Discrete Riemann surfaces and the Ising model: notes

Jack Ceroni*
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I. Introduction

The goal of these notes is to summarize and explain in greater detail the ideas outlined in Christian Mercat's paper *Discrete Riemann surfaces and the Ising model*. My main goal is for these notes to be *self-contained*, *exhaustive*, and *rigorous*. Ultimately, I want this to be a comprehensive deconstruction of a mathematical paper which can stand alone, and be understood by individuals with basic background in differential geometry.

II. Introducing the terminology of discrete surfaces

We begin by letting Σ be an oriented surface without boundary. In these notes, we will in addition assume that Σ is a smooth manifold (it has a smooth structure which has smooth transition maps).

A. Introducing cell complexes

We now come to the first set of definitions. In particular, we develop a means of placing a discrete, lattice-like structure on an otherwise continuous surface, in such a way that the underlying geometry of the surface is repsected.

Definition II.1 (Cellular decomposition)**.** Given Σ as defined above (an oriented surface without boundary), a *cellular decomposition* Γ of Σ is a partition of Σ into disjoint connected sets (which we call cells) of three different types:

- A discrete set of points. We call these the *vertices* of Γ, and denote them by Γ_0
- A collection of non-intersecting sets of the form $\gamma((0,1))$, where $\gamma : [0,1] \to \Sigma$ is a bijective path such that $\gamma(0)$ and $\gamma(1)$, the endpoints of the path, are contained in Γ_0. We will assume that any such γ is also smooth, in the sense that each $\varphi_\alpha \circ \gamma$ is smooth for $x \in \gamma((0,1)) \cap U_\alpha$, where $(U_\alpha, \varphi_\alpha)$ is a coordinate chart of the smooth structure on Σ. We call these the *edges* of Γ and denote them by Γ_1.
- A collection of topological discs of the form B (in other words, an embedding of an open ball B^2 in Σ) such that ∂B can be written as a finite union of elements of Γ_0 and Γ_1 (nodes and edges). We call these the *faces* of Γ, and denote them by Γ_2.

A cellular decomposition is said to be *locally finite* if every compact subset C of Σ intersects only a finite number of elements of Γ.

Remark II.1 (Parameterizing Γ)**.** Note that the vertices, edges and faces of a cellular decomposition Γ are all images of a 0-ball (a point), 1-ball (the interval $(0,1)$), and 2-ball (the set $\{x \in \mathbb{R}^2 \mid |x| < 1\}$), respectively, with respect to given parameterizations. This follows directly from the definition, with the edges being images of $(0,1)$ and faces being embeddings of B^2. Equivalently, we can choose parameterizations which map from each element of Γ to topological balls instead.

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* jackceroni@gmail.com

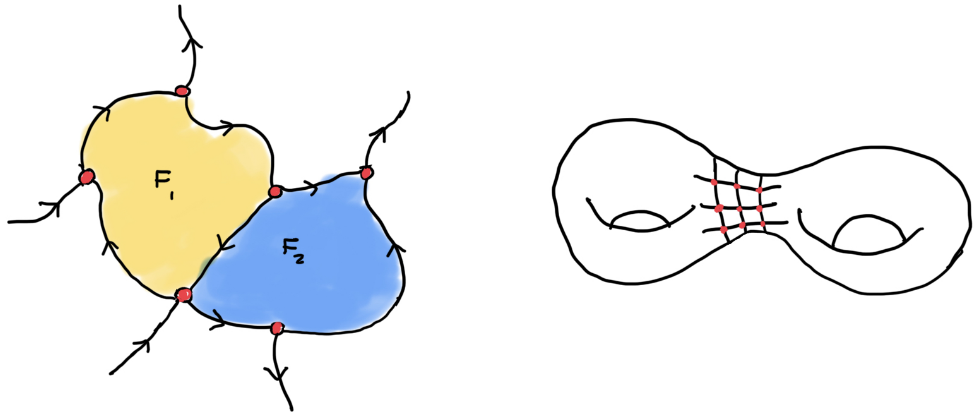

FIG. 1. The left image depicts two faces F_1 and F_2 and their bounding, oriented edges (and vertices) of some cellular decomposition Γ. The right picture shows some of the cells of a cellular decomposition Γ of a genus-2 surface Σ.

Remark II.2 (Orientation of Γ)**.** Since each face in Γ is an open subset of Σ, each will naturally inherit the orientation of the larger surface Σ. On the other hand, the edges of Γ are not open in Σ: they, along with their vertex endpoints, make up the boundaries of the faces. Thus, the edges are not equiped with a canonical orientation, so we instead arbitrarily choose one of the *two possible* orientations (each edge is an orientable and path-connected manifold, so there are precisely two choices of orientation).

We continue by translating a well-known construction from standard differential geometry to its discrete counterpart.

Definition II.2. We define the space of k-chains on Γ, $C_k(\Gamma)$, to be the $\mathbb{Z}$-module generated by taking formal linear cobminations of all dimension-k cells Γ. $C_k(\Gamma^*)$ is defined in the same way for the dual cells. This leads to a natural collection of boundary operators, $\partial_k : C_k(\Gamma) \to C_{k-1}(\Gamma)$ and $\partial_k : C_k(\Gamma^*) \to C_{k-1}(\Gamma^*)$, so that we have the following chain complexes

$$C_2(\Gamma) \xrightarrow{\quad \partial_2 \quad} C_1(\Gamma) \xrightarrow{\quad \partial_1 \quad} C_0(\Gamma), \tag{1}$$

$$C_2(\Gamma^*) \xrightarrow{\quad \partial_2 \quad} C_1(\Gamma^*) \xrightarrow{\quad \partial_1 \quad} C_0(\Gamma^*). \tag{2}$$

These boundary operators are defined in precisely the same way that they were defined in standard differential geometry (see *Spivak*, for instance). It follows immediately that $\partial_1 \circ \partial_2 = 0$, due to the signs that are picked up by endpoints when we take the boundary operation. We similarly let $C_k(\Lambda)$ be the spaces of chains generated both by elements of Λ and Λ^*. Note that $C(\Lambda) = C_k(\Gamma) \oplus C_k(\Gamma^*)$, and we can define a boundary operator on $C(\Lambda)$ which splits onto the two subspaces in the direct sum.

This allows us to define singular homology groups on the complex. we let $H_k(\Lambda) = \mathrm{Ker}(\partial_k)/\mathrm{Im}(\partial_{k-1})$.

III. Cochains

So far, we have been able to define notions of a chains and boundary on our discrete structure: constructions relatedf to *homology*. This raises a natural next question: how do we define *cohomology* in the discrete structure? Our strategy for doing this will be to make use of the isomorphismic nature of forms and dual maps on the spaces of chains, a fact which is true in the standard picture due to Hodge's theorem and is carried-over to the discrete picture *via definitions.*

Claim III.1 (Differential form intuition)**.** Generally speaking, when provided a k-form ω, the function that it serves is to be integrated over a k-chain (or a k-manifold, but integration over k-manifold can effectively be reduced to integrating locally over k-chains) to yield some number, $\int_c \omega$. By definition, integration is linear in c, $\int_{c_1+\lambda c_2} \omega := \int_{c_1} \omega + \lambda \int_{c_2} \omega$.

Taking this notion to its extreme: 0-forms (functions) should be thought of as "linear maps that take points and yield numbers". 1-forms should be thought of as "linear maps that take lines and yield numbers". 2-forms should be thought of as "linear maps that take areas an yield numbers". Generally, k-forms should be thought of as "things that eat k-dimensional regions and yield numbers". If true, this implies that k-forms truly are the *dual objects* to k-chains.

Theorem III.1 (Hodge's theorem)**.**

It follows from these facts that we *define* the space of forms $C^k(\Lambda)$ to be precisely the space of dual maps on $C_k(\Lambda)$, $C^k(\Lambda) = \mathrm{Hom}(C_k(\Lambda), \mathbb{C})$. We introduce the following notation, to make clear the connection between the forms in the discrete picture, and the evaluation of forms in the standard picture over a chain via integration. Let $c \in C_k(\Lambda)$, let $\omega \in C^k(\Lambda)$, we define

$$\omega(c) := \int_c \omega. \tag{3}$$

This is a *purely notational construction*: we don't have a notion of an "integral" in the discrete picture.

This notation is suggestive of a systematic way to define transformations on forms in the discrete picture. Suppose ω is a form in the standard picture, on the surface Σ, so $\omega \in \Omega^k(\Sigma)$. Suppose $F : \Omega^k(\Sigma) \to \Omega^\ell(\Sigma)$. Suppose further than we *know the map F induces on integral evaluations*: we have a formula of the form $\int_c F(\omega) = \int_{c'} \omega'$ for every $c \in \Omega^\ell(\Sigma)$, where c' and ω' depend on c and ω, and "make sense" in the discrete picture.

We can then simply define $F : C^k(\Lambda) \to C^\ell(\Lambda)$, the discrete analogue, as $F(\omega)(c) = \omega'(c')$ as well.

Example III.1 (The exterior derivative)**.** In the standard picture, $d : \Omega^k(\Sigma) \to \Omega^{k+1}(\Sigma)$ is defined via a pushforward. However, we also know from Stokes' theorem that for a chain c,

$$\int_c d\omega = \int_{\partial c} \omega. \tag{4}$$

We have a notion of boundary in the discrete picture, which immediately suggestes that for $\omega \in C^k(\Lambda)$, we should define $(d\omega)(c) := \omega(\partial c)$. Note that that implies that in this context, d is *precisely* the dual map of ∂, $d\omega = \partial^*(c)$ (we could have also used this connection to arrive at our definition).

IV. A discrete Hodge star

In a similar fashion, we define a discrete analogue of the Hodge star. In Λ, we have established that Γ effectively represents "horizontal grid lines" and Γ^* effectively represents "vertical grid lines".

V. A discrete Laplacian