Solutions to Hatcher's algebraic topology book

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I. Chapter 0

Solution I.1 (Problem 0.1)**.** Recall that $T^2 = S^1 \times S^1$, remove point $(0,1) \times (0,1)$. We can represent points in $S^1 \times S^1$ by a pair of angles in $S = [-\pi, \pi] \times [-\pi, \pi] - (0,0)$, the idea is to homotop this pair to the boundary of the square $[-\pi, \pi] \times [-\pi, \pi]$ (which we can do continuously as we omit the origin). The resulting boundary of the square is precisely the space $-1 \times S^1 \cup S^1 \times -1$, which is a pair of circles wedged at $(-1, -1)$.

Solution I.2 (Problem 0.2)**.** Simply use $F(x,t) = (1-t)x + \frac{tx}{||x||}$.

Solution I.3 (Problem 0.3)**.** If $X \to Y$ and $Y \to Z$ are homotopy equivalences (called f_1 and g_1 respectively, with corresponding backward maps f_2 and g_2), then

$$(g_1 \circ f_1) \circ (f_2 \circ g_2) \simeq g_1 \circ g_2 \simeq \text{id} \tag{1}$$

with similar logic showing that $(f_2 \circ g_2) \circ (g_1 \circ f_1) \simeq \text{id}$, where we construct a homotopy by concatenating the homotopy $f_1 \circ f_2 \simeq \text{id}$ and the homotopy $g_1 \circ g_2 \simeq \text{id}$. This same concatenation argument shows that if maps $f, g : X \to Y$ are homotopic via F and $g, h : X \to Y$ are homotopic via G, then f and h are homotopic via $G \star F$. Finally, if f is homotopic to $g : X \to Y$, which is a homotopy equivalence with backward map $h : Y \to X$, then by transitivity of homotopy proved earlier,

$$f \circ h \simeq g \circ h \simeq \text{id} \quad \text{and} \quad h \circ f \simeq h \circ g \simeq \text{id} \tag{2}$$

so that f is a homotopy equivalence.

Solution I.4 (Problem 0.4)**.** Let $r : X \to A$ be defined as $r(x) = f_1(x)$. Then note that $r \circ \iota : A \to A$ is homotopic to the identity via the homotopy $G : A \times [0,1] \to A$ given by $G(x,t) = f_t(x)$, which is well-defined at $f_t(A) \subset A$ for all t. Similarly, $\iota \circ r$ is homotopic to the identity via $F = f_t$.

II. Chapter 1

A. Section 1.1

Solution II.1 (Problem 1.1.17)**.** The idea is to loop the first circle of the wedge around the second one exactly n times. In particular, suppose $S^1 \vee S^1$ is realized as the space $S^1 \times \{1\} \cup \{1\} \times S^1$ (where we think of $S^1 \subset \mathbb{C}$. We define $r_n : S^1 \vee S^1 \to \{1\} \times S^1$ as

$$r_n(e^{i\theta} \times 1) = 1 \times e^{in\theta} \quad \text{and} \quad r_n(1 \times e^{i\theta}) = 1 \times e^{i\theta}. \tag{3}$$

It is clear that this function is continuous. This is clearly a retraction onto $\{1\} \times S^1 \simeq S^1$. To see that all of the r_n are not mutually homotopic, recall that each of the homotopy classes of S^1 based at 1 are represented by the loops $\omega_n : t \mapsto e^{2\pi i n t}$ for $t \in [0,1]$. If we had $r_n \simeq r_m$ via homotopy H for $m \neq n$, this would give a homotopy between ω_n and ω_m via

$$G(s,t) = (\pi_2 \circ H)(e^{2\pi i s} \times 1, t) \tag{4}$$

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where $\pi_2 : \{1\} \times S^1 \to S^1$ is projection. In particular,

$$G(s,0) = (\pi_2 \circ H)(e^{2\pi i s} \times 1, 0) = (\pi_2 \circ r_n)(e^{2\pi i s} \times 1) = e^{2\pi i n s} = r_n(s) \tag{5}$$

and

$$G(s,1) = (\pi_2 \circ H)(e^{2\pi i s} \times 1, 1) = (\pi_2 \circ r_m)(e^{2\pi i s} \times 1) = e^{2\pi i m s} = r_m(s) \tag{6}$$

which is a contradiction.

Solution II.2 (Problem 1.1.18)**.** Let $f : S^{n-1} \to A$ be the attaching map which attaches e^n to A to form $X \simeq D^n \sqcup_f A$. Then X is the union of open sets $e^n = \mathrm{Int}(D^n)$ and $X - \{p\}$ for some $p \in e^n$. The intersection of these open sets is $e^n - \{p\}$ which is path-connected (as $n \geq 2$), and from Lemma 1.15 we can write any loop in $\pi_1(X, x_0)$ for some $x_0 \in e^n - \{p\}$ as a product of loops contained in either of these open sets. Of course, any loop in e^n is nullhomotopic, so any loop is a product of loops contained in $X - \{p\}$.

Thus, the inclusion $\iota_* : \pi(X - \{p\}, x_0) \to \pi_1(X, x_0)$ is a surjection. We can choose some $y_0 \in A$, and note that since A (thus the whole space X and $X - \{p\}$ are, as we attach a cell of dimension 2 or more) is path-connected, a path from x_0 to y_0 will induce isomorphisms $\pi_1(X - \{p\}, x_0) \simeq \pi_1(X - \{p\}, y_0)$ and $\pi_1(X, x_0) \simeq \pi_1(X, y_0)$. The overall effect of composing these maps with ι_* is to take a loop in $X - \{p\}$ based at y_0, move its basepoint to x_0, map it via inclusion into X, and move the basepoint back to x_0. This is the same map on homotopy classes as the inclusion $\iota_* : \pi_1(X - \{p\}, y_0) \to \pi_1(X, y_0)$, so this map is an inclusion. Finally, note that $\iota_* : \pi_1(A, y_0) \to \pi_1(X - \{p\}, y_0)$ is an isomorphism, as $X - \{p\}$ deformation retracts to A. Thus, the inclusion of A in X induces a surjection on fundamental groups. From here:

- $S^1 \vee S^2$ is obtained by attaching a two-cell to S^1 at a single point, so $\iota_* : \pi_1(S^1, x_0) \to \pi_1(S^1 \vee S^2, x_0)$ is a surjection. Moreover, $S^1 \vee S^2$ retracts to S^1, so ι_* is also an injection. Thus, $\pi_1(S^1 \vee S^2, x_0) \simeq \mathbb{Z}$.
- Assuming that X contains finitely many cells, note that X can be constructed by repeatedly attaching cells e^n with $n \geq 2$ to X^1. Composing each of these inclusions gives the desired result. In the case that X contains infinitely many cells, we know that a compact subset of X is contained in a finite subset of the cells. Thus, given some $[\gamma] \in \pi_1(X, x_0)$, note that the image of γ is compact and therefore contained in some finite subsets of cell Y, which can be obtained by attaching finitely many cells to X^1. Define $\eta : [0,1] \to Y$ as $\eta(t) = \gamma(t)$, so that $j_*[\eta] = [\gamma]$ for inclusion $j : Y \to X$. From above, the inclusion $\iota_* : \pi_1(X^1, x_0) \to \pi_1(Y, x_0)$ will be a surjection, so pick θ where $\iota_*[\theta] = j_*[\eta] = [\gamma]$, and we have the desired result.

Solution II.3 (Problem 1.1.19)**.** Let γ be a loop in X, since γ is compact we can assume, without loss of generality, that X is made up of a finite number of cells (see Problem 1.1.18). Let $e_1^1, \ldots, e_k^1$ be the set of 1-cells in X. For the j-th cell e_j^1, let $\Phi_j : D^1 \to X$ be the corresponding characteristic map. Depending on whether both endpoints of $D^1 \simeq [0,1]$ are attached to $p \in X^0$ or just one, choose an open set U_j of one or both endpoints of the form $[0, \varepsilon)$, $(1-\varepsilon, 1]$, or $[0,\varepsilon) \cup (1-\varepsilon, 1]$. Then $\Phi_j(U_j)$ is a contractible open neighbourhood of p.

From here, we cover X with the 1-cells, along with all of the contractible $\Phi_j(U_j)$. Let $R_t^j : X \to X$ be the homotopy which shrinks $\Phi_j(U_j)$ to p and stretches $\Phi_j([0,1] - U_j)$ so that $[0,1] - U_j$ is mapped homeomorphically onto all of $\overline{e_j^1}$, with the endpoint (or endpoints) in ∂U_j taken to p. We note that we can partition $[0,1]$ into a finite collection on intervals $[s_j, s_{j+1}]$ taken by γ into one of the open sets described (by the Lebesgue number lemma). This allows us to write γ as a (finite) product of paths in each of the open sets, $\beta_1 \star \cdots \star \beta_n$.

Let R_t be the concatenation of all homotopies R_t^j. We note that R_t will fix the basepoint x_0 of γ, which is assumed to be a 0-cell, so $R_t \circ \gamma$ is a path-homotopy, so

$$\gamma \simeq R_1 \circ \gamma = R_1 \circ (\beta_1 \star \cdots \star \beta_n) = (R_1 \circ \beta_1) \star \cdots \star (R_1 \circ \beta_n) \tag{7}$$

which has the effect of shrinking the β_k contained in some $\Phi_j(U_j)$ to a point and stretching β_k in e_j^1 to the closure $\overline{e_j^1}$. Thus, without loss of generality, we can assume that γ is path-homotopic to a composition of paths each of which is contained in some $\overline{e_j^1}$. Moreover, by combining neighbouring paths in the composition which lie in the same cell-closure, we can assume that the endpoints of each β_k are the endpoints of a cell. Depending on whether these endpoints are the same or different, we can use a straight-line homotopy, which preserves endpoints, taking β_k to its single endpoint, or the path going from one endpoint to the other. This gives us the desired result: γ is path-homotopic to a finite composition of paths traversing the edges of the cell-complex.

B. Section 1.2

Before jumping into the exercises of this section, we find it necessary to provide our own brief proof of Seifert Van-Kampen, following both the detailed treatment of Munkres and the brief treatment of Hatcher.

Definition II.1. Let G be a group. A word in G is an element of the set of finite-length tuples of elements of $W(G)$, of the form $(g_1, \ldots, g_n)$. A word $(g_1, \ldots, g_n)$ represents $g \in G$ if $g_1 \cdots g_n = g$.

Definition II.2 (Free product)**.** Given a collection of subgroups $\{G_\alpha\}_{\alpha \in J}$, we say that a word $(g_1, \ldots, g_n)$ is a word in these subgroups if each g_j is in some G_α. We say that $(g_1, \ldots, g_n)$ is reduced if $g_j \neq 1$ for all j and adjacent g_j and g_{j+1} are contained in distinct G_α. We say that G is a *free product* of the G_α if $G_\alpha \cap G_\beta = \{1\}$ for $\alpha \neq \beta$ and each $g \in G$ is represented by a *unique* reduced word in the G_α.

Lemma II.1. Let G be a group with subgroups $\{G_\alpha\}_{\alpha \in J}$, let $(w_1, \ldots, w_\ell)$ be a word in the G_α representing w. Then there exists a reduced word representing w which can be obtained from $(w_1, \ldots, w_\ell)$ be removing all instances of 1 and then performing a finite sequence of mappings

$$(w_1, \ldots, w_\ell) \mapsto (w_1, \ldots, w_{n-1}, w_n w_{n+1}, w_{n+2}, \ldots, w_\ell) \quad \text{when } w_n, w_{n+1} \in G_\alpha \text{ and } w_n w_{n+1} \neq 1 \tag{8}$$
$$(w_1, \ldots, w_\ell) \mapsto (w_1, \ldots, w_{n-1}, w_{n+2}, \ldots, w_\ell) \quad \text{when } w_n, w_{n+1} \in G_\alpha \text{ and } w_n w_{n+1} = 1 \tag{9}$$

Proof. Of course, removing instances of 1 does not change the element of G that $(w_1, \ldots, w_\ell)$ represents, so without loss of generality, assume $w_j \neq 1$ for all j. Additionally, the above operations do not change the element of G that the word represents. If $(w_1, \ldots, w_\ell)$ is a word such that *neither* of the above operations can be performed, we must have all adjacent w_j in distinct G_α, so $(w_1, \ldots, w_\ell)$ is reduced. Since the above operations strictly reduce the length of the word, it follows by induction on the length of the word that the claim holds. □

Proposition II.1 (Extension property of free products)**.** Suppose G is a free product of $\{G_\alpha\}_{\alpha \in J}$. Then if H is another group, and $\varphi_\alpha : G_\alpha \to H$ are homomorphisms, there is a unique homomorphism $\Phi : G \to H$ such that $\Phi|_{G_\alpha} = \varphi_\alpha$.

Proof. If G is a free product of the G_α, given $g \in G$ there is a unique reduced word in the G_α, called $(g_1, \ldots, g_n)$ such that $g_1 \cdots g_n = g$. Define $\Phi(g) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$ where $g_j \in G_{\alpha_j}$. To check that this is a homomorphism, take some h represented by reduced word $(h_1, \ldots, h_m)$ with $h_j \in G_{\beta_j}$. We have

$$\Phi(g)\Phi(h) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n) \varphi_{\beta_1}(h_1) \cdots \varphi_{\beta_m}(h_m) \tag{10}$$

Consider the word $w = (g_1, \ldots, g_n, h_1, \ldots, h_m)$. This word represents gh, which has a unique reduced representation. As was explained earlier, we can obtain this unique reduced representation by performing a finite sequence of manipulations to w:

$$(w_1, \ldots, w_\ell) \mapsto (w_1, \ldots, w_{n-1}, w_n w_{n+1}, w_{n+2}, \ldots, w_\ell) \quad \text{when } \gamma_n = \gamma_{n+1} \text{ and } w_n w_{n+1} \neq 1 \tag{11}$$
$$(w_1, \ldots, w_\ell) \mapsto (w_1, \ldots, w_{n-1}, w_{n+2}, \ldots, w_\ell) \quad \text{when } \gamma_n = \gamma_{n+1} \text{ and } w_n w_{n+1} = 1 \tag{12}$$

It is easy to see that none of these moves

□

Remark II.1. Conversely, if G satisfies this extension property for some collection of homomorphisms $\varphi_\alpha : G_\alpha \to H$ from subgroups, then G is a free product of the G_α. This proof is harder.

Definition II.3 (External free products)**.**

We have defined what an external free product of a collection of groups is, but the question still remains as to whether they even *exist*. As it turns out, they do.

Proposition II.2 (Existence of external free products)**.**

Remark II.2.

Proposition II.3 (Extension property of external free products)**.**

III. Section 1.3

IV. Chapter 2

A. Section 2.2

Solution IV.1 (Problem 2.2.43)**.** First, let us state what it means for a chain complex to split into a direct sum. If we consider the chain complex of groups C_n, it splits into a direct sum of chain complexes of groups A_n and B_n if we have a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B_{n-1} \arrow[r] & B_n \arrow[r] & B_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

where each of the vertical sequences is short exact and splits, that is $C_n \simeq A_n \oplus B_n$ for all n. As a first step, consider the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\partial) \arrow[r, "\iota"] & C_n \arrow[r, "\partial"] & \mathrm{Im}(\partial) \arrow[r] & 0
\end{tikzcd}$$

where the C_n are free Abelian groups. Because C_n is free Abelian, it has a basis $\{x_\alpha^{(n)}\}_{\alpha \in J}$ where each subgroup $C_{n,\alpha} = \langle x_\alpha^{(n)} \rangle$ is infinite cyclic and C is the direct sum of the $C_{n,\alpha}$. Let $I \subset J$ be the collection of α such that $\partial x_\alpha^{(n)} = 0$ and let $K = J - I$. We define a family of homomorphisms $\phi_\alpha : C_{n,\alpha} \to \mathrm{Ker}(\partial)$ as follows: if $\alpha \in I$, then $\phi_\alpha = \iota$, inclusion, and if $\alpha \in K$, then $\phi_\alpha = 0$. It then follows from the extension property of the direct sum that we can extend this family of homomorphisms uniquely to a homomorphism $\Phi : C_n \to \mathrm{Ker}(\partial)$ which restricts to the ϕ_α on the subgroups. It is clear that $\Phi \circ \iota = \mathrm{id}$, so from the splitting lemma, our short exact sequence splits and we have $C_n \simeq \mathrm{Ker}(\partial) \oplus \mathrm{Im}(\partial)$.

This procedure gives us the following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(\partial_{n+1}) \arrow[r, "\iota"'] \arrow[d] & \mathrm{Ker}(\partial_n) \arrow[r] \arrow[d, "\iota"] & 0 & \\
0 \arrow[r] & \mathrm{Im}(\partial_{n+1}) \arrow[r, "\iota"] \arrow[d] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & \mathrm{Ker}(\partial_{n-1}) \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & \mathrm{Im}(\partial_n) \arrow[r, "\iota"] & \mathrm{Ker}(\partial_{n-1}) \arrow[r] & 0
\end{tikzcd}
$$

If we let $K_n = \mathrm{Ker}(\partial_n)$ and $L_{n+1} = \mathrm{Im}(\partial_{n+1})$, then this diagram becomes

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & L_{n+1} \arrow[r] \arrow[d] & K_n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & L_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & K_{n-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & L_n \arrow[r] & K_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

so that the direct sum of the complexes in the bottom and top row is $0 \to L_{n+1} \to C_n \to K_{n-1} \to 0$. We can repeat this process inductively to write the entire chain complex as an iterated direct sum of length-at-most-2 subcomplexes.

Moving on to Part B, let us assume that each of the C_n are finitely generated.