MAT436 problem set 7

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Contents

I. Problem 1 1

II. Problem 2 2

III. Problem 3 2

References 2

I. Problem 1

Definition I.1. Recall that if H is a Hilbert space and $V \in \mathcal{L}(H)$, then V is said to be a partial isometry if $||Vf|| = ||f||$ for all $f \in \text{Ker}(V)^{\perp}$.

Proposition I.1. Some $V \in \mathcal{L}(H)$ is a partial isometry, according to the above definition, if and only if V^*V is a projection.

Proof. In the case that V^*V is a projection, it follows immediately that any $f \in \text{Ker}(V)^{\perp}$ must be in the 1-eigenspace of V^*V, implying that

$$||Vf|| = \langle Vf, Vf \rangle^{1/2} = \langle V^*Vf, f \rangle^{1/2} = \langle f, f \rangle^{1/2} = ||f|| \tag{1}$$

Conversely, suppose $||Vf|| = ||f||$ for all $f \in \text{Ker}(V)^{\perp}$. Since $\text{Ker}(V) \oplus \text{Ker}(V)^{\perp}$ is dense in H, it follows that V is bounded on H. Hence, $\text{Ker}(V) = V^{-1}(0)$ is closed, and H is actually equal to $\text{Ker}(V) \oplus \text{Ker}(V)^{\perp}$. We have

$$||Vf||^2 = \langle Vf, Vf \rangle = \langle V^*Vf, f \rangle = ||f||^2 = \langle f, f \rangle \tag{2}$$

Note that $\langle V^*Vf, e \rangle = \langle Vf, Ve \rangle = 0$ for some $e \in \text{Ker}(V)$, so $V^*Vf \in \text{Ker}(V)^{\perp}$ for $f \in \text{Ker}(V)^{\perp}$. Since V^*V is self-adjoint and bounded, we have an orthonormal basis of eigenvectors, suppose e is a non-zero unit eigenvector, so $V^*Ve = \lambda e$, and

$$\lambda = \langle V^*Ve, e \rangle = \langle e, e \rangle = 1 \tag{3}$$

so it follows that the eigenvalues of V^*V are exactly 0 and 1, which implies that it is a projection. □

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II. Problem 2

Note that if V is a partial isometry, then given $f = f_1 + f_2$ with $f_1 \in \mathrm{Ker}(V)$ and $f_2 \in \mathrm{Ker}(V)^\perp$, we have

$$\langle (I - V^*V)f, f\rangle = \langle f, f\rangle - \langle f_2, f_2\rangle \geq 0 \tag{4}$$

which means that $I - V^*V$ is a positive operator, therefore has a unique positive square-root, $(1 - V^*V)^{1/2}$. Additionally,

Proposition II.1. If V is a partial isometry, then V^* is also a partial isometry.

Proof. Since V^*V is a projection onto $\mathrm{Ker}(V)^\perp$, it follows that if $f \in \mathrm{Ker}(V)^\perp$, then $V^*Vf = f$, so $VV^*(Vf) = V(V^*Vf) = Vf$. Moreover, if $e \in \mathrm{Im}(V)^\perp$, then

$$\langle VV^*e, VV^*e\rangle = \langle V(V^*VV^*e), e\rangle = 0 \tag{5}$$

which means that $VV^*e = 0$. Thus, VV^* is a projection onto $\mathrm{Im}(V)$, implying that V^* is a partial isometry. $\square$

III. Problem 3

Proposition III.1. If $T \in \mathcal{L}(H)$, then there exists a positive operator P and partial isometry V such that $T = VP$ and $\mathrm{Ker}(V) = \mathrm{Ker}(P)$.

Proof. For any $f \in H$,

$$|||T|f||^2 = \langle |T|f, |T|f\rangle = \langle |T|^2 f, f\rangle = \langle T^*Tf, f\rangle = ||Tf||^2 \tag{6}$$

Let us define $\widetilde{V} : \mathrm{Im}(|T|) \to H$ by $\widetilde{V}(|T|f) = Tf$. This map is well-defined because if $|T|f_1 = |T|f_2$, so $|T|(f_1 - f_2) = 0$, then from the above equation, $T(f_1 - f_2) = 0$ as well, so $Tf_1 = Tf_2$. Linearity is then easy to see. It also follows from the previous equation that $\widetilde{V}$ is isometric, and thus extends uniquely to an isometric map from $\overline{\mathrm{Im}(|T|)}$. We then define

$$Vf = \begin{cases} \widetilde{V}f & \text{if } f \in \overline{\mathrm{Im}(|T|)} \\ 0 & \text{if } f \in \mathrm{Im}(|T|)^\perp \end{cases} \tag{7}$$

It is easy to see that V is a partial isometry, as it preserves the norm of elements in the orthogonal complement of its image, so the proof is complete, as clearly $T = V|T|$.

To see that the kernels of V and $|T|$ are equal, simply note that

$$\mathrm{Ker}(V) = \mathrm{Im}(|T|)^\perp = \mathrm{Ker}(|T|) \tag{8}$$

$\square$