

MAT495 January report

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I. Introduction

I spent the majority of the month looking into the classification theorem for finite-dimensional C^* -algebras. I will summarize the material that I covered in this report.

II. Classification of finite-dimensional C^* -algebras

The classification theorem for finite-dimensional C^* -algebras says the following:

Theorem II.1 (Classification of finite-dimensional C^* -algebras). Every finite dimensional C^* -algebra is $*$ -isomorphic to a direct sum of matrix algebras over the complex numbers. In particular, if A is a finite-dimensional C^* -algebra, then for some collection of positive integers $\{n_1, \dots, n_k\}$,

$$A \simeq \bigoplus_{j=1}^N M_{n_j}(\mathbb{C}) \quad (1)$$

Since the matrix algebra $M_n(\mathbb{C})$ is non-commutative for any $n > 1$, we have the immediately corollary:

Corollary II.1.1. If A is a finite-dimensional, *commutative* C^* -algebra, then $A \simeq \mathbb{C} \oplus \dots \oplus \mathbb{C}$, the direct sum of k copies of the complex numbers, for some k .

A direct proof of this elementary result is more straightforward than the case of possibly non-Abelian C^* -algebras, so we will begin here. This proof, as well as the proof of the more general result will rely on the following:

Theorem II.2 (Gelfand). Every Abelian C^* -algebra is isometrically isomorphic to the algebra of continuous functions vanishing at infinity, $C_0(X)$, on some locally compact Hausdorff space X .

Making use of this result, we conclude that some finite-dimensional C^* -algebra A will be $*$ -isomorphic to $C_0(X)$ for some locally compact Hausdorff space X . Let us quickly prove a topological lemma:

Lemma II.1. Locally compact Hausdorff space separate points with continuous functions. That is, for $x_1, x_2 \in X$ distinct, there exists a continuous function $f : X \rightarrow \mathbb{R}$ such that $f(x_1) \neq f(x_2)$ (it is completely regular).

Proof. This follows from the fact that we can embed X into its one-point compactification, which is compact Hausdorff, and thus normal, a stronger condition than being completely regular (this follows from Urysohn's lemma). A subspace will also be completely regular, so X is. \square

From this fact, it immediately follows that when A is finite-dimensional, the corresponding X must only have a finite number of points. For, let consider some set of distinct points $\{x_1, \dots, x_n\} \in X$. Let f_k be a function which is 1 at x_k and 0 at all the other x_j . Clearly, the collection of functions f_1, \dots, f_n will be linearly independent. Since $C_0(X)$ is isomorphic to a finite-dimensional algebra A , we can have at most $\dim(A)$ such linearly independent functions, which implies that the number of distinct points in X can be at most $\dim(A)$.

Of course, a topological space with a finite number of points is compact, in this case, we can simply replace $C_0(X)$ with $C(X)$, the space of continuous functions. In fact, if $n = \dim(A)$, then the set of

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functions (f_1, \dots, f_n) described above will clearly form a basis for $C(X)$. We can then define the map $\Psi : C(X) \rightarrow \mathbb{C} \oplus \dots \oplus \mathbb{C}$, as the linear map such that $\Psi(f_k) = 0 \oplus \dots \oplus 1 \oplus \dots \oplus 0$, where the 1 is placed at the k -th (out of n) direct summands.

Verifying that this is a $*$ -isomorphism is easy: it is linear, since $f_j f_k = \delta_{jk} f_j$, we have $\Psi(f_j f_k) = \Psi(f_j) \Psi(f_k)$, and clearly $\Psi(f_k^*) = \Psi(f_k)^*$. Thus, we have proved that $A \simeq \mathbb{C} \oplus \dots \oplus \mathbb{C}$.

A. Proving the more general result

Now that we have proved the desired result for Abelian C^* -algebras, we must turn our attention to *arbitrary* finite-dimensional C^* -algebras. Let us briefly begin at the end: consider the specific case of a finite-dimensional C^* -algebra that is *a priori* the direct sum of matrix algebras, as in Eq. (??). A basis for this algebra is easy to write down: let $e(k, i, j)$ be the $k \times k$ matrix with 1 at the (i, j) -th entry and 0 elsewhere. In our case, the set

$$\mathcal{B} = \left\{ e_{ij}^{(k)} = 0 \oplus \dots \oplus e(n_k, i, j) \oplus \dots \oplus 0 \mid 1 \leq k \leq N, 1 \leq i, j \leq n_k \right\} \quad (2)$$

where in the direct sum for $e_i^{(k)} j$, $e(n_k, i, j)$ is the k -th direct summand. This collection will clearly form a basis. These elements have particular easy-to-verify properties, namely

1. $e_{ij}^{(k)} e_{jl}^{(k)} = e_{il}^{(k)}$
2. $e_{ij}^{(k)} e_{mn}^{(l)} = 0$ if $k \neq l$ or $j \neq m$
3. $(e_{ij}^{(k)})^* = e_{ji}^{(k)}$

The idea for the proof of the classification theorem is that for an arbitrary finite-dimensional C^* -algebra, we will identify elements of the algebra which have this same behaviour, and in turn define a $*$ -isomorphism from this collection of elements, to the matrix elements defined above, which form a basis for a direct sum of matrix algebras. Generally, we will call elements which satisfy the set of conditions of list *matrix units*. To be more specific, we have the following:

Claim II.1. If B is another C^* -algebra containing elements $f_{ij}^{(k)}$ which behave in the exact same way as the $e_{ij}^{(k)}$ (they are non-zero, have the same properties outlined in the list above, and they span the algebra B), then there is a unique $*$ -isomorphism $\varphi : A \rightarrow B$ such that $\varphi(e_{ij}^{(k)}) = f_{ij}^{(k)}$

Proof. We define φ as the unique linear extension of $\varphi(e_{ij}^{(k)}) = f_{ij}^{(k)}$. Checking whether multiplication is preserved amounts to verifying that $\varphi(e_{ij}^{(k)} e_{mn}^{(l)}) = \varphi(e_{ij}^{(k)}) \varphi(e_{mn}^{(l)}) = f_{ij}^{(k)} f_{mn}^{(l)}$, which can be immediately verified from the list of properties which the $e_{ij}^{(k)}$ and the $f_{ij}^{(k)}$ verify. In addition,

$$\varphi((e_{ij}^{(k)})^*) = \varphi(e_{ji}^{(k)}) = f_{ji}^{(k)} = (f_{ij}^{(k)})^* = \varphi(e_{ij}^{(k)})^* \quad (3)$$

so we have a $*$ -isomorphism. If the $f_{ij}^{(k)}$ span B , then we have a surjection, and if none are zero, we have an empty kernel, so φ is an injection. Thus, it is a bijection, so φ is a $*$ -isomorphism. \square

Now, let us begin our search for these $f_{ij}^{(k)}$. First, a small lemma:

Lemma II.2. If v is a partial isometry (so $v^* v$ is a projection) then $vv^* v = v$.

Proof. Let $z = v - vv^* v = (1 - vv^*)v$. Note that

$$z^* z = v^* (1 - vv^*)^* (1 - vv^*) v = v^* (1 - 2vv^* + vv^* vv^*) v = v^* v - 2v^* v + v^* v = 0 \quad (4)$$

where we use that $(v^* v)^2 = v^* v$. Thus, $\|z\|^2 = \|z^* z\| = 0$, so $\|z\| = 0$ and $v = vv^* v$, as desired. \square

Now, another lemma:

Lemma II.3. Suppose that $\{f_{ii}^{(k)} \mid 1 \leq k \leq r, 1 \leq i \leq n_k\}$ is a set of mutually orthogonal projections in C^* -algebra B and that

$$f_{11}^{(k)} \sim f_{22}^{(k)} \sim \dots \sim f_{n_k n_k}^{(k)} \quad (5)$$

for each k . Then there is a system of matrix units $\{f_{ij}^{(k)}\}$ extending $\{f_{ii}^{(k)}\}$.

The idea behind constructing systems of matrix units is, essentially, to have a “basis” for each component of the direct sum that we will eventually demonstrate characterizes the C^* -algebra B . Each of the sets $\{f_{ii}^{(k)}\}$ are analogous to matrix projections with 1 at the i -th slot on the diagonal, at the k -th slot in the direct sum. Let us now prove the lemma.

Proof. Of course, here, we will make use of the Murray-von Neumann equivalence. Namely,

$$f_{11}^{(k)} \sim f_{jj}^{(k)} \implies f_{11}^{(k)} = f_{1j}^{(k)} f_{1j}^{(k)*} \quad \text{and} \quad f_{jj}^{(k)} = f_{1j}^{(k)*} f_{1j}^{(k)} \quad (6)$$

This notation is consistent, as $f_{11}^{(k)}$ is self-adjoint, so setting $j = 1$ above causes no problems. Our claim is that if we set $\tilde{f}_{ij}^{(k)} = f_{1i}^{(k)*} f_{1j}^{(k)}$ then we will have the desired system of matrix units. This is in fact an extension of the system we are already provided. Namely, we have

$$\tilde{f}_{jj}^{(k)} = f_{1j}^{(k)*} f_{1j}^{(k)} = f_{jj}^{(k)} \quad (7)$$

by definition. In fact, we might as well denote \tilde{f}_{ij} by f_{ij} , as for $i = 1$, we have

$$\tilde{f}_{1j} = f_{11}^{(k)*} f_{1j}^{(k)} = f_{1j}^{(k)} f_{1j}^{(k)*} f_{1j}^{(k)} = f_{1j}^{(k)} \quad (8)$$

where we use the above lemma and the fact that $f_{1j}^{(k)*} f_{1j}^{(k)} = f_{jj}^{(k)}$ is a projection. Let us now complete our verification. Of course, we have

$$f_{pq}^{(k)} f_{qr}^{(k)} = f_{1p}^{(k)*} f_{1q}^{(k)} f_{1q}^{(k)*} f_{1r}^{(k)} = f_{1p}^{(k)*} f_{11}^{(k)} f_{1r}^{(k)} = f_{1p}^{(k)*} f_{1r}^{(k)} f_{1r}^{(k)*} f_{1r}^{(k)} = f_{1p}^{(k)*} f_{1r}^{(k)} = f_{pr}^{(k)} \quad (9)$$

where we use the first lemma to note that $f_{1r}^{(k)} f_{1r}^{(k)*} f_{1r}^{(k)} = f_{1r}^{(k)}$, as $f_{1r}^{(k)*} f_{1r}^{(k)} = f_{rr}^{(k)}$ is a projection. Next, note that

$$f_{pq}^{(k)} f_{rs}^{(\ell)} = f_{1p}^{(k)*} f_{1q}^{(k)} f_{1r}^{(\ell)*} f_{1s}^{(\ell)} \quad (10)$$

Once again using the first lemma, we have $f_{1q}^{(k)} = f_{1q}^{(k)} f_{1q}^{(k)*} f_{1q}^{(k)} = f_{1q}^{(k)} f_{qq}^{(k)}$ and $f_{1r}^{(\ell)} = f_{1r}^{(\ell)} f_{1r}^{(\ell)*} f_{1r}^{(\ell)} = f_{1r}^{(\ell)} f_{rr}^{(\ell)}$ so that $f_{1r}^{(\ell)*} = f_{rr}^{(\ell)} f_{1r}^{(\ell)*}$. We then use the fact that the projections in our set are mutually orthogonal to conclude that

$$f_{1p}^{(k)*} f_{1q}^{(k)} f_{1r}^{(\ell)*} f_{1s}^{(\ell)} = f_{1p}^{(k)*} f_{1q}^{(k)} (f_{qq}^{(k)} f_{rr}^{(\ell)}) f_{1r}^{(\ell)*} f_{1s}^{(\ell)} = 0 \quad (11)$$

which is 0 when $q \neq r$ or $k \neq \ell$, as in these cases, $f_{qq}^{(k)} f_{rr}^{(\ell)} = 0$. It is very immediately clear that $f_{ij}^{(k)*} = f_{1j}^{(k)*} f_{1i}^{(k)} = f_{ji}^{(k)}$, so we have verified the third condition, and it follows that our set of $f_{ij}^{(k)}$ is in fact a system of matrix units in B extending $\{f_{ii}^{(k)}\}$. \square

Going forward, our new strategy is to find the “diagonal matrix elements” $f_{ii}^{(k)}$, and then extend them to the full set of matrix units that we need. One nice fact we should notice is that they mutually commute, as they are mutually orthogonal. In fact, they are a *maximal Abelian subalgebra* (or, a *masa*) of B (if they exist): an Abelian subalgebra which is not contained in any larger Abelian subalgebra. Via properties of masas, we will effectively attempt to reduce the problem of classifying finite-dimensional C^* -algebras to the easier case discussed above: classifying finite-dimensional *Abelian* C^* -algebras.

Lemma II.4. A sub- C^* -algebra D of a C^* -algebra A is a masa if and only if $D' = D$, where D' is the set of all elements in A for such $da = ad$ for all $d \in D$.

Proof. We suppose first that D is a masa. Note that D' clearly contains D , as every element of D commutes with every other element of D , since D is Abelian. Suppose $y \in D'$ and $y \notin D$. Then $yd = dy$ for all elements of D . It follows that $C(D \cup \{y\})$ is an Abelian sub- C^* -algebra which contains D properly, a clear contradiction, so we must have $D' \subset D$. We have inclusion both ways, so $D = D'$.

Conversely, suppose $D = D'$. Pick $x, y \in D$. Note that $x \in D'$ commutes with every element of D , so $xy = yx$. Thus, D is an Abelian sub- C^* -algebra. Via Zorn's lemma, we know that every Abelian sub- C^* -algebra is in a masa. Let F be a masa containing D . Given some $f \in F$, note that f will commute with every element of D , so that $f \in D' = D$. Thus, we must have $D = F$, and the proof is complete. \square

Lemma II.5. Let D be a masa in a C^* -algebra A , then

1. If a is an element in A that commutes with every element in D , then a belongs to D .
2. If D is unital, then A is unital, and the unit of D is equal to the unit of A .
3. If p is a projection in D satisfying $pDp = \mathbb{C}p$, then $pAp = \mathbb{C}p$. In other words, a minimal projection in D is also a minimal projection in A .

Proof. The first claim follows trivially from the proof of the previous lemma. For the second claim, suppose 1_D is a unit in D . Let $z = a1_D - a$, so that $zd = ad - ad = 0$ for all $d \in D$. Moreover, $d^* \in D$ so $zd^* = (dz^*)^* = 0$, so $dz^* = 0$. It follows that $dz^*z = 0$ and $z^*zd = 0$, so $z^*z \in D$, as it commutes with all elements of D . In particular, $(z^*z)^2 = 0$, so that $\|(z^*z)^2\| = \|z^*z\|^4 = \|z\|^4 = 0$, so $\|z\| = 0$, implying $z = 0$, so $a1_D = a$. In addition, $1_D a = (a^*1_D)^* = (a^*)^* = a$. Thus, 1_D is a unit for A .

To conclude, let us consider the final point. Suppose $pDp = \mathbb{C}p$. Then since $p \in D$, $pd = dp = dp^2 = pdp = \lambda p$ for $\lambda \in \mathbb{C}$. Thus for $pap \in pAp$, we have $dpap = \lambda pap = papd$, so $pap \in D$, implying that $pAp \in pDp \in \mathbb{C}p$ as desired. \square

Now, we are finally in a position where we can prove the main theorem.

To start, let A be a finite-dimensional C^* -algebra, and let D be a masa in A . Using Gelfand's theorem again, we note that D will be $*$ -isomorphic to $C_0(X)$ for some locally compact space X , and as we argued earlier X can only have a finite number of points, as D is finite-dimensional.

Since X has a finite number of points, it is compact, so $C_0(X) = C(X)$ is unital (the constant 1-function is the unit). It follows that D must be unital, and via the lemma above where we showed that a unit in D is a unit in A , it follows that A is unital as well. Going forward, we let $X = \{x_1, \dots, x_n\}$, where we index the distinct points.

Denote the $*$ -isomorphism by $\Phi : D \rightarrow C(X)$. Let f_1, \dots, f_n denote functions in $C(X)$ where $f_k(x_k) = 1$ and $f_k(x_j) = 0$ for $j \neq k$. We know that continuous functions of this form exist from Lem. ???. Clearly, $f_k^2 = f_k = f_k^*$. Moreover, the set of f_k are mutually orthogonal, $f_k f_j = \delta_{jk} f_k$. We let $p_k = \Phi^{-1}(f_k)$, for each k . Since the f_k are projections in $C(X)$, the p_k will be projections in D (and thus A). Finally, note

$$f_1 + \dots + f_n = 1 \quad \text{and} \quad f_k C(X) f_k = \mathbb{C} f_k \quad (12)$$

for any f_k (as if $f \in C(X)$ and $f(x_k) = c$, then $f_k f f_k$ is just $c f_k$). It follows via Φ that

$$p_1 + \dots + p_n = 1 \quad \text{and} \quad p_k D p_k = \mathbb{C} p_k \quad (13)$$

It follows again from the earlier lemma that, since $p_k D p_k = \mathbb{C} p_k$, we have $p_k A p_k = \mathbb{C} p_k$.

Let us assume that $p_j A p_k$ (with j possibly not equal to k !) is non-zero, and choose some $v \in p_j A p_k$. Then $v/\|v\| \in p_j A p_k$ as well, so without loss of generality, we can choose v such that $\|v\| = 1$. It follows that $v^* v \in p_k A p_k$ is positive and $\|v^* v\| = \|v\|^2 = 1$. Since $v^* v$ is of the form $p_k a p_k$, and thus of the form λp_k for some $\lambda \in \mathbb{C}$, we must have $\|\lambda p_k\| = |\lambda| \|p_k\| = |\lambda| = 1$. Moreover, since $v^* v$ is positive, λ must be real, so $\lambda = 1$ and $v^* v = p_k$. A repeat of the same argument shows that $vv^* = p_j$. Thus, in this case, $p_j \sim p_k$.

Now, suppose $p_j A p_k$ is some other element of $p_j A p_k$. Note that $p_j a p_k = (p_j a p_k) p_k = (p_j a p_k v^*) v$. Since $v^* \in p_k A p_j$, it follows that $p_j a p_k v^* \in p_j A p_j = \mathbb{C} p_j$, so $p_j a p_k = \lambda p_j v = \lambda v$ (as $v \in p_j A p_k$, so $p_j v = v$). Thus, any element of $p_j A p_k$ is a scalar multiple of v . We have now show that two possibilities exist:

Either $p_j A p_k$ is empty, or if it is non-empty, $p_j \sim p_k$ and moreover, $p_j A p_k = \mathbb{C} v$, where v is a partial isometry with $v^ v = p_k$ and $vv^* = p_j$.*

From here, we partition of the set of projections $\{p_1, \dots, p_n\}$ into equivalence classes based on Murray von-Neumann equivalence. We let n_k denote the number of projections in the k -th equivalence class. We then label the elements of the k -th equivalence class in a suggestive way: $\{f_{11}^{(k)}, \dots, f_{n_k n_k}^{(k)}\}$. Our claim is that the resulting $f_{jj}^{(k)}$ will be a system of orthogonal projections we can extend to our system of matrix units.

This fact is fairly easy to verify. Clearly, these projections are mutually orthogonal, as for $k \neq l$, we showed that $f_{ii}^{(k)} A f_{jj}^{(l)} = \{0\}$, so in particular for $1 \in A$, we have $f_{ii}^{(k)} f_{jj}^{(l)} = 0$. Moreover, by definition, $f_{ii}^{(k)} \sim f_{jj}^{(k)}$. Thus, we can find a system of matrix units $f_{ij}^{(k)}$ extending this system. In fact, if we back-step to the proof of this extension lemma, we can see that v will be $f_{ij}^{(k)}$ for $f_{ii}^{(k)} = v^* v$ and $f_{jj}^{(k)} = v v^*$, so that $f_{ii}^{(k)} A f_{jj}^{(k)} = \mathbb{C} f_{ij}^{(k)}$ from above.

Thus, for $a \in A$, since $\sum_{k,i} f_{ii}^{(k)} = 1$, as was argued above,

$$a = \left(\sum_{k,i} f_{ii}^{(k)} \right) a \left(\sum_{k,i} f_{ii}^{(k)} \right) = \sum_k \sum_{ij} f_{ii}^{(k)} a f_{jj}^{(k)} = \sum_k \sum_{ij} \lambda_{ij}^{(k)} f_{ij}^{(k)} \quad (14)$$

for $\lambda_{ij}^{(k)} \in \mathbb{C}$. Thus, this system of matrix units spans A . They are all clearly non-zero, by definition. Thus, from Claim ??, we have the desired isomorphism between A and a direct sum of matrix algebras!