Lee's Introduction to Smooth Manifolds: Chapter 1 Assorted Notes, Proofs and Exercises

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I. Exercise 1.1

For both definitions, it is clear that the new subsumes the old, as both open balls and $\mathbb{R}^n$ are open neighbourhoods around every point of $\mathbb{R}^n$. Conversely, if we assume the original definition, then given some point $p \in M$ and local homeomorphism $\varphi : U \to \widehat{U}$, then we can find an open ball B around $\varphi(p)$, and note that the restriction $\varphi : \varphi^{-1}(B) \to B$ is a homeomorphism, so p has a neighbourhood homeomorphic to an open ball. As for the other case, this follows from the fact that open n-balls are homeomorphic to $\mathbb{R}^n$.

II. Example 1.4

We wish to show that the n-sphere is a manifold. The Hausdorff condition and second-countability are clearly inherited from the global space. As for the local Euclidean condition, the general strategy is to separate the sphere into different parts. Let $p = (p_1, \dots, p_{n+1})$ be a point on the n-sphere. Clearly, there must exist some coordinate $p_k \neq 0$. We let

$$U_p = \left\{(x_1, \dots, x_{n+1}) \mid x_k^2 = 1 - x_{n+1}^2 - \cdots - x_{k+1}^2 - x_{k-1}^2 - \cdots - x_1^2 \text{ and } \mathrm{sign}(x_k) = \mathrm{sign}(p_k)\right\} \tag{1}$$

Clearly, $U_p = S^n \cap \mathbb{R}_k^{\mathrm{sign}(p_k)}$, which is open in the subspace topology. Let $\pi_k : U_p \to \pi_k(U_p)$ be the projection onto all coordinates *except for the k-th*. Define

$$\pi_k^{-1}(x) = \pi_k^{-1}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}) = (x_1, \dots, f_k(x), \dots, x_{n+1}) \tag{2}$$

where $f_k(x) = \mathrm{sign}(p_k)\sqrt{1 - x_{n+1}^2 - \cdots - x_1^2}$ is clearly continuous. It is easy to check that π_k^{-1} is the well-defined inverse of π_k, and both maps are continuous, so $\pi_k(U_p)$ is open, and π_k is the desired local homeomorphism.

III. Lemma 1.10

Note that M is second-countable, let $\mathcal{B}$ be the countable basis. For each $p \in M$, note that there exists a local homeomorphism $\varphi : U \to \widehat{U}$, where U contains p. Since there exists some $B \in \mathcal{B}$ with $p \in B \subset U$, B is homeomorphic to an open subset of $\mathbb{R}^n$. Thus, we have a countable subcollection $\mathcal{B}'$ of elements of $\mathcal{B}$ such that each $p \in M$ is contained in some element of $\mathcal{B}'$ and each element of $\mathcal{B}'$ is homeomorphic (via some φ_B) to an open subset of $\mathbb{R}^n$.

For each $B \in \mathcal{B}'$, let S_B denote the countable collection of all open balls of rational radius contained in $\varphi_B(B)$, centred at rational points. Let $\varphi_B^{-1}(S_B)$ denote the inverse image of each of these balls, which are all open. Clearly, given some $p \in U$ open in M, p is contained in some $B \in \mathcal{B}'$, and $B \cap U$ is an open subset of B, so $\varphi_B(B \cap U)$ is open in $\varphi_B(B)$. We can pick an open ball in S_B containing $\varphi_B(p)$, so p is contained in some element of $\varphi_B^{-1}(S_B)$ which is contained in B. Let $S = \cup_{B \in \mathcal{B}'} S_B$. Suppose $\varphi_{B_1}^{-1}(B_1), \varphi_{B_2}^{-1}(B_2) \in S$ are two preimages of open balls with non-empty (thus open) intersection, U. Then the images $\varphi_{B_1}(U)$ and

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$\varphi_{B_2}(U)$ will intersect as and be open, so we can find some open ball B at a rational point, of rational radius, in this intersection, so $\varphi_{B_1}^{-1}(B)$ (and $\varphi_{B_2}^{-1}(B)$) are in U. Thus, S satisfies the intersection condition, and is in fact a countable basis of coordinate balls (as it is a countable intersection of countable sets).

Finally, note that since each φ_B is a homeomorphism, $\overline{\varphi_B^{-1}(B)} = \varphi_B^{-1}(\overline{B})$. Since $\overline{B}$ is compact, so too is $\varphi_B^{-1}(\overline{B})$ and thus $\overline{\varphi_B^{-1}(B)}$.

IV. Exercise 1.14

We begin with the first claim. Suppose $\mathcal{X}$ is locally finite, let x be a point of the topological space. Then there exists a neighbourhood U intersecting only $X_1, \ldots, X_N$. Suppose U intersects some $\overline{X_M}$ not in this set. Then by definition, U contains a point y where every neighbourhood of y intersects X_M, so U intersects X_M, a contradiction. Thus, $\overline{\mathcal{X}}$ is locally finite.

Next, note that $\cup_{X\mathcal{X}}\overline{X} \subset \overline{\cup_{X\in\mathcal{X}}X}$ trivially. To prove inclusion the other way, given $x \in \overline{\cup_{X\in\mathcal{X}}X}$, note that every open set of x intersects the union at a finite number of X. If there isn't some X' intersected by all neighbourhoods, we can pick some neighbourhood U which intersects $X_1, \ldots X_M$ and neighbourhood $U_1, \ldots, U_M$ where U_k does not intersect X_k. Then $U \cap U_1 \cap \cdots U_M$ does not intersect any element of $\mathcal{X}$, which is a contradiction.

V. Proposition 1.16

This result is very simple when it is understood diagramatically, which is basically as far as we made it in third-year algebraic topology! Nevertheless, a formal proof is required.

We first require a lemma.

Lemma V.1 (Lebesgue Number Lemma)**.** Suppose X is a compact set in a metric space and $\mathcal{A}$ is an open cover. Then there exists some $\delta > 0$ such that every subset of X having diameter less than δ is contained in an element of $\mathcal{A}$.

Proof. Let $A_1, \ldots, A_N$ be a finite subcover. The key idea is to use extreme value theorem. Define the function $f : X \to \mathbb{R}$ as

$$f(x) = \frac{1}{N}\sum_{k=1}^{N} d(x, X - A_k), \tag{3}$$

the average distance from x to the exterior of each A_k. Clearly, $x \in A_k$, for some A_k open, so the minimum value of f must be greater than 0. Set it to δ. Suppose U has diameter less than δ. Suppose U is not contained in a single A_k. Then, there exist points x and y of U not contained in a common A_k, with $d(x, y) \leq \delta$. Then, clearly, $d(x, y) \geq d(x, X - A_k)$ for each A_k. But then, $f(x) < \delta$, a contradiction. It follows that U must be contained in a single A_k, so the lemma holds. □

Let $\mathcal{B}$ be a countable basis for M, let $B, B' \in \mathcal{B}$ such that $B \cap B'$ is non-empty. Since $B \cap B'$ is a manifold, it has a finite number of connected components, which are also path-connected. Pick some x from each component. Then, for all pairs of $B, B' \in \mathcal{B}$, take all such points x and combine them into a set $\mathcal{X}$. For each pair of points $x, x' \in \mathcal{X}$ where $x, x' \in B$, let $h^B_{x,x'}(t)$ be a path between them.

Assume M is path-connected. Let us pick some $p \in \mathcal{X}$. Let $f : [0, 1] \to M$ be a loop in $\pi_1(M, p)$. The collection of $f^{-1}(B)$ for $B \in \mathcal{B}$. will clearly cover $[0, 1]$, so there exists a finite subcover, $f^{-1}(B_1), \ldots, f^{-1}(B_N)$. **WIP**
