# Fall 2023 MAT437 problem set 3

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## I. Problem 1

**Part 1.** Recall that for projections $p$ and $q$ in $M_n(\mathbb{C})$, the relation of *Murray-von Neumann equivalence* is that $p \sim q$ in $\mathcal{P}(M_n(\mathbb{C}))$ when there exists some $v \in M_n(\mathbb{C})$ such that $p = v^*v$ and $q = vv^*$

Let us assume condition (a) holds, so $p = v^*v$ and $q = vv^*$ for some $v$. Then

$$\text{Tr}(p) = \text{Tr}(v^*v) = \text{Tr}(vv^*) = \text{Tr}(v) \tag{1}$$

by the well-known internal commutativity of the trace, so (a) implies (b). Assuming (b), note that since $p$ and $q$ are projections, their spectra must lie entirely in $\{0, 1\}$ (we proved this in an earlier exercise, in a previous problem set). Thus, for $r$ a projection, $\text{Tr}(r)$ is precisely the multiplicity of the 1-eigenvalue, and is thus the dimension of the 1-eigenspace. Of course, the 0-eigenspace is the kernel, so the dimension of the 1-eigenspace is just the $\dim(\text{range}(p))$. If $\text{Tr}(p) = \text{Tr}(q)$, then it follows immediately that $\dim(\text{range}(p)) = \dim(\text{range}(q))$, and (b) implies (c).

Finally, assume (c) holds. From the same logic as above, both $p$ and $q$ must have the same multiplicity of their 0 and 1 eigenvalues. Moreover, since $p$ and $q$ are self-adjoint, they both have orthogonal bases of eigenvectors: $v_1, \dots, v_n$ and $w_1, \dots, w_n$, respectively. We define $u$ to be a linear map taking each 1-eigenvector of $p$ to a 1-eigenvector of $q$, and each 0-eigenvector of $p$ to a 0-eigenvector of $q$ (which we can do in bijective correspondence, as they have equal multiplicity).

~~Clearly, $p = u^{-1}qu$. Moreover, $u$ is unitary. Note that $\langle uv_j, uv_j \rangle = \langle u^*uv_j, v_j \rangle = 1$ for each $v_j$, and~~ $\overline{\langle uv_j, uv_j \rangle}$ ~~=~~ $\langle u^*v_j, u^*b_j \rangle = \langle uu^*v_j, v_j \rangle = 1$ for each $v_j$. Since the $v_j$ form a basis, $u^*u = uu^* = 1$. Thus, $u$ is in fact unitary. It follows thjat $p \sim_u q$, so since the underlying algebra is unital, from Proposition 2.2.2 of the book, $p \sim q$.

**Part 2.** Recall that $\mathcal{D}(\mathbb{C}) = \mathcal{P}_\infty(\mathbb{C}) / \sim_0$ where $\mathcal{P}_\infty(\mathbb{C}) = \cup_{n=1}^{\infty} \mathcal{P}(M_n(\mathbb{C}))$. As we remarked upon in the Part 1, the trace of any projection is the dimension of the 1-eigenspace, and is thus a non-negative integer. Thus, define $\text{Tr} : \mathcal{P}_\infty(\mathbb{C}) \to \mathbb{Z}_{\geq 0}$ be the standard trace map. Note that this map is surjective, as the $n \times n$ identity and zero matrices are elements of $\mathcal{P}_\infty(\mathbb{C})$, which have traces 0 and $n$ respectively, for all $n \geq 1$.

**Claim I.1.** Given $p, q \in \mathcal{P}_\infty(\mathbb{C})$, with $p$ an $m \times m$ matrix and $q$ an $n \times n$ matrix with $m \geq n$, then $p \sim_0 q$ if and only if $p \oplus 0_{m-n} \sim q$, where $0_{m-n}$ is the $(m - n) \times (m - n)$ zero-matrix.

*Proof.* Note from RLL Proposition 2.3.2 that $p \sim_0 p \oplus 0_{m-n}$. Since $\sim_0$ is an equivalence relation, it follows that if $p \sim_0 q$, then $q \sim_0 p \oplus 0_{m-n}$. Thus, by definition, since the left and right are the same size, $q \sim p \oplus 0_{m-n}$. The converse follows from $q \sim p \oplus 0_{m-n} \Rightarrow q \sim_0 p \oplus 0_{m-n} \Rightarrow q \sim_0 p$. □

It follows from this result that Tr is a valid map on $\mathcal{P}_\infty(\mathbb{C}) / \sim_0$, as if $p \sim_0 q$, then assuming WLOG that $q$ is a larger matrix than $p$, $p \oplus 0 \sim q$ for some zero-matrix 0, so $\text{Tr}(p) = \text{Tr}(p \oplus 0) = \text{Tr}(q)$, from Part 1 of this problem. We already claimed that Tr is surjective. It is also injective as if $\text{Tr}(p) = \text{Tr}(q) = \text{Tr}(p \oplus 0)$, then $q \sim p \oplus 0$, so $p \sim_0 q$. Finally, it is clear from the definition that $\text{Tr}(p \oplus q) = \text{Tr}(p) + \text{Tr}(q)$. Thus, Tr is an isomorphism of semigroups, so $\mathcal{D}(\mathbb{C}) \simeq \mathbb{Z}_{\geq 0}$: the image of Tr.

It follows immediately that $K_0(A) = G(\mathcal{D}(\mathbb{C})) \simeq G(\mathbb{Z}_{\geq 0}) = \mathbb{Z}$, where the fact that the semigroup isomorphism gives rise to a group isomorphism of Grothendieck groups and the fact that $G(\mathbb{Z}_{\geq 0}) = \mathbb{Z}$ is discussed

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in the textbook.

**Part 3.** I didn't have time to finish this part, will do so for next week's problems.

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## II. Problem 3

Note that $\nu(p \oplus 0) = \nu(p) + \nu(0) = \nu(p)$.

Suppose (i) holds. Suppose $p, q \in \mathcal{P}_n(A)$ are such that $p \sim_u q$, so $p = uqu^*$ for some $u \in \widetilde{A}$. Then, note from Proposition 2.2.8 in RLL that $p \oplus 0 \sim_h q \oplus 0$, where 0 is the $n \times n$ zero-matrix. It follows that $\nu(p \oplus 0) = \nu(q \oplus 0)$ by (i), so $\nu(p) = \nu(q)$ by assumption. Thus, (i) implies (ii).

Suppose (ii) holds. For $p, q \in \mathcal{P}_\infty(A)$, suppose $p \sim_0 q$. Suppose WLOG that $p$ is in $\mathcal{P}_m$ and $q$ is in $\mathcal{P}_n$ with $m \geq n$. It is easy to see that $p \sim q \oplus 0_{m-n}$: note that $p \sim_0 q$, $p = vv^*$ and $q = v^*v$ for $v \in M_{m,n}(A)$. Let $w$ be the element of $M_m(A)$ which adds $m - n$ zero-columns to the right side of $v$. It is easy to check that $ww^* = vv^* = p$ and $w^*w = q \oplus 0_{m-n}$. Thus, from RLL Proposition 2.2.8, $p \oplus 0 \sim_u q \oplus 0_{m-n} \oplus 0$, so from (ii), $\nu(p \oplus 0) = \nu(q \oplus 0_{m-n} \oplus 0)$, implying $\nu(p) = \nu(q)$, so (ii) implies (iii).

Suppose (iii) holds, suppose $p \sim_s q$. Then from RLL Definition 3.1.6, $p \oplus 1_n \sim_0 q \oplus 1_n$, so from (iii), $\nu(p \oplus 1_n) = \nu(q \oplus 1_n) \Rightarrow \nu(p) + \nu(1_n) = \nu(q) + \nu(1_n)$. Thus, $\nu(p) = \nu(q)$, so (iii) implies (iv).

Finally, let us assume (iv). Suppose $p \sim_h q$. Then from Proposition 2.2.7 of RLL, $p \sim q$, so immediately $p \sim_0 q$ and thus $p \sim_s q$, so from (iv), $\nu(p) = \nu(q)$, and (iv) implies (i).

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## III. Problem 4

**Part 1.** By definition, $e = ab$ and $f = ba$. Let $c = aba$ and $d = bab$. Then

$$cd = ababab = (ab)(ab)(ab) = (e)(e)(e) = e \tag{2}$$

as well as

$$dc = bababa = (ba)(ba)(ba) = (f)(f)(f) = f \tag{3}$$

Moreover, $cdc = e(aba) = e^2a = ea = aba = c$ and $dcd = f(bab) = f^2b = fb = bab = d$. This completes the proof.

**Part 2.** Reflexivity is trivial, as $f = ff$, so $f \approx_0 f$. Symmetry is also trivial as if $f \approx_0 e$, then $f = ab$, $e = ba$, so swapping $a$ and $b$ (and $n$ and $m$: the sizes of the matrices) implies $e \approx_0 f$. This relation is also transitive. Suppose we have $x, y, z$ idempotent with $x \approx_0 y$ and $y \approx_0 z$. Then there exists $a, b$ *with the properties of Part 1*, with $x = ab$ and $y = ba$, as well as $c$ and $d$ *with the properties of Part 1* such that $y = cd$ and $z = dc$.

Thus, $ba = cd$. It follows that $a = aba = acd$ and $c = cdc = bac$. Thus, $x = (ac)(db)$ and $z = (db)(ac)$ so $x \approx_0 z$. It follows that $\approx_0$ is an equivalence relation.

**Part 3.** Note that $e$ is a $m \times m$ matrix with elements in $R$. Let $f$ be the matrix which adds $n$ zero-columns to the right-hand side of $e$, so the resulting matrix is $m \times (m+n)$. Let $g$ be the matrix adding $n$ zero-rows to the bottom of $e$, so the resulting matrix is $(m+n) \times m$. We then have

$$fg = \begin{pmatrix} e & 0_n^c \end{pmatrix} \begin{pmatrix} e \\ 0_n^r \end{pmatrix} = e^2 = e \tag{4}$$

as well as

$$gf = \begin{pmatrix} e \\ 0_n^r \end{pmatrix} \begin{pmatrix} e & 0_n^c \end{pmatrix} = \begin{pmatrix} e^2 & 0 \\ 0 & 0 \end{pmatrix} = e \oplus 0_n \tag{5}$$

where the rightmost $0_n$ is the $n \times n$ zero-matrix, and $0_n^r$ and $0_n^c$ are the zero-rows and zero-columns described above. Thus, by definition, $e \approx_0 e \oplus 0_n$.

**Part 4.** To ensure that this operation is well-defined, we must demonstrate that if $e \approx_0 e'$ and $f \approx_0 f'$, then $e \oplus f \approx_0 e' \oplus f'$. This follows similarly from the proof of the previous result. In particular, we have $e = ab$, $e' = ba$, $f = cd$, $f' = dc$. It is easy to see that $e \oplus f = ab \oplus cd = \text{diag}(a, c)\text{diag}(b, d)$ and $e' \oplus f' = ba \oplus dc = \text{diag}(b, d)\text{diag}(a, c)$. Thus, the equivalence follows from the definition.

**Part 5.** To verify that $(V(R), +)$ is an Abelian semigroup, we require a well-defined closed addition (which we demonstrated in the previous exercise), associatitvity and commutativity. Associaitivty is trivial: this simply follows from the fact that $\oplus$ is associative.

$$([a]_V + [b]_V) + [c]_V = [a \oplus b]_V + [c]_V = [(a \oplus b) \oplus c]_V = [a \oplus (b \oplus c)]_V = [a]_V + [b \oplus c]_V = [a]_V + ([b]_V + [c]_V). \tag{6}$$

All that remains is verifying commutatitivty: this is equivalent to showing that $[a \oplus b]_V = [b \oplus a]_V$, or that $a \oplus b \approx_0 b \oplus a$. Note that

$$a \oplus b = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \quad \text{and} \quad b \oplus a = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \tag{7}$$

so the equivalence just follows from the definition.

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## IV. Problem 5

**Part 1.** Given idempotent $e \in A$, we define $x = e - e^*$ and $h = 1 + xx^*$. Note that $xx^*$ is positive, so $\text{sp}(xx^*) \in \mathbb{R}_{\geq 0}$. Suppose $1 + xx^*$ is not invertible: this would imply that $-1 \in \text{sp}(xx^*)$, a clear contradiction. Thus $1 + xx^*$ must be invertible. Now, note that

$$eh = e(1 + xx^*) = e + e(e - e^*)(e^* - e) = e + e(ee^* - e^2 + e^*e - (e^*)^2) = e + ee^* - e + ee^*e - ee^* = ee^*e \tag{8}$$

where we use $(e^*)^2 = e^*e^* = (e^2)^*$. In addition,

$$he = e + (e - e^*)(e^* - e)e = e + (ee^* - e^2 + e^*e - (e^*)^2)e = e + ee^*e - e + e^*e - e^*e = ee^*e \tag{9}$$

Since $h$ is self-adjoint, $he^* = (eh)^* = e^*ee^* = (he)^* = e^*h$ as well. We set $p = ee^*h^{-1}$. We claim that $p$ is a projection in $A$. Indeed, using the commutativity relations derived above, $p^* = (h^{-1})^*ee^* = h^{-1}ee^* = h^{-1}ee^*hh^{-1} = h^{-1}ehe^*h^{-1} = h^{-1}hee^*h^{-1} = ee^*h^{-1}$,so $p$ is self-adjoint (we also use that $(h^{-1})^* = (h^*)^{-1} = h^{-1}$ as $(hh^{-1}) = 1 \Rightarrow (h^{-1})^*h^* = 1$, so $(h^{-1})^* = (h^{-1})^*$). Finally, note that

$$p^2 = p^*p = (h^{-1}ee^*)(ee^*h^{-1}) = h^{-1}(ee^*e)e^*h^{-1} = h^{-1}hee^*h^{-1} = ee^*h^{-1} = p \tag{10}$$

so $p$ is a projection. It ids also easy to verify that $ep = eee^*h^{-1} = ee^*h^{-1} = p$ and $pe = p^*e = h^{-1}ee^*e = h^{-1}he = e$. Thus, by definition, $e \approx_0 p$, as we have found elements $e, p$ such that $p = ep$ and $e = pe$. This completes the proof.

**Part 2.** Clearly, if $p \sim_0 q$, then there exists $v$ such that $p = vv^*$ and $q = v^*v$, so it immediately follows that $p \approx_0 q$, as we can simply let $a = v$ and $b = v^*$ so that $p = ab$ and $q = ba$. Now, let us assume that $p \approx_0 q$. It follows from the solution to Part 1 of Problem 4 that we can choose $a$ and $b$ such that $p = ab$, $q = ba$, as well as $a = aba$ and $b = bab$. Note that $b^*b = b^*a^*b^*bab = (ab)^*b^*b(ab) = pb^*bp \in pAp$. It is straightforward to note that $||a^*a|| - a^*a = ||a||^2 - a^*a$ is positive, as $||a^*a|| = r(a^*a)$ (an upper bound on the spectrum). Thus, $x^*x = ||a||^2 - a^*a$ for some $x$, and $b^*x^*xb = (xb)^*xb$ is positive, so

$$b^*x^*xb = b^*(||a||^2 - a^*a)b = ||a||^2b^*b - b^*a^*ab = ||a||^2b^*b - p^*p = ||a||^2b^*b - p \geq 0 \Rightarrow ||a||^2b^*b \geq p \tag{11}$$

Since $b^*b$ is positive, it has a positive root $(b^*b)^{1/2}$. I couldn't figure out how to continue the argument past here in a timely fashion, so I will revisit this next week.

**Part 3.** I also didn't have time to finish this part, will do so for next week's problems.