An primer on topological K-theory and a topological proof of Bott periodicity

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I. Introduction

This essay was written for the Fall 2023 session of MAT437: K-Theory and C^*-Algebras, taught by Professor George Elliott, at the University of Toronto.

The goal of this essay is to discuss in greater detail than has been covered in class and in the text of Rordam, certain aspects of topological K-theory, and their relationship to the algebraic flavour of K-theory to which we have dedicated greater focus.

II. Basics

Let us begin by recalling a very basic definition:

Definition II.1 (Complex vector bundle)**.** A complex vector bundle over topological space X is defined to be a triple $\xi = (E, \pi, X)$, where E is a topological space and $\pi : E \to X$ is a continuous surjection from E to X, the so-called *base space*, where each *fibre* $\pi^{-1}(x)$ has the structure of a complex finite-dimensional vector space. We require that ξ has the following local compatibility property: for each $x \in X$, there must exist a neighbourhood U of x and a homeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{C}^n$ for some $n \in \mathbb{N}$ making the following diagram commute:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "h"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^n \arrow[dl, "\eta"] \\
& U &
\end{tikzcd}$$

where $\eta(x, v) = x$ is the projection onto the base, and, for each $y \in U$, the map $h : \pi^{-1}(y) \to \{y\} \times \mathbb{C}^n \simeq \mathbb{C}^n$ is a vector space isomorphism. Intuitively, this simply means that locally, in the base space, the corresponding bundle of fibres in E must resemble a trivial product space $U \times \mathbb{C}^n$, both in a topological sense and an algebraic sense.

The motivation for studying topological K-theory comes from the desire to classify all vector bundles over a given base space of a particular, fixed dimension n. To be more specific, we say that bundles $\xi = (E, \pi, X)$ and $\xi' = (E', \nu, X)$ are *isomorphic* if there is a homeomorphism $h : E \to E'$ making the following diagram commute:

$$\begin{tikzcd}
E \arrow[rr, "h"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\nu"] \\
& X &
\end{tikzcd}$$

such that the restricted map $h : \pi^{-1}(x) \to \nu^{-1}(x)$ is a vector space isomorphism for each $x \in X$. The task of computing the isomorphism classes of vector bundles (which we denote $\langle \xi \rangle$, where ξ is an equivalence class representative) is a highly non-trivial task, and can only be done in very basic, often low-dimensional cases. Thus, in order to make partial progress on solving this profoundly difficult question, a natural question to

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ask is: *are there weaker equivalence relations than the above notion of isomorphism which can more easily be computed, at the cost of ending up with a more crude classification?* The answer to this question is a resounding yes, via the construction of topological K-groups.

Treating vector bundle ξ now as fundamental objects, let us define algebraic operations between them.

Lemma II.1 (Direct sum of vector bundles)**.**

Lemma II.2 (Inner products on vector bundles)**.**

Lemma II.3 (Tensor products of vector bundles)**.**

Lemma II.4.

Theorem II.1 (Equivalence of algebraic and topological K-theory)**.** If X is a compact Hausdorff space, then $K_0(C(X))$, the algebraic K_0-group of the C^*-algebra of functions $C(X)$, and $K(X)$, the topological K^0-group, are isomorphic as Abelian groups.

Proof. □

III. Bott periodicity

IV. Characteristic classes