4.1 Transforms

The Concept

The solution of a differential equation requires that some function be found which, when substituted into the differential equation, differentiated, and/or integrated, multiplied by constants, and added to other derivatives of itself makes the original equation true. In Chapter 3 these operations were all handled by the calculator.

Even simple first order equations produce some rather involved results. Many interesting and useful systems combine multiple capacitors *and* inductors, or motors moving around multiple axes. Each energy storage element adds an order to the differential equation. Sixth order circuits are common. These differential equations very quickly become unmanageable.

The Laplace transform allow you to manipulate and solve differential equations with a table and some algebra. That's all. Look at Figure 4-1

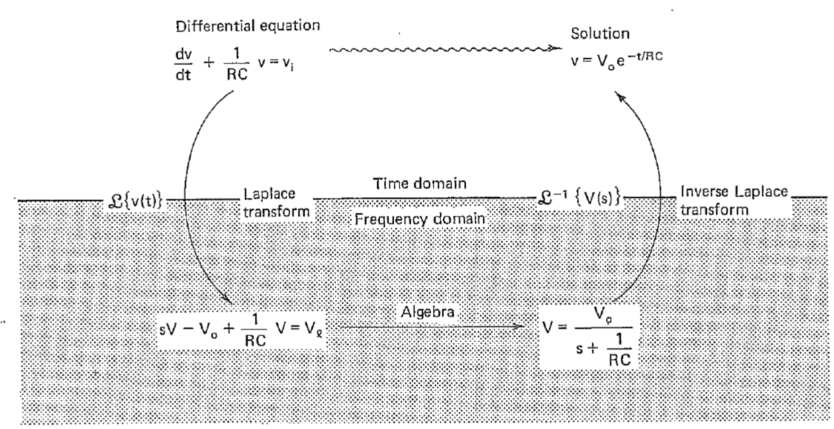

Figure 4-1 Laplace transformation solution of differential equations

Differential equations *might* be solved directly to produce the answer. That is the wavey line. Or, the equation could be transformed into the Laplace domain. All common function transformations are tabulated. So, just look up the transformed equation. The differential equation is in the *time* domain, where the independent variable is *t*. In the Laplace domain, *t* and its derivative and integral are replaced with *s* and 1/*s*. Just solve the equation using algebra. Then use the table again to get back into the time domain.

This might seem like "the long way around". But in the Laplace domain,

$$\frac{dv}{dt} \to s \times V$$

and

$$\int v dt \to \frac{V}{s}$$

Differentiation is replaced by multiplication by *s* and integration is replaced by division by *s*. Algebra replaces differential equations. In the Laplace domain, the math is far simpler.

Calculus

The proof of this concept and the transformations into and out of the Laplace domain are properly the contents of an advanced Calculus course. But, the conversion of a time domain function into a Laplace domain function is a straight forward integral, which the TI-Nspire can handle. Variables in the time domain are shown in lower case, while the same variables in then Laplace domain are upper case.

$$\mathcal{L}\{f(t)\} = F(s)$$

The transformation replaces *t* with *s* by integrating over *t* from 0 to ∞, i.e. integrating *t* out of the equation.

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

Example 4-1
Find the Laplace transform of

$$f(t) = Ae^{-\frac{t}{\tau}}$$

$$\mathcal{L}\left\{Ae^{-\frac{t}{\tau}}\right\} = \int_0^\infty Ae^{-\frac{t}{\tau}} \times e^{-st}dt$$

There are a few precautions to observe when completing this integration on the TI-Nspire.

- The ∞ symbol is on the menu accessed with the π key.
- If either exponent of *e* is positive, then integrating *t* from 0 to ∞ produces an infinite answer. To prevent that problem, conditions have been added after *dt*. The | is found under the book key, and can be interpreted as "for all". Then adding **s>0 and τ>0** assures that neither exponent goes to infinity.

$$\mathcal{L}\left\{Ae^{-\frac{t}{\tau}}\right\} = \frac{A\tau}{\tau s + 1}$$

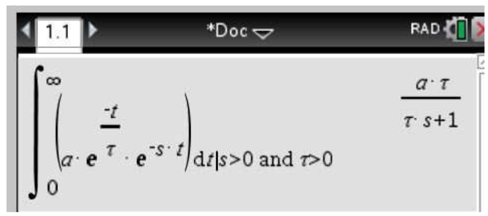

Figure 4-2 Calculator Laplace transformation of $Ae^{-\frac{t}{\tau}}$

Practice
Find

$$\mathcal{L}\{Asin(\omega t)\}$$

Answer

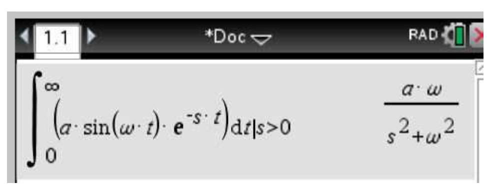

Figure 4-2 Calculator Laplace transformation of $Asin(\omega t)$

Going back from the Laplace domain, F(s), to the time domain, f(t), is an *Inverse* Laplace Transform.

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

This integration is more involved, far beyond this text.

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\sigma-j\omega}^{\sigma+j\omega} e^{st}F(s)ds$$

So, is there no practical way out of the Laplace domain? Since the

$$\mathcal{L}\{f(t)\} = F(s)$$

is relatively straightforward, one approach is to just create a table of the practical time domain funtions $f(t)$ and their F(s). Then going back and forth into and out of the Laplace domain is just a matter of looking up the correct entry in the table.

Table

Table 4-1 Laplace Tansforms

Floyd E Nixon, *Handbook of Laplace Transformations: Fundamentals, Applications, Tables, and Examples* 21e © Prentice Hall, Inc., Englewood Cliffs New Jersey

No.	$F(S)$	$f(t)$	Comments
1.	1	$\delta(t)$	Unit impulse
2.	$\frac{A}{s}$	$A(t) = \begin{cases} 0 & t < 0 \\ A & t \geqslant 0 \end{cases}$	Step
3.	$\frac{1}{s}$	$U(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geqslant 0 \end{cases}$	Unit step
4.	$\frac{A}{s^2}$	At	Ramp
5.	$\frac{2A}{s^3}$	At^2	Parabola
6.	$\frac{A\omega}{s^2+\omega^2}$	$A \sin \omega t$	Sine
7.	$\frac{As}{s^2+\omega^2}$	$A \cos \omega t$	Cosine
8.	$aF(s)$	$af(t)$	
9.	$\frac{n!}{s^{n+1}}$	t^n	
10.	$sF(s) - f(0)$	$\frac{df(t)}{dt}$	
11.	$s^2F(s) - sf(0) - \frac{df(0)}{dt}$	$\frac{d^2f(t)}{dt^2}$	
12.	$\frac{F(s)}{s}$	$\int f(t)\, dt$	

13a.	$\frac{A}{\tau s + 1}$	$\frac{A}{\tau} e^{-t/\tau}$	Free response of first-order system
13b.	$\frac{A}{s + a}$	Ae^{-at}	
14a.	$\frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)}$	$\frac{A}{\tau_1 - \tau_2}(e^{-t/\tau_1} - e^{-t/\tau_2})$	Free response of second-order system ($\zeta > 1$)
14b.	$\frac{A}{(s + a)(s + b)}$	$\frac{A}{b - a}(e^{-at} - e^{-bt})$	
15a.	$\frac{A}{(\tau s + 1)^2}$	$\frac{At}{\tau^2} e^{-t/\tau}$	Free response of second-order system ($\zeta = 1$)
15b.	$\frac{A}{(s + a)^2}$	Ate^{-at}	
16.	$\frac{A\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$	$\frac{A\omega_n e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_n\sqrt{1 - \zeta^2}t)$	Second-order system, free response ($\zeta < 1$)
17a.	$\frac{A}{s(\tau s + 1)}$	$A(1 - e^{-t/\tau})$	First-order system response to a step input
17b.	$\frac{A}{s(s + a)}$	$\frac{A}{a}(1 - e^{-at})$	
18a.	$\frac{A}{s^2(\tau s + 1)}$	$A\tau\left(e^{-t/\tau} + \frac{t}{\tau} - 1\right)$	First-order system response to a ramp input
18b.	$\frac{A}{s^2(s + a)}$	$\frac{A}{a^2}(e^{-at} + at - 1)$	

No.	$F(S)$	$f(t)$	Comments
19a.	$\dfrac{A\omega}{(s^2+\omega^2)(\tau s+1)}$	$\dfrac{A\omega\tau}{1+\omega^2\tau^2}e^{-t/\tau}+\dfrac{A}{\sqrt{1+\omega^2\tau^2}}\sin(\omega t-\psi)$ where $\psi=\tan^{-1}\omega\tau \quad (0<\psi<\pi)$	First-order system response to a sine input
19b.	$\dfrac{A\omega}{(s^2+\omega^2)(s+a)}$	$\dfrac{A\omega e^{-at}}{\omega^2+a^2}+\dfrac{A}{\sqrt{\omega^2+a^2}}\sin(\omega t-\psi)$ where $\psi=\tan^{-1}\omega/a \quad (0<\psi<\pi)$	
20a.	$\dfrac{A}{s(\tau_1 s+1)(\tau_2 s+1)}$	$A\left(1+\dfrac{\tau_1 e^{-t/\tau_1}-\tau_2 e^{-t/\tau_2}}{\tau_2-\tau_1}\right)$	Second-order system response to a step input ($\zeta>1$)
20b.	$\dfrac{A}{s(s+a)(s+b)}$	$\dfrac{A}{ab}\left(1+\dfrac{ae^{-bt}-be^{-at}}{b-a}\right)$	
21a.	$\dfrac{A}{s(\tau s+1)^2}$	$A\left(1-\dfrac{\tau+t}{\tau}e^{-t/\tau}\right)$	Second-order system response to a step input ($\zeta=1$)
21b.	$\dfrac{A}{s(s+a)^2}$	$\dfrac{A}{a^2}[1-(1+at)e^{-at}]$	
22.	$\dfrac{A\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$	$A\left[1+\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t-\psi)\right]$ where $\psi=\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{-\zeta} \quad (0<\psi<\pi)$	Second-order system response to a step input ($\zeta<1$)
23a.	$\dfrac{A}{s^2(\tau_1 s+1)(\tau_2 s+1)}$	$A\left(t-\tau_1-\tau_2-\dfrac{\tau_2^2 e^{-t/\tau_2}-\tau_1^2 e^{-t/\tau_1}}{\tau_1-\tau_2}\right)$	Second-order system response to a ramp input ($\zeta>1$)
23b.	$\dfrac{A}{s^2(s+a)(s+b)}$	$\dfrac{A}{ab}\left[t-\dfrac{a+b}{ab}-\dfrac{(b/a)e^{-bt}-(a/b)e^{-at}}{b-a}\right]$	
24a.	$\dfrac{A}{s^2(\tau s+1)^2}$	$A[t-2\tau+(t+2\tau)e^{-t/\tau}]$	Second-order system response to a ramp input ($\zeta=1$)
24b.	$\dfrac{A}{s^2(s+a)^2}$	$\dfrac{A}{a^2}\left[t-\dfrac{2}{a}+\left(t+\dfrac{2}{a}\right)e^{-at}\right]$	
25.	$\dfrac{A\omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)}$	$A\left[t-\dfrac{2\zeta}{\omega_n}+\dfrac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t-\psi)\right]$ where $\psi=2\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{-\zeta} \quad (0<\psi<\pi)$	Second-order system response to a ramp input ($\zeta<1$)

No.	$F(S)$	$f(t)$	Comments
26a.	$\dfrac{A\omega}{(s^2+\omega^2)(\tau_1 s+1)(\tau_2 s+1)}$	$A\left[\dfrac{\tau_1^2\omega e^{-t/\tau_1}}{(\tau_1-\tau_2)(1+\omega^2\tau_1^2)}+\dfrac{\tau_2^2\omega e^{-t/\tau_2}}{(\tau_2-\tau_1)(1+\omega^2\tau_2^2)}+\dfrac{\sin(\omega t-\psi)}{[(1+\omega^2\tau_1^2)(1+\omega^2\tau_2^2)]^{1/2}}\right]$ where $\psi=\tan^{-1}\omega\tau_1+\tan^{-1}\omega\tau_2$	Second-order system response to a sine input ($\zeta>1$)
26b.	$\dfrac{A\omega}{(s^2+\omega^2)(s+a)(s+b)}$	$A\left[\dfrac{\omega e^{-at}}{(b-a)(\omega^2+a^2)}+\dfrac{\omega e^{-bt}}{(a-b)(\omega^2+b^2)}+\dfrac{\sin(\omega t-\psi)}{[(\omega^2+a^2)(\omega^2+b^2)]^{1/2}}\right]$ where $\psi=\tan^{-1}\dfrac{\omega(a+b)}{ab-\omega^2}$ $(0<\psi<\pi)$	
27a.	$\dfrac{A\omega}{(s^2+\omega^2)(\tau s+1)^2}$	$\dfrac{A}{1+\omega^2\tau^2}\left[\dfrac{\omega t+2\omega\tau}{1+\omega^2\tau^2}e^{-t/\tau}+\sin(\omega t-\psi)\right]$ where $\psi=2\tan^{-1}\omega\tau$	Second-order system response to a sine input($\zeta=1$)
27b.	$\dfrac{A\omega}{(s^2+\omega^2)(s+a)^2}$	$\dfrac{A}{\omega^2+a^2}\left[\dfrac{a\omega(at+2)e^{-at}}{\omega^2+a^2}+\sin(\omega t-\psi)\right]$	
28.	$\dfrac{A\omega\omega_n^2}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)}$	$\dfrac{A\omega_n^2}{[(\omega_n^2-\omega^2)^2+4\zeta^2\omega^2\omega_n^2]^{1/2}}\cdot\left[\sin(\omega t-\psi_1)+\dfrac{\omega e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t-\psi_2)}{\omega_n\sqrt{1-\zeta^2}}\right]$ where $\psi_1=\tan^{-1}\dfrac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}$, $0<\psi_1<\pi$ and $\psi_2=\tan^{-1}-\dfrac{2\zeta\omega_n^2\sqrt{1-\zeta^2}}{\omega^2-\omega_n^2(1-2\zeta^2)}$, $0<\psi_2<\pi$	Second-order system response to a sine input ($\zeta<1$)

Source: Floyd E. Nixon, *Handbook of Laplace Transformation: Fundamentals, Applications, Tables and Examples*, 21e, © 1965, Referenced and adapted by permission of Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Operations

Because the Laplace transformation is an integration, the property of integrals, which allow their simplification and association, apply to the transformation.

The transform of a constant times a function is equal to that constant times the transform of the function.

Pull the constant out front before doing the transform.

$$\mathcal{L}\{A\times f(t)\}=A\times\mathcal{L}\{f(t)\}$$

The transformation is an integral, which is just infinitesimal additions.

$$\mathcal{L}\{f_1(t) + f_2(t) + f_3(t) + \cdots\} = \mathcal{L}\{f_1(t)\} + \mathcal{L}\{f_2(t)\} + \mathcal{L}\{f_3(t)\} + \cdots$$

Just like superposition!

This is just like superposition. If there is a complicated function made of pieces that are added together, then handle (transform) each separately, *then* add them up.

Even though the Laplace of a sum *is* the sum of the Laplace transforms, this does *not* hold true for multiplication

$$\mathcal{L}\{f_1(t) \times f_2(t) \times f_3(t) \times \ldots\} \neq \mathcal{L}\{f_1(t)\} \times \mathcal{L}\{f_2(t)\} \times \mathcal{L}\{f_3(t)\} \times \ldots$$

Transformation cannot be distributed across the product of terms.

Look at row 10 of Table 4-1.

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Derivative $\leftrightarrow \times \boldsymbol{s}$

A derivative in the time domain transforms to multiplying by *s*. This is *huge*. All of the derivatives have just become multiplications.

From row 12 of Table 4-1

$$\mathcal{L}\left\{\int f(t)dt\right\} = \frac{F(s)}{s}$$

Integal $\leftrightarrow \boldsymbol{\frac{1}{s}}$

Consistently, integrating (the compliment of differentiating) transforms to dividing by *s* (the compliment of multiplying).

With these operations, take another look at Table 3-1, the composite of the relationships of current and voltage for resistors, capacitors, and inductors. For now, assume that there is no initial current or voltage.

Let $I_o = 0$ and $V_o = 0$

Table 4-2 Laplace voltage, current, and impedance relationships

	unknown	R	C	L
time domain	i	$\frac{v}{R}$	$C\frac{dv}{dt}$	$\frac{1}{L}\int_0^t v\,dt + I_o$
	v	$i \times R$	$\frac{1}{C}\int_0^t i\,dt + V_o$	$L\frac{di}{dt}$
Laplace domain	I	$\frac{V}{R}$	sCV	$\frac{V}{sL}$
	V	$I \times R$	$\frac{I}{sC}$	sLI
Laplace impedance	Z	R	$\frac{1}{sC}$	sL

The last row makes circuit analysis *much* simpler. Remember, impedance is defined as

$$Z = \frac{V}{I}$$

So, Laplace impedances can be defined as well.

$$Z_R = \frac{V}{I} = \frac{I \times R}{I} = R$$

$$Z_C = \frac{V}{I} = \frac{\frac{I}{sC}}{I} = \frac{1}{sC}$$

$$Z_L = \frac{V}{I} = \frac{sLI}{I} = sL$$