Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 8 - solutions

November 3, 2020

Exercise 1. *Find a formula for $f(n)$, and prove it by induction, if*

1. $f(0) = 0$ *and* $f(n) = f(n-1) - 1$.

   We get $f(0) = 0, f(1) = -1, f(2) = -2$ etc. In general, we get the formula $f(n) = -n$. We will now prove this formula by induction.

   Statement: $P(n)$: $f(n) = -n$ for $n \geq 0$.

   Basis step: For $n = 0$, we get $f(0) = -0 = 0$, i.e., the formula holds. Hence $P(0)$ is true.

   Induction step $P(k) \to P(k+1)$: As induction hypothesis (IH), we assume that $f(k) = -k$ for an arbitrary $k \geq 0$. Then we get

   $$f(k+1) = f(k) - 1 \stackrel{\text{IH}}{=} -k - 1 = -(k+1).$$

   Thus, the formula holds for $k+1$ and $P(k+1)$ is true.

   Conclusion: Since $P(0)$ is true and $P(k+1)$ is true given that $P(k)$ holds for an arbitrary $k \geq 0$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 0$.

2. $f(0) = 0$, $f(1) = 1$ *and* $f(n) = 2f(n-2)$.

   We get $f(0) = 0, f(1) = 1, f(2) = 0, f(3) = 2, f(4) = 0, f(5) = 4$ etc. In general we get the formula

   $$f(n) = \begin{cases} 0 & \text{if n is even} \\ 2^{\frac{n-1}{2}} & \text{if n is odd} \end{cases}.$$

   We will now prove this formula by induction.

   Statement: $P(n)$:

   $$\text{for all } n \geq 0,\ f(n) = \begin{cases} 0 & \text{if n is even} \\ 2^{\frac{n-1}{2}} & \text{if n is odd} \end{cases}.$$

   Basis step: For $n = 0$ we get $f(0) = 0$, for $n = 1$ we get $f(1) = 1 = 2^{\frac{0}{2}}$, i.e., the formula holds for $n = 0, 1$. Hence $P(0)$ and $P(1)$ are true.

   Induction step $(P(k-1) \wedge P(k)) \to P(k+1)$: This only makes sense for $k \geq 1$, whereas we have to prove $P(k+1)$ for arbitrary $k \geq 0$. Note that the case $k = 1$ (i.e., proving $P(1)$) has already been taken care of in the basis step. As induction hypothesis (IH), we assume that the formula holds for $f(k)$ and $f(k-1)$ for an arbitrary $k \geq 1$. Then we get

   $$f(k+1) = 2f(k-1) \stackrel{\text{IH}}{=} 2 \left\{ \begin{array}{ll} 0 & \text{if } k-1 \text{ is even} \\ 2^{\frac{k-2}{2}} & \text{if } k-1 \text{ is odd} \end{array} \right\} = \begin{cases} 0 & \text{if } k+1 \text{ is even} \\ 2^{\frac{k}{2}} & \text{if } k+1 \text{ is odd} \end{cases},$$

   since $k+1$ is odd if and only if $k-1$ is odd. Thus, the formula holds for $k+1$ and $P(k+1)$ is true.

   Conclusion: Since $P(0)$ and $P(1)$ are true and $P(k+1)$ is true given that $P(k)$ and $P(k-1)$ holds for an arbitrary $k \geq 1$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 0$.

Exercise 2. *Use strong or mathematical induction to show the following statements.*

1. *Any postage of at least 8 cents can be formed using just 3 cents and 5 cents stamps.*
   We use strong induction to show the statement.

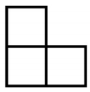
   Statement: $P(n)$: $\forall n \geq 8$ we can write n as a sum of 3s and 5s.

   Basis step: We can form $8 = 3 + 5$, $9 = 3 + 3 + 3$ and $10 = 5 + 5$ cents with just 3 and 5 cents stamps. Hence $P(8)$, $P(9)$ and $P(10)$ are true.

   Induction step $(P(k-2) \wedge P(k-1) \wedge P(k)) \rightarrow P(k+1)$: As strong induction hypothesis we assume that for an arbitrary $k \geq 10$, $P(l)$ is true for all $l \in \{k-2, k-1, k\}$. We want to prove $P(k+1)$, i.e., to form $k+1$ cents for $k \geq 10$. We know by the induction hypothesis that we can form the amount of $k-2$ cents with 3 and 5 cents stamps (note that $k - 2 \geq 8$ so the argument is valid in the boundary case). Thus, we can add a 3 cents stamp to form $k+1$. Hence $P(k+1)$ holds.

   Conclusion: Since $P(8)$, $P(9)$ and $P(10)$ are true and $P(k+1)$ is true given that $P(l)$ holds for all $l \in \{k-2, k-1, k\}$ for an arbitrary $k \geq 10$, by the strong induction principle we conclude that $P(n)$ is true for all $n \geq 8$.

2. *Consider a $2^n \times 2^n$ grid of 2^{2n} squares arranged in 2^n rows and 2^n columns. If we remove one square from this grid of 2^{2n} squares, we obtain a shape. Let C_n be the set of shapes obtained by removing one square from the grid of 2^{2n} squares. We say that a shape is* L-coverable *if it can be completely covered, without overlapping, with L-shaped tiles occupying exactly 3 squares, like this:*

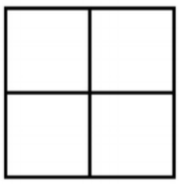
   Prove that, for any integer $n \geq 1$, any shape in C_n is L-coverable*.*

   We use mathematical induction to show the statement.

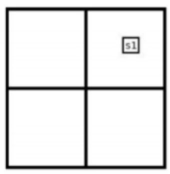
   Statement: $P(n)$: for any integer $n \geq 1$, any shape in C_n is *L-coverable.*

   Basis step: For $n = 1$ we have a 2×2 square. If we remove any one square, we have an L-shape consisting of thre squares, i.e., the obtained shape is L-coverable. $P(1)$ is true.

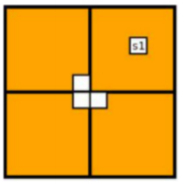
   Induction step $P(k) \rightarrow P(k+1)$: As induction hypothesis, we assume that the statement $P(k)$ holds for an arbitrary $k \geq 1$. Let $k \geq 1$, the induction hypothesis says that any shape in C_k is L-coverable. Consider the $2^{k+1} \times 2^{k+1}$ grid containing C_{k+1} and divide it into 4 grids of size $2^k \times 2^k$:

   For any shape in C_{k+1}, the removed square, let us call it s_1, is in one of the smaller grids.

By the induction hypothesis, this smaller grid without the removed square is in C_k and is hence L-coverable. For the other three grids we remove the square closest to the center of the large grid $2^{k+1} \times 2^{k+1}$:



Again, by induction hypothesis, we know that those other three grids without the removed squares are L-coverable. Since the respective three squares form an L-shape, we get that the big grid without s_1, i.e., the shape in C_{k+1}, is L-coverable. Hence $P(k+1)$ holds.

Conclusion: Since $P(1)$ is true and $P(k+1)$ is true given that $P(k)$ holds for an arbitrary $k \geq 1$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 1$.

Exercise 3. *Denote by f_n the nth Fibonacci number, i.e., $f_0 = 0, f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$.*

We prove the three statements by mathematical induction.

1. *Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, when n is a positive integer.*

Statement: $P(n)$: $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, when n is a positive integer.

Basis step: For $n = 1$ we get $f_1^2 = 1^2 = 1$ and $f_1 f_2 = 1$, hence the statement is true for $n = 1$: $P(1)$ is true.

Induction step $P(k) \to P(k+1)$: As induction hypothesis (IH), we assume that the statement $P(k)$ is true for an arbitrary integer $k \geq 1$. Then, for $k+1$ we get

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 &= (f_1^2 + f_2^2 + \cdots + f_k^2) + f_{k+1}^2 \\ &\overset{\text{IH}}{=} f_k f_{k+1} + f_{k+1}^2 \\ &= (f_k + f_{k+1}) f_{k+1} \\ &= f_{k+2} f_{k+1} \end{aligned}$$

Hence $P(k+1)$ is true.

Conclusion: Since $P(1)$ is true and $P(k+1)$ is true given that $P(k)$ holds for an arbitrary $k \geq 1$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 1$.

2. *Show that $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$, when n is a positive integer.*

Statement: $P(n)$: $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$, when n is a positive integer.

Basis step: For $n = 1$ we get $f_2^2 = 1^2 = 1$ and $f_0 f_1 + f_1 f_2 = 0 + 1 = 1$, hence the statement is true for $n = 1$: $P(1)$ holds.

Induction step $P(k) \to P(k+1)$: We assume that the statement $P(k)$ is true for an arbitrary $k \geq 1$ (IH). Then, for $k+1$ we get

$$\begin{aligned} f_0 f_1 + f_1 f_2 + \cdots + f_{2(k+1)-1} f_{2(k+1)} &= (f_0 f_1 + f_1 f_2 + \cdots + f_{2k-1} f_{2k}) + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\ &\overset{\text{IH}}{=} f_{2k}^2 + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} \\ &= f_{2k}(f_{2k} + f_{2k+1}) + f_{2k+1} f_{2k+2} \\ &= f_{2k} f_{2k+2} + f_{2k+1} f_{2k+2} \\ &= (f_{2k} + f_{2k+1}) f_{2k+2} \\ &= f_{2k+2}^2. \end{aligned}$$

Hence $P(k+1)$ is true.

Conclusion: Since $P(1)$ is true and $P(k+1)$ is true given that $P(k)$ holds for an arbitrary $k \geq 1$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 1$.

3. *Show that if* $a = (1+\sqrt{5})/2$ *and* $b = (1-\sqrt{5})/2$*, then*

$$f_n = (a^n - b^n)/(a-b).$$

[Hint: a and b are two solutions of the equation $x^2 = x + 1$*]*

Statement: $P(n)$: if $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$, then $f_n = (a^n - b^n)/(a-b)$, when $n \geq 0$.

Basis step: It is straightforward for $n = 0$ and $n = 1$. Statements $P(0)$ and $P(1)$ are true.

Induction step $P(k-1) \wedge P(k) \to P(k+1)$: As induction hypothesis, we assume that $P(k-1)$ and $P(k)$ are true for an arbitrary integer $k \geq 1$. Namely,

$$f_{k-1} = \frac{a^{k-1} - b^{k-1}}{a-b} \text{ and } f_k = \frac{a^k - b^k}{a-b}.$$

Then,

$$f_{k+1} = f_{k-1} + f_k = \frac{(a^{k-1} - b^{k-1}) + (a^k - b^k)}{a-b} = \frac{a^{k-1}(a+1) - b^{k-1}(b+1)}{a-b}.$$

Since $a + 1 = a^2$ and $b + 1 = b^2$, we obtain $f_{k+1} = (a^{k+1} - b^{k+1})(a-b)$. Hence, $P(k+1)$ is true.

Conclusion: Since $P(0)$ and $P(1)$ are true and $P(k+1)$ is true given that $P(k)$ and $P(k-1)$ holds for an arbitrary $k \geq 1$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 0$.

Exercise 4. *Prove that* $n! > 2^n$ *for* $n \geq 4$.

Statement: $P(n)$: $n! > 2^n$ for $n \geq 4$.

Basis step: We start with the first possible value of n which is $n = 4$: $4! = 24 > 16 = 2^4$, the statement $P(4)$ is true.

Induction step $P(k) \to P(k+1)$: We assume, as the induction hypothesis, that the statement is true for an arbitrary $k \geq 4$. Then

$$\begin{aligned}
(k+1)! &= k!(k+1) \\
&> 2^k(k+1) \text{ (by the induction hypothesis)} \\
&> 2^k \times 2 \text{ (because } k \geq 4 \text{ hence } k+1 > 2) \\
&= 2^{k+1}.
\end{aligned}$$

Thus $P(k+1)$ is true.

Conclusion: Since $P(4)$ is true and $P(k+1)$ is true given that $P(k)$ holds for an arbitrary $k \geq 4$, by the principle of induction, $P(n)$ is true for all $n \geq 4$.

Exercise 5. *Let* $P(n)$ *for* $n \in \mathbf{Z}_{\geq 0}$ *be the propositional function "all cardinality-n sets of integers consist of only even integers," which is proved using strong induction:*

Basis step $P(0)$ *is true, since if* S *is an empty set of integers the statement "*$\forall s\ s \in S \to s \text{ is even}$*" is true.*

Inductive step *Let $k \geq 0$ and assume that $P(i)$ is true for $0 \leq i \leq k$. To prove that $P(k+1)$ is true we use the following steps:*

1. *Let T be an arbitrary set of integers with $|T| = k+1$.*
2. *Write T as the disjoint union of sets T_1 and T_2 such that $|T_1| = k$ and $|T_2| = 1$.*
3. *Because $|T_1| < |T|$ and $|T_2| < |T|$ the induction hypothesis applies to both T_1 and T_2, implying that all elements of both T_1 and T_2 are even.*
4. *Because $T = T_1 \cup T_2$ it follows that all elements of T are even as well.*
5. *Because T was arbitrarily chosen as a set of integers of cardinality $k+1$, it follows that $P(k+1)$ is true.*

Because not all integers are even, the proof cannot be correct (unless the well-ordering principle is false). Find the mistake.

The problem of the proof is that $|T_2| < |T|$ is not necessarily the case because for $k = 0$, we have that $|T| = 1$ and that $|T_2| = 1$ as well, so the induction hypothesis does not apply to T_2. In particular, note that the basis step is correct. Also, make sure not to confuse arguments concerning the elements of the sets and the cardinalities of the sets—they have nothing to do with each other.

Exercise 6. *Let $P(n)$ for $n \in \mathbf{Z}_{>0}$ be the propositional function "all cardinality-n sets of integers consist of only odd integers," which is proved using strong induction:*

Basis Step *$P(1)$ is true because 1 is odd.*

Inductive step *Let $k > 0$ and assume that $P(i)$ is true for $0 < i \leq k$. To prove that $P(k+1)$ is true we use the following steps:*

1. *Let S be an arbitrary set of integers with $|S| = k+1$.*
2. *Write S as the disjoint union of sets S_1 and S_2 such that $|S_1| = k$ and $|S_2| = 1$.*
3. *Because $|S_1| < |S|$ and $|S_2| < |S|$ the induction hypothesis applies to both S_1 and S_2,*
4. *implying that all elements of both S_1 and S_2 are odd.*
5. *Because $S = S_1 \cup S_2$ it follows that all elements of S are odd as well.*
6. *Because S is an arbitrarily chosen set of integers with $|S| = k+1$, it follows that $P(k+1)$ is true.*

✓ *Only the basis step in the proof is incorrect.*

○ *The basis step and step (c) of the inductive step of the proof are incorrect.*

○ *Only step (c) of the inductive step of the proof is incorrect.*

○ *Only step (d) of the inductive step of the proof is incorrect.*

The statement $P(1)$ says that "all cardinality-1 sets of integers consist of only odd integers". Obviously, that is nonsense. If it were true, then indeed it is a simple matter to prove that $P(n)$ is true for any $n > 1$, because any finite set can be written as a disjoint union of singleton sets (sets containing a single element): $S = \cup_{s \in S}\{s\}$ (and indeed, the Inductive step in the above "proof" is entirely correct). Because $P(1)$ is also the basis of the induction, this is a strong indication that something must be wrong with the proof of the basis of the induction. Indeed, though it is true that 1 is odd, it does not follow from the fact that 1 is odd that $P(1)$ is true: $P(1)$ is a statement about the element of *any* set containing a single integer (say $T = \{t\}$), namely saying that that element (thus $t \in T$) is odd: $P(1)$ is not a statement about the cardinality of that set (T), which equals one ($|T| = 1$) and which is indeed odd, but that has nothing to do with the parity of the element (t).

That the statement $P(1)$ is incorrect follows by considering the set $S = \{0\}$: this is a set containing a single element, that element is an integer, and that integer (zero) is even. Thus $P(1)$ is incorrect.

Note that the reason why the induction proof that all integers are even (see above) fails (namely that the inductive step did not hold for $k = 0$, which was in the domain in that example) does not apply here: here the inductive step is for $k > 0$.

Exercise 7. *Suppose that* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are square matrices with the property* $\boldsymbol{AB} = \boldsymbol{BA}$. *Prove using induction that* $\boldsymbol{AB}^n = \boldsymbol{B}^n\boldsymbol{A}$ *for every positive integer* n.

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices with the property $\mathbf{AB} = \mathbf{BA}$.

Statement: $P(n): \ \mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.

Basis step: $P(1)$ is true by definition since $\mathbf{AB} = \mathbf{BA}$.

Induction step $P(k) \to P(k+1)$**:** We assume that the statement is true for an arbitrary $k \geq 1$. Then

$$\begin{aligned}
\mathbf{AB}^k &= \mathbf{B}^k\mathbf{A} \\
(\mathbf{AB}^k)\mathbf{B} &= (\mathbf{B}^k\mathbf{A})\mathbf{B} \\
\mathbf{A}(\mathbf{B}^k\mathbf{B}) &= \mathbf{B}^k(\mathbf{AB}) \\
\mathbf{AB}^{k+1} &= \mathbf{B}^k(\mathbf{BA}) \\
\mathbf{AB}^{k+1} &= (\mathbf{B}^k\mathbf{B})\mathbf{A} \\
\mathbf{AB}^{k+1} &= \mathbf{B}^{k+1}\mathbf{A}
\end{aligned}$$

Thus $P(k+1)$ is true.

Conclusion: Since $P(1)$ is true and $P(k+1)$ is true given that $P(k)$ holds for an arbitrary $k \geq 1$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 1$.

Note: Recall that given three matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, the products $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ are defined if and only if the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$ and the number of columns of $\mathbf{B}$ equals the number of rows of $\mathbf{C}$. In this case, one has the associative property: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.