Chen iterated integrals and beyond

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(Dated: Monday 9th September, 2024)

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I. Introduction

The goal of these lecture notes is to give a brief exposition to the *Chen iterated integral*, a powerful tool introduced by Chen in Ref. [**?**], utilized by Guggenheim [**?**] and Arias Abad-Schaetz [**?**] to construction an A_∞-generalization of De Rham's theorem. These notes will begin with a discussion of the Chen map, followed by a summary of its use in building an A_∞-quasi-isomorphism between differential graded algebras of k-forms and k-singular cochains.

II. The Chen map

Let M be a finite-dimensional, compact, orientable smooth manifold. The Chen iterated integral is a map

$$C : B(s\Omega^\bullet(M)) \to \Omega^\bullet(PM) \tag{1}$$

where s is the suspension of the graded algebra $\Omega^\bullet(M)$: the De Rham complex of piecewise smooth differential forms on M.

Remark II.1. Right away, one of the main characteristics of the Chen maps that makes it useful is that it is a natural mapping from a complex of data defined on a finite-dimensional manifold M, to data on an *infinite dimensional manifold* PM. It follows that the Chen map may help us deduce certain properties of PM which are difficult to access directly from definitions.

Definition II.1 (Suspension)**.** If V is a $\mathbb{Z}$-graded vector space, the suspension sV is the graded vector space with grading shifted by 1, so that $(sV)^k = V^{k+1}$. Thus, $v \in V^k$ is an element of $(sV)^{k-1}$ in sV, so the suspension decreases the degree of individual elements. We denote $v \in sV$ by sv, to emphasize the change in grading.

Note that B denotes the bar complex (for an algebra A, $BA = \bigoplus_{k\geq 1} A^{\otimes k}$) [**?**]. PM is the *piecewise-smooth path space of* M.

Definition II.2 (Piecewise-smooth path space)**.** Given smooth manifold M, the piecewise-smooth path space PM is the set of all piecewise smooth $\gamma : [0,1] \to M$. Let $PM^\infty = C^\infty([0,1], M) \subset PM$ be the subset consisting of smooth paths. We take the C^1-topology to be the initial topology of the map $\Gamma : C^\infty([0,1], M) \to C^\infty([0,1], M) \times C^\infty(T[0,1], TM)$ taking $\gamma \mapsto (\gamma, \gamma_*)$, where the range is endowed with the compact-open topology on each factor. We then define a topology on PM by taking the final topology of the inclusion $\iota : C^\infty([0,1], M) \to PM$.

Given finite-dimensional smooth manifold X, we say that a map $f : X \to PM$ is (piecewise) smooth if the map $\widetilde{f} : X \times [0,1] \to M$ given by $\widetilde{f}(x,t) = f(x)(t)$ is (piecewise) smooth.

With a definition for PM, it is now possible to define $\Omega^\bullet(PM)$: the De Rham complex of differential forms on the path space.

Definition II.3 (Differential forms on path space)**.** A differential k-form $\eta \in \Omega^k(PM)$ on path space PM is an association of each pair (X, f) of finite-dimensional smooth manifold and smooth map $f : X \to PM$ to a differential form $\eta(X, f) \in \Omega^k(X)$, such that if g is a smooth map between finite-dimensional manifolds, $g : Y \to X$, then

$$\eta(Y, f \circ g) = g^*\eta(X, f). \tag{2}$$

In light of this naturalness, we can simply denote $\eta(X, f)$ by $f^*\eta$. Technically, one should use the language of natural transformations to make this "association" completely rigorous, but this won't be needed for our current purposes.

It is important to note that there is a natural way to define the standard operations one can usually perform on differential forms, namely the exterior derivative and the wedge product. In particular, we simply use the pullback under any function $f : X \to PM$: for $\omega, \eta \in \Omega^\bullet(PM)$, we define

$$f^*(\eta \wedge \omega) \coloneqq f^*\eta \wedge f^*\omega \quad \text{ and } \quad f^*(d\eta) \coloneqq df^*\eta. \tag{3}$$

From here, let us notice that both $B(s\Omega^\bullet(M))$ and $\Omega(PM)$ are cochain complexes. We equip $\Omega(PM)$ with the coboundary defined above. For the case of the bar complex, given arbitrary DGA $(A, d, \wedge)$, we may define a couboundary $D : B(sA) \to B(sA)$ given by

$$\begin{aligned} D(sa_1 \otimes \cdots \otimes sa_n) &= \sum_{i=1}^{n} (-1)^{[a_1]+\cdots+[a_{i-1}]} sa_1 \otimes \cdots \otimes sa_{i-1} \otimes s(da_i) \otimes sa_{i+1} \otimes \cdots \otimes sa_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^{[a_1]+\cdots+[a_i]} sa_1 \otimes \cdots \otimes sa_{i-1} \otimes s(a_i \wedge a_{i+1}) \otimes sa_{i+2} \otimes \cdots \otimes sa_n. \qquad (4) \\ &\coloneqq d_\otimes(sa_1 \otimes \cdots \otimes sa_n) + d_\wedge(sa_1 \otimes \cdots \otimes sa_n) \qquad (5) \end{aligned}$$

where $[a_k]$ is the degree of $a_k \in A$ (a graded algebra). One can verify that $D \circ D = 0$. Note that D is a degree-1 map. Clearly, $d_\otimes$ is. As for $d_\wedge$, removing sa_i and sa_{i+1} from the tensor product decreases degree by $[a_i] + [a_{i+1}] - 2$, and adding $s(a_i \wedge a_{i+1})$ increases it by $[a_i \wedge a_{i+1}] - 1 = [a_i] + [a_{i+1}] - 1$. Thus, there is a net degree-increase of 1.

Going forward, let $\overline{D}$ be the bar differential associated to the DGA $(s\Omega^\bullet(M), -d, \wedge)$, where we flip the sign of d, which is itself given by $d(s\omega) = sd\omega$. Before defining the Chen map, we require first a bit more setup.

Definition II.4. Given a map $f : X \to PM$, where X is a finite-dimensional smooth manifold, recall that $\widetilde{f} : [0, 1] \times X \to M$ is defined as $\widetilde{f}(t, x) = f(x)(t)$. We then define $\widetilde{f}_{(m)} : \Delta^m \times X \to M^{\times m}$ as

$$\widetilde{f}_{(m)}(t_1, \ldots, t_m, x) = (\widetilde{f}(t_1, x), \ldots, \widetilde{f}(t_m, x)) \tag{6}$$

with Δ^m the standard m-simplex, $\Delta^m = \{(t_1, \ldots, t_m) \mid 0 \le t_1 \le \cdots \le t_m \le 1\}$.

Definition II.5 (Fibre integral)**.** Let $\pi : E \to M$ be a fibre bundle over M such that each fibre $\pi^{-1}(p)$ is compact and oriented. Let $\omega \in \Omega^k(E)$. We define $\pi_*\alpha \in \Omega^{k-n}(M)$, where n is the dimension of the fibres, as

$$\pi_*\alpha_p(v_{1,p}, \ldots, v_{k-n,p}) = \int_{\pi^{-1}(p)} \widetilde{\alpha} \tag{7}$$

where $\widetilde{\alpha} \in \Omega^n(\pi^{-1}(p))$ is the top-form defined as

$$\widetilde{\alpha}_q(w_{1,q}, \ldots, w_{n,q}) = \alpha_q(w_{1,q}, \ldots, w_{n,q}, \widetilde{v}_{1,q}, \ldots, \widetilde{v}_{k-n,q}) \tag{8}$$

where $\widetilde{v}_{j,q}$ is a lift of $v_j \in T_pM$ to an element of $\widetilde{v}_{j,q} \in T_q\pi^{-1}(p)$ (i.e. we must have $d\pi_q(\widetilde{v}_{j,q}) = v_j$). Note that this definition ends up being independent of the chosen lift.

III. Applications