Non-linear Katz-Oda

Jack Ceroni
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I. Introduction

Summarizing work done on the non-linear Katz-Oda Grothendieck p-conjecture problem.

II. Background

We begin by considering the ODE

$$\frac{dY}{dz} = A(z)Y \tag{1}$$

where $A(z)$ is a matrix whose entries are rational functions of z relative to some algebraic number field K (a field extension of $\mathbb{Q}$ such that $[K : \mathbb{Q}]$ is finite). For the sake of simplicity, let $K = \mathbb{Q}$. Some entry of $A(z)$ will be of the form

$$A_{ij}(z) = \frac{A_{ij}^{(n)} z^n + \cdots + A_{ij}^{(0)}}{B_{ij}^{(m)} z^m + \cdots + B_{ij}^{(0)}} \tag{2}$$

which of course may be reduced modulo some prime p in the ring of algebraic integers of K, for almost all primes (in this case, just the usual prime numbers in $\mathbb{Z}$). In particular, setting

$$\frac{a}{b} \mod p \equiv \frac{a \mod p}{b \mod p} \tag{3}$$

when $p \nmid b$. Thus, we may reduce the rational function's coefficients in the case when p doesn't divide any of the denominators of the rational coefficients $B_{ij}^{(k)}$. The result will be a differental equation over $\mathbb{F}_q[z]$ for some finite field $\mathbb{F}_q$. As a particular example, suppose $a \in \mathbb{Z} \subset \mathbb{Q}$ and we have the ODE given by

$$\frac{dy}{dz} = \frac{1}{az} y \tag{4}$$

We can reduce by $p \nmid a$. When we do such a reduction, we get the ODE over $\mathbb{F}_p[z]$ given by $y' = \frac{1}{(a \mod p)z}$. The equation admits solutions $y = z^b$ for any b such that $ab \equiv 1 \mod p$, as we will have

$$\frac{dy}{dz} = \frac{d}{dz} z^b = (b \mod p) z^{b-1} = \frac{ab \mod p}{(a \mod p)z} = \frac{1}{az} y \tag{5}$$

where we are interpreting the derivative in this case as the derivative of polynomials over a finite field, which is defined using the power rule. On the other hand, in the rational case, $y = z^{1/a}$ is the solution (up to a multiplicative factor).

Differential equations of this form, and reducation by a prime is related to the p-curvature conjecture. There is a result concerning a particular family of ODEs of this form (Picard-Fuchs), formulated by Katz and Oda, which we can restate here

Proposition II.1. The Picard-Fuchs differential equation is an ODE of the form in Eq. (1). It is an algebraic differential equation, meaning that its solution is an algebraic function. Moreover, there exists sufficiently large N such that for almost all primes p,

$$\left(\frac{d}{dz} - A(z)\right)^{Np} \equiv 0 \mod p \tag{6}$$

The sense in which we mean that iterating this differential operator yields $0 \mod p$ is as follows: differential operators which have been reduced relative to some prime are thought of as "discrete", and act only on rational functions, whose derivatives are defined using the product rule. Thus, to check if such a differential operator is 0, we merely must evaluate it on all powers of z, z^n. For example, consider the differential operator $z\partial_z$. Suppose we iterate it 3 times, we will get

$$\begin{aligned}(z\partial_z)^3 = z\partial_z(z\partial_z + z^2\partial_z^2) &= z\partial_z + z^2\partial_z^2 + 2z^2\partial_z^2 + z^3\partial_z^3 &(7)\\ &= z\partial_z + 3z^2\partial_z^2 + z^3\partial_z^3 &(8)\end{aligned}$$

From here, we may evaluate it on z^n:

$$(z\partial_z + 3z^2\partial_z^2 + z^3\partial_z^3)z^n = [n + 3n(n-1) + n(n-1)(n-2)]\, z^n \tag{9}$$

Note that

$$n + 3n(n-1) + n(n-1)(n-2) = n(1 + 3(n-1) + (n-1)(n-2)) = n(1 + (n+1)(n-1)) = n^3 \tag{10}$$

and from Fermat's little theorem, $n^3 = n \mod 3$. In other words,

$$(z\partial_z)^3 z^n = nz^n = (z\partial_z)z^n \mod 3 \tag{11}$$

so it follows that modulo the prime 3, we have $(z\partial_z)^3 = z\partial_z$.

III. The Non-linear problem

We wish to understand whether a similar result as Thm. **??** holds for the Schlesinger system, which is a non-linear system of partial differential equations, given by

$$\frac{\partial B_i}{\partial \lambda_j} = \frac{[B_i, B_j]}{\lambda_i - \lambda_j} \quad \text{for } i \neq j \tag{12}$$

$$\sum_j \frac{\partial B_i}{\partial \lambda_j} = 0 \quad \text{for all } i. \tag{13}$$

Note that each B_j is a function of variables $\lambda_1, \ldots, \lambda_n$, for j from 1 to m.

The first idea is to understand the vector field/phase portrait associated with each individual variable λ_j. At a high-level, this will be a vector field in which the vectors have entries which are matrices. In particular, we can identify $M_{N\times N}(\mathbb{R})$ with $\mathbb{R}^{N^2}$, so that we will have a vector field with vectors in $\mathbb{R}^{mN^2}$. For a given

λ_j, the system of differential equations is then given by

$$\frac{\partial}{\partial\lambda_j}\begin{pmatrix} B_1 \\ \vdots \\ B_{j-1} \\ B_j \\ B_{j+1} \\ \vdots \\ B_m \end{pmatrix} = \begin{pmatrix} \frac{[B_1,B_j]}{\lambda_1-\lambda_j} \\ \vdots \\ \frac{[B_{j-1},B_j]}{\lambda_i-\lambda_j} \\ -\sum_{i\neq j}\frac{[B_i,B_j]}{\lambda_i-\lambda_j} \\ \frac{[B_{j-1},B_j]}{\lambda_{j-1}-\lambda_j} \\ \vdots \\ \frac{[B_m,B_j]}{\lambda_m-\lambda_j} \end{pmatrix} \tag{14}$$

This will indeed yield a collection of n different vector fields, for j ranging from 1 to j. Under the identification of $M_{N\times N}(\mathbb{R})$ with $\mathbb{R}^{N^2}$, note that the variables with respect to which this vector field is a derivation are the matrix entries, $[B_k]_{ij}$ for k ranging from 1 to m and $1 \leq i, j \neq N$. In particular, at the point $(B_1, \ldots, B_m) \in \mathbb{R}^{mN^2}$, the vector field is given by

$$X^{(j)}_{B_1,\ldots,B_m} = \sum_{p,q=1}^{N}\sum_{i,i\neq j}\frac{[B_i,B_j]_{pq}}{\lambda_i-\lambda_j}\left(\frac{\partial}{\partial[B_i]_{pq}} - \frac{\partial}{\partial[B_j]_{pq}}\right) \tag{15}$$

Note that $X^{(j)}_{B_1,\ldots,B_m}$ takes some $f \in C^\infty(\mathbb{R}^{mN^2})$ of the form $f(M_1, \ldots, M_m)$ to a real number, so $X^{(j)}(f)$ is a smooth function which takes m matrices and outputs a real number. We will be interested in all possible products/compositions of these vector fields with themselves:

$$X^{(J)} = X^{(j_1\cdots j_N)} = \prod_{k=1}^{N} X^{(j_k)} \tag{16}$$

and, in particular, how large we need to make N such that for all tuples $(j_1, \ldots, j_N)$, the operator $X^{(J)}$ vanishes, modulo some prime p.

IV. Numerics

As a first example, let us test the claim of Proposition 2.1 of the notes, which states that for a Picard-Fuchs linear system of ODEs, given by

$$\frac{dy}{dz} = A(z)y = \sum_{i=1}^{n}\frac{A_i}{q_i z - p_i}y \tag{17}$$

where $q_i, p_i \in \mathbb{Z}$ and the entires of the A_i are also integers. The corresponding differential operator $\left(\frac{d}{dz} - A(z)\right)$ will be equal to 0 mod p for almost all primes p, when raised to a power Np for sufficiently large N. We are interested in the action of this differential operator on any $y(z)$, an n-dimensional vector whose entries are rational functions with rational coefficients (WLOG we can assume that the coefficients are integers). Suppose $Q(z)$ is a polynomial with integer coefficients. For some z_0 **rational** and not a root of Q (of which there are a finite number), so $Q(z_0) > 0$, it follows from continuity that we can choose some neighbourhood of z_0 such that

$$|Q(z) - Q(z_0)| < |Q(z_0)| \tag{18}$$

which means that $Q(z) \neq 0$ in this neighbourhood and

$$\frac{1}{Q(z)} = \frac{1}{Q(z_0) - (Q(z_0) - Q(z))} = \frac{1}{Q(z_0)}\frac{1}{1-\left(1-\frac{Q(z)}{Q(z_0)}\right)} = \sum_{j=0}^{\infty}\frac{1}{Q(z_0)^{j+1}}\left(Q(z_0) - Q(z)\right)^j \tag{19}$$

Since z_0 is rational, so is $\frac{1}{Q(z_0)}$ and all powers of it. It follows that the above is a power series with rational coefficients. It follows that in a neighbourhood of each point which isn't a pole, the rational function $\frac{P(z)}{Q(z)}$ with rational coefficients can be written as an absolutely convergent power series with rational coefficients. It follows from linearity of the Picard-Fuchs differential operator that if we verify that it is 0 mod p for power Np when acting on $y(z)$ of the form

$$y_p(z) = (z^{p_1}, z^{p_2}, \dots, z^{p_{m-1}}, z^{p_m}) \tag{20}$$

for $p = (p_1, \dots, p_m)$ with $p_j \in \mathbb{Z}_{\geq 0}$ then it will also be 0 for all rational functions with rational coefficients (of course this is an if and only if, as the above y *is* a vector of rational functions with rational coefficients). Ideally, given some fixed input numbers $q_i, p_i \in \mathbb{Z}$ and matrices A_i, would would like to output the coefficients of rational functions in the entries of the vector

$$\left(\frac{d}{dz} - \sum_{i=1}^{n} \frac{A_i}{q_i z - p_i}\right)^M y_p(z) \tag{21}$$

as a function of M and $p = (p_1, \dots, p_m)$. Note that the differential operator $O_k = (\frac{d}{dz} - A(z))^k$ will be of the form

$$O_k = \left(\frac{d}{dz} - A(z)\right)^k = (-1)^k A(z)^k + B_k^{(1)}(z)\frac{d}{dz} + \dots + B_k^{(k-1)}(z)\frac{d^{k-1}}{dz^{k-1}} + \frac{d^k}{dz^k} \tag{22}$$

Moreover, these operators will satisfy a recursion relation given by

$$O_{k+1} = \left(\frac{d}{dz} - A(z)\right) O_k = \frac{dO_k}{dz} + O_k \frac{d}{dz} - A(z) O_k \tag{23}$$

where $\frac{dO_z}{dz}$ denotes differentiation of the coefficients. It follows that the individual coefficients satisfy

$$B_{k+1}^{(0)}(z) = (-1)^{k+1} A(z) \tag{24}$$

$$B_{k+1}^{(j)}(z) = \frac{dB_k^{(j)}(z)}{dz} + B_k^{(j-1)}(z) - A(z) B_k^{(j)}(z) \qquad \text{for} \quad 1 \leq j \leq k \tag{25}$$

$$B_{k+1}^{(k+1)}(z) = 1 \tag{26}$$

Another possibility is attempting to solve for the coefficients using the Baker-Campbell-Hausdorff formula. In particular, let $Y(z) = \frac{d}{dz} - A(z)$, we then have

$$e^{tA(z)} e^{tY(z)} = e^{tZ(z)} \implies e^{tY(z)} = e^{-tA(z)} e^{tZ(z)} \tag{27}$$

where $Z(z)$ is given by the BCH formula. We can explicitly write it using Dynkin's combinatorial formula, and then look at the n-th order term in t:

$$tZ = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{\substack{r,s \\ r_j + s_j > 0}} \frac{t^{|r|+|s|}}{(|r| + |s|) \prod_{i=1}^{n} r_i! s_i!} [A(z)^{r_1} Y(z)^{s_1} \dots A(z)^{r_n} Y(z)^{s_n}] \tag{28}$$

Actually, this probably won't give anything more useful than the recursive formula. Nevermind...

When we apply O_k to $y_p(z)$, note that we will get

$$O_n(z) y_p(z) = \sum_{j=0}^{n} B_n^{(j)}(z) \frac{d^j y_p(z)}{dz^j} = \sum_{j=0}^{n} B_n^{(j)}(z) \begin{pmatrix} \frac{p_1!}{(p_1-j)!} z^{p_1 - j} \\ \vdots \\ \frac{p_m!}{(p_m-j)!} z^{p_m - j} \end{pmatrix} \tag{29}$$

So, to reiterate, if we can compute all of the coefficients $B_n^{(j)}(z)$ for a given n, for j from 0 to n, then we can use the above formula to get a symbolic expression for $O_n(z) y_p(z)$. The problem is that this **is not** an explicit formula in terms of the order n, so we can't use this formula to verify that $O_{Np} = 0$ mod p for almost all primes, we can only do it in specific cases.