ALGEBRAIC TOPOLOGY

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1. Introduction

Notes taken while studying algebraic topology in MAT327, in Fall 2021

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2. Lecture 1: November 4th

Remark 1. *Two spaces are equivalent if they have the same shape.*

Definition 1 (Homotopy Equivalence)**.** *Let f and f' be continuous maps from X to Y. Then f is said to be* ***homotopic*** *to f' if there is a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$, and $F(x, 1) = f'(x)$ for all $x \in X$. F is called a* ***homotopy****.*

Remark 2. *This can be visualized as a continuous deformation from one map to another.*

Definition 2. *If f' is constant, then f is called* ***nullhomotopic***

We are interested in situations in which f and f' are **paths**. More specifically, situations where f and f' have the same starting point, and the same endpoint. If there is a homotopy that preserves the endpoints, then f and f' are said to be path-homotopic: $f \cong f'$.

Lemma 1. *Path-homotopy and homotopy are equivalence relations. The equivalence classes are called homotopy classes and path-homotopy classes.*

Proof. Clearly, reflexivity and symmetry hold (we simply "run time in reverse").

Suppose $f \simeq f'$, and $f' \simeq f''$. Let F and G be corresponding homotopies. Then define $H(x,t) = F(x, 2t)$ for $t \in \left[0, \frac{1}{2}\right]$ and $H(x,t) = G(x, 2t-1)$ for $t \in \left[\frac{1}{2}, 1\right]$. The continuity of this map follows from the pasting lemma. Thus, H is a homotopy for f and f''. □

Example 1. *Consider continuous maps $f, g : X \to \mathbb{R}^2$, with $X \subset \mathbb{R}$. f is homotopic to g, as we can define the homotopy $F(x,t) = (1-t)f(x) + tg(x)$. This is true for any convex subset of $\mathbb{R}^n$.*

Example 2. *Consider the punctured plane $\mathbb{R}^2 - \{0\}$. Consider the paths $f(s) = (\cos \pi s, \sin \pi s)$, as well as $g(s) = (\cos \pi s, 2 \sin \pi 2)$ and $h(s) = (\cos \pi s, -\sin \pi s)$.*

Definition 3. *Let f be a path from x_0 to x_1, and g be a path from x_1 to x_2. Then:*

$$(f * g)(x) = \begin{cases} f(2s) & s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

We want to be able to define an operation $*$ between path-homotopy/homotopy classes such that $[f] * [g] = [f * g]$. We need to show that given f' such that $[f'] = [f]$, and g' such that $[g'] = [g]$, that $[f' * g'] = [f * g]$. In other words, if f' and f are path-homotopic, and g' and g are path-homotopic, then we need to show that $f' * g'$ and $f * g$ are path-homotopic.

Theorem 1. *The operation $*$ acting on homotopy classes has the following properties:*

1. *It is associative.*
2. *There exist right and left identities. In other words, we can find $[e_{x_1}]$ and $[e_{x_2}]$ such that $[f] * [e_{x_1}] = [f]$, and $[e_{x_2}] * [f] = [f]$.*
3. *There exist inverses. In other words, given $[f]$, there is $[\overline{f}]$ such that $[\overline{f}] * [f] = [e_{x_1}]$, and $[f] * [\overline{f}] = [e_{x_2}]$.*

Proof. If k is a continuous map between topological spaces X and Y, F is a path-homotopy in X between f and f', then $K \circ F$ is a path-homotopy in Y between $k \circ f$ and $k \circ f'$. □