MAT497 running notes

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(Dated: Monday 23rd September, 2024)

Contents

I. Introduction

These notes are not particularly organized but are simply meant to record information in preparation for making more structure write-ups of the things I'm reading.

II. Connections

The goal of this section is to review some background information on abstract connections, defined on Lie algebroids: a key ingredient in the Riemann-Hilbert correspondence.

We take the approach of defining abstract connections on Lie algebroids. The case of a connection on a vector bundle is a special case.

Definition II.1 (Lie algebroid)**.** A vector bundle $\pi_A : A \to M$ over a smooth manifold M, equipped with a bracket $[\cdot, \cdot]$ on the sections $\Gamma(A)$ and an anchor map $\rho : A \to TM$. The anchor map is a vector bundle morphism, meaning that $\pi_A = \pi \circ \rho$. Moreover, the bracket must satisfy a Leibniz rule:

$$[X, fY] = \mathcal{L}_{\rho(X)}(f) \cdot Y + f[X, Y] \tag{1}$$

where $\mathcal{L}_{\rho(X)}$ is the Lie derivative with respect to the vector field $\rho(X) \in \mathfrak{X}(M)$.

Example II.1 (Tangent bundle)**.** The most obvious example of a Lie algebroid is TM itself. The anchor is the identity map and the bracket is the standard Lie bracket between smooth vector fields $X, Y \in \mathfrak{X}(M)$. Indeed,

$$[X, fY] = \mathcal{L}_X(fY) = X(f) \cdot Y + f\mathcal{L}_X(Y) = X(f) \cdot Y + f[X, Y] \tag{2}$$

as required.

Definition II.2 (Connection)**.** Let $A \to M$ be a Lie algebroid, let E be a vector bundle over M. An E-connection relative to A is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(a, e) \mapsto \nabla_a e$ such that $\nabla_{fa} e = f \nabla_a e$ for all $f \in C^\infty(M)$, and

$$\nabla_a fe = \mathcal{L}_{\rho(a)}(f) \cdot e + f \nabla_a e \tag{3}$$

where ρ is the anchor of A. A connection is said to be flat if $\nabla_{[a,b]} = [\nabla_a, \nabla_b]$. Observe that fixing $a \in \Gamma(A)$, ∇_a is a linear map from $\Gamma(E)$ to itself satisfying the Leibniz rule of the above formula.

Example II.2. In the case that $A = TM$, we recover the definition of a connection on smooth manifold M.

To better contextualize a connection as a "derivative of a vector bundle", we can write down an explicit local form. For simplicity, we will assume that $A = TM$ for the time being.

Remark II.1 (Local form of connection)**.** Let M be a smooth manifold, let us consider first the case of the trivial bundle $E = M \times \mathbb{R}^n$. Any section $\sigma \in \Gamma(E) = \Gamma(M \times \mathbb{R}^n)$ will of course be of the form $\sigma : p \mapsto (f_1(p), \ldots, f_n(p)) = f_1(p)e_1 + \cdots + f_n(p)e_n$, for $f_j \in C^\infty(M)$. Pick some $X \in \mathfrak{X}(M)$. By linearity, we simply must determine $\nabla_X(f_j e_j)$ for each j. We have, by definition

$$\nabla_X(f_j e_j) = \mathcal{L}_X(f_j) \cdot e_j + f_j \nabla_X(e_j) = df_j(X) \cdot e_j + f_j \nabla_X(e_j) \tag{4}$$

Note that $\nabla_X(e_j) \in \Gamma(M \times \mathbb{R}^n)$ for each j. Thus, we let $\nabla_X(e_j)(p) = A_{j1}(p)e_1 + \cdots + A_{jn}(p)e_n$. We let $A(p)$ be the matrix with entries $A_{jk}(p)$. It follows from this that we can write ∇_X, with a slight abuse of notation, in the form $\iota_X d + A_X$, where d is the exterior derivative, ι_X is the inner product relative to X, and A_X is an element of $\Gamma(\mathrm{End}(E))$ over $C^\infty(M)$. Indeed, we have

$$\nabla_X(\sigma)(p) = \sum_{j=1}^n \nabla_X(f_j e_j)(p) = \sum_{j=1}^n df_j(X)(p)e_j + \sum_{j=1}^n \sum_{k=1}^n f_j(p)A_{jk}(p)e_k \tag{5}$$

$$= (df_1(X), \ldots, df_n(X))(p) + A(p) \cdot (f_1(p), \ldots, f_n(p)) \tag{6}$$

so this notation is justified.

Let us generalize to the case of a general vector bundle E. Let $(U_\alpha, \varphi_\alpha)$ be the local trivialization of E, so $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ is a homeomorphism and the fibre maps are linear isomorphisms between vector spaces. Then if $\sigma \in \Gamma(E)$, the restriction $\varphi_\alpha \circ \sigma|_{U_\alpha} : U_\alpha \to U_\alpha \times \mathbb{R}^n$ is well-defined, and is a section of the trivial bundle $U_\alpha \times \mathbb{R}^n$ over open submanifold U_α, as

$$\mathrm{proj} \circ (\varphi_\alpha \circ \sigma|_{U_\alpha}) = \pi \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \sigma|_{U_\alpha} = \pi \circ \sigma|_{U_\alpha} = \mathrm{id}|_{U_\alpha}. \tag{7}$$

It is equally easy to show that if $\sigma \in \Gamma(U_\alpha \times \mathbb{R})$, then $\varphi_\alpha^{-1} \circ \sigma$ is in $\Gamma(\pi^{-1}(U_\alpha))$.

It is important for us to show first that $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is a *local map.* In particular, if σ and σ' are elements of $\Gamma(E)$ which agree on an open subset V of M, then $\nabla_X(\sigma)(p) = \nabla_X(\sigma')(p)$ for all $p \in V$. To prove this, note that for some $p \in V$, we can always take a coordinate ball W with closure contained in V around p, as $V \subset M$, a smooth manifold. We pick a smooth bump function $\chi : M \to \mathbb{R}$ which is 1 inside $\overline{W}$ and vanishes outside V. Clearly, $\chi\sigma = \chi\sigma'$. We then have, from the Leibniz rule,

$$\mathcal{L}_X(\chi)(p) \cdot \sigma(p) + \nabla_X(\sigma)(p) = \nabla_X(\chi\sigma)(p) = \nabla_X(\chi\sigma')(p) = \mathcal{L}_X(\chi)(p) \cdot \sigma'(p) + \nabla_X(\sigma')(p) \tag{8}$$

which means that $\nabla_X(\sigma)(p) = \nabla_X(\sigma')(p)$ as desired. Moreover, suppose X and Y agree on some open neighbourhood V, we take W and χ the same as above, note that $\chi X = \chi Y$, and get

$$\chi(p)\nabla_X(\sigma)(p) = \nabla_{\chi X}(\sigma)(p) = \nabla_{\chi Y}(\sigma)(p) = \chi(p)\nabla_Y(\sigma)(p) \tag{9}$$

so that $\nabla_X(\sigma)(p) = \nabla_Y(\sigma)(p)$. It follows from these locality conditions that given the connection ∇, we can consider open submanifold U and subbundle $\pi^{-1}(U)$, and conclude that the restricted connection $\nabla|_U : \mathfrak{X}(U) \times \Gamma(\pi^{-1}(U)) \to \Gamma(\pi^{-1}(U))$ given by extending $X \in \mathfrak{X}(U)$ and $\sigma \in \Gamma(\pi^{-1}(U))$ to vector field and section in M and

$\Gamma(E)$ respectively and feeding to ∇, then restricting to U, is well-defined. This map can easily be verified to be a valid connection on the submanifold. Moreover, given $\sigma \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, the locality results imply that

$$\nabla_X(\sigma)|_U = (\nabla|_U)_{X|_U}(\sigma|_U). \tag{10}$$

It follows from this that to understand the local behaviour of ∇, we can look at $\nabla|_{U_\alpha}$ for each of the trivializations $(U_\alpha, \varphi_\alpha)$. In particular, we will consider the maps $\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}) : \mathfrak{X}(U_\alpha) \times \Gamma(U_\alpha \times \mathbb{R}^n) \to \Gamma(U_\alpha \times \mathbb{R}^n)$, where in this context, φ_α^{-1} is sending section $\sigma \in \Gamma(U_\alpha \times \mathbb{R}^n)$ to $\varphi_\alpha^{-1} \circ \sigma \in \Gamma(\pi^{-1}(U_\alpha))$, and φ_α is doing the opposite. Our claim is that $\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1})$ is itself a connection on the trivial bundle $U_\alpha \times \mathbb{R}^n$. Note that φ_α^{-1} is a linear isomorphism when restricted to fibres. Therefore,

$$\begin{aligned}
(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1})(X + fY, \sigma) &= \varphi_\alpha \circ (\nabla|_{U_\alpha})_{X+fY}\varphi_\alpha^{-1}\sigma & (11)\\
&= \varphi_\alpha \circ \left[(\nabla|_{U_\alpha})_X\varphi_\alpha^{-1}\sigma + f(\nabla|_{U_\alpha})_Y\varphi_\alpha^{-1}\sigma\right] & (12)\\
&= f(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(Y, \sigma) + (\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(X, \sigma) & (13)
\end{aligned}$$

Moreover, note that $(\varphi_\alpha^{-1}g\sigma)(p) = \varphi_\alpha^{-1}(p, g(p)\sigma(p)) = g(p)\varphi_\alpha^{-1}(p, \sigma(p))$, so $\varphi_\alpha^{-1}g\sigma = g\varphi_\alpha^{-1}\sigma$. It follows that

$$\begin{aligned}
(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(X, g\sigma) &= \varphi_\alpha \circ (\nabla|_{U_\alpha})_X g\varphi_\alpha^{-1}\sigma & (14)\\
&= \varphi_\alpha \circ \left[\mathcal{L}_X(g)\varphi_\alpha^{-1}\sigma + g(\nabla|_{U_\alpha})_X\varphi^{-1}\sigma\right] & (15)\\
&= \mathcal{L}_X(g)\sigma + g(\varphi_\alpha \circ (\nabla|_{U_\alpha}) \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(X, \sigma) & (16)
\end{aligned}$$

so we have the properties needed for a connection.

It follows from the previous argument that $\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1})$ can be written in the form $\iota_X d + A_{\alpha,X}$. In other words, up to linear isomorphisms of $\mathfrak{X}(U_\alpha) \times \Gamma(U_\alpha \times \mathbb{R}^n)$ and $\Gamma(U_\alpha \times \mathbb{R}^n)$, which are easy to characterize, we can describe the local form of the connection.

To find the matrix $A_{\alpha,X}$, let $\varphi_\alpha = (x_\alpha^1, \ldots, x_\alpha^n)$ be the coordinates associated to the chart. Let $e_1, \ldots, e_n$ be the standard global basis for the trivial bundle. Note that

$$(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))\left(\frac{d}{dx_\alpha^i}, x_\alpha^j e_k\right) = dx_\alpha^j\left(\frac{d}{dx_\alpha^i}\right)e_k + x_\beta^j A_{\alpha,X}e_k = \delta_{ij} + x_\beta^j A_{\alpha,X}e_k. \tag{17}$$

Thus, we can recover the linear transformation $A_{\alpha,X}$ from this data.

A. Parallel transport and holonomy

B. The Gauss-Manin connection

C. Relationship to sheaves

In many standard formulations of the Riemann-Hilbert correspondence, connections are described as maps of sheaves rather than sections of vector bundles.

III. Monodromy

The goal of this section is to exposit the idea of monodromy, also in preparation for a discussion of the Riemann-Hilbert correspondence. I'm not sure if this is the most general discussion of monodromy that is possible, as I'm not making reference to any sort of algebraic variety, I'm just working in the complex plane.

Definition III.1. If X is connected and locally simply connected, and $\pi : \widetilde{X} \to X$ is a universal cover, a multi-valued function f on X is a continuous function on $\widetilde{X}$. If U is a simply connected open subset of X and $s : U \to \widetilde{X}$ is a local section, then $f \circ s$ defines a branch of the multivalued function.

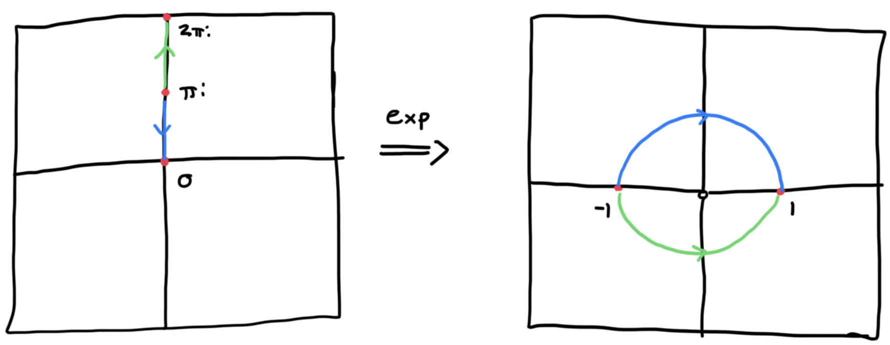
Example III.1 (Square root)**.** Suppose $X = \mathbb{C} - 0 \simeq \mathbb{R}^2 - 0$. We know that $\exp : \mathbb{C} \to X$ will be a universal cover. Of course, we cannot define a square root as a single-valued function on X. However, suppose we let $f(z) = e^{z/2}$ on $\widetilde{X} = \mathbb{C}$. Of course, for any $e^z \in X$, we will have $\pi^{-1}(e^z) = \{z + 2\pi n\}_{n\in\mathbb{Z}}$. Suppose we have U as a particular open set and s a local section, so $s(e^z) = z + 2\pi k$ for a particular k.

We then have $(f \circ s)(e^z) = e^{\pi k} e^{z/2}$, which is equal to $e^{z/2}$ if k is even and $-e^{z/2}$ is k is odd, both of which square to e^z.

Definition III.2. Given some disc D centred at $a \in \Omega$, where Ω is some complex domain, and f is a holomorphic function on D such that the Taylor series of f centred at a converges in D, we call the pair (D, f) an *analytic element*. Given analytic element (D, f) around a, we say that this analytic element has a prolongation to element (D', f') around b if we can find a curve γ such that $\gamma(0) = a$, $\gamma(1) = b$, and the image of γ can be covered by analytic elements centred at points on the curve, such that any two such elements will agree on their intersection.

It is very important to note that prolongations of an analytic element can differ on different paths. One way to see this is by considering some multivalued function extended along paths which wind around some singularity in a complex domain. Let us consider the example of the square root function, given above.

Example III.2 (Square root prolongation)**.** For example, suppose $\Omega = \mathbb{C} - 0$ and we look at the square root function. Take $z_0 = -1$, and let $z_1 = 1$. We can look at two paths, γ_1 and γ_2, semi-circles in the upper-half and lower-half plane respectively. We let $f : \mathbb{C} \to \mathbb{C}$ be the square root on the covering space. From here, consider paths $\widetilde{\gamma}_1, \widetilde{\gamma}_2 : [0, 1] \to \mathbb{C}$ in the universal cover, where $\widetilde{\gamma}_1$ is the straight line from πi to 0, and $\widetilde{\gamma}_2$ is the straight line from πi to $2\pi i$. Note that $f \circ \widetilde{\gamma}_1 = \gamma_1$ and $f \circ \widetilde{\gamma}_2 = \gamma_2$.



For each point $e^{i\theta} \in S^1 - (1, 0)$ with $\theta \in (0, 2\pi)$ unique, we define D_θ to be the radius-1 open disc around $e^{i\theta}$. We define $s_\theta : D_\theta \to \mathbb{C}$ as taking a point $z = re^{i\varphi}$ in D_θ to $\log(r) + i\varphi$, where φ is the unique angle representing z contained in $(\theta - \pi/2, \theta + \pi/2)$. From here, one can see that the sequence of analytic elements $(D_\theta, f \circ s_\theta)$ along γ_1 define a prolongation to $(D_0, f \circ s_0)$, while the same sequence taken along γ_2 defines a prolongation to $(D_{2\pi}, f \circ s_{2\pi})$. While it is true that $D_0 = D_{2\pi}$ are radius-1 discs around 1, note that $f \circ s_0 \neq f \circ s_{2\pi}$. In fact, one can check that

$$(f \circ s_0)(z) = -(f \circ s_{2\pi})(z) \tag{18}$$

so we have given prolongations which yield different results at the same endpoint.

Despite the fact that we just gave an example where prolongations to the same endpoint yield different results, there is a result (know to some as *the monodromy theorem*) which gives conditions for prolongations to be the same.

Theorem III.1 (Classical monodromy theorem)**.** If paths $\gamma_0, \gamma_1 \in \Omega$ joining a and b are path-homotopic via some homotopy γ_s for $s \in [0, 1]$, and we have prolongations of some analytic element (D, f) around a to analytic elements (D_s, f_s) along each γ_s (for a fixed s), then $f_0 = f_1$ on $D_0 \cap D_1$: the analytic elements agree.

Proof. □

Definition III.3 (Riemann surfaces from multivalued functions).

Definition III.4 (Monodromy action).

A. The Schlesinger system

IV. The Riemann-Hilbert correspondence

Goal is to discuss, at a basic level, the Riemann-Hilbert correspondence for vector bundles.

V. Algebraic geometry