MAT436 problem set 3

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I. Suggested Problem 1

Suppose first that U is unitary. Of course, U is bijective, so it is surjective. Moreover, since we are in a Hilbert space, we assume the norm is induced from the inner product and get

$$||Uv||_{H_2} = \sqrt{\langle Uv, Uv\rangle_{H_2}} = \sqrt{\langle U^*Uv, v\rangle_{H_1}} = \sqrt{\langle v, v\rangle_{H_1}} = ||v||_{H_1} \tag{1}$$

which implies that U is an isometry. Now, assume that U is a surjective isometry. We need to show that U is angle-preserving. Indeed, we have the polarization identity (assuming the inner product is anti-linear in the second argument, where we take the Hilbert space to be, generally, complex)

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k ||x + i^k y||^2 \tag{2}$$

so we have, since U is an isometry,

$$\langle Ux, Uy\rangle_{H_2} = \frac{1}{4}\sum_{k=0}^{3} i^k ||Ux + i^k Uy||_{H_2}^2 = \frac{1}{4}\sum_{k=0}^{3} i^k ||U(x + i^k y)||_{H_2}^2 = \frac{1}{4}\sum_{k=0}^{3} i^k ||x + i^k y||_{H_1}^2 = \langle x, y\rangle_{H_1} \tag{3}$$

as desired. Now, suppose the third statement holds, and we have U which is surjective and angle-preserving. U will be injective as if $Ux = 0$, then $\langle Ux, Ux\rangle_{H_2} = 0$, implying $\langle x, x\rangle_{H_1} = 0$, implying $x = 0$. Note that U^{-1} will preserve inner products as

$$\langle U^{-1}v, U^{-1}v\rangle_{H_1} = \langle UU^{-1}v, UU^{-1}v\rangle_{H_2} = \langle v, v\rangle_{H_2} \tag{4}$$

Thus, we have

$$\langle (U^{-1} - U^*)v, (U^{-1} - U^*)v\rangle = (\langle U^*v, U^*v\rangle - \langle U^{-1}v, U^*v\rangle) + (\langle U^{-1}v, U^{-1}v\rangle - \langle U^*v, U^{-1}v\rangle) \tag{5}$$

$$= \langle U^*v, U^*v\rangle - \langle v, UU^*v\rangle + \langle U^{-1}v, U^{-1}v\rangle - \langle v, v\rangle = 0 \tag{6}$$

so $U^{-1}v = U^*v$ for all v, implying $U^{-1} = U^*$, and U is unitary.

II. Suggested Problem 2

Part A. Note that S is not surjective as its image contains no element of the form $(1, x_1, x_2, \dots)$, so it cannot be unitary. On the other hand,

$$||Sx|| = \sqrt{0^2 + \sum_{j=1}^{\infty} |x_j|^2} = \sqrt{\sum_{j=1}^{\infty} |x_j|^2} = ||x|| \tag{7}$$

so the map is an isometry.

Part B. By definition, $\langle Sx, y\rangle = \langle x, S^*y\rangle$. Of course,

$$\sum_{j=1}^{\infty} x_j \overline{y_{j+1}} \tag{8}$$

so we have S^* as the left-shift operator which sends $(y_1, y_2, \dots)$ to $(y_2, y_3, \dots)$. To see that S^* has an uncountable number of eigenvalues, pick some $r \in \mathbb{R}$. Define $y = (y_1, y_2, \dots)$ with $y_k = r^k$. We then have

$$Sy = S(r, r^2, \dots) = (r^2, r^3, \dots) = r \cdot (r, r^2, \dots) = ry \tag{9}$$

so r is an eigenvalue. It follows that S^* is not diagonalizable, as by definition, diagonalizing an operator requires choosing a countable basis of eigenvectors, but there will always exist