

## Week 8

November 3, 2020

**Exercise 1.** Find a formula for  $f(n)$ , and prove it by induction, if

1.  $f(0) = 0$  and  $f(n) = f(n-1) - 1$ .
2.  $f(0) = 0$ ,  $f(1) = 1$  and  $f(n) = 2f(n-2)$ .

**Exercise 2.** Use strong<sup>1</sup> or mathematical induction to show the following statements.

1. Any postage of at least 8 cents can be formed using just 3 cents and 5 cents stamps.
2. Consider a  $2^n \times 2^n$  grid of  $2^{2n}$  squares arranged in  $2^n$  rows and  $2^n$  columns. If we remove one square from this grid of  $2^{2n}$  squares, we obtain a shape. Let  $C_n$  be the set of shapes obtained by removing one square from the grid of  $2^{2n}$  squares. We say that a shape is *L-coverable* if it can be completely covered, without overlapping, with L-shaped tiles occupying exactly 3 squares, like this:



Prove that, for any integer  $n \geq 1$ , any shape in  $C_n$  is *L-coverable*.

**Exercise 3.** Denote by  $f_n$  the  $n$ th Fibonacci number, i.e.,  $f_0 = 0$ ,  $f_1 = 1$  and for  $n \geq 2$ ,  $f_n = f_{n-1} + f_{n-2}$ .

1. Prove that  $f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}$ , when  $n$  is a positive integer.
2. Show that  $f_0 f_1 + f_1 f_2 + \dots + f_{2n-1} f_{2n} = f_{2n}^2$ , when  $n$  is a positive integer.
3. Show that if  $a = (1 + \sqrt{5})/2$  and  $b = (1 - \sqrt{5})/2$ , then

$$f_n = (a^n - b^n)/(a - b).$$

[Hint:  $a$  and  $b$  are two solutions of the equation  $x^2 = x + 1$ ]

**Exercise 4.** Prove that  $n! > 2^n$  for  $n \geq 4$ .

**Exercise 5.** Let  $P(n)$  for  $n \in \mathbf{Z}_{\geq 0}$  be the propositional function “all cardinality- $n$  sets of integers consist of only even integers,” which is proved using strong induction:

**Basis step**  $P(0)$  is true, since if  $S$  is an empty set of integers the statement “ $\forall s \in S \rightarrow s$  is even” is true.

**Inductive step** Let  $k \geq 0$  and assume that  $P(i)$  is true for  $0 \leq i \leq k$ . To prove that  $P(k+1)$  is true we use the following steps:

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<sup>1</sup>Mathematical induction is based on the tautology  $(P(0) \wedge \forall k [P(k) \rightarrow P(k+1)]) \rightarrow \forall n P(n)$ , strong induction is based on the tautology  $(P(0) \wedge \forall k [(\forall \ell \leq k P(\ell)) \rightarrow P(k+1)]) \rightarrow \forall n P(n)$ . Although the latter looks “stronger” than mathematical induction, the two notions are equivalent.

1. Let  $T$  be an arbitrary set of integers with  $|T| = k + 1$ .
2. Write  $T$  as the disjoint union of sets  $T_1$  and  $T_2$  such that  $|T_1| = k$  and  $|T_2| = 1$ .
3. Because  $|T_1| < |T|$  and  $|T_2| < |T|$  the induction hypothesis applies to both  $T_1$  and  $T_2$ , implying that all elements of both  $T_1$  and  $T_2$  are even.
4. Because  $T = T_1 \cup T_2$  it follows that all elements of  $T$  are even as well.
5. Because  $T$  was arbitrarily chosen as a set of integers of cardinality  $k + 1$ , it follows that  $P(k + 1)$  is true.

Because not all integers are even, the proof cannot be correct (unless the well-ordering principle is false). Find the mistake.

**Exercise 6.** Let  $P(n)$  for  $n \in \mathbf{Z}_{>0}$  be the propositional function “all cardinality- $n$  sets of integers consist of only odd integers,” which is proved using strong induction:

**Basis Step**  $P(1)$  is true because 1 is odd.

**Inductive step** Let  $k > 0$  and assume that  $P(i)$  is true for  $0 < i \leq k$ . To prove that  $P(k + 1)$  is true we use the following steps:

1. Let  $S$  be an arbitrary set of integers with  $|S| = k + 1$ .
2. Write  $S$  as the disjoint union of sets  $S_1$  and  $S_2$  such that  $|S_1| = k$  and  $|S_2| = 1$ .
3. Because  $|S_1| < |S|$  and  $|S_2| < |S|$  the induction hypothesis applies to both  $S_1$  and  $S_2$ ,
4. implying that all elements of both  $S_1$  and  $S_2$  are odd.
5. Because  $S = S_1 \cup S_2$  it follows that all elements of  $S$  are odd as well.
6. Because  $S$  is an arbitrarily chosen set of integers with  $|S| = k + 1$ , it follows that  $P(k + 1)$  is true.

- ☐ Only the basis step in the proof is incorrect.
- ☐ The basis step and step (c) of the inductive step of the proof are incorrect.
- ☐ Only step (c) of the inductive step of the proof is incorrect.
- ☐ Only step (d) of the inductive step of the proof is incorrect.

**Exercise 7.** Suppose that  $\mathbf{A}$  and  $\mathbf{B}$  are square matrices with the property  $\mathbf{AB} = \mathbf{BA}$ . Prove using induction that  $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$  for every positive integer  $n$ .