MAT436 problem set 5

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I. Problem 1 (Suggested Problem 6)

Part 1. Let us make note of the fact that if f is essentially bounded (in $L^{\infty}(\mathbb{T})$), and $x \in L^2(\mathbb{T})$ is square integrable, then $fx \in L^2(\mathbb{T})$ (the product is clearly square integrable). It follows that

$$||T_f x|| = ||P(fx)|| = \left|\left| P\left(\sum_{n<0} \langle e_n, fx \rangle e_n + \sum_{n \geq 0} \langle e_n, fx \rangle e_n \right)\right|\right| = \sum_{n \geq 0} \langle |e_n, fx \rangle| ||e_n|| = \sum_{n \geq 0} |\langle e_n, fx \rangle| \tag{1}$$

where $\langle fx, e_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} fxz^n \, d\mu$, where μ is the uniform Haar measure on the circle. Of course,

$$\sum_{n \geq 0} \langle |e_n, fx \rangle| \leq \sum_{n \in \mathbb{Z}} \langle |e_n, fx \rangle| = ||fx|| \leq ||f|| ||x|| \leq M||x|| \tag{2}$$

as f is essentially bounded by some M, so that $||T_f x|| \leq M||x||$, and by definition, T_f is bounded.

Part 2. In the above proof, we have $M = ||f||_{\infty}$. Thus, we have already shown that $\frac{||T_f x||}{||x||} \leq ||f||_{\infty}$ for all x. Thus,

$$||T_f|| = \sup_x \frac{||T_f x||}{||x||} \leq ||f||_{\infty} \tag{3}$$

as well. Let us also note that for basis vectors $e_j, e_k \in H^2(\mathbb{T})$ with $j, k \geq 0$, we have

$$\langle T_f e_j, e_k \rangle = \langle P(fe_j), e_k \rangle = \frac{1}{\sqrt{2\pi}} \sum_{i \in \mathbb{Z}} \langle f, e_i \rangle \langle e_{i+j}, e_k \rangle = \frac{1}{\sqrt{2\pi}} \langle f, e_{k-j} \rangle \tag{4}$$

where we make use of the fact that $e_a \cdot e_b = \frac{1}{2\pi} z^a \cdot z^b = \frac{1}{2\pi} z^{a+b} = \frac{1}{\sqrt{2\pi}} e_{a+b}$. Moreover, note that

$$\langle e_j, T_{\overline{f}} e_k \rangle = \langle e_j, P(\overline{f} e_k) \rangle = \frac{1}{\sqrt{2\pi}} \sum_{i \in \mathbb{Z}} \langle e_j, \overline{\langle f, e_i \rangle} e_{k-i} \rangle = \frac{1}{\sqrt{2\pi}} \langle f, e_{k-j} \rangle \tag{5}$$

where we use the fact that if $f = \sum_k \langle \langle f, e_k \rangle e_k$, then $\overline{f} = \sum_k \overline{\langle f, e_k \rangle} e_{-k}$ as $\overline{z^k} = z^{-k}$ for $k \in \mathbb{T}$. Thus, by definition, $T_f^* = T_{\overline{f}}$.

Part 3. We have

$$T_f e_n = P(fe_n) = \frac{1}{\sqrt{2\pi}} P\left(\sum_{m \in \mathbb{Z}} \langle f, e_m \rangle e_{m+n}\right) = \frac{1}{\sqrt{2\pi}} \sum_{m+n \geq 0} \langle f, e_m \rangle e_{m+n} = \frac{1}{\sqrt{2\pi}} \sum_{m \geq 0} \langle f, e_{m-n} \rangle e_m \tag{6}$$

which immediately yields the desired result, as clearly,

$$\langle f, e_{m-n} \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(z) z^{n-m} \, d\mu = \frac{1}{\sqrt{2\pi}} \int_0^1 f(e^{2\pi i\theta}) e^{-2\pi(m-n)\theta i} \, d\theta = \hat{f}(m-n) \tag{7}$$

is the $m-n$-th Taylor series coefficient.

Part 4. Note that $T_{e_k}(e_j) = \frac{1}{\sqrt{2\pi}} P(e_{k+j})$, which is $\frac{1}{\sqrt{2\pi}}$ for $k + j \geq 0$ and 0 otherwise. Thus, the induced operator $\widetilde{T}_{e_k}$ must send δ_n to $\frac{1}{\sqrt{2\pi}}\delta_{n+k}$, so it follows that this operator is precisely the k-th power of the unilateral shift, composed with multiplication by $\frac{1}{\sqrt{2\pi}}$.

Part 5. Clearly, if $f = 0$, the operator is compact. Now, conversely, suppose that T_f is compact. Recall that in a Hilbert space, a sequence x_n is said to converge weakly if for every $y \in H$, we have $\lim_{n\to\infty}\langle x_n, y\rangle = 0$. Since the e_n form a Hilbert space basis, $\sum_{n\in\mathbb{Z}} |\langle e_n, y\rangle|^2 = ||y||^2$ for some y. Thus, the series $\langle e_n, y\rangle$ converges absolutely, so $\langle e_n, y\rangle \to 0$, so the sequence of e_n converges weakly.

Since T_f is compact, $T_f e_n$ converges strongly, $||T_f e_n|| \to 0$. Thus, it follows from Part 3 that

$$||T_f e_n||^2 = \sum_{m=0}^{\infty} |\hat{f}(m-n)|^2 \to 0 \tag{8}$$

as we take $n \to \infty$. This mean that $\hat{f}(m-n) \to 0$ as we taken $n \to \infty$, for each $m \in \mathbb{Z}_+ \cup \{0\}$ uniformly over all m. Of course, this means that for some $\varepsilon > 0$, we can pick n big enough such that we have $|\widehat{f}(m-n)| \leq \varepsilon$ for each non-negative integer m. If we set $m = n + m_0$ for some $m_0 \in \mathbb{Z}$ (WLOG, we can assume n is large enough such that $n + m_0 \geq 0$), this shows that $|\widehat{f}(m_0)| \leq \varepsilon$. We can do this for any $\varepsilon > 0$ and any m_0, so that $\widehat{f} = 0$. This immediately implies that $f = 0$ as well.

Part 6. Suppose first that $f = e_k$ and g arbitrary. Note that, from Part 3,

$$\begin{aligned}(T_{e_k}T_g - T_gT_{e_k})e_n &= T_{e_k}\sum_{m\geq 0}\hat{g}(m-n)e_m - T_g\left(\frac{1}{\sqrt{2\pi}}e_{k+n}\right)\\ &= \frac{1}{\sqrt{2\pi}}\sum_{m\geq 0}\hat{g}(m-n)e_{m+k} - \frac{1}{\sqrt{2\pi}}\sum_{m\geq 0}\hat{g}(m-k-n)e_m = -\frac{1}{\sqrt{2\pi}}\sum_{m=-k}^{-1}\hat{g}(m-n)e_{m+k}\end{aligned} \tag{9}$$

Thus, the image of any of the basis vectors e_n under the commutator map can be written as a linear combination of the basis vectors $e_0, \dots, e_{k-1}$. It follows immediately that $[T_{e_k}, T_g]$ has finite rank. This implies that the operator is compact: given a sequence $\{x_n\}$ of functions, their image under this operator must be bounded (this comes from finite rank), so Bolzano-Weierstrass gives a convergent subsequence and the operator is compact.

Now, let us turn our attention to the general case. Since f is continuous on $\mathbb{T}$, Stone-Wierestrass implies that we can approximate it uniformaly with trig polynomials. Clearly, T_f is linear, in the sense that $T_{\lambda f+g} = \lambda T_f + T_g$. Thus, we can find a sequence $f_n = \sum_{|k|\leq n} c_k e_k$ which converges uniformly to f for $|k| \to \infty$. Each operator

$$[T_g, T_{f_n}] = \left[T_g, \sum_{|k|\leq n} c_k T_{e_k}\right] = \sum_{|k|\leq n} c_k [T_g, T_{e_k}] \tag{10}$$

is a finite sum of compact operators, and is thus compact. Since the compact operators form a sub-C^*-algebra, the limit point of this sequence, which is precisely $[T_g, T_f]$, will also be compact, and the proof is complete.

The proof for the operator $T_fT_g - T_{fg}$ carries forward in the exact same way.

Part 7. Recall that an operator T is Fredholm if we can find another operator $S \in B(H)$ such that $1 - ST$ and $1 - TS$ are both compact. In the case that f is non-zero, so that f^{-1} is well-defined, we have $T_{ff^{-1}} = T_1$, which is clearly the identity on $H^2(\mathbb{T})$. Moreover, from Part 6, we have that

$$T_{ff^{-1}} - T_fT_{f^{-1}} = 1 - T_fT_{f^{-1}} \quad \text{and} \quad T_{f^{-1}f} - T_{f^{-1}}T_f = 1 - T_{f^{-1}}T_f \tag{11}$$

are both compact, so T_f is automatically Fredholm. Since $|f|$ is simply a scalar, it factors out: $T_f = T_{|f|\cdot f/|f|} = |f|T_u$. Of course, multiplication by a (non-zero) scalar does not have an effect on index, as it has no influence on the dimension of the kernel or cokernel. Thus, $\text{index}(T_f) = \text{index}(T_u)$.

Part 8. Letting f_t be our desired homotopy, we know that small perturbations in operator norm to Fredholm operators preserve the Fredholm property, and the index remains the same. It follows that T_{f_t} and T_{f_s} will have the same index for s and t sufficiently close, as $||T_{f_t} - T_{f_s}||$ is clearly upper-bounded by the distance between functions f_t and f_s. To be more specific, note that f_t is unitary so it is invertible and $T_{f_s \circ f_t^{-1} \circ f_t} - T_{f_s \circ f_t^{-1}} T_{f_t}$ is compact, from above, so subtracting it from the operator T_{f_s} does not change the index.

Thus, we simply need to show that T_{f_t} and $T_{f_s \circ f_t^{-1}} T_{f_t}$ have the same index for t and s close, which boils down to showing that $||1 - T_{f_s \circ f_t^{-1}}||$ is small enough, for t and s close, which is striaghtforward from the definitions.

Since $[0, 1]$ is compact, we can choose a finite collection of $[s_i, t_i]$ covering the interval, on which the index of T_{f_t} does not change, impyling the index remains the same throughout the homtopy.

Part 9. We know that the indices of T_f and T_g for which f and g are homotopic are the same. Of course, f and g are homotopic if and only if they have the same winding number, so each $\text{index}(T_f)$ for f with a fixed winding number is a fixed integer value. Clearly, e_k has winding number k, and moreover, f has winding number k when f is homotopic to e_k (each f will be homotopic to some such e_k).

Thus, all we have to do to prove the claim is show that $\text{index}(T_{e_k}) = -k$. But this follows immediately from the definition. For some non-negative k, multiplying by e_k and projecting has an empty kernel. Menawhile, the cokernel is the codomian quotiented by the image, which is clearly all functions comprised of a linear combination of terms e_j with $j \geq k$. Thus, the dimension of the quotient is precisely k, as all contribution to any function in $H^2(\mathbb{T})$, e_j with $j \geq k$ are quotiented together.

It follows that $\text{index}(T_{e_k}) = \dim \ker(e_k) - \dim \text{coker}(e_k) = -k$. In addition, note that $T_{e_{-k}} = T_{\overline{e_k}} = T_{e_k}^*$ for $k \geq 0$. It is obvious that $\text{index}(T^*) = -\text{index}(T)$, as the kernel and cokernel trade places, so $\text{index}(T_{e_{-k}}) = k$, and the result holds for all integer k.