Chen iterated integrals and beyond

Jack Ceroni
(Dated: Thursday 12^{th} September, 2024)

Contents

I. Introduction

The goal of these lecture notes is to give a brief exposition to the *Chen iterated integral*, a powerful tool introduced by Chen in Ref. [?], utilized by Guggenheim [?] and Arias Abad-Schaetz [?] to construction an A_∞-generalization of De Rham's theorem. These notes will begin with a discussion of the Chen map, followed by a summary of its use in building an A_∞-quasi-isomorphism between differential graded algebras of k-forms and k-singular cochains.

II. The Chen map

Let M be a finite-dimensional, compact, orientable smooth manifold. The Chen iterated integral is a map

$$C : B(s\Omega^\bullet(M)) \to \Omega^\bullet(PM) \tag{1}$$

where s is the suspension of the graded algebra $\Omega^\bullet(M)$: the De Rham complex of piecewise smooth differential forms on M.

Remark II.1. Right away, one of the main characteristics of the Chen maps that makes it useful is that it is a natural mapping from a complex of data defined on a finite-dimensional manifold M, to data on an *infinite dimensional manifold* PM. It follows that the Chen map may help us deduce certain properties of PM which are difficult to access directly from definitions.

Definition II.1 (Suspension)**.** If V is a $\mathbb{Z}$-graded vector space, the suspension sV is the graded vector space with grading shifted by 1, so that $(sV)^k = V^{k+1}$. Thus, $v \in V^k$ is an element of $(sV)^{k-1}$ in sV, so the suspension decreases the degree of individual elements. We denote $v \in sV$ by sv, to emphasize the change in grading.

Note that B denotes the bar complex (for an algebra A, $BA = \bigoplus_{k \geq 1} A^{\otimes k}$) [?]. PM is the *piecewise-smooth path space of* M.

Definition II.2 (Piecewise-smooth path space)**.** Given smooth manifold M, the piecewise-smooth path space PM is the set of all piecewise smooth $\gamma : [0,1] \to M$. Let $PM^\infty = C^\infty([0,1], M) \subset PM$ be the subset consisting of smooth paths. We take the C^1-topology to be the initial topology of the map $\Gamma : C^\infty([0,1], M) \to C^\infty([0,1], M) \times C^\infty(T[0,1], TM)$ taking $\gamma \mapsto (\gamma, \gamma_*)$, where the range is endowed with the compact-open topology on each factor. We then define a topology on PM by taking the final topology of the inclusion $\iota : C^\infty([0,1],M) \to PM$.

Given finite-dimensional smooth manifold X, we say that a map $f : X \to PM$ is (piecewise) smooth if the map $\widetilde{f} : X \times [0,1] \to M$ given by $\widetilde{f}(x,t) = f(x)(t)$ is (piecewise) smooth.

With a definition for PM, it is now possible to define $\Omega^\bullet(PM)$: the De Rham complex of differential forms on the path space.

Definition II.3 (Differential forms on path space)**.** A differential k-form $\eta \in \Omega^k(PM)$ on path space PM is an association of each pair (X,f) of finite-dimensional smooth manifold and smooth map $f : X \to PM$ to a differential form $\eta(X,f) \in \Omega^k(X)$, such that if g is a smooth map between finite-dimensional manifolds, $g : Y \to X$, then

$$\eta(Y, f \circ g) = g^* \eta(X,f). \tag{2}$$

In light of this naturalness, we can simply denote $\eta(X,f)$ by $f^*\eta$. Technically, one should use the language of natural transformations to make this "association" completely rigorous, but this won't be needed for our current purposes.

It is important to note that there is a natural way to define the standard operations one can usually perform on differential forms, namely the exterior derivative and the wedge product. In particular, we simply use the pullback under any function $f : X \to PM$: for $\omega, \eta \in \Omega^\bullet(PM)$, we define

$$f^*(\eta \wedge \omega) \coloneqq f^*\eta \wedge f^*\omega \quad \text{ and } \quad f^*(d\eta) \coloneqq df^*\eta. \tag{3}$$

From here, let us notice that both $B(s\Omega^\bullet(M))$ and $\Omega(PM)$ are cochain complexes. We equip $\Omega(PM)$ with the coboundary defined above. For the case of the bar complex, given arbitrary DGA $(A, d, \wedge)$, we may define a couboundary $D : B(sA) \to B(sA)$ given by

$$\begin{aligned} D(sa_1 \otimes \cdots \otimes sa_n) &= \sum_{i=1}^{n} (-1)^{[a_1]+\cdots+[a_{i-1}]} sa_1 \otimes \cdots \otimes sa_{i-1} \otimes s(da_i) \otimes sa_{i+1} \otimes \cdots \otimes sa_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^{[a_1]+\cdots+[a_i]} sa_1 \otimes \cdots \otimes sa_{i-1} \otimes s(a_i \wedge a_{i+1}) \otimes sa_{i+2} \otimes \cdots \otimes sa_n. \qquad (4) \\ &\coloneqq d_\otimes(sa_1 \otimes \cdots \otimes sa_n) + d_\wedge(sa_1 \otimes \cdots \otimes sa_n) \qquad (5) \end{aligned}$$

where $[a_k]$ is the degree of $a_k \in A$ (a graded algebra). One can verify that $D \circ D = 0$. Note that D is a degree-1 map. Clearly, $d_\otimes$ is. As for $d_\wedge$, removing sa_i and sa_{i+1} from the tensor product decreases degree by $[a_i] + [a_{i+1}] - 2$, and adding $s(a_i \wedge a_{i+1})$ increases it by $[a_i \wedge a_{i+1}] - 1 = [a_i] + [a_{i+1}] - 1$. Thus, there is a net degree-increase of 1.

Going forward, let $\overline{D}$ be the bar differential associated to the DGA $(s\Omega^\bullet(M), -d, \wedge)$, where we flip the sign of d, which is itself given by $d(s\omega) = sd\omega$. Before defining the Chen map, we require first a bit more setup.

Definition II.4. Given a map $f : X \to PM$, where X is a finite-dimensional smooth manifold, recall that $\widetilde{f} : [0,1] \times X \to M$ is defined as $\widetilde{f}(t,x) = f(x)(t)$. We then define $\widetilde{f}_{(m)} : \Delta^m \times X \to M^{\times m}$ as

$$\widetilde{f}_{(m)}(t_1, \ldots, t_m, x) = (\widetilde{f}(t_1, x), \ldots, \widetilde{f}(t_m, x)) \tag{6}$$

with Δ^m the standard m-simplex, $\Delta^m = \{(t_1, \ldots, t_m) \mid 0 \le t_1 \le \cdots \le t_m \le 1\}$.

Definition II.5 (Fibre integral)**.** Let $\pi : E \to M$ be a fibre bundle over M such that each fibre $\pi^{-1}(p)$ is compact and oriented. Let $\omega \in \Omega^k(E)$. We define $\pi_*\alpha \in \Omega^{k-n}(M)$, where n is the dimension of the fibres, as

$$\pi_*\alpha_p(v_{1,p}, \ldots, v_{k-n,p}) = \int_{\pi^{-1}(p)} \widetilde{\alpha} \tag{7}$$

where $\widetilde{\alpha} \in \Omega^n(\pi^{-1}(p))$ is the top-form defined as

$$\widetilde{\alpha}_q(w_{1,q}, \dots, w_{n,q}) = \alpha_q(w_{1,q}, \dots, w_{n,q}, \widetilde{v}_{1,q}, \dots, \widetilde{v}_{k-n,q}) \tag{8}$$

where $\widetilde{v}_{j,q}$ is a lift of $v_j \in T_pM$ to an element of $\widetilde{v}_{j,q} \in T_q\pi^{-1}(p)$ (i.e. we must have $d\pi_q(\widetilde{v}_{j,q}) = v_j$). Note that this definition ends up being independent of the chosen lift.

With this machinery, we may now define the Chen map C. Recall that C will take some $s\omega_1 \otimes \cdots \otimes s\omega_m \in B(s\Omega^\bullet(M))$ and produces an element of $\Omega^\bullet(PM)$. Thus, we should be able to feed $C(s\omega_1 \otimes \cdots \otimes s\omega_m)$ some smooth map $f : X \to PM$ from a finite-dimensional manifold X to the path space to obtain a form of X, $f^*C(s\omega_1 \otimes \cdots \otimes s\omega_m)$.

To produce this form on X, given an f, we follow these steps:

1. Let us define the projection maps $\pi_k : M^{\times m} \to M$ which project onto the k-th factor. Note that $s\omega_1 \otimes \cdots \otimes s\omega_m$ is an element of degree $([\omega_1] - 1) + \cdots + ([\omega_m] - 1) = [\omega_1] + \cdots + [\omega_m] - m$ in the bar complex. We pull-back each ω_k by π_k, obtaining $\pi_k^*\omega_k \in \Omega^{[\omega_k]}(M^{\times m})$.

2. We take the wedge product of all the $\pi_k^*\omega_k$, to obtain

$$\pi_1^*\omega_1 \wedge \cdots \wedge \pi_m^*\omega_m \in \Omega^{[\omega_1]+\cdots+[\omega_m]}(M^{\times m}) \tag{9}$$

3. We take the function $f : X \to PM$ and produce the m-fold induced functions $\widetilde{f}_{(m)} : \Delta^m \times X \to M^{\times m}$ defined above. We then pullback to get

$$\widetilde{f}^*_{(m)}(\pi_1^*\omega_1 \wedge \cdots \wedge \pi_m^*\omega_m) \in \Omega^{[\omega_1]+\cdots+[\omega_m]}(\Delta^m \times X) \tag{10}$$

4. Note that $\Delta^m \times X$ is a trivial fibre bundle over X. Thus, we can take the fibre integral to produce

$$\pi_*\widetilde{f}^*_{(m)}(\pi_1^*\omega_1 \wedge \cdots \wedge \pi_m^*\omega_m) \in \Omega^{[\omega_1]+\cdots+[\omega_m]-m}(X) \tag{11}$$

5. Finally, we multiply by a sign, yielding the final definition of the Chen map:

$$f^*C(s\omega_1 \otimes \cdots \otimes s\omega_m) = (-1)^{\sum_{i=1}^m [\omega_i](m-i)}\pi_*\widetilde{f}^*_{(m)}(\pi_1^*\omega_1 \wedge \cdots \wedge \pi_m^*\omega_m) \tag{12}$$

Let us look at some basic examples to better understand why this definition makes sense.

Example II.1 (A single 1-form)**.** Let us look at the example when we have ω, a 1-form. In this case, $s\omega$ has degree 0 in the bar complex, and we expect to product an element of degree 0 in $\Omega^\bullet(X)$: a function on path space. We can of course identify smooth functions $f : \{p\} \to PM$ with smooth paths $\gamma : [0,1] \to M$. In particular, $\widetilde{f}_{(1)} = \gamma$ will be smooth path in M. In this case, we also have $\pi_1^*\omega = \omega$. Therefore, $\widetilde{f}^*_{(1)}\pi_1^*\omega = \gamma^*\omega \in \Omega^1([0,1] \times \{p\})$ is just a 1-form on the interval. The fibre integral is just integrating this 1-form on $[0,1]$, so that up to an overall sign, the Chen map yields

$$\int_{[0,1]} \gamma^*\omega = \int_\gamma \omega \tag{13}$$

when given some path γ. In other words, the real function on path space that is produced takes a path $\gamma \in PM$ and maps $\gamma \mapsto \int_\gamma \omega$.

Example II.2 (Many 1-forms)**.** Let us now consider the case that we have m individual 1-form $\omega_1, \dots, \omega_m$. Once again, we expect to produce a function on path space. Once again we consider functions $f : \{p\} \to PM$, which can be identified with paths $\gamma : [0,1] \to M$.

When we pullback under $\widetilde{f}_{(m)}$, we are effectively pulling back under $(t_1, \dots, t_m) \mapsto (\gamma(t_1), \dots, \gamma(t_m))$ for $(t_1, \dots, t_m) \in \Delta^m$. Of course, $(\pi_k \circ \widetilde{f}_{(m)})(t_1, \dots, t_m) = \gamma(t_k)$. Define $\gamma_k(t_1, \dots, t_m) = \gamma(t_k)$. We have

$$\widetilde{f}^*_{(m)}(\pi_1^*\omega_1 \wedge \cdots \wedge \pi_m^*\omega_m) = (\pi_1 \circ \widetilde{f}_{(m)})^*(\omega_1) \wedge \cdots \wedge (\pi_m \circ \widetilde{f}_{(m)})^*(\omega_m) \tag{14}$$

$$= \gamma_1^*\omega_1 \wedge \cdots \wedge \gamma_m^*\omega_m \in \Omega^m(\Delta^m). \tag{15}$$

The fibre integral simply amounts to integrating over the entire simplex. Thus, the Chen map produces the following function on path space, up to a sign:

$$\gamma \mapsto \int_{\Delta^m} \gamma_1^* \omega_1 \wedge \cdots \wedge \gamma_m^* \omega_m \tag{16}$$

If we consider the particular case where $M = \mathbb{R}$, and we let dt be the standard basis for $\Omega^1(\mathbb{R})$, then we can write each ω_k as $\omega_k = g_k dt$. In this case, we then have

$$\int_{\Delta^m} \gamma_1^* \omega_1 \wedge \cdots \wedge \gamma_m^* \omega_m = \int_{0 \le t_1 \le \cdots \le t_m \le 1} g_1(\gamma(t_1))\gamma'(t_1) \cdots g_m(\gamma(t_m))\gamma'(t_m)\ dt_1 \cdots dt_m \tag{17}$$

which is precisely the kind of integral that one might see in the Dyson series of quantum field theory.

Example II.3 (A single 2-form)**.** As a final example, consider what happens when ω is a 2-form, so that $C(s\omega)$ is a 1-form on the path space, meaning that each $f^*C(s\omega)$ is a 1-form. We expect to be able to integrate an element of $\Omega^1(PM)$ over a path in PM (i.e. a path of paths) to get a scalar. Suppose f is such a path of paths, so we have $f : [0,1] \to PM$. For simplicity, assume that $M = \mathbb{R}^2$, so we can write $\omega = F dx \wedge dy$. The projection π_1 is just the identity, so we simply pullback ω by $\widetilde{f} : [0,1] \times [0,1] \to \mathbb{R}^2$. We denote $\widetilde{f}(s,t) = (f_1(s,t), f_2(s,t))$. We have

$$\pi_* \widetilde{f}^*(\omega) = \left(\int_{[0,1]} F(\widetilde{f}(s,t)) \left(\frac{df_1}{ds}\frac{df_2}{dt} - \frac{df_1}{dt}\frac{df_2}{ds} \right)\ ds \right) dt \tag{18}$$

which is a 1-form on $[0,1]$.

One can show that the Chen map has many advantageous properties, in particular: it is natural, and it is *almost* a cochain map, in the sense that it commutes with the differentials on the bar complex and the De Rham complex of the path space up to the addition of a "boundary term" (this result is known as Chen's theorem).

Lemma II.1 (Naturality of the discrete Chen map)**.** If M and N are both smooth manifolds, let $F : M \to N$ be a smooth map. Then the following diagram commutes:

$$\begin{tikzcd}
B(sC^\bullet(N)) \arrow[r, "BsF^*"] \arrow[d, "C"'] & B(sC^\bullet(M)) \arrow[d, "C"] \\
\Omega^\bullet(PN) \arrow[r, "(PF)^*"'] & \Omega^\bullet(PM)
\end{tikzcd}$$

where $BsF^*(s\omega_1 \otimes \cdots \otimes s\omega_m) = sF^*\omega_1 \otimes \cdots \otimes sF^*\omega_m$ and $PF(\gamma) = F \circ \gamma \in PN$ for $\gamma \in PM$.

Theorem II.1 (Chen's theorem)**.** For any $s\omega_1 \otimes \cdots \otimes s\omega_m \in B(s\Omega^\bullet(M))$, we have

$$\begin{aligned}(d \circ C)(s\omega_1 \otimes \cdots \otimes s\omega_m) = (C \circ \overline{D})(s\omega_1 \otimes \cdots \otimes s\omega_m) + \\ \mathrm{ev}_1^*(\omega_1) \wedge C(s\omega_2 \otimes \cdots \otimes s\omega_m) - (-1)^{[\omega_1] + \cdots + [\omega_{m-1}]} C(s\omega_1 \otimes \cdots \otimes s\omega_{m-1}) \wedge \mathrm{ev}_0^*(\omega_m)\end{aligned} \tag{19}$$

where $\mathrm{ev}_i : PM \to M$ takes γ to $\gamma(i)$ for $i \in \{0,1\}$.

III. Application: an A_∞-De Rham theorem

Having defined the Chen map and described some of its properties, let us briefly move to a discussion of its broader applications. In particular, one can use the Chen map as an ingredient to formulate an A_∞ variant of De Rham's theorem in the work of Arias Abad and Schaetz.

To be more specific, the ordinary De Rham theorem gives a quasi-isomorphism (i.e. an isomorphism in cohomology) of the complex of k-singular cochains on some manifold M, and the k-th De Rham complex of differetial k-forms on the manifold. The claim of this result is that this isomorphism actually has more structure: it preserves the A_∞-structure of the DGAs $\Omega^\bullet(M)$ and $C^\bullet(M)$.

A. A_∞-algebras

To begin to make sense of this, let us introduce definitions relating to A_∞-algebras, following the work of Ref. [**?**]. To motivate the definition of an A_∞-algebra, we first consider that of an A_∞-space, which can be thought of as a space equipped with an operation which fails to be associative, but admits homotopies between "alternatively-bracketed" elements, satisfying certain conditions.

Let us consider the example of *based loop space*:

Example III.1. Let (X,p) be a pointed topological space, let ΩX be the space of all loops $f : S^1 \to X$ based at p (so $f(1,0)=p$). Given loops f_1 and f_2, there exists an operation $m_2(f_1,f_2) = f_1 * f_2 \in \Omega X$ which is f_1 on the upper-half of S^1 and f_2 on the lower-half. This operation is not associative, as $(f_1 * f_2) * f_3 \neq f_1 * (f_2 * f_3)$, but the two loops are clearly homotopic. Denote this homotopy between $m_2 \circ m_2 \times \mathrm{id}$ and $m_2 \circ \mathrm{id} \times m_2$ by $m_3 : [0,1] \times \Omega X^3 \to \Omega X$.

If we now turn our attention to 4-fold compositions, note that there are 5 possible ways to brackets the factors, namely:

$$f_1 * (f_2 * (f_3 * f_4)), \quad f_1 * ((f_2 * f_3) * f_4), \quad (f_1 * (f_2, * f_3)) * f_4, \tag{20}$$

$$((f_1 * f_2) * f_3) * f_4, \quad (f_1 * f_2) * (f_3 * f_4) \tag{21}$$

Note that each of the above loops is homotopic, via some application of m_3, to precisely two of the other loops. Thus, we define a map on the boundary of a pentagon, K_4, of the form $\partial m_4 : \partial K_4 \times \Omega X^4 \to \Omega X$ which is equal to each of the homotopies on each side. We then claim we can extend this map to a map $m_4 : K_4 \times \Omega X^4 \to X$, which is a sort-of "homotopy of homotopies". As it turns out, we can continue this procedure for all n-fold compositions, finding dimension $n-2$ polytopes K_n which allow us to define "n-fold homotopy". This is the A_∞-structure which makes the based loop space into a so-called A_∞-space.

This leads us to a "rough" definition of an A_∞-space, which will suffice for our purposes, for now:

Definition III.1 (A_∞-space (informal))**.** A topological space Y along with maps $m_n : K_n \times Y^n \to Y$ which satisfy certain compatibility conditions similar to the conditions above.

The algebraic analogue of A_∞-spaces are A_∞-algebras. In particular, the singular chain complex of an A_∞-space is automatically an A_∞-algebra.

Definition III.2 (Suspension)**.** If V is a $\mathbb{Z}$-graded vector space, the suspension sV is the graded vector space with grading shifted by 1, so that $(sV)^k = V^{k+1}$. Thus, $v \in V^k$ is an element of $(sV)^{k-1}$ in sV, so the suspension decreases the degree of individual elements. We denote $v \in sV$ by sv, to emphasize the change in grading.

Definition III.3 (A_∞-algebra)**.** It is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{n\in\mathbb{Z}} A^n$ with linear maps $m_n : (sA)^{\otimes n} \to sA$ for $n \geq 1$ of degree 1 which satisfy

$$\sum_{r+s+t=n} m_{r+t+1} \circ (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0 \tag{22}$$

for all $n \geq 1$. Note that when we apply these maps to elements of the algebra, we make use of the Koszul sign rule:

$$(f \otimes g)(x \otimes y) = (-1)^{[x][g]} f(x) \otimes g(y). \tag{23}$$

To be understand these relations, consider first the case of $n=1$, the only possible option is $s=1$ with $r,t=0$. It follows that we must have $m_1^2 = 0$, making (sA, m_1) a cochain complex. For $n=2$, we have

$$m_2 \circ (\mathrm{id} \otimes m_1 + m_1 \otimes \mathrm{id}) + m_1 \circ m_2 = 0 \tag{24}$$

which implies that if we think of m_2 as giving a multiplication for an algebra, m_1 is a graded derivation with respect to this algebra. For example, suppose $A = \Omega^\bullet(M)$. We can immediately set $m_1(s\omega) = s(d\omega)$ and $m_2(s\omega \otimes s\eta) = s(\omega \wedge \eta)$, and note that we will have $m_1^2 = 0$. We also have

$$(\mathrm{id} \otimes m_1 + m_1 \otimes \mathrm{id})(s\omega_1 \otimes s\omega_2) = (-1)^{[m_1][\omega_1]} s\omega_1 \otimes sd\omega_2 + (-1)^{[\mathrm{id}][\omega_1]} sd\omega_1 \otimes s\omega_2 \tag{25}$$

$$= (-1)^{[s\omega_1]} s\omega_1 \otimes sd\omega_2 + sd\omega_1 \otimes s\omega_2 \tag{26}$$

and upon applying m_2 we get $s(d\omega_1 \wedge \omega_2) + (-1)^{[\omega_1]} s(\omega_1 \wedge d\omega_2)$. The A_∞-relation says that this should be equal to $sd(\omega_1 \wedge \omega_2)$, which of course holds. More generally, a differential graded algebra is an A_∞-algebra where $m_n = 0$ for $n \geq 3$.

Definition III.4 (A_∞-homomophism)**.** Let A and A' be A_∞-algebras with maps m_n and m'_n. An A_∞-homomorphism is a collection of degree-0 maps $\psi_n : (sA)^{\otimes n} \to sA'$ such that

$$\sum_{i+j+k=n} \circ (\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k}) = \sum_{l_1 + \cdots + l_r = n} m'_r \circ (\psi_{l_1} \otimes \cdots \otimes \psi_{l_r}) \tag{27}$$

for all $n \geq 1$.

Definition III.5 (A_∞-quasi-isomorphism)**.** An A_∞-homomorphism where ψ_1 induces an isomorphism in cohomology, where m_1 is the coboundary as explained above.

As can clearly be seen in these definitions, a version of De Rham's theorem which is an A_∞-quasi-isomorphism requires the existence of a *collection* of maps $\psi_n : (sA)^{\otimes n} \to sA'$ obeying certain compatibility conditions, for all n. In other words, we require more structure to be preserved than the standard De Rham quasi-isomorphism.

B. The quasi-isomorphism

To construct the quasi-isomorphism, we must specify a collection of maps $\psi_n : (s\Omega^\bullet(M))^{\otimes n} \to sC^\bullet(M)$. Equivalently, we can define a single graded map on the bar complex, $\psi : B(s\Omega^\bullet(M)) \to C^\bullet(M)$, as of course the bar is just the direct sum of all tensor powers of the algebra in question. The main claim of the paper by Arias Abad and Schaetz is that this map can be constructed by concatenting the maps

$$C : B(s\Omega^\bullet(M)) \to \Omega^\bullet(PM) \quad \text{and} \quad I : \Omega^\bullet(PM) \to sC^\bullet(M) \tag{28}$$

where C is the Chen map constructed above and I is the so-called *Igusa map* (which is outside the scope of these notes). In particular, the A_∞-structure-preserving character of C follows directly from the results discussed in the previous section (naturality and Chen's theorem).