AP Calculus BC

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Question 1

$$\int_3^e \left(\frac{x^3 - x}{x^2}\right) dx$$

Proof:

$$\int_3^e \left(\frac{x^3}{x^2} - \frac{x}{x^2}\right) dx$$

Cancel the terms out:

$$\int_3^e \left(x - \frac{1}{x}\right)$$

Integrate:

$$= \left[\frac{x^2}{2} - \ln|x|\right]_3^e$$

Plug in for the definite integral: F(b) - F(a):

$$\left(\frac{e^2}{2} - lne\right) - \left(\frac{9}{2} - ln3\right)$$

Remember that $\ln \mathrm{e} = 1$

$$= \left(\frac{e^2}{2} - 1\right) - \frac{9}{2} + \ln 3$$

$$= \frac{e^2}{2} - \frac{2}{2} - \frac{9}{2} + \ln 3$$

$$= \boxed{\frac{e^2}{2} - \frac{11}{2} + \ln 3}$$

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Question 2

$$\int \tan^2 2x dx \tag{1}$$

Proof:

Remember that the $\int \tan^2 2x = 1 - \sec^2 2x$

Rewrite:

$$\int \left(1 - \sec^2 2x\right) dx$$

Let u = 2x

du = 2dx

$$\frac{du}{2} = dx$$

$$\int \left(1 - \sec^2(2x)\right) \frac{du}{2}$$

Separate into two integrals:

$$\int \left[1\right] - \int \left[-\sec^2(2x)\right] \frac{du}{2}$$

Remember that $\int \sec^2 u du = \tan u + C$

Evaluate:

$$x + \left[-\tan 2x + C\right] \frac{du}{2}$$

Note that the constant 1/2 is only applied to the tan

$$x - \frac{1}{2} \tan 2x + C$$

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REVIEW OF BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)du = k\int f(u)du$
2. $\int [f(u) \pm g(u)]du = \int f(u)du \pm \int g(u)du$
3. $\int du = u + C$
4. $\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int e^u du = e^u + C$
7. $\int a'' du = \left(\frac{1}{\ln a}\right) a^u + C$
8. $\int \sin u du = -\cos u + C$
9. $\int \cos u du = \sin u + C$
10. $\int \tan u du = -\ln|\cos u| + C$
11. $\int e^{ax}\, dx = \frac{1}{a} e^{ax}$

PROCEDURES FOR FITTING INTEGRANDS TO BASIC INTEGRATION RULE

Expand (numerator).

$$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$$

Separate numerator.

$$\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$$

Complete the square.

$$\frac{1}{\sqrt{2x - x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$$

Divide improper rational function.

$$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$$

Add and subtract terms in numerator.

$$\frac{2x}{x^2+2x+1} = \frac{2x+2-2}{x^2+2x+1} = \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}$$

Use trigonometric identities.

$$\cot^2 x = \csc^2 x - 1$$

Multiply and divide by Pythagorean conjugate.

$$\frac{1}{1+\sin x} = \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) = \frac{1-\sin x}{1-\sin^2 x}$$

$$= \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$$

Question 3

Evaluate:

$$\int_0^1 \frac{x+3}{\sqrt[2]{4-x^2}} dx \tag{2}$$

Proof:

Separate into two terms for integration:

$$\int_0^1 \left(\frac{x}{\sqrt{4-x^2}} + \frac{3}{\sqrt{4-x^2}}\right) dx$$

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NOTES FOR TEST

$$\boxed{\int e^{ax}\,dx = \frac{1}{a}e^{ax}}$$

$$\text{Total distance} = \int_a^b \left|v(t)\right|$$

$$\int \frac{1}{e} dt = \frac{1}{e} t + C, \text{ note that } \frac{1}{e} \text{ is a fraction}$$

$$\text{Average Value Formula: } \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example:

Find the average value of the function $f(x) = \frac{2}{1+x}$ from x = 1 to x = 7

Express your answer as a constant times ln2.

Proof:

$$\frac{1}{7-1}\int_{1}^{7}\frac{2}{1+x}\,dx$$

Take the anti-derivative, take constant out:

$$\frac{1}{6} * 2\ln|1+x|\Big|_{1}^{7}$$

Evaluate and plug in:

$$\frac{1}{3}\ln\left(\frac{8}{2}\right) = \frac{1}{3}\ln(4)$$

Rewrite the 4:

$$\frac{1}{3}\ln\left(2^2\right)$$

Take the power out according to Logarithic Rules:

$$\frac{1}{3} * 2 * \ln(2) = \frac{2}{3}\ln(2)$$

$$\frac{2}{3}\ln(2)$$

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Trig with Motion, Solving for primary zeroes:

A particle moves along the x-axis with position given by $\mathbf{x(t) = 2\sin\left(\frac{\pi}{5}\,t\right) + 3}$

Find all times in the interval $0 \leqslant t < 10$

Proof:

Find velocity function first, take the derivative of position function:

$$v(t) = 2\cos\left(\frac{\pi}{5}t\right) * \frac{\pi}{5}$$

Set the velocity function to zero and solve for t:

$$0 = \frac{2\pi}{5}\cos\frac{\pi}{5}t$$

Divide the constant out:

$$0 = \cos\frac{\pi}{5}t$$

Remember that cos is only equal to 0 at the muliples of π, starting at $\frac{\pi}{2}$, e.g.: $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$...

We need to make the term inside the cos to be equal to one of those values

$$\tfrac{\pi}{5}t = \tfrac{\pi}{2} \quad \tfrac{\pi}{5}t = \tfrac{3\pi}{2}$$

Divide:

$$t = \tfrac{5}{2} \quad t = \tfrac{15}{2}$$

$$\left(\tfrac{5}{2}, \tfrac{15}{2}\right)$$

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$$\frac{d}{dx} a^{bx} = a^{bx} * b * \ln a$$

$$\int a^{bx} = \frac{a^{bx}}{b \ln a}$$