Algebraic topology solutions

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I. Problem set 3

Problem I.1 (Hatcher 2.1.2)**.** See homework

Problem I.2 (Hatcher 2.1.4)**.** See homework

Problem I.3 (Hatcher 2.1.11)**.** See homework

Problem I.4 (Hatcher 2.1.16)**.** See homework

Problem I.5 (Hatcher 2.1.17)**.** See homework

Problem I.6 (Hatcher 2.1.22a)**.** For the case of $n = 0$, note that X will be a disjoint collection of points. Therefore, $H_0(X) = \oplus_\alpha H_0(x_\alpha) = \oplus_\alpha \mathbb{Z}$, so we have a free group. If $i > 0$, then we know $H_i(X) = 0$.

We proceed by induction. Suppose the claim holds in dimension n. Suppose X is dimension $n+1$, note that the n-skeleton X^n is of dimension at most n, and it is a good pair to X. We then have

$$\cdots \longrightarrow H_i(X^n) \longrightarrow H_i(X) \longrightarrow H_i(X, X^n) \longrightarrow \cdots \tag{1}$$

Note that $H_i(X, X^n) \simeq H_i(X/X^n) \simeq \bigoplus_\alpha H_i(S^{n+1}_\alpha)$, which is a free group when $i = n+1$ and 0 otherwise (for $i > 0$). In the first case, if $i > n+1 > n$, then $H_i(X^n) = 0$, so $H_i(X)$ is 0 as well. For $i = n+1$, we have a 0-map on the left, so the map on the right is injective. Thus, $H_i(X) \simeq \mathrm{Im}(\phi_*)$ (the map on the right), which is free as a subgroup of a free group.

Problem I.7 (Hatcher 2.1.22b)**.** Since X is a finite CW-complex, $X = X^k$ for some k. We then have

$$H_{n+1}(X^m, X^{m-1}) \longrightarrow H_n(X^{m-1}) \longrightarrow H_n(X^m) \longrightarrow H_n(X^m, X^{m-1}) \tag{2}$$

where on the left, the group is isomorphic to a direct sum of $H_{n+1}(S^m)$ and on the right, $H_n(S^m)$. Thus, for $m > n+1$, we have $H_n(X^{m-1}) \simeq H_n(X^m)$. In particular, setting $m = n+2$, we have $H_n(X^{n+1}) \simeq H_n(X^k) = H_n(X)$ when $k \geq n+1$. When $k \leq n+1$, we clearly have $H_n(X^{k+1}) = H_n(X)$ immediately. Since there are no $n+1$ cells, $X^{n+1} = X^n$. From here, we have

$$H_n(X^{n-1}) \longrightarrow H_n(X^n) \longrightarrow H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}) \tag{3}$$

Both groups on the left and right are 0, so $H_n(X) \simeq H_n(X^n) \simeq H_n(X^n, X^{n-1})$, so $H_n(X)$ is free with elements in bijective correpondence with the cells.

Problem I.8 (Hatcher 2.1.22c)**.** We proved above that $H_n(X) \simeq H_n(X^{n+1})$. Thus, we have

$$H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) \tag{4}$$

where the right-group is 0, so the map $H_n(X^n) \to H_n(X^{n+1}) = H_n(X)$ is surjective. We proved in Part A that $H_n(X^n)$ is isomorphic to a subgroup of a free Abelian group $H_n(X^n, X^{n-1})$, which is free on k elements, so $H_n(X^n)$ is free on at most k elements. Then $H_n(X) \simeq H_n(X^n)/\mathrm{Ker}(\varphi)$ is generated by at most k elements.

Problem I.9 (Hatcher 2.2.1)**.** The map $f : D^n \to D^n$ induces a map a hemisphere to a hemisphere. Let $\phi_\pm : D^n \to S^n_\pm$ be these homeomorphisms. Let $\widetilde{f} : S^n \to S^n$ be defined as $\phi_- \circ f \circ \phi_+^{-1}$ on the upper-hemisphere and $\phi_- \circ f \circ \phi_-^{-1}$ on the lower. The maps are equal on the equator, so $\widetilde{f}$ is well-defined. Note that $\widetilde{f}$ is not surjective, so $\deg(\widetilde{f}) = 0$. Thus, $\widetilde{f}$ must have a fixed point, otherwise it would have degree $(-1)^{n+1}$.

It follows that $\phi_- \circ f \circ \phi_-^{-1}(p) = p$ for some p in the lower hemisphere, so f fixes $\phi_-^{-1}(p)$.

Problem I.10 (Hatcher 2.2.2)**.** Let a be the antipodal map, so that $\deg(a \circ f) = \deg(a)\deg(f) = (-1)^{2n+1}\deg(f) = -\deg(f)$. Suppose f has no fixed point, then $\deg(f) = (-1)^{2n+1} = -1$, so $\deg(a \circ f) = 1$. Suppose that $a \circ f$ additionally has no fixed points, then we must have $\deg(a \circ f) = (-1)^{2n+1} = -1$, a contradiction. Thus, either f or $a \circ f$ must have a fixed point, so there exists x such that $f(x) = x$, or $f(x) = -x$.

Let $g : \mathbb{R}P^2 \to \mathbb{R}P^2$, consider $g \circ \pi : S^{2n} \to \mathbb{R}P^{2n}$. Then since $(g \circ \pi)_*(\pi_1(\mathbb{R}P^{2n})) \subset \pi_*(\pi_1(S^2))$, then $g \circ \pi$ lifts to $\widetilde{g} : S^2 \to S^2$ such that $\pi \circ \widetilde{g} = g \circ \pi$. We know $\widetilde{g}$ has some x such that $\widetilde{g}(x) = \pm x$. Thus, $\pi\widetilde{g}(x) = [x] = g\pi(x) = g([x])$, so g has a fixed point.

Consider the map $A : (x_1, y_1, \ldots, x_n, y_n) \mapsto (y_1, -x_1, \ldots, y_n, -x_n)$. This linear map clearly has no eigenvectors, and induces a map $\widetilde{A} : \mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$. Such a map has no fixed point, as if $\widetilde{A}(x) = x$ for some x, then we would necessarily have $Ax = \lambda x$ for some $\lambda \neq 0$, implying A has an eigenvalue.

Problem I.11 (Hatcher 2.2.3)**.** f must have a fixed point as $\deg(f) \neq (-1)^{n+1}$. Note that $a \circ f$ has degree 0 as well, so has a fixed point, so there is y with $f(y) = -y$.

Since $F(x) \neq 0$, $\widetilde{F}(x) = F(x)/||F(x)||$ is well-defined. Let $\phi : S^n \to D^n$ be $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$ and $\phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sqrt{1 - x_1^2 + \cdots + x_n^2})$, these are homeomorphisms of hemispheres with the n-disk. Then $G = \phi^{-1} \circ \widetilde{F} \circ \phi$. This is a degree-0 map, so $\widetilde{F} \circ \phi)(x) = \phi(x)$ for some x, meaning $F(\phi(x))/||F(\phi(x))|| = \phi(x)$ for some $\phi(x) \in D^n$. Since the LHS has norm-1, $\phi(x) \in \partial D^n$. Moreover, $\widetilde{F} \circ \phi)(y) = \phi(-y) = -\phi(y)$ for some $\phi(y) \in \partial D^n$. We have our inward/outward vectors on the boundary.

Problem I.12 (Hatcher 2.2.9)**.** Point-by-point, motherfuckaaaa

- This space is homotopy equivalent to $S^2 \vee S^1$, which has $H_0 = 0$ and $H_n(S^2 \vee S^1) \simeq H_n(S^2) \oplus H_n(S^1)$ for $n > 0$.
- Must use cellular homology. Clearly, this is a 2-dimensional cell complex, so $H_n(X) = 0$ for $n > 2$. Since we have a single 0-cell and the space is path-connected, $H_0(X) = \mathbb{Z}$ and $d_1 = 0$. The map d_2 connects two 2-cells via path $\alpha\beta\overline{\alpha}\overline{\beta}$ and $\gamma\beta\overline{\gamma}\overline{\beta}$, so the cellular boundary formula implies $d_2 = 0$. Thus, $H_2(X) = \mathrm{Ker}(d_2) = \mathbb{Z}^2$ and $H_1(X) = \mathrm{Ker}(d_1)/\mathrm{Im}(d_2) = \mathbb{Z}^3$.
- See homework, LETS GO!
- Attaching map is trivial, so the homology is the same as the torus.

Problem I.13 (Hatcher 2.2.30)**.** BRUH REVISIT

Problem I.14 (Hatcher 2.2.31)**.** See homework

II. Problem set 4

Problem II.1 (Hatcher 2.2.4)**.** In the case of $n = 1$, define f as the map which reflects the lower semi-circle across the x-axis, and fixed the upper semi-circle. Let $g(z) = z^2$. Then $g \circ f = h$ is a surjection, and $\deg(h) = 0$ as $\deg(f) = 0$, since it is not surjective. To get all of our maps for $n > 1$, take the k-fold suspension $S^k f : S^k S^1 \to S^k S^1$, which is a surjective map of degree-0 from S^{k+1} to S^{k+1}.

Problem II.2 (Hatcher 2.2.6)**.** If f has no fixed point, it is homotopic to the antipodal map via F_t. Let $r_t(x) = (\cos(\pi t)x_0 + \sin(\pi t)x_1, \cos(\pi t)x_1 - \sin(\pi t)x_0, x_2, x_3, \ldots, x_n \ldots)$. Note that $r_0 = \mathrm{id}$. Let $G_t = F_t \circ r_t$. We have $G_0 = f$ and $G_1 = F_1 \circ r_1 = a \circ r_1$. This map takes $(1, 0, \ldots, 0)$ to $(-1, 0, \ldots, 0)$, which is taken by a to $(1, 0, \ldots, 0)$, so G_1 fixes this point.

Problem II.3 (Hatcher 2.2.10)**.** See homework

Problem II.4 (Hatcher 2.2.12)**.** See homework

Problem II.5 (Hatcher 2.2.15)**.** REVISIT

Problem II.6 (Hatcher 2.2.19)**.** In this case, X is a CW-complex with 1 cell in dimensions n, $n-1$, $\vdots m+1$, and 0. In this case, the attaching map of e^{m+1} will amount to collapsing the boundary down to a single point, and will thus have 0-degree. The other attaching maps are unchanged (they are still 2-sheeted coverings, except along an equator). Thus, the attaching map of cell e^k, with $\partial e^k = S^{k-1}$ will still have degree $1 + (-1)^k$, for $k = m + 2$ and greater.

It follows the groups will alternate between being $\mathbb{Z}_2$ and 0. When m is even, H_{m+1} will be $\mathbb{Z}_2$, if m is odd, then it will be $\mathbb{Z}$. The rest of the groups remain unchanged.

Problem II.7 (Hatcher 2.2.20)**.** Note that if X has cells $e_\alpha^{n_\alpha}$ and Y has cells $e_\beta^{n_\beta}$, then $X \times Y$ has cells $e_{\alpha,\beta}^{n_\alpha + n_\beta} = e_\alpha^{n_\alpha} \times e_\beta^{n_\beta}$. Thus, the number of k-cells of $X \times Y$, $N_k(X \times Y)$ is given by $\sum_{(e_\alpha^{n_\alpha}, e_\beta^{n_\beta}),\ n_\alpha + n_\beta = k} N_{n_\alpha}(X) N_{n_\beta}(Y)$. We then have

$$\xi(X \times Y) = \sum_k (-1)^k N_k(X \times Y) = \sum_k \sum_{(e_\alpha^{n_\alpha}, e_\beta^{n_\beta}),\ n_\alpha + n_\beta = k} (-1)^{n_\alpha} (-1)^{n_\beta} N_{n_\alpha}(X) N_{n_\beta}(Y) \tag{5}$$

which is exactly the product of the Euler characteristics of the individual spaces.

Problem II.8 (Hatcher 2.2.26). See homework

Problem II.9 (Hatcher 2.2.28). For Part A, see homework!. For Part B, we can take a similar approach. As A, take the Mobius band plus a small neighbourhood of the boundary circle extending into $\mathbb{R}P^2$, similar for B being B plus a little bit. The intersection deformation retracts to a circle. The map from this circle to the Mobius band sends $1 \mapsto 2$, while in $\mathbb{R}P^2$, this circle is simply the generator $1 \in \mathbb{Z}_2$.

Thus, $H_3 = 0$, The map $H_2(X) \mapsto H_1(S^1)$ has trivial kernel, and since $H_1(S^1) \to H_1(M) \oplus H_1(\mathbb{R}P^2)$ has trivial kernel, it is an isomorphism, so $H_2(X) = \mathbb{Z}$. Moreover, $H_1(M) \oplus H_1(\mathbb{R}P^2) \to H_1(X) \to 0$ is surjective, so $H_1(X)$ is isomorphic to $H_1(M) \oplus H_1(\mathbb{R}P^2) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ with the kernel moded out, which is precisely the image of $1 \mapsto (2,1)$. Thus, the resulting group is $\langle (1,0),(0,1)\rangle / \langle (2,1)\rangle$ (where the second 1 is in $\mathbb{Z}_2$.

Note that $(2,0) = (2,1) + (0,1)$, so it follows that any equivalence class in the quotient is represented by $(0,0)$, $(1,0)$, $(0,1)$, or $(1,1)$. One can verify via checking the addition between these elements that the resulting group is isomorphic to $\mathbb{Z}_4$.

Problem II.10 (Hatcher 2.2.29). See homework

Problem II.11 (Hatcher 2.2.32). See homework

Problem II.12 (Hatcher 2.2.36). Obviously S^n retracts to a point, so $X \times S^n$ retracts to X. Thus, from the split exact sequence, $H_i(X \times S^n) \simeq H_i(X) \oplus H_i(X \times S^n, X)$. To see the latter isomorphism, note that we can break up $X \times S^n$ into the union of $X \times D^n_\pm$, where each of the $D^n_\pm$ is S^n minus the north/south pole, respectively. $X \times D^n_\pm$ deformation retracts to $X \times x_0$ in both cases, so the relative Mayer-Vietoris sequence yields $0 \to H_i(X \times S^n, X) \to H_{i-1}(X \times S^{n-1}, X) \to 0$, as the intersection of $D^n_\pm$ is S^{n-1}.

The isomorphism follows by induction. Note that if $i < n$, then in our induction, we eventually reach a step where the latter group is 0.

III. Problem set 5

Problem III.1 (Hatcher 2.B.3). As is suggested by the problem, we glue copies of (D^n, D) to the ends of $(S^{n-1} \times I, S \times I)$, to produce a pair (S^n, S'), where S' is homeomorphic to S^k. REVISIT

Problem III.2 (Hatcher 2.C.2). We will compute the Lefschetz number (showing it is non-zero). The homology is S^n is trivial in all dimensions except 0 and n, and the induced map in the 0-th dimension is the identity as the space is path-connected, so $\tau(f) = (-1)^n \mathrm{Tr}(f_* : H_n) + 1$. It follows that the map will have a fixed point unless $\mathrm{Tr}(f_* : H_n) = (-1)^{n+1}$ (the degree of the antipodal map).

By definition, $\mathrm{Tr}(f_*) = \deg(f)$ in this context, so we have proved the result.

Problem III.3 (Hatcher 2.C.5). Lefschetz number does not change under homotopy, and no fixed points implies Lefschetz number 0, so this is possibly only when r has Lefschetz number 0. Recall that the homology of M is $0 \to \mathbb{Z} \to \mathbb{Z}^{2g} \to \mathbb{Z} \to 0$. REVISIT

Problem III.4 (Hatcher 3.1.5). REVISIT

Problem III.5 (Hatcher 3.1.9). REVISIT

Problem III.6 (Hatcher 3.2.7). REVISIT

IV. Other stuff