MAT437 problem set 8

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I. Problem 1

The proof of the following lemma included in RLL excludes many key parts, so for one of my exercises, I complete it

We start by proving a collection of technical lemma, which will be required for the following proof:

Lemma I.1. If v is a partial isometry (so v^*v is a projection) then $vv^*v = v$.

Proof. Let $z = v - vv^*v = (1 - vv^*)v$. Note that

$$z^*z = v^*(1 - vv^*)^*(1 - vv^*) = v^*(1 - 2vv^* + vv^*vv^*)v = v^*v - 2v^*v + v^*v = 0 \tag{1}$$

where we use that $(v^*v)^2 = v^*v$. Thus, $||z||^2 = ||z^*z|| = 0$, so $||z|| = 0$ and $v = vv^*v$, as desired. □

Now, another lemma:

Lemma I.2. Suppose that $\{f_{ii}^{(k)} \mid 1 \le k \le r, 1 \le i \le n_k\}$ is a set of mutually orthogonal projections in C^*-algebra B and that

$$f_{11}^{(k)} \sim f_{22}^{(k)} \sim \cdots \sim f_{n_k n_k}^{(k)} \tag{2}$$

for each k. Then there is a system of matrix units $\{f_{ij}^{(k)}\}$ extending $\{f_{ii}^{(k)}\}$.

The idea behind constructing systems of matrix units is, essentially, to have a "basis" for each component of the direct sum that we will eventually demonstrate characterizes the C^*-algebra B. Each of the sets $\{f_{ii}^{(k)}\}$ are analogous to matrix projections with 1 at the i-th slot on the diagonal, at the k-th slot in the direct sum. Let us now prove the lemma.

Proof. Of course, here, we will make us of the Murray-von Neumann equivalence. Namely,

$$f_{11}^{(k)} \sim f_{jj}^{(k)} \Longrightarrow f_{11}^{(k)} = f_{1j}^{(k)} f_{1j}^{(k)^*} \quad \text{and} \quad f_{jj}^{(k)} = f_{1j}^{(k)^*} f_{1j}^{(k)} \tag{3}$$

This notation is consistent, as $f_{11}^{(k)}$ is self-adjoint, so setting $j = 1$ above causes no problems. Our claim is that if we set $\widetilde{f}_{ij}^{(k)} = f_{1i}^{(k)^*} f_{1j}^{(k)}$ then we will have the desired system of matrix units. This is in fact an extension of the system we are already provided. Namely, we have

$$\widetilde{f}_{jj}^{(k)} = f_{1j}^{(k)^*} f_{1j}^{(k)} = f_{jj}^{(k)} \tag{4}$$

by definition. In fact, we might as well denote $\widetilde{f}_{ij}$ by f_{ij}, as for $i = 1$, we have

$$\widetilde{f}_{1j} = f_{11}^{(k)^*} f_{1j}^{(k)} = f_{1j}^{(k)} f_{1j}^{(k)^*} f_{1j}^{(k)} = f_{1j}^{(k)} \tag{5}$$

where we use the above lemma and the fact that $f_{1j}^{(k)^*} f_{1j}^{(k)} = f_{jj}^{(k)}$ is a projection. Let us now complete our verification. Of course, we have

$$f_{pq}^{(k)} f_{qr}^{(k)} = f_{1p}^{(k)^*} f_{1q}^{(k)} f_{1q}^{(k)^*} f_{1r}^{(k)} = f_{1p}^{(k)^*} f_{11}^{(k)} f_{1r}^{(k)} = f_{1p}^{(k)^*} f_{1r}^{(k)} f_{1r}^{(k)^*} f_{1r}^{(k)} = f_{1p}^{(k)^*} f_{1r}^{(k)} = f_{pr}^{(k)} \tag{6}$$

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where we use the first lemma to note that $f_{1r}^{(k)} f_{1r}^{(k)^*} f_{1r}^{(k)} = f_{1r}^{(k)}$, as $f_{1r}^{(k)^*} f_{1r}^{(k)} = f_{rr}^{(k)}$ is a projection. Next, note that

$$f_{pq}^{(k)} f_{rs}^{(\ell)} = f_{1p}^{(k)^*} f_{1q}^{(k)} f_{1r}^{(\ell)^*} f_{1s}^{(\ell)} \tag{7}$$

Once again using the first lemma, we have $f_{1q}^{(k)} = f_{1q}^{(k)} f_{1q}^{(k)^*} f_{1q}^{(k)} = f_{1q}^{(k)} f_{qq}^{(k)}$ and $f_{1r}^{(\ell)} = f_{1r}^{(\ell)} f_{1r}^{(\ell)^*} f_{1r}^{(\ell)} = f_{1r}^{(\ell)} f_{rr}^{(\ell)}$ so that $f_{1r}^{(\ell)^*} = f_{rr}^{(\ell)} f_{1r}^{(\ell)^*}$. We then use the fact that the projections in our set are mutually orthogonal to conclude that

$$f_{1p}^{(k)^*} f_{1q}^{(k)} f_{1r}^{(\ell)^*} f_{1s}^{(\ell)} = f_{1p}^{(k)^*} f_{1q}^{(k)} (f_{qq}^{(k)} f_{rr}^{(\ell)}) f_{1r}^{(\ell)^*} f_{1s}^{(\ell)} = 0 \tag{8}$$

which is 0 when $q \neq r$ or $k \neq \ell$, as in these cases, $f_{qq}^{(k)} f_{rr}^{(\ell)} = 0$. It is very immediately clear that $f_{ij}^{(k)^*} = f_{1j}^{(k)^*} f_{1i}^{(k)} = f_{ji}^{(k)}$, so we have verified the third condition, and it follows that our set of $f_{ij}^{(k)}$ is in fact a system of matrix units in B extending $\{f_{ii}^{(k)}\}$. □

II. RLL Problem 7.4 (Suggested Problem 1)

Part 1. Suppose A and B both have the cancellation property, which means that the semigroups $\mathcal{D}(A)$ and $\mathcal{D}(B)$ have the cancellation property. Equivalently, some X has the cancellation property if andf only if, for each $p, q \in \mathcal{P}_\infty(X)$, then

$$[p]_0 = [q]_0 \Longleftrightarrow p \sim_0 q \tag{9}$$

Recall from earlier in RLL that if i_A and i_B are canonical inclusion maps of A and B into $A \oplus B$, then $\Phi = K_0(i_A) \oplus K_0(i_B)$ is a group isomorphism. Suppose $p, q \in \mathcal{P}_\infty(A \oplus B)$. Of course, $p \sim_0 q$ implies $[p]_{\mathcal{D}} = [q]_{\mathcal{D}}$, which implies that $[p]_0 = [q]_0$.

Conversely, suppose that $[p]_0 = [q]_0$, where p, q are projections in $\mathcal{P}_\infty(A \oplus B)$ It is easy to see in this case that $p = p_1 \oplus p_2$ and $q = q_1 \oplus q_2$ for $p_1, p_2 \in \mathcal{P}_\infty(A)$ and $q_1, q_2 \in \mathcal{P}_\infty(B)$. Then note that $\Phi^{-1}([p]_0) = [p_1]_0 \oplus [p_2]_0 = \Phi^{-1}([q]_0) = [q_1]_0 \oplus [q_2]_0$, so that $[p_1]_0 = [q_1]_0$ in $K_0(A)$ and $[p_2]_0 = [q_2]_0$ in $K_0(B)$. Thus, since we have the canellation property in these algebras, $p_1 \sim_0 q_1$ and $p_2 \sim_0 q_2$. It follows that $p_1 \oplus p_2 = p \sim_0 q_1 \oplus q_2 = q$, clearly. This completes the proof.

Part 2. Suppose the sequence of C^*-algebras has the the cancellation property. We will use continuity of K_0 to show that their inductive limit has the cancellation property. In particular, note that

$$\varinjlim K_0(A_n) \simeq K_0(\varinjlim A_n) = K_0(A) \tag{10}$$

where the isomorphism is clearly of Abelian groups. Suppose $p, q \in \mathcal{P}_\infty(A)$ with $[p]_0, [q]_0 \in K_0(A)$ are such that $[p]_0 = [q]_0$. Recall that from the continuity, we know that:

$$K_0(A) = \bigcup_{n=1}^{\infty} K_0(\mu_n)(K_0(A_n)) \tag{11}$$

so it follows that we can writen $[p]_0 = [\mu_n(p_n)]_0$ and $[q]_0 = [\mu_m(q_m)]_0$ for some n and m, as well as $p_n \in \mathcal{P}_\infty(A_n)$ and $q_m \in \mathcal{P}_\infty(A_m)$, where $(A, \{\mu_n\})$ is ther inductive limit of the algebras. Without loss of generality, we can assume $m = n$, as if $m < n$ for example, we can always note that

$$\mu_n(p_n) = (\varphi_{n,m} \circ \mu_m)(p_n) \tag{12}$$

via the connection maps. Thus, we will have $K_0(\mu_n)([p_n]_0 - [q_m]_0)$. Again from continuity, we know that $[p_n]_0 - [q_m]_0$ must be in the kernel of some connection map $\Phi = K_0(\varphi_{a,n})$, so that $[\Phi(p_n)]_0 = [\Phi(q_m)]_0$. Since each algebra A_n has the cancellation property, $\Phi(p_n) \sim_0 \Phi(q_m)$. Thus, $\mu_n(p_n) \sim_0 \mu_n(q_m)$, which implies that $p \sim_0 q$, and we have the desired cancellation property.

Part 3. Let us prove that any matrix algebra over $\mathbb{C}$ has the cancellation property. If we can do this, then since AF algebras are simply direct limits of direct sums of matrix algebras, we will have proved that all AF algebras have the cancellation property.

Lemma II.1. The algebra $M_n(\mathbb{C})$ has the cancellation property.

Proof. Recall that if $p, q \in M_n(\mathbb{C})$ are projections, then $p \sim q$ if and only if $\mathrm{Tr}(p) = \mathrm{Tr}(q)$. Moreover, recall that the trace map induces a map $K_0(\mathrm{Tr}) : K_0(M_n(\mathbb{C})) \to \mathbb{Z}$ which is in fact a group isomorphism. Given $[p]_0 = [q]_0$, it follows immediately that

$$K_0(\mathrm{Tr})([p]_0) = \mathrm{Tr}(p) = K_0(\mathrm{Tr})([q]_0) = \mathrm{Tr}(q)$$

Note that $p, q \in \mathcal{P}_\infty(M_n(\mathbb{C}))$, which means that they are projections in some $M_{n_1}(\mathbb{C})$ and $M_{n_2}(\mathbb{C})$ repsectivelty. Without loss of generality, suppose $n_1 \geq n_2$. We have $\mathrm{Tr}(q) = \mathrm{Tr}(q \oplus 0_{n_1 - n_2})$, so $q \sim q \oplus 0_{n_1 - n_2}$ from the first sentence, which means that $p \sim_0 q$. This completes the proof: the complex matrix algebra has the cancellation property. $\square$

To reiterate, if we have an AF algebra A, then

$$A = \varinjlim A_n = \varinjlim M_{j(n)_1}(\mathbb{C}) \oplus \cdots \oplus M_{j(n)_k}(\mathbb{C})$$

Since each $M_{j(n)_k}$ has the cancellation property, so does each direct sum, and thus so too does the direct limit, implying A has the cancellation property.