

# A Crash Course in Vector Calculus

## SCALAR AND VECTOR FIELDS

This introductory chapter is a review of mathematical concepts required for the course. It is assumed that the reader is already familiar with elementary vector analysis.

Physical quantities that we deal with in electromagnetism can be scalars or vectors.

A scalar is an entity which only has a magnitude. Examples of scalars are mass, time, distance, electric charge, electric potential, energy, temperature etc.

A vector is characterized by both magnitude and direction. Examples of vectors in physics are displacement, velocity, acceleration, force, electric field, magnetic field etc.

A field is a quantity which can be specified everywhere in space as a function of position. The quantity that is specified may be a scalar or a vector. For instance, we can specify the temperature at every point in a room. The room may, therefore, be said to be a region of “temperature field” which is a scalar field because the temperature  $T(x, y, z)$  is a scalar function of the position. An example of a scalar field in electromagnetism is the *electric potential*.

In a similar manner, a vector quantity which can be specified at every point in a region of space is a vector field. For instance, every point on the earth may be considered to be in the gravitational force field of the earth. we may specify the field by the magnitude and the direction of acceleration due to gravity (i.e. force per unit mass )  $g(x, y, z)$  at every point in space. As another example consider flow of water in a pipe. At each point in the pipe, the water molecule has a velocity  $\vec{v}(x, y, z)$ . The water in the pipe may be said to be in a velocity field. There are several examples of vector field in electromagnetism, e.g., the electric field  $\vec{E}$ , the magnetic flux density  $\vec{B}$  etc.

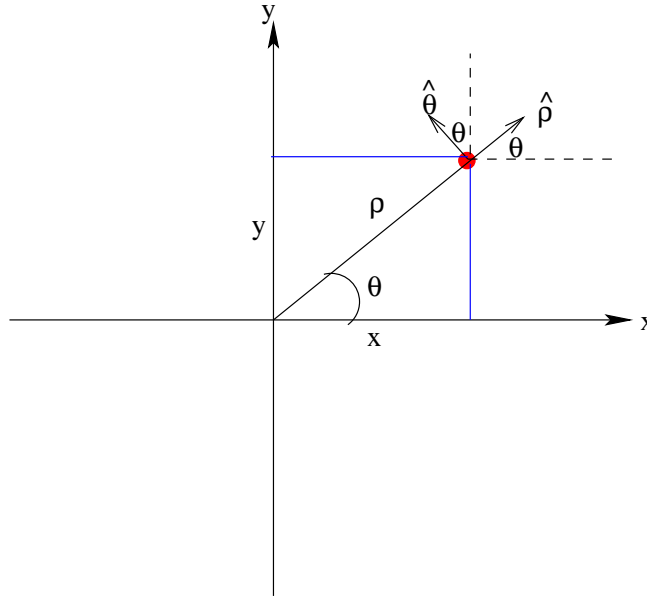
### 1.1 Coordinate Systems :

We are familiar with cartesian coordinate system. For systems exhibiting cylindrical or spherical symmetry, it is convenient to use respectively the cylindrical and spherical coordinate systems.

#### 1.1.1 Polar Coordinates :

In two dimensions one defines the polar coordinate  $(\rho, \theta)$  of a point by defining

$\rho$  as the radial distance from the origin  $O$  and  $\theta$  as the angle made by the radial vector with a reference line (usually chosen to coincide with the x-axis of the cartesian system). The radial unit vector  $\hat{\rho}$  and the tangential (or angular) unit vector  $\hat{\theta}$  are taken respectively along the direction of increasing distance  $\rho$  and that of increasing angle  $\theta$  respectively, as shown in the figure.



Relationship with the cartesian components are

$$\begin{aligned}x &= \rho \cos \theta \\y &= \rho \sin \theta\end{aligned}$$

so that the inverse relationships are

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1} \frac{y}{x}\end{aligned}$$

By definition, the distance  $\rho > 0$ . we will take the range of angles  $\theta$  to be  $0 \leq \theta < 2\pi$  (It is possible to define the range to be  $-\pi \leq \theta < +\pi$ ). One has to be careful in using the inverse tangent as the arc-tan function is defined in  $0 \leq \theta < \pi$ . If  $y$  is negative, one has to add  $\pi$  to the principal value of  $\theta$  calculated by the arc - tan function so that the point is in proper quadrant.

**Example 1 :**

A vector  $\vec{A}$  has cartesian components  $A_x$  and  $A_y$ . Write the vector in terms of its radial and tangential components.

**Solution :**

Let us write

$$\vec{A} = A_\rho \hat{\rho} + A_\theta \hat{\theta}$$

Since  $\hat{\rho}$  and  $\hat{\theta}$  are orthonormal basis vectors  $\hat{\rho} \cdot \hat{\rho} = \hat{\theta} \cdot \hat{\theta} = 1$  and  $\hat{\rho} \cdot \hat{\theta} = 0$ . Thus

$$A_\rho = \vec{A} \cdot \hat{\rho}, \quad A_\theta = \vec{A} \cdot \hat{\theta}$$

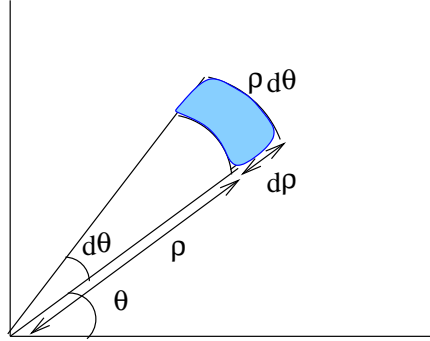
Note that (see figure)  $\hat{\rho}$  makes an angle  $\theta$  with the x-axis ( $\hat{i}$ ) and  $\pi/2 - \theta$  with the y-axis ( $\hat{j}$ ). Similarly, the unit vector  $\hat{\theta}$  makes  $\pi/2 + \theta$  with the x-axis and  $\theta$  with the y-axis. Thus

$$\begin{aligned} A_\rho = \vec{A} \cdot \hat{\rho} &= A_x \hat{i} \cdot \hat{\rho} + A_y \hat{j} \cdot \hat{\rho} \\ &= A_x \cos \theta + A_y \cos(\pi/2 - \theta) \\ &= A_x \cos \theta + A_y \sin \theta \\ A_\theta = \vec{A} \cdot \hat{\theta} &= A_x \hat{i} \cdot \hat{\theta} + A_y \hat{j} \cdot \hat{\theta} \\ &= A_x \cos(\pi/2 + \theta) + A_y \cos \theta \\ &= -A_x \sin \theta + A_y \cos \theta \end{aligned}$$

**The Jacobian :**

When we transform from one coordinate system to another, the differential element also transform.

For instance, in 2 dimension the element of an area is  $dx dy$  but in polar coordinates the element is not  $d\theta d\rho$  but  $(\rho d\theta) d\rho$ . This extra factor  $\rho$  is important when we wish to integrate a function using a different coordinate system.



If  $f(x, y)$  is a function of  $x, y$  we may express the function in polar coordinates and write it as  $g(\rho, \theta)$ . However, when we evaluate the integral  $\int f(x, y) dx dy$  in polar coordinates, the corresponding integral is  $\int g(r, \theta) \rho d\theta d\rho$ . In general, if  $x = f(u, v)$  and  $y = g(u, v)$ , then, in going from  $(x, y)$  to  $(u, v)$ , the differential

element  $dx dy \rightarrow |J| du dv$  where  $J$  is given by the determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The differentiations are partial, i.e., while differentiating  $\partial x / \partial u = \partial f(u, v) / \partial u$ , the variable  $v$  is treated as constant. An useful fact is that the Jacobian of the inverse transformation is  $1/J$  because the determinant of the inverse of a matrix is equal to the inverse of the determinant of the original matrix.

**Example 2 :** Show that the Jacobian of the transformation from cartesian to polar coordinates is  $\rho$ .

**Solution :**

We have

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

Using  $x = \rho \cos \theta$  and  $y = \rho \sin \theta$ , we have

$$J = \begin{vmatrix} \cos \theta & -\rho \sin \theta \\ \sin \theta & \rho \cos \theta \end{vmatrix} = \rho$$

**Exercise :**

Show that the Jacobian of the inverse transformation from polar to cartesian is  $1/\rho = 1/\sqrt{x^2 + y^2}$ .

**Example 3 :**

find the area of a circle of radius  $R$ .

**Solution :**

Take the origin to be at the centre of the circle and the plane of the circle to be the  $\rho - \theta$  plane. Since the area element in the polar coordinates is  $\rho d\theta d\rho$ , the area of the circle is

$$\int_0^{2\pi} d\theta \int_0^R \rho d\rho = 2\pi \left[ \frac{\rho^2}{2} \right]_0^R = \pi R^2$$

- a very well known result !

**Example 4 :**

Find the integral  $\int e^{-(x^2+y^2)} dx dy$  where the region of integration is a unit circle about the origin.

Using polar coordinates the integrand becomes  $e^{-\rho^2}$ . The range of  $\rho$  integration for  $\rho$  is from 0 to 1 and for  $\theta$  is from 0 to  $2\pi$ . The integral is given by

$$I = \int_0^{2\pi} d\theta \int_0^1 e^{-\rho^2} \rho d\rho = 2\pi \int_0^1 e^{-\rho^2} \rho d\rho$$

The radial integral is evaluated by substitution  $w = \rho^2$  so that  $\rho d\rho = dw/2$ . The value of the integral is

$$I = 2\pi \int_0^1 e^{-w} dw / 2 = \pi [-e^{-w}]_0^1 = \pi(1 - \frac{1}{e})$$

**Exercise :**

Evaluate  $\int \int xy dx dy$  where the region of integration is the part of the area between circles of radii 1 and 2 that lies in the first quadrant. (Ans. 15/8)

**Exercise :**

Evaluate the Gaussian integral  $I = \int_0^\infty e^{-x^2} dx$

[ Hint : The integration cannot be done using cartesian coordinates but is relatively easy using polar coordinates and properties of definite integrals. By changing the dummy variable  $x$  to  $y$ , one can write  $I = \int_0^\infty e^{-y^2} dy$ , so that we can write

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$$

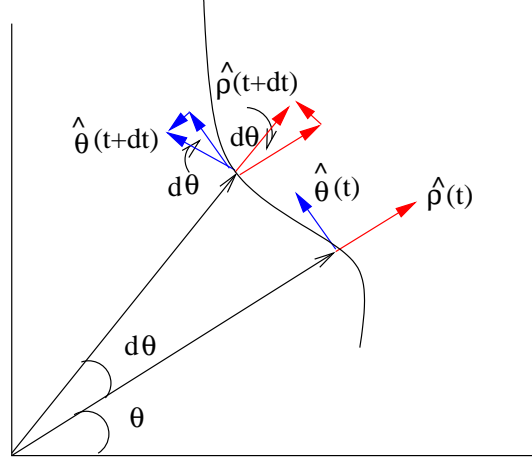
Transform this to polar. Range of integration for  $\rho$  is from 0 to  $\infty$  and that of  $\theta$  is from 0 to  $\pi/2$ (why ?) Answer :  $\sqrt{\pi}/2$  ]

**Differentiation of polar unit vectors with respect to time :**

It may be noted that the basis vectors  $\hat{\rho}$  and  $\hat{\theta}$ , unlike  $\hat{i}$  and  $\hat{j}$  are not constant vectors but depend on the position of the point. The time derivative of the unit vectors are defined as follows

$$\begin{aligned} \frac{d\hat{\rho}}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta \hat{\rho}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\hat{\rho}(t + \Delta t) - \hat{\rho}(t)}{\Delta t} \\ \frac{d\hat{\theta}}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta \hat{\theta}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\hat{\theta}(t + \Delta t) - \hat{\theta}(t)}{\Delta t} \end{aligned}$$

One can evaluate the derivatives by laborious process of expressing the unit vectors  $\hat{\rho}$  and  $\hat{\theta}$  in terms of constant unit vectors of cartesian system, differentiating the resulting expressions and finally transform back to the polar form. Alternatively, we can look at the problem geometrically, as shown in the following figure.



In the figure, the positions of a particle are shown at time  $t$  and  $t + dt$ . The unit vectors  $\hat{\rho}$  is shown in red while the unit vector  $\hat{\theta}$  is shown in blue. It can be easily seen by triangle law of addition of vectors that the magnitude of  $\Delta\hat{\rho}$  and  $\Delta\hat{\theta}$  is  $1 \cdot d\theta = d\theta$ . However, as the limit  $dt \rightarrow 0$ , the direction of  $\Delta\hat{\rho}$  is in the direction of  $\hat{\theta}$  while that of  $\Delta\hat{\theta}$  is in the direction of  $-\hat{\rho}$ . Thus

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta\hat{\rho}}{\Delta t} = \frac{d\theta}{dt} \hat{\theta} \\ \frac{d\hat{\theta}}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta\hat{\theta}}{\Delta t} = -\frac{d\theta}{dt} \hat{\rho}\end{aligned}$$

Now,  $d\theta/dt$  is the angular velocity of the point, which is usually denoted by  $\omega$ , Thus we have,

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \omega \hat{\theta} \\ \frac{d\hat{\theta}}{dt} &= -\omega \hat{\rho}\end{aligned}$$

**Exercise :**

Obtain the above relationships directly by using the expression for the unit vectors  $\hat{\rho}$  and  $\hat{\theta}$  in terms of the cartesian unit vectors, viz.,

$$\begin{aligned}\hat{\rho} &= \hat{i} \cos \theta + \hat{j} \sin \theta \\ \hat{\theta} &= -\hat{i} \sin \theta + \hat{j} \cos \theta\end{aligned}$$

### 1.1.2 Cylindrical coordinates :

Cylindrical coordinate system is obtained by extending the polar coordinates by adding a  $z$ -axis along the height of a right circular cylinder. The  $z$ -axis of the coordinate system is same as that in a cartesian system.

In the figure  $\rho$  is the distance of the foot of the perpendicular drawn from the point to the  $x - y(\rho, \theta)$  plane. Note that  $\rho$  here is *not* the distance of the point P from the origin, as is the case in polar coordinate systems. (Some texts use  $r$  to denote what we are calling as  $\rho$  here. However, we use  $\rho$  to denote the distance from the origin to the foot of the perpendicular to avoid confusion.) In terms of cartesian coordinates

$$\begin{aligned}x &= \rho \cos \theta \\y &= \rho \sin \theta \\z &= z\end{aligned}$$

so that the inverse relationships are

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1} \frac{y}{x} \\ z &= z\end{aligned}$$

#### Exercise :

Find the cylindrical coordinate of the point  $3\hat{i} + 4\hat{j} + \hat{k}$ . [Hint : Determine  $\rho$  and  $\tan \theta$  using above transformation]

(Ans.  $5\hat{\rho} + \tan^{-1}(4/3)\hat{\theta} + \hat{k}$ )

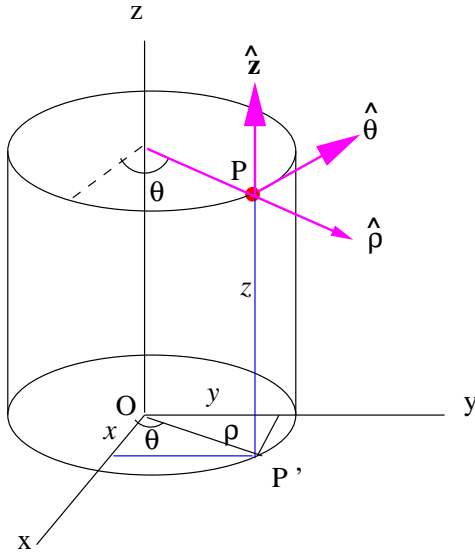
The line element in the system is given by

$$\vec{dl} = \hat{\rho}d\rho + \rho d\theta\hat{\theta} + dz\hat{z}$$

and the volume element is

$$dV = \rho d\theta d\rho dz$$

The Jacobian of transformation from cartesian to cylindrical is  $\rho$  as in the polar coordinates since  $z$  coordinate remains the same.

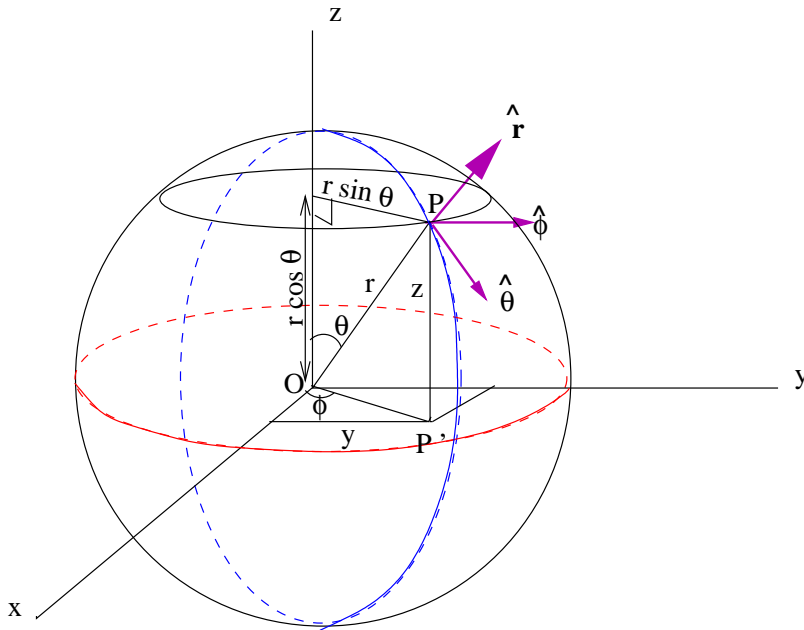


### 1.2.3 Spherical Polar Coordinates :

Spherical coordinates are useful in dealing with problems which possess spherical symmetry. The independent variables of the system are  $(r, \theta, \phi)$ . Here  $r$  is the distance of the point  $P$  from the origin. Angles  $\theta$  and  $\phi$  are similar to latitudes and longitudes.

Two mutually perpendicular lines are chosen, taken to coincide with the x-axis and z-axis of the cartesian system. We take angle  $\theta$  to be the angle made by the radius vector (i.e. the vector connecting the origin to  $P$ ) with the z-axis (the angle  $\theta$  is actually complementary to the latitude). The angle  $\phi$  is the angle between the x-axis and the line joining the origin to  $P'$ , the foot of the perpendicular from  $P$  to the x-y plane.

The unit vectors  $\hat{r}, \hat{\theta}$  and  $\hat{\phi}$  are respectively along the directions of increasing  $r, \theta$  and  $\phi$ .



The surface of constant  $r$  are spheres of radius  $r$  about the centre. the surface of constant  $\theta$  is a cone of semi-angle  $\theta$  about the z-axis. The reference for measuring  $\phi$  is the x-z plane of the cartesian system. A surface of constant  $\phi$  is a plane containing the z-axis which makes an angle  $\phi$  with the reference plane.

#### Example 5 :



Express unit vectors of spherical coordinate system in terms of unit vectors of cartesian system.

**Solution :**

From the point P drop a perpendicular on to the x-y plane. Denote  $\vec{OP'}$  by  $\vec{\rho}$ . The figure below shows the unit vectors in both the systems. By triangle law of vector addition,

$$\vec{r} = \vec{OP} = \vec{OP'} + \vec{P'P} = r \sin \theta \hat{\rho} + r \cos \theta \hat{k}$$

However,  $\hat{\rho} = \cos \phi \hat{i} + \sin \phi \hat{j}$ . Substituting this in the expression for  $\vec{r}$ , we get on dividing both sides by the magnitude of  $r$

$$\hat{r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$

The unit vector  $\hat{\theta}$  is perpendicular to OP in the direction of increasing  $\theta$ . The angle that  $\hat{\theta}$  makes with the positive z-axis is  $\pi/2 + \theta$ . Hence,

$$\hat{\theta} = \cos\left(\frac{\pi}{2} + \theta\right) \hat{k} + \sin\left(\frac{\pi}{2} + \theta\right) \hat{\rho} = -\sin \theta \hat{k} + \cos \theta \hat{\rho}$$

Substituting for  $\hat{\rho}$ , we get

$$\hat{\theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}$$

The azimuthal unit vector  $\hat{\phi}$  is in the direction of increasing angle  $\phi$ . It is perpendicular to  $\vec{\rho}$  and has no z- component. It can be easily seen that

$$\hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

**Transformation from spherical to cartesian :**

Using the expression for  $\vec{r}$  in terms of cartesian basis, it is seen that

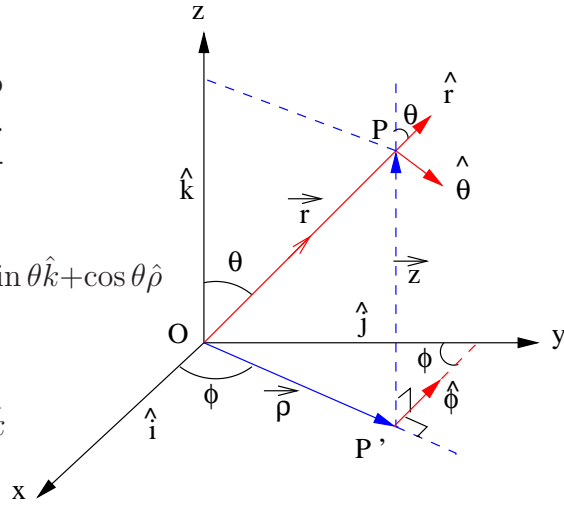
$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

and the inverse transformation

$$r = \sqrt{x^2 + y^2 + z^2}$$



$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \cos^{-1} \frac{z}{r}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

Range of the variables are as follows :

$$0 \leq r < \infty \quad 0 \leq \theta \leq \pi \quad 0 \leq \phi \leq 2\pi$$

**Exercise :**

A particle moves along a spherical helix. its position coordinate at time  $t$  is given by

$$x = \frac{\cos t}{\sqrt{1+t^2}}, \quad y = \frac{\sin t}{\sqrt{1+t^2}}, \quad z = \frac{t}{\sqrt{1+t^2}}$$

Express the equation of the path in spherical coordinates.

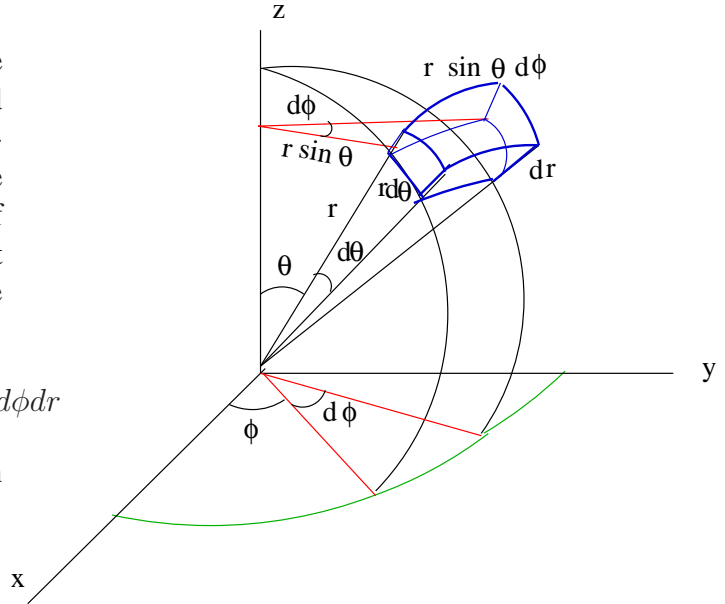
(Ans.

$$r = 1, \quad \cos \theta = t/\sqrt{1+t^2} \quad \phi(t) = t)$$

The differential element of volume is obtained by constructing a closed volume by extending  $r, \theta$  and  $\phi$  respectively by  $dr, d\theta$  and  $d\phi$ . The length elements in the direction of  $\hat{r}$  is  $dr$ , that along  $\hat{\theta}$  is  $r d\theta$  while that along  $\hat{\phi}$  is  $r \sin \theta d\phi$  (see figure). The volume element, therefore, is

$$dV = dr \cdot r d\theta \cdot r \sin \theta d\phi = r^2 \sin \theta d\theta d\phi dr$$

Thus the Jacobian of transformation is  $r^2 \sin \theta$ .



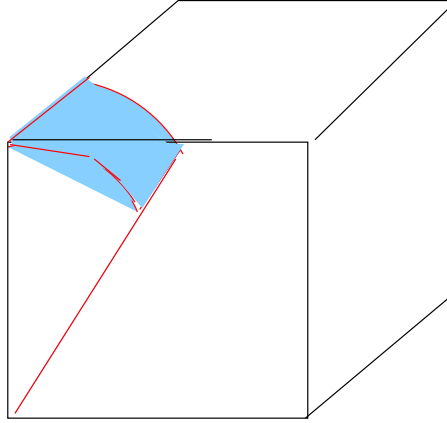
**Example 6:**

Find the volume of a solid region in the first octant that is bounded from above by the sphere  $x^2 + y^2 + z^2 = 9$  and from below by the cone  $x^2 + y^2 = 3z^2$ .

**Solution :**

Because of obvious spherical symmetry, the problem is best solved in spherical polar coordinates. The equation to sphere is  $r = 3$  so that the range of  $r$  variable for our solid is from 0 to 3.

The equation to the cone  $x^2 + y^2 = 3z^2$  becomes  $r^2 \sin^2 \theta = 3r^2 \cos^2 \theta$ . Solving, the semi-angle of cone is  $\tan \theta = \sqrt{3}$  i.e.  $\theta = \pi/3$ . Since the solid is restricted to the first octant, i.e.,  $(x, y, z \geq 0)$ , the range of the azimuthal angle  $\phi$  is from 0 to  $\pi/2$ .



The volume of the solid is

$$\begin{aligned} \int dV &= \int r^2 \sin \theta d\theta d\phi dr = \int_0^{\pi/2} d\phi \int_0^{\pi/3} \sin \theta d\theta \int_0^3 r^2 dr \\ &= \frac{\pi}{2} \cdot [-\cos \theta]_0^{\pi/3} \cdot \left[ \frac{r^3}{3} \right]_0^3 \\ &= \frac{9\pi}{4} \end{aligned}$$

### Exercise :

Using direct integration find the volume of the first octant bounded by a sphere  $x^2 + y^2 + z^2 = 9$ .

## 1.2 Line and Surface Integrals of a Vector Field :

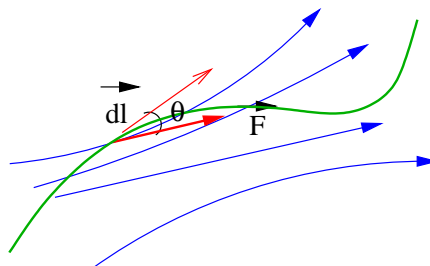
Since a vector field is defined at every position in a region of space, like a scalar function it can be integrated and differentiated. However, as a vector field has both magnitude and direction it is necessary to define operations of calculus to take care of both these aspects.

### 1.2.1 Line Integral :

If a vector field  $\vec{F}$  is known in a certain region of space, one can define a line integral of the vector function may be defined as

$$\int_C \vec{F} \cdot d\vec{l}$$

where  $C$  is the curve along which the integral is calculated. Like the integral of a scalar function the integral above is also interpreted as a limit of a sum. We first divide the curve  $C$  into a large number of infinitesimally small line segments such that the vector function is constant (in magnitude and direction) over each such line segment. The integrand is then equal to the product of the length of the line segment and the component of  $\vec{F}$  along the segment. The integration is defined as the limit of the sum of contributions from all such segments in the same manner as ordinary integration is defined.



The concept of line integral is very useful in many branches of physics. In mechanics, we define work done by a force  $\vec{F}$  in moving an object from an initial position A to a final position B as the line integral of the force along the curve joining the end points. Except in the case of conservative forces, the line integral depends on the actual path along which the particle moves under the force.

**Example 7:**

A force  $\vec{F} = zy\hat{i} + x\hat{j} + z^2x\hat{k}$  acts on a particle that travels from the origin to the point  $(1, 2, 3)$ . Calculate the work done if the particle travels (i) along the path  $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 2, 0) \rightarrow (1, 2, 3)$  along straightline segments joining each pair of end points (ii) along the straightline joining the initial and final points.

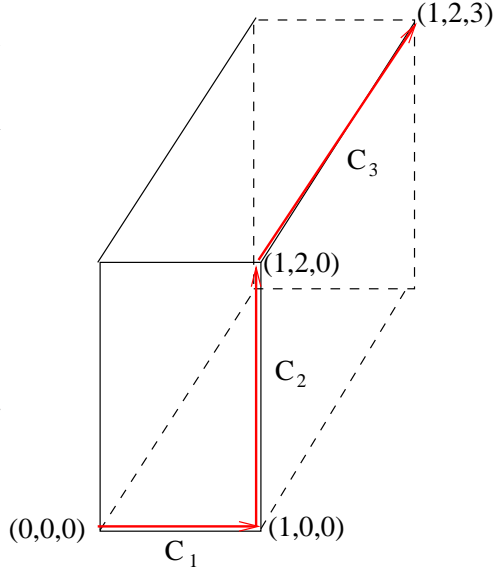
**Solution :**

Along the path  $(0, 0, 0) \rightarrow (1, 0, 0)$ ,  $\vec{dl} = \hat{i}dx$  and  $y = 0, z = 0$ . Since  $F_x = 0$  along this segment, the integral along  $C_1$  is zero. Along the path  $C_2$  joining  $(1, 0, 0)$  and  $(1, 2, 0)$ ,  $\vec{dl} = \hat{j}dy$  and  $x = 1$ . Along this path  $F_y = x = 1$ . The integral

$$\int_0^2 F_y dy = \int_0^2 dy = 2$$

Along the third path connecting  $(1, 2, 0)$  to  $(1, 2, 3)$ ,  $x = 1, y = 2$ .  $\vec{dl} = \hat{k}dz$  and  $F_z = z^2x = z^2$ . The line integral is

$$\int_0^3 z^2 dz = \frac{z^3}{3} \Big|_0^3 = 9$$



The work done is, therefore,  $0 + 2 + 9 = 11$ .

In order to calculate the work done when the particle moves along the straightline connecting the initial and final points, we need to write down the equation to the line in a parametric form. If  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  are the end points, the equation is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = m$$

Substituting for the coordinates, we get the equation to the line as

$$\begin{aligned} x &= m \\ y &= 2m \\ z &= 3m \end{aligned}$$

Thus the differential elements are  $dx = dm$ ,  $dy = 2dm$  and  $dz = 3dm$ . The line integral is given by

$$\begin{aligned} \int \vec{F} \cdot \vec{dl} &= \int (F_x dx + F_y dy + F_z dz) \\ &= \int_0^1 (6m^2 \cdot dm + m \cdot 2dm + 9m^3 \cdot 3dm) \end{aligned}$$

$$= 2m^3 + m^2 + 27\frac{m^4}{4} \Big|_0^1 = 9.75$$

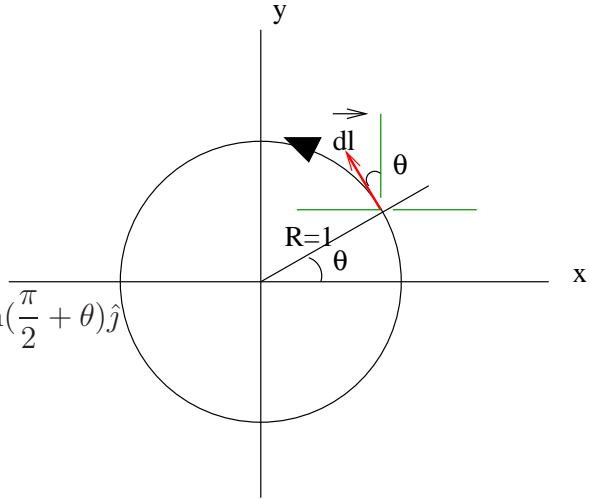
**Example 8 :**

Find the line integral of  $\vec{F} = y\hat{i} + \hat{j}$  over an anticlockwise circular loop of radius 1 with the origin as the centre of the circle.

**Solution :**

The length element  $dl$  has a magnitude  $1 \cdot d\theta = d\theta$ . Since the unit vector along  $\vec{dl}$  makes an angle of  $(\pi/2) + \theta$  with the positive  $x$ -axis,

$$\begin{aligned} \vec{dl} &= |dl| \cos\left(\frac{\pi}{2} + \theta\right)\hat{i} + |dl| \sin\left(\frac{\pi}{2} + \theta\right)\hat{j} \\ &= -\sin\theta\hat{i} + \cos\theta\hat{j} \end{aligned}$$



In polar coordinates,  $x = \cos\theta$  and  $y = \sin\theta$  (since the radius is 1). Thus

$$\begin{aligned} \int \vec{F} \cdot \vec{dl} &= \int F_x dx + F_y dy \\ &= -\int_0^{2\pi} \sin^2\theta d\theta + \int_0^{2\pi} \cos\theta d\theta \\ &= -\pi + 0 = -\pi \end{aligned}$$

**Exercise :**

A force  $\vec{F} = xy\hat{i} + (x^2 - z^2)\hat{j} - xz^2\hat{k}$  acts on a particle. Calculate the work done if the particle is taken from the point  $(0, 0, 0)$  to the point  $(2, 1, 3)$  along straight line segment connecting  $(0, 0, 0) \rightarrow (0, 1, 0) \rightarrow (2, 1, 0) \rightarrow (2, 1, 3)$ . What would be the work done if the particle directly moved to the final point along the straightline connecting to origin. (Ans.  $-16, -13.8$ .)

**Exercise :**

A vector field is given by

$$\vec{F} = (2x + 3y)\hat{i} + (3x + 2y)\hat{j}$$

Evaluate the line integral of the field around a circle of unit radius traversed in clockwise fashion. (Ans.  $6\pi$ )

**Exercise :**

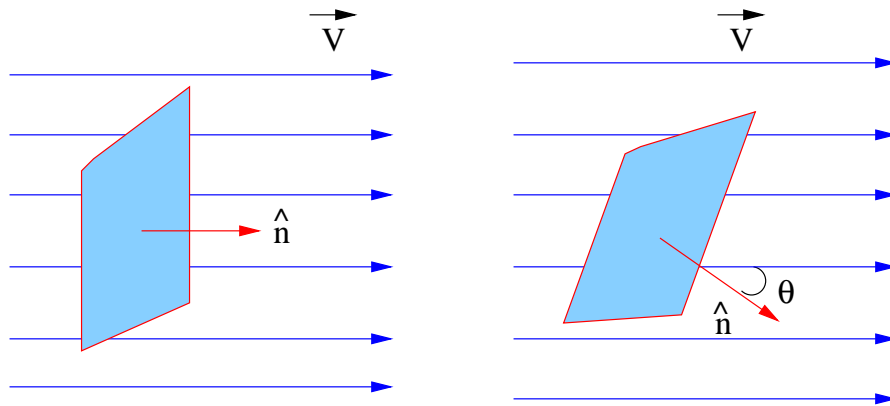
Evaluate the line integral of a *scalar function*  $xy$  along a parabolic path  $y = x^2$  connecting the origin to the point  $(1, 1)$ .

[Hint : Remember that the arc length along a curve is given by  $\sqrt{(dx)^2 + (dy)^2}$ .

The curve can be parametrized by  $x = t$  and  $y = t^2$ . Ans.  $(25\sqrt{5} + 1)/120$  ]

**1.2.3 Surface Integral :**

We have seen that an area element can be regarded as a vector with its direction being defined as the outward normal to the surface. The concept of a surface integral is related to flow. Suppose the vector field represents the rate at which water flows at a point in the region of flow. The flow may be measured in cubic meter of water flowing per square meter of area per second. If an area is oriented perpendicular to the direction of flow, as shown in the figure to the left, maximum amount of water would flow through the surface. The amount of water passing through the area is the **flux** (measured in cubic meter per second).



If, on the other hand, the surface is tilted relative to the flow, as shown to the right, the amount of flux through the area decreases. Clearly, only the part of the area that is perpendicular to the direction of flow will contribute to the flux.

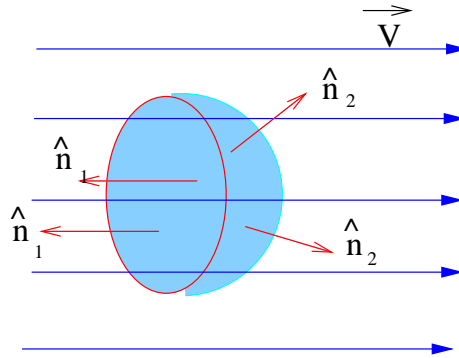
We define **flux** through an area element  $d\vec{S}$  as the dot product of the vector field  $\vec{V}$  with the area vector  $d\vec{S}$ . When  $\vec{V}$  is parallel to  $d\vec{S}$ , i.e. if the surface is oriented perpendicular to the direction of flow, the flux is maximum. On the other hand, a surface oriented parallel to the flow does not contribute to the flux.

**Example 9 :**

A hemispherical bowl of radius  $R$  is oriented such that the circular base is perpendicular to direction of flow. Calculate the flux through the curved surface of the bowl, assuming the flow vector  $\vec{V}$  to be constant.

**Solution :**

Since the curved surface makes different angles at different positions, it is somewhat difficult to calculate the flux through it. However, one can circumvent it by calculating the flux through the circular base.



Since the flow vector is constant all over the circular face which is oriented perpendicular to the direction of flow, the flux through the base is  $-\pi R^2 V$ . The minus sign is a result of the fact that the direction of the surface is opposite to the direction of flow. Thus we may call the flux through the base as *inward flux*. Since there is no source or sink of flow field (i.e. there is no accumulation of water) inside the hemisphere, whatever fluid enters through the base must leave through the curved face. Thus the *outward flux* from the curved face is  $+\pi R^2 V$ .

We may now generalize the above for a surface over which the field is not uniform by defining the flux through an area as the sum of contribution to the flux from infinitesimal area elements which comprises the total area by treating the field to be uniform over such area elements. Since the flux is a scalar, the surface integral, defined as the limit of the sum, is also a scalar.



If  $\Delta S_i$  is the  $i$ -th surface element, the normal  $\hat{n}_i$  to which makes an angle  $\theta_i$  with the vector field  $\vec{V}_i$  at the position of the element, the total flux

$$\Phi = \sum_i \vec{V}_i \cdot \hat{n}_i \Delta S_i = \sum_i V_i \delta S_i \cos \theta_i$$

In the limit of  $\Delta S_i \rightarrow 0$ , the sum above becomes a surface integral

$$\Phi = \int_S \vec{V} \cdot d\vec{S}$$

If the surface is closed, it encloses a volume and we define

$$\Phi = \oint_S \vec{V} \cdot d\vec{S}$$

to be the net outward flux. In terms of cartesian components

$$\Phi = \int_S (V_x dydz + V_y dxdz + V_z dxdy)$$

### Example 10 :

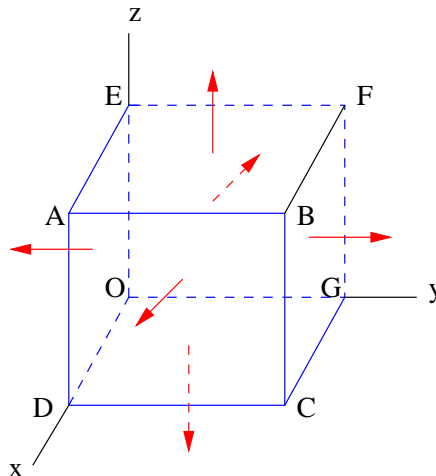
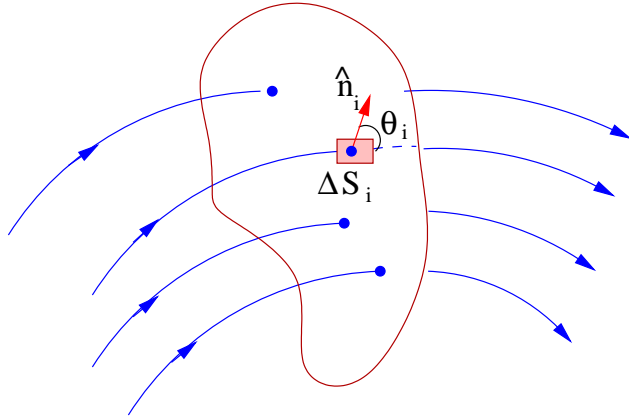
A vector field is given by  $\vec{B} = xy\hat{i} + yz\hat{j} + zx\hat{k}$ . Evaluate the flux through each face of a unit cube whose edges along the cartesian axes and one of the corners is at the origin.

**Solution :**

Consider the base of the cube (OGCD), which is the x-y plane on which  $z = 0$ . On this face  $\vec{B} = xy\hat{i}$ . The surface vector  $\hat{n}$  is along  $-\hat{k}$  direction. Thus on this surface flux  $\int \vec{B} \cdot \hat{n} dxdy = \int xy\hat{i} \cdot \hat{k} dxdy = 0$  since  $\hat{i} \cdot \hat{k} = 0$ . For the top surface (ABFE),  $z = 1$  and  $\hat{n} = \hat{k}$ . The flux from this surface is

$$\int V_z dxdy = \int_0^1 x dx \int_0^1 dy = \frac{1}{2}$$

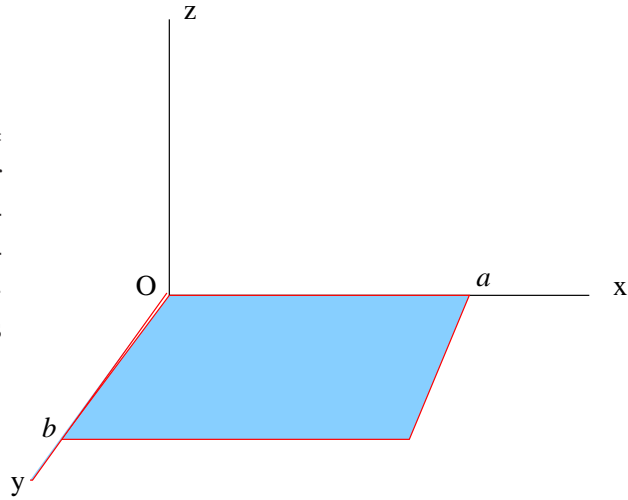
In a similar way one can show that flux from left side (AEOD) is zero while the contribution from the right side (BFGC) is 1/2. The back face (EFGO) contributes



zero while the front face (ABCD) contributes  $1/2$ . The net flux, therefore, is  $3/2$ .

**Exercise :**

Find the flux of the vector field  $\vec{V} = Ax\hat{i} + By^2\hat{j}$  through a rectangular surface in the x-y plane having dimensions  $a \times b$ . The origin of the coordinate system is at one of the corners of the rectangle and the x-axis along its length. (Ans.  $Bab^3/3$ )



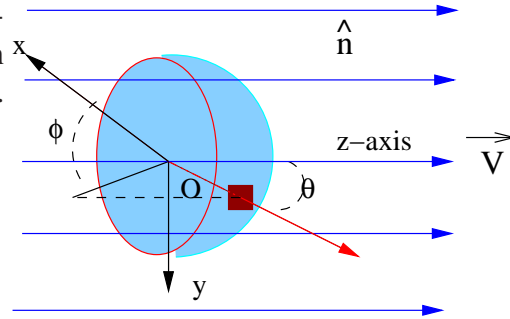
**Example 11 :**

Calculate directly the flux through the curved surface of the hemispherical bowl of Example 10.

**Solution :**

Use a spherical polar coordinates with the base of the hemisphere being the x-y plane and the direction of the vector field as the z-axis. We have seen that an area element on the surface is given by  $dS = R^2 \sin \theta d\theta d\phi$ . Thus the flux  $\Phi = \int \vec{V} \cdot \hat{n}$  is given by

$$\begin{aligned}\Phi &= |V| R^2 \int_0^2 \pi d\phi \int_0^{\pi/2} \cos \theta \sin \theta d\theta \\ &= |V| R^2 \cdot 2\pi \left[ \frac{\sin^2 \theta}{2} \right]_0^{\pi/2} \\ &= \pi |V| R^2\end{aligned}$$



**Exercise :**

Find the flux through a hemispherical bowl with its base on the x-y plane and the origin at the centre of the base. The vector field, in spherical polar coordinates is  $\vec{V} = r \sin \theta \hat{r} + \hat{\theta} + \hat{\phi}$ . (Ans.  $\pi(1 + \pi/2)$ )

**Example 12 :**

A cylindrical object occupies a volume defined by  $x^2 + y^2 \leq R^2$  and  $0 \leq z \leq h$ . Find the flux through each of the surfaces when the object is in a vector field  $\vec{V} = \hat{i}x + \hat{j}y + \hat{k}z$ .

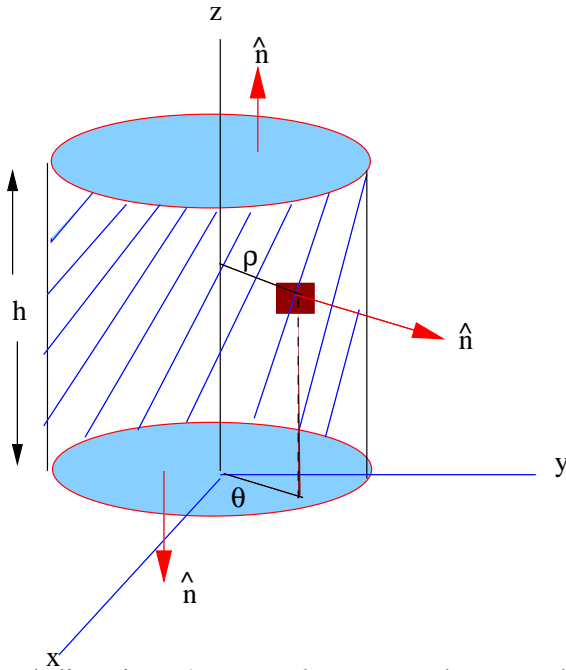
**Solution :**

Because of cylindrical symmetry, it is convenient to work in a cylindrical  $(\rho, \theta, z)$  coordinates. The vector field is given by

$$\vec{V} = \rho\hat{\rho} + z\hat{k}$$

Bottom face has  $z = 0$  and the normal to the face points in  $-\hat{k}$  direction. Thus the flux from this face  $\int \vec{V} \cdot \hat{n} = \int V_z \rho d\theta d\rho = 0$ . The top face has  $z = h$  and  $\hat{n} = \hat{k}$ . The flux is

$$\begin{aligned} \int V_z \rho d\theta d\rho &= h \int_0^R \rho d\rho \int_0^{2\pi} d\theta \\ &= h \frac{R^2}{2} \cdot 2\pi = \pi R^2 h \end{aligned}$$



The normal to the curved face is along  $\hat{\rho}$  direction. An area element on the curved face is  $R d\theta dz$ . Thus the flux from this face is

$$\int (R\hat{\rho} + z\hat{k}) \cdot \hat{\rho} R d\theta dz = R^2 \int_0^{2\pi} d\theta \int_0^h dz = 2\pi R^2 h$$

The net flux from the faces of the object is  $3\pi R^2 h$ .

**Exercise :**

Find the flux of the vector field  $\vec{V} = 2\hat{\rho} - 3r\hat{\theta} + z\rho\hat{k}$  through surfaces of a right cylinder of radius 1 and height 2. The base of the cylinder is in the  $z = 0$  plane with the origin at the centre of the base. (Ans.  $28\pi/3$ )

### 1.3 Gradient of a Scalar Function :

Consider a scalar field such as temperature  $T(x, y, z)$  in some region of space. The distribution of temperature may be represented by drawing isothermal surfaces or contours connecting points of identical temperatures,

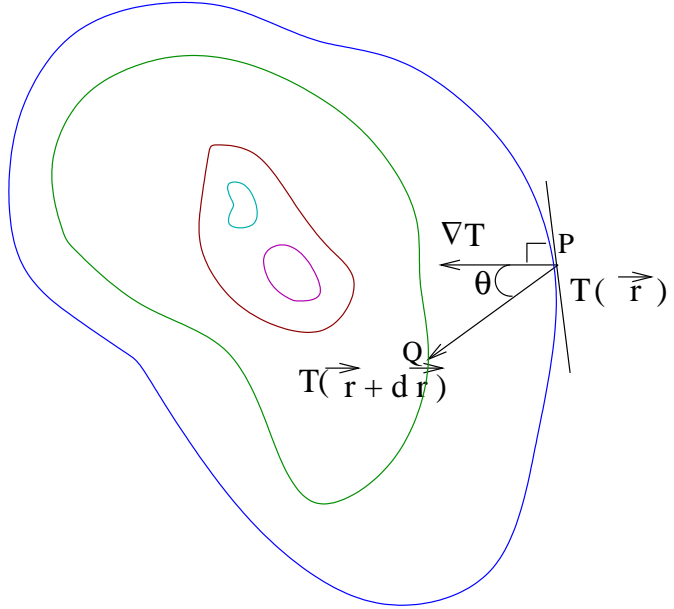
$$T(x, y, z) = \text{constant}$$

One can draw such contours for different temperatures. If we are located at a point  $\vec{r}$  on one of these contours and move away along any direction other than along the contour, the temperature would change.

The change  $\Delta T$  in temperature as we move away from a point  $P(x, y, z)$  to a point  $Q(x + \Delta x, y + \Delta y, z + \Delta z)$  is given by

$$\Delta T = \frac{\partial T}{\partial x} \Delta x + \frac{\partial T}{\partial y} \Delta y + \frac{\partial T}{\partial z} \Delta z$$

where the derivatives in the above expression are partial derivatives.



If the displacement from the initial position is infinitesimal, we get

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

Note that the change  $dT$  involves a change in temperature with respect to each of the three directions. We define a vector called the **gradient** of  $T$ , denoted by  $\nabla T$  or  $\text{grad } T$  as

$$\nabla T = \hat{i} \frac{\partial T}{\partial x} + \hat{j} \frac{\partial T}{\partial y} + \hat{k} \frac{\partial T}{\partial z}$$

using which, we get

$$dT = \nabla T \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz) = \nabla T \cdot d\vec{r}$$

Note that  $\nabla T$ , the gradient of a scalar  $T$  is itself a vector. If  $\theta$  is the angle between the direction of  $\nabla T$  and  $d\vec{r}$ ,

$$dT = |\nabla T| |d\vec{r}| \cos \theta = (\nabla T)_r |d\vec{r}|$$

where  $(\nabla T)_r$  is the component of the gradient in the direction of  $d\vec{r}$ . If  $d\vec{r}$  lies on an isothermal surface then  $dT = 0$ . Thus,  $\nabla T$  is perpendicular to the surfaces of constant  $T$ . When  $d\vec{r}$  and  $\nabla T$  are parallel,  $\cos \theta = 1$   $dT$  has maximum value. Thus the magnitude of the gradient is equal to the maximum rate of change of  $T$  and its direction is along the direction of greatest change.

The above discussion is true for any scalar field  $V$ . If a vector field can be written as a gradient of some scalar function, the latter is called the *potential* of the vector field. This fact is of importance in defining a conservative field of force in mechanics. Suppose we have a force field  $\vec{F}$  which is expressible as a gradient

$$\vec{F} = \nabla V$$

The line integral of  $\vec{F}$  can then be written as follows :

$$\begin{aligned} \int_i^f \vec{F} \cdot d\vec{l} &= \int_i^f \nabla V \cdot d\vec{l} \\ &= \int_i^f dV = V_f - V_i \end{aligned}$$

where the symbols  $i$  and  $f$  represent the initial and the final positions and in the last step we have used an expression for  $dV$  similar to that derived for  $dT$  above. Thus the line integral of the force field is independent of the path connecting the initial and final points. If the initial and final points are the same, i.e., if the particle is taken through a closed loop under the force field, we have

$$\oint \vec{F} \cdot d\vec{l} = 0$$

Since the scalar product of force with displacement is equal to the work done by a force, the above is a statement of conservation of mechanical energy. Because of this reason, forces for which one can define a potential function are called *conservative forces*.

**Example 13 :**

Find the gradient of the scalar function  $V = x^2y + y^2z + z^2x + 2xyz$ .

**Solution :**

$$\begin{aligned}\nabla V &= \hat{i} \frac{\partial V}{\partial x} + \hat{j} \frac{\partial V}{\partial y} + \hat{k} \frac{\partial V}{\partial z} \\ &= 2(xy + zx + yz)(\hat{i} + \hat{j} + \hat{k})\end{aligned}$$

**Exercise :** Find the gradients of

(i)  $xz - x^2y + y^2z^2$

(ii)  $x^3 + y^3 + z^3$

(iii)  $\ln \sqrt{x^2 + y^2 + z^2}$  (Ans.  $(x\hat{i} + y\hat{j} + z\hat{k})/(x^2 + y^2 + z^2)$ )

Gradient can be expressed in other coordinate systems by finding the length elements in the direction of basis vectors. For example, in cylindrical coordinates the length elements are  $d\rho$ ,  $\rho d\theta$  and  $dz$  along  $\hat{\rho}$ ,  $\hat{\theta}$  and  $\hat{k}$  respectively. The expression for gradient is

$$\nabla V = \hat{\rho} \frac{\partial V}{\partial \rho} + \hat{\theta} \frac{1}{\rho} \frac{\partial V}{\partial \theta} + \hat{k} \frac{\partial V}{\partial z}$$

The following facts may be noted regarding the gradient

1. The gradient of a scalar function is a vector
2.  $\nabla(U + V) = \nabla U + \nabla V$
3.  $\nabla(UV) = U\nabla(V) + V\nabla(U)$
4.  $\nabla(V^n) = nV^{n-1}\nabla V$

**Example 14 :**

Find the gradient of  $V = e^{-(x^2+y^2)}$  in cylindrical (polar) coordinates.

**Solution :**

In polar variables the function becomes  $V = e^{-\rho^2}$ . Thus

$$\begin{aligned}\nabla V &= \hat{\rho} \frac{\partial e^{-\rho^2}}{\partial \rho} \\ &= \hat{\rho} e^{-\rho^2} \cdot (-2)\rho = -2\hat{\rho}\rho V\end{aligned}$$

**Exercise :**

Find the gradient of the function  $V$  of Example 15 in cartesian coordinates and then transform into polar form to verify the answer.

**Exercise :**

Find the gradient of the function  $\ln \sqrt{\rho^2 + z^2}$  in cylindrical coordinates. (Ans.  $(\hat{\rho}\rho + \hat{k}z)/(\rho^2 + z^2)$ )

In spherical coordinates the length elements are  $dr, r d\theta$  and  $r \sin \theta d\phi$ . Hence the gradient of a scalar function  $U$  is given by

$$\nabla V = \hat{r} \frac{\partial V}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi}$$

**Exercise :**

Find the gradient of  $V = r^2 \cos \theta \cos \phi$  (Ans.  $2r \cos \theta \cos \phi \hat{r} - r \sin \theta \cos \phi \hat{\theta} - r \cos \theta \sin \phi \hat{\phi}$ .)

**Exercise :**

A potential function is given in cylindrical coordinates as  $k/\sqrt{\rho^2 + z^2}$ . Find the force field it represents and express the field in spherical polar coordinates. (Ans.  $-k\vec{r}/r^3$ )

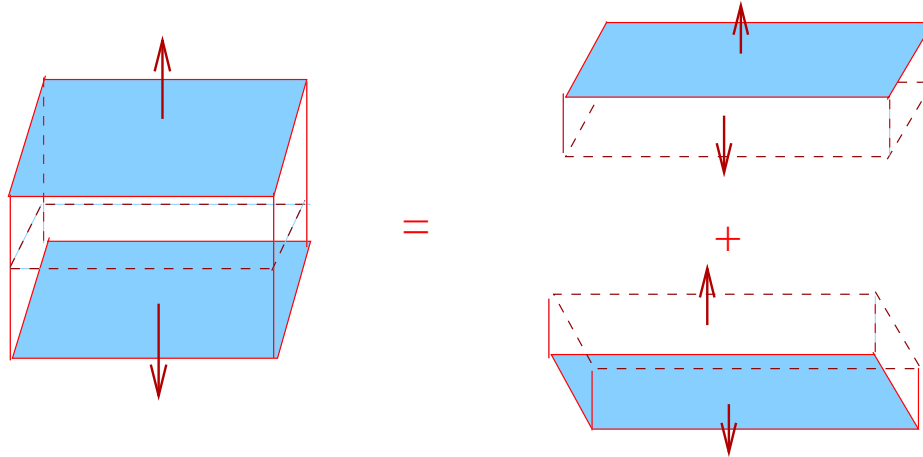
**1.4 Divergence of a Vector Field :**

Divergence of a vector field  $\vec{F}$  is a measure of net outward flux from a closed surface  $S$  enclosing a volume  $V$ , as the volume shrinks to zero.

$$\text{div} \vec{F} \equiv \vec{\nabla} \cdot \vec{F} = \lim_{\Delta V \rightarrow 0} \frac{\oint_S \vec{F} \cdot d\vec{S}}{\Delta V}$$

where  $\Delta V$  is the volume (enclosed by the closed surface  $S$ ) in which the point P at which the divergence is being calculated is located. Since the volume shrinks to zero, the divergence is a point relationship and is a scalar.

Consider a closed volume  $V$  bounded by  $S$ . The volume may be mentally broken into a large number of elemental volumes closely packed together. It is easy to see that the flux out of the boundary  $S$  is equal to the sum of fluxes out of the surfaces of the constituent volumes. This is because surfaces of boundaries of two adjacent volumes have their outward normals pointing opposite to each other. The following figure illustrates it.



We can generalize the above to closely packed volumes and conclude that the flux out of the bounding surface  $S$  of a volume  $V$  is equal to the sum of fluxes out of the elemental cubes. If  $\Delta V$  is the volume of an elemental cube with  $\Delta S$  as the surface, then,

$$\int_S \vec{F} \cdot d\vec{S} = \sum \int_{\Delta S} \vec{F} \cdot d\vec{S} = \lim_{\Delta V \rightarrow 0} \left( \frac{1}{\Delta V} \int \vec{F} \cdot d\vec{S} \right) \Delta V$$

The quantity in the bracket of the above expression was defined as the divergence of  $\vec{F}$ , giving

$$\boxed{\int_S \vec{F} \cdot d\vec{S} = \int_V \text{div} \vec{F} dV}$$

This is known as the **Divergence Theorem**.

We now calculate the divergence of  $\vec{F}$  from an infinitesimal volume over which variation of  $\vec{F}$  is small so that one can retain only the first order term in a Taylor expansion. Let the dimensions of the volume element be  $\Delta x \times \Delta y \times \Delta z$  and let the element be oriented parallel to the axes.

Consider the contribution to the flux from the two shaded faces. On these faces, the normal is along the  $+\hat{j}$  and  $-\hat{j}$  directions so that the contribution to the flux is from the y-component of  $\vec{F}$  only and is given by



$$[F_y(y + dy) - F_y(y)] dx dz$$

Expanding  $F_y(y + dy)$  in a Taylor series and retaining only the first order term

$$F(y + dy) = F_y + \frac{\partial F_y}{\partial y} dy$$

so that the flux from these two faces is

$$\frac{\partial F_y}{\partial y} dx dy dz = \frac{\partial F_y}{\partial y} dV$$

where  $dV = dx dy dz$  is the volume  $x$  of the cuboid.

Combining the above with contributions from the two remaining pairs of faces, the total flux is

$$\left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dV$$

Thus

$$\int_S \vec{F} \cdot d\vec{S} = \int \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dV$$

Comparing with the statement of the divergence theorem, we have

$$\text{div} \vec{F} = \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right)$$

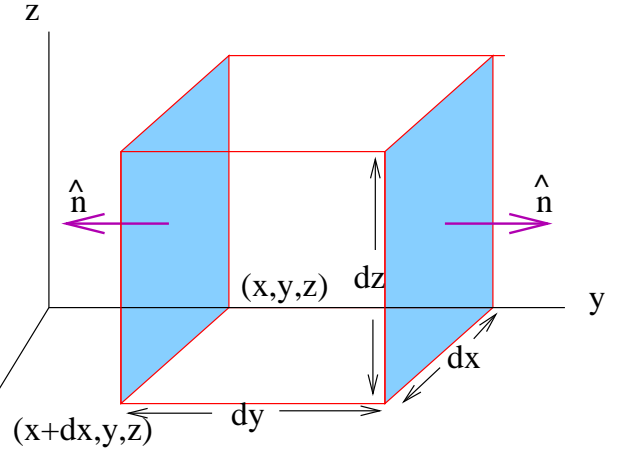
Recalling that the operator  $\nabla$  is given by

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

and using  $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$ , we can write  $\text{div} \vec{F} = \nabla \cdot \vec{F}$ .

The following facts may be noted :

1. The divergence of a vector field is a scalar
2.  $\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$



3.  $\nabla \cdot (\phi \vec{F}) = \phi \nabla \cdot \vec{F} + \nabla \phi \cdot \vec{F}$

4. In cylindrical coordinates

$$\nabla \cdot \vec{F} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \theta} F_\theta + \frac{\partial}{\partial z} F_z$$

5. In spherical polar coordinates

$$\nabla \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (F_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} F_\phi$$

6. The divergence theorem is

$$\oint_S \vec{F} \cdot d\vec{S} = \int_V \nabla \cdot \vec{F} dV$$

**Example 15 :** Divergence of  $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

Divergence of position vector  $\vec{r}$  is very useful to remember.

$$\nabla \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

One can also calculate easily in spherical coordinate since  $\vec{r}$  only has radial component

$$\nabla \cdot \vec{r} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \cdot r) = \frac{1}{r^2} \cdot 3r^2 = 3$$

**Exercise :**

Calculate the divergence of the vector field  $\vec{r}/r^3$  using all the three coordinate systems. (Ans. 0)

**Example 16 :**

A vector field is given by  $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ . Find the surface integral of the field from the surfaces of a unit cube bounded by planes  $x = 0, x = 1, y = 0, y = 1, z = 0$  and  $z = 1$ . Verify that the result agrees with the divergence theorem.

**Solution :**

Divergence of  $\vec{F}$  is

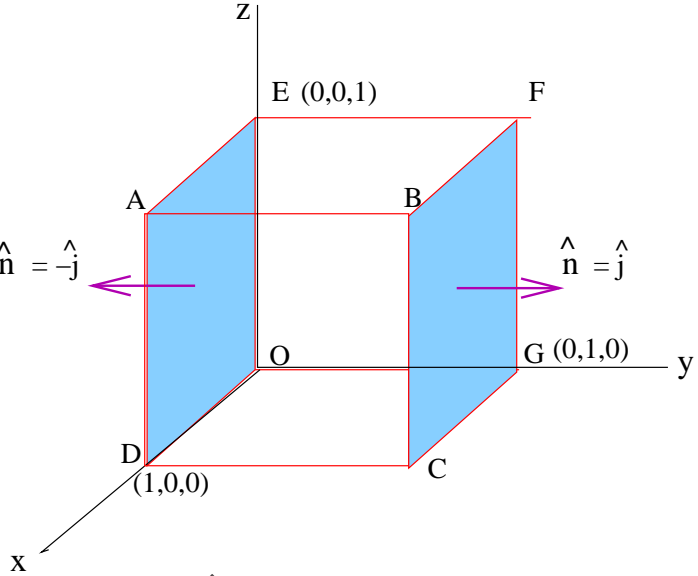
$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= \frac{\partial 4xz}{\partial x} + \frac{\partial (-y^2)}{\partial y} + \frac{\partial yz}{\partial z} \\ &= 4z - 2y + y = 4z - y \end{aligned}$$

The volume integral of above is

$$\int \nabla \cdot \vec{F} dV = \int_0^1 dx \int_0^1 dy \int_0^1 (4z - y) dx dy dz = \frac{3}{2}$$

Consider the surface integral from the six faces individually. For the face AEOD, the normal is along  $-\hat{j}$ . On this face  $y = 0$  so that  $\vec{F} = 4xz\hat{i}$ . Since  $\hat{i} \cdot \hat{j} = 0$ , the integrand is zero. For the surface BFGC, the normal is along  $\hat{j}$  and on this face  $y = 1$ .  $\hat{n} = \hat{j}$ . On this face the vector field is  $\vec{F} = 4xz\hat{i} - \hat{j} + z\hat{k}$ . The surface integral is

$$\begin{aligned} \int \vec{F} \cdot d\vec{S} &= \int \vec{F} \cdot \hat{j} dx dz \\ &= - \int_0^1 dx \int_0^1 dz = -1 \end{aligned}$$



Consider the top face (ABFE) for which the normal is  $\hat{k}$  so that the surface integral is  $\int F_z dy dx$ . On this face  $z = 1$  and  $F_z = y$ . The contribution to the surface integral from this face is

$$\int_0^1 dx \int_0^1 y dy = \frac{1}{2}$$

For the bottom face (DOGC) the normal is along  $-\hat{k}$  and  $z = 0$ . This gives  $F_z = 0$  so that the integral vanishes.

For the face EFGO the normal is along  $-\hat{i}$  so that the surface integral is  $-\int F_x dy dz$ . On this face  $x = 0$  giving  $F_x = 0$ . The surface integral is zero. For the front face ABCD, the normal is along  $\hat{i}$  and on this face  $x = 1$  giving  $F_x = 4z$ . The surface integral is

$$\int_0^1 dy \int_0^1 4z dz = 2$$

Adding the six contributions above, the surface integral is  $3/2$  consistent with the divergence theorem.

#### Exercise :

Verify the divergence theorem by calculating the surface integral of the vector field  $\vec{F} = \hat{i}x^3 + \hat{j}y^3 + \hat{k}z^3$  for the cubical volume of Example 17. (Ans. Surface

integral has value 3)

**Example 17 :**

In Example 13 we found that the surface integral of a vector field  $x\hat{i} + y\hat{j} + z\hat{k}$  over a cylinder of radius  $R$  and height  $h$  is  $3\pi R^2 h$ . Verify this result using the divergence theorem.

**Solution :**

In Example 16 we have seen that the divergence of the field vector is 3. Since the integrand is constant, the volume integral is  $3V = 3\pi R^2 h$ .

**Example 18 :**

A vector field is given by  $\vec{F} = x^3\hat{i} + y^3\hat{j}$ . Verify Divergence theorem for a cylinder of radius 2 and height 5. The origin of the coordinate system is at the centre of the base of the cylinder and z-axis along the axis.

**Solution :**

The problem is obviously simple in cylindrical coordinates. The divergence can be easily seen to be  $3(x^2 + y^2) = 3\rho^2$ . Recalling that the volume element is  $\rho d\rho d\theta dz$ , the integral is

$$\int \text{div} \vec{F} dV = \int 3\rho^2 dV = 3 \int_0^2 \rho^3 d\rho \int_0^{2\pi} d\theta \int_0^5 dz = 120\pi$$

In order to calculate the surface integral, we first observe that the end faces have their normals along  $\pm\hat{k}$ . Since the field does not have any z- component, the contribution to surface integral from the end faces is zero.

We will calculate the contribution to the surface integral from the curved surface. Using the coordinate transformation to cylindrical

$$x = \rho \cos \theta \quad y = \rho \sin \theta$$

and

$$\begin{aligned} \hat{i} &= \hat{\rho} \cos \theta - \hat{\theta} \sin \theta \\ \hat{j} &= \hat{\rho} \sin \theta + \hat{\theta} \cos \theta \\ \hat{k} &= \hat{k} \end{aligned}$$

Using these

$$\vec{F} = \rho^3(\cos^4 \theta + \sin^4 \theta)\hat{\rho} + \rho^3(\sin^3 \theta \cos \theta - \cos^3 \theta \sin \theta)\hat{\theta}$$

The area element on the curved surface is  $R d\theta dz \hat{\rho}$ , where  $R$  is the radius. Thus the surface integral is

$$\int F_{\rho} dS = R^4 \int_0^{2\pi} (\cos^4 \theta + \sin^4 \theta) d\theta \int_0^5 dz$$

$$= 16 \cdot \frac{3\pi}{2} \cdot 5 = 120\pi$$

where we have used  $\int_0^{2\pi} \sin^4 \theta d\theta = \int_0^{2\pi} \cos^4 \theta d\theta = 3\pi/4$ .

**Exercise :**

In the Exercise following Example 13, we had seen that surface integral of the vector field  $\vec{V} = 2\hat{\rho} - 3\rho\hat{\theta} + z\rho\hat{k}$  through the surface of a cylinder of radius 1 and height 2 is  $28\pi/3$ . Re-confirm the same result using divergence theorem.

**Example 19 :**

A hemispherical bowl of radius 1 lies with its base on the x-y plane and the origin at the centre of the circular base. Calculate the surface integral of the vector field  $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$  in the hemisphere and verify the divergence theorem.

**Solution :**

The divergence of  $\vec{F}$  is easily calculated

$$\begin{aligned}\nabla \cdot \vec{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ &= \frac{\partial x^3}{\partial x} + \frac{\partial y^3}{\partial y} + \frac{\partial z^3}{\partial z} \\ &= 3(x^2 + y^2 + z^2) = 3r^2\end{aligned}$$

where  $r$  is the distance from origin. The volume integral over the hemisphere is conveniently calculated in spherical polar using the volume element  $r^2 \sin \theta d\theta d\phi$ . Since it is a hemisphere with  $z = 0$  as the base, the range of  $\theta$  is 0 to  $\pi/2$ .

$$\begin{aligned}\nabla \cdot \vec{F} dV &= 3 \int r^2 dV = 3 \int_0^1 r^4 dr \int_0^{\pi/2} \sin \theta d\theta \int_0^{2\pi} d\phi \\ &= 3 \times \frac{1}{5} \times 1 \times 2\pi = \frac{6\pi}{5}\end{aligned}$$

The surface integrals are calculated conveniently in spherical polar. There is no contribution to the flux from the base because the outward normal points in the  $-z$  direction but the z-component of the field is zero because the base of the hemisphere is  $z = 0$ .

In order to calculate the flux from the curved face we need to express the force field and the unit vectors in spherical polar coordinates. Using the transformation properties given earlier and observing that we only require the radial component of the vector field since the area element is radially directed. Using  $R^2 \sin \theta d\theta d\phi$  as the area element, a bit of laborious algebra gives

$$\begin{aligned}\vec{F} \cdot d\vec{S} &= R^5 \int_0^{\pi/2} \sin^5 \theta d\theta \int_0^{2\pi} \cos^4 \phi d\phi + R^5 \int_0^{\pi/2} \sin^5 \theta d\theta \int_0^{2\pi} \sin^4 \phi d\phi \\ &+ R^5 \int_0^{\pi/2} \cos^4 \theta \sin \theta d\theta \int_0^{2\pi} d\phi\end{aligned}$$

Using  $\int_0^{\pi/2} \sin^5 \theta d\theta = 8/15$  and  $\int_0^{2\pi} \cos^4 \phi d\phi = 3\pi/4$ , the above integral can be seen to give the correct result.

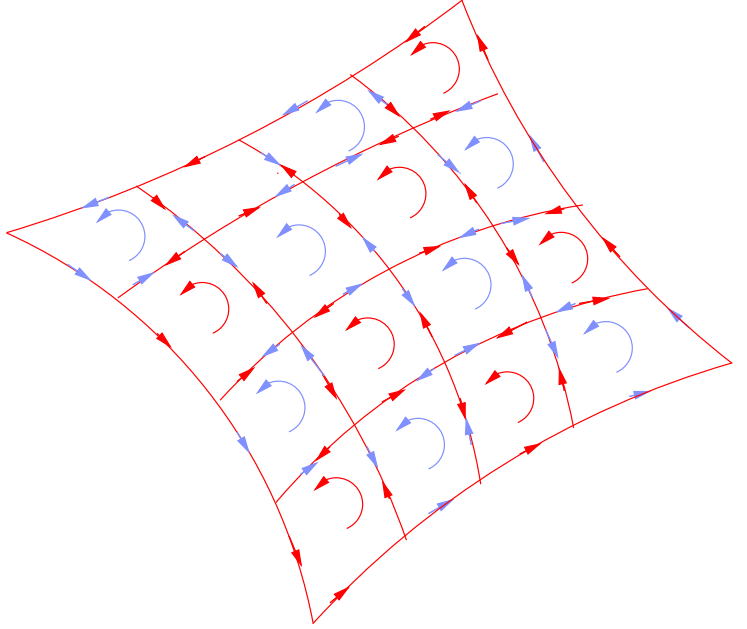
### 1.5 Curl of a Vector - Stoke's Theorem

We have seen that the line integral of a vector field  $\oint \vec{F} \cdot d\vec{l}$  is essentially a sum of the component of  $\vec{F}$  along the curve. If the line integral is taken over a closed path, we represent it as  $\oint \vec{F} \cdot d\vec{l}$ . If the vector field is conservative, i.e., if there exists a scalar function  $V$  such that one can write  $\vec{F}$  as  $\nabla V$ , the contour integral is zero. In other cases, it is, in general, non-zero.

Consider a contour  $\mathcal{C}$  enclosing a surface  $S$ . We may split the contour into a large number of elementary surface areas defined by a mesh of closed contours.

Since adjacent contours are traversed in opposite directions, the only non-vanishing contribution to the integral comes from the boundary of the contour  $\mathcal{C}$ . If the surface area enclosed by the  $i$ -th cell is  $\Delta S_i$ , then

$$\begin{aligned}\oint_{\mathcal{C}} \vec{F} \cdot d\vec{l} &= \sum_i \oint_{\mathcal{C}_i} \vec{F} \cdot d\vec{l} \\ &= \sum_i \frac{\oint_{\mathcal{C}_i} \vec{F} \cdot d\vec{l}}{\Delta S_i} \Delta S_i\end{aligned}$$



We define the quantity

$$\lim_{\Delta S_i \rightarrow 0} \frac{\oint_{\mathcal{C}_i} \vec{F} \cdot d\vec{l}}{\Delta S_i} \hat{n}_i$$

as the curl of the vector  $\vec{F}$  at a point P which lies on the surface  $\Delta S_i$ . Since the area  $\Delta S_i$  is infinitesimal it is a point relationship. The direction of  $\hat{n}_i$  is, as usual,

along the outward normal to the area element  $\Delta S_i$ . For instance, the x-component of the curl is given by

$$(\text{curl})_x = \lim_{\Delta y, \Delta z \rightarrow 0} \frac{\oint_{C_i} \vec{F} \cdot d\vec{l}}{\Delta y \Delta z} \hat{n}_i$$

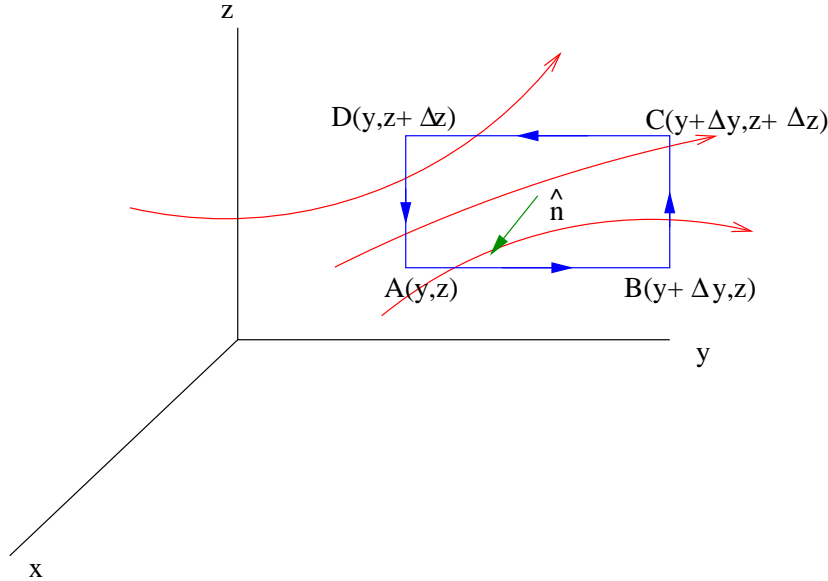
Thus

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S \text{curl} \vec{F} \cdot d\vec{S}$$

This is **Stoke's Theorem** which relates the surface integral of a curl of a vector to the line integral of the vector itself. The direction of  $d\vec{l}$  and  $d\vec{S}$  are fixed by the right hand rule, i.e. when the fingers of the right hand are curled to point in the direction of  $d\vec{l}$ , the thumb points in the direction of  $d\vec{S}$ .

### 1.5.1 Curl in Cartesian Coordinates :

We will obtain an expression for the curl in the cartesian coordinates. Let us consider a rectangular contour ABCD in the y-z plane having dimensions  $\Delta y \times \Delta z$ . The rectangle is oriented with its edges parallel to the axes and one of the corners is located at  $(y, z)$ . We will calculate the line integral of a vector field  $\vec{F}$  along this contour. We assume the field to vary slowly over the length (or the breadth) so that we may retain only the first term in a Taylor expansion in computing the field variation.



Contribution to the line integral from the two sides AB and CD are computed as follows.

On AB :  $\int \vec{F} \cdot d\vec{l} = \int \vec{F} \cdot \hat{j} dy = \int F_y dy$

ON CD :  $\int \vec{F} \cdot d\vec{l} = \int \vec{F} \cdot (-\hat{j}) dy = -\int F_y dy$

Using Taylor expansion (retaining only the first order term), we can write

$$F_y|_{CD} = F_y|_{AB} + \frac{\partial F_y}{\partial z} \Delta z$$

Thus the line integral from the pair of sides AB and CD is

$$-\int \frac{\partial F_y}{\partial z} \Delta z dy \approx -\frac{\partial F_y}{\partial z} \Delta z \Delta y$$

In a similar way one can calculate the contributions from the sides BC and DA and show it to be

$$\int \frac{\partial F_z}{\partial y} \Delta y dz \approx \frac{\partial F_z}{\partial y} \Delta y \Delta z$$

Adding up we get

$$(\text{curl } F)_x = \lim_{\Delta y, \Delta z \rightarrow 0} \frac{\left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \Delta y \Delta z}{\Delta y \Delta z} = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

In a very similar way, one can obtain expressions for the y and z components

$$\begin{aligned} (\text{curl } F)_y &= \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ (\text{curl } F)_z &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{aligned}$$

One can write the expression for the curl of  $\vec{F}$  by using the del operator as

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

### Expression for curl in Cylindrical and Spherical Coordinates :

In the cylindrical coordinates the curl is given by

$$\begin{aligned} \nabla \times \vec{F} &= \left( \frac{1}{\rho} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \hat{\rho} \\ &+ \left( \frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) \hat{\theta} + \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_\theta) - \frac{1}{\rho} \frac{\partial F_\rho}{\partial \theta} \right) \hat{k} \end{aligned}$$

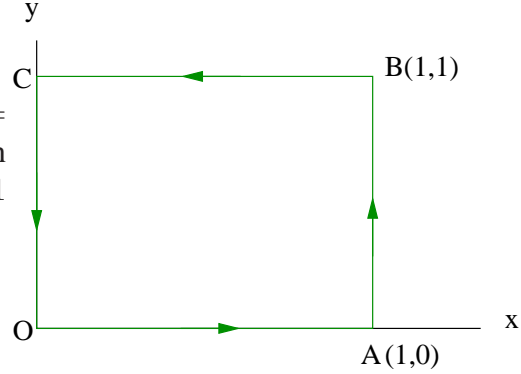


In the spherical coordinates the corresponding expression for the curl is

$$\begin{aligned}\nabla \times \vec{F} &= \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \theta} (F_\phi \sin \theta) - \frac{\partial F_\theta}{\partial \phi} \right) \hat{r} \\ &+ \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r} (r F_\phi) \right) \hat{\theta} + \frac{1}{r} \left( \frac{\partial}{\partial r} (r F_\theta) - \frac{\partial F_r}{\partial \theta} \right) \hat{\phi}\end{aligned}$$

**Example 20 :**

Verify Stoke's theorem for  $\vec{F} = x^2\hat{i} + 2x\hat{j} + z^2\hat{k}$  for the rectangle shown below, defined by sides  $x = 0, y = 0, x = 1$  and  $y = 1$ .



The line integrals along the four sides are

$$\begin{aligned}\oint \vec{F} \cdot d\vec{l} &= \int_0^1 F_x|_{y=0} dx + \int_0^1 F_y|_{x=0} dy + \int_1^0 F_x|_{y=1} dx + \int_0^1 F_y|_{x=1} dy \\ &= \int_0^1 x^2 dx + 0 + \int_1^0 x^2 dx + \int_0^1 2dy \\ &= \frac{1}{3} + 0 - \frac{1}{3} + 2 = 2\end{aligned}$$

Since the normal to the plane is along  $\hat{k}$ , we only need  $z$ - component of  $\nabla \times \vec{F}$  to calculate the surface integral. It can be checked that

$$(\nabla \times \vec{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 2 - 0 = 2$$

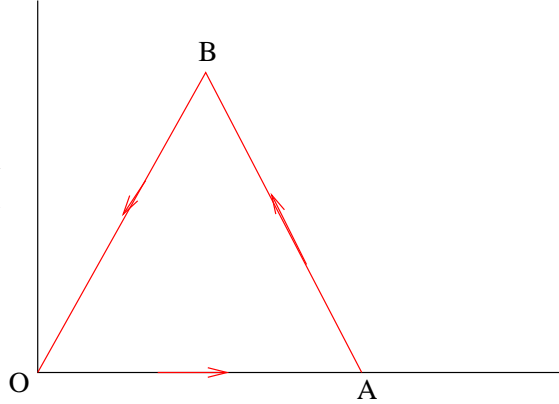
Thus

$$\int_S (\nabla \times \vec{F}) \cdot \hat{k} dS = \int_0^1 \int_0^1 2 dx dy = 2$$

which agrees with the line integral calculated.

**Exercise :**

A vector field is given by  $\vec{F} = -y\hat{i} + x\hat{j} + x^2\hat{k}$ . Calculate the line integral of the field along the triangular path shown above. Verify your result by Stoke's theorem. (Ans. 1)



(Hint : To calculate the line integral along a straightline, you need the equation to the line. For instance, the equation to the line BO is  $y = 2x$ . Check that  $\int_{AB} \vec{F} \cdot d\vec{l} = -\int ydx + \int xdy = 1$ .)

**Example 21 :**

A vector field is given by  $\vec{F} = -y\hat{i} + z\hat{j} + x^2\hat{k}$ . Calculate the line integral of the field along a circular path of radius  $R$  in the x-y plane with its centre at the origin. Verify Stoke's theorem by considering the circle to define (i) the plane of the circle and (ii) a cylinder of height  $z = h$ .

**Solution :**

The curl of  $\vec{F}$  may be calculated as

$$\nabla \times \vec{F} = -\hat{i} + 2x\hat{j} + \hat{k}$$

Because of symmetry, we use cylindrical (polar) coordinates. The transformations are  $x = \rho \cos \theta$ ,  $y = \rho \sin \theta$ ,  $z = z$ . The unit vectors are

$$\begin{aligned}\hat{i} &= \hat{\rho} \cos \theta - \hat{\theta} \sin \theta \\ \hat{j} &= \hat{\rho} \sin \theta + \hat{\theta} \cos \theta \\ \hat{k} &= \hat{k}\end{aligned}$$

Substituting the above, the field  $\vec{F}$  and its curl are given by

$$\begin{aligned}\vec{F} &= \hat{\rho}(-\rho \sin \theta \cos \theta + z \sin \theta) \\ &+ \hat{\theta}(\rho \sin^2 \theta + z \cos \theta) + \hat{k}\rho^2 \cos^2 \theta \\ \nabla \times \vec{F} &= \hat{\rho}(-\cos \theta + 2\rho \cos \theta \sin \theta) \\ &+ \hat{\theta}(\sin \theta + 2\rho \cos^2 \theta) + \hat{k}\end{aligned}$$

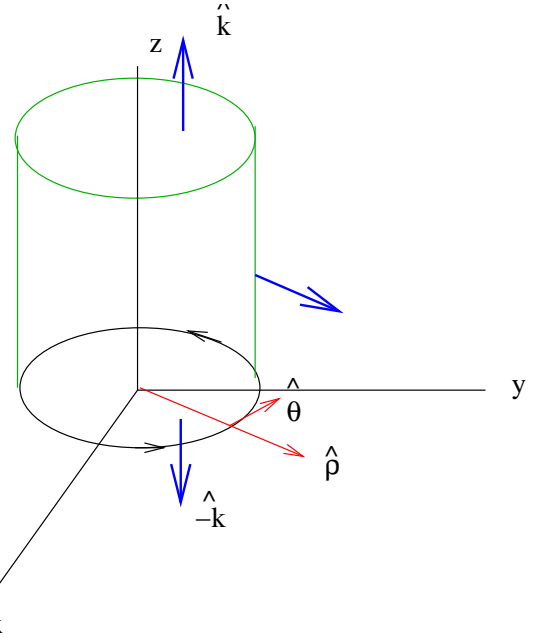
The line integral of  $\vec{F}$  around the circular loop :

Since the line element is  $d\vec{l} = R d\theta \hat{\theta}$ ,

$$\oint \vec{F} \cdot d\vec{l} = \int_0^{2\pi} (R \sin^2 \theta + z \cos \theta) R d\theta$$

On the circle  $z = 0$ . The integral over  $\sin^2 \theta$  gives  $1/2$ . Hence

$$\oint \vec{F} \cdot d\vec{l} = \pi R^2$$



Let us calculate the surface integral of the curl of the field over two surfaces bound by the circular curve.

(i) On the circular surface bound by the curve in the x-y plane, the outward normal is along  $\hat{k}$  (right hand rule). Thus

$$\int_S (\nabla \times \vec{F}) \cdot (\hat{k} dS) = \int_S dS = \pi R^2$$

(ii) For the cylindrical cup, we have two surfaces : the curved face of the cylinder on which  $\hat{n} = \hat{\rho}$  and the top circular face on which  $\hat{n} = \hat{k}$ . The contribution from the top circular cap is  $\pi R^2$ , as before because the two caps only differ in their  $z$  values (the  $z$ -component of the curl is independent of  $z$ ). The surface integral from the curved surface is (the area element is  $R d\theta dz \hat{\rho}$ )

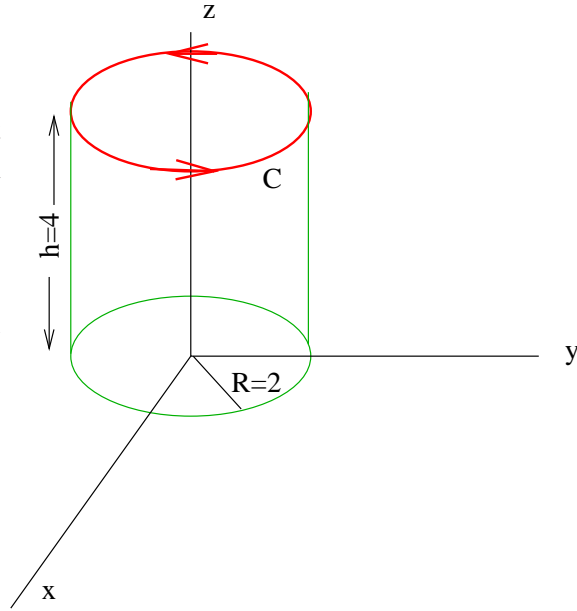
$$\int_0^{2\pi} R d\theta \int_0^h dz (-R \sin \theta \cos \theta + z \cos \theta)$$

For both the terms of the above integral, the angle integration gives zero. Thus the net surface integral is  $\pi R^2$ , as expected.

**Exercise :**

A vector field is given by  $\vec{F} = k\rho^2 z \hat{\theta}$ . Check the validity of the Stoke's theorem

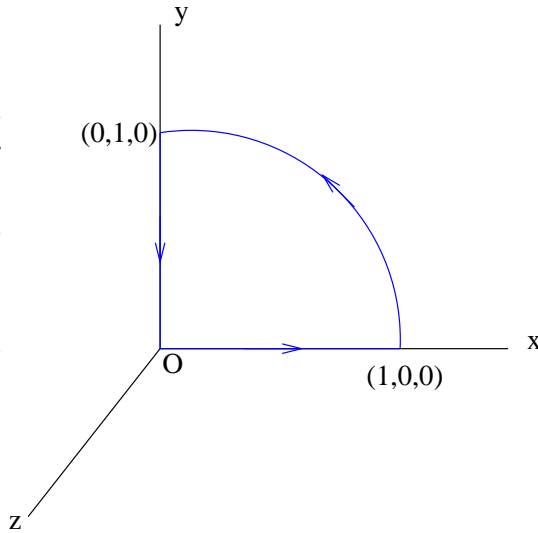
by calculating the line integral about the closed contour in the form of a circle at  $z = 4$  and also calculating the surface integral of the open surface of the cylinder below it, as shown. (Hint : Express the curl in cylindrical coordinates and take care of the signs of the surface elements from the curved surface and the bottom cap. Ans.  $64k\pi$  )



**Exercise :**

Let  $C$  be a closed curve in the  $x$ - $y$  plane in the shape of a quadrant of a circle of radius  $R$ .

If  $\vec{F} = \hat{i}y + \hat{j}z + \hat{k}x$ . calculate the line integral of the field along the contour shown in a direction which is anti-clockwise when looked from above the plane ( $z > 0$ ). Take the surface of the quadrant enclosed by the curve as the open surface bounded by the curve and verify Stoke's theorem. (Ans.  $-\pi R^2/4$ )



**Example 22 :**

A vector field is given by  $\vec{F}(r, \theta, \phi) = f(r)\hat{\phi}$  where  $\phi$  is the azimuthal angle variable of a spherical coordinate system. Calculate the line integral over a circle of radius  $R$  in the  $x$ - $y$  plane centered at the origin. Consider an open surface in the form of a hemispherical bowl in the northern hemisphere bounded by the circle.

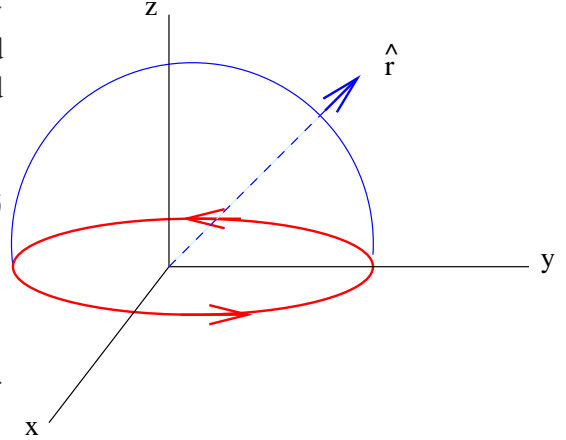
**Solution :**

On the equatorial circle  $\vec{dl} = R d\phi \hat{\phi}$ . Hence,

$$\oint \vec{F} \cdot \vec{dl} = \int_0^{2\pi} f(R) R d\phi = 2\pi R f(R)$$

The expression for curl in spherical coordinates may be used to calculate the curl of  $\vec{F}$ . Since the field only has azimuthal component, the curl has radial and polar ( $\theta$ ) components.

$$\begin{aligned} \nabla \times \vec{F} &= \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (f(r) \sin \theta) \hat{r} - \frac{1}{r} \frac{\partial}{\partial r} (r f(r)) \hat{\theta} \\ &= \frac{f(r) \cos \theta}{r \sin \theta} \hat{r} - \left( \frac{f(r)}{r} + \frac{\partial f(r)}{\partial r} \right) \hat{\theta} \end{aligned}$$



The area element on the surface of the northern hemisphere is  $R^2 \sin \theta d\theta d\phi \hat{r}$ . Hence the surface integral is

$$\begin{aligned} \int_0^{2\pi} d\phi \int_0^{\pi/2} \frac{f(r) \cos \theta}{R \sin \theta} R^2 \sin \theta d\theta &= 2\pi R f(R) \int_0^{\pi/2} \cos \theta d\theta \\ &= 2\pi R f(R) \end{aligned}$$

**Exercise :**

Verify Stoke's theorem for a vector field  $2z\hat{i} + 3x\hat{j} + 5y\hat{k}$  where the contour is an equatorial circle of radius  $R$  and is anticlockwise when viewed from above and the surface is the hemisphere shown in the preceding example.. (Ans.  $3R^2\pi$ )

**1.6 Laplacian :**

Since gradient of a scalar field gives a vector field, we may compute the divergence of the resulting vector field to obtain yet another scalar field. The operator  $\text{div}(\text{grad}) = \nabla \cdot \nabla$  is called the **Laplacian** and is written as  $\nabla^2$ .

If  $V$  is a scalar, then,

$$\begin{aligned} \nabla^2 V &= \nabla \cdot (\nabla V) \\ &= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left( \hat{i} \frac{\partial V}{\partial x} + \hat{j} \frac{\partial V}{\partial y} + \hat{k} \frac{\partial V}{\partial z} \right) \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \end{aligned}$$

**Example 23 :**

Calculate the Laplacian of  $1/r = 1/\sqrt{x^2 + y^2 + z^2}$ .

**Solution :**

$$\begin{aligned}\frac{\partial^2}{\partial x^2} \frac{1}{\sqrt{x^2 + y^2 + z^2}} &= \frac{\partial}{\partial x} \frac{-x}{(x^2 + y^2 + z^2)^{3/2}} \\ &= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \\ &= \frac{3x^2 - r^2}{r^5}\end{aligned}$$

Adding similar contributions from  $\partial^2/\partial y^2$  and  $\partial^2/\partial z^2$ , we get

$$\nabla^2 \frac{1}{r} = \frac{3(x^2 + y^2 + z^2) - 3r^2}{r^5} = \frac{3r^2 - 3r^2}{r^5} = 0$$

**Laplacian in cylindrical and spherical coordinates:**

In cylindrical :

$$\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

In spherical :

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

Frequently the Laplacian of a vector field is used. It is simply a short hand notation for the componentwise Laplacian

$$\nabla^2 \vec{F} = \hat{i} \nabla^2 F_x + \hat{j} \nabla^2 F_y + \hat{k} \nabla^2 F_z$$

**Exercise :**

Show that

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

**1.7 Dirac- Delta Function :**

In electromagnetism, we often come across use of a function known as Dirac-  $\delta$  function. The peculiarity of the function is that though the value of the function

is zero everywhere, other than at one point, the integral of the function over any region which includes this singular point is finite. We define

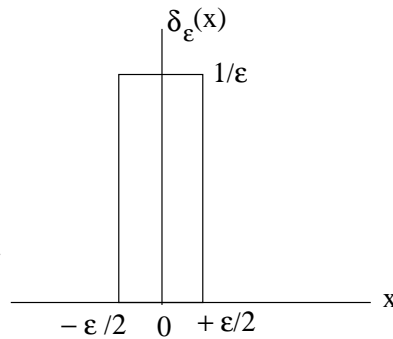
$$\delta(x - a) = 0 \text{ if } x \neq a$$

with

$$\int f(x)\delta(x - a)dx = f(a)$$

where  $f(x)$  is any function that is continuous at  $x = a$ , provided that the range of integration includes the point  $x = a$ . Strictly speaking,  $\delta(x - a)$  is not a function in the usual sense as Riemann integral of any function which is zero everywhere, excepting at discrete set of points should be zero. However, one can look at the  $\delta$  function as a limit of a sequence of functions. For instance, if we define a function  $\delta_\epsilon(x)$  such that

$$\begin{aligned} \delta_\epsilon(x) &= \frac{1}{\epsilon} \text{ for } -\frac{\epsilon}{2} < x < +\frac{\epsilon}{2} \\ &= 0 \text{ for } |x| > \frac{\epsilon}{2} \end{aligned}$$



Then  $\delta(x)$  can be thought of as the limit of  $\delta_\epsilon(x)$  as  $\epsilon \rightarrow 0$ .

One can easily extend the definition to three dimensions

$$\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

which has the property

$$\int f(\vec{r})\delta(\vec{r} - \vec{a}) = f(\vec{a})$$

provided, of course, the range of integration includes the point  $\vec{r} = \vec{a}$ .

**Example :**

Evaluate  $\int_0^5 \cos x \delta(x - \pi) dx$ .

**Solution :**

The range of integration includes the point  $x = \pi$  at which the argument of the delta function vanishes. Thus, the value of the integral is  $\cos \pi = -1$ .

**Exercise :**

Evaluate  $\int \vec{r} \cdot (\vec{a} - \vec{r}) \delta(\vec{r} - \vec{b}) d^3r$ , where  $\vec{a} = (1, 2, 3)$ ,  $\vec{b} = (3, 2, 1)$  and the integration is over a sphere of radius 1.5 centered at (2,2,2) (Ans. -4)

A physical example is the volume density of charge in a region which contains a point charge  $q$ . The charge density is zero everywhere except at the point where the charge is located. However, the volume integral of the density in any region which includes this point is equal to  $q$  itself. Thus if  $q$  is located at the point  $\vec{r} = \vec{a}$ , we can write

$$\rho(\vec{r}) = q\delta(\vec{r} - \vec{a})$$

**Example :**

Show that  $\nabla^2(1/r)$  is a delta function.

**Solution :**

As  $r = \sqrt{x^2 + y^2 + z^2}$ , we have

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

using this it is easy to show that

$$\frac{\partial^2}{\partial x^2} \left( \frac{1}{r} \right) = \frac{3x^2 - r^2}{r^5}$$

Thus

$$\nabla^2 \left( \frac{1}{r} \right) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left( \frac{1}{r} \right) = \frac{3(x^2 + y^2 + z^2) - 3r^2}{r^5} = 0$$

However, the above is not true at the origin as  $1/r$  diverges at  $r = 0$  and is not differentiable at that point. Interestingly, however, the integral of  $\nabla^2(1/r)$  over any volume which includes the point  $r = 0$  is not zero. As the value of the integrand is zero everywhere excepting at the origin, the point  $r = 0$  has to be treated with care.

Consider an infinitesimally small sphere of radius  $r_0$  with the centre at the origin. Using divergence theorem, we have,

$$\int_V \nabla^2 \left( \frac{1}{r} \right) d^3r = \int_V \nabla \cdot \nabla \left( \frac{1}{r} \right) d^3r = \int_S \nabla \left( \frac{1}{r} \right) \cdot d\vec{S}$$

where the last integral is over the surface  $S$  of the sphere defined above. As the gradient is taken at points on the surface for which  $r \neq 0$ , we may replace  $\nabla(1/r)$  with  $-1/r_0^2$  at all points on the surface. Thus the value of the integral is

$$-\frac{1}{r_0^2} \int_S dS = -\frac{1}{r_0^2} \cdot 4\pi r_0^2 = -4\pi$$

Hence

$$\nabla^2 \left( \frac{1}{r} \right) = -4\pi\delta(r)$$