Fall 2023 MAT437 problem set 2

Jack Ceroni*
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I. Problem 1

Part 1. Let A be a C^*-algebra, let $x \in A$. Define $a = \frac{1}{2}(x + x^*)$ and $b = \frac{1}{2i}(x - x^*)$. Clearly, $a^* = a$ and $b^* = b$. Note that

$$a + ib = \frac{1}{2}(x + x^*) + \frac{1}{2}(x - x^*) = x \tag{1}$$

We can also easily prove uniqueness: suppose $a' + ib' = x$ where a' and b' are self-adjoint, so we have $(a' - a) + i(b' - b) = 0$. We then note that $(a' - a)^* - i(b' - b)^* = (a' - a) - i(b' - b) = 0$, so $a' - a = 0$, and $a' = a$. Thus, $b' = b$ as well.

Part 2. Let us review briefly what it means to evaluate a function on an element $a \in A$. Recall the continuous functional calculus:

Lemma I.1 (The continuous function calculus)**.** Given a unital C^*-algebra A, associated to each normal element a is a unique $*$-isomorphism $\Phi_a : C(\text{sp}(a)) \to C^*(a, 1) \subset A$ such that when $p : \text{sp}(a) \to \mathbb{C}$ is a polynomial, $\Phi_a(p) = p(a)$ and when $p(s) = \overline{s}$, $\Phi_a(p) = a^*$.

Theorem I.1 (Spectral mapping theorem)**.** For every normal element a of a unital C^*-algebra A, and every continuous function $f : \text{sp}(a) \to \mathbb{C}$, $\text{sp}(\Phi_a(f)) = f(\text{sp}(a))$.

For $a \in A$ self-adjoint with $||a|| \leq 1$, it follows that $r(a) \leq 1$, so $f(x) = \sqrt{1 - x^2}$ is a real-valued continuous function on $\text{sp}(a)$, so there exists a $*$-homomorphism $\Phi_a(f)$ is defined to be the element $\sqrt{1 - a^2}$ which is referenced in the problem statement. In particular, note that

$$\left(\sqrt{1 - a^2}\right)^* = \Phi_a(f)^* = \Phi_a(\overline{f}) = \Phi_a(f) = \sqrt{1 - a^2} \tag{2}$$

as well as

$$\sqrt{1 - a^2}\sqrt{1 - a^2} = \Phi_a(f^2) = \Phi_a(x \mapsto 1 - x^2) = 1 - a^2. \tag{3}$$

Therefore,

$$\left(a + i\sqrt{1 - a^2}\right)^* \left(a + i\sqrt{1 - a^2}\right) = \left(a - i\sqrt{1 - a^2}\right) \left(a + i\sqrt{1 - a^2}\right) \tag{4}$$

$$= a^2 + 1 - a^2 = 1 = \left(a + i\sqrt{1 - a^2}\right) \left(a + i\sqrt{1 - a^2}\right)^* \tag{5}$$

with an analogous equation holding true for $a - i\sqrt{1 - a^2}$, clearly. Thus, both are unitary elements. Thus, given self-adjoint a, we can write $a = \frac{1}{2}\left[\left(a + i\sqrt{1 - a^2}\right) + \left(a - i\sqrt{1 - a^2}\right)\right]$, so it follows immediately from Part 1 that every element of a C^* algebra is a linear combination of four unitaries.

Part 3. No. Consider the C^*-algebra $C([0, 1])$: the set of complex continuous functions on $[0, 1]$, with the max-norm and the $*$-operation being complex conjugation on the image. Suppose $f \in C([0, 1])$ is a projection: then we must have $p^2 = p$, so $p(x)p(x) = p(x)$ for all $x \in [0, 1]$. It follows for a given x, $p(x) = 0$ or $p(x) = 1$. Since p is continuous, we have $p = 1$ or $p = 0$. Clearly, any non-constant continuous function cannot be written as a linear combination of the constant 1 and 0 functions, and thus cannot be written as a linear combination of projections.

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* jackceroni@gmail.com

II. Problem 2

Part 1. Suppose $||a-t|| \leq t$. Since a is self-adjoint, $a-t$ is as well as $t \in \mathbb{R}$, so $\mathrm{sp}(a-t) \leq t$. Thus, for some $\lambda \in \mathrm{sp}(a-t)$, $|\lambda| \leq t$, so $\lambda \in [-t, t]$. It is easy to see that $\lambda \in \mathrm{sp}(a-t)$ if and only if $\lambda + t \in \mathrm{sp}(a)$. But clearly, $-t \leq \lambda$, so $\lambda + t \geq 0$, and a is positive by definition. Conversely, if $a \geq 0$ and $||a|| \leq t$, then it must be true that $\mathrm{sp}(a) \in [0, t]$, so $\mathrm{sp}(a) \in [-t, 0]$. Thus, $r(a-t) = ||a-t|| \leq t$.

Part 2. Suppose a and b are positive, so $||a-t_1|| \leq t_1$ and $||b-t_2|| \leq t_2$ for $t_1, t_2 \in \mathbb{R}$. Then $||(a+b)-(t_1+t_2)|| \leq ||a-t_1|| + ||b-t_2|| \leq t_1 + t_2$, so from Part 1, $a+b$ is positive.

Part 3. This follows immediately from the definition: $(b+c)-(a+c) = b-a \geq 0$, so $a+c \leq b+c$. Moreover, if $a \leq b$, then $b-a$ is positive, so $a-b = x^*x$ for some x. It then follows that $c^*(a-b)c = c^*x^*xc = (xc)^*xc$ is positive. Thus, $c^*ac \leq c^*bc$.

Part 4. I haven't figured this one out yet.

III. Problem 3

Part 1. Suppose $p \perp q$, so $pq = 0$. Since p and q are projections, they are self-adjoint, so $(pq)^* = q^*p^* = qp = 0$. Then $(p+q)^2 = p^2 + q^2 + pq + qp = p + q$, so $p+q$ is a projection. Suppose $(p+q)$ is a projection. Then $(p+q)^2 = p+q = (p+q)^*$, and in the case that the algebra has unit, $(1-(p+q))^*(1-(p+q)) = (1-(p+q))^2 = 1 - 2(p+q) + (p+q) = 1-(p+q)$, so $1-(p+q) = x^*x$, where $x = 1-(p+q)$ and is thus positive, so $p+q \leq 1$. Clearly, the same logic holds when we unitize the algebra.

Finally, assuming $p+q \leq 1$ (where if the algebra doesn't have unit, we unitize), note that since if $a \leq b$, then $c^*ac \leq c^*bc$, we have

$$p+q \leq 1 \Longrightarrow p^*(p+q)p = p(p+q)p \leq p^2 = p \tag{6}$$

Thus, $p+pqp \leq p$, so in other words, $pqp \leq 0$. But since q is a projection (thus positive), and $p = p^*$, we must have $pqp = p^*qp \geq 0$ (as it is the conjugation of a positive element, so this is implied by Problem 2 Part 3). Since $pqp \leq 0$ and $pqp \geq 0$, the spectrum of pqp must be both in $(-\infty, 0]$ and $[0, \infty)$, so it is 0. Since pqp is self-adjoint, $||pqp|| = r(pqp) = 0$, so $pqp = 0$. Finally, note that $pqp = pq^2p = pq(qp) = pq(p^*q^*)^* = (pq)(pq)^*$. Therefore, $||(pq)(pq)^*|| = ||pq||^2 = 0$, so $||pq|| = 0$ and $pq = 0$ as well.

We have shown 1 implies 2, 2 implies 3, and 3 implies 1, so all the statements are equivalent.

Part 2. This part follows fairly straightforwardly from Part 1. In particular, we will use induction.

We have proved the case of $n = 2$. Let us assume the case of n holds, and prove the case of $n+1$. In particular, suppose $p_1, \ldots, p_n, p_{n+1}$ are mutually orthogonal. Then $p_1, \ldots, p_n$ are mutually orthogonal. Thus, $p' = p_1 + \cdots + p_n$ is a projection. Moreover, $p_{n+1}p' = p_{n+1}(p_1 + \cdots + p_n) = 0$, from the mutual orthogonality, so p' and p_{n+1} are mutually orthogonal. It follows from the case of two elements that $p' + p_{n+1}$ is a projection, so 1 implies 2.

Next, suppose $p = p_1 + \cdots + p_{n+1}$ is a projection: it follows immediately that the spectrum of p is in $\{0, 1\}$, so the spectrum of $1-p$ is in $\{0, 1\}$, and thus $1-p$ is positive, so $p \leq 1$. Thus, 2 implies 3.

Finally, suppose $p_1 + \cdots + p_{n+1} \leq 1$, so $1-(p_1+\cdots+p_{n+1})$ is positive. Since p_{n+1} is a projection, it is positive, and since the sum of positive elements is positive (Problem 2), $1-(p_1+\cdots+p_n) = 1-(p_1+\cdots+p_{n+1})+p_{n+1}$ is positive, so $p_1 + \cdots + p_n \leq 1$. Thus, $p_1, \ldots, p_n$ are mutually orthogonal. In fact, we can repeat this logic for every subset of n elements from $p_1, \ldots, p_{n+1}$. Thus, all the elements of the $n+1$ element list are also mutually orthogonal, as for any p_i, p_j with $i \neq j$, we can always find an n-element subset containing 2, so $p_ip_j = 0$. Thus, condition 3 implies condition 1.

IV. Problem 4

Part 1. Setting $z = v - vv^*v$, note that $z^*z = (v-vv^*v)^*(v-vv^*v) = v^*v - v^*vv^*v - v^*vv^*v + v^*vv^*vv^*v$. We know that $(v^*v)^2 = v^*v$, so simplifying this expression gives $z^*z = 0$. It follows that $||z^*z|| = ||z||^2 = 0$, so $||z|| = 0$ and thus $z = 0$. It follows immediately that $v = vv^*v$, as desired.

Part 2. From Part 1, $vv^* = vv^*vv^* = (vv^*)^2$, so $(v^*)^*v^*$ is a projection. Clearly, vv^* is self-adjoint as well. Thus, it is a projection, so v^* is a partial isometery by definition.

Part 3. We have already shown that $v = qv = vp$ above, as $qv = vv^*v$ and $vp = vv^*v$ as well. It follows immediately that $qv = q(vp) = qvp$, so $v = qv = vp = qvp$.

Part 4. In this case, $v^*v = v^*(vv^*v) = (v^*v)^2$. In addition, $(v^*v)^* = v^*v$. Thus, by definition, v^*v is a projection so v is a partial isometry.

Part 5. Clearly, $(vw)^*(vw) = w^*v^*vw = w^*(v^*v)w = w^*w = 1$, so vw is an isometry. In addition, $(vz)^*vz = z^*v^*vz = z^*z$. Since z is a partial isometry, z^*z is a projection, so $(vz)^*(vz)$ is a projection and by definition vz is a partial isometry.

Part 6. No, this isn't even true when v is more generally a projection (every projection is clearly a partial isometry, as for v a projection, $v^*v = v^2 = v$ is a projection).

Consider the C^*-algebra of 2×2 complex matrices, with the conjugate transpose being the $*$-operation and the usual matrix norm being the norm. Take $v = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Clearly, v is a projection and thus a partial isometry. Moreover, take $w = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Verifying that w^*w is a projection is trivial. Note that

$$(vw)^*vw = w^*v^*vw = w^*vw = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \tag{7}$$

which is not a projection, as it does not square to itself.

V. Problem 5

Since each v_j is a partial isometry, each $v_j^*v_j$ is a projection. We know from Problem 3 that if $p_1, \ldots, p_n$ is a collection of projections, such that $p_1 + \cdots + p_n \leq 1$, then $p_1, \ldots, p_n$ are mutually orthogonal. Thus, since $\sum_j v_j^*v_j = 1$, with each element of the sum being a projection, the elements $v_j^*v_j$ are all pairwise mutually orthogonal. It follows that for $i \neq j$, $(v_j^*v_j)(v_i^*v_i) = 0$. Therefore, $(v_jv_j^*v_j)(v_i^*v_iv_i^*) = (v_jv_j^*v_j)(v_iv_i^*v_i)^* = 0$. However, recall once again from Problem 4 that for v a partial isometry, $v = vv^*v$. Thus, $(v_jv_j^*v_j)(v_iv_i^*v_i)^* = v_jv_i^* = 0$ for $j \neq i$.

We can repeat the same logic to show that each $v_j^*v_i = 0$ as well. Since $\sum_j v_jv_j^* = 1$, we have $(v_jv_j^*)(v_iv_i^*) = 0$ for all $i \neq j$, so $(v_jv_j^*v_j)^*(v_iv_i^*v_i) = 0$ for all $i \neq j$, and since each v_i is a partial isometry, $v_j^*v_i = 0$ for all $j \neq i$. Thus, we have

$$\left(\sum_{j=1}^{n} v_j\right)^* \left(\sum_{j=1}^{n} v_j\right) = \sum_{j=1}^{n} v_j^*v_j + \sum_{i>j}(v_i^*v_j + v_j^*v_i) = \sum_{j=1}^{n} v_j^*v_j = 1 \tag{8}$$

as well as

$$\left(\sum_{j=1}^{n} v_j\right) \left(\sum_{j=1}^{n} v_j\right)^* = \sum_{j=1}^{n} v_jv_j^* + \sum_{i>j}(v_iv_j^* + v_jv_i^*) = \sum_{j=1}^{n} v_jv_j^* = 1 \tag{9}$$

It follows that $u = \sum_j v_j$ satisfies $u^*u = uu^* = 1$, so by definition, u is unitary.

VI. Problem 6

Part 1. Clearly, v is unitary as an element of the C^*-algebra $C(\mathbb{T})$. Taking the $*$-operation to be complex conjugation on the codomain of a given function, we note that for some $s \in \mathbb{T}$, $v(s)v^*(s) = v^*(s)v(s) =$

$v(s)\overline{v(s)} = s\bar{s} = 1$, as $||s|| = 1$. Thus, $vv^* = v^*v = 1$ (the constant 1-map, which is the multiplicative identity of the algebra), so v is unitary.

Part 2. Suppose there were a unitary u in $C(\mathbb{D})$ such that $\psi(u) = v$. This would of course mean that $u|_{\mathbb{T}}(s) = v(s) = s$ for each $s \in \mathbb{T}$. Moreover, since u is unitary, $u(s)u^*(s) = 1$ for each $s \in \mathbb{D}$, so $||u(s)|| = 1$ (where the norm is the supremum norm). In other words, the range of u is contained in $\mathbb{T}$. Thus, u can be thought of as a continuous function taking the disk to its boundary, such that the boundary remains fixed under u.

In other words: u would be a retraction of the disk to its boundary, a clear contradiction, as the non-existence of such a map is a fundamental result in algebraic topology.

Part 3. Recall that the set $\mathcal{U}_0(C(\mathbb{T}))$ is precisely the collection of all elements v of $C(\mathbb{T})$ for which $v \sim 1$ within $C(\mathbb{T})$. Suppose we did have such a homotopy between v and 1, so there exists continuous $F : \mathbb{T} \times [0, 1] \to \mathbb{T}$ such that $F(x, 1) = v(x)$ and $F(x, 0) = 1$, where $F(x, t) \in \mathbb{T}$ for all (x, t).

The main idea here is that F, it it were to exist, would define a lift u such that $\psi(u) = v$. Note that $\mathbb{D} = \mathbb{T} \times [0, 1]/\sim$, where $\sim$ is the equivalence relation which identifies all points $(s, 0)$. Let p be the corresponding quotient map. Since F is a constant on these points, there exists a well-defined continuous map $\widetilde{F} : \mathbb{D} \to \mathbb{T}$ such that $F = \widetilde{F} \circ p$, via the universal property of the quotient topology/quotient map (a standard fact in point-set topology).

In particular, note that since the codomain of $\widetilde{F}$ is $\mathbb{T}$, it is unitary, it is continuous, and restricted to points $(s, 1)$, it clearly agrees with F and is simply the unitary v. Thus, F would be a retraction of $\mathbb{D}$ onto $\mathbb{T}$, a contradiction. It follows that $v \nsim 1$, so $v \notin \mathcal{U}_0(C(\mathbb{T}))$.

Clearly, 1 and v are both unitary in $C(\mathbb{T})$, so this result also implies the existence of unitaries $v_1, v_2 \in C(\mathbb{T})$ which are not homotopic.

Finally, note that for any element $w \in C(\mathbb{T})$ such that $w = e^{ih}$ for $h \in C(\mathbb{T})$ self-adjoint, the map $F(x, t) = e^{ith(x)}$ is continuous, it lies in $C(\mathbb{T})$ as $F(x, t)^*F(x, t) = e^{-ith(x)}e^{ith(x)} = 1$ for any x and t, and has $F(x, 0) = 1$, $F(x, 1) = e^{ih(x)} = w(x)$, so we have a homotopy between 1 and w. Since $1 \nsim v$, it follows that we cannot have $v = e^{ih}$ for self-adjoint h.