WEEK 7 : HOBSON QUESTIONS

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Answer 14.2) Line element in 3-space is given as:

$0 \equiv t$
$1 \equiv r$
$2 \equiv \theta$
$3 \equiv \phi$

$$d\sigma^2 = B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

($c = 1$ unit)

$$g_{00} = -1; \quad g_{11} = B(r) \quad ; \quad g_{22} = r^2 \quad ; \quad g_{33} = r^2 \sin^2\theta$$

Ricci tensor is given as: $R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha} = -R^{\alpha}{}_{\mu\alpha\nu}$

$$\therefore \boxed{-R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu}} \quad \text{(I)}$$

Finding the Christoffel symbols:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu})$$

$$\Rightarrow \boxed{\begin{aligned} \Gamma^{\alpha}_{\mu\nu} = & \frac{1}{2} g^{\alpha 0}(\partial_\nu g_{0\mu} + \partial_\mu g_{0\nu} - \partial_0 g_{\mu\nu}) \\ & + \frac{1}{2} g^{\alpha 1}(\partial_\nu g_{1\mu} + \partial_\mu g_{1\nu} - \partial_1 g_{\mu\nu}) \\ & + \frac{1}{2} g^{\alpha 2}(\partial_\nu g_{2\mu} + \partial_\mu g_{2\nu} - \partial_2 g_{\mu\nu}) \\ & + \frac{1}{2} g^{\alpha 3}(\partial_\nu g_{3\mu} + \partial_\mu g_{3\nu} - \partial_3 g_{\mu\nu}) \end{aligned}} \quad \text{(II)}$$

$g^{00} = 1$
$g^{11} = \frac{1}{B(r)}$
$g^{22} = \frac{1}{r^2}$
$g^{33} = \frac{1}{r^2 \sin^2\theta}$

$$\Gamma^{0}_{00} = \frac{1}{2} g^{00}(2\partial_0 g_{00} - \partial_0 g_{00}) = \frac{1}{2} g^{00} \partial_0 g_{00}$$

$$g_{00} = 1 \quad ; \quad \partial_0(1) = 0$$

$\therefore \boxed{\Gamma^0_{00} = 0}$

$$\Gamma^0_{01} = \frac{1}{2} g^{00} \left(\cancelto{0}{\partial_1 g_{00}} + \cancelto{0}{\partial_0 g_{01}} - \cancelto{0}{\partial_0 g_{01}} \right)$$

$\therefore \boxed{\Gamma^0_{01} = 0}$

$$\Gamma^0_{10} = \frac{1}{2} g^{00} \left(\cancelto{0}{\partial_0 g_{01}} + \cancelto{0}{\partial_1 g_{00}} - \cancelto{0}{\partial_0 g_{10}} \right) = 0$$

$\therefore \boxed{\Gamma^0_{10} = 0}$

$$\Gamma^0_{11} = \frac{1}{2} g^{00} \left(\cancelto{0}{\partial_1 g_{01}} + \cancelto{0}{\partial_0 g_{01}} - \partial_0 g_{11} \right) = -\frac{1}{2} (1) \times \cancelto{0}{\partial_t B(r)}$$

$\therefore \boxed{\Gamma^0_{11} = 0}$

As, g_{22}, g_{33} also don't depend on time:

$\boxed{\Gamma^0_{22} = 0}$ $\boxed{\Gamma^0_{33} = 0}$

Also as g_{00} does not depend on r, θ, φ:

$\boxed{\Gamma^0_{02} = \Gamma^0_{20} = 0}$ $\boxed{\Gamma^0_{03} = \Gamma^0_{30} = 0}$

As only diagonal elements of metric is non zero:

$\boxed{\Gamma^0_{12} = \Gamma^0_{21} = 0}$ $\boxed{\Gamma^0_{13} = \Gamma^0_{31} = 0}$

$\boxed{\Gamma^0_{23} = \Gamma^0_{32} = 0}$

We will also use the fact that Christoffel symbol is symmetric in $\mu\nu$.

$$\Gamma^1_{11} = \frac{1}{2} g^{11} (\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = \frac{1}{2} g^{11} \partial_1 g_{11}$$

$$\therefore \boxed{\Gamma^1_{11} = \frac{1}{2B} \frac{\partial B}{\partial r}} \quad \text{also,} \quad \boxed{\Gamma^1_{00} = 0}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{2} g^{11} (\cancel{\partial_1 g_{01}}^{\,0} + \cancel{\partial_0 g_{11}}^{\,0} - \cancel{\partial_1 g_{01}}^{\,0}) = 0$$

$$\therefore \boxed{\Gamma^1_{01} = \Gamma^1_{01} = 0}$$

Also, $\boxed{\Gamma^1_{02} = \Gamma^1_{20} = 0}$ $\quad \boxed{\Gamma^1_{03} = \Gamma^1_{30} = 0}$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2} g^{11} (\cancel{\partial_2 g_{11}}^{\,0} + \cancel{\partial_1 g_{12}}^{\,0} - \cancel{\partial_1 g_{12}}^{\,0}) = 0$$

as $B(r)$ does not depend on θ

$$\therefore \boxed{\Gamma^1_{12} = \Gamma^1_{21} = 0}$$

Similarly, $\boxed{\Gamma^1_{13} = \Gamma^1_{31} = 0}$

$$\Gamma^1_{22} = \frac{1}{2} g^{11} (\cancel{\partial_2 g_{12}}^{\,0} + \cancel{\partial_2 g_{12}}^{\,0} - \partial_1 g_{22}) = -\frac{1}{2} g^{11} \partial_1 g_{22}$$

$$\Rightarrow \Gamma^1_{22} = -\frac{1}{2B} \frac{\partial (r^2)}{\partial r} = -\frac{1}{2B} (2r) = -\frac{r}{B}$$

$$\therefore \boxed{\Gamma^1_{22} = -\frac{r}{B}}$$

Also, $\boxed{\Gamma^1_{23} = \Gamma^1_{32} = 0}$

$$\Gamma^1_{33} = \frac{1}{2} g^{11} (\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}) = -\frac{1}{2} g^{11} \partial_1 g_{33}$$

$$\Rightarrow \Gamma^1_{33} = -\frac{1}{2B} \frac{\partial (r^2 \sin^2\theta)}{\partial r^2} = -\frac{r}{B} \sin^2\theta$$

$$\therefore \boxed{\Gamma^{1}_{33} = -\frac{r}{B}\sin^2\theta}$$

As g_{00} does not depend on r, θ, φ and $g_{\mu\nu}$ is diagonal, we can directly write:

$$\boxed{\Gamma^{2}_{00} = 0} \quad \boxed{\Gamma^{2}_{01} = \Gamma^{2}_{10} = 0} \quad \boxed{\Gamma^{2}_{03} = \Gamma^{2}_{30} = 0}$$

$$\boxed{\Gamma^{2}_{13} = \Gamma^{2}_{31} = 0} \quad \boxed{\Gamma^{3}_{00} = 0} \quad \boxed{\Gamma^{3}_{01} = \Gamma^{3}_{10} = 0}$$

$$\boxed{\Gamma^{3}_{02} = \Gamma^{3}_{20} = 0} \quad \boxed{\Gamma^{3}_{12} = \Gamma^{3}_{21} = 0}$$

Now as $g_{11} = B(r)$ does not depend on θ, φ:

$$\boxed{\Gamma^{2}_{11} = 0} \quad \boxed{\Gamma^{3}_{11} = 0}$$

Calculating the rest:

$$\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2} g^{22}\left(\cancelto{0}{\partial_2 g_{21}} + \partial_1 g_{22} - \cancelto{0}{\partial_2 g_{12}}\right)$$

$$\Rightarrow \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2} g^{22}\, \partial_1 g_{22} = \frac{1}{2r^2}\,\frac{\partial(r^2)}{\partial r}$$

$$\therefore \boxed{\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{r}}$$

$$\Gamma^{2}_{22} = \frac{1}{2} g^{22}\, \partial_2 g_{22} = \frac{1}{2} r^2 \frac{\partial(r^2)}{\partial \theta} = 0$$

$$\therefore \boxed{\Gamma^{2}_{22} = 0} \;;\; \Gamma^{2}_{23} = \Gamma^{2}_{32} = \frac{1}{2} g^{22}\, \partial_3 g_{22} = 0 \quad \therefore \boxed{\Gamma^{2}_{23} = \Gamma^{2}_{32} = 0}$$

$$\Gamma^{2}_{02} = \Gamma^{2}_{20} = \frac{1}{2} g^{22}\, \partial_0 g_{22} = 0$$

$$\therefore \boxed{\Gamma^{2}_{02} = \Gamma^{2}_{20} = 0} \qquad \Gamma^{2}_{33} = -\frac{1}{2} g^{22}\, \partial_2 g_{33} = -\frac{1}{2r^2}\, r^2\, 2\sin\theta\cos\theta$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{2} g^{33}\, \partial_0 g_{33} = 0$$

$$\therefore \boxed{\Gamma^3_{03} = \Gamma^3_{30} = 0} \qquad\qquad \therefore \boxed{\Gamma^2_{33} = -\sin\theta\cos\theta}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33} \left(\partial_3 \cancelto{0}{g_{31}} + \partial_1 g_{33} - \partial_3 \cancelto{0}{g_{13}}\right)$$

$$\Rightarrow \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33}\, \partial_1 g_{33} = \frac{1}{2r^2\sin^2\theta} \frac{\partial (r^2\sin^2\theta)}{\partial r}$$

$$\therefore \boxed{\Gamma^3_{13} = \Gamma^3_{31} = 1/r}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33}\, \partial_2 g_{33} = \frac{1}{2r^2\sin^2\theta} \frac{\partial (r^2\sin^2\theta)}{\partial \theta}$$

$$\therefore \boxed{\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta}$$

$$\Gamma^3_{33} = \frac{1}{2} g^{33}\, \partial_3 g_{33} = \frac{1}{2} (r^2\sin^2\theta) \frac{\partial (r^2\sin^2\theta)}{\partial \phi} = 0$$

$$\therefore \boxed{\Gamma^3_{33} = 0}$$

Now going back to expression of Ricci tensor (1):

$$-R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}$$

$$= \partial_0 \Gamma^0_{\mu\nu} + \partial_1 \Gamma^1_{\mu\nu} + \partial_2 \Gamma^2_{\mu\nu} + \partial_3 \Gamma^3_{\mu\nu}$$

$$- \partial_\nu \Gamma^0_{\mu 0} - \partial_\nu \Gamma^1_{\mu 1} - \partial_\nu \Gamma^2_{\mu 2} - \partial_\nu \Gamma^3_{\mu 3}$$

$$+ \Gamma^\beta_{\mu\nu}\Gamma^0_{\beta 0} + \Gamma^\beta_{\mu\nu}\Gamma^1_{\beta 1} + \Gamma^\beta_{\mu\nu}\Gamma^2_{\beta 2} + \Gamma^\beta_{\mu\nu}\Gamma^3_{\beta 3}$$

$$- \Gamma^\beta_{\mu 0}\Gamma^0_{\beta\nu} - \Gamma^\beta_{\mu 1}\Gamma^1_{\beta\nu} - \Gamma^\beta_{\mu 2}\Gamma^2_{\beta\nu} - \Gamma^\beta_{\mu 3}\Gamma^3_{\beta\nu}$$

(using the values of christoffel symbols calculated above)

$$
\begin{aligned}
\Rightarrow -R_{\mu\nu} = \; & \partial_0 \cancelto{0}{\Gamma^0_{\mu\nu}} + \partial_1 \Gamma^1_{\mu\nu} + \partial_2 \Gamma^2_{\mu\nu} + \partial_3 \Gamma^3_{\mu\nu} \\
& - \partial_\nu \cancelto{0}{\Gamma^0_{\mu 0}} - \partial_\nu \Gamma^1_{\mu 1} - \partial_\nu \Gamma^2_{\mu 2} - \partial_\nu \Gamma^3_{\mu 3} \\
& + \cancelto{0}{\Gamma^0_{\mu\nu}} \cancelto{0}{\Gamma^0_{00}} + \Gamma^1_{\mu\nu} \cancelto{0}{\Gamma^0_{10}} + \Gamma^2_{\mu\nu} \cancelto{0}{\Gamma^0_{20}} + \Gamma^3_{\mu\nu} \cancelto{0}{\Gamma^0_{30}} \\
& + \cancelto{0}{\Gamma^0_{\mu\nu}} \cancelto{0}{\Gamma^1_{01}} + \Gamma^1_{\mu\nu} \Gamma^1_{11} + \Gamma^2_{\mu\nu} \cancelto{0}{\Gamma^1_{21}} + \Gamma^3_{\mu\nu} \cancelto{0}{\Gamma^1_{31}} \\
& + \cancelto{0}{\Gamma^0_{\mu\nu}} \cancelto{0}{\Gamma^2_{02}} + \Gamma^1_{\mu\nu} \Gamma^2_{12} + \Gamma^2_{\mu\nu} \cancelto{0}{\Gamma^2_{22}} + \Gamma^3_{\mu\nu} \cancelto{0}{\Gamma^2_{32}} \\
& + \cancelto{0}{\Gamma^0_{\mu\nu}} \Gamma^3_{03} + \Gamma^1_{\mu\nu} \Gamma^3_{13} + \Gamma^2_{\mu\nu} \Gamma^3_{23} + \Gamma^3_{\mu\nu} \cancelto{0}{\Gamma^3_{33}} \\
& - \cancelto{0}{\Gamma^0_{\mu 0}} \cancelto{0}{\Gamma^0_{0\nu}} - \Gamma^1_{\mu 0} \cancelto{0}{\Gamma^0_{1\nu}} - \Gamma^2_{\mu 0} \cancelto{0}{\Gamma^0_{2\nu}} - \Gamma^3_{\mu 0} \cancelto{0}{\Gamma^0_{3\nu}} \\
& - \cancelto{0}{\Gamma^0_{\mu 1}} \Gamma^1_{0\nu} - \Gamma^1_{\mu 1} \Gamma^1_{1\nu} - \Gamma^2_{\mu 1} \Gamma^1_{2\nu} - \Gamma^3_{\mu 1} \Gamma^1_{3\nu} \\
& - \cancelto{0}{\Gamma^0_{\mu 2}} \Gamma^2_{0\nu} - \Gamma^1_{\mu 2} \Gamma^2_{1\nu} - \Gamma^2_{\mu 2} \Gamma^2_{2\nu} - \Gamma^3_{\mu 2} \Gamma^2_{3\nu} \\
& - \cancelto{0}{\Gamma^0_{\mu 3}} \Gamma^3_{0\nu} - \Gamma^1_{\mu 3} \Gamma^3_{1\nu} - \Gamma^2_{\mu 3} \Gamma^3_{2\nu} - \Gamma^3_{\mu 3} \Gamma^3_{3\nu}
\end{aligned}
$$

Thus, we obtain :

$$
\boxed{
\begin{aligned}
R_{\mu\nu} = \; & -\partial_1 \Gamma^1_{\mu\nu} - \partial_2 \Gamma^2_{\mu\nu} - \partial_3 \Gamma^3_{\mu\nu} + \partial_\nu \Gamma^1_{\mu 1} + \partial_\nu \Gamma^2_{\mu 2} + \partial_\nu \Gamma^3_{\mu 3} \\
& - \Gamma^1_{\mu\nu} \Gamma^1_{11} - \Gamma^1_{\mu\nu} \Gamma^2_{12} - \Gamma^1_{\mu\nu} \Gamma^3_{13} - \Gamma^2_{\mu\nu} \Gamma^3_{23} \\
& + \Gamma^1_{\mu 1} \Gamma^1_{1\nu} + \Gamma^2_{\mu 1} \Gamma^1_{2\nu} + \Gamma^3_{\mu 1} \Gamma^1_{3\nu} \\
& + \Gamma^1_{\mu 2} \Gamma^2_{1\nu} + \Gamma^2_{\mu 2} \Gamma^2_{2\nu} + \Gamma^3_{\mu 2} \Gamma^2_{3\nu} \\
& + \Gamma^1_{\mu 3} \Gamma^3_{1\nu} + \Gamma^2_{\mu 3} \Gamma^3_{2\nu} + \Gamma^3_{\mu 3} \Gamma^3_{3\nu}
\end{aligned}
}
$$

— (11)

Now we can calculate :

$$-R_{00} = \partial_1 \cancelto{0}{\Gamma^1_{00}} + \partial_2 \cancelto{0}{\Gamma^2_{00}} + \partial_3 \cancelto{0}{\Gamma^3_{00}} - \partial_0 \cancelto{0}{\Gamma^1_{01}} - \partial_0 \cancelto{0}{\Gamma^2_{02}} - \partial_0 \cancelto{0}{\Gamma^3_{03}}$$

$$+ \cancelto{0}{\Gamma^1_{00}} \Gamma^1_{11} + \cancelto{0}{\Gamma^1_{00}} \Gamma^2_{12} + \cancelto{0}{\Gamma^1_{00}} \Gamma^3_{13} + \cancelto{0}{\Gamma^2_{00}} \Gamma^3_{23}$$

$$- \cancelto{0}{\Gamma^1_{01}} \cancelto{0}{\Gamma^1_{10}} - \cancelto{0}{\Gamma^2_{01}} \cancelto{0}{\Gamma^1_{20}} - \cancelto{0}{\Gamma^3_{01}} \cancelto{0}{\Gamma^1_{30}}$$

$$- \cancelto{0}{\Gamma^1_{02}} \cancelto{0}{\Gamma^2_{10}} - \cancelto{0}{\Gamma^2_{02}} \cancelto{0}{\Gamma^2_{20}} - \cancelto{0}{\Gamma^3_{02}} \cancelto{0}{\Gamma^2_{30}}$$

$$- \cancelto{0}{\Gamma^1_{03}} \cancelto{0}{\Gamma^3_{10}} - \cancelto{0}{\Gamma^2_{03}} \cancelto{0}{\Gamma^3_{20}} - \cancelto{0}{\Gamma^3_{03}} \cancelto{0}{\Gamma^3_{30}}$$

$$\therefore \quad \boxed{R_{00} = 0}$$

$$-R_{01} = \partial_1 \cancelto{0}{\Gamma^1_{01}} + \partial_2 \cancelto{0}{\Gamma^2_{01}} + \partial_3 \cancelto{0}{\Gamma^3_{01}} - \partial_1 \cancelto{0}{\Gamma^1_{01}} - \partial_1 \cancelto{0}{\Gamma^2_{02}} - \partial_1 \cancelto{0}{\Gamma^3_{03}}$$

$$+ \cancelto{0}{\Gamma^1_{01}} \Gamma^1_{11} + \cancelto{0}{\Gamma^1_{01}} \Gamma^2_{12} + \cancelto{0}{\Gamma^1_{01}} \Gamma^3_{13} + \cancelto{0}{\Gamma^2_{01}} \Gamma^3_{23}$$

$$- \cancelto{0}{\Gamma^1_{01}} \Gamma^1_{11} - \cancelto{0}{\Gamma^2_{01}} \cancelto{0}{\Gamma^1_{21}} - \cancelto{0}{\Gamma^3_{01}} \cancelto{0}{\Gamma^1_{31}}$$

$$- \cancelto{0}{\Gamma^1_{02}} \cancelto{0}{\Gamma^2_{11}} - \cancelto{0}{\Gamma^2_{02}} \Gamma^2_{21} - \cancelto{0}{\Gamma^3_{02}} \cancelto{0}{\Gamma^2_{31}}$$

$$- \cancelto{0}{\Gamma^1_{03}} \cancelto{0}{\Gamma^3_{11}} - \cancelto{0}{\Gamma^2_{03}} \cancelto{0}{\Gamma^3_{21}} - \cancelto{0}{\Gamma^3_{03}} \Gamma^3_{31}$$

$$\therefore \quad \boxed{R_{01} = 0 = R_{10}}$$

$$-R_{02} = \partial_1 \cancelto{0}{\Gamma^1_{02}} + \partial_2 \cancelto{0}{\Gamma^2_{02}} + \partial_3 \cancelto{0}{\Gamma^3_{02}} - \partial_2 \cancelto{0}{\Gamma^1_{01}} - \partial_2 \cancelto{0}{\Gamma^2_{02}} - \partial_2 \cancelto{0}{\Gamma^3_{03}}$$

$$+ \cancelto{0}{\Gamma^1_{02}} \Gamma^1_{11} + \cancelto{0}{\Gamma^1_{02}} \Gamma^2_{12} + \cancelto{0}{\Gamma^1_{02}} \Gamma^3_{13} + \cancelto{0}{\Gamma^2_{02}} \Gamma^3_{23}$$

$$- \cancelto{0}{\Gamma^1_{01}} \Gamma^1_{12} - \cancelto{0}{\Gamma^2_{01}} \Gamma^1_{22} - \cancelto{0}{\Gamma^3_{01}} \Gamma^1_{32}$$

$$- \cancelto{0}{\Gamma^1_{02}} \Gamma^2_{12} - \cancelto{0}{\Gamma^2_{02}} \cancelto{0}{\Gamma^2_{22}} - \cancelto{0}{\Gamma^3_{02}} \Gamma^2_{32}$$

$$-\cancelto{0}{\Gamma^1_{03}}\,\Gamma^3_{12} - \cancelto{0}{\Gamma^2_{03}}\,\Gamma^3_{22} - \cancelto{0}{\Gamma^3_{03}}\,\Gamma^3_{32}$$

$$\therefore \quad \boxed{R_{02} = 0 = R_{20}}$$

$$-R_{03} = \partial_1 \cancelto{0}{\Gamma^1_{03}} + \partial_2 \cancelto{0}{\Gamma^2_{03}} + \partial_3 \cancelto{0}{\Gamma^3_{03}} - \partial_3 \cancelto{0}{\Gamma^1_{01}} - \partial_3 \cancelto{0}{\Gamma^2_{02}} - \partial_3 \cancelto{0}{\Gamma^3_{03}}$$

$$+\cancelto{0}{\Gamma^1_{03}}\,\Gamma^1_{11} + \cancelto{0}{\Gamma^1_{03}}\,\Gamma^2_{12} + \cancelto{0}{\Gamma^1_{03}}\,\Gamma^3_{13} + \cancelto{0}{\Gamma^2_{03}}\,\Gamma^3_{23}$$

$$-\cancelto{0}{\Gamma^1_{01}}\,\Gamma^1_{13} - \cancelto{0}{\Gamma^2_{01}}\,\Gamma^1_{23} - \cancelto{0}{\Gamma^3_{01}}\,\Gamma^1_{33}$$

$$-\cancelto{0}{\Gamma^1_{02}}\,\Gamma^2_{13} - \cancelto{0}{\Gamma^2_{02}}\,\Gamma^2_{23} - \cancelto{0}{\Gamma^3_{02}}\,\Gamma^2_{33}$$

$$-\cancelto{0}{\Gamma^1_{03}}\,\Gamma^3_{13} \quad \cancelto{0}{\Gamma^2_{03}}\,\Gamma^3_{23} - \cancelto{0}{\Gamma^3_{03}}\,\Gamma^3_{33}$$

$$\therefore \quad \boxed{R_{03} = 0 = R_{30}}$$

$$-R_{11} = \cancel{\partial_1 \Gamma^1_{11}} + \partial_2 \cancelto{0}{\Gamma^2_{11}} + \partial_3 \cancelto{0}{\Gamma^3_{11}} - \cancel{\partial_1 \Gamma^1_{11}} - \partial_1 \Gamma^2_{12} - \partial_1 \Gamma^3_{13}$$

$$+\cancel{\Gamma^1_{11}\Gamma^1_{11}} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^1_{11}\Gamma^3_{13} + \cancelto{0}{\Gamma^2_{11}}\,\Gamma^3_{23}$$

$$-\cancel{\Gamma^1_{11}\Gamma^1_{11}} - \cancelto{0}{\Gamma^2_{11}}\,\Gamma^1_{21} - \cancelto{0}{\Gamma^3_{11}}\,\Gamma^1_{31}$$

$$-\cancelto{0}{\Gamma^1_{12}}\,\cancelto{0}{\Gamma^2_{11}} - \Gamma^2_{12}\Gamma^2_{21} - \cancelto{0}{\Gamma^3_{12}}\,\Gamma^2_{31}$$

$$-\cancelto{0}{\Gamma^1_{13}}\,\cancelto{0}{\Gamma^3_{11}} - \cancelto{0}{\Gamma^2_{13}}\,\cancelto{0}{\Gamma^3_{21}} - \Gamma^3_{13}\Gamma^3_{31}$$

$$\Rightarrow -R_{11} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\partial}{\partial r}\left(1/r\right) + \frac{1}{2B}\frac{dB}{dr}\left(\frac{2}{r}\right) - \frac{2}{r^2}$$

$$\therefore \quad \boxed{R_{11} = \frac{-1}{rB}\frac{dB}{dr}}$$ Ans

$$-R_{12} = \partial_1 \cancelto{0}{\Gamma^1_{12}} + \cancel{\partial_2 \Gamma^2_{12}} + \partial_3 \cancelto{0}{\Gamma^3_{12}} - \partial_2 \cancelto{0}{\Gamma^1_{11}} - \cancel{\partial_2 \Gamma^2_{12}} - \cancelto{0}{\partial_2 \Gamma^3_{13}}$$

$$+ \cancelto{0}{\Gamma^1_{12}}\Gamma^1_{11} + \cancelto{0}{\Gamma^1_{12}}\Gamma^2_{12} + \cancelto{0}{\Gamma^1_{12}}\Gamma^3_{13} + \cancel{\Gamma^2_{12}\Gamma^3_{23}}$$

$$- \Gamma^1_{11}\cancelto{0}{\Gamma^1_{12}} - \cancelto{0}{\Gamma^2_{11}}\Gamma^1_{22} - \Gamma^3_{11}\cancelto{0}{\Gamma^1_{32}}$$

$$- \cancelto{0}{\Gamma^1_{12}}\Gamma^2_{12} - \Gamma^2_{12}\cancelto{0}{\Gamma^2_{22}} - \cancelto{0}{\Gamma^3_{12}}\Gamma^2_{32}$$

$$- \cancelto{0}{\Gamma^1_{13}}\Gamma^3_{12} - \cancelto{0}{\Gamma^2_{13}}\Gamma^3_{22} - \Gamma^3_{13}\cancel{\Gamma^3_{32}}$$

as Γ^1_{11}, Γ^3_{13} are independent of θ, and $\Gamma^2_{12} = \Gamma^3_{13}$ everything gets cancelled.

$$\therefore \quad \boxed{R_{12} = R_{21} = 0}$$

$$-R_{13} = \partial_1 \cancelto{0}{\Gamma^1_{13}} + \partial_2 \cancelto{0}{\Gamma^2_{13}} + \cancel{\partial_3 \Gamma^3_{13}} - \partial_3 \cancelto{0}{\Gamma^1_{11}} - \cancel{\partial_3 \Gamma^2_{12}} - \cancel{\partial_3 \Gamma^3_{13}}$$

$$+ \cancelto{0}{\Gamma^1_{13}}\Gamma^1_{11} + \cancelto{0}{\Gamma^1_{13}}\Gamma^2_{12} + \cancelto{0}{\Gamma^1_{13}}\Gamma^3_{13} + \cancelto{0}{\Gamma^2_{13}}\Gamma^3_{23}$$

$$- \Gamma^1_{11}\cancelto{0}{\Gamma^1_{13}} - \cancelto{0}{\Gamma^2_{11}}\Gamma^1_{23} - \cancelto{0}{\Gamma^3_{11}}\Gamma^1_{33}$$

$$- \Gamma^1_{12}\cancelto{0}{\Gamma^2_{13}} - \Gamma^2_{12}\cancelto{0}{\Gamma^2_{23}} - \cancelto{0}{\Gamma^3_{12}}\Gamma^2_{33}$$

$$- \cancelto{0}{\Gamma^1_{13}}\Gamma^3_{13} - \cancelto{0}{\Gamma^2_{13}}\Gamma^3_{23} - \Gamma^3_{13}\cancelto{0}{\Gamma^3_{33}}$$

$$\therefore \quad \boxed{R_{13} = R_{31} = 0}$$

$$-R_{22} = \partial_1 \Gamma^1_{22} + \partial_2 \cancelto{0}{\Gamma^2_{22}} + \partial_3 \cancelto{0}{\Gamma^3_{22}} - \partial_2 \cancelto{0}{\Gamma^1_{21}} - \partial_2 \cancelto{0}{\Gamma^2_{22}} - \partial_2 \Gamma^3_{23}$$

$$+ \Gamma^1_{22}\Gamma^1_{11} + \cancel{\Gamma^1_{22}\Gamma^2_{12}} + \cancel{\Gamma^1_{22}\Gamma^3_{13}} + \cancelto{0}{\Gamma^2_{23}}\Gamma^3_{23}$$

$$- \cancelto{0}{\Gamma^1_{21}}\cancelto{0}{\Gamma^1_{12}} - \cancel{\Gamma^2_{21}\Gamma^1_{22}} - \cancelto{0}{\Gamma^3_{21}}\cancelto{0}{\Gamma^1_{32}}$$

$$- \cancel{\Gamma^1_{22}\Gamma^2_{12}} - \cancel{\Gamma^2_{22}}^{\to 0}\cancel{\Gamma^2_{22}}^{\to 0} - \cancel{\Gamma^3_{22}}^{\to 0}\cancel{\Gamma^2_{32}}^{\to 0}$$

$$- \cancel{\Gamma^1_{23}}^{\to 0}\cancel{\Gamma^3_{12}}^{\to 0} - \cancel{\Gamma^2_{23}}^{\to 0}\cancel{\Gamma^3_{22}}^{\to 0} - \Gamma^3_{23}\Gamma^3_{32}$$

$$\Rightarrow R_{22} = -\frac{\partial}{\partial r}\left(-r/B\right) + \frac{\partial}{\partial \theta}(\cot\theta) - \left(\frac{-r}{B}\right)\left(\frac{1}{2B}\frac{dB}{dr}\right) + \cot^2\theta$$

$$\Rightarrow R_{22} = \frac{1}{B} - \operatorname{cosec}^2\theta + \cot^2\theta + \frac{r}{2B^2}\frac{dB}{dr} - \frac{r}{B^2}\frac{dB}{dr}$$

$$\therefore \boxed{R_{22} = \frac{1}{B} - 1 - \frac{r}{2B^2}\frac{dB}{dr}}$$ Ans

$$-R_{33} = \partial_1\Gamma^1_{33} + \partial_2\Gamma^2_{33} + \cancel{\partial_3\Gamma^3_{33}}^{\to 0} - \cancel{\partial_3\Gamma^1_{31}}^{\to 0} - \cancel{\partial_3\Gamma^2_{32}}^{\to 0} - \cancel{\partial_3\Gamma^3_{23}}^{\to 0}$$

$$+ \Gamma^1_{33}\Gamma^1_{11} + \cancel{\Gamma^1_{33}\Gamma^2_{12}} + \Gamma^1_{33}\Gamma^3_{13} + \cancel{\Gamma^2_{33}\Gamma^3_{23}}$$

$$- \cancel{\Gamma^1_{31}}^{\to 0}\cancel{\Gamma^1_{13}}^{\to 0} - \cancel{\Gamma^2_{31}}^{\to 0}\cancel{\Gamma^1_{23}}^{\to 0} - \cancel{\Gamma^3_{31}\Gamma^1_{33}}$$

$$- \cancel{\Gamma^1_{32}}^{\to 0}\cancel{\Gamma^2_{13}}^{\to 0} - \cancel{\Gamma^2_{32}}^{\to 0}\cancel{\Gamma^2_{23}}^{\to 0} - \Gamma^3_{32}\Gamma^2_{33}$$

$$- \cancel{\Gamma^1_{33}\Gamma^3_{13}} - \cancel{\Gamma^2_{33}\Gamma^3_{23}} - \cancel{\Gamma^3_{33}}^{\to 0}\cancel{\Gamma^3_{33}}^{\to 0}$$

$$\Rightarrow -R_{33} = \frac{\partial}{\partial r}\left(-r\sin^2\theta/B\right) + \frac{\partial}{\partial\theta}(-\sin\theta\cos\theta) - \frac{r\sin^2\theta}{B}\cdot\frac{1}{2B}\cdot\frac{dB}{dr} - (\cot\theta)(-\sin\theta\cos\theta)$$

$$\Rightarrow -R_{33} = \frac{-\sin^2\theta}{B} - \cos^2\theta + \sin^2\theta + \frac{r}{2B^2}\frac{dB}{dr}\sin^2\theta + \cos^2\theta$$

$$\Rightarrow R_{33} = \left(\frac{1}{B} - 1 - \frac{r}{2B^2}\frac{dB}{dr}\right)\sin^2\theta$$

$$\therefore \boxed{R_{33} = R_{22}\sin^2\theta} \quad \text{Ans}$$

For maximally symmetric metric:

$$R_{ij} = -2K g_{ij}$$

Thus,

$$R_{11} = -2K g_{11}$$

$$\Rightarrow -\frac{1}{rB}\frac{dB}{dr} = -2K \cdot B$$

$$\therefore \frac{1}{rB}\frac{dB}{dr} = 2KB \text{ — (IV)} \qquad \Rightarrow \frac{1}{2B^2}\frac{dB}{dr} = Kr$$

$$R_{22} = -2K g_{22}$$

$$\Rightarrow \frac{1}{B} - 1 - \frac{r}{2B^2}\frac{dB}{dr} = -2Kr^2$$

$$\therefore 2Kr^2 = 1 + \frac{r}{2B^2}\frac{dB}{dr} - \frac{1}{B} \Rightarrow 2Kr^2 = 1 + Kr^2 - \frac{1}{B}$$

$$\frac{dB}{B^2} = 2K r\, dr \Rightarrow -\frac{1}{B} = Kr^2 - A$$

$$\therefore \boxed{B(r) = \frac{1}{A - Kr^2}} \qquad \therefore \boxed{A = 1} \quad \text{Ans}$$

Answer 14.6) The Redshift is given as:

$$z = \frac{a(t_0)}{a(t)} - 1 \quad \text{— (i)}$$

$$a(t) = a[t_0 - (t_0 - t)]$$

Making a power series expansion:

$$a(t) = a(t_0) - (t_0 - t)\dot{a}(t_0) + \frac{1}{2}(t_0 - t)^2 \ddot{a}(t_0) \ldots$$

Now Hubble parameter is given as: $H(t) = \frac{\dot{a}(t)}{a(t)}$

$$\Rightarrow a(t) = a(t_0)\left\{1 - (t_0 - t)\frac{\dot{a}(t_0)}{a(t_0)} + \frac{1}{2}(t_0 - t)\frac{\ddot{a}(t_0)}{a(t_0)} \ldots\right\}$$

$$\Rightarrow a(t) = a(t_0)\left\{1 - (t_0 - t)H(t_0) - \frac{1}{2}(t_0 - t)H^2(t_0)\left(-\frac{\ddot{a}(t_0)a(t_0)}{\dot{a}^2(t_0)}\right)\ldots\right\}$$

Defining: $q(t) = \frac{-\ddot{a}(t)\,a(t)}{\dot{a}^2(t)}$

$$\therefore a(t) = a(t_0)\left\{1 - (t_0 - t)H(t_0) - \frac{1}{2}q(t_0)(t_0 - t)^2 H^2(t_0) \ldots\right\} \quad \text{— (ii)}$$

Using this in (i):

$$z = \left[1 - (t_0 - t)H(t_0) - \frac{1}{2}q(t_0)(t_0 - t)^2 H^2(t_0) \ldots\right]^{-1} - 1$$

For $t_0 - t \ll t_0$, we obtain:

(PTO)

$$z = \left\{ 1 + (t_0 - t) H(t_0) + (t_0 - t)^2 \left(1 + \frac{q_0}{2}\right) H^2(t_0) + \cdots \right\} - 1$$

let $H(t_0) \equiv H_0$

$$\Rightarrow z = (t_0 - t) H_0 + (t_0 - t)^2 \left(1 + \frac{q_0}{2}\right) H_0^2 + \cdots \quad \text{— (iii)}$$

$$\therefore \boxed{(t_0 - t) = H_0^{-1} \left\{ z - \left(\frac{1 + q_0}{2}\right) z^2 \cdots \right\}} \quad \underline{\text{Ans}}$$

Now we can write :

$$\frac{dz}{dt} = -H_0 - 2(t_0 - t)\left(1 + \frac{q_0}{2}\right) H_0^2 + \cdots$$

Using previous answer :

$$\frac{dz}{dt} = -H_0 \left\{ 1 + 2z\left(1 + \frac{q_0}{2}\right) - 2\left(1 + \frac{q_0}{2}\right)^2 z^2 \cdots \right\}$$

$$H(z) = \frac{\dot{a}}{a} = \frac{-\dot{z}(1+z)}{(1+z)^2} = \frac{-\dot{z}}{(1+z)} = -\dot{z}(1+z)^{-1}$$

For $z \ll 1$,

$$H(z) = -\dot{z}(1 - z)$$

$$-\dot{z}(1 - z) = H_0 \left\{ 1 + 2z\left(1 + \frac{q_0}{2}\right) \cdots \right\}(1 - z)$$

$$\Rightarrow -\dot{z}(1 - z) = H_0 \left\{ (1 - z) + 2z(1 - z)\left(1 + \frac{q_0}{2}\right) \cdots \right\}$$

$$\boxed{H(z) = H_0 \left[1 + (1 + q_0) z \cdots \right]} \quad \underline{\text{Ans}}$$

Answer 14.8) Given :

$$t_0 - t = \frac{d}{c} - \frac{H_0 d^2}{2c^2} + \cdots$$

Now from (iii) in question 14.6, we know:

$$z = (t_0 - t) H_0 + (t_0 - t)^2 \left(1 + \frac{q_0}{2}\right) H_0^2 + \cdots$$

Now, as distances are nearby,

$$v = cz \simeq H_0 d \quad \text{as } d \simeq c(t_0 - t), \quad z \simeq (t_0 - t) H_0$$

$$(t - t_0) \simeq \frac{d}{c} \; ; \quad (t - t_0)^2 \simeq \frac{d^2}{c^2} \quad \text{and so on:}$$

$$\therefore \boxed{z \simeq \frac{H_0 d}{c} + \left(1 + \frac{q_0}{2}\right) \frac{H_0^2 d^2}{c^2} + \cdots} \quad \text{Ans}$$

Notes ques (Pg18, week 7) we have:

$$x = \int_t^{t_0} c a_0^{-1} \left[1 - (t_0 - t) H_0 - \cdots \right]^{-1} dt$$

$$\Rightarrow \quad x = c a_0^{-1} \left\{ (t_0 - t) + \frac{1}{2} (t_0 - t)^2 H_0 + \cdots \right\}$$

$$\Rightarrow \quad x = c a_0^{-1} (t_0 - t) \left\{ 1 + \frac{1}{2} (t_0 - t) H_0 + \cdots \right\}$$

Now from ques 14.6, we know:

$$t_0 - t = H_0^{-1} \left\{ z - \left(1 + \frac{q_0}{2}\right) z^2 + \cdots \right\} \quad \text{for } z \ll 1$$

$$\Rightarrow (t_0 - t) = H_0^{-1} z \left\{ 1 - \left(1 + \frac{q_0}{2}\right) z + \cdots \right\}$$

$$\Rightarrow (t_0 - t)^2 \simeq H_0^{-2} z^2 \left\{ 1 - 2\left(1 + \frac{q_0}{2}\right) z + \cdots \right\}$$

Putting this in x:

$$\Rightarrow x = \frac{c}{a_0} \left[H_0^{-1} \left\{ z - \left(1 + \frac{q_0}{2}\right) z^2 + \cdots \right\} + \frac{H_0^{-1} z^2}{2} \left\{ 1 - 2\left(1 + \frac{q_0}{2}\right) z \cdots \right\} \right]$$

$$\Rightarrow x = \frac{c}{a_0 H_0} \left[z - \left(1 + \frac{q_0}{2}\right) z^2 + \frac{z^2}{2} - \left(1 + \frac{q_0}{2}\right) z^3 + \cdots \right]$$

$$\frac{z^2}{2} - \left(1 + \frac{q_0}{2}\right) z^2 = \frac{z^2}{2} - z^2 - \frac{q_0}{2} z^2 = -\frac{z^2}{2} - \frac{q_0}{2} z^2$$

$$\therefore \boxed{x = \frac{c}{a_0 H_0} \left\{ z - \frac{1}{2} (1 + q_0) z^2 + \cdots \right\}}$$ Ans