Fall 2023 MAT437 problem set 4

Jack Ceroni*

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I. Problem 1

Part 1. Do we need connectedness of X here? This is required in Example 3.5.5, we are supposed to use dim as it is defined in that example, so I assume it here.

Example 3.3.5 already proved dim is a surjective group homomorphism. Suppose $K_0(C(X)) \simeq \mathbb{Z}$, let φ be an isomorphism from $\mathbb{Z}$ to $K_0(C(X))$. Then $\phi = \dim \circ \varphi$ is a group homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$. Note that we must have $\phi(n) = \phi(1 + \cdots + 1) = \phi(1)n$, so ϕ is either identically 0 or injective. The former case is clearly not true, so ϕ is injective. It follows that $\dim = \phi \circ \varphi^{-1}$ is an injective group homomorphism, which is also surjective from Example 3.3.5, and is thus an isomorphism.

To find a generator for $K_0(C(X))$, we simply can take $\dim^{-1}(1)$, which can be represented by the element $[p]_0$ for which $\mathrm{Tr}(p(x)) = 1$, so p, a representative as a function in $C(X, M_n(\mathbb{C}))$ has eigenvalue 1 with multiplicity precisely 1.

II. Problem 2

To begin, let us assume (i), so that $\tau(uv) = \tau(vu)$ for all u and v. Of course, (ii) follows immediately.

Assuming (ii), note that $\tau(uau^*) = \tau(au^*u) = \tau(a)$, as u iis unitary, so $u^*u = 1$. Thus, (ii) implies (iii).

Finally, let us assume (iii). Given some x, we note that we are able to decompose x into a linear combination of self-adjoint elements $f(x) = x^* + x$ and $g(x) = i(x^* - x)$. Next, recall the definition $|a| = (a^*a)^{1/2}$, for some element of the algebra A. From the continuous function calculus, for self-adjoint a, this mapping will have the effect of taking the absolute value of the spectrum. More generally, from the continuous function calculus, for self-adjoint a, the map $a \mapsto a + |a|$ will enact the function $f(x) = x + |x|$ on the spectrum, which clearly is non-negative for all x. Moreover, $a \mapsto |a| - a$ will enact the function $f(x) = |x| - x$ on the spectrum, which is also non-negative for all x. Clearly,

$$a = \frac{a + |a|}{2} - \frac{|a| - a}{2} = p(a) + q(a) \tag{1}$$

so it follows that self-adjoint a can be written as a linear combination of positive elements $p(a)$ and $q(a)$. Combining the two results, and it follows that we can write any element of A as a linear combination of four positive elements, $x = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$. We then have, for any unitary u,

$$\tau(uxu^*) = \sum_{j=1}^{4} c_j \tau(ux_ju^*) = \sum_{j=1}^{4} c_j \tau(x_j) = \tau(x) \tag{2}$$

from the assumption of (iii). It followsd immediately that for any $x \in A$ and any unitary $u \in \widetilde{A}$, $\tau(ux) = \tau(u^*uxu) = \tau(xu)$. Finally, recall that we can write any element of A as a linear combination of four unitaries in the unitization $\widetilde{A}$ (we proved this is a previous problem set). Thus, given $y \in A$, we have $y = \sum_{j=1}^{4} d_j u_j$

* jackceroni@gmail.com

for unitaries $u_j \in \widetilde{A}$. In particular, we have

$$\tau(yx) = \sum_{j=1}^{4} d_j \tau(u_j x) = \sum_{j=1}^{4} d_j \tau(x u_j) = \tau(xy) \tag{3}$$

for any $x, y \in A$. Thus, (iii) implies (i).

III. Problem 3

Part 1. For notational ease, let $\varphi_t(a) = \varphi(t, a)$. Let S be the set of $a \in A$ for which $t \mapsto \varphi(t, a)$ is continuous. To demonstrate that S is a subalgebra, we merely must show that it is a submodule, so it must be closed under all arithmetic operations (and contains additive inverses and the additive identity, but this is implied by the previous conditions). Indeed, note that if $a, b \in S$, then $\varphi(t, a+b) = \varphi(t, a) + \varphi(t, b)$ is continuous, as it is the sum of two continuous maps. In addition, for scalar multiplication, $\varphi(t, \lambda a) = \lambda \varphi(t, a)$ is clearly continuous for $a \in S$ and $\varphi(t, ab) = \varphi(t, a)\varphi(t, b)$ is also continuous as it is a product of continuous functions for $a, b \in S$. The fact that S contains the identity and additive inverses follows from setting $\lambda = 0, -1$ and using the closure under scalar multiplication by elements of $\mathbb{C}$. Thus, S is a sub-$\mathbb{C}$ algebra.

To show that S is a $*$-algebra, we simply note that $\varphi(t, a^*) = \varphi(t, a)^*$, so if $a \in S$, then $\varphi(t, a)$ is continuous, so $\varphi(t, a)^*$ is continuous, as we assume the involution is continuous, and we have the composition of two continuous functions, so $\varphi(t, a^*)$ is continuous as well and $a^* \in S$.

Finally, to show that the sub-$*$ algebra is in fact a sub-C^* algebra, it is simply necessary to demonstrate that S is norm-closed with respect to the norm of the larger algebra A. We prove a lemma.

Lemma III.1. Suppose A and B are C^*-algebras and φ is a $*$-homomorphism from A and B. Then φ is a continuous function when A and B are treated as metric spaces.

Proof. First, we will show that $\mathrm{sp}(\varphi(a)) \subset \mathrm{sp}(a)$. Suppose $\lambda \in \mathrm{sp}(\varphi(a))$, so $\varphi(a) - \lambda \cdot 1_{\widetilde{B}}$ is not invertible, where $1_{\widetilde{B}}$ is the unit in the unitization of B. Suppose $a - \lambda 1_{\widetilde{A}}$ is invertible. Then so is $\widetilde{\varphi}(a - \lambda \cdot 1_{\widetilde{A}}) = \varphi(a) - \lambda 1_{\widetilde{B}}$ where $\widetilde{\varphi}$ is the lifting of φ to the $*$-homomorphism between $\widetilde{A}$ and $\widetilde{B}$. This is a contradiction, so we must have $\lambda \in \mathrm{sp}(a)$, and the claim is proved. This result implies that $r(\varphi(a)) \leq r(a)$. Thus,

$$||\varphi(a)||^2 = ||\varphi(a)^* \varphi(a)|| = r(\varphi(a)^* \varphi(a)) = r(\varphi(a^* a)) \leq r(a^* a) = ||a^* a|| = ||a||^2 \tag{4}$$

so $||\varphi(a)|| \leq ||a||$. Thus, $||\varphi(x) - \varphi(y)|| = ||\varphi(x - y)|| \leq ||x - y||$ so φ is Lipschitz, and is thus continuous. $\square$

Suppose $\{a_n\}$ is a sequence of points of A, so that each function $t \mapsto \varphi(t, a_n)$ is continuous. Because φ is continuous in its second argument, we have $\varphi(t, a_n) \to \varphi(t, a)$ as $a_n \to a$. In fact, the convergence is actually uniform. Note that $||\varphi(t, a) - \varphi(t, a_n)|| = ||\varphi(t, a - a_n)|| \leq ||a - a_n||$, so that we have a bounded independent of t. Since $a_n \to a$, if we choose n large enough we can make $\varphi(t, a_n)$ close to $\varphi(t, a)$ for all t. It follows from uniform limit theorem that $\varphi(t, a)$ is continuous, so $a \in S$, and $S = \overline{S}$ is closed in the metric topology.

Thus, S is, in fact, a sub-C^* algebra.

Part 2. Note that $F \subset S$, so since $C^*(F)$ is the minimal C^*-algebra generated by F, we must have $C^*(F) \subset S$ as S is a C^*-algebra. But $C^*(F) = A$ and $S \subset A$, so $S = A$, and it follows that for every $a \in A$, $\varphi_t(a)$ is continuous.

IV. Problem 4

Part 1. Recall that a separable C^*-algebra is one that contains a countable dense subset. Of course, we know that $K_{00}(A) = G(\mathcal{D}(A)) = G(\mathcal{P}_\infty(A)/\sim_0)$. The Grothendieck Abelian group is constructed via a quotient of $\mathcal{P}_\infty(A)/\sim_0 \times \mathcal{P}_\infty(A)/\sim_0$, and is therefore countable if and only if $\mathcal{P}_\infty(A)/\sim_0$ is countable.

Of course, note that if A has a countable dense subset, then so does $M_n(A)$ for some n, and so does the subset $\mathcal{P}(M_n(A))$. From RLL Proposition 2.2.4, we know that if p and q are projections in A, and $||p - q|| < 1$, then $p \sim_h q$. In addition, $p \sim_h q$ implies that $p \sim q$. Let S_n be a countable dense subset of $\mathcal{P}(M_n(A))$, given $p \in \mathcal{P}(M_n(A))$, we can clearly find $p' \in S_n$ such that $||p - p'|| < 1$ (in fact, we can find p' arbitrarily close to p), so $p \sim p'$ for some $p' \in S_n$. Let $S = S_1 \cup S_2 \cup \cdots$, a countable set, and recall that $\mathcal{P}_\infty(A) = \cup_{n=1}^\infty \mathcal{P}(M_n(A))$. The fact we have just proved implies that for any element $p \in \mathcal{P}_\infty(A)$, there is $p' \in S$ such that $p \sim p'$, so $p \sim_0 p'$. It follows immediately that $\mathcal{P}_\infty(A)/\sim_0$ must be countable. So, from the first argument, $K_{00}(A)$ is a countable Abelian group.

Of course, we know that for A a unital C^*-algebra, $K_{00}(A) = K_0(A)$, so the latter claim follows immediately from above.

Part 2. Recall the definition of $K_0(A)$ for non-unital C^*-algebras: we consider the split exact sequence associated with A and its unitization $\widetilde{A}$. We define K_0 as the kernel of the homomorphism $K_0(\pi) : K_0(\widetilde{A}) \to K_0(\mathbb{C})$ where $\pi : \widetilde{A} \to \mathbb{C}$ is the projection which takes $(a, \alpha) \in \widetilde{A}$ to $\alpha \in \mathbb{C}$. The previous result demonstrated that $K_{00}(\widetilde{A}) = K_0(\widetilde{A})$ is countable. The kernel of a map from this domain is of course a subset, and thus itself must be countable as well, so the proof is immediate.

V. RLL Exercise 3.13

I felt like doing a problem that was a bit more topological, so here is my solution to another problem from RLL!

The purpose of this exericse is to verify the claim from Example 3.3.6 that the map $t \mapsto \varphi_t(f)$ is continuous for each $f \in C(X)$, where X is a compact Hausdorff space, where $\alpha : [0,1] \times X \to X$ is a continuous map, and where $\varphi_t(f)(x) = f(\alpha(t,x))$. Let $t_0 \in [0,1]$ and $\varepsilon > 0$ be given. Set

$$W = \{(t,x) \mid |f(\alpha(t,x)) - f(\alpha(t_0,x))| < \varepsilon\} \tag{5}$$

Show there is $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \times X$ is contained in W, and conclude that $||\varphi_t(f) - \varphi_{t_0}(f)|| < \varepsilon$ for all $t \in (t_0 - \delta, t_0 + \delta)$.

To prove this claim, we will utilize a result known in certain topology texts as the *tube lemma*. As an exercise, I will attempt to re-derive it here.

Lemma V.1 (Tube lemma)**.** Let X and Y be topological spaces, suppose Y is compact. Suppose N is open in $X \times Y$, containing a slice $\{x_0\} \times Y$. Then there exists a tube, $W \times Y$, for W open in X, such that $\{x_0\} \times Y \subset W \times Y \subset N$.

Proof. Note that $\{x_0\} \times Y$ is compact. For each $x_0 \times y$, let us choose an open rectangle $A_y \times B_y$ (a basis element in the priduct topology) about this point which is contained in N. Since the slice is compact, we can choose a finite subcover, $A_{y_1} \times B_{y_1}, \dots, A_{y_n} \times B_{y_n}$. Let $W = A_{y_1} \cap \cdots \cap A_{y_n}$. Clearly, $W \times Y \subset N$, as the B_{y_k} cover Y. This completes the proof. □

Now that we have the tube lemma, we can prove the result. In particular, let us fix $t_0 \in [0,1]$. Clearly, the function $g : (x,t) \mapsto f(\alpha(t,x)) - f(\alpha(t_0,x))$ is continuous, as both f and α are. We must demonstrate that $W = \{(x,t) \mid |g(x,t)| < \varepsilon\}$ is an open set. Indeed, suppose $(x,t) \in W$, so $|g(x,t)| < \varepsilon$, or in other words, $g(x,t) \in (-\varepsilon, \varepsilon)$ Note that, $(x,t) \in g^{-1}(-\varepsilon, \varepsilon)$, an open set, since g is a continuous function. Clearly, $g^{-1}(-\varepsilon, \varepsilon) \subset W$. Therefore, we can find an open neighbourhood around each $(x,t) \in W$ that is itself in W, so W is open.

It follows immediately that, since the slice $\{t_0\} \times X$ is clearly contained in W, we can find via the tube lemma a tube $(t_0 - \delta, t_0 + \delta) \times X$ which is itself contained in W. In other words, for any $(t,x) \in (t_0 - \delta, t_0 + \delta) \times X$, we have $|f(\alpha(t,x)) - f(\alpha(t_0,x))| < \varepsilon$. So, for any $t \in (t_0 - \delta, t_0 + \delta)$,

$$||\varphi_t(f) - \varphi_{t_0}(f)|| = \sup_{x \in X} |f(\alpha(t,x)) - f(\alpha(t_0,x))| < \varepsilon \tag{6}$$

which completes the proof.