A primer on the Gelfand transformation and related topics

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I. Introduction

This essay was written for the Fall 2023 session of MAT437: K-Theory and C^*-Algebras, taught by Professor George Elliott, at the University of Toronto.

The goal of this paper is to introduce some of the main ideas related to the Gelfand transformation and Gelfand duality. We will discuss the implications of these results in the study of C^*-algebras, in particular how this allows for constructions such as the "continuous functional calculus" which is briefly discussed in the first Chapter of Rordam's K-theory textbook. We will conclude the essay by discussing some K-theoretic implications of these results.

II. Introducing the Gelfand transformation

We begin by providing some information on the main construction of interest: the Gelfand transformation, and some related results.

A. The main idea

Recall that a *Banach algebra* A is a norm-closed algebra satisfying sub-multiplicativity: $||xy|| \leq ||x||||y||$ for all $x, y \in A$. Suppose A is over the field $\mathbb{C}$ and is commutative. Let $\Delta(A)$ denote the set of all non-zero algebra homomorphisms $\chi : A \to \mathbb{C}$. We will call this the set of *characters* of A.

Definition II.1 (The weak-$*$ topology)**.**

B. Gelfand duality and the continuous function calculus

As it turns out, when the Banach algebra that we use to construct a Gelfand transformation is in fact a C^*-algebra, the Gelfand transformation induces an *isomorphism*.

Theorem II.1 (Gelfand duality)**.**

Theorem II.2 (The continuous function calculus)**.**

III. Implications in K-theory

We will dedicate the second part of this essay to discussing some of the implications of Gelfand duality in K-theory, in particular, for building a bridge which begins with an arbitrary algebraic K-theory (the lower K-groups), to a topological K-theory.

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A. A short aside: topological K-theory

Let us begin by recalling a very basic definition:

Definition III.1 (Complex vector bundle)**.** A complex vector bundle over topological space X is defined to be a triple $\xi = (E, \pi, X)$, where E is a topological space and $\pi : E \to X$ is a continuous surjection from E to X, the so-called *base space*, where each *fibre* $\pi^{-1}(x)$ has the structure of a complex finite-dimensional vector space. We require that ξ has the following local compatibility property: for each $x \in X$, there must exist a neighbourhood U of x and a homeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{C}^n$ for some $n \in \mathbb{N}$ making the following diagram commute:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "h"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^n \arrow[dl, "\eta"] \\ & U & \end{tikzcd}$$

where $\eta(x, v) = x$ is the projection onto the base, and, for each $y \in U$, the map $h : \pi^{-1}(y) \to \{y\} \times \mathbb{C}^n \simeq \mathbb{C}^n$ is a vector space isomorphism. Intuitively, this simply means that locally, in the base space, the corresponding bundle of fibres in E must resemble a trivial product space $U \times \mathbb{C}^n$, both in a topological sense and an algebraic sense.

The motivation for studying topological K-theory comes from the desire to classify all vector bundles over a given base space of a particular, fixed dimension n. To be more specific, we say that bundles $\xi = (E, \pi, X)$ and $\xi' = (E', \nu, X)$ are *isomorphic* if there is a homeomorphism $h : E \to E'$ making the following diagram commute:

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

such that the restricted map $h : \pi^{-1}(x) \to \nu^{-1}(x)$ is a vector space isomorphism for each $x \in X$. The task of computing the isomorphism classes of vector bundles (which we denote $\langle \xi \rangle$, where ξ is an equivalence class representative) is a highly non-trivial task, and can only be done in very basic, often low-dimensional cases. Thus, in order to make partial progress on solving this profoundly difficult question, a natural question to ask is: *are there weaker equivalence relations than the above notion of isomorphism which can more easily be computed, at the cost of ending up with a more crude classification?* The answer to this question is a resounding yes, via the construction of topological K-groups.

Treating vector bundle ξ now as fundamental objects, let us define algebraic operations between them.

Lemma III.1 (Direct sum of vector bundles)**.**

Lemma III.2 (Inner products on vector bundles)**.**

Lemma III.3 (Tensor products of vector bundles)**.**

Lemma III.4.

Theorem III.1 (Equivalence of algebraic and topological K-theory)**.** If X is a compact Hausdorff space, then $K_0(C(X))$, the algebraic K_0-group of the C^*-algebra of functions $C(X)$, and $K(X)$, the topological K^0-group, are isomorphic as Abelian groups.

Proof. □

B. The K-theoretic correspondence induced by Gelfand duality

Of course, any time that we induce a $*$-homomorphism between C^*-algebras, there will be an induced group homomorphism between the corresponding K_0-groups. Seeing as, in the case of C^* algebras we in fact have a

$*$-isomorphism from A to $C(\mathrm{Spec}(A))$, it follows immediately that their K_0-groups are isomorphic. Recall in addition that the K_0-group of the algebra $C(X)$ is precisely $K^0(X)$ when X is compact Hausdorff. It follows immediately that

$$K_0(A) \simeq K_0(C(\mathrm{Spec}(A))) \simeq K^0(\mathrm{Spec}(A)) \tag{1}$$

In other words, we have a well-behaved correspondence between an algebraic invariant of the space A, namely its K_0-group, and a purely topological object: the topological K_0-group of the topological space $\mathrm{Spec}(A)$. Carrying this same reasoning forward, we have

$$K_1(A) \simeq K_0(SA) \simeq K_0(C(\mathrm{Spec}(SA))) \simeq K^0(\mathrm{Spec}(SA)) \tag{2}$$

In particular, note that $\mathrm{Spec}(SA) = \mathrm{Hom}(SA, \mathbb{C})$: all $*$-homomorphisms (thus, automatically continuous) from SA to $\mathbb{C}$, with the weak-$*$ topology. In other words, the topology on $\mathrm{Hom}(SA, \mathbb{C})$ is the weakest topology such that all evaluation maps $\mathrm{eval}_f(\Phi) = \Phi(f)$ are continuous.

Theorem III.2 (Riesz-Markov)**.**

Example III.1 (Finite-dimensional algebras)**.** Suppose A is a finite-dimensional