An primer on C^*-dynamical systems

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I. Introduction

This essay was written for the Fall 2023 session of MAT437: K-Theory and C^*-Algebras, taught by Professor George Elliott, at the University of Toronto.

The goal of this essay is to introduce some of the main ideas relating to the study of C^*-dynamical systems, and associated constructions, such as crossed products, thermodynamics, and their relevance in K-theory.

II. C^*-dynamical systems and crossed-products

Going forward, we will let $\mathrm{Aut}(A)$ denote the set of $*$-automorphisms We begin with a definition.

Definition II.1 (C^*-dynamical system)**.** A C^*-dynamical system is defined to be a triple (G, A, α), consisting of a locally compact topological group G, a C^*-algebra A, and a continuous group action α of G on A. That is, $\alpha : G \to \mathrm{Aut}(A)$ is a group homomorphism, where for each $a \in A$, the map $g \mapsto \alpha(g)(a)$ is a continuous map, with A of course having its metric topology.

Note that if G is a discrete group, then the topology on G must be discrete, and any group homomorphism $\alpha : G \to \mathrm{Aut}(A)$ is a continuous group action. Going forward, let $\mathcal{H}$ denote a Hilbert space. We say that a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ in $B(\mathcal{H})$ is compact if it takes bounded subset of $\mathcal{H}$ to relatively compact subsets of $\mathcal{H}$ (subsets whose closure under the ambient metric topology are compact). We denote the set of all compact operators as $K(\mathcal{H})$. It is straightforward to see that $K(\mathcal{H})$ is a sub-C^*-algebra of $B(\mathcal{H})$.

Let us recall another fundamental definition:

Definition II.2 (Representations)**.** A representation of a group G (or a C^*-algebra A) on vector space V is a group (or a C^*-algebra) homomorphism $\pi : G \to \mathrm{GL}(V)$ (or $\pi : A \to \mathrm{GL}(V)$), where $\mathrm{GL}(V)$ is the general linear group over V. A *unitary* representation is a representation π on a complex Hilbert space $\mathcal{H}$ such that $\pi(x)$ is unitary for all $x \in G$ (or $x \in A$). A representation π is said to be *nondegenerate* if it satisfies the following property: if $v \in V$ is such that $\pi(x)v = 0$ for all $x \in G$ (or $x \in A$), then $v = 0$.

Given a discrete group G, as well as a unitary representation π of G, there is a natural action $\alpha : G \to \mathrm{Aut}(K(\mathcal{H}))$ given by

$$\alpha(g)(a) = \pi(g)a\pi(g)^* \qquad \text{for } a \in K(\mathcal{H}), g \in G, \tag{1}$$

as we have

$$\alpha(g_1 + g_2)(a) = \pi(g_1 + g_2)a\pi(g_1 + g_2)^* = \pi(g_1)\pi(g_2)a\pi(g_2)^*\pi(g_1) = (\alpha(g_1) \circ \alpha(g_2))(a) \tag{2}$$

which implies that $\alpha(g_1 + g_2) = \alpha(g_1) \circ \alpha(g_2)$, so α is a group homomorphism, and since G is discrete, this is a continuous group action. It follows that $(G, K(\mathcal{H}), \alpha)$ is a C^*-dynamical system. In particular, note that $\alpha(G) \subset \mathrm{Inn}(K(\mathcal{H}))$, the set of inner automorphisms of $K(\mathcal{H})$, as $\pi(g)$ is unitary.

Now, suppose $\xi = (G, A, \alpha)$ is a C^*-dynamical system. Suppose that π is a nondegenerate representation of A on $\mathcal{H}$, suppose U is a unitary representation of G on $\mathcal{H}$. If it holds for all $a \in A$ and $g \in G$ that

$$\pi(\alpha(g)(a)) = U(g)\pi(a)U(g)^* \tag{3}$$

then we call the pair (π, U) a *covariant representation* of ξ.

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Remark II.1. Intuitively, we know that inner automorphisms are particularly nice objects with which to work. The above construction attempts, in a certain sense, to associate a construction resembling an inner automorphism to an arbitrary continuous group action α, for each $g \in G$.

As it turns out, if we are given a C^*-dynamical system (G, A, α), in addition to a representation π of A on $\mathcal{H}$, there is a way to construct a canonical covariant representation which we call the left-regular representation associated with π. First, let us recall some basic facts about Haar integrals.

Remark II.2 (Haar integral over a topological group)**.** Given a locally compact Hausdorff topological group G, it is known that there is a unique left-invariant measure μ over G (up to constant multiplier, with a few other conditions) called the *left-Haar measure*. This allows us to define a Lebesgue integral over a topological group where $f : G \to A$ is some function from G to algebra A, $\int_G f d\mu$. For topological group G and algebra A, let $L^2(G, A)$ denote the space of all square-integrable functions $f : G \to A$, that is f for which

$$\langle f, f \rangle := \int_G f(h) f(h)^* d\mu(h) \tag{4}$$

exists. Let $L^1(G, A)$ denote the space of all integrable function, f for which $\int_G f(h) d\mu(h)$ exists. Note that both $L^2(G, A)$ and $L^1(G, A)$ are, clearly, vector spaces, with the addition/scalar multiplication on functions induced by the algebra A.

We can now define the left-regular representation:

Definition II.3 (Left-regular representation)**.** Given $\xi = (G, A, \alpha)$ and π representing A on $\mathcal{H}$, define $\widetilde{\pi}$ and $\widetilde{\lambda}$ as maps from G and A respectively, to $\mathrm{Aut}(L^2(G, \mathcal{H}))$, as

$$(\widetilde{\pi}(a) f)(g) = \pi(\alpha(g^{-1})(a)) f(g) \tag{5}$$

as well as

$$(\widetilde{\lambda}(g) f)(h) = f(g^{-1} h) \tag{6}$$

Claim II.1. $\widetilde{\pi}$ and $\widetilde{\lambda}$ are representations of G and A on $L^2(G, \mathcal{H})$, respectively. Moreover, $(\widetilde{\pi}, \widetilde{\lambda})$ is a covariant representation of (G, A, α).

Proof. □

From here, let us define a new $*$-algebra from a given C^*-dynamical system.

III. Thermodynamics of C^*-dynamical systems

Now that we have described some of the basic constructions relating to C^*-dynamical systems, we will move to a discussion of their associated *thermodynamic properties*.

IV. Relationship with K-theory