A brief course in basic complex analysis

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Contents

I. Introduction

The goal of these notes is to follow a very elementary first course in complex analysis, with similar structure to Edward Bierstone's course at the University of Toronto (MAT354), in the Fall of 2023. I will also be adding information from my own readings/other resources, and modifying/expanding upon certain explanations as I see fit. I hope that someone finds these notes useful. However, in the event that no one does, I certainly will! (I'm writing these notes as I am studying for the MAT354 exam!)

Here is a list of resources that I will draw from in these notes:

- Professor Edward Bierstone's lectures (MAT354, University of Toronto, Fall 2023)
- The MAT354 tutorial worksheets, prepared by Vicente Marin-Marquez

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- Cartan, *Elementary theory of analytic functions of one or several complex variables*
- Ahlfors, *Complex analysis*

II. Basics, the complex plane and the Riemann sphere

III. Introduction to complex functions

IV. Holomorphic functions

Definition IV.1 (Branch)**.**

Definition IV.2 (Branch cut)**.**

V. Power series and analytic functions

Theorem V.1. The zeros of an analytic function for a discrete set. That is, each is contained in a neighbourhood not containing any of the others.

Proof. Let f be an analytic function, let a be a zero of the function. Of course, we can expand f locally as

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k = (z-a)^n \sum_{k=0}^{\infty} c_{k+n}(z-a)^k = (z-a)^n g(z)$$

where we assume that $c_n \neq 0$. In particular, $g(a) \neq 0$. Therefore, since g is continuous, we can choose an open neighbourhood around a for which $g(z) \neq 0$. Thus, there is a neighbourhood around a for which a is the only root. □

Theorem V.2 (Analytic continuation)**.** Let f be an analytic function in connected open set $D \subset \mathbb{C}$. Let $x_0 \in D$. Then the following conditions are equivalent:

- $f^{(n)}(x_0) = 0$ for all non-negative integers n
- f is identically 0 in a neighbourhood of x_0
- f is identically 0 on the entire set D

Proof. □

This proof has a few immediate corollaries:

A. Interlude: exercises

Another exercise interlude:

Exercise V.1 (MAT354 6.2)**.** Recall that if all derivatives of an analytic function are equal to 0 anywhere, then the function is identically 0. Since f is 0 on $S = D \cap \{\mathrm{Im}(z) = 0\}$, it follows immediately that $\partial_x^{(n)} f(0) = 0$. Thus, since f is holomorphic, $\partial_y^{(n)} f(0) = 0$ as well. It follows that f has all of its derivatives equal to 0 at the origin, so it must be identically 0.

In the case that f is meromorphic, it can be expressed as a quotient g/h of analytic functions. If it is 0 on S, then g is 0 on S, so $g = 0$ on D, and f is 0 on D as well.

VI. Conformal maps

We conclude the first part of this course with a brief discussion of conformal mappings of the plane.

A. Interlude: exercises

VII. Complex integration

We now move into the next major topic in our study of complex analysis: integration over complex domains. The tools that we develop in this section will prove to be uniquely useful in proving many major results in complex analysis.

A. Basics and notation

Recall that a function $f : \mathbb{C} \to \mathbb{C}$ is, topologically, a map $f : \mathbb{R}^2 \to \mathbb{R}^2$. Of course, from our studies in standard analysis, we already know how to integrate differential 1-forms $\omega = p(x,y)dx + q(x,y)dy$ and 2-forms $\eta = f(x,y)dx \wedge dy$ over chains in $\mathbb{R}^2$, with $p, q, f \in C(\mathbb{R}^2, \mathbb{R})$. However, in the complex case, it will often be true that we will want to integrate *complex-valued* forms over complex domains. This leads us to the following, somewhat obvious, definitions:

$$\int_\gamma p(x,y)dx + q(x,y)dy := \int_\gamma \mathrm{Re}[p](x,y)dx + \mathrm{Re}[q](x,y)dy + i\int_\gamma \mathrm{Im}[p](x,y)dx + \mathrm{Im}[q](x,y)dy \tag{1}$$

$$\int_\Omega f(x,y)dx \wedge dy := \int_\Omega \mathrm{Re}[f](x,y)dx \wedge dy + i\int_\Omega \mathrm{Im}[f](x,y)dx \wedge dy \tag{2}$$

when $p, q, f \in C(\mathbb{R}^2, \mathbb{C})$. When the component functions of a form are continuous, we call the form continuous. Generally speaking, every form that we consider in this section will be continuous (at least). When the functions are continuously differentiable, we call the form continuously differentiable, or a C^1-form.

We make one more generalization. Generally, when integrating one-forms over paths γ, we make the implicit assumption that γ is continuously differentiable. Thus is required in the standard definition of the integral $\int_\gamma \omega$, as when the integral is pulled back to an integral over an interval, we must take a derivative of γ. Let us suppose instead that γ is piecewise-C^1, meaning that it is characterized by a finite collection of C^1 curves γ_k which, when appended together, form a continuous path. We define the integral over γ as

$$\int_\gamma \omega := \sum_{k=1}^{n} \int_{\gamma_k} \omega. \tag{3}$$

B. Closed and exact forms

We begin by recalling definitions from standard analysis:

Definition VII.1 (Closed and exact forms)**.** A continuously differentiable form ω is *closed* if $d\omega = 0$, where d is the standard exterior derivative, extended linearly from the case of real forms to complex forms in the obvious way. A form ω is called *exact* if there exists a C^1-form η such that $d\eta = \omega$. In particular, when ω is a 1-form, ω is exact if there exists a C^1-function F such that $dF = \omega$. We call such a function a *primitive* of ω.

We won't deal with closed forms too much in subsequent sections, to the exactness property will prove to be very important. We begin with the following equivalence:

Lemma VII.1. Let ω be a 1-form on a connected open set Ω in the complex plane. Then Ω is exact if and only if, for every piecewise-C^1 closed path γ lying in Ω (γ has the same start and end points), we have $\int_\gamma \omega = 0$.

Corollary VII.0.1. Let ω be a 1-form on a disk D in the complex plane. Then Ω is exact if and only if, for every piecewise-C^1 closed path η lying on the boundary of a rectangle in D, we have $\int_\eta \omega = 0$.

This leads us to the following result:

Lemma VII.2. If ω defined on open connected Ω is an exact C^1-form, then it is closed. If Ω is an open disc, and ω is closed, then it is exact.

C. Locally exact forms

Exact forms have useful properties, as we have seen, but exactness if a fairly strong condition. As it turns out, if we weaken the exactness condition such as to make it *local* rather than global, we achieve a much more broadly-useful class of forms, which still have some pretty nice properties.

Definition VII.2.

Remark VII.1. Some authors, such as Cartan, will call locally exact forms *closed forms*. As we will see below, this is because in the case of C^1-forms, the definitions of equvalent. However, for continuous forms which are not C^1, the definitions are different (the closed property isn't defined).

Lemma VII.3. A form ω is locally exact at x if and only if, for a sufficiently small rectangular boundary around x, γ, we have $\int_\gamma \omega = 0$. In addition, a C^1-form is locally exact if and only if it is closed.

Proof. □

D. Integrating locally exact forms

We now move on to the main technical lemma of this section, which will prove invaluable for our study of integration of locally exact forms. At a high-level, this result will tell us that given two paths γ_1 and γ_2 which are homotopic, we can always choose primitives along intermediate paths of the homotopy which vary continuously.

In the case of exact 1-forms, we know that we can always find a global primitive. Thus, it is easy to integrate exact forms, via the fundamental theorem of calculus. In particular, we have $\omega = dF$ for some F. If γ is a C^1 curve, we then have

$$\int_\gamma \omega = \int_a^b \gamma^*(dF) = \int_a^b \frac{d(F \circ \gamma)(t)}{dt}\, dt = f(b) - f(a) \tag{4}$$

where $f = F \circ \gamma$.

But what about the case when a form is locally exact? Integrals are local in the sense that they "sum" a form over a path by individually accumulating values at individual points along a curve, so it would make sense to assume that we can also integrate locally exact forms via the fundamental theorem of calculus. We cannot find a global primitive, but, as we will see, it suffices to find local primitives which vary continuously, and sum over them.

Definition VII.3 (Primitive along a path and primitive on a region)**.** Let $\gamma : [a, b] \to \Omega$ be a continuous path, let ω be locally exact. We say that a function $f : [a, b] \to \mathbb{C}$ is a primitive along the path γ if, for each point on the curve, $\gamma(t)$, there exists a function F such that in a neighbourhood of $\gamma(t)$, F is a primitive of ω, and for all x in some neighbourhood of t, $f(x) = (F \circ \gamma)(x)$.

This definition is effectively a "local" version of the f in the previous paragraph, which was the composition of the global primitive with the path.

Remark VII.2. Note that the definition of a primitive along a path is a special case of the primitive on a region, where the rectangle from which we map has one of its side-lengths equal to 0.

Lemma VII.4 (The general primitive lemma)**.** If ω is locally exact, and $\delta : [a,b]\times[a',b'] \to \Omega$ is a continuous mapping of the rectangle $R = [a,b]\times[a',b']$ into Ω, then there always exists a primitive $f(x,y)$ on the region. Moreover, it is unique up to addition of a constant.

Finally, if $S = \{F_1, F_2, \dots\}$ is a set of functions such that for any $x \in \Omega$, there exists $F_k \in S$ which is a local primitive of ω near x, and for any i,j,k, $F_i + (F_j - F_k) \in S$, then $f(x,y) = F_k(\gamma(x,y))$ for some k, for each point $(x,y) \in R$.

Proof. Clearly, R is compact, so its image under the continuous function δ is compact. Let us choose an open cover $\{A_k\}$ of $\delta(R)$, its inverse image is an open cover of R. It follows from the Lebesgue number lemma that there exists ε sufficiently small such that any rectangle with side lengths less than or equal to ε is contained in an element of $\{\delta^{-1}(A_k)\}$. □

E. An illustrative example: winding number

We dedicate a section of these notes to explaining a particular useful example of the integration of a locally exact form, $\omega = dz/z$.

Lemma VII.5 (Winding number for C^1-curves)**.** If $\gamma : [0,1] \to \Omega$ is a C^1 curve which does not pass through point a, then the integral

$$w(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \tag{5}$$

is an integer. We call $w(\gamma,a)$ the winding number of γ about a.

Proof. Of course, we have

$$\int_\gamma \frac{dz}{z-a} = \int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}\ dt \tag{6}$$

Let us define $f(s) = \int_0^s \frac{\gamma'(t)}{\gamma(t)-a}\ dt$. We know that such a function is differentiable, with

$$f'(s) = \frac{\gamma'(s)}{\gamma(s)-a} \implies (\gamma(s)-a)f'(s) = \gamma'(s) \tag{7}$$

Let $F(s) = e^{-f(s)}(\gamma(s)-a)$. From above, $F'(s) = 0$, so F is a constant. In particular, $F(s) = F(0) = \gamma(0)-a$. Thus, $e^{f(s)} = \frac{\gamma(s)-a}{\gamma(0)-a}$. We then have $e^{f(1)} = 1$, as $\gamma(0) = \gamma(1)$. This implies that f must be an integer multiple of $2\pi i$. Therefore, $w(\gamma,a)$ is an integer. □

We now prove the general version of the winding number lemma, where we only require that γ is continuous.

Lemma VII.6 (Adult winding-number lemma)**.** If $\gamma : [0,1] \to \Omega$ is a continuous curve which does not pass through point a, then the integral

$$w(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \tag{8}$$

is an integer. We call $w(\gamma,a)$ the winding number of γ about a.

F. Cauchy's theorem and its implications

The goal of this section is to prove the most broadly useful theorem in the study of complex integration: Cauchy's theorem. We will use this result to prove a plethora of further results, including incredibly deep statements such as the equivalence between holomorphic and analytic functions, and by extension Liouville's theorem and the fundamental theorem of algebra.

Let us begin by proving a baby version of the theorem:

Theorem VII.1 (C^1-Cauchy's theorem)**.** Recall that $dz := dx + idy$. If f is a holomorphic function with continuous complex derivative, then the form $\omega = f(z)dz$ is closed.

Proof. We have

$$d\omega = \left(\frac{\partial f}{\partial y} - i\frac{\partial f}{\partial y}\right) dx \wedge dy = 0 \tag{9}$$

where we simply used the characteristic identity of holomorphic functions, $\partial_x f = i\partial_y f$. $\square$

We will see, very soon, that holomorphic functions are smooth, so this version of the proof actually does hold for any holomorphic f. However, it is the following proof that we will *use* to show that holomorphic functions are analytic, and thus smooth!

Theorem VII.2 (Cauchy's theorem)**.** If f is a holomorphic function, then the form $\omega = f(z)dz$ is locally exact.

Proof. For this proof, we must take a more direct approach. $\square$

Theorem VII.3 (The equivalence theorem)**.** Let f be a function defined on connected open set Ω. The following properties are equivalent:

1. f is holomorphic in Ω.
2. The form $\omega = f(z)dz$ is locally exact in Ω.
3. Cauchy's integral formula holds. That is,

$$f(z) = \int_\gamma \frac{f(t)}{z - t} dz \tag{10}$$

   for any closed piecewise C^1 path around z in Ω.
4. f is analytic on Ω.

G. Interlude: exercises

We take a small break from presenting new material to solve a few problems from Cartan, Ahlfors, and the MAT354 tutorial worksheets. Note that we will use a few concepts introduced in subsequent sections in these problems (i.e. the definition of a meromorphic function).

Exercise VII.1 (Cartan 2.2)**.** Recall the definition of an integral of a locally exact form over a continuous path $\gamma : [a, b] \to \mathbb{C}$: it is the difference $f(b) - f(a)$, where f is the primitive along γ (unique up to a constant).

Of course, if ω_1 and ω_2 are locally exact, then their sum $\omega_1 + \omega_2$ is as well: given $x \in \Omega$, let F_1 and F_2 be local primitives of ω_1 and ω_2 on U_1 and U_2 respectively. Then $F_1 + F_2$ is a local primitive on $U_1 \cap U_2$, an open neighbourhood of x. Moreover, let f_1 be the primitive along γ of ω_1, let f_2 be the primitive along γ of ω_2. Let $f = f_1 + f_2$. By definition, given $t \in [a, b]$, we can choose $x \in (t - \varepsilon, t + \varepsilon)$.

Exercise VII.2 (Cartan 2.3)**.** Recall that if a sequence of continuous functions f_n converges uniformly to f (which is continuous via uniform limit theorem), then

$$\lim_{n\to\infty} \int_I f_n = \int_I f \tag{11}$$

Exercise VII.3 (MAT354 8.1 and 8.2)**.** Clearly, if γ is piecewise-C^1 with C^1-components γ_k, then $f \circ \gamma$ is piecewise-C^1 with C^1-components $f \circ \gamma_k$, so this construction is well-defined. By definition, we have

$$\alpha(f) = w(f \circ \sigma, 0) = \frac{1}{2\pi i} \int_{f \circ \sigma} \frac{dz}{z} = \frac{1}{2\pi i} \int_0^1 (f \circ \sigma)^*(z^{-1}) d(f \circ \sigma)^*(z) \tag{12}$$

$$= \frac{1}{2\pi i} \int_0^1 \sigma^*(f(z)^{-1}) d\sigma^*(f(z)) = \frac{1}{2\pi i} \int_\sigma \frac{df}{f} = \frac{1}{2\pi i} \int_\sigma \frac{f'}{f} dz \tag{13}$$

This immediately implies that

$$\alpha(fg) = \frac{1}{2\pi i} \int_\sigma \frac{(fg)'}{fg} dz = \frac{1}{2\pi i} \int_\sigma \frac{(f'g + fg'}{fg} dz = \frac{1}{2\pi i} \int_\sigma \frac{f'}{f} dz + \frac{1}{2\pi i} \int_\sigma \frac{g'}{g} dz = \alpha(f) + \alpha(g). \tag{14}$$

Clearly, for $c \neq 0$, $\alpha(c) = 0$. Thus, for f non-zero on S^1, we have

$$0 = \alpha(1) = \alpha\left(\frac{f}{f}\right) = \alpha(f) + \alpha\left(\frac{1}{f}\right) \Longrightarrow \alpha\left(\frac{1}{f}\right) = -\alpha(f). \tag{15}$$

Exercise VII.4 (MAT354 8.3)**.** Note that $f(z) = 2z^3 - 5z^2 + 2z = z(z-2)(2z-1)$. It follows that

$$\alpha(f) = \alpha(z) + \alpha(z-2) + \alpha(2z-1) = \alpha(z) + \alpha(z-2) + \frac{1}{2}\alpha\left(z - \frac{1}{2}\right).$$

Of course, for a function of the form $p(z) = z - c$, we have $\alpha(p) = w(\sigma, c)$ (from Problem 8.2 above). σ is a circular path around the origin of unit radius. Thus, if c lies in the unit disk, then $w(\sigma, c) = 1$ and otherwise, $w(\sigma, c) = 0$. Therefore,

$$\alpha(z) + \alpha(z-2) + \frac{1}{2}\alpha\left(z - \frac{1}{2}\right) = 1 + 0 + \frac{1}{2} = \frac{3}{2}. \tag{16}$$

We can compure $\alpha(4z^2+1)$ in an identical fashion. In particular, note that $f(z) = 4z^2+1 = (2z+i)(2z-i)$. Thus,

$$\alpha(f) = \alpha(2z+i) + \alpha(2z-i) = \frac{1}{2}\alpha\left(z + \frac{i}{2}\right) + \frac{1}{2}\alpha\left(z - \frac{i}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1. \tag{17}$$

It follows that $\alpha(1/f) = -\alpha(f) = -1$. Note that f is non-zero on S^1, so we can apply this identity.

Exercise VII.5 (MAT354 8.4)**.** Recall that if a form is locally exact on Ω, and γ_1 and γ_2 two free homotopic loops in Ω, then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$. Since f is non-zero on $\overline{D}$, f'/f is holomorphic on $\overline{D}$. Clearly, σ is free homotopic to the constant zero-path. By Cauchy's theorem, the form $f'dz/f$ is locally exact. Therefore,

$$\alpha(f) = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f} dz = 0. \tag{18}$$

Exercise VII.6 (MAT354 8.5)**.** Without loss of generality, we can assume that we never have $\nu_k = \mu_j$. Define the rational function

$$R(z) = \frac{\prod_{i=1}^s (z - \mu_i)^{m_i}}{\prod_{j=1}^r (z - \nu_i)^{n_i}} \tag{19}$$

and let $g(z) = R(z)f(z)$. Clearly, this function is meromorphic, as it is the product of meromorphic functions. Moreover, for a given root ν_k, note that we can write $f(z) = (z - \nu_k)^{n_k} f_k(z)$ where $f_k(z)$ does not have a root nor a pole at ν_k. Thus,

$$g(z) = \frac{\prod_{i=1}^s (z - \mu_i)^{m_i}}{\prod_{j=1}^r (z - \nu_j)^{n_j}} (z - \nu_k)^{n_k} f_k(z) = \frac{\prod_{i=1}^s (z - \mu_i)^{m_i}}{\prod_{j \neq k} (z - \nu_j)^{n_j}} f_k(z) \tag{20}$$

Clearly, the right-hand side is a meromorphic function with no root and no pole at ν_k. We can repeat this process for all ν_k, and similarly for all μ_k. Thus, g does not have roots or poles at any of the points $\{\nu_k\}$ or $\{\mu_k\}$. But of course, the only possible locations of roots and poles within the disk for $g(z) = R(z)f(z)$ are at these points, so we can conclude that g has no roots or poles in the disk. It follows that

$$f(z) = \frac{\prod_{j=1}^{r}(z-\nu_j)^{n_j}}{\prod_{i=1}^{s}(z-\mu_i)^{m_i}} g(z) \tag{21}$$

and, from the previous problems above, we have

$$\alpha(f) = \alpha\left(\frac{\prod_{j=1}^{r}(z-\nu_j)^{n_j}}{\prod_{i=1}^{s}(z-\mu_i)^{m_i}} g(z)\right) = \sum_j n_j \alpha\left((z-\nu_j)\right) - \sum_j m_j \alpha\left((z-\mu_j)\right) + \alpha(g) \tag{22}$$

where we use $\alpha(fg) = \alpha(f) + \alpha(g)$ and $\alpha(1/f) = -\alpha(f)$. Since each ν_j and each μ_j lies in the unit disk, each of the $\alpha(\cdot)$ expression above evaluates to 1, except for $\alpha(g)$, which is 0 from 8.4. Thus,

$$\alpha(f) = \sum_j n_j - \sum_j m_j \tag{23}$$

as desired.

Exercise VII.7 (MAT354 8.6 and 8.7)**.** We will once again use the equivalence of integrals around homotopic paths. In particular, we will demonstrate that the homotopy $F_t(x) = f(x) + tg(x)$ never crosses the origin. Indeed, for some $t \in [0,1]$, note that for any $x \in S^1$,

$$|F_t(x)| \geq ||f(x)| - t|g(x)|| \geq ||f(x)| - |g(x)||.$$

We know that $|f(x)| \geq m$, while $|g(x)| < m$, immediately implying that $|F_t(x)| > 0$. Thus, it is in fact true that the homotopy never crosses the origin. It follows immediately that the path $t \mapsto f(e^{2\pi i t})$ and $t \mapsto (f+g)(e^{2\pi i t})$ are free-homotopic, so

$$\alpha(f) = w(f\circ\sigma, 0) = \frac{1}{2\pi i}\int_{f\circ\sigma}\frac{dz}{z} = \frac{1}{2\pi i}\int_{(f+g)\circ\sigma}\frac{dz}{z} = w((f+g)\circ\sigma, 0) = \alpha(f+g). \tag{24}$$

This completes the proof.

Now, note that $h(z) = 7z^5 - 2e^z$ is holomorphic (and thus meromorphic with no poles). Clearly, $|7z^5| = 7|z|^5 = 7$ for $z \in S^1$, while $|-2e^z| \leq 2e^{|z|} = 2e < 6$ for $z \in S^1$. It follows from the previous problems that the number of roots of h is $\alpha(7z^5) = 5\alpha(z) = 5$.

Exercise VII.8 (MAT354 8.8)**.** Obviously, we have for $|z| = R$

$$\frac{|g(z)|}{R^n} \leq \sum_{k=0}^{n-1}|g_k|\frac{|z|^k}{R^n} = \sum_{k=0}^{n-1}|g_k|R^{k-n} \to 0 \quad \text{as } R\to\infty \tag{25}$$

So for large enough R, $|g(z)| < R^n$. It follows that $f(Rz) = (Rz)^n + g(Rz)$ satisfies the conditions to apply Problem 8.6 for sufficiently large R, so $\alpha(f(Rz)) = \alpha((Rz)^n) = n$. Since f has no roots as it is a polynomial, this implies that the number of zeros is $n \geq 1$, so $f(Rz)$ has a root in the unit disk.

H. The Cauchy inequalities and their implications

I. The mean value property and its implications

VIII. Meromorphic functions and Laurent expansions

Definition VIII.1 (Meromorphic function)**.** A function f taking values in the complex plane is said to be meromorphic on Ω if there exists a set of isolated points, $\{p_k\}$, such that f is well-defined and holomorphic on $\Omega - \{p_k\}$.

Theorem VIII.1. If f is meromorphic, then it can be expressed as the quotient of two entire functions.

Proof. Let f be meromorphic, let □

Remark VIII.1 (Functions of the Riemann sphere)**.**

This set of definitions allows us to prove theorems of the following form:

Theorem VIII.2. If an entire function has a pole at ∞, then it is a polynomial.

Theorem VIII.3. If a function f is meromorphic on S^2, then it is rational.

Proof. Recall that f is meromorphic on S^2 if and only if it is meromorphic on the plane, and $f(1/z)$ is meromorphic around the origin. First of all, $f(1/z)$ being meromorphic at the origin means that there exists some M such that $z^M f(1/z)$ can be extended to an analytic function at the origin. In particular, this function is bounded in a neighbourhood of the origin, so we have for $|z| \leq R$,

$$\left| z^M f\left(\frac{1}{z}\right) \right| \leq C \Longrightarrow |f(z)| \leq C|z|^M \quad \text{for } |z| \geq R. \tag{26}$$

Now, let us prove that f has a finite number of poles. Indeed, suppose not, so $\{\alpha_k\}$ is an infinite sequence of poles. Note that the set of $1/\alpha_k$ are poles of $f(1/z)$, which is meromorphic. If the sequence of unbounded, then $f(1/z)$ will have infinitely many poles arbitrarily close to 0. Thus, $f(1/z)$ cannot be analytic at 0, so it must have a pole here, but then this pole won't be isolated, a contradiction. Similarly, if the α_k are bounded, then they have a convergent subsequence. This point will necessarily be a pole, but also can't be isolated, a contradiction.

Thus, f must have a finite number of poles $\alpha_1, \ldots, \alpha_n$. Let

$$g(z) = \prod_j (z - \alpha_j)^{m_j} f(z) \tag{27}$$

which is analytic at all points in the plane. For $|z|$ large enough, we have

$$|g(z)| \leq C|z|^M \prod_j |z - \alpha_j|^{m_j} \tag{28}$$

which implies from the Cauchy inequalities that g must be a polynomial. Thus, f is rational, as desired. □

A. Interlude: exercises

It is time for another exercise interlude.

Exercise VIII.1.