## 3.6 DC Motor

http://www.leeson.com/Products/products/DCMotors/lowvoltage.html

**Figure 3-35** DC motor

Fractional horse power motors are used extensively, from the small motor that moves the nose on a toy dog to the prime mover in an industrial robot. For a simple brushed DC motor to accurately complete its tasks, it must be driven with the correct voltage; the shaft begins to move; then it takes time for the motor to achieve its final response.

The motor converts electrical energy into mechanical (rotational) energy. But it takes time for the motor's speed to build up or coast down to a new level. That's just like the capacitor storing energy in its charge (electrostatic field) or the inductor storing energy in its magnetic field. Similar differential equations can be written and the motor's performance determined just as accurately. The derivation of that equation belongs in a study of rotating machines. The result is

### Speed

$$\omega' + \frac{1}{\tau}\omega = \frac{m}{\tau}v$$

$\omega$ = speed of the motor

$\frac{\text{revolutions}}{\text{minute}}$ (RPM)

$\tau$ = time constant

second

$m$ = motor's gain

$\frac{\text{RPM}}{\text{V}}$

$v$ = applied voltage

Motor manufactures' data sheets provide information about full speed RPM, maximum voltage, torque provided, and current requirements. Unfortunately, the two performance specifications needed to calculate the motor's dynamic response, $m$ and $\tau$ are usually not specified. But, they can be *measured*.

Mount and load the motor as it will be used. Then apply a low voltage and record the speed once it has stabilized. Repeat this process across the range of the motor. An example tabulation is shown in Figure 3-36. Notice that it is typical for the motor not to start until there is 10% to 20% of the maximum voltage applied. This produces the nonlinearity and the offset shown in the scatter plot.

In Excel, right click on a data point in the scatter plot. In the menu provided, select **Trend Line**. Then pick **Linear** and **Display Equation on Chart.** This gives the equation of the best-fit straight line.

y = 20.57x - 231.64 i.e. y=mx+b

$$m = 20.57\ \frac{\text{RPM}}{\text{V}}$$

| V | RPM |
|---|---|
| 0 | 0 |
| 10 | 0 |
| 20 | 120 |
| 30 | 230 |
| 40 | 460 |
| 50 | 730 |
| 60 | 980 |
| 70 | 1210 |
| 80 | 1460 |
| 90 | 1750 |

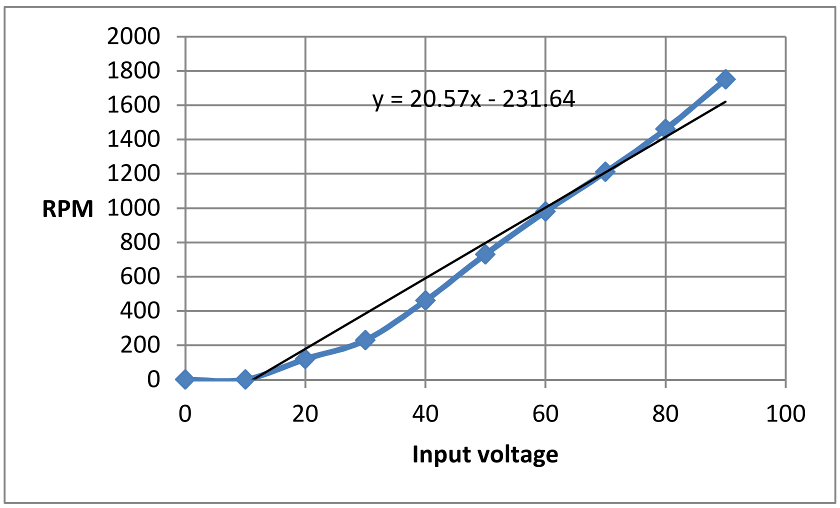



**Figure 3-36** Typical DC motor static performance analysis

$$\omega' + \frac{1}{\tau}\omega = \frac{m}{\tau}v$$

Measuring the time constant is a little more involved. The speed (RPM) must be captured as the motor changes speed. This can be done with an optical encoder. It is attached to the shaft and spins as the shaft spins. The output of the encoder is a frequency that is a direct measure of the speed. This frequency can be captured with a data acquisition card and displayed with Lab View. Or, the frequency can be converted to a voltage with a V-F IC, and the IC's output voltage captured with a digital oscilloscope. See Figure 3-37.

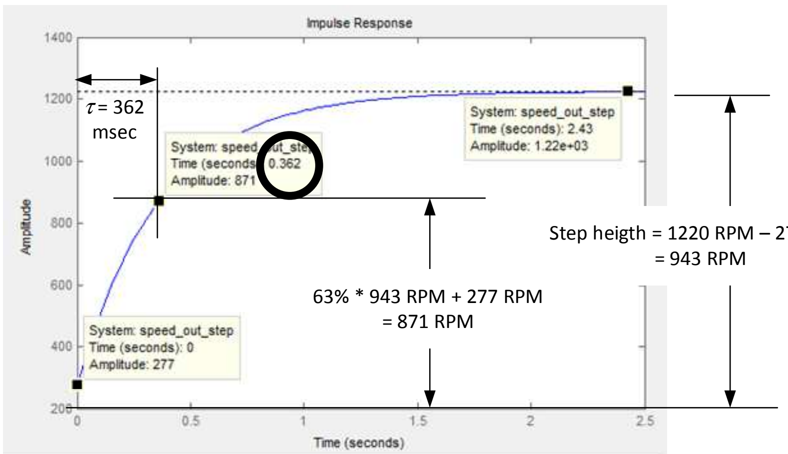



**Figure 3-37** Motor time constant measurement

The test begins by applying a voltage to the motor that spins it at about 1/3 of its full speed. Once the motor is spinning *steadily*, trigger the data acquisition system or oscilloscope to begin grabbing speed data, and step the motor voltage to about 2/3 of full speed. Continue catching speed data until the motor settles at its new level. This process may take several iterations to capture a full set of data, starting just before the voltage step and ending a little after the change is complete.

Print the data. A table is also useful.

- Find the step height, 1220 RPM – 277 RPM = 943 RPM
- The time constant is the time it takes to change by 63% of the maximum change. Calculate the change in one time constant. 63%*943 RPM = 871 RPM
- The time from when the step occurred until when the speed changes by 871 RPM = $\tau$ = 362 msec

$$\omega' + \frac{1}{0.362\ \text{sec}}\omega = 56.8\,\frac{\text{RPM}}{\text{v sec}}\,v$$

Speed is in RPM. Time is in seconds

**Example 3-8**

**a.** Setup and solve the differential equation for the motor whose parameters were developed in Figure 3-36 and 3-37. Apply the same initial conditions and voltage step as shown in Figure 3-37.

**b.** Plot the result and compare these calculations to the experimental data of Figure 3-37.

**Solution**

From Figure 3-37, the initial speed is $\omega(0) = 277$ RPM

The other end point of the step is at 1200 RPM. From Figure 3-36, the trendline linear approximation starts at 0 RPM, 10 V and reaches 1200 RPM at 70 V. So the *step* height is

$$v = 70\ \mathrm{V} - 10\ \mathrm{V} = 60\ \mathrm{V}$$

$$\omega' + \frac{1}{0.362\ \mathrm{sec}}\omega = 56.8\,\frac{\mathrm{RPM}}{\mathrm{V\ sec}}\,v$$

$$\omega' + \frac{1}{0.362}\omega = 56.8\ \times 60$$

Figure 3-38 gives the solution of the differential equation. This looks awkward, but perhaps a little familiar. A little algebra is needed.

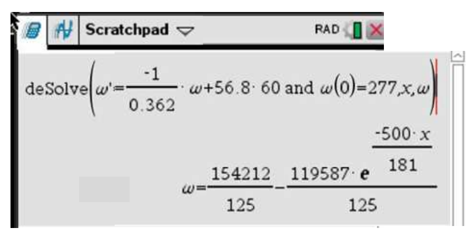


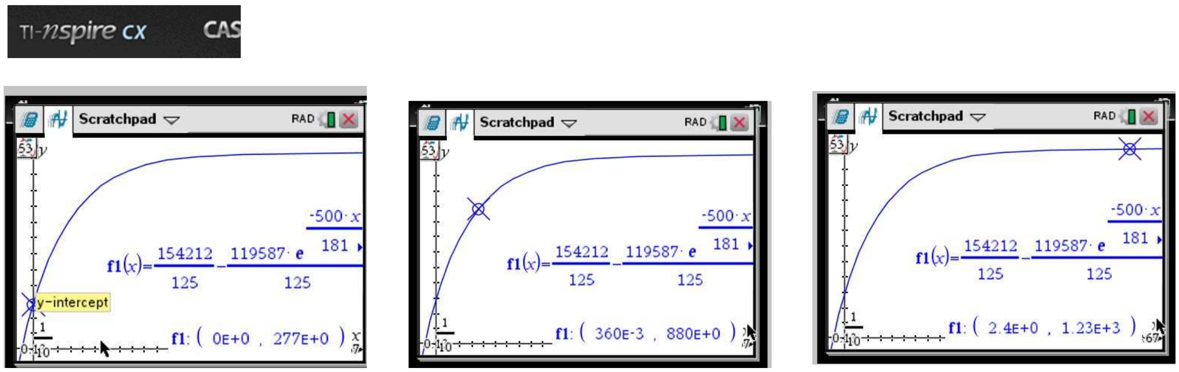

**Figure 3-38** Motor differential equation solution

$$\omega = 1234 - 957e^{-\frac{t}{0.362}}$$

$$\omega = 277 + 957\left(1 - e^{-\frac{t}{0.362}}\right)$$

At $t$ = 0 the second term falls out and the calculated speed is the measure initial speed.

The second term shows that the speed grows exponentially with a time constant of 0.362 seconds, exactly the results shown by the experimental step in Figure 3-37.




**(a)** Initial speed
$\omega(0)$ = 277 RPM

**(b)** Time constant
$\tau$ ~ 0.360 sec

**(c)** Final speed
$\omega$(2.4 sec) = 1230 RPM

**Figure 3-39** Calculated and plotted differential solution of the motor's response to a step

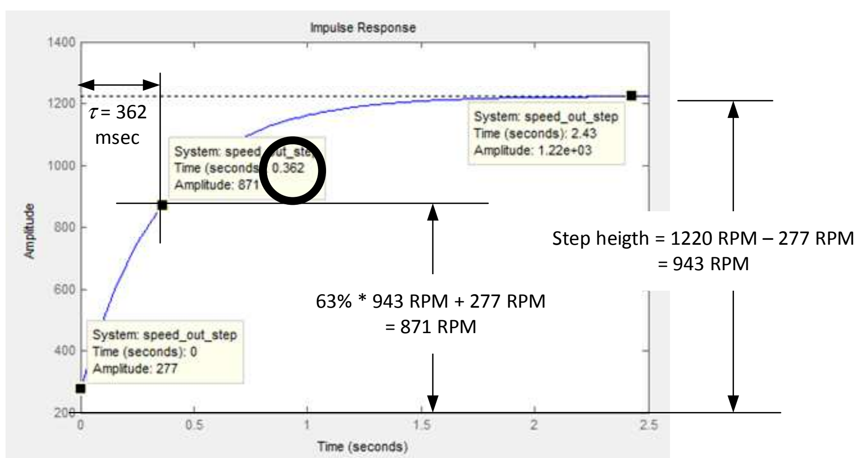



**Figure 3-37** (again) Measured motor performance

The shape and the three parameters calculated all match well with the experimental data.

## Position

Placing an element at a precise position is central to many manufacturing and nearly all robotics systems. Using the DC motor, with

$$\omega' + \frac{1}{\tau}\omega = \frac{m}{\tau}v$$

the relationship between shaft speed and its location is just

$$\omega = \frac{d\theta}{dt} = \theta'$$

where $\theta$ is the angular position.

$$\theta(t) = \int_0^t \omega\, dt + \theta_{t=0}$$

This looks straight forward, but remember, $\omega$ has been in RPM, i.e. revolutions per minute. Traditionally, time in an integration is in seconds and angle is in radians, though often degrees are preferred. The proper conversions must be included.

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**Example 3-9**

A 1:100 reduction gear is added to the shaft of the motor in Example 3-8. Plot the angular position of the gear's output in response to that step. Assume that at $t$ = 0 the shaft is passing through the origin $0^o$ even though it is moving at 277 RPM.

Scale the vertical axis in degrees and determine

$$\theta_{t=1\,sec}$$

**Solution**

From Example 3-8 the speed of the motor's shaft is

$$\omega_{\text{motor}} = 277\frac{\text{rev}}{\text{min}} + 957\frac{\text{rev}}{\text{min}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

The gear reduces these speeds by 1/100.

$$\omega_{\text{gear}} = 277\frac{\text{rev}}{\text{min}} \times \frac{1}{100} + 957\frac{\text{rev}}{\text{min}} \times \frac{1}{100}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

$$\omega_{\text{gear}} = 2.77\frac{\text{rev}}{\text{min}} + 9.57\frac{\text{rev}}{\text{min}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

The minutes now must be converted into seconds.

$$\omega_{\text{gear}} = 2.77\frac{\text{rev}}{\text{min}} \times \frac{1\text{ min}}{60\text{ sec}} + 9.57\frac{\text{rev}}{\text{min}} \times \frac{1\text{ min}}{60\text{ sec}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

$$\omega_{\text{gear}} = 0.0462\frac{\text{rev}}{\text{sec}} + 0.160\frac{\text{rev}}{\text{sec}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

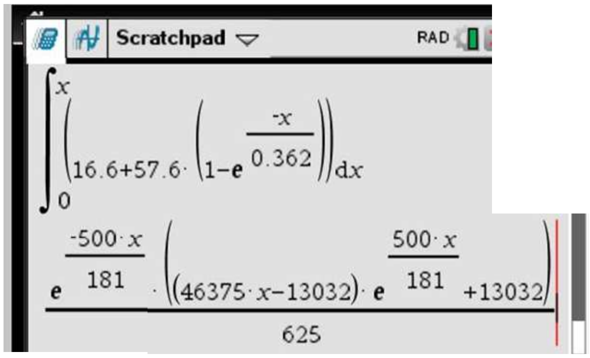
Now convert the revolutions to degrees.

$$\omega_{\text{gear}} = 0.0462\frac{\text{rev}}{\text{sec}} \times \frac{360\text{ deg}}{1\text{ rev}} + 0.160\frac{\text{rev}}{\text{sec}} \times \frac{360\text{ deg}}{1\text{ rev}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

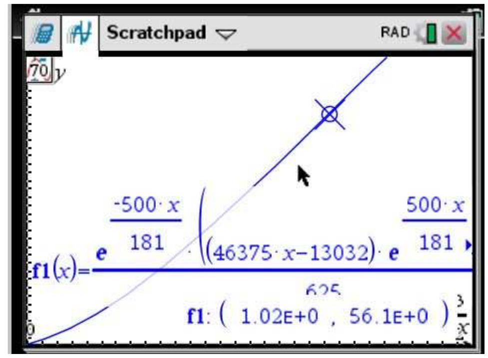
$$\omega_{\text{gear}} = 16.6\frac{\text{deg}}{\text{sec}} + 57.6\frac{\text{deg}}{\text{sec}}\left(1 - e^{-\frac{t}{0.362\,sec}}\right)$$

$$\theta(t) = \int_0^t \omega\, dt + \theta_{t=0}$$

The integral solution is shown in Figure 3-40 a, and the plot with a marker at 1 second in Figure 3-40 b.




**(a)** Integral solution




**(b)** Graph with trace After 1 second the gear has moved 56 degrees

**Figure 3-40** Example 3-9 position solution and plot

Since the problem asked for a plot and the angle at 1 second, the complicated form of the answer provided by the calculator's integration was just copied onto the Graph and then scaled. Initially, the gear's output moves nonlinearly, but eventually smoothes out to a steady rise. This makes physical sense. Eventually the shaft is turning at a new steady rate, sending the gear further and further along, at a steady rate. But, for the first several seconds, the shaft speed is rising exponentially. So position goes up, but slowly to start with and then faster and faster.

Simplifying the integral solution provides a position equation with a dominant first term that increases linearly with time, and a second term that goes to a constant.

$$\theta = 74.2\frac{\text{deg}}{\text{sec}}t - 20.9\ \text{deg}\left(1 - e^{-\frac{t}{0.362\ sec}}\right)$$

This is just like

$$y = \text{m}x + \text{b}$$

Although this constant is subtracted from the first term, it is a steady *offset* (b), caused by the shaft starting slowly. It does not alter the slope (m), i.e. how quickly the position increases.