MUNKRES TOPOLOGY SOLUTIONS

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1. Topological Groups

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2. Section 30

2.1. **Problem 16a.** *Show that the product space $\mathbb{R}^I$ with I the unit interval has a countable dense subset.*

We let S be the set of all points that are rational for finitely many coordinates, and 0 at all other coordinates. Clearly, such a set is countable, as the rationals are countable, and:

$$S = \bigcup_{n \in \mathbb{N}} S_n$$

where S_n is the set of all points such that n coordinates are rational, and the rest are 0. It is easy to check that this is a dense set: given some $\boldsymbol{x} \in \mathbb{R}^I$, and some basis element $U = \prod_{\alpha \in I} U_\alpha$, with U_α open in $\mathbb{R}$, we will have $U_\alpha = \mathbb{R}$ except for finitely many $\alpha \in \{\alpha_1, \ ..., \ \alpha_n\}$.

We then pick a rational point in each U_{α_j} for $\alpha_j \in \{\alpha_1, \ ..., \ \alpha_n\}$, and let $\boldsymbol{y}$ be equal to these rational points at the corresponding coordinates, and 0 otherwise. Clearly, $\boldsymbol{y} \in S$ and $\boldsymbol{y} \in U$. Thus, $\boldsymbol{x} \in \overline{S}$, so S is dense.

2.2. **Problem 16b.** *Show that if J has cardinality greater than $\mathcal{P}(\mathbb{Z}^+)$, then $\mathbb{R}^J$ does not have a countable dense subset as a product space.*

Our strategy is to show that if $\mathbb{R}^J$ has a countable dense subset, then there exists some injection of J into $\mathcal{P}(\mathbb{Z}^+)$.

Let A be the countable dense subset of $\mathbb{R}^J$. It follows that for every $\boldsymbol{x} \in \mathbb{R}^J$, and every neighbourhood U of $\boldsymbol{x}$, U intersects A. We define a map $g : J \to \mathcal{P}(A)$ as:

$$g(\alpha) = A \cap \pi_\alpha^{-1}((a, b))$$

where $(a, b) \in \mathbb{R}$ is chosen arbitrarily. Clearly, $g(\alpha) \in \mathcal{P}(A)$, for each α, as $g(\alpha)$ is a subset of A. We show that g is an injection. Suppose:

$$A \cap \pi_\alpha^{-1}((a, b)) = A \cap \pi_\beta^{-1}((a, b))$$

Suppose $\alpha \neq \beta$. Let $B = A \cap \pi_\alpha^{-1}((a, b)) \cap \pi_\beta^{-1}((c, d))$, where $(c, d) \cap (a, b) = \emptyset$. Clearly, B is non-empty as A is dense. Thus, there exists some $\boldsymbol{y} \in A$ such that $y_\alpha \in (a, b)$, but $y_\beta \notin (a, b)$, contradicting the above. It follows that $\alpha = \beta$, so our map is injective.

Finally, since A is countable, there is a bijection $h : A \to \mathbb{Z}^+$. Hence, there is a bijection $h' : \mathcal{P}(A) \to \mathcal{P}(\mathbb{Z}^+)$. Thus, letting $f = h' \circ g$, and we have our desired injection.

2.3. **Problem 17.** *Give* $\mathbb{R}^\omega$ *the box topology, and let* $\mathbb{Q}^\infty$ *be the set of all rational sequences which end in a string of* 0*s. Which of the four countability axioms does this space satisfy?*

Note that $\mathbb{Q}^\infty$ is itself countable. Thus, it has a countable dense subset (itself), and is clearly Lindelof.

However, this space is **not** first-countable, and is therefore not second-countable. Let $\boldsymbol{x}$ be an arbitrary point, and let $\mathcal{B}$ be a countable collection of non-empty open neighbourhoods of $\boldsymbol{x}$. We claim that this set is not a basis at $\boldsymbol{x}$.

Clearly, we will have $B_n = V_n \cap \mathbb{Q}^\infty$, for V_n open in $\mathbb{R}^\omega$, for each n. Clearly, we then have:

$$\prod_{k \in \mathbb{N}} (a_k^n, b_k^n) \subset V_n$$

by definition of the box topology. We define U open in $\mathbb{Q}^\infty$ as follows:

$$U = \prod_{n \in \mathbb{N}} U_n \cap \mathbb{Q}^\infty$$

where U_n is chosen such that U_n is an interval strictly contained in (a_n^n, b_n^n) (which was defined above). We can guarantee strict containment, as we are working in the box topology.

We claim that U contains no B_n. Indeed, suppose we have $B_N \subset U$. Then, from above, we must have $(a_k^N, b_k^N) \subset U_k$ for all k. But this is not true for $k = N$. Thus, U cannot contain any element of $\mathcal{B}$.

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3. Section 31

3.1. **Problem 9a.**