Keeping things staight in algebraic topology

Jack Ceroni*
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One of the central concepts in algebraic topology is that of *homotopy*: a continuous deformation of one continuous map into another. The notion of homotopy will allow us to define notions of "equivalence" for topological spaces that are weaker than homeomorphism, but still imply preservation of many nice properties, such as the *fundamental group*: something of fundamental (pun intended) importance that will be discussed in a later chapter.

The definition of homotopy is pretty much what was stated above:

Definition II.1 (Homotopy). A homotopy is a continuous map of the form $F : X \times [0,1] \to Y$, where X and Y are topological spaces. Maps $f, g : X \to Y$ are said to be *homotopic* if there exists a homotopy between them, that is, some homotopy $F : X \times [0,1] \to Y$ such that $f(x) = F(x,0)$ and $g(x) = F(x,1)$. If f and g are homotopic, we use notation $f \simeq g$. It is easy to check that homotopy is an equivalence relation.

Example II.1 (Path homotopy). We will often be interested in homotopy of paths (i.e. when $X = [0,1]$, and we have two paths $f, g : [0,1] \to Y$). Such a homotopy will be a continuous map of the form $F : [0,1] \times [0,1] \to Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We say that two paths f and g which share beginning and endpoints are *path homotopic* if they are homotopic with a homotopy which fixes the endpoints (i.e. $F(0,t) = f(0) = g(0)$ and $F(1,t) = f(1) = g(1)$ for all $t \in [0,1]$).

Remark II.1 (Relative homotopy). More generally, a homotopy F which fixes some $A \subset X$ is said to be a *homotopy relative to A*. A path homotopy is a homotopy relative to the two-point set $\{0,1\} \subset [0,1]$.

Note that a map $f : X \to Y$ that is homotopic to a constant map is said to be *nullhomotopic*.

A. Deformation retracts and the mapping cylinder

We will dedicate the first part of this section to discussion of a particular, important kind of homotopy: a deformation retract.

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* jack.ceroni@mail.utoronto.ca

Definition II.2 (Retraction)**.** A continuous map $r : X \to A$ is said to be a *retraction* of X onto some subspace $A \subset X$ if $r|_A = \mathrm{id}$.

In this sense, a retraction can be thought of as a map which "compresses" a space down to some principal subset.

Definition II.3. Given a space X and a subspace $A \subset X$, a deformation retract of X onto A is a homotopy relative to A of the identity map $\mathrm{id} : X \to X$ with a retraction r of X onto A (actually, $j \circ r$, the retraction embedded in the ambient space X, where $j : A \to X$ is inclusion). Given a space X and a subspace A, if a deformation retract exists, we say that X deformation retracts to A.

In this sense, a deformation retraction can be thought of a continuous, gradual "compression" of a space down to some principal subset. There is another way in which we can think of deformation retracts, which provides some nice geometric intuition: this is the mapping cylinder.

Definition II.4 (Mapping cylinder)**.** Given a continuous map $f : X \to Y$, we define the mapping cylinder M_f associated with f to be the quotient space $(X \times [0,1]) \sqcup Y/ \sim$, where we identify $(x,1) \sim f(x)$.

Intuitively, the points at the "end" of the "cylinder" $X \times [0,1]$ are glued to the point in Y to which they are sent by X. We have the following nice result, which makes intuitive sense if we think about "sliding" points of $X \times [0,1]$ along their trajectories, parameterized by $[0,1]$, to their corresponding points on Y:

Proposition II.1. Let $f : X \to Y$ be a continuous map, let M_f be the mapping cylinder. Then M_f deformation retracts to Y.

Proof. We simply have to write down such a deformation retract. Let $Z = (X \times [0,1]) \sqcup Y$. We start by defining $F : Z \times [0,1] \to M_f$ as

$$F((x,s),t) = [(x, 1 - (1-s)(1-t))] = [(x, s + t - st)] \tag{1}$$

on $X \times [0,1] \times [0,1]$ and $F(y,t) = [y]$ on $Y \times [0,1]$. Note that $[\cdot]$ denotes the equivalence classes of $\sim$. F is of course the composition of a continuous map from $Z \times [0,1]$ to Z with the quotient map $[\cdot] : Z \to M_f$: both maps are continuous, and thus F is as well.

By the universal property of the quotient, F descends to a map $F : M_f \to M_f$ which is also continuous. Moreover, note that $F([(x,s)], 0) = [(x,s)]$, so $F(z,0) = \mathrm{id}(z)$, and in addition,

$$F([(x,s)], 1) = [(x,1)] = [f(x)], \tag{2}$$

so $F(z,1)$ takes M_f to the copy of Y embedded in M_f. Note that $F([(x,1)], 1) = [(x,1)]$, so that $F(z,1)$ is the identity on the embedded copy of Y. Finally, note that F fixes the copy of Y embedded in M_f, so it is a homotopy relative to this subset.

Thus, by definition, F is a deformation retract of M_f to Y, and we are done. $\qquad\square$

The previous proof is one of the rare instances where we will take great care to actual explicitly write down a homotopy/check that it is continuous/etc. Standard practice is algebraic topology is usually to provide diagrammatic justification, and this is good enough (provided you know what you are doing, and what you are saying is correct).

We know that M_f deformation retracts to Y, what about X? The property of M_f deformation retracting to X definitely does **not** hold in general! For instance, consider the "dunce cap", where we define $f : S^1 \to \{x\}$ as the constant map, and take the mapping cylinder M_f. It is clear that M_f will deformation retract to the single point at the top of the cap, but we *cannot* deformation retract this space to a circle.

Why is this the case? Well, note that $M_f = (S^1 \times [0,1]) \sqcup \{x\}$ is homeomorphic to a disk. A retraction of the disk onto its boundary, the circle, cannot exist (we will show this later), so there certainly cannot be a *deformation* retract.

B. Homotopy equivalence

Now, let us move on to *homotopy equivalence*: the weaker notion of "equivalence of topological spaces" alluded to earlier.

Definition II.5 (Homotopy equivalence)**.** A continuous map $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity on Y and X respectively. If, for two spaces X and Y, there exists a homotopy equivalence $f : X \to Y$, we say that X and Y are homotopy equivalent.

Example II.2 (Deformation retract and homotopy equivalence)**.** Let us consider deformation retracts in the context of homotopy equivalence. Suppose X deformation retracts to A. Let $j : A \to X$ be the inclusion map of A in X, let $r : X \to A$ be the retraction. Note that, by definition, $r \circ j = \mathrm{id}|_A$. Moreover, note that by definition, since X deformation retracts to A, $j \circ r \simeq \mathrm{id}|_X$.

Thus, X and A are homotopy equivalent.

Example II.3. A space X is said to be *contractible* if it is homotopy equivalent to a single point. Note that X being contractible does not necessarily imply that X deformation retracts to a point. However, the converse is true (as we showed above that a deformation retract gives a homotopy equivalence).

The main idea with homotopy equivalence is that we can go back and forth between two spaces in such a way that this action of "going back and forth" can be continuously deformed to the identity map. The notion of homotopy equivalence gives rise to *homotopy type*, where we say that two spaces X and Y are of the samer homotopy type if there exists a homotopy equivalence between them. The spaces of the same homotopy type, as it turns out, are equivalence classes: homotopy equivalence is an equivalence relation. Reflexivity and symmetry are obvious, so it remains to show that it is transitive.

Proposition II.2. Composition of homotopy equivalences is a homotopy equivalence. It follows immediately that if X and Y are homotopy equivalent, and Y and Z are homotopy equivalent, then X and Z are homotopy equivalent.

Proof. Suppose $f : X \to Y$ and $g : Y \to Z$ are homotopy equivalences, let f' and g' be the corresponding maps in the other direction. Note that

$$(f \circ g) \circ (g' \circ f') = f \circ (g \circ g') \circ f' \simeq f \circ f' \simeq \mathrm{id} \tag{3}$$

and

$$(g' \circ f') \circ (f \circ g) = g' \circ (f' \circ f) \circ g \simeq g' \circ g \simeq \mathrm{id} \tag{4}$$

so the claim follows immediately. $\qquad\square$

If you found the definition of a homotopy equivalence a bit opaque, that's no problem, we are about to provide (half of) an alternate, equivalent definition of homotopy equivalence which is easier to visualize.

Proposition II.3. Let X and Y be topological spaces, let Z be a common space in which X and Y are embedded via embeddings $j_X : X \to Z$ and $j_Y : Y \to Z$. If Z deformation retracts to both $j_X(X)$ and $j_Y(Y)$, then X and Y are homotopy equivalent.

Proof. This follows immediately from the fact that deformation retracts are homotopy equivalences. In particular, $j_X(X)$ is homotopy equivalent to Z, which is homotopy equivalent to $j_Y(Y)$. Thus, $j_X(X)$ and $j_Y(Y)$ are homotopy equivalent. X and Y are homeomorphic to these spaces respectively, and are thus homotopy equivalent (it is trivial to check homotopy equivalence persists under homeomorphism). $\qquad\square$

As it turns out, the converse holds as well, in the following sense:

Claim II.1. Two topological spaces X and Y are homotopy equivalent if and only if there exists a common space Z and embeddings $j_X : X \to Z$ and $j_Y : Y \to Z$ such that Z deformation retracts to both $j_X(X)$ and $j_Y(Y)$.

We will prove this in the next section.

C.   Homotopy extension property

The goal of this (final) section is to prove a condition for homotopy equivalence which as a consequence will allow us to prove Claim. II.1. The main definition we will require is that of the *homotopy extension property*.

Definition II.6 (Homotopy extension property). Suppose $f : X \to Y$ is a continuous map, and suppose that $f_t : A \to Y$ is a homotopy with $f_0 = f|_A$, where $A \subset X$. If (X, A) has the property that for any f and any f_t as above, f_t can be extended to a homotopy on all of X, $f_t : X \to Y$ with $f_0 = f$, then (X, A) has the homotopy extension property.

Lemma II.1. A pair (X, A) (where A is closed) has the homotopy extension property if and only if $Z = X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof. Suppose $r : X \times I \to Z$ is a retract, let $F : A \times I \to Y$ be a homotopy with $F(x, 0) = f|_A(x)$ where $f : X \to Y$ is a continuous function. Since A is closed, Z is the union of closed sets $X \times \{0\}$ and $A \times I$. Define $F' : Z \to Y$ as F on $A \times I$ and $F'(x, 0) = f(x)$ on $X \times \{0\}$. This function is continuous by the pasting lemma of point-set topology. We then define $\widetilde{F} = (F' \circ r) : X \times I \to Y$. This is a continuous function, and since r is a retraction, it is an extension of F', so it is a homotopy extension of F as desired.

Conversely, suppose (X, A) has the homotopy extension property. Note that $\mathrm{id} : A \times I \to Z$, which is a valid homotopy, can be extended to $F : X \times I \to Z$ such that $F(x, 0) = \mathrm{id}$ and $F(a, t) = (a, t)$, so that $X \times I$ retracts to Z. $\qquad\square$

Example II.4. If (X, A) is a pair where A has a mapping cylinder neighbourhood, then the pair has the homotopy extension property. In particular, A has a mapping cylinder neighbourhood if there exists a closed neighbourhood N such that $\mathrm{Int}(N)$ is a neighbourhood of A with $f : \partial N \to A$ such that there is a homeomorphism $h : M_f \to N$ which fixes A and ∂N.

TODO: *Prove this statement*

III.   CW-complexes

CW-complexes are topological spaces which can be built recursively by gluing discs of dimensions 0, 1, 2, and so on, together. In particular, to define a CW-complex, we begin with a (finite) collection a points $\{x_1, \ldots, x_p\}$: a disjoint union of 0-disk. This is the 0-cell-structure, X^0.

Given the X^{n-1}-cell-structure, we define the n-cell-structure, X^n, recursively. We take a collection of closed n-disk, D_α^n and a corresponding collection of continuous *attaching maps*, $\varphi_\alpha : S^n \to X^{n-1}$. We then take X^n to be the quotient space

$$X^n = X^{n-1} \sqcup \left(\bigsqcup_\alpha D_\alpha^n \right) \Big/ \sim \tag{5}$$

where $\sim$ is the equivalence relation identifying $x \sim \varphi_\alpha(x)$ for $x \in \partial D_\alpha^n = S^n$. Topologically, X^n is a collection of open disks of dimensions $k \in \{0, 1, \ldots, n\}$, which we call k-cells, which we denote e_α^k. Note that the topology of X^n is defined reucrsively as well: at each step, each disk has its subspace topology as a subset of $\mathbb{R}^n$, taking the disjoint union simply involves taking the union of the constituent topologies, and quotienting by the continuous maps φ_α yields a quotient topology.

We can also define the space

$$X^\infty = \bigcup_{n=0}^\infty X^n \tag{6}$$

where U is open in X^∞ if and only if $U \cap X^n$ is open for each $n \in \mathbb{Z}_{\geq 0}$.