MUNKRES TOPOLOGY SOLUTIONS

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Contents

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1. Problem TG.1

Let H denote a group that is a topological space satyisfying the T_1 axiom. Show that H is a topological group if and only if the map f sending $x \times y$ to $x \cdot y^{-1}$ is continuous.

Suppose the original definition of a topological group holds. Then the above map is a composition of continuous maps, so it is continuous.

Suppose the alternate definition holds. Let e be the identity element of H. Then, the map:

$$y \mapsto e \times y \mapsto e \cdot y^{-1} = y^{-1}$$

is clearly continuous, as it is the composition of continuous maps. Then, the map:

$$x \times y \to x \times y^{-1} \mapsto x \cdot (y^{-1})^{-1} = x \cdot y$$

as inversion is continuous, and maps into product that are continuous are themselves continuous.

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2. Problem TG.3

Let H be a subspace of G. Show that if H is also a subgroup of G, then both H and $\overline{H}$ are topological groups.

Let $f : G \times G \to G$ be defined as $f(x, y) = x \cdot y$ and $g : G \to G$ be defined as $g(x) = x^{-1}$. Clearly, the restrictions $f|_{H \times H} : H \times H \to H$ and $g|_H : H \to H$ are well-defined, as H is a subgroup. In addition, they are continuous, as restrictions of continuous functions are continuous.

Finally, it is clear that a subspace of a T_1 space is T_1. Thus, H is a topological group.

It remains to show that $\overline{H}$ is a topological group. Clearly, $\overline{H}$ is T_1. We need to show that the restrictions of the binary operation and inversion maps are well-defined. Indeed, note that for continuous p, we have $h(\overline{A}) \subset \overline{h(A)}$, for any set A. Setting $A = H$, note that $g(H) \subset H$, so $\overline{g(H)} \subset \overline{H}$. Thus:

$$g(\overline{H}) \subset \overline{g(H)} \subset \overline{H}$$

In addition, recall that a product of closures is a closure of products, so:

$$f(\overline{H} \times \overline{H}) = f(\overline{H \times H}) \subset \overline{f(H \times H)} \subset \overline{H}$$

Thus, the restrictions of f and g to $\overline{H}$ are both well-defined and continuous, from the same logic as above, so $\overline{H}$ is a topological group as well.

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3. Problem TG.4

Let $\alpha \in G$. Show that the maps $f_\alpha, g_\alpha : G \to G$ defined by $f_\alpha(x) = \alpha \cdot x$ and $g_\alpha(x) = x \cdot \alpha$ are homeomorphisms of G.

Clearly, both maps are continuous, as the binary operation map is continuous, so this map is effectively $x \mapsto (\alpha, x) \mapsto \alpha \cdot x$ or $x \mapsto (x, \alpha) \mapsto x \cdot \alpha$.

Clearly, both these maps are bijective, as $f_\alpha^{-1}(x) = \alpha^{-1} \cdot x$ and $g_\alpha^{-1}(x) = x \cdot \alpha^{-1}$ are well-defined inverses of f_α and g_α. Finally, it is easy to see that both these maps are continuous, from the same logic as above.

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4. Problem TG.5

*Let H be a subgroup of G. If $x \in G$, define $xH = \{x \cdot h \mid h \in H\}$. This set is called a **left coset** of H in G. Let G/H denote the collection of left cosets of H in G: it is a partition of G. Give G/H the quotient topology.*

4.1. Part A.
Show that if $\alpha \in G$, the map f_α induces a homeomorphism of G/H carrying xH to $(\alpha \cdot x)H$.

Let p be the quotient map which sends elements of G to elements of G/H. Let $g : G \to G/H$ be defined as $g(x) = (p \circ f_\alpha)(x)$. Clearly, this is a quotient map, as both p and f_α are quotient maps.

Note that given some $xH \in G/H$, we have:

$$g^{-1}(\{xH\}) = (f_\alpha^{-1} \circ p^{-1})(\{xH\}) = f_\alpha^{-1}\{x \cdot h \mid h \in H\} = \{(\alpha^{-1} \cdot x) \cdot h \mid h \in H\} = (\alpha^{-1} \cdot x)H$$

Taking the collection of all such cosets clearly gives G/H, again.

Finally, let r be the map from G/H to G/H induced by p and $p \circ f_\alpha$ (in other words, $r \circ p = p \circ f_\alpha$), which we know exist from Corollary 22.3 of the previous section. We also know from this Corollary that this map will be a homeomorphism.

4.2. Part B.
Show that if H is a closed set in the topology of G, then one-point sets are closed in G/H.

Let p be the quotient map from G to G/H. Note that $p^{-1}(xH) = \{x \cdot h \mid h \in H\} = f_x(H)$. Since f_x is a homeomorphism and H is closed, it follows that $p^{-1}(xH)$ is closed. Thus, since p is a quotient map, $\{xH\}$ is also closed.

4.3. Part C.
Let U be open in G. It follows that

4.4. Part D.
First, we know from Part B that G/H satisfies the T_1 axiom. It remains to check that G/H is indeed a group, and the binary operation/inversion operations are continuous.

Since H is normal, we know that G/H is a group, under the operations $xH \cdot yH = (x \cdot y)H$ and $(xH)^{-1} = x^{-1}H$. This is more of an exercise in algebra, so we won't do it here, but we will sketch the proof at the end of the document.

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5. Problem TG.6

Quotienting $\mathbb{Z}$ out of $(\mathbb{R}, +)$ gives a familiar topological group. What is it?

This topological group is isomorphic to the circle group.

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6. Problem TG.7

6.1. Part A.

Show that there exists a symmetric neighbourhood V of e such that $V \cdot V \subset U$.

Note that given any neighbourhood W of e, the

6.2. Part B.

6.3. Part C.

6.4. Part D.
