Exercise - 5.2

1) find the centre of curvature at the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$.

$x = at^2$, $y = 2at$

$\frac{dx}{dt} = 2at$ $\frac{dy}{dt} = 2a$

$y_1 = \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right)$

$= \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx}$

$y_2 = \frac{-1}{t^2}\,\frac{1}{2at} = \frac{-1}{2at^3}$

$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$

$= at^2 - \frac{1/t}{-1/2at^3}\left(1+\frac{1}{t^2}\right)$

$= at^2 + 2a\cancel{t^2}\left(\frac{t^2+1}{\cancel{t^2}}\right)$

$= at^2 + 2at^2 + 2a$

$\bar{x} = 3at^2 + 2a$

$\bar{y} = 2at + \frac{1+\frac{1}{t^2}}{-1/2at^3}$

$= 2at - \left(\frac{t^2+1}{t^2}\right)(2at^3)$

$= 2at - (2at^3 + 2at)$

$$= \cancel{2at} - 2at^3 - \cancel{2at}$$

$$\bar{y} = -2at^3$$

Centre of curvature $(\bar{x}, \bar{y}) = (3at^2 + 2a, -2at^3)$.

2) Find the coordinates of centre of curvature at $(3, 27)$ on the curve $y = x^3$. Also find its circle of curvature.

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$\left.\frac{dy}{dx}\right|_{(3,27)} = 3(3)^2 = 27$$

$$y_1 = 27$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(3x^2) = 6x$$

$$y_2(3, 27) = 6(3) = 18$$

$$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)$$

$$= 3 - \frac{27}{18}(1 + 27^2)$$

$$= 3 - \frac{3}{2}(1 + 729)$$

$$= 3 - 1095$$

$$\bar{x} = -1092$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= 27 + \frac{1+729}{18}.$$

$$= \frac{486+730}{18} = \frac{1216}{18}$$

$$\bar{y} = \frac{608}{9}$$

Centre of curvature $(\bar{x}, \bar{y}) = \left(-1092, \frac{608}{9}\right)$

Circle of curvature $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$.

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+729)^{3/2}}{18}.$$

$$= \frac{(730)^{3/2}}{18}.$$

$$\rho^2 = \frac{(730)^3}{18^2}$$

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$(x+1092)^2 + \left(y - \frac{608}{9}\right)^2 = \frac{(730)^3}{324}$$

3) Find the centre of curvature

i) at 't' on $xy = c^2$

ii) at θ on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$xy = c^2$$

$x = ct$ $\quad\quad$ $y = \frac{c}{t}$

$\frac{dx}{dt} = c$ $\quad\quad$ $\frac{dy}{dt} = \frac{-c}{t^2}$

$$\frac{dy}{dx} = \frac{-c}{ct^2} = \frac{-1}{t^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{-1}{t^2}\right)\frac{dt}{dx}$$

$$= \frac{2}{t^3} \times \frac{1}{c} = \frac{2}{ct^3}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= ct - \frac{(-1/t^2)}{2/ct^3}\left(1+\frac{1}{t^4}\right)$$

$$= ct + \frac{ct}{2}\left(\frac{t^4+1}{t^4}\right)$$

$$= ct + \frac{c}{2t^3}(t^4+1)$$

$$= \frac{c}{2}\left(3t + \frac{1}{t^3}\right)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= \frac{c}{t} + \frac{\left(1+\frac{1}{t^4}\right)}{2/ct^3}$$

$$= \frac{c}{t} + \frac{t^4+1}{2t^4} \times ct^3$$

$$= \frac{c}{t} + c\left(\frac{t^4+1}{2t}\right)$$

$$\bar{y} = \frac{2c+ct^4+c}{2t}$$

$$\bar{y} = \frac{3c + ct^4}{2t}$$

$$= \frac{c}{2}\left[\frac{3}{t} + t^3\right]$$

Centre of curvature

$$(\bar{x}, \bar{y}) = \left(\frac{c}{2}\left(3t + \frac{1}{t^3}\right), \frac{c}{2}\left(\frac{3}{t} + t^3\right)\right).$$

ii) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$x = a\cos\theta \quad , \quad y = b\sin\theta.$$

$$\frac{dx}{d\theta} = -a\sin\theta \, , \quad \frac{dy}{d\theta} = b\cos\theta.$$

$$\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta.$$

$$\frac{d^2y}{dx^2} = \frac{b}{a}\operatorname{cosec}^2\theta \times \frac{-1}{a\sin\theta} = \frac{-b}{a^2 \cancel{co}\sin^3\theta}$$

$$\bar{x} = a\cos\theta - \frac{-b/a \cot\theta}{\frac{-b}{a^2}\operatorname{cosec}^3\theta}\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)$$

$$= a\cos\theta - \frac{a\cot\theta}{\operatorname{cosec}^3\theta}\left(\frac{a^2 + b^2\cot^2\theta}{a^2}\right)$$

$$= a\cos\theta - a\sin^2\theta\cos\theta\left(\frac{a^2 + b^2\cot^2\theta}{a^2}\right)$$

$$= a\cos\theta - a\sin^2\theta\cos\theta\left(\frac{a^2 + b^2\frac{\cos^2\theta}{\sin^2\theta}}{a^2}\right)$$

$$= a\cos\theta - a\sin^2\theta\cos\theta\left(1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta$$

$$\bar{x} = a\cos\theta\,(1-\sin^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$\bar{x} = \left(\frac{a^2-b^2}{a}\right)\cos^3\theta$$

$$\bar{y} = b\sin\theta + \frac{1+\frac{b^2}{a^2}\cot^2\theta}{\frac{-b}{a^2}\operatorname{cosec}^3\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta\left(1+\frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta\cos^2\theta$$

$$= b\sin\theta\,(1-\cos^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin^3\theta - \frac{a^2}{b}\sin^3\theta = \frac{b^2-a^2}{b}\sin^3\theta$$

4) Show that the coordinates of centre of curvature at the point $(2,4)$ on the parabola $y^2 = 4(x+2)$ are $(12,-16)$.

$$y^2 = 4(x+2)$$

$$2y\cdot y_1 = 4(1)$$

$$y_1 = \frac{2}{y}$$

$$y_1(2,4) = \frac{2}{4} = \frac{1}{2}$$

$$y_2 = \frac{d}{dy}\left(\frac{2}{y}\right)$$

$$= \frac{-2}{y^2} y_1$$

$$y_2(2,4) = \frac{-2}{16}\left(\frac{1}{2}\right) = \frac{-1}{16}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= 2 - \frac{1/2}{-1/16}\left(1+\frac{1}{4}\right)$$

$$= 2 + 8\left(\frac{5}{4}\right)$$

$$\bar{x} = 2 + 2(5) = 12.$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2}$$

$$= 4 + \frac{\left(1+\frac{1}{4}\right)}{-1/16} = 4 + \frac{5}{4}(-16)$$

$$= 4 + 5(-4) = -20 + 4$$

$$\bar{y} = -16.$$

$$(\bar{x}, \bar{y}) = (12, -16).$$

Centre of curvature $(\bar{x}, \bar{y}) = (12, -16)$.

Radius of Curvature - Problems

1. Find the radius of curvature at any general point.

i) $x^2 - y^2 = a^2$

ii) $x^{2/3} + y^{2/3} = a^{2/3}$.

i) $x^2 - y^2 = a^2$.

diff w.r.t to 'x'

$$2x - 2yy_1 = 0$$

$$y_1 = \frac{2x}{2y} = \frac{x}{y}$$

$$y_2 = \frac{(y - xy_1)}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2}$$

$$= \frac{y^2 - x^2}{y^2} = -\frac{a^2}{y^2}$$

$$y_2 = \frac{-a^2}{y^2}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{1 + \frac{x^2}{y^2}}{a^2/y^2} = \frac{(y^2+x^2)^{3/2}}{a^2 y^3} \times y^2$$

$$\rho = \frac{(x^2+y^2)^{3/2}}{a^2 y}.$$

ii) $x^{2/3} + y^{2/3} = a^{2/3}$.

$$x = a\cos^3\theta \quad , \quad y = a\sin^3\theta$$

diff w.r.t 'θ'

$$\frac{dx}{d\theta} = 3a\cos^2\theta\sin\theta \, , \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}$$

$$= -\sec^2\theta\left(\frac{-1}{3a\cos^2\theta\sin\theta}\right)$$

$$= \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = (1+\tan^2\theta)^{3/2}\, 3a\cos^4\theta\sin\theta$$

$$= \sec^3\theta \times 3a\cos^4\theta\sin\theta = 3a\cos\theta\sin\theta$$

$$\rho = 3a\left(\frac{x}{a}\right)^{1/3}\left(\frac{y}{a}\right)^{1/3} = 3(axy)^{1/3}$$

$$\rho = 3(axy)^{1/3}.$$

2) If ρ be the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S be its focus, then show that ρ^2 varies as $(SP)^3$.

$$x = at^2 \qquad y = 2at$$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx} = \frac{-1}{t^2}\,\frac{1}{2at} = \frac{-1}{2at^3}$$

$$\rho = \frac{\left(1+\frac{1}{t^2}\right)^{3/2}}{\left(\frac{-1}{2at^3}\right)}$$

$$= -\left(\frac{t^2+1}{t^2}\right)^{3/2} 2at^3$$

$$= -(t^2+1)^{3/2} 2a$$

$$\rho = -2a(t^2+1)^{3/2}$$

$S(a, 0)$, $P(at^2, 2at)$

$$SP = \sqrt{(2at-0)^2+(at^2-a)^2}$$

$$SP = \sqrt{4a^2t^2+a^2(1-t^2)^2}$$

$$= a\sqrt{1+t^4-2t^2+4t^2}$$

$$= a\sqrt{1+t^4+2t^2}$$

$$SP = a\sqrt{(1+t^2)^2}$$

$$= a(1+t^2)$$

$$SP^3 = a^3(1+t^2)^3 \rightarrow ①$$

$$\rho^2 = 4a^2(1+t^2)^3 \rightarrow ②$$

from ①

$$(1+t^2)^3 = \frac{SP^3}{a^3}$$

$$\rho^2 = 4a^2\frac{SP^3}{a^3}$$

$$\rho^2 = 4a^2 \cdot \frac{SP^3}{a^3}$$

$$\boxed{\rho^2 = \frac{4}{a} SP^3}$$

Hence the ρ^2 is proportional to SP^3.

3. Prove that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\rho = \frac{a^2 b^2}{p^3}$, where p being the perpendicular from the centre on the tangent at (x, y).

The ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

diff w.r.t 'x'

$$\frac{2x}{a^2} + \frac{2y y_1}{b^2} = 0$$

$$y_1 = \frac{-2xb^2}{a^2 2y} = \frac{-b^2}{a^2} \frac{x}{y}$$

$$y_2 = \frac{-b^2}{a^2}\left(\frac{y - xy_1}{y^2}\right) = \frac{-b^2}{a^2}\left(\frac{y - x\left(\frac{-b^2}{a^2}\frac{x}{y}\right)}{y^2}\right)$$

$$= \frac{-b^2}{a^2}\left(\frac{a^2y^2 + b^2x^2}{a^2y^3}\right)$$

$$= \frac{-b^2}{a^2y^3}\left(y^2 + \frac{b^2}{a^2}x^2\right) = \frac{-b^4}{a^2y^3}\left[\frac{y^2}{b^2} + \frac{x^2}{a^2}\right]$$

$$y_2 = \frac{-b^4}{a^2y^3} \quad \left[\because \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right]$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{b^4}{a^4}\frac{x^2}{y^2}\right)^{3/2}}{\left|\frac{-b^4}{a^2y^3}\right|}$$

$$= \left(\frac{a^4y^2+b^4x^2}{a^4y^2}\right)^{3/2} \times \frac{a^2y^3}{b^4}$$

$$= \frac{(a^4y^2+b^4x^2)^{3/2}}{a^6y^3b^4} \times a^2y^3$$

$$\rho = \frac{(a^4y^2+b^4x^2)^{3/2}}{a^4b^4}.$$

The tangent at (x,y) to the ellipse with perpendicular distance

$$p = \frac{1}{\sqrt{\left(\frac{x}{a^2}\right)^2+\left(\frac{y}{b^2}\right)^2}} = \frac{a^2b^2}{\sqrt{a^4y^2+b^4x^2}}$$

$$\frac{1}{p^3} = \frac{(a^4y^2+b^4x^2)^{3/2}}{a^6b^6} = \frac{(a^4y^2+b^4x^2)^{3/2}}{a^4b^4} \times \frac{1}{a^2b^2}$$

$$\frac{1}{p^3} = \frac{\rho}{a^2b^2}$$

$$\boxed{\rho = \frac{a^2b^2}{p^3}.}$$

4) If ρ_1 and ρ_2 are the radii of curvature at extremities of a focal chord of a parabola whose latus rectum is $2l$, show that $\rho_1^{-2/3} + \rho_2^{-2/3} = l^{-2/3}$

$$y^2 = 4ax$$

$$x = at^2, \quad y = 2at$$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx} = \frac{-1}{t^2}\cdot\frac{1}{2at} = \frac{-1}{2at^3}$$

$$\rho = \frac{\left(1+\frac{1}{t^2}\right)^{3/2}}{\frac{-1}{2at^3}} = -\left(\frac{t^2+1}{t^2}\right)^{3/2} 2at^3$$

$$= -(t^2+1)^{3/2}\, 2a$$

$$\rho = -2a(t^2+1)^{3/2}$$

$$\rho_1 = 2a(1+t_1^2)^{3/2}$$

$$\rho_2 = 2a(1+t_2^2)^{3/2}$$

$$(\rho_1)^{-2/3} = \frac{(2a)^{-2/3}}{1+t_1^2}, \quad (\rho_2)^{-2/3} = \frac{(2a)^{-2/3}}{1+t_2^2}$$

$$t_1 t_2 = -1$$

$$t_2 = \frac{-1}{t_1}$$

$$(\rho_2)^{-2/3} = (2a)^{-2/3}\left(1+\frac{1}{t_1^2}\right)^{-1}$$

$$= (2a)^{-2/3}\left(\frac{1+t_1^2}{t_1^2}\right)^{-1}$$

$$= (2a)^{-2/3}\left(\frac{t_1^2}{1+t_1^2}\right)$$

$$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (2a)^{-2/3}\left[\frac{1}{1+t_1^2}\right] + (2a)^{-2/3}\left(\frac{t_1^2}{1+t_1^2}\right)$$

$$= (2a)^{-2/3}\left[\cancel{\frac{1}{1+t_1^2} + \frac{t_1^2}{1+t_1^2}}\right]$$

$$= (2a)^{-2/3}$$

$$4a = 2l$$

$$a = l/2$$

$$l = 2a.$$

$$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (l)^{-2/3}.$$

5) If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1+\cos\theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$

The parametric form of a cardioid is

$r_1 = a(1+\cos\theta)$ $\quad r_2 = a(1-\cos\theta)$

$r_1' = -a\sin\theta$ $\quad r_2' = a\sin\theta$

$r_1'' = -a\cos\theta$ $\quad r_2'' = a\cos\theta$

$$\rho_1 = \frac{(r_1^2 + (r_1')^2)^{3/2}}{r_1^2 - r_1(r_1'') + 2(r_1')^2}$$

$$= \frac{(a^2(1+\cos\theta)^2 + (-a\sin\theta)^2)^{3/2}}{a^2(1+\cos\theta)^2 - a(1+\cos\theta)(-a\cos\theta) + 2a^2\sin^2\theta}$$

$$\rho_1 = \frac{(2a^2 + 2a^2\cos\theta)^{3/2}}{3a^2 + 3a^2\cos\theta}$$

$$\rho_1 = \frac{(2a^2)^{3/2}(1+\cos\theta)^{3/2}}{3a^2(1+\cos\theta)}$$

$$\rho_1 = \frac{2\sqrt{2}a^3}{3a^2}(1+\cos\theta)^{1/2}$$

$$\rho_1 = \frac{2\sqrt{2}}{3}a\left(\sqrt{2}\cos\frac{\theta}{2}\right)$$

$$\rho_1 = \frac{4}{3}a\cos\frac{\theta}{2}$$

$$\rho_1^2 = \frac{16}{9}a^2\cos^2\frac{\theta}{2}$$

$$\rho_2 = \frac{(r_2^2 + (r_2')^2)^{3/2}}{r_2^2 - r_2(r_2'') + 2(r_2')^2}$$

$$= \frac{\left(a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right)^{3/2}}{a^2(1-\cos\theta)^2 - a(1-\cos\theta)(a\cos\theta) + 2(a^2\sin^2\theta)}$$

$$\rho_2 = \frac{(2a^2 - 2a^2\cos\theta)^{3/2}}{3a^2 - 3a^2\cos\theta}$$

$$\rho_2 = \frac{(2a^2)^{3/2}(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$$

$$\rho_2 = \frac{2\sqrt{2}a^3}{3a^2}(1-\cos\theta)^{1/2}$$

$$\rho_2 = \frac{2\sqrt{2}a}{3}\left(\sqrt{2}\sin\frac{\theta}{2}\right)$$

$$\rho_2 = \frac{4}{3}a\sin\frac{\theta}{2}$$

$$\rho_2^2 = \frac{16}{9}a^2\sin^2\frac{\theta}{2}$$

$$\rho_1^2 + \rho_2^2 = \frac{16}{9}a^2\cos^2\frac{\theta}{2} + \frac{16}{9}a^2\sin^2\frac{\theta}{2}$$

$$= \frac{16}{9}a^2\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)$$

$$= \frac{16}{9}a^2(1)$$

$$\therefore \rho_1^2 + \rho_2^2 = \frac{16a^2}{9}.$$