APSTA Week 12 Exercises

1. Coin flipping (T)

Assume that you flipped a coin $N = 11$ times. You got tails 7 times and heads 4 times, after which the coin fell into a well and you lost it. After the experiment you started to wonder whether the coin was fair.
(a) Formulate test statistic, the null hypothesis and alternative hypothesis for testing the fairness of the coin.
(b) Was the coin fair? Assume 0.05 risk level.

Solution:
The test statistic is going to be number of heads, H, in this case, the realisation of H is $h = 4$. This is a random variable with a binomial distribution of n Bernoulli distributed random variables. In this case,

$$H = R_1 + R_2 + \cdots + R_{11}$$

where $R_i = 1$ if the coin flip resulted in a heads. Each flip R_i has probability p of being a heads.
The null hypothesis is that the coin is fair. That is, $H_0 : p = 0.5$, with the alternative hypothesis being that the coin is unfair. Specifically, I only care about the scenario that tails is more likely than heads, as that is what I see from my data. Therefore, $H_1 : p < 0.5$.
I can now test whether the coin is fair. Assuming H_0, $\mathrm{P}(R_i = 1) = 0.5$. For 11 coin flips, that is

$$\mathrm{P}(H = k) = \binom{11}{k} 0.5^{11}.$$

I don't want the probability of getting exactly k heads however. I want the probability of seeing a result at least as extreme as 4 heads. That is, I want $\mathrm{P}(H \leq 4)$.

$$\begin{aligned}
\mathrm{P}(H \leq 4) &= \mathrm{P}(H = 4) + \mathrm{P}(H = 3) + \cdots + \mathrm{P}(H = 0) \\
&= \binom{11}{4} 0.5^{11} + \binom{11}{3} 0.5^{11} + \cdots + \binom{11}{0} 0.5^{11} \\
&= 0.5^{11} \left(\binom{11}{4} + \binom{11}{3} + \cdots + \binom{11}{0} \right) \\
&= 0.5^{11}(330 + 165 + 55 + 11 + 1) \\
&= 0.5^{11}(562) \\
&\approx 0.274
\end{aligned}$$

The coin was most likely fair, as there is still a 27% chance to only get 4 heads in 11 flips.

$ 2. Car Problems (R)
Historically, a car from a given company has a 10% chance of having a significant mechanical problem during its warranty period. A new model of the car is being sold. Of the first $25,000$ sold, $2,700$ have had an issue. Perform a test of significance to see whether the proportion of these new cars that will have a problem is more than 10%. What is the p-value?

Solution:

```
cars_tot <- 25000
cars_bad <- 2700
prop.test(cars_bad, cars_tot, p = 0.1, alternative = "greater")
```

```
##
##  1-sample proportions test with continuity correction
##
## data:  cars_bad out of cars_tot, null probability 0.1
## X-squared = 17.689, df = 1, p-value = 1.301e-05
## alternative hypothesis: true p is greater than 0.1
## 95 percent confidence interval:
##  0.1047937 1.0000000
## sample estimates:
##     p
## 0.108
```

p-value of 0.108. The number of problematic cars is not unexpected.

3. Horse Race (T)

To investigate the hypothesis that a horse´s chances of winning an eight-horse race on a circular track are affected by its position in the starting lineup, the starting position of each of 144 winners was recorded. It turned out that 29 of these winners had starting position one (closest to the rail on the inside track). We model the number of winners with starting position one by a random variable T with a $Bin(144, p)$ distribution. We test the hypothesis $H_0 : p = 1/8$ against $H_1 : p > 1/8$ at the level $\alpha = 0.01$ with T as test statistic.
(a) Argue whether the test procedure involves a right critical value, a left critical value, or both.

Solution:
We want to know whether the true probability p is greater than 1/8. Therefore, we use a right-tail probability.

(b) Use the normal approximation to compute the critical value(s) corresponding to $\alpha = 0.01$, determine the critical region, and report your conclusion about the null hypothesis.

Solution:
Not sure what's meant by "normal approximation" when the distribution is binomial.
I use the exact same technique as above:

$$P(T = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

We want know how many winners can expect before we reach 0.01 probability. Therefore, we want to model $P(T \geq k) = 0.01$ and solve for k.

$$P(T \geq k) = 1 - P(T < k) = 1 - P(T \leq k-1) = 1 - F(k-1).$$

Since realisations of T are discrete real numbers, we change $T < k$ to $T \leq k-1$. Now we solve for k by setting it equal to 0.01.

$$\begin{aligned} 1 - F(k-1) = 0.01 \Leftrightarrow \\ F(k-1) = 0.99 \Leftrightarrow \\ F^{-1}(0.99) = k-1 \Leftrightarrow \\ F^{-1}(0.99) + 1 = k \Leftrightarrow \end{aligned}$$

```
qbinom(0.99, size=144, p=1/8) + 1
```

```
## [1] 142
```

Therefore, $k = 142$. ˆ this is probably wrong.

4. Insurance (R)

In the United States in 1998, the proportion of adults age 21˘24 who had no medical insurance was 34.4 percent, according to the Employee Benefit Research Institute. A survey of 75 recent college graduates in this age range finds that 40 are without insurance. Does this support a difference from the nationwide proportion? Perform a test of significance and report the p-value. Is it significant?

Solution:

```
students_tot <- 75
students_no_insurance <- 40
p <- 34.4/100
prop.test(students_no_insurance, students_tot, p=p, alternative="greater")
```

```
##
##  1-sample proportions test with continuity correction
##
## data:  students_no_insurance out of students_tot, null probability p
## X-squared = 11.09, df = 1, p-value = 0.0004341
## alternative hypothesis: true p is greater than 0.344
## 95 percent confidence interval:
##  0.4325686 1.0000000
## sample estimates:
##         p
## 0.5333333
```

p-value is 0.5, there's absolutely no reason to believe the students deviate from the nationwide proportion.

Type I and II errors (T)

Let T be a random variable following an $N(\mu, 1)$ distribution. Assume testing the hypothesis $H_0 : \mu = 0$ against the alternative hypothesis $H_1 : \mu \neq 0$ using the test statistic T.
(a) With the decision to reject the null hypothesis if the realisation $|t| > 1$ what is the probability of committing a type I error?

Solution:
We want to calculate the probability of getting $|t| > 1$ with a $N(0, 1)$ distribution. That is,

$$\mathrm{P}(|T| > 1)$$

We can use the following formula that makes use of critical values:

$$\mathrm{P}(-z_p \leq Z \leq z_p) = 1 - 2p$$

where Z is a $N(0, 1)$ distributed random variable. We just set $z_p = 1$ and calculate p:

$$\begin{aligned} \Phi(z_p) = 1 - p \Leftrightarrow \\ \Phi(1) = 1 - p \Leftrightarrow \\ 1 - \Phi(1) = p \end{aligned}$$

```
pnorm(1)
```

```
## [1] 0.8413447
```

$$p = 1 - 0.8413447 = 0.1586553$$

Therefore,

$$\text{P}(-z_p \leq Z \leq z_p) = 0.683$$

This is then the probability that T lies between -1 and 1, or $\text{P}(|T| < 1)$, but we want $\text{P}(|T| > 1)$. Therefore

$$\text{P}(|T| > 1) = 1 - \text{P}(|T| < 1) = 1 - 0.683 = 0.317$$

So there is a 31.7% probability of committing a type I error.

(b) Assuming that the true value of μ is 1, what is the probability of committing a type II error?

Solution:
The probability of committing a type II error is the probability of not rejecting H_0 despite H_1 being true. That is, what is the probability that $|t| < 1$ given $\mu = 1$.

$$\text{P}(|T| < 1) = \text{P}(-z_p \leq T \leq z_p) = 1 - 2p$$

where $z_p = 1$. This time, the distribution is $N(1, 1)$.

```
pnorm(1, mean=1)
```

```
## [1] 0.5
```

We can use the same formula as before

$$\begin{aligned}
\Phi(z_p) &= 1 - p \Leftrightarrow \\
\Phi(1) &= 1 - p \Leftrightarrow \\
1 - \Phi(1) &= p \Leftrightarrow \\
1 - 0.5 &= p \Leftrightarrow \\
p &= 0.5.
\end{aligned}$$

Therefore,

$$\text{P}(-1 \leq T \leq 1) = 0$$

This is clearly wrong :)