14.3 3-sphere of radius R (w, x, y, z)

$$w^2 + x^2 + y^2 + z^2 = a^2.$$

$w = a\cos\chi$ $\quad y = a\sin\chi\sin\theta\sin\phi$

$x = a\sin\chi\sin\theta\cos\phi$ $\quad z = a\sin\chi\cos\theta$

then, metric on the surface would be

$$d\sigma^2 = dw^2 + dx^2 + dy^2 + dz^2$$

$$dw^2 = a^2\sin^2\chi\, d\chi^2$$

$$dx^2 = \left(a\cos\chi\, d\chi\, \sin\theta\cos\phi \overset{d\chi}{\wedge} + a\sin\chi\cos\theta\, d\theta\cos\phi - a\sin\chi\sin\theta\sin\phi\, d\phi\right)^2$$

$$dy^2 = \left(a\cos\chi\, d\chi\, \sin\theta\sin\phi \overset{d\chi}{\wedge} + a\sin\chi\cos\theta\, d\theta\sin\phi + a\sin\chi\sin\theta\cos\phi\, d\phi\right)^2$$

$$dz^2 = \left(a\cos\chi\, d\chi - a\sin\chi\sin\theta\, d\theta\right)^2$$

$$\therefore\ dw^2 + dx^2 + dy^2 + dz^2$$

$$= \left[a^2\sin^2\chi + a^2\cos^2\chi\left(\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta\right)\right] d\chi^2$$

$$+\left[a^2 \sin^2\chi \cos^2\theta \left(\cos^2\phi + \sin^2\phi\right) + a^2 \sin^2\chi \sin^2\theta\right] d^2\theta$$

$$+\left[a^2 \sin^2\chi \sin^2\theta \left(\cos^2\phi + \sin^2\phi\right) d^2\phi\right]$$

$$= a^2 d^2\chi + a^2 \sin^2\chi \left(d^2\theta + \sin\theta \, d^2\phi\right)$$

(All the cross terms cancel with each other)

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh^2\chi - \cosh^2\chi = -1$$

$$V = \int_{x=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (a\,dx)\overline{(a\sin x\,d\theta)}(a\sin x\sin\theta\,d\phi)$$

$$V = a^3 \int_{x=0}^{\pi} \sin^2 x\,dx \underbrace{\int_{\theta=0}^{\pi} \sin\theta \left(\int_0^{2\pi} d\phi\right) d\theta}$$

$$V = a^3 \cdot 4\pi \int_{x=0}^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$\boxed{\because \int_0^{\pi} \sin\theta\,d\theta = \cos 0 - \cos\pi = 2}$$

$$V = 4\pi a^3 \left[\frac{\pi}{2} - \frac{1}{2}\int_{x=0}^{\pi} \cos 2x\,dx\right]$$

$$\sin(2\pi) - \sin 0 \to 0$$

$$V = 2\pi^2 a^3$$

$\to$ scale factor

14.4 + NOTES PROBLEM

3-hyperboloid sphere of radius R.

$$w^2 - x^2 - y^2 - z^2 = a^2$$

$w = a\cosh\chi$; $x = a\sinh\chi\sin\theta\cos\phi$

$z = a\sinh\chi\cos\theta$; $y = a\sinh\chi\sin\theta\sin\phi$

metric on it's surface.

$$d\sigma^2 = dw^2 - dx^2 - dy^2 - dz^2$$

$$dw^2 = a^2\sinh^2\chi\, d\chi^2$$

$$dx^2 = \left(a\cosh\chi\, d\chi\sin\theta\cos\phi + a\sinh\chi\cos\theta\cos\phi\, d\theta - a\sinh\chi\sin\theta\sin\phi\, d\phi\right)^2$$

$$dy^2 = \left(a\cosh\chi\sin\theta\sin\phi\, d\chi + a\sinh\chi\cos\theta\sin\phi\, d\theta + a\sinh\chi\sin\theta\cos\phi\, d\phi\right)^2$$

$$dz^2 = \left(a\cosh\chi\cos\theta\, d\chi - a\sinh\chi\sin\theta\, d\theta\right)^2$$

$$dw^2 - (dx^2 + dy^2 + dz^2)$$

$$= \left[-a^2 \cos^2 h\chi \left(\sin^2\theta (\sin^2\phi + \cos^2\phi) + \cos^2\theta\right) + a^2 \sin^2 h\chi\right] d\chi^2$$

$$- \left[a^2 \sin^2 h\chi \left(\cos^2\theta (\cos^2\phi + \sin^2\phi) + \sin^2\theta\right)\right] d^2\theta$$

$$- a^2 \sinh^2\chi \sin^2\theta (\sin^2\phi + \cos^2\phi)\, d\phi^2$$

$$= a^2(\sinh^2\chi - \cosh^2\chi) d\chi^2 - a^2 \sinh^2\chi (d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -a^2\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right.$$

↳ "-" sign shouldn't be there though!!!

NOTES
PROBLEM

Show that area of 2-hyperboloid surfaces $\chi = \text{constant}$ is

$$A = 4\pi a^2 \sinh^2 \chi$$

$$d\sigma^2 = a^2\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$$w = a \cosh\chi$$

$$x = a \sinh\chi \sin\theta \cos\phi$$

$$y = a \sinh\chi \sin\theta \sin\phi$$

$$z = a \sinh\chi \cos\theta$$

for 3-sphere.

$$A = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (a \sinh\chi\, d\theta)(a \sinh\chi \sin\theta\, d\phi)$$

$$A = a^2 \sinh^2\chi \int_{\theta=0}^{\pi} \sin\left(\int_{\phi=0}^{2\pi} d\phi\right) d\theta$$

$$A = a^2 \sinh^2\chi \cdot 4\pi = 4\pi a^2 \sinh^2\chi$$

$$V = \int_{\chi=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (a\,d\chi)(a\sinh\chi\,d\theta)(a\sinh\chi\sin\theta\,d\phi)$$

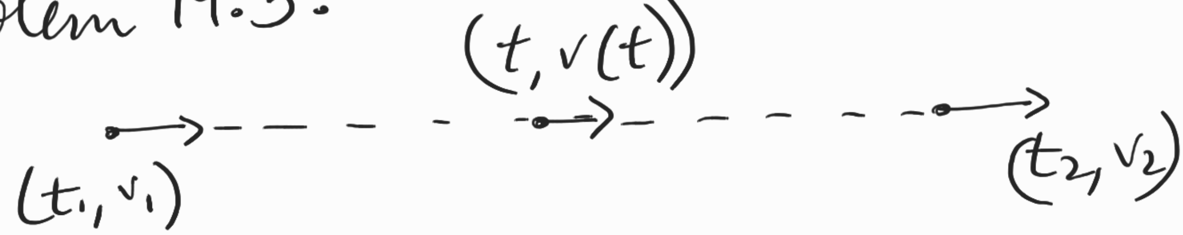
$$V = a^3 \int_{\chi=0}^{\infty} \sinh^2\chi\,d\chi \int_{\theta=0}^{\pi} \sin\theta \left(\int_{\phi=0}^{2\pi} d\phi \right) d\theta$$

$$V = 4\pi a^3 \int_{\chi=0}^{\infty} \sinh^2\chi\,d\chi$$

$$V = 4\pi a^3 \left[e^{2x} \cdots\cdots \right]_0^{\infty}$$

make it infinity

Problem 14.5.



$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - a^2(t)\left[d\chi^2 + s^2(\chi)\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right]$$

$\hookrightarrow$ FRW metric

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{a^2(t)}{c^2}\left(\frac{d\chi}{dt}\right)^2 \quad \left[\frac{d\theta}{dt} = \frac{d\phi}{dt} = 0 \text{ (can be taken)}\right]$$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{a^2(t)}{c^2}\dot{\chi}^2$$

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, dt = \frac{dt}{\gamma}$$

$$\cancel{1} - \frac{v^2(t)}{c^2} = \cancel{1} - \frac{a^2(t)}{c^2}\dot{\chi}^2$$

$$v(t) = a(t)\dot{\chi}$$

now, $$\frac{\gamma_{v_2} v_2}{\gamma_{v_1} v_1} = \left(\frac{\gamma_{v_2} \dot{\chi}_{t=t_2}}{\gamma_{v_1} \dot{\chi}_{t=t_1}}\right)\frac{a(t_2)}{a(t_1)}$$

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{1}{\gamma}\frac{dx}{d\tau}$$

& $$a^2(t)\frac{dx}{d\tau} = k$$

$$\Rightarrow \quad \left.\frac{dx}{d\tau}\right|_{t=t_1} = \frac{k}{a^2(t_1)} = \gamma_{v_1}\left.\frac{dx}{dt}\right|_{t=t_1}$$

$$\left.\frac{dx}{d\tau}\right|_{t=t_2} = \frac{k}{a^2(t_2)} = \gamma_{v_2}\left.\frac{dx}{dt}\right|_{t=t_2}$$

hence, $$\frac{\gamma_{v_2}\dot{x}_{t=t_2}}{\gamma_{v_1}\dot{x}_{t=t_1}} = \frac{a^2(t_1)}{a^2(t_2)}$$

and,

$$\frac{\gamma_{v_2} v_2}{\gamma_{v_1} v_1} = \frac{a^2(t_1)}{a^2(t_2)} \cdot \frac{a(t_2)}{a(t_1)} = \frac{a(t_1)}{a(t_2)}$$

NOTES

PROBLEM : ($E \propto \frac{1}{a}$ for massless particle)

$$\left(g_{\alpha\beta} p^\alpha p^\beta = -m^2 \ \& \ E \frac{\partial E}{\partial t} = -\underline{\underline{\delta_{ij}\, a\dot{a}\, p^i p^j}} \right.$$

$$g_{\alpha\beta} p^\alpha p^\beta = -m^2$$

$$\Rightarrow \quad g_{00} (p^0)^2 + \delta_{ij} a^2 p^i p^j = -m^2$$

$$\Rightarrow \quad -E^2 \frac{\dot{a}}{a} + \delta_{ij}\, \dot{a} a p^i p^j = -m^2 \frac{\dot{a}}{a}$$

$$\Rightarrow \quad -E^2 \frac{\dot{a}}{a} - E \frac{\partial E}{\partial t} = -m^2 \frac{\dot{a}}{a}$$

$$\Rightarrow \quad E \frac{dE}{dt} = (m^2 - E^2) \frac{1}{a} \frac{da}{dt}$$

$$\Rightarrow \quad \frac{da}{a} = \frac{E\, dE}{m^2 - E^2} = \frac{-1}{2} \frac{-2\, dE}{(m^2 - E^2)}$$

$$\Rightarrow \quad \int \frac{da}{a} = -\frac{1}{2} \int \frac{d(m^2 - E^2)}{(m^2 - E^2)}$$

$$\Rightarrow \ln(a C_1) = -\frac{1}{2} \ln(m^2 - E^2)$$

$$\underbrace{\ln(a C_1)}_{\text{Constat}}$$

$$\Rightarrow a C_1 = \frac{1}{\sqrt{m^2 - E^2}}$$

$$\Rightarrow m^2 - E^2 = \frac{1}{a^2 C_1^2}$$

$$\Rightarrow E^2 \propto \frac{1}{a^2} \quad \text{or} \quad E \propto \frac{1}{a}$$

NOTES

PROBLEM:

$$x = \int_t^{t_0} \frac{c\,dt}{a(t)} = \frac{c}{a(t_0)} \int_0^z \frac{dz}{H(z)}$$

$$a(t) = \frac{a(t_0)}{(1+z)} \Rightarrow d\,a(t) = -\frac{dz\, a(t_0)}{(1+z)^2}$$

$$\Rightarrow \quad \frac{da}{dt}\frac{dt}{a(t)} = \frac{-dz\, a(t_0)}{(1+z)^2 a(t)}$$

$$\Rightarrow \quad \dot{a}\frac{dt}{a(t)} = \frac{-dz\, \cancel{a(t_0)}}{a^2(t_0)} \cdot a(t)$$

$$\Rightarrow \quad \frac{dt}{a(t)} = -\frac{a(t)}{\dot{a}(t)}\frac{dz}{a(t_0)}$$

$$x = -\int_t^{t_0} \frac{c\, dz}{a(t_0)\,\frac{\dot{a}}{a}}$$

$t = t_0$
$\Rightarrow z = 0$
and t for any random z.

$$x = \frac{c}{a(t_0)}\int_0^z \frac{dz}{H(z)}$$