

MAT437 problem set 8

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I. Problem 1

The proof of the following lemma included in RLL excludes many key parts, so for one of my exercises, I complete it

We start by proving a collection of technical lemma, which will be required for the following proof:

Lemma I.1. If v is a partial isometry (so v^*v is a projection) then $vv^*v = v$.

Proof. Let $z = v - vv^*v = (1 - vv^*)v$. Note that

$$z^*z = v^*(1 - vv^*)^*(1 - vv^*)v = v^*(1 - 2vv^* + vv^*vv^*)v = v^*v - 2v^*vv + v^*v = 0 \quad (1)$$

where we use that $(v^*v)^2 = v^*v$. Thus, $\|z\|^2 = \|z^*z\| = 0$, so $\|z\| = 0$ and $v = vv^*v$, as desired. \square

Now, another lemma:

Lemma I.2. Suppose that $\{f_{ii}^{(k)} \mid 1 \leq k \leq r, 1 \leq i \leq n_k\}$ is a set of mutually orthogonal projections in C^* -algebra B and that

$$f_{11}^{(k)} \sim f_{22}^{(k)} \sim \dots \sim f_{n_k n_k}^{(k)} \quad (2)$$

for each k . Then there is a system of matrix units $\{f_{ij}^{(k)}\}$ extending $\{f_{ii}^{(k)}\}$.

The idea behind constructing systems of matrix units is, essentially, to have a “basis” for each component of the direct sum that we will eventually demonstrate characterizes the C^* -algebra B . Each of the sets $\{f_{ii}^{(k)}\}$ are analogous to matrix projections with 1 at the i -th slot on the diagonal, at the k -th slot in the direct sum. Let us now prove the lemma.

Proof. Of course, here, we will make use of the Murray-von Neumann equivalence. Namely,

$$f_{11}^{(k)} \sim f_{jj}^{(k)} \implies f_{11}^{(k)} = f_{1j}^{(k)} f_{1j}^{(k)*} \quad \text{and} \quad f_{jj}^{(k)} = f_{1j}^{(k)*} f_{1j}^{(k)} \quad (3)$$

This notation is consistent, as $f_{11}^{(k)}$ is self-adjoint, so setting $j = 1$ above causes no problems. Our claim is that if we set $\tilde{f}_{ij}^{(k)} = f_{1i}^{(k)*} f_{1j}^{(k)}$ then we will have the desired system of matrix units. This is in fact an extension of the system we are already provided. Namely, we have

$$\tilde{f}_{jj}^{(k)} = f_{1j}^{(k)*} f_{1j}^{(k)} = f_{jj}^{(k)} \quad (4)$$

by definition. In fact, we might as well denote $\tilde{f}_{ij}^{(k)}$ by $f_{ij}^{(k)}$, as for $i = 1$, we have

$$\tilde{f}_{1j}^{(k)} = f_{11}^{(k)*} f_{1j}^{(k)} = f_{1j}^{(k)} f_{1j}^{(k)*} f_{1j}^{(k)} = f_{1j}^{(k)} \quad (5)$$

where we use the above lemma and the fact that $f_{1j}^{(k)*} f_{1j}^{(k)} = f_{jj}^{(k)}$ is a projection. Let us now complete our verification. Of course, we have

$$f_{pq}^{(k)} f_{qr}^{(k)} = f_{1p}^{(k)*} f_{1q}^{(k)} f_{1q}^{(k)*} f_{1r}^{(k)} = f_{1p}^{(k)*} f_{11}^{(k)} f_{1r}^{(k)} = f_{1p}^{(k)*} f_{1r}^{(k)} f_{1r}^{(k)*} f_{1r}^{(k)} = f_{1p}^{(k)*} f_{1r}^{(k)} = f_{pr}^{(k)} \quad (6)$$

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where we use the first lemma to note that $f_{1r}^{(k)} f_{1r}^{(k)*} f_{1r}^{(k)} = f_{1r}^{(k)}$, as $f_{1r}^{(k)*} f_{1r}^{(k)} = f_{rr}^{(k)}$ is a projection. Next, note that

$$f_{pq}^{(k)} f_{rs}^{(\ell)} = f_{1p}^{(k)*} f_{1q}^{(k)} f_{1r}^{(\ell)*} f_{1s}^{(\ell)} \quad (7)$$

Once again using the first lemma, we have $f_{1q}^{(k)} = f_{1q}^{(k)} f_{1q}^{(k)*} f_{1q}^{(k)} = f_{1q}^{(k)} f_{qq}^{(k)}$ and $f_{1r}^{(\ell)} = f_{1r}^{(\ell)} f_{1r}^{(\ell)*} f_{1r}^{(\ell)} = f_{1r}^{(\ell)} f_{rr}^{(\ell)}$ so that $f_{1r}^{(\ell)*} = f_{rr}^{(\ell)} f_{1r}^{(\ell)*}$. We then use the fact that the projections in our set are mutually orthogonal to conclude that

$$f_{1p}^{(k)*} f_{1q}^{(k)} f_{1r}^{(\ell)*} f_{1s}^{(\ell)} = f_{1p}^{(k)*} f_{1q}^{(k)} (f_{qq}^{(k)} f_{rr}^{(\ell)}) f_{1r}^{(\ell)*} f_{1s}^{(\ell)} = 0 \quad (8)$$

which is 0 when $q \neq r$ or $k \neq \ell$, as in these cases, $f_{qq}^{(k)} f_{rr}^{(\ell)} = 0$. It is very immediately clear that $f_{ij}^{(k)*} = f_{1j}^{(k)*} f_{1i}^{(k)} = f_{ji}^{(k)}$, so we have verified the third condition, and it follows that our set of $f_{ij}^{(k)}$ is in fact a system of matrix units in B extending $\{f_{ii}^{(k)}\}$. \square

II. RLL Problem 7.4 (Suggested Problem 1)

Part 1. Suppose A and B both have the cancellation property, which means that the semigroups $\mathcal{D}(A)$ and $\mathcal{D}(B)$ have the cancellation property. Equivalently, some X has the cancellation property if and only if, for each $p, q \in \mathcal{P}_\infty(X)$, then

$$[p]_0 = [q]_0 \iff p \sim_0 q \quad (9)$$

Recall from earlier in RLL that if i_A and i_B are canonical inclusion maps of A and B into $A \oplus B$, then $\Phi = K_0(i_A) \oplus K_0(i_B)$ is a group isomorphism. Suppose $p, q \in \mathcal{P}_\infty(A \oplus B)$. Of course, $p \sim_0 q$ implies $[p]_{\mathcal{D}} = [q]_{\mathcal{D}}$, which implies that $[p]_0 = [q]_0$.

Conversely, suppose that $[p]_0 = [q]_0$, where p, q are projections in $\mathcal{P}_\infty(A \oplus B)$. It is easy to see in this case that $p = p_1 \oplus p_2$ and $q = q_1 \oplus q_2$ for $p_1, p_2 \in \mathcal{P}_\infty(A)$ and $q_1, q_2 \in \mathcal{P}_\infty(B)$. Then note that $\Phi^{-1}([p]_0) = [p_1]_0 \oplus [p_2]_0 = \Phi^{-1}([q]_0) = [q_1]_0 \oplus [q_2]_0$, so that $[p_1]_0 = [q_1]_0$ in $K_0(A)$ and $[p_2]_0 = [q_2]_0$ in $K_0(B)$. Thus, since we have the cancellation property in these algebras, $p_1 \sim_0 q_1$ and $p_2 \sim_0 q_2$. It follows that $p_1 \oplus p_2 = p \sim_0 q_1 \oplus q_2 = q$, clearly. This completes the proof.

Part 2. Suppose the sequence of C^* -algebras has the cancellation property. We will use continuity of K_0 to show that their inductive limit has the cancellation property. In particular, note that

$$\varinjlim K_0(A_n) \simeq K_0(\varinjlim A_n) = K_0(A) \quad (10)$$

where the isomorphism is clearly of Abelian groups. Suppose $p, q \in \mathcal{P}_\infty(A)$ with $[p]_0, [q]_0 \in K_0(A)$ are such that $[p]_0 = [q]_0$. Let Φ be the isomorphism so that

$$p' = \Phi([p]_0) = \Phi([q]_0) = q' \in \varinjlim K_0(A_n).$$

By definition, p' and q' are both equivalence classes in $\prod_k K_0(A_k) / \sum_k K_0(A_k)$.

Part 3. Let us prove that any matrix algebra over \mathbb{C} has the cancellation property. If we can do this, then since AF algebras are simply direct limits of direct sums of matrix algebras, we will have proved that all AF algebras have the cancellation property.

Indeed, suppose $[x]_{\mathcal{D}} + [y]_{\mathcal{D}}, [z]_{\mathcal{D}} + [y]_{\mathcal{D}} \in \mathcal{D}(M_n(\mathbb{C}))$, the semigroup of the matrix algebra. This implies that $x \oplus y \sim_0 z \oplus y$, so $x \oplus y = v^*v$ and $z \oplus y = vv^*$. Suppose x is size n , z is size m .