Lecturer Report Tutorial Questions

General statistics

- Number of conversations: 35

- Number of questions in this tutorial: 7

Overall insights

The student’s struggles across several topics boil down to three interrelated themes:

1. Unclear or mixed-up definitions

• Eigenvalues vs. eigenvectors (λ I, A v = λ v, characteristic polynomial)

• Symmetric matrix ($A = A^T$, $a_{ij} = a_{ji}$) vs. identity or commuting factors

• Orthogonal vectors vs. orthogonal matrix ($M^T M = I$) vs. linear independence

• Rotation matrix entries (linking cos θ, sin θ to geometric rotation)

2. Gaps in algebraic mechanics and notation

• Mixing scalars with matrices (when and why $\lambda \to \lambda$ I)

• Rearranging/factoring to get $(A – \lambda I)v = 0$ and using $\det(A – \lambda I)=0$

• Applying the transpose-of-a-product rule ($(AB)^T = B^T A^T$) and $\det(A^T)=\det(A)$

• Carrying out dot products correctly (sum vs. vector) and checking all pairwise products

• Tracking dimensions and correct order in matrix multiplication

3. Difficulty linking abstract statements to concrete checks

• Translating definitions into index or component form (e.g. $a_{ij} = a_{ji}$, $v \cdot w=0$)

• Working through small examples step by step (2×2 matrices, specific θ for rotations)

• Articulating each logical step (rather than skipping from premise to conclusion)

• Recognizing when counterexamples violate hypotheses (e.g. zero vector in an “orthogonal” set)

In short, the student needs clearer, self-generated definitions; guided practice on the algebraic rules (scalar/matrix interplay, transposes, determinants, dot products); and more worked-out examples that tie abstract properties to entrywise and geometric checks.

Question-specific insights

Question 1

The student's difficulties can be grouped into three main areas:

1. Notation and basic concepts

   - Misreading "{u, v}" as a system of equations rather than the set containing two vectors.

   - Unclear that independence/dependence is a property of a set of vectors, not of an individual vector or matrix.

2. Formal definition vs. intuition

   - Forgotten or confused the precise criterion: "a set is independent iff the only solution to $c_1v_1 + \dots + c_kv_k = 0$ is the trivial one (all $c_i = 0$)."

   - Initially equated dependence only with one vector being a scalar multiple of another, rather than any nontrivial linear combination.

3. Applying the definition

   - Struggled to translate $v_1c_1 + v_2c_2 = 0$ into componentwise equations and solve for c_1, c_2.

   - Misremembered given relations (e.g. accidentally introduced $u + v = 0$).

   - Didn't immediately see that a free variable in the three-vector test implies infinitely many nontrivial solutions—and thus dependence.

Question 2

The student's main challenges clustered around three areas:

1. Conceptual Foundations

   - Lacking a crisp definition of an orthogonal matrix ($M^T M = I$ or equivalently $M^{-1} = M^T$), and confusing that with the notion of orthogonal vectors.
   - Uncertainty about what a transpose is, what the identity matrix I is, and how matrix multiplication works.

2. Key Algebraic Properties

   - Applying the product–transpose rule $(AB)^T = B^T A^T$.
   - Seeing why, in $(AB)^T(AB) = B^T(A^T A)B = B^T B = I$, both A and B being orthogonal force AB to be orthogonal.

3. Application & Reasoning

   - Working through concrete examples (e.g. showing the zero matrix fails $M^T M = I$).
   - Interpreting "relation" among columns in terms of linear dependence and how dependence in B's columns carries over to AB.
   - Performing the actual matrix multiplications without arithmetic slips.
   - Articulating each logical step instead of jumping to "false" or accepting statements without justification.

Question 3

The student's difficulties cluster around two broad areas—basic eigen-concepts and the algebraic machinery used to derive the characteristic equation (and then extend it to A^T):

1. Misunderstanding of eigenvalues vs. eigenvectors

   - Could not clearly state or interpret $A v = \lambda v$

   - Confused eigenvectors (the nonzero v) with eigenvalues (the scalars λ)

2. Scalar-vs-matrix operations

   - Didn't see why λ becomes λI when mixing scalars and matrices

   - Mixed up left/right multiplication ($(\lambda I)v$ vs. $v(\lambda I)$)

3. Rearranging and factoring to get $(A - \lambda I)v = 0$

   - Struggled to move terms, factor out v, and recognize the zero-vector condition

4. Link between $\det(A - \lambda I)=0$ and nontrivial solutions

   - Didn't grasp that a zero determinant means non-invertibility, which in turn allows nonzero v

5. Transpose operation and its determinant

   - Unfamiliar with $(AB)^T = B^T A^T$ and the fact that $\det(A^T)=\det(A)$

   - Unsure how to show A and A^T share the same eigenvalues without inventing a new "transpose" eigenvector

In short, the student needs a firm review of (1) what eigenvalues/eigenvectors are, (2) how λI and determinants produce the characteristic polynomial, and (3) how transposes interact with determinants and matrix-vector products.

Question 4

The student's conceptual roadblocks boil down to three interrelated gaps:

1. Foundations of symmetry

- They haven't firmly grasped the definition "A is symmetric ⇔ $A^T = A$," nor why that forces $a_{ij} = a_{ji}$.
- They're unsure what, if any, constraints symmetry places on diagonal entries.

2. Transpose-and-product mechanics

- They don't see how $(AB)^T = B^TA^T$ combines with $A^T = A$, $B^T = B$ to yield $AB = BA$ as the only way to make $(AB)^T = AB$.
- They're unclear why two individually symmetric factors can fail to produce a symmetric product unless they commute.

3. Concrete examples and notation

- They need non-trivial examples of symmetric matrices (beyond "all-ones") to get intuition.
- They struggle to translate symmetry into index form ($a_{ij} = a_{ji}$) and to check off-diagonal entries in practice.

In sum, the student needs:

• A clear definition and consequences of symmetry (including diagonal vs. off-diagonal behavior)

• Step-by-step application of the transpose rule to AB

• Illustrative examples showing when $AB \neq BA$ spoils symmetry

• Practice with index notation to link the abstract identities to entrywise checks.

Question 6

The student's difficulties can be grouped into two broad, interrelated areas:

1. Fundamentals of the dot product and orthogonality

• They didn't consistently carry out the dot-product procedure: they stopped at pairwise multiplications and sometimes thought the result was a vector instead of summing to a scalar.

• They forgot that orthogonality means a dot product of zero (initially guessed “positive value”).

• They were unclear on what an “orthogonal set” entails—namely, that you must check every pair in a collection of vectors (there are $n(n–1)/2$ pairs) and confirm each dot product vanishes.

2. Structure and interpretation of the 2×2 rotation matrix

• They weren’t sure why its entries are exactly [cos θ –sin θ; sin θ cos θ], nor how cos θ and sin θ arise from rotating basis vectors by angle θ.

• They struggled to see how plugging in a specific angle (e.g. 90°) changes those entries and why the matrix’s columns remain orthonormal.

• They held the misconception that a “rotation matrix” must look identical after certain rotations, rather than understanding that its components vary continuously with θ.

• They needed help linking the geometric action (rotating (1,0) and (0,1)) to the algebraic form and verifying orthogonality of the resulting column vectors via the dot product.

Question 7

The student’s main challenges can be grouped into five areas:

1. Definitions and concrete meaning of symmetry

– Precisely recalling that “A is symmetric” means $A = A^T$ (equivalently $a_{ij} = a_{ji}$) rather than just “rows look like columns.”

– Seeing how this plays out in a simple 2×2 example (e.g. $a_{12} = a_{21}$).

2. Applying the transpose-of-a-product rule

– Remembering $(XYZ)^T = Z^T Y^T X^T$ in a multi-factor product.

– Choosing how to group factors (e.g. treating B^TA as one block) and tracking the reversal of order.

– Handling double transposes ($(B^T)^T = B$) to simplify the expression.

3. Step-by-step algebraic execution

– Writing out each intermediate transpose and substitution instead of leaping to the final result.

– Organizing the work so that one can clearly see $(B^T A B)^T \to B^T A^T (B^T)^T \to B^T A B$.

4. Basic matrix-multiplication conventions and misconceptions

– Ensuring dimensions match (columns of one matrix = rows of the next) and identifying the correct size of a product ($m\times q$ for an $m\times n$ times $n\times q$).

– Avoiding the false leap from “symmetric” (and square) straight to “identity,” and clarifying which matrices in the problem must be tested for which properties.

5. Articulation of understanding

– Restating definitions and properties in their own words when prompted.

– Connecting each algebraic step back to the core concept (showing $M^T = M$ is exactly what it means to be symmetric).

Question 8

The student’s core challenges revolved around the fine points of orthogonality versus linear independence:

• They had not internalized that an orthogonal set must consist of *distinct nonzero* vectors whose pairwise inner products vanish.

• They missed that the standard result “orthogonal ⇒ linearly independent” only holds when no vector in the set is the zero vector.

• They were puzzled by the fact that the zero vector is orthogonal to every vector (its inner product is zero) yet its inclusion automatically creates a linear dependency.

- As a result, they needed help seeing how the simple counterexample $\{0, e_1\}$ illustrates that an “orthogonal” set containing 0 can fail to be independent.