Video 23: Relations on a Set

- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations

Binary Relation on a Set

Definition: A **binary relation** R **on a set** A is a subset of $A \times A$ or a relation from A to A.

Example:

- Let $A = \{a, b, c\}$
  Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$
  R $= \{(a, b) \mid a \text{ divides } b\} = \{(1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$ is a relation on A.

Reflexive Relations

Definition: A relation R on a set A is **reflexive** iff $(a, a) \in R$ for every element $a \in A$.

R is reflexive iff $\forall x\,(x \in A \longrightarrow (x, x) \in R)$

Observation: The empty relation on an empty set is reflexive!

Example

$R_1 = \{(a, b) \mid a \leq b\}$	reflexive
$R_2 = \{(a, b) \mid a > b\}$	not reflexive (note that $3 \not> 3$)
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$	reflexive
$R_4 = \{(a, b) \mid a = b\}$	reflexive
$R_5 = \{(a, b) \mid a = b + 1\}$	not reflexive (note that $3 \neq 3 + 1$)
$R_6 = \{(a, b) \mid a + b \leq 3\}$	not reflexive (note that $4 + 4 \not\leq 3$)

Symmetric Relations

Definition: A relation R on a set A is **symmetric** iff $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

R is symmetric iff $\forall x \forall y ((x, y) \in R \longrightarrow (y, x) \in R)$

Example

$R_1 = \{(a, b) \mid a \leq b\}$	not symmetric (note that $3 \leq 4$, but $4 \nleq 3$)
$R_2 = \{(a, b) \mid a > b\}$	not symmetric (note that $4 > 3$, but $3 \ngtr 4$)
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$	symmetric
$R_4 = \{(a, b) \mid a = b\}$	symmetric
$R_5 = \{(a, b) \mid a = b + 1\}$	not symmetric (note that $4 = 3 + 1$, but $3 \neq 4 + 1$)
$R_6 = \{(a, b) \mid a + b \leq 3\}$	symmetric

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called **antisymmetric**.

R is antisymmetric iff $\forall x\, \forall y\, ((x, y) \in R \land (y, x) \in R \longrightarrow x = y)$

Note: symmetric and antisymmetric are not opposites of each other!

Example

$R_1 = \{(a, b) \mid a \leq b\}$	antisymmetric
$R_2 = \{(a, b) \mid a > b\}$	antisymmetric
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$	not antisymmetric (note $1 \neq -1$)
$R_4 = \{(a, b) \mid a = b\}$	antisymmetric
$R_5 = \{(a, b) \mid a = b + 1\}$	antisymmetric
$R_6 = \{(a, b) \mid a + b \leq 3\}$	not antisymmetric (note $2 + 1 = 1 + 2 \leq 3$)

Transitive Relations

Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

R is transitive if and only if $\forall x \, \forall y \, \forall z \, ((x, y) \in R \land (y, z) \in R \longrightarrow (x, z) \in R)$

Example

$R_1 = \{(a, b) \mid a \le b\}$	transitive
$R_2 = \{(a, b) \mid a > b\}$	transitive
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$	transitive
$R_4 = \{(a, b) \mid a = b\}$	transitive
$R_5 = \{(a, b) \mid a = b + 1\}$	not transitive (3,2) and (4,3) belong to R_5, but not (3,3)
$R_6 = \{(a, b) \mid a + b \le 3\}$	not transitive (2,1) and (1,2) belong to R_6, but not (2,2)

Number of Relations on a Set

How many relations are there on a set A?

$A \times A$ has $|A|^2$ elements when A has $/A/$ elements.

Every subset of $A \times A$ can be a relation

Therefore there are $2^{|A|^2}$ relations on a set A.

Summary

- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations

Equivalence Relations

Section 9.5

Video 24: Equivalence Relations

- Equivalence Relations
- Equivalence Classes
- Equivalence Classes and Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called **equivalent**.

The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example

$R_{minus} = \{ (a, b) \in \boldsymbol{R} \times \boldsymbol{R} \mid a - b \in \boldsymbol{Z} \}$

Is *R* an equivalence relation?

Reflexive: 0 – 0 = 0, 0 is an integer.

Symmetric: a – b = b – a, if a – b is in integer, then b – a is an integer.

Transitive: (a – b) + (b – c) = a – c, if a – b is an integer and b – c is an integer, then a – c is an integer.

Example

$R_{divides} = \{ (a, b) \in \mathbf{N} \times \mathbf{N} \mid a \text{ divides } b \} = \{ (a, b) \in \mathbf{N} \times \mathbf{N} \mid a \mid b \}$

Is R an equivalence relation?

No, it is not symmetric: 2 divides 4, but 4 does not divide 2

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.

The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$.

Note that $[a]_R = \{s \mid (a, s) \in R\}$.

If $b \in [a]_R$, then b is called a **representative** of this equivalence class.

Any element of a class can be used as a representative of the class.

Example

What is the equivalence class of R_{minus} = { *(a, b)* ∈ ***R*** ⨉ ***R*** | *a - b* ∈ ***Z*** } of element 0.

$$[0]_{R_{minus}} = \mathbb{Z}$$

Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) $R(a, b)$

(*ii*) $[a] = [b]$

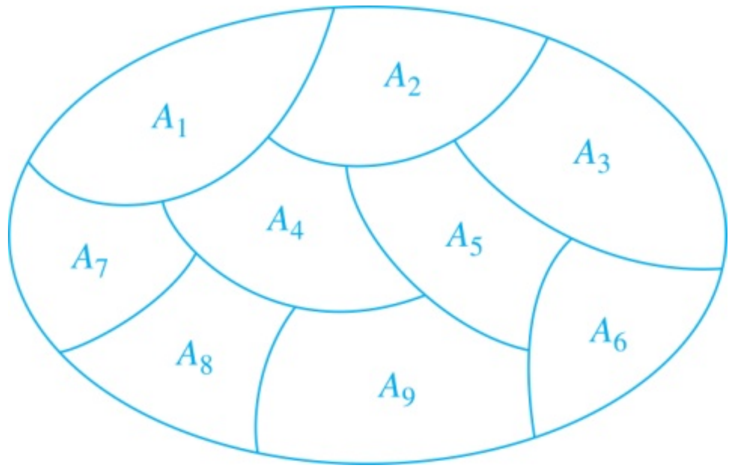
(*iii*) $[a] \cap [b] \neq \emptyset$

Partition of a Set

Definition: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

Formally, for an index set I the collection of subsets A_i, where $i \in I$ forms a partition of S if and only if

$A_i \neq \emptyset$ for $i \in I$	*non-empty subsets*
$A_i \cap A_j = \emptyset$ when $i \neq j$	*disjoint subsets*
and $\bigcup_{i \in I} A_i = S$	*union is S*



A Partition of a Set

An Equivalence Relation Partitions a Set

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Summary

- Equivalence Relations
- Equivalence Classes
- Partitions
- Equivalence Classes and Partitions

Partial Orderings

Section 9.6

Video 25: Partial Ordering

- Partial Orderings and Partially-ordered Sets
- Lexicographic Orderings
- Hasse Diagrams
- Lattices
- Topological Sorting

Partial Orderings

Definition 1: A relation *R* on a set S is called a **partial ordering***,* or **partial order***,* if it is reflexive, antisymmetric, and transitive.

A set together with a partial ordering *R* is called a **partially ordered set**, or **poset**, and is denoted by (*S*, *R*).

$(\mathbf{Z}, \geq)$ is a poset

Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

Reflexivity: $a \geq a$ for every integer a.

Antisymmetry: If $a \geq b$ and $b \geq a$, then $a = b$.

Transitivity: If $a \geq b$ and $b \geq c$, then $a \geq c$.

($\boldsymbol{Z}^+$, |) is a poset

The divisibility relation (|) is a partial ordering on the set of integers.

Reflexivity:
$a \mid a$ for all integers a.

Antisymmetry:
If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.

Transitivity:
Suppose that $a \mid b$ and $b \mid c$. Then there are positive integers k and l such that $b = ak$ and $c = bl$.
Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

($\mathcal{P}$(S), ⊆) is a poset

The inclusion relation (⊆) is a partial ordering on the power set of a set S.

Reflexivity:
$A \subseteq A$ whenever A is a subset of S.

Antisymmetry:
If A and B are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Transitivity:
If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Lattices

Definition: A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

Example: $(\mathcal{P}(S), \subseteq)$ is a lattice.

Proof: The least upper bound of two subsets A and B is $A \cup B$, the greatest lower bound is $A \cap B$

Partial Order on Cartesian Product

Definition: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the **lexicographic ordering** on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is,

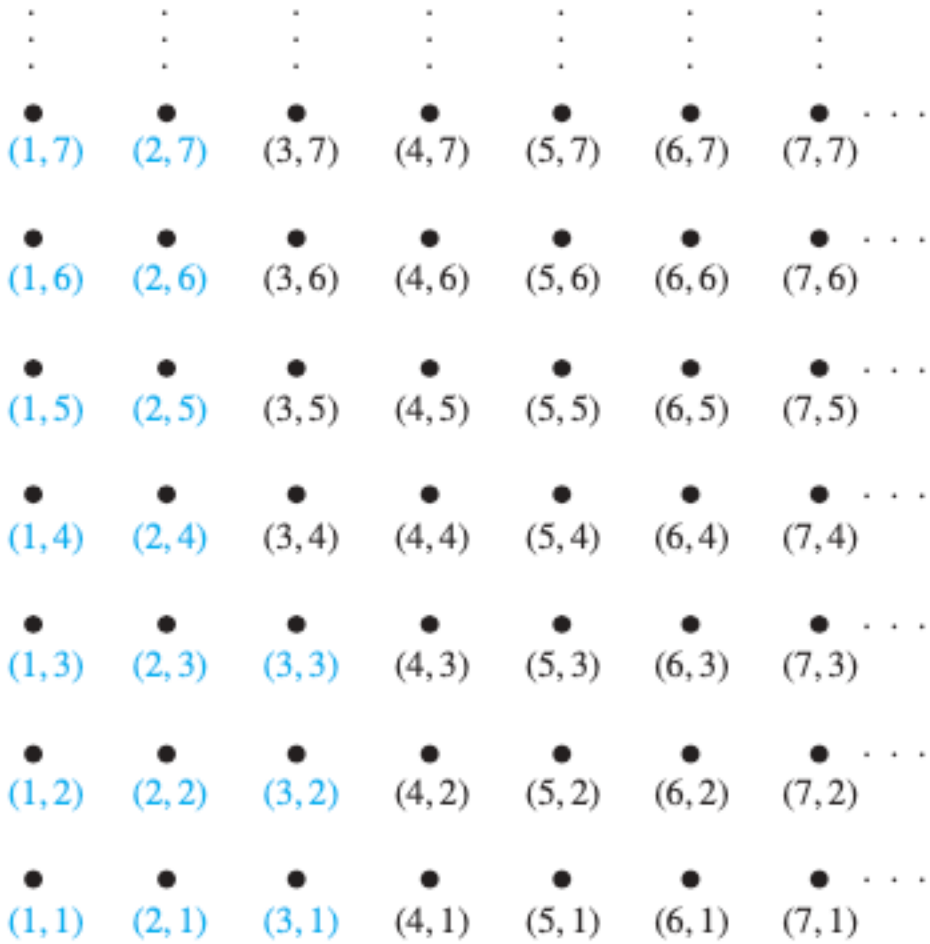
$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

This definition can be easily extended to a lexicographic ordering on n-ary Cartesian products

Example

$(\boldsymbol{Z} \times \boldsymbol{Z}, \prec)$

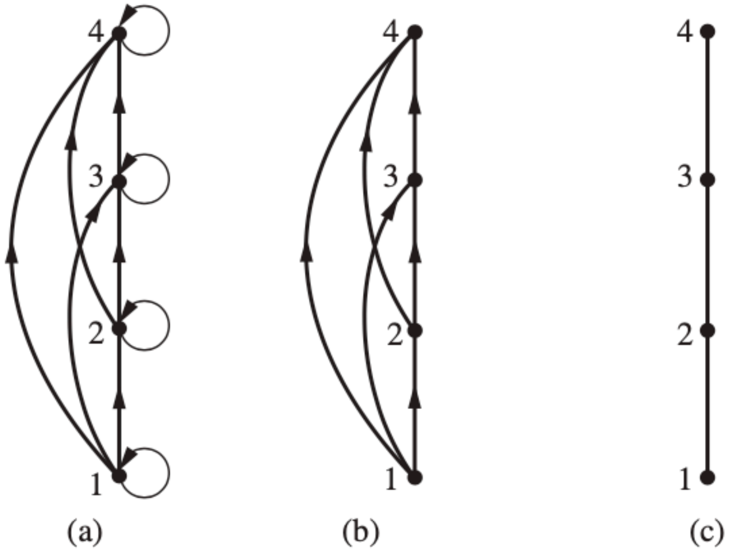


All ordered pairs less than (3, 4)

Hasse Diagrams

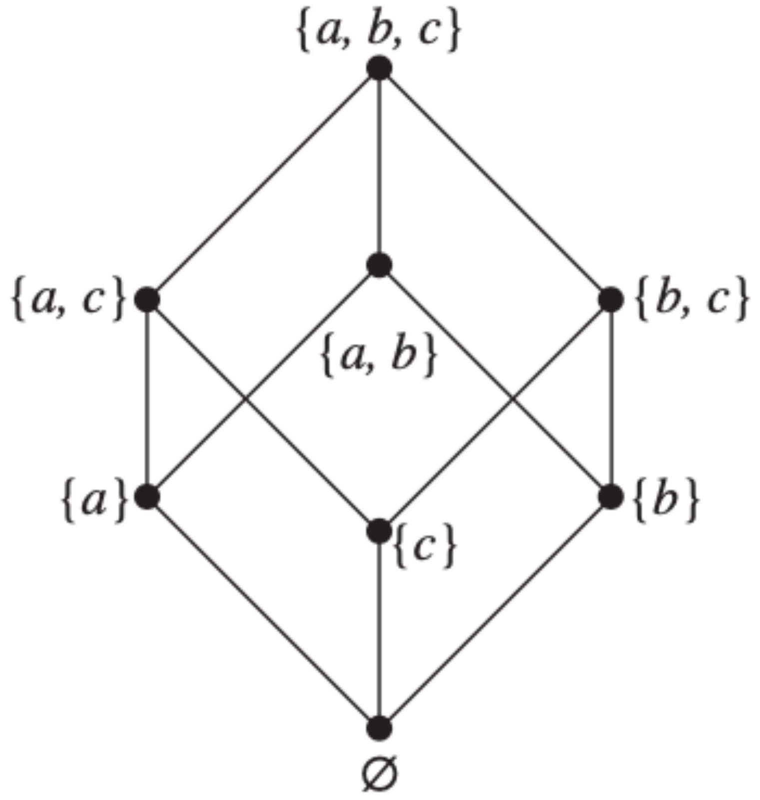
If a relation is reflexive and transitive, the representation as directed graph can be simplified

- If R is a partial order then we can (a) omit self-loops, (b) omit transitive edges and (c) assume that arrows point upwards



(c) Is a Hasse Diagram

Example



Hasse Diagram of $(P(\{a, b, c\}), \subseteq)$

Comparability

The symbol $\preccurlyeq$ is used to denote the relation in any poset

Definition 2: The elements a and b of a poset $(S,\preccurlyeq)$ are **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called **incomparable**.

Definition 3: If $(S,\preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a **totally ordered** or **linearly ordered set**, and $\preccurlyeq$ is called a **total order** or a **linear order**.

Definition 4: $(S,\preccurlyeq)$ is a **well-ordered set** if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Example

The poset ($\mathbf{Z}$, ≤) is totally ordered

For every two integers *a* and *b*, either $a \leq b$ or $b \leq a$ (or both)

The poset ($\boldsymbol{Z}^+$, |) is not totally ordered

For integers 5 and 7, 5 does not divide 7, and 7 does not divide 5

The poset ($\mathcal{P}$(S), ⊆) is not totally ordered if |S| > 1

Since there are at least two elements a and b in S, we have subsets {a} and {b} which are not comparable

Summary

- Partial Orderings and Partially-ordered Sets
  - Lexicographic Orderings
  - Lattices
- Visualization: Hasse Diagrams
- Total Orderings
- Well-ordered sets

Sequences and Summations

Section 2.4

Video 26: Sequences

- Sequences
- Examples of Sequences
- Recurrence relations

Introduction

Sequences are ordered lists of elements of a set

- 1, 2, 3, 5, 8
- c, o, m, p, u, t, e, r
- 1, 3, 9, 27, 81, …

Sequences arise throughout mathematics, computer science, and in many other sciences and arts, e.g. biology or music

Sequences

Definition: A **sequence** is a function from a subset of the integers to a set S.

Usually it is either the set $\mathbf{Z}^+$ or **N.**

Let $f : \mathbf{Z}^+ \rightarrow S$ be the function that defines a sequence.

We write a_n to denote the image $f(n)$ of the integer n.

The notation a_n is used to denote the image of the integer n.

We call a_n a **term** of the sequence.

Example

Let $\{a_n\}$ denote the sequence that is defined by $a_n = \frac{1}{n}$

The function defining the sequence is f : **N** → *S*, $f(n) = \frac{1}{n}$

Then $\{a_n\} = \{a_1, a_2, a_3, \dots\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

Arithmetic Progression

Definition: An **arithmetic progression** is a sequence of the form:

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the **initial term** *a* and the **common difference** *d* are real numbers.

An arithmetic progression is defined by the function $f : \mathbf{N} \rightarrow \mathbf{R}$, $f(n) = a + nd$

Examples

Let a = −1 and d = 4:

$$\{s_n\} \ = \ \{s_0, s_1, s_2, s_3, s_4, \ldots\} \ = \ \{-1, 3, 7, 11, 15, \ldots\}$$

Let a = 7 and d = −3:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

Let a = 1 and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Geometric Progression

Definition: A ***geometric progression*** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the **initial term** *a* and the **common ratio** *r* are real numbers.

An arithmetic progression is defined by the function
$f : \mathbf{Z}^+ \rightarrow \mathbf{R}$, $f(n) = ar^n$

Examples

Let a = 1 and r = −1. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

Let a = 2 and r = 5. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

Let a = 6 and r = 1/3. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Strings

Definition: A **string** is a finite sequence of characters from a finite set A (an alphabet).

A string is defined by a function

$f : \{1, \dots , n\} \rightarrow A$

Sequences of characters or bits are important in computer science.

- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Lexicographic Ordering on Strings

Consider strings of lowercase English letters.

A lexicographic ordering can be defined using the ordering of the letters in the alphabet.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

- Strings with lexicographic ordering are well-ordered sets.
- This is the same ordering as that used in dictionaries.

Recurrence Relations

Definition: A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of a finite number *k* of the preceding terms of the sequence, i.e.,

$$a_n = f(a_{n-1}, a_{n-2,} \dots, a_{n-k})$$

A sequence $\{a_n\}$ is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

The **initial conditions** for a sequence specify the terms a_0, a_1, …, a_{k-1}

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} + 3 \text{ for } n = 1, 2, 3, 4, \ldots.$$

and suppose that $a_0 = 2$.

Then

$$a_1 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} - a_{n-2} \text{ for } n = 2,3,4,\ldots.$$

and suppose that $a_0 = 3$ and $a_1 = 5$.

Then

$$a_2 = 5 - 3 = 2$$

$$a_3 = 2 - 5 = -3$$

$$a_4 = -3 - 2 = -5$$

$$a_5 = -5 + 3 = -2$$

Summary

- Sequences
- Examples of Sequences
  - Arithmetic progression
  - Geometric progression
  - Strings
- Recurrence relations

Video 27: Number Sequences

- Guessing sequences of numbers
- Modeling using number sequences
- Solving recurrence relations

Guessing Sequences of Numbers

Given a few terms of a sequence, try to identify the sequence. Conjecture a closed formula, recurrence relation, or some other pattern.

Some questions to ask?

- Are there repeated terms of the same value?
- Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
- Can you obtain a term by combining the previous terms in some way?
- Are there cycles among the terms?
- Do the terms match those of a well known sequence?

Example

Find a formulae for the sequence with the following first five terms:

1, 3, 5, 7, 9

We observe that each term is obtained by adding 2 to the previous term.

A possible formula is $a_n = a + 2n$

Since $a_0 = 1$ we conclude $a = 1$

This is an arithmetic progression with $a = 1$ and $d = 2$.

Example

Find a formulae for the sequence with the following first five terms:

1, ½, ¼, 1/8, 1/16

We observe that the denominators are powers of 2.

We guess that the sequence with $a_n = 1/2^n$ is a possible match.

This is a geometric progression with $a = 1$ and $r = ½$.

Example

Find a formulae for the sequence with the following first five terms:

1, -1, 1, -1,1

We observe that the terms alternate between 1 and -1.

A possible sequence is $a_n = (-1)^n$.

This is a geometric progression with $a = 1$ and $r = -1$.

Rabbits

A young pair of rabbits (one of each gender) is placed on an island.

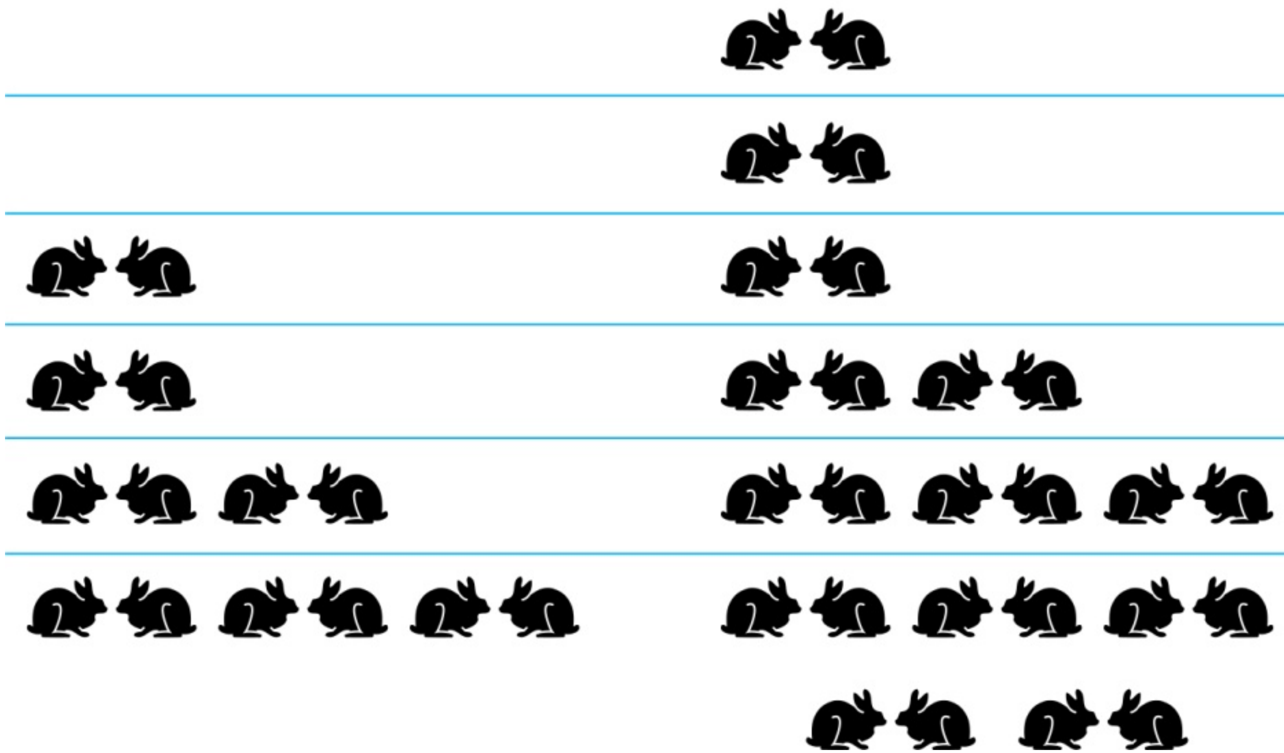
A pair of rabbits does not breed until they are 2 months old.

After they are 2 months old, each pair of rabbits produces another pair each month.

Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Modeling the Population Growth of Rabbits



Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Fibonacci Sequence

Definition: The **Fibonacci sequence** $f_0, f_1, f_2, \ldots$ is defined as:

Initial Conditions: $f_0 = 0,\ f_1 = 1$

Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

$f_2 = f_1 + f_0 = 1 + 0 = 1$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

$f_4 = f_3 + f_2 = 2 + 1 = 3$

$f_5 = f_4 + f_3 = 3 + 2 = 5$

$f_6 = f_5 + f_4 = 5 + 3 = 8$

Integer Sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Solving Recurrence Relations

Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called **solving the recurrence relation**.

- Such a formula is called a **closed formula**.
- Various methods for solving recurrence relations will be covered in Advanced Counting, where recurrence relations will be studied in greater depth.

Solving Recurrence Relations

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n\text{-}1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2.$

We may solve the recurrence relation by guessing the formula

Step 1: substitute repeatedly the recurrence

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot \mathbf{2} \\
&= (a_{n-3} + 3\,) + 3 \cdot 2 \;= a_{n-3} + 3 \cdot \mathbf{3} \\
&\quad \ldots \\
&= a_2 + 3(n-2) \;= (a_1 + 3) + 3(n-2) \;= 2 + 3\boldsymbol{(n-1)}
\end{aligned}
$$

Step 2: guess the formula: $a_n = 2 + 3(n-1)$

Step 3: verify that your guess is right:

$a_1 = 2 + 3*(1\text{-}1) = 2$, initial condition is ok

$a_n = 2 + 3*(n\text{-}1) = a_{n\text{-}1} + 3 = 2 + 3*(n—2) + 3,$ recurrence is ok

Summary

- Guessing sequences of numbers
- Modeling using number sequences
  - Fibonacci sequence
- Special integer sequences
- Solving recurrence relations

Video 28: Summations

- Sum and Product Notation
- Closed formula for geometric series
- Important summation formulae

Summation Notation

Given a sequence $\{a_n\} = \{a_1, a_2, a_3, \dots\}$

The notations

$$\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{j=m}^{n} a_j \qquad \sum_{m \le j \le n} a_j$$

denote the sum of the terms $a_m, a_{m+1}, \dots, a_n$

$$a_m + a_{m+1} + \cdots + a_n$$

The variable *j* is called the **index of summation**. It runs through all the integers starting with its **lower limit** *m* and ending with its **upper limit** *n*.

Example

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_0^n r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_1^\infty \frac{1}{i}$$

The upper limit can be infinite!

Summation over Sets

More generally for a set *S* we can denote

$$\sum_{j \in S} a_j$$

Example:

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation

Given a sequence $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$

The notations

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \prod_{m \le j \le n} a_j$$

denote the product of the terms $a_m, a_{m+1}, \ldots, a_n$

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Sums as Sequences

We may define a sequence *{s_n}* by a summation formula

$$s_n = \sum_{j=0}^{n} f(j)$$

An important task is to find a **closed formula** *$s(n)$* such that *$s(n) = s_n$*

Geometric Series

Theorem: If *a* and *r* are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof

Proof: Let

$$S_n = \sum_{j=0}^{n} ar^j.$$

To compute S, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$\begin{aligned}
rS_n &= r\sum_{j=0}^{n} ar^j && \text{substituting summation formula for } S \\
&= \sum_{j=0}^{n} ar^{j+1} && \text{by the distributive property} \\
&= \sum_{k=1}^{n+1} ar^k && \text{shifting the index of summation, with } k = j+1 \\
&= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a) && \text{removing } k = n+1 \text{ term and adding } k = 0 \text{ term} \\
&= S_n + (ar^{n+1} - a) && \text{substituting } S \text{ for summation formula}
\end{aligned}$$

From these equalities, we see that

$$rS_n = S_n + (ar^{n+1} - a).$$

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$. ◁

Important Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

Geometric Series: We just proved this.

We will be able to prove these using induction.

These proofs require analysis

Summary

- Sum and Product Notation
- Closed formula for geometric series
- Important summation formulae

Cardinality of Sets

Section 2.5

Video 29: Cardinality of Sets

- Cardinality
- Countable Sets

Cardinality

Definition: The **cardinality** of a set A is **equal** to the cardinality of a set B, denoted by $|A| = |B|$ iff there is a bijection from A to B.

If there is an injection from A to B, the **cardinality** of A is **less than or the same** as the cardinality of B and we write $|A| \leq |B|$.

When $|A| \leq |B|$ and A and B have different cardinality, we say that the **cardinality** of A is **less** than the cardinality of B and write $|A| < |B|$.

Countable Sets

Definition: A set that is either finite or has the same cardinality as the set of positive integers $\mathbf{Z}^+$ is called **countable**. A set that is not countable is **uncountable**.

When an infinite set is countable (**countably infinite**) its cardinality is $\aleph_0$.

We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Note: $\aleph$ is aleph, the 1st letter of the Hebrew alphabet

Showing that a Set is Countable

Theorem: An infinite set S is countable iff it is possible to list the elements of the set in a sequence indexed by the positive integers.

Proof:

If the set is countable, there exists a bijection from $\mathbf{Z}^+$ to S.

Therefore we can form the sequence $a_1, a_2, \ldots, a_n, \ldots$ where

$$a_1 = f(1),\ a_2 = f(2), \ldots,\ a_n = f(n), \ldots$$

If we can list the set in a sequence $\{a_n\}$ indexed by the positive integers, we can define the function

$$f(n) = a_n$$

which is a bijection.

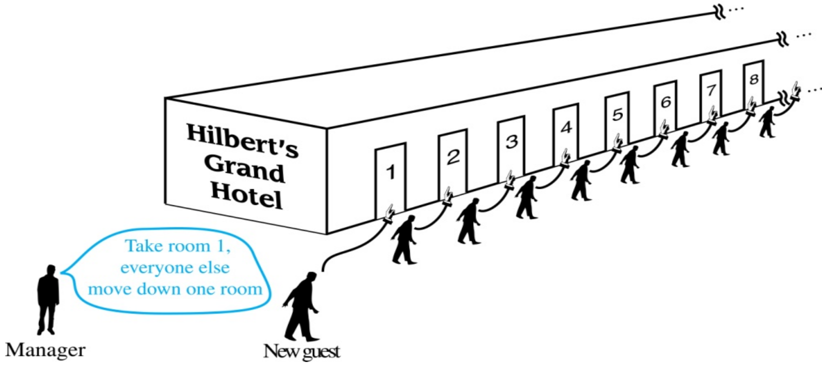
Hilbert’s Grand Hotel

David Hilbert

The Grand Hotel has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation:

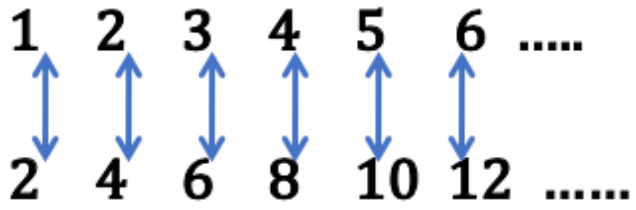
- Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on.
- When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room *n* to Room *n* + 1, for all positive integers *n*.
- This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.



Example

Show that the set of positive even integers E is countable set.

Let $f : \mathbf{Z}^+ \rightarrow E$, $f(x) = 2x$.



Then f is a bijection from $\mathbf{Z}^+$ to E since f is both injective and surjective.

Proof:

Suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. Therefore it is injective.

Suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$. Therefore it is surjective.

◄

Example

Show that the set of integers **Z** is countable.

We can define a bijection from **N** to **Z**

- When n is even: $f(n) = n/2$
- When n is odd: $f(\mathrm{n}) = -(n-1)/2$

Alternatively we can list the numbers in a sequence

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

The Positive Rational Numbers are Countable

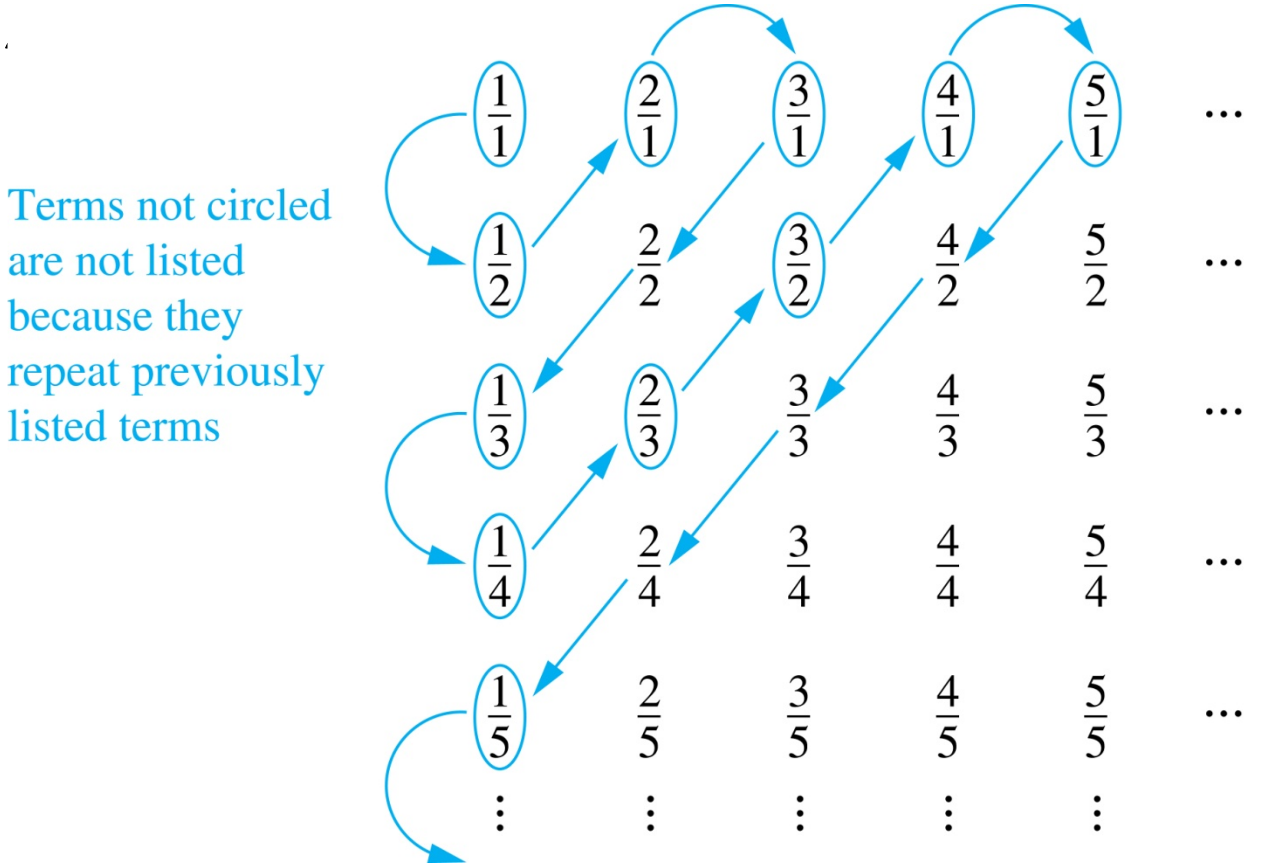
The positive rational numbers are countable since they can be arranged in a sequence r_1 , r_2 , r_3

Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

1, ½, 2, 3, 1/3,1/4, 2/3, ….



First row $q = 1$.
Second row $q = 2$.
etc.

The Set of Finite Strings is Countable

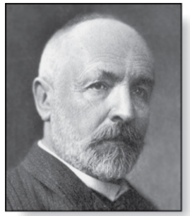
The set of finite strings S over a finite alphabet A is countably infinite.

Show that the strings can be listed in a sequence.

First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic order.
3. Then all the strings of length 2 in lexicographic order.
4. Etc.

The Set of Real Numbers **R** is Uncountable



Georg Cantor

Cantor diagnalization argument, a proof by contradiction.

Proof:

- Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable, as any subset of a countable set is countable.
- Then the real numbers between 0 and 1 can be listed as a sequence $r_1, r_2, r_3, \ldots$
- Let the decimal representation of this listing be

$$
\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots \\
r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots \\
r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\ldots \\
&\vdots
\end{aligned}
$$

Diagonalization

- Form a new real number with the decimal expansion

$$r = .r_1 r_2 r_3 r_4 \ldots$$

  where

$$r_i = 3 \text{ if } d_{ii} \neq 3 \ \text{ and } \ r_i = 4 \text{ if } d_{ii} = 3$$

- *r* is not equal to any of the r_1 , r_2 , r_3 ,...
  It differs from r_i in its i^{th} position after the decimal point.

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\ldots \\ &\ \vdots \end{aligned}$$

Contradiction

- Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion.
- Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
- Since a set with an uncountable subset is uncountable, the set of real numbers is uncountable.

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Summary

- Cardinality
- Countable Sets
- Proving countability
- Example of countable sets
  - Even numbers
  - Integers
  - Rational Numbers
- Uncountable sets
  - Real numbers