CSN 102: DATA STRUCTURES

Space and Time Complexity

Program Performance

- To compare different solution for the same functionality
- Performance is measured by the amount of space and time needed to run the program.
- Two ways to determine:
  - Analytically: Analysing the program without executing it
  - Experimentally: measure performance by executing program for multiple set of inputs

Criteria for Measurement

- Space
  - Amount of memory program occupies
  - Usually measured in Bytes, KB or MB
  - Instruction space, data space and stack space are parts of total space

- Time
  - Execution time of the program
  - Compilation time not included as it's a one time process
  - Usually measured in number of executions of various statements in the program

Analytical Measurement

- Measure the performance of program without execution
- Performance is measured in terms of the size of the input
- Upper and lower bounds of the program are determined
- Results are not actual
- Commonly used for comparison between algorithms

Experimental Measurements

- Run multiple instances of algorithm for various input sets in a controlled environment
- Underlying software and hardware impacts the results
- Results are actual for the given environment and constraints

Space Complexity

- Space complexity is defined as the amount of memory a program needs to complete execution.

- Why is this of concern?
  - We could be running on a multi-user system where programs are allocated a specific amount of space.
  - We may not have sufficient memory on our computer.
  - There may be multiple solutions, each having different space requirements.
  - The space complexity may define an upper bound on the data that the program might have to handle.

Components of Program Space

- Program Space = Instruction Space + Data Space + Stack Space

- Instruction Space
  - Total memory occupied by instructions
  - Dependent on compiler, compilation options and target system

```
printHello( int n)
{         printf("Hello World!");       }
```

- Constant space is required (not dependent on input size)

Components of Program Space

- Data Space:
  - Compile time allocated memory + dynamically allocated memory for data
  - Dependent on computer architecture and compiler

int arr[10][20]; ⇒ 200*sizeof(int) compile time

Eptr nNode = (Eptr)malloc(100*sizeof(Estruct));

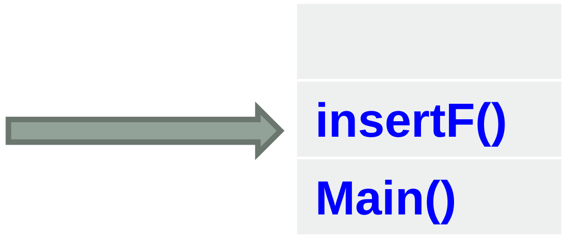
⇓

100*sizeof(Estruct) run time

Components of Program Space

- Stack Space
  - Every time a function is called, following data (activation record) of calling function is stored on stack
    - Return address
    - Values of local variables and formal parameters
    - Binding of reference parameters

```
Eptr insertF (Eptr start, int nData) {
        return newNode(nData, start);
}
```



Program Stack

Components of Program Space

- Identify instance characteristics (factors that determine the size of the problem instance e.g. the number of inputs, outputs, magnitude of numbers involved etc.)

- Total space needed by a program:
  - **Fixed part** – independent of instance characteristics [instruction space, space for simple variables, constants etc.]
  - **Variable part** – dynamically allocated space, recursion stack space[depends on space needed by local variables and formal parameters, maximum depth of recursion]

- Therefore, total space = C + S_p(instance characteristics)

Fixed Part **Variable Part**

Space Complexity (Examples)

```
int abc(int a, int b, int c) {
        return a + b * c;
}
```

Space required by variable part: $S_{abc}(ic) = 0$

```
int sum(int a[], int n) {
// return the sum of numbers a[0:n-1]
        int theSum = 0;
        for (int i=0; i<n; i++)
                theSum+=a[i];
        return theSum
}
```

Space required by variable part: $S_{sum}(n) = 0$

Space Complexity(Examples)

```
int rSum(int a[], int n) {
// return sum of numbers a[0:n-1] using recursion
        if (n>0)
                return (rSum(a, n-1) + a[n-1]);
        return 0;
}
```

$S_{rSum}(n) = 2n + 2n + 2n = 6n$

Reference of a

Value of n

Return address (assuming address of 2 Bytes)

Space Complexity(Examples)

```
int factorial( int n) {
// return n!
        if (n <= 1)
                return 1;
        else
                return (n*factorial(n-1));
}
```

$S_{factorial}(n) = 2*max\{n-1,1\} +2*max\{n-1,1\} = 4*max\{n-1,1\}$

Value of n

Return address (assuming address of 2 Bytes)

Time Complexity

- Time complexity is the total computer time a program needs to execute.

- Why is this a concern?
  - Some computers require upper limits for program execution times.
  - Some programs require a real-time response.
  - If there are many solutions to a problem, typically we'd like to choose the quickest.

Time Complexity

- How do we measure?
  1. Count a particular operation (operation counts)
  2. Count the number of steps (step counts)

Running Example: Insertion Sort

```
for (int i = 1; i < n; i++)                // n is the number of
{                                          // elements in array
  //insert a[i] into a[0:i-1]
  int t = a[i];
  int j;
  for (j = i - 1; j >= 0 && t < a[j]; j--)
          a[j + 1] = a[j];
  a[j + 1] = t;
}
```

Operation Count Method

- Pick an instance characteristic … n,
  n = the number of elements in case of insertion sort.

- Determine count as a function of this instance characteristic.

- Focus only on key operations and ignore all others

- Worst case count = maximum count
- Best case count = minimum count
- Average count

Operation Count : Comparison

```
for (int i = 1; i < n; i++)
{
  // insert a[i] into a[0:i-1]
  int t = a[i];
  int j;
  for (j = i - 1; j >= 0 && t < a[j]; j--)
          a[j + 1] = a[j];
  a[j + 1] = t;
}
```

Operation Count : Comparison

```
for (int i = 1; i < n; i++)
  for (j = i - 1; j >= 0 && t < a[j]; j--)
          a[j + 1] = a[j];
```

- How many comparisons are made?
- The number of compares depends on **a[ ]** and **t** as well as on **n**.

Worst Case Operation Count

- Only inner Loop

```
for (j = i - 1; j >= 0 && t < a[j]; j--)
        a[j + 1] = a[j];
```

a = [1,2,3,4] and t = 0 $\Rightarrow$ 4 compares

a = [1,2,3,4,…,i] and t = 0 $\Rightarrow$ i compares

Worst Case Operation Count

- Both the loops

```
for (int i = 1; i < n; i++)
  for (j = i - 1; j >= 0 && t < a[j]; j--)
          a[j + 1] = a[j];
```

total compares = 1+2+3+…+(n-1)

= (n-1)n/2

Step Count Method

- The operation-count method omits accounting for the time spent on all but the chosen operation
- The step-count method count for all the time spent in all parts of the program
- A program step is loosely defined to be a syntactically or semantically meaningful segment of a program for which the execution time is independent of the instance characteristics.
  - 100 adds, 100 subtracts, 1000 multiples can be counted as one step.
  - However, ***n*** adds cannot be counted as one step.

Step Count : insertion sort

```
                                                steps/execution (s/e)
for (int i = 1; i < n; i++)                             1
{                                                       0
  // insert a[i] into a[0:i-1]                          0
  int t = a[i];                                         1
  int j;                                                1
  for (j = i - 1; j >= 0 && t < a[j]; j--)              1
          a[j + 1] = a[j];                              1
  a[j + 1] = t;                                         1
}                                                       0
```

Step Count : Insertion Sort

	s/e	frequency
for (int i = 1; i < n; i++)	**1**	**n-1**
{	**0**	**0**
// insert a[i] into a[0:i-1]	**0**	**0**
int t = a[i];	**1**	**n-1**
int j;	**1**	**n-1**
for (j = i - 1; j >= 0 && t < a[j]; j--)	**1**	**(n-1)n/2**
a[j + 1] = a[j];	**1**	**(n-1)n/2**
a[j + 1] = t;	**1**	**n-1**
}	**0**	**0**

Step Count : Insertion Sort

Total step counts

$= (n-1) + 0 + 0 + (n-1) + (n-1) + (n-1)n/2 + (n-1)n/2 + (n-1) + 0$

$= n^2 + 3n – 4$

Operation and Step Count

- Two important reasons to determine operation and step counts
  1. To compare the time complexities of two programs that compute the same function
  2. To predict the **growth** in run time as the instance characteristic changes

Examples(1)

```
Main() {
        int n;
        scanf(“%d”,&n);
        printf(“\n Value of n is: %d”,n);
}
```

Examples(1)

Code	s/e	Frequency
Main() {		
int n;	1	1
scanf(“%d”,&n);	1	1
printf(“\n Value of n is: %d”,n);	1	1
}		

Time Complexity = 1*1 + 1*1 + 1*1

= 3

= **O(1)**

=>constant(independent of input size)

Examples(2)

```
Main() { //omitting print and scan statements
        int i, n, x=0;
        for (i=0; i<n; i++) {
                x=x+1;
        }
}
```

Examples(2)

Main() { //omitting trivial statements	s/e	Frequency
int i, n, x=0;	1	1
for (i=0; i<n; i++) {	1	n+1
x=x+1;	1	n
}		
}		

Time Complexity = 1*1 + 1*(n+1) + 1*n

= 2*n + 2

= **O(n)**

=>Linear(grows linearly with input size)

Examples(3)

```
Main() {
        int i, j, x=0;
        for (i=0; i<n; i++) {
                x=x+1; }
        for (i=0; i<n; i++) {
                for (j=0; j<n; j++) {
                        x=x+1; } }
}
```

Examples(3)

Code	s/e	Frequency
`Main() {`		
`int i, n, x=0;`	1	1
`for (i=0; i<n; i++) {`	1	n+1
`x=x+1; }`	1	n
`for (i=0; i<n; i++) {`	1	n+1
`for (j=0; j<n; j++) {`	1	(n+1)(n+1)
`x=x+1; } }`	1	n*n
`}`		

Time Complexity $= 2n^2 + 5n + 4$

$= \mathbf{O(n^2)}$

=>quadratic

Examples(4)

```
Main() {
        int n;
        while ( n > 1) {
                n = n/2;
        }
}
```

Examples(4)

```
Main() {
        int n;
        while (n > 1) {
                n = n/2;
        }
}
```

	if n=2	if n=4	if n=8
After 1st iteration	n=1	n=2	n=4
After 2nd iteration		n=1	n=2
After 3rd iteration			n=1
Frequency	1	2	3

Examples(4)

```
Main() {
        int n;
        while (n > 1) {
                n = n/2;
        }
}
```

	if n=2	if n=4	if n=8
After 1st iteration	n=1	n=2	n=4
After 2nd iteration		n=1	n=2
After 3rd iteration			n=1
Frequency	1	2	3

Time Complexity = **$O(\log_2 n)$** (grows logarithmically to input size)

Exercise

```
Main() {
    int i, j, k, n;
    for (i=0; i<n; i++) {
        for (j=0; j<i; j++) {
            if (j%n == 0) {
                for (k=0; k<n; k++) {
                    x=x+1;
                }
            }
        }
    }
}
```

Exercise

	s/e	frequency
`Main() {`		
`    int i, j, k, n, x=0;`	1	1
`    for (i=1; i<n; i++) {`	1	n
`        for (j=1; j<i; j++) {`	1	n*(n+1)/2
`            if (j%n == 0) {`	1	n*(n+1)/2
`                for (k=1; k<n; k++) {`	1	n
`                    x=x+1;`	1	n-1
`                }`		
`            }`		
`        }`		
`    }`		
`}`		

Time Complexity = **O(n^2)**

Asymptotic Notation

- Describes the behavior of the time or space complexity for large instance characteristic
- **Big O (O)** notation provides an **upper bound** for the function *f*
- **Big Omega (Ω)** notation provides a **lower-bound**
- **Theta (Θ)** notation is used when an algorithm can be **bounded both from above and below** by the same function
- **Little O (o)** defines a **loose upper bound**.

Upper Bounds

- Time complexity $T(n)$ is a function of the problem size n. The value of $T(n)$ is the running time of the algorithm in the **worst case**, i.e., the number of steps it requires ***at most*** with an arbitrary input.
- Example: the sorting algorithm Insertion Sort has a time complexity of $T(n) = n \cdot (n-1)/2$ comparison-exchange steps to sort a sequence of n data elements.
- Often, it is not necessary to know the exact value of $T(n)$, but only an **upper bound** as an estimate.
- e.g., an upper bound for time complexity $T(n)$ of insertion sort is the function $f(n) = n^2$, since $T(n) \leq f(n)$ for all n.

Big O (O) Notation

- In general, just the order of the asymptotic complexity is of interest, i.e., if it is a linear, quadratic, exponential function.

- Definition: $\boldsymbol{f(n) = O(g(n))}$ (read as "$f(n)$ is Big O of $g(n)$") iff positive constants c and n_0 exist such that $f(n) \leq cg(n)$ for all n, $n \geq n_0$.

- That is, $O(g(n))$ comprises all functions f, for which there exists a constant c and a number n_0, such that $f(n)$ is smaller or equal to $c \cdot g(n)$ for all n, $n \geq n_0$.

Big O Examples

- Example: the time complexity $T(n)$ of Insertion Sort lies in the complexity class $O(n^2)$.
- $O(n^2)$ is the complexity class of all functions that grow at most quadratically. Respectively, $O(n)$ is the set of all functions that grow at most linearly, $O(1)$ is the set of all functions that are bounded from above by a constant, $O(n^k)$ is the set of all functions that grow polynomially, etc.

Lower Bounds

- Once an algorithm for solving a specific problem is found, the question arises whether it is possible to design a faster algorithm or not.
- How can we know unless we have found such an algorithm? In most cases a lower bound for the problem can be given, i.e., **a certain number of steps that *every* algorithm has to execute *at least*** in order to solve the problem.
- e.g., In order to sort n numbers, every algorithm at least has to take a look at every number. So, it needs at least n steps. Thus, $f(n) = n$ is a lower bound for sorting

Omega (Ω) Notation

- Again, only the order of the lower bound is considered, namely if it is a linear, quadratic, exponential or some other function. This order is given by a function class using the Omega **(Ω)** notation.
- Definition: $\boldsymbol{f(n) = \Omega(g(n))}$ (read as "$f(n)$ is omega of $g(n)$") iff positive constants c and n_0 exist such that $f(n) \geq cg(n)$ for all n, $n \geq n_0$.
- That is, $\Omega(g)$ comprises all functions f, for which there exists a constant c and a number n_0, such that $f(n)$ is greater or equal to $c \cdot g(n)$ for all $n \geq n_0$.

Omega Examples

- Let $f(n) = 2n^2 + 7n - 10$ and $g(n) = n^2$. Since with $c = 1$ and for $n \geq n_0 = 2$ we have $2n^2 + 7n - 10 \geq c \cdot n^2$, thus $f(n) = \Omega(g)$.
- This example function $f(n)$ lies in $\Omega(n^2)$ as well as in $O(n^2)$, i.e., it grows at least quadratically, but also at most quadratically.
- In order to express the exact order of a function the class $\Theta(f)$ is introduced (Θ is the greek letter theta).

Theta (Θ) Notation

- Used when the function f can be **bounded both from above and below** by the same function g.

- Definition: $\boldsymbol{f(n) = \Theta(g(n))}$ (read as "$f(n)$ is theta of $g(n)$") iff positive constants c_1, c_2 and n_0 exist such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for all n, $n \ge n_0$.

- That is, f lies between c_1 times the function g and c_2 times the function g, except possibly when n is smaller than n_0.

Theta Examples

- As seen above the lower bound for the sorting problem lies in $\Omega(n)$. Its upper bound lies in $O(n^2)$, e.g., using Insertion Sort. Thus, time complexity for insertion sort in not $\Theta(n)$

- e.g., with Heapsort, the upper bound is $O(n.\log(n))$ and the lower bound can also be improved to $\Omega(n\cdot\log(n))$.

- Thus, Heapsort is an optimal sorting algorithm, since its upper bound matches the lower bound for the sorting problem and time complexity is $\Theta(n.\log(n))$

Little O (o) Notation

- Definition: ***f(n)=o(g(n))*** (read as "*f(n)* is little o of *g(n)*") iff *f(n)* = *O(g(n))* and *f(n)* ≠ Ω*(g(n))*.
- That is, *f* has a lower growth rate than *g*.
- Little oh is also called a loose upper bound.
- Hierarchy of growth rate functions:

  $1 < \log n < n < n\log n < n^2 < n^3 < 2^n < 3^n < n! < n^n$

Common Growth Rate Functions

- 1 (constant): growth is independent of the problem size n.
- $\log_2 n$ (logarithmic): growth increases slowly compared to the problem size (binary search)
- n (linear): directly proportional to the size of the problem.
- $n * \log_2 n$ (*n* log *n*): typical of some divide and conquer approaches (merge sort)
- n^2 (quadratic): typical in nested loops
- n^3 (cubic): more nested loops
- 2^n (exponential): growth is extremely rapid and possibly impractical.
- n! (factorial)

Practical Complexities

logn	n	nlogn	n^2	n^3	2^n
0	1	0	1	1	2
1	2	2	4	8	4
2	4	8	16	64	16
3	8	24	64	512	256
4	16	64	256	4096	65536
5	32	160	1024	32768	4294967296

Overly complex programs may not be practical given the computing power of the system.

Recurrence Relation

- A Recurrence Relation is an equation which is defined in terms of itself
- Eg. $A_n = A_{n-1} + 1$, $A_1 = 1$

  $A_n = 2*A_{n-1}$, $A_1 = 1$

  $A_n = n*A_{n-1}$, $A_1 = 1$

  Boundary Condition

- Recursive function can be expressed as recurrence relation to find total time required in the program
- Time required to execute recursive problem is dependent on time required by its sub-problem

Recurrence Relation: Sum of Array

```
int rSum(int a[], int size) {
        if (size > 0)
                return(rSum(a, size-1) + a[size-1]);
        else
                return 0;
}
```

$$\mathbf{rSum(n)} = \begin{cases} \mathbf{rSum(n-1)} + \mathbf{C_1}, & \mathbf{n>0} \\ \mathbf{C_2}, & \mathbf{n=0} \end{cases}$$

Recurrence Relation: Binary Search

```
int BS(int a[], int front, int rear, int key) {
        if (front == rear)
                if (a[front] == key)
                        return front;
                else
                        return -1;
        mid = (front + rear) / 2;
        if (a[mid] > key)
                return BS(a, front, mid, key);
        else
                return BS(a, mid+1, rear, key);
}
```

$$\mathbf{BS(n)} = \begin{cases} \mathbf{BS(n/2) + C_1}, & \mathbf{n>1} \\ \mathbf{C_2}, & \mathbf{n=1} \end{cases}$$

Recurrence Relation: Factorial

```
int factorial( int n) {
// return n!
        if (n <= 1)
                return 1;
        else
                return (n*factorial(n-1));
}
```

Recurrence Relation: Factorial

```
int fact( int n) {
// return n!
        if (n <= 1)
                return 1;
        else
                return (n*factorial(n-1));
}
```

$$\mathbf{fact(n)} = \mathbf{fact(n-1)} + \mathbf{C_1}, \quad \mathbf{n>1}$$

$$= \mathbf{C_2}, \quad \mathbf{n<=1}$$

Solve Recurrence Relation

- Three methods to solve
  - Substitution
  - Recursive Tree Method
  - Master Theorem

Solve Recurrence Relation

- Three methods to solve
  - **Substitution**
  - Recursive Tree Method
  - Master Theorem

Substitution

- Substitute the given function again and again until a pattern is visible
- Use Boundary/Terminating condition to end substitution
- Find the sum of the terms

Substitution Example(1)

$$\mathbf{rSum(n) = rSum(n-1) + C_1, \qquad n>0}$$
$$\mathbf{= C_2, \qquad n=0}$$

$rSum(n) = rSum(n-1) + C_1$

Substitution Example(1)

$$\mathbf{rSum(n) = rSum(n\text{-}1) + C_1, \quad n>0}$$
$$\mathbf{= C_2, \quad n=0}$$

$$rSum(n) = rSum(n-1) + C_1$$
$$= (rSum(n-2) + C_1) + C_1$$

Substitution Example(1)

$$\mathbf{rSum(n) = rSum(n\text{-}1) + C_1, \quad n>0}$$
$$\mathbf{= C_2, \quad n=0}$$

$$\begin{aligned} rSum(n) &= rSum(n-1) + C_1 \\ &= (rSum(n-2) + C_1) + C_1 \\ &= rSum(n-2) + 2*C_1 \end{aligned}$$

Substitution Example(1)

$$\mathbf{rSum(n) = rSum(n\text{-}1) + C_1, \quad n>0}$$
$$\mathbf{= C_2, \quad n=0}$$

$$\begin{aligned} rSum(n) &= rSum(n-1) + C_1 \\ &= (rSum(n-2) + C_1) + C_1 \\ &= rSum(n-2) + 2*C_1 \\ &= rSum(n-3) + 3*C_1 \end{aligned}$$

Substitution Example(1)

$$\mathbf{rSum(n)} = \mathbf{rSum(n-1)} + \mathbf{C_1}, \qquad \mathbf{n>0}$$
$$= \mathbf{C_2}, \qquad \mathbf{n=0}$$

$rSum(n) = rSum(n-1) + C_1$

$= (rSum(n-2) + C_1) + C_1$

$= rSum(n-2) + 2*C_1$

$= rSum(n-3) + 3*C_1$

Substituting **n** times

$= rSum(0) + n*C1$

Substitution Example(1)

$rSum(n) = rSum(n-1) + C_1, \quad n>0$

$ = C_2, \quad n=0$

$rSum(n) = rSum(n-1) + C_1$

$= (rSum(n-2) + C_1) + C_1$

$= rSum(n-2) + 2*C_1$

$= rSum(n-3) + 3*C_1$

Substituting **n** times

$= rSum(0) + n*C1$

$= C2 + n*C1$

$= O(n)$

Substitution Example(2)

$$\mathbf{BS(n) = BS(n/2) + C_1, \quad n>1}$$
$$\mathbf{= C_2, \quad n=1}$$

$BS(n) = BS(n/2) + C_1$

Substitution Example(2)

$$\mathbf{BS(n) = BS(n/2) + C_1, \quad n>1}$$
$$\mathbf{ = C_2, \quad n=1}$$

$$\begin{aligned} BS(n) &= BS(n/2) + C_1 \\ &= (BS(n/2^2) + C_1) + C_1 \\ &= BS(n/2^2) + 2*C_1 \end{aligned}$$

Substitution Example(2)

$BS(n) = BS(n/2) + C_1$, $\quad$ $n>1$

$\quad\quad\;\; = C_2$, $\quad$ $n=1$

$$
\begin{aligned}
BS(n) &= BS(n/2) + C_1 \\
&= (BS(n/2^2) + C_1) + C_1 \\
&= BS(n/2^2) + 2*C_1 \\
&= BS(n/2^3) + 3*C_1
\end{aligned}
$$

Substitution Example(2)

$$BS(n) = BS(n/2) + C_1, \quad n>1$$
$$= C_2, \quad n=1$$

$BS(n) = BS(n/2) + C_1$

$= (BS(n/2^2) + C_1) + C_1$

$= BS(n/2^2) + 2*C_1$

$= BS(n/2^3) + 3*C_1$

Substituting **$\log_2 n$** times

$= BS(1) + (\log_2 n)*C1$

Substitution Example(2)

$$\mathbf{BS(n) = BS(n/2) + C_1, \quad n>1}$$
$$\mathbf{= C_2, \quad n=1}$$

$BS(n) = BS(n/2) + C_1$

$= (BS(n/2^2) + C_1) + C_1$

$= BS(n/2^2) + 2*C_1$

$= BS(n/2^3) + 3*C_1$

Substituting $\mathbf{log_2\ n}$ times

$= BS(1) + (log_2 n)*C1$

$= C2 + (log_2 n)* C1$

$= O(log_2 n)$

Substitution Example(3)

$$T(n) = \begin{cases} T(n-1) + n, & n>1 \\ 1, & n=1 \end{cases}$$

$T(n) = T(n-1) + n$

Substitution Example(3)

$$T(n) = \begin{cases} T(n-1) + n, & n>1 \\ 1, & n=1 \end{cases}$$

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Substituting **n-1** times

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Substituting **n-1** times
= T(1) + n + (n-1) + (n-2) + … + (n-(n-k)) + 3 + 2
= 1 + 2 + 3 + … + (n-2) + (n-1) + n
= n*(n+1)/2 = $(n^2 + 2*n + 1)/2$
= $O(n^2)$

Exercise

- Find Time Complexity of below recurrence relation:

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T.C. of T(n) = **O(n)**

Solve Recurrence Relation

- Three methods to solve
  - Substitution
  - Recursive Tree Method
  - **Master Theorem**

Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function and let $T(n)$ be defined on the non-negative integers by the recurrence

$$T(n) = aT(n/b) + f(n)$$

Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) = O(n^{log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $\mathrm{T(n)} = O\left(n^{log_b a - \epsilon}\right)$
2. If $f(n) = \Theta(n^{log_b a})$, then $T(n) = \Theta(n^{log_b a}\ lg\ n)$
3. If $f(n) = \Omega(n^{log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$, and all sufficiently large n, then $\mathrm{T(n)} = \Theta(n^{log_b a + \epsilon})$