Classification of finite-dimensional C^*-algebras

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I. Introduction

The goal of these lecture notes is to provide supporing material for a talk that I will be giving at the Fields Institute operator algebra seminar in January 2024.

II. A few preliminary results

Before jumping into the main proof, we have to prove a few preliminary lemmas. The eventual classification theorem will show that all finite-dimensional C^*-algebras can be written as direct sums of matrix algebras over $\mathbb{C}$. To prove this, we will have to show that any finite-dimensional algebra behaves in a way analogous to a direct sum of matrix algebras.

The first lemma we will prove is in this general spirit: it proves that a "matrix-like" property holds for C^*-algebras.

Lemma II.1. If v is a partial isometry (so v^*v is a projection) then $vv^*v = v$.

Proof. Let $z = v - vv^*v = (1 - vv^*)v$. Note that

$$z^*z = v^*(1 - vv^*)^*(1 - vv^*) = v^*(1 - 2vv^* + vv^*vv^*)v = v^*v - 2v^*v + v^*v = 0 \tag{1}$$

where we use that $(v^*v)^2 = v^*v$. Thus, $||z||^2 = ||z^*z|| = 0$, so $||z|| = 0$ and $v = vv^*v$, as desired. □

Now, another lemma:

Lemma II.2. Suppose that $\{f_{ii}^{(k)} \mid 1 \leq k \leq r, 1 \leq i \leq n_k\}$ is a set of mutually orthogonal projections in C^*-algebra B and that

$$f_{11}^{(k)} \sim f_{22}^{(k)} \sim \cdots \sim f_{n_k n_k}^{(k)} \tag{2}$$

for each k. Then there is a system of matrix units $\{f_{ij}^{(k)}\}$ extending $\{f_{ii}^{(k)}\}$.

The idea behind constructing systems of matrix units is, essentially, to have a "basis" for each component of the direct sum that we will eventually demonstrate characterizes the C^*-algebra B. Each of the sets $\{f_{ii}^{(k)}\}$ are analogous to matrix projections with 1 at the i-th slot on the diagonal, at the k-th slot in the direct sum. Let us now prove the lemma.

Proof. Of course, here, we will make us of the Murray-von Neumann equivalence. Namely,

$$f_{11}^{(k)} \sim f_{jj}^{(k)} \Longrightarrow f_{11}^{(k)} = f_{1j}^{(k)} f_{1j}^{(k)^*} \quad \text{and} \quad f_{jj}^{(k)} = f_{1j}^{(k)^*} f_{1j}^{(k)} \tag{3}$$

This notation is consistent, as $f_{11}^{(k)}$ is self-adjoint, so setting $j = 1$ above causes no problems. Our claim is that if we set $\widetilde{f}_{ij}^{(k)} = f_{1i}^{(k)^*} f_{1j}^{(k)}$ then we will have the desired system of matrix units. This is in fact an extension of the system we are already provided. Namely, we have

$$\widetilde{f}_{jj}^{(k)} = f_{1j}^{(k)^*} f_{1j}^{(k)} = f_{jj}^{(k)} \tag{4}$$

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by definition. In fact, we might as well denote $\widetilde{f}_{ij}$ by f_{ij}, as for $i = 1$, we have

$$\widetilde{f}_{1j} = f_{11}^{(k)^*} f_{1j}^{(k)} = f_{1j}^{(k)} f_{1j}^{(k)^*} f_{1j}^{(k)} = f_{1j}^{(k)} \tag{5}$$

where we use the above lemma and the fact that $f_{1j}^{(k)^*} f_{1j}^{(k)} = f_{jj}^{(k)}$ is a projection. Let us now complete our verification. Of course, we have

$$f_{pq}^{(k)} f_{qr}^{(k)} = f_{1p}^{(k)^*} f_{1q}^{(k)} f_{1q}^{(k)^*} f_{1r}^{(k)} = f_{1p}^{(k)^*} f_{11}^{(k)} f_{1r}^{(k)} = f_{1p}^{(k)^*} f_{1r}^{(k)} f_{1r}^{(k)^*} f_{1r}^{(k)} = f_{1p}^{(k)^*} f_{1r}^{(k)} = f_{pr}^{(k)} \tag{6}$$

where we use the first lemma to note that $f_{1r}^{(k)} f_{1r}^{(k)^*} f_{1r}^{(k)} = f_{1r}^{(k)}$, as $f_{1r}^{(k)^*} f_{1r}^{(k)} = f_{rr}^{(k)}$ is a projection. Next, note that

$$f_{pq}^{(k)} f_{rs}^{(\ell)} = f_{1p}^{(k)^*} f_{1q}^{(k)} f_{1r}^{(\ell)^*} f_{1s}^{(\ell)} \tag{7}$$

Once again using the first lemma, we have $f_{1q}^{(k)} = f_{1q}^{(k)} f_{1q}^{(k)^*} f_{1q}^{(k)} = f_{1q}^{(k)} f_{qq}^{(k)}$ and $f_{1r}^{(\ell)} = f_{1r}^{(\ell)} f_{1r}^{(\ell)^*} f_{1r}^{(\ell)} = f_{1r}^{(\ell)} f_{rr}^{(\ell)}$ so that $f_{1r}^{(\ell)^*} = f_{rr}^{(\ell)} f_{1r}^{(\ell)^*}$. We then use the fact that the projections in our set are mutually orthogonal to conclude that

$$f_{1p}^{(k)^*} f_{1q}^{(k)} f_{1r}^{(\ell)^*} f_{1s}^{(\ell)} = f_{1p}^{(k)^*} f_{1q}^{(k)} (f_{qq}^{(k)} f_{rr}^{(\ell)}) f_{1r}^{(\ell)^*} f_{1s}^{(\ell)} = 0 \tag{8}$$

which is 0 when $q \neq r$ or $k \neq \ell$, as in these cases, $f_{qq}^{(k)} f_{rr}^{(\ell)} = 0$. It is very immediately clear that $f_{ij}^{(k)^*} = f_{1j}^{(k)^*} f_{1i}^{(k)} = f_{ji}^{(k)}$, so we have verified the third condition, and it follows that our set of $f_{ij}^{(k)}$ is in fact a system of matrix units in B extending $\{f_{ii}^{(k)}\}$. □

III. The main proof