Fall 2023 MAT437 problem set 6

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I. Problem 1

Recall that in Problem Set 2, we proved the following facts:

Claim I.1. For self adjoint h, h_1, h_2 in a unital C^*-algebra, $h \leq ||h|| \cdot 1$ and if $h_1 \leq h_2$, then $x^* h_1 x \leq x^* h_2 x$ for all x in the algebra.

Proof. Note that the spectrum of $h - ||h|| \cdot 1$ must be entirely non-positive, as for self-adjoint h, $||h|| = r(h)$. Thus, $h - ||h|| \cdot 1 \leq 1$ by definition. As for the latter fact, note that if $h_2 - h_1 \geq 0$, we can write $h_2 - h_1 = y^* y$ for some y. We subsequently note that

$$x^*(h_2 - h_1)x = x^* y^* y x = (yx)^*(yx) \geq 0 \Longrightarrow x^* h_1 x \leq x^* h_2 x$$

□

From here, note that if a is lef-tinvertible, $ba = 1$, then we have

$$1 = 1^* 1 = (ba)^*(ba) = a^* b^* b a \leq ||b^* b|| a^* a = ||b||^2 a^* a \tag{1}$$

Thus, for $||b|| > 0$, $a^* a - ||b||^{-2} \geq 0$, so $a^* a$ has a non-negative spectrum (the spectrum of $a^* a$ is already in $[0, \infty)$ as it is positive, so this fact implies it is in $[||b||^{-2}, \infty)$). Thus, $a^* a$ is invertible. Conversely, suppose $a^* a$ is invertible. Since it is positive, $(a^* a)^{-1/2}$ is well-defined. Letting $s = a(a^* a)^{-1/2}$, we have

$$s^* s = (a^* a)^{-1/2} (a^* a)(a^* a)^{-1/2} = 1$$

implying that s is invertible, so $a = s(a^* a)^{1/2}$ is also invertible. Proving the case of aa^*/a being right-invertible follows via an identical argument.

II. Problem 2

Part 1. Recall that the induced trace is simply the sum of the traces (via τ) of the diagonal entires of some element in the matrix algebra. It is quite clear that if $x \in M_n(A)$, then $x^* \in M_n(A)$ with entries $(x^*)_{ij} = (x_{ji})^*$, as the induced-$*$ is simply the conjugate transpose. Thus,

$$(x^* x)_{ij} = \sum_k (x^*)_{ik} x_{kj} = \sum_k (x_{ki})^* x_{kj} \Longrightarrow \tau_n(x^* x) = \sum_j \tau((x^* x)_{jj}) = \sum_{k,j} \tau(x_{kj}^* x_{kj})$$

as desired.

Part 2. We assumed τ is positive, so from the above formula, is is obvious that τ_n is positive (as each $x_{kj}^* x_{kj}$ is positive, so the sum of their traces will be non-negative).

Part 3. This also follows trivially from Part 1. Given positive a, we write $a = x^* x$, note from Part 1 that $\tau_n(a)$ is a sum of τ evaluated on positive elements, so since τ is faithful, each of these terms is greater than

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0, so their sum is as well.

Part 4. To show that A is stably finite, we must show that 1, the unit, is a finite projection in each induced matrix algebra $M_n(A)$. Suppose τ is a positive faithful trace, so the induced traces τ_n are also positive faithful, from Part 2 and Part 3. Thus, $\tau_n(0) = 0$ and $\tau(a) > 0$ for all positive a. Let us suppose that in $M_n(A)$, $1_n \sim p_n < 1_n$, where 1_n is the unit of $M_n(A)$. We must have $1_n = v_n^* v_n$ and $p_n = v_n v_n^*$. Since τ_n is a trace, we then have

$$\tau_n(1_n) = \tau_n(v_n^* v_n) = \tau_n(v_n v_n^*) = \tau(p_n) \Longrightarrow \tau(1 - p_n) = 0 \tag{2}$$

But, note that by assumption, $1 - p_n$ is positive (and not equal to 0), so since τ_n is positive faithful, $\tau(1 - p_n) > 0$, a clear contradiction. Thus, we cannot have $1_n \sim p_n$ for any n, so each $M_n(A)$ is finite, and A is stably finite.

III. Problem 3

Part 1. I'm assuming that we let the involution operation be the operator adjoint $\langle O^* a, b\rangle = \langle a, Ob\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product on H,

$$\langle f, g\rangle = \int f\overline{g}\ d\mu. \tag{3}$$

It follows that

$$\int O^*(f)\overline{g}\ d\mu = \int f\overline{O(g)}\ d\mu \tag{4}$$

For the particular case of u and v, we then have

$$\int u^*(f)(z_1, z_2)\overline{g(z_1, z_2)}\ d\mu = \int f(z_1, z_2)\overline{u(g)(z_1 z_2)}\ d\mu = \int \overline{z_1} f(z_1, z_2)\overline{g(z_1, z_2)}\ d\mu \tag{5}$$

$$\int v^*(f)(z_1, z_2)\overline{g(z_1, z_2)}\ d\mu = \int f(z_1, z_2)\overline{v(g)(z_1 z_2)}\ d\mu = \int \overline{z_2} f(z_1, z_2)\overline{g(\omega z_1, z_2)}\ d\mu \tag{6}$$

$$= \int \overline{z_2} f(\overline{\omega} z_1, z_2)\overline{g(z_1, z_2)}\ d\mu \tag{7}$$

so that $u^*(f)(z_1, z_2) = \overline{z_1} f(z_1, z_2)$ and $v^*(f)(z_1, z_2) = \overline{z_2} f(\overline{\omega} z_1, z_2)$. Note that we are allowed the make the substitution $z_1 \mapsto \overline{\omega} z_1$ in the integral as we are integrating with respect to the Haar measure, which will be invariant with respect to left-action by elements of $\mathbb{T}$. Thus, we have

$$(u^* u)(f)(z_1, z_2) = u^*\left(z_1 \cdot f(z_1, z_2)\right) = \overline{z_1} z_1 f(z_1, z_2) = f(z_1, z_2) = (uu^*)(f)(z_1, z_2) \tag{8}$$

as well as

$$(v^* v)(f)(z_1, z_2) = u^*\left(z_2 \cdot f(\omega z_1, z_2)\right) = \overline{z_2} z_2 f(\overline{\omega}\omega z_1, z_2) = f(z_1, z_2) = (vv^*)(f)(z_1, z_2) \tag{9}$$

as $\overline{z_1} z_1 = \overline{z_2} z_2 = \overline{\omega}\omega = 1$, since $z_1, z_2, \omega \in \mathbb{T}$. Thus, both $u, v \in B(H)$ are unitary. Moreover, it is easy for us to check that u and v almost commute: note that

$$(uv)(f)(z_1, z_2) = u\left(z_2 f(\omega z_1, z_2)\right) = z_1 z_2 f(z_1, z_2) \tag{10}$$

as well as

$$(vu)(f)(z_1, z_2) = v\left(z_1 f(z_1, z_2)\right) = z_2(\omega z_1) f(\omega z_1, z_2) \tag{11}$$

so $vu = \omega uv$.

Part 2. This follows immediately from the definition of $C^*(u,v)$. Recall that $C^*(u,v)$ is the closure over the span of all words generated by u, v, u^*, and v^*. Clearly, $u^* = u^{-1}$ and $v^* = v^{-1}$.

Note that $u^{-1}v^{-1} = (vu)^{-1} = (\omega uv)^{-1} = \overline{\omega}v^{-1}u^{-1}$. Then, note that $uv^{-1} = v^{-1}vuv^{-1} = \omega v^{-1}uvv^{-1} = \omega v^{-1}u$. This implies that $u^{-1}v = (v^{-1}u)^{-1} = (\overline{\omega}uv^{-1})^{-1} = \omega vu^{-1}$. Thus, swapping any two of the elements v, u, v^{-1}, u^{-1} induces multiplication by either ω or $\overline{\omega}$. It follows that any words of u, v, u^* and v^* can be re-written in the form $\beta u^m v^n$, for $m,n \in \mathbb{Z}$ and $\beta \in \mathbb{T}$. Thus, the span of all such words W is clearly precisely the set of Laurent polynomials in u and v, $\mathcal{A}_\theta$. Thus, $\mathcal{A}_\theta$ is a sub-$*$ algebra, as $\text{span}(W)$ is.

Note that $A_\theta = C^*(u,v) = \overline{\text{span}(W)} = \overline{\mathcal{A}_\theta}$. Thus, by definition, $\mathcal{A}_\theta$ is dense in A_θ.

Part 3. By definition, we note that

$$\tau\left(\sum_{n,m\in\mathbb{Z}} \alpha_{n,m}u^n v^m\right) = \sum_{n,m\in\mathbb{Z}} \alpha_{n,m}\tau(u^n v^m) \tag{12}$$

Clearly, by definition,

$$\tau(u^n v^m) = \langle u^n v^m \xi_0, \xi_0\rangle = \int (u^n v^m)(\xi_0)(z_1,z_2)\overline{\xi_0(z_1,z_2)}\, d\mu \tag{13}$$

Because $(vf)(z_1,z_2) = z_2 f(\omega z_1, z_2)$ and $(v^{-1}f)(z_1,z_2) = \overline{z_2}f(\omega^{-1}z_1,z_2)$, it is easy to see that $(v^m\xi_0)(z_1,z_2) = z_2^m\xi_0(\omega^m z_1, z_2)$. Then, using similar logic, $u^n(z_2^m\xi_0(\omega^m z_1,z_2)) = z_1^n z_2^m \xi_0(\omega^m z_1, z_2) = z_1^n z_2^m$, by definition of ξ_0. Thus,

$$\sum_{n,m\in\mathbb{Z}} \alpha_{n,m}\tau(u^n v^m) = \sum_{n,m\in\mathbb{Z}} \alpha_{n,m}\left(\int_{\mathbb{T}} z_1^n\, d\mu\right)\left(\int_{\mathbb{T}} z_2^m\, d\mu\right) = \alpha_{0,0} \tag{14}$$

as the integral over the unit circle with respect to the Haar measure of any non-zero power of z is clearly just 0.

Part 4. To demonstrate that τ is a tracial state, it is necessary to show that τ is a trace, it is positive, and it sends 1 to 1. Clearly, $\tau(1) = \langle\xi_0,\xi_0\rangle = \int 1\, d\mu = 1$. Now, given some $p \in \mathcal{A}_\theta$, note that

$$p = \sum_{m,n\in\mathbb{Z}} \alpha_{m,n}u^m v^n \quad \text{and} \quad p^* = \sum_{m,n\in\mathbb{Z}} \overline{\alpha_{m,n}}v^{-n}u^{-m} \tag{15}$$

which implies that

$$pp^* = \sum_{pqrs} \alpha_{p,q}\overline{\alpha_{r,s}}u^p v^{q-r}u^{-s} \quad \text{and} \quad p^*p = \sum_{pqrs} \overline{\alpha_{p,q}}\alpha_{r,s}v^{-p}u^{r-q}v^s \tag{16}$$

from the almost-commutation relations between u and v, it is clear that the degree-0 terms in pp^* and p^*p are precisely those such that $r = q$ and $p = s$. Thus, the degree-0 term of pp^* and p^*p, which we denote β, are clearly the same, and from Part 3, $\tau(pp^*) = \tau(p^*p)$ for all $p \in \mathcal{A}_\theta$. To show that this relation holds for all of A_θ, let us pick $p \in A_\theta$, and choose a sequence of $p_n \in \mathcal{A}_\theta$ which approach p. Note that the function τ is a continuous function on A_θ, as we have

$$|\tau(x) - \tau(y)| = |\tau(x-y)| = |\langle (x-y)\xi_0, \xi_0\rangle| \leq ||x-y|| \tag{17}$$

so τ is Lipschitz. The function $f(x) = \tau(xx^* - x^*x)$ is then clearly continuous. Thus, we have

$$f(p) = f\left(\lim_{n\to\infty} p_n\right) = \lim_{n\to\infty} f(p_n) = 0 \tag{18}$$

so $\tau(pp^*) = \tau(p^*p)$ here as well. It follows from Problem Set 4 Question 2 that τ is a valid trace map on A_θ. All that remains to check is that $\tau(a) \geq 0$ for all positive $a \in A_\theta$.

Claim III.1. If H is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, then if $A \in B(H)$ is positive, $\langle Av, v\rangle \geq 0$ for all $v \in H$.

Proof. Suppose A is positive, so it is of the form $A = X^*X$. Then $\langle Av, v\rangle = \langle Xv, Xv\rangle \geq 0$, by definition of the inner product. $\square$

It follows immediately that $\tau(a) = \langle a\xi_0, \xi_0\rangle$ must send positive $a \in A_\theta$ to non-negative real numbers, so τ is a positive trace. Thus, the proof is complete: τ is a tracial state.

Part 5. It is easy to see that $p = p^*$. Since $f, g : \mathbb{T} \to \mathbb{R}$, it follows that for some $s \in \mathbb{T}$, $\overline{f(s)} = f(s)$ and $\overline{g(s)} = g(s)$. It follows that if Φ is the $*$-isomorphism which assigns functions in $C(\mathrm{sp}(a))$ to elements of $C^*(a, 1)$ (via the continuous function calculus), we have $\Phi(f - \overline{f}) = 0$, so $f(a) = f(a)^*$, with the same logic showing that $g(a) = g(a)^*$. Thus,

$$p^* = (f(u)v^*) + g(u)^* + (vf(u))^* = vf(u)^* + g(u) + f(u)^*v = vf(u) + f(u) + f(u)v = p \tag{19}$$

To show that $\tau(p) = \int_{\mathbb{T}} g(z)\ dz$ (we replace the $d\mu$ notation with dz, to be consistent with the notation of the question), let's us begin by considering the case where f and g are Laurent polynomials. Clearly, both $f(u)v^*$ and $vf(u)$ will then be Laurent polynomials with no degree-0 term, so $\tau(f(u)v^*) = \tau(vf(u)) = 0$. It follows that

$$\tau(p) = \tau(g(u)) = \tau\left(\sum_{m\in\mathbb{Z}} \alpha_m u^m\right) = \alpha_0 \tag{20}$$

from Part 3 (note that only a finite number of the α_m are non-zero, by definition of a Laurent polynomial). Of course, note that

$$\int_{\mathbb{T}} g(z)\ dz = \sum_{m\in\mathbb{Z}} \alpha_m \int_{\mathbb{T}} z^m\ dz = \alpha_0 = \tau(p) \tag{21}$$

as integrating any power of z uniformly over the unit circle will be 0. Thus, we have proved the formula for the case of f and g being Laurent polynomials. To prove it in general, note that by Stone-Weierstrass Theorem, the set of Laurent polynomials is dense in the metric space of continuous function $C(\mathbb{T}, \mathbb{R})$ with the uniform metric. For a particular pair of f, g, let f_n and g_n be sequences of Laurent polynomials which converge uniformly to f and g. Let $p_n = f_n(u)v^* + g_n(u) + vf_n(u)$. Note that

$$\tau(p_n) = \int_{\mathbb{T}} g_n(z)\ dz \tag{22}$$

Because τ is continuous, $\lim_{n\to\infty} \tau(p_n) = \tau(p)$. Moreover, since g_n converges *uniformly* to g, we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} g_n(z)\ dz = \int_{\mathbb{T}} \lim_{n\to\infty} g_n(z)\ dz = \int_{\mathbb{T}} g(z)\ dz$$

so the formula $\tau(p) = \int_{\mathbb{T}} g(z)\ dz$ holds for all of A_θ, and we are done.

Part 6. Once again, let us begin with the case where h is a Laurent polynomial, so $h(z) = \sum_{m\in\mathbb{Z}} \alpha_m z^m$. We have $\varphi(u) = \omega u$. Therefore,

$$h(\varphi(u))v = \sum_{m\in\mathbb{Z}} \alpha_m \omega^m u^m v \tag{23}$$

Now, recall that $uv = \omega^{-1}vu$. Thus, inductively, we see that $u^m v = (u\cdots u)v = \omega^{-m}v(u\cdots u) = \omega^{-m}vu^m$, as we perform m swaps, this picking up m factors of ω^{-1}. Thus,

$$\sum_{m\in\mathbb{Z}} \alpha_m \omega^m u^m v = \sum_{m\in\mathbb{Z}} \alpha_m \omega^m \omega^{-m} v u^m = v \sum_{m\in\mathbb{Z}} \alpha_m u^m = vh(u) \tag{24}$$

so the formula holds when h is a Laurent polynomial. Similar to what we did previously, we note that we can choose a sequence of Laurent polynomials h_n converging uniformly to h. Obviously, $vh_n(u) - h_n(\omega u)v = 0$

for all n, so its limit for $n \to \infty$ is also 0, implying $vh(u) = h(\omega u)v = (h \circ \varphi)(u)v$.

Part 7. Let us first make note of the fact that $\varphi^{-1}(z) = \overline{\omega}z$. Suppose h is a continuous function with real range, as above. We then have $vh(u) = (h \circ \varphi)(u)v$, so $h(u)v^* = v^*(h \circ \varphi)(u)$. Then, since φ is invertible, we have $v(h \circ \varphi^{-1})(u) = h(u)v$ and $(h \circ \varphi^{-1})(u)v^* = v^*h(u)$. We then have

$$\begin{aligned}
p^2 &= (f(u)v^* + g(u) + vf(u))^2 = g(u)^2 + g(u)(f(u)v^* + vf(u)) + (f(u)v^* + vf(u))g(u) + (f(u)v^* + vf(u))^2 \\
&= (g^2 + f^2 + vf^2v^*) + (fv^*fv^* + vfvf) + (gfv^* + gvf + fv^*g + vfg) \qquad (25) \\
&= (g^2 + f^2 + (f \circ \varphi)^2) + (f \cdot (f \circ \varphi^{-1}))(v^*)^2 + v^2(f \cdot (f \circ \varphi^{-1})) + vf \cdot (g + (g \circ \varphi^{-1})) + f \cdot (g + (g \circ \varphi^{-1}))v^*
\end{aligned}$$

where we make the u-dependence implicit, to condense notation. Thus, it is clear that if the relations in the problem statement hold, then $p^2 = p$ (we can see this by direct comparison of the formulas). To prove the other direction, we consider the operator $p^2 - p$. In particular, note that

$$\begin{aligned}
p^2 - p &= (g^2 + f^2 + (f \circ \varphi)^2) - g + \left(v^2(f \cdot (f \circ \varphi^{-1})) + v(f \cdot (g + (g \circ \varphi^{-1})) - f) + \text{conj.}\right) \qquad (26) \\
&= a + \left(v^2 b + vc + cv^* + b(v^*)^2\right) \qquad (27)
\end{aligned}$$

where we use conj. to denote that we are also adding the $*$-conjugate of the terms inside the brackets I'm sorry if this is confusing, I couldn't think of a better way to condense notation. Now, note that evaluating $(p^2 - p)(\xi_0(z_1, z_2))$ will yield a function in z_1 and z_2. In fact, we will have

$$(p^2 - p)(\xi_0(z_1, z_2)) = a(z_1) + b'(z_1)z_2^2 + c'(z_1)z_2 + c(z_1)z_2^{-1} + b(z_1)z_2^{-2} \qquad (28)$$

where b' and c' are modified from b and c, via moving v to the right, with the almost-commutation relations. In order for $p^2 - p$ to be 0 for all z_1 and z_2, so for fixed z_1, all coefficients in the Laurent polynomial in z_2 must be 0 individually, for any choice of z_1. This will imply the desired relations between f and g. This is, of course, only a sketch of why this result holds, but we've essentially already proved all the necessary parts, so for the sake of saving time, I will skip over writing everything out explicitly.

Part 8. First, note that $\tau(p) = \int_{\mathbb{T}} g(z)\, dz$, independent of the choice of f. But clearly, g as we have defined it satisfies the requirement that $\tau(p) = \theta$, as

$$\int_{\mathbb{T}} g(z)\, dz = \varepsilon^{-1}\int_0^{\varepsilon} t\, dt + (\theta - \varepsilon) + \varepsilon^{-1}\int_{\theta}^{\theta+\varepsilon} (\theta + \varepsilon - t)\, dt = 2\varepsilon^{-1}\int_0^{\varepsilon} t\, dt + \theta - \varepsilon = \frac{2\varepsilon^2}{2\varepsilon} + \theta - \varepsilon = \theta \qquad (29)$$

Note that we are integrating with respect to the Haar measure on the unit circle, so we can pull-back and integrate *uniformly* with respect to the angle.
