Assignment 5

CPSC 302 - 2022W1

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Problem 1

Problem 1
Consider the 2×2 matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and suppose we are required to solve $A\vec{x} = \vec{b}$, where $\vec{b}$ is an arbitrary right-hand side vector. Clearly, solving a 2×2 linear system using a stationary scheme is an utterly ridiculous idea, but the following is helpful in understanding more about convergence of stationary methods.

(a) Find the spectral radius of the Jacobi and Gauss-Seidel iteration matrices and the asymptotic rate of convergence for these two schemes, namely $-\log_{10}\rho(T)$, where $\rho(T)$ denotes the spectral radius of the corresponding iteration matrix, T.

(b) How much faster does Gauss-Seidel converge compared to Jacobi for a fixed reduction in the relative residual norm, $\frac{\|\vec{r}_k\|_2}{\|\vec{b}\|_2}$, in terms of iteration counts?

(c) Write down the SOR iteration matrix as a function of the relaxation parameter, ω.

(d) Find the optimal SOR parameter, ω_{opt}, and the spectral radius of the corresponding iteration matrix.

(e) Approximately how much faster does SOR with ω_{opt} converge compared to Jacobi?

Solution (a).

Given A let us first calculate the Jacobi iteration matrix of A, denoted D. Since D all zeros along the non-diagonal entries and shares the same diagonal entries as A then we conclude that D is the following. We also indicate its inverse below.

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Since we have a computed value for D^{-1} we can now calculate the iteration matrix T_{Jacobi}.

$$\begin{aligned} T_{\text{Jacobi}} &= I - D^{-1}A \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \end{aligned}$$

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Now let us calculate the eigenvalues of T_{Jacobi} to obtain the spectral radius.

calculate the eigenvalues of T_{Jacobi} here

todo

We know that the largest absolute eigenvalue is $\frac{1}{2}$.

$$\rho\left(D^{-1}\right) = \lambda_1 = \frac{1}{2}$$

Now that we have a value for $\rho\left(D^{-1}\right)$, we can calculate the asymptotic rate of convergence for the Jacobi iteration matrix of A to be the following. *Note: The following results were calculated using Matlab.*

$$-\log_{10}\rho\left(D^{-1}\right) = -\log_{10}\left(\frac{1}{2}\right) \approx 0.3010$$

We have now calculated the asymptotic rate of convergence for the Jacobi iteration matrix of A. Now let us calculate the asymptotic rate of convergence and the spectral radius for the Gauss-Seidel iteration matrix of A.

finish this part here, after you finish the Jacobi part

thing

0.6021 for rate of convergence

Solution (b).

Gauss-Seidel will converge at twice the rate that Jacobi will converge. Jacobi will take twice as many iterations as Gauss-Seidel.

Solution (c).

We know that the SOR iteration matrix T_{SOR} is the following by definition.

$$I - \omega((1-\omega)D + \omega E)^{-1}A$$

Thus we can compose it as the following function.

$$W(\omega) = I - \frac{1}{\omega}((1-\omega)D + \omega E)A$$

write this up properly

Solution (d).

use page 235 ch7 from textbook to solve this

Solution (e).

compare both of the values from d and a then compare