Differential geometry: problems and solutions

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I. MAT367 tutorial 1

Problem I.1 (MAT367 problem 1.4)**.** Recall that a map $F : M \to N$ between smooth manifolds is smooth if, for each $p \in M$, we can find a chart (U_p, φ_p) around p and a chart (V_p, ψ_p) around $F(p)$ such that $F(U_p) \subset V_p$ and $\psi_p \circ F \circ \varphi_p^{-1} \varphi_p(U_p) \to \psi_p(V)$ is smooth. It is a diffeomorphism if it also has a smooth inverse.

Suppose, for each chart (V, ψ) of N, $(F^{-1}(V), \psi \circ F)$ is a chart of M. Then, note that if $F(p)$ is in the chart (V_p, ψ_p), p is in chart $(F^{-1}(V_p), \psi_p \circ F)$ where $F(F^{-1}(V_p)) = V_p$ and $\psi_p \circ F \circ (\psi_p \circ F)^{-1} = \mathrm{id}$ is a smooth function on $(\psi_p \circ F)(F^{-1}(V_p)) = \psi_p(V_p)$. Moreover, $(\psi_p \circ F) \circ F^{-1} \circ \psi_p^{-1} = \mathrm{id}$ is also smooth. Thus, by definition, F is a diffeomorphism.

Conversely, suppose F is a diffeomorphism. Then $F^{-1}(V)$ is smooth for open V. Pick arbitrary chart (V, ψ). To verify that $(F^{-1}(V), \psi \circ F)$ is a chart, we must verify it is smoothly-compatible with the smooth structure of M, which is to say that the transition functions

$$(\psi \circ F) \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \mathbb{R}^n \quad \text{and} \quad \varphi \circ (\psi \circ F)^{-1} : (\psi \circ F)(U \cap F^{-1}(V)) \to \mathbb{R}^n \tag{1}$$

are smooth. Pick some $p \in U \cap F^{-1}(V)$. Let (U_p, φ_p) and (V_p, ψ_p) be the charts relative to which F is smooth. Note that $U \cap F^{-1}(V) \cap U_p$ is a neighbourhood of p contained in U_p, and on this neighbourhood,

$$(\psi \circ F) \circ \varphi^{-1} = (\psi \circ \psi_p^{-1}) \circ (\psi_p \circ F \circ \varphi_p^{-1}) \circ (\varphi_p \circ \varphi^{-1}) \tag{2}$$

which is smooth, so $(\psi \circ F) \circ \varphi^{-1}$ is smooth at p, for every $p \in U \cap F^{-1}(V)$, and is thus smooth. We can apply the same logic to conclude that $\varphi \circ (\psi \circ F)^{-1}$ is smooth as well. It follows that $(F^{-1}(V), \psi \circ F)$ is smoothly-compatible with the atlas for M, and is thus a chart.

II. MAT367 tutorial 2

Problem II.1 (MAT367 problem 2.1)**.** If c is a constant function, $c(x) = c$, then $\nu(c) = c \cdot \nu(1)$, where $1(x) = 1$, by linearity. Since ν is a derivation, $\nu(1)\nu(1 \cdot 1) = 1 \cdot \nu(1) + 1 \cdot \nu(1) = 2\nu(1)$, which implies that $\nu(1) = 0$, so $\nu(c) = 0$ as well.

In this latter case, $\nu(fg) = f(p)\nu(g) + g(p)\nu(f) = 0$.

Problem II.2 (MAT367 problem 2.2)**.** This is trivial: $d[\mathrm{id}]_p(\nu)(f) = \nu(f \circ \mathrm{id}) = \nu(f)$. To prove the second thing, let us first make note of the chain rule for differentials. Note that if $F : M \to N$ and $G : N \to P$, then for some $p \in M$ and $\nu \in T_pM$, we have

$$d(G \circ F)_p(\nu)(f) = \nu(f \circ (G \circ F)) = dF_p(\nu)(f \circ G) = (dG_{F(p)} \circ dF_p)(\nu)(f) \tag{3}$$

so that $d(G \circ F)_p = dG_{F(p)} \circ dF_p$. It follows immediately that $\mathrm{id} = d(F^{-1} \circ F)_p = d[F^{-1}]_{F(p)} \circ dF_p = \mathrm{id}$, so that $d[F^{-1}]_{F(p)}$ is a left-inverse of dF_p. Moreover, $d(F \circ F^{-1})_{F(p)} = dF_p \circ d[F^{-1}]_{F(p)} = \mathrm{id}$, so $d[F^{-1}]_{F(p)}$ is a right-inverse of dF_p. Thus, dF_p is invertible with $[dF_p]^{-1} = d[F^{-1}]_{F(p)}$.

For the final thing, note that if F is the constant map, then $dF_p(\nu)(f) = \nu(f \circ F) = \nu(f(q))$ where $f(q)$ is the constant map equal to $f(q)$ on M. We showed above that derivations send constant maps to 0, so $\nu(f(q)) = 0$, and $dF_p(\nu) = 0$.

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Problem II.3 (MAT367 problem 2.3)**.** Note that we have

$$dF_p\left(\left.\frac{d}{dx}\right|_p\right)(f) = \frac{d(f\circ F)}{dx}(p) = \frac{dF^1}{dx}(p)\left.\frac{df}{du}\right|_{F(p)} + \frac{dF^2}{dx}(p)\left.\frac{df}{dv}\right|_{F(p)} + \frac{dF^3}{dx}(p)\left.\frac{df}{dw}\right|_{F(p)} \tag{4}$$

$$= \left.\frac{df}{du}\right|_{F(p)} + y\left.\frac{df}{dw}\right|_{F(p)} \tag{5}$$

Problem II.4 (MAT367 problem 2.4)**.** It is easy to verify that the projections are smooth maps. Thus, $d[\pi_1]_p \times d[\pi_2]_q : T_{p\times q}M \times N \to T_pM \times T_qN$ is a well-defined linear map. As vector spaces, the two spaces have the same dimension. Thus, we can prove either injecitivty or surjectivity to prove to prove bijectivity.

We'll prove surjectivity. Suppose $\nu_1 \in T_pM$ and $\nu_2 \in T_qN$. Let us define the smooth maps $\phi_1, \phi_2 : M, N \to M \times N$ as $\phi_1(x) = (x, q)$ and $\phi_2(y) = (p, y)$. Let us define $\nu = d[\phi_1]_p(\nu_1) + d[\phi_2]_q(\nu_2)$. Note that

$$d[\pi_1]_{p\times q}(\nu) = d[\pi_1 \circ \phi_1]_p(\nu_1) + d[\pi_1 \circ \phi_2]_q(\nu_2) = \nu_1 \tag{6}$$

as well as

$$d[\pi_2]_{p\times q}(\nu) = d[\pi_2 \circ \phi_1]_p(\nu_1) + d[\pi_2 \circ \phi_2]_q(\nu_2) = \nu_2 \tag{7}$$

so surjectivity follows.

III. Other stuff

Now, I will try to do a very concise derivation of the Lie derivative of a vector field just being the commutator of vector fields. To be more specific, a vector field X will take in some function f, and yield a new smooth function $p \mapsto X_p(f)$. We will just denote this function $X(f)$. It follows that if Y is another smooth vector field, then $Y(X(f))$ and $X(Y(f))$ are smooth functions.

Claim III.1. The Lie derivative of a vector field, $\mathcal{L}_XY$ (which is just another vector field), is $XY - YX$.

Proof. Note that

$$(\mathcal{L}_XY)_p(f) = \left.\frac{d}{dt}\right|_{t=0}\Phi_t^*(Y)_p = \left.\frac{d}{dt}\right|_{t=0} d\left[\Phi_t^{-1}\right]_{\Phi_t(p)}(Y_{\Phi_p(t)})(f) = \left.\frac{d}{dt}\right|_{t=0} Y_{\Phi_t(p)}(f\circ\Phi_{-t}) \tag{8}$$

$$= \left.\frac{d}{dt}\right|_{t=0}\Phi_t^*(Y(f\circ\Phi_{-t}))(p) \tag{9}$$

From here, let's use the definition of the derivative!

$$\left.\frac{d}{dt}\right|_{t=0}\Phi_t^*(Y(f\circ\Phi_{-t}))(p) = \lim_{t\to 0}\frac{\Phi_t^*(Y(f\circ\Phi_{-t}))(p) - Y_p(f)}{t} \tag{10}$$

$$= \lim_{t\to 0}\frac{\Phi_t^*(Y(f\circ\Phi_{-t}))(p) - \Phi_t^*(Y(f))(p) + \Phi_t^*(Y(f))(p) - Y_p(f)}{t} \tag{11}$$

Note that

$$\lim_{t\to 0}\frac{\Phi_t^*(Y(f))(p) - Y_p(f)}{t} = \left.\frac{d}{dt}\right|_{t=0}\Phi_t^*(Y(f))(p) = X_p(Y(f)) \tag{12}$$

Moreover, note that

$$\lim_{t\to 0}\frac{\Phi_t^*(Y(f\circ\Phi_{-t}))(p) - \Phi_t^*(Y(f))(p)}{t} = \lim_{t\to 0} Y_{\Phi_t(p)}\left(\frac{f\circ\Phi_{-t} - f}{t}\right) = -\lim_{t\to 0} Y_{\Phi_{-t}(p)}\left(\frac{\Phi_t^*(f) - f}{t}\right) \tag{13}$$

Now for a leap of faith: we assume the limit is nicely-behaved, so

$$-\lim_{t\to 0} Y_{\Phi_{-t}(p)}\left(\frac{\Phi_t^*(f) - f}{t}\right) = -Y_p\left(\left.\frac{d}{dt}\right|_{t=0}\Phi^*(f)\right) = -Y_p(X(f)) \tag{14}$$

Thus, the overall limit is $X_p(Y(f)) - Y_p(X(f))$, so that $\mathcal{L}_X(Y) = XY - YX$. $\square$