Universality and Renormalization

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PH5500 course project

Content

- some discussion on universality
- emergence of criticality in physical systems
- renormalization technique in 2D percolation

what is universality?

- mathematical similarities in behaviour for a large class of systems, independent of dynamic details
- systems may be fundamentally different, but appear indifferentiable at different length scales

Universality

- Universalities are characterized into classes, depending on system's *order parameter* and *critical exponent*
- universality classes include ising and percolation models
- Order parameter and critical exponent are defined from the relationship,
$$C = (T - T_C)^{-\alpha}$$
- These parameters depend, not on the system specifics, rather only the dimensionality and range of interaction.

Universality near criticality

- critical point - where system exhibits scale invariance i.e self similarity or fractal nature
- fluctuations occur at the scale of the entire system
- *correlation length* blows up; every constituent of the system can cause fluctuations at system level
- it is a widely observed phenomenon; striking example being **the human brain**

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In states of matter

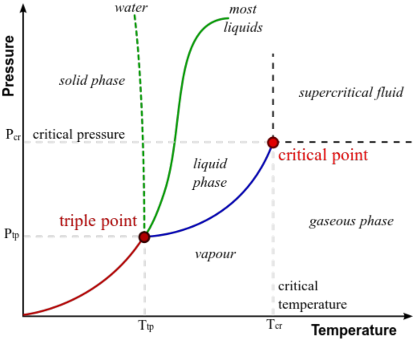


Figure: Phase diagram for different states of matter

The Ising model

- 2D lattice model introduced by Ising used to derive magnetisation
- criticality of magnetisation as a function of temperature: para to ferro transition
- critical point: scale invariance, self similarity/fractality in Ising model

The Ising model

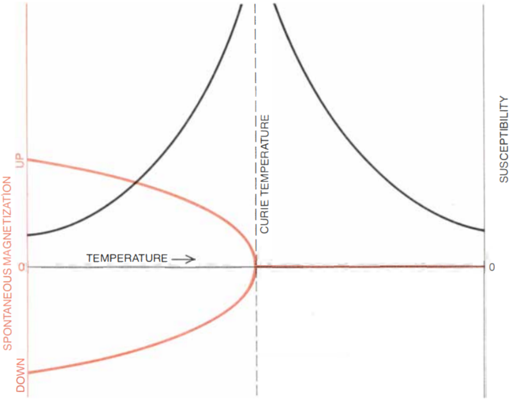


Figure: Sudden onset of Magnetisation at Curie Temprature. We essentially observe a Pitchfork bifurcation in spontaneous Magnetisation as temprature is varied

Ising model

- Numerical computations on the Ising model is a NP-hard problem classically
- number of spin configurations scale exponentially with size of lattice.
- thus, many mean field approaches have been introduced to reduce the size of Ising model, a popular one being the Block spin renormalization.

A simplistic Percolation scenario

- Consider a 2D lattice, as shown in figure below
- Each site has a probability ’p’ of being occupied.
- If two adjacent sites are occupied, they are referred to as ’connected’

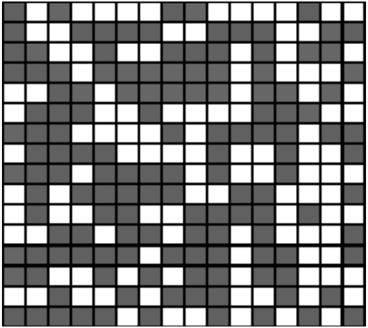


Figure: 16x16 lattice with p = 0.5

A simplistic Percolation scenario

- Now imagine a fluid entering the lattice from the topmost row, through shaded region
- what is the probability that it will 'percolate' through the grid?
- alternatively, we may ask, **Can we comment on existence of a spanning cluster in the grid?**

Renormalization Procedure

- What are the clusters that would be considered coloured for b=2?

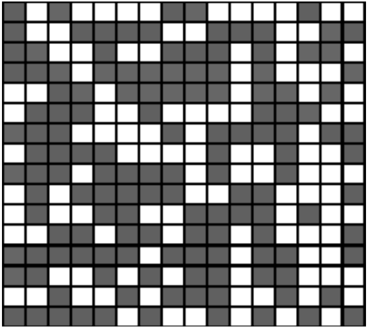
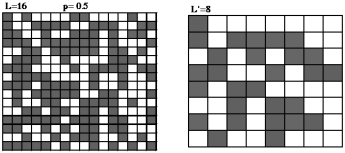


Figure: 16x16 lattice with p = 0.5

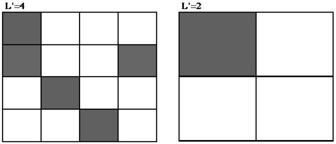
Renormalization Procedure

- First cluster has probability of occurance p^4
- second set of clusters have $4p^3(1-p)$
- last set of clusters have $2p^2(1-p)^2$

Renormalization Procedure



(a) b=2 type RNG- going from l=16 to 8



(b) the final two RNG process

Solving with RNG equation

- if sites are occupied with probability p, then the cells are occupied with probability p'. As a result, the Renormalization equation is

$$p' = p^4 + 4p^3(1-p) + 2p^2(1-p)^2$$

- For eg., continued application of RG at = p = 0.5 gives R(0.5) = 0.44, R(R(0.5) = 0.35.. and eventually yields 0

- Similarly, p = 0.7 goes to 1 upon successive RG applications.

Solving for the non-Trivial fixed points

- We want the non-trivial fixed point such that p* = R(p*).
- solving for this, we get
- p* = 0 and 1 (the trivial fixed points) and p* = 0.61804 which is it's critical value

Estimating critical point

- monte carlo simulations of above procedure were performed on python
- critical point was found to lie close to 0.62, thus *closely* agreeing with the calculations above

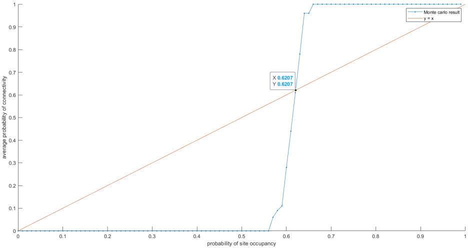


Figure: Phase transition observed as p is varied

Results and some thoughts

- below is a simulation result which compares above approach with directly checking percolation in a matrix:

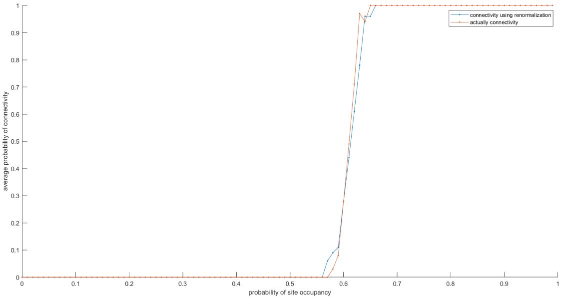


Figure: comparing renormalization to actual connectivity

- Empirically determined value of p_c for infinite lattice is 0.5927
- this value is slightly smaller than the p_c obtained above. **(Why?)**

Significance of the Percolation model

- Applications in tissue engineering with carbon nanoTube polymer composites
- Modelling forest fires
- Their Use in Flow and Transport Modeling

Some open problems

- Here is a small problem in two dimensions. Each edge of L^2 is oriented in a random direction, horizontal edges being oriented eastwards with probability p and westwards otherwise, and vertical edges being oriented northwards with probability p and southwards otherwise. Let $\eta(p)$ be the probability that there exists an infinite oriented path starting at the origin. It is not hard to see that $\eta(1/2) = 0$, and also that $\eta(p) = \eta(1 - p)$. Is it the case that $\eta(p) > 0$ if $p \neq 1/2$?

References

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- https://www.youtube.com/watch?v=ihymiZ4zhug&ab$_$channel=SamanthavonBibra
- http://schleife.web.engr.illinois.edu/teaching/mse485/lnotes/scaling.html
- https://blog.dougmet.net/2011/07/universality-at-the-critical-point/.