9

Active Filters

Electric filters are used in practically all circuits that require the separation of signals according to their frequencies. Applications include (but are certainly not limited to) noise rejection and signal separation in industrial and measurement circuits, feedback of phase and amplitude control in servoloops, smoothing of digitally generated analog (D-A) signals, audio signal shaping and sound reinforcement, channel separation, and signal enhancement in communications circuits.

Such filters can be built from passive RLC components, electromechanical devices, crystals, resistors, capacitors, and op amps (active filters). Active filters are applicable over a wide range of frequencies. They are also inexpensive and offer high input impedance, low output impedance, adjustable gain, and a variety of responses.

In this chapter, you will learn the characteristics, terminology, and mathematics of active filters. The analysis, design, and calculations behind several types of low and high pass filters, wide and narrow band pass filters, the notch filter, and the state variable filter are presented. Electronically controlled active filters allow adaptive and automatic filtering. You will see how to build filters that can be adjusted with an analog voltage, a data word, or a clock frequency.

Objectives

After studying this chapter, you should be able to do the following:

1. Describe in detail a frequency response plot.
2. Convert between ratio and dB gain.
3. Define the f_o, and f_{-3dB} and locate each on a frequency response plot.
4. Briefly state the purpose of Laplace transforms.
5. Analyze a wide variety of circuits to determine their transfer function, gain versus frequency response, and phase versus frequency response.
6. Draw the ideal and practical frequency response plots for a low pass filter, high pass filter, band pass filter, and notch filter.
7. Identify the stop band and the pass band of a given filter response.
8. Describe the roll-off rate, dB/decade, and dB/octave, and their relationship to filter order.

9. Given the response curve of a band pass or notch filter, determine the center frequency, low frequency cut-off, high frequency cut-off, bandwidth, Q, and notch depth.
10. Describe one application for the low pass, high pass, band pass, and notch filters.
11. List four disadvantages of passive filters and explain how active filters overcome them.
12. List four disadvantages of active filters.
13. Derive the transfer function and gain and phase equations for a first- and a second-order low pass, high pass, band pass, notch, voltage controlled, and state variable active filter.
14. Given filter specifications, determine type, order, and component values necessary to build the filter using a Sallen-Key, equal component, band pass, notch, state variable, voltage controlled or switched capacitor digitally controlled implementation.
15. Analyze a given active filter to determine f_o, f_{-3dB}, response shape, roll-off rate, and pass band gain.
16. List advantages, disadvantages, limitations, and precautions for designing and building each of these filters.

9-1 Introduction to Filtering

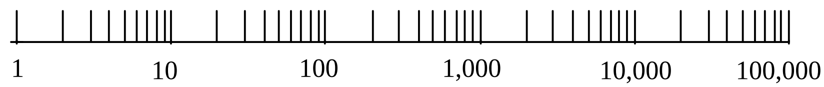
Filters are specified, analyzed, and designed somewhat differently than are the circuits discussed in the previous chapters. Performance is specified in terms of the frequency response, that is, how the gain and phase shift change with frequency. Analysis and design often use Laplace transforms and the circuit's transfer function. Angular frequency, f_o, f_{3dB}, bands, ripple, roll-off rate, center frequency, and Q are among the terms unique to filters. In this section, frequency response plots, basic transform math, and terminology common to most filters are presented. Later sections of the chapter apply these fundamentals to specific filter circuits.

9-1.1 Frequency Response

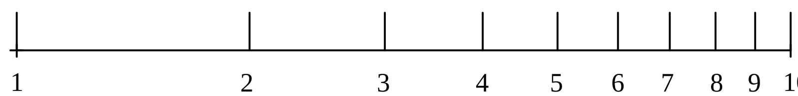
The gain and phase shift of a filter change as the frequency changes. Indeed, this is the purpose of a filter. Performance is often described with a graph of gain and phase versus frequency.

You have already seen such a frequency response when investigating the frequency response of an op amp, the gain bandwidth. The vertical axis is the gain, and the horizontal axis is the frequency. The frequency axis gives equal distance for each **decade** of frequency. The distance between 1Hz and 10Hz is the same as the distance between 100kHz and 1MHz. On a linear plot, the distance between 100kHz and 1MHz should be 100,000 larger. However, this would make it impossible to plot any large range of frequencies on a single graph, so the horizontal axis is scaled logarithmically.

A log scaled horizontal axis is shown in Figure 9-1. In Figure 9-1(a), each decade (factor of 10) increase or decrease moves you the same distance along the scale. There are two other key ideas. First, the starting point is not at zero. If you went farther left, each increment would lower the scale by 10 (to 0.1, 0.01, 0.001, …), but you would never reach zero. So, start your log divisions at the lowest frequency of interest, not zero (DC). Second, the divisions between decades are not uniform (linear). This is easily seen in Figure 9-1(b), an expansion of a single decade. Five does not fall halfway between 1 and 10; $3^1/_3$ does. Be very careful about this when interpolating between major scale divisions.



(a) 5 decade log scale



(b) 1 logarithmic decade expanded

Figure 9-1 Logarithmic scales

Gain, plotted on the vertical axis, is normally expressed in dB.

$$\mathrm{dB} = 10\ \log_{10} \frac{\text{power out}}{\text{power in}}$$

$$\mathrm{dB} = 10\ \log \frac{\dfrac{v_o^2}{R_{load}}}{\dfrac{e_{in}^2}{R_{in}}}$$

$$\mathrm{dB} = 10\ \log \left(\frac{v_o}{e_{in}}\right)^2 \left(\frac{R_{in}}{R_{load}}\right)$$

If you assume that

$$R_{in} = R_{load}$$

then

$$dB = 10 \log\left(\frac{v_o}{e_{in}}\right)^2$$

$$dB = 20 \log\frac{v_o}{e_{in}}$$

For this to be valid, the load resistance must equal the filter's input resistance. This is seldom the case. However, the definition is normally used anyway.

You can readily convert ratio gain (v_o/e_{in}) to dB gain with any scientific calculator. Notice that log base 10 is used, not the natural log (log base e or ln). It is handy to know several dB and ratio points. These are listed in Table 9-1. Decreasing the ratio gain by 10 subtracts 20dB; increasing it by a ratio of 10 adds 20dB. Doubling the gain adds 6dB; halving it subtracts 6dB. Cutting the gain by $\sqrt{2}$ gives -3dB. A ratio of 1 ($v_o = e_{in}$) is a gain of 0dB.

Table 9-1 Ratio and dB gain comparison

v_o/e_{in}	dB	v_o/e_{in}	dB
1000	60	0.707	-3
100	40	0.5	-6
10	20	0.1	-20
2	6	0.01	-40
1	0	0.001	-60

A typical low pass filter frequency response is given in Figure 9-2. The vertical axis is scaled in decibels. The highest gain is called the **pass band gain**, $\mathbf{A_o}$. In Figure 9-2, $A_o = +10dB$.

The horizontal axis is scaled logarithmically (two decades in the figure). By plotting dB versus log frequency, you are actually plotting log gain versus log frequency. So any linear relationship is displayed as a straight line.

A parameter of major importance is the **-3dB frequency**, f_{-3dB}. At that frequency, the gain has fallen 3dB below A_o. In Figure 9-2, the pass band gain A_o=10dB. At f_{-3dB} the gain falls to 7dB.

$$A_o = 10dB$$

$$f_{-3dB} = 2500Hz$$

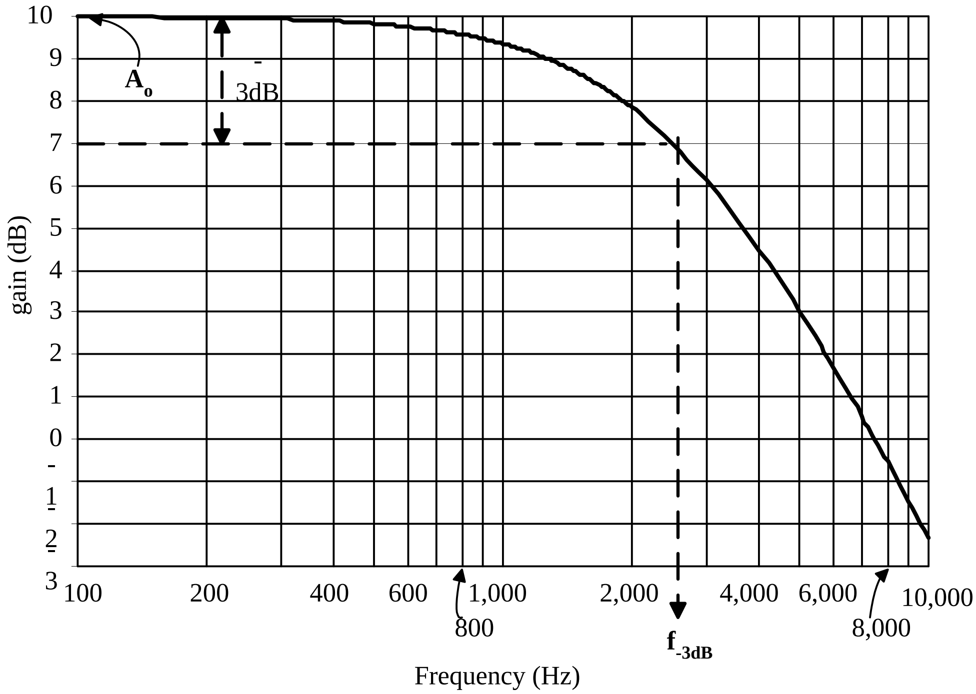


Figure 9-2 Typical low pass response curve

What does this -3dB reduction in the voltage gain mean about the power delivered to the load?

$$-3\text{dB} = 20\ \log(A_v)$$

$$A_v = 10^{-\frac{3}{20}} = 0.707$$

The voltage gain has fallen by 0.707 below its value at A_o. This means at f_{-3dB} the output voltage has fallen by 0.707.

$$v_{out\ -3dB} = 0.707 v_{out\ @Ao}$$

$$P_{out@f-3dB} = \frac{v^2}{R} = \frac{\left(0.707v_{out@Ao}\right)^2}{R}$$

$$P_{out@f-3dB} = \frac{0.5v^2_{out@Ao}}{R} = \tfrac{1}{2}P_{out@Ao}$$

Consequently, at the -3dB frequency, the power delivered to the load has been cut in half, and the voltage gain has fallen by 0.707.

As frequency changes, not only does the filter's output amplitude change, but its input-output phase relationship also shifts. At the **critical frequency**, $\mathbf{f_o}$, that shift is an integer multiple of 45°. For simple filters built with a single RC pair, there is a 45° shift in phase. For each additional pair you add to the circuit, the phase shifts another 45° at f_o. Two RC pairs produce a 90° shift at the critical frequency; three pairs cause a 135° shift at f_o.

In many filters the -3dB frequency, f_{-3dB}, and the critical frequency, f_o, occur at the same point. But two different effects are occurring at that frequency. The voltage gain has dropped -3dB (0.707), f_{-3dB}, and the phase has shifted the correct number of 45°, f_o. However, when you use an op amp to add gain and positive feedback to a filter, these two frequencies no longer occur at the same point.

A typical phase plot combined with the gain plot is given in Figure 9-3. The combination of the gain variation and the phase shift as a function of frequency forms a complete frequency response plot. Not only is plotting both functions on the same graph convenient, but it also allows you to determine if the filter (or any system) you anticipate building will be stable (i.e., not oscillate).

Analysis of circuits containing several reactive components is most easily handled by using Laplace transforms. It is far beyond the scope of this text to present a rigorous coverage of Laplace transform circuit analysis. However, a simplified touch is useful.

When the Laplace transform of an equation (or a circuit) is used, differential and integral terms are replaced by s and 1/s. You can then manipulate and simplify the equation using the rules of algebra rather than those of differential equations.

As far as active filters are concerned, the primary term is

$$i_C = C\frac{dv}{dt}$$

To take the Laplace transform of this, just replace the derivative with s. Uppercase letters are used to indicate that they belong to the Laplace domain. Lower case letters are used for variables in the time (normal) domain.

$$I = CsV$$

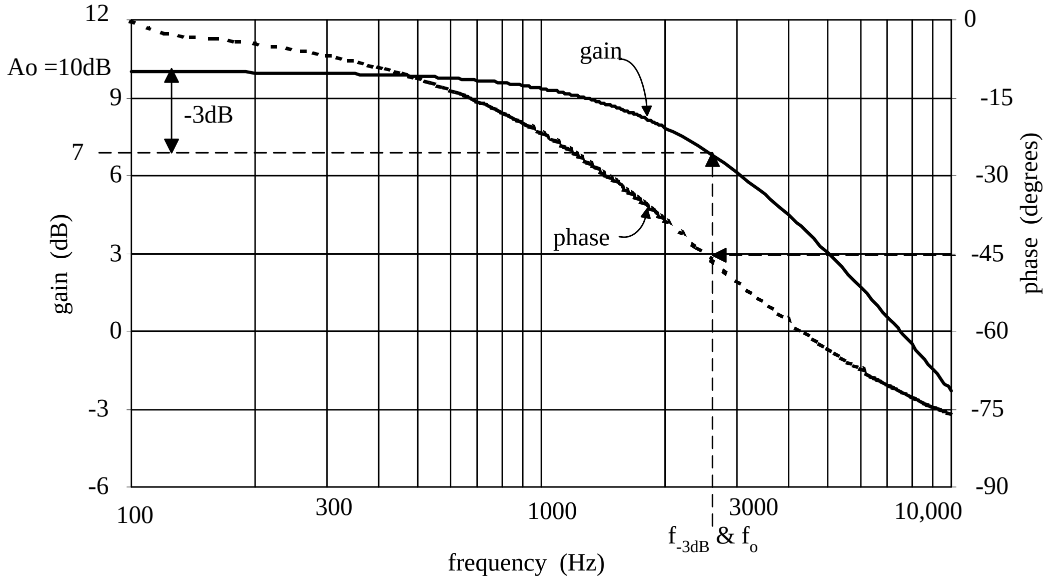


Figure 9-3 Composite gain and phase frequency response

The Laplace impedance, then, for a capacitor is

$$Z = \frac{V}{I} = \frac{1}{Cs}$$

The term s contains both the amplitude and the phase information about an equation's or a circuit's response. To obtain the frequency response from a Laplace equation, make the substitution

$$s = j\omega$$

where j is the imaginary number $\sqrt{-1}$ and $\omega = 2\pi f$.

Example 9-1

Verify that the Laplace form of capacitive impedance correctly converts back to the frequency domain.

Solution

$$Z = \frac{1}{Cs}$$

To convert from the Laplace domain (equation containing s) to the frequency domain (equation containing f or ω):

$$s = j\omega$$

$$\overline{Z} = \frac{1}{j\omega C}$$

$$\overline{Z} = \frac{1}{j2\pi fC}$$

$$\overline{Z} = 0 - j\frac{1}{2\pi fC}$$

$$\overline{Z} = 0 - jX_C$$

This is the phasor impedance of a capacitor, as you learned from AC circuits.

Using Laplace functions, you can determine the frequency response (both gain and phase) of a circuit. You must first determine the circuit's transfer function. The transfer function is just the gain (V_{out}/E_{in}) expressed in Laplace terms.

Example 9-2

a. Determine the transfer function of the circuit in Figure 9-4.

b. Calculate and plot the frequency response (both magnitude and phase) for R = 637Ω and C = 0.1μF.

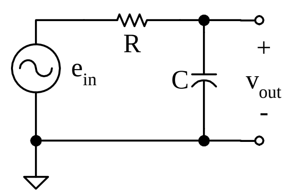


Figure 9-4 Circuit for Example 9-2

Solution

a. To determine the transfer function, apply the voltage divider law.

$$\overline{v_{out}} = \frac{\overline{Z_C}}{R + \overline{Z_C}} \overline{e_{in}}$$

Substitution:

$$\overline{e_{in}} \to E_{in} \qquad R \to R \qquad \overline{Z_C} \to \frac{1}{Cs} \qquad \overline{v_{out}} \to V_{out}$$

$$V_{out} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} E_{in}$$

$$\frac{V_{out}}{E_{in}} = \frac{\frac{1}{Cs}}{\frac{RCs+1}{Cs}}$$

$$\frac{V_{out}}{E_{in}} = \frac{1}{RCs+1}$$

b. To determine the frequency response, substitute

$$s = j\omega$$

$$\frac{V_{out}}{E_{in}} = \frac{1}{Rj\omega C + 1} = \frac{1}{1 + j\omega RC}$$

This is an equation with real and imaginary parts in its denominator. To separate these parts, multiply both the numerator and the denominator by the complex conjugate of the denominator.

$$\frac{V_{out}}{E_{in}} = \frac{1}{1 + j\omega RC} \times \frac{1 - j\omega RC}{1 - j\omega RC}$$

$$= \frac{1 - j\omega RC}{1 - j^2\omega^2 R^2 C^2}$$

$$= \frac{1 - j\omega RC}{1 + \omega^2 R^2 C^2}$$

$$= \frac{1}{1+\omega^2R^2C^2} - j\frac{\omega RC}{1+\omega^2R^2C^2}$$

$$\frac{V_{out}}{E_{in}} = \text{real} + j\ \text{imaginary}$$

$$\text{real} = \frac{1}{1+\omega^2R^2C^2} \qquad \text{imaginary} = \frac{-\omega RC}{1+\omega^2R^2C^2}$$

$$\left|\overline{G}\right| = \text{magnitude} = \sqrt{\text{real}^2 + \text{imaginary}^2}$$

$$= \sqrt{\left(\frac{1}{1+\omega^2R^2C^2}\right)^2 + \left(\frac{\omega RC}{1+\omega^2R^2C^2}\right)^2}$$

$$= \sqrt{\frac{1+\omega^2R^2C^2}{\left(1+\omega^2R^2C^2\right)^2}}$$

$$\left|\overline{G}\right| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$$

$$\phi = \text{phase shift} = \arctan\left(\frac{\text{imaginary}}{\text{real}}\right)$$

$$= \arctan\left(\frac{\dfrac{-\omega RC}{1+\omega^2R^2C^2}}{\dfrac{1}{1+\omega^2R^2C^2}}\right)$$

$$\phi = -\arctan(\omega RC)$$

Now that you have the two equations, the simplest next step is to use a spreadsheet to tabulate frequency, magnitude, and phase; then create the plot. It is similar to Figure 9-3. However, this circuit has a pass band gain, A_o, of 0dB, not the 10dB shown in Figure 9-3.

The techniques of Example 9-2 can be used to determine the frequency response of any network. However, each RC combination in the circuit increases the order of the denominator by 1. Two RC pairs cause $s^2 + bs + c$; three RC pairs cause s^3+bs^2+cs+d. Fortunately, the mathematics involved in breaking down and solving these higher-order equations has already been worked out for the popular, more useful circuits. Your job is to obtain the transfer function and recognize and extract key parameters. Actually, the major mathematical effort to convert back from the Laplace domain into the frequency domain may not have to be done at all.

9-1.2 Characteristics and Terminology

The frequency response that you have seen so far is for a low pass filter. The purpose of a low pass filter is to pass low frequency signals while stopping high frequency signals. Ideally, a low pass filter would have a frequency response as shown in Figure 9-5. All frequencies below the critical frequency would be uniformly passed. Any frequency above f_o would be completely stopped.

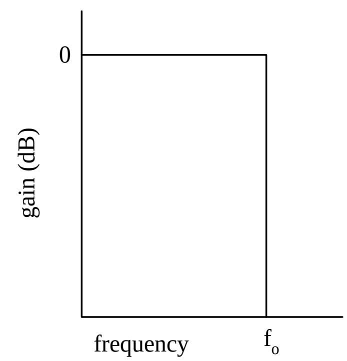


Figure 9-5 Ideal low pass filter

Of course, such a filter cannot be built. The response of a practical filter is divided into two bands as shown in Figure 9-6. The -3dB frequency, f_{-3dB}, forms the boundary between the pass band and the stop band. For some filters, gain may vary up and down (or ripple) in the pass band or in the stop band, or both. The amount of pass band ripple allowable (and, to a lesser degree, stop band ripple) is an important parameter to keep in mind when you are designing a filter. How rapidly the gain falls as the stop band is entered is called the roll-off. The first six roll-off rates are illustrated in Figure 9-7. The roll-off rate is determined by the filter's order (1, 2, 3, …). Each increase in order increases the roll-off by 20dB/decade. In turn, as was mentioned in the Laplace transform section, the order of the filter (and its transfer function's denominator) equals the number of resistor/capacitor pairs in the circuit. So increasing the circuit's complexity by adding an RC pair increases the circuit's order, and the difficulties of the mathematics, but also increases the roll-off rate by 20dB/decade.

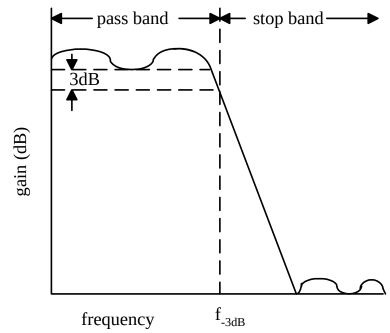


Figure 9-6 Practical low pass filter

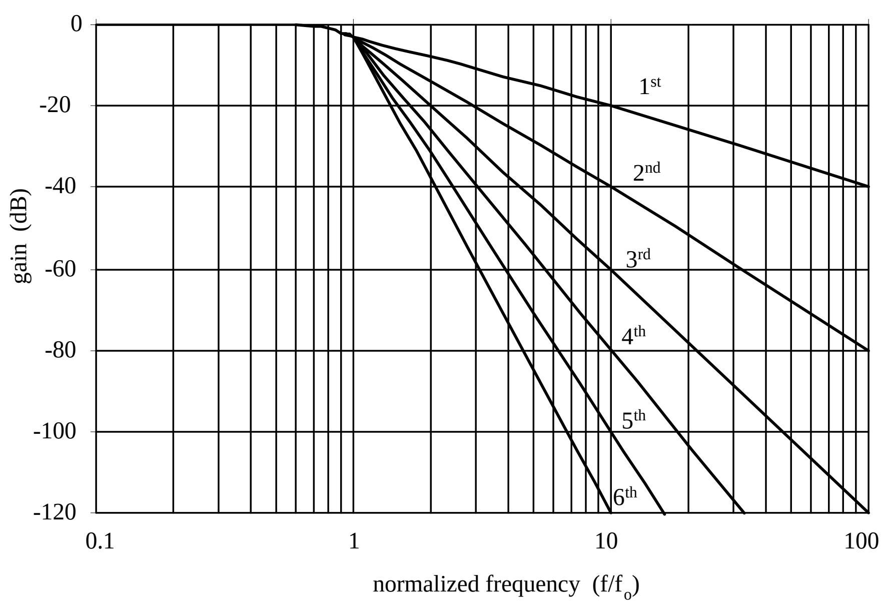


Figure 9-7 Roll-off rate comparison

You will hear roll off specified in dB/decade and dB/octave. A decade increase in frequency means that the frequency has changed by a factor of ten. An octave increase means that the frequency has doubled. Decibels per octave ratings are used primarily with audio/music applications. Table 9-2 correlates filter order (number of RC pairs and transfer function denominator order) with the roll off rate in dB/decade and dB/octave.

Table 9-2 Roll-off rate comparison

Order	dB/decade	dB/octave
1	20	6
2	40	12
3	60	18
4	80	24
5	100	30

The opposite of the low pass filter is the high pass filter. A high pass filter is illustrated in Figure 9-8. Low frequency signals and DC (which is 0Hz) are blocked, while high frequency signals are passed. Specifications, analysis, and design of high pass filters are closely analogous to those you have already seen for low pass filters.

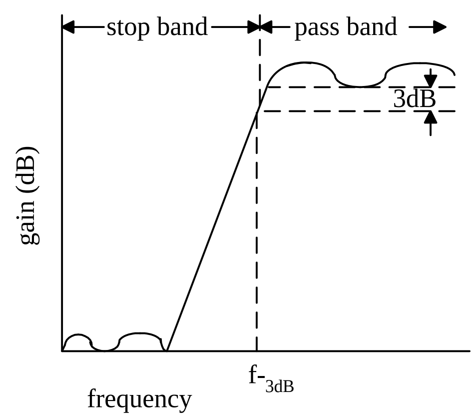
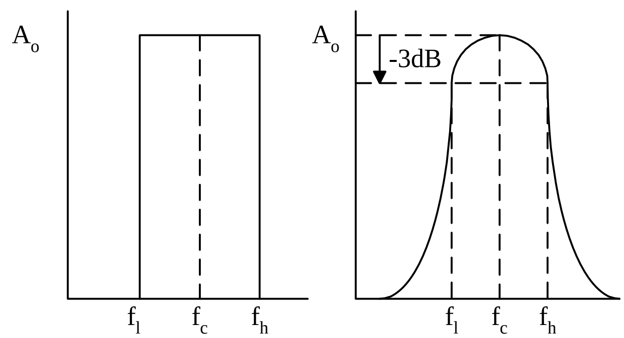


Figure 9-8 High pass filter

You have already used a simple high pass filter. The couplers used in single supply op amp amplifiers block the DC bias from previous stages or the signal generator, while passing the signal whose frequency is of interest. This RC coupler is the complement of the circuit analyzed in Example 9-2. With a single RC pair, the RC coupler has a 20dB/decade roll-off and a 45° phase shift at the critical frequency.

The band pass filter passes only those signals within a given band. Signals above and below that band are blocked. Figure 9-9(a) is the response of an ideal band pass filter, while Figure 9-9(b) is a more realistic response. Since the response rises, peaks, and then falls, there are three frequencies of interest. The **center frequency** is $\mathbf{f_c}$. Depending on the component configuration and values, there may be considerable gain at the center frequency (even for filters with no built-in amplifier). The **cut-off frequencies** ($\mathbf{f_l}$ and $\mathbf{f_h}$) occur where the gain has fallen 3dB below the center frequency gain.



(a) Ideal response (b) Actual response

Figure 9-9 Band pass filter response

Instead of specifying roll off-rate, **bandwidth** and **Q** are given. Bandwidth is the distance between the low frequency and high frequency cut-offs.

$$\Delta f = f_h - f_l$$

Q is the ratio of center frequency to bandwidth.

$$Q = \frac{f_c}{\Delta f}$$

This gives a measure of the sharpness or narrowness of the band pass filter. The higher the Q, the more selective the filter.

Simple band pass filters can be made with two RC pairs. This makes the transfer function second-order. However, since one pair creates the roll-off at high frequencies and the other pair handles the low frequencies, the eventual roll-off rate is half as steep as the same-order low pass filter or high pass filter.

Band pass filters are used in audio, communications, and instrumentation circuits. Equalizers and speech filters are audio band pass filters. Station tuning in radio and television uses band pass filters. Spectrum analyzers measure a circuit's frequency response with a band pass filter.

The notch filter is the complement of the band pass filter. It rejects those signals in a given band of frequencies and passes all others. The response of a notch filter is illustrated in Figure 9-10. As with the band pass filter, center frequency, low frequency cut-off, and high frequency cut-off are specified. Both bandwidth and Q are defined for the notch as they were for the band pass filter. The other specification needed is **notch depth**, which indicates how severely the signal at the center frequency is rejected.

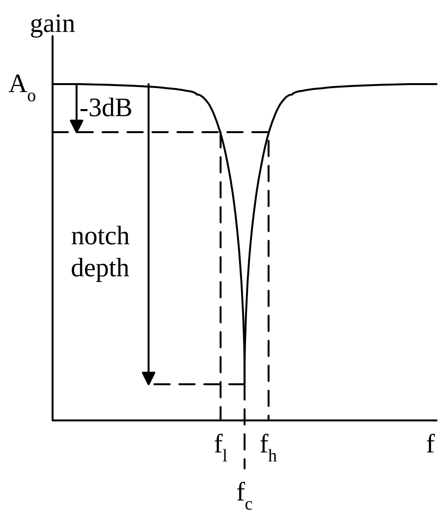


Figure 9-10 Notch filter response

9-1.3 Active versus Passive Filters

The descriptions of Sections 9-1.1 and 9-1.2 apply, more or less, to filters in general, independent of how they are built. The simplest approach to building a filter is with passive components (resistors, capacitors, and inductors). In the radio frequency range this works well. However, as the frequency comes down, inductors begin to have problems. Audio frequency inductors are physically large, heavy, and therefore expensive. To increase inductance for the lower frequency applications, more turns of wire must be added. This adds to the series resistance, degrading the inductor's performance.

Both input and output impedances of passive filters are a problem. The input impedance may be low, which loads down the source, and varies with frequency. The output impedance may be high, which limits the load impedance that the filter can drive. There is no isolation between the load impedance and the passive filter. This means that the load must be considered as a component of the filter and must be taken into consideration when you determine filter response or design. Any change in load impedance may significantly alter one or more of the filter's response characteristics.

Active filters incorporate an amplifier with resistor/capacitor networks to overcome these problems. Originally built with vacuum tubes and then with transistors, active filters now are normally centered around op amps. By enclosing a capacitor in a positive feedback loop, the inductor (with all of its low frequency problems) can be eliminated. If the op amp is properly configured, the input impedance can be increased. The load is driven from the output of the op amp, giving a very low output impedance. Not only does this improve load drive capability, but the load is now isolated from the frequency determining network. Variations in load have no effect on the active filter's characteristics.

The amplifier allows you to specify and easily adjust pass band gain, pass band ripple, cut-off frequency, and **initial** roll-off. Because of the high input impedance of the op amp, large value resistors can be used. This allows you to reduce the value (size, cost, and nonideal behavior) of the capacitors. By selecting a quad op amp IC, you can build steep roll-offs in very little space and for very little money.

Active filters also have limitations. High frequency response is limited by the gain bandwidth and slew rate of the op amp. High frequency op amps are more expensive, making passive filters a more economical choice for many rf applications. An op amp adds noise to any signal passing through it. So high quality audio applications may avoid those filter configurations that place the op amp in series with the main signal path. Active filters require a power supply. For op amps this may be two supplies. Variations in that power supply's output voltage show up, to some degree, in the signal output from the active filter. In multiple-stage applications, the common power supply provides a bus for high frequency signals. Feedback along these power supply lines can cause oscillations. Active devices, and therefore active filters, are much more susceptible to radio frequency interference and ionization than are passive RLC filters. Practical considerations limit the Q of the band pass and notch filters to less than 20. For circuits requiring very selective (narrow) filtering, a crystal filter may be a better choice.

9-2 Low Pass Filter

There are a wide variety of ways to implement a low pass filter. In this section you will first develop the response of a general Sallen-Key second-order circuit. This introduces the concepts of damping and types of response. The equal component implementation will be studied in more detail. Finally, you will see how to determine what order filter is required by a certain applications and how to cascade first- and second-order stages to build the required filter.

9-2.1 Second-Order Sallen-Key Filter

You saw the first-order filter in Example 9-2. With only rare exception, that simple, passive implementation is appropriate when you need a first-order (20dB/decade) response. The second-order filter consists of two RC pairs and has a roll-off rate of 40dB/decade. At the critical frequency the phase is shifted -90°. The denominator of the transfer function is a quadratic in s (s^2+as+b).

The schematic for the general second-order Sallen-Key filter is given in Figure 9-11. Impedances Z1, Z2, Z3, and Z4 may be either resistors or capacitors. This general form is being used so that the results are applicable to low pass or high pass filters.

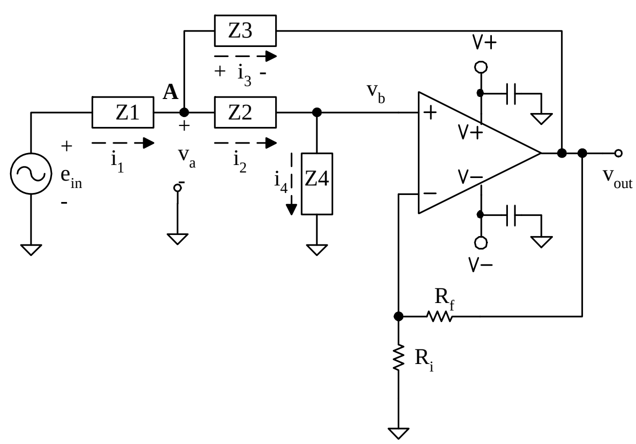


Figure 9-11 Second-order active filter model

The output, v_{out}, is determined by the signal at the op amp's noninverting input and the amplifier's gain.

$$v_{out} = A_o v_b$$

$$A_o = 1 + \frac{R_f}{R_i}$$

So $$v_b = \frac{v_{out}}{A_o}$$

Assuming that no current flows into the op amp,

$$i_4 = \frac{v_b}{Z4}$$

$$i_2 = i_4 = \frac{v_b}{Z4}$$

At node A, $$v_a = i_4\left(Z2 + Z4\right)$$

Combining these yields $$v_a = \frac{v_b}{Z4}\left(Z2 + Z4\right)$$

Current into the filter, i_1, is the difference in potential across Z1 divided by Z1.

$$i_1 = \frac{e_{in} - v_a}{Z1} = \frac{e_{in}}{Z1} - \frac{v_a}{Z1}$$

Substitute for v_a. $$i_1 = \frac{e_{in}}{Z1} - \frac{v_b\left(Z2 + Z4\right)}{Z1Z4}$$

The current through the feedback impedance, i_3, can be calculated by summing the currents at node A.

$$i_3 = i_1 - i_2$$

Combine the equations for i_1 and i_2.

$$i_3 = \frac{e_{in}}{Z1} - \frac{v_b\left(Z2 + Z4\right)}{Z1Z4} - \frac{v_b}{Z4}$$

Summing the loop from node A, Z3, and the output yields

$$v_a - i_3 Z3 - v_{out} = 0$$

$$v_{out} = v_a - i_3 Z3$$

Substitute for i_3.

$$v_{out} = \frac{v_b}{Z4}(Z2 + Z4) - \left[\frac{e_{in}}{Z1} - \frac{v_b(Z2 + Z4)}{Z1Z4} - \frac{v_b}{Z4}\right]Z3$$

Combine this with the initial relationship for the input to output voltages of the op amp.

$$v_{out} = \frac{v_{out}}{A_oZ4}(Z2 + Z4) - \frac{e_{in}Z3}{Z1} + \frac{v_{out}(Z2 + Z4)Z3}{A_oZ1Z4} + \frac{v_{out}Z3}{A_oZ4}$$

This is an expression in v_{out}, e_{in}, and circuit components. The circuit analysis is complete. To obtain the transfer function, you have to manipulate this equation to group and separate terms, isolating v_{out}/e_{in} on the left side of the equation. When this is done, you have

$$\frac{v_{out}}{e_{in}} = \frac{A_oZ3Z4}{Z1Z2 + Z2Z3 + Z3Z4 + Z1Z3 + Z1Z4(1 - A_o)}$$

9-2.2 Second-Order Low Pass Sallen-Key Characteristics

To convert Figure 9-11 into a second-order low pass active filter, the resistors of the RC pairs must be in series with the main signal path, and the capacitors are tied to ground (or to the output which is only a few ohms from ground). This is shown in Figure 9-12.

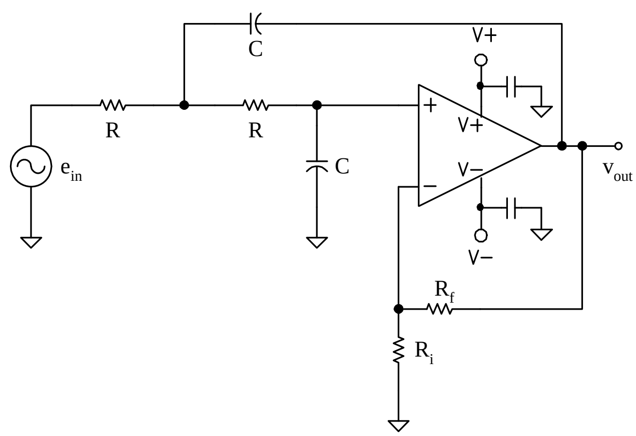


Figure 9-12 Sallen-Key, equal component, second-order, low pass, active filter

Compare Figure 9-12 to Figure 9-11. The equation developed for Figure 9-11 applies to the low pass filter of Figure 9-12 if

$$Z1 = Z2 = R \qquad Z3 = Z4 = \frac{1}{Cs}$$

Making these substitutions, you have

$$\frac{V_{out}}{E_{in}} = \frac{\dfrac{A_o}{C^2 s^2}}{R^2 + \dfrac{R}{Cs} + \dfrac{1}{C^2 s^2} + \dfrac{R}{Cs} + \dfrac{R}{Cs}\left(1 - A_o\right)}$$

$$= \frac{\dfrac{A_o}{C^2 s^2}}{\dfrac{R^2 C^2 s^2 + RCs + 1 + RCs + RCs\left(1 - A_o\right)}{C^2 s^2}}$$

$$= \frac{A_o}{R^2 C^2 s^2 + 2RCs + RCs\left(1 - A_o\right) + 1}$$

$$\frac{V_{out}}{E_{in}} = \frac{A_o}{R^2 C^2 s^2 + RC\left[2 + \left(1 - A_o\right)\right]s + 1}$$

The quadratic in the denominator is more easily solved if the coefficient of s^2 is 1. Dividing numerator and denominator by R^2C^2, you obtain

$$\frac{V_{out}}{E_{in}} = \frac{\dfrac{A_o}{R^2 C^2}}{s^2 + \left(\dfrac{3 - A_o}{RC}\right)s + \dfrac{1}{R^2 C^2}}$$

Second-order systems have been studied extensively. Mechanical and chemical as well as electrical second-order systems behave similarly. One transfer function is

$$\frac{A_o \omega_o^2}{s^2 + \alpha \omega_o s + \omega_o^2}$$

where

A_o = the gain
ω_o = the critical frequency in rad/s
α = the damping coefficient

This looks very similar to the second-order system response transfer function that you may have encountered in an advanced electrical networks or control systems course.

$$\frac{A_o \omega_o^2}{s^2 + 2\xi\omega_o s + \omega_o^2}$$

The damping factor (ξ) of the systems transfer function determines if the system is over-damped ($\xi > 1$), critically damped ($\xi = 1$), or under-damped ($\xi < 1$). By comparing the two transfer functions, you can see that

$$\alpha = 2\xi$$

So α plays a similar role for this active filter.

Further comparisons of the Sallen-Key, equal component, low pass filter transfer function with the general second-order form reveals

$$A_o = 1 + \frac{R_f}{R_i}$$

$$\omega_o^2 = \frac{1}{R^2 C^2}$$

$$\omega_o = \frac{1}{RC}$$

$$f_o = \frac{1}{2\pi RC}$$

$$\alpha\omega_o = \frac{3 - A_o}{RC}$$

$$\alpha = 3 - A_o$$

Since it is more convenient to work with a normalized frequency, substitute

$$\omega_o = 1$$

This scales the horizontal axis to make the critical frequency occur at 1. To return to the “real” world, you just multiply the horizontal axis by ω_o.

To obtain the gain and phase relationships, substitute $s = j\omega$ into the transfer function.

$$\frac{V_{out}}{E_{in}} = \frac{A_o}{(j\omega)^2 + \alpha j\omega + 1}$$

$$= \frac{A_o}{-\omega^2 + j\alpha\omega + 1}$$

$$\frac{V_{out}}{E_{in}} = \frac{A_o}{\left(1-\omega^2\right) + j\alpha\omega}$$

Now you must separate this into a real term and an imaginary term. Start by multiplying the numerator and the denominator by the complex conjugate of the denominator.

$$\frac{V_{out}}{E_{in}} = \frac{A_o}{\left(1-\omega^2\right) + j\alpha\omega} \times \frac{\left(1-\omega^2\right) - j\alpha\omega}{\left(1-\omega^2\right) - j\alpha\omega}$$

$$= \frac{A_o\left[\left(1-\omega^2\right) - j\alpha\omega\right]}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}$$

$$\frac{V_{out}}{E_{in}} = \frac{A_o\left(1-\omega^2\right)}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2} - j\frac{A_o\alpha\omega}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}$$

$$\text{Real} = \frac{A_o\left(1-\omega^2\right)}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2} \qquad \text{Imaginary} = -\frac{A_o\alpha\omega}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}$$

$$\left|\overline{G}\right| = \sqrt{\text{real}^2 + \text{imaginary}^2}$$

$$= \sqrt{\frac{A_o^2\left(1-\omega^2\right)^2}{\left[\left(1-\omega^2\right)^2 + \alpha^2\omega^2\right]^2} + \frac{A_o^2\alpha^2\omega^2}{\left[\left(1-\omega^2\right)^2 + \alpha^2\omega^2\right]^2}}$$

$$= \frac{A_o \sqrt{\left[\left(1-\omega^2\right)^2 + \alpha^2\omega^2\right]}}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}$$

$$\left|\overline{G}\right| = \frac{A_o}{\sqrt{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}}$$

The phase relationship is

$$\phi = \arctan \frac{\text{imaginary}}{\text{real}}$$

$$\phi = -\arctan \frac{\alpha\omega}{1-\omega^2}$$

Both the magnitude and the phase depend on the damping. And, the damping is set by the gain.

$$\alpha = 3 - A_o$$

or

$$A_o = 3 - \alpha$$

The three most commonly used damping coefficients are listed in Table 9-3. The Bessel filter has the heaviest damping and as such is both the most stable and the slowest to respond. It is often used to filter pulses because it does not overshoot or ring. It also provides the best phase (time) delay for sinusoidal signals. However, noticeable roll-off begins at $0.3\omega_o$, causing significant attenuation of the upper end of the pass band.

The Butterworth response provides the flattest frequency response. As such, it is most often the choice for audio and instrumentation circuits. It is underdamped, so there will be some overshoot when a pulse is applied. However it gives reasonable initial roll-off.

The Chebyshev filter is more lightly damped than the Butterworth. This causes the gain to increase with frequency at the upper end of the pass band, actually rising 3dB above A_o. This provides much faster initial roll-off than the Bessel or Butterworth. But this lowered damping also means that there is considerable ringing in response to a step.

Table 9-3 Second-order filter parameters

Filter Type	Damping (α)	Gain (A_o)	Correction (k_{lp})
Bessel	1.732	1.268	0.785
Butterworth	1.414	1.586	1.000
3dB Chebyshev	0.766	2.234	1.390

The normalized frequency response of the magnitude is given in Figure 9-13. The damping coefficient determines the shape of the frequency response plot **near** the critical frequency. At two octaves above or below the critical frequency (1 in Figure 9-13), the gain response of each type of filter is virtually identical.

Look carefully at these three responses. The Butterworth filter has a pass band gain of 4dB. That gain falls -3dB, to 1dB, at the critical frequency. So for Butterworth damping, $f_{3dB} = f_o$. However, the Bessel filter has a pass band gain of 2dB. Its f_{-3dB} occurs early, before f_o. The Chebyshev filter ripples **up** before it falls. Its -3dB frequency occurs where the gain has dropped 3dB below the **bottom** of the pass band (at A_o–3dB). So f_{-3dB} depends on the damping coefficient and is not necessarily equal to the critical frequency. That is where the k_{lp} correction factor in Table 9-3 comes in. It relates f_{-3dB} and f_o, depending on the damping coefficient.

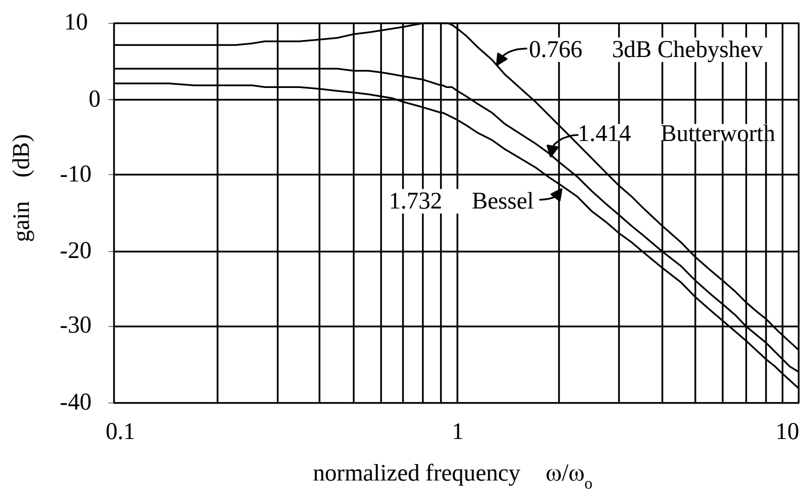


Figure 9-13 Normalized frequency response for a Sallen-Key, equal component, second-order, low pass filter

$$f_{-3dB} = k_{lp} f_o$$

Figure 9-14 shows the phase response of the second-order low pass filters for different damping coefficients. Each curve begins at 0°, passes through -90° at the critical frequency, and then asymptotes toward -180°. The heavier the damping (higher damping coefficient), the flatter the response.

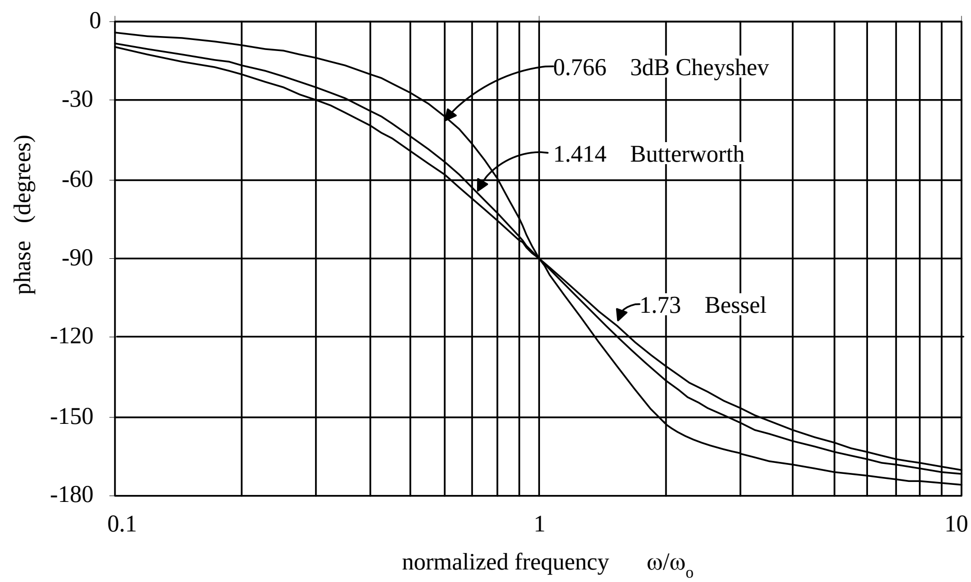


Figure 9-14 Phase shift of second-order, Sallen-Key, equal component, low-pass filter

Example 9-3

Design a Sallen-Key, equal component, second-order, low-pass filter to meet the following specifications:

f_{-3dB} = 2kHz A_o = 5 flattest possible pass band

Solution

Since the flattest possible pass band is required, you must use a Butterworth implementation.

$$f_o = \frac{1}{2\pi RC}$$

$$f_{-3dB} = k_{lp} f_o = \frac{k_{lp}}{2\pi RC}$$

$$R = \frac{k_{lp}}{2\pi f_{-3dB} C}$$

Pick C = 0.01μF.

$$R = \frac{1}{2\pi \times 2\text{kHz} \times 0.01\mu\text{F}} = 7.96\text{k}\Omega$$

Since this is not close to a standard value resistor, pick another standard capacitor, and repeat the calculation until you have discovered a standard capacitor and a standard resistor (in the 10kΩ range) which produce f_{-3dB} = 2kHz.

Pick C = 0.0068μF.

$$R = \frac{1}{2\pi \times 2\text{kHz} \times 0.0068\mu\text{F}} = 11.7\text{k}\Omega$$

Pick R = 12kΩ.

For the Butterworth filter,

$$A_o = 3 - \alpha = 3 - 1.414 = 1.586$$

$$A_o = 1 + \frac{R_f}{R_i} = 1.586$$

$$\frac{R_f}{R_i} = 0.586 \quad \text{or} \quad R_f = 0.586 R_i$$

However, to minimize the effect of offset current (which may be important in a **low** pass filter), the resistance at the inverting terminal must equal the resistance at the noninverting terminal.

$$2R = \frac{R_f R_i}{R_f + R_i}$$

Substituting for R_f, you have

$$2R = \frac{0.586R_i^2}{1.586R_i}$$

$$2 \times 12k\Omega = 0.369R_i$$

$$R_i = 65k\Omega$$

Pick $R_i = 68k\Omega$.

$$R_f = 0.586R_i = 39.8k\Omega$$

Pick $R_f = 39k\Omega$ with a series 820Ω resistor.

To obtain an overall gain of 5, you must add an amplifier after the filter. You can**not** just set the gain of the filter's amplifier to 5. The gain of the filter's amplifier must be set to 1.586 to assure that it behaves as a Butterworth filter. Any additional gain (or attenuation) must then be obtained from another amplifier.

$$A_{amp} = \frac{5}{1.586} = 3.15$$

So add a noninverting amplifier after the filter. Set its $R_{f\ amp} = 2.2k\Omega$ and $R_{i\ amp} = 1k\Omega$.

Example 9-4

Given a Sallen-Key, equal component, second-order, low pass filter, with the following values

$R = 22k\Omega$ $C = 0.47\mu F$ $R_f = 12k\Omega$ $R_i = 10k\Omega$

determine:

f_{-3dB} A_o the response type

Also, plot the magnitude's frequency response.

Solution

The simulation of this circuit using *Electronics Workbench*^{TM} is shown in Figure 9-15. The initial run indicates a Chebyshev type of response with a pass band gain of 6.8dB. Moving the cursor up the curve reveals a 2.7dB peak, not far from the 3dB Chebyshev. The f_{-3dB} point occurs at a little over 21Hz, and there is a -40dB/decade roll-off.

Now, let's look at what theory indicates that you can expect.

$$A_o = 1 + \frac{R_f}{R_i} = 1 + \frac{12\text{k}\Omega}{10\text{k}\Omega} = 2.2$$

$$A_{o\ \text{dB}} = 20\ \log(2.2) = 6.8\text{dB}$$

$$f_o = \frac{1}{2\pi RC} = \frac{1}{2\pi \times 22\text{k}\Omega \times 0.47\mu\text{F}} = 15.4\text{Hz}$$

$$f_{-3\text{dB}} = k_{lp} f_o = 1.39 \times 15.4\text{Hz} = 21.4\text{Hz}$$

This correlates well with the simulation.

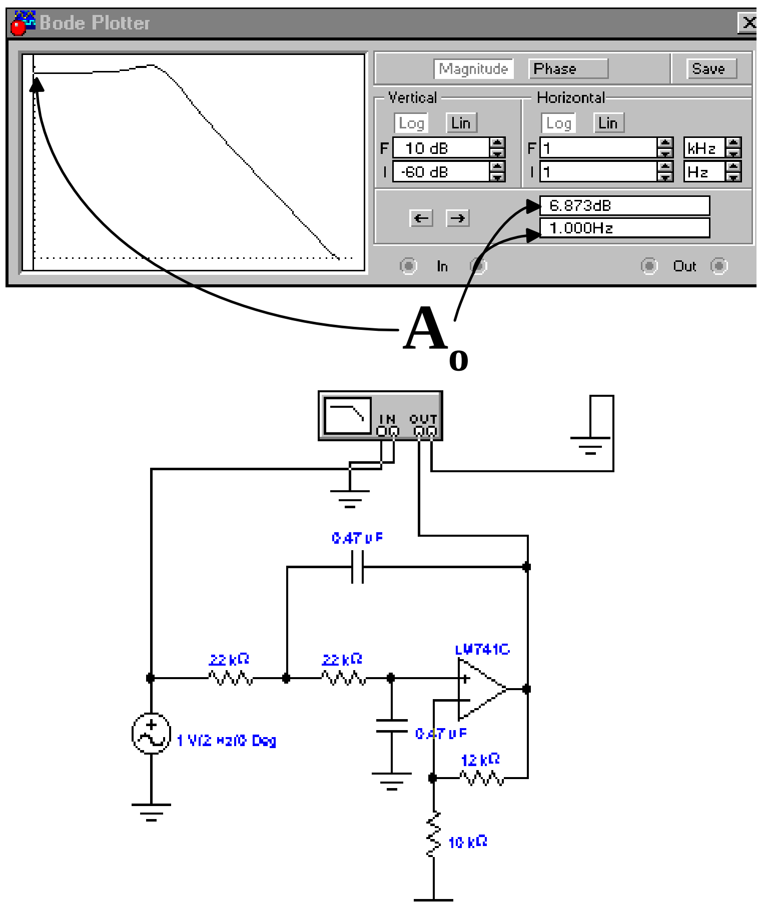


Figure 9-15 *Electronics Workbench*^{TM} simulation results for Example 9-4

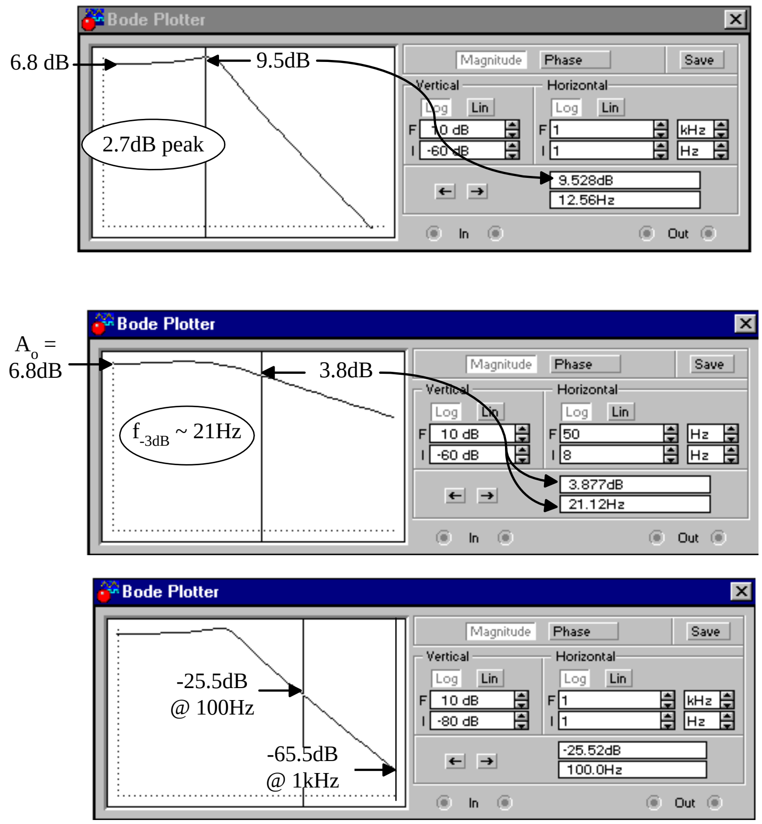


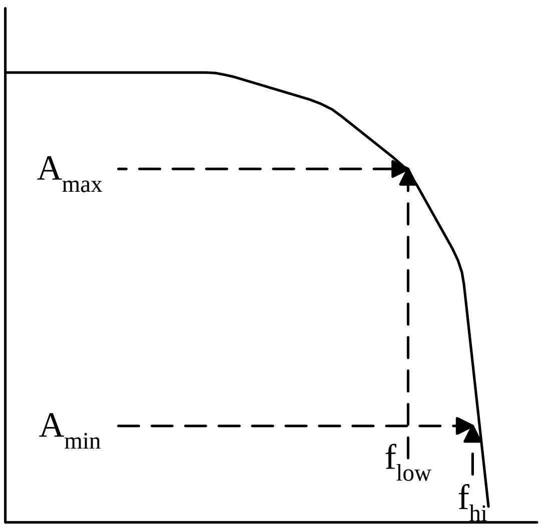
Figure 9-15 (cont.) *Electronics Workbench*^{TM} simulation results for Example 9-4

A second-order filter should have a roll-off of -40dB/decade. The simulation shows the gain dropping from -25.5dB at 100Hz to -65.5dB at 1kHz. That is -40dB/decade.

9-2.3 Higher-Order Filters

The first-order and second-order filters, though relatively easy to build, may not provide an adequate roll-off rate. The only way to improve the roll-off rate is to increase the order of the filter as illustrated in Figure 9-7. Each increase in order produces a 20dB/decade or 6dB/octave increase in the roll off-rate.

To determine what order you need, look at Figure 9-16. Figure 9-16 and the equation for n_B are for a Butterworth filter. The second equation, for n_C, is for a Chebyshev filter. The Bessel rolls off more slowly than the Butterworth, so you may need two orders higher.



$$n_B = \frac{\log\left[\left(10^{-0.1A_{min(dB)}} - 1\right) / \left(10^{-0.1A_{max(dB)}} - 1\right)\right]}{2\ \log\left(\frac{f_{hi}}{f_{low}}\right)}$$

$$n_C = \frac{\cosh^{-1}\sqrt{\left(10^{-0.1A_{min(dB)}} - 1\right) / \left(10^{-0.1A_{max(dB)}} - 1\right)}}{\cosh^{-1}\left(\frac{f_{hi}}{f_{low}}\right)}$$

Figure 9-16 Order selection

Example 9-5

It is necessary to build a filter with a -3dB frequency of 3kHz. At 10kHz the gain must be -40dB below the pass band gain. Determine the order of the filter.

Solution

You have to look carefully at the way the specifications are given. Gains are indicated with respect to the pass band gain. So we can assign an arbitrary value to it, $A_o = 0dB$. This means that

$A_{max} = -3dB \quad f_{low} = 3kHz \qquad A_{min} = -40dB \quad f_{hi} = 10kHz$

Apply the order selection equation:

$$n_B = \frac{\log\left[\left(10^{-0.1A_{min(dB)}} - 1\right) / \left(10^{-0.1A_{max(dB)}} - 1\right)\right]}{2\ \log\left(\dfrac{f_{hi}}{f_{low}}\right)}$$

$$n_B = \frac{\log\left[\left(10^{-0.1\times -40dB} - 1\right) / \left(10^{-0.1\times -3dB} - 1\right)\right]}{2\ \log\left(\dfrac{10kHz}{3kHz}\right)}$$

$$n_B = \frac{\log(9999/0.995)}{1.0458} = 3.83$$

Since you cannot make a 3.83-order filter, use a fourth-order.

Higher-order filters can be built by cascading the proper number of first- and second-order filter sections. For a fifth-order filter, this technique results in a transfer

$$\frac{A_o}{\underbrace{\left(s^2 + \alpha_1 s + \omega_1^2\right)}_{\text{second-order section}}\underbrace{\left(s^2 + \alpha_2 s + \omega_2^2\right)}_{\text{another second-order section}}\underbrace{\left(s + \omega_3\right)}_{\text{first-order section}}}$$

Each term in the denominator has its own damping coefficient and critical frequency. To obtain a given, well-defined response (Bessel, Butterworth, or Chebyshev), the transfer function, **as a whole,** must be solved and the appropriate coefficients determined.

$$\frac{A_o}{s^5 + as^4 + bs^3 + cs^2 + ds + e}$$

It is unreasonable to expect the α's and ω's of the cascaded filter transfer function to correlate in a simple way with the coefficients of the overall filter. You do **not** get a fifth-order, 1kHz Bessel filter by cascading two 1kHz, second-order Bessel filters and a first -order, RC passive stage.

The mathematics used to solve these higher-order polynomials is beyond the scope of this book. The results are presented in Table 9-4.

Example 9-6

Design a fourth-order Bessel filter with f_{-3dB} = 3kHz. Does its roll-off meet the requirements set by Example 9-5?

Table 9-4 Higher order damping and frequency correction factors

Filter Order	Section		Bessel	Butterworth	3dB Cheby
2	2	α	1.732	1.414	0.766
		k_{lp}	0.785	1.000	1.390
3	1	α	-	-	-
		k_{lp}	0.753	1.000	3.591
	2	α	1.447	1.000	0.326
		k_{lp}	0.687	1.000	1.172
4	2	α	1.916	1.848	0.929
		k_{lp}	0.696	1.000	2.349
	2	α	1.242	0.765	0.179
		k_{lp}	0.621	1.000	1.095
5	1	α	-	-	-
		k_{lp}	0.665	1.000	5.762
	2	α	1.775	1.618	0.468
		k_{lp}	0.641	1.000	1.670
	2	α	1.091	0.618	0.113
		k_{lp}	0.569	1.000	1.061
6	2	α	1.959	1.932	0.958
		k_{lp}	0.621	1.000	3.412
	2	α	1.636	1.414	0.289
		k_{lp}	0.590	1.000	1.408
	2	α	0.977	0.518	0.078
		k_{lp}	0.523	1.000	1.042

Solution

You can build a fourth-order filter by cascading two second-order stages. From Table 9-4, the first stage must have a damping coefficient of

$$\alpha_1 = 1.916$$

$$A_{o\,1} = 3 - 1.916 = 1.084$$

For that section, the frequency is set by

$$f_{-3dB} = \frac{k_{lp}}{2\pi RC}$$

$$k_{lp} = 0.696$$

Pick C1 = 0.01μF.

$$R1 = \frac{k_{lp}}{2\pi f_{-3dB} C} = \frac{0.696}{2\pi \times 3kHz \times 0.01\mu F} = 3.69k\Omega$$

You can build this with a 3.3kΩ resistor in series with a 330Ω resistor.

$$A_{o\,1} = 1 + \frac{R_{f1}}{R_{i1}} = 1.084$$

$$\frac{R_{f1}}{R_{i1}} = 0.084 \quad \text{or} \quad R_{f1} = 0.084 R_{i1}$$

But

$$2R1 = \frac{R_{f1} R_{i1}}{R_{f1} + R_{i1}}$$

$$7.38k\Omega = \frac{(0.084 R_{i1}) R_{i1}}{1.084 R_{i1}}$$

$$7.38k\Omega = 0.0775 R_{i1}$$

$$R_{i1} = 95.2k\Omega$$

$$R_{f1} = 0.084 \times 95.2k\Omega = 8.0k\Omega$$

The second stage is handled in the same way, by using:

$$\alpha_2 = 1.242 \qquad k_{lp\,2} = 0.621$$

This gives:

C2 = 0.01μF   R2 = 3.3kΩ R_{i2} = 15.3kΩ R_{f2} = 11.6kΩ

Neither section, **alone**, exhibits a Bessel response or has the correct cut-off. However, together the proper response is produced. The overall pass band filter gain is the product of each stage:

$$A_o = A_{o\,1} \times A_{o\,2} = 1.084 \times 1.76$$

$$A_o = 1.91 = 5.62dB$$

The simulation schematic and overall Bode plot are given in Figure 9-17. The frequency response is smooth and well behaved, suggesting a Bessel (or Butterworth) response. The pass band gain from the simulation matches the manual calculations, as does f_{-3dB}. The roll-off rate is -23.4dB/octave, a close match to the -24dB/octave predicted (6dB/octave/order; fourth order). The order selection from Example 9-5 indicated that a fourth-order (Butterworth) filter would cause the gain at 10kHz to be -40dB below A_o. But this filter falls only to -23dB at 10kHz. Sluggish early roll-off is typical of a Bessel filter. If that specification is critical, either increase the order, or use a Butterworth filter.

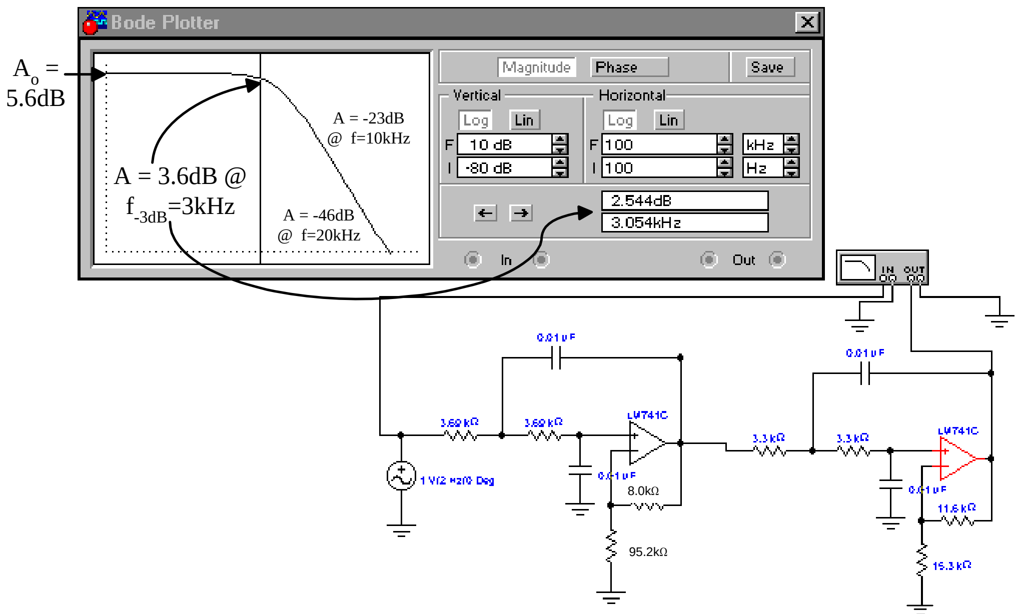


Figure 9-17 Simulation of Example 9-6

When cascading filter sections to produce higher-order filters, be sure to use the correct damping and filter correction factor from Table 9-4. Use the lowest-order filter that meets the given specifications. Damping coefficients (and therefore filter stability) become quite small as you increase the order. Use a Butterworth filter if possible. Go to Bessel if you need better transient response. But this gives poorer **initial** roll-off. Use the Chebyshev filter if initial roll-off of the Butterworth is not adequate. However, transient response and pass band flatness suffer.

9-3 High Pass Active Filter

The complement of the low pass filter is the high pass active filter. It is formed by exchanging the place of the resistors and capacitors in the frequency determining section of the filter. This is shown in Figure 9-18. Compare it carefully with Figure 9-12, the schematic of the second-order, low pass active filter. The general second order active filter is shown in Figure 9-11. Its transfer function was derived as

$$\frac{V_{out}}{E_{in}} = \frac{A_o Z3Z4}{Z1Z2 + Z2Z3 + Z3Z4 + Z1Z3 + Z1Z4(1 - A_o)}$$

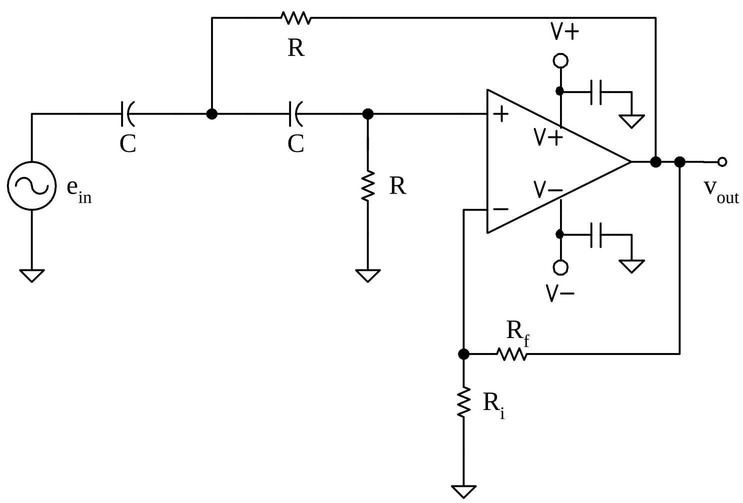


Figure 9-18 Sallen-Key, equal component, second-order, high pass, active filter

For the Sallen-Key, equal component, second order, high pass filter of Figure 9-18,

$$Z1 = \frac{1}{Cs} \qquad Z2 = \frac{1}{Cs} \qquad Z3 = R \qquad Z4 = R$$

Substituting these into the general transfer function gives:

$$\frac{V_{out}}{E_{in}} = \frac{A_o R^2}{\frac{1}{C^2 s^2} + \frac{R}{Cs} + R^2 + \frac{R}{Cs} + \frac{R}{Cs}(1 - A_o)}$$

$$= \frac{A_o R^2 C^2 s^2}{1 + RCs + R^2 C^2 s^2 + RCs + RCs(1 - A_o)}$$

$$= \frac{A_o R^2 C^2 s^2}{R^2 C^2 s^2 + RC(3 - A_o)s + 1}$$

$$\frac{V_{out}}{E_{in}} = \frac{A_o s^2}{s^2 + \frac{(3 - A_o)}{RC}s + \frac{1}{R^2 C^2}}$$

Second-order physical systems have been studied extensively for many years. Mechanical, hydraulic, and chemical, as well as electrical, second-order systems behave similarly. The transfer function for one group of such second-order systems is

$$\frac{A_o s^2}{s^2 + \alpha\omega_o s + \omega_o^2}$$

where A_o = the gain
ω_o = the critical frequency
α = the damping coefficient

Compare the general second-order transfer function to that for the Sallen-Key, equal component, second-order, high pass filter.

$$A_o = 1 + \frac{R_f}{R_i}$$

$$\omega_o = \frac{1}{RC} \qquad f_o = \frac{1}{2\pi RC}$$

$$\alpha = 3 - A_o$$

This is **identical** to the relationships developed for the Sallen-Key, equal component, second-order, low pass filter.

Normalizing the transfer function sets $\omega_o = 1$. It then becomes

$$\frac{A_o s^2}{s^2 + \alpha s + 1}$$

To determine the gain and phase relationships in the frequency domain, substitute s = jω into the transfer function.

$$\frac{V_{out}}{E_{in}} = \frac{-A_o\omega^2}{-\omega^2 + j\alpha\omega + 1}$$

$$= \frac{-A_o\omega^2}{\left(1-\omega^2\right) + j\alpha\omega} \times \frac{\left(1-\omega^2\right) - j\alpha\omega}{\left(1-\omega^2\right) - j\alpha\omega}$$

$$= \frac{-A_o\omega^2\left(1-\omega^2\right) + jA_o\alpha\omega^3}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}$$

$$\text{Real} = \frac{-A_o\omega^2\left(1-\omega^2\right)}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}$$

$$\text{Imaginary} = \frac{A_o\alpha\omega^3}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}$$

$$\left|\overline{G}\right| = \sqrt{\text{Real}^2 + \text{Imaginary}^2}$$

$$= \frac{\sqrt{A_o^2\omega^4\left(1-\omega^2\right)^2 + A_o^2\alpha^2\omega^6}}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}$$

$$= \frac{A_o\omega^2\sqrt{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}}{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}$$

$$\left|\overline{G}\right| = \frac{A_o\omega^2}{\sqrt{\left(1-\omega^2\right)^2 + \alpha^2\omega^2}}$$

Compare this to the gain magnitude of the Sallen-Key, second-order, low pass filter. The denominators are identical. However, for the high pass filter, the magnitude is **directly** proportional to the square of the frequency. An increase in frequency causes an increase in the gain. This is plotted in Figure 9-19. Compare these curves to the three for the low pass filter in Figure 9-13. The curves have just been mirrored around the $\omega = 1$ axis.

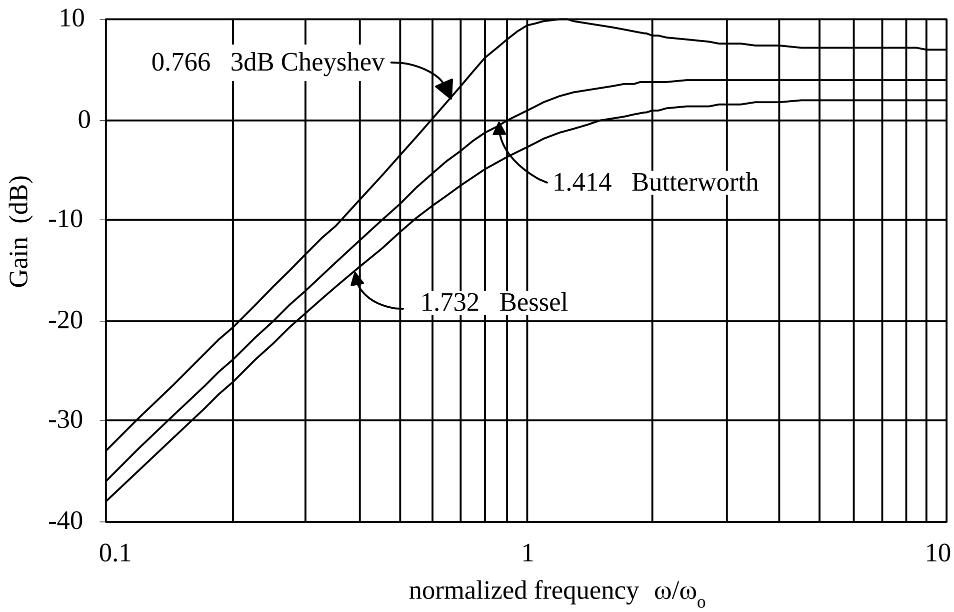


Figure 9-19 Frequency response of the Sallen-Key, equal component, second-order, high pass filter

The phase shift is

$$\phi = \arctan \frac{\text{imaginary}}{\text{real}}$$

$$\phi = -\arctan \frac{\alpha\omega}{1 - \omega^2}$$

This is the same as the equation developed for the Sallen-Key, equal component, second-order, low pass filter. The plot is shown in Figure 9-14.

Since the gain magnitude plots are rotated around the normalized axis, compared to the low pass plots, the correction factors of Table 9-3 must be changed for the high pass filter.

$$k_{hp} = \frac{1}{k_{lp}}$$

Example 9-7

For a high pass filter with the components listed below, calculate f_{-3dB}, the filter type, and A_o.

$R = 10k\Omega$ $C = 0.1\mu F$ $R_f = 5.8k\Omega$ $R_i = 10k\Omega$

Solution

$$A_o = 1 + \frac{R_f}{R_i} = 1 + \frac{5.8k\Omega}{10k\Omega}$$

$$A_o = 1.58$$

$$\alpha = 3 - A_o = 3 - 1.58$$

$$\alpha = 1.414$$

This is a Butterworth filter.

$$k_{hp} = \frac{1}{k_{lp}} = \frac{1}{1} = 1$$

$$f_{-3dB} = \frac{k_{hp}}{2\pi RC} = \frac{1}{2\pi \times 10k\Omega \times 0.1\mu F}$$

$$f_{-3dB} = 159Hz$$

The gain bandwidth for a 741C op amp is 1MHz. With a pass band gain of 1.58 (4dB), the filter's gain falls off at high frequencies.

$$f_h = \frac{GBW}{A_o} = \frac{1MHz}{1.58}$$

$$f_h = 633kHz$$

In addition, the slew rate limits the amplitude of the high frequency outputs. For full power, maximum output amplitude, the highest frequency is

Chapter 10
Introduction to Filters

CHAPTER LEARNING OBJECTIVES

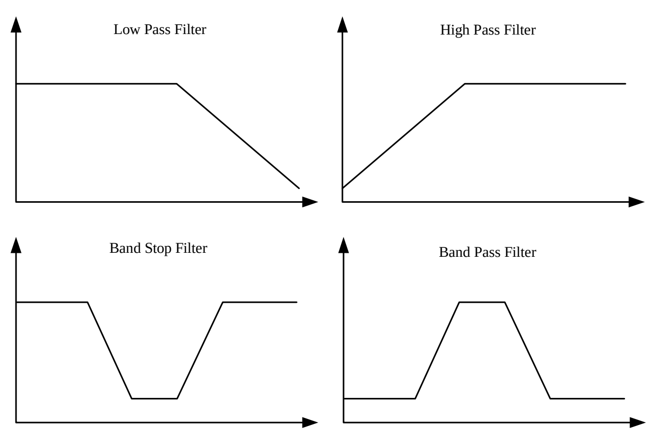
1. Describe each of the four basic responses for filters.
2. Calculate the roll off rate (in dB/decade or dB/octave) for any of the filter responses given the order or number of poles of a filter.
3. Determine the response, number of poles and roll off rate from a transfer function of a filter or a filter description.
4. Compare and contrast the filter approximations of Butterworth, Chebyshev and Bessel filters for trade-offs and output characteristics of gain and phase.
5. Calculate and plot the poles for any order Butterworth filter approximation.
6. List the trade offs between passive and active filters
7. Design Butterworth low pass filters using the LPP filter mask design procedure from any point in the process to meet specifications.
8. List the transfer functions for one pole active filters (inverting and non-inverting) and for a two pole active filter (Sallen-Key only).
9. Determine and implement the set of equations to balance the inputs of any active filter.

Why study filters? The obvious reason is that filters are used in many aspects of electronics engineering, such as instrumentation, amplifier design, controls, analog circuits, etc… Filters also complement the material that has been studied to date in this course. The material presented on filters incorporates Laplace transforms, Fourier analysis, transfer functions, poles and zeros, and bode plots. The concept of the filter design process also introduces the elements of system design.

What is a filter? A filter is simply a circuit that passes signals of only desired frequencies and attenuating (removing) undesired frequencies. There are many types of circuits to implement filters. The circuits are used to implement the filter transfer function. General filter information and transfer function analysis will be presented first, to better understand what those circuits will be doing and measuring their performance.

There are four basic *responses* of filters;

- Low Pass (LP)
- High Pass (HP)
- Band Pass (BP)
- Band Stop (BS) also known as a 'notch' filter.



The filter responses are plotted using the *Bode* plot introduced previously. A key parameter of the filter response is the critical frequency (ω_C, f_C), which is also known as the cutoff frequency or pass band frequency (be careful, as these are used interchangeably in this course and other texts that may be used as reference).

As noted previously, the roll off rates of the filters on the *Bode* plots are dictated by the number of poles or the order (n) of the transfer function (20dB/decade or 6dB/octave). This roll off rate is <u>*away*</u> from the critical frequency in order to be in the linear region of the plots.

$$n \cdot 20dB/decade = roll-off \;\; rate$$

For the band pass and band stop filters, the order is 'halved' for each side of the band. There is a 'low' side and a 'high' side to these filters. For this course, only even ordered band filters will be discussed. This makes the work more straight forward, i.e. a 2nd order band pass filter will have 1 order on the low side and 1 order on the high side, yielding a 20dB/dec roll-off rate to each side of the band. Of course, odd ordered band filters are possible and can be designed, but is beyond the scope of this course.

<u>Filter transfer functions:</u>

The filter response is determined by the transfer function of the filter. The transfer function output is dictated by its pole-zero relationship. How does the pole-zero relationship determine the filter response. It is important to not lose sight that the poles and zeros of the transfer function are really the reactive components of a circuit. The transfer function is just an efficient method of analyzing and extracting information. Consider the basic transfer functions for the responses of a 1st and 2nd order filter shown in the table.

The functions presented are in a general normalized form and are meant to be used in a descriptive manner for discussion purposes only. The functions are normalized to $\omega_o = 1$ rad/sec. It is important to normalize these functions as filters are designed to operate at all possible frequencies and it would be extremely difficult and tedious to be required to consider each specific frequency, so the transfer functions are frequency normalized for calculations and analysis, then denormalized to the specific frequency. More details on how to normalize will be presented in upcoming sections.

These transfer functions are independent of the kind of <u>filter approximation</u> (Butterworth, Chebyshev, etc… to be discussed in next section), it is only based on the <u>response</u> (HP, LP, BP, …).

Table of normalized filter transfer functions			
Response	Transfer Function	1 Pole – Normalized	2 pole – Normalized
Low Pass	$\frac{\omega_o^2}{s^2+2\zeta\omega_o s+\omega_o^2}$	$\frac{1}{s+1}$	$\frac{1}{s^2+s+1}$
High Pass	$\frac{s^2}{s^2+2\zeta\omega_o s+\omega_o^2}$	$\frac{s}{s+1}$	$\frac{s^2}{s^2+s+1}$
Band Pass	$\frac{\omega_o s}{s^2+2\zeta\omega_o s+\omega_o^2}$	X	$\frac{s}{s^2+s+1}$
Band Stop	$\frac{s^2+\omega_o^2}{s^2+2\zeta\omega_o s+\omega_o^2}$	X	$\frac{s^2+1}{s^2+s+1}$
All Pass	$\frac{s^2-2\zeta\omega_o s+\omega_o^2}{s^2+2\zeta\omega_o s+\omega_o^2}$	X	$\frac{s^2-s+1}{s^2+s+1}$

The denominator of the transfer function is called the characteristic equation. In the transfer function equation, ω_o, is the natural resonant frequency, which happens to correspond to the critical (or cutoff) frequency of the filter (ω_C). The parameter ζ (zeta) is the damping ratio, which has the same meaning as developed for the series RLC circuit, although the associated calculations are different. Another common notation used in the transfer functions of filters is the Q. What is the Q? In filter transfer functions, the Q is related by the following:

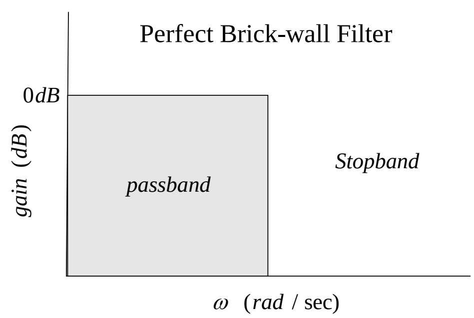
$$Q = \frac{1}{2\zeta}$$

The information has not changed, it is just another method of analyzing the filter transfer function. Many filter design texts and programs use Q as a design parameter. Q will be explained and studied further in upcoming chapters

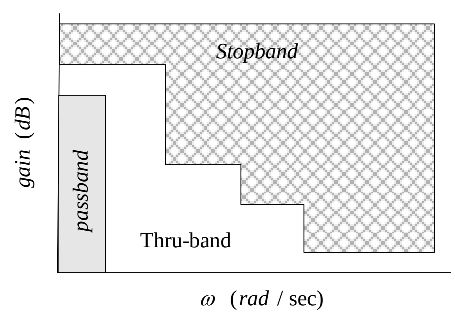
Filter Approximations:

The low pass filter will be used as the example for much of the following discussion, and all this information can be extended to the other filter types.

The ideal or perfect low pass filter is the 'Brick wall', where only the desired frequencies are passed and all others are removed entirely. This is obviously not realistic and there needs to exist a region of acceptable values. This possible range of acceptable values constitutes the 'thru-band'.



The filter response can have many 'shapes' that will 'fit-in' the available thru-band range and these shapes have a set of standard forms known as *filter approximations*.



There are many different approximations, the most common are:

- Butterworth
- Chebyshev
- Bessel
- Inverse Chebyshev (or Chebyshev II)
- Elliptic (or Cauer)

This course will deal only with the first three approximations which consist of the majority of filter work and design.

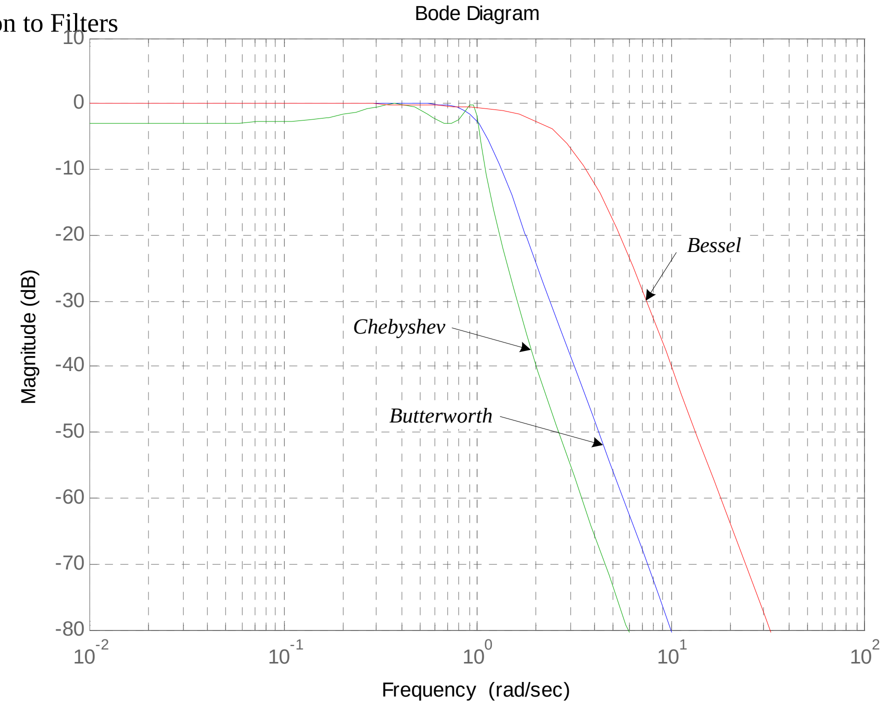
The Butterworth approximation is considered to offer the best all-around filter approximation. It has maximum flatness in the pass-band with moderate roll-off at the cutoff frequency, and shows only a very slight overshoot in response to a step input. It is just slightly underdamped in the time domain. The Butterworth phase response is fairly flat, except near the cutoff frequency.

The Chebyshev approximation sacrifices pass-band flatness for a high rate of attenuation **at** the cutoff frequency. It also exhibits large overshoot and ringing in response to a step input. It is underdamped in the time domain. The phase response of the Chebyshev approximation varies a large amount across the pass band is not recommended if phase is an important consideration.

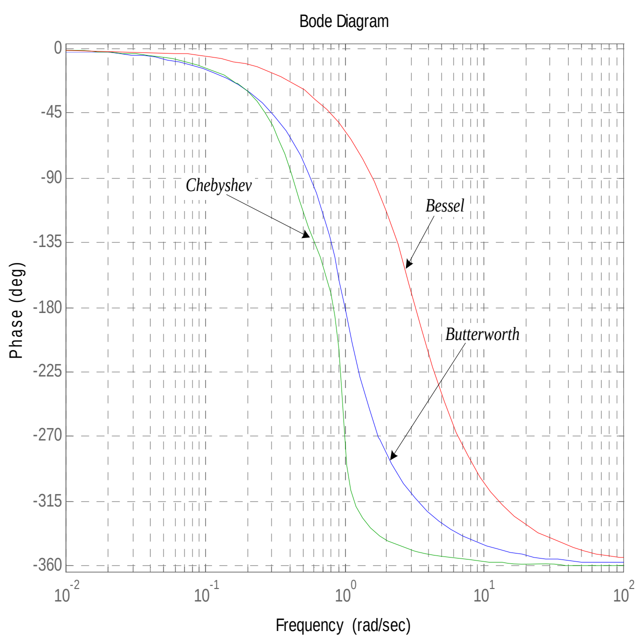
The Bessel approximation has an excellent phase response. The group delay, which is a measure of phase variation versus frequency, is flat or constant (this is the slope of the phase plot). This is important when signal-conditioning square-wave signals or other signal with 'sharp edges'. The constant group delay means that the square-wave signal is passed with minimum distortion (overshoot). This comes at the expense of slower rate of attenuation above cutoff. The gain in the pass band is linear in nature and the roll off at the cut off frequency is poor, especially as the filter order increases. The Bessel approximation is over damped in the time domain and has no overshoot in response to a step input.

Table: A summary of filter approximation trade-offs.

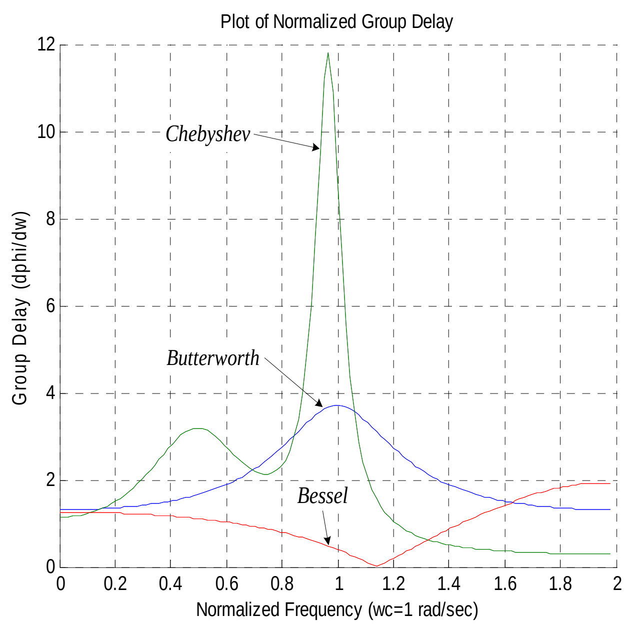
Advantages:	*Disadvantages:*
Butterworth	
• Maximally flat magnitude response in the pass-band. • Good all-around performance. • Step or Pulse response better than Chebyshev. • Rate of attenuation better than Bessel.	• Slight overshoot and ring in time domain step response.
Chebyshev	
• Better rate of attenuation at the pass-band cut off frequency than Butterworth.	• Ripple in pass-band. • Ringing in time domain step response (more than Butterworth)
Bessel	
• Phase Response (group delay) excellent • No Ringing and overshoot in time domain step response.	• Linear pass band gain – not flat • Slower initial rate of attenuation beyond the pass-band than Butterworth.



Normalized Frequency Response of the Three Filter Types for n=4 and $\omega_C = 1$.



(a)



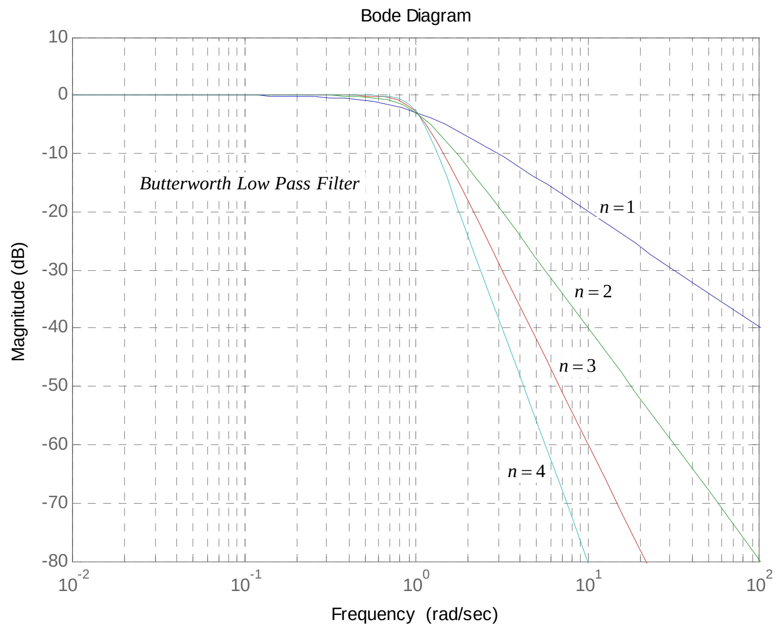
(b)

Plots of (a) Phase Response vs. Normalized Frequency and (b) Group Delay vs. Normalized Frequency for the three basic filter types (n=4, $\omega_C = 1$).

Group delay is $\dfrac{d\phi}{d\omega}$.

Butterworth:

The Butterworth filter is common to many applications that require a response that is flat in the pass-band but cuts off as sharply as possible afterwards. The roll-off is smooth and monotonic, with a low-pass or high-pass roll off rate of 20dB/decade or 6dB/octave for each pole. That response is obtained by arranging the poles of a low-pass filter with equal spacing around a semicircular radius, and the result is a Butterworth filter.

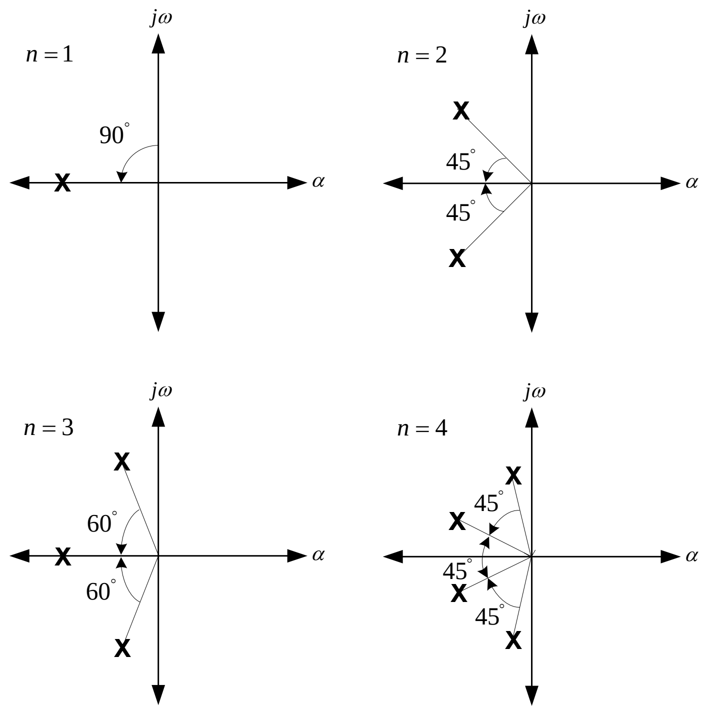


Butterworth low pass filter response normalized at various orders (ω_C =1) . Note the responses all intersect at ω_{-3dB} for all the orders. “n” is the order of the filter.

The Butterworth filter approximation is obtained by arranging the poles of the filter with equal spacing around a semicircular radius (on the left-hand side of the complex plane for stability). The radius of poles is ρ (rho). This arrangement allows the poles to have *optimal stability*. Each of the *n* poles is equally spaced between each other by an angle of:

$$\theta_n = \frac{180^\circ}{n}$$

The semicircle has a total of 180º available. The spacing from the pole to the $j\omega$ axis is the remaining angle equally divided. The figure shows the pole location of a low pass Butterworth filter for orders of 1-4.



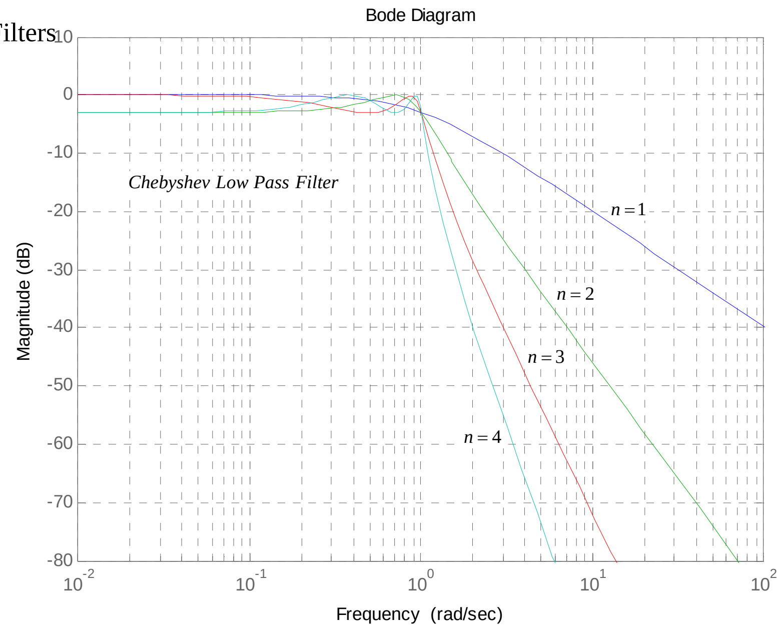
Pole locations for a low pass Butterworth filter approximation for orders of 1-4.

<u>Chebyshev:</u>

Placing the poles closer to the jω axis as compared to a Butterworth (increasing the 'Q'), creates a filter that has a frequency cutoff steeper than that of a Butterworth. This arrangement has a penalty: The effects of each pole will be visible in the filter response, giving a variation in amplitude known as ripple in the passband. The pole arrangement forms an ellipse, which results in a Chebyshev filter. The Chebyshev filter offers a trade-off between ripple and cutoff. In this respect, the Butterworth filter in which passband ripple has been set to zero is a special case of the Chebyshev.

The Chebyshev is also called the equal ripple response. This filter has a ripple in the passband amplitude response. The amount of passband ripple is one of the parameters used in specifying a Chebyshev filter. For a given filter order, the higher the passband ripple, the steeper the roll off at the cutoff frequency. The Chebyshev sacrifices pass-band flatness for a high rate of attenuation near cutoff. It also exhibits a large overshoot and ringing in response to a pulse input.

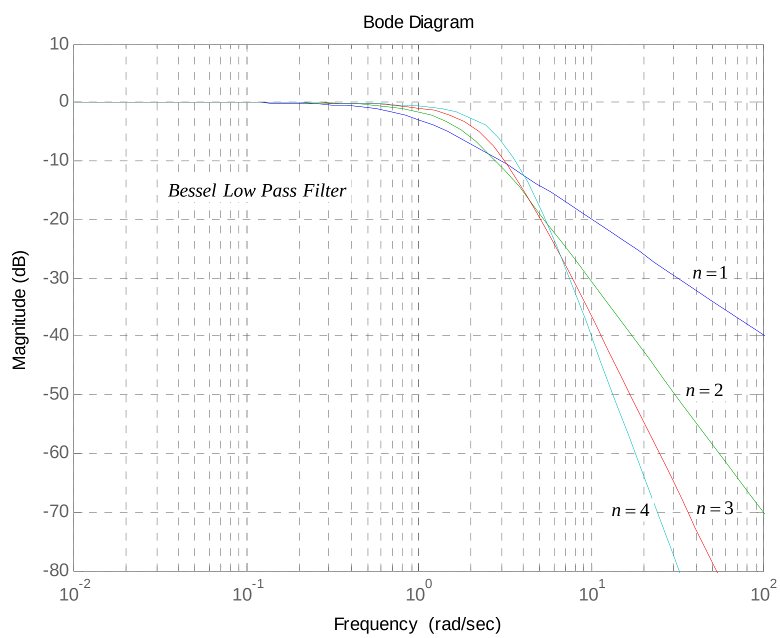
The Chebyshev filter will be developed more in the next chapter.



Chebyshev normalized low-pass filter response at various orders ($\omega_c = 1$). "n" is the order of the filter. Note that odd and even orders have different levels that the ripples originate.

Bessel:

All Filters exhibit phase shift that varies with frequency. This can cause problems in some applications. If the phase shift is not linear with frequency, components of the input signal at one frequency will appear at the output shifted in phase (or time) with respect to other frequencies. This will distort non-sinusoidal shapes. The Bessel filter represents a trade-off in the opposite direction from the Butterworth compared to the Chebyshev. The Bessel's poles lie on an ellipse further from the jω axis. Transient response (phase) is improved, but at the expense of a poor amplitude response at cutoff and in the stopband.



Bessel normalized low-pass filter response at various orders. "n" is the order of the filter. Note the responses do not intersect at ω_{-3dB} for all the orders.

NOTE:
1. The roll off rate is determined by the number of poles (or order) in the filter.
2. A 2nd order filter, whether Butterworth, Chebyshev or Bessel will always have a 40 dB/dec roll off rate away from the cutoff frequency in the linear region.

Using MATLAB, the pole-zero relationship in the s-domain (Laplace) can be examined. The pole location determine the type of filter approximation (Butterworth, Chebyshev, …) and the zeros will dictate the response (LP, HP, …).

MATLAB examples:

Basic Filter Design

There is already some familiarity from other courses in basic filter techniques. The filters that many are acquainted with use passive elements (R, L, and C) and surprisingly enough are known as passive filters.

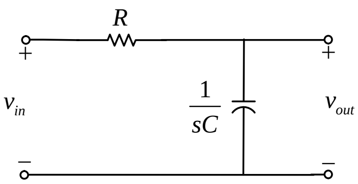
This course will deal mainly with filters that use op-amps and are called active filters. The op-amp essentially acts as a buffer to a passive RC filter with its high input impedance and low output impedance. However, it allows the use of feedback and feedforward circuit topologies thus giving a large range of possible circuit type and responses.

The op-amp allows second order filters using capacitors only, thus eliminating the use of inductors. The active filters can be cascaded in series allowing for complex transfer functions to be implemented using simpler, more manageable transfer functions. The methods presented can be used for both passive and active filters in very similar manner; the design techniques presented are valid for both.

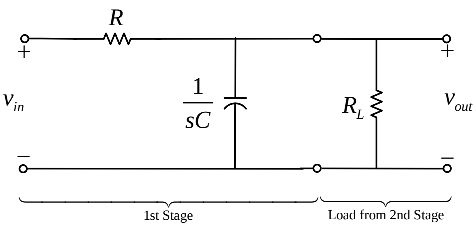
Table: A summary of trade-offs for active and passive filters	
Advantages:	*Disadvantages:*
Active	
• Excellent for multiple order of filters • Eliminates the use of inductors • Filters are 'independent' in each stage and can 'stand-alone' • Op-amp can add limited gain	• Op-amp can add gain and noise • Frequency response dependent on op-amp quality • Requires external power
Passive	
• Excellent for single-stage filters (1 or multiple orders) • Low noise • Requires no external power	• Complex for multiple stages – loading effects between stages or other circuits will change the transfer function.

Example:

$$G_1(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}}$$



$$G_2(s) = \frac{\left(\frac{1}{sC} \Big\| R_L\right)}{R + \left(\frac{1}{sC} \Big\| R_L\right)}$$



$$G_1(s) \neq G_2(s)$$

- Each filter needs to be 'tweaked' and tuned for each application (a custom circuit). The load may even vary with frequency as shown from the analysis of circuits using sinusoidal steady state.

Active Filter Design Using the Mask Procedure:

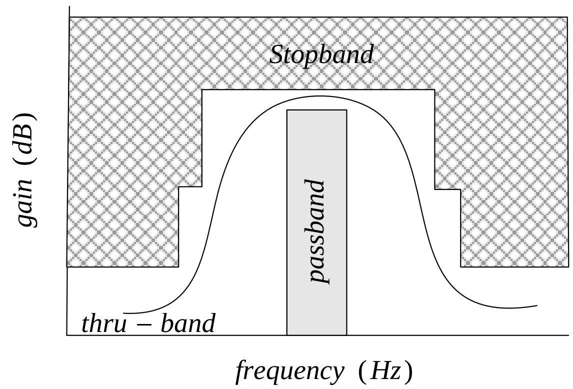
> **Important:**
> The filter design process consists of 3 main elements.
> 1. Determine the transfer function of the desired filter specification.
> 2. Determine the transfer function of the desired filter circuit.
> 3. Implement the circuit by equating both of transfer functions.
>
> Keep these elements separate as the following design procedures are presented.

Overview:

- Implementation of filters is achieved using a common Filter Mask design process, whether the implementation technology is Passive, Active, or DSP or Switched Capacitors.
- High order filters are frequently required to satisfy specifications.
- The Mask design procedure requires design of Low Pass Prototype (LPP) filter - the core of the process.

1. **Determine the specifications of the desired filter,**
   This information is determined from the application, interpreted from customer requirements, or is given as requirement.
   a. Which response? LP, HP, BP, BS
   b. Pass band frequencies and gain
   c. stop band frequencies and attenuation
   d. gains or attenuation at other frequencies

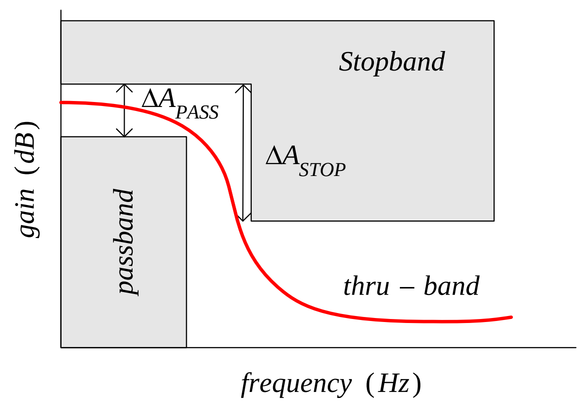
2. Summarize the specs with a filter mask diagram:



Necessary to relate specifications to a visual reference.

Creating a diagram of the filter mask is very important. It provides a visual presentation of the filter requirements. The filter mask will indicate the frequency response of the filter for the pass band gains and stop band attenuation. An accurate plot of the filter specifications is essential.

Consider the low pass filter shown to the right, the important requirements are the change in the pass band gain (ΔA_{PASS}) and the stop band points (ΔA_{STOP}). Theses changes in the gain outline the acceptable ranges of frequencies and amplitudes of the filter. This is the *thru-band*. The 'shape' of the output response can range anywhere in the thru-band as long as it does not touch the stop band – allowing unwanted frequencies or the pass band – attenuating desired frequencies.



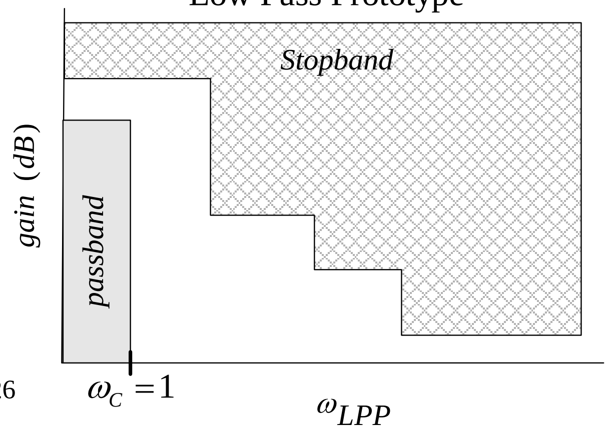
3. Transform the response to normalized Low Pass Prototype specifications:

The cutoff frequency (also known as the pass band edge frequency) is set to ω_C=1 rad/sec. The frequency is normalized by this step. Creation of the low pass prototype (LPP) is key to the design process. The LPP filter is the basis from which all the filter types and approximations will be analyzed and designed.

4. Choose an approximation:

- Butterworth
- Chebyshev
- Bessel

Low Pass Prototype



5. **Determine the order required for the filter.**
Calculate *n*, the order of the low pass prototype for type of filter response desired. Choose the minimum order (*n*) that satisfies all the specs from the filter type chosen in Step 4. Design the least complex filter to get the job done.

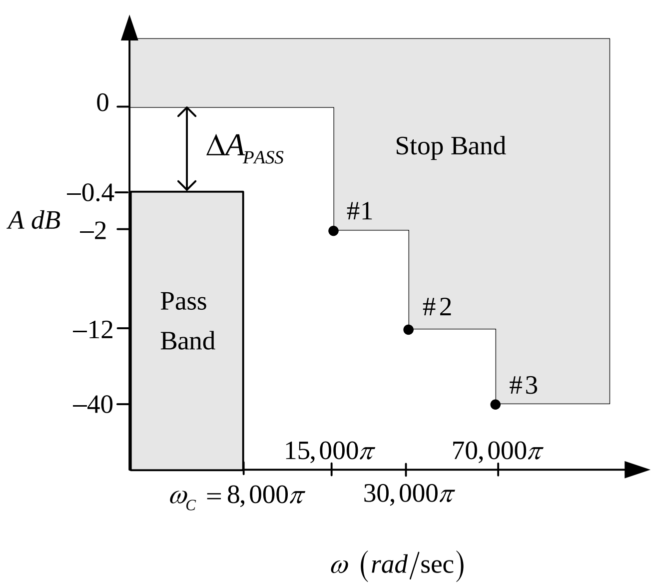
6. **Determine transfer function $G_{LPP}(s)$ for low pass prototype.**
If the critical or pass band edge frequency is given as the f_{-3dB} frequency, then go to tables found in the Appendices to obtain the $G_{LPP}(s)$. If another specification is given then calculations must be performed to determine the proper LPP transfer function.

7. **Denormalize $G_{LPP}(s)$ to the $G_{LP}(s)$, $G_{BP}(s)$ etc. with the original frequency scale.**
All this work was perform for ω_C=1 rad/sec, now the critical frequency must go back the original frequency required.

The best method to introduce the procedure is to work an example:

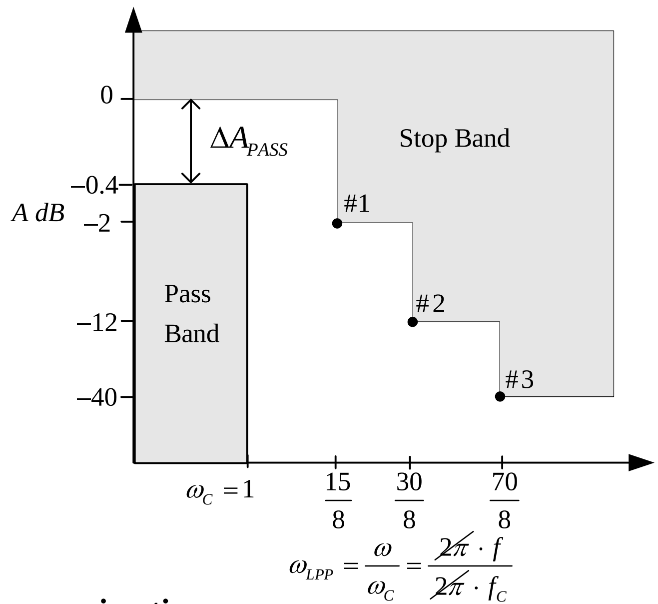
1. **Specifications**
Pass-band: All frequencies below 4 kHz must be passed with a gain between 0dB and -0.4dB
Stop-band: -2dB for frequencies between 7.5 kHz and 15 kHz (-2dB @ 7.5 kHz)
-12dB for frequencies between 15 kHz and 35 kHz (-12dB @ 15 kHz)
-40dB for frequencies greater than 35 kHz (-40dB @ 35 kHz)

2. **Summarize specifications with a filter mask**
Notice the diagram need not be scaled perfectly, but is required to show exact values. Notice the scaling factor for frequency units to keep in rad/sec.



3. Transform to the Low Pass Prototype (LPP)

This step is to normalize the original low pass filter to a LPP. This only requires a scaling (or normalization) of the frequency axis. The original specification has ω_C=8000π rad/sec and the goal is to have ω_C=1 rad/sec. This is easily accomplished by dividing the frequency axis by ω_C=8000π rad/sec. Note that the frequency axis is unitless after normalization, but continues to be labeled ω_{LPP}.



4. Choose an approximation:

This will be a Butterworth filter by decree.

5. Determine the required order of the filter.

In this example, there are 3 stop band specifications that must be checked against the passband specification. The following equation essentially determines the slope from the pass band frequency corner and the stop band point.

The number calculated will be rounded up to the nearest integer (no such animal as a fractional pole). Each stop point is checked and the largest overall order is used as the minimum required value of *n*.

$$n_{BW} = \frac{\log\left(\frac{\left(10^{-0.1\Delta A_{STOP}(dB)} - 1\right)}{\left(10^{-0.1\Delta A_{PASS}(dB)} - 1\right)}\right)}{2\log\left(\frac{\omega_{STOP}}{\omega_{PASS}}\right)} \quad \text{This is ONLY for Butterworth}$$

1st Stop point: $n_1 = \dfrac{\log\left(\dfrac{\left(10^{-0.1(-2)}-1\right)}{\left(10^{-0.1(-0.4)}-1\right)}\right)}{2\log\left(\dfrac{15/8}{1}\right)} = \dfrac{\log\left(\dfrac{0.5849}{0.0965}\right)}{0.5460} = \dfrac{0.7826}{0.5460} = 1.433 \Rightarrow 2$

2nd Stop point: $n_2 = \dfrac{\log\left(\dfrac{\left(10^{-0.1(-12)}-1\right)}{\left(10^{-0.1(-0.4)}-1\right)}\right)}{2\log\left(\dfrac{30/8}{1}\right)} = \dfrac{\log\left(\dfrac{14.8489}{0.0965}\right)}{1.1481} = \dfrac{2.1872}{1.1481} = 1.91 \Rightarrow 2$

3rd Stop Point: $n_3 = \dfrac{\log\left(\dfrac{\left(10^{-0.1(-40)}-1\right)}{\left(10^{-0.1(-0.4)}-1\right)}\right)}{2\log\left(\dfrac{70/8}{1}\right)} = \dfrac{\log\left(\dfrac{9,999}{0.0965}\right)}{1.8840} = \dfrac{5.0154}{1.8840} = 2.66 \Rightarrow 3$

The overall minimum order required to meet the specifications is 3.

It should be obvious that increasing the order even more will allow the specifications to be met and exceeded. The minimum order filter will be designed in this course. This is to reinforce using the minimum circuit to accomplish the required task.

It can be noted from the calculations of the order (*n*) as to what level of difficulty, one might expect on actually building the circuit and meeting the specification. If the fractional pole was, 2.3, which was round up to a whole order of 3, then this filter will be ‘easier’ to implement than a requirement that had a fractional pole of 2.9. There is more ‘room’ in the thru-band with the lower ordered fractional pole.

6. **Determine the transfer function ($G_{LPP}(s)$) for the Low Pass Prototype.**
The required order has been determined. The order is also the number of poles in the denominator of the transfer function. The angle of the Butterworth poles was presented earlier. Now the ‘length’ of the radius needs to be determined.

   a. Calculate the ripple factor ε:
   $$\varepsilon = \sqrt{\left(10^{-0.1\Delta A_{PASS}(dB)}-1\right)} = \sqrt{\left(10^{-0.1(-0.4)}-1\right)} = 0.3106$$

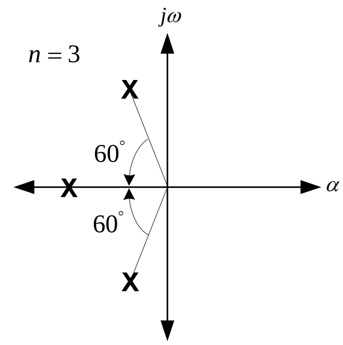
   b. Radius of poles:
   $$\rho = \varepsilon^{-\frac{1}{n}} = (0.3106)^{-\frac{1}{3}} = 1.4766$$

> **Note:**
> If the critical frequency given for the end of the pass band has an attenuation of -3dB, then ρ=1, and can immediately use the table to obtain the transfer function for the Low Pass Prototype.

c. These poles can now be accurately plotted and a transfer function determined from the pole-zero plot.

$$n = 3, \quad \rho = 1.4766 \qquad \theta = 0^\circ, \pm 60^\circ$$

3 poles are located as shown on the plot. Using trigonometry, the locations can be determined.



$$p_1 = -1.4766$$
$$p_2, p_3 = \rho \cdot \cos\theta^\circ_{axis} \pm j\rho \cdot \sin\theta^\circ_{axis}$$
$$p_2, p_3 = -1.4766 \cdot \cos 60^\circ \pm j1.4766 \cdot \sin 60^\circ$$

$$G_{LPP}(s) = \frac{1}{(s+1.4766)} \cdot \frac{1}{(s+0.7383+j1.2788)(s+0.7383-j1.2788)}$$
$$G_{LPP}(s) = \frac{1}{(s+1.4766)} \cdot \frac{1}{(s^2+1.4766s+2.1804)}$$

This is the transfer function for the Low Pass Prototype, $G_{LPP}(s)$, which has a normalized ω_C=1 rad/sec. $G_{LPP}(s)$ must be *denormalized* back to the original cutoff frequency of the low pass filter, $G_{LP}(s)$.

Note: Do not exceed a quadratic for the transfer function. Create a 2nd order (quadratic) for each complex pole pair. Create a 1st order for single pole.

7. Denormalize G_{LPP}(s) to the G_{LP}(s), G_{BP}(s) etc. with the original frequency scale. To denormalize, apply the following substitution in $G_{LPP}(s)$ (it is just algebra).

$$G_{LP}(s) = G_{LPP}(s)\Big|_{S=\frac{s}{\omega_C}} = G_{LPP}(s)\Big|_{S=\frac{s}{8000\pi}}$$

$$G_{LP}(s) = \frac{1}{\left(\left(\frac{s}{8000\pi}\right)+1.4766\right)} \cdot \frac{1}{\left(\left(\frac{s}{8000\pi}\right)^2+1.4766\left(\frac{s}{8000\pi}\right)+2.1804\right)}$$

$$G_{LP}(s) = \frac{8000\pi}{(s+1.4766\cdot(8000\pi))} \cdot \frac{(8000\pi)^2}{\left(s^2+1.4766\cdot(8000\pi)s+2.1804\cdot(8000\pi)^2\right)}$$

$$G_{LP}(s) = \frac{25{,}132}{(s+37{,}111)} \cdot \frac{631.7\times10^6}{\left(s^2+37{,}111s+1.377\times10^9\right)}$$

This is it. The transfer function for the filter that was specified using a Butterworth approximation. There are two separate functions (a 1st order and a 2nd order) that when cascaded yields the desired output response. Each function represents a 'stage'.

Now what? This transfer function can be placed in MATLAB or Simulink, and PSpice (yes, there is a 'component' called 'Laplace' to enter mathematical functions – check it out). These various simulations will give information and analysis to determine if the designed transfer function will provide the proper response.

How about **designing a circuit** that will implement this transfer function?

Implementation of a transfer function into a circuit:

The method of implementing a transfer function into circuit is called, *equating of coefficients*. Recall, that each circuit can be analyzed using Laplace techniques to obtain a transfer function for that *specific* circuit. Now, there are two transfer functions, the *first* for the filter, and a *second* for the circuit.

If the coefficients for each transfer function are equal then, the circuit will act as the designed filter, *becoming* the filter.

$$G_{LP}(s) = G_{circuit}(s)$$

Consider the four common low pass filter circuits in the following table.

Some design criteria required:

1. Resistors: $1k\Omega \le R \le 100k\Omega$
2. Capacitors: $1nF \le C \le 5\mu F$

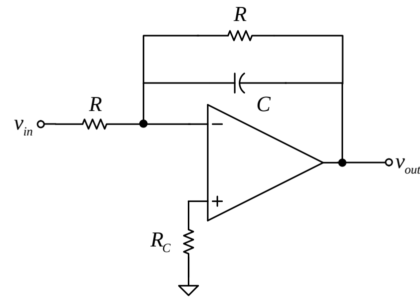
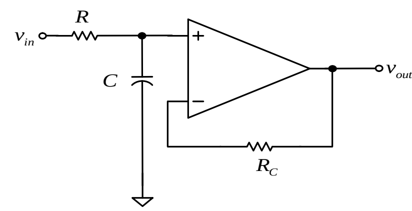
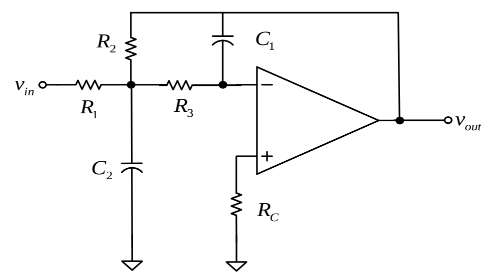
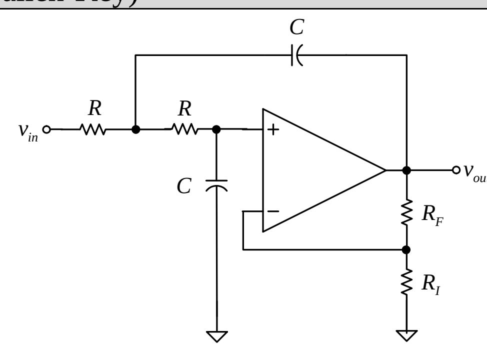
More detail in the following chapters on tolerance, types, material, etc…

For this example, there are 3 poles, so 2 cascaded circuits (or *stages*) are required. The first stage will be a single pole circuit and the second stage will be a two pole circuit. Does the position of the circuits matter, i.e, which is first or last? The answer is 'No'.

For this example, non-inverting circuits will be chosen (by decree). Does the choice of circuit matter, i.e. whether non-inverting or inverting? In general, the answer is 'No", but this choice will dictated by the application of the user. Later chapters will contain material to help in the decision making process.

EXTREMELY IMPORTANT NOTE:
In the solution of this example and all filter design problems, **ALWAYS** start with the **DENOMINATOR.**

Table of Low Pass Filter circuits

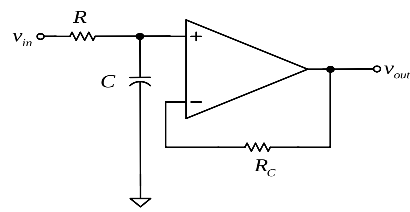
Transfer Function	Balance Inputs	Comments	Circuit Schematic
One Pole, Inverting			
$G(s)=\dfrac{-\,{}^{1}\!/\!{}_{R_I C}}{s+{}^{1}\!/\!{}_{R_F C}}$	$R_C = R_F \parallel R_I$	Unity gain	
One Pole, Non-inverting			
$G(s)=\dfrac{{}^{1}\!/\!{}_{RC}}{s+{}^{1}\!/\!{}_{RC}}$	$R_C = R$	Unity gain	
Two Pole, Inverting (known a equal component Multi-Feedback)			
$G(s)=\dfrac{-\left(\frac{R_2}{R_1}\right)\frac{1}{R_2R_3C_1C_2}}{s^2+\left(\frac{1}{R_1C_2}+\frac{1}{R_3C_2}+\frac{1}{R_2C_2}\right)s+\frac{1}{R_2R_3C_1C_2}}$	$R_C = R_3 + R_1 \parallel R_2$	See appendix for MFB	
Two Pole, Non-inverting (known a equal component Sallen-Key)			
$G(s)=\dfrac{A\cdot\frac{1}{R^2C^2}}{s^2+\left(\frac{3-A}{RC}\right)s+\frac{1}{R^2C^2}}$ where, $A=1+\dfrac{R_F}{R_I}$	$2R = R_F \parallel R_I$	Unstable when A>3	

<u>Note:</u>

1. The number of reactive components equals the number of poles equals the order of the circuit.
2. The first order circuit have a gain of 0 dB (A=1) and have a voltage follower topology.

Circuit Implementation:

Stage 1: 1 pole, non-inverting op-amp.



Set the filter transfer function equal to the filter transfer function.

$$G_{1^{st}\ \text{Pole}}(s) = G_{Circuit}(s)$$

$$\frac{25{,}132}{(s+37{,}111)} = \frac{1/RC}{s+1/RC}$$

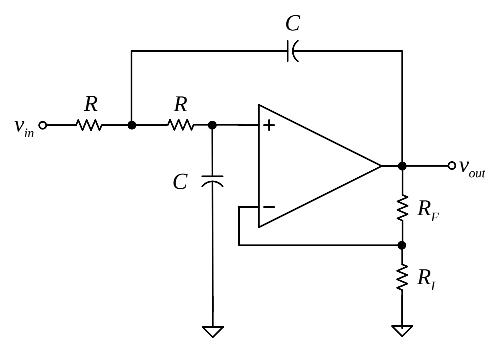
Starting with the denominator: $\frac{1}{RC} = 37{,}111$, choose a capacitance value, $C = 10nF$.

The resulting resistance is: $R = 2{,}694\Omega$.

All that is left is to balance the op-amp inputs (how to do this and why are at the end of this chapter).

$$R_C = R = 2{,}694\Omega$$

Stage 2: 2 pole, non-inverting op-amp.



$$G_{2\ \text{Pole}}(s) = G_{Circuit}(s)$$

$$\frac{631.7\times10^6}{\left(s^2+37{,}111s+1.377\times10^9\right)} = \frac{A\cdot\frac{1}{R^2C^2}}{s^2+\left(\frac{3-A}{RC}\right)s+\frac{1}{R^2C^2}}$$

Starting with the denominator:

1. $\frac{1}{R^2C^2} = 1.377\times10^9$, choose a capacitance value, $C = 10nF$

   $$R = \sqrt{\frac{1}{C^2\cdot1.377\times10^9}} = 2{,}695\Omega$$

2. $\frac{3-A}{RC} = 37{,}111 \quad \Rightarrow \quad A = 3 - RC\cdot37{,}111 = 1.9999 = 2$

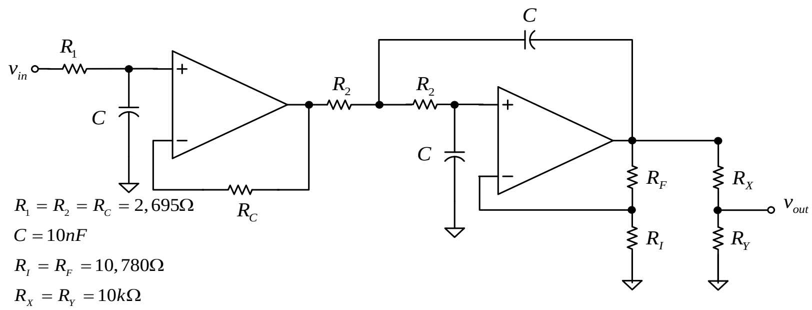
Balance Inputs:

1. $A = 1+\frac{R_F}{R_I} \quad \Rightarrow 2 = 1+\frac{R_F}{R_I} \quad \Rightarrow R_F = R_I$

2. $2R = R_F \| R_I \quad \Rightarrow 5{,}390 = \frac{R_F\cdot R_I}{R_F+R_I} = \frac{R_I\cdot R_I}{R_I+R_I} = \frac{R_I^{\,2}}{2R_I} = \frac{R_I}{2}$

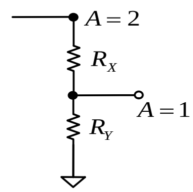
   $\Rightarrow R_I = R_F = 10{,}780\Omega$

At this point, all the component values have been determined.



Checking the specification (do lose sight of the original problem), it is noted that the gain should not exceed 0 dB. The gain of the 1st stage is 1 (unity gain), and the gain of the 2nd stage is 2, yielding a total cascade gain of 2. It is required to reduce the gain to 0 dB or A=1. A resistive voltage divider on the output will reduce the overall gain to the appropriate value.

$1 = \frac{R_Y}{R_X + R_Y} \cdot 2$, choose $R_Y = 10k\Omega$, yielding $R_X = 10k\Omega$



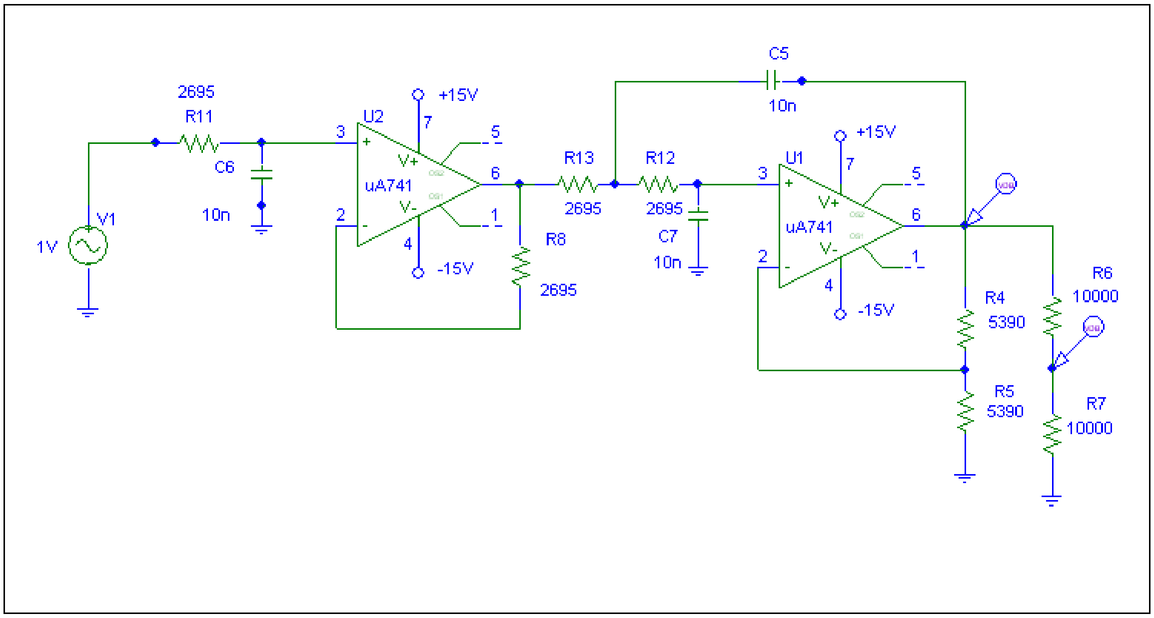
Any method can be used to reduce the overall gain to meet the specification. Resistive or capacitive voltage divider and additional op-amp stages can be used.

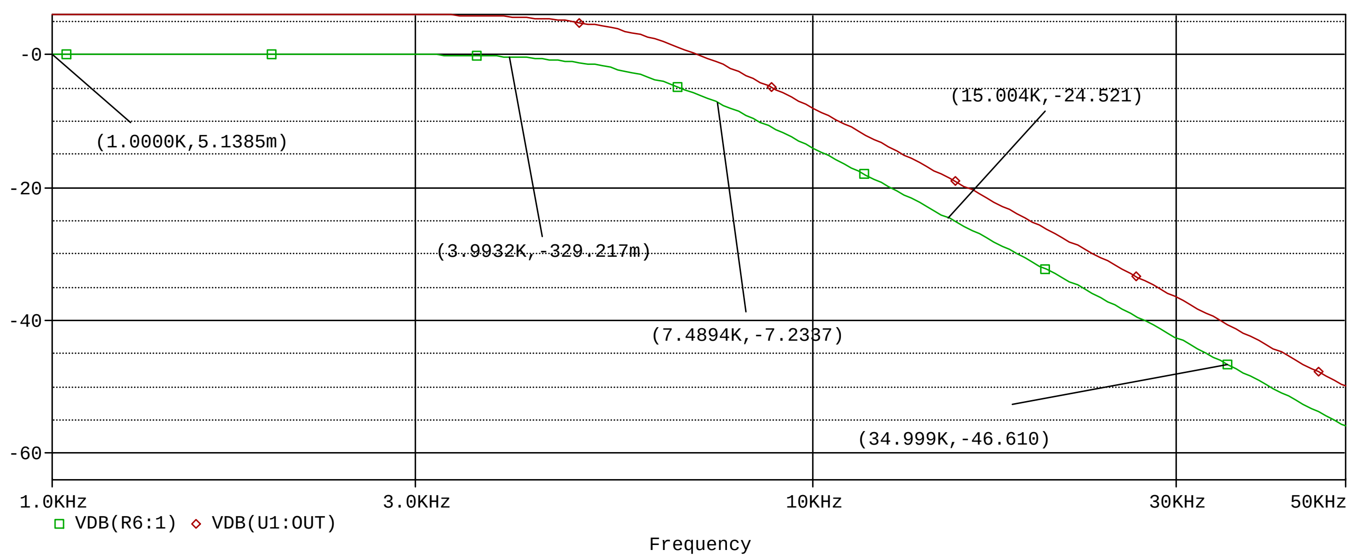
Note: The following are guidelines on how to handle the gain of these circuits.

1. Starting with the denominator of the transfer functions, choose C and calculate the resulting R's.
2. Calculate the required gain (A), if non-inverting.
   a. This A set the 'internal gain' of the circuit (will make it a BW, CH…)
3. Balance the inputs
4. Check the specifications for the 'outside gain' required by the outside world.
   a. Reduce (or increase) the gain as required **external** to the circuit.
5. If no outside gain is specified, then use the numerator to determine it.

This course will not use the numerator as a source of the specifications. It will be shown in Lab that the numerator will work out fine, but the design for the class work will always have some 'over riding' external specification.

PSpice Results:

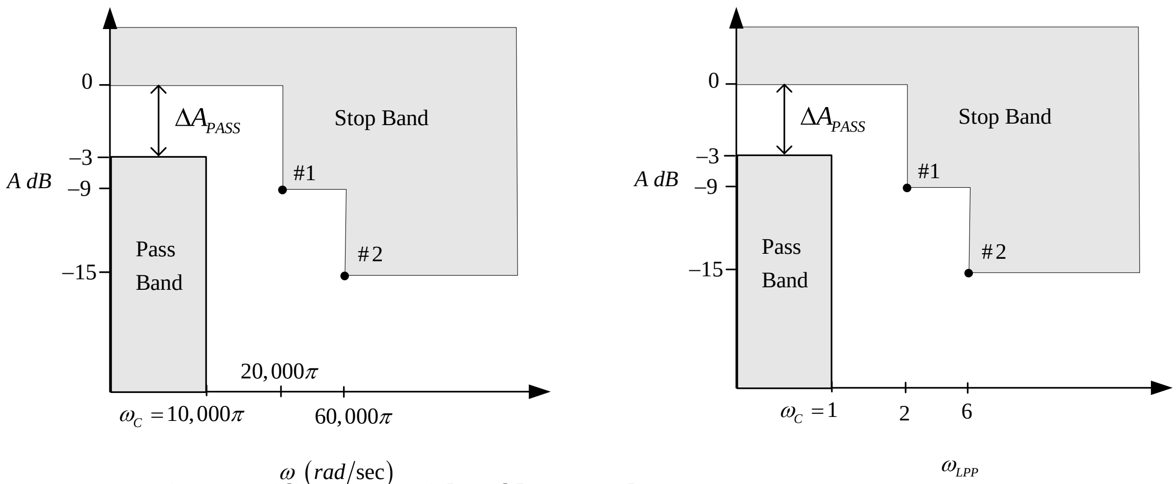




Note: The voltage divider acts as a level shifter.

Example #2

1. **Specifications**
   Pass-band: All frequencies <5 kHz must be passed with a gain between 0dB and -3dB
   Stop-band: -9dB @ 10 kHz
   -15dB @ 30 kHz



2. **Summarize specifications with a filter mask**

3. **Transform to the Low Pass Prototype (LPP)**
   ω_C=10,000π rad/sec

4. **Choose an approximation:**
   This will be a Butterworth filter by decree.

5. **Determine the required order of the filter.**
   2 stop band specifications that must be checked

1st Stop point:

$$n_1 = \frac{\log\left(\frac{\left(10^{-0.1(-9)}-1\right)}{\left(10^{-0.1(-3)}-1\right)}\right)}{2\log\left(\frac{2}{1}\right)} = \frac{\log\left(\frac{6.9433}{0.9953}\right)}{0.6021} = \frac{0.8436}{0.6021} = 1.401 \Rightarrow 2$$

2nd Stop point:

$$n_2 = \frac{\log\left(\frac{\left(10^{-0.1(-15)}-1\right)}{\left(10^{-0.1(-3)}-1\right)}\right)}{2\log\left(\frac{6}{1}\right)} = \frac{\log\left(\frac{30.6223}{0.9953}\right)}{1.5563} = \frac{1.4881}{1.5563} = 0.9562 \Rightarrow 1$$

The overall minimum order required to meet the specifications is 2.

6. Determine the transfer function ($G_{LPP}(s)$) for the Low Pass Prototype.

a. Calculate the ripple factor ε:

$$\varepsilon = \sqrt{\left(10^{-0.1\Delta A_{PASS}(dB)} - 1\right)} = \sqrt{\left(10^{-0.1(-3)} - 1\right)} = 0.9976$$

b. Radius of poles:

$$\rho = \varepsilon^{-\frac{1}{n}} = \left(0.9976\right)^{-\frac{1}{2}} = 1.001 \Rightarrow 1$$

A 'shortcut' can be used if the critical frequency or cutoff frequency is given at the ω_{-3dB} or f_{-3dB}. A set of LPP transfer functions have been developed using this common passband specification. The tables are located in the Appendix for a number of Approximations. As an example, for a Butterworth, the order needs to be determined then from the tables, $G(s)_{LPP}$ is quickly obtained. From the generic transfer function: $\omega_0 = \omega_{-3dB}$ in the denominator (only valid for Butterworth filters). The remaining denormalization is performed to obtain the overall filter transfer function.

$$G(s)_{LPP} = \frac{1}{s^2 + 1.4142s + 1}$$ Obtained from the Butterworth table, n=2.

7. Denormalize $G_{LPP}(s)$ to the $G_{LP}(s)$ with the original frequency scale.
To denormalize, apply the following substitution in $G_{LPP}(s)$.

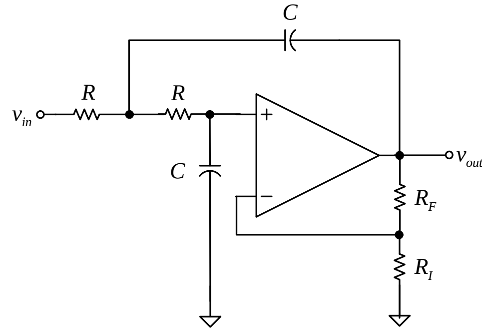
$$G_{LP}(s) = G_{LPP}(s)\Big|_{S=\frac{s}{\omega_C}} = G_{LPP}(s)\Big|_{S=\frac{s}{10{,}000\pi}}$$

$$G_{LP}(s) = \frac{1}{\left(\left(\frac{s}{10{,}000\pi}\right)^2 + 1.4142\left(\frac{s}{10{,}000\pi}\right) + 1\right)}$$

$$G_{LP}(s) = \frac{(10{,}000\pi)^2}{\left(s^2 + 1.4142\cdot(10{,}000\pi)s + (10{,}000\pi)^2\right)} = \frac{986.96\times10^6}{\left(s^2 + 44{,}428s + 986.96\times10^6\right)}$$

<u>Implementation of a transfer function into a circuit:</u>
For this example, the non-inverting Sallen-Key circuit will be chosen (by decree).

<u>Circuit Implementation:</u>



$$G_{2\ \text{Pole}}(s) = G_{Circuit}(s)$$

$$\frac{986.96\times10^6}{\left(s^2 + 44{,}428s + 986.96\times10^6\right)} = \frac{A\cdot\frac{1}{R^2C^2}}{s^2 + \left(\frac{3-A}{RC}\right)s + \frac{1}{R^2C^2}}$$

Starting with the denominator:

1. $\dfrac{1}{R^2C^2} = 986.96\times10^6$, choose a capacitance value, $C = 5nF$

   $$R = \sqrt{\frac{1}{C^2 \cdot 986.96\times10^6}} = 6,366\Omega$$

2. $\dfrac{3-A}{RC} = 44,428 \quad \Rightarrow \quad A = 3 - RC\cdot 44,428 = 1.5858$ - **<u>STOP!!!!</u>**

   It was determined earlier in this chapter that the 2 pole Butterworth filter must have fixed gain of A=1.586. This calculation proves it to be true --- **<u>A=1.586</u>**.

Balance Inputs:

1. $A = 1 + \dfrac{R_F}{R_I} \quad \Rightarrow 1.586 = 1 + \dfrac{R_F}{R_I} \quad \Rightarrow R_F = 0.586R_I$

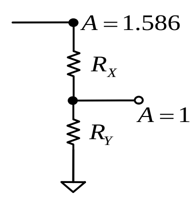
2. $2R = R_F \| R_I \quad \Rightarrow 12,732 = \dfrac{R_F \cdot R_I}{R_F + R_I} = \dfrac{0.586R_I \cdot R_I}{0.586R_I + R_I} = \dfrac{0.586{R_I}^2}{1.586R_I} = 0.3695R_I$

   $\Rightarrow R_I = 34,457\Omega$

   $\Rightarrow R_F = 20,192\Omega$

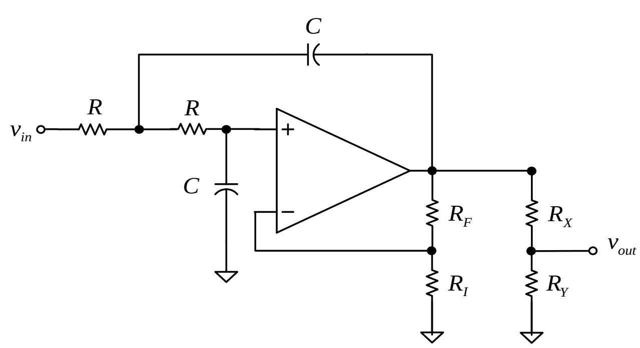
Checking the specification need 0dB on output (A=1), the output from the circuit is A=1.586.

$1 = \dfrac{R_Y}{R_X + R_Y}\cdot 1.586$, choose $R_Y = 10k\Omega$, yielding $R_X = 5.86k\Omega$



The final circuit is:

$R = 6,366\Omega$
$C = 5nF$
$R_I = 34,457\Omega$
$R_F = 20,192\Omega$
$R_X = 5.86k\Omega$
$R_Y = 10k\Omega$

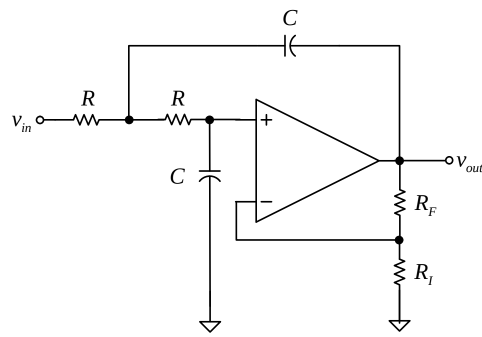


<u>PSpice Results:</u> Simulate to verify the result.

<u>Example #3 Implementation of a given transfer function into a circuit:</u>
Implement the following transfer function into a Sallen-Key equal component filter. Use C=10nF, the output gain should not exceed 0dB, balance inputs.

$$G(s) = \frac{324.45\times10^6}{s^2 + 9,559s + 216.3\times10^6}$$

<u>Circuit Implementation:</u>



$$G_{2\text{ Pole}}(s) = G_{Filter}(s)$$

$$\frac{324.45\times10^6}{s^2 + 9,559s + 216.3\times10^6} = \frac{A\cdot\frac{1}{R^2C^2}}{s^2 + \left(\frac{3-A}{RC}\right)s + \frac{1}{R^2C^2}}$$

Starting with the denominator:

1. $\frac{1}{R^2C^2} = 216.3\times10^6$, choose a capacitance value, $C = 10nF$

   $$R = \sqrt{\frac{1}{C^2\cdot216.3\times10^9}} = 6,800\Omega$$

2. $\frac{3-A}{RC} = 9,559 \qquad \Rightarrow \qquad A = 3 - RC\cdot9,559 = 2.35$ - **<u>STOP!!!!</u>**

   <u>If</u> the instructions called for this to be a Butterworth, the gain would need to be changed to A=1.586 (this is a 2nd order system). However, since no approximation was given, then the calculated gain must be used. It will be shown later that this is a Chebyshev filter and how it can be determined.

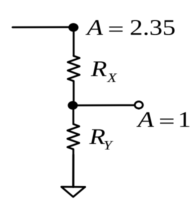
Balance Inputs:

1. $A = 1 + \frac{R_F}{R_I} \quad \Rightarrow 2.35 = 1 + \frac{R_F}{R_I} \quad \Rightarrow R_F = 1.35R_I$
2. $2R = R_F \| R_I \quad \Rightarrow 13,600 = \frac{R_F\cdot R_I}{R_F + R_I} = \frac{1.35R_I\cdot R_I}{1.35R_I + R_I} = \frac{1.35R_I^{\;2}}{2.35R_I} = 0.5745R_I$

   $\Rightarrow R_I = 23,673\Omega$
   $\Rightarrow R_F = 31,958\Omega$

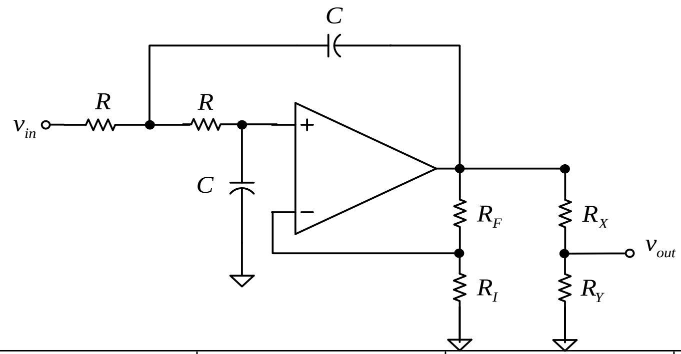
Checking the specification need 0dB on output (A=1), the gain from the circuit is A=2.35.

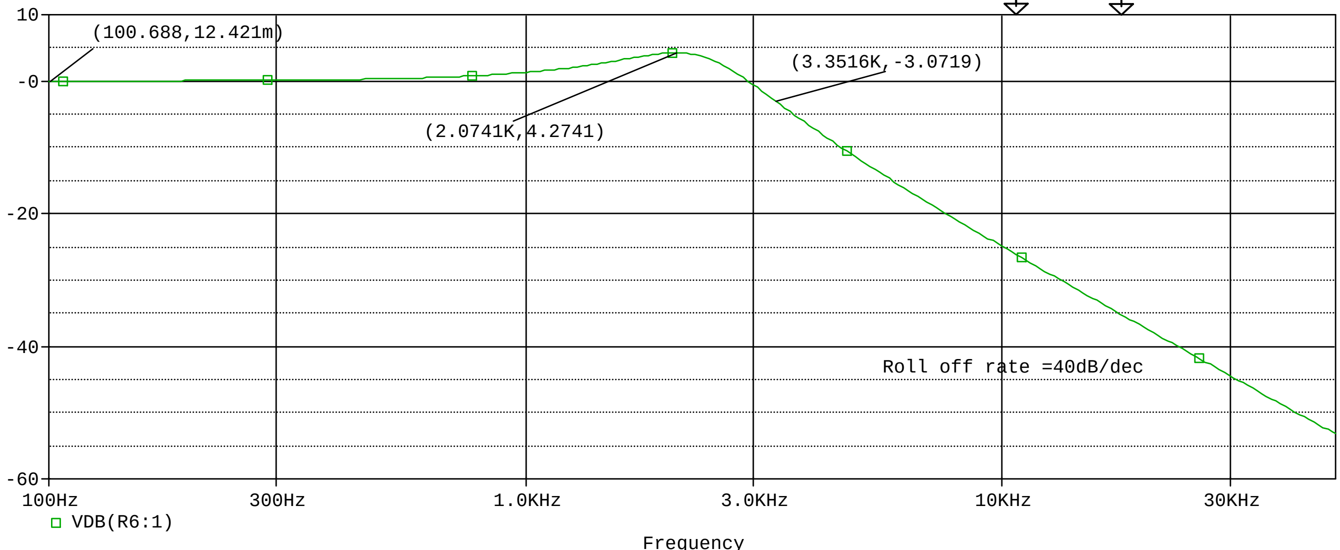
$1 = \frac{R_Y}{R_X + R_Y}\cdot2.35$, choose $R_Y = 10k\Omega$, yielding $R_X = 13.5k\Omega$



The final circuit is:

$R = 6,800\Omega$
$C = 10nF$
$R_I = 23,673\Omega$
$R_F = 31,958\Omega$
$R_X = 13.5k\Omega$
$R_Y = 10k\Omega$





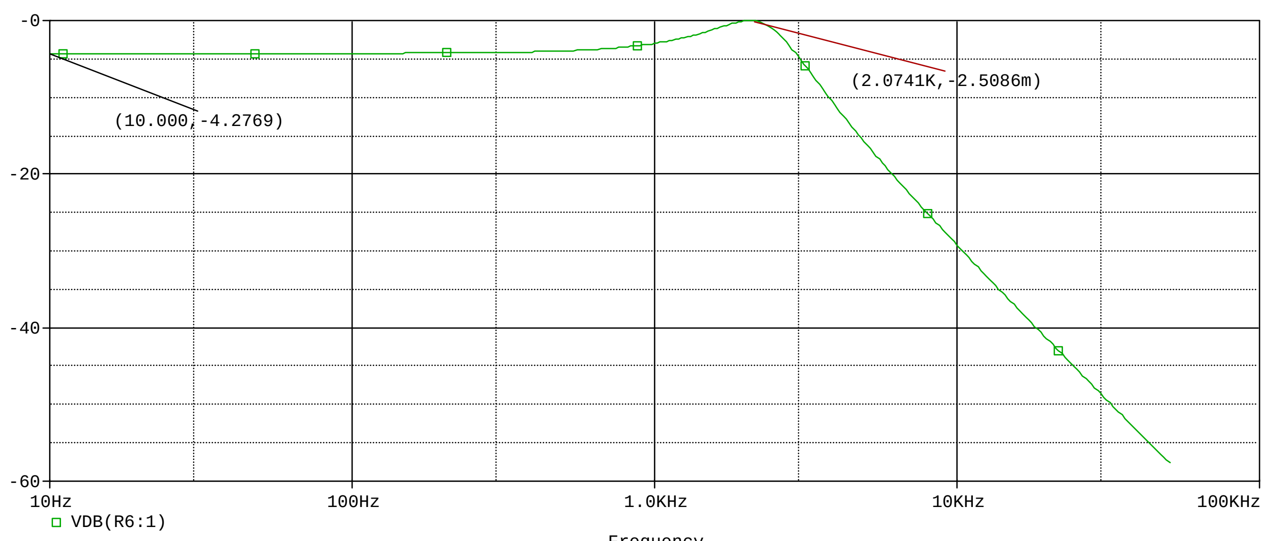
For the problem given: The above material and calculations is the correct answer. This is the answer expected on quiz or test, when only a transfer function is given. However, for the complete filter design, one more step is required.

Notice: The ripple goes above the 0dB point to 4.275dB even with the proper voltage divider to reduce the calculated gain. The problem is that even ordered Chebyshev filter ripple 'above' 0dB. The total gain of the circuit is the stage gain (A=2.35) and the ripple (A=1.636), yielding a total gain of 3.845. Odd order Chebyshev go ripple below 0dB and do not have this 'problem'.

$0dB \Rightarrow A = 1$

$4.275\text{dB} \Rightarrow A = 1.636$

$1 = \dfrac{R_Y}{R_X + R_Y} \cdot 3.845$, choose $R_Y = 10k\Omega$, yielding $R_X = 28.45k\Omega$



A word about balanced inputs of op-amps:

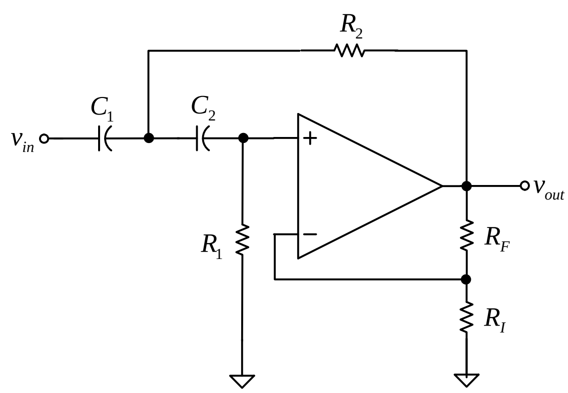
For proper operation of a circuit using an op-amp, it is required to compensate for the currents to the inputs of the op-amp. If the inputs are not balanced, then the op-amp will draw a compensating current between the positive and negative inputs and can generate offset voltages on the output and yielding poor results and reliability. Since the op-amp positive and negative inputs are 'virtually' tied together, it only required to insure that the resistance to each input is equal, thus equal and balanced input currents.

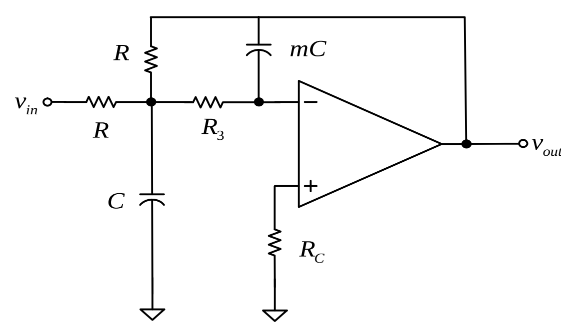
Is it really necessary to balance inputs to the circuit? In our class, the answer is 'no' and it not required to balance inputs in the laboratory, but you **WILL** be required to balance the inputs for the coursework and simulations. In real life, the answer is 'yes, unless'. Why the difference? In our labs, a sine wave is used as the test input, which is well balanced to begin with. In real life, unless the input is known (sine, square, etc…), balanced inputs are necessary.

To balance op-amp input:

- DC analysis
- Capacitors are open, inductors are shorts, all input and output nodes are grounded.
- Find the resistance at the positive input $(R+)$
- Find the resistance at the negative input $(R-)$
- Set $R+ = R-$, analyze the resistances to determine the values for balance.

Examples:





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