Discrete Riemann surfaces and the Ising model: notes

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I. Introduction

The goal of these notes is to summarize and explain in greater detail the ideas outlined in Christian Mercat's paper *Discrete Riemann surfaces and the Ising model*.

II. Introducing the terminology of discrete surfaces

We begin by letting Σ be an oriented surface without boundary. In these notes, we will in addition assume that Σ is a smooth manifold (it has a smooth structure which has smooth transition maps). Let us begin by recalling a basic definition.

Definition II.1 (Equivalent definitions of an orientation)**.** The most common definition for the orientation of a smooth manifold, and the one most likely known by the reader, is the following. Given a manifold M, an *oriented atlas* of M is an atlas of open neighbourhoods and associated coordinate charts, $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ such that each transition map $\varphi_\alpha \circ \varphi_\beta^{-1}$ (which is assumed to be C^∞ as M is smooth) has positive Jacobian determinant everywhere. An *orientation* on a manifold is a smooth structure (a maximal smooth atlas) which is oriented.

Equivalently, TODO: fill this in

A. Introducing cell complexes

We now come to the first set of definitions. In particular, we develop a means of placing a discrete, lattice-like structure on an otherwise continuous surface, in such a way that the underlying geometry of the surface is repsected.

Definition II.2 (Cellular decomposition)**.** Given Σ as defined above (an oriented surface without boundary), a *cellular decomposition* Γ of Σ is a partition of Σ into disjoint connected sets (which we call cells) of three different types:

- A discrete set of points. We call these the *vertices* of Γ, and denote them by Γ_0
- A collection of non-intersecting sets of the form $\gamma((0,1))$, where $\gamma : [0,1] \to \Sigma$ is a bijective path such that $\gamma(0)$ and $\gamma(1)$, the endpoints of the path, are contained in Γ_0. We will assume that any such γ is also smooth, in the sense that each $\varphi_\alpha \circ \gamma$ is smooth for $x \in \gamma((0,1)) \cap U_\alpha$, where $(U_\alpha, \varphi_\alpha)$ is a coordinate chart of the smooth structure on Σ. We call these the *edges* of Γ and denote them by Γ_1.
- A collection of topological discs of the form B (in other words, an embedding of an open ball B^2 in Σ) such that ∂B can be written as a finite union of elements of Γ_0 and Γ_1 (nodes and edges). We call these the *faces* of Γ, and denote them by Γ_2.

A cellular decomposition is said to be *locally finite* if every compact subset C of Σ intersects only a finite number of elements of Γ.

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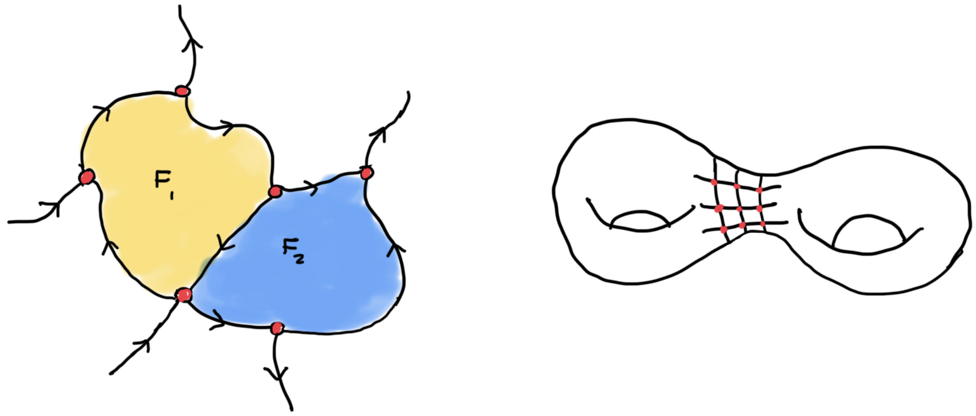

FIG. 1. The left image depicts two faces F_1 and F_2 and their bounding, oriented edges (and vertices) of some cellular decomposition Γ. The right picture shows some of the cells of a cellular decomposition Γ of a genus-2 surface Σ.

Remark II.1 (Parameterizing Γ)**.** Note that the vertices, edges and faces of a cellular decomposition Γ are all images of a 0-ball (a point), 1-ball (the interval $(0,1)$), and 2-ball (the set $\{x \in \mathbb{R}^2 \mid |x| < 1\}$), respectively, with respect to given parameterizations. This follows directly from the definition, with the edges being images of $(0,1)$ and faces being embeddings of B^2. Equivalently, we can choose parameterizations which map from each element of Γ to topological balls instead.

Remark II.2 (Orientation of Γ)**.** Since each face in Γ is an open subset of Σ, each will naturally inherit the orientation of the larger surface Σ. On the other hand, the edges of Γ are not open in Σ: they, along with their vertex endpoints, make up the boundaries of the faces. Thus, the edges are not equiped with a canonical orientation, so we instead arbitrarily choose one of the *two possible* orientations (each edge is an orientable and path-connected manifold, so there are precisely two choices of orientation).

We continue by translating a well-known construction from standard differential geometry to its discrete counterpart.

Definition II.3. We define the space of *chains* on Γ, $C(\Gamma)$, as the $\mathbb{Z}$-module generated by taking formal linear combinations of cells of Γ. We define the space of k-chains on Γ, $C_k(\Gamma)$, to be the $\mathbb{Z}$-module generate by all dimension-k cells (when each cell individually is treated as a manifold). This leads to a natural collection of boundary operators, $\partial : C_k(\Gamma) \to C_{k-1}(\Gamma)$, so that we have the following complex

$$C_2(\Gamma) \xrightarrow{\quad\partial\quad} C_1(\Gamma) \xrightarrow{\quad\partial\quad} C_0(\Gamma). \tag{1}$$

To be more specific, ∂ is a linear map which takes a formal linear combination of k-cells to a formal linear combination, with the same coefficients, of the corresponding boundaries

$$\partial(c_1 S_1 + \cdots + c_k S_k) = c_1 \partial S_k + \cdots + c_k \partial S_k \tag{2}$$

where $S_j \in \Gamma$ for each j. Note that $\partial \circ \partial = 0$, as given a formal sum of faces, edges and vertices, the boundary will be a formal sum of edges and vertices.

III. Discrete holomorphic functions

IV. A discrete deRham complex

V. A discrete Hodge star

VI. Hodge's theorem for discrete surfaces