4.3 DC Motor in the Laplace Domain

Speed

In section 3.6, the DC motor's performance in the time domain is given as

$$\omega' + \frac{1}{\tau}\omega = \frac{m}{\tau}v$$

ω = speed of the motor (RPM)
ω_0 = initial speed (RPM)
τ = time constant (sec)
m = motor's gain $\left(\frac{\text{RPM}}{\text{V}}\right)$
v = applied voltage (V)

This equation transforms into the Laplace domain as

$$sW - \omega_0 + \frac{1}{\tau}W = \frac{m}{\tau}V$$

Adding w_0 to both sides $\quad W + \frac{1}{\tau}W = \frac{m}{\tau}V + \omega_0$

Multiplying by τ $\quad s\tau W + W = mV + \tau\omega_0$

Factoring out W $\quad W(\tau s + 1) = mV + \tau\omega_0$

Dividing by the term in ( ) $\quad W = \frac{mV}{(\tau s+1)} + \frac{\tau\omega_0}{(\tau s+1)}$ **DC motor's first order response**

This looks like the RC first order equations $\quad V_{\text{out}}(s) = \frac{E_{\text{in}}}{\tau s+1}$
with a term for the applied input and another for the initial speed.

Example 4-2
Find the response of the motor from Section 3.6 with:
m = 20.57 RPM/V,
τ = 0.362 sec,
ω_0 = 277 RPM,
V = 60 V

These are the same conditions as Example 3-8, completed with differential equations.

Solution

$$W = \frac{mV}{(\tau s + 1)} + \frac{\tau \omega_0}{(\tau s + 1)}$$

For a step input voltage

$$V = \frac{E_{\text{in}}}{s}$$

$$W = \frac{mE_{in}}{(\tau s + 1)} + \frac{\tau \omega_0}{(\tau s + 1)}$$

The conversion back to the time domain can be done in two steps, because the transforms are just integral, infinitesimal summations.

$$\mathcal{L}\{f_1(t) + f_2(t) + f_3(t) + \cdots\} = \mathcal{L}\{f_1(t)\} + \mathcal{L}\{f_2(t)\} + \mathcal{L}\{f_3(t)\} + \cdots$$

So

$$\mathcal{L}^{-1}\{f_1(t) + f_2(t)\} = \mathcal{L}^{-1}\{f_1(t)\} + \mathcal{L}^{-1}\{f_2(t)\}$$

The first term uses line 17a, and the second term uses line 13a.

17a.	$\frac{A}{s(\tau s + 1)}$	$A(1 - e^{-t/\tau})$
13a.	$\frac{A}{\tau s + 1}$	$\frac{A}{\tau} e^{-t/\tau}$

The first term has $A = mE_{\text{in}} = 20.57 \frac{\text{RPM}}{\text{V}} \times 60\ \text{V} = 1234\ \text{RPM}$

The second term has
$A = \tau \omega_0 = 0.365\ \text{sec} \times 277\ \text{RPM} = 101\ \text{RPM sec}$

Combining these gives

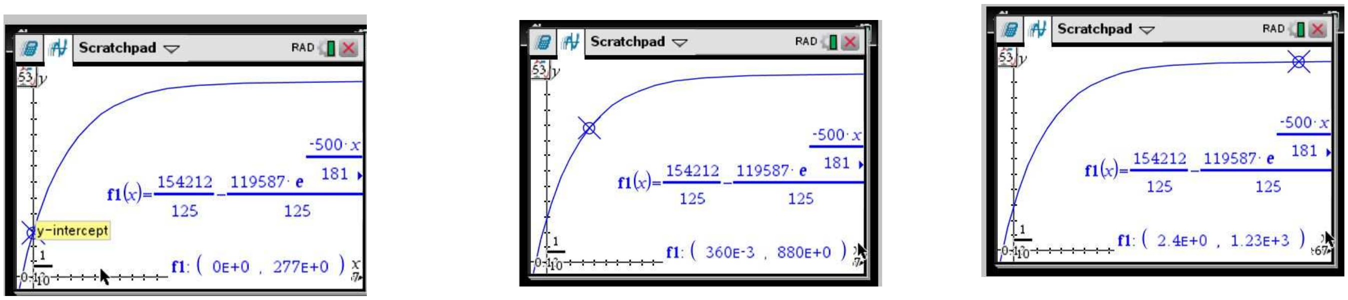
$$\omega = 1234\ \text{RPM} \left(1 - e^{-\frac{t}{0.365\, sec}}\right) + \frac{101\ \text{RPM sec}}{0.365\ \text{sec}} e^{-\frac{t}{0.365\, sec}}$$

$$\omega = 1234\ \text{RPM} \left(1 - e^{-\frac{t}{0.365\, sec}}\right) + 277\ \text{RPM}\ e^{-\frac{t}{0.365\, sec}}$$

At $t = 0$, the first term goes to 0, and the second exponential is 1, giving 277 RPM, which is ω_0.

As $t \to \infty$, both exponentials go to zero. The second term, the initial speed decays away. The first term goes to the final speed of 1234 RPM.

These are precisely the results from the differential equations solution in Figure 3-39, but obtained in *fewer* steps, using *algebra*.

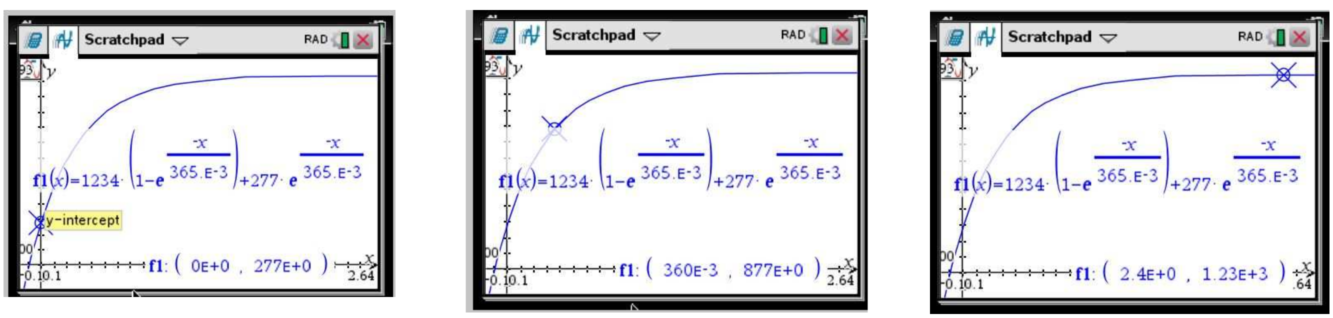


(a) Differential equations solution $\omega(0)$ = 277 RPM

(b) Differential equations solution $\tau \sim 0.360$ sec

(c) Differential equations solution ω(2.4 sec) = 1230 RPM

Figure 3-39 Calculated and plotted differential equations solution of the motor's step response



(a) Laplace domain solution $\omega(0)$ = 277 RPM

(b) Laplace domain solution $\tau \sim 0.360$ sec

(c) Laplace domain solution ω(2.4 sec) = 1230 RPM

Figure 4-9 Plot of Laplace solution of Example 4-2

The solution using Laplace can be even simpler. The Laplace domain equation can be entered into Matlab, and ltiview can plot the time domain result without your ever making the Laplace to time domain inversions. Look at Figure 4-10.

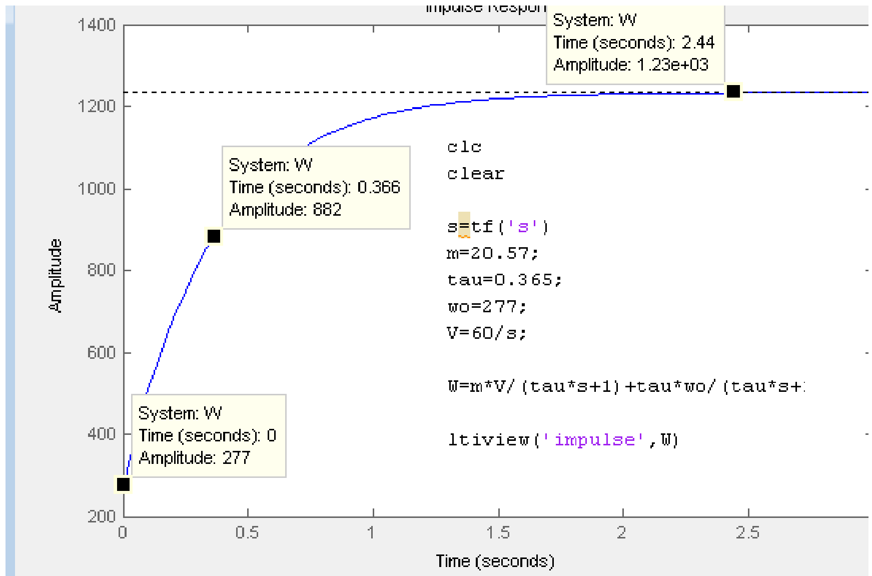


Figure 4-10 Motor speed response calculated and plotted with Matlab's ltiview

Position

To get position from speed, just integrate.

$$\theta = \int_0^t \omega(t)dt + \theta_0$$

Starting at the origin $\theta_0 = 0$, the Laplace domain, this becomes

$$\Theta = \frac{W(s)}{s}$$

So, just divide the Laplace speed by *s*, transform and clean up the algebra.

$$W = \frac{mE_{\text{in}}}{(\tau s + 1)} + \frac{\tau\omega_0}{(\tau s + 1)}$$

In order that this technique's results can be compared to those from section 3.6 the variables are replaced with constants. Remember, in addition to changing to deg/sec, there is a 1:100 reduction gear.

$$m = \frac{20.57}{100}\frac{\text{RPM}}{\text{V}} = 1.234\frac{\frac{\text{deg}}{\text{sec}}}{\text{V}}$$

$$E_{in} = \frac{60\ \text{V}}{s}$$

τ = 0.365 sec

$$\omega_0 = \frac{277}{100}\text{RPM} = 6.06\ \frac{\text{deg}}{\text{sec}}$$

$$W = \frac{74.0}{s(\tau s + 1)} + \frac{2.21}{(\tau s + 1)}$$

$$\Theta = \frac{W}{s} = \frac{74.0}{s^2(\tau s+1)} + \frac{2.21}{s(\tau s+1)}$$

The first term is transformed using line 18a, where A = 74 and τ = 0.365.

18a. $\dfrac{A}{s^2(\tau s + 1)}$ $\quad A\tau\left(e^{-t/\tau} + \dfrac{t}{\tau} - 1\right)$

The second term is transformed using line 17a, where A = 2.21 and τ = 0.365.

17a. $\dfrac{A}{s(\tau s + 1)}$ $\quad A(1 - e^{-t/\tau})$

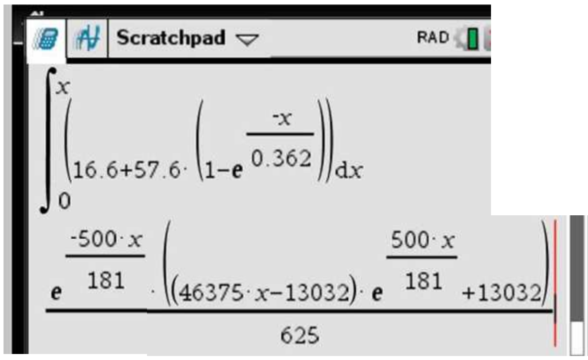
Making those substitutions

$$\theta = 27.01\left(e^{-\frac{t}{\tau}} + 2.74t - 1\right) + 2.21\left(1 - e^{-\frac{t}{\tau}}\right)$$

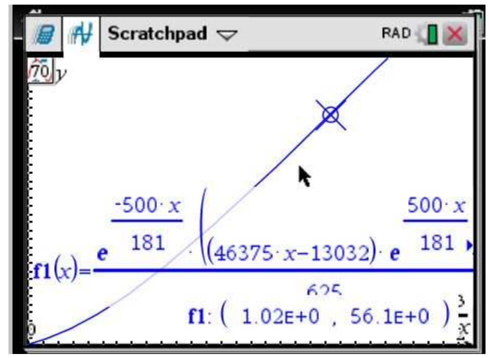
$$\theta = 27.01e^{-\frac{t}{\tau}} + 74t - 27 + 2.21 - 2.21e^{-\frac{t}{\tau}}$$

$$\theta = 74t - 20.9\left(1 - e^{-\frac{t}{\tau}}\right)$$

Below is the results using differential equations and the TI Nspire, shown in Figure 3-40.



(a) Integral solution



(b) Graph with trace After 1 second the gear has moved 56 degrees

Figure 3-40 Example 3-9 position solution and plot

The differential equations solution and the Laplace domain result simplifies to the same equation.

Figure 4-11 is the Matlab and ltiview plot of the Laplace domain equation, the same as the differential equation solution plot above. The additional *s* in each denominator is caused by the integration. The plot's *t* axis has been rescaled to match Figure 3-40.

```
clc
clear

s=tf('s')
m=1.234;
tau=0.365;
wo=6.06;
V=60/s;

position=m*V/(s*(tau*s+1))+tau*wo/(s*(tau*s+1));

ltiview('impulse',position)
```

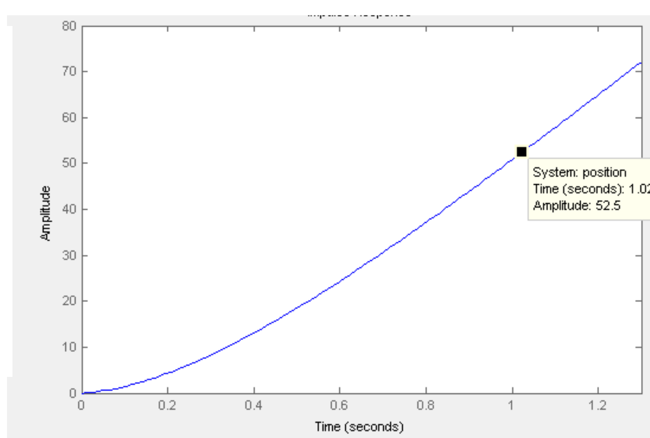


Figure 4-11 Matlab and ltiview plot