THE CAPACITOR PARADOX

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1 Introduction

The *Capacitor Paradox* is a widely known trick question commonly encountered in the field of electronics, especially on internet forums, and sometimes even in electrical engineering lectures, designed to throw people off.

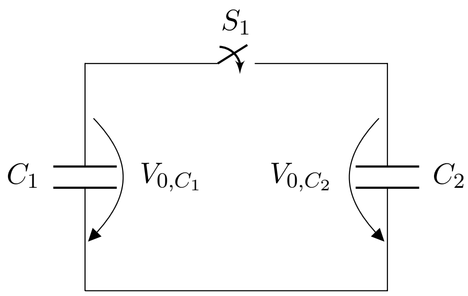

Figure 1

Consider two capacitors, $C_1 = 1\,\mu\text{F}$ and $C_2 = 1\,\mu\text{F}$, with initial voltages of $V_{0,C_1} = 10\,\text{V}$ and $V_{0,C_2} = 0\,\text{V}$, respectively.

The energy stored in C_1 is

$$e_{C_1} = \frac{1}{2}C_1 V_{0,C_1}^2 = 50\,\mu\text{J} \tag{1}$$

Now, close the switch S_1 and let the circuit settle. Since $C_1 = C_2$, the charge from C_1 will evenly distribute over both capacitors such that both voltages come to be $V_{settle} = 5\,\text{V}$.

Calculating the new energy stored in both capacitors reveals something strange:

$$\begin{aligned} e_{settle} &= 2\left(\frac{1}{2}C_1 V_{settle}\right) && (2)\\ &= 2 \cdot 12.5\,\mu\text{J} && (3)\\ &= 25\,\mu\text{J} && (4)\\ & && (5) \end{aligned}$$

The new total energy e_{settle} is **half** of what it was before. Where did the energy go?

2 A Closer Look

The circuit presented in figure 1 represents not only a physical impossibility, but a mathematical impossibility as well (more on the latter later). If one tried to simulate this circuit, one would realise that an infinite amount of current would have to flow in zero time, not to mention the switch would have to close infinitely quickly and have no resistive losses.

A more realistic model is one that models the imperfections of the switch and is shown in figure 2.

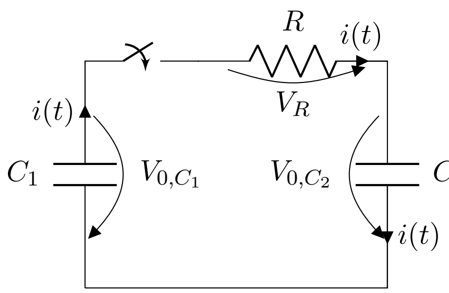

Figure 2

This new model at least gets rid of the mathematical impossibility and allows us to analyse the circuit in more detail. There are two approaches.

2.1 The Easy Way

Perhaps the easiest explanation of all is to realise that the model can be further simplified into an equivalent circuit. Consider the circuits shown in figures 3 and 4.

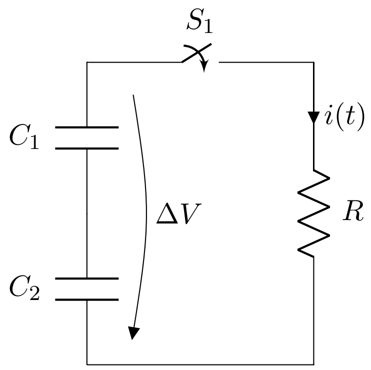

Figure 3: Re-arrangement of the same circuit seen in figure 2.

The equivalent circuit seen in figure 4 makes it very easy to see that all of the energy in C_{eq} will discharge over R and dissipate. The calculation is:

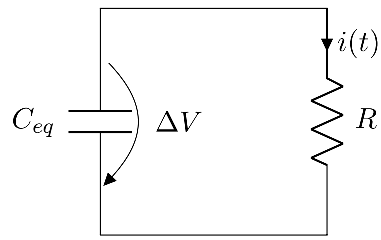

Figure 4: Equivalent circuit of figure 3

$$e_{loss} = \frac{1}{2} C_{eq} \Delta V^2 \tag{6}$$

$$= \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} \left(V_{0,C_1} - V_{0,C_2}\right)^2 \tag{7}$$

Using the same numbers from the first example, we find that $C_{eq} = 500\,\mathrm{nF}$, $\Delta V = 10\,\mathrm{V}$ and $e_{loss} = 25\,\mu\mathrm{J}$, as expected.

2.2 The Hard Way

In the interest of possibly figuring out what happens for $R \to 0$, a more complete analysis will be performed to predict the exact behaviour of the current $i(t)$ and the power $p_R(t)$ and energy $e_R(t)$ dissipated by the resistor.

2.2.1 Energy Dissipated

Closing the switch in circuit 2 at $t = 0$ completes the circuit and Kirchhoff's second law must hold true:

$$V_{C_1}(t) = V_R(t) + V_{C_2}(t) \tag{8}$$

Kirchoff's first law also states that the current through the resistor $i_R(t)$ must be equal to the current leaving and entering the two capacitors $i_{C_1}(t)$ and $i_{C_2}(t)$.

$$i(t) = i_R(t) = i_{C_1}(t) = i_{C_2}(t) \tag{9}$$

In order to solve for the unknown function $i(t)$, the voltage over each element can be expressed as a function of current:

Table 1: Relevant Laplace Transforms

Time function	Transformed	Validity
$af(t)$	$aF(s)$	$f(t<0)=0$
a	$\frac{a}{s}$	$a>0$
e^{-at}	$\frac{1}{s+a}$	$a>0$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	$f(\tau<0)=0$

$$V_R(t) = i(t)R \tag{10}$$

$$V_{C_1}(t) = -\frac{1}{C_1}\int_0^t i(\tau)\,d\tau + V_{0,C_1} \tag{11}$$

$$V_{C_2}(t) = \frac{1}{C_2}\int_0^t i(\tau)\,d\tau + V_{0,C_2} \tag{12}$$

Note the negation in equation 11, because the voltage V_{C_1} is drawn in the reverse direction in figure 2.

By substituting the equations 10, 11 and 12 into equation 8 we obtain the differential equation

$$\frac{1}{C_1}\int_0^t i(\tau)\,d\tau + \Delta V = i(t)R + \frac{1}{C_2}\int_0^t i(\tau)\,d\tau \tag{13}$$

where $\Delta V = V_{0,C_1} - V_{0,C_2}$.

One way to solve this differential equation is with the help of the Laplace transform, using the relationships listed in table 1.

Transforming equation 13 results yields:

$$\frac{I(s)}{sC_1} = I(s)R + \frac{I(s)}{sC_2} - \frac{\Delta V}{s} \tag{14}$$

$$\frac{\Delta V}{s} = I(s)\frac{sRC_2+1}{sC_2} + \frac{I(s)}{sC_1} \tag{15}$$

$$\Delta V = I(s)\frac{sRC_1C_2 + C_1 + C_2}{C_1C_2} \tag{16}$$

$$I(s) = \frac{\Delta V C_1 C_2}{sRC_1C_2 + C_1 + C_2} \tag{17}$$

$$I(s) = \frac{\Delta V}{R}\cdot\frac{1}{s+\frac{C_1+C_2}{RC_1C_2}} \tag{18}$$

Equation 18 can be transformed back into the time domain by using the relationship described by equation **??**, and so we obtain the well-known current formula for an RC-Network.

$$i(t) = I_0 \cdot e^{\frac{-t}{\tau}} \qquad \begin{aligned} I_0 &= \frac{V_{0,C_1} - V_{0,C_2}}{R} \\ \tau &= R\frac{C_1C_2}{C_1+C_2} \end{aligned} \tag{19}$$

This result is intuitive, because from the perspective of the resistor (with a closed switch), we effectively "see" a total capacitance of $\frac{C_1C_2}{C_1+C_2}$ charged to a voltage of $V_{0,C_1} - V_{0,C_2}$.

The power and energy dissipated by the resistor can be calculated from the current $i(t)$:

$$p_R(t) = i^2(t)R \tag{20}$$

$$e_R(t) = \int_0^t p_R(\tau)\,d\tau \tag{21}$$

Substituting the equation for $i(t)$ 19 into the power equation 20 yields:

$$p_R(t) = R\left(\frac{\Delta V}{R}e^{\frac{-t}{\tau}}\right)^2 \tag{22}$$

$$= \frac{\Delta V^2}{R}e^{\frac{-2t}{\tau}} \tag{23}$$

Substituting this equation into the energy equation 21 and solving the integral yields:

$$e_R(t) = \frac{\Delta V^2}{R}\int_0^t e^{\frac{-2z}{\tau}}\,dz \tag{24}$$

$$= \frac{\Delta V^2}{R}\left(-\frac{\tau}{2}e^{\frac{-2z}{\tau}}\Big|_0^t\right) \tag{25}$$

$$= \frac{\Delta V^2}{R}\frac{\tau}{2}\left(1 - e^{\frac{-2t}{\tau}}\right) \tag{26}$$

The resulting expression can further be simplified by substituting τ with the result from equation 19:

$$e_R(t) = \frac{\Delta V^2}{R}\frac{RC_1C_2}{2\,(C_1+C_2)}\left(1 - e^{\frac{-2t}{\tau}}\right) \tag{27}$$

$$= \frac{1}{2}\frac{C_1C_2}{C_1+C_2}\Delta V^2\left(1 - e^{\frac{-2t(C_1+C_2)}{RC_1C_2}}\right) \tag{28}$$

If we allow the circuit to settle for a significant amount of time, it is easy to see that the energy dissipated by the resistor **does not depend on the value of the resistor!**

$$\lim_{t\to\infty} \frac{1}{2}\frac{C_1 C_2}{C_1+C_2}\Delta V^2\left(1-e^{\frac{-2t(C_1+C_2)}{RC_1C_2}}\right) \quad (29)$$
$$= \frac{1}{2}\frac{C_1 C_2}{C_1+C_2}\Delta V^2 \quad (30)$$

2.2.2 Energy Transferred

The current function $i(t)$ from equation 19 can be re-inserted into the integral equations 11 and 12 to calculate the voltage over capacitor C_2:

$$v_{C_2}(t) = \frac{I_0}{C_2}\int_0^t e^{\frac{-z}{\tau}}\,dz + V_{0,C_2} \quad (31)$$
$$= -\frac{I_0\tau}{C_2}\cdot e^{\frac{-z}{\tau}}\Big|_0^t + V_{0,C_2} \quad (32)$$
$$= \frac{I_0\tau}{C_2}\left(1-e^{\frac{-t}{\tau}}\right) + V_{0,C_2} \quad (33)$$

The power is of course $v_{C_2}(t)\cdot i(t)$

$$p_{C_2}(t) = v_{C_2}(t)\cdot i(t) \quad (34)$$
$$= \left(\frac{I_0\tau}{C}\left(1-e^{\frac{-t}{\tau}}\right)+V_0\right)\cdot I_0\,e^{\frac{-t}{\tau}} \quad (35)$$
$$= \frac{I_0^2\tau}{C}\left(e^{\frac{-t}{\tau}}-e^{\frac{-2t}{\tau}}\right)+V_0 I_0\,e^{\frac{-t}{\tau}} \quad (36)$$

and the energy is therefore

$$e_{C_2}(t) = \int_0^z p_{C_2}(t)\,dz \quad (37)$$
$$= \frac{I_0^2\tau}{C_2}\int_0^t e^{\frac{-z}{\tau}}\,dz - \frac{I_0^2\tau}{C_2}\int_0^t e^{\frac{-2z}{\tau}}\,dz + U_0 I_0\int_0^t e^{\frac{-z}{\tau}}\,dz \quad (38)$$
$$= \frac{I_0^2\tau^2}{C_2}\left(1-e^{\frac{-t}{\tau}}\right) - \frac{I_0^2\tau^2}{2C_2}\left(1-e^{\frac{-2t}{\tau}}\right) + V_{0,C_2}I_0\tau\left(1-e^{\frac{-t}{\tau}}\right) \quad (39)$$

By substituting $\tau = R\frac{C_1C_2}{C_1+C_2}$ and $I_0 = \frac{\Delta V}{R}$ we can further simplify the equation. The steps are omitted, but the resulting final function for the energy entering the capacitor C_2 is:

$$e_{C_2}(t) = \frac{1}{2}\frac{C_1^2 C_2}{(C_1+C_2)^2}\Delta V^2\left(1-2e^{\frac{-t}{\tau}}-e^{\frac{-2t}{\tau}}\right) + \Delta V\cdot V_{0,C_2}\frac{C_1C_2}{C_1+C_2}\left(1-e^{\frac{-t}{\tau}}\right) \quad (40)$$

If we allow the circuit to settle for a significant amount of time, we obtain the total amount of energy transferred into capacitor C_2.

$$\lim_{t\to\infty} e_{C_2}(t) = \frac{1}{2}\frac{C_1^2C_2}{(C_1+C_2)^2}\Delta V^2 + \Delta V\cdot V_{0,C_2}\frac{C_1C_2}{C_1+C_2} \quad (41)$$

3 Results

The equation **??** from *The Hard Way* is consistent with the formula we derived by using *The Easy Way*.

If we assume that C_2 is completely discharged ($V_{0,C_2} = 0$) and that $C_1 = C_2$, we get the surprising result that a quarter of the transferred energy is dissipated on the resistor, **regardless of the resistor**. This turns out to be half of the total amount of initial energy, if you take into consideration that C_1 will have the same charge as C_2 after the circuit has settled.

$$\frac{e_{C_2}}{e_R} = \frac{\frac{1}{2}\frac{C_1^2 C_2}{(C_1+C_2)^2}\Delta V^2}{\frac{1}{2}\frac{C_1 C_2}{C_1+C_2}\Delta V^2} \tag{42}$$

$$= \frac{C_1}{C_1 + C_2} \tag{43}$$

$$= \frac{1}{2} \tag{44}$$

This, however, is only the case when $C_1 = C_2$.

To get an idea of how much energy is lost with different sized capacitors, we can calculate a "transfer efficiency" factor by dividing the energy that arrives at C_2, $e_{C_2}(t)\ |_{t\to\infty}$, by the total amount of transferred energy, $e_{C_2}(t)\ |_{t\to\infty} + e_R(t)\ |_{t\to\infty}$, for different ratios of $r = \frac{C_2}{C_1}$. That is:

$$\eta(r) = \frac{e_{C_2}(t)\ |_{t\to\infty}}{e_{C_2}(t)\ |_{t\to\infty} + e_R(t)\ |_{t\to\infty}} \tag{45}$$

The resulting efficiency function can be seen in figure 5.

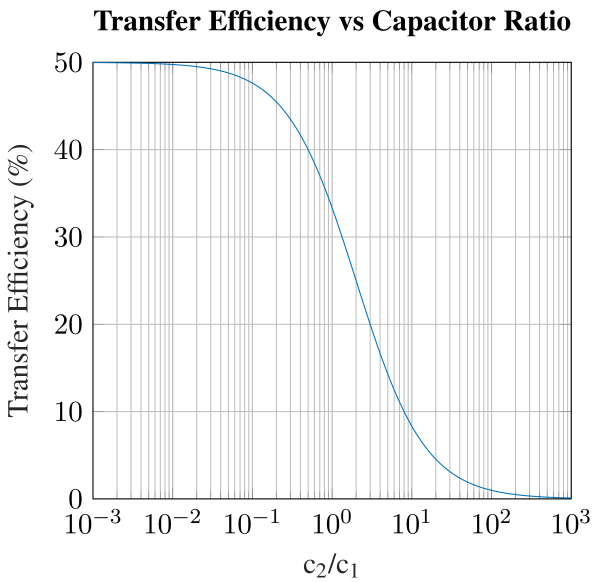

Figure 5: Efficiency of energy transfer for different C_1 to C_2 ratios.

From this graph we see that when $\frac{C_2}{C_1} = 1$ or $C_1 = C_2$, the transfer efficiency is 33.3 % (since $e_R\ |_{t\to\infty} = e_{C_2}\ |_{t\to\infty}$.

As the value of C_2 grows larger than C_1, more and more energy is dissipated on R versus being transferred into C_2.

As the value of C_2 grows smaller than C_1, the energy transfer gets more and more efficient, peaking at a theoretical maximum of 50 %.