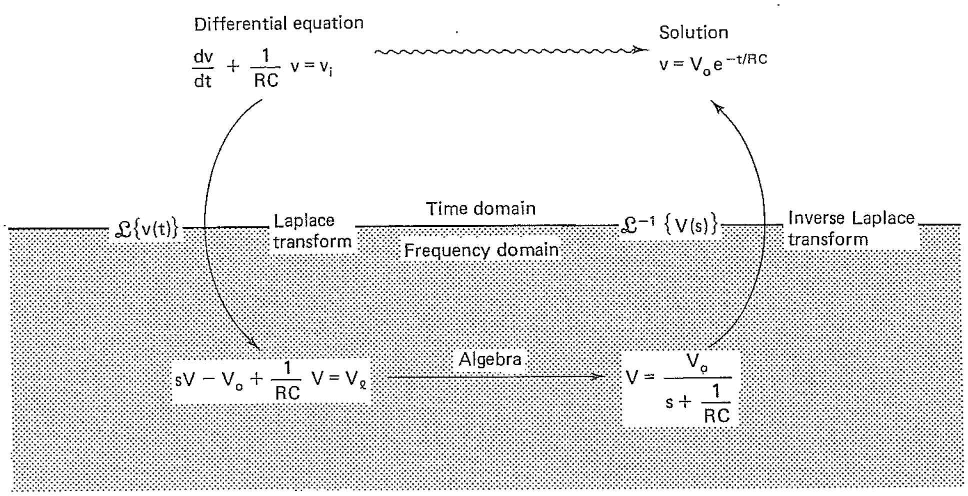

Figure A–1 Laplace transform solution of differential equation.

The inverse transformation, from a frequency-domain function, $F(s)$, to a time-domain function, $f(t)$, is

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$f(t) = \int_{a-i\infty}^{a+i\infty} F(s)e^{st}\, dt$$

Fortunately, a table of useful functions, with their transformations, has been developed. So you do not have to perform any integrations. Just use the table. Table A-1 gives those transforms that are found most often in process control applications. You can find much more extensive tables in any handbook of mathematical tables. The following examples illustrate how to use the techniques of Laplace transforms to solve differential equations.

TABLE A-1 LAPLACE TRANSFORMS

No.	$F(S)$	$f(t)$	Comments
1.	1	$\delta(t)$	Unit impulse
2.	$\frac{A}{s}$	$A(t) = \begin{cases} 0 & t < 0 \\ A & t \geqslant 0 \end{cases}$	Step
3.	$\frac{1}{s}$	$U(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geqslant 0 \end{cases}$	Unit step
4.	$\frac{A}{s^2}$	At	Ramp
5.	$\frac{2A}{s^3}$	At^2	Parabola
6.	$\frac{A\omega}{s^2 + \omega^2}$	$A \sin \omega t$	Sine

TABLE A-1 *(continued)*

No.	$F(S)$	$f(t)$	Comments
7.	$\dfrac{As}{s^2+\omega^2}$	$A\cos\omega t$	Cosine
8.	$aF(s)$	$af(t)$	
9.	$\dfrac{n!}{s^{n+1}}$	t^n	
10.	$sF(s)-f(0)$	$\dfrac{df(t)}{\mathrm{dt}}$	
11.	$s^2F(s)-sf(0)-\dfrac{df(0)}{dt}$	$\dfrac{d^2f(t)}{dt^2}$	
12.	$\dfrac{F(s)}{s}$	$\int f(t)\,dt$	
13a.	$\dfrac{A}{\tau s+1}$	$\dfrac{A}{\tau}e^{-t/\tau}$	Free response of first-order system
13b.	$\dfrac{A}{s+a}$	Ae^{-at}	
14a.	$\dfrac{A}{(\tau_1 s+1)(\tau_2 s+1)}$	$\dfrac{A}{\tau_1-\tau_2}(e^{-t/\tau_1}-e^{-t/\tau_2})$	Free response of second-order system ($\zeta>1$)
14b.	$\dfrac{A}{(s+a)(s+b)}$	$\dfrac{A}{b-a}(e^{-at}-e^{-bt})$	
15a.	$\dfrac{A}{(\tau s+1)^2}$	$\dfrac{At}{\tau^2}e^{-t/\tau}$	Free response of second-order system ($\zeta=1$)
15b.	$\dfrac{A}{(s+a)^2}$	Ate^{-at}	
16.	$\dfrac{A\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\dfrac{A\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t)$	Second-order system, free response ($\zeta<1$)
17a.	$\dfrac{A}{s(\tau s+1)}$	$A(1-e^{-t/\tau})$	First-order system response to a step input
17b.	$\dfrac{A}{s(s+a)}$	$\dfrac{A}{a}(1-e^{-at})$	
18a.	$\dfrac{A}{s^2(\tau s+1)}$	$A\tau\left(e^{-t/\tau}+\dfrac{t}{\tau}-1\right)$	First-order system response to a ramp input
18b.	$\dfrac{A}{s^2(s+a)}$	$\dfrac{A}{a^2}(e^{-at}+at-1)$	

TABLE A-1 *(continued)*

No.	$F(S)$	$f(t)$	Comments
19a.	$\dfrac{A\omega}{(s^2+\omega^2)(\tau s+1)}$	$\dfrac{A\omega\tau}{1+\omega^2\tau^2}e^{-t/\tau}+\dfrac{A}{\sqrt{1+\omega^2\tau^2}}\sin(\omega t-\psi)$ where $\psi=\tan^{-1}\omega\tau \quad (0<\psi<\pi)$	First-order system response to a sine input
19b.	$\dfrac{A\omega}{(s^2+\omega^2)(s+a)}$	$\dfrac{A\omega e^{-at}}{\omega^2+a^2}+\dfrac{A}{\sqrt{\omega^2+a^2}}\sin(\omega t-\psi)$ where $\psi=\tan^{-1}\omega/a \quad (0<\psi<\pi)$	
20a.	$\dfrac{A}{s(\tau_1 s+1)(\tau_2 s+1)}$	$A\left(1+\dfrac{\tau_1 e^{-t/\tau_1}-\tau_2 e^{-t/\tau_2}}{\tau_2-\tau_1}\right)$	Second-order system response to a step input $(\zeta>1)$
20b.	$\dfrac{A}{s(s+a)(s+b)}$	$\dfrac{A}{ab}\left(1+\dfrac{ae^{-bt}-be^{-at}}{b-a}\right)$	
21a.	$\dfrac{A}{s(\tau s+1)^2}$	$A\left(1-\dfrac{\tau+t}{\tau}e^{-t/\tau}\right)$	Second-order system response to a step input $(\zeta=1)$
21b.	$\dfrac{A}{s(s+a)^2}$	$\dfrac{A}{a^2}[1-(1+at)e^{-at}]$	
22.	$\dfrac{A\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$	$A\left[1+\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t-\psi)\right]$ where $\psi=\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{-\zeta} \quad (0<\psi<\pi)$	Second-order system response to a step input $(\zeta<1)$
23a.	$\dfrac{A}{s^2(\tau_1 s+1)(\tau_2 s+1)}$	$A\left(t-\tau_1-\tau_2-\dfrac{\tau_2^2 e^{-t/\tau_2}-\tau_1^2 e^{-t/\tau_1}}{\tau_1-\tau_2}\right)$	Second-order system response to a ramp input $(\zeta>1)$
23b.	$\dfrac{A}{s^2(s+a)(s+b)}$	$\dfrac{A}{ab}\left[t-\dfrac{a+b}{ab}-\dfrac{(b/a)e^{-bt}-(a/b)e^{-at}}{b-a}\right]$	
24a.	$\dfrac{A}{s^2(\tau s+1)^2}$	$A[t-2\tau+(t+2\tau)e^{-t/\tau}]$	Second-order system response to a ramp input $(\zeta=1)$
24b.	$\dfrac{A}{s^2(s+a)^2}$	$\dfrac{A}{a^2}\left[t-\dfrac{2}{a}+\left(t+\dfrac{2}{a}\right)e^{-at}\right]$	
25.	$\dfrac{A\omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)}$	$A\left[t-\dfrac{2\zeta}{\omega_n}+\dfrac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t-\psi)\right]$ where $\psi=2\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{-\zeta} \quad (0<\psi<\pi)$	Second-order system response to a ramp input $(\zeta<1)$

TABLE A-1 *(continued)*

No.	$F(S)$	$f(t)$	Comments
26a.	$\dfrac{A\omega}{(s^2+\omega^2)(\tau_1 s+1)(\tau_2 s+1)}$	$A\left[\dfrac{\tau_1^2\omega e^{-t/\tau_1}}{(\tau_1-\tau_2)(1+\omega^2\tau_1^2)}+\dfrac{\tau_2^2\omega e^{-t/\tau_2}}{(\tau_2-\tau_1)(1+\omega^2\tau_2^2)}+\dfrac{\sin(\omega t-\psi)}{[(1+\omega^2\tau_1^2)(1+\omega^2\tau_2^2)]^{1/2}}\right]$ where $\psi=\tan^{-1}\omega\tau_1+\tan^{-1}\omega\tau_2$	Second-order system response to a sine input ($\zeta>1$)
26b.	$\dfrac{A\omega}{(s^2+\omega^2)(s+a)(s+b)}$	$A\left[\dfrac{\omega e^{-at}}{(b-a)(\omega^2+a^2)}+\dfrac{\omega e^{-bt}}{(a-b)(\omega^2+b^2)}+\dfrac{\sin(\omega t-\psi)}{[(\omega^2+a^2)(\omega^2+b^2)]^{1/2}}\right]$ where $\psi=\tan^{-1}\dfrac{\omega(a+b)}{ab-\omega^2}\quad(0<\psi<\pi)$	
27a.	$\dfrac{A\omega}{(s^2+\omega^2)(\tau s+1)^2}$	$\dfrac{A}{1+\omega^2\tau^2}\left[\dfrac{\omega t+2\omega\tau}{1+\omega^2\tau^2}e^{-t/\tau}+\sin(\omega t-\psi)\right]$ where $\psi=2\tan^{-1}\omega\tau$	Second-order system response to a sine input($\zeta=1$)
27b.	$\dfrac{A\omega}{(s^2+\omega^2)(s+a)^2}$	$\dfrac{A}{\omega^2+a^2}\left[\dfrac{a\omega(at+2)e^{-at}}{\omega^2+a^2}+\sin(\omega t-\psi)\right]$	
28.	$\dfrac{A\omega\omega_n^2}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)}$	$\dfrac{A\omega_n^2}{[(\omega_n^2-\omega^2)^2+4\zeta^2\omega^2\omega_n^2]^{1/2}}\cdot\left[\sin(\omega t-\psi_1)+\dfrac{\omega e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t-\psi_2)}{\omega_n\sqrt{1-\zeta^2}}\right]$ where $\psi_1=\tan^{-1}\dfrac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}$ $0<\psi_1<\pi$ and $\psi_2=\tan^{-1}-\dfrac{2\zeta\omega_n^2\sqrt{1-\zeta^2}}{\omega^2-\omega_n^2(1-2\zeta^2)}$ $0<\psi_2<\pi$	Second-order system response to a sine input ($\zeta<1$)

Source: Floyd E. Nixon, *Handbook of Laplace Transformation: Fundamentals, Applications, Tables and Examples*, 21e, © 1965, Referenced and adapted by permission of Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

EXAMPLE A-1

Given a charged capacitor circuit with an initial charge of 5 V, $R = 10\ \mathrm{k\Omega}$, and $C = 0.1\ \mu\mathrm{F}$, plot the voltage versus time.

$$\frac{dv}{dt}+\frac{1}{RC}v=0\ \Bigg|\ v(t=0)=v_o$$