4.

$$n_i = \frac{g_i}{(2\pi)^3} \int d^3p \, f_i(\bar{p})$$

using

$$\rho_i = \frac{g_i}{(2\pi)^3} \int d^3p \, E_p \, f_i(\bar{p})$$

$$\& \quad P_i = \frac{g_i}{(2\pi)^3} \int d^3p \, \frac{|p|^2}{3E} \, f_i(\bar{p})$$

$\Rightarrow$

where for fermions $f_i(p) = \dfrac{1}{\exp\left(\frac{E_p - \mu_i}{T}\right) + 1}$

& bosons $f_i(p) = \dfrac{1}{\exp\left(\frac{E_p - \mu_i}{T}\right) - 1}$

for (

for bosons, under the limit $m, \mu \ll T$.

$$E^2 = p^2 + m^2$$
$$\Rightarrow E \approx p$$

$$\therefore n(T) = \frac{g}{(2\pi)^3} \int_0^\infty \frac{d^3\bar{p}}{\exp(P/T) - 1} = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^2 dp}{\exp(P/T) - 1}$$

~~$n(T) = a$~~ taking $x = P/T$ $\therefore \Rightarrow dp = T\,dx$ & $p = Tx$.

$$\Rightarrow n(T) = \frac{gT^3}{(2\pi)^3} \int_0^\infty \frac{4\pi x^2 dx}{e^x - 1} = \frac{gT^3}{2\pi^2} \int_0^\infty \frac{x^2 dx}{e^x - 1}$$

~~$n(T) =$~~ now, $\because \zeta(3) = \frac{1}{\Gamma(3)} \int_0^\infty \frac{x^2 dx}{e^x - 1}$

$$\Rightarrow \int_0^\infty \frac{x^2 dx}{e^x - 1} = \zeta(3) \cdot \Gamma(3)$$

$$\Rightarrow n(T) = \frac{gT^3}{2\pi^2} \cdot \Gamma(3) \cdot \zeta(3) \qquad \because \Gamma(3) = 2! = 2$$

$$n(T) = \frac{gT^3 \zeta(3)}{\pi^2}$$

now, $\rho(T) = \frac{g}{(2\pi)^3}\int_0^\infty \frac{d^3p \cdot E}{\exp\left(p/T\right)-1}$

$$\rho(T) = \frac{g \cdot 4\pi}{(2\pi)^3}\int_0^\infty \frac{p^2 \cdot p \cdot dp}{\exp\left(p/T\right)-1}$$

Similar procedure will produce,

$$\rho(T) = \frac{gT^4}{2\pi^2} \cancel{\frac{x^3 dp}{\exp(p}} \int_0^\infty \frac{x^3\, dx}{e^x-1}$$

$$\rho(T) = \frac{gT^4}{2\pi^2}\int_0^\infty \frac{x^3\, dx}{e^x-1}$$

using $\zeta(4) = \frac{1}{\Gamma(4)}\int_0^\infty \frac{x^3\, dx}{e^x-1} \Rightarrow \int_0^\infty \frac{x^3\, dx}{e^x-1} = \Gamma(4)\,\zeta(4)$

$$= 3!\,\zeta(4)$$

$$\int_0^\infty \frac{x^3\, dx}{e^x-1} = 6\,\zeta(4)$$

$$\Rightarrow \rho(T) = \frac{gT^4 \cdot 6 \cdot \zeta(4)}{2\pi^2} = \frac{3gT^4}{\pi^2}\zeta(4).$$

using $\zeta(4) = \frac{\pi^4}{90}$, we get $\rho(T) = \frac{gT^4}{\cancel{\pi^2}}\left(\frac{\pi^2}{30}\right)$.

now,

$$P_{pressure}(T) = \frac{g}{(2\pi)^3}\int_0^\infty \frac{p^2 \cdot 4\pi p^2\, dp}{3p \cdot \exp(P/T) - 1}.$$

$$\Rightarrow \text{Pressure}(T) = \cancel{\frac{g\,\pi\,(4)}{(2\pi)^3\,(3)}\int_0^\infty \frac{p^3}{e}}\ \frac{g}{2\pi^2 \cdot 3}\int_0^\infty \frac{p^3\, dp}{\exp(P/T)-1}$$

$$= \cancel{g}\ \frac{1}{3}\left(\frac{g}{2\pi^2}\int_0^\infty \frac{p^3\, dp}{\exp(P/T)-1}\right)$$

$$\text{Pressure}(T) = \frac{1}{3}\,\rho(T).$$

---

Now, for ~~fermion~~ the fermions.

Now, for ~~fermion~~ the fermions, similar procedure will be followed ~~and~~ but the $f_i(A,T)$ will be

$$f_i(T) = \frac{1}{\exp\left(\frac{E-\mu}{T}\right)+1}$$

$$\therefore n(T) = \frac{g}{(2\pi)^3}\int_0^\infty \frac{4\pi p^2 dp}{\exp\left(\frac{E-\mu}{T}\right)+1}$$

under these limits of

$T \gg m, \mu$

$E \approx p$

(4)

$$\Rightarrow n(T) = \frac{g\,4\pi}{(2\pi)^3}\int_0^\infty \frac{p^2 dp}{\exp\left(P/T\right)+1}$$

$x = P/T$

$\Rightarrow dp = T dx$

$$\Rightarrow n(T) = \frac{g\,T^3}{2\pi^2}\int_0^\infty \frac{x^2 dx}{e^x+1}$$

using $\zeta(3) = \frac{2}{3}\left(\int_0^\infty \frac{x^2}{e^x+1}\,dx\right)$, we get.

$$\Rightarrow \quad n(T) = \frac{gT^3}{2\pi^2} \times \frac{3}{2}\zeta(3) = \frac{3}{4}\left(\frac{g}{\pi^2}\zeta(3)\,T^3\right)$$

now, $$\rho(T) = \frac{g}{(2\pi)^3}\int_0^\infty \frac{4\pi p^2 \cdot p \cdot dp}{e^{(p/T)}+1}$$

$$\rho(T) = \frac{g\,4\pi}{(2\pi)^3}\int_0^\infty \frac{p^3\,dp}{e^{(p/T)}+1} = \frac{g\,T^4}{2\pi^2}\int_0^\infty \frac{x^3\,dx}{e^x+1}$$

$$\cancel{\zeta(4) = \frac{\pi^4}{90} = \frac{4}{21}\int_0^\infty \frac{x^3\,dx}{e^x+1}} \quad \Rightarrow \quad \int_0^\infty \frac{x^3\,dx}{e^x+1} = \frac{21}{4}\zeta(4)$$

$$= \frac{21}{4}\cdot\frac{\pi^4}{90}$$

$$\Rightarrow \quad \rho(T) = \frac{g\,T^4}{\cancel{\pi^2}} \times \left(\frac{7}{8}\right) \times \frac{\pi^2}{30}.$$

and $$P_{ress.}(T) = \frac{g}{(2\pi)^3}\int_0^\infty \frac{p^2 \cdot 4\pi p^2\,dp}{3p\cdot\left(e^{(p/T)}-1\right)} = \frac{1}{3}\left(\frac{g\,4\pi}{(2\pi)^3}\int_0^\infty \frac{p^3\,dp}{e^{p/T}+1}\right)$$

$$P_{ress.}(T) = \frac{\rho(T)}{3}.$$