Real analysis problems and solutions

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Contents

I. Week 1

Problem I.1 (Pugh Problem 2.13)**.** This statement is actually an if and only if. Suppose f is continuous. Suppose $p_n \to p$. We can choose δ such that $d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon$, for any ε. Since $p_n \to p$, take N such that $n \geq N$ implies $d(p_n, p) < \delta$, so for $n \geq N$, $d(f(p_n), f(p)) < \varepsilon$. Thus, $f(p_n) \to f(p)$.

Conversely, if p_n converging implies that $f(p_n)$ converges, then suppose for the sake of contradiction that f is not continuous at x. Thus, for some $\varepsilon > 0$, there exists some $y_n \in B_{1/n}(x)$ where $d(f(y_n), f(x)) \geq \varepsilon$. Define the sequence z_n which alternates between y_n and x, so $z = (y_1, x, y_2, x, y_3, x, \dots)$. Clearly, $z_n \to x$, so $f(z_n)$ must converge as well. But it cannot, as the y_n will always be ε-away from $f(x)$.

Problem I.2 (Pugh Problem 2.14)**.** Note that an isometry is Lipschitz, and has a Lipschitz inverse, so it is a homeomorphism. The reason why $[0,1]$ is not isometric to $[0,2]$ is because any bijection will necessarily have to send some points $x, y \in [0,1]$ to 0 and 2 respectively, and $d(x,y) \leq 1$ while $d(2,0) = 2$.

Problem I.3 (Pugh Problem 2.19)**.** I'm assuming we are topologizing these sets as subspaces of $\mathbb{R}$. No, they are not homeomorphic, as one-point sets are open in $\mathbb{N}$, and their inverse images, one point sets in $\mathbb{Q}$, are not open, as any open $U \subset \mathbb{R}$ will intersect an infinite number of elements of $\mathbb{Q}$.

Problem I.4 (Pugh Problem 2.20)**.** Take $f(x) = (1-x)^{-1} - 1$ on $[0,1)$ and $f(x) = 1 - (1+x)^{-1}$ on $(-1,0]$. This function is clearly continuous and invertible with continuous inverse. Moreover, it goes to $\pm\infty$ and $x \to \pm 1$, so its image is $\mathbb{R}$ by the intermediate value theorem. Thus, f is a homeomorphism.

This claim is true, we map $(a,b) \mapsto (0,1)$ via $f(x) = \frac{x-a}{b-a} + b$. Checking this is a homeomorphism is trivial.

Problem I.5 (Pugh Problem 2.22)**.** This is true. Let $\{x_n\}$ be a Cauchy sequence. Then it is clearly bounded, so that it is contained in a closed/bounded set C, which is compact by assumption. It follows that this sequence will contain a subsequence $y_k = x_{n_k}$ which will converge to $y \in C$. We claim that $x_n \to y$ as well.

For some $\varepsilon > 0$, we can pick N such that for $k \geq N$, we have $|y_k - y| = |x_{n_k} - y| < \varepsilon$. We can also choose M such that for $i, j \geq n_M$, we have $|x_i - x_j| < \varepsilon$. We take $R = \max\{N, M\}$, so that for $k \geq R$, we have

$$|x_k - y| \leq |x_k - y_R| + |y_R - y| = |x_k - x_{n_R}| + |y_R - y| \leq 2\varepsilon \tag{1}$$

where we use the fact that $n_R \geq n_M$ and $R \geq N$. It follows by definition that $x_k \to y$.

Problem I.6 (Pugh Problem 2.37)**.** We use the modified comb function. Let f be the function on $\mathbb{R}$ which takes all irrational numbers and integers to 0, and all other rational numbers r to $1/n$, where n is the smallest n such that $|r - N| \leq 1/n$, where $N \in \mathbb{Z}$.

Clearly, this function is not continuous at any non-integer rational, nor any irrational, but is arbitrarily close to 0 on neighbourhoods of integers, so it is continuous here.

Problem I.7 (Pugh Problem 2.41)**.** Since $||\cdot||:\mathbb{R}^m\to\mathbb{R}$ is continuous and $B=||\cdot||^{-1}([0,1])$, where $[0,1]$ is closed, it follows that B is closed. Now, let $K=\{||x||_2=1\}$, a compact set. Note that $||\cdot||$ will take on a minimum value on K, and it will be non-zero as $0\notin K$. Thus, for all $x\in K$, $m\le||x||$. Then, for any $x=||x||_2\widetilde{x}$ where $\widetilde{x}\in K$, we have $||x||=||x||_2||x||\ge m||x||_2$. Thus, if $||x||\le 1$, then $||x||_2\le 1/m$, so B is bounded

Thus, B is closed and bounded in Euclidean space, so it is compact.

Problem I.8 (Pugh Problem 2.43)**.** Projection maps are continuous. The image of compact sets under continuous maps are comapct. The claim follows immediately.

Problem I.9 (Pugh Problem 2.44)**.** Suppose f is continuous, we have $\Gamma_f=\{(x,f(x))\in M\times Y\mid x\in M\}$. For some $(x,y)\notin\Gamma_f$, we choose open U and V disjoint in Y (as the space is Hausdorff), containing y and $f(x)$ respectively. Since f is continuous, $W=f^{-1}(V)$ is open in M. $W\times U$ is open in $M\times Y$, it contains (x,y), and it is disjoint from Γ_f, as $(z,f(z))\in W\times U$ would imply that $f(z)\in V$, which is disjoint from U, a contradiction. Thus, $M\times Y-\Gamma_f$ is open, so Γ_f is closed.

If M is compact, $\Gamma_f=(\mathrm{id}\times f)(M)$ is the image of a compact set under a continuous map, and is thus compact.

Now, suppose Γ_f is compact. Let x_n be a sequence of points: this induces a sequence $(x_n,f(x_n))$ is the graph. Since Γ_f is compact, this sequence has a subsequence convering to some $(y,f(y))$ in the graph. Since $x_n\to x$, we must have $y=x$.

Now, suppose the entire sequence $(x_n,f(x_n))$ doesn't converge to $(x,f(x))$, so there is an ε where for any N, we can always choose $n\ge N$ where $d((x_n,f(x_n)),(x,f(x))\ge\varepsilon$. We form a sequence of elements of this form, and note that a subsequence must also converge to some $(z,f(z))$, but again, we must have $z=x$, so $f(z)=f(x)$, contradicting the fact that this sequence is always at least ε-separated from $(x,f(x))$.

Let $f:[0,1]\to\mathbb{R}$ be defined as $f(x)=1/x$ for $x\in(0,1]$ and $f(0)=0$. The set $f((0,1])$ is closed because for any point p in its complement, the value $\inf_{y\in\Gamma_f}d(p,y)$ is always non-zero, so p is in a ball in the complement. The single point $(0,0)$ is closed, so the union of the two sets is closed.

Problem I.10 (Pugh Problem 2.46)**.** Note that $d:A\times B\to\mathbb{R}$ is a continuous map from a compact set to $\mathbb{R}$, so extreme value theorem implies d has a minimum.

Problem I.11 (Pugh Problem 2.52)**.** For each n, pick some $x_n\in A_1\cap\cdots\cap A_n$ (there is an x_n because the sets are non-empty nested). Since $\mathrm{diam}(A_n)\to 0$, the sequence is Cauchy, as for $n,m\ge N$, $x_n,x_m\in A_N$, so $d(x_n,x_m)\le\mathrm{diam}(A_N)$. Thus, $x_n\to x$. Note that $x\in A$ as if $x\notin A_N$ for some N, then $x\in A_N^C$, an open set, so it is distance at least some ε from x_n for all $n\ge N$, a contradiction to $x_n\to x$. Finally, note that x is the only point in A as if there were two unique ones, $d(x,y)=\varepsilon>0$, but $\mathrm{diam}(A_n)\to 0$, so this can't be.

Problem I.12 (Pugh Problem 2.53)**.** Note that $d:K_n\times K_n\to\mathbb{R}$ is continuous from a compact, so it achieves its maximum at $(a_n,b_n)\in K_n\times K_n$. By definition, $\mathrm{diam}(K_n)=d(a_n,b_n)$. Since K_1 is compact, this sequence has a convergent subsequence, going to some (a,b). We must have $(a,b)\in K\times K$ as if it isn't in some $K_n\times K_n$, then it is in open $(K_n\times K_n)^C$, and thus separated from $(a_m,b_m)\in K_n\times K_n$ for $m\ge n$. We then have, by continuity, $d(a,b)=\lim_{n\to\infty}d(a_n,b_n)\ge\mu$, so $\mathrm{diam}(K)\ge\mu$.

Problem I.13 (Pugh Problem 2.54)**.** For $x\in\overline{S}-S$, we pick (any) sequence $x_n\to x$ with $x_n\in S$. Each $f(x_n)$ is in N.

By uniform continuity, for some $\varepsilon>0$, there exists some $\delta>0$ such that for $d(x,y)<\delta$, we have $d'(f(x)-f(y))<\varepsilon$. Thus, we pick N such that for $n,m\ge N$, $d(x_n,x_m)<\delta$, which implies $d'(f(x_n)-f(x_m))<\varepsilon$. Thus, $f(x_n)$ is Cauchy, so it converges to some limit y. We define $\overline{f}(x)=y$.

This function is uniformly continuous: if $\varepsilon>0$, pick δ such that $d(x,y)<\delta$ implies $d'(f(x)-f(y))<\varepsilon$ for $x,y\in S$. For $x',y'\in\overline{S}$, pick $x,y\in S$ which are δ-close to x' and y' respectively, and note

$$d'(f(x')-f(y'))\le d'(f(x')-f(x))+d'(f(y')-f(y))+d'(f(x)-f(y))<3\varepsilon$$

and we are done.

$\overline{f}$ is unique: if g is another one then $k(x)=d'(\overline{f}(x),g(x))$ sends S to 0. Given any $x\in\overline{S}$, let $x_n\to x$. By continuity, $k(x_n)\to k(x)$, but $k(x_n)=0$ for all x_n, so $k(x)=0$ as well and $\overline{f}=g$.

Problem I.14 (Pugh Problem 2.55)**.** Suppose $\mathrm{dist}(p,S)=0$. Then for each $1/n$, there must exists some $p_n\in S$ where $d(p,p_n)<1/n$. It follows immediately by definition that $p_n\to p$, so p is a limit. Conversely, if p is a limit of p_n, then for each $\varepsilon > 0$, there is N where $n \geq N$ implies $d(p, p_n) < \varepsilon$. In particular, we can find points of S arbitrarily close to p, so the infimum of distances over S must be 0.

Suppose $d(x, y) < \delta$. Then for any $p \in S$, we have $\text{dist}(x, S) \leq d(x, p) \leq d(y, p) + d(x, y) \leq d(y, p) + d(x, y)$. This holds for any $p \in S$, so $\text{dist}(x, S) \leq \text{dist}(y, S) + d(x, y) < \text{dist}(y, S) + \delta$. Thus, $|\text{dist}(x, S) - \text{dist}(y, S)| < \delta$, and uniform continuity is immediate.

Problem I.15 (Pugh Problem 2.58a)**.** No, $(0, 1) \cup (1, 2)$ is disconnected but its closure $[0, 2]$ is connected.

Problem I.16 (Pugh Problem 2.60)**.** Suppose f is continuous and f is non-constant. Suppose f takes on distinct values $p, q \in \mathbb{Z}$ with $p < q$. It is clear that $f(M) \cap (-\infty, p + (1/2))$ and $f(M) \cap (p + (1/2), \infty)$ form a separation of $f(M)$ with both non-empty as p is in the first and q is in the second, a contradiction, as M is connected so $f(M)$ must be as well.

If the values are all irrational and f is non-constant, we pick a rational p between two values that f takes on and use the same argument to derive a contradiction.

Problem I.17 (Pugh Problem 2.65)**.** Obviously, $\text{Int}(A) \subset A$ so $\overline{\text{Int}(A)} \subset \overline{A} = A$. To prove the other inclusion, note that A has non-empty interior, so we can pick $a \in \text{Int}(A)$. In fact, it can't just be a single point, as the interior is open and single-point sets are closed, so let $a, b \in \text{Int}(A)$ with $a \neq b$. Let $x \in A - \text{Int}(A)$. Then the triangle T formed by distinct points a, b, x is contained in A. It follows that a neighbourhood of x will intersect $\text{Int}(T) \subset \text{Int}(A)$, so $x \in \overline{\text{Int}(A)}$. Thus, we have shown $A = \overline{\text{Int}(A)}$.

To prove the second part, pick some $a \in A$, note that A is the union of all line segments extending from a to some other $b \in A$. Each line segment will lie on a ray passing through a. Note that since A is compact, each of these rays must contain a unique, single point of ∂A (which is a curve in the plane). Let $\gamma(\theta)$ be the Blah blah blah line argument

Problem I.18 (Pugh Problem 2.76 (a, b, c))**.** $S^1 - \{p\}$ and $S^1 - \{q\}$ for p and q distinct has intersection $S^1 - \{p, q\}$, which is not connected.

No, suppose we take $S_k = \{(x, y) \mid x = 0, 1\} \cup \{(x, y) \mid x \in [0, 1],\ y \geq k\}$. Of course, each S_k is closed and connected, and they are nested, but their infinite intersection is $\{x = 0, 1\}$ which is not connected.

If each S_k is additionally compact, **TODO**

Problem I.19 (Pugh Problem 2.96)**.** If $A \subset B \subset C$, then $\overline{A} = B$ and $\overline{B} = C$. But of course, the closure of the closure is the closure, as a closure is closed, so $\overline{A} = \overline{B} = C$, and A is dense in C by definition.

Problem I.20 (Pugh Problem 2.97)**.** Yes. For some positive integer n and some integer q, note that (via the division algorithm), we can always choose some k_n such that $|2^n - k_n q| < q$. It immediately follows that for some rational p/q,

$$\left| \frac{p}{q} - \frac{pk_n}{2^n} \right| = \frac{|p||2^n - k_n q|}{|q|2^n} < \frac{|p|}{2^n} \tag{2}$$

We can do this for any n, so we can find a dyadic which is arbitrarily close to p/q. Thus, they are dense. This immediately implies density in $\mathbb{R}$, as $\mathbb{Q}$ is dense in $\mathbb{R}$.

Problem I.21 (Pugh Problem 2.101)**.**

Problem I.22 (Pugh Problem 2.103)**.** This follows pretty easily from the space-filling curve that already exists from $[0, 1]$ to B^2.

Problem I.23 (Pugh Problem 2.106)**.** He won't ask this

Problem I.24 (Pugh Problem 2.107)**.** If a sequence $p_n \in \overline{M} \subset S$ is Cauchy, then p_n converges to some $p \in S$. Since $\overline{M}$ is closed, $p \in \overline{M}$, so any Cauchy p_n in $\overline{M}$ converges in $\overline{M}$. Clearly, $M \subset \overline{M}$, and $\overline{M}$ extends the metric on M, which is defined on all of S, so we have a completion.

Problem I.25 (Pugh Problem 2.108)**.** He won't ask this

Problem I.26 (Pugh Problem 2.115)**.** Part A is trivial: if it's rational then after a finite number of iterations, we will effectuate a map of the form $(1, \theta) \mapsto (1, \theta + 2\pi N)$ for some $N \in \mathbb{Z}$, which is simply the identity.

For Part B TODO

II. Week 2

Problem II.1 (Pugh Prelim 2.2)**.** Continuous functions on compacts are uniformly continuous. Thus, for some $\varepsilon > 0$, we can pick δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon < M|x - y| + \varepsilon$. for any choice of M. Additionally, the function from a compact set $X \times X \to \mathbb{R}$ defined by $(x, y) \mapsto |f(x) - f(y)|$ will take on a maximum value R. We then have for $|x - y| \geq \delta$, we have $|f(x) - f(y)| \leq R \leq R\delta^{-1}|x - y| + \varepsilon$, so we set $M = R\delta^{-1}$, and we are done.

Problem II.2 (Pugh Prelim 2.5)**.** Note that $[0, 1] \times [0, 1]$ is compact. Pick δ such that for $||(x, y) - (x', y')|| < \delta$, we have $|f(x, y) - f(x', y')| < \varepsilon$. Suppose $g(x) = f(x, y)$ for some y and $g(x') = f(x', y')$ for some y'. We have $|f(x, y) - f(x', y)| < \varepsilon$ and $|f(x, y') - f(x', y')| < \varepsilon$, so that $g(x') \geq f(x', y) > f(x, y) - \varepsilon = g(x) - \varepsilon$ and $g(x) \geq f(x, y') > f(x', y') - \varepsilon = g(x') - \varepsilon$.

Thus, $|g(x) - g(x')| < \varepsilon$. It follows that g is continuous.

Problem II.3 (Pugh Prelim 2.7)**.** To prove it is bounded, cover $[0, 1]$ in a finite number of δ-balls, $B_1, \ldots, B_n$, pick $x_k \in B_k$. Note that any $y \in [0, 1]$ is δ-close to some x_k, so $f(y) < f(x_k) + \varepsilon$. Thus, if M upper-bounds the sets of $f(x_k)$, then f is bounded by $M + \varepsilon$.

Now, let $A = f([0, 1])$. Consider $U_a = f^{-1}(-\infty, a)$ for some $a \in A$. To show that this set is open, note that if $x \in U_a$, then $a - f(x) = \varepsilon$. Pick δ so that $|y - x| < \delta$ implies $f(y) < f(x) + \varepsilon < a$ so $y \in (\infty, a)$. Clearly, the collection of all U_a will cover $[0, 1]$ if A has no maximum element. But then $[0, 1]$ is covered by a finite number of the U_{a_k}. Let U_{a_N} be the one with the largest a_N: then a_N is not in the union of the image of the U_{a_k}, contradicting the fact that they must cover A. So there must be a maximal element.

Problem II.4 (Pugh Prelim 2.11)**.** $\mathbb{R}^n$ has no proper clopen sets as it is connected, so we must prove U is closed. Suppose $x_n \to x$ in $\mathbb{R}^m$, so it is Cauchy. Since the f is uniformly continuous, $f(x_n)$ is also Cauchy (given ε choose $n, m \geq N$ such that $|x_n - x_m| < \delta$, which implies $|f(x_n) - f(x_m)| < \varepsilon$). Thus, $f(x_n) \to y$, so again by continuity, $x_n \to f^{-1}(y)$, so $x = f^{-1}(y)$, implying that $x \in U$.

Problem II.5 (Pugh Prelim 2.14)**.** Pick M such that if $|x - y| < 1$, then $|f(x) - f(y)| < M$. It follows that for integers $n \in \mathbb{Z}$, $|f(n) - f(0)| \leq \sum_{k=1}^{n} |f(k) - f(k - 1)| < M|n|$. For some arbitrary x, it follows that $|f(x) - f(0)| < |f(x) - f(n)| + M|n| < (M + 1)|n| \leq (M + 1)|x| + (M + 1)$, where n is the closest integer to x, so $|n| \leq |x| + 1$. Thus, $|f(x)| \leq |f(0)| + (M + 1) + (M + 1)|x|$.

Problem II.6 (Pugh Prelim 2.15)**.** Let $z_k = f(1 - (1/k))$. Note that this sequence is Cauchy: if ε is chosen, then there exists some $1/N$ such that for $|x - y| < 1/N$, $|f(x) - f(y)| < \varepsilon$. In particular, for $n, m \geq N$, $\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{N}$, so $|z_n - z_m| < \varepsilon$. Thus, $z_k \to z$ in $\mathbb{R}$. We define $g(1) = z$.

If we choose w which is $(1/N)$-close to 1, then we will have

$$|f(w) - g(1)| \leq |f(w) - f(1 - (1/N))| + |g(1) - f(1 - (1/N))| < 2\varepsilon \tag{3}$$

so g is continuous.

This function is the only possible continuous extension, as any such function would necessarily have $z = \lim_k z_k = \lim_k g(1 - (1/k)) = g(\lim_k(1 - (1/k))) = g(1)$

Problem II.7 (Pugh Problem 4.2)**.** Pick N such that $|f_N(x) - f(x)| < \varepsilon$ for all x. Pick δ such that $|x - a| < \delta$ implies $|f_N(x) - f_N(a)| < \varepsilon$. Then,

$$|f(x) - f(a)| < |f_N(x) - f(x)| + |f_N(x) - f_N(a)| + |f(a) - f_N(a)| < 3\varepsilon \tag{4}$$

so f is continuous at a, and is thus continuous everywhere.

Problem II.8 (Pugh Problem 4.3a)**.** It's another repeat of the above proof...

Problem II.9 (Pugh Problem 4.4)**.** This is true. As usual, for some ε, pick N such that $|f(x) - f_N(x)| < \varepsilon$ everywhere. Then for any $|x - y| < \delta$ with $|f_N(x) - f_N(y)| < \varepsilon$, we have $|f(x) - f(y)| < |f(x) - f_N(x)| + |f_N(y) - f(y)| + |f_N(x) - f_N(y)| < 3\varepsilon$.

This also works for arbitrary metric spaces!

Problem II.10 (Pugh Problem 4.6a and 4.6b)**.** The idea is that $[a, b]$ has a countable dense subset, $S = \mathbb{Q} \cap [a, b]$. The function $f \in C^0$ is completely determined by its values on S. Thus, f is determined by a countable collection of real numbers, which has the same cardinality as $\mathbb{R}$.

Actually, a better way to see this is via polynomial approximation. Every function is the limit of a polynomial sequence, which is specified by a countable collection of real numbers.

To construct this bijection, note that $\mathbb{R}$ has the same cardinality as $\{0, 1\}^\omega$. It follows that $\mathbb{R}^\omega$ has the same cardinality as $(\{0, 1\}^\omega)^\omega$

Problem II.11 (Pugh Problem 4.7a)**.** Suppose first that f_n converges uniformly to continuous f, which has compact graph G. For some ε, pick N where $n \geq N$ implies that $|f_n - f| < \varepsilon$. It follows that for $n \geq N$,

$$D(G, G_n) < \varepsilon \tag{5}$$

as $G \in M_\varepsilon G_n$ and $G_n \in M_\varepsilon G$, so that the sequence of G_n converges to G in $\mathcal{K}(\mathbb{R}^2)$. Conversely, if $G_n \to G$, with G the graph of a continuous function f, then for every δ, there exists some N such that for $n \geq N$, the sets G_n and G are δ-close.

Since f is continuous on a compact, it is uniformly continuous. Pick ε. There must exist some δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$, and

Pick N such that for $n \geq N$, the sets G_n and G are δ-close. For some x, note that $(x, f_n(x))$ must be δ-close to some $(y, f(y))$, so $|x - y| < \delta$ and $|f(y) - f(x)| < \varepsilon$ and $|f_n(x) - f(y)| < \delta$.

Problem II.12 (Pugh Problem 4.8)**.** Yes. It is trivial to see that sums of equicontinuous families are as well: given g_n and f_n, if we choose δ and δ' such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon$ and $|x - y| < \delta'$ gives $|g_n(x) - g_n(y)| < \varepsilon$, then $|x - y| < \min\{\delta, \delta'\}$ implies that $|(f_n + g_n)(x) - (f_n + g_n)(y)| \leq 2\varepsilon$.

Clearly, $\cos(n + x)$ is equicontinuous from mean value theorem, $|\cos(n + x) - \cos(n + y)| \leq |x - y|$. So is $h_n(x) = \log(1 + (n + 2)^{1/2} \sin^2(n^n x))$. For some ε, pick N where $n \geq N$ implies $h_n(x) < \varepsilon$. Then, for all the $n < N$, we can uniformly bound the derivatives of the h_n by some M, so that $|x - y| < \varepsilon/M$ implies that $|h_n(x) - h_n(y)| < \varepsilon$ for $n < N$, and automatically for $n > N$.

Thus, the sum is equicontinuous.

Problem II.13 (Pugh Problem 4.9)**.** f must be constant. Suppose $f(a) \neq f(b)$. WLOG, $f(b) - f(a) = \varepsilon > 0$. There must exist δ such that $|x - y| < \delta$ implies $|f(nx) - f(ny)| < \varepsilon$. Choose N such that $\frac{|a-b|}{N} < \delta$, which would imply that $f(b) - f(a) < \varepsilon$ when we plug-in a/N and b/N, a contradiction.

Problem II.14 (Pugh Problem 4.15)**.** Suppose f has modulus of continuity μ. Pick ε, pick δ such that $|x| < \delta$ implies $\mu(x) < \varepsilon$. Then $|s - t| < \delta$ implies $|f(s) - f(t)| \leq \mu(|s - t|) < \varepsilon$ so we have uniform continuity.

Conversely, suppose f is uniformly continuous. Define

$$\mu(t) = \sup\{|f(x) - f(y)| \mid |x - y| \leq t\} \tag{6}$$

Of course, this function is strictly increasing and $\mu(0) = 0$ as $|f(x) - f(y)| = 0$ if and only if $x = y$. Of course, $|f(x) - f(y)| \leq \mu(|x - y|)$. To see that this function is continuous, we use uniform continuity of f. Pick δ such that $|s - t| < \delta$ implies that $|f(s) - f(t)| < \varepsilon$.

Given s and t with $s < t + \delta$, pick $|x - y| < s$. Pick z such that $|x - z| < t$ and $|y - z| < \delta$, so that $|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| < |f(x) - f(y)| + \varepsilon$. Thus, for $|s - t| < \delta$, for any element of the set defining $\mu(s)$, there is an element of the set defining $\mu(t)$ which is at most ε-far from the element of $\mu(s)$. Thus, $\sup \mu(s) < \mu(t) + \varepsilon$, so $|\mu(s) - \mu(t)| < \varepsilon$ Heavy abuse of notation here, but the idea is clear

NOOOOONONONONON

Suppose $\mathcal{E}$ is an equicontinuous family, then there exists δ such that $|x - y| < \delta$ implies that $|f_n(x) - f_n(y)| < 1$. Thus, we define

$$\mu(\tag{7}$$

Problem II.15 (Pugh Problem 4.19)**.** Of course, this is trivial: putting δ-balls around every $a \in A$ covers A as given $x \in M$, the δ-ball around x contains $a \in A$, so the δ ball around a contains x. Taking a finite-subcover (from compactness) gives the desired set (the centres of the balls in the subcover).

Problem II.16 (Pugh Problem 4.21a, 4.21b)**.** Given ε, we can choose δ such that for every $f \in \mathcal{E}$, $|s-t| < \delta$ implies $|f(s) - f(t)| < \varepsilon$. Clearly, $\sup\{f(x)\} - \sup\{f(y)\} = \sup\{f(x) - f(y)\} = \sup\{|f(x) - f(y)|\}$, so if we simply pick $|x - y| < \delta$, we will have $\sup|f(x) - f(y)| \leq \varepsilon$.

To see this fails without equicontinuity, consider the case when $\mathcal{E}$ is the collection of $f_n(x) = nx$ on $[-1/n, 1/n]$ and 1 elsewhere on $[-1, 1]$ (we assume WLOG that $[a, b] = [-1, 1]$). Clearly, $\sup\{f(0)\} = 0$, but $f_n(1/n) = 1$ with $1/n \to 0$, so $\sup\{f(1/n)\} \geq 1$ for each n. These functions are obviously not equicontinuous.

Problem II.17 (Pugh Problem 4.22)**.** How about $f(x) = \frac{1}{n}\sin(n^3 x)$: its derivative is $3n\cos(n^3 x)$, which is clearly not uniformly bounded. Neverthless, given ε, we can choose N such that $2/N < \varepsilon$, so for $n \geq N$, $|f(x) - f(y)|$ is always less than ε. For $n < N$, we choose δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for this finite set of n. Then the family is equicontinuous.

Problem II.18 (Pugh Problem 4.24a)**.** For each $x \in [a, b]$, there is some $N(x)$ such that $f_n(x) - f(x) < \varepsilon$ for all $n \geq N(x)$. Also, pick δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$, pick δ_n such that $|x - y| < \delta_n$ implies $|f_n(x) - f_n(y)| < \varepsilon$. For each x, place a ball of radius $r(x) = \min(\delta_{N(x)}, \delta)$ around it, pick a finite subcover centred at $x_1, \ldots, x_m$. Let M be the largest of the $N(x_k)$.

Given y, note that y is in the $r(x_k)$-neighbourhood of some x_k. Then, for $n \geq M \geq N(x_k)$,

$$f_n(y) - f(y) \leq f_{N(x_k)}(y) - f(y) = \left[f_{N(x_k)}(y) - f_{N(x_k)}(x_k)\right] + \left[f_{N(x_k)}(x_k) - f(x_k)\right] + [f(x_k) - f(y)] \quad (8)$$

Note that $|y - x_k| < r(x_k) < \delta_{N(x_k)}$, so the first term is bounded above by ε. The second term is bounded above by ε, and so is the third. Thus, we ahve uniform continuity.

Problem II.19 (Pugh Problem 4.30)**.** Rotate the circle by a little bit!

Problem II.20 (Pugh Problem 4.32a)**.** We'll use Stone-Weirestrass, by showing that the space of Lipschitz functions forms a function algebra which vanishes nowhere. Note that the sum and scalar product of Lipschitz functions is clearly Lipschitz. As for the product,

$$|f(p)g(p) - f(q)g(q)| \leq |f(p)||g(p) - g(q)| + |g(q)||f(p) - f(q)| \leq Ad(p, q) \quad (9)$$

as f and g are bounded, since the domain is compact. To finally see that this collection of functions vanishes nowhere, note that the constant functions are Lipschitz. To see that it spearates points, note that the functions $f : x \mapsto d(p, x)$ is continuous and separated p and every other point. It is Lipschitz because $|f(x) - f(y)| = |d(p, x) - d(p, y)| \leq d(x, y)$ because $d(p, x) \leq d(p, y) + d(x, y)$ and $d(p, y) \leq d(p, x) + d(x, y)$.

Thus, by Stone-Weirestrass, this set of functions is dense.

Problem II.21 (Pugh Problem 4.37a)**.** Let β_k be a smooth bump function which is equal to 1 in a neighbourhood of the origin and is 0 outside $[-|a_k|^{-1}k^{-2}, |a_k|^{-1}k^{-2}]$ when $a_k \neq 0$, and let $\beta_k = 1$ otherwise. All derivatives of β_k at 0 are 0, so the power series will have the correct derivatives, assuming it converges.

To prove convergence, note that the r-th derivative of $\frac{1}{k!}$

Problem II.22 (Pugh Problem 4.38a)**.** Without loss of generality, suppose the cluster point is 0 (we can achieve this be horizontal shift of the functions). In a neighbourhood of 0, we can write $f = \sum_k a_k x^k$ and $g = \sum_k b_k x^k$. Suppose $f \neq g$, then there must exist some n such that $f - g = x^n h$ where $h(0) \neq 0$ (as equality of analytic functions on a neighbourhood implies global equality).

For each $1/m$ neighbourhood around 0, we know there exists an infinite number of points contained in this neighbourhood in T. For each m, pick one of these points $x_m \neq 0$. Then $x_m \to 0$, and $f(x_m) - g(x_m) = x_m^n h(x_m) = 0$ for each m. Since $x_m \neq 0$, $h(x_m) = 0$ for each x_m, and since h is continuous, $h(0) = 0$, contradicting the assumption that $h(0) \neq 0$. Thus, $f = g$ everywhere.

Problem II.23 (Pugh Problem 4.39)**.** Of course, $d(q, x) \leq d(p, q) + d(p, x)$ and $d(p, x) \leq d(q, p) + d(q, x)$, so $|d(q, x) - d(p, x)| \leq d(p, q)$ impyling boundedness. Also d is continuous so f_q is. Finally, note that if $d(q, r) = \delta$, then $|f_q(x) - f_r(x)| = |d(q, x) - d(r, x)|$. Of course, $|d(q, x) - d(r, x)| \leq d(r, q)$ via the same logic as above, which is saturated at $x = q$ or $x = r$, so $||f_q - f_r|| = \delta$ as well.

Of course, M_0 is dense in its closure, and since C_b^0 is complete and $\overline{M_0}$ is a closed subset, it is complete as well. In particular, $\overline{M_0}$ is a completion of M_0, which is a homeomorphic copy of M isometrically embedded in $\overline{M_0}$.

III. Week 3 solutions

Problem III.1 (Pugh Prelim 4.1)**.** Note that $f_n(x) \to f(x)$ for any x pointwise, as the constant sequence of x repeated converges to x. We want to show that $f(x_n) \to f(x)$. Note

$$|f(x) - f(x_m)| \leq |f(x) - f_n(x_m)| + |f(x_m) - f_n(x_m)| \tag{10}$$

for any m and n. Since we have pointwise convergence, for each k, pick some n_k such that $|f_{n_k}(x_k) - f(x_k)| < \varepsilon$. If we let y be the sequence of x_1 repeated n_1 times, x_2 repeated $n_2 - n_1$ times, and so on, so $y_{n_k} = x_k$, then since $f_k(y_k) \to f(x)$, we have $f_{n_k}(x_k) \to f(x)$ as we take $k \to \infty$. Thus, we take k large enough so that $|f_{n_k}(x_k) - f(x)| < \varepsilon$ and $|f(x_k) - f_{n_k}(x_k)| < \varepsilon$, so $|f(x) - f(x_k)| < 2\varepsilon$.

Problem III.2 (Pugh Prelim 4.3)**.** Clearly, this collection of functions in bounded as $|u(x)| = |u(x) - u(0)| \leq |x| \leq 1$. It is equicontinuous as the function satisfy a Lipschitz condition. It is closed because if $f_n \to f$, and $\frac{|f_n(x) - f_n(y)|}{|x-y|} \leq 1$ for any chosen x, y, then the limit for $n \to \infty$ must repsect this upper bound as well. It follows that the collection of functions we consider is compact. Thus, the function ϕ, which is continuous as

$$|\phi(u) - \phi(w)| \leq d(u, w) + \sup |u - w||u + w| \leq Cd(u, w) \tag{11}$$

is Lipschitz itself! Thus, ϕ attains a maximum on the compact domain.

Problem III.3 (Pugh Problem 4.4)**.** Obviously the functions are continuous. Uniformly bounded derivatives implies that the sequence is equicontinuous. To see that the sequence is also bounded, note that $|g_n(0)| = |g_n'(0)| \leq 1$ for all n. Thus, $|g_n(x)| \leq |g_n(x) - g_n(0)| + |g_n(0)| \leq |x| + 1 \leq 2$ for all x, so the sequence is bounded.

Problem III.4 (Pugh Problem 4.6)**.** Since $f'(0) > 0$, there exists a neighbourhood $(0, \varepsilon)$ such that $f(x)/x > 0$, so $f(x) > 0$ for x in this set. Suppose there exists some $y \geq \varepsilon$ such that $f(y) \leq 0$. By IVT, there must be some $z \geq \varepsilon$ such that $f(z) = 0$.

Let S be the set of all x in $[\varepsilon, \infty)$ such that $f(x) = 0$. This set is non-empty by assumption, and is bounded below, so $r = \inf(S)$ exists, and by continuity, $f(r) = 0$. Note that f is positive on $(0, r)$. It follows from MVT that there exists some $w \in (0, r)$ for which $f'(w) = 0$, so $f(w) < 0$, a contradiction to this claim.

Thus, there cannot exist any $y > 0$ such that $f(y) \leq 0$, so $f(y) > 0$ for all $y > 0$.

Problem III.5 (Pugh 4.12)**.** Trivial via Cauchy-Schwarz:

$$|f(x)| = \left| \int_0^x f'(t)\ dt \right| \leq \left(\int_0^x (f'(t))^2\ dt \right)^{1/2} \tag{12}$$

Taking the supremum over all $x \in [0, 1]$ gives the inequality.

Problem III.6 (Pugh 4.26)**.** Also trivial via Cauchy-Schwarz. Note that

$$\left| \int_0^1 \sqrt{x+y} f_n(y)\ dy \right| \leq \left(\int_0^1 x + y\ dy \right)^{1/2} \left(\int_0^1 (f_n(y))^2 \right)^{1/2} \leq \sqrt{5} \left(x + \frac{1}{2} \right) \leq \frac{3\sqrt{5}}{2} \tag{13}$$

To prove equicontinuity, note that

$$|g_n(x) - g_n(z)| = \left| \int_0^1 (\sqrt{x+y} f_n(y) - \sqrt{z+y} f_n(y)\ dy \right| \leq M \sup |\sqrt{x+y} - \sqrt{z+y}| \leq M \sup |(\sqrt{x+y} + \sqrt{y})(\sqrt{x+1} - \sqrt{z})| \tag{14}$$

$$= M|(x+y) \tag{15}$$

Blah blah blah

Problem III.7 (Pugh Prelim 4.34)**.** Trivial.

Problem III.8 (Pugh Prelim 4.37)**.**

Problem III.9 (Pugh Prelim 4.39)**.** Note that the collection real trigonometric polynomials $\sum_{k=-n}^{n} c_k e^{ikx}$ as a non-vanishing function algebra in the continuous functions which separates points. Hence, we can ε-approximate f with a trig polynomial P. Since P is real, $c_{-n} = \overline{c_n}$. The integral condition implies that $c_{-n} = c_n$ and $c_{-n} = -c_n$ for all $n \neq 0$, so the claim follows. Indeed:

$$\int f(f - P) \tag{16}$$

Problem III.10 (Pugh Prelim 4.41)**.**