Exercise : 5.3

(1) evolute of Parabola $y^2 = 4ax$

$x = at^2 \quad y = 2at$

$\frac{dx}{dt} = 2at \quad \frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{2a}{2at} = 1/t$

$\frac{d^2y}{dx^2} = -1/t^2 \left(1/2at\right)$

$= -1/2at^3$

$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$= at^2 - \frac{(1/t)}{(-1/2at^3)}\left(1 + 1/t^2\right)$

$= at^2 + \frac{2at^{\cancel{3}}}{\cancel{t}}\left(\frac{t^2+1}{\cancel{t^2}}\right)$

$= at^2 + 2at^2 + 2a$

$= 3at^2 + 2a$

$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$

$= 2at + \frac{(1+1/t^2)}{(-1/2at^3)}$

$= 2at + \frac{(t^2+1)(-2at^3)}{t^2}$

$= \cancel{2at} - 2at^3 - \cancel{2at}$

$\bar{y} = -2at^3 \Rightarrow t^3 = -\bar{y}/2a \Rightarrow t^6 = (-\bar{y}/2a)^2$

$\bar{x} = 3at^2 + 2a \Rightarrow t^2 = \dfrac{\bar{x} - 2a}{3a} \Rightarrow t^6 = \left(\dfrac{\bar{x} - 2a}{3a}\right)^3$

$$\left(\frac{\bar{x} - 2a}{3a}\right)^3 = \frac{\bar{y}^2}{4a^2}$$

$$4a^2(\bar{x} - 2a)^3 = 27a^3\bar{y}^2$$

$$4(\bar{x} - 2a)^3 = 27a\bar{y}^2$$

Locus of $(\bar{x}, \bar{y})$ is $4(\bar{x} - 2a)^3 = 27ay^2$

(2) evolute of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$x = a\cos\theta \quad y = b\sin\theta$

$\dfrac{dx}{d\theta} = -a\sin\theta \quad \dfrac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = -\cot\theta \; b/a$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\cot\theta \; b/a)$$

$$= \frac{d}{d\theta}(-\cot\theta \; b/a) \cdot \frac{d\theta}{dx}$$

$$= \frac{b\operatorname{cosec}^2\theta}{a} \cdot \frac{1}{-a\sin\theta}$$

$$= \frac{-b\operatorname{cosec}^3\theta}{a^2}$$

$\cancel{\bar{x}} = \cancel{a\cos\theta} +$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= a\cos\theta - \frac{\left(\frac{\cot\theta\, b}{a}\right)}{\left(\frac{-b\,\mathrm{cosec}^3\theta}{a^2}\right)}\left(1+\cot^2\theta\frac{b^2}{a^2}\right)$$

$$= a\cos\theta - \frac{a\cot\theta}{\mathrm{cosec}^3\theta}\left(1+\frac{\cot^2\theta}{a^2}b^2\right)$$

$$= a\cos\theta - \frac{a\cot\theta}{\mathrm{cosec}^3\theta} - \frac{\cancel{a}\cot^3\theta\, b^2}{a^2\,\mathrm{cosec}^3\theta}$$

$$= a\cos\theta - \frac{a\cot\theta}{\mathrm{cosec}^3\theta} - \frac{b^2\cot^3\theta}{a\,\mathrm{cosec}^3\theta}$$

$$= a\cos\theta - a\cos\theta\sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\left[1-\sin^2\theta\right] - \frac{b^2}{a}\cos^3\theta$$

$$= \cos^3\theta\left[\frac{a^2-b^2}{a}\right]$$

$$\bar{y} = \cancel{bs}\ y + \frac{(1+y_1^2)}{y_2}$$

$$= b\sin\theta + \frac{(1+\cot^2\theta\, b^2/a^2)}{\left(\frac{-b\,\mathrm{cosec}^3\theta}{a^2}\right)}$$

$$= b\sin\theta - \frac{(1+\cot^2\theta\, b^2/a^2)\,a^2}{b\,\mathrm{cosec}^3\theta}$$

$$= b\sin\theta - (a^2/b)\sin^3\theta - \frac{b\cos^2\theta}{\sin^2\theta}\sin\theta$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\cos^2\theta\sin\theta$$

$$= b\sin\theta\,[1-\cos^2\theta] - \frac{a^2}{b}\sin^3\theta$$

$$= \sin^3\theta\left[\frac{b^2-a^2}{b}\right]$$

$$\bar{y} = -\sin^3\theta\left(\frac{a^2-b^2}{b}\right) \Rightarrow \sin^2\theta = \left[\frac{\bar{y}b}{b^2-a^2}\right]^{2/3} \quad -①$$

$$\bar{x} = \cos^3\theta\left(\frac{a^2-b^2}{a}\right) \Rightarrow \cos^2\theta = \left[\frac{\bar{x}a}{a^2-b^2}\right]^{2/3} \quad -②$$

①+②

$$\sin^2\theta + \cos^2\theta = \left[\frac{\bar{y}b}{a^2-b^2}\right]^{2/3} + \left[\frac{\bar{x}a}{a^2-b^2}\right]^{2/3}$$

$$\frac{(\bar{x}a)^{2/3} + (\bar{y}b)^{2/3}}{[a^2-b^2]^{2/3}} = 1$$

$$(\bar{x}a)^{2/3} + (\bar{y}b)^{2/3} = [a^2-b^2]^{2/3}$$

(3) $x = a(\theta+\sin\theta)$ $\quad$ $y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1+\cos\theta) \qquad \frac{dy}{d\theta} = a(\sin\theta)$$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}{\cancel{2}\cos^2\theta/2} = \tan\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d(\tan\theta/2)}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{1}{2}\sec^2\theta/2 \times \frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2\theta/2 \times \frac{1}{2a\cos^2\theta/2}$$

$$= \frac{1}{a(2\cos^2\theta/2)^2}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= a(\theta+\sin\theta) - \frac{\tan\theta/2}{1/a(2\cos^2\theta/2)^2} \times (1+\tan^2\theta/2)$$

$$= a\theta + a\sin\theta - 4a\tan\theta/2\cos^4\theta/2\,(\sec^2\theta/2)$$

$$= a\theta + a\sin\theta - 4a\frac{\sin\theta/2}{\cancel{\cos\theta/2}}\cos^{\cancel{4}}\theta/2$$

$$= a\theta + a\sin\theta - 4a\sin\theta/2\cos\theta/2$$

$$= a\theta + a\sin\theta - 2a\sin\theta$$

$$= a\theta - a\sin\theta$$

$$= a(\theta - \sin\theta) \quad \text{--- (1)}$$

$$\bar{y} = y + \frac{(y_1^2+1)}{y_2}$$

$$= a(1-\cos\theta) + (\tan^2\theta/2 + 1) \times (4a\cos^4\theta/2)$$

$= a - a\cos\theta + \sec^2\theta/_2 \,(4a\cos^4\theta/_2)$

$= a - a\cos\theta + 4a\cos^2\theta/_2$

$= a - a\cos\theta + (2a)(1+\cos\theta)$

$= a - a\cos\theta + 2a + 2a\cos\theta$

$\bar{y} = 3a + a\cos\theta$

$\bar{y} - 2a = a(1+\cos\theta) \quad - ②$

~~ele~~ elimination of θ from ① and ② is not easy

$\therefore$ Locus of $(\bar{x}, \bar{y})$ is given by Parametric equation

$x = a(\theta - \sin\theta) \qquad y - 2a = a(1+\cos\theta)$

(4) $x = a(\cos\theta + \theta\sin\theta) \qquad y = a(\sin\theta - \theta\cos\theta)$

$\frac{dx}{d\theta} = a(\cancel{-\sin\theta} + \theta\cos\theta + \cancel{\sin\theta})$

$= a\theta\cos\theta$

$\frac{dy}{d\theta} = a(\cancel{\cos\theta} - \cancel{\cos\theta} + a\sin\theta)$

$= a\theta\sin\theta$

$\frac{dy}{dx} = \tan\theta$

$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(\tan\theta) \cdot \frac{d\theta}{dx}$

$$= \sec^2\theta \cdot \frac{1}{a\theta\cos\theta}$$

$$= \frac{1}{a\theta\cos^3\theta}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= a\cos\theta + a\theta\sin\theta - \frac{\tan\theta}{1/a\theta\cos^3\theta}(1+\tan^2\theta)$$

$$= a\cos\theta + a\theta\sin\theta - a\theta\cancel{\cos^3\theta} \cdot \frac{\sin\theta}{\cancel{\cos\theta}} \frac{1}{\cancel{\cos^2\theta}}$$

$$= a\cos\theta \cancel{+ a\theta\sin\theta} \cancel{- a\theta\sin\theta}$$

$$= a\cos\theta \quad \text{— (1)}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= a\sin\theta - a\theta\cos\theta + \frac{(1+\tan^2\theta)}{1/a\theta\cos^3\theta}$$

$$= a\sin\theta - a\theta\cos\theta + \frac{1}{\cancel{\cos^2\theta}} a\theta\cos^3\theta$$

$$= a\sin\theta - \cancel{a\theta\cos\theta} \cancel{+ a\theta\cos\theta}$$

$$= a\sin\theta \quad \text{— (2)}$$

$(1)^2 + (2)^2$

$$\bar{x}^2 + \bar{y}^2 = a^2(\sin^2\theta + \cos^2\theta)$$

$$\bar{x}^2+\bar{y}^2 = a^2$$

$\therefore$ Locus of $(\bar{x},\bar{y})$ is given by $x^2+y^2=a^2$

Problems on Evolute and Envelope

(1) find the envelope of family of lines $\frac{x}{a}+\frac{y}{b}=1$, where Parameters a and b are connected by relation $a^2+b^2=c^2$

$$\frac{x}{a}+\frac{y}{b}=1 \quad — ①$$

$$a^2+b^2=c^2 \quad — ②$$

differentiate eq ① wrt a, consider $b=f(a)$

$$-\frac{x}{a^2}-\frac{y}{b^2}\left(\frac{db}{da}\right)=0$$

$$\frac{db}{da}=-\frac{x}{a^2}\cdot\frac{b^2}{y} \quad — ③$$

differentiate eq ② wrt a, consider $b=f(a)$

$$2a+2b\cdot\frac{db}{da}=0$$

$$\frac{db}{da}=-\frac{a}{b} \quad — ④$$

$$\frac{x}{a}=\frac{y}{b}=l=\frac{x+y}{a+b}$$

$x=al$, $y=bl$

$x+y=(a+b)l$

$$l=\frac{x+y}{a+b}$$

equate ③ & ④

$$\frac{x}{a^2}\frac{b^2}{y}=\frac{a}{b}$$

$$\frac{x}{a^3}=\frac{y}{b^3}$$

$$\frac{\frac{x}{a}}{a^2}=\frac{\frac{y}{b}}{y^2}=\frac{\frac{x}{a}+\frac{y}{b}}{a^2+y^2}=\frac{1}{c^2}$$

$$(\bar{x}+\bar{y})^{2/3} - (\bar{x}-\bar{y})^{2/3} = \left(\frac{c}{2}\right)^{2/3}[t+1/t]^2 - \left(-\frac{c}{2}\right)^{2/3}[t-1/t]^2$$

$$(\bar{x}+\bar{y})^{2/3} - (\bar{x}-\bar{y})^{2/3} = (4c)^{2/3}$$

(3) find envelope of family of lines $\frac{x}{a}+\frac{y}{b}=1$, where the Parameters a and b are connected by relation $ab=c^2$

$$\frac{x}{a}+\frac{y}{b}=1 \quad -① \qquad ab=c^2 \quad -②$$

differentiate eq ① wrt a, consider $b=f(a)$

$$-\frac{x}{a^2} - \frac{y}{b^2}\left(\frac{db}{da}\right) = 0$$

$$\frac{db}{da} = \frac{-xb^2}{ya^2} \quad -③$$

differentiate eq ② wrt a, consider $b=f(a)$

$$b + a\frac{db}{da} = 0$$

$$\frac{db}{da} = -b/a \quad -④$$

equate eq ③ & ④

$$\frac{\cancel{-}xb^2}{ya^2} = \cancel{-}b/a$$

$$\frac{x}{y} = \frac{a}{b}$$

$$\frac{\frac{x}{a}}{1} = \frac{\frac{y}{b}}{1} = \frac{\frac{x}{a}+\frac{y}{b}}{1+1} = \frac{1}{2}$$

$$\frac{x}{a} = \frac{1}{2} \qquad \frac{y}{b} = \frac{1}{2}$$

$$a = 2x \qquad b = 2y$$

Substitute eq ①

$$ab = c^2$$

$$(2x)(2y) = c^2$$

$$\boxed{4xy = c^2}$$

(4) Find envelope of family of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where

Parameters a and b connected by $a+b=c$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad -①$$

$$a+b=c \quad -②$$

differentiate eq ① wrt a, consider $b = f(a)$

$$-\frac{x^2}{a^3} - \frac{y^2}{b^3}\frac{db}{da} = 0$$

$$\frac{db}{da} = -\frac{x^2 b^3}{y^2 a^3} \quad -③$$

differentiate eq ② wrt a, consider $b = f(a)$

$$1 + \frac{db}{da} = 0$$

$$\frac{db}{da} = -1 \quad -④$$

Equate ③ & ④

$$\frac{x^2 b^3}{y^2 a^3} = 1$$

$$\frac{x^2}{a^3} = \frac{y^2}{b^3}$$

$$\frac{\frac{x^2}{a^2}}{a} = \frac{\frac{y^2}{b^2}}{b} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2}}{a+b} = \frac{1}{c}$$

$$\frac{x^2}{a^3} = \frac{1}{c} \qquad \frac{y^2}{b^3} = 1/c$$

$$a = (x^2 c)^{1/3} \qquad b = (y^2 c)^{1/3}$$

Substitute in eq ②

$$(x^2 c)^{1/3} + (y^2 c)^{1/3} = c$$

$$c^{1/3}(x^{2/3} + y^{2/3}) = c$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

Assignment 2

if the ~~curve~~ center of the curvature of curve at variable point t on it is $(2a+3at^2, -2at^3)$, find the evolute of the curve.

Solution :-

$$\bar{x} = 2a + 3at^2 \quad - (1)$$

$$\bar{y} = -2at^3 \quad - (2)$$

from eqn (2)

$$\left(\frac{-y}{2a}\right)^{1/3} = t \quad - (3)$$

Substitute (3) in (1)

$$\bar{x} = 2a + 3a\left(\frac{-y}{2a}\right)^{\frac{2}{3}}$$

$$(x - 2a) = 3a\left(\frac{y}{2a}\right)^{\frac{2}{3}}$$

$$(x-2a)^{3/2} = (3a)^{3/2}\left(\frac{y}{2a}\right)^{2/3 \times 3/2}$$

square on both sides

$$(x-2a)^3 = 27a^3\left(\frac{y^2}{4a^2}\right)$$

$$\boxed{4(x-2a)^3 = 27ay^2}$$

↳ required evolute

Q if center of curvature of a curve at a variable point 't' on it is $\left[\frac{c}{a}\cos^3 t, -\frac{c}{b}\sin^3 t\right)$

find evolute.

Solu:-

$\bar{x} = \frac{c}{a}\cos^3 t$

$\bar{y} = \frac{-c}{b}\sin^3 t$

$\left(\frac{a\bar{x}}{c}\right)^{1/3} = \cos t$ — ① ; $\left(\frac{-by}{c}\right)^{1/3} = \sin t$ — ②

we knowthat $\sin^2 t + \cos^2 t = 1$.

∴ square and add both ① & ②

$$\left(\frac{ax}{c}\right)^{2/3} + \left(\frac{-by}{c}\right)^{2/3} = \sin^2 t + \cos^2 t = 1$$

$$\boxed{(ax)^{2/3} + (by)^{2/3} = (c)^{2/3}}$$

↳ Required evolute.

If the center of curvature of a curve at a variable point 'θ' on it is $\left[a\log\cot\theta/2, \frac{a}{\sin\theta}\right)$, find the evolute

Solution:-

$x = a \log \cot \theta/2 \Rightarrow e^{\frac{x}{a}} = \cot \theta/2$

$y = \frac{a}{\sin\theta} \Rightarrow \sin\theta = \frac{a}{y}.$

We know that $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}.$

$e^{-x/a} = \tan\theta/2$

$\sin\theta = \frac{a}{y}.$

$$\frac{2\tan\theta/2}{1+\tan^2\theta/2} = \frac{a}{y}$$

$$\frac{2 \cdot e^{-x/a}}{1+e^{-2x/a}} = \frac{a}{y} \rightarrow$$ Reciprocal this equation

$$\frac{y}{a} = \frac{1+e^{-\frac{2x}{a}}}{2e^{-x/a}}.$$

$$\frac{2y}{a} = \frac{1}{e^{-x/a}} + \frac{e^{-\frac{2x}{a}}}{e^{-x/a}}$$

$$\frac{y}{a} = \boxed{\frac{e^{x/a}+e^{-x/a}}{2}} \rightarrow$$ formula for $\cosh\left(\frac{x}{a}\right)$

$\boxed{\because \cosh x = \frac{e^x+e^{-x}}{2}}$

$$\boxed{\frac{y}{a} = \cosh\left(\frac{x}{a}\right)}$$

↳ required evolute.

Find the envelope of the family of lines

$y = mx \pm a\sqrt{1+m^2}$ m being parameter

Solution :-

$y = \cancel{mx \pm a\sqrt{1+m^2}}$ → diff this eqn.

$0 = x \pm a(1+m^2)^{-1/2} \cdot \frac{1}{2}$

$y = mx \pm a\sqrt{1+m^2}$ — ①

diff above equation

$0 = x \pm a \frac{1}{\cancel{2}}(1+m^2)^{-1/2} \cdot \cancel{2}m$

$x = \pm a(1+m^2)^{-1/2} m$

$\frac{x}{a} = (1+m^2)^{-1/2} \cdot m$

$\frac{x}{a} = \frac{m}{\sqrt{1+m^2}} \Rightarrow \frac{x}{a} = \frac{1}{\sqrt{\frac{1}{m^2}+1}}$

$\frac{a}{x} = \sqrt{\frac{1}{m^2}+1} \Rightarrow \frac{a^2}{x^2} - 1 = \frac{1}{m^2}$

$\frac{a^2-x^2}{x^2} = \frac{1}{m^2} \Rightarrow \frac{x^2}{a^2-x^2} = m^2$

$$\boxed{\sqrt{\frac{x^2}{a^2-x^2}} = m} \longrightarrow ②$$

substitute ② in ~~one~~ ①

$$y = \left(\sqrt{\frac{x^2}{a^2-x^2}}\right)x \pm a\sqrt{1+\left(\frac{x^2}{a^2-x^2}\right)^{1/2\times 2}}$$

$$y = \left(\sqrt{\frac{x^2}{a^2-x^2}}\right)x \pm a\sqrt{1+\frac{x^2}{a^2-x^2}}$$

$$y = \left(\sqrt{\frac{x^2}{a^2-x^2}}\right)x \pm a\sqrt{\frac{a^2-\cancel{x^2}+\cancel{x^2}}{a^2-x^2}}$$

$$y = \left(\sqrt{\frac{x^2}{a^2-x^2}}\right)x \pm a\sqrt{\frac{a^2}{a^2-x^2}}$$

$$y = \frac{x^2 \pm a^2}{\sqrt{a^2-x^2}}$$

$$y\sqrt{a^2-x^2} = a^2-x^2$$

$$y^2 = a^2-x^2$$

$\boxed{x^2+y^2=a^2}$ $\longrightarrow$ required envelope

find the envelope of the family of lines

$y = mx + \frac{a}{m}$ m - parameter

solution:-

$$y = mx + \frac{a}{m} \quad -(1)$$

~~Find~~ differentiate the above equation.

$$0 = x + \frac{-a}{m^2}$$

$$x = \frac{a}{m^2} \Rightarrow m = \sqrt{\frac{a}{x}} \rightarrow (2).$$

substitute (2) in equation (1).

$$y = \sqrt{\frac{a}{x}}\,x + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$y = \sqrt{ax} + \sqrt{ax}$$

$$y = 2\sqrt{ax} \Rightarrow \boxed{y^2 = 4ax}$$ required envelope.

13. Find the envelope of the family of lines $y = mx + am^2$ m - parameter:

Solution:

$$y = mx + am^2 \quad — (1).$$

Substitute (2) in (1)

diff above eqn

$$0 = x + 2am.$$

$$\boxed{x = -2am} \Rightarrow \boxed{\frac{-x}{2a} = m} \quad - (2)$$

$$y = \left(\frac{-x}{2a}\right)x + a\left(\frac{-x}{2a}\right)^2$$

$$4a^2y = 2a(-x^2) + ax^2$$

$$4a^2y = -ax^2 \Rightarrow ax^2 + 4a^2y = 0$$

required envelope. $\Leftarrow$ $\boxed{x^2 + 4ay = 0}$

Find the envelope of the family of lines

$y = mx \pm \sqrt{a^2m^2+b^2}$ m - parameter

Solution:

$\Rightarrow y = mx \pm \sqrt{a^2m^2+b^2}$

differentiate the equ.

$0 = x \pm \frac{1}{\cancel{2}}(a^2m^2+b^2)^{-1/2} \cdot \cancel{2}am$

$x = \pm \frac{am}{\sqrt{a^2m^2+b^2}}$

$x = \frac{1}{\sqrt{1+\frac{b^2}{a^2m^2}}}$

$\frac{1}{x} = \sqrt{1+\frac{b^2}{a^2m^2}}$

$\frac{1}{x^2} - 1 = \frac{b^2}{a^2m^2} \Rightarrow \frac{a^2}{b^2}\left[\frac{1}{x^2}-1\right] = \frac{1}{m^2}$

$\frac{a^2}{b^2}\left[\frac{1}{x^2}-1\right] = m^2$

$\sqrt{\frac{a^2}{b^2}\left[\frac{1}{x^2}-1\right]} = m.$

$y = \left(\sqrt{\frac{a^2}{b^2}\left[\frac{1}{x^2}-1\right]}\right)x \pm \sqrt{a^2\left(\frac{a^2}{b^2}\right.}$

$$\left(\frac{1}{x^2} - 1\right)\frac{a^2}{b^2} = \frac{1}{m^2}$$

$$\cancel{\frac{b^2}{a^2}}\left[\frac{1-x^2}{x^2}\right]\frac{a^2}{b^2} = \frac{1}{m^2}$$

$$\frac{x^2 b^2}{(1-x^2)\,a^2} = m^2 \Rightarrow m = \sqrt{\frac{x^2 b^2}{(1-x^2)\,a^2}}$$

$$y = mx \pm \sqrt{a^2 m^2 + b^2}$$

$$y = \left(\sqrt{\frac{x^2 b^2}{(1-x^2)\,a^2}}\right)x \pm \sqrt{\cancel{a^2}\left(\frac{x^2 b^2}{(1-x^2)\,\cancel{a^2}}\right) + b^2}$$

$$y = \left(\sqrt{\frac{x^2 b^2}{(1-x^2)\,a^2}}\right)x \pm \sqrt{\frac{\cancel{x^2 b^2} + b^2 - \cancel{x^2 b^2}}{(1-x^2)}}$$

$$y = \frac{x^2 b}{a\sqrt{1-x^2}} \pm \frac{b}{(1-x^2)^{1/2}}$$

$$\sqrt{1-x^2}\; y = b\left[\frac{x^2}{a} \pm 1\right]$$

$$\sqrt{1-x^2}\;\frac{y}{b} = \left[\frac{x^2}{a} \pm 1\right]$$

$$(1-x^2)\,\frac{y^2}{b^2}$$

Find envelope of the family of line $\frac{x}{t} + yt = 2c$

t = parameter

$$\frac{x}{t} + yt = 2c$$

→ diff above eqn.

$$-\frac{x}{t^2} + y = 0.$$

$$\frac{x}{t^2} = +y$$

$$\frac{x}{y} = t^2$$

$$\sqrt{\frac{x}{y}} = t.$$

Substitute in (1).

$$\frac{x}{\sqrt{\frac{x}{y}}} + y \cdot \sqrt{\frac{x}{y}} = 2c$$

$$\sqrt{xy} + \sqrt{xy} = 2c$$

$$\cancel{2}\sqrt{xy} = \cancel{2}c$$

$$\boxed{xy = c^2}$$

Q. Find envelope of family of Lines $x\cos\alpha + y\sin\alpha = p$ α - parameter

Solution:

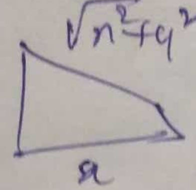
$$x\cos\alpha + y\sin\alpha = p \quad (1)$$

diff above equation

$$x\,(-\sin\alpha) + y\cos\alpha = 0$$

$$x\sin\alpha = y\cos\alpha$$

$$\tan\alpha = \frac{y}{x}$$



$$\cancel{x \times \frac{x}{\sqrt{x^2+y^2}}} + \cancel{y}.$$

Now $\sin\alpha = \dfrac{y}{\sqrt{x^2+y^2}}$

$\cos\alpha = \dfrac{x}{\sqrt{x^2+y^2}}$

Substitute in (1)

$$x \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}} \cdot y = p$$

$$x^2+y^2 = p\sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} = p$$

$$\boxed{x^2+y^2 = p^2}$$

Q Find the envelope of family of lines

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

θ = parameter

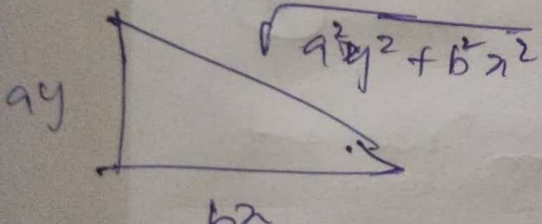
<u>Solution</u> :-

$$\frac{x}{a} - \sin\alpha + \frac{y}{b}\cos\alpha = 0$$

$$\frac{x}{a}\sin\alpha = \frac{y}{b}\cos\alpha$$

$$\tan\alpha = \frac{y}{b} \cdot \frac{a}{x}$$

$$\tan\alpha = \frac{y}{x} \cdot \frac{a}{b}$$



$$\left.\begin{aligned} \sin\alpha &= \frac{ay}{\sqrt{a^2y^2 + b^2x^2}} \\ \cos\alpha &= \frac{bx}{\sqrt{a^2y^2 + b^2x^2}} \end{aligned}\right\}$$

Substitute in eqn (1).

$$\frac{x}{a} \cdot \frac{bx}{\sqrt{a^2y^2 + b^2x^2}} + \frac{y}{b} \frac{ay}{\sqrt{a^2y^2 + b^2x^2}} = 1$$

$$\frac{b}{a}x^2 + \frac{a}{b}y^2 = \sqrt{a^2y^2 + b^2x^2}$$

$$b^2x^2 + a^2y^2 = ab\sqrt{a^2y^2 + b^2x^2}$$

$$\sqrt{b^2x^2 + a^2y^2} = ab$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \longrightarrow$$ ~~Find the~~ ∴ Required envelope.

Find the envelope of the lines

$$\frac{x}{a}\sec\theta + \frac{y}{b}\tan\theta = 1 \qquad \theta = \text{parameter}$$

Solution

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1. \quad \text{--- (1)}$$

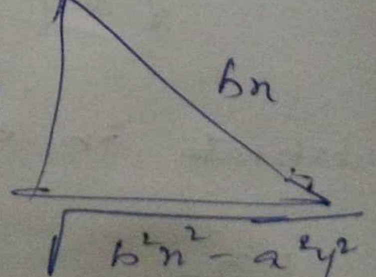
differentiating above equation we get

$$\frac{x}{a}\cdot\sec\theta\cdot\tan\theta - \frac{y}{b}\cdot\sec^2\theta = 0$$

$$\frac{x}{a}\cancel{\sec\theta}\cdot\tan\theta = \frac{y}{b}\sec^2\theta$$

$$\frac{\sin\theta}{\cancel{\cos\theta}}\left(\frac{x}{a}\right) = \frac{y}{b}\frac{1}{\cancel{\cos\theta}}$$

$$\sin\theta = \frac{a}{b}\cdot\frac{y}{x}.$$



$$\sec\theta = \frac{bx}{\sqrt{b^2x^2 - a^2y^2}}$$

$$\tan\theta = \frac{ay}{\sqrt{b^2x^2 - a^2y^2}}$$

substitute in eqn ①.

$$\frac{x}{a} \cdot \frac{bx}{\sqrt{b^2x^2 - a^2y^2}} - \frac{y}{b} \frac{ay}{\sqrt{b^2x^2 - a^2y^2}} = 1$$

$$\frac{bx^2}{a} - \frac{y^2 a}{b} = \sqrt{b^2x^2 - a^2y^2}$$

$$b^2x^2 - a^2y^2 = ab\sqrt{b^2x^2 - a^2y^2}$$

$$\sqrt{b^2x^2 - a^2y^2} = ab$$

$$b^2x^2 - a^2y^2 = a^2b^2$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

required envelope.

Find the envelope of the fan lines $x\sec\theta - y\tan\theta = a$.

Solution:-

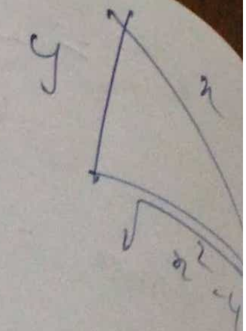
$x\sec\theta - y\tan\theta = a.$ ⟶ diff this we get

$x \cdot \sec\theta \cdot \tan\theta - y \cdot \sec^2\theta = 0.$

$x\,\cancel{\sec\theta} \cdot \tan\theta = y\sec^{\cancel{2}}\theta$

$$x \cdot \frac{\sin\theta}{\cancel{\cos\theta}} = y \cdot \frac{1}{\cancel{\cos\theta}}$$

$$\sin\theta = \frac{y}{x}$$



$$\sec\theta = \frac{x}{\sqrt{x^2-y^2}}$$

$$\tan\theta = \frac{y}{\sqrt{x^2-y^2}}$$

substitute this in $x\sec\theta - y\tan\theta$.

$$x \cdot \frac{x}{\sqrt{x^2-y^2}} - y \cdot \frac{y}{\sqrt{x^2-y^2}} = a.$$

$$x^2 - y^2 = a\sqrt{x^2-y^2}$$

$$\sqrt{x^2-y^2} = a.$$

$$\boxed{x^2 - y^2 = a^2} \longrightarrow \text{required envelope.}$$

Find the envelope of the lines

$x\,\mathrm{cosec}\,\theta - y \cot\theta = a \qquad \theta = \text{parameter}$

Solution :-

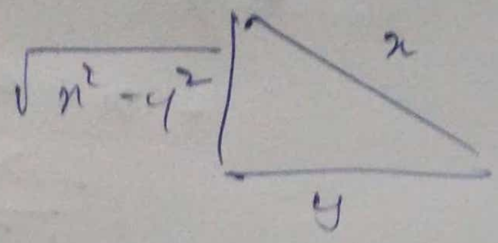
$$x\,\mathrm{cosec}\,\theta - y\cot\theta = a \quad \text{—— (1)}$$

differentiating the above equation we get

$$x - \mathrm{cosec}\,x \cdot \cot x - y(-\mathrm{cosec}^2\theta) = 0$$

$x \cdot \cancel{\operatorname{cosec} x} \cdot \cot x = y \cdot \operatorname{cosec}^2 x$

$x \cdot \frac{\cos\theta}{\cancel{\sin\theta}} = y \cdot \frac{1}{\cancel{\sin\theta}}$



$\cos\theta = \frac{y}{x}$

$$\left.\begin{aligned} \operatorname{cosec}\theta &= \frac{x}{\sqrt{x^2-y^2}} \\ \cot\theta &= \frac{y}{\sqrt{x^2-y^2}} \end{aligned}\right\}$$ substitute in eqn ①.

$$x \cdot \frac{x}{\sqrt{x^2-y^2}} - y \cdot \frac{y}{\sqrt{x^2-y^2}} = a.$$

$$x^2 - y^2 = a \cdot \sqrt{x^2-y^2}$$

$$\sqrt{x^2-y^2} = a$$

$$\boxed{x^2 - y^2 = a^2} \longrightarrow$$ required envelope.

Show that the family of circles $(x-a)^2 + y^2 = a^2$ has no envelope.

$\boxed{a = \text{parameter}}$

Solution

$$(x-a)^2 + y^2 = a^2$$

differentiate this equation wrt to a we get

$$2(x-a)^{\cancel{2}} \cdot (-1) + 0 = 2a$$

$$-\cancel{2}(x-a) = \cancel{2}a$$

$$-x+a = a$$

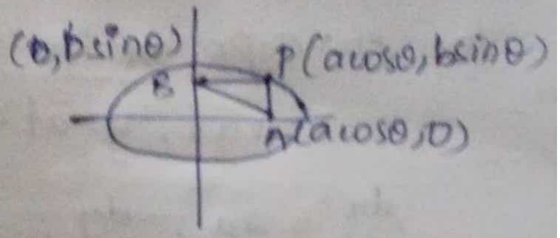
$$-x = a-a$$

$$\boxed{x=0}.$$

As here while differentiating ~~the~~ with parameter $\underline{a}$ gets ~~no value~~ cancelled

Hence it is proved that the family of circle $(x-a)^2+y^2=a^2$ has no envelope.

Q) From a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, perpendiculars are drawn to the axis and the feet of these perpendiculars are formed. Find the envelope of the line thus formed.



$$\frac{x}{a\cos\theta}+\frac{y}{b\sin\theta}=1 \rightarrow (1)$$

$$\frac{x}{a}\sec\theta+\frac{y}{b}\operatorname{cosec}\theta=1 \rightarrow (2)$$

differentiate eq (2) w.r.t 'θ'

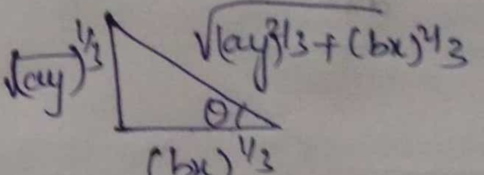
$$\frac{x}{a}(\sec\theta\tan\theta)+\frac{y}{b}(-\operatorname{cosec}\theta\cot\theta)=0$$

$$\frac{x}{a}\left(\frac{\sin\theta}{\cos^2\theta}\right)=\frac{y}{b}\left(\frac{\cos\theta}{\sin^2\theta}\right)$$

$$\frac{\sin^3\theta}{\operatorname{Tan}^3\theta}=\frac{ay}{bx} \Rightarrow \operatorname{Tan}^3\theta=\frac{ay}{bx}$$

$$\operatorname{Tan}\theta=\left(\frac{ay}{bx}\right)^{1/3}$$

From (2)



$$\frac{x}{a}\sec\theta+\frac{y}{b}\operatorname{cosec}\theta=1$$

$$\frac{x}{a}\left(\frac{\sqrt{(ay)^{2/3}+(bx)^{2/3}}}{(bx)^{1/3}}\right)+\frac{y}{b}\left(\frac{\sqrt{(ay)^{2/3}+(bx)^{2/3}}}{(ay)^{1/3}}\right)=1$$

$$\sqrt{(ay)^{2/3}+(bx)^{2/3}}\left(\frac{x^{2/3}}{ab^{1/3}}+\frac{y^{2/3}}{a^{1/3}b}\right)=1$$

$$\frac{\sqrt{(ay)^{2/3}+(bx)^{2/3}}}{a^{1/3}b^{1/3}}\left(\frac{x^{2/3}}{a^{2/3}}+\frac{y^{2/3}}{b^{2/3}}\right)=1$$

$$\frac{\sqrt{(ay)^{2/3}+(bx)^{2/3}}}{(ab)^{1/3}}\left(\frac{(bx)^{2/3}+(ay)^{2/3}}{(ab)^{2/3}}\right)=1$$

$$\left((ay)^{2/3}+(bx)^{2/3}\right)^{3/2}=ab$$

$$(ay)^{2/3}+(bx)^{2/3}=ab^{2/3} \Rightarrow \left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$$

Q) Find the ~~envelope~~ evolute of the parabola $x^2 = 4ay$ treating it as envelope of its normal.

$$x = 2at, \quad y = at^2 \qquad x^2 = 4ay \rightarrow (1)$$

$$\frac{dx}{dt} = 2a \quad \frac{dy}{dt} = 2at$$

$$\frac{dy}{dx} = t$$

Equation of normal to (1) is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

$$y - at^2 = \frac{-1}{t}(x - 2at)$$

$$yt - at^3 = -x + 2at$$

$$-at^3 + x - 2at + yt = 0 \rightarrow (2)$$ Evolute of eq (1) is same as envelope of eq (2)

diff eq (2) w.r.t 't'

$$-3at^2 + 0 - 2a + y = 0$$

$$t = \left(\frac{y - 2a}{3a}\right)^{1/2}$$

$$(2) \Rightarrow (y - 2a)t - at^3 + x = 0$$

$$(y - 2a)\left(\frac{y - 2a}{3a}\right)^{1/2} - a\left(\frac{y - 2a}{3a}\right)^{3/2} + x = 0$$

$$\frac{(y - 2a)^{3/2}}{(3a)^{1/2}} - \frac{(y - 2a)^{3/2}}{(3a)^{3/2}} \cdot a + x = 0$$

$$x = a\frac{(y - 2a)^{3/2}}{(3a)^{3/2}} - \frac{(y - 2a)^{3/2}}{(3a)^{1/2}}$$

$$= \frac{a(y - 2a)^{3/2} - 3a(y - 2a)^{3/2}}{(3a)^{3/2}}$$

$$x = \frac{(y - 2a)^{3/2}(-2a)}{(3a)^{3/2}}$$

$$x^2 = \frac{(y - 2a)^3(4a^2)}{(3a)^3}$$

$$27ax^2 = 4(y - 2a)^3$$

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