Complex Analysis

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Contents

I. Introduction

The goal of these notes is to follow a basic first course in complex analysis, following Edward Bierstone's course at the University of Toronto (MAT354), in the Fall of 2023. I will also be adding information from my own readings/other resources. I hope that someone finds these notes useful. However, in the event that no one does, I certainly will!

Here is a list of other resources that I will draw from in these notes:

- Edward Bierstone's lectures (MAT354, University of Toronto, Fall 2023)

- Cartan, *Elementary theory of analytic functions of one or several complex variables*

- Ahlfors, *Complex analysis*

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II. Basics and notation

We begin our discussion of complex analysis by introducing the complex plane, $\mathbb{C}$. Of course, $\mathbb{C}$ can be constructed via taking $\mathbb{R}^2$ and imposing a particular complex structure over it, which tells us how to multiply by the imaginary unit i.

Definition II.1 (Complex structure).

We can also construct $\mathbb{C}$ via taking $\mathbb{R}[x]$ and quotienting by the ideal $(x^2 + 1)$. For the purposes of these introductory notes, it is not particularly important to dwell on this point: all that we need to know is that there are many different ways to construct $\mathbb{C}$, each of which are equivalent. They are all topologically identical to $\mathbb{R}^2$, and posses the desired "complex arithmetic" that we are familiar with and want to use.

We will assume basic familiarity with fundamental notions related to the manipulation of complex numbers. As for notation, we let $\overline{z}$ denote the complex conjugate of $z \in \mathbb{C}$. We let $|z|$ denote the absolute value or modulus. We let $\Re[z]$ and $\Im[z]$ denote the real and complex components in $\mathbb{R}$ of $z \in \mathbb{C}$, respectively. Assuming that we have already constructed the function $\sin(t)$ and $\cos(t)$, and know their basic properties, it is straightforward to show that any $(x, y) \in \mathbb{C}$ can be expressed as $(r\cos(\theta), r\sin(\theta))$: we simply set $r = \sqrt{x^2 + y^2}$ and apply the Pythagorean identity to $\cos(\theta)$ and $\sin(\theta)$, which allows us to choose $\theta \in [0, 2\pi)$ satisfying the desired equality.

Definition II.2 (Polar coordinates). The map going from $\mathbb{R}^2$ to $\mathbb{R} \times [0, 2\pi)$ which sends $(x, y) \mapsto (r, \theta)$ via the above protocol is a bijection where the imagine of (x, y) is refered to as the *polar coordinate representation* of a pair of coordinates (x, y).

Definition II.3 (Complex exponential). We define the map $t \mapsto \cos(t) + i\sin(t) := e^{it}$ from $\mathbb{R}$ to $\mathbb{C}$ and refer to it as the *complex exponential*. It is easy to demonstrate that the complex exponential has all the standard properties of a regular exponential function, when it comes to the inverse, multiplication, powers, etc. This definition also agrees with the series definition of the exponential function mapping from $\mathbb{R}$ to $\mathbb{R}$, where we substitute an imaginary number, and utilize the multiplication rule for repeatedly multiplying i with itself.

Note: I may revise this section in the future, if I think of any extra, basic information that must be exposited/notation that must be explained.

III. The Riemann sphere and complex projective space

We are now able to move on to a discussion of different representations of the complex plane.

Definition III.1 (Riemann sphere). Rather than strictly working with the standard complex plane $\mathbb{C}$, we will often consider its one-point compactification $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where ∞ is refered to as the "point at infinity". $\overline{\mathbb{C}}$ is called the Riemann sphere, as it is homeomorphic to a sphere (topologically, the one-point compactification of $\mathbb{R}^2$ is homeomorphic to S^2).

Of course, the function addition $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and multiplication, $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ must be extended to functions from $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ (to the greatest extent possible). We take

$$a + \infty = \infty + a = \infty \quad \text{for all } a \neq \infty \tag{1}$$
$$b \cdot \infty = \infty \cdot b = \infty \quad \text{for all } b \neq 0 \tag{2}$$

Now, when we say "to the greatest extent possible" it is clear what we mean: we cannot define $0 \cdot \infty$ or $\infty + \infty$, or else we will violate the standard laws of addition and multiplication that we wish to satisfy with our new, extended arithmetic. For example, one may be inclined to say that $\infty + \infty = \infty$, but then if additive cancellation holds, $\infty = 0$, a contradiction.

Claim III.1 (Stereographic projection). We can define a map f (in fact, a homeomorphism) from $S^2 - (0, 0, 1)$ to $\mathbb{R}^2$ called the *stereographic projection*. This map is given by

$$f : (x, y, z) \mapsto w = \frac{1}{1 - z}(x, y) \tag{3}$$

Proof. Clearly, this is a continuous function. Moreover, given $(x, y, z) \in S^2 - (0, 0, 1)$, note that $x^2 + y^2 + z^2 = 1$. It follows immediately that if identify $x + iy$ with (x, y), then we will have

$$\frac{x + iy}{1 - z} = w \implies \frac{x^2 + y^2}{(1 - z)^2} = \frac{1 - z^2}{(1 - z)^2} = \frac{1 + z}{1 - z} = |w|^2 \tag{4}$$

for some $w \in \mathbb{C}$. Or, if w is in the image of the stereographic projection, then we have $z = \frac{|w|^2 - 1}{|w|^2 + 1}$. Moreover,

$$x = \frac{1}{2}(1 - z)(w + \overline{w}) = \frac{w + \overline{w}}{|w|^2 + 1} \quad \text{and} \quad y = \frac{1}{2i}(1 - z)(w - \overline{w}) = i\frac{\overline{w} - w}{|w|^2 + 1}. \tag{5}$$

Thus, we define

$$f^{-1} : w \mapsto \left(\frac{w + \overline{w}}{|w|^2 + 1}, i\frac{\overline{w} - w}{|w|^2 + 1}, \frac{|w|^2 - 1}{|w|^2 + 1} \right). \tag{6}$$

as a function from $\mathbb{R}^2$ to S^2, where (x, y) is identified with $w = x + iy$. Clearly, this function is continuous, and it is easy to verify that it is in fact the desired inverse of f, so f is a homeomorphism. $\square$

It follows from basic point-set topology that we can extend this homeomorphism to a homeomorphism between the one-point compactifications, S^2 and $\mathbb{R}^2 \cup \{\infty\}$, where $(0, 0, 1)$ is sent to ∞, the so-called point at infinity.

Fact III.1. The one-point compactification of $\mathbb{R}^2$ is homeomorphic to S^2 via the unique extension of the stereographic projection.

Definition III.2 (Complex projective space)**. TODO**: Fill this in

Now, let us move back to the discussion of the Riemann sphere, which is the representation of the complex numbers we will work with the most. It is natural to ask what the correspondence is between basic objects in $\overline{\mathbb{C}}$ and S^2, via the stereographic projection. We begin with a result.

Theorem III.1. Via stereographic projection, all circles in $S^2 - (0, 0, 1)$ correspond to lines or circles in $\mathbb{R}^2$. Conversely, all lines and circles in $\mathbb{R}^2$ correspond to circles in the sphere minus a point.

Proof. Let us suppose that we have a circle $C \subset S^2$. Such a circle corresponds to a collection of points which lie on the intersection of a plane with S^2. Thus, $C = \{(x, y, z) \in S^2 \mid ax + by + cz = d\}$ for some real constants a, b, c, d. Let f be the stereographic projection, we wish to find $f(C)$.

Let us assume that $w \in f(C)$ with $w = x + iy$ identified with (x, y) (we will often perform this identification use complex arithmetic more naturally). Moreover, suppose $f(x_0, y_0, z_0) = w$. Then we know that

$$|w|^2 = x^2 + y^2 = \frac{1 + z_0}{1 - z_0} \implies cx^2 + cy^2 = \frac{c + d - ax_0 - by_0}{1 - z_0} \implies cx^2 + cy^2 + ax + by = \frac{c + d}{1 - z_0} \tag{7}$$

Recall that $1 - z_0 = \frac{2}{|w|^2 + 1}$, so we then have

$$cx^2 + cy^2 + ax + by = \frac{c + d}{2}(x^2 + y^2 + 1) \tag{8}$$

which, upon simplification, if the equation of a circle or a line (a line when $c - d = 0$ and a circle otherwise). Conversely, given the circle or line in the plane, it is not difficult to verify that we can find constants a, b, c, d such that the above equation yields the equations of the circle/line. Thus, the inverse of the stereographic projection takes this points to the original set C: a circle. $\square$

Verifying that this construction extends to circles on the Riemann sphere and circles/lines in $\mathbb{R}^2 \cup \{\infty\}$ is straightforward and just requires some casework.

IV. Introduction to complex functions

A. Linear, polynomial, and rational functions

Now that we have described the arena in which we perform calculations (the complex plane/Riemann sphere), we can move on to understanding corresponding functional transformations.

Definition IV.1 (Linear functions)**.** When dealing with the complex plane, we can split notions of real functions into two categories: *real linear functions* and *complex linear functions.* Real linear functions are the functions which are linear when $\mathbb{C} \simeq \mathbb{R}^2$ is treated as a two-dimensional vector space over $\mathbb{R}$ and complex linear functions are those which are linear when $\mathbb{C}$ is treated as a one-dimensional vector space over $\mathbb{C}$.

Claim IV.1. Any complex linear function is of the form $w(z) = az$ for some $a \in \mathbb{C}$. Any real linear function that preserves angles is of the form $w(z) = az$ for some $a \in \mathbb{C}$, or $w(z) = a\overline{z}$.

Proof. Obviously, if f is complex linear, then $f(c) = f(1) \cdot c$ for any $c \in \mathbb{C}$, so $f(z) = f(1)z$.

Now, let g be a real linear function which preserves angles. Clearly, $g(1) = w$. Let $s(z) = w^{-1}z$, so $f = s \circ g$ fixes 1. Since f preserves angles, it follows that $f(0,1) = f(i) = ci$ as 1 is fixed, for some $c \in \mathbb{R}$. Note from linearity, we must have $f(1,1) = (1,c)$. But if the angle between $(1,0)$ and $(1,1)$ is to be preserved, $c = \pm 1$. So it follows that $f(0,1) = (0, \pm 1)$ and $f(1,0) = (1,0)$. We know f on a basis, so it follows that $f(z) = z$ or $f(z) = \overline{z}$. It follows that $g(z) = wf(z)$, which is either wz or $w\overline{z}$, as desired. $\square$

Definition IV.2. Given a complex rational function $R : \mathbb{C} \to \overline{\mathbb{C}}$, we can define an extension of R to $\overline{\mathbb{C}}$ where $R(\infty) = \lim_{z \to 0} R\left(\frac{1}{z}\right)$. Clearly, $R\left(\frac{1}{z}\right)$ is a well-defined rational function on $\mathbb{C}$, so $\lim_{z \to 0} R\left(\frac{1}{z}\right)$ is a well-defined point in $\overline{\mathbb{C}}$.

Given a rational function, it is possible to quickly sketch a rough portrait of its behaviour via analyzing its roots and poles.

Definition IV.3 (Roots and poles)**.** Given a rational function $R(z) = \frac{P(z)}{Q(z)}$ in reduced form, defined on $\overline{\mathbb{C}}$ (so $R(\infty)$ is well-defined) the roots of R are the collection of all z such that $P(z) = 0$. The poles are the collection of all z such that $Q(z) = 0$. Provided $z \neq \infty$, both $P(z)$ and $Q(z)$ will be finite, and since they are in reduced form, we will have $R(z) = 0$ at the roots and $R(z) = \infty$ at the poles (as common roots of P and Q cannot exist in reduced form). This also holds in the case that $z = \infty$, due to the fact that $R(\infty) = \lim_{z \to 0} R\left(\frac{1}{z}\right)$, where $R\left(\frac{1}{z}\right)$ is also a rational function in reduced form. The *order* or a finite root of $R(z)$ is the multiplicity of the root in $P(z)$. The order of a finite pole of $Q(z)$ is the multiplicity of the root in $Q(z)$. The order of a root/pole at infinite is the order of the 0-root/pole for the rational function $R\left(\frac{1}{z}\right)$.

Suppose we are provided a rational function $R(z) = \frac{P(z)}{Q(z)}$ in reduced form, such that $\deg(P) = m$ and $\deg(Q) = n$. A natural question to ask is the total number of roots and poles that a particular rational function has (where we count roots/pole swith multiplicity). Of course, by the fundamental theorem of algebra, R has m finite roots and n finite poles, so the only remaining question is how many infinite roots/poles the function has. Suppose $P(z) = \sum_{j=0}^{m} p_j z^j$ and $Q(z) = \sum_{j=0}^{n} q_j z^j$. Then note that

$$R\left(\frac{1}{z}\right) = \frac{P\left(\frac{1}{z}\right)}{Q\left(\frac{1}{z}\right)} = \frac{\sum_{j=0}^{m} p_j z^{-1}}{\sum_{j=0}^{n} q_j z^{-1}} = z^{n-m} \frac{\sum_{j=0}^{m} p_{m-j} z^j}{\sum_{j=0}^{n} q_{n-j} z^j} \tag{9}$$

Note that p_m and q_n are non-zero, as we have assumed that the polynomials are simplified. Therefore, the numerator and denominator of this rational function are both non-zero at $z = 0$, so the only roots/poles at $z = 0$ are given by the leading z^{n-m} term. In particular, if $n = m$, we have no roots/poles. If $n > m$, we have $n - m$ roots and no poles. And if $n < m$, we have $m - n$ poles and no roots.

It follows that in each of the three cases, we can write down the number of roots/poles: when $n = m$, we have $m = n$ roots and poles. When $n > m$, we have $m + (n - m) = n$ roots and n poles. Finally, when $n < m$, we have m roots and $(m - n) + n = m$ poles. We can easily theoremize this result:

Theorem IV.1. Given a rational function $R = P/Q$ in reduced form, where $m = \deg(P)$ and $n = \deg(Q)$, the number of roots is equal to the number of poles, with both numbers being equal to $k = \max\{m, n\}$. We call k the *degree* of a rational function.

Remark IV.1. This result pretty much matches our intuition from calculus: given a rational function of real polynomials, for large enough x, their quotient eventually behaves like a polynomial of degree x^{m-n} when $m > n$, an inverse polynomial of degree $n - m$ when $m < n$, and some constant when $m = n$. Treating this asymptotic behaviour as the "order" of an ∞-root/pole yields the same casework and the same result as above.

This result allows us to also prove some corollaries fairly immediately. For example:

Corollary IV.1.1. Given a rational function $R(z)$, the equation $R(z) = a$ for some $a \in \mathbb{C}$ has $\max\{m, n\}$ solutions (where m and n are the degrees of the reduced numerator and denominator).

Proof. $R(z) - a$ is a rational function, and thus has exactly $\max\{m, n\}$ roots. $\qquad\square$

Aside from the fairly low-hanging fruit of computing the number of roots/poles of a rational function, with a bit more effort, it is possible to *put a rational function in a form where its behaviour near poles is clearly exposed*. We call this form the **partial fraction decomposition** of a rational function.

Theorem IV.2 (Representation of complex rational functions by partial fractions)**.** Given a rational function R with unique poles at $\beta_1, \ldots, \beta_n$ and possibly at ∞, there exist polynomials $G, G_1, \ldots, G_n$ such that

$$R(z) = G(z) + \sum_{j=1}^{n} G_j\left(\frac{1}{z - \beta_j}\right) \tag{10}$$

Proof. Let us begin by considering the case of a pole at ∞. In particular, suppose R has a pole at ∞. It follows that if $R = \frac{P}{Q}$, then $\deg(P) > \deg(Q)$. Suppose we perform polynomial division, so that $P(z) = G(z)Q(z) + R(z)$, where $\deg(R) \leq \deg(Q)$ and G has no constant term. It follows that $R = G + H$, where $H = \frac{R}{Q}$. Clearly, H is finite at ∞, since the degree of the numerator is less than or equal to the degree of the denomninator, and G is simply a polynomial (this polynomial will in fact dictate the "behaviour" of the rational function as $z \to \infty$.

Let us now consider the case of a finite pole. As it turns out, we can easily reduce this case to the previous case of a pole at infinity. Let us define $z = \beta_j + \zeta^{-1}$, so that $\zeta = (z - \beta_j)^{-1}$. Clearly,

$$\widetilde{R}(\zeta) = R\left(\beta_j + \frac{1}{\zeta}\right) = \frac{\widetilde{P}(\zeta)}{\widetilde{Q}(\zeta)} \tag{11}$$

is a rational function in the variable ζ. Moreover, since R has a pole at β_j, R' will have a pole at $\zeta = \infty$. Thus, we can repeat the above procedure for the polynomial P' to get $P' = G'_j Q' + R'$ and $R'(\zeta) = G'_j(\zeta) + H'(\zeta)$, where H' is finite at $\zeta = \infty$. We then take $G_j(z) = G'_j\left(\frac{1}{z - \beta_j}\right)$, switching back to the variable z.

Our claim is now that the sum of all the G_j along with G is the original rational function R. Indeed, consider the rational function $F = R - G - \sum_j G_j$. Clearly, the only possible poles of this rational function are at the β_j and at ∞. Except for each β_i, note that $F = (R - G_i) - G - \sum_{j \neq i} G_j$. We know that $R - G_i$ is finite at β_i, by construction, and that each of the other polynomials will also be finite. This also holds true at the ∞ pole.

Thus, the rational function F can have no poles, so it must be a constant c (since the number of roots/poles is the maximum of the numerator and denominators' degrees). We can easily perform the shift $G \mapsto G - c$ without changing the asymptotic behaviour of G, so that it is in fact true that $F = 0$ for all z, and we have the desired equality. $\qquad\square$

A nice feature of this theorem is that it allows us to prove the partial fraction decomposition in the case of real rational functions.

Theorem IV.3 (Representation of real rational functions by partial fractions)**.** Given a real rational function $R(z) = \frac{P(z)}{Q(z)}$ (so P and Q have real coefficients), there exists a decomposition of R of the form

$$R(z) = G(z) + \sum_i A_i \left(\frac{1}{z - a_i} \right) + \sum_j z B_j \left(\frac{1}{z^2 + b_j z + c_j} \right) + C_j \left(\frac{1}{z^2 + b_j z + c_j} \right) \tag{12}$$

where G, A_i, B_j, and C_j are all real polynomials, and $a_i, b_j, c_j \in \mathbb{R}$, where $z - a_i$ and $z^2 + b_j z + c_j$ are the unique irreducible factors of $Q(z)$ as a real polynomial.

Proof. **TODO**: Add this from problem set solutions $\qquad\square$

B. Fractional linear transformations and classifying rational functions

As it turns out, degree-1 rational functions are not only particularly well-behaved, but they are a very special class of functions on the complex plane. In fact, they are so special that we give them their own name: *fractional linear transformation (FLTs)*. We begin by making note of a key fact

Fact IV.1. Fractional linear transformations are invertible.

Proof. If $R = \frac{P(z)}{Q(z)}$ is an FLT, it is order-1. From Cor. IV.1.1, for any $a \in \mathbb{C}$, the equation $R(z) = a$ has a single solution. Moreover, $R(z)$ has a single pole, so there is a unique z_0 sent to ∞ by R. Thus, R is bijective and invertible. $\qquad\square$

Clearly, given an FLT, $R(z) = \frac{P(z)}{Q(z)}$, at least one of P and Q are degree-1 polynomials, so we can write $R(z) = \frac{az+b}{cz+d}$. However, it is not the case that every such set of (a, b, c, d) will yield an FLT. Recall that a for $\frac{az+b}{cz+d}$ to be of order-1, it must either have degree-1 numerator and denominator *in reduced form*. Given arbitrary (a, b, c, d), the quotient $\frac{az+b}{cz+d}$ does not necessarily have this property when put into reduced form. This is a particularly subtle point which must be stressed: *the degree of a rational function can only be defined with respect to a rational function in reduced form.*

Claim IV.2. Given (a, b, c, d), the rational function $f(z) = \frac{az+b}{cz+d}$ is an FLT if and only if $ad - bc \neq 0$.

Proof. Suppose $ad = bc$. Then either $a = b = 0$, $a = c = 0$, $d = b = 0$, $d = c = 0$, or a, b, c, d are all non-zero. In each of the first four cases, it is easy to check that f is not an FLT (as in reduced form, it is a constant). In the final case, we have

$$f(z) = \frac{1}{d} \cdot \frac{adz + bd}{cz + d} = \frac{bcz + bd}{cz + d} = \frac{b}{d} \frac{cz + d}{cz + d}. \tag{13}$$

Thus, in reduced form, f is a constant here as well. It follows that if f is an FLT, $ad \neq bc$, so $ad - bc \neq 0$. Now, suppose f is not an FLT, so it is degree-0 in reduced form (a constant). This implies that we must have $f(0) = f(\infty)$, so $a/c = b/d$, implying $ad - bc = 0$. Thus, if $ad - bc \neq 0$, then f is an FLT. This completes the proof. $\qquad\square$

Note that we can represent FLTs using matrices

$$\frac{az + b}{cz + d} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{14}$$

Claim IV.3. The composition of fractional linear transformations yields a fractional linear transformation whose associated matrix is precisely the the product of the associated matrices of the two constituent fractional linear transformations of the composition.

This proof is just tedious computation, so I won't do it. This result has the nice corollary:

Corollary IV.3.1. The inverse of a fractional linear transform is the fractional linear transform described by the entries in the matrix obtained from inverting the matrix of the original fractional linear transform

As has been suggested, fractional linear transformations are particularly well-behaved, so much so that it makes sense to consider rational functions which are equal, up to composition with fractional linear transformations, to be "equivalent in a sense". More formally,

Fact IV.2. The criterion of "equal up to fractional linear transformations" (which we will call *FLT equivalent* going forward) defines an equivalence relation on the set of complex rational functions. We say that $R \sim R'$ is there exist fractional linear transformations f and g such that $R' = f \circ R \circ g$.

Of course, since we have an equivalence relation, we automatically can ask what the associated equivalence classes are. Beginning with the simple case of a rational function of order 2, let us determine both how many equivalence classes are generated by this relation and a "nice" representative of each equivalence class.

Theorem IV.4. Every fractional linear transformation of order 2 is FLT equivalent to $a(z) = z^2$ or $b(z) = z + z^{-1}$.

Proof. Let us begin with the case where $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

C. The cross-ratio

Now, we can move on to particularly important type of FLT: the cross-ratio. The cross-ratio is an answer to the question of whether, given three *distinct* points $z_1, z_2, z_3 \in \overline{\mathbb{C}}$, we can construct a fractional linear transformation which performs the mapping $(z_1, z_2, z_3) \mapsto (0, 1, \infty)$. Let us first suppose that z_1, z_2 and z_3 are all finite. Clearly, the FLT $\frac{z - z_1}{z - z_3}$ will send z_3 to ∞ and z_1 to 0. To ensure that z_1 is sent to 1, we simply multiply by the reciprocal of this function by its evaluation at z_2, giving:

$$S(z, z_1, z_2, z_3) = \frac{z_2 - z_3}{z_2 - z_1} \frac{z - z_1}{z - z_3} \tag{15}$$

we call S the *cross-ratio*. We can also define the cross-ration when one of z_1, z_2 or z_3 is equal to ∞: this just requires some casework so we will leave it as an exercise. We arrive next at a very important result:

Lemma IV.1. A fractional linear transformation is uniquely determined by where it sends any three distinct points of $\overline{\mathbb{C}}$.

Proof. Suppose h is an FLT taking $0, 1, \infty \mapsto 0, 1, \infty$. We can write $h(z) = \frac{az+b}{cz+d}$. Clearly, if $h(0) = 0$, then we must have $b/d = 0$, so $b = 0$. If $h(\infty) = \infty$, we must have $c = 0$. Thus, $h(z) = (a/d)z$. Since $h(1) = 1$, we must have $h(z) = z$, so h is the identity
  Now, suppose f is an FLT with $f(z_1), f(z_2), f(z_3) = a, b, c$. Let p be the cross-ratio of a, b, c, such that $p(0), p(1), p(\infty) = a, b, c$. Suppose g is another FLT such that $g(z_1), g(z_2), g(z_3) = a, b, c$. Then $p^{-1} \circ f \circ g^{-1} \circ p$ is an FLT which takes $0, 1, \infty \mapsto 0, 1\infty$. From above, this FLT is the identity, so $f \circ g^{-1}$ is the identity, so $f = g$. $\qquad\qquad\qquad$ $\square$

Using this result allows us quickly prove the following result:

Lemma IV.2. The cross-ratio is invariant under application of an FLT. In particular, if z_1, z_2, z_3, z_4 are distinct points, and T is an FLT, then $S(Tz_1, Tz_2, Tz_3, Tz_4) = S(z_1, z_2, z_3, z_4)$.

Proof. Note that by definition, $Q(z) = S(z, Tz_2, Tz_3, Tz_4)$ sends $(Tz_2, Tz_3, Tz_4) \mapsto (0, 1, \infty)$. Thus, $Q \circ T$ sends $(z_2, z_3, z_4) \mapsto (0, 1, \infty)$, so $Q \circ T = S$ from the above theorem. But clearly, $(Q \circ T)(z) = S(Tz, Tz_1, Tz_2, Tz_3)$, so the proof is complete. $\qquad\qquad\qquad$ $\square$

Lemma IV.3. The cross ratio, $S(z_1, z_2, z_3, z_4)$ is real if and only if z_1, z_2, z_3, z_4 lie on a circle or a line.

Proof. Let $T(z) = \frac{az+b}{cz+d}$. $T(z)$ is real if and only if

$$T(z) = \overline{T(z)} \iff (az + b)(\overline{c}\overline{z} + \overline{d}) = (\overline{a}\overline{z} + \overline{b})(cz + d) \tag{16}$$

$$\iff (a\overline{c} - \overline{a}c)|z|^2 + (b\overline{c} - \overline{a}d)\overline{z} + (a\overline{d} - \overline{b}c)z + (b\overline{d} - \overline{b}d) = 0 \tag{17}$$

for $z = x + iy$, we have

$$\alpha(x^2 + y^2) + (\beta - \overline{\beta})x + i(\beta + \overline{\beta})y + \gamma = 0 \tag{18}$$

where $\alpha = a\overline{c} - \overline{a}c$, $\beta = a\overline{d} - \overline{b}c$ and $\gamma = b\overline{d} - \overline{b}d$. Clearly, α and γ are imaginary. In addition, $\beta - \overline{\beta}$ is imaginary, and $i(\beta + \overline{\beta})$ is as well. Thus, dividing both sides by i, we have

$$\alpha'(x^2 + y^2) + \beta_1' x + \beta_2' y + \gamma' = 0 \tag{19}$$

for real $\alpha', \beta_1', \beta_2'$, and γ'. This is the equation of a circle or a line (if $\alpha = 0$). Thus, an FLT (any FLT!) is real at z if and only if z lies on a particular circle/line determined by the FLT. In particular, the cross-ratio $z \mapsto S(z, z_2, z_3, z_4)$ is real at $z = z_2, z_3, z_4$, and is real if and only if z lies on a particular circle/line determined by S. Thus, $S(z, z_2, z_3, z_4)$ is real if and only if (z, z_2, z_3, z_4) lie on the same circle/line. $\square$

Note that since FLTs are bijective, the cross-ratio $z \mapsto S(z, z_2, z_3, z_4)$ must send the circle/line determined by z_2, z_3, z_4 to the real line. We can now use the above two results to prove the following *very important* claim:

Theorem IV.5. Fractional linear transformations take circles and lines to circles and lines. Moreover, given any pair of circles/lines, there exists a unique FLT sending one to the other.

Proof. First, note that a circle or a line in the plane can be determined by precisely three distinct points. Two points will determine a circle *and* a line passing through them, a third point will then distinguish whether the points do lie on the corresponding circle, *or* the line. Let A be a circle/line determined by z_1, z_2, and z_3. Since $S(z, z_1, z_2, z_3)$ will be real if and only if z, z_1, z_2, z_3 lie on a circle/line, $S(z, z_1, z_2, z_3)$ will be real if and only if $z \in A$. Since S is bijective, this means that $S(A) = \mathbb{R}$.

Now, suppose T is an arbitrary FLT. Consider the cross-ratio F sending $T(z_1), T(z_2), T(z_3) \mapsto 0, 1\infty$. Note that $F \circ T = S$. Thus, $T = F^{-1} \circ S$, and $T(A) = F^{-1}(\mathbb{R})$. Again, we know that F will take the circle/line A' defined by $T(z_1), T(z_2), T(z_3)$ to $\mathbb{R}$ bijectively, so $F^{-1}(\mathbb{R}) = A'$, and the proof is complete. $\square$

The importance of this result lies in the fact that it allows us to think about FLTs as a very general-purpose tool which we can use to be back and forth between different regions in a particularly nice fashion. In particular, since an FLT is bijective and continuous in both directions, it is a homeomorphism (in fact, it is a diffeomorphism which is everywhere conformal: we will discuss this later).

TODO: Fill in with more examples of why this result is useful.

V. Holomorphic functions

Now that we have introduced some basic theory related to complex functions, we can move to explaining one of the fundamental concepts underlying complex analysis: *holomorphic functions*. Before proceeding, we make an important remark

Remark V.1. As a high-level, when we do complex analysis, we are in some sense doing calculus in two variables, and in another sense, doing calculus in a single variable. The complex plane is defined as $\mathbb{R}^2$, with a particular complex structure imposed, but we then go on to defining a collection of arithmetic operations which allow us to perform all of the standard operations one would expect to be able to apply to real numbers, to pairs of numbers (x, y) which we call *complex numbers* and denote $z = x + iy$.

As a result of this fundamental fact, great care must be taken when defining particular constructions in complex analysis. Oftentimes, we will define something "with respect to x and/or y". In other situations, we will define something "with respect to $z = x + iy$". This vague assertion will make more sense as we begin to encounter concrete examples of it throughout the following sections, but it is worthwhile to make note of it early, so that one may be aware and on the lookout for situations in which this slight ambiguity may arise.

A. Introducing and motivating holomorphic functions

The main idea motivating the definition of a holomorphic function is the desire for a function of the variables (x, y), such that standard differentiation of a two-variable function on $\mathbb{R}^2$ is "compatible" with the notion of differentiability that we expect to arise for a single-variable function, *with the added structure of complex arithmetic.*

As was stated in Rem. V.1, we can think about a complex function f either as a two-variable function, $f(x, y)$ on $\mathbb{R}^2$, or as a single-variable function $f(z)$ on $\mathbb{C}$. We know how to differentiate function on $\mathbb{R}^2$ from standard calculus. We do not know how to differentiate a function on $\mathbb{C}$ (yet), however, we do know how to differentiate single-variable functions on $\mathbb{R}$. Thus, if we wish to treat the complex plane similar to how we treat $\mathbb{R}$, with the addition of complex arithmetic, it may make sense to define the derivative of a complex function on the same way as it is defined in the real-single variable case.

Definition V.1 (Complex derivative)**.** Given a function $f : \mathbb{C} \to \mathbb{C}$ written in terms of the complex variable $z = x + iy$, we say that f is complex differentiable or *holomorphic* at z if the limit

$$f'(z) = \lim_{|h| \to 0} \frac{f(z+h) - f(z)}{h} \tag{20}$$

exists, calling this limit $f'(z)$ the complex derivative of f in this case. Note that this limit is well-defined, as division by a complex number is well-defined.

Now, suppose we are given a function of the form $g : \mathbb{R}^2 \to \mathbb{R}^2$. This function both has a well-defined *real derivative* and a 2×2 Jacobian of partial derivatives, but it also has an interpretation as a complex function, when we put the associated complex structure (and thus, complex arithmetic) on the domain and co-domain. However, it is **not** the case that the standard, real derivative of g will agree with its associated *complex derivative.* In fact it is even worse than this: *there will be cases where the real derivative exists but the complex derivative does not.*

Example V.1. Consider the function $f(x, y) = (x^2 + y^2, 0)$. When $z = x + iy$, this function can also be written as $f(z) = \bar{z}z$. This function is differentiable as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, but is not complex differentiable. The first fact is trivial. But, note that

$$f(z+h) - f(z) = (x+h_1)^2 + (y+h_2)^2 - x^2 - y^2 = 2h_1 x + h_1^2 + 2h_2 y + h_2^2 \tag{21}$$

As an example, in the case that $h_1 = 0$, then $\frac{f(z+h) - f(z)}{h} = 2y + h_2$ and if $h_2 = 0$, then $\frac{f(z+h) - f(z)}{h} = 2x + h_1$. In other words, in arbitrarily small neighbourhoods around z, there exists points such the quotient inside the limit is arbitrarily close to $2y$ and $2x$. Thus, if $x \neq y$, then the limit cannot possibly exist at z. In fact, using similar arguments when $x = y$, we can show that this function is nowhere holomorphic.

Examples like this lead us to the following question:

Question V.1. If a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable in the usual, real sense, under what conditions is it also differentiable when interpreted as a complex function (holomorphic)?

The answer to this question leads us to the famous Cauchy-Riemann equations.

Theorem V.1. A function $g : \mathbb{R}^2 \to \mathbb{R}^2$ with $g(x, y) = (g_1(x, y), g_2(x, y))$ is complex differentiable/holomorphic, when interpreted as a function from $\mathbb{C}$ to $\mathbb{C}$ (in other words, when written as $g(x + iy) := g_1(x, y) + ig_2(x, y)$), if and only if it is differentiable as a real function, and satisfies the *Cauchy-Riemann equations*

$$\frac{\partial g_1}{\partial x} = \frac{\partial g_2}{\partial y} \tag{22}$$

$$\frac{\partial g_1}{\partial y} = -\frac{\partial g_2}{\partial x} \tag{23}$$

In particular, note that the condition of being holomorphic is stronger than that of being real-differentiable.

Proof. Suppose that g is complex differentiable, so that the limit of Eq. (20) exists. It then follows that there exists a neighbourhood around z for which

$$g(z+h) = g(z) + g'(z)h + o(h) \tag{24}$$

where $o(h)$ is a function such that $\lim_{|h|\to 0} \frac{o(h)}{h} = 0$. $g'(z) = u + iv$ and $z = x + iy$ are complex numbers. In addition, $h = h_1 + ih_2$ and when interpreted as a complex function, $g(x + iy) := g_1(x, y) + ig_2(x, y)$. It follows that

$$g(z+h) = g\left((x+h_1) + i(y+h_2)\right) = g_1(x+h_1, y+h_2) + ig_2(x+h_1, y+h_2) \tag{25}$$

$$g(z) = g_1(x,y) + ig_2(x,y) \quad \text{and} \quad g'(z)h = (uh_1 - vh_2) + i(uh_2 + vh_1) \tag{26}$$

Therefore, it follows that

$$g_1(x+h_1, y+h_2) = g_1(x,y) + (uh_1 - vh_2) + \mathrm{Re}[o(h)] = g_1(x,y) + \begin{pmatrix} u & -v \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \mathrm{Re}[o(h)] \tag{27}$$

as well as

$$g_2(x+h_1, y+h_2) = g_2(x,y) + (uh_2 + vh_1) + \mathrm{Im}[o(h)] = g_2(x,y) + \begin{pmatrix} v & u \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \mathrm{Im}[o(h)] \tag{28}$$

It follows immediately that

$$g((x,y) + (h_1, h_2)) = g(x,y) + \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + o(h_1, h_2) \tag{29}$$

in a neighbourhood around (x, y). It follows by definition that g is real-differentiable, and has (unique) derivative

$$Dg(z) = \begin{pmatrix} \partial_x g_1 & \partial_y g_1 \\ \partial_x g_2 & \partial_y g_2 \end{pmatrix} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \tag{30}$$

which of course means that $\partial_x g_1 = \partial_y g_2$ and $\partial_y g_1 = -\partial_x g_2$, precisely the Cauchy-Riemann equations. Conversely, let us suppose that g is real-differentiable with partial derivatives satisfying the Cauchy-Riemann equations: we effectively just trace the derivation above in reverse to recover Eq. (24), which implies that g is complex-differentiable. $\qquad\square$

Remark V.2. An import artifact of the above proof is that for a holomorphic function f, matrix multiplication of the standard differential matrix against some $h \in \mathbb{R}^2$ is precisely the same as multiplication of the complex derivative $f'(z)$ against the complex number $h = h_1 + ih_2$.

Another important artifact of the above proof is handful of formulas for the complex derivative, namely

$$g'(z) = u + iv = \frac{\partial g_1}{\partial x} - i\frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial y} + i\frac{\partial g_2}{\partial x} = \frac{1}{2}\left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) \tag{31}$$

$$= \frac{\partial g_1}{\partial x} + i\frac{\partial g_2}{\partial x} = \frac{\partial}{\partial x}\left(g_1(x,y) + ig_2(x,y)\right) = \frac{\partial g(x,y)}{\partial x} \tag{32}$$

$$= -i\left(\frac{\partial g_1}{\partial y} + i\frac{\partial g_2}{\partial y}\right) = -i\frac{\partial}{\partial y}\left(g_1(x,y) + ig_2(x,y)\right) = -i\frac{\partial g(x,y)}{\partial y} \tag{33}$$

as well as its squared magnitude

$$|g'(z)|^2 = u^2 + v^2 = \frac{\partial g_1}{\partial x}\frac{\partial g_2}{\partial y} - \frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial x} = \det Dg(x,y) = \left|\frac{\partial g(x,y)}{\partial x}\right|^2 = \left|\frac{\partial g(x,y)}{\partial y}\right|^2. \tag{34}$$

While being explicit and easy to use, these equations are suggestive: there are more succinct ways of expressing the holomorphic condition. The condition of a function being holomorphic may better be expressed as *a*

function which is constant in a particular direction in the complex plane. In particular, let us define the following differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{35}$$

This allows us to rewrite the complex derivative in Eq. (31) in a much nicer way:

$$g'(z) = \frac{1}{2}\left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(g_1 + ig_2) = \frac{\partial g(z)}{\partial z} \tag{36}$$

It should now make sense why we defined $\frac{\partial}{\partial z}$ the way we did! In addition, note that

$$\frac{\partial g}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(g_1 + ig_2) = \frac{1}{2}\left(\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial g_2}{\partial x} + \frac{\partial g_1}{\partial y}\right) = 0 \tag{37}$$

This leads us, easily, to the following result:

Theorem V.2. A differentiable function g is holomorphic if and only if $\frac{\partial g}{\partial \overline{z}} = 0$.

The definitions of these differential operators effectively arise from a change of basis in the tangent space, when we think about $\mathbb{R}^2$ as a *complex manifold*. Let's think through what this means: There exists a global identifcation $\iota : \mathbb{R}^2 \to \mathbb{C}$, which we can think of as a global coordinate chart for thinking of $\mathbb{R}^2$ as a one-dimensional complex manifold. **TODO:** Need to finish this.

B. A more sophisticated perspective

TODO: This section is incomplete and needs to be finished.

Let us begin with a brief review of differential geometry.

Remark V.3 (Tangent space notation)**.** Recall that for a vector space V, we define $\bigwedge^k(V)$ to be the set of alternating k-tensors on V. We define the tangent space of $\mathbb{R}^n$ at point p, $T_p\mathbb{R}^n$, to be the vector space $\{p\} \times \mathbb{R}^n$, with elements (p, v) and $(p, v_1) + \lambda(p, v_2) \coloneqq (p, v_1 + \lambda v_2)$. A differential k-form ω is a function from $\mathbb{R}^n$ to the disjoint union of all $\bigwedge^k(T_p\mathbb{R}^n)$ over $p \in \mathbb{R}^n$ such that $\omega(p) \in \bigwedge^k(T_p\mathbb{R}^n)$. Let $\Omega^k(\mathbb{R}^n)$ denote the space of all k-forms. We define the differential $d : C^\infty(\mathbb{R}^n, \mathbb{R}) \to \Omega^1(\mathbb{R}^n)$ as taking f to the differential form ω such that $\omega(p)(p, v) = [Df(p)]v$. Note that $C^\infty(\mathbb{R}^n, \mathbb{R}) = \Omega^0(\mathbb{R}^n)$. It follows that

Definition V.2.

Remark V.4. Another, more formal way, of stating this fact is that holomorphic functions are precisely the functions g from $(\mathbb{R}^2, J)$ to $(\mathbb{R}^2, J)$ whose derivative commutes with the complex structure,

$$J \circ Dg(z) = Dg(z) \circ J. \tag{38}$$

Intuitively, this means that the map which turns $\mathbb{R}^2$ into $\mathbb{C}$ is compatible with taking the derivative in the usual sense.

Definition V.3 (Holomorphic and anti-holomorphic functions)**.**

C. Basic properties and implications

Having introduced the notion of a holomorphic function, let us discuss some basic properties and implications related to functions of this form.

Claim V.1. If f is differentiable, then $\frac{\partial f}{\partial \overline{z}} = \frac{\overline{\partial f}}{\partial z}$.

Proof. Note that

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = \overline{\frac{\partial \overline{f}}{\partial x}} + i\overline{\frac{\partial \overline{f}}{\partial y}} = \overline{\frac{\partial \overline{f}}{\partial \overline{z}}}. \tag{39}$$

$\square$

Thanfkully, we can also prove the following result:

Claim V.2 (Complex derivatives behave like normal derivatives)**.** The complex derivative $\frac{\partial}{\partial z}$ satisfies the chain rule and product rule.

Proof. Let's consider the chain rule, for example. We want to show that $\partial_z f(g(z)) = \partial_z f(g(z) \cdot \partial_z g(z) = f'(g(z)) \cdot g'(z)$. Recall that for a holomorphic function, multiplication via $f'(z)$ is the same as matrix multiplication by the differential. In other words,

$$Df(g(z)) \cdot g'(z) = f'(g(z)) \cdot g'(z) \implies Df(g(z)) \cdot \partial_x g(x,y) = f'(g(z)) \cdot g'(z) \tag{40}$$
$$\implies \partial_x f(g(x,y)) = \partial_z f(g(z)) = f'(g(z)) \cdot g'(z) \tag{41}$$

so we have chain rule. Of course, we used the fact that $\partial_z f(z) = \partial_x f(x,y)$ for holomorphic f, and the regular two-variable chain rule. Product rule is more annyoing, but we can still do it. We have

$$f'(z)g(z) + g'(z)f(z) = \partial_x f(x,y)g(z) + \partial_x g(x,y)f(z) \tag{42}$$
$$= ((\partial_x f_1)g_1 - (\partial_x f_2)g_2 + (\partial_x g_1)f_1 - (\partial_x g_2)f_2) + i((\partial_x f_1)g_2 + (\partial_x f_2)g_1 + (\partial_x g_1)f_2 + (\partial_x g_2)f_1)$$
$$= \partial_x(f_1 g_1 - f_2 g_2) + i\partial_x(f_1 g_2 + f_2 g_1) = \partial_x(fg) = (fg)'(z) \tag{43}$$

$\square$

Claim V.3 (Basic properties of holomorphic functions)**.** Let f be a holomorphic function on a connected open set Ω. Then

1. If $f'(z) = 0$ on Ω, then f is constant.

2. If $\mathrm{Re}[f]$ is constant on Ω, then f is constant.

3. If $|f|$ is constant, then f is constant.

Proof.   1. If $f'(z) = 0$, then by definition $\frac{\partial f}{\partial z} = 0$. Since f is holomorphic, $\frac{\partial f}{\partial \overline{z}} = 0$. Therefore,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}} = 0 \tag{44}$$
$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial z} - i\frac{\partial f}{\partial \overline{z}} = 0 \tag{45}$$

   which implies that $Df = 0$ so f is constant.

2. Since f is holomorphic, it follows from the Cauchy-Riemann equations that $\partial_x \mathrm{Im}[f] = \partial_y \mathrm{Im}[f] = 0$, so $\mathrm{Im}[f]$ is constant, and f is therefore constant.

3. $|f|$ being constant is equivalent to $f\overline{f}$ being constant. Thus,

$$\frac{\partial}{\partial z}(f\overline{f}) = \overline{f}\frac{\partial f}{\partial z} + f\frac{\partial \overline{f}}{\partial z} = 0 \implies \overline{f}\frac{\partial f}{\partial z} = 0 \tag{46}$$

   where we know that $\frac{\partial \overline{f}}{\partial z} = \overline{\frac{\partial f}{\partial \overline{z}}} = 0$ as f is holomorphic. Thus, at each point in Ω, we either have $\overline{f}(z) = 0$ or $\partial_z f = 0$. If $|f|$ is constant and $\overline{f} = 0$ at a single point, then $|f| = 0$ everywhere and thus $f = 0$ everywhere. If $\overline{f}$ is nowhere 0, then $\partial_z f = 0$ on Ω, so from the first point, f is constant.

$\square$

Now, let us look at a few more sophisiticated results.

Theorem V.3. Suppose f is a smooth function defined on connected domain Ω such that $|\det Df(x,y)| \neq 0$ on Ω. Then if f is angle-preserving (i.e. the angles between tangent lines are preserved), it will either be holomorphic or anti-holomorphic ($\partial_z f = 0$).

Proof. Because Ω is connected and $|\det D(x,y)|$ is continuous, it cannot change sign, otherwise it would equal 0 at some point by intermediate value theorem. If f preserves angles, it follows by definition that $Df(x,y)$ is an isometry. Thus, we must have

$$Df(x,y) = \begin{pmatrix} u(x,y) & -v(x,y) \\ v(x,y) & u(x,y) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -u(x,y) & v(x,y) \\ v(x,y) & u(x,y) \end{pmatrix} \tag{47}$$

depending on whether $\det Df(x,y)$ is positive or negative on Ω, which corresponds to f being holomorphic or anti-holomorphic. We could have also seen this by noting that a real linear function which preserves angles must be of the form $f(z) = wz$ or $f(z) = w\bar{z}$. Since $f(z) = Df(z_0)z$ preserves angles and is real-linear, it must be of either of these forms. It is easy to check that the former case corresponds to f being holomorphic and the latter corresponds to f being anti-holomorphic. $\qquad\square$

Now, let's consider an even more advanced result, which should be familiar from calculus.

Theorem V.4 (Inverse function theorem). Suppose f is holomorphic on an open set U containing z_0 with $f'(z_0) \neq 0$. Then there exists a local holomorphic inverse around $f(z_0)$

Proof. If f is holomorphic, then $f'(z_0) \neq 0 \Rightarrow |\det Df| \neq 0$, so we just need to show that if f is holomorphic, then is is continuously differentiable, to show that we have a continuously differentiable local inverse. The fact that this local inverse is holomorphic itself follows from the fact that we can write down exactly what the inverse differential must be, namely

$$Df^{-1}(f(z_0)) = Df(z_0)^{-1} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}^{-1} = \frac{1}{u^2 + v^2} \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \tag{48}$$

where it is easy to check from here that f^{-1} must be holomorphic. We will prove that holomorphic implies continuously differentiable **later**! $\qquad\square$

VI. Examples of holomorphic functions

Now that we have outlined a few basic results related to holomorphic functions, let us discuss a few examples of families of functions which are holomorphic. We can begin with the obvious: rational functions (away from their poles). To prove this, note that

$$\frac{\partial z}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x + iy) = 0 \tag{49}$$

In addition, note that since $\frac{\partial}{\partial \bar{z}}$ satisfies chain and product rule, for $z \neq 0$, note that $z^{-1} = \frac{x-iy}{x^2+y^2}$ is (real) differentiable and

$$\frac{\partial}{\partial \bar{z}}(zz^{-1}) = z^{-1}\frac{\partial z}{\partial \bar{z}} + z\frac{\partial z^{-1}}{\partial \bar{z}} = z\frac{\partial z^{-1}}{\partial \bar{z}} = 0 \Rightarrow \frac{\partial z^{-1}}{\partial \bar{z}} = 0 \tag{50}$$

so z^{-1} is holomorphic as well. Product and chain rule imply that all rational functions are holomorphic, after proving these facts.

A. Power series

The next example of holomorphic functions that we will look at are *power series*: the limits of sequences of complex polynomials.

Theorem VI.1 (Abel). Given a complex power series $f(z) = \sum_{j=0}^{\infty} a_j z^j$, there exists a *radius of convergence* $R \in [0, \infty]$ such that

1. for $r < R$, f is uniformly and absolutely convergent in $B_r(0)$: the closed ball of radius r around 0.

2. For $|z| > R$, $\sum_{j=0}^{\infty} a_j z^j$ diverges (in fact, the terms $a_j z^j$ are unbounded in magnitude).

3. The derived series $\sum_{j=1}^{\infty} j a_j z^{j-1}$ has the same radius of convergence as f. Moreover, f is holomorphic and $f'(z)$ (the complex derivative) is precisely the derived series. Note that this result immediately implies that a power series is smooth.

In fact, we can compute the radius of convergence via the *Cauchy-Hadamard formula*: $R^{-1} = \limsup \sqrt[n]{a_n}$.

Proof. **TODO**: Fill this in. This is an important result! $\qquad\square$

Equipped with the main theorem characterizing power series as functions, we can begin to discuss manipulations of complex power series. Of course, all complex power series are determined by a sequence of complex coefficients.

Remark VI.1 (Composition of power series). Suppose we have two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. We consider the question of when it makes sense to write down the composition of two power series. It turns out that this will be precisely when $b_0 = 0$, as here, formally composing the power series yields a new, well-defined power series where the coefficients are *finite* sums rather than *infinite* ones.

Theorem VI.2 (Formal inverse function theorem for power series).

B. The complex exponential

Let us now consider, arguably, the most important holomorphic complex function determined via power series: the complex exponential.

Definition VI.1 (Complex exponential). We define the complex exponential function, $\exp(z)$, via the power series

$$\exp(z) = e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} \tag{51}$$

This definition agrees with the standard definition of the exponential function on $\mathbb{R}$, as this is simply the Taylor expansion: we now take it as the *definition* of the function on all complex numbers.

Theorem VI.3. The complex exponential behaves as we would expect an exponential to behave. In particular, $e^a e^b = e^{a+b}$ for all $a, b \in \mathbb{C}$.

Proof. We will use a slick proof. Define $f(z) = e^{-z} e^{z+w} - e^w$ for fixed w. Since the exponential is defined via a power series and converges on all of $\mathbb{C}$, note that we can compute its (complex) derivative via the term-wise derivative, as well as use chain rule (since we showed it applies to complex derivatives). In particular, it is easy to see that $\partial_z e^z = e^z$. Thus,

$$\partial_z f(z) = -e^{-z} e^{z+w} + e^{-z} e^{z+w} = 0 \tag{52}$$

so $f(z)$ must be constant on connected domain $\mathbb{C}$. Setting $z = 0$, we can see that this constant must be 0. Thus, for any z and w, $e^{-z} e^{z+w} = e^w$, which implies that $e^a e^b = e^{a+b}$ for any $a, b \in \mathbb{C}$. $\qquad\square$

Let us define the function $g : \mathbb{R} \to \mathbb{T}$ where $g(y) = e^{iy}$. The reason why this is a mapping into the unit circle follows from Thm. VI.3, $|e^{iy}| = e^{-iy} e^{iy} = 1$. By definition,

$$e^{iy} = \sum_{j=0}^{\infty} \frac{(iy)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j y^{2j}}{(2j)!} + i \sum_{j=0}^{\infty} \frac{(-1)^j y^{2j+1}}{(2j+1)!} = \cos(y) + i \sin(y) \tag{53}$$

where we use the standard real-valued Taylor series for cos and sin functions. As you may have guessed, we will define cos and sin *on all of* $\mathbb{C}$ with these power series expansions! This allows us to note that

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \tag{54}$$

From here,

C. The Arg function

In the previous section, we defined the exponential function via a power series on the entirety of the complex domain. This allows us to define a function from $\mathbb{R}$ to $\mathbb{T}$, $y \mapsto e^{iy}$.
We will also invoke a very important result:

Theorem VI.4 (Universal property of quotient topology). Let $g : X \to Y$ be a surjective continuous map, let X^* be the collection of fibres of g, let $\pi : X \to X^*$ be the corresponding quotient map. g induces a bijective continuous map $f : X^* \to Z$ which is a homeomorphism if and only if g is a quotient map.

Proof. **TODO:** This is definitely worth proving, I'll do this after I finish filling in the rest of the unfinished sections. □

Let $\sim$ be the equivalence relation which identifies points differing up to integer multiple of 2π. We already demonstrated that the fibres of $y \mapsto e^{iy}$ are precisely the equivalence classes generated by $\sim$, so $X^* = \mathbb{R}/\sim =$ $\mathbb{R}/2\pi\mathbb{Z}$, in the notation of the above theorem. Thus, if we show that $g : y \mapsto e^{iy}$ is a quotient map, we will have a homeomorphism between $\mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{T}$. Indeed, g is continuous, open, and surjective, so it is a quotient map. Thus, the map induced by $y \mapsto e^{iy}$ such that $[y] \mapsto e^{iy}$ is a homeomorphism.
   The most important implication of this result is that $[y] \mapsto e^{iy}$ *has a continuous inverse.* We will call this inverse arg, the argument function.

Definition VI.2 (The argument function). The function $\arg : \mathbb{T} \to \mathbb{R}/2\pi\mathbb{Z}$ is called the argument function and is defined as the continuous inverse of the induced map $[y] \mapsto e^{iy}$ constructed above. In fact, we can extend arg to a function on $\mathbb{C} - (0,0)$ by re-defining $\arg(z)$ as $\arg(z/|z|)$, where $z/|z| \in \mathbb{T}$, clearly.

The argument function is effectively a function assigning an "angle" to points on the complex plane. However, working with equivalence classes isn't always desirable.

TODO: Fill in the part about branches

Definition VI.3 (Branch). A branch is a continuous function $f(z)$ defined on some connected set Ω such that $g(f(z)) = 0$, for some constraint $g(x) = 0$.

D. The complex logarithm and power functions

In addition to defining a complex exponential, we can also define an inverse function: a complex logarithm. We want the complex logarithm to be a function which satisfies $e^{f(z)} = z$. If $f(z) = f_1(z) + if_2(z)$, then we must have $e^{f_1(z)}e^{if_2(z)} = |z|e^{i\arg(z)}$. For real z, we must have $e^{f_1(z)} = |z|$, so we should set $f_1(z) = |z|$. Then, we must have $e^{if_2(z)} = e^{i\arg(z)}$.

Claim VI.1. If f is a branch of $\log(z)$ on Ω, then it is of the form $\log|z| + i\mathrm{Arg}(z) + 2\pi ik$, where $\mathrm{Arg}(z)$ is the principal argument (defined above). Moreover, f is holomorphic on Ω and has complex derivative $1/z$.

Proof. Suppose $e^{f(z)} = z$. We must have $f(z) = f_1(z) + if_2(z)$, where both f_1 and f_2 are real functions. In particular, we must have $e^{f_1(z)}e^{if_2(z)} = z$, so $e^{f_1(z)} = |z|$. This immediately implies that $f_1(z) = \log|z|$. Note that $z = |z|e^{i\mathrm{Arg}(z)}$, so we must have $e^{if_2(z)} = e^{i\mathrm{Arg}(z)}$. In other words, $e^{i(f_2(z) - \mathrm{Arg}(z))} = 1$ for all z. We know that $e^{ix} = 1$ if and only if $x = 2\pi ik$ for some $k \in \mathbb{Z}$. Thus, we must have $f_2(z) - \mathrm{Arg}(z) = 2\pi ik(z)$, where $k(z)$ is a *continuous* function from $\mathbb{C}$ to $\mathbb{R}$ which is integer-valued. Thus, it must be constant, so $f_2(z) = \mathrm{Arg}(z) + 2\pi ik$ for some constant $k \in \mathbb{Z}$.

To prove that a branch $f(z)$ is holomorphic with the desired derivative, note that

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{(z+h) - h} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{e^{f(z+h)} - e^{f(z)}} = \left(\lim_{h \to 0} \frac{e^{f(z+h)} - e^{f(z)}}{f(z+h) - f(z)} \right)^{-1} \tag{55}$$

$$= \left(\lim_{f(r) \to f(z)} \frac{e^{f(r)} - e^{f(z)}}{f(r) - f(z)} \right)^{-1} = e^{-f(z)} = \frac{1}{z} \tag{56}$$

and the proof is complete. $\qquad\square$

E. Riemann surfaces

Riemann surfaces are, at a high-level, a nice way of dealing with classes of complex-valued functions which have multiple branches

VII. Conformal maps

We now move to a discussion of conformal maps, which roughly speaking, can be thought of as maps which preserve angles. We did briefly mention maps of this form in previous sections, but here, we will elaborate.

Theorem VII.1. Holomorphic functions are conformal.

Corollary VII.1.1. Fractional linear transformations preserve angles on $\mathbb{C}$.