6

Active Filters

Introduction

Electric filters are used in practically all circuits that require the separation of signals according to their frequencies. Applications include (but are certainly not limited to) noise rejection and signal separation in industrial and measurement circuits, feedback of phase and amplitude control in servoloops, smoothing of digitally generated analog (D-A) signals, audio signal shaping and sound reinforcement, channel separation, and signal enhancement in communications circuits.

Such filters can be built from passive RLC components, electromechanical devices, crystals, resistors, capacitors, and op amps (active filters). Active filters are applicable over a wide range of frequencies. They are also inexpensive and offer high input impedance, low output impedance, adjustable gain, and a variety of responses.

In this chapter, you will learn the characteristics, terminology, and mathematics of active filters. The analysis, design, and calculations behind several types of low and high pass filters, wide and narrow band pass filters, are presented.

Objectives

6.1 Introduction to Filtering

Filters are specified, analyzed, and designed somewhat differently than are the circuits discussed in the previous chapters. Performance is specified in terms of the frequency response, that is, how the gain and phase shift change with frequency. Analysis and design often use Laplace transforms and the circuit's transfer function. Angular frequency, f_o, f_{-3dB}, bands, ripple, roll-off rate, center frequency, and Q are among the terms unique to filters. In this section, frequency response plots, basic transform math, and terminology common to most filters are presented. Later sections of the chapter apply these fundamentals to specific filter circuits.

6-1.1 Frequency Response

The gain and phase shift of a filter change as the frequency changes. Indeed, this is the purpose of a filter. Performance is often described with a graph of gain and phase versus frequency.

The vertical axis is the gain, and the horizontal axis is the frequency. The frequency axis gives equal distance for each **decade** of frequency. The distance between 1 Hz and 10 Hz is the same as the distance between 100 kHz and 1 MHz. On a linear plot, the distance between 100 kHz and 1 MHz should be 100,000 larger. However, this would make it impossible to plot any large range of frequencies on a single graph, so the horizontal axis is scaled logarithmically.

A log scaled horizontal axis is shown in Figure 6-1. In Figure 6-1(a), each decade (factor of 10) increase or decrease moves the same distance along the scale. There are two other key ideas. First, the starting point is not at zero. Moving left, each increment would lower the scale by 10 (to 0.1, 0.01, 0.001, …), but zero is never reached. So, start the log divisions at the lowest frequency of interest, not zero (DC). Second, the divisions between decades are not uniform (linear). This is easily seen in Figure 6-1(b), an expansion of a single decade. Five does not fall halfway between 1 and 10; $3^1/_3$ does. Be very careful about this when interpolating between major scale divisions.

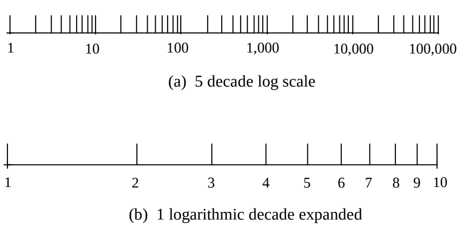


Figure 6-1 Logarithmic scales

Gain, plotted on the vertical axis, is normally expressed in dB.

$$dB = 10\log_{10}\frac{power_{\text{out}}}{power_{\text{in}}}$$

$$dB = 10\log\frac{\dfrac{v_{\text{o}}^2}{\text{R}_{\text{load}}}}{\dfrac{e_{\text{in}}^2}{R_{\text{in}}}}$$

$$dB = 10\log\left(\frac{v_{\text{o}}}{e_{\text{in}}}\right)^2\left(\frac{R_{\text{in}}}{\text{R}_{\text{load}}}\right)$$

Assuming that

$$R_{\text{in}} = \text{R}_{\text{load}}$$

Gives

$$dB = 10\log\left(\frac{v_{\text{o}}}{e_{\text{in}}}\right)^2$$

$$dB = 20\log\frac{v_{\text{o}}}{e_{\text{in}}}$$

$$\boldsymbol{dB = 20\log\frac{v_{\text{o}}}{e_{\text{in}}}}$$

For this to be valid, the load resistance must equal the filter's input resistance. This is seldom the case. However, the definition is normally used anyway.

Notice that log base 10 is used, not the natural log (log base e or ln). It is handy to know several dB and ratio points. These are listed in Table 6-1. Decreasing the ratio gain by 10 subtracts 20 dB; increasing it by a ratio of 10 adds 20 dB. Doubling the gain adds 6 dB; halving it subtracts 6 dB. Cutting the gain by $\sqrt{2}$ gives -3 dB. A ratio of 1 ($v_{\text{o}} = e_{\text{in}}$) is a gain of 0 dB.

Table 6-1 Ratio and dB gain comparison

v_o/e_{in}	dB	v_o/e_{in}	dB
1000	60	0.707	-3
100	40	0.5	-6
10	20	0.1	-20
2	6	0.01	-40
1	0	0.001	-60

A typical low pass filter frequency response is given in Figure 6-2. The vertical axis is scaled in decibels. The highest gain is called the **pass band gain**, A_o. In Figure 6-2, $A_o = +10$ dB.

The horizontal axis is scaled logarithmically (two decades in the figure). By plotting dB versus log frequency, log gain versus log frequency is actually being plotted. So any linear relationship is displayed as a straight line.

A parameter of major importance is the **-3 dB frequency**, f_{-3dB}. At that frequency, the gain has fallen 3 dB below A_o. In Figure 6-2, the pass band gain A_o=10 dB. At f_{-3dB} the gain falls to 7 dB.

$$A_o = 10 \text{ dB}$$

$$f_{-3dB} = 2500 \text{ Hz}$$

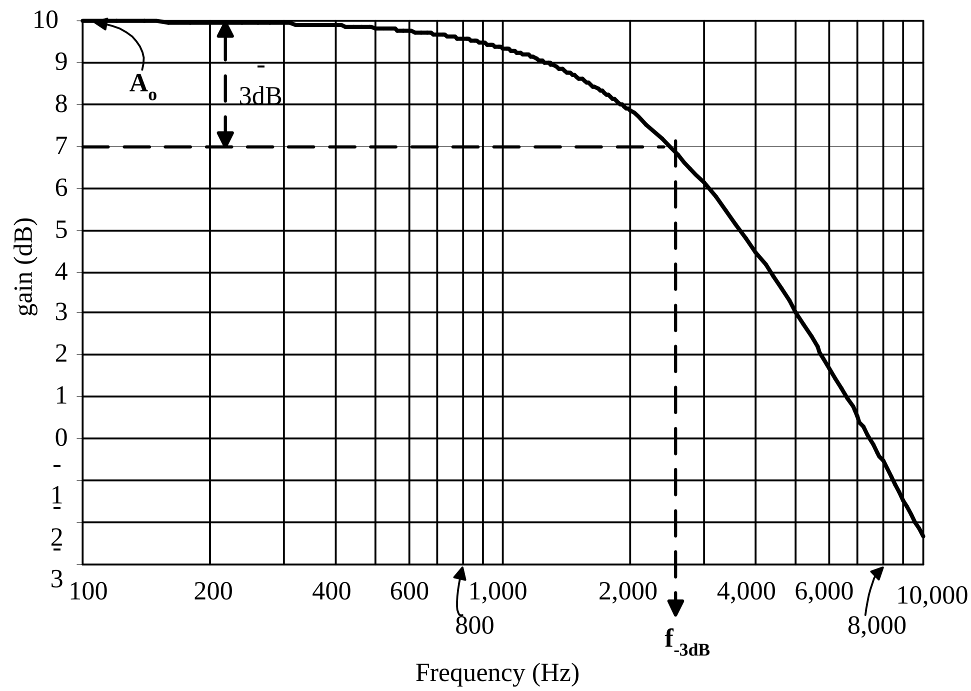


Figure 6-2 Typical low pass response curve

What does this -3 dB reduction in the voltage gain mean about the power delivered to the load?

$$-3\,\text{dB} = 20\log\left(A_v\right)$$

$$A_v = 10^{-\frac{3}{20}} = 0.707$$

The voltage gain has fallen by 0.707 below its value at A_o. This means at $f_{-3\text{dB}}$ the output voltage has fallen by 0.707.

$$v_{\text{out}-3\text{dB}} = 0.707 \times v_{\text{out@Ao}}$$

$$P_{\text{out@f}\,-3\text{dB}} = \frac{v^2}{R} = \frac{\left(0.707 \times v_{\text{out@Ao}}\right)^2}{R}$$

$$P_{\text{out@f}\,-3\text{dB}} = \frac{0.5 \times v_{\text{out@Ao}}^2}{R} = \tfrac{1}{2} P_{\text{out@Ao}}$$

At $f_{-3\text{ dB}}$:
v_{out} has dropped 0.707

P_{out} is cut in *half.*

As frequency changes, not only does the filter's output amplitude change, but its input-output phase relationship also shifts. At the **critical frequency**, f_o, that shift is an integer multiple of 45°. For simple filters built with a single RC pair, there is a 45° shift in phase. For each additional pair you add to the circuit, the phase shifts another 45° at f_o. Two RC pairs produce a 90° shift at the critical frequency; three pairs cause a 135° shift at f_o.

At f_o,:
the *phase* of v_{out} has shifted

$$\pm n \times 45°$$

***n* = circuit's Laplace domain order**

In many filters the -3 dB frequency, $f_{-3\text{dB}}$, and the critical frequency, f_o, occur at the same point. But two different effects are occurring at that frequency. The voltage gain has dropped -3 dB (0.707), $f_{-3\text{ dB}}$, and the phase has shifted the correct number of 45°, f_o. However, when using an op amp to add gain and positive feedback to a filter, these two frequencies often no longer occur at the same point.

A typical phase plot combined with the gain plot is given in Figure 6-3. The combination of the gain variation and the phase shift as a function of frequency forms a complete frequency response plot. Not only is plotting both functions on the same graph convenient, but it also allows you to determine if the filter (or any system) you anticipate building will be stable (i.e., not oscillate).

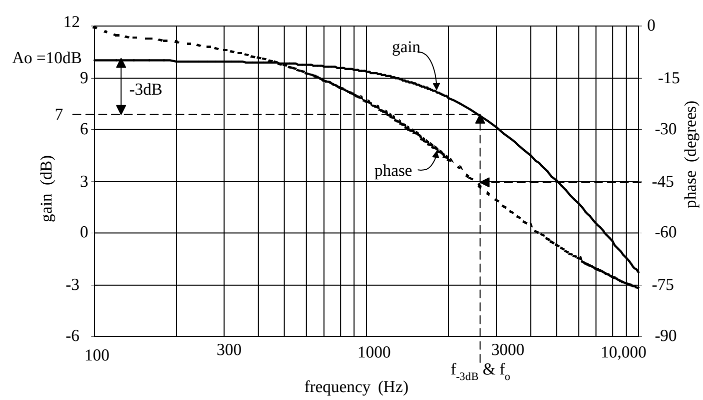


Figure 6-3 Composite gain and phase frequency response

Analysis of circuits containing several reactive components is most easily handled by using Laplace transforms. As far as active filters are concerned, the primary term is

$$i_C = C\frac{dv}{dt}$$

To take the Laplace transform of this, just replace the derivative with *s*. Uppercase letters are used to indicate that they belong to the Laplace domain. Lower case letters are used for variables in the time (normal) domain.

$$I = CsV$$

The Laplace impedance, then, for a capacitor is

$$Z = \frac{V}{I} = \frac{1}{Cs}$$

The term *s* contains both the amplitude and the phase information about an equation's or a circuit's response. To obtain the frequency response from a Laplace equation, make the substitution

s = jω

$$s = j\omega$$

where j is the imaginary number $\sqrt{-1}$ and $\omega = 2\pi f$.

Example 6-1

Verify that the Laplace form of capacitive impedance correctly converts back to the frequency domain.

Solution

$$Z = \frac{1}{Cs}$$

To convert from the Laplace domain (equation containing *s*) to the frequency domain (equation containing *f* or ω):

$$s = j\omega$$

$$\overline{Z} = \frac{1}{j\omega C}$$

$$\overline{Z} = \frac{1}{\mathrm{j}\,2\pi f\ \mathrm{C}}$$

$$\overline{Z} = 0 - \mathrm{j}\frac{1}{2\pi f\ \mathrm{C}}$$

$$\overline{Z} = 0 - \mathrm{j}X_{\mathrm{C}}$$

This is the phasor impedance of a capacitor.

Using Laplace functions, the frequency response (both gain and phase) of a circuit can be determined. First determine the circuit's transfer function. The transfer function is just the gain ($V_{\mathrm{out}}/E_{\mathrm{in}}$) expressed in Laplace terms.

Example 6-2

a. Determine the transfer function of the circuit in Figure 6-4.

b. Calculate the frequency response (both magnitude and phase) for R = 637Ω and C = 0.1μF.

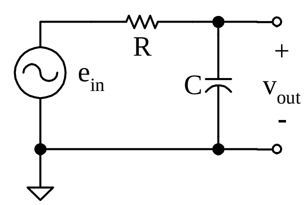


Figure 6-4 Circuit for Example 6-2

Solution

a. To determine the transfer function, apply the voltage divider law.

$$\overline{v_{\mathrm{out}}} = \frac{\overline{Z_{\mathrm{C}}}}{R + \overline{Z_{\mathrm{C}}}}\overline{e_{\mathrm{in}}}$$

Substitution:

$$\overline{e_{\mathrm{in}}} \rightarrow E_{\mathrm{in}} \qquad \mathrm{R} \rightarrow \mathrm{R} \qquad \overline{Z_{\mathrm{C}}} \rightarrow \frac{1}{\mathrm{Cs}} \qquad \overline{v_{\mathrm{out}}} \rightarrow V_{\mathrm{out}}$$

$$V_{\mathrm{out}} = \frac{\frac{1}{\mathrm{C}s}}{R + \frac{1}{\mathrm{C}s}}E_{\mathrm{in}}$$

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{\dfrac{1}{\text{C}s}}{\dfrac{\text{RC}s+1}{\text{C}s}}$$

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{1}{\text{RC}s+1}$$

This looks just like the standard first order solution from Chapter 4.

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{1}{\tau s+1}$$

where $\tau = \text{RC}$

b. To determine the frequency response, substitute

$$s = \text{j}\omega$$

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{1}{\text{Rj}\omega\text{C}+1} = \frac{1}{1+\text{j}\omega\text{RC}}$$

This is an equation with real and imaginary parts in its denominator. To separate these parts, multiply both the numerator and the denominator by the complex conjugate of the denominator.

$$\frac{V_{\text{out}}}{E_{\text{in}}} = \frac{1}{1+\text{j}\omega\text{RC}} \times \frac{1-\text{j}\omega\text{RC}}{1-\text{j}\omega\text{RC}}$$

$$= \frac{1-\text{j}\omega\text{RC}}{1-\text{j}^2\omega^2\text{R}^2\text{C}^2}$$

$$= \frac{1-\text{j}\omega\text{RC}}{1+\omega^2\text{R}^2\text{C}^2}$$

$$= \frac{1}{1+\omega^2\text{R}^2\text{C}^2} - \text{j}\frac{\omega\text{RC}}{1+\omega^2\text{R}^2\text{C}^2}$$

$$\frac{V_{\text{out}}}{E_{\text{in}}} = real + \text{j}\ \ imaginary$$

$$real = \frac{1}{1+\omega^2 R^2 C^2} \qquad imaginary = \frac{-\omega RC}{1+\omega^2 R^2 C^2}$$

$$\left|\overline{G}\right| = magnitude = \sqrt{real^2 + imaginary^2}$$

$$= \sqrt{\left(\frac{1}{1+\omega^2 R^2 C^2}\right)^2 + \left(\frac{\omega RC}{1+\omega^2 R^2 C^2}\right)^2}$$

$$= \sqrt{\frac{1+\omega^2 R^2 C^2}{\left(1+\omega^2 R^2 C^2\right)^2}}$$

$$\left|\overline{G}\right| = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}$$

$$\phi = phaseshift = \arctan\left(\frac{imaginary}{real}\right)$$

$$= \arctan\left(\frac{\dfrac{-\omega RC}{1+\omega^2 R^2 C^2}}{\dfrac{1}{1+\omega^2 R^2 C^2}}\right)$$

$$\phi = -\arctan(\omega RC)$$

Now that there are two equations, the simplest next step is to use a spreadsheet to tabulate frequency, magnitude, and phase; then create the plot. It is similar to Figure 6-3. However, this circuit has a pass band gain, A_o, of 0 dB, not the 10 dB shown in Figure 6-3.

The techniques of Example 6-2 can be used to determine the frequency response of any network. However, each RC combination in the circuit increases the order of the denominator by 1. Two RC pairs cause

$s^2 + bs + c$; three RC pairs cause s^3+bs^2+cs+d. Fortunately, the mathematics involved in breaking down and solving these higher-order equations has already been worked out for the popular, more useful circuits. Your job is to obtain the transfer function and recognize and extract key parameters. Actually, the major mathematical effort to convert back from the Laplace domain into the frequency domain may not have to be done at all.

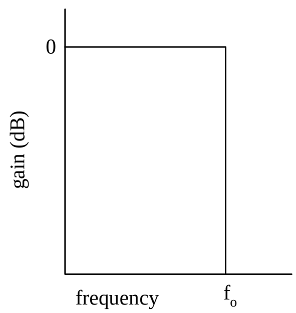


Figure 6-5 Ideal low pass filter

6-1.2 Characteristics and Terminology

The frequency response shown so far is for a **low pass filter**. The purpose of a low pass filter is to pass low frequency signals while stopping high frequency signals. Ideally, a low pass filter would have a frequency response as shown in Figure 6-5. All frequencies below the critical frequency would be uniformly passed. Any frequency above f_o would be completely stopped.

Of course, such a filter cannot be built. The response of a practical filter is divided into two bands as shown in Figure 6-6. The -3 dB frequency, f_{-3dB}, forms the boundary between the pass band and the stop band. For some filters, gain may vary up and down (or ripple) in the pass band or in the stop band, or both. The amount of pass band ripple allowable (and, to a lesser degree, stop band ripple) is an important parameter to keep in mind when designing a filter.

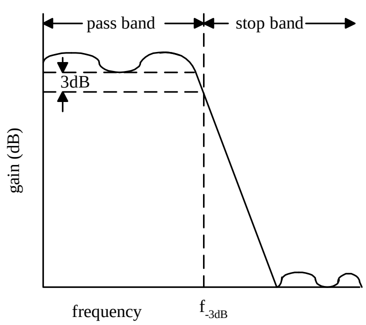
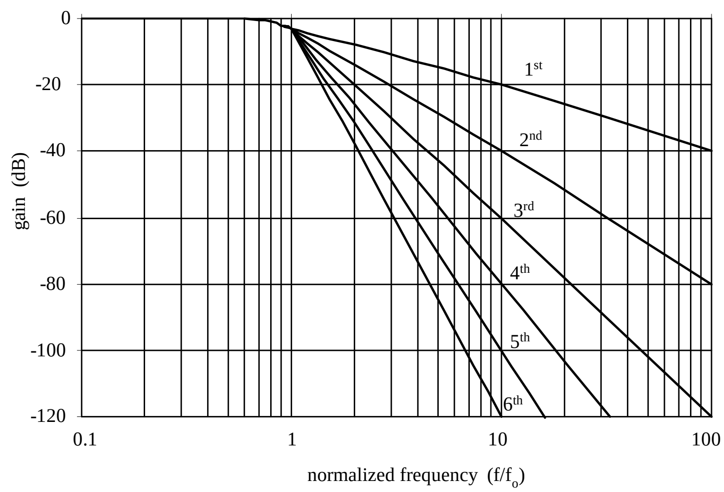


Figure 6-6 Practical low pass filter

How rapidly the gain falls as the stop band is entered is called the roll-off. The first six roll-off rates are illustrated in Figure 6-7. The roll-off rate is determined by the filter's order (1, 2, 3, …). Each increase in order increases the roll-off by 20 dB/decade or 6 dB/octave. In turn, as was mentioned in the Laplace transform section, the order of the filter (and its transfer function's denominator) equals the number of resistor/capacitor pairs in the circuit. So increasing the circuit's complexity by adding an RC pair increases the circuit's order, and the difficulties of the mathematics, but also increases the roll-off rate.



Roll-odd rate:

$$\boldsymbol{n \times 20 \frac{\text{dB}}{\text{decade}} / \text{order}}$$

or

$$\boldsymbol{n \times 6 \frac{\text{dB}}{\text{octave}} / \text{order}}$$

Where ***n*** **= the filter's order**

Decade **= × 10 change in frequency (1 to 10, or 10 to 100)**
Octave **= × 2 change in frequency (1 to 2, 2 to 4, 3 to 6, …)**

Figure 6-7 Roll-off rate comparison

Roll off is specified in dB/decade and dB/octave. A decade increase in frequency means that the frequency has changed by a factor of ten. An octave increase means that the frequency has doubled. Decibels per octave ratings are used primarily with audio/music applications. Table 6-2 correlates filter order (number of RC pairs and transfer function denominator order) with the roll off rate in dB/decade and dB/octave.

Table 6-2 Roll-off rate comparison

Order	dB/decade	dB/octave
1	20	6
2	40	12
3	60	18
4	80	24
5	100	30

The opposite of the low pass filter is the high pass filter. A high pass filter is illustrated in Figure 6-8. Low frequency signals and DC (which is 0 Hz) are blocked, while high frequency signals are passed. Specifications, analysis, and design of high pass filters are closely analogous to those you have already seen for low pass filters.

The couplers used in single supply amplifiers block the DC bias from previous stages or the signal generator, while passing the signal whose frequency is of interest. This RC coupler is the complement of the circuit analyzed in Example 6-2. With a single RC pair, the RC coupler has a 20 dB/decade roll-off and a +45° phase shift at the critical frequency.

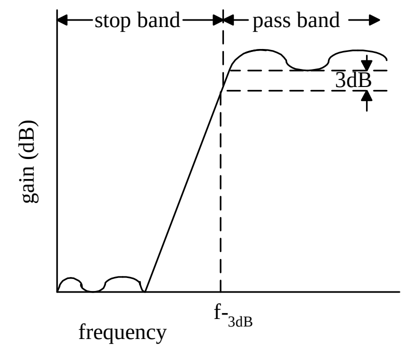
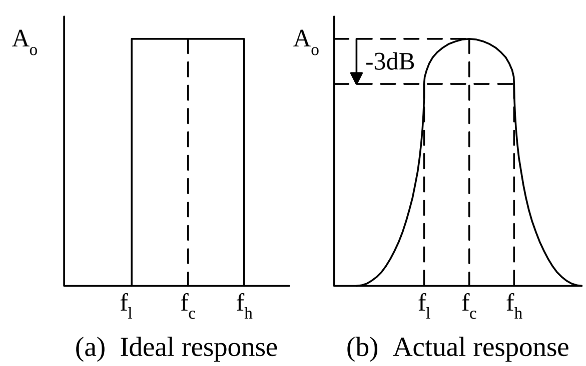


Figure 6-8 High pass filter

The band pass filter passes only those signals within a given band. Signals above and below that band are blocked. Figure 6-9(a) is the response of an ideal band pass filter, while Figure 6-9(b) is a more realistic response. Since the response rises, peaks, and then falls, there are three frequencies of interest. The **center frequency** is $\boldsymbol{f_c}$**.** Depending on the component configuration and values, there may be considerable gain at the center frequency (even for filters with no built-in amplifier). The **cut-off frequencies** ($\boldsymbol{f_l}$ and $\boldsymbol{f_h}$) occur where the gain has fallen 3 dB below the center frequency gain.

Figure 6-9 Band pass filter response



Instead of specifying roll off-rate, ***bandwidth*** and ***Q*** are given. Bandwidth is the distance between the low frequency and high frequency cut-offs.

$$\Delta f = f_{\mathrm{h}} - f_{\mathrm{l}}$$

Q is the ratio of center frequency to bandwidth.

$$Q = \frac{f_{\mathrm{c}}}{\Delta f}$$

This gives a measure of the sharpness or narrowness of the band pass filter. The higher the Q, the more selective the filter.

Simple band pass filters can be made with two RC pairs. This makes the transfer function second-order. However, since one pair creates the roll-off at high frequencies and the other pair handles the low frequencies, the eventual roll-off rate is half as steep as the same-order low pass filter or high pass filter.

Band pass filters are used in audio, communications, and instrumentation circuits. Equalizers and speech filters are audio band pass filters. Station tuning in broadcast radio and television uses band pass filters. Spectrum analyzers measure a circuit's frequency response with a band pass filter.

6-1.3 Active versus Passive Filters

The descriptions of Sections 6-1.1 and 6-1.2 apply, more or less, to filters in general, independent of how they are built. The simplest approach to building a filter is with passive components (resistors, capacitors, and inductors). In the radio frequency range this works well. However, as the frequency comes down, inductors begin to have problems. Audio frequency inductors are physically large, heavy, and therefore expensive. To increase inductance for the lower frequency applications, more turns of wire must be added. This adds to the series resistance, degrading the inductor's performance.

Both input and output impedances of passive filters are a problem. The input impedance may be low, which loads down the source, and varies with frequency. The output impedance may be high, which limits the load impedance that the filter can drive. There is no isolation between the load impedance and the passive filter. This means that the load must be considered as a component of the filter and must be taken into consideration when determining the filter's response or design. Any change in load impedance may significantly alter one or more of the filter's response characteristics.

Active filters incorporate an amplifier with resistor/capacitor networks to overcome these problems. Originally built with vacuum tubes and then with transistors, active filters now are normally centered around op amps. By enclosing a capacitor in a *positive* feedback loop, the inductor (with all of its low frequency problems) can be eliminated. If the op amp is properly configured, the input impedance can be increased. The load is driven from the output of the op amp, giving a very low output impedance. Not only does this improve load drive capability, but the load is now isolated from the frequency determining network. Variations in load have no effect on the active filter's characteristics.

The amplifier allows specification and easy adjustment of the pass band gain, A_o, pass band ripple, cut-off frequency,$f_{-3\ dB}$ and f_o, and **initial** roll-off. Because of the high input impedance of the op amp, large value resistors can be used. This reduces the value (size, cost, and nonideal behavior) of the capacitors. By selecting a quad op amp IC, filters can be built with steep roll-offs in very little space and for very little money.

Active filters also have limitations. High frequency response is limited by the gain bandwidth and slew rate of the op amp. High frequency op amps are more expensive, making passive filters a more economical choice for many rf applications. An op amp adds noise to any signal passing through it. So high quality audio applications may avoid those filter configurations that place the op amp in the main signal path. Active filters require a power supply. For op amps this may be two supplies. Variations in that power supply's output voltage show up, to some degree, in the signal output from the active filter. In multiple-stage applications, the common power supply provides a bus for high frequency signals. Feedback along these power supply lines can cause oscillations. Active devices, and therefore active filters, are much more susceptible to radio frequency interference and ionization than are passive RLC filters. Practical considerations limit the Q of the band pass and notch filters to less than 20. For circuits requiring very selective (narrow) filtering, a crystal filter may be a better choice.

6.2 The Sallen-Key Active Filter

There are a wide variety of ways to implement an active filter. In this section the response of a general Sallen-Key second-order circuit is developed. This introduces the concepts of damping and types of response. The equal component implementation is studied in more detail. Finally, filters of higher order filter are developed by cascade first- and second-order stages.

6-2.1 Second-Order Sallen-Key Filter

The first-order filter was shown in Example 6-2. With only rare exception, that simple, passive implementation is appropriate when a first-order (20 dB/decade) response is needed. The second-order filter consists of two RC pairs and has a roll-off rate of 40 dB/decade. At the critical frequency the phase is shifted $\pm 90°$. The denominator of the transfer function is a quadratic in s (s^2+as+b).

The schematic for the general second-order Sallen-Key filter is given in Figure 6-10. Impedances Z1, Z2, Z3, and Z4 may be either resistors or capacitors. This general form is being used so that the results are applicable to both low pass and high pass filters.

The output, v_ξ, is determined by the signal at the op amp's noninverting input and the amplifier's gain.

$$v_\xi = A_\xi v_b$$

$$A_\xi = 1 + \frac{R_{f\,\xi}}{R_{i\,\xi}}$$

So

$$v_b = \frac{v_\xi}{A_\xi}$$

Assuming that no current flows into the op amp,

$$i_4 = \frac{v_b}{Z4}$$

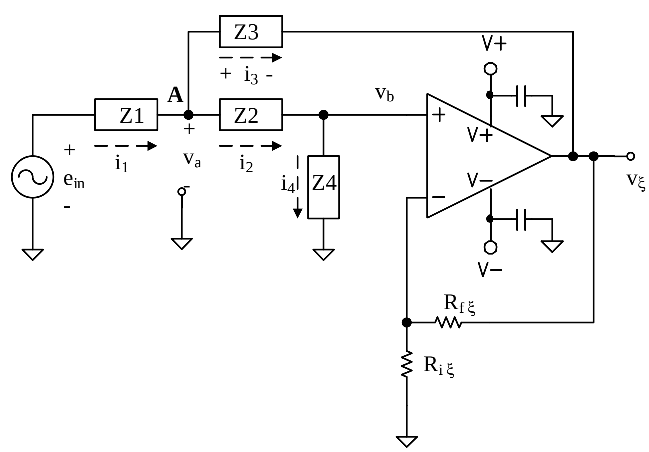


Figure 6-10 Second-order active filter model

$$i_2 = i_4 = \frac{v_b}{Z4}$$

At node A, $$v_a = i_4(Z2 + Z4)$$

Combining yields $$v_a = \frac{v_b}{Z4}(Z2 + Z4)$$

Current into the filter, i_1, is the difference in potential across Z1 divided by Z1.

$$i_1 = \frac{e_{in} - v_a}{Z1} = \frac{e_{in}}{Z1} - \frac{v_a}{Z1}$$

Substitute for v_a. $$i_1 = \frac{e_{in}}{Z1} - \frac{v_b(Z2 + Z4)}{Z1Z4}$$

The current through the feedback impedance, i_3, can be calculated by summing the currents at node A.

$$i_3 = i_1 - i_2$$

Combine the equations for i_1 and i_2.

$$i_3 = \frac{e_{in}}{Z1} - \frac{v_b(Z2 + Z4)}{Z1Z4} - \frac{v_b}{Z4}$$

Summing the loop from node **A**, Z3, and the output yields

$$v_a - i_3 Z3 - v_\xi = 0$$

$$v_\xi = v_a - i_3 Z3$$

Substitute for i_3.

$$v_\xi = \frac{v_b}{Z4}(Z2 + Z4) - \left[\frac{e_{in}}{Z1} - \frac{v_b(Z2 + Z4)}{Z1Z4} - \frac{v_b}{Z4}\right]Z3$$

Combine this with the initial relationship for the input to output voltages of the op amp.

$$v_\xi = \frac{v_\xi}{A_\xi Z4}(Z2 + Z4) - \frac{e_{in} Z3}{Z1} + \frac{v_\xi(Z2 + Z4)Z3}{A_\xi Z1Z4} + \frac{v_\xi Z3}{A_\xi Z4}$$

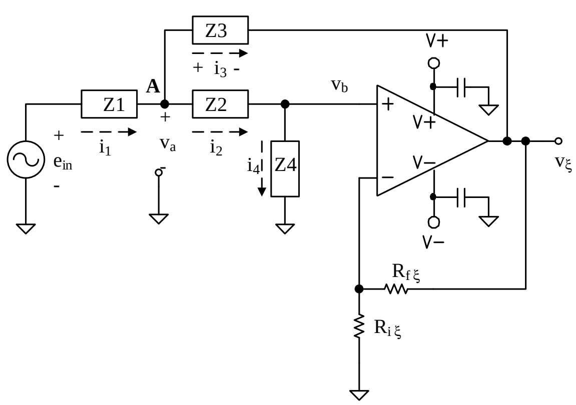


Figure 6-10 Second-order active filter model

This is an expression in v_ξ, e_{in}, and circuit components. The circuit analysis is complete. To obtain the transfer function, you have to manipulate this equation to group and separate terms, isolating v_ξ / e_{in} on the left side of the equation. When this is done, you have

$$\frac{v_\xi}{e_{in}} = \frac{A_\xi Z3\,Z4}{Z1Z2 + Z2Z3 + Z3Z4 + Z1Z3 + Z1Z4(1 - A_\xi)}$$

6-2.2 Second-Order Low Pass Sallen-Key Characteristics

To convert Figure 6-10 into a second-order low pass active filter, the resistors of the RC pairs must be in series with the main signal path, and the capacitors are tied to ground (or to the output which is only a few ohms from ground). This is shown in Figure 6-11.

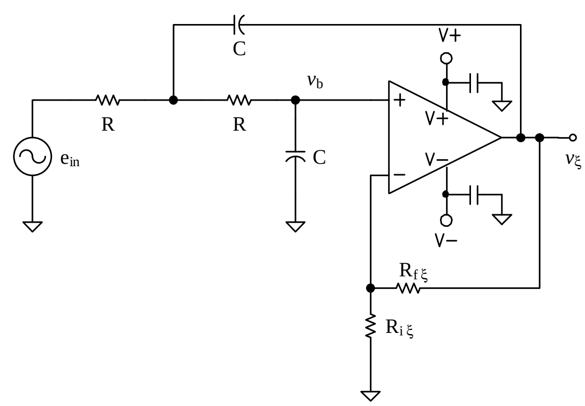


Figure 6-11 Second-order Sallen-Key low-pass active filter

Compare Figure 6-11 to Figure 6-10. The equation developed for Figure 6-10 applies to the low pass filter of Figure 6-11 if

$$Z1 = Z2 = R \qquad Z3 = Z4 = \frac{1}{Cs}$$

Making these substitutions, gives

$$\frac{V_\xi}{E_{in}} = \frac{\dfrac{A_\xi}{C^2 s^2}}{R^2 + \dfrac{R}{Cs} + \dfrac{1}{C^2 s^2} + \dfrac{R}{Cs} + \dfrac{R}{Cs}(1 - A_\xi)}$$

$$= \frac{\dfrac{A_\xi}{C^2 s^2}}{\dfrac{R^2C^2s^2 + RCs + 1 + RCs + RCs(1 - A_\xi)}{C^2 s^2}}$$

$$= \frac{A_\xi}{R^2C^2s^2 + 2RCs + RCs(1 - A_\xi) + 1}$$

$$\frac{V_\xi}{E_{in}} = \frac{A_\xi}{R^2C^2s^2 + RC[2 + (1 - A_\xi)]s + 1}$$

The quadratic in the denominator is more easily solved if the coefficient of s^2 is 1. Dividing numerator and denominator by R^2C^2, gives

$$\frac{V_\xi}{E_{in}} = \frac{\dfrac{A_\xi}{R^2C^2}}{s^2 + \left(\dfrac{3 - A_\xi}{RC}\right)s + \dfrac{1}{R^2C^2}}$$

Second-order systems were studied extensively in Chapters 4 and 5. Mechanical and chemical as well as electrical second-order systems behave similarly. One transfer function is

$$\frac{V_\xi}{E_{in}} = \frac{\dfrac{A_\xi}{R^2C^2}}{s^2 + \left(\dfrac{3 - A_\xi}{RC}\right)s + \dfrac{1}{R^2C^2}}$$

$$\frac{A_o \omega_o^2}{s^2 + \alpha\omega_o s + \omega_o^2}$$

where

A_o = the gain
ω_o = the critical frequency in rad/s
α = the damping coefficient

This looks very similar to the second-order system response transfer function from Chapters 4 and 5.

$$\frac{A_o \omega_o^2}{s^2 + 2\xi\omega_o s + \omega_o^2}$$

The damping factor (ξ) of the systems transfer function determines if the system is over-damped ($\xi > 1$), critically damped ($\xi = 1$), or under-damped ($\xi < 1$). Comparing the two functions gives

$$\alpha = 2\xi$$

Further comparisons of the Sallen-Key, equal component, low pass filter transfer function with the general second-order form reveals

$$\omega_o^2 = \frac{1}{R^2C^2} \qquad \omega_o = \frac{1}{RC}$$

$$f_o = \frac{1}{2\pi RC}$$

Critical frequency

$$A_o = A_\xi = 1 + \frac{R_{f\,\xi}}{R_{i\,\xi}}$$

Pass band gain

$$\alpha\omega_o = \frac{3 - A_\xi}{RC}$$

$$\alpha = 3 - A_\xi$$

Damping

The resistor and capacitor set the critical frequency. The gain of the op amp, A_ξ, sets *both* the filter's gain, A_o, *and* its damping, α. Changing one of these parameters changes the other. They are *not* independent. This is a big problem! But, that is where most Sallen-Key designs stop.

The solution is shown in Figure 6-12.

$$v_\xi = A_\xi \times v_b$$

$$v_b = \frac{v_\xi}{A_\xi}$$

$$V_\xi = E_{in} \times \frac{\dfrac{A_\xi}{R^2C^2}}{s^2 + \left(\dfrac{3 - A_\xi}{RC}\right)s + \dfrac{1}{R^2C^2}}$$

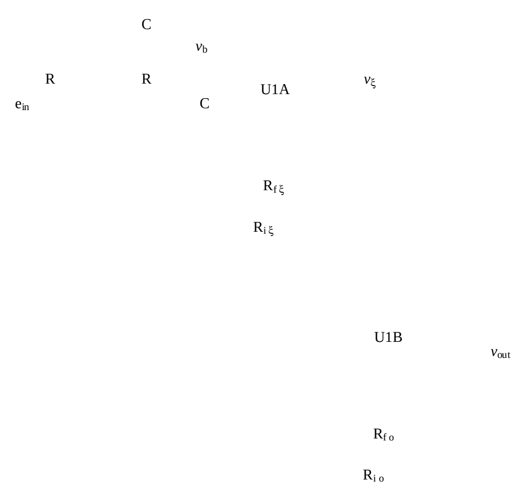


Figure 6-12 Sallen-Key with independent variable adjustment

$$v_b = E_{in} \times \frac{\frac{A_\xi}{R^2C^2}}{s^2 + \left(\frac{3 - A_\xi}{RC}\right)s + \frac{1}{R^2C^2}} \times \frac{1}{A_\xi}$$

$$v_b = E_{in} \times \frac{\frac{1}{R^2C^2}}{s^2 + \left(\frac{3 - A_\xi}{RC}\right)s + \frac{1}{R^2C^2}}$$

This voltage is sent to U1B, a noninverting amplifier with its gain set by R_{fo} and R_{io}.

$$v_{out} = v_b \left(1 + \frac{R_{fo}}{R_{io}}\right)$$

$$v_{out} = E_{in} \times \frac{\frac{1 + \frac{R_{fo}}{R_{io}}}{R^2C^2}}{s^2 + \left(\frac{3 - A_\xi}{RC}\right)s + \frac{1}{R^2C^2}}$$

$$\frac{v_{out}}{E_{in}} = \frac{\frac{A_o}{R^2C^2}}{s^2 + \left(\frac{3 - A_\xi}{RC}\right)s + \frac{1}{R^2C^2}}$$

Pass band gain
Set with R_{fo} and R_{io}

$$A_o = \left(1 + \frac{R_{fo}}{R_{io}}\right)$$

Critical frequency
Set with R and C

$$f_o = \frac{1}{2\pi RC}$$

Damping
Set with $R_{f\xi}$ and $R_{i\xi}$

$$\alpha = 3 - A_\xi \qquad A_\xi = \left(1 + \frac{R_{f\xi}}{R_{i\xi}}\right)$$

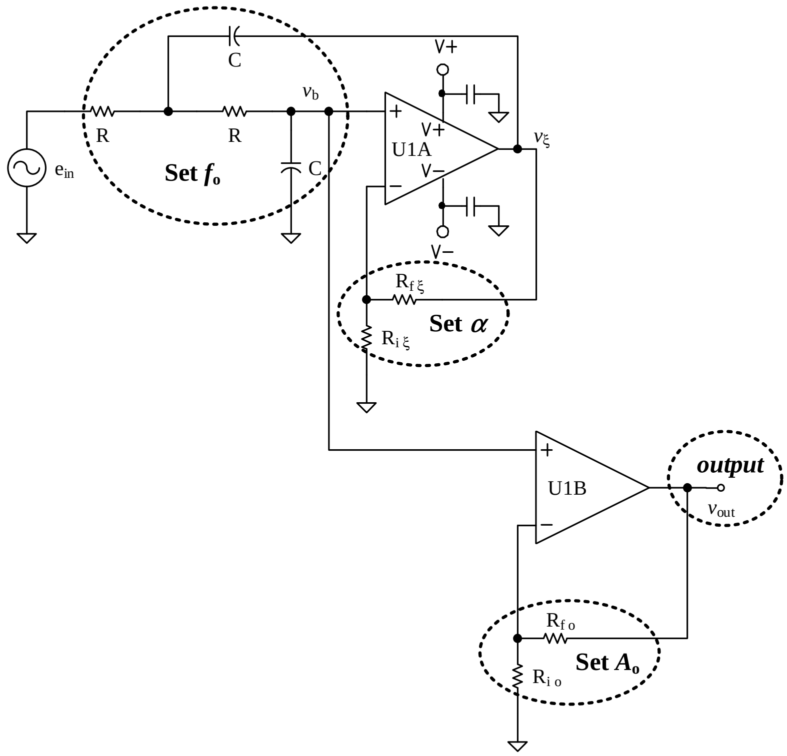


Figure 6-13 Sallen-Key with independent variable adjustment

The three most commonly used damping coefficients are listed in Table 6-3. The Bessel filter has the heaviest damping and as such is both the most stable and the slowest to respond. It is often used to filter pulses because it does not overshoot or ring. It also provides the best phase (time) delay for sinusoidal signals. However, noticeable roll-off begins at $0.3\omega_o$, causing attenuation of the upper end of the pass band.

The Butterworth response provides the flattest frequency response. As such, it is most often the choice for audio and instrumentation circuits. It is underdamped, so there will be some overshoot when a pulse is applied. However it gives reasonable initial roll-off.

The Chebyshev filter is more lightly damped than the Butterworth. This causes the gain to increase with frequency at the upper end of the pass band, actually rising as much as 3 dB above A_o. This provides much faster initial roll-off than the Bessel or Butterworth. But this lowered damping also means that there is considerable ringing in response to a step.

Table 6-3 Second-order filter parameters

$f_{-3\text{ dB}} = k_{lp}\, f_o$

Filter type	Damping (α)	Correction (k_{lp})
Bessel	1.732	0.785
Butterworth	1.414	1.000
3 dB Chebyshev	0.766	1.390

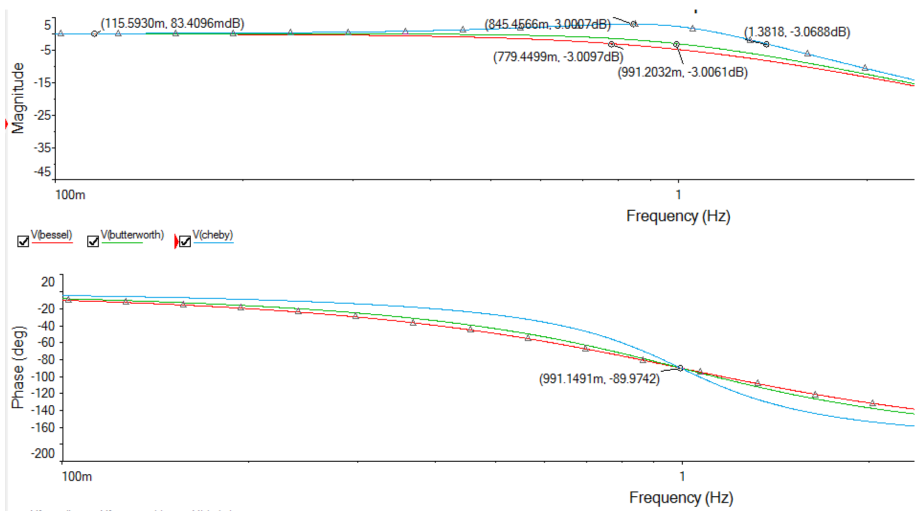


Figure 6-13 Sallen-Key responses for Bessel, Butterworth, and Chebyshev

The normalized frequency response are given in Figure 6-13. The damping coefficient determines the shape of the frequency response plot **near** the critical frequency.

All three responses have the same f_o = 1 Hz, where there is a -90^o phase shift.

The Butterworth filter's gain is down -3 dB at 1 Hz. But the Bessel filter's gain falls off early, and the Chebyshev filter peaks *up* + 3dB before falling. Finally, it falls -3 dB at 1.38 Hz.

That is where the k_{lp} correction factor in Table 6-3 comes in. It relates $f_{-3\text{ dB}}$ and f_o, depending on the damping coefficient.

Example 6-3

Design a Sallen-Key, equal component, second-order, low-pass filter to meet the following specifications:

f_{-3dB} = 2 kHz A_o = 5   flattest possible pass band

Solution

Since the flattest possible pass band is required, you must use a Butterworth implementation.

$$f_o = \frac{1}{2\pi RC}$$

$$f_{-3dB} = k_{lp} f_o = \frac{k_{lp}}{2\pi RC}$$

$$R = \frac{k_{lp}}{2\pi f_{-3dB} C}$$

Pick C = 10 nF.

$$R = \frac{1}{2\pi \times 2\,kHz \times 0.01\mu.} = 7.96k\Omega$$

Since this is not close to a standard value resistor, pick another standard capacitor, and repeat the calculation until you have discovered a standard capacitor and a standard resistor (in the 10kΩ range) which produce f_{-3dB} = 2 kHz.

Pick C = 6.8 nF.

$$R = \frac{1}{2\pi \times 2\,kHz \times 6.8\,nF} = 11.7\,k\Omega$$

Pick R = 12 kΩ.

For the Butterworth filter,

$$A_\xi = 3 - \alpha = 3 - 1.414 = 1.586$$

$$A_\xi = 1 + \frac{R_{f\,\xi}}{R_{i\ x}} = 1.586$$

$$\frac{R_{f\,\xi}}{R_{i\,\xi}} = 0.586 \quad \text{or} \quad R_{f\,\xi} = 0.586\ R_{i\ \xi}$$

Pick $R_{i\,\xi} = 10\ \mathrm{k\Omega}$ $\qquad R_{f\,\xi} = 5.8\ \mathrm{k\Omega}$

$$A_o = \left(1 + \frac{R_{f\,o}}{R_{i\,o}}\right) = 5$$

$$\frac{R_{f\,o}}{R_{i\,o}} = 4$$

Pick $R_{i\,o} = 20\ \mathrm{k\Omega}$ $\quad R_{f\,o} = 80\ \mathrm{k\Omega}$ (82 kΩ is standard)

The Multisim schematic and AC Sweep analysis results are shown in Figure 6-14.

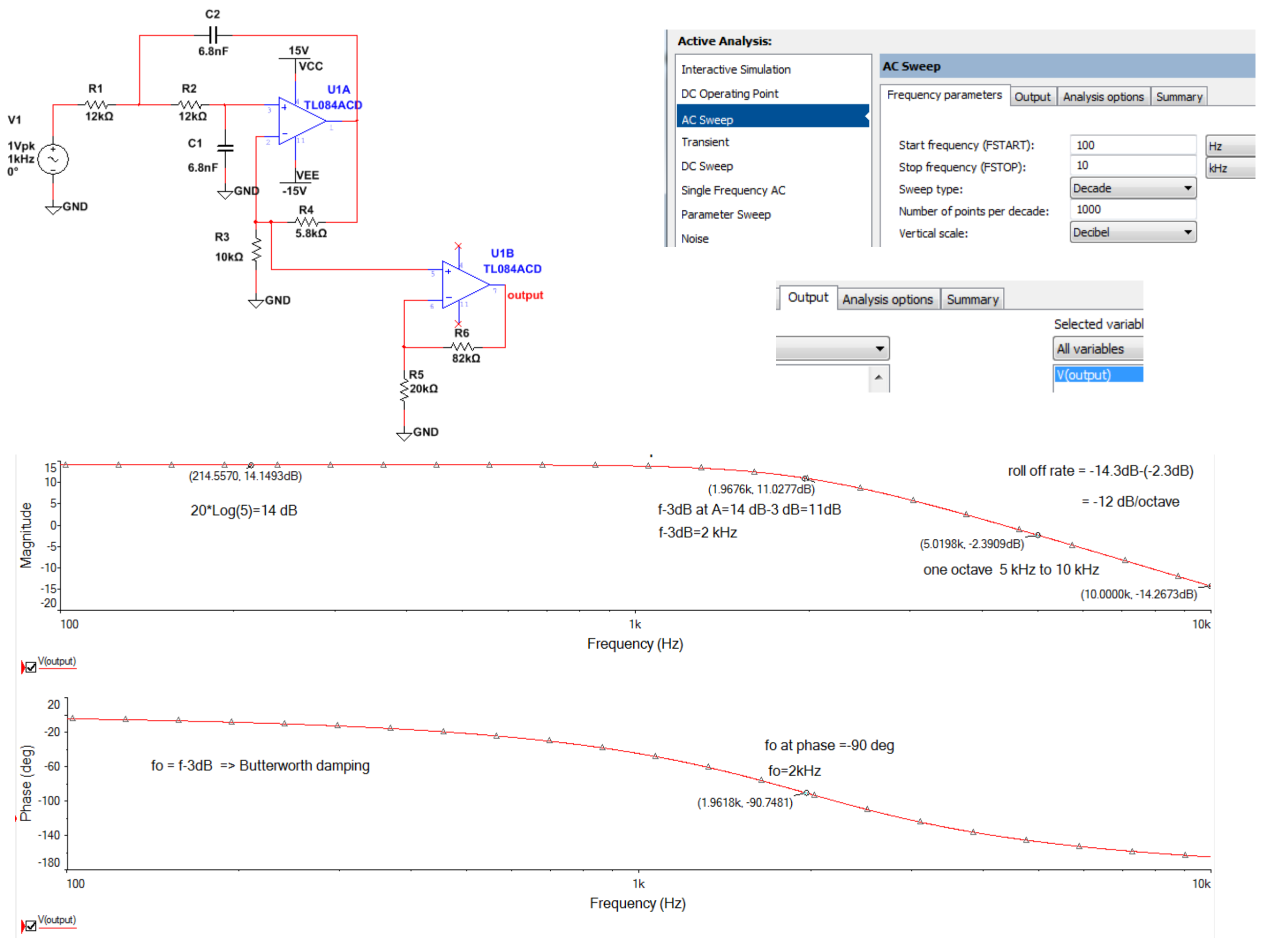


Figure 6-14 Example 6-3 simulation results

The schematic and the AC Sweep set up are in the upper half of the figure. The AC analysis shown in the lower half indicates that $A_o = 20\ \mathrm{Log}_{10}(5) = 14$ dB. f_{-3dB} occurs where the gain has fallen -3 dB from 14 dB to 11 dB. This happens at 2 kHz. An octave is a change from 5 kHz to 10 kHz. The gain falls -12 dB. On the phase plot, the shift is -90º at 2 kHz. This is f_o. So this is, indeed, a Butterworth filter, because $f_o = f_{-3dB}$.

6-2.3 Higher-Order Low Pass Sallen-Key Characteristics

The first-order and second-order filters, though relatively easy to build, may not provide an adequate roll-off rate. The only way to improve the roll-off rate is to increase the order of the filter as illustrated in Figure 6-7. Each increase in order produces a 20dB/decade or 6dB/octave increase in the roll off-rate.

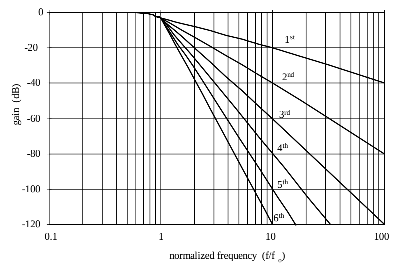


Figure 6-7 Roll-off rate comparison

Higher-order filters can be built by cascading the proper number of first- and second-order filter sections. For a fifth-order filter, this technique results in a transfer

$$\frac{A_o}{\underbrace{\left(s^2 + \alpha_1 s + \omega_1^2\right)}_{\text{second-order section}}\underbrace{\left(s^2 + \alpha_2 s + \omega_2^2\right)}_{\text{another second-order section}}\underbrace{\left(s + \omega_3\right)}_{\text{first-order section}}}$$

Each term in the denominator has its own damping coefficient and critical frequency. To obtain a given, well-defined response (Bessel, Butterworth, or Chebyshev), the transfer function, **as a whole,** must be solved and the appropriate coefficients determined.

$$\frac{A_o}{s^5 + as^4 + bs^3 + cs^2 + ds + e}$$

It is unreasonable to expect the α's and ω's of the cascaded filter transfer function to correlate in a simple way with the coefficients of the overall filter. You do **not** get a fifth-order, 1kHz Bessel filter by cascading two 1kHz, second-order Bessel filters and a first -order, RC passive stage.

The mathematics used to solve these higher-order polynomials is beyond the scope of this book. The results are presented in Table 6-4.