Lectures in Differential Topology

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Contents

I. Introduction

Some results in differential geometry and topology that I never want to forget. These notes are not exactly meant to be used as an introduction to the subject, although they contain many notions which one would cover in an introductory course. I'm merely trying to write down things which I don't think are articulated quite in the way that I prefer (or at all) in most of the textbooks on the subject. I also expect to be teaching geometry/topology one day, and these notes will be a good reference.

II. Basic results

III. Smooth manifolds

Remark III.1. A good (perhaps essential) mantra that one should always repeat when doing differential geometry is that we can produce many examples of the main objects of interest to starting with a collection of local things, and identifying them in a consistent way to produce something global. This will be evidenced in the next two lemmas, which give two techniques for producing smooth manifolds. The first one involves picking a candidate smooth atlas for an arbitrarily set, and imposing a compatible topology/smooth structure. The second one involves picking some

Lemma III.1 (Chart lemma for smooth manifolds)**.** Let M be a set, let $(U_\alpha, \varphi_\alpha)$ be a collection of subsets, such that as countable subcollection covers M, where each map $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a bijection of U_α with an open set in $\mathbb{H}^n$. Moreover, suppose all transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ defined on intersections of the cover are smooth, with $\varphi_\beta(U_\alpha \cap U_\beta)$ and $\varphi_\alpha(U_\alpha \cap U_\beta)$ both open sets in $\mathbb{H}^n$. Finally, suppose that for $p \neq q$, either

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there exists some U_α containing both p and q, or disjoint U_α and U_β containing p and q respectively (this is to ensure that we get the Hausdorff property). Then there exists a unique topology and smooth structure on M turning it into a smooth n-manifold.

Proof. Our first claim is that the collection of all $\varphi_\alpha^{-1}(V)$ for V open in $\varphi_\alpha(U_\alpha)$ forms a basis for a topology on M. Indeed, note that each p is contained in $\varphi_\alpha^{-1}(\varphi_\alpha(U_\alpha))$. Moreover, given $\varphi_\alpha^{-1}(V)$ and $\varphi_\beta^{-1}(W)$ with non-empty intersection,

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) \cap (U_\alpha \cap U_\beta) \tag{1}$$

$$= \varphi_\alpha^{-1}\left(V \cap (\varphi_\alpha \circ \varphi_\beta^{-1})(W \cap \varphi_\beta(U_\alpha \cap U_\beta))\right) \tag{2}$$

which is of course an element of our collection of sets, so we do in fact have a basis. From here, second-countability follows from the fact that a countable collection of the U_α cover M, and each is homeomorphic to a second-countable set (an open subset of $\mathbb{H}^n$). We also immediately have the local Euclidean property: each $(U_\alpha, \varphi_\alpha)$ is a chart and the transition functions are smooth. To finally prove the Hausdorff property, we note that any two points are either in disjoint U_α and U_β, or are contained in U_α which is homeomorphic to a Hausdorff space.

To see uniqueness, note that if M had any other topology and smooth structure making the $(U_\alpha, \varphi_\alpha)$ smooth charts, then each set $\varphi_\alpha^{-1}(V)$ would necessarily have to be open, so that the topology defined above is coarser than the new topology. However, given open set U in the new topology, note that given $p \in U$, we can pick U_α with $p \in U_\alpha$, and note that $p \in \varphi_\alpha^{-1}(\varphi_\alpha(U_\alpha)) \cap U \subset U$, which means that U is open in the topology generated by the basis (the old topology). Hence, the two topologies are equal. A choice of smooth atlas uniquely determines a smooth structure, so we immediately also have uniqueness of smooth structure as well. $\square$

Example III.1 (Circle). Let us construct the circle via the gluing construction. Let $M = \mathbb{R} \sqcup \mathbb{R}$: two disjoint copies of the real line.

Lemma III.2 (Gluing lemma for smooth manifolds). Let U_α be a countable collection of open subsets of $\mathbb{R}^n$. Let S be a collection of $\varphi_{\alpha\beta} : U_\alpha \to U_\beta$: smooth maps between pairs U_α and U_β (not necessarily between all of the open sets) which satisfies the following criteria:

  1.

Proof. $\square$

Proposition III.1 (Constant rank theorem).

IV.   Bordism

V.   Bundles

Lemma V.1 (Chart lemma for vector bundles).

Lemma V.2 (Gluing lemma for vector bundles).

Remark V.1. A smooth fibre bundle $\pi : E \to B$ is a submersion. We have $\pi = \text{proj} \circ \varphi$ locally, where φ is a diffeomorphism and proj is obviously a submersion (as a projection $U \times F \to U$), so it follows that π is a submersion.

A.   Subbundles

Lemma V.3 (Smooth local coframe criterion).

B. Fibre integration

Consider a fibre bundle $\pi : E \to B$ with compact oriented fibres, in the smooth category so π is a smooth map of manifolds and we have a trivialization of B by open sets U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times F$.

Lemma V.4. Let $\pi : E \to B$ be a smooth submersion, let X be a smooth vector field on B. Then there exists a smooth vector field $\widetilde{X}$ on E such that $\pi_{*,p}(\widetilde{X}_p) = X_{\pi(p)}$ for all $p \in E$.

Proof. Given $p \in E$, pick coordinates $(U, \varphi) = (U, x^1, \ldots, x^m)$ and $(V, \psi) = (V, y^1, \ldots, y^m)$ so that $\psi \circ \pi \circ \varphi^{-1}$ is a projection. Write X in V as $\sum_{j=1}^{m} X^j \frac{d}{dy^j}$ and define $\widetilde{X}|_U$ as $\sum_{j=1}^{m} X^j \frac{d}{dx^j}$. We do this for some open cover U_α and then combine via a partition of unity subordinate to the cover φ_α to obtain $\widetilde{X}$.
From here, given $p \in$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Using this lemma, it follows that given some tangent vectors $v_1, \ldots, v_{k-m} \in T_p B$, we can always pull-back to a smooth vector field on the fibre $\pi^{-1}(b)$.

VI. Riemannian geometry