Fall 2023 MAT437 problem set 5

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I. Problem 1

Part 1. Verification that μ preserves all of the algebraic operations of the algebra is simple:

$$\mu(\lambda a + b) = s(\lambda a + b)s^* = \lambda sas^* + sbs^* = \lambda\mu(a) + \mu(b) \tag{1}$$

$$\mu(ab) = sabs^* = sas^*sbs^* = \mu(a)\mu(b) \tag{2}$$

as s is an isometry, so $s^*s = 1$. It is also easy to check that involution is preserved, as $\mu(a^*) = sa^*s^* = (sas^*)^* = \mu(a)^*$. Note that this map is automatically injective: suppose $sas^* = 0$. Then $s^*sas^*s = s^*0s \Rightarrow a = 0$. Thus, μ is in fact an endomorphism of C^*-algebras. Moreover, μ has a left-inverse, but clearly this map is not a right-inverse! It is clear that the map $a \mapsto s^*as$ will also be a right-inverse when s is *unitary*, and as we will show below, this is not only a sufficient condition for μ to be an isomorphism, but is also necessary!

We claim that μ is an isomorphism if and only if s is not only just an isometry, but is in fact *unitary*, so $ss^* = 1$. Clearly, if s is unitary, then μ is invertible: $\mu^{-1}(a) = s^*as$.

If μ is an isomorphism, it clearly must be both surjective and injective. In particular, since $s^* \in A$, there must exist some $k \in A$ such that $sks^* = s^*$, so $sk = sks^*s = s^*s = 1$. Thus, s has a right-inverse. Clearly, it has a left-inverse, s^*. We then have $k = s^*sk = s^*$. Thus, $s^*s = ss^* = 1$, so s is unitary.

Part 2. *To jog my own memory, I'm going to begin this question by briefly recalling the functoriality of K_0 in inducing maps between K_0-groups, from maps between C^*-algebras.*

Indeed, suppose we have a $*$-homomorphism $\varphi : A \to B$ between C^*-algebras A and B. Then there is an induced map $\widetilde{\varphi} : \mathcal{P}_\infty(A) \to \mathcal{P}_\infty(B)$ which is achieved via evaluating φ on the entries of a matrix of $\mathcal{P}_\infty(B)$. We then can recall that for maps from $\mathcal{P}_\infty(A)$ to an Abelian group A, with a particular set of properties, there is a universal property inducing a map from $K_0(A)$ to G. Thus, we can consider the map $\phi(p) = [\widetilde{\varphi}(p)]_0 = \gamma([\widetilde{\varphi}(p)]_{\mathcal{D}}) \in K_0(B)$, where $K_0(B)$ is an Abelian group. If this map has these "nice properties" to which we refer (which, as we can verify, it does), then we will find that we can induce a map $\widetilde{\phi}$ from $K_0(A)$ to $K_0(B)$ via the map we just defined! Such a map will be unique and satisfy $\widetilde{\phi}([p]_0) = \phi(p) = [\widetilde{\varphi}(p)]_0$. We will write $K_0(\varphi) = \widetilde{\phi}$.

Having completed our brief recollection, let us consider $K_0(\mu)$. The map induced by μ on $\mathcal{P}_\infty(A)$ takes (i, j)-th entry of a matrix a_{ij} to $s^*a_{ij}s$. Thus, if a is an $n \times n$ matrix, we have

$$\widetilde{\mu} : a \mapsto SaS^* = \text{diag}(s, \dots, s) \cdot a \cdot \text{diag}(s^*, \dots, s^*) \tag{3}$$

where s^* and s are repeated n-times on the diagonal. We already know such a map will induce a $*$-homomorphism. From here, let us consider the mapping $p \mapsto \gamma([\widetilde{\mu}(p)]_{\mathcal{D}})$. We claim that $\widetilde{\mu}(p) \sim_0 p$: indeed, note that $\widetilde{\mu}(p) = SpS^* = Spp^*S^* = (Sp)(Sp)^*$ and $p = p^2 = p^*S^*Sp = (Sp)^*(Sp)$. Thus, by definition, they are equivalent. It follows that $K_0(\mu)([p]_0) = \gamma([p]_{\mathcal{D}}) = [p]_0$ for all $p \in \mathcal{P}_\infty(A)$. Since a generic element of $K_0(A)$ is of the form $[p]_0 - [q]_0$, we have

$$K_0(\mu)([p]_0 - [q]_0) = K_0(\mu)([p]_0) - K_0(\mu)([q]_0) = [p]_0 - [q]_0 \tag{4}$$

so $K_0(\mu)$ is in fact the identity, as desired.

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II. Problem 2

Part 1. TODO

Note: We will refer to the results outlined in the question, generally, as the characterization of Cuntz algebras (CCA).

Part 2. This follows immediately from the CCA. Note that $\mathcal{O}_n$ is itself a unital C^*-algebra (as it contains isometry s_1, so it contains $s_1^* s_1 = 1$ of $B(H)$). Suppose $s_1, \ldots, s_n$ are the elements generating $\mathcal{O}_n$. Let $t_j = us_j$. Note that

$$t_j^* t_j = s_j^* u^* u s_j = u s_j^* s_j u^* = uu^* = 1.$$

Also, note that

$$\sum_{j=1}^{n} t_j t_j^* = u \left(\sum_j s_j s_j^* \right) u^* = uu^* = 1 \tag{5}$$

Thus, from CCA, there is a unique $*$-endomorphism $\varphi_u : \mathcal{O}_n \to \mathcal{O}_n$ such that $\varphi_u(s_j) = t_j = us_j$, as desired. Finally, it is clear that

$$u = \sum_j \varphi_u(s_j) s_j^* = \sum_j u s_j s_j^* = u \sum_j s_j s_j^* = u \tag{6}$$

as desired.

Part 3. Let φ be a unital endomorphism on $\mathcal{O}_n$. We have the following claim:

Claim II.1. The elements s_j satisfy the orthogonality relation $s_j^* s_i = \delta_{ij}$.

Proof. Note that $(s_j s_j^*)^2 = s_j s_j^* = (s_j s_j^*)^*$, so each element $s_j s_j^*$ is a projection. Recall from an early homework exercise that since we have a sum of projections $s_1 s_1^* + \cdots + s_n s_n^* = 1$, then the projections must be mutually orthogonal. This then implies that

$$s_j^* s_i = s_j^* (s_j s_j^*)(s_i s_i^*) s_i = 0 \tag{7}$$

and we have the desired orthogonality. □

Now, let us set $u = \sum_j \varphi(s_j) s_j^*$. We have $us_j = \varphi(s_j)$ from the orthogonality. Moreover, it is easy to verify that u is unitary:

$$u^* u = \sum_{ij} \varphi(s_i^* s_j) s_i s_j^* = \sum_j \varphi(1) s_j s_j^* = 1. \tag{8}$$

Checking that $uu^* = 1$ is identical. Thus, φ_u and φ agree on all s_j. It follows from CCA that we must have $\varphi_u = \varphi$.

Part 4. Clearly, this map is linear and $*$-preserving. Moreover, it is multiplicative as

$$\lambda(a)\lambda(b) = \sum_{ij} s_j a s_j^* s_i b s_i^* = \sum_j s_j ab s_j^* = \lambda(ab). \tag{9}$$

Thus, λ is an endomorphism. In fact, each term in the sum defines an endomorphism: this was shown in Problem 1. From Problem 1, we immediately have

$$K_0 \left(\sum_j s_j(\cdot) s_j^* \right)(g) = \sum_j K_0(s_j(\cdot)s_j^*)(g) = \sum_{j=1}^{n} \mathrm{id}(g) = ng \tag{10}$$

as desired.

Part 5. We already found a systemic method for constructing the unitary u such that $\varphi_u = \lambda$. In particular, we note that λ is unital, so we can immediately apply Part 3 to get

$$u = \sum_j \lambda(s_j)s_j^* = \sum_j \sum_i s_i s_j s_i^* s_j^* = \sum_{ij} (s_i s_j)(s_i s_j)^*. \tag{11}$$

Clearly, $u = u^*$ with this definition, so u is self-adjoint. To conclude that $u \sim_h 1$, we define $\gamma(t) = i\sin(t)u + \cos(t)$, which is clearly always in $\mathcal{O}_n$. Moreover,

$$v(t)^*v(t) = (-i\sin(t)u + \cos(t))(\sin(t)u + \cos(t)) = \sin^2(t)u^2 + \cos^2(t) = 1 = v(t)v(t)^* \tag{12}$$

with $v(0) = 1$ and $v(1) = iu$. From here, we can connect iu to u via the path $\mu(t) = e^{it\pi/2}u$, which is clearly also in $\mathcal{O}_n$. Thus, $u \sim_h 1$. We will let ξ denote the composite path. Next, we show that $\lambda \sim_h \mathrm{id}$. Indeed, we will construct a homotopy F_t such that $F_0 = \mathrm{id}$ and $F_1 = \lambda$. Let $F_t(p) = \varphi_{\gamma(t)}(p)$. Clearly, $F_0 = \mathrm{id}$ (this follows from CCA, as we must have $\varphi_1(s_j) = s_j$ on all s_j) and $F_1 = \varphi_u = \lambda$. We simply must verify that $t \mapsto F_t(p)$ is a continuous function for each p in the algebra $\mathcal{O}_n$.

Indeed, we can verify that $t \mapsto F_t(s_j)$ is continuous for each s_j in the generating set, as $\varphi_{\gamma(t)}(s_j) = \gamma(t)s_j$, which is clearly continuous in t, as γ is. Thus, by using Problem 3 from the 4th problem set, we note that since $t \mapsto F_t(s_j)$ is continuous for each s_j, the map $t \mapsto F_t(s_j)$ is continuous for each $a \in C(s_1, \ldots, s_n) = \mathcal{O}_n$, and we have a homotopy, so $\lambda \sim_h \mathrm{id}$.

Note: *I didn't realize this until now, but RLL uses a wekaer version of homotopy than I am used to: they only require that the functions $t \mapsto F_t(x)$ are pointwise continuous for each x, rather than requiring that the map $(x, t) \mapsto F_t(x)$ be continuous.*

All that remains now is to show that $K_0(\lambda) = \mathrm{id}$, but of course this follows from the fact that if $f \sim_h g$, then $K_0(f) = K_0(g)$. We know that $K_0(\mathrm{id}) = \mathrm{id}$, so the proof is immediate.

Part 6. In the previous two parts, we showed that $K_0(\lambda)(g) = ng$ and that $K_0(\lambda)(g) = g$, for $g \in K_0(\mathcal{O}_n)$. Thus, we must have $(n-1)g = 0$ for each $g \in K_0(\mathcal{O}_n)$. In particular, $K_0(\mathcal{O}_2) = 0$.

III. Problem 3

Part 1. This is quite clear: it follows from Problem 1 Part 1: we showed that the endomorphism on $\widetilde{A}$ defined by $a \mapsto sas^*$ is an automorphism precisely when s is unitary. The reason why the resulting map restricts to an automorphism of the possibly non-unital algebra A is because if $a \in A$, then we can write $a \simeq (a, 0) \in \widetilde{A}$, and note clearly that $u(a, 0)u^* \in A$ as well.

Part 2. I'm assuming that the group operation here is multiplication. In this case, Note that

$$\mathrm{Ad}(uv)(a) = (uv)a(uv)^* = u(vav^*)u^* = (\mathrm{Ad}(u) \circ \mathrm{Ad}(v))(a)$$

Thus, $\mathrm{Ad}(uv) = \mathrm{Ad}(u) \circ \mathrm{Ad}(v)$, and we have the desired group homomorphism.

Part 3. Let $\mathrm{Ad}(u) \in \mathrm{Inn}(A)$, let $f \in \mathrm{Aut}(A)$. Let $\varphi = f \circ \mathrm{Ad}(u) \circ f^{-1}$, and note that

$$\varphi(a) = (f \circ \mathrm{Ad}(u) \circ f^{-1})(a) = f(uf^{-1}(a)u^*) = f(u)f(u)f(f^{-1}(a))f(u^*) = f(u)af(u)^*$$

Since f is a $*$-automorphism, it follows that if u is unitary, then $f(u)$ is unitary in $\widetilde{A}$, so $\varphi = \mathrm{Ad}(f(u))$, and is this an element of $\mathrm{Inn}(A)$. It follows by definition that $\mathrm{Inn}(A) \trianglelefteq \mathrm{Aut}(A)$.

Part 4. Let φ be an inner automorphism, so that $\varphi = \text{Aut}(u, \mu)$, for some $(u, \mu) \in \widetilde{A}$. $(u, \mu)^*(u, \mu) = (u, \mu)(u, \mu)^* = (0, 1)$. Thus, $u^*u + \mu u^* + \overline{\mu} u = uu^* + \mu u^* + \overline{\mu} u = 0$ and $\mu\overline{\mu} = 1$. Let $v = u + \mu 1_A$, where 1_A is the unit in A. Such an element is unitary in A from the previous equations. Moreover,

$$(u + \mu 1_A)a(u + \mu 1_A)^* = uau^* + \mu au^* + \overline{\mu} ua + \overline{\mu}\mu a = (u, \mu)(a, 0)(u, \mu)^* \tag{13}$$

so $\text{Ad}(v) = \text{Ad}(u + \mu 1_A) = \text{Ad}(u, \mu)$, and the proof is complete.

Part 5. This proof is essentially identical to that of Problem 1 Part 2. All we need to do is verify that for $u \in \mathcal{U}(\widetilde{A})$, and $p \in \mathcal{P}_\infty(A)$, that $Up = \text{diag}(u, \ldots, u) \cdot p$ is in fact itself and element of $\mathcal{P}_\infty(A)$ (this will allow us to directly translate the proof of Problem 1 Part 2). Indeed, if $(p_{ij}, 0) \in A$, and $(u, \mu) \in \widetilde{A}$, then the product $(u, \mu)(p_{ij}, 0)$ will be an element of A, so this fact is immediate.
