L^p-spaces, operator theory, and duality

Jack Ceroni[1, *]

[1] *Department of Mathematics, University of Toronto*

(Dated: Sunday 1st December, 2024)

Contents

I. Introduction

The goal of this essay is to explore some of the fundamental results of the theory of L^p-spaces and their relationship to operator theory, discussed in Chapter 6.4 of Pedersen's *Analysis Now* [1], as well as Folland's *Real Analysis* [2]. The goal of the essay is to culminate in proving the duality theorem of L^p-spaces.

II. L^p-spaces, operator theory, and duality

Let us fix a measure space $(X, \mathcal{M}, \mu)$, and recall the definition of the p-norm for some measurable function f:

$$||f||_p = \left[\int |f|^p \ d\mu\right]^{1/p} \tag{1}$$

for $0 < p < \infty$. We then take L^p to be the set of all measurable functions such that $||f||_p < \infty$, identifying functions which are equal almost everywhere. In addition, we use the special notation ℓ^p to denote the L^p space associated with the counting measure on $(X, \mathcal{P}(X))$. It is very easy to see that L^p is a vector space (in fact, we will show that it is a Banach space for $p \geq 1$). To begin, note that if $f, g \in L^p$, then

$$|f+g|^p \leq (|f| + |g|)^p = \sum_{k=0}^{p} \binom{p}{k} |f|^k |g|^{p-k} \leq 2^p \max(|f|, |g|)^p \leq 2^p (|f|^p + |g|^p) \tag{2}$$

which implies that

$$\int |f+g|^p \ d\mu \leq 2^p \left(\int |f|^p \ d\mu + \int |g|^p \ d\mu \right) < \infty \tag{3}$$

so $f + g \in L^p$. It is trivial to check that if $f \in L^p$, then $\lambda f \in L^p$. To verify that $|| \cdot ||_p$ is a norm, so that we in fact have a *normed vector space*, we first note that it is clear $||f||_p = 0$ if and only if $f = 0$ almost everywhere. Thus, all that remains to check is the triangle inequality, *which is only valid for* $p \geq 1$. This will be shown via *Minkowski's inequality*, which is a consequence of *Holder's inequality*. We begin with a lemma:

* jack.ceroni@mail.utoronto.ca

Lemma II.1. Given $a, b \geq 0$ and $\lambda \in (0,1)$, then $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$, with equality if and only if $a = b$.

Proof. If either $a = 0$ or $b = 0$, the inequality is trivial. Otherwise, assume $a, b > 0$. Note that the function $x \mapsto \log(x)$ is concave, as $\frac{d^2}{dx^2}\log(x) = -\frac{1}{x^2}$. It follows that $\lambda \log(a) + (1-\lambda)\log(b) \leq \log(\lambda a + (1-\lambda)b)$. If we take the exponential of both sides, we obtain the desired inequality. $\square$

This generalization of the AM-GM inequality immediately allows us to prove Holder's inequality:

Theorem II.1 (Holder)**.** Given $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$ (i.e. p and q are *conjugate*). If f and g are measurable functions on X, then

$$||fg||_1 \leq ||f||_p ||g||_q \tag{4}$$

This immediately implies that if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and the inequality becomes an equality if and only if $\alpha|f|^p = \beta|g|^q$, where neither α nor β are 0.

Proof. In the case that $||f||_p = 0$ or $||g||_q = 0$ or $||f||_p = \infty$ or $||g||_q = \infty$, the inequality is trivial. Otherwise, we can assume WLOG that $||f||_p = ||g||_q = 1$, as we can always normalize. Note that

$$||fg||_1 = \int_X |fg| \; d\mu = \int_X (|f|^p)^{1/p}(|g|^q)^{1/q} \leq \frac{1}{p}\int_X |f|^p \; d\mu + \frac{1}{q}\int_X |g|^q \; d\mu = \frac{1}{p} + \frac{1}{q} = 1 \tag{5}$$

where we are simply applying the previous lemma to $a = |f|^p$, $b = |g|^q$, and $\lambda = 1/p$ (so $1 - \lambda = 1/q$). Note that the inequality used above is an equality if and only if $|f|^p = |g|^q$, in the case that they are normalized, so in the general case, we can have the factors α and β. $\square$

This result immediately allows us to prove the Minkwoski inequality, which shows that $||\cdot||_p$ is a valid norm on L^p (i.e. it satisfies the triangle inequality).

Theorem II.2 (Minkowski)**.** For any $p \in [1, \infty)$, we have $||f+g||_p \leq ||f||_p + ||g||_p$.

Proof. Once again, we can immediately disregard the trivial cases of $p = 1$ (as this follows from the standard triangle inequality) and when $f + g = 0$ almost everywhere. Otherwise, we note that

$$|f+g|^p \leq (|f| + |g|)|f+g|^{p-1} \tag{6}$$

where q is conjugate to p, so $q = p/(p-1)$. It then follows from Holder's inequality that

$$\int_X (|f| + |g|)|f+g|^{p-1} \; d\mu \leq ||f||_p |||f+g|^{p-1}||_q + ||g||_p |||f+g|^{p-1}||_q \tag{7}$$

Note that

$$|||f+g|^{p-1}||_q = \left(\int_X |f+g|^{(p-1)q} \; d\mu\right)^{1/q} = \left(\int_X |f+g|^p \; d\mu\right)^{1/q} = ||f+g||_p^{p/q}. \tag{8}$$

It follows immediately that

$$||f+g||_p^p \leq (||f||_p + ||g||_p)||f+g||_p^{p/q} \tag{9}$$

and since $p - \frac{p}{q} = p - (p-1) = 1$, we have $||f+g||_p \leq ||f||_p + ||g||_p$, as desired. $\square$

We have shown that we do, in fact, have a norm, so L^p is a normed vector space. To prove that our L^p space equipped with this norm is a Banach space, we simply must verify that it is complete. To do so, we begin with an important lemma concerning normed vector spaces:

Lemma II.2. A normed vector space is complete if and only if every absolutely converging sequences converges.

Proof. Suppose to begin that X is complete and $\sum_{k=1}^{\infty} ||x_k|| < \infty$. Let $S_N = \sum_{k=1}^{N} x_k$. It follows that

$$||S_N - S_M|| \left|\left| \sum_{k=M+1}^{N} x_k \right|\right| \leq \sum_{k=M+1}^{N} ||x_k|| \tag{10}$$

which goes to 0 for $M, N \to \infty$, so we have a Cauchy sequence, so S_N converges as X is complete. Conversely, suppose x_n is a Cauchy sequence. We let n_j be an ascending sequence of numbers such that for $n, m \geq n_j$, we have $||x_m - x_n|| \leq 2^{-j}$. We then note that if we set $y_j = x_{n_{j+1}} - x_{n_j}$, then we will have

$$\sum_{j=1}^{\infty} ||y_j|| = \sum_{j=1}^{\infty} ||x_{n_{j+1}} - x_{n_j}|| < \sum_{j=1}^{\infty} 2^{-j} = 1 \tag{11}$$

which means that $y = \sum_{j=1}^{\infty} y_j$ exists. Note that

$$||x_{n_j} - y|| = \left|\left| \left(\sum_{k=1}^{j-1} y_k \right) - y \right|\right| \tag{12}$$

so for large enough j, we can make this difference arbitrarily small. Hence, the subsequence x_{n_j} of x_k converges to y. But, we know that if a Cauchy sequence has a convergent subsequence, the whole sequence converges to the same limit, so we are done. □

Theorem II.3. For $p \in [1, \infty)$, L^p is a Banach space when equipped with the norm $|| \cdot ||_p$.

Proof. We will make use of the previous lemma. Suppose we pick a sequence of functions $f_p \in L^P$, and suppose $\sum_{k=1}^{\infty} ||f_k||_p = C < \infty$. We let $S_N = \sum_{k=1}^{N} |f_k|$, and note that $||S_N||_p \leq B$, from the triangle inequality, for every N. It follows immediately from monotone convergence theorem that

$$\int S_{\infty}^p = \lim_{N \to \infty} \int S_N \leq C^p \tag{13}$$

Thus, the function S_{∞} is in L^p and $S_{\infty}(x) < \infty$ a.e. so it must be true that $\sum_{k=1}^{\infty} f_k$ converges a.e. (we know convergence at points because the reals are obviously complete). If we let $F = \sum_{k=1}^{\infty} f_k$, then we of course have $|F| \leq S_{\infty}$, so $|F| \in L^P$. Finally, note that

$$\left| F - \sum_{k=1}^{N} f_k \right|^p \leq \left(|F| + \sum_{k=1}^{N} |f_k| \right)^p \leq (2S_N)^p \tag{14}$$

which means that by dominated convergence theorem:

$$\left|\left| F - \sum_{k=1}^{N} f_k \right|\right|_p^p = \int \left| F - \sum_{k=1}^{N} f_k \right|^p \to 0 \tag{15}$$

as $N \to \infty$., as the function inside the integral goes to 0 a.e. It follows by definition that $\sum_{k=1}^{\infty} f_k$ converges in L^p. Thus, L^p is complete, by the previous lemma. □

There is one particular case that we have yet to discuss, namely when $p = \infty$. In this particular case, we define

$$||f||_{\infty} = \inf \{a > 0 \mid \mu\{x \mid |f(x)| > a\} = 0\} \tag{16}$$

It is easy to incorporate the ∞-norm into our theory for L^p spaces: the results which we proved above all also hold in the infinite case.

Our main operator-theoretic point of interest, with respect to L^p-spaces, will be the dual L^p-spaces, $(L^p)^*$. In particular, given some $g \in L^q$, let us define the Banach space functional $\phi_g : L^p \to \mathbb{C}$, where $p^{-1}+q^{-1} = 1$, as

$$\phi_g(f) = \int fg \tag{17}$$

Note that this operator is bounded by Holder's inequality, as

$$|\phi_g(f)| = \left|\int fg\right| \leq \int |fg| = ||fg||_1 \leq ||f||_p ||g||_q \tag{18}$$

implying that

$$||\phi_g|| = \sup_f \frac{|\phi_g(f)|}{|||f||_p} \leq ||g||_q \tag{19}$$

Lemma II.3. Suppose $p^{-1} + q^{-1} = 1$, with $q \in [1, \infty)$, then if $g \in L^q$, we have

$$||g||_q = ||\phi_g|| = \sup \left\{ \left|\int fg\right| \;\middle|\; ||f||_p = 1 \right\} \tag{20}$$

In the case that μ is semifinite, it also holds for $q = \infty$.

Proof. We already saw that $||\phi_g|| \leq ||g||_q$, equality is immediate when $g = 0$ a.e., clearly. Otherwise, we define

$$f = \frac{|g|^{q-1}\overline{\text{sign}(g)}}{||g||_q^{q-1}} \tag{21}$$

and note that since $p^{-1} + q^{-1} = 1$, we have $q + p = pq$, so $pq - p = q$ and

$$||f||_p^p = \frac{1}{||g||_q^{pq-p}} \int |g|^{pq-p} = 1 \tag{22}$$

which means that

$$||\phi_g|| \geq \int fg = \frac{1}{||g||_q^{q-1}} \int |g|^q \overline{\text{sign}(g)}\text{sign}(g) = \frac{||g||_q^q}{||g||_q^{q-1}} = ||g||_q \tag{23}$$

so we have both sides of the inequality, implying $||\phi_g|| = ||g||_q$. In the particular case that $q = \infty$, then for $\varepsilon > 0$, we set $S = \{x \mid |g(x)| > ||g||_\infty - \varepsilon\}$, and by definition of the infinity-norm, $\mu(S) > 0$, so in the case μ is semi-finite, we can pick some $B \subset S$ where $0 < \mu(B) < \infty$. We then define $f = \mu(B)^{-1}\chi_B\overline{\text{sign}(g)}$: obviously $||f||_1 = 1$, so

$$||\phi_g|| \geq \int fg = \frac{1}{\mu(B)} \int |g| \geq ||g||_\infty - \varepsilon \tag{24}$$

where we are using the fact that $B \subset S$: a set on which $|g| > ||g||_\infty - \varepsilon$. Since this holds for any ε, $||\phi_g|| \geq ||g||_\infty$, and we are done. $\square$

In order to prove our main theorem, we require one more technical result, which we can now state:

Lemma II.4. Suppose $p^{-1} + q^{-1} = 1$ and g is measurable on X where $fg \in L^1$ for all f in Σ: the space of simple functions which vanish outside a set of finite measure, and suppose

$$M_q(g) = \sup \left\{ \left|\int fg\right| \mid f \in \Sigma \text{ and } ||f||_p = 1 \right\} \tag{25}$$

is finite. We also assume that $S_g = \{x \mid g(x) \neq 0\}$ is σ-finite and that μ is semifinite, then $g \in L^q$ and $M_q(g) = ||g||_q$.

For the sake of brevity, I will skip the proof of this fact: it boils down to demonstrating that one can approximate the norm of the operator ϕ_g sufficiently well be only considering a particular family of simple functions.

The main theorem of this essay is the *duality theorem for L^p spaces*:

Theorem II.4. Suppose $p^{-1} + q^{-1} = 1$, suppose μ is a σ-finite measure. If $1 < p < \infty$, then for $\phi \in (L^p)^*$, there exists some $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$, so L^q is isometrically isomorphic to $(L^q)^*$ via the map $g \mapsto \phi_g$ (we already proved equality of norms earlier).

Proof. As a first step, suppose μ is finite, which means all simple functions will be in. L^p. We define a measure as follows, let $\nu(E) = \phi(\chi_E)$ for some given $\phi \in (L^p)^*$. If E_j is a disjoint sequence of measurable sets and E is their union, then $\chi_E = \sum_{j=1}^{\infty} \chi_{E_j}$ where we can see that we have convergence in L^p easily:

$$\left|\left|\chi_E - \sum_{j=1}^{n} \chi_{E_j}\right|\right|_p = \left|\left|\sum_{j=n+1}^{\infty} \chi_{E_j}\right|\right|_p = \mu\left(\bigcup_{j=n+1}^{\infty} E_j\right)^{1/p} \to 0 \tag{26}$$

as we take $n \to \infty$, from lower measurable continuity. Therefore,

$$\nu(E) = \sum_{j=1}^{\infty} \phi(\chi_{E_j}) = \sum_{j=1}^{\infty} \nu(E_j) \tag{27}$$

so ν is, in fact, a valid complex measure. Obviously if $\mu(E) = 0$, then $\chi_E = 0$ a.e. so $\nu(E) = 0$ as well. Thus, $\nu << \mu$, so it follows from Radon-Nikodym there is $g \in L^1(\mu)$ with $\phi(\chi_E) = \nu(E) = \int_E g \, d\mu$ for all E. It follows immediately that if f is a simple function, then by linearity, $\phi(f) = \int fg \, d\mu$. We also know that $|\int fg| \leq ||\phi|| ||f||_p$ for such functions, so it follows from the technical lemma that $g \in L^q$. We can then use the fact that simple functions are *dense* in L^p spaces to conclude that $\phi(f) = \int fg \, d\mu$ for all $f \in L^p$.

Now let us move to the general case where μ is σ-finite. We let E_n be an increasing sequence of sets where $X = \cup_{n=1}^{\infty} E_n$, with each set having finite, non-zero measure. The argument from above demonstrates that we can pick $g_n \in L^q(E_n)$, for each n, such that

$$\phi(f) = \int g_n f \, d\mu \tag{28}$$

for all $finL^p(E_n)$, and $||g_n||_q = ||\phi|_{L^p(E_n)}|| \leq ||\phi||$. Of course, the g_n functions are unique up to possible changes on a nullset, so for $n < m$, $g_n = g_m$ on their common domain, which is E_n. We thus have a well-defined function g on X given by $g = g_n$ on E_n. It follows from monotone convergence theorem that $||g||_q = \lim_n ||g_n||_q \leq ||\phi||$, so $g \in L^q$ on the entire space X. If we then pick some $f \in L^p$, then dominated convergence theorem implies that $f\chi_{E_n} \to f$ in L^p, implying

$$\phi(f) = \lim_n \phi(f\chi_{E_n}) = \lim_n \int_{E_n} fg = \int fg \tag{29}$$

which is exactly what we wanted to prove. □

[1] Gert K Pedersen. *Analysis now*, volume 118. Springer Science & Business Media, 2012.
[2] Gerald B Folland. *Real analysis: modern techniques and their applications*, volume 40. John Wiley & Sons, 1999.