# MAT477 running research notes

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## I. Non-linear Katz-Oda calculations

I will begin these notes by discussing some of the prior art. We begin by considering the ODE

$$\frac{dY}{dz} = A(z)Y \tag{1}$$

where $A(z)$ is a matrix whose entries are rational functions of $z$ relative to some algebraic number field $K$ (a field extension of $\mathbb{Q}$ such that $[K : \mathbb{Q}]$ is finite). Some entry of $A(z)$ will be of the form

$$A_{ij}(z) = \frac{A_{ij}^{(n)} z^n + \cdots + A_{ij}^{(0)}}{B_{ij}^{(m)} z^m + \cdots + B_{ij}^{(0)}} \tag{2}$$

which of course may be reduced modulo some prime $p \in K$ for almost all primes (namely those which don't divide any of the coefficients $B_{ij}^{(k)}$). The result will be a differental equation over $\mathbb{F}_q[z]$ for some finite field $\mathbb{F}_q$. As a particular example, suppose $a \in \mathbb{Z} \subset \mathbb{Q}$ and we have the ODE

$$\frac{dy}{dz} = \frac{1}{az} y \tag{3}$$

We can reduce by $p \nmid a$. When we do such a reduction, the equation admits solutions $y = z^b$ for any $b$ such that $ab \equiv 1 \mod p$, clearly.

There is a result concerning ODEs of this form, formulated by Katz and Oda, which we can restate here

**Theorem I.1.** The ODE of Eq. (1) is an algebraic differential equation, meaning that its solution is an algebraic function. Moreover, there exists sufficiently large $N$ such that for almost all primes $p$,

$$\left( \frac{d}{dz} - A(z) \right)^{Np} \equiv 0 \mod p \tag{4}$$

Our goal is to establish a similar result for the non-inear Schlesinger system, which is defined to be the system of ODEs given by

$$\frac{\partial B_i}{\partial \lambda_j} = \frac{[B_i, B_j]}{\lambda_i - \lambda_j} \quad \text{for } i \neq j \tag{5}$$

$$\sum_j \frac{\partial B_i}{\partial \lambda_j} = 0 \quad \text{for all } i. \tag{6}$$

Of course, any the solutions to any partial differential equation can be expressed as the nullspace of some operator acting on some function space. In this particular case, note that we can rewrite the first equation as

$$\frac{\partial B_i}{\partial \lambda_j} - \frac{1}{\lambda_i - \lambda_j} (B_i B_j - B_j B_i) = \left( \frac{\partial}{\partial \lambda_j} \circ \pi_i - \frac{1}{\lambda_i - \lambda_j} (\pi_{ij} - \pi_{ji}) \right) (B_1, \dots, B_n) = 0. \tag{7}$$

for all $i \neq j$. The operator $\pi_i$ is projection onto $B_i$, and $\pi_{ij}$ is projection and multiplication to obtain $B_iB_j$. Moreover, the final equation is simply the nullspace of the operator $\sum_j \frac{\partial}{\partial \lambda_j} \circ \pi_i$ for all $i$. In total, there are $n^2$ different equations with $n$ different functions $B_j$. This is a clear problem, as if we think of the system of PDEs as an operator, they are sending $n$ matrix-valued functions to $n^2$ matrix-valued functions.