# MAT436 Problem Set 1

Jack Ceroni
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## I. Suggested Problem 1

**Part A.** Suppose every absolutely summable sequence is summable. Let $x_n \in X$ be Cauchy. For each $n$, pick some $M_n$ such that for $j, k \geq M_n$, $||x_j - x_k|| < 1/2^n$. Assume WLOG $M_1, M_2, \dots$ is strictly increasing. Define $y_1 = x_{M_1}$ and $y_n = x_{M_n} - x_{M_{n-1}}$ for $n \geq 2$. Note that $||y_n|| < 1/2^n$, so that the sequence of $y_n$ is absolutely summable, and thus summable. Let $y$ be this sum. Of course, the partial sums $y_1 + \cdots + y_n = x_{M_n}$ get arbitrarily close to $y$. Since the sequence of $M_n$ approaches $\infty$, it follows that $x_n$ is a Cauchy sequence with a convergent subsequence, so it converges, and $X$ is complete.

Conversely, suppose $X$ is complete. Suppose $y_n$ is absolutely summable. Let $x_n = y_n + \cdots + y_1$, and note that for $n > m$,

$$||x_n - x_m|| = ||y_n + \cdots + y_{m+1}|| \leq ||y_n|| + \cdots + ||y_{m+1}|| \tag{1}$$

Thus, for $n$ and $m$ chosen sufficiently large, the tails of the absolute sum of the above form can be made arbitrarily small (since $y_n$ is absolutely summable), so that $x_n$ is Cauchy, and thus converges to a limit in $X$. This immediately means that $y_n$ is summable.

**Part B.** Note that $\text{Ker}(T) = T^{-1}(\{0\})$. Since normed vector spaces are obviously Hausdorff, $\{0\}$ is closed, so if $T$ is continuous, $\text{Ker}(T)$ must be closed. Conversely, suppose the kernel is closed. Let $e_1, \dots, e_n$ be a basis for $Y$. We pick $f_1, \dots, f_n$ such thqat $Tf_j = e_j$ for each $j$. Note that for any $x \in X$, we have $Tx = \sum_j c_j e_j$, so that $T\left(\sum_j c_j f_j\right) = Tx$. It follows that $x - \sum_j c_j f_j \in \text{Ker}(T)$. Since $x = x - \sum_j c_j f_j + \sum_j c_j f_j$, it follows that any element of $X$ can be written as some linear combination of the $f_1, \dots, f_n$, plus an element of the kernel.

Since $\text{Ker}(T)$ is closed, $X/\text{Ker}(T)$ is a normed vector space with the infimum norm, and from the above argument, it is finite-dimensional as it is spanned by $[f_1], \dots, [f_n]$. Let us define $\widetilde{T} : X/\text{Ker}(T) \to Y$ as $\widetilde{T}[x] = Tx$. This is a linear map between finite-dimensional vector spaces, and is thus continuous. It follows that $T = \widetilde{T} \circ \pi$, where $\pi$ is the quotient map, is continuous as well.

**Part C.** Suppose $X$ is finite-dimensional. Then $X$ is homeomorphic to $\mathbb{R}^n$ for some $n$ via a linear map. Let $\Phi$ be the linear homeomorphism, let $B$ be the unit ball in $X$. Then $\Phi^{-1}(B)$ is closed in $\mathbb{R}^n$ as $\Phi$ is continuous. Moreover, since $\Phi^{-1} : X \to \mathbb{R}^n$ is continuous, it is bounded, so $||\Phi^{-1}(x)|| \leq M||x||$ for some $M$, implying that $\Phi^{-1}(B)$ is bounded in $\mathbb{R}^n$. Thus, $\Phi^{-1}(B)$ is closed and bounded in Euclidean space, so it compact. Then $B = \Phi\Phi^{-1}(B)$ is the continuous image of a compact space and thus is compact.

Conversely, suppose $B \subset X$ is compact. Suppose $B$ is not finite-dimensional, so we can choose a sequence $x_1, x_2, \dots$ such that for each set $S_n = \{x_1, \dots, x_n\}$, we cannot write $x_{n+1}$ as a linear combination of the elements of $S_n$. I want to show that such a sequence, constructed n the right way, cannot have a convergent subsequence, but I'm a bit stumped right now. I will keep thinking about this and if I make more progress will include in the next problem set submission.

## II. Suggested Problem 2

**Part A.** Since $X$ is Banach, note that $B(X)$ is also Banach. Let $S_n = \sum_{k=0}^{n} T^k$. Note that for $m > n$, we have

$$||S_m - S_n|| = \left|\left|\sum_{k=n+1}^{m} T^k\right|\right| \leq \sum_{k=n+1}^{m} ||T||^k = ||T||^{n+1} \sum_{k=0}^{m-n-1} ||T||^k \leq \frac{||T||^{n+1}}{1 - ||T||} \tag{2}$$

which can be made arbitrarily small for choice of $n$ (and thus $m$) sufficiently large. It follows that the sequence of partial sums $S_n$ is Cauchy, this converges. In addition, note that

$$S_n(I-T) = (I-T)S_n = S_n - S_{n+1} + I \tag{3}$$

It is then easy to see that $\lim S_n \cdot (I-T) = (I-T) \cdot \lim S_n = I$, as desired.

**Part B.** Consider $\mathrm{GL}(B(X)) \subset B(X)$. Given invertible $Y \in B(X)$, suppose $||T-Y|| < ||Y^{-1}||^{-1}$, so that $||I - Y^{-1}T|| \leq ||Y^{-1}|| ||Y-T|| < 1$, implying that $Y^{-1}T$ is invertible. Thus, $T$ is invertible. It follows that there is an open ball around $Y$ which is contained in $\mathrm{GL}(B(X))$, so it is an open set.

## III. Suggested Problem 3

**Part A.** First, note that

$$T(v+w) = T\left(\frac{2v+2w}{2}\right) = \frac{T(2v)+T(2w)}{2} \tag{4}$$

Moreover, since $T(0) = 0$,

$$T(2^n v) = T\left(\frac{2^{n+1}v + 0}{2}\right) = \frac{T(2^{n+1}v) + T(0)}{2} \Longrightarrow T(2^{n+1}v) = 2T(2^n v) \tag{5}$$

for any $n$. In the case that $n = 0$, we have $T(2v) = 2T(v)$, so that $T(v+w) = T(v) + T(w)$. Moreover, via induction, it is easy to see that $T(2^n v) = 2^n T(v)$ for any $n \in \mathbb{Z}$. Let $\alpha$ be a rational number with a finite binary expansion, so that $\alpha = \sum_{n\in\mathbb{Z}} x_n 2^n$ where $x_n = 1$ for finitely many $n$ and is 0 for the remaining $n$. We then have

$$T(\alpha v) = T\left(\sum_{n\in\mathbb{Z}} x_n 2^n v\right) = \sum_{n\in\mathbb{Z}} x_n 2^n T(v) = \alpha T(v) \tag{6}$$

Clearly, the collection of $\alpha$ with this property is dense in the reals, so since $T$ is continuous, we have $T(\alpha v) = \alpha T(v)$ for all $\alpha \in \mathbb{R}$. It follows that $T$ is linear.

**Part B.** Note that

$$\begin{aligned}
\mathrm{mdef}_{v_1,v_2}(A) &= \left|\left|\frac{1}{2}A\left(\frac{v_1+v_2}{2}\right) - \frac{A(v_1)}{2} + \frac{1}{2}A\left(\frac{v_1+v_2}{2}\right) - \frac{A(v_2)}{2}\right|\right| \qquad (7)\\
&\leq \left|\left|\frac{1}{2}A\left(\frac{v_1+v_2}{2}\right) - \frac{A(v_1)}{2}\right|\right| + \left|\left|\frac{1}{2}A\left(\frac{v_1+v_2}{2}\right) - \frac{A(v_2)}{2}\right|\right| \qquad (8)\\
&= \frac{1}{2}\left|\left|\frac{v_1+v_2}{2} - \frac{2v_1}{2}\right|\right| + \frac{1}{2}\left|\left|\frac{v_1+v_2}{2} - \frac{2v_2}{2}\right|\right| = \frac{1}{2}||v_1 - v_2||. \qquad (9)
\end{aligned}$$

which completes the proof.

**Part C.** Note that $R_z(v) = z + (z - v) = 2z - v$. Since $A$ is an isometry, it is injective, as if $Av = Aw$, then $Av - Aw = 0$, so $v - w = 0$. Since it is also surjective, it is a bijection. Moreover, $A^{-1}$ is also an isometry, as $||A^{-1}v - A^{-1}w|| = ||AA^{-1}v - AA^{-1}w|| = ||v-w||$. It follows that

$$B(v) = A^{-1}R_z A(v) = A^{-1}\left(A(v_1) + A(v_2) - A(v)\right) \tag{10}$$

which immediately means that

$$B\left(\frac{v_1+v_2}{2}\right) = A^{-1}\left(A(v_1) + A(v_2) - A\left(\frac{v_1+v_2}{2}\right)\right) \tag{11}$$

as well as

$$\frac{B(v_1)+B(v_2)}{2} = \frac{1}{2}\left[A^{-1}(A(v_2)) + A^{-1}(A(v_1))\right] = \frac{v_1+v_2}{2} = A^{-1}\left(A\left(\frac{v_1+v_2}{2}\right)\right). \tag{12}$$

We then have

$$\mathrm{mdef}_{(v_1,v_2)}(B) = \left|\left|B\left(\frac{v_1+v_2}{2}\right) - \frac{B(v_1)+B(v_2)}{2}\right|\right| \tag{13}$$

$$= \left|\left|A^{-1}\left(A(v_1)+A(v_2)-A\left(\frac{v_1+v_2}{2}\right)\right) - A^{-1}\left(A\left(\frac{v_1+v_2}{2}\right)\right)\right|\right| \tag{14}$$

$$= \left|\left|A(v_1)+A(v_2)-2A\left(\frac{v_1+v_2}{2}\right)\right|\right| = 2\mathrm{mdef}_{(v_1,v_2)}(A) \tag{15}$$

as desired, and the proof is complete.

**Part D.** Part B implied that the midpoint defect is bounded above by the quantity $\frac{1}{2}||v_1+v_2||$. However, it also follows from Part C that given surjective isometry $A$, we can let $A_1 = A$ and choose a sequence $A_k$ of surjective isometries such that

$$\mathrm{mdef}_{(v_1,v_2)}(A_{k+1}) = 2\mathrm{mdef}_{(v_1,v_2)}(A_k) \Longrightarrow \mathrm{mdef}_{(v_1,v_2)}(A_{k+1}) = 2^k\mathrm{mdef}_{(v_1,v_2)}(A) \tag{16}$$

If $\mathrm{mdef}_{(v_1,v_2)}(A) \neq 0$, then it follows that $\mathrm{mdef}_{(v_1,v_2)}(A_{k+1})$ can be made arbitrarily large for sufficiently large $k$, thus contradicting the fact that each element of this sequence must be bounded above by $\frac{1}{2}||v_1+v_2||$. It follows that $\mathrm{mdef}_{(v_1,v_2)}(A) = 0$.

**Part E.** Note that since $A$ is an isometry, it is automatically continuous. Let $w_0 = A(0)$. Then it is clear that $A'(v) = A(v) - w_0$ is a surjective isometry with $A'(0) = 0$. Since $A'$ is also continuous, and it preserves midpoints, from Part D, it follows from Part A that it is linear, and the proof is complete.

## IV. Suggested Problem 4

We must show that the "derivative norm" is both a valid norm, and that it makes $C_b^k(U)$ into a Banach space. First, note that

$$||f+g|| = \max_{|\alpha|\leq k} ||\partial_\alpha(f+g)||_\infty = \max_{|\alpha|\leq k} ||\partial_\alpha f + \partial_\alpha g||_\infty \tag{17}$$

$$\leq \max_{|\alpha|\leq k} ||\partial_\alpha f||_\infty + \max_{|\alpha|\leq k} ||\partial_\alpha g||_\infty \tag{18}$$

$$= ||f|| + ||g|| \tag{19}$$

as well as

$$||\beta f|| = \max_{|\alpha|\leq k} ||\partial_\alpha(\beta f)||_\infty = |\beta| \max_{|\alpha|\leq k} ||\partial_\alpha f||_\infty = |\beta| ||f||. \tag{20}$$

Both of these results follow from the fact that we know the standard uniform norm on functions is in fact a norm. Finally, note that if $f = 0$, then $||f|| = 0$. Moreover, if $||f|| = 0$, then from the case $\alpha = (0, \dots, 0)$, we have

$$||\partial_{(0,\dots,0)} f||_\infty = ||f||_\infty \leq \max_{|\alpha|\leq k} ||\partial_\alpha f||_\infty = ||f|| = 0 \tag{21}$$

which means that $||f||_\infty = 0$, so $f = 0$. It follows that $||\cdot||$ is in fact a norm.

To prove that $(C_b^k(U), ||\cdot||)$ is in fact a Banach space, let $f_j$ be a Cauchy sequence. We already know that $(C_b(U), ||\cdot||_\infty)$ is a Banach space, which is the $k = 0$ case of the statement we are trying to prove. If $f_j$ is Cauchy realative to $||\cdot||$, it follows that each of the sequences of derivatives $\partial_\alpha f_j$ for $|\alpha| \leq k$ is Cauchy relative to the standard infinity-norm, as

$$||\partial_\alpha f_m - \partial_\alpha f_n||_\infty \leq \max_{|\alpha|\leq k} ||\partial_\alpha f_m - \partial_\alpha f_n||_\infty = ||f_m - f_n||. \tag{22}$$

Thus, each of the sequences $\partial_\alpha f_j$ converge uniformly to a bounded continuous function $f^{(\alpha)}$. Let $f = f^{(0\dots0)}$. It is a basic fact from real analysis that given a sequence of $C^k$ functions converging uniformly, with uniformly converging derivatives of all orders, the resulting limit, in this case $f$, is also $C^k$, with its derivatives given by the limits of the derivative sequences.

Thus, it follows that the Cauchy sequence $f_j$ does in fact converge to a $C^k$ function $f$ in the norm $||\cdot||$ (as we showed all derivatives $\partial_\alpha f_j$ converge uniformly to derivatives $\partial_\alpha f$, which exist). It follows immediately that our space is Banach, and we are done.

## V. Suggested Problem 5

The existence of $\beta$ follows immediately. Let $e_1, \dots, e_d$ be the standard normalized basis for $\mathbb{R}^d$:

$$\left|\left|\sum_{j=1}^d \alpha_j e_j\right|\right| \leq \sum_{j=1}^d |\alpha_j| ||e_j|| \leq M \sum_{j=1}^d |\alpha_j| = M \left|\left|\sum_{j=1}^d \alpha_j e_j\right|\right|_1 \tag{23}$$

where $M = \max_j ||e_j||$. This immediately implies that the function $F : \mathbb{R}^d \to \mathbb{R}$ given by $F(x) = ||x||$ is continuous in the topology generated by $||\cdot||_1$, as fixing some $y$ and some $\varepsilon > 0$, note that if we set $||x-y||_1 < \varepsilon/M$, then

$$|F(x) - F(y)| = |||x|| - ||y||| \leq ||x-y|| \leq M||x-y||_1 < \varepsilon \tag{24}$$

as desired.

Of course, the unit ball $B$ in the norm $||\cdot||_1$ is compact. It follows from the extreme value theorem that $F = ||\cdot||$ takes on a minimum. Moreover, this minimum must be greater than 0, as $0 \notin B$. In other words, there exists $\alpha > 0$ such that $\alpha \leq ||x||$ for all $x$ with $||x||_1 = 1$. It then follows that for arbitrary $x$,

$$||x|| = ||x||_1 \cdot \left|\left|\frac{x}{||x||_1}\right|\right| \geq \alpha ||x||_1. \tag{25}$$

We have therefore shown that $\alpha||x||_1 \leq ||x|| \leq M||x||_1$ for all $x \in \mathbb{R}^d$.

## VI. Suggested Problem 6

**Part A.** This follows immediately from the Leibniz integral rule. Indeed,

$$f'(x) = -\sin(x) + \int_0^x \cos(x-t)g(t)\ dt \tag{26}$$

and

$$f''(x) = -\cos(x) + \cos(0)g(x) - \int_0^x \sin(x-t)g(t)\ dt \tag{27}$$

so that

$$f''(x) + f(x) = -\cos(x) + g(x) - \int_0^x \sin(x-t)g(t)\ dt + \cos(x) + \int_0^x \sin(x-t)g(t)\ dt = g(x) \tag{28}$$

as desired. Checking that $f$ satisfies the initial conditions is trivial.

**Part B.** Suppose $f$ is a function satisfying the equation

$$f(x) = \cos(x) + \int_0^x \sin(x-t)\sigma(t)f(t)\ dt \tag{29}$$

Then, from Part A, $f$ solves the given ODE with the given initial conditions.

Thus, we let $u(x) = \cos(x)$ (which is smooth) and we define $K$ as the operator taking $f(x)$ to $\int_0^x \sin(x-t)\sigma(t)f(t)\ dt$. Clearly, the new function is twice-differentiable. Of course, $K$ is linear, and it is also bounded. Let $||\cdot||$ denote the usual uniform norm on function space, we have

$$||Kf|| = \sup_{x\in[0,1]} \left|\left|\int_0^x \sin(x-t)\sigma(t)f(t)\right|\right| \le \sup_{x\in[0,1]} \int_0^x |\sin(x-t)\sigma(t)f(t)|\ dt \le \sup_{t\in[0,1]} |\sigma(t)||f(t)| \le M||f|| \tag{30}$$

where $M = \sup_{t\in[0,1]} |\sigma(t)|$.

**Part C.** It is easy to see that

$$(K^n f)(x) = \int_{0\le t_n \le \cdots \le t_t \le x} \sin(x-t_1)\sin(t_1-t_2)\cdots\sin(t_{n-1}-t_n)\sigma(t_1)\cdots\sigma(t_n)f(t_n)\ dt_1\cdots dt_n \tag{31}$$

which means that, since $\sin(x) \le x$ for $x \in [0,1]$, we will have

$$|(K^n f)(x)| \le \int_{0\le t_n \le \cdots \le t_t \le x} |\sin(x-t_1)\sin(t_1-t_2)\cdots\sin(t_{n-1}-t_n)\sigma(t_1)\cdots\sigma(t_n)f(t_n)|\, dt_1\cdots dt_n \tag{32}$$

$$\le M^n||f|| \int_{0\le t_n \le \cdots \le t_t \le x} |\sin(x-t_1)(t_1-t_2)\cdots(t_{n-1}-t_n)|\, dt_1\cdots dt_n \tag{33}$$

$$\le M^n||f|| \int_{0\le t_n \le \cdots \le t_t \le x} t_1\cdots t_{n-1}\ dt_1\cdots dt_n \tag{34}$$

$$\le M^n||f|| \int_{0\le t_n \le \cdots \le t_1 \le x} t_1^{n-1}\ dt_1\cdots dt_n \tag{35}$$

$$\le M^n||f|| \int_0^x \cdots \int_0^x t_1^{n-1} dt_1\cdots dt_n \tag{36}$$

$$= \frac{M^n||f||t_1^n x^{n-1}}{n!} \le \frac{M^n||f||}{n!} \tag{37}$$

where $M = \max_{[0,1]} \sigma$, which achieves its maximum as a continuous function on a compact domain. We also know that $0 \le x \le 1$ (we use this fact in the last inequality). It follows that $||K^n f|| \le \frac{M^n||f||}{n!}$, so immediately we have $||K^n|| \le \frac{M^n}{n!}$, by definition of the operator norm.

**Part D.** Note that $f = u + Kf$ if and only if $(1-K)f = u$. Moreover, the operator $K' = \sum_{n=0}^\infty K^n$ is well-defined as the space of bounded operators $B(C[0,1])$ is Banach, and the partial sums $S_N = \sum_{n=0}^M K^n$ are Cauchy, since

$$||S_M - S_N|| = \left|\left|\sum_{n=N+1}^M K^n\right|\right| \le ||K^{N+1}|| \sum_{n=0}^{M-N-1} ||K^n|| \le \frac{C^{N+1}}{(N+1)!}\sum_{n=0}^\infty \frac{C^n}{n!} = \frac{\exp(C)C^{N+1}}{(N+1)!} \tag{38}$$

which eventually becomes arbitrarily small for sufficiently large $N$ and thus $M \ge N$. Thus, the limit $K'$ is well-defined, and by the same logic as Problem 2, $K'$ is the inverse of $1-K$. It follows immediately that setting $f = K'u$ is a well-defined solution to $f = u + Kf$. We showed that a solution to this equation solves the desired ODE with the desired initial conditions, so the proof is complete.