Homework #3

Wednesday, 20 November 2024 20:23

1.1

$$P(X=+1 \mid Y>0) = \frac{P(Y>0 \mid X=+1)\, P(X=+1)}{P(Y>0)}$$

$$= \frac{P(Y>0 \mid X=+1)}{2\, P(Y>0)}$$

Similarly: $P(X=-1 \mid Y>0) = \dfrac{P(Y>0 \mid X=-1)}{2\, P(Y>0)}$

We have: $P(Y>0 \mid X=+1) = P(Z>-1) = \Phi(1)$ $\left(\begin{array}{l} Y \sim N(1,1) \\ Z \sim N(0,1) \end{array}\right)$

$P(Y>0 \mid X=-1) = P(Z>1) = 1-\Phi(1)$ $\left(\begin{array}{l} Y \sim N(-1,1) \\ Z \sim N(0,1) \end{array}\right)$

$$P(Y>0) = P(Y>0 \mid X=+1) \cdot P(X=+1) + P(Y>0 \mid X=-1) \cdot P(X=-1)$$

$$= \frac{1}{2}\Phi(1) + \frac{1}{2}(1-\Phi(1)) = \frac{1}{2}$$

With $\Phi(1) = 0{,}8413$

$$\Rightarrow \begin{cases} P(X=+1 \mid Y>0) = \dfrac{\Phi(1)}{\Phi(1) + (1-\Phi(1))} = 0{,}8413 \\ P(X=-1 \mid Y>0) = 1-\Phi(1) = 0{,}1587 \end{cases}$$

1.2.

Let: A: switch to IT

B: has those characteristic

We have:

$P(A) = 1/20 = 0{,}05 \Rightarrow P(\bar{A}) = 0{,}95$

$P(B \mid A) = 0{,}7$

$P(B \mid \bar{A}) = 0{,}1$

$$P(B) = P(B \mid A) \cdot P(A) + P(B \mid \bar{A}) \cdot P(\bar{A})$$

$$= 0{,}7 \cdot 0{,}05 + 0{,}1 \cdot 0{,}95$$

$$= 0{,}13$$

$$\Rightarrow P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

$$= \frac{0{,}7 \cdot 0{,}05}{0{,}13} \approx 0{,}2692$$

$$= 26{,}92\%$$

1.3

b/

We have: $P(X_n = x_n) = \dfrac{\lambda^{x_n} e^{-\lambda}}{x_n!}$

and $Y_n = \begin{cases} 1, & X_n \geq 1 \\ 0, & X_n = 0 \end{cases}$

$\Rightarrow$ PMF of Y_n : $P(Y_n = 1) = 1 - P(X_n = 0)$

$$= 1 - \frac{\lambda^0 e^{-\lambda}}{0!} = 1 - e^{-\lambda}$$

$$P(Y_n = 0) = P(X_n = 0)$$
$$= e^{-\lambda}$$

a/

$$L(\lambda) = \prod_{n=1}^{N} P(Y_n = y_n) = \prod_{Y_n=1} (1-e^{-\lambda})^{y_n} \prod_{Y_n=0} (e^{-\lambda})^{(1-y_n)}$$

$$\log L(\lambda) = \sum_{n=1}^{T} \left[y_n \log(1-e^{-\lambda}) + (1-y_n)(-\lambda) \right]$$

$$\frac{\partial \log L(\lambda)}{\partial \lambda} = \sum_{n=1}^{T} \frac{y_n e^{-\lambda}}{1-e^{-\lambda}} - T e^{-\lambda} = 0 \quad \text{(maximize)}$$

$$\Leftrightarrow \frac{\sum_{n=1}^{T} y_n}{T} = 1 - e^{-\lambda}$$

$$\Leftrightarrow \lambda = -\log\left(1 - \frac{\sum_{n=1}^{T} y_n}{T}\right)$$

1.4

We have: $p(\theta) = \mathcal{N}(\mu_0, \Sigma_0)$

$$\Leftrightarrow \log p(\theta) = -\frac{1}{2}(\theta-\mu_0)^T \Sigma_0^{-1} (\theta-\mu_0).$$

$\Rightarrow$ Likelihood:

$$p(y|\theta) = \mathcal{N}(X\theta, \sigma^2 I)$$

$$\Leftrightarrow \log p(y|\theta) = -\frac{1}{2\sigma^2}(y - X\theta)^T (y - X\theta)$$

Bayes :

$$\log p(y|\theta) = \log p(y|\theta) + \log p(\theta) + C$$

$$\Rightarrow \log p(y|\theta) = \frac{1}{2\sigma^2}(y-X\theta)^T(y-X\theta) - \frac{1}{2}(\theta-\mu_0)^T \Sigma_0^{-1}(\theta-\mu_0)$$

$$= \frac{1}{2\sigma^2}(y^T y - 2y^T X\theta + \theta^T X^T X\theta)$$

$$= \frac{1}{2\sigma^2}\left(y^T y - 2y^T X\theta + \theta^T X^T X\theta\right)$$

$$= \frac{1}{2}\left(\theta^T \Sigma_0^{-1}\theta - 2\mu_0^T \Sigma_0^{-1}\theta + \mu_0^T \Sigma_0^{-1}\mu_0\right)$$

$$= \frac{1}{2}\left[\theta^T \underbrace{\left(\Sigma_0^{-1} + \frac{1}{\sigma^2}X^T X\right)}_{\Sigma_1^{-1}}\theta - 2\theta^T \underbrace{\left(\Sigma_0^{-1}\mu_0 + \frac{1}{\sigma^2}X^T y\right)}_{\Sigma_1^{-1}\mu_1}\right]$$

$\Rightarrow$ Posterior: $p(\theta|y) = \mathcal{N}(\mu_1, \Sigma_1)$, $\begin{cases} \Sigma_1^{-1} = \Sigma_0^{-1} + \frac{1}{\sigma^2}X^T X \\ \mu_1 = \Sigma_1\left(\Sigma_0^{-1}\mu_0 + \frac{1}{\sigma^2}X^T y\right) \end{cases}$