Capacitor Paradox

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1 Introduction

The *Capacitor Paradox* is a widely known trick question commonly encountered in the field of electronics, especially on internet forums[1], and sometimes even in electrical engineering lectures, designed to throw people off.

Consider two capacitors, $C_1 = 1\,\mu F$ and $C_2 = 1\,\mu F$, with initial voltages of $V_{0,C_1} = 10\,V$ and $V_{0,C_2} = 0\,V$, respectively (see figure 1).

The energy stored in C_1 is

$$e_{C_1} = \frac{1}{2} C_1 V_{0,C_1}^2 = 50\,\mu J \tag{1}$$

Now, close the switch S_1 and let the circuit settle. Since $C_1 = C_2$, the charge from C_1 will evenly distribute over both capacitors such that both voltages settle at $V_{settle} = 5\,V$.

Calculating the new energy stored in both capacitors reveals something strange:

$$\begin{aligned} e_{settle} &= 2\left(\frac{1}{2} C_1 V_{settle}\right) && (2) \\ &= 2 \cdot 12.5\,\mu J && (3) \\ &= 25\,\mu J && (4) \\ & && (5) \end{aligned}$$

The new total energy e_{settle} is **half** of what it was before. Where did the energy go?

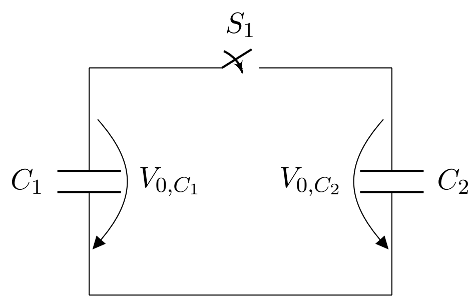


Figure 1: Closing the switch discharges C_1 into C_2. Calculating the new total amount of energy reveals that half of the energy has gone missing. Why?

2 A Closer Look

The circuit presented in figure 1 represents not only a physical impossibility, but a mathematical impossibility as well (more on the latter later). If one tried to simulate this circuit, one would realise that an infinite amount of current would have to flow in zero time, not to mention the switch would have to close infinitely quickly and have no resistive losses.

A more realistic model is one that models the imperfections of the switch and is shown in figure 2.

This new model at least gets rid of the mathematical impossibility and allows us to analyse the circuit in more detail. There are two approaches.

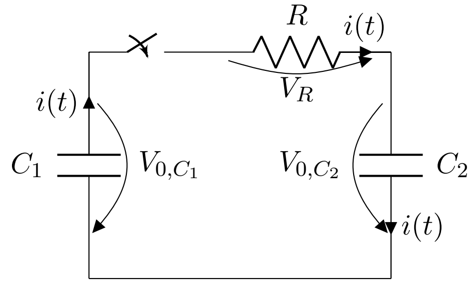


Figure 2

2.1 The Easy Way

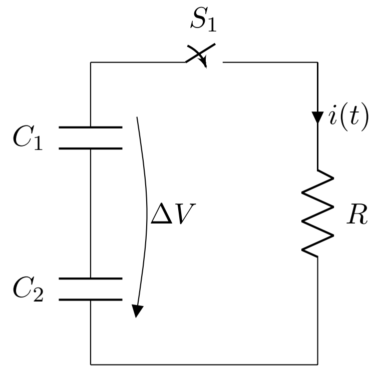
Perhaps the easiest explanation of all is to realise that the model can be further simplified into an equivalent circuit. Consider the circuits shown in figures 3a and 3b.

The equivalent circuit seen in figure 3b makes it very easy to see that all of the energy in C_{eq} will discharge over R and dissipate. The calculation is:

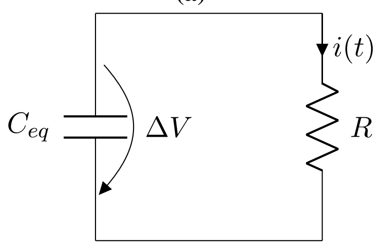
$$e_{loss} = \frac{1}{2} C_{eq} \Delta V^2 \tag{6}$$

$$= \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} \left(V_{0,C_1} - V_{0,C_2}\right)^2 \tag{7}$$

Using the same numbers from the first example, we find that $C_{eq} = 500\,\mathrm{nF}$, $\Delta V = 10\,\mathrm{V}$ and $e_{loss} = 25\,\mu\mathrm{J}$, as expected.



(a)



(b)

Figure 3: Shows **(a)** a re-arrangement of the circuit from figure 2 and **(b)** a simplified version of the circuit.

2.2 The Hard Way

In the interest of possibly figuring out what happens for $R \to 0$, a more complete analysis will be performed to predict the exact behaviour of the current $i(t)$ and the power $p_R(t)$ and energy $e_R(t)$ dissipated by the resistor.

2.2.1 Energy Dissipated

Closing the switch in circuit 2 at $t = 0$ completes the circuit and Kirchhoff's second law must hold true:

$$V_{C_1}(t) = V_R(t) + V_{C_2}(t) \tag{8}$$

Kirchoff's first law also states that the current through the resistor $i_R(t)$ must be equal to the current leaving and entering the two capacitors $i_{C_1}(t)$ and $i_{C_2}(t)$.

$$i(t) = i_R(t) = i_{C_1}(t) = i_{C_2}(t) \tag{9}$$

In order to solve for the unknown function $i(t)$, the voltage over each element can be expressed as a function of current:

$$V_R(t) = i(t)R \tag{10}$$
$$V_{C_1}(t) = -\frac{1}{C_1}\int_0^t i(\tau)\,d\tau + V_{0,C_1} \tag{11}$$
$$V_{C_2}(t) = \frac{1}{C_2}\int_0^t i(\tau)\,d\tau + V_{0,C_2} \tag{12}$$

Note the negation in equation 11, because the voltage V_{C_1} is drawn in the reverse direction in figure 2.

By substituting the equations 10, 11 and 12 into equation 8 we obtain the differential equation

$$\frac{1}{C_1}\int_0^t i(\tau)\,d\tau + \Delta V = i(t)R + \frac{1}{C_2}\int_0^t i(\tau)\,d\tau \tag{13}$$

where $\Delta V = V_{0,C_1} - V_{0,C_2}$.

One way to solve this differential equation is with the help of the Laplace transform, using the relationships listed in table 1.

Table 1: Relevant Laplace Transforms

Time function	Transformed	Validity
$af(t)$	$aF(s)$	$f(t<0)=0$
a	$\frac{a}{s}$	$a>0$
e^{-at}	$\frac{1}{s+a}$	$a>0$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	$f(\tau<0)=0$

Transforming equation 13 results yields:

$$\frac{I(s)}{sC_1} = I(s)R + \frac{I(s)}{sC_2} - \frac{\Delta V}{s} \tag{14}$$
$$\frac{\Delta V}{s} = I(s)\frac{sRC_2+1}{sC_2} + \frac{I(s)}{sC_1} \tag{15}$$
$$\Delta V = I(s)\frac{sRC_1C_2 + C_1 + C_2}{C_1C_2} \tag{16}$$
$$I(s) = \frac{\Delta V C_1 C_2}{sRC_1C_2 + C_1 + C_2} \tag{17}$$
$$I(s) = \frac{\Delta V}{R}\cdot\frac{1}{s + \frac{C_1+C_2}{RC_1C_2}} \tag{18}$$

Equation 18 can be transformed back into the time domain by using the third relationship listed in table 1, and so we obtain the well-known current formula for an RC-Network.

$$i(t) = I_0 \cdot e^{\frac{-t}{\tau}} \qquad \begin{aligned} I_0 &= \frac{V_{0,C_1} - V_{0,C_2}}{R} \\ \tau &= R\frac{C_1C_2}{C_1+C_2} \end{aligned} \tag{19}$$

This result is intuitive, because from the perspective of the resistor (with a closed switch), we effectively "see" a total capacitance of $\frac{C_1C_2}{C_1+C_2}$ charged to a voltage of $V_{0,C_1} - V_{0,C_2}$.

The power and energy dissipated by the resistor can be calculated from the current $i(t)$:

$$p_R(t) = i^2(t)R \tag{20}$$
$$e_R(t) = \int_0^t p_R(\tau)\,d\tau \tag{21}$$

Substituting the equation for $i(t)$ 19 into the power equation 20 yields:

$$p_R(t) = R\left(\frac{\Delta V}{R}e^{\frac{-t}{\tau}}\right)^2 \tag{22}$$
$$= \frac{\Delta V^2}{R}e^{\frac{-2t}{\tau}} \tag{23}$$

Substituting this equation into the energy equation 21 and solving the integral yields:

$$e_R(t) = \frac{\Delta V^2}{R}\int_0^t e^{\frac{-2z}{\tau}}\,dz \tag{24}$$
$$= \frac{\Delta V^2}{R}\left(-\frac{\tau}{2}e^{\frac{-2z}{\tau}}\Big|_0^t\right) \tag{25}$$
$$= \frac{\Delta V^2}{R}\frac{\tau}{2}\left(1-e^{\frac{-2t}{\tau}}\right) \tag{26}$$

The resulting expression can further be simplified by substituting τ with the result from equation 19:

$$e_R(t) = \frac{\Delta V^2}{R}\frac{RC_1C_2}{2\,(C_1+C_2)}\left(1-e^{\frac{-2t}{\tau}}\right) \tag{27}$$
$$= \frac{1}{2}\frac{C_1C_2}{C_1+C_2}\Delta V^2\left(1-e^{\frac{-2t(C_1+C_2)}{RC_1C_2}}\right) \tag{28}$$

If we allow the circuit to settle for a significant amount of time, it is easy to see that the energy dissipated by the resistor **does not depend on the value of the resistor!**

$$\lim_{t\to\infty}\frac{1}{2}\frac{C_1C_2}{C_1+C_2}\Delta V^2\left(1-e^{\frac{-2t(C_1+C_2)}{RC_1C_2}}\right) \tag{29}$$
$$= \frac{1}{2}\frac{C_1C_2}{C_1+C_2}\Delta V^2 \tag{30}$$

2.2.2 Energy Transferred

The current function $i(t)$ from equation 19 can be re-inserted into the integral equations 11 and 12 to calculate the voltage over capacitor C_2:

$$v_{C_2}(t) = \frac{I_0}{C_2}\int_0^t e^{\frac{-z}{\tau}}\,dz + V_{0,C_2} \tag{31}$$
$$= -\frac{I_0\tau}{C_2}\cdot e^{\frac{-z}{\tau}}\Big|_0^t + V_{0,C_2} \tag{32}$$
$$= \frac{I_0\tau}{C_2}\left(1-e^{\frac{-t}{\tau}}\right) + V_{0,C_2} \tag{33}$$

The power is of course $v_{C_2}(t)\cdot i(t)$

$$p_{C_2}(t) = v_{C_2}(t)\cdot i(t) \tag{34}$$
$$= \left(\frac{I_0\tau}{C}\left(1-e^{\frac{-t}{\tau}}\right)+V_0\right)\cdot I_0\,e^{\frac{-t}{\tau}} \tag{35}$$
$$= \frac{I_0^2\tau}{C}\left(e^{\frac{-t}{\tau}}-e^{\frac{-2t}{\tau}}\right)+V_0I_0\,e^{\frac{-t}{\tau}} \tag{36}$$

and the energy is therefore

$$e_{C_2}(t) = \int_0^z p_{C_2}(t)\,dz \tag{37}$$
$$= \frac{I_0^2\tau}{C_2}\int_0^t e^{\frac{-z}{\tau}}\,dz - \frac{I_0^2\tau}{C_2}\int_0^t e^{\frac{-2z}{\tau}}\,dz + U_0I_0\int_0^t e^{\frac{-z}{\tau}}\,dz \tag{38}$$
$$= \frac{I_0^2\tau^2}{C_2}\left(1-e^{\frac{-t}{\tau}}\right) - \frac{I_0^2\tau^2}{2C_2}\left(1-e^{\frac{-2t}{\tau}}\right) + V_{0,C_2}I_0\tau\left(1-e^{\frac{-t}{\tau}}\right) \tag{39}$$

By substituting $\tau = R\frac{C_1C_2}{C_1+C_2}$ and $I_0 = \frac{\Delta V}{R}$ we can further simplify the equation. The steps are omitted, but the resulting final function for the energy entering the capacitor C_2 is:

$$e_{C_2}(t) = \frac{1}{2}\frac{C_1^2C_2}{(C_1+C_2)^2}\Delta V^2\left(1-2e^{\frac{-t}{\tau}}-e^{\frac{-2t}{\tau}}\right) + \Delta V\cdot V_{0,C_2}\frac{C_1C_2}{C_1+C_2}\left(1-e^{\frac{-t}{\tau}}\right) \tag{40}$$

If we allow the circuit to settle for a significant amount of time, we obtain the total amount of energy transferred into capacitor C_2.

$$\lim_{t\to\infty} e_{C_2}(t) = \frac{1}{2}\frac{C_1^2 C_2}{(C_1+C_2)^2}\Delta V^2 + \Delta V \cdot V_{0,C_2}\frac{C_1 C_2}{C_1+C_2} \tag{41}$$

3 Results

The two final formulas we derived describe the *energy transferred* and the *energy dissipated* and are listed here again, respectively.

$$e_{C_2} = \frac{1}{2}\frac{C_1^2 C_2}{(C_1+C_2)^2}\Delta V^2 + \Delta V \cdot V_{0,C_2}\frac{C_1 C_2}{C_1+C_2} \tag{42}$$

$$e_R = \frac{1}{2}\frac{C_1 C_2}{C_1+C_2}\Delta V^2 \tag{43}$$

If we assume that C_2 is discharged ($V_{0,C_2} = 0$) and that $C_1 = C_2$, we get the surprising result that a quarter of the transferred energy is dissipated on the resistor, **regardless of the resistor**. This turns out to be half of the total amount of initial energy, if you take into consideration that C_1 will have the same charge as C_2 after the circuit has settled.

$$\frac{e_{C_2}}{e_R} = \frac{\frac{1}{2}\frac{C_1^2 C_2}{(C_1+C_2)^2}\Delta V^2}{\frac{1}{2}\frac{C_1 C_2}{C_1+C_2}\Delta V^2} \tag{44}$$

$$= \frac{C_1}{C_1+C_2} \tag{45}$$

$$= \frac{1}{2} \tag{46}$$

This is an efficiency of 33 %. However, this is only the case when $C_1 = C_2$.

To get an idea of how much energy is lost with different sized capacitors, we can calculate a "transfer efficiency" factor by dividing the energy that arrives at C_2 (e_{C_2}) by the total amount of transferred energy ($e_{C_2} + e_R$) for different ratios of $r = \frac{C_2}{C_1}$. That is:

$$\eta(r) = \frac{e_{C_2}}{e_{C_2} + e_R} \tag{47}$$

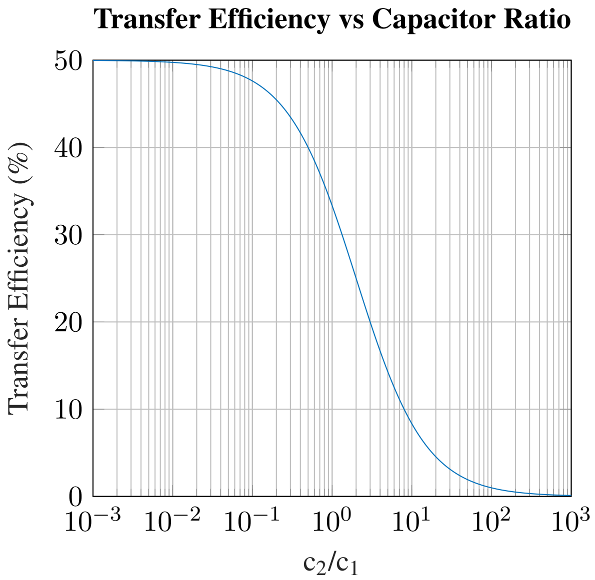


Figure 4: Efficiency of energy transfer for different C_1 to C_2 ratios.

The resulting efficiency function can be seen in figure 4.

From this graph we see that for $C_1 = C_2$, the resistor dissipates 33 % of the energy. As the value of C_2 grows larger than C_1, more and more energy is dissipated on R versus being transferred into C_2.

As the value of C_2 grows smaller than C_1, the energy transfer gets more and more efficient, peaking at a theoretical maximum of 50 %.

4 Discussion

It is worth noting that the calculations in this report assume the worst possible case (the voltage is a step function) and this may not be an assumption you can make in real world applications. A stepped voltage can appear for example in a sample and hold (S&H) circuit of an ADC converter, so in that situation – if efficiency is a concern – the calculations in this report are relevant.

The transfer efficiency will **increase** substantially for slower changes in the input voltage ($\frac{dv}{dt}$). Bypass capacitors, for example, are powered by voltage regulators that typically have fairly long ramp-up times (in the order of multiple milli-seconds), so the transfer efficiency in these cases will be a lot better and circuits such as the one seen in figure 5 won't dissipate any significant amount of energy.

It is also worth pointing out that adding a series inductor to the circuit, as seen in figure 6 has no influence on the total amount of dissipated energy, even if the circuit oscillates.

Letting $R \to 0$ in equations 43 and 42 does not converge to any limit. It is still an interesting thought experiment to think about what might happen if one had two superconducting capacitors and were able to close a switch close to the speed of light. An educated guess might be that some of the energy is dissipated over the electric arc formed between the switch contacts as it closes, and, assuming the switch is able to close before all of the energy can be transferred (limited by the transmission speed of electrons), the remaining energy would then oscillate back and forth between the two capacitors, assuming the superconducting wire connecting them is slightly inductive. It's possible that the energy would slowly decay over time as electromagnetic radiation.

A far more complete explanation of what happens for $R \to 0$ as well as analogies to water tanks and thought experiments where the capacitor plates are stretched to decrease their voltages can be found here[2].

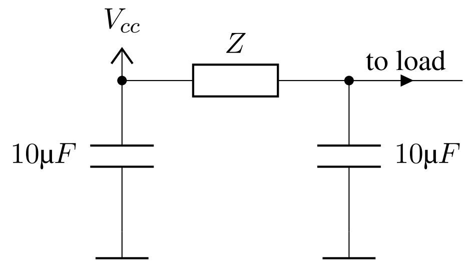


Figure 5: A common way to bypass a load: Two capacitors and a ferrite bead (sometimes, inductors or even resistors are used in place of the ferrite bead).

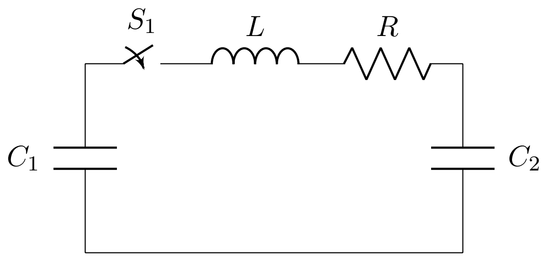


Figure 6: Inductor L has no influence on any of the previous calculations.

References

[1] *Capacitor paradox*. https://www.physicsforums.com/threads/capacitor-paradox.321298/. Accessed 2017-11-24.

[2] Ashok K. Singal. "The Paradox of Two Charged Capacitors – A New Perspective". In: *Phys. Education, 31, No. 4, 2* (2015).