Fall 2023 MAT437 problem set 1

Jack Ceroni*
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I. Problem 1

Part 1. It is clear that $\widetilde{A}$ is an algebra over the commutative ring $\mathbb{C}$, with the defined operations. Verification that $\widetilde{A}$ is a $*$-algebra over $\mathbb{C}$ follows from verifying that the $*$-operation on $\widetilde{A}$ is in fact a valid involution and anti-automorphism on the ring $\widetilde{A}$, as well as that $(\lambda a)^* = \overline{\lambda} a^*$, for $\lambda \in \mathbb{C}$, $a \in \widetilde{A}$. Indeed, the latter property is easy to verify,

$$(\lambda(a,\alpha))^* = (\lambda a, \lambda\alpha)^* = ((\lambda a)^*, (\lambda\alpha)^*) = (\overline{\lambda} a^*, \overline{\lambda}\overline{\alpha}) = \overline{\lambda}(a^*, \overline{\alpha}) = \overline{\lambda}(a,\alpha)^*. \tag{1}$$

All that remains is to check that the desired properties of $*$ hold. Indeed, almost trivially,

$$(a+b, \alpha+\beta)^* = ((a+b)^*, \overline{\alpha+\beta}) = (a^* + b^*, \overline{\alpha} + \overline{\beta}) = (a,\alpha)^* + (b,\beta)^*, \tag{2}$$

$$((a,\alpha)^*)^* = (a^*, \overline{\alpha})^* = (a,\alpha). \tag{3}$$

The only condition which is slightly non-trivial is checking the reversal of multiplication of elements of $\widetilde{A}$ under the $*$-operation. This follows from the definition,

$$((a,\alpha)\cdot(b,\beta))^* = (ab + \beta a + \alpha b, \alpha\beta)^* = ((ab + \beta a + \alpha b)^*, \overline{\alpha}\overline{\beta}) = ((ab)^* + \overline{\beta}a^* + \overline{\alpha}b^*, \overline{\alpha}\overline{\beta}) \tag{4}$$

$$= (b^*a^* + \overline{\alpha}b^* + \overline{\beta}a^*, \overline{\beta}\overline{\alpha}) = (b,\beta)^* \cdot (a,\alpha)^*. \tag{5}$$

This completes the proof that $\widetilde{A}$ is a $*$-algebra. Clearly, $(0,1)\cdot(a,\alpha) = (a,\alpha) = (a,\alpha)\cdot(0,1)$, immediately from the definition of the multiplication, and $(0,1)^* = (0,1)$.

Clearly, ι and π are linear. They are also multiplicative:

$$\iota(ab) = (ab, 0) = (a,0)\cdot(b,0) = \iota(a)\cdot\iota(b) \tag{6}$$

$$\pi((a,\alpha)\cdot(b,\beta)) = \pi(ab + \alpha b + \beta a, \alpha\beta) = \alpha\beta = \pi(a,\alpha)\pi(b,\beta) \tag{7}$$

In addition, $\iota(a^*) = (a^*, 0) = (a,0)^*$, and it also has trivial kernel, so it is an injective $*$-homomorphism. Similarly, $\pi((a,\alpha)^*) = \pi(a^*, \overline{\alpha}) = \overline{\alpha}$, and given $\alpha \in \mathbb{C}$, $\pi(a,\alpha) = \alpha$ for some $a \in A$ (of course, assuming A is non-empty), so π is a surjective $*$-homomorphism.

Part 2. Note that given $a \in A$ and $x \in \widetilde{A}$, we define $||ax||_A$ as $||ax_1||_A$, where $x = (x_1, x_2)$, as $ax = (ax_1, 0) \simeq ax_1$. Note that

$$||a||_{\widetilde{A}} = \max\left\{|||a||||_{\widetilde{A}}, |\pi(a)|\right\} = \max\left\{|||a||||_{\widetilde{A}}, 0\right\} \tag{8}$$

where we also know that

$$|||a||||_{\widetilde{A}} = \sup\{||a'a||_A,\ a' \in A, ||a'||_A \leq 1\} = \sup(S_a). \tag{9}$$

Note that $||a'a||_A \leq ||a'||_A||a||_A \leq ||a||_A$ for any $||a'|| \leq 1$, so $||a||_A$ is an upper-bound on S_a. In case $a = 0$, then automatically $|||a|||_{\widetilde{A}} = ||a||_A = 0$ and $||a||_{\widetilde{A}} = ||a||_A$. Otherwise, when $a \neq 0$, note that $b = a^*/||a||_A$ is contained in A and has unit norm. In addition,

$$\left|\left|\left(\frac{a^*}{||a||_A}\right)a\right|\right| = \frac{1}{||a||_A}||a^*a||_A = \frac{||a||_A^2}{||a||_A} = ||a||_A \tag{10}$$

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* jackceroni@gmail.com

so $||a||_A \in S_a$. Thus, $|||a||||_{\widetilde{A}} = \sup(S_a) = ||a||_A > 0$, so $||a||_{\widetilde{A}} = \max\{||a||_A, 0\} = ||a||_A$, and the proof is complete.

Part 3. If $||x||_{\widetilde{A}} = 0$, then $\pi(x) = 0$, so $x \in A$. Thus, $||x||_{\widetilde{A}} = ||x||_A = 0$. Since $||\cdot||_A$ is a valid norm, $x = 0$.

Part 4. We have shown positive-definiteness of the norm above (clearly, $||0||_{\widetilde{A}} = ||0||_A = 0$ as well, so $||x||_{\widetilde{A}} = 0 \Leftrightarrow x = 0$). Even easier, non-negativity follows from the fact that $|\pi(x)| \geq 0$, and $||x||_{\widetilde{A}} \geq |\pi(x)| \geq 0$. Continuing on, note that $|\pi(\lambda x)| = |\lambda||\pi(x)|$ for $\lambda \in \mathbb{C}$, and since $||a(\lambda x)||_A = ||\lambda(ax)||_A = |\lambda|||ax||_A$, it clearly follows that $\sup(S_{\lambda x}) = |\lambda| \sup(S_x)$. Thus, since we take the maximum of these two quantities, it is easy to see that $||\lambda x||_{\widetilde{A}} = |\lambda|||x||_{\widetilde{A}}$.

Finally, note that $|\pi(x+y)| = |\pi(x)+\pi(y)| \leq |\pi(x)|+|\pi(y)|$. In addition, we know that for $a \in A$, $x, y \in \widetilde{A}$, $||a(x+y)||_A = ||ax + ay||_A \leq ||ax||_A + ||ay||_A$. Therefore,

$$\sup(S_{x+y}) = \sup\{||a(x+y)||_A,\ a \in A, ||a||_A \leq 1\} \leq \sup\{||ax||_A + ||ay||_A,\ a \in A, ||a||_A \leq 1\} \tag{11}$$
$$\leq \sup(S_x) + \sup(S_y) \tag{12}$$

so it follows that $|||\cdot|||_{\widetilde{A}}$ also satisfies the triangle inequality. Thus,

$$||x+y||_{\widetilde{A}} = \max\left\{|\pi(x+y)|, |||x+y|||_{\widetilde{A}}\right\} \leq \max\left\{|\pi(x)| + |\pi(y)|, |||x|||_{\widetilde{A}} + |||y|||_{\widetilde{A}}\right\} \tag{13}$$
$$\leq \max\left\{|\pi(x)|, |||x|||_{\widetilde{A}}\right\} + \max\left\{|\pi(y)|, |||y|||_{\widetilde{A}}\right\} \tag{14}$$
$$= ||x||_{\widetilde{A}} + ||y||_{\widetilde{A}}. \tag{15}$$

Therefore, $||\cdot||_{\widetilde{A}}$ is a valid norm.

To verify the other properties of this norm (so that $\widetilde{A}$ is a valid C^*-algebra), we note that $|\pi(xy)| = |\pi(x)\pi(y)| = |\pi(x)||\pi(y)|$, as well as $|\pi(x^*x)| = |\pi(x^*)\pi(x)| = |\pi(x)^*\pi(x)| = |\pi(x)|^2$, as π is a $*$-homomorphism.

Additionally, note that for $x, y \in \widetilde{A}$,

$$\sup(S_{xy}) = \sup\{||axy||_A,\ a \in A, ||a||_A \leq 1\} = |||x|||_{\widetilde{A}} \sup\{\left|\left|\frac{axy}{|||x|||_{\widetilde{A}}}\right|\right|_A,\ a \in A, ||a||_A \leq 1\} \tag{16}$$

Note that

$$\left|\left|\frac{ax}{|||x|||_{\widetilde{A}}}\right|\right|_A = \frac{||ax||_A}{\sup(S_x)} \leq 1 \tag{17}$$

for some particular a such that $||a||_A \leq 1$, as $\sup(S_x)$ is the supremum of this quantity over *all* such a, so it follows from above that

$$|||x|||_{\widetilde{A}} \sup\{\left|\left|\frac{axy}{|||x|||_{\widetilde{A}}}\right|\right|_A,\ a \in A, ||a||_A \leq 1\} \leq |||x|||_{\widetilde{A}} \sup\{||ay||_A,\ a \in A, ||a||_A \leq 1\} = |||x|||_{\widetilde{A}}|||y|||_{\widetilde{A}}. \tag{18}$$

Thus, via an identical argument to the triangle inequality proof, it follows from the inequalities proved for $|\pi(xy)|$ and $|||xy|||_{\widetilde{A}}$ that $||xy||_{\widetilde{A}} \leq ||x||_{\widetilde{A}}||y||_{\widetilde{A}}$ as well. Finally, it is clearly true that $|\pi(x^*x)| = |\pi(x)|^2$. In addition,

$$\sup(S_x)^2 = \sup\{||ax||_A^2,\ a \in A, ||a||_A \leq 1\} = \sup\{||axx^*a^*||_A,\ a \in A, ||a||_A \leq 1\} \tag{19}$$
$$\leq \sup\{||axx^*||_A||a^*||_A,\ a \in A, ||a||_A \leq 1\} \tag{20}$$
$$\leq \sup\{||axx^*||_A,\ a \in A, ||a||_A \leq 1\} = \sup(S_{xx^*}) \tag{21}$$

so $||||x|||_{\widetilde{A}}^2 \leq |||xx^*|||_{\widetilde{A}}$. Using the previous submultiplicative property we proved, $|||xx^*|||_{\widetilde{A}} \leq |||x|||_{\widetilde{A}}|||x^*|||_{\widetilde{A}}$. Thus, combing the two inequalities, $|||x|||_{\widetilde{A}} \leq |||x^*|||_{\widetilde{A}}$. Taking the conjugate of x, $|||x^*|||_{\widetilde{A}} \leq |||x|||_{\widetilde{A}}$, so $|||x|||_{\widetilde{A}} = |||x^*|||_{\widetilde{A}}$, the two previous inequalities become equalities, giving $||||x|||_{\widetilde{A}}^2 = |||xx^*|||_{\widetilde{A}}$. Therefore, combining this with the property of π mentioned above the multi-line equation above, $||x^*x||_{\widetilde{A}} = ||x||_{\widetilde{A}}$, as we are just taking the max of these two quantities.

The last claim that remains to be verified is that the space $\widetilde{A}$ is complete. Let us assume that the sequence $x_n \in \widetilde{A}$ is Cauchy with respect to $||\cdot||_{\widetilde{A}}$. Then it follows that both $|||x_n - x_m|||_{\widetilde{A}}$ and $|\pi(x_n - x_m)| = |\pi(x_n) - \pi(x_m)|$ become arbitrarily small, for m, n sufficiently large (as their max does). It follows that the sequence of complex numbers $\pi(x_n)$ is Cauchy, so it converges to some point x^*, as $\mathbb{C}$ is complete. Thus, we define $x'_n = x_n - \pi(x_n) \in A$, so $x_n = x'_n + \pi(x_n)$, and we have

$$||x'_n - x'_m||_A = ||x'_n - x'_m||_{\widetilde{A}} \leq ||x_n - x_m||_{\widetilde{A}} + |\pi(x_n) - \pi(x_m)| \tag{22}$$

Since with sufficiently large m and n, we can make $||x_n - x_m||_{\widetilde{A}}$ and $|\pi(x_n) - \pi(x_m)|$ arbitrarily small, it follows that the sequence of $x'_n \in A$ is Cauchy as well. Since A is complete, the sequence x'_n converges, $x'_n \to x'$. We claim that $x_n \to x' + x^*$. Indeed, note that

$$||x_n - (x' + x^*)||_{\widetilde{A}} = ||(x'_n + \pi(x_n)) - (x' + x^*)||_{\widetilde{A}} \leq ||x'_n - x'||_A + |\pi(x_n) - x^*| \tag{23}$$

where we can make both terms of the last sum arbitrarily small with large enough n. It follows by definition that $x_n \to x' + x^*$, so the Cauchy sequence converges, and $\widetilde{A}$ is complete with respect to $||\cdot||_{\widetilde{A}}$.

Part 5. This sequence is clearly exact: we know that ι is injective, so it has trivial kernel, so $0 \subset \ker(\iota)$. The image of the inclusion ι is $A \times \{0\}$, which is sent to 0 by π, so $\mathrm{Im}(\iota) \subset \ker(\pi)$. Finally, the image of π is the complex numbers, which are all sent to 0 by the zero map, so $\mathrm{Im}(\pi) \subset \ker(0) = \mathbb{C}$. To verify that the sequence is split exact, we must show the existence of a $*$-homomorphism λ such that $\pi \circ \lambda = \mathrm{id}_{\mathbb{C}}$. Indeed, let $\lambda(b) = (0, b) \in \widetilde{A}$. It is clear (nearly trivial to verify) that such a map is a $*$-homomorphism, and moreover, it is clear that $\pi \circ \lambda$ is the identity on $\mathbb{C}$. Thus, the sequence is split exact.

Part 6. Verifying that $A \oplus \mathbb{C}$ is a C^*-algebra with respect to the max norm of RLL follows similarly from the above arguments, so we will not repeat them.

Suppose A is not unital. Then $A \oplus \mathbb{C}$ is not unital, as if $a \in A \oplus \mathbb{C}$ were a unit, then given $a' \in A$, we would have

$$(a', 0) \cdot a = (a', 0) \cdot (a_1, a_2) = (a'a_1, 0) = (a_1, a_2) = a \cdot (a', 0) \tag{24}$$

so $a'a_1 = a_1 = a_1 a'$, and a_1 would be a unit in A, a contradiction.

Suppose A is unital. Define the map φ going from $A \oplus \mathbb{C}$ to $\widetilde{A}$ taking $(a, \alpha) \in A \oplus \mathbb{C}$ to $a + \alpha p = a + \alpha(1_{\widetilde{A}} + 1_A)$, where $1_{\widetilde{A}}$ is the unit in $\widetilde{A}$ and 1_A is the unit in A. Verifying that this map is linear is trivial. Note that $1_{\widetilde{A}} \cdot 1_A = 1_A$, $1_A^2 = 1_A$, and $1_{\widetilde{A}}^2 = 1_{\widetilde{A}}$. Thus, $p^2 = p$, when we expand and apply these identities. Also note that for $x \in A$, $px = p(x_1, 0) = 0 = xp$. It follows that

$$\varphi((a, \alpha) \cdot (b, \beta)) = \varphi(ab, \alpha\beta) = ab + \alpha\beta p = ab + \alpha pb + \beta ap + \beta\alpha p^2 = (a + \alpha p)(b + \beta p) = \varphi(a, \alpha)\varphi(b, \beta). \tag{25}$$

Thus, the map is multiplicative. Note that φ is a $*$-homomorphism, as $\varphi((a, \alpha)^*) = \varphi(a^*, \overline{\alpha}) = a^* + \overline{\alpha}p = (a + \alpha p)^* = \varphi(a, \alpha)^*$, where we know that $1_{\widetilde{A}}^* = 1_{\widetilde{A}}$ and $1_A^* = 1_A$, so $p^* = p$. If $a + \alpha p = 0$, then $a - \alpha 1_A = -\alpha 1_{\widetilde{A}}$ where the LHS is in A and the RHS is not in A, so $\alpha = 0$, and thus $a = 0$, so φ is injective (As it has trivial kernel). It is also obviously surjective. Thus, we have a $*$-isomorphism between C^*-algebras, so the two are isomorphic, as desired.

I was attempting a to prove a claim which was left unproven in the book. I didn't manage to succeed (yet), but I've turned my (potentially misguided) strategy into another (likely useless) result, because I thought the proof method was somewhat interesting. I also realized after reading a bit of Chapter 2 of RLL, after doing this proof, that this sort of proof technique may actually come up again (as there seemed to be some kind of theorem which uses similar logic).

Proposition I.1. Given a two-sided algebraic ideal I of a C^*-algebra A (I is a sub-algebra over $\mathbb{C}$ and closed under multiplication from both sides by elements of A) which is also topologically closed and $a \in I$ with $||a|| < 1$, there exists an element $\widetilde{a} \in I$ such that $\widetilde{a}a = \widetilde{a} - a$.

Proof. Recall that the space is topologized with metric $d(a, b) = ||a - b||$, where $||\cdot||$ is the norm on the space.

Now, suppose x is some element of the ideal such that $||x|| < 1$. In this case, note that the geometric series $\sum_{j=1}^{\infty} ||x||^j$ converges to some constant C. Note that $x_n = x + x^2 + \cdots + x^n$ are elements of I, for all $n \geq 1$. In addition, assume that $n > m$, and note that

$$||x_n - x_m|| = ||x^n + x^{n-1} + \cdots + x^{m+1}|| \leq ||x||^m ||x + \cdots + x^{n-m}|| \tag{26}$$

$$\leq ||x||^m (||x|| + \cdots + ||x||^{n-m}) \leq C||x||^m. \tag{27}$$

It follows that the sequence (x_n) is Cauchy, and since the underlying space is complete, $x_n \to \widetilde{x}$ (the sequence has a (unique, as we're in a metric space) limit point $\widetilde{x}$). Since I is topologically closed, $\widetilde{x} \in I$.

Let us now return back to our original problem. Since $||a|| < 1$, the element $\widetilde{a}$ will be in I as well. Note that $a \cdot a_n = a_{n+1} - a$. Since the map $a_n \mapsto a \cdot a_n$ is continuous, it follows that the sequence of $a \cdot a_n$ has $a \cdot \widetilde{a}$ as its unique limit point, while $a_{n+1} - a$ has $\widetilde{a} - a$ as its unique limit point, so $a\widetilde{a} = \widetilde{a} - a$. □

II. Problem 2

Part A. In the case that A does not have unit, we adjoin one. The element p is of course identified with $p + 0 \cdot 1$ is the larger space. Suppose $\lambda \notin \{0, 1\}$. It is easy to demonstrate that $p - \lambda \cdot 1$ is invertible: note that $\lambda(1 - \lambda) \neq 0$, so

$$(p - \lambda \cdot 1) \left(\frac{1}{\lambda(1-\lambda)} p - \frac{1}{\lambda} \cdot 1 \right) = \left(\frac{p^2 - \lambda p - (1-\lambda)p}{\lambda(1-\lambda)} \right) + 1 = 1 \tag{28}$$

as $p^2 = p$.

Part B. Consider the operator $p' = p^2 - p$. Note from the quadratic formula that for some $\lambda \in \mathbb{C}$,

$$p' - \lambda = p^2 - p - \lambda = \left(p - \frac{1 + \sqrt{1 + 4\lambda}}{2} \right) \left(p - \frac{1 - \sqrt{1 + 4\lambda}}{2} \right) = (p - h_+(\lambda))(p - h_-(\lambda)) \tag{29}$$

It is easy to see that when $\lambda \neq 0$, $h_\pm(\lambda) \notin \{0, 1\}$. Since $\mathrm{sp}(p) \subset \{0, 1\}$, it follows that $p - h_+(\lambda)$ and $p - h_-(\lambda)$ are both invertible, so $p' - \lambda$ is invertible. Thus, $\mathrm{sp}(p') \subset \{0\}$. Since p is normal, it is easy to verify that $p' = p^2 - p$ is as well. It follows that $||p'|| = r(p') = 0$, where r is the spectral radius (this is a result from RLL). It follows that $p' = 0$, by definition of the norm, so $p^2 = p$.

From RLL, since we know that p is a positive operator (as it is normal with spectrum in $\mathbb{R}^+$), it follows that $p = x^* x$ for some $x \in A$. Thus, $p^* = (x^* x)^* = x^* x = p$. We have $p^2 = p = p^*$, so p is a projection, as desired.

Part C. First, note that $||u||^2 = ||u^* u|| = ||1|| = 1$, so $||u|| = 1$. It follows that $r(u) = 1$, as u is clearly normal, so for any $\lambda \in \mathrm{sp}(\lambda)$, we have $|\lambda| \leq 1$. Clearly, u is invertible, as $uu^* = u^* u = 1$, so $0 \notin \mathrm{sp}(u)$. Thus, if $\lambda \in \mathrm{sp}(u)$, $|\lambda| \in (0, 1]$. Suppose λ is in the spectrum with $0 < |\lambda| < 1$. Then $u - \lambda$ is not invertible. Since u^* is invertible, $(u - \lambda)u^*$ is not invertible. We have

$$(u - \lambda)u^* = uu^* - \lambda u^* = 1 - \lambda u^* = -\lambda \left(u^* - \frac{1}{\lambda} \right) \tag{30}$$

Thus, $u^* - \lambda^{-1}$ is not invertible, so $\lambda^{-1} \in \mathrm{sp}(u^*)$. Of course, this implies that $u - (\lambda^{-1})^*$ is non-invertible, so $(\lambda^{-1})^* \in \mathrm{sp}(u)$. But this can't be, as $|(\lambda^{-1})^*| = |\lambda^{-1}| > 1$, a clear contradiction. Thus, if $\lambda \in \mathrm{sp}(u)$, we must have $|\lambda| = 1$, so by definition, $\mathrm{sp}(u) \subset \mathbb{T}$, as desired.

Before proceeding, let us make a brief interlude to present important results from RLL with some notation that we will make use of going forward.

Lemma II.1 (The continuous function calculus)**.** Given a unital C^*-algebra A, associated to each normal element a is a unique $*$-isomorphism $\Phi_a : C(\mathrm{sp}(a)) \to C^*(a, 1) \subset A$ such that when $p : \mathrm{sp}(a) \to \mathbb{C}$ is a polynomial, $\Phi_a(p) = p(a)$ and when $p(s) = \overline{s}$, $\Phi_a(p) = a^*$.

Theorem II.1 (Spectral mapping theorem)**.** For every normal element a of a unital C^*-algebra A, and every continuous function $f : \mathrm{sp}(a) \to \mathbb{C}$, $\mathrm{sp}(\Phi_a(f)) = f(\mathrm{sp}(a))$.

Theorem II.2. Let K be a non-empty subset of $\mathbb{C}$, let $f : K \to \mathbb{C}$ be a continuous function. Let A be a unital C^*-algebra, let Ω_K be the set of normal elements with spectrum in K. Then the function $\Phi : \Omega_K \to A$ where $\Phi(a) = \Phi_a(f)$ is continuous.

Now, we can continue with the problem.

Part D. We can use Thm. II.1. In particular, define $g(z) = \overline{z}z - 1 = h(z) \cdot \mathrm{id}(z) - 1$ acting on $\mathrm{sp}(u) = \mathbb{T}$. From Lem. II.1, $\Phi_u(g) = \Phi_u(h)\Phi_u(\mathrm{id}) - \Phi_u(1) = u^*u - 1$. Clearly, g is continuous, so by Thm. II.1, $\mathrm{sp}(\Phi_u(g)) = \mathrm{sp}(u^*u - 1) = g(\mathbb{T}) = \{0\}$. Thus, since $u^*u - 1$ is normal, $||u^*u - 1|| = r(u^*u - 1) = 0$, so $u^*u = 1$. Since u is normal, $u^*u = uu^* = 1$, and u is unitary, as desired.

III. Problem 3

Part 1. If a is invertible, it follows that a^* is invertible: note that $(a^{-1})^*a^* = (aa^{-1})^* = 1^* = 1$ and $a^*(a^{-1})^* = (a^{-1}a)^* = 1^* = 1$, so $(a^{-1})^* = (a^*)^{-1}$. Thus, both a^*a and aa^* must be invertible as well, as they are products of invertible elements.

Conversely, if a^*a and aa^* are invertible, then $[(a^*a)^{-1}a^*]a = 1$ and $a[a^*(aa^*)^{-1}] = 1$ (by associativity), so we have left and right-inverses for a. Of course, if $ab = ca = 1$, then $b = (ca)b = c(ab) = c$, so there exists a single element a^{-1} which is the inverse of a, and $a^{-1} = a^*(aa^*)^{-1} = (a^*a)^{-1}a^*$ (this equality, of course, holds in the former case as well, when we begin by assuming a is invertible).

Part 2. This result follows from the use of Lem. II.1. We must show that for invertible a, there exists a function $f \in C(\mathrm{sp}(a))$ such that $\Phi_a(f) = a^{-1}$. Let $f(x) = x^{-1}$. Since a is invertible, $0 \notin \mathrm{sp}(a)$, so f is well-defined and continuous. Our claim is that $\Phi_a(f) = a^{-1}$. Indeed, note that $\Phi_a(f)a = \Phi_a(f)\Phi_a(\mathrm{id}) = \Phi_a(f \cdot \mathrm{id}) = \Phi_a(1) = 1$, so this is in fact true.

Part 3. Note that for some $a \in A$, a^*a is self-adjoint, and thus clearly normal. It follows that $\Phi_{a^*a} : C(\mathrm{sp}(a^*a)) \to C^*(a^*a, 1)$ is well-defined, and using the same notation as above, $\Phi_{a^*a}(f) = (a^*a)^{-1}$ and is contained in $C^*(a^*a, 1)$. Since a is invertible, $a^{-1} = (a^*a)^{-1}a^*$, and is contained in $C^*(a^*a, 1) \cdot a^*$: the smallest C^*-algebra generated by a^*a and 1, right-multiplied by a^*. Recall that $C^*(a^*a, 1)$ is the closure of all words in a^*a, $(a^*a)^* = a^*a$, and 1, $\overline{W}$. The map $x \mapsto x \cdot a^*$ is clearly continuous, so

$$C^*(a^*a, 1) \cdot a^* = f(C^*(a^*a, 1)) = f(\overline{W}) \subset \overline{f(W)} \tag{31}$$

which is the closure of all words generated by a^*a and 1, left-multiplied by a^*, which is clearly the closure over all words generated by a^*aa^* and a^*. This set is, of course, contained in the closure over all words generated by a and a^*, which is equal to $C^*(a)$, so $a^{-1} \in C^*(a)$.

IV. Problem 4

Part 1. Suppose $\lambda \in \mathrm{sp}(\varphi(a))$. Then $\varphi(a) - \lambda \cdot 1 = \varphi(a - \lambda \cdot 1)$ is non-invertible. Suppose that $a - \lambda \cdot 1$ were invertible with inverse b. Then $\varphi(b)\varphi(a - \lambda \cdot 1) = \varphi(b(a - \lambda \cdot 1)) = \varphi(1) = 1$, contradicting the non-invertibility of $\varphi(a) - \lambda \cdot 1$. Thus, $a - \lambda$ must be non-invertible and $\lambda \in \mathrm{sp}(a)$. It follows that $\mathrm{sp}(\varphi(a)) \subset \mathrm{sp}(a)$.

I haven't quite been able to demonstrate that when φ is injective, the two spectra are equal. For the subsequent sections of this problem, I will assume this condition holds, and continue working on my proof, with the goal of submitting it next week.

Part 2. Note that a^*a is self-adjoint and thus clearly normal. We then have

$$||\varphi(a)||^2 = ||\varphi(a)^*\varphi(a)|| = ||\varphi(a^*)\varphi(a)|| = ||\varphi(a^*a)|| = r(\varphi(a^*a)). \tag{32}$$

Because $\mathrm{sp}(\varphi(a^*a)) \subset \mathrm{sp}(a^*a)$, it follows from the definition that $r(\varphi(a^*a)) \leq r(a^*a) = ||a^*a|| = ||a||^2$. In other words, $||\varphi(a)||^2 \leq ||a||^2$, so $||\varphi(a)|| \leq ||a||$.

In the case that φ is injective, $\mathrm{sp}(\varphi(a^*a)) = \mathrm{sp}(a^*a)$ and $r(\varphi(a^*a)) = r(a^*a)$ and the above inequality becomes an equality.

Part 3. In this case, we form the unitizations $\widetilde{A}$ of A and $\widetilde{B}$ of B. Via RLL, there is a unique lifting of $*$-homomorphism φ to $\widetilde{\varphi}$ between the unitized C^*-algebras, such that $\widetilde{\varphi}(a + \alpha \cdot 1) = \varphi(a) + \alpha \cdot 1$. Note that we will have $||\widetilde{\varphi}(x)||_{\widetilde{A}} \leq ||x||_{\widetilde{A}}$ for all $x \in \widetilde{A}$. From Problem 1 Part 2, if $a \in A$ is identified with $a + 0 \cdot 1 \in \widetilde{A}$, we have $||a||_{\widetilde{A}} = ||a||_A$. Thus, for $a \in A$,

$$||\varphi(a)||_B = ||\varphi(a)||_{\widetilde{B}} \leq ||a||_{\widetilde{A}} = ||a||_A \tag{33}$$

We use the exact same argument to prove equality when φ is injective: clearly $\widetilde{\varphi}$ will also be injective as if $\widetilde{\varphi}(x) = \varphi(a) + \alpha \cdot 1 = 0$, then $\alpha = 0$ and $a = 0$. Thus, $||x||_{\widetilde{A}} = ||\widetilde{\varphi}(x)||_{\widetilde{A}}$, and the inequalities in the above equation become equalities.

Part 4. Clearly, $\varphi(A)$ is closed under $*$, addition, multiplication, and scaling, so $\varphi(A)$ is clearly a $*$-algebra.

Consider the quotient $A/\ker(\varphi)$: from RLL, we can make this space a C^*-algebra with the norm $||a + \ker(\varphi)|| = \inf\{||a + x||,\ x \in \ker(\varphi)\}$. From the first isomorphism theorem, there exists a unique $*$-homomorphism $\varphi_0 : A/\ker(\varphi) \to B$ between C^*-algebra $A/\ker(\varphi)$ and C^*-algebra B such that φ_0 is injective. Thus, $||\varphi_0(x)|| = ||x||$. Note that $||\varphi_0(x) - \varphi_0(y)|| = ||\varphi_0(x - y)|| = ||x - y||$. Suppose $\varphi_0(x_n)$ is a sequence of points of $\varphi_0(A/\ker(\varphi))$ which converges, so it is Cauchy, so by the isometric nature of φ, the sequence of $x_n \in A/\ker(\varphi)$ is Cauchy, convering to $x \in A/\ker(\varphi)$, as this space is a C^* algebra. Again from isometry, it is easy to see that $\varphi_0(x_n) \to \varphi_0(x)$, so $\varphi_0(A/\ker(\varphi_0))$ is closed in B.

By definition of the induced map, it is easy to check that $\varphi_0(A/\ker(\varphi_0)) = \varphi(A)$. Thus, $\varphi(A)$ is norm-closed and a $*$-algebra, so it is a C^*-algebra.

V. Problem 6

Part 1. We can use the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{34}$$

It is a well-known result (in quantum information theory, specifically, and probably in algebra broadly) that these matrices obey the desired anti-commutation relation.

Part 2. This follows essentially from the definition. Note that

$$e^2 - e = e(e-1) = \left(\frac{1+F}{2}\right)\left(\frac{F-1}{2}\right) = \frac{F^2 - 1}{4} \tag{35}$$

Note from the anti-commutation relation that $\sigma_j^2 = \mathrm{id}$. Thus,

$$F^2 = (x_1^2 + x_2^2 + x_3^2) + x_1x_2\{\sigma_1, \sigma_2\} + x_2x_3\{\sigma_2, \sigma_3\} + x_1x_3\{\sigma_1, \sigma_3\} = x_1^2 + x_2^2 + x_3^2 = 1 \tag{36}$$

where $\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i$. It follows that $e^2 - e = (F^2 - 1)/4 = 0$, so e is idempotent.

Part 3. Clearly, each vector space associated with some x, of the bundle E, has dimension $2 = \dim(\mathbb{C}^2)$. Because e is idempotent, $e(e-1) = 0$, so since the minimal polynomial divides $p(x) = x(x-1)$, e must be diagonalizable with eigenvalues 0 and 1 (the sizes of the Jordan blocks associated with 0 and 1, if they exist, can be at most one). Thus e can either be 0, the identity, or have both 0 and 1 as eigenvalues.

Claim V.1. Any matrices $\sigma_1, \sigma_2, \sigma_3$ satisfying the anti-commutation relation are linearly independent along with the identity matrix id.

Proof. Suppose $c_0 + c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3 = 0$. Then $c_0\sigma_1 + c_1 + c_2\sigma_2\sigma_1 + c_3\sigma_3\sigma_1 = 0$ and $c_0\sigma_1 + c_1 + c_2\sigma_1\sigma_2 + c_3\sigma_1\sigma_3 = 0$. Summing these two equations, and applying the anti-commutation relation, $2c_0\sigma_1 + 2c_1 = 0$, so $c_0\sigma_1 = -c_1$. Note that $\sigma_1 \neq c \cdot \mathrm{id}$ for some constant c, as this would imply that $\sigma_2 = \sigma_3 = 0$ (by the anti-commutation relations), which would violate the other anti-commutation relations. Thus, $c_1 = 0$, and we have $c_0 + c_2\sigma_2 + c_3\sigma_3 = 0$.

We repeat the same argument to show that $c_2 = c_3 = 0$. Thus, $c_0 = 0$ as well, and the matrices are linearly independent. $\square$

From here, it is clear that $e(x_1, x_2, x_3) \neq 0, \mathrm{id}$ for any $x \in S^2$, as if $e = 0$, then $F = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = -\mathrm{id}$, where at least one $x_j \neq 0$, violating linear independence. Similarly, if $e = 1$, then $F = 1$, and the same logic applied. Thus, e has 0 and 1 as eigenvalues, and since it is a 2×2 matrix, it follows that its rank is precisely 1, as the dimension of its kernel is 1.

For some x, $1 - e(x)$ is a map from $\mathbb{C}^2$ to itself. In fact, $E = \ker(1 - e(x))$, where the map $1 - e(x)$ also clearly has one eigenvalue equal to 1, and one equal to 0, so $\dim\ker(1 - e(x)) = 1 = \dim(E)$.

I ran out of time, and didn't finish Part 4 and Part 5, but I will try to finish and submit these with next week's problems.