MAT436 problem set 2

Jack Ceroni
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I. Problem 1 (First Suggested Problem)

Part A. We can make use of the formula for the sum of a finite geometric series in the case that $e^{2\pi ix} \neq 0$ (this formula holds over the unit-norm complex numbers, as is easy proved via induction). More specifically,

$$D_N(x) = \sum_{n=-N}^{N} \chi_n(x) = \chi_0(x) + \sum_{n=1}^{N} \chi_n(x) + \chi_{-n}(x) \tag{1}$$

$$= 1 + \sum_{n=1}^{N} e^{2\pi inx} + e^{-2\pi inx} \tag{2}$$

$$= 1 + \frac{e^{2\pi ix}(1 - e^{2\pi iNx})}{1 - e^{2\pi ix}} + \frac{e^{-2\pi ix}(1 - e^{-2\pi iNx})}{1 - e^{-2\pi ix}} \tag{3}$$

$$= 1 - \frac{e^{\pi ix}(1 - e^{2\pi iNx})}{2i\sin(\pi x)} + \frac{e^{-\pi ix}(1 - e^{2\pi iNx})}{2i\sin(\pi x)} \tag{4}$$

$$= 1 + \frac{2i\sin\left(\left(N + \frac{1}{2}\right)2\pi x\right) - 2i\sin(\pi x)}{2i\sin(\pi x)} \tag{5}$$

$$= \frac{\sin\left(\left(N + \frac{1}{2}\right)2\pi x\right)}{\sin(\pi x)} \tag{6}$$

as desired. When $x = 0$ and $e^{2\pi ix} = 1$, we obviously have $\sum_{n=-N}^{N} e^{2\pi inx} = 2N + 1$. From here, note that

$$\int_{[0,1]} e^{2\pi nix} \ dx = 0 \quad \text{for } n \neq 0 \tag{7}$$

and is 1 for $n = 0$. Thus, distributing the integral over terms of the sum yields $\int_{[0,1]} D_N(x) \ dx = 1$ as desired.

Part B. Of course, we have

$$T_N f = \sum_{n=-N}^{N} \int_{[0,1]} f(x)\chi_n(x) \ dx = \sum_{n=-N}^{N} \int_{[0,1]} f(x)e^{2\pi inx} \ dx = \sum_{n=-N}^{N} a_n^f \chi_n(0) = s_N^f(0) \tag{8}$$

as desired.

Part C. We have

$$|T_N f| = \left| \int_{\mathbb{T}} f(x) D_N(x) \right| \leq \int_{\mathbb{T}} |f(x)||D_N(x)| \ dx \leq ||f||_\infty \int_{\mathbb{T}} |D_N(x)| \ dx. \tag{9}$$

It follows that if we can saturate this bound, then we will have shown that $||T_N|| = \int_{\mathbb{T}} |D_N(x)| \ dx$. Indeed, this is the case. To see this, note that for any N, there exist a finite number of roots $x_1 < x_2 < \cdots < x_{n-1} < x_n$ where $D_N(x)$ changes sign (we set $x_0 = 0$ and $x_{n+1} = 1$). Thus, the function $F_N(x) = |D_N(x)| D_N(x)^{-1}$ is, almost-everywhere, a well-defined continuous function (i.e. everywhere except for the x_j). Since integrals are invariant under altering a function on a measure-0 set, we can, for our purposes, think of F as being defined on the entire domain $\mathbb{T}$, setting it to 0, perhaps, at the x_j.

From here, note that it is a well-known result in Fourier theory that we may L^1-approximate step functions with continuous functions. In particular, let us define

$$S^{(j)}(x) = \begin{cases} F_N(x) = |D_N(x)|D_N(x)^{-1} & \text{if } x \in [x_j, x_{j+1}] \\ 0 & \text{if } x \notin [x_j, x_{j+1}] \end{cases} \tag{10}$$

We then define $S_\varepsilon^{(j)}$ to be the ε-continuous approximation in L^1 norm, which is to say that

$$\int_{\mathbb{T}} |S_\varepsilon^{(j)}(x) - S^{(j)}(x)|\ dx < \varepsilon \tag{11}$$

We then let $P_\varepsilon(x) = S_\varepsilon^{(1)}(x) + \cdots + S_\varepsilon^{(n)}(x)$, also a continuous function. Note that $||P_\varepsilon||_\infty = 1$, of course, as the x_j form a partition of $[0, 1]$. We then have

$$\begin{aligned}
|T_N P_\varepsilon| = \left| \int_{\mathbb{T}} D_N(x) P_\varepsilon(x) \right| &= \left| \int_{\mathbb{T}} D_N(x) \left(P_\varepsilon(x) - F_N(x) + F_N(x)\right)\ dx \right| &(12)\\
&= \left| \int_{\mathbb{T}} |D_N(x)|\ dx + \int_{\mathbb{T}} D_N(x)(P_\varepsilon(x) - F_N(x))\ dx \right| &(13)\\
&\geq \left| \int_{\mathbb{T}} |D_N(x)|\ dx - \left| \int_{\mathbb{T}} D_N(x)(P_\varepsilon(x) - F_N(x)) \right| \right| &(14)\\
&\geq \int_{\mathbb{T}} |D_N(x)| - \int_{\mathbb{T}} |D_N(x)||P_\varepsilon(x) - F_N(x)|\ dx &(15)\\
&\geq \int_{\mathbb{T}} |D_N(x)| - ||D_N||_\infty \sum_{j=0}^{n+2} ||S_\varepsilon^{(j)}(x) - S^{(j)}(x)||_1 &(16)\\
&\leq \int_{\mathbb{T}} |D_N(x)|||D_N||_\infty (n+2)\varepsilon &(17)
\end{aligned}$$

and since ε was arbitrary, we can find continuous functions f such that

$$|T_N f| \geq \int_{\mathbb{T}} |D_N(x)|\ dx - \varepsilon \tag{18}$$

for any $\varepsilon > 0$. It follows immediately that $||T_N|| = \int_{\mathbb{T}} |D_N(x)|\ dx$, as desired.

Part C. Note that

$$\begin{aligned}
||T_N|| = \int_{\mathbb{T}} |D_N(x)|\ dx &= \int_{[0,1]} \left| \frac{\sin\left(\left(N + \frac{1}{2}\right) 2\pi x\right)}{\sin(\pi x)} \right|\ dx &(19)\\
&\geq \int_{[0,1]} \frac{\left|\sin\left(\left(N + \frac{1}{2}\right) 2\pi x\right)\right|}{\pi x}\ dx &(20)\\
&= \int_{[0,2N+1]} \frac{|\sin(\pi x)|}{\pi x}\ dx &(21)
\end{aligned}$$

Note that on each interval $[k + 1/4, k + 3/4]$, for integer k, we have $|\sin(\pi x)| \geq \frac{1}{\sqrt{2}}$. Thus,

$$\int_{[0,2N+1]} \frac{|\sin(\pi x)|}{\pi x}\ dx \geq \frac{1}{\sqrt{2}\pi} \sum_{k=0}^{2N} \frac{1}{k+1} = \frac{1}{\sqrt{2}\pi} \sum_{k=1}^{2N} \frac{1}{k} \tag{22}$$

which will diverge as $N \to \infty$. Thus, $||T_N|| \to \infty$ for $N \to \infty$.

Part D. Let us recall the principle of uniform boundedness. Given X Banach and Y a normed vector space, and $B(X,Y)$ the set of bounded linear operators from X to Y. Given some collection $F \subset B(X,Y)$, if $\sup_{T\in F} ||Tx|| < \infty$ for all $x \in X$, then

$$\sup_{T\in F} ||T|| = \sup_{T\in F, ||x||\leq 1} ||Tx|| < \infty \tag{23}$$

In particular, suppose F is the collection of T_N. Suppose there existed no such function with diverging Fourier series at $x = 0$. Since $T_N f = s_N(0)$, it would immediately follow that $\sup_N |T_N f| < \infty$ for every f. Thus, we would necessarily have $\sup_N ||T_N|| < \infty$ as well, which we have shown not to be the case. Hence, there must exist some f where $|T_N f| = |s_N(0)|$ approaches ∞ for $N \to \infty$, implying the Fourier series diverges at 0.

A. Problem 2

As a problem, I thought I would try to prove an interesting lemma from Pedersen independently (without consulting his proof).

Lemma I.1. If X is a normed vector space and D is a closed subspace such that both D and X/D are Banach spaces, then X is a Banach space.

Proof. Recall that if D is a closed subpsace, then the infimum seminorm on X/D becomes a true norm. In particular, if $\pi : X \to X/D$ is the quotient, we define

$$||\pi(x)|| = \inf_{y\in D} ||x - y|| \tag{24}$$

Let x_n be a Cauchy sequence in X. Consider the sequence $\pi(x_n)$. Note that

$$||\pi(x_n) - \pi(x_m)|| = \inf_{y\in D} ||x_n - x_m - y|| \leq ||x_n - x_m|| \tag{25}$$

as $0 \in D$, so it follows that the sequence $\pi(x_n)$ is Cauchy. Thus, since X/D is complete, this sequence converges to some $\pi(x) \in X/D$. In particular, the quantity

$$||\pi(x_n) - \pi(x)|| = \inf_{y\in D} ||x_n - x - y|| := C_n \tag{26}$$

will become arbitrarily small, as we make n large. For each n, let us choose some $y_n \in D$ such that $||x_n - x - y_n|| - C_n \leq \frac{1}{n}$ (we can do this as C_n is the infimum). Let us now consider the sequence of y_n. Note that

$$\begin{aligned} ||y_n - y_m|| &= ||(y_n + x - x_n) + (x_m - x - y_m) + (x_n - x_m)|| && (27)\\ &\leq ||x_n - x - y_n|| + ||x_m - x - y_m|| + ||x_n - x_m|| && (28)\\ &\leq C_n + \frac{1}{n} + C_m + \frac{1}{m} + ||x_n - x_m|| && (29) \end{aligned}$$

Since $C_n, C_m \to 0$ for $n, m \to \infty$, it follows that we can choose n and m sufficiently large such that the above sum is arbitrarily small. Thus, y_n is a Cauchy sequence, so it converges to some $y^* \in D$. Finally, we claim that x_n converges to $x + y^*$. Indeed,

$$||x_n - x - y^*|| \leq ||x_n - x - y_n|| + ||y_n - y^*|| \leq C_n + \frac{1}{n} + ||y_n - y^*|| \tag{30}$$

which goes to 0 for $n \to \infty$, so it follows that $x_n \to x + y^* \in X$, so X is complete and thus by definition a Banach space. $\square$

B. Problem 3 (Suggested Problem 2)

Recall that $L^\infty(X,\mu)$ is the set of all functions which are essentially bounded relative to μ (i.e. those which are bounded except possibly on a set of measure-0). We assume $gf \in L^2(X,\mu)$ for all $f \in L^2(X,\mu)$. Suppose $g \notin L^\infty(X,\mu)$. It follows that $\mu(\{x \mid |g(x)| \geq N\}) = \mu(|g|^{-1}([N,\infty)) > 0$ for each integer N, where we know g is measurable, so $|g|^{-1}([N,\infty))$ is measurable.

From here, note that the statement that $gf \in L^2(X,\mu)$ and $f \in L^2(X,\mu)$ is precisely the statement that

$$\int |gf|^2 \; d\mu < \infty \quad \text{and} \quad \int |f|^2 \; d\mu < \infty \tag{31}$$

for all f. Let $X_N = |g|^{-1}([N,\infty))$. We define the simple function ϕ_N to be $\mu(X_N)^{-1/2}N^{-1} < \infty$ on X_N and 0 everywhere else. Obviously, $\int |\phi_N|^2 \; d\mu = \frac{1}{N^2}$. If we then let $\phi = \phi_1 + \phi_2 + \cdots$, note that we will have

$$\int |\phi|^2 \leq \sum_j \int |\phi_j|^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty \tag{32}$$

so this function is L^2. Moreover,

$$\int |\phi g|^2 \; d\mu \leq \sum_{j=1}^{\infty} \mu(X_j)^{-1} j^{-2} \int_{X_j} |g|^2 \; d\mu \geq \sum_{j=1}^{\infty} \mu(X_j)^{-1} j^{-2} \cdot j^2 \mu(X_j) = \sum_{j=1}^{\infty} 1 = \infty \tag{33}$$

so it follows that $fg \notin L^2$, a contradiction. Thus, we must have $g \in L^\infty$. From here, we have $M_g(f) = gf$. Obviously it is linear. Note that each set $X_n = |g|^{-1}([n,\infty))$ for $n < ||g||_\infty$ will have positive measure. Let $Y_n = X_{||g||_\infty - n^{-1}}$. We then let ψ_n be the simple function which is $\mu(Y_n)^{-1/2}$ on X_n. We have

$$||M_g(\psi_n)|| = \left(\int |\psi_n g|^2 \; d\mu \right)^{1/2} = \left(\mu(X_n)^{-1} \int_{X_n} |g|^2 \right)^{1/2} \geq ||g||_\infty - \frac{1}{n}. \tag{34}$$

where this holds for any n, so $||M_g|| \geq ||g||_\infty$. Of course, we can also see that

$$\left(\int |fg|^2 \; d\mu \right)^{1/2} \leq ||g||_\infty \left(\int |f|^2 \; d\mu \right)^{1/2} = ||g||_\infty ||f|| \tag{35}$$

so that $||M_g|| \leq ||g||_\infty$. Thus, we have inequalities both ways, so $||M_g|| = ||g||_\infty$, and the proof is complete.