Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 4 — Solutions

October 12, 2020

Exercise 1.

1. *Let f be a function mapping set X to set Y and let g be a function from set Y to set Z. For each statement below, prove it if it is true and give a counterexample otherwise.*

   (a) *If f or g is injective, then $g \circ f$ is injective.*

   FALSE. Let $X = Y = \{0, 1\}$ and $Z = \{0\}$. Let $f(0) = 0$ and $f(1) = 1$. The function f is injective. Let $g(0) = g(1) = 0$. Then $g \circ f(0) = g \circ f(1)$, therefore $g \circ f$ is not injective.

   (b) *If f or g is surjective, then $g \circ f$ is surjective.*

   FALSE. Let $X = Y = \{0\}$ and $Z = \{0, 1\}$. Let $f(0) = 0$ and $g(0) = 0$. The function f is surjective, but $g \circ f$ is not. No element of X maps to $1 \in Z$.

   (c) *If f and g are injective, then $g \circ f$ is injective.*

   TRUE. Let $g \circ f(x) = g \circ f(y)$. Because g is injective, $f(x) = f(y)$. Then $x = y$ by injectivity of f.

   (d) *If f and g are surjective, then $g \circ f$ is surjective.*

   TRUE. Let $z \in Z$. Because g is surjective, there exists $y \in Y$ such that $g(y) = z$. Now by surjectivity of f there is $x \in X$ such that $f(x) = y$. Clearly $g \circ f(x) = g(y) = z$. We have shown that $g \circ f$ is surjective.

   (e) *If $g \circ f$ is injective, then f is injective.*

   TRUE. Let $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$ and injectivity of $g \circ f$ implies $x = y$. Therefore f is injective.

   (f) *If $g \circ f$ is injective, then g is injective.*

   FALSE. Let $X = Z = \{0\}$ and $Y = \{0, 1\}$. Let $f(0) = 0$ and let $g(0) = g(1) = 0$. Then g is not injective, but $g \circ f$ is.

   (g) *If $g \circ f$ is surjective, then g is surjective.*

   TRUE. Let $z \in Z$. Because $g \circ f$ is surjective, there exists $x \in X$ such that $g \circ f(x) = z$. But then $g(f(x)) = z$, therefore g is surjective.

   (h) *If $g \circ f$ is surjective, then f is surjective.*

   FALSE. Let $X = Z = \{0\}$ and $Y = \{0, 1\}$. Let $f(0) = 0$ and $g(0) = g(1) = 0$. The function $g \circ f$ is surjective, because $g \circ f(0) = 0$ and there are no other elements in Z. There is no $x \in X$ such that $f(x) = 1$, therefore f is not surjective.

   (i) *If $g \circ f$ is bijective, then f is bijective.*

   FALSE. Let $X = Z = \{0\}$ and $Y = \{0, 1\}$. Let $f(0) = 0$ and $g(0) = g(1) = 0$. The function $g \circ f$ is a bijection, but f is not surjective. There is no $x \in X$ such that $f(x) = 1$. Therefore f is not bijective.

   (j) *If $g \circ f$ is bijective, then g is bijective.*

   FALSE. Let $X = Z = \{0\}$ and $Y = \{0, 1\}$. Let $f(0) = 0$ and $g(0) = g(1) = 0$. The function $g \circ f$ is a bijection, but g is not injective as $g(0) = g(1) = 0$. Therefore g is not bijective.

2. *For each false implication above, determine if it is always false irrespective of the choices of f and g (in which case it would be called a* contradiction*) or if it may be true or false depending on the particular choices of f and g (in which case it would be called a* contingency*).*

   All the false statements are contingencies: they are all of the form $A \to B$, where A is not necessarily true.

Exercise 2. (From last year's midterm exam)

(*français*) *Soit* $f : \{x \mid x \in \mathbf{R}, -2 \le x \le 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \textit{pour} \quad -2 \le x \le 0 \\ \lfloor x \rfloor & \textit{pour} \quad 0 \le x < 2 \\ x^2 & \textit{pour} \quad 2 \le x \le 5. \end{cases}$$

(*English*) *Let* $f : \{x \mid x \in \mathbf{R}, -2 \le x \le 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \textit{for} \quad -2 \le x \le 0 \\ \lfloor x \rfloor & \textit{for} \quad 0 \le x < 2 \\ x^2 & \textit{for} \quad 2 \le x \le 5. \end{cases}$$

- ○ $\begin{cases} f \textit{ est injective mais } f \textit{ n'est pas surjective.} \\ f \textit{ is injective but not surjective.} \end{cases}$
- ○ $\begin{cases} f \textit{ est surjective mais } f \textit{ n'est pas injective.} \\ f \textit{ is surjective but not injective.} \end{cases}$
- ○ $\begin{cases} f \textit{ est bijective.} \\ f \textit{ is bijective.} \end{cases}$
- ✓ $\begin{cases} f \textit{ n'est pas une fonction.} \\ f \textit{ is not a function.} \end{cases}$

For $-2 \le x \le 0$ we have that x is mapped to $3 + \frac{3}{2}x$, which equals 3 for $x = 0$. But for $0 \le x < 2$ we have that x is mapped to $\lfloor x \rfloor$, which equals 0 for $x = 0$. Thus $x = 0$ is mapped by f to both 3 and to 0, which implies that the definition of f violates the definition of a function, namely that each value of the domain has a single function value.

It follows that the last circle must be ticked.

Exercise 3. Let $f : \{x \mid x \in \mathbf{R}, 0 < x < 1\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 2 - \dfrac{1}{x} & \text{if} \quad 0 < x < 1/2 \\[2ex] \dfrac{1}{1-x} - 2 & \text{if} \quad \frac{1}{2} \le x < 1. \end{cases}$$

- ○ *f is not injective and not surjective.*
- ○ *f is injective but not surjective.*
- ○ *f is surjective but not injective.*
- ✓ *f is bijective.*

Proving injectivity: $\forall x_1, x_2 \; f(x_1) = f(x_2) \to x_1 = x_2$.
Assume $f(x_1) = f(x_2)$ and consider every possible combination of x_1, x_2.

- $0 < x_1, x_2 < 1/2$.

  $2 - \frac{1}{x_1} = 2 - \frac{1}{x_2}$ obviously implies $x_1 = x_2$.

- $0 < x_1 < 1/2$ and $1/2 \leq x_2 < 1$ (or $1/2 \leq x_1 < 1$ and $0 < x_2 < 1/2$).

  In this case,

  $$\begin{aligned} 2 - \frac{1}{x_1} &= \frac{1}{1-x_2} - 2 \\ \frac{1}{1-x_2} + \frac{1}{x_1} &= 4. \end{aligned}$$

  However, since $0 < x_1 < 1/2$ and $1/2 \leq x_2 < 1$, we have

  $$\begin{cases} \dfrac{1}{x_1} > 2 \\ \dfrac{1}{1-x_2} \geq 2. \end{cases}$$

  Hence, $\frac{1}{x_1} + \frac{1}{1-x_2} > 4$, so $f(x_1) \neq f(x_2)$ (which still makes the statement true).

- $1/2 \leq x_1, x_2 < 1$.

  $\frac{1}{1-x_1} - 2 = \frac{1}{1-x_2} - 2$ obviously implies $x_1 = x_2$.

Therefore, $f(x)$ is *injective*.

Proving surjectivity: $\forall y \, \exists x \; f(x) = y$.
Let $y \in \mathbf{R}$ and consider the two following cases.

- $y < 0$.

  In this case, for $0 < x < 1/2$, we have

  $$\begin{aligned} 2 - \frac{1}{x} &= y \\ x(2-y) &= 1 \\ x &= \frac{1}{2-y} \end{aligned}$$

  which respects the constraints on x and, hence, validates the statement.

- $y \geq 0$.

  In this case, for $1/2 \leq x < 1$, we have

  $$\begin{aligned} \frac{1}{1-x} - 2 &= y \\ 1 - 2(1-x) &= y(1-x) \\ x(y+2) &= y+1 \\ x &= \frac{y+1}{y+2} \end{aligned}$$

  which respects the constraints on x and, hence, validates the statement.

Therefore, $f(x)$ is *surjective*.

Exercise 4.

(*français*) *Pour un $\delta \in \mathbf{R}$ arbitraire, soient f_δ et g_δ les deux fonctions de $\mathbf{R}$ vers $\mathbf{R}$ suivantes*

$$f_\delta(x) = \begin{cases} x+\delta & \text{si } x \in \mathbf{Z} \\ -x+\delta & \text{si } x \notin \mathbf{Z}, \end{cases} \qquad g_\delta(x) = \begin{cases} x+\delta & \text{si } x \in \mathbf{Z} \\ -x-\delta & \text{si } x \notin \mathbf{Z}. \end{cases}$$

Considérez les deux propositions

$$\forall \delta \in \mathbf{R} \;\; f_\delta \text{ est une bijection} \qquad \text{et} \qquad \forall \delta \in \mathbf{R} \;\; g_\delta \text{ est une bijection.}$$

(*English*) *For any $\delta \in \mathbf{R}$ let f_δ and g_δ be the following two functions from $\mathbf{R}$ to $\mathbf{R}$*

$$f_\delta(x) = \begin{cases} x+\delta & \text{if } x \in \mathbf{Z} \\ -x+\delta & \text{if } x \notin \mathbf{Z}, \end{cases} \qquad g_\delta(x) = \begin{cases} x+\delta & \text{if } x \in \mathbf{Z} \\ -x-\delta & \text{if } x \notin \mathbf{Z}. \end{cases}$$

Consider the two statements

$$\forall \delta \in \mathbf{R} \;\; f_\delta \text{ is a bijection} \qquad \text{and} \qquad \forall \delta \in \mathbf{R} \;\; g_\delta \text{ is a bijection.}$$

○ *Seule la seconde proposition est vraie.* / *Only the second statement is true.*

✓ *Seule la première proposition est vraie.* / *Only the first statement is true.*

○ *Elles sont vraies toutes les deux.* / *They are both true.*

○ *Elles sont fausses toutes les deux.* / *They are both false.*

Proving injectivity: $\forall x_1, x_2 \; f(x_1) = f(x_2) \to x_1 = x_2$.

- Given any distinct $x_1, x_2 \in \mathbf{R}$, if $x_1 \in \mathbf{Z}$ and $x_2 \in \mathbf{Z}$, then $f_\delta(x_1) = x_1 + \delta \neq x_2 + \delta = f_\delta(x_2)$, otherwise if $x_1 \in \mathbf{Z}$ and $x_2 \notin \mathbf{Z}$, then also $-x_2 \notin \mathbf{Z}$ so that $x_1 \neq -x_2$ and $f_\delta(x_1) = x_1 + \delta \neq -x_2 + \delta = f_\delta(x_2)$, and otherwise if $x_1 \notin \mathbf{Z}$ and $x_2 \notin \mathbf{Z}$, then $f_\delta(x_1) = -x_1 + \delta \neq -x_2 + \delta = f_\delta(x_2)$.

  Because the remaining case $x_1 \notin \mathbf{Z}$ and $x_2 \in \mathbf{Z}$ is equivalent to the case $x_1 \in \mathbf{Z}$ and $x_2 \notin \mathbf{Z}$ that was treated already, we find that for any $x_1 \neq x_2$ we have that $f_\delta(x_1) \neq f_\delta(x_2)$.

  This argument works for any δ.
- Since $g_{(1/3)}(0) = \frac{1}{3} = \frac{2}{3} - \frac{1}{3} = g_{(1/3)}(-\frac{2}{3})$ we find that there exists a $\delta \in \mathbf{R}$ such that g_δ is not injective and therefore not bijective.

Proving surjectivity: $\forall y \, \exists x \; f(x) = y$.

- Given an arbitrary $y \in \mathbf{R}$, if $y - \delta \in \mathbf{Z}$ then $f_\delta(y-\delta) = (y-\delta)+\delta = y$,

  and otherwise if $y - \delta \notin \mathbf{Z}$ then $-(y-\delta) \notin \mathbf{Z}$ and $f_\delta(-(y-\delta)) = -(-(y-\delta)) + \delta = (y-\delta)+\delta = y$.

  This argument works for any δ.

Based on the above, it follows that for any δ the function f_δ is bijective while the function g_δ is not bijective. The negation of the second statement (namely "$\exists \delta \in \mathbf{R} \;\; g_\delta$ is not a bijection") is correct, implying that the second statement must be ticked. (Note that there exists at least one value for δ such that g_δ is bijective, namely $\delta = 0$ because $f_0 = g_0$.)

Exercise 5.

(*français*) *Soit $\mathcal{P}(X)$ l'ensemble des parties d'un ensemble X (c'est-à-dire le "power set" de X) et soit $\emptyset$ l'ensemble vide. Soient les propositions ci-dessous*

pour tous ensembles A et B, si $\mathcal{P}(A) = \mathcal{P}(B)$, alors $A = B$;

et

il existe un ensemble C tel que $\mathcal{P}(C) = \emptyset$.

(*English*) *Let $\mathcal{P}(X)$ denote the power set of a set X and let $\emptyset$ denote the empty set. Consider the two statements*

for any sets A and B, if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$;

and

there exists a set C such that $\mathcal{P}(C) = \emptyset$.

○ $\left\{\begin{array}{l}\textit{Elles sont vraies toutes les deux.}\\ \textit{They are both true.}\end{array}\right.$

✓ $\left\{\begin{array}{l}\textit{Seulement la première est vraie.}\\ \textit{Only the first is true.}\end{array}\right.$

○ $\left\{\begin{array}{l}\textit{Seulement la seconde est vraie.}\\ \textit{Only the second is true.}\end{array}\right.$

○ $\left\{\begin{array}{l}\textit{Elles sont fausses toutes les deux.}\\ \textit{They are both false.}\end{array}\right.$

For any set C it is the case that $C = \bigcup_{c \in \mathcal{P}(C)} c$. It immediately follows that if $\mathcal{P}(A) = \mathcal{P}(B)$ for two sets A and B, then $A = B$. Thus the first statement is true.

For any set C it is the case that $\emptyset \subseteq C$ (because $\forall x\ x \in \emptyset \to x \in C$), so that $\emptyset \in \mathcal{P}(C)$ and thus $\mathcal{P}(C) \neq \emptyset$. Therefore there is no set C such that $\mathcal{P}(C) = \emptyset$ and the second statement is false.

It follows that the second circle must be ticked.

Exercise 6.

(*français*) *Soient $X = \{1, 2, 3, 4, 5\}$ et $\mathcal{P}(X)$ l'ensemble des parties de X (c'est-à-dire le "power set" de X). Soient les propositions ci-dessous*

(*English*) *Let $X = \{1, 2, 3, 4, 5\}$ and let $\mathcal{P}(X)$ denote the power set of X. Given the statements*

$$\emptyset \in \mathcal{P}(X) \qquad\qquad \{\emptyset\} \in \mathcal{P}(X)$$

✓ $\left\{\begin{array}{l}\textit{Seulement la première est vraie.}\\ \textit{Only the first is true.}\end{array}\right.$

○ $\left\{\begin{array}{l}\textit{Elles sont vraies toutes les deux.}\\ \textit{They are both true.}\end{array}\right.$

○ $\left\{\begin{array}{l}\textit{Seulement la seconde est vraie.}\\ \textit{Only the second is true.}\end{array}\right.$

○ $\left\{\begin{array}{l}\textit{Elles sont fausses toutes les deux.}\\ \textit{They are both false.}\end{array}\right.$

The power set $\mathcal{P}(X)$ is defined as the set that has all subsets of X as its elements. Furthermore, the statement

$$\forall x \in \emptyset\ x \in X$$

consists of a universal quantifier that ranges over an empty set and is thus true. According to the definition of "subset" it follows that $\emptyset \subseteq X$ so that it follows, using the definition of $\mathcal{P}(X)$, that $\emptyset \in \mathcal{P}(X)$.

If $\{\emptyset\} \in \mathcal{P}(X)$, then (using the definition of $\mathcal{P}(X)$), the set $\{\emptyset\}$ must be a subset of X, implying (according to the definition of a subset) that all elements of the set $\{\emptyset\}$ must also be elements of X. The set $\{\emptyset\}$ has just a single element, namely $\emptyset$, and $\emptyset$ is not one of the elements of X, because X just consists of the elements 1, 2, 3, 4, and 5. Thus $\{\emptyset\} \notin \mathcal{P}(X)$ and only the first answer is correct.

Exercise 7. *Find a bijection between* $(0,1) \subset \mathbf{R}$ *and* $(0,1] \subset \mathbf{R}$ *or show that it cannot exist.*

$$f : x \mapsto \begin{cases} \dfrac{1}{2^n} & \text{if } x = \dfrac{1}{2^{n+1}} \text{ for } n \in \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$