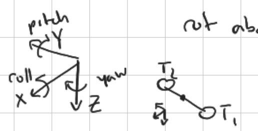
MAE144 HW2 Brian Kong A16996735

from above:

	CW	CCW
T_1	BWD	FWD
T_2	FWD	BWD

Q1)



rot about: $x \rightarrow \gamma \rightarrow I_1$
$y \rightarrow \theta \rightarrow I_2$
$z \rightarrow \alpha \rightarrow I_3$

we want control of $\alpha(t)$.

$\frac{d\alpha}{dt} > 0$ for T_2 moving back & T_1 moving fwd

$u_{yaw} = \frac{1}{2}(u_{T_1} + u_{T_2})$, $u_{yaw} > 0$ for when both T_1 & T_2 are CCW.

wheels stay in contact for $\alpha(t) \in [-10°, 10°]$

$\therefore$ motion: $I_3 \frac{d^2\alpha}{dt^2} \approx u_{yaw}$, $I_3 \approx \frac{1}{12} m w^2$, m: mass of whole bot, w: dist. btwn walls.

a) $U_{yaw} \rightarrow \boxed{G(s)} \rightarrow \alpha(s)$

$I_3 \frac{d^2\alpha}{dt^2} = u_{yaw} \xrightarrow{\mathcal{L}} I_3\left[s^2\alpha(s) - s\,\alpha(0) - \frac{d\alpha}{dt}(0)\right] = U_{yaw}(s)$

$s^2\alpha(s) = I_3^{-1} U_{yaw}(s) + \cancel{\alpha(0)s + \frac{d\alpha}{dt}(0)}$ → IC's = 0

$G(s) = \frac{\alpha(s)}{U_{yaw}(s)} = \frac{1}{I_3 s^2}$

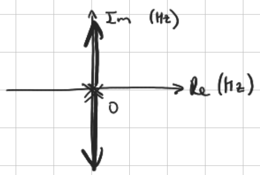
assuming sys starts at zero rotation and zero rot. velocity.

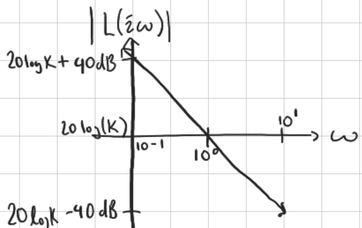
b) assuming $D(s) = K$, $T(s) = \frac{GD}{1+GD} = \frac{L(s)}{1+L(s)}$, $L(s) = G(s)D(s) = \frac{K}{I_3 s^2}$

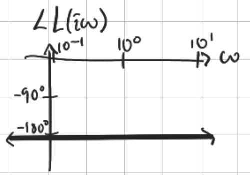
$n = 2$ poles, $m = 0$ zeros $\rightarrow l = 1, 2$

the two poles at $s=0$ approach asymptote at $s = \frac{\sum p_i - \sum z_i}{2} = 0$

departing angles: $\theta_1 = \frac{180° + 0}{2} = 90°$
$\theta_2 = \frac{180 + 360}{2} = -90°$
for $\underline{K > 0}$







$|L(i\omega)| \Rightarrow 20\log|L(i\omega)| = 20\log(K) - 20(2)\log(i\omega)$

↓ slope = -40 dB/decade

$|L(i\omega)|_{\omega=1} = 20\log(K) - 0$

$\angle L(i\omega) = \angle K - 2\angle i\omega = 0 - 2(90°) = -180°$

The resulting closed loop system is on the boundary, with all the possible CL systems for $K>0$ being marginally stable / on the boundary as shown by the RL plot.

the bode plot shows that it can be unstable at a ω s.t. $\begin{cases} |L(j\omega)| = 0\,dB \\ \angle L(j\omega) = 180° \end{cases}$

c) for an undetermined $D(s)$ w/ $t_r = 0.18$ sec, $M_p = 10\%$, I can find good pole locations

by using design guidelines such as: $\omega_n \geq 1.8/t_r$, $\zeta = \frac{PM}{100}$ $\begin{cases} \zeta \geq 0.5 \text{ for } M_p \leq 15\% \\ \zeta \geq 0.7 \text{ for } M_p \leq 5\% \end{cases}$

we interpolate for $M_p = 10\%$ to get a design guide of $\zeta = 0.6$, and $\omega_n = 1.8/0.18 = 10$ Hz.

This is for 2nd order systems, which matches our $G(s)$.

these correspond with poles for the char. eq. of $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$\rightarrow s_1, s_2 = -\omega_n\zeta \pm \omega_n\sqrt{\zeta^2 - 1} = -10\left(\frac{3}{5}\right) \pm 10\sqrt{\frac{9}{25} - 1}$

$= -6 \pm 10(4)i = -6 \pm 40i$

d) $D(s) = K\frac{s+z}{s+p}$, $z = 1$, $p = 10$. $L(s) = D(s)G(s) = \frac{s+z}{I_3 s^2 (s+p)}$

$n = 3$ poles, $m = 1$ zero, $l = 1, 2$

RL: asymptote centered at $\alpha = \frac{\sum p_i - \sum z_i}{n-m} = \frac{-10+1}{3-1} = -9/2$

$\theta_{l=1} = (180° + 0)/2 = 90°$

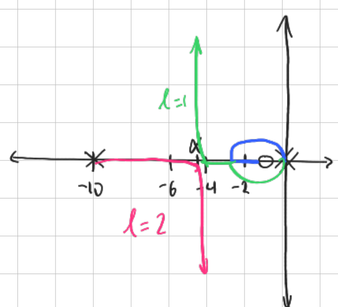
$\theta_2 = (180° + 360°)/2 = -90°$

departure from poles: $\psi^{l,l}_{dep} = \left[\sum \varphi_i - \sum_{i \neq l} \psi_i + 180° + (l-1)180°\right]/q_l$

$\psi^{1,1} = [180 - 180 + 180 + 0]/2 = 90°$

$\psi^{2,2} = [0 - 0 + 180 + 180]/1 = 0°$

arrival to $z=1$: $\varphi = \frac{0 - (0 + 180) + 0(360°)}{1} = 180°$



from this we can see the system being stable for $K > 0$.
$\because$ no p&z in RHP for $K > 0$.

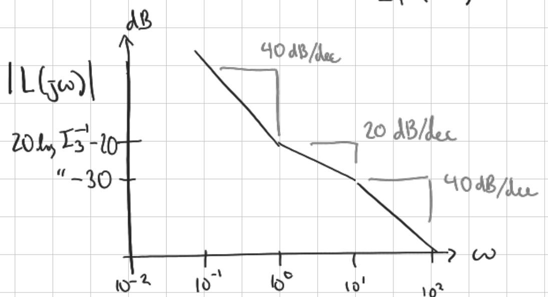
Bode: $L(s) = \dfrac{s+z}{I_3 s^2 (s+p)} = \dfrac{1}{I_3} \dfrac{s+1}{s^2(s+10)} = \dfrac{1}{I_3 \cdot 10} \dfrac{(1+s)}{s^2(1+s/10)}$

$20\log_{10}|L(j\omega)| = 20\log_{10}\left(\frac{1}{10 I_3}\right) - 20(2)\log_{10}(j\omega) - 20\log_{10}(1+j\omega/10) \quad + 20\log_{10}(1+j\omega)$

$= 20\log_{10}(I_3^{-1}) - 20 \quad - 40\log_{10}|\omega| \quad - \begin{cases} 0 & \text{for } |\omega| < 10 \\ 20\log_{10}\left|\frac{j\omega}{10}\right| & \text{for } |\omega| > 10 \end{cases} + \begin{cases} 0 & \text{for } |\omega| < 1 \\ 20\log_{10}(j\omega) & \text{for } |\omega| > 1 \end{cases}$

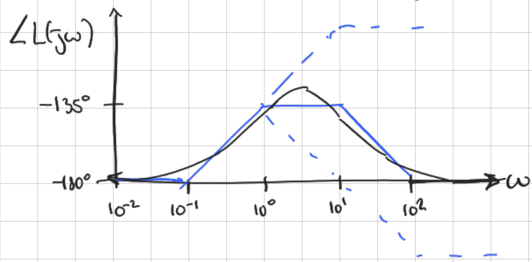
$\angle L(j\omega) = \angle\left(\frac{1}{10 I_3}\right) \quad - 2\angle j\omega \quad - \angle(1 + \tfrac{1}{10}j\omega) + \angle(1+j\omega)$

$= 0 \quad - 2(90°) \quad - \begin{cases} 0 & \text{for } |\omega| < \frac{1}{10}(1/10)^{-1} = 10^0 \\ 90° & \text{for } |\omega| > 10(1/10)^{-1} = 10^2 \\ 45° \log(\frac{10}{10}\omega) & \text{for } |\omega| \in [10^0, 10^2] \end{cases} + \begin{cases} 0 & \text{for } |\omega| < \frac{1}{10}(1) = 10^{-1} \\ 90° & \text{for } |\omega| > 10(1) = 10^1 \\ 45° \log(10\omega) & \text{for } |\omega| \in [10^{-1}, 10^1] \end{cases}$



might've not stated it before, but we look at $L(s=j\omega_r) = -1$ for resonant freq. $\omega_r \rightarrow |L(j\omega_r)| = 1$
$\angle L(j\omega_r) = \pm 180°$

our $\angle L(j\omega)$ doesn't intersect $\pm 180°$ for an undefined gain margin, so we can conclude our system is stable for any real values of K



e) with $M_p = 10\%$, $t_r = 0.18 \rightarrow \zeta = 0.6$, $\omega_n = 10$ Hz. from 1c),

we use $\zeta = PM/100$ to get $PM = 60$, and $\omega_g = \omega_n = 10$ Hz.

we then tune the phase such that the $PM = 60°$ (from $-180°$) by shifting the pole $p=10$ further into the LHP and/or the zero $z=1$ closer to zero to allow the phase gain from our zero to reach a higher value $(-180° + PM)$ at the crossover freq. ω_g

before being negated by the additional pole over the freq. range.

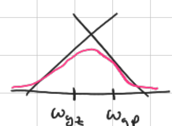
after we tuned our p and z, we tune our K to achieve $M_p = 10\%$
which is done by adjusting our K to scale our controller such that $L(j\omega_g) = 1.10$

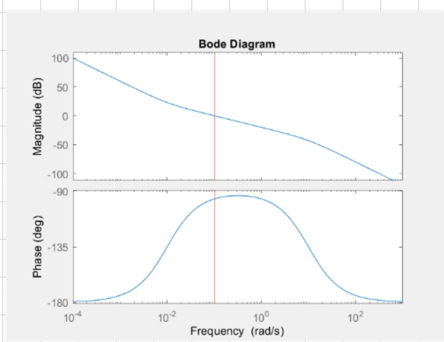
f) to improve upon this controller to better track at $\omega \lesssim 0.1\text{Hz} = 0.2\pi$ rad/s,

we change our current $D(s) = D_1$ by add a lead controller to reduce our phase delay:

$D(s) = D_1(s) \cdot D_{lead}(s)$, where $D_{lead}(s) = \dfrac{s + z_\ell}{s + p_\ell}$, $p_\ell > z_\ell > 0$, and set z_ℓ and p_ℓ

such that the maximum PM occurs at/near $\omega = 0.1\text{Hz} = 10^{-1}$
this is achieved by setting z_ℓ's ω_g a similar distance from ω as btwn ω and p_ℓ's ω_g in the log plot: e.g.
$\omega_{gz} = 10^{-2}, \rightarrow z_\ell = 1/100$
$\omega_{gp} = 10^0 \rightarrow p_\ell = 1$





⟸ did NOT tune my K after; just used it to demonstrate phase.

we still need to keep our new additions in the LHP for stability, and also need to adjust the K after adding our new controller, which added its own gain to the output. otherwise our system could be unstable or have steady state error and such.

might want higher multiplicity for the lead for our $\omega = 10^{-1}$.

g) to better reject disturbances for $\omega \geq 100$ Hz, we can add a low pass filter;

$D(s) = D_1(s) \cdot D_{lead}(s) \cdot D_{LPF}(s)$, and use $\omega_g = 100$ Hz or slightly more to attenuate the higher freq. we don't desire. We can set it slightly higher than 100 Hz because the LPF has slightly noticeable attenuation before the ω_g we desire, if the signals close to 100 Hz matter. otherwise, we might see less gain than desired near 100 Hz.

$$D_{LPF}(s) = \frac{\omega_g}{s + \omega_g}$$

Q2) Obj: convert CT controller to DT.

a) for a given crossover freq. ω_g, we want to have ω_{nyq} an order of magnitude over it;
$\omega_{nyq} = 10\omega_g \rightarrow$ our sampling freq $\omega_s = 2\omega_{nyq} \rightarrow h = \frac{1}{2\omega_{nyq}} = \frac{1}{20\omega_g}$

b) the (ZOH of a DAC)'s delay is $h/2$. find time delay & phase loss

$$\text{delay} = h/2 = \frac{1}{40\omega_g}$$

$$\text{Phase loss} = \underset{\text{full cycle}}{2\pi}\left(\frac{\text{delay}}{\text{wave period}}\right) = 2\pi \frac{h/2}{2\pi/\omega} = h\omega/2 = \frac{\omega}{40\omega_g}$$

the delay is $h/2$ as the DT signal maintains the value from the CT signal for each sampling period, resulting in it holding each value for up to the middle of each sampling interval,

c) the biggest issue w/ the ZOH delay is that at high ω, the DT system isn't stable when the CT can be as a result of its phase delay. we can try mitigating this by adding a greater phase margin in our $D(s)$ to make the PM of the $D(z)$ still be enough to be stable for the desired range of ω after the effects of the ZOH.

d) Using a $D(s)$ w/ the previous steps, we use Tustin's method, using $z \simeq \frac{1+sh/2}{1-sh/2} \rightarrow s = \frac{2}{h}\left(\frac{z-1}{z+1}\right)$

to convert $D(s) \rightarrow D(z)$: $D(z) = D(s)\Big|_{s=\frac{2}{h}\left(\frac{z-1}{z+1}\right)}$,

and convert it into a difference equation in the form of
$y[n] = \sum_{i=0}^{N} b_i u[n-i] - \sum_{j=1}^{N} a_j y[n-j]$, where we determine the coeff b_i and a_j from manipulating $D(z)$ to be in terms of z^{-k} and using coefficients in the $D(z) = \frac{Y(z)}{X(z)}$ form where b are coeff for $Y(z)$ and a are coeff in $X(z)$.

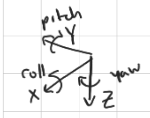
$Y(z)z^{-k} \xrightarrow{k\text{-sample delay}} y[n-k]$. make sure it's causal (no $y[n+k]$)

throw it into code, and do embedded hell, and you'll have it do something lol.
it never works the first time ::

Q3) We have a u_{thrust} ; $u_{thrust}(t) \geq 0$ moves it fwd and tends to increase $\theta(t)$ when $l > 0$

from above:

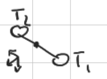
	CW	CCW
T_1	BWD	FWD
T_2	FWD	BWD



rot about: $x \to \gamma \to I_1$ we want control of $\alpha(t)$.

$y \to \theta \to I_2$ $\quad \frac{d\alpha}{dt} > 0$ for T_2 moving back & T_1 moving fwd

$z \to \alpha \to I_3$



$u_{yaw} = \frac{1}{2}(u_{T_1} + u_{T_2})$, $u_{yaw} > 0$ for when both T_1 & T_2 are CCW.

wheels stay in contact for $\alpha(t) \in [-10°, 10°]$

$f_{th} = \frac{u_{th}}{r_{wheel}}$. coupled eq:

"body rotation" dynamics: $(I_p + m_p \ell^2)\frac{d^2\theta}{dt^2} - m_p g \ell\, \theta = m_p \ell \frac{d^2 x}{dt^2}$, (1a)

"translational" dynamics: $(m_c + m_p)\frac{d^2 x}{dt^2} = m_p \ell \frac{d^2\theta}{dt^2} + f_{\text{thrust}}$, (1b)

a) compute consts in $G_1(s) = \frac{\theta(s)}{F_{th}(s)} = \frac{b_0}{s^2 - a_0}$, $G_2(s) = \frac{X(s)}{\theta(s)} = \frac{\bar{b}_2 s^2 - \bar{b}_0}{s^2}$

Assuming IC's $= 0$:

$$(1a): (I_p + m_p\ell^2)s^2\theta(s) - m_p g \ell \theta(s) = m_p \ell s^2 X(s)$$

$$[(I_p + m_p \ell^2)s^2 - m_p g \ell]\theta(s) = [m_p \ell s^2] X(s)$$

$$G_2(s) = \frac{X(s)}{\theta(s)} = \frac{(I_p + m_p\ell^2)s^2 - m_p g \ell}{m_p \ell s^2} = \frac{(I_p/m_p\ell + \ell)s^2 - g}{s^2}$$

$$\bar{b}_2 = (I_p + m_p\ell^2)/m_p\ell = \frac{I_p}{m_p \ell} + \ell$$

$$\bar{b}_0 = m_p g \ell / m_p \ell = g$$

(1b):

$$(m_c + m_p)s^2 X(s) = m_p \ell s^2 \theta(s) + F_{th}(s)$$

$$\left[(m_c + m_p)s^2 \frac{(I_p + m_p\ell^2)s^2 - m_p g \ell}{m_p \ell s^2} - m_p \ell s^2\right]\theta(s) = F_{th}(s)$$

$$\frac{(m_c + m_p)(I_p + m_p \ell^2)s^2 - m_p g \ell - m_p^2 \ell^2}{m_p \ell}\theta(s) = F_{th}(s)$$

$$G_1(s) = \frac{\theta(s)}{F_{th}(s)} = \frac{m_p\ell / [(m_c + m_p)(I_p + m_p\ell^2)]}{s^2 - \frac{m_p \ell (g + m_p \ell)}{(m_c + m_p)(I_p + m_p \ell^2)}} = \frac{b_0}{s^2 - a_0}; \quad b_0 = \frac{m_p \ell}{(m_c + m_p)(I_p + m_p\ell^2)}, \quad a_0 = \frac{m_p \ell (g + m_p \ell)}{(m_c + m_p)(I_p + m_p \ell^2)}$$

b) $D_1(s) = K\frac{s + z}{s + p}$, $p > z$. I state $L(s) = D_1(s)G_1(s)$, and we modify K for our $1 + KL(s) = 0$ to generate the RL plot. to perform pole zero cancellation, I set $z = \gtrsim a_0$ or slightly to the left of the complex pole to cancel one, and place $p > a_0$ such that the asymptote the two approach is in the LHP. This should work if we are certain about the poles for the plant, and if the potential instability is dominated by stronger poles and even $G_2(s)$ later on.

I can fine tune it w/ its phase delay & margin by adding lead/lag compensation to the system while using Bode plot, deciding the t_r for a ω by changing the PM for said ω, and then scale the controller to match a M_p at the frequency. By changing the order of components and their ω_j, I can control their influence on magnitude and phase in their region of influence. Generally I want to avoid having an ω where $|L(j\omega)| = 1$ & $\angle L(j\omega) = \pm 180°$ as this causes instability via resonance.

c) we treat the inner loop $T_1(s) \simeq 1$ at our desired freq, then use $D_2(s)$ to slowly stabilize the outer loop. using $D_2(s) = -K$, $K > 0$ then you do analysis on the RL plot of the idealized outer loop $D(s)T_1(s) \approx 1$.

we notice that for lower freq. and lower values of K, it behaves similarly to the inner loop $T_1(s)$, which is stable

for a better response, we apply it to the tf of $T_1(s) = P\left(\frac{L_1(s)}{1 + L_1(s)}\right)$, where P is the pregain to have $T_1(s) \approx 1$, then manipulate the Bode plot, adjusting poles to reduce our phase delay for desired frequencies, adding a LPF, and a lead component to also reduce the phase delay.