Fall 2023 MAT437 problem set 7

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I. Problem 1

*This is a theorem in the book: I thought that I would try to prove it myself **without** looking at the proof provided in the book (until close to the end), to make sure that I understand the inductive limit construction well.*

I also tried my best to explain my thought process, including a couple incorrect initial guesses I had while doing this proof, and why they eventually failed.

Proposition I.1. Every inductive sequence of C^*-algebras, $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$, has an inductive limit $(A, \{\mu_n\})$.

Proof. First, let us consider the following naive, *incorrect* approach:

The wrong approach: Consider the sequence $\{a_{n,m}\}_{m=n+1}^{\infty}$ where $a_{n,m} = \varphi_{m,n}(a)$, for some $a \in A_n$. We may be inclined, at first glance, to take $\mu_n(a)$ to be the "limit" of this sequence. However, this wouldn't really make sense: each element of this sequence may be an element of a different algebra, with a different norm, meaning there isn't a good way to compare elements of this sequence. Even worse, the norms of the individual elements of this sequence may go to infinity!
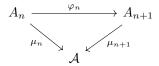
We have to begin by defining the space A at the end of the sequence. Intuitively, we want this space to reflect what each element of the complex of C^*-algebras eventually becomes as it is mapped through the infinite sequence of φ_k maps. This implies that sequences which *eventually* become identical, should be identified together in the "eventual space", A. This leads us to define

$$\mathcal{A} = \prod_{n \in \mathbb{N}} A_n \Big/ \sum_{n \in \mathbb{N}} A_n \tag{1}$$

as RLL showed that $\sum_n A_n$ is a closed ideal in $\prod_n A_n$, so $\mathcal{A}$ is a valid C^*-algebra. We define $\mu_n : A_n \to \mathcal{A}$ as follows: we simply map $a \in A_n$ to $[(0, \ldots, 0, a, \varphi_{n+1,n}(a), \varphi_{n+2,2}(a), \ldots)]$: an equivalence class. Note that $\xi \in \mathcal{A}$ because $\alpha = (0, \ldots, 0, a, \varphi_{n+1,n}(a), \varphi_{n+2,2}(a), \ldots) \in \prod_n A_n$. This is because $\sup_n \alpha_n$ is finite: recall that we proved in an earlier problem set that if φ is a $*$-homomorphism between C^*-algebras A and B, then $||\varphi(a)|| \leq ||a||$. It follows that for some $m > n$, $||\varphi_{m,n}(a)|| \leq ||a||$.

Our claim now is that the complex $(\mathcal{A}, \{\mu_n\})$ constitutes the desired inductive limit (This is actually slightly incorrect, as we will show).
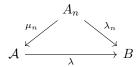
The first step is to show that the diagram:

commutes. Indeed, this is a triviality, as $(\mu_{n+1} \circ \varphi_n)(a) = [(0, \ldots, 0, \varphi_n(a), \varphi_{n+1,n}(a), \ldots)]$ and $\mu_n(a) = [(0, \ldots, 0, a, \varphi_n(a), \ldots)]$ which are clearly the same equivalence class as their difference is $(0, \ldots, 0, a, 0, \ldots)$:

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a sequence which is eventually 0. As the second and final step, we must show that if $(B, \{\lambda_n\})$ is a system in which $\lambda_n : A_n \to B$ with $\lambda_n = \lambda_{n+1} \circ \varphi_n$, then there is precisely one λ making the diagram:

$$
\begin{array}{ccc}
 & A_n & \\
 {\scriptstyle \mu_n}\swarrow & & \searrow{\scriptstyle \lambda_n} \\
\mathcal{A} & \xrightarrow{\ \ \lambda\ \ } & B
\end{array}
$$

commute for each n. As it turns out, *we won't be able to do this*, because $\mathcal{A}$ is "too big" to guarantee uniqueness of λ. Indeed, the collection of all $\mu_n(A_n)$ could be a small subset of $\mathcal{A}$, and λ could vary drastically on the remainder of the codomain $\mathcal{A}$. This leads us to make the codomain "as small as possible". We define

$$
A = \overline{\bigcup_{n \in \mathbb{N}} \mu_n(A_n)} = \overline{A'} \subset \mathcal{A} \tag{2}
$$

Clearly, A is a C^*-algebra (note that it is closed in the metric topology). Note that replacing $\mathcal{A}$ with A in the first commutative diagram does not alter the proof. We will define our map λ as follows.

Let's assume that such a system $(B, \{\lambda_n\})$ exists. Suppose $x \in A_n$ and $y \in A_m$ are two elements which are mapped to the same element of A. In other words, there will exist some N such that for $k \geq N$, $\varphi_{k,n}(x) = \varphi_{k,m}(y)$ (the sequences generated by these elements must eventually become equal). It follows that for some such m

$$
\lambda_n(x) = (\lambda_{n+1} \circ \varphi_n)(x) = (\lambda_{k+1} \circ \varphi_k \circ \cdots \circ \varphi_n)(x) = \lambda_{k+1}(\varphi_{k,n}(x)) = \lambda_{k+1}(\varphi_{k,m}(y)) = \lambda_m(y) \tag{3}
$$

It follows that $\lambda = \lambda_n \circ \mu_n^{-1}$ constitutes a well-defined mapping: given some $a \in \bigcup_n \mu_n(A_n) = A'$, we note that $a = \mu_n(a_n)$ for some $n \in \mathbb{N}$ and some $a_n \in A_n$, and we take $\lambda(a) = \lambda_n(a_n)$. The above logic showed that λ is in fact independent of the choice of both n and a_n.

We have successfully defined a map from A' to B. Note that A' is a sub-$*$ algebra of the C^* algebra A. In addition, note that $\lambda : A' \to B$ is a $*$-homomorphism from the sub-$*$ algebra A' into B. For $a, b \in A$, note that $\mu_n^{-1}(a) + \gamma \mu_n^{-1}(b) \subset \mu_n^{-1}(a + \gamma b)$, so that $\lambda_n(\mu_n^{-1}(a) + \gamma \mu_n^{-1}(b)) = \lambda(a) + \gamma \lambda(b) \subset \lambda(a + \gamma b)$. But of course these are both one-point sets, so we must have $\lambda(a) + \gamma \lambda(b) \subset \lambda(a + \gamma b)$. Using the same argument for the multiplicativity and $*$-preserving properties, we can conclude that λ is a $*$-homomorphism.

We have shown that there exists a $*$-homomorphism from A' to B which satisfies the desired properties. Note that this $*$-homomorphism is the unique one from A' to B satisfying the desired properties: suppose $\lambda' : A' \to B$ also satisfied $\lambda' \circ \mu_n = \lambda_n$ for each n. Since $\lambda = \lambda_n \circ \mu_n^{-1}$, we would have $\lambda = \lambda' \circ \mu_n \circ \mu_n^{-1} = \lambda'$. If there *did* exist an extension of λ to a $*$-homomorphism from A (the closure of A') to the closure of B, then note that it would necessarily be unique, as such a function would be continuous, and continuous functions agreeing on a dense subset are equal.

Thus, the only remaining task is to show that **we can extend $\lambda : A' \to B$ to a $*$-homomorphism from $A = \overline{A'}$ to B**. At this point, I couldn't quite finish the proof myself, so I prompted myself by consulting the book. Annoyingly, the book had practically nothing to say on this matter, it simply asserted that "we can extend $\lambda : A' \to B$ by uniform continuity to $\lambda : A \to B$".

$\square$

Proposition I.2. Given an inductive sequence of C^*-algebras $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$ with inductive limit $(A, \{\mu_n\})$, the following facts holds:

1. $\|\mu_n(a)\| = \lim_{m \to \infty} \|\varphi_{m,n}(a)\|$ for all $n \in \mathbb{N}$ and $a \in A_n$. This immediately implies that $\mathrm{Ker}(\mu_n) = \{a \in A_n \mid \lim_{m \to \infty} \|\varphi_{m,n}(a)\| = 0\}$.

2. If $(B, \{\lambda_n\})$ and $\lambda : A \to B$ are as in Def. 6.2.2 of RLL, then $\mathrm{Ker}(\mu_n) \subset \mathrm{Ker}(\lambda_n)$ for all $n \in \mathbb{N}$, λ is injective if and only if $\mathrm{Ker}(\lambda_n) \subset \mathrm{Ker}(\mu_n)$ for all $n \in \mathbb{N}$, and λ is surjective if and only if $B = \overline{\bigcup_n \lambda_n(A_n)}$.

Proof. To prove the first claim, lt us recall Lemma 6.1.3 of RLL, namely that if we let $\pi : \prod_n A_n \to \prod_n A_n \big/ \sum_n A_n$ be the quotient map, then $||\pi(a)|| = \limsup_{n\to\infty} ||a_n||$. Note that

$$\mu_n(a) = [(0,\ldots,0,a,\varphi_n(a),\varphi_{n+1,n}(a),\ldots)] = \pi(0,\ldots,0,a,\varphi_n(a),\varphi_{n+1,n}(a),\ldots).$$

Thus,

$$||\mu_n(a)|| = \limsup_{m\to\infty} ||(0,\ldots,0,a,\varphi_n(a),\varphi_{n+1,n}(a),\ldots)_m|| = \limsup_{m\to\infty} ||\varphi_{m,n}(a)|| \tag{4}$$

Note that the sequence $\varphi_{m,n}(a)$ is strictly decreasing in norm, as for $*$-homomorphisms, we know that $||\varphi_{m+1,n}(a)|| = ||(\varphi_{m+1} \circ \varphi_{m,n})(a)|| \leq ||\varphi_{m,n}(a)||$. Thus, the limit superior is actually just the limit (in particular, the limit exists), and we have the desired result.

To prove the second claim, it is clear that if $\mu_n(a) = 0$, then $\lambda_n(a) = (\lambda \circ \mu_n)(a) = \lambda(0) = 0$, so $\mathrm{Ker}(\mu_n) \subset \mathrm{Ker}(\lambda_n)$. Clearly, a $*$-homomorphism is injective if and only if it has trivial kernel, so $\lambda(a) = 0$ must imply $a = 0$. In particular, if $\lambda_n(a) = \lambda(\mu_n(a)) = 0$, then $\mu_n(a) = 0$, so $\mathrm{Ker}(\lambda_n) \subset \mathrm{Ker}(\mu_n)$.

Finally, note that since λ is continuous, we have

$$\mathrm{Im}(\lambda) = \lambda(A) = \lambda\left(\overline{\bigcup_{n\in\mathbb{N}} \mu_n(A_n)}\right) \subset \overline{\bigcup_{n\in\mathbb{N}} (\lambda \circ \mu_n)(A_n)} = \overline{\bigcup_{n\in\mathbb{N}} \lambda_n(A_n)} \subset B \tag{5}$$

Of course, if λ is surjective, then $\mathrm{Im}(\lambda) = B$, so since it lies between the two sets, we must have $\overline{\bigcup_{n\in\mathbb{N}} \lambda_n(A_n)} = B$ as well. I got stuck at this point again (proving the converse). From consulting the book again, the converse seems to follow from the extension theorem invoked earlier in the proof. $\square$

II. RLL Problem 6.1 (Suggested Problem 1)

Equipped with the basic knowledge of inductive limits, let us answer a few questions regarding the calculation of simple inductive limits.

Part 1. Of course, in this exercise, we are tasked with computing the inductive limit of a sequence of Abelian groups. Homomorphism from the group of integers to some other group are completely determined by where 1 is sent. Let us define $\mu_n : \mathbb{Z} \to \mathbb{Q}$ as $\mu_n(1) = \frac{1}{(n-1)!}$. We can easily verify that $(\{\mu_n\}, \mathbb{Q})$ is the inductive limit of the sequence of $\mathbb{Z}$ with $\varphi_n = n$.

First, note that

$$(\mu_{n+1} \circ \varphi_n)(k) = \frac{nk}{n!} = \frac{k}{(n-1)!} = \mu_n(k) \tag{6}$$

so we have verified the first condition. Moreover, given B and λ_n such that $\lambda_{n+1} \circ \mu_n = \lambda_n$, we can simply define $\lambda : \mathbb{Q} \to B$ as $\lambda(k) = \lambda_n((n-1)!k)$. Obviously, $\lambda \circ \mu_n = \lambda_n$. In addition, λ does not depend on n because we have

$$\lambda_n((n-1)!k) = (\lambda_m \circ \varphi_{m-1,n})((n-1)!k) = \lambda_m((m-1)!k) \tag{7}$$

for some $m > n$. To verify uniqueness, note that if λ' also satisfies the desired property, then

$$\lambda'\left(\frac{a}{b}\right) = \lambda'\left(\frac{(b-1)!}{(b-1)!}\frac{a}{b}\right) = \lambda'\left(\frac{(b-1)!a}{b!}\right) = (\lambda' \circ \mu_{b+1})((b-1)!a) \tag{8}$$

$$= \lambda_{b+1}((b-1)!a) = (\lambda \circ \mu_{b+1})((b-1)!a) = \lambda\left(\frac{a}{b}\right) \tag{9}$$

Thus, by definition, $(\{\mu_n\}, \mathbb{Q})$ is in fact the inductive limit of this sequence.

Part 2. We take a similar approach to before. In particular, we define $\mu_n(1) = \frac{1}{2^{n-1}}$. In this case, we will clearly have $(\varphi_n \circ \mu_{n+1})(k) = \varphi_n\left(\frac{k}{2^n}\right) = \frac{k}{2^{n-1}} = \mu_n(k)$. If the maps μ_n are indeed the morphisms giving rise to the inductive limit, then we know that we must have

$$G_2 = \bigcup_{n \in \mathbb{N}} \mu_n(\mathbb{Z}) = \left\{ \frac{p}{2^n} \,\middle|\, p, n \in \mathbb{Z} \right\} \subset \mathbb{Q} \tag{10}$$

It is clear that G_2 is an additive subgroup of $\mathbb{Q}$. We claim that $(\{\mu_n\}, G_2)$ is in fact the desired inductive limit. We verified the first required property of μ_n above. To verify the second, we proceed similarly to Part 1. Given B and λ_n such that $\lambda_{n+1} \circ \mu_n = \lambda_n$, we can simply define $\lambda : G_2 \to B$ as $\lambda(k) = \lambda_n(2^{n-1}k)$. Obviously, $\lambda \circ \mu_n = \lambda_n$. In addition, λ does not depend on n because we have

$$\lambda_n(2^{n-1}k) = (\lambda_m \circ \varphi_{m-1,n})(2^{n-1}k) = \lambda_m(2^{m-1}k) \tag{11}$$

for some $m > n$. To verify uniqueness, note that if λ' also satisfies the desired property, then

$$\lambda'\left(\frac{p}{2^n}\right) = (\lambda' \circ \mu_{n+1})(p) = \lambda_{n+1}(p) = (\lambda \circ \mu_{n+1})(p) = \lambda\left(\frac{p}{2^n}\right) \tag{12}$$

Thus, by definition, $(\{\mu_n\}, G_2)$ is in fact the inductive limit of this sequence.

III.   Topological Inverse Limit and Inductive Limit Duality (Suggested Problem 5)

Note that the definition of an inverse limit provided in the problem set is incorrect. I will complete the problem assuming the correct definition.

Part 1. Let us take

$$X = \left\{ (x_1, x_2, \dots) \,\middle|\, (x_1, x_2, \dots) \in \prod_{n \in \mathbb{N}} X_n, \ f_i(x_{i+1}) = x_i \ \forall i \in \mathbb{N} \right\}$$

where we give $\prod_{n \in \mathbb{N}} X_n$ the usual product topology, and X inherits the subspace topology. We simply take $\pi_i : X \to X_i$ to be the projection onto the i-th coordinate. Automatically, it is clear that $(f_{i-1} \circ \pi_i)(x_1, x_2, \dots) = f_{i-1}(x_i) = x_{i-1} = \pi_{i-1}(x_1, x_2, \dots)$.

In addition, suppose continuous functions $\lambda_i : Y \to X_i$ satisfy $f_{i-1} \circ \lambda_i = \lambda_{i-1}$. We wish to define continuous $\lambda : B \to A$ such that $\lambda_i = \pi_i \circ \lambda$. For a particular $y \in Y$, we define $\lambda(y) = (\lambda_1(y), \lambda_2(y), \dots)$. Note that $\lambda(y) \in X$ because we have assumed that

$$f_i(\lambda(y)_{i+1}) = f_i(\lambda_{i+1}(y)) = \lambda_i(y) = \lambda(y)_i$$

Clearly, λ is a continuous function, as it is a product of continuous functions, and clearly $\lambda_i = \pi_i \circ \lambda$. To show uniqueness of λ, note that if λ' is also a continuous function with the same properties, we have

$$\lambda'(y)_i = (\pi_i \circ \lambda')(y) = \lambda_i(y) = \lambda(y)_i \tag{13}$$

for each i, so $\lambda'(y) = \lambda(y)$. Thus, we have shown that inverse limits exist. To show that an inverse limit is unique, suppose $(\{\pi'_i\}, X')$ is also an inverse limit. It follows from the definition of the inverse limit that the following diagram will commute:

$$X' \xrightarrow{\ \Pi_1\ } X \xrightarrow{\ \Pi_2\ } X'$$
$$\pi'_i \searrow \quad \downarrow \pi_i \quad \swarrow \pi'_i$$
$$X_i$$

where $\Pi_1 : X' \to X$ and $\Pi_2 : X \to X'$ are continuous. Thus, X' and X are homeomorphic, so up to homeomorphism, the inverse limit is unique.

Part 2. Since each X_j is Hausdorff, it follows that $\prod_{n\in\mathbb{N}} X_n$ is Hausdorff (given $x \neq y$, x and y must differ at coordinate j, so choose disjoint open set U and V in X_j of x_j and y_j, then note that $x \in X_1 \times \dots U \times X_{j+1} \times \cdots$ and $y \in X_1 \times \dots V \times X_{j+1} \times \cdots$ are disjoint). Thus, X, and subspace of $\prod_{n\in\mathbb{N}} X_n$ is also Hausdorff.

Tychonoff's theorem tells us that arbitrary products of compact spaces are compact, so $\prod_{n\in\mathbb{N}} X_n$ is compact. We will show that X is closed in the product space. Indeed, by definition, note that:

$$X = \bigcap_{n\in\mathbb{N}} \left\{ (x_1, x_2, \dots) \mid f_n(x_{n+1}) = x_n \right\} = \bigcap_{n\in\mathbb{N}} X_1 \times \cdots \times X_{n-1} \times \Gamma_{f_n}(X_{n+1}) \times X_{n+2} \times \cdots = \bigcap_{n\in\mathbb{N}} C_n \quad (14)$$
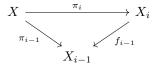
where $\Gamma_{f_n}(X_{n+1}) = \{ (f_n(x), x) \mid x \in X_{n+1} \}$ is the graph of f_n. If we can show that each C_n is closed, then X will also be closed, as arbitrary intersections of closed sets are closed. From here, we can conclude that X is compact, as closed subspaces of compact spaces are compact.

In fact, it is easy to show that the graph is closed. Given a point $(y, x) \in X_n \times X_{n+1}$ in the graph's complement, we simply take open set U in X_n containing y and open set W containing $f(x)$, such that U and W are disjoint (we can make this choice as X_n is Hausdorff). Note that the open set $U \times f^{-1}(W)$ contains (y, x), and does not contain any point of the graph, as any $x \in f^{-1}(W)$ is mapped by f to W disjoint from U. Thus, the complement of the graph is open, so the graph is closed.

Part 3. This result follows from carefully making use of the contravariant nature of C (I guess this could be considered a functor? I'm not completely sure.) Also note that all the spaces we are working with are compact, so in this context $C = C_0$. To begin, let

$$X = \varprojlim X_i.$$

By definition, the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\;\pi_i\;} & X_i \\
& \searrow_{\pi_{i-1}} \quad \swarrow_{f_{i-1}} & \\
& X_{i-1} &
\end{array}
$$

which immediately implies that this diagram commutes:

$$
\begin{array}{ccc}
C(X) & \xleftarrow{\;\pi_{i+1}^*\;} & C(X_{i+1}) \\
& \nwarrow_{\pi_i^*} \quad \nearrow_{f_i^*} & \\
& C(X_i) &
\end{array}
$$

where we shift i to be 1. Thus, we have maps $\varphi_i = f_i^* : C(X_i) \to C(X_{i+1})$ and $\mu_i = \pi_i^*$ which satisfy the first condition required for $(C(X), \{\pi_i^*\})$ to be the inductive limit of the sequence of $f_i^* : C(X_i) \to C(X_{i+1})$. If we can verify that the second condition holds, it will follows immediately that $(C(X), \{\pi_i^*\})$ is the inductive limit, and by uniqueness, we then must have

$$C(X) = C(\varprojlim X_i) \simeq \varinjlim C(X_i).$$

Indeed, it is easy to show that the second condition holds. Suppose $(B, \{\lambda_i\})$ is another system where $\lambda_i : C(X_i) \to B$ and $\lambda_i = \lambda_{i+1} \circ f_i^*$. By Gelfand's theorem, B is isomorphic to $C_0(Y)$ for some locally compact Hausdorff space Y. Thus, WLOG, we can assume $\lambda_i : C(X_i) \to C_0(Y)$. Moreover, again from Gelfand, we know that there exists some continuous $\gamma_i : Y \to X_i$ such that $\gamma_i^* = \lambda_i$, so that $\gamma_i^* = \gamma_{i+1}^* \circ f_i^*$, which implies that $\gamma_i = f_i \circ \gamma_{i+1}$

From the characterization of the inverse limit, we know that there is a unique continuous function $\gamma : Y \to X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \gamma\ } & X \\
& \searrow^{\gamma_i} & \downarrow{\pi_i} \\
& & X_i
\end{array}
$$

which implies immediately that there is a unique $*$-homomorphism (again following from Gelfand's theorem) $\gamma^* : C(X) \to C_0(Y) \simeq B$ making the diagram commute:

$$
\begin{array}{ccc}
B & \xleftarrow{\ \gamma^*\ } & C(X) \\
& \nwarrow^{\lambda_i} & \uparrow{\pi_i^*} \\
& & C(X_i)
\end{array}
$$

where we use the fact that $\gamma_i^* = \lambda_i$. Thus, it follows from the definition that $C(X)$ is isomorphic to the inverse limit of the sequence of $C(X_i)$, and the proof is complete.

Part 4. This result follows from Part 3, as well as the fact that the K_0 functor "commutes" with inverse limits, and that if $A \simeq B$, then $K_0(A) \simeq K_0(B)$. In particular, note that

$$
K_0(C(\Sigma_2)) \simeq K_0(\varinjlim(C(S^1), \varphi_n^*)) \simeq \varinjlim(K_0\left(C(S^1)\right), K_0(\varphi_n^*)) \simeq \varinjlim(\mathbb{Z}, K_0(\varphi_n^*)) \tag{15}
$$

where the morphisms φ_n are simply $\varphi_n(z) = z^2$ for all n. This means that $\varphi_n^*(f) = f \circ z^2$ for $f \in C(S^1)$.