MAT437 problem set 9

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I. RLL Problem 8.2

Recall that $K_1(C(X)) \simeq K_0(SC(X))$, where $SC(X)$ is the set of suspensions over $C(X)$, namely,

$$SC(X) = \{f \in C([0,1], C(X)) \mid f(0) = f(1) = 0\} \tag{1}$$

where 0 is the zero-function.

Lemma I.1. Let A and B be topological spaces with B compact. Let R be a C^*-algebra. Then, as topological spaces with the uniform metric, $C(A, C(B, R))$ and $C(A \times B, R)$ are homeomorphic. In fact, they are $*$-isomorphic as C^*-algebras.

Proof. Let $\Phi : C(A, C(B, R)) \to C(A \times B, R)$ be the map such that $\Phi(f)(a, b) = f(a)(b)$. We need to show that this map is actually well-defined. Indeed, suppose f is continuous in $C(A, C(B))$. We need to show that $\Phi(f)$ is continuous.

Let W_ε be the ε-ball around $f(a) \in C(B)$. Of course, since $f : A \to C(B)$ is continuous, we can pick open U in A around a such that $f(U) \subset W_\varepsilon$. In particular, this means that for any $x \in U$ and $f(x) \in f(U) \subset W_\varepsilon$, we have $\sup_{y \in B} ||f(x)(y) - f(a)(y)|| < \varepsilon$. We then pick V open in B such that $f(a)(V) \subset (f(a)(b) - \varepsilon, f(a)(b) + \varepsilon)$. Thus, if we pick some $(x, y) \in U \times V$, then

$$||\Phi(f)(x, y) - \Phi(f)(a, b)|| = ||f(x)(y) - f(a)(b)|| \leq ||f(x)(y) - f(a)(y)|| + ||f(a)(y) - f(a)(b)|| \leq 2\varepsilon \tag{2}$$

Since we can do this for arbitrary $\varepsilon > 0$, $\Phi(f)$ must be a continuous function. Conversely, define $\Phi^{-1}(f)(a)(b) = f(a, b)$. It is clear that $\Phi^{-1}(f)(a)$ is an element of $C(B)$, as it is the restriction of a continuous function to one input variable. All that remains is to show that $\Phi^{-1}(f) : A \to C(B)$ is continuous.

For fixed a, if we choose for each point (a, b) for some $b \in B$ some $U_b \times V_b$ which is mapped by f into $B_\varepsilon(f(a, b))$ (the ε-ball about $f(a, b)$), then since B is compact, we can take a finite subcover of V_b, and intersect all of the U_b associated to get an open set U such that for some $x \in U$ and any $b \in B$, $||f(x, b) - f(a, b)|| < \varepsilon$. Thus, for $x \in U$,

$$||\Phi^{-1}(f)(x) - \Phi^{-1}(f)(a)||_\infty = \sup_{b \in B} ||f(x, b) - f(a, b)|| \leq \varepsilon \tag{3}$$

so it follows immediately that $\Phi^{-1}(f)$ is continuous.

It is obvious that Φ and Φ^{-1} are inverses of each other. Clearly, Φ is linear and preserves multiplication. Moreover, $\Phi(f^*)(x, y) = f(x)(y)^* = \Phi(f)(x, y)^*$, with the same of course holding true for Φ^{-1}. Thus, we do in fact have a well-defined $*$-isomorphism between the two spaces. □

Now, it is very clear that the subset of $f \in C([0,1], C(X))$ for which $f(0) = f(1) = 0$ is a sub-C^*-algebra of $C([0,1], C(X))$. It is also clear that this function will be in bijective correspondence with the functions $g \in C([0,1] \times X, R)$ for which $g(0, x) = 0$ and $g(1, x) = 0$ for all $x \in X$. Clearly, this is also a sub-C^* algebra, this time of the algebra $C([0,1] \times X)$.

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From this result, it follows immediately that for compact Hausdorff space X,

$$\begin{aligned} SC(X) \simeq \{f \in C([0,1] \times X) \mid f(0,x) = f(1,x) = 0 \ \text{ for all } \ x \in X\} := A \\ \Longrightarrow K_1(C(X)) \simeq K_0(SC(X)) \simeq K_0(A) \end{aligned} \tag{4}$$

Of course, $[0,1] \times X$ is compact Hausdorff contractible. But we already know that for compact Hausdorff contractible space Y, $K_0(C(Y)) = 0$ (this was proved in an earlier chapter). Thus, $K_0(C([0,1] \times X)) = 0$. Since A is a sub-$*$-algebra, we must also have $K_0(A) = 0$.

II. RLL Problem 8.3

This problem will carry forward similarly to the previous problem. In particular, let us make note of the fact that $K_1(CA) \simeq K_0(SCA)$. Note that via the above proof (in the previous problem),

$$SCA \simeq F = \{f : [0,1] \times [0,1] \to A \mid f(0,t) = 0 \quad \text{and} \quad f(x,0) = f(x,1) = 0\} \tag{5}$$

Of course, the right-hand side is a sub-C^*-algebra of $C([0,1] \times [0,1] \to A)$. Since $[0,1] \times [0,1]$ is compact Hausdorff contractible, the K_0-group of the space of fucntion is 0. Thus, $K_0(F) = 0$, and $K_1(CA) = 0$ as well.

III. RLL Problem 8.5

Let us recall the identification of Proposition 8.1.6 of RLL. Namely, when we have a unital C^*-algebra A, then there exists an isomorphism $\rho : K_1(A) \to \mathcal{U}_\infty(A)/\sim_1$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{U}_\infty(\widetilde{A}) \arrow[r, "\mu"] \arrow[d, "{[\cdot]_1}"'] & \mathcal{U}_\infty(A) \arrow[d, "{[\cdot]_1}"] \\ K_1(A) \arrow[r, "\rho"'] & \mathcal{U}_\infty(A)/\sim_1 \end{tikzcd}$$

where $\mu(a + \alpha(1_{\widetilde{A}} - 1_A)) = a$ is the projection map. Now, consider $\varphi : A \to B$. We can extend φ to a map $\widetilde{\varphi} : \mathcal{U}_\infty(\widetilde{A}) \to \mathcal{U}_\infty(\widetilde{B})$ between the matrix algebras of the unitizations, in the usual, element-wise way. In fact, we have $K_1(\varphi)([u]_1) = [\widetilde{\varphi}(u)]_1$ for $u \in \mathcal{U}_\infty(\widetilde{A})$, as was shown in RLL. Pick $u \in \mathcal{U}_n(\mathcal{A})$. Let $1_{\widetilde{A}}$ be the unit in $\widetilde{A}$. Let 1_A be the unit in A. Note that Note that

$$\varphi(u + (1_{\widetilde{A}} - 1_A)) = u \quad \text{and} \quad (u + (1_{\widetilde{A}} - 1_A))^*(u + (1_{\widetilde{A}} - 1_A)) = u^*u - 1_A + 1_{\widetilde{A}} = 1_{\widetilde{A}} \tag{6}$$

so $u + (1_{\widetilde{A}} - 1_A)$ is unitary in $\widetilde{A}$, the unitization. Via the above commutative diagram, we identify $[u]_1$ (one the right-hand side) with $[u + (1_{\widetilde{A}} - 1_A)]_1$ (on the left-hand side). We then have

$$K_1(\varphi)([u + (1_{\widetilde{A}} - 1_A)]_1) = [\widetilde{\varphi}(u + (1_{\widetilde{A}} - 1_A))]_1 = [\varphi(u) + 1_{\widetilde{A}} - \varphi(1_A)]_1 \tag{7}$$

To find the equivalence class of $\mathcal{U}_\infty(A)/\sim_1$ to which this element of the K_1-group corresponds, we map via ρ, which via the above diagram, is equivalent to mapping the unitary representative $\varphi(u) + 1_{\widetilde{A}} - \varphi(1_A)$ via μ, and then by $[\cdot]_1$ (the projection on the right). We have

$$\mu(\varphi(u) + 1_{\widetilde{A}} - \varphi(1_A)) = \mu(\varphi(u) - \varphi(1_A) + 1_A + (1_{\widetilde{A}} - 1_A)) = \varphi(u) + 1_A - \varphi(1_A) \tag{8}$$

so the corresponding equivalence class is $[\varphi(u) + 1_A - \varphi(1_A)]_1$, as desired. Thus, if φ is unital, then the equivalence class will be $[\varphi(u)]_1$, and the $*$-homomorphism will respect the structure of the unitization.

IV. RLL Problem 8.8

Part 1. Let $\alpha \in \mathrm{Inn}(A)$, so that $\alpha(x) = uxu^{-1}$ for some unitary $u \in A$. Of course, we have $K_1(\alpha)([v]_1) = [\alpha(v)]_1 = [uvu^{-1}]_1$. From Whitehead lemma, we have

$$\begin{pmatrix} uvu^{-1} & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu^{-1} & 0 \\ 0 & u \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

Therefore, $uvu^{-1} \sim_1 v$, so $[uvu^{-1}]_1 = [v]_1$. Therefore, $K_1(\alpha) = \mathrm{id}$.

Part 3. Let us pick $\alpha \in \mathrm{Aut}(A)$. Clearly, $K_1(\alpha) : K_1(A) \to K_1(A)$ is group homomorphism. We need to show that this homomorphism is an isomorphism. In particular, note that if $K_1(\alpha)([u]_1) = [1]_1$, then $[\alpha(u)]_1 = [1]_1$ so that $\alpha(u) \sim_1 1$. Since α is invertible and unit preserving, and we know that $\alpha(u) \oplus 1_m \sim_h 1$ via homotopy f, then the homotopy $\alpha^{-1}(f(t))$ will connect $u \oplus 1_m$ and $\alpha^{-1}(1) = 1$, so that $[u]_1 = [1]_1$. It follows that $K_1(\alpha)$ must be an injection. In additin, it is obviously a surjection as $K_1(\alpha)([\alpha^{-1}(u)]_1) = [u]_1$ for all $[u]_1 \in K_1(A)$. Thus, $K_1(\alpha) \in \mathrm{Aut}(K_1(A))$ as desired.

Part 4. Of course, we showed that $K_1(\alpha) \in \mathrm{Aut}(K_1(A))$. To show that this is a group homomorphism, we require that $K_1(\alpha \circ \beta) = K_1(\alpha) \circ K_1(\beta)$. In particular,

$$K_1(\alpha \circ \beta)([u]_1) = [(\alpha \circ \beta)(u)]_1 = K_1(\alpha)([\beta(u)]_1) = (K_1(\alpha) \circ K_1(\beta))([u]_1) \tag{10}$$

and the proposition is proved.