MAT436 problem set 4

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I. Problem 1 (Suggested Problem 4)

Let S and T be self-adjoint compact operators on a separable Hilbert space. It follows from the spectral theorem that S admits a basis of orthonormal eigenvectors $\{s_n\}$ with eigenvalues λ_n such that $\lambda_n \to 0$. We then have

$$STs_n = TSs_n = \lambda_n Ts_n \tag{1}$$

which implies that Ts_n is an eigenvector of S with eigenvalue λ_n. In the case that $\lambda_n \neq 0$, there are only a finite number of s_k in the list of eigenvectors $\{s_n\}$ which have eigenvalue λ_n, so it follows that we must be able to express Ts_n exactly as a linear combination of these elements. Thus, let $(s_{j_1}, \dots, s_{j_m})$ be the collection of s_k having λ_n as their eigenvalue, and we have

$$Ts_{j_p} = \sum_{k=1}^{m} c_{pk} s_{j_k} \tag{2}$$

In other words, T takes $s_J = (s_{j_1}, \dots, s_{j_m})$ to $C \cdot (s_{j_1}, \dots, s_{j_m})$, where C is the matrix with entries c_{pk}. Note that C is invertible, as T restricted to the subspace of s_J is invertible. We can perform an eigendecomposition on C to get $C = U\Lambda U^{\dagger}$, where Λ is diagonal with all non-zero entires on the diagonal. We then note that

$$CU(s_{j_1}, \dots, s_{j_m}) = U\Lambda(s_{j_1}, \dots, s_{j_m}) = U(\lambda'_1 s_{j_1}, \dots, \lambda'_m s_{j_m}) \tag{3}$$

so that if we replace $(s_{j_1}, \dots, s_{j_m})$ in the list of eigenvectors $\{s_n\}$ for s with the rotated subspace $U(s_{j_1}, \dots, s_{j_m})$, then this will be a simultaneous eigenspace for S and T. We can repeat this for all λ_n, thus yielding a simultaneous eigenbasis for S and T.

II. Problem 2 (Suggested Problem 5)

Part A. Let $A = \frac{1}{2}(T + T^{\dagger})$ and $B = \frac{1}{2i}(T - T^{\dagger})$. Note that $T = A + iB$, $A^{\dagger} = A$ and $B^{\dagger} = B$. Moreover,

$$[A, B] = \frac{1}{4i}[T + T^{\dagger}, T - T^{\dagger}] = \frac{1}{4i}([T^{\dagger}, T] - [T, T^{\dagger}]) = \frac{1}{2i}[T^{\dagger}, T] = 0 \tag{4}$$

as T is normal, so it commutes with its adjoint. Finally, to see that A and B are compact, note from Schauder's theorem that $T^{\dagger}$ will be compact if T is, so it follows that $T + T^{\dagger}$ and $T - T^{\dagger}$ will be compact. Thus, we have written T as a linear combination of two self-adjoint, commuting, compact operators.

Part B. We know from the usual spectral theorem that since A and B are compact self-adjoint, they will be diagonalizable. Moreover, since they commute, they are simultaneously diagonalizable, by Problem 1. It follows immediately follows that $A + iB$ can diagonalized, as we simply choose a simultaneous orthonormal eigenbasis.

III. Problem 3

My goal in this problem is to re-derive Hahn-Banach theorem myself

Let us begin by recalling that a sublinear functional on X, a normed vector space, is a map $p : X \to \mathbb{R}$, such that $p(\lambda x) = \lambda p(x)$ for all $x \in X$ and $\lambda \geq 0$, as well as $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Theorem III.1 (Hahn-Banach Theorem)**.** Let X be a real vector space, p a sublinear functional on X, M a subspace of X, and f a linear functional on M such that $f(x) \le p(x)$ for all $x \in M$. Then there exists a linear functional F on X such that $F(x) \le p(x)$ for all $x \in X$ and $F|_M = f$.

Proof. The first step is to show that if $x \in X - M$, then we can extend f to a linear functional g on $M + \mathbb{R}x$ satisfying $g(y) \le p(y)$ for $y \in M + \mathbb{R}x$. We will do this. The idea is to "approximate" the value that g should take on for vectors of the form $m + rx$ for $m \in M$ and $r \in \mathbb{R}$ by looking at their value relative to p. In particular, given $y_1, y_2 \in M$, we note that

$$f(y_1) + f(y_2) = f(y_1 + y_2) \le p(y_1 + y_2) \le p(y_1 - x) + p(x + y_2) \tag{5}$$

Therefore, it follows immediately that

$$f(y_1) - p(y_1 - x) \le p(x + y_2) - f(y_2) \tag{6}$$

for all $y_1, y_2 \in M$. Since this holds for *all* y_1 and y_2, we can take the infimum/supremum to get

$$\sup\{f(y_1) - p(y_1 - x) \mid y_1 \in M\} \le \inf\{p(x + y_2) - f(y_2) \mid y_2 \in M\} \tag{7}$$

we then choose some a lying between these values, and we set $g(m + rx) = f(m) + ra$. Clearly, $g|_M = f$. With this definition, note that if $r > 0$,

$$\begin{aligned} g(m + rx) = f(m) + ra \le f(m) + r\inf\{p(x + y_2) - f(y_2)\} &= f(m) + \inf\{p(rx + ry_2) - f(ry_2)\} && (8) \\ &\le f(m) + p(rx + m) - f(m) = p(m + rx) && (9) \end{aligned}$$

Similarly, if $-r > 0$, then we have

$$\begin{aligned} g(m - rx) = f(m) - ra \le f(m) - r\inf\{p(x + y_1) - f(y_1)\} &= f(m) + \sup\{p(-rx - ry_2) - f(-ry_2)\} && (10) \\ &\le f(m) + p(m - rx) - f(m) = p(m - rx) && (11) \end{aligned}$$

which is exactly what we want. From here, the main idea is to repeat this procedure indefinitely, extending to all other x. Of course, this will involve extension to a potentially uncountable number of x, which we can't do without extra work. In particular, we must make use of Zorn's lemma. To be more specific, let $\mathcal{F}$ denote the family of all linear extensions of f satisfying $F \le p$: we can think of these elements as being subsets of $X \times \mathbb{R}$, and note that they are partially ordered by inclusion. This this family has a maximal element by Zorn's lemma, which necessarily will have domain all of X, otherwise we could extend it to a larger domain using the same procedure invoked above. This maximal element will be precisely the extension we are looking for. ☐

As an immediate corollary, we ge the following result:

1. If M is a closed subspace of X and $x \in X - M$, there exists $f \in X^*$ such that $f(x) \neq 0$ and $f|_M = 0$. In addition, if $\delta = \inf_{y \in M} ||x - y||$, f can be made to satisfy $||f|| = 1$ and $f(x) = \delta$. To see this, Define f on $M + \mathbb{R}x$ by $f(m + rx) = r\delta$. Note that

$$f(m + rx) = r\delta \le |r|\delta = \inf_{y \in M} ||rx - ry|| \le ||rx + m|| \tag{12}$$

   which immediately means we can apply Hahn-Banach with $p(x) = ||x||$ to get precisely the f that we want. To see that $||f|| = 1$, note that by construction, $|f(x)| \le ||x||$, so $||f|| \le 1$. To see that this bound is saturated, pick y such that $||x - y||$ is arbitrarily close to δ.

2. If $x \neq 0$, then there exists some $f \in X^*$ such that $||f|| = 1$ and $f(x) = ||x||$. To see this, we simply set $M = 0$ in Part A.

3. The bounded linear functionals on X separate points. To see this, pick $x, y \in X$, choose f so that f is 0 on the subspace spanned by y and $f(x) \neq 0$.

4. Define $\widehat{x} : X^* \to \mathbb{R}$ taking $\widehat{x}(f) = f(x)$. The map $x \mapsto \widehat{x}$ from X to X^{**} is a linear isometry. To see this, note

$$|\widehat{x}(f)| = |f(x)| \leq ||f||||x|| \tag{13}$$

which means that $||\widehat{x}|| \leq ||x||$. On the other hand, we know that there exists $f \in X^*$ with $||f|| = 1$ and $f(x) = ||x||$, so $\widehat{x}(f) = f(x) = ||x||$, so $||x|| \leq ||\widehat{x}||||f|| = ||\widehat{x}||$. We have the inequality both ways, so $||x|| = ||\widehat{x}||$.