Assignment 5

CPSC 302 - 2022W1

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Problem 1

Problem 1

Consider the 2×2 matrix

$$A = \left(\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array}\right) ,$$

and suppose we are required to solve $A\vec{x} = \vec{b}$, where \vec{b} is an arbitrary right-hand side vector. Clearly, solving a 2×2 linear system using a stationary scheme is an utterly ridiculous idea, but the following is helpful in understanding more about convergence of stationary methods.

- (a) Find the spectral radius of the Jacobi and Gauss-Seidel iteration matrices and the asymptotic rate of convergence for these two schemes, namely $-\log_{10}\rho(T)$, where $\rho(T)$ denotes the spectral radius of the corresponding iteration matrix, T.
- (b) How much faster does Gauss-Seidel converge compared to Jacobi for a fixed reduction in the relative residual norm, $\frac{\|\vec{r}_k\|_2}{\|\vec{b}\|_2}$, in terms of iteration counts?
- (c) Write down the SOR iteration matrix as a function of the relaxation parameter, ω .
- (d) Find the optimal SOR parameter, $\omega_{\rm opt}$, and the spectral radius of the corresponding iteration matrix.
- (e) Approximately how much faster does SOR with $\omega_{\rm opt}$ converge compared to Jacobi?

Solution (a).

Given A let us first calculate the Jacobi iteration matrix of A, denoted D. Since D all zeros along the non-diagonal entries and shares the same diagonal entries as A then we conclude that D is the following. We also indicate its inverse below.

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \qquad D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Since we have a computed value for D^{-1} we can now calculate the iteration matrix T_{Jacobi} .

$$T_{\text{Jacobi}} = I - D^{-1}A$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

Now let us calculate the eigenvalues of T_{Jacobi} to obtain the spectral radius.

$$\det (T_{\text{Jacobi}} - I\lambda) = 0$$

$$\begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \frac{1}{4} = 0$$

$$\lambda = \pm \frac{1}{2}$$

We know that the largest absolute eigenvalue is $\frac{1}{2}$.

$$\rho\left(D^{-1}\right) = \lambda_1 = \frac{1}{2}$$

Now that we have a value for $\rho(D^{-1})$, we can calculate the asymptotic rate of convergence for the Jacobi iteration matrix of A to be the following. Note: The following results were calculated using Matlab.

$$-\log_{10} \rho\left(D^{-1}\right) = -\log_{10}\left(\frac{1}{2}\right) \approx \mathbf{0.3010}$$

We have now calculated the asymptotic rate of convergence for the Jacobi iteration matrix of A. Now let us calculate the asymptotic rate of convergence and the spectral radius for the Gauss-Seidel iteration matrix of A.

$$E = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \qquad E^{-1} = \frac{1}{2 \cdot 2 - 0 \cdot (-1)} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix}$$

Since we now have a value for E^{-1} we can now calculate the iteration matrix.

$$T_{GS} = I - E^{-1}A$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1/2 \\ 0 & 3/4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/2 \\ 0 & 1/4 \end{pmatrix}$$

Now that we have a value for $T_{\rm GS}$ let us calculate its eigenvalues.

$$\det (T_{\text{GS}} - I\lambda) = 0$$

$$\begin{vmatrix} -\lambda & \frac{1}{2} \\ 0 & 1/4 - \lambda \end{vmatrix} = 0$$

$$-\lambda \cdot \left(\frac{1}{4} - \lambda\right) - \frac{1}{2} \cdot 0 = 0$$

$$\lambda = 0, \frac{1}{4}$$

We know that the largest absolute eigenvalue is $\frac{1}{4}$.

$$\rho\left(E^{-1}\right) = \lambda_1 = \frac{1}{4}$$

Now let us calculate the asymptotic rate of convergence.

$$-\log_{10} \rho\left(E^{-1}\right) = -\log_{10}\left(\frac{1}{4}\right) \approx \mathbf{0.6021}$$

Solution (b).

Jacobi will take twice as many iterations as Gauss-Seidel since the asymptotic rate of Jacobi is roughly double that of Gauss-Seidel, thus we can say that Gauss-Seidel will converge at twice the rate that Jacobi will converge.

Solution (c).

We know that the following given an SOR iteration.

$$M = \frac{1 - \omega}{\omega} D + E$$

Let us substitute M into the function T_{SOR} where $T_{SOR}: \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$.

$$T_{\text{SOR}}(\omega) = I - M^{-1}A$$

= $I - \left(\frac{1 - \omega}{\omega}D + E\right)^{-1}A$

Thus we have the following function.

$$T_{\text{SOR}} = I - \left(\frac{1-\omega}{\omega}D + E\right)^{-1}A$$

Solution (d).

Because we already have a value for ρ_{Jacobi} , let us substitute it into the ω_{opt} equation from the textbook.

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - (\rho_J)^2}}$$
$$= \frac{2}{1 + \sqrt{1 - (\frac{1}{2})^2}}$$
$$\approx 1.0718$$

Thus we obtain $\omega_{opt} \approx 1.0718$.

Solution (e).

From the textbook we know that $\rho_{SOR} = \omega_{opt} - 1$. Thus we know that $\rho_{SOR} = 0.0718$. Let us now calculate an asymptotic rate of convergence.

$$-\log_{10}\left(
ho_{\mathrm{SOR}}\right) pprox \mathbf{1.1439}$$

Let us note that the asymptotic rate of convergence for SOR is around 4 times that of Jacobi's. Thus we can say that SOR uses 4 times fewer iterations than Jacobi when $\omega = \omega_{opt}$ is used for SOR. We can conclude that SOR converges 4 times faster than Jacobi.

Problem 2

(a) Download from the assignment webpage the file J_GS.m. In your MATLAB command line, run the command: J_GS(100); Include in your assignment solution the convergence graph that you are seeing. This is a linear system with the two-dimensional Laplacian of size 10,000×10,000

This is a linear system with the two-dimensional Laplacian of size $10,000 \times 10,000$ (we have $n = 100, n^2 = 10,000$), and we are plotting the norm of the relative residual after 10,000 iterations for Jacobi and Gauss-Seidel. There is no reason to be impressed with this graph; convergence here is slow. But we are going to see the effect of using SOR.

(b) Given the eigenvalues of the Laplacian in the slides, show that the optimal SOR parameter can be expressed as

$$\omega_{\text{opt}} = \frac{2}{1 + \sin\left(\frac{\pi}{n+1}\right)}.$$

(c) Modify the MATLAB function so that in addition to the Jacobi and Gauss-Seidel graphs (for the same matrix) a convergence plot for the SOR method is included as well. Use the same initial guess (the zero vector) and the same stopping criterion: $\frac{\|\vec{r}_k\|_2}{\|\vec{b}\|_2} < 10^{-6}$.

 $\overline{\text{Tip}}$: You should be seeing an *extremely dramatic* improvement in convergence. If you are not seeing such an improvement, then you must have done something wrong.

(d) Explain your results.

Solution (a).

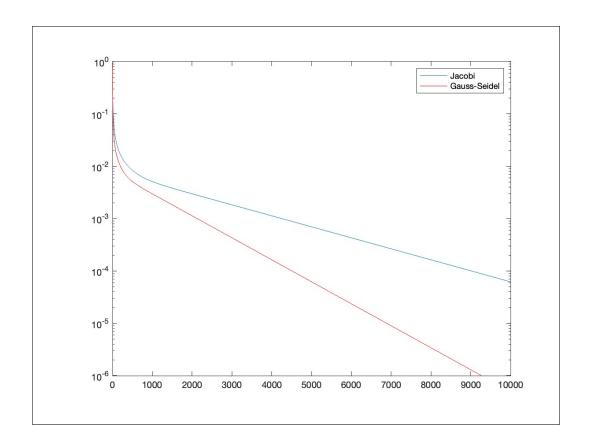
Note: The line

```
saveas(gcf, "DevamSisodraker_2a.jpg", "jpg");
```

was appended to the end of J_GS.m to ease the exporting process.

```
File: J_GS.m
function J_GS(n)
close all
\% Apply a stationary method to a linear system involving the n^2-by-n^2
% Laplacian
A=delsq(numgrid('S',n+2));
E=tril(A); % M=E for Gauss-Seidel itermax=10000; % maximum number of iterations
resvecJ=zeros(itermax,1); % allocate initial space for Jacobi residual norm vector
resvecGS=zeros(itermax,1); % allocate initial space for Gauss-Seidel residual vector
% Jacobi
xJ=zeros(n^2,1);
b=A*ones(n^2,1); % generate a solution of all 1s and a right-hand side
nb=norm(b);
r=b;
for i=1:10000
   resvecJ(i)=norm(r)/nb; % relative residual norm
   if resvecJ(i)<1e-6, break,end % terminate loop if stopping criterion is satisfied
   xJ=xJ+D\r;
                         % next iterate
   r=b-A*xJ;
                         % update residual
% Gauss-Seidel
xGS=zeros(n^2,1);
r=b;
for i=1:10000
   resvecGS(i)=norm(r)/nb; % relative residual norm
   if resvecGS(i)<1e-6, break,end % terminate loop if stopping criterion is satisfied
   xGS=xGS+E\r;
                         % next iterate
   r=b-A*xGS;
                         % update residual
end
hold on
semilogy(resvecGS, 'r');
                         % plot relative residual norm for Gauss-Seidel
legend('Jacobi','Gauss-Seidel')
saveas(gcf, "DevamSisodraker_2a.jpg", "jpg");
```

```
File: DevamSisodraker_2a_out.txt
>> J_GS(100)
ans =
 Figure (1) with properties:
     Number: 1
       Name: ''
      Color: [0.9400 0.9400 0.9400]
   Position: [584 595 560 420]
      Units: 'pixels'
 Show all properties
```



Solution (b).

The following is given in the lecture slides.

$$\lambda_{i,j} = 4 - 2\left(\cos\frac{i\pi}{n+1} + \cos\frac{j\pi}{n+1}\right)$$

Before we continue, let us first recall that the spectral radius of T is given by $\mu_{1,1}$. Thus it follows that $\rho(T) = \mu_{i,i} = \cos \frac{\pi}{n+1}$. Then let us substitute this $\rho(T)$ into the equation which defines ω_{opt} .

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - (\rho(T))^2}}$$

$$= \frac{2}{1 + \sqrt{1 - (\cos \frac{\pi}{n+1})^2}}$$
substitution of $\rho(T)$

$$= \frac{2}{1 + \sqrt{\left(\sin \frac{\pi}{n+1}\right)^2}}$$
by Pythagorean trigonometric identity
$$= \frac{2}{1 + \sin \frac{\pi}{n+1}}$$

Solution (c).

```
File: DevamSisodraker_2c.m
function J GS(n)
close all
% Apply a stationary method to a linear system involving the n^2-by-n^2
% Laplacian
A=delsq(numgrid('S',n+2));
D=diag(diag(A)); % M=D for Jacovi
E=tril(A); % M=E for Gauss-Seidel
                                  % maximum number of iterations
itermax=10000;
resvecJ=zeros(itermax,1); % allocate initial space for Jacobi residual norm vector
resvecGS=zeros(itermax,1); % allocate initial space for Gauss-Seidel residual vector
{\tt resvecSOR=zeros(itermax,1);~\%~allocate~initial~space~for~SOR~residual~vector}
% Jacobi
xJ=zeros(n^2,1);
b=A*ones(n^2,1); % generate a solution of all 1s and a right-hand side
nb=norm(b);
r=b;
for i=1:10000
       resvecJ(i)=norm(r)/nb; % relative residual norm
        if resvecJ(i)<1e-6, break,end % terminate loop if stopping criterion is satisfied
                                            % next iterate
                                                          % update residual
semilogy(resvecJ);
                                                       % plot relative residual norm for Jacobi
xGS=zeros(n^2,1);
r=b;
for i=1:10000
        resvecGS(i)=norm(r)/nb; % relative residual norm
        if resvecGS(i)<1e-6, break, end % terminate loop if stopping criterion is satisfied
        xGS=xGS+E\r; % next iterate
       r=b-A*xGS;
                                                         % update residual
end
hold on
% SOR
xSOR=zeros(n^2,1);
r=b;
for i=1:10000
        wOpt = 2 / (1 + sin(pi / (n + 1)));
        resvecSOR(i)=norm(r)/nb; % relative residual norm
        resvecSoR(i)-10-10, % / **constant ** constant ** cons
        xSOR=xSOR+((1 - wOpt)*D + wOpt*E)\r;
        r=b-A*xSOR;
                                                            % update residual
hold on
legend('Jacobi','Gauss-Seidel','SOR')
saveas(gcf, "DevamSisodraker_2c.jpg", "jpg");
```

```
File: DevamSisodraker_2c_out.txt

>> DevamSisodraker_2c(100)

ans =

Figure (1) with properties:

Number: 1
Name: ''
Color: [0.9400 0.9400 0.9400]
Position: [584 595 560 420]
Units: 'pixels'

Show all properties

>>
```

Solution (d).

write some bs explanation after you see a, b,

Problem 3

Problem 3

- (a) Suppose A is an n-by-n orthogonal matrix. Show that all its singular values are equal to 1.
- (b) Recall that an orthogonal projector is a symmetric matrix P for which $P^2 = P$. What are the eigenvalues of an orthogonal projector?

Solution (a).

Let us suppose that A is an $n \times n$ orthogonal matrix. This means that $A^TA = I$. Because the singular values of A are simply the square roots of the eigenvalues of A^TA then the singular values of A are equal to the eigenvalues of I. Since all eigenvalues of I are 1 and since $\sqrt{1} = 1$ then it holds that all $n \times n$ orthogonal matrices have all singular values equal to 1.

Solution (b).

The eigenvalues of P are represented by λ in the equation $P\vec{v} = \lambda \vec{v}$. Since $P = P^2$ then it holds that $\lambda^2 \vec{v} = P^2 \vec{v} = P \vec{v} = \lambda \vec{v}$. We can manipulate it to obtain the following.

$$P^{2}\vec{v} = P\vec{v}$$
$$\lambda^{2}\vec{v} = \lambda\vec{v}$$
$$\lambda^{2} = \lambda$$
$$\lambda^{2} - \lambda = 0$$
$$\lambda(1 - \lambda) = 0$$
$$\lambda = 0, 1$$

Thus we conclude that orthogonal projectors can only have values that are either 0 or 1. However, since orthogonal projectors are also symmetric $n \times n$ matrices, then it holds that the only eigenvalue that an orthogonal projector can have is $\lambda = 1$.

Problem 4

Problem 4

Load the following .mat matrix that appears on the assignment page:

load powerMatrix;

If you hit whos you should be seeing a matrix called A, of size 100×100 . To validate any of your results below, you may run the MATLAB command eig, as long as you understand that in typical eigenvalue computations (in a potentially more challenging computational environment) we generally do not have the luxury of running eig to check ourselves.

(a) Apply the power method. Terminate the iteration once the iterates satisfy

$$\lambda_1^{(k)} - \lambda_1^{(k-1)} | < 10^{-4}.$$

Your program should print out the value of the final iterate and a graph of the absolute errors: $|\lambda_1^{(k)} - \lambda_{\max}|$. For better visualization, use semilogy for your graphs when necessary. As an initial guess for the eigenvector use a vector produced by the MATLAB command randn. (When you repeat your experiments, the number of iterations may slightly vary due to the random initial guess.)

- (b) Repeat your computations with the *inverse* power iteration, with a shift $\alpha = 4$, and produce the same graph as you did for the power method.
- (c) Discuss the differences between the performance of the power and the inverse power methods in terms of the cost of single iterations and t overall computational cost.
- (d) Suppose now that we know that A has an eigenvalue close to 3 and we are interested to compute it to six correct decimal digits. Suggest an efficient procedure for doing so. Implement your suggested algorithm and compute the eigenvalue.

Solution (a).

```
File: DevamSisodraker_4a.m
gcf
hold on;
clear;
load('powerMatrix.mat')
k = 0;
vector_k = randn(100, 1);
lambda_k = transpose(vector_k)*A*vector_k;
lambda_k_1 = transpose(vector_k)*A*vector_k;
lambda_audit = [];
lambda_delta_audit = [];
eigval = max(eig(A));
while true
   % statements here
    \mbox{\it \%} if \mbox{\it `WhileCondition, break} ; end
    lambda_k_1 = lambda_k;
    vector_k = A*vector_k;
    vector_k = vector_k/norm(vector_k);
    lambda_k = transpose(vector_k)*A*vector_k;
    lambda_audit = [lambda_audit, lambda_k];
    lambda_delta_audit = [lambda_delta_audit, abs(lambda_k - eigval)];
    if abs(lambda_k_1 - lambda_k) < 10^-4
        break;
    end
end
plot(1:1:k, lambda_audit);
plot(1:1:k, lambda_delta_audit);
plot([1, k], [eigval, eigval]);
title("4a");
legend({ ...
    '\lambda_k', ...
    '| \lambda_k - \lambda_{MAX} |', ...
    '\lambda_{MAX}', ...
});
xlabel("k");
ylabel("Value");
hold off;
saveas(gcf, "DevamSisodraker_4a.jpg", "jpg");
```

```
File: DevamSisodraker_4a_out.txt

>> DevamSisodraker_q4a

ans =

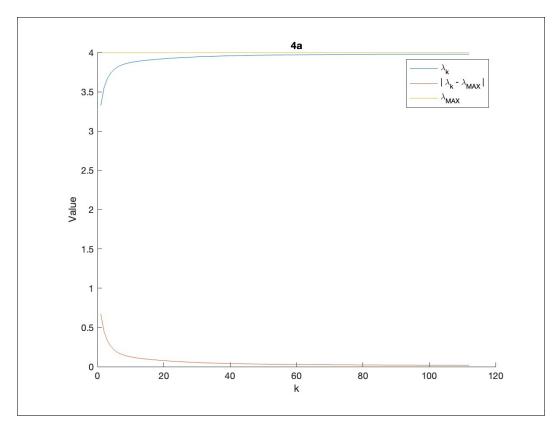
Figure (1) with properties:

Number: 1
Name: ''
Color: [0.9400 0.9400 0.9400]
Position: [584 595 560 420]
Units: 'pixels'

Show all properties

lambda_k =

3.9804
```



Solution (b).

```
File: DevamSisodraker_4b.m
gcf
hold on;
load('powerMatrix.mat');
k = 0;
vector_k = randn(100, 1);
vector_k_1 = vector_k;
lambda_k = transpose(vector_k)*A*vector_k;
lambda_k_1 = transpose(vector_k)*A*vector_k;
lambda_audit = [];
lambda_delta_audit = [];
eigval = max(eig(A));
alpha = 4;
while true
    % statements here
    % if ~WhileCondition, break; end
   lambda_k_1 = lambda_k;
    vector_k_1 = vector_k;
```

```
vector_k = (A - alpha * eye(size(A)))/transpose(vector_k_1);
    vector_k = vector_k/norm(vector_k);
    lambda_k = transpose(vector_k)*A*vector_k;
    lambda_audit = [lambda_audit, lambda_k];
    lambda_delta_audit = [lambda_delta_audit, abs(lambda_k - eigval)];
    k = k + 1;
    if abs(lambda_k_1 - lambda_k) < 10^-4
        break;
    \quad \text{end} \quad
end
plot(1:1:k, lambda_audit);
plot(1:1:k, lambda_delta_audit);
plot([1, k], [eigval, eigval]);
title("4b");
legend({ ...
    '\lambda_k', ...
'| \lambda_k - \lambda_{MAX} |', ...
    '\lambda_{MAX}', ...
});
xlabel("k");
ylabel("Value");
hold off;
saveas(gcf, "DevamSisodraker_4b.jpg", "jpg");
lambda_k
```

```
File: DevamSisodraker_4b_out.txt

>> DevamSisodraker_4b

ans =

Figure (1) with properties:

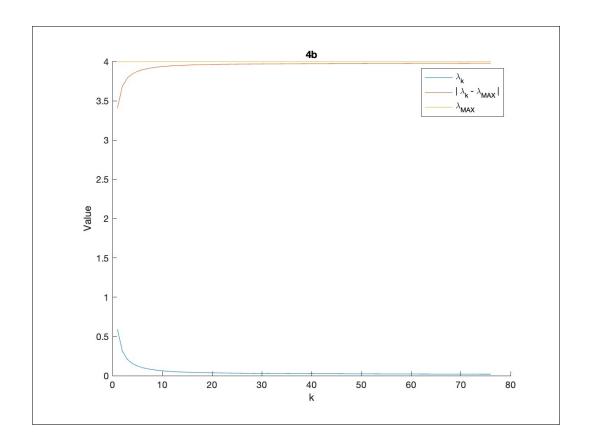
Number: 1
Name: ''
Color: [0.9400 0.9400 0.9400]
Position: [584 595 560 420]
Units: 'pixels'

Show all properties

lambda_k =

0.0202

>>
```



Solution (c).

bullshit at a later time

Solution (d). We will use the inverse power method with $\alpha=3$ to find the eigenvalue close to 3. continued on next page...

```
File: DevamSisodraker_4d.m
gcf
hold on;
clear;
load('powerMatrix.mat');
k = 0;
vector_k = randn(100, 1);
vector_k_1 = vector_k;
lambda_k = transpose(vector_k)*A*vector_k;
lambda_k_1 = transpose(vector_k)*A*vector_k;
lambda_audit = [];
alpha = 3;
while true
    % statements here
    % if ~WhileCondition, break; end
    lambda_k_1 = lambda_k;
    vector_k_1 = vector_k;
    vector_k = (A - alpha * eye(size(A)))\vector_k_1;
    vector_k = vector_k/norm(vector_k);
    lambda_k = transpose(vector_k)*A*vector_k;
    lambda_audit = [lambda_audit, lambda_k];
    k = k + 1;
    if abs(lambda_k_1 - lambda_k) < 10^-4
        break:
end
plot(1:1:k, lambda_audit);
title("4d");
legend({ ...
    '\lambda_k', ...
xlabel("k");
ylabel("Value");
hold off;
saveas(gcf, "DevamSisodraker_4d.jpg", "jpg");
lambda_k
```

```
File: DevamSisodraker_4d_out.txt

>> DevamSisodraker_4d

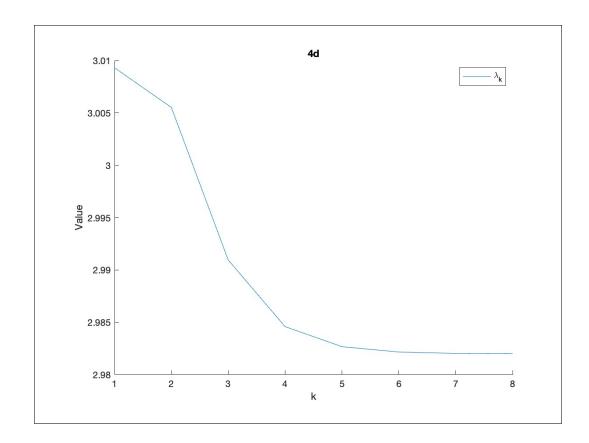
ans =

Figure (1) with properties:

    Number: 1
        Name: ''
        Color: [0.9400 0.9400 0.9400]
    Position: [584 595 560 420]
        Units: 'pixels'

Show all properties

lambda_k =
    2.9820
>>
```

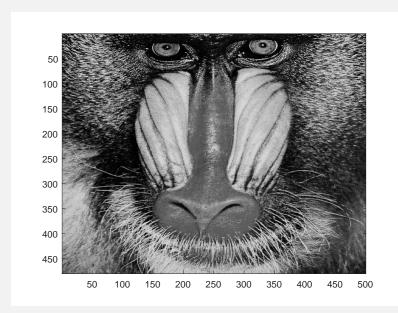


Problem 5

Problem 5

The following picture appears in MATLAB'S repository of images, and can be retrieved by entering

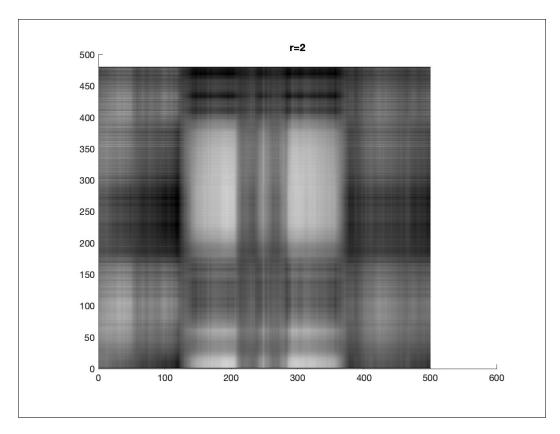
```
load mandrill;
colormap('gray');
image(X);
```

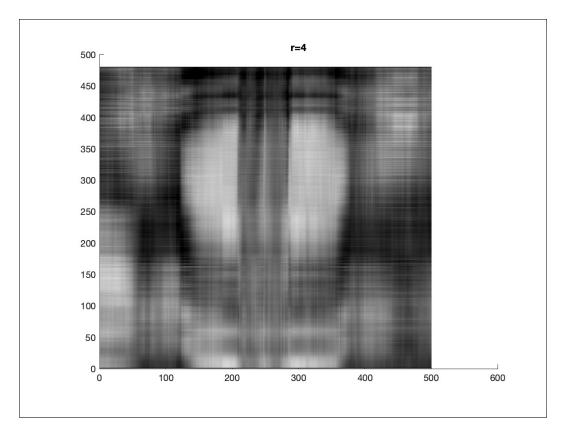


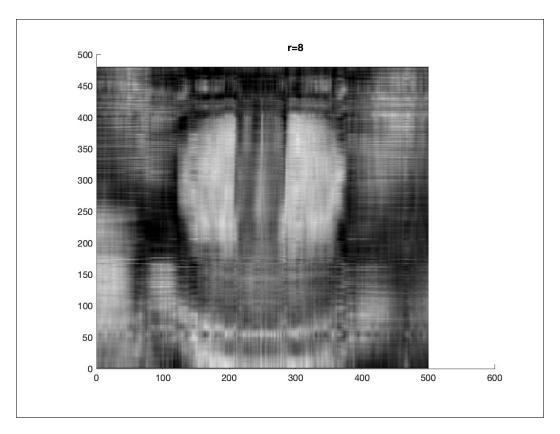
- (a) Print out the images generated by the truncated SVD. Start with r=2 and go up by powers of 2, to $r=2^6=64$ (six plots in total). For a compact presentation of your figures, you may use the command subplot(3,2). (Check out help subplot.)
- (b) Comment briefly on the quality of the images as a function of r. For what value of r would you say that the quality of the image is acceptable, in that we can be rather confident of what we are seeing? (We are not looking for a specific "correct answer" here just make your subjective observation.)
- (c) For the value of r you stated in part (b), how much storage is required? Compare it to the storage required for the original image.

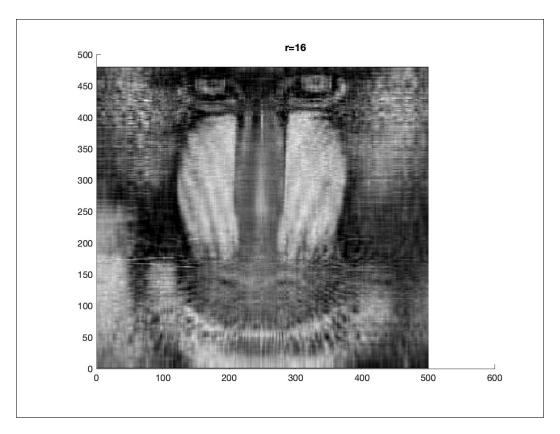
```
File: DevamSisodraker_5a.m
clear;
load mandrill;
colormap("gray");
[U,S,V] = svd(X);
image(U*S*V');
close all;
for i = 1:1:6
   r = 2^i;
   dims = size(X);
    S_trunc = diag(S);
   S_{trunc}((r + 1):min(dims(1), dims(2))) = 0;
   S_trunc = diag(S_trunc);
   S_{trunc(dims(1), dims(2))} = 0;
    gcf
    hold on;
    Xout = U * S_trunc * V';
    colormap("gray");
    image(flipud(Xout));
    title(strcat("r=", string(r)));
    saveas(gcf, strcat("DevamSisodraker_5a_", string(r), ".jpg"), "jpg");
end
```

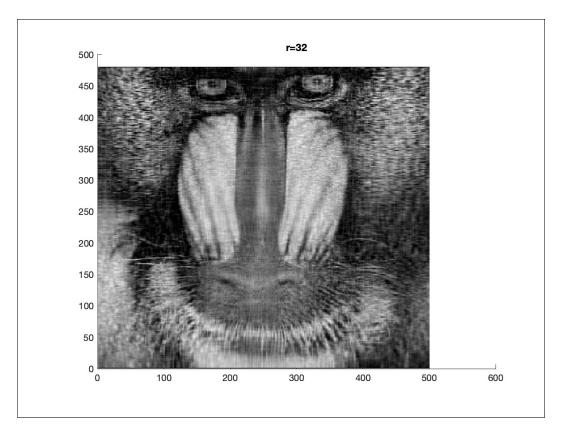
```
File: DevamSisodraker_5a_out.txt
>> DevamSisodraker_5a
ans =
 Figure (1) with properties:
     Number: 1
       Name: ''
      Color: [0.9400 0.9400 0.9400]
   Position: [584 595 560 420]
      Units: 'pixels'
 Show all properties
ans =
 Figure (1) with properties:
      Number: 1
       Name: ''
      Color: [0.9400 0.9400 0.9400]
   Position: [584 587 560 420]
      Units: 'pixels'
 Show all properties
ans =
 Figure (1) with properties:
      Number: 1
       Name: ''
      Color: [0.9400 0.9400 0.9400]
   Position: [584 587 560 420]
      Units: 'pixels'
 Show all properties
 Figure (1) with properties:
     Number: 1
       Name: ''
      Color: [0.9400 0.9400 0.9400]
   Position: [584 587 560 420]
      Units: 'pixels'
 Show all properties
ans =
 Figure (1) with properties:
```

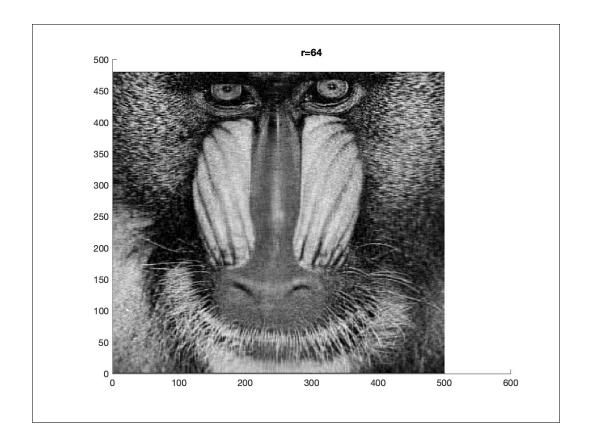












Devam Sisodraker	Assignment 5	CPSC 302 - 2022W1	
Solution (b).		-	bullshit
Solution (c).			bullshit