# A mathematical theory of topological invariants of quantum spin systems

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## I. Introduction

The goal of this essay is to outline some of the main results in the recent paper (**very recent**, in fact: this paper was released on October 25th), *A mathematical theory of topological invariants of quantum spin systems* (Ref. [1]). In particular, this paper uses a combination of operator/$C^*$-algebraic and geometric tools to prove certain results about the existence of properties of topological invariants which can be used to classify quantum spin systems.

The main goal of this essay is to describe the operator-theoretic formulation of quantum many-body/lattice systems, and some of the fundamental constructions which are used in this paper.

## II. Quantum lattice systems

We make use of an $\ell^{\infty}$ metric defined on Euclidean space, $\mathbb{R}^n$, which is of course given by $d(x,y) = \max_j |x_j - y_j|$. We also speak of distances associated with sets:

$$\text{diam}(U) = \sup_{x,y \in U} d(x,y) \quad \text{and} \quad d(U,V) = \inf_{x \in U, y \in V} d(x,y) \tag{1}$$

Moreover, we define $U^r$, for some non-empty $U$ as

$$U^r = \{x \in \mathbb{R}^n \mid d(x,U) \leq r\} \tag{2}$$

**Definition II.1.** A quantum lattice system is taken to be a subset $\Lambda \subset \mathbb{R}^n$ along with a finite-dimensional complex Hilbert space $V_j$ associated with each $j \in \Lambda$. We assume that there exists $C_\Lambda > 0$ such that the number of $v \in \Lambda$ contained inside a hypercube of diameter $d$ is upper-bounded by $C_\Lambda (d+1)^n$.

From here, we define an algebra of observables localized to a particular bounded $X \subset \mathbb{R}^n$ as

$$\mathcal{A}(X) = \bigotimes_{j \in X \cap \Lambda} \text{End}_{\mathbb{C}}(V_j) \tag{3}$$

Of course, when $X \subset Y$, there is a natural inclusion of $\mathcal{A}(X) \to \mathcal{A}(Y)$, where an elementary tensor $\sigma_{j_1} \otimes \cdots \otimes \sigma_{j_n}$ is sent to $\sigma_{i_1} \otimes \cdots \otimes \eta_{i_m}$ by $\iota_{XY}$, where the $i_k$ range over the sites in $Y \cap \Lambda$, $\eta_{i_k} = \sigma_{j_\ell}$ when $i_k = j_\ell$ for some $j_\ell \in X \cap \Lambda$, and $\eta_{i_k} = \text{id}$ otherwise.

**Proposition II.1.** Each of the local algebras $\mathcal{A}(X)$ are a finite-dimensional $C^*$-algebra. Moreover, the norm associated with these $C^*$-algebras is preserved under the canonical inclusion described above.

*Proof.* Since each $V_j$ is a finite-dimensional complex Hilbert space, each will have some inner product $\langle \cdot, \cdot \rangle_j$ and a norm $|| \cdot ||_j$ relative to which $V_j$ is a complete vector space. We can define an inner product on $V_X = \bigotimes_{j \in X \cap \Lambda} V_j$ via

$$\langle v_{j_1} \otimes \cdots \otimes v_{j_n}, w_{j_1} \otimes \cdots \otimes w_{j_n} \rangle_X = \prod_{k=1}^{n} \langle v_{j_k}, w_{j_k} \rangle \tag{4}$$

where it is easy to check that this is a valid, well-defined inner product. Note that since $X$ is bounded, and due to the upper-bound on the number of points in $\Lambda$ that may be contained inside bounded sets, the number of $j \in X \cap \Lambda$ will be finite.

This of course defines a norm, $|| \cdot ||_X$ as well. Then, via the identification of $\mathcal{A}(X)$ with $\mathrm{End}(V_X)$, we can define a norm on the local algebra as

$$||O|| = \sup_{v \in V_X, v \neq 0} \frac{||Ov||_X}{||v||_X} \tag{5}$$

In general, we know that operator norms on a finite-dimensional Hilbert spaces makes a space of operators into a $C^*$-algebra. Since $(V_X, \langle \cdot, \cdot \rangle_X)$ is a finite-dimensional inner-product space, it is complete, so it is a Hilbert space. To see that $\iota_{XY}$ preserves norm, simply recall that the operator norm on the finite-dimensional Hilbert space is precisely the largest singular value of the operator therefore,

$$||A \otimes B|| = ||A|| ||B|| \tag{6}$$

which means that given $O \in \mathcal{A}(X)$, padding it with identities to make it an element of $\mathcal{A}(Y)$ won't affect the norm, as the tensor product of identities have all of their singular values equal to 1. $\square$

**Definition II.2** (Direct limit of $C^*$-algebras)**.** Let $I$ be an index set, let $\leq$ be a preorder on $I$ such that every pair of elements has an upper-bound. Recall that a direct system of $C^*$-algebras is defined to be a collection of $C^*$-algebras $\{A_i\}_{i \in I}$ and $*$-homomorphisms $\varphi_{ij} : A_i \to A_j$ for $i \leq j$, such that $f_{ii} = \mathrm{id}$ and $f_{ik} = f_{jk} \circ f_{ij}$ when $i \leq j \leq k$. Given a direct system, we define the corresponding *direct limit* to be

$$\mathcal{A}_l = \varinjlim A_i = \bigsqcup_{i \in I} A_i \Big/ \sim \tag{7}$$

where $a \sim b$ for $a \in A_i$ and $b \in A_j$ if there exists some $k$ with $i \leq k$ and $j \leq k$ such that $\varphi_{ik}(a) = \varphi_{jk}(b)$. In other words, $a$ and $b$ are "eventually equal" as they are pushed further and further down the direct sequence, via the connecting maps. Note that the direct limit of a collection of $C^*$-algebras is a $*$-algebra: this follows easily from the fact that the connecting maps are $*$-homomorphisms. For example, we define $[a] \cdot [b]$, for representatives $a \in A_i$ and $b \in A_j$ to be

$$[a] \cdot [b] = [\varphi_{ik}(a) \cdot \varphi_{jk}(b)] \tag{8}$$

where $k$ is a common upper-bound for both $i$ and $j$. Checking that this operation is well-defined is simple.

Note that in our case, the index set is the collection of all $X \subset \mathbb{R}^n$, and the preorder is set inclusion. Of course, every pair of subset in $\mathbb{R}^n$ is contained in a common superset, namely their union. Clearly, the canonical inclusions $\iota_{XY}$ for $X \subset Y$ described above satisfy the necessary axioms to make them, along with the $\mathcal{A}(X)$, a direct system of $C^*$-algebras. Moreover, because

$$||a||_X = ||\iota_{XY}(a)||_Y \quad \text{for } X \subset Y \tag{9}$$

it follows that $\lim \mathcal{A}(X)$ has a norm given by

$$||[a]|| = ||a||_X \quad \text{for } a \in \mathcal{A}(X) \tag{10}$$

To see that this is well-defined, if $b \sim a$ (i.e. $[a] = [b]$), and $b \in Y$, then there is $Z$ which contains both $X$ and $Y$ where $\iota_{XZ}(a) = \iota_{YZ}(b)$, and we have

$$||[b]|| = ||b||_Y = ||\iota_{YZ}(b)||_Z = ||\iota_{XZ}(a)||_Z = ||a||_X = ||[a]|| \tag{11}$$

which completes the proof. This norm satisfies the desired properties of a norm because each $||\cdot||_X$ does. It follows that our direct system yields a direct limit which is a normed $*$-algebra.

**Definition II.3.** We take the algebra of quasi-local observables, $\mathcal{A}$ to be the norm completion of the normed $*$-algebra $\lim \mathcal{A}(X)$. Of course, this will be a $C^*$-algebra, by definition.

From here, for som e bounded $X \subset \mathbb{R}^n$, let us consider the *normalized trace*, which is taken to be the map $\overline{\mathrm{Tr}} : \mathcal{A}(X) \to \mathbb{C}$ defined by

$$\overline{\mathrm{Tr}}(A) = \frac{\mathrm{Tr}(A)}{\sqrt{\dim(\mathcal{A}(X))}} \tag{12}$$

It is clear that we can also uniquely define a partial trace operation $\overline{\mathrm{Tr}}_{X^C} : \mathcal{A}(Y) \to \mathcal{A}(X)$ which takes

$$\overline{\mathrm{Tr}}_{X^C}(A \otimes B) = \overline{\mathrm{Tr}}(A)B \qquad A \in B(Y - X),\ B \in B(X) \tag{13}$$

for any bounded $X \subset Y$. Note that $\overline{\mathrm{Tr}}$ extends to a positive linear functional on all of $\mathcal{A}$ as the maps $\overline{\mathrm{Tr}}$ define an inverse system relative to the $\mathcal{A}(X)$. We denote the space of traceless anti-Hermitian elements of $\mathcal{A}(X)$ by $\mathfrak{d}_l(X)$, and we let its corresponding direct limit be denoted $\mathfrak{d}_l$. It is easy to see that given some $A \in \mathcal{A}_l$, we have $A = A - \overline{\mathrm{Tr}}(A)\mathbb{I} + \overline{\mathrm{Tr}}(A)\mathbb{I} = A' + \overline{\mathrm{Tr}}(A)\mathbb{I}$, where $A' \in \mathfrak{d}_l$, which implies that $\mathcal{A}_l = \mathbb{C}\mathbb{I} \oplus \mathbb{C} \otimes \mathfrak{d}_l$. Note that $\mathfrak{d}_l(X)$ is a Lie algebra equipped with the usual commutator, as

$$\overline{\mathrm{Tr}}([A, B]) = \overline{\mathrm{Tr}}(AB - BA) = \overline{\mathrm{Tr}}(AB) - \overline{\mathrm{Tr}}(AB) = 0 \tag{14}$$

from the cyclic property of the trace.

**Definition II.4** (Brick)**.** A brick in $\mathbb{R}^n$ is a non-empty subset of the form

$$Y = \{(x_1, \dots, x_n) \mid \ell_i \leq x_i \leq m_i,\ i = 1, \dots, n\} \tag{15}$$

where $(\ell_1, \dots, \ell_n)$ and $(m_1, \dots, m_n)$ are tuples of integers, let $\mathbb{B}_n$ denote the set of all $n$-bricks in $\mathbb{R}^n$.

We can show that the bricks satisfy the following:

**Lemma II.1.** For any $j \in \mathbb{R}^n$, we have

$$\sum_{Y \in \mathbb{B}_n} (1 + \mathrm{diam}(Y) + d(Y, j))^{-2n-2} \leq \frac{\pi^4 4^n (n+1)^2}{36} \tag{16}$$

*Proof.* For some $x, y \in \mathbb{Z}^n$, note that there is an associated (non-unique) brick with these points on its corners. If we let $X$ denote the brick of $x$ and $y$, note that

$$D = \max(d(x, j), d(y, j)) \leq \mathrm{diam}(X) + d(X, j) \tag{17}$$

for any $j \in \mathbb{Z}^n$, by the triangle inequality. It follows immediately that

$$(1 + d(x, j))(1 + d(y, j)) \leq (1 + D)^2 \leq (1 + \mathrm{diam}(X) + d(X, j))^2 \tag{18}$$

This clearly means that

$$(1 + \mathrm{diam}(X) + d(X, j))^{-2n-2} \leq (1 + d(x, j))^{-n-1}(1 + d(y, j))^{-n-1} \tag{19}$$

so we can take the sum, getting

$$\sum_{Y\in\mathbb{B}_n}(1+\mathrm{diam}(Y)+d(Y,j))^{-2n-2} \le \sum_{x,y\in\mathbb{Z}^n}(1+d(x,j))^{-n-1}(1+d(y,j))^{-n-1} \tag{20}$$

$$\le \left(\sum_{x\in\mathbb{Z}^n}(1+d(x,j))^{-n-1}\right)^2 \tag{21}$$

To bound the above sum, let $f(k)=(1+k)^{-n-1}$ and let $g(k)=\#(\mathbb{Z}^n\cap B_k(j))$, where $B_k(j)$ is the radius-$k$ ball around $j$. Note that $g(k)\le(1+2k)^n$, as $1+2k>\mathrm{diam}(B_k(j))$, so the ball is contained in the $n$-cube with these side-lengths. We then have, using summation by parts:

$$\sum_{j\in\Lambda}(1+d(x,j))^{-n-1} \le \sum_{k\ge 0} f(k)(g(k+1)-g(k)) \tag{22}$$

$$= \lim_{k\to\infty} f(k)g(k) - \sum_{k\ge 0} g(k)(f(k+1)-f(k)) \tag{23}$$

Clearly, $f(k)g(k)\to 0$ for $k\to\infty$, as $f$ dominates $g$ asymptotically in $k$. Moreover, it is an easy calculation to show that

$$-(f(k+1)-f(k)) \le (n+1)(1+k)^{-n-2} \tag{24}$$

which means that

$$\sum_{j\in\Lambda}(1+d(x,j))^{-n-1} \le 2^n(n+1)\sum_{k\ge 0}(1+k)^{-2} \le \frac{\pi^2 2^n(n+1)}{6} \tag{25}$$

and the proof is complete. □

One of the particular objects of interest in this work are the traceless elements which also have trace vanishing on any set not contained in brick $Y$. In particular, given brick $Y$, let

$$\mathfrak{d}_l^Y = \{A\in\mathfrak{d}_l(Y) \mid \overline{\mathrm{Tr}}_{X^C}(A)=0 \text{ for all bricks } X\cap Y^C\ne\emptyset\} \tag{26}$$

Let us consider *derivations* on $\mathcal{A}$, which are operators of the form

$$F(A) = \sum_{Y\in\mathbb{B}_n}[F^Y, A] \tag{27}$$

where $F^Y\in\mathfrak{d}_l(Y)$. Given $U\subset\mathbb{R}^n$, let us define $\mathcal{D}_{al}(U)$ as the collection of derivationsw which are approximately localized to the set $U$. In particular,

**Definition II.5.** For any element $F$, which is uniquely determined by a collection of $F^Y$ for all $Y\in\mathbb{B}_n$, let us define, given some $U\subset\mathbb{R}^n$,

$$||F||_{U,k} = \sup_{Y\in\mathbb{B}_n} ||F^Y||(1+\mathrm{diam}(Y)+d(U,Y))^k \tag{28}$$

and define $\mathcal{D}_{al}(U)$ to be the set of all derivations $F$ such that $||F||_{U,k}<\infty$ for all $k\ge 0$.

We are able to prove some important regularity conditions about our space of derivations, similarly to the regularity result about bricks proved earlier. First, note that each $||\cdot||_{U,k}$ is a norm on $\mathcal{D}_{al}(U)$, for all $k\ge 0$: this is more or less trivial to check from the definition. Thus, we endow $\mathcal{D}_{al}(U)$ with the locally convex topology given by all of the norms.

**Definition II.6** (Frechet space)**.** A Frechet space is a topological vector space when it is Hausdorff and when its topology can be generated by a countable family of semi-norms with respect to which it is complete.

This leads us to a very important result in our setup:

**Theorem II.1.** $\mathcal{D}_{al}(U)$ is a Frechet space.

*Proof.* We can immediately get the Hausdorff property from the fact that if $||F||_{U,k} = 0$, then $F = 0$, as this will imply that $||F^Y|| = 0$ for all $Y$, from the formula. In particular, this means that distinct points are necessarily separated by open balls in the norms. To show completeness, pick some Cauchy sequence $F_m$. In other words, for $k \geq 0$ and some $\varepsilon > 0$, there exists $N$ such that for $m, q \geq N$, we have

$$||F_m - F_q||_{U,k} < \varepsilon \tag{29}$$

For a fixed brick $Y$, it follows that $F_n^Y$ is Cauchy in $\mathfrak{d}_l^Y$ (as $\sup_Y ||F_m^Y - F_q^Y||$ becomes arbitrarily small). We know that $\mathfrak{d}_l^Y$ is a closed subspace of $\mathcal{A}$ (it is a Lie algebra), so it follows that $F_n^Y$ converges to some $F^Y$, as this subspace will be complete.

We let $F$ be the derivation determined by all the limits $F^Y$ obtained via the above method. Fix some $k$ and some $\varepsilon > 0$ and pick $N_\ell$ for all $\ell = 0, 1, 2, \ldots$ where if $m, q \geq N_\ell$, then

$$||F_m - F_q|| < 2^{-\ell-1}\varepsilon \tag{30}$$

It follows that for any brick $Y$, $\geq N_1$, and $M > 1$, we have

$$||F_m^Y - F^Y|| \leq ||F_m^Y - F_{N_1}^Y|| + \sum_{i=1}^{M-1} ||F_{N_i}^Y - F_{N_{i+1}}^Y|| + ||F_{N_M}^Y - F^Y|| \tag{31}$$

$$\leq \varepsilon(1 + \operatorname{diam}(Y) + d(U, Y))^{-k} + ||F_{N_M}^Y - F^Y|| \tag{32}$$

from here, if we take $M \to \infty$, then we will have

$$||F_m^Y - F^Y||(1 + \operatorname{diam}(Y) + d(U, Y))^k < \varepsilon \tag{33}$$

for any $\varepsilon$, as $||F_{N_M}^Y - F^Y|| \to 0$, so we can multiply both sides of the above inequality by $(1 + \operatorname{diam}(Y) + d(U, Y))^k$ to upper bound it by $2\varepsilon$, for instance. Since $Y$ was chosen arbitrarily, we then have $||F_m - F||_{U,k} < \varepsilon$. Moreover, $k$ was arbitrarily, so $F_n \to F$ in the topology on $\mathcal{D}_{al}(U)$ (i.e. the locally convex topology induced by the norms). $\square$

[1] Adam Artymowicz, Anton Kapustin, and Bowen Yang. A mathematical theory of topological invariants of quantum spin systems, 2024.