Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

# Week 5

October 20, 2020

**Exercise 1.** Let $X = \mathscr{P}(\mathbf{Q})$ be the set of subsets of $\mathbf{Q}$. Determine whether or not the following relations $\sim_i$ on $X$ are a) reflexive, b) symmetric, c) transitive. Let $A$ and $B$ be arbitrary elements of $X$.

1. $A \sim_1 B$ if and only if $A \subseteq B$.
2. $A \sim_2 B$ if and only if $A \cap B = \emptyset$.
3. $A \sim_3 B$ if and only if $A \oplus B$ is finite[1].
4. $A \sim_4 B$ if and only if there exists a $c \in \mathbf{R}$ such that for any $x \in A \oplus B$, we have $|x| < c$.
5. $A \sim_5 B$ if and only if $A$ and $B$ contain the same number of integers (potentially infinite).

**Exercise 2.** Let $\sim$ be the relation on $\mathbf{R} \times \mathbf{R}$ defined by $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

1. Prove that it is an equivalence relation.
2. Prove that the set of equivalence classes of $\sim$ is uncountable.

**Exercise 3.** Let $n$ be an integer. Consider the relation $R$ on $\mathbf{Z}$ defined by $xRy$ if and only if $x - y = n$. Prove that the relation $\sim$ on $\mathbf{Z}$, defined by $x \sim y$ if and only if $n$ divides $x - y$, is the smallest equivalence relation containing the relation $R$.

**Exercise 4.** A relation $R$ on a finite set $X$ can be represented by a directed graph: the elements of $X$ are vertices, and there is an edge from a vertex $a \in X$ to $b \in X$ if and only if $aRb$. A path from $a$ to $b$ in the graph is a sequence $a = x_0, x_1, x_2, \dots, x_{k-1}, x_k = b$ such that $x_i R x_{i+1}$ for any $0 \le i < k$. Such a path is of length $k$. The distance $d(a, b)$ from $a$ to $b$ is the length of the shortest path from $a$ to $b$ (the distance from $a$ to $a$ is 0).

1. Prove that if $R$ is symmetric, then $d(a, b) = d(b, a)$ for any $a, b \in X$.
2. Prove that if $R$ is transitive, then $d(a, b) \in \{0, 1\}$ for any $a, b \in X$.

**Exercise 5.** Draw the Hasse diagram for divisibility on the set:

1. {1, 2, 3, 4, 5, 6, 7, 8}
2. {1, 2, 3, 5, 7, 11, 13}
3. {1, 2, 4, 8, 16, 32, 64}

**Exercise 6.** Suppose that $(S, \preceq_1)$ and $(T, \preceq_2)$ are posets. Show that $(S \times T, \preceq)$ is a poset where $(s, t) \preceq (u, v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$.

**Exercise 7.** Determine whether these posets are lattices.

1. $(1, 3, 6, 9, 12, |)$

[1] the set $A \oplus B = (A \cup B) \setminus (A \cap B)$ is the *symmetric difference* of $A$ and $B$.

2. $(1, 5, 25, 125, |)$
3. $(Z, \geq)$
4. $(P(S), \supseteq)$, where $P(S)$ is the power set of a set $S$

**Exercise 8.** Find the lexicographic ordering of these n-tuples:

1. (1, 1, 2), (1, 2, 1)
2. (0, 1, 2, 3), (0, 1, 3, 2)
3. (1, 0, 1, 0, 1), (0, 1, 1, 1, 0)

**Exercise 9.** Hilbert's Grand Hotel has a countably infinite number of rooms, and each room is occupied by a single guest.

1. A new guest arrives. Since every room is occupied, if the hotel was finite, the new guest could not be accommodated without evicting a current guest. How can a new guest be accommodated in Hilbert's Grand Hotel? The hotel can ask current guests to change room.
2. How can a finite number of new guests, say n, be accommodated[2]?
3. A bus carrying a countably infinite number of guests arrives. Can they all be accommodated?
4. A countably infinite number of such buses arrives. Can the guests all be accommodated?
5. A bus carrying an uncountable number of guests arrives. Can the guests all be accommodated?

**Exercise 10.** Which of the following statements is incorrect?

- ◯ The Cartesian product of finitely many countable sets is countable.
- ◯ Any subset of infinite cardinality of an uncountable set is uncountable.
- ◯ $N \cup \{x \in \mathbf{R}, 0 < x < 1\}$ is uncountable.
- ◯ The intersection of two uncountable sets can be countably infinite.

**Exercise 11.**

(francais) Soit B l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation binaire, et soit D l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation décimale. Laquelle des propositions suivantes est correcte?

(English) Let B be the set of real numbers with a finite number of ones in their binary representation, and let D be the set of real numbers with a finite number of ones in their decimal representation. Which of the following statements is correct?

- ◯ $\left\{ \begin{array}{l} \text{B est dénombrable et D ne l'est pas.} \\ \text{B is countable and D is uncountable.} \end{array} \right.$
- ◯ $\left\{ \begin{array}{l} \text{B et D sont dénombrables tous les deux.} \\ \text{B and D are both countable.} \end{array} \right.$

[2]Accommodating a guest means that the guest gets a room after waiting a period of time that has a finite length.

○ $\begin{cases} \text{B et D ne sont pas dénombrables.} \\ \text{B and D are both uncountable.} \end{cases}$

○ $\begin{cases} \text{B nést pas dénombrable mais D est dénombrable.} \\ \text{B is uncountable but D is countable.} \end{cases}$

**Exercise 12.** Let F be the set of real numbers with decimal representation consisting of all fours (and possiby a single decimal point). Examples of numbers contained in F are 4, 44, 4444444, 44.4, 4.444444, 444.44444,... etc.

Let G be the set of real numbers with decimal representation consisting of all fours or sixes (and possiby a single decimal point). Examples of numbers contained in G are 4, 6, 44, 66, 46, 64, 4464464, 46.46, 6.644464, 646.64646464, 446.6666666, . . . etc.

- ○ The set F is countable and the set G is not countable.
- ○ The sets F and G are both countable.
- ○ The set G is countable and the set F is not countable.
- ○ The sets F and G are both not countable.

**Exercise 13.** Let $S = \{0, 1\}$. Let $A = \bigcup_{i=1}^{\infty} S^i$, and let $B = S^*$ be the set of infinite sequences of bits. Which of the following statements is correct?

- ○ A is countable and B is not countable.
- ○ A and B are both countable.
- ○ A and B are both uncountable.
- ○ A is uncountable but B is countable.

**Exercise 14.** Let $r \geq -1$ be an integer and let $S(r) = \sum_{i=-r}^{r+2} (i-2)^{i-2}$.

- ○ $\forall r \; S(r) > 0 \leftrightarrow r \geq 0$.
- ○ $\forall r \; S(r) > 0 \leftrightarrow r \geq -1$.
- ○ $\forall r \; S(r) > 0 \leftrightarrow r \geq 1$.
- ○ $\forall r \; S(r) > 0$ only if $r \geq 1$.

**Exercise 15.** Let $S(n) = \displaystyle\sum_{i=1}^{n} \frac{1}{i(i+2)}$ for $n > 64$.

- ○ $S(n) = \dfrac{n+1}{2n+4}$.
- ○ $S(n) = \dfrac{3n+4}{4n+8}$.
- ○ $S(n) = \dfrac{3n^2+5n}{4n^2+12n+8}$.
- ○ $S(n) = \dfrac{3n^2-4n-3}{4n^2-4}$.

**Exercise 16.** Which of the following statements is correct?

- ◯ $\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = 8.$
- ◯ $\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = 4.$
- ◯ $\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = 16.$
- ◯ $\prod_{i=1}^{\infty} 16^{1/(i(i+2))} = \infty.$

**Exercise 17.** Suppose that the number of bacteria in a colony triples every hour.

1. Set up a recurrence relation for the number of bacteria after $n$ hours have elapsed
2. If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?