HATCHER ALGEBRAIC TOPOLOGY: CHAPTER 0

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1. Quotient Spaces

Strictly speaking, this is not a section in Hatcher. However, quotient spaces are so important that we will spend some time going over the basic results outlined in Munkres.

2. Cell Complexes

A cell-complex is a way that we can build a space by succesively gluing together n-dimensional disks. This can be accomplished as follows:

- We start with a collection of points, which we call X^0. We can regard each of these points as a 0-cell.
- Glue lines (which we call 1-cells) to the points in X^0. Formally, we denote these 1-cells by e^1_α. For each 1-cell, we can define a corresponding attaching map $\phi_\alpha : S^0 \to X^0$, which simply takes the boundary points of e^1_α and sends them to points in X^0. In this case, this will simply be the end points of the line. We then can define X^1 as:

$$X^1 = \left(X^0 \bigsqcup_\alpha e^1_\alpha \right) / \sim$$

  where $\sim$ is the equivalence relation defined by identifying point $x \in X^0$ with all y such that $\phi_\alpha(y) = x$, for some α.
- Inductively continue the above procedure, for the case of n-cells, defined to by n-dimensional disks. Thus, a 2-cell is an open disk, a 3-cell an open, solid ball, and so on. We define each X^n an an analogous way.
- Terminate this procedure after a finite number of steps (say, N), and take our final topological space to be X^N, or continue indefinitely and take the final topological space to be $\cup_{n\in\mathbb{Z}^+} X^n$.

One of the more interesting examples of cell-complexes in action is constructing projective spaces.

We define the real projective space $\mathbb{R}P^n$ to be the quotient space formed by taking all lines through the origin in $\mathbb{R}^{n+1} - \{0\}$. That is to say, $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \neq 0$. Clearly, this topological space is homeomorphic to S^n with antipodal points identified, take:

$$f : \mathbb{R}^{n+1} - \{0\} \to S^n \qquad f(x) = \frac{x}{||x||}$$

and let $\tilde{f} : \mathbb{R}P^n \to S^n/\sim$ be defined as $\tilde{f}([x]) = [f(x)]$, for $x \in \mathbb{R}P^n$, where we take $S^n/\sim$ to be a subspace of the projective space. It is easy to check this is a homeomorphism, using the facts about quotient spaces proved in the first section.

Now, brining together the antipodal points of S^n, it is clear that $S^n/\sim$ is homeomorphic to the disk B^n with antipodal boundary points identified (we will not formally write out why this is the case, but it is geometrically obvious).