Rough notes

(Dated: Thursday 6^{th} June, 2024)

Contents

I. Basic constructs

We begin with some basic constructs utilized throughout the notes.

A. Lie groupoids and Lie algebroids

Definition I.1 (Groupoid)**.** A category in which all arrows are reversible.

Definition I.2 (Lie groupoid)**.** A groupoids in which the arrows form a smooth manifold, the objects are an embedded submanifold under the map sending each object to its identity arrow, and the source and target maps are surjective submersions.

Definition I.3 (Lie algebroid)**.** A vector bundle $\pi_A : A \to M$ over a smooth manifold M, equipped with a bracket $[\cdot,\cdot]$ on the sections $\Gamma(A)$ and an anchor map $\rho : A \to TM$. The anchor map is a vector bundle morphism, meaning that $\pi_A = \pi \circ \rho$. Moreover, the bracket must satisfy a Leibniz rule:

$$[X, fY] = \mathcal{L}_{\rho(X)}(f) \cdot Y + f[X, Y] \tag{1}$$

where $\mathcal{L}_{\rho(X)}$ is the Lie derivative with respect to the vector field $\rho(X) \in \mathfrak{X}(M)$.

Remark I.1. I very much like Abad and Crainic's assertion in their paper *Representations up to homotopy of Lie algebroids* that Lie algebroids are to be thought of as "generalized tangent bundles associated to various geometric situations".

Example I.1 (Tangent bundle)**.** The most obvious example of a Lie algebroid is TM itself. The anchor is the identity map and the bracket is the standard Lie bracket between smooth vector fields $X, Y \in \mathfrak{X}(M)$. Indeed,

$$[X, fY] = \mathcal{L}_X(fY) = X(f) \cdot Y + f\mathcal{L}_X(Y) = X(f) \cdot Y + f[X, Y] \tag{2}$$

as required.

Example I.2 (Lie algebra)**.** One should think of a Lie algebroid as a generalized Lie algebra, in which the vector space is replaced with a vector bundle over a base space and the bracket of vectors is replaced by a bracket of sections of the bundle.

It is in this sense that a Lie algebra is a Lie algebroid: if $\mathfrak{g}$ is a Lie algebra, take $A = \mathfrak{g} \times M$ for some M, let $\pi : A \to M$ be the projection, let the anchor be trivial, $\rho(e) = 0_{\pi(e)} \in T_{\pi(e)}M$ for all e, and let the bracket between sections be defined as $[X, Y](p) = [X(p), Y(p)]_{\mathfrak{g}}$, where $[\cdot, \cdot]_{\mathfrak{g}}$ is the bracket of the Lie algebra. A is then a Lie algebroid.

Much like the case of Lie groups/Lie algebras, if given a Lie groupoid G, we can product a Lie algebroid, $\mathrm{Lie}(G)$. This construction is somewhat analogous to the Lie group to Lie algebra map, where we take the tangent space at the identity. For a Lie groupoid, each $p \in M$ (the base manifold of objects) has an associated identity in G (the self-referential arrow), e_p. TODO: revisit, stress that the main difficulty comes from constructing the bracket

B. Some categorical constructions

Categorical language is required for many of the results we wish to discuss. We review briefly.

Definition I.4 (Simplicial sets, face and degeneracy maps)**.** Let Δ be the category of sets $[n] = \{0, \ldots, n\}$ for $n \geq 0$ (the simplex category). The morphisms are non-order-decreasing set maps. A simplicial set is a contravariant functor $X : \Delta \to \mathtt{Set}$. We denote $X([n])$ by X_n. Corresponding to simplicial set X, we have face maps for each image X_n, defined by $d_i : X_n \to X_{n-1}$ for i from 0 to n. In particular, we let $\delta_i : [n-1] \to [n]$ be the unique non-decreasing map which doesn't hit $i \in [n]$. We then let $d_i = X(\delta_i)$. Similarly, if we let $\sigma_i : [n+1] \to [n]$ be the unique non-decreasing map hitting i twice and let $s_i = X(\sigma_i)$ be a map from X_n to X_{n+1}, we call these the degeneracy maps.

Remark I.2. The face and degeneracy maps will satisfy certain compatibility conditions which follow immediately from the definitions and the contravariance of X:

1. $d_i d_j = d_{j-1} d_i$ for $i < j$.
2. $d_i s_j = s_{j-1} d_i$ for $i < j$.
3. $d_i s_j = \mathrm{id}$ if $i = j$ or $i = j + 1$.
4. $d_i s_j = s_j d_{i-1}$ for $i > j + 1$.
5. $s_i s_j = s_{j+1} s_i$ for $i \leq j$.

It is in fact true that if given a collection of sets X_n and maps $d_n : X_n \to X_{n-1}$ and $s_n : X_n \to X_{n+1}$ which satisfy the above conditions, then there is a unique corresponding simplicial set.

Definition I.5 (Nerve of a category)**.** Let C be a small category. The nerve of C, NC, is a simplicial set which is constructed as follows. Let $X_0 = \mathrm{ob}(C)$, $X_1 = \hom(C)$, and more generally, X_n is the collection of all n-fold compositions of arrows in the category. Each of the X_n is an object of $\mathtt{Set}$. We define maps d_i and s_i on each X_n as follows: d_i takes a chain of n arrows, $A_0 \to \cdots \to A_n$ and composes the arrows going in and out of A_i into a single arrow, leaving a chain of $n - 1$ arrows. s_i takes $A_0 \to \cdots \to A_n$ and adds a self-referential arrow at the i-th object, yielding a chain of $n + 1$ arrows. One can easily verify that these maps satisfy the conditions of the previous remark, so we let NC be the unique simplicial set having d_i and s_i as face and degeneracy maps.

Why do we do this? Because simplicial sets have a nice associated homotopy theory, and our hope is that we can say things about C by looking at homotopy of NC (as it turns out, we can!).

Definition I.6 (Natural transformation)**.** A natural transformation between functors $F, G : C \to D$ (both covariant or contravariant) is a family of morphisms $\eta_X : F(X) \to G(X)$ for all $X \in C$, such that if $f : X \to Y$ is in $\hom(C)$, then $\eta_Y \circ F(f) = G(f) \circ \eta_X$ for covariant functors and $\eta_X \circ F(f) = G(f) \circ \eta_Y$ for contravariant functors.

C. Representations up to homotopy preliminaries

We will soon begin our discussion of representations up to homotopy. Let us build to this by describing rerpesentations in a series of increasing generality.

> *Categories of representations are like how you can't see the face of God, you can only see Him through all of His actions.*

If you have a group or an algebra, its category of representations are all the ways that it can act on different things (vector spaces).

Definition I.7 (Lie algebra representation)**.** Let A be a Lie algebra with bracket $[\cdot,\cdot]_A$. A Lie algebra representation on vector space V is the pair (V,ρ) where $\rho : A \to \mathrm{End}(V)$ is a linear map such that

$$\rho([g,h]_A) = [\rho(g),\rho(h)] \tag{3}$$

for all $g,h \in A$ and $[\cdot,\cdot]$ is the standard commutator of linear maps on V.

Now, as an interlude, let us say some things about connections and parallel transport. This will motivate the definition of a Lie algebroid representation.

Definition I.8 (Connection)**.** Let $A \to M$ be a Lie algebroid, let E be a vector bundle over M. An E-connection relative to A is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(a,e) \mapsto \nabla_a e$ such that $\nabla_{fa} e = f\nabla_a e$ for all $f \in C^\infty(M)$, and

$$\nabla_a fe = \mathcal{L}_{\rho(a)}(f)\cdot e + f\nabla_a e \tag{4}$$

where ρ is the anchor of A. A connection is said to be flat if $\nabla_{[a,b]} = [\nabla_a,\nabla_b]$. Observe that fixing $a \in \Gamma(A)$, ∇_a is a linear map from $\Gamma(E)$ to itself satisfying the Leibniz rule of the above formula. In this sense, the flatness condition is very much a generalization of Eq. (3) in the definition of a Lie algebra representation.

Example I.3. In the case that $A = TM$, we recover the definition of a connection on smooth manifold M.

Example I.4. Suppose $A = \mathfrak{g} \times M$ for some Lie algebra $\mathfrak{g}$. Suppose (V,ρ) is a Lie algebra representation. Let $E = V \times M$ be the trivial bundle. Define $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ as $\nabla_a e = \rho(a)\cdot e$. Clearly, this map is bilinear, and is $C^\infty(M)$-linear in a and e. It follows since $\rho = 0$ that the required conditions are satisfied and ∇ defined directly from ρ is a connection. In fact, it is a flat connection, as

$$\nabla_{[a,b]} = \rho([a,b]) = [\rho(a),\rho(b)] = [\nabla_a,\nabla_b]. \tag{5}$$

A connection gives us the machinery required to talk about parallel transport and holonomy in a vector bundle. We begin by restricting our attention to connections on TM. In particular, given some E-connection ∇ on TM, suppose γ is a path in M. Let $E_{\gamma(0)}$ and $E_{\gamma(1)}$ denote the fibres over the endpoints. Our goal is to define an isomorphism between the two fibres.

Definition I.9. We say that a section $\sigma \in \Gamma(E)$ is flat relative to ∇ along the path γ if $\nabla_{\dot\gamma(t)}(\sigma)(\gamma(t)) = 0$ for all t.

Claim I.1. Given ∇ and curve γ, there exists a unique flat section $\sigma \in \Gamma(E)$ relative to ∇ along γ.

We need a local form for ∇:

Remark I.3. TODO: Should probably revisit and make this explanation a bit more clear/sequential Let ∇ be an E-connection (where we assume E to be an n-dimensional real vector bundle) on TM. Let $(U_\alpha,\varphi_\alpha)$ be the local trivialization of E, so $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ is a homeomorphism. Then if $\sigma \in \Gamma(E)$, we have $\varphi_\alpha \circ \sigma|_{U_\alpha} : U_\alpha \to U_\alpha \times \mathbb{R}^n$ is well-defined, and is a section of the trivial bundle $U_\alpha \times \mathbb{R}^n$ over U_α, as

$$\mathrm{proj} \circ (\varphi_\alpha \circ \sigma|_{U_\alpha}) = \pi \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \sigma|_{U_\alpha} = \pi \circ \sigma|_{U_\alpha} = \mathrm{id}|_{U_\alpha}. \tag{6}$$

It is equally easy to show that if $\sigma \in \Gamma(U_\alpha \times \mathbb{R})$, then $\varphi_\alpha^{-1} \circ \sigma$ is in $\Gamma(\pi^{-1}(U_\alpha))$. Let us consider for some fixed $v \in \mathfrak{X}(M)$ the connection relative to v on the trivial bundle, $\nabla_v|_{U_\alpha} \circ \varphi_\alpha^{-1}$. Note that ∇_v is defined on $\Gamma(E)$, the global sections. However, note that we can define a smooth partition of unity vanishing outside U_α, implying we may extend any $\sigma \in \Gamma(\pi^{-1}(U_\alpha))$ to a global section $\widetilde{\sigma}$. In addition, we may prove that ∇_v is a local operation: if σ and σ' are in $\Gamma(E)$ and are the same on a neighbourhood U, then for $p \in U$, there is a neighbourhood V of p, with $p \in V \subset U$ such that $\nabla_v(\sigma)(p) = \nabla_v(\sigma')(p)$. To prove this, note that since M is a smooth manifold, we can always take V to be a coordinate ball with closure inside U. We let χ be a smooth bump function equal to 1 on $\overline{V}$ and vanishing outside U. We have $\chi\sigma = \chi\sigma'$. Then, for any $q \in V$,

$$\nabla_v(\chi\sigma)(q) = \mathcal{L}_v(\chi)(q) \cdot \sigma(q) + \nabla_v(\sigma)(q) \tag{7}$$

as well as

$$\nabla_v(\chi\sigma')(q) = \mathcal{L}_v(\chi)(q) \cdot \sigma(q) + \nabla_v(\sigma)(q) \tag{8}$$

which implies that $\nabla_v(\sigma)(q) = \nabla_v(\sigma')(q)$. It follows that $\nabla_v|_U$ is well-defined, and can easily be seen to be a connection on $\Gamma(\pi^{-1}(U))$.

The map $\nabla_v|_U \circ \varphi^{-1}$ is from $\Gamma(U \times \mathbb{R}^n)$ to $\Gamma(U)$. Note that any $\sigma \in \Gamma(U \times \mathbb{R}^n)$ is necessarily of the form $p \mapsto (f_1(p), \ldots, f_n(p)) = f_1(p)e_1 + \cdots + f_n(p)e_n$. From linearity, we only need to know the action of $\nabla_v \circ \varphi^{-1}$ on $f_k \cdot e_k$. Of course,

$$(\varphi^{-1} \circ (f_k e_k))(p) = \varphi^{-1}(p, f_k(p)e_k) = f_k(p)\varphi^{-1}(p, e_k) \tag{9}$$

so that $\varphi^{-1} \circ f_k e_k = f_k \cdot \varphi^{-1}(\cdot, e_k)$. It follows that

$$(\nabla_v|_U \circ \varphi^{-1})(f_k e_k) = \nabla_v(f_k \cdot \varphi^{-1}(\cdot, e_k)) = \mathcal{L}_v(f_k) \cdot \varphi^{-1}(\cdot, e_k) + f_k \cdot \nabla_v \varphi^{-1}(\cdot, e_k) \tag{10}$$

We define A_{jk} by

$$\nabla_v \varphi^{-1}(\cdot, e_j)(p) = \sum_k A_{jk}(p)\varphi^{-1}(p, e_k) \tag{11}$$

It follows immediately that we can write $\nabla_v \circ \varphi^{-1}$ as $\mathcal{L}_v + A$, where we take the Lie derivative of the components of σ, and A is a linear function depending on p. The condition of flatness along some curve γ immediately reduces, locally, to solving a linear ODE. Local existence and uniqueness of solutions to ODEs of this form gives us the existence of uniqueness of a flat section TODO: elaborate.

Definition I.10 (Parallel transport operation)**.** We define the parallel transport operation of ∇ along γ to be the isomorphism between $E_{\gamma(0)}$ and $E_{\gamma(1)}$ induced by the flat section. TODO: prove this is an isomorphism

Now, let us consider the added condition that ∇ is a flat connection.

Claim I.2. If ∇ is a flat connection, then $P_\gamma = P_{\gamma'}$ for γ and γ' which are path-homotopic.

Proof. TODO: Spend some time trying to fill in this proof, seems probably of non-trivial difficulty. ☐

Definition I.11 (Lie algebroid representation)**.** We replace (V, ρ) with (E, ∇), where E is a vector bundle over M, the base manifold of the Lie algebroid $A \to M$, and ∇ is a flat connection.

Remark I.4. Just as a Lie algebra representation gives us a map from the Lie algebra to operators on some vector space, the Lie algebroid representation gives a map ∇ which takes a section $\Gamma(A)$ and induces a map from $\Gamma(E)$ to itself. Moreover, such a map induces isomorphisms between all fibres of the vector bundle E via parallel transport.

II. Representations up to homotopy

Now, let us begin our discussion of representation up to homotopy.

Remark II.1. Again drawing from Abad and Crainic, we note the main idea of representation up to homotopy: *"the idea is to represent Lie algebroids in cochain complexes of vector bundles, rather than in vector bundles"*.

Note that given a Lie algebroid A, since it is a vector bundle, there is an associated De Rham complex $\Omega^\bullet(A) = \Gamma(\wedge^\bullet A^*)$ where the differential operator $d_A : \Omega^k(A) \to \Omega^{k+1}(A)$ defined by the Koszul formula

$$d_A\omega(\alpha_1, \ldots, \alpha_{k+1}) = \sum_{i<j}(-1)^{i+j}\omega([\alpha_i, \alpha_j]_A, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_{k+1}) \tag{12}$$

$$+ \sum_i (-1)^i \mathcal{L}_{\rho(\alpha_i)}\omega(\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_{k+1}) \tag{13}$$

where each $\alpha_k \in \Gamma(A)$. One can easily check that in the case $A = TM$, d_A is the standard exterior derivative and we are left with the standard De Rham cohomology.

Remark II.2. Note that this formula is actually a valid definition of a differential form, which takes elements of the vector bundle as arguments (not necessarily vector fields). One can show that given $\alpha_1, \ldots, \alpha_{k+1}$ in some fibre A_p, then any extension to smooth vector fields $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{k+1}$ will yield the same value $d_A\omega(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{k+1})_p$.

Let us also make note of the fact that if ∇ is an E-connection for A, then we can define *another* operator d_∇ on $\Omega^\bullet(A, E) = \Gamma(\wedge^\bullet A^* \otimes E)$ given by a similar formula

$$d_\nabla\omega(\alpha_1, \ldots, \alpha_{k+1}) = \sum_{i<j}(-1)^{i+j}\omega([\alpha_i, \alpha_j]_A, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_{k+1}) \tag{14}$$

$$+ \sum_i (-1)^i \mathcal{L}_{\rho(\alpha_i)}\omega(\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_{k+1}) \tag{15}$$

Note that d_∇ satisfies the Leibniz rule always, but squares to 0 only when ∇ is flat.

Definition II.1 (Representation up to homotopy)**.** A representation up to homotopy of A on a graded vector bundle E is a differential operator $D : \Omega^k(A, E) \to \Omega^{k+1}(A, E)$ (where degree is computed by taking the sum of the degrees in $\wedge^\bullet A^*$ and $E^\bullet$). In particular, it satisfies

$$D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (D\beta) \tag{16}$$

as well as $D^2 = 0$.

If (E, ∇) is a representation of Lie algebroid A, so ∇ is flat, is (E, d_∇) a representation up to homotopy? I still don't feel like I really appreciate this definition. What does this have to do with deformation cohomology, I keep seeing this in other papers...

Now we move in to the contents of the paper, *The A_∞ de Rham theorem and integration of representations up to homotopy*. We start by defining a representation up to homotopy. Going to have to revisit after I get a better grasp of what representations up to homotopy are.

III. Chen's iterated integral and the Igusa map

Sec. 3 of Ref. [**?**] begins by stating its intention to construct an A_∞-quasi-isomorphism of DG-algebras, from $(\Omega(M), -d, \wedge)$ to $(C(M), \delta, \cup)$. We start by defining the *bar complex* as the graded algebra of DG-algebra $(A, d, \wedge)$, given by

$$B(sA) = \bigoplus_{k\geq 1}(sA)^{\otimes k} \tag{17}$$

where sA is the suspension: it shifts the index of the grading of A by 1 forward, so $(sA)^k = A^{k+1}$. $B(sA)$ is graded because

$$(a_1 \otimes \cdots \otimes a_j) \otimes (b_1 \otimes \cdots \otimes b_k) \in (sA)^{\otimes(j+k)}. \tag{18}$$

clearly. Why do we even need the suspension? Note that the bar complex carries coboundary D (formula in the paper).

Definition III.1 (Path space)**.** Let M be a smooth manifold, the path space PM is $C^\infty([0,1], M)$ equipped with the C^1-topology. We say that a map $f : X \to PM$ is smooth if $(t, x) \mapsto f(x)(t)$ is smooth. The C^1-topology is defined as follows: we take the coarsest topology making the map from $C^\infty([0,1], M)$ to $C(T[0,1], TM) = C([0,1] \times \mathbb{R}, TM)$ with the compact-open topology, which sends $f \mapsto f_*$, continuous. Note that the compact-open topology is the topology on a function space where we take the collection of all $V(K, U)$, sets of functions from a compact subset to an open subset, as a subbasis for the topology.

At a high-level, we want the differential to be the piece of data which determines the topology. The compact-open topology for the codomain is also a natural choice. why? After some investigation, it appears as though path space doesn't really have a natural smooth manifold structure. This means that in order to define a "differential form on path space", we will have to take a different approach.

Definition III.2 (Forms on PM)**.** The idea with defining a differential form on PM is by specifying its "pull-back" to every smooth manifold under every possible smooth map $f : X \to PM$. To be more specific, we have data which associated a differential form $f^*\alpha$ on X with a pair (X, f): this allows us to "recover α", which can be thought of as the differential form on PM.

To be more precise, the paper takes $C^\infty(-, PM)$ to be the category of all (X, f) where X is a finite dimensional smooth manifold and $f : X \to PM$ is smooth (with smoothness defined above). The morphisms from (X, f) to (Y, g) are smooth maps $h : X \to Y$ such that $f = h^*(g) = g \circ h$. One can easily check this constitutes a valid category.

We then let $\underline{\mathbb{R}}(-)$ be the trivial functor from $C^\infty(-, PM)$ to `Vect`, the category of real vector space, which collapses all objects to $\mathbb{R}$ and all morphisms to id. Finally, we define functor $\underline{\Omega}(-)$ from $C^\infty(-, PM)$ to `Vect` taking (X, f) to $\Omega(X)$ and $h \mapsto h^*$.

We say that *a form on PM is a natrual transformation from $\underline{\mathbb{R}}(-)$ to $\underline{\Omega}(-)$*. Let us unpack this: we must have a family of morphisms $\eta_{(X,f)} : \underline{\mathbb{R}}(X, f) = \mathbb{R} \to \Omega(X)$ such that for $h : (X, f) \to (Y, g)$, we have

$$\eta_{(X,f)} = \eta_{(X,h^*g)} = h^* \circ \eta_{(Y,g)} \tag{19}$$

Still figuring out why this definition coincides with the intuition of the first few sentences.

IV. Resources

- *The A_∞ de Rham theorem and integration of representations up to homotopy:* https://arxiv.org/pdf/1011.4693
- *Representations up to homotopy of Lie algebroids:* https://arxiv.org/pdf/0901.0319
- *Deformations of Lie brackets: cohomological aspects:* https://arxiv.org/pdf/math/0403434
- *Pursuing stacks:* https://thescrivener.github.io/PursuingStacks/ps-online.pdf
- *Camilo Arias Abad's thesis on representations up to homotopy*

V. Questions

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