Counting

Chapter 6

The Basics of Counting

Section 6.1

Counting

- How many different passwords exist?
- How many outcomes does an experiment have?
- How many steps an algorithm performs?
- How many moves are possible in a game?

- Counting is ubiquitous in computer science and mathematics

Video 61: The Product Rule

- Product Rule
- Applications of the Product Rule

Basic Counting Principles: The Product Rule

The Product Rule: Assume there are two tasks A and B. There are n_1 ways to do A and n_2 ways to do B. Then there are $n_1 \cdot n_2$ ways to do **both** tasks.

Example: How many pairs of dices can be rolled?

Since each dice has 6 faces, we can roll 6 * 6 = 36 pairs

Example

How many different license plates can be made if each plate contains a sequence of two uppercase letters followed by five digits?

By the product rule, there are $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 =$ 67’600’000 different possible license plates.

Counting Functions

How many functions are there from a set with *m* elements to a set with *n* elements?

Since a function represents a choice of one of the *n* elements of the codomain for each of the *m* elements in the domain, the product rule tells us that there are $n \cdot n \cdot \ldots \cdot n = n^m$ such functions.

Counting One-to-one Functions

How many one-to-one functions are there from a set with m elements to one with n elements?

Suppose the elements in the domain are a_1, a_2,…, a_m.

There are n ways to choose the value of a_1, $n-1$ ways to choose a_2, etc.

The product rule tells us that there are $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-m+1)$ such functions.

Counting Subsets of a Finite Set

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Proof:

- When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$.
- When the i^{th} element is in the subset, the bit string has a 1 in the i^{th} position and a 0 otherwise.
- By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets. ◀

Counting Cartesian Products

If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

Proof:

The task of choosing an element in the Cartesian product

$A_1 \times A_2 \times \cdots \times A_m$ is done by

choosing an element in A_1,

then an element in A_2, …, and finally an element in A_m.

By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|. \qquad \blacktriangleleft$$

Summary

- Product Rule
- Applications of the Product Rule
  - Counting functions
  - Counting subsets
  - Counting tuples

Video 62: The Sum Rule

- Sum Rule
- Subtraction Rule

Basic Counting Principles: The Sum Rule

The Sum Rule: Assume there are two tasks A and B. There are n_1 ways to do A and n_2 ways to do B and none of the set of n_1 ways is the same as any of the set of n_2 ways. Then there are $n_1 + n_2$ ways to do task A or B.

Example: A student can choose a semester project from one of three laboratories. The three laboratories offer 5, 3, and 7 possible projects, respectively. No project is offered by several laboratories. How many possible projects are there to choose from?

By the sum rule it follows that there are 5+3+7 = 15 ways to choose a project.

The Sum Rule in Terms of Sets

The sum rule can be phrased as

$|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

or more generally,

$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$

when $A_i \cap A_j = \emptyset$ for all i, j.

The case where the sets have elements in common is different!

Combining the Sum and Product Rule

Example: Suppose variable names in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible names.

Use the product rule.

$$26 + 26 \cdot 10 = 286$$

Counting Passwords

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit.

How many possible passwords are there?

- Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.
  - By the sum rule $P = P_6 + P_7 + P_8$.
  - To find each of P_6, P_7, and P_8, we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters

    $P_6 = 36^6 - 26^6$

    $P_7 = 36^7 - 26^7$

    $P_8 = 36^8 - 26^8$

  Consequently, $P = P_6 + P_7 + P_8 = (36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8) = 2{,}684{,}483{,}063{,}360$

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the **principle of inclusion-exclusion**:

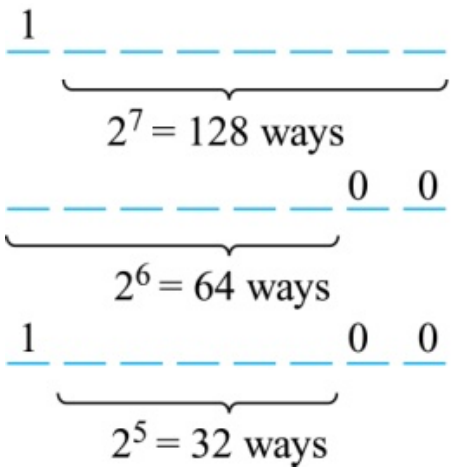
$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Use the principle of inclusion-exclusion.

- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$



Hence, the number is $128 + 64 - 32 = 160$.

Summary

- Sum Rule
- Subtraction Rule
- Applications to counting strings

The Pigeonhole Principle

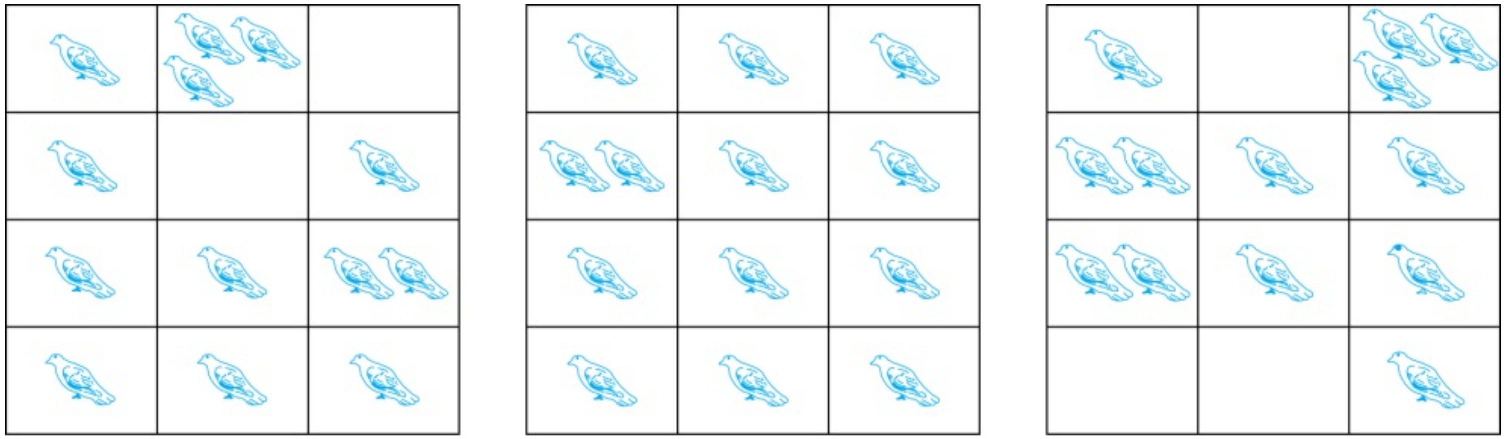
Section 6.2

Video 63: The Pigeonhole Principle

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle

If a flock of 13 pigeons lives in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(a) (b) (c)

The Pigeonhole Principle

Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition.

- Suppose none of the k boxes has more than one object.
- Then the total number of objects would be at most k.
- This contradicts the assumption that we have $k + 1$ objects. ◀

Using the Pigeonhole Principle

Corollary: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the codomain of f .
- Put in the box for y all of the elements x from the domain such that $f(x) = y$.
- Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one. ◀

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contraposition.

- Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects.
- Since $\lceil N/k \rceil < N/k + 1$, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

- This is a contradiction because there are a total of N objects . ◀

Example

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example

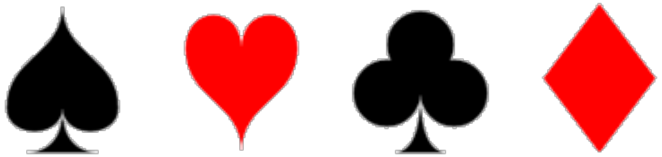
How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

We assume four boxes; one for each suit.

Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards.

At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$.

The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$.



The 4 suits of cards

Summary

- The Pigeonhole Principle
    - Counting functions
- The Generalized Pigeonhole Principle

Permutations and Combinations

Section 6.3

Video 64: Permutations and Combinations

- Permutations
- Combinations

Permutations

Definition: A **permutation** of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an **r-permutation**.

The number of r-permutations of a set with n elements is denoted by **P(n, r).**

Example: Let $S = \{1, 2, 3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

Counting the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element. Note that $P(n, 0) = 1$, since there is only one way to order zero elements. ◀

Corollary: If n and r are integers with $1 \leq r \leq n$, then $P(n,r) = \frac{n!}{(n-r)!}$

Example

How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$

Example

How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

We count the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Definition: An **r-combination** of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

The number of *r*-combinations of a set with n distinct elements is denoted by **C(n, r)** or $\binom{n}{r}$

Example: Let *S* be the set {*a*, *b*, *c*, *d*}.

{*a*, *c*, *d*} is a 3-combination from S.

It is the same as {*d*, *c*, *a*} since the order does not matter

Counting Combinations

Theorem 2: The number of *r*-combinations of a set with *n* elements, where *n* ≥ *r* ≥ 0, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule *P*(*n*, *r*) = *C*(*n*,*r*) · *P*(*r*,*r*). Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}\ .$$

◀

Example

How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

How many ways are there to select 47 cards from a deck of 52 cards?

The number of different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

Combinations

Corollary: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} \; .$$

Hence, $C(n, r) = C(n, n - r)$. ◀

Example: Full House

How many poker hands of five cards with a full house (three of a kind and a pair) can be dealt?

13 kind of cards to select the three, e.g. Aces

4 ways to select three Aces (we have to skip one color)

12 kind of cards left for the pair

6 ways to select two cards out of 4 – (1,2)(1,3)(1,4)(2,3)(2,4)(3,4)

So in total $13 \cdot 4 \cdot 12 \cdot 6 = 3744$ ways to select a full house

Summary

- Permutations n!
- Combinations $\binom{n}{r}$

Binomial Coefficients and Identities

Section 6.4

Video 65: The Binomial Theorem

- The Binomial Theorem
- Pascal's Identity and Triangle

Example

Expanding $(x + y)^3$

$(x + y)$ $(x + y)$ $(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.

Terms of the form x^3, x^2y, $x\,y^2$, y^3 arise.

What are the coefficients?

- To obtain x^3 , an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
- To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this, so the coefficient of x^2y is 3.
- To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{2}$ ways to do this and so the coefficient of xy^2 is 3.
- To obtain y^3 , a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.

Binomial Theorem

Binomial Theorem: Let *x* and *y* be variables, and *n* a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

- The coefficients of the expansion of the powers of (x+y) are thus related to the number of combinations

Proof of Binomial Theorem

Proof: We use combinatorial reasoning.

The terms in the expansion of $(x + y)^n$ are of the form

$$x^{n-j}y^j \text{ for } j = 0, 1, 2, \ldots, n.$$

To form the term $x^{n-j}y^j$, it is necessary to choose $n - j$ times an x from the n sums.

Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$. ◀

Using the Binomial Theorem

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Since $(2x - 3y)^{25} = ((2x) + (-3y))^{25}$.

by the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}.$$

Pascal's Identity

Pascal's Identity: If *n* and *k* are integers with $n \geq k \geq 0$, then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Proof (*algebraic*):

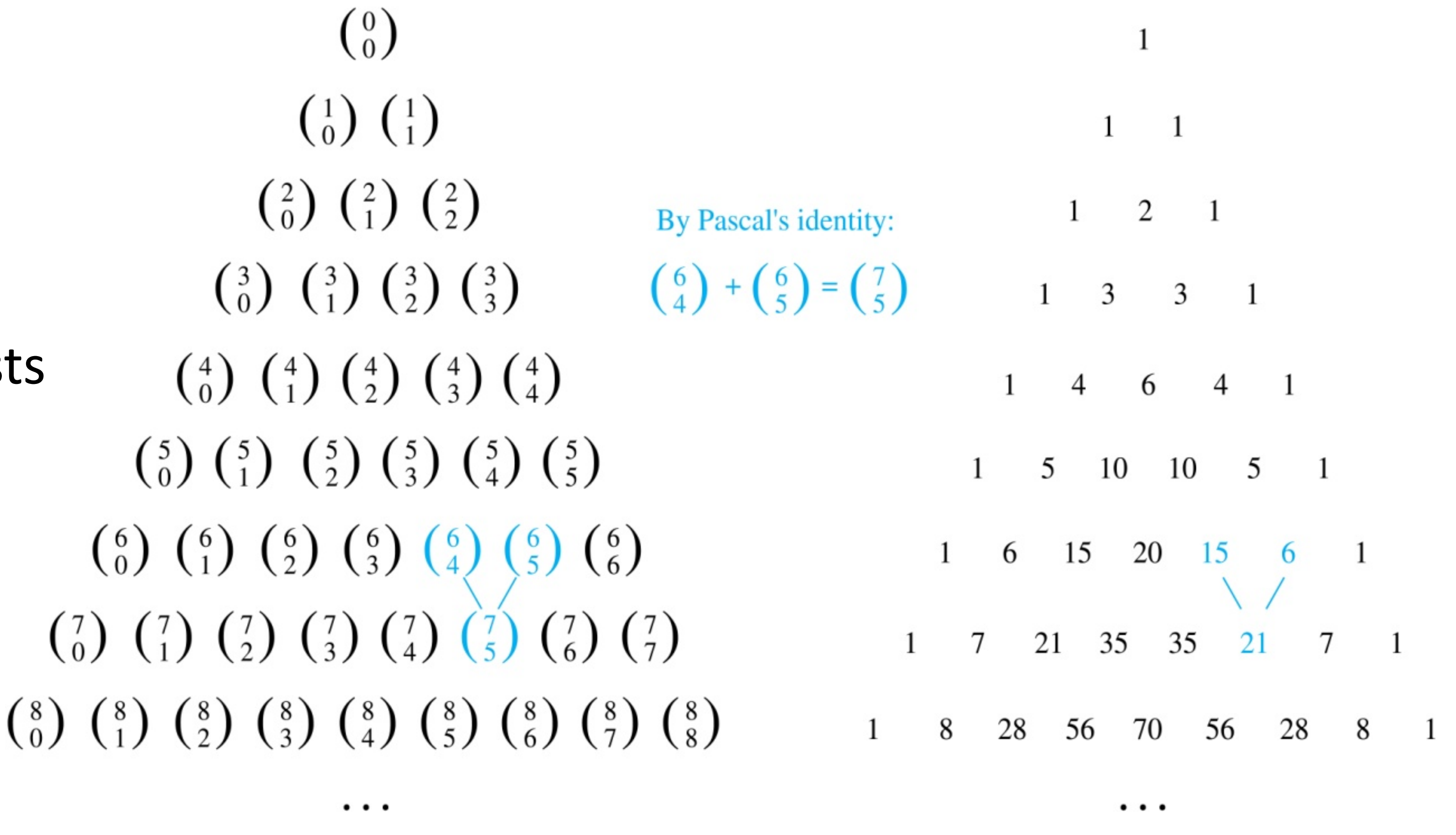
$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!} = \frac{n!\,k}{k!\,(n-k+1)!} + \frac{n!\,(n-k+1)}{k!\,(n-k+1)!}$$

$$= \frac{n!\,(k+n-k+1)}{k!\,(n-k+1)!} = \frac{(n+1)!}{k!\,(n-k+1)!} = \binom{n+1}{k}$$

◀

Pascal’s Triangle

The n^{th} row in the triangle consists of the binomial coefficients $\binom{k}{n}$, $k = 0,1,\ldots.,n$.



(a) (b)

By Pascal’s identity, adding two adjacent binomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Summary

- The Binomial Theorem
  - Binomial expansion
- Pascal's Identity and Triangle

Generalized Permutations and Combinations

Section 6.5

Video 66: Counting with Repetitions

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects

Permutations with Repetition

Definition: An **r-permutation** with repetition of a set of distinct objects is an ordered arrangement of r elements from the set, where elements can occur multiple times.

Theorem 3: The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.

Proof: There are *n* ways to select an element of the set for each of the *r* positions in the *r*-permutation when repetition is allowed.

Hence, by the product rule there are n^r *r*-permutations with repetition.

◀

Example

How many strings of length r can be formed from the uppercase letters of the English alphabet?

The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

r-combinations with Repetition

Definition: An **r-combination** with repetition of elements of a set is an unordered selection of *r* elements from the set, where elements can occur multiple times

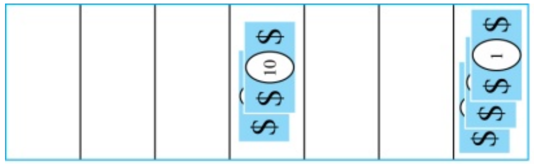
Example: How many ways are there to select four pieces of apples, oranges, and pears if the order does not matter and the fruit are indistinguishable?

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

r-Combinations with Repetition

Example: How many ways are there to select five bills of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

Place the selected bills in the appropriate position of a cash box as illustrated below:



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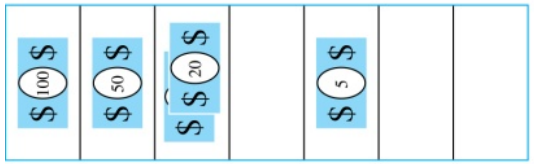
Seven bills need six bars to separate six bars and five stars in a row



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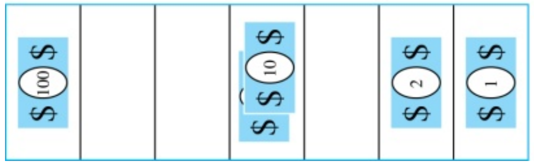
Putting the bills corresponds to selecting 5 stars from 11 possible positions



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Therefore $C(11,5) = \frac{11!}{5!6!} = 462$ possible ways to choose the bills

Combinations with Repetition

Theorem 4: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Proof:

Each r-combination of a set with n elements with repetition allowed can be represented by a list of $n - 1$ bars and r stars.

The bars mark the n cells containing a star for each time the i^{th} element of the set occurs in the combination.

The number of such lists is $C(n + r - 1, r)$: each list is a choice of the r positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars.

This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars. ◀

Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1 , x_2 and x_3 are nonnegative integers?

Each solution corresponds to a way to select 11 items from a set with three elements: x_1 elements of type one, x_2 of type two, and x_3 of type three.

By Theorem 4 it follows that there are

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$$

solutions.

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7, 3)$ different ways, leaving four positions free.
- Then the two Cs can be placed in $C(4, 2)$ different ways, leaving two positions free.
- Then the U can be placed in $C(2, 1)$ different ways, leaving one position free.
- Then the E can be placed in $C(1, 1)$ ways.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

Permutations with Indistinguishable Objects

Theorem 5: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}\ .$$

Proof: By the product rule the total number of permutations is:

$C(n, n_1)\ C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since

- The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- This is repeated, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.

Then

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}\ . \qquad \blacktriangleleft$$

Summary: Permutations and Combinations

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$