
Assignment 5

CPSC 302 - 2022W1

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Problem 1

Problem 1

Consider the 2×2 matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and suppose we are required to solve $A\vec{x} = \vec{b}$, where \vec{b} is an arbitrary right-hand side vector. Clearly, solving a 2×2 linear system using a stationary scheme is an utterly ridiculous idea, but the following is helpful in understanding more about convergence of stationary methods.

- Find the spectral radius of the Jacobi and Gauss-Seidel iteration matrices and the asymptotic rate of convergence for these two schemes, namely $-\log_{10} \rho(T)$, where $\rho(T)$ denotes the spectral radius of the corresponding iteration matrix, T .
- How much faster does Gauss-Seidel converge compared to Jacobi for a fixed reduction in the relative residual norm, $\frac{\|\vec{r}_k\|_2}{\|\vec{b}\|_2}$, in terms of iteration counts?
- Write down the SOR iteration matrix as a function of the relaxation parameter, ω .
- Find the optimal SOR parameter, ω_{opt} , and the spectral radius of the corresponding iteration matrix.
- Approximately how much faster does SOR with ω_{opt} converge compared to Jacobi?

Solution (a).

Given A let us first calculate the Jacobi iteration matrix of A , denoted D . Since D all zeros along the non-diagonal entries and shares the same diagonal entries as A then we conclude that D is the following. We also indicate its inverse below.

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Since we have a computed value for D^{-1} we can now calculate the iteration matrix T_{Jacobi} .

$$\begin{aligned} T_{\text{Jacobi}} &= I - D^{-1}A \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \end{aligned}$$

continued on next page...

Now let us calculate the eigenvalues of T_{Jacobi} to obtain the spectral radius.

todo

calculate the
eigenvalues
of T_{Jacobi}
here

We know that the largest absolute eigenvalue is $\frac{1}{2}$.

$$\rho(D^{-1}) = \lambda_1 = \frac{1}{2}$$

Now that we have a value for $\rho(D^{-1})$, we can calculate the asymptotic rate of convergence for the Jacobi iteration matrix of A to be the following. *Note: The following results were calculated using Matlab.*

$$-\log_{10} \rho(D^{-1}) = -\log_{10} \left(\frac{1}{2} \right) \approx 0.3010$$

We have now calculated the asymptotic rate of convergence for the Jacobi iteration matrix of A . Now let us calculate the asymptotic rate of convergence and the spectral radius for the Gauss-Seidel iteration matrix of A .

thing

finish this
part here,
after you
finish the
Jacobi part

0.6021 for rate of convergence

Solution (b).

Gauss-Seidel will converge at twice the rate that Jacobi will converge. Jacobi will take twice as many iterations as Gauss-Seidel.

Solution (c).

We know that the SOR iteration matrix T_{SOR} is the following by definition.

$$I - \omega((1 - \omega)D + \omega E)^{-1}A$$

Thus we can compose it as the following function.

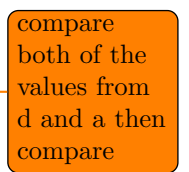
$$W(\omega) = I - \frac{1}{\omega}((1 - \omega)D + \omega E)A$$

write this up properly

Solution (d).

use page 235
ch7 from
textbook to
solve this

Solution (e).



compare
both of the
values from
d and a then
compare