Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 5

October 20, 2020

Exercise 1. Let $X = \mathscr{P}(\mathbf{Q})$ be the set of subsets of $\mathbf{Q}$. Determine whether or not the following relations $\sim_i$ on X are a) reflexive, b) symmetric, c) transitive. Let A and B be arbitrary elements of X.

1. $A \sim_1 B$ if and only if $A \subseteq B$.
2. $A \sim_2 B$ if and only if $A \cap B = \emptyset$.
3. $A \sim_3 B$ if and only if $A \oplus B$ is finite[1].
4. $A \sim_4 B$ if and only if there exists a $c \in \mathbf{R}$ such that for any $x \in A \triangle B$, we have $|x| < c$.
5. $A \sim_5 B$ if and only if A and B contain the same number of integers (potentially infinite).

Exercise 2. Let $\sim$ be the relation on $\mathbf{R} \times \mathbf{R}$ defined by $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

1. Prove that it is an equivalence relation.
2. Prove that the set of equivalence classes of $\sim$ is uncountable.

Exercise 3. Let n be an integer. Consider the relation R on $\mathbf{Z}$ defined by xRy if and only if $x - y = n$. Prove that the relation $\sim$ on $\mathbf{Z}$, defined by $x \sim y$ if and only if n divides $x - y$, is the smallest equivalence relation containing the relation R.

Exercise 4. A relation R on a finite set X can be represented by a directed graph: the elements of X are vertices, and there is an edge from a vertex $a \in X$ to $b \in X$ if and only if aRb. A path from a to b in the graph is a sequence $a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b$ such that $x_i R x_{i+1}$ for any $0 \leq i < k$. Such a path is of length k. The distance $d(a, b)$ from a to b is the length of the shortest path from a to b (the distance from a to a is 0).

1. Prove that if R is symmetric, then $d(a, b) = d(b, a)$ for any $a, b \in X$.
2. Prove that if R is transitive, then $d(a, b) \in \{0, 1\}$ for any $a, b \in X$.
3. Adapt Warshall's algorithm to compute the matrix of distances of the graph of any relation R (the matrix with entry $d(a, b)$ on row a, column b).

Exercise 5. Draw the Hasse diagram for divisibility on the set:

1. {1, 2, 3, 4, 5, 6, 7, 8}.
2. {1, 2, 3, 6, 12, 24, 36, 48}
3. {1, 2, 4, 8, 16, 32, 64}
4. {1, 2, 3, 5, 7, 11, 13}

[1] the set $A \oplus B = (A \cup B) \setminus (A \cap B)$ is the *symmetric difference* of A and B.

Exercise 6. Suppose that $(S, \preceq_1)$ and $(T, \preceq_2)$ are posets. Show that $(S \times T, \preceq)$ is a poset where $(s, t) \preceq (u, v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$.

Exercise 7. Determine whether these posets are lattices.

1. $(1, 3, 6, 9, 12, |)$
2. $(1, 5, 25, 125, |)$
3. $(Z, \geq)$
4. $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S

Exercise 8. Find the lexicographic ordering of these n-tuples:

1. (1, 1, 2), (1, 2, 1)
2. (0, 1, 2, 3), (0, 1, 3, 2)
3. (1, 0, 1, 0, 1), (0, 1, 1, 1, 0)

Exercise 9. Which of these pairs of elements are comparable in the poset $(Z+, |)$?

- ○ 5, 15
- ○ 6, 9
- ○ 8, 16
- ○ 7, 7

Exercise 10. Hilbert's Grand Hotel has a countably infinite number of rooms, and each room is occupied by a single guest.

1. A new guest arrives. Since every room is occupied, if the hotel was finite, the new guest could not be accommodated without evicting a current guest. How can a new guest be accommodated in Hilbert's Grand Hotel? The hotel can ask current guests to change room.
2. How can a finite number of new guests, say n, be accommodated[2]?
3. A bus carrying a countably infinite number of guests arrives. Can they all be accommodated?
4. A countably infinite number of such buses arrives. Can the guests all be accommodated?
5. A bus carrying an uncountable number of guests arrives. Can the guests all be accommodated?

Exercise 11. Which of the following statements is incorrect?

- ○ The Cartesian product of finitely many countable sets is countable.
- ○ Any subset of infinite cardinality of an uncountable set is uncountable.
- ○ $N \cup \{x \in \mathbf{R}, 0 < x < 1\}$ is uncountable.
- ○ The intersection of two uncountable sets can be countably infinite.

[2]Accommodating a guest means that the guest gets a room after waiting a period of time that has a finite length.

Exercise 12.

(francais) Soit B l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation binaire, et soit D l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation décimale. Laquelle des propositions suivantes est correcte?

(English) Let B be the set of real numbers with a finite number of ones in their binary representation, and let D be the set of real numbers with a finite number of ones in their decimal representation. Which of the following statements is correct?

- ○ B est dénombrable et D ne l'est pas. / B is countable and D is uncountable.
- ○ B et D sont dénombrables tous les deux. / B and D are both countable.
- ○ B et D ne sont pas dénombrables. / B and D are both uncountable.
- ○ B nést pas dénombrable mais D est dénombrable. / B is uncountable but D is countable.

Exercise 13. Let F be the set of real numbers with decimal representation consisting of all fours (and possiby a single decimal point). Examples of numbers contained in F are 4, 44, 4444444, 44.4, 4.444444, 444.44444,... etc.

Let G be the set of real numbers with decimal representation consisting of all fours or sixes (and possiby a single decimal point). Examples of numbers contained in G are 4, 6, 44, 66, 46, 64, 4464464, 46.46, 6.644464, 646.64646464, 446.6666666, . . . etc.

- ○ The set F is countable and the set G is not countable.
- ○ The sets F and G are both countable.
- ○ The set G is countable and the set F is not countable.
- ○ The sets F and G are both not countable.

Exercise 14. Let $S = \{0, 1\}$. Let $A = \bigcup_{i=1}^{\infty} S^i$, and let $B = S^*$ be the set of infinite sequences of bits. Which of the following statements is correct?

- ○ A is countable and B is not countable.
- ○ A and B are both countable.
- ○ A and B are both uncountable.
- ○ A is uncountable but B is countable.