Algebraic topology

Jack Ceroni[*]
(Dated: Thursday 12$^{\text{th}}$ June, 2025)

Contents

I. Introduction

II. Hatcher's book

III. May's book

A. Chapter 1

1. Solutions

Solution III.1 (May Problem 1.1)**.** Suppose $\deg(\widehat{p}) = n = \deg(z^n)$. It follows that $\widehat{p} \simeq z^n$. Therefore, if we also have $\deg(\widehat{q}) = m$, then $\widehat{p}\widehat{q} \simeq z^n z^m = z^{n+m}$, so $\deg(\widehat{p}\widehat{q}) = n + m = \deg(\widehat{p}) + \deg(\widehat{q})$. We know that p has n roots (counted with multiplicity), so we can write

$$\widehat{p}(z) = \frac{p(z)}{|p(z)|} = \prod_{k=1}^{n} \frac{z - c_k}{|z - c_k|} \tag{1}$$

which means that

$$\deg(\widehat{p}) = \sum_{k=1}^{n} \deg\left(\frac{z - c_k}{|z - c_k|}\right) \tag{2}$$

From the proof of the fundamental theorem of algebra, we know that the degree of each summand above is 1 if $c_k \in D$ and 0 otherwise, which completes the proof.

Solution III.2 (May Problem 1.2)**.** Note that $\frac{f(z)}{z}$ is also a map from S^1 to S^1. If $g : S^1 \to S^1$ is a map, and g is *not* surjective, it is clear that we can homotop the image of g to a single point, which means that $\deg(g) = 0$. Thus, if $\deg(g) \neq 0$, then g is surjective. We then note (using the multiplicative property of degree we proved in the previous solution),

$$\deg\left(\frac{f(z)}{z}\right) = \deg(f) - 1 \neq 0 \tag{3}$$

[*] jceroni@uchicago.edu

as $\deg(f) \neq 1$, so $\frac{f(z)}{z}$ is surjective. In particular, there is some z_0 where $\frac{f(z_0)}{z_0} = 1$, so $f(z_0) = z_0$, as desired.

Solution III.3 (May Problem 1.3)**.** Let us consider the composition of loops based at e. It is easy to see that $(\beta \cdot \alpha)(\beta' \cdot \alpha') = (\beta\beta') \cdot (\alpha\alpha')$. Thus,

$$\beta\alpha \simeq (\beta \cdot c_e)(c_e \cdot \alpha) = \beta \cdot \alpha \tag{4}$$

From here, we want to show that $\alpha(t)\beta(t)$ and $\beta(t)\alpha(t)$ are homotopic loops. Consider $H(t,s) = \alpha(st)\beta(t)\alpha(st)^{-1}$. Note that $H(t,0) = \beta(t)$, $H(t,1) = \alpha(t)\beta(t)\alpha(t)^{-1}$, $H(0,s) = e = \alpha(s)\alpha(s)^{-1} = H(1,s)$, so H is a homotopy of loops based at e. By multiplying both of these homotopic loops aon the right by $\alpha(t)$, we get $\alpha\beta \simeq \beta\alpha$. Thus,

$$[\alpha] \cdot [\beta] = [\alpha\beta] = [\beta\alpha] = [\beta] \cdot [\alpha] \tag{5}$$

as desired.

IV. Chapter 2

V. Chapter 3

I want to present the proof of the fundamental theorem of covering groupoids in full detail.

Theorem V.1 (Fundamental theorem)**.** Let $p : E \to B$ be a covering map of groupoids, let $f : H \to B$ be a functor between groupoids. Pick some $x_0 \in H$, let $b_0 = f(x_0)$ and choose e_0 with $p(e_0) = b_0$. Then there exists a functor $g : H \to E$ such that $g(x_0) = e_0$ and $p \circ g = f$ if and only if

$$f(\pi(H, x_0)) \subset p(\pi(E, e_0)) \tag{6}$$

where we recall that $\pi(G, g)$ is the subcategory consisting of all automorphisms of $g \in \mathrm{Obj}(g)$, where G is a groupoid.

Proof. Suppose we have functor g, then given some $\alpha \in \pi(H, x_0)$, note that $f(\alpha) = p(g(\alpha))$. Of course, $g(\alpha) \in \pi(E, e_0)$ as $g(x_0) = e_0$, and we have the deisred inclusion of sets.

On the other hand, because p is a covering map, it is surjective on objects and restricts to a bijection $p : \mathrm{st}(e_0) \to \mathrm{st}(b_0)$. This means that given $\alpha \in \mathrm{st}(x_0)$, so that $f(\alpha) \in \mathrm{st}(b_0)$, there exists unique $\widetilde{\alpha} \in \mathrm{st}(e_0)$ such that $p(\widetilde{\alpha}) = f(\alpha)$. Given some $\alpha \in \mathrm{Mor}_H(x, y)$, the idea is to choose some $\beta \in \mathrm{Mor}_H(x_0, x)$ and define $g(\alpha) = (\widetilde{\beta})^{-1} \cdot \widetilde{\beta \cdot \alpha}$. To show that this is well-defined, ∎