Quiz Questions: Induction and Recursion

1. Suppose you want to use mathematical induction to prove that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all positive integers n. Which of these is the correct statement to be proved in the basis step.
    a. $1 + 2^1 = 2^{1+1} - 1$
    b. $1 + 2 + 2^2 + \cdots + 2^n$
    c. $1 = 2^{0+1} - 1$
    d. $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

    Explanation: Since we prove the equation for all positive integers, the base case is n = 1. Substituting n=1 into the equation gives $1 + 2^1 = 2^{1+1} - 1$.

2. Let $P(n) = n! < n^n$, where $n > 1$. What is the statement $P(n+1)$?
    a. $(n+1)n! < (n+1)^{n+1}$
    b. $n! < n^{n+1}$
    c. $(n+1)! < (n+1)^n$
    d. $(n+1)n! < (n+1)^n$

    Explanation: we have to substitute $n+1$ for n in the inequality $P(n)$. Considering that $(n+1)n! = (n+1)!$ We see that $(n+1)\,n! < (n+1)^{n+1}$ corresponds to $P(n+1)$.

3. Suppose $f(n)$ has the recursive definition $f(n) = \frac{3}{f(n-1)} + 1$ and you know that $f(2) = 1/2$. What is the value $f(1)$?
    a. 2/3
    b. 1/2
    c. -6
    d. -6/5

    Explanation: If we chose $f(1) = -6$, we can compute $f(2) = \frac{3}{-6} + 1 = -\frac{1}{2} + 1 = \frac{1}{2}$

4. Which of the following functions is not defined recursively?
    a. $f(n+1) = n * f(n-1) + 1$
    b. $f(n) = n * f(n-1) + 1$
    c. $f(n) = (n-1) * f(n) + 1$
    d. $f(n+1) = (n+1) * f(n) + 1$

    Explanation: $f(n) = (n-1) * f(n) + 1$ is not a recursive definition for f, since $f(n)$ is expressed in terms of $f(n)$ itself, rather than $f(k)$ for $k < n$. In fact, this is an equation that could be solved for $f(n)$.

5. Let $2 \epsilon S$ and if x, y in S then $x * y - 1$ in S
    a. S = {2, 3, 5, 8, 9, …}
    b. S = {2, 3, 4, 5, 6, …}
    c. S = {2, 4, 8, 16, …}
    d. S = {2, 3, 5, 7, 11, …}

Explanation: Initially S = {2}. Then in the first recursion we obtain S = {2,3}, since 2*2-1 = 3. In the second recursion we obtain S = {2,3,5,8} since 2*3 – 1 = 5 and 3*3 – 1 = 8. In the third recursion we obtain, among others, S = {2,3,5,8,9,…} since 2*5 – 1 = 9.

6. Which of the following formulae is not well-formed?
   a. $(T \leftrightarrow F)$
   b. $T \leftrightarrow T$
   c. $(T \rightarrow F)$
   d. $(F \rightarrow T)$

Explanation: in the recursive definition of well-formed formulae (see slide 41 of Week 8) it is required that a well-formed formulae is always enclosed in parenthesis (except individual variables T,F,s)

7. Which expression does fast recursive exponentiation evaluate when computing a^5?
   a. $a * (a * (a * (a * (a * 1)))))$
   b. $a^0 * (a^{-1} * (a^3)^2)$
   c. $a^0 * \left(a^1 * (a^2)\right)^2$
   d. $a^0 * (a^1 * (a^2)^2)$

Explanation: Applying the recursive exponentation algorithm
$fast\_power(a, 5) =$
$a^1 * (fast\_power(a, 2))^2 =$
$a^1 * (a^0 * (fast\_power(a, 1))^2)^2 =$
$a^1 * (a^0 * (a^1 * fast\_power(a, 0))^2)^2 =$
$a^1 * (a^0 * (a^1 * 1)^2)^2 =$
$a^1 * \left(a^0 * (a^2)\right)^2$
Note: the “correct” answer is actually not correct. The a^0 and a^1 have been unfortunately switched.

8. In merge sort when merging the lists 1,3,4 and 2,5,6 after 4 steps the partial result is
   a. 1,2,3,4
   b. 1,2,3,5
   c. 3,4,5,6
   d. 1,3,2,5

Explanation: After four steps, when merging the two lists, always the 4 smallest elements with have been selected, however they were distributed over the original lists.