Crash course in algebraic geometry

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I. Introduction

God help me....

II. Affine algebraic sets

Definition II.1 (Affine algebraic set)**.** A simultaneous zero-set of a collection of polynomials over some field k. If S is a collection of $k[X_1, \ldots, X_n]$, then we let $V(S)$ denote their zero-set/corresponding algebraic set.

It is worthwhile to note that for $F, G \in k[X_1, \ldots, X_n]$, we have $V(FG) = V(F) \cup V(G)$, which is trivial to verify.

Definition II.2. Given some $X \subset \mathbb{A}^n(k)$, let $I(X)$ be the ideal of $F \in k[X_1, \ldots, X_n]$ such that $F|_X = 0$. It is an ideal because it is clearly subring, and given F vanishing on X, so does GF for any $G \in k[X_1, \ldots, X_n]$.

Remark II.1. If X is a set and $F^n \in I(X)$, then $F \in I(X)$. Note that if $F(X_1, \ldots, X_n)^n = 0$ for all $(X_1, \ldots, X_n) \in X$, then $F(X_1, \ldots, X_n) = 0$ for all such points as well. Thus, $I(X)$ is a *radical ideal*, in the sense that it is equal to its radical: the set of all n-th roots of an ideal, which we denote $\mathrm{Rad}(I)$ for arbitrary ideal I.

Theorem II.1 (Hilbert basis theorem)**.** If R is Noetherian (i.e. every ideal is finitely-generated), then $R[X_1, \ldots, X_n]$ is Noetherian.

Corollary II.1.1. Every algebraic set is the intersection of a finite set of hypersurfaces (that is, $V(F)$ for a single polynomial F).

Proof. Every algebraic set $V(S)$ is equal to $V(I)$ for an ideal I, so if $I \subset k[X_1, \ldots, X_n]$ is finitely-generated, then $I = (F_1, \ldots, F_m)$ and $V(I) = V(F_1) \cap \cdots \cap V(F_m)$. Since k is a field, it is a PID so obvious Noetherian and Hilbert basis theorem implies I is Noetherian. $\square$

We call an affine algebraic set V *reducible* if it can be written as a union of proper algebraic subsets of V.

Proposition II.1. An algebraic set is irreducible if and only if $I(V)$ is a prime ideal.

Proof. If $I(V)$ is not prime, so $FG \in I(V)$ with $F, G \notin I(V)$. Then $V = (V \cap V(F)) \cup (V \cap V(G))$ with both subsets being proper algebraic subsets so V is reducible. Conversely, if $V = V_1 \cup V_2$ for proper algebraic subsets, then there necessarily exists some $F \in V_1$ which is not in V_2 and $G \in V_2$ which is not in V_1. Note that FG vanishes on V_1 and V_2, thus on V, then $FG \in I(V)$ with $F, G \notin I(V)$ so $I(V)$ is not prime. $\square$

Theorem II.2. Any affine algebraic set V is the unique union of a finite number of irreducible algebraic subsets $V_1, \ldots, V_m$ such that $V_i \cap V_j^C \neq \emptyset$ for each i and j. We refer to an irreducible algebraic set as an *affine algebraic variety.*

Definition II.3. If V is a variety, then $I(V)$ is prime from Prop. II.1 which implies that $k[X_1, \ldots, X_n]/I(V)$ is a domain (easy algebra fact). We define $\Gamma(V)$ to be this domain and call it the *coordinate ring* of V. It is immediately obvious that we can identify $\Gamma(V)$ with the collection of polynomial functions on V, as two formal polynomials determine the same function if and only if their difference vanishes on V (i.e. the difference is in $I(V)$).

Definition II.4. A map $\varphi : V \to W$ between varieties in $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively is a *polynomial map* if it can be written as $(T_1, \ldots, T_m)$ for $T_j \in k[X_1, \ldots, X_n]$.

Given some map $\varphi : V \to W$, let $\varphi^* : \mathcal{F}(W, k) \to \mathcal{F}(V, k)$ denote the induced homomorphism of rings of functions going from W and V to the field k. This homomorphism has the property that it sends the copy of k inside $\mathcal{F}(W, k)$: the subring of constant functions, to k in $\mathcal{F}(V, k)$. In the specific case that φ is a polynomial map, then $\varphi^*(\Gamma(W)) \subset \Gamma(W)$, when we identify the coordinate ring with the polynomial functions. This means that $\varphi^* : \Gamma(W) \to \Gamma(V)$ is a well-defined ring homomorphism.

Proposition II.2. In the specific case that $V = \mathbb{A}^n$ and $W = \mathbb{A}^m$, and $T_1, \ldots, T_m \in k[X_1, \ldots, X_n]$ determine a polynomial map T, then we can recover the T_j from T (i.e. they are uniquely determined by T).

Proposition II.3. There is a natural 1-to-1 correspondence between polynomial maps $\varphi : V \to W$ between varieties and the homomorphisms $\psi : \Gamma(W) \to \Gamma(V)$ via φ^*.

Proof. **TODO** □

Given a variety V and its coordinate ring $\Gamma(V)$, since it is a domain, we can consider the quotient field $k(V)$ of *rational functions*. Given some $p \in V$, we take $\mathcal{O}_p(V)$ to be the set of rational functions that are defined at p (i.e. f is such that $f = a/b$ for $a, b \in \Gamma(V)$ and $b(p) \neq 0$ for some a and b). One can verify that $\mathcal{O}_p(V)$ is a subring of $k(V)$ which contains $\Gamma(V)$, the polynomial functions.

Definition II.5. We call $\mathcal{O}_p(V)$ the *local ring* of V at p. We also call the ideal $\mathfrak{m}_p(V) = \{f \in \mathcal{O}_p(V) \mid f(p) = 0\}$ inside the local ring the *maximal ideal* of V at p.

Note that $\mathcal{O}_p(V)/\mathfrak{m}_p(V)$ is isomorphic to k, as the maximal ideal is the kernel of the evaluation map $f \mapsto f(p)$, so this follows from the first isomorphism theorem. In addition, note that $f \in \mathcal{O}_p(V)$ is a unit if and only if $f(p) \neq 0$, so $\mathfrak{m}_p(V)$ consists of all non-units of the local ring.

Proposition II.4. The following are equivalent:

1. The set of non-units of ring R form an ideal.

2. R has a unique maximal ideal that contains every proper ideal of R.

When a ring satisfies either of these equivalent criteria, we call it *local*. This justifies our calling $\mathcal{O}_p(V)$ *the* local ring of V and p and $\mathfrak{m}_p(V)$ the maximal ideal: the above proposition implies that $\mathfrak{m}_p(V)$ is the unique maximal ideal of the ring $\mathcal{O}_p(V)$.

Remark II.2. All of the properties of V which depend only on a neighbourhood of p are reflected in $\mathcal{O}_p(V)$, hence the name.

Proposition II.5. Let R be a domain (but not a field), then the following are equivalent:

1. R is Noetherian and local and the maximal ideal is principal.

2. There is an irreducible $t \in R$