Meromorphic connections and the Stokes phenomenon

Jack Ceroni*
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I. Introduction

This is a set of lecture notes written while taking Marco Gualtieri's course at the University of Toronto in Winter 2024, *Topics in geometry: Meromorphic connections and the Stokes phenomenon.*

II. Lecture 1

The first goal is to describe the Riemann-Hilbert correspondence. We let X be a complex manifold. Let $V \to X$ denote a holomorphic vector bundle, which is to say a vector bundle where the transition functions $\varphi_U \circ \varphi_V^{-1}$ of the local trivialization are holomorphic. Let $\mathrm{Tot}(V)$ be the total space of the bundle.

We begin with providing two definitions for a *connection*. Let $\mathcal{V}$ be the space of all local sections of $V \to X$ (which are simply continuous right inverses of $\pi : \mathrm{Tot}(V) \to X$). We define a connection $\nabla : \mathcal{V} \to \Omega^1(X) \otimes \mathcal{V}$ to be a $\mathbb{C}$-linear map such that

$$\nabla(fh) = df \otimes h + f \nabla h \tag{1}$$

for all holomorphic functions f on X and holomorphic sections $h \in \mathcal{V}$. We emphasize that addition makes sense here, as $h, h' \in \mathcal{V}$ take values $h(x), h'(x)$ inside the same fibre above some point x, which has vector space structure, so $\Omega^{-1}(X) \otimes \mathcal{V}$ has vector space structure.

A connection allows us to define the notion of a derivative along a vector field $Y \in \mathfrak{X}(X)$. Namely, given Y, a choice of a tangent vector for each tangent space, there is of course a natural induced map $Y : \Omega^1(X) \otimes \mathcal{V} \to \mathcal{V}$. We define $\nabla_Y = Y \circ \nabla : \mathcal{V} \to \mathcal{V}$. Going forward, let $\mathrm{Conn}(V)$ denote the space of connections. Note that this is an affine space for $\Omega^1(X, \mathrm{End}(V))$, which is taken to be the space of 1-forms which take on values in $\mathrm{End}(V)$ upon evaluation on tanget spaces. This is due to the fact that if we have $\nabla, \nabla' \in \mathrm{Conn}(V)$, then

$$\nabla - \tag{2}$$

* jack.ceroni@mail.utoronto.ca