# MAT497 running notes

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**Contents**



## I. Connections

*The goal of this section is to review some background information on abstract connections, defined on Lie algebroids: a key ingredient in the Riemann-Hilbert correspondence.*

### A. Basic definitions and local description

We take the approach of defining abstract connections on Lie algebroids. The case of a connection on a vector bundle is a special case.

**Definition I.1** (Lie algebroid)**.** A vector bundle $\pi_A : A \to M$ over a smooth manifold $M$, equipped with a bracket $[\cdot, \cdot]$ on the sections $\Gamma(A)$ and an anchor map $\rho : A \to TM$. The anchor map is a vector bundle morphism, meaning that $\pi_A = \pi \circ \rho$. Moreover, the bracket must satisfy a Leibniz rule:

$$[X, fY] = \mathcal{L}_{\rho(X)}(f) \cdot Y + f[X, Y] \tag{1}$$

where $\mathcal{L}_{\rho(X)}$ is the Lie derivative with respect to the vector field $\rho(X) \in \mathfrak{X}(M)$.

**Example I.1** (Tangent bundle)**.** The most obvious example of a Lie algebroid is $TM$ itself. The anchor is the identity map and the bracket is the standard Lie bracket between smooth vector fields $X, Y \in \mathfrak{X}(M)$. Indeed,

$$[X, fY] = \mathcal{L}_X(fY) = X(f) \cdot Y + f\mathcal{L}_X(Y) = X(f) \cdot Y + f[X, Y] \tag{2}$$

as required.

**Definition I.2** (Connection)**.** Let $A \to M$ be a Lie algebroid, let $E$ be a vector bundle over $M$. An $E$-connection relative to $A$ is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(a, e) \mapsto \nabla_a e$ such that $\nabla_{fa} e = f\nabla_a e$ for all $f \in C^\infty(M)$, and

$$\nabla_a fe = \mathcal{L}_{\rho(a)}(f) \cdot e + f\nabla_a e \tag{3}$$

where $\rho$ is the anchor of $A$. A connection is said to be flat if $\nabla_{[a,b]} = [\nabla_a, \nabla_b]$. Observe that fixing $a \in \Gamma(A)$, $\nabla_a$ is a linear map from $\Gamma(E)$ to itself satisfying the Leibniz rule of the above formula.

**Example I.2.** In the case that $A = TM$, we recover the definition of a connection on smooth manifold $M$.

To better contextualize a connection as a "derivative of a vector bundle", we can write down an explicit local form. For simplicity, we will assume that $A = TM$ for the time being.

**Remark I.1** (Local form of connection)**.** Let $M$ be a smooth manifold, let us consider first the case of the trivial bundle $E = M \times \mathbb{R}^n$. Any section $\sigma \in \Gamma(E) = \Gamma(M \times \mathbb{R}^n)$ will of course be of the form $\sigma : p \mapsto (f_1(p), \ldots, f_n(p)) = f_1(p)e_1 + \cdots + f_n(p)e_n$, for $f_j \in C^\infty(M)$. Pick some $X \in \mathfrak{X}(M)$. By linearity, we simply must determine $\nabla_X(f_j e_j)$ for each $j$. We have, by definition

$$\nabla_X(f_j e_j) = \mathcal{L}_X(f_j) \cdot e_j + f_j \nabla_X(e_j) = df_j(X) \cdot e_j + f_j \nabla_X(e_j) \tag{4}$$

Note that $\nabla_X(e_j) \in \Gamma(M \times \mathbb{R}^n)$ for each $j$. Thus, we let $\nabla_X(e_j)(p) = A_{j1}(p)e_1 + \cdots + A_{jn}(p)e_n$. We let $A(p)$ be the matrix with entries $A_{jk}(p)$. It follows from this that we can write $\nabla_X$, with a slight abuse of notation, in the form $\iota_X d + A_X$, where $d$ is the exterior derivative, $\iota_X$ is the inner product relative to $X$, and $A_X$ is an element of $\Gamma(\mathrm{End}(E))$ over $C^\infty(M)$. Indeed, we have

$$\nabla_X(\sigma)(p) = \sum_{j=1}^n \nabla_X(f_j e_j)(p) = \sum_{j=1}^n df_j(X)(p)e_j + \sum_{j=1}^n \sum_{k=1}^n f_j(p)A_{jk}(p)e_k \tag{5}$$

$$= (df_1(X), \ldots, df_n(X))(p) + A(p) \cdot (f_1(p), \ldots, f_n(p)) \tag{6}$$

so this notation is justified.

Let us generalize to the case of a general vector bundle $E$. Let $(U_\alpha, \varphi_\alpha)$ be the local trivialization of $E$, so $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ is a homeomorphism and the fibre maps are linear isomorphisms between vector spaces. Then if $\sigma \in \Gamma(E)$, the restriction $\varphi_\alpha \circ \sigma|_{U_\alpha} : U_\alpha \to U_\alpha \times \mathbb{R}^n$ is well-defined, and is a section of the trivial bundle $U_\alpha \times \mathbb{R}^n$ over open submanifold $U_\alpha$, as

$$\mathrm{proj} \circ (\varphi_\alpha \circ \sigma|_{U_\alpha}) = \pi \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \sigma|_{U_\alpha} = \pi \circ \sigma|_{U_\alpha} = \mathrm{id}|_{U_\alpha}. \tag{7}$$

It is equally easy to show that if $\sigma \in \Gamma(U_\alpha \times \mathbb{R})$, then $\varphi_\alpha^{-1} \circ \sigma$ is in $\Gamma(\pi^{-1}(U_\alpha))$.

It is important for us to show first that $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is a *local map.* In particular, if $\sigma$ and $\sigma'$ are elements of $\Gamma(E)$ which agree on an open subset $V$ of $M$, then $\nabla_X(\sigma)(p) = \nabla_X(\sigma')(p)$ for all $p \in V$. To prove this, note that for some $p \in V$, we can always take a coordinate ball $W$ with closure contained in $V$ around $p$, as $V \subset M$, a smooth manifold. We pick a smooth bump function $\chi : M \to \mathbb{R}$ which is 1 inside $\overline{W}$ and vanishes outside $V$. Clearly, $\chi\sigma = \chi\sigma'$. We then have, from the Leibniz rule,

$$\mathcal{L}_X(\chi)(p) \cdot \sigma(p) + \nabla_X(\sigma)(p) = \nabla_X(\chi\sigma)(p) = \nabla_X(\chi\sigma')(p) = \mathcal{L}_X(\chi)(p) \cdot \sigma'(p) + \nabla_X(\sigma')(p) \tag{8}$$

which means that $\nabla_X(\sigma)(p) = \nabla_X(\sigma')(p)$ as desired. Moreover, suppose $X$ and $Y$ agree on some open neighbourhood $V$, we take $W$ and $\chi$ the same as above, note that $\chi X = \chi Y$, and get

$$\chi(p)\nabla_X(\sigma)(p) = \nabla_{\chi X}(\sigma)(p) = \nabla_{\chi Y}(\sigma)(p) = \chi(p)\nabla_Y(\sigma)(p) \tag{9}$$

so that $\nabla_X(\sigma)(p) = \nabla_Y(\sigma)(p)$. It follows from these locality conditions that given the connection $\nabla$, we can consider open submanifold $U$ and subbundle $\pi^{-1}(U)$, and conclude that the restricted connection $\nabla|_U : \mathfrak{X}(U) \times \Gamma(\pi^{-1}(U)) \to \Gamma(\pi^{-1}(U))$ given by extending $X \in \mathfrak{X}(U)$ and $\sigma \in \Gamma(\pi^{-1}(U))$ to vector field and section in $M$ and $\Gamma(E)$ respectively and feeding to $\nabla$, then restricting to $U$, is well-defined. This map can easily be verified to be a valid connection on the submanifold. Moreover, given $\sigma \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, the locality results imply that

$$\nabla_X(\sigma)|_U = (\nabla|_U)_{X|_U}(\sigma|_U). \tag{10}$$

It follows from this that to understand the local behaviour of $\nabla$, we can look at $\nabla|_{U_\alpha}$ for each of the trivializations $(U_\alpha, \varphi_\alpha)$. In particular, we will consider the maps $\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}) : \mathfrak{X}(U_\alpha) \times \Gamma(U_\alpha \times \mathbb{R}^n) \to \Gamma(U_\alpha \times \mathbb{R}^n)$, where in this context, $\varphi_\alpha^{-1}$ is sending section $\sigma \in \Gamma(U_\alpha \times \mathbb{R}^n)$ to $\varphi_\alpha^{-1} \circ \sigma \in \Gamma(\pi^{-1}(U_\alpha))$, and $\varphi_\alpha$ is doing the opposite. Our claim is that $\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1})$ is itself a connection on the trivial bundle $U_\alpha \times \mathbb{R}^n$. Note that $\varphi_\alpha^{-1}$ is a linear isomorphism when restricted to fibres. Therefore,

$$
\begin{aligned}
(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1})(X + fY, \sigma) &= \varphi_\alpha \circ (\nabla|_{U_\alpha})_{X+fY} \varphi_\alpha^{-1} \sigma && (11)\\
&= \varphi_\alpha \circ \left[(\nabla|_{U_\alpha})_X \varphi_\alpha^{-1}\sigma + f(\nabla|_{U_\alpha})_Y \varphi_\alpha^{-1}\sigma\right] && (12)\\
&= f(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(Y, \sigma) + (\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(X, \sigma) && (13)
\end{aligned}
$$

Moreover, note that $(\varphi_\alpha^{-1} g\sigma)(p) = \varphi_\alpha^{-1}(p, g(p)\sigma(p)) = g(p)\varphi_\alpha^{-1}(p, \sigma(p))$, so $\varphi_\alpha^{-1} g\sigma = g\varphi_\alpha^{-1}\sigma$. It follows that

$$
\begin{aligned}
(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(X, g\sigma) &= \varphi_\alpha \circ (\nabla|_{U_\alpha})_X g \varphi_\alpha^{-1}\sigma && (14)\\
&= \varphi_\alpha \circ \left[\mathcal{L}_X(g)\varphi_\alpha^{-1}\sigma + g(\nabla|_{U_\alpha})_X \varphi^{-1}\sigma\right] && (15)\\
&= \mathcal{L}_X(g)\sigma + g(\varphi_\alpha \circ (\nabla|_{U_\alpha}) \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(X, \sigma) && (16)
\end{aligned}
$$

so we have the properties needed for a connection.

It follows from the previous argument that $\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1})$ can be written in the form $\iota_X d + A_{\alpha,X}$. In other words, up to linear isomorphisms of $\mathfrak{X}(U_\alpha) \times \Gamma(U_\alpha \times \mathbb{R}^n)$ and $\Gamma(U_\alpha \times \mathbb{R}^n)$, which are easy to characterize, we can describe the local form of the connection.

To find the matrix $A_{\alpha,X}$, let $\varphi_\alpha = (x_\alpha^1, \ldots, x_\alpha^n)$ be the coordinates associated to the chart. Let $e_1, \ldots, e_n$ be the standard global basis for the trivial bundle. Note that

$$
(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))\left(\frac{d}{dx_\alpha^i}, x_\alpha^j e_k\right) = dx_\alpha^j \left(\frac{d}{dx_\alpha^i}\right) e_k + x_\beta^j A_{\alpha,X} e_k = \delta_{ij} + x_\beta^j A_{\alpha,X} e_k. \quad (17)
$$

Thus, we can recover the linear transformation $A_{\alpha,X}$ from this data.

### B. A digression on parallel transport and holonomy

### C. Relationship to sheaves

In many standard formulations of the Riemann-Hilbert correspondence, connections are described as maps of sheaves rather than sections of vector bundles.

## II. Results of Katz-Oda

I will begin these notes by discussing some of the prior art. We begin by considering the ODE

$$
\frac{dY}{dz} = A(z)Y \quad (18)
$$

where $A(z)$ is a matrix whose entries are rational functions of $z$ relative to some algebraic number field $K$ (a field extension of $\mathbb{Q}$ such that $[K : \mathbb{Q}]$ is finite). Some entry of $A(z)$ will be of the form

$$
A_{ij}(z) = \frac{A_{ij}^{(n)} z^n + \cdots + A_{ij}^{(0)}}{B_{ij}^{(m)} z^m + \cdots + B_{ij}^{(0)}} \quad (19)
$$

which of course may be reduced modulo some prime $p \in K$ for almost all primes (namely those which don't divide any of the coefficients $B_{ij}^{(k)}$). The result will be a differental equation over $\mathbb{F}_q[z]$ for some finite field $\mathbb{F}_q$. As a particular example, suppose $a \in \mathbb{Z} \subset \mathbb{Q}$ and we have the ODE

$$
\frac{dy}{dz} = \frac{1}{az} y \quad (20)
$$

We can reduce by $p \nmid a$. When we do such a reduction, the equation admits solutions $y = z^b$ for any $b$ such that $ab \equiv 1 \mod p$, clearly.

There is a result concerning ODEs of this form, formulated by Katz and Oda, which we can restate here

**Theorem II.1.** The ODE of Eq. (18) is an algebraic differential equation, meaning that its solution is an algebraic function. Moreover, there exists sufficiently large $N$ such that for almost all primes $p$,

$$\left(\frac{d}{dz} - A(z)\right)^{Np} \equiv 0 \mod p \tag{21}$$

## III. Monodromy basics

*The goal of this section is to exposit the idea of monodromy, also in preparation for a discussion of the Riemann-Hilbert correspondence*

### A. The Schlesinger system

## IV. Introduction to the Riemann-Hilbert correspondence

*Goal is to discuss, at a basic level, the Riemann-Hilbert correspondence for vector bundles.*