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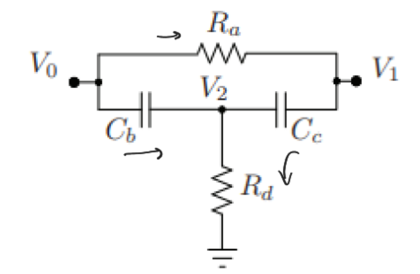
Q1: **1.** Consider first the circuit shown at left in Figure 1, assuming that:
(a) the voltage $V_0(t)$ at the input node is specified precisely, regardless of the current $I_0(t)$ drawn by the circuit, and
(b) the current out the output node (if that node is connected to anything, to the right) is negligible, $I_1(t) = 0$.



1a. Write down the component equations across each of the 4 components, as well as KCL at the 3 "T" junctions (that is, at the left, at the right, and in the middle). Note that we have seven unknowns: $\{V_1, V_2, I_0, I_a, I_b, I_c, I_d\}$.

$$V_0 - V_1 = I_a R_a \quad \cdots(1), \qquad I_0 = I_a + I_b \quad \cdots(5)$$

$$I_a = I_c \quad \cdots(6)$$

$$C_b \frac{d(V_0 - V_2)}{dt} = I_b \quad \cdots(2) \qquad I_d = I_b + I_c = I_0 \text{ (dependent, not needed)} \quad \cdots(7)$$

$$C_c \frac{d(V_1 - V_2)}{dt} = I_c \quad \cdots(3)$$

$$V_2 - 0 = I_d R_d \quad \cdots(4)$$

1b. Take the Laplace transform of the 7 equations in question 1a, assuming zero initial conditions on all variables. Combine the 7 equations in 7 unknowns, by hand, to eliminate $\{V_2, I_0, I_a, I_b, I_c, I_d\}$, thereby developing a single equation for $V_1(s)/V_0(s)$ in terms of the 4 parameters $\{R_a, C_b, C_c, R_d\}$.

$$(2) \xrightarrow{\mathcal{L}} C_b s(V_0 - V_2) = I_b(s)$$

$$(3) \longrightarrow C_c s(V_1 - V_2) = I_c(s)$$

$$V_1 - V_2 = \frac{1}{C_c s} I_c$$

$$I_0 = I_a + I_b = I_b + I_c \overset{(2)}{=} C_b s(V_0 - V_2) + \overset{(3)}{C_c s(V_1 - V_2)}$$

$$(7) \quad I_d = C_b s V_0 + C_c s V_1 - (C_b + C_c) s V_2$$

$$\left(\frac{1}{R_d} + (C_b + C_c)s\right) V_2(s) = (C_b V_0 + C_c V_1)s$$

$$(7) \quad I_d = \frac{V_2}{R_d} = \frac{V_0 - V_1}{R_a} + C_b s(V_0 - V_2)$$

$$\left(\frac{1}{R_d} + C_b s\right) V_2 = \frac{V_0 - V_1}{R_a} + C_b V_0 s$$

$$\left(\frac{1}{R_d} + (C_b + C_c)s\right)\left(\frac{\frac{V_0 - V_1}{R_a} + C_b V_0 s}{1/R_d + C_b s}\right) = (C_b V_0 + C_c V_1)s$$

$$\left(\frac{R_d}{R_d}\right)\frac{1/R_d + (C_b + C_c)s}{1/R_d + C_b s}\left((1/R_a + C_b s)V_0 - \frac{1}{R_a} V_1\right) = C_b s V_0 + C_c s V_1$$

$$\frac{1 + (C_b + C_c)R_d s}{1 + C_b R_d s} \cdot \left((1 + C_b R_a s)V_0 - V_1\right) = C_b R_a s V_0 + C_c R_a s V_1$$

$$(1 + (C_b + C_c)R_d s)\left[(1 + C_b R_a s)V_0 - V_1\right] = (1 + C_b R_d s)\left[C_b R_a s V_0 + C_c R_a s V_1\right]$$

$$(1 + (C_b + C_c)R_d s)V_1 + (1 + C_b R_d s)C_c R_a s V_1 = +(1 + (C_b + C_c)R_d s)(1 + C_b R_a s)V_0 + (1 + C_b R_d s)C_b R_a s V_0$$

$$\frac{V_1}{V_0} = + \frac{(1 + (C_b + C_c)R_d s)(1 + C_b R_a s) + (1 + C_b R_d s)C_b R_a s}{(1 + (C_b + C_c)R_d s) + (1 + C_b R_d s)C_c R_a s}$$

$$= \frac{+1 - C_b R_a s + C_b^2 R_a R_d s^2 + C_b C_c R_a R_d s^2 + (C_b + C_c)R_d s + C_b R_a s + C_b^2 R_a R_d s^2}{1 + (C_b + C_c)R_d s + C_c R_a s + C_b C_c R_a R_d s^2}$$

$$\frac{V_1(s)}{V_0(s)} = \frac{C_b C_c R_a R_d s^2 + (C_b + C_c)R_d s + 1}{C_b C_c R_a R_d s^2 + [(C_b + C_c)R_d + C_c R_a]s + 1}$$

1c. Defining $R_a = 2R$, $C_b = C_c = C$, and $R_c = R/2$, where R and C are kept as symbolics, simplify the transfer function $V_1(s)/V_0(s)$ determined in question 1b. Sketch its Bode plot. What is this circuit useful for? What is its characteristic frequency?

$$\frac{V_1(s)}{V_0(s)} = \frac{C_b C_c R_a R_d s^2 + (C_b + C_c)R_d s + 1}{C_b C_c R_a R_d s^2 + [(C_b + C_c)R_d + C_c R_a]s + 1}$$

$$= \frac{C^2 R^2 s^2 + CRs + 1}{C^2 R^2 s^2 + 3CRs + 1} = \frac{s^2 + \frac{1}{CR}s + \frac{1}{C^2R^2}}{s^2 + \frac{3}{CR}s + \frac{1}{C^2R^2}}$$

$$\omega = \frac{1}{CR} \qquad = \frac{s^2 + \omega s + \omega^2}{s^2 + 3\omega s + \omega^2}$$

$$3\omega = 2\zeta\omega$$

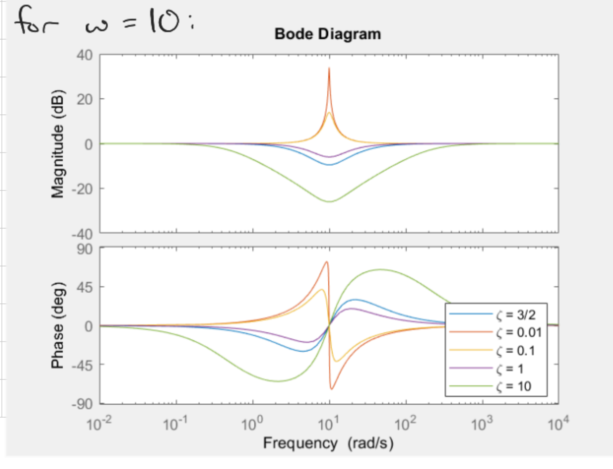
$$\zeta = 3/2$$

Quadratic formula:

$m=2$: zeros $s = \frac{\omega}{2}(-1 \pm i\sqrt{3})$

$n=2$: poles $s = \frac{\omega}{2}(-3 \pm \sqrt{5})$

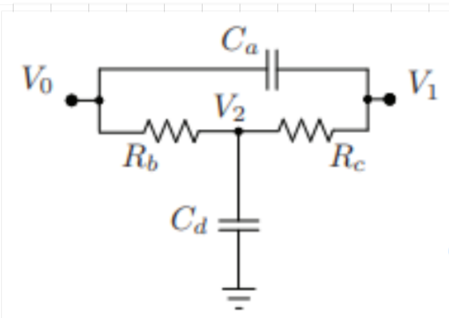
for $\omega = 10$:



looking at how it rejects frequencies similar to the freq. ω (so the tf lowers the gain of an input signal w/ freq. close to ω), it looks useful for filtering out noise near a given freq., or in a niche scenario, selectively amplifying it.

1d) turned everything in 1a&b into matrix form (check code for further comments.)

2. Repeat all four parts of question 1, in their entirety, for the circuit shown at right in Figure 1, subject to the same two assumptions.
For question 2a, note that our seven unknowns are again denoted $\{V_1, V_2, I_0, I_a, I_b, I_c, I_d\}$.
For question 2b, develop a single equation for $V_1(s)/V_0(s)$ in terms of the 4 new parameters $\{C_a, R_b, R_c, C_d\}$.
For question 2c, simplify by defining $C_a = C/2$, $R_b = R_c = R$, and $C_d = 2C$, where R and C are kept as symbolics. 7
For question 2d, now take $\{s, V_0\}$ and $\{C_a, R_b, R_c, C_d\}$ as symbolics.



→ means plug into

2a&b)

$(1)\cdots I_0 = I_a + I_b$

$(2)\cdots I_a = I_c$

$(3)\cdots I_b + I_c = I_d = I_0$

$(4)\cdots C_a\frac{d(V_0-V_1)}{dt} = I_a = C_a s(V_0 - V_1)$

$(5)\cdots V_0 - V_2 = I_b R_b$

$(6)\cdots V_1 - V_2 = I_c R_c$

$(7)\cdots C_d\frac{d}{dt}(V_2 - 0) = I_d = C_d s V_2$

$(7,5,6) \rightarrow 3$

$$C_d s V_2 = \underbrace{R_b^{-1}(V_0 - V_2)}_{5} + \underbrace{R_c^{-1}(V_1 - V_2)}_{6}$$

$$V_2(C_d s + R_c^{-1} + R_b^{-1}) = R_b^{-1}V_0 + R_c^{-1}V_1$$

$(4,6) \rightarrow 2$

$$(V_1 - V_2)R_c^{-1} = C_a s(V_0 - V_1)$$

$$V_2 = R_c C_a s(V_1 - V_0) + V_1$$

$$[R_c C_a s(V_1 - V_0) + V_1](C_d s + R_c^{-1} + R_b^{-1}) = R_b^{-1}V_0 + R_c^{-1}V_1$$

$$V_1 R_b^{-1} + V_1 R_c^{-1} - C_a V_0 s + C_a V_1 s + C_d V_1 s - C_a C_d R_c V_0 s^2 + C_a C_d R_c V_1 s^2 - C_a R_c R_b^{-1} V_0 s + C_a R_c R_b^{-1} V_1 s = R_b^{-1}V_0 + R_c^{-1}V_1$$

$$V_1[R_b^{-1} + s(C_a + C_d + C_a R_b^{-1} R_c) + s^2(C_a C_d R_c)] = V_0[R_b^{-1} + s(C_a + C_a R_b^{-1} R_c) + s^2(C_a C_d R_c)]$$

$$V_1[1 + s(C_a R_c + R_b(C_a + C_d)) + s^2 C_a C_d R_b R_c] = V_0[1 + s C_a(R_b + R_c) + s^2 C_a C_d R_b R_c]$$

$$\frac{V_1(s)}{V_0(s)} = \frac{C_a C_d R_b R_c s^2 + C_a(R_b + R_c)s + 1}{C_a C_d R_b R_c s^2 + (C_a R_c + R_b(C_a + C_d))s + 1},$$

$C_a = C/2,\ C_d = 2C,\ R_b = R_c = R$

2c) $$\frac{V_1(s)}{V_0(s)} = \frac{C^2R^2s^2 + CRs + 1}{C^2R^2s^2 + 3CRs + 1} = \frac{s^2 + \frac{1}{CR}s + \frac{1}{C^2R^2}}{s^2 + \frac{3}{CR}s + \frac{1}{C^2R^2}} \underset{\omega = \frac{1}{CR}}{=} \frac{s^2 + \omega s + \omega^2}{s^2 + 3\omega s + \omega^2},$$

$3\omega = 2\zeta\omega$, $\zeta = 3/2$

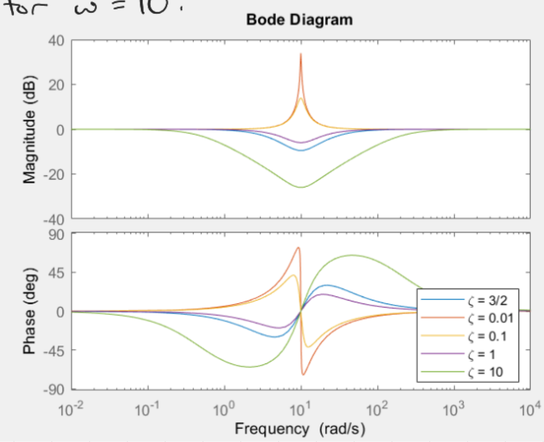
quadratic formula again:

zero: $s = \frac{\omega}{2}(-1 \pm i\sqrt{3})$

poles: $s = \frac{\omega}{2}(-3 \pm \sqrt{5})$

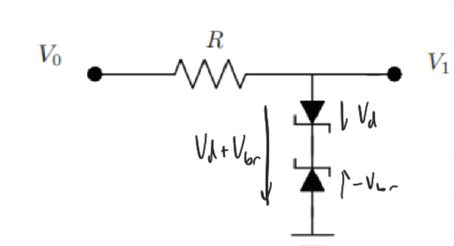
We notice that this is an equivalent circuit:

for $\omega = 10$:



everything mentioned in (1c) still applies, and the deciding factor between the two would most likely be size/geometry constraints.

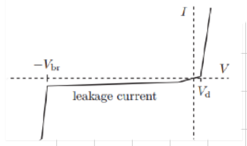
2d) same as above... everything is commented on in the code.



3. Consider a circuit with two zener diodes, attached "back-to-back" as illustrated at left in Figure 2. Assume both zener diodes have a breakdown voltage of $V_{br} = 7.5\,\text{V}$, a cut-in voltage of $V_d = 0.7\,\text{V}$, negligible leakage current, and a very steep I/V response outside the range $V \in [-V_{br}, V_d]$, as illustrated at right in Figure 2. Note that a zener diode is *not* a linear component, so we can *not* analyze a circuit like this with Laplace transforms and Bode plots. For the purpose of this exam, we will assume that $V_0(t) = A \sin(\omega t)$. We also assume, again, that:
(a) the voltage $V_0(t)$ at the input node is specified precisely, regardless of the current $I_0(t)$ drawn by the circuit, and
(b) the current out the output node (if that node is connected to anything, to the right) is negligible, $I_1(t) = 0$.

Figure 2: (left) The back-to-back zener filter considered in Problem 3, and (right) the current-voltage relationship of the zener diodes used, with $V_{br} = 7.5\,\text{V}$, $V_d = 0.7\,\text{V}$, and negligible leaking current.

3a. Subject to the assumptions listed above, if A is less than some threshold A_0, the output is the same as the input, $V_1(t) = A \sin(\omega t)$. Calculate A_0. What is the current through the Zener diodes in this case?



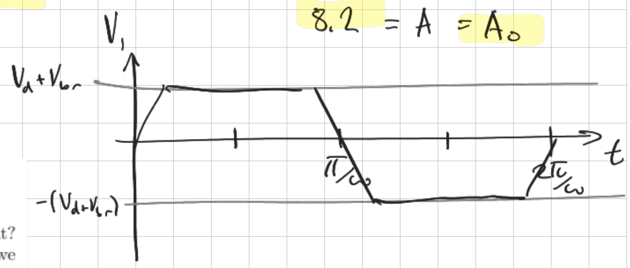
as $A < A_0$, the current through R is 0 → $\underbrace{V_0 - V_1 = IR}_{=0}$ → $V_1 = V_0$

, $V_1 = A\sin(\omega t)$

at the peaks:
$V_1 = V_d + V_{br} = A \cancel{\sin(\omega t)}^{\,1}$
$8.2 = A = A_0$

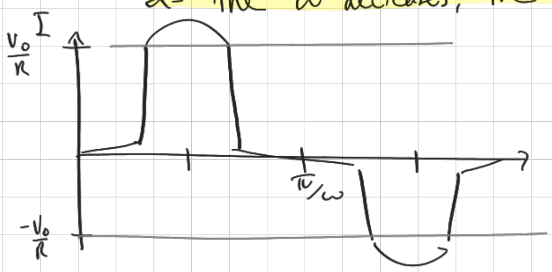
in the case of $A < A_0$, there's no current flow through the zener diodes

in the case of $A = A_0$, the current is determined by $I_R = I_z$ → $\frac{V_0}{R} = I_z$



3b. For A greater than the threshold A_0 computed in question 3a, answer the following:
- Plot $V_1(t)$ in this case.
- Discuss how the shape of this output changes for different values of ω.
- Plot the current through the Zener diodes, as a function of time, in this case. What is the value of the peak current?
- Discuss how to select R. Specifically, what would happen if we got rid of the resistor in this circuit (that is, if we took $R \to 0$)?

as the ω decreases, the slope between each plateau is more gradual, and as ω increases, the wave becomes more similar to a square wave.



the $I_{max} = \frac{V_{0\,max}}{R} = \frac{8.2}{R}$

as $R \to 0$, $I_{peak} = \frac{V_0}{R} \to \infty$, which can damage the circuit and its elements.