14.8.21

ASSIGNMENT : WEEK 1

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Question 1.) Show in Natural Units : $1eV = 11605K \simeq 10^4 K$

Answer 1.) Starting from universal constants :

$$G = 6.7 \times 10^{-11} \, m^3 kg^{-1} s^{-2}$$
$$c = 3.0 \times 10^{8} \, ms^{-1}$$
$$\hbar = 6.6 \times 10^{-16} \, eV \, s$$

Planck length : $l_p \equiv \left(\frac{G\hbar}{c^3}\right)^{1/2} = 1.6 \times 10^{-35} m$

Planck Mass : $M_P \equiv \left(\frac{\hbar c}{G}\right)^{1/2} = 2.2 \times 10^{-8} kg$

Planck Time : $t_P \equiv \left(\frac{G\hbar}{c^5}\right)^{1/2} = 5.4 \times 10^{-44} s$

Planck Energy : $E_P = M_P c^2 = 1.2 \times 10^{28} eV$

Planck Temperature : $T_P = \frac{E_P}{k_B} = 1.4 \times 10^{32} K$

$k_B = 8.6 \times 10^{-5} \, eV \, K^{-1}$ (Boltzmann constant)

So, using E_P and T_P :

$$1eV = \frac{E_P}{1.2 \times 10^{28}}$$

Now using Natural units : $c = \hbar = k_B = G = 1$

$$T_p = E_p \implies 1\,eV = \frac{T_p}{1.2\times10^{28}} = \frac{1.4\times10^{32}\,K}{1.2\times10^{28}}$$

$$\therefore \boxed{1\,eV = 11666.67\,K \simeq 10^4\,K}$$ Ans.

Question 2.) Convert the following by inserting appropriate factors of c, $\hbar$ and k_B.

a. $T_\gamma = 2.725\,K \longrightarrow eV$

Answer 2a.) In Natural units, $T_\gamma = E_\gamma$
with k_B : $T_\gamma k_B = E_\gamma$

$$\implies E_\gamma = 8.6\times10^{-5}\,eV\,K^{-1} \times 2.725\,K$$
$$\therefore E_\gamma = 2.3435\times10^{-4}\,eV$$

$$\therefore \boxed{2.725\,K \longrightarrow 2.3435\times10^{-4}\,eV}$$ Ans.

b. $\rho_c = \frac{3H_0^2}{8\pi G}$ in (i.) $kg\,m^{-3}$ (ii.) $(GeV/c^2)\,m^{-3}$ (iii.) $M_\odot\,Mpc^{-3}$

$H_0 = 70\,km\,s^{-1}\,Mpc^{-1}$
$1\,Mpc$ (Megaparsec) $= 10^6\,pc = 3.1\times10^{22}\,m = 3.1\times10^{19}\,km$
$1\,M_\odot$ (Solar Mass) $= 2\times10^{30}\,kg$
$G = 6.7\times10^{-11}\,m^3\,kg^{-1}\,s^{-2}$

Answer 2b.) i. $$\rho_c = \frac{3H_0^2}{8\pi G} = \frac{3\times(70)^2\,km^2\,\cancel{s^{-2}}\,Mpc^{-2}}{8\pi\times6.7\times10^{-11}\,m^3\,kg^{-1}\,\cancel{s^{-2}}}$$

$$= \frac{8.73\times10^{12}\,\cancel{km^2}\,kg\,m^{-3}}{(3.1\times10^{19})^2\,\cancel{km^2}}$$

$$\therefore \boxed{\rho_c = 0.9084 \times 10^{-26}\ \text{kg m}^{-3}}$$ Ans

ii. $\rho_c = 0.9084 \times 10^{-26}$ kg m^{-3}

$$\left(\frac{\text{GeV}}{c^2}\right) = \frac{10^9\ \text{eV}}{(3\times10^8)^2\ \text{m}^2\ \text{s}^{-2}}$$

$$\Rightarrow \left(\frac{\text{GeV}}{c^2}\right)\text{m}^{-3} = \frac{10^9 \times 1.6\times10^{-19}}{9\times10^{16}}\ \text{J m}^{-2}\,\text{s}^{2}\,.\text{m}^{-3}$$

$$\Rightarrow \left(\frac{\text{GeV}}{c^2}\right)\text{m}^{-3} = 1.78\times10^{-27}\ \text{kg}\,\cancel{\text{m}^{2}}\,\cancel{\text{s}^{-2}}.\cancel{\text{m}^{-2}}\,\cancel{\text{s}^{2}}.\ \text{m}^{-3}$$

$$\therefore 1\left(\frac{\text{GeV}}{c^2}\right)\text{m}^{-3} = 1.78\times10^{-27}\ \text{kg m}^{-3}$$

$$\Rightarrow 1\ \text{kg m}^{-3} = \frac{1}{1.78}\times10^{27}\left(\frac{\text{GeV}}{c^2}\right)\text{m}^{-3}$$

$$\Rightarrow \rho_c = 0.9084\times10^{-26}\ \text{kg m}^{-3} = \frac{0.9084\times10^{-26}\times10^{27}}{1.78}\left(\frac{\text{GeV}}{c^2}\right)\text{m}^{-3}$$

$$\therefore \boxed{\rho_c = 5.103\left(\frac{\text{GeV}}{c^2}\right)\text{m}^{-3}}$$ Ans

iii. $\rho_c = 0.9084 \times 10^{-2}$ kg m^{-3}

$1\ M_\odot = 2\times10^{30}$ kg

$1\ \text{Mpc} = 3.1\times10^{22}$ m

$$\Rightarrow 1\ M_\odot\ \text{Mpc}^{-3} = \frac{2\times10^{30}}{(3.1\times10^{22})^3}\ \frac{\text{kg}}{\text{m}^3}$$

$\therefore$ $1 M_\odot \, Mpc^{-3} = 6.713 \times 10^{-38} \, kg \, m^{-3}$

$\Rightarrow$ $1 \, kg \, m^{-3} = \dfrac{10^{38}}{6.713} \, M_\odot \, Mpc^{-3}$

$\Rightarrow$ $\rho_c = 0.9084 \times 10^{-26} \, kg \, m^{-3} = \dfrac{0.9084 \times 10^{-26} \times 10^{38}}{6.713} \, M_\odot Mpc^{-3}$

$\therefore$ $\boxed{\rho_c = 1.353 \times 10^{11} \, M_\odot Mpc^{-3}}$ Ans

c. $1/H_0 \longrightarrow cm$

Answer 2c.) $\dfrac{1}{H_0} = \dfrac{1}{70} \, km^{-1} \, s \, Mpc$

$= \dfrac{1}{70} \times \dfrac{1}{1000} \times 3.1 \times 10^{22} \, m^{-1} \, s \, m$

$\therefore$ $\dfrac{1}{H_0} = 4.429 \times 10^{17} \, s$

To convert $1/H_0$ to cm, we need to multiply with speed of light.

$d_H = cH_0^{-1} = 3.0 \times 10^{8} \times 4.429 \times 10^{17} \, m \, s^{-1} s$

$= 3 \times 10^{10} \times 4.429 \times 10^{17} \, cm$

$\therefore$ $d_H = 1.3287 \times 10^{28} \, cm$

$\therefore$ $\boxed{H_0^{-1} \longrightarrow 1.3287 \times 10^{28} \, cm}$ Ans

d. $m_{Pl} = 1.2 \times 10^{19} \, GeV$
Convert to (i) K (ii) cm (iii) sec.

Answer 2d.) As this is given in Natural units,

$$m_{PI} \cdot c^2 = 1.2\times10^{19}\ \text{GeV} = E_{PI}$$

i. In Planck's system, $T_{P1} = \dfrac{E_{P1}}{k_B}$

$k_B = 8.6\times10^{-5}\ \text{eV K}^{-1}$

$$\Rightarrow T_{PI} = \frac{1.2\times10^{19}\times10^{9}}{8.6\times10^{-5}}\ \text{eV eV}^{-1}\ \text{K}$$

$$\Rightarrow T_{PI} = 1.395\times10^{32}\ \text{K}$$

$$\therefore \boxed{m_{PI} = 1.2\times10^{19}\ \text{GeV} \longrightarrow 1.395\times10^{32}\ \text{K}}$$ Ans

ii. $m_{PI} c^2 = 1.2\times10^{19}\ \text{GeV} = 1.2\times10^{19}\times10^{9}\ \text{eV}$

$\hbar = 6.6\times10^{-16}\ \text{eV s}$

$c = 3\times10^{8}\ \text{m/s}$

Now,

$$\frac{m_{PI} c^2}{\hbar c} = \frac{1.2\times10^{19}\times10^{9}}{6.6\times10^{-16}\times3\times10^{8}} \cdot \frac{\text{eV}}{\text{eV s}\cdot\text{ms}^{-1}}$$

$$\Rightarrow \frac{m_{PI} c^2}{\hbar c} = 6.061\times10^{34}\ \text{m}^{-1}$$

$$\Rightarrow \frac{m_{PI} c^2}{\hbar c} = 6.061\times10^{34}\times\frac{1}{100}\ \text{cm}^{-1}$$

$$\therefore \frac{m_{PI} c^2}{\hbar c} = 6.061\times10^{32}\ \text{cm}^{-1}$$

$$\therefore \boxed{m_{PI} = 1.2\times10^{19}\ \text{GeV} \longrightarrow 6.061\times10^{32}\ \text{cm}^{-1}}$$ Ans

iii. In the previous part we obtained :

$$\frac{m_{PI}\, c^2}{\hbar c} = 6.061 \times 10^{32}\ \text{cm}^{-1}$$

Multiplying with c :

$$\frac{m_{PI}c^2}{\hbar c} \cdot c = 6.061 \times 10^{32} \times 3 \times 10^{10}\ \text{cm} \cdot s^{-1}\ \text{cm}^{-1}$$
$$= 1.818 \times 10^{43}\ \text{sec}^{-1}$$

$$\therefore \boxed{m_{PI} = 1.2 \times 10^{19}\ \text{GeV} \longrightarrow 1.818 \times 10^{43}\ \text{sec}^{-1}}$$ Ans

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Question 3:) Assuming expansion rate to be :

$$H_0 = 70\ \text{km}\ s^{-1}\ \text{Mpc}^{-1}$$

Show,

expansion rate : $t_H \equiv H_0^{-1} = 13.87\ \text{Gyr}$

Answer 3:) $$t_H \equiv H_0^{-1} = \frac{1}{70}\ \text{km}^{-1}\ s\ \text{Mpc}^{-1}$$
$$= \frac{1}{70} \times 3.1 \times 10^{19}\ \text{km}^{-1} \cdot s\ \text{km}$$

$$\therefore\ t_H = 4.4286 \times 10^{17}\ s$$

$$1\ \text{Gyr} = 10^9\ \text{yr} = 3.2 \times 10^{16}\ s$$

$$\Rightarrow 4.4286 \times 10^{17}\ s = \frac{4.4286 \times 10^{17}}{3.2 \times 10^{16}}\ \text{Gyr} = 13.84\ \text{Gyr}$$

$$\therefore \boxed{t_H = 13.84\ \text{Gyr}}$$ Ans

Derivation of $t_H = H_0^{-1}$ relation :

Let the receding velocity of galaxies is v. Let their separation be r.

Then : Hubble's constant

$$v = H_0 r$$

Assuming the separation rate to be a constant, the time elapsed from galaxies being at one point to their current separation can be given as :

$$t = \frac{r}{v} \Rightarrow t = \frac{r}{H_0 r}$$

$$\therefore \boxed{t = \frac{1}{H_0} = H_0^{-1}}$$

Question 4.)

Limits	Particles	n	ρ	P
a. $T \gg m, \mu$	Bosons	$g\left(\frac{\zeta(3)}{\pi^2}\right)T^3$	$\left(\frac{\pi^2}{30}\right)gT^4$	$\rho/3$
	Fermions	$g\left(\frac{3\zeta(3)}{4\pi^2}\right)T^2$	$\left(\frac{7\pi^2}{g}\times 30\right)gT^4$	$\rho/3$
b. $\mu \gg T \gg m$	Fermions	$\frac{g\mu^3}{6\pi^2}$	$\frac{g\mu^3}{8\pi^2}$	$\rho/3$
c. $T \ll m$	Bosons & Fermions	$g\left(\frac{mT}{2\pi}\right)^{3/2}\exp\left[\frac{\mu-m}{T}\right]$	$\left(\frac{m+3T}{2}\right)n$	nT

$\zeta(3) \simeq 1.202$ = Reimann zeta function of order 3.

Calculate and verify n, ρ, P for different limits.

Answers 4.)

$$F_A(E,T) = \frac{g}{(2\pi)^3}\frac{1}{\exp\left[\frac{E-\mu}{T_A(t)}\right] \pm 1}$$

$$n_A = \int F_A \, d^3p \qquad ; \quad E\,dE = p\,dp$$
$$d^3p = 4\pi p^2 dp$$

$$\rho_A(t) = \int F_A E(p)\, d^3p$$

$$P_A(t) = \int F_A \frac{p^2}{3E}\, d^3p$$

a. $T \gg m, \mu$; $E^2 = p^2 + m^2$

Taking m to be negligible, $E = p$

$$\Rightarrow \exp\left[\frac{E-\mu}{T}\right] = \exp\left[\frac{p-\mu}{T}\right] = \exp\left(\frac{p}{T}\right)\exp\left(-\frac{\mu}{T}\right)$$

As $\frac{\mu}{T} << 1$, $\exp\left(\frac{\mu}{T}\right) \longrightarrow 1$

$$F_A = \frac{g}{(2\pi)^3} \frac{1}{\exp\left(\frac{p}{T}\right) \pm 1} ;$$

Let $\frac{p}{T} = x$; $\frac{dp}{T} = dx$

For Bosons:

$$n_A = \frac{g}{(2\pi)^3} \int \frac{1}{\exp(x)-1} d^3p = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^2 dp}{\exp(x)-1}$$

$$= 4\pi \cdot \frac{g}{(2\pi)^3} \int_0^\infty \frac{x^2 T^2 \cdot T dx}{\exp(x)-1}$$

$$n_A = \frac{g}{2\pi^2} \cdot T^3 \int_0^\infty \frac{x^2}{\exp(x)-1} dx = \frac{g T^3}{\pi^2} \underbrace{\left[\frac{1}{2} \int_0^\infty \frac{x^3}{\exp(x)-1} dx\right]}_{\zeta(3)}$$

$$\therefore \boxed{n_A = \frac{g}{\pi^2} T^3 \zeta(3) . T^3}$$

$T >> \mu, m$ Bosons

Ans