Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 5 — solutions

October 20, 2020

Exercise 1. *Let $X = \mathscr{P}(\mathbf{Q})$ be the set of subsets of $\mathbf{Q}$. Determine whether or not the following relations $\sim_i$ on X are a) reflexive, b) symmetric, c) transitive. Let A and B be arbitrary elements of X.*

1. *$A \sim_1 B$ if and only if $A \subseteq B$.*
   (a) ✓ Reflexive: $A \sim_1 A$ since $A \subseteq A$
   (b) ☒ Symmetric: If $A \subset B$ then $A \sim_1 B$, but not $B \sim A$
   (c) ✓ Transitive: $A \subseteq B$ and $B \subseteq C \to A \subseteq C$
2. *$A \sim_2 B$ if and only if $A \cap B = \emptyset$.*
   (a) ☒ Reflexive: $A \cap A \neq \emptyset$
   (b) ✓ Symmetric: $A \cap B = \emptyset \leftrightarrow B \cap A = \emptyset$
   (c) ☒ Transitive: Let $A \subset C$ and $A \sim_2 B$ and $B \sim_2 C$, then $A \cap C \neq \emptyset$
3. *$A \sim_3 B$ if and only if $A \oplus B$ is finite.*
   (a) ✓ Reflexive: $A \oplus A = \emptyset$, hence finite
   (b) ✓ Symmetric: $A \oplus B = B \oplus A$
   (c) ✓ Transitive: the symmetric difference has the property $(A \oplus B) \oplus (B \oplus C) = A \oplus C$. If $(A \oplus B)$ and $(B \oplus C)$ are both finite sets, then their symmetric difference is finite as well, hence $A \oplus C$ is finite and $A \sim_3 C$
4. *$A \sim_4 B$ if and only if there exists a $c \in \mathbf{R}$ such that for any $x \in A \oplus B$, we have $|x| < c$.*
   (a) ✓ Reflexive: $A \oplus A = \emptyset$
   (b) ✓ Symmetric: $A \oplus B = B \oplus A$
   (c) ✓ Transitive: Let $A \sim_4 B$ and $B \sim_4 C$. Then we have:

   $$\exists c_1 \in \mathbf{R} \text{ s.t. } \forall x \in A \oplus B\ |x| < c_1 \text{ and } \exists c_2 \in \mathbf{R} \text{ s.t. } \forall x \in B \oplus C\ |x| < c_2,$$

   $$\forall x \in (A \oplus B) \cup (B \oplus C) \quad |x| < c = \max(c_1, c_2).$$

   Since $A \oplus C \subseteq (A \oplus B) \cup (B \oplus C)$, we have

   $$\forall x \in (A \oplus C) \quad |x| < c = \max(c_1, c_2).$$

   Hence, $A \sim_4 C$
5. *$A \sim_5 B$ if and only if A and B contain the same number of integers (potentially infinite).*
   (a) ✓ Reflexive: $|A| = |A|$.
   (b) ✓ Symmetric: If $|A| = |B|$, then $|B| = |A|$.
   (c) ✓ Transitive: If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Exercise 2. *Let $\sim$ be the relation on $\mathbf{R} \times \mathbf{R}$ defined by $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.*

1. *Prove that it is an equivalence relation.*

   Define the function $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by $f(a, b) = a - b$. Then, for any two pairs $p, q \in \mathbf{R} \times \mathbf{R}$, we have $p \sim q$ if and only if $f(p) = f(q)$. It is then easy to see that the relation is reflexive (because $f(p) = f(p)$), symmetric (because $f(p) = f(q)$ implies $f(q) = f(p)$), and transitive (because $f(p) = f(q)$ and $f(q) = f(r)$ implies $f(p) = f(r)$).

2. *Prove that the set of equivalence classes of $\sim$ is uncountable.*

   Let $\mathscr{Q}$ be the partition of $\mathbf{R} \times \mathbf{R}$ induced by the relation $\sim$ (i.e., $\mathscr{Q}$ is the set of equivalence classes). We define a function $F : \mathscr{Q} \to \mathbf{R}$ as follows: for any equivalence class $C \in \mathscr{Q}$, choose $p \in C$ and let $F(C) = f(p)$. This function is well defined because if $q \in C$ is an other element of the class then $f(q) = f(p)$. In other words,

   $$F : \mathscr{Q} \longrightarrow \mathbf{R} : C \longmapsto f(p) \text{ for any } p \in C.$$

   It is surjective: for any $x \in \mathbf{R}$, let C be the class of $(x, 0)$, then $F(C) = f(x, 0) = x$. It is injective: if $F(C) = F(D)$ for $C, D \in \mathscr{Q}$, then for any pairs $p \in C$ and $q \in D$ we have $f(p) = f(q)$. Then $p \sim q$, and therefore p and q must be in the same equivalence class, hence $C = D$. So F is a bijection. Since $\mathbf{R}$ is uncountable, $\mathscr{Q}$ is also uncountable.

Exercise 3. *Let n be an integer. Consider the relation R on $\mathbf{Z}$ defined by xRy if and only if $x - y = n$. Prove that the relation $\sim$ on $\mathbf{Z}$, defined by $x \sim y$ if and only if n divides $x - y$, is the smallest equivalence relation containing the relation R.*

We have the following two relations defined on $\mathbf{Z}$:

$$R = \{(x, y) \mid xRy \leftrightarrow x - y = n\}$$

$$R_\sim = \{(x, y) \mid x \sim y \leftrightarrow x - y = k \cdot n \text{ for some } k \in \mathbf{Z}\}$$

Let us first find the transitive closure S of R, by computing $S = \bigcup_{i=1}^{\infty} S_i$.

$$S_1 = R$$

$$\begin{aligned} S_2 = S_1^2 &= \{(x, y) \mid \exists z \in \mathbf{Z} : \ (x, z) \in S_1 \wedge (z, y) \in S_1\} \\ &= \{(x, y) \mid \exists z \in \mathbf{Z} : \ x - z = n \wedge z - y = n\} \\ &= \{(x, y) \mid \exists z \in \mathbf{Z} : \ x - y = 2n\} \end{aligned}$$

$$\begin{aligned} S_3 = S_1^3 &= \{(x, y) \mid \exists z \in \mathbf{Z} : \ (x, z) \in S_1 \wedge (z, y) \in S_2\} \\ &= \{(x, y) \mid \exists z \in \mathbf{Z} : \ x - z = n \wedge z - y = 2n\} \\ &= \{(x, y) \mid \exists z \in \mathbf{Z} : \ x - y = 3n\} \end{aligned}$$

$$\vdots$$

We now have that $S = \bigcup_{i=1}^{\infty} S_i = \{(x, y) \mid x - y = kn \text{ for some } k \in \mathbf{Z}_{\geq 1}\}$. Let's now form the reflexive closure S_r of S and the symmetric closure S_{rs} of S_r:

$$S_r = S \cup \{(x, y) \mid x = y\}$$

$$S_{rs} = S_r \cup \{(x, y) \mid (y, x) \in S_r\}$$

We can notice that:

$$S_{rs} = \{(x, y) \mid x - y = kn \text{ for some } k \in \mathbf{Z}\},$$

which is exactly the initial relation $R_\sim$. Since S is the transitive closure of R, it is the smallest transitive relation which contains R. Furthermore, in the previous exercises we showed that $R_\sim$ is an equivalence relation. Hence, we can say that $R_\sim$ is the smallest equivalence relation which contains R.

Exercise 4. *A relation R on a finite set X can be represented by a directed graph: the elements of X are vertices, and there is an edge from a vertex $a \in X$ to $b \in X$ if and only if aRb. A path from a to b in the graph is a sequence $a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b$ such that $x_i R x_{i+1}$ for any $0 \leq i < k$. Such a path is of length k. The distance $d(a, b)$ from a to b is the length of the shortest path from a to b (the distance from a to a is 0).*

1. *Prove that if R is symmetric, then $d(a, b) = d(b, a)$ for any $a, b \in X$.*

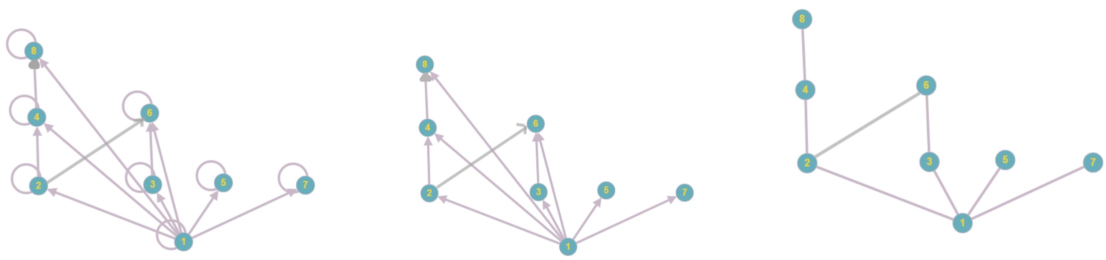
   Suppose that R is symmetric and that $d(a, b) = k$ for some $a, b \in X$. Then there is a path $a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b$ such that $x_i R x_{i+1}$ for any $0 \leq i < k$. Since R is symmetric, we also have that $x_{i+1} R x_i$ for any $0 \leq i < k$. Hence, there exists a path from b to a of length $k = d(a, b)$. Let's assume now that there exists a path from b to a of length $k_1 < k$, namely $d(b, a) = k_1$. Following the same reasoning, we conclude that then there must be a path from a to b of length $k_1 < d(a, b)$. Since $d(a, b)$ is by definition the shortest path from a to b, we conclude that $d(b, a) = d(a, b)$.

2. *Prove that if R is transitive, then $d(a, b) \in \{0, 1\}$ for any $a, b \in X$.*

   Suppose that R is transitive and that there is a path from a to b with $d(a, b) = k \geq 2$. Then we have $a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b$ such that $x_i R x_{i+1}$ for any $0 \leq i < k$. Since R is transitive we also have that if $x_i R x_{i+1}$ and $x_{i+1} R x_{i+2}$, then $x_i R x_{i+2}$. If we apply this property to our sequence we get that $x_0 R x_k$. Therefore, if there is a path from a to b of length $k \geq 1$, then there exists a path from a to b of length 1 and we have that $d(a, b) = \{0, 1\}$.

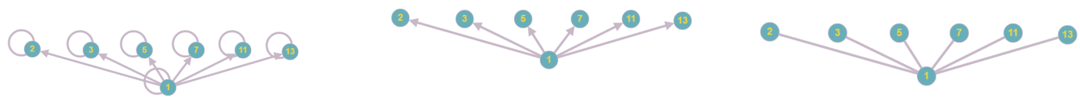
Exercise 5. *Draw the Hasse diagram for divisibility on the set:*

1. $\{1, 2, 3, 4, 5, 6, 7, 8\}$



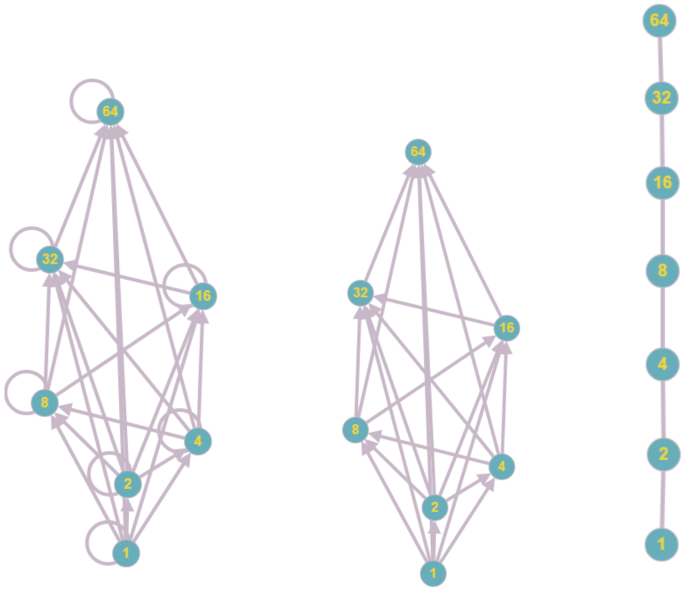
(a) (b) (c)

2. $\{1, 2, 3, 5, 7, 11, 13\}$



(a) (b) (c)

3. $\{1, 2, 4, 8, 16, 32, 64\}$



(a) (b) (c)

Exercise 6. *Suppose that $(S, \preceq_1)$ and $(T, \preceq_2)$ are posets. Show that $(S \times T, \preceq)$ is a poset where $(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$.*

A relation R on a set A is a partial ordering if the relation R is reflexive, antisymmetric, and transitive. (S, R) is then called a poset.

PROOF $(S \times T, \preceq)$ is a poset if and only if the relation $R = \{((s,t),(u,v))|(s,t) \preceq (u,v)\}$ is a partial ordering.

$(S, \preceq_1)$ and $(T, \preceq_2)$ are posets, thus the relations $R_1 = \{(s,u)|s \preceq_1 u\}$ and $R_2 = \{(t,v)|t \preceq_1 v\}$ are both reflexive, antisymmetric and transitive.

Reflexive
Let $(s,t) \in S \times T$, where $s \in S$ and $t \in T$.
Since $R_1 = \{(s,u)|s \preceq_1 u\}$ and $R_2 = \{(t,v)|t \preceq_1 v\}$ are both reflexive:

$$s \preceq_1 s$$
$$t \preceq_2 t$$

$(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$

$$(s,t) \preceq (s,t)$$

which implies $((s,t),(s,t)) \in R$ and thus R is reflexive.

Antisymmetric
Let $((s,t),(u,v)) \in R$ and $((u,v),(s,t)) \in R$

$$(s,t) \preceq (u,v)$$
$$(u,v) \preceq (s,t)$$

$(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$

$$s \preceq_1 u$$
$$u \preceq_1 s$$
$$t \preceq_2 v$$
$$v \preceq_2 t$$

Since $R_1 = \{(s,u) | s \preceq_1 u\}$ and $R_2 = \{(t,v) | t \preceq_2 v\}$ are both antisymmetric:

$$s = u,\, t = v$$

which implies:

$$(s,t) = (u,v)$$

Thus R is antisymmetric.

Transitive
Let $((s,t),(u,v)) \in R$ and $((u,v),(w,x)) \in R$

$$\begin{gathered} (s,t) \preceq (u,v) \\ (u,v) \preceq (w,x) \end{gathered}$$

$(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$

$$\begin{gathered} s \preceq_1 u \\ u \preceq_1 w \\ t \preceq_2 v \\ v \preceq_2 x \end{gathered}$$

Since $R_1 = \{(s,u) | s \preceq_1 u\}$ and $R_2 = \{(t,v) | t \preceq_2 v\}$ are both transitive:

$$\begin{gathered} s \preceq_1 w \\ t \preceq_2 x \end{gathered}$$

$(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$

$$(d,t) \preceq (w,x)$$

Thus R is transitive.
Conclusion: R is reflexive, antisymmetric and transitive. Then R is a partial ordering and $(S \times T, R)$ is a poset.

Exercise 7. *Determine whether these posets are lattices.*

1. $(1,3,6,9,12,|)$: The poset does not form a lattice. There is no least upper bound for 9 and 12.

2. $(1,5,25,125,|)$: The poset forms a lattice, because the greatest lower bound of any two elements $a \in S$ and $b \in S$ is their minimum and the least upper bound is their maximum.

3. $(Z, \geq)$: The poset forms a lattice, because the greatest lower bound of any two elements $a \in Z$ and $b \in Z$ is their minimum and the least upper bound is their maximum.

4. $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S: The poset forms a lattice, because the greatest lower bound of any two elements $B \in Z$ and $C \in Z$ is their intersection and the least upper bound is their union.

Exercise 8. *Find the lexicographic ordering of these n-tuples:*

1. $(1,1,2) \prec (1,2,1)$

2. $(0,1,2,3) \prec (0,1,3,2)$

3. $(1,0,1,0,1) \succ (0,1,1,1,0)$

Exercise 9. *Hilbert's Grand Hotel has a countably infinite number of rooms, and each room is occupied by a single guest.*

1. *A new guest arrives. Since every room is occupied, if the hotel was finite, the new guest could not be accommodated without evicting a current guest. How can a new guest be accommodated in Hilbert's Grand Hotel? The hotel can ask current guests to change room.*

   Since Hilbert's Grand Hotel has countably infinite number of rooms, we can list the rooms as Room 1, Room 2, Room 3, and so on. When a new guest arrives, hotel asks the guest in Room 1 to move to Room 2, in Room 2 to Room 3, and in general in Room k to Room $k+1$ for all positive integers k. After the moving, Room 1 will be free and the hotel can accommodate the new guest in it.

2. *How can a finite number of new guests, say n, be accommodated?*

   A finite number n of guests can be accommodated simply by repeating n times the procedure from the previous point.

3. *A bus carrying a countably infinite number of guests arrives. Can they all be accommodated?*

   Yes. In this case, we can ask the current guests to change rooms in the following way:

   $$\begin{array}{ccc} \text{Room } 1 & \to & \text{Room } 2 \\ \text{Room } 2 & \to & \text{Room } 4 \\ & \dots & \\ \text{Room } n & \to & \text{Room } 2n \\ & \dots & \end{array}$$

   After the moving, all the current guests will be in even-numbered rooms, which means that all the odd-numbered rooms will be empty. Since there are infinitely many odd integers we can accommodate all the arriving guests in odd-numbered rooms.

4. *A countably infinite number of such buses arrives. Can the guests all be accommodated?*

   Yes, they can. Let $S = \{(a, b) : a, b \in \mathbf{N}\}$ be the set of pairs of integers. Since $\mathbf{N}$ is countable, S is also countable so we can enumerate its elements $s_1, s_2, ...$ Let $s_n = (a, b)$ and assign the bth guest of the ath bus to Room n (assume the current guests are in the bus number 0). Hence we have a function which maps each person to a room, so we can accommodate new guests.

   One concrete example of how to accommodate new guests - ask the current guests to change rooms in the following way:

   $$\begin{array}{ccc} \text{Room } 1 & \to & \text{Room } 3 \\ \text{Room } 2 & \to & \text{Room } 5 \\ & ... & \\ \text{Room } n & \to & \text{Room } 2n+i \\ & ... & \end{array}$$

   After the moving, accommodate all the new guests in the following way: the bth guest of the ath bus goes to Room $2^a(2b+1)$.

5. *A bus carrying an uncountable number of guests arrives. Can the guests all be accommodated?*

   No, they cannot. Since one-to-one mapping between countable and uncountable sets can't be made, we can't accommodate all the arriving guests.

Exercise 10. *Which of the following statements is* **incorrect***?*

- ○ *The Cartesian product of finitely many countable sets is countable.*
- ✓ *Any subset of infinite cardinality of an uncountable set is uncountable.*
- ○ $\mathbf{N} \cup \{x \mid x \in \mathbf{R}, 0 < x < 1\}$ *is uncountable.*
- ○ *The intersection of two uncountable sets can be countably infinite.*

The set **Z** of integers is a countable subset of infinite cardinality of the uncountable set **R** of real numbers, implying that the second statement is incorrect. The other statements are correct.

Exercise 11.

(*français*) *Soit B l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation binaire, et soit D l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation décimale.*
Laquelle des propositions suivantes est correcte?

(*English*) *Let B be the set of real numbers with a finite number of ones in their binary representation, and let D be the set of real numbers with a finite number of ones in their decimal representation.*
Which of the following statements is correct?

✓ $\left\{ \begin{array}{l} \textit{B est dénombrable et D ne l'est pas.} \\ \textit{B is countable and D is uncountable.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{B et D sont dénombrables tous les deux.} \\ \textit{B and D are both countable.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{B et D ne sont pas dénombrables.} \\ \textit{B and D are both uncountable.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{B n'est pas dénombrable mais D est dénombrable.} \\ \textit{B is uncountable but D is countable.} \end{array} \right.$

- Concerning B, for any finite number of ones, the different ways the ones can be "located" are countable (because there is never a choice for the complement of the ones: they must be zeros). So B is a countable collection (because the number of ones is countable) of countable sets and thus countable.
- Concerning D, consider its subset of numbers consisting of a decimal point followed by an infinite sequence of 1s or 2s. The assumption that this subset is countable leads to an immediate contradiction (use Cantor diagonalization: the assumed-to-exist enumeration does not contain the number x that has digit $3 - d \in \{1, 2\}$ in its i-th position when the i-th number in the assumed-to-exist enumeration has digit $d \in \{1, 2\}$ in its i-th position – because $3 - d \neq d$ the number x is not in the enumeration), so D is uncountable.

If follows that (only) the first answer is correct.

Exercise 12. *Let F be the set of real numbers with decimal representation consisting of all fours (and possiby a single decimal point). Examples of numbers contained in F are* 4, 44, 4444444, 44.4, 4.444444, 444.44444, … *etc.*

Let G be the set of real numbers with decimal representation consisting of all fours or sixes (and possiby a single decimal point). Examples of numbers contained in G are 4, 6, 44, 66, 46, 64, 4464464, 46.46, 6.644464, 646.64646464, 446.6666666, … *etc.*

✓ *The set F is countable and the set G is not countable.*

○ *The sets F and G are both countable.*

○ *The set G is countable and the set F is not countable.*

○ *The sets F and G are both not countable.*

- Concerning F, note that its only elements with a non-terminating decimal expansion are the numbers $4\frac{4}{9} = 4.444444\ldots$, $44\frac{4}{9} = 44.444444\ldots$, $444\frac{4}{9} = 444.444444\ldots$, ... All other elements of F have a finite decimal expansion. It follows that the elements of F can be enumerated as follows: 4, $4\frac{4}{9}$, 44, 4.4, $44\frac{4}{9}$, 444, 44.4, 4.44, $444\frac{4}{9}$, 4444, 444.4, 44.44, 4.444, $4444\frac{4}{9}$, 44444, 4444.4, 444.44, 44.444, 4.4444, ... (note that in this enumeration the "$\frac{4}{9}$" is just a placeholder for the infinite decimal expansion ".444444..."). Because this enumeration eventually reaches any element of F, it follows that F is countable.

- Concerning G, looking at just the subset $\widehat{G}$ of elements of G that have an infinite decimal expansion and that are at least 4 and less than 6 (thus elements of $\widehat{G}$ look like 4.4... or 4.6... with any infinite sequence of fours or sixes replacing the ...), it follows from Cantor's diagonalization argument that $\widehat{G}$ is not countable: assuming an enumeration, switch the fours and sixes on the diagonal of the enumeration to find an element of $\widehat{G}$ that does not belong to the enumeration. Because G contains a non-countable subset, G itself is not countable either.

It follows that the first answer must be ticked.

Exercise 13. *Let $S = \{0, 1\}$. Let $A = \bigcup_{i=1}^{\infty} \mathbf{S}^i$, and let $B = \mathbf{S}^*$ be the set of infinite sequences of bits. Which of the following statements is correct?*

✓ *A is countable and B is not countable.*

○ *A and B are both countable.*

○ *A and B are both uncountable.*

○ *A is uncountable but B is countable.*

- Concerning A, each set $\{0, 1\}^i$ is finite and thus countable, implying that A as the countable union of countable sets is countable.

- Concerning B, it follows from Cantor's diagonalization argument that the set of infinite sequences over the set $\{0, 1\}$ of bits is uncountable.

It follows that, once again, the first answer is the correct one.

Exercise 14. *Let $r \geq -1$ be an integer and let $S(r) = \sum_{i=-r}^{r+2} (i-2)^{i-2}$.*

✓ $\forall r\ S(r) > 0 \leftrightarrow r \geq 0$.

○ $\forall r\ S(r) > 0 \leftrightarrow r \geq -1$.

○ $\forall r\ S(r) > 0 \leftrightarrow r \geq 1$.

○ $\forall r\ S(r) > 0$ only if $r \geq 1$.

Let's first compute $S(-1)$:

$$S(-1) = \sum_{i=1}^{1} (i-2)^{i-2} = (-1)^{-1} = -1.$$

So, $S(-1) < 0$ and obviously the second answer is not correct.
Now, let's start with $S(0)$:

$$S(0) = \sum_{i=0}^{2} (i-2)^{i-2} = \frac{1}{4} > 0$$

Let's consider now what happens with $S(r+1)$ for $r \geq 0$:

$$
\begin{aligned}
S(r+1) &= \sum_{i=-r-1}^{r+1+2} (i-2)^{i-2} \\
&= \sum_{i=-r}^{r+2} (i-2)^{i-2} + (-r-1-2)^{-r-1-2} + (r+2+1-2)^{r+2+1-2} \\
&= S(r) + (-r-3)^{-r-3} + (r+1)^{r+1} \\
&= S(r) + (r+1)^{r+1} + (-1)^{r+3}\frac{1}{(r+3)^{r+3}}
\end{aligned}
$$

By rearranging the terms, we have that:

$$S(r+1) - S(r) = (r+1)^{r+1} + (-1)^{r+3}\frac{1}{(r+3)^{r+3}}.$$

Because the equation involves a factor of $(-1)^{r+3}$, the result depends on the parity of r:

- If r is odd, then we have that $r+3$ is even and:

$$S(r+1) - S(r) = (r+1)^{r+1} + \frac{1}{(r+3)^{r+3}} > 0.$$

- If r is even, then we have $r+3$ odd and:

$$S(r+1) - S(r) = (r+1)^{r+1} - \frac{1}{(r+3)^{r+3}} > 0.$$

So, we conclude that:

$$\forall r_{\geq 0}\ S(r+1) - S(r) > 0 \leftrightarrow \forall r_{\geq 0}\ S(r+1) > S(r).$$

Together with the fact that $S(0) > 0$ we can conclude that:

$$\forall r \geq 0\ S(r) > 0.$$

Exercise 15. *Let* $S(n) = \sum_{i=1}^{n} \frac{1}{i(i+2)}$ *for* $n > 64$.

○ $S(n) = \dfrac{n+1}{2n+4}$.

○ $S(n) = \dfrac{3n+4}{4n+8}$.

✓ $S(n) = \dfrac{3n^2+5n}{4n^2+12n+8}$.

○ $S(n) = \dfrac{3n^2-4n-3}{4n^2-4}$.

First, solve for x and y the following equation:

$$\frac{1}{i(i+2)} = \frac{x}{i} + \frac{y}{i+2}.$$

We find $1 = (i+2)x + iy$ from which it follows that $x = \frac{1}{2}$ and $y = -\frac{1}{2}$ and therefore that

$$S(n) = \sum_{i=1}^{n} \left(\frac{1}{2i} - \frac{1}{2(i+2)} \right).$$

This suggests a "telescope", which means that a few back-of-the-envelope experiments should give more insight (not paying attention to the weird "'$n > 64$" restriction). We find that $S(1) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$ and that $S(2) = \frac{1}{2} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} = \frac{1}{3} + \frac{1}{8} = \frac{11}{24}$, both still without any telescoping effects—but it is interesting to see that only the third answer "survives" these experiments.

For larger n values, the telescope should kick in: $S(3) = \frac{1}{2} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \frac{1}{6} - \frac{1}{10} = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \frac{1}{10}$ and $S(4) = \frac{1}{2} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \frac{1}{6} - \frac{1}{10} + \frac{1}{8} - \frac{1}{12} = \frac{1}{2} + \frac{1}{4} - \frac{1}{10} - \frac{1}{12}$. This suggests that

$$S(n) = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

because apparently only the first two positive terms and the last two negative terms (when $S(n)$ is written as a sum of n positive and n negative terms) are not cancelled by the telescope. Because

$$\frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} = \frac{2(n+1)(n+2) + (n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$

which equals

$$\frac{3(n^2+3n+2) - 4n - 6}{4n^2+12n+8} = \frac{3n^2+5n}{4n^2+12n+8}.$$

This informal telescope-waving argument suggests that only the third answer is correct ("only" because the expressions in the other answers are different).

A formal argument follows that uses summation index manipulations to prove that

$$S(n) = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

so that indeed the third answer is proved to be correct.

From $S(n) = \sum_{i=1}^{n} \left(\frac{1}{2i} - \frac{1}{2(i+2)} \right)$ it follows that $S(n) = \sum_{i=1}^{n} \frac{1}{2i} - \sum_{i=1}^{n} \frac{1}{2(i+2)}$ and thus (with $i+2 = j$ so that $j = 3$ when $i = 1$ and $j = n+2$ when $i = n$) $S(n) = \sum_{i=1}^{n} \frac{1}{2i} - \sum_{j=3}^{n+2} \frac{1}{2j}$. Splitting off the first two terms of the first summation, and the last two terms of the second summation, and switching back from j to i, we find that

$$S(n) = \frac{1}{2} + \frac{1}{4} + \left(\sum_{i=3}^{n} \frac{1}{2i} \right) - \left(\sum_{i=3}^{n} \frac{1}{2i} \right) - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

as desired (note that we implicitly assumed that $n \geq 3$, which follows from $n > 64$).

Note, however, that this second "formal" part is by no means required to find the correct solution: the first informal argumentation more than sufficed.

Exercise 16. *Which of the following statements is correct?*

✓ $\displaystyle\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = 8.$

○ $\displaystyle\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = 4.$

○ $\displaystyle\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = 16.$

$\bigcirc$ $\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = \infty.$

With the following:

$$\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = 16^{\sum_{i=1}^{\infty} 1/(i(i+2))},$$

and the fact that (see previous exercise):

$$\sum_{i=1}^{\infty} \frac{1}{i(i+2)} = \frac{1}{2} + \frac{1}{4} + \lim_{n\to\infty} \left(\frac{1}{2(n+1)} - \frac{1}{2(n+2)} \right) = \frac{3}{4},$$

it follows that $\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = 16^{3/4} = (2^4)^{3/4} = 2^3 = 8.$

Another way to see that $\sum_{i=1}^{\infty} \frac{1}{i(i+2)} = \frac{3}{4}$ is to derive it as follows:

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{1}{i(i+2)} &= \sum_{i=1}^{\infty} \left(\frac{1}{2i} - \frac{1}{2(i+2)} \right) \\
&= \sum_{i=1}^{\infty} \frac{1}{2i} - \sum_{i=1}^{\infty} \frac{1}{2(i+2)} \\
&= \frac{1}{2} + \frac{1}{4} + \sum_{i=3}^{\infty} \frac{1}{2i} - \sum_{i=1}^{\infty} \frac{1}{2(i+2)} \\
&= \frac{3}{4} + \sum_{i=3}^{\infty} \frac{1}{2i} - \sum_{j=3}^{\infty} \frac{1}{2j} \\
&= \frac{3}{4} + \sum_{i=3}^{\infty} \frac{1}{2i} - \sum_{i=3}^{\infty} \frac{1}{2i} \\
&= \frac{3}{4},
\end{aligned}$$

where for the second summation first $i+2$ is replaced by j (so $j = 3$ when $i = 1$), after which j is called i again.

Exercise 17. *Suppose that the number of bacteria in a colony triples every hour.*

1. *Set up a recurrence relation for the number of bacteria after n hours have elapsed*
   Let a_n represents the number of bacteria after n hours have elapsed. Every hour, the number of bacteria triples. Thus the number of bacteria is the number of bacteria at an hour ago multiplied by 3.

   $$a_n = 3a_{n-1}$$

2. *If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?*
   Given:

   $$\begin{aligned} a_n &= 3a_{n-1} \\ a_0 &= 100 \end{aligned}$$

   We successively apply the recurrence relation:

   $$\begin{aligned} a_n &= 3a_{n-1} = 3^1 a_{n-1} \\ &= 3(3a_{n-2}) = 3^2 a_{n-2} \\ &= 3^2(3a_{n-3}) = 3^3 a_{n-3} \\ &= 3^3(3a_{n-4}) = 3^4 a_{n-4} \end{aligned}$$

$$
\begin{aligned}
&\dots \\
&= 3^n a_{n-n} \\
&= 3^n a_0 \\
&= 100 \cdot 3^n
\end{aligned}
$$

Evaluate the found expression at $n = 10$:

$$a_{10} = 100 \cdot 3^{10} \approx 5,904,900$$

Thus there are 5,904,900 bacteria after 100 hours.