Exercises Week 01 solutions

1. Let E and F be two events for which one knows that the probability that at least one of them occurs is $3/4$. What is the probability that neither E nor F occurs? Hint: use one of DeMorgan's laws: $E^{\mathrm{C}} \cap F^{\mathrm{C}} = (E \cup F)^{\mathrm{c}}$.
Solution: The probability that that either E, F or $E \cup F$ occurs is $\frac{3}{4}$. Therefore, $(E \cup F)^{\mathrm{C}} = \frac{1}{4}$.

2.

a) Let A and B be two events in a sample space for which $P(A) = 1/3$, $P(B) = 1/2$, and $P(A \cup B) = 3/4$. What is $P(A \cap B)$?

Solution:
The intersection of A and B can be calculated by subtracting $A \cup B$ from $A + B$, as $A + B$ contains $A \cap B$ twice, but $A \cup B$ only contains it once.

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 1/3 + 1/2 - 3/4 \\ &= 1/12 \end{aligned}$$

b) Let C and D be two events for which one knows that $P(C) = 0.1$, $P(D) = 0.3$, and $P(C \cap D) = 0.05$. What is $P(C^{\mathrm{C}} \cap D^{\mathrm{C}})$?

Solution:
DeMorgan's law states that: $A^{\mathrm{C}} \cap B^{\mathrm{C}} = (A \cup B)^{\mathrm{C}}$. Therefore it is first necessary to find $C \cup D$. $C \cup D = C + D - C \cap D$ for the same reason as a. Therefore

$$\begin{aligned} P(C^{\mathrm{C}} \cap D^{\mathrm{C}}) &= P((C \cup D)^{\mathrm{C}}) \\ &= 1 - (P(C) + P(D) - P(C \cap D)) \\ &= 1 - (0.1 + 0.3 - 0.05) \\ &= 1 - 0.45 \\ &= 0.55 \end{aligned}$$

3. Consider tossing a fair coin for three times.

a) Write down the sample space Ω.

Solution:
$\Omega = \{HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT\}$.

Then, write down the set of outcomes and probabilities for the the events

b) "We throw tails exactly two times",

Solution:
$P(\{HTT,\ THT,\ TTH\}) = 3/8$

c) "We throw heads at least twice,

Solution:
$P(\{HHH,\ HHT,\ HTH,\ THH\}) = 4/8 = 1/2$

d) "Both the first and last throws is heads",

Solution:
$P(\{HHH,\ HTH\}) = 2/8 = 1/4$

e) "We get no tails at all".

Solution:
$P(\{HHH\}) = 1/8$

4. Consider rolling a fair die as many times until the first six will turn up. Write down the probability that

a) it takes exactly three rolls to get the first six.

Solution:
The sample space in three rolls is $\Omega = \{111,\ 112,\ 113,\ 114,\ 115,\ 116,\ 121,\ ...\}$.
Firstly, we need to find the amount of outcomes where 6 is the final number. We can start by modeling this with 2 throws. The first number can be any integer in $[1, 5]$, so 5 different outcomes. With another number, there are now 2 numbers that can be integers in $[1, 5]$, so there are 5^2 or 25 different outcomes.
Now we need to figure out the total number of possible outcomes. Each number can be an integer in $[1, 6]$ now, and there are 3 numbers that fall in to this range, thus there are 6^3 different outcomes, or 216.
The probability that the last of three rolls is a 6 is therefore $\frac{25}{216}$ or roughly 11.6%.

b) you need to roll the die more than three times to get the first six.

Solution:
First we need to find the probability that you do get a 6 within 3 rolls. To get a six, either the first, second or third roll has to be a six. In all three cases, the other two rolls can be any integer in $[1, 5]$, so there are $3 \cdot 5^2$ or 75 different outcomes where 6 is one of the rolls.
This however does not take into account rolls with more than 1 six. With 2 sixes the non-six value can show up in either the first, second or third roll, and be an integer in [1,5], so there are $3 \cdot 5 = 15$ outcomes where this happens. Finally, there is only one possible outcome with 3 sixes. This means the total number of outcomes with at least one six is $75 + 15 + 1 = 91$ different outcomes.
As we learned previously, there are 216 total outcomes, which means the probability that there was a six within 3 rolls is $\frac{91}{216} \approx 42.1\%$. That means that the probability you need more than 3 rolls is $1 - \frac{91}{216} = \frac{125}{216} \approx 57.9\%$.

5. Use R as you would use a calculator to find numeric answers to the following expressions

a) $1 + 2(3 + 4)$

Solution:

```r
1+2*(3+4)
```

```
## [1] 15
```

b) $4^3 + 3^{2+1}$

Solution:

```r
4^3+3^(2+1)
```

```
## [1] 91
```

c) $\sqrt{(4 + 3)(2 + 1)}$

Solution:

```r
sqrt((4+3)*(2+1))
```

```
## [1] 4.582576
```

d) $\frac{1+2\cdot 3^4}{5/6-7}$

Solution:

```r
(1+2*3^4)/(5/6-7)
```

```
## [1] -26.43243
```

e) $\frac{0.25-0.2}{\sqrt{0.2\cdot(1-0.2)/100}}$

Solution:

```r
(0.25-0.2)/sqrt(0.2*(1-0.2)/100)
```

```
## [1] 1.25
```