

## Week 7

November 3, 2020

### Exercise 1.

1. Show that  $5x$  is  $o(x^2)$ .
2. Show that  $2x^2$  is not  $o(x^2)$ .
3. Show that  $1/x$  is  $o(x)$ .
4. Show that if  $f(x)$  is  $o(g(x))$ , then  $f(x)$  is  $O(g(x))$ .

### Exercise 2.

Which function below grows fastest when  $n$  goes to infinity?

- ☐  $(\log_3(33))^{n-3}$
- ☐  $3^n$
- ☐  $n^{3\log_3(n)}$
- ☐  $n^3 \log_3(n)$

### Exercise 3.

Consider the two statements below, where  $c$  and  $k$  are constants with  $k \geq 2$ :

$$n^k \text{ is } O(k^n) \qquad (\log n)^k e^{(c+o(1))(\log n)^{1/3}(\log(\log(n)))^{2/3}} \text{ is } e^{(c+o(1))(\log n)^{1/3}(\log(\log(n)))^{2/3}}$$

- ☐ They are both False.
- ☐ Only the first is True.
- ☐ Only the second is True.
- ☐ They are both True.

### Exercise 4.

Let  $f$  be arbitrary functions from  $\mathbf{N}$  to  $\mathbf{R}_{>0}$ .

Let  $g_1, g_2$  be two functions from  $\mathbf{N}$  to  $\mathbf{R}_{>0}$  such that  $g_1$  and  $g_2$  are both  $\Theta(f)$ .

1. Show that the function  $g_1 + g_2$  is  $\Theta(f)$  or provide a counterexample.
2. Show that the function  $g_1 g_2$  is  $\Theta(f^2)$  or provide a counterexample.

Let  $g_3, g_4$  be two functions from  $\mathbf{N}$  to  $\mathbf{R}$  such that  $g_3$  and  $g_4$  are both  $\Theta(f)$ .

3. Show that the function  $g_3 + g_4$  is  $\Theta(f)$  or provide a counterexample.
4. Show that the function  $g_3 g_4$  is  $\Theta(f^2)$  or provide a counterexample.

Let  $g$  be a function from  $\mathbf{N}$  to  $\mathbf{R}_{>0}$  such that  $g$  is  $O(f)$ .

5. Show that  $2^g$  is  $O(2^f)$ , or provide a counterexample.

**Exercise 5.** Consider the two statements below, where  $k$  and  $\ell$  are constants with  $k > \ell \geq 2$  and  $m \rightarrow \infty$ :

$$\log_m(k) \text{ is } \Theta(\log_m(\ell)) \quad k^{\log_\ell(m)} \text{ is } O(\ell^{\log_k(m)}).$$

- ☐ They are both false.
- ☐ Only the first is true.
- ☐ Only the second is true.
- ☐ They are both true.

**Exercise 6.** Consider the following two statements:

$$(f \text{ is } o(f)) \quad \text{and} \quad (f \text{ is } o(g) \text{ implies } f \text{ is } O(g)).$$

- ☐ Only the second is true.
- ☐ They are both false.
- ☐ Only the first is true.
- ☐ They are both true.

**Exercise 7.** Given the two statements below, where  $d > 0$  is an integer constant and  $a_i$  for all  $i \in \mathbf{Z}$  are positive integers with  $\max_{i \in \mathbf{Z}}(a_i) = D$  for a constant  $D > 0$ ,

$$\sum_{i=0}^n a_i i^d \text{ is } \Theta(n^{d+1}) \quad \sum_{i=0}^d a_i n^i \text{ is } \Theta(n^d)$$

- ☐ They are both true.
- ☐ Only the first is true.
- ☐ Only the second is true.
- ☐ They are both false.

**Exercise 8.** Construct two functions  $f$  and  $g$  from  $\mathbf{N}$  to  $\mathbf{R}_{>0}$  such that  $f$  is not  $O(g)$  and  $g$  is not  $O(f)$  or prove that such functions are impossible to find.

**Exercise 9.** Consider the following algorithm, which takes as input a sequence of  $n$  integers  $a_1, a_2, \dots, a_n$  and produces as output a matrix  $M = \{m_{ij}\}$  where  $m_{ij}$  is the minimum term in the sequence of integers  $a_i, a_{i+1}, \dots, a_j$  for  $j \geq i$  and  $m_{ij} = 0$  otherwise.

initializ **M** so that  $m_{ij} = a_i$  if  $j \geq i$  and  $m_{ij} = 0$  otherwise

**for**  $i := 1$  to  $n$

**for**  $j := i + 1$  to  $n$

**for**  $k := i + 1$  to  $j$

$m_{ij} := \min(m_{ij}, a_k)$

**return** **M** =  $\{m_{ij}\}$   $\{m_{ij}$  is the minimum term of  $a_i, a_{i+1}, \dots, a_j\}$

1. Show that this algorithm uses  $O(n^3)$  comparisons to compute the matrix  $M$ .
2. Show that this algorithm uses  $\Omega(n^3)$  comparisons to compute the matrix  $M$ . Using this fact and part (a), conclude that the algorithm uses  $\Theta(n^3)$  comparisons.

**Exercise 10.** What is the largest  $n$  for which one can solve within a minute using an algorithm that requires  $f(n)$  bit operations, where each bit operation is carried out in  $10^{-12}$  seconds, with these functions  $f(n)$ ?

- a.  $\log n$
- b.  $1,000,000n$
- c.  $n^2$