MAT327 PROBLEM SET 1

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1. Problem 1

Since the sequence is Cauchy, it converges to some point x. It may be the case that x is in $\{x_k\}$, or it may be the case that it is not. Either way, we can take $B = A \cup \{x\}$. Pick an open cover of A called $\mathcal{U}$. Let $V \in \mathcal{U}$ contain $\{x\}$. Since the sequence converges to $\{x\}$, all but finitely many of the x_k must be in V. Taking elements of $\mathcal{U}$ that contain all other elements of x_k not in V, and we have a finite subcover, so A is compact.

Take the sequence $x_k = k$. Clearly, this sequence is not Cauchy, as $d(x_m, x_n) = |m - n| \geq 1$ for all m and n not equal. In addition, this set is not compact as it is not bounded (and it will clearly not be bounded even if we add a point).

2. Problem 2

It is easy to see from the definition that $\lambda(0) = 0$.

First consider $\mu \circ \lambda$. We pick $\epsilon > 0$, and choose δ small enough such that $|\lambda(x)| \leq C|x|$, and $|x| < \delta \Rightarrow |\mu(x)| < \frac{\epsilon}{C}|x|$. We then note that:

$$|x| < \min\left(\delta, \frac{\delta}{C}\right) \Rightarrow |\lambda(x)| \leq C|x| < \delta \Rightarrow |\mu(\lambda(x))| < \frac{\epsilon}{C}|\lambda(x)| \leq \epsilon|x|$$

so $\mu \circ \lambda$ is tiny. Now, we can consider $\lambda \circ \mu$. We choose $\epsilon > 0$, and δ small enough such that $|\lambda(x)| \leq C|x|$, and $|x| < \delta$ implies $|\mu(x)| < \frac{\epsilon}{C}|x|$. It follows that:

$$|x| < \min\left(\delta, \frac{\delta C}{\epsilon}\right) \Rightarrow |\mu(x)| < \frac{\epsilon}{C}|x| \Rightarrow |\lambda(\mu(x))| \leq C|\mu(x)| < \epsilon|x|$$

where we know that $|\lambda(\mu(x))| \leq C|\mu(x)|$ as $|\mu(x)| < \delta$.

3. Problem 4

Recall the axis-crawl method. Let U be an open set inside which all the partial derivatives exist and are bounded. We let $h = (h_1, \ ..., \ h_n)$ be in U, and define the sequence of points $t_0 = (0, \ ..., \ 0), t_1 = (h_1, 0, \ ..., \ 0), \ ..., \ t_n = h$. We then note that:

$$f(h) - f(0) = \sum_{j=1}^{n} f(t_i) - f(t_{i-1}) = \sum_{i=1}^{n} f(t_{i-1} + h_i e_i) - f(t_{i-1})$$

Note that the functions obtained by setting all h_j constant, except for the k-th spot are differentiable in U, as the partial derivatives exist. Hence, we know from the mean value theorem that there exists some $c_i \in (0, h_i)$ for which $f(t_{i-1} + h_i e_i) - f(t_{i-1}) = D_i f(c_i) h_i$. Thus:

$$f(h) - f(0) = \sum_{j=1}^{n} D_i f(c_i) h_i \Rightarrow |f(h) - f(0)| = \left|\sum_{j=1}^{n} D_i f(c_i) h_i\right| \leq \sum_{j=1}^{n} |D_i f(c_i) h_i| \leq M \sum_{j=1}^{n} |h_i|$$

as each partial derivative is bounded. Taking the limit as $h \to 0$ allows us to conclude from squeeze theorem that $\lim_{h \to 0} |f(h) - f(0)| = 0$, so by definition, f is continuous at 0.

4. Problem 5

Part A

From chain rule:

$$[Dh](x, y) = [Df](x, y, g(x, y)) \circ [Dk](x, y)$$

where $k(x, y) = (x, y, g(x, y))$. Clearly:

$$[Df](x, y, g(x, y)) = \begin{pmatrix} D_1 f(x, y, g(x, y)) & D_2 f(x, y, g(x, y)) & D_3 f(x, y, g(x, y)) \end{pmatrix}$$

and:

$$[Dk](x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ D_1 g(x, y) & D_2 g(x, y) \end{pmatrix}$$

which implies that:

$$[Dh](x, y) = \begin{pmatrix} D_1 f(x, y, g(x, y)) + D_3 f(x, y, g(x, y)) D_1 g(x, y) & D_2 f(x, y, g(x, y)) + D_3 f(x, y, g(x, y)) D_2 g(x, y) \end{pmatrix}$$