Complex Analysis

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Contents

I. Introduction

The goal of these notes is to follow a basic first course in complex analysis, following Edward Bierstone's course at the University of Toronto (MAT354), in the Fall of 2023. I will also be adding information from my own readings/other resources. I hope that someone finds these notes useful. However, in the event that no one does, I certainly will!

Here is a list of other resources that I will draw from in these notes:

- Edward Bierstone's lectures (MAT354, University of Toronto, Fall 2023)

- Cartan, *Elementary theory of analytic functions of one or several complex variables*

- Ahlfors, *Complex analysis*

II. Basics and notation

We begin our discussion of complex analysis by introducing the complex plane, $\mathbb{C}$. Of course, $\mathbb{C}$ can be constructed via taking $\mathbb{R}^2$ and imposing a particular complex structure over it, which tells us how to multiply by the imaginary unit i.

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Definition II.1 (Complex structure)**.**

We can also construct $\mathbb{C}$ via taking $\mathbb{R}[x]$ and quotienting by the ideal $(x^2 + 1)$. For the purposes of these introductory notes, it is not particularly important to dwell on this point: all that we need to know is that there are many different ways to construct $\mathbb{C}$, each of which are equivalent. They are all topologically identical to $\mathbb{R}^2$, and posses the desired "complex arithmetic" that we are familiar with and want to use.

We will assume basic familiarity with fundamental notions related to the manipulation of complex numbers. As for notation, we let $\overline{z}$ denote the complex conjugate of $z \in \mathbb{C}$. We let $|z|$ denote the absolute value or modulus. We let $\Re[z]$ and $\Im[z]$ denote the real and complex components in $\mathbb{R}$ of $z \in \mathbb{C}$, respectively. Assuming that we have already constructed the function $\sin(t)$ and $\cos(t)$, and know their basic properties, it is straightforward to show that any $(x, y) \in \mathbb{C}$ can be expressed as $(r\cos(\theta), r\sin(\theta))$: we simply set $r = \sqrt{x^2 + y^2}$ and apply the Pythagorean identity to $\cos(\theta)$ and $\sin(\theta)$, which allows us to choose $\theta \in [0, 2\pi)$ satisfying the desired equality.

Definition II.2 (Polar coordinates)**.** The map going from $\mathbb{R}^2$ to $\mathbb{R} \times [0, 2\pi)$ which sends $(x, y) \mapsto (r, \theta)$ via the above protocol is a bijection where the imagine of (x, y) is refered to as the *polar coordinate representation* of a pair of coordinates (x, y).

Definition II.3 (Complex exponential)**.** We define the map $t \mapsto \cos(t) + i\sin(t) := e^{it}$ from $\mathbb{R}$ to $\mathbb{C}$ and refer to it as the *complex exponential*. It is easy to demonstrate that the complex exponential has all the standard properties of a regular exponential function, when it comes to the inverse, multiplication, powers, etc. This definition also agrees with the series definition of the exponential function mapping from $\mathbb{R}$ to $\mathbb{R}$, where we substitute an imaginary number, and utilize the multiplication rule for repeatedly multiplying i with itself.

Note: I may revise this section in the future, if I think of any extra, basic information that must be exposited/notation that must be explained.

III. The Riemann sphere and complex projective space

We are now able to move on to a discussion of different representations of the complex plane.

Definition III.1 (Riemann sphere)**.** Rather than strictly working with the standard complex plane $\mathbb{C}$, we will often consider its one-point compactification $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where ∞ is refered to as the "point at infinity". $\overline{\mathbb{C}}$ is called the Riemann sphere, as it is homeomorphic to a sphere (topologically, the one-point compactification of $\mathbb{R}^2$ is homeomorphic to S^2).

Of course, the function addition $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and multiplication, $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ must be extended to functions from $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ (to the greatest extent possible). We take

$$a + \infty = \infty + a = \infty \quad \text{for all } a \neq \infty \tag{1}$$
$$b \cdot \infty = \infty \cdot b = \infty \quad \text{for all } b \neq 0 \tag{2}$$

Now, when we say "to the greatest extent possible" it is clear what we mean: we cannot define $0 \cdot \infty$ or $\infty + \infty$, or else we will violate the standard laws of addition and multiplication that we wish to satisfy with our new, extended arithmetic. For example, one may be inclined to say that $\infty + \infty = \infty$, but then if additive cancellation holds, $\infty = 0$, a contradiction.

Definition III.2 (Stereographic projection)**. TODO**: Fill this in

Claim III.1. The one-point compactification of $\mathbb{R}^2$ is homeomorphic to S^2 via the stereographic projection.

Definition III.3 (Complex projective space)**. TODO**: Fill this in

Now, let us move back to the discussion of the Riemann sphere, which is the representation of the complex numbers we will work with the most. It is natural to ask what the correspondence is between basic objects in $\overline{\mathbb{C}}$ and S^2, via the stereographic projection. We begin with a result.

Theorem III.1. All circles in the Riemann sphere correspond to lines or circles in $\overline{\mathbb{C}}$. Conversely, all lines and circles in $\overline{\mathbb{C}}$ correspond to circles in the Riemann sphere.

IV. Introduction to complex functions

A. Linear, polynomial, and rational functions

Now that we have described the arena in which we perform calculations (the complex plane/Riemann sphere), we can move on to understanding corresponding functional transformations.

Claim IV.1. Any real linear function is of the form $w(z) = az$ for some $a \in \mathbb{C}$, or $w(z) = a\overline{z}$.

Proof. **TODO**: Fill this in $\qquad\square$

Definition IV.1. Given a complex rational function $R : \mathbb{C} \to \overline{\mathbb{C}}$, we can define an extension of R to $\overline{\mathbb{C}}$ where $R(\infty) = \lim_{z \to 0} R\left(\frac{1}{z}\right)$. Clearly, $R\left(\frac{1}{z}\right)$ is a well-defined rational function on $\mathbb{C}$, so $\lim_{z\to 0} R\left(\frac{1}{z}\right)$ is a well-defined point in $\overline{\mathbb{C}}$.

Given a rational function, it is possible to quickly sketch a rough portrait of its behaviour via analyzing its roots and poles.

Definition IV.2 (Roots and poles)**.** Given a rational function $R(z) = \frac{P(z)}{Q(z)}$ in reduced form, defined on $\overline{\mathbb{C}}$ (so $R(\infty)$ is well-defined) the roots of R are the collection of all z such that $P(z) = 0$. The poles are the collection of all z such that $Q(z) = 0$. Provided $z \neq \infty$, both $P(z)$ and $Q(z)$ will be finite, and since they are in reduced form, we will have $R(z) = 0$ at the roots and $R(z) = \infty$ at the poles (as common roots of P and Q cannot exist in reduced form). This also holds in the case that $z = \infty$, due to the fact that $R(\infty) = \lim_{z\to 0} R\left(\frac{1}{z}\right)$, where $R\left(\frac{1}{z}\right)$ is also a rational function in reduced form. The *order* or a finite root of $R(z)$ is the multiplicity of the root in $P(z)$. The order of a finite pole of $Q(z)$ is the multiplicity of the root in $Q(z)$. The order of a root/pole at infinite is the order of the 0-root/pole for the rational function $R\left(\frac{1}{z}\right)$.

Suppose we are provided a rational function $R(z) = \frac{P(z)}{Q(z)}$ in reduced form, such that $\deg(P) = m$ and $\deg(Q) = n$. A natural question to ask is the total number of roots and poles that a particular rational function has (where we count roots/pole swith multiplicity). Of course, by the fundamental theorem of algebra, R has m finite roots and n finite poles, so the only remaining question is how many infinite roots/poles the function has. Suppose $P(z) = \sum_{j=0}^{m} p_j z^j$ and $Q(z) = \sum_{j=0}^{n} q_j z^j$. Then note that

$$R\left(\frac{1}{z}\right) = \frac{P\left(\frac{1}{z}\right)}{Q\left(\frac{1}{z}\right)} = \frac{\sum_{j=0}^{m} p_j z^{-1}}{\sum_{j=0}^{n} q_j z^{-1}} = z^{n-m} \frac{\sum_{j=0}^{m} p_{m-j} z^j}{\sum_{j=0}^{n} q_{n-j} z^j} \tag{3}$$

Note that p_m and q_n are non-zero, as we have assumed that the polynomials are simplified. Therefore, the numerator and denominator of this rational function are both non-zero at $z = 0$, so the only roots/poles at $z = 0$ are given by the leading z^{n-m} term. In particular, if $n = m$, we have no roots/poles. If $n > m$, we have $n - m$ roots and no poles. And if $n < m$, we have $m - n$ poles and no roots.

It follows that in each of the three cases, we can write down the number of roots/poles: when $n = m$, we have $m = n$ roots and poles. When $n > m$, we have $m + (n - m) = n$ roots and n poles. Finally, when $n < m$, we have m roots and $(m - n) + n = m$ poles. We can easily theoremize this result:

Theorem IV.1. Given a rational function $R = P/Q$ in reduced form, where $m = \deg(P)$ and $n = \deg(Q)$, the number of roots is equal to the number of poles, with both numbers being equal to $k = \max\{m, n\}$. We call k the *degree* of a rational function.

Remark IV.1. This result pretty much matches our intuition from calculus: given a rational function of real polynomials, for large enough x, their quotient eventually behaves like a polynomial of degree x^{m-n} when $m > n$, an inverse polynomial of degree $n - m$ when $m < n$, and some constant when $m = n$. Treating this asymptotic behaviour as the "order" of an ∞-root/pole yields the same casework and the same result as above.

This result allows us to also prove some corollaries fairly immediately. For example:

Corollary IV.1.1. Given a rational function $R(z)$, the equation $R(z) = a$ for some $a \in \mathbb{C}$ has $\max\{m, n\}$ solutions (where m and n are the degrees of the reduced numerator and denominator).

Proof. $R(z) - a$ is a rational function, and thus has exactly $\max\{m, n\}$ roots. $\qquad \square$

Aside from the fairly low-hanging fruit of computing the number of roots/poles of a rational function, with a bit more effort, it is possible to *put a rational function in a form where its behaviour near poles is clearly exposed*. We call this form the **partial fraction decomposition** of a rational function.

Theorem IV.2 (Representation of complex rational functions by partial fractions)**.** Given a rational function R with unique poles at $\beta_1, \ldots, \beta_n$ and possibly at ∞, there exist polynomials $G, G_1, \ldots, G_n$ such that

$$R(z) = G(z) + \sum_{j=1}^{n} G_j \left(\frac{1}{z - \beta_j} \right) \tag{4}$$

Proof. Let us begin by considering the case of a pole at ∞. In particular, suppose R has a pole at ∞. It follows that if $R = \frac{P}{Q}$, then $\deg(P) > \deg(Q)$. Suppose we perform polynomial division, so that $P(z) = G(z)Q(z) + R(z)$, where $\deg(R) \leq \deg(Q)$ and G has no constant term. It follows that $R = G + H$, where $H = \frac{R}{Q}$. Clearly, H is finite at ∞, since the degree of the numerator is less than or equal to the degree of the denomninator, and G is simply a polynomial (this polynomial will in fact dictate the "behaviour" of the rational function as $z \to \infty$.

Let us now consider the case of a finite pole. As it turns out, we can easily reduce this case to the previous case of a pole at infinity. Let us define $z = \beta_j + \zeta^{-1}$, so that $\zeta = (z - \beta_j)^{-1}$. Clearly,

$$\widetilde{R}(\zeta) = R\left(\beta_j + \frac{1}{\zeta}\right) = \frac{\widetilde{P}(\zeta)}{\widetilde{Q}(\zeta)} \tag{5}$$

is a rational function in the variable ζ. Moreover, since R has a pole at β_j, R' will have a pole at $\zeta = \infty$. Thus, we can repeat the above procedure for the polynomial P' to get $P' = G'_j Q' + R'$ and $R'(\zeta) = G'_j(\zeta) + H'(\zeta)$, where H' is finite at $\zeta = \infty$. We then take $G_j(z) = G'_j \left(\frac{1}{z - \beta_j} \right)$, switching back to the variable z.

Our claim is now that the sum of all the G_j along with G is the original rational function R. Indeed, consider the rational function $F = R - G - \sum_j G_j$. Clearly, the only possible poles of this rational function are at the β_j and at ∞. Except for each β_i, note that $F = (R - G_i) - G - \sum_{j \neq i} G_j$. We know that $R - G_i$ is finite at β_i, by construction, and that each of the other polynomials will also be finite. This also holds true at the ∞ pole.

Thus, the rational function F can have no poles, so it must be a constant c (since the number of roots/poles is the maximum of the numerator and denominators' degrees). We can easily perform the shift $G \mapsto G - c$ without changing the asymptotic behaviour of G, so that it is in fact true that $F = 0$ for all z, and we have the desired equality. $\qquad \square$

A nice feature of this theorem is that it allows us to prove the partial fraction decomposition in the case of real rational functions.

Theorem IV.3 (Representation of real rational functions by partial fractions)**.** Given a real rational function $R(z) = \frac{P(z)}{Q(z)}$ (so P and Q have real coefficients), there exists a decomposition of R of the form

$$R(z) = G(z) + \sum_i A_i \left(\frac{1}{z - a_i} \right) + \sum_j z B_j \left(\frac{1}{z^2 + b_j z + c_j} \right) + C_j \left(\frac{1}{z^2 + b_j z + c_j} \right) \tag{6}$$

where G, A_i, B_j, and C_j are all real polynomials, and $a_i, b_j, c_j \in \mathbb{R}$, where $z - a_i$ and $z^2 + b_j z + c_j$ are the unique irreducible factors of $Q(z)$ as a real polynomial.

Proof. **TODO**: Add this from problem set solutions $\qquad \square$

B. Fractional linear transformations and classifying rational functions

As it turns out, degree-1 rational functions are not only particularly well-behaved, but they are a very special class of functions on the complex plane. In fact, they are so special that we give them their own name: *fractional linear transformation (FLTs)*. We begin by making note of a key fact

Fact IV.1. Fractional linear transformations are invertible.

Proof. If $R = \frac{P(z)}{Q(z)}$ is an FLT, it is order-1. From Cor. IV.1.1, for any $a \in \mathbb{C}$, the equation $R(z) = a$ has a single solution. Moreover, $R(z)$ has a single pole, so there is a unique z_0 sent to ∞ by R. Thus, R is bijective and invertible. $\square$

Clearly, given an FLT, $R(z) = \frac{P(z)}{Q(z)}$, at least one of P and Q are degree-1 polynomials, so we can write $R(z) = \frac{az+b}{cz+d}$. However, it is not the case that every set of (a, b, c, d) will yield an FLT. Recall that a for $\frac{az+b}{cz+d}$ to be of order-1, it must either have degree-1 numerator and denominator *in reduced form*. Given arbitrary (a, b, c, d), the quotient $\frac{az+b}{cz+d}$ does not necessarily have this property when put into reduced form.

Claim IV.2. The rational function $f(z) = \frac{az+b}{cz+d}$ is an FLT if and only if $ad - bc \neq 0$.

Proof. Suppose $ad = bc$. Then either $a = b = 0$, $a = c = 0$, $d = b = 0$, $d = c = 0$, or a, b, c, d are all non-zero. In each of the first four cases, it is easy to check that f is not an FLT (as in reduced form, it is a constant). In the final case, we have

$$f(z) = \frac{1}{d} \cdot \frac{adz + bd}{cz + d} = \frac{bcz + bd}{cz + d} = \frac{b}{d} \frac{cz + d}{cz + d}. \tag{7}$$

Thus, in reduced form, f is a constant here as well. It follows that if f is an FLT, $ad \neq bc$, so $ad - bc \neq 0$. Now, suppose $ad - bc \neq 0$. We must show that $\frac{az+b}{cz+d}$ is $\square$

Claim IV.3. The composition of fractional linear transformations yields a fractional linear transformation whose associated matrix is precisely the the product of the associated matrices of the two constituent fractional linear transformations of the composition.

Proof. **TODO**: Fill this in $\square$

As has been suggested, fractional linear transformations are particularly well-behaved, so much so that it makes sense to consider rational functions which are equal, up to composition with fractional linear transformations, to be "equivalent in a sense". More formally,

Fact IV.2. The criterion of "equal up to fractional linear transformations" (which we will call *FLT equivalent* going forward) defines an equivalence relation on the set of complex rational functions. We say that $R \sim R'$ is there exist fractional linear transformations f and g such that $R' = f \circ R \circ g$.

Of course, since we have an equivalence relation, we automatically can ask what the associated equivalence classes are. Beginning with the simple case of a rational function of order 2, let us determine 1. how mny equivalence classes are generated by this relation and 2. a "nice" representative of each equivalence class.

Theorem IV.4. Every fractional linear transformation of order 2 is FLT equivalent to $a(z) = z^2$ or $b(z) = z + z^{-1}$.

Proof. **TODO**: Fill this in $\square$

C. The cross-ratio

Lemma IV.1. A fractional linear transformation is uniquely determined by where it sends any three distinct points of $\overline{\mathbb{C}}$.

Proof. **TODO**: Copy over from problem set $\square$

Lemma IV.2. The cross-ratio is invariant under application of an FLT. In particular, if z_1, z_2, z_3, z_4 are distinct points, and T is an FLT, then $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.

Proof. **TODO**: Fill this in $\qquad\qquad\square$

Lemma IV.3. The FLT induced by the cross ratio, $z \mapsto (z, z_2, z_3, z_4)$ is real if and only if z_2, z_3, z_4 lie on a circle or a line.

Proof. **TODO**: Fill this in $\qquad\qquad\square$

Theorem IV.5. Fractional linear transformations take circles and lines to circles and lines. Moreover, given any pair of circles/lines, there exists a unique FLT sending one to the other.

Proof. **TODO**: Fill this in $\qquad\qquad\square$

V. Holomorphic functions

Now that we have introduced some basic theory related to complex functions, we can move to explaining one of the fundamental concepts underlying complex analysis: *holomorphic functions*. Before proceeding, we make an important remark

Remark V.1. As a high-level, when we do complex analysis, we are in some sense doing calculus in two variables, and in another sense, doing calculus in a single variable. The complex plane is defined as $\mathbb{R}^2$, with a particular complex structure imposed, but we then go on to defining a collection of arithmetic operations which allow us to perform all of the standard operations one would expect to be able to apply to real numbers, to pairs of numbers (x, y) which we call *complex numbers* and denote $z = x + iy$.

As a result of this fundamenntal fact, great care must be taken when defining particular constructions in complex analysis. Oftentimes, we will define something "with respect to x and/or y". In other situations, we will define something "with respect to $z = x + iy$". This vague assertion will make more sense as we begin to encounter concrete examples of it throughout the following sections, but it is worthwhile to make note of it early, so that one may be aware and on the lookout for situations in which this slight ambiguity may arise.

A. Introducing and motivating holomorphic functions

The main idea motivating the definition of a holomorphic function is the desire for a function of the variables (x, y), such that standard differentiation of a two-variable function on $\mathbb{R}^2$ is "compatible" with the notion of differentiability that we expect to arise for a single-variable function, *with the added structure of complex arithmetic.*

As was stated in Rem. V.1, we can think about a complex function f either as a two-variable function, $f(x, y)$ on $\mathbb{R}^2$, or as a single-variable function $f(z)$ on $\mathbb{C}$. We know how to differentiate function on $\mathbb{R}^2$ from standard calculus. We do not know how to differentiate a function on $\mathbb{C}$ (yet), however, we do know how to differentiate single-variable functions on $\mathbb{R}$. Thus, if we wish to treat the complex plane similar to how we treat $\mathbb{R}$, with the addition of complex arithmetic, it may make sense to define the derivative of a complex function on the same way as it is defined in the real-single variable case.

Definition V.1 (Complex derivative). Given a function $f : \mathbb{C} \to \mathbb{C}$ of a single, complex variable z, we say that f is complex differentiable or *holomorphic* at a if the limit

$$f'(a) = \lim_{|h| \to 0} \frac{f(a+h) - f(a)}{h} \tag{8}$$

exists, calling this limit $f'(a)$ the complex derivative of f in this case. Note that this limit is well-defined, as division by a complex number is well-defined.

Remark V.2 (Formalizing)**.** Suppose we wanted to be really *really* formal.

Now, suppose we are given a function of the form $g : \mathbb{R}^2 \to \mathbb{R}^2$. This function both has a well-defined *real derivative* and a 2×2 Jacobian of partial derivatives, but it also has an interpretation as a complex function, when we put the associated complex structure (and thus, complex arithmetic) on the domain and co-domain. However, it is **not** the case that the standard, real derivative of g will agree with its associated *complex derivative*. In fact it is even worse than this: *there will be cases where the real derivative exists but the complex derivative does not.*

Question V.1. If a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable in the usual, real sense, under what conditions is it also differentiable when interpreted as a complex function (holomorphic)?

The answer to this question leads us to the famous Cauchy-Riemann equations.

Theorem V.1. A function $g : \mathbb{R}^2 \to \mathbb{R}^2$ with $g(x, y) = (g_1(x, y), g_2(x, y))$ is complex differentiable/holomorphic, when interpreted as a function from $\mathbb{C}$ to $\mathbb{C}$ (defined as $g(x + iy) := g_1(x, y) + ig_2(x, y)$), if and only if it is differentiable as a real function, and satisfies the *Cauchy-Riemann equations*

$$\frac{\partial g_1}{\partial x} = \frac{\partial g_2}{\partial y} \tag{9}$$

$$\frac{\partial g_1}{\partial y} = -\frac{\partial g_2}{\partial x} \tag{10}$$

Proof. Suppose that g is complex differentiable, so that the limit of Eq. (8) exists. It then follows that

$$g(a + h) = g(a) + Lh + o(h) \tag{11}$$

where $o(h)$ is a function such that $\lim_{|h| \to 0} \frac{o(h)}{h} = 0$. $L = p + iq$ and $a = r + is$ are complex numbers. In addition, $h = t + iu$ and when interpreted as a complex function, $g(x + iy) := g_1(x, y) + ig_2(x, y)$. It follows that

$$g(a + h) = g\left((r + t) + i(s + u)\right) = g_1(r + t, s + u) + ig_2(r + t, s + u) \tag{12}$$

$$g(a) = g_1(r, s) + ig_2(r, s) \tag{13}$$

$$Lh = (pt - qu) + i(pu + qt) \tag{14}$$

Therefore, it follows that

$$g_1(r + t, s + u) = g_1(r, s) + (pt - qu) + \text{Re}[o(h)] = g_1(r, s) + \begin{pmatrix} p & -q \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix} + \text{Re}[o(h)] \tag{15}$$

as well as

$$g_2(r + t, s + u) = g_2(r, s) + (pu + qt) + \text{Im}[o(h)] = g_1(r, s) + \begin{pmatrix} q & p \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix} + \text{Im}[o(h)] \tag{16}$$

where both $o_1(t, u) := \text{Re}[o(t + iu)]$ and $o_2(t, u) := \text{Im}[o(t + iu)]$ are real-valued functions such that □

B. A more sophisticated perspective

Let us begin with a brief review of differential geometry.

Remark V.3 (Tangent space notation)**.** Recall that for a vector space V, we define $\bigwedge^k(V)$ to be the set of alternating k-tensors on V. We define the tangent space of $\mathbb{R}^n$ at point p, $T_p\mathbb{R}^n$, to be the vector space $\{p\} \times \mathbb{R}^n$, with elements (p, v) and $(p, v_1) + \lambda(p, v_2) := (p, v_1 + \lambda v_2)$. A differential k-form ω is a function from $\mathbb{R}^n$ to the disjoint union of all $\bigwedge^k(T_p\mathbb{R}^n)$ over $p \in \mathbb{R}^n$ such that $\omega(p) \in \bigwedge^k(T_p\mathbb{R}^n)$. Let $\Omega^k(\mathbb{R}^n)$ denote the space of all k-forms. We define the differential $d : C^\infty(\mathbb{R}^n, \mathbb{R}) \to \Omega^1(\mathbb{R}^n)$ as taking f to the differential form ω such that $\omega(p)(p, v) = [Df(p)]v$. Note that $C^\infty(\mathbb{R}^n, \mathbb{R}) = \Omega^0(\mathbb{R}^n)$. It follows that

Definition V.2.

Remark V.4. Another, more formal way, of stating this fact is that holomorphic functions are precisely the functions g from $(\mathbb{R}^2, J)$ to $(\mathbb{R}^2, J)$ whose derivative commutes with the complex structure,

$$J \circ Dg(z) = Dg(z) \circ J. \tag{17}$$

Intuitively, this means that the map which turns $\mathbb{R}^2$ into $\mathbb{C}$ is compatible with taking the derivative in the usual sense.

Definition V.3 (Holomorphic and anti-holomorphic functions)**.**

C. Basic properties and implications

Claim V.1 (Basic properties of holomorphic functions)**.**

VI. Conformal maps

VII. Power series

We dedicate this section to providing a brief review of power series, re-proving some familar results from introductory calculus in the complex regime.

VIII. Complex exponential and logarithm