Lee's Introduction to Smooth Manifolds: Chapter 1 Assorted Notes, Proofs and Exercises

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I. Chapter 1

Exercise I.1 (Lee Exercise 1.1)**.** For both definitions, it is clear that the new subsumes the old, as both open balls and $\mathbb{R}^n$ are open neighbourhoods around every point of $\mathbb{R}^n$. Conversely, if we assume the original definition, then given some point $p \in M$ and local homeomorphism $\varphi : U \to \widehat{U}$, then we can find an open ball B around $\varphi(p)$, and note that the restriction $\varphi : \varphi^{-1}(B) \to B$ is a homeomorphism, so p has a neighbourhood homeomorphic to an open ball. As for the other case, this follows from the fact that open n-balls are homeomorphic to $\mathbb{R}^n$.

Example I.1 (Lee Example 1.4: the Sphere)**.** We wish to show that the n-sphere is a manifold. The Hausdorff condition and second-countability are clearly inherited from the global space. As for the local Euclidean condition, the general strategy is to separate the sphere into different parts. Let $p = (p_1, \dots, p_{n+1})$ be a point on the n-sphere. Clearly, there must exist some coordinate $p_k \neq 0$. We let

$$U_p = \left\{(x_1, \dots, x_{n+1}) \mid x_k^2 = 1 - x_{n+1}^2 - \cdots - x_{k+1}^2 - x_{k-1}^2 - \cdots - x_1^2 \text{ and } \text{sign}(x_k) = \text{sign}(p_k)\right\} \quad (1)$$

Clearly, $U_p = S^n \cap \mathbb{R}_k^{\text{sign}(p_k)}$, which is open in the subspace topology. Let $\pi_k : U_p \to \pi_k(U_p)$ be the projection onto all coordinates *except for the k-th.* Define

$$\pi_k^{-1}(x) = \pi_k^{-1}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}) = (x_1, \dots, f_k(x), \dots, x_{n+1}) \quad (2)$$

where $f_k(x) = \text{sign}(p_k)\sqrt{1 - x_{n+1}^2 - \cdots - x_1^2}$ is clearly continuous. It is easy to check that π_k^{-1} is the well-defined inverse of π_k, and both maps are continuous, so $\pi_k(U_p)$ is open, and π_k is the desired local homeomorphism.

Example I.2 (Lee Example 1.5: Projective spaces)**.** Another example of an n-manifold is a projective space. Indeed, define $\mathbb{R}P^n$ to be the collection of all 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$. For example, $\mathbb{R}P$ is merely the collection of all lines passing through the origin in $\mathbb{R}^2$, which can be identified with the upper half-open circle, which itself can be identified with S^1, which we already know is a manifold.

A bit more formally, we must prove Hausdorff, second-countability, as well as the existence of an atlas. We topologize $\mathbb{R}P^n$ by taking the quotient topology on $\mathbb{R}^{n+1} - \{0\}$ via the quotient map π which sends x to its corresponding linear subspace.

Lemma I.1 (Lee Lemma 1.10)**.** Note that M is second-countable, let $\mathcal{B}$ be the countable basis. For each $p \in M$, note that there exists a local homeomorphism $\varphi : U \to \widehat{U}$, where U contains p. Since there exists some $B \in \mathcal{B}$ with $p \in B \subset U$, B is homeomorphic to an open subset of $\mathbb{R}^n$. Thus, we have a countable subcollection $\mathcal{B}'$ of elements of $\mathcal{B}$ such that each $p \in M$ is contained in some element of $\mathcal{B}'$ and each element of $\mathcal{B}'$ is homeomorphic (via some φ_B) to an open subset of $\mathbb{R}^n$.

For each $B \in \mathcal{B}'$, let S_B denote the countable collection of all open balls of rational radius contained in $\varphi_B(B)$, centred at rational points. Let $\varphi_B^{-1}(S_B)$ denote the inverse image of each of these balls, which are all open. Clearly, given some $p \in U$ open in M, p is contained in some $B \in \mathcal{B}'$, and $B \cap U$ is an open subset of B, so $\varphi_B(B \cap U)$ is open in $\varphi_B(B)$. We can pick an open ball in S_B containing $\varphi_B(p)$, so p is contained in some element of $\varphi_B^{-1}(S_B)$ which is contained in B. Let $S = \cup_{B \in \mathcal{B}'} S_B$. Suppose $\varphi_{B_1}^{-1}(B_1), \varphi_{B_2}^{-1}(B_2) \in S$ are two preimages of open balls with non-empty (thus open) intersection, U. Then the images $\varphi_{B_1}(U)$ and

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$\varphi_{B_2}(U)$ will intersect as and be open, so we can find some open ball B at a rational point, of rational radius, in this intersection, so $\varphi_{B_1}^{-1}(B)$ (and $\varphi_{B_2}^{-1}(B)$) are in U. Thus, S satisfies the intersection condition, and is in fact a countable basis of coordinate balls (as it is a countable intersection of countable sets).

Finally, note that since each φ_B is a homeomorphism, $\overline{\varphi_B^{-1}(B)} = \varphi_B^{-1}(\overline{B})$. Since $\overline{B}$ is compact, so too is $\varphi_B^{-1}(\overline{B})$ and thus $\overline{\varphi_B^{-1}(B)}$.

Exercise I.2 (Lee Exercise 1.14)**.** We begin with the first claim. Suppose $\mathcal{X}$ is locally finite, let x be a point of the topological space. Then there exists a neighbourhood U intersecting only $X_1, \ldots, X_N$. Suppose U intersects some $\overline{X_M}$ not in this set. Then by definition, U contains a point y where every neighbourhood of y intersects X_M, so U intersects X_M, a contradiction. Thus, $\overline{\mathcal{X}}$ is locally finite.

Next, note that $\cup_{X\mathcal{X}}\overline{X} \subset \overline{\cup_{X\in\mathcal{X}}X}$ trivially. To prove inclusion the other way, given $x \in \overline{\cup_{X\in\mathcal{X}}X}$, note that every open set of x intersects the union at a finite number of X. If there isn't some X' intersected by all neighbourhoods, we can pick some neighbourhood U which intersects $X_1, \ldots X_M$ and neighbourhood $U_1, \ldots, U_M$ where U_k does not intersect X_k. Then $U \cap U_1 \cap \cdots U_M$ does not intersect any element of $\mathcal{X}$, which is a contradiction.

Proposition I.1 (Lee Proposition 1.16)**.** *This result is very simple when it is understood diagramatically, which is basically as far as we made it in third-year algebraic topology! Nevertheless, a formal proof is required.*

We first require a lemma.

Lemma I.2 (Lebesgue Number Lemma)**.** Suppose X is a compact set in a metric space and $\mathcal{A}$ is an open cover. Then there exists some $\delta > 0$ such that every subset of X having diameter less than δ is contained in an element of $\mathcal{A}$.

Proof. Let $A_1, \ldots, A_N$ be a finite subcover. The key idea is to use extreme value theorem. Define the function $f : X \to \mathbb{R}$ as

$$f(x) = \frac{1}{N}\sum_{k=1}^{N} d(x, X - A_k), \tag{3}$$

the average distance from x to the exterior of each A_k. Clearly, $x \in A_k$, for some A_k open, so the minimum value of f must be greater than 0. Set it to δ. Suppose U has diameter less than δ. Suppose U is not contained in a single A_k. Then, there exist points x and y of U not contained in a common A_k, with $d(x, y) \leq \delta$. Then, clearly, $d(x, y) \geq d(x, X - A_k)$ for each A_k. But then, $f(x) < \delta$, a contradiction. It follows that U must be contained in a single A_k, so the lemma holds. □

Let $\mathcal{B}$ be a countable basis for M, let $B, B' \in \mathcal{B}$ such that $B \cap B'$ is non-empty. Since $B \cap B'$ is a manifold, it has a finite number of connected components, which are also path-connected. Pick some x from each component. Then, for all pairs of $B, B' \in \mathcal{B}$, take all such points x and combine them into a set $\mathcal{X}$. For each pair of points $x, x' \in \mathcal{X}$ where $x, x' \in B$, let $h^B_{x,x'}(t)$ be a path between them.

Assume M is path-connected. Let us pick some $p \in \mathcal{X}$. Let $f : [0,1] \to M$ be a loop in $\pi_1(M, p)$. The collection of $f^{-1}(B)$ for $B \in \mathcal{B}$. will clearly cover $[0,1]$, so there exists a finite subcover, $f^{-1}(B_1), \ldots, f^{-1}(B_N)$.
WIP: will finish soon, apologies

Lemma I.3. If $f : U \to \mathbb{R}^n$ is smooth and injective with non-singular Jacobian at each point, then $f : U \to f(U)$ is a diffeomorphism. *Note: this was a homework question in MAT257!*

Proof. Because of the fact that f has non-singular Jacobian, it is locally invertible at every point, via inverse function theorem, and in fact, this inverse function is itself C^∞ with $Df^{-1}(x) = [Df](f^{-1}(x))^{-1}$. Let $f(x) \in f(U)$, by uniqueness of inverses, there must exist an open set $V_x \subset f(U)$ around $f(x)$ in $\mathbb{R}$ such that $f^{-1} : V_x \to W_x$, with W_x open in U is smooth. This holds for every point of $f(U)$. Taking the union of all such open sets implies $f(U)$ is open, and that $f : U \to f(U)$ is a diffeomorphism. □

A smooth atlas is said to be *maximal* if it is not properly contained within any larger smooth atlas.

Example I.3 (Lee Example 1.36: Grassmann manifolds)**.** *My goal is to reconstruct the technique of Lee utilized to prove that $G_k(V)$ is in fact a smooth manifold of dimension $k(n-k)$.*

The main idea is to construct a smooth atlas and topologize the space with respect to this atlas by considering direct sum decompositions of V. Indeed, if P is a dimension-k vector subspace of V, then there exists a space of dimension $n-k$ such that $V = P \oplus Q$. Each $v \in V$ can be written uniquely as $v = p + q$.

Given a linear map $X : P \to Q$, the graph of X can be identified with the space $\{v + Xv \mid v \in P\}$ of dimension-k. Clearly, this subspace intersects Q only trivially, as if it had a non-trivial intersection with Q, then we would have $Xv \neq 0$ when $v = 0$, a contradiction. Given any dimension-k subspace $S \subset V$ which intersects Q trivially, it is possible to show that S is the graph of a map from P to Q. Let π_P and π_Q be projections onto P and Q. Clearly, $\pi_Q(S) = 0$. Thus, $\pi_P : S \to P$ must be an injection, as if $\pi_1(v) = \pi_1(v')$, then $v = p \oplus q$ and $v' = p \oplus q'$ are in S, implying $q - q'$ is as well, so $q = q'$ and $v = v'$. Rank-nullity implies π_P is a vector space isomorphism.

From here, define $X(p) = \pi_Q \circ \pi_P^{-1}$, where π_P is restricted to domain S. It is clear that the graph of X is S. Now, consider the collection of all dimension-k subspaces which intersect Q trivially, U_Q: these are in bijective correspondence with linear maps from P to Q, for a particular pair of P and Q. More formally, let Γ send an element of $\mathcal{L}(P, Q)$ to its graph, which can be identified with U_Q. We can then define

$$\varphi_Q = \Gamma : U_Q \to \mathcal{L}(P, Q) \simeq M((n-k) \times k, \mathbb{R}) \simeq \mathbb{R}^{k(n-k)} \tag{4}$$

by inverting Γ. Repeating this process for each Q gives a collection of φ_Q: we take these to define our charts.

Our goal is to show that we can apply Lemma 1.35, to define a smooth structure, using the collection of charts (U_Q, φ_Q). We will go through the necessary conditions point-by-point. **WIP**

Proposition I.2 (Lee Proposition 1.38)**.** We go point-by-point:

- Note that for $p \in \mathrm{Int}(M)$, there exists U open in M such that U is homeomorphic via φ with an open subset of $\mathbb{R}^n$. Given any other $q \in U$, note that q is also an interior point via the same local homeomorphism. Thus, $U \subset \mathrm{Int}(M)$, so $\mathrm{Int}(M)$ is open (here, we did invoke invariance of interior/boundary). Clearly, this set is an n-manifold.

- ∂M is the complement in M of $\mathrm{Int}(M)$ via invariance of boundary/interior, so it is closed. This set is an $(n-1)$-manifold via local homeomorphisms from open sets in the subspace topology induced on ∂M and $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$.

  More specifically, take $p \in \partial M$. Note that p is contained in U open in M where $\varphi(U)$ is open in $\mathbb{H}^n$ with non-trivial intersection with $\partial \mathbb{H}$ and $\varphi(p) \in \partial \mathbb{H}^n$. Then the open set $U \cap \partial M$ in the subspace topology is sent to $\varphi(U) \cap \mathbb{H}^n$, which is open in $\mathbb{H}^n = \mathbb{R}^{n-1}$.

- If $\partial M = \emptyset$, $M = \mathrm{Int}(M)$ and the result follows. If M is a manifold, every point is in neighbourhood locally homeomorphic to an open set in $\mathbb{R}$, meaning there is no open set which is homeomorphic to open set in $\mathbb{H}^n$ with non-trivial intersection with $\partial \mathbb{H}^n$, as this set will not be open in $\mathbb{R}^n$.

- Trivial.

Theorem I.1 (Lee Theorem 1.46: Smooth invariance of boundary)**.** Let us assume that we are given a point p such that (U, φ) is a chart having $p \in U$ with $\varphi(p) \in \partial \mathbb{H}^n$ and $\varphi(U) \subset \mathbb{H}^n$. Suppose we also have a chart (V, ψ) for which $\psi(p) \in \mathrm{Int}(\mathbb{H}^n)$.

Let $x = \psi(p)$, note that $f = \varphi \circ \psi^{-1}$ is a diffeomorphism between $\psi(U \cap V)$ and $\varphi(U \cap V)$. Clearly, since $x = \psi(p)$ is in the interior of $\mathbb{H}^n$, there exists an open ball B around x for which f is smooth. Moreover, $f(x) = \varphi(p)$ is on the boundary, so there exists an open set W in $\mathbb{R}^n$ and an extension $\tau : W \to \mathbb{R}^n$ for which τ is smooth, an such that τ agrees with f^{-1} on $W \cap \varphi(U \cap V)$.

From here, note that $\tau(W)$ contains x, so without loss of generality, we can assume $B \subset \tau(W)$. Thus, B is a subset of $f^{-1}(W \cap \varphi(U \cap V))$. The map from B to W to $\mathbb{R}^n$ given by $\tau \circ f$ is therefore equal to $f^{-1} \circ f$, as any element of B can be written as $f^{-1}(z)$, implying that Df is non-singular on B. Thus, f is an open map, so $f(B)$ is open in $\mathbb{R}^n$ and contains $f(x)$. Moreover, it must be contained in $\psi(U \cap V)$. But this contradicts the fact that $\psi(U \cap V)$ is contained in $\mathbb{H}^n$, as any open neighbourhood of $f(x) \in \partial \mathbb{H}^n$ will intersect a point outside $\mathbb{H}^n$.