

L^p -spaces, operator theory, and duality

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I. Introduction

The goal of this essay is to explore some of the fundamental results of the theory of L^p -spaces and their relationship to operator theory, discussed in Chapter 6.4 of Pedersen's *Analysis Now* [1], as well as other sources. The goal of the essay is to culminate in proving the duality theorem of L^p -spaces.

II. L^p -spaces, operator theory, and duality

Let us fix a measure space (X, \mathcal{M}, μ) , and recall the definition of the p -norm for some measurable function f :

$$\|f\|_p = \left[\int |f|^p d\mu \right]^{1/p} \quad (1)$$

for $0 < p < \infty$. We then take L^p to be the set of all measurable functions such that $\|f\|_p < \infty$, identifying functions which are equal almost everywhere. In addition, we use the special notation ℓ^p to denote the L^p space associated with the counting measure on $(X, \mathcal{P}(X))$. It is very easy to see that L^p is a vector space (in fact, we will show that it is a Banach space for $p \geq 1$). To begin, note that if $f, g \in L^p$, then

$$|f + g|^p \leq (|f| + |g|)^p = \sum_{k=0}^p \binom{p}{k} |f|^k |g|^{p-k} \leq 2^p \max(|f|, |g|)^p \leq 2^p (|f|^p + |g|^p) \quad (2)$$

which implies that

$$\int |f + g|^p d\mu \leq 2^p \left(\int |f|^p d\mu + \int |g|^p d\mu \right) < \infty \quad (3)$$

so $f + g \in L^p$. It is trivial to check that if $f \in L^p$, then $\lambda f \in L^p$. To verify that $\|\cdot\|_p$ is a norm, so that we in fact have a *normed vector space*, we first note that it is clear $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere. Thus, all that remains to check is the triangle inequality, *which is only valid for $p \geq 1$* . This will be shown via *Minkowski's inequality*, which is a consequence of *Holder's inequality*. We begin with a lemma:

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Lemma II.1. Given $a, b \geq 0$ and $\lambda \in (0, 1)$, then $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$, with equality if and only if $a = b$.

Proof. If either $a = 0$ or $b = 0$, the inequality is trivial. Otherwise, assume $a, b > 0$. Note that the function $x \mapsto \log(x)$ is concave, as $\frac{d^2}{dx^2} \log(x) = -\frac{1}{x^2}$. It follows that $\lambda \log(a) + (1-\lambda) \log(b) \leq \log(\lambda a + (1-\lambda)b)$. If we take the exponential of both sides, we obtain the desired inequality. \square

This generalization of the AM-GM inequality immediately allows us to prove Holder's inequality:

Theorem II.1 (Holder). Given $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$ (i.e. p and q are *conjugate*). If f and g are measurable functions on X , then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad (4)$$

This immediately implies that if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and the inequality becomes an equality if and only if $\alpha|f|^p = \beta|g|^q$, where neither α nor β are 0.

Proof. In the case that $\|f\|_p = 0$ or $\|g\|_q = 0$ or $\|f\|_p = \infty$ or $\|g\|_q = \infty$, the inequality is trivial. Otherwise, we can assume WLOG that $\|f\|_p = \|g\|_q = 1$, as we can always normalize. Note that

$$\|fg\|_1 = \int_X |fg| \, d\mu = \int_X (|f|^p)^{1/p} (|g|^q)^{1/q} \leq \frac{1}{p} \int_X |f|^p \, d\mu + \frac{1}{q} \int_X |g|^q \, d\mu = \frac{1}{p} + \frac{1}{q} = 1 \quad (5)$$

where we are simply applying the previous lemma to $a = |f|^p$, $b = |g|^q$, and $\lambda = 1/p$ (so $1 - \lambda = 1/q$). Note that the inequality used above is an equality if and only if $|f|^p = |g|^q$, in the case that they are normalized, so in the general case, we can have the factors α and β . \square

This result immediately allows us to prove the Minkowski inequality, which shows that $\|\cdot\|_p$ is a valid norm on L^p (i.e. it satisfies the triangle inequality).

Theorem II.2 (Minkowski). For any $p \in [1, \infty)$, we have $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. Once again, we can immediately disregard the trivial cases of $p = 1$ (as this follows from the standard triangle inequality) and when $f + g = 0$ almost everywhere. Otherwise, we note that

$$|f + g|^p \leq (|f| + |g|)|f + g|^{p-1} \quad (6)$$

where q is conjugate to p , so $q = p/(p-1)$. It then follows from Holder's inequality that

$$\int_X (|f| + |g|)|f + g|^{p-1} \, d\mu \leq \|f\|_p \|f + g|^{p-1}\|_q + \|g\|_p \|f + g|^{p-1}\|_q \quad (7)$$

Note that

$$\|f + g|^{p-1}\|_q = \left(\int_X |f + g|^{(p-1)q} \, d\mu \right)^{1/q} = \left(\int_X |f + g|^p \, d\mu \right)^{1/q} = \|f + g\|_p^{p/q}. \quad (8)$$

It follows immediately that

$$\|f + g\|_p^p \leq (\|f\|_p + \|g\|_p) \|f + g\|_p^{p/q} \quad (9)$$

and since $p - \frac{p}{q} = p - (p-1) = 1$, we have $\|f + g\|_p \leq \|f\|_p + \|g\|_p$, as desired. \square

We have shown that we do, in fact, have a norm, so L^p is a normed vector space. To prove that our L^p space equipped with this norm is a Banach space, we simply must verify that it is complete. To do so, we begin with an important lemma concerning normed vector spaces:

Lemma II.2. A normed vector space is complete if and only if every absolutely converging sequences converges.

Proof. Suppose to begin that X is complete and $\sum_{k=1}^{\infty} \|x_k\| < \infty$. Let $S_N = \sum_{k=1}^N x_k$. It follows that

$$\|S_N - S_M\| = \left\| \sum_{k=M+1}^N x_k \right\| \leq \sum_{k=M+1}^N \|x_k\| \quad (10)$$

which goes to 0 for $M, N \rightarrow \infty$, so we have a Cauchy sequence, so S_N converges as X is complete. Conversely, suppose x_n is a Cauchy sequence. We let n_j be an ascending sequence of numbers such that for $n, m \geq n_j$, we have $\|x_m - x_n\| \leq 2^{-j}$. We then note that if we set $y_j = x_{n_{j+1}} - x_{n_j}$, then we will have

$$\sum_{j=1}^{\infty} \|y_j\| = \sum_{j=1}^{\infty} \|x_{n_{j+1}} - x_{n_j}\| < \sum_{j=1}^{\infty} 2^{-j} = 1 \quad (11)$$

which means that $y = \sum_{j=1}^{\infty} y_j$ exists. Note that

$$\|x_{n_j} - y\| = \left\| \left(\sum_{k=1}^{j-1} y_k \right) - y \right\| \quad (12)$$

so for large enough j , we can make this difference arbitrarily small. Hence, the subsequence x_{n_j} of x_k converges to y . But, we know that if a Cauchy sequence has a convergent subsequence, the whole sequence converges to the same limit, so we are done. \square

Theorem II.3. For $p \in [1, \infty)$, L^p is a Banach space when equipped with the norm $\|\cdot\|_p$.

Proof. We will make use of the previous lemma. Suppose we pick a sequence of functions $f_p \in L^p$, and suppose $\sum_{k=1}^{\infty} \|f_k\|_p = C < \infty$. We let $S_N = \sum_{k=1}^N |f_k|$, and note that $\|S_N\|_p \leq B$, from the triangle inequality, for every N . It follows immediately from monotone convergence theorem that

$$\int S_{\infty}^p = \lim_{N \rightarrow \infty} \int S_N \leq C^p \quad (13)$$

Thus, the function S_{∞} is in L^p and $S_{\infty}(x) < \infty$ a.e. so it must be true that $\sum_{k=1}^{\infty} f_k$ converges a.e. (we know convergence at points because the reals are obviously complete). If we let $F = \sum_{k=1}^{\infty} f_k$, then we of course have $|F| \leq S_{\infty}$, so $|F| \in L^p$. Finally, note that

$$\left| F - \sum_{k=1}^N f_k \right|^p \leq \left(|F| + \sum_{k=1}^N |f_k| \right)^p \leq (2S_N)^p \quad (14)$$

which means that by dominated convergence theorem:

$$\left\| F - \sum_{k=1}^N f_k \right\|_p^p = \int \left| F - \sum_{k=1}^N f_k \right|^p \rightarrow 0 \quad (15)$$

as $N \rightarrow \infty$, as the function inside the integral goes to 0 a.e. It follows by definition that $\sum_{k=1}^{\infty} f_k$ converges in L^p . Thus, L^p is complete, by the previous lemma. \square

There is one particular case that we have yet to discuss, namely when $p = \infty$. In this particular case, we define

$$\|f\|_{\infty} = \inf \{a > 0 \mid \mu\{x \mid |f(x)| > a\} = 0\} \quad (16)$$

It is easy to incorporate the ∞ -norm into our theory for L^p spaces: the results which we proved above all also hold in the infinite case.

Our main operator-theoretic point of interest, with respect to L^p -spaces, will be the dual L^p -spaces, $(L^p)^*$. In particular, given some $g \in L^q$, let us define the Banach space functional $\phi_g : L^p \rightarrow \mathbb{C}$, where $p^{-1} + q^{-1} = 1$, as

$$\phi_g(f) = \int fg \quad (17)$$

Note that this operator is bounded by Holder's inequality, as

$$|\phi_g(f)| = \left| \int fg \right| \leq \int |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q \quad (18)$$

implying that

$$\|\phi_g\| = \sup_f \frac{|\phi_g(f)|}{\|f\|_p} \leq \|g\|_q \quad (19)$$

Lemma II.3. Suppose $p^{-1} + q^{-1} = 1$, with $q \in [1, \infty)$, then if $g \in L^q$, we have

$$\|g\|_q = \|\phi_g\| = \sup \left\{ \left| \int fg \right| \mid \|f\|_p = 1 \right\} \quad (20)$$

In the case that μ is semifinite, it also holds for $q = \infty$.

Proof. We already saw that $\|\phi_g\| \leq \|g\|_q$, equality is immediate when $g = 0$ a.e., clearly. Otherwise, we define

$$f = \frac{|g|^{q-1} \overline{\text{sign}(g)}}{\|g\|_q^{q-1}} \quad (21)$$

and note that since $p^{-1} + q^{-1} = 1$, we have $q + p = pq$, so $pq - p = q$ and

$$\|f\|_p^p = \frac{1}{\|g\|_q^{pq-p}} \int |g|^{pq-p} = 1 \quad (22)$$

which means that

$$\|\phi_g\| \geq \int fg = \frac{1}{\|g\|_q^{q-1}} \int |g|^q \overline{\text{sign}(g)} \text{sign}(g) = \frac{\|g\|_q^q}{\|g\|_q^{q-1}} = \|g\|_q \quad (23)$$

so we have both sides of the inequality, implying $\|\phi_g\| = \|g\|_q$. In the particular case that $q = \infty$, then for $\varepsilon > 0$, we set $S = \{x \mid |g(x)| > \|g\|_\infty - \varepsilon\}$, and by definition of the infinity-norm, $\mu(S) > 0$, so in the case μ is semi-finite, we can pick some $B \subset S$ where $0 < \mu(B) < \infty$. We then define $f = \mu(B)^{-1} \chi_B \overline{\text{sign}(g)}$: obviously $\|f\|_1 = 1$, so

$$\|\phi_g\| \geq \int fg = \frac{1}{\mu(B)} \int |g| \geq \|g\|_\infty - \varepsilon \quad (24)$$

where we are using the fact that $B \subset S$: a set on which $|g| > \|g\|_\infty - \varepsilon$. Since this holds for any ε , $\|\phi_g\| \geq \|g\|_\infty$, and we are done. \square

In order to prove our main theorem, we require one more technical result, which we can now state:

Lemma II.4. Suppose $p^{-1} + q^{-1} = 1$ and g is measurable on X where $fg \in L^1$ for all f in Σ : the space of simple functions which vanish outside a set of finite measure, and suppose

$$M_q(g) = \sup \left\{ \left| \int fg \right| \mid f \in \Sigma \text{ and } \|f\|_p = 1 \right\} \quad (25)$$

is finite. We also assume that $S_g = \{x \mid g(x) \neq 0\}$ is σ -finite and that μ is semifinite, then $g \in L^q$ and $M_q(g) = \|g\|_q$.

For the sake of brevity, I will skip the proof of this fact: it boils down to demonstrating that one can approximate the norm of the operator ϕ_g sufficiently well by only considering a particular family of simple functions.

The main theorem of this essay is the *duality theorem for L^p spaces*:

Theorem II.4. Suppose $p^{-1} + q^{-1} = 1$, suppose μ is a σ -finite measure. If $1 < p < \infty$, then for $\phi \in (L^p)^*$, there exists some $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$, so L^q is isometrically isomorphic to $(L^p)^*$ via the map $g \mapsto \phi_g$ (we already proved equality of norms earlier).

Proof. As a first step, suppose μ is finite, which means all simple functions will be in L^p . We define a measure as follows, let $\nu(E) = \phi(\chi_E)$ for some given $\phi \in (L^p)^*$. If E_j is a disjoint sequence of measurable sets and E is their union, then $\chi_E = \sum_{j=1}^{\infty} \chi_{E_j}$ where we can see that we have convergence in L^p easily:

$$\left\| \chi_E - \sum_{j=1}^n \chi_{E_j} \right\|_p = \left\| \sum_{j=n+1}^{\infty} \chi_{E_j} \right\|_p = \mu \left(\bigcup_{j=n+1}^{\infty} E_j \right)^{1/p} \rightarrow 0 \quad (26)$$

as we take $n \rightarrow \infty$, from lower measurable continuity. Therefore,

$$\nu(E) = \sum_{j=1}^{\infty} \phi(\chi_{E_j}) = \sum_{j=1}^{\infty} \nu(E_j) \quad (27)$$

so ν is, in fact, a valid complex measure. Obviously if $\mu(E) = 0$, then $\chi_E = 0$ a.e. so $\nu(E) = 0$ as well. Thus, $\nu \ll \mu$, so it follows from Radon-Nikodym there is $g \in L^1(\mu)$ with $\phi(\chi_E) = \nu(E) = \int_E g \, d\mu$ for all E . It follows immediately that if f is a simple function, then by linearity, $\phi(f) = \int fg \, d\mu$. We also know that $|\int fg| \leq \|\phi\| \|f\|_p$ for such functions, so it follows from the technical lemma that $g \in L^q$. We can then use the fact that simple functions are *dense* in L^p spaces to conclude that $\phi(f) = \int fg \, d\mu$ for all $f \in L^p$.

Now let us move to the general case where μ is σ -finite. We let E_n be an increasing sequence of sets where $X = \bigcup_{n=1}^{\infty} E_n$, with each set having finite, non-zero measure. The argument from above demonstrates that we can pick $g_n \in L^q(E_n)$, for each n , such that

$$\phi(f) = \int g_n f \, d\mu \quad (28)$$

for all $f \in L^p(E_n)$, and $\|g_n\|_q = \|\phi|_{L^p(E_n)}\| \leq \|\phi\|$. Of course, the g_n functions are unique up to possible changes on a nullset, so for $n < m$, $g_n = g_m$ on their common domain, which is E_n . We thus have a well-defined function g on X given by $g = g_n$ on E_n . It follows from monotone convergence theorem that $\|g\|_q = \lim_n \|g_n\|_q \leq \|\phi\|$, so $g \in L^q$ on the entire space X . If we then pick some $f \in L^p$, then dominated convergence theorem implies that $f\chi_{E_n} \rightarrow f$ in L^p , implying

$$\phi(f) = \lim_n \phi(f\chi_{E_n}) = \lim_n \int_{E_n} fg = \int fg \quad (29)$$

which is exactly what we wanted to prove. □

[1] Gert K Pedersen. *Analysis now*, volume 118. Springer Science & Business Media, 2012.