MAT436 problem set 3

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I. Problem 1 (First suggested problem)

Suppose first that U is unitary. Of course, U is bijective, so it is surjective. Moreover, since we are in a Hilbert space, we assume the norm is induced from the inner product and get

$$||Uv||_{H_2} = \sqrt{\langle Uv, Uv \rangle_{H_2}} = \sqrt{\langle U^* Uv, v \rangle_{H_1}} = \sqrt{\langle v, v \rangle_{H_1}} = ||v||_{H_1} \tag{1}$$

which implies that U is an isometry. Now, assume that U is a surjective isometry. We need to show that U is angle-preserving. Indeed, we have the polarization identity (assuming the inner product is anti-linear in the second argument, where we take the Hilbert space to be, generally, complex)

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k ||x + i^k y||^2 \tag{2}$$

so we have, since U is an isometry,

$$\langle Ux, Uy \rangle_{H_2} = \frac{1}{4} \sum_{k=0}^{3} i^k ||Ux + i^k Uy||_{H_2}^2 = \frac{1}{4} \sum_{k=0}^{3} i^k ||U(x + i^k y)||_{H_2}^2 = \frac{1}{4} \sum_{k=0}^{3} i^k ||x + i^k y||_{H_1}^2 = \langle x, y \rangle_{H_1} \tag{3}$$

as desired. Now, suppose the third statement holds, and we have U which is surjective and angle-preserving. U will be injective as if $Ux = 0$, then $\langle Ux, Ux \rangle_{H_2} = 0$, implying $\langle x, x \rangle_{H_1} = 0$, implying $x = 0$. Note that U^{-1} will preserve inner products as

$$\langle U^{-1}v, U^{-1}v \rangle_{H_1} = \langle UU^{-1}v, UU^{-1}v \rangle_{H_2} = \langle v, v \rangle_{H_2} \tag{4}$$

Thus, we have

$$\langle (U^{-1} - U^*)v, (U^{-1} - U^*)v \rangle = (\langle U^*v, U^*v \rangle - \langle U^{-1}v, U^*v \rangle) + (\langle U^{-1}v, U^{-1}v \rangle - \langle U^*v, U^{-1}v \rangle) \tag{5}$$

$$= \langle U^*v, U^*v \rangle - \langle v, UU^*v \rangle + \langle U^{-1}v, U^{-1}v \rangle - \langle v, v \rangle = 0 \tag{6}$$

so $U^{-1}v = U^*v$ for all v, implying $U^{-1} = U^*$, and U is unitary.

II. Problem 2 (Second suggested problem)

Part A. Note that S is not surjective as its image contains no element of the form $(1, x_1, x_2, \dots)$, so it cannot be unitary. On the other hand,

$$||Sx|| = \sqrt{0^2 + \sum_{j=1}^{\infty} |x_j|^2} = \sqrt{\sum_{j=1}^{\infty} |x_j|^2} = ||x|| \tag{7}$$

so the map is an isometry.

Part B. By definition, $\langle Sx, y \rangle = \langle x, S^*y \rangle$. Of course,

$$\sum_{j=1}^{\infty} x_j \overline{y_{j+1}} \tag{8}$$

so we have S^* as the left-shift operator which sends $(y_1, y_2, \dots)$ to $(y_2, y_3, \dots)$. To see that S^* has an uncountable number of eigenvalues, pick some $r \in \mathbb{R}$. Define $y = (y_1, y_2, \dots)$ with $y_k = r^k$. We then have

$$Sy = S(r, r^2, \dots) = (r^2, r^3, \dots) = r \cdot (r, r^2, \dots) = ry \tag{9}$$

so r is an eigenvalue. It follows that S^* is not diagonalizable, as for this to be the case, it must be true the case, every vector v can only have non-zero inner product with a countable subset of the eigenvectors of the operator. Clearly, $(1, 0, 0, \dots)$ has non-zero inner product with all $(r, r^2, \dots)$ for $r \neq 0$.

Part C. Note that U is clearly surjective, as given x if we let y be the sequence where $y_n = x_{n-1}$, then $(Uy)_n = x_n$, so $Uy = x$. Moreover, $||Ux|| = ||x||$, clearly. Thus, from Problem 1, we know U is unitary.

Part D. Suppose $Ux = \lambda x$, then we must have $(Ux)_n = x_{n+1} = \lambda x_n$ for all n. It follows by induction that we would have to have $x_n = \lambda^n x_0$. If $\lambda = 0$ or $x_0 = 0$, then we would have $x = 0$. If λ and x_0 are non-zero, then we would have

$$||x||^2 = \sum_{n \in \mathbb{Z}} |x_n|^2 = x_0 + \sum_{n=1}^{\infty} \left(\lambda^n + \frac{1}{\lambda^n} \right) x_0 \tag{10}$$

where $\lambda^n + \lambda^{-n}$ is eventually greater than 1, so the sum does not converge. Thus, $x \notin \ell^2(\mathbb{Z})$. It follows that U has no eigenvectors.

III. Problem 3 (Suggested Problem 3)

Part A. To see that each $T^n V$ is a closed subspace, let $T^n x_j$ be a Cauchy sequence. Then since T is an isometry, $||T^n x_i - T^n x_j|| = ||x_i - x_j||$, so the sequence of x_j is Cauchy as well, in $V = \text{Im}(T)^\perp$. From here, define $\varphi_v(x) = \langle x, v \rangle$. We know that this is a continuous function as a bounded linear map. It is bounded via Cauchy-Schwarz:

$$|\varphi_v(x)| = |\langle x, v \rangle| \leq ||v|| ||x|| \tag{11}$$

Given $v \in \text{Im}(T)$, note that $\varphi_v^{-1}(0) \subset H$ is closed, and is the set of all $x \in H$ which are orthogonal to v. It follows that the intersection $\cap_{v \in \text{Im}(T)} \varphi_v^{-1}(0)$ is closed as well, as the intersection of closed sets. But, of course, this is precisely $\text{Im}(T)^\perp$. Thus, the original sequence x_j converges to $x \in V$. So, since $||T^n x_j - T^n x|| = ||x_j - x||$, it is easy to see that $T^n x_j \to T^n x$, so $T^n V$ is closed as well.

In addition, note that for $n > m$, from the polarization identity,

$$\langle T^n x, T^m y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k ||T^n x + i^k T^m y||^2 = \frac{1}{4} \sum_{k=0}^{3} i^k ||T^{n-m} x + i^k y||^2 = \langle T^{n-m} x, y \rangle = 0 \tag{12}$$

as $y \in \text{Im}(T)^\perp$ and $T^{n-m} x \in \text{Im}(T)$ for $n > m$.

Part B. First, let us show that $\text{Im}(T)$ is closed. Indeed, let Tx_n be a Cauchy sequence. Then since $||Tx_n - Tx_m|| = ||x_n - x_m||$, it follows that x_n is Cauchy as well and this converges to some x. Then since $||Tx_n - Tx|| = ||x_n - x||$, it is easy to see that $Tx_n \to Tx$ as well. Thus, $H = \text{Im}(T) \oplus \text{Im}(T)^\perp$. Suppose $v \in H_u$, so it is perpendicular to every $T^n V = T^n(\text{Im}(T)0^\perp$, including the $n = 0$ case. It follows that we must have $v \in \text{Im}(T)$, so $v = Tw$ for some w. Note that we must also have $w \in H_u$, as if we had $\langle w, Tz \rangle \neq 0$ for some z, then we would have

$$\langle Tw, T^2 z \rangle = \langle v, T^2 z \rangle \neq 0 \tag{13}$$

as T preserves the inner product, as we proved earlier, so $v \notin H_u = H_s^\perp$, as $T^2 z \in H_s$, a contradiction. Thus, $H_u \subset T(H_u)$. To prove that $T(H_u) \subset H_u$, simply pick some $v \in H_u$, and look at $\langle Tv, T^n z\rangle$ for each $n \geq 0$ and $z \in V$. For $n \geq 1$, this is equal to $\langle v, T^{n-1}z\rangle = 0$, as $v \in H_s^\perp$. As for $n = 0$, then $T^0 z = z \in V = (\mathrm{Im}(T))^\perp$, so

$$\langle Tv, T^0 z\rangle = \langle Tv, z\rangle = 0 \tag{14}$$

as $Tv \in \mathrm{Im}(T)$. Thus, we have inclusion both ways so $T(H_u) = H_u$. We have $T(H_s) \subset H_s$ immediately from the definition of H_s, clearly. To see that $T|_{H_u} : H_u \to H_u$ (which we showed is well-defined, as $T(H_u) = H_u$) is unitary, simply note that it is surjective and an isometry, so this follows from what we proved in Question 1!

Part C. The idea is to choose some orthonormal basis $\{e_i\}_{i\in I}$ for V. From here, let $W_j \subset H_s$ be the Hilbert space generated by all powers of e_j, given by $e_j, Te_j, T^2e_j, \ldots, T^m e_j, \ldots$. Note that from earlier, we have $\langle T^m e_j, T^n e_j\rangle = 0$ for $m \neq n$ and 1 for $m = n$, so these vectors will all be linearly independent, and thus a basis for the space they generate, W_j.

From here, define $A_j : W_j \to \ell^2(\mathbb{N})$ via setting it on our basis to be $A(T^n e_j) = (0, 0, \ldots, 1, 0, \ldots)$, where 1 is at the n-th slot. This map is clearly well-defined. To see that it is a unitary, note that it is a surjective isometry, as $\langle A(T^n e_j), A(T^n e_j)\rangle = 1$. Finally, note that

$$SA_j(T^n e_j) = S(0, 0, \ldots, 0, 1, 0, \ldots) = (0, 0, \ldots, 0, 0, 1, \ldots) = A_j(T^{n+1}e_j) = A_j T|_{W_j}(T^n e_j) \tag{15}$$

so that $SA_j = A_j T|_{W_j}$. Finally, it is obvious that $H_s = \oplus_{i\in I} W_i$: the W_i spaces span all of H_s and the sum of all of them is direct because any two basis elements $T^m e_j$ and $T^n e_i$ will be orthogonal unless $m = n$ and $j = i$, so they are all mutually linearly independent.

IV. Problem 4 (Suggested Problem 6)

Part A. This is basically the mean value property for holomorphic functions. We can write $f(x) = \sum_{n=0}^\infty a_n(x - z)^n$: a series expansion centred at $z \in D$, then using absolute convergence of this sum we have

$$\int_{B(z;r)} f(x+iy)\; dx \wedge dy = \int_0^r \int_0^{2\pi} f(z + se^{it})\,(\cos(t)ds - s\sin(t)dt) \wedge (\sin(t)ds + s\cos(t)dt) \tag{16}$$

$$= \int_0^r \int_0^{2\pi} sf(z+se^{it})dt\; ds = \sum_{n=0}^\infty a_n \int_0^r \int_0^{2\pi} s^{n+1}e^{int}dr \wedge dt = \frac{2\pi r^2 a_0}{2} = \pi r^2 a_0 = \pi r^2 f(z) \tag{17}$$

where $a_0 = f(z)$ due to the form of the power series representation. Thus,

$$f(z) = \frac{1}{\pi r^2}\int_{B(z;r)} f(x+iy)\; dx \wedge dy \tag{18}$$

as desired.

Part B. Consider the inner product

$$\langle f, g\rangle = \int_{B(z;r)} f(x)\overline{g(x)}\; dx \tag{19}$$

It follows that from Cauchy-Schwarz, we have

$$\int_{B(z;r)} |f(x)|\; dx = \langle |f|, 1\rangle \leq ||f||||1|| = \sqrt{\int_{B(z;r)} |f(x)|^2\; dx}\sqrt{\int_{B(z;r)} 1\; dx} = \sqrt{\pi}r||f||_{L^2(B(z;r))} \tag{20}$$

which means that

$$|f(z)| = \left| \frac{1}{\pi r^2} \int_{B(z;r)} f(x+iy)\ dx \wedge dy \right| \leq \frac{1}{\pi r^2} \int_{B(z;r)} |f| \leq \frac{1}{\sqrt{\pi} r} ||f||_{L^2(B(z;r))} \tag{21}$$

as desired.

Part C. We need to show that this space is complete. Let f_n be a Cauchy sequence in L^2-norm. We have, for above, $|f_n(z) - f_m(z)| \leq C||f_n - f_m||_{L^2(C)}$ for all $z \in C$ a compact set. Thus, in arbitrary $\overline{B_\rho(0)} \subset D$, closed balls in D with $\rho < 1$, the sequence of f_n is Cauchy is uniform norm, and thus converges to a square-integrable function in $A^2(D)$.

Part D. Note that these functions f_n have dense span as every holomorphic function has a power series representation in terms of these powers. We also note that

$$\langle f_n, f_m \rangle = \frac{((m+1)(n+1))^{1/2}}{\pi} \int_Z z^{n-m} dA \tag{22}$$

which will clearly be 0 for $n \neq m$, via the same reasoning/calculation as Part A, and for $n = m$, will be $\sqrt{n+1}$.