# EMTH118-21S2, Engineering Mathematics 1A

## Tutorial 7 - Fundamentals of Differentiation 1 (based on Week 7 lectures)

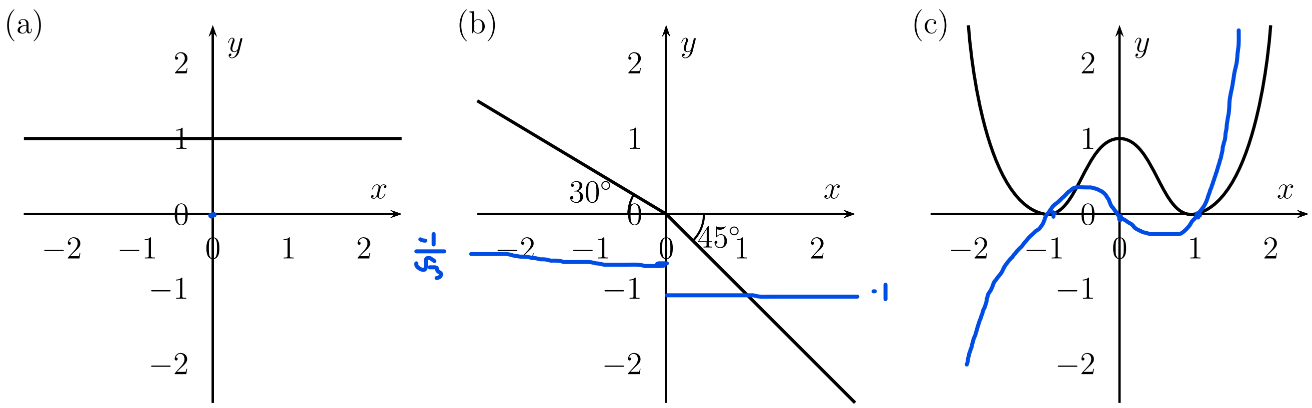
The lectures in week 7 covered the limit definition of the derivative and the concept of differentiability; rules of differentiation including the product rule, the quotient rule, and the chain rule. You should be able to

- understand the concept of differentiability,
- describe the situations where a function is not differentiable,
- given the graph of a function, identify or sketch the graph of the derivative,
- use the limit definition of the derivative to find derivatives of functions,
- apply the correct rules of differentiation to find derivatives, including the product rule, quotient rule, and chain rule,
- find the equations of tangent lines to curves.

The relevant sections from Stewart's *Calculus* are 2.7 - 2.8, 3.1 - 3.4.

**Homework Problems: To be done before your tutorial. See selected final answers at the end of this tutorial for checking purposes.**

**1**. Sketch the graph of the derivative of the functions shown below.




$f'(a) = na^{n-1}$

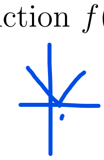
**2**. (a) Write down the definition of the derivative of a function $f(x)$ at a point $x = a$. $f'(x) = nx^{n-1}$

(b) Compute the derivative of $f(x) = x^2 + 3$ at $x = -1$ using the definition from (a). $f'(x) = 2x$ $f'(-1) = -2$

(c) Use the definition of the derivative to show that $f(x) = |x - 1|$ is not differentiable at $x = 1$.

$f'(x) = \frac{1}{x-1}$ $f'(1) = \frac{1}{0}$ does not exist

(d) Graph the function $f(x) = |x - 1|$ and give a geometric explanation of (c).



point on graph so undifferentiable

**3**. Find the derivative of the function

$$f(x) = 2x^2 \quad f'(x)\ 4x$$

according to the limit definition of the derivative (by first principles).

**4**. Find the derivatives of the following functions using the rules of differentiation.

(a) $f(x) = 3x^7 + 4 + \dfrac{5}{x^2}$

$f'(x) = 21x^6 - \frac{10}{x^3}$

(b) $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}} + a$

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

(c) $f(t) = t^5(3t + 5t^3)$

$f'(t) = 15t^5 + 25t^7$

(d) $g(x) = \dfrac{4x + 1}{x^2 - 5}$

$\frac{4(x^2-5) - 2x(4x+1)}{(x^2-5)^2}$

$\frac{-4x^2-2x-20}{(x^2-5)^2}$

(e) $g(x) = 4x^5 \sin x$

$20x^4 \cos x$

$4x^5 x \cos x + 20x^4 x \cos x$

(f) $g(y) = \dfrac{\sqrt{y}}{3\pi - y}$

$\frac{3\sqrt{\pi} - 3y}{2\sqrt{y}} \to \frac{3(\pi - y)}{2\sqrt{y}}$

$\left(\frac{1}{2\sqrt{y}}\right)(3\pi - y) - 1(\sqrt{y})$

$\frac{3\pi - y}{2\sqrt{y}} - \sqrt{y}$

**5**. (a) Find the equation of the tangent line to $y = \sqrt{x}$ at $x = 1$.

$y'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2} \quad y = \frac{1}{2}x + \frac{1}{2}$

(b) Find an equation for the tangent line to the graph of $y = x\cos(3x)$ at $x = \pi$.

$-3\sin(3x)$

$y' = \cos(3x) - 3x\sin(3x)$

$\cos(3\pi) - 3\pi\sin(3\pi) = -1$

$y = -x$

**6**. By rewriting the following functions in terms of $\sin x$ and $\cos x$, verify the following derivatives.

(a) $\dfrac{d}{dx}(\sec x) = \sec x \tan x.$

$\frac{1}{\cos x} = \frac{1}{\cos x} \quad \frac{\sin x}{\cos x} \quad \frac{0\cos x - -\sin x}{(\cos x)^2} \quad \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$

$= \frac{\sin x}{\cos^2 x}$

(b) $\dfrac{d}{dx}(\cot x) = -\csc^2 x.$

$\frac{1}{\tan x} = \frac{1}{\sin x} \quad \frac{\cos x}{\sin x} \quad \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x}$

$+ \sin^2 x + \cos^2 x = \frac{-1}{\sin x} \quad \frac{1}{\sin^2 x} = \frac{-1}{\sin x}$

(c) $\dfrac{d}{dx}(\tan x) = \sec^2 x.$

**7**. Find the following derivatives.

(a) Find $\dfrac{d^2y}{dx^2}$ where $y = \dfrac{1 + x}{1 - x}$.

(b) Find $\dfrac{d^2y}{d\theta^2}$ where $y = \sin^2(\pi - \theta)$.

(c) Find $\dfrac{dy}{dx}$ when $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

**8**. Find $g^{(4)}(x)$ where $g(x) = \dfrac{1}{x - 1}$.

Optional Extra: Look up factorials and see if you can find a general formula for $g^{(n)}(x)$.

**9**. Use the chain rule to find $f'(x)$ for the following functions.

(a) $f(x) = \sqrt{x^3 - 4}$

(b) $f(x) = \left(x^3 - \dfrac{7}{x}\right)^{-2}$

(c) $f(x) = \dfrac{1}{(x^5 - x + 1)^9}$

(d) $f(x) = \cos(\cos x)$

(e) $f(x) = \ln(\sin(x^2))$

(f) $f(x) = x\, e^{-x^2}$

**10**. Let $f(x) = x^2 + 1$ and $g(x) = 2x + 3$. Verify the chain rule by computing the following.

(a) $f(g(x))$

(b) The derivative of the function obtained in (a)

(c) $f'(g(x))g'(x)$

**11**. Recall that $\left.\dfrac{dy}{dx}\right|_{x=a}$ means the rate of change of $y$ with respect to $x$ at $x = a$.

(a) Find $\left.\dfrac{dy}{dx}\right|_{x=2}$ where $y = 3x^4 + 2x - 1$.

(b) Find $\left.\dfrac{dx}{dt}\right|_{t=\pi}$ where $x = \sin t + 2$.

(c) Use the chain rule to find $\left.\dfrac{dy}{dt}\right|_{t=\pi}$.

**12**. More differentiation practice: Find $\dfrac{dy}{dx}$ for the following functions.

(a) $y = \sqrt{\ln(x)}$

(b) $y = \dfrac{x^2}{1 + \ln(x)}$

(c) $y = \ln(\sin^2(x))$

(d) $y = e^{\sin(x)}$

(e) $y = \cos(e^{2x+1})$

(f) $y = e^{1/x}$

Please see over for In-Tutorial problems.

## In-Tutorial Problems

**13**. Consider the piecewise function $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ x + 2, & x > 1 \end{cases}$.

Show that $f$ is continuous but not differentiable at $x = 1$ using the definitions of continuity and differentiability.

**14**. Find the derivatives of the functions $f(x) = \cos^2 x$ and $g(x) = \sin^2 x$. What does this tell you about the function $f(x) + g(x)$? Does this make sense?

**15**. Find the derivatives of the following functions.

(a) $y = \sin^3 x$

(b) $y = \sin(x^3)$

(c) $h(x) = \tan^2(\sqrt{x})$

(d) $f(x) = \sec^2(3x) - \tan^2(3x)$

**16**. Using the same technique that was used in the lectures to differentiate $f(x) = \sin x$ using the limit definition of the derivative, show that

$$\frac{d}{dx}(\cos x) = -\sin x.$$

**Hint:** You will need to use the trig identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$. You will also need to know the two trig limits

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1 \qquad \text{and} \qquad \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0.$$

**17**. **Extension**: Find the derivative of the function

$$f(x) = \sqrt{x} \qquad x > 0$$

according to the limit definition of the derivative (by first principles).

(Hint: Think back to the limits topic to recall how we approached limits involving a radical term.)

**Final answers to selected problems:**

You should also mark your work thoroughly using the full tutorial solutions (available in Learn on Thursdays at 4pm.)

4(a) $f'(x) = 21x^6 - \dfrac{10}{x^3}$

4(b) $f'(x) = \dfrac{1}{2}\left(\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{x^3}}\right)$

4(c) $f'(t) = 18t^5 + 40t^7$

4(d) $g'(x) = \dfrac{-4x^2 - 2x - 20}{(x^2-5)^2}$

4(e) $g'(x) = 20x^4 \sin(x) + 4x^5 \cos(x)$

4(f) $g'(x) = \dfrac{3\pi y^{-1/2} + y^{1/2}}{2(3\pi - y)^2}$

5(a) $y = \dfrac{1}{2}x + \dfrac{1}{2}$

5(b) $y = -x$

7(a) $\dfrac{d^2y}{dx^2} = \dfrac{4}{(1-x)^3}$

7(b) $\dfrac{d^2y}{d\theta^2} = 2\cos^2(\pi - \theta) - 2\sin^2(\pi - \theta)$

7(c) $\dfrac{dy}{dx} = \dfrac{4}{(e^x + e^{-x})^2}$

8 $g^{(4)}(x) = \dfrac{24}{(x-1)^5} = \dfrac{4!}{(x-1)^5}$

9(c) $f'(x) = \dfrac{-9(5x^4 - 1)}{(x^5 - x + 1)^{10}}$

9(f) $f'(x) = e^{-x^2} - 2x^2 e^{-x^2}$

12(a) $\dfrac{dy}{dx} = \dfrac{1}{2x\sqrt{\ln x}}$

12(b) $\dfrac{dy}{dx} = \dfrac{x(1 + 2\ln x)}{(1 + \ln x)^2}$