3.4 RC & RL Circuit Differential Equations

With derivatives and integrals, a capacitor's or inductor's current and voltage can be calculated. When those components are put into a circuit, then these derivatives and integrals must be incorporated with normal circuit analysis laws: Kirchhoff's Voltage and Current Laws, Thevenin's Theorem, Superposition, Impedance Combination, even Mesh Analysis.

Table 3-1 Voltage and current relationships

unknown	R	C	L
i	$\frac{v}{R}$	$C\frac{dv}{dt}$	$\frac{1}{L}\int_0^t v\,dt + I_o$
v	$i \times R$	$\frac{1}{C}\int_0^t i\,dt + V_o$	$L\frac{di}{dt}$

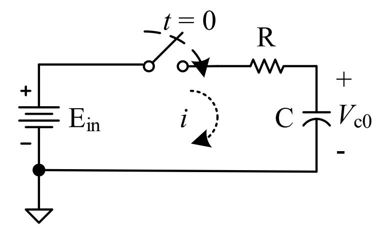


Figure 3-16 RC circuit with a step input

Figure 3-16 is a series circuit. Initially, the capacitor's charge (if any) is V_{c0}. At $t = 0$, the switch is closed, applying a set voltage to the circuit, creating current i. The voltages and current change with time, as the capacitor changes its charge. The key voltage and current relationships are shown in Table 3-1 (again).

To begin, assume that the capacitor is initially uncharged.

$$V_{c0+} = 0 \text{ V}$$

Remember, a capacitor cannot change its voltage (charge) instantly. It takes time for the charge to flow from the source and to build up on the plates, creating an electrostatic field and voltage. So, just after the switch is closed $t = 0^+$, there is still no voltage across the capacitor. All of the source voltage must be across the resistor.

$$V_{R0+} = E_{in}$$

The resistor sets (limits) that initial current.

$$i_{0^+} = \frac{V_{R0^+}}{R} = \frac{E_{\text{in}}}{R}$$

By recognizing that a capacitor opposes a change in voltage, cannot change its voltage instantly, the *initial* ($t = 0^+$) voltages and current have been calculated.

As time goes on, performance becomes more involved. Begin by writing Kirchhoff's Voltage Law around the circuit.

$$E_{\text{in}} = v_{\text{R}} + v_{\text{C}}$$

To solve this single loop there can be only one variable. Both v_{R} and v_{C} can be written in terms of the circuit current, i, and know component values, E_{in}, R, and C. Look at Table 3-1.

$$E_{\text{in}} = i\text{R} + \frac{1}{\text{C}}\int_0^t i\,dt + V_o$$

Since the initial capacitor voltage is 0 V, the circuit equation becomes

$$E_{\text{in}} = i\text{R} + \frac{1}{\text{C}}\int_0^t i\,dt$$

There is a whole body of mathematics called Differential Equations. Those procedures can be used to solve this equation for i. But first, the integral must be dealt with. Differentiate both sides of the equation.

$$\frac{d}{dt}(E_{in}) = \frac{d}{dt}(i\text{R}) + \frac{d}{dt}\left(\frac{1}{\text{C}}\int_0^t i\,dt\right)$$

Since the voltage is a constant, its time derivative is 0. The derivative of an integral of a function is just the function.

$$0 = \text{R}\frac{di}{dt} + \frac{1}{\text{C}}i$$

This helps quite a bit. It is common to denote the time derivative of a variable as that variable followed by a prime, $\frac{di}{dt} = i'$

$$0 = \text{R}i' + \frac{1}{\text{C}}i$$

Finally, the equation is usually rearranged to put the highest derivative of the variable on the left of the equation, alone.

$$i' = -\frac{1}{\mathrm{RC}}i$$

This is the simplest form of the first order (one derivative) differential equation, and the simplest form of the equation that describes the performance of the series RC circuit to a step input.

The question now becomes, "What function can be differentiated and equal itself times a constant?" That is answered by *entire* mathematics courses on Differential Equations, way outside the content of this discussion of *electronics*. Fortunately, the TI-Nspire calculator can solve differential equations of electronics interest.

deSolve

Begin by adjusting the settings as shown in Figure 3-17.

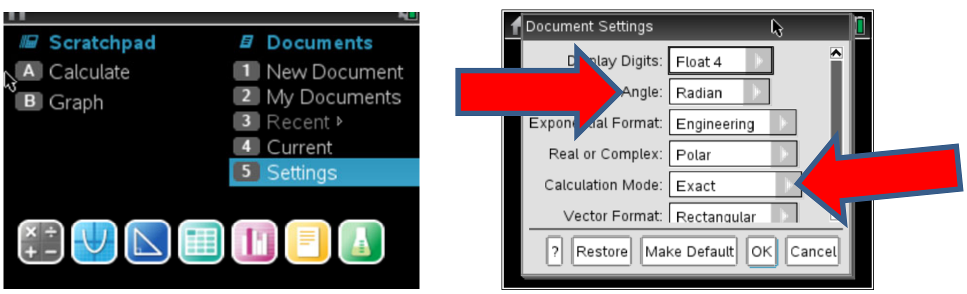


Figure 3-17 Calculator settings for differential equations solution

Then, from the Menu, select **4:Calculus** and **D:Differential Solver**

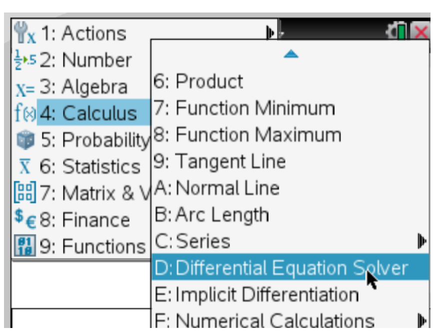


Figure 3-18 Menu selection to start the Different Solver

TI-nspire CX CAS

$$i' = -\frac{1}{\mathrm{RC}} i$$

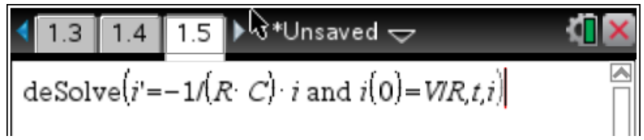


Figure 3-19 Entering the differential equation

$$i_{0^+} = \frac{V_{R0^+}}{R} = \frac{E_{\text{in}}}{R}$$

When the Differential Solver is selected, **deSolve()** Appears. Enter the differential equation:

- The prime is on the π menu.
- Spaces before and after **and** are necessary.
- Then enter the initial conditions equation,
- Followed by a comma, the *x* variable, and the y variable.

$$i = \frac{E_{\text{in}}}{\mathrm{R}} \mathrm{e}^{-\frac{t}{RC}}$$

$$i(t) = Ae^{\frac{-t}{\tau}}$$

$$A = \frac{E_{\text{in}}}{R}$$

$$\tau = RC$$

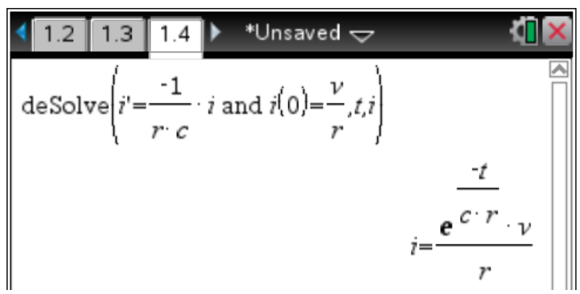


Figure 3-20 Calculator solution of the differential equation

The solution, Figure 3-20, looks a little different from the *standard* forms, but a little are the same. The current spikes up to i_{0+}, and then falls, with a time constant of tl So, limit the current that the circuit must produce with R. Once R is chosen, the cap mines how rapidly the circuit responds.

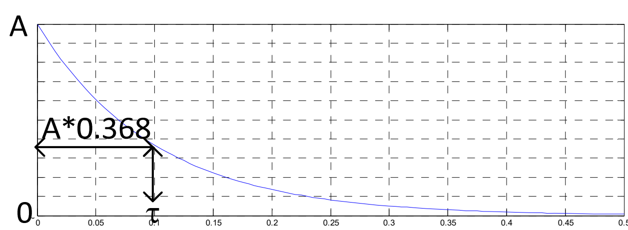


Figure 3-8 Exponential fall with amplitude (A), and time constant (τ) defined (again)

Once the circuit current is known, the voltage across the resistor and capacitor can be calculated.

$$v_{\mathrm{R}} = i \times \mathrm{R}$$

$$i = \frac{E_{\mathrm{in}}}{\mathrm{R}} \mathrm{e}^{-\frac{t}{RC}}$$

$$v_R = i \times R = \left(\frac{E_{\mathrm{in}}}{\mathrm{R}} \mathrm{e}^{-\frac{t}{RC}}\right) \times R$$

$$v_{\mathrm{R}} = E_{\mathrm{in}}\, \mathrm{e}^{-\frac{t}{RC}}$$

$$v_{\mathrm{C}} = \frac{1}{C} \int_0^t i\, dt + V_o$$

$$v_{\mathrm{C}} = \frac{1}{\mathrm{C}} \int_0^t \left(\frac{E_{\mathrm{in}}}{\mathrm{R}} \mathrm{e}^{-\frac{t}{RC}}\right) dt$$

$$v_{\mathrm{C}} = \frac{E_{\mathrm{in}}}{\mathrm{RC}} \int_0^t \mathrm{e}^{-\frac{t}{RC}}\, dt \qquad\qquad \int_0^t \left(Ae^{\frac{-t}{\tau}}\right) dt = A\tau\left(1 - e^{\frac{-t}{\tau}}\right)$$

$$v_C = \frac{E_{\mathrm{in}}}{\mathrm{RC}} \times \mathrm{RC}\left(1 - e^{\frac{-t}{RC}}\right)$$

$$v_C = E_{\mathrm{in}}\left(1 - e^{\frac{-t}{RC}}\right)$$

If all these calculations are correct, then the original Kirchhoff's Voltage Law should work with these equations.

$$E_{\mathrm{in}} = v_{\mathrm{R}} + v_{\mathrm{C}}$$

$$E_{\mathrm{in}} \mathrel{?=} E_{\mathrm{in}}\, \mathrm{e}^{-\frac{t}{RC}} + E_{\mathrm{in}}\left(1 - e^{\frac{-t}{RC}}\right)$$

$$E_{\mathrm{in}} \mathrel{?=} E_{\mathrm{in}}\, \mathrm{e}^{-\frac{t}{RC}} + E_{\mathrm{in}} - E_{\mathrm{in}} e^{\frac{-t}{RC}}$$

$$E_{\mathrm{in}} \equiv E_{\mathrm{in}}$$

Example 3-7

Given:

$E_{\text{in}} = 10$ V
R = 1 kΩ
C = 10 nF

Calculate:

Initial current
Time constant
Final voltage across R
Final voltage across C

Plot v_R and v_C.

TI-nspire CX CAS

Solutions

$$i_{t=0} = \frac{E_{\text{in}}}{\text{R}}\text{e}^{-\frac{t}{RC}} = \frac{10\ V}{1\ \text{k}\Omega}\text{e}^{-\frac{0}{RC}}$$

$$i_{t=0} = 10\ \text{mA}$$

$$\tau = \text{RC} = 1\text{k}\Omega \times 10\text{nF} = 10\ \mu\text{sec}$$

$$v_{\text{R}} = E_{\text{in}}\ \text{e}^{-\frac{t}{RC}} = 10\ \text{V}\ \text{e}^{-\frac{\infty}{RC}}$$

$v_{\text{R}} = 0\ V$ Eventually there is *no* voltage across the resistor.

$$v_C = E_{\text{in}}\left(1 - e^{\frac{-t}{RC}}\right) = 10\ \text{V}\left(1 - e^{\frac{-\infty}{RC}}\right)$$

$$v_C = 10\ \text{V}(1 - 0) = 10\ \text{V}$$

Eventually *all* of the voltage is across the capacitor. It is fully charged.

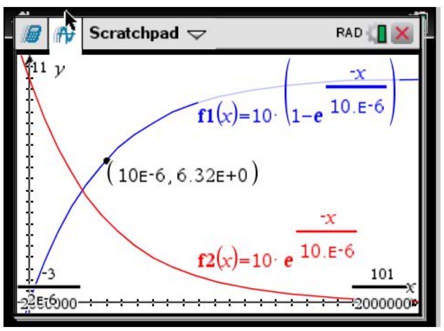


Figure 3-21 RC circuit voltage waveforms

The plots are shown in Figure 3-21. Initially, all of the voltage appears across the resistor [f2(x)], and the voltage across the capacitor is 0 V, [f1(x)]. As time goes on, the voltage across the resistor drops and the voltage across the capacitor rises, as it charges. Eventually, all of the input is across the capacitor, the capacitor is fully charged, there is no current, and so there is no voltage across the resistor.

After one time constant, 10 μsec, 63.2 % of the voltage is across the capacitor, leaving 36.8% of the voltage across the resistor.

The Time Constant, τ

For all *exponential* responses, the key timing parameter is the time constant, τ. For the rising waveforms, time t only appears in the exponent of the exponential e.

$$v(t) = A\left(1 - e^{\frac{-t}{\tau}}\right)$$

At $t = \tau$, the voltage has changed 63.2% of the maximum possible change.

$$v(t) = A(1 - e^{-1}) = 0.632 \times A$$

The rising wave starts at 0 V. So its maximum possible change is to increase to 63.2% of A. See Figure 3-22(a).

The interesting thing is that during the second time constant, the wave changes by 63.2% of the remaining maximum possible change. The rising wave starts at 0.632 * A. During the second time constant, the maximum remaining change is 0.368 * A. It changes 63.2% * 0.368 * A = 0.233 * A. Since it started the second time constant at 0.632 * A the voltage will arrive at 0.632 * A + 0.233 * A = 0.865 * A, Figure 3-22(b).

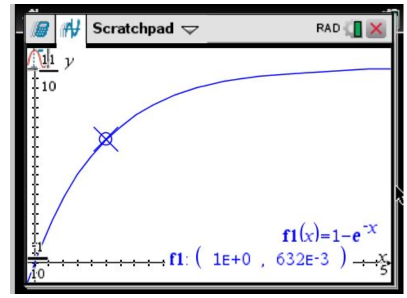
During the *third* time constant, there is only *A-0.865* * A = 0.135 * A for the voltage to rise. It will only rise 63.2% of this (63.2%*0.135 * A = 0.086 * A), arriving at 0.95 * A, Figure 3-22 (c).

Following this logic, as the voltage gets closer and closer to A, the function increases less and less. Taken to the extreme, the function can *never* reach A. But, after five time constants, the voltage rises to 0.993 * A, which is traditionally considered "close enough".

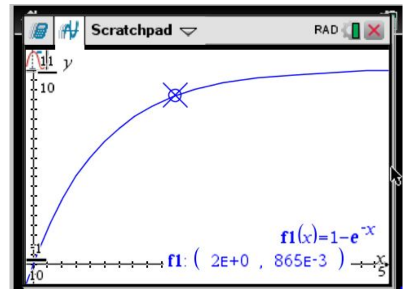
It takes 5τ for an RC circuit to fully (99.3%) respond.

A more useful measure is the *rise time*, t_{rise}. The rise time is the time it takes a wave to rise from 10% to 90%. This gets around any delays in getting started and the slow approach at the end. The plot in Figure 3-23 shows that the wave passes 10% (101E-3) at 0.1 τ (106E-3). The wave passes 90% (900E-3) at 2.3 τ. So, it takes 2.2 τ for the wave to rise from 10% to 90%.

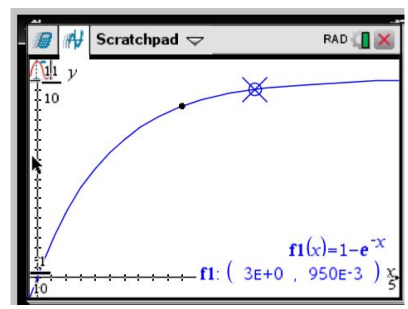
It takes 2.2τ for an RC circuit to complete its *rise time*.



(a) Rise after one time constant



(b) Rise after two time constants



(c) Rise after three time constants

Figure 3-22 Exponential rise

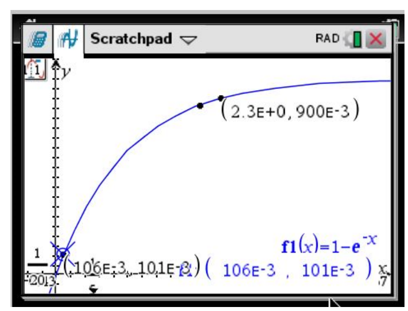


Figure 3-23 Exponential rise time

RL Differential Equation Circuit Analysis

Replacing the capacitor in Figure 3-16 with an inductor produces a complimentary response. The schematic is in Figure 3-24.

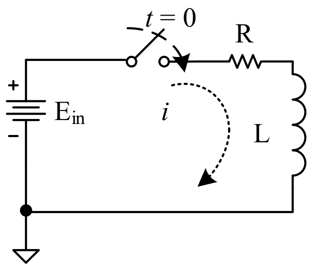


Figure 3-24 RL circuit with a step

Table 3-1 Voltage and current relationships

unknown	R	C	L
i	$\frac{v}{R}$	$C\frac{dv}{dt}$	$\frac{1}{L}\int_0^t v\,dt + I_o$
v	$i \times R$	$\frac{1}{C}\int_0^t i\,dt + V_o$	$L\frac{di}{dt}$

Magnetic fields building and cutting segments of the inductor induces current into those segments, establishing the current through the inductor. These magnetic fields cannot appear instantaneously. It takes time for them to build. So, the induced current cannot change instantly either.

Inductors oppose a change in current.

Starting with an open circuit the instant *before* the switch is closed gives $i_{t=0-} = 0$ A. The inductor opposes any change in current. So

$$i_{t=0+-} = 0\text{ A}$$

the instant after the switch is closed, the initial current is also 0A.

Applying Kirchhoff's Voltage Law, just as with the RC circuit gives

$$E_{\text{in}} = v_{\text{R}} + v_L$$

Putting this equation in terms of current

$$E_{\text{in}} = i \times \text{R} + \text{L}\frac{di}{dt}$$

This is different from the RC circuit because there is no integral to get rid of, but now E_{in} remains part of the equation. A little algebra puts this circuit equation into the standard differential equations format,

$$E_{\text{in}} = i\text{R} + \text{L}i'$$

$$i' = -\frac{\text{R}}{\text{L}}i + \frac{E_{\text{in}}}{\text{L}}$$

The TI-Nspire solution is shown in Figure 3-25.

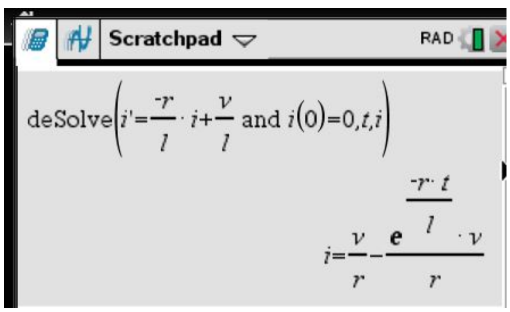


Figure 3-25 Calculator solution of the RL differential equation

This looks a clumsy. But, a little algebra cleans this up into one of the standard forms. Begin by factoring *v*/*r* out. Then clean up the exponent.

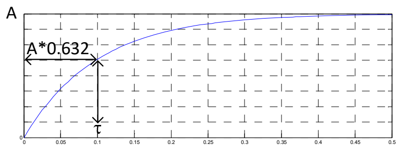
$$i = \frac{E_{\text{in}}}{\text{R}}\left(1 - e^{-\frac{t}{L/R}}\right)$$

$$i(t) = A\left(1 - e^{\frac{-t}{\tau}}\right)$$

Standard form

$$A = \frac{E_{\text{in}}}{\text{R}}$$

$$\tau = \frac{\text{L}}{\text{R}}$$



The voltage across the resistor is

$$v_{\mathrm{R}} = i \times \mathrm{R} = \left[\frac{E_{\mathrm{in}}}{\mathrm{R}}\left(1 - e^{-\frac{t}{L/R}}\right)\right] \times \mathrm{R}$$

$$v_{\mathrm{R}} = E_{\mathrm{in}}\left(1 - e^{-\frac{t}{L/R}}\right)$$

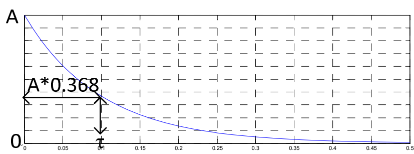
The voltage across the inductor is

$$v_{\mathrm{L}} = L\frac{di}{dt}$$

$$\frac{d}{dt}\left[A\left(1 - e^{\frac{-t}{\tau}}\right)\right] = \frac{A}{\tau}e^{\frac{-t}{\tau}}$$

$$v_{\mathrm{L}} = \mathrm{L}\frac{d}{dt}\left[\frac{E_{\mathrm{in}}}{\mathrm{R}}\left(1 - e^{-\frac{t}{L/R}}\right)\right]$$

$$v_{\mathrm{L}} = \frac{E_{in}\frac{\mathrm{L}}{\mathrm{R}}}{\frac{\mathrm{L}}{\mathrm{R}}}e^{-\frac{t}{L/R}}$$



$$v_{\mathrm{L}} = E_{in}e^{-\frac{t}{L/R}}$$

The TI-Nspire produces the same result, shown in Figre 2-26.

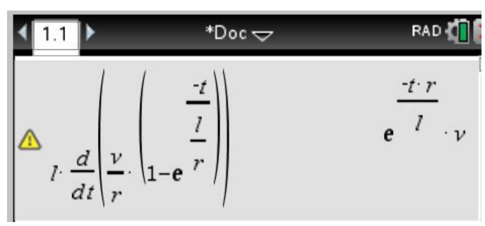
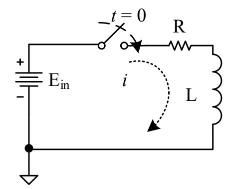


Figure 3-26 Calculator verification of the voltage across the inductor

$$i = \frac{E_{\text{in}}}{\text{R}}\left(1 - e^{-\frac{t}{L/R}}\right) \qquad v_{\text{R}} = E_{\text{in}}\left(1 - e^{-\frac{t}{L/R}}\right) \qquad v_{\text{L}} = E_{in}e^{-\frac{t}{L/R}}$$



Does all this math produce results that match what experience shows happens in the circuit?

The inductor opposes a change in current. So, the instant after the switch closes ($t = 0^+$), there will be *no* current in the circuit. Substituting $t = 0$ into the current equation produces $i = 0$.

With no current, the resistor can drop no voltage. Substituting $t = 0$ into the v_R equation gives $v_R = 0$.

With no voltage across the resistor, all of E_{in} must show up across the inductor. Substituting $t = 0$ in the the v_L equation gives $v_L = E_{in}$.

As time goes on and on, eventually the inductor becomes fully charged (i.e. the magnetic fields are fully established), and it looks like a piece of wire. At that time ($t = \infty$), the inductor (as a piece of wire) drops *no* voltage. Substituting $t = \infty$ into the v_L equation gives $v_L = 0$.

With no voltage across the inductor, all of E_{in} must appear across the resistor. Substituting $t = \infty$ into the v_R equation gives $v_R = E_{in}$.

Since there is E_{in} across the resistor, $i = E_{in}/\text{R}$. Substituting $t = \infty$ into the i equation gives that result.

3.5 Initial Conditions and Other Inputs

When the circuit does not start from rest, or when the input is something more complicated than the simple step used in the preceding section, the *algebra* becomes more involved. But the electronics and the procedures are the same.

Initial Conditions

For the RL circuit in Figure 3-24, how does the current behave differently, if at $t = 0^+$, $i = -5$ mA (the negative indicates it is flowing up, opposite to the indicated direction)?

The derivation of the circuit's differential equation is the same as originally done.

$$E_{\text{in}} = v_{\text{R}} + v_L$$

$$E_{\text{in}} = i \times \text{R} + \text{L}\frac{di}{dt}$$

$$E_{\text{in}} = i\text{R} + \text{L}i'$$

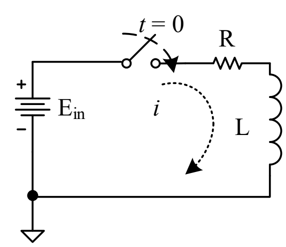


Figure 3-24 RL circuit with a step input

$$i' = -\frac{\mathrm{R}}{\mathrm{L}}i + \frac{E_{\mathrm{in}}}{\mathrm{L}}$$

Entering this into the calculator for a solution, however requires that the initial current be included.

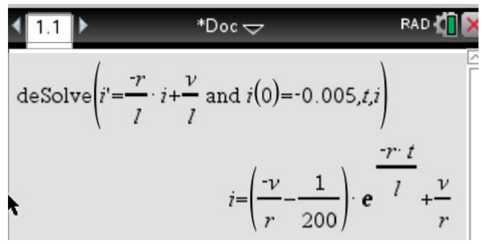


Figure 3-27 Calculator solution of RL circuit with initial current

Although pieces of this result have been seen before, as a whole this is not familiar. Algebra is needed. Begin by distributing the exponent.

$$i = -\frac{V}{R}e^{-\frac{t}{L/R}} - 5\text{ mA}\,e^{-\frac{t}{L/R}} + \frac{V}{R}$$

Both the first and the last term have V/R, which is the eventual current. So, combine those.

$$i = \frac{V}{R} - \frac{V}{R}e^{-\frac{t}{L/R}} - 5\text{ mA}\,e^{-\frac{t}{L/R}}$$

$$i = \frac{V}{R}\left(1 - e^{-\frac{t}{L/R}}\right) - 5\text{ mA}\,e^{-\frac{t}{L/R}}$$

Now this looks like pieces seen before. The second term is just the exponential decay of the initial current. Then, the first term is the performance expected with no initial current.

It's just superposition, a voltage source and a current source. The voltage source produces a buildup in current, just as it did in the previous section (without the initial current). The (initial) current decays as its magnetic fields collapse.