Non-linear Grothendieck-Katz and the p-curvature conjecture

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Contents

I. Introduction

The goal of these notes is to document my progress on the non-linear Katz-Oda problem

II. Prior work

I will begin these notes by discussing some of the prior art. We begin by considering the ODE

$$\frac{dY}{dz} = A(z)Y \tag{1}$$

where $A(z)$ is a matrix whose entries are rational functions of z relative to some algebraic number field K (a field extension of $\mathbb{Q}$ such that $[K : \mathbb{Q}]$ is finite). For the sake of simplicity, let $K = \mathbb{Q}$. Some entry of $A(z)$ will be of the form

$$A_{ij}(z) = \frac{A_{ij}^{(n)} z^n + \cdots + A_{ij}^{(0)}}{B_{ij}^{(m)} z^m + \cdots + B_{ij}^{(0)}} \tag{2}$$

which of course may be reduced modulo some prime p in the ring of algebraic integers of K, for almost all primes (in this case, just the usual prime numbers in $\mathbb{Z}$). In particular, setting

$$\frac{a}{b} \mod p \equiv \frac{a \mod p}{b \mod p} \tag{3}$$

when $p \nmid b$. Thus, we may reduce the rational function's coefficients in the case when p doesn't divide any of the denominators of the rational coefficients $B_{ij}^{(k)}$. The result will be a differental equation over $\mathbb{F}_q[z]$ for some finite field $\mathbb{F}_q$. As a particular example, suppose $a \in \mathbb{Z} \subset \mathbb{Q}$ and we have the ODE

$$\frac{dy}{dz} = \frac{1}{az} y \tag{4}$$

We can reduce by $p \nmid a$. When we do such a reduction, the equation admits solutions $y = z^b$ for any b such that $ab \equiv 1 \mod p$, clearly.

Differential equations of this form, and reducation by a prime is related to the p-curvature conjecture. There is a result concerning ODEs of this form, formulated by Katz and Oda, which we can restate here

Theorem II.1. The ODE of Eq. (1) is an algebraic differential equation, meaning that its solution is an algebraic function. Moreover, there exists sufficiently large N such that for almost all primes p,

$$\left(\frac{d}{dz} - A(z)\right)^{Np} \equiv 0 \mod p \tag{5}$$

III. Non-linear problem

We wish to understand whether a similar result as Thm. II.1 holds for the Schlesinger system, which is a non-linear system of partial differential equations, given by

$$\frac{\partial B_i}{\partial \lambda_j} = \frac{[B_i, B_j]}{\lambda_i - \lambda_j} \quad \text{for } i \neq j \tag{6}$$

$$\sum_j \frac{\partial B_i}{\partial \lambda_j} = 0 \quad \text{for all } i. \tag{7}$$

Note that each B_j is a function of variables $\lambda_1, \ldots, \lambda_n$, for j from 1 to m. The first idea is to understand the vector field/phase portrait associated with each individual variable λ_j. As insane as this sounds, this will in fact be a vector field in which the vectors have entries which are matrices. In particular,

$$\frac{\partial}{\partial \lambda_j} \begin{pmatrix} B_1 \\ \vdots \\ B_{j-1} \\ B_j \\ B_{j+1} \\ \vdots \\ B_m \end{pmatrix} = \begin{pmatrix} \frac{[B_1, B_j]}{\lambda_1 - \lambda_j} \\ \vdots \\ \frac{[B_{j-1}, B_j]}{\lambda_i - \lambda_j} \\ -\sum_{i\neq j} \frac{[B_i, B_j]}{\lambda_i - \lambda_j} \\ \frac{[B_{j-1}, B_j]}{\lambda_{j-1} - \lambda_j} \\ \vdots \\ \frac{[B_m, B_j]}{\lambda_m - \lambda_j} \end{pmatrix} \tag{8}$$

This will indeed yield a collection of n different vector fields, for j ranging from 1 to j,

$$X^{(j)} = \sum_{i \neq j} \frac{[B_i, B_j]}{\lambda_i - \lambda_j} \left(\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j}\right) \tag{9}$$

We will be interested in all possible products of these vector fields with themselves:

$$X^{(J)} = X^{(j_1 \cdots j_N)} = \prod_{k=1}^{N} X^{(j_k)} \tag{10}$$

and, in particular, how large we need to make N such that for all tuples $(j_1, \ldots, j_N)$, the operator $X^{(J)}$ vanishes, modulo some prime p. We already know that this will be the case for $N = p^2$ (where we modulo by p). We want to see if it is ever true for some smaller power than this. To shed light on what is meant by reducing a differential operator by some prime, consider the particular example of $z\partial_z$. Note that

$$(z\partial_z)^3 = z\partial_z(z\partial_z + z^2\partial_z^2) = z\partial_z + z^2\partial_z^2 + 2z^2\partial_z^2 + z^3\partial_z^3 \tag{11}$$

$$= z\partial_z + 3z^2\partial_z^2 + z^3\partial_z^3 \tag{12}$$

We will assume that we are working with **analytic functions** here, so that we can just check whatever value this operator takes on, on powers of z. Indeed,

$$(z\partial_z + 3z^2\partial_z^2 + z^3\partial_z^3)z^n = [n + 3n(n-1) + n(n-1)(n-2)]\, z^n \tag{13}$$

Note that

$$n + 3n(n-1) + n(n-1)(n-2) = n(1 + 3(n-1) + (n-1)(n-2)) = n(1 + (n+1)(n-1)) = n^3 \tag{14}$$

and from Fermat's little theorem, $n^3 = n \mod 3$. In other words,

$$(z\partial_z)^3 z^n = nz^n = (z\partial_z)z^n \mod 3 \tag{15}$$

so it follows that modulo the prime 3, we have $(z\partial_z)^3 = z\partial_z$. We want to perform a similar product on our collection of differential operators. We will make a similar assumption that the matrices have analytic entries. Note also that our ability to repeatedly compose differential operators only makes sense when the corresponding coefficients multiply with respect to some kind of well-defined multiplication. I'm assuming that that will be matrix multiplication for the case of our "matrix differential operators".