
Assignment 5

CPSC 302 - 2022W1

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Problem 1

Problem 1

Consider the 2×2 matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and suppose we are required to solve $A\vec{x} = \vec{b}$, where \vec{b} is an arbitrary right-hand side vector. Clearly, solving a 2×2 linear system using a stationary scheme is an utterly ridiculous idea, but the following is helpful in understanding more about convergence of stationary methods.

- Find the spectral radius of the Jacobi and Gauss-Seidel iteration matrices and the asymptotic rate of convergence for these two schemes, namely $-\log_{10} \rho(T)$, where $\rho(T)$ denotes the spectral radius of the corresponding iteration matrix, T .
- How much faster does Gauss-Seidel converge compared to Jacobi for a fixed reduction in the relative residual norm, $\frac{\|\vec{r}_k\|_2}{\|\vec{b}\|_2}$, in terms of iteration counts?
- Write down the SOR iteration matrix as a function of the relaxation parameter, ω .
- Find the optimal SOR parameter, ω_{opt} , and the spectral radius of the corresponding iteration matrix.
- Approximately how much faster does SOR with ω_{opt} converge compared to Jacobi?

Solution (a).

Given A let us first calculate the Jacobi iteration matrix of A , denoted D . Since D all zeros along the non-diagonal entries and shares the same diagonal entries as A then we conclude that D is the following. We also indicate its inverse below.

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Since we have a computed value for D^{-1} we can now calculate the iteration matrix T_{Jacobi} .

$$\begin{aligned} T_{\text{Jacobi}} &= I - D^{-1}A \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \end{aligned}$$

continued on next page...

Now let us calculate the eigenvalues of T_{Jacobi} to obtain the spectral radius.

$$\begin{aligned}\det(T_{\text{Jacobi}} - I\lambda) &= 0 \\ \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} &= 0 \\ \lambda^2 - \frac{1}{4} &= 0 \\ \lambda &= \pm \frac{1}{2}\end{aligned}$$

We know that the largest absolute eigenvalue is $\frac{1}{2}$.

$$\rho(D^{-1}) = \lambda_1 = \frac{1}{2}$$

Now that we have a value for $\rho(D^{-1})$, we can calculate the asymptotic rate of convergence for the Jacobi iteration matrix of A to be the following. *Note: The following results were calculated using Matlab.*

$$-\log_{10} \rho(D^{-1}) = -\log_{10} \left(\frac{1}{2} \right) \approx \mathbf{0.3010}$$

We have now calculated the asymptotic rate of convergence for the Jacobi iteration matrix of A . Now let us calculate the asymptotic rate of convergence and the spectral radius for the Gauss-Seidel iteration matrix of A .

$$E = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \quad E^{-1} = \frac{1}{2 \cdot 2 - 0 \cdot (-1)} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix}$$

Since we now have a value for E^{-1} we can now calculate the iteration matrix.

$$\begin{aligned}T_{\text{GS}} &= I - E^{-1}A \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1/2 \\ 0 & 3/4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1/2 \\ 0 & 1/4 \end{pmatrix}\end{aligned}$$

continued on next page...

Now that we have a value for T_{GS} let us calculate its eigenvalues.

$$\begin{aligned}\det(T_{\text{GS}} - I\lambda) &= 0 \\ \begin{vmatrix} -\lambda & \frac{1}{2} \\ 0 & 1/4 - \lambda \end{vmatrix} &= 0 \\ -\lambda \cdot \left(\frac{1}{4} - \lambda\right) - \frac{1}{2} \cdot 0 &= 0 \\ \lambda &= 0, \frac{1}{4}\end{aligned}$$

We know that the largest absolute eigenvalue is $\frac{1}{4}$.

$$\rho(E^{-1}) = \lambda_1 = \frac{1}{4}$$

Now let us calculate the asymptotic rate of convergence.

$$-\log_{10} \rho(E^{-1}) = -\log_{10} \left(\frac{1}{4}\right) \approx \mathbf{0.6021}$$

Solution (b).

Jacobi will take twice as many iterations as Gauss-Seidel since the asymptotic rate of Jacobi is roughly double that of Gauss-Seidel, thus we can say that Gauss-Seidel will converge at twice the rate that Jacobi will converge.

Solution (c).

We know that the following given an SOR iteration.

$$M = \frac{1-\omega}{\omega}D + E$$

Let us substitute M into the function T_{SOR} where $T_{\text{SOR}} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$.

$$\begin{aligned} T_{\text{SOR}}(\omega) &= I - M^{-1}A \\ &= I - \left(\frac{1-\omega}{\omega}D + E \right)^{-1} A \end{aligned}$$

Thus we have the following function.

$$T_{\text{SOR}} = I - \left(\frac{1-\omega}{\omega}D + E \right)^{-1} A$$

Solution (d).

Because we already have a value for ρ_{Jacobi} , let us substitute it into the ω_{opt} equation from the textbook.

$$\begin{aligned}\omega_{opt} &= \frac{2}{1 + \sqrt{1 - (\rho_J)^2}} \\ &= \frac{2}{1 + \sqrt{1 - \left(\frac{1}{2}\right)^2}} \\ &\approx \mathbf{1.0718}\end{aligned}$$

Thus we obtain $\omega_{opt} \approx \mathbf{1.0718}$.

Solution (e).

From the textbook we know that $\rho_{\text{SOR}} = \omega_{\text{opt}} - 1$. Thus we know that $\rho_{\text{SOR}} = 0.0718$. Let us now calculate an asymptotic rate of convergence.

$$-\log_{10}(\rho_{\text{SOR}}) \approx \mathbf{1.1439}$$

Let us note that the asymptotic rate of convergence for SOR is around 4 times that of Jacobi's. Thus we can say that SOR uses 4 times fewer iterations than Jacobi when $\omega = \omega_{\text{opt}}$ is used for SOR. We can conclude that SOR converges 4 times faster than Jacobi.

Problem 2

Problem 2

- (a) Download from the assignment webpage the file `J_GS.m`. In your MATLAB command line, run the command: `J_GS(100)`;
Include in your assignment solution the convergence graph that you are seeing. This is a linear system with the two-dimensional Laplacian of size $10,000 \times 10,000$ (we have $n = 100, n^2 = 10,000$), and we are plotting the norm of the relative residual after 10,000 iterations for Jacobi and Gauss-Seidel. There is no reason to be impressed with this graph; convergence here is slow. But we are going to see the effect of using SOR.
- (b) Given the eigenvalues of the Laplacian in the slides, show that the optimal SOR parameter can be expressed as

$$\omega_{\text{opt}} = \frac{2}{1 + \sin\left(\frac{\pi}{n+1}\right)}.$$

- (c) Modify the MATLAB function so that in addition to the Jacobi and Gauss-Seidel graphs (for the same matrix) a convergence plot for the SOR method is included as well. Use the same initial guess (the zero vector) and the same stopping criterion: $\frac{\|\vec{r}_k\|_2}{\|\vec{b}\|_2} < 10^{-6}$.
- Tip: You should be seeing an *extremely dramatic* improvement in convergence. If you are not seeing such an improvement, then you must have done something wrong.
- (d) Explain your results.

Solution (a).

Note: The line

```
saveas(gcf, "DevamSisodraker_2a.jpg", "jpg");
```

was appended to the end of J_GS.m to ease the exporting process.

```
File: J_GS.m

function J_GS(n)
close all
gcf
% Apply a stationary method to a linear system involving the n^2-by-n^2
% Laplacian

A=delsq(numgrid('S',n+2));
D=diag(diag(A)); % M=D for Jacobi
E=tril(A); % M=E for Gauss-Seidel
itermax=10000; % maximum number of iterations
resvecJ=zeros(itermax,1); % allocate initial space for Jacobi residual norm vector
resvecGS=zeros(itermax,1); % allocate initial space for Gauss-Seidel residual vector

% Jacobi
xJ=zeros(n^2,1);
b=A*ones(n^2,1); % generate a solution of all 1s and a right-hand side
nb=norm(b);
r=b;
for i=1:10000
    resvecJ(i)=norm(r)/nb; % relative residual norm
    if resvecJ(i)<1e-6, break,end % terminate loop if stopping criterion is satisfied
    xJ=xJ+D\r; % next iterate
    r=b-A*xJ; % update residual
end
semilogy(resvecJ); % plot relative residual norm for Jacobi

% Gauss-Seidel
xGS=zeros(n^2,1);
r=b;
for i=1:10000
    resvecGS(i)=norm(r)/nb; % relative residual norm
    if resvecGS(i)<1e-6, break,end % terminate loop if stopping criterion is satisfied
    xGS=xGS+E\r; % next iterate
    r=b-A*xGS; % update residual
end
hold on
semilogy(resvecGS,'r'); % plot relative residual norm for Gauss-Seidel
legend('Jacobi','Gauss-Seidel')
saveas(gcf, "DevamSisodraker_2a.jpg", "jpg");
```

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```
File: DevamSisodraker_2a.out.txt

>> J_GS(100)

ans =

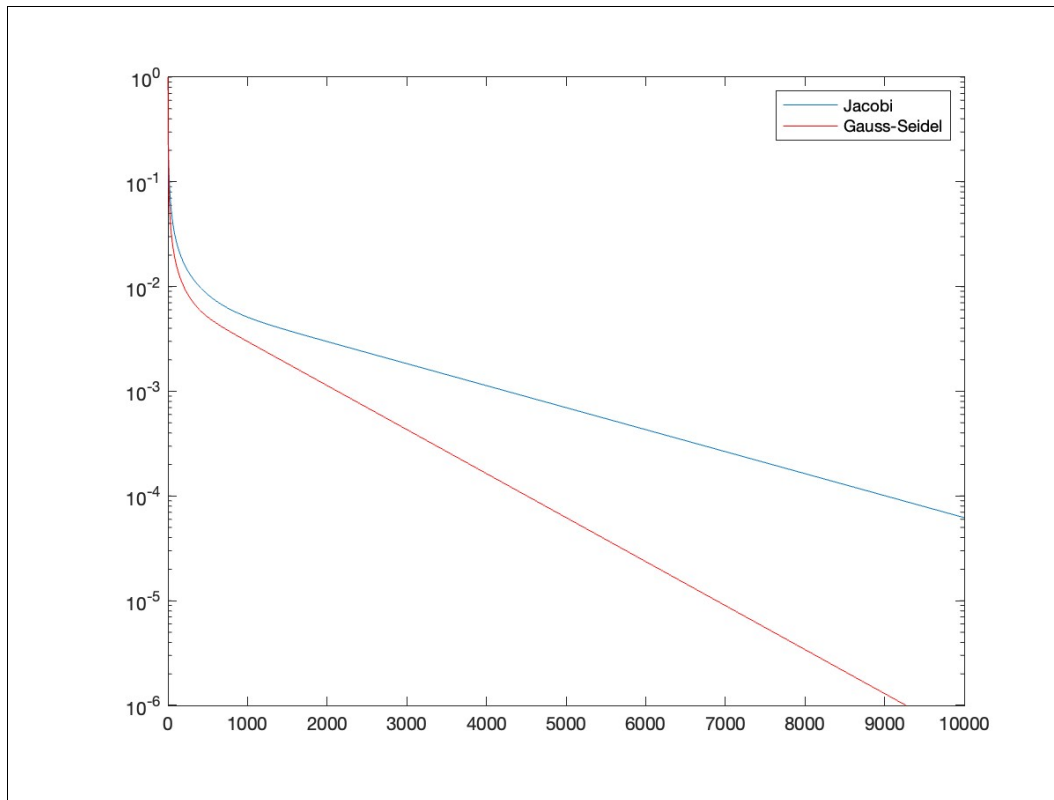
Figure (1) with properties:

    Number: 1
    Name: ''
    Color: [0.9400 0.9400 0.9400]
    Position: [584 595 560 420]
    Units: 'pixels'

Show all properties

>>
```

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Solution (b).

The following is given in the lecture slides.

$$\lambda_{i,j} = 4 - 2 \left(\cos \frac{i\pi}{n+1} + \cos \frac{j\pi}{n+1} \right)$$

Before we continue, let us first recall that the spectral radius of T is given by $\mu_{1,1}$. Thus it follows that $\rho(T) = \mu_{1,1} = \cos \frac{\pi}{n+1}$. Then let us substitute this $\rho(T)$ into the equation which defines ω_{opt} .

$$\begin{aligned} \omega_{opt} &= \frac{2}{1 + \sqrt{1 - (\rho(T))^2}} \\ &= \frac{2}{1 + \sqrt{1 - \left(\cos \frac{\pi}{n+1}\right)^2}} && \text{substitution of } \rho(T) \\ &= \frac{2}{1 + \sqrt{\left(\sin \frac{\pi}{n+1}\right)^2}} && \text{by Pythagorean trigonometric identity} \\ &= \frac{2}{1 + \sin \frac{\pi}{n+1}} \end{aligned}$$

Solution (c).

```

File: DevamSisodraker_2c.m

function J_GS(n)
close all
gcf
% Apply a stationary method to a linear system involving the n^2-by-n^2
% Laplacian

A=delsq(numgrid('S',n+2));
D=diag(diag(A)); % M=D for Jacobi
E=tril(A); % M=E for Gauss-Seidel
itermax=10000; % maximum number of iterations
resvecJ=zeros(itermax,1); % allocate initial space for Jacobi residual norm vector
resvecGS=zeros(itermax,1); % allocate initial space for Gauss-Seidel residual vector
resvecSOR=zeros(itermax,1); % allocate initial space for SOR residual vector

% Jacobi
xJ=zeros(n^2,1);
b=A*ones(n^2,1); % generate a solution of all 1s and a right-hand side
nb=norm(b);
r=b;
for i=1:10000
    resvecJ(i)=norm(r)/nb; % relative residual norm
    if resvecJ(i)<1e-6, break,end % terminate loop if stopping criterion is satisfied
    xJ=xJ+D\r; % next iterate
    r=b-A*xJ; % update residual
end
semilogy(resvecJ); % plot relative residual norm for Jacobi

% Gauss-Seidel
xGS=zeros(n^2,1);
r=b;
for i=1:10000
    resvecGS(i)=norm(r)/nb; % relative residual norm
    if resvecGS(i)<1e-6, break,end % terminate loop if stopping criterion is satisfied
    xGS=xGS+E\r; % next iterate
    r=b-A*xGS; % update residual
end
hold on
semilogy(resvecGS,'r'); % plot relative residual norm for Gauss-Seidel

% SOR
xSOR=zeros(n^2,1);
r=b;
for i=1:10000
    wOpt = 2 / (1 + sin(pi / (n + 1)));
    resvecSOR(i)=norm(r)/nb; % relative residual norm
    if resvecSOR(i)<1e-6, break,end % terminate loop if stopping criterion is satisfied
    xSOR=xSOR+((1 - wOpt)*D + wOpt*E)\r; % next iterate
    r=b-A*xSOR; % update residual
end
hold on
semilogy(resvecSOR,'g'); % plot relative residual norm for Gauss-Seidel

legend('Jacobi','Gauss-Seidel','SOR')
saveas(gcf, "DevamSisodraker_2c.jpg", "jpg");

```

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```
File: DevamSisodraker_2c.out.txt

>> DevamSisodraker_2c(100)

ans =

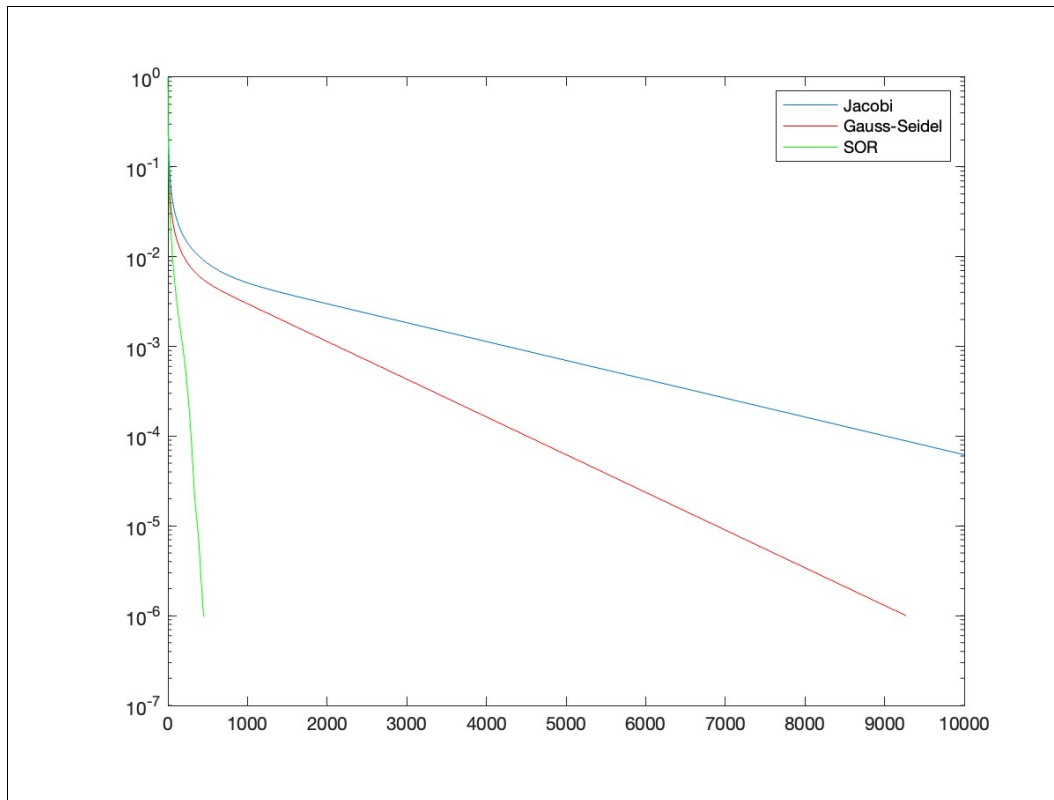
Figure (1) with properties:

    Number: 1
    Name: ''
    Color: [0.9400 0.9400 0.9400]
    Position: [584 595 560 420]
    Units: 'pixels'

Show all properties

>>
```

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Solution (d).

write some
bs explana-
tion after
you see a, b,
c

Problem 3

Problem 3

- (a) Suppose A is an n -by- n orthogonal matrix. Show that all its singular values are equal to 1.
- (b) Recall that an orthogonal projector is a symmetric matrix P for which $P^2 = P$. What are the eigenvalues of an orthogonal projector?

Solution (a).

hard bullshit

Solution (b).

hard bullshit

Problem 4

Problem 4

Load the following .mat matrix that appears on the assignment page:

```
load powerMatrix;
```

If you hit `whos` you should be seeing a matrix called A , of size 100×100 . To validate any of your results below, you may run the MATLAB command `eig`, as long as you understand that in typical eigenvalue computations (in a potentially more challenging computational environment) we generally do not have the luxury of running `eig` to check ourselves.

- (a) Apply the power method. Terminate the iteration once the iterates satisfy

$$|\lambda_1^{(k)} - \lambda_1^{(k-1)}| < 10^{-4}.$$

Your program should print out the value of the final iterate and a graph of the absolute errors: $|\lambda_1^{(k)} - \lambda_{\max}|$. For better visualization, use `semilogy` for your graphs when necessary. As an initial guess for the eigenvector use a vector produced by the MATLAB command `randn`. (When you repeat your experiments, the number of iterations may slightly vary due to the random initial guess.)

- (b) Repeat your computations with the *inverse* power iteration, with a shift $\alpha = 4$, and produce the same graph as you did for the power method.
- (c) Discuss the differences between the performance of the power and the inverse power methods in terms of the cost of single iterations and the overall computational cost.
- (d) Suppose now that we know that A has an eigenvalue close to 3 and we are interested to compute it to six correct decimal digits. Suggest an efficient procedure for doing so. Implement your suggested algorithm and compute the eigenvalue.

Solution (a).

```
File: DevamSisodraker_4a.m

gcf
hold on;

clear;
load('powerMatrix.mat')

k = 0;
vector_k = randn(100, 1);
lambda_k = transpose(vector_k)*A*vector_k;
lambda_k_1 = transpose(vector_k)*A*vector_k;
lambda_audit = [];
lambda_delta_audit = [];
eigval = max(eig(A));

while true
    % statements here
    % if ~WhileCondition, break ; end
    lambda_k_1 = lambda_k;
    vector_k = A*vector_k;
    vector_k = vector_k/norm(vector_k);
    lambda_k = transpose(vector_k)*A*vector_k;
    lambda_audit = [lambda_audit, lambda_k];
    lambda_delta_audit = [lambda_delta_audit, abs(lambda_k - eigval)];
    k = k + 1;
    if abs(lambda_k_1 - lambda_k) < 10^-4
        break;
    end
end

plot(1:1:k, lambda_audit);

plot(1:1:k, lambda_delta_audit);

plot([1, k], [eigval, eigval]);

title("4a");
legend({ ...
    '\lambda_k', ...
    '| \lambda_k - \lambda_{MAX} |', ...
    '\lambda_{MAX}', ...
});
xlabel("k");
ylabel("Value");

hold off;
saveas(gcf, "DevamSisodraker_4a.jpg", "jpg");
```

continued on next page...

```
File: DevamSisodraker_4a.out.txt

>> DevamSisodraker_q4a

ans =

Figure (1) with properties:

    Number: 1
    Name: ''
    Color: [0.9400 0.9400 0.9400]
    Position: [584 595 560 420]
    Units: 'pixels'

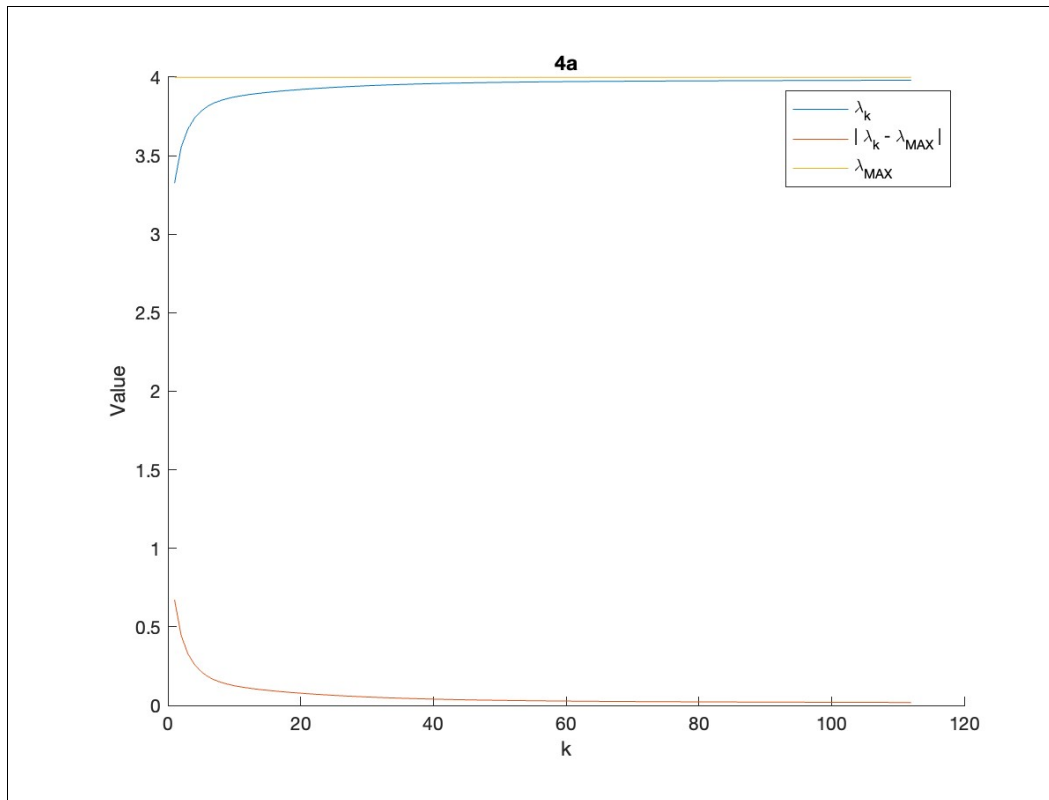
Show all properties

lambda_k =

    3.9804

>>
```

continued on next page...



Solution (b).

File: DevamSisodraker_4b.m

```
gcf
hold on;

clear;
load('powerMatrix.mat');

k = 0;
vector_k = randn(100, 1);
vector_k_1 = vector_k;
lambda_k = transpose(vector_k)*A*vector_k;
lambda_k_1 = transpose(vector_k)*A*vector_k;
lambda_audit = [];
lambda_delta_audit = [];
eigval = max(eig(A));
alpha = 4;

while true
    % statements here
    % if ~WhileCondition, break ; end
    lambda_k_1 = lambda_k;
    vector_k_1 = vector_k;
```

```
vector_k = (A - alpha * eye(size(A)))/transpose(vector_k_1);
vector_k = vector_k/norm(vector_k);
lambda_k = transpose(vector_k)*A*vector_k;

lambda_audit = [lambda_audit, lambda_k];
lambda_delta_audit = [lambda_delta_audit, abs(lambda_k - eigval)];
k = k + 1;

if abs(lambda_k_1 - lambda_k) < 10^-4
    break;
end
end

plot(1:1:k, lambda_audit);
plot(1:1:k, lambda_delta_audit);
plot([1, k], [eigval, eigval]);

title("4b");
legend({ ...
    '\lambda_k', ...
    '| \lambda_k - \lambda_{MAX} |', ...
    '\lambda_{MAX}', ...
});
xlabel("k");
ylabel("Value");

hold off;
saveas(gcf, "DevamSisodraker_4b.jpg", "jpg");
lambda_k
```

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```
File: DevamSisodraker_4b.out.txt

>> DevamSisodraker_4b

ans =

Figure (1) with properties:

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    Color: [0.9400 0.9400 0.9400]
    Position: [584 595 560 420]
    Units: 'pixels'

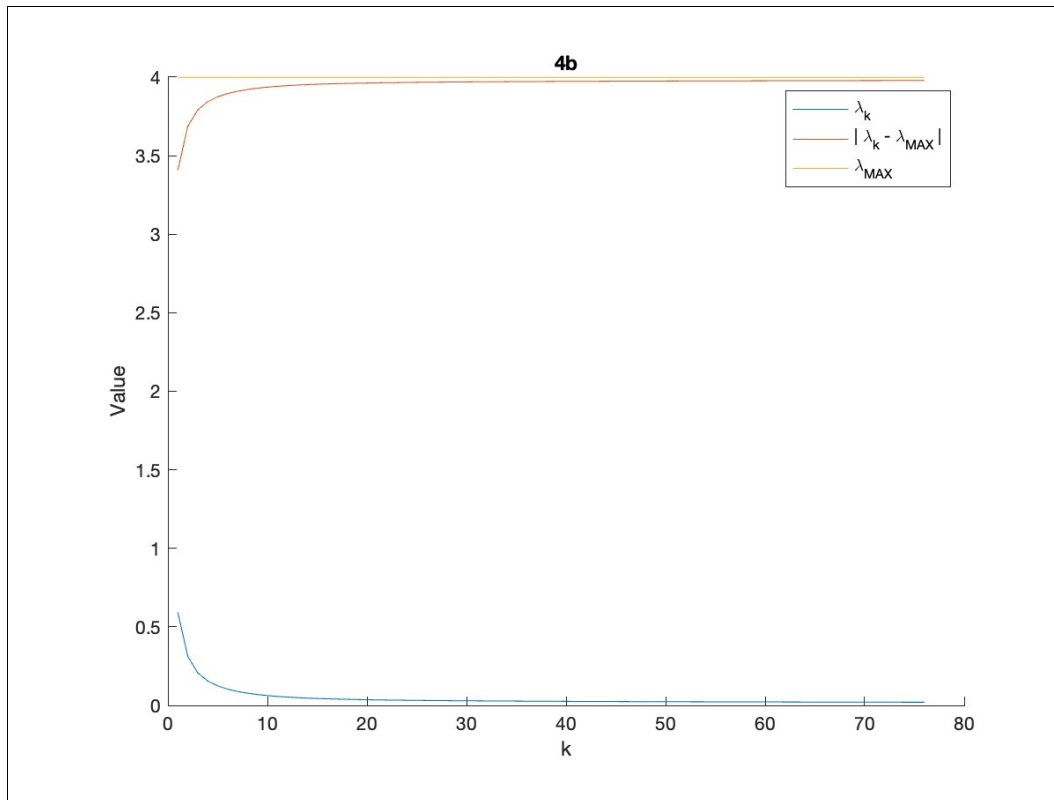
Show all properties

lambda_k =

    0.0202

>>
```

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Solution (c).

bullshit at a
later time

Solution (d).

We will use the inverse power method with $\alpha = 3$ to find the eigenvalue close to 3.

continued on next page...

```
File: DevamSisodraker_4d.m

gcf
hold on;

clear;
load('powerMatrix.mat');

k = 0;
vector_k = randn(100, 1);
vector_k_1 = vector_k;
lambda_k = transpose(vector_k)*A*vector_k;
lambda_k_1 = transpose(vector_k)*A*vector_k;
lambda_audit = [];
alpha = 3;

while true
    % statements here
    % if ~WhileCondition, break ; end
    lambda_k_1 = lambda_k;
    vector_k_1 = vector_k;

    vector_k = (A - alpha * eye(size(A)))\vector_k_1;
    vector_k = vector_k/norm(vector_k);
    lambda_k = transpose(vector_k)*A*vector_k;

    lambda_audit = [lambda_audit, lambda_k];
    k = k + 1;

    if abs(lambda_k_1 - lambda_k) < 10^-4
        break;
    end
end

plot(1:1:k, lambda_audit);

title("4d");
legend({ ...
    '\lambda_k', ...
});
xlabel("k");
ylabel("Value");

hold off;
saveas(gcf, "DevamSisodraker_4d.jpg", "jpg");
lambda_k
```

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```
File: DevamSisodraker_4d.out.txt

>> DevamSisodraker_4d

ans =

Figure (1) with properties:

    Number: 1
    Name: ''
    Color: [0.9400 0.9400 0.9400]
    Position: [584 595 560 420]
    Units: 'pixels'

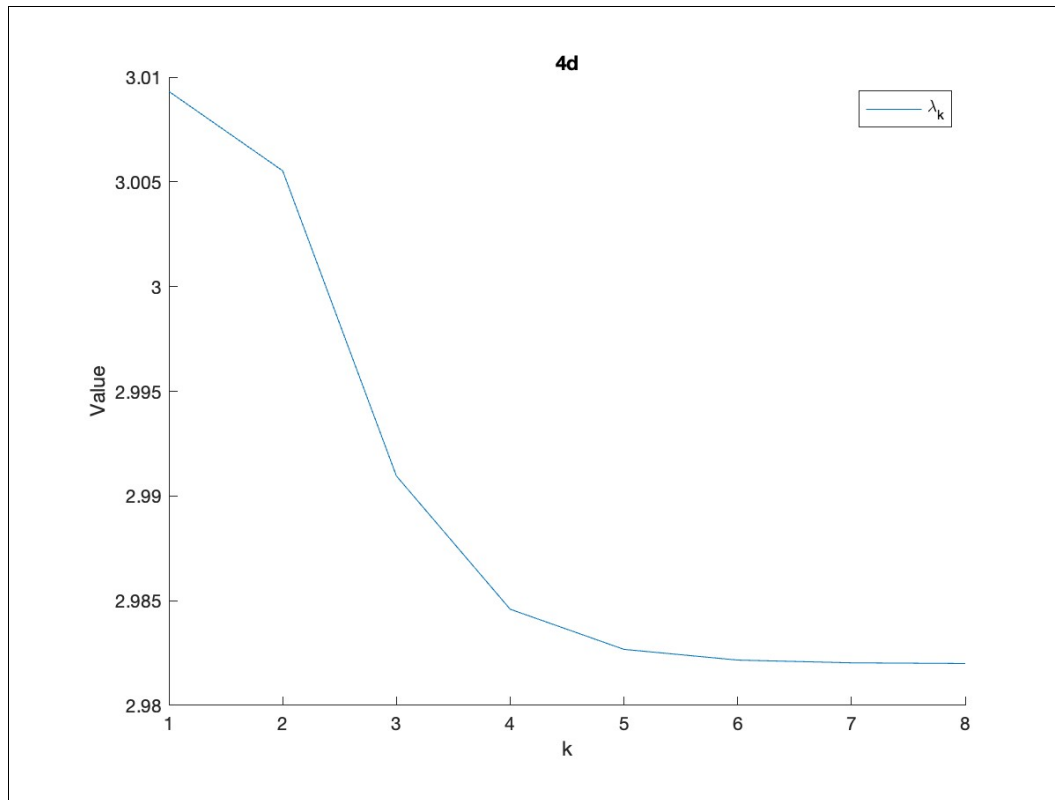
Show all properties

lambda_k =

    2.9820

>>
```

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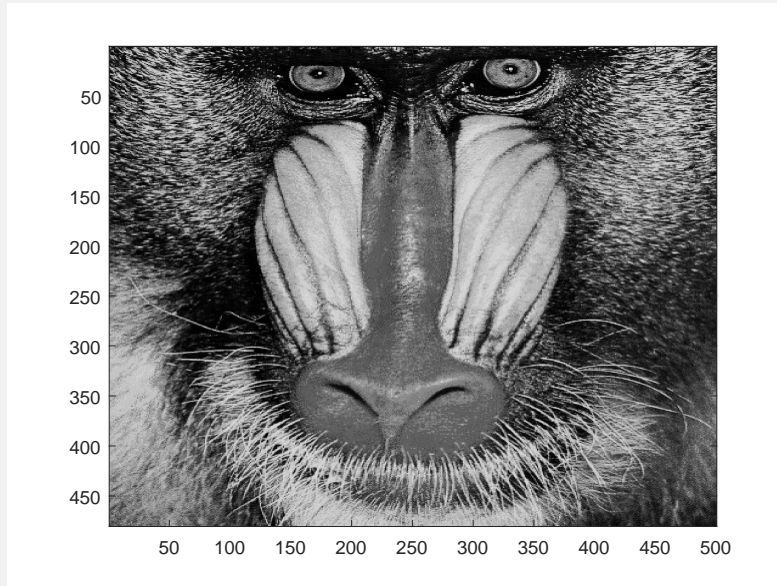


Problem 5

Problem 5

The following picture appears in MATLAB's repository of images, and can be retrieved by entering

```
load mandrill;  
colormap('gray');  
image(X);
```



- (a) Print out the images generated by the truncated SVD. Start with $r = 2$ and go up by powers of 2, to $r = 2^6 = 64$ (six plots in total). For a compact presentation of your figures, you may use the command `subplot(3,2)`. (Check out `help subplot`.)
- (b) Comment briefly on the quality of the images as a function of r . For what value of r would you say that the quality of the image is acceptable, in that we can be rather confident of what we are seeing? (We are not looking for a specific “correct answer” here - just make your subjective observation.)
- (c) For the value of r you stated in part (b), how much storage is required? Compare it to the storage required for the original image.

Solution (a).

File: DevamSisodraker_5a.m

```
clear;
load mandrill;
colormap("gray");
[U,S,V] = svd(X);
image(U*S*V');

close all;
for i = 1:1:6
    r = 2^i;

    dims = size(X);
    S_trunc = diag(S);
    S_trunc((r + 1):min(dims(1), dims(2))) = 0;
    S_trunc = diag(S_trunc);
    S_trunc(dims(1), dims(2)) = 0;

   (gcf
    hold on;
    Xout = U * S_trunc * V';
    colormap("gray");
    image(flipud(Xout));
    title(strcat("r=", string(r)));
    saveas(gcf, strcat("DevamSisodraker_5a_", string(r), ".jpg"), "jpg");
    hold off;
end
```

continued on next page...

```
File: DevamSisodraker_5a.out.txt

>> DevamSisodraker_5a

ans =

Figure (1) with properties:

    Number: 1
    Name: ''
    Color: [0.9400 0.9400 0.9400]
    Position: [584 595 560 420]
    Units: 'pixels'

Show all properties

ans =

Figure (1) with properties:

    Number: 1
    Name: ''
    Color: [0.9400 0.9400 0.9400]
    Position: [584 587 560 420]
    Units: 'pixels'

Show all properties

ans =

Figure (1) with properties:

    Number: 1
    Name: ''
    Color: [0.9400 0.9400 0.9400]
    Position: [584 587 560 420]
    Units: 'pixels'

Show all properties

ans =

Figure (1) with properties:
```

```
    Number: 1
    Name: ''
    Color: [0.9400 0.9400 0.9400]
    Position: [584 587 560 420]
    Units: 'pixels'

Show all properties

ans =

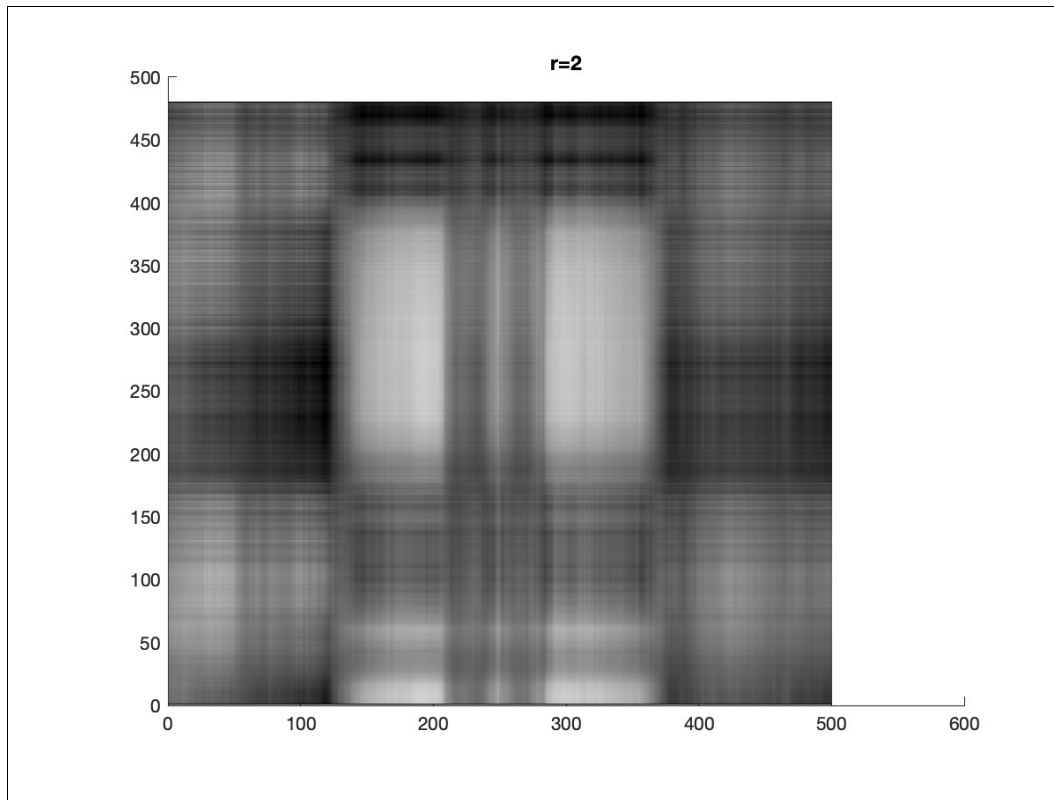
Figure (1) with properties:

    Number: 1
    Name: ''
    Color: [0.9400 0.9400 0.9400]
    Position: [584 587 560 420]
    Units: 'pixels'

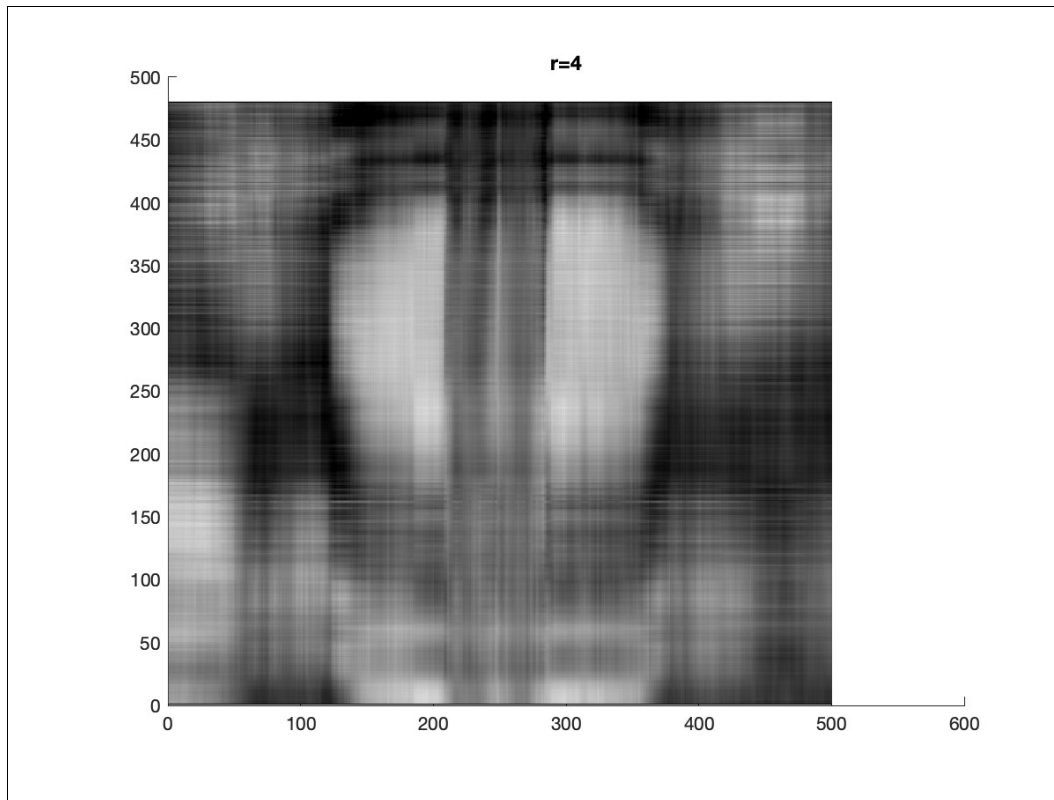
Show all properties

>> >
```

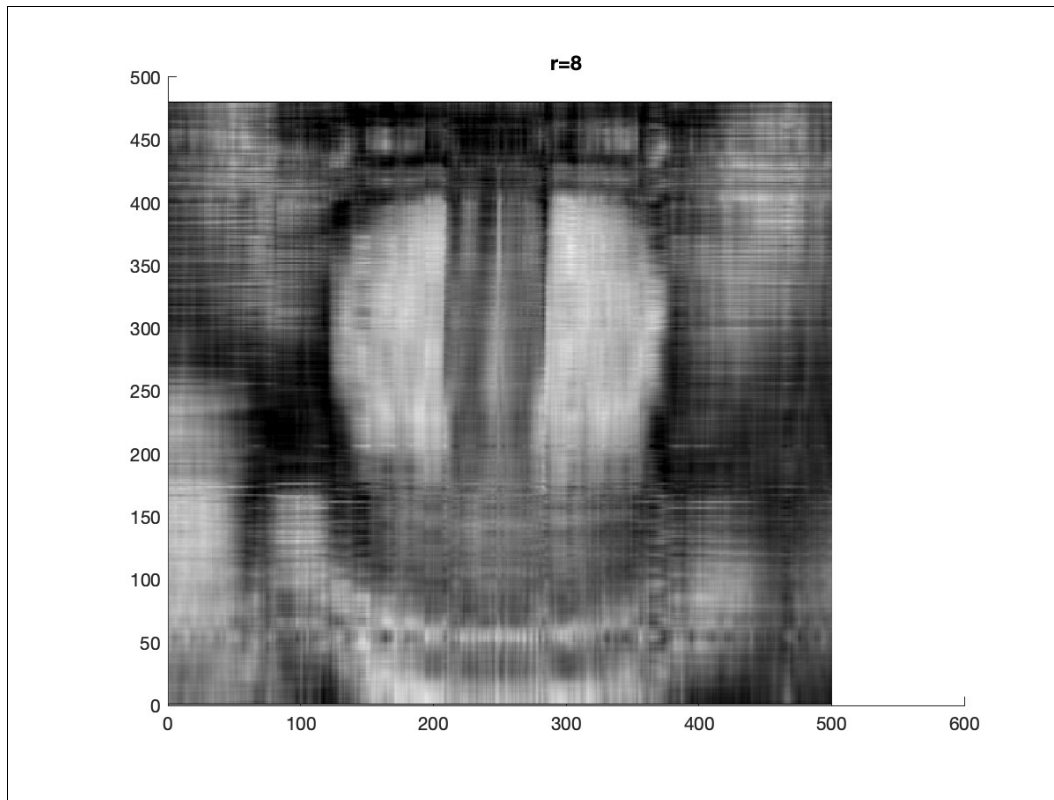
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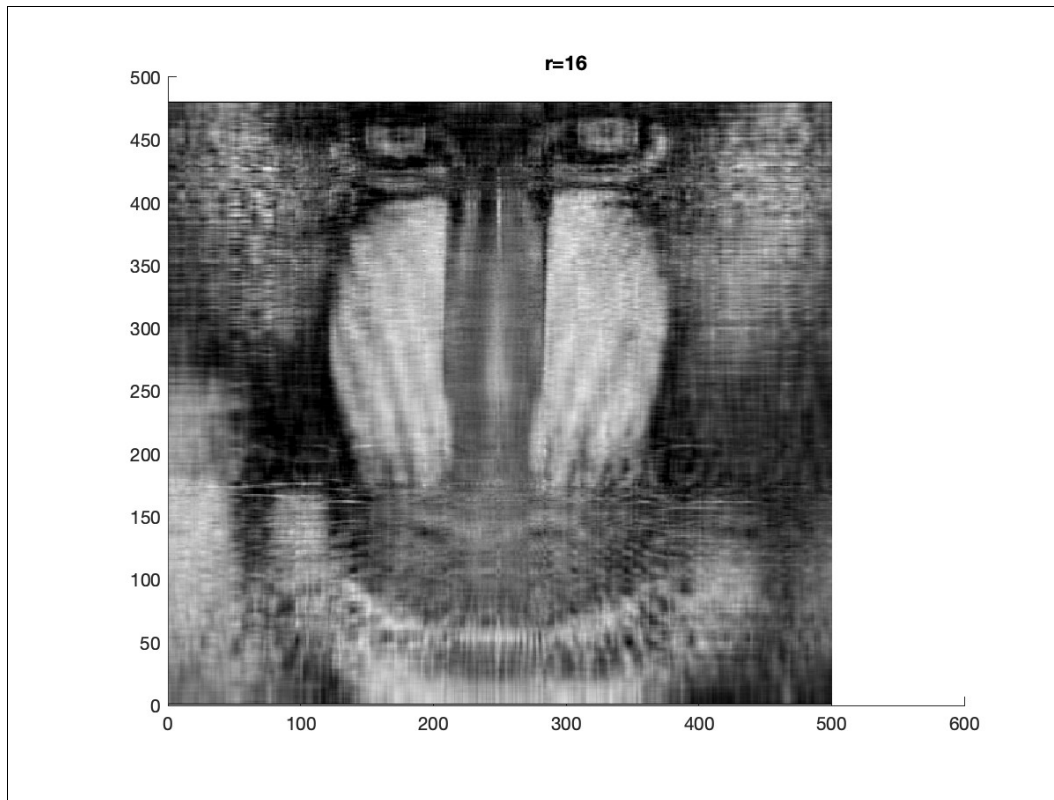
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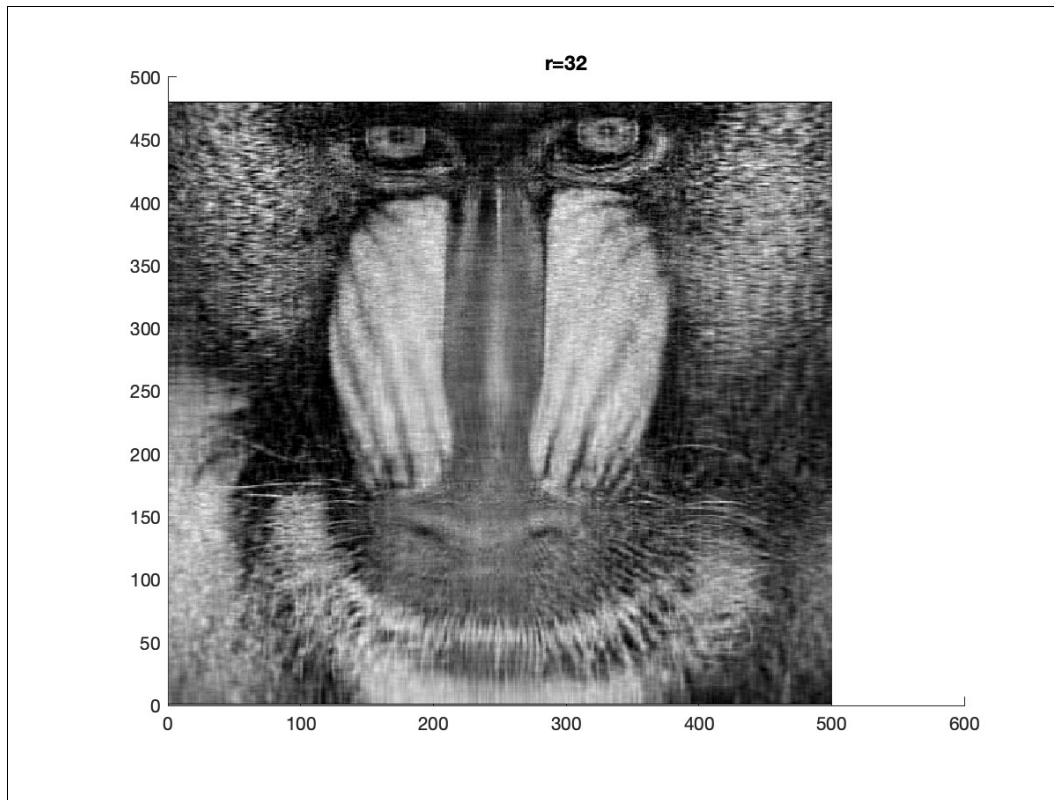
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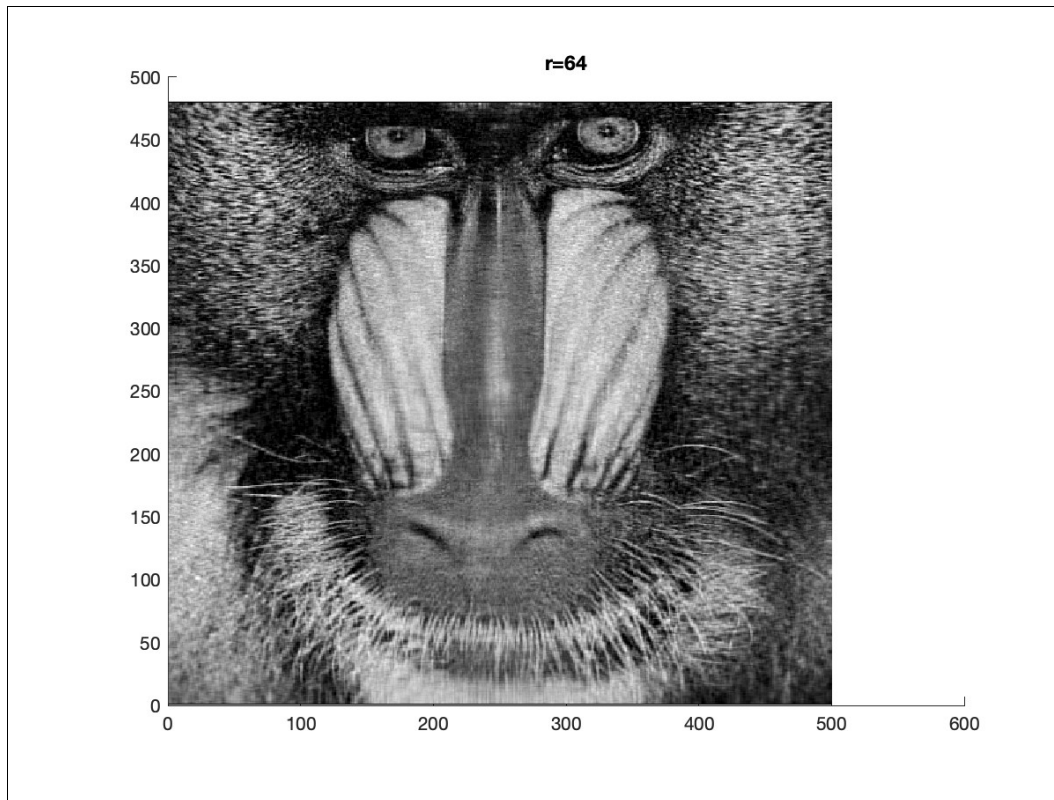
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Solution (b).

bullshit

Solution (c).

bullshit