MAT495 essay 3: Non-commutative geometry and solid-state physics, and the index map

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I. Introduction

The goal of this essay is to outline and explain some of the key ideas in the formulation of solid-state physics in terms of non-commutative geometry and C^*-algebras.

II. Non-commutative geometry and solids

We begin by defining an appropriate model for a system of atoms $\mathcal{L}$ in ambient space $\mathbb{R}^D$, making up a solid. We assume that the solid is in idealized conditions, at 0-temperature, so that the atoms are fixed in place. We naturally expect $\mathcal{L}$ to have certain properties: in particular, we expect that atoms are not spaced arbitrarily close or arbitrarily far away from each other.

Definition II.1. We say that $\mathcal{L} \subset \mathbb{R}^D$ is r-discrete is every open ball of radius-r contains at most one point of $\mathcal{L}$. We say that $\mathcal{L}$ is R-dense if every ball of radius R contains at least one point of $\mathcal{L}$. If $\mathcal{L}$ is r-discrete and R-dense, it is called an (r, R)-Delone set. A Delone set is of finite type when the set of relative displacements, $\mathcal{L} - \mathcal{L}$ is locally finite, it is a Meyer set if $\mathcal{L} - \mathcal{L}$ is a Delone set itself.

The condition of "not arbitrarily close and not arbitrarily far" is axiomatized as the condition of being a Delone set, so most solid systems considered will be modelled as Delone sets.

Definition II.2. Given $\mathcal{L}$ which is r-discrete, let $\nu^{\mathcal{L}}$ denote the counting measure of $\mathcal{L}$ on the the Borel sets of $\mathbb{R}^D$ as the measure which counts the number of elements of $\mathcal{L}$ contained inside some Borel set A:

$$\nu^{\mathcal{L}}(A) = \sum_{y \in \mathcal{L}} \sum_{a \in A} \delta(a - y) \tag{1}$$

Note that this is a Radon measure on $\mathbb{R}^D$, due to the discreteness of $\mathcal{L}$: clearly it is finite on compact sets C as we can cover C by a finite number of r-balls. Inner and outer regularity are also easy to check.

Remark II.1. Since $\nu^{\mathcal{L}}$ is a Radon measure, it can be considered as an element of the dual space of $C_c(\mathbb{R}^D)$. In this case, it is easy to see what this element will be:

$$\nu^{\mathcal{L}}(f) = \sum_{y \in \mathcal{L}} f(y) \tag{2}$$

where the sum is finite as f has compact support.

Definition II.3. We call ν a counting measure on $\mathbb{R}^D$ if it is Radon and assigns an integer value to open balls $B \subset \mathbb{R}^D$. We say that ν is r-discrete is $\nu(B) \leq 1$ for balls B of radius less than or equal to r. We say that ν is R-dense if $\nu(B) \geq 1$ for balls of radius greater than or equal to R. If ν is r-discrete and R-dense, we call it (r, R)-Delone.

There is a clear 1-to-1 correspondence between sets $\mathcal{L}$ and counting measures, and moreover, the properties of r-discreteness and R-density carry over exactly under this correspondence. Thus, we often will represent a system of atoms by a counting measure.

If we let $\mathfrak{M}(\mathbb{R}^D)$ denote the space of Radon measures on $\mathbb{R}^D$, we can endow it with the weak-$*$ topology: a sequence μ_n converges to μ if and only if $\mu_n(f) \to \mu(f)$ for any $f \in C_c(\mathcal{R}^D)$. Then, closed sets are those which contain their limit points, and we have a topology.

Definition II.4. Given $a \in \mathbb{R}^D$, define the translation map $\tau^a : x \mapsto x + a$. Clearly, these maps form a group, and they act on $C_c(\mathbb{R}^D)$ via $\tau^a f(x) = f(x-a)$ as

$$\tau^{a+b} f(x) = f(x-(a+b)) = (\tau^a \circ \tau^b) f(x). \tag{3}$$

Moreover, the translations act on $\mathfrak{M}(\mathbb{R}^D)$ via $\tau^a \mu(f) = \mu(\tau^{-a} f)$, via noting that a counting measure can be represented as in Eq. (2) for some discrete $\mathcal{L}$.

Claim II.1. Let $QD(\mathbb{R}^D)$ denote the set of counting measures in $\mathfrak{M}(\mathbb{R}^D)$. This set is invariant under the action of τ^a.

Proof. Note that if ν is a counting measure, $\nu(f) = \sum_{y\in\mathcal{L}} f(y)$ for some r-discrete $\mathcal{L}$, so

$$\tau^a \nu(f) = \nu(\tau^{-a} f) = \sum_{y\in\mathcal{L}} f(y+a) = \sum_{y\in\mathcal{L}+a} f(y)$$

where it is easy to see that $\mathcal{L}+a$ is r-discrete: if some ball $B_{r'}(s)$ of radius r' less than or equal to r contains $\ell + a$ and $\ell' + a$ for $\ell, \ell' \in \mathcal{L}$ and $\ell \neq \ell'$, then $B_{r'}(s-a)$ contains ℓ, ℓ', contradicting the fact that $\mathcal{L}$ is r-discrete. $\square$

Note that the above implies that each set $UD_r(\mathbb{R}^D)$ of r-discrete counting measures (for fixed r) is individually invariant, a stronger result. It is equally easy to see that $\mathrm{Del}_{r,R}(\mathbb{R}^D)$, the set of (r,R)-Delone counting measures is invariant as well.

Claim II.2. Finite-type $\mathcal{L}$ will be sent by translation to finite-type sets.

Proof. Translation doesn't change the set of relative displacements! $\square$

Definition II.5. Given some r-discrete $\mathcal{L}$, we define the *hull* of $\mathcal{L}$ to be the dynamical systems $(\Omega, \mathbb{R}^D, \tau)$, where $\tau : (t, x) \mapsto \tau^t(x)$ and Ω is the closure of the orbit of counting measure $\nu^{\mathcal{L}}$ under the action of translation.

Remark II.2. It is possible to demonstrate that $UD_r(\mathbb{R}^D)$ as well as $\mathrm{Del}_{r,R}(\mathbb{R}^D)$ are both closed and compact sets in the weak-$*$ topology on the space of measures. Equipped with this fact, note that the orbit of $\nu^{\mathcal{L}}$ will be contained in $UD_r(\mathbb{R}^D)$ for some r, as $\mathcal{L}$ is r-discrete for some r, and we know that we have translation invariance under the action described above. Thus, the closure of this orbit must also be contained in $UD_r(\mathbb{R}^D)$, and moreover, this orbit is compact, as it is a closed subset of a compact space.

It follows that given $\omega \in \Omega$, we know that ω will be a r-discrete measure. Thus, ω will describe an r-discrete set. In particular, we know that $\omega = \sum_{y\in\mathcal{L}_\omega} f(y)$ for some r-discrete $\mathcal{L}_\omega$.

Definition II.6. Given $\mathcal{L}$ and its hull Ω, define the canonical transversal to be the subset $X \subset \Omega$ of ω such that $\mathcal{L}_\omega$ contains the origin.

III. Crossed products

Now that we have outlined some of the basic terminology related to the mathematical model of solids which we will make use of, we must understand the notion of a *crossed product*, which will allow us to assign a C^*-algebra to the dynamical system $(\Omega, \mathbb{R}^D, \tau)$ discussed above.

Definition III.1 (Representation)**.** Recall that a group representation of group G on vector space V is a group homomorphism $\pi : G \to \mathrm{GL}(V)$. A unitary representation is a group representation on Hilbert space H such that $\pi(g)$ is unitary for each $g \in G$. Going forward, we will assume that all unitary representations we deal with are *strongly continuous*, meaning that G is a topological group (locally compact), and each map $g \mapsto \pi(g)v$ is norm-continuous for every $v \in H$.

First, let A be a C^*-algebra, and let $\alpha : G \to \mathrm{Aut}(A)$ be a group action on A by group G. We assume that G is a locally compact topological group. Let us package together the triple (G, A, α). We then define a *covariant representation* for the triple on Hilbert space H to be a pair (π, v) such that $\pi : A \to B(H)$ is a representation taking values in the *bounded* operators on H, and $v : G \to U(H)$ is a unitary representation. In addition, the pair must satisfy the following condition:

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a)) \tag{4}$$

for all $g \in G$ and $a \in A$. The intuition behind this definition is clear: we want the action of the group to be represented via unitary conjugation in a way that is "compatible" with the operator representation of $a \in A$.

Example III.1 (Quantum)**.** Here is an illuminating example. Suppose (A, σ_t) is a C^*-dynamical system, then $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$ with $t \mapsto \sigma_t$ is a group action. In the context of quantum mechanics, the time-dynamics are often given by Heisenberg-picture evolution of observables, $\sigma_t(a) = e^{iht}ae^{-iht}$, where h is self-adjoint, and e^{iht} is thus unitary. In the case that $A = B(H)$, then h self-adjoint in $B(H)$ immediately means that $e^{iht} \in U(H)$. Thus, setting $v(t) = e^{iht}$ and $\pi(a) = a$ immediately yields a covariant representation of $(\mathbb{R}, A, \sigma)$: clearly

$$v(t+s) = e^{ih(t+s)} = e^{iht} \circ e^{ihs} = v(t) \circ v(s) \quad \text{and} \quad \alpha(a \circ b) = \alpha(a) \circ \alpha(b) \tag{5}$$

so these maps are homomorphisms, and thus valid representations.

Remark III.1. We will assume that representations of C^*-algebras are $*$-representations going forward, meaning that $\pi(a^*) = \pi(a)^*$ (where we assume that this is a representation on a Hilbert space, so we have an involution in the codomain).

Definition III.2. Recall that a *Haar measure* on locally compact topological group G is a measure which is left-invariant, finite of compacts, outer regular on Borel sets, and inner regular on open sets. Up to multiplicative constant, Haar measure is unique.

With this in mind, let μ be a fixed Haar-measure. Let $C_c(G, A, \alpha)$ be the algebra of compactly-supported continuous functions $f : G \to A$. The multiplication is defined using the Haar-measure, we take a "twisted convolution":

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))d\mu(h). \tag{6}$$

The involution is defined via

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*) \tag{7}$$

where Δ is the *modular function* associated with the group G. This function is defined via the following result:

Claim III.1. Given locally compact Hausdorff topological group G with Haar-measure μ, there exists a continuous homomorphism $\Delta : G \to \mathbb{R}^+$ such that for all $t \in G$ and all Borel subsets S, we have $\mu(St) = \Delta(t^{-1})\mu(S)$. In other words, we should have

$$\mu(St^{-1})\mu(S)^{-1} = \Delta(t) \tag{8}$$

for all S such that $\mu(S) > 0$. To see that this holds, note that the function $\nu_t(S) = \mu(St)$ is a measure (for any t). Moreover, it is left-invariant, as $\nu_t(gS) = \mu(gSt) = \mu(St) = \nu_t(S)$. It follows by uniqueness of Haar-measure, up to scalar multiple, that $\mu(S) = \Delta(t)\nu_t(S)$ for some $\Delta(t)$ multiple. We then clearly have $\mu(S) = \Delta(t)\mu(St)$, so $\Delta(t)^{-1}\mu(S) = \mu(St)$.

Moreover, note that we have

$$\Delta(st)^{-1}\mu(S) = \mu(Sst) = \Delta(t)^{-1}\mu(Ss) = \Delta(t)^{-1}\Delta(s)^{-1}\mu(S) \tag{9}$$

which implies that $\Delta(st) = \Delta(s)\Delta(t)$.In other words, Δ is a homomorphism.

Claim III.2. The involution defined above is in fact an involution

Proof. Note that

$$(a^*)^*(g) = \Delta(g)^{-1}\alpha_g(a^*(g^{-1})^*) = \Delta(g)^{-1}\alpha_g\left(\left[\Delta(g^{-1})^{-1}\alpha_{g^{-1}}(a(g)^*)\right]^*\right) \tag{10}$$

$$= \Delta(g)^{-1}\Delta(g)(\alpha_g \circ \alpha_{g^{-1}})(a(g)) = a(g) \tag{11}$$

and we are done. □

With this structure, we know that $C_c(G, A, \alpha)$ is a $*$-algebra. To define a norm, we let

$$||a|| = \int_G ||a(g)||d\mu(g) \tag{12}$$

When we combine this norm with the involution defined above, and then take the completion of $C_c(G, A, \alpha)$, which we denote $L^1(G, A, \alpha)$, we obtain a Banach $*$-algebra.

Having discussed representations of the original algebra A, let us now consider a representation of $C_c(G, A, \alpha)$. Let (π, v) be a covariant representation of (G, A, α) on H, then we define *the integrated form* of (v, π) to be the representation σ of $C_c(G, A, \alpha)$ on H sending elements of $C_c(G, A, \alpha)$ into $B(H)$ such that

$$\sigma(a)\xi = \int_G \pi(a(g))v(g)\xi \tag{13}$$

From here, we *extend* this representation to a representation for the completion, $L^1(G, A, \alpha)$ with repsect to the norm given above. Given (G, A, α), we define the *universal representation* to be the direct sum of all non-degenerate representations of $L^1(G, A, \alpha)$. We denote this map σ We then define the *crossed product* $A \rtimes_\alpha G$ to be the norm closure of $\sigma(C_c(G, A, \alpha))$. Note that if we are given a non-degenerate covariant representation of (G, A, α), we automatically get a representation for the crossed product: we just take the integrated form, which is a representation of $C_c(G, A, \alpha)$. Given element in the image of universal representation, we restrict to a direct summand, to get the image of $C_c(G, A, \alpha)$ under the integrated form (as this is a representation in the direct sum).

IV. The Brillouin zone and non-commutative Brillouin zone

Now that we have defined the crossed product, we are able to combine the machinery developed in the first section of this essay with this construction in order to formulate a non-commutative geometric theory of aperiodic solids.

To be more specific, recall that the *hull* of an r-discrete set $\mathcal{L}$ naturally pairs with action under translation to form a dynamical system $(\Omega, \mathbb{R}^D, \tau)$. Given a dynamical system, as was discussed in the previous section, we are able to form a natural corresponding C^*-algebra: the crossed product $C(\Omega) \rtimes \mathbb{R}^D$. Note that we already showed that Ω is compact in the first section, so $C_c(\Omega) = C(\Omega)$ in this case.

From here, we define the *non-commutative Brillouin zone* to be the topological manifold associate with the C^*-algebra $\Omega \rtimes \mathbb{R}^D$, where we assume the influence of no external magnetic field. It is via understanding the translation symmetry of the underlying system as being engrained in the C^*-algebra describing the system's state space that we are able to arrive at non-trivial facts about the system's dynamics (*most of these results are incredibly complex, and I have realized lie very far outside my current capabilities of understanding*).

V. A few thoughts on the index map

This is a completely different topic than the rest of this essay, but I was also thinking about the index map in the last month of the course, so here are a few notes on it.

The main idea with the index map is to construct a bridge between short exact sequence of K_0 and K_1 -groups. In particular, we begin with a short exact sequence of algebras:

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{14}$$

Because K_0 and K_1 are covariant functors, we get analogous short exact sequences when we map everything under either functor:

$$0 \longrightarrow K_0(I) \xrightarrow{K_0(\varphi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B) \longrightarrow 0 \tag{15}$$

as well as

$$0 \longrightarrow K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\psi)} K_1(B) \longrightarrow 0. \tag{16}$$

We desire an index map $\delta_1 : K_1(B) \to K_0(I)$, which pastes together the two short exact sequence into a single sequence of six different groups: we cycle through all the K_1 groups, then map under δ_1 and cycle through all of the K_0 groups.

With this, let us recall what $K_1(B)$ and $K_0(I)$ actually are. The K_0 -group, in the case that the underlying algebra is not necessarily unital is slightly more complicated to construct than in the unital case: $K_0(I)$ is precisely $\mathrm{Ker}(K_0(\pi))$ where π is the scalar projection associated with the short exact sequence

$$0 \longrightarrow I \xrightarrow{\text{inclusion}} \widetilde{I} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0 \tag{17}$$

where $\widetilde{I}$ is the unitization (i.e. the algebra of formal sums $a + \alpha 1$ for $a \in I$ and $\alpha \in \mathbb{C}$ with arithmetic defined in the obvious way). In the case of $K_0(\pi)$, K_0 is the functor as it is defined in the unital sense (I won't review this here, as this definition of lengthy and is outlined in Chapters 2-3 of Rordam). It follows that for some $[(p_{ij} + \alpha_{ij}1)_{ij}]_0 \in K_0(\widetilde{I})$ where $(p_{ij} + \alpha_{ij}1)_{ij} \in \mathcal{P}_\infty(\widetilde{I})$, we have

$$K_0(\pi)([(p_{ij} + \alpha_{ij}1)_{ij}]_0) = [(\alpha_{ij})_{ij}]_0 \in K_0(\mathbb{C}) = \sum_i \alpha_{ii} \in \mathbb{Z} \tag{18}$$

where $K_0(\mathrm{Tr})([p]_0) = \mathrm{Tr}(p)$ for $p \in \mathcal{P}_\infty(\mathbb{C})$ is a group isomorphism. Indeed, since $(p_{ij} + \alpha_{ij}1)_{ij}$ is necessarily a projection, it is easy to check that $(\alpha_{ij})^2_{ij} = (\alpha_{ij})_{ij}$, so α is a projection matrix with entries in $\mathbb{C}$.

A generic element of $K_0(\widetilde{I})$ is of the form $[p + \alpha 1]_0 - [q + \beta 1]_0$ where $p + \alpha 1, q + \beta 1 \in \mathcal{P}_\infty(\widetilde{I})$. Thus, it follows that:

$$K_0(I) = \mathrm{Ker}(K_0(\pi)) = \Big\{ [p + \alpha 1]_0 - [q + \beta 1]_0 \;\Big|\; p + \alpha 1, q + \beta 1 \in \mathcal{P}_\infty(\widetilde{I}), \text{ where } \alpha, \beta = 0 \text{ or } \alpha, \beta = 1 \Big\} \tag{19}$$

An equivalent definition, as is outlined is Rordam is the following:

$$K_0(I) = \Big\{ [p]_0 - [s(p)]_0 \;\Big|\; p \in \mathcal{P}_\infty(\widetilde{I}) \Big\} \tag{20}$$

where $s = \text{inclusion} \circ \pi$ is a map from $\widetilde{I}$ to itself: it strips off the scalar part of some element of the unitization. (this proof is also in Rordam). Having reviewed what $K_0(I)$ means, let us turn our attention to $K_1(B)$. This is somewhat easier: $K_1(B) = \mathcal{U}_\infty(\widetilde{B})/\sim_1$, where $\mathcal{U}_\infty(\widetilde{B})$ is the space of all unitaries in each of the matrix algebras over $\widetilde{B}$, and $\sim_1$ is homotopy equivalence (under the assumption we are allowed to embed unitaries u and v as blocks in larger matrix $u \oplus \mathbb{I}$ and $v \oplus \mathbb{I}$). We also have,

$$K_1(B) = \Big\{ [u]_1 \;\Big|\; u \in \mathcal{U}_\infty(\widetilde{B}) \Big\} \tag{21}$$

which is slightly nicer behaviour than the K_0 -group.

As is usually the case when constructing maps between "quotients", we'll have to define a map between the un-quotiented spaces, and then hope that the map is well-defined under the quotient. In the case of δ_1, we have to achieve a map of the form

$$\delta_1 : [u]_1 \mapsto [p]_0 - [s(p)]_0. \tag{22}$$

To do so, we will try to construct a mapping to $p \in \mathcal{P}_\infty(\widetilde{I})$, from some $u \in \mathcal{U}_\infty(\widetilde{B})$, and then check that this map lifts to a well-defined δ_1. Let us be more specific: starting with some $u \in \mathcal{U}_n(\widetilde{B})$, we can show that there exists $v \in \mathcal{U}_{2n}(\widetilde{A})$ and $p \in \mathcal{P}_{2n}(\widetilde{I})$ such that $\widetilde{\psi}(v) = \mathrm{diag}(u, u^*)$, $\widetilde{\varphi}(p) = v(\mathbb{I}_n \oplus 0_n)v^*$, and $s(p) = \mathbb{I}_n \oplus 0_n$. Recall that ψ and φ are the maps associated with the short exact sequence.

Moreover, if w and q are chosen analogously to u and p, and satisfy the first two equations, then automatically, $s(q) = \mathbb{I}_n \oplus 0_n$ and $p \sim_u q$ (i.e. they are unitary equivalent). This uniqueness result (modulo a certain equivalence) will allow us to actually show that the maps we construct are well-defined.

The proof of this result is straightforward: we simply use Lemma 2.1.7 of Rordam's book (which itself relies on a bunch of other lemmas about equivalences of unitaries), which immediately gives the element $v \in \mathcal{U}_{2n}(\widetilde{A})$ that we want. Suppose we are able to find the desired p, then it is obvious that $s(p) = \mathbb{I}_n \oplus 0_n$, as

$$\widetilde{\varphi}(p) = \widetilde{\varphi}((a)_{ij} + (\alpha 1)_{ij}) = \varphi((a)_{ij}) + (\alpha 1)_{ij} = v(\mathbb{I}_n \oplus 0_n)v^* \tag{23}$$

so $s(p) = s(\widetilde{\varphi}(p)) = s(v(\mathbb{I}_n \oplus 0_n)v^*)$.