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3

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Geometry

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Chapter 2

of

GEOMETRY

ALGEBRA IN GEOMETRY

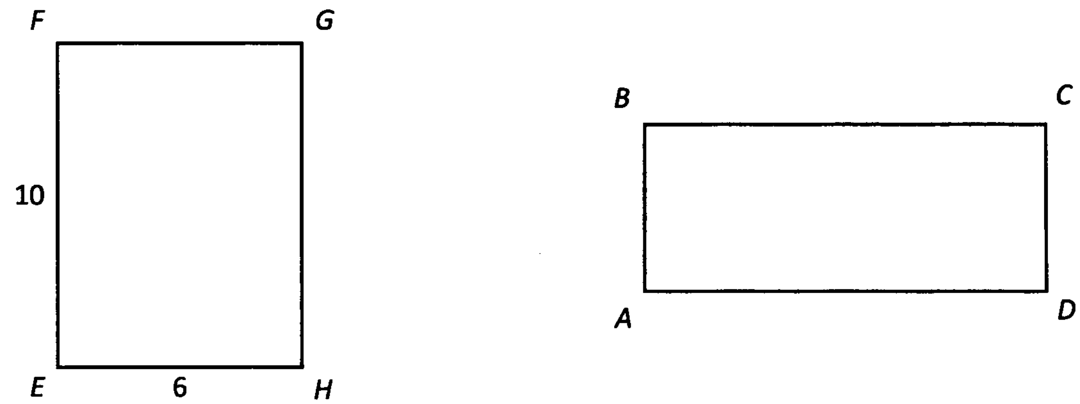
In This Chapter . . .

- Using Equations to Solve Geometry Problems

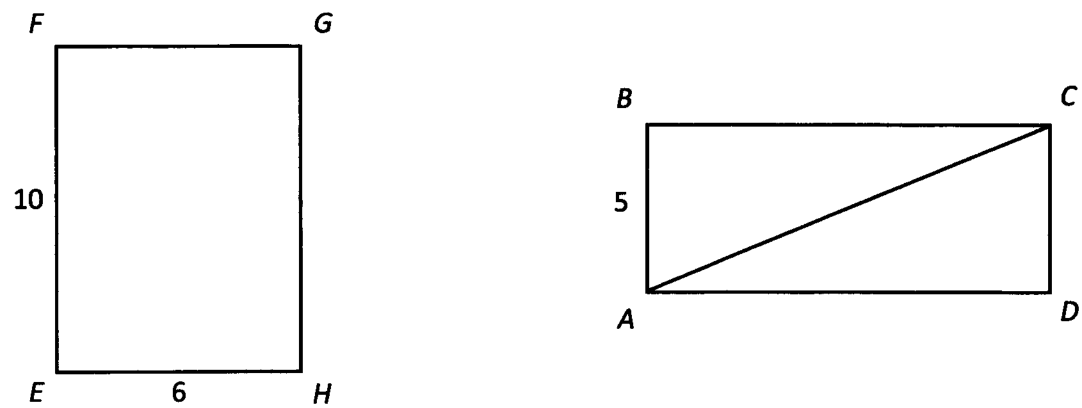
Using Equations to Solve Geometry Problems

Before we dive into the specific properties of the many shapes tested on the GRE, it's important to establish a foundation of translating the information presented in questions into algebraic equations. This will allow us to more easily, and quickly, solve even the most complex geometry problems. To start, let's do the following problem together.

> **Rectangles *ABCD* and *EFGH*, shown below, have equal areas. The length of side *AB* is 5. What is the length of diagonal *AC*?**



The first step in any geometry question involving shapes is to draw your own copies of the shapes on your note paper and fill in everything you know. In this problem in particular, you would want to redraw both rectangles and add to your picture the information that side *AB* has a length of 5. Also, make note of what you're looking for—in this case we want the length of diagonal *AC*. So your new figures would look like this:

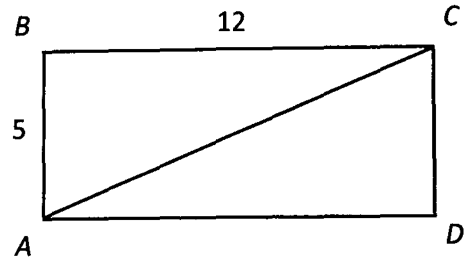


Now that we have redrawn our figures and filled in all the given information, it's time to begin answering the question.

So now the question becomes—has the question provided us any information that can be expressed mathematically? In other words, can we create equations? Well, they did tell us one thing that we can use—the two rectangles have equal areas. So we can say that $\text{Area}_{ABCD} = \text{Area}_{EFGH}$. But we can do better than that. The formula for area of a rectangle is Area = (length) × (width). So our equation can be rewritten as $(\text{length}_{ABCD}) \times (\text{width}_{ABCD}) = (\text{length}_{EFGH}) \times (\text{width}_{EFGH})$.

The length and width of rectangle *EFGH* are 6 and 10 (it doesn't matter which is which) and the length of *AB* is 5. So our equation becomes $(5) \times (\text{width}_{ABCD}) = (6) \times (10)$. So $(5) \times (\text{width}_{ABCD}) = 60$, which means that the width of rectangle *ABCD* equals 12.

Any time you learn a new piece of information (in this case the width of rectangle *ABCD*) you should put that information into your picture. So our picture of rectangle *ABCD* now looks like this:



To recap what we've done so far, we started this problem by redrawing the shapes described in the question and filling in all the information (such as side lengths, angles, etc.) that we knew, and made note of the value the question was asking us for. The first step for geometry problems is to **draw or redraw figures and fill in all given information**. Of course, we should also confirm what we're being asked!

Next we made use of additional information provided in the question. The question stated that the two rectangles had equal areas. We created an equation to express this relationship, and then plugged in the values we knew (length and width of rectangle *EFGH* and length of rectangle *ABCD*) and solved for the width of rectangle *ABCD*. We **identified relationships and created equations.** After that, we **solved the equations for the missing value** (in this case, the width of *ABCD*).

In some ways, all we have done so far is set up the problem. In fact, aside from noting that we need to find the length of diagonal *AC*, nothing we have done so far seems to have directly helped us actually solve for that value. The result of the work we've done to this point is to find that the width of rectangle *ABCD* is 12.

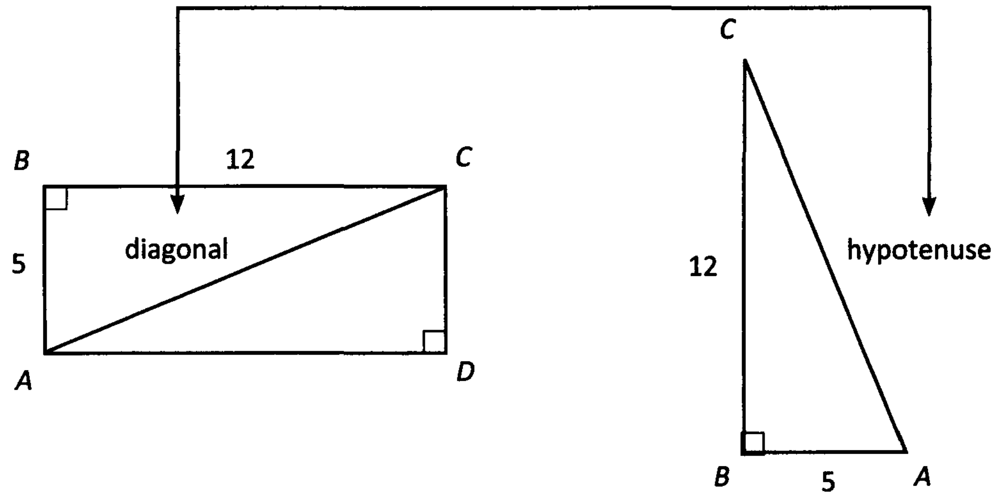
So why did we bother solving for the width of rectangle *ABCD* when we didn't even know why we would need it? The answer is that there is a very good chance that we will need that value in order to answer the question.

There was no way initially to find the length of diagonal *AC*. We simply did not have enough information. The question did, however, provide us enough information to find the width of rectangle *ABCD*. More often than not, if you have enough information to solve for a value, you need that value to answer the question.

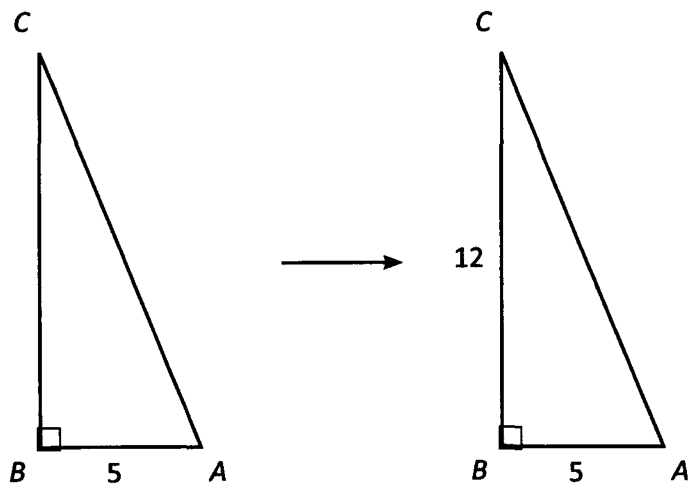
So the question now becomes, what can we do now that we know the width of *ABCD* that we couldn't do before? To answer that, let's take another look at the value we're looking for: the length of *AC*.

As mentioned earlier, an important part of problem solving is to identify relationships. We already identified the relationship mentioned in the question—that both rectangles have equal areas. But for many geometry problems there are additional relationships that aren't as obvious.

The key to this problem is to recognize that *AC* is not only the diagonal of rectangle *ABCD*, but is also the hypotenuse of a right triangle. We know this because one property of rectangles is that all four interior angles are right angles.



Now that we know AC is the hypotenuse of a right triangle, we can use the Pythagorean Theorem to find the length of the hypotenuse using the two side lengths.



Sides BC and AB are the legs of the triangle, and AC is the hypotenuse, so:

$$(BC)^2 + (AB)^2 = (AC)^2$$
$$(12)^2 + (5)^2 = (AC)^2$$
$$144 + 25 = (AC)^2$$
$$169 = (AC)^2$$
$$13 = AC$$

Alternatively, you can avoid that work by recognizing that this triangle is one of the Pythagorean triplets: a 5–12–13 triangle. Either way, the answer to the question is $AC = 13$.

Let's recap what we did in the last portion of this question. The process that allowed us to solve for the width of $ABCD$ was based on information explicitly presented to us in the question. To proceed from there, however, required a different sort of process. The key insight was that the diagonal of rectangle $ABCD$ was also the hypotenuse of right triangle ABC. Additionally, we needed to know that, in order to find the length of AC, we needed the lengths of the other two sides of the triangle. The last part of this problem required us to **make inferences from the figures.** Sometimes these inferences require you to make a jump from one shape to another through a common element. For instance, we needed to see AC as both a diagonal of a rectangle and as a hypotenuse of a right triangle. Here AC was the common element in both a rectangle and a right triangle. Other times, these inferences make you think about what information you would need in order to find another value.

In a moment, we'll go through another sample problem, but before we do, let's revisit the important steps to answering geometry problems.

Recap:

Step 1: **Draw or redraw figures and fill in all given information.**
Fill in all known angles and lengths and make note of any equal sides or angles.

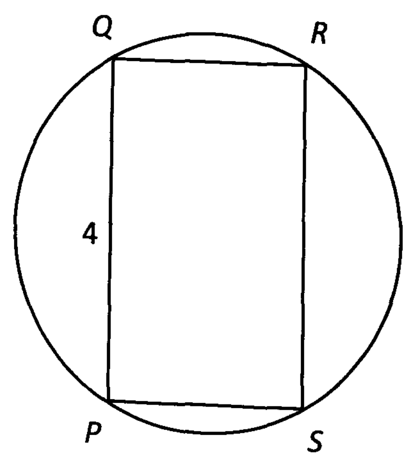
Step 2: **Identify relationships and create equations.**
Often these relationships will be explicitly stated in the question.

Step 3: **Solve the equations for the missing value.**
If you can solve for a value, you will often need that value to answer the question.

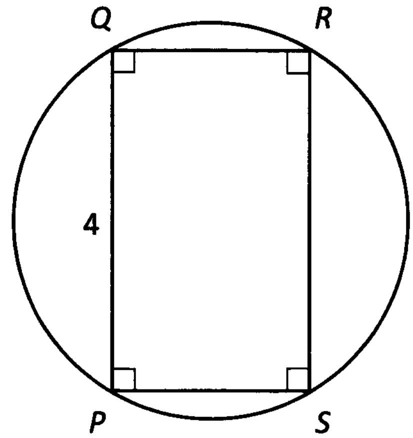
Step 4: **Make inferences from the figures.**
You will often need to make use of relationships that are not explicitly stated.

Now that we've got the basic process down, let's do another problem. Try it on your own first, then we'll walk through it together.

Rectangle *PQRS* is inscribed in Circle *O* pictured below. The center of Circle *O* is also the center of Rectangle *PQRS*. If the circumference of Circle *O* is 5π, what is the area of Rectangle *PQRS*?



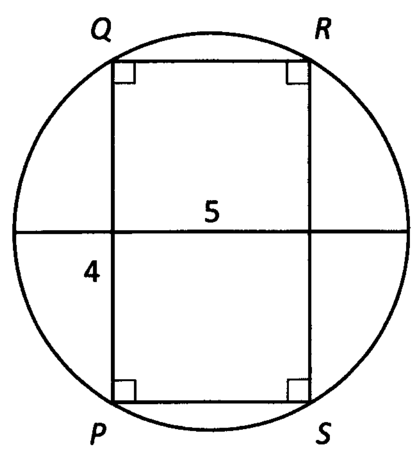
The first thing you should have done is **redraw the figure** on whatever note paper you are using and **fill in all the given information.** The question didn't explicitly give us the value of any side lengths or angles, but it did say that *PQRS* is a rectangle. That means all 4 internal angles are right angles. So when you redraw the figure, it might look like this.



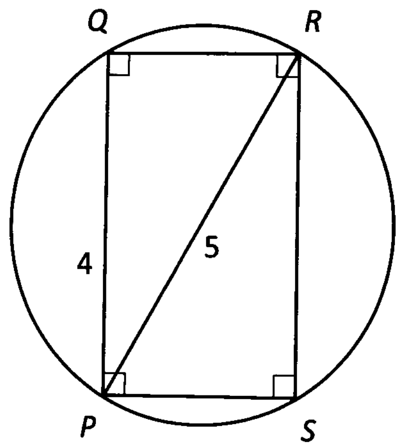
Now it's time to **identify relationships and create equations.** The question stated that the circumference of Circle O is 5π, and we know the formula for circumference. Circumference equals $2\pi r$, so $5\pi = 2\pi r$. Now that we know the circumference, there's only one unknown (r), so we should **solve the equation for the missing value** and find the radius, which turns out to be 2.5. We also know that $d = 2r$, so the diameter of Circle O is 5.

As with the previous problem, we are now left with the question—why did we find the radius and diameter? We were able to solve for them, which is a very good clue that we need one of them to answer the question. Now is the time to **make inferences from the figures.**

Ultimately, this question is asking for the area of rectangle *PQRS*. What information do we need to find that value? We have the length of *QP*, which means that if we can find the length of either *QR* or *PS*, we can find the area of the rectangle. So we need to somehow find a connection between the rectangle and the radius or diameter. Let's put a diameter into the circle.



That didn't really seem to help much, because we still have no way to make the connection between the diameter and the rectangle. It's important to remember, though, that *any* line that passes through the center is a diameter. What if we drew the diameter so that it passed through the center but touched the circle at points P and R? We know that the line connecting points P and R will be a diameter because we know that the center of the circle is also the center of the rectangle. Our circle now looks like this:



What was the advantage of drawing the diameter so that it connected points P and R? Now the diameter of the circle is also the diagonal of the rectangle. The circle and the rectangle have a common element.

Where do we go from here? We still need the length of either QR or PS. Do we have a way to get either one of those values? As a matter of fact, we do. PQR is a right triangle. It's not oriented the way we are used to seeing it, but all the important elements are there. It's a triangle, and one of its internal angles is a right angle. Additionally, we know the lengths of 2 of the sides: QP and PR. That means we can use the Pythagorean Theorem to find the length of the third side: QR.

$(QR)^2 + (QP)^2 = (PR)^2$

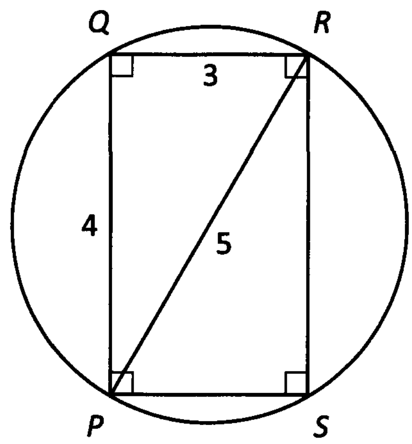
$(QR)^2 + (4)^2 = (5)^2$

$(QR)^2 + 16 = 25$

$(QR)^2 = 9$

$QR = 3$

Alternatively, we could have recognized the Pythagorean triplet—triangle PQR is a 3–4–5 triangle. Either way we arrive at the conclusion that the length of QR is 3. Our circle now looks like this:



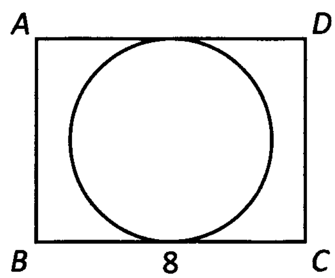
Now we have what we need to find the area of rectangle *PQRS*. Area = (length) × (width) = (4) × (3) = 12. So the answer to the question is 12.

What did we need to do in order to arrive at that answer? For starters, we needed to make sure that we had an accurate
figure to work with, and that we populated that figure with all the information that had been given to us. Next we had to realize that knowing the circumference of the circle allowed us to find the diameter of the circle.

After that came what is often the most difficult part of the process—we had to make inferences based on the figure. The key insight in this problem was that we could draw a diameter in our figure that could also act as the diagonal of the rectangle. As if that wasn't difficult enough, we then had to recognize that *PQR* was a right triangle, even though it was rotated in a way that made it difficult to see. It is these kinds of insights that are going to be crucial to success on the GRE—recognizing shapes when they're presented in an unfamiliar format and finding connections between different shapes.

Check Your Skills

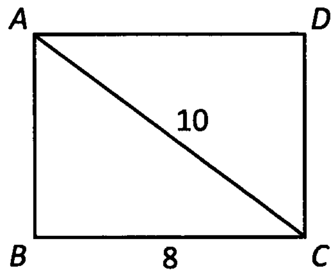
1. In rectangle *ABCD*, the distance from *A* to *C* is 10. What is the area of the circle inside the rectangle, if this circle touches both *AD* and *BC*? (This is known as an inscribed circle).



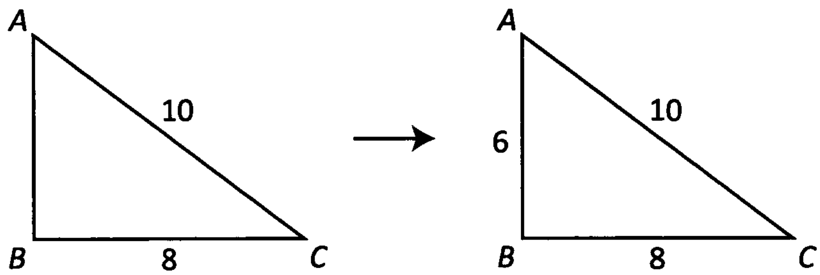
The answer can be found on page 35.

Check Your Skills Answers

1. **9π:** Redraw the diagram *without* the circle, so you can focus on the rectangle. Add in the diagonal AC, since we're given its length.



Now we look at right triangle ABC. AC functions not only as the diagonal of rectangle $ABCD$ but also as the hypotenuse of right triangle ABC. So now we find the third side of triangle ABC, either using the Pythagorean Theorem or recognizing a Pythagorean triplet (6–8–10).



Side AB has length 6.

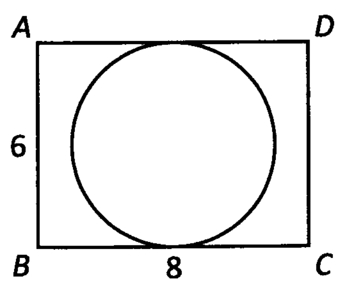
$(AB)^2 + 8^2 = 10^2$

$(AB)^2 + 64 = 100$

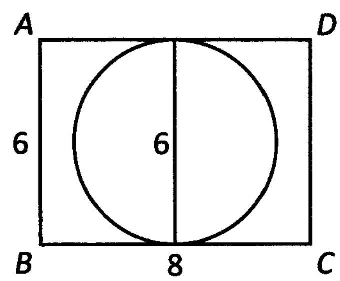
$(AB)^2 = 36$

$AB = 6$

Now, we redraw the diagram *with* the circle but without the diagonal, since we've gotten what we needed from that: the other side of the rectangle.



Since the circle touches both AD and BC, we know that its diameter must be 6.



Finally, we find the radius and compute the area:

$d = 6 = 2r$

$3 = r$

$\text{Area} = \pi r^2$

$= \pi 3^2$

$\text{Area} = 9\pi$

Problem Set

1.

The "aspect ratio" of a rectangular TV screen is the ratio of its height to its width.

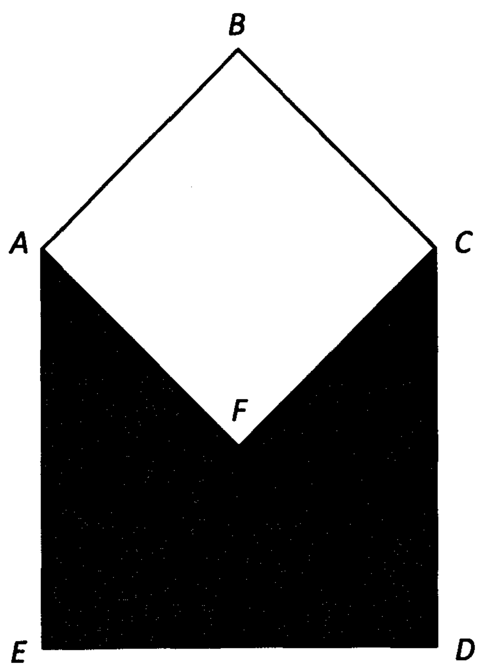
Quantity A	Quantity B
The area of a rectangular TV screen with an aspect ratio of 3:4 and a diagonal of 25"	The area of a rectangular TV screen with an aspect ratio of 9:16 and a diagonal of 25"

2.

Ten 8-foot long poles will be arranged in a rectangle to surround a flowerbed.

Quantity A	Quantity B
The area in square feet of the flowerbed	300

3.

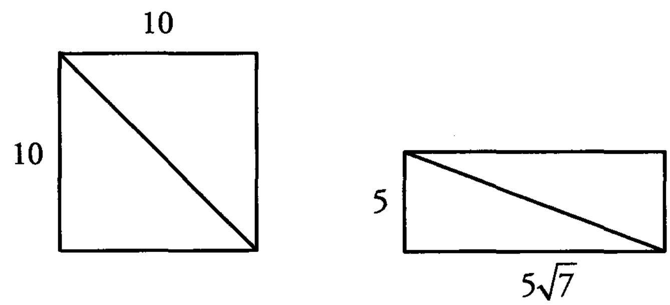


ABCF and *ACDE* are squares.

Quantity A	Quantity B
Twice the area of the shaded region	Three times the area of *ABCF*

1. **A:** For any fixed diagonal, as for any fixed perimeter, the rectangle with the greatest area is a square. Given two rectangles with equal diagonals, the one that is more nearly square has a greater area. A 3 : 4 ratio is more nearly square (closer to 1) than is a 9 : 16 ratio.

Each of the following rectangles has the same diagonal. Compare their areas.



The area of the square is larger, and is the largest possible area for the diagonal of that length. As the diagonal rotates, the area of the rectangle gets progressively smaller. Thus **Quantity A is larger.**

2. **D:** There are two possible ways to arrange the poles into rectangles, 2 poles × 3 poles, and 1 pole × 4 poles on each side. The 2 × 3 arrangement will enclose an area 16 × 24, which is more than 300 square feet. The 1 × 4 arrangement will enclose an area 8 × 32, which is less than 300 square feet.

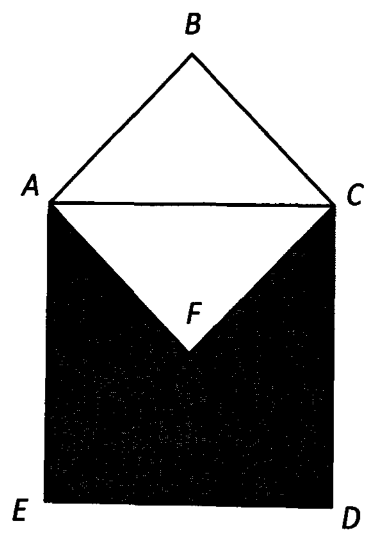
Ten 8-foot long poles will be arranged in a rectangle to surround a flower-bed.

Quantity A	Quantity B
The area in square feet of the flowerbed =	300

$16 \times 24 = \mathbf{384}$ OR
$8 \times 32 = \mathbf{256}$

Therefore **we do not have enough information** to determine which quantity is greater.

3. **C:** The simplest solution is to draw the diagonals of the larger square and add the line AC. This yields a combined figure cut into five equal triangles:



The shaded region is three such triangles, so Quantity A represents the area of six such triangles. $ABCF$ is two such triangles, so Quantity B represents the area of six such triangles.

There are also algebraic solutions to this problem, although they're all reasonably complicated. Here's one: Since the ratio of the diagonal of a square to its side is $\sqrt{2}:1$, $AC = \sqrt{2}AB$.

So the area of the large square is $(\sqrt{2}AB)^2$, or $2(AB)^2$.

The area of the smaller square is simply $(AB)^2$.

The shaded region represents 3/4 the area $2(AB)^2$, or $\frac{3}{2}(AB)^2$.

We can rewrite the quantities thusly:

<u>Quantity A</u>	<u>Quantity B</u>
Twice the area of the shaded region $= 2 \cdot \frac{3}{2}(AB)^2 = \mathbf{3(AB)^2}$	Three times the area of $ABCF = 3 \cdot (AB)^2 = \mathbf{3(AB)^2}$

Therefore **the two quantities are equal.**

Chapter 3

of

GEOMETRY

TRIANGLES & DIAGONALS

In This Chapter . . .

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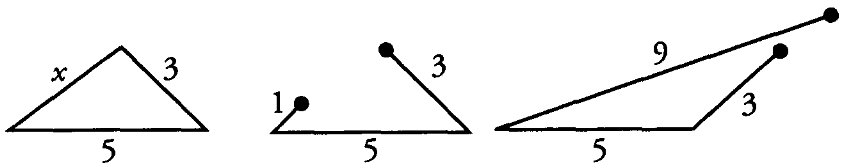
- The Basic Properties of a Triangle
- Perimeter and Area
- Right Triangles
- Pythagorean Triplets
- Isosceles Triangles and the 45–45–90 Triangle
- Equilateral Triangles and the 30–60–90 Triangle
- Diagonals of Other Polygons

The Basic Properties of a Triangle

Triangles show up all over the GRE. You'll often find them hiding in problems that seem to be about rectangles or other shapes. Of the basic shapes, triangles are perhaps the most challenging to master. One reason is that several properties of triangles are tested.

Let's start with some general comments on triangles:

The sum of any two side lengths of a triangle will always be greater than the third side length. This is because the shortest distance between two points is a straight line. At the same time, the third side length will always be greater than the difference of the other two side lengths. The pictures below illustrate these two points.



What is the largest number x could be? What's the smallest? Could it be 9? 1?

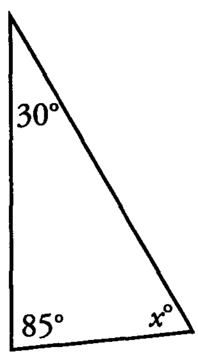
x must be less than $3 + 5 = 8$
x must be greater than $5 - 3 = 2$
$2 < x < 8$

Check Your Skills

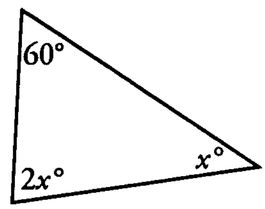
1. Two sides of a triangle have lengths 5 and 19. Can the third side have a length of 13?
2. Two sides of a triangle have lengths 8 and 17. What is the range of possible values of the length of the third side?

Answers can be found on page 55.

The internal angles of a triangle must add up to 180°. This rule can sometimes allow us to make inferences about angles of unknown size. It means that if we know the measures of 2 angles in the triangle, we can determine the measure of the third angle. Take a look at this triangle:



The 3 internal angles must add up to 180°, so we know that $30 + 85 + x = 180$. Solving for x tells us that $x = 65$. So the third angle is 65°. The GRE can also test your knowledge of this rule in more complicated ways. Take a look at this triangle:



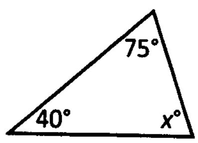
In this situation, we only know one of the angles. The other 2 are given in terms of x. Even though we only know one angle, we can still determine the other 2. Again, we know that the 3 angles will add up to 180. So $60 + x + 2x = 180$. That means that
$3x = 120$. So $x = 40$. Thus the angle labeled $x°$ has a measure of 40° and the angle labeled $2x°$ has a measure of 80°.

The GRE will not always draw triangles to scale, so don't try to guess angles from the picture, which could be distorted. Instead, solve for angles mathematically.

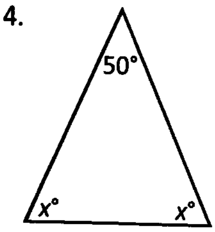
Check Your Skills

Find the missing angle(s).

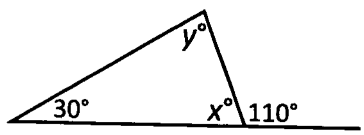
3.



4.

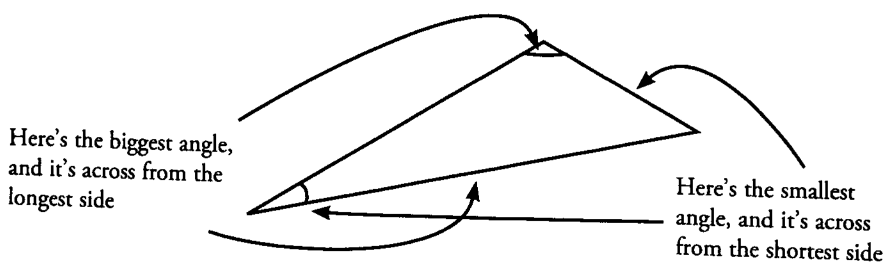


5.

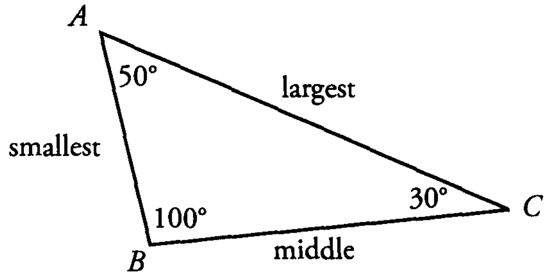


Answers can be found on pages 55.

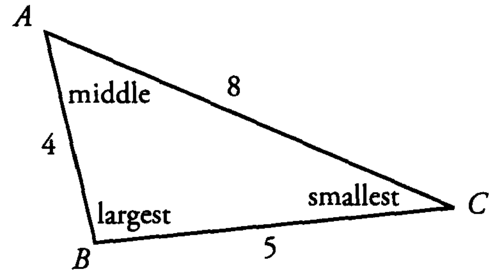
Internal angles of a triangle are important on the GRE for another reason. Sides correspond to their opposite angles. This means that the longest side is opposite the largest angle, and the smallest side is opposite the smallest angle. Think about an alligator opening its mouth, bigger and bigger… as the angle between its upper and lower jaws increases, the distance between the front teeth on the bottom and top jaws would get longer and longer.



One important thing to remember about this relationship is that it works both ways. If we know the sides of the triangle, we can make inferences about the angles. If we know the angles, we can make inferences about the sides.



$AC > BC > AB$



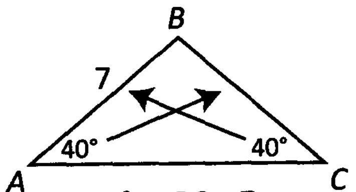
$\angle ABC > \angle BAC > \angle ACB$

Although we can determine from the angle measures which sides are longer, which sides are shorter, and which sides are equal, we cannot determine how MUCH greater or shorter. For instance, in the triangle to the above left, $\angle ABC$ is twice as large as $\angle BAC$, but that does not mean that AC is twice as large as BC.

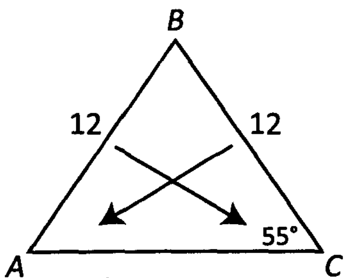
Things get interesting when a triangle has sides that are the same length or angles that have the same measure. We can classify triangles by the number of equal sides that they have.

- A triangle that has 2 equal angles and 2 equal sides (opposite the equal angles) is an **isosceles triangle.**
- A triangle that has 3 equal angles (all 60°) and 3 equal sides is an **equilateral triangle.**

Once again, it is important to remember that this relationship between equal angles and equal sides works in both directions. Take a look at these isosceles triangles, and think about what additional information we can infer from them.

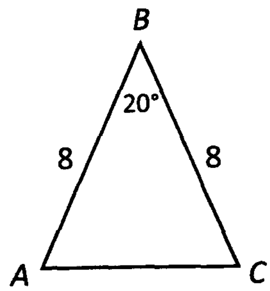


Infer: $BC = 7$



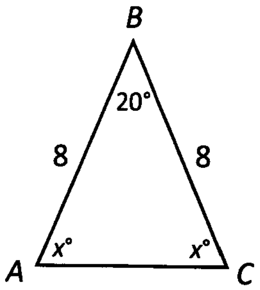
Infer: $\angle A = 55°$

The GRE loves isosceles triangles and uses them in a variety of ways. The following is a more challenging application of the equal sides/equal angles rule.



Take a look at the triangle and see what other information you can fill in. Specifically, do you know the degree measure of either BAC or BCA?

Because side *AB* is the same length as side *BC*, we know that *BAC* has the same degree measure as *BCA*. For convenience we could label each of those angles as $x°$ on our diagram.

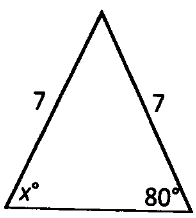


We also know that the 3 internal angles will add up to 180. So $20 + x + x = 180$. $2x = 160$, and $x = 80$. So *BAC* and *BCA* each equal 80°. We can't find the side length *AC* without more advanced math, but the GRE wouldn't ask you for this side length for that very reason.

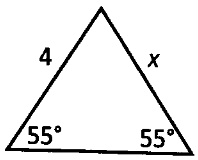
Check Your Skills

Find the value of *x*.

6.



7.



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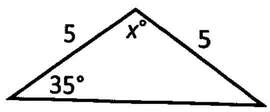
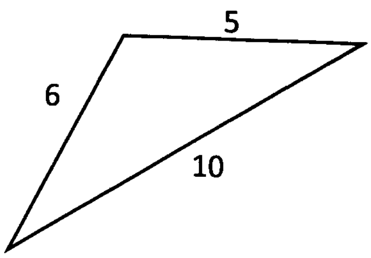


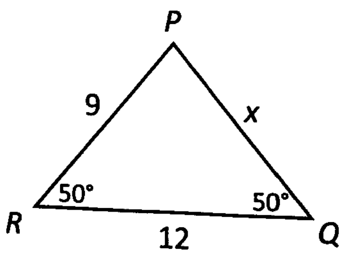
Answers can be found on pages 55–56.

Perimeter and Area

The **perimeter** of a triangle is the sum of the lengths of all 3 sides.



In this triangle, the perimeter is $5 + 6 + 10 = 21$. This is a relatively simple property of a triangle, so often it will be used in combination with another property. Try this next problem. What is the perimeter of triangle *PQR*?

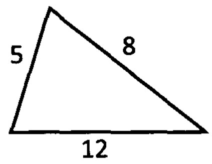


To solve for the perimeter, we will need to determine the value of *x*. Because angles *PQR* and *PRQ* are both 50°, we know that their opposite sides will have equal lengths. That means sides *PR* and *PQ* must have equal lengths, so we can infer that side *PQ* has a length of 9. The perimeter of triangle *PQR* is $9 + 9 + 12 = 30$.

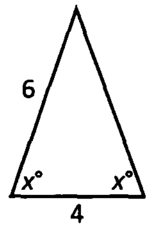
Check Your Skills

What is the perimeter of each triangle?

9.



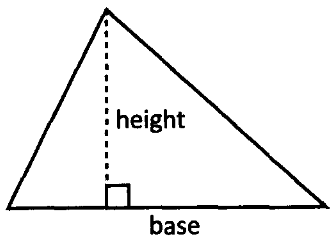
10.



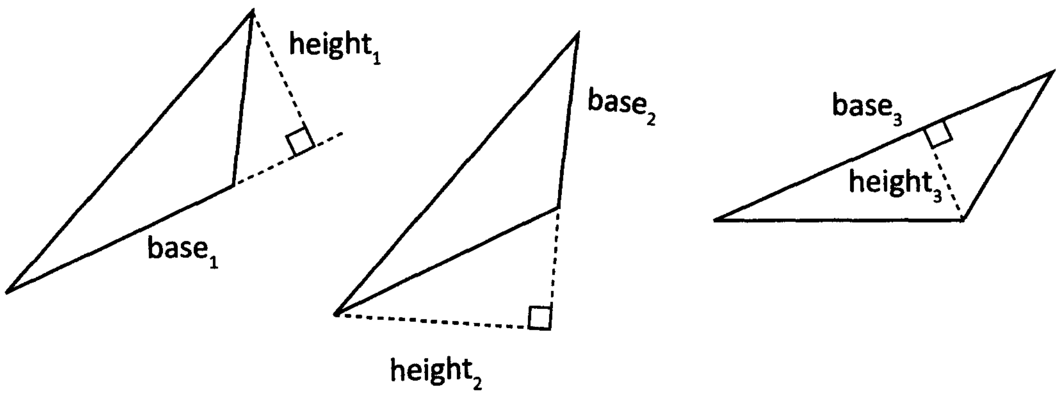
Answers can be found on page 56.

Note: Figures not drawn to scale. You need to be ready to solve geometry problems without depending on exactly accurate figures.

The final property of a triangle we will discuss is area. You may be familiar with the equation Area = $\frac{1}{2}$ (base) × (height). One very important thing to understand about the area of a triangle (and area in general) is the relationship between the base and the height. The base and the height MUST be perpendicular to each other. In a triangle, one side of the triangle is the base, and the height is formed by dropping a line from the third point of the triangle straight down towards the base, so that it forms a 90° angle with the base. The small square located where the height and base meet (in the figure below) is a very common symbol used to denote a right angle.



An additional challenge on the GRE is that problems will ask you about familiar shapes but present them to you in orientations you are not accustomed to. Even the area of a triangle is affected. Most people generally think of the base of the triangle as the bottom side of the triangle, but in reality, any side of the triangle could act as a base. In fact, depending on the orientation of the triangle, there may not actually be a bottom side. The three triangles below are all the same triangle, but in each one we have made a different side the base, and drawn in the corresponding height.

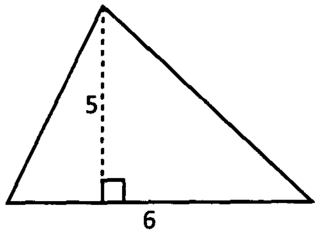


As it turns out, not only can any side be the base, but the height doesn't even need to appear in the triangle! The only thing that matters is that the base and the height are perpendicular to each other.

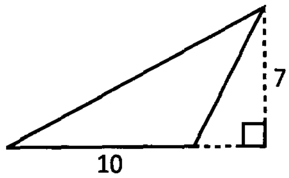
Check Your Skills

What are the areas of the following triangles?

11.



12.

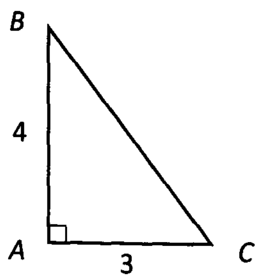


Answers can be found on page 56–57.

Right Triangles

There is one more class of triangle that is very common on the GRE: the **right triangle.** A right triangle is any triangle in which one of the angles is a right angle. The reason they are so important will become more clear as we attempt to answer the next question.

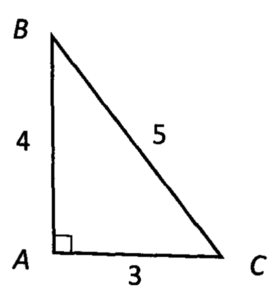
What is the perimeter of triangle *ABC*?



Normally we would be unable to answer this question. We only have two sides of the triangle, but we need all three sides to calculate the perimeter.

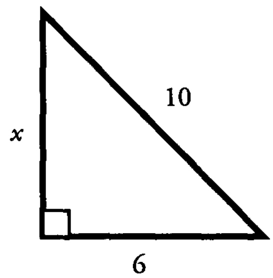
The reason we can answer this question is that right triangles have an additional property that the GRE likes to make use of: there is a consistent relationship among the lengths of its sides. This relationship is known as the **Pythagorean Theorem.** For *any* right triangle, the relationship is $a^2 + b^2 = c^2$, where a and b are the lengths of the sides touching the right angle, also known as **legs**, and c is the length of the side opposite the right angle, also known as the **hypotenuse.**

In the above triangle, sides *AB* and *AC* are a and b (it doesn't matter which is which) and side *BC* is c. So $(3)^2 + (4)^2 = (BC)^2$. $9 + 16 = (BC)^2$, so $25 = (BC)^2$, and the length of side *BC* is 5. Our triangle really looks like this:



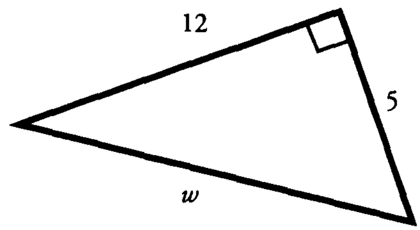
Finally, the perimeter = 3 + 4 + 5 = 12.

Pythagorean Theorem: $a^2 + b^2 = c^2$



What is x?

$$a^2 + b^2 = c^2$$
$$x^2 + 6^2 = 10^2$$
$$x^2 + 36 = 100$$
$$x^2 = 64$$
$$x = 8$$



What is w?

$$a^2 + b^2 = c^2$$
$$5^2 + 12^2 = w^2$$
$$25 + 144 = w^2$$
$$169 = w^2$$
$$13 = w$$

Pythagorean Triplets

As mentioned above, right triangles show up in many problems on the GRE, and many of these problems require the Pythagorean Theorem. But there is a shortcut that we can use in many situations to make the calculations easier.

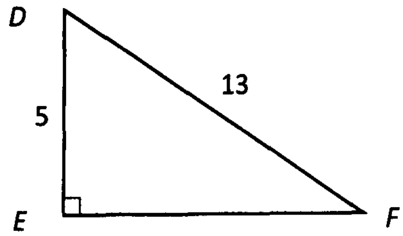
The GRE favors a certain subset of right triangles in which all three sides have lengths that are integer values. The triangle we saw above was an example of that. The lengths of the sides were 3, 4 and 5—all integers. This group of side lengths is a **Pythagorean triplet**—a 3–4–5 triangle. Although there is an infinite number of Pythagorean triplets, a few are likely to appear on the test and should be memorized. For each triplet, the first two numbers are the lengths of the sides that *touch the right angle*, and the third (and largest) number is the *length of the hypotenuse*. They are:

Common Combinations	**Key Multiples**
3–4–5 The most popular of all right triangles $3^2 + 4^2 = 5^2$ $(9 + 16 = 25)$	6–8–10 9–12–15 12–16–20
5–12–13 Also quite popular on the GRE $5^2 + 12^2 = 13^2$ $(25 + 144 = 169)$	10–24–26
8–15–17 This one appears less frequently $8^2 + 15^2 = 17^2$ $(64 + 225 = 289)$	None

Warning! Even as you memorize these triangles, don't assume that all triangles fall into these categories. When using common combinations to solve a problem, be sure that the triangle is a right triangle, and that the largest side (hypotenuse) corresponds to the largest number in the triplet. For example, if you have a right triangle with one side measuring 3 and the hypotenuse measuring 4, DO NOT conclude that the remaining side is 5.

That being said, let's look at a practice question to see how memorizing these triplets can save us time on the GRE.

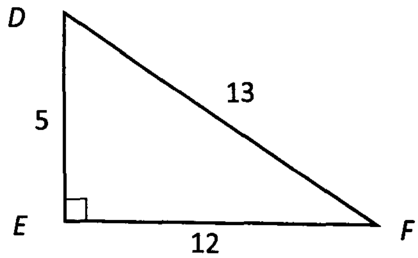
What is the area of triangle *DEF*?



What do we need in order to find the area of triangle *DEF*? Like any triangle, the formula is area = $\frac{1}{2}$ (base) × (height), so we need a base and a height. This is a right triangle, so sides *DE* and *EF* are perpendicular to each other, which means that if we can figure out the length of side *EF*, we can calculate the area.

The question then becomes, how do we find the length of side *EF*? First, realize that we can ***always*** find the length of the third side of a right triangle if we know the lengths of the other two sides. That's because we know the Pythagorean Theorem. In this case, the formula would look like this: $(DE)^2 + (EF)^2 = (DF)^2$. We know the lengths of two of those sides, so we could rewrite the equation as $(5)^2 + (EF)^2 = (13)^2$. Solving this equation, we get $25 + (EF)^2 = 169$, so $(EF)^2 = 144$, which means $EF = 12$. But these calculations are unnecessary; once you see a right triangle in which one of the legs has a length of 5 and the hypotenuse has a length of 13, you should recognize the Pythagorean triplet. The length of the other leg must be 12.

However you find the length of side *EF*, our triangle now looks like this:

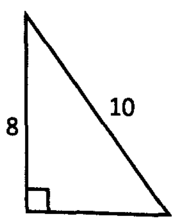


Now we have what we need to find the area of triangle *DEF*. Area = $\frac{1}{2}(12) \times (5) = \frac{1}{2}(60) = 30$. Note that in a right triangle, you can consider one leg the base and the other leg the height.

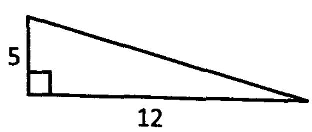
Check Your Skills

For #13–14, what is the length of the third side of the triangle? For #15, find the area.

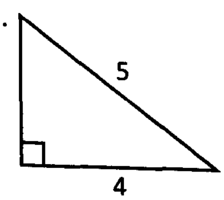
13.



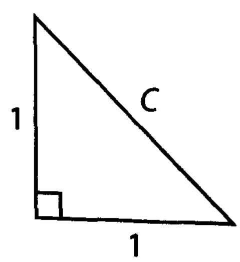
14.



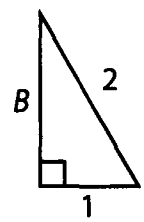
15.



16. What is the value of hypotenuse *C*? (pictured right)



17. What is the value of leg *B*? (pictured right)



18. Triangle *ABC* is isosceles. If $AB = 3$, and $BC = 4$, what are the possible lengths of *AC*?

Answers can be found on page 57.

Isosceles Triangles and the 45–45–90 Triangle

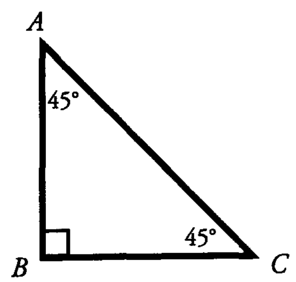
As previously noted, an isosceles triangle is one in which two sides are equal. The two angles opposite those two sides will also be equal. The most important isosceles triangle on the GRE is the isosceles right triangle.

An isosceles right triangle has one 90° angle (opposite the hypotenuse) and two 45° angles (opposite the two equal legs). This triangle is called the 45–45–90 triangle.

The lengths of the legs of every 45–45–90 triangle have a specific ratio, which you must memorize:

45°	→	45°	→	90°
leg		leg		hypotenuse
1	:	1	:	$\sqrt{2}$
x	:	x	:	$x\sqrt{2}$

What does it mean that the sides of a 45–45–90 triangle are in a $1 : 1 : \sqrt{2}$ ratio? It doesn't mean that they are actually 1, 1, or $\sqrt{2}$ (although that's a possibility). It means that the sides are some multiple of $1 : 1 : \sqrt{2}$. For instance, they could be 2, 2, and $2\sqrt{2}$, or 5.5, 5.5, and $5.5\sqrt{2}$. In the last two cases, the number we multiplied the ratio by—either 2 or 5.5—is called the "multiplier." Using a multiplier of 2 has the same effect as doubling a recipe—each of the ingredients gets doubled. Of course you can also triple a recipe or multiply it by any other number, even a fraction.



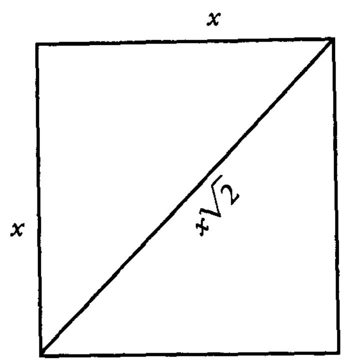
Given that the length of side *AB* is 5, what are the lengths of sides *BC* and *AC*?

Since *AB* is 5, we use the ratio $1 : 1 : \sqrt{2}$ for sides $AB : BC : AC$ to determine that the multiplier x is 5. We then find that the sides of the triangle have lengths $5 : 5 : 5\sqrt{2}$ Therefore, the length of side $BC = 5$, and the length of side $AC = 5\sqrt{2}$ Using the same figure, let's discuss the following problem.

Given that the length of side *AC* is $\sqrt{18}$, what are the lengths of sides *AB* and *BC*?

Since the hypotenuse AC is $\sqrt{18} = x\sqrt{2}$, we find that $x = \sqrt{18} \div \sqrt{2} = \sqrt{9} = 3$. Thus, the sides AB and BC are each equal to x, or 3.

One reason that the 45–45–90 triangle is so important is that this triangle is exactly half of a square! That is, two 45–45–90 triangles put together make up a square. Thus, if you are given the diagonal of a square, you can use the 45–45–90 ratio to find the length of a side of the square.



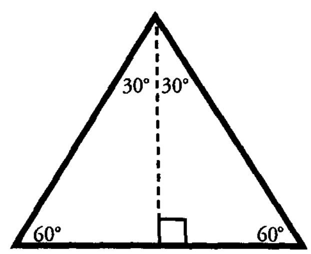
Check Your Skills

19. What is the area of a square with diagonal of 6?
20. What is the diagonal of a square with an area of 25?

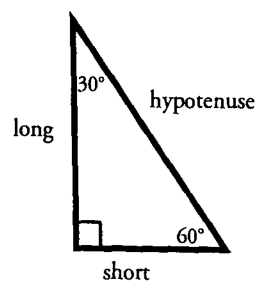
Answers can be found on pages 57.

Equilateral Triangles and the 30–60–90 Triangle

An equilateral triangle is one in which all three sides (and all three angles) are equal. Each angle of an equilateral triangle is 60° (because all 3 angles must sum to 180°). A close relative of the equilateral triangle is the 30–60–90 triangle. Notice that two of these triangles, when put together, form an equilateral triangle:



EQUILATERAL TRIANGLE



30–60–90 TRIANGLE

The lengths of the legs of every 30–60–90 triangle have the following ratio, which you must memorize:

30°	→	60°	→	90°
short leg		long leg		hypotenuse
1	**:**	$\sqrt{3}$	**:**	**2**
x	**:**	$x\sqrt{3}$	**:**	**2x**

Given that the short leg of a 30–60–90 triangle has a length of 6, what are the lengths of the long leg and the hypotenuse?

The short leg, which is opposite the 30 degree angle, is 6. We use the ratio $1 : \sqrt{3} : 2$ to determine that the multiplier x is 6. We then find that the sides of the triangle have lengths $6 : 6\sqrt{3} : 12$. The long leg measures $6\sqrt{3}$ and the hypotenuse measures 12.

Given that an equilateral triangle has a side of length 10, what is its height?

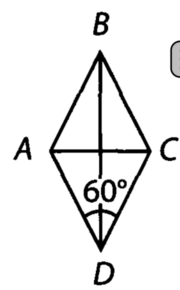
Looking at the equilateral triangle above, we can see that the side of an equilateral triangle is the same as the hypotenuse of a 30–60–90 triangle. Additionally, the height of an equilateral triangle is the same as the long leg of a 30–60–90 triangle.

Since we are told that the hypotenuse is 10, we use the ratio $x : x\sqrt{3} : 2x$ to set $2x = 10$ and determine that the multiplier x is 5. We then find that the sides of the 30–60–90 triangle have lengths $5 : 5\sqrt{3} : 10$. Thus, the long leg has a length of $5\sqrt{3}$ which is the height of the equilateral triangle.

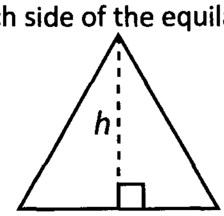
If you get tangled up on a 30–60–90 triangle, try to find the length of the short leg. The other legs will then be easier to figure out.

Check Your Skills

21. Quadrilateral *ABCD* (to the right) is composed of four 30–60–90 triangles. If $BD = 10(\sqrt{3})$, what is the perimeter of *ABCD*?



22. Each side of the equilateral triangle below is 2. What is the height of the triangle?



Answers can be found on page 58.

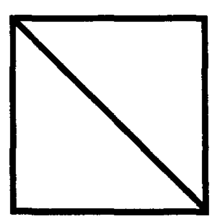
Diagonals of Other Polygons

Right triangles are useful for more than just triangle problems. They are also helpful for finding the diagonals of other polygons, specifically squares, cubes, rectangles, and rectangular solids.

The diagonal of a square can be found using this formula: $d = s\sqrt{2}$, where s is a side of the square. This is also the face diagonal of a cube.

Alternatively, you can recall that any square can be divided into two 45–45–90 triangles, and you can use the ratio $1 : 1 : \sqrt{2}$ to find the diagonal. You can also always use the Pythagorean Theorem.

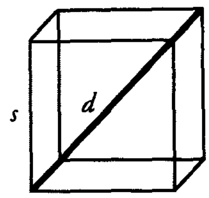
Given a square with a side of length 5, what is the length of the diagonal of the square?



Using the formula $d = s\sqrt{2}$, we find that the length of the diagonal of the square is $5\sqrt{2}$.

The main diagonal of a cube can be found using this formula: $d = s\sqrt{3}$, where s is an edge of the cube.

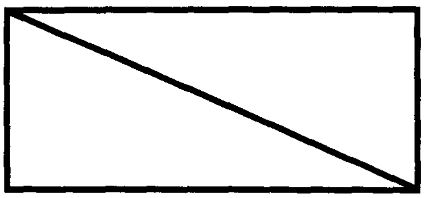
What is the measure of an edge of a cube with a main diagonal of length $\sqrt{60}$?



Again, using the formula $d = s\sqrt{3}$, we solve as follows:

$$\sqrt{60} = s\sqrt{3} \rightarrow s = \frac{\sqrt{60}}{\sqrt{3}} = \sqrt{20}$$

Thus, the length of the edge of the cube is $\sqrt{20} = 2\sqrt{5}$.



To find the diagonal of a rectangle, you must know EITHER the length and the width OR one dimension and the proportion of one to the other.

We will use the rectangle to the left for the next two problems.

If the rectangle above has a length of 12 and a width of 5, what is the length of the diagonal?

Using the Pythagorean Theorem, we solve:

$$5^2 + 12^2 = c^2 \rightarrow 25 + 144 = c^2 \rightarrow c = 13$$

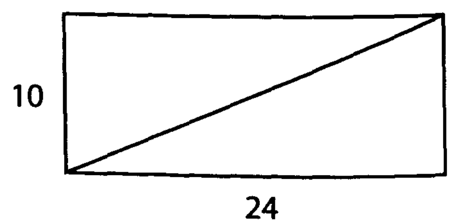
The diagonal length is 13.

If the rectangle above has a width of 6, and the ratio of the width to the length is 3 : 4, what is the diagonal?

In this problem, we can use the ratio to find the value of the length. Using the ratio of 3 : 4 given in this problem, we find that the length is 8. Then we can use the Pythagorean Theorem. Alternatively, we can recognize that this is a 6–8–10 triangle. Either way, we find that the diagonal length is 10.

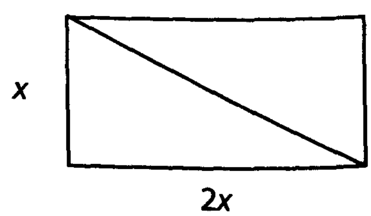
Check Your Skills

23. What is the diagonal of the rectangle to the right?



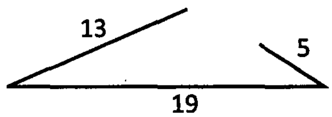
24. If the rectangle to the right has a perimeter of 6, what is its diagonal?

Answers can be found on page 58.



Check Your Skills Answers

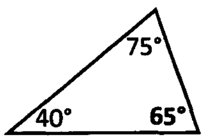
1. **No:** If the two known sides of the triangle are 5 and 19, then the third side of the triangle cannot have a length of 13, because that would violate the rule that any two sides of a triangle must add up to greater than the third side. 5 + 13 = 18, and 18 < 19.



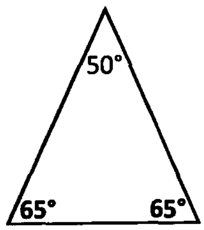
No possible triangle with these lengths.

2. **9 < third side < 25:** If the two known sides of the triangle are 8 and 17, then the third side must be less than the sum of the other 2 sides. 8 + 17 = 25, so the third side must be less than 25. The third side must also be greater than the difference of the other two sides. 17 – 8 = 9, so the third side must be greater than 9. That means that 9 < third side < 25.

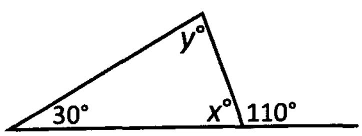
3. **65°:** The internal angles of a triangle must add up to 180°, so we know that $40 + 75 + x = 180$. Solving for x gives us $x = 65°$.



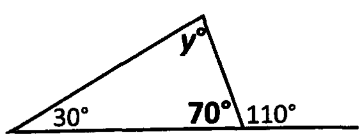
4. **65°:** The 3 internal angles of the triangle must add up to 180°, so $50 + x + x = 180$. That means that $2x = 130$, and $x = 65$.



5. $\boldsymbol{x = 70°, y = 80°}$**:** In order to determine the missing angles of the triangle, we need to do a little work with the picture. We can figure out the value of x, because straight lines have a degree measure of 180, so $110 + x = 180$, which means $x = 70$.

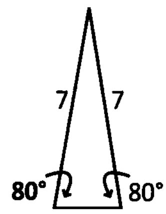


That means our picture looks like this:

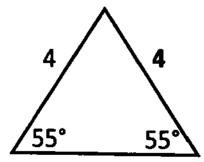


Now we can find y, because $30 + 70 + y = 180$. Solving for y gives us $y = 80$.

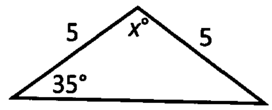
6. **80°:** In this triangle, two sides have the same length, which means this triangle is isosceles. We also know that the two angles opposite the two equal sides will also be equal. That means that x must be 80.



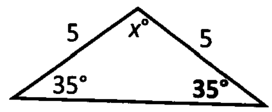
7. **4:** In this triangle, two angles are equal, which means this triangle is isosceles. We also know that the two sides opposite the equal angles must also be equal, so x must equal 4.



8. **110°:** This triangle is isosceles, because two sides have the same length. That means that the angles opposite the equal sides must also be equal.

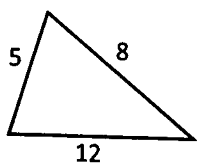


That means our triangle really looks like this:



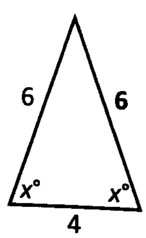
Now we can find x, because we know $35 + 35 + x = 180$. Solving for x gives us $x = 110$.

9. **25:**



To find the perimeter of the triangle, we add up all three sides. $5 + 8 + 12 = 25$, so the perimeter is 25.

10. **16:** To find the perimeter of the triangle, we need the lengths of all three sides. This is an isosceles triangle, because two angles are equal. That means that the sides opposite the equal angles must also be equal. So our triangle looks like this:



So the perimeter is $6 + 6 + 4$, which equals 16. The perimeter is 16.

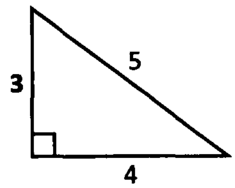
11. **15:** The area of a triangle is $\frac{1}{2}b \times h$. In the triangle shown, the base is 6 and the height is 5. So the area is $\frac{1}{2}(6) \times 5$, which equals 15.

12. **35:** In this triangle, the base is 10 and the height is 7. Remember that the height must be perpendicular to the base—it doesn't need to lie within the triangle. So the area is $\frac{1}{2}(10) \times 7$, which equals 35. The area of the triangle is 35.

13. **6:** This is a right triangle, so we can use the Pythagorean Theorem to solve for the length of the third side. The hypotenuse is the side with length 10, so the formula is $(8)^2 + b^2 = (10)^2$. $64 + b^2 = 100$. $b^2 = 36$, which means $b = 6$. So the third side of the triangle has a length of 6. Alternatively, you could recognize that this triangle is one of the Pythagorean triplets—a 6–8–10 triangle, which is just a doubled 3–4–5 triangle.

14. **13:** This is a right triangle, so we can use the Pythagorean Theorem to solve for the length of the third side. The hypotenuse is the unknown side, so the formula is $(5)^2 + (12)^2 = c^2$. $25 + 144 = c^2$. $c^2 = 169$, which means $c = 13$. So the third side of the triangle has a length of 13. Alternatively, you could recognize that this triangle is one of the Pythagorean triplets—a 5–12–13 triangle.

15. **6:** This is a right triangle, so we can use the Pythagorean Theorem to solve for the third side, or alternatively recognize that this is a 3–4–5 triangle. Either way, the result is the same: The length of the third side is 3.



Now we can find the area of the triangle. Area of a triangle is $\frac{1}{2} b \times h$, so the area of this triangle is $\frac{1}{2}(3) \times (4)$, which equals 6. The area of the triangle is 6.

16. $\sqrt{2}$**:** Apply the Pythagorean Theorem directly, substituting 1 for A and B,

$$1^2 + 1^2 = C^2$$
$$2 = C^2$$
$$C = \sqrt{2}$$

17. $\sqrt{3}$**:** Apply the Pythagorean Theorem directly, substituting 1 for A and 2 for C,

$$1^2 + B^2 = 2^2$$
$$1 + B^2 = 4$$
$$B^2 = 3$$
$$B = \sqrt{3}$$

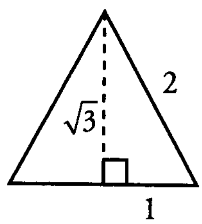
18. **3 and 4:** Since an isosceles triangle has two equal sides, the third side must be equal to one of the two named sides.

19. **18:** Let's call the side length of the square x. Thus, the diagonal would be $x\sqrt{2}$. We know the diagonal is 6, so $x\sqrt{2} = 6$. This means $x = \frac{6}{\sqrt{2}}$. The area is $x \cdot x$, or $\frac{6}{\sqrt{2}} \times \frac{6}{\sqrt{2}} = \frac{36}{2} = 18$.

20. $5\sqrt{2}$**:** If the area is 25, the side length x is 5. Since the diagonal is $x\sqrt{2}$, the diagonal is $5\sqrt{2}$.

21. **40:** The long diagonal *BD* is the sum of two long legs of the 30–60–90 triangle, so each long leg is $5\sqrt{3}$. The leg–leg–hypotenuse ratio of a 30–60–90 triangle is $x : x\sqrt{3} : 2x$, which means that $5\sqrt{3} = x\sqrt{3}$. Therefore $x = 5$, so the length of the short leg is 5 and the length of the hypotenuse is 10. Since the perimeter of the figure is the sum of four hypotenuses, the perimeter of this figure is 40.

22. $\sqrt{3}$**:** The line along which the height is measured in the figure bisects the equilateral triangle, creating two identical 30–60–90 triangles, each with a base of 1. The base of each of these triangles is the short leg of a 30–60–90 triangle. Since the leg : leg : hypotenuse ratio of a 30–60–90 triangle is $1 : \sqrt{3} : 2$, the long leg of each 30–60–90 triangle, and the height of the equilateral triangle, is $\sqrt{3}$.



23. **26:** The diagonal of the rectangle is the hypotenuse of a right triangle whose legs are the length and width of the rectangle. In this case that means that the legs of the right triangle are 10 and 24. Plug these leg lengths into the Pythagorean Theorem:

$$A^2 + B^2 = C^2$$
$$10^2 + 24^2 = C^2$$
$$C^2 = 100 + 576 = 676$$
$$C = \sqrt{676} = 26$$

You could use the calculator to take this big square root.

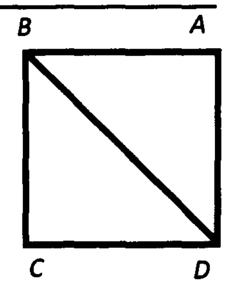
Alternatively, you could recognize the 10 : 24 : 26 triangle (a multiple of the more common 5 : 12 : 13 triangle), and save yourself the trouble.

24. $\sqrt{5}$: The perimeter of a rectangle is 2(length + width). In this case, that means $2(x + 2x)$, or $6x$. We are told the perimeter equals 6, so $6x = 6$, and $x = 1$. Therefore the length $(2x)$ is 2 and the width (x) is 1. The diagonal of the rectangle is the hypotenuse of a right triangle whose legs are the length and width of the rectangle. Plug the leg lengths into the Pythagorean theorem:

$$A^2 + B^2 = C^2$$
$$1^2 + 2^2 = C^2$$
$$C^2 = 1 + 4 = 5$$
$$C = \sqrt{5}$$

Problem Set (Note: Figures are not drawn to scale.)

1. A square is bisected into two equal triangles (see figure to the right). If the length of BD is $16\sqrt{2}$ inches, what is the area of the square?



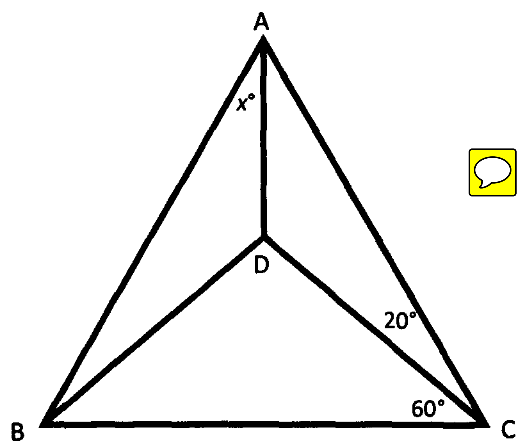
2. Beginning in Town A, Biker Bob rides his bike 10 miles west, 3 miles north, 5 miles east, and then 9 miles north, to Town B. How far apart are Town A and Town B? (Ignore the curvature of the earth.)

3. Now in Town B, Biker Bob walks due west, and then straight north to Town C. If Town B and Town C are 26 miles apart, and Biker Bob went 10 miles west, how many miles north did he go? (Again, ignore the curvature of the earth.)

4. The longest side of an isosceles right triangle measures $20\sqrt{2}$. What is the area of the triangle?

5. A square field has an area of 400 square meters. Posts are set at all corners of the field. What is the longest distance between any two posts?

6. In Triangle *ABC*, $AD = DB = DC$ (see figure to the right). Given that angle *DCB* is 60° and angle *ACD* is 20°, what is the measure of angle *x*?

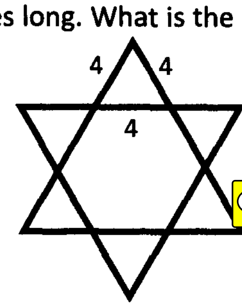


7. Two sides of a triangle are 4 and 10. If the third side is an integer *x*, how many possible values are there for *x*?

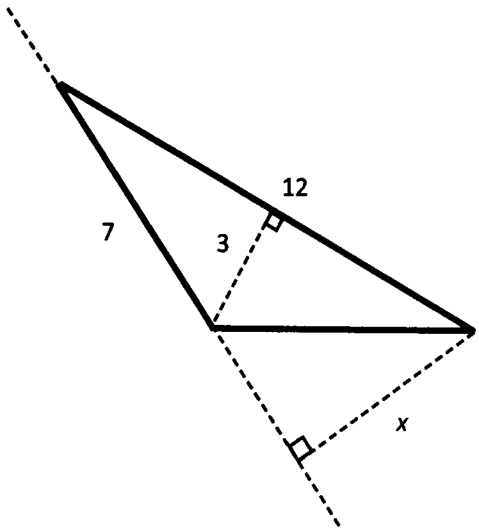
8. Jack makes himself a clay box in the shape of a cube, the edges of which are 4 inches long. What is the longest object he could fit inside the box (i.e., what is the diagonal of the cube)?

9. What is the area of an equilateral triangle whose sides measure 8 cm long?

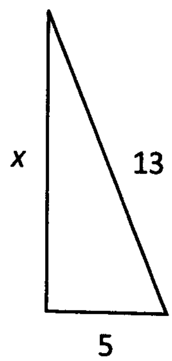
10. The points of a six-pointed star consist of six identical equilateral triangles, with each side 4 cm (see figure). What is the area of the entire star, including the center?



11. What is x in the diagram below?

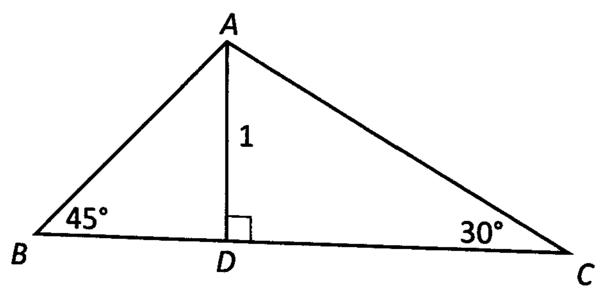


12.



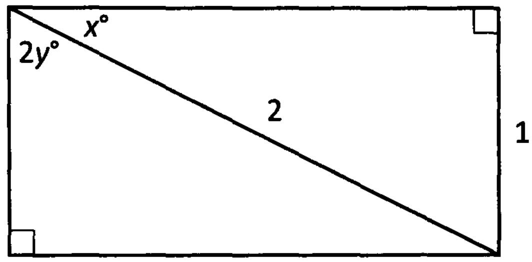
Quantity A	Quantity B
x	12

13.



Quantity A	Quantity B
The perimeter of triangle ABC above	5

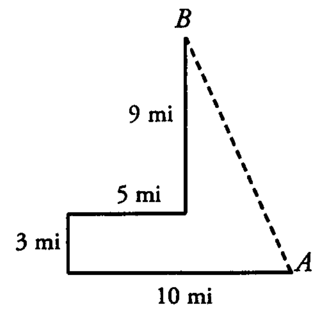
14.



Column A	Column B
x	y

1. **256 square units:** The diagonal of a square is $s\sqrt{2}$; therefore, the side length of square ABCD is 16. The area of the square is s^2, or $16^2 = 256$.

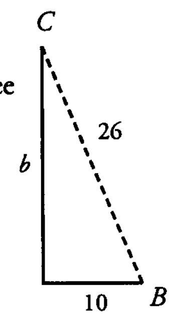
2. **13 miles:** If you draw a rough sketch of the path Biker Bob takes, as shown to the right, you can see that the direct distance from A to B forms the hypotenuse of a right triangle. The short leg (horizontal) is $10 - 5 = 5$ miles, and the long leg (vertical) is $9 + 3 = 12$ miles. Therefore, you can use the Pythagorean Theorem to find the direct distance from A to B:



$$\begin{aligned} 5^2 + 12^2 &= c^2 \\ 25 + 144 &= c^2 \\ c^2 &= 169 \\ c &= 13 \end{aligned}$$

You might recognize the common right triangle: 5–12–13.

3. **24 miles:** If you draw a rough sketch of the path Biker Bob takes, as shown to the right, you can see that the direct distance from B to C forms the hypotenuse of a right triangle.



$$\begin{aligned} 10^2 + b^2 &= 26^2 \\ 100 + b^2 &= 676 \\ b^2 &= 576 \\ b &= 24 \end{aligned}$$

You might also recognize this as a multiple of the common 5–12–13 triangle.

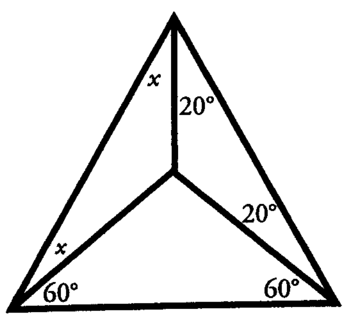
4. **200:** An isosceles right triangle is a 45–45–90 triangle, with sides in the ratio of $1 : 1 : \sqrt{2}$. If the longest side, the hypotenuse, measures $20\sqrt{2}$, the two other sides each measure 20. Therefore, the area of the triangle is:

$$A = \frac{b \times h}{2} = \frac{20 \times 20}{2} = 200$$

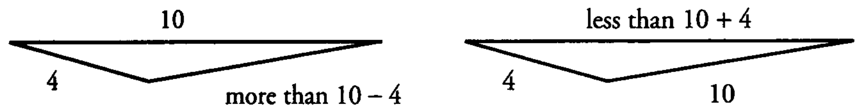
5: **$20\sqrt{2}$**: The longest distance between any two posts is the diagonal of the field. If the area of the square field is 400 square meters, then each side must measure 20 meters. Diagonal $= d = s\sqrt{2}$, so $d = 20\sqrt{2}$.

6. **10:** If $AD = DB = DC$, then the three triangular regions in this figure are all isosceles triangles. Therefore, we can fill in some of the missing angle measurements as shown to the right. Since we know that there are 180° in the large triangle ABC, we can write the following equation:

$$\begin{aligned} x + x + 20 + 20 + 60 + 60 &= 180 \\ 2x + 160 &= 180 \\ x &= 10 \end{aligned}$$

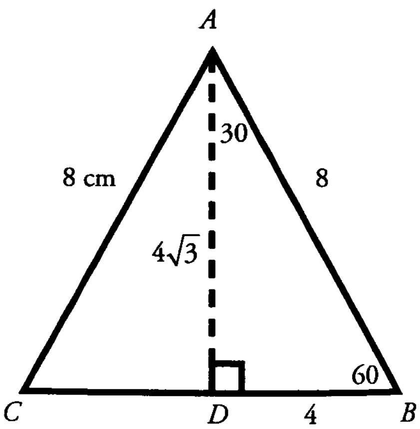


7. **7:** If two sides of a triangle are 4 and 10, the third side must be greater than 10 – 4 and smaller than 10 + 4. Therefore, the possible values for x are {7, 8, 9, 10, 11, 12, and 13}. You can draw a sketch to convince yourself of this result:



8. **$4\sqrt{3}$:** The diagonal of a cube with side s is $s\sqrt{3}$ Therefore, the longest object Jack could fit inside the box would be $4\sqrt{3}$ inches long.

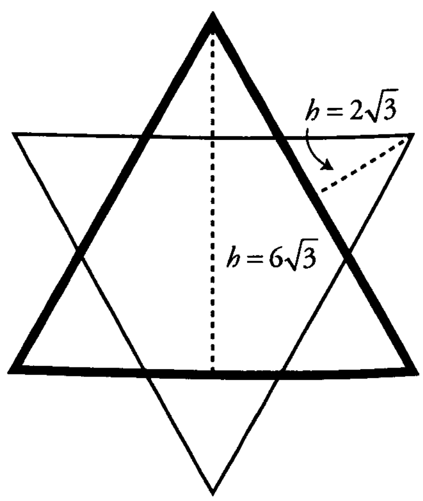
9. **$16\sqrt{3}$:** Draw in the height of the triangle (see figure). If triangle ABC is an equilateral triangle, and ABD is a right triangle, then ABD is a 30–60–90 triangle. Therefore, its sides are in the ratio of $1:\sqrt{3}:2$. If the hypotenuse is 8, the short leg is 4, and the long leg is $4\sqrt{3}$ This is the height of triangle ABC. Find the area of triangle ABC with the formula for area of a triangle:



$$A = \frac{b \times h}{2} = \frac{8 \times 4\sqrt{3}}{2} = 16\sqrt{3}$$

10. **$48\sqrt{3}$cm²:** We can think of this star as a large equilateral triangle with sides 12 cm long, and three additional smaller equilateral triangles with sides 4 inches long. Using the same 30–60–90 logic we applied in problem #9, we can see that the height of the larger equilateral triangle is $6\sqrt{3}$, and the height of the smaller equilateral triangle is $2\sqrt{3}$.

Therefore, the areas of the triangles are as follows:



Large triangle: $$A=\frac{b\times h}{2}=\frac{12\times 6\sqrt{3}}{2}=36\sqrt{3}$$

Small triangles: $$A=\frac{b\times h}{2}=\frac{4\times 2\sqrt{3}}{2}=4\sqrt{3}$$

The total area of three smaller triangles and one large triangle is:

$$36\sqrt{3}+3(4\sqrt{3})=48\sqrt{3}\ \text{cm}^2.$$

11. **36/7:** We can calculate the area of the triangle, using the side of length 12 as the base:

$$\frac{1}{2}(12)(3) = 18$$

Next, we use the side of length 7 as the base and write the equation for the area:

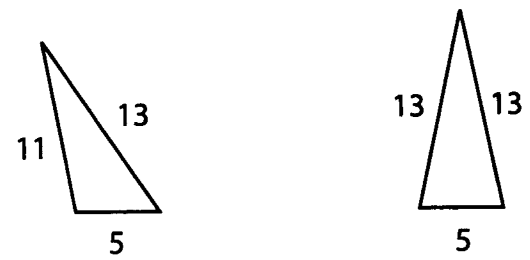
$$\frac{1}{2}(7)(x) = 18$$

Now solve for x, the unknown height.

$$7x = 36$$
$$x = \frac{36}{7}$$

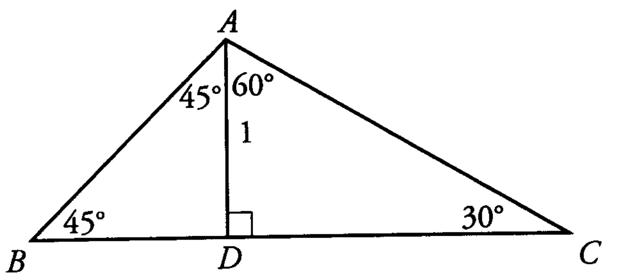
You could also solve this problem using the Pythagorean Theorem, but the process is *much* harder.

12. **D:** Although this appears to be a 5 : 12 : 13 triangle, we do not know that it is a right triangle. There is no "right triangle" symbol in the diagram. Remember, Don't Trust The Picture! Below are a couple of possible triangles:

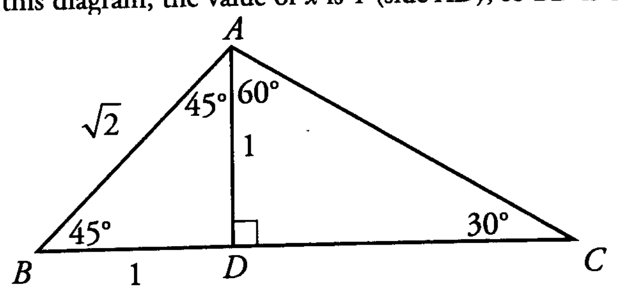


Therefore **we do not have enough information** to answer the question.

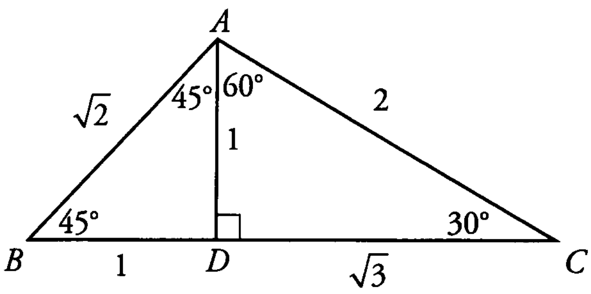
13. **A:** Although there seems to be very little information here, the two small triangles that comprise *ABC* may seem familiar. First, fill in the additional angles in the diagram.



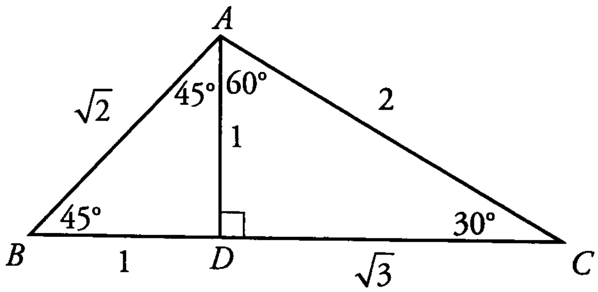
With the additional angles filled in, it is clear that the two smaller triangles are special right triangles: a 45–45–90 triangle and a 30–60–90 triangle. We know the ratios of the side lengths for each of these triangles. For a 45–45–90 triangle the ratio is $x : x : x\sqrt{2}$. In this diagram, the value of x is 1 (side AD), so BD is 1 and AB is $\sqrt{2}$.



For a 30–60–90 triangle, the ratio is $x : x\sqrt{3} : 2x$. In this diagram, x is 1 (side AD), so CD is $\sqrt{3}$ and AC is 2.



Now we can calculate the perimeter of triangle ABC.



Quantity A	Quantity B
The perimeter of triangle ABC above = $\mathbf{1 + 2 + \sqrt{2} + \sqrt{3}}$	5

Now we need to compare this sum to 5. A good approximation of $\sqrt{2}$ is 1.4 and a good approximation of $\sqrt{3}$ is 1.7.

Quantity A	Quantity B
$1 + 2 + \sqrt{2} + \sqrt{3} \approx$ $1 + 2 + 1.4 + 1.7 = \mathbf{5.1}$	5

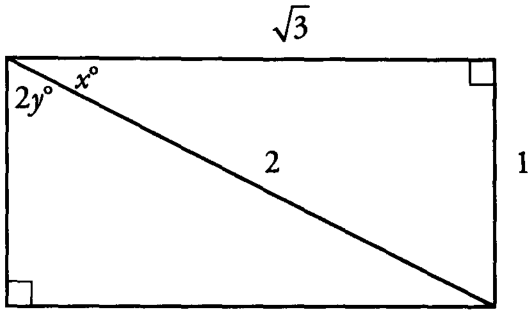
Therefore **Quantity A is larger.**

Alternatively, you could use the calculator to compute Quantity A.

14. **C:** The diagonal of the rectangle is the hypotenuse of a right triangle whose legs are the length and width of the rectangle. In this case we are given the width and diagonal. Plug those into the Pythagorean Theorem to determine the length:

$$A^2 + B^2 = C^2$$
$$1^2 + B^2 = 2^2$$
$$1 + B^2 = 4$$
$$B^2 = 3$$
$$B = \sqrt{3}$$

Plug this value into the diagram.



The key to this question is recognizing that each of the triangles are 30–60–90 triangles. Any time you see a right triangle and one of the sides has a length of $\sqrt{3}$ or a multiple of $\sqrt{3}$, you should check to see if it is a 30–60–90 triangle. Another clue is a right triangle in which the hypotenuse is twice the length of one of the sides.

Now, in addition to the side lengths, you can fill in the values of the angles in this diagram. Angle x is opposite the short leg, which means it has a degree measure of 30. Similarly, $2y$ is opposite the long leg, which means it has a degree measure of 60.

$2y = 60$
$y = 30$

Quantity A	Quantity B
$x = 30$	$y = 30$

Therefore **the two quantities are the same.**

Chapter 4
of
GEOMETRY

POLYGONS

In This Chapter . . .

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- Quadrilaterals: An Overview
- Polygons and Interior Angles
- Polygons and Perimeter
- Polygons and Area
- 3 Dimensions: Surface Area
- 3 Dimensions: Volume
- Quadrilaterals
- Maximum Area of Polygons

POLYGONS

Polygons are a very familiar sight on the GRE. As you saw in the last chapter, many questions about triangles will often involve other polygons, most notably quadrilaterals. Mastery of polygons will ultimately involve understanding the basic properties, such as perimeter and area, and will also involve the ability to distinguish polygons from other shapes in diagrams that include other polygons or circles.

A polygon is defined as a closed shape formed by line segments. The polygons tested on the GRE include the following:

- Three-sided shapes (Triangles)
- Four-sided shapes (Quadrilaterals)
- Other polygons with *n* sides (where *n* is five or more)

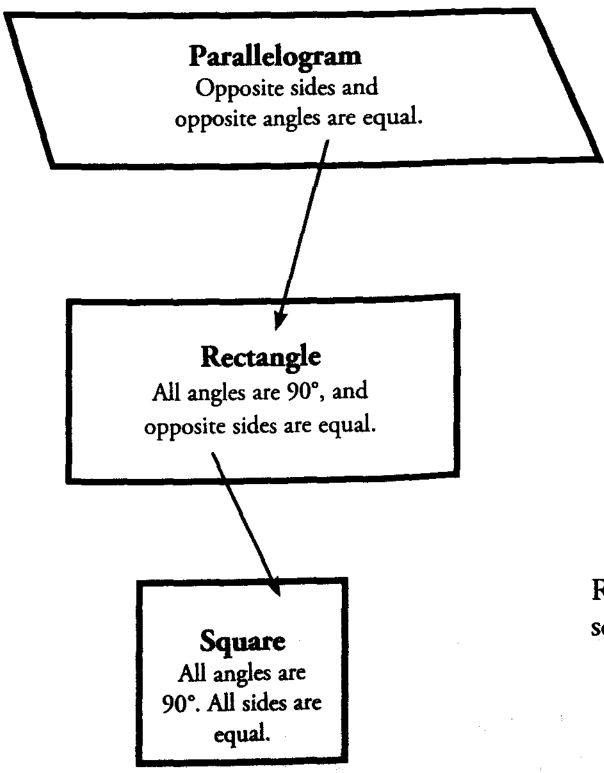
This section will focus on polygons of four or more sides. In particular, the GRE emphasizes quadrilaterals—four-sided polygons—squares, rectangles, parallelograms, and trapezoids.

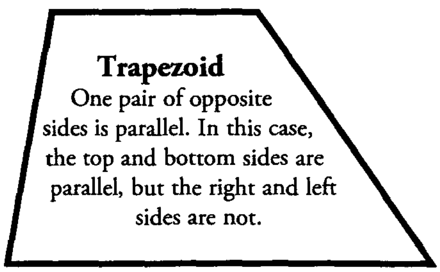
Polygons are two-dimensional shapes—they lie in a plane. The GRE tests your ability to work with different measurements associated with polygons. The measurements you must be adept with are (1) interior angles, (2) perimeter, and (3) area.

The GRE also tests your knowledge of three-dimensional shapes formed from polygons, particularly rectangular solids and cubes. The measurements you must be adept with are (1) surface area and (2) volume.

Quadrilaterals: An Overview

The most common polygon tested on the GRE, aside from the triangle, is the quadrilateral (any four-sided polygon). Almost all GRE polygon problems involve the special types of quadrilaterals shown below.





Rectangles are special types of parallelograms, and squares are special types of rectangles.

Polygons and Interior Angles

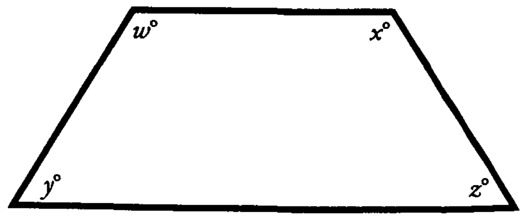
The sum of the interior angles of a given polygon depends on the **number of sides in the polygon.** The following chart displays the relationship between the type of polygon and the sum of its interior angles.

The sum of the interior angles of a polygon follows a specific pattern that depends on n, the number of sides that the polygon has. This sum is always $(n-2) \times 180$ (where n is the number of sides), because the polygon can be cut into $(n-2)$ triangles, each of which contains 180°.

Polygon	# of Sides	Sum of Interior Angles
Triangle	3	180°
Quadrilateral	4	360°
Pentagon	5	540°
Hexagon	6	720°

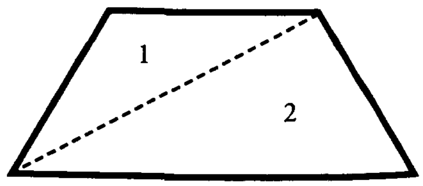
$$\mathbf{(n-2) \times 180 = \text{Sum of Interior Angles of a Polygon}}$$

If you forget this formula, you can always say "okay, a triangle has 180°, a rectangle has 360°," and so on. Add 180° for each new side.

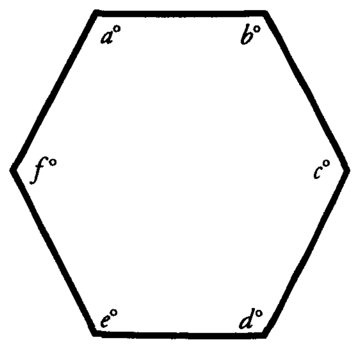


Since this polygon has four sides, the sum of its interior angles is $(4-2)180 = 2(180) = 360°$.

Alternatively, note that a quadrilateral can be cut into two triangles by a line connecting opposite corners. Thus, the sum of the angles = $2(180) = 360°$.

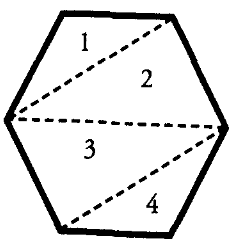


If a polygon has six sides, as in the figure below, the sum of its interior angles would be $(6-2)180 = 4(180) = 720°$.



Alternatively, note that a hexagon can be cut into four triangles by three lines connecting corners.

Thus, the sum of the angles = $4(180) = 720°$.



By the way, the corners of polygons are also known as vertices (singular: vertex).

Check Your Skills

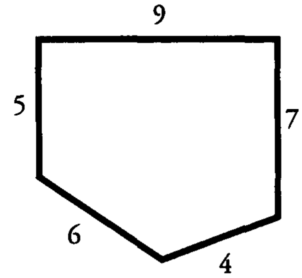
1. What is the sum of the interior angles of an octagon (eight-sided polygon)?
2. A regular polygon is a polygon in which every line is of equal length and every interior angle is equal. What is the degree measure of each interior angle in a regular hexagon (six-sided polygon)?

Answers can be found on page 81.

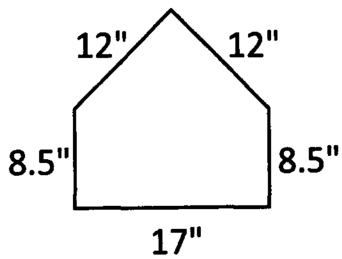
Polygons and Perimeter

The perimeter refers to the distance around a polygon, or the sum of the lengths of all the sides. The amount of fencing needed to surround a yard would be equivalent to the perimeter of that yard (the sum of all the sides).

The perimeter of the pentagon to the right is:
$9 + 7 + 4 + 6 + 5 =$ **31.**



Check Your Skills



3. The figure above represents a standard baseball home plate. What is the perimeter of this figure?

Answers can be found on page 81.

Polygons and Area

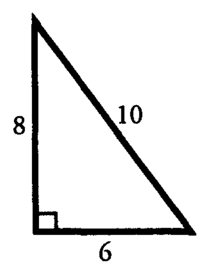
The area of a polygon refers to the space inside the polygon. Area is measured in square units, such as cm^2 (square centimeters), m^2 (square meters), or ft^2 (square feet). For example, the amount of space that a garden occupies is the area of that garden.

On the GRE, there are two polygon area formulas you MUST know:

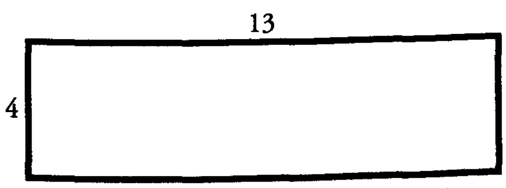
1) Area of a TRIANGLE $= \dfrac{\textbf{Base} \times \textbf{Height}}{\textbf{2}}$

The height ALWAYS refers to a line that is perpendicular (at a 90° angle) to the base.

In this triangle, the base is 6 and the height (perpendicular to the base) is 8.
The area $= (6 \times 8) \div 2 = 48 \div 2 = 24$.



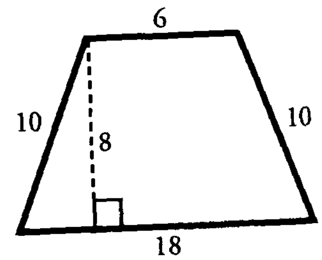
2) Area of a RECTANGLE = **Length** × **Width**



The length of this rectangle is 13, and the width is 4. Therefore, the area = $13 \times 4 = 52$.

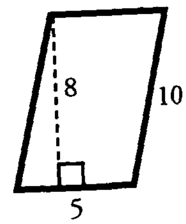
The GRE will occasionally ask you to find the area of a polygon more complex than a simple triangle or rectangle. The following formulas can be used to find the areas of other types of quadrilaterals:

3) Area of a TRAPEZOID = $\frac{(\textbf{Base}_1 + \textbf{Base}_2) \times \textbf{Height}}{\textbf{2}}$



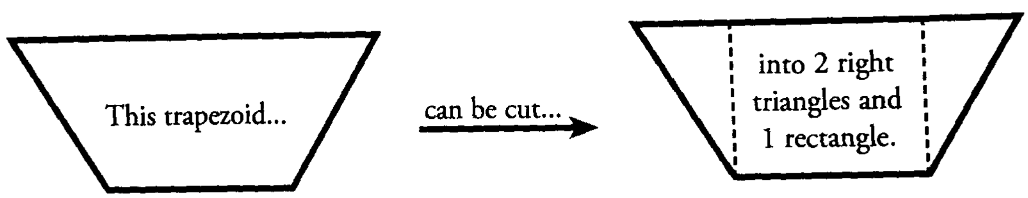
Note that the height refers to a line perpendicular to the two bases, which are parallel. (You often have to draw in the height, as in this case.) In the trapezoid shown, base_1 = 18, base_2 = 6, and the height = 8. The area = (18 + 6) × 8 ÷ 2 = 96. Another way to think about this is to take the *average* of the two bases and multiply it by the height.

4) Area of any PARALLELOGRAM = **Base** × **Height**



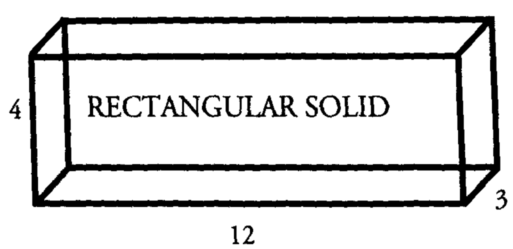
Note that the height refers to the line perpendicular to the base. (As with the trapezoid, you often have to draw in the height.) In the parallelogram shown, the base = 5 and the height = 8. Therefore, the area is 5 × 8 = 40.

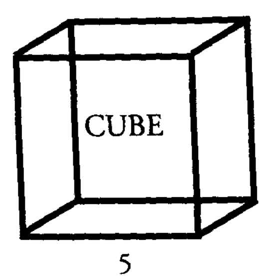
Although these formulas are very useful to memorize for the GRE, you may notice that all of the above shapes can actually be divided into some combination of rectangles and right triangles. Therefore, if you forget the area formula for a particular shape, simply cut the shape into rectangles and right triangles, and then find the areas of these individual pieces. For example:



3 Dimensions: Surface Area

The GRE tests two particular three-dimensional shapes formed from polygons: the rectangular solid and the cube. Note that a cube is just a special type of rectangular solid.





The surface area of a three-dimensional shape is the amount of space on the surface of that particular object. For example, the amount of paint that it would take to fully cover a rectangular box could be determined by finding the surface area of that box. As with simple area, surface area is measured in square units such as inches^2 (square inches) or ft^2 (square feet).

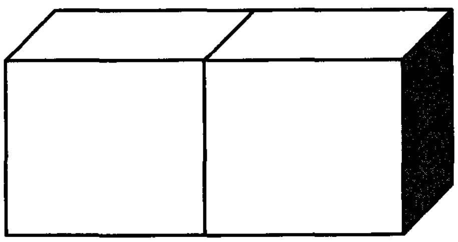
Surface Area = the SUM of the areas of ALL of the faces

Both a rectangular solid and a cube have **six faces.**

To determine the surface area of a rectangular solid, you must find the area of each face. Notice, however, that in a rectangular solid, the front and back faces have the same area, the top and bottom faces have the same area, and the two side faces have the same area. In the solid above, the area of the front face is equal to $12 \times 4 = 48$. Thus, the back face also has an area of 48. The area of the bottom face is equal to $12 \times 3 = 36$. Thus, the top face also has an area of 36. Finally, each side face has an area of $3 \times 4 = 12$. Therefore, the surface area, or the sum of the areas of all six faces equals $48(2) + 36(2) + 12(2) = 192$.

To determine the surface area of a cube, you only need the length of one side. We can see from the cube above that a cube is made of six identical square surfaces. First, find the area of one face: $5 \times 5 = 25$. Then, multiply by six to account for all of the faces: $6 \times 25 = 150$.

Check Your Skills

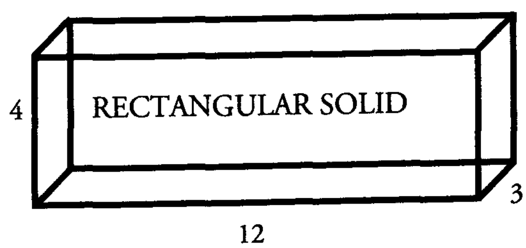


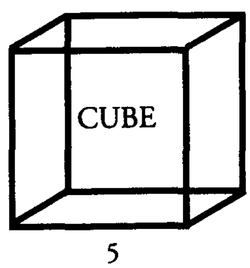
4. The figure to the left shows two wooden cubes joined to form a rectangular solid. If each cube has a surface area of 24, what is the surface area of the resulting rectangular solid?

Answers can be found on page 81.

3 Dimensions: Volume

The volume of a three-dimensional shape is the amount of "stuff" it can hold. For example, the amount of liquid that a rectangular milk carton holds can be determined by finding the volume of the carton. Volume is measured in cubic units such as inches^3 (cubic inches) or ft^3 (cubic feet).





Volume = Length × Width × Height

By looking at the rectangular solid above, we can see that the length is 12, the width is 3, and the height is 4. Therefore, the volume is $12 \times 3 \times 4 = 144$.

In a cube, all three of the dimensions—length, width, and height—are identical. Therefore, knowing the measurement of just one side of the cube is sufficient to find the volume. In the cube above, the volume is $5 \times 5 \times 5 = 125$.

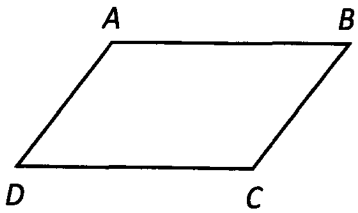
Check Your Skills

5. The volume of a rectangular solid with length 8, width 6, and height 4 is how many times the volume of a rectangular solid with length 4, width 3, and height 2?

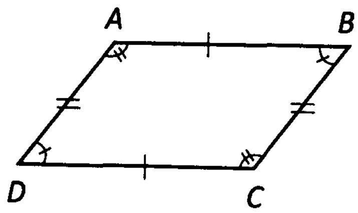
Answers can be found on page 81.

Quadrilaterals

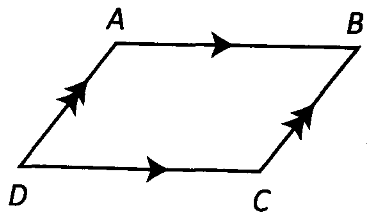
A quadrilateral is any figure with 4 sides. The GRE largely deals with one class of quadrilaterals known as **parallelograms.** A parallelogram is any 4 sided figure in which the opposite sides are parallel and equal and opposite angles are equal. This is an example of a parallelogram.



In this figure, sides *AB* and *CD* are parallel and have equal lengths, sides *AD* and *BC* are parallel and equal length, angles *ADC* and *ABC* are equal and angles *DAB* and *DCB* are equal.



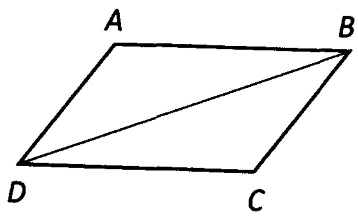
Use hash marks to indicate equal lengths or equal angles.



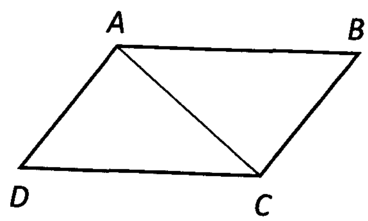
Use arrows to indicate parallel lines.

Any quadrilateral with two sets of opposite and equal sides is a parallelogram, as is any quadrilateral with two sets of opposite and equal angles.

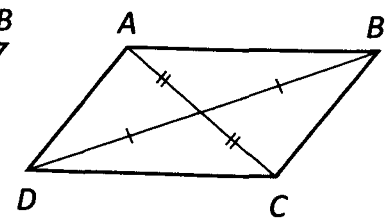
An additional property of any parallelogram is that the diagonal will divide the parallelogram into 2 equal triangles.



Triangle *ABD* = Triangle *BCD*

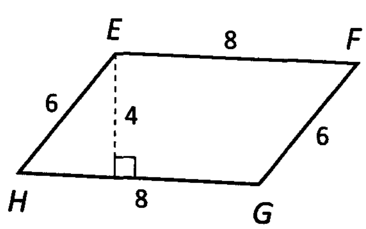


Triangle *ADC* = Triangle *ABC*



The diagonals also cut each other in half (bisect each other)

For any parallelogram, the perimeter is the sum of the lengths of all the sides and the area is equal to (base) × (height). With parallelograms, as with triangles, it is important to remember that the base and the height MUST be perpendicular to one another.



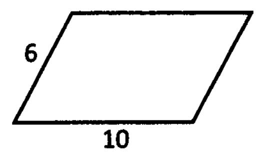
In this parallelogram, what is the perimeter, and what is the area? The perimeter is the sum of the sides, so it's 6 + 8 + 6 + 8 = 28. Alternatively, you can use one of the properties of parallelograms to calculate the perimeter in a different way.

We know that parallelograms have two sets of equal sides. In this parallelogram, two of the sides have a length of 6, and two of the sides have a length of 8. So the perimeter equals $2 \times 6 + 2 \times 8$. We can factor out a 2, and say that perimeter $= 2 \times (6 + 8) = 28$.

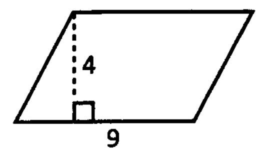
To calculate the area, we need a base and a height. It might be tempting to say that the area is $6 \times 8 = 48$. But the two sides of this parallelogram are not perpendicular to each other. The dotted line drawn into the figure, however, is perpendicular to side *HG*. The area of parallelogram *EFGH* is $8 \times 4 = 32$.

Check Your Skills

6. What is the perimeter of the parallelogram?



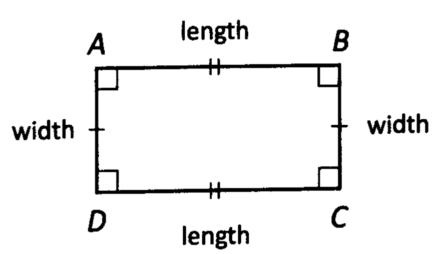
7. What is the area of the parallelogram?



Answers can be found on page 81.

Rectangles

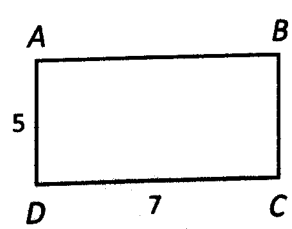
Rectangles are a specific type of parallelogram. Rectangles have all the same properties as parallelograms, with one additional property—all four internal angles of a rectangle are right angles. Additionally, with rectangles, we refer to one pair of sides as the length and one pair of sides as the width.



The formula for the perimeter of a rectangle is the same as for the perimeter of a parallelogram—either sum the lengths of the four sides or add the length and the width and multiply by 2.

The formula for the area of a rectangle is also the same as for the area of a parallelogram, but for any rectangle, the length and width are by definition perpendicular to each other, so you don't need a separate height. For this reason, the area of a rectangle is commonly expressed as (length) × (width).

Let's practice. For the following rectangle, find the perimeter and the area.



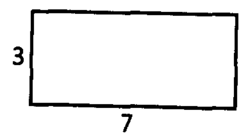
Let's start with the perimeter. Again, we can either fill in the missing sides and add them all up, or recognize that we have two sides with a length of 5 and two sides with a length of 7. Therefore, perimeter = $2 \times (5 + 7)$, which equals 24. Alternatively, $5 + 5 + 7 + 7$ also equals 24.

Now to find the area. The formula for area is (length) × (width). For the purposes of finding the area, it is irrelevant which side is the length and which side is the width. If we make *AD* the length and *DC* the width, then the area = $(5) \times (7) = 35$. If, instead, we make *DC* the length and *AD* the width, then we have area = $(7) \times (5) = 35$. The only thing that matters is that we choose two sides that are perpendicular to each other.

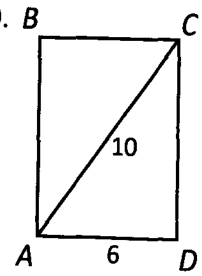
Check Your Skills

Find the area and perimeter of each rectangle.

8.



9.

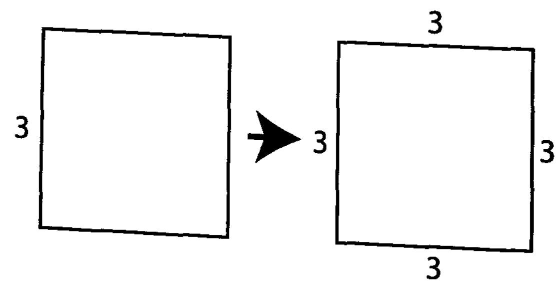


Answers can be found on pages 81–82.

Squares

One particular type of rectangle warrants mention—the square. A square is a rectangle in which the lengths of all four sides are equal. Everything that is true of rectangles is true of squares as well. What this means is that knowing only one side of a square is enough to determine the perimeter and area of a square.

For instance, if we have a square, and we know that the length of one of its sides is 3, we know that all 4 sides have a length of 3.



The perimeter of the square is $3 + 3 + 3 + 3$, which equals 12. Alternatively, once you know the length of one side of a square, you can multiply that length by 4 to find the perimeter. $3 \times 4 = 12$.

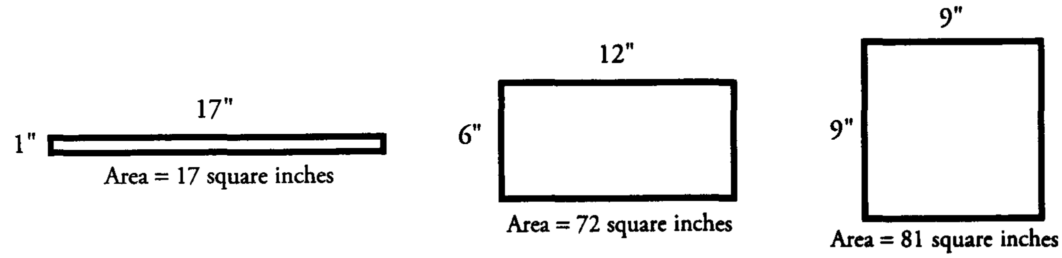
To find the area, we use the same formula as for a rectangle—Area = (length) × (width). But, because the shape is a square, we know that the length and the width are equal. Therefore, we can say that the area of a square is Area = $(\text{side})^2$. In this case, Area = $(3)^2 = 9$.

Maximum Area of Polygons

In some problems, the GRE may require you to determine the maximum or minimum area of a given figure. Following a few simple shortcuts can help you solve certain problems quickly.

Maximum Area of a Quadrilateral

Perhaps the best-known maximum–area problem is one which asks you to maximize the area of a *quadrilateral* (usually a rectangle) with a *fixed perimeter*. If a quadrilateral has a fixed perimeter, say, 36 inches, it can take a variety of shapes:



Of these figures, the one with the largest area is the square. This is a general rule: **Of all quadrilaterals with a given perimeter, the SQUARE has the largest area.** This is true even in cases involving non-integer lengths. For instance, of all quadrilaterals with a perimeter of 25 feet, the one with the largest area is a square with 25/4 = 6.25 feet per side.

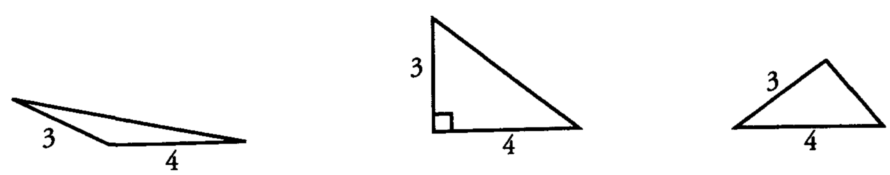
This principle can also be turned around to yield the following corollary: **Of all quadrilaterals with a given area, the SQUARE has the minimum perimeter.**

Both of these principles can be generalized for polygons with n sides: **a regular polygon with all sides equal and all angles equal will maximize area for a given perimeter and minimize perimeter for a given area.**

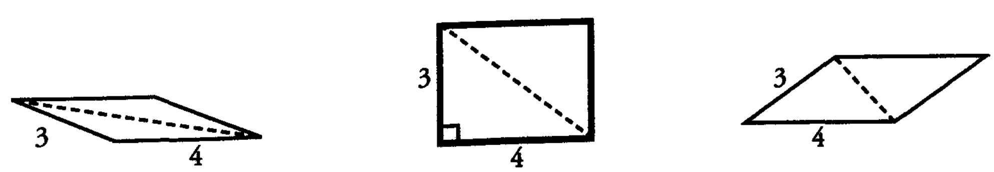
Maximum Area of a Parallelogram or Triangle

Another common optimization problem involves maximizing the area of a *triangle or parallelogram with given side lengths*.

For instance, there are many triangles with two sides 3 and 4 units long. Imagine that the two sides of length 3 and 4 are on a hinge. The third side can have various lengths:



There are many corresponding parallelograms with two sides 3 and 4 units long:



The area of a triangle is given by $A = \frac{1}{2}bh$, and the area of a parallelogram is given by $A = bh$. Because both of these formulas involve the perpendicular height h, the maximum area of each figure is achieved when the 3-unit side is perpendicular to the 4-unit side, so that the height is 3 units. All the other figures have lesser heights. (Note that in this case, the triangle of maximum area is the famous 3–4–5 right triangle.) If the sides are not perpendicular, then the figure is squished, so to speak.

The general rule is this: **if you are given two sides of a triangle or parallelogram, you can maximize the area by placing those two sides PERPENDICULAR to each other.**

Check Your Skills Answers

1. **1,080:** One way to calculate the sum of the interior angles of a polygon is by applying the formula $(n - 2)180$ = Sum of the interior angles, where n is the number of sides. Substituting 8 for n yields:

$$\begin{aligned}\text{Sum of the interior angles} &= (8 - 2)180 \\ &= (6)180 \\ &= 1{,}080\end{aligned}$$

2. **120:** Since each interior angle is the same, we can determine the angle of any one by dividing the sum of the interior angles by 6 (the number of interior angles). Use the formula $(n - 2)180$ = Sum of the interior angles, where n is the number of sides. Substituting 6 for n yields: Sum = $(4)180 = 720$. Divide 720 by 6 to get 120.

3. **58:** The sum of the five sides is 58". It is simplest to arrange them as $12 + 12 + 17 + (8\frac{1}{2} + 8\frac{1}{2}) = 12 + 12 + 17 + 17 = 58$.

4. **40:** Since the surface area of a cube is 6 times the area of one face, each square face of each cube must have an area of 4. One face of each cube is lost when the two cubes are joined, so the total surface area of the figure will be the sum of the surface areas of both cubes minus the surface areas of the covered faces.

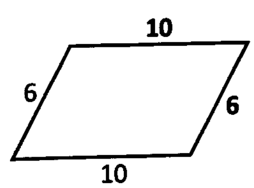
Each cube has surface area of 24, so the total surface area is 48. Subtract the surface area of each covered face (4). $48 - 2(4) = 40$.

5. **8:** The volume of a rectangular solid is the product of its three dimensions, length, width, and height.

$$8 \times 6 \times 4 = 192 \text{ and } 4 \times 3 \times 2 = 24$$

$\frac{192}{24} = 8$, so the volume of the larger cube is 8 times the volume of the smaller cube.

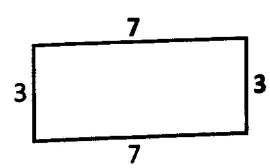
6. **32:** In parallelograms, opposite sides have equal lengths, so we know that two of the sides of the parallelogram have a length of 6 and two sides have a length of 10.



So the perimeter is 6 + 10 + 6 + 10, which equals 32.

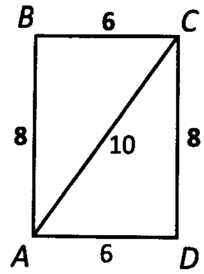
7. **36:** Area of a parallelogram is $b \times h$. In this parallelogram, the base is 9 and the height is 4, so the area is $(9) \times (4)$, which equals 36. The area of the parallelogram is 36.

8. **20, 21:** In rectangles, opposite sides have equal lengths, so our rectangle looks like this:



So the perimeter is 3 + 7 + 3 + 7, which equals 20. The area of a rectangle is $b \times h$, so the area is (7) × (3), which equals 21. So the perimeter is 20, and the area is 21.

9. **28, 48:** To find the area and perimeter of the rectangle, we need to know the length of either side *AB* or side *CD*. The diagonal of the rectangle creates a right triangle, so we can use the Pythagorean Theorem to find the length of side *CD*. Alternatively, we can recognize that triangle *ACD* is a 6–8–10 triangle, and thus the length of side *CD* is 8. Either way, our rectangle now looks like this:



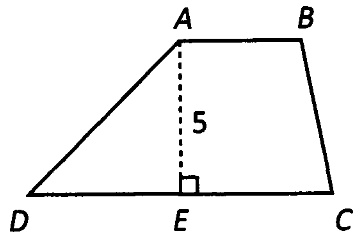
So the perimeter of the rectangle is 6 + 8 + 6 + 8, which equals 28. The area is (6) × (8), which equals 48.

Problem Set (Note: Figures are not drawn to scale.)

1. Frank the Fencemaker needs to fence in a rectangular yard. He fences in the entire yard, except for one 40-foot side of the yard. The yard has an area of 280 square feet. How many feet of fence does Frank use?

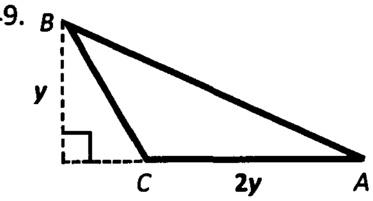
2. A pentagon has three sides with length x, and two sides with the length 3*x*. If *x* is $\frac{2}{3}$ of an inch, what is the perimeter of the pentagon?

3. *ABCD* is a quadrilateral, with *AB* parallel to *CD* (see figure). *E* is a point between *C* and *D* such that *AE* represents the height of *ABCD*, and *E* is the midpoint of *CD*. If *AB* is 4 inches long, *AE* is 5 inches long, and the area of triangle *AED* is 12.5 square inches, what is the area of *ABCD*?



4. A rectangular tank needs to be coated with insulation. The tank has dimensions of 4 feet, 5 feet, and 2.5 feet. Each square foot of insulation costs $20. How much will it cost to cover the surface of the tank with insulation?

5. Triangle *ABC* (see figure) has a base of 2*y*, a height of *y*, and an area of 49. What is *y*?



6. 40 percent of Andrea's living room floor is covered by a carpet that is 4 feet by
9 feet. What is the area of her living room floor?

7. If the perimeter of a rectangular flower bed is 30 feet, and its area is 44 square feet, what is the length of each of its shorter sides?

8. There is a rectangular parking lot with a length of 2x and a width of *x*. What is the ratio of the perimeter of the parking lot to the area of the parking lot, in terms of *x*?

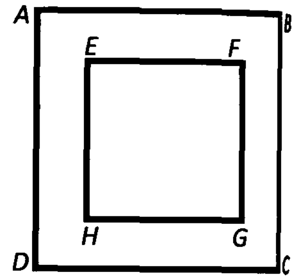
9. A rectangular solid has a square base, with each side of the base measuring 4 meters. If the volume of the solid is 112 cubic meters, what is the surface area of the solid?

10. A swimming pool has a length of 30 meters, a width of 10 meters, and an average depth of 2 meters. If a hose can fill the pool at a rate of 0.5 cubic meters per minute, how many hours will it take the hose to fill the pool?

11. A Rubix cube has an edge 5 inches long. What is the ratio of the cube's surface area to its volume?

12. If the length of an edge of Cube A is one third the length of an edge of Cube B, what is the ratio of the volume of Cube A to the volume of Cube B?

13. *ABCD* is a square picture frame (see figure). *EFGH* is a square inscribed within *ABCD* as a space for a picture. The area of *EFGH* (for the picture) is equal to the area of the picture frame (the area of *ABCD* minus the area of *EFGH*). If $AB = 6$, what is the length of *EF*?



14. What is the maximum possible area of a quadrilateral with a perimeter of 80 centimeters?

15. What is the minimum possible perimeter of a quadrilateral with an area of 1,600 square feet?

16. What is the maximum possible area of a parallelogram with one side of length 2 meters and a perimeter of
24 meters?

17. What is the maximum possible area of a triangle with a side of length 7 units and another side of length
8 units?

18. The lengths of the two shorter legs of a right triangle add up to 40 units. What is the maximum possible area of the triangle?

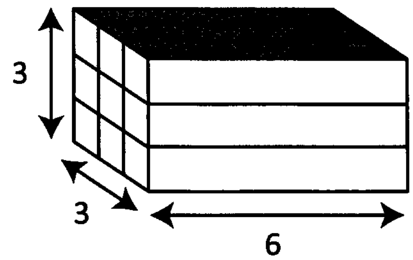
19.

Quantity A	Quantity B
The surface area in square inches of a cube with edges of length 6	The volume in cubic inches of a cube with edges of length 6

20.

Quantity A	Quantity B
The total volume of 3 cubes with edges of length 2	The total volume of 2 cubes with edges of length 3

21.



The large rectangular solid above
is formed by binding together nine
identical rectangular rods, as shown.

Quantity A	Quantity B
The combined surface area of four of the individual, identical rectangular rods.	The surface area of the large rectangular solid

1. **54 feet:** We know that one side of the yard is 40 feet long; let us call this the length. We also know that the area of the yard is 280 square feet. In order to determine the perimeter, we must know the width of the yard.

$$A = l \times w$$
$$280 = 40w$$
$$w = 280 \div 40 = 7 \text{ feet}$$

Frank fences in the two 7-foot sides and one of the 40-foot sides. 40 + 2(7) = 54.

2. **6 inches:** The perimeter of a pentagon is the sum of its five sides: $x + x + x + 3x + 3x = 9x$. If x is $\frac{2}{3}$ of an inch, the perimeter is $9\frac{2}{3}$, or 6 inches.

3. **35 in²:** If E is the midpoint of CD, then $CE = DE = x$. We can determine the length of x by using what we know about the area of triangle AED.

$$A = \frac{b \times h}{2} \quad 12.5 = \frac{5x}{2}$$
$$25 = 5x$$
$$x = 5$$

Therefore, the length of CD is $2x$, or 10.

To find the area of the trapezoid, use the formula: $A = \frac{b_1 + b_2}{2} \times h$

$$= \frac{4+10}{2} \times 5$$
$$= 35 \text{ in}^2$$

4. **$1,700:** To find the surface area of a rectangular solid, sum the individual areas of all six faces:

Top and Bottom:	$5 \times 4 = 20$	$\rightarrow$	$2 \times 20 = 40$
Side 1:	$5 \times 2.5 = 12.5$	$\rightarrow$	$2 \times 12.5 = 25$
Side 2:	$4 \times 2.5 = 10$	$\rightarrow$	$2 \times 10 = 20$

$$40 + 25 + 20 = 85 \text{ ft}^2$$

Covering the entire tank will cost $85 \times \$20 = \$1,700$.

5. **7:** The area of a triangle is equal to half the base times the height. Therefore, we can write the following relationship:

$$\frac{2y(y)}{2} = 49$$
$$y^2 = 49$$
$$y = 7$$

6. **90 ft²:** The area of the carpet is equal to $l \times w$, or 36 ft². Set up a percent table or a proportion to find the area of the whole living room floor:

$$\frac{40}{100} = \frac{36}{x}$$ Cross-multiply to solve.

$$40x = 3600$$

$$x = 90 \text{ ft}^2$$

7. **4:** Set up equations to represent the area and perimeter of the flower bed:

$$A = l \times w \qquad\qquad P = 2(l + w)$$

Then, substitute the known values for the variables A and P:

$$44 = l \times w \qquad\qquad 30 = 2(l + w)$$

Solve the two equations with the substitution method:

$$l = \frac{44}{w}$$

$$30 = 2\left(\frac{44}{w} + w\right)$$

$$30 = \frac{88}{w} + 2w$$

Multiply the entire equation by $\frac{w}{2}$.

$$15w = 44 + w^2$$

$$w^2 - 15w + 44 = 0$$

$$(w - 11)(w - 4) = 0$$

$$w = \{4,\ 11\}$$

Solving the quadratic equation yields two solutions: 4 and 11. Since we are looking only for the length of the shorter side, the answer is 4.

Alternatively, you can arrive at the correct solution by picking numbers. What length and width add up to 15 (half of the perimeter) and multiply to produce 44 (the area)? Some experimentation will demonstrate that the longer side must be 11 and the shorter side must be 4.

8. $\frac{3}{x}$**:** If the length of the parking lot is $2x$ and the width is x, we can set up a fraction to represent the ratio of the perimeter to the area as follows:

$$\frac{\text{perimeter}}{\text{area}} = \frac{2(2x + x)}{(2x)(x)} = \frac{6x}{2x^2} = \frac{3}{x}$$

9. **144 m²:** The volume of a rectangular solid equals (length) × (width) × (height). If we know that the length and width are both 4 meters long, we can substitute values into the formulas as shown:

$$112 = 4 \times 4 \times h$$
$$h = 7$$

To find the surface area of a rectangular solid, sum the individual areas of all six faces:

Top and Bottom:	$4 \times 4 = 16$	$\rightarrow$	$2 \times 16 = 32$
Sides:	$4 \times 7 = 28$	$\rightarrow$	$4 \times 28 = 112$

$$32 + 112 = 144\text{ m}^2$$

10. **20 hours:** The volume of the pool is (length) $\times$ (width) $\times$ (height), or $30 \times 10 \times 2 = 600$ cubic meters. Use a standard work equation, $RT = W$, where W represents the total work of 600 m^3.

$0.5t = 600$

$t = 1{,}200$ minutes Convert this time to hours by dividing by 60: $1{,}200 \div 60 = 20$ hours.

11. $\frac{6}{5}$**:** To find the surface area of a cube, find the area of 1 face, and multiply that by 6: $6(5^2) = 150$.

To find the volume of a cube, cube its edge length: $5^3 = 125$.

The ratio of the cube's surface area to its volume, therefore, is $\frac{150}{125}$, or $\frac{6}{5}$.

12. **1 to 27:** First, assign the variable x to the length of one side of Cube A. Then the length of one side of Cube B is $3x$. The volume of Cube A is x^3. The volume of Cube B is $(3x)^3$, or $27x^3$.

Therefore, the ratio of the volume of Cube A to Cube B is $\frac{x^3}{27x^3}$, or 1 to 27. You can also pick a number for the length of a side of Cube A and solve accordingly.

13. $3\sqrt{2}$**:** The area of the frame and the area of the picture sum to the total area of the image, which is 6^2, or 36. Therefore, the area of the frame and the picture are each equal to half of 36, or 18. Since $EFGH$ is a square, the length of EF is $\sqrt{18}$, or $3\sqrt{2}$.

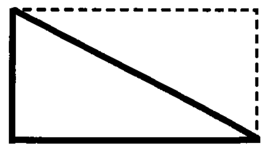
14. **400 cm^2:** The quadrilateral with maximum area for a given perimeter is a square, which has four equal sides. Therefore, the square that has a perimeter of 80 centimeters has sides of length 20 centimeters each. Since the area of a square is the side length squared, the area = (20 cm)(20 cm) = 400 cm^2.

15. **160 ft:** The quadrilateral with minimum perimeter for a given area is a square. Since the area of a square is the side length squared, we can solve the equation $x^2 = 1{,}600$ ft^2 for the side length x, yielding $x = 40$ ft. The perimeter, which is four times the side length, is (4)(40 ft) = 160 ft.

16. **20 m^2:** If one side of the parallelogram is 2 meters long, then the opposite side must also be 2 meters long. We can solve for the unknown sides, which are equal in length, by writing an equation for the perimeter: $24 = 2(2) + 2x$, with x as the unknown side. Solving, we get $x = 10$ meters. The parallelogram with these dimensions and maximum area is a *rectangle* with 2-meter and 10-meter sides. Thus the maximum possible area of the figure is (2 m)(10 m) = 20 m^2.

17. **28 square units:** A triangle with two given sides has maximum area if these two sides are placed at right angles to each other. For this triangle, one of the given sides can be considered the base, and the other side can be considered the height (because they meet at a right angle). Thus we plug these sides into the formula $A=\frac{1}{2}bh$: $A=\frac{1}{2}(7)(8)=28$.

18. **200 square units:** You can think of a right triangle as half of a rectangle. Constructing this right triangle with legs adding to 40 is equivalent to constructing the rectangle with a perimeter of 80. Since the area of the triangle is half that of the rectangle, you can use the previously mentioned technique for maximizing the area of a rectangle: of all rectangles with a given perimeter, the *square* has the greatest area. The desired rectangle is thus a 20 by 20 square, and the right triangle has area $\frac{1}{2}(20)(20)=200$ units.



19. **C:** The surface area of a cube is 6 times e^2, where e *is* the length of each edge (that is, the surface area is the number of faces times the area of each face). Apply this formula to Quantity A.

Quantity A	Quantity B
The surface area in square inches of a cube with edges of length 6 = **6 × (6 × 6)**	The volume in cubic inches of a cube with edges of length 6.

The volume of a cube is e^3, where e is the length of each edge. Apply this formula to Quantity B.

Quantity A	Quantity B
6 × (6 × 6)	The volume in cubic inches of a cube with edges of length 6 = **6 × 6 × 6**

It is not generally the case that the volume of a cube in cubic units is equal to the surface area of the cube in square inches; they are only equal when the edge of the cube is of length 6. In this case, **the two quantities are equal.**

20. **B:** The volume of a cube is e^3, where e is the length of each edge. Apply this formula to each quantity.

Quantity A	Quantity B
The total volume of 3 cubes with edges of length 2 = $3 \times (2^3) =$ **24**	The total volume of 2 cubes with edges of length 3 = $2 \times (3^3) =$ **54**

Quantity B is larger.

21. **A:** A rectangular solid has three pairs of opposing equal faces, each pair representing two of the dimensions of the solid (length × width; length × height; height × width). The total surface area of a rectangular solid is the sum of the surface areas of those three pairs of opposing sides.

According to the diagram, the dimensions of each rod must be $1 \times 1 \times 6$. So each of the rods described in Quantity A has a surface area of:

$$2(1 \times 1) + 2(1 \times 6) + 2(1 \times 6), \quad \text{or} \quad 2[(1 \times 1) + (1 \times 6) + (1 \times 6)]$$

That is, each rod has a total surface area of 26, and the four rods together have a surface area of $4 \times 26 = 104$.

Quantity A	Quantity B
The combined surface area of four of the identical rectangular rods = **104**	The surface area of the large rectangular solid.

The large rectangular solid has a total surface area of:

$$2(3 \times 3) + 2(3 \times 6) + 2(3 \times 6), \text{ or } 90.$$

Quantity A	Quantity B
104	The surface area of the large rectangular solid = **90**

Quantity A is larger.

Chapter 5

of

GEOMETRY

CIRCLES & CYLINDERS

In This Chapter . . .

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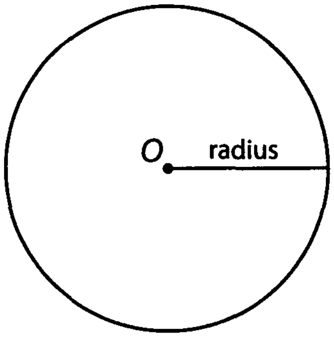
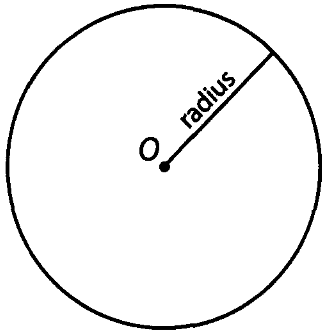
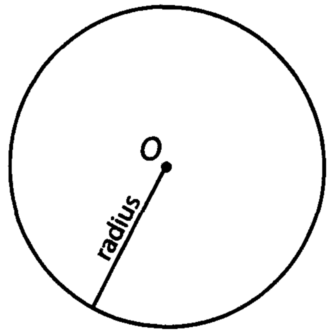
- The Basic Elements of a Circle
- Sectors
- Inscribed vs. Central Angles
- Inscribed Triangles
- Cylinders and Surface Area
- Cylinders and Volume

The Basic Elements of a Circle

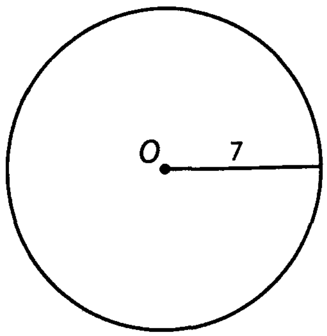
A circle is a set of points that are all the same distance from a central point. By definition, every circle has a center. Although the center is not itself a point on the circle, it is nevertheless an important component of the circle. The **radius** of a circle is defined as the distance between the center of the circle and a point on the circle. The first thing to know about radii is that *any* line segment connecting the center of the circle (usually labeled *O*) and *any* point on the circle is a radius (usually labeled *r*). All radii in the same circle have the same length.



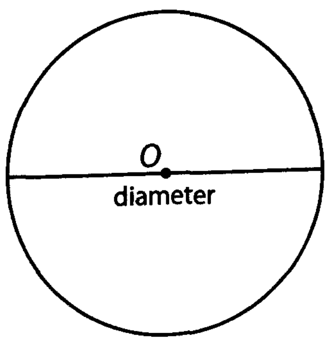




We'll discuss the other basic elements by dealing with a particular circle. Our circle will have a radius of 7, and we'll see what else we can figure out about the circle based on that one measurement. As you'll see, we'll be able to figure out quite a lot.

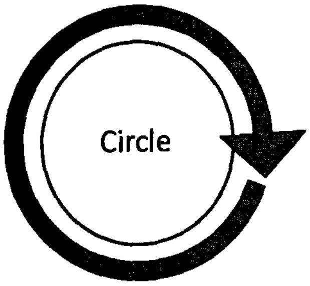


Once we know the radius, the next easiest piece to figure out is the **diameter.** The **diameter** passes through the center of a circle and connects 2 opposite points on the circle.



One way of thinking about the diameter (usually referred to as d) is that it is 2 radii laid end to end. The diameter will always be exactly twice the length of the radius. This relationship can be expressed as $d = 2r$. That means that our circle with radius 7 has a **diameter** of 14.

Now it's time for our next important measurement—the **circumference.** Circumference (usually referred to as C) is a measure of the distance around a circle. One way to think about circumference is that it's the perimeter of a circle.

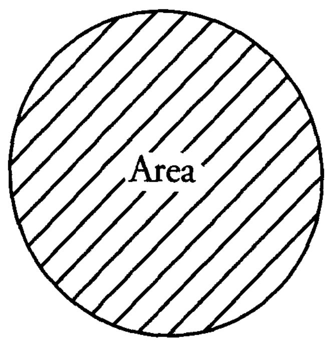


As it happens, there is a consistent relationship between the circumference and the diameter of any circle. If you were to divide the circumference by the diameter, you would always get the same number—3.14… (the number is actually a non-repeating decimal, so it's usually rounded to the hundredths place). You may be more familiar with this number as the Greek letter π (pi). To recap:

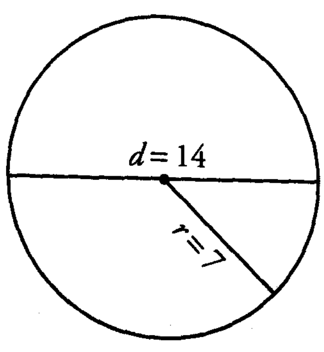
$$\frac{\text{circumference}}{\text{diameter}} = \pi. \text{ Or } \pi d = C.$$

In our circle with a diameter of 14, the circumference is $\pi(14) = 14\pi$. The vast majority of questions that involve circles and π will use the Greek letter rather than the decimal approximation for π. Suppose a question about our circle with radius 7 asked for the circumference. The correct answer would read 14π, rather than 43.96 (which is 14×3.14). It's worth mentioning that another very common way of expressing the circumference is that twice the radius times π also equals C, because the diameter is twice the radius. This relationship is commonly expressed as $C = 2\pi r$. As you prepare for the GRE, you should be comfortable with using either equation.

There is one more element of a circle that you'll need to be familiar with, and that is **area.** The area (usually referred to as A) is the space inside the circle.



Once again, it turns out that there is a consistent relationship between the area of a circle and its diameter (and radius). If you know the radius of the circle, then the formula for the area is $A = \pi r^2$. For our circle of radius 7, the area is $\pi(7)^2 = 49\pi$. To recap, once we know the radius, we are able to determine the diameter, the circumference, and the area.



$C = 14\pi$

$A = 49\pi$

These relationships are true of any circle. What's more, if you know *any* of these values, you can determine the rest. In fact, the ability to use one element of a circle to determine another is one of the most important skills for answering questions about circles.

To demonstrate, we'll work through another circle, but this time we know that the area of the circle is 36π. Well, we know the formula for the area, so let's start by plugging this value into the formula.

$36\pi = \pi r^2$

Now we can solve for the radius by isolating r.

C = **circumference**
the distance around a circle

d = **diameter**
the distance across a circle

r = **radius**
half the distance across a circle

A = **area**
area of the circle

$C = \pi \times d$

$d = 2 \times r$

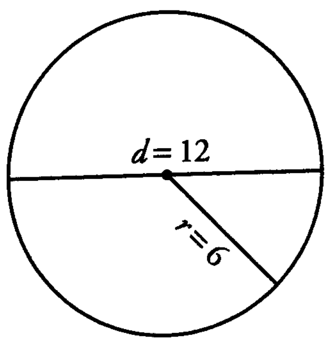
$A = \pi \times r^2$

$36\pi = \pi r^2$ Divide by π

$36 = r^2$ Take the square root of both sides

$6 = r$

Now that we know the radius, we can simply multiply it by 2 to get the diameter, so our diameter is 12. Finally, to find the circumference, simply multiply the diameter by π, which gives us a circumference of 12π.



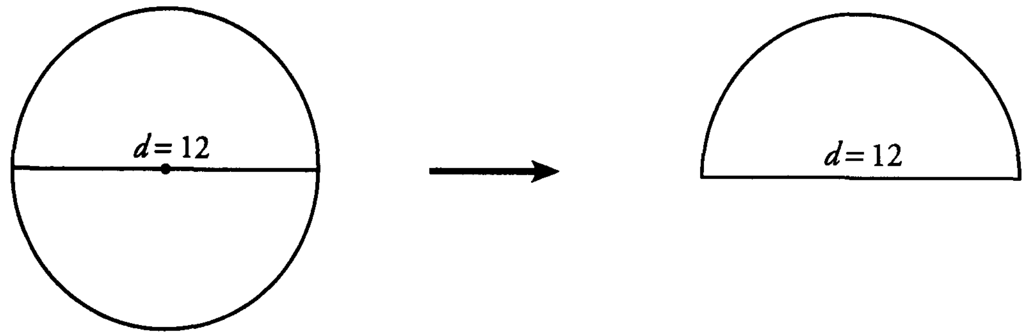
Check Your Skills

1. The radius of a circle is 7. What is the area?
2. The circumference of a circle is 17π. What is the diameter?
3. The area of a circle is 25π. What is the circumference?

Answers can be found on page 103.

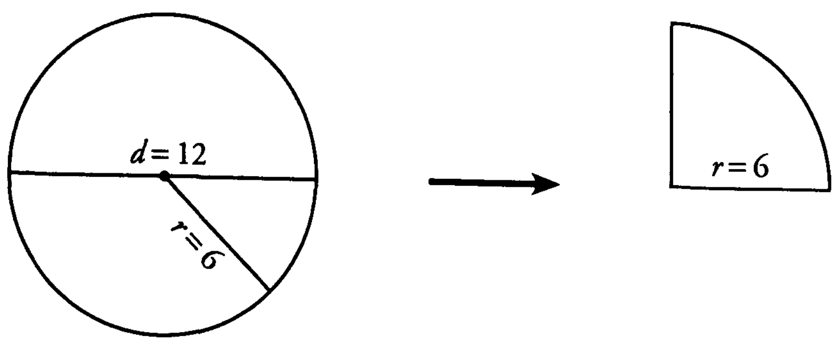
Sectors

Let's continue working with our circle that has an area of 36π. But now, let's cut it in half and make it a semicircle. Any time you have a fractional portion of a circle, it's known as a **sector.**



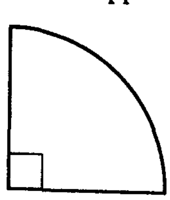
What effect does cutting the circle in half have on the basic elements of the circle? The diameter stays the same, as does the radius. But what happened to the area and the circumference? They're also cut in half. So the area of the semicircle is 18π and the circumference is 6π. When dealing with sectors, we call the portion of the circumference that remains the **arc length.** So the arc length of this sector is 6π.

In fact, this rule applies even more generally to circles. If, instead of cutting the circle in half, we had cut it into 1/4's, each piece of the circle would have 1/4 the area of the entire circle and 1/4 the circumference.



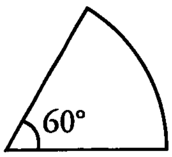
1/4 of whole circle

Now, on the GRE, you're unlikely to be told that you have 1/4th of a circle. There is one more basic element of circles that becomes relevant when you are dealing with sectors, and that is the **central angle.** The central angle of a sector is the degree measure between the two radii. Take a look at the quarter circle. Normally, there are 360° in a full circle. What is the degree measure of the angle between the 2 radii? The same thing that happens to area and circumference happens to the central angle. It is now 1/4th of 360°, which is 90°.



$$\frac{1}{4} = \frac{90^\circ}{360^\circ}$$

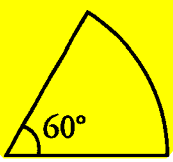
Let's see how we can use the central angle to determine sector area and arc length. For our next example, we will still use the circle with area 36π, but now the sector will have a central angle of 60°.



We need to figure out what fractional amount of the circle remains if the central angle is 60°. If 360° is the whole amount, and 60° is the part, then 60/360 is the fraction we're looking for. 60/360 reduces to 1/6. That means a sector with a central angle of 60° is 1/6th of the entire circle. If that's the case, then the sector area is $\frac{1}{6} \times$ (Area of circle) and arc length is $\frac{1}{6} \times$ (Circumference of circle). So:

$$\text{Sector Area} = \frac{1}{6} \times (36\pi) = 6\pi$$

$$\text{Arc Length} = \frac{1}{6} \times (12\pi) = 2\pi$$



$$\frac{1}{6} = \frac{60^\circ}{360^\circ} = \frac{\text{Sector Area}}{\text{Circule Area}} = \frac{\text{Arc Length}}{\text{Circumference}}$$

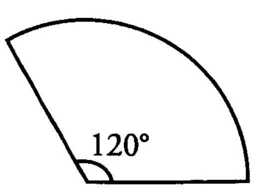
In our last example, we used the central angle to find what fractional amount of the circle the sector was. But any of the three properties of a sector (central angle, arc length and area) could be used if you know the radius.

Let's look at an example.

A sector has a radius of 9 and an area of 27π. What is the central angle of the sector?

We still need to determine what fractional amount of the circle the sector is. This time, however, we have to use the area to figure that out. We know the area of the sector, so if we can figure out the area of the whole circle, we can figure out what fractional amount the sector is.

We know the radius is 9, so we can calculate the area of the whole circle. Area $= \pi r^2$, so Area $= \pi(9)^2 = 81\pi$. $\frac{27\pi}{81\pi} = \frac{1}{3}$, so the sector is 1/3 of the circle. The full circle has a central angle of 360, so we can multiply that by 1/3. $1/3 \times 360 = 120$, so the central angle of the sector is 120°.



$$\frac{1}{3} = \frac{120^\circ}{360^\circ} = \frac{27\pi \text{ (sector area)}}{81\pi \text{ (circle area)}}$$

Let's recap what we know about sectors. Every question about sectors involves determining what fraction of the circle the sector is. That means that every question about sectors will provide you with enough info to calculate one of the following fractions:

$$\frac{\text{central angle}}{360} \quad \frac{\text{sector area}}{\text{circle area}} \quad \frac{\text{arc length}}{\text{circumference}}$$

Once you know any of those fractions, you know them all, and you can find the value of any piece of the sector or the original circle.

Check Your Skills

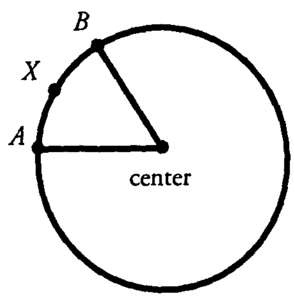
4. A sector has a central angle of 270° and a radius of 2. What is the area of the sector?
5. A sector has an arc length of 4π and a radius of 3. What is the central angle of the sector?
6. A sector has an area of 40π and a radius of 10. What is the arc length of the sector?

Answers can be found on page 103.

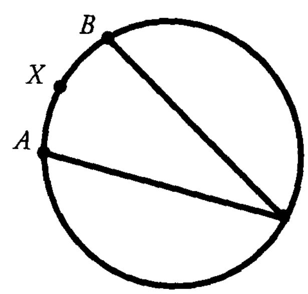
Inscribed vs. Central Angles

Thus far, in dealing with arcs and sectors, we have referred to the concept of a **central angle.** A central angle is defined as an angle whose vertex lies at the center point of a circle. As we have seen, a central angle defines both an arc and a sector of a circle.

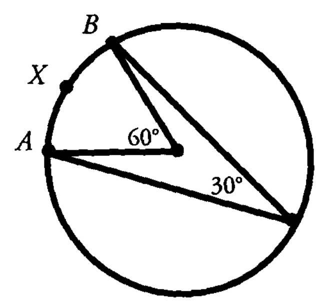
Another type of angle is termed an **inscribed angle.** An inscribed angle has its vertex *on the circle itself* (rather than on the center of the circle). The following diagrams illustrate the difference between a central angle and an inscribed angle.



CENTRAL ANGLE



INSCRIBED ANGLE

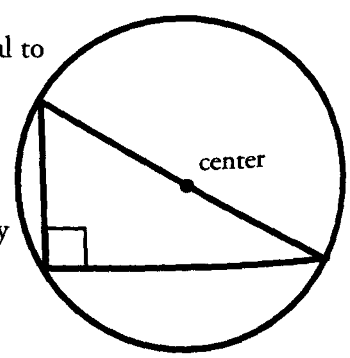


Notice that, in the circle at the far right, there is a central angle and an inscribed angle, both of which intercept arc AXB. It is the central angle that defines the arc. That is, the arc is 60° (or one sixth of the complete 360° circle). **An inscribed angle is equal to half of the arc it intercepts**, in degrees. In this case, the inscribed angle is 30°, which is half of 60°.

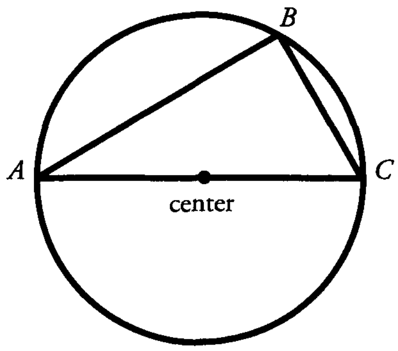
Inscribed Triangles

Related to this idea of an inscribed angle is that of an **inscribed triangle.** A triangle is said to be inscribed in a circle if all of the vertices of the triangle are points on the circle.

To the right is a special case of the rule mentioned above (that an inscribed angle is equal to half of the arc it intercepts, in degrees). In this case, the right angle (90°) lies opposite a semicircle, which is an arc that measures 180°.



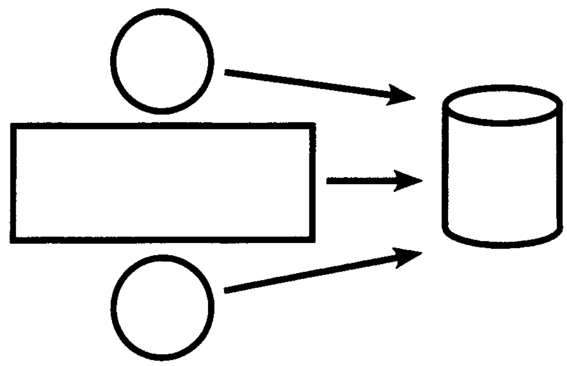
The important rule to remember is: **if one of the sides of an inscribed triangle is a DIAMETER of the circle, then the triangle MUST be a right triangle.** Conversely, any right triangle inscribed in a circle must have the diameter of the circle as one of its sides (thereby splitting the circle in half).



In the inscribed triangle to the left, triangle ABC must be a right triangle, since AC is a diameter of the circle.

Cylinders and Surface Area

Two circles and a rectangle combine to form a three-dimensional shape called a right circular cylinder (referred to from now on simply as a **cylinder**). The top and bottom of the cylinder are circles, while the middle of the cylinder is formed from a rolled-up rectangle, as shown in the diagram below:



In order to determine the surface area of a cylinder, sum the areas of
the 3 surfaces: The area of each circle is πr^2, while the area of the rectangle is length $\times$ width.
Looking at the figures on the right, we can see that the length of the rectangle is equal to the circumference of the circle ($2\pi r$), and the
width of the rectangle is equal to the height of the cylinder (h). Therefore, the area of the rectangle is $2\pi r \times h$. To find the total surface area of a cylinder, add the area of the circular top and bottom,
as well as the area of the rectangle that wraps around the outside.

$$\boxed{SA = 2 \text{ circles} + \text{rectangle} = 2(\pi r^2) + 2\pi rh}$$

The only information you need to find the surface area of a cylinder is (1) the radius of the cylinder and (2) the height of the cylinder.

Cylinders and Volume

The volume of a cylinder measures how much "stuff" it can hold inside. In order to find the volume of a cylinder, use the following formula.

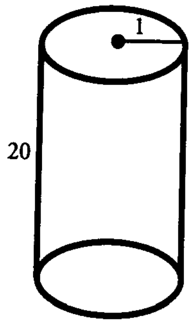
$$\boxed{V = \pi r^2 h}$$

V is the volume, r is the radius of the cylinder, and h is the height of the cylinder.

As with finding surface area, determining the volume of a cylinder requires two pieces of information: (1) the radius of the cylinder and (2) the height of the cylinder.

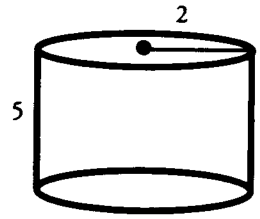
One way to remember this formula is to think of a cylinder as a stack of circles, each with an area or πr^2. Just multiply $\pi r^2 \times$ the height (h) of the shape to find the area.

The diagram below shows that two cylinders can have the same volume but different shapes (and therefore each fits differently inside a larger object).



$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(1)^2 20 \\ &= 20\pi \end{aligned}$$

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(2)^2 5 \\ &= 20\pi \end{aligned}$$



Check Your Skills Answers

1. **49π:** The formula for area is $A = \pi r^2$. The radius is 7, so Area $= \pi(7)^2 = 49\pi$.

2. **17:** Circumference of a circle is either $C = 2\pi r$ or $C = \pi d$. The question asks for the diameter, so we'll use the latter formula. $17\pi = \pi d$. Divide by π, and we get $17 = d$. The diameter is 17.

3. **10π:** The link between area and circumference of a circle is that they are both defined in terms of the radius. Area of a circle is $A = \pi r^2$, so we can use the area of the circle to find the radius. $25\pi = \pi r^2$, so $r = 5$. If the radius equals 5, then the circumference is $C = 2\pi(5)$, which equals 10π. The circumference is 10π.

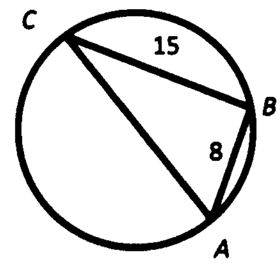
4. **3π:** If the central angle of the sector is 270°, then it is 3/4 of the full circle, because $\frac{270^\circ}{360^\circ} = \frac{3}{4}$. If the radius is 2, then the area of the full circle is $\pi(2)^2$, which equals 4π. If the area of the full circle is 4π, then the area of the sector will be $3/4 \times 4\pi$, which equals 3π.

5. **240°:** To find the central angle, we first need to figure out what fraction of the circle the sector is. We can do that by finding the circumference of the full circle. The radius is 3, so the circumference of the circle is $2\pi(3) = 6\pi$. That means the sector is 2/3 of the circle, because $\frac{4\pi}{6\pi} = \frac{2}{3}$. That means the central angle of the sector is $2/3 \times 360°$, which equals 240°.

6. **8π:** We can begin by finding the area of the whole circle. The radius of the circle is 10, so the area is $\pi(10)^2$, which equals 100π. That means the sector is 2/5 of the circle, because $\frac{40\pi}{100\pi} = \frac{4}{10} = \frac{2}{5}$. We can find the circumference of the whole circle using $C = 2\pi r$. The circumference equals 20π. $2/5 \times 20\pi = 8\pi$. The arc length of the sector is 8π.

Problem Set (Note: Figures are not drawn to scale.)

1. Triangle ABC is inscribed in a circle, such that AC is a diameter of the circle (see figure). If AB has a length of 8 and BC has a length of 15, what is the circumference of the circle?

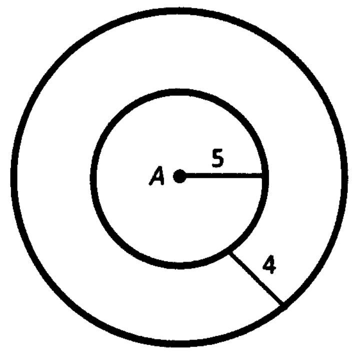


2. A cylinder has a surface area of 360π, and is 3 units tall. What is the diameter of the cylinder's circular base?

3. Randy can run π meters every 2 seconds. If the circular track has a radius of 75 meters, how many minutes does it take Randy to run twice around the track?

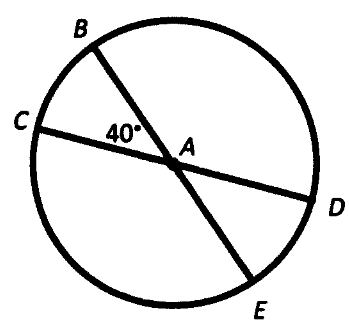
4. Randy then moves on to the Jumbo Track, which has a radius of 200 meters (as compared to the first track, with a radius of 75 meters). Ordinarily, Randy runs 8 laps on the normal track. How many laps on the Jumbo Track would Randy have to run in order to have the same work-out?

5. A circular lawn with a radius of 5 meters is surrounded by a circular walkway that is 4 meters wide (see figure). What is the area of the walkway?



6. A cylindrical water tank has a diameter of 14 meters and a height of 20 meters. A water truck can fill π cubic meters of the tank every minute. How long in hours and minutes will it take the water truck to fill the water tank from empty to half-full?

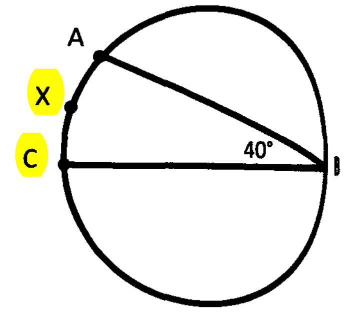
7. BE and CD are both diameters of Circle A (see figure). If the area of Circle A is 180 units^2, what is the area of sector ABC + sector ADE?



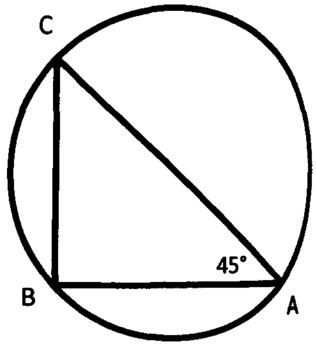
8. Jane has to paint a cylindrical column that is 14 feet high and that has a circular base with a radius of 3 feet. If one bucket of paint will cover 10π square feet, how many buckets does Jane need to buy in order to paint the column, including the top and bottom?

9. A circular flower bed takes up half the area of a square lawn. If an edge of the lawn is 200 feet long, what is the radius of the flower bed? (Express the answer in terms of π.)

10. If angle ABC is 40 degrees (see figure), and the area of the circle is 81π, how long is arc AXC?

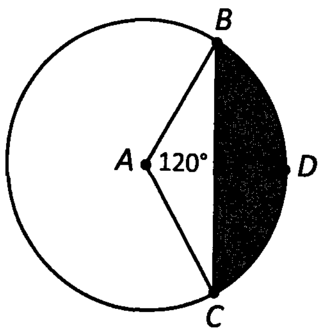


11. Triangle ABC is inscribed in a circle, such that AC is a diameter of the circle and anglde BAC is 45° (see figure). If the area of triangle ABC is 72 square units, how much larger is the area of the circle than the area of triangle ABC?



12. Triangle ABC is inscribed in a circle, such that AC is a diameter of the circle and angle BAC is 45°. (Refer to the same figure as for problem #11.) If the area of triangle ABC is 84.5 square units, what is the length of arc BC?

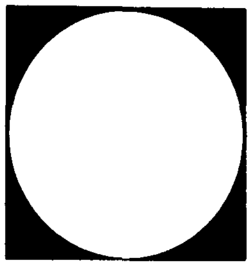
13.



A is the center of the circle above.

Quantity A	Quantity B
The perimeter of triangle ABC	The perimeter of the shaded region

14.



In the figure above, a circle with area π is inscribed in a square.

Quantity A	Quantity B
The combined area of the shaded regions	1

15.

Quantity A	Quantity B
The combined area of four circles, each with radius 1	The area of a circle with radius 2

1. **17π:** If AC is a diameter of the circle, then inscribed triangle ABC is a right triangle, with AC as the hypotenuse. Therefore, we can apply the Pythagorean Theorem to find the length of AC.

$8^2 + 15^2 = c^2$
$64 + 225 = c^2$ You might recognize the common 8–15–17 right triangle.
$c^2 = 289$
$c = 17$

The circumference of the circle is $2\pi r$, or 17π.

2. **24:** The surface area of a cylinder is the area of the circular top and bottom, plus the area of its wrapped-around rectangular third face. We can express this in formula form as:

$SA = 2(\pi r^2) + 2\pi rh$

Substitute the known values into this formula to find the radius of the circular base:

$360\pi = 2(\pi r^2) + 2\pi r(3)$
$360\pi = 2\pi r^2 + 6\pi r$
$2\pi r^2 + 6\pi r - 360\pi = 0$ Divide by 2π
$r^2 + 3r - 180 = 0$
$(r + 15)(r - 12) = 0$

$r + 15 = 0$ OR $r - 12 = 0$
$r = \{-15, 12\}$

Use only the positive value of r: 12. If $r = 12$, the diameter of the cylinder's circular base is 24.

3. **10 minutes:** The distance around the track is the circumference of the circle:

$C = 2\pi r$
$= 150\pi$

Running twice around the circle would equal a distance of 300π meters. If Randy can run π meters every 2 seconds, he runs 30π meters every minute. Therefore, it will take him 10 minutes to run around the circular track twice.

4. **3 laps:** 8 laps on the normal track is a distance of $1{,}200\pi$ meters. (Recall from problem #3 that the circumference of the normal track is 150π meters.) If the Jumbo Track has a radius of 200 meters, its circumference is 400π meters. It will take
3 laps around this track to travel $1{,}200\pi$ meters.

5. **56πm^2:** The area of the walkway is the area of the entire image (walkway + lawn) minus the area of the lawn. To find the area of each circle, use the formula:

Large circle: $A = \pi r^2 = \pi(9)^2 = 81\pi$
Small circle: $A = \pi r^2 = \pi(5)^2 = 25\pi$

$81\pi - 25\pi = 56\pi\text{m}^2$

6. **8 hours and 10 minutes:** First find the volume of the cylindrical tank:

$V = \pi r^2 \times h$
$= \pi(7)^2 \times 20$
$= 980\pi$

If the water truck can fill π cubic meters of the tank every minute, it will take 980 minutes to fill the tank completely; therefore, it will take $980 \div 2 = 490$ minutes to fill the tank halfway. This is equal to 8 hours and 10 minutes.

7. **40 units²:** The two central angles, *CAB* and *DAE*, describe a total of 80°. Simplify the fraction to find out what fraction of the circle this represents:

$\frac{80}{360} = \frac{2}{9}$ $\frac{2}{9}$ of 180 units² is 40 units².

8. **11 buckets:** The surface area of a cylinder is the area of the circular top and bottom, plus the area of its wrapped-around rectangular third face.

Top & Bottom: $A = \pi r^2 = 9\pi$
Rectangle: $A = 2\pi r \times h = 84\pi$

The total surface area, then, is $9\pi + 9\pi + 84\pi = 102\pi$ ft². If one bucket of paint will cover 10π ft², then Jane will need 0.2 buckets to paint the entire column. Since paint stores do not sell fractional buckets, she will need to purchase 11 buckets.

9. $\sqrt{\frac{\mathbf{20,000}}{\pi}}$: The area of the lawn is $(200)^2 = 40,000$ ft².

Therefore, the area of the flower bed is $40,000 \div 2 = 20,000$ ft².

$A = \pi r^2 = 20,000$ The radius of the flower bed is equal to $\sqrt{\frac{20,000}{\pi}}$.

10. **4π:** If the area of the circle is 81π, then the radius of the circle is 9 ($A = \pi r^2$). Therefore, the total circumference of the circle is 18π ($C = 2\pi r$). Angle *ABC*, an inscribed angle of 40°, corresponds to a central angle of 80°. Thus, arc *AXC* is equal to 80/360 = 2/9 of the total circumference:

$\frac{2}{9}(18\pi) = 4\pi$.

11. **72π − 72:** If *AC* is a diameter of the circle, then angle *ABC* is a right angle. Therefore, triangle *ABC* is a 45–45–90 triangle, and the base and height are equal. Assign the variable x to represent both the base and height:

$A = \frac{bh}{2}$ $\frac{x^2}{2} = 72$

$x^2 = 144$
$x = 12$

The base and height of the triangle are equal to 12, and so the area of the triangle is $\frac{12 \times 12}{2} = 72$.

The hypotenuse of the triangle, which is also the diameter of the circle, is equal to $12\sqrt{2}$. Therefore, the radius is equal to $6\sqrt{2}$ and the area of the circle, πr^2, $= 72\pi$. The area of the circle is $72\pi - 72$ square units larger than the area of triangle *ABC*.

12. $\frac{\mathbf{13\sqrt{2} \times \pi}}{\mathbf{4}}$: We know that the area of triangle *ABC* is 84.5 square units, so we can use the same logic as in the previous problem to establish the base and height of the triangle:

$A = \frac{bh}{2}$ $\frac{x^2}{2} = 84.5$

$x^2 = 169$
$x = 13$

The base and height of the triangle are equal to 13. Therefore, the hypotenuse, which is also the diameter of the circle, is equal to $13\sqrt{2}$, and the circumference ($C = \pi d$) is equal to $13\sqrt{2} \times \pi$. Angle A, an inscribed angle, corresponds to a central angle of 90°. Thus, arc BC = 90/360 = 1/4 of the total circumference:

$$\frac{1}{4} \text{ of } 13\sqrt{2} \times \pi \text{ is } \frac{13\sqrt{2} \times \pi}{4}.$$

13. **B:** Since the two perimeters share the line BC, we can recast this question as

Quantity A	**Quantity B**
The combined length of two radii (AB and AC)	The length of arc BDC

The easiest thing to do in this situation is use numbers. Assume the radius of the circle is 2.

If the radius is 2, then we can rewrite Quantity A.

Quantity A	**Quantity B**
The combined length of two radii (AB and AC) = 4	The length of arc BDC

Now we need to figure out the length of arc BDC if the radius is 2. We can set up a proportion, because the ratio of central angle to 360 will be the same as the ratio of the arc length to the circumference.

$$\frac{\text{Arc Length}}{\text{Circumference}} = \frac{120^\circ}{360^\circ} = \frac{1}{3}$$

Circumference is $2\pi r$, so

$$C = 2\pi(2) = 4\pi$$

Rewrite the proportion.

$$\frac{\text{Arc Length}}{4\pi} = \frac{1}{3}$$

$$\text{Arc Length} = \frac{4\pi}{3}$$

Rewrite Quantity B.

Quantity A	**Quantity B**
4	The length of arc $BDC = \frac{4\pi}{3}$

Compare 4 to $4\pi/3$. π is greater than 3, so $\frac{4\pi}{3}$ is slightly greater than 4.

14. **B:** Use the area of the circle to determine the area of the square, then subtract the area of the circle from the area of the square to determine the shaded region. The formula for area is $A = \pi r^2$. If we substitute the area of this circle for A, we can determine the radius:

$$\pi = \pi r^2$$
$$1 = r^2$$
$$1 = r$$

Since the radius of the circle is 1, the diameter of the circle is 2, as is each side of the square. A square with sides of 2 has an area of 4. Rewrite Quantity A.

Quantity A	Quantity B
The combined area of the shaded regions = $\text{Area}_{\text{Square}} - \text{Area}_{\text{Circle}} =$ $\mathbf{4 - \pi}$	1

π is greater than 3, so $4 - \pi$ is less than 1. Therefore **Quantity B is greater.**

15. **C:** First, evaluate Quantity A. Plug 1 in for r in the formula for the area of a circle:

$$A = \pi r^2$$
$$A = \pi(1)^2$$
$$A = \pi$$

Each circle has an area of π, and the four circles have a total area of 4π.

Quantity A	Quantity B
The combined area of four circles, each with radius 1 = $\mathbf{4\pi}$	The area of a circle with radius 2

For Quantity B, plug 2 in for r in the formula for the area of a circle:

$$A = \pi r^2$$
$$A = \pi(2)^2$$
$$A = 4\pi$$

Quantity A	Quantity B
$\mathbf{4\pi}$	The area of a circle with radius 2 = $\mathbf{4\pi}$

Therefore **the two quantities are equal.**

g

Chapter 6

of

GEOMETRY

LINES & ANGLES

In This Chapter . . .

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- Intersecting Lines
- Exterior Angles of a Triangle
- Parallel Lines Cut By a Transversal

LINES & ANGLES

A straight line is 180°. Think of a line as half of a circle.

Parallel lines are lines that lie in a plane and that never intersect. No matter how far you extend the lines, they never meet. Two parallel lines are shown below:

Perpendicular lines are lines that intersect at a 90° angle. Two perpendicular lines are shown below:

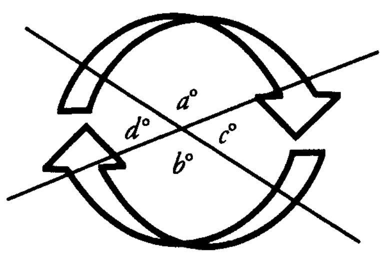
There are two major line-angle relationships that you must know for the GRE:

(1) The angles formed by any intersecting lines.
(2) The angles formed by parallel lines cut by a transversal.

Intersecting Lines

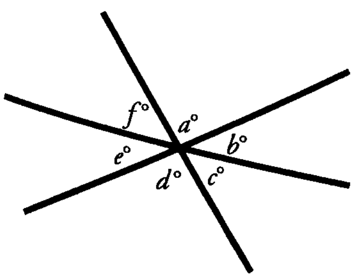
Intersecting lines have three important properties.

First, the interior angles formed by intersecting lines form a circle, so the sum of these angles is 360°. In the diagram shown, $a + b + c + d = 360$.



Second, interior angles that combine to form a line sum to 180°. These are termed **supplementary angles.** Thus, in the diagram shown, $a + d = 180$, because angles a and d form a line together. Other supplementary angles are $b + c = 180$, $a + c = 180$, and $d + b = 180$.

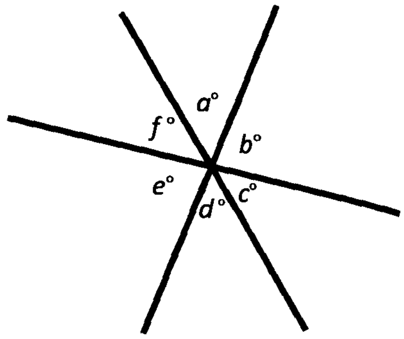
Third, angles found opposite each other where these two lines intersect are equal. These are called **vertical angles.** Thus, in the diagram above, $a = b$, because these angles are opposite one another, and are formed from the same two lines. Additionally, $c = d$ for the same reason.



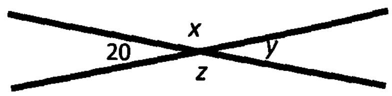
Note that these rules apply to more than two lines that intersect at a point, as shown to the left. In this diagram, $a + b + c + d + e + f = 360$, because these angles combine to form a circle. In addition, $a + b + c = 180$, because these three angles combine to form a line. Finally, $a = d$, $b = e$, and $c = f$, because they are pairs of vertical angles.

Check Your Skills

1. If $b + f = 150$, what is angle d?



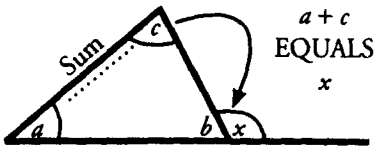
2. What is $x - y$?



Answers can be found on page 119.

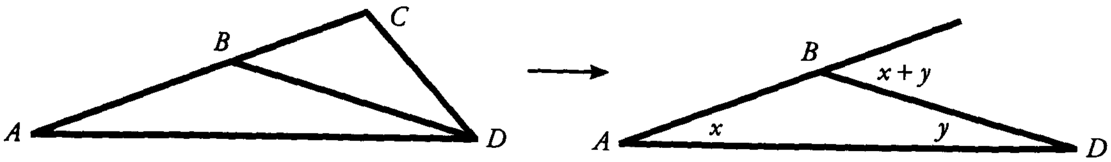
Exterior Angles of a Triangle

An **exterior angle** of a triangle is equal in measure to the sum of the two non-adjacent (opposite) **interior angles** of the triangle. For example:



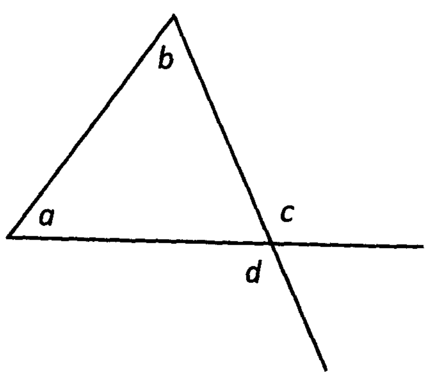
$a + b + c = 180$ (sum of angles in a triangle).
$b + x = 180$ (supplementary angles).
Therefore, $x = a + c$.

This property is frequently tested on the GRE! In particular, look for exterior angles within more complicated diagrams. You might even redraw the diagram with certain lines removed to isolate the triangle and exterior angle you need.



Check Your Skills

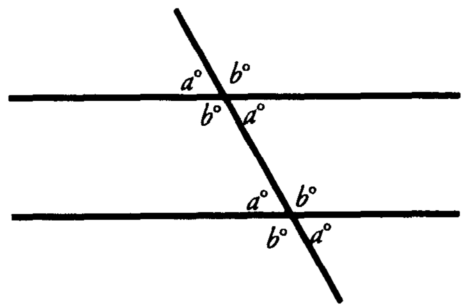
3. If $c + d = 200$, what is $a + b$?



Answers can be found on page 119.

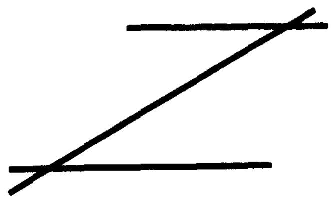
Parallel Lines Cut By a Transversal

The GRE makes frequent use of diagrams that include parallel lines cut by a **transversal.**



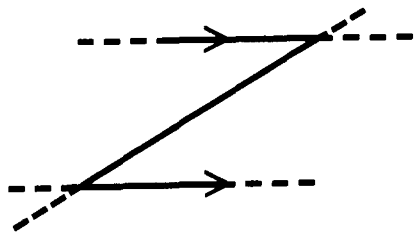
Notice that there are 8 angles formed by this construction, but there are only TWO different angle measures (a and b). All the **acute** angles (less than 90°) in this diagram are equal. Likewise, all the **obtuse** angles (more than 90° but less than 180°) are equal. Any acute angle is supplementary to any obtuse angle. Thus, $a + b = 180°$.

When you see a third line intersecting two lines that you know to be parallel, fill in all the a (acute) and b (obtuse) angles, just as in the diagram above.



Sometimes the GRE disguises the parallel lines and the transversal so that they are not readily apparent, as in the diagram pictured to the right.

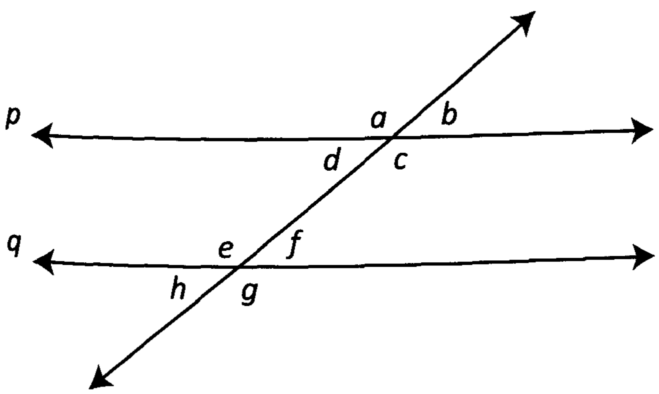
In these disguised cases, it is a good idea to extend the lines so that you can easily see the parallel lines and the transversal. Just remember always to be on the lookout for parallel lines. When you see them, extend lines and label the acute and obtuse angles.



You might also mark the parallel lines with arrows.

Check Your Skills

Refer to the following diagram for questions 4–5.



(lines p and q are parallel)

4. If angle $g = 120$, what is a?
5. If angle $g = 120$, what is $a + b + c$?

Answers can be found on page 119.

Check Your Skills Answers

1. **30:** Because they are vertical angles, angle a is equal to angle d.

Because they add to form a straight line, $a + b + f = 180$.

Substitute d for a to yield, $(d) + b + f = 180$. Substitute 150 for $b + f$ to yield $d + (150) = 180$. So $d = 180 - 150 = 30$.

2. **140:** Because x and 20 are supplementary, $x = 180 - 20 = 160$. Because y and 20 are vertical, $y = 20$. So $x - y = 160 - 20 = 140$.

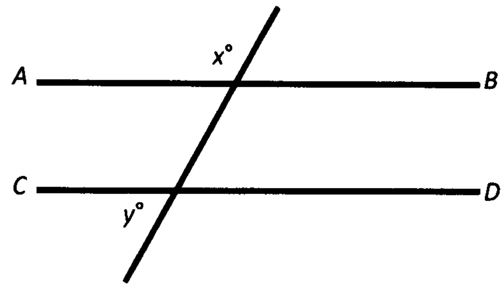
3. **100:** Since c and d are vertical angles, they are equal. Since they sum to 200, each must be 100. $a + b = c$, because c is an exterior angle of the triangle shown, and a and b are the two non-adjacent interior angles. $a + b = c = 100$.

4. **120:** In a system of parallel lines cut by a transversal, opposite exterior angles (like a and g) are equal. $g = a = 120$.

5. **300:** From question 4, we know that $a = 120$. Since $a = 120$, its supplementary angle $d = 180 - 120 = 60$. Since $a + b + c + d = 360$, and $d = 60$, $a + b + c = 300$.

Problem Set (Note: Figures are not drawn to scale.)

Problems 1–4 refer to the diagram on the right, where line *AB* is parallel to line *CD*.



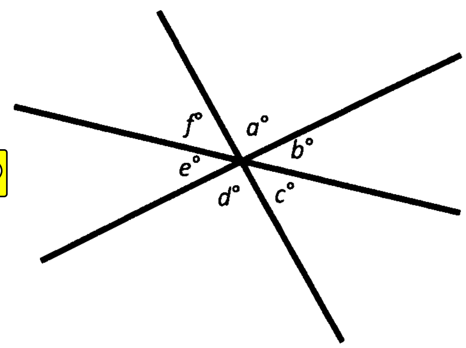
1. If $x - y = 10$, what is x?

2. If the ratio of x to y is 3 : 2, what is y?

3. If $x + (x + y) = 320$, what is x?

4. If $\frac{x}{x - y} = 2$, what is x?

Problems 5–8 refer to the diagram on the right.



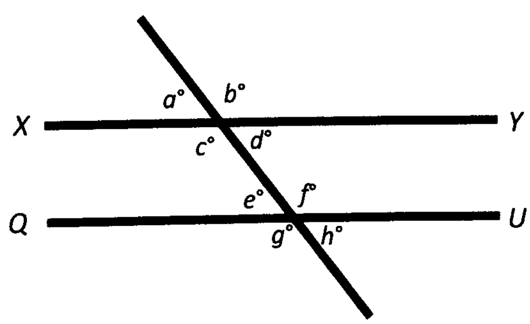
5. If a is 95, what is $b + d - e$?

6. If $c + f = 70$, and $d = 80$, what is b?

7. If a and b are **complementary angles** (they sum to 90°), name three other pairs of complementary angles.

8. If e is 45, what is the sum of all the other angles?

Problems 9–12 refer to the diagram on the right, where line *XY* is parallel to line *QU*.



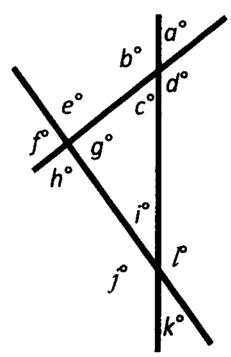
9. If $a + e = 150$, find f.

10. If $a = y$, $g = 3y + 20$, and $f = 2x$, find x.

11. If $g = 11y$, $a = 4x - y$, and $d = 5y + 2x - 20$, find h.

12. If $b = 4x$, $e = x + 2y$, and $d = 3y + 8$, find h.

Problems 13–15 refer to the diagram to the right.

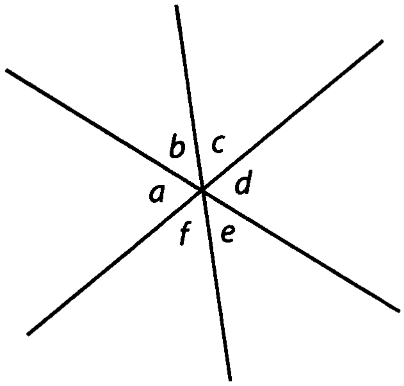


13. If $c + g = 140$, find k.

14. If $g = 90$, what is $a + k$?

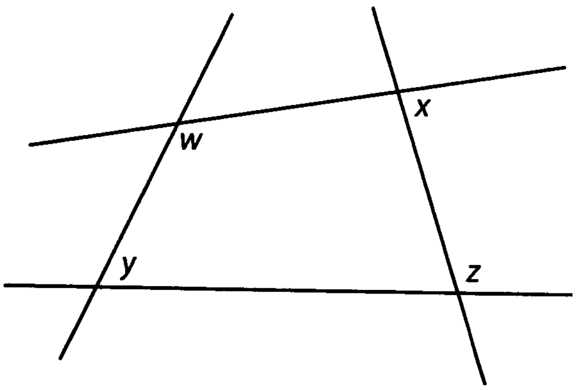
15. If $f + k = 150$, find b.

16.



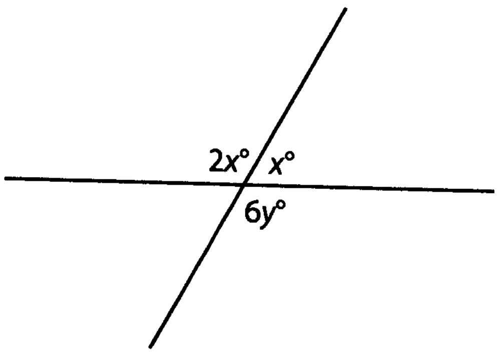
Quantity A	Quantity B
$a+f+b$	$c+d+e$

17.



Quantity A	Quantity B
$w+y$	$x+z$

18.



Quantity A	Quantity B
y	10

1. **95:** We know that $x + y = 180$, since any acute angle formed by a transversal that cuts across two parallel lines is supplementary to any obtuse angle. Use the information given to set up a system of two equations with two variables:

$$\begin{aligned} x + y &= 180 \\ x - y &= \ \ 10 \\ \hline 2x &= 190 \\ x &= \ \ 95 \end{aligned}$$

2. **72:** Set up a ratio, using the unknown multiplier, a.

$$\frac{x}{y} = \frac{3a}{2a}$$

$$180 = x + y = 3a + 2a = 5a$$
$$180 = 5a$$
$$a = 36$$
$$y = 2a = 2(36) = 72$$

3. **140:** Use the fact that $x + y = 180$ to set up a system of two equations with two variables:

$$x + y = 180 \quad \rightarrow \quad \begin{aligned} -x - y &= -180 \\ +\quad 2x + y &= \ \ 320 \\ \hline x &= \ \ 140 \end{aligned}$$

4. **120:** Use the fact that $x + y = 180$ to set up a system of two equations with two variables:

$$\frac{x}{x - y} = 2 \quad \rightarrow \quad \begin{aligned} x - 2y &= 0 \\ -\quad x + \ \ y &= 180 \\ \hline -3y &= -180 \end{aligned}$$

$$y = 60 \quad \rightarrow \quad \text{Therefore, } x = 120.$$

5. **95:** Because a and d are vertical angles, they have the same measure: $a = d = 95°$. Likewise, since b and e are vertical angles, they have the same measure: $b = e$. Therefore, $b + d - e = d = 95°$.

6. **65:** Because c and f are vertical angles, they have the same measure: $c + f = 70$, so $c = f = 35$. Notice that b, c, and d form a straight line: $b + c + d = 180$. Substitute the known values of c and d into this equation:

$$\begin{aligned} b + 35 + 80 &= 180 \\ b + 115 &= 180 \\ b &= 65 \end{aligned}$$

7. ***b* and *d*, *a* and *e*, & *d* and *e*:** If a is complementary to b, then d (which is equal to a, since they are vertical angles), is also complementary to b. Likewise, if a is complementary to b, then a is also complementary to e (which is equal to b, since they are vertical angles). Finally, d and e must be complementary, since $d = a$ and $e = b$. You do not need to know the term
"complementary," but you should be able to work with the concept (two angles adding up to 90°).

8. **315°:** If $e = 45$, then the sum of all the other angles is $360° — 45° = 315°$.

9. **105:** We are told that $a + e = 150$. Since they are both acute angles formed by a transversal cutting across two parallel lines, they are also congruent. Therefore, $a = e = 75$. Any acute angle in this diagram is supplementary to any obtuse angle, so $75 + f = 180$, and $f = 105$.

10. **70:** We know that angles a and g are supplementary; their measures sum to 180. Therefore:

$$y + 3y + 20 = 180$$
$$4y = 160$$
$$y = 40$$

Angle f is congruent to angle g, so its measure is also $3y + 20$. The measure of angle $f = g = 3(40) + 20 = 140$. If $f = 2x$, then $140 = 2x \rightarrow x = 70$.

11. **70:** We are given the measure of one acute angle (a) and one obtuse angle (g). Since any acute angle in this diagram is supplementary to any obtuse angle, $11y + 4x - y = 180$, or $4x + 10y = 180$. Since angle d is congruent to angle a, we know that $5y + 2x - 20 = 4x - y$, or $2x - 6y = -20$. We can set up a system of two equations with two variables:

$$2x - 6y = -20 \quad \rightarrow \quad \begin{array}{r} -4x + 12y = 40 \\ 4x + 10y = 180 \\ \hline 22y = 220 \\ y = 10;\ x = 20 \end{array}$$

Since h is one of the acute angles, h has the same measure as a: $4x - y = 4(20) - 10 = 70$.

12. **68:** Because b and d are supplementary, $4x + 3y + 8 = 180$, or $4x + 3y = 172$. Since d and e are congruent, $3y + 8 = x + 2y$, or $x - y = 8$. We can set up a system of two equations with two variables:

$$x - y = 8 \quad \rightarrow \quad \begin{array}{rl} 4x + 3y & = 172 \\ 3x - 3y & = 24 \\ \hline 7x & = 196 \\ x & = 28;\ y = 20 \end{array}$$

Since h is congruent to d, $h = 3y + 8$, or $3(20) + 8 = 68$.

13. **40:** If $c + g = 140$, then $i = 40$, because there are 180° in a triangle. Since k is vertical to i, k is also $= 40$. Alternately, if $c + g = 140$, then $j = 140$, since j is an exterior angle of the triangle and is therefore equal to the sum of the two remote interior angles. Since k is supplementary to j, $k = 180 - 140 = 40$.

14. **90:** If $g = 90$, then the other two angles in the triangle, c and i, sum to 90. Since a and k are vertical angles to c and i, they sum to 90 as well.

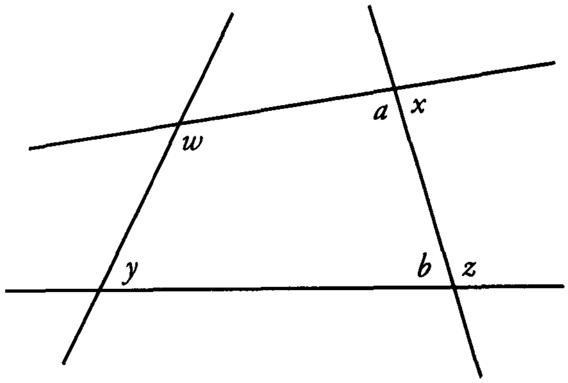
15. **150:** Angles f and k are vertical to angles g and i. These two angles, then, must also sum to 150. Angle b, an exterior angle of the triangle, must be equal to the sum of the two remote interior angles g and i. Therefore, $b = 150$.

16. **C:** You can substitute each of the values in Quantity A for a corresponding value in Quantity B. $a = d$, $c = f$, and $b = e$, in each case because the equal angles are vertical angles. Rewrite Quantity A.

Quantity A	Quantity B
$a + f + b = (d) + (c) + (e)$	$c + d + e$

Therefore **the two quantities are equal.**

17. **C:** To see why the sums in the two quantities are equal, label the remaining two interior angles of the quadrilateral a and b.



Quantity A	Quantity B
$w + y$	$x + z$

There are several relationships we can describe based on the diagram. For instance, we know the sum of the four internal angles of the quadrilateral is 360.

$w + y + a + b = 360$

We also have two pairs of supplementary angles.

$a + x = 180$
$b + z = 180$

Add the two equations together:

$a + b + x + z = 360$

$w + y + a + b$ sum to 360, as do $a + b + x + z$. Therefore the two sums equal each other.

$w + y + a + b = a + b + x + z$ Subtract $a + b$ from both sides
$w + y = x + z$

The **two quantities are equal.**

18. **A:** First solve for x. The two angles x and $2x$ are supplementary.

$x + 2x = 180$
$3x = 180$
$x = 60$

Next note that $2x = 6y$, because $2x$ and $6y$ are vertical angles. Plug in 60 for x and solve for y.

$2(60) = 6y$
$120 = 6y$
$20 = y$

Quantity A	Quantity B
$y = \mathbf{20}$	10

Therefore **Quantity A is larger.**

g

Chapter 7

of

GEOMETRY

THE COORDINATE PLANE

In This Chapter . . .

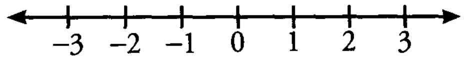
g

- Knowing Just One Coordinate
- Knowing Ranges
- Reading a Graph
- Plotting a Relationship
- Lines in the Plane
- The Intercepts of a Line
- The Intersection of Two Lines
- The Distance Between 2 Points

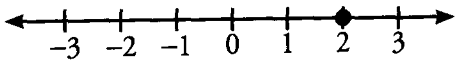
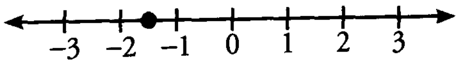
COORDINATE PLANE

Before we discuss the coordinate plane, let's review the number line.

The Number Line



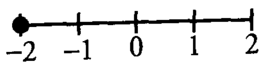
The number line is a ruler or measuring stick that goes as far as we want in both directions. With the number line, we can say where something is positioned with a single number. In other words, we can link a position with a number.

Position	Number	Number Line
"Two units right of zero"	2	
"One and a half units left of zero"	−1.5	

We use both positive and negative numbers, because we want to indicate positions both left and right of zero.

You might be wondering "The position of what?" The answer is, a **point**, which is just a dot. When we are dealing with the number line, a point and a number mean the same thing.

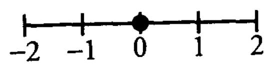
If you show me where the point is on the number line, I can tell you the number.



→ *the point is at −2*

If you tell me the number, I can show you where the point is on the number line.

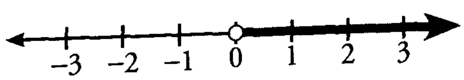
the point is at 0 →



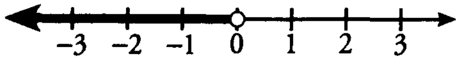
This works even if we only have partial information about our point. If you tell me *something* about where the point is, I can tell you *something* about the number, and vice-versa.

For instance, if I say that the number is positive, then I know that the point lies somewhere to the right of 0 on the number line. Even though I don't know the exact location of the point, I do know a range of potential values.

The number is positive.
In other words, the number is greater than (>) 0.

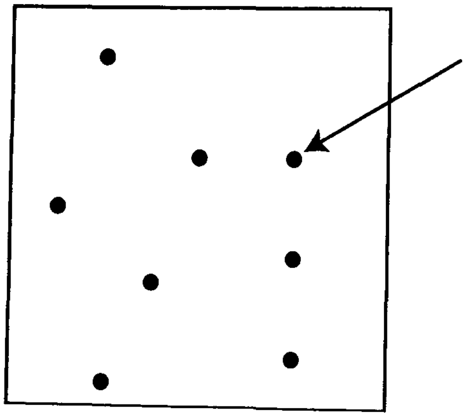


This also works in reverse. If I see a range of potential positions on a number line, I can tell you what that range is for the number.



The number is less than (<) 0.

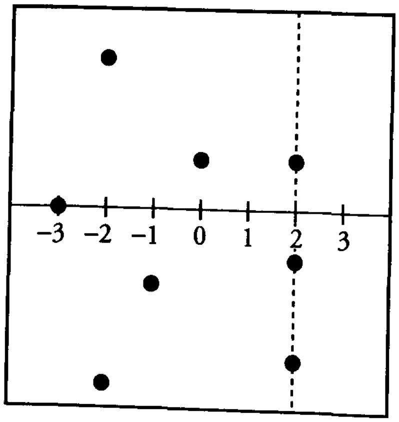
Let's make things more complicated. What if we want to be able to locate a point that's not on a straight line, but on a page?



The point we want

Now one number line won't be enough to tell us where the point is.

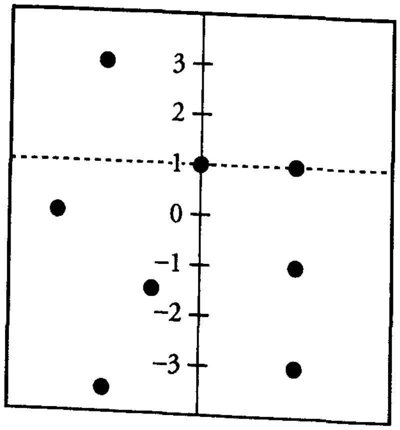
Let's begin by inserting our number line into the picture. This will help us determine how far to the right or left of 0 our point is.



The point is two units to the right of zero.

But all three points that touch the dotted line are two units to the right of zero. We don't have enough information to determine the unique location of our point.

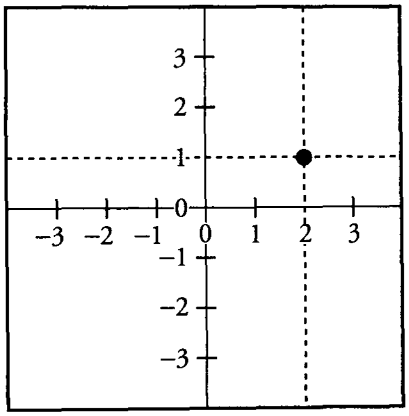
In order to know the location of our point, we also need to know how far up or down the dotted line we need to go. To determine how far up or down we need to go, we're going to need another number line. This number line, however, is going to be vertical. Using this vertical number line, we will be able to measure how far above or below 0 a point is.



The point is one unit above zero.

Notice that this number line by itself also does not provide enough information to determine the unique location of the point.

But, if we combine the information from the two number lines, we can determine both how far left or right *and* how far up or down the point is.



The point is 2 units to the right of 0, because a page has two dimensions.

AND

The point is 1 unit above 0.

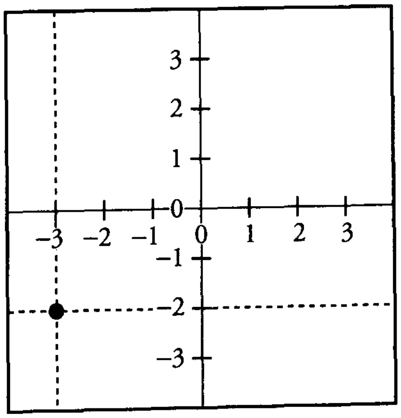
Now we have a unique description of the point's position. There is only one point on the page that is BOTH 2 units to the right of 0 AND 1 unit above 0. So, on a page, we need two numbers to indicate position.

Just as with the number line, information can travel in either direction. If we know the two numbers that give the location, we can place that point on the page.

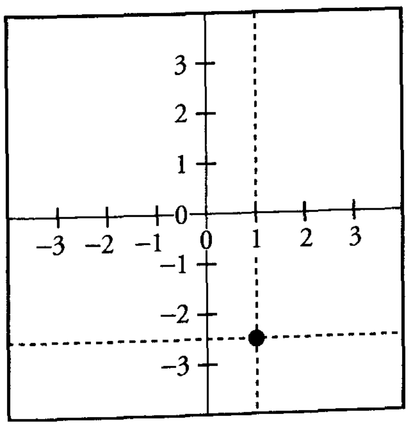
The point is 3 units to the left of 0.

AND

The point is 2 units below 0.



If, on the other hand, we see a point on the page, we can identify its location and determine the two numbers.



The point is 1 unit to the right of 0.

AND

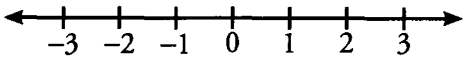
The point is 2.5 units below 0.

Now that we have two pieces of information for each point, we need to keep straight which number is which. In other words, we need to know which number gives the left-right position and which number gives the up-down position.

To represent the difference, we use some technical terms:

The ***x*-coordinate** is the left right-number.

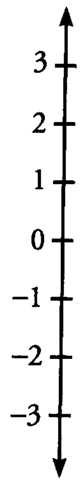
Numbers to the right of 0 are positive.
Numbers to the left of 0 are negative.



This number line is the ***x*-axis.**

The ***y*-coordinate** is the up-down number.

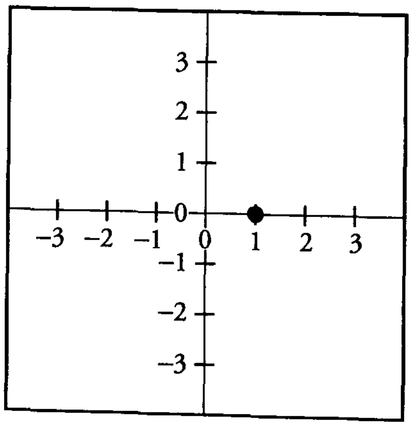
Numbers above 0 are positive.
Numbers below 0 are negative.



This number line is the ***y*-axis.**

Now, when describing the location of a point, we can use the technical terms.

The x-coordinate of the point is 1 and the y-coordinate of the point is 0.

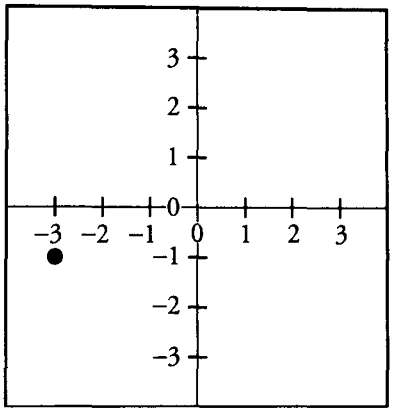


We can condense this and say that, for this point, $x = 1$ and $y = 0$. In fact, we can go even further. We can say that the point is at (1, 0). This shorthand always has the same basic layout. The first number in the parentheses is the x-coordinate, and the second number is the y-coordinate. One easy way to remember this is that x comes before y in the alphabet.

The point is at (−3, −1),

OR

the point has an x-coordinate of −3 and a y-coordinate of −1.



Now we have a fully functioning **coordinate plane:** an x-axis and a y-axis drawn on a page. The coordinate plane allows us to determine the unique position of any point on a **plane** (essentially, a really big and flat sheet of paper).

And in case you were ever curious about what one-dimensional and two-dimensional mean, now you know. A line is one dimensional, because you only need *one* number to identify a point's location. A plane is two-dimensional because you need *two* numbers to identify a point's location.

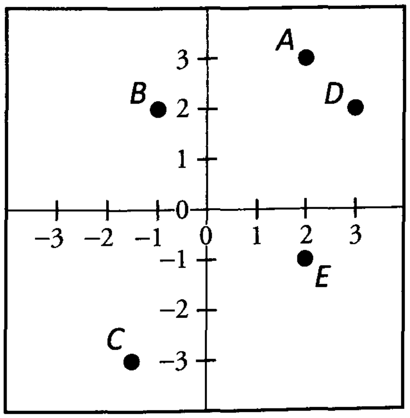
Check Your Skills

1. Draw a coordinate plane and plot the following points:

1. (3, 1) 2. (−2, 3.5) 3. (0, −4.5) 4. (1, 0)

2. Which point on the coordinate plane below is indicated by the following coordinates?

1. (2, −1) 2. (−1.5, −3) 3. (−1, 2) 4. (3, 2)

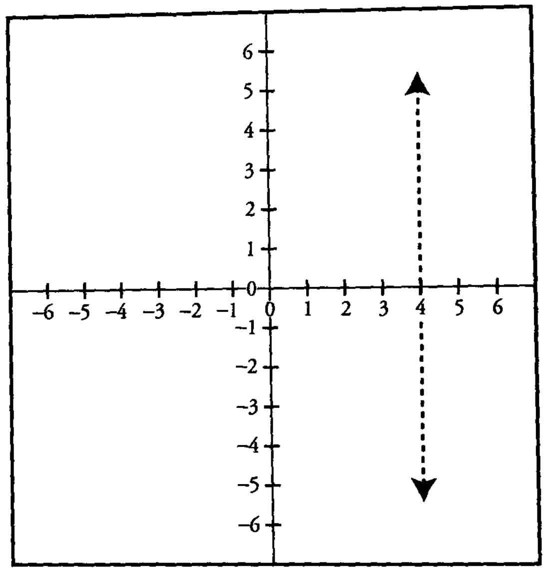


Answers can be found on page 151.

Knowing Just One Coordinate

As we've just seen, we need to know both the x-coordinate and the y-coordinate to plot a point exactly on the coordinate plane. If we only know one coordinate, we can't tell precisely where the point is, but we can narrow down the possibilities.

Consider this situation. Let's say that this is all we know: the point is 4 units to the right of 0.

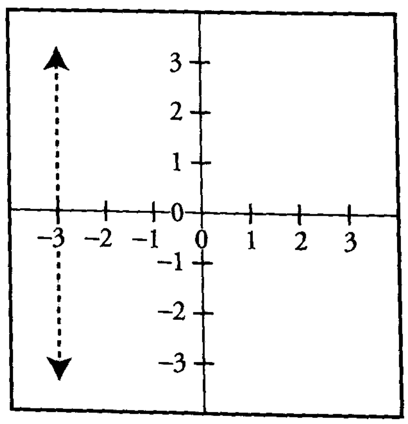


As we saw earlier, any point along the vertical dotted line is 4 units to the right of 0. In other words, every point on the dotted line has an x-coordinate of 4. We could shorten that and say $x = 4$. We don't know anything about the y-coordinate, which could be any number. All the points along the dotted line have different y-coordinates but the same x-coordinate, which equals 4.

So, if we know that $x = 4$, then our point can be anywhere along a vertical line that crosses the x-axis at (4, 0). Let's try with another example.

If we know that $x = -3$...

Then we know

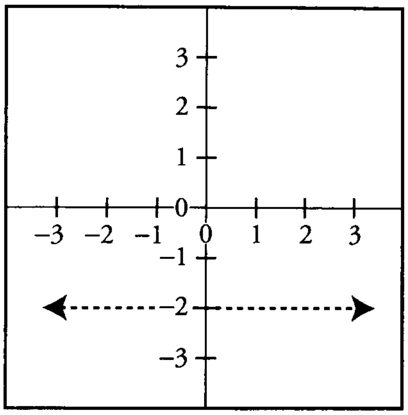


Every point on the dotted line has an x-coordinate of -3.

Points on the dotted line include (–3, 1), (–3, –7), (–3, 100) and so on. In general, if we know the x-coordinate of a point and not the y-coordinate, then all we can say about the point is that it lies on a vertical line.

The x-coordinate still indicates left-right position. If we fix that position but not the up-down position, then the point can only move up and down—forming a vertical line.

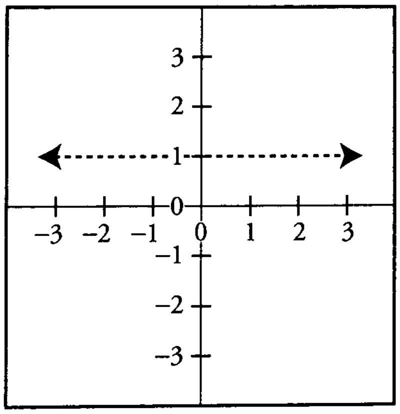
Now imagine that all we know is the y-coordinate of a number. Let's say we know that $y = -2$. How could we represent this on the coordinate plane? In other words, what are all the points for which $y = -2$?



Every point 2 units below 0 fits this condition. These points form a horizontal line. We don't know anything about the x-coordinate, which could be any number. All the points along the horizontal dotted line have different x-coordinates but the same y-coordinate, which equals -2. For instance, $(-3, -2)$, $(-2, -2)$, $(50, -2)$ are all on the line.

Let's try another example. If we know that $y = 1$...

Then we know



Every point on the dotted line has an y-coordinate of 1.

If we know the y-coordinate but not the x-coordinate, then we know the point lies somewhere on a horizontal line.

Check Your Skills

Draw a coordinate plane and plot the following lines.

3. $x = 6$

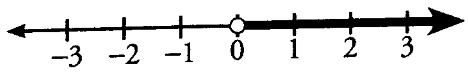
4. $y = -2$

5. $x = 0$

Answers can be found on pages 151–152.

Knowing Ranges

Now let's provide even less information. Instead of knowing the actual x-coordinate, let's see what happens if all we know is a range of possible values for x. What do we do if all we know is that $x > 0$? To answer that, let's return to the number line for a moment. As we saw earlier, if $x > 0$, then the target is anywhere to the right of 0.



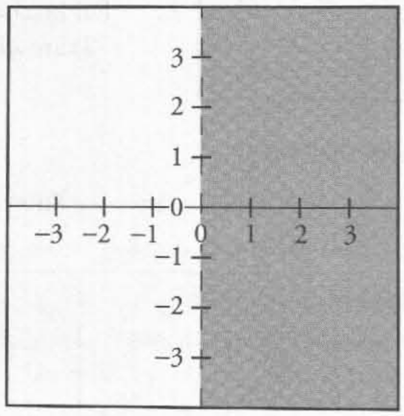
$x > 0$

Now let's look at the coordinate plane. All we know is that x is greater than 0. And we don't know *anything* about y, which could be any number.

How do we show all the possible points? We can shade in part of the coordinate plane: the part to the right of 0.

If we know that $x > 0$…

Then we know:

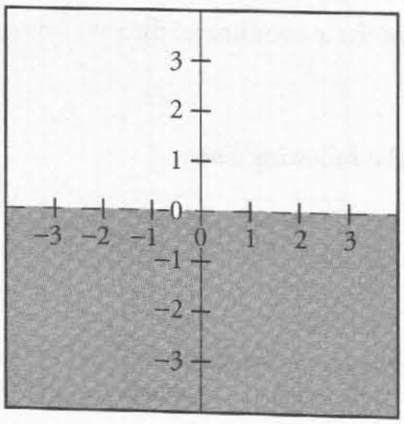


Every point in the shaded region has an x-coordinate greater than 0.

Now let's say that all we know is $y < 0$. Then we can shade in the bottom half of the coordinate plane—where the y-coordinate is less than 0. The x-coordinate can be anything. Notice that the dashed line indicates that x cannot be zero. It must be below the dashed line.

If we know that $y < 0$…

Then we know:

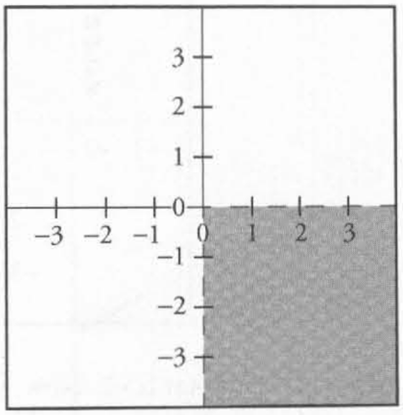


Every point in the shaded region has a y-coordinate less than 0.

Finally, if we know information about both x and y, then we can narrow down the shaded region.

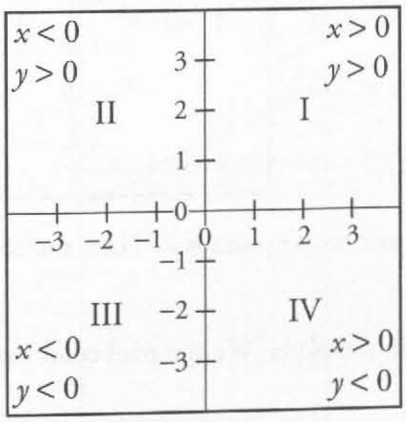
If we know that $x > 0$ AND $y < 0$…

Then we know:



The only place where x is greater than 0 AND y is less than 0 is the bottom right quarter of the plane. So we know that the point lies somewhere in the bottom right quarter of the coordinate plane.

The four quarters of the coordinate plane are called **quadrants**. Each quadrant corresponds to a different combination of signs of x and y. The quadrants are always numbered as shown below, starting on the top right and going counter-clockwise.



Check Your Skills

6. Which quadrant do the following points lie in?

1. (1, −2) 2. (−4.6, 7) 3. (−1, −2.5) 4. (3, 3)

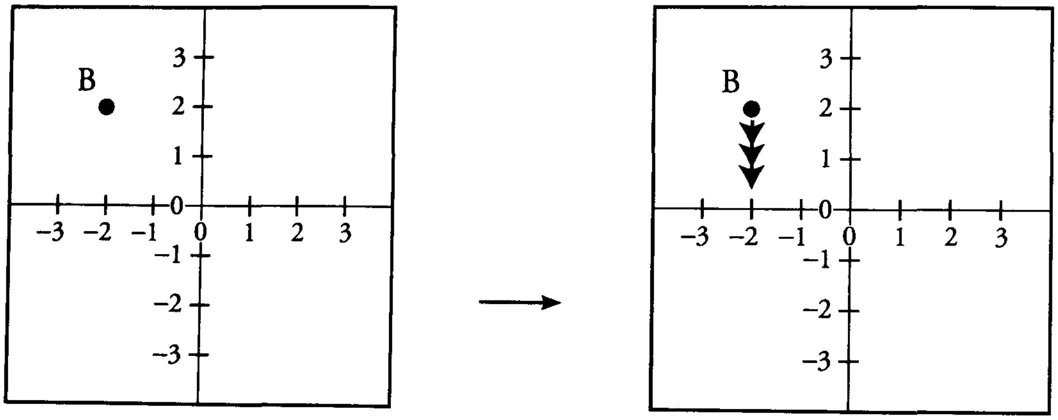
7. Which quadrant or quadrants are indicated by the following?

1. $x < 0, y > 0$ 2. $x < 0, y < 0$ 3. $y > 0$ 4. $x < 0$

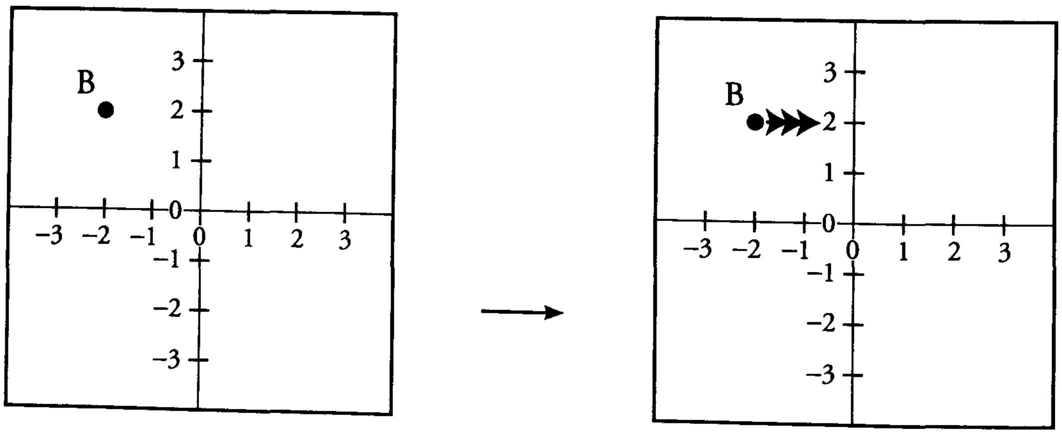
Answers can be found on page 152.

Reading a Graph

If we see a point on a coordinate plane, we can read off its coordinates as follows. To find an x-coordinate, drop an imaginary line down to the x-axis (if the point is above the x-axis) or draw a line up to the x-axis (if the point is below the x-axis) and read off the number.



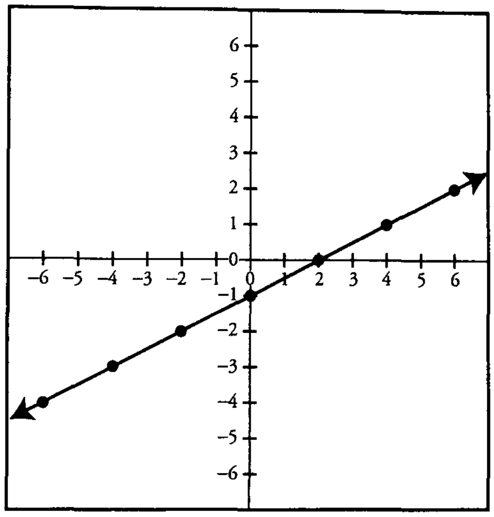
The line hit the x-axis at −2, which means the x-coordinate of our point is −2. Now, to find the y-coordinate, we employ a similar technique, only now we draw a horizontal line instead of a vertical line.



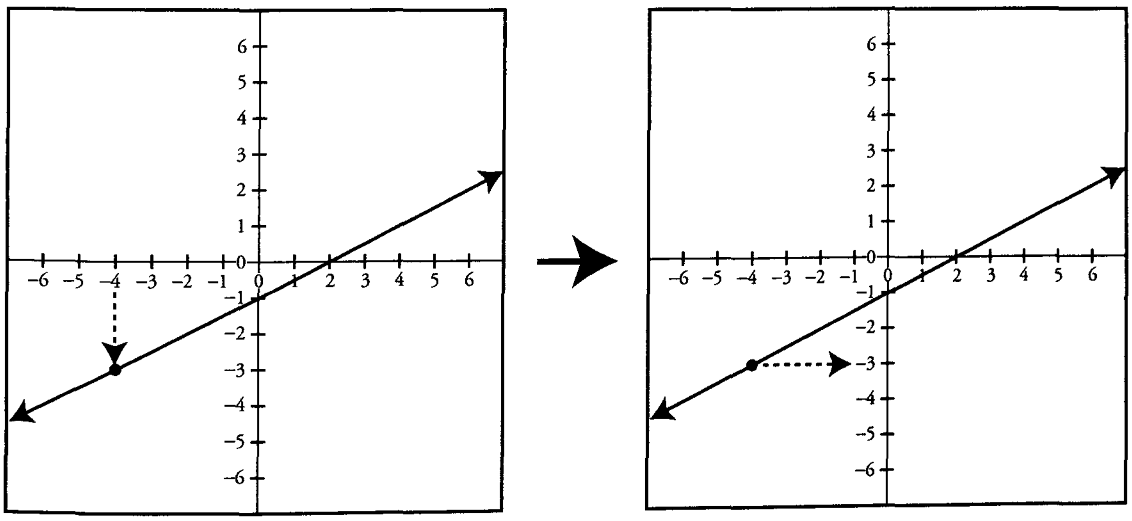
Our line touched the y-axis at 2, which means the y-coordinate of our point is 2. Thus, the coordinates of point B are (−2, 2).

Now suppose that we know the target is on a slanted line in the plane. We can read coordinates off of this slanted line. Try this problem on your own first.

On the line shown, what is the y-coordinate of the point that has an x-coordinate of −4?



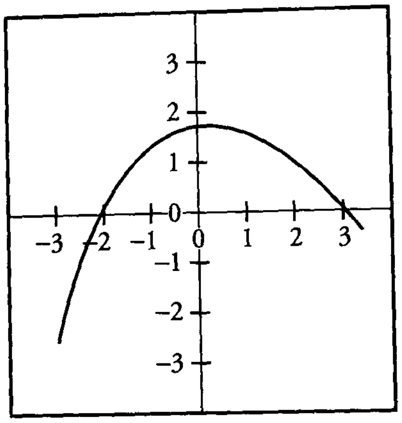
To answer this question, think about reading the coordinates of a point. We went from the point to the axes. Here, we will go from the axis that we know (here, the *x*-axis) to the line that contains the point, and then to the *y*-axis (the axis we don't know).



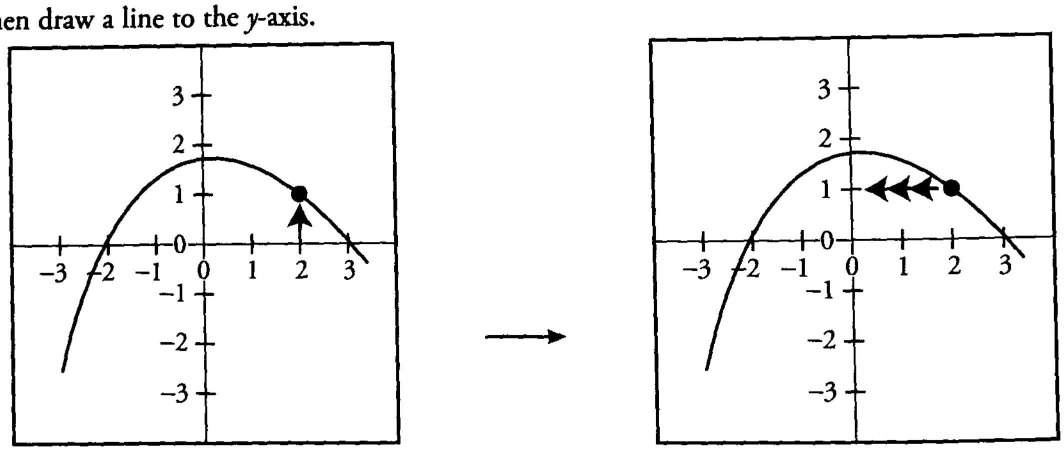
So the point on the line that has an *x*-coordinate of -4 has a *y*-coordinate of -3.

This method of locating points applies equally well to any shape or curve you may encounter on a coordinate plane. Try this next problem.

On the curve shown, what is the value of y when $x = 2$?



Once again, we know the x-coordinate, so we draw a line from the x-axis (where we know the coordinate) to the curve, and then draw a line to the y-axis.

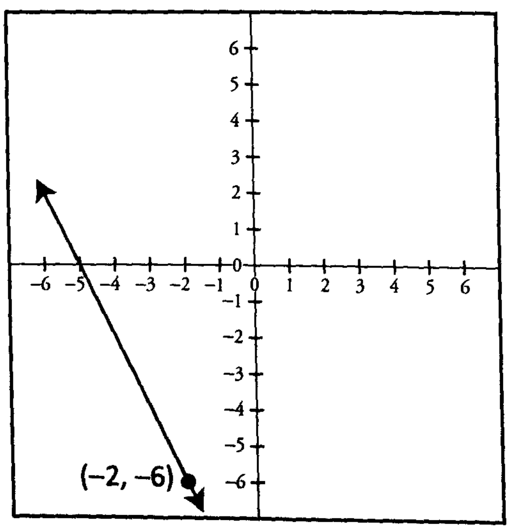


On the curve shown, the point that has an x-coordinate of 2 has a y-coordinate of 1.

Note that the GRE will mathematically define each line or curve, so you will never be forced to guess visually where a point falls. In fact, if more specific information is not given for a coordinate problem on the GRE, you cannot infer the location of a point based solely on visual cues. This discussion is **only** meant as an exercise to convey how to use any graphical representation.

Check Your Skills

8. On the following graph, what is the y-coordinate of the point on the line that has an x-coordinate of −3?



The answer can be found on page 152.

Plotting a Relationship

The most frequent use of the coordinate plane is to display a relationship between x and y. Often, this relationship is expressed this way: if you tell me x, I can tell you y.

As an equation, this sort of relationship looks like this:

y = some expression involving x — Another way of saying this is we have y "in terms of" x

Examples: $y = 2x + 1$

$y = x^2 - 3x + 2$

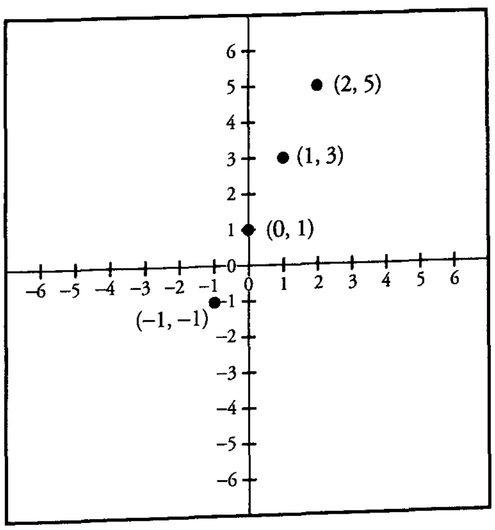
$y = \dfrac{x}{x+2}$

If you plug a number in for x in any of these equations, you can calculate a value for y.

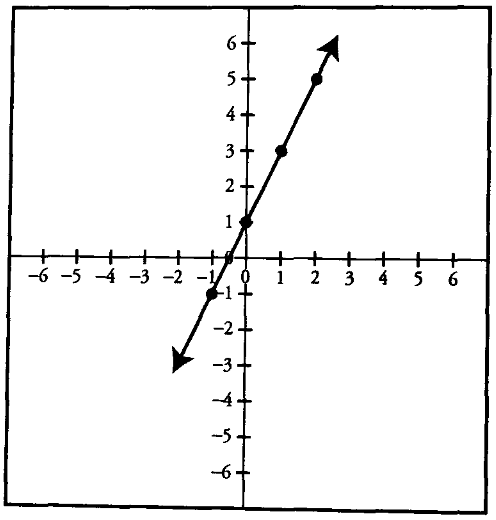
Let's take $y = 2x + 1$. We can generate a set of y's by plugging in various values of x. Start by making a table.

x	$y = 2x + 1$
-1	$y = 2(-1) + 1 = -1$
0	$y = 2(0) + 1 = 1$
1	$y = 2(1) + 1 = 3$
2	$y = 2(2) + 1 = 5$

Now that we have some values, let's see what we can do with them. We can say that when x equals 0, y equals 1. These two values form a pair. We express this connection by plotting the point (0, 1) on the coordinate plane. Similarly, we can plot all the other points that represent an x-y pair from our table:

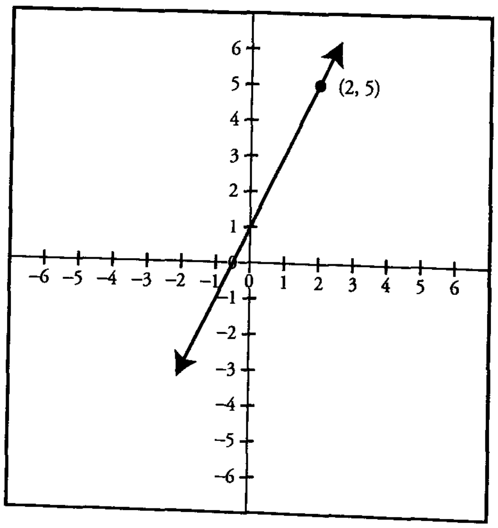


You might notice that these points seem to lie on a straight line. You're right—they do. In fact, any point that we can generate using the relationship $y = 2x + 1$ will also lie on the line.



This line is the graphical representation of $y = 2x + 1$

So now we can talk about equations in visual terms. In fact, that's what lines and curves on the coordinate plane are—they represent all the *x-y* pairs that make an equation true. Take a look at the following example:



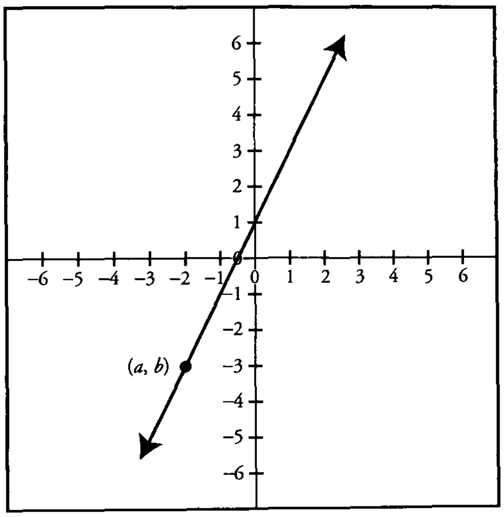
$\longleftrightarrow$

$y = 2x + 1$

$5 = 2(2) + 1$

The point (2, 5) lies on the line $y = 2x + 1$ $\longleftrightarrow$ If we plug in 2 for x in $y = 2x + 1$, we get 5 for y

We can even speak more generally, using variables.



$\longleftrightarrow$ $y = 2x + 1$

$b = 2(a) + 1$

The point (a, b) lies on the line $y = 2x + 1$ $\longleftrightarrow$ If we plug in a for x in $y = 2x + 1$, we get b for y

Check Your Skills

9. True or False? The point (9, 21) is on the line $y = 2x + 1$

10. True or False? The point (4, 14) is on the curve $y = x^2 - 2$

Answers can be found on page 153.

Lines in the Plane

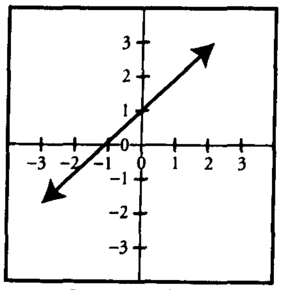
The relationship $y = 2x + 1$ formed a line in the coordinate plane, as we saw. We can actually generalize this relationship. *Any* relationship of the following form represents a line:

$y = mx + b$ m and b represent numbers (positive or negative)

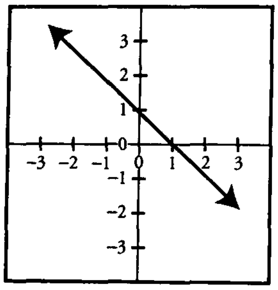
For instance, in the equation $y = 2x + 1$, we can see that $m = 2$ and $b = 1$.

Lines		Not Lines
$y = 3x - 2$	$m = 3$, $b = -2$	$y = x^2$
$y = -x + 4$	$m = -1$, $b = 4$	$y = \frac{1}{x}$
These are called linear equations.		These equations are not linear.

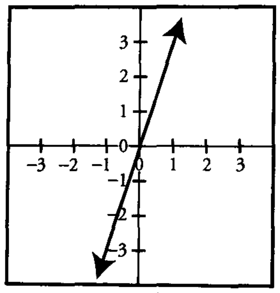
The numbers m and b have special meanings when we are dealing with linear equations. m = **slope.** This tells us how steep the line is and whether the line is rising or falling.



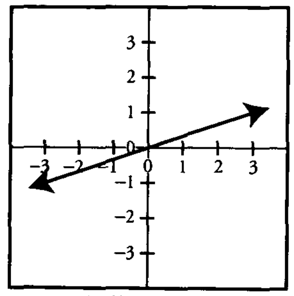
Positive Slope
$m > 0$



Negative Slope
$m < 0$

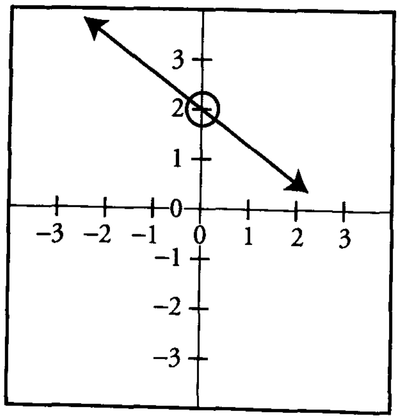


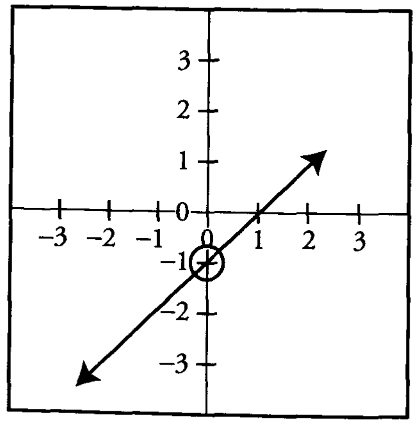
Steep Slope
$m > 1$



Shallow Slope
$0 < m < 1$

b = ***y*-intercept.** This tells you where the line crosses the y-axis. Any line or curve crosses the y-axis when $x = 0$. To find the y-intercept, plug in 0 for x into the equation.





By recognizing linear equations and identifying m and b, we can plot a line more quickly than by plotting several points on the line.

Check Your Skills

What are the slope and y-intercept of the following lines?

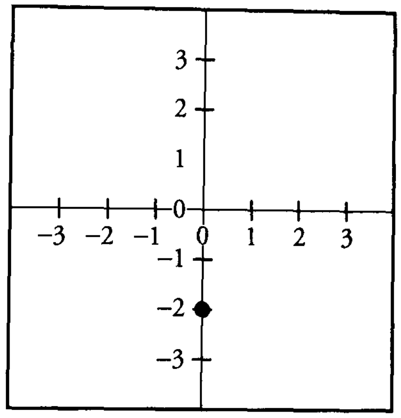
11. $y = 3x + 4$

12. $2y = 5x - 12$

Answers can be found on page 153.

Now the question becomes, how do we use m and b to sketch a line? Let's plot the line $y = \frac{1}{2}x - 2$.

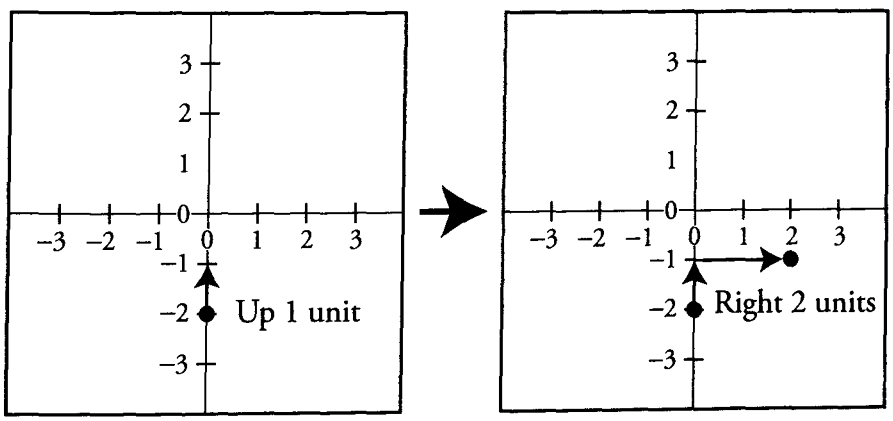
The easiest way to begin graphing a line is to begin with the y-intercept. We know that the line crosses the y-axis at $y = -2$, so let's begin by plotting that point on our coordinate plane.



Now we need to figure out how to use slope in order to finish drawing our line. Every slope, whether an integer or a fraction, should be thought of as a fraction. In this equation, our m is $\frac{1}{2}$. Let's look at the parts of the fraction and see what they can tell us about our slope.

$$\frac{1}{2} \rightarrow \frac{\text{Numerator}}{\text{Denominator}} \rightarrow \frac{\text{Rise}}{\text{Run}} \rightarrow \frac{\text{Change in } y}{\text{Change in } x}$$

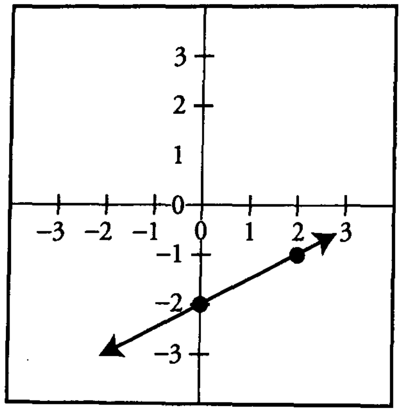
The numerator of our fraction tells us how many units we want to move in the y direction—in other words, how far up or down we want to move. The denominator tells us how many units we want to move in the x direction—in other words, how far left or right we want to move. For this particular equation, the slope is $\frac{1}{2}$, which means we want to move up 1 unit and right 2 units.



After we went up 1 unit and right 2 units, we ended up at the point (2, −1). What that means is that the point (2, −1) is also a solution to the equation $y = \frac{1}{2}x - 2$. In fact, we can plug in the x value and solve for y to check that we did this correctly.

$$y = \frac{1}{2}x - 2 \rightarrow y = \frac{1}{2}(2) - 2 \rightarrow y = 1 - 2 \rightarrow y = -1$$

What this means is that we can use the slope to generate points and draw our line. If we go up another 1 unit and right another 2 units, we will end up with another point that appears on the line. Although we could keep doing this indefinitely, in reality, with only 2 points we can figure out what our line looks like. Now all we need to do is draw the line that connects the 2 points we have, and we're done.

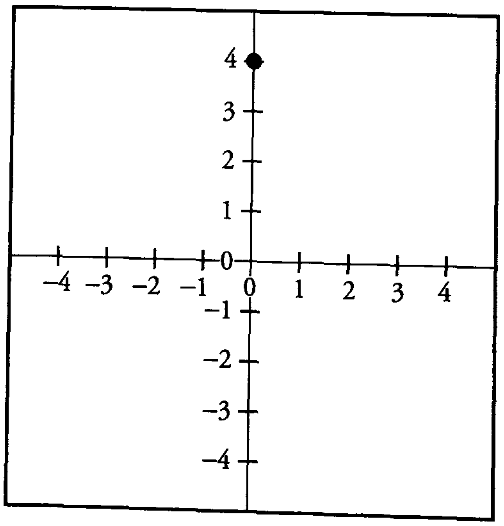


$$y = \frac{1}{2}x - 2$$

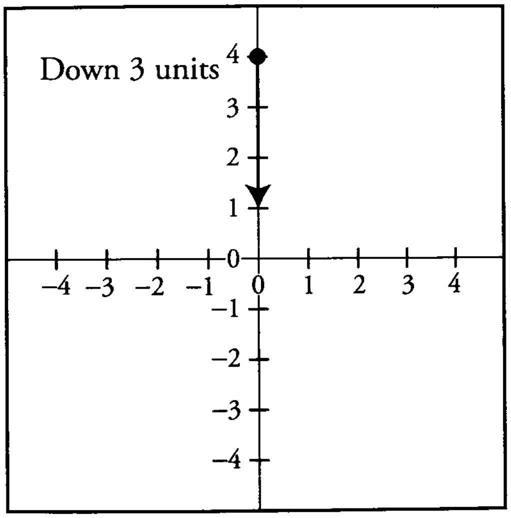
That means that this line is the graphical representation of $y = \frac{1}{2}x - 2$.

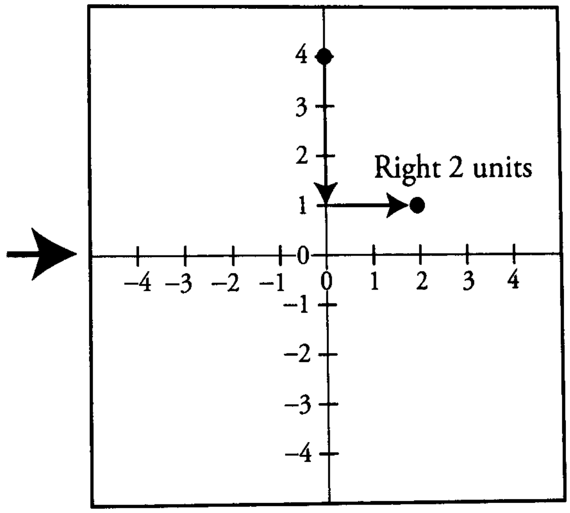
Let's try another one. Graph the equation $y = \left(-\frac{3}{2}\right)x + 4$.

Once again, the best way to start is to plot the y-intercept. In this equation, $b = 4$, so we know the line crosses the y-axis at the point (0, 4):

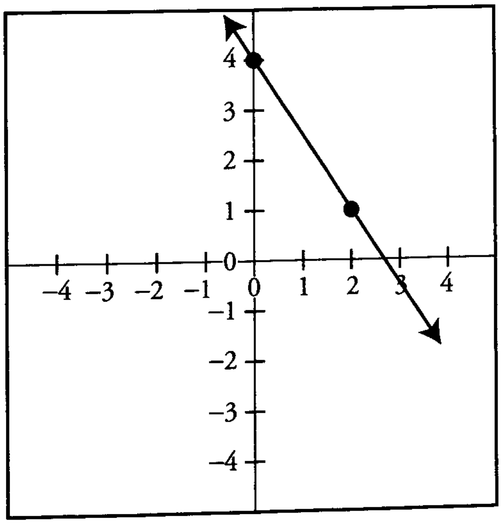


Now we can use the slope to find a second point. This time, the slope is $-\frac{3}{2}$, which is a negative slope. While positive slopes go up and to the right, negative slopes go down and to the right. Now, to find the next point, we need to go *down* 3 units and right 2 units.





That means that (2, 1) is another point on the line. Now that we have 2 points, we can draw our line.



$$y = \left(-\frac{3}{2}\right)x + 4$$

Check Your Skills

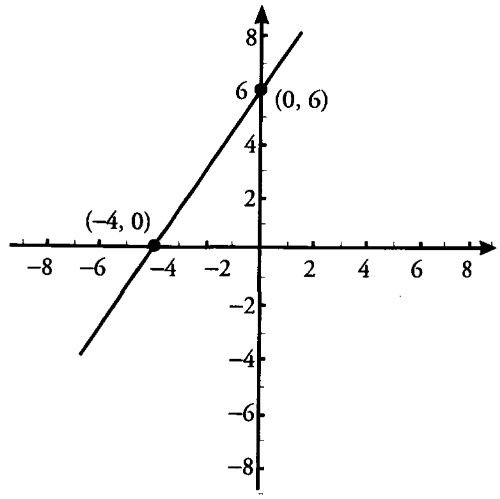
13. Draw a coordinate plane and graph the line $y = 2x - 4$. Identify the slope and the y-intercept.

The answer can be found on page 153.

The Intercepts of a Line

A point where a line intersects a coordinate axis is called an **intercept.** There are two types of intercepts: the x-intercept, where the line intersects the x-axis, and the y-intercept, where the line intersects the y-axis.

The x-intercept is expressed using the ordered pair $(x, 0)$, where x is the point where the line intersects the x-axis. **The x-intercept is the point on the line at which $y = 0$.** In this diagram, the x-intercept is -4, as expressed by the ordered pair $(-4, 0)$.



The y-intercept is expressed using the ordered pair $(0, y)$, where y is the point where the line intersects the y-axis. **The y-intercept is the point on the line at which $x = 0$.** In this diagram, the y-intercept is 6, as expressed by the ordered pair $(0, 6)$.

To find x-intercepts, plug in 0 for y. To find y-intercepts, plug in 0 for x.

Check Your Skills

14. What are the x- and y-intercepts of the equation $x - 2y = 8$?

Answers can be found on page 153–154.

The Intersection of Two Lines

Recall that a line in the coordinate plane is defined by a linear equation relating x and y. That is, if a point (x, y) lies on the line, then those values of x and y satisfy the equation. For instance, the point $(3, 2)$ lies on the line defined by the equation $y = 4x - 10$, since the equation is true when we plug in $x = 3$ and $y = 2$:

$$y = 4x - 10$$
$$2 = 4(3) - 10 = 12 - 10$$
$$2 = 2 \quad \text{TRUE}$$

On the other hand, the point $(7, 5)$ does not lie on that line, because the equation is false when we plug in $x = 7$ and $y = 5$:

$$y = 4x - 10$$
$$5 = 4(7) - 10 = 28 - 10 = 18? \quad \text{FALSE}$$

So, what does it mean when two lines intersect in the coordinate plane? It means that at the point of intersection, BOTH equations representing the lines are true. That is, the pair of numbers (x, y) that represents the point of intersection solves BOTH equations. Finding this point of intersection is equivalent to solving a system of two linear equations. You can find the intersection by using algebra more easily than by graphing the two lines.

At what point does the line represented by $y = 4x - 10$ intersect the line represented by $2x + 3y = 26$?

Since $y = 4x - 10$, replace y in the second equation with $4x - 10$ and solve for x:

$$
\begin{aligned}
2x + 3(4x - 10) &= 26 \\
2x + 12x - 30 &= 26 \\
14x &= 56 \\
x &= 4
\end{aligned}
$$

Now solve for y. You can use either equation, but the first one is more convenient:

$$
\begin{aligned}
y &= 4x - 10 \\
y &= 4(4) - 10 \\
y &= 16 - 10 = 6
\end{aligned}
$$

Thus, the point of intersection of the two lines is (4, 6).

If two lines in a plane do not intersect, then the lines are parallel. If this is the case, there is NO pair of numbers (x, y) that satisfies both equations at the same time.

Two linear equations can represent two lines that intersect at a single point, or they can represent parallel lines that never intersect. There is one other possibility: the two equations might represent the same line. In this case, infinitely many points (x, y) along the line satisfy the two equations (which must actually be the same equation in disguise).

The Distance Between 2 Points

The distance between any two points in the coordinate plane can be calculated by using the Pythagorean Theorem. For example:

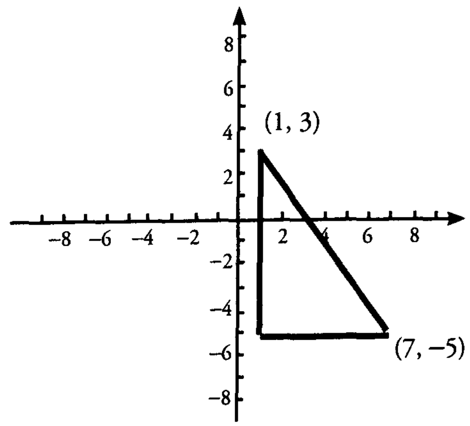
What is the distance between the points (1, 3) and (7, –5)?

(1) Draw a right triangle connecting the points.

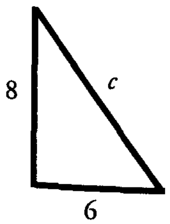
(2) Find the lengths of the two legs of the triangle by calculating the rise and the run.

The y-coordinate changes from 3 to –5, a difference of 8 (the vertical leg).

The x-coordinate changes from 1 to 7, a difference of 6 (the horizontal leg).



(3) Use the Pythagorean Theorem to calculate the length of the diagonal, which is the distance between the points.



$$
\begin{aligned}
6^2 + 8^2 &= c^2 \\
36 + 64 &= c^2 \\
100 &= c^2 \\
c &= 10
\end{aligned}
$$

The distance between the two points is 10 units.

Alternatively, to find the hypotenuse, we might have recognized this triangle as a variation of a 3–4–5 triangle (specifically, a 6–8–10 triangle).

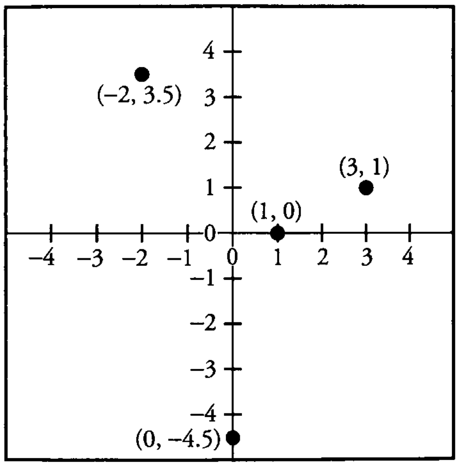
Check Your Skills

15. What is the distance between (−2, −4) and (3, 8)?

Answers can be found on page 154.

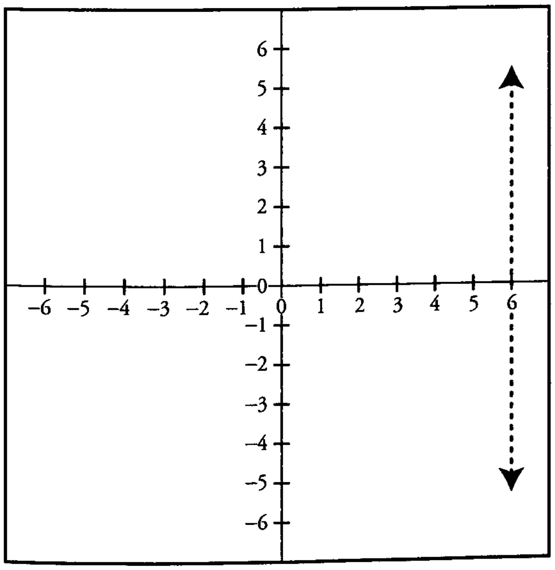
Check Your Skills Answers

1.



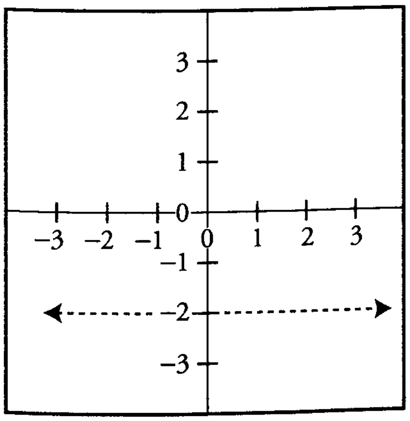
2.
1. (2, −1): **E**
2. (−1.5, −3): **C**
3. (−1, 2): **B**
4. (3, 2): **D**

3.

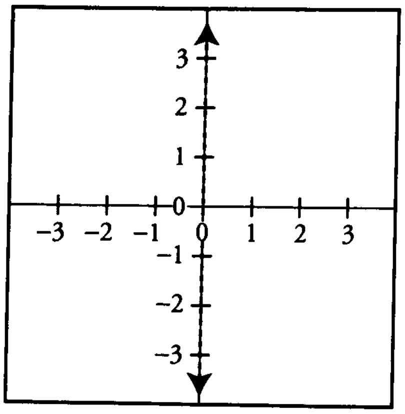


$x = 6$

4.

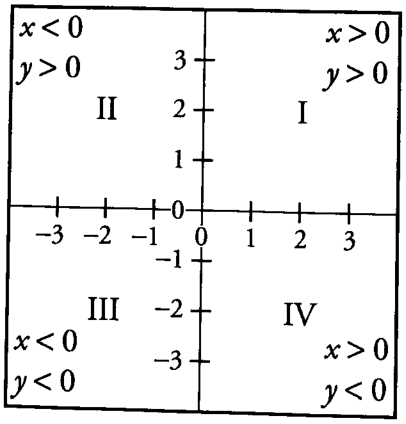


$y = -2$

5. 

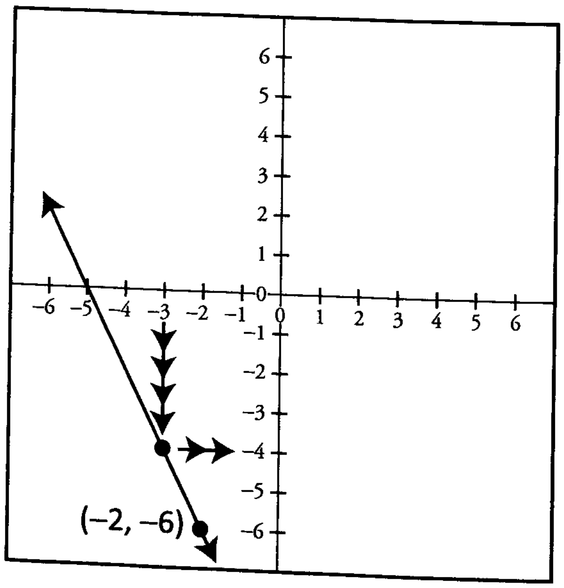
$x = 0$ is the y-axis.

6. **1. (1, −2) is in Quadrant IV**
2. (−4.6, 7) is in Quadrant II
3. (−1, −2.5) is in Quadrant III
4. (3, 3) is in Quadrant I



7. **1. $x < 0, y > 0$ indicates Quadrant II**
2. $x < 0, y < 0$ indicates Quadrant III
3. $y > 0$ indicates Quadrants I and II
4. $x < 0$ indicates Quadrants II and III

8. The point on the line with $x = -3$ has a y-coordinate of -4.



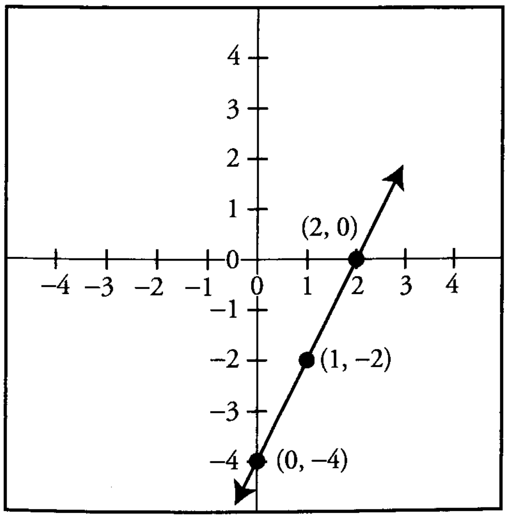
9. **False.** The relationship is $y = 2x + 1$, and the point we are testing is (9, 21). So we plug in 9 for x and see if we get 21 for y. $y = 2(9) + 1 = 19$. The point (9, 21) does not lie on the line.

10. **True.** The relationship is $y = x^2 - 2$, and the point we are testing is (4, 14). So we plug in 4 for x and see if we get 14 for y. $y = (4)^2 - 2 = 14$. The point (4, 14) lies on the curve.

11. **Slope is 3, *y*-intercept is 4.** The equation $y = 3x + 4$ is already in $y = mx + b$ form, so we can directly find the slope and y-intercept. The slope is 3, and the y-intercept is 4.

12. **Slope is 2.5, *y*-intercept is –6.** To find the slope and y-intercept of a line, we need the equation to be in $y = mx + b$ form. We need to divide our original equation by 2 to make that happen. So $2y = 5x - 12$ becomes $y = 2.5x - 6$. So the slope is 2.5 (or 5/2) and the y-intercept is –6.

13.

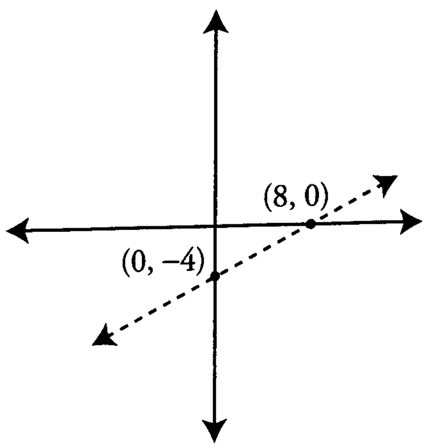


$y = 2x - 4$

slope = 2

y-intercept = –4

14. ***x*-intercept is 8, *y*-intercept is –4:**



We've illustrated the line on the coordinate plane above, but you can also answer this question using algebra.

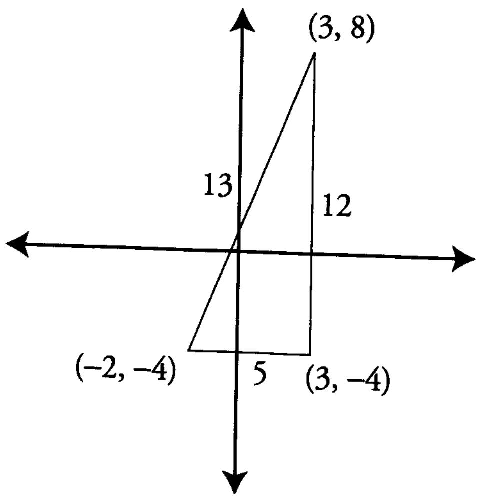
To determine the x-intercept, set y equal to 0, then solve for x:

$$x - 2y = 8$$
$$y = 0$$
$$x - 0 = 8$$
$$x = 8$$

To determine the y-intercept, set x equal to 0, then solve for y:

$$x - 2y = 8$$
$$x = 0$$
$$0 - 2y = 8$$
$$-2y = 8$$
$$y = -4$$

15. **13:** Answer: 13



The illustration above shows the two points. We have constructed a right triangle by finding a point directly below (3, 8) and directly to the right of (−2, −4). This right triangle has legs of 5 (the change from −2 to 3) and 12 (the change from −4 to 8). We can plug those values into the Pythagorean Theorem and solve for the hypotenuse:

$$A^2 + B^2 = C^2$$
$$5^2 + 12^2 = C^2$$
$$25 + 144 = C^2$$
$$C^2 = 169$$
$$C = \sqrt{169} = 13$$

Alternatively, we could recognize the common Pythagorean triplet 5, 12, 13.

Problem Set

1. A line has the equation $y = 3x + 7$. At which point will this line intersect the y-axis?

2. A line has the equation $x = \frac{y}{80} - 20$. At which point will this line intersect the x-axis?

3 A line has the equation $x = -2y + z$. If (3, 2) is a point on the line, what is z?

4. A line is represented by the equation $y = zx + 18$. If this line intersects the x-axis at (–3, 0), what is z?

5. A line has a slope of 1/6 and intersects the x-axis at (–24, 0). Where does this line intersect the y-axis?

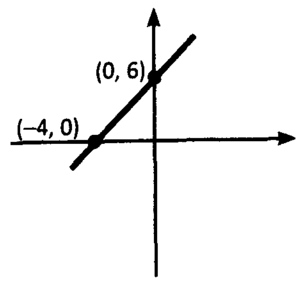
6. Which quadrants, if any, do not contain any points on the line represented by $x - y = 18$?

7. Which quadrants, if any, do not contain any points on the line represented by $x = 10y$?

8. Which quadrants contain points on the line $y = \frac{x}{1{,}000} + 1{,}000{,}000$?

9. Which quadrants contain points on the line represented by $x + 18 = 2y$?

10. What is the equation of the line shown to the right?



11. What is the intersection point of the lines defined by the equations $2x + y = 7$ and $3x - 2y = 21$?

12.

Quantity A	Quantity B
The y-intercept o\f the line $y = \frac{3}{2}x - 2$	The x-intercept of the line $y = \frac{3}{2}x - 2$

13.

Quantity A	Quantity B
The slope of the line $2x + 5y = 10$	The slope of the line $5x + 2y = 10$

14.

Quantity A	Quantity B
The distance between points (0, 9) and (–2, 0)	The distance between points (3, 9) and (10, 3)

1. **(0, 7):** A line intersects the y-axis at the y-intercept. Since this equation is written in slope-intercept form, the y-intercept is easy to identify: 7. Thus, the line intersects the y-axis at the point (0, 7).

2. **(–20, 0):** A line intersects the x-axis at the x-intercept, or when the y-coordinate is equal to zero. Substitute zero for y and solve for x:

$$x = 0 - 20$$
$$x = -20$$

3. **7:** Substitute the coordinates (3, 2) for x and y and solve for z.

$$3 = -2(2) + z$$
$$3 = -4 + z$$
$$z = 7$$

4. **6:** Substitute the coordinates (–3, 0) for x and y and solve for z.

$$0 = z(-3) + 18$$
$$3z = 18$$
$$z = 6$$

5. **(0, 4):** Use the information given to find the equation of the line:

$$y = \frac{1}{6}x + b$$
$$0 = \frac{1}{6}(-24) + b$$
$$0 = -4 + b$$
$$b = 4$$

The variable b represents the y-intercept. Therefore, the line intersects the y-axis at (0, 4).

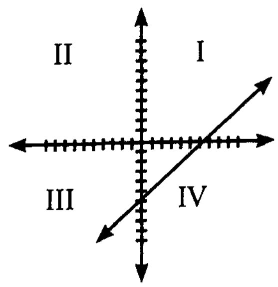
6. **II:** First, rewrite the line in slope-intercept form:

$$y = x - 18$$

Find the intercepts by setting x to zero and y to zero:

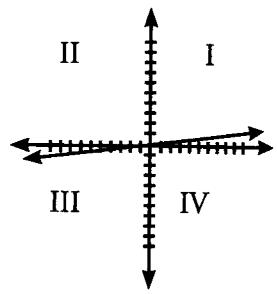
$$y = 0 - 18 \qquad 0 = x - 18$$
$$y = -18 \qquad x = 18$$

Plot the points: (0, –18), and (18, 0). From the sketch, we can see that the line does not pass through quadrant II.



7. **II and IV:** First, rewrite the line in slope-intercept form:

$$y = \frac{x}{10}$$



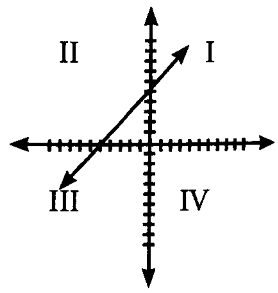
Notice from the equation that the y-intercept of the line is (0,0). This means that the line crosses the y-intercept at the origin, so the x- and y-intercepts are the same. To find another point on the line, substitute any convenient number for x; in this case, 10 would be a convenient, or "smart," number.

$$y = \frac{10}{10} = 1$$ The point (10, 1) is on the line.

Plot the points: (0, 0) and (10, 1). From the sketch, we can see that the line does not pass through quadrants II and IV.

8. **I, II, and III:** The line is already written in slope-intercept form:

$$y = \frac{x}{1,000} + 1,000,000$$



Find the intercepts by setting x to zero and y to zero:

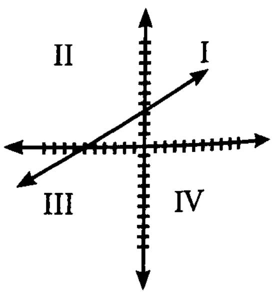
$$0 = \frac{x}{1,000} + 1,000,000 \qquad y = \frac{0}{1,000} + 1,000,000$$

$$x = -1,000,000,000 \qquad y = 1,000,000$$

Plot the points: (–1,000,000,000, 0) and (0, 1,000,000). From the sketch, we can see that the line passes through quadrants I, II, and III.

9. **I, II, and III:** First, rewrite the line in slope-intercept form:

$$y = \frac{x}{2} + 9$$



Find the intercepts by setting x to zero and y to zero:

$$0 = \frac{x}{2} + 9 \qquad y = \frac{0}{2} + 9$$

$$x = -18 \qquad y = 9$$

Plot the points: (–18, 0) and (0, 9). From the sketch, we can see that the line passes through quadrants I, II, and III.

10. $\boldsymbol{y = \frac{3}{2}x + 6}$**:** First, calculate the slope of the line:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{6-0}{0-(-4)} = \frac{6}{4} = \frac{3}{2}$$

We can see from the graph that the line crosses the y-axis at (0, 6). The equation of the line is:

$$y = \frac{3}{2}x + 6$$

11. **(5, –3):** To find the coordinates of the point of intersection, solve the system of 2 linear equations. You could turn both equations into slope-intercept form and set them equal to each other, but it is easier is to multiply the first equation by 2 and then add the second equation:

$2x + y = 7$ (first equation)
$4x + 2y = 14$ (multiply by 2)
$3x - 2y = 21$ (second equation)

$7x = 35$ (sum of previous two eqations)
$x = 5$

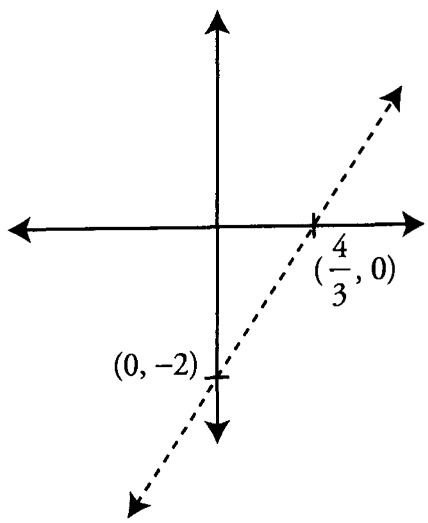
Now plug $x = 5$ into either equation:

$2x + y = 7$ (first equation)
$2(5) + y = 7$

$10 + y = 7$
$y = -3$

Thus, the point (5, –3) is the point of intersection. There is no need to graph the two lines to find the point of intersection.

12. **B:**



We've illustrated the line on the coordinate plane above. Because the equation is already in slope intercept form ($y = mx + b$), you can read the y-intercept directly from the b position, and use the slope to determine the x-intercept. A slope of 3/2 means that the line corresponding to this equation will rise 3 for every 2 that it runs. You don't need to determine the exact x-intercept to see that it is positive, and so greater than –2.

Alternatively, you could set each variable equal to 0, and determine the intercepts.

To determine the y-intercept, set x equal to 0, then solve for y:

$$y = \frac{3}{2}x - 2$$

$$y = \frac{3}{2}(0) - 2$$

$$y = 0 - 2 = -2$$

To determine the x-intercept, set y equal to 0, then solve for x:

$$y = \frac{3}{2}x - 2$$

$$(0) = \frac{3}{2}x - 2$$

$$2 = \frac{3}{2}x$$

$$\frac{4}{3} = x$$

Quantity A	Quantity B
The y-intercept of the line	The x-intercept of the line
-2	$\frac{4}{3}$

Therefore **Quantity B is greater.**

13. **A:** The best method would be to put each equation into slope-intercept form ($y = mx + b$), and see which has the greater value for m, which represents the slope. Start with the equation in **Quantity A:**

$$2x + 5y = 10$$
$$5y = -2x + 10$$
$$y = -\frac{2}{5}x + 2$$

Quantity A	Quantity B
The slope of the line $2x + 5y = 10$ is $-\frac{2}{5}$	The slope of the line $5x + 2y = 10$

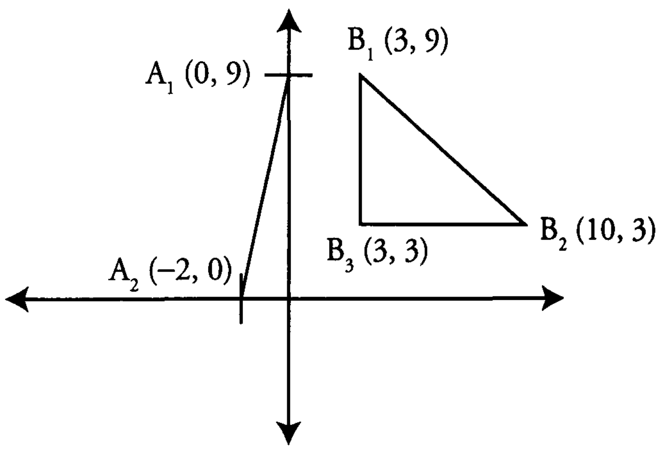
Now find the slope of the equation in **Quantity B:**

$$5x + 2y = 10$$
$$2y = -5x + 10$$
$$y = -\frac{5}{2}x + 5$$

Quantity A	Quantity B
−2/5	The slope of the line $5x + 2y = 10$ is $-\frac{5}{2}$

Be careful. Remember that $-\frac{2}{5} > -\frac{5}{2}$. Therefore **Quantity A is larger.**

14. **C:**



The illustration above shows the two points from **Quantity A,** here labeled A_1 and A_2, and the two points from **Quantity B,** here labeled B_1 and B_2. We have constructed a right triangle from the A values by finding a point (0, 0) directly below A_1 and directly to the right of A_2. This right triangle has legs of 2 (the change from −2 to 0) and 9 (the change from 0 to 9). We can plug those values into the Pythagorean Theorem and solve for the hypotenuse:

$A^2 + B^2 = C^2$
$(2)^2 + (9)^2 = C^2$
$4 + 81 = C^2$
$C^2 = 85$
$C = \sqrt{85}$

Quantity A	Quantity B
The distance between points (0, 9) and (−2, 0) = $\sqrt{85}$	The distance between points (3, 9) and (10, 3)

We have constructed a right triangle from the B values by finding a point (3, 3) directly below B_1 and directly to the right of... left of B_2. This right triangle has legs of 7 (the change from 3 to 10) and 6 (the change from 3 to 9). We can plug those values into the Pythagorean Theorem and solve for the hypotenuse:

$A^2 + B^2 = C^2$
$(7)^2 + (6)^2 = C^2$
$49 + 36 = C^2$
$C^2 = 85$
$C = \sqrt{85}$

Quantity A	Quantity B
$\sqrt{85}$	The distance between points (3, 9) and (10, 3) = $\sqrt{85}$

Therefore **the two quantities are equal.**

Chapter 8
of
GEOMETRY

DRILL SETS

In This Chapter . . .

g

- Geometry Drill Sets

Geometry Drill Sets

DRILL SET 1:

Drill 1

1. The radius of a circle is 4. What is its area?
2. The diameter of a circle is 7. What is its circumference?
3. The radius of a circle is 3. What is its circumference?
4. The area of a circle is 36π. What is its radius?
5. The circumference of a circle is 18π. What is its area?

Drill 2

1. The area of a circle is 100π. What is its circumference?
2. The diameter of a circle is 16. Calculate its radius, circumference, and area.
3. Which circle has a larger area? Circle A has a circumference of 6π and Circle B has an area of 8π.
4. Which has a larger area? Circle C has a diameter of 10 and Circle D has a circumference of 12π.
5. A circle initially has an area of 4π. If the radius is doubled, the new area is how many times as large as the original area?

Drill 3

1. A sector has a central angle of 90°. If the sector has a radius of 8, what is the area of the sector?
2. A sector has a central angle of 30°. If the sector has a radius of 6, what is the arc length of the sector?
3. A sector has an arc length of 7π and a radius of 7. What is the central angle of the sector?
4. A sector has a central angle of 270°. If the sector has a radius of 4, what is the area of the sector?
5. A sector has an area of 24π and a radius of 12. What is the central angle of the sector?

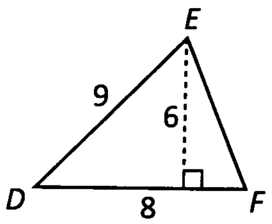
Drill 4

1. The area of a sector is $\frac{1}{10}$th the area of the full circle. What is the central angle of the sector?
2. What is the perimeter of a sector with a radius of 5 and a central angle of 72°?
3. A sector has a radius of 8 and an area of 8π. What is the arc length of the sector?
4. A sector has an arc length of $\frac{\pi}{2}$ and a central angle of 45°. What is the radius of the sector?
5. Which of the following two sectors has a larger area? Sector A has a radius of 4 and a central angle of 90°. Sector B has a radius of 6 and a central angle of 45°.

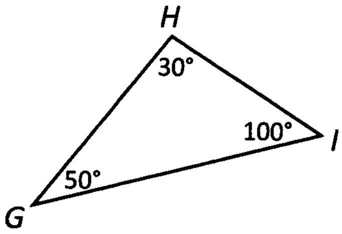
DRILL SET 2:

Drill 1

1. A triangle has two sides with lengths of 5 and 11, respectively. What is the range of values for the length of the third side?
2. In a right triangle, the length of one of the legs is 3 and the length of the hypotenuse is 5. What is the length of the other leg?
3. What is the area of Triangle DEF?

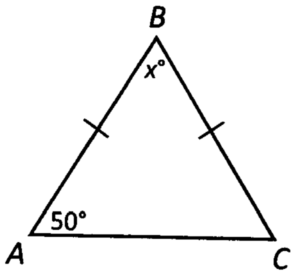


4. Which side of Triangle *GHI* has the longest length?



Not drawn to scale.

5. What is the value of *x*?



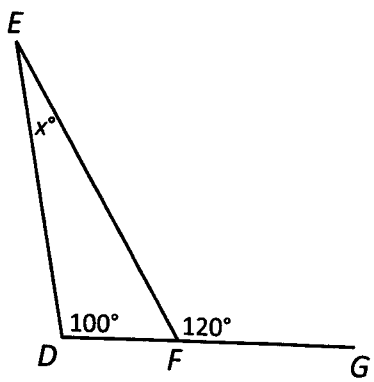
Not drawn to scale.

Drill 2

1. Two sides of a triangle have lengths 4 and 8. Which of the following are possible side lengths of the third side? (More than one may apply.)

   a. 2 b. 4 c. 6 d. 8

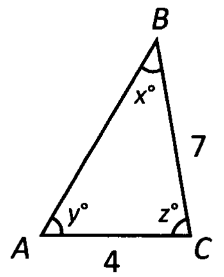
2. *DFG* is a straight line. What is the value of *x*?



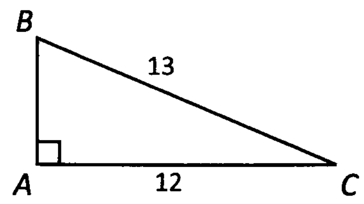
3. Isosceles triangle *ABC* has two sides with lengths 3 and 9. What is the length of the third side?

4. Which of the following could be the length of side AB, if $x < y < z$?

a. 6   b. 10   c. 14

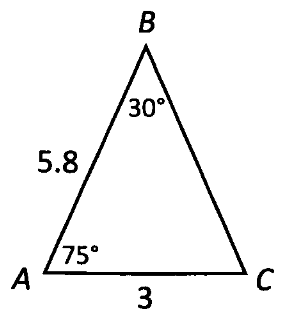


5. What is the area of right triangle ABC?

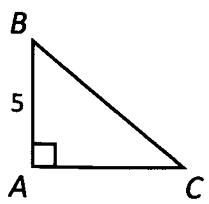


Drill 3

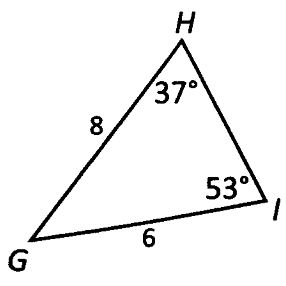
1. What is the perimeter of triangle ABC?



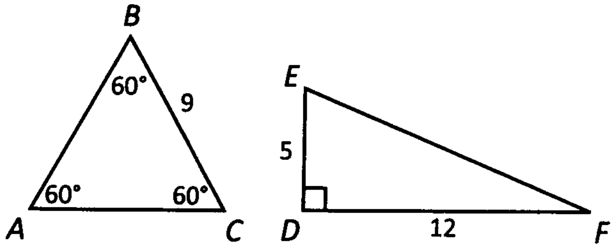
2. The area of right triangle ABC is 15. What is the length of hypotenuse BC?



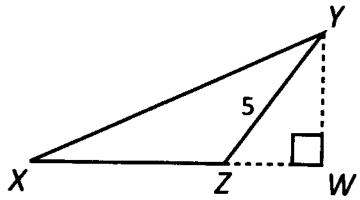
3. What is the length of side HI?



4. Which triangle has the greatest perimeter?



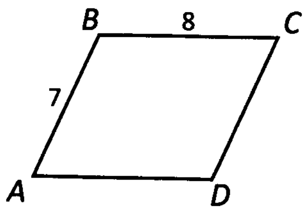
5. *WZ* has a length of 3 and *ZX* has a length of 6. What is the area of Triangle *XYZ*?



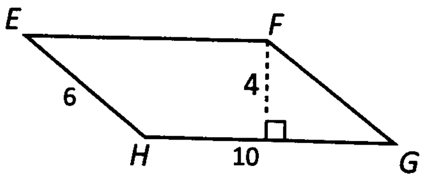
DRILL SET 3

Drill 1:

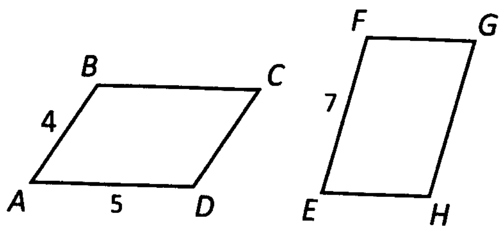
1. What is the perimeter of parallelogram *ABCD*?



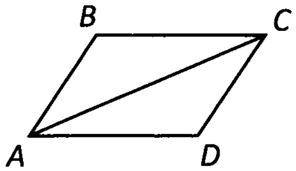
2. What is the area of parallelogram *EFGH*?



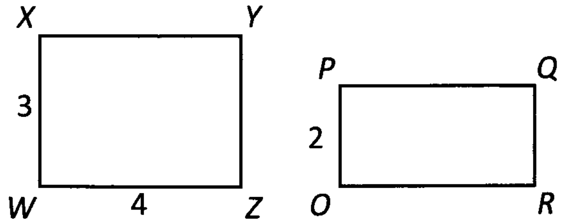
3. The two parallelograms pictured below have the same perimeter. What is the length of side *EH*?



4. In Parallelogram *ABCD*, Triangle *ABC* has an area of 12. What is the area of Triangle *ACD*?

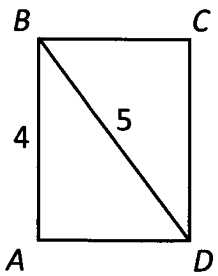


5. Rectangle *WXYZ* and Rectangle *OPQR* have equal areas. What is the length of side *PQ*?

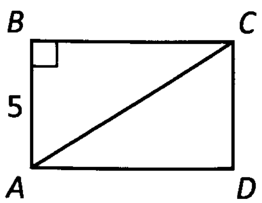


Drill 2

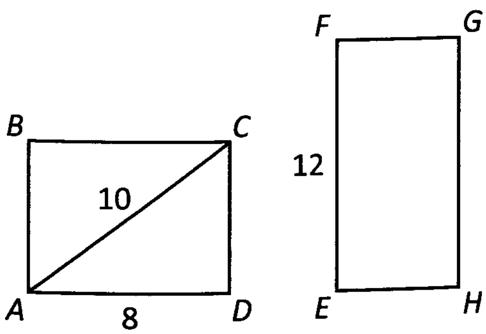
1. What is the area of Rectangle *ABCD*?



2. In Rectangle *ABCD*, the area of Triangle *ABC* is 30. What is the length of diagonal *AC*?

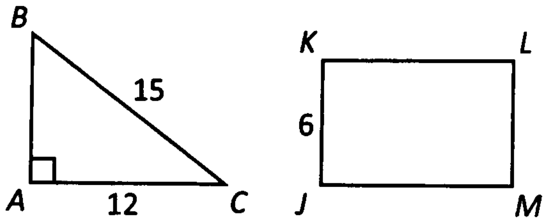


3. Rectangles *ABCD* and *EFGH* have equal areas. What is the length of side *FG*?



4. A rectangle has a perimeter of 10 and an area of 6. What are the length and width of the rectangle?

5. Triangle *XYZ* and Rectangle *JKLM* have equal areas. What is the perimeter of Rectangle *JKLM*?

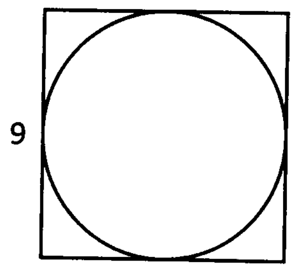


Drill 3

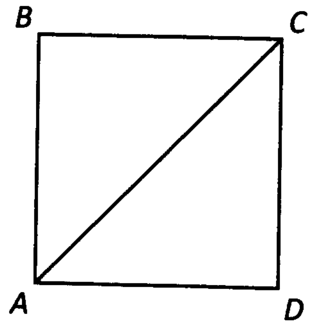
1. What is the perimeter of a square with an area of 25?

2. A rectangle and a square have the same area. The square has a perimeter of 32 and the rectangle has a length of 4. What is the width of the rectangle?

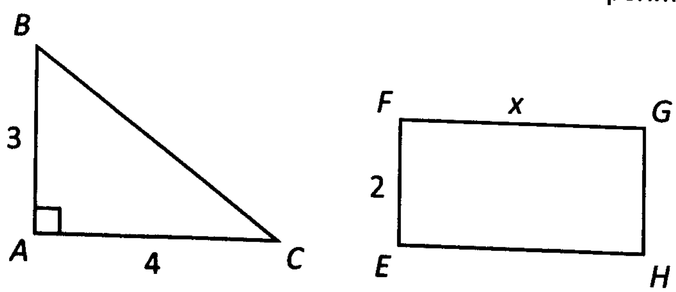
3. A circle is inscribed inside a square, so that the circle touches all four sides of the square. The length of one of the sides of the square is 9. What is the area of the circle?



4. Square *ABCD* has an area of 49. What is the length of diagonal *AC*?



5. Right Triangle *ABC* and Rectangle *EFGH* have the same perimeter. What is the value of *x*?

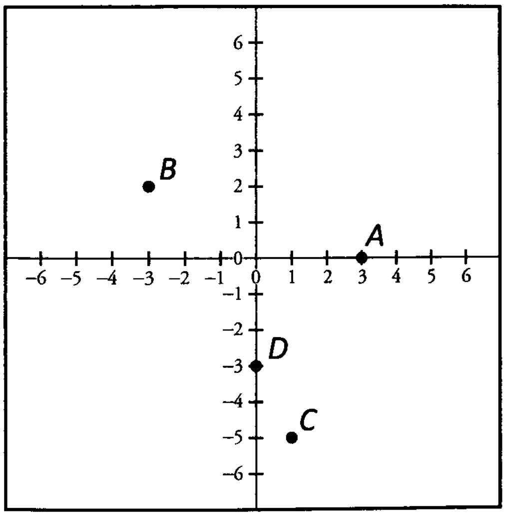


DRILL SET 4:

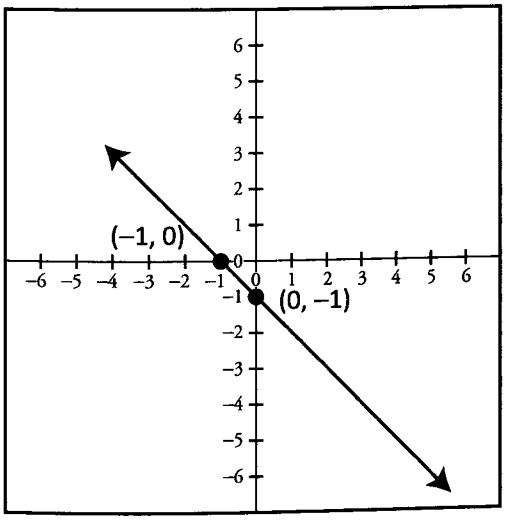
Drill 1

1. Draw a coordinate plane and plot the following points:
   1. (2, 3) 2. (−2, −1) 3. (−5, −6) 4. (4, −2.5)

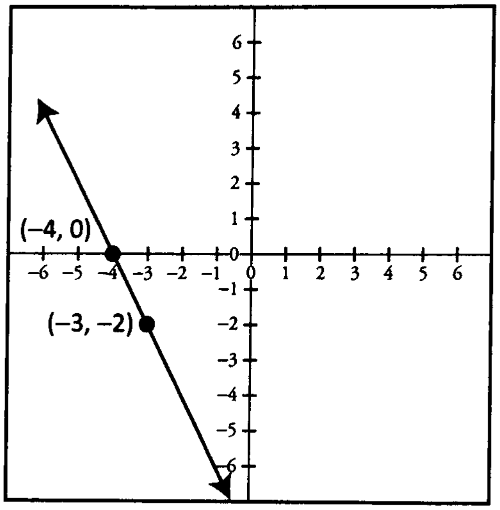
2. What are the *X*- and *Y*-coordinates of the following points?



3. What is the *y*-coordinate of the point on the line that has an *x*-coordinate of 3?



4. What is the *x*-coordinate of the point on the line that has a *y*-coordinate of −4?



5. Does the point (3, −2) lie on the line $y = 2x - 8$?

Drill 2

1. Does the point (−3, 0) lie on the curve $y = x^2 - 3$?
2. For the line $y = 4x + 2$, what is the y-coordinate when $x = 3$?
3. What is the y-intercept of the line $y = -2x - 7$?
4. Graph the line $y = \frac{1}{3}x - 4$.
5. Graph the line $\frac{1}{2}y = -\frac{1}{2}x + 1$.

Drill Set Answers

DRILL SET 1:

Set 1, Drill 1

1. **16π:** The radius of a circle is 4. What is its area?

   Area of a circle is πr^2, so the area of the circle is $\pi(4)^2$, which equals 16π.

2. **7π:** The diameter of a circle is 7. What is its circumference?

   Circumference of a circle is $2\pi r$, or πd. We have the diameter, so the circumference equals $\pi(7)$, which equals 7π.

3. **6π:** The radius of a circle is 3. What is its circumference?

   Circumference of a circle is $2\pi r$, or πd. We have the radius, so circumference equals $2\pi(3)$, which equals 6π.

4. **6:** The area of a circle is 36π. What is its radius?

   Area of a circle is πr^2, so $36\pi = \pi r^2$. We need to solve for r. Divide both sides by π, so $36 = r^2$. Take the square root of both sides, and $6 = r$. We can ignore the negative solution because distances cannot be negative.

5. **81π:** The circumference of a circle is 18π. What is its area?

   The connection between circumference and area is radius. We can use the circumference to solve for the radius. $18\pi = 2\pi r$, which means that $9 = r$. That means that area = $\pi(9)^2$, which equals 81π.

Set 1, Drill 2

1. **20π:** The area of a circle is 100π. What is its circumference?

   The connection between circumference and area is radius. $100\pi = \pi r^2$, and solving for r gives us $r = 10$. That means that Circumference = $2\pi(10)$, which equals 20π.

2. **8, 16π, 64π:** The diameter of a circle is 16. Calculate its radius, circumference, and area.

   $d = 2r$, so $16 = 2r$. Radius = 8. Circumference = $2\pi r$, so Circumference = $2\pi(8) = 16\pi$. Area = πr^2, so Area = $\pi(8)^2 = 64\pi$.

3. **Circle *A*:** Which circle has a larger area? Circle A has a circumference of 6π and Circle *B* has an area of 8π.

   To figure out which circle has a larger area, we need to find the area of Circle *A*. If we know the circumference, then $6\pi = 2\pi r$, which means $r = 3$. If $r = 3$, then Area = $\pi(3)^2 = 9\pi$. $9\pi > 8\pi$, so Circle *A* has a larger area.

4. **Circle *D*:** Which has a larger area? Circle *C* has a diameter of 10 and Circle *D* has a circumference of 12π.

   We need to find the area of both circles. Let's start with Circle *C*. If the diameter of Circle *C* is 10, then the radius is 5. That means that Area = $\pi(5)^2 = 25\pi$.

   If the circumference of Circle *D* is 12π, then $12\pi = 2\pi r$. $r = 6$. If $r = 6$, then Area = $\pi(6)^2 = 36\pi$. $36\pi > 25\pi$, so Circle *D* has the larger area.

5. **4 times:** A circle initially has an area of 4π. If the radius is doubled, the new area is how many times as large as the original area?

To begin, we need to find the original radius of the circle. $4\pi = \pi r^2$, so $r = 2$. If we double the radius, the new radius is 4. A circle with a radius of 4 has an area of 16π. 16π is 4 times 4π, so the new area is 4 times the original area.

Set 1, Drill 3

1. **16π:** A sector has a central angle of 90°. If the sector has a radius of 8, what is the area of the sector?

If the sector has a central angle of 90°, then the sector is 1/4 of the circle, because $\frac{90}{360} = \frac{1}{4}$. To find the area of the sector, we need to find the area of the whole circle first. The radius is 8, which means the area is $\pi(8)^2 = 64\pi$. $\frac{1}{4} \times 64\pi = 16\pi$. The area of the sector is 16π.

2. **π:** A sector has a central angle of 30°. If the sector has a radius of 6, what is the arc length of the sector?

If the sector has a central angle of 30°, then it is 1/12th of the circle, because $\frac{30}{360} = \frac{1}{12}$. To find the arc length of the sector, we need to know the circumference of the entire circle. The radius of the circle is 6, so the circumference is $2\pi(6) = 12\pi$. That means that the arc length of the sector is $\frac{1}{12} \times 12\pi = \pi$.

3. **180°:** A sector has an arc length of 7π and a radius of 7. What is the central angle of the sector?

To find the central angle of the sector, we first need to find what fraction of the full circle the sector is. We have the arc length, so if we can find the circumference of the circle, we can figure out what fraction of the circle the sector is. The radius is 7, so the circumference is $2\pi(7) = 14\pi$. $\frac{7\pi}{14\pi} = \frac{1}{2}$. So the sector is 1/2 the full circle. That means that the central angle of the sector is $\frac{1}{2} \times 360° = 180°$. So the central angle is 180°.

4. **12π:** A sector has a central angle of 270°. If the sector has a radius of 4, what is the area of the sector?

The sector is 3/4 of the circle, because $\frac{270°}{360°} = \frac{3}{4}$. To find the area of the sector, we need the area of the whole circle. The radius of the circle is 4, so the area is $\pi(4)^2 = 16\pi$. That means the area of the circle is $\frac{3}{4} \times 16\pi = 12\pi$.

5. **60°:** A sector has an area of 24π and a radius of 12. What is the central angle of the sector?

We first need to find the area of the whole circle. The radius is 12, which means the area is $\pi(12)^2 = 144\pi$. $\frac{24\pi}{144\pi} = \frac{1}{6}$, so the sector is 1/6th of the entire circle. That means that the central angle is 1/6th of 360. $\frac{1}{6} \times 360 = 60$, so the central angle is 60°.

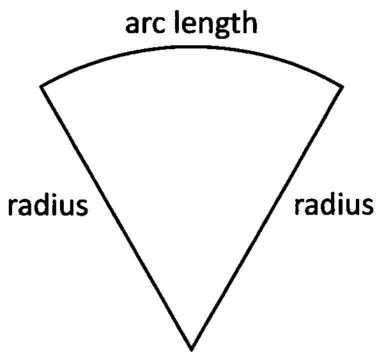
Set 1, Drill 4

1. **36°:** The area of a sector is 1/10th the area of the full circle. What is the central angle of the sector?

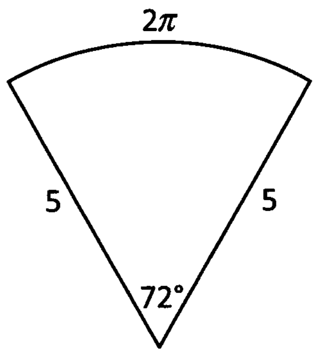
   If the area of the sector is 1/10th of the area of the full circle, then the central angle will be 1/10th of the degree measure of the full circle. $\frac{1}{10} \times 360 = 36$, so the central angle of the sector is 36°.

2. **$10 + 2\pi$:** What is the perimeter of a sector with a radius of 5 and a central angle of 72°?

   To find the perimeter of a sector, we need to know the radius of the circle and the arc length of the sector.



We know the radius is 5, so now we need to find the arc length. Let's begin by determining what fraction of the circle the sector is. The central angle of the sector is 72°, so the sector is 1/5th of the circle, because $\frac{72}{360} = \frac{1}{5}$. Now we need to find the circumference. The radius is 5, so the circumference of the circle is $2\pi(5) = 10\pi$. The arc length of the sector is 1/5th the circumference. $\frac{1}{5} \times 10\pi = 2\pi$. So now our sector looks like this. The perimeter of the sector is $10 + 2\pi$.



3. **2π:** A sector has a radius of 8 and an area of 8π. What is the arc length of the sector?

   We first need to find what fraction of the circle the sector is. We can do this by comparing areas. The radius of the circle is 8, so the area of the circle is $\pi(8)^2 = 64\pi$. That means the sector is 1/8th of the circle, because $\frac{8\pi}{64\pi} = \frac{1}{8}$. If we want to find the arc length of the sector, we need to know the circumference. The radius is 8, so the circumference is $2\pi(8) = 16\pi$. The sector is 1/8th of the circle, so the arc length will be 1/8th of the circumference. $1/8 \times 16\pi = 2\pi$. The arc length of the sector is 2π.

4. **2:** A sector has an arc length of $\pi/2$ and a central angle of 45°. What is the radius of the sector?

If the sector has a central angle of 45°, then the sector is 1/8th of the circle, because $\frac{45}{360} = \frac{1}{8}$. If the sector is 1/8th of the circle, then that means the arc length of the sector is 1/8th of the circumference of the circle. That means that $\pi/2$ is 1/8th of the circumference. If we designate x as the circumference of the circle, then we can say that $\frac{\pi}{2} = \frac{1}{8}x$. Multiply both sides by 8, and we get $4\pi = x$. That means the circumference is 4π. We know the formula for circumference, so we know that $4\pi = 2\pi r$. Divide both sides by 2π and we get $r = 2$. The radius of the sector is 2.

5. **Sector *B*:** Which of the following two sectors has a larger area? Sector A has a radius of 4 and a central angle of 90°. Sector B has a radius of 6 and a central angle of 45°.

We need to find the area of each circle. Sector A is 1/4th of the circle, because $\frac{90}{360} = \frac{1}{4}$. The radius is 4, so the area of the circle is $\pi(4)^2 = 16\pi$. That means the area of Sector A is 1/4th of 16π. $1/4 \times 16\pi = 4\pi$, so the area of Sector A is 4π.

Sector B is 1/8th of the circle, because $\frac{45}{360} = \frac{1}{8}$. The radius of Sector B is 6, so the area of the full circle is $\pi(6)^2 = 36\pi$. Sector B is 1/8th of the circle, so the area of Sector B is $\frac{1}{8} \times 36\pi = 4.5\pi$. The area of Sector B is 4.5π.

$4.5\pi > 4\pi$, so the area of Sector B is greater than the area of Sector A.

DRILL SET 2:

Set 2, Drill 1

1. **6 < third side < 16.:** A triangle has two sides with lengths of 5 and 11, respectively. What is the range of values for the length of the third side?

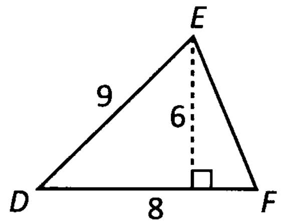
The lengths of any two sides of a triangle must add up to more than the length of the third side. The third side must be less than $5 + 11 = 16$. It must also be greater than $11 - 5 = 6$. Therefore, $6 <$ third side < 16.

2. **4:** In a right triangle, the length of one of the legs is 3 and the length of the hypotenuse is 5. What is the length of the other leg?

If you know the lengths of two sides of a right triangle, you can use the Pythagorean Theorem to solve for the length of the third side. Remember that the hypotenuse must be the side labeled c in the equation $a^2 + b^2 = c^2$. That means that $(3)^2 + (b)^2 = (5)^2$. $9 + b^2 = 25$. $b^2 = 16$, so $b = 4$.

Alternatively, you can recognize the Pythagorean triplet. This is a 3–4–5 triangle.

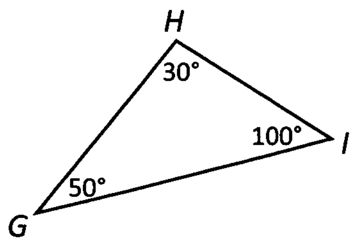
3. **24:** What is the area of Triangle *DEF*?



The area of a triangle is $\frac{1}{2}$(base) × (height). Remember that the base and the height must be perpendicular to each other. That means that in Triangle *DEF*, side *DF* can act as the base, and the line dropping straight down from point *E* to touch side *DF* at a right angle can act as the height. Therefore

Area = $\frac{1}{2}(8) \times (6) = 24$.

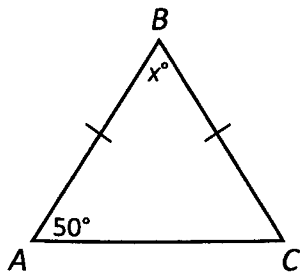
4. **Side *GH*:** Which side of Triangle *GHI* has the longest length?



Not drawn to scale.

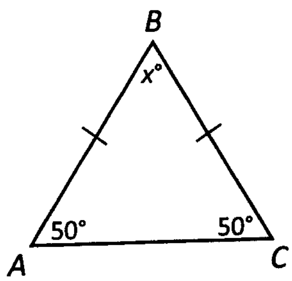
Although *GI* looks like the longest side, remember that you can't trust what the picture looks like when the question states the picture is not drawn to scale. In any triangle, the longest side will be opposite the largest angle. Angle *GIH* is the largest angle in the triangle, and side *GH* is thus the longest side.

5. **80:** What is the value of *x*?



Not drawn to scale.

If you know the other 2 angles in a triangle, then you can find the third, because all 3 angles must add up to 180. In Triangle *ABC*, sides *AB* and *BC* are equal. That means their opposite angles are also equal. That means that angle *ACB* is also 50°.



Now that we know the other 2 angles, we can find angle *x*. We know that $50 + 50 + x = 180$, so $x = 80$.

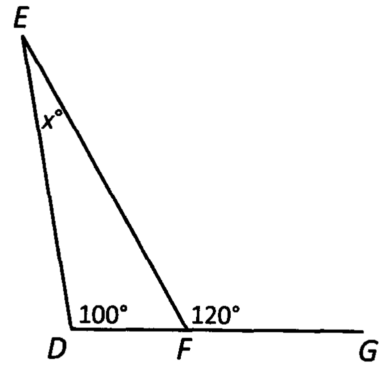
Set 2, Drill 2

1. **c & d:** Two sides of a triangle have lengths 4 and 8. Which of the following are possible side lengths of the third side? (More than one may apply.)

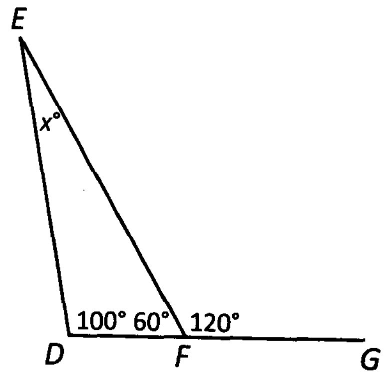
a. 2 b. 4 c. 6 d. 8

The lengths of any two sides of a triangle must add up to more than the length of the third side. The third side must be less than $4 + 8 = 12$ and greater than $8 - 4 = 4$. So $4 <$ third side < 12. Only choices c. and d. are in that range.

2. **20:** *DFG* is a straight line. What is the value of x?



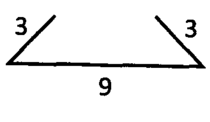
To find the value of x, we need to find the degree measures of the other two angles in Triangle *DEF*. We can make use of the fact that *DFG* is a straight line. Straight lines have a degree measure of 180, so angle $DFE + 120 = 180$, which means angle $DFE = 60$.



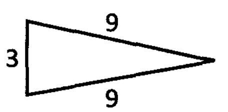
Now we can solve for x, because $100 + 60 + x = 180$. Solving for x, we get $x = 20$.

3. **9:** Isosceles triangle *ABC* has two sides with lengths 3 and 9. What is the length of the third side?

It may at first appear like we don't have enough information to answer this question. If all we know is that the triangle is isosceles, then all we know is that two sides have equal length, which means the third side has a length of either 3 or 9. But if the third side were 3, then the lengths of two of the sides would not add up to greater than the length of the third side, because $3 + 3$ is not greater than 9.



Can't make a triangle.

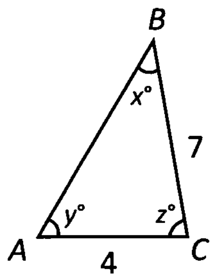


Can make a triangle.

That means that the length of the third side must be 9.

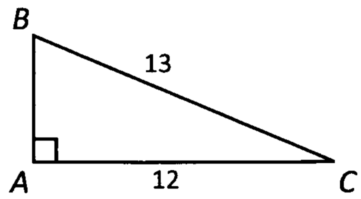
4. **b:** Which of the following could be the length of side *AB*, if $x < y < z$?

a. 6 b. 10 c. 14



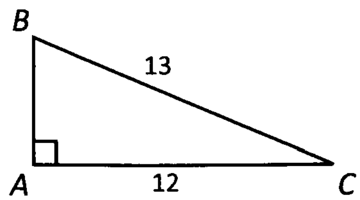
There are two properties of a triangle at play here. The lengths of any two sides of a triangle must add up to greater than the length of the third side. Also, longer sides must be opposite larger angles. Answer choice a. is out because side AB is opposite the largest angle, so side AB must have a length greater than 7. Answer choice c. is out, because $4 + 7 = 11$, so the third side has to be less than 11. The only remaining possibility is b. 10.

5. **30:** What is the area of right triangle ABC?



To find the area, we need a base and a height. If we can find the length of side AB, then AB can be the height and AC can be the base, because the two sides are perpendicular to each other.

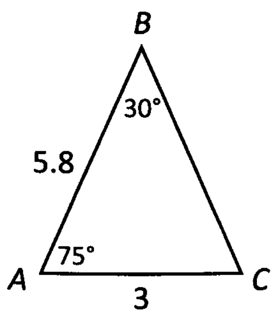
We can use the Pythagorean Theorem to find the length of side AB. $(a)^2 + (12)^2 = (13)^2$. $a^2 + 144 = 169$. $a^2 = 25$. $a = 5$. Alternatively, we could recognize that the triangle is a Pythagorean triplet 5–12–13.



Now that we know the length of side AB we can find the area. Area $= \frac{1}{2}(12) \times (5) = 30$.

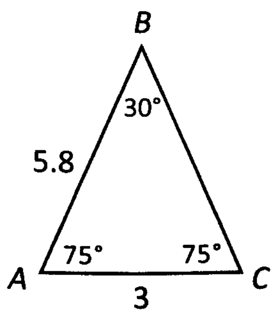
Set 2, Drill 3

1. **14.6:** What is the perimeter of triangle ABC?

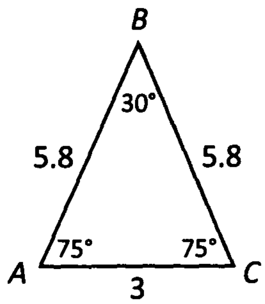


To find the perimeter of Triangle ABC, we need the lengths of all 3 sides. There is no immediately obvious way to find the length of side BC, so let's see what inferences we can make from the information the question gave us.

We know the degree measures of two of the angles in Triangle ABC, so we can find the degree measure of the third. We'll label the third angle x. We know that $30 + 75 + x = 180$. Solving for x we find that $x = 75$.

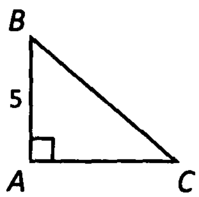


Angle BAC and angle BCA are both 75, which means Triangle ABC is an isosceles triangle. If those two angles are equal, we know that their opposite sides are also equal. Side AB has a length of 5.8, so we know that BC also has a length of 5.8.

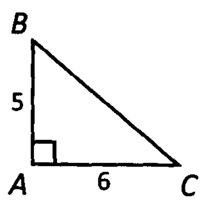


To find the perimeter, we add up the lengths of the three sides. $5.8 + 5.8 + 3 = 14.6$.

2. $\sqrt{61}$: The area of right triangle *ABC* is 15. What is the length of hypotenuse *BC*?

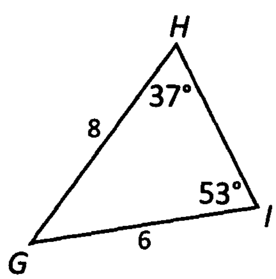


To find the length of the hypotenuse, we need the lengths of the other two sides. Then we can use the Pythagorean Theorem to find the length of the hypotenuse. We can use the area formula to find the length of *AC*. Area = $\frac{1}{2}$(base) × (height), and we know the area and the height. So $15 = \frac{1}{2}$(base) × (5). When we solve this equation, we find that the base = 6.

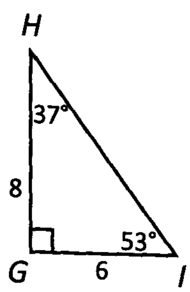


Now we can use the Pythagorean Theorem. $(5)^2 + (6)^2 = c^2$. $25 + 36 = c^2$. $61 = c^2$. $\sqrt{61} = c$. Since 61 is not a perfect square, we know that c will be a decimal. 61 is also prime, so we cannot simplify $\sqrt{61}$ any further. (It will be a little less than $\sqrt{64} = 8$.)

3. **10:** What is the length of side *HI*?



There is no immediately obvious way to find the length of side *HI*, so let's see what we can infer from the picture. We know two of the angles of Triangle *GHI*, so we can find the third. We'll label the third angle x. $37 + 53 + x = 180$. That means $x = 90$. So really our triangle looks like this:

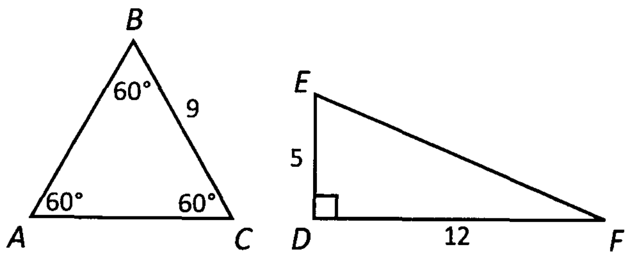


You should definitely redraw once you discover the triangle is a right triangle!

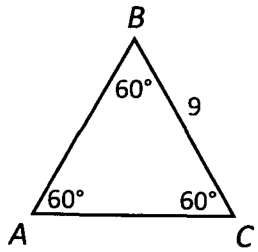
Now that we know Triangle *GHI* is a right triangle, we can use the Pythagorean Theorem to find the length of *HI*. *HI* is the hypotenuse, so $(6)^2 + (8)^2 = c^2$. $36 + 64 = c^2$. $100 = c^2$. $10 = c$. The length of *HI* is 10.

Alternatively, we could have recognized the Pythagorean triplet. Triangle *GHI* is a 6–8–10 triangle.

4. **Triangle *DEF*:** Which triangle has the greater perimeter?



To determine which triangle has the greater perimeter, we need to know the side lengths of all three sides of both triangles. Let's begin with Triangle *ABC.*



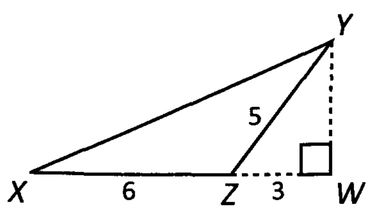
All three angles in Triangle *ABC* are 60°. If all three angles are equal, that means all three sides are equal in this equilateral triangle. So every side of Triangle *ABC* has a length of 9. That means the perimeter = 9 + 9 + 9 = 27.

Now let's look at Triangle *DEF.* Triangle *DEF* is a right triangle, so we can use the Pythagorean Theorem to find the length of side *EF. EF* is the hypotenuse, so $(5)^2 + (12)^2 = c^2$. $25 + 144 = c^2$. $169 = c^2$. $13 = c$. That means the perimeter is 5 + 12 + 13 = 30. Alternatively, 5–12–13 is a Pythagorean triplet.

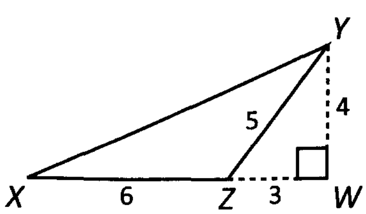
30 > 27, so Triangle *DEF* has a greater perimeter than Triangle *ABC.*

5. **12:** *WZ* has a length of 3 and *ZX* has a length of 6. What is the area of Triangle *XYZ*?

Let's start by filling in everything we know about Triangle *XYZ.*



To find the area of Triangle *XYZ*, we need a base and a height. If Side *XZ* is a base, then *YW* can act as a height. We can find the length of *YW* because Triangle *ZYW* is a right triangle, and we know the lengths of two of the sides. *YZ* is the hypotenuse, so $(a)^2 + (3)^2 = (5)^2$. $a^2 + 9 = 25$. $a^2 = 16$. $a = 4$.



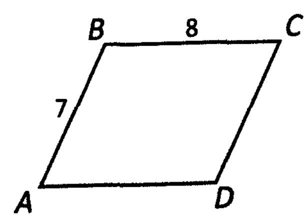
Alternatively, we could recognize the Pythagorean triplet: *ZYW* is a 3–4–5 triangle.

Now we know that the area of Triangle *XYZ* is $\frac{1}{2}(b) \times (h) = \frac{1}{2}(6) \times (4) = 12$.

DRILL SET 3:

Set 3, Drill 1

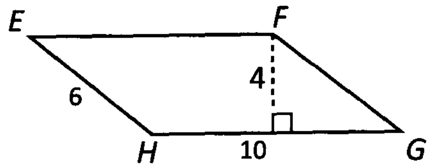
1. **30:** What is the perimeter of parallelogram *ABCD*?



Opposite sides of a parallelogram are equal, so we know that side *CD* has a length of 7 and side *AD* has a length of 8. So the perimeter is 7 + 8 + 7 + 8 = 30.

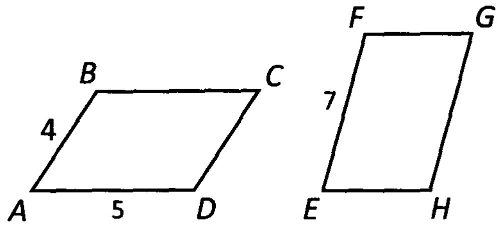
Alternatively, the perimeter is $2 \times (7 + 8) = 30$. We can say this because we know that 2 sides have a length of 7 and 2 sides have a length of 8.

2. **40:** What is the area of parallelogram *EFGH*?

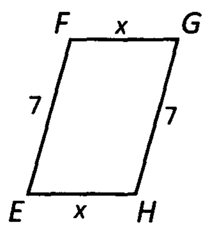


The area of a parallelogram is base × height. In this parallelogram, the base is 10 and the height is 4 (remember, base and height need to be perpendicular). So the area is $10 \times 4 = 40$.

3. **2:** The two parallelograms pictured below have the same perimeter. What is the length of side *EH*?



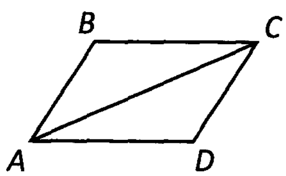
First we can find the perimeter of Parallelogram *ABCD*. We know that 2 sides have a length of 4, and 2 sides have a length of 5. The perimeter is $2 \times (4 + 5) = 18$. That means Parallelogram *EFGH* also has a perimeter of 18. We know side *GH* also has a length of 7. We don't know the lengths of the other 2 sides, but we know they have the same length, so for now let's say the length of each side is x. Our parallelogram now looks like this:



So we know that $7 + x + 7 + x = 18 \rightarrow 2x + 14 = 18 \rightarrow 2x = 4 \rightarrow x = 2$

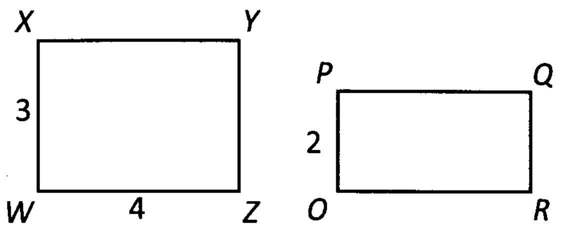
The length of side *EH* is 2.

4. **12:** In Parallelogram *ABCD*, Triangle *ABC* has an area of 12. What is the area of Triangle *ACD*?



One property that is true of any parallelogram is that the diagonal will split the parallelogram into two equal triangles. If Triangle *ABC* has an area of 12, then Triangle *ACD* must also have an area of 12.

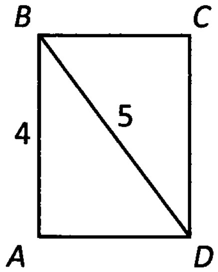
5. **6:** Rectangle *WXYZ* and Rectangle *OPQR* have equal areas. What is the length of side *PQ*?



We can start by finding the area of Rectangle *WXYZ*. Area of a rectangle is length × width, so the area of Rectangle *WXYZ* is $3 \times 4 = 12$. So Rectangle *OPQR* also has an area of 12. We know the length of side *OP*, so that is the length of Rectangle *OPQR*. So now we know the area, and we know the width, so we can solve for the length. $l \times 2 = 12 \rightarrow l = 6$. The length of side *PQ* is 6.

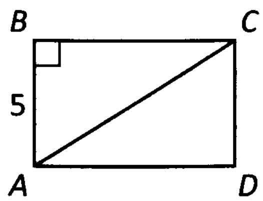
Set 3, Drill 2

1. 12: What is the area of Rectangle *ABCD*?

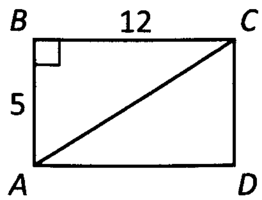


To find the area of Rectangle *ABCD*, we need to know the length of *AD* or *BC*. In a rectangle, every internal angle is 90 degrees, so Triangle *ABD* is actually a right triangle. That means we can use the Pythagorean Theorem to find the length of side *AD*. Actually, this right triangle is one of the Pythagorean Triplets—a 3–4–5 triangle. The length of side *AD* is 3. That means the area of Rectangle *ABCD* is $3 \times 4 = 12$.

2. 13: In Rectangle *ABCD*, the area of Triangle *ABC* is 30. What is the length of diagonal *AC*?

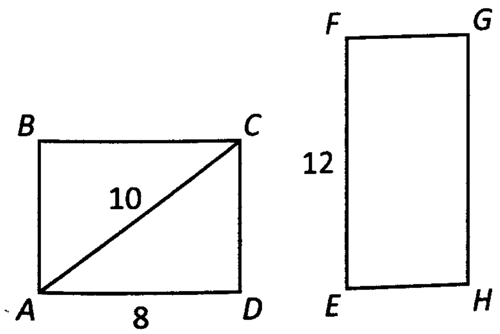


We know the area of Triangle *ABC* and the length of side *AB*. Because side *BC* is perpendicular to side *AB*, we can use those as the base and height of Triangle *ABC*. So we know that $\frac{1}{2}(5) \times (BC) = 30$. That means the length of side *BC* is 12.



Now we can use the Pythagorean Theorem to find the length of diagonal *AC*, which is the hypotenuse of right triangle *ABC*. We can also recognize that this is a Pythagorean Triplet—a 5–12–13 triangle. The length of diagonal *AC* is 13.

3. 4: Rectangles *ABCD* and *EFGH* have equal areas. What is the length of side *FG*?

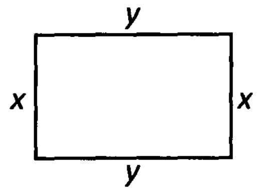


The first thing to notice in this problem is that we can find the length of side *CD*. Triangle *ACD* is a right triangle, and we know the lengths of two of the sides. We can either use the Pythagorean Theorem or recognize that this is one of our Pythagorean Triplets—a 6–8–10 triangle. The length of side *CD* is 6. Now we can find the area of Rectangle *ABCD*. Side *AD* is the length and side *CD* is the width.
$8 \times 6 = 48$.

That means that the area of Rectangle *EFGH* is also 48. We can use the area and the length of side *EF* to solve for the length of side *FG*. $12 \times (FG) = 48$. The length of side *FG* is 4.

4. **length and width are 2 and 3:** A rectangle has a perimeter of 10 and an area of 6. What are the length and width of the rectangle?

In order to answer this question, let's begin by drawing a rectangle. In this rectangle, we'll make one pair of equal sides have a length of x, and the other pair of equal sides has a length of y.



Using the lengths x and y, we know the perimeter of the rectangle is $2x + 2y$. So we know that:

$2x + 2y = 10$ This can be simplified to $\boldsymbol{x + y = 5}$.

We also know the area of the rectangle is $xy = 6$.

$\boldsymbol{xy = 6}$ Area of the rectangle $= l \times w = 6$

Now we can use substitution to solve for the values of our variables. In the first equation, we can isolate x.

$x = 5 - y$

Substitute $(5 - y)$ for x in the second equation.

$(5 - y)y = 6$

$5y - y^2 = 6$ This is a quadratic, so we need to get everything on one side.

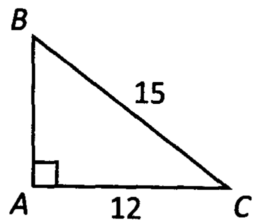
$y^2 - 5y + 6 = 0$ Now we can factor the equation.

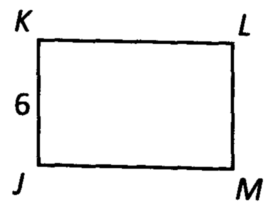
$(y - 3)(y - 2) = 0$

So $y = 2$ or 3.

When we plug in these values to solve for x, we find something a little unusual. When $y = 2$, $x = 3$. When $y = 3$, $x = 2$.What that means is that either the length is 2 and the width is 3, or the length is 3 and the width is 2. Both of these rectangles are identical, so we have our answer.

5. **30:** Triangle *XYZ* and Rectangle *JKLM* have equal areas. What is the perimeter of Rectangle *JKLM*?





If we can find the length of side *AB*, then we can find the area of Triangle *ABC*. We can use the Pythagorean Theorem to find the length of side *AB*. $(12)^2 + (AB)^2 = (15)^2 \rightarrow 144 + AB^2 = 225 \rightarrow AB^2 = 81 \rightarrow AB = 9$. (A 9–12–15 triangle is a 3–4–5 triangle, with all the measurements tripled.)

Now that we know AB, we can find the area of Triangle ABC. It's $\frac{1}{2}(12) \times 9 = 54$.

That means that Rectangle $JKLM$ also has an area of 54. We have one side of the rectangle, so we can solve for the other. $6 \times (JM) = 54$. So the length of side JM is 9. That means that the perimeter is $2 \times (6 + 9) = 30$.

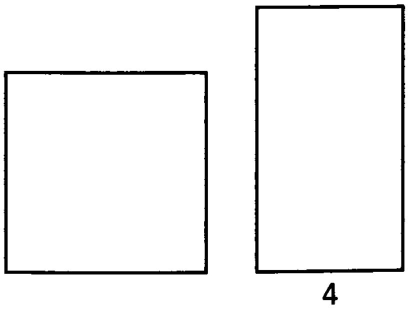
Set 3, Drill 3

1. **20:** What is the perimeter of a square with an area of 25?

   A square has four equal sides, so the area of a square is the length of one side squared. That means the lengths of the sides of the square are 5. If each of the four sides has a length of 5, then the perimeter is $4 \times (5) = 20$.

2. **16:** A rectangle and a square have the same area. The square has a perimeter of 32 and the rectangle has a length of 4. What is the width of the rectangle?

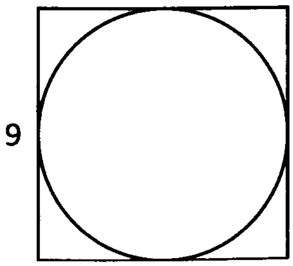
   We should start by drawing the shapes that they describe.



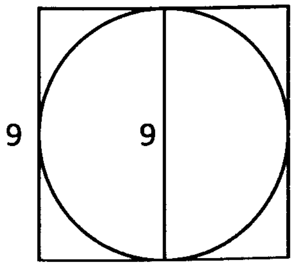
   The square has four equal sides, so that means that the perimeter is 4 times the length of one side. If we designate the length of the sides of the square s, then the perimeter is $4s = 32$. That means that s is 8. Now that we know the length of the sides, we can figure out the area of the square. Area $= 8^2$. So the area of the square is 64.

   That means that the area of the rectangle is also 64. We know the length of the rectangle is 4, so we can solve for the width. $4 \times (\text{width}) = 64$. The width is 16.

3. **20.25π:** A circle is inscribed inside a square, so that the circle touches all four sides of the square. The length of one of the sides of the square is 9. What is the area of the circle?

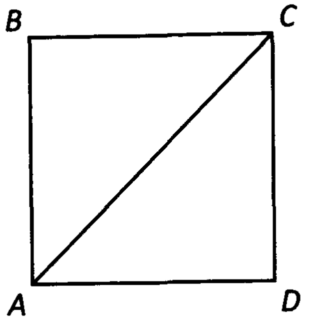


   We need to find a common link between the square and the circle, so that we can find the area of the circle. We know that the length of the sides of the square is 9. We can draw a new line in our figure that has the same length as the sides AND is the diameter of the circle.

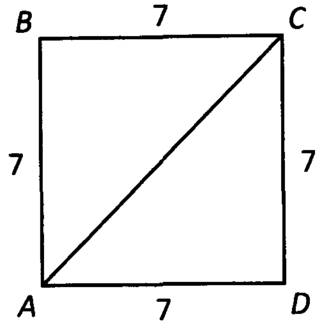


   That means that the diameter of the circle is 9. If the diameter is 9, then the radius is 4.5. That means the area of the circle is $\pi(4.5)^2$, which equals 20.25π.

4. $7\sqrt{2}$: Square *ABCD* has an area of 49. What is the length of diagonal *AC*?

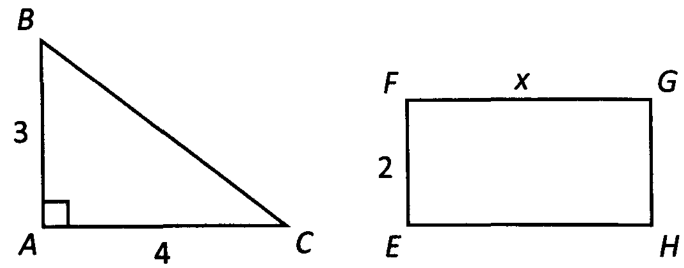


If the square has an area of 49, then $(\text{side})^2 = 49$. That means that the length of the sides of the square is 7.



Now we can use the Pythagorean Theorem to find the length of diagonal *AC*, which is also the hypotenuse of Triangle *ACD*. $7^2 + 7^2 = (AC)^2 \rightarrow 98 = (AC)^2 \rightarrow \sqrt{98} = AC$. But this can be simplified.
$AC = \sqrt{2 \times 49} = \sqrt{2 \times 7 \times 7} = 7\sqrt{2}$.

5. **4:** Right Triangle *ABC* and Rectangle *EFGH* have the same perimeter. What is the value of x?



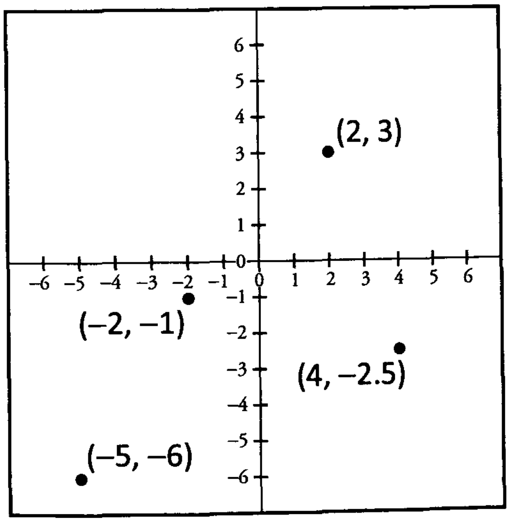
Triangle *ABC* is a right triangle, so we can find the length of hypotenuse *BC*. This is a 3–4–5 triangle, so the length of side *BC* is 5. That means the perimeter of Triangle *ABC* is $3 + 4 + 5 = 12$.

That means the perimeter of Rectangle *EFGH* is also 12. That means that $2 \times (2 + x) = 12$. So $4 + 2x = 12 \rightarrow 2x = 8 \rightarrow x = 4$.

DRILL SET 4:

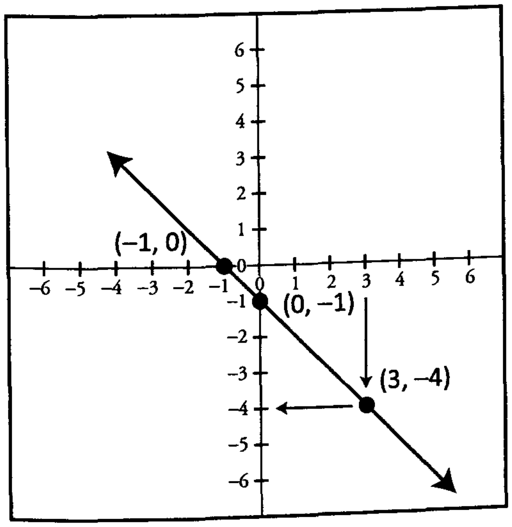
Set 4, Drill 1

1. Draw a coordinate plane and plot the following points:
1. (2, 3) 2. (−2, −1) 3. (−5, −6) 4. (4, −2.5)

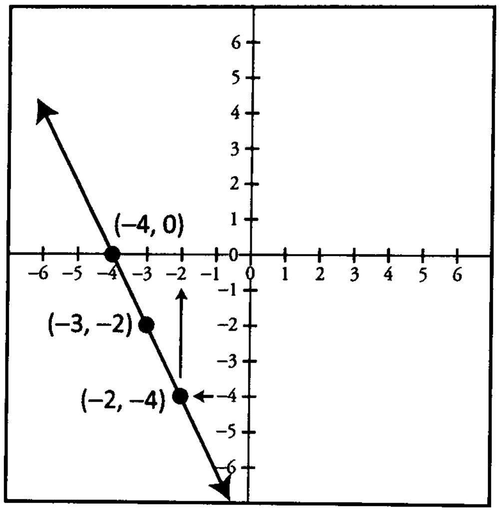


2. ***A:* (3, 0) *B:* (−3, 2) *C:* (1, −5) *D:* (0, −3)**

3. **−4:** The *y*-coordinate of the point on the line that has an *x*-coordinate of 3 is −4. The point is (3, −4). If you want, you can determine that the line has a slope of −1 from the two labeled points that the line intercepts, (−1, 0) and (0, −1).



4. **−2:** The x-coordinate of the point on the line that has a y-coordinate of −4 is −2. The point is (−2, −4). If you want, you can determine that the line has a slope of −2 from the two labeled points that the line intercepts, (−4, 0) and (−3, −2).



5. **Yes:** For the point (3, −2) to lie on the line $y = 2x - 8$, y needs to equal −2 when we plug in 3 for x.

$y = 2(3) - 8$
$y = 6 - 8 = -2$
y does equal −2 when x equals 3, so the point does lie on the line.

Set 4, Drill 2

1. **No:** For the point (−3, 0) to lie on the curve $y = x^2 - 3$, y needs to equal 0 when we plug in −3 for x.

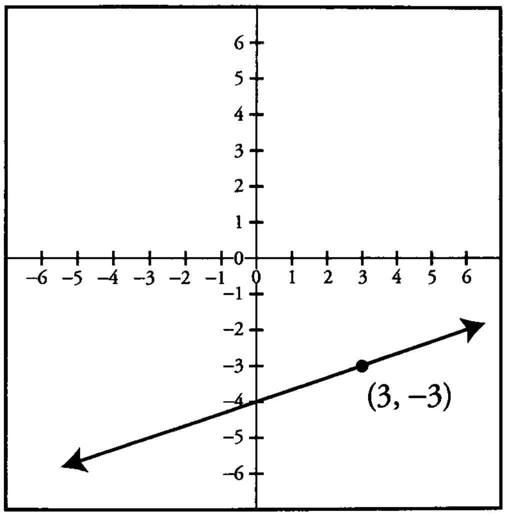
$y = (-3)^2 - 3$
$y = 9 - 3 = 6$
y does not equal 0 when x equals −3, so the point does not lie on the curve.

2. **14:** To find the y-coordinate, we need to plug in 3 for x and solve for y.

$y = 4(3) + 2$
$y = 12 + 2 = 14$
The y-coordinate is 14. The point is (3, 14).

3. **−7:** The equation of the line is already in $y = mx + b$ form, and b stands for the y-intercept, so we just need to look at the equation to find the y-intercept. The equation is $y = -2x - 7$. That means the y-intercept is −7. The point is (0, −7).

4. Graph the line $y = \frac{1}{3}x - 4$



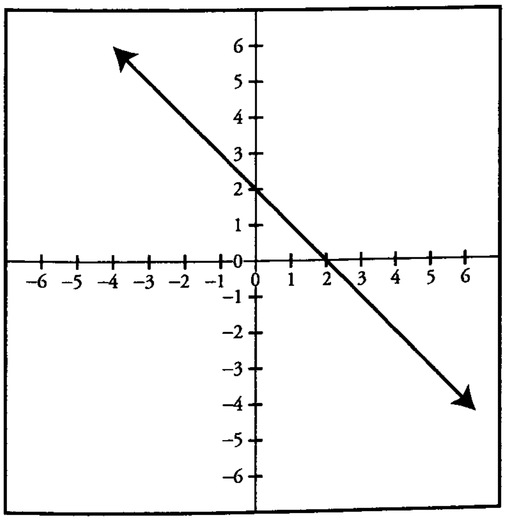
The slope (m) is 1/3, so the line slopes gently up to the right, rising only 1 unit for every 3 units of run.

The y-intercept (b) is −4, so the line crosses the y-axis at (0, −4).

5. Graph the line $\frac{1}{2}y = \frac{1}{2}x + 1$.

Before we can graph the line, we need to put the equation into $y = mx + b$ form. Multiply both sides by 2.

$y = -x + 2$



The slope (m) is −1, so the line drops to the right, falling 1 unit for every unit of run.

The y-intercept is 2, so the line crosses the y-axis at (0, 2).

Chapter 9

of

GEOMETRY

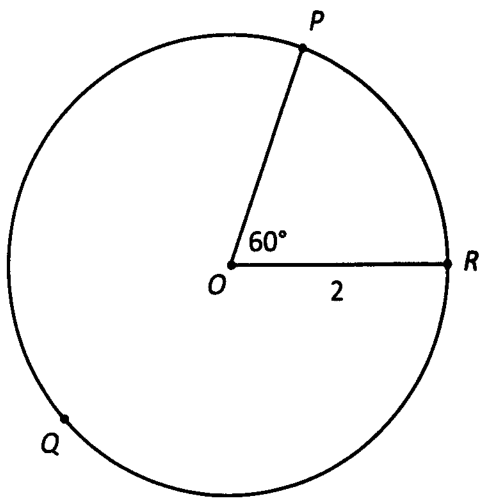
GEOMETRY PRACTICE QUESTION SETS

In This Chapter . . .

- Easy Practice Question Set
- Medium Practice Question Set
- Hard Practice Question Set
- Easy Practice Question Solutions
- Medium Practice Question Solutions
- Hard Practice Question Solutions

Geometry: Easy Practice Question Set

1.



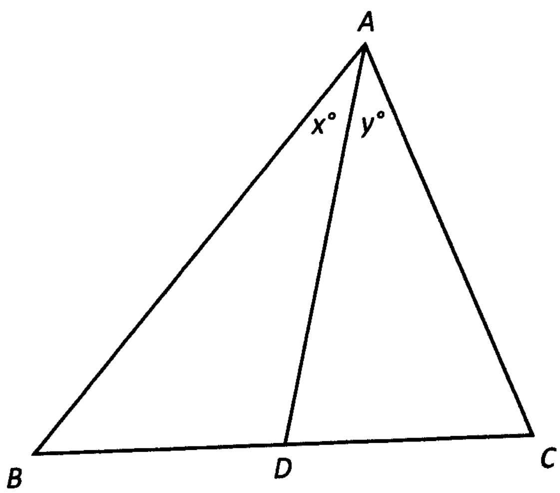
Point O is the center of the circle PQR.

Quantity A	Quantity B
The length of arc PQR	3π

2. If a triangle has sides measuring 5 inches and 12 inches, which of the following could *not* be the measure of the third side?

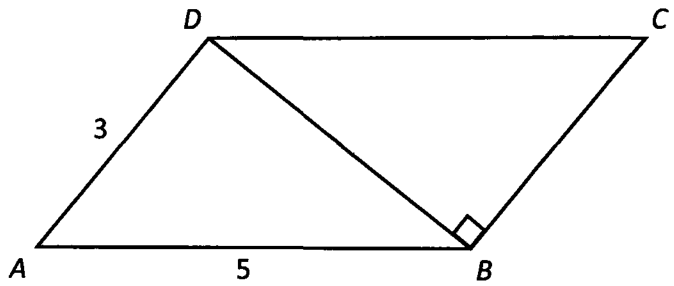
(A) 7.5 inches
(B) 10 inches
(C) 12.5 inches
(D) 15 inches
(E) 17.5 inches

3.



$BD = CD$

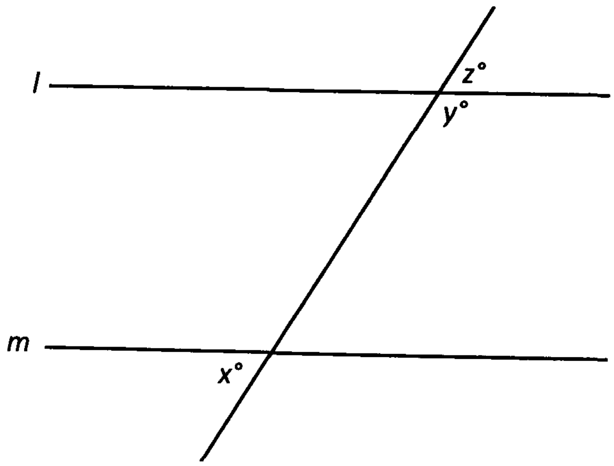
Quantity A	Quantity B
x	y



4. The area of parallelogram *ABCD* equals

(A) 6
(B) 7.5
(C) $5\sqrt{3}$
(D) 12
(E) 15

5.



Lines *l* and *m* are parallel.

Quantity A	Quantity B
$y - z$	x

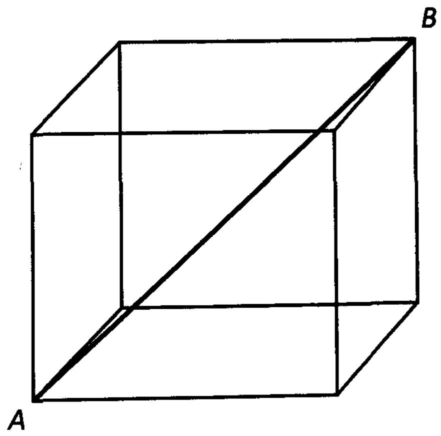
6. What is the *x*-intercept of the line given by $2x + 5y = 7$?

(A) 1
(B) 1.4
(C) 2
(D) 3
(E) 3.5

7. Swimming Pool A has a perimeter of 100 meters. Swimming Pool B has a perimeter of 80 meters. Both swimming pools are rectangular.

Quantity A	Quantity B
The area of Swimming Pool *A*, in square meters	The area of Swimming Pool *B*, in square meters

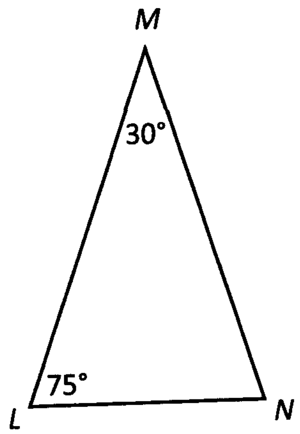
8.



The cube has an edge of length 10.

Quantity A	Quantity B
The length of the diagonal from point *A* to point *B*	17

9.



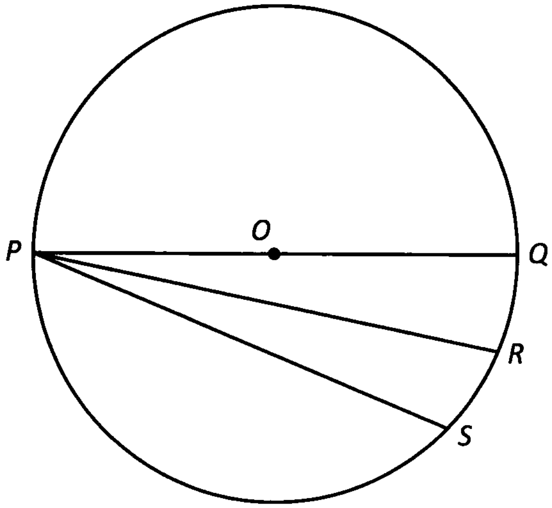
Quantity A	Quantity B
The length of *LM*	The length of *MN*

10.

In the xy-plane, the equation of line k is $4x + 5y = 3$.

Quantity A	**Quantity B**
The x-intercept of line k	The y-intercept of line k

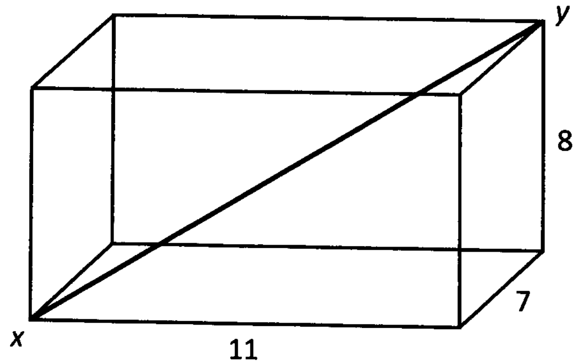
11.



PQ is a diameter of the circle above.

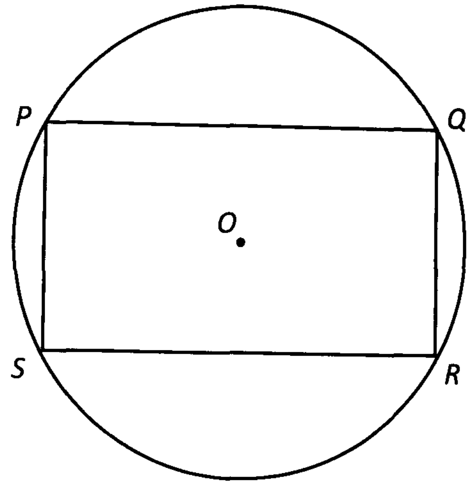
Quantity A	**Quantity B**
The length of PR	The average (arithmetic mean) of the lengths of PQ and PS

12.



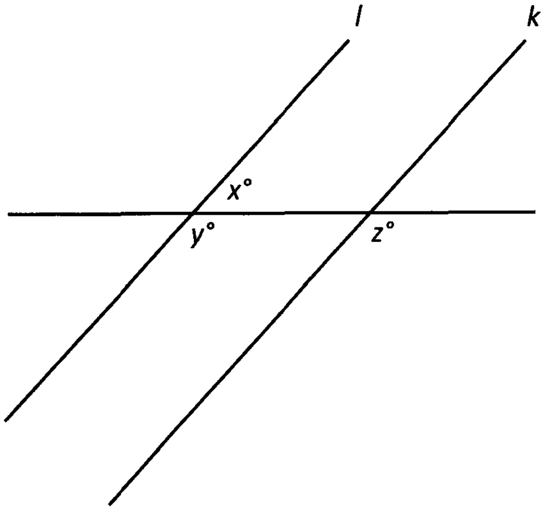
For the rectangular solid above, the square of the length of the diagonal XY is what?

13.



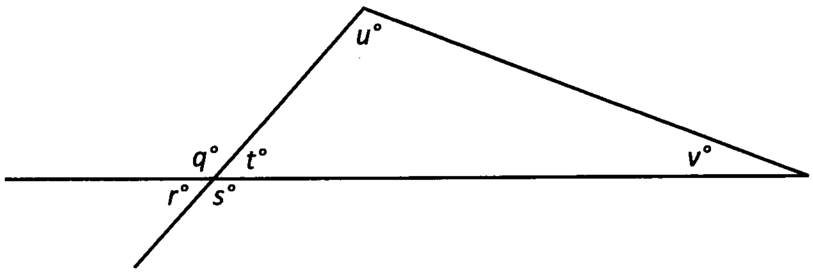
In the figure above, the diameter of the circle is 12.

Quantity A	Quantity B
The area of rectangle *PQRS*	80



14. In the above diagram, lines *l* and *k* are parallel. If $y - x = 30$, what is z?

(A) 60
(B) 75
(C) 90
(D) 105
(E) 120



15. In the above diagram, which of the following quantities would be sufficient to solve for the value of q?

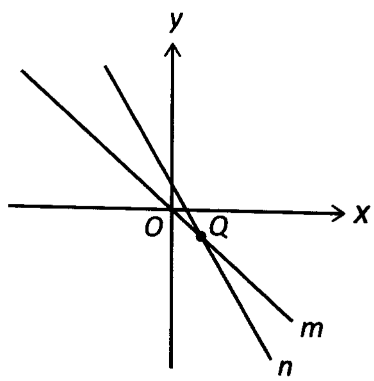
Indicate all such quantities:

- A r
- B s
- C t
- D u
- E v
- F $t + u$
- G $u + v$

16. If two lines intersect such that one of the angles formed at the intersection measures 20°, which of the following is the closest approximation to the product of the degree measures of all four angles formed at the intersection?

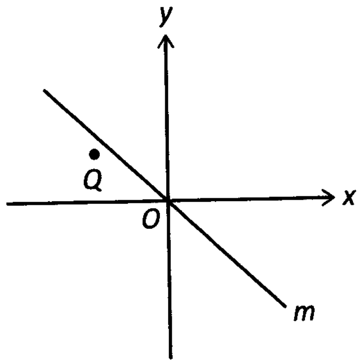
(A) 1×10^7
(B) 2×10^7
(C) 4×10^7
(D) 2×10^8
(E) 4×10^8

17.



Quantity A	Quantity B
The slope of line m	The slope of line n

18.



Point Q lies at (–3, 2).

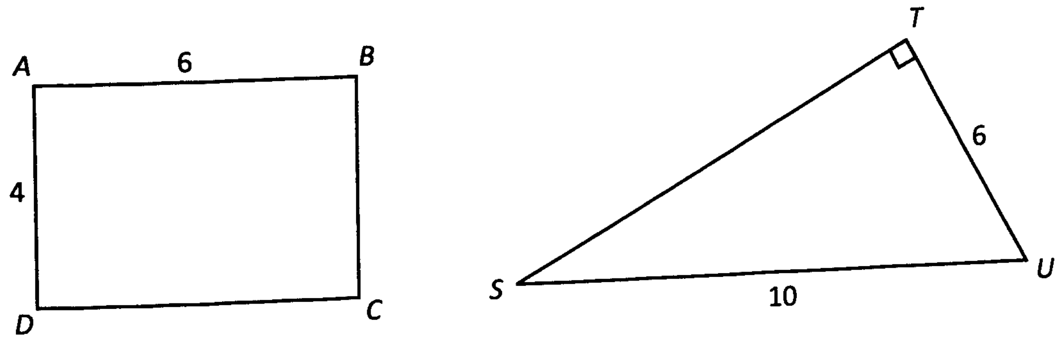
Quantity A	Quantity B
The slope of line m	–1

19. Line k in the xy-plane goes through the point (1, 1) and has a negative slope. Which of the following points could lie on line k?

Indicate all such points:

- A (1, 2)
- B (2, 0)
- C (–2, 0)
- D (–2, 2)

20.



Quantity A	Quantity B
The area of rectangle $ABCD$	The area of triangle STU

Geometry: Medium Practice Question Set

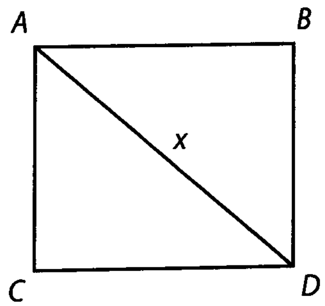
1. In the coordinate plane, for which of the following values of x would the graph of the equation $y = x^3 - x^2 - 6x$ touch the x axis?

(A) 2
(B) −3
(C) −2
(D) 1
(E) 6

2. Points P, Q and R lie in the coordinate plane. If $P = (1, 5)$, $Q = (1, 1)$, and $R = (7, y)$, how many different integer values for y could be chosen to form triangle ΔPQR, where none of the angles in ΔPQR is greater than 90°?

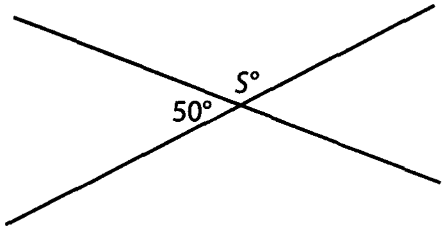
(A) 0
(B) 3
(C) 5
(D) 7
(E) It cannot be determined from the information given.

3.



Polygon $ABCD$ is a square.

Quantity A	Quantity B
$0.6x$	The length of side BD



4. In the diagram above $S = xy$, where x and y are positive integers. Which of the following could equal x?

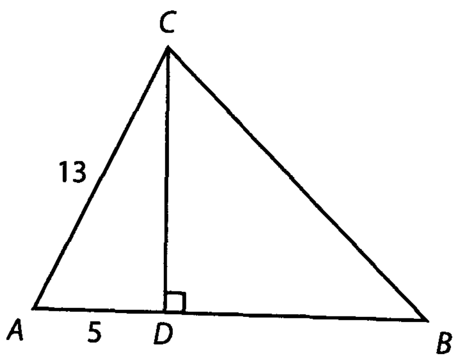
Indicate all such possible values for x:

[A] 26
[B] 25
[C] 10
[D] 5
[E] 4
[F] 0
[G] −10

5. Right triangle *PRS* has sides of length 6, 8 and x.

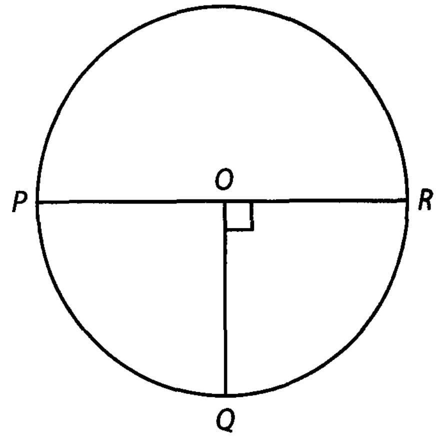
Quantity A	Quantity B
x	10

6.



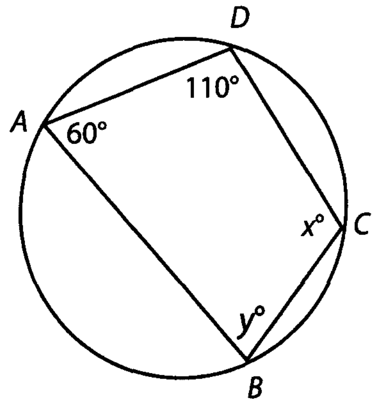
In the figure above, $\angle DBC = \angle DCB$. What is the area of triangle ABC?

7.



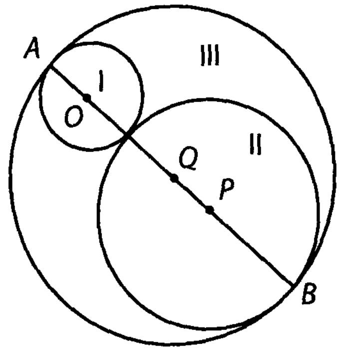
PR is a diameter of the circle centered at *O*. If the area of sector *OPQ* equals 4π, then the ratio of the circumference of the circle to π equals what?

8.



In the figure above, quadrilateral *ABCD* is inscribed in a circle. What is $x - y$?

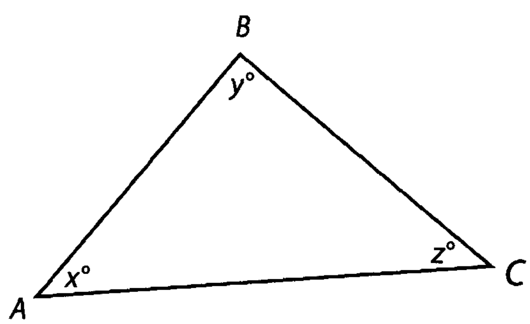
9.



Circles *I*, *II* and *III* are mutually tangent with centers *O*, *P*, and *Q*, respectively. *O*, *P* and *Q* lie on line segment *AB*.

Quantity A	**Quantity B**
The sum of the circumferences of circles *I* and *II*	The circumference of circle *III*

10. A beekeeper hosts two rectangular bee colonies in separate regions on her property. The first colony has an area of 600 square feet and a length of 40 feet. If the second colony has a width twice that of the first colony, but only $\frac{1}{2}$ the area, the ratio of the perimeter of the first colony to that of the second colony equals what?

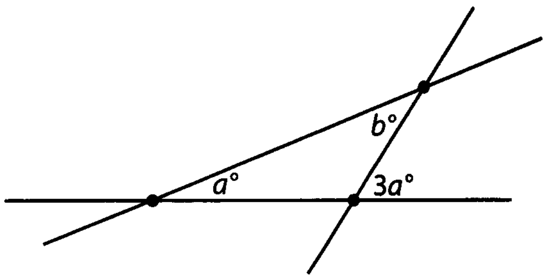


11. Triangle ABC's longest side is length 10. If $x \neq y \neq z$ and each side has an integer length, which of the following could be the length of its shortest side?

Indicate <u>all</u> such lengths.

- [A] 4
- [B] 5
- [C] 6
- [D] 8
- [E] 9
- [F] 10

12.

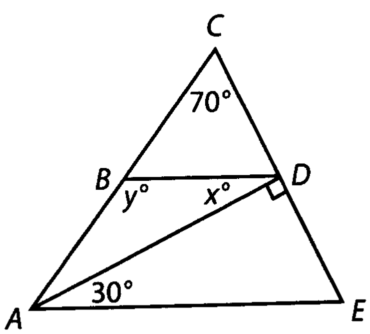


Quantity A	Quantity B
a	b

13. A circular garden is surrounded by a fence (the width of the fence is negligible) along its boundary. If the length of the fence is $\frac{1}{2}$ the area of the garden, what is the radius of the circular garden?

(A) 1
(B) 2
(C) 4
(D) 8
(E) 16

14.



If $AB = BD$, then $\frac{x}{y} = \frac{\square}{\square}$.

15. What is the perimeter, in *inches*, of a rectangular sandbox 5 feet long that has twice the area of a rectangular sandbox 20 feet long and 5 feet wide?

(A) 600
(B) 800
(C) 960
(D) 1,080
(E) 1,200

16. The relationship between the area A of a square and its perimeter is given by the formula $A = nP^2$, where n is a constant. What is the value of n^2?

(A) $\frac{1}{4}$

(B) $\frac{1}{8}$

(C) $\frac{1}{16}$

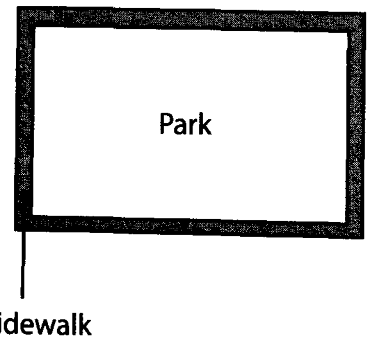
(D) $\frac{1}{64}$

(E) $\frac{1}{256}$

17. P, Q, and R are each rectangles. The length and width of rectangle P are 30 percent less and 20 percent greater, respectively, than the length and width of rectangle R. The length and width of rectangle Q are 40 percent greater and 40 percent less, respectively, than the length and width of rectangle R.

Quantity A	Quantity B
The area of rectangle P	The area of rectangle Q

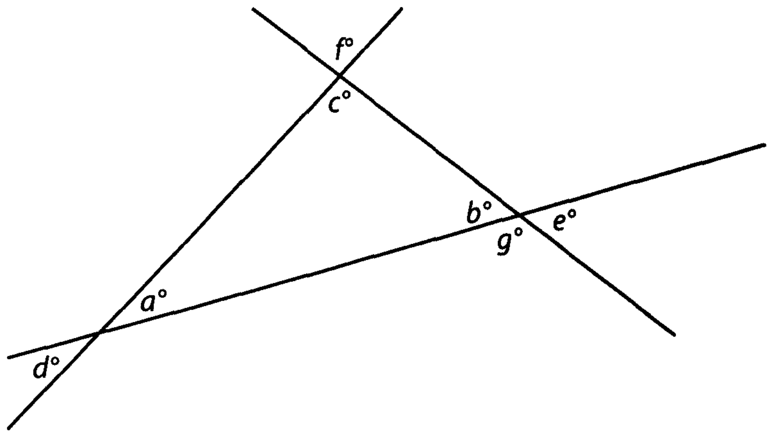
18.



The diagram above represents a rectangular park with a sidewalk surrounding it. The park is 150 feet long and 90 feet wide, not including the sidewalk. The sidewalk is 5 feet wide all the way around. The area of the sidewalk alone is how many square feet?

[ ] square feet

19.

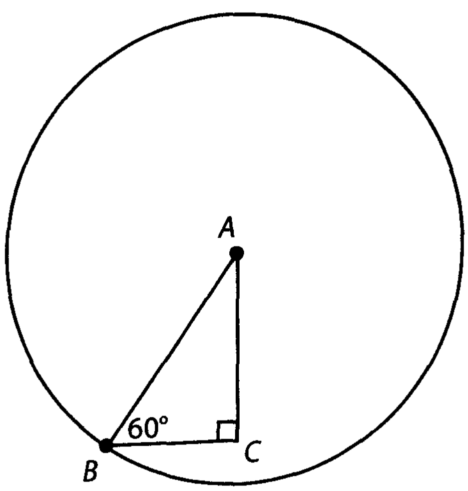


In the above diagram, which of the following quantities would be sufficient to solve for the value of *b*?

Indicate all such quantities.

- A a
- B c
- C e
- D $a + c$
- E $d + e$
- F $d + f$
- G g

20.



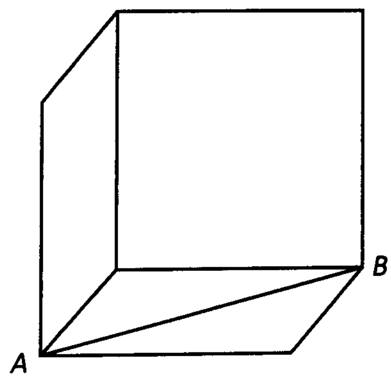
The circumference of the circle centered at point *A* is 12π.

Quantity A	Quantity B
14	The perimeter of triangle *ABC*

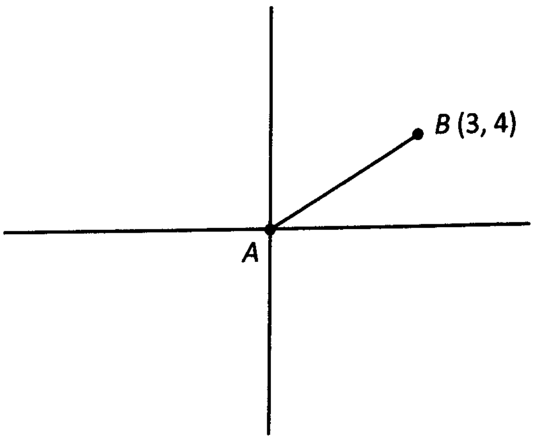
Geometry: Hard Practice Question Set

CAUTION: These problems are *very difficult*—more difficult than many of the problems you will likely see on the GRE. Conside these "Challenge Problems." Have fun!

1.



In the cube above, the length of line segment AB is 8. The surface area of the cube equals what?



2.

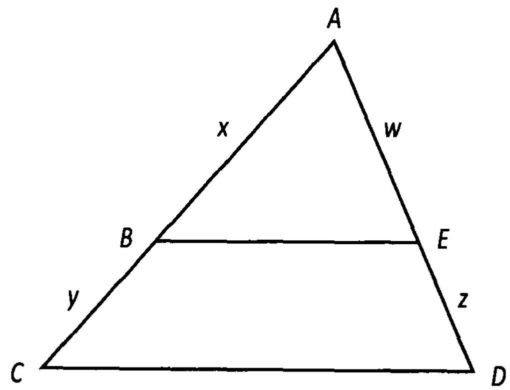
In the above diagram, point C is not displayed. If the length of line segment $\overline{BC}$ is *twice* the length of line segment $\overline{AB}$, which of the following could *not* be the coordinates of point C?

(A) (–5, –2)
(B) (9, 12)
(C) (10, 11)
(D) (11, 10)
(E) (13, 4)

3. Line M is described by the equation $y = 3x + 10$. Line N is described by the equation $2y = 5x - 6$. Line P has a y-axis intercept of 6, and the point (6, 4) lies on line P.

Quantity A	Quantity B
The measure of the largest angle created by the intersection of line M and line N	The measure of the largest angle created by the intersection of line M and line P

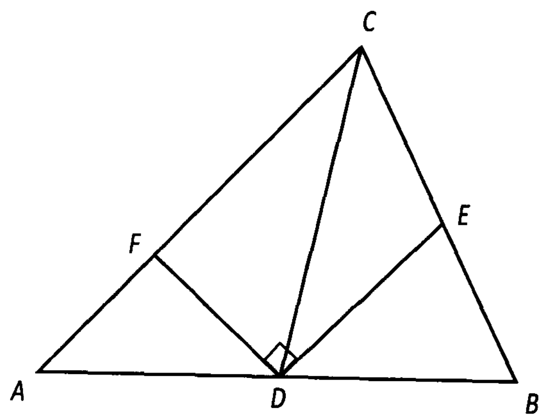
4.



BE is parallel to *CD*.

Quantity A	Quantity B
xz	yw

5.



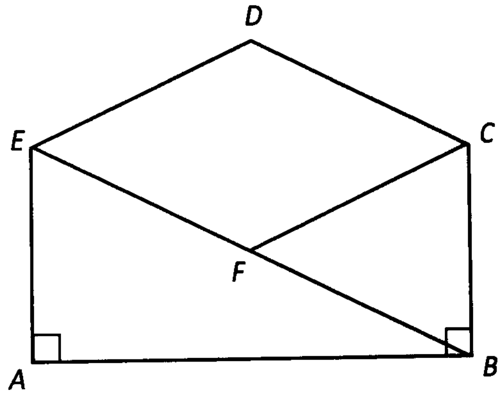
D is the midpoint of *AB*, and *E* is the midpoint of *BC*.

Which of the following statements MUST be true?

Indicate all such statements:

- [A] *ED* is parallel to *AC*
- [B] The area of triangle *EDB* = the area of triangle *CDE*
- [C] The area of triangle *ADF* < the area of triangle *EDB*
- [D] Angle *ADF* = angle *CDF*
- [E] The area of triangle *ABC* = $AC \times DF$

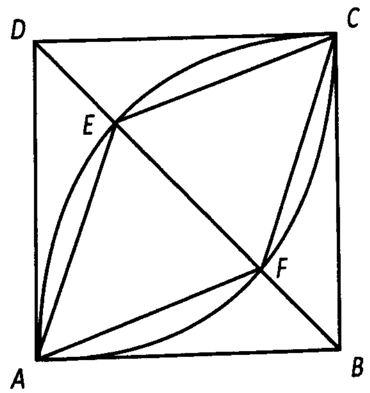
6.



$AE = BC$, $ED = CD$, EB is parallel to CD, and ED is parallel to CF

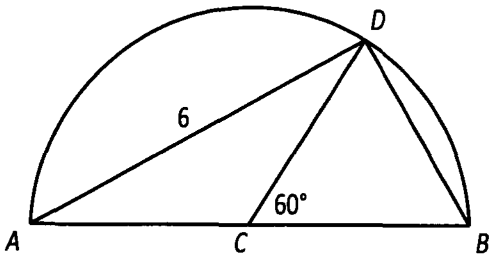
Quantity A	**Quantity B**
The area of triangle ABE	The area of quadrilateral $EFCD$

7.



In the figure above, $ABCD$ is a square with sides equal to 1, AFC is an arc of a circle centered at D, and AEC is an arc of a circle centered at B. What is the area of rhombus $AFCE$?

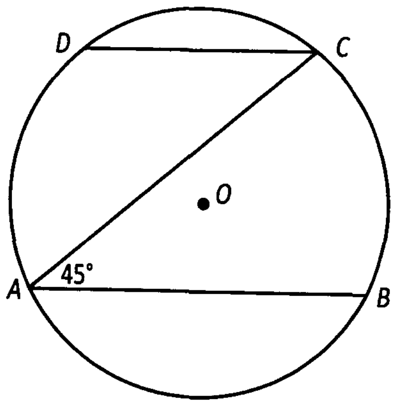
(A) $2-\sqrt{2}$
(B) $\sqrt{2}-1$
(C) $\sqrt{2}\left(2-\sqrt{2}\right)$
(D) $\sqrt{2}$
(E) $1+\sqrt{2}$



8. Triangle *ABD* is inscribed in a semicircle centered at *C*. What is the area of triangle *ABD*?

(A) $\frac{12}{\sqrt{3}}$
(B) $6\sqrt{3}$
(C) 12
(D) $12\sqrt{3}$
(E) $18\sqrt{3}$

9.



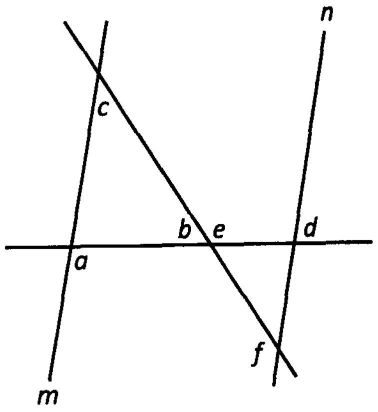
In the figure above, circle *O* has radius 8, and *AB* is parallel to *CD*. If the length of minor arc *AB* is twice the length of minor arc *CD*, what is the length of minor arc *CD*?

(A) 2π
(B) $\frac{8\pi}{3}$
(C) 3π
(D) 4π
(E) $\frac{16\pi}{3}$

10.

Quantity A	Quantity B
The volume of a right circular cylinder with radius and height each equal to 3	84

11.



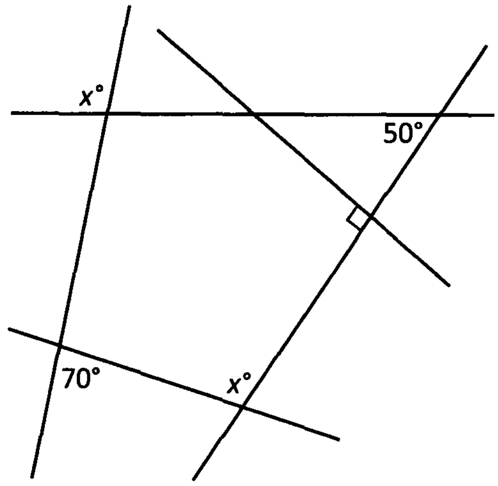
Lines m and n are parallel.

The measures of which of the following angles or pairs of angles, by themselves, are sufficient to determine the measure of angle a?

Indicate all that apply.

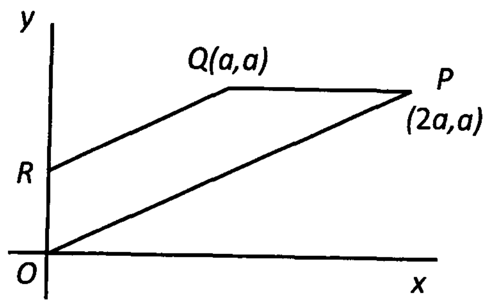
- A b
- B c
- C d
- D b and c
- E c and e
- F b and f
- G c and f

12.



In the figure above, what is x?

13.



Trapezoid *OPQR* has one vertex at the origin. What is the area of *OPQR*?

(A) $\frac{a^2}{4}$

(B) $\frac{a^2}{2}$

(C) $\frac{3a^2}{4}$

(D) $\frac{3a^2}{2}$

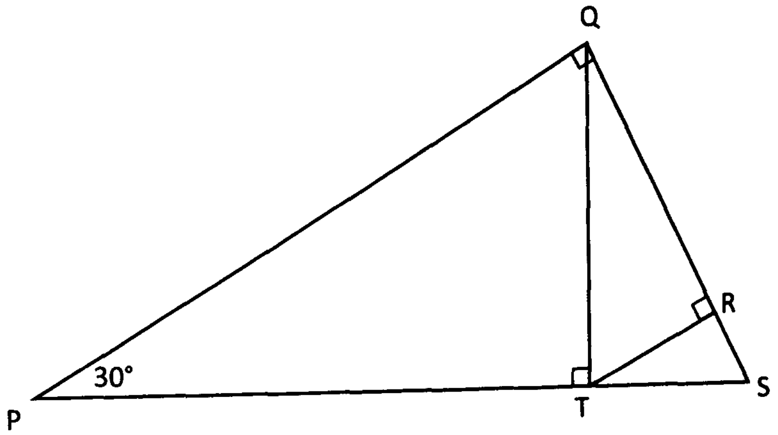
(E) $2a^2$

14. Perpendicular lines *m* and *n* intersect at point (a, b), where $a > b > 0$. The slope of line *m* is between 0 and 1. Which of the following statements MUST be true?

Indicate all that apply.

- A The *x*-intercept of line *m* is positive.
- B The *y*-intercept of line *m* is negative.
- C The *x*-intercept of line *n* is positive.
- D The *y*-intercept of line *n* is negative.
- E The product of the *x*- and *y*-intercepts of line *m* is negative.
- F The sum of the *x*-intercepts of lines *m* and *n* is positive.

15.



If $PQ = 1$, what is the length of RS?

(A) $\frac{1}{12}$

(B) $\frac{\sqrt{3}}{12}$

(C) $\frac{1}{6}$

(D) $\frac{2}{3\sqrt{3}}$

(E) $\frac{2}{\sqrt{12}}$

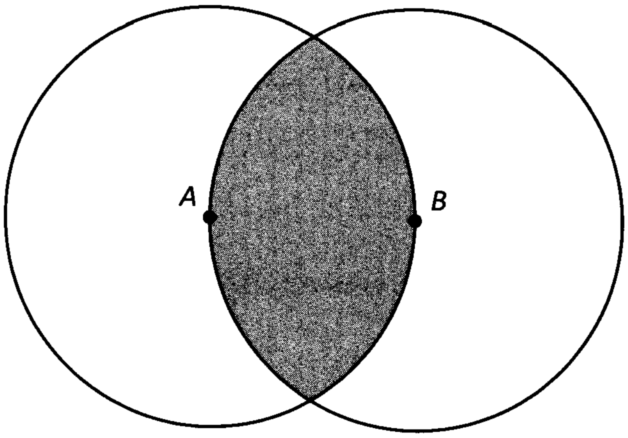
16. If the length of one side of a regular hexagon (all sides and angles equal) is 8, what is the area of the hexagon?

(A) 48
(B) $36\sqrt{3}$
(C) $48\sqrt{2}$
(D) $96\sqrt{3}$
(E) $128\sqrt{2}$

17. The average measure of the interior angles of an *n*-sided polygon is divisible by 10. For which values of *n* could this be true?

Indicate <u>all</u> such possible values for *n*.

[A] 4
[B] 5
[C] 6
[D] 8
[E] 9
[F] 10



18. In the figure above, *A* and *B* are the centers of the two circles. If each circle has radius *x*, what is the area of the shaded region?

(A) $\dfrac{\left(2\pi-\sqrt{3}\right)x^2}{6}$

(B) $\dfrac{\left(4\pi-3\sqrt{3}\right)x^2}{12}$

(C) $\dfrac{\left(4\pi-3\sqrt{3}\right)x^2}{6}$

(D) $\frac{(4\pi - \sqrt{3})x^2}{6}$

(E) $\frac{(6\pi - 1)x^2}{6}$

19. In the xy-plane, line n is a line that passes through the origin.

Which of the following statements individually provide (s) sufficient additional information to determine whether the slope of line n is greater than 1?

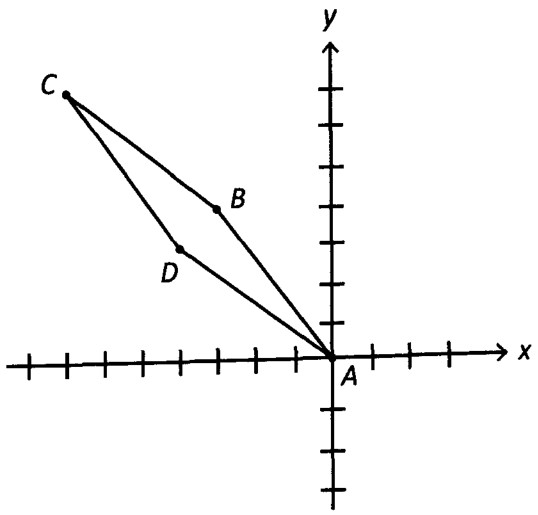
Indicate all such statements.

[A] Line n does not pass through any point (a, b) where a and b are positive and $a > b$.

[B] Line m is perpendicular to line n and has a slope of -1.

[C] Line n passes through the point $(c, d + 1)$ where c and d are consecutive integers and $c > d$.

20.



Parallelogram $ABCD$ lies in the xy-plane, as shown in the diagram above. The coordinates of point B are $(-3, -4)$ and the coordinates of point C are $(-7, -7)$. What is the area of the parallelogram?

(A) 1
(B) $2\sqrt{7}$
(C) 7
(D) 8
(E) $7\sqrt{2}$

Geometry: Easy Practice Question Solutions

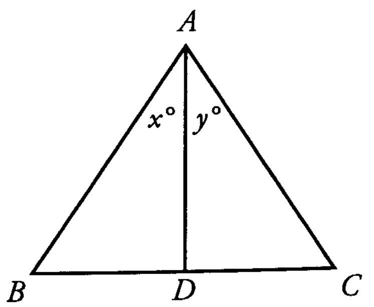
1. **A:** To find the length of arc *PQR*, we first need to calculate the circumference of the circle. This is found by the equation $C = \pi \times d = \pi \times 2 \times r$. Because O is the center of the circle, we can use $OR = 2$ as the radius. Therefore $C = \pi \times 2 \times 2 = 4\pi$. Because $\frac{1}{6}$ of the circle is ***not*** included in arc *PQR* (60° is $\frac{1}{6}$ of 360°), the remaining $\frac{5}{6}$ of the circumference represents the length of arc *PQR*. Hence arc $PQR = \frac{5}{6}(4\pi) = \frac{20}{6\pi} = \frac{10}{3\pi}$. Or, put simply:

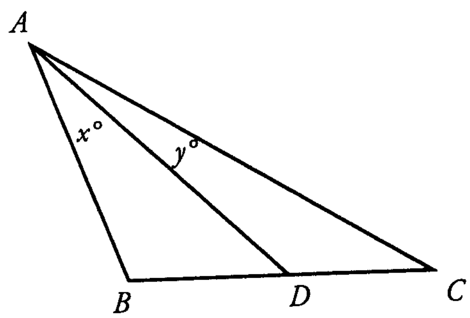
$\frac{10}{3\pi}$	3π

Therefore **Quantity A is greater.**

2. **E:** Any side x of a triangle must be *greater* than the difference between the lengths of the two other sides, and *less* than the sum of the two other sides. In this case, the third side must be between $(12 - 5) = 7$ inches and $(12 + 5) =$ 17 inches. Therefore $7 < x < 17$. Only 17.5 inches is outside this range.

3. **D:** Even though line segments *BD* and *CD* have equal lengths, the relationship between angle measures x and y is indeterminate. If *AD* is perpendicular to *BC*, as shown in the figure to the left (below), then $x = y$. However, if point *A* is skewed to one side, then the angle on that side becomes larger than the other. For example, in the figure to the right, $x > y$.





We cannot infer anything from the appearance of the drawing, so **we do not have enough information** to determine which quantity is greater.

4. **D:** Parallelogram *ABCD* is comprised of two congruent right triangles, *ABD* and *BCD*. The length of diagonal *BD* is 4, as can be determined from the Pythagorean Theorem or the 3–4–5 triangle rule. Thus, the area of each of the right triangles is $\frac{1}{2} \times 3 \times 4 = 6$, and the area of *ABCD* is 12.

5. **D:** In the figure, $x = z$ (alternate exterior angles of a transversal) and $y + z = 180$ (supplementary angles). Combining, $y + z = 180$ so that $y = 180 - x$. The comparison is therefore between $y - z = (180 - x) - x$ and x. Depending on the value of x, either $180 - 2x$ or x may be greater, or the two quantities may be equal. **We do not have enough information** to determine which is the case.

6. **E:** The x-intercept of a line is the value of x at which the line crosses the x-axis. When a line crosses the x-axis, the value of y equals zero. Thus the x-intercept can be determined by setting y equal to zero in the equation for the line: $2x + 0 = 7$, so $x = 3.5$.

7. **D:** Even though both swimming pools are rectangular and Swimming Pool A has a larger perimeter, it is possible for Swimming Pool B to cover a larger area or the same area as Swimming Pool A. For example, Swimming Pool A could have a length of 40 meters and a width of 10 meters (Perimeter = 2 × length + 2 × width for rectangles). This would give Swimming Pool A an area of 40 × 10 = 400 square meters. If Swimming Pool B is square, its area would be $\left(\frac{80}{4}\right)^2 = 20^2 = 400$, equal to that of Swimming Pool A. If instead Swimming Pool B had a length of 30 meters and a width of 10 meters, its area would equal 30 × 10 = 300 square meters. It is also possible to construct examples in which Swimming Pool B has a larger area. Thus **we do not have enough information** to determine which is a larger quantity.

8. **A:** The formula for the length of a diagonal across opposite ends of a rectangular solid is as follows: $d^2 = l^2 + w^2 + h^2$, where l, w, and h are the length, width, and height of the rectangular solid. Since $l = w = h$ for a square, we can simplify this to: $d^2 = 3l^2$. Since $l = 10$, $d^2 = 3(10)^2$, and $d = 10\sqrt{3}$. Using the GRE on-screen calculator, you can determine that $\sqrt{3}$ is slightly larger than 1.7, so d is slightly larger than 17. Thus **Quantity A is larger.**

9. **C:** Because the interior angles of a triangle must sum to 180°, angle LNM must equal 180° – 75° – 30° = 75°. Therefore triangle LMN is an isosceles triangle, and sides LM and MN must be equal length. Therefore **the two quantities are equal.**

10. **A:** Rewriting the equation in standard form ($y = mx + b$), we get $5y = -4x + 3$, so $y = -\frac{4}{5}x + \frac{3}{5}$. To find the x-intercept, we set $y = 0$ and solve for x. Similarly, to find the y-intercept, we set $x = 0$ and solve for y.

x-intercept: $0 = -\frac{4}{5}x + \frac{3}{5} \rightarrow \frac{4}{5}x = \frac{3}{5} \rightarrow x = \frac{3}{4}$

y-intercept: $y = -\frac{4}{5}(0) + \frac{3}{5} \rightarrow y = \frac{3}{5}$

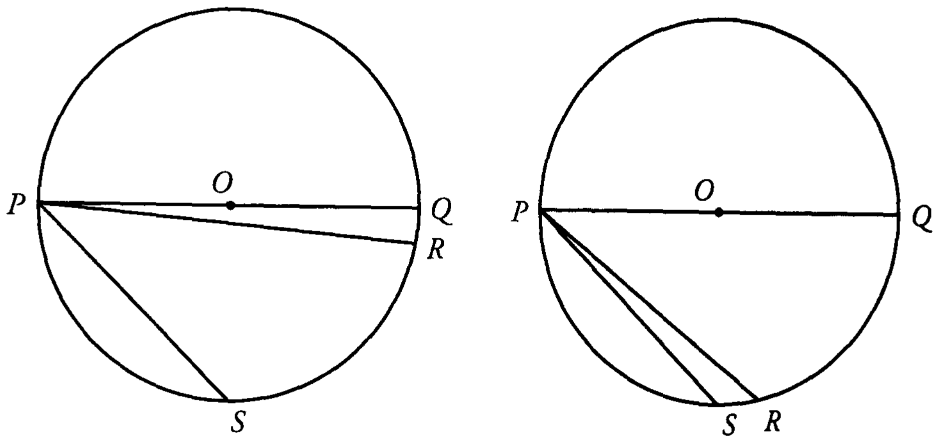
Therefore **Quantity A is greater.**

11. **D:** Notice that a diameter (such as PQ in the diagram) is the largest chord that can be drawn through a circle. The farther a chord passes from the center, the smaller the chord will be. Therefore, $PQ > PR > PS$. The question asks us to compare the length of PR, the middle-length chord, to the average of PQ and PS:

Is $PR > \frac{PQ + PS}{2}$?

Multiplying by 2, we get, is $2PR > PQ + PS$?

By moving PR and PS around, we can see that in some cases, $2PR$ will be larger; in others, $PQ + PS$ will be larger.



$2PR > PQ + PS$ $2PR < PQ + PS$

Therefore, **there is not enough information** in the diagram to determine which quantity is greater.

12. **234:** To calculate the diagonal of a rectangular solid, can simply take the sum of the squares of the length, width, and height, and take the square root. (This is the mathematical equivalent of using the Pythagorean Theorem twice in succession.) For this problem, we need only report the square of the length, so we do not need to find the square root of the sum of the squared sides.

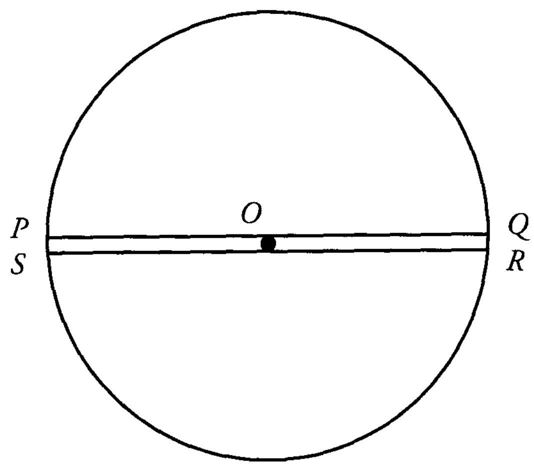
$$XY^2 = 11^2 + 7^2 + 8^2 = 121 + 49 + 64 = 234.$$

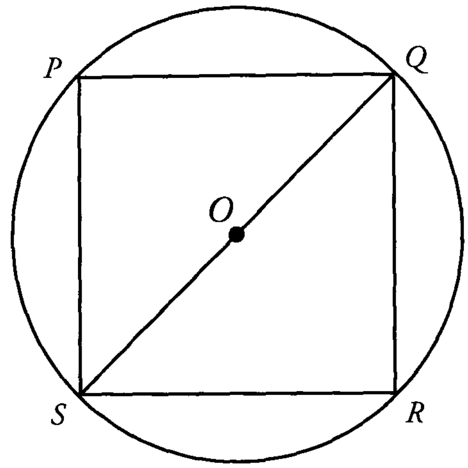
13. **B:** Given that the diameter of the circle is 12, the area of the circle is given by $\pi r^2 = \pi\left(\frac{d}{2}\right)^2 = \pi\left(\frac{12}{2}\right)^2 = 36\pi$.

The inscribed rectangle *PQRS* must be smaller than that. The question is, what are the limitations on its size?

To picture the smallest possible rectangle, envision one extremely short and wide, as in the figure at left, below. This rectangle would have a width approaching that of the circle's diameter (12), but a height approaching 0. Thus the area would approach 0.

To picture the largest possible rectangle, envision perfect square. As in the figure at right, the diagonal $QR = 12$, which is also a diameter of the circle. This diameter cuts the square into two 45–45–90 triangles, which have sides in the proportion $1:1:\sqrt{2}$. Therefore the side of the square would equal $\frac{12}{\sqrt{2}} = 6\sqrt{2}$, and the area would equal $\left(6\sqrt{2}\right)^2 = 72$, which is less than Quantity B (80). Therefore, the rectangle will always have an area smaller than Quantity B. **Quantity B is larger.**





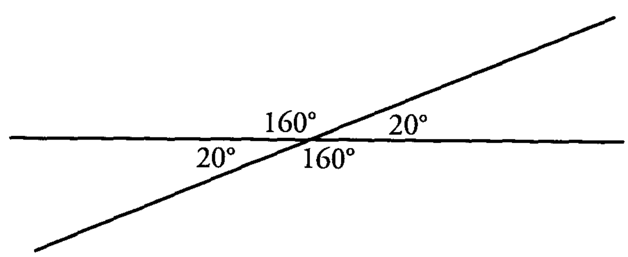
14. **D:** Because l and k are parallel, the horizonal line creates a transversal. As a result, $y = z$. In addition, $x + y = 180$, because x and y are supplementary angles. The question stem tells us that $y - x = 30$. If we add these two equations together, we get $2y = 210$, or $y = 105$. Therefore $z = 105$.

15. **A, B, C, and G:** Because t, u, and v are the interior angles of a triangle, $t + u + v = 180$. Also, since q and t are supplementary angles, $q + t = 180$. Therefore $t + u + v = q + t$, which implies that $u + v = q$. Therefore, knowing the value of $u + v$ would enable us to solve for q. Thus Choice G is correct. Knowing u or v individually would not enable us to solve for q, nor would knowing the sum $t + u$. (That sum would enable us to solve for v, which we have already established is not sufficient to solve for q.)

As previously stated, $q + t = 180$, so knowing t would enable us to solve for q. Thus Choice C is correct.

Finally, q and s are opposite angles, as are r and t, so $q = s$, and $r = t$. Thus knowing s gives us q directly, and knowing r gives us t directly, which we have already demonstrated is sufficient. Thus Choices A and B are correct.

16. **A:** Because one of the angles formed equals 20°, the other angles must measure 20°, 160°, and 160° (see diagram).

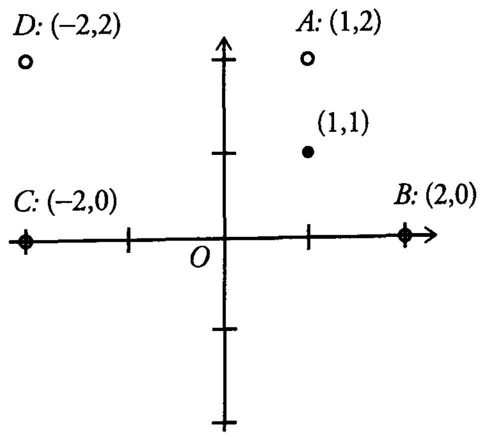


Therefore the product of all four degree measures = $(20)^2(160)^2 = 400 \times 25{,}600 = 10{,}240{,}000$. This is very close to 10,000,000, or 1×10^7.

17. **A:** Slope is defined as $\frac{\text{change in } y}{\text{change in } x}$, or $\frac{\text{rise}}{\text{run}}$. In the given diagram, both lines have a negative slope, but for any given change in x, the negative change in y is larger for line n than for line m. (In other words, line n is "falling" faster than line m). Therefore the ***absolute value*** of the slope of line n is larger than that of line m, but since ***both slopes are negative***, the slope of line m is the larger quantity. Therefore **Quantity A is larger.**

18. **D:** Because the point Q lies at $(-3, 2)$, the slope of the line containing both Q and the origin (O) has a slope of $-\frac{2}{3}$. Line m is steeper than that, with a negative slope. Therefore the slope of line m is $< -\frac{2}{3}$. It is possible for the slope to be $-\frac{3}{4}$, -1, or -2, for example. Therefore **we do not have enough information** to determine which quantity is larger.

19. **B and D:** The easiest way to solve this problem is to draw the xy-plane with each of the points in the answer choices plotted:



Drawing a line between (1, 1) and Choice A would produce a vertical line (infinite or undefined slope).

Drawing a line between (1, 1) and Choice B would produce a line with negative slope (-1).

Drawing a line between (1, 1) and Choice C would produce a line with positive slope $\left(\frac{1}{3}\right)$.

Drawing a line between (1, 1) and Choice D would produce a line with negative slope $\left(-\frac{1}{3}\right)$.

Therefore Choices B and D satisfy the conditions in the problem.

20. **C:** The formula for the area of rectangle $ABCD = l \times w = 6 \times 4 = 24$. To figure out the area of triangle STU, we need to use the Pythagorean Theorem to solve for ST (the base of the triangle): $6^2 + ST^2 = 10^2$. Therefore, $ST^2 = 64$ and $ST = 8$. Alternatively, we could recognize that triangle STU is a 6–8–10 triangle.

The area of a triangle is given by the formula $\frac{1}{2}bh$, which in this case equals $\frac{1}{2}(8)(6) = 24$.

The two quantities are equal.

Geometry: Medium Practice Question Solutions

1. **C**: In the standard Cartesian coordinate plane, a line crosses the x-axis at values of x that return a y value of 0. These points are known as the "roots" of the equation. To find the roots of the equation $y = x^3 - x^2 - 6x$, we set $y = 0$ and factor the right side of the equation as follows:

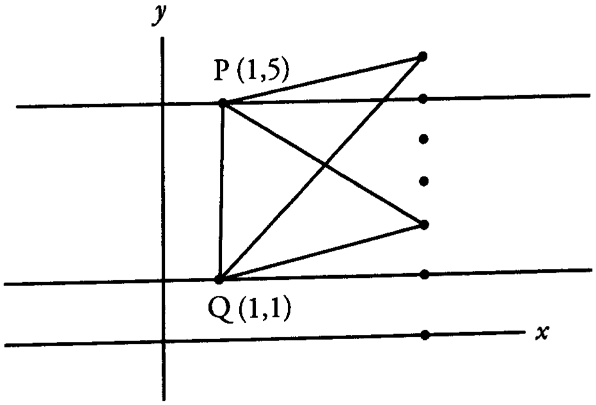
$$y = x^3 - x^2 - 6x$$

$$y = x(x^2 - x - 6)$$

$$y = x(x + 2)(x - 3)$$

Hence, $y = 0$ when $x = 0, -2$, or 3. Only Choice C matches one of these roots.

2. **C**: An obtuse angle is an angle with degree measure higher that 90°. As the following diagram shows, if y is greater than 5 or less than 1, it will cause ΔPQR to have an obtuse angle. Therefore, y must be an integer between 1 and 5, inclusive.



3. **B**: The ratio of the length of a side of a square to the length of its diagonal (labeled x in this diagram) is $1:\sqrt{2}$, because the diagonal divides the square into two right isosceles (45–45–90) triangles. We may find the length of a side of the square by dividing x by $\sqrt{2}$. Since $\sqrt{2}$ is roughly equal to 1.4 (or $\frac{7}{5}$), side BD equals approximately $\frac{x}{\frac{7}{5}}$, or $\frac{5x}{7}$. This equals approximately $0.7x$. **Quantity B is greater.**

4. **A, C, and D**: Since opposite angles are equal, we know that there are two 50° angles and two angles of $S°$ in this diagram. Since the set of angles forms a complete circle, the total degree measure must be 360°. Therefore, $2S° = 360° - 2 \times 50° = 260°$, and $S = 130°$. Since $S = xy$, it must be true that $xy = 130$.

From there, we need to determine which integers could multiply to produce 130. 130 can be broken down into the prime factors $2 \times 5 \times 13$, so any factor of 130 must be composed of *only some subset* of these prime factors. 25 and 4 are not composed strictly of 2, 5, and 13. Furthermore 0 cannot be possible, because if x were equal to 0, S would have to equal 0.

Note finally that -10 is not a possible value of x, because the question stem stipulates that both x and y be positive integers.

5. **D:** The trap in this problem is the common (but faulty) assumption that 6 and 8 are the lengths of the two perpendicular sides of a right triangle, because 6–8–10 triangles are common on the GRE.

If that were the case, the Pythagorean theorem or the 3–4–5 triangle rule could be used to determine that the length x of the hypotenuse equals 10: $6^2 + 8^2 = 10^2$. However, it is also possible that the side with length 8 is the hypotenuse of the triangle. In that case, $6^2 + x^2 = 8^2$, and $x = \sqrt{28} = 2\sqrt{7} \neq 10$.

6. **102:** The right triangle on the left (triangle ACD) is a 5–12–13 triangle, which is one of the common right triangles with integer side lengths. (Note that we could also use the Pythagorean Theorem to determine that the length of CD equals 12: $5^2 + 12^2 = 13^2$.) CD is the height of triangle ABC.

Because, $\angle DBC = \angle DCB$, triangle BCD is isosceles, and the length of BD must also equal 12. Thus the base of triangle ABC has length 5 + 12 = 17, and its area is given by $\frac{1}{2}bh = \frac{1}{2}(17)(12) = 17 \times 6 = 102$.

7. **8:** Because angle QOR is a right angle and POR is a straight line, angle POQ must also be a right angle. Angle POQ is the central angle of sector OPQ. Therefore, the area of sector OPQ must be one-fourth of the entire circle (90° is one fourth of 360°). The area of the entire circle is found as follows: $\frac{4\pi}{\frac{1}{4}} = 16\pi$. Using the area formula for a circle, we obtain $\pi r^2 = 16\pi$, so $r = \sqrt{16} = 4$. Finally, the circumference of the circle is found from the formula $c = 2\pi r = 8\pi$. The ratio of the circumference to π equals $\frac{8\pi}{\pi} = 8$.

8. **50:** When a quadrilateral is inscribed in a circle, opposite angles must add up to 180 degrees. This is because all angles of such a quadrilateral are inscribed angles of a circle. For example, angle ADC intercepts arc ABC, and angle ABC intercepts arc ADC. The two arcs constitute the entire circle. Thus, the sum of the arcs intercepted by these angles is a whole circle, or 360°, and the angles must sum to $\frac{1}{2}$(360°) = 180°. (This follows from the rule that the measure of an inscribed angle of a circle is one half the measure of the corresponding central angle.) This gives $x = 180° - 60° = 120°$ and $y = 180° - 110° = 70°$. Combining, $x - y = 120° - 70° = 50°$.

9. **C:** Because O, P, and Q all lie on line segment AB, the diameter of circle I (call it D_{I}) plus the diameter of circle II (D_{II}) must equal the diameter of circle III (D_{III}): $D_{I} + D_{II} = D_{III}$. Because the circumference of a circle equals the diameter times π, i.e. $c = \pi d$, we can write $\pi(D_{I} + D_{II}) = \pi D_{III}$, or (letting C denote circumference), $(C_{I} + C_{II}) = C_{III}$. **The two quantities are equal.**

10. $\mathbf{\frac{110}{80}}$ **(or any equivalent):** According to the problem, the first colony has an area of 600 square feet and a length of 40 feet. Therefore the width of the colony is $\frac{600}{40} = 15$ feet. The second colony has a width twice that amount (30 feet), and an area only half that of the first colony ($\frac{1}{2}$ of 600 square feet = 300 square feet). Therefore the length of the second colony is $\frac{300}{30} = 10$ feet.

The perimeter of the first colony is 2 × length + 2 × width = 2(40) + 2(15) = 110.

The perimeter of the second colony is 2 × length + 2 × width = 2(10) + 2(30) = 80.

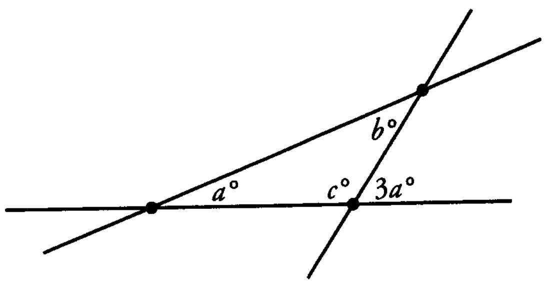
Thus the correct ratio is $\frac{110}{80}$ (or the mathematical equivalent).

11. **A, B, C, D:** According to the problem, the longest side of the triangle is 10, each side has an integer length, and none of the angles are equal. Since none of the angles are equal, none of the sides can be equal. Therefore Choice F can be eliminated—if the shortest side were 10, then all 3 sides would have to equal 10, yielding an equilateral triangle (all angles equal).

Similarly, if the shortest side were length 9, then the third side would have to equal 9 or 10. This would yield an isosceles triangle, and two of the angles would have to be equal. Choice E can thus be eliminated.

All of the other Choices are possible. For example, the sides could be 4–7–10 (satisfying Choice A), 5–7–10 (satisfying Choice B), 6–7–10 (satisfying Choice C), or 8–9–10 (satisfying Choice D).

12. **B:** Because $3a$ is an exterior angle to the triangle enclosed by the three lines in the diagram, $3a$ must equal $a + b$. This is easiest to see by adding a label to the third angle in the triangle:



$a + b + c = 180$ (interior angles of a triangle)
$c + 3a = 180$ (supplementary angles)

Therefore, $a + b + c = c + 3a$, and $a + b = 3a$. Thus $b = 2a$, and since angles must have a positive value, $b > a$. Thus **Quantity B is greater.**

13. **C:** If a circular garden is surrounded by a fence of negligible width, the fence will have a length equal to the circumference of the garden. Thus, the length of the fence is given by $C = 2\pi r$. The area of the garden is given by the formula for the area of a circle, $A = \pi r^2$. Finally, the problem tells us that the length of the fence is $\frac{1}{2}$ the area of the garden, so we may write our equation as:

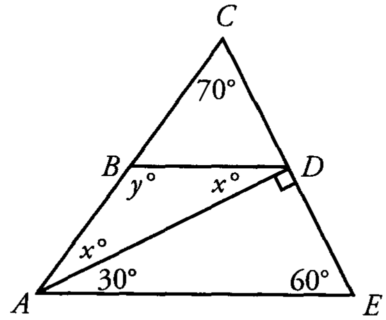
$$C = \left(\frac{1}{2}\right)(A)$$

$$2\pi r = \frac{1}{2}\pi r^2$$

$$4\pi r = \pi r^2$$
$$4r = r^2$$
$$r^2 - 4r = 0$$
$$r(r - 4) = 0$$
$$r = 0 \text{ and } r = 4$$

The garden cannot have a zero radius, so the radius of the garden must be 4. The correct answer is Choice C.

14. $\frac{\mathbf{20}}{\mathbf{140}}$ **(or any equivalent):** According to the problem, $AB = BD$. Therefore triangle ABD is an isosceles triangle, and angle BAD is equal to $x°$. Additionally, since we know angle $ADE = 90°$ and angle $DAE = 30°$, we can determine that angle $DEA = 180° - 90° - 30° = 60°$ (since these three angles form the triangle ADE). We can update the diagram as follows:



The sum of angles BCD, DEA, BAD, and BAE must be 180° because they form the angles of the large triangle, ACE. Therefore, $70 + 60 + x + 30 = 180$, and $x = 20$. Since y, x and x must sum to 180 (they form the angles of triangle BAD), $y = 180 - 2x$, so $y = 140$.

Thus the correct ratio is $\frac{20}{140}$ (or the mathematical equivalent).

15. **D:** If the sandbox with a length of 5 feet has twice the area of the other sandbox (which has an area of $20 \times 5 = 100$ square feet), it must have an area of 200 square feet. Therefore its width must equal $\frac{200}{5} = 40$ feet. Converting its measurements to inches, the sandbox is $5 \times 12 = 60$ inches long and $40 \times 12 = 480$ inches wide. Since the formula for the perimeter of a rectangle is $2l + 2w$, the correct perimeter is $2(60) + 2(480) = 120 + 960 = 1{,}080$ inches.

16. **A:** The area of a square is given by the formula $A = s^2$, where s is the length of the square's side. The perimeter is given by $P = 4s$. Plugging these values into the equation given in the question, we can solve for n:

$$(s^2) = n \times (4s)^2$$

$$s^2 = 16s^2n$$

$$n = \frac{s^2}{16s^2} = \frac{1}{16}$$

Thus $n^2 = \left(\frac{1}{16}\right)^2 = \frac{1}{256}$.

17. **C:** In this question you are asked to compare the area of rectangle P with that of rectangle Q. Since the information given is the relative size of the length and width of rectangles P and Q with respect to those of rectangle R, we should try to evaluate the relative areas using rectangle R's dimensions as the starting point.

If we assign x to the length of rectangle R, and y to its width, we get an area for rectangle R of xy. Because the length and width of rectangle P are 30 percent less and 20 percent greater, respectively, than those of rectangle R, we can see that the length and width of rectangle P equal $(1 - 0.3)x = 0.7x$ and $(1 + 0.2)y = 1.2y$. Thus the area of rectangle P is $(0.7x)(1.2y) = 0.84xy$.

Similarly, the length and width of rectangle Q are 40 percent greater and 40 percent less, respectively, than those of rectangle R, so we can see that the length and width of rectangle Q equal $(1 + 0.4)x = 1.4x$ and $(1 - 0.4)y = 0.6y$. Thus the area of rectangle Q is $(1.4x)(0.6y) = 0.84xy$.

The rectangles P and Q have equal area. **The two quantities are therefore equal.**

18. **2,500:** The region represented by the smaller rectangle is the park. The sidewalk is the shaded region all around it. Because the sidewalk is 5 feet wide, the larger rectangle (including the sidewalk) is *ten* feet longer and wider than the park itself. Therefore the larger rectangle, which includes both the sidewalk and the park, is 160 feet long and 100 feet wide. The area covered by the sidewalk alone can be found by subtracting the area of the larger rectangle from that of the smaller rectangle:

Larger rectangle area: $160 \times 100 = 16{,}000$
Smaller rectangle area: $150 \times 90 = 13{,}500$

The sidewalk therefore covers a total area of $16{,}000 - 13{,}500 = 2{,}500$.

19. **C, D, F, and G:** Because a, b, and c are the interior angles of a triangle, $a + b + c = 180$. Therefore, knowing the value of $a + c$ would enable us to solve for b. Thus Choice D is correct. Similarly, b and g are supplementarty angles, so $b + g = 180$. Knowing g would enable us to solve for b, so Choice G is correct.

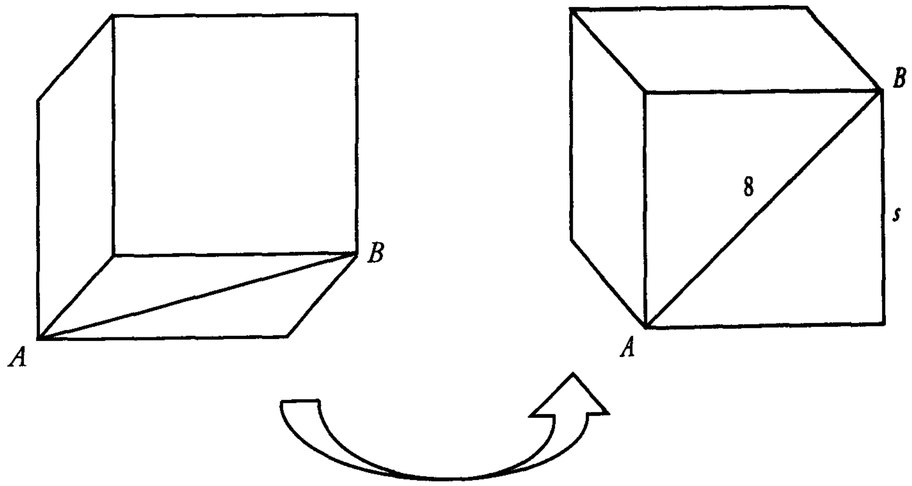
In addition, because a and d are opposite angles, as are b and e, and as are c and f, we know that $a = d$, $b = e$, and $c = f$. Thus if $a + c$ is sufficient to solve for b, so is $d + f$. Thus Choice F is correct. Finally, directly because $b = e$, Choice C is correct.

20. **B:** Plugging the circumference of the circle into the formula $C = 2\pi r$, we get $(12\pi) = 2\pi r$, so $r = 6$ and $AB = 6$. Triangle ABC is a 30–60–90 triangle, so the proportions of the sides must be $1 : \sqrt{3} : 2$. Therefore, $BC = (6) \times \frac{1}{2} = 3$ and $AC = (6)\times\frac{\sqrt{3}}{2} = 3\sqrt{3}$. The perimeter of Triangle ABC is therefore $6 + 3 + 3\sqrt{3} = 9 + 3\sqrt{3}$. Since $\sqrt{3} \approx 1.7$, we can estimate the perimeter to be $9 + 3(1.7) = 14.1$, which is larger than 14. (Note $\sqrt{3}$ that is actually larger than 1.7, so the perimeter will actually be slightly larger than 14.1.) This would be a good problem on which to use the GRE on-screen calculator.

Geometry: Hard Practice Question Solutions

1. **192:** Using the formula for the surface area of a cube (Area = $6s^2$), we can find the surface area by simply finding the length of one of the sides of the cube.

Flipping the cube so that the bottom of it is visible (and noting that this bottom surface is a square), we can see that line segment *AB* divides the square into two equal 45:45:90 triangles. Let's label one of the sides *s*:



Applying the rule that the sides of a 45:45:90 triangle are in the proportion $1:1:\sqrt{2}$, we can derive a formula to solve for *s*:

$$\frac{s}{8} = \frac{1}{\sqrt{2}}$$

$$\sqrt{2}s = 8$$

$$s = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

Therefore the area of the bottom square is $\left(4\sqrt{2}\right)^2 = 32$ and the surface area of the entire cube is 32 × 6 = 192.

2. **C**: Since point *A* is at the origin (0,0) and point *B* is at (3, 4), we can determine that *AB* = 5 by applying the distance formula (Pythagorean Theorem):

$$\text{Distance} = \sqrt{(\text{Change in } x \text{ coordinates})^2 + (\text{Change in } y \text{ coordinates})^2}.$$

In this case, Distance $= \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$. (We could also note that *AB* forms the hypotenuse of a 3–4–5 right triangle, by filling in a line segment from *B* to the *x*-axis.)

Since *AB* = 5, we know that *BC* = 2(5) = 10. Now we need to determine which of the listed points are **not** 10 units away from (3,4).

Choice E is the easiest to eliminate, as we can simply add 10 the *x* value of point *B* to arrive at the location of *C*. That line segment will be 10 units long.

To handle the other Choices, we need to calculate their distances from (3,4). Again, we can use the distance formula. For answer Choice A, $\sqrt{(3-(-5))^2 + (4-(-2))^2} = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$. (Note: this is a 6–8–10 right triangle. Recognizing this pattern saves a lot of calculation time!)

For answer Choice B, $\sqrt{(3-9)^2 + (4-12)^2} = \sqrt{6^2+8^2} = \sqrt{36+64} = \sqrt{100} = 10$. (Note: this also is a 6–8–10 right triangle!)

For answer Choice C, $\sqrt{(3-10)^2 + (4-11)^2} = \sqrt{7^2+7^2} = \sqrt{49+49} = \sqrt{98} = 7\sqrt{2}$. This does not equal 10, and so Choice C is the correct answer.

For Choice D, $\sqrt{(3-11)^2 + (4-10)^2} = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$. (Note: this is the same 6–8–10 triangle as in Choice B, but the legs have been switched!)

3. **A:** The measure of the angles created at the intersection of any two lines is a function of the relative slopes of the lines. Since we are given enough information to solve for the slope of each of the lines involved, we can eliminate Choice D immediately—we have enough information for a solution.

The slopes of lines M and N can be calculated directly since we are provided with equations that describe the lines. In the generic equation $y = mx + b$, m is the slope of the line and b is the y-intercept. Thus the slope of M is 3 and the slope of N is $\frac{5}{2}$ (we obtain the slope of line N by dividing both sides of the equation for N by 2). Since we know that line P contains the points (0,6) and (6,4), we can compute the slope of the line as follows:

Slope $\frac{\Delta y}{\Delta x} = \frac{(6-4)}{(0-6)} = -\frac{1}{3}$.

The slope of line P is the negative reciprocal of the slope of line M; *by definition this means that the lines are perpendicular*. It follows that every angle created by the intersection of lines M and P must be 90 degrees.

Since lines M and N are ***not*** perpendicular, we know that the intersection creates two angles of less than 90° and two angles of greater than 90°. Thus **Quantity A is larger.**

4. **C:** Because BE is parallel to CD, angle ABE is equal to angle ACD, and angle AEB is equal to angle ADC. This implies that triangles ABE and ACD are similar. Similar triangles have the property that ratios of the lengths of corresponding sides are equal. Consider the left sides: the length of the left side of triangle ABE is x, whereas the length of the left side of triangle ACD is $x + y$. Likewise, the right sides have lengths w and $w + z$, respectively. Therefore:

$$\frac{x+y}{x} = \frac{w+z}{w}$$

$$1+\frac{y}{x} = 1+\frac{z}{w}$$

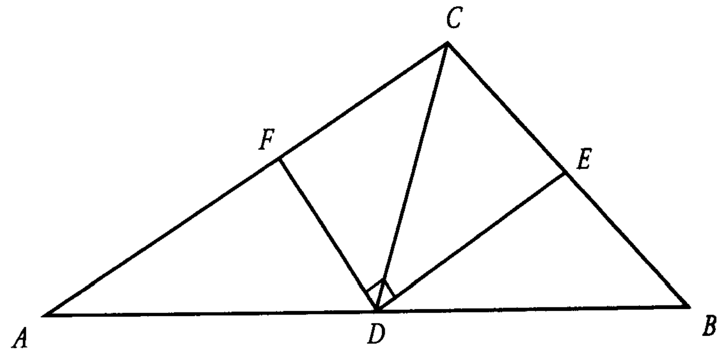
$$\frac{y}{x} = \frac{z}{w}$$

Cross-multiplying yields $xz = yw$. **The two quantities are equal.**

5. **A, B, and E:** First, because BE is $\frac{1}{2}$ of BC and BD is $\frac{1}{2}$ of AB (i.e., corresponding sides have proportional lengths), and angle ABC is shared between the two triangles, we can see that triangles ABC and DBE must be similar. Similar triangles have the further property that corresponding angles are of equal measure. Thus, for example, angle EDB equals angle CAB, and so ED is parallel to AC.

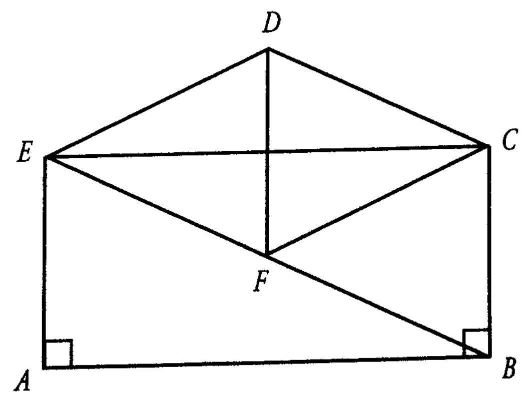
Next, looking at the two smaller triangles in the right half of the figure, we can see that triangle EDB and triangle CDE have collinear and equal "bases" (CE and BE), and share their third vertex (D). Due to the shared vertex, triangles EDB and CDE have the same height relative to bases CE and BE. This implies that EDB and CDE must have the same area. (For a similar reason, triangles ADC and BDC must have equal areas; more on that later.)

Choices C and D need not be true. The figure has been redrawn below so as to serve as a counterexample to both. In general, trying to "deform" the figure while remaining within the specified constraints is an effective way to discover which statements remain true and which do not.



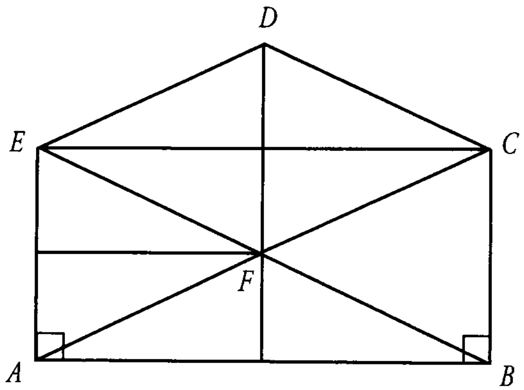
Lastly, because triangles ADC and BDC must have equal areas as indicated above, we can see that the area of triangle ABC must be twice that of triangle ACD. Note that ED is parallel to AC due to (true) Choice A, and DF is perpendicular to ED. DF must therefore be perpendicular to AC as well. Put differently, AC can be regarded as the base, and DF the height, of triangle ADC. The area of triangle ADC equals $\frac{1}{2}$ times $AC \times DF$. The area of triangle ABC, which is twice that of ADC, must therefore equal $AC \times DF$.

6. **C:** Because of the various parallel and equal length constraints, we can see that quadrilateral $EFCD$ is a rhombus. The area of a rhombus is $\frac{1}{2}$ the product of its diagonals, which are perpendicular to each other. Consider the picture below, where the two diagonals have been added:



We can see that $EC = AB$. Furthermore, $BCDF$ is a parallelogram, such that $DF = BC$, and since $BC = AE$, it must be true that $DF = AE$. The area of triangle ABE is given by $\frac{1}{2}(AB)(AE)$. The area of rhombus $EFCD$ is given by $\frac{1}{2}(EC)(DF)$. The two areas must therefore be equal.

A purely graphical explanation is also possible: Below is the same figure, augmented by additional lines drawn through triangle ABE. We can see that ABE and $EFCD$ are both comprised of four smaller, equal right triangles, so that they must have the same area.



7. **B:** The area of a rhombus is equal to $\frac{1}{2}$ times the product of its diagonals. One diagonal of $AFCE$ is AC, which is also a diagonal of the square, and has length $\sqrt{2}$. (We can use either the Pythagorean Theorem or the 45–45–90 triangle rule to prove this.) The length of the other diagonal, EF, can be determined as follows: both DF and BE are radii of quarter-circles, each of which has a side on the square which is also a radius. The sides of the square equal 1, so DF and BE must equal 1. Therefore, $DF + BE = 2$. DF and BE also lie along the other diagonal of the square, BD. The length of BD is equal to $\sqrt{2}$. The reason $DF + BE$ is greater than BD is that DF and BE overlap; the length of the overlap EF is counted twice. Thus, we can find the length of EF by subtracting the length of BD from the sum of DF and BE: $EF = 2 - \sqrt{2}$. Finally, we compute the area of $AFCE$ as follows:

$$\text{Area} = \frac{1}{2}(AC)(EF) = \frac{1}{2}\left(\sqrt{2}\right)\left(2-\sqrt{2}\right) = \frac{1}{2}\left(2\sqrt{2}-2\right) = \sqrt{2}-1.$$

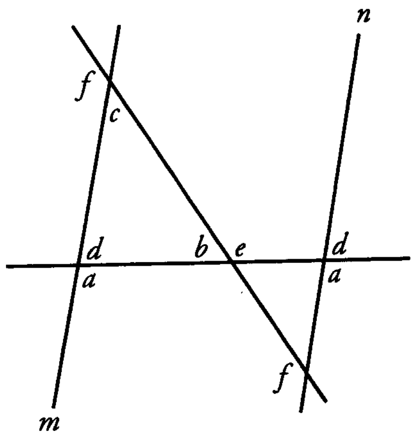
An alternative is to eliminate answer choices based on estimation. The area of $AFCE$ appears to be around one-half of the area of the square, which equals 1. Thus we would expect the value of the answer to be around 0.5. Seen in that light, only Choices A and B make sense.

8. **B:** First, we note that triangle ABD is a right triangle, because any triangle inscribed in a semicircle is a right triangle. Thus the area of ABD is one half the product of AD and BD, which are its perpendicular sides. AD is already given. In order to find BD, we can use the fact that the measure of angle DAB has to equal one half of the measure of angle DCB. This is because angle DAB is an inscribed angle of the circle and angle DCB is its corresponding central angle. Once we determine that angle DAB measures 30 degrees, we can identify triangle ABD as a 30–60–90 triangle. The ratio of AD to BD must therefore equal $\sqrt{3}:1$. Forming a proportion, we obtain $\frac{6}{BD} = \frac{\sqrt{3}}{1}$ or $BD = \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$. The area of ABD equals $\frac{1}{2}(6)\left(2\sqrt{3}\right) = 6\sqrt{3}$.

9. **B:** Because AB is parallel to CD, the measure of angle ACD is also 45 degrees. Angles CAB and ACD are both inscribed angles of the circle. The measures of the corresponding central angles are twice 45°, or 90° each. Therefore, taken together, minor arcs AD and BC make up 180° of the entire circle, leaving 180° for minor arcs AB and CD—and because AB is twice the length of CD, CD must measure 60°, while AB measures 120°. Therefore minor arc CD is $\frac{60}{360}=\frac{1}{6}$ of the entire circumference of the circle, which equals $2\pi r = 16\pi$. The length of CD is thus $\frac{16\pi}{6}=\frac{8\pi}{3}$.

10. **A:** The volume of a right circular cylinder is given by the formula $V = \pi r^2 h$. In this case, the volume equals $\pi(3^2)(3) = 27\pi$. Using the approximate value of $\pi = 3.14$ on the GRE Calculator, the volume is computed as $V = 3.14 \times 27 = 84.78$, which is greater than 84. (Note that since π is slightly larger than 3.14, the volume will be slightly larger than 84.78.)

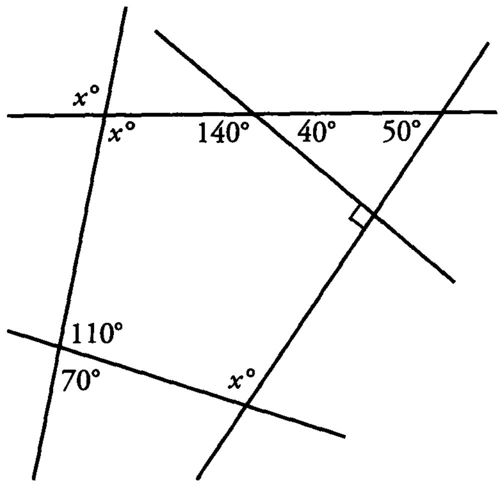
11. **C, D, E, and F:** Refer to the figure below, in which some of the angle labels have been replicated. This updated figure is based on the various angular equalities associated with parallel lines (lines m and n) being intersected by a transversal.



It can be seen that angles b and c individually have no relation to a, but they are "remote interior angles" in a triangle of which a is an exterior angle. Thus, by the exterior angle rule, the measure of a is equal to the sum of the measures of b and c. (This is true because $a + d = 180°$, and $b + c + d = 180°$, so $a = b + c$.) Thus, knowing the values of b and c is sufficient to find a, so Choice D is correct.

It can also be seen from the figure that a and d, b and e, and c and f are pairs of supplementary angles (they sum to 180°). In other words, knowing one angle's value provides the value of the other. Thus: d is sufficient to find a. Also, c and e is sufficient to find a (because e is sufficient to find b), and knowing b and f is sufficient (because f is sufficient to find c). Hence, Choices C, E, and F are correct.

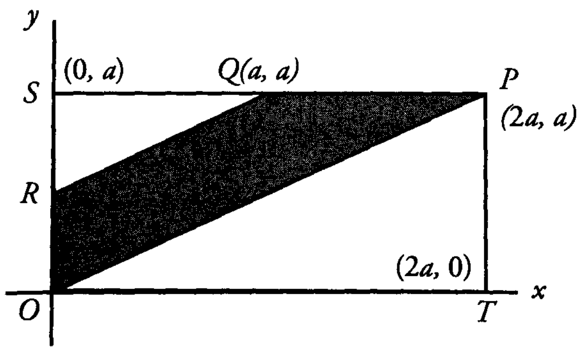
12. **100:** Some additional angle measures have been entered into the figure below.



The polygon in the center of the figure is a pentagon. The sum of the interior angles of an n-sided polygon is given by $(n - 2) \times 180°$. For a pentagon, this yields $(5 - 2) \times 180° = 540°$.

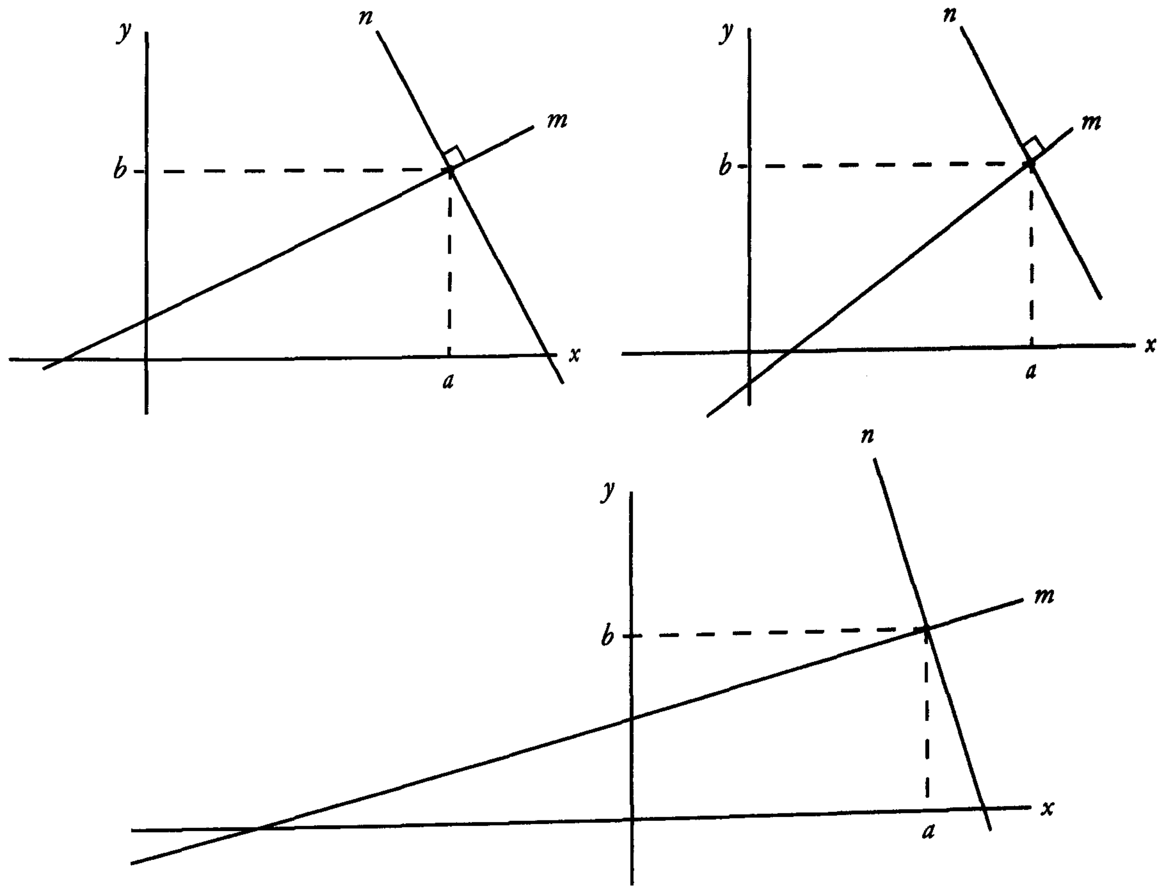
Thus: $x + x + 110 + 140 + 90 = 540$, or $2x = 200$. Therefore $x = 100$.

13. **C:** Consider the figure below, which has additional lines, labels, and coordinates filled in:

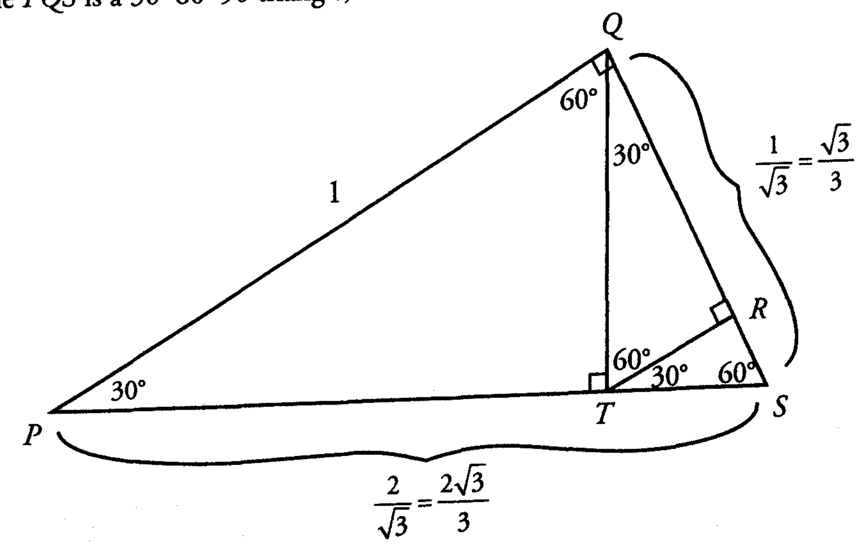


The area of trapezoid *OPQR* can be found by subtracting the areas of right triangles *SRQ* and *OTP* from the area of rectangle *OTPS*. (This allows us to arrive at the value of the shaded area in the figure, which is the original trapezoid.) Because *OPQR* is a trapezoid, *QR* is parallel to *OP* and has a slope equal to $\frac{a}{2a} = \frac{1}{2}$. The y-coordinate of *R* must therefore equal $\frac{a}{2}$. The area of rectangle *OTPS* is $2a \times a = 2a^2$, the area of triangle *SRQ* is $\frac{1}{2}(SR)(SQ) = \frac{1}{2}\left(\frac{a}{2}\right)(a) = \frac{a^2}{4}$, and the area of triangle *OTP* is $\frac{1}{2}(OT)(TP) = \frac{1}{2}(2a)(a) = a^2$. Thus the area of *OPQR* is $2a^2 - \frac{a^2}{4} - a^2 = \frac{3a^2}{4}$.

14. **C and E:** Consider the figures below, which illustrate various arrangements of the lines that satisfy the given constraints. It can be seen that the x- and y-intercepts of line n will always be positive. Those of line m can be positive or negative; however they always have opposite signs. That is why their product has to be negative. Lastly, even though it first appears that the x-intercept of line n would always be farther from zero than the x-intercept of line m, the last figure illustrates that this need not always be the case. Thus Choice F is not necessarily true.



15. **B:** Because angle $QPS = 30°$ and Triangle PQS is a right triangle, angle $RST = 60°$. Triangles PQT, QST, QRT, and RST are all also right triangles, and the angles at each point can be filled in as in the diagram below. Furthermore, triangle PQS is a 30–60–90 triangle, so the sides must be in the proportion $1:\sqrt{3}:2$:



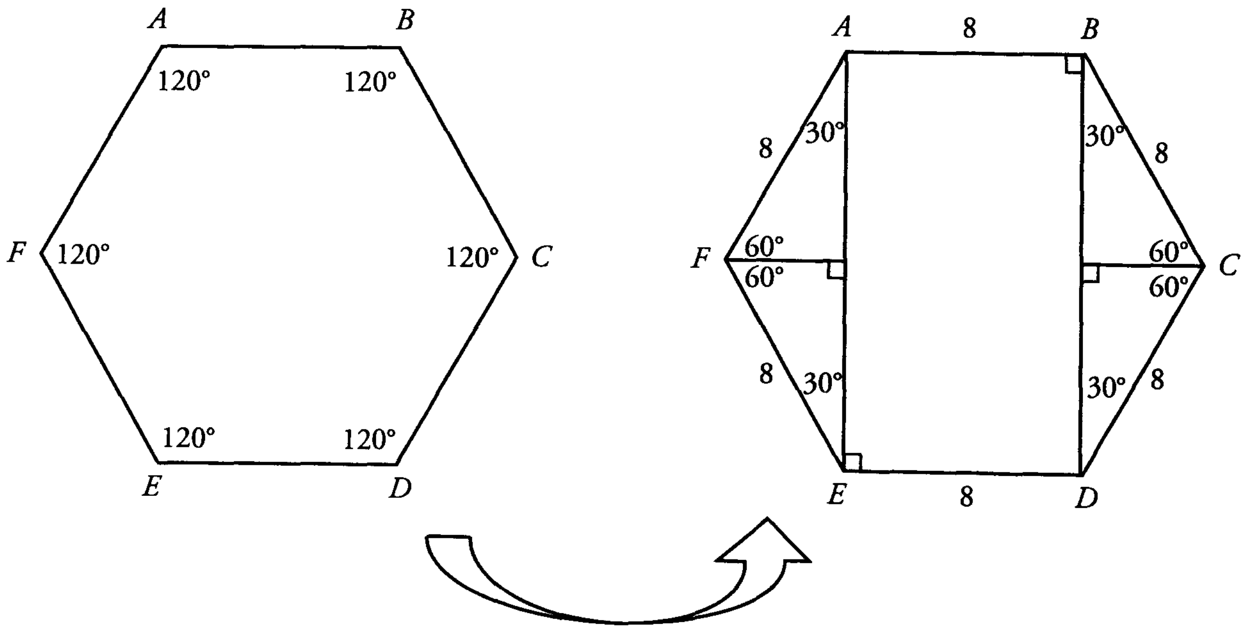
Next, each of these triangles share multiple sides and are all 30–60–90 triangles. Therefore, triangles *PQS, PQT, QST, QRT,* and *RST* are similar triangles, implying that the ratios of the respective sides (opposite the 30°, 60°, and 90° angles) are all in equal proportion:

$$\frac{PS}{PQ}=\frac{QS}{QT}\text{, so } QT=\frac{QS\times PQ}{PS}=\frac{\left(\frac{\sqrt{3}}{3}\right)\times(1)}{\left(\frac{2\sqrt{3}}{3}\right)}=\frac{1}{2}$$

$$\frac{PQ}{QT}=\frac{QS}{ST}\text{, so } ST=\frac{QS\times QT}{PQ}=\frac{\left(\frac{\sqrt{3}}{3}\right)\left(\frac{1}{2}\right)}{1}=\frac{\sqrt{3}}{6}$$

$$\frac{QS}{ST}=\frac{ST}{RS}\text{, so } RS=\frac{ST\times ST}{QS}=\frac{\left(\frac{\sqrt{3}}{6}\right)^2}{\left(\frac{\sqrt{3}}{3}\right)}=\frac{\frac{3}{36}}{\frac{\sqrt{3}}{3}}=\frac{1}{12}\times\frac{3}{\sqrt{3}}=\frac{1}{4\sqrt{3}}=\frac{4\sqrt{3}}{48}=\frac{\sqrt{3}}{12}.$$

16. **D:** We can start by drawing a regular hexagon. Since the sum of the interior angles of an *n*-sided polygon must equal $(n-2)\times 180°$, the sum of the angles will equal $(6-2)\times 180° = 720°$ and each angle will equal 120°. We can divide the hexagon into a rectangular piece and 4 right triangles as follows:



From this diagram we can determine that rectangle ABDE has a width of 8 and the 4 right triangles have hypotenuses of length 8. Because they are 30-60-90 triangles, the sides must be in the ratio of $1:\sqrt{3}:2$, so the short legs of the right triangles equal 4 and the long legs equal $4\sqrt{3}$. Thus the rectangle ABDE has a height of $2(4\sqrt{3})=8\sqrt{3}$ and thus an area of $8\times 8\sqrt{3}=64\sqrt{3}$.

Each of the four right triangles have a base of 4 and a height of $4\sqrt{3}$, so their area is $\frac{1}{2}(4)(4\sqrt{3})=8\sqrt{3}$, and since there are 4 of them, they contribute $4(8\sqrt{3})=32\sqrt{3}$ to the area of the hexagon.

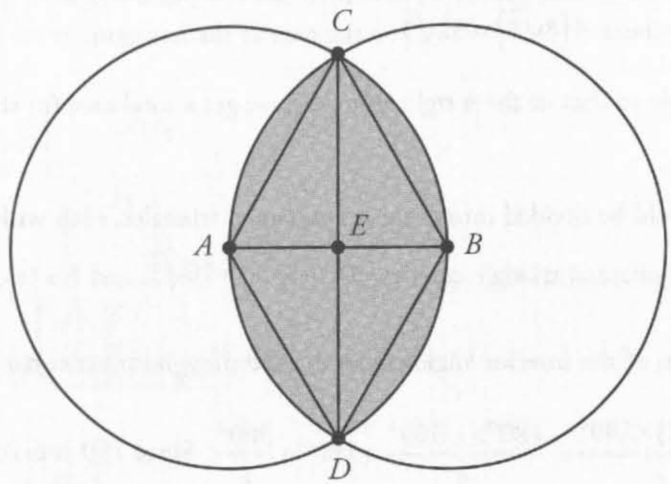
Adding the area of the rectangle to that of the 4 right triangles, we get a total area for the hexagon of $64\sqrt{3}+32\sqrt{3}=96\sqrt{3}$.

Note also that the hexagon could be divided into 6 equal equilateral triangles, each with a base of 8 and a height of $4\sqrt{3}$. The area of each such equilateral triangle equals $\frac{1}{2}(8)(4\sqrt{3})=16\sqrt{3}$, and $6\times16\sqrt{3}=96\sqrt{3}$.

17. **A, C, and E:** Since the sum of the interior angles of an *n*-sided polygon must equal $(n - 2) \times 180°$, the average of the angles will equal $\frac{(n-2)\times180°}{n}=\frac{180°n-360°}{n}=180°-\frac{360°}{n}$. Since 180 is divisible by 10, we need only evaluate whether $\frac{360}{n}$ is divisible by 10 for each answer choice:

- $\frac{360}{4}=90$, which is divisible by 10. In other words, 4-sided polygons will have an average angle of $180° - 90° = 90°$, which is divisible by 10.
- $\frac{360}{5}=72$, which is NOT divisible by 10. In other words, 5-sided polygons will have an average angle of $180° - 72° = 108°$, which is NOT divisible by 10.
- $\frac{360}{6}=60$, which is divisible by 10. In other words, 6-sided polygons will have an average angle of $180° - 60° = 120°$, which is divisible by 10.
- $\frac{360}{8}=45$, which is NOT divisible by 10. In other words, 8-sided polygons will have an average angle of $180° - 45° = 135°$, which is NOT divisible by 10.
- $\frac{360}{9}=40$, which is divisible by 10. In other words, 9-sided polygons will have an average angle of $180° - 40° = 140°$, which is divisible by 10.
- $\frac{360}{10}=36$, which is NOT divisible by 10. In other words, 10-sided polygons will have an average angle of $180° - 36° = 144°$, which is NOT divisible by 10.

18. **C**: The easiest way to solve this problem is to add a few lines to the diagram:



Because *AB, AC, AD, BC,* and *BD* are all radii of one of the circles, they are of equal length (all equal to *x*) and form two equilateral triangles: *ABC* and *ABD*. Therefore angles *ACB* and *ADB* are 60° each, and angles *CAD* and *CBD* all equal 120°. Line segment *CD* bisects angles *ACB* and *ADB* and is perpendicular to line segment *AB*. Therefore triangles *ACE, ADE, BCE,* and *BDE* are all 30–60–90 triangles.

Because angle *CAD* = 120°, it intercepts an arc equal to $\frac{1}{3}$ of the area of Circle *A*. The area of Circle *A* is given by πx^2, so the area of the sector intercepted by that angle is $\frac{\pi x^2}{3}$. Similarly, the area of the arc intercepted by angle *CBD* is $\frac{\pi x^2}{3}$. Adding these together, we get $\frac{2\pi x^2}{3}$. However, this measure double counts the four 30–60–90 triangles in the middle of the region. Therefore we must subtract out their areas.

Each of the 30–60–90 triangles has a hypotenuse of *x* and a short leg of $\frac{x}{2}$. Using the proportion $1:\sqrt{3}:2$ for the sides of a 30-60-90 triangle, the long legs of these 4 triangles equal $\frac{x}{\sqrt{3}} = \frac{x\sqrt{3}}{3}$. Therefore the area of each triangle equals $\frac{1}{2}bh = \frac{1}{2}\left(\frac{x\sqrt{3}}{2}\right)\left(\frac{x}{2}\right) = \frac{x^2\sqrt{3}}{8}$. Since there are 4 of them, the total area of the 30–60–90 triangles is

$\frac{4x^2\sqrt{3}}{8} = \frac{x^2\sqrt{3}}{2}$.

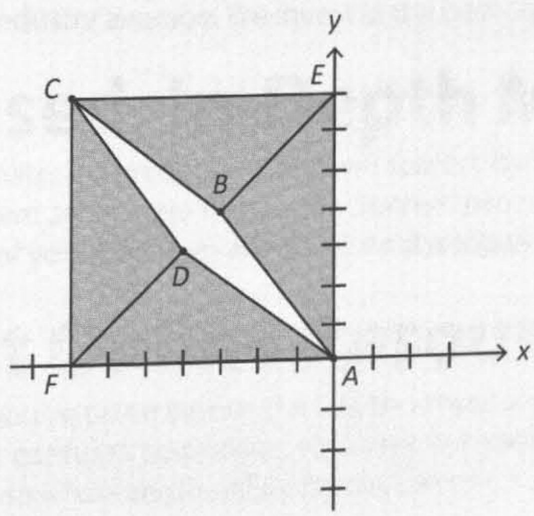
Therefore the total area of the shaded region equals $\frac{2\pi x^2}{3} - \frac{x^2\sqrt{3}}{2} = \frac{4\pi x^2 - 3\sqrt{3}x^2}{6} = \frac{\left(4\pi - 3\sqrt{3}\right)x^2}{6}$.

19. **B and C:** Choice A tells us that line n does not pass through any point that has a larger x-coordinate than its y-coordinate. Because line n passes through the origin, this means the line does not have a slope of $\frac{b}{a}$, where $a > b$—in other words, line n does not have a slope of *less* than 1. This does not answer the question, because line n could still have a slope of *exactly* 1, or a slope *greater than* 1.

Choice B, however, answers the question, because the slope of line n must equal the negative reciprocal of perpendicular line m, which is -1. Thus, line n has a slope of $-\left(\frac{1}{-1}\right) = 1$.

Choice C also answers the question—if c and d are consecutive integers and c is larger than d, then $c = d + 1$. Thus line n passes through the point (c, c), and thus has a slope of $\frac{c}{c} = 1$.

20. **C:** Perhaps the easiest way to solve this problem is to draw a rectangle around the parallelogram, find its area, and subtract out the area of the triangles that emerge around the parallelogram, within the rectangle (but that are not part of the parallelogram):



Since $ABCD$ is a parallelogram, line segments AB and CD have the same length and the same slope. Therefore, in the diagram above, point D is at $(-4, -3)$. Triangle ADF has a height of 3 and a base of 7. The same is true of triangles ABE, BCE, and CDF. Therefore each triangle has an area of $\frac{3 \times 7}{2} = \frac{21}{2}$. Since point C is at $(-7, -7)$ and A is at the origin $(0, 0)$, rectangle $AFCE$ has an area of $7 \times 7 = 49$. Therefore the area of parallelogram $ABCD$ equals

$$49 - 4\left(\frac{21}{2}\right) = 49 - 42 = 7.$$

Alternatively, you could note that parallelogram is a rhombus, with each side equal to $\sqrt{3^2 + 4^2} = 5$. The diagonals of the rhombus are of length $\sqrt{7^2 + 7^2} = 7\sqrt{2}$ and $\sqrt{1^2 + 1^2} = \sqrt{2}$. Using the formula for the area of a rhombus, $\frac{d_1 \times d_2}{2}$, we get $\frac{7\sqrt{2}}{2} \times \sqrt{2} = 7$.