《前沿数学方法及其在航天工程中的应用》

张量在流体力学中的应用

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第二部分：张量在流体力学中的应用

一．场论与笛卡尔张量分析

二．流体力学控制方程

三．正交曲线坐标系下的控制方程

三、正交曲线坐标系下的控制方程

- 航天工程中的几个流场实例
- 正交曲线坐标系(*自学)
- 流体力学控制方程(*自学)

※ 重点掌握：

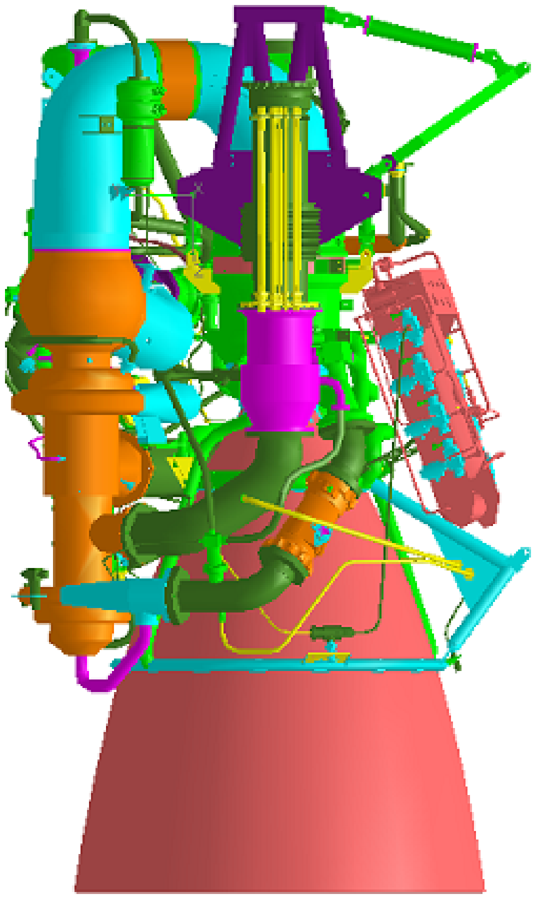
正交曲线坐标系**基矢量与拉梅系数**，**圆柱坐标系**下的**微分算子**及**流体力学控制方程**推导。

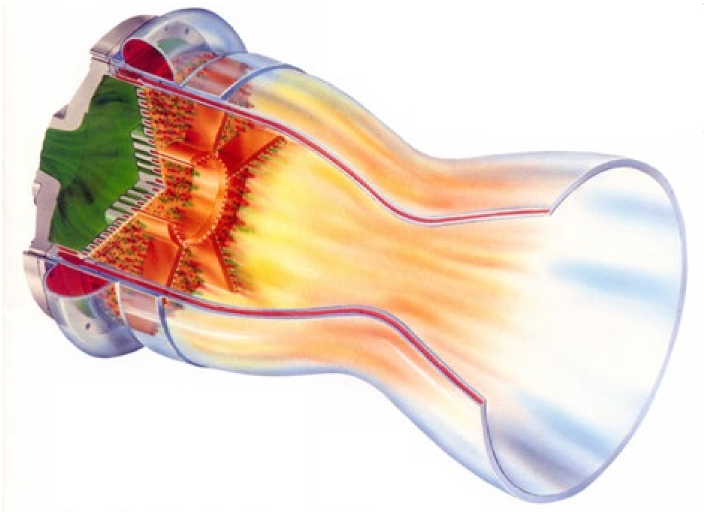
3.1 航天工程中的几个流场实例

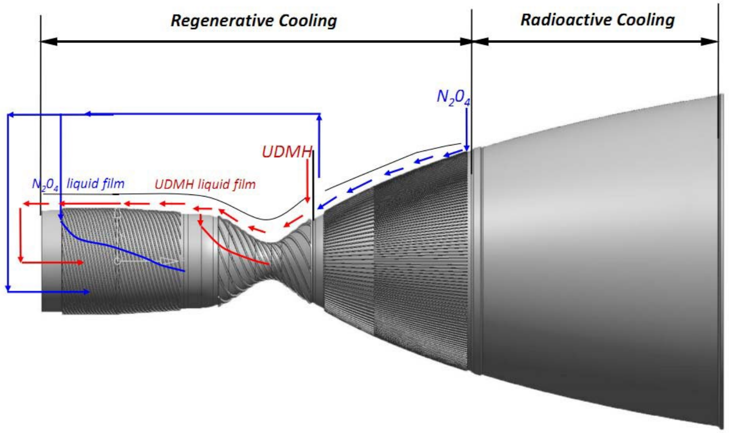
- 液体火箭发动机
- 旋转流体机械
- 飞行器外流场

3.1 航天工程中的几个流场实例

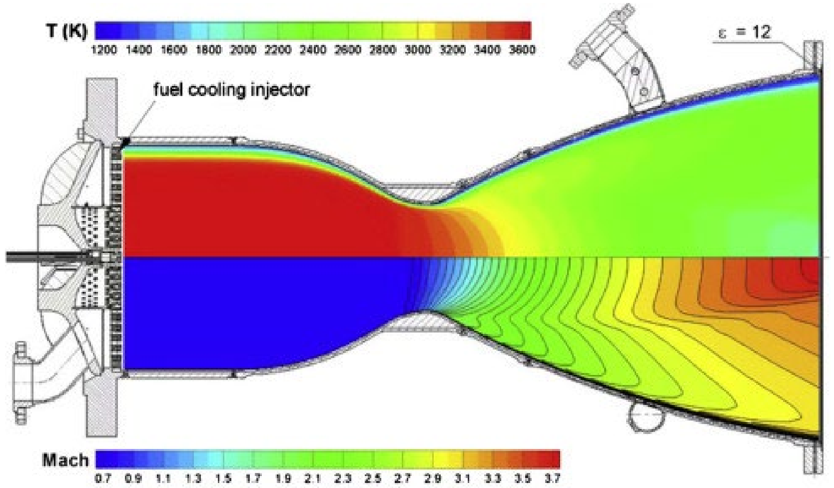
① 液体火箭发动机

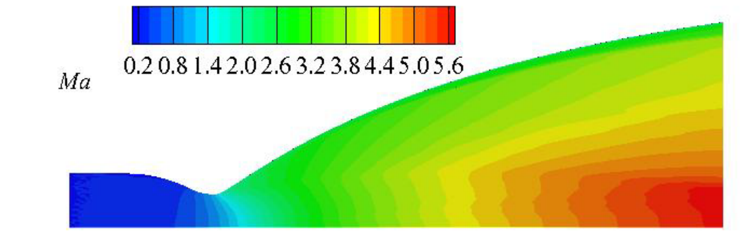


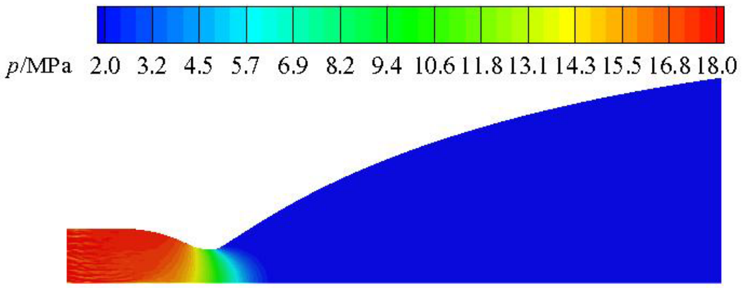




典型液体火箭发动机构型



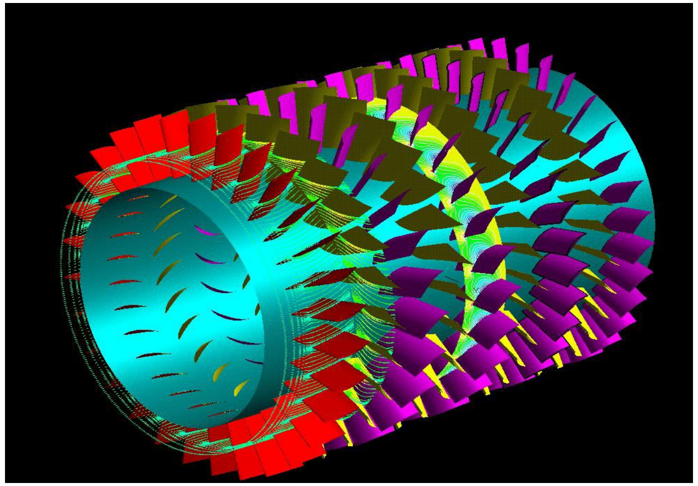


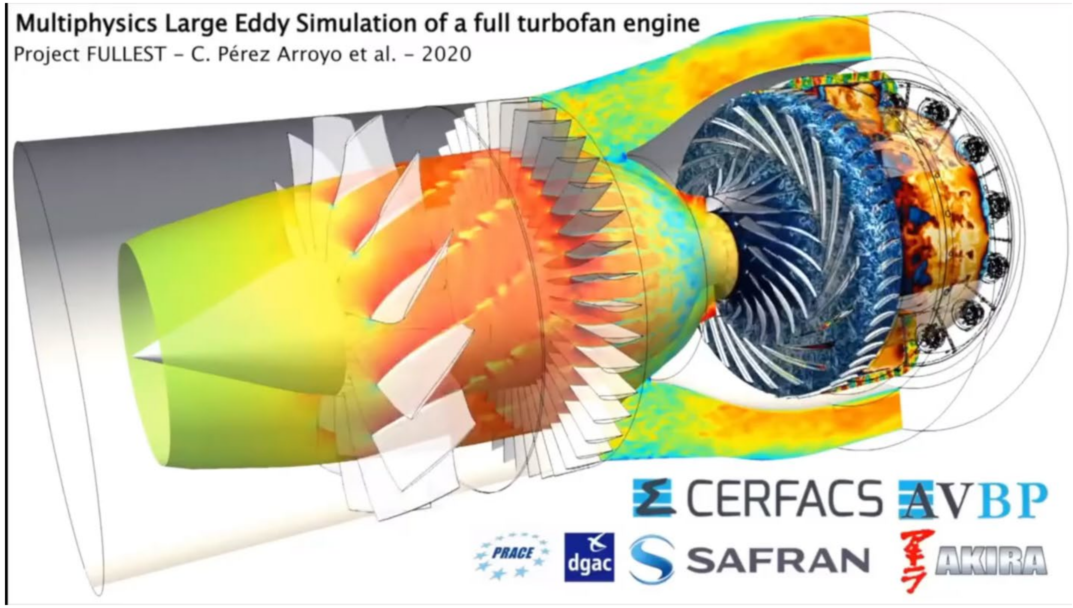


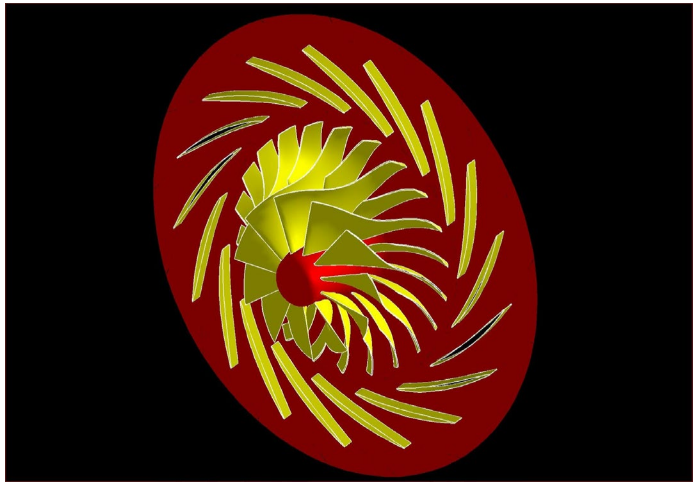
典型流场参数分布

3.1 航天工程中的几个流场实例

② 旋转流体机械





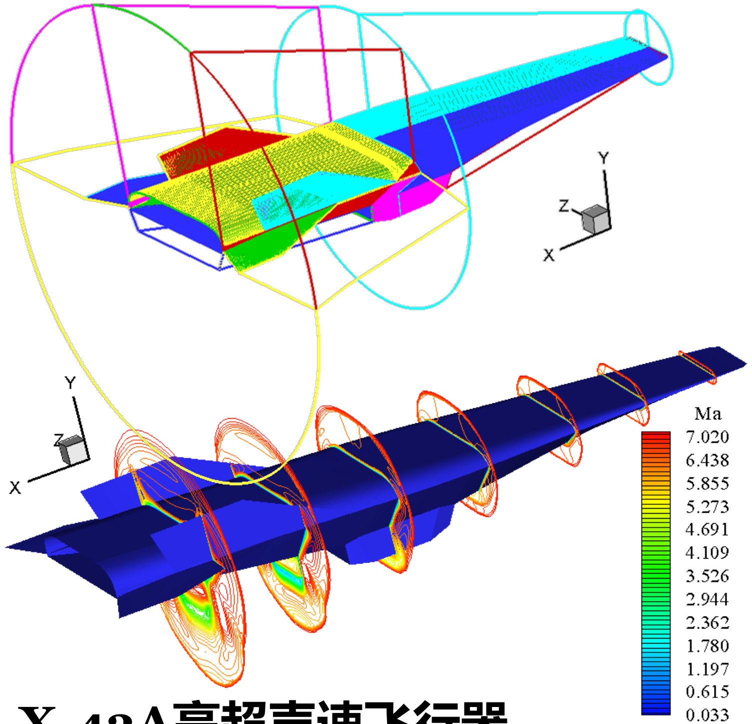


典型叶轮机械中流体流动

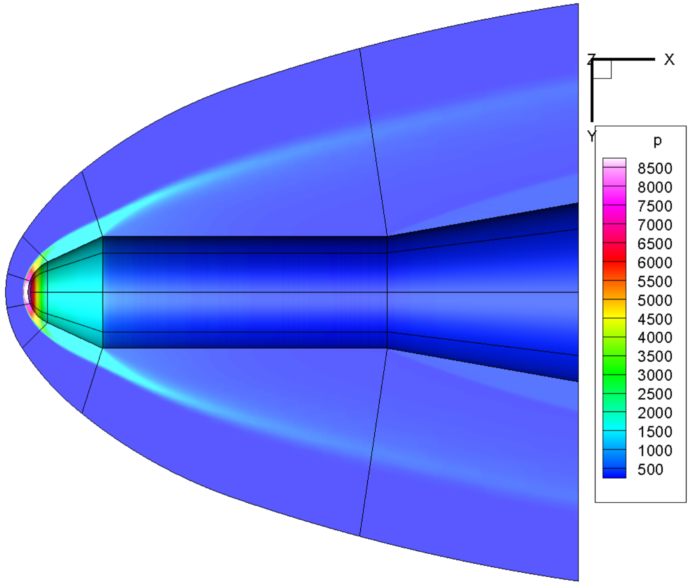
3.1 航天工程中的几个流场实例

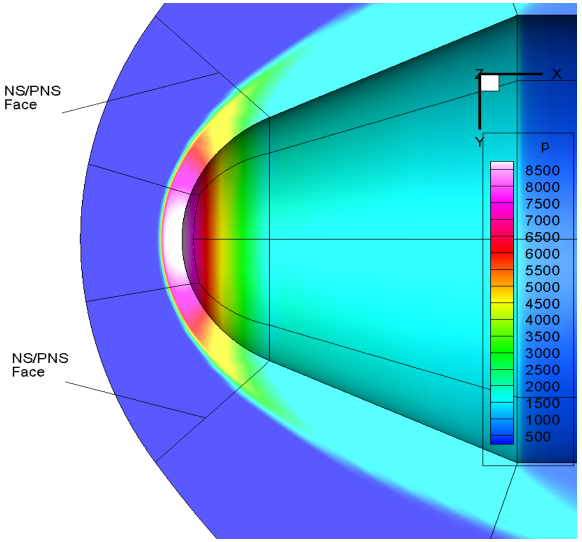
③ 飞行器外流场

（a）X-43A 飞行效果图



X-43A高超声速飞行器





球-锥-柱融合体

3.2* 正交曲线坐标系

- 曲线坐标系与正交曲线坐标系
- 基矢量与拉梅系数
- 基矢量对坐标的偏导数
- 正交曲线坐标系下的微分算子

3.2 正交曲线坐标系

① 曲线坐标系与正交曲线坐标系

■ 曲线坐标系

M点的坐标可以在笛卡尔坐标系$Oxyz$中表示为

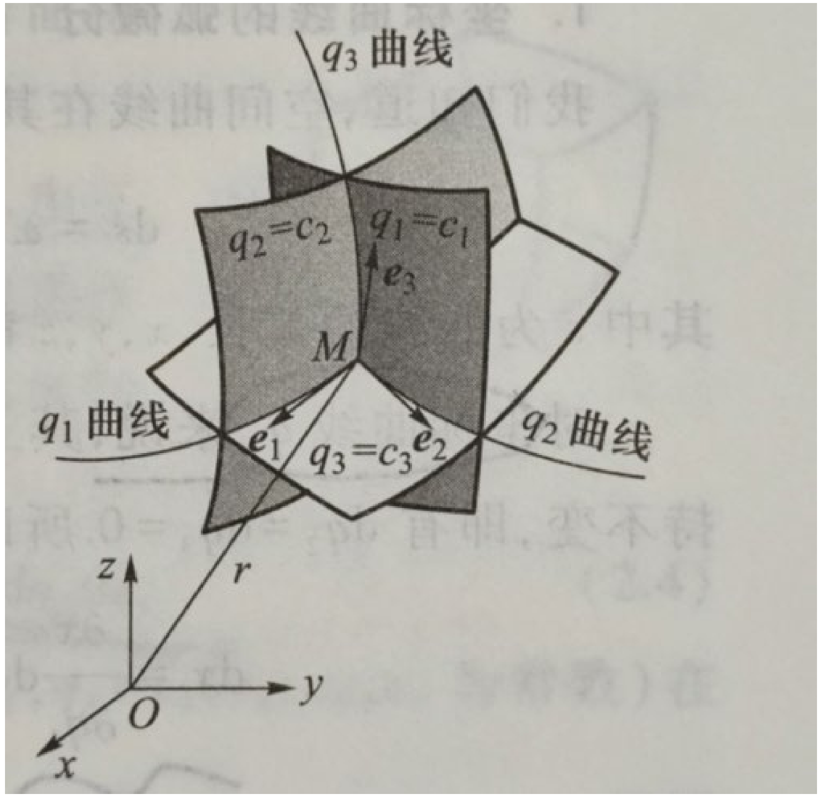
$$M(x,y,z)$$

也可以用曲线坐标系$Oq_1q_2q_3$表示为

$$M(q_1,q_2,q_3)$$

因为M是空间中的同一个点，因此两个坐标之间满足如下关系

$$\begin{cases} q_1 = q_1(x,y,z) \\ q_2 = q_2(x,y,z) \\ q_3 = q_3(x,y,z) \end{cases} \qquad \begin{cases} x = x(q_1,q_2,q_3) \\ y = y(q_1,q_2,q_3) \\ z = z(q_1,q_2,q_3) \end{cases}$$



曲线坐标系

3.2 正交曲线坐标系

① 曲线坐标系与正交曲线坐标系

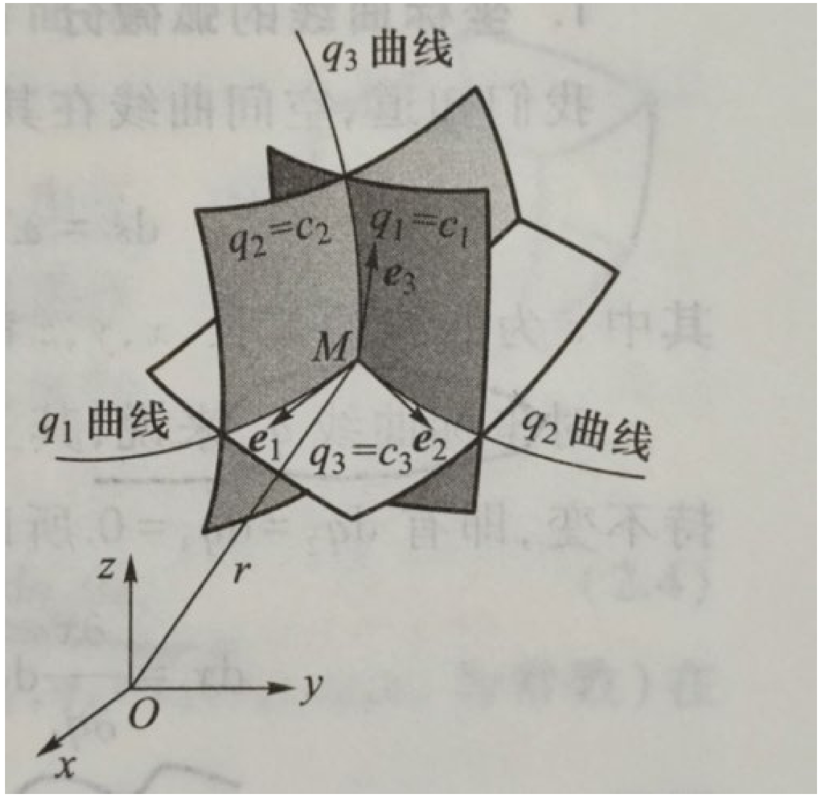
■ 正交曲线坐标系

在笛卡尔坐标系中，作曲面： $q_i = c_i$，即为曲线坐标系下的坐标面。在坐标线q_1, q_2, q_3上做单位向量$\vec{e}_1, \vec{e}_2, \vec{e}_3$。如果$\vec{e}_1, \vec{e}_2, \vec{e}_3$相互正交，则称为正交曲线坐标系；反之，称为斜交曲线坐标系。在曲线坐标系中，矢量表示为

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$$

曲线坐标系与笛卡尔坐标系最大的区别：

曲线坐标系三个方向的单位矢量，随着空间点的不同是不同的，方向发生变化。



曲线坐标系

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① 曲线坐标系与正交曲线坐标系

■ 正交曲线坐标系

常见的正交曲线坐标系：

- 柱坐标系

$q_1=r,\ q_2=\theta,\ q_3=z$

$x=r\cos\theta, y=r\sin\theta, z=z$

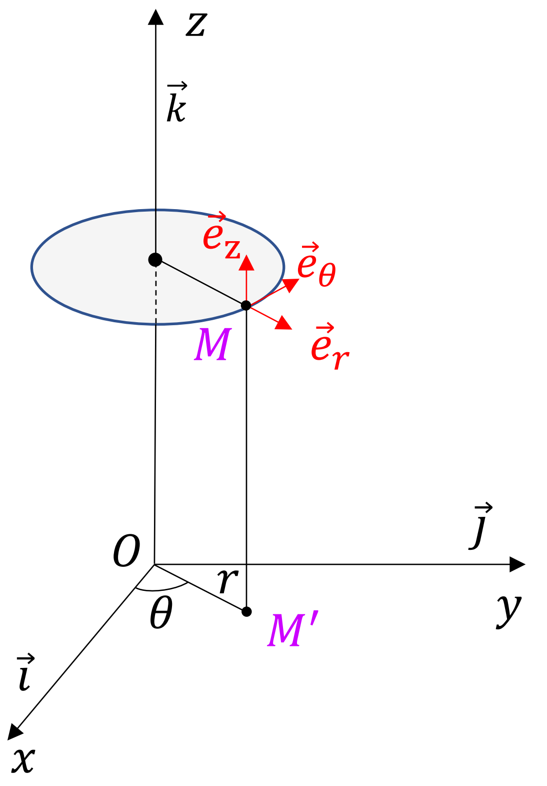
$\vec{e}_\theta$ 和$\vec{e}_r$的**方向在变化。**

- 球坐标系

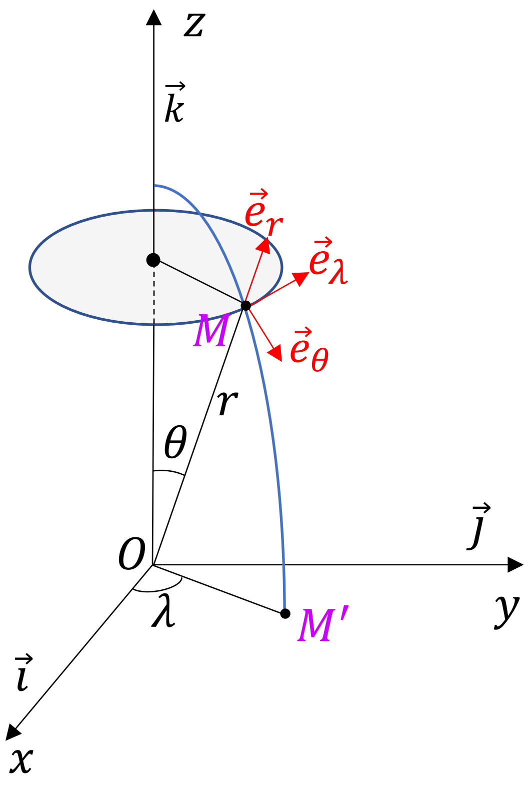
$q_1=r,\ q_2=\theta,\ q_3=\lambda$

$x=r\sin\theta\cos\lambda, y=r\sin\theta\sin\lambda, z=r\cos\theta$

$\vec{e}_r$、$\vec{e}_\theta$ 和$\vec{e}_\lambda$的**方向都在变化。**



柱坐标系



球坐标系

3.2 正交曲线坐标系

② 基矢量与拉梅系数

■ 基矢量

曲线坐标系上的**弧元矢量**为：

$$d\vec{r}=\frac{\partial\vec{r}}{\partial q_i}dq_i=d\vec{s}_i \qquad\qquad d\vec{s}_i=\frac{\partial\vec{r}}{\partial q_i}dq_i$$

于是，坐标轴q_i方向的单位向量$\vec{e}_i$为

$$\vec{e}_i=\frac{d\vec{s}_i}{|d\vec{s}_i|}=\frac{\frac{\partial\vec{r}}{\partial q_i}dq_i}{\left|\frac{\partial\vec{r}}{\partial q_i}dq_i\right|}=\frac{\frac{\partial\vec{r}}{\partial q_i}dq_i}{\left|\frac{\partial\vec{r}}{\partial q_i}\right|dq_i}=\frac{\frac{\partial\vec{r}}{\partial q_i}}{\left|\frac{\partial\vec{r}}{\partial q_i}\right|}$$

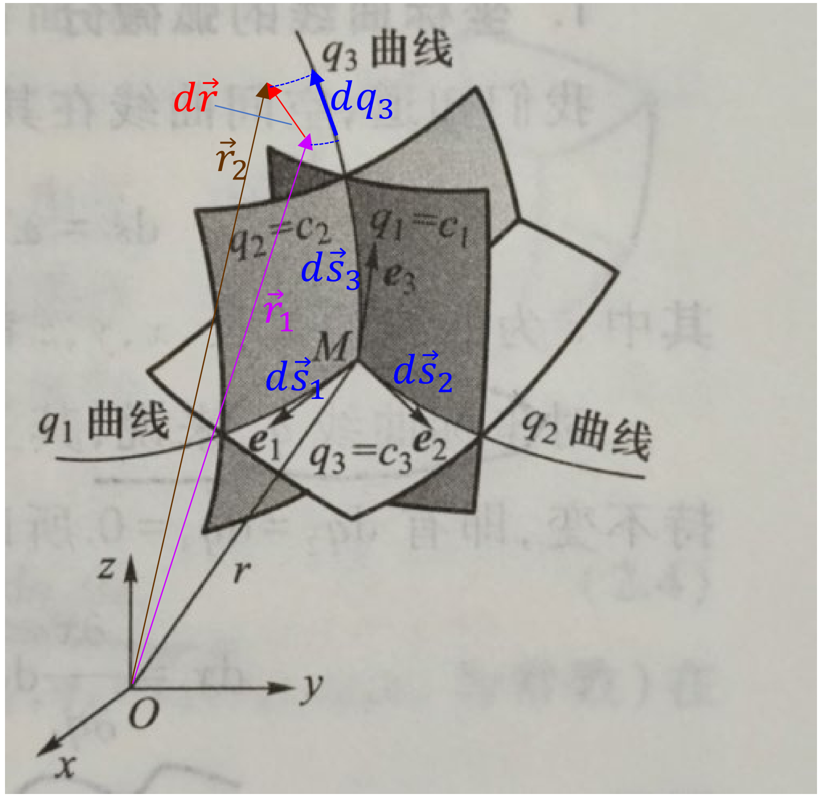
令$h_i=\left|\frac{\partial\vec{r}}{\partial q_i}\right|$，则有

$$\vec{e}_i=\frac{d\vec{s}_i}{|d\vec{s}_i|}=\frac{\frac{\partial\vec{r}}{\partial q_i}}{h_i}=\frac{1}{h_i}\left(\frac{\partial x}{\partial q_i}\vec{i}+\frac{\partial y}{\partial q_i}\vec{j}+\frac{\partial z}{\partial q_i}\vec{k}\right)$$

$$d\vec{s}_i=h_i dq_i\,\vec{e}_i$$

$$d\vec{r}=\vec{r}_2-\vec{r}_1$$

$$d\vec{r}\cdot\vec{e}_i=h_i dq_i$$



曲线坐标系

② 基矢量与拉梅系数

■ 拉梅系数

h_i称为**拉梅系数**，表示沿着坐标轴的弧长$|\partial\vec{r}|$与坐标增量∂q_i的比值

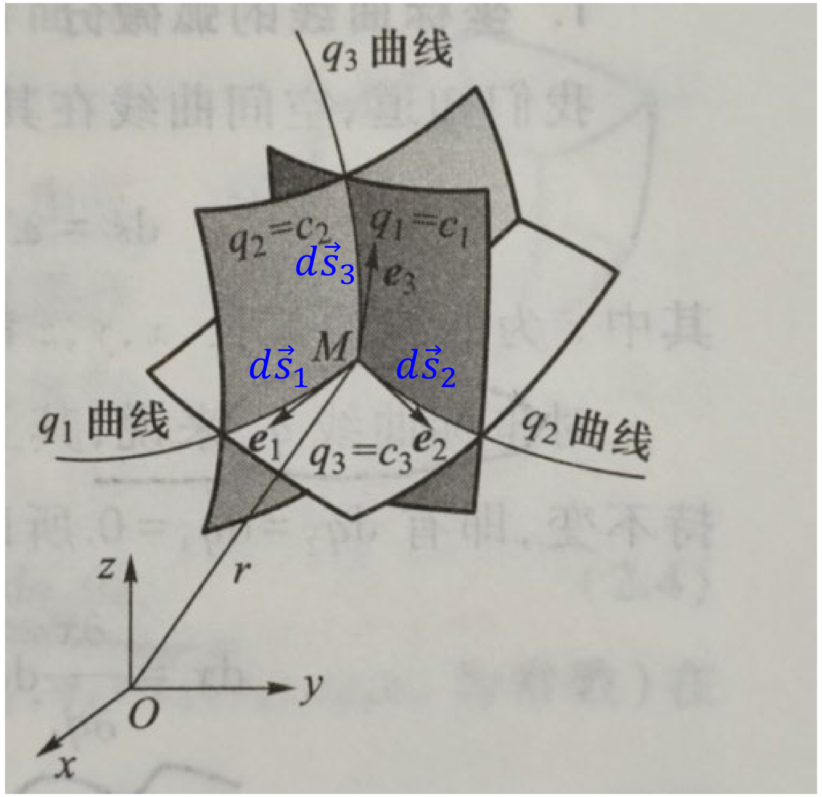
$$h_i = \left|\frac{\partial\vec{r}}{\partial q_i}\right| = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}$$

于是有

弧长 $$d\vec{r} = \frac{\partial\vec{r}}{\partial q_1}dq_1 + \frac{\partial\vec{r}}{\partial q_2}dq_2 + \frac{\partial\vec{r}}{\partial q_3}dq_3 = \underbrace{h_1 dq_1\vec{e}_1}_{d\vec{s}_1} + \underbrace{h_2 dq_2\vec{e}_2}_{d\vec{s}_2} + \underbrace{h_3 dq_3\vec{e}_3}_{d\vec{s}_3}$$

面积 $$\begin{cases} dS_1 = |h_2 dq_2\vec{e}_2 \times h_3 dq_3\vec{e}_3| = h_2 h_3 dq_2 dq_3 \\ dS_2 = h_1 h_3 dq_1 dq_3 \\ dS_3 = h_1 h_2 dq_1 dq_2 \end{cases}$$

体积 $$dV = h_1 dq_1\vec{e}_1 \cdot (h_2 dq_2\vec{e}_2 \times h_3 dq_3\vec{e}_3) = h_1 h_2 h_3 dq_1 dq_2 dq_3$$



曲线坐标系

② 基矢量与拉梅系数

■ 柱坐标系和球坐标系

$q_1 = r,\ q_2 = \theta,\ q_3 = z$

$x = r\cos\theta, y = r\sin\theta, z = z$

$\vec{e}_\theta$ 和$\vec{e}_r$的方向在变化。

$q_1 = r,\ q_2 = \theta,\ q_3 = \gamma$

$x = r\sin\theta\cos\lambda, y = r\sin\theta\sin\lambda, z = r\cos\theta$

$\vec{e}_r$、$\vec{e}_\theta$ 和$\vec{e}_\lambda$的方向都在变化。

$$\vec{e}_i = \frac{1}{h_i}\left(\frac{\partial x}{\partial q_i}\vec{i} + \frac{\partial y}{\partial q_i}\vec{j} + \frac{\partial z}{\partial q_i}\vec{k}\right)$$

- 柱坐标系

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1,\ h_2 = r,\quad h_3 = 1$$

$$\vec{e}_r = \cos\theta\vec{i} + \sin\theta\vec{j}$$

$$\vec{e}_\theta = -\sin\theta\vec{i} + \cos\theta\vec{j}$$

$$\vec{e}_z = \vec{k}$$

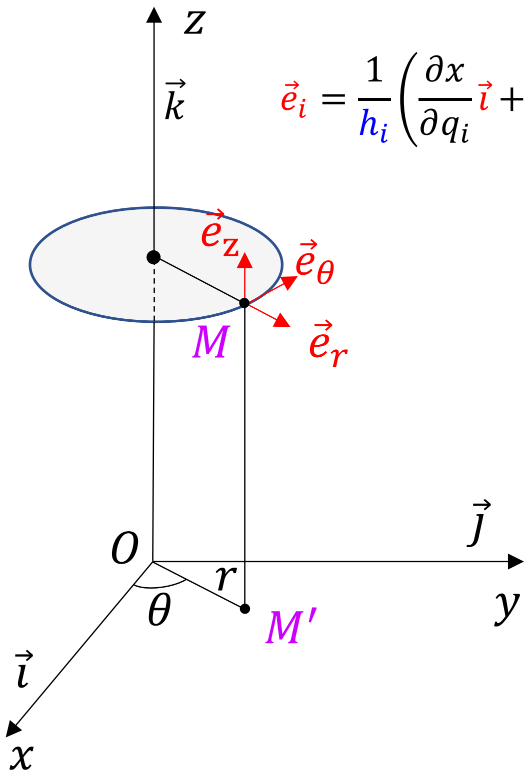
- 球坐标系

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1,\ h_2 = r,\ \ h_3 = r\sin\theta$$

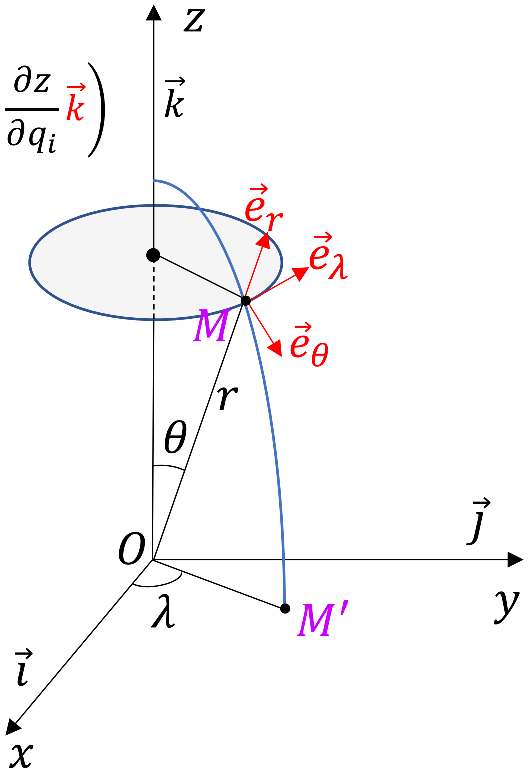
$$\vec{e}_r = \sin\theta\cos\lambda\vec{i} + \sin\theta\sin\lambda\vec{j} + \cos\theta\vec{k}$$

$$\vec{e}_\theta = \cos\theta\cos\lambda\vec{i} - \cos\theta\sin\lambda\vec{j} + \sin\theta\vec{k}$$

$$\vec{e}_\lambda = -\sin\lambda\vec{i} + \cos\lambda\vec{j}$$



柱坐标系



球坐标系

③ 基矢量对坐标的偏导数

当$i \neq j \neq k$时，基矢量偏导数

$$\frac{\partial \vec{e}_i}{\partial q_i} = -\frac{1}{h_j}\frac{\partial h_i}{\partial q_j}\vec{e}_j - \frac{1}{h_k}\frac{\partial h_i}{\partial q_k}\vec{e}_k \qquad (\text{当}i,j,k\text{轮换})$$

比如

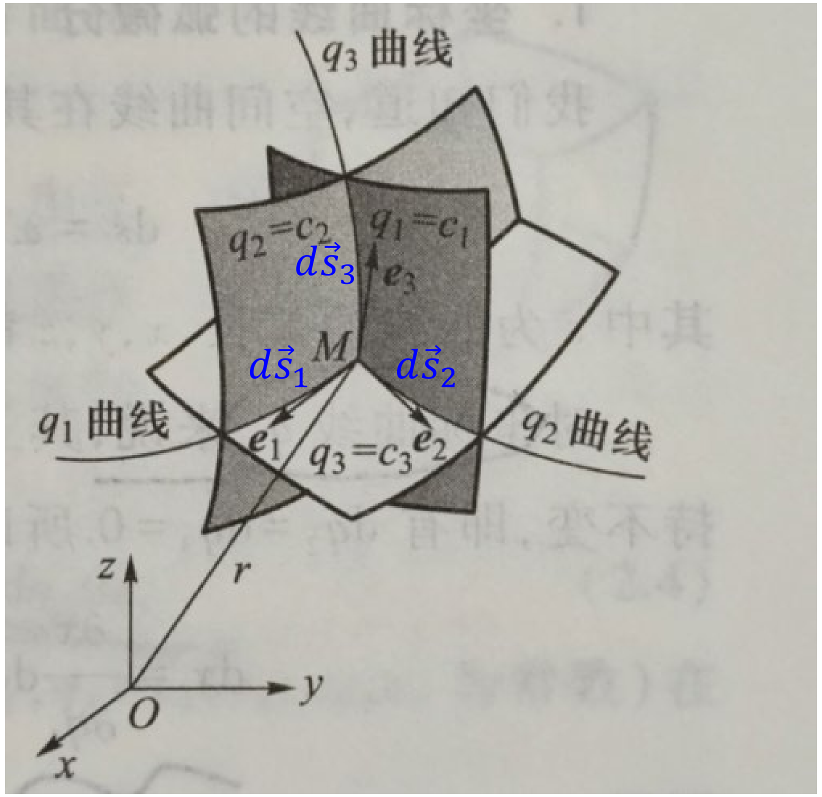
$$\frac{\partial \vec{e}_1}{\partial q_1} = -\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\vec{e}_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial q_3}\vec{e}_3$$

如果$i \neq j$，则基矢量偏导数

$$\frac{\partial \vec{e}_i}{\partial q_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial q_i}\vec{e}_j$$

比如

$$\frac{\partial \vec{e}_1}{\partial q_2} = \frac{1}{h_1}\frac{\partial h_2}{\partial q_1}\vec{e}_2$$



曲线坐标系

④ 正交曲线坐标系下的微分算子

■ 梯度

这里仅给出标量ϕ的梯度的表达式。由梯度的性质可知，梯度$grad\phi$在曲线坐标$\vec{e}_1$、$\vec{e}_2$和$\vec{e}_3$三个方向的投影，即为三个方向的方向导数，分别为

$$\frac{\partial\phi}{\partial\vec{e}_1} = grad\phi\cdot\vec{e}_1 = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1}$$
$$\frac{\partial\phi}{\partial\vec{e}_2} = grad\phi\cdot\vec{e}_2 = \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}$$
$$\frac{\partial\phi}{\partial\vec{e}_3} = grad\phi\cdot\vec{e}_3 = \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}$$

$$\Rightarrow\quad grad\phi = \nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}\vec{e}_3$$

□ 柱坐标系

$$grad\phi = \nabla\phi = \frac{\partial\phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta + \frac{\partial\phi}{\partial z}\vec{e}_z$$

□ 球坐标系

$$grad\phi = \nabla\phi = \frac{\partial\phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\lambda}\vec{e}_\lambda$$

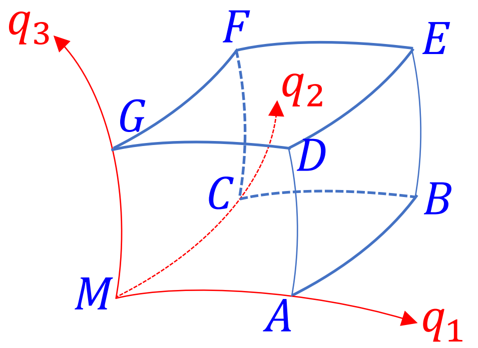
3.2 正交曲线坐标系

④ 正交曲线坐标系下的微分算子

■ 散度

这里仅给出矢量$\vec{a}$的散度的表达式。在曲线坐标系中，矢量$\vec{a}$为

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$$



曲线坐标系六面体单元

我们用散度定义来推导表达式。显然，分量a_1 、a_2和a_3在面元上的通量，只分别与$\vec{e}_1$ 、 $\vec{e}_2$和$\vec{e}_3$三个方向的面元分量有关，由前述面积和体积的表达式，可得

$$div\vec{a} = \lim_{V\to 0}\frac{\oint_S a_n ds}{V} = \frac{\left[\frac{\partial(a_1h_2h_3)}{\partial q_1} + \frac{\partial(a_2h_1H_3)}{\partial q_2} + \frac{\partial(a_3h_1h_2)}{\partial q_3}\right]dq_1dq_2dq_3}{h_1h_2h_3dq_1dq_2dq_3} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(a_1h_2h_3)}{\partial q_1} + \frac{\partial(a_2h_1h_3)}{\partial q_2} + \frac{\partial(a_3h_1h_2)}{\partial q_3}\right]$$

□ 柱坐标系

$$div\vec{a} = \nabla\cdot\vec{a} = \frac{1}{r}\frac{\partial(ra_r)}{\partial r} + \frac{1}{r}\frac{\partial a_\theta}{\partial\theta} + \frac{\partial a_z}{\partial z}$$

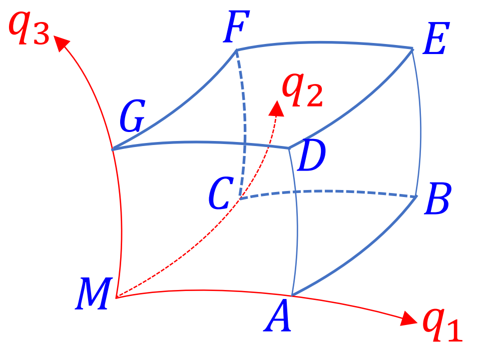
□ 球坐标系

$$div\vec{a} = \nabla\cdot\vec{a} = \frac{1}{r^2}\frac{\partial(r^2a_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta a_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\lambda}{\partial\lambda}$$

④ 正交曲线坐标系下的微分算子

■ 散度

这里再给出二阶张量P的散度的表达式 $P=\{p_{ij}\}=p_{ij}=\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$



曲线坐标系六面体单元

在曲线坐标系中，矢量P的散度为

$$(\nabla\cdot\boldsymbol{P})_i=\frac{1}{h_1h_2h_3}\sum_{j=1}^{3}\frac{\partial}{\partial q_j}\left(\frac{h_1h_2h_3}{h_j}p_{ij}\right)+\sum_{j=1}^{3}\frac{1}{h_ih_j}\left(\frac{\partial h_i}{\partial q_j}p_{jj}-\frac{\partial h_j}{\partial q_i}p_{ij}\right)$$

□ 柱坐标系

$$(\nabla\cdot\boldsymbol{P})_r=\frac{1}{r}\frac{\partial(rp_{rr})}{\partial r}+\frac{1}{r}\frac{\partial p_{r\theta}}{\partial\theta}+\frac{\partial p_{rz}}{\partial z}-\frac{p_{\theta\theta}}{r}$$

$$(\nabla\cdot\boldsymbol{P})_\theta=\frac{1}{r}\frac{\partial(rp_{\theta r})}{\partial r}+\frac{1}{r}\frac{\partial p_{\theta\theta}}{\partial\theta}+\frac{\partial p_{\theta z}}{\partial z}+\frac{p_{r\theta}}{r}$$

$$(\nabla\cdot\boldsymbol{P})_z=\frac{1}{r}\frac{\partial(rp_{zr})}{\partial r}+\frac{1}{r}\frac{\partial p_{z\theta}}{\partial\theta}+\frac{\partial p_{zz}}{\partial z}$$

$h_1=1$

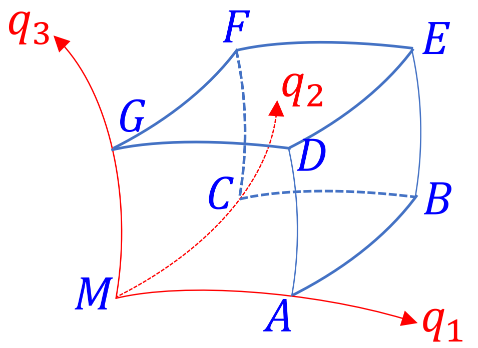
$h_2=r$

$h_3=1$

④ 正交曲线坐标系下的微分算子

■ 散度

这里再给出二阶张量**P**的散度的表达式 $P=\{p_{ij}\}=p_{ij}=\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$



曲线坐标系六面体单元

在曲线坐标系中，矢量**P**的散度为

$$(\nabla\cdot\boldsymbol{P})_i=\frac{1}{h_1h_2h_3}\sum_{j=1}^{3}\frac{\partial}{\partial q_j}\left(\frac{h_1h_2h_3}{h_j}p_{ij}\right)+\sum_{j=1}^{3}\frac{1}{h_ih_j}\left(\frac{\partial h_i}{\partial q_j}p_{jj}-\frac{\partial h_j}{\partial q_i}p_{ij}\right)$$

□ 球坐标系

$$(\nabla\cdot\boldsymbol{P})_r=\frac{1}{r^2}\frac{\partial(r^2p_{rr})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta p_{\theta r})}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial p_{\lambda r}}{\partial\lambda}-\frac{p_{\theta\theta}+p_{\lambda\lambda}}{r}$$

$$(\nabla\cdot\boldsymbol{P})_\theta=\frac{1}{r^2}\frac{\partial(r^2p_{r\theta})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta p_{\theta\theta})}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial p_{\lambda\theta}}{\partial\lambda}+\frac{p_{\theta r}}{r}-\frac{\cot\theta}{r}p_{\lambda\lambda}$$

$$(\nabla\cdot\boldsymbol{P})_\lambda=\frac{1}{r^2}\frac{\partial(r^2p_{r\lambda})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta p_{\theta\lambda})}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial p_{\lambda\lambda}}{\partial\lambda}+\frac{p_{\lambda r}}{r}+\frac{\cot\theta}{r}p_{\lambda\theta}$$

$h_1=1$

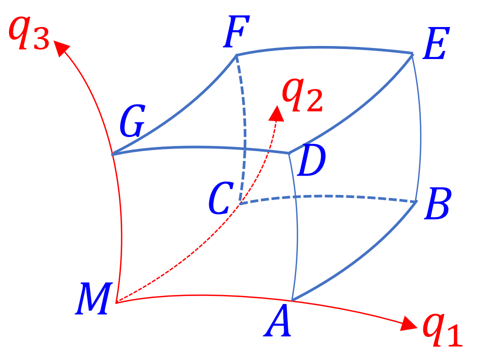
$h_2=r$

$h_3=r\sin\theta$

④ 正交曲线坐标系下的微分算子

■ 旋度

梯度$rot\vec{a}$在曲线坐标$\vec{e}_1$ 、 $\vec{e}_2$和$\vec{e}_3$三个方向的投影，相当于在三个坐标面内求环量密度。经过推导可以得到各分量分别为



曲线坐标系六面体单元

$$(rot\vec{a})_1=\frac{1}{h_2h_3}\left[\frac{\partial(a_3h_3)}{\partial q_2}-\frac{\partial(a_2h_2)}{\partial q_3}\right]$$

$$(rot\vec{a})_2=\frac{1}{h_1h_3}\left[\frac{\partial(a_1h_1)}{\partial q_3}-\frac{\partial(a_3h_3)}{\partial q_1}\right]$$

$$(rot\vec{a})_3=\frac{1}{h_1h_2}\left[\frac{\partial(a_2h_2)}{\partial q_1}-\frac{\partial(a_1h_1)}{\partial q_3}\right]$$

$$\Rightarrow rot\vec{a}=\nabla\times\vec{a}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix}$$

□ 柱坐标系

$$rot\vec{a}=\nabla\times\vec{a}=\left(\frac{1}{r}\frac{\partial a_z}{\partial\theta}-\frac{\partial a_\theta}{\partial z}\right)\vec{e}_r+\left(\frac{\partial a_r}{\partial z}-\frac{\partial a_z}{\partial r}\right)\vec{e}_\theta+\frac{1}{r}\left[\frac{\partial(ra_\theta)}{\partial r}-\frac{\partial a_r}{\partial\theta}\right]\vec{e}_z$$

□ 球坐标系

$$rot\vec{a}=\nabla\times\vec{a}=\frac{1}{r\sin\theta}\left[\frac{\partial(a_\lambda\sin\theta)}{\partial\theta}-\frac{\partial a_\theta}{\partial\lambda}\right]\vec{e}_r+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\lambda}-\frac{\partial(ra_\lambda)}{\partial r}\right]\vec{e}_\theta+\frac{1}{r}\left[\frac{\partial(ra_\theta)}{\partial r}-\frac{\partial a_r}{\partial\theta}\right]\vec{e}_\lambda$$

3.3* 流体力学控制方程

- 柱坐标系N-S方程
- 球坐标系N-S方程
- 一般正交曲线坐标系N-S方程

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推导过程

张量实体形式N-S方程

$$\frac{\partial \rho}{\partial t} + div(\rho\vec{V}) = 0$$

$$\frac{\partial}{\partial t}(\rho\vec{V}) + div(\rho\vec{V}\vec{V}) = \rho\vec{R} + divP$$

$$\frac{\partial}{\partial t}(\rho e_t) + div(\rho e_t\vec{V}) = \rho\vec{R}\cdot\vec{V} + div(P\cdot\vec{V}) + div(kgradT) + \rho q$$

$$P = -pI + 2\mu\left(S - \frac{1}{3}Idiv\vec{V}\right)$$

$$e_t = \left(\frac{1}{2}V^2 + u\right)$$

按对应坐标系下的微分算子展开

对应坐标系下的方程
(比如笛卡尔坐标系等正交曲线坐标系)

① 柱坐标系N-S方程

微分算子

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+v_z\frac{\partial}{\partial z}$$

$$grad\phi=\nabla\phi=\frac{\partial\phi}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta+\frac{\partial\phi}{\partial z}\vec{e}_z$$

$$div\vec{V}=\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}$$

$$rot\vec{a}=\left(\frac{1}{r}\frac{\partial a_z}{\partial\theta}-\frac{\partial a_\theta}{\partial z}\right)\vec{e}_r+\left(\frac{\partial a_r}{\partial z}-\frac{\partial a_z}{\partial r}\right)\vec{e}_\theta+\frac{1}{r}\left[\frac{\partial(ra_\theta)}{\partial r}-\frac{\partial a_r}{\partial\theta}\right]\vec{e}_z$$

$$(\nabla\cdot\boldsymbol{P})_r=\frac{1}{r}\frac{\partial(rp_{rr})}{\partial r}+\frac{1}{r}\frac{\partial p_{r\theta}}{\partial\theta}+\frac{\partial p_{rz}}{\partial z}-\frac{p_{\theta\theta}}{r}$$

$$(\nabla\cdot\boldsymbol{P})_\theta=\frac{1}{r}\frac{\partial(rp_{\theta r})}{\partial r}+\frac{1}{r}\frac{\partial p_{\theta\theta}}{\partial\theta}+\frac{\partial p_{\theta z}}{\partial z}+\frac{p_{r\theta}}{r}$$

$$(\nabla\cdot\boldsymbol{P})_z=\frac{1}{r}\frac{\partial(rp_{zr})}{\partial r}+\frac{1}{r}\frac{\partial p_{z\theta}}{\partial\theta}+\frac{\partial p_{zz}}{\partial z}$$

□ 连续方程

$$\frac{\partial\rho}{\partial t}+div(\rho\vec{V})=0$$

$$div\vec{a}=\frac{1}{r}\frac{\partial(ra_r)}{\partial r}+\frac{1}{r}\frac{\partial a_\theta}{\partial\theta}+\frac{\partial a_z}{\partial z}$$

$$\Downarrow$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}=0$$

□ 动量方程

$$\frac{\partial}{\partial t}(\rho\vec{V})+div(\rho\vec{V}\vec{V})=\rho\vec{R}+divP$$

$$(\nabla\cdot\boldsymbol{P})_r=\frac{1}{r}\frac{\partial(rP_{rr})}{\partial r}+\frac{1}{r}\frac{\partial P_{r\theta}}{\partial\theta}+\frac{\partial P_{rz}}{\partial z}-\frac{P_{\theta\theta}}{r}$$

$$\Downarrow$$

$$\rho\left(\frac{dv_r}{dt}-\frac{v_\theta^2}{r}\right)=\rho F_r+\frac{1}{r}\left[\frac{\partial(rp_{rr})}{\partial r}+\frac{\partial p_{r\theta}}{\partial\theta}+\frac{\partial(rp_{zr})}{\partial z}\right]-\frac{p_{\theta\theta}}{r}$$

$$\rho\left(\frac{dv_\theta}{dt}+\frac{v_rv_\theta}{r}\right)=\rho F_\theta+\frac{1}{r}\left[\frac{\partial(rp_{r\theta})}{\partial r}+\frac{\partial p_{\theta\theta}}{\partial\theta}+\frac{\partial(rp_{\theta z})}{\partial z}\right]+\frac{p_{r\theta}}{r}$$

$$\rho\frac{dv_z}{dt}=\rho F_z+\frac{1}{r}\left[\frac{\partial(rp_{zr})}{\partial r}+\frac{\partial p_{\theta z}}{\partial\theta}+\frac{\partial(rp_{zz})}{\partial z}\right]$$

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① 柱坐标系N-S方程

微分算子

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+v_z\frac{\partial}{\partial z}$$

$$grad\phi=\nabla\phi=\frac{\partial\phi}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta+\frac{\partial\phi}{\partial z}\vec{e}_z$$

$$div\vec{V}=\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}$$

$$rot\vec{a}=\left(\frac{1}{r}\frac{\partial a_z}{\partial\theta}-\frac{\partial a_\theta}{\partial z}\right)\vec{e}_r+\left(\frac{\partial a_r}{\partial z}-\frac{\partial a_z}{\partial r}\right)\vec{e}_\theta+\frac{1}{r}\left[\frac{\partial(ra_\theta)}{\partial r}-\frac{\partial a_r}{\partial\theta}\right]\vec{e}_z$$

- Φ为函数函数，表示粘性耗散掉的机械能

$$\Phi=-\frac{2}{3}\mu\left(div\vec{V}\right)^2+2\mu S{:}S$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}=0$$

$$\rho\left(\frac{dv_r}{dt}-\frac{v_\theta^2}{r}\right)=\rho F_r+\frac{1}{r}\left[\frac{\partial(rp_{rr})}{\partial r}+\frac{\partial p_{r\theta}}{\partial\theta}+\frac{\partial(rp_{zr})}{\partial z}\right]-\frac{p_{\theta\theta}}{r}$$

$$\rho\left(\frac{dv_\theta}{dt}+\frac{v_rv_\theta}{r}\right)=\rho F_\theta+\frac{1}{r}\left[\frac{\partial(rp_{r\theta})}{\partial r}+\frac{\partial p_{\theta\theta}}{\partial\theta}+\frac{\partial(rp_{\theta z})}{\partial z}\right]+\frac{p_{r\theta}}{r}$$

$$\rho\frac{dv_z}{dt}=\rho F_z+\frac{1}{r}\left[\frac{\partial(rp_{zr})}{\partial r}+\frac{\partial p_{\theta z}}{\partial\theta}+\frac{\partial(rp_{zz})}{\partial z}\right]$$

$$\rho T\frac{ds}{dt}=\Phi+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(k\frac{\partial T}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(rk\frac{\partial T}{\partial z}\right)\right]+\rho q$$

$$p_{rr}=-p+2\mu\frac{\partial v_r}{\partial r}-\frac{2}{3}\mu div\vec{V}$$

$$p_{\theta\theta}=-p+2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)-\frac{2}{3}\mu div\vec{V}$$

$$p_{zz}=-p+2\mu\frac{\partial v_z}{\partial z}-\frac{2}{3}\mu div\vec{V}$$

$$p_{r\theta}=\mu\left(\frac{\partial v_\theta}{\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right)$$

$$p_{\theta z}=\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}\right)$$

$$p_{zr}=\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)$$

Ref.: 吴望一. 流体力学(第二版). 北京：北京大学出版社，2021（**p.137-138**）

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② 球坐标系N-S方程

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+\frac{v_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}$$

$$grad\phi=\nabla\phi=\frac{\partial\phi}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\lambda}\vec{e}_\lambda$$

$$div\vec{v}=\nabla\cdot\vec{v}=\frac{1}{r^2}\frac{\partial(r^2v_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\sin\theta v_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda}$$

$$rot\vec{a}=\nabla\times\vec{a}=\frac{1}{r\sin\theta}\left[\frac{\partial(a_\lambda\sin\theta)}{\partial\theta}-\frac{\partial a_\theta}{\partial\lambda}\right]\vec{e}_r+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\lambda}-\frac{\partial(ra_\lambda)}{\partial r}\right]\vec{e}_\theta+\frac{1}{r}\left[\frac{\partial(ra_\theta)}{\partial r}-\frac{\partial a_r}{\partial\theta}\right]\vec{e}_\lambda$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial(\rho r^2v_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(\rho\sin\theta v_\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial(\rho v_\lambda)}{\partial\lambda}=0$$

$$\rho\left(\frac{dv_r}{dt}-\frac{v_\theta^2+v_\lambda^2}{r}\right)=\rho F_r+\frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta p_{rr})}{\partial r}+\frac{\partial(r\sin\theta p_{\theta r})}{\partial\theta}+\frac{\partial(rp_{\lambda r})}{\partial\lambda}\right]-\frac{p_{\theta\theta}+p_{\lambda\lambda}}{r}$$

$$\rho\left(\frac{dv_\theta}{dt}+\frac{v_rv_\theta}{r}-\frac{v_\lambda^2\cot\theta}{r}\right)=\rho F_\theta+\frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta p_{r\theta})}{\partial r}+\frac{\partial(r\sin\theta p_{\theta\theta})}{\partial\theta}+\frac{\partial(rp_{\theta\lambda})}{\partial\lambda}\right]+\frac{p_{r\theta}}{r}-\frac{p_{\lambda\lambda}\cot\theta}{r}$$

$$\left(\frac{dv_\lambda}{dt}+\frac{v_rv_\lambda}{r}+\frac{v_\theta v_\lambda\cot\theta}{r}\right)=\rho F_\lambda+\frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta p_{\lambda r})}{\partial r}+\frac{\partial(r\sin\theta p_{\theta\lambda})}{\partial\theta}+\frac{\partial(rp_{\lambda\lambda})}{\partial\lambda}\right]+\frac{p_{\lambda r}}{r}+\frac{p_{\theta\lambda}\cot\theta}{r}$$

$$\rho T\frac{ds}{dt}=\Phi+\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta k\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta k\frac{\partial T}{\partial\theta}\right)+\frac{\partial}{\partial\lambda}\left(\frac{1}{\sin\theta}k\frac{\partial T}{\partial\lambda}\right)\right]+\rho q$$

$$p_{rr}=-p+2\mu\frac{\partial v_r}{\partial r}-\frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$p_{\theta\theta}=-p+2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)-\frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$p_{\lambda\lambda}=-p+2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right)-\frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$p_{r\theta}=p_{\theta r}=\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)$$

$$p_{\theta\lambda}=p_{\lambda\theta}=\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\lambda}+\frac{1}{r}\frac{\partial v_\lambda}{\partial\theta}-\frac{v_\lambda\cot\theta}{r}\right)$$

$$p_{\lambda r}=p_{r\lambda}=\mu\left(\frac{\partial v_\lambda}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\lambda}-\frac{v_\lambda}{r}\right)$$

Ref.: 吴望一. 流体力学(第二版). 北京：北京大学出版社，2021（**p.138-139**）

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③ 一般正交曲线坐标系N-S方程

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\frac{v_1}{h_1}\frac{\partial}{\partial q_1}+\frac{v_2}{h_2}\frac{\partial}{\partial q_2}+\frac{v_3}{h_3}\frac{\partial}{\partial q_3}$$

$$div\vec{V}=\frac{1}{h_1h_2h_3}\left[\frac{\partial(v_1h_2h_3)}{\partial q_1}+\frac{\partial(v_2h_1h_3)}{\partial q_2}+\frac{\partial(v_3h_1h_2)}{\partial q_3}\right]$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{h_1h_2h_3}\left[\frac{\partial(\rho h_2h_3v_1)}{\partial q_1}+\frac{\partial(\rho h_3h_1v_2)}{\partial q_2}+\frac{\partial(\rho h_1h_2v_3)}{\partial q_3}\right]=0$$

$$\begin{aligned}\rho\left(\frac{dv_1}{dt}+\frac{v_1v_2}{h_1h_2}\frac{\partial h_1}{\partial q_2}+\frac{v_1v_3}{h_1h_3}\frac{\partial h_1}{\partial q_3}-\frac{v_2^2}{h_1h_2}\frac{\partial h_2}{\partial q_1}-\frac{v_3^2}{h_3h_1}\frac{\partial h_3}{\partial q_1}\right)\\=\rho F_1+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3p_{11})+\frac{\partial}{\partial q_2}(h_3h_1p_{12})+\frac{\partial}{\partial q_3}(h_1h_2p_{31})\right]+\frac{p_{12}}{h_1h_2}\frac{\partial h_1}{\partial q_2}\\+\frac{p_{13}}{h_1h_3}\frac{\partial h_1}{\partial q_3}-\frac{p_{22}}{h_1h_2}\frac{\partial h_2}{\partial q_1}-\frac{p_{33}}{h_3h_1}\frac{\partial h_3}{\partial q_1}\end{aligned}$$

$$\begin{aligned}\rho\left(\frac{dv_2}{dt}+\frac{v_1v_2}{h_1h_2}\frac{\partial h_2}{\partial q_1}+\frac{v_2v_3}{h_2h_3}\frac{\partial h_2}{\partial q_3}-\frac{v_3^2}{h_2h_3}\frac{\partial h_3}{\partial q_2}-\frac{v_1^2}{h_1h_2}\frac{\partial h_1}{\partial q_2}\right)\\=\rho F_2+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3p_{12})+\frac{\partial}{\partial q_2}(h_3h_1p_{22})+\frac{\partial}{\partial q_3}(h_1h_2p_{23})\right]+\frac{p_{12}}{h_1h_2}\frac{\partial h_2}{\partial q_1}\\+\frac{p_{23}}{h_2h_3}\frac{\partial h_2}{\partial q_3}-\frac{p_{33}}{h_2h_3}\frac{\partial h_3}{\partial q_2}-\frac{p_{11}}{h_1h_2}\frac{\partial h_1}{\partial q_2}\end{aligned}$$

$$\begin{aligned}\rho\left(\frac{dv_3}{dt}+\frac{v_3v_1}{h_3h_1}\frac{\partial h_3}{\partial q_1}+\frac{v_2v_3}{h_2h_3}\frac{\partial h_3}{\partial q_2}-\frac{v_1^2}{h_3h_1}\frac{\partial h_1}{\partial q_3}-\frac{v_2^2}{h_3h_2}\frac{\partial h_2}{\partial q_3}\right)\\=\rho F_3+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3p_{31})+\frac{\partial}{\partial q_2}(h_3h_1p_{23})+\frac{\partial}{\partial q_3}(h_1h_2p_{33})\right]+\frac{p_{31}}{h_1h_3}\frac{\partial h_3}{\partial q_1}\\+\frac{p_{23}}{h_2h_3}\frac{\partial h_3}{\partial q_2}-\frac{p_{11}}{h_3h_1}\frac{\partial h_1}{\partial q_3}-\frac{p_{22}}{h_2h_3}\frac{\partial h_2}{\partial q_3}\end{aligned}$$

$$\rho T\frac{ds}{dt}=\Phi+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}k\frac{\partial T}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}k\frac{\partial T}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}k\frac{\partial T}{\partial q_3}\right)\right]+\rho q$$

$$p_{11}=-p+2\mu\left(\frac{1}{h_1}\frac{\partial v_1}{\partial q_1}+\frac{v_2}{h_1h_2}\frac{\partial h_1}{\partial q_2}+\frac{v_3}{h_1h_3}\frac{\partial h_1}{\partial q_3}\right)-\frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$p_{22}=-p+2\mu\left(\frac{1}{h_2}\frac{\partial v_2}{\partial q_2}+\frac{v_3}{h_2h_3}\frac{\partial h_2}{\partial q_3}+\frac{v_1}{h_1h_2}\frac{\partial h_2}{\partial q_1}\right)-\frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$p_{33}=-p+2\mu\left(\frac{1}{h_3}\frac{\partial v_3}{\partial q_3}+\frac{v_1}{h_3h_1}\frac{\partial h_3}{\partial q_1}+\frac{v_2}{h_2h_3}\frac{\partial h_3}{\partial q_2}\right)-\frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$p_{12}=\mu\left(\frac{1}{h_2}\frac{\partial v_1}{\partial q_2}+\frac{1}{h_1}\frac{\partial v_2}{\partial q_1}-\frac{v_1}{h_1h_2}\frac{\partial h_1}{\partial q_2}-\frac{v_2}{h_1h_2}\frac{\partial h_2}{\partial q_1}\right)$$

$$p_{23}=\mu\left(\frac{1}{h_3}\frac{\partial v_2}{\partial q_3}+\frac{1}{h_2}\frac{\partial v_3}{\partial q_2}-\frac{v_2}{h_2h_3}\frac{\partial h_2}{\partial q_3}-\frac{v_3}{h_2h_3}\frac{\partial h_3}{\partial q_2}\right)$$

$$p_{31}=\mu\left(\frac{1}{h_1}\frac{\partial v_3}{\partial q_1}+\frac{1}{h_3}\frac{\partial v_1}{\partial q_3}-\frac{v_3}{h_3h_1}\frac{\partial h_3}{\partial q_1}-\frac{v_1}{h_3h_1}\frac{\partial h_1}{\partial q_3}\right)$$

Ref.: 吴望一. 流体力学(第二版). 北京：北京大学出版社，2021 (**p.135-137**)

第二部分教学内容的知识图谱

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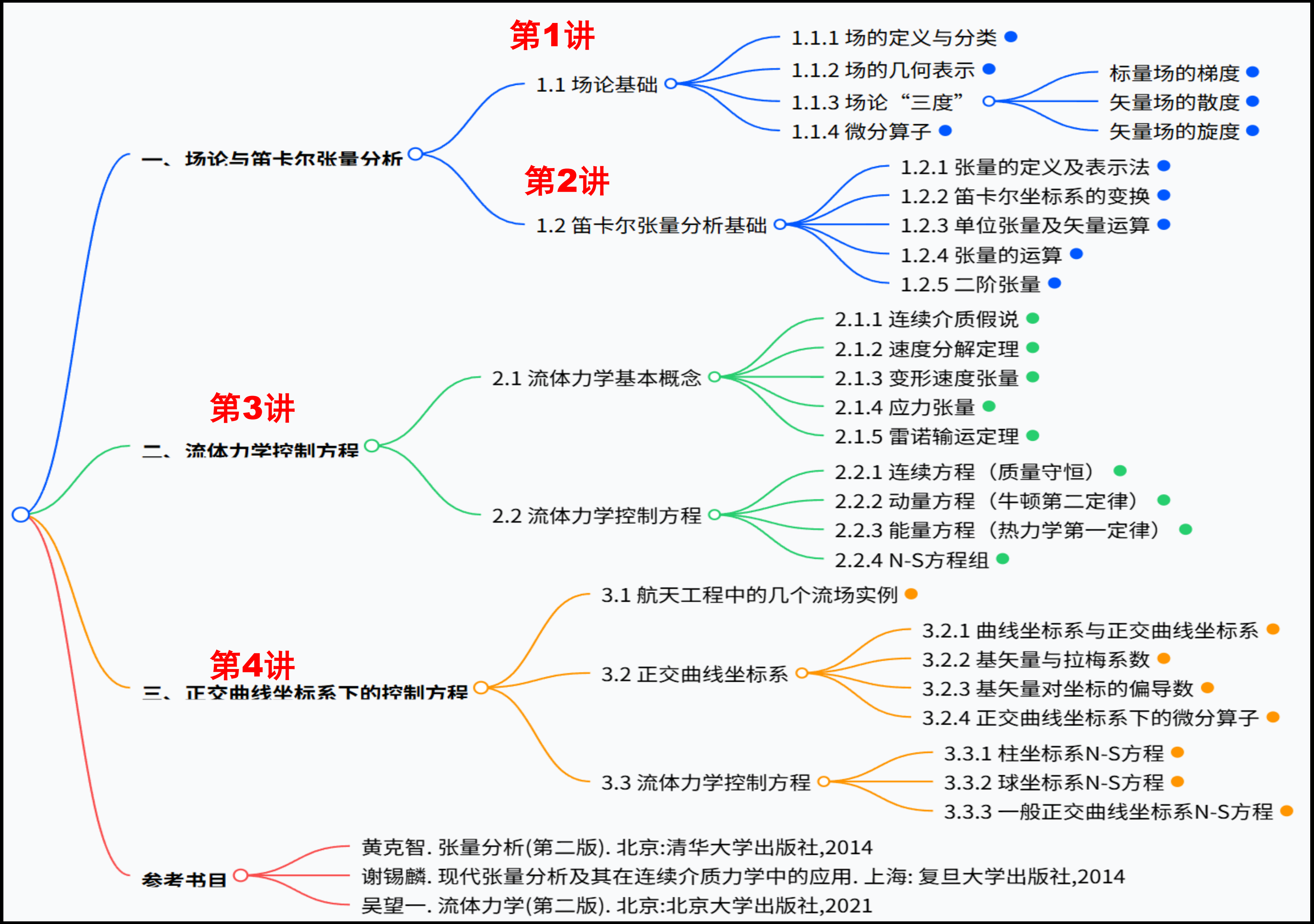

第四次课：

习题一： 13题

补充题：

1. 已知流体质点的速度分量为$v_x = x + 2y$，$v_y = 2x - y$，$v_z = 0$（直角坐标系），试推导该流场的速度梯度张量$\frac{\partial v_i}{\partial x_j}$，并将其分解为对称张量$S$（变形速度张量）和反对称张量$A$，同时计算旋转角速度$\vec{\omega}$的各分量。
2. 已知柱坐标系与笛卡尔坐标系的变换关系为$x = r\cos\theta, y = r\sin\theta, z = z$，试推导柱坐标系的拉梅系数$h_r$、$h_\theta$、$h_z$。
3. 请写出张量形式的流体力学控制方程组，并将其转化为圆柱坐标系下的形式（写出转换推导过程）。

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The End