Physics 202 Problem Set 1

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1. Taylor Series Review: For each function find the lowest few terms in the Taylor Series around $x = 0$, stopping when you reach the lowest nonzero, nonconstant term.

   a. $\sin(x)$

$$f(0) = \sin(0) = 0$$
$$f'(0) = \cos(0) = 1$$

$$\sin(x) \approx 0 + 1(x) = x$$

   b. $\cos(x)$

$$f(0) = \cos(0) = 1$$
$$f'(0) = -\sin(0) = 0$$
$$f''(0) = -\cos(0) = -1$$

$$\cos(x) \approx 1 + 0x - \frac{x^2}{2} = 1 - \frac{x^2}{2}$$

   c. e^x

$$f(0) = e^0 = 1$$
$$f'(0) = e^0 = 1$$

$$e^x \approx 1 + x$$

   d. $(1+x)^n \quad (n \neq 0)$

$$f(0) = (1+0)^n = 1$$
$$f'(0) = n(1+0)^{n-1} = n$$

$$(1+x)^n \approx 1 + nx$$

2. Exercise with complex numbers, a math review.

a. Consider the number $z = 5 + 2i$. Rewrite z is polar form (ie. in the form $z = Ae^{i\phi}$)

$$A = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$\phi = \arctan\frac{2}{5} = 0.38 \quad \text{(5, 2) is in the first quadrant and so is 0.38 radians}$$

$$\boxed{z = \sqrt{29}e^{i0.38}}$$

b. What is $\sqrt{i}$?

The polar form of $\sqrt{i}$ is the same as the polar form of i to the one half power, which is $(e^{i\pi/2})^{1/2}$. Therefore,

$$\sqrt{i} = \boxed{e^{i\pi/4}} = \boxed{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i}$$ [1]

c. What is $\left(\frac{1+i}{1-i}\right)^2$?

$$\left(\frac{1+i}{1-i}\right)^2 = \left(\frac{1+2i-1}{1-2i-1}\right) = \frac{2i}{-2i} = \boxed{-1}$$

d. What is $\left|\frac{1+i}{1-i}\right|^2$?

The definition of the absolute value of a complex number z is $|z| = \sqrt{zz^*}$. We also know that $(z_1/z_2)^* = z_1^*/z_2^*$.[2] Hence,

$$\left|\frac{1+i}{1-i}\right|^2 = \frac{1+i}{1-i}\left(\frac{1+i}{1-i}\right)^* = \frac{1+i}{1-i}\left(\frac{1-i}{1+i}\right) = \boxed{1}$$

[1]There is also the other root $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

[2]We proved this on our 210 problem set this week.

3. Solving the SHO. Suppose that I prefer sines to cosines, so I replace Morin eq. 1-9 with $x(t) = A\sin(\omega t + \phi)$. Show that this too solves the equation of motion (Morin eq. 1-8) for a harmonic oscillator (with $\omega\sqrt{k/m}$ as before) and find the equation that replaces Morin 1-38.

   We want to show that $x(t) = A\sin(\omega t + \phi)$ solves the equation $-kx = m\frac{d^2x}{dt^2}$ with $\omega = \sqrt{\frac{k}{m}}$. Therefore,

$$m\frac{d^2x}{dt^2} = -mA\omega^2\sin(\omega t + \phi) = -mA\frac{k}{m}\sin(\omega t + \phi)$$
$$-kA\sin(\omega t + \phi) = -kx(t)$$

   Now to find the equation that replaces Morin eq. 1-38, we note that $v(t) = \omega A\cos(\omega t + \varphi)$. Hence,

$$x_0 = x(0) = A\sin(\omega(0) + \phi) = A\sin(\phi)$$
$$v_0 = v(t) = \omega A\cos(\omega(0) + \varphi) = \omega A\cos(\phi)$$

   Therefore, rearranging our equations for x_0 and v_0, we have that

$$\frac{x_0}{\sin\phi} = \frac{v_0}{\omega\cos\phi}$$
$$\frac{\sin\phi}{\cos\phi} = \frac{\omega x_0}{v_0}$$
$$\tan\phi = \frac{\omega x_0}{v_0}$$

   Now, multiplying x_0 by ω and then squaring and adding this equation with v_0^2 yields

$$\omega^2 x_0^2 + v_0^2 = \omega^2 A^2\sin^2\phi + A^2\omega^2\cos^2\phi = A^2\omega^2$$

   Thus,

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$$

   which leads to our modified verision of Morin eq. 1-38

$$\boxed{x(t) = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}\sin\left(\omega t + \arctan\left(\frac{\omega x_0}{v_0}\right)\right)}$$

4. Imagine that there is a straight hole that passes all the way through Earth. If you drop a quarter into the hole, does it execute simple harmonic motion? If so, what is its frequency in Hz?

   First, I will approximate the Earth as a sphere of radius R_E, with mass M_E, and uniform density ρ. Also, let r be the distance the quarter is from the center of the Earth and $m_{quarter}$ be the mass of the quarter. Since the equation for universal gravitation is the same form as Coloumb's law, we could derive a Gauss's Law for gravitation. Hence, when the quarter is inside the Earth, only mass at at radii nearer to the center than the quarter will create a net gravitational force towards the center of the Earth. Hence, there is a force that gets stronger when the quarter is further away from the center of the Earth, with a direction towards the center of the Earth. This is our restoring force. And it is linear. With the origin at the center of the Earth,

$$F(r) = -G\frac{4}{3}\pi(r)^3\rho\frac{m_{quarter}}{r^2} = -G\frac{4}{3}\pi r\rho m_{quarter}$$

   So the quarter, assuming is not thrown through the Earth at a ridiculous velocity, will exhibit simple harmonic motion.

   To find the frequency, we not that from our equation for force, $k = G\frac{4}{3}\pi\rho m_{quarter}$. We know that for simple harmonic motion $\omega = \sqrt{k/m}$. Therefore,

$$f = \frac{1}{2\pi}\sqrt{G\frac{4}{3}\pi\rho m_{quarter}/m_{quarter}} = \frac{1}{2\pi}\sqrt{G\frac{4}{3}\pi\rho}$$

   Now choosing $M_E = 5.97 \times 10^{24}$ kg and $R_E = 6378000$ m, we find that $\rho = 4600$ kg m^{-3}. Therefore,

$$f = \frac{1}{2\pi}\sqrt{G\frac{4}{3}\pi(4600 \text{ kg m}^{-3})} = \boxed{1.8 \times 10^{-4} \text{ Hz}}$$

5. French 3.4. A cylinder of a diameter d floats with l of its length submerged. The total height is L. Assume no damping. At time $t = 0$ the cylinder is pushed down a distance B and released.

   a. What is the frequency of the oscillation? By the Archimedes principle we know that weight of the object is equal to the buoyant force.

   $$mg = g\rho\pi l d^2/4$$
   $$m = l\rho\pi d^2/4$$

   From French, we know that

   $$ma = g\rho\frac{\pi}{4}d^2\Delta y$$

   Where ρ is the density of the liquid, m is the mass of the cylinder, and Δy is the distance away from its normal floating point

   Therefore,

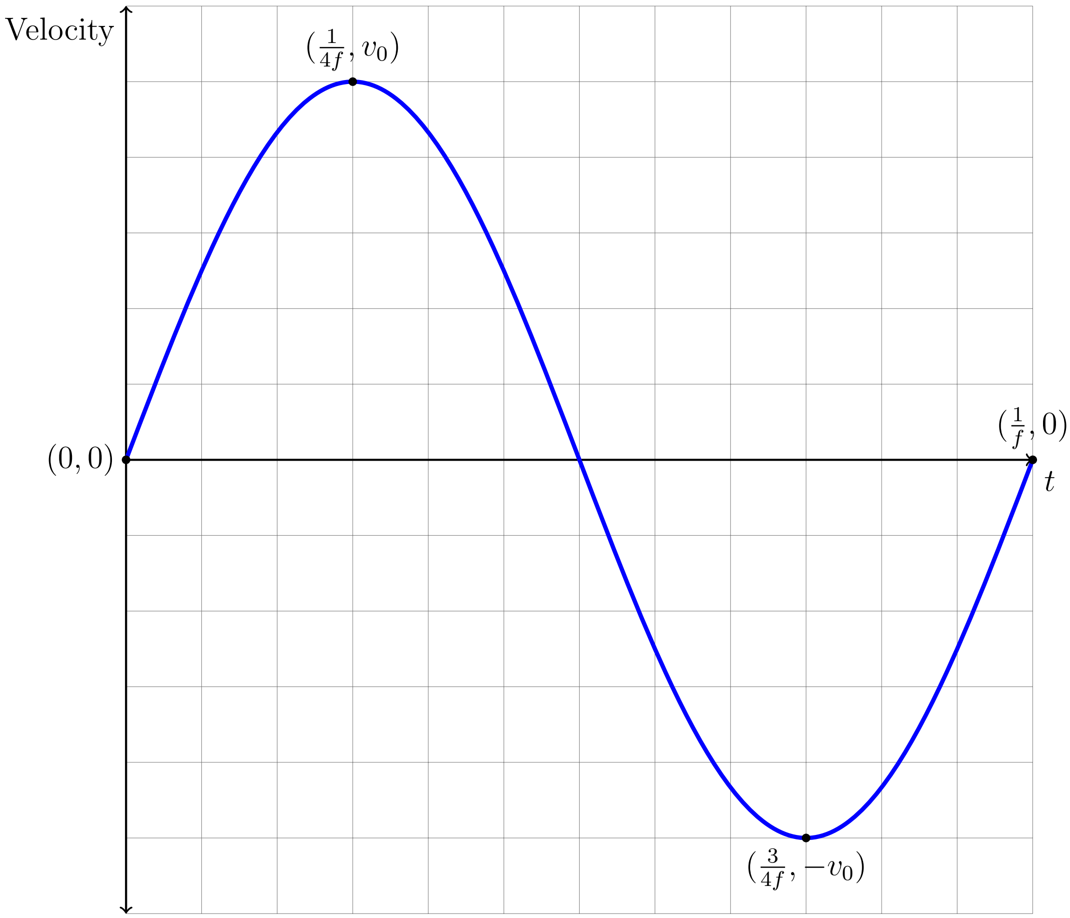
   $$f = \frac{1}{2\pi}\sqrt{\frac{g\rho\pi d^2}{4m}} = \boxed{\frac{1}{2\pi}\sqrt{\frac{g}{l}}}$$

   b. Draw a graph of velocity versus time from $t = 0$ to $t = \textit{one}$ period. The correct amplitude and phase should be included.

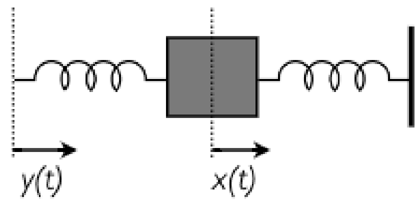
   Let the cylinder's position be indicated by measuring the displacement of the bottom most face of the cylinder compared to its equilibrium position with l of the cylinder submerged. Hence, the bottom face of the cylinder starts B beneath its equilibrium position ($y = 0$). Let down be the negative direction.

   $$x(t) = -B\cos\left(\sqrt{\frac{g}{l}}t\right)$$
   $$x'(t) = B\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}t\right)$$

   Let $v_0 = B\sqrt{\frac{g}{l}}$


Velocity
$(\frac{1}{4f}, v_0)$
$(0,0)$
$(\frac{1}{f}, 0)$
t
$(\frac{3}{4f}, -v_0)$

6. Driven SHO.



$$y(t) = y_0 \cos(\omega_d t)$$

Let us define the positive direction to be going to the right.

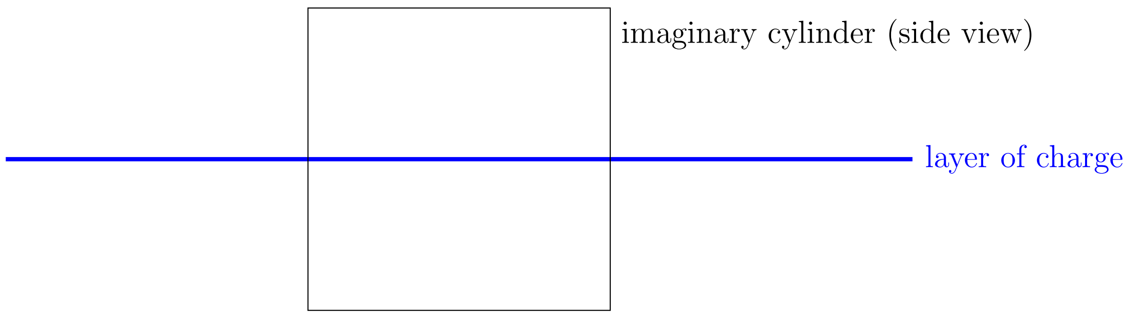
From this figure we can see that the force on the mass due to the left spring is $s(y(t) - x(t))$. Additionally, the force on the mass due to the right spring is $-sx(t)$. Therefore, the net force is

$$m\ddot{x} = s(y(t) - x(t)) - s(x(t)) = -2sx(t) - sy(t)$$

This is of the form , $\ddot{x} = -kx - F_0 \cos(\omega_d t)$, with $k = 2s$ and $F_0 = sy_0$

7. Smith 2.1

a. We imagine a cylinder placed around one of the layers of charge. Let this cylinder have a radius r and a height h. Center the cylinder vertically on the layer of charge.



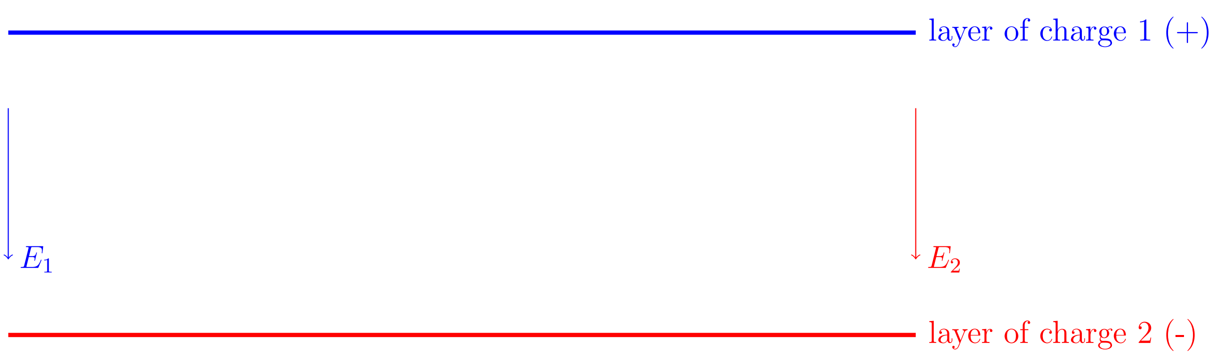
Hence the charge enclosed by this cylinder is $ne\pi r^2 x$. Additionally, by symmetry the electric field flux is 0 through the sides of the cylinder and it is $E\pi r^2$ through both the top and bottom of the cylinder.

Therefore, by Gauss's Law

$$2E\pi r^2 = \frac{ne\pi r^2 x}{\varepsilon_0}$$

$$E = \frac{nex}{2\varepsilon_0}$$

However, when we consider the contributions from the electric fields of both layers, they are pointed in the same direction but when between them, a test charge is below one layer and above another. Like the diagram below



Hence,

$$E_{total} = \frac{nex}{2\varepsilon_0} + \frac{nex}{2\varepsilon_0} = \boxed{\frac{nex}{\varepsilon_0}}$$

b. Because $E = \frac{nex}{\varepsilon_0}$, then the force on the electrons is $-Enel^3$, (l^3 being the volume of the cube). Hence, the restoring force is

$$F = -\frac{n^2e^2l^3}{\varepsilon_0}x$$

c. Because there is a linear restoring force, we can use what we know about simple harmonic motion. Namely that with a linear restoring force $F = -kx$, then $\omega = \sqrt{\frac{k}{m}}$.

Thus, we can state that for the restoring force on the electrons, $k = \frac{n^2e^2l^3}{\varepsilon_0}$. Additionally, we must consider the mass of all the electrons. Therefore, $m = nm_el^3$. Hence,

$$\omega = \sqrt{\frac{n^2e^2l^3}{nm_el^3\varepsilon_0}} = \boxed{\sqrt{\frac{ne^2}{m_e\varepsilon_0}}}$$

d. Estimate the plasma frequency of solid copper assuming one free electron per atom. What kind of radiation does this correspond to?

We can look up the density of copper, which is 8.96g/cm^{-3}, and its molar mass, which is 63.546g. Assuming one free electron per atom, then

$$n = \frac{8.96\text{g/cm}^{-3}6.022 \times 10^{23}}{63.546\text{g}} = 8.5 \times 10^{22} \text{ carriers/cm}^{-3}$$

$$= 8.5 \times 10^{28} \text{ carriers/m}^{-3}$$

$$\omega = \sqrt{\frac{8.5 \times 10^{28} \text{ carriers m}^{-3}(1.6 \times 10^{-19}\text{C})^2}{(9.11 \times 10^{-31}\text{kg})(8.85 \times 10^{-12}\text{m}^{-3}\text{kg}^{-1}\text{s}^4\text{A}^2)}}$$

$$\omega = 1.6 \times 10^{16}\text{rad s}^{-1}$$

$$f = 2.6 \times 10^{15}\mathrm{Hz}$$

This is in the ultraviolet range.