4.4

for all possible curvatures (flat, open & closed universe)

let's consider $\Omega_\Lambda = 0$,

< 1

> 1

(i) $\Omega_\Lambda = 0$, $\Omega_T \simeq \Omega_M$.

hence, friedmann's equation becomes,

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_M}{a^3} + \cancelto{0}{\frac{\Omega_R}{a^4}} + \cancelto{0}{\Omega_\Lambda} + \frac{(1-\Omega_T)}{a^2}\right)$$

now, there are three subcases

$\underbrace{\Omega_T = 1}_{\text{flat}}$, $\underbrace{\Omega_T < 1}_{\text{open}}$, $\underbrace{\Omega_T > 1}_{\text{closed.}}$

for flat i.e. $\Omega_T = 1$ & $\Omega_M \simeq 1$

$$\frac{\dot{a}^2}{a^2} = H_0^2\left(\frac{\Omega_M}{a^3}\right) \Rightarrow \frac{da}{dt} = H_0 \frac{\sqrt{\Omega_M}}{\sqrt{a}}$$

or $$\int \sqrt{a}\, da = H_0 \sqrt{\Omega_M} \int dt$$

$$\Rightarrow \quad a^{3/2} \propto H_0 t$$

or $$a(t) \propto H_0 t^{2/3}$$

So, this condition is similar to matter dominated epoch.

for $\Omega_T < 1$ i.e. open.

$$\frac{\dot{a}^2}{a^2} = H_0^2\left(\frac{\Omega_M}{a^3} + \frac{1-\Omega_T}{a^2}\right)$$

$$\Rightarrow \frac{da}{dt} = H_0 \sqrt{\frac{\Omega_M}{a} + 1 - \Omega_T}\,.$$

$$\Rightarrow a(t) > 0$$

i.e. the open universe is expanding all the time.

when the $\frac{\Omega_M}{a^3} > \frac{1-\Omega_T}{a^2}$ (initially)

then there is matter epoch

but when $\frac{\Omega_M}{a^3} < \frac{1-\Omega_T}{a^2}$

$$\text{or} \quad a > \frac{\Omega_M}{1-\Omega_T} \simeq \frac{\Omega_T}{1-\Omega_T}$$

then $\frac{da}{dt} = H_0 \sqrt{1-\Omega_T}$

$$\text{or} \quad a(t) \propto t$$

this is curvature dominated epoch followed after matter epoch.

for $\Omega_T > 1$ i.e. closed universe

$$1 - \Omega_T < 0$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0 \left(\frac{\Omega_M}{a^3} + \frac{1-\Omega_T}{a^2}\right)$$

$$\frac{da}{dt} = H_0 \sqrt{\frac{\Omega_M}{a^3} + \frac{1-\Omega_T}{a^2}}$$

$\because \quad \frac{1-\Omega_T}{a^2} < 0 \quad \& \quad \frac{\Omega_M}{a^3} > 0$

this $\frac{da}{dt} = 0$ will occur at

$$\frac{\Omega_M}{a^3} = \frac{\Omega_T - 1}{a^2} \Rightarrow a = \frac{\Omega_M}{\Omega_T - 1}.$$

after initial accelerated expansion matter dominated epoch.

after $a = \dfrac{\Omega_M}{\Omega_T - 1}$, $\dfrac{da}{dt} < 0$.

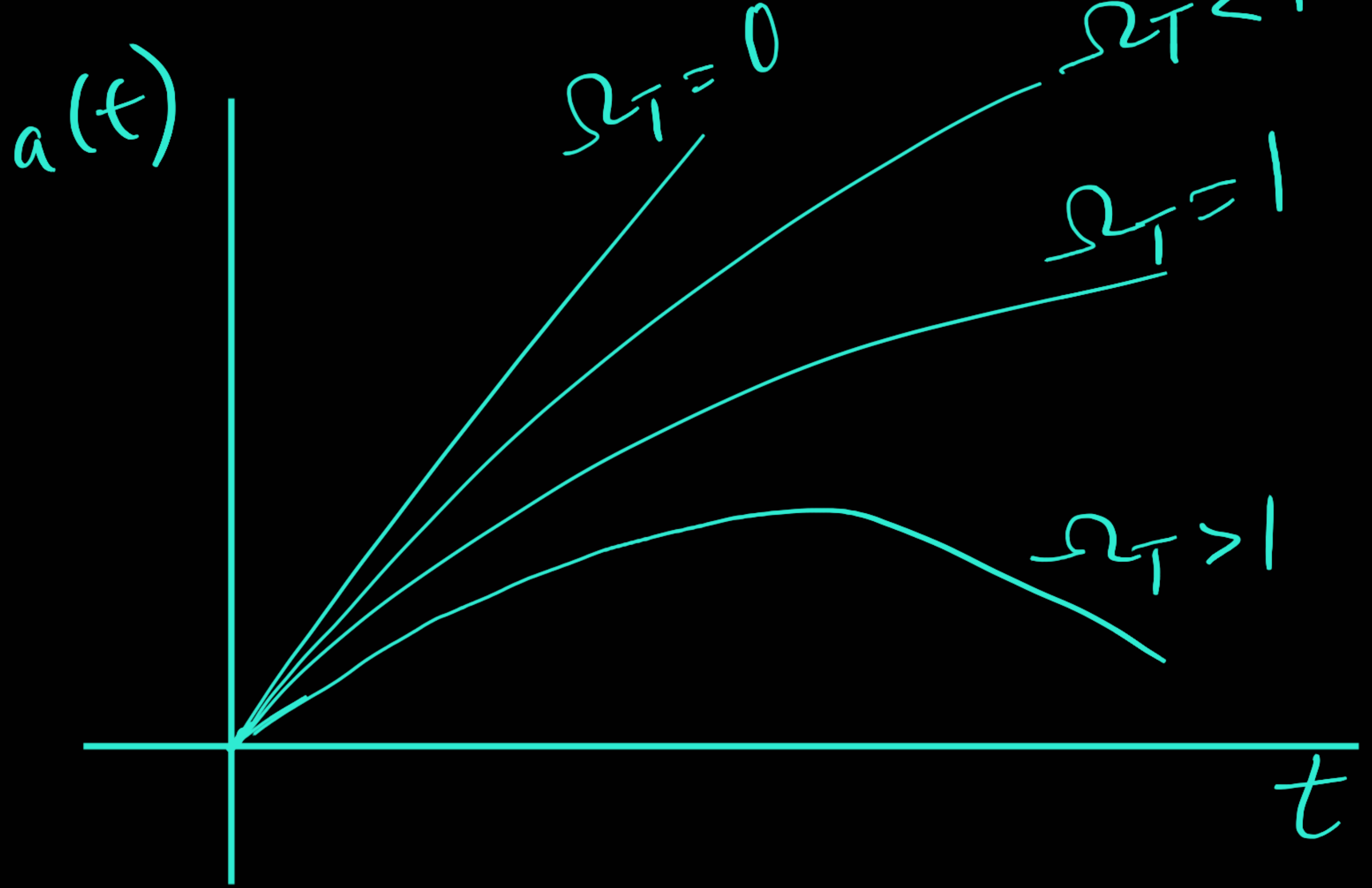
the expansion will be decelerated.

for empty universe,

$\Omega_M \simeq \Omega_T = 0$

$\therefore \left(\dfrac{\dot{a}}{a}\right)^2 = H_0^2 \dfrac{1}{a^2}$ or $\dfrac{da}{dt} = H_0$

$\Rightarrow a(t) = t$



Case (ii) $\Omega_\Lambda > 0$

freidmann's equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_M}{a^3} + \Omega_\Lambda + \frac{1-\Omega_T}{a^2}\right)$$

now, taking similar cases as above.

$\Omega_T = 1$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_M}{a^3} + \Omega_\Lambda\right)$$

or $\dot{a}^2 = H_0^2\left(\frac{\Omega_M}{a} + \Omega_\Lambda a^2\right)$

or, $\dfrac{da}{dt} = H_0 \sqrt{\dfrac{\Omega_M}{a} + \Omega_\Lambda a^2}$.

universe will be expanding

$\dfrac{da}{dt} > 0$

initially matter dominated epoch will follow

but after $\dfrac{\Omega_M}{a} < \Omega_\Lambda a^2$

or $a^3 > \dfrac{\Omega_M}{\Omega_\Lambda}$; vacuum epoch will follow from book

this $a(t) \propto \exp\left(H_0 \Omega_\Lambda^{1/2} t\right)$

now, for $\Omega_T < 1$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\frac{\Omega_M}{a^3} + \Omega_\Lambda + \frac{1-\Omega_T}{a^2}\right)$$

or $$\frac{da}{dt} = H_0 \sqrt{\frac{\Omega_M}{a} + \Omega_\Lambda a^2 + 1 - \Omega_T}$$

$\forall\, t,\ \frac{da}{dt} > 0$ i.e.

universe is

expanding

In this case

matter epoch $\frac{\Omega_M}{a} > 1 - \Omega_T > \Omega_\Lambda a^2$

then vacuum epoch will follow

$$\Omega_\Lambda a^2 > 1 - \Omega_T > \frac{\Omega_M}{a}$$

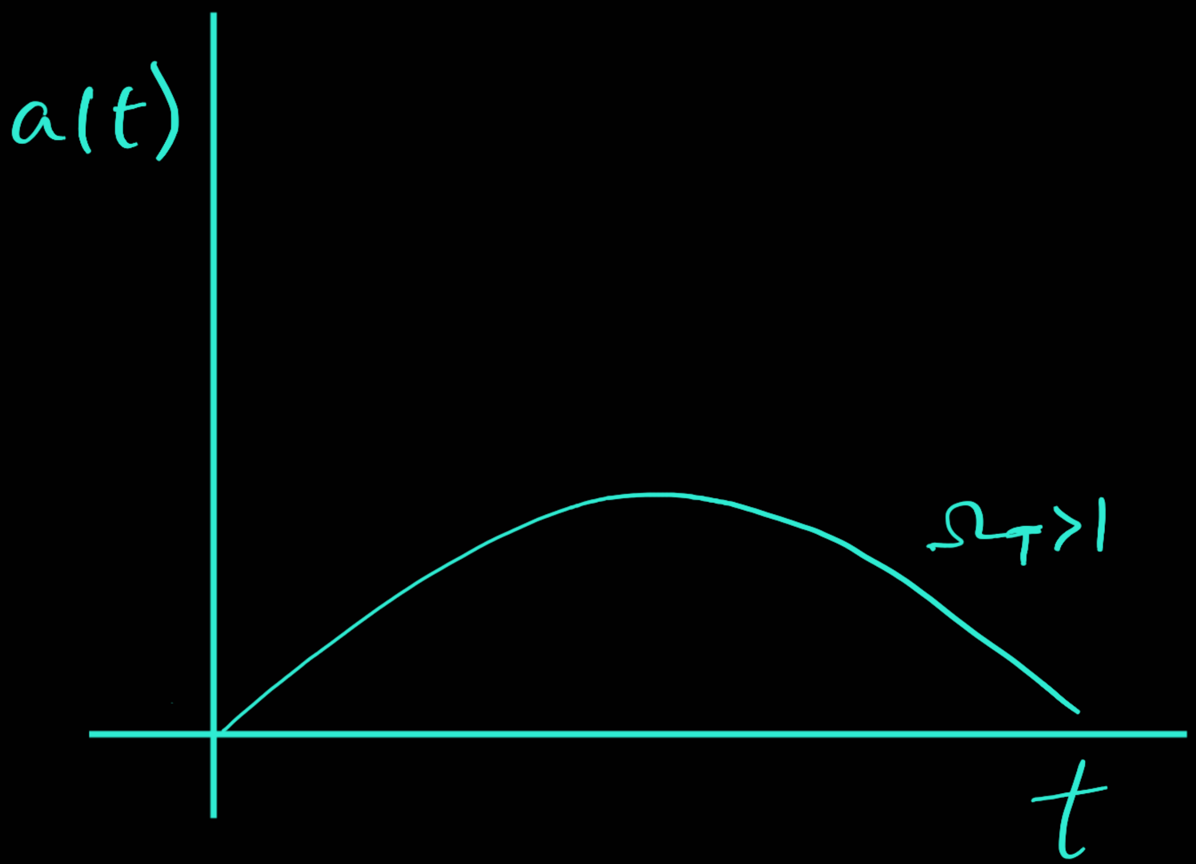
or after $a > \left(\frac{1-\Omega_T}{\Omega_\Lambda}\right)^{1/2}$.

for closed universe i.e. $\Omega_T > 1$

$$\frac{da}{dt} = H_0 \sqrt{\frac{\Omega_M}{a} + \Omega_\Lambda a^2 + 1 - \Omega_T}$$

initially, universe will expand.

- then, in the approximation of $\frac{\Omega_M}{a} \ll \Omega_\Lambda a^2$, derivative will have a root at $a^2 = \frac{\Omega_T - 1}{\Omega_M}$



Case (iii) $\Omega_\Lambda < 0$

for $\Omega_T = 1$, $\frac{da}{dt} = H_0 \sqrt{\underbrace{\frac{\Omega_M}{a}}_{+ve} + \underbrace{\Omega_\Lambda a^2}_{-ve}}$

$\therefore \frac{da}{dt} = 0$ at $\frac{\Omega_M}{a} = -\Omega_\Lambda a^2$

or $a = \left(\frac{\Omega_M}{-\Omega_\Lambda}\right)^{1/3}$

expansion will be accelerated initially but decelerated later on.

Now, for $\Omega_T > 1$ & $\Omega_T < 1$

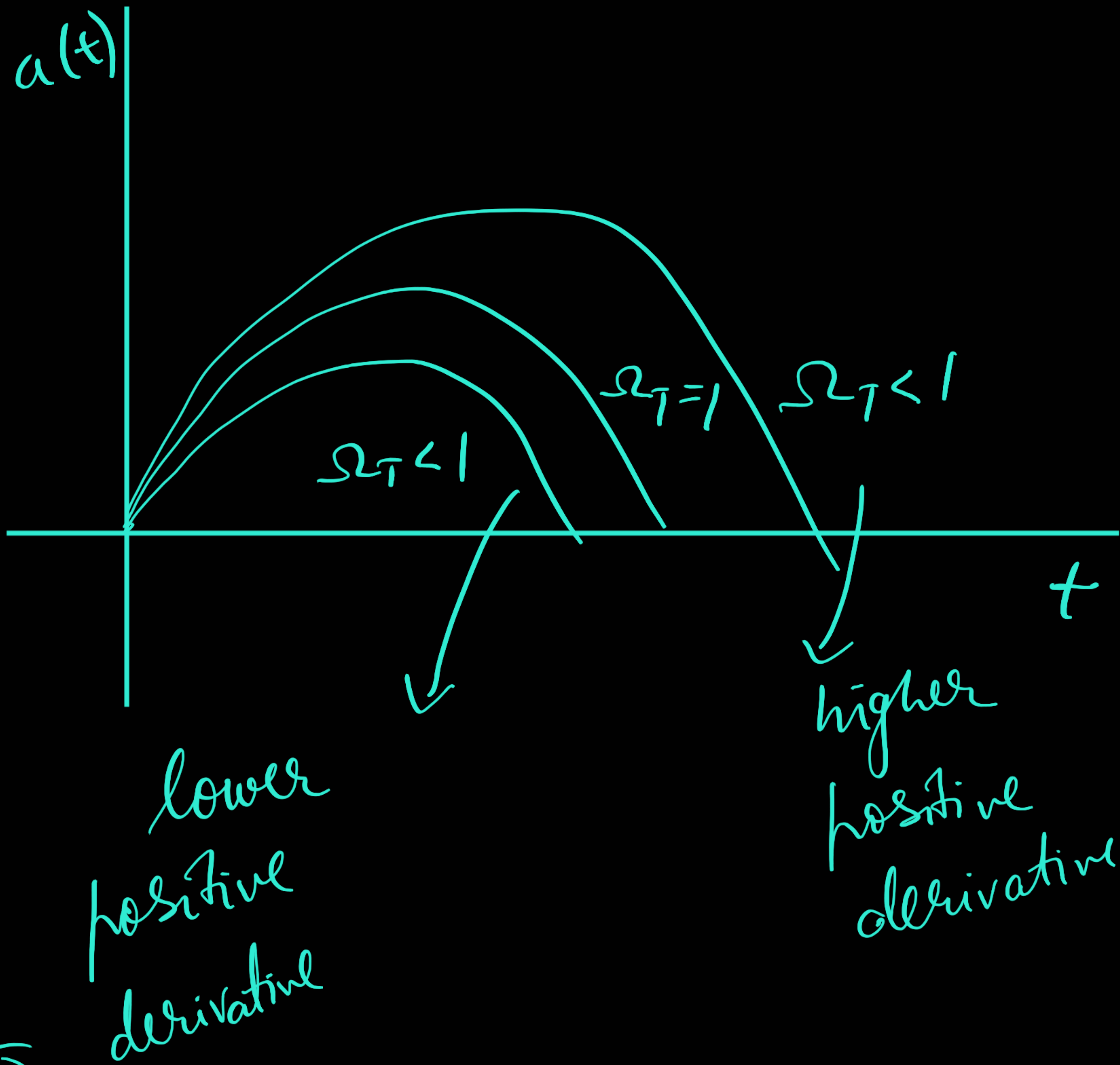
$$\frac{da}{dt} = H_0 \sqrt{\underbrace{\frac{\Omega_M}{a}}_{+ve} + \underbrace{\Omega_\Lambda a^2}_{-ve} + \underbrace{(1-\Omega_T)}_{-ve}}$$

for $\Omega_T > 1$, it has a root and accelerated expansion will be followed by decelerated expansion.

similarly in case of $\Omega_T < 1$

$$\frac{da}{dt} = H_0 \sqrt{\underbrace{\frac{\Omega_M}{a}}_{+ve} + \underbrace{\Omega_\Lambda a^2}_{-ve} + \underbrace{(1-\Omega_T)}_{+ve}}$$

in all the case of $\Omega_\Lambda < 0$, the universe has a maxima of expansion.



4.5

$a_{eq.}$ is given by $\dfrac{\Omega_M}{a^3} = \dfrac{\Omega_R}{a^4}$

$\therefore\ a_{eq.} = \dfrac{\Omega_R}{\Omega_M}$ $\qquad \Omega_R = 1.68\,\Omega_\gamma$

$= 8.4 \times 10^{-5}$

& $\Omega_M = 0.27 \Rightarrow a_{eq.} = \frac{8.4 \times 10^{-5}}{0.27}$

$$a_{eq.} \simeq 3 \times 10^{-4}$$

using $a_{eq.} = \frac{1}{1 + z_{eq.}}$

or $z_{eq.} = \frac{1}{a_{eq.}} - 1 \approx \frac{1}{a_{eq.}}$

$$\approx 0.33 \times 10^{4}$$