AP Calculus BC

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$$\int_3^e \left(\frac{x^3 - x}{x^2} \right) dx$$

Proof:

$$\int_3^e \left(\frac{x^3}{x^2} - \frac{x}{x^2} \right) dx$$

Cancel the terms out:

$$\int_3^e \left(x - \frac{1}{x} \right)$$

Integrate:

$$= \left[\frac{x^2}{2} - \ln|x| \right]_3^e$$

Plug in for the definite integral: F(b) - F(a):

$$\left(\frac{e^2}{2} - \ln e \right) - \left(\frac{9}{2} - \ln 3 \right)$$

Remember that $\ln e = 1$

$$= \left(\frac{e^2}{2} - 1 \right) - \frac{9}{2} + \ln 3$$

$$= \frac{e^2}{2} - \frac{2}{2} - \frac{9}{2} + \ln 3$$

$$= \frac{e^2}{2} - \frac{11}{2} + \ln 3$$

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Question 2

$$\int \tan^2 2x \, dx \tag{1}$$

Proof:

Remember that the $\int \tan^2 2x = 1 - \sec^2 2x$

Rewrite:

$$\int \left(1 - \sec^2 2x\right) dx$$

Let u = 2x

du = 2dx

$$\frac{du}{2} = dx$$

$$\int \left(1 - \sec^2(2x)\right) \frac{du}{2}$$

Separate into two integrals:

$$\int \left[1\right] - \int \left[-\sec^2(2x)\right] \frac{du}{2}$$

Remember that $\int \sec^2 u \, du = \tan u + C$

Evaluate:

$$x + \left[-\tan 2x + C\right] \frac{du}{2}$$

Note that the constant 1/2 is only applied to the tan

$$x - \frac{1}{2} \tan 2x + C$$

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REVIEW OF BASIC INTEGRATION RULES $(a > 0)$

1. $\int kf(u)du = k \int f(u)du$

2. $\int [f(u) \pm g(u)]du = \int f(u)du \pm \int g(u)du$

3. $\int du = u + C$

4. $\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$

5. $\int \frac{du}{u} = \ln |u| + C$

6. $\int e^u du = e^u + C$

7. $\int a'' du = \left(\frac{1}{\ln a}\right) a^u + C$

8. $\int \sin u\, du = -\cos u + C$

9. $\int \cos u\, du = \sin u + C$

10. $\int \tan u\, du = -\ln |\cos u| + C$

11. $\boxed{\int e^{ax}\, dx = \frac{1}{a}e^{ax}}$

PROCEDURES FOR FITTING INTEGRANDS TO BASIC INTEGRATION RULE

Expand (numerator).
$$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$$

Separate numerator.
$$\frac{1 + x}{x^2 + 1} = \frac{1}{x^2 + 1} + \frac{x}{x^2 + 1}$$

Complete the square.
$$\frac{1}{\sqrt{2x - x^2}} = \frac{1}{\sqrt{1 - (x - 1)^2}}$$

Divide improper rational function.
$$\frac{x^2}{x^2 + 1} = 1 - \frac{1}{x^2 + 1}$$

Add and subtract terms in numerator.
$$\frac{2x}{x^2 + 2x + 1} = \frac{2x + 2 - 2}{x^2 + 2x + 1} = \frac{2x + 2}{x^2 + 2x + 1} - \frac{2}{(x + 1)^2}$$

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Use trigonometric identities.

$$\cot^2 x = \csc^2 x - 1$$

Multiply and divide by Pythagorean conjugate.

$$\frac{1}{1 + \sin x} = \left(\frac{1}{1 + \sin x}\right)\left(\frac{1 - \sin x}{1 - \sin x}\right) = \frac{1 - \sin x}{1 - \sin^2 x}$$

$$= \frac{1 - \sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$$

Question 3

Evaluate:

$$\int_0^1 \frac{x + 3}{\sqrt[2]{4 - x^2}}\,dx \tag{2}$$

Proof:

Separate into two terms for integration:

$$\int_0^1 \left(\frac{x}{\sqrt{4 - x^2}} + \frac{3}{\sqrt{4 - x^2}}\right) dx$$

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NOTES FOR TEST

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\text{Total distance} = \int_a^b |v(t)|$$

$$\int \frac{1}{e}\,dt = \frac{1}{e}t + C, \text{ note that } \frac{1}{e} \text{ is a fraction}$$

$$\text{Average Value Formula: } \frac{1}{b - a}\int_a^b f(x)\,dx$$

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Example:

Find the average value of the function $f(x) = \dfrac{2}{1+x}$ from x = 1 to x = 7

Express your answer as a constant times ln2.

Proof:

$$\frac{1}{7-1} \int_1^7 \frac{2}{1+x}\, dx$$

Take the anti-derivative, take constant out:

$$\frac{1}{6} * 2\ln\left|1+x\right| \Bigg|_1^7$$

Evaluate and plug in:

$$\frac{1}{3}\ln\left(\frac{8}{2}\right) = \frac{1}{3}\ln(4)$$

Rewrite the 4:

$$\frac{1}{3}\ln\left(2^2\right)$$

Take the power out according to Logarithic Rules:

$$\frac{1}{3} * 2 * \ln(2) = \frac{2}{3}\ln(2)$$

$$\frac{2}{3}\ln(2)$$

Trig with Motion, Solving for primary zeroes:

A particle moves along the x-axis with position given by $\mathbf{x(t)} = \mathbf{2}\sin\left(\frac{\pi}{5}\,\mathbf{t}\right) + \mathbf{3}$

Find all times in the interval $0 \leqslant t < 10$

Proof:

Find velocity function first, take the derivative of position function:

$$v(t) = 2\cos\left(\frac{\pi}{5}t\right) * \frac{\pi}{5}$$

Set the velocity function to zero and solve for t:

$$0 = \frac{2\pi}{5}\cos\frac{\pi}{5}t$$

Divide the constant out:

$$0 = \cos\frac{\pi}{5}t$$

Remember that cos is only equal to 0 at the muliples of π, starting at $\frac{\pi}{2}$, e.g.: $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}\ldots$

We need to make the term inside the cos to be equal to one of those values

$$\tfrac{\pi}{5}t = \tfrac{\pi}{2} \quad \tfrac{\pi}{5}t = \tfrac{3\pi}{2}$$

Divide:

$$t = \tfrac{5}{2} \quad t = \tfrac{15}{2}$$

$$\left(\tfrac{5}{2}, \tfrac{15}{2}\right)$$

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