MAT437 problem set 11

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I. RLL Problem 9.3

Part 1. By Stone-Weirestrass theorem, for some continuous function $f : \mathbb{R}^+ \to \mathbb{C}$, we can approximate f to arbitrary precision with a polynomial, p. In other words, we can choose p such that $|f(x) - p(x)| < \varepsilon$ for self-adjoint $x \in A$ with $||x|| \leq M$ (which holds, as we are given $||x|| \leq 1$). For some $a \in A$, aa^* and a^*a are self-adjoint. We then note that

$$\begin{aligned}
|af(a^*a) - f(aa^*)a| &\leq |af(a^*a) - ap(a^*a)| + |ap(a^*a) - p(aa^*)a| + |p(aa^*)a - f(aa^*)a| && (1)\\
&\leq 2||a||\varepsilon + |ap(a^*a) - p(aa^*)a| && (2)\\
&\leq 2\varepsilon + \left| \sum_{k=0}^{n} p_k a(a^*a)^k - \sum_{k=0}^{n} p_k (aa^*)^k a \right| && (3)\\
&= 2\varepsilon + \left| \sum_{k=0}^{n} p_k a(a^*a)^k - \sum_{k=0}^{n} p_k a(a^*a)^k \right| = 2\varepsilon && (4)
\end{aligned}$$

Thus, for any $\varepsilon > 0$, we have $|af(a^*a) - f(aa^*)a| < \varepsilon$, so it follows that this difference must be 0, and we have $af(a^*a) = f(aa^*)a$, as desired.

From here, it follows immediately that if $f(x) = (1-x)^{1/2}$, we have $a(1 - a^*a)^{1/2} = (1 - aa^*)^{1/2}a$. Let

$$v = \begin{pmatrix} a & (1-aa^*)^{1/2} \\ -(1-a^*a)^{1/2} & a^* \end{pmatrix} \quad \text{so that} \quad v^* = \begin{pmatrix} a^* & -(1-a^*a)^{1/2} \\ (1-aa^*)^{1/2} & a \end{pmatrix} \tag{5}$$

which immediately gives

$$vv^* = \begin{pmatrix} 1 & f(aa^*)a - af(a^*a) \\ a^*f(aa^*) - f(a^*a)a^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

where we note that $a^*f(aa^*) = (f(aa^*)a)^* = (af(a^*a))^* = f(a^*a)a^*$. It is easy to see, via identical logic that v^*v is also the identity, so that v is a unitary element, as desired.

Part 2. Given an ideal I in A, we let $\pi : A \to A/I$ be the quotient map. We let u be a unitary in A/I. We know, from Rordam, that there is an element a in A such that $||a|| = 1$ and $\pi(a) = u$. We define v as in Part 1. We have

$$\pi(v) = \begin{pmatrix} \pi(a) & \pi((1-aa^*)^{1/2}) \\ \pi(-(1-a^*a)^{1/2}) & \pi(a^*) \end{pmatrix} = \begin{pmatrix} u & \pi((1-aa^*)^{1/2}) \\ -\pi((1-a^*a)^{1/2}) & u^* \end{pmatrix} \tag{7}$$

Since $u = u_0 + I$ is unitary in A/I, it follows that $u^*u = u_0^*u_0 + I = 1 + I$. Similarly, $uu^* = u_0u_0^* + I = 1 + I$. Thus, both $1 - u_0^*u_0$ and $1 - u_0u_0^*$ are in I, so that

$$(1 - u_0^*u_0) + I = (1 - u_0u_0^*) + I = 0 \tag{8}$$

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In addition, via the same logic as before (Stone-Weirestrass), continuous functions of these elements will be equal to 0 as well:

$$g(1-u_0^*u_0)+I=g(1-u_0u_0^*)+I=0 \tag{9}$$

Thus, $\pi((1-a^*a)^{1/2})=(1-u_0^*u_0)^{1/2}+I=0$, and the same holds for $(1-aa^*)^{1/2}$. Thus,

$$\begin{pmatrix} u & \pi((1-aa^*)^{1/2}) \\ -\pi((1-a^*a)^{1/2}) & u^* \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \tag{10}$$

as desired.

Part 3. We let $\delta_1 : K_1(A/I) \to K_0(I)$ be the index map associated with

$$0 \longrightarrow I \xrightarrow{\text{inclusion}} A \xrightarrow{\pi} A/I \longrightarrow 0 \tag{11}$$

We let u be some unitary in A/I. Let us recall the standard picture of the index map: if we have a short exact sequence of the form

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{12}$$

and we have $u \in \mathcal{U}_n(\widetilde{B})$, $v \in \mathcal{U}_{2n}(\widetilde{A})$ and $p \in \mathcal{P}_{2n}(\widetilde{I})$ satifying

$$\widetilde{\varphi}(p) = v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^* \quad \text{and} \quad \widetilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \tag{13}$$

then $\delta_1([u]_1) = [p]_0 - [s(p)]_0$. It follows that if we choose a as before, with $\pi(a) = u$, then we have already shown that the latter condition holds for the quotient map π. We should set

$$p = j(p) = v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^* = \begin{pmatrix} a & (1-aa^*)^{1/2} \\ -(1-a^*a)^{1/2} & a^* \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^* & -(1-a^*a)^{1/2} \\ (1-aa^*)^{1/2} & a \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} a & (1-aa^*)^{1/2} \\ -(1-a^*a)^{1/2} & a^* \end{pmatrix} \begin{pmatrix} a^* & -(1-a^*a)^{1/2} \\ 0 & 0 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} aa^* & -a(1-a^*a)^{1/2} \\ -(1-a^*a)^{1/2}a^* & 1-a^*a \end{pmatrix} \tag{16}$$

We can easily verify that this is a projection: indeed it is self-adjoint, and moreover,

$$p^2 = \begin{pmatrix} aa^* & -a(1-a^*a)^{1/2} \\ -(1-a^*a)^{1/2}a^* & 1-a^*a \end{pmatrix}^2 \tag{17}$$

$$= \begin{pmatrix} aa^*aa^* + a(1-a^*a)a^* & -aa^*a(1-a^*a)^{1/2} - a(1-a^*a)^{3/2} \\ -(1-a^*a)^{1/2}a^*aa^* - (1-a^*a)^{3/2}a^* & (1-a^*a)^{1/2}a^*a(1-a^*a)^{1/2} + (1-a^*a)^2 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} aa^* & -a(1-a^*a)^{1/2} \\ -(1-a^*a)^{1/2}a^* & 1-a^*a \end{pmatrix} = p \tag{19}$$

Thus, this is precisely the desired projection: the index map is given by $[\delta_1(u)]_0 = [p] - [s(p)]_0$. Recall, from RLL, that $s(p) = \operatorname{diag}(1,0)$, so

$$[\delta_1(u)]_0 = [p] - [s(p)]_0 = \left[\begin{pmatrix} aa^*-1 & -a(1-a^*a)^{1/2} \\ -(1-a^*a)^{1/2}a^* & 1-a^*a \end{pmatrix} \right]_0 . \tag{20}$$

Part 4. We once again let u be a unitary in A/I, and let a be the lift of u with $||a|| = 1$. We let v be the partial isometry in $M_2(A)$ such that v lifts $\operatorname{diag}(u,0)$ (we know this exists from RLL). We know the explicit form of v, from the proof of Lemma 9.2.1 in RLL:

$$v = \begin{pmatrix} a & 0 \\ (1-a^*a)^{1/2} & 0 \end{pmatrix} \tag{21}$$

It follows that we can easily compute $p = j(p) = 1 - v^*v$ and $q = j(q) = 1 - vv^*$ (where j is the inclusion map). In particular,

$$p = 1 - v^*v = \mathbb{I} - \begin{pmatrix} a^* & (1-a^*a)^{1/2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ (1-a^*a)^{1/2} & 0 \end{pmatrix} \tag{22}$$

$$= \mathbb{I} - \begin{pmatrix} a^*a + (1-a^*a) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{23}$$

and, in addition,

$$q = 1 - vv^* = \mathbb{I} - \begin{pmatrix} a & 0 \\ (1-a^*a)^{1/2} & 0 \end{pmatrix} \begin{pmatrix} a^* & (1-a^*a)^{1/2} \\ 0 & 0 \end{pmatrix} \tag{24}$$

$$= \mathbb{I} - \begin{pmatrix} aa^* & a(1-a^*a)^{1/2} \\ (1-a^*a)^{1/2}a^* & 1-a^*a \end{pmatrix} = \begin{pmatrix} 1-aa^* & a(1-a^*a)^{1/2} \\ (1-a^*a)^{1/2}a^* & a^*a \end{pmatrix} \tag{25}$$

Recall the alternative definition of the index map, $\delta_1([u]_1) = [p]_0 - [q]_0$. From the above calculations, we have

$$[p]_0 - [q]_0 = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]_0 - \left[\begin{pmatrix} 1-aa^* & a(1-a^*a)^{1/2} \\ (1-a^*a)^{1/2}a^* & a^*a \end{pmatrix}\right]_0 \tag{26}$$

$$= \left[\begin{pmatrix} aa^*-1 & -a(1-a^*a)^{1/2} \\ -(1-a^*a)^{1/2}a^* & 1-a^*a \end{pmatrix}\right]_0 \tag{27}$$

which agrees with the previously derived value of $[\delta_1(u)]_0$.

II. Problem Set 10 Suggested Problem 4 (RLL Problem 10.1)

Part 1. We let $\mathbb{T}A = C(\mathbb{T}, A)$. We must construct a split exact sequence

$$0 \longrightarrow SA \longrightarrow \mathbb{T}A \longleftrightarrow A \longrightarrow 0 \tag{28}$$

Recall that SA is the suspension of A, $SA = \{f \in C([0,1], A) \mid f(0) = f(1) = 0\}$. Of course, a map is this form can be naturally sent to an element of $\mathbb{T}A$: define $\phi : SA \to \mathbb{T}A$ as

$$\phi(f)(e^{2\pi i\theta}) = f(\theta) \tag{29}$$

If course, this is an injective $*$-homomorphism. Now, let us define $\pi : \mathbb{T}A \to A$ as $\pi(f) = f(1)$. Of course, this is a surjective $*$-homomorphism, as we can always find some $f \in \mathbb{T}A$ whose value at $1 \in \mathbb{T}$ is any $a \in A$ that we desire.

We must now show that this sequence is exact, and that it splits (i.e. π has a right-inverse $*$-homomorphism). It is easy to check that it is exact, note that $\mathrm{Ker}(\pi)$ is precisely all $f \in \mathbb{T}A$ such that $f(1) = 0$. In addition, note that $\mathrm{Im}(\phi)$ is precisely all maps of the unit circle into A (which can all be written as $g(\theta) = f(e^{2\pi i\theta})$), such that $g(0) = g(1) = f(1) = 0$, by definition of SA. Thus, $\mathrm{Im}(\phi) = \mathrm{Ker}(\pi)$, and the sequence of exact.

To show that it splits, define $\lambda : A \to \mathbb{T}A$ as $\lambda(a)(e^{2\pi i\theta}) = a$ for all θ. Note that $(\pi \circ \lambda)(a) = \lambda(a)(1) = a$ for all a, so λ is a right-inverse for π and the sequence splits.

Part 2. Because the above sequence of split exact, it follows immediately that mapping everything under the K_n-functor will yield a split exact sequence as well. In particular, it follows that

$$K_n(\mathbb{T}A) \simeq K_n(SA) \oplus K_n(A) \tag{30}$$

By Bott periodicity, we know that $K_{n+1}(A) \simeq K_n(SA)$. Thus, $K_n(\mathbb{T}A) \simeq K_{n+1}(A) \oplus K_n(A)$ as desired.

Part 3. To start, we must show that $\mathbb{T}^n\mathbb{C}$ is isomorphic to $C(\mathbb{T}^n)$. In the case of $n=1$, we have $\mathbb{T}^n\mathbb{C} = \mathbb{T}\mathbb{C} = C(\mathbb{T},\mathbb{C}) = C(\mathbb{T})$. Let us assume the claim holds for the case of $n-1$. For the case of n, we have

$$\mathbb{T}^n\mathbb{C} = \mathbb{T}(\mathbb{T}^{n-1}\mathbb{C}) \simeq \mathbb{T}C(\mathbb{T}^{n-1}) = C(\mathbb{T}, C(\mathbb{T}^{n-1})) \simeq C(\mathbb{T}^n) \tag{31}$$

and we are done: the claim holds by induction. It follows immediately from this fact and Part 2 that we have expressions for $K_0(C(\mathbb{T}^n))$ and $K_1(C(\mathbb{T}^n))$. In particular, we note that for some m,

$$K_m(C(\mathbb{T}^n)) \simeq K_m(\mathbb{T}^n\mathbb{C}) = K_m(\mathbb{T}^{n-1}\mathbb{C}) \oplus K_{m+1}(\mathbb{T}^{n-1}\mathbb{C}) \simeq K_m(C(\mathbb{T}^{n-1})) \oplus K_{m+1}(C(\mathbb{T}^{n-1})) \tag{32}$$

It follows that we have a recursive relation for $K_m(C(\mathbb{T}^n))$ for some n. If we repeatedly use this recursion, we will eventually be able to express $K_m(C(\mathbb{T}^n))$ as a direct sum of $K_m(\mathbb{C})$, for $m \in \mathbb{Z}^+ \cup \{0\}$. In particular, it is easy to see (via induction) that

$$K_m((C(\mathbb{T}^n))) \simeq K_m(\mathbb{C}) \oplus \left(\bigoplus_{k=1}^{n} K_{m+k}(\mathbb{C})^{n-k+1}\right) \tag{33}$$

where we are taking the $(n-k+1)$-fold direct sum/Cartesian product of $K_{m+k}(\mathbb{C})$). It follows immediately that

$$K_0((C(\mathbb{T}^n))) \simeq K_0(\mathbb{C}) \oplus \left(\bigoplus_{k=1}^{n} K_{k}(\mathbb{C})^{n-k+1}\right) \tag{34}$$

and

$$K_1((C(\mathbb{T}^n))) \simeq K_1(\mathbb{C}) \oplus \left(\bigoplus_{k=1}^{n} K_{k+1}(\mathbb{C})^{n-k+1}\right). \tag{35}$$

III. RLL Problem 11.6

We have

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{36}$$

Note that a is nilpotent with $a^2 = 0$. Thus, for all $k \geq 2$, $a^k = 0$. Since $f \in \mathcal{H}(\Omega)$ (it is holomorphic in the domain Ω), it has a power series development about 0 (as $\mathrm{sp}(a) = \{0\}$, so we take Ω a neighbourhood about 0) $f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) z^k$. It follows from the holomorphic function calculus that

$$f(a) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) a^k = f(0)\mathbb{I} + f'(0)a = \begin{pmatrix} f(0) & f'(0) \\ 0 & f(0) \end{pmatrix} \tag{37}$$

as all the higher-order terms vanish, and we are done.