ASSIGNMENT WEEK 4

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Answer 1.) $\hat{a}(t) = \dfrac{a(t)}{a_0}$ [Recreate Fig 1.1 of James Rich Book]

Friedman equation :

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_M \hat{a}^{-3} + \Omega_R \hat{a}^{-4} + \Omega_\Lambda + (1-\Omega_T)\hat{a}^{-2}\right]$$

For a matter dominated Universe, $\Omega_R = 0$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_M\left\{\frac{a_0}{a(t)}\right\}^3 + \Omega_\Lambda + (1-\Omega_T)\left\{\frac{a_0}{a(t)}\right\}^2\right]$$

a.) Let $y(t) = \dfrac{a(t)}{a_0} \Rightarrow \dot{y} = \dfrac{\dot{a}(t)}{a_0} \Rightarrow \dfrac{\dot{y}}{y} = \dfrac{\dot{a}}{a}$

$$\Rightarrow \left(\frac{\dot{y}}{y}\right)^2 = H_0^2\left[\frac{\Omega_M}{y^3} + \Omega_\Lambda + \frac{(1-\Omega_T)}{y^2}\right]$$

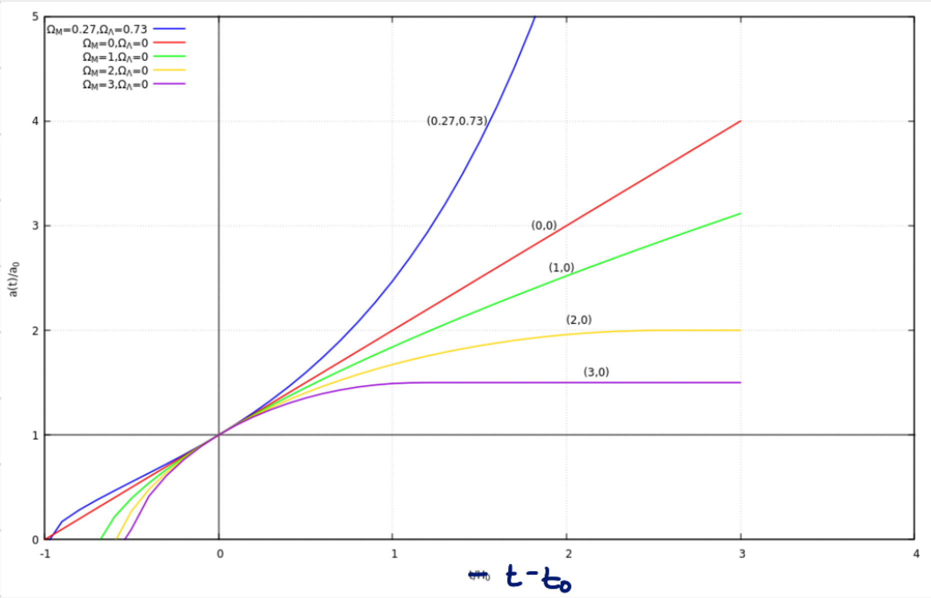
$$\Rightarrow \dot{y}^2 = H_0^2\left[\frac{\Omega_M}{y} + \Omega_\Lambda y^2 + (1-\Omega_T)\right]$$

$$\therefore \dot{y} = \left[\frac{\Omega_M}{y} + \Omega_\Lambda y^2 + (1-\Omega_T)\right]^{1/2}$$

We have used RK4 method to numerically solve the above equation.

$$\dot{y} = \frac{dy}{d(t/H_0)}$$

We then plotted y vs $t - t_0$ in Gnuplot.



For $\Omega > 1$, the Universe will collapse after some time due to the action of gravity.

Answer 3.) $$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_M \hat{a}^{-3} + \Omega_R \hat{a}^{-4} + \Omega_\Lambda + (1-\Omega_T)\hat{a}^{-2}\right]$$

$$\hat{a} = \frac{a(t)}{a_0}$$

For a Universe dominated with cosmological constant and matter :

$$\Omega_M = 0.30, \quad \Omega_\Lambda = 0.70, \quad \Omega_R = 0$$

$$\Omega_T = \Omega_M + \Omega_\Lambda + \Omega_R = 1$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[0.30\,\hat{a}^{-3} + 0 + 0.70 + 0\right]$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[0.30\,\frac{a_0^3}{a^3} + 0.70\right]$$

$$\Rightarrow \dot{a}^2 = H_0^2\left[0.30 \times \frac{a_0^3}{a} + 0.70 \times a^2\right]$$

$$a_0 = a(t_0) = 1$$

$$\Rightarrow \dot{a}^2 = H_0^2\left[\frac{0.3}{a} + 0.7a^2\right]$$

$$\therefore \frac{da}{dt} = H_0\left(\frac{0.3}{a} + 0.7a^2\right)^{1/2}$$

Integrating this we obtain :

$$\left\{\frac{2\sqrt{10}\,\ln\left[\sqrt{7a^3+3} + \sqrt{7}\,a^{3/2}\right]}{3\sqrt{7}}\right\}_0^{a_0=1} = H_0\left[t\right]_0^{t_0}$$

At $t = t_0$, $a(t) = a_0 = 1$. Putting these values :-

$$\Rightarrow 0.964 = H_0 t_0$$

$$\Rightarrow t_0 = \frac{1}{H_0} \times 0.964 = 13.6 \times 0.964 = 13.11 \text{ Byr}$$

$\therefore$ $\boxed{\text{Age of the Universe} = 13.11 \text{ Byr}}$ Ans

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Answer 2a) [Question 1.1 of James Rich book.]

$$\Omega_I(t) = \frac{\rho_I(t)}{3\left(\frac{\dot{a}}{a}\right)^2/8\pi^2 G}$$ [Eq. 1.101 of James Rich]

$I = T, M, R, \Lambda \ldots$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_M \hat{a}^{-3} + \Omega_R \hat{a}^{-4} + \Omega_\Lambda + (1-\Omega_T)\hat{a}^{-2}\right]$$

Case I. $\boxed{\Omega_M = \Omega_T = 1}$ $\Omega_R = 0, \; \Omega_\Lambda = 0$

$$\therefore \left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \hat{a}^{-3}$$

$$\Rightarrow \Omega_I(t) = \frac{\rho_I(t)}{3\left(\frac{H_0^2}{\hat{a}^3}\right)/8\pi G} = \hat{a}^3 \frac{\rho_I}{\rho_c}$$

Now :

i. $\Omega_M(a) = \hat{a}^3 \frac{\rho_M(a)}{\rho_c}$; $\rho_M(a) = \Omega_M \frac{3H_0^2}{2\pi G} \hat{a}^{-3}$

$$\Rightarrow \Omega_M(a) = \hat{a}^3 \cdot \frac{1}{\rho_c} \cdot \Omega_M \rho_c \hat{a}^{-3} = 1$$

$$\therefore \boxed{\Omega_M(a) = 1}$$

ii. $\Omega_\Lambda(a) = \hat{a}^3 \frac{\rho_\Lambda(a)}{\rho_c}$; $\rho_\Lambda(a) = \frac{3H_0^2}{8\pi G}\Omega_\Lambda = \rho_c \Omega_\Lambda = 0$

$$\Rightarrow \Omega_\Lambda(a) = \hat{a}^3 \Omega_\Lambda = 0$$

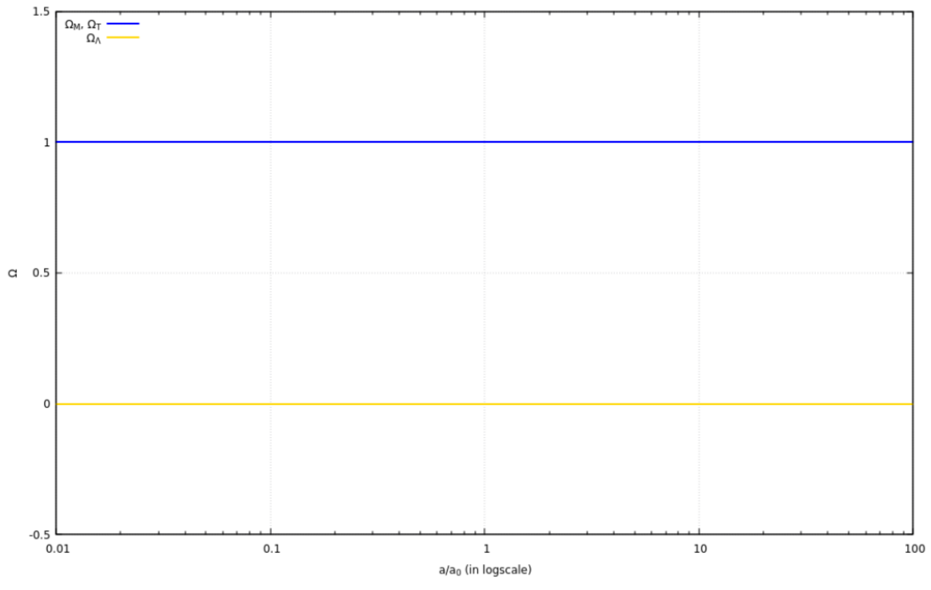
$$\therefore \boxed{\Omega_\Lambda(a) = 0}$$

iii. $\Omega_T(a) = \hat{a}^3 \frac{\rho_T(a)}{\rho_c}$; $\rho_T(a) = \rho_M(a) + \rho_\Lambda(a) = \rho_M(a)$

$$\Rightarrow \Omega_T(a) = \hat{a}^3 \frac{\rho_M(a)}{\rho_c} = \hat{a}^3 \cdot \frac{1}{\rho_c} \cdot \rho_c \Omega_M \hat{a}^{-3} = \Omega_M = 1$$

$$\therefore \boxed{\Omega_M(a) = 1}$$

Plot for $\Omega_M = \Omega_T = 1$



Case II. $\left[\Omega_M = \Omega_T = 0.3\right]$ $\quad \Omega_R = \Omega_T = 0$

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left[0.3\,\hat a^{-3} + 0.7\,\hat a^{-2}\right]$$

$$\Rightarrow \quad \Omega_I(t) = \frac{\rho_I(t)}{3H_0^2\left(0.3\hat a^{-3} + 0.7\hat a^{-2}\right)/8\pi G}$$

$$\therefore \quad \Omega_I(t) = \left(0.3\hat a^{-3} + 0.7\hat a^{-2}\right)^{-1}\frac{\rho_I(t)}{\rho_c}$$

i. $\rho_M(a) = \hat a^{-3}\Omega_M\rho_c = 0.3\hat a^{-3}\rho_c$

$$\Omega_M(t) = \frac{\hat a^{-3}\Omega_M}{0.3\hat a^{-3} + 0.7\hat a^{-2}} = \frac{0.3\,\hat a^{-3}}{0.3\hat a^{-3} + 0.7\hat a^{-2}}$$

$$\Rightarrow \quad \Omega_M(t) = \frac{0.3\,\hat a^{-3}}{\frac{0.3}{\hat a^3} + \frac{0.7}{\hat a^2}} = \frac{0.3\,\hat a^{-3}\cdot\hat a^3}{0.3 + 0.7\hat a} = \frac{0.3}{0.3 + 0.7\hat a}$$

$$\therefore \quad \boxed{\Omega_M(a) = \frac{0.3}{0.3 + 0.7a}}$$

ii. $\rho_\Lambda(a) = \rho_c \Omega_\Lambda = 0$

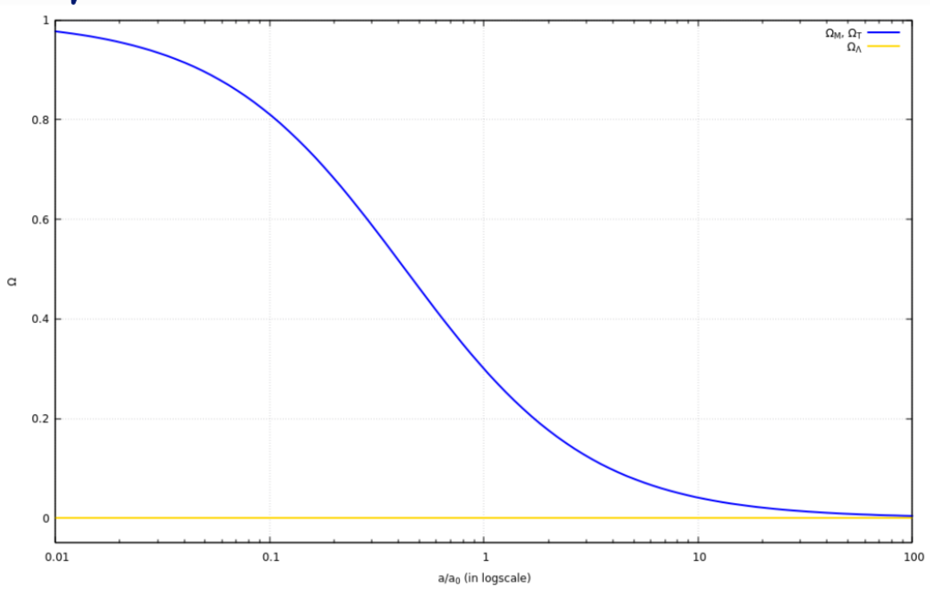
$\therefore$ $\boxed{\Omega_\Lambda(a) = 0}$

iii. $\rho_T(a) = \rho_M(a) + \rho_\Lambda(a) = \rho_M(a) = 0.3\hat{a}^{-3}\rho_c$

$$\Omega_T(a) = \frac{\rho_M}{\rho_c} \cdot (0.3\hat{a}^{-3} + 0.7\hat{a}^{-2})^{-1}$$

$\therefore$ $$\boxed{\Omega_T(a) = \Omega_M(a) = \frac{0.3}{0.3+0.7a}}$$

Plot for $\Omega_M = \Omega_T = 0.3$



Case III. $[\Omega_M = 0.3, \Omega_\Lambda = 0.7]$ $\Omega_T = 1$; $\Omega_R = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[0.3\hat{a}^{-3} + 0.7\right]$$

$\therefore$ $$\Omega_I(t) = \frac{\rho_I(t)}{3H_0^2[0.3\hat{a}^{-3}+0.7]/8\pi G} = (0.3\hat{a}^{-3}+0.7)^{-1}\frac{\rho_I(t)}{\rho_c}$$

i. $\rho_M(a) = \hat{a}^{-3}\Omega_M \rho_c$

$\therefore$ $\rho_M(a) = 0.3\hat{a}^{-3}\rho_c$

$$\Rightarrow \quad \Omega_M(a) = \frac{0.3\hat{a}^{-3}}{0.3\hat{a}^{-3} + 0.7} = \frac{0.3\hat{a}^{-3} \cdot \hat{a}^{3}}{0.3 + 0.7\hat{a}^{3}}$$

$$\therefore \quad \boxed{\Omega_M(a) = \frac{0.3}{0.3 + 0.7\hat{a}^{3}}}$$

ii. $\rho_\Lambda(a) = \rho_c \Omega_\Lambda = 0.7\rho_c$

$$\Rightarrow \quad \Omega_\Lambda(a) = \frac{0.7}{0.3\hat{a}^{-3} + 0.7} = \frac{0.7\hat{a}^{3}}{0.3 + 0.7\hat{a}^{3}}$$

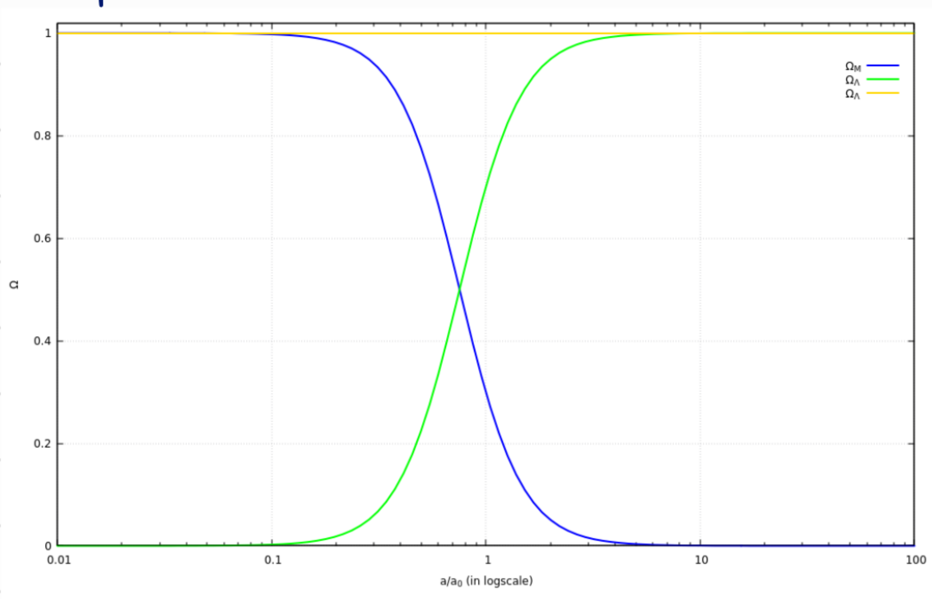
$$\therefore \quad \boxed{\Omega_\Lambda(a) = \frac{0.7\hat{a}^{3}}{0.3 + 0.7\hat{a}^{3}}}$$

iii. $\rho_T(a) = \rho_\Lambda(a) + \rho_M(a) = 0.7\rho_c + 0.3\hat{a}^{-3}\rho_c$

$$\Omega_T(a) = \frac{0.3\hat{a}^{-3} + 0.7}{0.3\hat{a}^{-3} + 0.7} = 1$$

$$\therefore \quad \boxed{\Omega_T(a) = 1}$$

Plot for $\Omega_M = 0.3$, $\Omega_\Lambda = 0.7$



Now we have to calculate value of $a(t)$ when $\Omega_M(a) = 0.5$. $[a_0 = 1, \text{ so } \hat{a} = a(t)]$

Case I. For $\Omega_M = \Omega_T = 1$:

In this case, $\Omega_M(a)$ and $\Omega_T(a)$ are constants. They donot depend on a.

$$\Omega_M(a) = \Omega_T(a) = 1$$

So, $\Omega_M(a)$ never becomes 0.5 and structure formation never ceases. Ans

Case II. For $\Omega_M = 0.3 = \Omega_T$:

$$\Omega_M(a) = \frac{0.3}{0.3+0.7a} \quad ; \text{ Now } \Omega_M(a) = 0.5 :$$

$$\Rightarrow \quad 0.5 = \frac{0.3}{0.3+0.7a} \quad \Rightarrow \quad a(t) = 0.3/0.7$$

$$\therefore \quad \boxed{a(t) = 0.4286} \quad \text{Ans}$$

Case III. For $\Omega_M = 0.3, \Omega_\Lambda = 0.7, \Omega_\Lambda = 1$:

$$\Omega_M(a) = \frac{0.3}{0.3+0.7a^3} \quad ; \text{ Now } \Omega_M(a) = 0.5$$

$$\Rightarrow \quad 0.5 = \frac{0.3}{0.3+0.7a^3} \quad \Rightarrow \quad a^3 = 0.4286$$

$$\therefore \quad \boxed{a(t) = 0.754} \quad \text{Ans}$$