1. Practice Exam

1.1. **Problem 1.** *Let J be uncountable. Is $\mathbb{R}^J$ with the product topology metrizable?*

$\mathbb{R}^J$ with the product topology is not metrizable. This is due to the fact that we can form a contradiction to the sequence lemma, which in turn implies that the space is not metrizable.

We let A be the set of all points $\boldsymbol{x}$ in $\mathbb{R}^J$ that are 1 at all except finitely many of their coordinates. We let $\mathbf{0}$ be the point with all coordinates equal to 0.

Given some neighbourhood of the form $U = \prod_\alpha U_\alpha$ around $\mathbf{0}$, note that $U_\alpha \neq \mathbb{R}$ for only finitely many α, so the point $\boldsymbol{y}$, with $y_\beta = 0$ for this finite collection of indices and 1 otherwise is in U and A. Thus, $\mathbf{0} \in \bar{A}$.

We assert that there is no sequence of points of A that converges to $\mathbf{0}$. This is due to the fact that for any sequence $\boldsymbol{a}_n$, the set of coordinates that differ from 1 for some element of the sequence is finite. Taking the union of all such indices over n yields a countable set. But the set of all indices, J, is uncountable, so we can choose some coordinate β such that every $a \in \boldsymbol{a}_n$ is 1 at that coordinate.

We then take the open set $\pi_\beta^{-1}(-1,1)$ in $\mathbb{R}^J$. This contains $\mathbf{0}$, but no element of $\boldsymbol{a}_n$. Thus, $\boldsymbol{a}_n$ does not converge to $\mathbf{0}$.

1.2. **Problem 2.** *Let A be countable. Is $\mathbb{R}^\omega - A$ path-connected, where $\mathbb{R}^\omega$ is given the product topology?*

Let a and b be points of $\mathbb{R}^\omega$. Let $S_x = \{L_x^m \mid m \in \mathbb{R}\}$ be the set of all lines passing through x, so:

$$L_x^m = \{x + (t, mt, 0, 0, \ ...) \mid t \in \mathbb{R}\}$$

Clearly, given $m \neq n$, we have $L_x^m \cap L_x^n = \{x\}$. Since A is countable, and the sets S_a, S_b are uncountable, it follows that there exist infinite subsets $V_a \subset S_a$ and $V_b \subset S_b$ of lines that do not intersect A. Pick $L_a^m \in V_a$, and $L_b^n \in V_b$ that is not parallel to L_a^m. It follows that $L_a^m \cap L_b^n = \{c\}$.

We then define $F : [0,1] \to \mathbb{R}^\omega - A$ as:

$$F = \begin{cases} (1-t)a + tc & t \in [0,1] \\ tc + (t-1)b & t \in [1,2] \end{cases}$$

It is easy to see that $F([0,1]) \subset L_a^m$ and $F([1,2]) \subset L_b^n$, so $F([0,2])$ lies in $\mathbb{R}^\omega - A$ and F is well-defined. In addition, it is clear that F is a path from a to b, so the space is path-connected.

1.3. **Problem 3.** *Suppose A is a subset of X, a compact metric space, and every continuous function $f : A \to \mathbb{R}$ has a maximum in A. Prove that A is compact.*

We will show that A is closed, and therefore compact (as we are operating in a compact metric space).

Suppose A is not closed. Then $X - A$ is not open. Thus, there must exist some $a \in X - A$ such that $a \in \bar{A}$.

Define $f : A \to \mathbb{R}$ such that $f(x) = \frac{1}{d(x,a)}$. Clearly, this function is continuous on A, as $g(x) = d(x,a)$ is continuous. However, it is obvious that f is not bounded, as for any N, we can choose a point $y \in A$ such that $y \in B_{1/N}(a)$, so $f(y) > N$. Thus, we have a contradiction, so A must be closed, and is therefore compact.

1.4. **Problem 4.** *Calculate the fundamental group of S^1 with the north and south poles identified*

It is easy to see through a sketch that this space is homotopy equivalent to a sphere with a line connecting the north and south poles. This space is itself homotopy equivalent to $S^2 \vee S^1$, which can been seen be gradually sliding the end points of the diameter together.

Let x_0 be the attachment point. We proceed via Seifert-van Kampen. Removing a point from S^1 gives an open set U that deform retracts to S^2, so $\pi_1(U, x_0)$ is trivial. Removing a point from S^2 gives an open set V that deform retracts to S^1, so $\pi_1(V, x_0) = \langle a \rangle$.

The intersection $U \cap V$ gives a space which deform retracts to a point, and therefore has a trivial fundamental group.

Thus, by Seifert-van Kampen, $\pi_1(X, x_0) = \langle a \mid \emptyset \rangle = \mathbb{Z}$.

1.5. **Problem 5.** *Describe the universal covering space and covering map of the above space.*

Don't know, universal covering spaces were never once mentioned in class, so I'm not sure why this is on the practice exam...

1.6. **Problem 6. Part A**

Find the fundamental group of the doubly-punctured torus.

It is easy to see that the doubly-punctured torus deform retracts to the boundary of the rectangle defining it, with a line passing down the middle. This is itself clearly the wedge of three circles: $X = S^1 \vee S^1 \vee S^1$.

We once again use Seifert-van Kampen. Let $a, b, c \in X$ be distinct, lying on each of the circles in the wedge (not at the attachment point x_0). Let $U = X - \{a\}$ and $V = X - \{b, c\}$. Clearly, U deform retracts to $S^1 \vee S^1$, so $\pi_1(U, x_0) = \langle a, b \rangle$. In addition, V clearly deform retracts to S^1, so $\pi_1(V, x_0) = \langle c \rangle$. It follows that:

$$\pi_1(X, x_0) = \pi_1(U \cup V, x_0) = \langle a, b, c \rangle = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

as $U \cap V$ deform retracts to a point, and therefore is simply connected.

Part B

Find the fundamental group of $\mathbb{R}^3$ with n lines through the origin removed.

Proposition 1. *$\mathbb{R}^2$ with n points removed (which we call R_n) has a fundamental group $\mathbb{Z} * \cdots * \mathbb{Z}$, repeated n time.*

Proof. We proceed by induction. For $n = 1$, the claim clearly holds. Assume the case of n. We prove the case of $n+1$. Choose open sets U and V, with U containing the $n+1$-th hole, and V containing the other n holes, such that $U \cup V = X$, and $U \cap V$ is some unpunctured open set, and is therefore simply connected.

It follows that $\pi_1(U, x_0)$ is the free group with 1 generator, while $\pi_1(V, x_0)$ is the free group of n generators, from the inductive hypothesis. Thus, $\pi_1(R_n, x_0)$ is the free group with $n+1$ generators. □

$\mathbb{R}^3 - X$ clearly deform retracts to S^2, with $2n$ holes. We know that $S^2 - \{p\}$ is homeomorphic to $\mathbb{R}^2$. Thus, subtracting an additional $2n-1$ points implies that our space is homeomorphic to $\mathbb{R}^2$ with $2n-1$ distinct points removed, so from above:

$$\pi_1(\mathbb{R}^3 - X, x_0) = \pi(R_{2n-1}, x_0) = \mathbb{Z} * \cdots * \mathbb{Z}$$

repeated $2n-1$ times.

1.7. **Problem 7.** *Show that every continuous map $f : \mathbb{R}P^2 \to S^1$ is nulhomotopic.*

We utilize the general lifting lemma. Consider a continuous map $f : Y \to B$, and covering map $p : E \to B$, with $p(e_0) = b_0$ and $f(y_0) = b_0$. Suppose Y is path-connected locally path-connected. Then there exists a lift of f to $\tilde{f}$ with $\tilde{f}(y_0) = e_0$ if and only if:

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$$

We know that, given $y_0 \in \mathbb{R}P^2$, we have $\pi_1(\mathbb{R}P^2, y_0) = \mathbb{Z}/\mathbb{Z}2$. We also know that $\pi_1(S^1, b_0) = \mathbb{Z}$. Clearly, if ϕ is a homomorphism from $\mathbb{Z}/\mathbb{Z}2$ to $\mathbb{Z}$, then $\phi(0) = 0$. Suppose $\phi(1) = n \neq 0$. Then:

$$\phi(0) = \phi(1+1) = 2\phi(1) = 2n \neq 0$$

a contradiction, so we must have $\phi(1) = 0$. Thus, f_* must be the trivial homomorphism, so we clearly have $f_*(\pi_1(\mathbb{R}P^2, y_0)) \subset p_*(\pi_1(\mathbb{R}, e_0))$, where $p : \mathbb{R} \to S^1$ is the usual covering map.

From the general lifting lemma, we can lift f to $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R}$. Let $F(x, t) = (1-t)\tilde{f}(x) + te_0$. Clearly, this is a homotopy of $\tilde{f}(x)$ and e_0. Then $p \circ F$ is a homtopy of $p \circ \tilde{f} = f$ and $p(e_0)$, so f is nulhomotopic.

1.8. **Problem 8.** *Prove that there exists no retraction from the solid torus to its boundary torus.*

Recall that we proved that there exists no retraction from D^2 to S^1.

Suppose there is a retraction r from $S^1 \times D^2$ to $S^1 \times S^1$. Then define $g : D^2 \to S^1$ as:

$$g(x) = (\pi_2 \circ r)(x_0, x)$$

where $x_0 \in S^1$ and $\pi_2 : S^1 \times S^1 \to S^1$ is a projection. Clearly, this map is continuous, as both π_2 and r are continuous. Note that for $x \in S^1$, we have $(x_0, x) \in S^1 \times S^1$, so $r(x_0, x) = (x_0, x)$ (by definition of r), so:

$$g(x) = (\pi_2 \circ r)(x_0, x) = \pi_2(x_0, x) = x$$

so g is a retraction from D^2 to S^1, a contradiction. It follows that there is no retraction from $S^1 \times D^2$ to $S^1 \times S^1$.

Alternatively, we could have completed this proof through more algebraic means. Note that $S^1 \times S^1$ has fundamental group $\mathbb{Z} \times \mathbb{Z}$, while $S^1 \times D^2$ deform retracts to S^1, so it has fundamental group S^1.

If a retract exists, then inclusion of $S^1 \times S^1$ into $S^1 \times D^2$ induces an injective homomorphism of fundamental groups. However, there is no injective homomorphism of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z}$. Suppose ϕ is such a homomorphism. Then let:

$$\phi(1,0) = a \quad \text{and} \quad \phi(0,1) = b$$

But we then have $\phi(b,0) = ab$ and $\phi(0,a) = ab$, so $\phi(b,0) = \phi(0,a)$, a clear contradiction to injectivity. Thus, not such retraction can exist.

1.9. **Problem 9.** *Let $F_1 \cup F_2 \cup F_3 = S^2$, where each F_i is closed. Show that some F_i contains a pair of antipodal points.*

Lemma 1. *Let X be a metric space and $f(x) = d(x, A)$. If A is closed, $f(x) = 0$ implies that $x \in A$.*

Proof. Suppose $f(x) = 0$, so $\inf_{a \in A} d(x,a) = 0$. Suppose $x \notin A$, so $x \notin \bar{A}$. Then there exists some ϵ-ball around x disjoint from A, so $d(x,a) \geq \epsilon$ for all $a \in A$. This is a contradiction, so $x \in A$. □

We wish to apply Borsuk-Ulam. Consider the functions $f(x) = (d(x, F_1), d(x, F_2))$. Borsuk-Ulam says that there is x such that $f(x) = f(-x)$. If $f(x)_1 = 0$ or $f(x)_2 = 0$, then we automatically have $x, -x \in F_1$ or $x, -x \in F_2$. If both components are non-zero, then we must have $x, -x \in F_3$, as they are not in F_1 nor F_2.

1.10. **Problem 10.** *Let x_0 and x_1 be points of a path-connected space. Show that $\pi(X, x_0)$ is abelian if and only if $\hat{\alpha} = \hat{\beta}$ for every pair of paths α, β from x_0 to x_1. Recall that $\hat{\alpha}([f]) = [\overline{\alpha} * f * \alpha]$.*

First, let us suppose that $\pi_1(X, x_0)$ is abelian. Let α and β be paths from x_0 to x_1. Then $\alpha * \overline{\beta}$ and $\beta * \overline{\alpha}$ are loops based as x_0. Let f be a loop at x_0. We therefore have:

$$[\alpha * \overline{\beta}] * [f] * [\beta * \overline{\alpha}] = [\alpha * \overline{\beta}] * [\beta * \overline{\alpha}] * [f] = [f]$$

where each element of the product is a loop in $\pi_1(X, x_0)$. It follows that:

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha] = [\overline{\alpha}] * [\alpha * \overline{\beta}] * [f] * [\beta * \overline{\alpha}] * [\alpha] = [\overline{\beta}] * [f] * [\beta] = \hat{\beta}([f])$$

from above.

Now, suppose that $\hat{\alpha} = \hat{\beta}$ for every pair of paths. Suppose $[f], [g] \in \pi_1(X, x_0)$. Let α be a path from x_0 to x_1. We also have $h = f * \alpha$ a path from x_0 to x_1. Thus:

$$\hat{\alpha}([g]) = [\overline{\alpha} * g * \alpha] = \hat{h}([g]) = [\overline{\alpha} * \overline{f} * g * f * \alpha]$$

This then implies that $[g] = [\overline{f} * g * f]$, so $[f] * [g] = [g] * [f]$. Thus, the group is abelian.
