MAT327 EXAM PREP

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CONTENTS

1. Practice Exam

1.1. **Problem 1.** *Let J be uncountable. Is $\mathbb{R}^J$ with the product topology metrizable?*

$\mathbb{R}^J$ with the product topology is not metrizable. This is due to the fact that we can form a contradiction to the sequence lemma, which in turn implies that the space is not metrizable.

We let A be the set of all points $\boldsymbol{x}$ in $\mathbb{R}^J$ that are 1 at all except finitely many of their coordinates. We let $\boldsymbol{0}$ be the point with all coordinates equal to 0.

Given some neighbourhood of the form $U = \prod_\alpha U_\alpha$ around $\boldsymbol{0}$, note that $U_\alpha \neq \mathbb{R}$ for only finitely many α, so the point $\boldsymbol{y}$, with $y_\beta = 0$ for this finite collection of indices and 1 otherwise is in U and A. Thus, $\boldsymbol{0} \in \bar{A}$.

We assert that there is no sequence of points of A that converges to $\boldsymbol{0}$. This is due to the fact that for any sequence $\boldsymbol{a}_n$, the set of coordinates that differ from 1 for some element of the sequence is finite. Taking the union of all such indices over n yields a countable set. But the set of all indices, J, is uncountable, so we can choose some coordinate β such that every $a \in \boldsymbol{a}_n$ is 1 at that coordinate.

We then take the open set $\pi_\beta^{-1}(-1, 1)$ in $\mathbb{R}^J$. This contains $\boldsymbol{0}$, but no element of $\boldsymbol{a}_n$. Thus, $\boldsymbol{a}_n$ does not converge to $\boldsymbol{0}$.

1.2. **Problem 2.** *Let A be countable. Is $\mathbb{R}^\omega - A$ path-connected, where $\mathbb{R}^\omega$ is given the product topology?*

Let a and b be points of $\mathbb{R}^\omega$. Let $S_x = \{L_x^m \mid m \in \mathbb{R}\}$ be the set of all lines passing through x lying on the plane defined by a and b, so:

$$L_x^m = \{x + t(a + mb) \mid t \in \mathbb{R}\}$$

Clearly, given $m \neq n$, we have $L_x^m \cap L_x^n = \{x\}$. Since A is countable, and the sets S_a, S_b are uncountable, it follows that there exist infinite subsets $V_a \subset S_a$ and $V_b \subset S_b$ of lines that do not intersect A. Pick $L_a^m \in V_a$, and $L_b^n \in V_b$ that is not parallel to L_a^m. It follows that $L_a^m \cap L_b^n = \{c\}$.

We then define $F : [0, 2] \to \mathbb{R}^\omega - A$ as:

$$F = \begin{cases} (1 - t)a + tc & t \in [0, 1] \\ tc + (t - 1)b & t \in [1, 2] \end{cases}$$

It is easy to see that $F([0, 1]) \subset L_a^m$ and $F([1, 2]) \subset L_b^n$, so $F([0, 2])$ lies in $\mathbb{R}^\omega - A$ and F is well-defined. In addition, it is clear that F is a path from a to b, so the space is path-connected.

1.3. **Problem 3.** *Suppose A is a subset of X, a compact metric space, and every continuous function $f : A \to \mathbb{R}$ has a maximum in A. Prove that A is compact.*

We will show that A is closed, and therefore compact (as we are operating in a compact metric space).

Suppose A is not closed. Then $X - A$ is not open. Thus, there must exist some $a \in X - A$ such that $a \in \bar{A}$.

Define $f : A \to \mathbb{R}$ such that $f(x) = \frac{1}{d(x,a)}$. Clearly, this function is continuous on A, as $g(x) = d(x,a)$ is continuous. However, it is obvious that f is not bounded, as for any N, we can choose a point $y \in A$ such that $y \in B_{1/N}(a)$, so $f(y) > N$. Thus, we have a contradiction, so A must be closed, and is therefore compact.

1.4. **Problem 4.** *Calculate the fundamental group of S^1 with the north and south poles identified*

It is easy to see through a sketch that this space is homotopy equivalent to a sphere with a line connecting the north and south poles. This space is itself homotopy equivalent to $S^2 \vee S^1$, which can been seen be gradually sliding the end points of the diameter together.

Let x_0 be the attachment point. We proceed via Seifert-van Kampen. Removing a point from S^1 gives an open set U that deform retracts to S^2, so $\pi_1(U, x_0)$ is trivial. Removing a point from S^2 gives an open set V that deform retracts to S^1, so $\pi_1(V, x_0) = \langle a \rangle$.

The intersection $U \cap V$ gives a space which deform retracts to a point, and therefore has a trivial fundamental group.

Thus, by Seifert-van Kampen, $\pi_1(X, x_0) = \langle a \mid \emptyset \rangle = \mathbb{Z}$.

1.5. **Problem 5.** *Describe the universal covering space and covering map of the above space.*

Don't know, universal covering spaces were never once mentioned in class, so I'm not sure why this is on the practice exam...

1.6. **Problem 6. Part A**

Find the fundamental group of the doubly-punctured torus.

It is easy to see that the doubly-punctured torus deform retracts to the boundary of the rectangle defining it, with a line passing down the middle. This is itself clearly the wedge of three circles: $X = S^1 \vee S^1 \vee S^1$.

We once again use Seifert-van Kampen. Let $a, b, c \in X$ be distinct, lying on each of the circles in the wedge (not at the attachment point x_0). Let $U = X - \{a\}$ and $V = X - \{b, c\}$. Clearly, U deform retracts to $S^1 \vee S^1$, so $\pi_1(U, x_0) = \langle a, b \rangle$. In addition, V clearly deform retracts to S^1, so $\pi_1(V, x_0) = \langle c \rangle$. It follows that:

$$\pi_1(X, x_0) = \pi_1(U \cup V, x_0) = \langle a, b, c \rangle = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

as $U \cap V$ deform retracts to a point, and therefore is simply connected.

Part B

Find the fundamental group of $\mathbb{R}^3$ with n lines through the origin removed.

Proposition 1. $\mathbb{R}^2$ *with n points removed (which we call R_n) has a fundamental group $\mathbb{Z} * \cdots * \mathbb{Z}$, repeated n time.*

Proof. We proceed by induction. For $n = 1$, the claim clearly holds. Assume the case of n. We prove the case of $n + 1$. Choose open sets U and V, with U containing the $n + 1$-th hole, and V containing the other n holes, such that $U \cup V = X$, and $U \cap V$ is some unpunctured open set, and is therefore simply connected.

It follows that $\pi_1(U, x_0)$ is the free group with 1 generator, while $\pi_1(V, x_0)$ is the free group of n generators, from the inductive hypothesis. Thus, $\pi_1(R_n, x_0)$ is the free group with $n + 1$ generators. $\square$

$\mathbb{R}^3 - X$ clearly deform retracts to S^2, with $2n$ holes. We know that $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$. Thus, subtracting an additional $2n - 1$ points implies that our space is homeomorphic to $\mathbb{R}^2$ with $2n - 1$ distinct points removed, so from above:

$$\pi_1(\mathbb{R}^3 - X, x_0) = \pi(R_{2n-1}, x_0) = \mathbb{Z} * \cdots * \mathbb{Z}$$

repeated $2n - 1$ times.

1.7. **Problem 7.** *Show that every continuous map $f : \mathbb{R}P^2 \to S^1$ is nulhomotopic.*

We utilize the general lifting lemma. Consider a continuous map $f : Y \to B$, and covering map $p : E \to B$, with $p(e_0) = b_0$ and $f(y_0) = b_0$. Suppose Y is path-connected locally path-connected. Then there exists a lift of f to $\tilde{f}$ with $\tilde{f}(y_0) = e_0$ if and only if:

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$$

We know that, given $y_0 \in \mathbb{R}P^2$, we have $\pi_1(\mathbb{R}P^2, y_0) = \mathbb{Z}/\mathbb{Z}2$. We also know that $\pi_1(S^1, b_0) = \mathbb{Z}$. Clerarly, if ϕ is a homomorphism from $\mathbb{Z}/\mathbb{Z}2$ to $\mathbb{Z}$, then $\phi(0) = 0$. Suppose $\phi(1) = n \neq 0$. Then:

$$\phi(0) = \phi(1 + 1) = 2\phi(1) = 2n \neq 0$$

a contradiction, so we must have $\phi(1) = 0$. Thus, f_* must be the trivial homomorphism, so we clearly have $f_*(\pi_1(\mathbb{R}P^2, y_0)) \subset p_*(\pi_1(\mathbb{R}, e_0))$, where $p : \mathbb{R} \to S^1$ is the usual covering map.

From the general lifting lemma, we can lift f to $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R}$. Let $F(x, t) = (1 - t)\tilde{f}(x) + te_0$. Clearly, this is a homotopy of $\tilde{f}(x)$ and e_0. Then $p \circ F$ is a homtopy of $p \circ \tilde{f} = f$ and $p(e_0)$, so f is nulhomotopic.

1.8. **Problem 8.** *Prove that there exists no retraction from the solid torus to its boundary torus.*

Recall that we proved that there exists no retraction from D^2 to S^1.

Suppose there is a retraction r from $S^1 \times D^2$ to $S^1 \times S^1$. Then define $g : D^2 \to S^1$ as:

$$g(x) = (\pi_2 \circ r)(x_0, x)$$

where $x_0 \in S^1$ and $\pi_2 : S^1 \times S^1 \to S^1$ is a projection. Clearly, this map is continuous, as both π_2 and r are continuous. Note that for $x \in S^1$, we have $(x_0, x) \in S^1 \times S^1$, so $r(x_0, x) = (x_0, x)$ (by definition of r), so:

$$g(x) = (\pi_2 \circ r)(x_0, x) = \pi_2(x_0, x) = x$$

so g is a retraction from D^2 to S^1, a contradiction. It follows that there is no retraction from $S^1 \times D^2$ to $S^1 \times S^1$.

Alternatively, we could have completed this proof through more algebraic means. Note that $S^1 \times S^1$ has fundamental group $\mathbb{Z} \times \mathbb{Z}$, while $S^1 \times D^2$ deform retracts to S^1, so it has fundamental group S^1.

If a retract exists, then inclusion of $S^1 \times S^1$ into $S^1 \times D^2$ induces an injective homomorphism of fundamental groups. However, there is no injective homomorphism of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z}$. Suppose ϕ is such a homomorphism. Then let:

$$\phi(1,0) = a \quad \text{and} \quad \phi(0,1) = b$$

But we then have $\phi(b,0) = ab$ and $\phi(0,a) = ab$, so $\phi(b,0) = \phi(0,a)$, a clear contradiction to injectivity. Thus, not such retraction can exist.

1.9. **Problem 9.** *Let $F_1 \cup F_2 \cup F_3 = S^2$, where each F_i is closed. Show that some F_i contains a pair of antipodal points.*

Lemma 1. *Let X be a metric space and $f(x) = d(x, A)$. If A is closed, $f(x) = 0$ implies that $x \in A$.*

Proof. Suppose $f(x) = 0$, so $\inf_{a \in A} d(x, a) = 0$. Suppose $x \notin A$, so $x \notin \bar{A}$. Then there exists some ϵ-ball around x disjoint from A, so $d(x,a) \geq \epsilon$ for all $a \in A$. This is a contradiction, so $x \in A$. $\square$

We wish to apply Borsuk-Ulam. Consider the functions $f(x) = (d(x, F_1), d(x, F_2))$. Borsuk-Ulam says that there is x such that $f(x) = f(-x)$. If $f(x)_1 = 0$ or $f(x)_2 = 0$, then we automatically have $x, -x \in F_1$ or $x, -x \in F_2$. If both components are non-zero, then we must have $x, -x \in F_3$, as they are not in F_1 nor F_2.

1.10. **Problem 10.** *Let x_0 and x_1 be points of a path-connected space. Show that $\pi(X, x_0)$ is abelian if and only if $\hat{\alpha} = \hat{\beta}$ for every pair of paths α, β from x_0 to x_1. Recall that $\hat{\alpha}([f]) = [\bar{\alpha} * f * \alpha]$.*

First, let us suppose that $\pi_1(X, x_0)$ is abelian. Let α and β be paths from x_0 to x_1. Then $\alpha * \bar{\beta}$ and $\beta * \bar{\alpha}$ are loops based as x_0. Let f be a loop at x_0. We therefore have:

$$[\alpha * \bar{\beta}] * [f] * [\beta * \bar{\alpha}] = [\alpha * \bar{\beta}] * [\beta * \bar{\alpha}] * [f] = [f]$$

where each element of the product is a loop in $\pi_1(X, x_0)$. It follows that:

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha] = [\bar{\alpha}] * [\alpha * \bar{\beta}] * [f] * [\beta * \bar{\alpha}] * [\alpha] = [\bar{\beta}] * [f] * [\beta] = \hat{\beta}([f])$$

from above.

Now, suppose that $\hat{\alpha} = \hat{\beta}$ for every pair of paths. Suppose $[f], [g] \in \pi_1(X, x_0)$. Let α be a path from x_0 to x_1. We also have $h = f * \alpha$ a path from x_0 to x_1. Thus:

$$\hat{\alpha}([g]) = [\bar{\alpha} * g * \alpha] = \hat{h}([g]) = [\bar{\alpha} * \bar{f} * g * f * \alpha]$$

This then implies that $[g] = [\bar{f} * g * f]$, so $[f] * [g] = [g] * [f]$. Thus, the group is abelian.

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2. Extra Hatcher Problems

2.1. **Problem 0.1.** *Construct an explicit deformation retraction from the torus T with a point removed to its boundary circles.*

Not really sure what this problem wants, literally an explicit formula for a deformation retraction?

2.2. **Problem 0.3. Part A**

Show that the composition of homotopy equivalences is a homotopy equivalence.

Let $f : X \to Y$ and $g : Y \to Z$ be homotopy equivalences. We wish to show that $g \circ f : X \to Z$ is a homotopy equivalence. Note that there exist maps $f' : Y \to X$ and $g' : Z \to Y$ such that ff', $f'f$, gg', and $g'g$ are homotopic to the identity map on their respective domains/codomains.

We claim that $(g \circ f) \circ (f' \circ g') \simeq \mathrm{id}_Z$ and $(f' \circ g') \circ (g \circ f) \simeq \mathrm{id}_X$ (we don't assume transitivity of homotopy, this is Part B of the question, so we have to construct an explicit homotopy).

If F is a homotopy between $f \circ f'$ and id_Y, and G is a homotopy between $g \circ g'$ and id_Z, define:

$$H(x,t) = \begin{cases} g \circ F(x, 2t) \circ g' & t \in \left[0, \frac{1}{2}\right] \\ G(x, 2t - 1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

This function is continuous, by the pasting lemma. Also, note that $H(x, 0) = g \circ F(x, 0) \circ g' = g \circ f \circ f' \circ g'$, and $H(x, 1) = G(x, 1) = \mathrm{id}_Z$. Almost identical reasoning shows $(f' \circ g') \circ (g \circ f) \simeq \mathrm{id}_X$, so it follows that $g \circ f$ is a homotopy equivalence.

Part B

Show that the relation of homotopy among maps is an equivalence relation.

Given maps $f \simeq g$ and $g \simeq h$, we wish to show that $f \simeq h$. This is easy. If F is a homotopy between the first pair and G is a homotopy between the second pair, then define H like we did above:

$$H(x,t) = \begin{cases} F(x, 2t) & t \in \left[0, \frac{1}{2}\right] \\ G(x, 2t - 1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Clearly, $H(x, 0) = F(x, 0) = f(x)$ and $H(x, 1) = G(x, 1) = h(x)$. Continuity is given by the pasting lemma, at $t = \frac{1}{2}$. Thus, homotopy is an equivalence relation, as maps are clearly homotopic to themselves, and if $f \simeq g$ with homotopy $F(x, t)$, then $F(x, 1-t)$ is a homotopy from g to f, so $g \simeq f$.

Part C

Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Let $f : X \to Y$ be a homotopy equivalence, so there is $f' : Y \to X$ with ff' and $f'f$ homotopic to the identity. Let us consider $g \simeq f$. We prove a small lemma:

Lemma 2. *If $k_1 \simeq k_2$ (with $k_i : Y \to Z$), then given continuous $f : Z \to W$ and $g : X \to Y$, then:*

$$f \circ k_1 \circ g \simeq f \circ k_2 \circ g$$

Proof. Since $k_1 \simeq k_2$, there exists a homotopy F between them. Define $G(x, t) = (f \circ F)(g(x), t)$. We claim that this is a homotopy between $f \circ k_1 \circ g$ and $f \circ k_2 \circ g$. Clearly, this function is continuous, as f and g are. Note that:

$$G(x, 0) = (f \circ F)(g(x), 0) = (f \circ k_1 \circ g)(x) \quad \text{and} \quad G(x, 1) = (f \circ F)(g(x), 1) = (f \circ k_2 \circ g)(x)$$

and we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

The proof is now trivial, we have:

$$gf' \simeq ff' \simeq \mathrm{id}_Y \quad \text{and} \quad f'g \simeq f'f \simeq \mathrm{id}_X$$

so g is a homotopy equivalence, and we are done.

2.3. **Problem 0.4.** *A deformation retraction in the weak sense of a space X to a subspace A is a homotopy such that $F(x,0) = x$, $F(x,1) \subset A$, and $F(A,t) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion from A to X is a homotopy equivalence.*

Let $j : A \to X$ denote inclusion. Let $r(x) = F(x,1)$. We claim that jr and rj are homotopic to the identity. Indeed, let $r : X \to A$ be defined as $r(x) = F(x,1)$. Indeed, this is well-defined as $F(x,1) \in A$ for all $x \in X$.

To show $r \circ j \simeq \mathrm{id}_A$, we let $G : A \times I \to A$ be defined as $G(x,t) = F(x,t)$, which is well defined as $G(x,t) \in A$ for $x \in A$. It is easy to check this is a homotopy. Similarly, to show $j \circ r \simeq \mathrm{id}_X$, we use the original homotopy F.

Thus, j is a homotopy equivalence.

2.4. **Problem 0.9.** *Show that a retract of a contractible space is contractible.*

Let A be a retract of a contractible space X. We know that $\mathrm{id}_X \simeq e_c$, where e_c is the identity map. If r is our retract, then note that:

$$\mathrm{id}_A = r \circ \mathrm{id}_X \circ j$$

If F is a homotopy between id_X and e_c, we claim that $G : A \to A$ defined as $G(x,t) = (r \circ F)(j(x),t)$ is a homotopy between id_A and $e_{r(c)}$. Indeed, note that G is continuous, well-defined, $G(x,0) = (r \circ j)(x) = \mathrm{id}_A(x)$ (from above), and $G(x,1) = r(c)$. Thus, $\mathrm{id}_A \simeq e_{r(c)}$, so A is contractible.

2.5. **Problem 0.10.** *Show that a space X is contractible iff every map from X to Y for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map from Y to X is nullhomotopic.*

First, suppose that every $f : X \to Y$, for arbitrary Y, is nulhomotopic. Then setting $Y = X$ and $f = \mathrm{id}_X$ shows that X is contractible. Conversely, suppose X is contractible. Pick some Y and some f. Note that:

$$f = f \circ \mathrm{id}_X \simeq f \circ e_c = e_{f(c)}$$

so f is nulhomotopic.

Now, suppose every $f : Y \to X$ is nulhomotopic. Again, setting $Y = X$ and $f = \mathrm{id}_X$ shows that X is contractible. Conversely, pick some $f : Y \to X$. Let F be a homotopy between id_X and e_c. Then define $G(x,t) = F(f(x),t)$. We claim that this is a homotopy between f and $e_c : Y \to X$.

Indeed, note that G is clearly continuous and well-defined, $G(x,0) = f(x)$, and $G(x,1) = c$.

This proof is also a trivial consequence of the lemma given in in Problem 0.3: $\mathrm{id}_X \simeq e_c$ implies $f \circ \mathrm{id}_X \simeq f \circ e_c$, and $\mathrm{id}_x \circ f \simeq e_c \circ f$.

2.6. **Problem 1.10.** *Show that loops of $\pi_1(X \times Y, x_0 \times y_0)$ whose images are contained in $X_0 \times \{y_0\}$ and $\{x_0\} \times Y_0$ respectively commute.*

Loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ are of the form $f = (f_1, y_0)$ and $g = (x_0, g_2)$. We can think of $f * g$ as a loop which traverses a path in X, while remaining stationary at y_0, then traverses a path in Y while remaining stationary at x_0.

We wish to show that $[f] * [g] = [g] * [f]$. We define a homotopy between $f * g = (f_1 * x_0, y_0 * g_2)$ and $g * f = (x_0 * f_1, g_2 * y_0)$. We define a homotopy between $f_1 * x_0$ and $x_0 * f_1$ as follows: we let α and β be defined as:

$$\alpha(s) = \begin{cases} 2s & s \in \left[0, \frac{1}{2}\right] \\ 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases} \quad \text{and} \quad \beta(s) = \begin{cases} 0 & s \in \left[0, \frac{1}{2}\right] \\ 2s - 1 & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

We then define $F(x, t) = f_1((1 - t)\alpha(x) + t\beta(x))$. Clearly, this function is continuous, as it is the composition of continuous functions. Note that:

$$F(x, 0) = f_1(\alpha(x)) = \begin{cases} f_1(2x) & x \in \left[0, \frac{1}{2}\right] \\ f_1(1) = x_0 & x \in \left[\frac{1}{2}, 1\right] \end{cases} \quad \text{and} \quad F(x, 1) = f_1(\beta(x)) = \begin{cases} f_1(0) = x_0 & x \in \left[0, \frac{1}{2}\right] \\ f_1(2x - 1) & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

so $F(x, 0) = f_1 * x_0$ and $F(x, 1) = x_0 * f_1$. Finally, note that:

$$F(0, t) = f_1(0) = x_0 \quad \text{and} \quad F(1, t) = f_1(1) = x_0$$

for all t. Thus, F is a path-homotopy between $f_1 * x_0$ and $x_0 * f_1$. Identical reasoning yields a path-homotopy G between $g_2 * y_0$ and $y_0 * g_2$. Finally, we define a homotopy H between $f * g$ and $g * f$ as $H(x, t) = (F(x, t), G(x, 1 - t))$. Clearly, H is continuous as a map into $X \times Y$. In addition, we have:

$$H(x, 0) = (F(x, 0), G(x, 1)) = (f_1 * x_0, y_0 * g_2) \quad \text{and} \quad H(x, 1) = (F(x, 1), G(x, 0)) = (x_0 * f_1, g_2 * y_0)$$

$$H(0, t) = (F(0, t), G(0, 1 - t)) = (x_0, y_0) \quad \text{and} \quad H(1, t) = H(1, t) = (F(1, t), G(1, 1 - t)) = (x_0, y_0)$$

so H is a path-homotopy between $f * g$ and $g * f$.

2.7. **Problem 1.16.** *Show that no retract exists in the following cases: (a) $\mathbb{R}^3$ to a space homeomorphic to S^1, (b) Solid torus to boundary torus, (c) Weird image in Hatcher (ignore), (d) Wedge of disks to wedge of boundary disks, (e) Disk with two points identified to boundary $S^1 \vee S^1$*

Part A

Clearly, $X = \mathbb{R}^3$ is simply connected while A homeomorphic to S^1 has fundamental group $\mathbb{Z}$, so there can exist no injective homomorphism from $\pi_1(A)$ to $\pi_1(\mathbb{R}^3)$, and therefore no retract.

Part B

See practice exam.

Part C

This is tough to see, and frankly, I hate how "hand-wavy" this question is. Both spaces have the same fundamental group, but inclusion of loops in A into the solid torus are actually trivial, as we can "unlink" the part of the circle at which they joint, so the homomorphism of fundamental groups given by inclusion is actually trivial, therefore not injective, so there is no retract.

Part D

Clearly, $D^2 \vee D^2$ has a trivial fundamental group, as it deform retracts to a point. We know that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, so there is no injection of fundamental groups, and therefore no retract.

Part E

It is easy to see (through drawings), that X is homotopy equivalent to a circle, so it has fundamental group $\mathbb{Z}$. Clearly, there is no injection from $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$ to $\mathbb{Z}$. Let a and b be the generators, then for some homomorphism ϕ, we have $\phi(ab) = \phi(a)\phi(b) = \phi(b)\phi(a) = \phi(ba)$, but $ab \neq ba$, by definition.

2.8. **Problem 2.2.** *If X is the union of n convex open subsets of $\mathbb{R}^m$, denoted X_1, ..., X_n, such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all i, j, k, then X is simply connected.*

We proceed by induction. Clearly, this holds for the case of $n = 1$ and $n = 2$ (by a trivial application of Seifert van-Kampen). Suppose it holds for the case of n (with $n \geq 2$). We prove that it holds for the case of $n + 1$.

Clearly, the intersection of finitely many convex sets is convex, as if we have A, B convex, and $x, y \in A \cap B$, then all points on the line between x and y will lie in both A and B, and therefore in their intersection.

Let our sets be denoted X_1, ..., X_{n+1}. We know that $Y = X_1 \cup \cdots \cup X_n$ is simply connected. We also know that X_{n+1} is simply-connected, as it is convex. Finally, note that $X_{n+1} \cap Y$ is path-connected: if this weren't the case then there would exist some X_i and X_j with $i, j \leq n$ such that $X_{n+1} \cap X_i$ and $X_{n+1} \cap X_j$ are disjoint, but this can't be the case, by assumption.

It follows that $X_1 \cap \cdots \cap X_{n+1} = Y \cap X_{n+1}$ is path-connected convex, so it is simply connected. Thus, by Seifert-van Kampen:

$$\pi_1(X, x_0) = \pi_1(X_1 \cup \cdots \cup X_n, x_0) * \pi_1(X_{n+1}, x_0) = \langle \emptyset \rangle$$

as we are taking the free product of two trivial groups.

2.9. **Problem 2.4.** Already done in practice exam!

2.10. **Problem 3.1.** *For a covering space $p : \tilde{X} \to X$ and a subspace $A \subset X$, let $\tilde{A} = p^{-1}(A)$. Show that the restriction $p_{\tilde{A}} : \tilde{A} \to A$ is a covering map.*

Clearly, $p|_{\tilde{A}}$ is still continuous and surjective. Take $x \in A$. Pick U around x open in X that is evenly covered by p. We claim that $U \cap A$ open in A, which contains x, is evenly covered by $p|_{\tilde{A}}$. Indeed, note that:

$$p|_{\tilde{A}}^{-1}(U \cap A) = \{a \in \tilde{A} \mid p(a) \in U,\ p(a) \in A\} = p^{-1}(U) \cap \tilde{A} = \bigcup_{\alpha}(V_\alpha \cap \tilde{A})$$

where each V_α is disjoint, open in $\tilde{X}$, and locally homeomorphic to U via p. It follows that each $V_\alpha \cap \tilde{A}$ is disjoint and open in $\tilde{A}$.

In addition, note that each restriction of $p|_{\tilde{A}}$ to $p|_{\tilde{A} \cap V_\alpha} : \tilde{A} \cap V_\alpha \to A$ is a local homeomorphism with $U \cap A$. Indeed, $p|_{\tilde{A} \cap V_\alpha}$ is injective, as $p|_{V_\alpha}$ is, and as a result:

$$p|_{\tilde{A} \cap V_\alpha}(\tilde{A} \cap V_\alpha) = p(\tilde{A}) \cap p(V_\alpha) = A \cap U$$

so $p|_{\tilde{A} \cap V_\alpha}$ bijectivelty maps to $A \cap U$. Finally, it is clear that this restriction is continuous, as p is continuous, and in addition, since $p^{-1}|_U : U \to V_\alpha$ is continuous, so too is $p^{-1}|_{U \cap A} : U \cap A \to \tilde{A} \cap V_\alpha$. Clearly, this is the inverse of the restricted map. Thus, it is a local homeomorphism.

It follows that p is a covering map.

2.11. **Problem 3.2.** *Show that a product of covering maps is a covering map.*

Suppose $p_1 : \tilde{X}_1 \to X_1$ and $p_2 : \tilde{X}_2 \to X_2$ are covering maps. Clearly, their product is continuous and surjective.

For some $x_1 \times x_2 \in X_1 \times X_2$, we pick U_1 open in X_1 evenly covered by p_1 and U_2 open in X_2 evenly covered by p_2. Then $U = U_1 \times U_2$ is open in $X_1 \times X_2$. We claim that $U_1 \times U_2$ is evenly covered by $p_1 \times p_2$. Indeed, note that:

$$(p_1 \times p_2)^{-1}(U_1 \times U_2) = \{t_1 \times t_2 \in \tilde{X}_1 \times \tilde{X}_2 \mid p_1(t_1) \in U_1,\ p_2(t_2) \in U_2\} = \left[p_1^{-1}(U_1) \times \tilde{X}_2\right] \cap \left[\tilde{X}_1 \times p_2^{-1}(U_2)\right]$$

$$= \left[\bigcup_\alpha V_\alpha \times \tilde{X}_1\right] \cap \left[\bigcup_\alpha W_\beta \times \tilde{X}_2\right] = \bigcup_{\alpha,\beta} V_\alpha \times W_\beta$$

Clearly, each open set in this union is disjoint. In addition, we note that $(p_1 \times p_2)|_{V_\alpha \times W_\beta}$ is a local homeomorphism from $V_\alpha \times W_\beta$ to $U_1 \times U_2$. Indeed, it is clear that $(p_1 \times p_2)|_{V_\alpha \times W_\beta}$ is injective and surjective, as its component functions are. Since $p_1 \times p_2$ is continuous, this restriction is as well.

Finally, we note that the restricted map is a homeomorphism. Since $p_1^{-1}|_{U_1} : U_1 \to V_\alpha$ and $p_2^{-1}|_{U_2} : U_2 \to W_\beta$ are continuous, so too is their product, which is clearly the inverse of $(p_1 \times p_2)|_{V_\alpha \times W_\beta}$. Thus, $p_1 \times p_2$ is a local homeomorphism.

2.12. **Problem 3.4.** Already done

2.13. Problem 1.5. Note that this was done in Munkres, but I'll attempt to reconstruct the proof from scratch.

Suppose (a) holds. We know that there exists a continuous $F : S^1 \times I \to X$ such that $F(x, 0) = f(x)$ and $F(x, 1) = c$. It is easy to see that the quotient space Y obtained from identifying every point of $S^1 \times \{1\}$ is homeomorphic to B^2. Hence, we have a quotient map p going from $S^1 \times I$ to B^2.

We claim that there exists a well-defined, continuous map k from B^2 to X induced by F and p. Indeed, since F is constant in each point in $S^1 \times \{1\} = p^{-1}(0)$, it follows that F extends to a map k from B^2 to X such that $k \circ g = F$.

Now, we show that (b) implies (c).

We can now make the deduction. Suppose X is simply-connected. Then we immediately have all maps $S^1 \to X$ being homotopic to a constant map, so they are homotopic to each other. Conversely, if all maps are homotopic, then they are all homotopic to a constant map, so the fundamental group is trivial.

2.14. Problem 1.6.
Part F

Let x_0 be a point on the boundary circle. Let x_1 be a point on the "middle circle" M of the Mobius strip. Note that X deform retracts onto M, so the inclusion map $j_* : \pi_1(M, x_1) \to \pi_1(X, x_1)$ as an isomorphism. Since M is a circle, $\pi_1(M, x_1) = \mathbb{Z}$. Since X is path-connected, the change-of-basepoint map from x_0 to x_1 as an isomorphism of fundamental groups. Thus, we have an isomorphism $k : \pi_1(X, x_0) \to \pi_1(M, x_1)$.

Let $[\omega]$ be the generator of $\pi_1(\partial X, x_0)$. Let $i_* : \pi_1(\partial X, x_0) \to \pi_1(X, x_0)$ be the inclusion map. Suppose there exists a retract from X to ∂X, so i_* is a homomorphism.

We have a homomorphism $k \circ i_*$ from $\pi_1(\partial X, x_0)$ to $\pi_1(M, x_1)$. Taking $[\omega]$, we note that mapping this loop into X and changing the basepoint with $k \circ i_*$ is path-homotopic to a loop which goes twice around M. In other words, $(k \circ i_*)([\omega]) = [\delta]^2$, where δ is the generator of $\pi_1(M, x_1)$. Since we identify $[\omega]$ and $[\delta]$ with $1 \in \mathbb{Z}$, we get $(k \circ i_*)(1) = 2$.

Note that $r_* k^{-1} k i_*$ is the identity map on $\pi_1(\partial X, x_0)$. Thus, $(r_* k^{-1} k i_*)(1) = (r_* k^{-1})(2) = 1$. But this is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$, so $2(r_* k^{-1})(1) = 1$, which is impossible. Thus, no such r can exist.

3. Seifert van-Kampen Notes and Examples

3.1. Theory. In this section, we outline the theory behind Seifert-van Kampen theorem.

At a high-level, the utility of Seifert-van Kampen relates to computing the fundamental group of a space by breaking it up into smaller pieces, of which we already know the fundamental groups.

We begin with a smaller theorem, which is a special case of Seifert-van Kampen:

Theorem 1. *Suppose that U and V are open sets, with $X = U \cup V$ such that the intersection $U \cap V$ is path-connected. Then given $x_0 \in U \cap V$, with i and j the inclusions of U and V in X, then the*

images of the induced homomorphisms of i and j based at x_0 generate the fundamental group of X at x_0.

Proof. The above statement is another way of stating that every element of $\pi_1(X, x_0)$ can be expressed as a product of elements in $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$.

Let $f : [0, 1] \to X$ be a loop in $X = U \cup V$. Note that $f^{-1}(U)$ and $f^{-1}(V)$ are on open cover for $[0, 1]$. Thus, by the Lebesgue number lemma, there exists some δ such that any interval $[a, b] \subset [0, 1]$ with $b - a < \delta$ is contained in either $f^{-1}(U)$ or $f^{-1}(V)$. Take the collection of intervals:

$$[0, \delta], \ [\delta, 2\delta], \ ..., \ [1 - \delta, \delta]$$

It then follows that for each interval r_k, we have $f(r_k) \subset U$ or $f(r_k) \subset V$. Now, we paste together intervals to get a new set of intervals $[s_1, s_2], \ ..., \ [s_{n-1}, s_n]$ to get a new set of interval q_k, with endpoints in $U \cap V$ (which we can clearly do, as we have a finite collection).

Clearly, $[f] = [q_1] * \cdots * [q_k]$. Finally, since $U \cap V$ is path connected, for each path q_k, choose a path α_k from x_0 to the start point of q_k and β_k to the endpoint. Note that we can have $\alpha_{k+1} = \beta_k$ and $\beta_{k-1} = \alpha_k$. Hence, inserting these products between each element of the product gives a product of loops entirely in U and V. $\square$

3.2. **Example: The Torus.** We begin by noting that we can express the torus as a union of two open set U and V with path-connected intersection: the torus with a point removed, and the torus without its boundary, respectively.

Clearly, U deform retracts to the boundary, which is itself a wedge of circles. Hence, the inclusion-induced homomorphism $i_* : \pi_1(\partial U, x_1) \to \pi_1(U, x_1)$ is an isomorphism, where x_1 is some point in ∂U. We know that $\pi_1(\partial U, x_1) \simeq \mathbb{Z} * \mathbb{Z}$. Finally, we know that the change-of-basepoint map $\hat{\alpha}$ is an isomorphism from $\pi_1(U, x_0)$ to $\pi_1(U, x_1)$.

We also note that V is simply connected, so $\pi_1(V, x_0) \simeq \emptyset$, where x_0 is our base-point.

Finally, note that $U \cap V$ is the torus with a point and its boundary removed, which deform retracts to a circle S pasing through x_0. Thus, $j_* : \pi_1(S, x_0) \to \pi_1(U \cap V, x_0)$ is an isomorphism, so $\pi_1(U \cap V, x_0) \simeq \mathbb{Z}$.

Now, we can construct $\pi_1(U \cup V, x_0)$ using Seifert Van-Kampen theorem. We know that this group will be the free group of generators given by generators of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$, along with their relations, and the set of relations R_S.

Clearly, the generators of $\pi_1(U, x_0)$ are single loops around each loop in the wedge (up to a change of basepoint), and $\pi_1(V, x_0)$ has no generators. In addition, the only relations arise from R_S, in which we identify $(\phi_1)_*([s]) = (\phi_2)_*([s])$, where $[s] \in \pi_1(U \cap V, x_0)$, and the two ϕ_1 and ϕ_2 maps are inclusions of $U \cap V$ into U and V respectively.

Let $[\omega]$ be the generator of $\pi_1(U \cap V, x_0)$ send to 1 by the bijection between this group and $\mathbb{Z}$, which is the loop that goes once around the central hole. Clearly, we will have $[\omega]$ sent to the trivial loop in $\pi_1(V, x_0)$, as this is a trivial group.

It is also clear that sending $[\omega]$ into U will yield a loop that is path-homotopic to a loop which starts at x_0, goes to x_1, goes around the boundary once, then goes back to x_0. This is isomorphic to the loop in $\pi_1(\partial U, x_1)$ which goes around the loop a, around b, then around a^{-1}, then around b^{-1}. Thus, we can write $aba^{-1}b^{-1} = 1$, where 1 is the trivial loop.

It follows that:

$$\pi_1(T, x_0) = \pi_1(U \cup V, x_0) = \langle a, b \mid ab = ba \rangle = \mathbb{Z} \times \mathbb{Z}$$

4. Covering Spaces Notes and Examples

These notes are on universal covering spaces.

4.1. **Steps for Drawing a Universal Covering Space.** Below is a method for computing the universal covering space of a wedge of two spaces:
   (1) Let $\tilde{X}$ and $\tilde{Y}$ be universal covering spaces of X and Y. Let x be the point at which X and Y are wedged.
   (2) Take a copy of $\tilde{X}$, and note the position of each element of $p^{-1}(x)$. At each of these points, join a copy of $\tilde{Y}$ at some $q^{-1}(x)$ in $\tilde{Y}$.
   (3) For all the unidentified points in each copy of $\tilde{Y}$, join a copy of $\tilde{X}$ at some $p^{-1}(x)$.
   (4) Repeat until termination of infinity!
To find the universal covering spaces of a sphere with a diameter, it is simply a matter of pushing the diameter out onto the surface of the sphere.

5. Other Practice Problems

5.1. Munkres: Problem 57.2.