# Every exercise in the first three chapters of Hartshorne

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(Dated: Wednesday 14th May, 2025)

**Contents**



## I. Introduction

Every exercise in the first three chapters of Hartshorne, no excuses ("every" means **every single one**).

### Current tally

- Chapter 1: 11/90
- Chapter 2: 1/134
- Chapter 3: 0/88

## II. Chapter 1

## III. Section 1.1

**Solution III.1** (Problem 1.1.1)**.** There are a few parts:

1. Of course, $A(Y) = k[x, y]/(y - x^2)$. We can define $\varphi : k[x, y] \to k[x]$ as $\varphi(p)(x) = p(x, x^2)$. Verification that this is a ring homomorphism is trivial. It is obviously surjective, as $k[x] \subset k[x, y]$, and $\varphi|_{k[x]} = \text{id}$. In addition, $\varphi(y - x^2) = 0$. Moreover, if $\varphi(p) = 0$, then $p(x, x^2) = 0$. Define $h(x, y) = p(x, y + x^2) \in k[x, y] = k[x][y]$. Of course, we may write $h(x, y) = h_0(x) + h_1(x)y + h_2(x)y^2 + \cdots$, by definition, and $p(x, x^2) = h(x, 0) = h_0(x)$, so $h(x, y) = yg(x, y)$ for some $g$. Thus, $p(x, y) = (y - x^2)g(x, y - x^2)$, which means $p \in (y - x^2)$. It follows that $\text{Ker}(\varphi) = (y - x^2)$, so by the first isomorphism theorem, $k[x, y]/(y - x^2) \simeq \text{Im}(\varphi) = k[x]$, so $A(Y) \simeq k[x]$, as desired.

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2. Suppose $\varphi : k[x,y]/(xy-1) \to k[x]$ is a ring homomorphism. Since $[x][y] = [xy] = [1]$ in the domain, $\varphi([x])\varphi([y]) = \varphi([x][y]) = \varphi([1]) = 1$. Thus, $\varphi([x]) = a \in k$ and $\varphi([y]) = a^{-1} \in k$. Given some $b \in k$, with $b \neq 0$, note that we must also have $\varphi([b])\varphi([b^{-1}]) = 1$, so $\varphi([b])$ is a unit in $k[x]$, thus in $k$. It follows that $\mathrm{Im}(\varphi) = k$, which means that $\varphi$ cannot be an isomorphism.

3. **(Starred)** The idea is to make use of a sequence of automorphisms of $k[x,y]$, which descend to automorphisms of the quotient. In particular, a general quadratic polynomial is of the form

$$p(x,y) = ax^2 + by^2 + cxy + dx + ey + f \tag{1}$$

TODO: Finish this

**Solution III.2** (Problem 1.1.2)**.** Clearly, $Y = V(y-x^2, z-x^3)$. We want to show that $(y-x^2, z-x^3)$. To do so, define the map $\varphi : k[x,y,z] \to k[x]$ which takes $p(x,y,z)$ to $p(x,x^2,x^3)$. It is clear that this is a surjective ring homomorphism and $(y-x^2, z-x^3) \subset \mathrm{Ker}(\varphi)$. Moreover, suppose $\varphi(p) = 0$. Define $h(x,y,z) = p(x, y+x^2, z+x^3)$. Of course, $h(x,0,0) = 0$, so we can write

$$h(x,y,z) = yg_1(x,y,z) + zg_2(x,y,z) \tag{2}$$

as each term is divisible by $z$ or $y$. We then have

$$p(x,y,z) = h(x, y-x^2, z-x^3) = (y-x^2)g_1(x, y-x^2, z-x^3) + (z-y^3)g_2(x, y-x^2, z-x^3) \tag{3}$$

which means that $p \in (y-x^2, z-x^3)$. Thus, $k[x] \simeq k[x,y,z]/(y-x^2, z-x^3)$ so $(y-x^2, z-x^3)$ is prime, so $Y$ is an affine variety and is 1-dimensional as $\dim(k[x]) = 1$. We have $I(Y) = (y-x^2, z-x^3)$, so we automatically have a pair of generators (this ideal clearly cannot be generated by one of these generators).

**Solution III.3** (Problem 1.1.3)**.** We have

$$\begin{aligned}
V(x^2-yz, xz-x) &= V(x^2-yz) \cap [V(x) \cup V(z-1)] && (4)\\
&= V(x^2-yz, x) \cup V(x^2-yz, z-1) && (5)\\
&= V(yz, x) \cup V(x^2-y, z-1) && (6)\\
&= ([V(y) \cup V(z)] \cap V(x)) \cup V(x^2-y, z-1) && (7)\\
&= V(y,x) \cup V(z,x) \cup V(y-x^2, z-1) && (8)
\end{aligned}$$

Showing that each of the ideals $(y,x)$, $(z,x)$ and $(y-x^2, z-1)$ can be done easily via the same method as Problem 1.1.2. Thus, $Y$ is the union of three irreducible components. It is easy to see that they each are 1-dimensional (as their coordinate rings are isomorphic to polynomials in a single variable).

**Solution III.4** (Problem 1.1.4)**.** Every open set in the product topology for $\mathbb{A}^1 \times \mathbb{A}^1$ can be written as the union of a collection of $U \times V$, with $U, V \subset \mathbb{A}^1$ open. Of course, $U$ and $V$ will be $\mathbb{A}^1$ with some finite collection of points removed (when they are non-empty). Thus, $U \times V$ is $\mathbb{A}^2$ with some finite collection of vertical/horizontal lines removed. All of these sets are open in the Zariski topology for $\mathbb{A}^2$. However, the Zariski topology is strictly finer than the product topology.

In particular, consider the open set $Y = V(x-y)^C$ in the Zariski topology. Given some point $p$ in $Y$, if $Y$ is open in the product topology we must be able to choose some $U \times V$ which is a neighbourhood of $p$ and contained in $Y$. However, any complement of a finite collection of horizontal/vertical lines will clearly contain some point on the diagonal $V(x-y)$.

**Solution III.5** (Problem 1.1.5)**.** First, note that the coordinate ring of some algebraic set $V$ is of the form $A(V) = k[x_1, \ldots, x_n]/I(V)$, where $I(V)$ is a radical ideal. It follows immediately that if $[p]^n = [p^n] = 0$ in $A(V)$, then $p^n \in I(V)$, so $p \in I(V)$ and $[p] = 0$. This means $A(V)$ has no nilpotent elements. Moreover, it is immediately clear that $A(V)$ is a finitely-generated $k$-algebra, generated by $x_1, \ldots, x_n$.

Conversely, suppose $B$ is a finitely-generated $k$-algebra with no nilpotent elements. Let $b_1, \ldots, b_n$ be a generating set for $B$, then the evaluation map $\mathrm{ev} : k[x_1, \ldots, x_n] \to B$ sending $p$ to $p(b_1, \ldots, b_n)$ is surjective.

It follows that $B \simeq k[x_1, \ldots, x_n]/\mathrm{Ker}(\varphi)$. Denote the ideal $\mathrm{Ker}(\varphi)$ by $I$. Since $B$ has no nilpotent elements, it follows that if $p^n \in I$ so that $p(b_1, \ldots, b_n)^n = 0$, then $p(b_1, \ldots, b_n) = 0$, so $p \in I$. Thus, $I$ is a radical ideal and $I(V(I)) = I$. It follows that $B \simeq A(V(I))$.

**Solution III.6** (Problem 1.1.6)**.** If $X$ is a topological space with non-empty open set $U$ and $\overline{U}$ is proper, then $X = \overline{U} \cup (X - U)$, so $X$ is reducible. Thus, if $X$ is irreducible, every non-empty open set is dense. For the second part, suppose $C_1$ and $C_2$ are closed in $\overline{Y}$ with $C_1 \cup C_2 = \overline{Y}$. Then since $\overline{Y}$ is closed in $X$, $C_1$ and $C_2$ are as well. Thus, $C_1 \cap Y$ and $C_2 \cap Y$ is closed in $Y$. Since $Y$ is irreducible and the union of these closed sets is $Y$, either $Y \subset C_1$ or $Y \subset C_2$. Thus, $\overline{Y} \subset C_1$ or $\overline{Y} \subset C_2$, so $\overline{Y}$ is irreducible.

**Solution III.7** (Problem 1.1.7)**.** This is a long problem:

1. Given some non-empty family $\mathcal{F}$ of closed sets, suppose there is no minimal element. Let $C_1 \in \mathcal{F}$, we can choose $C_2 \in \mathcal{F}$ which is a proper subset of $C_1$. We continue on inductively, choosing $C_{j+1} \in \mathcal{F}$ a proper subset of $C_j$. At follows that we have a descending chain in $X$ which does not terminate, so $X$ is not Noetherian. On the other hand, given a descending chain $C_1 \supset C_2 \supset \cdots$, there must be a minimal element, hence some $C_n$ such that if $C_j \subset C_n$, then $C_n = C_j$, so the chain terminates after a finite number of steps. The open set conditions can easily be seen to be equivalent to the conditions above by taking complements.

2. Let $\{U_\alpha\}_\alpha$ be an open cover for $X$. Suppose there is no finite subcover. Pick $U_1$ in the cover arbitrarily. Then pick some $x_2 \in U_1^C$ and pick a neighbourhood of $x_2$ in the cover, label it $U_2$. We proceed inductively, choosing $x_{n+1} \in (U_1 \cup \cdots \cup U_n)^C$, letting $U_{n+1} \in \{U_\alpha\}$ being a neighbourhood of $x_{n+1}$. Then the ascending chain of open set $V_n = U_1 \cup \cdots \cup U_n$ never terminates, as $V_{n+1}$ contains $x_{n+1}$ while $V_n$ does not. Thus, $X$ is not Noetherian. It follows that if $X$ is Noetherian, then it must be quasi-compact.

3. Any descending chain $C_1 \supset C_2 \supset \cdots$ of closed sets of $Y \subset X$ implies that $C_j = K_j \cap Y$ with $K_j$ a closed set. We then define $K'_n = K_1 \cap \cdots \cap K_n$, which gives a descending chain $K'_1 \supset K'_2 \supset \cdots$ of $X$. This chain eventually terminates, so $K'_n = K'_{n+1} = \cdots$. Note that
$$K'_j \cap Y = (K_1 \cap Y) \cap \cdots \cap (K_j \cap Y) = C_1 \cap \cdots \cap C_j = C_j \tag{9}$$
which implies that $C_n = C_{n+1} = \cdots$, so $Y$ is Noetherian.

4. Recall that any quasi-compact subset of a quasi-compact Hausdorff space is closed. If $X$ is Noetherian, then every subset is Noetherian, thus quasi-compact, thus closed. Every subset being closed implies that $X$ has the discrete topology. Quasi-compactness of the whole space then implies that it is a finite set of points (if we had an infinite number of points, with each single-point set open, we could not have a finite subcover).

**Solution III.8** (Problem 1.1.8)**.** Before jumping into this solution, we must remark on something: when we decompose an affine algebraic set into its irreducible components, $Y = Y_1 \cup \cdots \cup Y_m$, each $Y_j$ is maximal in the sense that it is not contained in any strictly larger irreducible algebraic subset $Y' \subset Y$. Clearly, $Y' = \bigcup_j (Y' \cap Y_j)$, so if $Y'$ is irreducible, $Y' \cap Y_j = Y'$ for at least one of the $j$. It can't be $j = 2, \ldots, m$ as we would then have $Y_1 \subset Y' \subset Y_j$. Thus, $Y' = Y_1$. This means that when given some affine algebraic set $V$ with irreducible decomposition $V = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_m)$ with each $\mathfrak{p}_j$ a prime ideal, then each of these primes is minimal in the sense that they contain no smaller prime which contains $I(V)$.

We write $Y = V(I)$, where $I$ is prime, and $H = V(f)$, where $f$ is some irreducible polynomial, as well as $Y \cap H = V(I, f) = V(J)$ where $J = \mathrm{Rad}(I, f)$. Let $V(\mathfrak{p}_j)$ be one of the irreducible components of $V(J)$, so $\mathfrak{p}_j \supset (I, f)$. We note that

$$\begin{aligned}
\dim V(\mathfrak{p}_j) = \dim k[x_1, \ldots, x_n]/\mathfrak{p}_j &= \dim(k[x_1, \ldots, x_n]/I)/(\mathfrak{p}_j/I) && (10)\\
&= \dim(k[x_1, \ldots, x_n]/I) - \mathrm{height}(\mathfrak{p}_j/I) && (11)\\
&= \dim V(I) - \mathrm{height}(\mathfrak{p}_j/I) && (12)\\
&= r - \mathrm{height}(\mathfrak{p}_j/I) && (13)
\end{aligned}$$

where $\mathfrak{p}_j/I$ is a prime ideal of $k[x_1, \ldots, x_n]/I$. Suppose $\mathfrak{q}$ were another prime ideal such that $\mathfrak{q} \subset \mathfrak{p}_j/I$, then $I \subset \pi^{-1}(\mathfrak{q}) \subset \mathfrak{p}_j$, so by minimality of $\mathfrak{p}_j$, $\mathfrak{q} = \mathfrak{p}_j/I$. In addition, $f + I \in \mathfrak{p}_j/I$. We can't have $f \in I$, as then $V(f) \supset V(I)$, which we assumed is not the case. Clearly $f + I$ is not a zero-divisor as $k[x_1, \ldots, x_n]/I$ is an integral domain, as $I$ is prime. Moreover, if $fg = 1 + I$, so $fg + i = 1$ for some $i \in I$ and $g \in k[x_1, \ldots, x_n]$, then $(f, I) = (1)$, so $V(J) = \emptyset$. Thus, we actually require the addition assumption that $Y$ and $H$ intersect at all!

With this new assumption, we can apply Krull's Haupidealsatz to see that $\text{height}(\mathfrak{p}_j/I) = 1$, so that $\dim V(\mathfrak{p}_j) = r - 1$ for all $j$.

**Solution III.9** (Problem 1.1.9)**.**

**Solution III.10** (Problem 1.1.10)**.** This problem has multiple parts:

1. Of course, any chain $Z_0 \subset \cdots \subset Z_n$ of distinct, closed, irreducible subsets of $Y$ is a chain of distinct, closed, irreducible subsets of $X$, so $\dim(Y) \leq \dim(X)$.

2. From the first part, $\dim U_i \leq \dim X$ for all $i$, so $\sup_i \dim U_i \leq \dim X$. Conversely, let $Z_0 \subset \cdots \subset Z_n$ be a chain of distinct, closed, irreducible subsets of $X$. For each $Z_{j-1} \subset Z_j$, there must exist some $U_{i_j}$ such that $Z_{j-1} \cap U_{i_j}$ is a proper subset of $Z_j \cap U_{i_j}$. Thus, if we let $U = U_{i_1} \cup \cdots \cup U_{i_n}$, then

3.

4.

**Solution III.11** (Problem 1.1.11)**.**

**Solution III.12** (Problem 1.1.12)**.** Consider

$$(x^2 + iy^2 - 1)(x^2 - iy^2 - 1) = x^4 + y^4 - 2x^2 + 1 \tag{14}$$

in $\mathbb{R}[x, y]$, which vanishes only at points $(\pm 1, 0) \in \mathbb{A}^2(\mathbb{R})$. Each of these single points is itself a closed proper subset of the vanishing set of $p(x, y) = x^4 + y^4 - 2x^2 + 1$, so the vanishing set is not irreducible. To see that this polynomial is irreducible in $\mathbb{R}[x, y]$, suppose that it was reducible. Since any facotirzation of $p$ in $\mathbb{R}[x, y]$ would be a factorization in $\mathbb{C}[x, y]$, and $\mathbb{C}[x, y]$ is a UFD, it follows that one of the factors in Eq. (14) must be reducible. Each factor must be degree-1, so it follows that if a factor is reducible, it will vanish precisely on the union of two straight lines. However, note that $x^2 \pm iy^2 - 1$ vanishes on the embedded circle $(\cos(\theta), (\mp i)^{1/2} \sin(\theta))$, where given any two lines, we can always find a point on the circle not contained in these lines.

Thus, the decomposition in Eq. (14) is into irreducible factors, so $p$ is irreducible in $\mathbb{R}[x, y]$.

### A. Section 1.2

Even after taking a course in algebraic curves, my relative comfort-level when working with projective and quasi-projective varieties is significantly lower than when working with their affine counterparts. Hopefully I'll fix this deficiency in the process of doing these problems.

**Solution III.13** (Problem 1.2.1)**.** Let $\pi : \mathbb{A}^{n+1}(k) - (0, \ldots, 0) \to \mathbb{P}^n(k)$ be the quotient map. Recall that

$$V_p(\mathfrak{a}) = \{y \in \mathbb{P}^n \mid \text{there exists } x \in \mathbb{A}^{n+1} - (0, \ldots, 0) \text{ where } \pi(x) = y, f(x) = 0 \text{ for all } f \in \mathfrak{a}_h\} \tag{15}$$

where $\mathfrak{a}_h$ is the set of all homogeneous elements of $\mathfrak{a}$. It follows that if $x \in V_a(\mathfrak{a}) - (0, \ldots, 0)$, then $\pi(x) \in V_p(\mathfrak{a})$. Similarly, if $x \in \pi^{-1}(V_p(\mathfrak{a}))$, then all homogeneous element of $\mathfrak{a}$ vanish at $x$, and since $\mathfrak{a}$ is generated by homogeneous elements, it follows that all $f \in \mathfrak{a}$ vanish at $x$, so $x \in V_a(\mathfrak{a}) - (0, \ldots, 0)$. We have therefore shown that

$$V_a(\mathfrak{a}) - (0, \ldots, 0) = \pi^{-1}(V_p(\mathfrak{a})) \tag{16}$$

It follows that if $f$ is homogeneous and vanishes on $V_p(\mathfrak{a})$, it vanishes on any homogeneous coordinates for any point in $V_p(\mathfrak{a})$, thus any point in $\pi^{-1}(V_p(\mathfrak{a})) = V_a(\mathfrak{a}) - (0, \ldots, 0)$. If $\deg(f) > 0$, then $f$ in fact vanishes on $V_a(\mathfrak{a})$, and $f \in I_a(V_a(\mathfrak{a})) = \mathrm{Rad}(\mathfrak{a})$. We have thus shown that $I_p(V_p(\mathfrak{a})) \cap S_{>0} \subset \mathrm{Rad}(\mathfrak{a})$. Note that any element of $S_0 = k$ clearly cannot be in $I_p(V_p(\mathfrak{a}))$ if $V_p(\mathfrak{a}) \neq \emptyset$, so in this case, $I_p(V_p(\mathfrak{a})) \subset \mathrm{Rad}(\mathfrak{a})$.

**Solution III.14** (Problem 1.2.2)**.** Of course, if $V_p(\mathfrak{a}) = \emptyset$, then $S \supset \mathrm{Rad}(\mathfrak{a}) \supset I_p(V_p(\mathfrak{a})) \cap S_{>0} = I_p(\emptyset) \cap S_{>0} = S_{>0}$. It follows that if $\mathrm{Rad}(\mathfrak{a})$ does not contain some $c \neq 0$ in $S_0 = k$, then it is $S_{>0}$, and if it does, it is all of $S$. If $\mathrm{Rad}(\mathfrak{a})$ is either $S$ or $S_{>0}$, then in either case, it contains each of the elements $x_0, \ldots, x_n$. It follows that for each $j$ from 0 to $n$, there is some $d_j$ where $x_j^{d_j} \in \mathfrak{a}$. Then if $D = d_0 \cdots d_n$, each $x_j^D$ is in $\mathfrak{a}$. Let $M = (n+1)D$, and note that any generator $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ will have some $\alpha_j \geq D$, so the generator is in $\mathfrak{a}$. Therefore, $S_M \subset \mathfrak{a}$. If $S_M \subset \mathfrak{a}$ for some $M$, then $V_p(\mathfrak{a}) \subset V_p(S_M)$. If $[x_0, \ldots, x_n] \in V_p(S_M)$, then $x_0^M = \cdots = x_n^M = 0$, which is not the case for any point of projective space, so $V_p(S_M) = \emptyset$ and $V_p(\mathfrak{a}) = \emptyset$ as well.

**Solution III.15** (Problem 1.2.3)**.** We go point-by-point:

1. This is a trivial application of the definitions.

2. Again, trivial.

3. Clearly, if $f$ is a homogeneous polynomial which vanishes on $Y_1 \cup Y_2$, it vanishes on $Y_1$ and $Y_2$ individually. The converse is also true, this proves the claim.

4. From Problem 1.2.1, we already know that if $V_p(\mathfrak{a}) \neq \emptyset$, then $I_p(V_p(\mathfrak{a})) \subset \mathrm{Rad}(\mathfrak{a})$. To show the reverse inclusion, let us first show that $I_p(X)$: the ideal generated by all homogeneous polynomials vanishing on $X \subset \mathbb{P}^n$, is radical. Suppose $f^m \in I_p(X)$, where $f = f_0 + f_1 + \cdots + f_d$ is the decomposition into homogeneous components. Since $I_p(X)$ is a homogeneous ideal, $f_d^m \in I_p(X)$, so $f_d \in I_p(X)$, as it vanishes on $X$. It follows that $(f - f_d)^m \in I_p(X)$ (use the binomial expansion) and $f - f_d = f_0 + \cdots + f_{d-1}$. Continue this process inductively to see that $f_j \in I_p(X)$ for each $j$, so $f \in I_p(X)$ as well.

   Clearly, if $f \in \mathfrak{a}$, with $f = f_0 + \cdots + f_d$, then each homogeneous component $f_j$ is also in $\mathfrak{a}$ (as it is a homogeneous ideal). Thus, obviously $f_j$ is homogeneous and vanishes on $V_p(\mathfrak{a})$, so $f_j \in I_p(V_p(\mathfrak{a}))$, so $f \in I_p(V_p(\mathfrak{a}))$ as well. Then, since $I_p(V_p(\mathfrak{a}))$ is radical, $\mathrm{Rad}(\mathfrak{a}) \subset I_p(V_p(\mathfrak{a}))$, and we have the desired equality.

5. Clearly $V_p(I_p(Y))$ is a closed set. Any homogeneous polynomial in $I_p(Y)$ will vanish on $Y$, so $Y \subset V_p(I_p(Y))$, thus implying that $\overline{Y} \subset V_p(I_p(Y))$. To show that this inclusion is an equality, suppose $Y \subset V_p(T)$, where $T$ is some collection of homogeneous polynomials. Any $f \in T$ vanishes on $Y$, so $f \in I_p(Y)^h$, the set of homogeneous elements in $I_p(Y)$. Thus, $T \subset I_p(Y)^h$, and

   $$V_p(I_p(Y)) = V_p(I_p(Y)^h) \subset V_p(T) \tag{17}$$

   It follows that $\overline{Y} = V_p(I_p(Y))$, as desired.

**Solution III.16** (Problem 1.2.4)**.** Another multi-part question, another test of my will:

1. Note that if $V_p(\mathfrak{a})$ is an algebraic/closed subset of $\mathbb{P}^n$, in the case that $V_p(\mathfrak{a}) = \emptyset$, then $I_p(V_p(\mathfrak{a})) = S$ and $V_p(S) = \emptyset$. If $V_p(\mathfrak{a}) \neq \emptyset$, then $I_p(V_p(\mathfrak{a})) = \mathrm{Rad}(\mathfrak{a})$ is a radical homogeneous ideal. It is not equal to $S_+$, as if this were the case, then by Problem 1.2.2, $V_p(\mathfrak{a}) = \emptyset$. We also have

   $$V_p(I_p(V_p(\mathfrak{a}))) = \overline{V_p(\mathfrak{a})} = V_p(\mathfrak{a}) \tag{18}$$

   Now, assume that $\mathfrak{a}$ is a radical homogeneous ideal not equal to $S_+$, then if $\mathfrak{a} = S$, $V_p(\mathfrak{a}) = \emptyset$, and $I_p(V_p(\mathfrak{a})) = I_p(\emptyset) = S$. Otheriwse, $V_p(\mathfrak{a})$ is a non-empty closed subset of $\mathbb{P}^n$, and $I_p(V_p(\mathfrak{a})) = \mathrm{Rad}(\mathfrak{a}) = \mathfrak{a}$, as desired.

2. If $Y \subset \mathbb{P}^n$ is irreducible, then given $fg \in I_p(Y)$, let us decompose into homogeneous components $f = f_0 + \cdots + f_r$ and $g = g_0 + \cdots + g_q$. Then $f_r g_q \in I_p(Y)$, so $Y \subset V_p(f_r g_q) = V_p(f_r) \cup V_p(g_q)$. We then have $Y = (Y \cap V_p(f_r)) \cup (Y \cap V_p(g_q))$, so either $Y \subset V_p(f_r)$ or $Y \subset V_p(g_q)$, so either $f_r \in I_p(Y)$ or $g_q \in I_p(Y)$. Then, either $(f - f_r)g \in I_p(Y)$ or $f(g - g_q) \in I_p(Y)$. We repeat this same argument inductively by looking at the top-degree homogeneous component of the new polynomial, until we eventually conclude that either $f_0, \ldots, f_r \in I_p(Y)$ or $g_0, \ldots, g_q \in I_p(Y)$. Thus, $f \in I_p(Y)$ or $g \in I_p(Y)$, and $I_p(Y)$ is therefore prime. On the other hand, suppose $Y = V_p(T_1) \cup V_p(T_2)$ where both closed sets are proper. Since there is some $x \in V_p(T_1)$ which is not in $V_p(T_2)$, there must exist some $f \in T_2$ which does not vanish at $x$, and is thus not in $I_p(V_p(T_1))$. We can choose a similar $g \in T_1$. Note that $fg$ is homogeneous and vanishes on $Y$, so is in $I_p(Y)$. However, neither $f$ nor $g$ is in $I_p(Y) = I_p(V_p(T_1)) \cap I_p(V_p(T_2))$, so $I_p(Y)$ is not prime.

3. Note that if $f \in I_p(\mathbb{P}^n(k))$ is homogeneous, then it necessarily vanishes on $\mathbb{A}^{n+1}(k) - (0, \ldots, 0)$. But $k$ is algebraically closed, so $f$ has a finite number of roots if it is non-zero. Thus, $I_p(\mathbb{P}^n(k)) = (0)$, which is prime, so $\mathbb{P}^n(k)$ is irreducible.

**Solution III.17** (Problem 1.2.5)**.**

## IV. Chapter 2

### A. Section 2.1

**Solution IV.1** (Problem 2.1.1)**.** Let $\mathcal{F}$ be a presheaf, recall that its sheafification $\mathcal{F}^+$ is obtained by setting $\mathcal{F}^+(U)$ to be all functions $s : U \to \sqcup_{p \in U} \mathcal{F}_p$ such that $s(p) \in \mathcal{F}_p$ and for each $p \in U$, there is a neighbourhood $V$ of $p$ and $t \in \mathcal{F}(V)$ such that $s(q) = t_q$ for each $q \in V$ (where $t_q$ is the germ of $t$ at $q$). The arrows are simply restriction of functions. The morphism $\theta(U) : \mathcal{F}(U) \to \mathcal{F}^+(U)$ of Abelian groups is given by $\theta(U)(s)(p) = s_p$.

If $\mathcal{F}$ is the constant presheaf associated to $A$ on $X$, note that $\mathcal{F}(U) = A$, so all of the stalks of $\mathcal{F}$ is isomorphic to $A$, and the Abelian group $\mathcal{F}^+(U)$ is isomorphic to the group of functions $s : U \to A$ such that for each $p \in U$, there is a neighbourhood $V$ of $p$ and $t \in A$ such that $s(q) = t$ for each $q \in V$. In other words, $s : U \to A$ is a locally constant function, which is the case if and only if it is continuous when $A$ is given the discrete topology. Hence, $\mathcal{F}^+(U)$ is isomorphic to $\mathcal{G}(U)$, where $\mathcal{G}$ is the constant sheaf associated to $A$ on $X$. We have such an isomorphism for all $U$, and the isomorphisms are compatible with the restriction maps, so we have an isomorphism of sheaves One can imagine working out the details of the isomorphism described in more detail, but I personally can't be bothered as it is clear what's going on.

**Solution IV.2** (Problem 2.1.2)**.** There are three parts to this question:

1. Once again, recall that the kernel presheaf of a presheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is given by $U \mapsto \mathrm{Ker}(\varphi(U)) \subset \mathcal{F}(U)$. Of course, it follows that every section of $\mathrm{Ker}(\varphi(U))$ is a section of $\mathcal{F}(U)$, so it follows that if $\varphi$ is a morphism of *sheaves*, and $s \in \mathrm{Ker}(\varphi(U))$ restricts to 0 on a cover of $U$, then $s = 0$. Moreover, given a collection of sections of $\mathrm{Ker}(\varphi(U))$ defined on a cover $\{V_i\}$ for $U$ which agree on overlaps, we can construct $s \in \mathcal{F}(U)$ which restricts to each of these sections. Then, we note that

$$\varphi(U)(s)|_{V_i} = \varphi(V_i)(s|_{V_i}) = \varphi(V_i)(0) = 0 \tag{19}$$

for each $V_i$, so since $\mathcal{G}$ is a sheaf, it follows that $\varphi(U)(s) = 0$, which means that $s \in \mathrm{Ker}(\varphi(U))$. Thus, in this case, the kernel presheaf is a sheaf.

### B. Section 2.2

## V. Chapter 3