$$x = -50\ \mathrm{k} \pm 43\ \mathrm{k}$$

x = -7 k x = -93 k

The denominator factors into

$$(s + 7k)(s + 93k)$$

a = 7 k b = 93 k

For the 1 V step input

$$V_{\mathrm{out}} = \frac{644.7\ \mathrm{M}}{s(s^2 + 100\ \mathrm{k}\ s + 644.7\ \mathrm{M})}$$

$$\frac{644{,}7\ \mathrm{M}}{s(s^2 + 100\ \mathrm{k}\ s + 644.7\ \mathrm{M})} = \frac{A}{s(s + 7k)(s + 93k)} \rightarrow \frac{644.7\mathrm{M}}{7\mathrm{k} \times 93\mathrm{k}}\left(1 + \frac{7\mathrm{k}e^{-93kt} - 93\mathrm{k}e^{-7kt}}{93\mathrm{k} - 7\mathrm{k}}\right)$$

$$v_{out} = 0.99\left(1 + 0.081e^{-\frac{t}{11\mu sec}} - 1.08e^{-\frac{t}{143\ \mu sec}}\right)$$

The output steps up nearly all of the input step, but does it exponentially. Although there are two exponentials, the 143 μsec time constant will dominates.

TI-nspire CX CAS

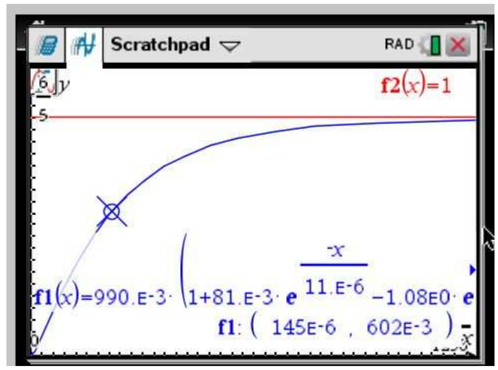


Figure 4-13 Overdamped response 143 μsec dominant time constant

Example 4-4

Find the response of the series RLC circuit in Figure 4-12, with:

R = 1.68 kΩ   L = 33 mH   C = 47 nF

This is the same as Example 4-3, with the resistor, and damping, reduced.

- **a.** Calculate the three parameters, A , ω_n and ξ.
- **b.** Given a 1 V input step, determine the time domain response.
- **c.** Plot that response.

Solutions

a. $A = 1$

$$V_{\text{out}} = \frac{\frac{1}{LC}}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{33\text{ mH} \times 47\text{ nF}}}$$

$$V_{\text{out}} = \frac{\frac{1}{33\text{ mH} \times 47\text{ nF}}}{s\left(s^2 + \frac{1.68\,k}{33\,mH}s + \frac{1}{33\text{ mH} \times 47\text{ nF}}\right)}$$

$$\omega_n = 25.39\,k\,\frac{\text{radians}}{\text{sec}}$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{1.68\text{ k}\Omega}{2}\sqrt{\frac{47\text{ nF}}{33\text{ mH}}}$$

$$V_{\text{out}} = \frac{644{,}7\text{ M}}{s(s^2 + 50.78\text{ k}\,s + 644.7\text{ M})}$$

$\xi = 1.00$ The circuit it *critically* damped.

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

where

$$\mathrm{A} = 1 \qquad \mathrm{B} = 2\xi\omega_\mathrm{n} \qquad C = \omega_n{}^2$$

$$\mathrm{B} = 2\xi\omega_\mathrm{n} = 2 \times 1.00 \times 25.39\ \mathrm{k} = 50.78\ \mathrm{k}$$

$$C = \omega_n{}^2 = (25.39\ k)^2 = 644.7\ \mathrm{M}$$

$$x = \frac{-50.78\ \mathrm{k} \pm \sqrt{(50.78\ \mathrm{k})^2 - 4 \times 1 \times 644.7\ \mathrm{M}}}{2 \times 1}$$

$$x = -25.4\ \mathrm{k} \pm 0\ \mathrm{k}$$

The denominator factors into two equal roots, the characteristic of a critically damped circuit.

$$(s + 25.4\ \mathrm{k})(s + 25.4\ \mathrm{k})$$

For the 1 V step input

$$V_\mathrm{out} = \frac{644.7\ \mathrm{M}}{s(s^2 + 50.8\ \mathrm{k}\ s + 644.7\ \mathrm{M})} = \frac{644.7\ \mathrm{M}}{s(s + 25.4\ \mathrm{k})^2}$$

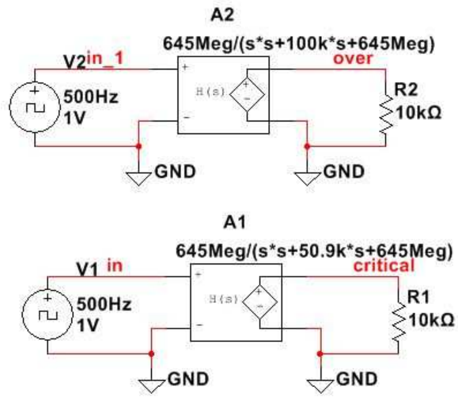
$$V_\mathrm{out} = \frac{A}{s(s+a)^2} \quad \rightarrow \quad \frac{A}{a^2}\ \left[1 - (1 + at)e^{-at}\right]$$

$$v_{out} = \frac{644\ \mathrm{M}}{(25.4\ \mathrm{k})^2}\ \left[1 - (1 + 25.4\ \mathrm{k}t)e^{-25.4\ kt}\right]$$

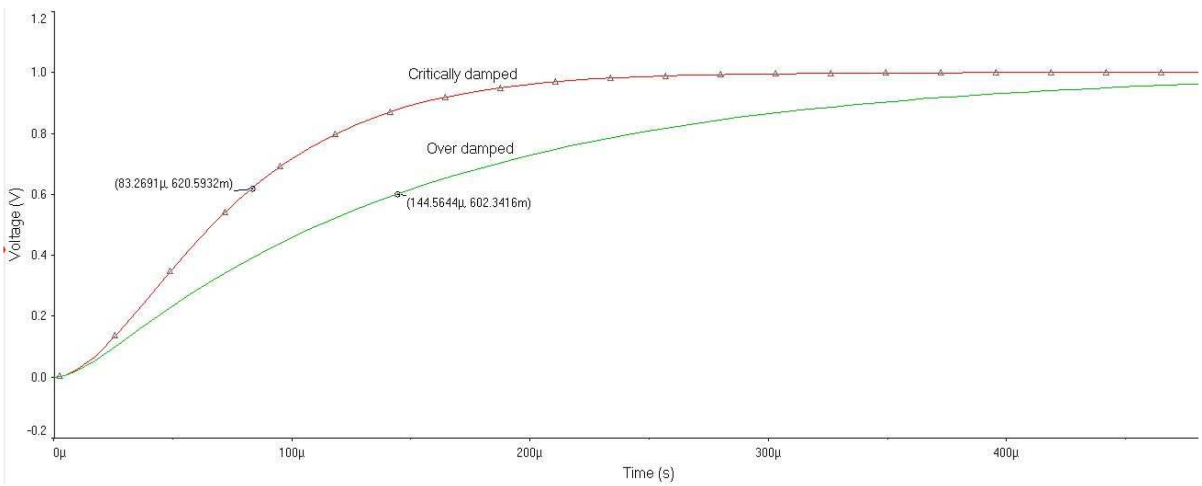
$$v_{out} = 0.99\ \left[1 - (1 + 25.4\ \mathrm{k}\ t)e^{-\frac{t}{39\ \mu sec}}\right]$$

Changing the resistor to make the circuit critically damped produces the same size step, with an exponential response. But the time constant of the exponential is significantly smaller. The circuit will settle much more quickly.

MULTISIM



(a) Multisim Laplace block implementation



(b) Transient responses

Figure 4-14 Multisim implementation of Example 4-4

In Figure 4-14 (a) notice that $645 \times 10^6$is entered as **645 Meg**. Entering **645 M** is interpreted as 645×10^{-3} even though the M is upper case.

Both circuits eventually reach 1, but the critically damped circuit responds much more quickly.

Example 4-5

Find the response of the series RLC circuit in Figure 4-12, with:

R = 820 Ω L = 33 mH C = 47 nF

This is the same as Example 4-3, with the resistor, and damping, reduced.

a. Calculate the three parameters, A , ω_n and ξ.
b. Given a 1 V input step, determine the time domain response.
c. Plot that response.

Solutions

a. $A = 1$

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{33\text{ mH} \times 47\text{ nF}}}$$

$$\omega_n = 25.39\,k\frac{\text{radians}}{\text{sec}}$$

$$\xi = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{820\ \Omega}{2}\sqrt{\frac{47\text{ nF}}{33\text{ mH}}}$$

$\xi = 0.49$ The circuit it *under* damped.

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

where

A = 1 $\mathrm{B} = 2\xi\omega_\mathrm{n}$ $C = {\omega_n}^2$

$$\mathrm{B} = 2\xi\omega_\mathrm{n} = 2 \times 0.49 \times 25.39\text{ k} = 24.88\text{ k}$$

$$C = {\omega_n}^2 = (25.39\,k)^2 = 644.7\text{ M}$$

$$x = \frac{-24.88\text{ k} \pm \sqrt{(24.88\text{ k})^2 - 4 \times 1 \times 644.7\text{ M}}}{2 \times 1}$$

$$x = -25.4\text{ k} \pm i44.27\text{ k}$$

The denominator factors into complex roots. There is no *real* solution. That verifies this is an underdamped circuit.

$$V_\text{out} = \frac{\frac{1}{LC}}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

$$V_\text{out} = \frac{\frac{1}{33\text{ mH} \times 47\text{ nF}}}{s\left(s^2 + \frac{820}{33\,mH}s + \frac{1}{33\text{ mH} \times 47\text{ nF}}\right)}$$

$$V_\text{out} = \frac{644.7\text{ M}}{s(s^2 + 24.8\text{ k}\,s + 644.7\text{ M})}$$

$$V_{\text{out}} = \frac{644.7\text{ M}}{s(s^2 + 24.8\text{ k}\,s + 644.7\text{ M})}$$

$$\frac{A\omega_n}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} \rightarrow A\left[1 + \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} sin\left(\omega_n\sqrt{1-\xi^2}\,t - \psi\right)\right]$$

$$\text{where } \psi = tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{-\xi}\right) \quad (0 < \psi < \pi)$$

$$v_{out} = 1\left[1 + \frac{e^{-0.49\times 25.4\,kt}}{\sqrt{1-0.49^2}} sin\left(25.4\,k\sqrt{1-0.49^2}\,t - \psi\right)\right]$$

$$v_{out} = 1 + 1.147e^{-12.48\,kt} sin(22.1\,kt - \psi)$$

$$v_{out} = 1 + 1.147e^{-\frac{t}{81\,\mu sec}} sin(2\pi \times 3.5\text{ kHz}t - \psi)$$

Changing the resistor to make the circuit under damped produces the same size step, (that's the 1 out front). Before it settles down, the output oscillates (the sine term). Notice that the frequency of the damped oscillations is set by the natural frequency *and* the damping.

$$\omega_{\text{damped oscillation}} = \omega_n\sqrt{1-\xi^2}$$

That sinusoidal dies out exponentially (the $e^{-\xi\omega_n t}$ term). Both the natural frequency and damping set that time constant.

The Matlab ltiview plot of all three circuit responses is shown in Figure 4-15. The under damped circuit climbs more quickly, reaching 63% in 60 µsec, while the critically damped circuit take 84 µsec, and the over damped circuit takes 156 µsec.

But, the under damped circuit overshoots, 17%, then rings back down before finally settling. Even though the under damped circuit rises far more quickly, for these values, the critically damped circuit actually settles while the under damped is still ringing.

The lower the damping, the quicker the rise. But an under damped circuit may take longer to settle than one that is critically damped, depending on how "settle" is defined.

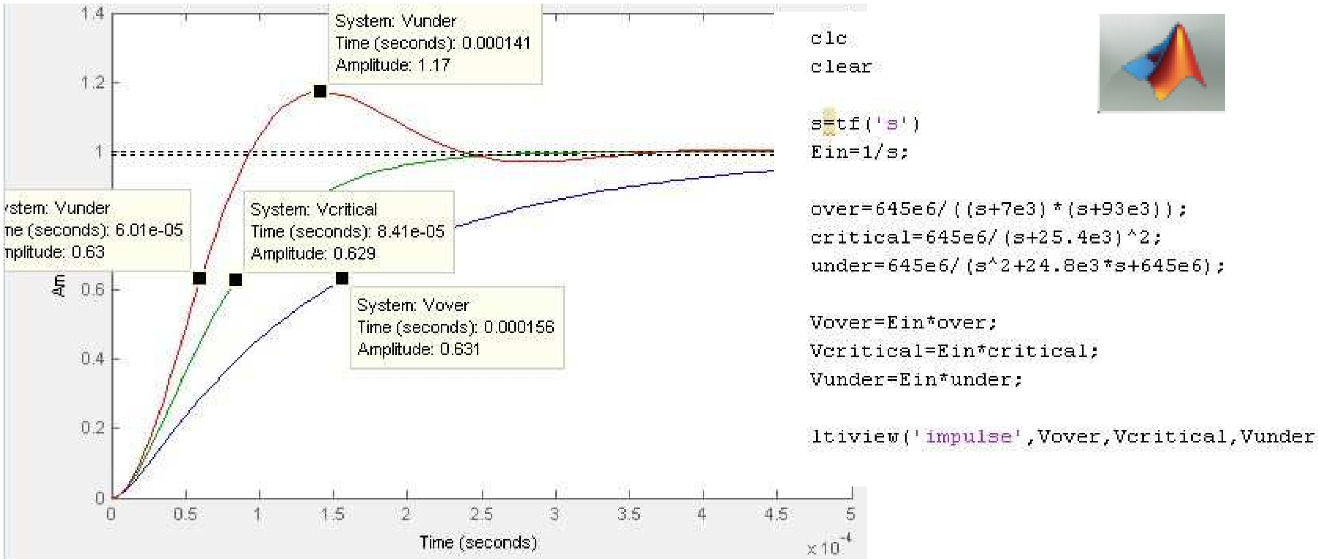


Figure 4-15 Matlab and ltiview step response of all three dampings

4.5 The LC Second Order Low Pass Filter

Two of the major uses of RLC passive circuits are in switching power supplies and in Class D audio amplifiers. In each, transistor switches pass large pulses of current to a resistive load. But, it is the *average* value of the pulse width modulated square wave that is wanted. All of the high frequency edges must be rejected, passing only the relatively slow, low frequency or DC average value.

That is the role of the low pass filter. An entire later chapter is dedicated to the details of building a host of sophisticated, op amp based, active filters. For now, a simple, very effective, *passive* low pass filter can be built with an inductor and a capacitor. It is shown in Figure 4-16.