Principal Curvature and Harmonic Analysis

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1. Introduction. Although the Fourier transform on $\mathbf{R}^n$ is defined in terms of the linear (group) structure, results of recent years have demonstrated a remarkable connection between harmonic analysis and a highly nonlinear object: the unit sphere S^{n-1}. Stein (see Fefferman [2]) was the first to notice that the Fourier transform of an L^p function has a well-defined restriction in $L^2(S^{n-1})$ if p is close enough to 1, even though S^{n-1} has measure 0 in $\mathbf{R}^n$ and $\hat{f}$ is, a priori, defined only almost everywhere. The proofs of this and related results all used explicit formulas for the Fourier transform of the surface measure on S^{n-1}, even though arguments in low dimensions made it clear that the crucial factor was the curvature of S^{n-1} (see Tomas [10], Strichartz [9], Fefferman [2] and Zygmund [11]).

In another direction, Stein and Wainger [7] showed that the maximal operator associated with averages over dilates of S^{n-1} is bounded on L^p, $p > n/(n-1)$, $n \geq 3$. Once again an exact formula, this time for the Kernels of the Bochner-Riesz partial summation operators, was needed in the proof.

The purpose of this paper is to show that the exact formulas can either be dispensed with (for the restriction theorem) or generalized (for the maximal theorem), thereby allowing us to obtain similar results for essentially arbitrary smooth hypersurfaces. The quality of the theorem, as measured by how large p is in the restriction theorem and how small it is in the maximal theorem, turns out to be a function of the number of principal curvatures which are nonzero at each point of the hypersurface. This is shown in Sections 2 and 3. In Section 4 we revert to the hypothesis of nonzero Gaussian curvature, but now allow the surfaces to vary smoothly from point to point and behave only asymptotically like dilations. Applications to wave equations in three space variables are given in Section 5.

Other approaches to results related to those of Section 4 have been carried out independently by Coifman, El-Kohen, Meyer, Nagel, Stein and Wainger in unpublished form.

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2. Restriction theorems.

Theorem 1. *Let $S \subset \mathbf{R}^{n+m}$ be a smooth n-dimensional submanifold with a smooth measure $d\mu$ compactly supported away from the boundary of S. Suppose that for some C, $r > 0$, $|\widehat{d\mu}(\xi)| \leq C(1 + |\xi|)^{-r}$. Then*

$$\|\hat{f}\,|_S\|_{L^2(S,\,d\mu)} \leq C'\|f\|_{L^p(\mathbf{R}^{m+n},dx)},\quad p = \frac{2(m + r)}{2m + r},\quad f \in \mathscr{S}.$$

Proof. Let $\sum \psi_k \equiv 1$ be a smooth partition of unity on a neighborhood of S such that on each $\mathrm{supp}(\psi_k)$, S can be represented as the graph of m functions. It suffices to show that $\|\hat{f}\,|_S\|_{L^2(S,\psi_k d\mu)} \leq C_k\|f\|_{L^p(\mathbf{R}^{n+m},dx)}$ for each k, since only finitely many of the $\psi_k d\mu$'s are nonzero.

Hence, we may assume that S is given near 0 by $x_{n+1} = \Phi_{n+1}(x'), \ldots, x_{n+m} = \Phi_{n+m}(x')$ where $x' = (x_1,\ldots,x_n)$, $\Phi_j(0) = \Phi_j'(0) = 0$, $j = n + 1, \ldots, n +$ m, and $\psi_k d\mu = d\mu_k = \varphi_k(x_1,\ldots,x_n)dx_1 \ldots dx_n$, where $\varphi_k \in C^\infty(\mathbf{R}^n)$. Note that since $d\mu_k = \psi_k d\mu$, $\widehat{d\mu}_k(\xi) = \hat{\psi}_k * \widehat{d\mu}(\xi)$ has the same rate of decrease as $\widehat{d\mu}(\xi)$, since $\hat{\psi}_k \in \mathscr{S}$. Thus $|\widehat{d\mu}_k(\xi)| \leq C_k(1 + |\xi|)^{-r}$.

Now, by a lemma of Tomas (see Strichartz [9]), it suffices to show $\|\widehat{d\mu}_k * g\|_{p'} \leq C_k\|g\|_p$ with p as in the statement of the theorem, $(1/p) + (1/p') = 1$ as usual, and $g \in \mathscr{S}$. Following Stein (see Strichartz [9]), we will embed $\widehat{d\mu}_k$ in an analytic family of Kernels. First, some elementary distribution theory:

Consider in $\mathbf{R}^m$ the distribution $r^{2\lambda} = \left(\sum_{j=1}^m x_j^2\right)^\lambda$. For $\mathrm{Re}(\lambda) > -m/2$, it is locally integrable. For $\psi \in \mathscr{S}$,

$$\begin{aligned}\int r^{2\lambda}\psi(x)\,dx &= \int_{|x|\leq 1} r^{2\lambda}(\psi(x) - \psi(0))\,dx + \psi(0)\int_{|x|\leq 1} r^{2\lambda}\,dx + \int_{|x|\geq 1} r^{2\lambda}\psi(x)\,dx\\ &= \int_{|x|\leq 1} r^{2\lambda}(\psi(x) - \psi(0))\,dx + \psi(0)\cdot\frac{\omega_{m-1}}{2\lambda + m} + \int_{|x|\geq 1} r^{2\lambda}\psi(x)\,dx,\end{aligned}$$

where ω_{m-1} = surface measure of S^{m-1}. Since $|\psi(x) - \psi(0)| \leq C|x|$, this allows us to extend the definition of $r^{2\lambda}$ to a meromorphic function in $\{\mathrm{Re}(\lambda) > -(m + 1)/2\}$ with a pole at $-m/2$. In fact, by replacing $\psi(x) - \psi(0)$ by $\psi(x) - \sum_{j=0}^N \psi^{(j)}(x)$, $\psi^{(j)}(x) = j^{\text{th}}$ order terms of the Taylor series of ψ about 0, we can continue $r^{2\lambda}$ to a meromorphic function of λ with simple poles at $\lambda = -m/2, -1 - (m/2), \ldots$.

Now, note that the residue at $\lambda = -m/2$ is, up to a constant, $\psi(0) = \langle\psi,\delta\rangle$, δ = Dirac distribution.

Hence, if we let $G_z(x) = \varphi_k(x_1,\ldots,x_n)/\Gamma(z + m/2)\cdot((x_{n+1} - \Phi_{n+1}(x'))^2 +$

$\ldots + (x_{n+m} - \Phi_{n+m}(x'))^2)^z$ in $\mathbf{R}^{n+m}$, G_z is an entire, distribution-valued function of z and $G - (m/2) = d\mu_k$. Let

$$T_z f(x) = \hat{G}_z * f(x), \qquad f \in \mathscr{S}.$$

On $\{\mathrm{Re}(z) = 0\}$, G_z is bounded, so $\|T_z f\|_{L^2} \le C_z \|f\|_{L^2}$, where C_z has at most exponential growth in $|\mathrm{Im}\, z|$. We next examine $\hat{G}_z$ on the line $\{\mathrm{Re}(z) = -(m+r)/2\}$. To calculate the Fourier transform, we first integrate with respect to $x_{n+1}, \ldots, x_{n+m}$. (For $\mathrm{Re}(z) > -m/2$, this is valid since $G_z(z)$ is locally integrable. For $\mathrm{Re}(z) \le -m/2$, the formula for $\hat{G}_z$ is still valid by analytic continuation.) By the formula on page 363 of Gelfand-Shilov [3],

$$\hat{r}^{2z} = 2^{z+m} \pi \frac{m}{2} \frac{\Gamma\left(z + \frac{m}{2}\right)}{\Gamma(-z)} |\xi|^{-2z-m}.$$

Letting $\xi'' = (\xi_{n+1}, \ldots, \xi_{n+m})$,

$$\hat{G}_z(\xi) = C'_z |\xi''|^{-2z-m} \cdot \left(\int \cdots \int e^{i(x_1\xi_1 + \ldots + x_n\xi_n + \Phi_{n+1}(x')\xi_{n+1} + \ldots + \Phi_{n+m}(x')\xi_{n+m})} \varphi_k(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \right)$$
$$= C'_z |\xi'|^{-2z-m} d\mu_k(\xi).$$

So, for $\mathrm{Re}(z) = -(m+r)/2$, $|\hat{G}_z(\xi)| \le C'_z |\xi'|^{-2\mathrm{Re}(z) - m - r} \le C'_z$ has at most exponential growth in $|\mathrm{Im}\, z|$ and so we may apply analytic interpolation. Since $\|T_z f\|_{L^\infty} \le C'_z \|f\|_{L^1}$ on $\{\mathrm{Re}(z) = -(m+r)/2\}$ and $\|T_z f\|_{L^2} \le C_z \|f\|_{L^2}$ on $\{\mathrm{Re}(z) = 0\}$, we obtain $\|T_{-m/2} f\|_{p'} \le C_k \|f\|_p$, $p = 2(m+r)/(2m+r)$. But, since $T_{-m/2} f = \widehat{d\mu}_k * f$, we are done. Q.E.D.

Corollary 1. *Let $S \subset \mathbf{R}^{n+1}$ be a smooth hypersurface with a smooth measure $d\mu$ supported compactly away from the boundary. Suppose that, at each point of* $\mathrm{supp}(d\mu)$, *at least K principal curvatures are nonzero. Then*

$$\| \hat{f} | S \|_{L^2(S, d\mu)} \le C \|f\|_{L^p(\mathbf{R}^{n+1}, dx)}, \quad p = \frac{2(k+2)}{k+4}, \quad f \in \mathscr{S}.$$

Proof. Littman [5] has proven that for such a $d\mu$, $|\widehat{d\mu}(\xi)| \le C(1 + |\xi|)^{-k/2}$. We apply Theorem 1 with $m = 1$, $r = k/2$. Thus

$$\| \hat{f} | S \|_{L^2} \le C \|f\|_{L^p}, \quad p = \frac{2\left(1 + \frac{k}{2}\right)}{2 + \frac{k}{2}} = \frac{2(k+2)}{k+4}.$$

Q.E.D.

Remark. If $S \subset \mathbf{R}^n$ has nonzero Gaussian curvature at each point, all $n - 1$ principal curvatures are nonzero and hence $p = 2(n+1)/(n+3)$, which in the case of S^{n-1} gives the well-known result of Tomas and Stein (see Tomas [10]). Note also that if there exists an open subset of S which is a piece of k-surface with nonzero Gaussian curvature cross a piece of $(n-k)$-dimensional Euclidean space, an argument of Knapp (see Strichartz [9]) shows that p is sharp. In general though, high orders of contact of S with its tangent planes will contribute to the decay of $\widehat{d\mu}$, so p will not be sharp.

3. Constant coefficient maximal theorems. We now generalize the maximal theorems of Stein and Wainger [7] pertaining to spherical averages. Recall that they proved that

$$\left\| \sup_{0<t<\infty} \left| \int_{S^{n-1}} f(x+ty)\,d\sigma(y) \right| \right\|_{L^p} \leq C_p \|f\|_{L^p}, \quad p > \frac{n}{n-1},$$

$$n \geq 3, \quad d\sigma = \text{surface measure on } S^{n-1}.$$

In this section we replace S^{n-1} by a piece of hypersurface with at least k principal curvatures nonzero and allow $y \to ty$ to be replaced by an arbitrary family of nonisotropic dilations.

Theorem 2. *Let S be a smooth hypersurface in* $\mathbf{R}^n$ *with a smooth measure $d\mu$ compactly supported away from the boundary. Suppose that at each point of* $\operatorname{supp}(d\mu)$, *at least $K \geq 2$ principal curvatures are nonzero. Let $\delta_t(x) = (t^{a_1}x_1, \dots, t^{a_n}x_n)$, $a_j \geq 1$ be a family of dilations transverse (defined below) to S on* $\operatorname{supp}(d\mu)$. *Then,*

$$\left\| \sup_{0<t<\infty} \left| \int_{S} f(x - \delta_t(y))\,d\mu(y) \right| \right\|_{L^p} \leq C_p \|f\|_{L^p}, \quad p > \frac{k+1}{k}.$$

Remarks. *1.* For $S = S^{n-1}$, $k = n-1$, $(k+1)/k = n/(n-1)$, so this theorem is a natural generalization of Stein and Wainger's.

2. δ_t transverse to S means that $S \times \mathbf{R}^+ \to \mathbf{R}^n - \{0\}$ by $(y,t) \to \delta_t(y)$ is a diffeomorphism onto its image. A condition of this sort is to be expected, since we want integrals of our surface averaged over intervals of t's to be dominated by some classical average over a solid region (in our case, the strong maximal function).

Proof of Theorem 2. First, by a partition of unity argument, we may assume that S (after a translation and rotation) can be written as the graph of a smooth function on the $(n-1)$-ball. Now, $d\mu = \omega(x)\,d\sigma$, where ω is a smooth function of compact support on S and $d\sigma$ is induced surface measure. We extend $\omega(x)$ to be δ_t-homogeneous of degree 0 and C^∞ away from the origin; this extension is also denoted by $\omega(x)$.

By the assumption of transversality, there exists a "norm" Φ for $\operatorname{supp}(d\mu)$.

That is, $\Phi(x) > 0$, δ_t-homogeneous of degree 1, C^∞ away from 0, and $\operatorname{supp}(d\mu) \subset \{\Phi(x) = 1\}$. Define

$$m_\alpha(x) = \frac{\omega(x)}{\Gamma(\alpha)}(1 - \Phi(x)^2)_+^{\alpha-1}, \quad \operatorname{Re}(\alpha) > 0.$$

We will show how to analytically continue m_α as a distribution and obtain the asymptotics of its Fourier transform. Toward this end, let us introduce $\chi \in C_0(\mathbf{R})$, $\chi \equiv 0$ near 0, $\chi \equiv 1$ near 1, χ supported near 1. We write $m_\alpha(x) = (1 - \chi(\Phi(x)))m_\alpha(x) + \chi(\Phi(x))m_\alpha(x)$. We will continue each of the terms on the right-hand side of this last equation to entire functions of α. The first term is easy: On the support of $1 - \chi(\Phi(x))$, $1 - \Phi(x)^2$ is uniformly bounded away from $0 \Rightarrow$

$$\frac{(1 - \chi(\Phi(x)))\omega(x)}{\Gamma(\alpha)}(1 - \Phi(x)^2)_+^{\alpha-1}$$

is an entire, distribution-valued function of α.

To continue $\chi(\Phi(x))m_\alpha(x)$, we consider its Fourier transform. Recalling that S can be written as the graph of a function and noting that $1 - \Phi(x)^2$ has simple zeros on S, we see that for $\operatorname{Re}(\alpha) > 0$, (after a translation and rotation)

$$\chi(\Phi(x))m_\alpha(x) = \frac{1}{\Gamma(\alpha)}\varphi(x_1,\dots,x_n)L^{\alpha-1}(x_1,\dots,x_n)(x_n - \tilde{\Phi}(x_1,\dots,x_{n-1}))_+^{\alpha-1}$$

where $\varphi \in C_0^\infty$ supported in the $(n-1)$ ball, $L \in C^\infty$, real, nonzero, and S is the graph of $x_n = \tilde{\Phi}(x')$, $\tilde{\Phi}(0) = \tilde{\Phi}'(0) = 0$. To calculate the Fourier transform of this, we first integrate in the x_n direction. (As in Section 2, this is valid for $\operatorname{Re}(\alpha) > 0$ since everything is integrable.) The classical identity

$$\int_0^\infty x^{\alpha-1}e^{-ixy}\,dx = \Gamma(\alpha)\cdot ie^{i(\pi/2)(\alpha-1)}(-y + i0)^{-\alpha}$$

yields

$$(\chi(\Phi)m_\alpha)\hat{\ }(\xi) = ie^{i(\pi/2)(\alpha-1)}\cdot\int\dots\int e^{-i(x_1\xi_1+\dots+\tilde{\Phi}(x')\xi_n)}$$
$$\cdot\,[(-\xi_n + i0)^{-\alpha} * (\varphi L^{\alpha-1})\tilde{\ }(x_1,\dots,x_{n-1},\xi_n)]\,dx_1,\dots,dx_{n-1}$$

where $(\varphi L^{\alpha-1})\tilde{\ }$ = Partial Fourier transform with respect to x_n and the convolution is in ξ_n. If we let

$$s_\alpha(x_1,\dots,x_{n-1};\xi_n) = (-\xi_n + i0)^{-\alpha} * ((\varphi L^{\alpha-1})\tilde{\ }(x_1,\dots,x_{n-1},\xi_n)),$$

then s_α is a smooth, compactly supported function from the $(n-1)$-ball to $S^{-\alpha}(\mathbf{R})$, the symbols of order $-\alpha$ on $\mathbf{R}$. If we then define $\widetilde{d\mu}_\alpha =$

$s_\alpha(x')\,dx_1, \ldots, dx_{n-1}$ (a smooth, compactly-supported vector-valued measure on S) then $(\chi(\Phi)m_\alpha)\hat{}\,(\xi) = ie^{i(\pi/2)(\alpha-1)}\widehat{d\mu}_\alpha(\xi)$. But, since s_α is an entire function of α (since $(-\xi_n + i0)^{-\alpha}$ is) so is this last expression, giving us our continuation of $\chi(\Phi)m_\alpha$ to all α.

To obtain the asymptotics of $(\chi(\Phi)m_\alpha)\hat{}\,(\xi)$, we simply note that Littman's result on the decay of $\widehat{d\mu}$ goes over easily to the case when the measure is vector valued. Hence,

$$\begin{aligned} |(\chi(\Phi)m_\alpha)\hat{}\,(\xi)| &\le c\, e^{c'|\operatorname{Im} z|}_{(|\xi_n|)^{-\operatorname{Re}(\alpha)}(1+|\xi|)^{-k/2},\quad \operatorname{Re}(\alpha)\le 0} \\ &\le c\, e^{c'|\operatorname{Im} z|}_{(1+|\xi|)^{-\operatorname{Re}(\alpha)-k/2},\quad \operatorname{Re}(\alpha)\le 0}. \end{aligned}$$

With our analytic continuations and asymptotics in hand, we can now proceed with the proof of our maximal theorem.

First, we define analytic families of operators $(1 - \chi(\Phi))M_t^\alpha$ and $\chi(\Phi)M_t^\alpha$ as follows: For $f \in \mathcal{S}$,

$$(1 - \chi(\Phi))M_t^\alpha f(x) = f * t^{-Q}m_\alpha(\delta_{t^{-1}}(x)), \quad Q = \sum_{j=1}^n a_j = \text{homog. dimension}$$

$$(\chi(\Phi)M_t^\alpha f)\hat{}\,(\xi) = (\chi(\Phi)m_\alpha)\hat{}\,(\delta_t(\xi))\hat{f}(\xi).$$

We let $M_t^\alpha f(x) = (1 - \chi(\Phi))M_t^\alpha + \chi(\Phi)M_t^a$. Then, by the study of the one-dimensional distribution $x_+^{\alpha-1}$, done in much that same way as $r^{2\lambda}$, we see that

$$M_t^0 f(x) = \int_S f(x - \delta_t(y))\,d\mu(y).$$

Second, for each α fix $\varphi \in \mathcal{S}$ such that φ is radially decreasing, $\int\varphi(x) = \hat{\varphi}(0) = (\chi(\Phi)m_\alpha)\hat{}\,(0)$. Define

$$g_\alpha(f)(x)^2 = \int_0^\infty |\chi(\Phi)M_t^\alpha f(x) - \varphi_t * f(x)|^2 \frac{dt}{t}, \quad f \in \mathcal{S},$$

where $\varphi_t(x) = t^{-Q}\varphi(\delta_{t^{-1}}(x))$. We now reprove the sequence of lemmas in Stein and Wainger [7].

Lemma 1. $\|g_\alpha(f)\|_{L^2} \le C_\alpha\|f\|_{L^2}$, $-k/2 < \operatorname{Re}(\alpha) \le 0$. *Furthermore, C_α exhibits at most exponential growth in $|\operatorname{Im}\alpha|$.*

Proof.

$$\begin{aligned} \|g_\alpha(f)\|_{L^2}^2 &= \int_0^\infty \int_{\mathbf{R}^n} |\chi(\Phi)M_t^\alpha f(x) - \varphi_t * f(x)|^2\,dx\,\frac{dt}{t} \\ &= \int_0^\infty \int_{\mathbf{R}^n} |(\chi(\Phi)m_\alpha)\hat{}\,(\delta_t(\xi)) - \hat{\varphi}(\delta_t(\xi))|^2\,d\xi\,\frac{dt}{t} \quad \text{by Plancherel} \end{aligned}$$

$$= \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 \cdot \int_0^\infty |(\chi(\Phi)m_\alpha)\hat{\ }(\delta_t(\xi)) - \hat{\varphi}(\delta_t(\xi))|^2 \frac{dt}{t}\, d\xi .$$

So it remains to show that $\int_0^\infty |(\chi(\Phi)m_\alpha)\hat{\ }(\delta_t(\xi)) - \hat{\varphi}(\delta_t(\xi))|^2 (dt/t) \leq C_a^2$ independent of $\xi \in \mathbf{R}^n$. Let $\| \ \ \|$ be any (fixed) norm for the δ_t's. That is, it is smooth away from the origin and $\|\delta_t(x)\| = t\|x\|$, $t > 0$, $x \in \mathbf{R}^n$. We split $(0,\infty)$ into $(0,1/\|\xi\|) \cup (1/\|\xi\|,\infty)$. Since $(\chi(\Phi)m_\alpha)\hat{\ }$ and $\hat{\varphi}$ are smooth and equal at 0,

$$\int_0^{1/\|\xi\|} |(\chi(\Phi)m_\alpha)\hat{\ }(\delta_t(\xi)) - \hat{\varphi}(\delta_t(\xi))|^2 \frac{dt}{t} \leq c \int_0^{1/\|\xi\|} |\delta_t(\xi)|^2 \frac{dt}{t}$$

$$\leq c \int_0^{1/\|\xi\|} \sum t^{2aj}\xi_j^2 \frac{dt}{t} \leq c' \left[\sum t^{2aj}\xi_j^2 \right]_{t=0}^{t=1/\|\xi\|} = c'|\delta_{1/\|\xi\|}(\xi)|.$$

But, $\delta_{1/\|\xi\|}(\xi)$ lies on $\{\|x\| = 1\}$ which is compact, so $|\delta_{1/\|\xi\|}(\xi)| \leq c''$. As for the $(1/\|\xi\|,\infty)$ term, $\int_{1/\|\xi\|}^\infty |\hat{\varphi}(\delta_t(\xi))|^2\, dt/t$ and $\int_{1/\|\xi\|}^\infty |(\chi(\Phi)m_\alpha)\hat{\ }(\delta_t(\xi))|^2\, dt/t$ are controlled in the same manner, which we show for the $\chi(\Phi)m_\alpha$ term.

$$\int_{1/\|\xi\|}^\infty |(\chi(\Phi)m_\alpha)\hat{\ }(\delta_t(\xi))|^2 \frac{dt}{t} \leq c_\alpha'^2 \int_{1/\|\xi\|}^\infty (1 + |\delta_t(\xi)|)^{-2\mathrm{Re}(\alpha)-k} \frac{dt}{t},$$

since $|(\chi(\Phi)m_\alpha)\hat{\ }(\xi)| \leq c_\alpha'(1 + |\xi|)^{-\mathrm{Re}(\alpha)-k/2}$, $\mathrm{Re}(\alpha) \leq 0$. We now let $t' = t\|\xi\|$. Our integral is $\leq C_\alpha'^2 \int_1^\infty (1 + |\delta_{t'}(\delta_{1/\|\xi\|}(\xi))|)^{-2\mathrm{Re}(\alpha)-k}(dt'/t')$. But, $\delta_{1/\|\xi\|}(\xi)$ lies on a compact set bounded away from the origin, so this is $\leq cC_\alpha^2 \int_1^\infty (1 + t')^{-2\mathrm{Re}(\alpha)-k}(dt'/t') \leq C_\alpha^2$ if $-k/2 < \mathrm{Re}(\alpha) \leq 0$. The growth of C_α is controlled by the growth of C_α', which we saw was at most exponential in $|\mathrm{Im}\,\alpha|$. Q.E.D.

Lemma 2.

$$\left\| \sup_{0<t<\infty} \left(\frac{1}{t} \int_0^t |M_s^\alpha f(x)|^2 ds \right)^{1/2} \right\|_{L^2} \leq C_\alpha \|f\|_{L^2}, \quad -\frac{k}{2} < \mathrm{Re}(\alpha) \leq 0.$$

Proof.

$$\frac{1}{t}\int_0^t |\chi(\Phi)M_s^\alpha f(x)|^2 ds \leq \frac{1}{t}\int_0^t |\chi(\Phi)M_s^\alpha f(x) - \varphi_s * f(x)|^2 ds$$

$$+ \frac{1}{t}\int_0^t |\varphi_s * f(x)|^2 ds \leq \int_0^\infty |\chi(\Phi)M_s^\alpha f(x) - \varphi_s * f(x)|^2 \frac{ds}{s}$$

$$+ \sup_{0<s<t} |\varphi_s * f(x)|^2 \leq g\alpha(f)(x)^2 + c_\alpha^2 M_S(f)(x)^2,$$

$M_S(f)(x)$ = strong maximal function. Thus, we have proven the lemma for $\chi(\Phi)M_t^\alpha$. Now $\sup_{0<t<\infty} ((1/t)\int_0^t |(1 - \chi(\Phi))M_s^\alpha f(x)|^2 ds^{1/2} \leq \sup_{0<t<\infty} |(1 - \chi(\Phi))$

$M_t^\alpha f(x)|$. But, $(1-\chi(\Phi))M_t^\alpha f(x)$ is given as the convolution of f with δ_t-dilate of a bounded function (with bound having at most exponential growth in $|\operatorname{Im}\alpha|$). Thus, $\sup_{0<t<\infty} |(1-\chi(\Phi))M_t^\alpha f(x)| \leq c_\alpha M_s f(x)$, which is bounded on L^2. Q.E.D.

Lemma 3. *If* $\operatorname{Re}(\alpha) > \operatorname{Re}(\alpha')$, $f \in \mathscr{S}$, *then*

$$M_t^\alpha f(x) = \frac{2}{\Gamma(\alpha-\alpha')} \int_0^1 M_{st}^{\alpha'} f(x)(1-s^2)^{\alpha-\alpha'-1} s^{Q+2\alpha'-1}\, ds.$$

Proof. This follows from the identity, for $\operatorname{Re}(\alpha') > 0$,

$$m_t^\alpha(x) = \frac{2}{\Gamma(\alpha-\alpha')} \int_0^1 m_{st}^{\alpha'}(x)(1-s^2)^{\alpha-\alpha'-1} s^{Q+2\alpha'-1}\, ds,$$

where $m_t^\alpha(x) = t^{-Q} m_\alpha(\delta_{t^{-1}}(x))$. This is proven as in Stein [8], using only the fact that Φ is δ_t-homogeneous of degree 1. For general α, α' the identity is still valid by analytic continuation.

Lemma 4. $f \in \mathscr{S}$, $\operatorname{Re}(\alpha) > (1/2) - (k/2) \Rightarrow \| \sup_{0<t<\infty} |M_t^\alpha f(x)| \|_{L^2} \leq c_\alpha \|f\|_{L^2}$.

Proof.

$$|M_t^\alpha f(x)| \leq \frac{2}{\Gamma(\alpha-\alpha')} \left(\int_0^1 |M_s^{\alpha'} f(x)|^2 ds \right)^{1/2} \cdot \left(\int_0^1 (1-s^2)^{2(\alpha-\alpha'-1)} s^{2(Q+2\alpha'-1)} ds \right)^{1/2}.$$

We pick α' such that $\operatorname{Re}(\alpha-\alpha') > 1/2$, $\operatorname{Re}(\alpha') > -k/2$. Then the first integral is dominated by the expression in Lemma 2 and the second converges, since $Q \geq n$ and $k \leq n-1$. Q.E.D.

Once we have linearized $M^\alpha(x) = \sup_{t>0} M_t^\alpha f(x)$, we can use analytic interpolation. (This is why we kept track of the $c\alpha$'s.) We use the fact that we have L^2 boundedness for $\operatorname{Re}(\alpha) > (1/2) - (k/2)$ and L^p boundedness for $\operatorname{Re}(\alpha) = 1$, $1 < p < \infty$, since M^α is dominated by M_S. Thus, for $1 < p \leq 2$, M^α is bounded on L^p, $\operatorname{Re}(\alpha) > -k + ((k+1)/p)$. This gives L^p boundedness of $M = M^0 = \sup_{0<t<\infty} |\int_S f(x-\delta_t(y))\, d\mu(y)|$ for $p > (k+1)/k$. Similarly, for $2 \leq p \leq \infty$, M^α is bounded on L^p for $\operatorname{Re}(\alpha) > (1/p)(1-k)$, so M^0 is bounded on L^p, $(k+1)/k < p < \infty$. Q.E.D.

Remark. As for the restriction theorem, if there exists an open subset of S which is a piece of k-surface with nonzero Gaussian curvature cross a piece of $(n-k-1)$ dimensional Euclidean space, the range of p is optimal, as can be shown by modifying an example of Stein and Wainger [7].

4. Variable coefficient maximal theorems. Let $r : \mathbf{R}^n \times [0,\infty) \times S^{n-1} \to \mathbf{R}^+$ be a smooth function, such that for each x, t, $\{r(x,t,y)y : y \in S^{n-1}\}$ has nonzero Gaussian curvature and is transverse to the $\delta_{t's}$. We would like to study the averages

$$M_t f(x) = \int_{S^{n-1}} f(x - \delta_t(r(x,t,y)y))\, d\sigma(y),$$

$d\sigma$ = surface measure on S^{n-1}, and the associated maximal function.

Theorem 3. *For each compact set K, $\exists t_0 > 0$ such that*

$$\|\sup_{0<t<t_0} |M_t f(x)|\|_{L^p(K)} \le c_{p,K} \|f\|_{L^p}, \quad p > \frac{n}{n-1}, \quad n \ge 3.$$

Remark. The reason we must now restrict the radii to be small is that, in contrast to the constant coefficient case, M_t is not necessarily a bounded operator on L^p. For instance, if we let, for the standard dilations, for x near 0, and every t,

$$\{x - r(x,t,y)y : y \in S^{n-1}\} = \text{the unit sphere } \textit{centered at 0,}$$

then M_1 assigns to each point near 0 the average of f over S^{n-1}, and this is certainly not bounded on any L^p, $p < \infty$.

Proof of Theorem 3. We try to mimic the proof of Theorem 2 as far as possible. In particular, we attempt to embed $M_t = M_t^0$ in an analytic family of operators satisfying

$$(*) \qquad M_t^\alpha f(x) = \frac{2}{\Gamma(\alpha - \alpha')} \int_0^1 M_{st}^{\alpha'} f(x)(1 - s^2)^{\alpha - \alpha' - 1} s^{Q + 2\alpha' - 1}\, ds.$$

Thinking of this as a self-fulfilling prophecy, we set, for $\operatorname{Re}(\alpha) > 0$,

$$M_t^\alpha f(x) = \frac{2}{\Gamma(\alpha)} \int_0^1 M_{st}^0 f(x)(1 - s^2)^{\alpha - 1} s^{Q-1}\, ds, \quad f \in S.$$

Lemma 5. *The $M_{t's}^\alpha$ so defined satisfy* $(*)$, $\operatorname{Re}(\alpha') > 0$.

Proof. The right-hand side of $(*)$ is

$$\frac{4}{\Gamma(\alpha - \alpha')\Gamma(\alpha')}$$

$$\cdot \int_0^1 \int_0^1 M_{ss't}^0 f(x)(1 - s'^2)^{\alpha' - 1}(1 - s^2)^{\alpha - \alpha' - 1} s^{Q + 2\alpha' - 1}(s')^{Q-1}\, ds\, ds'$$

$$= \int_0^1 M_{s''t}^0 f(x) I_{\alpha,\alpha'}(s'')\, ds''$$

where $I_{\alpha,\alpha'}(s'')$ is the Radon-Nikodym derivative with respect to ds'' of the measure on $[0,1]$ induced from $(4/(\Gamma(\alpha-\alpha')\Gamma(\alpha')))(1-s'^2)^{\alpha'-1}s'^{Q-1}(1-s^2)^{\alpha-\alpha'-1}s^{Q+2\alpha'-1}ds\,ds'$ under the map $(s,s')\to ss'$. But $I_{\alpha,\alpha'}(s'')$ is independent of the M_t^0's, so to evaluate it we may use the identity of Lemma 3. Thus $I_{\alpha,\alpha'(s'')}=(4/\Gamma(\alpha))(1-(s''^2))^{\alpha-1}(s'')^{Q-1}$ and hence our expression is

$$\frac{2}{\Gamma(\alpha)}\int_0^1 M_{s''t}f(x)(1-(s'')^2)^{\alpha-1}(s'')^{Q-1}ds''=M_t^\alpha f(x)$$

Q.E.D.

Note that when the $M_{t's}^\alpha$ are analytically continued, the identity (*) will still hold.

We now let χ be as before. That is, $\chi\in C_0^\infty(\mathbf{R})$, $\chi\equiv 0$ near 0, $\chi\equiv 1$ near 1, χ supported near 1. Let

$$(1-\chi)M_t^\alpha f(x)=\frac{2}{\Gamma(\alpha)}\int_0^1(1-\chi(s))M_{st}^0 f(x)(1-s^2)^{\alpha-1}s^{Q-1}ds.$$

This is an entire, operator-valued function of α, since $1-s^2$ is bounded away from 0 on $\operatorname{supp}(1-\chi)$. To continue χM_t^α, (= obvious thing for $\operatorname{Re}(\alpha)>0$), though, we will have to restrict t to be small. In fact, for $\operatorname{Re}(\alpha)>0$,

$$\begin{aligned}
&\chi M_t^\alpha f(x)\\
&=\frac{2}{\Gamma(\alpha)}\int_0^1\chi(s)M_{st}^0 f(x)(1-s^2)^{\alpha-1}s^{Q-1}ds\\
&=\int_{\mathbf{R}^n}e^{ix\cdot\xi}\frac{2}{\Gamma(\alpha)}\int_0^1\chi(s)\widehat{d\mu}_{x,t}(\delta_{st}(\xi))(1-s^2)^{\alpha-1}s^{Q-1}ds\hat f(\xi)d\xi\\
&=\int_{\mathbf{R}^n}e^{ix\cdot\xi}\left(\frac{2}{\Gamma(\alpha)}\int_0^1\chi(s)[d\mu_{x,t}]_s(1-s^2)^{\alpha-1}s^{Q-1}ds\right)^{\wedge}(\delta_t(\xi))\hat f(\xi)d\xi
\end{aligned}$$

where $d\mu_{x,t}$ = measure on $\{r(x,t,y)y:y\in S^{n-1}\}$ induced from $d\sigma$ on S^{n-1} and $[d\mu_{x,t}]_s$ = measure on $\{\delta_s(r(x,t,y)y):y\in S^{n-1}\}$ induced from $d\sigma$. The inner integral in the last line can be thought of as a vector-valued integral (with value in $\mathscr{B}(\mathbf{R}^n)$ = Borel measures), but for small t it is actually an absolutely continuous measure (for $\operatorname{Re}(\alpha)>0$). To see this, note that there exists a $t_0>0$ such that for $x\in K$, $(t,y)\to\delta_t(r(x,t,y)y)$ is a diffeomorphism of $(0,t_0)\times S^{n-1}\to$ a punctured neighborhood of the origin in $\mathbf{R}^n$. Hence, for $0<t<t_0$, $d\nu_{x,t}^\alpha=(2/\Gamma(\alpha))\int_0^1\chi(s)[d\mu_{x,t}]_s(1-s^2)^{\alpha-1}s^{Q-1}ds$ has Radon-Nikodym derivative $=(2/\Gamma(\alpha)\varphi(y_1,\dots,y_n)K^{\alpha-1}(y_1,\dots,y_n)(\rho_{x,t}(y))_+^{\alpha-1}$ where φ is a smooth cutoff function living near $S_{x,t}=\{r(x,t,y)y:y\in S^{n-1}\}$, K is C^∞, real, nonzero, and $\rho_{x,t}$ is a defining function for $S_{x,t}$. As in the constant coefficient case, we continue χM_t^α by continuing the Fourier transform (in y) of $d\nu_{x,t}^\alpha$.

To calculate $\widehat{d\nu}^{\alpha}_{x,t}(\xi)$, we may assume, after a rotation, that $\xi = (0,\ldots,0,\ldots,-u)$, $u > 0$. A translation in the $(y_1,\ldots,y_{n-1})$ plane (which does not affect $\widehat{d\nu}^{\alpha}_{x,t}(\xi)$) lets us assume that the unique points on $S_{x,t}$ with normal proportional to ξ are $(0,\ldots,0,\pm c_{x,t,\xi})$, since, as we shall see later, we may assume that $S_{x,t} = -S_{x,t}$. We split $d\nu^{\alpha}_{x,t}$ smoothly into pieces supported near $(0,\ldots,0,\pm c_{x,t,\xi})$ and away from those points. An integration by parts shows that the contribution from the latter term is in $\mathscr{S}$ (in terms of $u = |\xi|$) and is smoothly varying in x, t and ξ. The piece near $(0,\ldots,0,-c_{x,t,\xi})$ is given by the density function

$$\frac{2}{\Gamma(\alpha)}\varphi(x,t,y_1,\ldots,y_n)L^{\alpha-1}(x,t,y_1,\ldots,y_n)(y_n - \Phi(y_1,\ldots,y_{n-1})_+^{\alpha-1}$$

where $\varphi \in C_0^\infty$ in y, depending smoothly on x, t, L is C^∞ in y, real, and nonzero, depending smoothly on x, t, and $S_{x,t}$ is given locally at the graph of $y_n = \Phi(y_1,\ldots,y_{n-1})$. (In particular, $\Phi(0) = -c_{x,t,\xi}$.) We suppress the smooth dependence of Φ on x, t. The contribution from near $(0,\ldots,c_{x,t,\xi})$ is given by a similar expression.

This looks very much like the function whose Fourier transform we computed during the proof of Theorem 2. There we used Littman's asymptotics for the *size* of $d\mu$. Here, since we want to apply the machinery of Fourier Integral Operators, we need exact formulas for the amplitude and oscillation of $\widehat{d\mu}_{x,t}$. Hence, a brief digression.

Lemma 6. *Let $S \subset \mathbf{R}^n$ be a smooth compact hypersurface bounding a domain containing the origin. Suppose that S has nonzero Gaussian curvature. Then, if $d\mu$ is a smooth measure on S, and $S = -S$,*

$$\widehat{d\mu}(\xi) = c_n \cdot d(\xi)^{n-2}\,[r^{-(n-2)/2}J_{(n-2)/2}(r) * \hat{g}(\xi;r)]_{r=d(\xi)|\xi|}$$

where d is smooth and homogeneous of degree 0, $g(\xi;r)$ is in $C_0^\infty(\mathbf{R})$ as a function of r and is smooth and homogeneous of degree 0 in ξ. $*$ denotes 1-dimensional convolution.

Proof. By rotation we can assume $\xi = (u,0,\ldots,0)$, $u > 0$; a translation in the $(x_2,\ldots,x_n)$ plane insures that the unique point on S with normal $(1,0,\ldots,0)$ is $(d(\xi),0,\ldots,0)$. $d(\xi)$ is clearly smooth and homogeneous of degree 0. Because S has nonzero Gaussian curvature, we may parametrize it using cylindrical coordinates in the following way:

$$S = \{(x_1,(d(\xi)^2 - x_1^2)^{1/2}\cdot e(\xi;x_1,\ldots,x_n)\cdot(x_2,\ldots,x_n)) : (x_2,\ldots,x_n) \in S^{n-2}, x_1 \in [-d,d]\}$$

where e is smooth on $[-d,d] \times S^{n-2}$ in x and is smooth and homogeneous of degree 0 in ξ. The Jacobian of this parameterization is $f(\xi,x_1,x_2,\ldots,x_n)\cdot(d^2(\xi) - x_1^2)^{(n-3)/2}$, f smooth as e, f nonzero. Hence

$$\widehat{d\mu}(\xi) = \int_S e^{-ix\cdot\xi}\,d\mu(x)$$

$$= \int_{-d(\xi)}^{d(\xi)} e^{-iux_1} \int_{S^{n-2}} [f(\xi, x_1, \ldots, x_n)(d(\xi)^2 - x_1^2)^{(n-3)/2}]^{-1} \frac{d\mu}{d\nu}\, d\sigma\, dx_1$$

$$= \int_{-d(\xi)}^{d(\xi)} e^{-ix\cdot\xi}(d(\xi) - x_1^2)^{-(n-3)/2} \tilde{f}(\xi; x_1)\, dx_1$$

$$= d(\xi)^{n-2} \int_{-1}^{1} e^{-ix_1 d(\xi)u} \cdot \tilde{f}(\xi; d(\xi)x_1) \cdot (1 - x_1^2)^{-(n-3)/2} dx_1$$

$$= \Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right) d(\xi)^{n-2} [r^{-(n-2)/2} J_{(n-2)/2}(r) * \hat{g}(\xi; r)]_{r=d(\xi)|\xi|}$$

where: $d\nu$ = induced surface measure on S, $d\sigma$ = surface measure on S^{n-2}, $\tilde{f}(\xi; x_1)$ is smooth on $[-d(\xi), d(\xi)]$ in x_1, homogeneous of degree 0 in ξ, and $g(\xi; r)$ is an extension of $\tilde{f}(\xi; d(\xi)r)$ to an element of $C_0^\infty(\mathbf{R})$. Q.E.D.

Return to Proof of Theorem 3. Note that by a partition of unity argument, we could have assumed all along that the $S_{x,t}$'s satisfy $S_{x,t} = -S_{x,t}$, for this simply means that $r(x,t,y) = r(x,t,-y)$.

Recall that we wish to calculate the F. T. of

$$\psi \cdot d\nu_{x,t}^{\alpha} = \frac{2}{\Gamma(\alpha)} \varphi(x,t; y_1, \ldots, y_n) L^{\alpha-1}(x,t; y_1, \ldots, y_n)(y_n - \Phi(y_1, \ldots, y_{n-1}))_+^{\alpha-1};$$

ψ is a cutoff function. Using the same classical identity as in Theorem 2, we find that

$$(\psi d\nu_{x,t}^{\alpha})\hat{\ }(\xi) = ie^{i(\pi/2)(\alpha-1)} \int \cdots \int e^{-i(y_1\xi_1 + \ldots + \Phi(y')\xi_n)}$$
$$\cdot\ [(-\xi_n + i0)^{-\alpha} * (\varphi L^{\alpha-1})\tilde{\ }(x,t; y_1, \ldots, y_{n-1}, \xi_n)]\, dy_1 \ldots dy_{n-1}$$

where $y' = (y_1, \ldots, y_{n-1})$ and $\tilde{\ }$ denotes the partial Fourier transform with respect to y_n. Let $s^\alpha(x,t; y_1, \ldots, y_{n-1}, \xi_n) = (-\xi_n + i0)^{-\alpha} * (\varphi L^{\alpha-1})\tilde{\ }(x,t; y_1, \ldots, y_{n-1}, \xi_n)$. Then $s^\alpha : K \times [0, t_0) \times \mathbf{R}^{n-1} \to S^{-\alpha}(\mathbf{R})$ smoothly, with compact support in a small neighborhood of 0 in $\mathbf{R}^{n-1}$. Thus,

$$(\psi d\nu_{x,t}^{\alpha})\hat{\ }(\xi) = ie^{i(\pi/2)(\alpha-1)} \int_{S_{x,t}} e^{-iy\cdot\xi} \widetilde{d\mu}_{x,t}^{\alpha}(y)|_{\eta=\xi_n=|\xi|},$$

where $\widetilde{d\mu}_{x,t}^{\alpha} = s^\alpha dy_1, \ldots, dy_{n-1}$ is *vector valued,* taking values in $S^{-\alpha}(\mathbf{R}_\eta)$, and is a smooth measure on $S_{x,t}$. A moment's thought will convince the reader that Lemma 6 is still valid for vector-valued $d\mu$, with g now vector valued. Hence, for $\mathrm{Re}(\alpha) > 0$,

$$\left(\frac{2}{\Gamma(\alpha)}\int_0^1 \chi(s)\,[d\mu_{x,t}]_s(1-s^2)^{\alpha-1}s^{Q-1}\,ds\right)^{\wedge}(\xi)$$
$$= C_\alpha d(x,t;\xi)^{n-2}\,[r^{(n-2)/2}J_{(n-2)/2}(r) * \hat{g}^{\alpha}(x,t;\xi,r)]\Big|_{\substack{r=d(x,t;\xi)|\xi|\\ \eta=|\xi|}} + \mathscr{S}$$

where d is smooth on $K \times [0,t_0)$ and smooth and homogeneous of degree 0 in ξ; g_α is smooth on $K \times [0,t_0)$, smooth and homogeneous of degree 0 in ξ, in $C_0^\infty(\mathbf{R})$ in r, and takes values in $S^{-\alpha}(\mathbf{R}_\eta)$; the $\mathscr{S}$-errors are smooth in x and t and entire in α. We now simply note that s^α (and hence g^α) is actually entire in α (since $(-\xi_n + i0)^{-\alpha}$ is). Thus, if we denote the last line by $\widehat{\chi m}_\alpha(x,t;\xi)$, we may define

$$\chi M_t^\alpha f(x) = \int e^{ix\cdot\xi}\widehat{\chi m}_\alpha(x,t;\delta_t(\xi))\hat{f}(\xi)\,d\xi, \quad f \in \mathscr{S} \text{ supported in } K.$$

We next show that $\widehat{\chi m}_\alpha$ can be written as the product of an amplitude and a phase, thereby allowing us to express the χM_t^α's as Fourier Integral Operators. Since $r^{-(n-2)/2}J_{(n-2)/2}(r) = a(r)e^{-ir}$, $a \in S^{-(n-1)/2}(\mathbf{R})$ (actually, this is false: $r^{-(n-2)/2}J_{(n-2)/2}(r) = a_1(r)e^{ir} + a_2(r)e^{-ir}$, but the first term is handled in the same way as the second), we have

$$\begin{aligned}\widehat{\chi m}_\alpha(x,t;\xi) &= c_\alpha d(x,t;\xi)^{n-2}(a(r)e^{-ir} * \hat{g}^\alpha(x,t;\xi;r))|_{r=d(x,t;\xi)|\xi|,\eta=|\xi|} + \mathscr{S}\\ &= c_\alpha d(x,t;\xi)^{n-2}(a(r) * (\hat{g}^\alpha(x,t;\xi;r)e^{ir}))|_{r=d(x,t;\xi)|\xi|,\eta=|\xi|}\\ &\quad \cdot e^{id(x,t;\xi)} + \mathscr{S}.\end{aligned}$$

It is immediate that $d(x,t;\xi)^{n-2}$ is in $S^0_{1,0}(K \times \mathbf{R}^n)$, smoothly in t, since it is homogeneous of degree 0 in ξ and smooth. We claim that $b^\alpha(x,t;\xi) = (a(r) * (\hat{g}^\alpha(x,t;\xi,r)e^{ir}))\Big|_{\substack{r=d(x,t;\xi)|\xi|\\ \eta=|\xi|}}$ is in $S_{1,0}^{-((n-1)/2)-\alpha}(K \times \mathbf{R}^n)$ smoothly in t. In fact $b^\alpha(x,t;\xi) = \tilde{b}^\alpha(x,t;\xi;r;\eta)\Big|_{\substack{r=d(x,t;\xi)|\xi|\\ \eta=|\xi|}}$ where $\tilde{b}(x,t;\xi;r;\eta) = a(r) * (\hat{g}^\alpha(x,t;\xi;r)(\eta)e^{ir})$, $*$ = convolution in r. (Recall that g^α takes values in $S^{-\alpha}$ $(\mathbf{R}_\eta$.) Hence

$$\frac{\partial b^\alpha}{\partial \xi j} = \frac{\partial \tilde{b}^\alpha}{\partial \xi_j} + \frac{\partial \tilde{b}^\alpha}{\partial r}\cdot\frac{d(x,t;\xi)\xi_j}{|\xi|} + \frac{\partial \tilde{b}^\alpha}{\partial \eta}\cdot\frac{\xi_j}{|\xi|}\Bigg|_{r=d(x,t;\xi)|\xi|,\eta=|\xi|}.$$

The first term is the derivative of a smooth function $K \times [0,t_0) \times (\mathbf{R}^n - \{0\}) \to \tilde{S}^{-(n-1)/2}(\mathbf{R}_r)$ (where $\tilde{S}^{-(n-1)/2}$ denotes symbols taking values in $S^{-\alpha}(\mathbf{R}_\eta)$) evaluated at $r = d(x,t;\xi)|\xi|$, $\eta = |\xi|$. Since this function is homogeneous of degree 0 in ξ, its derivative is homogeneous of degree -1, so

$$\left|\frac{\partial \tilde{b}^\alpha}{\partial \xi_j}\right|_{\substack{r=d\cdot|\xi|\\ \eta=|\xi|}} \le C(1+d(x,t;\xi)|\xi|)^{-(n-1)/2}|\xi|^{-1}(1+|\xi|)^{-\mathrm{Re}(\alpha)}$$

$$\le C(1+|\xi|)^{-((n-1)/2)-\mathrm{Re}(\alpha)-1}, \quad x \in K.$$

In the second term, we are differentiating something in $\tilde{S}^{-(n-1)/2}(\mathbf{R}_r)$, so it is $\le c(1+d(x,t;\xi)|\xi|)^{-((n-1)/2)-1}(1+|\xi|)^{-\mathrm{Re}(\alpha)} \le C(1+|\xi|)^{-((n-1)/2)-\mathrm{Re}(\alpha)-1}$, $x \in K$. In the third, we are differentiating something in $S^{-\alpha}(\mathbf{R}_\eta)$, so it is $\le c(1+d(x,t;\xi)|\xi|)^{-((n-1)/2)-1}(1+|\xi|)^{-\mathrm{Re}(\alpha)} \le c(1+|\xi|)^{-((n-1)/2)-\mathrm{Re}(\alpha)-1}$, $x \in K$. On the other hand,

$$\frac{\partial b^\alpha}{\partial x_j} = \frac{\partial \tilde{b}^\alpha}{\partial x_j}\bigg|_{\substack{r=d(x,t;\xi)|\xi|\\ \eta=|\xi|}} + \frac{\partial \tilde{b}^\alpha}{\partial r}\bigg|_{\substack{r=d\cdot|\xi|\\ \eta=|\xi|}} \frac{\partial d}{\partial x_j}(x,t;\xi)|\xi|$$

and hence is $\le c(1+|\xi|)^{-((n-1)/2)-\mathrm{Re}(\alpha)}$. Similar reasoning for higher derivatives shows that

$$|\partial_\xi^\beta \partial_x^\gamma b^\alpha(x,t;\xi)| \le C_{\alpha,\beta,\gamma}(1+|\xi|)^{-((n-1)/2)-\mathrm{Re}(\alpha)-|\beta|}, \quad x \in K.$$

That is, $b^\alpha \in S_{1,0}^{-((n-1)/2)-\alpha}(K\times \mathbf{R}^n)$. The smoothness in t is clear from the proof.

If we now set $a^\alpha(x,t;\xi) = c_\alpha d(x,t;\xi)^{n-2} b^\alpha(x,t;\xi)$, then $a^\alpha \in S_{1,0}^{-((n-1)/2)-\alpha}(K\times\mathbf{R}^n)$ smoothly in t and we have

$$\chi M_t^\alpha f(x) = \int e^{ix\cdot\xi} a^\alpha(x,t;\delta_t(\xi)) e^{id(x,t;\delta_t(\xi))|\delta_t(\xi)|}\hat{f}(\xi)\,d\xi \quad \text{modulo } \mathcal{S}$$

$$= \int e^{i((x-y)\cdot\xi + d(x,t;\delta_t(\xi))|\delta_t(\xi)|)} a^\alpha(x,t;\delta_t(\xi)) f(y)\,dy\,d\xi \quad \text{modulo } \mathcal{S}.$$

We are thus naturally led to consider the phase function

$$\varphi_t(x,y;\xi) = (x-y)\cdot\xi + d(x,t;\delta_t(\xi))|\delta_t(\xi)|.$$

Claim. *$\exists t_0' > 0$ (which we will denote t_0 to spare notation) such that for $0 \le t < t_0$, φ_t is a nondegenerate operator phase function in the sense of Hörmander* [4].

Proof. *Fix x.* $\nabla\varphi_t = (-\xi, \dots) \ne 0$ if $\xi \ne 0$.

$$\nabla \frac{\partial \varphi_t}{\partial \xi_j}$$

$$= \nabla\left(x_j - y_j + t^{a_j}|\delta_t(\xi)|\frac{\partial d}{\partial \xi_j}(x,t;\delta_t(\xi)) + d(x,t;\delta_t))\frac{t^{2a_j}\xi_j}{|\delta_t(\xi)|}\right) = (-e_j, \dots)$$

where $e_j = (0,\dots,1,0,\dots,0)$, where the 1 is in the j^{th} place, and these are linearly independent.

Fix y. $\nabla\varphi_t = (\xi + |\delta_t(\xi)|\,\nabla_x d(x,t;\delta_t(\xi)),\ldots)$. If we pick

$$y < \min\left\{1, \frac{1}{2}\sup_{\substack{x\in K\\ \xi\in\mathbf{R}^n\\ t\in[0,t_0)}} (|\nabla_x d(x,t;\xi)|^{-1})\right\},$$

then this is $\neq 0$.

$$\nabla\frac{\partial\varphi_t}{\partial\xi_j} = \nabla\left(x_j - y_j + t^{a_j}|\delta_t(\xi)|\frac{\partial d}{\partial\xi_j}(x,t;\delta_t(\xi)) + d(x,t;\delta_t))\frac{t^{2a_j}\xi_j}{|\delta_t(\xi)|}\right)$$

$$= \left(t^{a_j}|\delta_t(\xi)|\frac{\partial^2 d}{\partial x_1\partial\xi_j}(x,t;\delta_t(\xi)) + \frac{t^{3a_j}\xi_j}{|\delta_t(\xi)|}\frac{\partial d}{\partial x_1}(x,t;\delta_t(\xi)),\ldots,\right.$$

$$\left. 1 + t^{a_j}|(\delta_t(\xi)|\frac{\partial^2 d}{\partial x_j\partial\xi_j}(x,t;\delta_t(\xi)) + \frac{t^{3a_j}\xi_j}{|\delta_t(\xi)|}\frac{\partial d}{\partial x_j}(x,t;\delta_t(\xi)),\ldots\right).$$

If

$$t < \min\left\{1, \frac{1}{10}\sup_{\substack{x\in K\\ \xi\in\mathbf{R}^n\\ i,j=1,\ldots,n,\\ t\in[0,t_0)}}\left(\left|\frac{\partial^2 d}{\partial x_i\partial\xi_j}(x,t;\delta_t(\xi))\right|^{-1}\right),\right.$$

$$\left.\frac{1}{10}\sup\left(\left|\frac{\partial d}{\partial x_j}(x,t;\delta_t(\xi))\right|^{-1}\right)\right\}$$

then these are linearly independent. Thus, if we adjust t_0, $0 \le t < t_0 \Rightarrow \varphi_t$ is a nondegenerate operator phase function. Q.E.D.

We now let

$$g_\alpha(f)(x)^2 = \int_0^{t_0} |\chi M_t^\alpha f(x) - c(x,t,\alpha)\chi M_t^1 f(x)|^2 \frac{dt}{t}$$

where $c(x,t,\alpha) = \widehat{\chi m}_\alpha(x,t;0)/\widehat{\chi m}_1(x,t;0)$. (This quantity is well-defined since χm_α is real-analytic in ξ for $\mathrm{Re}(\alpha) > 0$ and this property persists under analytic continuation.)

Lemma 7. $\|g_\alpha(f)\|_{L^2(K)} \le C_\alpha\|f\|_{L^2}$, $0 \ge \mathrm{Re}(\alpha) > -(n-1)/2$.

Proof. Let F_t be the elliptic Fourier Integral Operator corresponding to φ_t. F_t is an isomorphism of L^2. Modulo lower order operators, $\chi M_t^\alpha f(x) = F_t a^\alpha(x,t;\delta_t(D))f(x)$. In what follows, the contributions from the lower order terms are dominated by the $a^\alpha(x,t;\delta_t(\xi))$ term, so we ignore them. Hence,

$$\int g_\alpha(f)(x)^2\,dx = \int_0^{t_0} |F_t a^\alpha(x,t;\delta_t(D))F_t^{-1}F_t f(x)$$

$$- c(x,t,\alpha)F_t a^1(x,t;\delta_t(D))F_t^{-1}F_t f(x)|^2 \frac{dt}{t}.$$

By Egorov's Theorem, $F_t a^\alpha(x,t;\delta_t(D))F_t^{-1} = \tilde{a}^\alpha(x,t;D)$ where the leading symbol of $\tilde{a}^\alpha(x,t;D)$ is

$$\tilde{a}^\alpha(x,t;\xi) = a^\alpha(x,t;\delta_t(\xi)) \circ \Lambda_t,$$

Λ_t = Lagrangian manifold associated with φ_t. But, if $C_{\varphi_t} = \{(x,y,\xi): \nabla_\xi \varphi_t(x,y,\xi) = 0\}$, then Λ_t is the image of C_{φ_t} under the map $(x,y,\xi) \to (x, \nabla_x\varphi_t, y, \nabla_y\varphi_t) = (x, \xi + |\delta_t(\xi)|\,\nabla_x d(x,t;\delta_t(\xi)), y, -\xi)$. For $0 \le t < t_0$, $\xi \to \xi + |\delta_t(\xi)|\,\nabla_x d(x,t;\delta_t(\xi))$ is a diffeomorphism, homogeneous of degree 1, and has an inverse $\xi = g_t(x, \xi + |\delta_t(\xi)|\,\nabla_x d(x,t;\xi)))$.

Claim. $|\delta_t(\xi)| \simeq |\delta_t(g_t(x,\xi))|$. *In fact,* $|\delta_t(\xi + |\xi_t(\xi)|\nabla_x d(x,t;\delta_t(\xi)))| \simeq |\delta_t(\xi)|(1 + |\delta_t(\nabla_x d(x,t;\delta_t(\xi)))|) \simeq |\delta_t(\xi)|$ *for t small enough (we might have to adjust t_0 again). Thus, φ_t corresponds to a canonical transformation which changes the x coordinate homogeneously of degree* 0 *in ξ and leaves $|\delta_t(\xi$-coordinate$)|$ essentially unchanged. Thus, in what follows we may assume that* $\tilde{a}^\alpha(x,t;\xi) = a^\alpha(x,t;\delta_t(\xi))$.

We are reduced to showing that

$$\int\int_0^{t_0} |a^\alpha(x,t;\delta_t(D))f(x) - c(x,t,\alpha)a^1(x,t;\delta_t(D))f(x)|^2 \frac{dt}{t}\,dx \le C_\alpha \|f\|_{L^2}.$$

We now invoke the Wave-Packet theory of A. Córdoba and C. Fefferman [1]. Briefly, there exists a transform $f(x) \to Wf(x,\xi)$ such that

1) $\int |f(x)|^2 dx \simeq \int\int |Wf(x,\xi)|^2 dx\,d\xi$
2) $W(p(x,D)f)(x,\xi) = p(x,\xi)\,Wf(x,\xi)$ modulo lower terms.

(The contributions from the lower order terms are gain dominated by that of the leading term and are ignored.) Thus,

$$\int g_\alpha(f)(x)^2\,dx$$

$$\simeq \int_0^{t_0} |W(a^\alpha(x,t;\delta_t(D))f)(x,\xi) - c(x,t,\alpha)\,W(a^1(x,t;\delta_t(D)f)(x,\xi)|^2 dx\,d\xi\,\frac{dt}{t}$$

$$\simeq \int_0^{t_0} |a^\alpha(x,t;\delta_t(\xi))\,Wf(x,\xi) - c(x,t,\alpha)a^1(x,t;\delta_t(\xi))\,Wf(x,\xi)|^2 dx\,d\xi\,\frac{dt}{t}$$

$$\simeq \int |Wf(x,\xi)|^2 \int_0^{t_0} |a^\alpha(x,t;\delta_t(\xi)) - c(x,t,\alpha)\,a^1(x,t;\delta_t(\xi))|^2 \frac{dt}{t}\,dx\,d\xi.$$

Hence, to prove Lemma 7 it suffices to show that $\int_0^{t_0} |a^\alpha(x,t;\delta_t(\xi)) -$

$c(x,t,\alpha)\, a^1(x,t;\delta_t(\xi))|^2\, dt/t \le c_\alpha^2 < \infty$ uniformly in (x,ξ). But the proof of this is the same as in the constant coefficient case (Lemma 1), since our symbols satisfy the same estimates *uniformly* in $(x,t) \in K \times [0,t_0)$.

Q.E.D.

As in the constant coefficient case, we define

$$M_t^\alpha f(x) = (1-\chi) M_t^\alpha f(x) + \chi M_t^\alpha f(x).$$

Lemma 8.

$$\left\| \sup_{0<t<t_0} \left(\frac{1}{t} \int_0^t |M_s f(x)|^2 ds \right)^{1/2} \right\|_{L^2(K)} \le c_\alpha \|f\|_{L^2}, \quad 0 \ge \operatorname{Re}(\alpha) > -\frac{(n-1)}{2}.$$

Proof. Exactly as in Lemma 2, noting that $(1-\chi)M_t$ is again dominated by the strong maximal function.

Since the identity (*) holds, the rest of the proof is as in the constant coefficient case. This finishes the proof of Theorem 3. Q.E.D.

Remarks. 1. Theorem 3 clearly remains true when we consider the averages

$$M_t f(x) = \int_{S^{n-1}} f(x + \delta_t(r(x,t,y)y))\psi(x,t,y)\, d\sigma(y)$$

where $\psi \in C^\infty$ represents a smooth measure on each $S_{x,t}$.

2. A modification of the proof of Theorem 2 in the context of Theorem 3 shows that the $S_{x,t}$'s may be replaced with hypersurfaces with compactly supported smooth measures and nonzero Gaussian curvature. This allows $S_{x,t}$ to have both positive and negative principal curvatures.

5. Variable coefficient wave equations. Let M be a 4-dimensional manifold with real analytic Lorentzian metric $ds^2 = \sum g_{ij}(x)\, dx_i dx_j$. Riesz [6] studied the "wave operator" given by the Laplace-Beltrami operator of this metric. That is,

$$\Box u = \operatorname{div}(\operatorname{grad} u) = \sum_{j,k=1}^{4} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} \left(\sqrt{g}\, g^{jk} \frac{\partial u}{\partial x_k} \right).$$

Let $S \subset M$ be a noncharacteristic, oriented hypersurface. If u solves the Cauchy problem

1) $\Box u = 0$

2) $u|_S = 0, \quad \partial u/\partial n|_S = f,$

then Riesz showed that $u(x) = \int_{S^x} f(y)\, V^2(x,y)\, ds(y)$, where $s = s(x,y) =$ geodesic distance between x and y, $V^\alpha(x,y) = \Gamma((\alpha/2)-1)^{-1} s_+^{\alpha-4} G_\alpha(x,y)$, G_α real analytic in x, y and

$$\begin{aligned} S^x &= \{\text{points on } S \text{ at distance} \geq 0 \text{ from } x\} \\ &= \{y \in S : s(x,y) \geq 0\}. \end{aligned}$$

Let $\gamma : S \times \mathbf{R} \to M$ be smooth, $\gamma(x,0) = x$, $(d\gamma/d\gamma)(x,0)$ transverse to S and noncharacteristic. Then, for x restricted to a compact $K \subseteq S$ and t sufficiently small, $\partial S^{\gamma(x,t)}$ is a smooth compact manifold with nonzero Gaussian curvature bounding a domain about x and hence transverse to the standard dilations (when expressed in local coordinates in S). By the same distribution theory as before, we see that $V^\alpha(\gamma(x,t),\cdot) \to$ a smooth measure on $\partial S^{\gamma(x,t)}$ as $\alpha \to 2$, since s_+^2 is the restriction to $S^{\gamma(x,t)}$ of a defining function for $\partial S^{\gamma(x,t)}$. Thus, if we write $\partial S^{\gamma(x,t)} = \{x + tr(x,t,y)y : y \in S^2\}$, then

$$u(\gamma(x,t)) = s(x,\gamma(x,t)) \int_{S^2} f(x + tr(x,t,y)y)\psi(x,t,y)\,d\sigma(y)$$

where r and ψ are smooth on $K \times [0,t_0] \times S^2$. Hence, by the remark after Theorem 3,

$$\left\| \sup_{0<t<t_0} \frac{u(\gamma(x,t))}{s(x,\gamma(x,t))} \right\|_{L^p(K)} \leq C_p \|f\|_{L^p}, \quad p > \frac{3}{2}.$$

A limiting argument gives:

Theorem 4. *If $f \in L^p_{\mathrm{loc}}(S)$, $p > 3/2$, and u solves the Cauchy problem* $\Box u = 0$, $u|_S = 0$, $\partial u/\partial n|_S = f$, then

$$\frac{u(\gamma(x,t))}{s(x,\gamma(x,t))} \to c(x)f(x) \quad \text{a.e.,} \qquad c \in C^\infty(s).$$

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