

MUNKRES TOPOLOGY: CHAPTER 2

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CONTENTS

1. Section 13: Bases	2
1.1. Theorems	2
1.2. Problems and Solutions	2
2. Section 16: Different Topologies	3
2.1. Theorems	3
3. Section 17: Closures, Closed Sets	3
4. Section 18: Continuous Functions	4
5. Section 20: Metric Topology	7
6. Section 21	10

1. SECTION 13: BASES

1.1. Theorems.

1.2. Problems and Solutions.

Proposition 1. *If \mathcal{A} is a basis for a topology on X , then the topology generated by \mathcal{A} is the intersection of all topologies containing \mathcal{A} . Same holds true if \mathcal{A} is a subbasis.*

Proof. Let $\mathcal{T}_{\mathcal{A}}$ be the topology generated by \mathcal{A} . If $U \in \mathcal{T}_{\mathcal{A}}$, then it is a union of elements of \mathcal{A} . Clearly, all topologies containing \mathcal{A} must contain all unions of elements of \mathcal{A} , so U is in the intersection of all such topologies and $\mathcal{T}_{\mathcal{A}}$ is in the intersection. Similarly, note that $\mathcal{T}_{\mathcal{A}}$ contains \mathcal{A} , so it contains the intersection of all topologies containing \mathcal{A} . Thus, $\mathcal{T}_{\mathcal{A}}$ is equal to the intersection.

If \mathcal{A} is a subbasis, we recycle the above proof except note that some $U \in \mathcal{T}_{\mathcal{A}}$ is a union of intersections of elements of \mathcal{A} , which must be in each topology containing \mathcal{A} , so it is in the intersection. \square

Proposition 2. *The topologies \mathbb{R}_{ℓ} and \mathbb{R}_K are not comparable.*

Proof. Pick the basis element $B = (-1, 1) - K$ of the K -topology. Every basis element of the lower-limit topology containing 0 contains some element of K . Hence, the K -topology is not contained in the lower limit topology. Consider the basis element $B' = [0, 1)$ of the lower limit topology. There is no basis element of the K -topology containing 0 which is in B' . Hence, the lower limit topology is not in the K -topology. \square

Proposition 3. *If T_{α} is a family of topologies, then $\cap T_{\alpha}$ is also a topology. Moreover, there exists a unique smallest topology containing all T_{α} , and a unique largest topology contained in all T_{α} .*

Proof. Checking that the intersection of arbitrarily many topologies is easy. Clearly, $X, \emptyset \in \cap T_{\alpha}$. In addition, note that unions and finite intersections of some subcollection of elements contained in all T_{α} must also be in all T_{α} , by definition of a topology. Thus, $\cap T_{\alpha}$ is a topology.

We assert that the unique smallest topology is the topology B having $\cup T_{\alpha}$ as a subbasis. Indeed, this is a valid basis. Clearly, this topology contains all T_{α} . Finally, suppose R is another topology that contains each T_{α} . If U is open in B , it is a union of finite intersections of elements of $\cup T_{\alpha}$, so $U \subset R$ as well. Thus, R is finer than T_{α} .

We assert that the unique largest topology is $\cap T_{\alpha}$. Clearly, $\cap T_{\alpha}$ is contained in each T_{α} . In addition, suppose R is another topology contained in each T_{α} . Then it follows that R is in their intersection, so $\cap T_{\alpha}$ is finer than R . \square

Proposition 4. *Classify the standard topology, K -topology, finite complement, upper limit, and upper-ray basis topology.*

Proof. Clearly, $\mathcal{T}_5 \subset \mathcal{T}_1$, where inclusion is proper, and $\mathcal{T}_1 \subset \mathcal{T}_2$, where inclusion is also proper (Consider the basis element $S = (-1, 1) - K$. The point 0 is not contained in any basis element of the standard topology that is itself contained in S). It's also pretty clear that \mathcal{T}_4 is finer than \mathcal{T}_1 (properly). In addition, \mathcal{T}_4 is finer than \mathcal{T}_2 (properly, consider then interval $(2, 3]$, clearly this interval is not open in the K -topology).

Now, note that \mathcal{T}_1 is finer than \mathcal{T}_3 , as if $X - U$ is a finite set of points, then U can be written as a union of intervals, so U is open in \mathcal{T}_1 . This inclusion is clearly proper. We also assert that \mathcal{T}_3 and \mathcal{T}_5 are not comparable. Clearly, some interval of the form $(-\infty, a)$ is not in \mathcal{T}_3 . Similarly, note that the point 2 in the open set $(-\infty, 1) \cup (1, \infty)$ in \mathcal{T}_3 is not contained in some basis element of the form $(-\infty, a)$, so it is not open in \mathcal{T}_5 .

Thus, we have:

$$\mathcal{T}_5, \mathcal{T}_3 \subset \mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_4$$

□

2. SECTION 16: DIFFERENT TOPOLOGIES

2.1. Theorems.

Lemma 1. *The collection of:*

$$S = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology.

Proof. Let T_S be the topology generated by the subbasis and T be the product topology. Note that each element of S is in T , so unions of finite intersections of elements of S will be in T as well. Thus, $T_S \subset T$.

Now, pick basis element $U \times V$ in T . Note that:

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

which is in T_S . Hence $U \times V$ is in T_S . It follows unions of such elements will be in T_S as well, so every open U in T is also open in T_S . This implies that $T \subset T_S$.

We have inclusion both ways, so $T_S = T$. □

Lemma 2. *The product and the subspace topologies are equivalent.*

Proof. Let $(U \times V) \cap (A \times B)$ be a basis element for the subspace topology. We have:

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

so this is also a basis element for the product topology on $A \times B$. Similarly, if $(U \cap A) \times (V \cap B)$ is a basis element for the product topology, it is also a basis element for the subspace topology, by the above equation.

The two topologies have the same basis, so they are equal. □

Lemma 3. *Let X be an ordered set in the order topology, let Y be a convex subset. The order topology on Y is the same as the topology inherited by Y as a subspace of X .*

Proof. First, note that each basis element for the order topology on Y is open in □

2.2. Problems and Solutions.

Proposition 5. *The dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where \mathbb{R}_d is the discrete topology on \mathbb{R} . Compare this with the product topology on \mathbb{R}^2 .*

Proof. First, consider some basis element $\{x\} \times (a, b)$ of $\mathbb{R}_d \times \mathbb{R}$. Clearly, this is also open in the dictionary topology. Thus, the basis that generates $\mathbb{R}_d \times \mathbb{R}$ is in the dictionary topology, so the product topology on $\mathbb{R}_d \times \mathbb{R}$ is in the dictionary topology.

Conversely, pick the basis element $B = (a \times b, c \times d)$ in the dictionary topology. For some $x \times y \in B$ note that we can choose a basis element of $\mathbb{R}_d \times \mathbb{R}$ around $x \times y$ in B of the form $\{x\} \times (a, b)$. Thus, the product topology is finer than the dictionary topology.

We have inclusion both ways, so the topologies are equal. Comparing to the product topology $\mathbb{R} \times \mathbb{R}$, we note that this product topology is coarser than the above topology. Given some basis element $(a, b) \times (c, d)$ of $\mathbb{R} \times \mathbb{R}$, it is clear that this basis element is open with respect to $\mathbb{R}_d \times \mathbb{R}$ as well. However, note that $\{x\} \times (a, b)$ open in $\mathbb{R}_d \times \mathbb{R}$ is not open in \mathbb{R}^2 , as it contains no basis element of the form $(q, r) \times (s, t)$. \square

3. SECTION 17: CLOSURES, CLOSED SETS

Proposition 6. $\text{Int}(A)$ and $\partial(A)$ are disjoint. Moreover, $\overline{A} = \text{Int}(A) \cup \partial(A)$.

Proof. If $x \in \text{Int}(A)$, then it follows that it is contained in an open set that is contained in A , and therefore does not intersect $X - A$, so $x \notin \partial(A)$. Thus, $\text{Int}(A)$ contains no point that is also in $\partial(A)$, so the two sets must be disjoint.

Clearly, $\text{Int}(A) \cup \partial(A) \subset \overline{A}$ (this follows directly from the definitions). Moreover, picking some $x \in \overline{A}$, we know that every neighbourhood of x intersects A . Note that either there exists a neighbourhood of x is contained in A , so it is in the interior, or every neighbourhood of x intersects $X - A$, so it is in the boundary. Thus, $\overline{A} \subset \text{Int}(A) \cup \partial(A)$. Equality follows. \square

Proposition 7. $\partial(A) = \emptyset \Leftrightarrow A$ is both open and closed.

Proof. Suppose first that $\partial(A) = \emptyset$. It follows that $\overline{A} \cap \overline{X - A} = \emptyset$. Thus, every point $x \in A$ is contained in a neighbourhood that is contained entirely in A , so A is open. In addition, every point $x \in X - A$ is contained in a neighbourhood entirely in $X - A$, so $X - A$ is open and A is closed.

Conversely, suppose A is both open and closed. Since A is closed, $\overline{A} = A$ and since it is open, $\overline{X - A} = X - A$. Thus:

$$\partial(A) = \overline{A} \cap \overline{X - A} = A \cap (X - A) = \emptyset$$

\square

Proposition 8. U open $\Leftrightarrow \partial(U) = \overline{U} - U$.

Proof. Suppose U is open. Then $X - U$ is closed, so:

$$\partial(U) = \overline{U} \cap \overline{X - U} = \overline{U} \cap (X - U) = \overline{U} - U$$

Conversely, suppose that $\partial(U) = \overline{U} - U$. We then have:

$$\partial(U) = (\text{Int}(U) \cup \partial(U)) \cap (X - U) = \partial(U) \cap (X - U)$$

Thus, $\partial(U) \subset X - U$, which implies that if $x \in U$, then it must be contained in a neighbourhood that does not intersect $X - U$. This implies that U is open. \square

Proposition 9. It is not necessarily true that $U = \text{Int}(\overline{U})$ when U is open.

Proof. Consider $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology. Consider the open set $U = (a, b) \times (c, d)$. It is easy to see that:

$$\overline{(a, b) \times (c, d)} = [a, b] \times [c, d]$$

From here, one can check that $\text{Int}([a, b] \times [c, d]) = [a, b] \times (c, d)$, when we work in the order topology, which is clearly different from U itself. \square

4. SECTION 18: CONTINUOUS FUNCTIONS

Proposition 10. *If f and g are continuous, then the set $U = \{x \mid f(x) \leq g(x)\}$ is closed. Then, show that $h(x) = \min(f(x), g(x))$ is continuous.*

Proof. □

Proposition 11. *Let $\{A_\alpha\}$ be a collection of subsets of X whose union is X . Let $f : X \rightarrow Y$, with $f|_{A_\alpha}$ continuous for each α . If the collection is finite and each set is closed, then f is continuous.*

Proof. The proof of this statement is similar to the proof that “local continuity” implies continuity. Note that for some closed set C in Y :

$$f^{-1}(C) = \bigcup_{\alpha} (f^{-1}(C) \cap A_\alpha) = \bigcup_{\alpha} (f|_{A_\alpha})^{-1}(C) = U$$

Since U is a finite union of closed sets, it is closed, so f is continuous. Alternatively, this follows directly from the pasting lemma. □

Proposition 12. *Find an example where $\{A_\alpha\}$ is countable and each A_α is closed by f is not continuous.*

Proof. For each x , let U_x be a neighbourhood around x that only intersects finitely many A_α . Considering the finite collection of closed sets of the form $U_x \cap A_\alpha$ in the subspace topology for U_x , we note that since each $f|_{A_\alpha}$ is continuous, so too is each $f|_{(U_x \cap A_\alpha)}$, as this amounts to a domain restriction. By the above proposition, it follows that each $f|_{U_x}$ is continuous. Then, by the local formulation of continuity, f is continuous as well. □

Proposition 13. *Suppose $F : X \times Y \rightarrow A$ is continuous. Then it is continuous in both arguments separately.*

Proof. Pick $y_0 \in Y$. Note that the map g from X to $X \times Y$ such that $x \rightarrow (x, y_0)$ is continuous, as each component function is continuous (identity and constant functions). Thus, $F \circ g$ is continuous, as it is the composition of continuous functions, and it is precisely the function f such that $f(x) = F(x \times y_0)$. The claim follows. □

Proposition 14. *Let $A \subset X$. Let $f : A \rightarrow Y$ be continuous, Y be Hausdorff. Show that if f may be extended to continuous $g : \overline{A} \rightarrow Y$, then g is uniquely determined by f .*

Proof. Suppose there exist two distinct continuous functions h and g such that $h(x) = g(x) = f(x)$ for all $x \in A$. Let x be a point such that $h(x) \neq g(x)$, so $x \notin A$. Since Y is Hausdorff, we choose disjoint neighbourhoods U and V around $h(x)$ and $g(x)$ respectively. Since h and g are continuous, $h^{-1}(U)$ and $g^{-1}(V)$ are open and contain x . Since $x \in \overline{A}$, both must intersect A . Clearly, there cannot exist some point $a \in A$ that is in both $h^{-1}(U)$ and $g^{-1}(V)$, as we would have $h(a) = g(a) = f(a)$ in both U and V , contradicting the fact that they are disjoint. Thus, $h^{-1}(U) \cap g^{-1}(V)$ does not intersect A , contradicting the fact that $x \in \overline{A}$.

It follows that distinct h and g cannot exist and the function is uniquely determined by f . □

Proposition 15. *In both the box and product topologies:*

$$\prod_{\alpha} \overline{A_\alpha} = \overline{\prod_{\alpha} A_\alpha}$$

Proof. Suppose $\mathbf{x} \in \prod_{\alpha} \overline{A_\alpha}$. Pick some open neighbourhood U of \mathbf{x} . In both the box and product topologies, we have:

$$\mathbf{x} \in B = \prod_{\alpha} U_\alpha \subset U$$

where each U_α is open. Since $x_\alpha \in \overline{A_\alpha}$, it follows that U_α intersects A_α for each α . Thus, B must intersect $\prod_\alpha A_\alpha$, so U does as well. It follows that $\mathbf{x} \in \overline{\prod_\alpha A_\alpha}$. Conversely, suppose $\mathbf{x} \in \overline{\prod_\alpha A_\alpha}$. Picking some x_β , we pick some neighbourhood U . Then, consider the set open in both the product and box topologies:

$$V = \prod_\alpha V_\alpha \quad V_\alpha = U \text{ for } \alpha = \beta, \quad X_\alpha \text{ otherwise}$$

This open set must intersect $\prod_\alpha A_\alpha$, so it follows that U intersects A_β . Thus, $x_\beta \in \overline{A_\beta}$. It follows that $\mathbf{x} \in \prod_\alpha \overline{A_\alpha}$. \square

Proposition 16. *A component-wise map into a product space (in the product topology) is continuous if and only if each component function is continuous.*

Proof. One direction is easy: we know that $f_\alpha = \pi_\alpha \circ f$, so if f is continuous, f_α is the composition of two continuous functions, making it continuous.

Conversely, suppose each f_α is continuous. Picking some basis element B in $\prod_\alpha X_\alpha$, we note that:

$$\begin{aligned} B = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n}) &\Rightarrow f^{-1}(B) = (f^{-1} \circ \pi_{\alpha_1}^{-1})(U_{\alpha_1}) \cap \cdots \cap (f^{-1} \circ \pi_{\alpha_n}^{-1})(U_{\alpha_n}) \\ &= (\pi_{\alpha_1} \circ f)^{-1}(U_{\alpha_1}) \cap \cdots \cap (\pi_{\alpha_n} \circ f)^{-1}(U_{\alpha_n}) = f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f_{\alpha_n}^{-1}(U_{\alpha_n}) \end{aligned}$$

which is a finite product of open sets, and is therefore open. It follows that f is continuous and the proof is complete. Note that this proof will sometimes fail for the product topology, as the above intersection is not guaranteed to be finite (if we were to formulate basis elements of the box topology in this form), and we could end up with a set that is not open. \square

Proposition 17. *Let \mathbb{R}^∞ be the subset of \mathbb{R}^ω consisting of all sequences that are eventually 0. What is the closure of \mathbb{R}^∞ in the box and product topologies?*

Proof. In the box topology, the closure of \mathbb{R}^∞ is \mathbb{R}^∞ . Picking some \mathbf{x} in the complement of \mathbb{R}^∞ , we note that \mathbf{x} is not eventually 0. Thus, for each non-zero x_α , we choose an open interval U_α around x_α that does not contain 0. The open set $U = \prod_\alpha U_\alpha$ then does not intersect \mathbb{R}^∞ , so $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$.

In the product topology, the closure of \mathbb{R}^∞ is \mathbb{R}^ω . Picking any $\mathbf{x} \in \mathbb{R}^\omega$, note that \mathbf{x} is contained in a basis element of the form $\prod_\alpha U_\alpha$, where U_α eventually equals \mathbb{R} . Thus, since \mathbb{R} contains 0, U will contain a sequence that is eventually equal to 0, so $\mathbf{x} \in \overline{\mathbb{R}^\infty}$. \square

Proposition 18. *Let $f_\alpha : A \rightarrow X_\alpha$. Show that there exists a unique coarsest topology relative to which each f_α is continuous.*

Proof. For fun, let's do this in a non-constructive fashion. Let \mathcal{T}_a and \mathcal{T}_b be two topologies such that each f_α is continuous. Let $\mathcal{T} = \mathcal{T}_a \cap \mathcal{T}_b$. It is easy to see that \mathcal{T} is also a topology. Moreover, for some open set $V \in X_\alpha$, note that $f_\alpha^{-1}(V)$ must be in both \mathcal{T}_a and \mathcal{T}_b , by definition, so it is in \mathcal{T} . Thus, each f_α is continuous with respect to \mathcal{T} .

Taking the intersection of all topologies with respect to which each f_α is continuous (which we denote \mathcal{T}'), and we are left with the unique coarsest topology, as desired. \square

Proposition 19. *Show that the set:*

$$S = \bigcup_\beta \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$$

is a subbasis for the above topology.

Proof. Clearly, this subbasis generates a topology with respect to each f_β is continuous, so $\mathcal{T}' \subset \mathcal{T}_S$. Conversely, note that we must have $f_\beta^{-1}(V)$ in \mathcal{T}' for all β , where V is open in X_β . Thus, all finite intersections and unions of such elements must also be in \mathcal{T}' . Thus, every set generated by the subbasis is in \mathcal{T}' , so $\mathcal{T}_S \subset \mathcal{T}'$. Equality follows. \square

Proposition 20. *Show that a map $g : Y \rightarrow A$ is continuous relative to \mathcal{T}' if and only if each map $f_\alpha \circ g$ is continuous.*

Proof. Suppose g is continuous. Then $f_\alpha \circ g$ is a composition of continuous maps, so it is continuous.

Conversely, suppose each $f_\alpha \circ g$ is continuous. A general subbasis element of A is of the form $f_\beta^{-1}(U_\beta)$. Hence, we will have:

$$g^{-1}(f_\beta^{-1}(U_\beta)) = (f_\beta \circ g)^{-1}(U_\beta)$$

is open, as $f_\beta \circ g$ is continuous. It follows that g is continuous as well. \square

Proposition 21. *Let $f : A \rightarrow \prod_\alpha X_\alpha$. Show that the image of each open set in A under f is open in $f(A)$.*

Proof. It is enough to show that this is true for all basis elements. A general basis element of A will be of the form $V = \cap_{j=1}^n f_{\alpha_j}^{-1}(U_{\alpha_j})$. Note that:

$$V = \bigcap_{j=1}^n f_{\alpha_j}^{-1}(U_{\alpha_j}) = f^{-1} \left(\prod_{\alpha} U_{\alpha} \right)$$

where $U_\alpha = X_\alpha$ for all α not in the set $\alpha_1, \dots, \alpha_n$. It follows that:

$$f(V) = f(A) \cap \prod_{\alpha} U_{\alpha}$$

which is open in the subspace topology on $f(A)$. \square

5. SECTION 20: METRIC TOPOLOGY

Proposition 22. *The uniform topology on \mathbb{R}^J is finer than the product topology and coarser than the box topology.*

Proof. Recall that the uniform topology is the metric topology generated by the metric $\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_i, y_i) \mid i \in J\}$. Suppose we have some point \mathbf{x} and a basis element containing \mathbf{x} in the product topology of the form $B = \prod_{\alpha \in J} U_\alpha$, where $U_\alpha = \mathbb{R}$ for all but finite α . It is easy to pick $\epsilon_\alpha < 1$ (for the finite set of α where $U_\alpha \neq \mathbb{R}$ such that $(x - \epsilon_\alpha, x + \epsilon_\alpha) \subset U_\alpha$. Letting $\epsilon = \min_\alpha \epsilon_\alpha$, we note that the basis element $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ is contained in B . Thus, the uniform topology is finer than the product topology.

Now, pick basis element $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ in the uniform topology. Note that the basis element:

$$B = \prod_{\alpha \in J} (x_\alpha - \epsilon, x_\alpha + \epsilon)$$

in the box topology is contained in $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$. \square

Interestingly enough, I think there is some intuition to this. Basically, the product topology only gives us information about a finite number of the sets in some product space, whereas the box topology gives us information about all of the sets. The reason that the uniform metric is finer than the product topology is because it can always encode information about a finite number of sets. However, there may be cases where the distances involved in computing the uniform topology converge to some undesirable place, which is the reason that it is coarser than the box topology.

Proposition 23. Let \bar{d} be the standard bounded metric on \mathbb{R} . On \mathbb{R}^ω , define the metric:

$$D(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$$

This metric gives rise to the product topology on \mathbb{R}^ω .

Proof. First pick a basis element of the form $B = \prod_{\alpha} U_{\alpha}$, where $U_{\alpha} = \mathbb{R}$ for all but finite α containing some \mathbf{x} . We know that for each U_{α} , we have $x_{\alpha} \in (x_{\alpha} - \epsilon_{\alpha}, x_{\alpha} + \epsilon_{\alpha}) \subset U_{\alpha}$, for some $\epsilon_{\alpha} < 1$. Now, let $\epsilon = \min_i \{\epsilon_i/i\}$. Consider the set $B_D(\mathbf{x}, \epsilon)$. Clearly, for $\mathbf{y} \in B_D$, we will have:

$$D(\mathbf{x}, \mathbf{y}) < \epsilon \Rightarrow \frac{\bar{d}(x_i, y_i)}{i} < \frac{\epsilon_i}{i} \Rightarrow \bar{d}(x_i, y_i) < \epsilon_i \Rightarrow d(x_i, y_i) < \epsilon_i$$

for all i . Thus, $\mathbf{y} \in U$, so the topology generated by D is finer than the product topology. Conversely, consider the basis element $B_D(\mathbf{x}, \epsilon)$ of the metric topology. Without loss of generality, let $\epsilon < 1$. Let us choose natural N such that $\frac{1}{N} < \epsilon$. We consider the basis element in the product topology where:

$$B = \prod_{j=1}^{N-1} (x_j - \epsilon, x_j + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \cdots$$

and note that for $\mathbf{y} \in B$, for $i < N$, we will have:

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \bar{d}(x_i, y_i) = d(x_i, y_i) < \epsilon$$

and for $i \geq N$, we will have:

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{i} \leq \frac{1}{N} < \epsilon$$

Thus, $D(\mathbf{x}, \mathbf{y}) < \epsilon$, so $\mathbf{y} \in B_D$. This implies that the product topology is finer than the metric topology. Equality follows. \square

The reason why this topology gives rise to the product topology is because beyond a certain point, the $1/i$ part of the elements of the set describing D dominate over the numerator, and we are effectively left with a “finite amount” of information describing our space.

Proposition 24. Let X be a metric space. A metric $d : X \times X \rightarrow \mathbb{R}$ is continuous, where $X \times X$ is given the product topology.

Proof. The idea here is to exploit the triangle inequality in a slick way. Suppose we have some open $U \in \mathbb{R}$ with $d(x, y) \in U$. Pick some $A = (d(x, y) - \epsilon, d(x, y) + \epsilon) \subset U$. We then consider the open sets $B_d(x, \epsilon/2)$ and $B_d(y, \epsilon/2)$ in X . It follows that $B = B_d(x) \times B_d(y)$ is open in $X \times X$. Suppose $(a, b) \in B$. It follows that:

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, b) \leq d(a, x) + d(x, y) + d(y, b) < d(x, y) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = d(x, y) + \epsilon \\ d(a, b) &\geq d(x, b) - d(a, x) \geq d(x, y) - d(b, y) - d(a, x) > d(x, y) - \epsilon \end{aligned}$$

Thus, $d(a, b) \in A \subset U$. It follows by definition that $d : X \times X \rightarrow \mathbb{R}$ is continuous. \square

Proposition 25. Suppose X' is a topological space with the same underlying set as X but possibly a different topology. If $d : X' \times X' \rightarrow \mathbb{R}$ is continuous, then the topology of X' is finer than the metric topology imposed on X .

Proof. This follows almost immediately from the fact that if a map is continuous, then it is continuous in each of its arguments separately. Picking some y , we note that $d_y : X \rightarrow \mathbb{R}$ with $d_y(x) = d(y, x)$ is continuous. Thus, picking some $B_{\epsilon}(y)$ (a basis element for X), we note that $d_y^{-1}(-\epsilon, \epsilon)$ is open in X' and is equal to $B_{\epsilon}(y)$. Thus, the topology on X' is finer than the metric topology. \square

Proposition 26. *If the space \mathbb{R}^ω is given the uniform topology, then the closure of $\mathbb{R}^\infty \subset \mathbb{R}^\omega$ is precisely all x such that $x_i \rightarrow 0$ as elements of \mathbb{R} .*

Proof. Suppose x is such that $x_i \rightarrow 0$. Pick some open neighbourhood U of x . We then know that we have $\epsilon < 1$ such that $x \in B_{\bar{\rho}}(x, \epsilon) \subset U$. By definition, there must exist some N such that for $i > N$, we have $|x_i| < \delta$, where we choose $\delta < \epsilon$. We let $y = (x_1, \dots, x_N, 0, 0, \dots)$, which is clearly an element of \mathbb{R}^ω . Note that:

$$\begin{aligned}\bar{\rho}(x, y) &= \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in \mathbb{N}\} = \sup\{|x_\alpha - y_\alpha| \mid \alpha \in \mathbb{N}\} \\ &= \sup(\{|x_\alpha - y_\alpha| \mid 1 \leq \alpha \leq N\} \cup \{|x_\alpha| \mid \alpha > N\}) \leq \delta < \epsilon\end{aligned}$$

as $|x_\alpha - y_\alpha| = 0$ for $1 \leq \alpha \leq N$ and $|x_i| < \delta$ otherwise. Thus, $y \in B_{\bar{\rho}}$, so $x \in \overline{\mathbb{R}^\infty}$.

Conversely, suppose $x \in \overline{\mathbb{R}^\infty}$. It follows that every $B_{\bar{\rho}}(x, \epsilon)$ around x intersects \mathbb{R}^∞ . Now, let us consider the interval $(-\delta, \delta)$ around 0 in \mathbb{R} . Assume WLOG that $\delta < 1$. Let $y \in B_{\bar{\rho}}(x, \delta)$ and in \mathbb{R}^∞ . We have:

$$\bar{\rho}(x, y) < \delta \Rightarrow |x_i - y_i| < \delta \quad \text{for all } i$$

We also know that there exists some N such that for $i > N$, we have $y_i = 0$, by definition. Thus, for $i > N$, we have $|x_i| < \delta$, so all such x_i are in $(-\delta, \delta)$. It follows by definition that $x_i \rightarrow 0$. \square

Proposition 27. *Letting X be the subset of \mathbb{R}^ω such that $\sum_i x_i^2$ converges, show that:*

$$\text{box topology} \supset \ell^2 \text{ topology} \supset \text{uniform topology}$$

Proof. Suppose $B_{\bar{\rho}}(x, \epsilon)$ is a basis element in the product topology. Note that for $y \in B_d(x, \delta)$ where we choose $\delta < \epsilon$, we have:

$$d(y, x) < \delta \Rightarrow \sqrt{\sum_i (x_i - y_i)^2} < \delta \Rightarrow |x_i - y_i| < \delta \text{ for all } i$$

Thus, the supremum over i is less than or equal to δ , so it less than ϵ . This implies that $y \in B_{\bar{\rho}}$, so $B_{\bar{\rho}} \supset B_d$. It follows from an earlier result in Munkres that the ℓ^2 topology is finer than the uniform topology.

Consider the basis element B of the ℓ^2 topology. For some $x \in B$, we have $x \in B_d(x, \epsilon) \subset B$. Consider the following basis element in the box topology:

$$U = \prod_i \left(x_i - \frac{\epsilon}{2^i}, x_i + \frac{\epsilon}{2^i} \right)$$

for some $y \in U$, note that:

$$d(x, y) = \sqrt{\sum_i (x_i - y_i)^2} < \sqrt{\epsilon^2 \sum_{i=1}^{\infty} \frac{1}{4^i}} = \frac{\epsilon}{\sqrt{3}} < \epsilon$$

so $y \in B_d \subset B$. Thus, the box topology is finer than the ℓ^2 topology. \square

Proposition 28. *Clearly, $\mathbb{R}^\infty \subset X$. Show that the subspace topology for \mathbb{R}^∞ from the product, box, ℓ^2 , and uniform topologies are distinct.*

Proof. \square

Proposition 29. *In the box topology, no function of the form $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ with $f(t) = (a_1 t, a_2 t, \dots)$ is continuous, where $a_i > 0$.*

Proof. Pick the open set:

$$U = \prod_n \left(-\frac{a_n}{n}, \frac{a_n}{n} \right)$$

in the box topology. We assert that $f^{-1}(U)$ is not open in \mathbb{R} . Suppose it is. Note that $0 \in f^{-1}(U)$. Thus, there must be some $(-\delta, \delta)$ such that $0 \in (-\delta, \delta) \subset f^{-1}(U)$, by definition of the standard topology. However, this would imply that for all n :

$$(-\delta a_n, \delta a_n) \subset \left(-\frac{a_n}{n}, \frac{a_n}{n} \right)$$

which clearly is not the case. Thus, $f^{-1}(U)$ can't be open. It follows that f is not continuous. \square

Proposition 30. *In the uniform topology, a function of the form $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ with $f(t) = (a_1 t, a_2 t, \dots)$ is continuous, where $a_i > 0$ is continuous if and only if the set of a_i is bounded in \mathbb{R} .*

Proof. Suppose we are given some f with the set $\{a_i\}$ bounded. Let $a = \sup\{a_i\}$. Suppose U is open in the uniform topology and contains $f(x)$. It follows that there is some $B_{\bar{\rho}}(f(x), \epsilon)$ with $\epsilon < 1$ such that $f(x) \in B_{\bar{\rho}}(f(x), \epsilon) \subset U$. Choose $\delta = \frac{\epsilon}{s}$. We then note that:

$$\begin{aligned} y \in B(x, \delta) &\Rightarrow |y - x| < \delta = \frac{\epsilon}{a} \\ \Rightarrow \bar{\rho}(f(x), f(y)) &= \sup\{|a_i x - a_i y|\} = \sup\{|a_i||x - y|\} < \sup\left\{\frac{|a_i|\epsilon}{a}\right\} \leq \epsilon \end{aligned}$$

Hence, $f(y) \in B_{\bar{\rho}}(f(x), \epsilon)$, so $f(B) \subset B_{\bar{\rho}}(f(x), \epsilon)$. By definition, f is continuous.

Conversely, suppose f is continuous. Consider the open set $B_{\bar{\rho}}(f(x), \epsilon)$ in \mathbb{R}^ω , with $\epsilon < 1$. Since f is continuous, we can choose some $U = (x - \delta, x + \delta)$ such that $f(U) \subset B_{\bar{\rho}}$. Thus, we have:

$$y \in U \Rightarrow \sup\{\bar{d}(f(x)_i, f(y)_i)\} = \sup\{\min(|a_i||x - y|, 1)\} < \epsilon < 1$$

Since $\epsilon < 1$, we must have $|a_i||x - y| < 1$ for all i and all $y \in U$. Since we know there exists some $y_0 \neq x$ in U , we note that $|a_i| < \frac{1}{|x - y_0|}$ for all i , so the set of a_i is bounded. \square

From the above two propositions, we immediately arrive at the solution to Munkres Exercise 2.20.4. All of the maps are continuous with respect to the product topology, none are continuous with respect to the box topology, and the last two are continuous with respect to the uniform topology.

6. SECTION 21

Proposition 31. *Let X be a set, and let $f_n : X \rightarrow \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be a uniform metric on the space \mathbb{R}^X . The sequence (f_n) converges uniformly to f if and only if the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$.*

Proof. Firstly, note that the uniform metric on \mathbb{R}^X is given by:

$$\bar{\rho}(f, g) = \sup\{\bar{d}(f(x), g(x)) \mid x \in X\}$$

Suppose to begin that f_n converges uniformly to f . Pick some open U around f . Clearly, we will have $f \in B_{\bar{\rho}}(f, \epsilon/2) \subset U$, for some $\epsilon/2$ (we let $\epsilon/2 < 1$). We can choose N such that for $n > N$, we have $\bar{d}(f_n(x), f(x)) < \epsilon/2$ for all x . It then follows that:

$$\bar{\rho}(f, f_n) = \sup\{\bar{d}(f(x), f_n(x)) \mid x \in X\} \leq \frac{\epsilon}{2} < \epsilon$$

so $f_n \in B_{\bar{\rho}}$, for $n > N$. It follows that f_n converges to f in the uniform metric space. Conversely, suppose f_n converges to f in the uniform metric. For some ϵ (WLOG, we let $\epsilon < 1$), we can choose N such that for $n > N$, we have:

$$\sup \{ \bar{d}(f(x), f_n(x)) \mid x \in X \} < \epsilon$$

implying that $d(f(x), f_n(x)) < \epsilon$ for all x , for all $n > N$. The result follows. \square

Proposition 32. *Suppose $x_n \rightarrow x$ and f_n is a sequence of continuous function that converges uniformly to f . It follows that $f_n(x_n)$ converges to $f(x)$.*

Proof. By uniform limit theorem, f is continuous. It follows that $f(x_n)$ converges to $f(x)$. Pick some open interval $U = (f(x) - \epsilon, f(x) + \epsilon)$ around $f(x)$. There exists some M such that for $n > M$, we have $|f(x) - f(x_n)| < \epsilon/2$. By uniform continuity, we can choose N such that for $n > N$, we have $|f(x_n) - f_n(x_n)| < \epsilon/2$. Thus, for $n > \max\{M, N\}$, we have:

$$|f_n(x_n) - f(x)| < |f(x) - f(x_n)| + |f(x_n) - f_n(x_n)| < \epsilon$$

so $f_n(x_n) \in U$. It follows that $f_n(x_n) \rightarrow f(x)$. \square

Now, we will look at a few problems involving infinite series:

Proposition 33.