CALCULUS ON MANIFOLDS: INTEGRATION

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Contents

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1. Problem 1

Let the collection of points for which $f \neq g$ be denoted as $\{x_1,\ ...,\ x_n\}$. Let $m = g(x_1) + \cdots + g(x_n)$.

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2. Problem 2

2.1. Part A.

Pick some $\epsilon > 0$. Since C is content-0, we can choose a finite collection of closed rectangles R_k such that the sum of all $\text{vol}(R_k)$ is less than ϵ. We claim that ∂C is covered by the collection of R_k as well. Let $R = \cup R_k$. Indeed, given some $c \in \partial C$, note that every neighbourhood of c must intersect both C and C^C, so $c \notin R^C$, as this is an open set that does not intersect C.

It follows by definition that ∂C is also content-0.

2.2. Part B.

Take the rationals in $[0, 1]$, in the context of $\mathbb{R}$. This set is countable, so it is measure-0, but its boundary is clearly $[0, 1]$, which is not measure-0.

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3. Problem 3

Since f is integrable, it is continuous except on a set of measure-0, which we denote S. Since g is continuous, it follows that the composite $g \circ f$ is continuous for all $x \notin S$. Thus, $f \circ g$ is continuous except on a measure-0 set, so it is integrable.

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4. Problem 4

Since A is Jordan-measureable, the indicator function χ_A is integrable on some rectangle K containing A. Thus, by the Reimann criteria for integration, we can pick some partition P of K such that:

$$U(\chi_A, P) - L(\chi_A, P) = \sum_{S \in P} M_S(\chi_A)\text{vol}(S) - \sum_{S \in P} m_S(\chi_A)\text{vol}(S) < \frac{1}{257}$$

Clearly, for some $S \in P$ such that $S \subset A$, we will have $M_S(\chi_A) = m_s(\chi_A) = 1$, and similarly, for $S \subset A^C$, we will have both M_S and m_S equal to 0. Finally, for the remaining S, which intersect both A and A^C, we have $M_S = 1$ and $m_S = 0$, so:

$$\sum_{S \in P} M_S(\chi_A)\text{vol}(S) - \sum_{S \in P} m_S(\chi_A)\text{vol}(S) = \sum_{S \in P'} \text{vol}(S) < \frac{1}{257}$$

where P' is the set of all rectangles intersecting both A and A^C. Thus, taking $R = P$ and $R' = P'$, the proof is complete.

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5. Problem 5

5.1. Part A.

Since χ_B is integrable, it follows that ∂B is measure-0, so the closed set $\overline{B} = B \cup \partial B$ containing B is measure-0. Since this set is also bounded, it follows that it is compact, so it is content-0.

Thus, for any $\epsilon > 0$, we can pick some finite collection S of rectangles covering $\overline{B}$, with sum of volumes less than ϵ. We can take intersections of these rectangles, and extend the resulting set to a partition P of a rectangle R containing $\overline{B}$.

Clearly, the upper sum of χ_B on R will be the sums of the volumes of rectangles intersecting B, which is precisely the subset of P of rectangles obtained from intersecting elements of S. Clearly, the sum of volumes of these rectangles will be less than ϵ. Since $\mathrm{vol}(B) \leq U(\chi_B, P)$, for all P, it follows that $\mathrm{vol}(B)$ must equal 0, as we can make $U(\chi_B, P)$ arbitrarily small.

5.2. **Part B.** Let A be the set of rationals in $[0, 1]$. Checking that χ_A satisfies the criteria is a simple exercise.

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6. Problem 6

Suppose is not equal to 0. Then there exists some point a at which $f(a) \neq 0$. Since f is continuous, there is a neighbourhood of this point on which $f > 0$. Inside this neighbourhood, we can pick a rectangle R.

Thus, picking a partition containing R, we get a lower sum that is greater than 0, implying the integral itself must be greater than 0, which is a contradiction. Thus, we must have $f = 0$.

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7. Problem 8

Since A and B are Jordan-measureable, it follows that χ_A and χB are integrable. In addition, we have assumed that the functions $f_t(x) = \chi_{A_t}(x)$ and $g_t(x) = \chi_{B_t}(x)$, where A_t and B_t are the slices of A and B, are integrable as well.

Recall that for any t, we have:

$$\int \chi_{A_t}(x) = \int \chi_{B_t}(x)$$

by assumption. Clearly, for some t, we have $\chi_{S_t}(x) = \chi_S(x, t)$. Thus, by Fubini's theorem:

$$\mathrm{vol}(A) = \int \chi_A = \int_{\mathbb{R}} \int_{R_A} \chi_A(x, t)\ dx\ dt = \int_{\mathbb{R}} \int_{R_B} \chi_B(x, t)\ dx\ dt = \mathrm{vol}(B)$$

and we are done. **Note:** I'm being quite sloppy with my notation in a few places, but the idea should be clear.

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8. Problem 11

Let E be the ellipsoid in question. This is a standard change of vbariables. Let:

$$f(x, y, z) = \left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{3}}, \frac{z}{\sqrt{5}} \right)$$

Obviously, this function is bijective, and differentiable, with its differential having a non-zero determinant:

$$Df(x, y, z) = \mathrm{diag}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{5}} \right)$$

Finally, it obvious that:

$$f(C) = f(\{(x, y, z) \mid x^2 + y^2 + z^2 \leq \}) = \left\{ \left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{3}}, \frac{z}{\sqrt{5}} \right) \;\middle|\; x^2 + y^2 + z^2 \leq 1 \right\} = E$$

Thus, by change of variables:

$$\mathrm{vol}(E) = \mathrm{vol}(f(C)) = \int_{f(C)} 1 = \int_C |\det Df|$$

From above, $|\det Df| = \frac{1}{\sqrt{30}}$. Thus:

$$\mathrm{vol}(E) = \frac{1}{\sqrt{30}} \int_C 1$$

Since C is simply a sphere with radius 1, we know $\int_C 1 = \frac{4}{3}\pi$ (we could also show this with a spherical coordinate transform, but we are lazy). Thus, the volume of the ellipsoid is $\frac{1}{\sqrt{30}} \frac{4\pi}{3}$.

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9. Problem 12

9.1. **Part A.** This is an immediate consequence of the fundamental theorem of calculus. Let $g_1(x, y) = \partial_x \partial_y f(x, y)$, and let $g_2(x, y) = \partial_y \partial_x f(x, y)$. We know both of these functions are continuous. Hence, by Fubini's theorem:

$$\int_R g_1(x, y) = \int_{[c,d]} \int_{[a,b]} \partial_x \partial_y f(x, y)\, dx\, dy = \int_{[c,d]} \left(f(b, y) - f(a, y)\right)\, dy = f(b, d) - f(a, d) - f(b, c) + f(a, c)$$

An almost identical calculation shows that $\int_R g_2$ yields the same result. Thus, we have the desired equality.

9.2. **Part B.** Suppose there is some (a, b) at which $\partial_x \partial_y f - \partial_y \partial_x f > 0$. Since this function is continuous, there must be a neighbourhood around (a, b) on which it is positive. Taking the integral on a rectangle contained in this neighbourhood gives some positive number, but this contradicts Part A. Thus, not (a, b) exists.

Identical logic shows that an (a, b) at which the difference is negative cannot exist. Thus, $\partial_x \partial_y f = \partial_y \partial_x f$ for all points.

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10. Problem 13

Let $f(x, y) = \sqrt{x^2 + y^2}$. Note that $f \circ T_\theta = f$:

$$(f \circ T_\theta)(x, y) = f(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta) = \sqrt{(x^2 + y^2)(\sin^2\theta + \cos^2\theta)} = \sqrt{x^2 + y^2} = f(x, y)$$

Since B is Jordan-measureable, it bounded, so it is contained in some rectangle R. Since f is continuous, it has a maximum value on R. Since f is the radial distance of a point (x, y) from the origin, it therefore follows there is some circle C containing R, and thus B. It is easy to see that $T_\theta(C) = C$.

Now, all that is left to do is a change of variables. Clearly, T_θ is a diffeomorphism, and has determinant 1. Thus:

$$\mathrm{vol}(B) = \int_C \chi_B = \int_{T_\theta(C)} \chi_B = \int_C \chi_B \circ T_\theta = \int_C \chi_{T_\theta(B)} = \mathrm{vol}(T_\theta B)$$

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11. Problem 14

This is a straightforward application of definitions.

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12. Problem 15

Easy

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13. Problem 16

Easy

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14. Problem 17

Easy
