Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 8

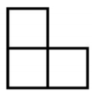
November 3, 2020

Exercise 1. Find a formula for $f(n)$, and prove it by induction, if

1. $f(0) = 0$ and $f(n) = f(n-1) - 1$.
2. $f(0) = 0$, $f(1) = 1$ and $f(n) = 2f(n-2)$.

Exercise 2. Use strong[1] or mathematical induction to show the following statements.

1. Any postage of at least 8 cents can be formed using just 3 cents and 5 cents stamps.
2. Consider a $2^n \times 2^n$ grid of 2^{2n} squares arranged in 2^n rows and 2^n columns. If we remove one square from this grid of 2^{2n} squares, we obtain a shape. Let C_n be the set of shapes obtained by removing one square from the grid of 2^{2n} squares. We say that a shape is *L-coverable* if it can be completely covered, without overlapping, with L-shaped tiles occupying exactly 3 squares, like this:



Prove that, for any integer $n \geq 1$, any shape in C_n is *L-coverable*.

Exercise 3. Denote by f_n the nth Fibonacci number, i.e., $f_0 = 0, f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$.

1. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, when n is a positive integer.
2. Show that $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$, when n is a positive integer.
3. Show that if $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$, then

$$f_n = (a^n - b^n)/(a-b).$$

[Hint: a and b are two solutions of the equation $x^2 = x + 1$]

Exercise 4. Prove that $n! > 2^n$ for $n \geq 4$.

Exercise 5. Let $P(n)$ for $n \in \mathbf{Z}_{\geq 0}$ be the propositional function "all cardinality-n sets of integers consist of only even integers," which is proved using strong induction:

Basis step $P(0)$ is true, since if S is an empty set of integers the statement "$\forall s\ s \in S \rightarrow s$ is even" is true.

Inductive step Let $k \geq 0$ and assume that $P(i)$ is true for $0 \leq i \leq k$. To prove that $P(k+1)$ is true we use the following steps:

[1]Mathematical induction is based on the tautology $\Big(P(0) \wedge \forall k\ [\underline{P(k)} \rightarrow P(k+1)]\Big) \rightarrow \forall n\ P(n)$, strong induction is based on the tautology $\Big(P(0) \wedge \forall k\ [\underline{(\forall \ell \leq k\ \ P(\ell))} \rightarrow P(k+1)]\Big) \rightarrow \forall n\ P(n)$. Although the latter looks "stronger" than mathematical induction, the two notions are equivalent.

1. Let T be an arbitrary set of integers with $|T| = k + 1$.
2. Write T as the disjoint union of sets T_1 and T_2 such that $|T_1| = k$ and $|T_2| = 1$.
3. Because $|T_1| < |T|$ and $|T_2| < |T|$ the induction hypothesis applies to both T_1 and T_2, implying that all elements of both T_1 and T_2 are even.
4. Because $T = T_1 \cup T_2$ it follows that all elements of T are even as well.
5. Because T was arbitrarily chosen as a set of integers of cardinality $k + 1$, it follows that $P(k + 1)$ is true.

Because not all integers are even, the proof cannot be correct (unless the well-ordering principle is false). Find the mistake.

Exercise 6. Let $P(n)$ for $n \in \mathbf{Z}_{>0}$ be the propositional function "all cardinality-n sets of integers consist of only odd integers," which is proved using strong induction:

Basis Step $P(1)$ is true because 1 is odd.

Inductive step Let $k > 0$ and assume that $P(i)$ is true for $0 < i \leq k$. To prove that $P(k + 1)$ is true we use the following steps:

1. Let S be an arbitrary set of integers with $|S| = k + 1$.
2. Write S as the disjoint union of sets S_1 and S_2 such that $|S_1| = k$ and $|S_2| = 1$.
3. Because $|S_1| < |S|$ and $|S_2| < |S|$ the induction hypothesis applies to both S_1 and S_2,
4. implying that all elements of both S_1 and S_2 are odd.
5. Because $S = S_1 \cup S_2$ it follows that all elements of S are odd as well.
6. Because S is an arbitrarily chosen set of integers with $|S| = k + 1$, it follows that $P(k + 1)$ is true.

- ◯ Only the basis step in the proof is incorrect.
- ◯ The basis step and step (3) of the inductive step of the proof are incorrect.
- ◯ Only step (3) of the inductive step of the proof is incorrect.
- ◯ Only step (4) of the inductive step of the proof is incorrect.

Exercise 7. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices with the property $\mathbf{AB} = \mathbf{BA}$. Prove using induction that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.