Classification of C^*-algebras: February report

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I. Introduction

I will discuss the reading I have completed in the month of February. This set of notes is entirely dedicated to exploration of C^*-dynamical systems, in addition to a paper related to the subject.

II. Introduction to C^*-dynamical systems

Let us begin by defining a C^*-dynamical system:

Definition II.1 (C^*-dynamical system)**.** A C^*-dynamical system is a pair (A, σ), where A is a C^*-algebra and $\sigma : \mathbb{R} \to \text{Aut}(A)$ is a map from the real numbers to the automorphism group of A, such that the semigroup condition is satisified, $\sigma_t \circ \sigma_s = \sigma_{s+t}$ and $\sigma_0 = \text{id}$. Moreover, the map $t \mapsto \sigma_t(a)$ is assumed to be continuous as a function from $\mathbb{R}$ to A, for each $a \in A$.

In the context of quantum mechanics, A might be a subalgebra of the algebra of bounded operators on some Hilbert space $\mathcal{H}$ of quantum states, where the operators in $A \subset B(\mathcal{H})$ are observables, and σ is the time-evolution induced by the Schrodinger equation corresponding to some Hamiltonian $H \in B(\mathcal{H})$, namely

$$\sigma_t(a) = e^{iHt} a e^{-iHt}. \tag{1}$$

Checking that this is a valid C^*-dynamical system is easy.

We will need some basic terminology for working with C^*-dynamical systems:

Definition II.2 (State)**.** A *state* on a C^*-algebra A is a linear map $\rho : A \to \mathbb{C}$ such that $\rho(a^* a) \geq 0$ and $\rho(1) = 1$. The idea with this definition is to generalize the notion of a state in quantum mechanics. We have observables which are evolving in time, and $\rho(\sigma_t(a))$ is the *expectation value* of a at time t with respect to the state ρ (if σ_t are the dynamics of some C^*-dynamical system).

Example II.1 (States of matrix algebras)**.** Suppose $A = M_{n\times n}(\mathbb{C})$, with the Hilbert-Schmidt inner product, $\langle a, b\rangle = \text{Tr}(a^* b)$, and the corresponding norm inducing a C^*-algebra structure (recall all matrix norms yield equivalent metric spaces). Recall the Riesz representation theorem for finite-dimensional vector spaces: given a linear function $\phi : A \to \mathbb{C}$ on an inner product space, there exists a unique $\rho \in A$ such that $\phi(a) = \langle \rho, a\rangle$, and thus a unique ρ such that $\phi(a) = \text{Tr}(\rho a)$ for all a. It follows that in this setting, an algebra state can be uniquely specified in terms of an *element of the algebra* $\rho \in A$, such that $\phi(1) = \text{Tr}(\rho) = 1$, and such that ρ has a non-negative spectrum, as if v is an eigenvector of ρ, then the eigenvalue projection p_v is positive, and we must have $\text{Tr}(\rho p_v) = \lambda_v \geq 0$, with λ_v the eigenvalue corresponding to v.

These conditions are also clearly sufficient: given non-negative $\rho \in A$ with $\text{Tr}(\rho) = 1$, clearly $a \mapsto \text{Tr}(\rho a)$ is a state (we have the positivity condition because if ρ is positive, then $\rho = s^* s$ and $\text{Tr}(\rho a^* a) = \text{Tr}(s^* s a^* a) = \text{Tr}(as^*)(as^*)^*) \geq 0$).

This is why, in quantum mechanics, we will call a positive operator ρ with unit trace a *quantum state*: because there is a bijection between the collection of all such quantum states, and algebra states $\phi \in A$!

Definition II.3. Given a C^*-dynamical system (A, σ), some $a \in A$ is said to be σ-analytic if the map $t \mapsto \sigma_t(a)$ extends to an A-valued entire function $z \mapsto \sigma_z(a)$ (in other words, this function is everywhere analytic/has a convergent power series representation).

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Definition II.4 (Types of states)**.** A state ϕ on A is said to be a ground state with respect to σ if, for every $a \in A$ and every σ-analytic $b \in A$, the function $z \mapsto \phi(a\sigma_z(b))$ is bounded in the region $\text{Im}(z) \geq 0$. A state ϕ is said to be σ-invariant if $\phi(\sigma_t(b)) = \phi(b)$ for all $b \in A$ and all $t \in \mathbb{R}$. A state ϕ is said to be a KMS_β state with respect to σ, at inverse temperature $\beta \in [0, \infty)$ if $\phi(ab) = \phi(b\sigma_{i\beta}(a))$ for all σ-analytic a and b.

Example II.2 (KMS states in quantum theory)**.** Suppose $A = M_{n\times n}(\mathbb{C})$: a C^*-algebra of $n \times n$ complex matrices. In quantum mechanics, one may think of this as the algebra of observables corresponding to a collection of qubits. We let σ_t be the *Heisenberg time-dynamics* of a given *Hamiltonian* $h \in A$, where h is self-adjoint, with

$$\sigma_t(a) = e^{iht} a e^{-iht} \tag{2}$$

Verifying that σ_t gives rise to a valid C^*-dynamical system is easy: note that

$$\sigma_{s+t}(a) = e^{ih(t+s)} a e^{-ih(t+s)} = e^{iht}(e^{ihs} a e^{-ihs})e^{-iht} = \sigma_t(\sigma_s(a)) \quad \text{and} \quad \sigma_0(a) = e^{ih(0)} a e^{-ih(0)} = a. \tag{3}$$

Moreover, given $a \in A$, the function $t \mapsto \sigma_t(a) = e^{iht} a e^{-iht}$ is norm-continuous due to the fact that the function $t \mapsto e^{iht} = \cos(th) + i\sin(th)$ is norm-continuous, via the continuous functional calculus of the underlying algebra.

In this setting, we can explicitly write down a KMS_β state, namely,

$$\rho_\beta(a) = \text{Tr}\left(\frac{e^{-\beta h}}{\text{Tr}\left[e^{-\beta h}\right]} a\right) \tag{4}$$

Verifying that this map is a KMS_β state is trivial: we have

$$\rho_\beta(b\sigma_{i\beta}(a)) = \rho_\beta(b e^{-\beta h} a e^{\beta h}) = \text{Tr}\left(\frac{1}{\text{Tr}\left[e^{-\beta h}\right]} b e^{-\beta h} a e^{\beta h} e^{-\beta h}\right) = \text{Tr}\left(\frac{a b e^{-\beta h}}{\text{Tr}\left[e^{-\beta h}\right]}\right) = \rho_\beta(ab) \tag{5}$$

It is moreover possible to see that this is the only possible KMS_β state. Recall we explained that every state $\phi \in A$ has a unique representation as $a \mapsto \text{Tr}(\rho a)$ for some positive, unit trace $\rho \in A$. It follows that if ϕ_β is a KMS_β state, we must have $\phi_\beta(a) = \text{Tr}(\eta a)$ for some η (which is positive and has unit-trace). The KMS-condition implies that we must have

$$\text{Tr}(b\eta a) = \text{Tr}(\eta ab) = \text{Tr}(\eta b e^{-\beta h} a e^{\beta h}) \tag{6}$$

Since h is Hermitian, it follows from spectral theorem that we have an eigenvector basis of h. Let $p_{v,w}$ be a map which sends eigenvector v to eigenvector w, and sends everything else to 0. Let $a_v = \sum_k p_{v,v_k}$, where k sums over all eigenvectors. Note that

$$\text{Tr}(\eta a_v a_w^*) = \text{Tr}(\eta a_w^* e^{-\beta h} a_v e^{\beta h}) = e^{\beta\lambda_v} \text{Tr}\left[e^{-\beta h}\right] \text{Tr}\left(\eta p_{v,w}\right) \tag{7}$$

for $v \neq w$, the left-hand side of the equation is 0, so $\text{Tr}(\eta p_{v,w}) = 0$. For $v = w$, we have

$$e^{\beta\lambda_v} \text{Tr}\left[e^{-\beta h}\right] \text{Tr}\left(\eta p_{v,v}\right) = \text{Tr}(\eta a_v a_v^*) = \sum_{jk} \text{Tr}\left(\eta p_{v_j,v_k}\right) = \text{Tr}(\eta) = 1 \tag{8}$$

which implies that $\text{Tr}(\eta p_{v,v}) = \text{Tr}[e^{-\beta h}]e^{-\beta\lambda_v}$. It follows immediately from these facts that we must have $\eta = \frac{1}{\text{Tr}[e^{-\beta h}]} e^{-\beta h}$, as desired.

Often, we wish to use an equivalent formulation of the KMS condition, which does not involve analytic elements of the algebra:

Proposition II.1. A state ϕ is a KMS_β state if and only if for any $a, b \in A$, there exists a function $F(a, b, z)$ analytic when $0 < \text{Im}[z] < \beta$, continuous on the closure of this strip, which satisfies the following conditions:

$$F(a, b, t) = \phi(a\sigma_t(b)), \qquad F(a, b, t + i\beta) = \phi(\sigma_t(b)a) \tag{9}$$

where $t \in \mathbb{R}$.

Proof. Suppose this condition holds, so we have a function $F(a,b,z)$ satisfying the conditions. Let $a,b \in A$ be σ-analytic. It follows that the function $t \mapsto \sigma_t(b)$ can be extended analytically to $G(z) = \sigma_z(b)$, and similarly, the function $H(z) = \phi(a\sigma_z(b))$ is analytic. Clearly, for any $z = t \in \mathbb{R}$, we have

$$P(t) = F(a,b,t) - H(t) = \phi(a\sigma_t(b)) - \phi(a\sigma_t(b)) = 0 \tag{10}$$

and moreover, since $P(z)$ is an analytic function on the domain $S_\beta = \{z|0 < \mathrm{Im}[z] < \beta\}$ which can be extended to a continuous function on the closure, taking on real values on the real line, by the Schwarz reflection principle, we can extend P to an analytic function $\widetilde{P}$ on $\widetilde{S}_\beta = \{z| -\beta < \mathrm{Im}[z] < \beta\}$. In particular, since $\widetilde{P}$ is analytic on the real line and vanishes, it must vanish on the entire domain, which means $F(a,b,z) = H(z)$ in the upper-strip. In particular,

$$H(i\gamma) = \phi(a\sigma_{i\gamma}(b)) = F(a,b,i\gamma) \tag{11}$$

for all $\gamma < \beta$. It follows from continuity that taking $\gamma \to \beta$, we have $H(i\beta) \to F(a,b,i\beta) = \phi(ba)$. This is precisely the desired condition.

To prove the converse, suppose we have $\phi(ab) = \phi(b\sigma_{i\beta}(a))$ for all σ-analytic a and b. Again, for such σ-analytic b, the function $F(a,b,z) := \phi(a\sigma_z(b))$ is analytic. The goal is to show that this function satisfies the required conditions of the proposition. Indeed, we have

$$F(a,b,t) = \phi(a\sigma_t(b)) \tag{12}$$

as well as

$$F(a,b,t+i\beta) = \phi(a\sigma_{t+i\beta}(a)) = \phi(a\sigma_{i\beta}(\sigma_t(b))) = \phi(\sigma_t(b)a) \tag{13}$$

where we know that $\sigma_{t+i\beta} = \sigma_{i\beta} \circ \sigma_t$ as a byproduct of the analytic extension. To define a function on all elements of the algebra, not just the σ-analytic ones, we will show that there is a dense $*$-subalgebra of analytic elements, from which we can extend F to the entire algebra. Indeed, suppose $a \in A$, we define

$$a_m = \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \sigma_t(a) e^{-mt^2}\, dt \tag{14}$$

Of course, this integral (this is the Bochner integral: an analogue of Lebesgue integration defined when the domain being considered is an arbitrary Banach space) converges as $||\sigma_t(a)|| \leq ||a||$. We claim that a_m is σ-analytic, and moreover, taking all a_m for all $a \in A$ yields a dense subset of A. Moreover,

$$||a_m|| = \left|\left|\sqrt{\frac{m}{\pi}} \int_{-\infty}^{\infty} a e^{-mt^2}\, dt\right|\right| \leq \sqrt{\frac{m}{\pi}} \int_{-\infty}^{\infty} ||\sigma_t(a)|| e^{-mt^2}\, dt \leq ||a|| \tag{15}$$

for each m. Due to the fact that $\sigma_s \in \mathrm{Aut}(a)$, we have

$$\sigma_s(a_m) = \sqrt{\frac{m}{\pi}} \int_{-\infty}^{\infty} \sigma_{s+t}(a) e^{-mt^2}\, dt = \sqrt{\frac{m}{\pi}} \int_{-\infty}^{\infty} \sigma_t(a) e^{-m(t-s)^2}\, dt \tag{16}$$

The right-hand side of the above extends to an analytic function on the complex plane in the variable s. Indeed, note that wehen we naively substitute a complex variable $z = x + iy$ for the place of s in the above expression, we get

$$e^{-m(t-x-iy)^2} = e^{-m(t-x)^2} e^{my^2} e^{2iy(t-x)} \tag{17}$$

which implies that the integrand still converges, as we are effectively multiplying by an overall norm-increasing factor of the form e^{my^2}, a constant. It therefore follows that the collection of a_m are σ-analytic. To see that the collection of a_m, for all $a \in A$, forms a dense subset of A, note that

$$||a - a_m|| = \left|\left|\int_{-\infty}^{\infty} (a - \sigma_t(a)) e^{-mt^2}\, dt\right|\right| \leq \int_{-\infty}^{\infty} ||a - \sigma_t(a)|| e^{-mt^2}\, dt \tag{18}$$

If we take $m \to \infty$, the integral above approaches $||a - \sigma_0(a)|| = ||a - a|| = 0$, which means we can make a_m arbitrarily close to a. For σ-analytic elements, note that

$$||F(a, b, z)|| = ||\phi(a\sigma_z(b))|| \leq ||a\sigma_z(b)|| \leq ||a||||\sigma_{i\mathrm{Im}[z]}(b)|| \tag{19}$$

where we *don't* necessarily know that σ_z for general complex numbers is a $*$-automorphism, so we cannot use the fact that $*$-automorphisms are norm decreasing. Nevertheless, this function will still be bounded on S_β. It follows from Hadamard's three-lines theorem that

$$\sup_{z \in \overline{S}_\beta} F(a, b, z) \leq ||\phi(a\sigma_{i\beta}(b))|| = ||\phi(ba)|| \leq ||a||||b|| \tag{20}$$

Now, let us pick $a, b \in A$ and sequences a_n and b_n of σ-analytic elements converging to each, respectively. We will have

$$\begin{aligned} ||F(a_n, b_n, z) - F(a_m, b_m, z)|| &= ||\phi(a_n\sigma_z(b_n)) - \phi(a_m\sigma_z(b_m))|| && (21)\\ &= ||\phi((a_n - a_m)\sigma_z(b_n)) - \phi(a_m\sigma_z(b_m - b_n))|| && (22)\\ &\leq ||\phi((a_n - a_m)\sigma_z(b_n))|| + ||\phi(a_m\sigma_z(b_m - b_n))|| && (23)\\ &= ||F(a_n - a_m, b_n)|| + ||F(a_m, b_m - b_n)|| && (24)\\ &\leq ||a_n - a_m||||b_n|| + ||b_m - b_n||||a_m|| && (25) \end{aligned}$$

which implies that the sequence of (a_n, b_n) is Cauchy in $\overline{S}_\beta$, so we have a well-defined continuous extension of F to all $a, b \in A$, completing the proof. □

A particularly nice property of KMS states is their invariance under time-evolution.

Proposition II.2. KMS$_\beta$ states with respect to the dynamical system σ are σ-invariant: they do not change under time-evolution with respect to σ.

Proof. Let ϕ be a KMS state. Let a be a σ-analytic element of the underlying algebra. The KMS condition implies that

$$\phi(\sigma_t(a)) = \phi(\sigma_{t+i\beta}(a)) \tag{26}$$

It follows that the entire function $z \mapsto \phi(\sigma_z(a))$ is β-periodic. Note that $||\sigma_t(a)|| \leq ||a||$ for all t. It follows that for any $z = x + iy$,

$$||\phi(\sigma_z(a))|| \leq ||\sigma_z(a)|| = ||\sigma_x(\sigma_{iy}(a))|| \leq ||\sigma_{iy}(a)|| \leq \max_{y \in [0,\beta]} ||\sigma_{iy}(\sigma)|| \tag{27}$$

where the maximum exists as the function $y \mapsto ||\sigma_{iy}(\sigma)||$ on $[0, \beta]$ is clearly continuous. Thus, we have a bounded entire function, which, by Liouville's theorem, is constant. In particular, $\phi(\sigma_t(a)) = \phi(a)$ for all t. To see that this claim is also true for arbitrary $a \in A$, we once again use the density of the σ-analytic elements, and note that if $a_n \to a$ with each a_n σ-analytic, then $\phi(\sigma_t(a_n)) \to \phi(\sigma_t(a))$ and $\phi(\sigma(a_n)) \to \phi(\sigma(a))$ by continuity, so $\phi(\sigma_t(a)) = \phi(a)$. □

III. Crystallization of C^*-algebras

The paper begins by defining what the crystal of a C^*-algebra is. Intuitively, given a C^*-dynamical system, we define the crystal as the quotient algebra which collapses all non-zero-energy eigenstates of the dynamics down to a point.

Definition III.1. Let A be a C^*-algebra, let σ_t for $t \in \mathbb{R}$ be a one-parameter group of algebra automorphisms for M. Given $\lambda \in \mathbb{R}$, define

$$A_\lambda = \{a \in A \mid \sigma_t(a) = e^{it\lambda}a \text{ for all } t \in \mathbb{R}\}. \tag{28}$$

We assume that the dynamics are almost-periodic, meaning that the $*$-subalgebra spanned by all the A_λ is dense in A. Let $I_\lambda = \overline{A_\lambda A_\lambda^*}$, which is an algebra ideal in A_0, and let

$$I = \overline{\sum_{\lambda>0} I_\lambda} \tag{29}$$

The C^*-algebra $A_c = A_0/I$ is called the *crystal* of (A, σ).

Lemma III.1. A unital $*$-homomorphism $\varphi : A \to B$ between C^*-algebras is norm-decreasing.

Proof. First, note that if $\lambda \in \mathrm{sp}(\varphi(a))$, then $\varphi(a) - \lambda \cdot 1$ is not invertible. Suppose $a - \lambda \cdot 1$ were invertible: then $\varphi(a - \lambda \cdot 1) = \varphi(a) - \lambda \cdot 1$ would be, a contradiction. Thus, $\mathrm{sp}(\varphi(a)) \subset \mathrm{sp}(a)$. Now, note that

$$||\varphi(a)|| = \sup |\mathrm{sp}(\varphi(a))| \leq \sup |\mathrm{sp}(a)| = ||a||. \tag{30}$$

and we're done. □

Claim III.1. The above definition of the crystal makes sense.

Proof. First, let us check that each A_λ is a sub-algebra. Clearly, it is closed under sums/products/scalar product/inversion and has the additive identity, since σ_t is an algebra automorphism, and is thus linear/multiplicative. In addition, suppose $a \in A_\lambda$, then $\sigma_t(a^*) = \sigma_t(a)^* = e^{-it\lambda}a^*$, so A_λ is not a sub-$*$-algebra unless $\lambda = 0$.

To see that A_0 is a C^*-algebra, note that $\sigma_t(1) = \sigma_t(1)\sigma_t(1)$ implying $\sigma_t(1) = 1$ as σ_t is an automorphism, so σ_t is a unital $*$-isomorphism, meaning that it is unit-preserving. Thus, $||\sigma_t(a)|| \leq ||a||$, so σ_t is continuous, implying that if $a_n \to a$ with $a_n \in A_0$, then $\sigma_t(a_n) \to 0$ and $\sigma_t(a) = 0$, so $a \in A_0$. To see that $A_\lambda A_\lambda^*$ is in A_0 is easy, for $a \in A_\lambda$, we have

$$\sigma_t(aa^*) = \sigma_t(a)\sigma_t(a^*) = e^{i\lambda t}ae^{-i\lambda t}a^* = aa^* \tag{31}$$

so $aa^* \in A_0$. Thus, the span of all such aa^* is in A_0 so $A_\lambda A_\lambda^* \subset A_0$. Since A_0 is closed, $\overline{A_\lambda A_\lambda^*} \subset A_0$. Clearly, I_λ is closed under the star. When the authors identify I_λ with $\overline{A_\lambda A_\lambda^*}$, I'm assuming that they mean I_λ is the two-sided ideal generated by the elements of this set. □

Proposition III.1. The elements $\lambda \in \mathbb{R}$ such that $A_\lambda \neq 0$ form a group under addition.

Proof. Note that if $A_c, A_d \neq 0$, then we can pick $a \in A_c$ and $b \in A_d$, so that

$$\sigma_t(ab) = \sigma_t(a)\sigma_t(b) = e^{i(c+d)t}ab \tag{32}$$

for all t, so $ab \in A_{c+d}$. In addition, note that if $a \in A_c$, then $\sigma_t(a^*) = \sigma_t(a)^* = e^{-ict}a^*$, so $a^* \in A_{-c}$. We assume $A_0 \neq 0$, or else the entire construction of the crystal becomes trivial. It follows immediately that we have a group. □

With this fact, let $\Gamma \subset \mathbb{R}$ be the group defined above. It is of course true that we have a natural embedding $\jmath : \Gamma \to \mathbb{R}$ by inclusion, so there is a homomorphism between Pontryagin duals, $j_* : \widehat{R} \to \widehat{\Gamma}$.

Claim III.2. There is a natural pairing between $\mathbb{R}$ and $\widehat{\mathbb{R}}$.

Proof. Recall that $\widehat{\mathbb{R}} = \mathrm{Hom}(\mathbb{R}, \mathbb{T})$. Suppose $\phi \in \mathrm{Hom}(\mathbb{R}, \mathbb{T})$, so there $\phi(y) = e^{i\varphi(y)}$, where $\varphi : \mathbb{R} \to [0, 2\pi)$. We must have $\phi(y + z) = \phi(y) \cdot \phi(z)$, so $e^{i\varphi(x+y)} = e^{i\varphi(x)+\varphi(y)}$. Thus,

$$e^{i[\varphi(x+y)-\varphi(x)-\varphi(y)]} = 1 \Longrightarrow \varphi(x + y) = \varphi(x) + \varphi(y) \tag{33}$$

as $\varphi(x), \varphi(y), \varphi(x + y) \in [0, 2\pi)$. Thus, φ must be linear, so $\phi(y) = e^{ixy}$ for all y, where $x \in \mathbb{R}$. Of course, this x is unique. Suppose $e^{ixy} = e^{ix'y}$ for all y, then $e^{i[x-x']y} = 1$ for all y, which would imply that $(x - x')y$ is always a multiply of 2π, for any y. The only way which this is possible is the case that the multiple is 0, and $x = x'$. □

It follows that we have a natural mapping $\mathbb{R} \to \widehat{\Gamma}$, taking $x \in \mathbb{R}$, to the map $y \mapsto e^{ixy} \in \widehat{\mathbb{R}}$, to the map $e^{ixy} \circ j$, where j is the embedding of Γ into $\mathbb{R}$. The image of $\mathbb{R} \mapsto \widehat{\Gamma}$ is dense, which implies that we should be able to extend a map $t \mapsto \sigma_t(a)$ to a map on all of $\widehat{\Gamma}$. This allows us to define $E : A \to A_0$, as

$$E(a) = \int_{\widehat{\Gamma}} \sigma_\chi(a) d\chi \tag{34}$$

where we are taking the integral with respect to the Haar-measure on $\widehat{\Gamma}$, and integral being taken is again the Bochner, operator-valued integral. The reason why this map goes into A_0 is because

$$\sigma_t(E(a)) = \int_{\widehat{\Gamma}} \sigma_{t+\chi}(a) d\chi = \int_{\widehat{\Gamma}} \sigma_\chi(a) d\chi \tag{35}$$

as the Haar measure is invariant under action by the group $\widehat{\Gamma}$, which amounts to addition by some scalar. Moreover, note that for $a \in A_\lambda$ with $\lambda \neq 0$, we will have

$$E(a) = \int_{\widehat{\Gamma}} \sigma_\chi(a) d\chi = e^{-it\lambda} \int_{\widehat{\Gamma}} \sigma_\chi(\sigma_t(a)) d\chi = e^{-i\lambda t} \int_{\widehat{\Gamma}} \sigma_\chi(a) d\chi = e^{-i\lambda t} E(a) \tag{36}$$

which means that $(1 - e^{-i\lambda t})E(a) = 0$ for all t. If $\lambda \neq 0$, $E(a) = 0$.

This observation immediately establishes an important fact: there is a natural way in which we can "collapse" the entire algebra A down to the crystal A_0, by simply performing the above "averaging" operation, such that everything outside of A_0 is collapsed to a point. **This implies that the above map E can actually be thought of as a contraction map to the quotient,** A_c This will allow us to draw connections between ground states of A, and ground states of A_c.

Proposition III.2. Suppose the subalgebra spanned by the A_λ is dense in A. Then there is a completely positive contraction onto the crystal A_c obtained via the averaging map defined above, $\vartheta : A \to A_c = A_0/I$, where $\vartheta(a) = E(a) + I$. In the case that $A_c \neq 0$, then the map $\psi \mapsto \psi \circ \vartheta$ is a bijection of the state-space of A_c and the σ-ground states of A. If $A_c = 0$, then there are no σ-ground states of A.

Proof. The first claim, that we have a contraction map, is true from the definition, as we explained in the previous paragraph. Recall that a ground state of A is precisely a character map $\phi : A \to \mathbb{C}$ such that for σ-analytic b and $a \in A$, the function $z \mapsto \phi(a\sigma_z(b))$ is bounded in the region of $\mathrm{Im}[z] \geq 0$. Since the algebra generated by the A_λ is dense, this is equivalent to the function $z \mapsto \phi(a\sigma_z(b))$ being bounded for all σ-analytic $b \in A_\lambda$ (in fact, all such b are σ-analytic), and all $a \in A$. Note that $\phi(a\sigma_z(b)) = e^{i\lambda z}\phi(ab)$. This function is bounded if and only if $\phi(ab) = 0$ for all $a \in A$, and $b \in A_\lambda$ for $\lambda < 0$, as in this case $i\lambda z$ has real component going $+\infty$ for $\mathrm{Im}[z] \geq 0$. By Cauchy-Schwarz,

$$||\phi(ab)||^2 \leq ||\phi(a^*a)||||\phi(b^*b)|| \tag{37}$$

so if $\phi(b^*b) = 0$, then $\phi(ab) = 0$. Moreover, if $\phi(ab) = 0$ for all a, then in particular, $\phi(b^*b) = 0$ for all b, we have $\phi(ab) = 0$ if and only if $\phi(b^*b) = 0$ for all $b \in A_\lambda$ with $\lambda < 0$. Recall from earlier that $\overline{A_\lambda^*} = A_{-\lambda}$. Thus, the above condition is equivalent to ϕ being a state which factor through $\overline{A_\lambda^* A_\lambda} = \overline{A_{-\lambda} A_{-\lambda}^*} = I_{-\lambda}$ for all $\lambda < 0$, so ϕ factors through I_λ for all $\lambda > 0$, which is true if and only if ϕ is a state which factors through I. Hence, $\phi \circ \theta$ is clearly a state for A_c. Moreover, given some state on A_c, we can think of it as a state $\phi \circ \vartheta$, where ϕ just collapses everything not in A_0 to 0, which in turn, by the above, means that ϕ will be a ground state for A. $\square$

Now that I have a better understanding of C^*-dynamical systems, for my March report, I will be diving deeper into some of the other theorems outlined in this work, which explore further properties of the algebra crystal.
