Algebraic topology: problems and solutions

Jack Ceroni*
(Dated: Friday 23rd February, 2024)

Contents

All solutions in reccomended problems which are omitted here are in the homework.

I. Chapter 0

Problem I.1 (Hathcer 0.1). SEE HOMEWORK

Problem I.2 (Hathcer 0.5). SEE HOMEWORK

Problem I.3 (Hatcher 0.10). Suppose X is contractible, let $f : X \to \{p\}$ be a homotopy equivalence with homotopy inverse g. Let F be the homotopy connecting id_X and $gf : X \to X$, where gf is of course just a constant map on X. Note that for any $h : X \to Y$, $h \circ F : X \times [0, 1] \to Y$ is a homotopy of $h(x)$ with a constant map, so h is nullhomotopic. Similarly, for any $h : Y \to X$, $H : Y \times [0, 1] \to X$ defined as $H(y, t) = F(h(y), t)$ is a homotopy of $F(h(y), 0) = h(y)$ and $F(h(y), 1) = p$, so h is nullhomotopic as well.

Conversely, if $f : X \to Y$ is nullhomotopic for every f and Y (or, if $f : Y \to X$ is nullhomotopic for every f and Y), then in particular $\text{id}_X : X \to X$ is homotopic to some constant map $f(x) = c$ so the constant map $h : X \to \{p\}$ is a homotopy equivalence with homotopic inverse $g(p) = c$ as $h \circ g = \text{id}_p$ and $(g \circ h)(x) = g(p) = c = f(x)$ so $g \circ h = f \simeq \text{id}_X$.

Problem I.4 (Hatcher 0.17a). Recall the definition of the mapping cylinder: given $f : X \to Y$, it is precisely the space $X \times [0, 1] \sqcup Y/\sim$ where $(x, 1) \sim f(x)$. In the case that $f : S^1 \to S^1$, we construct a CW-complex as follows:

- Begin with two points
- Attach a three 1-cells, one between the two points, and two with both endpoints attached to each of the respective points.

---

* jack.ceroni@mail.utoronto.ca

- We attach a 2-cell as follows. Up to homeomorphism, the 1-skeleton is two circles S^1_X and S^1_Y connected by a unit interval $[0,1]$. Take the first endpoint to be $1 \in S^1$, and without loss of generality, take the second endpoint to be $f(1)$. Define a map on the boundary of a rectangle

$$\phi(x,y) = \begin{cases} e^{2\pi i y} \in S^1_X & \text{for } (x,y) \in \{0\} \times [0,1] \\ x \in [0,1] & \text{for } (x,y) \in [0,1] \times \{1\} \\ f(e^{2\pi i y}) \in S^1_Y & \text{for } (x,y) \in \{1\} \times [0,1] \\ x \in [0,1] & \text{for } (x,y) \in [0,1] \times \{1\} \end{cases} \tag{1}$$

Verifying that this map is continuous is trivial. If we look at the space $(S^1_X \vee [0,1] \vee S^1_Y) \sqcup R/\sim$ where we identify $z \in \partial R$ with $\phi(z)$, one can check that the resulting space is precisely the mapping cylinder. To make this a true CW-complex, we define a homeomorphism h of B^2 to R, thus a homeomorphism of ∂B^2 to ∂R, and let $\varphi = \phi \circ h|_{\partial B^2}$ be the attaching map of a 2-cell B^2 to the 1-skeleton.

Problem I.5 (Hatcher 0.17b)**.** First of all, note that the Mobius band deformation retracts onto its centre circle. Namely, if we define the Mobius band via $M = [0,1] \times [0,1]/\sim$ with $\{0\} \times y \sim \{1\} \times (1-y)$, the map $r_t : [0,1] \times [0,1] \to [0,1] \times \{1/2\}/\sim$ which takes $r_t(x,y) = [x \times t/2 + (1-t)y]$ descends to a map from $r_t : M \to [0,1] \times \{1/2\} \simeq S^1$. This is a deformation retract.

To define the CW-complex, we will build the annulus and Mobius band separately, but include their central circles as 1-cells, and in particular, we attach these circles along a line L. We can then attach a rectangular 2-cell with two of its sides being attached along the two central circles, and the other two sides attached to L. Clearly, on either end of the cylinder, the annulus/Mobius band will deformation retract to its central circles, and the cylinder will deformation retract to the central circle on the other side, so that the overall CW-complex deformation retracts to both the annulus and Mobius band.

Problem I.6 (Hatcher 0.18)**.** Recall that the $*$ operation is the "join" operation. Recall that we showed $B^1 * B^1 * \cdots * B^1$ repeated n-times is homeomorphic to S^{n-1}. Note that joins are invariant under homeomorphism: if $X' \simeq X$ and $Y' \simeq Y$ then $X * Y \simeq X' * Y'$, as the join simply involves taking the product and a quotient. Moreover, we can prove that joins are associative as follows: $(X * Y) * Z$ is precisely all formal linear combinations $tp + (1-t)z$ where $p \in X * Y$ is $sx + (1-s)y$, which we can identify with the element $tsx + (1-ts)\left[\frac{t(1-s)}{1-ts}y + \frac{1-t}{1-ts}z\right] \in X * (Y * Z)$. Showing that this identification is a homeomorphism is straightforward.

Thus, since $S^n \simeq B^1 * \cdots * B^1$ repeated n-times and $S^m \simeq B^1 * \cdots * B^1$ repeated m-times, then $S^n * S^m \simeq B^1 * \cdots * B^1$ repeated $(n+m)$-times, which is homeomorphic to S^{m+n-1}, as desired.

Another, more rigorous proof that I like better goes as follows. Define $f : S^n \times S^m \times [0,1] \to S^{n+m+1}$ as

$$f((x_0, \ldots, x_n), (y_0, \ldots, y_m), t) = (\sqrt{1-t}x_0, \ldots, \sqrt{1-t}x_n, ty_0, \ldots, ty_m)$$

Of course, this map is continuous and surjective. It fails to be injective at $t = 1$ and $t = 0$. At $t = 0$, we have $f(x,y,0) = f(x,y',0)$ and at $t = 1$, we have $f(x,y,1) = f(x',y,1)$. Thus, under the quotient map which takes $S^n \times S^m \times [0,1]$ to $S^n * S^m$, we induce a continuous bijection $\widetilde{f} : S^n * S^m \to S^{n+m+1}$ from a compact to a Hausdorff space, thus a homeomorphism.

Problem I.7 (Hathcer 0.24)**.** This problem is mostly a mtter of careful bookkeeping. To begin, note that the space $X * Y$ is precisely $X \times Y \times I$ where we identify $(x,y,0) \sim (x,y',0)$ as well as $(x,y,1) \sim (x',y,1)$. Introducing the quotient of $X * \{y_0\} \sqcup \{x_0\} * Y$ further collapses all points of the form (x, y_0, t) and all points (x_0, y, t) to points p and q. Note that we will then have $(x,y,0) \sim (x,y_0,0) \sim p$ and $(x,y,1) \sim (x_0,y,1) \sim q$, so the faces $X \times Y \times \{0\}$ and $X \times Y \times \{1\}$ are entirely collapsed.

On the other hand, in $S(X \wedge Y) = [X \times Y \times I/(X \times \{y_0\} \cup \{x_0\} \times Y) \times I]$, we collapse all points (x, y_0, t) and (x_0, y, t), as we did above, and additionally collapse $(x,y,0)$ and $(x,y,1)$. Quotienting by $S(x_0 \wedge y_0)$ is redundant, as this will simply collapse (x_0, y_0, t).

Thus, both spaces are formed from $X \times Y \times I$ from performing the same identifications of points, so the spaces are homeomorphic. Since the subsets we are quotienting by can be thought of as subcomplexes in the CW-structure of each, both of which are contractible, collapsing is a homotopy equivalence, so $S(X \wedge Y) \simeq X * Y$.

II. Chapter 1.1

Problem II.1 (Hatcher 1.1.3). Suppose $\pi_1(X)$ is Abelian. Let β_h and β_g be basepoint changing homomorphisms. If h and g both begin at x_0 and end at x_1, then $\overline{h} * g$ is a loop based at x_1, so $[\overline{h} * g] \in \pi_1(X, x_1)$. Thus, $\beta_{\overline{h}}\beta_g[f] = [h * \overline{g} * f * g * \overline{f}] = [f]$, from commutativity, so $\beta_h[f] = \beta_g[f]$ for all $[f] \in \pi_1(X, x_1)$, so any two h and g with the same endpoints will induce the same basepoint-preserving map.

Conversely, if $\beta_h = \beta_g$ for h and g with the same endpoints, note that for any two loops f and g at x_1, the trivial loop at x_0, e, has the same endpoints as $\overline{f}$, so

$$\beta_{\overline{f}}[f * g] = \beta_e[f * g] \Rightarrow [\overline{f} * f * g * f] = [g * f] = [f * g] \tag{2}$$

so we have commutativity of $\pi_1(X, x_1)$ (and thus commutativity of $\pi_1(X)$ everywhere, since all the other anchored fundamental groups are isomorphic to this one).

Problem II.2 (Hatcher 1.1.5). SEE HOMEWORK

Problem II.3 (Hatcher 1.1.6). Given a path γ, let γ_t be the path between $\gamma(0)$ and $\gamma(t)$ for $t \in [0, 1]$.

Firstly, suppose X is path-connected. Let $[\Xi] \in [S^1, X]$ be a free homotopy class of maps $S^1 \to X$. Let Ξ be a representative, let y be some point in $\Xi(S^1)$. If X is path connected, let γ be a path from y to x_0. If we identify Ξ with a loop ξ based at y, then $\overline{\gamma} * \xi * \gamma$ is a loop based at x_0. It is free homotopic to ξ via homotopy $\overline{\gamma_t} * \xi * \gamma_t$, so $\Phi([\gamma * \xi * \overline{\gamma}]) = [\Xi]$.

Suppose $\Phi([f]) = \Phi([g])$. Let F be a free homotopy connecting the loops f and g. In particular, note that $f(0) = g(0) = x_0$, as f and g are loops at x_0. Let $\gamma(s) = F_s(0)$. Clearly, $\gamma(0) = \gamma(1) = x_0$. We then consider the homotopy $G_t(x) = (\gamma_t * F_t * \overline{\gamma_t})(x)$. Clearly, for any time t, $G_t(0) = \gamma_t(0) = x_0$ and $G_t(1) = \overline{\gamma_t}(1) = \gamma_t(0) = x_0$. Moreover, at $t = 0$, $G_0(x) = F_0(x) = f(x)$ and $G_1(x) = \gamma * g * \overline{\gamma}$, where γ is a loop at x_0, so $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

Conversely, if $[f]$ and $[g]$ are conjugate, then f is path-homotopic to $\overline{\gamma} * g * \gamma$ for some loop γ. Define $F_t(x) = \overline{\gamma_t} * g * \gamma_t$. This path begins at $\gamma(t)$ goes to x_0, goes around g, then returns to γ_t. Note that at $t = 0$, $F_0(x) = g$ and at $t = 1$, $F_1(x) = \overline{\gamma} * g * \gamma \simeq f$, so F induces a free-homotopy between f and g, impyling $\Phi([f]) = \Phi([g])$.

Problem II.4 (Hatcher 1.1.8). No, of course not! Consider the map $f : S^1 \to \mathbb{R}^2$ where $x + iy \in S^1$ is sent to (x, y). Of course, if $f(z) = f(-z)$, we would have $(x, y) = 0$, contradicting the fact that $x^2 + y^2 = 1$ for all x and y in the image. Then just define $F : S^1 \times S^1$ as $F(z_1, z_2) = f(z_1)$. Again, if $F(-z_1, -z_2) = F(z_1, z_2)$, then $f(z_1) = f(-z_1)$, which we have already shown can't happen.

Problem II.5 (Hatcher 1.1.16). This problem is a monster. Recall that the existence of a retraction $r : X \to A$ implies $r \circ i = \mathrm{id}_A$, which means that the homomorphism $i_* : \pi_1(A) \to \pi_1(X)$ must be an injection, as it has a left-inverse. Let's look at each case:

1. i_* can't be an injection as $\pi_1(X)$ is trivial and $\pi_1(A)$ isn't.

2. Note that $\pi_1(X) = \mathbb{Z}$ (via deformation retract) while $\pi_1(A) = \mathbb{Z} \times \mathbb{Z}$. A homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ can't be an injection as if $\phi(1, 0) = a$, $\phi(0, 1) = b$, then $\phi(b, 0) = \phi(0, a) = ab$ with $(b, 0) \neq (0, a)$, so injectivity fails unless $a = b = 0$, in which case ϕ is the zero-map.

3. Note that the inclusion $i_* : \pi_1(A) \to \pi_1(S^1 \times D^2)$ is the zero-map, as any loop can be homotoped to a point, but $\pi_1(A)$ is non-trivial, so i_* is not injective, contradicting the existence of a retract.

4. $D^2 \vee D^2$ can be deformation retracted to a point, while $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, so inclusion can't be injective.

5. This space deformation retracts to S^1. There cannot be an injective homomorphism from $\mathbb{Z} * \mathbb{Z}$ to $\mathbb{Z}$ as if $\alpha \mapsto a$ and $\beta \mapsto b$, then $\alpha^b = \beta^a = ab$, but $\alpha^b \neq \beta^a$ for non-zero a and b.

6. Last one. In this case, suppose there was such a retraction. Let $[\omega]$ be the 1-fold loop in A. Note that when considered in the Mobius band, this loop is homotopic to the two-fold loop around the central circle of the Mobius band. Thus, j_* must send $[\omega]$ to $[\omega^2]$. Note that $r_* \circ j_*$ is the identity, so r_* must be a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ sending 2 to 1. But no such homomorphism exists, as $\phi(1) = a$ implies $\phi(2) = 2a$, so $2a = 1$, which is not satisfied for any a.

Problem II.6 (Hatcher 1.1.18). REVISIT

III. Chapter 1.2

Problem III.1 (Hatcher Problem 1.2.4). Note that this space deformation retracts to the sphere S^2 with $2n$ points removed. This space, via opening up one of the holes, is homotopy equivalent to the disk D^2 minus $2n-1$ points. Our claim is that this space is the wedge of $2n-1$ circles. Clearly this is true for $n=1$, as we can deformation retract to a circle. Suppose it is true for n, for $n+1$, we are going from $2n-1$ to $2n+1$ holes. Pick neighbourhoods around $2n-1$ of the holes, and around the other two holes individually whose intersection is simply-connected (this is easy to do). Then the resulting fundamental group is $\mathbb{Z}^{*(2n-1)} * \mathbb{Z} * \mathbb{Z}$ as desired.

Problem III.2 (Hatcher Problem 1.2.6). Since the set C is discrete, this space is homotopy equivalent to $\mathbb{R}^3$ minus a collection of disjoint 3-cells (we can choose these 3-cells to be disjoint as the points must be separated be some positive distance, or else there would exist a one-point set which is not open in the subspace topology). Thus, $\mathbb{R}^3 - C$ is homotopy equivalent to a space which yields $\mathbb{R}^3$ after attaching 3-cells. Thus, we have an isomorphism of fundamental groups from the proposition.

Problem III.3 (Hatcher Problem 1.2.7). Yesssss, love this problem. Start with a point x_0, attach a single one-cell (circle). Let γ be the one-fold path around this circle, clockwise. Then, attach a two-cell by taking the rectangle, collapsing two of its opposite boundaries to the 0-cell x_0, and attaching its sides along the circle. The resulting loop induced by the attaching map is clearly $\gamma\overline{\gamma}$, which is trivial, so we get an isomorphism of fundamental groups betwen the 1-skeleton and the 2-skeleton, which is the desired space, namely $\pi_1(X, x_0) \simeq \mathbb{Z}$.

Problem III.4 (Hatcher Problem 1.2.8). This problem is also super lads. We will use the same strategy of attaching 2-cells. Start with a circle, label the one-fold loop γ. We attach two circles, with loops labelled α and β. We then attach two rectangles (2-cells). For the first one, two sides are attached to α and two to γ. For the second, two to β and two to γ. Winding around the boundary of the rectangle, the resulting induced loops are $\gamma\alpha\overline{\gamma}\overline{\alpha}$ and $\gamma\beta\overline{\gamma}\overline{\beta}$. Thus, the resulting fundamental group is

$$\langle \alpha, \beta, \gamma \mid \gamma\alpha\overline{\gamma}\overline{\alpha} = \gamma\beta\overline{\gamma}\overline{\beta} = 1 \rangle \tag{3}$$

Problem III.5 (Hatcher Problem 1.2.11). SEE HOMEWORK

Problem III.6 (Hatcher Problem 1.2.12).

IV. Chapter 1.3

Problem IV.1 (Hatcher Problem 1.3.4). SEE HOMEWORK

Problem IV.2 (Hatcher Problem 1.3.9). This problem is slick as fuck. Note that $\pi_1(S^1) \simeq \mathbb{Z}$ via ϕ. Thus, we have an induced homomorphism $\psi = \phi \circ f_* : \pi_1(X) \to \mathbb{Z}$. Since $\pi_1(X)$ is finite, consisting of elements $g_1, \dots, g_N$, let $\psi(g_k)$ be the integer with largest absolute value in the set of all $\psi(g_1), \dots, \psi(g_N)$. Then $\psi(g_k + g_k) = 2\psi(g_k)$. But clearly $g_k + g_k$ is one of the elements in the list, so we must have $2|\psi(g_k)| \leq |\psi(g_k)|$, so $\psi(g_k) = 0$, and it follows that ψ is the 0-homomorphism.

Thus, $f_*(\pi_1(X)) \subset p_*(\pi_1(\mathbb{R}))$, so f lifts to a map $\widetilde{f} : X \to \mathbb{R}$. Note that $\widetilde{f}$ is nullhomotopic, so $f = p \circ \widetilde{f}$ is nullhomotopic as well.

Problem IV.3 (Hatcher Problem 1.3.15). First, let us show that $p : \widetilde{A} \to A$ is a covering space. To begin, note that $p(\widetilde{A}) = A$. We can see this is follows: pick some point $x_0 \in \widetilde{A}$ and note that there is a path γ in A connecting $p(x_0)$ to another point $y \in A$ (as A is path-connected). This path lifts to a nique $\widetilde{\gamma}$ starting at x_0. This γ must lie entirely inside the path-component $\widetilde{A}$, so in particular, $p(\widetilde{\gamma}(1)) = \gamma(1) = y$.

Pick $x \in A$, then there is open U in X that is evenly covered by p (covered by the union of open sets V_β in $\widetilde{X}$). Since A is locally path-connected, we can choose $U' \subset U \cap A$ that is path-connected and open in A.

so $U \cap A$ open in A will be covered by the disjoint collection of $V_\beta \cap \widetilde{A}$ open in $\widetilde{A}$.

We need to show that $p_*(\pi_1(\widetilde{A})) \subset \pi_1(X)$ is precisely $\mathrm{Ker}(j_*)$ with $j : A \to X$ the inclusion.

V. Chapter 2.1

Problem V.1 (Hatcher Problem 2.1.4)**.** SEE HOMEWORK

Problem V.2 (Hatcher Problem 2.1.3)**.** Let $\pm e_k$ denote the positive/negative unit vectors. To construct $S^n \subset \mathbb{R}^{n+1}$ as a Δ-complex, we consider each of the faces $[\pm e_1, \ldots, \pm e_{n+1}]$, of which there are 2^{n+1}. We glue these faces along the boundary edges $[\pm e_1, \ldots, \widehat{\pm e_k}, \ldots, \pm e_{n+1}]$ for all k. The resulting structure is isomorphic to the join of $n+1$ copies of S^0, which is homeomorphic to S^n.

To then get $\mathbb{R}P^n$, we simply identify the antipodal faces: $[\pm e_1, \ldots, e_k, \ldots, \pm e_n] \sim [\pm e_1, \ldots, -e_k, \ldots, e_n]$. This is precisely equivalent, via the homeomorphism, to identifying antipodal points of S^n, and thus yields $\mathbb{R}P^n$.

Problem V.3 (Hatcher Problem 2.1.5)**.** We cut the Klein bottle down the its diagonal, to get two 2-simplices which are identified. Since this Δ-complex contains no n simplices for $n > 2$, $H_n^{\Delta}(X)$ is trivial for $n > 2$. For the case of $n = 2$, the group of simplical chains $\Delta_2(X)$ is generated by the two 2-simplices. In addition, we have 3 edges $\{a, b, c\}$ generating $\Delta_1(X)$ and a single vertex generating $\Delta_0(X)$.

The orientations on the edges determine the ordering of vertices in the Δ-complex. In particular, note that $\partial_2 U = b + a - c$ and $\partial_2 L = a + c - b$. Thus, $\mathrm{Im}(\partial_2) = \langle b + a - c, a + c - b\rangle$ and $\mathrm{Ker}(\partial_2) = 0$, so $H_2^{\Delta}(X) = 0$. Note that $\mathrm{Ker}(\partial_1)$ consists of all the edges, as all vertices are identified, so

$$H_1^{\Delta}(X) = \langle b + a - c, b, c\rangle / \langle b + a - c, a + c - b\rangle = \langle b, c - b\rangle / \langle 2(c - b)\rangle \simeq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

Finally, note that $\mathrm{Im}(\partial_1) = 0$, so $H_0^{\Delta}(X) = \mathbb{Z}$.

VI. Relative homology review

We will review the calculation of relative simplical homology, for the case probably not worth typing this tbh...

VII. Lifting correspondence and covering space review

I want to re-prove some stuff related to lifting correspondences.

VIII. Seifert van-Kampen review

Theorem VIII.1 (Baby SvK)**.** Suppose a space X can be written as the union $U \cup V$ of open sets containing some basepoint $x_0 \in U \cap V$, where we also assume that $U \cap V$ is path-connected. Then the fundamental group $\pi_1(X, x_0)$ is generated by the image of the inclusion-induced homomorphisms

$$j_{1*} : \pi_1(U, x_0) \to \pi_1(X, x_0) \quad \text{and} \quad j_{2*} : \pi_1(V, x_0) \to \pi_1(X, x_0) \tag{4}$$

This baby result will allow us to prove the real Seifert van-Kampen.

Remark VIII.1 (Free products)**.** Let us briefly discuss free products. Given two group G and H, we will let $G * H$ denote the external free product of groups. Technically the external free product isn't unique, but it is up to isomorphism, so we can use the word "the" without much trouble.

One particular external free product which one can use as a model is the space of all *reduced words* of elements in G and H, which are simply ordered tuples $(r_1, \ldots, r_k)$ of arbitrary length, such that neighbouring elements are not contained in the same group. The multiplication is concatenation of two words, followed by pairwise reduction of neighbouring elements in the same group. This is what we are speaking of when we talk about "the external free product".

There is an important result relating to external free products that we will use: if G is the external free product of groups G_α where $i_\alpha : G_\alpha \to G$ are the monomorphisms which send elements of G_α to length-1 words, and if $h_\alpha G_\alpha \to H$ are homomorphisms, there exists a unique homomorphism $h : G \to H$ such that $h_\alpha = h \circ i_\alpha$ for all α. What will this homomorphism look like? It will simply take some reduced word $(r_1, \ldots, r_k)$ to $h_1(r_1) \cdots h_k(r_k)$ (of course, one must verify this is actually a homomorphism).

Theorem VIII.2 (Seifert-van Kampen)**.** From the previous discussion, if we take the two groups $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ and take their external direct product $\pi_1(U, x_0) * \pi_1(V, x_0)$, there will exist a unique homomorphism $j_* : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ extending the inclusions, so that

$$j([f], [g]) = [j_1 \circ f] * [j_2 \circ g]. \tag{5}$$

Our claim is that j is surjective, and has kernel being the smallest normal subgroup N containing the words $(i_1([f])^{-1}, i_2([f]))$, where $[f] \in \pi_1(U \cap V, x_0)$ and i_1 and i_2 are the inclusion homomorphisms of this fundamental group into $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ respectively. It follows that $\pi_1(U, x_0) * \pi_1(V, x_0)/N \simeq \pi_1(X, x_0)$.

IX. Fundamental groups of surfaces