The Capacitor Paradox

Alexander Murray

alexander.murray@students.fhnw.ch

alex.murray@gmx.ch

November 20, 2017

1 Introduction

The *Capacitor Paradox* is a widely known trick question commonly encountered in the field of electronics, especially on internet forums, and sometimes even in electrical engineering lectures, designed to throw people off.

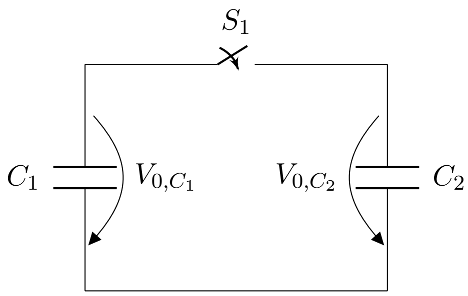

Figure 1

Consider two capacitors, $C_1 = 1\,\mu\text{F}$ and $C_2 = 1\,\mu\text{F}$, with initial voltages of $V_{0,C_1} = 10\,\text{V}$ and $V_{0,C_2} = 0\,\text{V}$, respectively.

The energy stored in C_1 is

$$e_{C_1} = \frac{1}{2} C_1 V_{0,C_1}^2 = 50\,\mu\text{J} \tag{1}$$

Now, close the switch S_1 and let the circuit settle. Since $C_1 = C_2$, the charge from C_1 will evenly distribute over both capacitors such that both voltages come to be $V_{settle} = 5\,\text{V}$.

Calculating the new energy stored in both capacitors reveals something strange:

$$\begin{aligned} e_{settle} &= 2\left(\frac{1}{2} C_1 V_{settle}\right) && (2) \\ &= 2 \cdot 12.5\,\mu\text{J} && (3) \\ &= 25\,\mu\text{J} && (4) \\ & && (5) \end{aligned}$$

The new total energy e_{settle} is **half** of what it was before. Where did the energy go?

2 A Closer Look

The circuit presented in figure 1 represents not only a physical impossibility, but a mathematical impossibility as well (more on the latter later). If one tried to simulate this circuit, one would realise that an infinite amount of current would have flow in zero time, not to mention the switch would have to close infinitely quickly and have no resistive losses.

A more realistic model is one that at least tries to model the imperfections of the switch and is shown in figure 2.

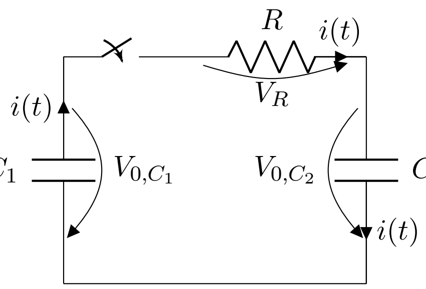

Figure 2

This new model at least gets rid of the mathematical impossibility and allows us to analyse the circuit in more detail. There are two approaches.

2.1 The Easy Way

Perhaps the easiest explanation of all is to realise that the model can be further simplified into an equivalent circuit. Consider the circuits shown in figures 3 and 4.

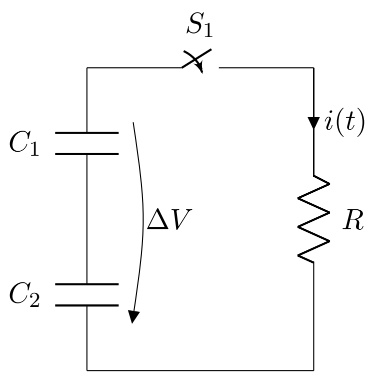

Figure 3: Re-arrangement of the same circuit seen in figure 2.

The equivalent circuit seen in figure 4 makes it very easy to see that all of the energy in C_{eq} will discharge over R and dissipate. The calculation is:

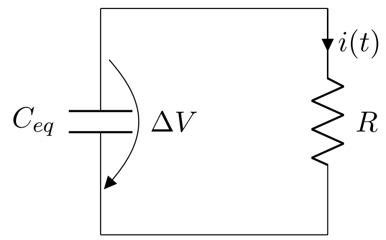

Figure 4: Equivalent circuit of figure 3

$$e_{loss} = \frac{1}{2} C_{eq} \Delta V^2 \tag{6}$$
$$= \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} \left(V_{0,C_1} - V_{0,C_2}\right)^2 \tag{7}$$

Using the same numbers from the first example, we find that $C_{eq} = 500\,\mathrm{nF}$, $\Delta V = 10\,\mathrm{V}$ and $e_{loss} = 25\,\mu\mathrm{J}$, as expected.

2.2 The Hard Way

In the interest of possibly figuring out what happens for $R \to 0$, a more complete analysis will be performed to predict the exact behaviour of the current $i(t)$ and the power $p_R(t)$ and energy $e_R(t)$ dissipated by the resistor.

Closing the switch in circuit 2 at $t = 0$ completes the circuit and Kirchhoff's second law must hold true:

$$\sum_{k=1}^{N} V_k = 0 \tag{8}$$
$$V_{C_1}(t) = V_R(t) + V_{C_2}(t) \tag{9}$$

Kirchoff's first law also states that the current through the resistor $i_R(t)$ must be equal to the current leaving and entering the two capacitors $i_{C_1}(t)$ and $i_{C_2}(t)$.

$$i(t) = i_R(t) = i_{C_1}(t) = i_{C_2}(t) \tag{10}$$

In order to solve for the unknown function $i(t)$, the voltage over each element can be expressed as a function of current:

$$V_R(t) = i(t)R \tag{11}$$

$$V_{C_1}(t) = -\frac{1}{C_1}\int_0^t i(\tau)\,d\tau + V_{0,C_1} \tag{12}$$

$$V_{C_2}(t) = \frac{1}{C_2}\int_0^t i(\tau)\,d\tau + V_{0,C_2} \tag{13}$$

Note the negation in equation 12, because the voltage V_{C_1} is drawn in the reverse direction in figure 2.

By substituting the equations 11, 12 and 13 into equation 9 we obtain the differential equation

$$\frac{1}{C_1}\int_0^t i(\tau)\,d\tau + \Delta V = i(t)R + \frac{1}{C_2}\int_0^t i(\tau)\,d\tau \tag{14}$$

where $\Delta V = V_{0,C_1} - V_{0,C_2}$.

One way to solve this differential equation is with the help of the Laplace transform, using the following relationships.

$$\mathscr{L}\left\{af(t)\right\} = aF(s), \quad f(t<0) = 0 \tag{15}$$

$$\mathscr{L}\left\{a\right\} = \frac{a}{s}, \quad a > 0 \tag{16}$$

$$\mathscr{L}\left\{e^{-at}\right\} = \frac{1}{s+a}, \quad a > 0 \tag{17}$$

$$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}, \quad f(\tau<0) = 0 \tag{18}$$

Transforming equation 14 results yields:

$$\frac{I(s)}{sC_1} = I(s)R + \frac{I(s)}{sC_2} - \frac{\Delta V}{s} \tag{19}$$

$$\frac{\Delta V}{s} = I(s)\frac{sRC_2+1}{sC_2} + \frac{I(s)}{sC_1} \tag{20}$$

$$\Delta V = I(s)\frac{sRC_1C_2 + C_1 + C_2}{C_1C_2} \tag{21}$$

$$I(s) = \frac{\Delta V C_1 C_2}{sRC_1C_2 + C_1 + C_2} \tag{22}$$

$$I(s) = \frac{\Delta V}{R}\cdot\frac{1}{s + \frac{C_1+C_2}{RC_1C_2}} \tag{23}$$

Equation 23 can be transformed back into the time domain by using the relationship described by equation 17, and so we obtain the well-known current formula for an RC-Network.

$$i(t) = I_0 \cdot e^{\frac{-t}{\tau}} \qquad \begin{aligned} I_0 &= \frac{V_{0,C_1} - V_{0,C_2}}{R} \\ \tau &= R\frac{C_1C_2}{C_1+C_2} \end{aligned} \tag{24}$$

This result is intuitive, because from the perspective of the resistor (with a closed switch), we effectively "see" a total capacitance of $\frac{C_1C_2}{C_1+C_2}$ charged to a voltage of $V_{0,C_1} - V_{0,C_2}$.

The power and energy dissipated by the resistor can be calculated from the current $i(t)$:

$$p_R(t) = i^2(t)R \tag{25}$$

$$e_R(t) = \int_0^t p_R(\tau)\,d\tau \tag{26}$$

Substituting the equation for $i(t)$ 24 into the power equation 25 yields:

$$p_R(t) = R\left(\frac{\Delta V}{R}e^{\frac{-t}{\tau}}\right)^2 \tag{27}$$

$$= \frac{\Delta V^2}{R}e^{\frac{-2t}{\tau}} \tag{28}$$

Substituting this equation into the energy equation 26 and solving the integral yields:

$$e_R(t) = \frac{\Delta V^2}{R}\int_0^t e^{\frac{-2z}{\tau}}\,dz \tag{29}$$

$$= \frac{\Delta V^2}{R}\left(-\frac{\tau}{2}e^{\frac{-2z}{\tau}}\Big|_0^t\right) \tag{30}$$

$$= \frac{\Delta V^2}{R}\frac{\tau}{2}\left(1 - e^{\frac{-2t}{\tau}}\right) \tag{31}$$

The resulting expression can further be simplified by substituting τ with the result from equation 24:

$$e_R(t) = \frac{\Delta V^2}{R}\frac{RC_1C_2}{2\,(C_1+C_2)}\left(1-e^{\frac{-2t}{\tau}}\right) \tag{32}$$
$$= \frac{1}{2}\frac{C_1C_2}{C_1+C_2}\Delta V^2\left(1-e^{\frac{-2t(C_1+C_2)}{RC_1C_2}}\right) \tag{33}$$

If we allow the circuit to settle for a significant amount of time, it is easy to see that the energy dissipated by the resistor **does not depend on the value of the resistor!**

$$\lim_{R\to\infty}\frac{1}{2}\frac{C_1C_2}{C_1+C_2}\Delta V^2\left(1-e^{\frac{-2t(C_1+C_2)}{RC_1C_2}}\right) \tag{34}$$
$$= \frac{1}{2}\frac{C_1C_2}{C_1+C_2}\Delta V^2 \tag{35}$$

2.3 Energy Transfer of the Capacitors

The current function $i(t)$ from equation 24 can be re-inserted into the integral equations 12 and 13 to calculate the voltages on both capacitors.

$$v(t) = \frac{I_0}{C}\int_0^t e^{\frac{-z}{\tau}}\,dz + V_0 \tag{36}$$
$$= -\frac{I_0\tau}{C}\cdot e^{\frac{-z}{\tau}}\Big|_0^t + V_0 \tag{37}$$
$$= \frac{I_0\tau}{C}\left(1-e^{\frac{-t}{\tau}}\right) + V_0 \tag{38}$$

The power is of course

$$p(t) = u(t)\cdot i(t) \tag{39}$$
$$= \left(\frac{I_0\tau}{C}\left(1-e^{\frac{-t}{\tau}}\right)+V_0\right)\cdot I_0\,e^{\frac{-t}{\tau}} \tag{40}$$
$$= \frac{I_0^2\tau}{C}\left(e^{\frac{-t}{\tau}}-e^{\frac{-2t}{\tau}}\right)+V_0I_0\,e^{\frac{-t}{\tau}} \tag{41}$$

and the energy is therefore

$$e(t) = \int_0^z p(t)\,dz \tag{42}$$
$$= \frac{I_0^2\tau}{C}\int_0^t e^{\frac{-z}{\tau}}\,dz - \frac{I_0^2\tau}{C}\int_0^t e^{\frac{-2z}{\tau}}\,dz + U_0I_0\int_0^t e^{\frac{-z}{\tau}}\,dz \tag{43}$$
$$= \tag{44}$$

2.4 eh

In fact, the resulting equation 35 is identical to the formula used to calculate the *initial energy difference of the capacitors!* For $t < 0$, the switch is open and both capacitors are connected in series. The total capacitance is therefore

$$C_{tot} = \frac{C_1C_2}{C_1+C_2} \tag{45}$$

The capacitors are charged with an initial voltage of V_{0,C_1} and V_{0,C_2}. Because V_{0,C_1} is drawn in figure 2 with opposite polarity to V_{0,C_2}, the total initial voltage over both capacitors is $\Delta V = V_{0,C_1} - V_{0,C_2}$ and the total initial energy difference is therefore:

$$e_{C_{tot}} = \frac{1}{2}C_{tot}\Delta V^2 \tag{46}$$

The surprising result is that $e_{C_{tot}} = e_R(t)\Big|_{t\to\infty}$ or in other words: The initial energy stored in both capacitors is equal to the energy dissipated by the resistor after closing the switch and letting the circuit settle.

3 Discussion