MAT436 problem set 6

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Contents

I. Problem 1 (Suggested Problem 1)

It is easy to see that the set of functions $C_b(X)$ forms an algebra with the usual addition and multiplication, in fact is is trivial to check that it is also a $*$-algebra with respect to complex conjugation, and we know that the supremum norm is a valid norm and is clearly submultiplicative, $||fg|| \leq ||f|| ||g||$. To check that this is a C^*-algebra, we must verify completeness of the metric space induced by the norm and $||g^* g|| = ||g||^2$.

The latter of these claims follows immediately from the fact that $|z^* z| = |z|^2$ for z a complex number.

To see completeness, note that if the sequence f_n is Cauchy, then $\sup_x |f_n(x) - f_m(x)|$ becomes arbitrarily small. Since the complex numbers are complete, this sequence will converge to some $f(x)$ for each x. To see that this assignment $x \mapsto f(x)$ is a valid element of $C_b(x)$, note that boundedness follows from the fact that we have convergence in supremum, so $\sup_x |f_n(x) - f(x)|$ becomes arbitrarily small and f_n is bounded, thus f must be as well. To see continuity, note that if y_n is a sequence converging to y, then

$$|f(y) - f(y_n)| \leq |f(y) - f_m(y)| + |f_m(y) - f_m(y_n)| + |f_m(y_n) - f(y_n)| \tag{1}$$

which can be made arbitrarily small for n and m large enough, as $f_m \to f$ pointwise and the functions f_m are continuous.

II. Problem 2 (Suggested Problem 3)

Part A. To prove that φ is a unitl $*$-homomorphism, begin by noting that if λ is a constant,

$$\varphi^*(\lambda f + g) = \lambda \varphi^* f + \varphi^* g \quad \text{and} \quad \varphi^*(fg) = \varphi^* f \varphi^* g \tag{2}$$

also, $\varphi^*(1) = 1$, clearly, and moreover, $\varphi^*(f^*) = f(\varphi(x))^* = (\varphi^* f)^*$, so we do in fact have a unital $*$-homomorphism. The fact that φ is itself continuous implies that the map is well-defined, between spaces of continuous functions.

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Part B. In the case that $\varphi_\rho : X \to Y$ is surjective, we have $\varphi_\rho^*(f) = f \circ \varphi_\rho$, suppose $f \circ \varphi_\rho = 0$, so $(f \circ \varphi_\rho)(x) = 0$ for all x. Since φ_ρ is surjective, it follows that for any y we have $\varphi_\rho(x) = y$ for some x, so $f(y) = f(\varphi_\rho(x)) = 0$ for all y, implying $f = 0$. Hence, $\rho = \varphi_\rho^*$ is injective.

Part C. Suppose $\varphi_\rho : X \to Y$ is injective, it follows that this function is a bijection with its image. Given some function g, it follows that we can consider $g \circ \varphi_\rho^{-1}$, where φ_ρ^{-1} is an extension of $\varphi_\rho^{-1} : \varphi_\rho(X) \to X$ to all of Y (in particular, we can find such an extension as we are working in a compact Hausdorff space). It follows that $\varphi_\rho^*(g \circ \varphi_\rho^{-1}) = g$, so ρ is surjective.

Part D. This follows immediately from the definition.

III. Problem 3

Proposition III.1. If f is a linear functional on a normed vector space X, then it is bounded if and only if $\mathrm{Ker}(f) = f^{-1}(0)$ is closed.

Proof. In the case that $f : X \to \mathbb{C}$ is bounded, it is automatically continuous, so it follows that $f^{-1}(0)$ is closed as the inverse image of a closed set by a continuous function.

On the other hand, if $f^{-1}(0)$ is closed, recall the Riesz lemma, which states that for any $\alpha \in (0, 1)$ and closed subspace $E \subset X$, we may choose some $x \in X$ such that $d(x, E) = \inf_{y \in E} ||x - y|| \geq \alpha$ and $||x|| = 1$. Choose some z such that $d(z, f^{-1}(0)) \geq 1/2$, so obviously $f(z) \neq 0$. Note that for any $x \in X - \mathrm{Ker}(f)$, we have

$$f(x) = f\left(\frac{f(x)}{f(z)} z\right) \Longrightarrow f\left(x - \frac{f(x)}{f(z)} z\right) = 0 \tag{3}$$

which means that

$$x - \frac{f(x)}{f(z)} z = k \in \mathrm{Ker}(f) \tag{4}$$

so we can re-arrange to get

$$\frac{f(z)}{f(x)} x = \frac{f(z)}{f(x)} k + z = z - \left(-\frac{f(z)}{f(x)} k\right) \tag{5}$$

where obviously $-\frac{f(z)}{f(x)} k \in \mathrm{Ker}(f)$, so by assumption

$$\left|\left|\frac{f(z)}{f(x)} x\right|\right| \geq \frac{1}{2} \Longrightarrow \frac{|f(x)|}{||x||} \leq 2|f(z)| \tag{6}$$

which implies that f is bounded, as desired. $\square$