# A basic course in differential geometry

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**Contents**



## I. Introduction

These notes should outline the bare minimum needed to understand differential geometry.

## II. Basics of smooth manifolds

We begin with a definition:

**Definition II.1** (Topological manifold)**.** A space $M$ is said to be a topological manifold if it is second-countable Hausdorff and is locally Euclidean.

TODO: Fill in more basic info on smooth manifolds, partitions of unity

**Proposition II.1** (Building a manifold)**.** This is sometimes called the smooth manifold chart lemma. Let $M$ be a set (not necessarily having any manifold structure), let $\{U_\alpha\}$ be a collection of subsets of $M$ together with maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ where:

1. $\varphi_\alpha$ is a bijection between $U_\alpha$ and an open subset of $\mathbb{R}^n$.
2. $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.
3. Each $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth, in the Euclidean sense.
4. Countably many $U_\alpha$ cover $M$

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5. For distinct $p$ and $q$ in $M$, $p, q \in U_\alpha$ or $p$ and $q$ lie in disjoint $U_\alpha$ and $U_\beta$.

Then, $M$ has a unique smooth manifold structure where each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

*Proof.* The idea is to take all sets of the form $\varphi_\alpha^{-1}(V)$ for $V$ open in $\mathbb{R}^n$ as a basis. Note that since each $\varphi_\alpha(U)$ is open with $\varphi_\alpha$ a bijection to its image, each $U_\alpha$ is in the basis, so every $p \in M$ is in a basis element. Moreover, suppose we choose two basis elements $\varphi_\alpha^{-1}(V)$ and $\varphi_\beta^{-1}(W)$. Note that

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}\left(V \cap (\varphi_\alpha \circ \varphi_\beta^{-1})(W)\right) \tag{1}$$

where $V \cap (\varphi_\alpha \circ \varphi_\beta^{-1})(W)$ is open from the smoothness condition. In particular, $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth and both the domain and codomain are themselves open. Thus, we have a valid basis. This basis can be made countable as for each of the countably many $U_\alpha$ which cover $M$, we take $\varphi_\alpha^{-1}$ of all rational radius balls centred at rational points in $\varphi_\alpha(U_\alpha)$. The Hausdorff condition is trivial from the final criterion. Of course, $(U_\alpha, \varphi_\alpha)$ forms a smooth coordinate chart. □

### A. Mappings on manifolds

**Definition II.2** (Smooth function)**.** Let $M$ be a smooth manifold with smooth structure $\mathcal{A}$. Let $f : M \to \mathbb{R}$ be a function. We say that $f$ is smooth at $p \in M$ if there exists a coordinate chart $(U, \varphi) \in \mathcal{A}$ around $p$ such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth (in the Euclidean sense, as $\varphi(U) \subset \mathbb{R}^n$).

**Definition II.3** (Smooth map)**.** Let $M$ and $N$ be smooth manifolds with structures $\mathcal{A}$ and $\mathcal{A}'$. Let $F : M \to N$ be a map, we say that $F$ is smooth at $p$ if we can choose chart $(U, \varphi) \in \mathcal{A}$ and $(V, \theta) \in \mathcal{A}'$ around $p$ and $F(p)$ repsectively, such that $F(U) \subset V$ and $\theta \circ F \circ \varphi^{-1} : \varphi(U) \to \theta(V)$ is smooth (in the Euclidean sense).

Note that if we think of $\mathbb{R}$ as having its trivial smooth structure, $(\mathbb{R}, \mathrm{id})$, then a smooth function (the first definition) is in reality a smooth map (the second definition) where $N = \mathbb{R}$ and $\mathcal{A}' = (\mathbb{R}, \mathrm{id}$, so we only really need the second definition.

Let us note in addition that if a map $F$ is smooth at $p$ with respect to one pair of charts $(U, \varphi)$, $(V, \theta)$, then it is smooth with respect to *any* pair of charts $(U', \varphi')$, $(V', \theta')$ such that $F(U') \subset V'$, $U' \subset U$, and $V' \subset V$. This follows from the fact that

$$\theta' \circ F \circ (\varphi')^{-1} = (\theta' \circ \theta^{-1}) \circ (\theta \circ F \circ \varphi^{-1}) \circ (\varphi \circ (\varphi')^{-1}) \tag{2}$$

The transition functions $\varphi \circ (\varphi')^{-1}$, which is well-defined on $\varphi'(U')$ and $\theta' \circ \theta^{-1}$, which is well-defined on $\theta(V')$, is a composition of three smooth functions, in the Euclidean sense, which is also smooth in the Euclidean sense.

**Remark II.1** (Diffeomorphism)**.** A diffeomorphism $F : M \to N$ is a smooth bijection between manifolds with a smooth inverse. This is of course true if and only if $F$ is a bijection and we can find charts $(U, \varphi)$ and $(V, \theta)$ as above where $\theta \circ F \circ \varphi^{-1}$ and $\varphi \circ F^{-1} \circ \theta^{-1}$ are smooth in the Euclidean sense. Thus, the above proof carries over to show that a diffeomorphism with respect to one pair of charts is a diffeomorphism with respect to all charts, under certain constraints.

## III. The tangent space

Let us now introduce the tangent space. The goal with this section will be to provide a very careful motivation for why the tangent space is defined the way that it is.

### A. Basics

At a high-level, the idea of the tangent space is that we want to "attach" a tangent plane to each point on a manifold $M$. In the case $M = \mathbb{R}^n$, this becomes easy: we simply assign a vector space $\mathbb{R}^n_a$ to each $a \in \mathbb{R}^n$. We call this the *geometric tangent space of* $\mathbb{R}^n$. Of course, we don't always have such a convenient correspondence: general manifolds are much more abstract objects. Thus, we require a more abstract definition:

**Definition III.1** (Tangent space, first definition)**.** Given smooth manifold $M$ and $p \in M$, the tangent space $T_pM$ is the space of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form

$$v(f) = (f \circ \gamma)'(0) \tag{3}$$

where $\gamma : [-1, 1] \to M$ is a smooth curve on $M$ with $\gamma(0) = p$.

In other words, a tangent vector specifies a curve along which to differentiate $f$, resulting in some number representing the "speed" of $f$ in the direction of the curve. Now, we will prove a lemma which will help us to gain more geometric intuition for why we define tangent space this way. As it turns out, the geometric intuition of considering $T_pM$ simply as a tangent plane, which is attached at each point $p \in M$, is quite accurate.

**Lemma III.1.** Let $p \in M$ be a point, let $(U, \varphi)$ be a chart around $p$. A linear map $v : C^\infty(M) \to \mathbb{R}$ is in $T_pM$ if and only if $v(f) = \sum_{i=1}^n w^i \frac{d(f\circ\varphi^{-1})}{dx^i}(\varphi(p))$, where $w \in \mathbb{R}^n$.

*Proof.* To begin, suppose $v \in T_pM$, so $v(f) = (f \circ \gamma)'(0)$ for some $\gamma$ with $\gamma(0) = p$. Then $\varphi \circ \gamma$ is a map from $[-1, 1]$ to $\mathbb{R}^n$. It follows from chain rule that

$$(f \circ \gamma)'(0) = \big((f \circ \varphi^{-1}) \circ (\varphi \circ \gamma)\big)'(0) = \sum_{i=1}^n \frac{d(f \circ \varphi^{-1})}{dx^i}(\varphi(p))\frac{d(\varphi \circ \gamma)_i}{dt}(0) = \sum_{i=1}^n w_i \frac{d(f \circ \varphi^{-1})}{dx^i}(\varphi(p)) \tag{4}$$

where we have set $w_i = (\varphi\circ\gamma)_i'(0)$. Conversely, suppose $v$ is of the form of a directional derivative with respect to chart $(U, \varphi)$. Let $w$ be the corresponding vector. We define the curve $\gamma : [-1, 1] \to M$ in the obvious way. Firstly, note that $\varphi : U \to \varphi(U)$ is a homeomorphism with $\varphi(U) \subset \mathbb{R}^n$. Let $\alpha : [-1, 1] \to \mathbb{R}^n$ be the map $\alpha(t) = \varphi(p) + wt$ Let $[-\varepsilon, \varepsilon]$ be an interval such that $\alpha([-\varepsilon, \varepsilon]) \subset \varphi(U)$. We now use a smooth bump function: let $\psi : [-1, 1] \to \varphi(U)$ be the smooth bump function between 0 and 1 which is 1 on $[-\varepsilon/2, \varepsilon/2]$ and 0 outside $[-\varepsilon, \varepsilon]$. We then let $\widetilde{\alpha}(t) = \varphi(p) + w\psi(t)t$. Note that $\widetilde{\alpha}$ has image in $\varphi(U)$. Moreover, $\widetilde{\alpha}'(0) = w\psi(0) = w$.

We finally take $\gamma : [-1, 1] \to M$ as $\gamma(t) = \varphi^{-1}(\widetilde{\alpha}(t))$. This map is well-defined as $\widetilde{\alpha}(t) \in \varphi(U)$. We then have

$$(f \circ \gamma)'(0) = ((f \circ \varphi^{-1}) \circ (\varphi \circ \gamma))'(0) = \sum_{i=1}^n w_i \frac{d(f \circ \varphi^{-1})}{dx^i}(\varphi(p)) = v(f) \tag{5}$$

as $(\varphi \circ \gamma)'(0) = \widetilde{\alpha}'(0) = w$, so $v \in T_pM$ and the proof is complete. □

In other words, all tangent vectors amount to maps which pull a function $f$ (locally) back into its local representation as a function on Euclidean space, and then take a direction derivative! From here, there are a few interesting facts that we can prove. The main motivation for defining the tangent space in the first place was to attach a vector space to each point of the manifold $M$. The vector space operations on $T_pM$ are simply taking sums/scalar products of linear maps. However, we want to better understand the structure of this vector space.

To do this, we prove the following:

**Lemma III.2.** Let $(U, \varphi)$ be a fixed chart around $p$. Define $\Phi : \mathbb{R}^n \to T_pM$ as

$$\Phi(v)(f) = \sum_{i=1}^n v^i \frac{d(f \circ \varphi^{-1})}{dx^i}(\varphi(p)) \tag{6}$$

is an isomorphism of vector spaces.

*Proof.* Linearity of this map is clear. Surjectivity follows from the fact that every element of $T_pM$ can be expressed as a directional derivative with respect to $(U, \varphi)$, from the previous result. Thus, we just need injectivity. Note that the component function $\varphi^j \in C^\infty(M)$ (after we smoothly extend at to all of $M$ via a partition of unity). Of course,

$$\Phi(v)(\varphi^j) = \sum_{i=1}^n v^i \frac{d(\varphi^j \circ \varphi^{-1})}{dx^i}(\varphi(p)) = v^j \tag{7}$$

so that if $\Phi(v) = 0$, then $v^j = 0$ for all $j$, so $v = 0$, and the map is injective. □

**Corollary III.0.1.** The set of maps $f \mapsto \frac{d(f\circ\varphi^{-1})}{dx^i}(\varphi(p))$ for each $i$ from 1 to $n$ form a basis for the $n$-dimensional tangent space $T_pM$. In the case when $M = U \subset \mathbb{R}^n$, and $\varphi$ can simply be taken as id, the set of $f \mapsto \frac{df}{dx^i}(p)$ form the basis for $T_pU$.

Now, let us throw away what we have done. There is a much nicer, chart-independent version of the tangent space that we can use, which we will now present

**Definition III.2** (Tangent space, second definition)**.** We take $T_pM$ to be the space of all derivations at $p$, that if linear maps $v$ from $C^\infty(M)$ to $\mathbb{R}$ which satisfy

$$v(fg) = f(p)v(g) + g(p)v(f) \tag{8}$$

for $f, g \in C^\infty(M)$ (we call this the Leibniz rule).

**Claim III.1.** The new and old definitions of the tangent space are equivalent.

It is easy to see that every $v \in T_pM$ relative to the old definition is a derivation:

$$v(fg) = ((f\circ\gamma)(g\circ\gamma))'(0) = f(p)(g\circ\gamma)'(0) + g(p)(f\circ\gamma)'(0) = f(p)v(g) + g(p)v(f) \tag{9}$$

However, seeing that each derivation of of the form of a derivative along a curve (or equivalently a directional derivative) is harder. Nevertheless, the proof is quite intuitive once we get into it. To begin, we'll need some machinery.

**Lemma III.3** (Hadamard's lemma)**.** Let $U \subset \mathbb{R}^n$ be a star-convex neighbourhood of $a$, let $f : U \to \mathbb{R}$ be smooth. Then there exist smooth functions $g_i : U \to \mathbb{R}$ such that for $x \in U$,

$$f(x) = f(a) + \sum_{i=1}^{n} g_i(x)(x_i - a_i) \tag{10}$$

*Proof.* Since $U$ is star-convex around $a$, the line segment $t \mapsto (1-t)a + tx$ is contained in $U$. Note that, by the fundamental theorem of calculus,

$$\int_0^1 \frac{dF(t)}{dt} = F(1) - F(0) = f(x) - f(a) \tag{11}$$

where $F(t) = f((1-t)(a) + tx)$. Note that

$$\frac{dF(t)}{dt} = \sum_{i=1}^{n} \frac{df^i}{dx^i}((1-t)(a) + tx)(x_i - a_i) \tag{12}$$

which implies

$$f(x) = f(a) + \sum_{i=1}^{n} \left( \int_0^1 \frac{df^i}{dx^i}((1-t)(a) + tx)\ dt \right)(x_i - a_i) \tag{13}$$

Thus, we simply must verify that $g_i : x \mapsto \int_0^1 \frac{df^i}{dx^i}((1-t)(a) + tx)\ dt$ are smooth, a fact which follows from basic calculus. $\square$

Now, a technical lemma related to derivations:

**Lemma III.4.** Derivations are local. In particular, if $v$ if a derivation at $p$, and $f$ and $g$ agree on a neighbourhood $V$ of a point $p$, then $v(f) = v(g)$.

*Proof.* This proof relies on partitions of unity. In particular, let $\psi$ be a smooth bump function with support in $V$. It follows that $(f-g)\psi = 0$ (globally). Thus,

$$v((f-g)\psi) = (f-g)(p)v(\psi) + v(f-g)\psi(p) = v(f-g) = 0$$

which implies that $v(f) = v(g)$. $\square$

The idea for the main proof is to make use of Hadamard's lemma, which will allow us to show that $v$ sends a function $f$ to the desired directional derivative. In particular, let $(U, \varphi)$ be a coordinate chart around $p$. Let $g$ be the smooth function defined on $U$ as $f \circ \varphi^{-1} \circ \varphi$, which we then extend to a smooth function on all of $M$. Thus, $f$ and $g$ agree on an open neighbourhood of $p$ (WLOG, we take this neighbourhood to be $V = \varphi^{-1}(B)$, where $B$ is a ball around $\varphi(p)$).

From here, by Hadamard's lemma, for $x \in V$,

$$g(x) = f(p) + \sum_{i=1}^{n} g_i(\varphi(x))(\varphi^i(x) - \varphi^i(p)) \tag{14}$$

which implies that $v(f) = v(g) = \sum_{i=1}^{n} g_i(\varphi(p))v(\varphi^i(x))$. But simply note that

$$g_i(\varphi(p)) = \int_0^1 \frac{d(f \circ \varphi^{-1})^i}{dx^i}(\varphi(p))\ dt = \frac{d(f \circ \varphi^{-1})^i}{dx^i}(\varphi(p)) \tag{15}$$

which completes the proof: $v$ is a directional derivative and thus in $T_pM$.

## B. The differential map

Let us move on to a new, broadly powerful construction: *the differential map*. This is a map is, arguably, the "most natural" map between tangent spaces of manifolds which is induced by a smooth map between manifolds.

**Definition III.3** (The differential)**.** Let $F : M \to N$ be a smooth function between smooth manifolds $M$ and $N$ (not necessarily of the same dimension). We define $dF_p : T_pM \to T_{F(p)}N$ as the (linear) map $dF_p(v)(f) = v(f \circ F)$, where $f \in C^\infty(N)$, so $f \circ F \in C^\infty(M)$.

Of course, we still need to check that this map is well-defined. Namely, we must show that $dF_p(v) \in T_{F(p)}N$. This amounts to shwoing that the product rule holds:

$$\begin{aligned} dF_p(v)(fg) = v(fg \circ F) = v((f \circ F)(g \circ F)) &= f(F(p))v(g \circ F) + g(F(p))v(f \circ F) && (16)\\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f) && (17)\end{aligned}$$

so we have a valid derivation, and $dF_p(v) \in T_{F(p)}N$.

**Example III.1** (Euclidean space)**.** Suppose $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. Recall that the partial derivatives $g \mapsto \frac{dg}{dx^i}(p)$ for $i \in \{1, \dots, m\}$ form a basis for $T_p\mathbb{R}^m$ and similarly, the partial derivatives $g \mapsto \frac{dg}{dy^i}(F(p))$ for $i \in \{1, \dots, n\}$ form a basis for $T_{F(p)}\mathbb{R}^n$.

Suppose $F : \mathbb{R}^m \to \mathbb{R}^n$ is smooth. By definition, we then have

$$dF_p\left(\left.\frac{d}{dx^i}\right|_p\right)(f) = \left.\frac{d(f \circ F)}{dx^i}\right|_p = \left(\sum_j \frac{dF^j}{dx^i}(p)\left.\frac{d}{dx^j}\right|_{F(p)}\right)(f) \tag{18}$$

from the chain rule. It follows, via the isomorphism $\Phi$ between the tangent spaces and Euclidean space, the basis vector $e_i \simeq \frac{d}{dx^i}$ is mapped to the vector $\sum_j \frac{dF^j}{dx^i}(p)e_j = DF(p)e_i$, where $D$ is the usual derivative. In other words, when we're in Euclidean space, the differential behaves just like the Jacobian of a smooth function.

As it turns out, this exact same logic can be applied to *any pair of manifolds $M$ and $N$, locally.* This is of course due to the fact that $M$ and $N$ look like $\mathbb{R}^n$ and $\mathbb{R}^m$, locally. We will revisit this point later, and be more precise.

At this point, let us prove a few more basic facts about the differential map:

**Proposition III.1.** The differential respect the chain rule: $d(F \circ G)_p = dF_{G(p)} \circ dG_p$.

*Proof.* We have

$$d(F\circ G)_p(v)(f) = v(f\circ F\circ G) = dG_p(v)(f\circ F) = dF_{G(p)}(dG_p(v))(f) \tag{19}$$

which implies the desired result. □

**Proposition III.2.** If $F : M \to N$ is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

*Proof.* Note that $F\circ F^{-1} = \mathrm{id}_N$ and $F^{-1}\circ F = \mathrm{id}_M$. It's easy to see that $d\mathrm{id} = \mathrm{id}$. Thus, from the chain rule,

$$d(F^{-1}\circ F)_p = d(F^{-1})_{F(p)}\circ dF_p = \mathrm{id} \tag{20}$$

and

$$d(F\circ F^{-1})_{F(p)} = dF_p\circ d(F^{-1})_{F(p)} = \mathrm{id} \tag{21}$$

which means that $dF_p$ is invertible with $d(F^{-1})_{F(p)}$ as its inverse. □

**Proposition III.3.** Let $M$ be a smooth manifold, let $U$ be an open subset around $p$. Let $j : U \to M$ be the inclusion map. $dj_p : T_pU \to T_pM$ is an isomorphism of vector spaces.

*Proof.* We already know this map is linear.

Let $v \in T_pU$, suppose $dj_p(v) = 0$. Thus, for any $f \in C^\infty(M)$, we have $dj_p(v)(f) = v(f\circ j_p) = 0$, where $f\circ j_p : U \to \mathbb{R}$. Suppose $g \in C^\infty(U)$. Using the locally Euclidean property, we can always choose some smaller neighbourhood of $p$, $V$, such that $\overline{V} \subset U$. Let $\widetilde{g}$ be a smooth extension of $g$ supported on $V$ to all of $M$. Note that $g$ and $\widetilde{g}\circ j$ agree on the open set $V$, so $v(g) = v(\widetilde{g}\circ j) = 0$, by assumption, so $v = 0$ and we have injectivity.

As for surjectivity, given some $v \in T_pM$, we take $w \in T_pU$ to be $w(f) = v(\widetilde{f})$, where $f \mapsto \widetilde{f}$ is a map taking $f$ to an extension supported on $V$. Note that two different extensions will yield the same value $v(\widetilde{f})$, so this map is well-defined. Moreover, note that this map is a derivation, as $f \mapsto \widetilde{f}$ is linear (for any chosen extension), and

$$w(fg) = v(\widetilde{fg}) = v(\widetilde{f}\widetilde{g}) = \widetilde{f}(p)v(\widetilde{g}) + \widetilde{g}(p)v(\widetilde{f}) = f(p)w(g) + g(p)w(f). \tag{22}$$

Finally, note that

$$dj_p(w)(f) = w(f\circ j) = v(\widetilde{f\circ j})$$

Clearly, $\widetilde{f\circ j}$ and $f$ agree on $V$, so $v(\widetilde{f\circ j}) = v(f)$, and $dj_p(w) = w$. □

Let's now revisit the stuff we were talking about before, when we explicitly computed the differential when $M$ and $N$ are Euclidean space. In particular, consider two charts, one for each manifold $(U,\phi)$ and $(V,\psi)$ (with codomains $\mathbb{R}^m$ and $\mathbb{R}^n$) around points $p$ and $F(p)$. As we have discussed a few times, $T_pM$ and $T_{F(p)}N$ are the spaces of all pulled-back directional derivatives with respect to any coordinate coordinate chart around $p$. Thus, $dF_p : T_pM \to T_{F(p)}N$ is performing a transformation of the form

$$dF_p : \sum_j v_j \frac{d}{dx^j}\bigg|_p \mapsto \sum_j w_j \frac{d}{dx^j}\bigg|_{F(p)}. \tag{23}$$

where $\frac{d}{dx^i}\Big|_p$ is the element of $T_pM$ such that

$$\left(\frac{d}{dx^i}\bigg|_p\right)(f) = \frac{d(f\circ\phi^{-1})}{dx^i}(\phi(p)) \quad \text{and} \quad \left(\frac{d}{dx^j}\bigg|_{F(p)}\right)(f) = \frac{d(f\circ\psi^{-1})}{dx^j}(\psi(F(p))).$$

As we discussed earlier, these maps form a basis for each of the tangent spaces, so it will be our goal to find the matrix representing $dF_p$, with respect to these bases.

But of course, this task is identical to what was shown in the Euclidean case. We simply have, by definition

$$dF_p\left(\left.\frac{d}{dx^i}\right|_p\right)(f) = \frac{d(f\circ F\circ\phi^{-1})}{dx^i}(\phi(p)) = \frac{d(f\circ\psi^{-1}\circ\psi\circ F\circ\phi^{-1})}{dx^i}(\phi(p)) \tag{24}$$

Note that $\Psi = \psi\circ F\circ\phi^{-1}$ is a map from $\mathbb{R}^m$ to $M$ to $N$ to $\mathbb{R}^n$. We have, via chain rule

$$\frac{d(f\circ\psi^{-1}\circ\Psi)}{dx^i}(\phi(p)) = \sum_j \frac{d\Psi^j}{dx^i}(\phi(p))\frac{d(f\circ\psi^{-1})}{dx^j}(\Psi(\phi(p))) \tag{25}$$

where $\Psi(\phi(p)) = \psi(F(p))$, so that

$$dF_p\left(\left.\frac{d}{dx^i}\right|_p\right)(f) = \left(\sum_j \frac{d\Psi^j}{dx^i}(\phi(p))\left.\frac{d}{dx^j}\right|_{F(p)}\right)(f) \tag{26}$$

It follows immediately that the $(j,i)$-th entry of the desired matrix is $\frac{d\Psi^j}{dx^i}(\phi(p))$, so the differential map, when written as a matrix, if just the Jacobian of the function $\Psi = \psi\circ F\circ\phi^{-1}$ evaluated at $\phi(p)$!

As a particular example when this is useful, consider the following. Suppose $(\varphi, U)$ and $(\psi, V)$ are two smoothly-compatible coordinate charts of manifold $M$, and $p\in U\cap V$. We know that we can write some element $\nu\in T_pM$ uniquely in terms of either the coordinate chart $\varphi$ or $\psi$. In other words,

$$\nu(f) = \sum_k a_k\left.\frac{(f\circ\varphi^{-1})}{\partial x^k}\right|_{\varphi(p)} \quad\text{and}\quad \nu(f) = \sum_k b_k\left.\frac{(f\circ\psi^{-1})}{\partial x^k}\right|_{\psi(p)} \tag{27}$$

We would like to express $v_k = f\mapsto \left.\frac{(f\circ\varphi^{-1})}{\partial x^i}\right|_{\varphi(p)}$ in terms of the basis of $w_k = f\mapsto \left.\frac{(f\circ\psi^{-1})}{\partial x^i}\right|_{\psi(p)}$. What is a simple way to do this? Well, from above, we know that we can write $v_k = d[\mathrm{id}]_p(v_k)$ in terms of the $w_k$, as these form a basis for the tangent space of the image: the coefficients will just be given by the Jacobian of the transition function $\varphi^{-1}\circ\psi$!

**Example III.2.** Let us consider a basic example. Suppose $M = \mathbb{R}^2 - [0,\infty)\times\{0\}$ (i.e. we are given the complex plane without the non-negative $x$-axis). There are couple different natural ways that we can think about a smooth structure on this space. The first is the obvious one: we use the single chart $(M, \mathrm{id})$. The second is a bit less trivial but still natural: we use the chart $(M, r\times\theta)$, where $r : M\to(0,\infty)$ and $\theta : M\to(0,2\pi)$ send a point of $M$ to a unique radius and angle from the origin. One can check these functions are well-defined and smoothly compatible with the trivial chart.

We have $\varphi^{-1}(x^1,x^2) = (x^1\cos(x^2), x^1\sin(x^2))$

$$\left.\frac{d(f\circ\varphi^{-1})}{dx^1}\right|_{r(x,y)} = \cos(x^2)\frac{df}{dx^1} + \sin(x^2)\frac{df}{dx^2} \tag{28}$$

**Example III.3.** Let us consider a basic example. Suppose $M = \mathbb{R}^2 - [0,\infty)\times\{0\}$ (i.e. we are given the complex plane without the non-negative $x$-axis). There are couple different natural ways that we can think about a smooth structure on this space. The first is the obvious one: we use the single chart $(M, \mathrm{id})$. The second is a bit less trivial but still natural: we use the chart $(M, r\times\theta)$, where $r : M\to(0,\infty)$ and $\theta : M\to(0,2\pi)$ send a point of $M$ to a unique radius and angle from the origin. One can check these functions are well-defined and smoothly compatible with the trivial chart.

Suppose $f : M\to\mathbb{R}$ is a smooth function on $M$. We can easily pass this $f$ as an argument to some tangent vector in $T(x,y)M$. Suppose we "rewrite $f$ in polar coordinates", which is to say that we now consider the function $\widetilde{f} = f\circ(r\times\theta)^{-1}$ on $(0,\infty)\times(0,2\pi)$. If $\nu$ is some tangent vector epxressed in terms of the polar chart, so

$$\nu(g) = a_1\left.\frac{d(g\circ(r\times\theta)^{-1}}{dr}\right|_{(r\times\theta)(x,y)} + a_2\left.\frac{d(g\circ(r\times\theta)^{-1}}{d\theta}\right|_{(r\times\theta)(x,y)} \tag{29}$$

then it is super easy to evaluate $\nu(f)$ if we know $\widetilde{f}$, it will of course just be

$$\nu(f) = a_1 \frac{d\widetilde{f}}{dr}\bigg|_{(r\times\theta)(x,y)} + a_2 \frac{d\widetilde{f}}{d\theta}\bigg|_{(r\times\theta)(x,y)} \tag{30}$$

### C. The tangent bundle

It is very useful and natural to treat all the tangent spaces as a single, unified object which "lies above" a manifold. We call this the tangent bundle.

**Definition III.4.** For a manifold $M$, define the tangent bundle $TM$ by

$$TM = \bigsqcup_{p\in M} T_pM \tag{31}$$

Of course, there is a natural projection $\pi : TM \to M$ from the tangent space to the base manifold, which sends some tangent vector $(p, v)$ to $p$, its basepoint. In general, since $T_pM$ for each $p \in M$ for $M$ an $n$-dimensional manifold is isomorphic to $\mathbb{R}^n$, it follows that $T_pM$ is in bijective correspondence with $M \times \mathbb{R}^n$, the so-called *trivial bundle* over $M$. However, $TM$ will often look drastically different from $M \times \mathbb{R}^n$ when we consider its *topological* and *smooth* properties.
To be more precise,

**Claim III.2.** $TM$ is itself a smooth manifold, where it inherits a natural topology and smooth structure from $M$.

We further claim that this inherited topology/smooth strucutre is often very different from the trivial topology/structure assigned to $M \times \mathbb{R}^n$ as a product manifold.

We will define a topology on $TM$ using coordinate charts. In particular, suppose $(U, \varphi)$ is a chart for $M$ around $p$. We define an induced map $\widetilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^n$. First, consider the map $d\varphi_p : T_pM \to T_p\mathbb{R}^n$. Note:

**Claim III.3.** The map $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism of vector spaces.

*Proof.* We proved earlier that $T_pM$ and $T_{\varphi(p)}\mathbb{R}^n$ are isomorphic to $T_pU$ and $T_{\varphi(p)}\varphi(U)$. Note that $\varphi : U \to \varphi(U)$ is a diffeomorphism, so it is an isomorphism between $T_pU$ and $T_{\varphi(p)}\varphi(U)$ (with respect to the inherited smooth structure). □

Of course, every element of $T_p\mathbb{R}^n$ can be written uniquely as $\sum_j v_j \frac{d}{dx_i}|_p$: we define the bijection taking this element to $v \in \mathbb{R}^n$ as $\Phi$. We then let

$$\widetilde{\varphi}(p, v) = (\varphi(p), \Phi(d\varphi_p(v))) \tag{32}$$

We attempt to define a smooth structure on the entirety of $TM$ by using the pairs of sets/maps $(\pi^{-1}(U_\alpha), \widetilde{\varphi}_\alpha)$ and Prop. II.1: we simply must check that our chosen collection of sets/maps satisfies all of the required conditions. Of course, we argued above that $\widetilde{\varphi}$ is a bijection between $\pi^{-1}(U_\alpha)$ and $\varphi_\alpha(U_\alpha) \times \mathbb{R}^n$, an open set of $\mathbb{R}^{2n}$. Note that $\pi^{-1}(U_\alpha \cap U_\beta)$ is sent to $\varphi_\gamma(U_\alpha \cap U_\beta) \times \mathbb{R}^n$ for $\gamma = \alpha, \beta$, which are both open. Smoothness of $\widetilde{\varphi}_\alpha \circ \widetilde{\varphi}_\beta^{-1}$ follows from the fact that

$$(\widetilde{\varphi}_\alpha \circ \widetilde{\varphi}_\beta^{-1})(p, v) = \Big((\varphi_\alpha \circ \varphi_\beta^{-1})(p), \left((\Phi \circ [d\varphi_\alpha]) \circ (\Phi \circ [d\varphi_\beta])^{-1}\right)(v)\Big) \tag{33}$$

$$= \Big((\varphi_\alpha \circ \varphi_\beta^{-1})(p), \Big(\Phi \circ d(\varphi_\alpha \circ \varphi_\beta^{-1}) \circ \Phi^{-1}\Big)(v)\Big) \tag{34}$$

$$= \Big((\varphi_\alpha \circ \varphi_\beta^{-1})(p), D\Phi v\Big) \tag{35}$$

where, as we proved in the previous section, $D\Phi$ is the derivative map of the transition function between charts. Finally, it is of course true that a countable number of the $\pi^{-1}(U_\alpha)$ cover $TM$. Note that $M$ is second-countable, so it has a countable basis $\mathcal{B}$. For each $x \in M$, pick some coordinate chart $U_x$ containing

$x$ and pick $B_x \subset U_x$ with $x \in B_x \subset U_x$. Take a countable subcover of the corresponding collection of $B_x$, and let $\mathcal{U}$ be the corresponding countable collection of $U_x$ containing these $B_x$. The collection $\pi^{-1}(\mathcal{U})$ is then a countable cover for $TM$. The final, Hausdorff condition is trivial to see, following from the Hausdorff condition for $M$. Thus, from Prop. II.1, we have a smooth structure on $TM$!

Let us now consider some examples:

**Proposition III.4.** The map $\pi : TM \to M$ is a smooth map.

*Proof.* Note that $\varphi \circ \pi \circ \widetilde{\varphi}^{-1} : \mathbb{R}^{2n} \to \mathbb{R}^n$ is precisely of the form $(p, v) \mapsto (\varphi \circ \pi \circ \widetilde{\varphi}^{-1})(p, v) = p$, which is of course smooth. □

**Claim III.4.** Any $n$-dimensional manifold $M$ which is covered by a single coordinate chart has $TM$ diffeomorphic to $M \times \mathbb{R}^n$.

*Proof.* Let $(\varphi, M)$ be the chart, so $\varphi : M \to \varphi(M)$ is a map with $\varphi(M)$ open in $\mathbb{R}^n$. Note that the induced map $\widetilde{\varphi}$ is defined on all of $TM$ and is a diffeomorphism with its image $\varphi(M) \times \mathbb{R}^n$, as it is a coordinate chart. Moreover, $\varphi : M \to \varphi(M)$ is a diffeomorphism, so $TM \simeq M \times \mathbb{R}^n$. □

Now, something a bit less trivial:

**Claim III.5.** $TS^1$ is diffeomorphic to $S^1 \times \mathbb{R}$.

The idea here is to show that for the two charts involved in the standard smooth atlas for $S^1$, the corresponding induced maps will induce a diffeomorphism. Let us recall an important fact:

**Fact III.1.** Diffeomorphisms are local. Suppose $M$ and $N$ are smooth manifolds, and $M$ is covered by some collection of open sets $U_\alpha$, and suppose that there is a smooth map $F_\alpha : U_\alpha \to N$ for each $U_\alpha$. Moreover, suppose $F_\alpha$ and $F_\beta$ agree on the intersection $U_\alpha \cap U_\beta$. Then the collection of $F_\alpha$ defines a unique smooth function $F : M \to N$ where $F|_{U_\alpha} = F_\alpha$ for each $U_\alpha$. This is very easy to prove, following directly from definitions.

Recall the standard coordinate chart for $S^1$: we take $\widehat{U}_1 = (0, 2\pi)$ and $\widehat{U}_2 = (-\pi, \pi)$. Note that each point in $S^1$ can be uniquely represented as $(\cos(\theta), \sin(\theta))$ for $\theta \in [0, 2\pi)$ or $\theta \in [-\pi, \pi)$. We therefore define $\varphi_1 : U_1 = S^1 - (1, 0) \to \hat{U}_1$ as $\varphi_1(\cos(\theta), \sin(\theta)) = \theta \in \hat{U}_1$ and $\varphi_2 : U_2 = S^1 - (-1, 0) \to \hat{U}_2$ in the same way, but with $\theta \in \hat{U}_2$. Smoothness of these maps follows from implicit function theorem.

The charts $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ form an atlas for $S^1$ (the standard atlas). Note that $\widetilde{\varphi}_1 : TU_1 \to \widehat{U}_1 \times \mathbb{R}$ and $\widetilde{\varphi}_2 : TU_2 \to \widehat{U}_2 \times \mathbb{R}$ are diffeomorphisms. Moreover, $\varphi_1^{-1} : \widehat{U}_1 \to U_1$ and $\varphi_2^{-1} : \widehat{U}_2 \to U_2$ are diffeomorphisms.

Let $\psi_1 = (\varphi_1^{-1} \times \mathrm{id}) \circ \widetilde{\varphi}_1$ and $\psi_2 = (\varphi_2^{-1} \times \mathrm{id}) \circ \widetilde{\varphi}_2$, where $\psi_1$ goes from $TU_1$ to $U_1 \times \mathbb{R}$ and $\psi_2$ goes from $TU_2$ to $U_2 \times \mathbb{R}$. The goal is to show that these two diffeomorphisms agree on $TU_1 \cap TU_2$. Indeed, suppose $(p, \gamma) \in TU_1 \cap TU_2$. We can write

$$\gamma(f) = v \frac{d(f \circ \varphi_1^{-1})}{dx}\bigg|_{\varphi_1(p)} \quad \text{and} \quad \gamma(f) = w \frac{d(f \circ \varphi_2^{-1})}{dx}\bigg|_{\varphi_2(p)} \tag{36}$$

for some $w, v \in \mathbb{R}$. As we proved earlier, these representations will be related by $D(\varphi_2 \circ \varphi_1^{-1})(\varphi_1(p))$, the Jacobian of the transition function between the charts. This function is defined on the domain $\varphi_1(U_1 \cap U_2) = (0, \pi) \cup (\pi, 2\pi)$. $\varphi_2 \circ \varphi_1^{-1}$ will send $\theta \in (0, \pi)$ to itself, and $\theta \in (\pi, 2\pi)$ to $\theta - 2\pi$. Thus, the Jacobian of the transition function is simply 1, so $v = w$, so it follows that $\psi_1$ and $\psi_2$ will agree on the required domain (as these maps will send $\gamma$ to the same element of $\mathbb{R}$).

Thus, we have a global diffeomorphism between $TS^1$ and $S^1 \times \mathbb{R}$.

Even more generally,

**Proposition III.5.** Let $M$ be a smooth manifold. Suppose $(U_\alpha, \varphi_\alpha)$ is an atlas for $M$ such that on each of the overlaps $U_\alpha \cap U_\beta$, the Jacobian of the transition function $\varphi_\beta \circ \varphi_\alpha^{-1}$ is the identity. Then $TM$ is diffeomorphic to $M \times \mathbb{R}^n$.

The proof of this claim is identical to the proof which showed $TS^1 \simeq S^1 \times \mathbb{R}$, with a tiny bit of trivial generalization.

Moving on, note that with the global structure we have defined on the tangent bundle, it is now possible to define a global differential map, $dF$. We simply let $dF|_{T_pM} = dF_p$, for each $p \in M$, where the tangent spaces $T_pM$ make up the entire tangent bundle.

**Claim III.6.** If $F : M \to N$ is a smooth map between manifolds, then $dF : TM \to TN$ is a smooth map between tangent bundles (treated as manifolds).

*Proof.* This follows from the fact that $dF$ is locally smooth. In particular, for some $(p, \gamma) \in TM$, let $(\pi^{-1}(U), \widetilde{\varphi})$ be a chart around this point, let $(\pi^{-1}(V), \widetilde{\psi})$ be a chart around $dF(p, \gamma)$. Note that

$$\left(\widetilde{\psi} \circ dF \circ \widetilde{\varphi}^{-1}\right)(p, v) = \left((\psi \circ F \circ \widetilde{\varphi}^{-1})(p), v'\right) \tag{37}$$

where

$$v' = (\Phi \circ d\psi \circ dF \circ d\widetilde{\varphi}^{-1} \circ \Phi^{-1})(v) = (\Phi \circ d(\psi \circ F \circ \varphi^{-1}) \circ \Phi^{-1})(v) = D\Phi v \tag{38}$$

where $D\Phi$ is the Jacobian of the transition function. Of course, it follows that the entire function is smooth (locally), so it is smooth globally as smoothness is a local property. $\square$

It is easy to see in this case that if $F : M \to N$ is a diffeomorphism, then $dF : TM \to TN$ is a diffeomorphism. The global differential has a lot of the properties of the local differential, proved above.

## IV. Vector fields, Lie groups, and the Lie bracket

### A. Vector fields

Finally, a new topic: vector fields.

**Definition IV.1.** Given manifold $M$, define a vector field on $M$ as a map $X : M \to TM$ such that $X_p :- X(p) \in T_pM$. If this map is a smooth map between manifolds, then $X$ is said to be a smooth vector field. Let $\mathfrak{X}(M)$ denote the space of all vector fields on $M$.

Thinking about vector fields locally can give us a better impression of what exactly is happening with this definition. In particular, note that for some $(U, \varphi)$ chart in $M$ around $p$, $X : M \to TM$ is of the form

$$X(p) = \left(p, \sum_i w^i(p) \frac{d}{dx^i}\bigg|_{\varphi(p)}\right) \tag{39}$$

for $p \in U$, as in this neighbourhood, any element of the tangent space can be represented as a directional derivative in this coordinate chart. Let us think about what it means for $X$ to be smooth at some point $p$. We must have charts $(U, \varphi)$ and $(\widetilde{U}, \widetilde{\varphi})$ about the point $p \in M$ such that the function $\widetilde{\varphi} \circ X \circ \varphi^{-1}$ is smooth. This function is of the form

$$(\widetilde{\phi} \circ X \circ \phi^{-1})(x) = \widetilde{\phi}(X_{\phi^{-1}(x)}) = (x, \Phi(d\phi_{\varphi^{-1}(x)}[X_{\phi^{-1}(x)}])) \tag{40}$$

where

$$d\phi_{\varphi^{-1}(x)}[X_{\phi^{-1}(x)}])(f) = \sum_k c_k(\varphi^{-1}(x)) \frac{df}{dx^k}(x) \tag{41}$$

which implies that $\Phi(d\phi_{\varphi^{-1}(x)}[X_{\phi^{-1}(x)}]) = (c_1(\varphi^{-1}(x)), \ldots, c_n(\varphi^{-1}(x))) = c(\varphi^{-1}(x))$. Thus, smoothness of $X$ amounts to requring that the function $p \mapsto (p, c(p))$ is smooth in charts around $p$. Thus, $X$ being smooth means that its coefficient functions must be smooth for all charts in some atlas for $M$. It follows immediately that:

1. If $X$ and $Y$ are smooth vector fields, then $X + \lambda Y$ is a smooth vector field. This follows from the fact that in each chart, the coefficients of $X + \lambda Y$ are of the form $c_k(p) + \lambda d_k(p)$, which will be smooth functions.

2. $X$ is a smooth vector field if and only if, for each $f \in C^\infty(M)$, the function $p \mapsto X_p(f)$ is a smooth function. If $X$ is smooth, then

$$X_p(f) = \sum_k c_k(p) \frac{d(f \circ \phi^{-1})}{dx^k}\bigg|_{\phi(p)} \tag{42}$$

for all $p$ is some chart $(U, \phi)$. Of course, $d(f \circ \phi^{-1})/dx^i$ evaluated at $\phi(p)$ is smooth, and the $c_k$ are smooth from above, so $p \mapsto X_p(f)$ also must be smooth. Conversely, if $p \mapsto X_p(f)$ is smooth for each $f \in C^\infty(M)$, then this will hold for the coordinate functions $\phi^i$, so we will have

$$X_p(\phi^i) = \sum_k c_k(p) \frac{d(\phi^i \circ \phi^{-1})}{dx^k}\bigg|_{\phi(p)} = c_i(p) \tag{43}$$

so that $c_i(p)$ is smooth inside of a chart. It follows from the definition that $X$ is a smooth vector field.

It will often be useful to think of a vector field as acting on a function $f$, and inducing a function $p \mapsto X_p(f)$. In particular, if a vector field is visualized as some directional derivative being taken at every point on $M$, then the function $p \mapsto X_p(f)$ is precisely the varying directional derivative *of the function* $f$, at each $p \in M$.

From here, we should suspect that $X$ has all of the characteristics of a derivative. Indeed, note that $X$ is a global derivation: if $f, g \in C^\infty(M)$, then $X_p(fg) = f(p)X_p(g) + g(p)X_p(g)$. This is trivial from the definition.

### B. Lie groups and the Lie bracket

Let us now move to describing Lie groups. At a high-level, a Lie group is a smooth manifold that is also a group, where the group operation $G \times G \to G$ with $(g, h) \mapsto g \cdot h$ and the inverse $g \mapsto g^{-1}$ are smooth functions. From here, we can reason that left and right multiplication by a group element are smooth maps. Let $L_g$ and $R_g$ be the maps which left and right-multiply by an element $g \in G$.

**Remark IV.1** (Vector field on a Lie group)**.** Given a Lie group $G$, let us define a vector field as follows: fix some $v \in T_eG$: the tangent space at the identity. We then consider the map $g \mapsto d(L_g)_e(v) \in T_gG$, which we denote by $X_g$. To see that this is a smooth map, note that of $f$ is a smooth curve in $G$, then

$$X_g(f) = d(L_g)_e(v)(f) = [f \circ L_g \circ \gamma]'(0) = \frac{d}{dt}\bigg|_{t=0} f(g \cdot \gamma(t)) \tag{44}$$

Clearly, $(g, t) \mapsto f(g \cdot \gamma(t))$ is smooth, so $\frac{d}{dt} f(g \cdot \gamma(t))$ is also smooth. Thus, fixing $t = 0$ yields a smooth function in $g$. Since a vector field is smooth if and only if its action on smooth functions yields a smooth function, $X$ is smooth.

**Definition IV.2.** We say that a vector field $X$ on Lie group $G$ is left $G$-invariant if for every $h, g \in G$, we have $(dL_h)_g(X_g) = X_{hg}$.

**Claim IV.1.** Let $\mathfrak{g}$ be the set of all left $G$-invariant vector fields on $G$. Note that any linear combination of left $G$-invariant vector fields is a left $G$-invariant vector field, so $\mathfrak{g}$ has vector space structure. We claim that there is a vector space isomorphism between $T_eG$ (where $e \in G$ is the identity) and $\mathfrak{g}$, via the map taking $v \in T_eG$ to vector field $X_g = d(L_g)_e(v)$

*Proof.* We proved above that $X_g$ is in fact a vector field. To see that it is left-invariant, note that

$$(dL_h)_g(X_g) = [(dL_h)_g \circ d(L_g)_e](v) = d(L_h \circ L_g)_e(v) = d(L_{hg})_e(v) = X_{gh} \tag{45}$$

where we used the chain rule for differentials. Clearly, the map from $v$ to $X_g$ is linear. To show injectivity, note that if $X_g = 0$, then $v(f \circ L_g) = 0$ for all functions $f \in C^\infty(G)$. If $f \in C^\infty(G)$, define $h(x) = h(g^{-1} \cdot x)$ which is also a smooth function, so $v(h \circ L_g) = v(f) = 0$, implying $v = 0$. To show surjectivity, suppose $X$ is a left $G$-invariant vector field. By definition, $X_g = (dL_g)_e(X_e)$ for each $g \in G$, where $X_e \in T_eG$. Thus, we have the desired linear isomorphism! $\square$

**TODO**: Add more here

## V. Flows on a manifold

Let us now discuss a new topic: flows on manifolds. First, we need a definition:

**Definition V.1** (Velocity vector for a curve)**.** Let $\gamma : (-1,1) \to M$ be a smooth curve in $M$. We define the velocity vector of $\gamma$ at time $t$ as $\dot{\gamma}(t) \in T_{\gamma(t)}M$ given by $\dot{\gamma}(t) = d\gamma_t \left( \frac{d}{ds}\Big|_{s=t} \right)$ where $\frac{d}{ds}\Big|_{s=t}$ is of course the usual derivative with respect to the single coordinate of the manifold $(-1,1) \subset \mathbb{R}$.

Note that

$$\dot{\gamma}(t)(f) = \frac{d}{ds}\Big|_{s=t} f(\gamma(s)) \tag{46}$$

so $\dot{\gamma}(t)$ gives the directional derivative of $f$ along some curve in the manifold. If $X$ is a smooth vector field, we say that $\gamma(t)$ is an integral curve for $X$ if $\dot{\gamma}(t) = X_{\gamma(t)}$ for all $t$.

**Example V.1.** Consider the smooth vector field on $\mathbb{R}^2$ given by

$$X_{x\times y} = y\frac{d}{dx} - x\frac{d}{dy} \tag{47}$$

In coordinates, some $\gamma(t)$ will yield velocity vector

$$\dot{\gamma}(t)(f) = d\gamma_t \left( \frac{d}{ds} \right)(f) = \frac{d(f\circ\gamma)}{ds}\Big|_{s=t} = \gamma_1'(t)\frac{df}{dx}(\gamma(t)) + \gamma_2'(t)\frac{df}{dy}(\gamma(t)) \tag{48}$$

so that for $\gamma$ to be an integral curve, we must have $\gamma_1'(t) = \gamma_2(t)$ and $\gamma_2(t) = -\gamma_1(t)$. The solutions to this system of ODEs will of course be circular trajectories in $\mathbb{R}^2$.

We're going to need some results about systems of ODEs in Euclidean space (more specifically, Picard-Lindeloff theorem).

**Theorem V.1.**

We won't prove this here.

## VI. Submersions and imersions