CURVATURE

Let P be any point on a given curve and Q a neighbouring point. Let arc $AP = s$ and arc $PQ = \delta s$. Let the tangents at P and Q make angle ψ and $\psi + \delta\psi$ with the x-axis, so that the angle between the tangents at P and $Q = \delta\psi$ (Fig. 4.9).

In moving from P to Q through a distance δs, the tangent has turned through the angle $\delta\psi$. This is called the *total bending or total curvature* of the arc PQ.

$\therefore$ The *average curvature* of arc $PQ = \dfrac{\delta\psi}{\delta s}$

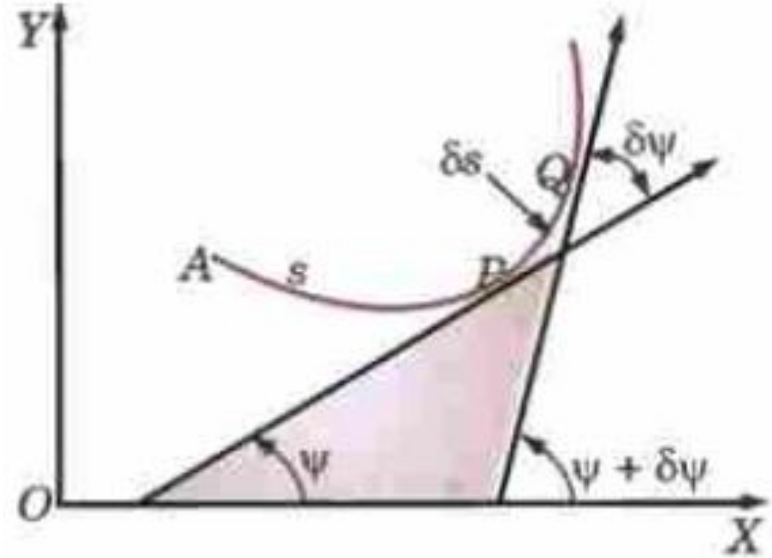


Fig. 4.9

The limiting value of average curvature when Q approaches P (i.e., $\delta s \to 0$) is defined as the curvature of the curve at P.

Thus **curvature** K (at P) $= \dfrac{d\psi}{ds}$

(2) **Radius of curvature.** *The reciprocal of the curvature of a curve at any point P is called the* **radius of curvature** *at P and is denoted by* ρ, so the $\rho = \mathbf{ds/d\psi}$.

(3) **Centre of curvature.** *A point C on the normal at any point P of a curve distant* ρ *from it, is called the* **centre of curvature** *at P.*

(4) **Circle of curvature.** *A circle with centre C (centre of curvature at P) and radius* ρ *is called the* **circle of curvature** *at P.*

RADIUS OF CURVATURE FOR CARTESIAN CURVE y = f (x), is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$\tan\psi = dy/dx = y_1$ or $\psi = \tan^{-1}(y_1)$

Differentiating both sides w.r.t. x,

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2} \cdot \frac{d(y_1)}{dx} = \frac{y_2}{1+y_1^2}$$

$$\therefore \quad \rho = \frac{ds}{d\psi} = \frac{ds}{dx} \cdot \frac{dx}{d\psi} = \sqrt{(1+y_1^2)} \cdot \frac{1+y_1^2}{y_2} = \frac{(1+y_1^2)^{3/2}}{y_2}$$

unit I Differential calculus

- R.C
  - Carte...
  - para...
  - polar
- Evolutes
- Envelope
- Max. Min of 2.

$$\rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''}$$

1. $y =$
2. $y' =$
3. $(y')_{(\ ,\ )}$
4. y''
5. $(y'')_{(\ ,\ )}$

6. $\rho =$

① Find ρ

$xy = c^2 \quad (c, c).$

sol

$$y = \frac{c^2}{x}$$

$$y' = -\frac{c^2}{x^2}$$

$$(y')_{(c,c)} = -\frac{c^2}{c^2} = -1$$

$$y'' = \frac{2c^2}{x^3}$$

$$(y'')_{(c,c)} = \frac{2c^2}{c^3} = \frac{2}{c}$$

$$\rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''}$$

$$(\rho)_{(c,c)} = \frac{\left[1+(-1)^2\right]^{3/2}}{2/c}$$

$$= 2^{3/2} \times \frac{c}{2}$$

$$= \frac{2\sqrt{2} \times c}{2}$$

$$= \sqrt{2}\, c$$

(2) Find ρ $y^2 = 4ax$ at $y = 2a$ $\underline{\underline{4\sqrt{2}a}}$

<u>Sol</u>

$$2yy' = 4a$$

$$y' = \frac{4a}{2y} = \frac{2a}{y}$$

$$(y')_{y=2a} = \frac{2a}{2a} = 1$$

$$yy' = 2a$$

$$yy'' + y'y' = 0$$

at $y = 2a$

$$(2a)y'' + (1)^2 = 0$$

$$y'' = -\frac{1}{2a}$$

$$\rho = \frac{\left[1 + (y')^2\right]^{3/2}}{y''}$$

$$= \frac{[1+1]^{3/2}}{-1/2a}$$

$$|\rho| = 2\sqrt{2} \times 2a$$

$$= \underline{\underline{4a\sqrt{2}}}$$

Note:- As curvature (and hence radius of curvature) of a curve at any pt. is independent of the choice of x and y-axis, so x and y can be interchanged in the formula.

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

you can use this formula when $\frac{dy}{dx}$ is infinite at a pt.

3 Find φ $\quad xy^2 = a^3 - x^3 \quad (a, 0)$

sol

$$y^2 = \frac{a^3 - x^3}{x} \rightarrow 1$$

$$2yy' = \frac{x(-3x^2) - (a^3 - x^3)(1)}{x^2}$$

$$= \frac{-3x^3 - a^3 + x^3}{x^2}$$

$$y' = \frac{-(a^3 + 2x^3)}{2x^2 y} \rightarrow 2$$

$$(y')_{(a,0)} = \infty$$

$$\frac{dx}{dy} = \frac{-2x^2y}{2x^3+a^3} \rightarrow 3$$

$$\left(\frac{dx}{dy}\right)_{(a,0)} = 0$$

Diff 3 w.r.t. y

$$\frac{d^2x}{dy^2} = \frac{-2\left[(2x^3+a^3)\left[x^2\cdot 1 + y\cdot 2x\frac{dx}{dy}\right] - (x^2y)\left(6x^2\frac{dx}{dy}\right)\right]}{(2x^3+a^3)^2}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = \frac{-2\left[(2a^3+a^3)(a^2+0) - 0\right]}{(2a^3+a^3)^2}$$

$$= \frac{-2a^2(3\cancel{a^3})}{(3a^3)^{\cancel{2}}}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = -\frac{2}{3a}$$

$$\rho = \frac{(1+0)^{3/2}}{-2/3a}$$

$$|\rho| = \frac{3a}{2}$$

(4) If ρ is the radius of curvature at any pt (x, y) on the curve $y = \frac{ax}{a+x}$, S.T

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

y'

y''

Sol

$$y = \frac{ax}{a+x}$$

$$y' = \frac{(a+x)a - (ax)(1)}{(a+x)^2} = \frac{a^2}{(a+x)^2}$$

$$y'' = \frac{-2a^2}{(a+x)^3}$$

$$\rho = \frac{\left(1+(y')^2\right)^{3/2}}{y''} = \frac{\left[1+\dfrac{a^4}{(a+x)^4}\right]^{3/2}}{-\dfrac{2a^2}{(a+x)^3}}$$

$$|\rho| = \left[\frac{(a+x)^4 + a^4}{(a+x)^4}\right]^{3/2} \times \frac{(a+x)^3}{2a^2}$$

$$\left[(a+x)^{\cancel{4}\,2}\right]^{3/\cancel{2}}$$

$$\rho = \frac{\left[(a+x)^4 + a^4\right]^{3/2}}{(a+x)^{\cancel{6}\,3}} \times \frac{\cancel{(a+x)^3}}{2a^2}$$

$$\frac{2\rho}{a} = \frac{\left[(a+x)^4 + a^4\right]^{3/2}}{(a+x)^3\, a^3}$$

$$\frac{2\rho}{a} = \frac{\left[(a+x)^4 + a^4\right]^{3/2}}{(a+x)^3\, a^3}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \frac{(a+x)^4 + a^4}{\left[(a+x)^3\right]^{2/3} (a^3)^{2/3}}$$

$$= \frac{(a+x)^4 + a^4}{(a+x)^2\, a^2}$$

$$= \frac{(a+x)^2}{a^2} + \frac{a^2}{(a+x)^2}$$

$$= \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

$$y = \frac{ax}{a+x}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

$$\frac{y}{x} = \frac{a}{a+x}$$

(5) Show that the curves $y = c\cosh\frac{x}{c}$ and $x^2 = 2c(y-c)$ have the same curvature, where they cross the y-axis.

For the 1st curve, $x = 0$, $y = c\,x\,1 = c$

$$\cosh x = \frac{e^x + e^{-x}}{e}$$

$$\cosh 1 = \frac{e^0 + e^{-0}}{2}$$

For the second curve,

at $x = 0$, $\quad 0 = 2c(y-c)$

$\Rightarrow y - c = 0 \Rightarrow y = c$

Thus the common pt is $(0, c)$

First curve, $y = c\cosh\frac{x}{c}$

$y' = c\sinh\frac{x}{c} \times \frac{1}{c} = \sinh\frac{x}{c}$

$(y')_{(0,c)} = \sinh 0 = 0$ $\qquad \frac{e^0 - e^{-0}}{2}$

$\frac{1-1}{2}$

$y'' = \cosh\frac{x}{c} \times \frac{1}{c}$

$(y'')_{(0,c)} = 1 \times \frac{1}{c} = \frac{1}{c}$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

$$y = \frac{ax}{a+x}$$

$$\frac{y}{x} = \frac{a}{a+x}$$

(5) Show that the curves $y = c\cosh\frac{x}{c}$ and $x^2 = 2c(y-c)$ where they cross the y-axis have the same curvature.

For the 1st curve, $x = 0$, $y = c\,x\,1 = c$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\cosh 1 = \frac{e^0 + e^{-0}}{2}$$

$x^2 = 2c(y-c)$

For the second curve,

at $x=0$, $0 = 2c(y-c)$

$\Rightarrow y-c=0 \Rightarrow y=c$

Thus the common pt is $(0, c)$

First curve, $y = c\cosh\frac{x}{c}$

$y' = c\sinh\frac{x}{c} \times \frac{1}{c} = \sinh\frac{x}{c}$

$(y')_{(0,c)} = \sinh 0 = 0$

$\frac{e^0 - e^{-0}}{2}$

$\frac{1-1}{2}$

$y'' = \cosh\frac{x}{c} \times \frac{1}{c}$

$(y'')_{(0,c)} = 1 \times \frac{1}{c} = \frac{1}{c}$

$$\rho = \frac{[1+(y')^2]^{3/2}}{y''} = \frac{[1+0]^{3/2}}{1/c} = c$$

Second Curve $x^2 = 2c(y-c)$

$$\frac{x^2}{2c} = y - c \Rightarrow y = c + \frac{x^2}{2c}$$

$$y' = \frac{2x}{2c} = \frac{x}{c}$$

$$(y')_{(0,c)} = 0$$

$$y'' = \frac{1}{c}$$

$$\rho = \frac{(1+0)^{3/2}}{1/c} = c$$

K_1 $\Downarrow$ $1/c$

K_2 $\Downarrow$ $1/c$

① Find ρ for the curve $y = x^2(x-3)$ at the pts where the tangent is parallel to the x-axis.

$y = c$

$\frac{dy}{dx} = 0$

Sol

$$y = x^3 - 3x^2$$

$$y' = 3x^2 - 6x$$

Given: tangent is parallel to the x-axis

$$\Rightarrow \frac{dy}{dx} = y' = 0$$

$$3x^2 - 6x = 0$$

$$3x(x-2) = 0$$

$$\Rightarrow x = 0, \quad x = 2$$

when $x = 0 \Rightarrow y = 0$

when $x = 2 \Rightarrow y = 8 - 12 = -4$

$y'' = 6x - 6$

$(y'')_{(0,0)} = -6$

$(y'')_{(2,-4)} = 12 - 6 = 6$

$\rho = \dfrac{(1+0)^{3/2}}{\pm 6} \Rightarrow |\rho| = \dfrac{1}{6}$

② Find ρ, $y = c\cosh\dfrac{x}{c}$ at the pt where it is minimum.

Sol

$y' = \cancel{c}\sinh\dfrac{x}{c} \times \dfrac{1}{\cancel{c}}$

$y' = 0 \Rightarrow \sinh\dfrac{x}{c} = 0$

$\Rightarrow \boxed{x = 0}$

$$y'' = \cosh\frac{x}{c} \times \frac{1}{c}$$

$$(y'')_{x=0} = 1 \times \frac{1}{c} = \frac{1}{c}$$

$$\rho = \frac{(1+0)^{3/2}}{1/c} = c.$$

(3) Find ρ, $x^2 = 4ay$ at the pt where the slope of the tangent is $\tan\theta$.

Sol

$$x^2 = 4ay$$

$$2x = 4ay'$$

$$\Rightarrow y' = \frac{2x}{4a} = \frac{x}{2a}$$

Given $y' = \tan\theta$

$$\frac{x}{2a} = \tan\theta$$

$$\Rightarrow \quad x = 2a\tan\theta \checkmark$$

$$y' = x/2a$$

$$y'' = 1/2a$$

$$\rho = \frac{(1+\tan^2\theta)^{3/2}}{1/2a}$$

$$= [\sec^2\theta]^{3/2} \times 2a$$

$$\rho = \underline{2a\sec^3\theta}$$

① $x^3 + y^3 = 3axy \quad \left(\frac{3a}{2}, \frac{3a}{2} \right)$

<u>sol</u> $\quad y' = \dfrac{ay - x^2}{y^2 - ax}$

$$(y') = \frac{ax - x^2}{x^2 - ax} = -1$$

$$y'' = \frac{(y^2 - ax)(ay' - 2x) - [ay - x^2][2yy' - a]}{(y^2 - ax)^2}$$

$$(y'') = \frac{(x^2 - ax)(-a - 2x) - (ax - x^2)(-2x - a)}{(x^2 - ax)^2}$$

$$y'' = -\frac{(x^2-ax)(a+2x) - (x^2-ax)(a+2x)}{(x^2-ax)}$$

$$y'' = -\frac{(a+2x) - (a+2x)}{(x^2-ax)}$$

$$y'' = -\frac{2(a+2x)}{x^2-ax}$$

$x = \frac{3a}{2}$, $y'' = -\frac{32a}{3a}$

$$\rho = \frac{3\sqrt{2}\,a}{16}$$

① Find ρ, $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ at any pt.

Sol

$x = a\cos^4 t$

$y = b\sin^4 t$

$$\frac{1}{\sqrt{a}}\sqrt{x} + \frac{1}{\sqrt{b}}\sqrt{y} = 1 \longrightarrow (1)$$

$$\frac{1}{\sqrt{a}}\frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{b}}\frac{1}{2\sqrt{y}}y' = 0$$

$$\frac{1}{\sqrt{a}}\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{b}}\frac{1}{\sqrt{y}}y' = 0$$

$$\frac{y'}{\sqrt{b}\sqrt{y}} = -\frac{1}{\sqrt{a}\sqrt{x}}$$

$$y' = -\frac{\sqrt{b}}{\sqrt{a}}\frac{\sqrt{y}}{\sqrt{x}} \longrightarrow (2)$$

$$y' = -\frac{\sqrt{b}}{\sqrt{a}} \frac{\sqrt{y}}{\sqrt{x}}$$

$$y'' = -\frac{\sqrt{b}}{\sqrt{a}} \left[\frac{\sqrt{x}\,\frac{1}{2\sqrt{y}}\,y' - \sqrt{y}\,\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} \right]$$

$$= -\frac{\sqrt{b}}{\sqrt{a}} \left[\frac{\frac{\cancel{\sqrt{x}}}{2\cancel{\sqrt{y}}} \left(-\frac{\sqrt{b}}{\sqrt{a}} \frac{\cancel{\sqrt{y}}}{\cancel{\sqrt{x}}} \right) - \sqrt{y}\,\frac{1}{2\sqrt{x}}}{x} \right]$$

$$= -\frac{\sqrt{b}}{\sqrt{a}} \left[\frac{-\frac{\sqrt{b}}{2\sqrt{a}} - \frac{\sqrt{y}}{2\sqrt{x}}}{x} \right]$$

$$= + \frac{\sqrt{b}}{\sqrt{a}\,x}\left[\frac{\sqrt{bx}+\sqrt{ay}}{2\sqrt{a}\sqrt{x}}\right]$$

$$= \frac{\sqrt{b}}{2ax^{3/2}}\left[\sqrt{ay}+\sqrt{bx}\right]$$

$$= \frac{\sqrt{b}}{2ax^{3/2}}\sqrt{ab}$$

$$y'' = \frac{b}{2\sqrt{a}\,x^{3/2}}$$

$$\left\{\begin{array}{l}\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}} \\ \qquad = 1 \\ \times\sqrt{ab}, \\ \sqrt{xb}+\sqrt{ya} \\ \quad = \sqrt{ab}\end{array}\right.$$

$$y' = -\frac{\sqrt{b}}{\sqrt{a}} \frac{\sqrt{y}}{\sqrt{x}}, \quad y'' = \frac{b}{2\sqrt{a}\, x^{3/2}}$$

$$\rho = \frac{\left[1 + \frac{b}{a}\frac{y}{x}\right]^{3/2}}{b} \times 2\sqrt{a}\, x^{3/2}$$

$$= \frac{(ax+by)^{3/2}}{(ax)^{3/2}} \times \frac{2\sqrt{a}\, x^{3/2}}{b}$$

$$\rho = \frac{2(ax+by)^{3/2}}{ab}$$

(1) find ρ, $x = a(\cos t + t\sin t)$

$y = a(\sin t - t\cos t)$

$$\frac{dx}{dt} = a[-\cancel{\sin t} + t\cos t + \cancel{\sin t}]$$

$$= at\cos t$$

$$\frac{dy}{dt} = a[\cancel{\cos t} - (t(-\sin t) + \cancel{\cos t})]$$

$$= at\sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\{\tan t\}$$

$$= \frac{d}{dt}(\tan t)\,\frac{dt}{dx}$$

$$= [\sec^2 t]\,\frac{1}{at\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{at\cos^3 t}$$

$$\rho = [1 + \tan^2 t]^{3/2} \times at\cos^3 t$$

$$= [\sec^2 t]^{3/2} \times at\cos^3 t$$

$$= \underline{\underline{at}}$$

3.1.3 Formula for Radius of Curvature in Parametric Co-ordinates

Let the parametric equations of the curve be

$$x = f(t) \quad \text{and} \quad y = g(t).$$

Then $\quad \dot{x} = \dfrac{dx}{dt} = f'(t) \quad$ and $\quad \dot{y} = \dfrac{dy}{dt} = g'(t).$

$\therefore$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \checkmark$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \times \frac{dt}{dx}$$

$$= \left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}\right) \cdot \frac{1}{\dot{x}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3} \checkmark$$

$$\frac{dy/dt}{dx/dt} = \dot{y}/\dot{x}$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

Now

$$\rho = \frac{\left\{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{3/2}}{\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)}$$

$$= \frac{\left\{1+\left(\frac{\dot{y}}{\dot{x}}\right)^2\right\}^{3/2}}{\left(\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^3}\right)}$$

$$\frac{\left(1+\frac{\dot{y}^2}{\dot{x}^2}\right)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} \times \dot{x}^3$$

$$\frac{\frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{(\dot{x}^2)^{3/2}} \; \dot{x}^3}{[\dot{x}\ddot{y}-\dot{y}\ddot{x}]}$$

$$= \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

$$= \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}^3} \times \frac{\dot{x}^3}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

Radius of Curvature in terms of parametric eqn

① Find ρ, $x = t^2$, $y = t$ at $t = 1$

Sol

$\dot{x}$
$(\dot{x})(1)$
$\dot{y}$
$(\dot{y})(1)$
$\ddot{x}$
$\ddot{y}$

$\dot{x} = \frac{dx}{dt} = 2t$

$(\dot{x})_{t=1} = 2$

$\ddot{x} = 2$

$\dot{y} = \frac{dy}{dt} = 1$

$(\dot{y})_{t=1} = 1$

$\ddot{y} = 0$

$$\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} = \frac{\left[(2)^2 + 1^2\right]^{3/2}}{(2)(0) - (2)(1)} = \frac{5^{3/2}}{-2}$$

$$= \frac{5\sqrt{5}}{2}$$

① Find ρ $x = a\log\left(\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right)$, $y = a\sec\theta$ at θ

$$\dot{x} = \frac{a}{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)} \sec^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{a}{2} \frac{\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)} \frac{1}{\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}$$

$$= \frac{a}{2\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}$$

$$= \frac{a}{\sin\left(\frac{\pi}{4}+\frac{\theta}{2}+\frac{\pi}{4}+\frac{\theta}{2}\right)}$$

$$= \frac{a}{\sin\left(\frac{\pi}{2} + \theta\right)} = \frac{a}{\cos\theta}$$

$$\checkmark\ \dot{x} = a\sec\theta$$

$$\checkmark\ \dot{x} = a\sec\theta\tan\theta$$

$$y = a\sec\theta$$

$$\checkmark\ \dot{y} = a\sec\theta\tan\theta$$

$$\checkmark\ \ddot{y} = a\sec^3\theta + a\sec\dot{\theta}\tan^2\theta$$

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}$$

$$Nr = (\dot{x}^2 + \dot{y}^2)^{3/2} = \left[a^2 \sec^2\theta + a^2 \sec^2\theta \tan^2\theta\right]^{3/2}$$

$$= \left[a^2 \sec^2\theta \left[\underbrace{1 + \tan^2\theta}\right]\right]^{3/2}$$

$$= a^3 \sec^3\theta \sec^3\theta = a^3 \sec^6\theta$$

$$Dr = \dot{x}\ddot{y} - \ddot{x}\dot{y} = a\sec\theta\left[a\sec^3\theta + a\sec\theta\tan^2\theta\right]$$

$$- (a\sec\theta\tan\theta)(a\sec\theta\tan\theta)$$

$$= a^2\sec^4\theta + a^2\sec^2\theta\tan^2\theta - a^2\sec^2\theta\tan^2\theta$$

$$= a^2\sec^4\theta$$

$$\rho = \frac{Nr}{Dr} = \frac{a^3\sec^6\theta}{a^2\sec^4\theta} = a\sec^2\theta$$

(2) Find ρ $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

Cycloid

[illegible]

$y = c\cosh\frac{x}{c}$

$\downarrow$

Catenary

Sol

$$\frac{dx}{d\theta} = a[1 - \cos\theta]$$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)}$$

$$= \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$\frac{dy}{dx} = \cot\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\frac{d\theta}{dx}$$

$$= \left[-\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{2}\right] \times \frac{1}{a(1-\cos\theta)}$$

$$= -\frac{1}{2a}\,\frac{1}{\sin^2\theta/2}\,\frac{1}{2\sin^2\theta/2}$$

$$= -\frac{1}{4a}\,\frac{1}{\sin^4\theta/2}$$

$$\rho = \frac{\left[1+\cot^2\theta/2\right]^{3/2}}{-1/4a\sin^4\theta/2} = \left[\operatorname{cosec}^2\theta/2\right]^{3/2} \times 4a\sin^4\frac{\theta}{2}$$

$$= \frac{1}{\sin^3\theta/2} \times 4a\sin^4\theta/2$$

$$= 4a\sin\theta/2$$

(3) Find ρ $x = 3a\cos\theta - a\cos 3\theta$

$y = 3a\sin\theta - a\sin 3\theta$

$$\frac{dx}{d\theta} = -3a\sin\theta + 3a\sin 3\theta$$
$$= 3a[\sin 3\theta - \sin\theta]$$

$$\frac{dy}{d\theta} = 3a\cos\theta - 3a\cos 3\theta$$
$$= 3a[\cos\theta - \cos 3\theta]$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{3a}[\cos\theta - \cos 3\theta]}{\cancel{3a}[\sin 3\theta - \sin\theta]}$$
$$= \frac{-2\sin\left(\frac{\theta+3\theta}{2}\right)\sin\left(\frac{\theta-3\theta}{2}\right)}{2\cos\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)}$$

$$= -\frac{\sin 2\theta \sin(-\theta)}{\cos 2\theta \sin\theta}$$

$$\frac{dy}{dx} = \frac{\sin 2\theta \sin\theta}{\cos 2\theta \sin\theta} = \tan 2\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\tan 2\theta\right)\frac{d\theta}{dx}$$

$$= 2\sec^2 2\theta \times \frac{1}{3a[\sin 3\theta - \sin\theta]}$$

$$= \frac{2\sec^2 2\theta}{3a[2\cos 2\theta \sin\theta]}$$

$$= \frac{\sec^3 2\theta}{3a\sin\theta}$$

$$\rho = \frac{\left[1 + \tan^2 2\theta\right]^{3/2}}{\sec^3 2\theta} \times 3a\sin\theta$$

$$= \frac{\sec^3 2\theta}{\sec^3 2\theta} \times 3a\sin\theta = \underline{3a\sin\theta}$$

(3) Find the radius of curvature at the point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3}+y^{2/3}=a^{2/3}$

Sol

$$x = a\cos^3\theta$$

$$\dot{x} = 3a\cos^2\theta(-\sin\theta)$$

✓ $$\dot{x} = -3a\cos^2\theta\sin\theta$$

$$\ddot{x} = -3a\left[\cos^2\theta\cos\theta + 2\cos\theta(-\sin\theta)\sin\theta\right]$$

$$\ddot{x} = -3a\left[\cos^3\theta - 2\sin^2\theta\cos\theta\right]$$

✓ $$\ddot{x} = -3a\left[\cos^3\theta\right.$$

$$y = a\sin^3\theta$$

✓ $$\dot{y} = 3a\sin^2\theta\cos\theta$$

$$\ddot{y} = 3a\left[\sin^2\theta[-\sin\theta] + 2\sin\theta\cos\theta\cos\theta\right]$$

✓ $$\ddot{y} = 3a\left[-\sin^3\theta + 2\sin\theta\cos^2\theta\right]$$

$$(\dot{x}^2+\dot{y}^2)^{3/2} = \left[9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta\right]^{3/2}$$

$$= \left[9a^2\sin^2\theta\cos^2\theta\right]^{3/2}\left[\cos^2\theta+\sin^2\theta\right]^{3/2}$$

$$= 9\sqrt{9}\,a^3\sin^3\theta\cos^3\theta$$

$$= 27a^3\sin^3\theta\cos^3\theta$$

$$\dot{x}\ddot{y}-\ddot{x}\dot{y} = \left[-3a\cos^2\theta\sin\theta\right]3a\left[-\sin^3\theta+2\sin\theta\cos^2\theta\right]$$

$$-\left[\cos^3\theta-2\sin^2\theta\cos\theta\right]\left[3a\sin^2\theta\cos\theta\right]$$

$$+3a\left[\cos^3\theta-2\sin^2\theta\cos\theta\right] \ldots$$

$$= 9a\sin^4\theta\cos^2\theta - 18a^2\sin^2\theta\cos^4\theta - 18a^2\sin^4\theta\cos^2\theta$$

$$+ 9a^2\sin^2\theta\cos^4\theta - 9a^2\sin^2\theta\cos^4\theta$$

$$+9a^2\sin^4\theta\cos^2\theta$$

$$= -9a^2\sin^4\theta\cos^2\theta - 9a^2\sin^2\theta\cos^4\theta$$

$$\rho = \frac{27a^3 \sin^3\theta \cos^3\theta}{-9a^2 \sin^2\theta \cos^2\theta\,[\sin^2\theta + \cos^2\theta]}$$

$$\rho = 3a \sin\theta \cos\theta$$

(2) Find ρ for the curve $x = a(\cos t + t \sin t)$
$y = a[\sin t - t \cos t)$

Sol

$$x = a[\cos t + t \sin t]$$

$$\frac{dx}{dt} = a\,[-\cancel{\sin t} + t \cos t + \cancel{\sin t}]$$

$$\frac{dx}{dt} = at \cos t$$

$$y = a(\sin t - t\cos t)$$

$$\frac{dy}{dt} = a\left[\cancel{\cos t} - \left(t(-\sin t) + \cancel{\cos t}\right)\right]$$

$$= at\sin t$$

$$\frac{dy}{dt} = at\sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\tan t\right)\frac{dt}{dx}$$

$$= \sec^2 t \, \frac{1}{at\cos t}$$

$$= \frac{1}{at\cos^3 t}$$

$$\rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''} = \frac{\left[1+\tan^2 t\right]^{3/2}}{1/at\cos^3 t}$$

$$= \left[\sec^{\cancel{2}} t\right]^{3/\cancel{2}} \times at\cos^3 t$$

$$= \left(\sec^3 t\right) at\cos^3 t$$

$$\boxed{\rho = at}$$

(2) Find ρ at $(ct, c/t)$ on $xy = c^2$ and p.T $\rho = \frac{r^3}{2c^2}$, where r is the distance of any pt (x, y) on the curve from the origin. Hence deduce that ρ at (c, c).

Sol

$x = ct$ | $y = c/t$

$\dot{x} = c$ | $\dot{y} = -c/t^2$

$\ddot{x} = 0$ | $\ddot{y} = \frac{2c}{t^3}$

$$\rho = \frac{\left[(\dot{x})^2 + (\dot{y})^2\right]^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$

$$= \frac{\left[c^2 + \left(-c/t^2\right)^2\right]^{3/2}}{c\left(\frac{2c}{t^3}\right) - 0}$$

$$= \frac{\left[c^2 + \frac{c^2}{t^4}\right]^{3/2}}{2c^2} \times t^3$$

$$= \frac{\left(c^2 t^4 + c^2\right)^{3/2}}{\left(t^{\cancel{4}\,2}\right)^{3/\cancel{4}}} \times \frac{t^3}{2c^2}$$

$$= \frac{\left(c^4\right)^{3/\cancel{4}}\left[t^4 + 1\right]^{3/2}}{\cancel{t^2}\, t^3} \times \frac{\cancel{t^3}}{2c^2}$$

$$\rho = \frac{c^3 [1+t^4]^{3/2}}{2c^2 t^3} \rightarrow 1$$

$$(x-0)^2 + (y-0)^2 = r^2$$

$$(ct-0)^2 + (c/t - 0)^2 = r^2$$

$$c^2t^2 + \frac{c^2}{t^2} = r^2$$

$$c^2 \left[t^2 + \frac{1}{t^2} \right] = r^2$$

$$r^2 = c^2 \left[\frac{t^4+1}{t^2} \right]$$

$$\rho = \frac{r^3}{2c^2}$$

$r : (x, y)$

$(0, 0)$

$$(r^{\cancel{2}})^{3/\cancel{2}} = (c^{\cancel{2}})^{3/\cancel{2}} \left[\frac{1+t^4}{t^2} \right]^{3/2}$$

$$r^3 = c^3 \left[\frac{1+t^4}{t^2} \right]^{3/2}$$

$$(r^4)^{3/4} = \frac{(c^4)^{3/4}(t^4+1)^{3/2}}{(t^4)^{3/4}} = \frac{c^3(1+t^4)^{3/2}}{t^3}$$

$$r^3 = \frac{c^3(1+t^4)^{3/2}}{t^3}$$

$$1 \Rightarrow \rho = \frac{c^3(1+t^4)^{3/2}}{t^3} \times \frac{1}{2c^2}$$

$$\rho = r^3 \times \frac{1}{2c^2} = \frac{r^3}{2c^2}$$

$$\rho = r^3 / 2c^2$$

$$\left(ct, c/t \right) \xrightarrow{t=1} (c, c)$$

$$1 \Longrightarrow \rho = \frac{c\,(1+t^4)^{3/2}}{2t^3}$$

$$(\rho)_{t=1} = \frac{c\,[1+1]^{3/2}}{2} = \frac{c\,2^{3/2}}{2} = \underline{\underline{c\sqrt{2}}}$$

Formula for radius of Curvature in polar Co-ordinates

$$\rho = \frac{\left[r^2 + r'^2\right]^{3/2}}{r^2 - r r'' + 2r'^2}$$

$$r' = \frac{dr}{d\theta}$$

$$r'' = \frac{d^2 r}{d\theta^2}$$

$$\boxed{\begin{matrix} r \\ r' \\ r'' \end{matrix}}$$

(1) Find ρ for the curve $r = a[1 + \cos\theta]$ at the pt $\theta = \pi/2$.

Sol

$$r = a[1 + \cos\theta] \Rightarrow (r)_{\theta = \frac{\pi}{2}} = a$$

$$r' = -a\sin\theta \Rightarrow (r')_{\theta = \pi/2} = -a$$

$$r'' = -a\cos\theta \Rightarrow (r'')_{\theta = \frac{\pi}{2}} = 0$$

$$\rho = \frac{[r^2 + r'^2]^{3/2}}{r^2 - rr'' + 2r'^2}$$

$$= \frac{[a^2 + (-a)^2]^{3/2}}{a^2 - 0 + 2a^2} = \frac{[2a^2]^{3/2}}{3a^2} = \frac{2\sqrt{2}\, a^3}{3a^2}$$

$$= \frac{2\sqrt{2}}{3} a$$

2. Find ρ at the pt (r, θ) on the curve $r^2 \cos 2\theta = a^2$

Sol

$$r^2 = \frac{a^2}{\cos 2\theta} \Rightarrow r^2 = a^2 \sec 2\theta$$

$$2 \log r = 2 \log a + \log \sec 2\theta \quad \rightarrow (1)$$

Diff (1) w.r.t θ, $\frac{\cancel{2}}{r} r' = 0 + \frac{\cancel{2}}{\cancel{\sec 2\theta}} \cancel{\sec 2\theta} \tan 2\theta$

$$\boxed{r' = r\tan 2\theta}$$

$$r'' = r\left(\sec^2 2\theta\right]2 + r'\tan 2\theta$$

$$= 2r\sec^2 2\theta + \left(r\tan 2\theta\right)\tan 2\theta$$

$$r'' = 2r\sec^2 2\theta + r\tan^2 2\theta$$

$$N = \left[r^2 + r'^2\right]^{3/2} = \left[r^2 + r^2\tan^2 2\theta\right]^{3/2}$$

$$= \left(r^2\right)^{3/2}\left[1 + \tan^2 2\theta\right]^{3/2}$$

$$= r^3\left[\sec^{\cancel{2}} 2\theta\right]^{3/\cancel{2}}$$

$$= r^3\sec^3 2\theta \longrightarrow (2)$$

$$Dr = r^2 - r r'' + 2r'^2$$

$$= r^2 - r\left[r\tan^2 2\theta + 2r\sec^2 2\theta\right] + 2r^2\tan^2 2\theta$$

$$= r^2 - r^2\tan^2 2\theta \underbrace{- 2r^2\sec^2 2\theta + 2r^2\tan^2 2\theta}$$

$$= r^2 + r^2\tan^2 2\theta - 2r^2\sec^2 2\theta$$

$$= r^2\sec^2 2\theta - 2r^2\sec^2 2\theta$$

$$Dr = -r^2\sec^2 2\theta$$

$$\rho = \frac{Nr}{Dr} = \frac{r^3\sec^3 2\theta}{-r^2\sec^2 2\theta} = r\sec 2\theta$$

$$\rho = r\sec 2\theta$$

$$= r \times \frac{r^2}{a^2}$$

$$\rho = \frac{r^3}{a^2}$$

$\because r^2\cos 2\theta = a^2$

$\Rightarrow r^2 = a^2\sec 2\theta$

$$\rho = \frac{\left(r^2 + r'^2\right)^{3/2}}{r^2 - r r'' + 2r'^2} \qquad r' = \frac{dr}{d\theta}, \quad r'' = \frac{d^2 r}{d\theta^2}$$

① Find ρ, equiangular spiral $r = a e^{\theta \cot\alpha}$

$\underline{r, \theta}$

sol

$$r' = a e^{\theta \cot\alpha} \cot\alpha = r \cot\alpha$$

$$r'' = a e^{\theta \cot\alpha} \cot^2\alpha = r \cot^2\alpha$$

$$\rho = \frac{\left[r^2 + r^2 \cot^2\alpha\right]^{3/2}}{r^2 - r^2 \cot^2\alpha + 2r^2 \cot^2\alpha}$$

$$= \frac{r^3 \left[\operatorname{cosec}^2\alpha\right]^{3/2}}{r^2 + r^2 \cot^2\alpha} = \frac{r^3 \operatorname{cosec}^3\alpha}{r^2 \operatorname{cosec}^2\alpha}$$

$$= \underline{\underline{r \operatorname{cosec}\alpha}}$$

① P.T the radius of Curvature at any pt of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ is three times the length of ⊥r from the origin to the tangent at that point.

Sol

we know that $\rho = 3a\sin\theta\cos\theta$.

$x = a\cos^3\theta$, $y = a\sin^3\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

Eqn of tangent is $y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$

$x\tan\theta + y - a\sin^3\theta - a\tan\theta\cos^3\theta = 0$

$x\tan\theta + y - a\sin^3\theta - a\sin\theta\cos^2\theta = 0$

$x\tan\theta + y - a\sin\theta\,[\sin^2\theta + \cos^2\theta] = 0$

$x\tan\theta + y - a\sin\theta \Rightarrow (1)$

p = length of the $\perp r$ from $(0,0)$ on (1)

$$= \frac{|0+0-a\sin\theta|}{\sqrt{\tan^2\theta+1}} = \frac{a\sin\theta}{\sec\theta}$$

$p = a\sin\theta\cos\theta$

$\Rightarrow \varrho = 3p$ //

(2) Find the radius of curvature for the curve $r^n = a^n \cos n\theta$

Sol

$$r^n = a^n \cos n\theta$$

$$\log r^n = \log(a^n \cos n\theta)$$

$$n \log r = n \log a + \log \cos n\theta \rightarrow (1)$$

Diff (1) w.r.t θ,

$$\frac{n}{r} r' = \frac{1}{\cos n\theta}(-\sin n\theta)\, n$$

$$r' = -r \tan n\theta$$

$$r'' = -\left[r \sec^2 n\theta\, (n) + r' \tan n\theta\right]$$

$$r'' = -rn\sec^2 n\theta - (-r\tan n\theta)\tan n\theta$$

$$r'' = -rn\sec^2 n\theta + r\tan^2 n\theta$$

$$Nr = (r^2 + r'^2)^{3/2}$$

$$= [r^2 + r^2\tan^2 n\theta]^{3/2}$$

$$= (r^2)^{3/2}[1 + \tan^2 n\theta]^{3/2}$$

$$= r^3 (\sec^2 n\theta)^{3/2}$$

$$= r^3 \sec^3 n\theta$$

$$Dr = r^2 - rr'' + 2r'^2$$

$$= r^2 - r\left[-rn\sec^2 n\theta + r\tan^2 n\theta\right] + 2r^2\tan^2 n\theta$$

$$= r^2 + r^2 n\sec^2 n\theta - r^2\tan^2 n\theta + 2r^2\tan^2 n\theta$$

$$= r^2 + r^2 n\sec^2 n\theta + r^2\tan^2 n\theta$$

$$= r^2\sec^2 n\theta + r^2 n\sec^2 n\theta$$

$$= r^2\sec^2 n\theta\left[1+n\right]$$

$$\rho = \frac{(r^2+r'^2)^{3/2}}{r^2 - rr'' + 2r'^2} = \frac{r^3\sec^3 n\theta}{r^2\sec^2 n\theta\,(n+1)} = \frac{r\sec n\theta}{n+1}$$

$$\rho = \frac{r \sec n\theta}{n+1}$$

$$= \frac{r}{n+1} \times \frac{a^n}{r^n}$$

$$\rho = \frac{a^n}{(n+1)\, r^{n-1}}$$

$r^n = a^n \cos n\theta$

$\sec n\theta = \frac{a^n}{r^n}$

③ Find ρ, $r^n = a^n \sin n\theta \rightarrow \rho = \frac{a^n}{(n+1)\, r^{n-1}}$

④ Find ρ, $\sqrt{r} \cos\frac{\theta}{2} = \sqrt{a} \longrightarrow \frac{2 r^{3/2}}{\sqrt{a}}$

(5) Find the pts on the parabola $y^2 = 4x$ at which the radius of curvature is $4\sqrt{2}$.

Curve, pt

$\rho = ?$,

Curve, ρ

pt = ?,

Sol

$$y^2 = 4x \rightarrow (1)$$

$$2yy' = 4 \Rightarrow y' = \frac{4}{2y} = \frac{2}{y}$$

$$y' = 2/y \rightarrow (2)$$

$$y'' = -2/y^2 \, y' = -\frac{2}{y^2}\left(\frac{2}{y}\right) = -\frac{4}{y^3}$$

$$y'' = -\frac{4}{y^3} \rightarrow (3)$$

Let (a, b) be the pt on the curve $y^2 = 4x$ at which radius of curvature is $4\sqrt{2}$.

at (a, b), $\quad (y')_{(a,b)} = \dfrac{2}{b}$, $(y'')_{(a,b)} = -\dfrac{4}{b^3}$

Also we have $b^2 = 4a \quad [\because y^2 = 4x]$

$$\rho = \frac{[1+(y')^2]^{3/2}}{y''} = \frac{\left[1 + \frac{4}{b^2}\right]^{3/2}}{-4/b^3} = \frac{(b^2+4)^{3/2}}{(b^{\cancel{2}})^{3/\cancel{2}}} \times -\frac{b^3}{4}$$

$$\rho = \frac{(b^2+4)^{3/2}}{4}$$

$\because \quad \rho = 4\sqrt{2}$ (Given)

$$\Rightarrow \quad 4\sqrt{2} = \frac{(b^2+4)^{3/2}}{4}$$

$$\Rightarrow (b^2+4)^{3/2} = 16\sqrt{2}$$

$$\Rightarrow \quad (4a+4)^{3/2} = 16\sqrt{2}$$

$$4^{3/2}(a+1)^{3/2} = 16\sqrt{2}$$

$$8(a+1)^{3/2} = 16\sqrt{2}$$

$$(a+1)^{3/2} = 2\sqrt{2}$$

$$(a+1)^3 = 4 \times 2 = 8 = 2^3$$

$\Rightarrow a + 1 = 2$

$\Rightarrow a = 1$

$\because b^2 = 4a \Rightarrow b^2 = 4 \Rightarrow b = \pm 2$

$\therefore$ The pts are $(1, 2)$ and $(1, -2)$

6) Find the pts on the parabola $y^2 = 8x$, $P = \frac{125}{16}$

③ S.T the pts of intersection of the curves $r = a\theta$ and $r = a/\theta$, their curvatures are in the ratio $3:1$

<u>Sol</u>

$r = a\theta, \quad r = a/\theta$

$\Rightarrow \cancel{a}\theta = \cancel{a}/\theta \Rightarrow \theta^2 = 1$

$\Rightarrow \boxed{\theta = \pm 1}$

<u>For the first curve</u>

$r = a\theta \Rightarrow (r)_{\theta = \pm 1} = \pm a$

$r' = a \Rightarrow (r')_{\theta = \pm 1} = a$

$r'' = 0 \Rightarrow (r'')_{\theta = \pm 1} = 0$

$$\rho_1 = \frac{(r^2 + (r')^2)^{3/2}}{r^2 - rr'' + 2r'^2} = \frac{(a^2 + a^2)^{3/2}}{a^2 - 0 + 2a^2}$$

$$\rho_1 = \frac{(2a^2)^{3/2}}{3a^2} = \frac{2\sqrt{2}\,a^3}{3a^2} = \frac{2\sqrt{2}\,a}{3}$$

For the Second Curve

$$r = a/\theta$$

$$r' = -a/\theta^2$$

$$r'' = \frac{2a}{\theta^3}$$

$$\rho_2 = \frac{\left[\left(a/\theta\right)^2 + \left(\frac{-a}{\theta^2}\right)^2\right]^{3/2}}{a^2/\theta^2 - \frac{2a^2}{\theta^4} + 2\frac{a^2}{\theta^4}}$$

$$(\rho_2)_{\theta=\pm 1} = \frac{(a^2 + a^2)^{3/2}}{a^2 - \cancel{2a^2} + \cancel{2a^2}}$$

$$(\rho_2)_{\theta=\pm 1} = \frac{(2a^2)^{3/2}}{a^2} = \frac{2\sqrt{2}\,a^3}{a^2} = 2\sqrt{2}\,a$$

$$\rho_1 : \rho_2 = \frac{\cancel{2\sqrt{2}}\,\cancel{a}}{3} : \cancel{2\sqrt{2}}\,\cancel{a}$$

$$\rho_1 : \rho_2 = \frac{1}{3} : 1$$

$$\Rightarrow \underline{\underline{k_1 : k_2 = 3 : 1}} \quad \checkmark$$

Centre of Curvature and Circle of Curvature

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$

$$\bar{y} = y + \frac{\left(1 + y_1^2\right)}{y''}$$

① Find the centre of curvature and Circle of Curvature of the parabola $y^2 = 12x$ at $(3, 6)$

y'

y''

Sol

$$y^2 = 12x$$

$$2yy' = 12 \Rightarrow y' = \frac{6}{y}$$

$$\left[y'\right]_{(3,6)} = \frac{6}{6} = 1$$

$$y'' = -\frac{6}{y^2}\, y'$$

$$(y'')_{[3,6]} = -\frac{6}{36}(1) = -\frac{1}{6}.$$

$$\rho = \frac{(1+1)^{3/2}}{-1/6} = -2\sqrt{2}\,6$$

$$\underline{\underline{\rho = 12\sqrt{2}}}$$

$$\bar{x} = x - \frac{y'}{y''}(1+y'^2) = 3 - \frac{1(1+1)}{-1/6}$$

$$= 3 + 12 = 15$$

$$\bar{y} = y + \frac{(1+y'^2)}{y''}$$

$$= 6 + \frac{(1+1)}{-1/6} = 6 - 12 = -6$$

$$C(\bar{x}, \bar{y}) = (15, -6)$$

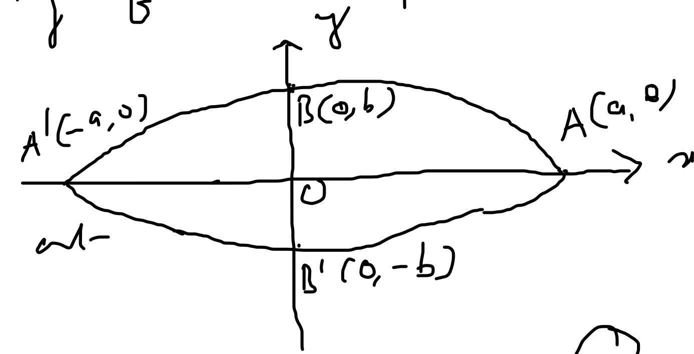
Circle of Curvature

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

$$(x - 15)^2 + (y + 6)^2 = (12\sqrt{2})^2$$

$$x^2 + y^2 - 30x + 12y - 27 = 0$$

① If the centre of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at one end of the minor axis lies at the other end, p.T the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$.



Sol

Given :- The centre of curvature of the ellipse at B(0, b) lies at B'(0, -b).

$BB' = 2b = \rho$ = radius of curvature. → ①

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Diff w.r.t x, $\frac{\cancel{2}x}{a^2}+\frac{\cancel{2}yy'}{b^2}=0$

$$\frac{x}{a^2} + \frac{y y'}{b^2} = 0$$

$$\frac{y y'}{b^2} = -\frac{x}{a^2}$$

$$\Rightarrow y' = -\frac{b^2}{a^2}\frac{x}{y} \longrightarrow (2)$$

$$(y')_{(0,b)} = -\frac{b^2}{a^2}(0) = 0$$

$$y'' = -\frac{b^2}{a^2}\left[\frac{y(1) - x y'}{y^2}\right]$$

$$(y'')_{(0,b)} = -\frac{b^2}{a^2}\left[\frac{b - 0}{b^2}\right] = -\frac{b}{a^2}$$

$$(\rho)_{(0,b)} = \frac{1+0}{-b/a^2} \Rightarrow |\rho| = \frac{a^2}{b} \longrightarrow (3)$$

From (1), (3), $p = 2b = \frac{a^2}{b}$

$$\Rightarrow a^2 = 2b^2 \rightarrow (4)$$

The eccentricity e of the ellipse is given by

$$b^2 = a^2(1-e^2)$$

$$b^2 = a^2 - a^2e^2$$

$$a^2e^2 = a^2 - b^2$$

$$e^2 = \frac{a^2 - b^2}{a^2}$$

$$e^2 = \frac{2b^2 - b^2}{2b^2}$$

$$e^2 = b^2/2b^2$$

$$e^2 = 1/2$$

$$e = 1/\sqrt{2}$$

(2) Find the centre of curvature at $\theta = \pi/2$ on the curve $x = 2\cos t + \cos 2t$, $y = 2\sin t + \sin 2t$

Sol

$$\frac{dx}{dt} = -2\sin t - 2\sin 2t$$

$$\frac{dy}{dt} = 2\cos t + 2\cos 2t$$

$$\frac{dy}{dx} = \frac{\cancel{2}(\cos t + \cos 2t)}{-\cancel{2}(\sin t + \sin 2t)}$$

$$\frac{dy}{dx} = \frac{-\left(2\cos\frac{3t}{2}\cos t/2\right)}{2\sin\frac{3t}{2}\cos t/2}$$

$$= -\frac{\cos 3t/2}{\sin 3t/2} = -\cot\frac{3t}{2}$$

$\cos C + \cos D$
$= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\sin C + \sin D$
$= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

y' $\quad y_1$

$-\cot \frac{3t}{2}$

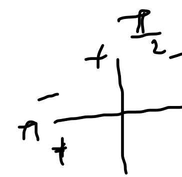
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(-\cot\frac{3t}{2}\right)\frac{dt}{dx}$$

$$= -\left(-\operatorname{cosec}^2\frac{3t}{2}\right)\left(\frac{3}{2}\right)\frac{1}{-2[\sin t + \sin 2t]}$$

$$= -\frac{3}{4}\operatorname{cosec}^2\frac{3t}{2}\,\frac{1}{2\sin\frac{3t}{2}\cos\frac{t}{2}}$$

$$= -\frac{3}{8}\,\frac{1}{\sin^3\frac{3t}{2}\cos\frac{t}{2}}$$



$$(y')_{t=\frac{\pi}{2}} = -\cot 3\frac{\pi}{4} = -\cot\left(\pi - \frac{\pi}{4}\right)$$

$$= -\left(-\cot\frac{\pi}{4}\right) = \cot\frac{\pi}{4}$$

$$= \underline{\underline{1}}$$

$$(y'')_{t=\frac{\pi}{2}} = -\frac{3}{8}\,\frac{1}{\sin^3\frac{3\pi}{4}\,\cos\frac{\pi}{4}}$$

$$= -\frac{3}{8}\,\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^3\left(\frac{1}{\sqrt{2}}\right)}$$

$$= -\frac{3}{8}\,\frac{1}{\frac{1}{2}\,\frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}}}$$

$$= -\frac{3}{8}\,\frac{1}{1/4}$$

$$= \underline{\underline{-3/2}}$$

$$\sin 3\frac{\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

✓ $(x)_{t=\frac{\pi}{2}} = 2\cos\frac{\pi}{2} + \cos \cancel{2}\frac{\pi}{\cancel{2}} = 0 - 1 = -1$

✓ $(y)_{t=\frac{\pi}{2}} = 2\sin\frac{\pi}{2} + \sin \cancel{2}\frac{\pi}{\cancel{2}} = 2 + 0 = 2$

✓ $$\bar{x} = x - \frac{y'(1+(y')^2)}{y''}$$

$$= -1 - \frac{(1)(1+1)}{-3/2} = -1 + \frac{4}{3} = \frac{1}{3}$$

✓ $$\bar{y} = y + \frac{1+(y')^2}{y''} = 2 + \frac{1+1}{-3/2} = 2 - \frac{4}{3} = \frac{2}{3}$$

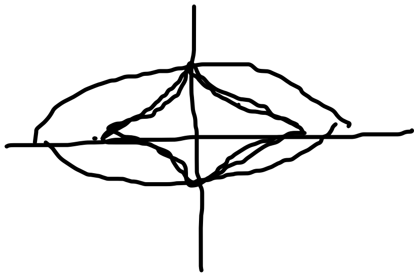
$$\therefore C(\bar{x}, \bar{y}) = \left(\frac{1}{3}, \frac{2}{3}\right) //$$

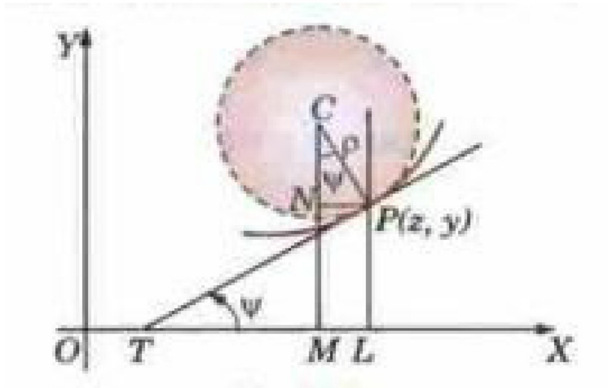
Equation of circle of curvature at P is

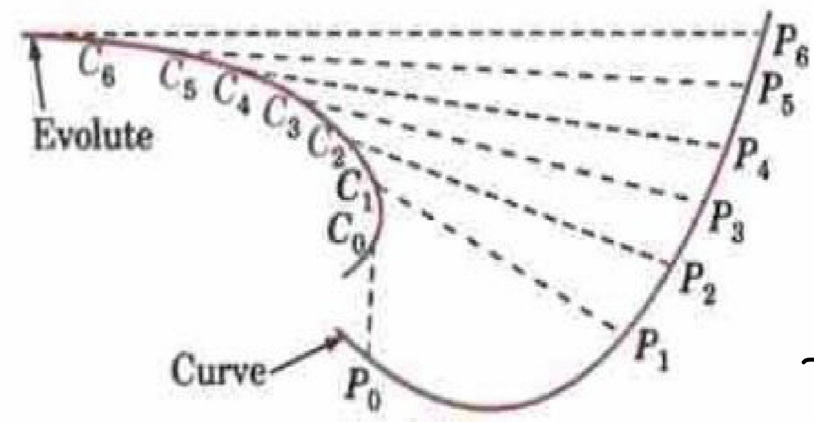
$(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$, where ρ is radius of curvature.

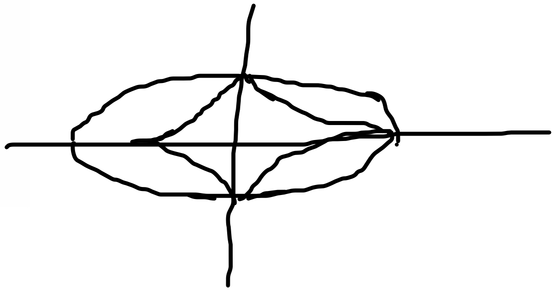
Evolute

The locus of centre of curvature for a curve is called its evolute and the curve is called an involute of its evolute.









1. $x =$ $y =$

2. $\dfrac{dx}{dt}$ $\dfrac{dy}{dt}$

3. dy/dx

4. $\dfrac{d^2y}{dx^2}$

5. $\bar{x} =$

6. $\bar{y}$

7. Eliminate the parameter t from $\bar{x}$ and $\bar{y}$ by using the known identities.

① Find the evolute of the parabola $y^2 = 4ax$.

Sol parametric equation

$x = at^2, \quad y = 2at$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx}$$

$$= -\frac{1}{t^2} \times \frac{1}{2at}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2at^3}$$

$$\bar{x} = x - \frac{y'(1+y'^2)}{y''}$$

$$= at^2 - \frac{\frac{1}{t}\left[1+\frac{1}{t^2}\right]}{\left(-1/2at^3\right)}$$

$$= at^2 + \left(\frac{1}{t}\right)\left(2at^3\right)\left[1+\frac{1}{t^2}\right]$$

$$= at^2 + 2at^2\left[1+\frac{1}{t^2}\right]$$

$$= at^2 + 2at^2 + 2a$$

$$\bar{x} = 3at^2 + 2a$$

$$\bar{y} = y + \frac{1 + y'^2}{y''}$$

$$= 2at + \frac{(1 + 1/t^2)}{-1/2at^3}$$

$$= 2at - 2at^3 \left[\frac{t^2 + 1}{t^2} \right]$$

$$= 2at - 2at \left[t^2 + 1 \right]$$

$$= 2at - 2at^3 - 2at$$

$$\bar{y} = -2at^3$$

$\bar{x} = 3at^2 + 2a$, $\quad \bar{y} = -2at^3$

$t^2 = \dfrac{\bar{x} - 2a}{3a}$ $\quad t^3 = -\dfrac{\bar{y}}{2a}$

$\Downarrow$

$t^6 = \left[\dfrac{\bar{x} - 2a}{3a}\right]^3$ $\quad t^6 = \left(-\dfrac{\bar{y}}{2a}\right)^2$

$$\frac{(\bar{x} - 2a)^3}{27a^{\cancel{3}1}} = \frac{(\bar{y})^2}{4a^2}$$

$$4a^2(\bar{x} - 2a)^3 = 27a\bar{y}^2$$

Locus of $(\bar{x}, \bar{y})$ is
$4(x - 2a)^3 = 27ay^2$, which is the evolute of the parabola $y^2 = 4ax$

Find the evolute of the parabola $x^2 = 4ay$:

$x = 2at, \quad y = at^2$

Ans: $4(y-2a)^3 = 27ax^2$

2. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol Its parametric equs are

$x = a\cos\theta, \quad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\frac{d\theta}{dx}$$

$$= -\frac{b}{a}\left(-\operatorname{cosec}^2\theta\right)\frac{1}{-a\sin\theta}$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a^2}\,\frac{1}{\sin^3\theta}$$

$$\bar{x} = x - \frac{y'(1+y'^2)}{y''}$$

$$= a\cos\theta - \frac{\left(-\frac{b}{a}\cot\theta\right)\left[1+\frac{b^2}{a^2}\cot^2\theta\right]}{-\frac{b}{a^2}\,\frac{1}{\sin^3\theta}}$$

$$= a\cos\theta - a\sin^3\theta\,\frac{\cos\theta}{\sin\theta}\left[1+\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= a\cos\theta - a\sin^2\theta\cos\theta\left[1+\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\left[1-\sin^2\theta\right] - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\bar{x} = \left[\frac{a^2-b^2}{a}\right]\cos^3\theta$$

Find the evolute of the parabola $x^2 = 4ay$:

$x = 2at, \quad y = at^2$

Ans: $4(y - 2a)^3 = 27ax^2$

(2) Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol Its parametric equs are

$x = a\cos\theta, \quad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = -\frac{b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\frac{d\theta}{dx}$$

$$= -\frac{b}{a}\left(-\operatorname{cosec}^2\theta\right)\frac{1}{-a\sin\theta}$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a^2}\frac{1}{\sin^3\theta}$$

$$\bar{x} = x - \frac{y'(1+y'^2)}{y''}$$

$$= a\cos\theta - \frac{\left(-\frac{b}{a}\cot\theta\right)\left[1+\frac{b^2}{a^2}\cot^2\theta\right]}{-\frac{b}{a^2}\frac{1}{\sin^3\theta}}$$

$$= a\cos\theta - a\sin^3\theta\,\frac{\cos\theta}{\sin\theta}\left[1 + \frac{b^2}{a^2}\,\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= a\cos\theta - a\sin^2\theta\cos\theta\left[1 + \frac{b^2}{a^2}\,\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\left[1 - \sin^2\theta\right] - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\bar{x} = \left[\frac{a^2 - b^2}{a}\right]\cos^3\theta$$

$$\bar{y} = y + \frac{(1+y'^2)}{y''}$$

$$= b\sin\theta + \frac{\left[1+\frac{b^2}{a^2}\cot^2\theta\right)}{-\frac{b}{a^2\sin^3\theta}}$$

$$= b\sin\theta - \frac{a^2\sin^3\theta}{b}\left[1+\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= b\sin\theta - \frac{a^2\sin^3\theta}{b} - b\sin\theta\cos^2\theta$$

$$= b\sin\theta\left[1-\cos^2\theta\right] - \frac{a^2\sin^3\theta}{b}$$

$$= b\sin^3\theta - \frac{a^2\sin^3\theta}{b}$$

$$\bar{y} = \left[\frac{b^2 - a^2}{b}\right] \sin^3\theta$$

$x = a\cos\theta$
$y = b\sin\theta$

$$\bar{y} = -\frac{(a^2 - b^2)}{b}\sin^3\theta$$

$$\bar{x} = \left[\frac{a^2 - b^2}{a}\right]\cos^3\theta, \quad \bar{y} = -\frac{(a^2-b^2)}{b}\sin^3\theta$$

$$(\bar{x})^{2/3} = \frac{(a^2-b^2)^{2/3}}{a^{2/3}}\cos^2\theta$$

$$(\bar{y})^{2/3} = (-1)^{2/3}\frac{(a^2-b^2)^{2/3}}{b^{2/3}}\sin^2\theta$$

$$\cos^2\theta = \frac{(a\bar{x})^{2/3}}{(a^2-b^2)^{2/3}}$$

$$\sin^2\theta = \frac{(b\bar{y})^{2/3}}{(a^2-b^2)^{2/3}}$$

$$\cos^2\theta + \sin^2\theta = \frac{(a\bar{x})^{2/3}}{(a^2-b^2)^{2/3}} + \frac{(b\bar{y})^{2/3}}{(a^2-b^2)^{2/3}}$$

$$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2 - b^2)^{2/3}$$

$\therefore$ Locus of $(\bar{x}, \bar{y})$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

(3) Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Sol:

$x = a\sec\theta, \quad y = b\tan\theta$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \quad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\frac{\sec\theta}{\tan\theta}$$

$$= \frac{b}{a} \times \frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}$$

$$\frac{dy}{dx} = \frac{b}{a\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(\frac{b}{a\sin\theta}\right)\frac{d\theta}{dx}$$

$$= -\frac{b}{a}\frac{1}{\sin^2\theta}\cos\theta \times \frac{1}{a\sec\theta\tan\theta}$$

$$= -\frac{b}{a^2}\frac{\cos^2\theta}{\sin^2\theta} \times \frac{\cos\theta}{\sin\theta}$$

$$= -\frac{b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$$

$$\bar{x} = x - \frac{y'(1+y'^2)}{y''}$$

$$= a\sec\theta - \frac{\frac{b}{a\sin\theta}\left[1 + \frac{b^2}{a^2\sin^2\theta}\right]}{-\frac{b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}}$$

$$= \frac{a}{\cos\theta} + \frac{b}{a\sin\theta} \times \frac{a^2}{b}\frac{\sin^3\theta}{\cos^3\theta}\left[1 + \frac{b^2}{a^2\sin^2\theta}\right]$$

$$= \frac{a}{\cos\theta} + \frac{a\sin^2\theta}{\cos^3\theta}\left(1 + \frac{b^2}{a^2\sin^2\theta}\right)$$

$$= \frac{a}{\cos\theta} + \frac{a\sin^2\theta}{\cos^3\theta} + \frac{b^2}{a\cos^3\theta}$$

$$= \frac{a^2\cos^2\theta + a^2\sin^2\theta + b^2}{a\cos^3\theta}$$

$$=$$

$$\bar{x} = \frac{a^2+b^2}{a\cos^3\theta} = \frac{(a^2+b^2)}{a}\sec^3\theta \longrightarrow (1)$$

$$x = a\sec\theta$$

$$y = b\tan\theta$$

$$\text{III}^{ly} \quad \bar{y} = -\frac{(a^2+b^2)}{b}\tan^3\theta \longrightarrow (2)$$

$(1) \Rightarrow$

$$(\bar{x})^{2/3} = \frac{(a^2+b^2)^{2/3}}{a^{2/3}}\sec^2\theta$$

$$\sec^2\theta = \frac{(a\bar{x})^{2/3}}{(a^2+b^2)^{2/3}} \longrightarrow (3)$$

$(2) \Rightarrow$

$$(\bar{y})^{2/3} = (-1)^{2/3}\frac{(a^2+b^2)^{2/3}}{b^{2/3}}\tan^2\theta$$

$$\tan^2\theta = \frac{(b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}} \longrightarrow (4)$$

$(3) - (4) \Rightarrow$

$$\sec^2\theta - \tan^2\theta = \frac{(a\bar{x})^{2/3}}{(a^2+b^2)^{2/3}} - \frac{(b\bar{y})^{2/3}}{(a^2+b^2)^{2/3}}$$

$$(a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}$$

$\therefore$ The locus of $(\bar{x}, \bar{y})$ is

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$

(4) Find the evolute of the rectangular hyperbola $xy = c^2$.

Sol

$x = ct$, $y = c/t$

$\frac{dx}{dt} = c$, $dy/dt = -c/t^2$

$$\frac{dy}{dx} = \frac{-c/t^2}{c} = -1/t^2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(-\frac{1}{t^2}\right)\frac{dt}{dx}$$

$$= \frac{2}{t^3} \times \frac{1}{c} = \frac{2}{ct^3}$$

$$\bar{x} = x - \frac{y'(1+y'^2)}{y''}$$

$$= ct - \frac{\left(-1/t^2\right)\left[1 + 1/t^4\right]}{2/ct^3}$$

$$= ct + \frac{1}{t^2} \times \frac{ct^3}{2}\left[1 + \frac{1}{t^4}\right]$$

$$= ct + \frac{ct}{2}\left[1 + \frac{1}{t^4}\right]$$

$$= ct + \frac{ct}{2} + \frac{c}{2t^3}$$

$$\bar{x} = \frac{3}{2}ct + \frac{c}{2t^3}$$

$$\bar{x} = \frac{c}{2}\left[3t + \frac{1}{t^3}\right]$$

$$\bar{y} = y + \frac{(1 + y'^2)}{y''}$$

$$= \frac{c}{t} + \frac{\left[1 + \frac{1}{t^4}\right]}{2/ct^3}$$

$$= \frac{c}{t} + \frac{ct^3}{2}\left[1 + \frac{1}{t^4}\right]$$

$$= \frac{c}{t} + \frac{ct^3}{2} + \frac{c}{2t}$$

$$= \frac{3c}{2t} + \frac{ct^3}{2} \qquad \boxed{\bar{y} = \frac{c}{2}\left[\frac{3}{t} + t^3\right]}$$

$$\frac{2c + c}{2t}$$

$$\bar{x} = \frac{c}{2}\left[3t + \frac{1}{t^3}\right] \qquad \bar{y} = \frac{c}{2}\left[\frac{3}{t} + t^3\right]$$

$$\bar{x} + \bar{y} = \frac{c}{2}\left[3t + \frac{1}{t^3} + \frac{3}{t} + t^3\right] = \frac{c}{2}\left[t + \frac{1}{t}\right]^3$$

$$\bar{x} - \bar{y} = \frac{c}{2}\left[3t + \frac{1}{t^3} - \frac{3}{t} - t^3\right]$$

$$= -\frac{c}{2}\left[t^3 - 3t + \frac{3}{t} - \frac{1}{t^3}\right] = -\frac{c}{2}\left[t - \frac{1}{t}\right]^3$$

$$(\bar{x} + \bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left(t + \frac{1}{t}\right)^2 \longrightarrow 1$$

$$(\bar{x} - \bar{y})^{2/3} = (-1)^{2/3}\left(\frac{c}{2}\right)^{2/3}\left(t - \frac{1}{t}\right)^2 \rightarrow 2$$

$$1 - 2 \Rightarrow (\bar{x} + \bar{y})^{2/3} - (\bar{x} - \bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}\left[\left(t^2 + \frac{1}{t^2} + 2\right) - \left(t^2 + \frac{1}{t^2} - 2\right)\right]$$

$$(\bar{x}+\bar{y})^{2/3}-(\bar{x}-\bar{y})^{2/3} = \frac{c^{2/3}}{2^{2/3}} \times 2^2$$

$$= c^{2/3} \times 2^{2-2/3}$$

$$= c^{2/3}\, 2^{4/3}$$

$$= c^{2/3}\left[(2)^2\right]^{2/3}$$

$$(\bar{x}+\bar{y})^{2/3}-(\bar{x}-\bar{y})^{2/3} = (4c)^{2/3}$$

$\therefore$ Locus of $(\bar{x},\bar{y})$ is $(x+y)^{2/3}-(x-y)^{2/3} = (4c)^{2/3}$

(4) Find the evolute of the astroid. $x^{2/3} + y^{2/3} = a^{2/3}$

Sol $x = a\cos^3\theta, \quad y = a\sin^3\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta, \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}$$

$$= -\sec^2\theta \times \frac{1}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{1}{3a\sin\theta\cos^4\theta}$$

$$\bar{x} = x - \frac{y'}{y''}\left(1 + y'^2\right)$$

$$= a\cos^3\theta - \frac{(-\tan\theta)\left[1 + \tan^2\theta\right]}{\frac{1}{3a\cos^4\theta\sin\theta}}$$

$$= a\cos^3\theta + \left[3a\cos^4\theta\sin\theta\right]\frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos^2\theta}$$

$$\boxed{\bar{x} = a\cos^3\theta + 3a\sin^2\theta\cos\theta}$$

$$\bar{y} = y + \frac{1 + y'^2}{y''}$$

$$= a\sin^3\theta + \frac{[1 + \tan^2\theta]}{\frac{1}{3a\sin\theta\cos^4\theta}}$$

$$= a\sin^3\theta + \frac{1}{\cos^2\theta} \times 3a\sin\theta\cos^4\theta$$

$$\boxed{\bar{y} = a\sin^3\theta + 3a\sin\theta\cos^2\theta}$$

$$\boxed{\bar{x} = a\cos^3\theta + 3a\sin^2\theta\cos\theta}$$

$$\bar{x}+\bar{y} = a[\cos^3\theta + 3\sin^2\theta\cos\theta + 3\sin\theta\cos^2\theta + \sin^3\theta]$$

$$\bar{x}+\bar{y} = a[\cos\theta+\sin\theta]^3$$

$$\bar{x}-\bar{y} = a[\cos^3\theta + 3\sin^2\theta\cos\theta - \sin^3\theta - 3\sin\theta\cos^2\theta]$$

$$\bar{x}-\bar{y} = a[\cos\theta-\sin\theta]^3$$

$$(\bar{x}+\bar{y})^{2/3} = a^{2/3}[\cos\theta+\sin\theta]^2 \to (1)$$

$$(\bar{x}-\bar{y})^{2/3} = a^{2/3}(\cos\theta-\sin\theta)^2 \to (2)$$

$$(1)+(2) \Rightarrow (\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = a^{2/3}[\cos^2\theta+\sin^2\theta+\cancel{2\cos\theta\sin\theta} + \cos^2\theta+\sin^2\theta-\cancel{2\cos\theta\sin\theta}]$$

$$(\bar{x}+\bar{y})^{2/3} + (\bar{x}-\bar{y})^{2/3} = 2a^{2/3}$$

$\therefore$ The locus of $(\bar{x}, \bar{y})$ is

$$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$$

① S.T the evolute of the ~~cyctoid~~ cycloid, given by
$x = a(\theta + \sin\theta)$
$y = a(1 - \cos\theta)$ is another cycloid.
$x = a(\theta - \sin\theta)$
$y - 2a = a(1 + \cos\theta)$

② S.T the evolute of the cycloid
$x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another cycloid

Ans :- $\bar{x} = a(\theta + \sin\theta)$
$\bar{y} = -a(1 - \cos\theta)$

(1) Find the evolute of tractrix $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$

$y = a\sin\theta$

Sol

$$x = a\left[\cos\theta + \log\tan\frac{\theta}{2}\right]$$

$$\frac{dx}{d\theta} = a\left[-\sin\theta + \frac{1}{\tan\frac{\theta}{2}}\sec^2\frac{\theta}{2}\,\frac{1}{2}\right]$$

$$= a\left[-\sin\theta + \frac{\cos\theta/2}{\sin\theta/2} \times \frac{1}{2\cos^2\theta/2}\right]$$

$$= a\left[-\sin\theta + \frac{1}{2\sin\theta/2\cos\theta/2}\right]$$

$$= a\left[-\sin\theta + \frac{1}{\sin\theta}\right]$$

$$= a\left[\frac{1-\sin^2\theta}{\sin\theta}\right]$$

$$\frac{dx}{d\theta} = \frac{a\cos^2\theta}{\sin\theta}$$

$$y = a\sin\theta$$

$$\frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{\cancel{a}\cos\theta}{\frac{\cancel{a}\cos^2\theta}{\sin\theta}} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx} = \sec^2\theta \times \frac{\sin\theta}{a\cos^2\theta}$$

$$= \frac{\sin\theta}{a\cos^4\theta}$$

$$\bar{x} = x - \frac{y'(1+y'^2)}{y''}$$

$$= a\left[\cos\theta + \log\tan\frac{\theta}{2}\right] - \frac{\tan\theta\left[1+\tan^2\theta\right]}{\frac{\sin\theta}{a\cos^4\theta}}$$

$$= a\cos\theta + a\log\tan\frac{\theta}{2} - \frac{a\cos^4\theta}{\sin\theta} \times \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos^2\theta}$$

$$= a\cos\theta + a\log\tan\frac{\theta}{2} - a\cos\theta$$

$$\boxed{\bar{x} = a\log\tan\frac{\theta}{2}}$$

$$\bar{y} = y + \frac{(1 + y'^2)}{y''}$$

$$= a\sin\theta + \frac{[1 + \tan^2\theta]}{\frac{\sin\theta}{a\cos^4\theta}}$$

$$= a\sin\theta + \frac{a\cos^4\theta}{\sin\theta} \times \frac{1}{\cos^2\theta}$$

$$= a\sin\theta + \frac{a\cos^2\theta}{\sin\theta}$$

$$= \frac{a[\sin^2\theta + \cos^2\theta]}{\sin\theta}$$

$$\bar{y} = \frac{a}{\sin\theta}$$

$$\bar{x} = a \log \tan \theta/2$$

$$\frac{\bar{x}}{a} = \log \tan \theta/2$$

$$e^{\bar{x}/a} = \tan \theta/2$$

$$\bar{y} = \frac{a}{\sin\theta} = \frac{\frac{a}{2\tan\theta/2}}{1+\tan^2\theta/2} = \frac{a(1+\tan^2\theta/2)}{2\tan\theta/2}$$

$$= \frac{a}{2}\left[\frac{1+e^{2\bar{x}/a}}{e^{\bar{x}/a}}\right]$$

$$= \frac{a}{2}\left[\frac{1}{e^{\bar{x}/a}} + \frac{e^{2\bar{x}/a}}{e^{\bar{x}/a}}\right]$$

$$= \frac{a}{2}\left[e^{-\bar{x}/a} + e^{\bar{x}/a}\right]$$

$$= a\left[\frac{e^{\bar{x}/a} + e^{-\bar{x}/a}}{2}\right]$$

$$\bar{y} = a \cosh \frac{\bar{x}}{a}$$

Locus of $(\bar{x}, \bar{y})$ is $\underline{\underline{y = a \cosh \frac{x}{a}}}$.

(2) S.T the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ is another cycloid.

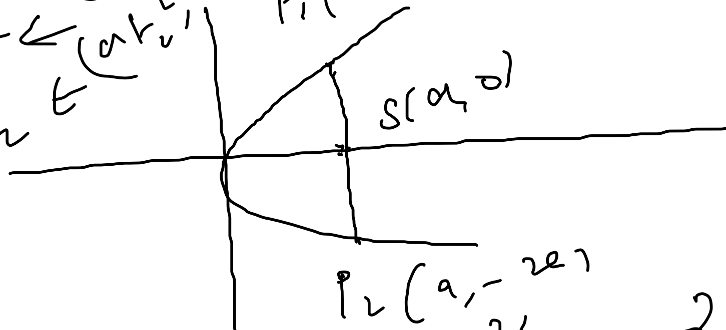
$$\rho_1 = \frac{(1+y'^2)^{3/2}}{y''}$$

$\rho_2 =$

(4)

$(at_1^2, 2at_1)$

$(at_2^2, 2at_2)$



$\rho_1 = 2a(1+t_1^2)^{3/2}$

$\rho_2 = 2a(1+t_2^2)^{3/2}$

$t_1 t_2 = -1$

$2l = 4a \Rightarrow a = \frac{2l}{4} \Rightarrow a = \frac{l}{2}$

$$\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}\left[\left[(1+t_1^2)^{3/2}\right]^{-2/3} + \left[(1+t_2^2)^{3/2}\right]^{-2/3}\right]$$

$$\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}\left[(1+t_1^2)^{-1} + (1+t_2^2)^{-1}\right]$$

$$= (2a)^{-2/3}\left[\frac{1}{1+t_1^2} + \frac{1}{1+t_2^2}\right]$$

$$= (2a)^{-2/3}\left[\frac{1+t_2^2 + 1+t_1^2}{(1+t_1^2)(1+t_2^2)}\right]$$

$$= (2a)^{-2/3}\left[\frac{t_1^2+t_2^2+2}{1+t_2^2+t_1^2+t_1^2t_2^2}\right]$$

$$= (2a)^{-2/3}\left[\frac{t_1^2+t_2^2+2}{t_1^2+t_2^2+2}\right]$$

$t_1 t_2 = -1$

$$= (2a)^{-2/3}$$

$$= [\cancel{2} \times l/\cancel{2}]^{-2/3}$$

$$\rho_1^{-2/3} + \rho_2^{-2/3} = l^{-2/3}$$

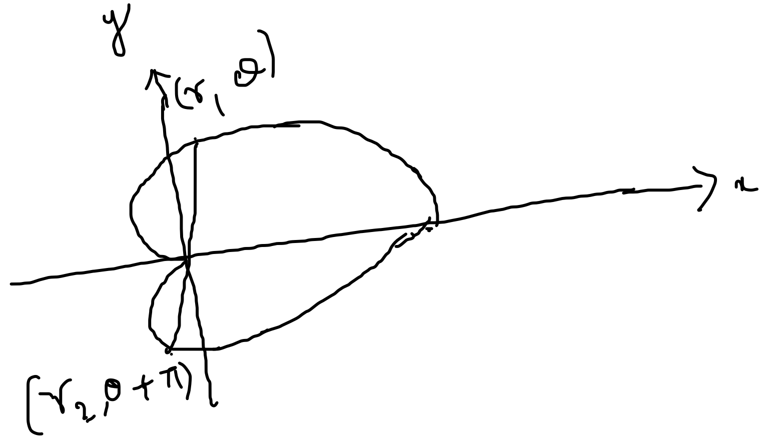
$$\boxed{a = l/2}$$

$4c = 2l$

(5) $\checkmark$ $r_1 = a[1 + \cos\theta]$

$r_2 = a[1 + \cos(\pi + \theta)]$

$r_2 = a[1 - \cos\theta]$



$$r_1 = a[1+\cos\theta] = a + a\cos\theta$$
$$r_1' = -a\sin\theta$$
$$r_1'' = -a\cos\theta$$

$$\rho_1 = \frac{(r_1^2 + r_1'^2)^{3/2}}{r_1^2 - r_1 r_1'' + 2r_1'^2}$$

$$= \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a^2(1+\cos\theta)^2 + a^2(1+\cos\theta)\cos\theta + 2a^2\sin^2\theta}$$

$$= \frac{(a^2)^{3/2}\left[1+\cos^2\theta+2\cos\theta+\sin^2\theta\right]^{3/2}}{a^2\left[1+\cos^2\theta+2\cos\theta+\cos\theta+\cos^2\theta+2\sin^2\theta\right]}$$

$$= \frac{a^3\left[2+2\cos\theta\right]^{3/2}}{a^2\left[3+3\cos\theta\right]}$$

$$= \frac{2\sqrt{2}\,a^{\cancel{3}\,1}\left[1+\cos\theta\right]^{\cancel{3/2}\;1/2}}{3\cancel{a^2}\left[1+\cancel{\cos\theta}\right]}$$

$$= \frac{2\sqrt{2}\, a \,(1+\cos\theta)^{1/2}}{3}$$

$$= \frac{2\sqrt{2}\, a}{3}\left[2\cos^2\theta/2\right]^{1/2}$$

$$= \frac{2\sqrt{2}\, a}{3}\sqrt{2}\,\cos\theta/2$$

$$\rho_1 = \frac{4a}{3}\cos\theta/2$$

$$r_2 = a[1 - \cos\theta] = a - a\cos\theta$$

$$r_2' = a\sin\theta$$

$$r_2'' = a\cos\theta$$

$$\rho_2 = \frac{\left[r_2^2 + (r_2')^2\right]^{3/2}}{r_2^2 - r_2 r_2'' + 2{r_2'}^2}$$

. . . .

$$\rho_2 = \frac{4a}{3}\sin\frac{\theta}{2}$$

$$\rho_1^2 + \rho_2^2$$

$$= \frac{16a^2}{9}\cos^2\theta/2$$

$$+ \frac{16a^2}{9}\sin^2\theta/2$$

$$= \frac{16a^2}{9}$$

Chapter 8

ENVELOPE

Consider a family of curves C represented by the equation $f(x,y,\alpha)=0$. Here α is called a parameter. For different values of α, we get different curves. In such a case, $f(x,y,\alpha)=0$ is called a one parameter family of curves. The following equations are examples of one parameter family of curves.

i) $y = mx + \dfrac{a}{m}$, m being a parameter

ii) $(x-\alpha)^2 + y^2 = a^2$, α being a parameter.

iii) $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$, θ being a parameter.

The following is an example of two parameter family of curves.

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$, μ and σ being parameters.

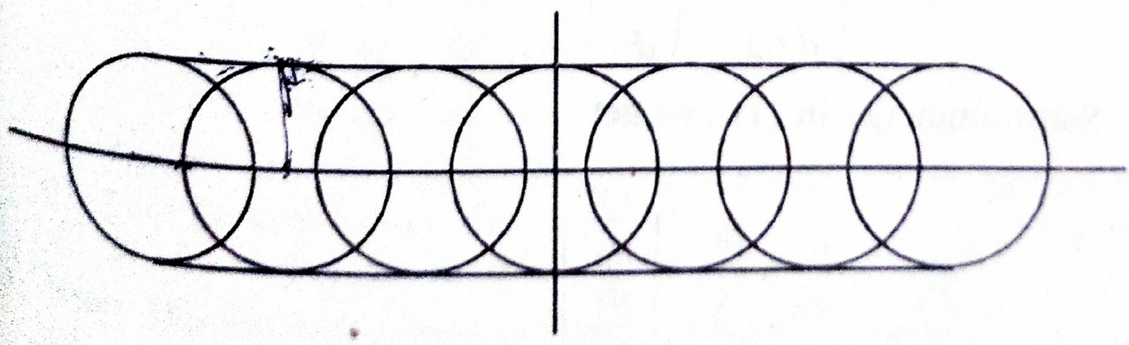
Envelope of the one parameter family of Curves

Consider the equation $(x-\alpha)^2 + y^2 = a^2$ where α is a parameter and a is a fixed quality.

This equation represents a circle whose centre is $(\alpha, 0)$ and whose radius is a.

For different values of α, we get different circles with centres at different points on the x- axis. Hence we get a family of circles considering different values of α.

It is clear that all these circles touch the lines $y = \pm a$.



Then we say that the pair of lines $y = \pm a$ is the envelope of the family of circles $(x-\alpha)^2 + y^2 = a^2$ Here we also note that each point on the envelope is the point of contact of some circle of the family

$(x-\alpha)^2 + y^2 = a^2$.

Hence we formally define an envelope as follows:

Definition:

A curve K which touches each member of the family of curves C and at each point on K is touched by some members of the family C is called the envelope.

The necessary and sufficient condition for the existence of an envelope for a family of curves $f(x, y, \alpha) = 0$ where α is a parameter is

$$\frac{\partial}{\partial \alpha} f(x, y, \alpha) = 0.$$

Envelope

Procedure to find envelope of a given family

Consider the family $f(x, y, c) = 0 \rightarrow (1)$ where c is a parameter.

Diff (1) partially w.r.t c,

$$\frac{\partial f}{\partial c}(x, y, c) = 0 \longrightarrow (2)$$

Eliminate c between (1) and (2), we get an envelope of (1).

Envelope of the family $Am^2 + Bm + C = 0$ where m is a parameter and A, B, C are functions of x and y.

Given: $Am^2 + Bm + C = 0 \rightarrow (1)$

Here m is a parameter.

Diff (1) p.w.r.t m, we get

$$2Am + B = 0 \rightarrow (2)$$

$$2Am = -B$$

$$m = \frac{-B}{2A}$$

put $m = \frac{-B}{2A}$ in (1), we get

$$A\left(\frac{-B}{2A}\right)^2 + B\left(\frac{-B}{2A}\right) + C = 0$$

$$A\left(\frac{B^2}{4A^2}\right) + B\left(\frac{-B}{2A}\right) + C = 0$$

$$\frac{B^2}{4A} - \frac{B^2}{2A} + C = 0$$

$$-\frac{B^2}{4A} + C = 0$$

$$\Rightarrow C = \frac{B^2}{4A}$$

$\Rightarrow B^2 - 4AC = 0$ is the envelope of (1).

(1) Find the envelope of the families of curves $y = mx + \frac{a}{m}$, m is a parameter.

Sol Method 1

$$y = mx + \frac{a}{m} \rightarrow (1)$$

Diff (1) p.w.r.t m,

$$0 = x - \frac{a}{m^2}$$

$$\Rightarrow x = \frac{a}{m^2} \Rightarrow m^2 = \frac{a}{x} \Rightarrow \boxed{m = \pm\sqrt{\frac{a}{x}}}$$

$$y = mx + \frac{a}{m}$$

$$y = \pm\sqrt{\frac{a}{x}}\,x + \frac{a}{\pm\sqrt{\frac{a}{x}}}$$

$xm^2 - ym + a = 0$

$$y = \pm\sqrt{ax} \pm \sqrt{ax}$$

$$y = \pm 2\sqrt{ax}$$

$y^2 = 4ax$ is the envelope of (i).

Method 2:- $y = mx + \frac{a}{m}$

$$my = m^2x + a$$

$xm^2 - ym + a = 0$, which is quadratic eq^n in m.

$$x m^2 - y m + a = 0$$

$$A = x, \quad B = -y, \quad C = a$$

Envelope is $B^2 - 4AC = 0$

$$y^2 - 4xa = 0$$

(2) Find the envelope of the family of st. lines

$$y = mx \pm \sqrt{a^2 m^2 - b^2}, \quad m \text{ is a parameter.}$$

Sol

$$y - mx = \pm \sqrt{a^2 m^2 - b^2}$$

$$(y - mx)^2 = a^2 m^2 - b^2$$

$$y^2 - 2mxy + m^2 x^2 - a^2 m^2 + b^2 = 0$$

$$(x^2 - a^2) m^2 - 2xym + y^2 + b^2 = 0,$$

is a quadratic in m.

$$(x^2-a^2)m^2 - 2xym + y^2+b^2 = 0$$

Envelope is $B^2 - 4AC = 0$

$$(-2xy)^2 - 4(x^2-a^2)(y^2+b^2) = 0$$

$$(-2xy)^2 - 4\left[x^2y^2 + x^2b^2 - a^2y^2 - a^2b^2\right] = 0$$

$$\cancel{4}x^2y^2 - \cancel{4}\left[x^2y^2 + x^2b^2 - a^2y^2 - a^2b^2\right] = 0$$

$$x^2b^2 - a^2y^2 - a^2b^2 = 0$$

$\div\, a^2b^2$, $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ is the envelope of (1).

③ $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1 \to$ ①   θ is a parameter

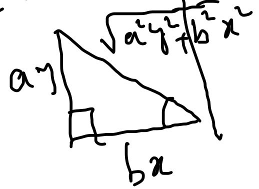
Diff ① p.w.r.t θ,

$$\frac{x}{a}(-\sin\theta) + \frac{y}{b}(\cos\theta) = 0 \Rightarrow ②$$

$$①^2 + ②^2 \Rightarrow \left[\frac{x^2}{a^2}\cos^2\theta + \frac{y^2}{b^2}\sin^2\theta + \cancel{\frac{2xy}{ab}\cos\theta\sin\theta}\right] + \left[\frac{x^2}{a^2}\sin^2\theta + \frac{y^2}{b^2}\cos^2\theta - \cancel{\frac{2xy}{ab}\sin\theta\cos\theta}\right] = 1^2 + 0^2$$

$\dfrac{y}{b}\cos\theta = \dfrac{x}{a}\sin\theta$

$\dfrac{ay}{bx} = \tan\theta$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$ is the envelope of ①

(4) $x\cos\alpha + y\sin\alpha = a\sec\alpha \longrightarrow (1)$

Sol Divide (1) by $\cos\alpha$,

$$x + y\tan\alpha = a\frac{\sec\alpha}{\cos\alpha}$$

$$x + y\tan\alpha = a(1+\tan^2\alpha)$$

$$x + y\tan\alpha - a(1+\tan^2\alpha) = 0$$

$$-a\tan^2\alpha + y\tan\alpha + (x-a) = 0, \text{ which}$$

is quadratic eqn in $\tan\alpha$.

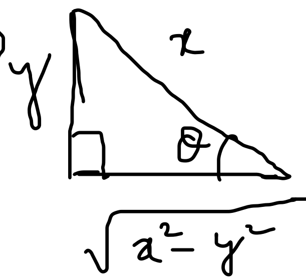
Envelope is $y^2 - 4(-a)(x-a) = 0$

$y^2 + 4a(x-a) = 0$ is the envelope of (1).

(5) $x \sec\theta - y\tan\theta = a$, θ - parameter.
$\hookrightarrow$ (1)

Sol Diff (1) p.w.r. θ,

$$x(\sec\theta\tan\theta) - y\sec^2\theta = 0 \rightarrow (2)$$



$$x\frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta} = \frac{y}{\cos^2\theta}$$

$$x\sin\theta = y$$

$$\sin\theta = y/x$$

$$\cos\theta = \frac{\sqrt{x^2-y^2}}{x}, \quad \tan\theta = \frac{y}{\sqrt{x^2-y^2}}$$

$$(1) \Rightarrow x \times \frac{x}{\sqrt{x^2-y^2}} - y \times \frac{y}{\sqrt{x^2-y^2}} = a$$

$$\frac{x^2-y^2}{\sqrt{x^2-y^2}} = a$$

$$\Rightarrow \sqrt{x^2-y^2} = a$$

$$\Rightarrow \underline{\underline{x^2-y^2 = a^2}}$$

⑥ Find the envelope of the family of parabolas

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha, \quad \alpha - \text{parameter}$$

$$y = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$$

$\underline{\underline{\tan\alpha = t}}$

$$y = xt - \frac{gx^2}{2u^2}(1+t^2) \qquad \alpha.$$

$$\frac{gx^2}{2u^2}(1+t^2) - xt + y = 0$$

$\times 2u^2$,

$$gx^2(1+t^2) - 2u^2xt + 2u^2y = 0$$

$$gx^2 + gx^2t^2 - 2u^2xt + 2u^2y = 0$$

$$gx^2t^2 - 2u^2xt + (gx^2 + 2u^2y) = 0 \quad \text{quadratic in } t$$

$$(-2u^2x)^2 - 4(gx^2)(gx^2+2u^2y) = 0$$

Envelope is $(-2u^2x)^2 - 4(gx^2)(gx^2+2u^2y) = 0$

$$\cancel{4}u^4\cancel{x^2} - \cancel{4}g\cancel{x^2}(gx^2+2u^2y)$$

$$u^4 - g^2x^2 - 2gu^2y = 0$$

$$g^2x^2 + 2gu^2y - u^4 = 0$$

$$gx^2 = u^4 - 2gu^2y$$

$$x^2 = \frac{1}{g^2}\left[u^4 - 2u^2gy\right]$$

$$\boxed{x^2 = -\frac{2u^2}{g}\left[y - \frac{u^2}{2g}\right]}$$

is envelope of (1).

1. $x\cos\theta + y\sin\theta = p$, θ - parameter

$$x^2+y^2=p^2$$

2. $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$, θ - parameter

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

3. $x\operatorname{cosec}\theta - y\cot\theta = a$, θ - parameter

$$x^2-y^2=a^2$$

(1) $x\cos\alpha + y\sin\alpha = c\sin\alpha\cos\alpha$, α - parameter

Sol $\div \sin\alpha\cos\alpha$

$$\frac{x}{\sin\alpha} + \frac{y}{\cos\alpha} = c \rightarrow (1)$$

Diff (1) p. w.r.t α,

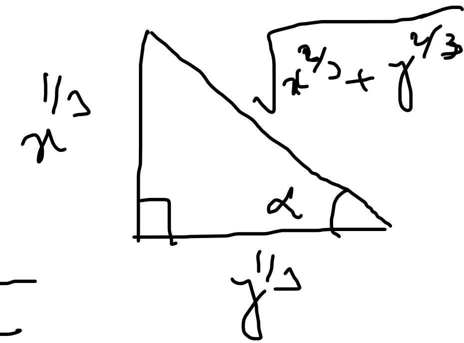
$$-\frac{x}{\sin^2\alpha}\cos\alpha - \frac{y}{\cos^2\alpha}(-\sin\alpha) = 0$$

$$\frac{y\sin\alpha}{\cos^2\alpha} = \frac{x\cos\alpha}{\sin^2\alpha}$$

$$\frac{\sin^3\alpha}{\cos^3\alpha} = \frac{x}{y}$$

$$\tan^3\alpha = \frac{x}{y}$$

$$\Rightarrow \tan\alpha = \frac{x^{1/3}}{y^{1/3}}$$



$$\sin\alpha = \frac{x^{1/3}}{\sqrt{x^{2/3}+y^{2/3}}} \; ; \; \cos\alpha = \frac{y^{1/3}}{\sqrt{x^{2/3}+y^{2/3}}}$$

$$(1) \Rightarrow \frac{x}{x^{1/3}} \times \sqrt{x^{2/3}+y^{2/3}} + \frac{y}{y^{1/3}} \sqrt{x^{2/3}+y^{2/3}} = c$$

$$\sqrt{x^{2/3}+y^{2/3}} \left[x^{1-1/3} + y^{1-1/3} \right] = c$$

$$\sqrt{x^{2/3}+y^{2/3}} \left[x^{2/3} + y^{2/3} \right] = c$$

$$\left(x^{2/3}+y^{2/3}\right)^{3/2}=c$$

$$\Rightarrow \quad x^{2/3}+y^{2/3}=c^{2/3}$$

2) Find the envelope of $y\cos\theta - x\sin\theta = a\cos 2\theta$, θ - is a parameter.

~~Sol~~

$$y\cos\theta - x\sin\theta = a\cos 2\theta \longrightarrow (1)$$

Diff (1) p.w.r.t θ,

$$-y\sin\theta - x\cos\theta = -2a\sin 2\theta \longrightarrow (2)$$

$(1)\times\sin\theta + (2)\times\cos\theta$

$$\left[\cancel{y\sin\theta\cos\theta} - x\sin^2\theta\right] - \cancel{y\sin\theta\cos\theta} - x\cos^2\theta$$

$$= a\cos 2\theta\sin\theta - 2a\sin 2\theta\cos\theta$$

$$\Rightarrow -x = a\cos 2\theta\sin\theta - 2a\sin 2\theta\cos\theta$$

$$\Rightarrow x = 2a\sin 2\theta\cos\theta - a\cos 2\theta\sin\theta$$

$$= 2a(2\sin\theta\cos\theta)\cos\theta - a(\cos^2\theta - \sin^2\theta)\sin\theta$$

$$= 4a\sin\theta\cos^2\theta - a\cos^2\theta\sin\theta + a\sin^3\theta$$

$$\boxed{x = a\sin^3\theta + 3a\sin\theta\cos^2\theta} \longrightarrow (3)$$

$$(1)\times\cos\theta - (2)\times\sin\theta$$

$$\left[y\cos^2\theta - x\cancel{\sin\theta}\cos\theta\right] + y\sin^2\theta + x\sin\theta\cancel{\cos\theta} = a\cos 2\theta\cos\theta + 2a\sin 2\theta\sin\theta$$

$$\Rightarrow y = a[\cos^2\theta - \sin^2\theta]\cos\theta + 2a(2\sin\theta\cos\theta)\sin\theta$$

$$y = a\cos^3\theta - a\sin^2\theta\cos\theta + 4a\sin^2\theta\cos\theta$$

$$y = a\cos^3\theta + 3a\sin^2\theta\cos\theta \longrightarrow (4)$$

$$(3) + (4) \Rightarrow x + y = a[\sin\theta + \cos\theta]^3 \longrightarrow (5)$$

$$(3) - (4) \Rightarrow x - y = -a[\cos\theta - \sin\theta]^3 \longrightarrow (6)$$

$$(x+y)^{2/3} + (x-y)^{2/3} = a^{2/3}\left[(\sin\theta + \cos\theta)^2 + (\cos\theta - \sin\theta)^2\right]$$

$$(x+y)^{2/3} + (x-y)^{2/3} = \underline{\underline{2a^{2/3}}}$$

① $x\sec^2\theta + y\,\mathrm{cosec}^2\theta = c$, θ – parameter

Ans: $\sqrt{x} + \sqrt{y} = \sqrt{c}$

② $x\cos^n\theta + y\sin^n\theta = a$, θ – parameter.

$\tan^2\theta = \frac{x}{y}$

$\tan\theta = \frac{x^{1/n-2}}{y^{1/n-2}}$

Ans:– $x \cdot y^{\frac{n}{n-2}} + y \cdot x^{\frac{n}{n-2}} = a\left[x^{2/n-2} + y^{2/n-2}\right]^{\frac{n}{2}}$

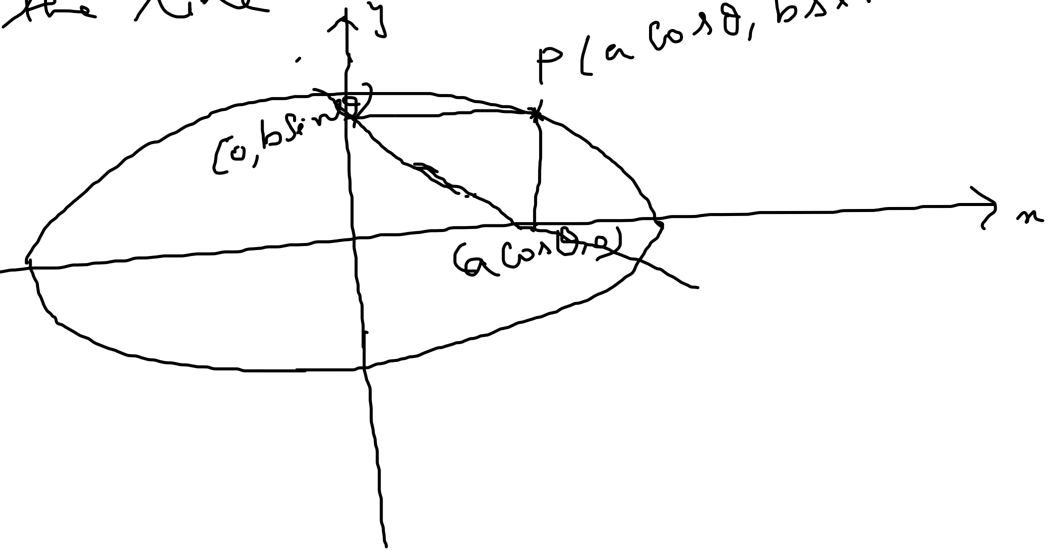
③ $y = mx + am^3 \rightarrow 4x^3 + 27ay^2 = 0$

④ $y = mx + 2am - am^3 \rightarrow 27ay^2 = 4(x+2a)^3$

⑤ $tx^3 + t^2y = a$, t – par. $\longrightarrow x^6 + 4ay = 0$

(6) From a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, perpendiculars are drawn to the axis and feet of these perpendiculars are joined. Find the envelope of the line thus formed.

$$\boxed{\frac{x}{a\cos\theta}+\frac{y}{b\sin\theta}=1}$$



$$\frac{x}{a\cos\theta} + \frac{y}{b\sin\theta} = 1 \longrightarrow (1), \theta - \text{parameter}$$

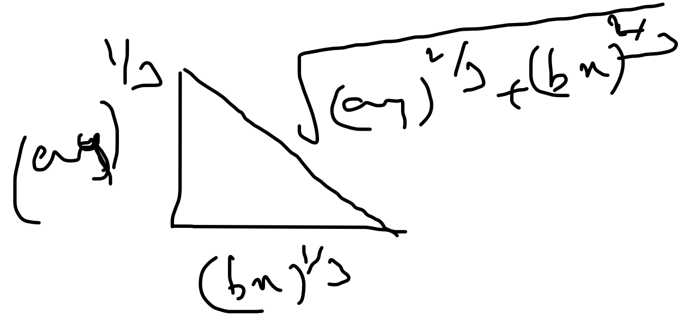
$$-\frac{x}{a\cos^2\theta}(-\sin\theta) - \frac{y}{b\sin^2\theta}\cos\theta = 0$$

$$\frac{x\sin\theta}{a\cos^2\theta} = \frac{y\cos\theta}{b\sin^2\theta}$$

$$\frac{\sin^3\theta}{\cos^3\theta} = \frac{ya}{xb}$$

$$\tan^3\theta = \frac{ay}{bx}$$

$$\tan\theta = \frac{(ay)^{1/3}}{(bx)^{1/3}}$$



$$\sin\theta = \frac{(ay)^{1/3}}{\sqrt{(ay)^{2/3}+(bx)^{2/3}}}$$

$$\cos\theta = \frac{(bx)^{1/3}}{\sqrt{(ay)^{2/3}+(bx)^{2/3}}}$$

(1) $\Rightarrow$ $\dfrac{x}{a\,(bx)^{1/3}}\sqrt{(ay)^{2/3}+(bx)^{2/3}} + \dfrac{y}{b\,(ay)^{1/3}}\sqrt{(ay)^{2/3}+(bx)^{2/3}} = 1$

$$\frac{\sqrt{(ay)^{2/3}+(bx)^{2/3}}}{a^{1/3}\,b^{1/3}}\left[\frac{x}{a^{2/3}\,x^{1/3}} + \frac{y}{b^{2/3}\,y^{1/3}}\right] = 1$$

$$\frac{\sqrt{(ay)^{2/3}+(bx)^{2/3}}}{a^{1/3}\,b^{1/3}}\left[\frac{x^{2/3}}{a^{2/3}} + \frac{y^{2/3}}{b^{2/3}}\right] = 1$$

$$\frac{\sqrt{(ay)^{2/3}+(bx)^{2/3}}}{a^{1/3}\,b^{1/3}}\left[\frac{(bx)^{2/3}+(ay)^{2/3}}{a^{2/3}\,b^{2/3}}\right] = 1$$

$$\Rightarrow \frac{\left[(ay)^{2/3} + (bx)^{2/3}\right]^{3/2}}{ab} = 1$$

$$\Rightarrow (ay)^{2/3} + (bx)^{2/3} = (ab)^{2/3}$$

$$\Rightarrow \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$

Envelope of Two parameter family

1. Find the envelope of the $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $a^n+b^n=c^n$, c - constant.

$a \to$, $b \to$

Sol

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \to (1)$$

Relation between the parameters $a^n+b^n=c^n \to (2)$

Diff (1) w.r.t a,

$$x^2\left(-\frac{\cancel{2}}{a^3}\right)+y^2\left(-\frac{\cancel{2}}{b^3}\frac{db}{da}\right)=0$$

$$-\frac{y^2}{b^3}\frac{db}{a}=\frac{x^2}{a^3} \Rightarrow \boxed{\frac{db}{da}=-\frac{x^2b^3}{y^2a^3}} \to (3)$$

Diff ② w.r.t a,

$$\cancel{n} a^{n-1} + \cancel{n} b^{n-1} \frac{db}{da} = 0$$

$$\frac{db}{da} = -\frac{a^{n-1}}{b^{n-1}} \longrightarrow ④$$

From ③ and ④

$$-\frac{x^2 b^3}{y^2 a^3} = -\frac{a^{n-1}}{b^{n-1}}$$

$$\frac{x^2}{a^{n+2}} = \frac{y^2}{b^{n+2}}$$

$$\frac{x^2/a^2}{a^n} = \frac{y^2/b^2}{b^n} = \frac{x^2/a^2 + y^2/b^2}{a^n + b^n} = \frac{1}{c^n}$$

Envelope of Two parameter family

(1) Find the envelope of the $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameters a and b are connected by the relation $a^n+b^n=c^n$, c – constant.

$a \to$ (1)
$b \to$

Sol

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \to (1)$$

Relation between the parameters $a^n+b^n=c^n \to (2)$

Diff (1) w.r.t a,

$$x^2\left(-\frac{\cancel{2}}{a^3}\right)+y^2\left(-\frac{\cancel{2}}{b^3}\frac{db}{da}\right)=0$$

$$-\frac{y^2}{b^3}\frac{db}{a}=\frac{x^2}{a^3} \Rightarrow \boxed{\frac{db}{da}=\frac{-x^2b^3}{y^2a^3}} \to (3)$$

Diff ② w.r.t a,

$$\cancel{n} a^{n-1} + \cancel{n} b^{n-1} \frac{db}{da} = 0$$

$$\frac{db}{da} = -\frac{a^{n-1}}{b^{n-1}} \longrightarrow 4$$

$$\frac{x}{a} = \frac{y}{b} = l$$

$$x = al$$

$$y = bl$$

$$x + y = (a+b)l$$

$$\frac{x+y}{a+b} = l$$

From 3 and 4

$$-\frac{x^2 b^3}{y^2 a^3} = -\frac{a^{n-1}}{b^{n-1}}$$

$$\frac{x^2}{a^{n+2}} = \frac{y^2}{b^{n+2}}$$

$$\frac{x^2/a^2}{a^n} = \frac{y^2/b^2}{b^n} = \frac{x^2/a^2 + y^2/b^2}{a^n + b^n} = \frac{1}{c^n}$$

$$\frac{x^2}{a^{n+2}} = \frac{1}{c^n} \Rightarrow a^{n+2} = c^n x^2$$

$$\Rightarrow a = (c^n x^2)^{1/n+2}$$

$$\frac{y^2}{b^{n+2}} = \frac{1}{c^n} \Rightarrow b^{n+2} = c^n y^2$$

$$\Rightarrow b = (c^n y^2)^{1/n+2}$$

Substitute the values of a and b in (2),

$$(c^n x^2)^{n/n+2} + (c^n y^2)^{n/n+2} = c^n$$

$$(c^n)^{n/n+2}\left[x^{2n/n+2} + y^{2n/n+2}\right] = c^n$$

$$\Rightarrow x^{\frac{2n}{n+2}} + y^{\frac{2n}{n+2}} = (c^n)(c^n)^{-n/n+2}$$

$$= [c^n]^{1-\frac{n}{n+2}}$$

$$= (c^n)^{\frac{\cancel{n}+2-\cancel{n}}{n+2}}$$

$$x^{\frac{2n}{n+2}} + y^{\frac{2n}{n+2}} = c^{2n/n+2}$$

(2) Find the envelope of family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are connected by the relation $ab = c^2$, c is a constant.

Sol

$$\frac{x}{a} + \frac{y}{b} = 1 \longrightarrow (1)$$

$$ab = c^2 \longrightarrow (2)$$

Diff (1) w.r.t a,

$$-\frac{x}{a^2} - \frac{y}{b^2}\frac{db}{da} = 0$$

$$\Rightarrow -\frac{y}{b^2}\frac{db}{da} = \frac{x}{a^2}$$

$$\Rightarrow \frac{db}{da} = -\frac{xb^2}{ya^2} \longrightarrow (3)$$

Diff (2) w.r.t a,

$$a\frac{db}{da} + b \cdot 1 = 0$$

$$\Rightarrow \frac{db}{da} = -\frac{b}{a} \longrightarrow (4)$$

From 3 and 4, $-\dfrac{xb^2}{ya^2} = -\dfrac{b}{a}$

$$\Rightarrow \frac{x}{y} = \frac{a^2 b}{b^2 a}$$

$$\Rightarrow \frac{x}{y} = \frac{a}{b}$$

$$\Rightarrow x/a = y/b$$

$$\Rightarrow \frac{x/a}{1} = \frac{y/b}{1} = \frac{\frac{x}{a} + y/b}{1+1} = \frac{1}{2}$$

$$\Rightarrow \frac{x}{a} = \frac{1}{2} \Rightarrow \boxed{a = 2x}$$

$$\frac{y}{b} = \frac{1}{2} \Rightarrow \boxed{b = 2y}$$

(2) $\Rightarrow (2x)(2y) = c^2$

$\Rightarrow 4xy = c^2$ is envelope of (1)

(3) Find the envelope of the system of lines $\frac{x}{l} + \frac{y}{m} = 1$, where l and m are connected by the relation $\frac{l}{a} + \frac{m}{b} = 1$, $l, m \rightarrow$ parameters

Sol

$$\frac{x}{l} + \frac{y}{m} = 1 \rightarrow (1), \quad \frac{l}{a} + \frac{m}{b} = 1 \rightarrow (2)$$

Diff (1) w.r.t l, $-\frac{x}{l^2} - \frac{y}{m^2}\frac{dm}{dl} = 0$

$$-\frac{y}{m^2}\frac{dm}{dx} = +\frac{x}{l^2}$$

$$\Rightarrow \frac{dm}{dl} = -\frac{xm^2}{yl^2} \rightarrow (3)$$

$$\frac{x}{l} + \frac{y}{m} = 1$$

$$\frac{l}{a} + \frac{m}{b} = 1$$

Diff (2) w.r.t l,

$$\frac{1}{a} + \frac{dm/dl}{b} = 0$$

$$\Rightarrow \frac{dm}{dl} = -\frac{b}{a} \rightarrow (4)$$

From (3) and (4) we have

$$-\frac{xm^2}{yl^2} = -\frac{b}{a}$$

$$\frac{xa}{l^2} = \frac{yb}{m^2}$$

$$\frac{x/l}{l/a} = \frac{y/m}{m/b} = \frac{\frac{x}{l} + \frac{y}{m}}{\frac{l}{a} + \frac{m}{b}} = \frac{1}{1} = 1$$

$$l = \sqrt{ax}$$

$$\frac{xa}{l^2} = 1 \Rightarrow l^2 = ax \Rightarrow$$

$$m = \sqrt{by}$$

$$m^2 = yb \Rightarrow$$

$$\frac{yb}{m^2} = 1 \Rightarrow$$

$$(2) \Rightarrow \frac{\sqrt{ax}}{a} + \frac{\sqrt{by}}{b} = 1$$

$$\Rightarrow \sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$$

(3) Find the envelope of a system of concentric ellipses with their axes along the co-ordinate axes and of constant area.

Hint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow (1)$$

$$\pi ab = c \longrightarrow (2)$$

Ans

$$2\pi xy = c$$

Evolute as the envelope of Normals

property: The evolute of a curve is the envelope of the normals of that curve.

$(\bar{x}, \bar{y})$

Normal eqn

$\text{Normal eqn} \xrightarrow{\text{envelope}} \text{evolute}$

$(\bar{x}, \bar{y}) \longrightarrow \text{evolute}$

① Find the evolute of the parabola $y^2 = 4ax$, Considering it as the envelope of its normal.

Sol

$$y^2 = 4ax \longrightarrow ①$$

$$x = at^2, \quad y = 2at$$

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t} = m$$

Eqn of normal to ① is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 2at = -\frac{1}{(1/t)}(x - at^2)$$

$$\Rightarrow y - 2at = -t(x - at^2)$$

$$y - 2at + xt - at^3 = 0 \longrightarrow (2)$$

Evolute of (1) is same as envelope of (2)

Diff (2) w.r.t 't',

$$-2a + x - 3at^2 = 0 \longrightarrow (3)$$

$$(3) \Rightarrow x - 2a = 3at^2$$

$$\Rightarrow t^2 = \frac{x - 2a}{3a}$$

$$\Rightarrow t = \left(\frac{x - 2a}{3a}\right)^{1/2}$$

2 $\Rightarrow (x-2a)t + y - at^3 = 0$

$$\Rightarrow (x-2a)\left(\frac{x-2a}{3a}\right)^{1/2} + y - a\left(\frac{x-2a}{3a}\right)^{3/2} = 0$$

$$\Rightarrow \frac{(x-2a)^{3/2}}{(3a)^{1/2}} + y - \frac{a(x-2a)^{3/2}}{(3a)^{3/2}} = 0$$

$$y = \frac{a(x-2a)^{3/2}}{(3a)^{3/2}} - \frac{(x-2a)^{3/2}}{(3a)^{1/2}}$$

$$y = \frac{a(x-2a)^{3/2} - 3a(x-2a)^{3/2}}{(3a)^{3/2}}$$

$$y = -\frac{2a(x-2a)^{3/2}}{(3a)^{3/2}}$$

$$y^2 = \frac{4a^2(x-2a)^3}{(3a)^3}$$

$$27a^3y^2 = 4a^2(x-2a)^3$$

$$\Rightarrow 27ay^2 = 4(x-2a)^3 \text{ is}$$

evolute of (1).

(2) Find ~~the~~ evolute of the tractrix $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$, $y = a\sin\theta$, treating it as the envelope of its normals.

Sol

$$x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right), \quad y = a\sin\theta$$

$$\frac{dx}{d\theta} = a\left[-\sin\theta + \frac{1}{\tan\theta/2}\sec^2\frac{\theta}{2}\cdot\frac{1}{2}\right]$$

$$= a\left[-\sin\theta + \frac{\cos\cancel{\theta/2}}{2\sin\theta/2} \times \frac{1}{\cos^{\cancel{2}}\theta/2}\right]$$

$$= a\left[-\sin\theta + \frac{1}{\sin\theta}\right] = \frac{a\cos^2\theta}{\sin\theta}$$

$$y = a\sin\theta$$

$$\frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{\cancel{a}\cos\theta}{\cancel{a}\cos^2\theta} \times \sin\theta = \tan\theta.$$

Eqn. of the normal at θ is

$$y - a\sin\theta = -\frac{1}{\tan\theta}\left(x - a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)\right)$$

$$y - a\sin\theta = -x\cot\theta + a\cot\theta\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$$

$\hookrightarrow$ (1)

(1) represents the family of normals of the tractrix, θ - is a parameter.

$$y - a\sin\theta = -x\cot\theta + a\cot\theta\left[\cos\theta + \log\tan\frac{\theta}{2}\right] \quad \text{L}_2 \; (1)$$

$$-a\cos\theta = -x\left(-\operatorname{cosec}^2\theta\right) + a\cot\theta\left[-\sin\theta + \frac{1}{\tan\frac{\theta}{2}} \times \sec^2\frac{\theta}{\cancel{2}} \times \frac{1}{2}\right]$$

$$- a\operatorname{cosec}^2\theta\left[\cos\theta + \log\tan\frac{\theta}{2}\right]$$

$$= \frac{x}{\sin^2\theta} + \frac{a\cos\theta}{\sin\theta}\left[-\sin\theta + \frac{\cos\frac{\theta}{2}}{2\sin\theta/2} \cancel{\times} \frac{1}{\cos^2\frac{\theta}{2}}\right]$$

$$- \frac{a}{\sin^2\theta}\cos\theta - \frac{a}{\sin^2\theta}\log\tan\frac{\theta}{2}$$

(2) Find ~~the~~ evolute of the tractrix $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$, $y = a\sin\theta$, treating it as the envelope of its normals.

Sol

$$x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right), \quad y = a\sin\theta$$

$$\frac{dx}{d\theta} = a\left[-\sin\theta + \frac{1}{\tan\theta/2}\sec^2\frac{\theta}{2}\cdot\frac{1}{2}\right]$$

$$= a\left[-\sin\theta + \frac{\cancel{\cos\theta/2}}{2\sin\theta/2} \times \frac{1}{\cos^{\cancel{2}}\theta/2}\right]$$

$$= a\left[-\sin\theta + \frac{1}{\sin\theta}\right] = \frac{a\cos^2\theta}{\sin\theta}$$

$$y = a\sin\theta$$

$$\frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{\cancel{a}\cos\theta}{\cancel{a}\cos^2\theta} \times \sin\theta = \tan\theta .$$

Eqn. of the normal at θ is

$$y - a\sin\theta = -\frac{1}{\tan\theta}\left(x - a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)\right)$$

$$y - a\sin\theta = -x\cot\theta + a\cot\theta\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$$

$\hookrightarrow$ (1)

(1) represents the family of normals of the tractrix, θ - is a parameter.

$$y - a\sin\theta = -x\cot\theta + a\cot\theta\left[\cos\theta + \log\tan\frac{\theta}{2}\right]$$

L_2 (1)

$$-a\cos\theta = -x\left(-\operatorname{cosec}^2\theta\right) + a\cot\theta\left[-\sin\theta + \frac{1}{\tan\frac{\theta}{2}} \times \sec^2\frac{\theta}{2} \times \frac{1}{2}\right]$$

$$-a\operatorname{cosec}^2\theta\left[\cos\theta + \log\tan\frac{\theta}{2}\right]$$

$$= \frac{x}{\sin^2\theta} + \frac{a\cos\theta}{\sin\theta}\left[-\sin\theta + \frac{\cos\frac{\theta}{2}}{2\sin\theta/2} \times \frac{1}{\cos^2\frac{\theta}{2}}\right]$$

$$-\frac{a}{\sin^2\theta}\cos\theta - \frac{a}{\sin^2\theta}\log\tan\frac{\theta}{2}$$

$$= \frac{x}{\sin^2\theta} + \frac{a\cos\theta}{\sin\theta}\left[-\sin\theta + \frac{1}{\sin\theta}\right]$$

$$-\frac{a\cos\theta}{\sin^2\theta} - \frac{a}{\sin\theta}\log\tan\frac{\theta}{2}$$

$$= \frac{x}{\sin^2\theta} + \frac{a\cos^3\theta}{\sin^2\theta} - \frac{a\cos\theta}{\sin^2\theta} - \frac{a}{\sin^2\theta}\log\tan\frac{\theta}{2}$$

$$= \frac{x}{\sin^2\theta} - \frac{a\cos\theta}{\cancel{\sin^2\theta}}\left[\cancel{1-\cos^2\theta}\right] - \frac{a}{\sin^2\theta}\log\tan\frac{\theta}{2}$$

$$-\cancel{a\cos\theta} = \frac{x}{\sin^2\theta} - \cancel{a\cos\theta} - \frac{a}{\sin^2\theta}\log\tan\frac{\theta}{2}$$

$$\Rightarrow \boxed{x = a\log\tan\frac{\theta}{2}} \rightarrow (2)$$

Rewriting 1, we have

$$y = a\sin\theta - x\cot\theta + a\cot\theta\cos\theta + a\cot\theta\log\tan\frac{\theta}{2} \longrightarrow 3$$

sub $x = a\log\tan\frac{\theta}{2}$ in 3

$$y = a\sin\theta - \cancel{a\log\tan\frac{\theta}{2}\cot\theta} + a\frac{\cos^2\theta}{\sin\theta} + \cancel{a\cot\theta\log\tan\frac{\theta}{2}}$$

$$y = \frac{a\left[\sin^2\theta + \cos^2\theta\right]}{\sin\theta}$$

$$\Rightarrow \quad y = \frac{a}{\sin\theta}$$

$$y = \frac{a}{2\tan\frac{\theta}{2}} \times \left(1 + \tan^2\frac{\theta}{2}\right)$$

$$= \frac{a}{2}\left[\frac{1}{\tan\frac{\theta}{2}} + \tan\frac{\theta}{2}\right]$$

$$y = \frac{a}{2}\left[e^{-x/a} + e^{x/a}\right]$$

$$y = a\left[\frac{e^{x/a} + e^{-x/a}}{2}\right]$$

$$y = a\cosh\frac{x}{a}$$

$$x = a\log\tan\frac{\theta}{2}$$

$$\frac{x}{a} = \log\tan\frac{\theta}{2}$$

$$e^{\frac{x}{a}} = \tan\frac{\theta}{2}$$

1. Find the evolute of the parabola $x^2 = 4ay$, treating it as the envelope of its normals.

2. hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

3. ellipse $\frac{x^2}{a^2} + \frac{y^2}{b} = 1$.

Maxima and Minima of functions of Two variables

A function $f(x, y)$ is said to have a relative maximum (or simply maximum) at $x = a$ and $y = b$ if $f(a, b) > f(a+h, b+k)$ for all small values of h and k.

$f(x, y)$ relative minimum at $x = a$ and $y = b$ if $f(a, b) < f(a+h, b+k)$ for all small values of h and k.

* A maximum (or) a minimum value of a function is called its extreme value.

Working rule to **find** the **extreme values of a function $f(x,y)$**

1. Find $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$

2. Solve the equs $f_x = 0$ and $f_y = 0$ simultaneously.
   Let the sols be (a,b), (c,d), ...

3. For each solution in step 2, find the values of
   $A = f_{xx} = \frac{\partial^2 f}{\partial x^2}$; $B = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$;
   $C = f_{yy} = \frac{\partial^2 f}{\partial y^2}$ and
   $\Delta = AC - B^2$

(i) (i) If $\Delta = AC - B^2 > 0$ and A (or) $C < 0$ for the sol (a, b), then $f(x, y)$ has maximum value at (a, b).

(ii) If $\Delta > 0$ and A (or) $C > 0$ for the sol. (a, b) then $f(x, y)$ has minimum value at (a, b)

(iii) If $\Delta < 0$ for the solution (a, b), $f(x, y)$ has neither a maximum nor a minimum value at (a, b). In this case, the pt (a, b) is called a saddle point of the function $f(x, y)$.

(iv) If $\Delta = 0$ (or) $A = 0$, the case is doubtful and further investigations are required to decide the nature of the extreme values of the function $f(x,y)$.

→ ① Examine $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ for extreme values.

Sol

$$f_x = 3x^2 + 3y^2 - 30x + 72$$

$$f_y = 6xy - 30y$$

$$f_{xx} = 6x - 30$$

$$f_{xy} = 6y$$

$$f_{yy} = 6x - 30$$

Note: The points like (a,b) at which $f_x = 0$ and $f_y = 0$ are called stationary points of the function $f(x,y)$.

The values $f(x,y)$ at the stationary points are called stationary values of $f(x,y)$.

$f_x = 0$ and $f_y = 0$

$\Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$

$3(x^2 + y^2 - 10x + 24) = 0 \longrightarrow (1)$

$f_y = 0 \Rightarrow 6y(x-5) = 0 \longrightarrow (2)$

From (2), we have $x = 5$ (or) $y = 0$

when $x = 5$, $25 + y^2 - 50 + 24 = 0$

$y^2 - 1 = 0$

$y^2 = 1$

$y = \pm 1$

$(5, 1), (5, -1)$

when $y = 0$,

$① \Rightarrow (x^2 + 0 - 10x + 24) = 0$

$x^2 - 10x + 24 = 0$

$(x-6)(x-4) = 0$

$\boxed{x = 4, 6}$

$(4,0), (6,0)$

The stationary pts are $(5,1), (5,-1), (4,0), (6,0)$

<u>At $(5,1)$</u>

$A = f_{xx} = 6(5) - 30 = 0$

$B = f_{xy} = 6(1) = 6$

$C = f_{yy} = 6(5) - 30 = 0$

$\Delta = AC - B^2 = 0 - 36$
$= -36$

$\Rightarrow \Delta < 0$

$\Rightarrow (5, 1)$ is a saddle point. Nothing can be said about maxima (or) minima of $f(x, y)$ at $(5, 1)$.

At $(5, -1)$:-

$$A = f_{xx} = 0$$

$$B = -6$$

$$C = 0$$

$$\Delta = AC - B^2 = -36 < 0$$

$\Rightarrow (5, -1)$ is a saddle pt.

At $(4,0)$:-

$$A = f_{xx} = -6$$

$$B = f_{xy} = 0$$

$$C = f_{yy} = -6$$

$$\Delta = AC - B^2 = 36 > 0$$

$\Delta > 0$, $A < 0$ $\Rightarrow$ $f(x,y)$ is maximum at the point $(4,0)$.

$\Rightarrow$ Max. value of $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$= 64 + 0 - 240 + 0 + 288$$

$$= 112$$

At $(6, 0)$

$$A = f_{xx} = 6$$

$$B = f_{xy} = 0$$

$$C = 6$$

$$\Delta = AC - B^2 = 36$$

$\Delta > 0$ and $A > 0$

$\Rightarrow f(x, y)$ is minimum at $(6, 0)$

The min. value of $f(x, y) = \underline{\underline{108}}$

① Examine the function $f(x, y) = x^3 y^2 (12 - x - y)$ for extreme values.

Sol

$$f(x, y) = 12x^3y^2 - x^4y^2 - x^3y^3$$

$$\frac{\partial f}{\partial x} = f_x = 36x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 24x^3y - 2x^4y - 3x^3y^2$$

$$A = f_{xx} = 72xy^2 - 12x^2y^2 - 6xy^3$$

$$B = f_{xy} = 72x^2y - 8x^3y - 9x^2y^2$$

$$C = f_{yy} = 24x^3 - 2x^4 - 6x^3y$$

$$f_x = 0 \Rightarrow x^2y^2(36 - 4x - 3y) = 0 \longrightarrow (1)$$

$$f_y = 0 \Rightarrow x^3y(24 - 2x - 3y) = 0 \longrightarrow (2)$$

$$\Delta = AC - B^2$$
$$= 0 \times C - 0$$
$$= 0$$

① $\Rightarrow$ $x = 0, \quad y = 0, \quad 36 - 4x - 3y = 0$

② $\Rightarrow$ $x = 0, \quad y = 0, \quad 24 - 2x - 3y = 0$

$(0, 0), \quad (0, 8), \quad (12, 0), \quad (0, 12), \quad (9, 0), \quad (6, 4)$.

stationary pts $\quad \Delta = AC - B^2 = 0$ for the first 5 pts.

At these 5 pts, further investigation is required to investigate the extremum at these pts.

<u>At (6,4)</u>

$A = f_{xx} = -2304$

$B = f_{xy} = -1728$

$C = f_{yy} = -2592$

$\Delta = AC - B^2 > 0$

$\Delta > 0, \quad A < 0, \quad (6, 4)$ is max pt.

Maximum value of $f(x, y) = 6^3 4^2 (12 - 6 - 4)$

$= (216) \times 16 \times 2$

$= \underline{\underline{6912}}$

② Examine the function $f(x,y) = x^3 + y^3 - 3axy$ for extreme values.

Sol

$$f_x = 3x^2 - 3ay$$

$$f_y = 3y^2 - 3ax$$

$$f_{xx} = 6x$$

$$f_{xy} = -3a$$

$$f_{yy} = 6y$$

$f_x = 0 \Rightarrow 3(x^2 - ay) = 0 \Rightarrow x^2 - ay = 0$ ⟶ ①

$f_y = 0 \Rightarrow 3(y^2 - ax) = 0 \Rightarrow y^2 - ax = 0$ ⟶ ②

$$① - ② \Rightarrow x^2 - ay - (y^2 - ax) = 0$$

$$x^2 - y^2 + ax - ay = 0$$

$$(x-y)(x+y) + a(x-y) = 0$$

$$(x-y)[x+y+a] = 0$$

$$x - y = 0 \quad (\text{or}) \quad x+y+a=0$$

$$\Rightarrow x = y$$

Substitute $x = y$ in ①, $x^2 - ax = 0$

$$x(x-a) = 0$$

$$\Rightarrow x = 0, \; x = a$$

$$\Rightarrow (0,0), (a,a)$$

stationary pts

$$x+y+a=0$$

$$y = -x-a$$

$$y^2 - ax = 0$$

$$(-x-a)^2 - ax = 0$$

$$x^2 + a^2 + 2ax - ax = 0$$

$$x^2 + ax + a^2 = 0$$

$$x = \frac{-a \pm \sqrt{a^2 - 4a^2}}{2} = \frac{-a \pm \sqrt{-3a^2}}{2} = \frac{-a \pm i\sqrt{3}a}{2}$$

At $(0,0)$

$A = 0, \quad B = -3a, \quad C = 0$

$\Delta = AC - B^2 = 0 - 9a^2 = -9a^2 < 0$

$\Delta < 0 \Rightarrow (0, 0)$ is saddle pt.

At (a, a)

$A = 6a, \quad B = -3a$

$C = 6a$

$\Delta = AC - B^2 = 36a^2 - 9a^2$

$= 27a^2 > 0$

l max
o min

$\Delta > 0$

$\Delta = 0 \rightarrow$ ...

$\Delta < 0 \rightarrow$
n max
no min

Min. at (a, a) if $a > 0$ $(A = 6a > 0$
$\Delta > 0)$

$\boxed{\text{1. min}}$

and the min. value of

$$f(x, y) = a^3 + a^3 - 3a\,a\,a$$
$$= 2a^3 - 3a^3$$
$$= -a^3$$

Max at (a, a) if $a < 0$ $(A = 6a < 0,\ \Delta > 0)$

and the max. value of

$$f(x, y) = -a^3$$

③ Discuss ~~the~~ function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$f_x = 4x^3 - 4x + 4y$

$f_y = 4y^3 + 4x - 4y$

$f_{xx} = 12x^2 - 4$

$f_{xy} = 4$

$f_{yy} = 12y^2 - 4$

$f_x = 0 \Rightarrow \cancel{4}(x^3 - x + y) = 0 \longrightarrow (1)$

$f_y = 0 \Rightarrow \cancel{4}(y^3 + x - y) = 0 \longrightarrow (2)$

$① + ② \Rightarrow x^3 + y^3 = 0$

$(x+y)(x^2 - xy + y^2) = 0$

$x^2 - xy + y^2 = 0$

$x + y = 0,$

$\Downarrow$

$x = -y \longrightarrow ③$

Sub ③ in ②

$y^3 - y - y = 0 \Rightarrow y^3 - 2y = 0$

$y(y^2 - 2) = 0$

$y = 0, \quad y = +\sqrt{2}, \quad y = -\sqrt{2}$

$\downarrow \quad \downarrow \quad \downarrow$

$x = 0 \quad x = -\sqrt{2} \quad x = \sqrt{2}$

Stationary pts are $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

At $(0,0)$

$A = -4$, $B = 4$, $C = -4$

$\Delta = AC - B^2 = 16 - 16 = 0$

The case is doubtful and further investigation are required to decide the nature of the extreme values of the function $f(x, y)$

At $(\sqrt{2}, -\sqrt{2})$

$A = 20$, $B = 4$, $C = 20$

$\Delta = AC - B^2 = 400 - 16 = 384 > 0$

$\Delta > 0$, $A > 0$, $(\sqrt{2}, -\sqrt{2})$ is min. pt.

Min. value

$$f(x,y) = (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(-\sqrt{2})(\sqrt{2}) - 2(\sqrt{2})^2$$

$$= \cancel{4} + \cancel{4} - \cancel{4} - 8 - \cancel{4}$$

$$= -8$$

$A(-(\sqrt{2}, -\sqrt{2})$

$A = 20, \quad B = 4, \quad C = 20$

$\Delta = AC - B^2 = 384 > 0$

$\Rightarrow (\sqrt{2}, -\sqrt{2})$ is min pt.

Min. value of $f(x,y) = -8$

1. $x^3 + y^3 - 12x - 3y + 20$

2. $x^4 + 2x^2y - x^2 + 3y^2$

3. $x^3y - 3x^2 - 2y^2 - 4y - 3$

4. $x^3y^2(a - x - y)$

5. $x^3y^2(12 - 3x - 4y)$

6. $xy + 27\left(\frac{1}{x} + \frac{1}{y}\right)$

(1) Examine the extrema of

$$f(x, y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$$

Sol

$$f_x = 2x + y - \frac{1}{x^2}$$

$$f_y = x + 2y - \frac{1}{y^2}$$

$$f_{xx} = 2 + \frac{2}{x^3}$$

$$f_{xy} = 1$$

$$f_{yy} = 2 + \frac{2}{y^3}$$

$$f_x = 0 \Rightarrow 2x + y - \frac{1}{x^2} = 0 \longrightarrow (1)$$

$$f_y = 0 \Rightarrow x + 2y - \frac{1}{y^2} = 0 \longrightarrow (2)$$

. . . . .
(or)
Critical pts
(or)
Turning pt

$$① - ② \Rightarrow x - y + \frac{1}{y^2} - \frac{1}{x^2} = 0$$

$$x - y + \frac{x^2 - y^2}{x^2 y^2} = 0$$

$$\frac{(x-y)x^2y^2 + (x-y)(x+y)}{x^2y^2} = 0$$

$$\Rightarrow (x-y)\left[x^2y^2 + x + y\right] = 0$$

$$\Rightarrow (x-y) = 0 \Rightarrow \boxed{x = y} \longrightarrow ③$$

Sub ③ in ①

$$2y + y - \frac{1}{y^2} = 0$$

$$2x + y - \frac{1}{x^2}$$

$$\Rightarrow 3y - 1/y^2 = 0$$

$$\Rightarrow 3y^3 - 1 = 0$$

$\Rightarrow y^3 = \frac{1}{3} \Rightarrow y = \left(\frac{1}{3}\right)^{1/3}$

$\Rightarrow x = \left(\frac{1}{3}\right)^{1/3}$

Stationary pt is $\left(\left(\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right)$

At $\left(\left(\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right)$

$A = 2 + \frac{2}{1/3} = 2 + 6 = 8 > 0$

$B = 1$

$C = 2 + \frac{2}{1/3} = 2 + 6 = 8$

$\Delta = AC - B^2 = 64 - 1 = 63 > 0$

$\Delta > 0$ and $A > 0 \Rightarrow \left(\left(\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3} \right)$ is min pt.

The min. value of $f(x, y) =$

$$\left(\frac{1}{3}\right)^{2/3} + \left(\frac{1}{3}\right)^{2/3} + \left(\frac{1}{3}\right)^{2/3} + 3^{1/3} + 3^{1/3}$$

$$= 3\left(\frac{1}{3}\right)^{2/3} + 2 \cdot 3^{1/3}$$

$$= 3^{1} \times 3^{-2/3} + 2 \cdot 3^{1/3}$$

$$= 3^{1/3} + 2 \cdot 3^{1/3}$$

$$= 3^{1} \cdot 3^{1/3} = \underline{\underline{3^{4/3}}}$$

* $f(x, y)$ — relative maximum at $x = a$ and $y = b$

if $f(a, b) > f(a+h, b+k)$ for all <u>small values</u> of h and k.

* $f(x, y)$ — relative minimum at $x = a$ and $y = b$

if $f(a, b) < f(a+h, b+k)$ for all small values of h and k.

(2) Discuss the extreme of the function

$f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^4$ at the origin.

Sol

$f_x = 2x - 2y + 3x^2 + 4x^3$

$f_y = -2x + 2y - 3y^2$

$f_{xx} = 2 + 6x + 12x^2$

$f_{xy} = -2$

$f_{yy} = 2 - 6y$

The origin $(0, 0)$ satisfies the eqns

$f_x = 0$ and $f_y = 0$.

At $(0, 0)$

$A = 2, \quad B = -2, \quad C = 2$

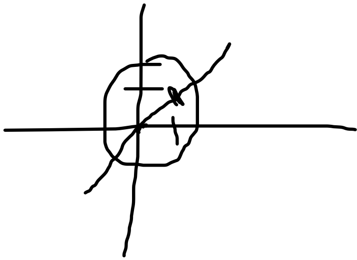
$$\Delta = AC - B^2 = 4 - 4 = 0$$

$\Delta = 0 \Rightarrow$ The case is doubtful and further investigations are required to decide the nature of $f(x, y)$ at $(0, 0)$.

Let us consider the values of $f(x, y)$ at 3 pts close to $(0, 0)$, namely at $(h, 0)$, $(0, k)$ and (h, h) which lie on the x-axis, the y-axis and the line $y = x$ resp.

$$f(h, 0) = h^2 - 0 + 0 + h^3 - 0 + h^4$$

$$= h^2 + h^3 + h^4 > 0$$



$$f(0, k) = 0 - 0 + k^2 + 0 - k^3 + 0$$

$$= k^2 - k^3$$

$$= k^2(1 - k) > 0 \text{ if } 0 < k < 1$$

$$f(h, h) = \cancel{h^2} - \cancel{2h^2} + \cancel{h^2} + \cancel{h^3} - \cancel{h^3} + h^4$$

$$= h^4 > 0$$

$\Rightarrow \quad f(x, y) > f(0, 0)$ in the neighbourhood of $(0, 0)$.

$\therefore (0,0)$ is a min pt of the function $f(x,y)$

and min. value of $f(x,y) = 0$.

p.p ① $x^4 + x^2 y + y^2$ at the origin

③ Examine the extreme of the function $\sin x + \sin y + \sin(x+y)$, $0 \le x, y \le \frac{\pi}{2}$

Sol

$$f_x = \cos x + \cos(x+y)$$

$$f_y = \cos y + \cos(x+y)$$

$$f_{xx} = -\sin x - \sin(x+y)$$

$$f_{xy} = -\sin(x+y)$$

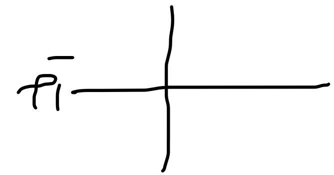
$$f_{yy} = -\sin y - \sin(x+y)$$

$$f_x = 0 \Rightarrow \cos(x+y) = -\cos x \longrightarrow (1)$$

$$f_y = 0 \Rightarrow \cos(x+y) = -\cos y \longrightarrow (2)$$

$$\Rightarrow -\cos x = -\cos y$$

$$\Rightarrow \boxed{x = y} \longrightarrow (3)$$



$$(1) \Rightarrow \cos(x+x) = -\cos x$$

$$\Rightarrow \cos 2x = -\cos x$$

$$\cos 2x = \cos(\pi - x)$$

$$\Rightarrow 2x = \pi - x$$

$$\Rightarrow 3x = \pi$$

$$\Rightarrow x = \frac{\pi}{3} \Rightarrow y = \frac{\pi}{3}$$

Stationary pt is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$A = -\sin\frac{\pi}{3} - \sin\frac{2\pi}{3}$$

$$= -\frac{\sqrt{3}}{2} - \sin\frac{\pi}{3}$$

$$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$$

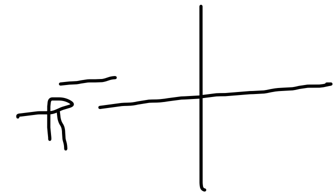
$$B = -\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$C = -\sin\frac{\pi}{3} - \sin\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

$$\Delta = AC - B^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

$$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right)$$

$$= \sin\frac{\pi}{3}$$



$\Rightarrow \quad \Delta > 0$ and $A < 0$

$\Rightarrow \quad \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is max point.

Max. value of $f(x, y) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3}$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2}$$

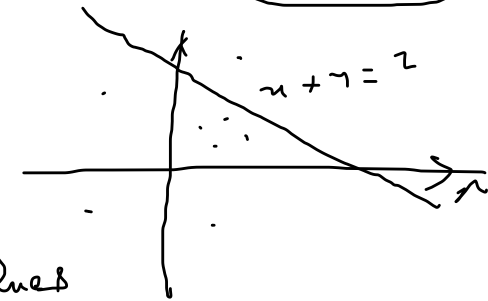
P.P 2 Find the max. and min. value of

$f(x,y) = \sin x \sin y \sin(x+y)$; $0 < x, y < \pi$

Constrained Maxima and Minima

$x = 0$, $y = 0$

$x + y = 2$



3 (or) more variables

$f(x, y, z)$

$\phi(x, y, z) = 0$

Constrained extreme values

(1) Find the min. value of $x^2+y^2+z^2$ when $x+y+z=3a$.

<u>Sol</u>

$$z = 3a - x - y.$$

$$f(x,y) = x^2+y^2+(3a-x-y)^2$$

$$f_x = 2x - 2(3a-x-y)$$

$$f_y = 2y - 2(3a-x-y)$$

$$f_{xx} = 2+2 = 4$$

$$f_{xy} = 2$$

$$f_{yy} = 2+2 = 4$$

$f_x = 0 \Rightarrow \cancel{2}x - \cancel{2}(3a - x - y) = 0$

$x = 3a - x - y$

$\Rightarrow 2x + y = 3a \rightarrow (1)$

$f_y = 0 \Rightarrow \cancel{2}y - \cancel{2}(3a - x - y) = 0$

$x + 2y = 3a \longrightarrow (2)$

$x = a, \ y = a \Rightarrow (a, a)$ is stationary pt.

<u>At (a, a)</u>

$A = 4, \ B = 2, \ C = 4$

$\Delta = 16 - 4 = 12 > 0$

$\Rightarrow \quad \Delta > 0, \quad A > 0$

$\Rightarrow (a, a)$ is min pt.

$\therefore$ The min. value of $f(x, y)$ at $(0, 0)$ is

$\cancel{f(x, y)} = a^2 + a^2 + a^2 = 3a^2$

① A retangular box, open at ~~the~~ top, is to have a volume of 32 cc. Find the dimensions of the box requiring least material for its construction.

Sol

$$V = 32 \text{ cc}$$

$lb + 2lh + 2bh$

$$\Rightarrow lbh = 32 \Rightarrow b = \frac{32}{lh}$$

surface area $= S = 2lh + 2bh + lb$

$$= 2h(l+b) + lb$$

$$= 2h\left(l + \frac{32}{lh}\right) + l\left[\frac{32}{lh}\right]$$

$$S = 2lh + \frac{64}{l} + \frac{32}{h} \rightarrow ①$$

$$S = 2lh + \frac{64}{l} + \frac{32}{h}$$

$$\frac{\partial S}{\partial l} = 2h - \frac{64}{l^2}$$

$$\frac{\partial S}{\partial h} = 2l - \frac{32}{h^2}$$

$$\frac{\partial^2 S}{\partial l^2} = \frac{128}{l^3}$$

$$\frac{\partial^2 S}{\partial l \partial h} = 2$$

$$\frac{\partial^2 S}{\partial h^2} = \frac{64}{h^3}$$

$$\frac{\partial S}{\partial l} = 0 \Rightarrow 2h - \frac{64}{l^2} = 0$$

$$\Rightarrow h = \frac{32}{l^2}$$

$$\frac{\partial S}{\partial h} = 0 \Rightarrow 2l - \frac{32}{h^2} = 0$$

$$\Rightarrow l = \frac{16}{h^2}$$

$$\Rightarrow l = \frac{16}{\left(\frac{32}{l^2}\right)^2} = \frac{16}{32 \times 32} \times l^4$$

$$l = \frac{l^4}{64}$$

$$\Rightarrow l^3 = 64 \Rightarrow \boxed{l = 4}$$

$$l^4 - 64l = 0$$

$$\underline{l}(l^3 - 64) = 0$$

$$\therefore h = \frac{32}{l^2} = \frac{32}{16} = 2$$

$$b = \frac{32}{lh} = \frac{32}{4\times 2} = 4$$

Stationary pt $(l, h) = (4, 2)$

At $(4, 2)$

$$A = \frac{\partial^2 S}{\partial l^2} = \frac{128}{4^3} = 2$$

$$B = \frac{\partial^2 S}{\partial l \partial h} = 2$$

$$C = \frac{\partial^2 S}{\partial h^2} = \frac{64}{h^3} = \frac{64}{2^3} = 8$$

$$\Delta = AC - B^2$$
$$= 16 - 4 = 12$$

$\therefore$ S is min when $l = 4$, $h = 2$, $b = 4$.

$\downarrow$

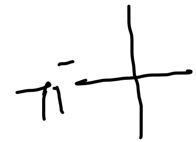
surface area

② In a triangle ABC, find the max. value of $\cos A \cos B \cos C$.

Sol In a triangle ABC, $A + B + C = \pi$

$$\cos A \cos B \cos(\pi - (A+B))$$

$$f(A, B) = \cos A \cos B \left(- \cos(A+B)\right)$$



$$f(A, B) = \cos A \cos B$$

$$f(A,B) = -\cos A \cos B \cos(A+B) \rightarrow (1)$$

$$f_A = -\cos B \left[\cos A \left(-\sin(A+B)\right) - \sin A \cos(A+B)\right]$$

$$f_A = \cos B \left[\cos A \sin(A+B) + \sin A \cos(A+B)\right]$$

$$f_A = \cos B \sin(2A+B)$$

$$f_B = -\cos A \left[\cos B \left(-\sin(A+B)\right) - \sin B \cos(A+B)\right]$$

$$= \cos A \left[\sin(A+B) \cos B + \cos(A+B) \sin B\right]$$

$$f_B = \cos A \sin(A+2B)$$

$$f_{AA} = 2 \cos B \sin(2A+B)$$

$$f_{AB} = \cos B \cos(2A+B) - \sin B \sin(2A+B) = \cos(2A+2B)$$

$f_{AB} = \cos(2A+2B)$

$f_{BB} = 2\cos A \cos(A+2B)$

$f_A = 0 \Rightarrow \cos B \sin(2A+B) = 0 \longrightarrow (2)$

$\cos A \sin(A+2B) = 0 \longrightarrow (3)$

$f_B = 0 \Rightarrow$

$(2) \Rightarrow \cos B = 0$ (or) $\sin(2A+B) = 0$

$\Rightarrow B = \frac{\pi}{2}$ (or) $2A+B = 0$ (or) π

$(3) \Rightarrow \cos A = 0$ (or) $\sin(A+2B) = 0$

$\Rightarrow A = \frac{\pi}{2}$ (or) $A+2B = 0$ (or) π

$2A + B = \pi$ and $A + 2B = \pi$

$$A = \frac{\pi}{3} \qquad B = \frac{\pi}{3}$$

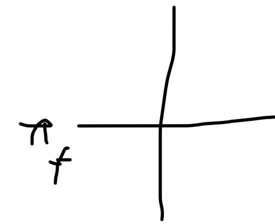
stationary pt $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$A = 2\cos B \cos(2A + B)$$

$$= 2\cos\frac{\pi}{3} \cos\pi$$

$$= 2 \cdot \frac{1}{2} \cdot (-1) = -1$$



$$B = \cos(2A + 2B) = \cos\frac{4\pi}{3}$$

$$= \cos\left(\pi + \frac{\pi}{3}\right)$$

$$= -\cos\pi/3$$

$$= -1/2$$

$$C = 2\cos A \cos(A + 2B)$$

$$= 2\cos\frac{\pi}{3}\cos(\pi)$$

$$= 2 \times \frac{1}{2} \times (-1) = -1$$

$$\Delta = AC - B^2 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$$

$\Delta > 0$, $A < 0 \Rightarrow \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is max. pt.

Max. value of $f(x, y) = \cos\frac{\pi}{3}\cos\frac{\pi}{3}\cos\frac{\pi}{3}$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{8} \,//$$

P.P S.T if the perimeter of a triangle is constant, its area is maximum when it is equilateral!

Lagrange's method

Let $u(x, y, z)$ be the function whose extreme values are required to be found subject to the condition $\varphi(x, y, z) = 0$.

The neccessary condition for the extreme values of the function $g = f + \lambda \varphi$, where $f = u(x, y, z)$, $\varphi(x, y, z)$ is the condition and λ is a Lagrange's multiplier (also a variable) are given by $\frac{\partial g}{\partial x} = 0$, $\frac{\partial g}{\partial y} = 0$, $\frac{\partial g}{\partial z} = 0$, $\frac{\partial g}{\partial \lambda} = 0$

Solving the above 4 equns, we get the values x, y, z and λ which give the extreme values of the function.

Note:- Lagrange's method does not specify whether the extreme values found out is a max (or) min value. It is decided from the physical consideration of the problem.

1. Find the max. value of $x^m y^n z^p$, when $x+y+z=a$

Sol

Let $f = x^m y^n z^p$

$\phi = x+y+z-a$

Let $g = f + \lambda \phi$

$g = x^m y^n z^p + \lambda(x+y+z-a)$

Stationary values are given by

$g_x = 0, \quad g_y = 0, \quad g_z = 0, \quad g_\lambda = 0$

$g_x = 0 \Rightarrow m x^{m-1} y^n z^p + \lambda = 0 \rightarrow (1)$

$g_y = 0 \Rightarrow n x^m y^{n-1} z^p + \lambda = 0 \rightarrow (2)$

$g_z = 0 \Rightarrow p\, x^m y^n z^{p-1} + \lambda = 0 \rightarrow (3)$

$g_\lambda = 0 \Rightarrow x + y + z - a = 0 \longrightarrow (4)$

From (1), (2), (3) $-\lambda = m x^{m-1} y^n z^p = n x^m y^{n-1} z^p = p x^m y^n z^{p-1}$

$$\Rightarrow \frac{m x^m y^n z^p}{x} = \frac{n x^m y^n z^p}{y} = \frac{p x^m y^n z^p}{z}$$

$$\Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$$

$$x = \frac{am}{m+n+p}, \quad y = \frac{an}{m+n+p}, \quad z = \frac{ap}{m+n+p}$$

$\therefore$ The max value of $f = x^m y^n z^p$ is given by

$$\left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$\frac{a^{m+n+p} \; m^m \; n^n \; p^p}{(m+n+p)^{m+n+p}}$$

$$= \left(\frac{a}{m+n+p}\right)^{m+n+p} m^m n^n p^p$$

p.p

1

① The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature at the surface of a unit sphere $x^2 + y^2 + z^2 = 1$

Sol Consider the Lagrange's function

$$g = T + \lambda \phi$$

$$= 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

For stationary values, $g_x = 0$, $g_y = 0$, $g_z = 0$, $g_\lambda = 0$

$g_x = 0 \Rightarrow 400yz^2 + 2\lambda x = 0 \rightarrow$ ①

$g_y = 0 \Rightarrow 400xz^2 + 2\lambda y = 0 \rightarrow$ ②

$g_z = 0 \Rightarrow 800xyz + 2\lambda z = 0 \rightarrow$ ③

$g_\lambda = 0 \Rightarrow x^2 + y^2 + z^2 - 1 = 0 \rightarrow$ ④

1, 2, 3, we have

$$\lambda = f\,\frac{\cancel{400}\,yz^2}{\cancel{y}x} = f\,\frac{\cancel{400}\,xz^2}{\cancel{x}y} = f\,\frac{\cancel{800}^{2}\,xy\cancel{z}}{\cancel{z}\cancel{z}}$$

$$\lambda = \frac{yz^2}{x} = \frac{xz^2}{y} = 2xy$$

$$\frac{y\cancel{z^2}}{x} = \frac{x\cancel{z^2}}{y} \Rightarrow \boxed{x^2 = y^2}$$

$$\frac{\cancel{x}z^2}{y} = 2\cancel{x}y \Rightarrow \boxed{2y^2 = z^2}$$

$$\boxed{x^2 = y^2 = \frac{z^2}{2}}$$

$\Rightarrow \quad x = y = \frac{z}{\sqrt{2}} \qquad z = \sqrt{2}\, x$

$4 \Rightarrow x^2 + x^2 + 2x^2 = 1$

$4x^2 = 1 \Rightarrow x^2 = \frac{1}{4}$

$\Rightarrow x = \pm \frac{1}{2}$

$\Rightarrow y = \pm \frac{1}{2}$

$\Rightarrow z = \sqrt{2}\left[\pm \frac{1}{2}\right]$

$z = \pm \frac{1}{\sqrt{2}}$

Substitute the values of x, y, z in T,

$$T = 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2$$

$$T = \frac{400}{8} = 50.$$

$T = 50$ is the highest temperature.

(2) S.T the rectangular solid of max volume that can be inscribed in a given sphere is a cube.

Sol Let $2x$, $2y$, $2z$ be ~~the~~ the length, breadth and height of the rectangular solid and R be the radius of the sphere.

$$x^2+y^2+z^2=R^2$$

$$V=(2x)(2y)(2z)=8xyz$$

Lagrange's function

$$g=V+\lambda(x^2+y^2+z^2-R^2)$$

For stationary values,

$g_x = 0, \quad g_y = 0, \quad g_z = 0$ and $g_\lambda = 0$

$g_x = 0 \Rightarrow 8yz + 2\lambda x = 0 \longrightarrow (1) \Rightarrow -\cancel{2}\lambda x = +\overset{4}{\cancel{8}}yz$

$-\lambda = \dfrac{4yz}{x}$

$g_y = 0 \Rightarrow 8zx + 2\lambda y = 0 \longrightarrow (2)$

$g_z = 0 \Rightarrow 8xy + 2\lambda z = 0 \longrightarrow (3)$

$g_\lambda = 0 \Rightarrow x^2 + y^2 + z^2 - R^2 = 0 \longrightarrow (4)$

(1), (2), (3) $\Rightarrow -\lambda = \dfrac{4yz}{x} = \dfrac{4zx}{y} = \dfrac{4xy}{z}$

$\dfrac{\cancel{4}y\cancel{z}}{x} = \dfrac{\cancel{4}\cancel{z}x}{y} \Rightarrow x^2 = y^2$; $\quad \dfrac{\cancel{4}z\cancel{x}}{y} = \dfrac{\cancel{4}\cancel{x}y}{z} \Rightarrow y^2 = z^2$

$\Rightarrow \quad x^2 = y^2 = z^2$

$\Rightarrow \quad x = y = z$

Hence rectangular Solid is a cube.

③ Split 24 into three parts such that the continued product of the first, square of the second and cube of the third may be minimum.

Sol Let the 3 parts be x, y and z

$$\Rightarrow x + y + z = 24$$

$$\text{Min } f = x^3 y^2 z$$

$$g = x^3 y^2 z + \lambda (x + y + z - 24)$$

$$g_x = 0 \Rightarrow 3x^2 y^2 z + \lambda = 0 \rightarrow (1)$$

$$g_y = 0 \Rightarrow 2x^3 y z + \lambda = 0 \rightarrow (2)$$

$$g_z = 0 \Rightarrow x^3 y^2 + \lambda = 0 \rightarrow (3)$$

$$g_\lambda = 0 \Rightarrow x + y + z - 24 = 0 \rightarrow (4)$$

from 1, 2, 3,

$$-\lambda = 3x^2y^2z = 2x^3yz = x^3y^2$$

$$3x^2y^2\cancel{z} = 2x^3y\cancel{z} \Rightarrow 3y = 2x \Rightarrow \boxed{y = \frac{2x}{3}}$$

$$2\cancel{x}yz = \cancel{x}y^2 \Rightarrow 2z = y$$

$$\Rightarrow y = 2z \Rightarrow z = \frac{y}{2} = \frac{2x}{6}$$

$$\boxed{z = \frac{x}{3}}$$

$$4 \Rightarrow x + \frac{2x}{3} + \frac{x}{3} = 24$$

$$\frac{3x + 2x + x}{3} = 24$$

$$\Rightarrow \frac{\cancel{6}x}{\cancel{3}} = 24$$

$$\Rightarrow x = 12$$

$\therefore y = \frac{2x}{3} = \frac{2}{\cancel{3}} \times \cancel{12}^{4} = 8$

$z = \frac{x}{3} = \frac{12}{3} = 4$

$x = 12, \quad y = 8, \quad z = 4.$

$x + y + z = 24$

$z = 24 - x - y$

$f = x^3 y^2 (24 - x - y)$

4. S.T ~~the~~ min. value of $a^3x^2 + b^3y^2 + c^3z^2$, when $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{k}$ is $k^2(a+b+c)^3$

Sol

$$g = a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{k}\right)$$

$$g_x = 0 \Rightarrow 2a^3x - \frac{\lambda}{x^2} = 0 \rightarrow (1) \Rightarrow 2a^3x = \lambda/x^2$$

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$$g_y = 0 \Rightarrow 2b^3y - \frac{\lambda}{y^2} = 0 \rightarrow (2)$$

$$g_z = 0 \Rightarrow 2c^3z - \frac{\lambda}{z^2} = 0 \rightarrow (3)$$

$$g_\lambda = 0 \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{k} = 0 \rightarrow (4)$$

$$(1), (2), (3) \Rightarrow \lambda = 2a^3x^3 = 2b^3y^3 = 2c^3z^3$$

$$x^3 = \frac{\lambda}{2a^3} \Rightarrow x = \left(\frac{\lambda}{2}\right)^{1/3}\frac{1}{a}$$

|||ly $\quad y = \left(\frac{\lambda}{2}\right)^{1/3} \frac{1}{b}, \quad z = \left(\frac{\lambda}{2}\right)^{1/3} \frac{1}{c}$

$$x = \left(\frac{\lambda}{2}\right)^{1/3} \frac{1}{a}$$

$$(4) \Rightarrow \left(\frac{2}{\lambda}\right)^{1/3} \left[a + b + c\right] = \frac{1}{k}$$

$$\frac{2}{\lambda} \left[a + b + c\right]^3 = \frac{1}{k^3}$$

$$\boxed{\frac{\lambda}{2} = k^3 \left[a + b + c\right]^3}$$

$$x = \frac{k(a+b+c)}{a}, \quad y = \frac{k(a+b+c)}{b}, \quad z = \frac{k(a+b+c)}{c}$$

Min. value of the function

$a^3x^2 + b^3y^2 + c^3z^2$ is

$$a^3\left[\frac{k(a+b+c)}{a}\right]^2 + b^3\left[\frac{k(a+b+c)}{b}\right]^2 + c^3\left[\frac{k(a+b+c)}{c}\right]^2$$

$$= k^2(a+b+c)^2\left[a+b+c\right]$$

$$= k^2(a+b+c)^3$$

(5) Find the volume of the greatest rectangular parallelopiped inscribed in the ellipsoid whose equ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular parallelopiped inscribed in the ellipsoid.

volume $V = (2x)(2y)(2z)$

$= 8xyz$

Lagrange's function $g = V + \lambda \phi$

$$g = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

for stationary values

$$g_x = 0 \Rightarrow 8yz + \frac{2x\lambda}{a^2} = 0 \longrightarrow (1)$$

$$g_y = 0 \Rightarrow 8xz + \frac{2y\lambda}{b^2} = 0 \longrightarrow (2)$$

$$g_z = 0 \Rightarrow 8xy + \frac{2\lambda z}{c^2} = 0 \longrightarrow (3)$$

$$g_\lambda = 0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \longrightarrow (4)$$

$$① \times x \Rightarrow \frac{2\lambda x^2}{a^2} = -8xyz$$

$$② \times y \Rightarrow \frac{2\lambda y^2}{b^2} = -8xyz$$

$$③ \times z \Rightarrow \frac{2\lambda z^2}{c^2} = -8xyz$$

$$\frac{2\lambda x^2}{a^2} = \frac{2\lambda y^2}{b^2} = \frac{2\lambda z^2}{c^2}$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c} = k$$

$$④ \Rightarrow k + k + k = 1 \Rightarrow 3k = 1$$

$$\Rightarrow k = 1/3$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{1}{3} \Rightarrow x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

$$\Rightarrow \frac{y^2}{b^2} = \frac{1}{3} \Rightarrow y^2 = \frac{b^2}{3} \Rightarrow y = \frac{b}{\sqrt{3}}$$

$$\text{III}^{ly} \quad \frac{z^2}{c^2} = \frac{1}{3} \Rightarrow z^2 = \frac{c^2}{3} \Rightarrow z = \frac{c}{\sqrt{3}}$$

$\therefore$ Max. vol $= 8xyz$

$$= 8 \frac{a}{\sqrt{3}} \frac{b}{\sqrt{3}} \frac{c}{\sqrt{3}}$$

$$= \frac{8abc}{3\sqrt{3}} \quad //$$

① Find the maximum and minimum distances from the point $(3, 4, 12)$ to the sphere $x^2+y^2+z^2=1$.

Sol Let (x, y, z) be any point on the sphere.

Distance of the point (x, y, z) from $(3, 4, 12)$ is $d = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$

we have to find the max and min values of d

(or) equivalently " " " " d^2

$d^2 = f = (x-3)^2 + (y-4)^2 + (z-12)^2$

subject to the condition that

$\phi(x, y, z) = x^2+y^2+z^2-1 = 0$

$$g = f + \lambda \varphi$$

$$g = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2+y^2+z^2-1)$$

$g_x = 0 \Rightarrow 2(x-3) + 2\lambda x = 0 \to 1$

$g_y = 0 \Rightarrow 2(y-4) + 2\lambda y = 0 \to 2$

$g_z = 0 \Rightarrow 2(z-12) + 2\lambda z = 0 \to 3$

$g_\lambda = 0 \Rightarrow x^2+y^2+z^2 = 1 \to 4$

$1 \Rightarrow \cancel{2}(x-3) = -\cancel{2}\lambda x$

$\lambda x + x = 3$

$\Rightarrow (\lambda+1)x = 3 \Rightarrow \boxed{x = \dfrac{3}{\lambda+1}}$

(2) $\Rightarrow$ $\cancel{2}(y-4) = -\cancel{2}\lambda y$

$$\lambda y + y = 4 \Rightarrow y(\lambda+1) = 4$$

$$\Rightarrow y = \frac{4}{\lambda+1}$$

(3) $\Rightarrow$ $\cancel{2}(z-12) = -\cancel{2}\lambda z$

$$\lambda z + z = 12 \Rightarrow z(\lambda+1) = 12$$

$$z = \frac{12}{\lambda+1}$$

(4) $\Rightarrow$ $\left(\frac{3}{\lambda+1}\right)^2 + \left(\frac{4}{\lambda+1}\right)^2 + \left(\frac{12}{\lambda+1}\right)^2 = 1$

$$9 + 16 + 144 = (\lambda+1)^2$$

$$\Rightarrow (\lambda+1)^2 = 169$$

$\Rightarrow \lambda + 1 = \pm 13$

$\Rightarrow \lambda = 13 - 1$ (or) $\lambda = -13 - 1$

$\lambda = 12$ (or) -14

$x = \frac{3}{\lambda+1}$

$y = \frac{4}{\lambda+1}$

$z = \frac{12}{\lambda+1}$

when $\lambda = 12$, the pt on ~~the~~ sphere is

$P\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$

when $\lambda = -14$, the pt on the sphere is

$Q\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$

Let $P\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ and $Q\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$ be pts on the sphere which are at a max (or) min distance from the given point $A(3, 4, 12)$.

$$AP = \sqrt{\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2}$$

$$= 12$$

$$AQ = \sqrt{\left(3+\frac{3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2}$$

$$= 14$$

(2) Find the max and min distances from the origin to the curve $x^2 + 4xy + 6y^2 = 140$.

Sol Let (x, y) be any point on the curve.

Let $A(0, 0)$.

Distance of $A(0,0)$ from (x, y) is

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

$f = x^2 + y^2$ and $\phi = x^2 + 4xy + 6y^2 - 140$

$$g = f + \lambda \phi$$

$$g = x^2 + y^2 + \lambda (x^2 + 4xy + 6y^2 - 140)$$

$$g =$$

$$g_x = 0 \Rightarrow 2x + \lambda(2x + 4y) = 0 \rightarrow (1)$$

$$g_y = 0 \Rightarrow 2y + \lambda(4x + 12y) = 0 \rightarrow (2)$$

$$g_\lambda = 0 \Rightarrow x^2 + 4xy + 6y^2 - 140 = 0 \rightarrow (3)$$

$(1) \times x + (2) \times y \Rightarrow$

$$2x^2 + \lambda(2x^2 + 4xy) + 2y^2 + \lambda(4xy + 12y^2) = 0$$

$$2x^2 + 2y^2 + \lambda(2x^2 + 8xy + 12y^2) = 0$$

$$\cancel{2}x^2 + \cancel{2}y^2 + \cancel{2}\lambda(x^2 + 4xy + 6y^2) = 0$$

$$x^2 + y^2 + 140\lambda = 0$$

$$\boxed{\lambda = -\frac{(x^2 + y^2)}{140}}$$

$$① \Rightarrow 2x - \frac{(x^2+y^2)}{140}(2x+4y) = 0$$

$$2x - \frac{(x^2+y^2)}{70}(x+2y) = 0$$

$$2x - \frac{f}{70}(x+2y) = 0$$

$$140x - fx - 2fy = 0$$

$$x(140-f) = 2fy$$

$$\boxed{\frac{x}{y} = \frac{2f}{140-f}} \rightarrow ④$$

$$② \Rightarrow 2y - \frac{(x^2+y^2)}{140}(4x+12y) = 0$$

$$2y - \frac{(x^2+y^2)}{35}(x+3y) = 0$$

$$70y - f(x+3y) = 0$$

$$-fx + y(70-3f) = 0$$

$$fx = y(70-3f)$$

$$\Rightarrow \boxed{\frac{x}{y} = \frac{70-3f}{f}} \rightarrow ⑤$$

from ④ and ⑤, $\dfrac{2f}{140-f} = \dfrac{70-3f}{f}$

$2f^2 = (70-3f)(140-f)$

$2f^2 = 9800 - 70f - 420f + 3f^2$

$f^2 - 490 + 9800 = 0$

$f = 469.1093$

(or)

$f = 20.8907$

Max distance $= 21.6589$

Min distance $= 4.5706$

245 +

$x^2 + y^2$

$f = 469.109$

$f = 20.89$

f

③ Find the greatest and least values of z, where (x, y, z) lies on the ellipse formed by the intersection of the plane $x+y+z=1$ and the ellipsoid $16x^2+4y^2+z^2=16$.

Sol

$$g = z + \lambda_1(x+y+z-1) + \lambda_2(16x^2+4y^2+z^2-16)$$

$g_x = 0 \Rightarrow \lambda_1 + 32\lambda_2 x = 0 \rightarrow (1)$

$g_y = 0 \Rightarrow \lambda_1 + 8\lambda_2 y = 0 \rightarrow (2)$

$g_z = 0 \Rightarrow 1 + \lambda_1 + 2\lambda_2 z = 0 \rightarrow (3)$

$g_{\lambda_1} = 0 \Rightarrow x+y+z = 1 \rightarrow (4)$

$g_{\lambda_2} = 0 \Rightarrow 16x^2+4y^2+z^2 = 16$

$ax + by = 0$

$cx + dy = 0$

Non-triv...

if

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

Eliminating λ_1, λ_2 from ①, ②, ③,

we have

$$\begin{vmatrix} 0 & 1 & 32x \\ 0 & 1 & 8y \\ 1 & 1 & 2z \end{vmatrix} = 0$$

$$1(8y - 32x) = 0$$

$$8y = 32x$$

$$\boxed{y = 4x}$$

sub $y = 4x$ in ④ $\Rightarrow$ $x + y + z = 1$

$$x + 4x + z = 1$$

$$\boxed{z = 1 - 5x}$$

5 $\Rightarrow$ $16x^2 + 4y^2 + z^2 = 16$

$$16x^2 + 4(4x)^2 + (1-5x)^2 = 16$$

$$16x^2 + 64x^2 + 1 + 25x^2 - 10x - 16 = 0$$

$$105x^2 - 10x - 15 = 0$$

$$x = \frac{3}{7} \text{ (or) } -\frac{1}{3}$$

If $x = \frac{3}{7} \Rightarrow y = 4x = 4 \times \frac{3}{7} = \frac{12}{7}$

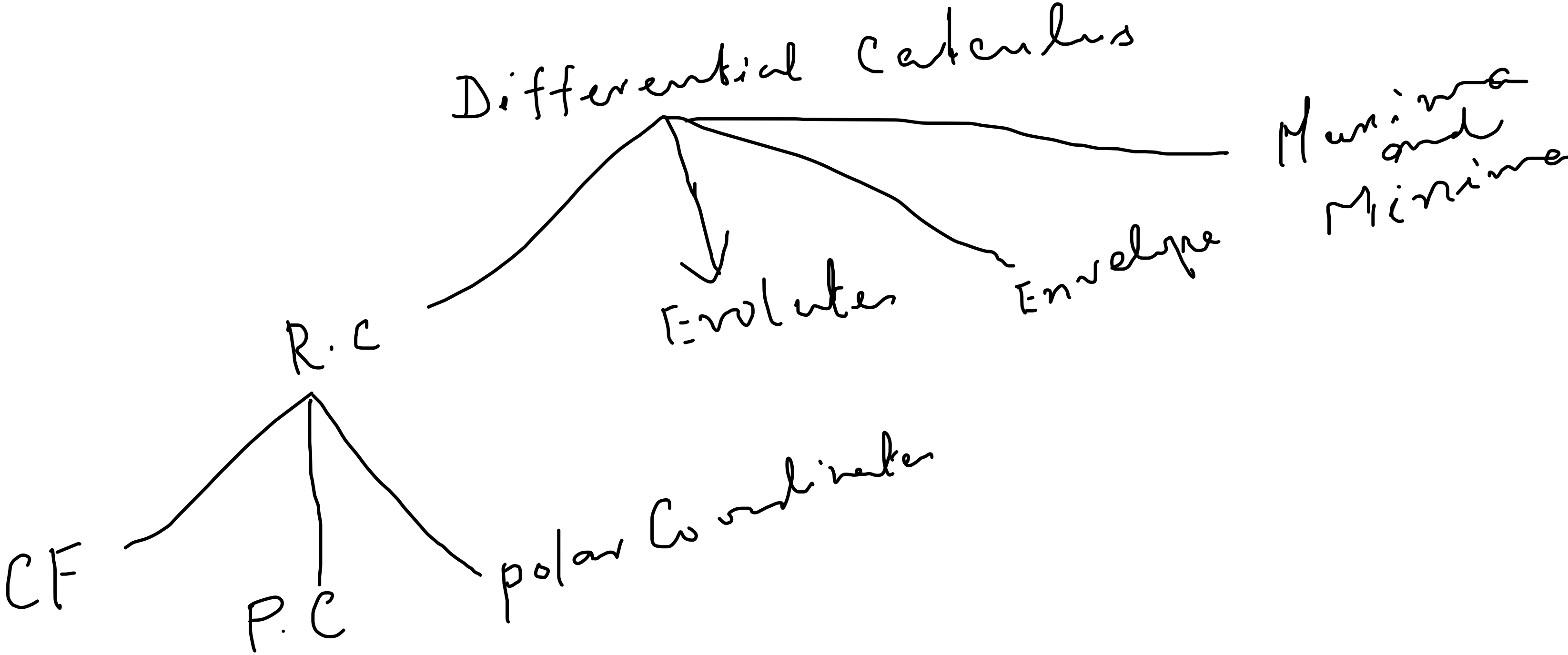
$$z = 1 - 5x = 1 - \frac{5 \times 3}{7} = 1 - \frac{15}{7} = -\frac{8}{7}$$

$$\left(\frac{3}{7}, \frac{12}{7}, -\frac{8}{7}\right)$$

If $x = -\frac{1}{3} \rightarrow y = -\frac{4}{3}, \quad z = \frac{8}{3}$

$$\left(-\frac{1}{3}, -\frac{4}{3}, \frac{8}{3}\right)$$

Max of $z = \frac{8}{3}$ and Min of $z = -\frac{8}{3}$


Differential Calculus
R.C
Evolutes
Envelope
Maxima and Minima
CF
P.C
polar Coordinates

Formula for Radius of Curvature in Cartesian Co-ordinates

CURVATURE

$$\lim_{\delta s \to 0} \frac{\delta \psi}{\delta s} = \frac{d\psi}{ds}$$

Let P be any point on a given curve and Q a neighbouring point. Let arc $AP = s$ and arc $PQ = \delta s$. Let the tangents at P and Q make angle ψ and $\psi + \delta\psi$ with the x-axis, so that the angle between the tangents at P and $Q = \delta\psi$ (Fig. 4.9).

In moving from P to Q through a distance δs, the tangent has turned through the angle $\delta\psi$. This is called the *total bending or total curvature* of the arc PQ.

$\therefore$ The *average curvature* of arc $PQ = \dfrac{\delta\psi}{\delta s}$

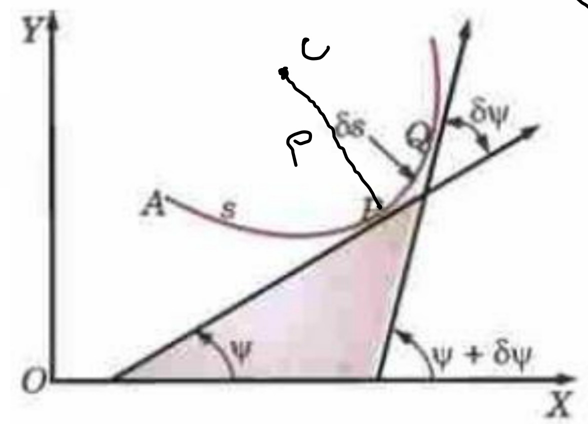


Fig. 4.9

ρ

The limiting value of average curvature when Q approaches P (i.e., $\delta s \to 0$*) is defined as the curvature of the curve at P.*

Thus **curvature** K (at P) $= \dfrac{\mathbf{d}\psi}{\mathbf{d}s} \rightarrow \dfrac{d\psi}{ds}$

(2) **Radius of curvature.** *The reciprocal of the curvature of a curve at any point P is called the* **radius of curvature** *at P and is denoted by* ρ, so the $\rho = \mathbf{ds/d\psi}$.

(3) **Centre of curvature.** *A point C on the normal at any point P of a curve distant* ρ *from it, is called the* **centre of curvature** *at P.*

(4) **Circle of curvature.** *A circle with centre C (centre of curvature at P) and radius* ρ *is called the* **circle of curvature** *at P.*

RADIUS OF CURVATURE FOR CARTESIAN CURVE y = f (x), is given by

$\cos\psi = \frac{dx}{ds}$

$\sin\psi = dy/ds$

$$\rho = \frac{(1 + y_1^{\ 2})^{3/2}}{y_2}$$

$\tan\psi = dy/dx = y_1 \quad \text{or} \quad \psi = \tan^{-1}(y_1)$

Differentiating both sides w.r.t. x,

$$\frac{d\psi}{dx} = \frac{1}{1 + y_1^{\ 2}} \cdot \frac{d(y_1)}{dx} = \frac{y_2}{1 + y_1^{\ 2}}$$

$$\therefore \quad \rho = \frac{ds}{d\psi} = \frac{ds}{dx} \cdot \frac{dx}{d\psi} = \sqrt{(1 + y_1^{\ 2})} \cdot \frac{1 + y_1^{\ 2}}{y_2} = \frac{(1 + y_1^{\ 2})^{3/2}}{y_2}$$

$\rho = \frac{ds}{d\psi} = \frac{ds}{dx} \cdot \boxed{\frac{dx}{d\psi}}$

$\frac{d\psi}{dx} = \frac{y_2}{1+y_1^2}$

$\frac{ds}{dx} = \sec\psi = \frac{1}{\cos\psi} = [\sec^2\psi]^{1/2} = [1 + \tan^2\psi]^{1/2} = [1 + y_1^2]^{1/2}$

① Find ρ, $xy = c^2$ at (c, c)

Sol

$$y = \frac{c^2}{x}$$

$$y' = -\frac{1}{x^2} c^2$$

$$(y')_{(c,c)} = \frac{-c^2}{c^2} = -1$$

$$y'' = \frac{2}{x^3} c^2$$

$$(y'')_{(c,c)} = \frac{2c^2}{c^3} = \frac{2}{c}$$

$$\rho = \frac{[1+(y')^2]^{3/2}}{y''} = \frac{[1+(-1)^2]^{3/2}}{2/c}$$

$$= 2\sqrt{2} \times \frac{c}{2}$$

$$= \underline{\underline{\sqrt{2}\, c}}$$

1. $y =$
2. y'
3. $(y')_{(\ )}$
4. y''
5. $(y'')_{(\ )}$
6. $(\rho)_{(\ )}$

② Find ρ $y^2 = 4ax$ at $y = 2a$

<u>Sol</u>

$$2yy' = 4a$$

$$y' = \frac{4a}{2y} = \frac{2a}{y}$$

$$(y')_{2a} = \frac{2a}{2a} = 1$$

$$yy' = 2a$$

$$yy'' + y'y' = 0$$

at $y = 2a$, $(2a)y'' + (1)^2 = 0$

$$y'' = -\frac{1}{2a}$$

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''} = \frac{\left[1+(1)^2\right]^{3/2}}{-1/2a} = 2\sqrt{2} \times -2a$$

$$\boxed{|\rho| = 4\sqrt{2}\,a}$$

Note :- As curvature (and hence radius of curvature) of a curve at any point is independent of the choice of x and y-axis, in the formula so x and y can be interchanged for ρ.

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

you can use this formula if $\frac{dy}{dx}$ is infinite at a pt.

3 Find ρ $xy^2 = a^3 - x^3$ at $(a, 0)$

Sol

$$y^2 = \frac{a^3 - x^3}{x} \longrightarrow 1$$

$$2yy' = \frac{x[-3x^2] - [a^3 - x^3](1)}{x^2}$$

$$= \frac{-3x^3 - a^3 + x^3}{x^2}$$

$$\frac{dy}{dx} = y' = \frac{-(a^3 + 2x^3)}{2x^2 y} \longrightarrow 2$$

$$(y')\Big|_{(a, 0)} = \infty$$

$$\frac{dx}{dy} = -\frac{2x^2 y}{2x^3 + a^3} \longrightarrow (3)$$

$$\left(\frac{dx}{dy}\right)_{(a,0)} = 0$$

Diff (3) w.r.t. y

$$\frac{d^2x}{dy^2} = \frac{-2\left[(2x^3+a^3)\left[x^2\cdot 1 + y\cdot 2x\frac{dx}{dy}\right] - \left[x^2 y \times 6x^2\frac{dx}{dy}\right]\right]}{\left[2x^3+a^3\right]^2}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = \frac{-2\left[(2a^3+a^3)\left[a^2+0\right] - 0\right]}{\left[2a^3+a^3\right]^2}$$

$$= \frac{-2a^2\,\cancel{(3a^3)}}{(3a^3)^{\cancel{2}\,1}}$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = -\frac{2}{3a}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{(1+0)^{3/2}}{-2/3a}$$

$$|\rho| = \frac{3a}{2}$$

(4) $y = e^x$ at $x = 0$

(5) $y = e^{\sqrt{3}x}$ at $x = 0$

(6) $y = \log \sec x$ at any pt on it

(7) S.T the curves $y = c \cosh \frac{x}{c}$ and $x^2 = 2c(y - c)$ have the same curvature where they cross the y-axis.

(8) at the pts $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ $x^3 + y^3 = 3axy$