Non-linear Katz-Oda

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I. Introduction

Summarizing work done on the non-linear Katz-Oda Grothendieck p-conjecture problem.

II. Background

We begin by considering the ODE

$$\frac{dY}{dz} = A(z)Y \tag{1}$$

where $A(z)$ is a matrix whose entries are rational functions of z relative to some algebraic number field K (a field extension of $\mathbb{Q}$ such that $[K : \mathbb{Q}]$ is finite). For the sake of simplicity, let $K = \mathbb{Q}$. Some entry of $A(z)$ will be of the form

$$A_{ij}(z) = \frac{A_{ij}^{(n)} z^n + \cdots + A_{ij}^{(0)}}{B_{ij}^{(m)} z^m + \cdots + B_{ij}^{(0)}} \tag{2}$$

which of course may be reduced modulo some prime p in the ring of algebraic integers of K, for almost all primes (in this case, just the usual prime numbers in $\mathbb{Z}$). In particular, setting

$$\frac{a}{b} \mod p \equiv \frac{a \mod p}{b \mod p} \tag{3}$$

when $p \nmid b$. Thus, we may reduce the rational function's coefficients in the case when p doesn't divide any of the denominators of the rational coefficients $B_{ij}^{(k)}$. The result will be a differental equation over $\mathbb{F}_q[z]$ for some finite field $\mathbb{F}_q$. As a particular example, suppose $a \in \mathbb{Z} \subset \mathbb{Q}$ and we have the ODE given by

$$\frac{dy}{dz} = \frac{1}{az} y \tag{4}$$

We can reduce by $p \nmid a$. When we do such a reduction, we get the ODE over $\mathbb{F}_p[z]$ given by $y' = \frac{1}{(a \mod p)z}$. The equation admits solutions $y = z^b$ for any b such that $ab \equiv 1 \mod p$, as we will have

$$\frac{dy}{dz} = \frac{d}{dz} z^b = (b \mod p) z^{b-1} = \frac{ab \mod p}{(a \mod p)z} = \frac{1}{az} y \tag{5}$$

where we are interpreting the derivative in this case as the derivative of polynomials over a finite field, which is defined using the power rule. On the other hand, in the rational case, $y = z^{1/a}$ is the solution (up to a multiplicative factor).

Differential equations of this form, and reducation by a prime is related to the p-curvature conjecture. There is a result concerning a particular family of ODEs of this form (Picard-Fuchs), formulated by Katz and Oda, which we can restate here

Proposition II.1. The Picard-Fuchs differential equation is an ODE of the form in Eq. (1). It is an algebraic differential equation, meaning that its solution is an algebraic function. Moreover, there exists sufficiently large N such that for almost all primes p,

$$\left(\frac{d}{dz} - A(z)\right)^{Np} \equiv 0 \mod p \tag{6}$$

The sense in which we mean that iterating this differential operator yields $0 \mod p$ is as follows: differential operators which have been reduced relative to some prime are thought of as "discrete", and act only on rational functions, whose derivatives are defined using the product rule. Thus, to check if such a differential operator is 0, we merely must evaluate it on all powers of z, z^n. For example, consider the differential operator $z\partial_z$. Suppose we iterate it 3 times, we will get

$$(z\partial_z)^3 = z\partial_z(z\partial_z + z^2\partial_z^2) = z\partial_z + z^2\partial_z^2 + 2z^2\partial_z^2 + z^3\partial_z^3 \tag{7}$$

$$= z\partial_z + 3z^2\partial_z^2 + z^3\partial_z^3 \tag{8}$$

From here, we may evaluate it on z^n:

$$(z\partial_z + 3z^2\partial_z^2 + z^3\partial_z^3)z^n = [n + 3n(n-1) + n(n-1)(n-2)]\, z^n \tag{9}$$

Note that

$$n + 3n(n-1) + n(n-1)(n-2) = n(1 + 3(n-1) + (n-1)(n-2)) = n(1 + (n+1)(n-1)) = n^3 \tag{10}$$

and from Fermat's little theorem, $n^3 = n \mod 3$. In other words,

$$(z\partial_z)^3 z^n = nz^n = (z\partial_z)z^n \mod 3 \tag{11}$$

so it follows that modulo the prime 3, we have $(z\partial_z)^3 = z\partial_z$.

III. The Non-linear problem

We wish to understand whether a similar result as Thm. **??** holds for the Schlesinger system, which is a non-linear system of partial differential equations, given by

$$\frac{\partial B_i}{\partial \lambda_j} = \frac{[B_i, B_j]}{\lambda_i - \lambda_j} \quad \text{for } i \neq j \tag{12}$$

$$\sum_j \frac{\partial B_i}{\partial \lambda_j} = 0 \quad \text{for all } i. \tag{13}$$

Note that each B_j is a function of variables $\lambda_1, \dots, \lambda_n$, for j from 1 to m.

The first idea is to understand the vector field/phase portrait associated with each individual variable λ_j. At a high-level, this will be a vector field in which the vectors have entries which are matrices. In particular, we can identify $M_{N\times N}(\mathbb{R})$ with $\mathbb{R}^{N^2}$, so that we will have a vector field with vectors in $\mathbb{R}^{mN^2}$. For a given

λ_j, the system of differential equations is then given by

$$\frac{\partial}{\partial \lambda_j}\begin{pmatrix} B_1 \\ \vdots \\ B_{j-1} \\ B_j \\ B_{j+1} \\ \vdots \\ B_m \end{pmatrix} = \begin{pmatrix} \frac{[B_1,B_j]}{\lambda_1-\lambda_j} \\ \vdots \\ \frac{[B_{j-1},B_j]}{\lambda_i-\lambda_j} \\ -\sum_{i\neq j}\frac{[B_i,B_j]}{\lambda_i-\lambda_j} \\ \frac{[B_{j-1},B_j]}{\lambda_{j-1}-\lambda_j} \\ \vdots \\ \frac{[B_m,B_j]}{\lambda_m-\lambda_j} \end{pmatrix} \tag{14}$$

This will indeed yield a collection of n different vector fields, for j ranging from 1 to j. Under the identification of $M_{N\times N}(\mathbb{R})$ with $\mathbb{R}^{N^2}$, note that the variables with respect to which this vector field is a derivation are the matrix entries, $[B_k]_{ij}$ for k ranging from 1 to m and $1 \leq i, j \neq N$. In particular, at the point $(B_1, \ldots, B_m) \in \mathbb{R}^{mN^2}$, the vector field is given by

$$X^{(j)}_{B_1,\ldots,B_m} = \sum_{p,q=1}^{N}\sum_{i,i\neq j}\frac{[B_i,B_j]_{pq}}{\lambda_i-\lambda_j}\left(\frac{\partial}{\partial[B_i]_{pq}} - \frac{\partial}{\partial[B_j]_{pq}}\right) \tag{15}$$

Note that $X^{(j)}_{B_1,\ldots,B_m}$ takes some $f \in C^\infty(\mathbb{R}^{mN^2})$ of the form $f(M_1,\ldots,M_m)$ to a real number, so $X^{(j)}(f)$ is a smooth function which takes m matrices and outputs a real number. We will be interested in all possible products/compositions of these vector fields with themselves:

$$X^{(J)} = X^{(j_1\cdots j_N)} = \prod_{k=1}^{N} X^{(j_k)} \tag{16}$$

and, in particular, how large we need to make N such that for all tuples $(j_1,\ldots,j_N)$, the operator $X^{(J)}$ vanishes, modulo some prime p.