Differential geometry exam prep

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I. Introduction

It's now or never...

II. Submanifolds, immersions, submersions

Claim II.1. Smooth embeddings are submanifolds

Proof. Suppose $F : M \to N$ is a smooth embedding. Pick some $F(x) \in F(N)$, pick a chart (V, ψ) around $F(x)$ and (U, φ) around x such that $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is inclusion. Note that since F is an embedding, $F(U) = F(N) \cap \widetilde{V}$, where $\widetilde{V}$ is open in N. Let $V' = V \cap \widetilde{V}$, we restrict ψ to this new open set.

Note that $F(U) \subset V'$ and $F(N) \cap V' = F(U)$. We then have $\psi(V' \cap F(N)) = \psi(F(U)) = j(\varphi(U)) = \varphi(U) \times 0$. To conclude, we set $V'' = \psi^{-1}(\varphi(U) \times \mathbb{R}^{n-k}) \cap V'$. We then have

$$\psi(V'' \cap F(N)) = \psi(V'' \cap F(U)) = (\varphi(U) \times 0) \cap \psi(V'') \tag{1}$$

Note that $\psi(V'') \subset \varphi(U) \times \mathbb{R}^{n-k}$ by construction, so it then follows that $(\varphi(U) \times 0) \cap \psi(V'') = \psi(V'') \cap \mathbb{R}^k$, and we are done. $\square$

III. Differential k-forms

Claim III.1. The wedge product of two alternating tensors $f \in A_k(V)$ and $g \in A_\ell(V)$ is itself an alternating tensor in $A_{k+\ell}(V)$.

Proof. By definition,

$$(f \wedge g)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \tag{2}$$

To check that this map is alternating, let $\tau_{ab} \in S_{k+\ell}$ be a permutation which swaps a and b, and leaves everything else fixed. We of course have

$$(f \wedge g)(v_{\tau_{ab}(1)}, \dots, v_{\tau_{ab}(k+\ell)}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma f(v_{\sigma(\tau(1))}, \dots, v_{\sigma(\tau(k))}) g(v_{\sigma(\tau(k+1))}, \dots, v_{\sigma(\tau(k+\ell))}) \tag{3}$$

$$= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^{\sigma \circ \tau} f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \tag{4}$$

$$= (-1)^\tau (f \wedge g)(v_1, \dots, v_{k+\ell}) = -(f \wedge g)(v_1, \dots, V_{k+\ell}) \tag{5}$$

and we are done. $\square$

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