C*-Algebra K-theory: Suggested Problems

Problem Set 1

1. (RLL Exercise 1.3) **Unitization.** Let A be a C^*-algebra (with or without a unit). Set

$$\tilde{A} := \{(a, \alpha) : a \in A, \alpha \in \mathbb{C}\}$$

and define addition and scalar multiplication coordinate-wise on $\tilde{A}$, and define mutliplication and involution by

$$(a, \alpha) \cdot (b, \beta) = (ab + \beta a + \alpha b, \alpha\beta), \qquad (a, \alpha)^* = (a^*, \overline{\alpha})$$

Let $\iota : A \to \tilde{A}$ and $\pi : \tilde{A} \to \mathbb{C}$ be given by $\iota(a) = (a, 0)$ and $\pi(a, \alpha) = \alpha$ respectively.

(i) Justify that $\tilde{A}$ is a $*$-algebra. Show that the element $(0, 1)$ in $\tilde{A}$ is a unit for $\tilde{A}$, that ι is an injective $*$-homomorphism, and that π is a surjective $*$-homomorphism.
We shall hereafter denote the unit element $(0, 1)$ in $\tilde{A}$ by $1_{\tilde{A}}$ or simply by 1.

Suppressing ι we obtain that A is contained in $\tilde{A}$ and that

$$\tilde{A} = \{a + \alpha \cdot 1 : a \in A, \alpha \in \mathbb{C}\}$$

Let $\|\cdot\|_A$ be the norm on A. For each $x \in \tilde{A}$, set

$$|||x|||_{\tilde{A}} = \sup\{\|ax\|_A : a \in A, \|a\|_A \leq 1\}$$

and define $\|x\|_{\tilde{A}} = \max\{|||x|||_{\tilde{A}}, |\pi(x)|\}$.

(ii) Show that $\|a\|_{\tilde{A}} = \|a\|_A$ for all $a \in A$.

(iii) Let $x \in \tilde{A}$. Show that $x = 0$ if $\|x\|_{\tilde{A}} = 0$.

(iv) Show that $\tilde{A}$ is a unital C^*-algebra. More specifically, show that $\|\cdot\|_{\tilde{A}}$ is a norm, that $\|xy\|_{\tilde{A}} \leq \|x\|_{\tilde{A}} \|y\|_{\tilde{A}}$, and that $\|x^*x\|_{\tilde{A}} = \|x\|_{\tilde{A}}^2$ for all $x, y \in \tilde{A}$, and show that $\tilde{A}$ is complete with respect to $\|\cdot\|_{\tilde{A}}$.

(v) Show that the sequence

$$0 \longrightarrow A \xrightarrow{\iota} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

is split exact. Conclude that A is an ideal in $\tilde{A}$.

(vi) Show that $\tilde{A}$ is isomorphic to $A \oplus \mathbb{C}$ (as C^*-algebras) if and only if A is unital.

2. (RLL Exercise 1.4) Recall that an element p in a C^*-algebra A is a *projection* if $p = p^* = p^2$, and an element u in a unital C^*-algebra is said to be *unitary* if $uu^* = u^*u = 1$.

Let A be a (not necessarily unital) C^*-algebra.

(i) Let $p \in A$ be a projection. Show that $\mathrm{sp}(p) \subset \{0, 1\}$.

(ii) Suppose $p \in A$ is normal and $\mathrm{sp}(p) \subset \{0, 1\}$. Show that p is a projection.

(iii) Assume A is unital and let $u \in A$ be unitary. Show that $\mathrm{sp}(u) \subset \mathbb{T}$. Recall

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

(iv) Assume A is unital and $u \in A$ is a normal element such that $\mathrm{sp}(u) \subset \mathbb{T}$. Show that u is unitary.

3. (RLL Exercise 1.6) Let A be a unital C^*-algebra, and let $a \in A$.

   (i) Show that a is invertible if and only if both aa^* and a^*a are invertible, and show that $a^{-1} = (a^*a)^{-1}a^* = a^*(aa^*)^{-1}$ in this case.

   (ii) Suppose that b is an invertible and normal element in A. Show that $b^{-1} = f(b)$ for some function $f \in C(\mathrm{sp}(b))$.

   (iii) Show that if a is invertible, then $a^{-1} \in C^*(a)$. [Hint: Use (i) and (ii).]

4. (RLL Exercise 1.8) Let A and B be C^*-algebras, and let $\varphi : A \to B$ be a $*$-homomorphism.

   (i) Assume A and B are unital and that φ is a unital $*$-homomorphism. Show that $\mathrm{sp}(\varphi(a)) \subset \mathrm{sp}(a)$ for each $a \in A$; and show that these two sets are equal if φ is injective.

   (ii) Still assuming A and B are unital and that φ is a unital $*$-homomorphism, show that $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$ (i.e. φ is *contractive*); and show that if in addition φ is injective, then $\|\varphi(a)\| = \|a\|$ for all $a \in A$ (i.e. φ is *isometric*). [Hint: Use (i) applied to a^*a together with the fact that $r(c) = \|c\|$ for every normal element $c \in A$.]

   (iii) Show that $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$; and show that $\|\varphi(a)\| = \|a\|$ for all $a \in A$ if φ is injective. Notice we are no longer assuming A and B are unital.

   (iv) Conclude that the image of φ is a C^*-algebra.

5. (Based on the first lecture.) Let R be a unital ring. Consider the map

$$\begin{array}{ccccc} \Phi : \left\{ \begin{array}{c} \text{Idempotents in } M_n(R) \\ \text{for some } n \end{array} \right\} & \to & \left\{ \begin{array}{c} \text{Isomorphism classes of finitely} \\ \text{generated projective right } R\text{-modules} \end{array} \right\} \\ e \in M_n(R) & \mapsto & [eR^n] \end{array}$$

   where $[M]$ denotes the isomorphism class of a right R-module M. (Assume for the moment that the codomain of this map is really a set)

   (i) Show that Φ is a well-defined surjection.

   (ii) Show that if $e \in M_n(R)$ and $f \in M_m(R)$, then $\Phi(e) = \Phi(f)$ if and only if e and f are *Murray-von Neumann equivalent*, meaning that there exist $u \in \mathrm{Hom}_R(R^m, R^n)$ and $v \in \mathrm{Hom}_R(R^n, R^m)$ such that $e = uv$ and $f = vu$.

   (iii) Show, in ZFC, that the codomain of Φ is really a set (and not a proper class). Moreover, show that $[M] + [N] := [M \oplus N]$ is a well-defined binary operation on the codomain of Φ giving it the structure of an abelian monoid.

   (iv) Show that Φ is a homomorphism of abelian monoids. That is, if $e \in M_n(R)$ and $f \in M_m(R)$, then the idempotent

$$e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in M_{n+m}(R)$$

   satisfies $\Phi(e \oplus f) = \Phi(e) + \Phi(f)$.

6. (An application to geometry: the Hopf line bundle on S^2.) Let σ_1, σ_2, and σ_3 be three matrices in $M_2(\mathbb{C})$ satisfying the *canonical anticommutation relations*, i.e.

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$$

for all $i, j = 1, 2, 3$ (where δ is Kronecker's delta).

(i) Find explicitly three matrices σ_1, σ_2, and σ_3 in $M_2(\mathbb{C})$ satisfying the above relations.

(ii) Viewing S^2 as the level set of $x_1^2 + x_2^2 + x_3^2 - 1$ in $\mathbb{R}^3$, define a continuous function $F : S^2 \to M_2(\mathbb{C})$ by $F(x_1, x_2, x_3) = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$. Show that

$$e := \frac{1+F}{2}$$

defines an idempotent in $M_2(C(S^2))$ (which we naturally identify with $C(S^2, M_2(\mathbb{C}))$).

(iii) As we will see later, by the Serre-Swan theorem, e gives rise to a (smooth) subbundle E of the trivial bundle $S^2 \times \mathbb{C}^2$ given by

$$E = \{(x, v) \in S^2 \times \mathbb{C}^2 : e(x)v = v\}$$

Show that $\operatorname{rank}(e(x)) = 1$ for all $x \in S^2$ and conclude that the rank of E is one.

(iv) Let $f : S^2 \to \mathbb{C}P^1$ be the (smooth) map given by $f(x) = \operatorname{im}(e(x))$. Recall that the *canonical line bundle* on $\mathbb{C}P^1$ is the (holomorphic) line bundle L on $\mathbb{C}P^1$ given by

$$L := \{(x, v) \in \mathbb{C}P^1 \times \mathbb{C}^2 : v \in x\}$$

Show that f is a homeomorphism (in fact it is a diffeomorphism) and that $E = f^*L$.

(v) Let us view S^3 as the level set of $|z_1|^2 + |z_2|^2 - 1$ in $\mathbb{C}^2$. Recall that the line bundle associated to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ is the line bundle $(S^3 \times \mathbb{C})/S^1$ where $((z_1, z_2), w) \sim ((e^{it}z_1, e^{it}z_2), e^{-it}w)$. Show E is isomorphic to the line bundle associated to the Hopf fibration. [Hint: First define a map $S^3 \times \mathbb{C} \to L$ and show it descends to an isomorphism of the associated bundle with L. Then use part (iv).]