Summer 2024 math notes

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Contents

I. Introduction

The main goal of this report is to summarize what I have learned during the summer of 2024 while being advised by Prof. Marco Gualtieri. One of our main goals was to read the paper by Arias Abad and Schatz, *The A_∞ de Rham theorem and integration of representations up to homotopy*, Ref. [**?**]. This lead to numerous tangents and digressions, as I had to fill in gaps in my knowledge related to homological algebra, simplicial complexes, Lie algebroids and their representations, parallel transport, and more. I will discuss all of these topics in these notes, drawing from many different sources.

I have tried to order the sections as naturally as possible, but a wide-range of ideas are introduced, some of which are fairly unrelated, so certain neighbouring sections may feel somewhat disjointed.

II. Background

The main goal of the *first part* of the paper, Ref. [**?**], is to construct an A_∞-quasi-isomorphism between differential graded algebras $(\Omega^\bullet(M), -d, \wedge)$, (where $\Omega^\bullet(M)$ is the usual de Rham complex, d is the usual exterior derivative and $\wedge$ is the wedge product), and $C^\bullet(M), \delta, \cup)$ (where $C^\bullet(M)$ is the space of singular cochains on M over the reals, δ is the coboundary, and $\cup$ is the cup product of cochains). This A_∞-quasi-isomorphism is stronger than the usual quasi-isomorphism between the DGAs which is present in de Rham's theorem due to the fact that we are preserving the A_∞-structure of these algebras, and thus require the existence of many higher-order maps (what is meant by this will become clear soon).

III. A_∞-algebras

To begin to make sense of this, let us introduce definitions relating to A_∞-algebras, following the work of Ref. [**?**]. To motivate the definition of an A_∞-algebra, we first consider that of an A_∞-space, which can be thought of as a space equipped with an operation which fails to be associative, but admits homotopies between "alternatively-bracketed" elements, satisfying certain conditions.

Let us consider the example of *based loop space*:

Example III.1. Let (X, p) be a pointed topological space, let ΩX be the space of all loops $f : S^1 \to X$ based at p (so $f(1, 0) = p$). Given loops f_1 and f_2, there exists an operation $m_2(f_1, f_2) = f_1 * f_2 \in \Omega X$ which is f_1 on the upper-half of S^1 and f_2 on the lower-half. This operation is not associative, as $(f_1 * f_2) * f_3 \neq f_1 * (f_2 * f_3)$, but the two loops are clearly homotopic. Denote this homotopy between $m_2 \circ m_2 \times \mathrm{id}$ and $m_2 \circ \mathrm{id} \times m_2$ by $m_3 : [0, 1] \times \Omega X^3 \to \Omega X$.

If we now turn our attention to 4-fold compositions, note that there are 5 possible ways to brackets the factors, namely:

$$f_1 * (f_2 * (f_3 * f_4)), \qquad f_1 * ((f_2 * f_3) * f_4), \qquad (f_1 * (f_2, *f_3)) * f_4, \tag{1}$$

$$((f_1 * f_2) * f_3) * f_4, \qquad (f_1 * f_2) * (f_3 * f_4) \tag{2}$$

Note that each of the above loops is homotopic, via some application of m_3, to precisely two of the other loops. Thus, we define a map on the boundary of a pentagon, K_4, of the form $\partial m_4 : \partial K_4 \times \Omega X^4 \to \Omega X$ which is equal to each of the homotopies on each side. We then claim we can extend this map to a map $m_4 : K_4 \times \Omega X^4 \to X$, which is a sort-of "homotopy of homotopies". As it turns out, we can continue this procedure for all n-fold compositions, finding dimension $n - 2$ polytopes K_n which allow us to define "n-fold homotopy". This is the A_∞-structure which makes the based loop space into a so-called A_∞-space.

This leads us to a "rough" definition of an A_∞-space, which will suffice for our purposes, for now:

Definition III.1 (A_∞-space (informal))**.** A topological space Y along with maps $m_n : K_n \times Y^n \to Y$ which satisfy certain compatibility conditions similar to the conditions above.

The algebraic analogue of A_∞-spaces are A_∞-algebras. In particular, the singular chain complex of an A_∞-space is automatically an A_∞-algebra.

Definition III.2 (Suspension)**.** If V is a $\mathbb{Z}$-graded vector space, the suspension sV is the graded vector space with grading shifted by 1, so that $(sV)^k = V^{k+1}$. Thus, $v \in V^k$ is an element of $(sV)^{k-1}$ in sV, so the suspension decreases the degree of individual elements. We denote $v \in sV$ by sv, to emphasize the change in grading.

Definition III.3 (A_∞-algebra)**.** It is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{n \in \mathbb{Z}} A^n$ with linear maps $m_n : (sA)^{\otimes n} \to sA$ for $n \geq 1$ of degree 1 which satisfy

$$\sum_{r+s+t=n} m_{r+t+1} \circ (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0 \tag{3}$$

for all $n \geq 1$. To be understand these relations, consider first the case of $n = 1$, the only possible option is $s = 1$ with $r, t = 0$. It follows that we must have $m_1^2 = 0$, making (sA, m_1) a cochain complex. For $n = 2$, we have

$$m_2 \circ (\mathrm{id} \otimes m_1 + m_1 \otimes \mathrm{id}) + m_1 \circ m_2 = 0 \tag{4}$$

which implies that if we think of m_2 as giving a multiplication for an algebra, m_1 is a graded derivation with respect to this algebra. For example, suppose $A = \Omega^\bullet(M)$. We can immediately set $m_1(\omega) = (-1)^{\deg(\omega)} d$ and $m_2 = \wedge$, and note that we will have $m_1^2 = 0$ as well as

Definition III.4 (A_∞-homomophism)**.**

Definition III.5 (Differential graded algebra)**.**

From here, we may construct an

IV. The Chen map

Let M be a finite-dimensional, compact, orientable smooth manifold. The Chen iterated integral is a map

$$C : B(s\Omega^{\bullet}(M)) \to \Omega^{\bullet}(PM) \tag{5}$$

where s is the suspension of the graded algebra $\Omega^{\bullet}(M)$: the De Rham complex of piecewise smooth differential forms on M (if A is a graded algebra, then the suspension sA is the graded algebra with the grading increased by 1, $(sA)^k = A^{k+1}$). B denotes the bar complex (for an algebra A, $BA = \bigoplus_{k\geq 1} A^{\otimes k}$) [**?**]. PM is the *piecewise-smooth path space of* M.

Definition IV.1 (Piecewise-smooth path space)**.** Given smooth manifold M, the piecewise-smooth path space PM is the set of all piecewise smooth $\gamma : [0,1] \to M$. Let $PM^{\infty} = C^{\infty}([0,1], M) \subset PM$ be the subset consisting of smooth paths. We take the C^1-topology to be the initial topology of the map $\Gamma : C^{\infty}([0,1], M) \to C^{\infty}([0,1], M) \times C^{\infty}(T[0,1], TM)$ taking $\gamma \mapsto (\gamma, \gamma_*)$, where the range is endowed with the compact-open topology on each factor. We then define a topology on PM by taking the final topology of the inclusion $\iota : C^{\infty}([0,1], M) \to PM$.

Given finite-dimensional smooth manifold X, we say that a map $f : X \to PM$ is (piecewise) smooth if the map $\widetilde{f} : X \times [0,1] \to M$ given by $\widetilde{f}(x,t) = f(x)(t)$ is (piecewise) smooth.

V. The Igusa map

VI. An A_∞-de Rham theorem

Let us begin by recalling the original de Rham theorem: the cochain complexes $H^n_{\text{dR}}(M)$ and $H^n(M)$ associated to de Rham and singular cohomology are quasi-isomorphic via the integration map $\int : H^n_{\text{dR}}(M) \to H^n(M)$ where $[\delta] = \int([\omega])$ is given by

$$\delta(c) = \int_c \omega \tag{6}$$

Proving that this map is well-defined is simple. Note that if we add an exact form,

$$\int_c \omega + d\eta = \int_c \omega + \int_{\partial c} \eta = \int([\omega + \partial^* \eta])(c) \tag{7}$$

where we use the fact that

VII. Discrete path space and Chen map

One question that has caught my attention over the past few months is the determination of the properties of a “combinatorial analogue” of the path space of a manifold. The path space, as we have mentioned, is relevant in the context of the Feynman path integral of quantum field theory (as we “sum over” all possible paths). On the other hand, one way that we attempt to make sense of the Feynman path integral is by considering its discrete analogue on graphs. We are able to sum over all paths between two vertices in a graph in a manner which is well-defined, as there is of course a finite collection of such paths. In light of these parallel lines of inquiry, it might be fruitful to consider whether machinery which is defined relative to continuum path space can easily be transported back and forth to and from the discrete.

We refer to our discrete model for the path space as the *Stone complex*, with the namesake originating from work of Stone (Ref. [**?**]).

Definition VII.1.

VIII. Lie algebroids, connections, and parallel transport

The goal of this section is to explain some of the key ideas surrounding Lie algebroids and parallel transport, so we may eventually discuss superconnections and representations up to homotopy.

Definition VIII.1 (Groupoid)**.** A category in which all arrows are reversible.

Definition VIII.2 (Lie groupoid)**.** A groupoids in which the arrows form a smooth manifold, the objects are an embedded submanifold under the map sending each object to its identity arrow, and the source and target maps are surjective submersions.

Definition VIII.3 (Lie algebroid)**.** A vector bundle $\pi_A : A \to M$ over a smooth manifold M, equipped with a bracket $[\cdot, \cdot]$ on the sections $\Gamma(A)$ and an anchor map $\rho : A \to TM$. The anchor map is a vector bundle morphism, meaning that $\pi_A = \pi \circ \rho$. Moreover, the bracket must satisfy a Leibniz rule:

$$[X, fY] = \mathcal{L}_{\rho(X)}(f) \cdot Y + f[X, Y] \tag{8}$$

where $\mathcal{L}_{\rho(X)}$ is the Lie derivative with respect to the vector field $\rho(X) \in \mathfrak{X}(M)$.

Remark VIII.1. I very much like Abad and Crainic's assertion in their paper Ref. [**?**] that Lie algebroids are to be thought of as "generalized tangent bundles associated to various geometric situations".

Example VIII.1 (Tangent bundle)**.** The most obvious example of a Lie algebroid is TM itself. The anchor is the identity map and the bracket is the standard Lie bracket between smooth vector fields $X, Y \in \mathfrak{X}(M)$. Indeed,

$$[X, fY] = \mathcal{L}_X(fY) = X(f) \cdot Y + f\mathcal{L}_X(Y) = X(f) \cdot Y + f[X, Y] \tag{9}$$

as required.

Example VIII.2 (Lie algebra)**.** One should think of a Lie algebroid as a generalized Lie algebra, in which the vector space is replaced with a vector bundle over a base space and the bracket of vectors is replaced by a bracket of sections of the bundle.

It is in this sense that a Lie algebra is a Lie algebroid: if $\mathfrak{g}$ is a Lie algebra, take $A = \mathfrak{g} \times M$ for some M, let $\pi : A \to M$ be the projection, let the anchor be trivial, $\rho(e) = 0_{\pi(e)} \in T_{\pi(e)}M$ for all e, and let the bracket between sections be defined as $[X, Y](p) = [X(p), Y(p)]_{\mathfrak{g}}$, where $[\cdot, \cdot]_{\mathfrak{g}}$ is the bracket of the Lie algebra. A is then a Lie algebroid.

Much like the case of Lie groups/Lie algebras, if given a Lie groupoid G, we can product a Lie algebroid, Lie(G). This construction is somewhat analogous to the Lie group to Lie algebra map, where we take the tangent space at the identity. For a Lie groupoid, each $p \in M$ (the base manifold of objects) has an associated identity in G (the self-referential arrow), e_p. **TODO:** revisit, stress that the main difficulty comes from constructing the bracket

We will eventually be interested in representations up to homotopy of Lie algebroids. First, we must discuss Lie algebra and Lie algebroid representations.

Definition VIII.4 (Lie algebra representation)**.** Let A be a Lie algebra with bracket $[\cdot, \cdot]_A$. A Lie algebra representation on vector space V is the pair (V, ρ) where $\rho : A \to \mathrm{End}(V)$ is a linear map such that

$$\rho([g, h]_A) = [\rho(g), \rho(h)] \tag{10}$$

for all $g, h \in A$ and $[\cdot, \cdot]$ is the standard commutator of linear maps on V.

Now, as an interlude, let us say some things about connections and parallel transport. This will motivate the definition of a Lie algebroid representation.

Definition VIII.5 (Connection)**.** Let $A \to M$ be a Lie algebroid, let E be a vector bundle over M. An E-connection relative to A is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(a, e) \mapsto \nabla_a e$ such that $\nabla_{fa} e = f\nabla_a e$ for all $f \in C^\infty(M)$, and

$$\nabla_a fe = \mathcal{L}_{\rho(a)}(f) \cdot e + f\nabla_a e \tag{11}$$

where ρ is the anchor of A. A connection is said to be flat if $\nabla_{[a,b]} = [\nabla_a, \nabla_b]$. Observe that fixing $a \in \Gamma(A)$, ∇_a is a linear map from $\Gamma(E)$ to itself satisfying the Leibniz rule of the above formula. In this sense, the flatness condition is very much a generalization of Eq. (10) in the definition of a Lie algebra representation.

Example VIII.3. In the case that $A = TM$, we recover the definition of a connection on smooth manifold M.

Example VIII.4. Suppose $A = \mathfrak{g} \times M$ for some Lie algebra $\mathfrak{g}$. Suppose (V, ρ) is a Lie algebra representation. Let $E = V \times M$ be the trivial bundle. Define $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ as $\nabla_a e = \rho(a) \cdot e$. Clearly, this map is bilinear, and is $C^\infty(M)$-linear in a and e. It follows since $\rho = 0$ that the required conditions are satisfied and ∇ defined directly from ρ is a connection. In fact, it is a flat connection, as

$$\nabla_{[a,b]} = \rho([a, b]) = [\rho(a), \rho(b)] = [\nabla_a, \nabla_b]. \tag{12}$$

To better contextualize a connection as a "derivative of a vector bundle", we can write down an explicit local form. For simplicity, we will assume that $A = TM$ for the time being.

Remark VIII.2 (Local form of connection)**.** Let M be a smooth manifold, let us consider first the case of the trivial bundle $E = M \times \mathbb{R}^n$. Any section $\sigma \in \Gamma(E) = \Gamma(M \times \mathbb{R}^n)$ will of course be of the form $\sigma : p \mapsto (f_1(p), \ldots, f_n(p)) = f_1(p)e_1 + \cdots + f_n(p)e_n$, for $f_j \in C^\infty(M)$. Pick some $X \in \mathfrak{X}(M)$. By linearity, we simply must determine $\nabla_X(f_j e_j)$ for each j. We have, by definition

$$\nabla_X(f_j e_j) = \mathcal{L}_X(f_j) \cdot e_j + f_j \nabla_X(e_j) = df_j(X) \cdot e_j + f_j \nabla_X(e_j) \tag{13}$$

Note that $\nabla_X(e_j) \in \Gamma(M \times \mathbb{R}^n)$ for each j. Thus, we let $\nabla_X(e_j)(p) = A_{j1}(p)e_1 + \cdots + A_{jn}(p)e_n$. We let $A(p)$ be the matrix with entries $A_{jk}(p)$. It follows from this that we can write ∇_X, with a slight abuse of notation, in the form $\iota_X d + A_X$, where d is the exterior derivative, ι_X is the inner product relative to X, and A_X is an element of $\Gamma(\mathrm{End}(E))$ over $C^\infty(M)$. Indeed, we have

$$\nabla_X(\sigma)(p) = \sum_{j=1}^n \nabla_X(f_j e_j)(p) = \sum_{j=1}^n df_j(X)(p)e_j + \sum_{j=1}^n \sum_{k=1}^n f_j(p)A_{jk}(p)e_k \tag{14}$$

$$= (df_1(X), \ldots, df_n(X))(p) + A(p) \cdot (f_1(p), \ldots, f_n(p)) \tag{15}$$

so this notation is justified.

Let us generalize to the case of a general vector bundle E. Let $(U_\alpha, \varphi_\alpha)$ be the local trivialization of E, so $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ is a homeomorphism and the fibre maps are linear isomorphisms between vector spaces. Then if $\sigma \in \Gamma(E)$, the restriction $\varphi_\alpha \circ \sigma|_{U_\alpha} : U_\alpha \to U_\alpha \times \mathbb{R}^n$ is well-defined, and is a section of the trivial bundle $U_\alpha \times \mathbb{R}^n$ over open submanifold U_α, as

$$\mathrm{proj} \circ (\varphi_\alpha \circ \sigma|_{U_\alpha}) = \pi \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \sigma|_{U_\alpha} = \pi \circ \sigma|_{U_\alpha} = \mathrm{id}|_{U_\alpha}. \tag{16}$$

It is equally easy to show that if $\sigma \in \Gamma(U_\alpha \times \mathbb{R})$, then $\varphi_\alpha^{-1} \circ \sigma$ is in $\Gamma(\pi^{-1}(U_\alpha))$.

It is important for us to show first that $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is a *local map*. In particular, if σ and σ' are elements of $\Gamma(E)$ which agree on an open subset V of M, then $\nabla_X(\sigma)(p) = \nabla_X(\sigma')(p)$ for all $p \in V$. To prove this, note that for some $p \in V$, we can always take a coordinate ball W with closure contained in V around p, as $V \subset M$, a smooth manifold. We pick a smooth bump function $\chi : M \to \mathbb{R}$ which is 1 inside $\overline{W}$ and vanishes outside V. Clearly, $\chi\sigma = \chi\sigma'$. We then have, from the Leibniz rule,

$$\mathcal{L}_X(\chi)(p) \cdot \sigma(p) + \nabla_X(\sigma)(p) = \nabla_X(\chi\sigma)(p) = \nabla_X(\chi\sigma')(p) = \mathcal{L}_X(\chi)(p) \cdot \sigma'(p) + \nabla_X(\sigma')(p) \tag{17}$$

which means that $\nabla_X(\sigma)(p) = \nabla_X(\sigma')(p)$ as desired. Moreover, suppose X and Y agree on some open neighbourhood V, we take W and χ the same as above, note that $\chi X = \chi Y$, and get

$$\chi(p)\nabla_X(\sigma)(p) = \nabla_{\chi X}(\sigma)(p) = \nabla_{\chi Y}(\sigma)(p) = \chi(p)\nabla_Y(\sigma)(p) \tag{18}$$

so that $\nabla_X(\sigma)(p) = \nabla_Y(\sigma)(p)$. It follows from these locality conditions that given the connection ∇, we can consider open submanifold U and subbundle $\pi^{-1}(U)$, and conclude that the restricted connection $\nabla|_U : \mathfrak{X}(U) \times \Gamma(\pi^{-1}(U)) \to \Gamma(\pi^{-1}(U))$ given by extending $X \in \mathfrak{X}(U)$ and $\sigma \in \Gamma(\pi^{-1}(U))$ to vector field and section in M and $\Gamma(E)$ respectively and feeding to ∇, then restricting to U, is well-defined. This map can easily be verified to be a valid connection on the submanifold. Moreover, given $\sigma \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, the locality results imply that

$$\nabla_X(\sigma)|_U = (\nabla|_U)_{X|_U}(\sigma|_U). \tag{19}$$

It follows from this that to understand the local behaviour of ∇, we can look at $\nabla|_{U_\alpha}$ for each of the trivializations $(U_\alpha, \varphi_\alpha)$. In particular, we will consider the maps $\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}) : \mathfrak{X}(U_\alpha) \times \Gamma(U_\alpha \times \mathbb{R}^n) \to \Gamma(U_\alpha \times \mathbb{R}^n)$, where in this context, φ_α^{-1} is sending section $\sigma \in \Gamma(U_\alpha \times \mathbb{R}^n)$ to $\varphi_\alpha^{-1} \circ \sigma \in \Gamma(\pi^{-1}(U_\alpha))$, and φ_α is doing the opposite. Our claim is that $\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1})$ is itself a connection on the trivial bundle $U_\alpha \times \mathbb{R}^n$. Note that φ_α^{-1} is a linear isomorphism when restricted to fibres. Therefore,

$$\begin{aligned}
(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1})(X + fY, \sigma) &= \varphi_\alpha \circ (\nabla|_{U_\alpha})_{X+fY}\varphi_\alpha^{-1}\sigma &(20)\\
&= \varphi_\alpha \circ \left[(\nabla|_{U_\alpha})_X\varphi_\alpha^{-1}\sigma + f(\nabla|_{U_\alpha})_Y\varphi_\alpha^{-1}\sigma\right] &(21)\\
&= f(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(Y, \sigma) + (\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(X, \sigma) &(22)
\end{aligned}$$

Moreover, note that $(\varphi_\alpha^{-1}g\sigma)(p) = \varphi_\alpha^{-1}(p, g(p)\sigma(p)) = g(p)\varphi_\alpha^{-1}(p, \sigma(p))$, so $\varphi_\alpha^{-1}g\sigma = g\varphi_\alpha^{-1}\sigma$. It follows that

$$\begin{aligned}
(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(X, g\sigma) &= \varphi_\alpha \circ (\nabla|_{U_\alpha})_X g\varphi_\alpha^{-1}\sigma &(23)\\
&= \varphi_\alpha \circ \left[\mathcal{L}_X(g)\varphi_\alpha^{-1}\sigma + g(\nabla|_{U_\alpha})_X\varphi^{-1}\sigma\right] &(24)\\
&= \mathcal{L}_X(g)\sigma + g(\varphi_\alpha \circ (\nabla|_{U_\alpha}) \circ (\mathrm{id} \times \varphi_\alpha^{-1}))(X, \sigma) &(25)
\end{aligned}$$

so we have the properties needed for a connection.

It follows from the previous argument that $\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1})$ can be written in the form $\iota_X d + A_{\alpha,X}$. In other words, up to linear isomorphisms of $\mathfrak{X}(U_\alpha) \times \Gamma(U_\alpha \times \mathbb{R}^n)$ and $\Gamma(U_\alpha \times \mathbb{R}^n)$, which are easy to characterize, we can describe the local form of the connection.

To find the matrix $A_{\alpha,X}$, let $\varphi_\alpha = (x_\alpha^1, \dots, x_\alpha^n)$ be the coordinates associated to the chart. Let $e_1, \dots, e_n$ be the standard global basis for the trivial bundle. Note that

$$(\varphi_\alpha \circ \nabla|_{U_\alpha} \circ (\mathrm{id} \times \varphi_\alpha^{-1}))\left(\frac{d}{dx_\alpha^i}, x_\alpha^j e_k\right) = dx_\alpha^j\left(\frac{d}{dx_\alpha^i}\right)e_k + x_\beta^j A_{\alpha,X}e_k = \delta_{ij} + x_\beta^j A_{\alpha,X}e_k. \tag{26}$$

Thus, we can recover the linear transformation $A_{\alpha,X}$ from this data.

A connection gives us the machinery required to talk about parallel transport and holonomy in a vector bundle. As above, we will continue restricting our attention to connections on TM. In particular, given some E-connection ∇ on TM, suppose γ is a path in M. Let $E_{\gamma(0)}$ and $E_{\gamma(1)}$ denote the fibres over the endpoints. Our goal is to define an isomorphism between the two fibres (as of course they are vector spaces on the same dimension).

Definition VIII.6 (Pullback bundle and pullback connection)**.** Suppose E is a vector bundle over smooth manifold M. Suppose $F : N \to M$ is a smooth map. We may define a vector bundle over N called the *pullback bundle*, F^*E, by taking

$$F^*E = \{(p, v) \in N \times E \mid F(p) = \pi(v)\} \tag{27}$$

where $\pi : E \to M$ is the projection for the given bundle. Verifying that F^*E is in fact a vector bundle over N is easy. Now, if ∇ is an E-connection relative to TM, we may define a F^*E-connection $F^*\nabla$ relative to TN. In particular, $F^*\nabla$ is a map from $\mathfrak{X}(N) \times \Gamma(F^*E) \to \Gamma(F^*E)$ defined as

$$F^*\nabla_X(\sigma) = \tag{28}$$

Let us consider the case of $\gamma : (-1,1) \to M$, a smooth path.

Remark VIII.3 (Flat sections)**.** We say that a section $\sigma \in \Gamma(E)$ is flat relative to ∇ along the path γ if $\nabla_{\dot{\gamma}(t)}(\sigma)(\gamma(t)) = 0$ for all t.

Claim VIII.1. If ∇ is a flat connection, the associated parallel transport operator will agree for path-homotopic curves.

A. Review of simplicial complexes

B. Review of algebraic topology

Definition B.1 (Cellular chains and cochains)**.** Suppose X is a CW-complex. Let X^n denote the n-skeleton. We define the group of cellular n-chains to be $C_n^{\mathrm{CW}}(X) = H_n(X^n, X^{n-1})$, the n-th singular homology group of X^n relative to X^{n-1}. We similarly define a cellular n-cochain as an element of the $\mathbb{R}$-module $C_n^{\mathrm{CW}}(X) = \mathrm{Hom}(C^n_{\mathrm{CW}}(X), \mathbb{R})$.

It is a fact that the group of cellular chains is generated by elements in 1-to-1 correspondence with the cells of the complex. To be more specific,

Proposition B.1. Suppose X is a CW-complex. Denote the n-cells by e^n_α indexed by α. We have characteristic maps $\Phi^n_\alpha : D^n \to X^n$ corresponding to each e^n_α. Let us fix a generator $[\sigma_n] \in H_n(D^n, \partial D^n) \simeq \mathbb{Z}$. Of course, Φ^n_α takes ∂D^n to X^{n-1}, so $\Phi^n_{\alpha,*}[\sigma_n] \in H_n(X^n, X^{n-1})$. The collection of all $[e^n_\alpha] := \Phi^n_{\alpha,*}[\sigma_n]$ generates $H_n(X^n, X^{n-1}) = C_n^{\mathrm{CW}}(X)$.

It is possible to turn the sets of cellular chains into a chain complex via a boundary d_n^{CW} defined as

$$\begin{tikzcd} H_n(X^n, X^{n-1}) \arrow[rr, "d_n^{\mathrm{CW}}"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\ & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & \end{tikzcd}$$

with ∂_n and j_{n-1} the usual relative boundary and quotient maps found the long exact sequence of homology for pairs. We get a coboundary $(d_n^{\mathrm{CW}})^*$ on $C^n_{\mathrm{CW}}(X)$ be dualizing the above diagram. The explicit generating set for the cellular chains yields a convenient formula for calculating d_n^{CW}.

Proposition B.2 (Cellular boundary formula)**.** Let X be a CW-complex. Fix a generator $[\sigma_n]$ for each n, and let $[e^n_\alpha]$ denote the corresponding generators, as above. We have the following formula:

$$d_n^{\mathrm{CW}}[e^n_\alpha] = \sum_\beta \deg(\chi_{\alpha\beta})[e^{n-1}_\beta], \tag{B1}$$

where $\chi_{\alpha\beta}$ is the map obtained from including S^{n-1} in D^n as the boundary, sending to X^n via Φ^n_α, collapsing all of X^n except for e^{n-1}_β to a point, and then sending $X^n/(X^n - e^{n-1}_\beta)$ to S^{n-1} via $(\Phi^{n-1}_\beta)^{-1}$. The index β sums over all $(n-1)$-cells in X.

It is important to note that if X and Y are CW-complexes, then $X \times Y$ is a CW-complex with cells $e^n_\alpha \times e^m_\beta$, where e^n_α a cell in X and e^m_β a cell in Y. The characteristic maps are the products of characteristic maps in X and Y. We also have the following result.

Lemma B.1. If $X_1, \dots, X_m$ are CW-complexes and d^{CW} is the cellular boundary operator, then for cells $e^{n_j}_{\alpha_j}$ in X_j for each j, we have

$$d^{\mathrm{CW}}([e^{n_1}_{\alpha_1}] \times \cdots \times [e^{n_m}_{\alpha_m}]) = \sum_{j=1}^m (-1)^{n_1 + \cdots + n_{j-1}} [e_{\alpha_1^{n_1}}] \times \cdots \times d^{\mathrm{CW}}[e^{n_j}_{\alpha_j}] \times \cdots \times [e^{n_m}_{\alpha_m}]. \tag{B2}$$

TODO: Need to discuss this more/prove the result.

Like simplicial maps, there is a natural homomorphism-inducing type of map between CW-complexes.

Definition B.2 (Cellular map)**.** If X and Y are CW-complexes, a map $f : X \to Y$ is said to be cellular if the n-skeleton of X is sent into the n-skeleton of Y, $f(X^n) \subset Y^n$. Similar to the case with simplicial maps, cellular maps induce homomorphisms of cellular chains and cochains, where $[\sigma] \in H_n(X^n, X^{n-1})$ with $\sigma \in C_n(X^n, X^{n-1})$ is sent to $[f \circ \sigma]$.

From this definition, we have an important result.

Theorem B.1 (Cellular approximation theorem)**.** If X and Y are CW-complexes, any continuous map $f : X \to Y$ is homotopic to a cellular map $g : X \to Y$. Moreover, if f is already cellular on a subcomplex $A \subset X$, then we can take the homotopy to be trivial on A.

One particularly useful application of the cellular approximation theorem is that it allows us to define a cup product on the cellular cohomology of a CW-complex. First note that there is an isomorphism

$$\phi : C_\bullet^{\mathrm{CW}}(X \times Y) \to C_\bullet^{\mathrm{CW}}(X) \otimes C_\bullet^{\mathrm{CW}}(Y) \tag{B3}$$

for CW-complexes X and Y. To be more specific, $C_\bullet^{\mathrm{CW}}(X \times Y)$ has, as a generating set, all $[x \times y]$, for x a cell of X and y a cell of Y. We take

$$\phi([x \times y]) = [x] \otimes [y]. \tag{B4}$$

It follows that we have an isomorphism

$$\phi^* : C^\bullet_{\mathrm{CW}}(X) \otimes C^\bullet_{\mathrm{CW}}(Y) \to C^\bullet_{\mathrm{CW}}(X \times Y) \tag{B5}$$

of cochain complexes. Set $X = Y$. The diagonal map $\Delta : X \to X \times X$ is not cellular, but we can use the cellular approximation theorem to find some $\widetilde{\Delta}$ homotopic to Δ which is cellular. Therefore, we can define a map between relative singular chains,

$$\widetilde{\Delta}_\# : C_n(X^n, X^{n-1}) \to C_n((X \times X)^n, (X \times X)^{n-1}). \tag{B6}$$

Such a map descends to relative singular homology,

$$\widetilde{\Delta}_* : H_n(X^n, X^{n-1}) \to H_n((X \times X)^n, (X \times X)^{n-1}), \tag{B7}$$

or in other words, a map from $C_n^{\mathrm{CW}}(X)$ to $C_n^{\mathrm{CW}}(X \times X)$. It is of *critical* importance to note that $\widetilde{\Delta}_*$ is well-defined *independent* of the chosen $\widetilde{\Delta}$ homotopic to Δ, as homotopic maps yield the same map when passed to (relative singular) homology. The final step is to take the dual, $\widetilde{\Delta}^*$, yielding a map from $C^\bullet_{\mathrm{CW}}(X \times X)$ to $C^\bullet_{\mathrm{CW}}(X)$. We define the cellular cup product as

$$\omega \cup \eta \coloneqq \cup(\omega \otimes \eta) = (\widetilde{\Delta}^* \circ \phi^*)(\omega \otimes \eta). \tag{B8}$$

Note that the tensor product of a p-cochain and a q-cochain is identified with a $(p+q)$-cochain in $C^\bullet_{\mathrm{CW}}(X \times X)$, so that their cup product will also be a $(p+q)$-cochain, as is expected from the cup product.