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5 GRE® Quantitative Reasoning

Your goals for this chapter	
	Learn the four types of *GRE®* Quantitative Reasoning questions
	Get tips for answering each question type
	Study sample Quantitative Reasoning questions with solutions
	Learn how to use the on-screen calculator

Overview of the Quantitative Reasoning Measure

The Quantitative Reasoning measure of the GRE revised General Test assesses your:

- basic mathematical skills
- understanding of elementary mathematical concepts
- ability to reason quantitatively and to model and solve problems with quantitative methods

Some of the questions in the measure are posed in real-life settings, while others are posed in purely mathematical settings. The skills, concepts, and abilities are tested in the four content areas below.

Arithmetic topics include properties and types of integers, such as divisibility, factorization, prime numbers, remainders, and odd and even integers; arithmetic operations, exponents, and roots; and concepts such as estimation, percent, ratio, rate, absolute value, the number line, decimal representation, and sequences of numbers.

Algebra topics include operations with exponents; factoring and simplifying algebraic expressions; relations, functions, equations, and inequalities; solving linear and quadratic equations and inequalities; solving simultaneous equations and inequalities; setting up equations to solve word problems; and coordinate geometry, including graphs of functions, equations, and inequalities, intercepts, and slopes of lines.

Geometry topics include parallel and perpendicular lines, circles, triangles—including isosceles, equilateral, and 30°-60°-90° triangles—quadrilaterals, other polygons, congruent and similar figures, three-dimensional figures, area, perimeter, volume, the Pythagorean theorem, and angle measurement in degrees. The ability to construct proofs is not tested.

Data analysis topics include basic descriptive statistics, such as mean, median, mode, range, standard deviation, interquartile range, quartiles, and percentiles; interpretation of data in tables and graphs, such as line graphs, bar graphs, circle graphs, boxplots, scatterplots, and frequency distributions; elementary probability, such as probabilities of compound events and independent events; random variables and probability distributions, including normal distributions; and counting methods, such as combinations, permutations, and Venn diagrams. These topics are typically taught in high school algebra courses or introductory statistics courses. Inferential statistics is not tested.

The content in these areas includes high school mathematics and statistics at a level that is generally no higher than a second course in algebra; it does not include trigonometry, calculus, or other higher-level mathematics. The publication *Math Review for the GRE revised General Test*, which is available at **www.ets.org/gre/prepare**, provides detailed information about the content of the Quantitative Reasoning measure. The *Math Review* is Chapter 7 in this book.

The mathematical symbols, terminology, and conventions used in the Quantitative Reasoning measure are those that are standard at the high school level. For example, the positive direction of a number line is to the right, distances are nonnegative, and prime numbers are greater than 1. Whenever nonstandard notation is used in a question, it is explicitly introduced in the question.

In addition to conventions, there are some assumptions about numbers and geometric figures that are used in the Quantitative Reasoning measure. Two of these assumptions are (i) all numbers used are real numbers and (ii) geometric figures are not necessarily drawn to scale. More about conventions and assumptions appears in the publication *Mathematical Conventions for the GRE revised General Test*, which is available at **www.ets.org/gre/prepare** and at the end of this chapter.

Quantitative Reasoning Question Types

The Quantitative Reasoning measure has four types of questions:

- Quantitative Comparison questions
- Multiple-choice questions—Select One Answer Choice
- Multiple-choice questions—Select One or More Answer Choices
- Numeric Entry questions

Each question appears either independently as a discrete question or as part of a set of questions called a Data Interpretation set. All of the questions in a Data Interpretation set are based on the same data presented in tables, graphs, or other displays of data.

In the computer-based test, you are allowed to use a basic calculator—provided on-screen—on the Quantitative Reasoning measure. Information about using the calculator appears later in this chapter.

For those taking the paper-based test, handheld calculators will be provided at the test center for use during the test. Information about using the handheld calculator to help you answer questions appears in the free *Practice Book for the Paper-based GRE revised General Test,* which is available at **www.ets.org/gre/prepare.**

Quantitative Comparison Questions

Description

Questions of this type ask you to compare two quantities—Quantity A and Quantity B—and then determine which of the following statements describes the comparison.

- Quantity A is greater.
- Quantity B is greater.
- The two quantities are equal.
- The relationship cannot be determined from the information given.

Tips for Answering

- **Become familiar with the answer choices**. Quantitative Comparison questions always have the same answer choices, so get to know them, especially the last choice, "The relationship cannot be determined from the information given." Never select this last choice if it is clear that the values of the two quantities can be determined by computation. Also, if you determine that one quantity is greater than the other, make sure you carefully select the corresponding choice so as not to reverse the first two choices.
- **Avoid unnecessary computations.** Don't waste time performing needless computations in order to compare the two quantities. Simplify, transform, or estimate one or both of the given quantities only as much as is necessary to compare them.
- **Remember that geometric figures are not necessarily drawn to scale**. If any aspect of a given geometric figure is not fully determined, try to redraw the figure, keeping those aspects that are completely determined by the given information fixed but changing the aspects of the figure that are not determined. Examine the results. What variations are possible in the relative lengths of line segments or measures of angles?
- **Plug in numbers**. If one or both of the quantities are algebraic expressions, you can substitute easy numbers for the variables and compare the resulting quantities in your analysis. Consider all kinds of appropriate numbers before you give an answer: e.g., zero, positive and negative numbers, small and large numbers, fractions and decimals. If you see that Quantity A is greater than Quantity B in one case and Quantity B is greater than Quantity A in another case, choose "The relationship cannot be determined from the information given."
- **Simplify the comparison.** If both quantities are algebraic or arithmetic expressions and you cannot easily see a relationship between them, you can try to simplify the comparison. Try a step-by-step simplification that is similar to the steps involved when you solve the equation $5 = 4x + 3$ for x, or similar to the steps involved when you determine that the inequality $\frac{3y+2}{5} < y$ is equivalent to the simpler inequality $1 < y$. Begin by setting up a comparison involving the two quantities, as follows:

  $$\text{Quantity A} \boxed{?} \text{Quantity B}$$

  where $\boxed{?}$ is a "placeholder" that could represent the relationship *greater than* ($>$), *less than* ($<$), or *equal to* ($=$) or could represent the fact that the relationship cannot be determined from the information given. Then try to simplify the comparison, step by step, until you can determine a relationship between simplified quantities. For example, you may conclude after the last step that $\boxed{?}$ represents equal to ($=$). Based on this conclusion, you may be able to compare Quantities A and B. To understand this strategy more fully, see sample questions 6 to 9.

Sample Questions

Compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given, and select one of the following four answer choices:

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B
1.	The least prime number greater than 24	The greatest prime number less than 28

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the information given.

Explanation

For the integers greater than 24, note that 25, 26, 27, and 28 are not prime numbers, but 29 is a prime number, as are 31 and many other greater integers. Thus, 29 is the least prime number greater than 24, and Quantity A is 29. For the integers less than 28, note that 27, 26, 25, and 24 are not prime numbers, but 23 is a prime number, as are 19 and several other lesser integers. Thus, 23 is the greatest prime number less than 28, and Quantity B is 23. **The correct answer is Choice A, Quantity A is greater.**

Lionel is younger than Maria.

	Quantity A	Quantity B
2.	Twice Lionel's age	Maria's age

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the information given.

Explanation

If Lionel's age is 6 years and Maria's age is 10 years, then Quantity A is greater, but if Lionel's age is 4 years and Maria's age is 10 years, then Quantity B is greater. Thus, the relationship cannot be determined.

The correct answer is Choice D, the relationship cannot be determined from the information given.

	Quantity A	Quantity B
3.	54% of 360	150

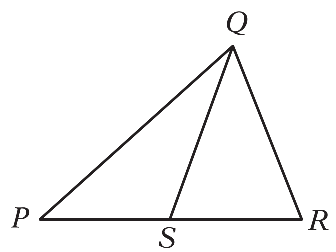
- (A) Quantity A is greater.
- (B) Quantity B is greater.
- (C) The two quantities are equal.
- (D) The relationship cannot be determined from the information given.

Explanation

Without doing the exact computation, you can see that 54 percent of 360 is greater than $\frac{1}{2}$ of 360, which is 180, and 180 is greater than Quantity B, 150.

Thus, the correct answer is Choice A, Quantity A is greater.

Figure 1



$PQ = PR$

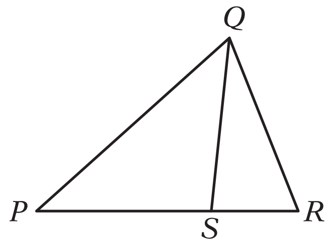
	Quantity A	Quantity B
4.	PS	SR

- (A) Quantity A is greater.
- (B) Quantity B is greater.
- (C) The two quantities are equal.
- (D) The relationship cannot be determined from the information given.

Explanation

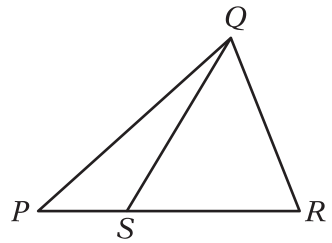
From Figure 1, you know that PQR is a triangle and that point S is between points P and R, so $PS < PR$ and $SR < PR$. You are also given that $PQ = PR$. However, this information is not sufficient to compare PS and SR. Furthermore, because the figure is not necessarily drawn to scale, you cannot determine the relative sizes of PS and SR visually from the figure, though they may appear to be equal. The position of S can vary along side PR anywhere between P and R. Following are two possible variations of Figure 1, each of which is drawn to be consistent with the information $PQ = PR$.

Figure 2



$PQ = PR$

Figure 3



$PQ = PR$

Note that Quantity A is greater in Figure 2 and Quantity B is greater in Figure 3.

Thus, the correct answer is Choice D, the relationship cannot be determined from the information given.

$$y = 2x^2 + 7x - 3$$

	Quantity A	Quantity B
5.	x	y

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

Explanation

If $x = 0$, then $y = 2(0^2) + 7(0) - 3 = -3$, so in this case, $x > y$; but if $x = 1$, then $y = 2(1^2) + 7(1) - 3 = 6$, so in that case, $y > x$.

Thus, the correct answer is Choice D, the relationship cannot be determined from the information given.

Note that plugging numbers into expressions *may not* be conclusive. It *is* conclusive, however, if you get different results after plugging in different numbers: the conclusion is that the relationship cannot be determined from the information given. It is also conclusive if there are only a small number of possible numbers to plug in and all of them yield the same result, say, that Quantity B is greater.

Now suppose that there are an infinite number of possible numbers to plug in. If you plug many of them in and each time the result is, for example, that Quantity A is greater, you still cannot conclude that Quantity A is greater for every possible number that could be plugged in. Further analysis would be necessary and should focus on whether Quantity A is greater for all possible numbers or whether there are numbers for which Quantity A is not greater.

The following sample questions focus on simplifying the comparison.

$$y > 4$$

	Quantity A	Quantity B
6.	$\frac{3y+2}{5}$	y

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the information given.

Explanation

Set up the initial comparison:

$$\frac{3y+2}{5} \boxed{?}\, y$$

Then simplify:

Step 1: Multiply both sides by 5 to get

$$3y + 2 \boxed{?}\, 5y$$

Step 2: Subtract $3y$ from both sides to get

$$2 \boxed{?}\, 2y$$

Step 3: Divide both sides by 2 to get

$$1 \boxed{?}\, y$$

The comparison is now simplified as much as possible. In order to compare 1 and y, note that you are given the information $y > 4$ (above Quantities A and B). It follows from $y > 4$ that $y > 1$, or $1 < y$, so that in the comparison $1 \boxed{?}\, y$, the placeholder $\boxed{?}$ represents *less than* ($<$): $1 < y$.

However, the problem asks for a comparison between Quantity A and Quantity B, not a comparison between 1 and y. To go from the comparison between 1 and y to a comparison between Quantities A and B, start with the last comparison, $1 < y$, and carefully consider each simplification step in reverse order to determine what each comparison implies about the preceding comparison, all the way back to the comparison between Quantities A and B if possible. Since step 3 was "*divide* both sides by 2," *multiplying* both sides of the comparison $1 < y$ by 2 implies the preceding comparison $2 < 2y$, thus reversing step 3. Each simplification step can be reversed as follows:

- Reverse step 3: *multiply* both sides by 2.
- Reverse step 2: *add* $3y$ to both sides.
- Reverse step 1: *divide* both sides by 5.

When each step is reversed, the relationship remains *less than* ($<$), so Quantity A is less than Quantity B.

Thus, the correct answer is Choice B, Quantity B is greater.

While some simplification steps like subtracting 3 from both sides or dividing both sides by 10 are always reversible, it is important to note that some steps, like squaring both sides, may not be reversible.

Also, note that when you simplify an *inequality*, the steps of multiplying or dividing both sides by a negative number change the direction of the inequality; for example, if $x < y$, then $-x > -y$. So the relationship in the final, simplified inequality may be the *opposite* of the relationship between Quantities A and B. This is another reason to consider the impact of each step carefully.

	Quantity A	Quantity B
7.	$\dfrac{2^{30} - 2^{29}}{2}$	2^{28}

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

Explanation

Set up the initial comparison:

$$\frac{2^{30} - 2^{29}}{2} \boxed{?} 2^{28}$$

Then simplify:

Step 1: Multiply both sides by 2 to get

$$2^{30} - 2^{29} \boxed{?} 2^{29}$$

Step 2: Add 2^{29} to both sides to get

$$2^{30} \boxed{?} 2^{29} + 2^{29}$$

Step 3: Simplify the right-hand side using the fact that $(2)(2^{29}) = 2^{30}$ to get

$$2^{30} \boxed{?} 2^{30}$$

The resulting relationship is *equal to* (=). In reverse order, each simplification step implies *equal to* in the preceding comparison. So Quantities A and B are also equal.

Thus, the correct answer is Choice C, the two quantities are equal.

	Quantity A	Quantity B
8.	$x^2 + 1$	$2x - 1$

- (A) Quantity A is greater.
- (B) Quantity B is greater.
- (C) The two quantities are equal.
- (D) The relationship cannot be determined from the information given.

Explanation

Set up the initial comparison:

$$x^2 + 1 \; \boxed{?} \; 2x - 1$$

Then simplify by noting that the quadratic polynomial $x^2 - 2x + 1$ can be factored:

Step 1: Subtract $2x$ from both sides to get

$$x^2 - 2x + 1 \; \boxed{?} \; -1$$

Step 2: Factor the left-hand side to get

$$(x - 1)^2 \; \boxed{?} \; -1$$

The left-hand side of the comparison is the square of a number. Since the square of a number is always greater than or equal to 0, and 0 is greater than -1, the simplified comparison is the inequality $(x - 1)^2 > -1$ and the resulting relationship is *greater than* ($>$). In reverse order, each simplification step implies the inequality *greater than* ($>$) in the preceding comparison. Therefore, Quantity A is greater than Quantity B.

The correct answer is Choice A, Quantity A is greater.

$$w > 1$$

	Quantity A	Quantity B
9.	$7w - 4$	$2w + 5$

- (A) Quantity A is greater.
- (B) Quantity B is greater.
- (C) The two quantities are equal.
- (D) The relationship cannot be determined from the information given.

Explanation

Set up the initial comparison:

$$7w - 4 \boxed{?} 2w + 5$$

Then simplify:

Step 1: Subtract $2w$ from both sides and add 4 to both sides to get

$$5w \boxed{?} 9$$

Step 2: Divide both sides by 5 to get

$$w \boxed{?} \frac{9}{5}$$

The comparison cannot be simplified any further. Although you are given that $w > 1$, you still don't know how w compares to $\frac{9}{5}$, or 1.8. For example, if $w = 1.5$, then $w < 1.8$, but if $w = 2$, then $w > 1.8$. In other words, the relationship between w and $\frac{9}{5}$ cannot be determined. Note that each of these simplification steps is reversible, so in reverse order, each simplification step implies that the *relationship cannot be determined* in the preceding comparison. Thus, the relationship between Quantities A and B cannot be determined.

The correct answer is Choice D, the relationship cannot be determined from the information given.

The strategy of simplifying the comparison works most efficiently when you note that a simplification step is reversible while actually taking the step. Here are some common steps that are always reversible:

- Adding any number or expression to both sides of a comparison
- Subtracting any number or expression from both sides
- Multiplying both sides by any nonzero number or expression
- Dividing both sides by any nonzero number or expression

Remember that if the relationship is an inequality, multiplying or dividing both sides by any *negative* number or expression will yield the opposite inequality. Be aware that some common operations like squaring both sides are generally not reversible and may require further analysis using other information given in the question in order to justify reversing such steps.

Multiple-choice Questions—Select One Answer Choice

Description

These questions are multiple-choice questions that ask you to select only one answer choice from a list of five choices.

Tips for Answering

- **Use the fact that the answer is there.** If your answer is not one of the five answer choices given, you should assume that your answer is incorrect and do the following:
  - Reread the question carefully—you may have missed an important detail or misinterpreted some information.
  - Check your computations—you may have made a mistake, such as mis-keying a number on the calculator.
  - Reevaluate your solution method—you may have a flaw in your reasoning.
- **Examine the answer choices.** In some questions you are asked explicitly which of the choices has a certain property. You may have to consider each choice separately, or you may be able to see a relationship between the choices that will help you find the answer more quickly. In other questions, it may be helpful to work backward from the choices, say, by substituting the choices in an equation or inequality to see which one works. However, be careful, as that method may take more time than using reasoning.
- **For questions that require approximations, scan the answer choices to see how close an approximation is needed.** In other questions, too, it may be helpful to scan the choices briefly before solving the problem to get a better sense of what the question is asking. If computations are involved in the solution, it may be necessary to carry out all computations exactly and round only your final answer in order to get the required degree of accuracy. In other questions, you may find that estimation is sufficient and will help you avoid spending time on long computations.

Sample Questions

Select a single answer choice.

1. If $5x + 32 = 4 - 2x$, what is the value of x ?

   (A) -4
   (B) -3
   (C) 4
   (D) 7
   (E) 12

Explanation

Solving the equation for x, you get $7x = -28$, and so $x = -4$. **The correct answer is Choice A, -4.**

2. Which of the following numbers is farthest from the number 1 on the number line?

Ⓐ −10
Ⓑ −5
Ⓒ 0
Ⓓ 5
Ⓔ 10

Explanation

Circling each of the answer choices in a sketch of the number line (Figure 4) shows that of the given numbers, −10 is the greatest distance from 1.

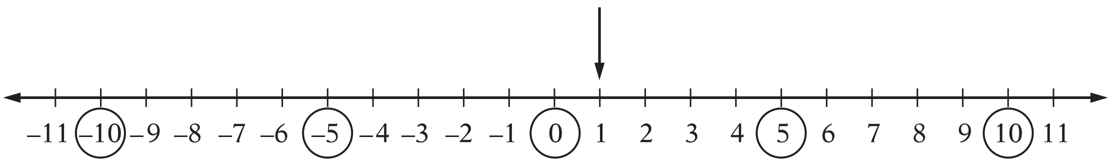


Figure 4

Another way to answer the question is to remember that the distance between two numbers on the number line is equal to the absolute value of the difference of the two numbers. For example, the distance between −10 and 1 is $|-10 - 1| = 11$, and the distance between 10 and 1 is $|10 - 1| = |9| = 9$. **The correct answer is Choice A, −10.**

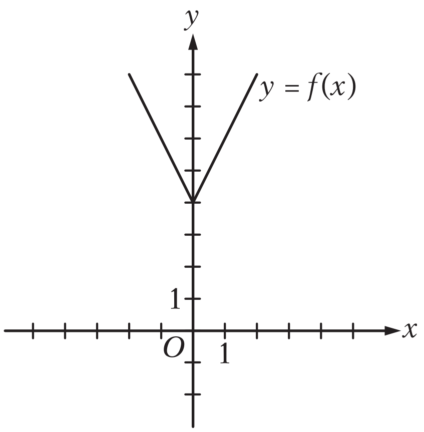


Figure 5

3. The figure above shows the graph of the function f defined by $f(x) = |2x| + 4$ for all numbers x. For which of the following functions g, defined for all numbers x, does the graph of g intersect the graph of f ?

Ⓐ $g(x) = x - 2$
Ⓑ $g(x) = x + 3$
Ⓒ $g(x) = 2x - 2$
Ⓓ $g(x) = 2x + 3$
Ⓔ $g(x) = 3x - 2$

Explanation

You can see that all five choices are linear functions whose graphs are lines with various slopes and y-intercepts. The graph of Choice A is a line with slope 1 and y-intercept -2, shown in Figure 6.

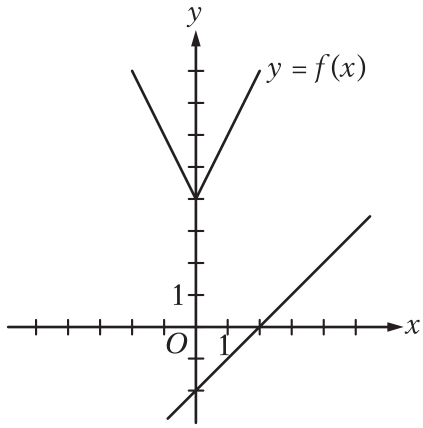


Figure 6

It is clear that this line will not intersect the graph of f to the left of the y-axis. To the right of the y-axis, the graph of f is a line with slope 2, which is greater than slope 1. Consequently, as the value of x increases, the value of y increases faster for f than for g, and therefore the graphs do not intersect to the right of the y-axis. Choice B is similarly ruled out. Note that if the y-intercept of either of the lines in Choices A and B were greater than or equal to 4 instead of less than 4, they would intersect the graph of f.

Choices C and D are lines with slope 2 and y-intercepts less than 4. Hence, they are parallel to the graph of f (to the right of the y-axis) and therefore will not intersect it. Any line with a slope greater than 2 and a y-intercept less than 4, like the line in Choice E, will intersect the graph of f (to the right of the y-axis).

The correct answer is Choice E, $g(x) = 3x - 2$.

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4. A car got 33 miles per gallon using gasoline that cost \$2.95 per gallon. Approximately what was the cost, in dollars, of the gasoline used in driving the car 350 miles?

   (A) \$10
   (B) \$20
   (C) \$30
   (D) \$40
   (E) \$50

Explanation

Scanning the answer choices indicates that you can do at least some estimation and still answer confidently. The car used $\frac{350}{33}$ gallons of gasoline, so the cost was $\left(\frac{350}{33}\right)(2.95)$ dollars. You can estimate the product $\left(\frac{350}{33}\right)(2.95)$ by estimating $\frac{350}{33}$ a little low, 10, and estimating 2.95 a little high, 3, to get approximately $(10)(3) = 30$ dollars. You can also use the calculator to compute a more exact answer and then round the answer to the nearest 10 dollars, as suggested by the answer choices. The calculator yields the decimal 31.287..., which rounds to 30 dollars.

Thus, the correct answer is Choice C, $30.

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5. A certain jar contains 60 jelly beans—22 white, 18 green, 11 yellow, 5 red, and 4 purple. If a jelly bean is to be chosen at random, what is the probability that the jelly bean will be neither red nor purple?

   Ⓐ 0.09
   Ⓑ 0.15
   Ⓒ 0.54
   Ⓓ 0.85
   Ⓔ 0.91

Explanation

Since there are 5 red and 4 purple jelly beans in the jar, there are 51 that are neither red nor purple, and the probability of selecting one of these is $\frac{51}{60}$. Since all of the answer choices are decimals, you must convert the fraction to its decimal equivalent, 0.85.

Thus, the correct answer is Choice D, 0.85.

Multiple-choice Questions—Select One or More Answer Choices

Description

These questions are multiple-choice questions that ask you to select one or more answer choices from a list of choices. A question may or may not specify the number of choices to select. These questions are marked with square boxes beside the answer choices, not circles or ovals.

Tips for Answering

- **Note whether you are asked to indicate a specific number of answer choices or all choices that apply.** In the latter case, be sure to consider all of the choices, determine which ones are correct, and select all of those and only those choices. Note that there may be only one correct choice.
- **In some questions that involve conditions that limit the possible values of numerical answer choices, it may be efficient to determine the least and/or the greatest possible value.** Knowing the least and/or greatest possible value may enable you to quickly determine all of the choices that are correct.
- **Avoid lengthy calculations by recognizing and continuing numerical patterns.**

Sample Questions

Select one or more answer choices according to the specific question directions.

If the question does not specify how many answer choices to select, select all that apply.

- **The correct answer may be just one of the choices or as many as all of the choices, depending on the question.**
- **No credit is given unless you select all of the correct choices and no others.**

If the question specifies how many answer choices to select, select exactly that number of choices.

1. Which two of the following numbers have a product that is between -1 and 0 ?

   Indicate both of the numbers.

   A -20
   B -10
   C 2^{-4}
   D 3^{-2}

Explanation

For this question, you must select a pair of answer choices. The product of the pair must be negative, so the possible products are $(-20)(2^{-4})$, $(-20)(3^{-2})$, $(-10)(2^{-4})$, and $(-10)(3^{-2})$. The product must also be greater than -1. The first product is $\frac{-20}{2^4} = -\frac{20}{16} < -1$, the second product is $\frac{-20}{3^2} = -\frac{20}{9} < -1$, and third product is $\frac{-10}{2^4} = -\frac{10}{16} > -1$, so you can stop there.

The correct answer consists of Choices B (-10) and C (2^{-4}).

2. Which of the following integers are multiples of both 2 and 3 ?

   Indicate all such integers.

   A 8
   B 9
   C 12
   D 18
   E 21
   F 36

Explanation

You can first identify the multiples of 2, which are 8, 12, 18, and 36, and then among the multiples of 2 identify the multiples of 3, which are 12, 18, and 36. Alternatively, if you realize that every number that is a multiple of 2 and 3 is also a multiple of 6, you can check which choices are multiples of 6.

The correct answer consists of Choices C (12), D (18), and F (36).

3. Each employee of a certain company is in either Department X or Department Y, and there are more than twice as many employees in Department X as in Department Y. The average (arithmetic mean) salary is \$25,000 for the employees in Department X and \$35,000 for the employees in Department Y. Which of the following amounts could be the average salary for all of the employees of the company?

   Indicate all such amounts.

   A \$26,000
   B \$28,000
   C \$29,000
   D \$30,000
   E \$31,000
   F \$32,000
   G \$34,000

Explanation

One strategy for answering this kind of question is to find the least and/or greatest possible value. Clearly the average salary is between \$25,000 and \$35,000, and all of the answer choices are in this interval. Since you are told that there are more employees with the lower average salary, the average salary of all employees must be less than the average of \$25,000 and \$35,000, which is \$30,000. If there were exactly twice as many employees in Department X as in Department Y, then the average salary for all employees would be, to the nearest dollar, the following weighted mean,

$$\frac{(2)(25{,}000) + (1)(35{,}000)}{2 + 1} \approx 28{,}333 \text{ dollars}$$

where the weight for \$25,000 is 2 and the weight for \$35,000 is 1. Since there are *more* than twice as many employees in Department X as in Department Y, the actual average salary must be even closer to \$25,000 because the weight for \$25,000 is greater than 2. This means that \$28,333 is the greatest possible average. Among the choices given, the possible values of the average are therefore \$26,000 and \$28,000.

Thus, the correct answer consists of Choices A (\$26,000) and B (\$28,000).

Intuitively, you might expect that any amount between \$25,000 and \$28,333 is a possible value of the average salary. To see that \$26,000 is possible, in the weighted mean above, use the respective weights 9 and 1 instead of 2 and 1. To see that \$28,000 is possible, use the respective weights 7 and 3.

4. Which of the following could be the units digit of 57^n, where n is a positive integer?

   Indicate <u>all</u> such digits.

   $\boxed{\text{A}}$ 0
   $\boxed{\text{B}}$ 1
   $\boxed{\text{C}}$ 2
   $\boxed{\text{D}}$ 3
   $\boxed{\text{E}}$ 4
   $\boxed{\text{F}}$ 5
   $\boxed{\text{G}}$ 6
   $\boxed{\text{H}}$ 7
   $\boxed{\text{I}}$ 8
   $\boxed{\text{J}}$ 9

Explanation

The units digit of 57^n is the same as the units digit of 7^n for all positive integers n. To see why this is true for $n = 2$, compute 57^2 by hand and observe how its units digit results from the units digit of 7^2. Because this is true for every positive integer n, you need to consider only powers of 7. Beginning with $n = 1$ and proceeding consecutively, the units digits of 7, 7^2, 7^3, 7^4, and 7^5 are 7, 9, 3, 1, and 7, respectively. In this sequence, the first digit, 7, appears again, and the pattern of four digits, 7, 9, 3, 1, repeats without end. Hence, these four digits are the only possible units digits of 7^n and therefore of 57^n.

The correct answer consists of Choices B (1), D (3), H (7), and J (9).

Numeric Entry Questions

Description

Questions of this type ask you either to enter your answer as an integer or a decimal in a single answer box or to enter it as a fraction in two separate boxes—one for the numerator and one for the denominator. In the computer-based test, use the computer mouse and keyboard to enter your answer.

Tips for Answering

- **Make sure you answer the question that is asked.** Since there are no answer choices to guide you, read the question carefully and make sure you provide the type of answer required. Sometimes there will be labels before or after the answer box to indicate the appropriate type of answer. Pay special attention to units such as feet or miles, to orders of magnitude such as millions or billions, and to percents as compared with decimals.
- **If you are asked to round your answer, make sure you round to the required degree of accuracy.** For example, if an answer of 46.7 is to be rounded to the nearest integer, you need to enter the number 47. If your solution strategy involves intermediate computations, you should carry out all computations exactly and round only your final answer in order to get the required degree of accuracy. If no rounding instructions are given, enter the exact answer.

- **Examine your answer to see if it is reasonable with respect to the information given**. You may want to use estimation or another solution path to double-check your answer.

Sample Questions

Enter your answer as an integer or a decimal if there is a single answer box OR as a fraction if there are two separate boxes—one for the numerator and one for the denominator.

To enter an integer or a decimal, either type the number in the answer box using the keyboard or use the Transfer Display button on the calculator.

- **First, click on the answer box—a cursor will appear in the box—and then type the number.**
- **To erase a number, use the Backspace key.**
- **For a negative sign, type a hyphen. For a decimal point, type a period.**
- **To remove a negative sign, type the hyphen again and it will disappear; the number will remain.**
- **The Transfer Display button on the calculator will transfer the calculator display to the answer box.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct.**
- **Enter the exact answer unless the question asks you to round your answer.**

To enter a fraction, type the numerator and the denominator in the respective boxes using the keyboard.

- **For a negative sign, type a hyphen; to remove it, type the hyphen again. A decimal point cannot be used in a fraction.**
- **The Transfer Display button on the calculator cannot be used for a fraction.**
- **Fractions do not need to be reduced to lowest terms, though you may need to reduce your fraction to fit in the boxes.**

1. One pen costs \$0.25 and one marker costs \$0.35. At those prices, what is the total cost of 18 pens and 100 markers?

$ [ ]

Explanation

Multiplying \$0.25 by 18 yields \$4.50, which is the cost of the 18 pens; and multiplying \$0.35 by 100 yields \$35.00, which is the cost of the 100 markers. The total cost is therefore \$4.50 + \$35.00 = \$39.50. Equivalent decimals, such as \$39.5 or \$39.500, are considered correct.

Thus, the correct answer is \$39.50 (or equivalent).

Note that the dollar symbol is in front of the answer box, so the symbol \$ does not need to be entered in the box. In fact, only numbers, a decimal point, and a negative sign can be entered in the answer box.

2. Rectangle R has length 30 and width 10, and square S has length 5. The perimeter of S is what fraction of the perimeter of R ?

Explanation

The perimeter of R is $30 + 10 + 30 + 10 = 80$, and the perimeter of S is $(4)(5) = 20$. Therefore, the perimeter of S is $\frac{20}{80}$ of the perimeter of R. To enter the answer $\frac{20}{80}$, you should enter the numerator 20 in the top box and the denominator 80 in the bottom box. Because the fraction does not need to be reduced to lowest terms, any fraction that is equivalent to $\frac{20}{80}$ is also considered correct, as long as it fits in the boxes. For example, both of the fractions $\frac{2}{8}$ and $\frac{1}{4}$ are considered correct.

Thus, the correct answer is $\frac{20}{80}$ (or any equivalent fraction).

RESULTS OF A USED-CAR AUCTION

	Small Cars	Large Cars
Number of cars offered	32	23
Number of cars sold	16	20
Projected sales total for cars offered (in thousands)	\$70	\$150
Actual sales total (in thousands)	\$41	\$120

Figure 7

3. For the large cars sold at an auction that is summarized in the table above, what was the average sale price per car?

\$

Explanation

From Figure 7, you see that the number of large cars sold was 20 and the sales total for large cars was \$120,000 (not \$120). Thus the average sale price per car was $\frac{\$120{,}000}{20} = \$6{,}000$.

The correct answer is \$6,000 (or equivalent).

(Note that the comma in 6,000 will appear automatically in the answer box in the computer-based test.)

4. A merchant made a profit of \$5 on the sale of a sweater that cost the merchant \$15. What is the profit expressed as a percent of the merchant's cost?

Give your answer to the nearest whole percent.

[ ] %

Explanation

The percent profit is $\left(\frac{5}{15}\right)(100) = 33.333\ldots = 33.\overline{3}$ percent, which is 33%, to the nearest whole percent.

Thus, the correct answer is 33% (or equivalent).

If you use the calculator and the Transfer Display button, the number that will be transferred to the answer box is 33.333333, which is incorrect since it is not given to the nearest whole percent. You will need to adjust the number in the answer box by deleting all of the digits to the right of the decimal point (using the Backspace key).

Also, since you are asked to give the answer as a percent, the decimal equivalent of 33 percent, which is 0.33, is incorrect. The percent symbol next to the answer box indicates that the form of the answer must be a percent. Entering 0.33 in the box would give the erroneous answer 0.33%.

---

5. Working alone at its constant rate, machine A produces k car parts in 10 minutes. Working alone at its constant rate, machine B produces k car parts in 15 minutes. How many minutes does it take machines A and B, working simultaneously at their respective constant rates, to produce k car parts?

[ ] minutes

Explanation

Machine A produces $\frac{k}{10}$ parts per minute, and machine B produces $\frac{k}{15}$ parts per minute. So when the machines work simultaneously, the rate at which the parts are produced is the sum of these two rates, which is $\frac{k}{10} + \frac{k}{15} = k\left(\frac{1}{10} + \frac{1}{15}\right) = k\left(\frac{25}{150}\right) = \frac{k}{6}$ parts per minute. To compute the time required to produce k parts at this rate, divide the amount k by the rate $\frac{k}{6}$ to get $\frac{k}{\frac{k}{6}} = 6$.

Therefore, the correct answer is 6 minutes (or equivalent).

One way to check that the answer of 6 minutes is reasonable is to observe that if the slower rate of machine B were the same as machine A's faster rate of k parts in 10 minutes, then the two machines, working simultaneously, would take half the time, or 5 minutes, to produce the k parts. So the answer has to be *greater than 5 minutes*. Similarly, if the faster rate of machine A were the same as machine B's slower rate of k parts in 15 minutes, then the two machines would take half the time, or 7.5 minutes, to produce the k parts. So the answer has to be *less than 7.5 minutes*. Thus, the answer of 6 minutes is reasonable compared to the lower estimate of 5 minutes and the upper estimate of 7.5 minutes.

Data Interpretation Sets

Description

Data Interpretation questions are grouped together and refer to the same table, graph, or other data presentation. These questions ask you to interpret or analyze the given data. The types of questions may be Multiple-choice (both types) or Numeric Entry.

Tips for Answering

- **Scan the data presentation briefly to see what it is about, but do not spend time studying all of the information in detail.** Focus on those aspects of the data that are necessary to answer the questions. Pay attention to the axes and scales of graphs; to the units of measurement or orders of magnitude (such as *billions*) that are given in the titles, labels, and legends; and to any notes that clarify the data.
- **Bar graphs and circle graphs, as well as other graphical displays of data, are drawn to scale, so you can read or estimate data visually from such graphs.** For example, you can use the relative sizes of bars or sectors to compare the quantities that they represent, but be aware of broken scales and of bars that do not start at 0.
- **The questions are to be answered only on the basis of the data presented, everyday facts (such as the number of days in a year), and your knowledge of mathematics.** Do not make use of specialized information you may recall from other sources about the particular context on which the questions are based unless the information can be derived from the data presented.

Sample Questions

Questions 1 to 3 are based on the following data.

ANNUAL PERCENT CHANGE IN DOLLAR AMOUNT OF SALES AT FIVE RETAIL STORES FROM 2006 TO 2008

Store	Percent Change from 2006 to 2007	Percent Change from 2007 to 2008
P	10	−10
Q	−20	9
R	5	12
S	−7	−15
T	17	−8

Figure 8

1. If the dollar amount of sales at Store *P* was $800,000 for 2006, what was the dollar amount of sales at that store for 2008 ?

   Ⓐ $727,200
   Ⓑ $792,000
   Ⓒ $800,000
   Ⓓ $880,000
   Ⓔ $968,000

Explanation

According to Figure 8, if the dollar amount of sales at Store P was \$800,000 for 2006, then it was 10 percent greater for 2007, which is 110 percent of that amount, or \$880,000. For 2008 the amount was 90 percent of \$880,000, which is \$792,000.

The correct answer is Choice B, \$792,000.

Note that an increase of 10 percent for one year and a decrease of 10 percent for the following year does not result in the same dollar amount as the original dollar amount because the base that is used in computing the percents is \$800,000 for the first change but \$880,000 for the second change.

---

2. At Store T, the dollar amount of sales for 2007 was what percent of the dollar amount of sales for 2008 ?

   Give your answer to the nearest 0.1 percent.

   [ ] %

Explanation

If A is the dollar amount of sales at Store T for 2007, then 8 percent of A, or $0.08A$, is the amount of decrease from 2007 to 2008. Thus $A - 0.08A = 0.92A$ is the dollar amount for 2008. Therefore, the desired percent can be obtained by dividing A by $0.92A$, which equals $\frac{A}{0.92A} = \frac{1}{0.92} = 1.0869565\ldots$. Expressed as a percent and rounded to the nearest 0.1 percent, this number is 108.7%.

Thus, the correct answer is 108.7% (or equivalent).

---

3. Based on the information given, which of the following statements must be true?

   Indicate all such statements.

   A. For 2008 the dollar amount of sales at Store R was greater than that at each of the other four stores.
   B. The dollar amount of sales at Store S for 2008 was 22 percent less than that for 2006.
   C. The dollar amount of sales at Store R for 2008 was more than 17 percent greater than that for 2006.

Explanation

For Choice A, since the only data given in Figure 8 are percent changes from year to year, there is no way to compare the actual dollar amount of sales at the stores for 2008 or for any other year. Even though Store R had the greatest percent increase from 2006 to 2008, its actual dollar amount of sales for 2008 may have been much smaller than that for any of the other four stores, and therefore Choice A is not necessarily true.

For Choice B, even though the sum of the two percent decreases would suggest a 22 percent decrease, the bases of the percents are different. If B is the dollar amount of sales at Store S for 2006, then the dollar amount for 2007 is 93 percent of B, or $0.93B$, and the dollar amount for 2008 is given by $(0.85)(0.93)B$, which is $0.7905B$. Note that this represents a percent decrease of $100 - 79.05 = 20.95$ percent, which is less than 22 percent, and so Choice B is not true.

For Choice C, if C is the dollar amount of sales at Store R for 2006, then the dollar amount for 2007 is given by $1.05C$ and the dollar amount for 2008 is given by $(1.12)(1.05)C$, which is $1.176C$. Note that this represents a 17.6 percent increase, which is greater than 17 percent, so Choice C must be true.

Therefore, the correct answer consists of only Choice C (The dollar amount of sales at Store *R* for 2008 was more than 17 percent greater than that for 2006).

Using the Calculator

Sometimes the computations you need to do in order to answer a question in the Quantitative Reasoning measure are somewhat time-consuming, like long division, or involve square roots. For such computations, you can use the on-screen calculator provided in the computer-based test. The on-screen calculator is shown in Figure 9.

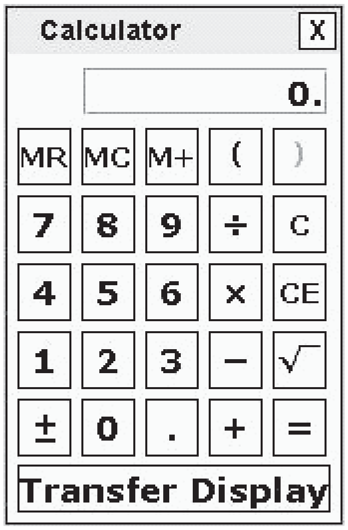


Figure 9

Although the calculator can shorten the time it takes to perform computations, keep in mind that the calculator provides results that supplement, but do not replace, your knowledge of mathematics. You must use your mathematical knowledge to determine whether the calculator's results are reasonable and how the results can be used to answer a question.

Here are some general guidelines for calculator use in the Quantitative Reasoning measure:

- Most of the questions don't require difficult computations, so don't use the calculator just because it's available.
- Use it for calculations that you know are tedious, such as long division, square roots, and addition, subtraction, or multiplication of numbers that have several digits.
- Avoid using it for simple computations that are quicker to do mentally, such as $10 - 490$, $(4)(70)$, $\frac{4{,}300}{10}$, $\sqrt{25}$, and 30^2.
- Avoid using it to introduce decimals if you are asked to give an answer as a fraction.
- Some questions can be answered more quickly by reasoning and estimating than by using the calculator.

- If you use the calculator, estimate the answer beforehand so that you can determine whether the calculator's answer is "in the ballpark." This may help you avoid key-entry errors.

The following guidelines are specific to the on-screen calculator in the computer-based test:

- When you use the computer mouse or the keyboard to operate the calculator, take care not to mis-key a number or operation.
- Note all of the calculator's buttons, including Transfer Display.
- The Transfer Display button can be used on Numeric Entry questions with a single answer box. This button will transfer the calculator display to the answer box. You should check that the transferred number has the correct form to answer the question. For example, if a question requires you to round your answer or convert your answer to a percent, make sure that you adjust the transferred number accordingly.
- Take note that the calculator respects order of operations, as explained below.

A mathematical convention called *order of operations* establishes which operations are performed before others in a mathematical expression that has more than one operation. The order is as follows: parentheses, exponentiation (including square roots), multiplications and divisions (from left to right), additions and subtractions (from left to right). With respect to order of operations, the value of the expression $1 + 2 \times 4$ is 9 because the expression is evaluated by first multiplying 2 and 4 and then by adding 1 to the result. This is how the on-screen calculator in the Quantitative Reasoning measure performs the operations. (Note that many basic calculators follow a different convention, whereby they perform multiple operations in the order that they are entered into the calculator. For such calculators, the result of entering $1 + 2 \times 4$ is 12. To get this result, the calculator adds 1 and 2, displays a result of 3, then multiplies 3 and 4, and displays a result of 12.)

- In addition to parentheses, the on-screen calculator has one memory location and three memory buttons that govern it: memory recall $\boxed{\text{MR}}$, memory clear $\boxed{\text{MC}}$, and memory sum $\boxed{\text{M+}}$. These buttons function as they normally do on most basic calculators.
- Some computations are not defined for real numbers: for example, division by zero or taking the square root of a negative number. If you enter $6\ \boxed{\div}\ 0\ \boxed{=}$, the word **Error** will be displayed. Similarly, if you enter $1\ \boxed{\pm}\ \boxed{\sqrt{\ }}$, then **Error** will be displayed. To clear the display, you must press the clear button $\boxed{\text{C}}$.
- The calculator displays up to eight digits. If a computation results in a number greater than 99,999,999, then **Error** will be displayed. For example, the calculation $10{,}000{,}000\ \boxed{\times}\ 10\ \boxed{=}$ results in **Error**. The clear button $\boxed{\text{C}}$ must be used to clear the display. If a computation results in a positive number less than 0.00000001, or 10^{-8}, then 0 will be displayed.

Below are some examples of computations using the calculator.

1. Compute $4+\dfrac{6.73}{2}$.

Explanation

Enter 4 $\boxed{+}$ 6.73 $\boxed{\div}$ 2 $\boxed{=}$ to get 7.365. Alternatively, enter 6.73 $\boxed{\div}$ 2 $\boxed{=}$ to get 3.365, and then enter $\boxed{+}$ 4 $\boxed{=}$ to get **7.365**.

---

2. Compute $-\dfrac{8.4+9.3}{70}$.

Explanation

Since division takes precedence over addition in the order of operations, you need to override that precedence in order to compute this fraction. Here are two ways to do that. You can use the parentheses for the addition in the numerator, entering $\boxed{(}$ 8.4 $\boxed{+}$ 9.3 $\boxed{)}$ $\boxed{\div}$ 70 $\boxed{=}$ $\boxed{\pm}$ to get -0.2528571. Or you can use the equals sign after 9.3, entering 8.4 $\boxed{+}$ 9.3 $\boxed{=}$ $\boxed{\div}$ 70 $\boxed{=}$ $\boxed{\pm}$ to get the same result. In the second way, note that pressing the first $\boxed{=}$ is essential, because without it, 8.4 $\boxed{+}$ 9.3 $\boxed{\div}$ 70 $\boxed{=}$ $\boxed{\pm}$ would erroneously compute $-\left(8.4+\dfrac{9.3}{70}\right)$ instead.

Incidentally, the exact value of the expression $-\dfrac{8.4+9.3}{70}$ is the repeating decimal $-0.25\overline{285714}$, where the digits 285714 repeat without end, but the calculator rounds the decimal to $-$**0.2528571.**

---

3. Find the length, to the nearest 0.01, of the hypotenuse of a right triangle with legs of length 21 and 54; that is, use the Pythagorean theorem and calculate $\sqrt{21^2+54^2}$.

Explanation

Enter 21 $\boxed{\times}$ 21 $\boxed{+}$ 54 $\boxed{\times}$ 54 $\boxed{=}$ $\boxed{\sqrt{\ }}$ to get 57.939624. Again, pressing the $\boxed{=}$ before the $\boxed{\sqrt{\ }}$ is essential because 21 $\boxed{\times}$ 21 $\boxed{+}$ 54 $\boxed{\times}$ 54 $\boxed{\sqrt{\ }}$ $\boxed{=}$ would erroneously compute $21^2+54\sqrt{54}$. This is because the square root would take precedence over the multiplication in the order of operations. Note that parentheses could be used, as in $\boxed{(}$ 21 $\boxed{\times}$ 21 $\boxed{)}$ $\boxed{+}$ $\boxed{(}$ 54 $\boxed{\times}$ 54 $\boxed{)}$ $\boxed{=}$ $\boxed{\sqrt{\ }}$, but they are not necessary because the multiplications already take precedence over the addition.

Incidentally, the exact answer is a nonterminating, nonrepeating decimal, or an irrational number, but the calculator rounds the decimal to 57.939624. Finally, note that the problem asks for the answer to the nearest 0.01, so the correct answer is **57.94**.

---

4. Compute $(-15)^3$.

Explanation

Enter 15 $\boxed{\pm}$ $\boxed{\times}$ 15 $\boxed{\pm}$ $\boxed{\times}$ 15 $\boxed{\pm}$ $\boxed{=}$ to get $-$**3,375.**

5. Convert 6 miles per hour to feet per second.

Explanation

The solution to this problem uses the conversion factors 1 mile = 5,280 feet and 1 hour = 3,600 seconds as follows:

$$\left(\frac{6\text{ miles}}{1\text{ hour}}\right)\left(\frac{5{,}280\text{ feet}}{1\text{ mile}}\right)\left(\frac{1\text{ hour}}{3{,}600\text{ seconds}}\right) = ?\ \frac{\text{feet}}{\text{second}}$$

Enter 6 $\boxed{\times}$ 5280 $\boxed{\div}$ 3600 $\boxed{=}$ to get 8.8. Alternatively, enter 6 $\boxed{\times}$ 5280 $\boxed{=}$ to get the result 31,680, and then enter $\boxed{\div}$ 3600 $\boxed{=}$ to get **8.8 feet per second.**

---

6. At a fund-raising event, 43 participants donated \$60 each, 21 participants donated \$80 each, and 16 participants donated \$100 each. What was the average (arithmetic mean) donation per participant, in dollars?

Explanation

The solution to this problem is to compute the weighted mean $\frac{(43)(60) + (21)(80) + (16)(100)}{43 + 21 + 16}$. You can use the memory buttons and parentheses for this computation as follows:

Enter 43 $\boxed{\times}$ 60 $\boxed{=}$ $\boxed{\text{M+}}$ 21 $\boxed{\times}$ 80 $\boxed{=}$ $\boxed{\text{M+}}$ 16 $\boxed{\times}$ 100 $\boxed{=}$ $\boxed{\text{M+}}$ $\boxed{\text{MR}}$ $\boxed{\div}$ $\boxed{(}$ 43 $\boxed{+}$ 21 $\boxed{+}$ 16 $\boxed{)}$ $\boxed{=}$ to get 73.25, or **\$73.25 per participant**.

When the $\boxed{\text{M+}}$ button is first used, the number in the calculator display is stored in memory and an **M** appears to the left of the display to show that the memory function is in use. Each subsequent use of the $\boxed{\text{M+}}$ button adds the number in the current display to the number stored in memory and replaces the number stored in memory by the sum. When the $\boxed{\text{MR}}$ button is pressed in the computation above, the current value in memory, 5,860, is displayed. To clear the memory, use the $\boxed{\text{MC}}$ button, and the **M** next to the display disappears.

Mathematical Conventions for the Quantitative Reasoning Measure of the GRE revised General Test

The mathematical symbols and terminology used in the Quantitative Reasoning measure of the test are conventional at the high school level, and most of these appear in the *Math Review* (Chapter 7). Whenever nonstandard or special notation or terminology is used in a test question, it is explicitly introduced in the question. However, there are some particular assumptions about numbers and geometric figures that are made throughout the test. These assumptions appear in the test at the beginning of the Quantitative Reasoning sections, and they are elaborated below.

Also, some notation and terminology, while standard at the high school level in many countries, may be different from those used in other countries or from those used at higher or lower levels of mathematics. Such notation and terminology are clarified below. Because it is impossible to ascertain which notation and terminology should be clarified for an individual test taker, more material than necessary may be included.

Finally, there are some guidelines for how certain information given in test questions should be interpreted and used in the context of answering the questions—information such as certain words, phrases, quantities, mathematical expressions, and displays of data. These guidelines appear at the end.

Numbers and Quantities

- All numbers used in the test questions are real numbers. In particular, integers and both rational and irrational numbers are to be considered, but imaginary numbers are not. This is the main assumption regarding numbers. Also, all quantities are real numbers, although quantities may involve units of measurement.
- Numbers are expressed in base 10 unless otherwise noted, using the 10 digits 0 through 9 and a period to the right of the ones digit, or units digit, for the decimal point. Also, in numbers that are 1,000 or greater, commas are used to separate groups of three digits to the left of the decimal point.
- When a positive integer is described by the number of its digits, e.g., a two-digit integer, the digits that are counted include the ones digit and all the digits further to the left, where the left-most digit is not 0. For example, 5,000 is a four-digit integer, whereas 031 is not considered to be a three-digit integer.
- Some other conventions involving numbers: *one billion* means 1,000,000,000, or 10^9 (not 10^{12}, as in some countries); *one dozen* means 12; the Greek letter π represents the ratio of the circumference of a circle to its diameter and is approximately 3.14.
- When a positive number is to be rounded to a certain decimal place and the number is halfway between the two nearest possibilities, the number should be rounded to the greater possibility. For example, 23.5 rounded to the nearest integer is 24, and 123.985 rounded to the nearest 0.01 is 123.99. When the number to be rounded is negative, the number should be rounded to the lesser possibility. For example, -36.5 rounded to the nearest integer is -37.
- Repeating decimals are sometimes written with a bar over the digits that repeat, as in $\frac{25}{12} = 2.08\overline{3}$ and $\frac{1}{7} = 0.\overline{142857}$.
- If r, s, and t are integers and $rs = t$, then r and s are *factors*, or *divisors*, of t; also, t is a *multiple* of r (and of s) and t is *divisible* by r (and by s). The factors of an integer include positive and negative integers. For example, -7 is a factor of

35, 8 is a factor of -40, and the integer 4 has six factors: -4, -2, -1, 1, 2, and 4. The terms *factor*, *divisor*, and *divisible* are used only when r, s, and t are integers. However, the term *multiple* can be used with any real numbers s and t provided r is an integer. For example, 1.2 is a multiple of 0.4, and -2π is a multiple of π.

- The *least common multiple* of two nonzero integers a and b is the least positive integer that is a multiple of both a and b. The *greatest common divisor* (or *greatest common factor*) of a and b is the greatest positive integer that is a divisor of both a and b.
- If an integer n is divided by a nonzero integer d resulting in a quotient q with remainder r, then $n = qd + r$, where $0 \le r < |d|$. Furthermore, $r = 0$ if and only if n is a multiple of d. For example, when 20 is divided by 7, the quotient is 2 and the remainder is 6; when 21 is divided by 7, the quotient is 3 and the remainder is 0; and when -17 is divided by 7, the quotient is -3 and the remainder is 4.
- A *prime number* is an integer greater than 1 that has only two positive divisors: 1 and itself. The first five prime numbers are 2, 3, 5, 7, and 11. A *composite number* is an integer greater than 1 that is not a prime number. The first five composite numbers are 4, 6, 8, 9, and 10.
- Odd and even integers are not necessarily positive; for example, -7 is odd, and -18 and 0 are even.
- The integer 0 is neither positive nor negative.

Mathematical Expressions, Symbols, and Variables

- As is common in algebra, italic letters like x are used to denote numbers, constants, and variables. Letters are also used to label various objects, such as line ℓ, point P, function f, set S, list T, event E, random variable X, Brand X, City Y, and Company Z. The meaning of a letter is determined by the context.
- When numbers, constants, or variables are given, their possible values are all real numbers unless otherwise restricted. It is common to restrict the possible values in various ways. Here are some examples: n is a nonzero integer; $1 \le x < \pi$; and T is the tens digits of a two-digit positive integer, so T is an integer from 1 to 9.
- Standard mathematical symbols at the high school level are used. These include the arithmetic operations $+, -, \times$, and $\div$, though multiplication is usually denoted by juxtaposition, often with parentheses, e.g., $2y$ and $(3)(4.5)$; and division is usually denoted with a horizontal fraction bar, e.g., $\frac{w}{3}$. Sometimes mixed numbers, or mixed fractions, are used, like $4\frac{3}{8}$ and $-10\frac{1}{2}$. These two numbers are equal to $\frac{35}{8}$ and $-\frac{21}{2}$, respectively. Exponents are also used, e.g., $2^{10} = 1{,}024$, $10^{-2} = \frac{1}{100}$, and $x^0 = 1$ for all nonzero numbers x.
- Mathematical expressions are to be interpreted with respect to *order of operations*, which establishes which operations are performed before others in an expression. The order is as follows: parentheses; exponentiation; negation; multiplication and division (from left to right); addition and subtraction (from left to right). For example, the value of the expression $1 + 2 \times 4$ is 9, because the expression is evaluated by first multiplying 2 and 4 and then adding 1 to the result. Also, -3^2 means "the negative of '3 squared'" because exponentiation takes precedence over negation. Therefore, $-3^2 = -9$, but $(-3)^2 = 9$ because parentheses take precedence over exponentiation.

- Here are examples of other standard symbols with their meanings:

$x \le y$	x is less than or equal to y
$x \ne y$	x and y are not equal
$x \approx y$	x and y are approximately equal
$\lvert x \rvert$	the absolute value of x
$\sqrt{x}$	the nonnegative square root of x, where $x \ge 0$
$-\sqrt{x}$	the nonpositive square root of x, where $x \ge 0$
$n!$	the product of all positive integers less than or equal to n, where n is any positive integer and, as a special definition, $0! = 1$.
$\ell \parallel m$	lines ℓ and m are parallel
$\ell \perp m$	lines ℓ and m are perpendicular

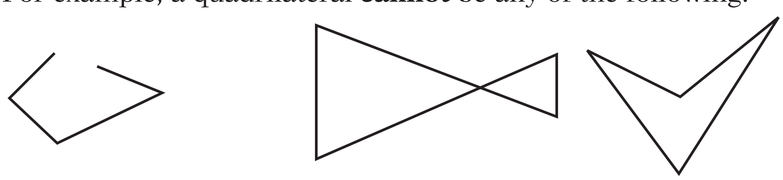
- Because all numbers are assumed to be real, some expressions are not defined. For example, for every number x, the expression $\frac{x}{0}$ is not defined; if $x < 0$, then $\sqrt{x}$ is not defined; and 0^0 is not defined.
- Sometimes special symbols or notation are introduced in a question. Here are two examples:

  The operation $\Diamond$ is defined for all integers r and s by $r \Diamond s = \frac{rs}{1+r^2}$.

  The operation $\sim$ is defined for all nonzero numbers x by $\sim x = -\frac{1}{x}$.
- Sometimes juxtaposition of letters does *not* denote multiplication, as in "consider a three-digit integer denoted by XYZ, where X, Y, and Z are digits." The meaning is taken from the context.
- Standard function notation is used in the test. For example, "the function g is defined for all $x \ge 0$ by $g(x) = 2x + \sqrt{x}$." If the domain of a function f is not given explicitly, it is assumed to be the set of all real numbers x for which $f(x)$ is a real number. If f and g are two functions, then the *composition* of g with f is denoted by $g(f(x))$.

Geometry

- In questions involving geometry, the conventions of plane (or Euclidean) geometry are followed, including the assumption that the sum of the measures of the interior angles of a triangle is 180 degrees.
- Lines are assumed to be "straight" lines that extend in both directions without end.
- Angle measures are in degrees and are assumed to be positive and less than or equal to 360 degrees.
- When a square, circle, polygon, or other closed geometric figure is described in words but not shown, the figure is assumed to enclose a convex region. It is also assumed that such a closed geometric figure is not just a single point or a line segment. For example, a quadrilateral **cannot** be any of the following:

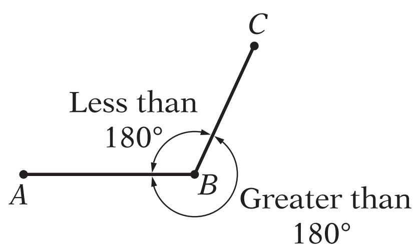


Not closed Not convex

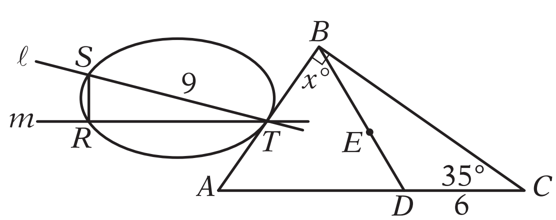
- The phrase *area of a rectangle* means the area of the region enclosed by the rectangle. The same terminology applies to circles, triangles, and other closed figures.
- The *distance between a point and a line* is the length of the perpendicular line segment from the point to the line, which is the shortest distance between the point and the line. Similarly, the *distance between two parallel lines* is the distance between a point on one line and the other line.
- In a geometric context, the phrase *similar triangles* (or other figures) means that the figures have the same shape. See the Geometry section of the *Math Review* for further explanation of the terms *similar* and *congruent*.

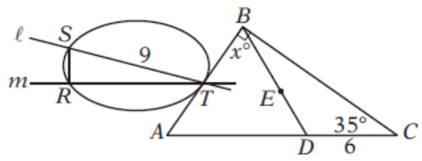
Geometric Figures

- Geometric figures consist of points, lines, line segments, curves (such as circles), angles, and regions; also included are labels, and markings or shadings that identify these objects or their sizes. A point is indicated by a dot, a label, or the intersection of two or more lines or curves. Points, lines, angles, etc., that are shown as distinct are indeed distinct. All figures are assumed to lie in a plane unless otherwise indicated.
- If points A, B, and C do not lie on the same line, then line segments AB and BC form two angles with vertex B—one angle with measure less than 180° and the other with measure greater than 180°, as shown below. Unless otherwise indicated, angle ABC, also denoted by $\angle ABC$ or $\angle B$, refers to the *smaller* of the two angles.



- The notation AB may mean the line segment with endpoints A and B, or it may mean the length of the line segment. The meaning can be determined from the context.
- Geometric figures **are not necessarily** drawn to scale. That is, you should **not** assume that quantities such as lengths and angle measures are as they appear in a figure. However, you should assume that lines shown as straight are actually straight, and when curves are shown, you should assume they are not straight. Also, assume that points on a line or a curve are in the order shown, points shown to be on opposite sides of a line or curve are so oriented, and more generally, assume all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.
- To illustrate some of these conventions, consider the geometric figure below.





The following can be determined from the figure.

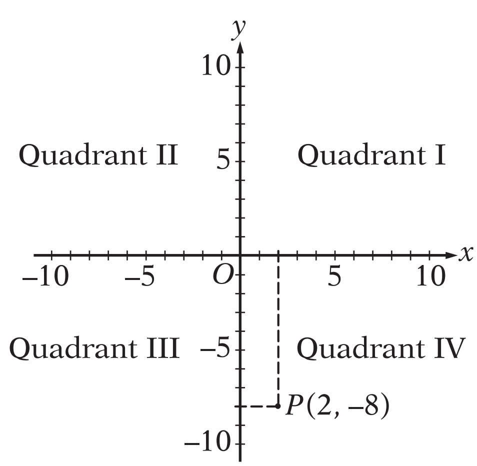
- ABD and DBC are triangles, and points R, S, and T lie on the closed curve.
- Points A, D, and C lie on a straight line, so ABC is a triangle with sides AB, BC, and AC.
- Point D is a distinct point between points A and C.
- Points A and S are on opposite sides of line m.
- Point E is on BD.
- $AD < AC$
- $ST = 9$, $DC = 6$, and the measure of angle C is 35 degrees.
- Angle ABC is a right angle, as indicated by the small square symbol at point B.
- The measure of angle ABD is x degrees, and $x < 90$.
- Line ℓ intersects the closed curve at points S and T, and the curve is tangent to AB at T.
- The area of the region enclosed by the curve is greater than the area of triangle RST.

The following **cannot** be determined from the figure.

- $AD > DC$
- The measures of angles BAD and BDA are equal.
- The measure of angle DBC is less than x degrees.
- The area of triangle ABD is greater than the area of triangle DBC.
- Angle SRT is a right angle.
- Line m is parallel to line AC.

Coordinate Systems

- Coordinate systems, such as xy-planes and number lines, **are** drawn to scale. Therefore, you can read, estimate, or compare quantities in such figures by sight or by measurement, including geometric figures that appear in coordinate systems.
- The positive direction of a number line is to the right.
- As in geometry, distances in a coordinate system are nonnegative.
- The rectangular coordinate plane, or rectangular coordinate system, commonly known as the xy-plane, is shown below. The x-axis and y-axis intersect at the origin O, and they partition the plane into four quadrants. Each point in the xy-plane has coordinates (x, y) that give its location with respect to the axes; for example, the point $P(2, -8)$ is located 2 units to the right of the y-axis and 8 units below the x-axis. The units on the x-axis have the same length as the units on the y-axis, unless otherwise noted.



- Intermediate grid lines or tick marks in a coordinate system are evenly spaced unless otherwise noted.
- The term *x-intercept* refers to the x-coordinate of the point at which a graph in the xy-plane intersects the x-axis. The term *y-intercept* is used analogously. Sometimes the terms *x-intercept* and *y-intercept* refer to the actual intersection points.

Sets, Lists, and Sequences

- Sets of numbers or other elements appear in some questions. Some sets are infinite, such as the set of integers; other sets are finite and may have all of their elements listed within curly brackets, such as the set $\{2, 4, 6, 8\}$. When the elements of a set are given, repetitions are *not* counted as additional elements and the order of the elements is *not* relevant. Elements are also called *members*. A set with one or more members is called *nonempty*; there is a set with no members, called the *empty set* and denoted by $\varnothing$. If A and B are sets, then the *intersection* of A and B, denoted by $A \cap B$, is the set of elements that are in both A and B, and the *union* of A and B, denoted by $A \cup B$, is the set of elements that are in A or B, or both. If all of the elements in A are also in B, then A is a *subset* of B. By convention, the empty set is a subset of every set. If A and B have no elements in common, they are called *disjoint* sets or *mutually exclusive* sets.
- Lists of numbers or other elements are also used in the test. When the elements of a list are given, repetitions *are* counted as additional elements and the order of the elements *is* relevant. For example, the list 3, 1, 2, 3, 3 contains five numbers, and the first, fourth, and last numbers in the list are each 3.
- The terms *data set* and *set of data* are not sets in the mathematical sense given above. Rather they refer to a list of data because there may be repetitions in the data, and if there are repetitions, they would be relevant.
- Sequences are lists that often have an infinite number of elements, or terms. The terms of a sequence are often represented by a fixed letter along with a subscript that indicates the order of a term in the sequence. For example, $a_1, a_2, a_3, \ldots, a_n, \ldots$ represents an infinite sequence in which the first term is a_1, the second term is a_2, and more generally, the nth term is a_n for every positive integer n. Sometimes the nth term of a sequence is given by a formula, such as $b_n = 2^n + 1$. Sometimes the first few terms of a sequence are given explicitly, as in the following sequence of consecutive even negative integers: $-2, -4, -6, -8, -10, \ldots$.
- Sets of consecutive integers are sometimes described by indicating the first and last integer, as in "the integers from 0 to 9, inclusive." This phrase refers to 10 integers, with or without "inclusive" at the end. Thus, the phrase "during the years from 1985 to 2005" refers to 21 years.

Data and Statistics

- Numerical data are sometimes given in lists and sometimes displayed in other ways, such as in tables, bar graphs, or circle graphs. Various statistics, or measures of data, appear in questions: measures of central tendency—mean, median, and mode; measures of position—quartiles and percentiles; and measures of dispersion—standard deviation, range, and interquartile range.
- The term *average* is used in two ways, with and without the qualification "(arithmetic mean)." For a list of data, the *average* (*arithmetic mean*) of the data is

the sum of the data divided by the number of data. The term *average* does not refer to either *median* or *mode* in the test. Without the qualification of "arithmetic mean," *average* can refer to a rate or the ratio of one quantity to another, as in "average number of miles per hour" or "average weight per truckload."

- When *mean* is used in the context of data, it means *arithmetic mean*.
- The *median* of an odd number of data is the middle number when the data are listed in increasing order; the *median* of an even number of data is the arithmetic mean of the two middle numbers when the data are listed in increasing order.
- For a list of data, the *mode* of the data is the most frequently occurring number in the list. Thus, there may be more than one mode for a list of data.
- For data listed in increasing order, the *first quartile*, *second quartile*, and *third quartile* of the data are three numbers that divide the data into four groups that are roughly equal in size. The first group of numbers is from the least number up to the first quartile. The second group is from the first quartile up to the second quartile, which is also the median of the data. The third group is from the second quartile up to the third quartile, and the fourth group is from the third quartile up to the greatest number. Note that the four groups themselves are sometimes referred to as quartiles—*first quartile*, *second quartile*, *third quartile*, and *fourth quartile*. The latter usage is clarified by the word "in" as in the phrase "the cow's weight is *in* the third quartile of the weights of the herd."
- For data listed in increasing order, the *percentiles* of the data are 99 numbers that divide the data into 100 groups that are roughly equal in size. The 25th percentile equals the first quartile; the 50th percentile equals the second quartile, or median; and the 75th percentile equals the third quartile.
- For a list of data, where the arithmetic mean is denoted by m, the *standard deviation* of the data refers to the nonnegative square root of the mean of the squared differences between m and each of the data. This statistic is also known as the *population standard deviation* and is not to be confused with the *sample standard deviation*.
- For a list of data, the *range* of the data is the greatest number in the list minus the least number. The *interquartile range* of the data is the third quartile minus the first quartile.

Data Distributions and Probability Distributions

- Some questions display data in *frequency distributions*, where discrete data values are repeated with various frequencies, or where preestablished intervals of possible values are assigned frequencies corresponding to the numbers of data in the intervals. For example, the lifetimes, rounded to the nearest hour, of 300 lightbulbs could be in the following 10 intervals: 501–550 hours, 551–600 hours, 601–650 hours, . . . , 951–1,000 hours; consequently, each of the intervals would have a number, or frequency, of lifetimes, and the sum of the 10 frequencies is 300.
- Questions may involve *relative frequency distributions*, where each frequency of a frequency distribution is divided by the total number of data in the distribution, resulting in a relative frequency. In the example above, the 10 frequencies of the 10 intervals would each be divided by 300, yielding 10 relative frequencies.
- When a question refers to a random selection or a random sample, all possible samples of equal size have the same probability of being selected unless there is information to the contrary.

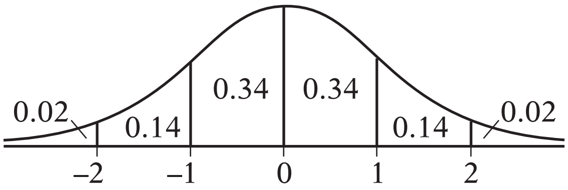
- Some questions describe *probability experiments*, or *random experiments*, that have a finite number of possible *outcomes*. In a random experiment, any particular set of outcomes is called an *event*, and every event E has a *probability*, denoted by $P(E)$, where $0 \le P(E) \le 1$. If each outcome of an experiment is equally likely, then the probability of an event E is defined as the following ratio:

$$P(E) = \frac{\text{the number of outcomes in the event } E}{\text{the number of possible outcomes in the experiment}}$$

- If E and F are two events in an experiment, then "E and F" is an event, which is the set of outcomes that are in the intersection of events E and F. Another event is "E or F," which is the set of outcomes that are in the union of events E and F.
- If E and F are two events and E and F are mutually exclusive, then $P(E \text{ and } F) = 0$.
- If E and F are two events such that the occurrence of either event does not affect the occurrence of the other, then E and F are said to be *independent* events. Events E and F are independent if and only if $P(E \text{ and } F) = P(E)P(F)$.
- A *random variable* is a variable that represents values resulting from a random experiment. The values of the random variable may be the actual outcomes of the experiment if the outcomes are numerical, or the random variable may be related to the outcomes more indirectly. In either case, random variables can be used to describe events in terms of numbers.
- A random variable from an experiment with only a finite number of possible outcomes also has only a finite number of values and is called a *discrete random variable*. When the values of a random variable form a continuous interval of real numbers, such as all of the numbers between 0 and 2, the random variable is called a *continuous random variable*.
- Every value of a discrete random variable X, say $X = a$, has a probability denoted by $P(a)$. A histogram (or a table) showing all of the values of X and their probabilities $P(X)$ is called the *probability distribution* of X. The *mean of the random variable* X is the sum of the products $XP(X)$ for all values of X.
- The mean of a random variable X is also called the *expected value* of X or the *mean of the probability distribution* of X.
- For a continuous random variable X, every interval of values, say $a \le X \le b$, has a probability, which is denoted by $P(a \le X \le b)$. The *probability distribution* of X can be described by a curve in the xy–plane that mimics the tops of the bars of a histogram, only smoother. The curve is the graph of a function f whose values are nonnegative and whose graph is therefore above the x-axis. The curve $y = f(x)$ is related to the probability of each interval $a \le X \le b$ in the following way: $P(a \le X \le b)$ is equal to the area of the region that is below the curve, above the x-axis, and between the vertical lines $x = a$ and $x = b$. The area of the entire region under the curve is 1.
- The *mean of a continuous random variable* X is the point m on the x-axis at which region under the distribution curve would perfectly balance if a fulcrum were placed at $x = m$. The *median* of X is the point M on the x-axis at which the line $x = M$ divides the region under the distribution curve into two regions of equal area.
- The *standard deviation of a random variable* X is a measure of dispersion, which indicates how spread out the probability distribution of X is from its mean. It is also called the *standard deviation of the probability distribution* of X.
- The most important probability distribution is the *normal distribution*, whose distribution curve is shaped like a bell. A random variable X with this distribution

is called *normally distributed*. The curve is symmetric about the line $x = m$, where m is the mean as well as the median. The right and left tails of the distribution become ever closer to the x-axis but never touch it.

- The *standard normal distribution* has mean 0 and standard deviation 1. The following figure shows the distribution, including approximate probabilities corresponding to the six intervals shown.



Graphical Representations of Data

- Graphical data presentations, such as bar graphs, circle graphs, and line graphs, **are** drawn to scale; therefore, you can read, estimate, or compare data values by sight or by measurement.
- Standard conventions apply to graphs of data unless otherwise indicated. For example, a circle graph represents 100 percent of the data indicated in the graph's title, and the areas of the individual sectors are proportional to the percents they represent. Scales, grid lines, dots, bars, shadings, solid and dashed lines, legends, etc., are used on graphs to indicate the data. Sometimes, scales that do not begin at 0 are used, and sometimes broken scales are used.
- In Venn diagrams, various sets of objects are represented by circular regions and by regions formed by intersections of the circles. In some Venn diagrams, all of the circles are inside a rectangular region that represents a universal set. A number placed in a region is the number of elements in the subset represented by the smallest region containing the number, unless otherwise noted. Sometimes a number is placed above a circle to indicate the number of elements in the entire circle.

Miscellaneous Guidelines for Interpreting and Using Information in Test Questions

- Numbers given in a question are to be used as exact numbers, even though in some real-life settings they are likely to have been rounded. For example, if a question states that "30 percent of the company's profit was from health products," then 30 is to be used as an exact number; it is not to be treated as though it were a nearby number, say 29 or 30.1, that has been rounded up or down.
- An integer that is given as the number of certain objects, whether in a real-life or pure-math setting, is to be taken as the total number of such objects. For example, if a question states that "a bag contains 50 marbles, and 23 of the marbles are red," then 50 is to be taken as the total number of marbles in the bag and 23 is to be taken as the total number of red marbles in the bag, so that the other 27 marbles are not red. Fractions and percents are understood in a similar way, so "one-fifth, or 20 percent, of the 50 marbles in the bag are green" means 10 marbles in the bag are green and 40 marbles are not green.

- When a multiple-choice question asks for an approximate quantity without stipulating a degree of approximation, the correct answer is the choice that is closest in value to the quantity that can be computed from the information given.
- Unless otherwise indicated, the phrase "difference between two quantities" is assumed to mean "positive difference," that is, the greater quantity minus the lesser quantity. For example, "for which two consecutive years was the difference in annual rainfall least?" means "for which two consecutive years was the *absolute value of the difference* in annual rainfall least?"
- When the term *profit* is used in a question, it refers to *gross profit*, which is the sales revenue minus the cost of production. The profit does not involve any other amounts unless they are explicitly given.
- The common meaning of terms such as *months* and *years* and other everyday terms are assumed in questions where the terms appear.
- In questions involving real-life scenarios in which a variable is given to represent a number of existing objects or another nonnegative amount, the context implies that the variable is greater than 0. For example, "Jane sold x rugs and deposited her profit of y dollars into her savings account" implies that x and y are greater than 0.
- Some quantities may involve units, such as inches, pounds, and Celsius degrees, while other quantities are pure numbers. Any units of measurement, such as English units or metric units, may be used. However, if an answer to a question requires converting one unit of measurement to another, then the relationship between the units is given in the question, unless the relationship is a common one, such as the relationships between minutes and hours, dollars and cents, and metric units like centimeters and meters.
- In any question, there may be some information that is not needed for obtaining the correct answer.
- When reading questions, do not introduce unwarranted assumptions. For example, if a question describes a trip that begins and ends at certain times, the intended answer will assume that the times are unaffected by crossing time zones or by changes to the local time for daylight savings, unless those matters are explicitly mentioned. As another example, do not consider sales taxes on purchases unless explicitly mentioned.
- The display of data in a Data Interpretation set of questions is the same for each question in the set. Also, the display may contain more than one graph or table. Each question will refer to some of the data, but it may happen that some part of the data will have no question that refers to it.
- In a Data Interpretation set of questions, each question should be considered separately from the others. No information except what is given in the display of data should be carried over from one question to another.
- In many questions, mathematical expressions and words appear together in a phrase. In such a phrase, each mathematical expression should be interpreted *separately* from the words before it is interpreted *along with* the words. For example, if n is an integer, then the phrase "the sum of the first two consecutive integers greater than $n + 6$" means $(n + 7) + (n + 8)$; it does not mean "the sum of the first two consecutive integers greater than n" plus 6, or $(n + 1) + (n + 2) + 6$. That is, the expression $n + 6$ should be interpreted first, separately from the words. However, in a phrase like "the function g is defined for all $x \geq 0$," the phrase "for all $x \geq 0$" is a mathematical shorthand for "for all numbers x such that $x \geq 0$."

6 GRE® Quantitative Reasoning Practice Questions

Your goals for this chapter

- Practice answering *GRE*® Quantitative Reasoning questions on your own
- Study answers and explanations, particularly for questions you answered incorrectly

This chapter contains four sets of GRE Quantitative Reasoning practice questions. Each of the first three practice sets consists of Quantitative Comparison questions, both types of Multiple-choice questions, and Numeric Entry questions. These three sets are arranged in order of increasing difficulty. The first is easy, the second is medium, and the third is hard. The fourth practice set consists of Data Interpretation questions of varying levels of difficulty.

Following the last set is an answer key for quick reference. Then, at the end of the chapter, you will find complete explanations for every question. Each explanation is presented with the corresponding question, so that you can easily see what was asked and what the various answer choices or Numeric Entry answer boxes were.

Sharpen your GRE Quantitative Reasoning skills by working your way through these question sets. For the Discrete question sets, begin with the easy sets and then move on to the medium and hard sets. Review the answers and explanations carefully, paying particular attention to explanations for questions that you answered incorrectly.

For the practice questions in this chapter, use the directions that begin on the following page.

General Directions

For each question, indicate the best answer, using the directions given.

Notes: All numbers used are real numbers.

All figures are assumed to lie in a plane unless otherwise indicated.

Geometric figures, such as lines, circles, triangles, and quadrilaterals, **are not necessarily** drawn to scale. That is, you should **not** assume that quantities such as lengths and angle measures are as they appear in a figure. You should assume, however, that lines shown as straight are actually straight, points on a line are in the order shown, and more generally, all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.

Coordinate systems, such as *xy*-planes and number lines, **are** drawn to scale; therefore, you can read, estimate, or compare quantities in such figures by sight or by measurement.

Graphical data presentations, such as bar graphs, circle graphs, and line graphs, **are** drawn to scale; therefore, you can read, estimate, or compare data values by sight or by measurement.

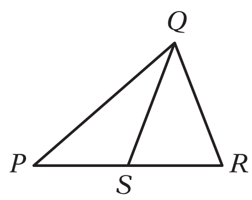
Directions for Quantitative Comparison questions

Compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

	Quantity A	Quantity B	Correct Answer
Example 1:	(2)(6)	$2 + 6$	(A) (B) (C) (D)



	Quantity A	Quantity B	Correct Answer
Example 2:	PS	SR	(A) (B) (C) (D) (since equal lengths cannot be assumed, even though PS and SR appear equal)

Directions for Numeric Entry questions

Enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms.**
- **Enter the exact answer unless the question asks you to round your answer.**

SET 1. Discrete Questions: Easy

Quantitative Comparison

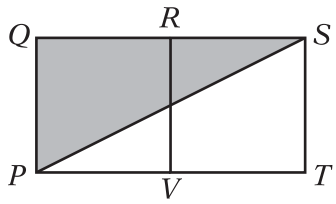
For Questions 1 to 6, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

Emma spent \$75 buying a used bicycle and \$27 repairing it. Then she sold the bicycle for 40 percent more than the total amount she spent buying and repairing it.

	Quantity A	Quantity B	
1.	The price at which Emma sold the bicycle	\$140	(A) (B) (C) (D)



In the figure above, squares $PQRV$ and $VRST$ have sides of length 6.

	Quantity A	Quantity B	
2.	The area of the shaded region	36	Ⓐ Ⓑ Ⓒ Ⓓ

In 2009 the property tax on each home in Town X was p percent of the assessed value of the home, where p is a constant. The property tax in 2009 on a home in Town X that had an assessed value of \$125,000 was \$2,500.

	Quantity A	Quantity B	
3.	The property tax in 2009 on a home in Town X that had an assessed value of \$160,000	\$3,000	Ⓐ Ⓑ Ⓒ Ⓓ

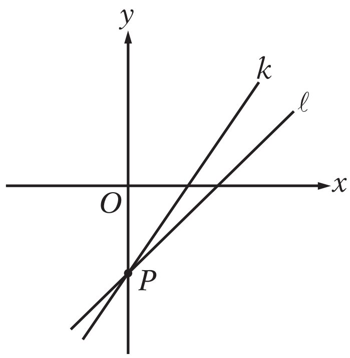
$$x + y = -1$$

	Quantity A	Quantity B	
4.	x	y	Ⓐ Ⓑ Ⓒ Ⓓ

---

r, s, and t are three consecutive odd integers such that $r < s < t$.

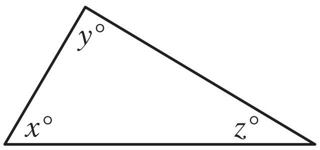
	Quantity A	Quantity B	
5.	$r + s + 1$	$s + t - 1$	Ⓐ Ⓑ Ⓒ Ⓓ



	Quantity A	Quantity B	
6.	The slope of line k	The slope of line ℓ	Ⓐ Ⓑ Ⓒ Ⓓ

Multiple-choice Questions—Select One Answer Choice

For Questions 7 to 11, select a single answer choice.



7. In the figure above, what is the value of $\dfrac{x+y+z}{45}$?

Ⓐ 2
Ⓑ 3
Ⓒ 4
Ⓓ 5
Ⓔ 6

8. A certain store sells two types of pens: one type for \$2 per pen and the other type for \$3 per pen. If a customer can spend up to \$25 to buy pens at the store and there is no sales tax, what is the greatest number of pens the customer can buy?

Ⓐ 9
Ⓑ 10
Ⓒ 11
Ⓓ 12
Ⓔ 20

9. If $y = 3x$ and $z = 2y$, what is $x + y + z$ in terms of x ?

Ⓐ $10x$
Ⓑ $9x$
Ⓒ $8x$
Ⓓ $6x$
Ⓔ $5x$

10. A certain shipping service charges an insurance fee of \$0.75 when shipping any package with contents worth \$25.00 or less and an insurance fee of \$1.00 when shipping any package with contents worth over \$25.00. If Dan uses the shipping company to ship three packages with contents worth \$18.25, \$25.00, and \$127.50, respectively, what is the total insurance fee that the company charges Dan to ship the three packages?

Ⓐ \$1.75
Ⓑ \$2.25
Ⓒ \$2.50
Ⓓ \$2.75
Ⓔ \$3.00

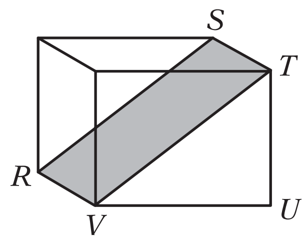
11. If 55 percent of the people who purchase a certain product are female, what is the ratio of the number of females who purchase the product to the number of males who purchase the product?

Ⓐ 11 to 9
Ⓑ 10 to 9
Ⓒ 9 to 10
Ⓓ 9 to 11
Ⓔ 5 to 9

Numeric Entry

For Questions 12 and 13, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms.**
- **Enter the exact answer unless the question asks you to round your answer.**



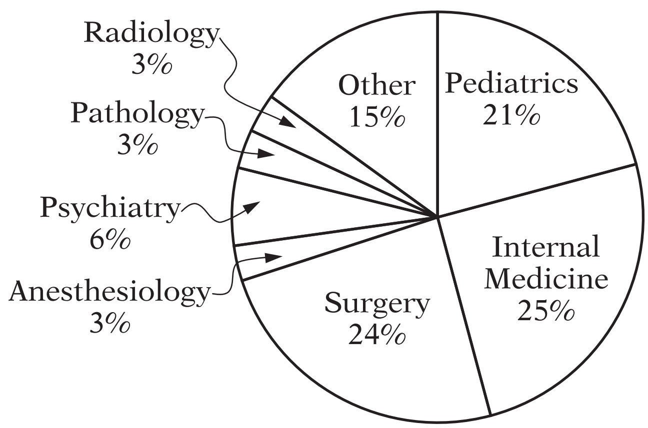
12. In the rectangular solid above, $TU = 3$, $UV = 4$, and $VR = 2$. What is the area of the shaded rectangular region?

13. A list of numbers has a mean of 8 and a standard deviation of 2.5. If x is a number in the list that is 2 standard deviations above the mean, what is the value of x ?

$x =$

Multiple-choice Questions—Select One or More Answer Choices

For Question 14, select all the answer choices that apply.



14. The circle graph above shows the distribution of 200,000 physicians by specialty. Which of the following sectors of the circle graph represent more than 40,000 physicians?

Indicate all such sectors.

A Pediatrics
B Internal Medicine
C Surgery
D Anesthesiology
E Psychiatry

SET 2. Discrete Questions: Medium

Quantitative Comparison

For Questions 1 to 5, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

Machine *R*, working alone at a constant rate, produces *x* units of a product in 30 minutes, and machine *S*, working alone at a constant rate, produces *x* units of the product in 48 minutes, where *x* is a positive integer.

	Quantity A	Quantity B	
1.	The number of units of the product that machine *R*, working alone at its constant rate, produces in 3 hours	The number of units of the product that machine *S*, working alone at its constant rate, produces in 4 hours	Ⓐ Ⓑ Ⓒ Ⓓ

Frequency Distribution for List *X*

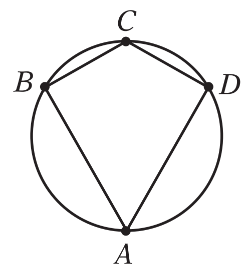
Number	1	2	3	5
Frequency	10	20	18	12

Frequency Distribution for List *Y*

Number	6	7	8	9
Frequency	24	17	10	9

List *X* and list *Y* each contain 60 numbers. Frequency distributions for each list are given above. The average (arithmetic mean) of the numbers in list *X* is 2.7, and the average of the numbers in list *Y* is 7.1. List *Z* contains 120 numbers: the 60 numbers in list *X* and the 60 numbers in list *Y*.

	Quantity A	Quantity B	
2.	The average of the 120 numbers in list *Z*	The median of the 120 numbers in list *Z*	Ⓐ Ⓑ Ⓒ Ⓓ



In the figure above, the diameter of the circle is 10.

	Quantity A	Quantity B	
3.	The area of quadrilateral *ABCD*	40	Ⓐ Ⓑ Ⓒ Ⓓ

$$x^2y > 0$$
$$xy^2 < 0$$

	Quantity A	Quantity B	
4.	x	y	Ⓐ Ⓑ Ⓒ Ⓓ

Among the 9,000 people attending a football game at College C, there were x students from College C and y students who were not from College C.

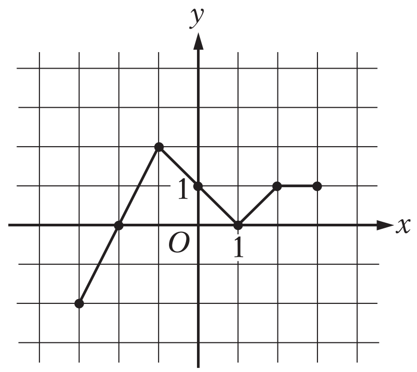
	Quantity A	Quantity B	
5.	The number of people attending the game who were not students	$9{,}000 - x - y$	Ⓐ Ⓑ Ⓒ Ⓓ

Multiple-choice Questions—Select One Answer Choice

For Questions 6 to 10, select a single answer choice.

6. If $x \neq 0$, which of the following is equivalent to $\frac{x(x^2)^3}{x^2}$?

Ⓐ x^2
Ⓑ x^3
Ⓒ x^4
Ⓓ x^5
Ⓔ x^6



7. The figure above shows the graph of the function f in the xy-plane. What is the value of $f(f(-1))$?

Ⓐ -2
Ⓑ -1
Ⓒ 0
Ⓓ 1
Ⓔ 2

8. If $\dfrac{d-3n}{7n-d} = 1$, which of the following statements describes d in terms of n?

Ⓐ d is 4 less than n.
Ⓑ d is 4 more than n.
Ⓒ d is $\frac{3}{7}$ of n.
Ⓓ d is 2 times n.
Ⓔ d is 5 times n.

9. By weight, liquid A makes up 8 percent of solution R and 18 percent of solution S. If 3 grams of solution R are mixed with 7 grams of solution S, then liquid A accounts for what percent of the weight of the resulting solution?

Ⓐ 10%
Ⓑ 13%
Ⓒ 15%
Ⓓ 19%
Ⓔ 26%

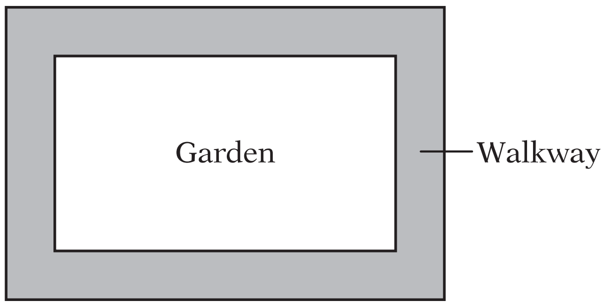
10. Of the 700 members of a certain organization, 120 are lawyers. Two members of the organization will be selected at random. Which of the following is closest to the probability that <u>neither</u> of the members selected will be a lawyer?

Ⓐ 0.5
Ⓑ 0.6
Ⓒ 0.7
Ⓓ 0.8
Ⓔ 0.9

Numeric Entry

For Questions 11 and 12, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms.**
- **Enter the exact answer unless the question asks you to round your answer.**



11. The figure above represents a rectangular garden with a walkway around it. The garden is 18 feet long and 12 feet wide. The walkway is uniformly 3 feet wide, and its edges meet at right angles. What is the area of the walkway?

[ ] square feet

12. Line k lies in the xy-plane. The x-intercept of line k is -4, and line k passes through the midpoint of the line segment whose endpoints are $(2, 9)$ and $(2, 0)$. What is the slope of line k ?

Give your answer as a fraction.

Multiple-choice Questions—Select One or More Answer Choices

For Questions 13 and 14, select all the answer choices that apply.

13. If the lengths of two sides of a triangle are 5 and 9, respectively, which of the following could be the length of the third side of the triangle?

    Indicate <u>all</u> such lengths.

    [A] 3
    [B] 5
    [C] 8
    [D] 15

14. On the number line shown above, the tick marks are equally spaced. Which of the following statements about the numbers x, y, and z must be true?

    Indicate <u>all</u> such statements.

    [A] $xyz < 0$
    [B] $x + z = y$
    [C] $z(y - x) > 0$

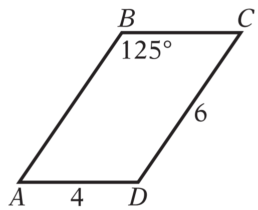
SET 3. Discrete Questions: Hard

Quantitative Comparison

For Questions 1 to 6, compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

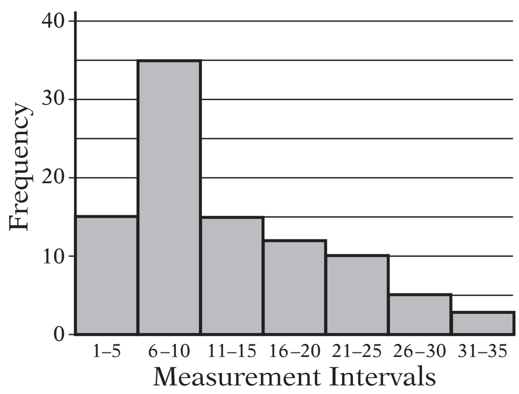
Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ The two quantities are equal.
Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.



In the figure above, *ABCD* is a parallelogram.

	Quantity A	Quantity B	
1.	The area of *ABCD*	24	Ⓐ Ⓑ Ⓒ Ⓓ



In the course of an experiment, 95 measurements were recorded, and all of the measurements were integers. The 95 measurements were then grouped into 7 measurement intervals. The graph above shows the frequency distribution of the 95 measurements by measurement interval.

	Quantity A	Quantity B	
2.	The average (arithmetic mean) of the 95 measurements	The median of the 95 measurements	Ⓐ Ⓑ Ⓒ Ⓓ

x is an integer greater than 1.

	Quantity A	Quantity B	
3.	3^{x+1}	4^x	(A) (B) (C) (D)

A, B, and C are three rectangles. The length and width of rectangle A are 10 percent greater and 10 percent less, respectively, than the length and width of rectangle C. The length and width of rectangle B are 20 percent greater and 20 percent less, respectively, than the length and width of rectangle C.

	Quantity A	Quantity B	
4.	The area of rectangle A	The area of rectangle B	(A) (B) (C) (D)

The random variable X is normally distributed. The values 650 and 850 are at the 60th and 90th percentiles of the distribution of X, respectively.

	Quantity A	Quantity B	
5.	The value at the 75th percentile of the distribution of X	750	(A) (B) (C) (D)

Set S consists of all positive integers less than 81 that are <u>not</u> equal to the square of an integer.

	Quantity A	Quantity B	
6.	The number of integers in set S	72	(A) (B) (C) (D)

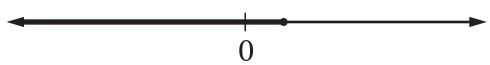
Multiple-choice Questions—Select One Answer Choice

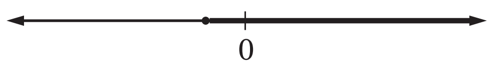
For Questions 7 to 12, select a single answer choice.

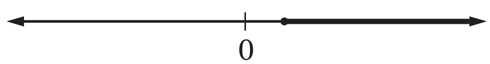
7. A manager is forming a 6-person team to work on a certain project. From the 11 candidates available for the team, the manager has already chosen 3 to be on the team. In selecting the other 3 team members, how many different combinations of 3 of the remaining candidates does the manager have to choose from?

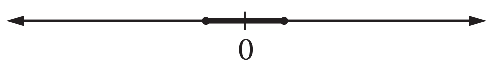
   Ⓐ 6
   Ⓑ 24
   Ⓒ 56
   Ⓓ 120
   Ⓔ 462

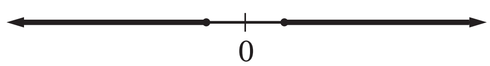
8. Which of the following could be the graph of all values of x that satisfy the inequality $2 - 5x \le -\frac{6x - 5}{3}$?

   Ⓐ 

   Ⓑ 

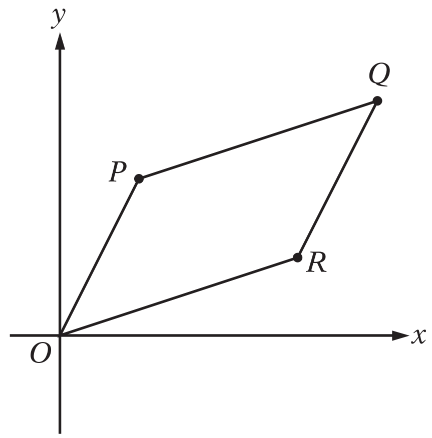
   Ⓒ 

   Ⓓ 

   Ⓔ 

9. If $1 + x + x^2 + x^3 = 60$, then the average (arithmetic mean) of x, x^2, x^3, and x^4 is equal to which of the following?

   Ⓐ $12x$
   Ⓑ $15x$
   Ⓒ $20x$
   Ⓓ $30x$
   Ⓔ $60x$



10. Parallelogram $OPQR$ lies in the xy-plane, as shown in the figure above. The coordinates of point P are $(2, 4)$ and the coordinates of point Q are $(8, 6)$. What are the coordinates of point R ?

(A) $(3, 2)$
(B) $(3, 3)$
(C) $(4, 4)$
(D) $(5, 2)$
(E) $(6, 2)$

11. The relationship between the area A of a circle and its circumference C is given by the formula $A = kC^2$, where k is a constant. What is the value of k ?

(A) $\frac{1}{4\pi}$
(B) $\frac{1}{2\pi}$
(C) $\frac{1}{4}$
(D) 2π
(E) $4\pi^2$

12. The sequence of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ is defined by $a_n = \frac{1}{n} - \frac{1}{n+2}$ for each integer $n \geq 1$. What is the sum of the first 20 terms of this sequence?

(A) $\left(1 + \frac{1}{2}\right) - \frac{1}{20}$
(B) $\left(1 + \frac{1}{2}\right) - \left(\frac{1}{21} + \frac{1}{22}\right)$
(C) $1 - \left(\frac{1}{20} + \frac{1}{22}\right)$
(D) $1 - \frac{1}{22}$
(E) $\frac{1}{20} - \frac{1}{22}$

Numeric Entry

For Question 13, enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms.**
- **Enter the exact answer unless the question asks you to round your answer.**

Y	Frequency
$\frac{1}{2}$	2
$\frac{3}{4}$	7
$\frac{5}{4}$	8
$\frac{3}{2}$	8
$\frac{7}{4}$	9

13. The table above shows the frequency distribution of the values of a variable Y. What is the mean of the distribution?

Give your answer to the nearest 0.01.

Multiple-choice Questions—Select One or More Answer Choices

For Questions 14 and 15, select all the answer choices that apply.

14. Let S be the set of all positive integers n such that n^2 is a multiple of both 24 and 108. Which of the following integers are divisors of every integer n in S ?

Indicate all such integers.

A 12
B 24
C 36
D 72

15. The range of the heights of the female students in a certain class is 13.2 inches, and the range of the heights of the male students in the class is 15.4 inches.

Which of the following statements individually provide(s) sufficient additional information to determine the range of the heights of all the students in the class?

Indicate all such statements.

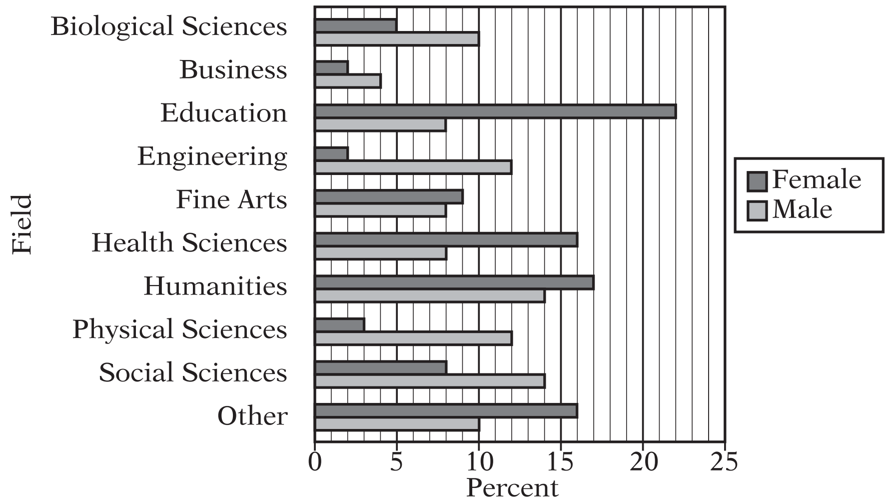
- A The tallest male student in the class is 5.8 inches taller than the tallest female student in the class.
- B The median height of the male students in the class is 1.1 inches greater than the median height of the female students in the class.
- C The average (arithmetic mean) height of the male students in the class is 4.6 inches greater than the average height of the female students in the class.

SET 4. Data Interpretation Sets

For Questions 1 to 7, select a single answer choice unless otherwise directed.

Questions 1 to 3 are based on the following data.

PERCENT OF FEMALE FACULTY AND PERCENT OF MALE FACULTY
AT UNIVERSITY *X*, BY FIELD
Total female faculty: 200
Total male faculty: 250



Medium Question

1. There are 275 students in the field of engineering at University *X*. Approximately what is the ratio of the number of students in engineering to the number of faculty in engineering?

   Ⓐ 8 to 1
   Ⓑ 10 to 1
   Ⓒ 12 to 1
   Ⓓ 14 to 1
   Ⓔ 20 to 1

Medium Question

2. Approximately what percent of the faculty in humanities are male?

   Ⓐ 35%
   Ⓑ 38%
   Ⓒ 41%
   Ⓓ 45%
   Ⓔ 51%

For Question 3, use the directions for Numeric Entry questions.

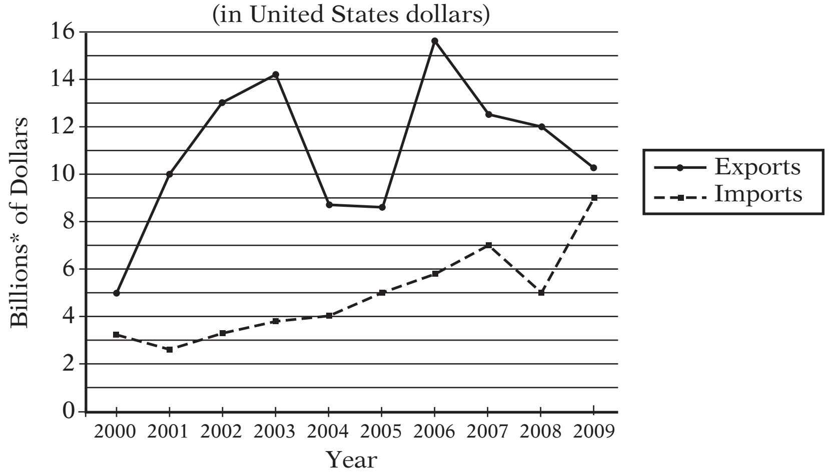
Hard Question

3. For the biological sciences and health sciences faculty combined, $\frac{1}{3}$ of the female and $\frac{2}{9}$ of the male faculty members are tenured professors. What fraction of all the faculty members in those two fields combined are tenured professors?

$$\frac{\boxed{}}{\boxed{}}$$

Questions 4 to 7 are based on the following data.

VALUE OF IMPORTS TO AND EXPORTS FROM COUNTRY *T*, 2000–2009
(in United States dollars)



*1 billion = 1,000,000,000

For Question 4, select all the answer choices that apply.

Easy Question

4. For which of the eight years from 2001 to 2008 did exports exceed imports by more than $5 billion?

Indicate all such years.

- A 2001
- B 2002
- C 2003
- D 2004
- E 2005
- F 2006
- G 2007
- H 2008

Medium Question

5. Which of the following is closest to the average (arithmetic mean) of the 9 changes in the value of imports between consecutive years from 2000 to 2009 ?

   Ⓐ $260 million
   Ⓑ $320 million
   Ⓒ $400 million
   Ⓓ $480 million
   Ⓔ $640 million

Medium Question

6. In 2008 the value of exports was approximately what percent greater than the value of imports?

   Ⓐ 40%
   Ⓑ 60%
   Ⓒ 70%
   Ⓓ 120%
   Ⓔ 140%

Hard Question

7. If it were discovered that the value of imports shown for 2007 was incorrect and should have been $5 billion instead, then the average (arithmetic mean) value of imports per year for the 10 years shown would have been approximately how much less?

   Ⓐ $200 million
   Ⓑ $50 million
   Ⓒ $20 million
   Ⓓ $7 million
   Ⓔ $5 million

ANSWER KEY

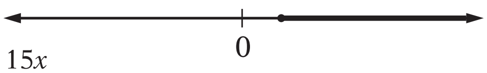
SET 1. Discrete Questions: Easy

1. **Choice A**: Quantity A is greater.
2. **Choice C**: The two quantities are equal.
3. **Choice A**: Quantity A is greater.
4. **Choice D**: The relationship cannot be determined from the information given.
5. **Choice B**: Quantity B is greater.
6. **Choice A**: Quantity A is greater.
7. **Choice C**: 4
8. **Choice D**: 12
9. **Choice A**: $10x$
10. **Choice C**: $2.50
11. **Choice A**: 11 to 9
12. **10**
13. **13**
14. **Choice A**: Pediatrics
    AND
    Choice B: Internal Medicine
    AND
    Choice C: Surgery

SET 2. Discrete Questions: Medium

1. **Choice A**: Quantity A is greater.
2. **Choice B**: Quantity B is greater.
3. **Choice D**: The relationship cannot be determined from the information given.
4. **Choice B**: Quantity B is greater.
5. **Choice C**: The two quantities are equal.
6. **Choice D**: x^5
7. **Choice D**: 1
8. **Choice E**: d is 5 times n.
9. **Choice C**: 15%
10. **Choice C**: 0.7
11. **216**
12. $\mathbf{\frac{3}{4}}$ (or any equivalent fraction)
13. **Choice B**: 5
    AND
    Choice C: 8
14. **Choice A**: $xyz < 0$
    AND
    Choice B: $x + z = y$
    AND
    Choice C: $z(y - x) > 0$

SET 3. Discrete Questions: Hard

1. **Choice B**: Quantity B is greater.
2. **Choice A**: Quantity A is greater.
3. **Choice D**: The relationship cannot be determined from the information given.
4. **Choice A**: Quantity A is greater.
5. **Choice B**: Quantity B is greater.
6. **Choice C**: The two quantities are equal.
7. **Choice C**: 56
8. **Choice C**: 

9. **Choice B**: $15x$
10. **Choice E**: $(6, 2)$
11. **Choice A**: $\frac{1}{4\pi}$
12. **Choice B**: $\left(1+\frac{1}{2}\right)-\left(\frac{1}{21}+\frac{1}{22}\right)$
13. **1.29**
14. **Choice A**: 12
    AND
    Choice C: 36
15. **Choice A**: The tallest male student in the class is 5.8 inches taller than the tallest female student in the class.

SET 4. Data Interpretation Sets

1. **Choice A**: 8 to 1
2. **Choice E**: 51%
3. $\mathbf{\frac{24}{87}}$ (or any equivalent fraction)
4. **Choice A**: 2001
   AND
   Choice B: 2002
   AND
   Choice C: 2003
   AND
   Choice F: 2006
   AND
   Choice G: 2007
   AND
   Choice H: 2008
5. **Choice E**: $640 million
6. **Choice E**: 140%
7. **Choice A**: $200 million

Answers and Explanations

For the practice questions in this chapter, use the following directions.

General Directions

For each question, indicate the best answer, using the directions given.

Notes: All numbers used are real numbers.

All figures are assumed to lie in a plane unless otherwise indicated.

Geometric figures, such as lines, circles, triangles, and quadrilaterals, **are not necessarily** drawn to scale. That is, you should **not** assume that quantities such as lengths and angle measures are as they appear in a figure. You should assume, however, that lines shown as straight are actually straight, points on a line are in the order shown, and more generally, all geometric objects are in the relative positions shown. For questions with geometric figures, you should base your answers on geometric reasoning, not on estimating or comparing quantities by sight or by measurement.

Coordinate systems, such as *xy*-planes and number lines, **are** drawn to scale; therefore, you can read, estimate, or compare quantities in such figures by sight or by measurement.

Graphical data presentations, such as bar graphs, circle graphs, and line graphs, **are** drawn to scale; therefore, you can read, estimate, or compare data values by sight or by measurement.

Directions for Quantitative Comparison questions

Compare Quantity A and Quantity B, using additional information centered above the two quantities if such information is given. Select one of the following four answer choices and fill in the corresponding oval to the right of the question.

Ⓐ Quantity A is greater.

Ⓑ Quantity B is greater.

Ⓒ The two quantities are equal.

Ⓓ The relationship cannot be determined from the information given.

A symbol that appears more than once in a question has the same meaning throughout the question.

Directions for Numeric Entry questions

Enter your answer in the answer box(es) below the question.

- **Your answer may be an integer, a decimal, or a fraction, and it may be negative.**
- **If a question asks for a fraction, there will be two boxes—one for the numerator and one for the denominator.**
- **Equivalent forms of the correct answer, such as 2.5 and 2.50, are all correct. Fractions do not need to be reduced to lowest terms.**
- **Enter the exact answer unless the question asks you to round your answer.**

SET 1. Discrete Questions: Easy

Quantitative Comparison

For Questions 1 to 6, use the directions for Quantitative Comparison questions.

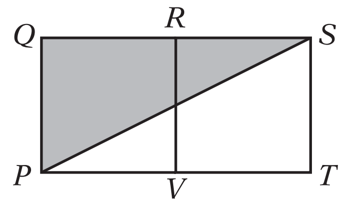
Emma spent \$75 buying a used bicycle and \$27 repairing it. Then she sold the bicycle for 40 percent more than the total amount she spent buying and repairing it.

	Quantity A	Quantity B	
1.	The price at which Emma sold the bicycle	\$140	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

In this question you are asked to compare the price at which Emma sold the bicycle with \$140. From the information given, you can conclude that Emma spent a total of $75 + 27 = 102$ dollars buying and repairing the bicycle and that she sold it for 40 percent more than the \$102 she spent buying and repairing it. If you notice that 140 is 40 percent more than 100, you can conclude that 40 percent more than 102 is greater than 40 percent more than 100, and therefore, Quantity A is greater than Quantity B. The correct answer is Choice A. (If you solve the problem in this way, you do not have to calculate the value of Quantity A.)

Another way to solve the problem is by explicitly calculating the value of Quantity A and comparing the result with \$140 directly. Since 40 percent of 102 is $(0.4)(102) = 40.8$, it follows that Quantity A, the price at which Emma sold the bicycle, is $102.00 + 40.80 = 142.80$ dollars. Thus Quantity A, \$142.80, is greater than Quantity B, \$140, and the correct answer is **Choice A**.



In the figure above, squares $PQRV$ and $VRST$ have sides of length 6.

	Quantity A	Quantity B	
2.	The area of the shaded region	36	(A) (B) (C) (D)

Explanation

In this question you are asked to compare the area of the shaded region with 36. You are given that both $PQRV$ and $VRST$ are squares with sides of length 6. Therefore, you can conclude that the length of QS is 12, and the area of the shaded right triangle PQS is $\frac{1}{2}(12)(6)$, or 36. Thus Quantity A is equal to Quantity B, and the correct answer is **Choice C**.

In 2009 the property tax on each home in Town X was p percent of the assessed value of the home, where p is a constant. The property tax in 2009 on a home in Town X that had an assessed value of \$125,000 was \$2,500.

	Quantity A	Quantity B	
3.	The property tax in 2009 on a home in Town X that had an assessed value of \$160,000	\$3,000	(A) (B) (C) (D)

Explanation

Before making the comparison in this problem, you need to analyze the information given to see what it tells you about the value of Quantity A, the property tax in 2009 on a home in Town X that had an assessed value of \$160,000. One way of doing this is to determine the value of the constant p and then use that value to calculate the tax on the home that had an assessed value of \$160,000.

Since it is given that a home that had an assessed value of \$125,000 had a property tax of \$2,500, you can conclude that p is equal to $\frac{2{,}500}{125{,}000}$, or 2%. Once you know that the property tax is 2% of the assessed value, you can determine that tax on the home that had an assessed value of \$160,000 was 2% of 160,000, or 3,200. The correct answer is **Choice A**.

Another way to calculate the property tax on a home with an assessed value of $160,000 is by setting up a proportion. Because the tax rate is the same for each home in Town X, you can let the variable x represent the tax for the home assessed at $160,000 and solve for x as follows.

$$\begin{aligned}\frac{x}{160{,}000} &= \frac{2{,}500}{125{,}000}\\ 125{,}000x &= (160{,}000)(2{,}500)\\ x &= \frac{(160{,}000)(2{,}500)}{125{,}000}\\ x &= 3{,}200\end{aligned}$$

The correct answer is **Choice A**.

$$x + y = -1$$

	Quantity A	Quantity B	
4.	x	y	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

One way to approach this question is to plug in values for one of the variables and determine the corresponding value for the other variable.

One way to plug in: Plug in easy values. For example, you can plug in $x = 0$ and find that the corresponding value of y is -1; then you can plug in $y = 0$ and find that the corresponding value of x is -1. Since in the first case x is greater than y and in the second case y is greater than x, the correct answer is **Choice D**, the relationship cannot be determined from the information given.

A second way to plug in: If you prefer to always plug in values of x to determine corresponding values of y, you can begin by writing the equation $x + y = -1$ as $y = -x - 1$. Writing it in this form makes it easier to find the corresponding values of y.

You can start by plugging in the value $x = 0$. For this value of x, the corresponding value of y is $y = -1$, and therefore, x is greater than y. If you continue plugging in a variety of values of x, some negative and some positive, you will see that sometimes x is greater than y and sometimes y is greater than x.

If you inspect the equation $y = -x - 1$, you can conclude that since there is a negative sign in front of the x but not in front of the y, for each value of x that is greater than 0, the corresponding value of y is less than 0; therefore, for each $x > 0$, x is greater than y.

What about negative values of x? A quick inspection of the equation $y = -x - 1$ allows you to conclude that if $x < -1$, then $y > 0$, so y is greater than x.

So for some values of x and y that satisfy the equation, x is greater than y; and for other values, y is greater than x. Therefore, the relationship between the two quantities x and y cannot be determined from the information given, and the correct answer is **Choice D**.

r, s, and t are three consecutive odd integers such that $r < s < t$.

	Quantity A	Quantity B	
5.	$r + s + 1$	$s + t - 1$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

You are given that three numbers, r, s, and t, are consecutive odd integers and that $r < s < t$. This means that if you express the three consecutive odd integers in terms of r, they are r, $r + 2$, and $r + 4$.

One way to approach this problem is to set up a placeholder relationship between the two quantities and simplify it to see what conclusions you can draw.

Simplification 1: Begin simplifying by expressing s and t in terms of r. The steps in this simplification can be done as follows.

$$r + s + 1 \boxed{?} s + t - 1$$
$$r + (r + 2) + 1 \boxed{?} (r + 2) + (r + 4) - 1$$
$$2r + 3 \boxed{?} 2r + 5$$
$$3 \boxed{?} 5$$

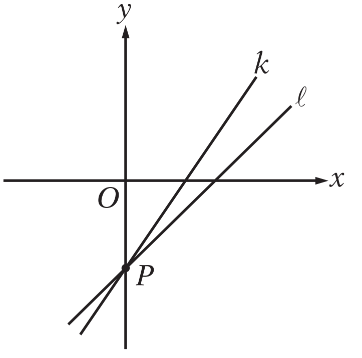
In the last step of the simplification, you can easily see that $3 < 5$. If you follow the simplification steps in reverse, you can see that the placeholder in each step remains unchanged, so you can conclude that Quantity B is greater than Quantity A, and the correct answer is **Choice B**.

Simplification 2: Since the number s appears in both quantities, you can begin the simplification by subtracting s from both sides of the relationship and then express t in terms of r. The steps in this simplification can be done as follows.

$$r + s + 1 \boxed{?} s + t - 1$$
$$r + 1 \boxed{?} t - 1$$
$$r + 1 \boxed{?} (r + 4) - 1$$
$$r + 1 \boxed{?} r + 3$$
$$1 \boxed{?} 3$$

In the last step of the simplification, you can easily see that $1 < 3$. If you follow the simplification steps in reverse, you can see that the placeholder in each step remains unchanged, so you can conclude that Quantity B is greater than Quantity A, and the correct answer is **Choice B**.

Note that in this solution, the fact that r is odd is not used; what is used is the fact that the consecutive odd integers differ by 2.



	Quantity A	Quantity B	
6.	The slope of line k	The slope of line ℓ	Ⓐ Ⓑ Ⓒ Ⓓ

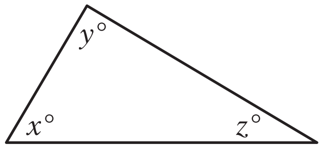
Explanation

Note that the slope of each of the lines is positive, since each line rises as it goes to the right. Since the slopes of both lines are positive and line k rises faster (or is steeper) than line ℓ, line k has the greater slope, and the correct answer is **Choice A**.

You can also use the definition of the slope to arrive at the correct answer. Slope can be defined as the ratio of "rise" to "run" between any two points on a line, where the rise is the vertical distance between the points and the run is the horizontal distance, and the slope is respectively positive or negative depending on whether the line rises or falls when viewed from left to right. Because both lines pass through point P on the y-axis, they have the same rise from P to the x-axis. However, line ℓ intersects the x-axis at a greater value than line k. Thus, the run of line ℓ from the y-axis to the x-intercept is greater than the run of line k. When the slope is expressed as a ratio, both lines have the same numerator (rise), but line ℓ has a greater denominator (run). The greater denominator results in a lesser fraction and a lesser slope for line ℓ. Therefore, the correct answer is **Choice A**.

Multiple-choice Questions—Select One Answer Choice

For Questions 7 to 11, select a single answer choice.



7. In the figure above, what is the value of $\frac{x+y+z}{45}$?

   (A) 2
   (B) 3
   (C) 4
   (D) 5
   (E) 6

Explanation

The sum of the measures, in degrees, of the three interior angles of any triangle is 180°. As shown in the figure, the three angles of the triangle have measures of $x°$, $y°$, and $z°$, so $x + y + z = 180$. Therefore, $\frac{x+y+z}{45} = \frac{180}{45} = 4$, and the correct answer is **Choice C**.

8. A certain store sells two types of pens: one type for \$2 per pen and the other type for \$3 per pen. If a customer can spend up to \$25 to buy pens at the store and there is no sales tax, what is the greatest number of pens the customer can buy?

Ⓐ 9
Ⓑ 10
Ⓒ 11
Ⓓ 12
Ⓔ 20

Explanation

It is fairly clear that the greatest number of pens that can be bought for \$25 will consist mostly, if not entirely, of \$2 pens. In fact, it is reasonable to begin by looking at how many of the \$2 pens the customer can buy if the customer does not buy any \$3 pens. It is easy to see that the customer could buy 12 of the \$2 pens, with \$1 left over.

If the customer bought 11 of the \$2 pens, there would be \$3 left over with which to buy a \$3 pen. In this case, the customer could still buy 12 pens.

If the customer bought 10 of the \$2 pens, there would be \$5 left over. Only 1 of the \$3 pens could be bought with the \$5, so in this case, the customer could buy only 11 pens.

As the number of \$2 pens decreases, the total number of pens that the customer can buy with \$25 decreases as well. Thus the greatest number of pens the customer can buy with \$25 is 12. The correct answer is **Choice D**.

9. If $y = 3x$ and $z = 2y$, what is $x + y + z$ in terms of x ?

Ⓐ $10x$
Ⓑ $9x$
Ⓒ $8x$
Ⓓ $6x$
Ⓔ $5x$

Explanation

It is not necessary to find the individual values of x, y, and z to answer the question. You are asked to rewrite the expression $x + y + z$ as an equivalent expression in terms of x. This means that you need to use the information provided about y and z to express them in terms of the variable x. The variable y is already given in terms of x; that is, $y = 3x$; and because $z = 2y$, it follows that $z = (2)(3x) = 6x$. Using substitution, you can rewrite the expression as follows.

$$\begin{aligned} x + y + z &= x + (3x) + (6x) \\ &= (1 + 3 + 6)x \\ &= 10x \end{aligned}$$

The correct answer is **Choice A**.

10. A certain shipping service charges an insurance fee of \$0.75 when shipping any package with contents worth \$25.00 or less and an insurance fee of \$1.00 when shipping any package with contents worth over \$25.00. If Dan uses the shipping company to ship three packages with contents worth \$18.25, \$25.00, and \$127.50, respectively, what is the total insurance fee that the company charges Dan to ship the three packages?

Ⓐ \$1.75
Ⓑ \$2.25
Ⓒ \$2.50
Ⓓ \$2.75
Ⓔ \$3.00

Explanation

Note that two of the packages being shipped have contents that are worth \$25.00 or less. Therefore, each of them has an insurance fee of \$0.75, for a total of \$1.50. The third package has contents worth over \$25.00, and it has an insurance fee of \$1.00. Therefore, the total insurance fee for the three packages is $\$1.50 + \$1.00 = \$2.50$, and the correct answer is **Choice C**.

11. If 55 percent of the people who purchase a certain product are female, what is the ratio of the number of females who purchase the product to the number of males who purchase the product?

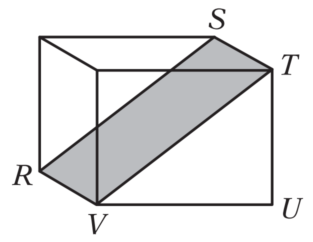
Ⓐ 11 to 9
Ⓑ 10 to 9
Ⓒ 9 to 10
Ⓓ 9 to 11
Ⓔ 5 to 9

Explanation

Note that because 55 percent of the people who purchase the product are females, it follows that 45 percent of the people who purchase the product are males. Therefore, the ratio of the number of females who purchase the product to the number of males who purchase the product is 55 to 45, or 11 to 9, and the correct answer is **Choice A**.

Numeric Entry

For Questions 12 and 13, use the directions for Numeric Entry questions.



12. In the rectangular solid above, $TU = 3$, $UV = 4$, and $VR = 2$. What is the area of the shaded rectangular region?

Explanation

To find the area of the shaded rectangular region, you need to multiply the length of the rectangular region by its width. In this question you are given the lengths of three edges: $TU = 3$, $UV = 4$, and $VR = 2$. Note that VR is the length of the shaded rectangle. To find the width of the shaded rectangle, you need to find either RS or VT. Note that VT lies on the front face of the rectangular solid. It is the hypotenuse of right triangle VUT. You know that $UV = 4$ and $TU = 3$, so by the Pythagorean theorem you can conclude that $VT = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. Therefore, the area of the shaded rectangular region is $(5)(2) = 10$. The correct answer is **10**.

13. A list of numbers has a mean of 8 and a standard deviation of 2.5. If x is a number in the list that is 2 standard deviations above the mean, what is the value of x ?

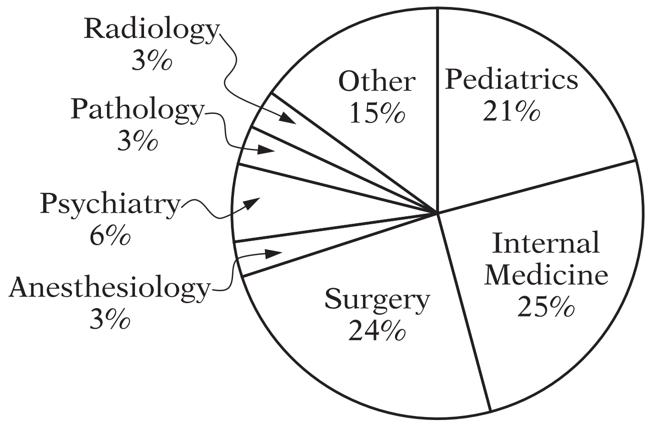
$x =$

Explanation

You are given that x is 2 standard deviations above the mean, 8. Because the standard deviation of the numbers in the list is 2.5, it follows that x is $(2)(2.5)$, or 5 units above the mean 8. Therefore, $x = 8 + 5 = 13$, and the correct answer is **13**.

Multiple-choice Questions—Select One or More Answer Choices

For Question 14, select all the answer choices that apply.



14. The circle graph above shows the distribution of 200,000 physicians by specialty. Which of the following sectors of the circle graph represent more than 40,000 physicians?

Indicate all such sectors.

- A Pediatrics
- B Internal Medicine
- C Surgery
- D Anesthesiology
- E Psychiatry

Explanation

One approach to solve this problem is to find out what percent of 200,000 is 40,000 and then compare this percent with the percents given in the circle graph. Because $\frac{40,000}{200,000} = 0.2$, it follows that 40,000 is 20% of 200,000, and any specialty that has more than 20% of the distribution has more than 40,000 physicians. This is true for the specialties of pediatrics, internal medicine, and surgery. The correct answer consists of **Choices A, B, and C**.

SET 2. Discrete Questions: Medium

Quantitative Comparison

For Questions 1 to 5, use the directions for Quantitative Comparison questions.

Machine R, working alone at a constant rate, produces x units of a product in 30 minutes, and machine S, working alone at a constant rate, produces x units of the product in 48 minutes, where x is a positive integer.

	Quantity A	Quantity B	
1.	The number of units of the product that machine R, working alone at its constant rate, produces in 3 hours	The number of units of the product that machine S, working alone at its constant rate, produces in 4 hours	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

In this question you are given that machine R, working alone at its constant rate, produces x units of a product in 30 minutes. Since it is easy to see that 3 hours is 6 times 30 minutes, you can conclude that Quantity A is $6x$.

You can compare $6x$ with Quantity B in two ways.

One: In the additional information centered above the quantities, you are given that machine S, working alone at its constant rate, produces x units of the product in 48 minutes, so you can conclude that machine S can produce $6x$ units of the product in (6)(48) minutes, or 4.8 hours. So in 4 hours, machine S produces less than $6x$ units, and Quantity B is less than $6x$.

Two: First, convert 48 minutes to $\frac{4}{5}$ hour, then find the number of 48-minute periods there are in 4 hours by computing $\frac{4}{\left(\frac{4}{5}\right)} = (4)\left(\frac{5}{4}\right) = 5$. Thus, Quantity B is $5x$.

Either way, Quantity A is greater than Quantity B, and the correct answer is **Choice A**.

Frequency Distribution for List X

Number	1	2	3	5
Frequency	10	20	18	12

Frequency Distribution for List Y

Number	6	7	8	9
Frequency	24	17	10	9

List X and list Y each contain 60 numbers. Frequency distributions for each list are given above. The average (arithmetic mean) of the numbers in list X is 2.7, and the average of the numbers in list Y is 7.1. List Z contains 120 numbers: the 60 numbers in list X and the 60 numbers in list Y.

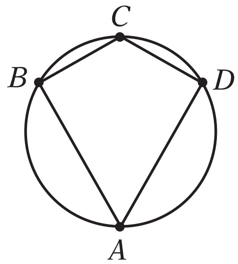
	Quantity A	Quantity B	
2.	The average of the 120 numbers in list Z	The median of the 120 numbers in list Z	(A) (B) (C) (D)

Explanation

In this problem you are asked to compare the average with the median of the 120 numbers in list Z. Since list Z consists of the numbers in lists X and Y combined, it is reasonable to try to use the information about lists X and Y to calculate the average and the median of the numbers in list Z.

To determine the average of the 120 numbers in list Z, you can use the information given about the individual averages of the numbers in lists X and Y. Because lists X and Y each contain 60 numbers, the average of the numbers in list Z is the average of the individual averages of the numbers in lists X and Y. Thus, the average of the numbers in list Z is $\frac{2.7 + 7.1}{2}$, or 4.9.

To determine the median of the 120 numbers in list Z, first note that list Z contains an even number of numbers, so the median of the numbers in list Z is the average of the middle two numbers when the numbers are listed in increasing order. If you look at the numbers in the two lists, you will see that the 60 numbers in list X are all less than or equal to 5, and the 60 numbers in list Y are all greater than or equal to 6. Thus, the two middle numbers in list Z are 5 and 6, and the average of these numbers is $\frac{5 + 6}{2}$, or 5.5. Therefore, the median of the numbers in list Z is 5.5, and this is greater than the average of 4.9. The correct answer is **Choice B**.



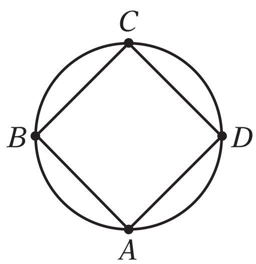
In the figure above, the diameter of the circle is 10.

	Quantity A	Quantity B	
3.	The area of quadrilateral *ABCD*	40	Ⓐ Ⓑ Ⓒ Ⓓ

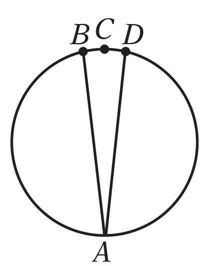
Explanation

You are given that the circle has a diameter of 10, and from the figure you can assume that points *A*, *B*, *C*, and *D* lie on the circle in the order shown. However, because figures are not necessarily drawn to scale, you cannot assume anything else about the positions of points *A*, *B*, *C*, and *D* on the circle. Therefore, to get an idea of how various possible positions of these four points could affect the area of quadrilateral *ABCD*, it is a good idea to see how the figure can vary but still have points *A*, *B*, *C*, and *D* in the same order as in the figure above.

One way that you might vary the figure is to evenly space the four points along the circle, as shown below.



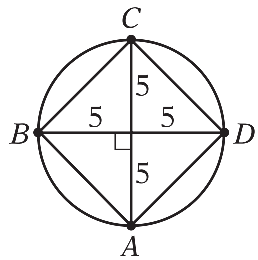
Another way is to draw points *A* and *C* opposite each other, with points *B* and *D* close to point *C*, as shown below.



From these figures you can draw some basic conclusions about the area of *ABCD*.

If points *A* and *C* are opposite each other, with points *B* and *D* very close to point *C*, the area of quadrilateral *ABCD* is very close to 0. Clearly, the area can be less than 40 (Quantity B).

If points *A*, *B*, *C*, and *D* are evenly spaced, the area is not close to 0. How does the area compare with 40? To calculate the area of *ABCD*, draw the diameters AC and BD in the figure. The two diameters are perpendicular bisectors of each other, so they divide $ABCD$ into four right triangles, as shown.



The area of each of the right triangles is $\left(\frac{1}{2}\right)(5)(5)$, or 12.5. Thus, the area of $ABCD$ is (4)(12.5), or 50.

Since the area of the quadrilateral in the first figure is less than 40 and the area of the quadrilateral in the second figure is greater than 40, the relationship cannot be determined from the information given. The correct answer is **Choice D**.

$$x^2y > 0$$
$$xy^2 < 0$$

	Quantity A	Quantity B	
4.	x	y	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

You are given that $x^2y > 0$, which means that the product of the two numbers x^2 and y is positive. Recall that the product of two numbers is positive only if both numbers are positive or both numbers are negative. The square of a number is always greater than or equal to 0. In this case, x^2 cannot equal 0 because the product x^2y is not 0. Thus, x^2 is positive and it follows that y is also positive.

You are also given that $xy^2 < 0$, which means that the product of the two numbers x and y^2 is negative. The product of two numbers is negative only if one of the numbers is negative and the other number is positive. In this case, y^2 cannot be negative because it is the square of a number, and it cannot be 0 because the product x^2y is not 0. Thus, y^2 is positive and so x must be negative.

Because x is negative and y is positive, y must be greater than x, and the correct answer is **Choice B**.

Among the 9,000 people attending a football game at College C, there were x students from College C and y students who were not from College C.

	Quantity A	Quantity B	
5.	The number of people attending the game who were not students	$9{,}000 - x - y$	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

In this question you are not told whether all of the 9,000 people attending the game were students. Let z be the number of people attending the game who were not students. The people attending the game can be broken down into three groups: students from College C, students not from College C, and people who were not students. This can be expressed algebraically as $9{,}000 = x + y + z$, where x represents the number of students from College C attending the game and y represents the number of students attending the game who were not from College C. Therefore, $9{,}000 - x - y = z$ is the number of people attending the game who were not students. The correct answer is **Choice C**.

Multiple-choice Questions—Select One Answer Choice

For Questions 6 to 10, select a single answer choice.

6. If $x \neq 0$, which of the following is equivalent to $\frac{x(x^2)^3}{x^2}$?

Ⓐ x^2
Ⓑ x^3
Ⓒ x^4
Ⓓ x^5
Ⓔ x^6

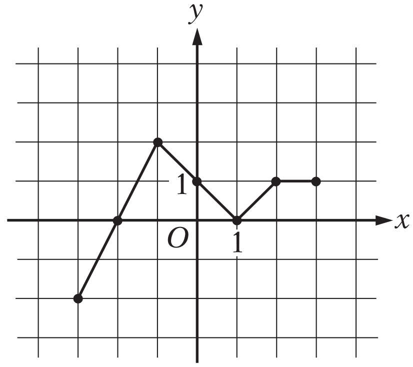
Explanation

To simplify $\frac{x(x^2)^3}{x^2}$, it can be helpful to write $(x^2)^3$ as $(x^2)(x^2)(x^2)$ in the given expression; that is, $\frac{x(x^2)^3}{x^2} = \frac{x(x^2)(x^2)(x^2)}{x^2}$. Because $x \neq 0$, both numerator and denominator can be divided by x^2, and the expression simplifies to $x(x^2)(x^2)$, which, by the rules of exponents, is equal to x^5.

Another way to simplify the expression using the rules of exponents directly is as follows.

$$\frac{x(x^2)^3}{x^2} = \frac{x(x^6)}{x^2} = \frac{x^7}{x^2} = x^5$$

The correct answer is **Choice D**.



7. The figure above shows the graph of the function f in the xy-plane. What is the value of $f(f(-1))$?

Ⓐ -2
Ⓑ -1
Ⓒ 0
Ⓓ 1
Ⓔ 2

Explanation

Note that to find $f(f(-1))$, you must apply the function f twice, first to find the value of $f(-1)$ and then to find the value of $f(f(-1))$. To find the value of $f(-1)$, find the point on the graph of the function f whose x-coordinate is $x = -1$. This point has y-coordinate $y = 2$. Therefore, the value of $f(-1)$ is 2, and $f(f(-1)) = f(2)$. Next you need to find the value of $f(2)$. To find the value of $f(2)$, find the point on the graph whose x-coordinate is $x = 2$. This point has y-coordinate $y = 1$. Therefore, $f(2) = 1$, and because $f(f(-1)) = f(2)$, you can conclude that $f(f(-1)) = 1$. The correct answer is **Choice D**.

8. If $\dfrac{d-3n}{7n-d} = 1$, which of the following statements describes d in terms of n?

Ⓐ d is 4 less than n.
Ⓑ d is 4 more than n.
Ⓒ d is $\dfrac{3}{7}$ of n.
Ⓓ d is 2 times n.
Ⓔ d is 5 times n.

Explanation

To describe d in terms of n, you need to solve the equation $\dfrac{d-3n}{7n-d} = 1$ for d. To simplify the equation, you can begin by multiplying both sides by $7n - d$ and then proceed as follows.

$$(7n-d)\left(\frac{d-3n}{7n-d}\right) = (7n-d)(1)$$
$$d - 3n = 7n - d$$
$$d = 10n - d$$
$$2d = 10n$$
$$d = 5n$$

The correct answer is **Choice E**.

9. By weight, liquid A makes up 8 percent of solution R and 18 percent of solution S. If 3 grams of solution R are mixed with 7 grams of solution S, then liquid A accounts for what percent of the weight of the resulting solution?

Ⓐ 10%
Ⓑ 13%
Ⓒ 15%
Ⓓ 19%
Ⓔ 26%

Explanation

Liquid A makes up 8 percent of the weight of solution R and 18 percent of the weight of solution S. Therefore, 3 grams of solution R contain $(0.08)(3)$, or 0.24 gram of liquid A, and 7 grams of solution S contain $(0.18)(7)$, or 1.26 grams of liquid A. When the two solutions are mixed, the resulting solution weighs $3 + 7$, or 10 grams and contains $0.24 + 1.26$, or 1.5 grams of liquid A. This means that liquid A makes up $\frac{1.5}{10}$, or $\frac{15}{100}$, or 15 percent of the weight of the resulting solution. The correct answer is **Choice C**.

10. Of the 700 members of a certain organization, 120 are lawyers. Two members of the organization will be selected at random. Which of the following is closest to the probability that <u>neither</u> of the members selected will be a lawyer?

Ⓐ 0.5
Ⓑ 0.6
Ⓒ 0.7
Ⓓ 0.8
Ⓔ 0.9

Explanation

The probability that neither of the members selected will be a lawyer is equal to the fraction

$$\frac{\text{the number of ways 2 members who are not lawyers can be selected}}{\text{the number of ways 2 members can be selected}}$$

where the order of selection does not matter.

Since there are 120 members who are lawyers, there must be $700 - 120$, or 580 members who are not lawyers. There are 580 ways of selecting a first member who is not a lawyer and 579 ways of selecting a second member who is not a lawyer. Multiplying these two numbers gives the number of ways to select 2 members who are not lawyers. However, in the $(580)(579)$ ways, each group of 2 members who are not lawyers is counted twice. You can see this by considering 2 members, A and B. The 2 members can be chosen in 2 ways: A first, followed by B, and B first, followed by A. To adjust for double counting, you need to divide $(580)(579)$ by 2.

Similarly, the number of ways 2 members can be selected from among the 700 members is (700)(699) divided by 2. Thus, the desired probability is

$$\frac{\frac{(580)(579)}{2}}{\frac{(700)(699)}{2}} = \frac{(580)(579)}{(700)(699)}$$

Since the answer choices are all tenths, you need to approximate the value of this fraction to the nearest tenth. There are several ways to do this approximation. One way is to use your calculator to convert the fraction to a decimal and round the decimal to the nearest tenth.

Another way is to approximate the value of the fraction as follows.

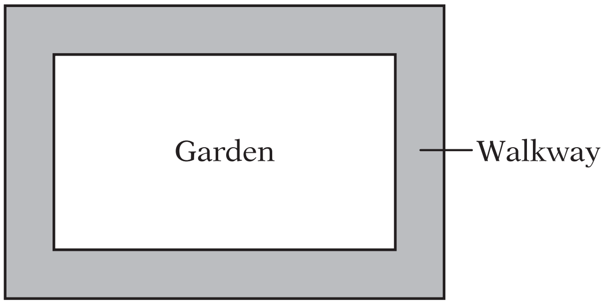
$$\frac{(580)(579)}{(700)(699)} \approx \frac{(600)(600)}{(700)(700)} = \left(\frac{6}{7}\right)^2 = \frac{36}{49} \approx \frac{36}{50} = 0.72$$

Either way, the answer choice that is closest to the value of the fraction is 0.7. The correct answer is **Choice C**.

Another approach to this problem is to consider the random selections as two separate but successive events. The probability of selecting a first member who is not a lawyer is $\frac{580}{700}$, because there are 580 members out of the 700 members who are not lawyers. For the second selection, there are only 699 members left to select from, because one member has already been selected. If the first member selected is not a lawyer, then there are only 579 members left who are not lawyers. So the probability of selecting a second member who is not a lawyer, given the condition that the first member selected was not a lawyer, is $\frac{579}{699}$. The probability that both members selected will not be lawyers is the product of the two probabilities, or $\left(\frac{580}{700}\right)\left(\frac{579}{699}\right)$, which is approximated above as 0.72. The correct answer is **Choice C.**

Numeric Entry

For Questions 11 and 12, use the directions for Numeric Entry questions.



11. The figure above represents a rectangular garden with a walkway around it. The garden is 18 feet long and 12 feet wide. The walkway is uniformly 3 feet wide, and its edges meet at right angles. What is the area of the walkway?

[ ] square feet

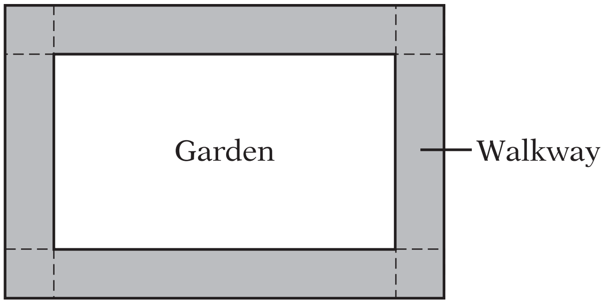
Explanation

You can see from the figure that the shaded region is the region between the two rectangles. Looking at the shaded region in this way suggests that the area of the walkway can be calculated as the difference between the area of the larger rectangle and the area of the smaller rectangle.

The region represented by the smaller rectangle is the garden. Since the garden is 18 feet long and 12 feet wide, its area is (18)(12), or 216 square feet.

The region represented by the larger rectangle is the garden and the walkway combined. The length of the region is the length of the garden plus twice the width of the walkway, or $18 + (2)(3) = 24$ feet. The width of the region is the width of the garden plus twice the width of the walkway, or $12 + (2)(3) = 18$ feet. Therefore, the area of the region represented by the larger rectangle is (24)(18), or 432 square feet, and the area of the walkway is $432 - 216$, or 216 square feet.

Another way to approach this problem is to think of the walkway as being composed of four rectangles and four squares, as shown in the figure below.



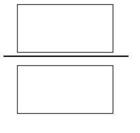
Each of the four squares is 3 feet long and 3 feet wide. The two rectangles running along the length of the garden are 18 feet long and 3 feet wide, and the two rectangles running along the width of the garden are 12 feet long and 3 feet wide. Thus, the area of the walkway is

$$4(3)(3) + 2(18)(3) + 2(12)(3) = 36 + 108 + 72 = 216 \text{ square feet}$$

The correct answer is **216**.

12. Line k lies in the xy-plane. The x-intercept of line k is -4, and line k passes through the midpoint of the line segment whose endpoints are $(2, 9)$ and $(2, 0)$. What is the slope of line k ?

    Give your answer as a fraction.



Explanation

You can calculate the slope of a line if you know the coordinates of two points on the line. In this question you are given information about two points on line k, namely,

- the point at which line k crosses the x-axis has x-coordinate -4;
- the midpoint of the line segment with endpoints at $(2, 9)$ and $(2, 0)$ is on line k.

The coordinates of the first point are $(-4, 0)$, since the x-coordinate is -4 and the y-coordinate of every point on the x-axis is 0. For the second point, the midpoint of the line segment is halfway between the endpoints $(2, 9)$ and $(2, 0)$. Thus, the midpoint has x-coordinate 2 and y-coordinate $\frac{9}{2}$, the number halfway between 9 and 0. Based on the coordinates $(-4, 0)$ and $\left(2, \frac{9}{2}\right)$, the slope of line k is

$$\frac{\frac{9}{2} - 0}{2 - (-4)} = \frac{\frac{9}{2}}{6} = \frac{3}{4}$$

The correct answer is $\mathbf{\frac{3}{4}}$ (or any equivalent fraction).

Multiple-choice Questions—Select One or More Answer Choices

For Questions 13 and 14, select all the answer choices that apply.

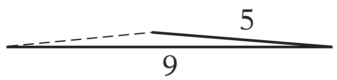
13. If the lengths of two sides of a triangle are 5 and 9, respectively, which of the following could be the length of the third side of the triangle?

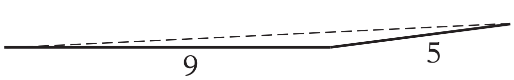
    Indicate all such lengths.

    A 3
    B 5
    C 8
    D 15

Explanation

A good way to approach this problem is to think about how much the length of the third side of a triangle with two fixed side lengths can vary. If you think about it a bit, you will see that the smaller the interior angle between the two sides of the triangle is, the smaller the length of the third side is; and the larger the interior angle between the two sides of the triangle is, the larger the length of the third side is. This suggests drawing two triangles, one in which the angle between the two sides is close to 0 degrees and one in which the angle between the two sides is close to 180 degrees, like the triangles below.

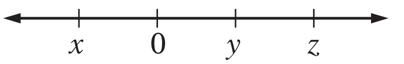




In the triangle in which the angle between the sides of length 5 and 9 is small, you can see that the length of the third side is a bit greater than $9 - 5$, or 4. If it were equal to 4, the triangle would degenerate into a line segment.

In the triangle in which the angle between the sides of length 5 and 9 is large, you can see that the length of the third side is a bit less than $9 + 5$, or 14. If it were equal to 14, the triangle would degenerate into a line segment.

Therefore, the length of the third side of the triangle must be greater than 4 and less than 14. Furthermore, it is intuitive that any length between these two numbers can be achieved by some triangle. The correct answer consists of **Choices B and C**.



14. On the number line shown above, the tick marks are equally spaced. Which of the following statements about the numbers x, y, and z must be true?

Indicate <u>all</u> such statements.

[A] $xyz < 0$

[B] $x + z = y$

[C] $z(y - x) > 0$

Explanation

You can see from their positions on the number line that x is less than 0 and both y and z are greater than 0. Because the tick marks are equally spaced, you can also see that $x = -y$ and $z = 2y$. You need to evaluate each answer choice separately to determine whether it must be true.

Choice A says that the product of the three numbers x, y, and z is less than 0. Recall that the product of three numbers is negative under either of the following two conditions.

- All three numbers are negative.
- One of the numbers is negative and the other two numbers are positive.

Choice A must be true, since x is negative and y and z are positive.

Choice B is the equation $x + z = y$. To see whether the equation must be true, it is a good idea to express two of the variables in terms of the third (that is, to "get rid of" two of the variables). The equations $x = -y$ and $z = 2y$ give x and z in terms of y, so the equation $x + z = y$ can be rewritten, substituting $-y$ for x and $2y$ for z, as $-y + 2y = y$. In this form you can quickly conclude that the equation must be true.

Choice C says that the product of the two numbers z and $y - x$ is greater than 0. Recall that the product of two numbers is positive under either of the following two conditions.

- Both numbers are positive.
- Both numbers are negative.

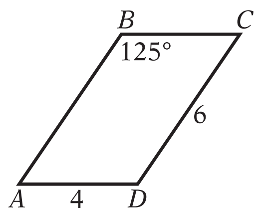
Since you already know that z is positive, you can conclude that the product $z(y - x)$ will be positive if $y - x$ is positive. By adding x to both sides of the inequality $y - x > 0$, you can see that it is equivalent to the inequality $y > x$, which is clearly true from the number line. Since $y - x$ is positive, the product $z(y - x)$ must be positive.

Therefore, the correct answer consists of **Choices A, B, and C**.

SET 3. Discrete Questions: Hard

Quantitative Comparison

For Questions 1 to 6, use the directions for Quantitative Comparison questions.



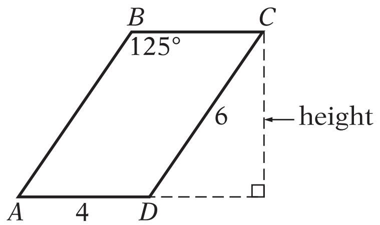
In the figure above, *ABCD* is a parallelogram.

	Quantity A	Quantity B	
1.	The area of *ABCD*	24	Ⓐ Ⓑ Ⓒ Ⓓ

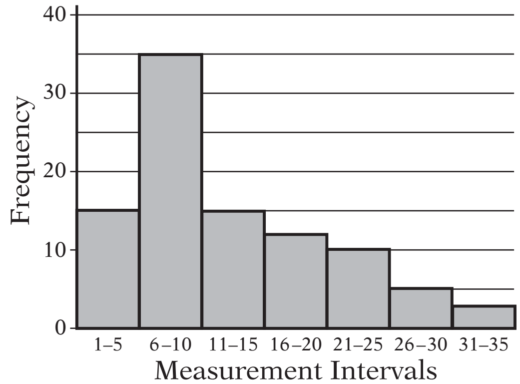
Explanation

In this question you are asked to compare the area of a parallelogram with an area of 24, given two side lengths and the measure of one interior angle of the parallelogram. Since the measure of the interior angle given is 125°, you can conclude that the parallelogram is not a rectangle.

Recall that the area of a parallelogram is found by multiplying the length of a base by the height corresponding to the base. It is helpful to draw the vertical height from vertex *C* to base *AD* of the parallelogram, as shown in the figure below.



Note that the newly drawn height is a leg in a newly formed right triangle. The hypotenuse of the triangle is a side of the parallelogram and has length 6. Thus, the leg of the triangle, which is the height of the parallelogram, must be less than the hypotenuse 6. The area of the parallelogram is equal to the length of base *AD*, which is 4, times the height, which is less than 6. Since the product of 4 and a number less than 6 must be less than 24, the area of the parallelogram must be less than 24. Quantity B is greater than Quantity A, and the correct answer is **Choice B.**



In the course of an experiment, 95 measurements were recorded, and all of the measurements were integers. The 95 measurements were then grouped into 7 measurement intervals. The graph above shows the frequency distribution of the 95 measurements by measurement interval.

	Quantity A	Quantity B	
2.	The average (arithmetic mean) of the 95 measurements	The median of the 95 measurements	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

From the histogram, you can observe that

- all of the measurement intervals are the same size,
- the distribution has a peak at the measurement interval 6–10, and
- more of the measurement intervals are to the right of the peak than are to the left of the peak.

Since in the histogram the 95 measurements have been grouped into intervals, you cannot calculate the exact value of either the average or the median; you must compare them without being able to determine the exact value of either one.

The median of the 95 measurements is the middle measurement when the measurements are listed in increasing order. The middle measurement is the 48th measurement. From the histogram, you can see that the measurement interval 1–5 contains the first 15 measurements, and the measurement interval 6–10 contains the next 35 measurements (that is, measurements 16 through 50). Therefore, the median is in the measurement interval 6–10 and could be 6, 7, 8, 9, or 10.

Estimating the average of the 95 measurements is more complicated.

Since you are asked to compare the average and the median, not necessarily to calculate them, you may ask yourself if you can tell whether the average is greater than or less than the median. Note that visually the measurements in the first three measurement intervals are symmetric around the measurement interval 6–10, so you would expect the average of the measurements in just these three measurement intervals to lie in the 6–10 measurement interval. The 30 measurements in the remaining four measurement intervals are all greater than 10, some significantly greater than 10. Therefore, the average of the 95 measurements is greater than the average of the measurements in the first three measurement intervals, probably greater than 10. At this point it seems likely that the average of the 95 measurements is greater than the median of the 95 measurements. It turns out that this is true.

To actually show that the average must be greater than 10, you can make the average as small as possible and see if the smallest possible average is greater than 10. To make the average as small as possible, assume that all of the measurements in each interval are as small as possible. That is to say, all 15 measurements in the measurement interval 1–5 are equal to 1, all 35 measurements in the measurement interval 6–10 are equal to 6, etc. Under this assumption, the average of the 95 measurements is

$$\frac{(1)(15)+(6)(35)+(11)(15)+(16)(12)+(21)(10)+(26)(5)+(31)(3)}{95}=\frac{1{,}015}{95}$$

The value of the smallest possible average, $\frac{1{,}015}{95}$, is greater than 10.

Therefore, since the average of the 95 measurements is greater than 10 and the median is in the measurement interval 6–10, it follows that the average is greater than the median, and the correct answer is **Choice A**.

x is an integer greater than 1.

	Quantity A	Quantity B	
3.	3^{x+1}	4^x	(A) (B) (C) (D)

Explanation

One way to approach this question is to plug in numbers for the variables and see what the relationship between the two quantities is for each of the numbers you plug in.

If you plug in $x = 2$, you see that Quantity A is $3^{x+1} = 3^3$, or 27, and Quantity B is $4^x = 4^2$, or 16. In this case, Quantity A is greater than Quantity B.

If you plug in $x = 3$, you see that Quantity A is $3^{x+1} = 3^4$, or 81, and Quantity B is $4^x = 4^3$, or 64. In this case, Quantity A is greater than Quantity B.

If you plug in $x = 4$, you see that Quantity A is $3^{x+1} = 3^5$, or 243, and Quantity B is $4^x = 4^4$, or 256. In this case, Quantity B is greater than Quantity A. Since for $x = 2$ and for $x = 3$, Quantity A is greater than Quantity B, and for $x = 4$, Quantity B is greater than Quantity A, it follows that the relationship between the two quantities cannot be determined. The correct answer is **Choice D**.

Since both quantities are algebraic expressions, another way to approach this problem is to set up a placeholder relationship between the two quantities and simplify it to see what conclusions you can draw.

$$3^{x+1} \;\boxed{?}\; 4^x$$

$$3(3^x) \;\boxed{?}\; 4^x$$

$$\frac{3(3^x)}{3^x} \;\boxed{?}\; \frac{4^x}{3^x}$$

$$3 \;\boxed{?}\; \left(\frac{4}{3}\right)^x$$

For any value of x, the value of 3^x is positive, so dividing by 3^x does not change any inequality that could be put in the placeholder. Since each step in this simplification is reversible, this reduces the problem to comparing 3 with $\left(\frac{4}{3}\right)^x$.

You can see that because $\frac{4}{3}$ is greater than 1, the value of $\left(\frac{4}{3}\right)^x$ becomes greater as x becomes larger. In particular, it is greater than 3 for large enough values of x. For the smallest value of x, $x = 2$, the relationship is $\left(\frac{4}{3}\right)^2 = \frac{16}{9} < 3$.

Since for $x = 2$, Quantity A is greater than Quantity B and for large values of x, Quantity B is greater than Quantity A, it follows that the relationship between the two quantities cannot be determined. The correct answer is **Choice D**.

A, *B*, and *C* are three rectangles. The length and width of rectangle *A* are 10 percent greater and 10 percent less, respectively, than the length and width of rectangle *C*. The length and width of rectangle *B* are 20 percent greater and 20 percent less, respectively, than the length and width of rectangle *C*.

	Quantity A	Quantity B	
4.	The area of rectangle *A*	The area of rectangle *B*	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

In this question you are asked to compare the area of rectangle *A* and the area of rectangle *B*. Since the information given relates the dimensions of both rectangle *A* and rectangle *B* to the corresponding dimensions of rectangle *C*, you can try to use the relationships to make the desired comparison.

If ℓ represents the length of rectangle *C* and w represents its width, then the length and width of rectangles *A* and *B* can be translated into algebraic expressions as follows.

- The length of rectangle *A* is 10 percent greater than the length of rectangle *C*, or 1.1ℓ.
- The width of rectangle *A* is 10 percent less than the width of rectangle *C*, or $0.9w$.
- The length of rectangle *B* is 20 percent greater than the length of rectangle *C*, or 1.2ℓ.
- The width of rectangle *B* is 20 percent less than the width of rectangle *C*, or $0.8w$.

In terms of ℓ and w, the area of rectangle *A* is $(1.1\ell)(0.9w)$, or $0.99\ell w$.

In terms of ℓ and w, the area of rectangle *B* is $(1.2\ell)(0.8w)$, or $0.96\ell w$.

Since $0.99\ell w$ is greater than $0.96\ell w$, Quantity A is greater than Quantity B, and the correct answer is **Choice A**.

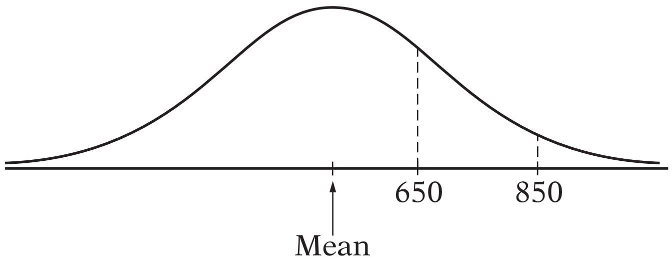
The random variable X is normally distributed. The values 650 and 850 are at the 60th and 90th percentiles of the distribution of X, respectively.

	Quantity A	Quantity B	
5.	The value at the 75th percentile of the distribution of X	750	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

You are given that the distribution of random variable X is normal and that the values 650 and 850 are at the 60th and 90th percentiles of the distribution, respectively.

Both of the values 650 and 850 are greater than the mean of the distribution. If you draw a rough sketch of the graph of the normal distribution, the sketch could look something like the one below. Note that it is not necessary to know the exact location of 650 and 850, just that both values are above the mean.



To say that the value 650 is at the 60th percentile of the distribution means, graphically, that 60 percent of the area between the normal curve and the horizontal axis lies to the left of the vertical line segment at 650. To say that 850 is at the 90th percentile of the distribution means that 90 percent of the area between the normal curve and the horizontal axis lies to the left of the vertical line segment at 850.

The value 750 is halfway between 650 and 850. However, because the curve is decreasing in that interval, the area between 650 and 750 is greater than the area between 750 and 850. Since the value at the 75th percentile should divide in half the area between the value at the 60th percentile (650) and the value at the 90th percentile (850), this value is closer to 650 than to 850. Thus you can conclude that Quantity A, the value at the 75th percentile of the distribution of X, is less than Quantity B. The correct answer is **Choice B**.

Set S consists of all positive integers less than 81 that are not equal to the square of an integer.

	Quantity A	Quantity B	
6.	The number of integers in set S	72	Ⓐ Ⓑ Ⓒ Ⓓ

Explanation

Set S consists of all integers from 1 to 80, except those that are equal to the square of an integer. So, Quantity A, the number of integers in set S, is equal to the number of positive integers that are less than 81 minus the number of positive integers less than 81 that are equal to the square of an integer.

Clearly, there are 80 positive integers that are less than 81.

One way to determine the number of positive integers less than 81 that are squares of integers is by noticing that 81 is equal to 9^2 and concluding that the squares of the integers from 1 to 8 are all positive integers that are less than 81.

You can also draw this conclusion by squaring each of the positive integers, beginning with 1, until you get to an integer n such that n^2 is greater than or equal to 81. Either way, there are 8 positive integers less than 81 that are squares of integers.

Therefore, the number of integers in set S is $80 - 8$, or 72, which is equal to Quantity B. So Quantity A is equal to Quantity B, and the correct answer is **Choice C**.

Multiple-choice Questions—Select One Answer Choice

For Questions 7 to 12, select a single answer choice.

7. A manager is forming a 6-person team to work on a certain project. From the 11 candidates available for the team, the manager has already chosen 3 to be on the team. In selecting the other 3 team members, how many different combinations of 3 of the remaining candidates does the manager have to choose from?

Ⓐ 6
Ⓑ 24
Ⓒ 56
Ⓓ 120
Ⓔ 462

Explanation

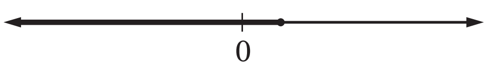
To determine the number of different combinations of 3 of the remaining candidates that the manager has to choose from, you first have to know the number of remaining candidates. Since you know that the manager has already chosen 3 of the 11 candidates to be on the team, it is easy to see that there are 8 remaining candidates. Now you need to count how many different combinations of 3 objects can be chosen from a group of 8 objects.

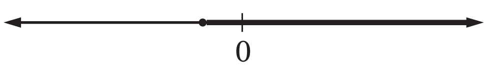
If you remember the combinations formula, you know that the number of combinations is $\frac{8!}{3!(8-3)!}$ (which is denoted symbolically as $\binom{8}{3}$ or ${}_8C_3$). You can then calculate the number of different combinations of 3 of the remaining candidates as follows.

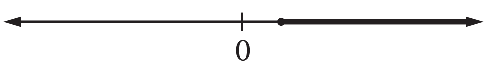
$$\frac{8!}{3!(8-3)!} = \frac{(8)(7)(6)(5!)}{(3!)(5!)} = \frac{(8)(7)(6)}{6} = 56$$

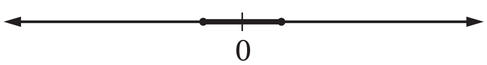
The correct answer is **Choice C**.

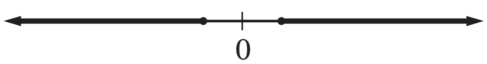
8. Which of the following could be the graph of all values of x that satisfy the inequality $2 - 5x \le -\frac{6x-5}{3}$?

Ⓐ 

Ⓑ 

Ⓒ 

Ⓓ 

Ⓔ 

Explanation

To determine which of the graphs is the correct answer, you first need to determine all values of x that satisfy the inequality. To do that you need to simplify the inequality until you isolate x.

You can begin by multiplying both sides of the inequality by 3 to obtain $(3)(2-5x) \le -(6x-5)$. Note that when you multiply by 3, the right-hand side of the inequality becomes $-(6x-5)$, not $-6x-5$.

The rest of the simplification is as follows.

$$\begin{aligned} (3)(2-5x) &\le -6x+5 \\ 6-15x &\le -6x+5 \\ -15x &\le -6x-1 \\ -9x &\le -1 \\ x &\ge \frac{1}{9} \end{aligned}$$

Note that when an inequality is multiplied (or divided) by a negative number, the direction of the inequality reverses.

The graphs in the answer choices are number lines on which only the number 0 is indicated. Therefore, you do not need to locate $\frac{1}{9}$ on the number line; it is enough to know that $\frac{1}{9}$ is a positive number. Choice C is the only choice in which the shaded part of the line is equal to or greater than a positive number. Therefore, the correct answer is **Choice C**.

9. If $1+x+x^2+x^3=60$, then the average (arithmetic mean) of x, x^2, x^3, and x^4 is equal to which of the following?

   (A) $12x$
   (B) $15x$
   (C) $20x$
   (D) $30x$
   (E) $60x$

Explanation

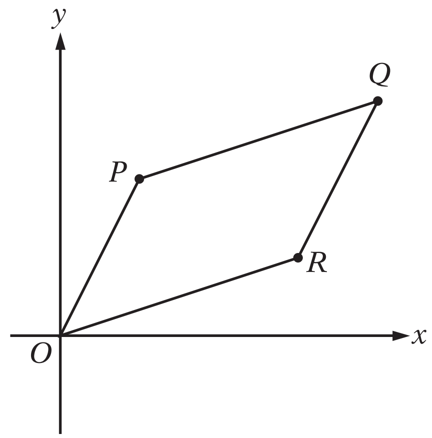
A quick inspection of the answer choices shows that it is not necessary to solve the equation $1+x+x^2+x^3=60$ for x to answer this question. You are being asked to express the average of the four quantities x, x^2, x^3, and x^4 in terms of x. To express this average in terms of x, you need to add the 4 quantities and divide the result by 4; that is, $\frac{x+x^2+x^3+x^4}{4}$.

The only information given in the question is that the sum of the 4 quantities, $1+x+x^2+x^3$, is 60, so you need to think of a way to use this information to simplify the expression $\frac{x+x^2+x^3+x^4}{4}$.

Note that the numerator of the fraction is a sum of 4 quantities, each of which has an x term raised to a power. Thus, the expression in the numerator can be factored as $x + x^2 + x^3 + x^4 = x(1 + x + x^2 + x^3)$. By using the information in the question, you can make the following simplification.

$$\frac{x + x^2 + x^3 + x^4}{4} = \frac{x(1 + x + x^2 + x^3)}{4} = \frac{x(60)}{4} = 15x$$

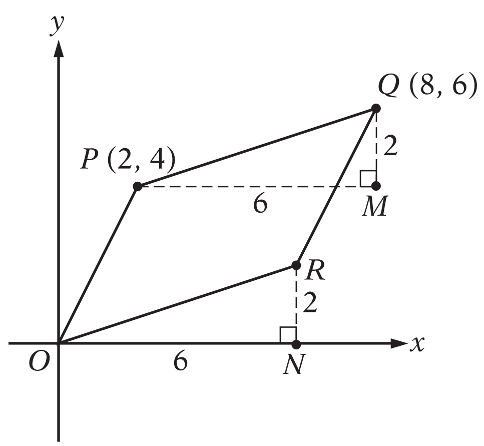
Therefore, the correct answer is **Choice B**.



10. Parallelogram $OPQR$ lies in the xy-plane, as shown in the figure above. The coordinates of point P are $(2, 4)$ and the coordinates of point Q are $(8, 6)$. What are the coordinates of point R ?

Ⓐ $(3, 2)$
Ⓑ $(3, 3)$
Ⓒ $(4, 4)$
Ⓓ $(5, 2)$
Ⓔ $(6, 2)$

Explanation



Since $OPQR$ is a parallelogram, line segments PQ and OR have the same length and the same slope. Therefore, in the figure above, PQM and ORN are congruent right triangles. From the coordinates of P and Q, the lengths of the legs of triangle PQM are $PM = 8 - 2 = 6$ and $QM = 6 - 4 = 2$. Thus, the lengths of the legs ON and RN of triangle ORN are also 6 and 2, respectively. So the coordinates of point R are $(6, 2)$. The correct answer is **Choice E**.

11. The relationship between the area A of a circle and its circumference C is given by the formula $A = kC^2$, where k is a constant. What is the value of k ?

(A) $\frac{1}{4\pi}$

(B) $\frac{1}{2\pi}$

(C) $\frac{1}{4}$

(D) 2π

(E) $4\pi^2$

Explanation

One way to approach this problem is to realize that the value of the constant k is the same for all circles. Therefore, you can pick a specific circle and substitute the circumference and the area of that particular circle into the formula and calculate the value of k.

Say, for example, that you pick a circle with radius 1. The area of the circle is π and the circumference of the circle is 2π. Inserting these values into the formula gives $\pi = k(2\pi)^2$. Solving this equation for k gives $k = \frac{1}{4\pi}$, and the correct answer is **Choice A**.

Another way to approach the problem is to express A and C in terms of a common variable and then solve the resulting equation for k. Recall the commonly used formulas for the area and the circumference of a circle: $A = \pi r^2$ and $C = 2\pi r$. Note that in these formulas, both A and C are expressed in terms of the radius r. So, in the formula $A = kC^2$, you can substitute expressions for A and C in terms of r.

Substituting πr^2 for A and $2\pi r$ for C gives $\pi r^2 = k(2\pi r)^2$.

Now you can determine the value of k by solving the equation for k as follows.

$$
\begin{aligned}
\pi r^2 &= k(2\pi r)^2 \\
\pi r^2 &= k(4\pi^2 r^2) \\
\pi &= k(4\pi^2) \\
\frac{1}{4\pi} &= k
\end{aligned}
$$

The correct answer is **Choice A**.

12. The sequence of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ is defined by $a_n = \frac{1}{n} - \frac{1}{n+2}$ for each integer $n \geq 1$. What is the sum of the first 20 terms of this sequence?

Ⓐ $\left(1 + \frac{1}{2}\right) - \frac{1}{20}$

Ⓑ $\left(1 + \frac{1}{2}\right) - \left(\frac{1}{21} + \frac{1}{22}\right)$

Ⓒ $1 - \left(\frac{1}{20} + \frac{1}{22}\right)$

Ⓓ $1 - \frac{1}{22}$

Ⓔ $\frac{1}{20} - \frac{1}{22}$

Explanation

This question asks for the sum of the first 20 terms of the sequence. Obviously, it would be very time-consuming to write out the first 20 terms of the sequence and add them together, so it is reasonable to try to find a more efficient way to calculate the sum. Questions involving sequences can often be answered by looking for a pattern. Scanning the answer choices and noting that they contain fractions with denominators 2, 20, 21, and 22, and nothing in between, seems to confirm that looking for a pattern is a good approach to try.

To look for a pattern, begin by adding the first two terms of the sequence.

$$\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) = \left(\frac{1}{1} + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{4}\right)$$

Now, if you add the first three terms of the sequence, you get

$$\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)$$

Note that you can simplify the sum by canceling the fraction $\frac{1}{3}$; that is, the sum of positive $\frac{1}{3}$ and negative $\frac{1}{3}$ is 0.

$$\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) = \left(\frac{1}{1} + \frac{1}{2}\right) - \left(\frac{1}{4} + \frac{1}{5}\right)$$

If you add the first four terms, you get

$$\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)$$

Again, you can simplify the sum by canceling. This time, you can cancel the fractions $\frac{1}{3}$ and $\frac{1}{4}$.

$$\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) = \left(\frac{1}{1} + \frac{1}{2}\right) - \left(\frac{1}{5} + \frac{1}{6}\right)$$

If you write out the next two sums and simplify them, you will see that they are

$$\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{6}+\frac{1}{7}\right) \text{ and } \left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{7}+\frac{1}{8}\right)$$

Working with the sums makes it clear that this pattern continues to hold as you add more and more terms of the sequence together and that a formula for the sum of the first k terms of the sequence is

$$\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{k+1}+\frac{1}{k+2}\right)$$

Therefore, the sum of the first 20 terms of the sequence is equal to

$$\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{20+1}+\frac{1}{20+2}\right)=\left(1+\frac{1}{2}\right)-\left(\frac{1}{21}+\frac{1}{22}\right)$$

The correct answer is **Choice B**.

Numeric Entry

For Question 13, use the directions for Numeric Entry questions.

Y	Frequency
$\frac{1}{2}$	2
$\frac{3}{4}$	7
$\frac{5}{4}$	8
$\frac{3}{2}$	8
$\frac{7}{4}$	9

13. The table above shows the frequency distribution of the values of a variable Y. What is the mean of the distribution?

    Give your answer to the nearest 0.01.

Explanation

The mean of distribution of the variable Y is the sum of all the values of Y divided by the number of values of Y. However, before you begin the summing process, you need to understand how the information is presented in the question. Information about the variable is given in a table, where any repetitions of values have been summarized in the column labeled "Frequency." Reading from the table, you can see that the value $\frac{1}{2}$ occurs twice, the value $\frac{3}{4}$ occurs seven times, and so on. To sum all the values of Y, you could add the value $\frac{1}{2}$ twice, add the value $\frac{3}{4}$ seven times, and continue the addition process in this manner. It is easier, however, to multiply the values by their corresponding frequencies and then sum the individual products, as shown below.

$$\begin{aligned}(2)\left(\frac{1}{2}\right)+(7)\left(\frac{3}{4}\right)+(8)\left(\frac{5}{4}\right)+(8)\left(\frac{3}{2}\right)+(9)\left(\frac{7}{4}\right) &= \frac{4}{4}+\frac{21}{4}+\frac{40}{4}+\frac{48}{4}+\frac{63}{4}\\ &= \frac{176}{4}\\ &= 44\end{aligned}$$

To find the average, you need to divide the sum, 44, by the number of values of Y. The number of values can be found by looking at the column of frequencies in the table. The sum of the numbers in this column, $2 + 7 + 8 + 8 + 9$, or 34, is the number of values of Y. Thus, the mean of the distribution is $\frac{44}{34}$, which, as a decimal, equals 1.2941.... Rounded to the nearest 0.01, the correct answer is **1.29**.

Multiple-choice Questions—Select One or More Answer Choices

For Questions 14 and 15, select all the answer choices that apply.

14. Let S be the set of all positive integers n such that n^2 is a multiple of both 24 and 108. Which of the following integers are divisors of every integer n in S ?

    Indicate all such integers.

    A 12
    B 24
    C 36
    D 72

Explanation

To determine which of the integers in the answer choices is a divisor of every positive integer n in S, you must first understand the integers that are in S. Note that in this question you are given information about n^2, not about n itself. Therefore, you must use the information about n^2 to derive information about n.

The fact that n^2 is a multiple of both 24 and 108 implies that n^2 is a multiple of the least common multiple of 24 and 108. To determine the least common multiple of 24 and 108, factor 24 and 108 into prime factors as $(2^3)(3)$ and $(2^2)(3^3)$, respectively. Because these are prime factorizations, you can conclude that the least common multiple of 24 and 108 is $(2^3)(3^3)$.

Knowing that n^2 must be a multiple of $(2^3)(3^3)$ does not mean that every multiple of $(2^3)(3^3)$ is a possible value of n^2, because n^2 must be the square of an integer. The prime factorization of a square number must contain only even exponents. Thus, the least multiple of $(2^3)(3^3)$ that is a square is $(2^4)(3^4)$. This is the least possible value of n^2, and so the least possible value of n is $(2^2)(3^2)$, or 36. Furthermore, since every value of n^2 is a multiple of $(2^4)(3^4)$, the values of n are the positive multiples of 36; that is, $S = \{36, 72, 108, 144, 180, \ldots\}$.

The question asks for integers that are divisors of every integer n in S, that is, divisors of every positive multiple of 36. Since Choice A, 12, is a divisor of 36, it is also a divisor of every multiple of 36. The same is true for Choice C, 36. Choices B and D, 24 and 72, are not divisors of 36, so they are not divisors of every integer in S. The correct answer consists of **Choices A and C**.

15. The range of the heights of the female students in a certain class is 13.2 inches, and the range of the heights of the male students in the class is 15.4 inches.

    Which of the following statements individually provide(s) sufficient additional information to determine the range of the heights of all the students in the class?

    Indicate all such statements.

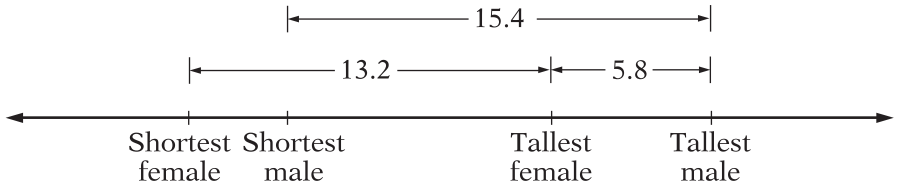
    A The tallest male student in the class is 5.8 inches taller than the tallest female student in the class.

    B The median height of the male students in the class is 1.1 inches greater than the median height of the female students in the class.

    C The average (arithmetic mean) height of the male students in the class is 4.6 inches greater than the average height of the female students in the class.

Explanation

Choice A tells you that the tallest male student is 5.8 inches taller than the tallest female student. You can combine this information with the given information about the male and female height ranges to place four students—the shortest male, the shortest female, the tallest male, and the tallest female—in relative order according to height, as shown in the figure below.



You can see from the figure that the tallest student must be a male and the shortest student must be a female. You can also see the difference in height between those two students, which is the range of the heights of the entire class. Therefore, Choice A provides sufficient additional information to determine the range.

Choice B provides information about one of the centers of the data—the median; it does not say anything about how spread out the data are around that center. You are given that the median height of the males is 1.1 inches greater than that of the females. First note that it is possible for two different sets of data to have the same median but have very different ranges. Choice B gives the difference between the medians of the male heights and the female heights, without giving the actual medians. However, even if you knew the medians, the fact that the ranges can vary widely indicates that the range of the heights of the entire class can also vary widely.

It is possible to construct examples of heights of students that satisfy all of the information in the question and in Choice B but have different ranges for the heights of the entire class. Here are two such examples, each of which has only three females and three males. Although the examples are small, they illustrate the fact that the range of the heights of the entire class can vary. In both examples, the range of female heights is 13.2, the range of male heights is 15.4, and the difference between the median heights is 1.1 inches.

<u>Example 1</u>

Female heights:	50.0	56.6	63.2	which have a median of 56.6
Male heights:	50.0	57.7	65.4	which have a median of 57.7

Range of heights of entire class: 15.4

<u>Example 2</u>

Female heights:	50.0	56.6	63.2	which have a median of 56.6
Male heights:	51.0	57.7	66.4	which have a median of 57.7

Range of heights of entire class: 16.4

Therefore, Choice B does not provide sufficient additional information to determine the range of the heights of the entire class.

Choice C provides information about another center of the data—the average. You are given that the average height of the males is 4.6 inches greater than that of the females. However, like Choice B, the statement gives no information about how spread out the data are around that center. Again, it is possible for two different sets of data to have the same average but have very different ranges. Examples similar to the two examples above can be constructed that satisfy all of the information in the question and in Choice C but have different ranges for the heights of the entire class. Therefore, Choice C does not provide sufficient additional information to determine the range of the heights of the entire class.

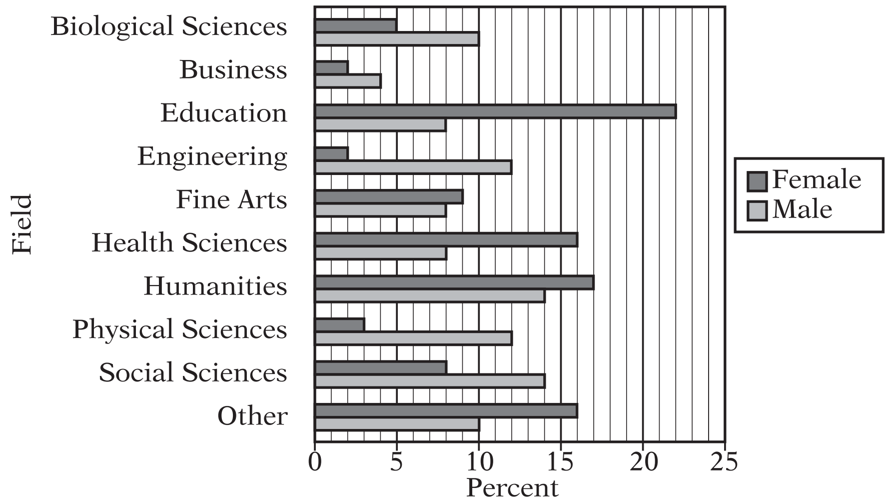
The correct answer consists of **Choice A**.

SET 4. Data Interpretation Sets

For Questions 1 to 7, select a single answer choice unless otherwise directed.

Questions 1 to 3 are based on the following data.

PERCENT OF FEMALE FACULTY AND PERCENT OF MALE FACULTY
AT UNIVERSITY *X*, BY FIELD
Total female faculty: 200
Total male faculty: 250



Medium Question

1. There are 275 students in the field of engineering at University *X*. Approximately what is the ratio of the number of students in engineering to the number of faculty in engineering?

Ⓐ 8 to 1
Ⓑ 10 to 1
Ⓒ 12 to 1
Ⓓ 14 to 1
Ⓔ 20 to 1

Explanation

According to the graph, 2 percent of the female faculty and 12 percent of the male faculty are in the engineering field. To determine the total number of faculty members in engineering, you need to add 2 percent of 200, which is 4, to 12 percent of 250, which is 30, to get 34. Thus, the ratio of the numbers of students to faculty in engineering is 275 to 34, which is approximately equal to 280 to 35, or 8 to 1. The correct answer is **Choice A**.

Medium Question

2. Approximately what percent of the faculty in humanities are male?

   (A) 35%
   (B) 38%
   (C) 41%
   (D) 45%
   (E) 51%

Explanation

You need to determine the numbers of female and male faculty in the humanities field. According to the graph, 17 percent of the 200 females, or 34, and 14 percent of the 250 males, or 35, are in humanities. Thus, the fraction of humanities faculty who are male is $\frac{35}{34+35} = \frac{35}{69}$, or approximately 0.507. As a percent, the answer choice that is closest to 0.507 is 51 percent. The correct answer is **Choice E**.

For Question 3, use the directions for Numeric Entry questions.

Hard Question

3. For the biological sciences and health sciences faculty combined, $\frac{1}{3}$ of the female and $\frac{2}{9}$ of the male faculty members are tenured professors. What fraction of all the faculty members in those two fields combined are tenured professors?

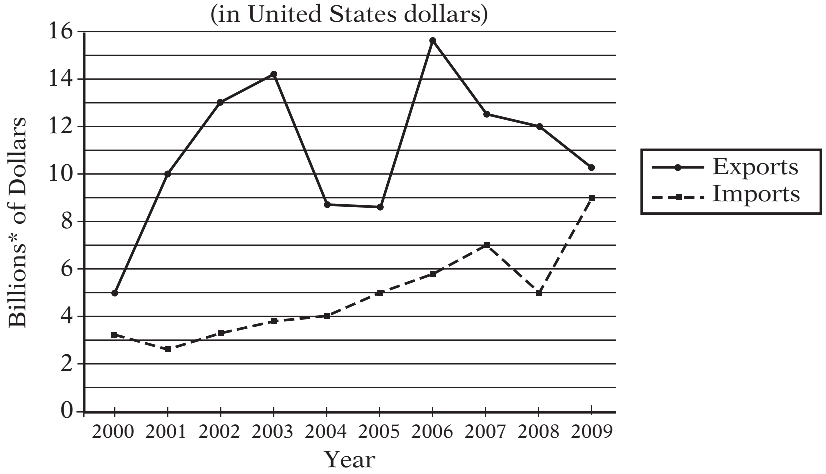
$$\frac{\boxed{}}{\boxed{}}$$

Explanation

You need to determine the number of female faculty and the number of male faculty in the combined group. According to the graph, 5 percent of the female faculty, or 10, and 10 percent of the male faculty, or 25, are in the biological sciences. Similarly, 16 percent of the female faculty, or 32, and 8 percent of the male faculty, or 20, are in the health sciences. When you combine the groups, you get a total of 42 females (10 + 32) and 45 males (25 + 20), which is a total of 87 faculty. The tenured faculty are $\frac{1}{3}$ of the 42 females, or 14 females, and $\frac{2}{9}$ of the 45 males, or 10 males. Thus, there are 24 tenured faculty, and the fraction that are tenured professors is $\frac{24}{87}$. The correct answer is $\mathbf{\frac{24}{87}}$ (or any equivalent fraction).

Questions 4 to 7 are based on the following data.

VALUE OF IMPORTS TO AND EXPORTS FROM COUNTRY *T*, 2000–2009
(in United States dollars)



*1 billion = 1,000,000,000

For Question 4, select all the answer choices that apply.

Easy Question

4. For which of the eight years from 2001 to 2008 did exports exceed imports by more than $5 billion?

Indicate all such years.

- A 2001
- B 2002
- C 2003
- D 2004
- E 2005
- F 2006
- G 2007
- H 2008

Explanation

Note that for all years shown, the dollar value of exports is greater than the dollar value of imports. For each year, the difference between the dollar value of exports and the dollar value of imports can be read directly from the graph. The difference was more than $5 billion for each of the years 2001, 2002, 2003, 2006, 2007, and 2008. The correct answer consists of **Choices A, B, C, F, G, and H**.

Medium Question

5. Which of the following is closest to the average (arithmetic mean) of the 9 changes in the value of imports between consecutive years from 2000 to 2009 ?

Ⓐ $260 million
Ⓑ $320 million
Ⓒ $400 million
Ⓓ $480 million
Ⓔ $640 million

Explanation

The average of the 9 changes in the value of imports between consecutive years can be represented as follows, where the function $v(\text{year})$ represents the value of imports for the indicated year.

$$\frac{(v(2001) - v(2000)) + (v(2002) - v(2001)) + (v(2003) - v(2002)) + \cdots + (v(2009) - v(2008))}{9}$$

Note that in the numerator of the fraction, each term, with the exception of $v(2000)$ and $v(2009)$, appears first as positive and then again as negative. The positive and negative pairs sum to 0, and the fraction simplifies to $\frac{v(2009) - v(2000)}{9}$.

Reading the values from the graph, you can approximate the value of the simplified fraction as $\frac{9.0 - 3.2}{9} = \frac{5.8}{9} \approx 0.644$ billion dollars. The answer choice that is closest to $0.644 billion is $640 million. The correct answer is **Choice E**.

Medium Question

6. In 2008 the value of exports was approximately what percent greater than the value of imports?

Ⓐ 40%
Ⓑ 60%
Ⓒ 70%
Ⓓ 120%
Ⓔ 140%

Explanation

The difference between the value of exports and the value of imports expressed as a percent of the value of imports is

$$\left(\frac{(\text{value of exports}) - (\text{value of imports})}{\text{value of imports}}\right)(100\%)$$

In 2008 the value of imports was approximately $5 billion and the value of exports was approximately $12 billion, so the value of the fraction is approximately $\frac{12 - 5}{5}$, or $\frac{7}{5}$.

Since the fraction is greater than 1, expressing it as a percent will give a percent greater than 100. The fraction is equal to 1.4, or 140 percent. The correct answer is **Choice E**.

Hard Question

7. If it were discovered that the value of imports shown for 2007 was incorrect and should have been \$5 billion instead, then the average (arithmetic mean) value of imports per year for the 10 years shown would have been approximately how much less?

   (A) \$200 million
   (B) \$50 million
   (C) \$20 million
   (D) \$7 million
   (E) \$5 million

Explanation

To answer this question, you do not need to compute either of the two 10-year averages referred to in the question; you just need to calculate the difference between the two averages.

The average value of imports for the 10 years shown in the graph is found by adding the 10 values and then dividing the sum by 10. The value of imports in 2007 is \$7 billion. If that amount were \$5 billion instead, then the sum of the values would be \$2 billion less. If the sum were \$2 billion less than what it was, then the average would decrease by 2 billion divided by 10, or $\frac{2{,}000{,}000{,}000}{10} = 200{,}000{,}000$. The average would therefore be \$200 million less, and the correct answer is **Choice A**.

A more algebraic approach to the problem is to let S represent the sum, in billions, of the 10 values of imports in the graph. The average of the 10 values is $\frac{S}{10}$. Note that $S - 2$ represents the sum, in billions, of the 10 values adjusted for the \$2 billion correction for 2007. The average of the adjusted sum is $\frac{S-2}{10}$. The difference between the two averages is

$$\begin{aligned}\frac{S}{10} - \frac{S-2}{10} &= \frac{S-(S-2)}{10}\\ &= \frac{S-S+2}{10}\\ &= \frac{2}{10}\end{aligned}$$

The difference is 0.2 billion dollars, or \$200 million. The correct answer is **Choice A**.

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7 GRE®
Math Review

Your goal for this chapter	
	Review the math topics likely to appear on the *GRE®* revised General Test
	Study examples with worked-out solutions
	Test your skills with practice exercises

This Math Review will familiarize you with the mathematical skills and concepts that are important to understand in order to solve problems and to reason quantitatively on the Quantitative Reasoning measure of the GRE revised General Test. The following material includes many definitions, properties, and examples, as well as a set of exercises (with answers) at the end of each review section. Note, however, that this review is not intended to be all-inclusive—there may be some concepts on the test that are not explicitly presented in this review. Also, if any topics in this review seem especially unfamiliar or are covered too briefly, we encourage you to consult appropriate mathematics texts for a more detailed treatment.

The Math Review covers the following topics:

1. Arithmetic
1.1 Integers
1.2 Fractions
1.3 Exponents and Roots
1.4 Decimals
1.5 Real Numbers
1.6 Ratio
1.7 Percent

2. Algebra
2.1 Operations with Algebraic Expressions
2.2 Rules of Exponents
2.3 Solving Linear Equations
2.4 Solving Quadratic Equations
2.5 Solving Linear Inequalities
2.6 Functions
2.7 Applications
2.8 Coordinate Geometry
2.9 Graphs of Functions

3. Geometry
3.1 Lines and Angles
3.2 Polygons
3.3 Triangles
3.4 Quadrilaterals
3.5 Circles
3.6 Three-dimensional Figures

4. Data Analysis
4.1 Graphical Methods for Describing Data
4.2 Numerical Methods for Describing Data
4.3 Counting Methods
4.4 Probability
4.5 Distributions of Data, Random Variables, and Probability Distributions
4.6 Data Interpretation Examples

1. ARITHMETIC

The review of arithmetic begins with integers, fractions, and decimals and progresses to the set of real numbers. The basic arithmetic operations of addition, subtraction, multiplication, and division are discussed, along with exponents and roots. The section ends with the concepts of ratio and percent.

1.1 Integers

The **integers** are the numbers 1, 2, 3, and so on, together with their negatives, −1, −2, −3,…, and 0. Thus, the set of integers is {…, −3, −2, −1, 0, 1, 2, 3,…}.

The positive integers are greater than 0, the negative integers are less than 0, and 0 is neither positive nor negative. When integers are added, subtracted, or multiplied, the result is always an integer; division of integers is addressed below. The many elementary number facts for these operations, such as $7 + 8 = 15$, $78 - 87 = -9$, $7 - (-18) = 25$, and $(7)(8) = 56$, should be familiar to you; they are not reviewed here. Here are some general facts regarding multiplication of integers.

- The product of two positive integers is a positive integer.
- The product of two negative integers is a positive integer.
- The product of a positive integer and a negative integer is a negative integer.

When integers are multiplied, each of the multiplied integers is called a **factor** or **divisor** of the resulting product. For example, $(2)(3)(10) = 60$, so 2, 3, and 10 are factors of 60. The integers 4, 15, 5, and 12 are also factors of 60, since $(4)(15) = 60$ and $(5)(12) = 60$. The positive factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. The negatives of these integers are also factors of 60, since, for example, $(-2)(-30) = 60$. There are no other factors of 60. We say that 60 is a **multiple** of each of its factors and that 60 is **divisible** by each of its divisors. Here are some more examples of factors and multiples.

- The positive factors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, and 100.
- 25 is a multiple of only six integers: 1, 5, 25, and their negatives.
- The list of positive multiples of 25 has no end: 25, 50, 75, 100, 125, 150, etc.; likewise, every nonzero integer has infinitely many multiples.
- 1 is a factor of every integer; 1 is not a multiple of any integer except 1 and −1.
- 0 is a multiple of every integer; 0 is not a factor of any integer except 0.

The **least common multiple** of two nonzero integers a and b is the least positive integer that is a multiple of both a and b. For example, the least common multiple of 30 and 75 is 150. This is because the positive multiples of 30 are 30, 60, 90, 120, 150, 180, 210, 240, 270, 300, etc., and the positive multiples of 75 are 75, 150, 225, 300, 375, 450, etc. Thus, the *common* positive multiples of 30 and 75 are 150, 300, 450, etc., and the least of these is 150.

The **greatest common divisor** (or **greatest common factor**) of two nonzero integers a and b is the greatest positive integer that is a divisor of both a and b. For example, the greatest common divisor of 30 and 75 is 15. This is because the positive divisors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30, and the positive divisors of 75 are 1, 3, 5, 15, 25, and 75. Thus, the *common* positive divisors of 30 and 75 are 1, 3, 5, and 15, and the greatest of these is 15.

When an integer a is divided by an integer b, where b is a divisor of a, the result is always a divisor of a. For example, when 60 is divided by 6 (one of its divisors), the result is 10, which is another divisor of 60. If b is *not* a divisor of a, then the result can be viewed in three different ways. The result can be viewed as a fraction or as a decimal, both of which are discussed later, or the result can be viewed as a **quotient** with a **remainder**, where both are integers. Each view is useful, depending on the context. Fractions and decimals are useful when the result must be viewed as a single number, while **quotients** with **remainders** are useful for describing the result in terms of integers only.

Regarding quotients with remainders, consider two positive integers a and b for which b is *not* a divisor of a; for example, the integers 19 and 7. When 19 is divided by 7, the result is greater than 2, since $(2)(7) < 19$, but less than 3, since $19 < (3)(7)$. Because 19 is 5 more than $(2)(7)$, we say that the result of 19 divided by 7 is the quotient 2 with remainder 5, or simply "2 remainder 5." In general, when a positive integer a is divided by a positive integer b, you first find the greatest multiple of b that is less than or equal to a. That multiple of b can be expressed as the product qb, where q is the quotient. Then the remainder is equal to a minus that multiple of b, or $r = a - qb$, where r is the remainder. The remainder is always greater than or equal to 0 and less than b.

Here are examples that illustrate a few different cases of division resulting in a quotient and remainder.

- 100 divided by 45 is 2 remainder 10, since the greatest multiple of 45 that's less than or equal to 100 is $(2)(45)$, or 90, which is 10 less than 100.
- 24 divided by 4 is 6 remainder 0, since the greatest multiple of 4 that's less than or equal to 24 is 24 itself, which is 0 less than 24. In general, the remainder is 0 if and only if a is divisible by b.
- 6 divided by 24 is 0 remainder 6, since the greatest multiple of 24 that's less than or equal to 6 is $(0)(24)$, or 0, which is 6 less than 6.

Here are some other examples.

- 100 divided by 3 is 33 remainder 1, since $100 = (33)(3) + 1$.
- 100 divided by 25 is 4 remainder 0, since $100 = (4)(25) + 0$.
- 80 divided by 100 is 0 remainder 80, since $80 = (0)(100) + 80$.
- When you divide 100 by 2, the remainder is 0.
- When you divide 99 by 2, the remainder is 1.

If an integer is divisible by 2, it is called an **even integer**; otherwise it is an **odd integer**. Note that when a positive odd integer is divided by 2, the remainder is always 1. The set of even integers is $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$, and the set of odd integers is $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$. There are several useful facts regarding the sum and product of even and odd integers.

- The sum of two even integers is an even integer.
- The sum of two odd integers is an even integer.
- The sum of an even integer and an odd integer is an odd integer.
- The product of two even integers is an even integer.
- The product of two odd integers is an odd integer.
- The product of an even integer and an odd integer is an even integer.

A **prime number** is an integer greater than 1 that has only two positive divisors: 1 and itself. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. The integer 14 is not a prime number, since it has four positive divisors: 1, 2, 7, and 14. The integer 1 is not a prime number, and the integer 2 is the only prime number that is even.

Every integer greater than 1 either is a prime number or can be uniquely expressed as a product of factors that are prime numbers, or **prime divisors**. Such an expression is called a **prime factorization**. Here are several examples of prime factorizations.

$$
\begin{aligned}
12 &= (2)(2)(3) = (2^2)(3)\\
14 &= (2)(7)\\
81 &= (3)(3)(3)(3) = 3^4\\
338 &= (2)(13)(13) = (2)(13^2)\\
800 &= (2)(2)(2)(2)(2)(5)(5) = (2^5)(5^2)\\
1{,}155 &= (3)(5)(7)(11)
\end{aligned}
$$

An integer greater than 1 that is not a prime number is called a **composite number**. The first ten composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, and 18.

1.2 Fractions

A **fraction** is a number $\frac{a}{b}$, of the form where a and b are integers and $b \neq 0$. The integer a is called the **numerator** of the fraction, and b is called the **denominator**. For example, $\frac{-7}{5}$ is a fraction in which -7 is the numerator and 5 is the denominator. Such numbers are also called **rational numbers**.

If both the numerator a and denominator b are multiplied by the same nonzero integer, the resulting fraction will be equivalent to $\frac{a}{b}$. For example,

$$\frac{-7}{5} = \frac{(-7)(4)}{(5)(4)} = \frac{-28}{20}$$

$$\frac{-7}{5} = \frac{(-7)(-1)}{(5)(-1)} = \frac{7}{-5}$$

A fraction with a negative sign in either the numerator or denominator can be written with the negative sign in front of the fraction; for example, $\frac{-7}{5} = \frac{7}{-5} = -\frac{7}{5}$.

If both the numerator and denominator have a common factor, then the numerator and denominator can be factored and reduced to an equivalent fraction. For example,

$$\frac{40}{72} = \frac{(8)(5)}{(8)(9)} = \frac{5}{9}$$

To add two fractions with the same denominator, you add the numerators and keep the same denominator. For example,

$$-\frac{8}{11} + \frac{5}{11} = \frac{-8+5}{11} = \frac{-3}{11} = -\frac{3}{11}$$

To add two fractions with different denominators, first find a **common denominator**, which is a common multiple of the two denominators. Then convert both fractions to equivalent fractions with the same denominator. Finally, add the numerators and keep the common denominator. For example, to add the fractions $\frac{1}{3}$ and $-\frac{2}{5}$, use the common denominator 15:

$$\frac{1}{3} + \frac{-2}{5} = \left(\frac{1}{3}\right)\left(\frac{5}{5}\right) + \left(\frac{-2}{5}\right)\left(\frac{3}{3}\right) = \frac{5}{15} + \frac{-6}{15} = \frac{5+(-6)}{15} = -\frac{1}{15}$$

The same method applies to subtraction of fractions.

To multiply two fractions, multiply the two numerators and multiply the two denominators. For example,

$$\left(\frac{10}{7}\right)\left(\frac{-1}{3}\right) = \frac{(10)(-1)}{(7)(3)} = \frac{-10}{21} = -\frac{10}{21}$$
$$\left(\frac{8}{3}\right)\left(\frac{7}{3}\right) = \frac{56}{9}$$

To divide one fraction by another, first **invert** the second fraction—that is, find its **reciprocal**—then multiply the first fraction by the inverted fraction. For example,

$$\frac{17}{8} \div \frac{3}{4} = \left(\frac{17}{8}\right)\left(\frac{4}{3}\right) = \left(\frac{4}{8}\right)\left(\frac{17}{3}\right) = \left(\frac{1}{2}\right)\left(\frac{17}{3}\right) = \frac{17}{6}$$
$$\frac{\frac{3}{10}}{\frac{7}{13}} = \left(\frac{3}{10}\right)\left(\frac{13}{7}\right) = \frac{39}{70}$$

An expression such as $4\frac{3}{8}$ is called a **mixed number**. It consists of an integer part and a fraction part; the mixed number $4\frac{3}{8}$ means $4 + \frac{3}{8}$. To convert a mixed number to an ordinary fraction, convert the integer part to an equivalent fraction and add it to the fraction part. For example,

$$4\frac{3}{8} = 4 + \frac{3}{8} = \left(\frac{4}{1}\right)\left(\frac{8}{8}\right) + \frac{3}{8} = \frac{32}{8} + \frac{3}{8} = \frac{35}{8}$$

Note that numbers of the form $\frac{a}{b}$, where either a or b is not an integer and $b \neq 0$, are fractional expressions that can be manipulated just like fractions. For example, the numbers $\frac{\pi}{2}$ and $\frac{\pi}{3}$ can be added together as follows.

$$\frac{\pi}{2} + \frac{\pi}{3} = \left(\frac{\pi}{2}\right)\left(\frac{3}{3}\right) + \left(\frac{\pi}{3}\right)\left(\frac{2}{2}\right) = \frac{3\pi}{6} + \frac{2\pi}{6} = \frac{5\pi}{6}$$

And the number $\frac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}}$ can be simplified as follows.

$$\frac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{5}}{3}\right) = \frac{\sqrt{5}}{3\sqrt{2}}$$

1.3 Exponents and Roots

Exponents are used to denote the repeated multiplication of a number by itself; for example, $3^4 = (3)(3)(3)(3) = 81$ and $5^3 = (5)(5)(5) = 125$. In the expression 3^4, 3 is called the **base**, 4 is called the **exponent**, and we read the expression as "3 to the fourth power." So 5 to the third power is 125. When the exponent is 2, we call the process **squaring**. Thus, 6 squared is 36, $6^2 = (6)(6) = 36$, and 7 squared is 49, $7^2 = (7)(7) = 49$.

When negative numbers are raised to powers, the result may be positive or negative. For example, $(-3)^2 = (-3)(-3) = 9$, while $(-3)^5 = (-3)(-3)(-3)(-3)(-3) = -243$. A negative number raised to an even power is always positive, and a negative number raised to an odd power is always negative. Note that without the parentheses, the expression -3^2 means "the negative of '3 squared'"; that is, the exponent is applied before the negative sign. So $(-3)^2 = 9$, but $-3^2 = -9$.

Exponents can also be negative or zero; such exponents are defined as follows.

- For all nonzero numbers a, $a^0 = 1$. The expression 0^0 is undefined.
- For all nonzero numbers a, $a^{-1} = \frac{1}{a}$, $a^{-2} = \frac{1}{a^2}$, $a^{-3} = \frac{1}{a^3}$, etc. Note that $(a)(a^{-1}) = (a)\left(\frac{1}{a}\right) = 1$.

A **square root** of a nonnegative number n is a number r such that $r^2 = n$. For example, 4 is a square root of 16 because $4^2 = 16$. Another square root of 16 is -4, since $(-4)^2 = 16$. All positive numbers have two square roots, one positive and one negative. The only square root of 0 is 0. The symbol $\sqrt{n}$ is used to denote the *nonnegative* square root of the nonnegative number n. Therefore, $\sqrt{100} = 10$, $-\sqrt{100} = -10$, and $\sqrt{0} = 0$. Square roots of negative numbers are not defined in the real number system.

Here are some important rules regarding operations with square roots, where $a > 0$ and $b > 0$.

Rule	Examples	
$\left(\sqrt{a}\right)^2 = a$	$\left(\sqrt{3}\right)^2 = 3$	$\left(\sqrt{\pi}\right)^2 = \pi$
$\sqrt{a^2} = a$	$\sqrt{4} = 2$	$\sqrt{\pi^2} = \pi$
$\sqrt{a}\sqrt{b} = \sqrt{ab}$	$\sqrt{3}\sqrt{10} = \sqrt{30}$	$\sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$
$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$	$\frac{\sqrt{5}}{\sqrt{15}} = \sqrt{\frac{5}{15}} = \sqrt{\frac{1}{3}}$	$\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3$

A square root is a root of order 2. Higher-order roots of a positive number n are defined similarly. For orders 3 and 4, the **cube root** $\sqrt[3]{n}$ and **fourth root** $\sqrt[4]{n}$ represent numbers such that when they are raised to the powers 3 and 4, respectively, the result is n. These roots obey rules similar to those above (but with the exponent 2 replaced by 3 or 4 in the first two rules). There are some notable differences between odd-order roots and even-order roots (in the real number system):

- For odd-order roots, there is *exactly one* root for *every* number n, even when n is negative.
- For even-order roots, there are *exactly two* roots for every *positive* number n and *no* roots for any *negative* number n.

For example, 8 has exactly one cube root, $\sqrt[3]{8} = 2$, but 8 has two fourth roots: $\sqrt[4]{8}$ and $-\sqrt[4]{8}$; and -8 has exactly one cube root, $\sqrt[3]{-8} = -2$, but -8 has no fourth root, since it is negative.

1.4 Decimals

The decimal number system is based on representing numbers using powers of 10. The place value of each digit corresponds to a power of 10. For example, the digits of the number 7,532.418 have the following place values.

Thousands		Hundreds	Tens	Ones or Units		Tenths	Hundredths	Thousandths
7	,	5	3	2	.	4	1	8

That is,

$$\begin{aligned} 7{,}532.418 &= 7(1{,}000) + 5(100) + 3(10) + 2(1) + 4\left(\frac{1}{10}\right) + 1\left(\frac{1}{100}\right) + 8\left(\frac{1}{1{,}000}\right) \\ &= 7(10^3) + 5(10^2) + 3(10^1) + 2(10^0) + 4(10^{-1}) + 1(10^{-2}) + 8(10^{-3}) \end{aligned}$$

If there are a finite number of digits to the right of the decimal point, converting a decimal to an equivalent fraction with integers in the numerator and denominator is a straightforward process. Since each place value is a power of 10, every decimal can be converted to an integer divided by a power of 10. For example,

$$\begin{aligned} 2.3 &= 2 + \frac{3}{10} = \frac{23}{10} \\ 90.17 &= 90 + \frac{17}{100} = \frac{9{,}000 + 17}{100} = \frac{9{,}017}{100} \\ 0.612 &= \frac{612}{1{,}000} = \frac{153}{250} \end{aligned}$$

Conversely, every fraction with integers in the numerator and denominator can be converted to an equivalent decimal by dividing the numerator by the denominator using long division (which is not in this review). The decimal that results from the long division will either **terminate**, as in $\frac{1}{4} = 0.25$ and $\frac{52}{25} = 2.08$, or the decimal will **repeat** without end, as in $\frac{1}{9} = 0.111\ldots$, $\frac{1}{22} = 0.0454545\ldots$, and $\frac{25}{12} = 2.08333\ldots$. One way to indicate the repeating part of a decimal that repeats without end is to use a bar over the digits that repeat. Here are some examples of fractions converted to decimals.

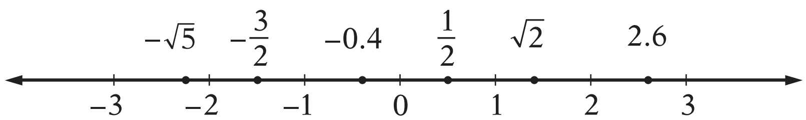
$$\begin{aligned} \frac{3}{8} &= 0.375 \\ \frac{259}{40} &= 6 + \frac{19}{40} = 6.475 \\ -\frac{1}{3} &= -0.\overline{3} \\ \frac{15}{14} &= 1.0\overline{714285} \end{aligned}$$

Every fraction with integers in the numerator and denominator is equivalent to a decimal that terminates or repeats. That is, every rational number can be expressed as a terminating or repeating decimal. The converse is also true; that is, every terminating or repeating decimal represents a rational number.

Not all decimals are terminating or repeating; for instance, the decimal that is equivalent to $\sqrt{2}$ is $1.41421356237\ldots$, and it can be shown that this decimal does not terminate or repeat. Another example is $0.010110111011110111110\ldots$, which has groups of consecutive 1's separated by a 0, where the number of 1's in each successive group increases by one. Since these two decimals do not terminate or repeat, they are not rational numbers. Such numbers are called **irrational numbers**.

1.5 Real Numbers

The set of **real numbers** consists of all rational numbers and all irrational numbers. The real numbers include all integers, fractions, and decimals. The set of real numbers can be represented by a number line called the **real number line**.



Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number. On the number line, all numbers to the left of 0 are negative and all numbers to the right of 0 are positive. Only the number 0 is neither negative nor positive.

A real number x is **less than** a real number y if x is to the left of y on the number line, which is written as $x < y$. A real number y is **greater than** x if y is to the right of x on the number line, which is written as $y > x$. For example,

$$-\sqrt{5} < -2$$
$$\frac{1}{2} > 0$$
$$1 < \sqrt{2} < 2$$

To say that a real number x is between 2 and 3 on the number line means that $x > 2$ and $x < 3$, which can also be written as the double inequality $2 < x < 3$. The set of all real numbers that are between 2 and 3 is called an **interval**, and the double inequality $2 < x < 3$ is often used to represent that interval. Note that the endpoints of the interval, 2 and 3, are not included in the interval. Sometimes one or both of the endpoints are to be included in an interval. The following inequalities represent four types of intervals, depending on whether the endpoints are included.

$$2 < x < 3$$
$$2 \le x < 3$$
$$2 < x \le 3$$
$$2 \le x \le 3$$

There are also four types of intervals with only one endpoint, each of which consists of all real numbers to the right or to the left of the endpoint, perhaps including the endpoint. The following inequalities represent these types of intervals.

$$x < 4$$
$$x \le 4$$
$$x > 4$$
$$x \ge 4$$

The entire real number line is also considered to be an interval.

The distance between a number x and 0 on the number line is called the **absolute value** of x, written as $|x|$. Therefore, $|3| = 3$ and $|-3| = 3$ because each of the numbers 3 and -3 is a distance of 3 from 0. Note that if x is positive, then $|x| = x$; if x is negative, then $|x| = -x$; and lastly, $|0| = 0$. It follows that the absolute value of any nonzero number is positive. Here are some examples.

$$|\sqrt{5}| = \sqrt{5}$$
$$|-23| = -(-23) = 23$$
$$|-10.2| = 10.2$$

There are several general properties of real numbers that are used frequently. If a, b, and c are real numbers, then

- $a + b = b + a$ and $ab = ba$.
  For example, $8 + 2 = 2 + 8 = 10$ and $(-3)(17) = (17)(-3) = -51$.
- $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
  For example, $(7 + 3) + 8 = 7 + (3 + 8) = 18$ and $\left(7\sqrt{2}\right)\sqrt{2} = 7\left(\sqrt{2}\sqrt{2}\right) = (7)(2) = 14$.
- $a(b + c) = ab + ac$
  For example, $5(3 + 16) = (5)(3) + (5)(16) = 95$.
- $a + 0 = a$, $(a)(0) = 0$, and $(a)(1) = a$.
- If $ab = 0$, then $a = 0$ or $b = 0$, or both.
  For example, if $-2b = 0$, then $b = 0$.
- Division by 0 is not defined; for example, $5 \div 0$, $\frac{-7}{0}$, and $\frac{0}{0}$ are undefined.
- If both a and b are positive, then both $a + b$ and ab are positive.
- If both a and b are negative, then $a + b$ is negative and ab is positive.
- If a is positive and b is negative, then ab is negative.
- $|a + b| \le |a| + |b|$. This is known as the **triangle inequality.**
  For example, if $a = 5$ and $b = -2$, then $|5 + (-2)| = |5-2| = |3| = 3$ and $|5| + |-2| = 5 + 2 = 7$. Therefore, $|5 + (-2)| \le |5| + |-2|$.
- $|a||b| = |ab|$. For example, $|5||-2| = |(5)(-2)| = |-10| = 10$.
- If $a > 1$, then $a^2 > a$. If $0 < b < 1$, then $b^2 < b$.
  For example, $5^2 = 25 > 5$, but $\left(\frac{1}{5}\right)^2 = \frac{1}{25} < \frac{1}{5}$.

1.6 Ratio

The **ratio** of one quantity to another is a way to express their relative sizes, often in the form of a fraction, where the first quantity is the numerator and the second quantity is the denominator. Thus, if s and t are positive quantities, then the ratio of s to t can be written as the fraction $\frac{s}{t}$. The notation "s to t" or "$s{:}t$" is also used to express this ratio. For example, if there are 2 apples and 3 oranges in a basket, we can say that the ratio of the number of apples to the number of oranges is $\frac{2}{3}$ or that it is 2 to 3 or that it is 2:3. Like fractions, ratios can be reduced to lowest terms. For example, if there are 8 apples and 12 oranges in a basket, then the ratio of the numbers of apples to oranges is still 2 to 3. Similarly, the ratio 9 to 12 is equivalent to the ratio 3 to 4.

If three or more positive quantities are being considered, say r, s, and t, then their relative sizes can also be expressed as a ratio with the notation "r to s to t." For example, if there are 5 apples, 30 pears, and 20 oranges in a basket, then the ratio of the numbers of apples to pears to oranges is 5 to 30 to 20. This ratio can be reduced to 1 to 6 to 4 by dividing each number by the greatest common divisor of 5, 30, and 20, which is 5.

A **proportion** is an equation relating two ratios; for example, $\frac{9}{12} = \frac{3}{4}$. To solve a problem involving ratios, you can often write a proportion and solve it by **cross multiplication**.

Example 1.6.1: To find a number x so that the ratio of x to 49 is the same as the ratio of 3 to 21, you can write

$$\frac{x}{49} = \frac{3}{21}$$

Then cross multiply to get $21x = (3)(49),$ and solve for x to get $x = \frac{(3)(49)}{21} = 7.$

1.7 Percent

The term **percent** means *per hundred*, or *hundredths*. Percents are ratios that are often used to represent *parts of a whole*, where the whole is considered as having 100 parts.

- 1 percent means 1 part out of 100 parts, or $\frac{1}{100}$.
- 32 percent means 32 parts out of 100 parts, or $\frac{32}{100}$.
- 50 percent means 50 parts out of 100 parts, or $\frac{1}{2}$.

Note that the *part* is the numerator of the ratio and the *whole* is the denominator. Percents are often written with the % symbol; fractional and decimal equivalents are often used as well but without the % symbol, as follows.

$$1\% = \frac{1}{100} = 0.01$$
$$100\% = \frac{100}{100} = 1$$
$$32\% = \frac{32}{100} = 0.32$$
$$50\% = \frac{50}{100} = 0.5$$
$$0.3\% = \frac{0.3}{100} = 0.003$$

Be careful not to confuse 0.01 with 0.01%. The percent symbol matters. For example, $0.01 = 1\%$ but $0.01\% = \frac{0.01}{100} = 0.0001.$

- To compute a *percent*, given the *part* and the *whole*, divide the part by the whole. The result will be the decimal equivalent, so multiply the result by 100 to convert to percent.

Example 1.7.1: If the whole is 20 and the part is 13, you can find the percent as follows.

$$\frac{part}{whole} = \frac{13}{20} = 0.65 = 65\%$$

Example 1.7.2: What percent of 150 is 12.9?

Solution: Here the whole is 150 and the part is 12.9.

$$\frac{part}{whole} = \frac{12.9}{150} = 0.086 = 8.6\%$$

- To find the *part* that is a certain *percent* of a *whole*, you can either multiply the *whole* by the decimal equivalent of the percent or set up a proportion to find the part.

Example 1.7.3: To find 30% of 350, multiply 350 by the decimal equivalent of 30%, or 0.3, as follows.

$$x = (350)(0.3) = 105$$

To use a proportion, you need to find the number of parts of 350 that yields the same ratio as 30 out of 100 parts. You want a number x that satisfies the proportion

$$\frac{part}{whole} = \frac{30}{100}$$
$$\frac{x}{350} = \frac{30}{100}$$

Solving for x yields $x = \frac{(30)(350)}{100} = 105$, so 30% of 350 is 105.

- Given the *percent* and the *part*, you can calculate the *whole*. To do this you can either use the decimal equivalent of the percent or you can set up a proportion and solve it.

Example 1.7.4: 15 is 60% of what number?

Solution: Use the decimal equivalent of 60%. Because 60% of some number z is 15, multiply z by the decimal equivalent of 60%, or 0.6.

$$0.6z = 15$$

Now solve for z by dividing both sides of the equation by 0.6 as follows.

$$z = \frac{15}{0.6} = 25$$

Using a proportion, look for a number z such that

$$\frac{part}{whole} = \frac{60}{100}$$
$$\frac{15}{z} = \frac{60}{100}$$

Hence, $60z = (15)(100)$, and therefore, $z = \frac{(15)(100)}{60} = \frac{1,500}{60} = 25$. That is, 15 is 60% of 25.

Although the discussion about percent so far assumes a context of a *part* and a *whole*, it is not necessary that the part be less than the whole. In general, the whole is called the **base** of the percent. When the numerator of a percent is greater than the base, the percent is greater than 100%. For example, 15 is 300% of 5, since

$$\frac{15}{5} = \frac{300}{100}$$

and 250% of 16 is $\left(\frac{250}{100}\right)(16) = (2.5)(16) = 40$. Note that the decimal equivalent of 250% is 2.5.

It is also not necessary for the part to be related to the whole at all, as in the question, "a teacher's salary is what percent of a banker's salary?"

When a quantity changes from an initial positive amount to another positive amount, for example, an employee's salary that is raised, you can compute the amount of change as a percent of the initial amount. This is called **percent change**. If a quantity increases from 600 to 750, then the **percent increase** is found by dividing the amount of increase, 150, by the base, 600, which is the initial number given:

$$\frac{\textit{amount of increase}}{\textit{base}} = \frac{750 - 600}{600} = \frac{150}{600} = \frac{25}{100} = 0.25 = 25\%$$

We say the percent increase is 25%. Sometimes this computation is written as

$$\left(\frac{750 - 600}{600}\right)(100\%) = \left(\frac{150}{600}\right)(100\%) = 25\%$$

If a quantity doubles in size, then the percent increase is 100%. For example, if a quantity changes from 150 to 300, then the percent increase is

$$\frac{\textit{change}}{\textit{base}} = \frac{300 - 150}{150} = \frac{150}{150} = 100\%$$

If a quantity decreases from 500 to 400, calculate the **percent decrease** as follows.

$$\frac{\textit{change}}{\textit{base}} = \frac{500 - 400}{500} = \frac{100}{500} = \frac{20}{100} = 0.20 = 20\%$$

The quantity decreased by 20%.

When computing a percent *increase*, the base is the *smaller* number. When computing a percent *decrease*, the base is the *larger* number. In either case, the base is the initial number, before the change.

Example 1.7.5: An investment in a mutual fund increased by 12% in a single day. If the value of the investment before the increase was \$1,300, what was the value after the increase?

Solution: The percent increase is 12%. Therefore, the value of the increase is 12% of \$1,300, or, using the decimal equivalent, the increase is (0.12)(\$1,300) = \$156. Thus, the value of the investment after the change is

$$\$1{,}300 + \$156 = \$1{,}456$$

Because the final result is the sum of the initial investment—100% of \$1,300—and the increase—12% of \$1,300—the final result is 100% + 12% = 112% of \$1,300.

Thus, another way to get the final result is to multiply the value of the investment by the decimal equivalent of 112%, which is 1.12:

$$(\$1{,}300)(1.12) = \$1{,}456$$

A quantity may have several successive percent changes. The base of each successive percent change is the result of the preceding percent change.

Example 1.7.6: The monthly enrollment at a preschool decreased by 8% during one month and increased by 6% during the next month. What was the cumulative percent change for the two months?

Solution: If E is the enrollment before the first month, then the enrollment as a result of the 8% decrease can be found by multiplying the base E by the decimal equivalent of $100\% - 8\% = 92\%$, which is 0.92:

$$0.92E$$

The enrollment as a result of the second percent change—the 6% increase—can be found by multiplying the *new* base $0.92E$ by the decimal equivalent of $100\% + 6\% = 106\%$, which is 1.06:

$$(1.06)(0.92)E = 0.9752E$$

The percent equivalent of 0.9752 is 97.52%, which is 2.48% less than 100%. Thus, the cumulative percent change in the enrollment for the two months is a 2.48% decrease.

ARITHMETIC EXERCISES

1. Evaluate the following.

   (a) $15 - (6 - 4)(-2)$
   (b) $(2 - 17) \div 5$
   (c) $(60 \div 12) - (-7 + 4)$
   (d) $(3)^4 - (-2)^3$
   (e) $(-5)(-3) - 15$
   (f) $(-2)^4(15 - 18)^4$
   (g) $(20 \div 5)^2(-2 + 6)^3$
   (h) $(-85)(0) - (-17)(3)$

2. Evaluate the following.

   (a) $\frac{1}{2} - \frac{1}{3} + \frac{1}{12}$
   (b) $\left(\frac{3}{4} + \frac{1}{7}\right)\left(\frac{-2}{5}\right)$
   (c) $\left(\frac{7}{8} - \frac{4}{5}\right)^2$
   (d) $\left(\frac{3}{-8}\right) \div \left(\frac{27}{32}\right)$

3. Which of the integers 312, 98, 112, and 144 are divisible by 8?

4. (a) What is the prime factorization of 372?
   (b) What are the positive divisors of 372?
5. (a) What are the prime divisors of 100?
   (b) What are the prime divisors of 144?
6. Which of the integers 2, 9, 19, 29, 30, 37, 45, 49, 51, 83, 90, and 91 are prime numbers?
7. What is the prime factorization of 585?

8. Which of the following statements are true?

(a) $-5 < 3.1$
(b) $\sqrt{16} = 4$
(c) $7 \div 0 = 0$
(d) $0 < \left|-\frac{1}{7}\right|$
(e) $0.3 < \frac{1}{3}$
(f) $(-1)^{87} = -1$
(g) $\sqrt{(-3)^2} < 0$
(h) $\frac{21}{28} = \frac{3}{4}$
(i) $-\left|-23\right| = 23$
(j) $\frac{1}{2} > \frac{1}{17}$
(k) $(59)^3(59)^2 = 59^6$
(l) $-\sqrt{25} < -4$

9. Find the following.

(a) 40% of 15
(b) 150% of 48
(c) 0.6% of 800
(d) 15 is 30% of which number?
(e) 11 is what percent of 55?

10. If a person's salary increased from \$200 per week to \$234 per week, what was the percent increase in the person's salary?
11. If an athlete's weight decreased from 160 pounds to 152 pounds, what was the percent decrease in the athlete's weight?
12. A particular stock is valued at \$40 per share. If the value increases by 20 percent and then decreases by 25 percent, what will be the value of the stock per share after the decrease?
13. If the ratio of the number of men to the number of women on a committee of 20 members is 3 to 2, how many members of the committee are women?
14. The integer a is even and the integer b is odd. For each of the following integers, indicate whether the integer is even or odd.

Integer	Even	Odd
$a + 2b$		
$2a + b$		
ab		
a^b		
$(a + b)^2$		
$a^2 - b^2$		

15. When the positive integer n is divided by 3, the remainder is 2 and when n is divided by 5, the remainder is 1. What is the least possible value of n?

ANSWERS TO ARITHMETIC EXERCISES

1. (a) 19
(b) -3
(c) 8
(d) 89
(e) 0
(f) 1,296
(g) 1,024
(h) 51

2. (a) $\frac{1}{4}$
   (b) $-\frac{5}{14}$
   (c) $\frac{9}{1,600}$
   (d) $-\frac{4}{9}$

3. 312, 112, and 144
4. (a) $372 = (2^2)(3)(31)$
   (b) The positive divisors of 372 are 1, 2, 3, 4, 6, 12, 31, 62, 93, 124, 186, and 372.
5. (a) $100 = (2^2)(5^2)$, so the prime divisors are 2 and 5.
   (b) $144 = (2^4)(3^2)$, so the prime divisors are 2 and 3.
6. 2, 19, 29, 37, and 83
7. $585 = (3^2)(5)(13)$
8. (a) True
   (b) True
   (c) False; division by 0 is undefined
   (d) True
   (e) True
   (f) True
   (g) False; $\sqrt{(-3)^2} = \sqrt{9} = 3 > 0$
   (h) True
   (i) False; $-|-23| = -23$
   (j) True
   (k) False; $(59)^3(59)^2 = 59^{3+2} = 59^5$
   (l) True

9. (a) 6
   (b) 72
   (c) 4.8
   (d) 50
   (e) 20%

10. 17%
11. 5%
12. \$36 per share
13. 8 women
14.

Integer	Even	Odd
$a + 2b$	✓	
$2a + b$		✓
ab	✓	
a^b	✓	
$(a + b)^2$		✓
$a^2 - b^2$		✓

15. 11

2. ALGEBRA

Basic algebra can be viewed as an extension of arithmetic. The main concept that distinguishes algebra from arithmetic is that of a **variable**, which is a letter that represents a quantity whose value is unknown. The letters x and y are often used as variables, although any letter can be used. Variables enable you to present a word problem in terms of unknown quantities by using algebraic expressions, equations, inequalities, and functions. This section reviews these algebraic tools and then progresses to several examples of applying them to solve real-life word problems. The section ends with coordinate geometry and graphs of functions as other important algebraic tools for solving problems.

2.1 Operations with Algebraic Expressions

An **algebraic expression** has one or more variables and can be written as a single **term** or as a sum of terms. Here are some examples of algebraic expressions.

$$2x \qquad y - \frac{1}{4} \qquad w^3z + 5z^2 - z^2 + 6 \qquad \frac{8}{n+p}$$

In the examples above, $2x$ is a single term, $y - \frac{1}{4}$ has two terms, $w^3z + 5z^2 - z^2 + 6$ has four terms, and $\frac{8}{n+p}$ has one term. In the expression $w^3z + 5z^2 - z^2 + 6$, the terms $5z^2$ and $-z^2$ are called **like terms** because they have the same variables, and the corresponding variables have the same exponents. A term that has no variable is called a **constant** term. A number that is multiplied by variables is called the **coefficient** of a term. For example, in the expression $2x^2 + 7x - 5$, 2 is the coefficient of the term $2x^2$, 7 is the coefficient of the term $7x$, and -5 is a constant term.

The same rules that govern operations with numbers apply to operations with algebraic expressions. One additional rule, which helps in simplifying algebraic expressions, is that like terms can be combined by simply adding their coefficients, as the following examples show.

$$2x + 5x = 7x$$

$$w^3z + 5z^2 - z^2 + 6 = w^3z + 4z^2 + 6$$

$$3xy + 2x - xy - 3x = 2xy - x$$

A number or variable that is a factor of each term in an algebraic expression can be factored out, as the following examples show.

$$4x + 12 = 4(x + 3)$$

$$15y^2 - 9y = 3y(5y - 3)$$

$$\frac{7x^2 + 14x}{2x + 4} = \frac{7x(x + 2)}{2(x + 2)} = \frac{7x}{2} \quad \text{(where } x \neq -2\text{, since division by 0 is not defined)}$$

To multiply two algebraic expressions, each term of the first expression is multiplied by each term of the second expression, and the results are added, as the following examples show.

$$\begin{aligned}(x+2)(3x-7) &= x(3x)+x(-7)+2(3x)+2(-7)\\ &= 3x^2-7x+6x-14\\ &= 3x^2-x-14\end{aligned}$$

A statement of equality between two algebraic expressions that is true for all possible values of the variables involved is called an **identity**. All of the preceding equality statements in this section are identities. Here are some standard identities that are useful.

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$$

$$a^2-b^2 = (a+b)(a-b)$$

All of the identities above can be used to modify and simplify algebraic expressions. For example, the identity $a^2-b^2=(a+b)(a-b)$ can be used to simplify the following algebraic expression.

$$\frac{x^2-9}{4x-12} = \frac{(x+3)(x-3)}{4(x-3)} = \frac{x+3}{4} \qquad (\text{where } x \neq 3)$$

A statement of equality between two algebraic expressions that is true for only certain values of the variables involved is called an **equation**. The values are called the **solutions** of the equation.

The following are examples of some basic types of equations.

$3x+5=-2$	A **linear equation in one variable,** x
$x-3y=10$	A **linear equation in two variables,** x and y
$20y^2+6y-17=0$	A **quadratic equation in one variable,** y

2.2 Rules of Exponents

In the algebraic expression x^a, where x is raised to the power a, x is called the **base** and a is called the **exponent**. For some equations involving bases and exponents, the following property is very useful: if $x^a = x^b$, then $a = b$. This is true for all positive numbers x, except $x = 1$, and for all integers a and b. For example, if $2^y = 64$, then since 64 is 2^6, you have $2^y = 2^6$, and you can conclude that $y = 6$.

Here are the basic rules of exponents, where the bases x and y are nonzero real numbers and the exponents a and b are integers.

1. $x^{-a} = \dfrac{1}{x^a}$

   Examples: $4^{-3} = \dfrac{1}{4^3} = \dfrac{1}{64}$, $x^{-10} = \dfrac{1}{x^{10}}$, and $\dfrac{1}{2^{-a}} = 2^a$

2. $(x^a)(x^b) = x^{a+b}$

   Examples: $(3^2)(3^4) = 3^{2+4} = 3^6 = 729$ and $(y^3)(y^{-1}) = y^2$

3. $\dfrac{x^a}{x^b} = x^{a-b} = \dfrac{1}{x^{b-a}}$

   Examples: $\dfrac{5^7}{5^4} = 5^{7-4} = 5^3 = 125$ and $\dfrac{t^3}{t^8} = t^{-5} = \dfrac{1}{t^5}$

4. $x^0 = 1$

   Examples: $7^0 = 1$ and $(-3)^0 = 1$. Note that 0^0 is not defined.

5. $(x^a)(y^a) = (xy)^a$

   Examples: $(2^3)(3^3) = 6^3 = 216$ and $(10z)^3 = 10^3z^3 = 1{,}000z^3$

6. $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$

   Examples: $\left(\dfrac{3}{4}\right)^2 = \dfrac{3^2}{4^2} = \dfrac{9}{16}$ and $\left(\dfrac{r}{4t}\right)^3 = \dfrac{r^3}{64t^3}$

7. $(x^a)^b = x^{ab}$

   Examples: $(2^5)^2 = 2^{10} = 1{,}024$ and $(3y^6)^2 = (3^2)(y^6)^2 = 9y^{12}$

The rules above are identities that are used to simplify expressions. Sometimes algebraic expressions look like they can be simplified in similar ways, but in fact they cannot. Here are several pairs of expressions that are *commonly mistaken* to be identities.

- $x^a y^b \neq (xy)^{a+b}$

  Note that the bases are not the same.
- $(x^a)^b \neq x^a x^b$

  Instead, $(x^a)^b = x^{ab}$ and $x^a x^b = x^{a+b}$; for example, $(4^2)^3 = 4^6$ and $4^2 4^3 = 4^5$.
- $(x+y)^a \neq x^a + y^a$

  Recall that $(x+y)^2 = x^2 + 2xy + y^2$; that is, the correct expansion contains terms such as $2xy$.
- $(-x)^2 \neq -x^2$

  Instead, $(-x)^2 = x^2$. Note carefully where each minus sign appears.
- $\sqrt{x^2 + y^2} \neq x + y$
- $\dfrac{a}{x+y} \neq \dfrac{a}{x} + \dfrac{a}{y}$

  But it *is* true that $\dfrac{x+y}{a} = \dfrac{x}{a} + \dfrac{y}{a}$.

2.3 Solving Linear Equations

To **solve an equation** means to find the values of the variables that make the equation true, that is, the values that **satisfy the equation**. Two equations that have the same solutions are called **equivalent equations**. For example, $x + 1 = 2$ and $2x + 2 = 4$ are equivalent equations; both are true when $x = 1$ and are false otherwise. The general method for solving an equation is to find successively simpler equivalent equations so that the simplest equivalent equation makes the solutions obvious.

The following rules are important for producing equivalent equations.

- When the same constant is added to or subtracted from both sides of an equation, the equality is preserved and the new equation is equivalent to the original equation.
- When both sides of an equation are multiplied or divided by the same nonzero constant, the equality is preserved and the new equation is equivalent to the original equation.

A **linear equation** is an equation involving one or more variables in which each term in the equation is either a constant term or a variable multiplied by a coefficient. None of the variables are multiplied together or raised to a power greater than 1. For example, $2x + 1 = 7x$ and $10x - 9y - z = 3$ are linear equations, but $x + y^2 = 0$ and $xz = 3$ are not.

Linear Equations in One Variable

To solve a linear equation in one variable, simplify each side of the equation by combining like terms. Then use the rules for producing simpler equivalent equations.

Example 2.3.1:

$$
\begin{aligned}
11x - 4 - 8x &= 2(x + 4) - 2x \\
3x - 4 &= 2x + 8 - 2x && \text{(like terms combined)} \\
3x - 4 &= 8 && \text{(simplified)} \\
3x - 4 + 4 &= 8 + 4 && \text{(4 added to both sides)} \\
3x &= 12 \\
\frac{3x}{3} &= \frac{12}{3} && \text{(both sides divided by 3)} \\
x &= 4
\end{aligned}
$$

You can always check your solution by substituting it into the original equation.

Note that it is possible for a linear equation to have no solutions. For example, the equation $2x + 3 = 2(7 + x)$ has no solution, since it is equivalent to the equation $3 = 14$, which is false. Also, it is possible that what looks to be a linear equation turns out to be an identity when you try to solve it. For example, $3x - 6 = -3(2 - x)$ is true for all values of x, so it is an identity.

Linear Equations in Two Variables

A linear equation in two variables, x and y, can be written in the form

$$ax + by = c$$

where a, b, and c are real numbers and a and b are not both zero. For example, $3x + 2y = 8$ is a linear equation in two variables.

A solution of such an equation is an **ordered pair** of numbers (x, y) that makes the equation true when the values of x and y are substituted into the equation. For example, both $(2, 1)$ and $\left(-\frac{2}{3}, 5\right)$ are solutions of the equation $3x + 2y = 8$, but $(1, 2)$ is not a solution. A linear equation in two variables has infinitely many solutions. If another linear equation in the same variables is given, it is usually possible to find a unique solution of both equations. Two equations with the same variables are called a **system of equations**, and the equations in the system are called **simultaneous equations**. To solve a system of two equations means to find an ordered pair of numbers that satisfies *both* equations in the system.

There are two basic methods for solving systems of linear equations, by **substitution** or by **elimination**. In the substitution method, one equation is manipulated to express one variable in terms of the other. Then the expression is substituted in the other equation. For example, to solve the system of equations

$$
\begin{aligned}
4x + 3y &= 13 \\
x + 2y &= 2
\end{aligned}
$$

you can express x in the second equation in terms of y as $x = 2 - 2y$. Then substitute $2 - 2y$ for x in the first equation to find the value of y.

$$\begin{aligned}
4(2-2y)+3y &= 13 \\
8-8y+3y &= 13 \\
-8y+3y &= 5 && \text{(8 subtracted from both sides)} \\
-5y &= 5 && \text{(like terms combined)} \\
y &= -1 && \text{(both sides divided by } -5\text{)}
\end{aligned}$$

Then -1 can be substituted for y in either equation to find the value of x. We use the second equation:

$$\begin{aligned}
x+2y &= 2 \\
x+2(-1) &= 2 \\
x-2 &= 2 \\
x &= 4 && \text{(2 added to both sides)}
\end{aligned}$$

In the elimination method, the object is to make the coefficients of one variable the same in both equations so that one variable can be eliminated either by adding the equations together or by subtracting one from the other. In the example above, multiplying both sides of the second equation by 4 yields $4(x+2y)=4(2)$, or $4x+8y=8$. Now you have two equations with the same coefficient of x.

$$\begin{aligned}
4x+3y &= 13 \\
4x+8y &= 8
\end{aligned}$$

If you subtract the second equation from the first, the result is $-5y=5$. Thus, $y=-1$, and substituting -1 for y in either of the original equations yields $x=4$.

By either method, the solution of the system is $x=4$ and $y=-1$, or $(x,y)=(4,-1)$.

2.4 Solving Quadratic Equations

A **quadratic equation** in the variable x is an equation that can be written in the form

$$ax^2+bx+c=0$$

where a, b, and c are real numbers and $a \neq 0$. When such an equation has solutions, they can be found using the **quadratic formula**:

$$x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

where the notation $\pm$ is shorthand for indicating two solutions—one that uses the plus sign and the other that uses the minus sign.

Example 2.4.1: In the quadratic equation $2x^2-x-6=0$, we have $a=2$, $b=-1$, and $c=-6$. Therefore, the quadratic formula yields

$$\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2-4(2)(-6)}}{2(2)} \\
&= \frac{1 \pm \sqrt{49}}{4} \\
&= \frac{1 \pm 7}{4}
\end{aligned}$$

Hence the two solutions are $x=\frac{1+7}{4}=2$ and $x=\frac{1-7}{4}=-\frac{3}{2}$.

Quadratic equations have at most two real solutions, as in the example above. However, some quadratic equations have only one real solution. For example, the quadratic equation $x^2 + 4x + 4 = 0$ has only one solution, which is $x = -2$. In this case, the expression under the square root symbol in the quadratic formula is equal to 0, and so adding or subtracting 0 yields the same result. Other quadratic equations have no real solutions; for example, $x^2 + x + 5 = 0$. In this case, the expression under the square root symbol is negative, so the entire expression is not a real number.

Some quadratic equations can be solved more quickly by factoring. For example, the quadratic equation $2x^2 - x - 6 = 0$ in example 2.4.1 can be factored as $(2x + 3)(x - 2) = 0$. When a product is equal to 0, at least one of the factors must be equal to 0, which leads to two cases: either $2x + 3 = 0$ or $x - 2 = 0$. Therefore,

$$\begin{aligned} 2x + 3 &= 0 \\ 2x &= -3 \\ x &= -\frac{3}{2} \end{aligned} \qquad \text{OR} \qquad \begin{aligned} x - 2 &= 0 \\ x &= 2 \end{aligned}$$

and the solutions are $-\frac{3}{2}$ and 2.

Example 2.4.2: Here is another example of a quadratic equation that can be easily factored.

$$\begin{aligned} 5x^2 + 3x - 2 &= 0 \\ (5x - 2)(x + 1) &= 0 \end{aligned}$$

Therefore,

$$\begin{aligned} 5x - 2 &= 0 \\ x &= \frac{2}{5} \end{aligned} \qquad \text{OR} \qquad \begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned}$$

2.5 Solving Linear Inequalities

A mathematical statement that uses one of the following inequality signs is called an **inequality**.

$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

Inequalities can involve variables and are similar to equations, except that the two sides are related by one of the inequality signs instead of the equality sign used in equations. For example, the inequality $4x - 1 \leq 7$ is a linear inequality in one variable, which states that "$4x - 1$ is less than or equal to 7." To **solve an inequality** means to find the set of all values of the variable that make the inequality true. This set of values is also known as the **solution set** of an inequality. Two inequalities that have the same solution set are called **equivalent inequalities**.

The procedure used to solve a linear inequality is similar to that used to solve a linear equation, which is to simplify the inequality by isolating the variable on one side of the inequality, using the following two rules.

- When the same constant is added to or subtracted from both sides of an inequality, the direction of the inequality is preserved and the new inequality is equivalent to the original.
- When both sides of the inequality are multiplied or divided by the same nonzero constant, the direction of the inequality is *preserved if the constant is positive* but the direction is *reversed if the constant is negative*. In either case, the new inequality is equivalent to the original.

Example 2.5.1: The inequality $-3x + 5 \le 17$ can be solved as follows.

$$-3x + 5 \le 17$$

$$-3x \le 12 \quad \text{(5 subtracted from both sides)}$$

$$\frac{-3x}{-3} \ge \frac{12}{-3} \quad \text{(both sides divided by } -3\text{, which reverses the direction of the inequality)}$$

$$x \ge -4$$

Therefore, the solution set of $-3x + 5 \le 17$ consists of all real numbers greater than or equal to -4.

Example 2.5.2:

$$\frac{4x+9}{11} < 5$$

$$4x + 9 < 55 \quad \text{(both sides multiplied by 11)}$$

$$4x < 46 \quad \text{(9 subtracted from both sides)}$$

$$x < \frac{46}{4} \quad \text{(both sides divided by 4)}$$

$$x < 11.5$$

Therefore, the solution set of $\frac{4x+9}{11} < 5$ consists of all real numbers less than 11.5.

2.6 Functions

An algebraic expression in one variable can be used to define a **function** of that variable. Functions are usually denoted by letters such as f, g, and h. For example, the algebraic expression $3x + 5$ can be used to define a function f by

$$f(x) = 3x + 5$$

where $f(x)$ is called the value of f at x and is obtained by substituting the value of x in the expression above. For example, if $x = 1$ is substituted in the expression above, the result is $f(1) = 8$.

It might be helpful to think of a function f as a machine that takes an input, which is a value of the variable x, and produces the corresponding output, $f(x)$. For any function, each input x gives exactly one output $f(x)$. However, more than one value of x can give the same output $f(x)$. For example, if g is the function defined by $g(x) = x^2 - 2x + 3$, then $g(0) = 3$ and $g(2) = 3$.

The **domain** of a function is the set of all permissible inputs, that is, all permissible values of the variable x. For the functions f and g defined above, the domain is the set of all real numbers. Sometimes the domain of the function is given explicitly and is restricted to a specific set of values of x. For example, we can define the function h by $h(x) = x^2 - 4$ for $-2 \le x \le 2$. Without an explicit restriction, the domain is assumed to be the set of all values of x for which $f(x)$ is a real number.

Example 2.6.1: Let f be the function defined by $f(x) = \frac{2x}{x-6}$. In this case, f is not defined at $x = 6$ because $\frac{12}{0}$ is not defined. Hence, the domain of f consists of all real numbers except for 6.

Example 2.6.2: Let g be the function defined by $g(x) = x^3 + \sqrt{x+2} - 10$. In this case, $g(x)$ is not a real number if $x < -2$. Hence, the domain of g consists of all real numbers x such that $x \ge -2$.

Example 2.6.3: Let h be the function defined by $h(x) = |x|$, the **absolute value** of x, which is the distance between x and 0 on the number line (see section 1.5). The domain of h is the set of all real numbers. Also, $h(x) = h(-x)$ for all real numbers x, which reflects the property that on the number line the distance between x and 0 is the same as the distance between $-x$ and 0.

2.7 Applications

Translating verbal descriptions into algebraic expressions is an essential initial step in solving word problems. Some examples are given below.

- If the square of the number x is multiplied by 3, and then 10 is added to that product, the result can be represented by $3x^2 + 10$.
- If John's present salary s is increased by 14 percent, then his new salary is $1.14s$.
- If y gallons of syrup are to be distributed among 5 people so that one particular person gets 1 gallon and the rest of the syrup is divided equally among the remaining 4, then each of those 4 people will get $\frac{y-1}{4}$ gallons of syrup.

Here are several examples of using algebraic techniques to solve word problems.

Example 2.7.1: Ellen has received the following scores on 3 exams: 82, 74, and 90. What score will Ellen need to receive on the next exam so that the average (arithmetic mean) score for the 4 exams will be 85?

Solution: Let x represent the score on Ellen's next exam. This initial step of assigning a variable to the quantity that is sought is an important beginning to solving the problem. Then in terms of x, the average of the 4 exams is

$$\frac{82 + 74 + 90 + x}{4}$$

which is supposed to equal 85. Now simplify the expression and set it equal to 85:

$$\frac{82 + 74 + 90 + x}{4} = \frac{246 + x}{4} = 85$$

Solving the resulting linear equation for x, you get

$$\begin{aligned} 246 + x &= 340 \\ x &= 94 \end{aligned}$$

Therefore, Ellen will need to attain a score of 94 on the next exam.

Example 2.7.2: A mixture of 12 ounces of vinegar and oil is 40 percent vinegar, where all of the measurements are by weight. How many ounces of oil must be added to the mixture to produce a new mixture that is only 25 percent vinegar?

Solution: Let x represent the number of ounces of oil to be added. Then the total number of ounces of the new mixture will be $12 + x$, and the total number of ounces of vinegar in the new mixture will be $(0.40)(12)$. Since the new mixture must be 25 percent vinegar,

$$\frac{(0.40)(12)}{12 + x} = 0.25$$

Therefore,

$$\begin{aligned} (0.40)(12) &= (12 + x)(0.25) \\ 4.8 &= 3 + 0.25x \\ 1.8 &= 0.25x \\ 7.2 &= x \end{aligned}$$

Thus, 7.2 ounces of oil must be added to produce a new mixture that is 25 percent vinegar.

Example 2.7.3: In a driving competition, Jeff and Dennis drove the same course at average speeds of 51 miles per hour and 54 miles per hour, respectively. If it took Jeff 40 minutes to drive the course, how long did it take Dennis?

Solution: Let x be the time, in minutes, that it took Dennis to drive the course. The distance d, in miles, is equal to the product of the rate r, in miles per hour, and the time t, in hours; that is,

$$d = rt$$

Note that since the rates are given in miles per *hour*, it is necessary to express the times in hours; for example, 40 minutes equals $\frac{40}{60}$ of an hour. Thus, the distance traveled by Jeff is the product of his speed and his time, $(51)\left(\frac{40}{60}\right)$ miles, and the distance traveled by Dennis is similarly represented by $(54)\left(\frac{x}{60}\right)$ miles. Since the distances are equal,

$$\begin{aligned} (51)\left(\frac{40}{60}\right) &= (54)\left(\frac{x}{60}\right) \\ (51)(40) &= 54x \\ x &= \frac{(51)(40)}{54} \approx 37.8 \end{aligned}$$

Thus, it took Dennis approximately 37.8 minutes to drive the course.

Example 2.7.4: Working alone at its constant rate, machine A takes 3 hours to produce a batch of identical computer parts. Working alone at its constant rate, machine B takes 2 hours to produce an identical batch of parts. How long will it take the two machines, working simultaneously at their respective constant rates, to produce an identical batch of parts?

Solution: Since machine A takes 3 hours to produce a batch, machine A can produce $\frac{1}{3}$ of the batch in 1 hour. Similarly, machine B can produce $\frac{1}{2}$ of the batch in 1 hour. If we let x represent the number of hours it takes both machines, working simultaneously, to produce the batch, then the two machines will produce $\frac{1}{x}$ of the job in 1 hour. When the two machines work together, adding their individual production rates, $\frac{1}{3}$ and $\frac{1}{2}$, gives their combined production rate $\frac{1}{x}$. Therefore,

$$\begin{aligned}
\frac{1}{3}+\frac{1}{2}&=\frac{1}{x}\\
\frac{2}{6}+\frac{3}{6}&=\frac{1}{x}\\
\frac{5}{6}&=\frac{1}{x}\\
\frac{6}{5}&=x
\end{aligned}$$

Thus, working together, the machines will take $\frac{6}{5}$ hours, or 1 hour 12 minutes, to produce a batch of parts.

Example 2.7.5: At a fruit stand, apples can be purchased for \$0.15 each and pears for \$0.20 each. At these rates, a bag of apples and pears was purchased for \$3.80. If the bag contained 21 pieces of fruit, how many of the pieces were pears?

Solution: If a represents the number of apples purchased and p represents the number of pears purchased, the information can be translated into the following system of equations.

$$\begin{aligned}
0.15a+0.20p&=3.80 &&\text{(total cost)}\\
a+p&=21 &&\text{(total number of fruit)}
\end{aligned}$$

From the second equation, $a = 21 - p$. Substituting $21 - p$ into the first equation for a gives

$$\begin{aligned}
0.15(21-p)+0.20p&=3.80\\
(0.15)(21)-0.15p+0.20p&=3.80\\
3.15-0.15p+0.20p&=3.80\\
0.05p&=0.65\\
p&=13
\end{aligned}$$

Thus, of the 21 pieces of fruit, 13 were pears.

Example 2.7.6: To produce a particular radio model, it costs a manufacturer \$30 per radio, and it is assumed that if 500 radios are produced, all of them will be sold. What must be the selling price per radio to ensure that the profit (revenue from the sales minus the total production cost) on the 500 radios is greater than \$8,200 ?

Solution: If y represents the selling price per radio, then the profit is $500(y - 30)$. Therefore, we set

$$500(y - 30) > 8{,}200$$

Solving the inequality, we get

$$\begin{aligned} 500y - 15{,}000 &> 8{,}200 \\ 500y &> 23{,}200 \\ y &> 46.4 \end{aligned}$$

Thus, the selling price must be greater than \$46.40 to ensure that the profit is greater than \$8,200.

Some applications involve computing **interest** earned on an investment during a specified time period. The interest can be computed as simple interest or compound interest.

Simple interest is based only on the initial deposit, which serves as the amount on which interest is computed, called the **principal**, for the entire time period. If the amount P is invested at a *simple annual interest rate of r percent*, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{rt}{100}\right)$$

where P and V are in dollars.

In the case of **compound interest**, interest is added to the principal at regular time intervals, such as annually, quarterly, and monthly. Each time interest is added to the principal, the interest is said to be compounded. After each compounding, interest is earned on the new principal, which is the sum of the preceding principal and the interest just added. If the amount P is invested at an *annual interest rate of r percent, compounded annually*, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{r}{100}\right)^t$$

If the amount P is invested at an *annual interest rate of r percent, compounded n times per year*, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{r}{100n}\right)^{nt}$$

Example 2.7.7: If \$10,000 is invested at a simple annual interest rate of 6 percent, what is the value of the investment after half a year?

Solution: According to the formula for simple interest, the value of the investment after $\frac{1}{2}$ year is

$$\$10{,}000\left(1 + 0.06\left(\frac{1}{2}\right)\right) = \$10{,}000(1.03) = \$10{,}300$$

Example 2.7.8: If an amount P is to be invested at an annual interest rate of 3.5 percent, compounded annually, what should be the value of P so that the value of the investment is \$1,000 at the end of 3 years?

Solution: According to the formula for 3.5 percent annual interest, compounded annually, the value of the investment after 3 years is

$$P(1 + 0.035)^3$$

and we set it to be equal to \$1,000

$$P(1 + 0.035)^3 = \$1{,}000$$

To find the value of P, we divide both sides of the equation by $(1 + 0.035)^3$.

$$P = \frac{\$1{,}000}{(1 + 0.035)^3} \approx \$901.94$$

Thus, an amount of approximately \$901.94 should be invested.

Example 2.7.9: A college student expects to earn at least \$1,000 in interest on an initial investment of \$20,000. If the money is invested for one year at interest compounded quarterly, what is the least annual interest rate that would achieve the goal?

Solution: According to the formula for r percent annual interest, compounded quarterly, the value of the investment after 1 year is

$$\$20{,}000\left(1 + \frac{r}{400}\right)^4$$

By setting this value greater than or equal to \$21,000 and solving for r, we get

$$\$20{,}000\left(1 + \frac{r}{400}\right)^4 \geq \$21{,}000$$
$$\left(1 + \frac{r}{400}\right)^4 \geq 1.05$$

We can use the fact that taking the positive fourth root of each side of an inequality preserves the direction of the inequality. This is also true for the positive square root or any other positive root.

$$1 + \frac{r}{400} \geq \sqrt[4]{1.05}$$
$$r \geq 400\left(\sqrt[4]{1.05} - 1\right)$$

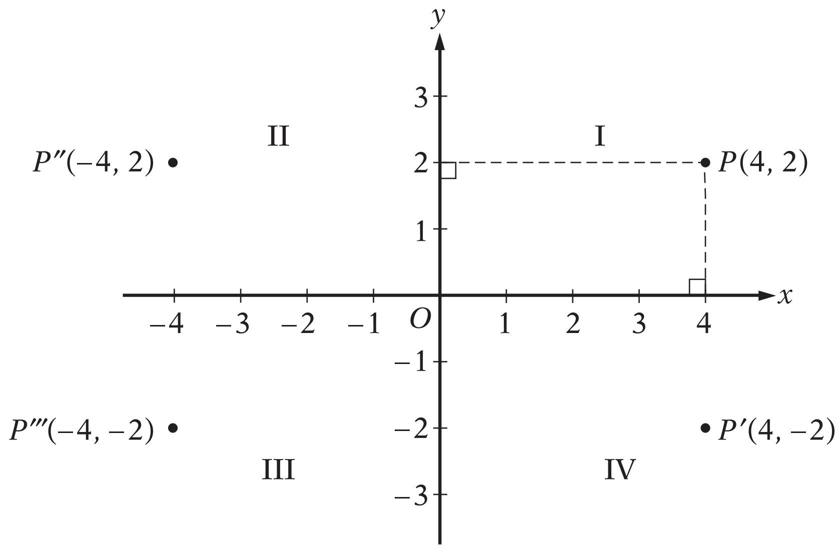
To compute the fourth root, we can use the fact that $\sqrt[4]{x} = \sqrt{\sqrt{x}}$ for $x \geq 0$; that is, we can compute a fourth root by taking a square root twice:

$$r \geq 400\left(\sqrt[4]{1.05} - 1\right) = 400\left(\sqrt{\sqrt{1.05}} - 1\right) \approx 4.91$$

So the least annual interest rate is approximately 4.91 percent.

2.8 Coordinate Geometry

Two real number lines that are perpendicular to each other and that intersect at their respective zero points define a **rectangular coordinate system**, often called the ***xy*-coordinate system** or ***xy*-plane**. The horizontal number line is called the ***x*-axis** and the vertical number line is called the ***y*-axis**. The point where the two axes intersect is called the **origin**, denoted by O. The positive half of the x-axis is to the right of the origin, and the positive half of the y-axis is above the origin. The two axes divide the plane into four regions called **quadrants I, II, III**, and **IV**, as shown in the figure below.



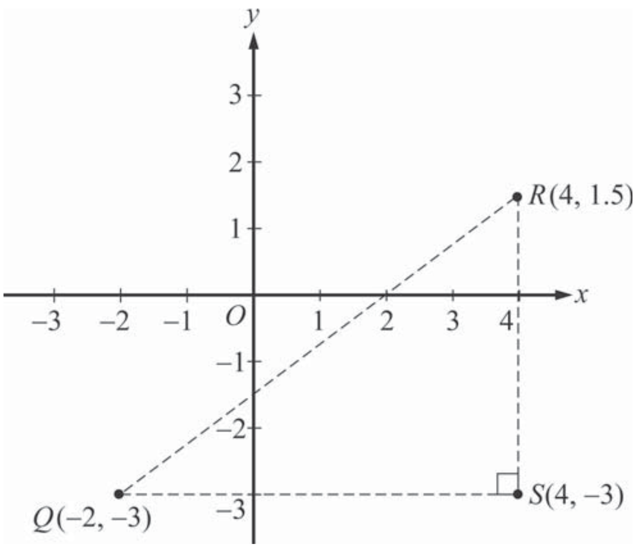
Each point P in the xy-plane can be identified with an ordered pair (x, y) of real numbers and is denoted by $P(x, y)$. The first number is called the ***x*-coordinate**, and the second number is called the ***y*-coordinate**. A point with coordinates (x, y) is located $|x|$ units to the right of the y-axis if x is positive or to the left of the y-axis if x is negative. Also, the point is located $|y|$ units above the x-axis if y is positive or below the x-axis if y is negative. If $x = 0$, the point lies on the y-axis, and if $y = 0$, the point lies on the x-axis. The origin has coordinates $(0, 0)$. Unless otherwise noted, the units used on the x-axis and the y-axis are the same.

In the figure above, the point $P(4, 2)$ is 4 units to the right of the y-axis and 2 units above the x-axis, and the point $P'''(-4, -2)$ is 4 units to the left of the y-axis and 2 units below the x-axis.

Note that the three points $P'(4, -2)$, $P''(-4, 2)$, and $P'''(-4, -2)$ have the same coordinates as P except for the sign. These points are geometrically related to P as follows.

- P' is the **reflection of *P* about the *x*-axis**, or P' and P are **symmetric about the *x*-axis**.
- P'' is the **reflection of *P* about the *y*-axis**, or P'' and P are **symmetric about the *y*-axis**.
- P''' is the **reflection of *P* about the origin**, or P''' and P are **symmetric about the origin**.

The distance between two points in the xy-plane can be found by using the Pythagorean theorem. For example, the distance between the two points $Q(-2, -3)$ and $R(4, 1.5)$ in the figure at the top of the following page is the length of line segment QR. To find this distance, construct a right triangle (indicated by the dashed lines) and then note that the two shorter sides of the triangle have lengths $QS = 4 - (-2) = 6$ and $RS = 1.5 - (-3) = 4.5$.



Since line segment QR is the hypotenuse of the triangle, you can apply the Pythagorean theorem:

$$QR = \sqrt{6^2 + 4.5^2} = \sqrt{56.25} = 7.5$$

(For a discussion of right triangles and the Pythagorean theorem, see section 3.3.)

Equations in two variables can be represented as graphs in the coordinate plane. In the xy-plane, the **graph of an equation** in the variables x and y is the set of all points whose ordered pairs (x, y) satisfy the equation.

The graph of a linear equation of the form $y = mx + b$ is a straight line in the xy-plane, where m is called the **slope** of the line and b is called the ***y*-intercept**.

The ***x*-intercepts** of a graph are the x-coordinates of the points at which the graph intersects the x-axis. Similarly, the ***y*-intercepts** of a graph are the y-coordinates of the points at which the graph intersects the y-axis. Sometimes the terms ***x*-intercept** and ***y*-intercept** refer to the actual intersection points.

The slope of a line passing through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$, where $x_1 \neq x_2$, is defined as

$$\frac{y_2 - y_1}{x_2 - x_1}$$

This ratio is often called "rise over run," where *rise* is the change in y when moving from Q to R and *run* is the change in x when moving from Q to R. A horizontal line has a slope of 0, since the rise is 0 for any two points on the line. So the equation of every horizontal line has the form $y = b$, where b is the y-intercept. The slope of a vertical line is not defined, since the run is 0. The equation of every vertical line has the form $x = a$, where a is the x-intercept.

Two lines are **parallel** if their slopes are equal. Two lines are **perpendicular** if their slopes are negative reciprocals of each other. For example, the line with equation $y = 2x + 5$ is perpendicular to the line with equation $y = -\frac{1}{2}x + 9$.

Example 2.8.1: In the xy-plane above, the slope of the line passing through the points $Q(-2, -3)$ and $R(4, 1.5)$ is

$$\frac{1.5 - (-3)}{4 - (-2)} = \frac{4.5}{6} = 0.75$$

Line QR appears to intersect the y-axis close to the point $(0, -1.5)$, so the y-intercept of the line must be close to -1.5. To get the exact value of the y-intercept, substitute the coordinates of any point on the line, say $Q(-2, -3)$, into the equation $y = 0.75x + b$, and solve it for b as follows.

$$\begin{aligned} y &= 0.75x + b \\ -3 &= (0.75)(-2) + b \\ b &= -3 + (0.75)(2) \\ b &= -1.5 \end{aligned}$$

Therefore, the equation of line QR is $y = 0.75x - 1.5$.

You can see from the graph that the x-intercept of line QR is 2, since QR passes through the point $(2, 0)$. More generally, you can find the x-intercept by setting $y = 0$ in an equation of the line and solving it for x as follows.

$$\begin{aligned} 0 &= 0.75x - 1.5 \\ 1.5 &= 0.75x \\ x &= \frac{1.5}{0.75} = 2 \end{aligned}$$

Graphs of linear equations can be used to illustrate solutions of systems of linear equations and inequalities.

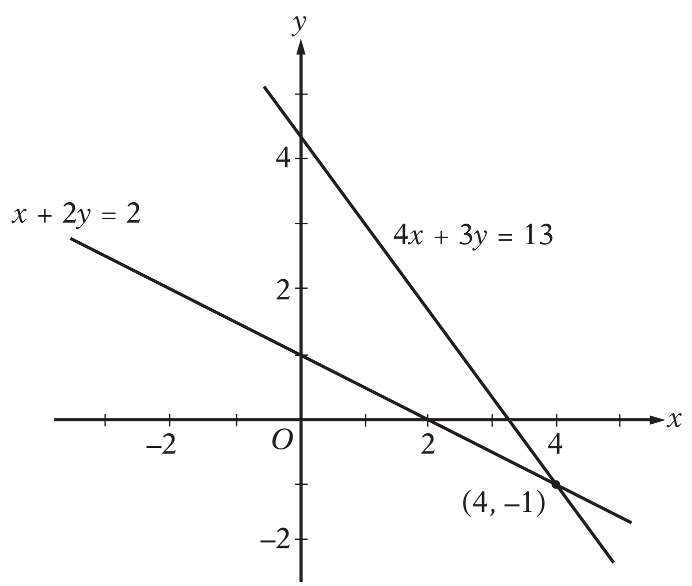
Example 2.8.2: Consider the system of linear equations in two variables in section 2.3:

$$\begin{aligned} 4x + 3y &= 13 \\ x + 2y &= 2 \end{aligned}$$

Solving each equation for y in terms of x yields

$$\begin{aligned} y &= -\frac{4}{3}x + \frac{13}{3} \\ y &= -\frac{1}{2}x + 1 \end{aligned}$$

The graphs of the two equations are below, and the solution of the system of equations is the point at which the two graphs intersect, which is $(4, -1)$.



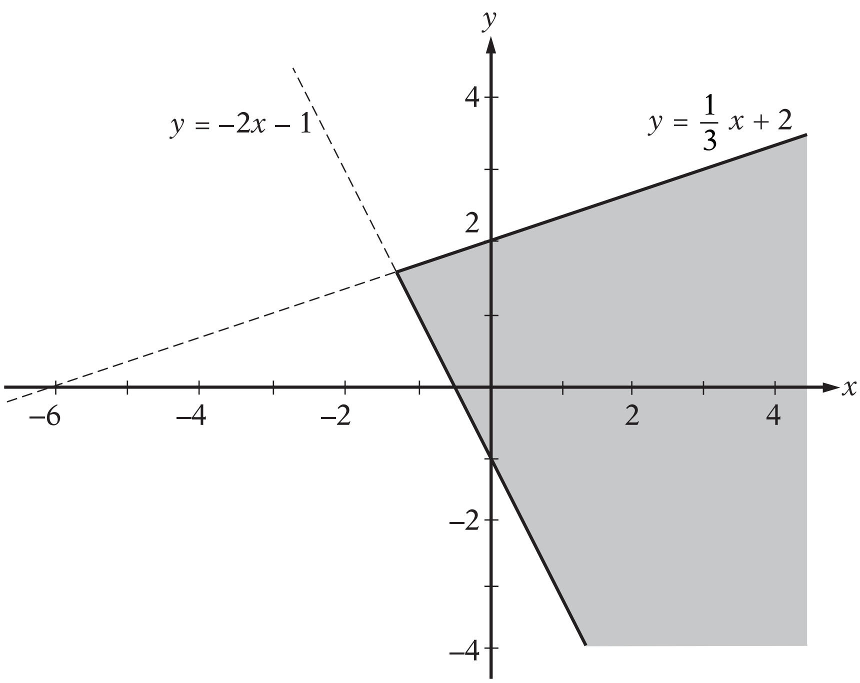
Example 2.8.3: Consider the following system of linear inequalities.

$$x - 3y \geq -6$$
$$2x + y \geq -1$$

Solving each inequality for y in terms of x yields

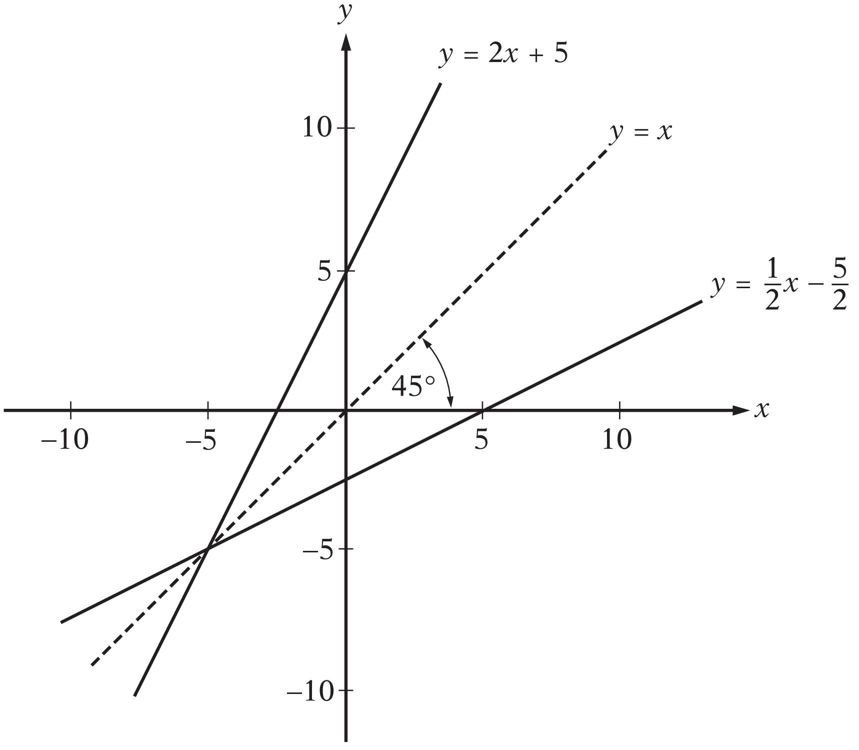
$$y \leq \frac{1}{3}x + 2$$
$$y \geq -2x - 1$$

Each point (x, y) that satisfies the first inequality $y \leq \frac{1}{3}x + 2$ is either on the line $y = \frac{1}{3}x + 2$ or *below* the line because the y-coordinate is either equal to or *less than* $\frac{1}{3}x + 2$. Therefore, the graph of $y \leq \frac{1}{3}x + 2$ consists of the line $y = \frac{1}{3}x + 2$ and the entire region below it. Similarly, the graph of $y \geq -2x - 1$ consists of the line $y = -2x - 1$ and the entire region *above* it. Thus, the solution set of the system of inequalities consists of all of the points that lie in the shaded region shown in the figure below, which is the intersection of the two regions described.



Symmetry with respect to the x-axis, the y-axis, and the origin is mentioned above. Another important symmetry is symmetry with respect to the line with equation $y = x$. The line $y = x$ passes through the origin, has a slope of 1, and makes a 45-degree angle with each axis. For any point with coordinates (a, b), the point with interchanged coordinates (b, a) is the reflection of (a, b) about the line $y = x$; that is, (a, b) and (b, a) are symmetric about the line $y = x$. It follows that interchanging x and y in the equation of any graph yields another graph that is the reflection of the original graph about the line $y = x$.

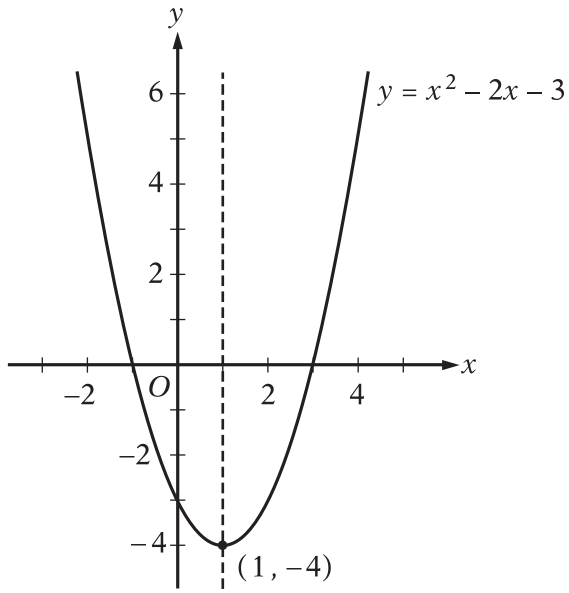
Example 2.8.4: Consider the line whose equation is $y = 2x + 5$. Interchanging x and y in the equation yields $x = 2y + 5$. Solving this equation for y yields $y = \frac{1}{2}x - \frac{5}{2}$. The line $y = 2x + 5$ and its reflection $y = \frac{1}{2}x - \frac{5}{2}$ are graphed below.



The line $y = x$ is a **line of symmetry** for the graphs of $y = 2x + 5$ and $y = \frac{1}{2}x - \frac{5}{2}$.

The graph of a quadratic equation of the form $y = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$, is a **parabola**. The x-intercepts of the parabola are the solutions of the equation $ax^2 + bx + c = 0$. If a is positive, the parabola opens upward and the **vertex** is its lowest point. If a is negative, the parabola opens downward and the vertex is the highest point. Every parabola is symmetric with itself about the vertical line that passes through its vertex. In particular, the two x-intercepts are equidistant from this line of symmetry.

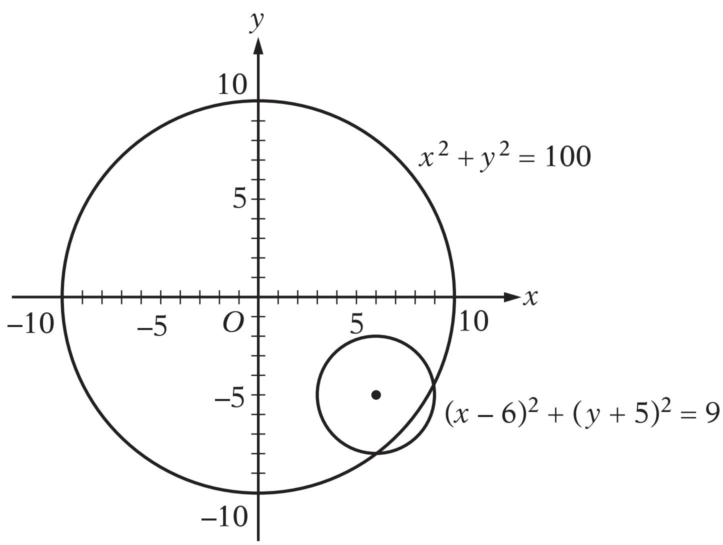
Example 2.8.5: The equation $y = x^2 - 2x - 3$ has the following graph.



The graph indicates that the x-intercepts of the parabola are -1 and 3. The values of the x-intercepts can be confirmed by solving the quadratic equation $x^2 - 2x - 3 = 0$ to get $x = -1$ and $x = 3$. The point $(1, -4)$ is the vertex of the parabola, and the line $x = 1$ is its line of symmetry. The y-intercept is the y-coordinate of the point on the parabola at which $x = 0$, which is $y = 0^2 - 2(0) - 3 = -3$.

The graph of an equation of the form $(x - a)^2 + (y - b)^2 = r^2$ is a **circle** with its center at the point (a, b) and with radius r.

Example 2.8.6: The graph of $x^2 + y^2 = 100$ is a circle with its center at the origin and with radius 10, as shown in the figure below. The smaller circle has center $(6, -5)$ and radius 3, so its equation is $(x - 6)^2 + (y + 5)^2 = 9$.



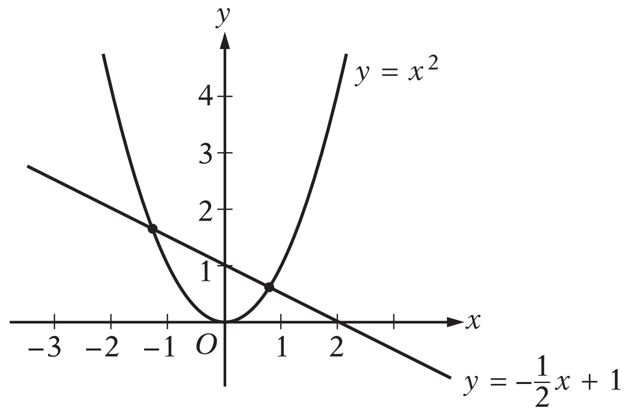
2.9 Graphs of Functions

The coordinate plane can be used for graphing functions. To graph a function in the xy-plane, you represent each input x and its corresponding output $f(x)$ as a point (x, y), where $y = f(x)$. In other words, you use the x-axis for the input and the y-axis for the output.

Below are several examples of graphs of elementary functions.

Example 2.9.1: Consider the linear function defined by $f(x) = -\frac{1}{2}x + 1$. Its graph in the xy-plane is the line with the linear equation $y = -\frac{1}{2}x + 1$, as shown in the figure below.

Example 2.9.2: Consider the quadratic function defined by $g(x) = x^2$. The graph of g is the parabola with the quadratic equation $y = x^2$, as shown in the figure below.



Note that the graphs of f and g from the two examples above intersect at two points. These are the points at which $g(x) = f(x)$. We can find these points algebraically by setting

$$g(x) = f(x)$$
$$x^2 = -\frac{1}{2}x + 1$$

and solving for x, using the quadratic formula, as follows.

$$x^2 = -\frac{1}{2}x + 1$$
$$x^2 + \frac{1}{2}x - 1 = 0$$
$$2x^2 + x - 2 = 0$$

We get $x = \dfrac{-1 \pm \sqrt{1 + 16}}{4}$, which represent the x-coordinates of the two solutions

$$x = \frac{-1 + \sqrt{17}}{4} \approx 0.78 \quad \text{and} \quad x = \frac{-1 - \sqrt{17}}{4} \approx -1.28$$

With these input values, the corresponding y-coordinates can be found using either f or g:

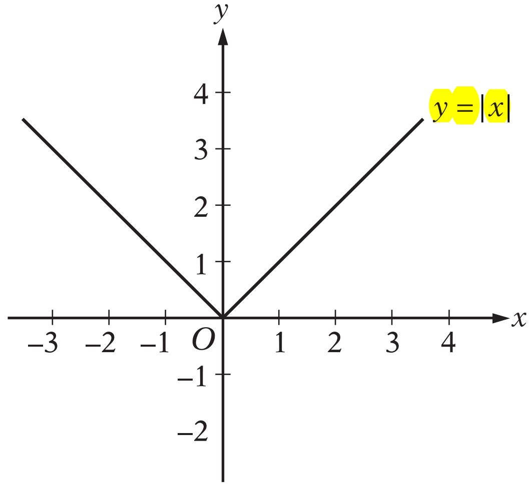
$$g\left(\frac{-1+\sqrt{17}}{4}\right)=\left(\frac{-1+\sqrt{17}}{4}\right)^2\approx 0.61 \quad \text{and} \quad g\left(\frac{-1-\sqrt{17}}{4}\right)=\left(\frac{-1-\sqrt{17}}{4}\right)^2\approx 1.64$$

Thus, the two intersection points can be approximated by $(0.78, 0.61)$ and $(-1.28, 1.64)$.

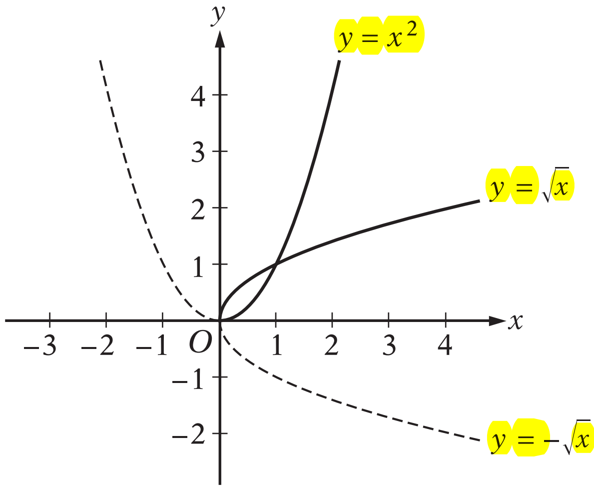
Example 2.9.3: Consider the absolute value function defined by $h(x) = |x|$. By using the definition of absolute value (see section 1.5), h can be expressed as a **piecewise-defined** function:

$$h(x)=\begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The graph of this function is V-shaped and consists of two linear pieces, $y = x$ and $y = -x$, joined at the origin, as shown in the figure below.



Example 2.9.4: Consider the positive square-root function defined by $j(x) = \sqrt{x}$ for $x \geq 0$, whose graph is half of a parabola lying on its side. Also consider the negative square-root function defined by $k(x) = -\sqrt{x}$ for $x \geq 0$, whose graph is the other half of the parabola lying on its side—the dashed curve below the x-axis. Both graphs are shown in the figure below, along with the parabola $y = x^2$ (with its left half dashed).



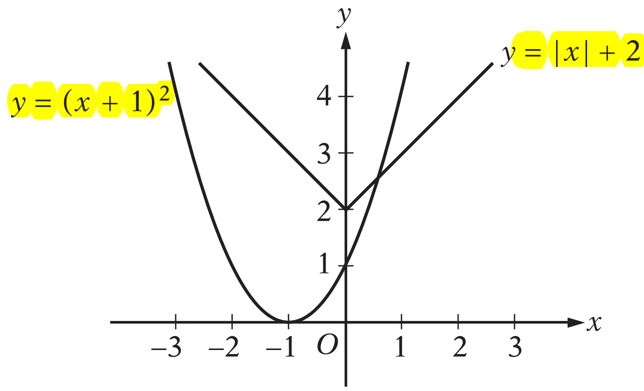
The graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ are halves of a parabola because they are reflections of the right and left halves, respectively, of the parabola $y = x^2$ about the line $y = x$. This follows from squaring both sides of the two square root equations to get $y^2 = x$ and then interchanging x and y to get $y = x^2$.

Also note that $y = -\sqrt{x}$ is the reflection of $y = \sqrt{x}$ about the x-axis. In general, for any function h, the graph of $y = -h(x)$ is the **reflection** of the graph of $y = h(x)$ about the x-axis.

Example 2.9.5: Consider the functions defined by $f(x) = |x| + 2$ and $g(x) = (x + 1)^2$. These functions are related to the absolute value function $|x|$ and the quadratic function x^2, respectively, in simple ways.

The graph of f is the graph of $|x|$ shifted upward by 2 units, as shown in the figure below. Similarly, the graph of the function $|x| - 5$ is the graph of $|x|$ shifted downward by 5 units (not shown).

The graph of g is the graph of x^2 shifted to the left by 1 unit, as shown in the figure below. Similarly, the graph of the function $(x - 4)^2$ is the graph of x^2 shifted to the right by 4 units (not shown). To double-check the direction of the shift, you can plot some corresponding values of the original function and the shifted function.



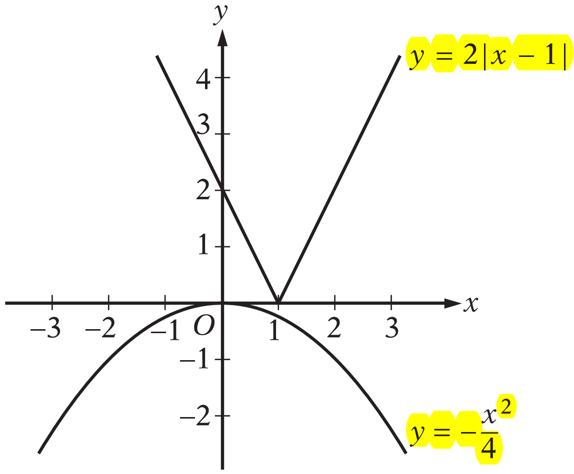
In general, for any function $h(x)$ and any positive number c, the following are true.

- The graph of $h(x) + c$ is the graph of $h(x)$ **shifted upward** by c units.
- The graph of $h(x) - c$ is the graph of $h(x)$ **shifted downward** by c units.
- The graph of $h(x + c)$ is the graph of $h(x)$ **shifted to the left** by c units.
- The graph of $h(x - c)$ is the graph of $h(x)$ **shifted to the right** by c units.

Example 2.9.6: Consider the functions defined by $f(x) = 2|x - 1|$ and $g(x) = -\frac{x^2}{4}$. These functions are related to the absolute value function $|x|$ and the quadratic function x^2, respectively, in more complicated ways than in the preceding example.

The graph of f is the graph of $|x|$ shifted to the right by 1 unit and then stretched vertically away from the x-axis by a factor of 2, as shown in the figure on the next page. Similarly, the graph of the function $\frac{1}{2}|x - 1|$ is the graph of $|x|$ shifted to the right by 1 unit and then shrunk vertically toward the x-axis by a factor of $\frac{1}{2}$ (not shown).

The graph of g is the graph of x^2 shrunk vertically by a factor of $\frac{1}{4}$ and then reflected in the x-axis, as shown in the figure below.



In general, for any function $h(x)$ and any positive number c, the following are true.

- The graph of $ch(x)$ is the graph of $h(x)$ **stretched vertically** by a factor of c if $c > 1$.
- The graph of $ch(x)$ is the graph of $h(x)$ **shrunk vertically** by a factor of c if $0 < c < 1$.

ALGEBRA EXERCISES

1. Find an algebraic expression to represent each of the following.
   (a) The square of y is subtracted from 5, and the result is multiplied by 37.
   (b) Three times x is squared, and the result is divided by 7.
   (c) The product of $(x + 4)$ and y is added to 18.

2. Simplify each of the following algebraic expressions.
   (a) $3x^2 - 6 + x + 11 - x^2 + 5x$
   (b) $3(5x - 1) - x + 4$
   (c) $\frac{x^2 - 16}{x - 4}$, where $x \neq 4$
   (d) $(2x + 5)(3x - 1)$

3. (a) What is the value of $f(x) = 3x^2 - 7x + 23$ when $x = -2$?
   (b) What is the value of $h(x) = x^3 - 2x^2 + x - 2$ when $x = 2$?
   (c) What is the value of $k(x) = \frac{5}{3}x - 7$ when $x = 0$?

4. If the function g is defined for all nonzero numbers y by $g(y) = \frac{y}{|y|}$, find the value of each of the following.
   (a) $g(2)$
   (b) $g(-2)$
   (c) $g(2) - g(-2)$

5. Use the rules of exponents to simplify the following.

(a) $(n^5)(n^{-3})$

(b) $(s^7)(t^7)$

(c) $\dfrac{r^{12}}{r^4}$

(d) $\left(\dfrac{2a}{b}\right)^5$

(e) $(w^5)^{-3}$

(f) $(5^0)(d^3)$

(g) $\dfrac{(x^{10})(y^{-1})}{(x^{-5})(y^5)}$

(h) $\left(\dfrac{3x}{y}\right)^2 \div \left(\dfrac{1}{y}\right)^5$

6. Solve each of the following equations for x.
(a) $5x - 7 = 28$
(b) $12 - 5x = x + 30$
(c) $5(x + 2) = 1 - 3x$
(d) $(x + 6)(2x - 1) = 0$
(e) $x^2 + 5x - 14 = 0$
(f) $x^2 - x - 1 = 0$

7. Solve each of the following systems of equations for x and y.

(a) $\begin{aligned} x + y &= 24 \\ x - y &= 18 \end{aligned}$

(b) $\begin{aligned} 3x - y &= -5 \\ x + 2y &= 3 \end{aligned}$

(c) $\begin{aligned} 15x - 18 - 2y &= -3x + y \\ 10x + 7y + 20 &= 4x + 2 \end{aligned}$

8. Solve each of the following inequalities for x.
(a) $-3x > 7 + x$
(b) $25x + 16 \geq 10 - x$
(c) $16 + x > 8x - 12$

9. For a given two-digit positive integer, the tens digit is 5 more than the units digit. The sum of the digits is 11. Find the integer.

10. If the ratio of $2x$ to $5y$ is 3 to 4, what is the ratio of x to y?

11. Kathleen's weekly salary was increased by 8 percent to \$237.60. What was her weekly salary before the increase?

12. A theater sells children's tickets for half the adult ticket price. If 5 adult tickets and 8 children's tickets cost a total of \$27, what is the cost of an adult ticket?

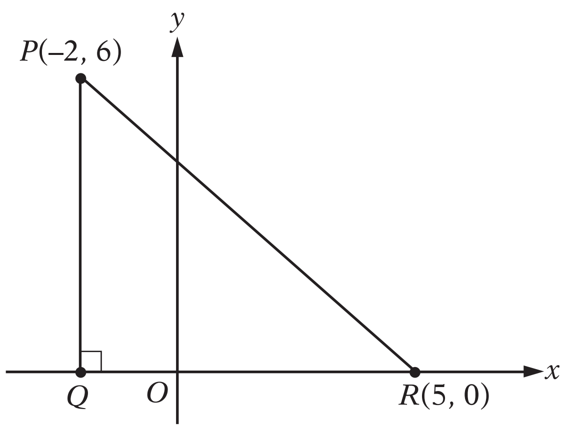
13. Pat invested a total of \$3,000. Part of the money was invested in a money market account that paid 10 percent simple annual interest, and the remainder of the money was invested in a fund that paid 8 percent simple annual interest. If the interest earned at the end of the first year from these investments was \$256, how much did Pat invest at 10 percent and how much at 8 percent?

14. Two cars started from the same point and traveled on a straight course in opposite directions for exactly 2 hours, at which time they were 208 miles apart. If one car traveled, on average, 8 miles per hour faster than the other car, what was the average speed of each car for the 2-hour trip?

15. A group can charter a particular aircraft at a fixed total cost. If 36 people charter the aircraft rather than 40 people, then the cost per person is greater by \$12.
(a) What is the fixed total cost to charter the aircraft?
(b) What is the cost per person if 40 people charter the aircraft?

16. An antiques dealer bought c antique chairs for a total of x dollars. The dealer sold each chair for y dollars.
(a) Write an algebraic expression for the profit, P, earned from buying and selling the chairs.
(b) Write an algebraic expression for the profit per chair.

17. In the coordinate system below, find the following.
(a) Coordinates of point Q
(b) Lengths of PQ, QR, and PR
(c) Perimeter of $\triangle PQR$
(d) Area of $\triangle PQR$
(e) Slope, y-intercept, and equation of the line passing through points P and R



18. In the xy-plane, find the following.
(a) Slope and y-intercept of the line with equation $2y + x = 6$
(b) Equation of the line passing through the point $(3, 2)$ with y-intercept 1
(c) The y-intercept of a line with slope 3 that passes through the point $(-2, 1)$
(d) The x-intercepts of the graphs in (a), (b), and (c)

19. For the parabola $y = x^2 - 4x - 12$ in the xy-plane, find the following.
(a) The x-intercepts
(b) The y-intercept
(c) Coordinates of the vertex

20. For the circle $(x - 1)^2 + (y + 1)^2 = 20$ in the xy-plane, find the following.
(a) Coordinates of the center
(b) Radius
(c) Area

21. For each of the following functions, give the domain and a description of the graph $y = f(x)$ in the xy-plane, including its shape, and the x- and y-intercepts.
(a) $f(x) = -4$
(b) $f(x) = 100 - 900x$
(c) $f(x) = 5 - (x + 20)^2$
(d) $f(x) = \sqrt{x + 2}$
(e) $f(x) = x + |x|$

ANSWERS TO ALGEBRA EXERCISES

1. (a) $37(5 - y^2)$, or $185 - 37y^2$
(b) $\frac{(3x)^2}{7}$, or $\frac{9x^2}{7}$
(c) $18 + (x + 4)(y)$, or $18 + xy + 4y$

2. (a) $2x^2 + 6x + 5$
(b) $14x + 1$
(c) $x + 4$
(d) $6x^2 + 13x - 5$

3. (a) 49 (b) 0 (c) -7

4. (a) 1 (b) -1 (c) 2

5. (a) n^2
(b) $(st)^7$
(c) r^8
(d) $\frac{32a^5}{b^5}$
(e) $\frac{1}{w^{15}}$
(f) d^3
(g) $\frac{x^{15}}{y^6}$
(h) $9x^2y^3$

6. (a) 7
(b) -3
(c) $-\frac{9}{8}$
(d) $-6, \frac{1}{2}$
(e) $-7, 2$
(f) $\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}$

7. (a) $x = 21$, $y = 3$ (b) $x = -1$, $y = 2$ (c) $x = \frac{1}{2}$, $y = -3$

8. (a) $x < -\frac{7}{4}$
(b) $x \geq -\frac{3}{13}$
(c) $x < 4$

9. 83

10. 15 to 8

11. \$220

12. \$3

13. \$800 at 10% and \$2,200 at 8%

14. 48 mph and 56 mph

15. (a) \$4,320 (b) \$108

16. (a) $P = cy - x$ (b) Profit per chair: $\frac{P}{c} = \frac{cy - x}{c} = y - \frac{x}{c}$

17. (a) $(-2, 0)$
(b) $PQ = 6$, $QR = 7$, $PR = \sqrt{85}$
(c) $13 + \sqrt{85}$
(d) 21
(e) Slope: $-\frac{6}{7}$; y-intercept: $\frac{30}{7}$; equation of line: $y = -\frac{6}{7}x + \frac{30}{7}$, or $7y + 6x = 30$

18. (a) Slope: $-\frac{1}{2}$; y-intercept: 3
(b) $y = \frac{x}{3} + 1$
(c) 7
(d) $6, -3, -\frac{7}{3}$

19. (a) $x = -2$ and $x = 6$
(b) $y = -12$
(c) $(2, -16)$

20. (a) $(1, -1)$
(b) $\sqrt{20}$
(c) 20π

21. (a) Domain: the set of all real numbers. The graph is a horizontal line with y-intercept -4 and no x-intercept.
(b) Domain: the set of all real numbers. The graph is a line with slope -900, y-intercept 100, and x-intercept $\frac{1}{9}$.
(c) Domain: the set of all real numbers. The graph is a parabola opening downward with vertex at $(-20, 5)$, line of symmetry $x = -20$, y-intercept -395, and x-intercepts $-20 \pm \sqrt{5}$.
(d) Domain: the set of numbers greater than or equal to -2. The graph is half a parabola opening to the right with vertex at $(-2, 0)$, x-intercept -2, and y-intercept $\sqrt{2}$.
(e) Domain: the set of all real numbers. The graph is two half-lines joined at the origin: one half-line is the negative x-axis and the other is a line starting at the origin with slope 2. Every nonpositive number is an x-intercept, and the y-intercept is 0. The function is equal to the following piecewise-defined function

$$f(x) = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

3. GEOMETRY

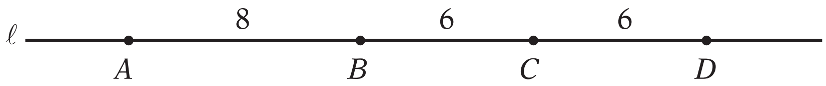
The review of geometry begins with lines and angles and progresses to other plane figures, such as polygons, triangles, quadrilaterals, and circles. The section ends with some basic three-dimensional figures. Coordinate geometry is covered in the Algebra section.

3.1 Lines and Angles

Plane geometry is devoted primarily to the properties and relations of plane figures, such as angles, triangles, other polygons, and circles. The terms "point," "line," and "plane" are familiar intuitive concepts. A **point** has no size and is the simplest geometric figure. All geometric figures consist of points. A **line** is understood to be a straight line that extends in both directions without ending. A **plane** can be thought of as a floor or a tabletop, except that a plane extends in all directions without ending and has no thickness.

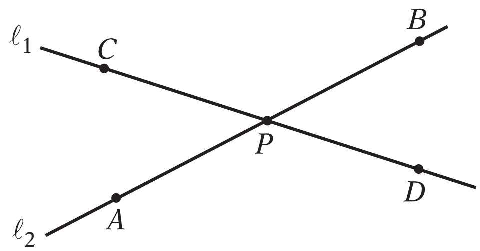
Given any two points on a line, a **line segment** is the part of the line that contains the two points and all the points between them. The two points are called **endpoints**. Line segments that have equal lengths are called **congruent line segments**. The point that divides a line segment into two congruent line segments is called the **midpoint** of the line segment.

In the figure below, A, B, C, and D are points on line ℓ.



Line segment AB consists of points A and B and all the points on the line between A and B. Sometimes the notation AB denotes line segment AB, and sometimes it denotes the **length** of line segment AB. The meaning of the notation can be determined from the context. According to the figure above, the lengths of line segments AB, BC, and CD are 8, 6, and 6, respectively. Hence, line segments BC and CD are congruent. Since C is halfway between B and D, point C is the midpoint of line segment BD.

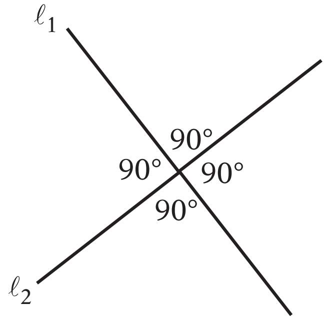
When two lines intersect at a point, they form four **angles**, as indicated below. Each angle has a **vertex** at point P, which is the point of intersection of the two lines.



In the figure, angles APC and BPD are called **opposite angles**, also known as **vertical angles**. Angles APD and CPB are also opposite angles. Opposite angles have equal measures, and angles that have equal measures are called **congruent angles**. Hence, opposite angles are congruent. The sum of the measures of the four angles is 360°.

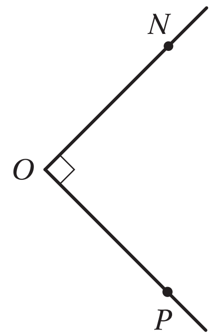
Sometimes the angle symbol $\angle$ is used instead of the word "angle." For example, angle APC can be written as $\angle APC$.

Two lines that intersect to form four congruent angles are called **perpendicular lines**. Each of the four angles has a measure of 90°. An angle with a measure of 90° is called a **right angle**. The figure below shows two lines, ℓ_1 and ℓ_2, that are perpendicular, denoted by $\ell_1 \perp \ell_2$.



$\ell_1 \perp \ell_2$

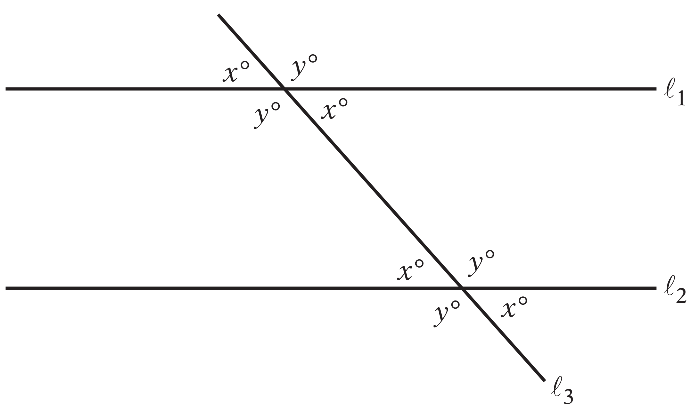
A common way to indicate that an angle is a right angle is to draw a small square at the vertex of the angle, as shown below, where *PON* is a right angle.



$ON \perp OP$

An angle with a measure less than 90° is called an **acute angle**, and an angle with a measure between 90° and 180° is called an **obtuse angle**.

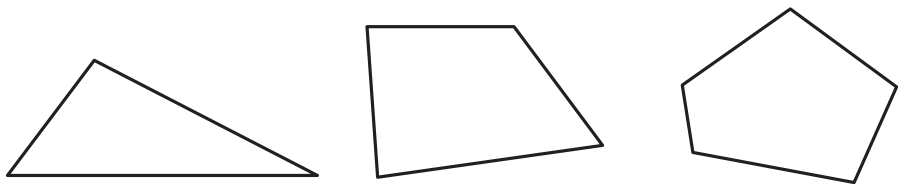
Two lines in the same plane that do not intersect are called **parallel lines**. The figure below shows two lines, ℓ_1 and ℓ_2, that are parallel, denoted by $\ell_1 \parallel \ell_2$. The two lines are intersected by a third line, ℓ_3, forming eight angles. Note that four of the angles have the measure $x°$, and the remaining four angles have the measure $y°$, where $x + y = 180$.



$\ell_1 \parallel \ell_2$

3.2 Polygons

A **polygon** is a closed figure formed by three or more line segments, called **sides**. Each side is joined to two other sides at its endpoints, and the endpoints are called **vertices**. In this discussion, the term "polygon" means "convex polygon," that is, a polygon in which the measure of each interior angle is less than 180°. The figures below are examples of such polygons.



Triangle (3 sides)   Quadrilateral (4 sides)   Pentagon (5 sides)

The simplest polygon is a **triangle**. Note that a **quadrilateral** can be divided into 2 triangles, and a **pentagon** can be divided into 3 triangles, as shown below.

If a polygon has n sides, it can be divided into $n - 2$ triangles. Since the sum of the measures of the interior angles of a triangle is 180°, it follows that the sum of the measures of the interior angles of an n-sided polygon is $(n - 2)(180°)$. For example, the sum for a quadrilateral $(n = 4)$ is $(4 - 2)(180°) = 360°$, and the sum for a **hexagon** $(n = 6)$ is $(6 - 2)(180°) = 720°$.

A polygon in which all sides are congruent and all interior angles are congruent is called a **regular polygon**. For example, in a **regular octagon** (8 sides), the sum of the measures of the interior angles is $(8 - 2)(180°) = 1{,}080°$. Therefore, the measure of each angle is $\frac{1{,}080°}{8} = 135°$.

The **perimeter** of a polygon is the sum of the lengths of its sides. The **area** of a polygon refers to the area of the region enclosed by the polygon.

In the next two sections, we will look at some basic properties of triangles and quadrilaterals.

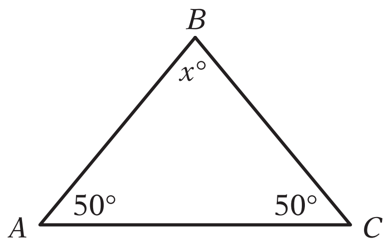
3.3 Triangles

Every triangle has three sides and three interior angles. The measures of the interior angles add up to 180°. The length of each side must be less than the sum of the lengths of the other two sides. For example, the sides of a triangle could not have the lengths 4, 7, and 12 because 12 is greater than 4 + 7.

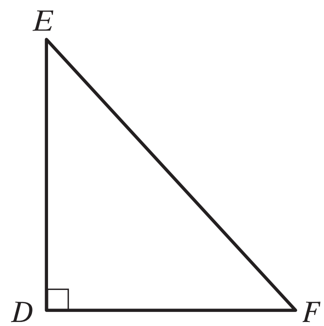
The following are special triangles.

- A triangle with three congruent sides is called an **equilateral triangle**. The measures of the three interior angles of such a triangle are also equal, and each measure is 60°.
- A triangle with at least two congruent sides is called an **isosceles triangle**. If a triangle has two congruent sides, then the angles opposite the two sides are

congruent. The converse is also true. For example, in $\triangle ABC$ below, since both $\angle A$ and $\angle C$ have measure 50°, it follows that $AB = BC$. Also, since $50 + 50 + x = 180$, the measure of $\angle B$ is 80°.



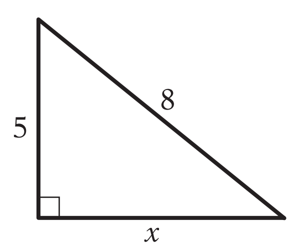
- A triangle with an interior right angle is called a **right triangle**. The side opposite the right angle is called the **hypotenuse**; the other two sides are called **legs**.



In right triangle DEF above, EF is the hypotenuse and DE and DF are legs. The **Pythagorean theorem** states that in a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. Thus, for triangle DEF above,

$$(EF)^2 = (DE)^2 + (DF)^2$$

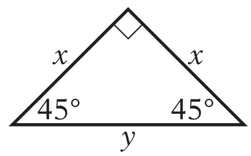
This relationship can be used to find the length of one side of a right triangle if the lengths of the other two sides are known. For example, if one leg of a right triangle has length 5 and the hypotenuse has length 8, then the length of the other side can be determined as follows.



$$\begin{aligned} 8^2 &= 5^2 + x^2 \\ 64 &= 25 + x^2 \\ 39 &= x^2 \end{aligned}$$

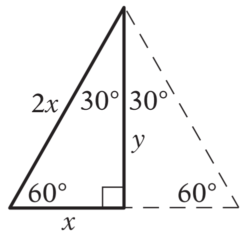
Since $x^2 = 39$ and x must be positive, it follows that $x = \sqrt{39}$, or approximately 6.2.

The Pythagorean theorem can be used to determine the ratios of the lengths of the sides of two special right triangles. One special right triangle is an isosceles right triangle. Applying the Pythagorean theorem to such a triangle shows that the lengths of its sides are in the ratio 1 to 1 to $\sqrt{2}$, as indicated below.



$$y^2 = x^2 + x^2$$
$$y^2 = 2x^2$$
$$y = \sqrt{2}x$$

The other special right triangle is a 30°- 60°- 90° right triangle, which is half of an equilateral triangle, as indicated below.

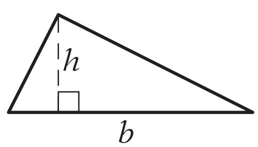


Note that the length of the shortest side, x, is one-half the length of the longest side, $2x$. By the Pythagorean theorem, the ratio of x to y is 1 to $\sqrt{3}$ because

$$x^2 + y^2 = (2x)^2$$
$$x^2 + y^2 = 4x^2$$
$$y^2 = 4x^2 - x^2$$
$$y^2 = 3x^2$$
$$y = \sqrt{3}x$$

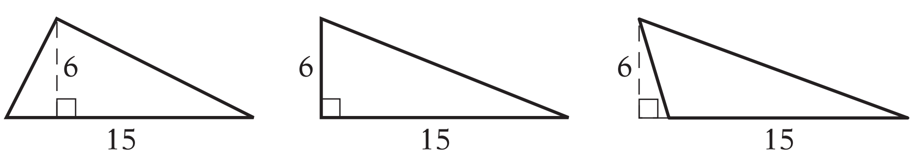
Hence, the ratio of the lengths of the three sides of such a triangle is 1 to $\sqrt{3}$ to 2.

The **area** A of a triangle equals one-half the product of the length of a base and the height corresponding to the base. In the figure below, the base is denoted by b and the corresponding height is denoted by h.



$$A = \frac{bh}{2}$$

Any side of a triangle can be used as a base; the height that corresponds to the base is the perpendicular line segment from the opposite vertex to the base (or to an extension of the base). The examples below show three different configurations of a base and the corresponding height.



In all three triangles above, the area is $\frac{(15)(6)}{2}$, or 45.

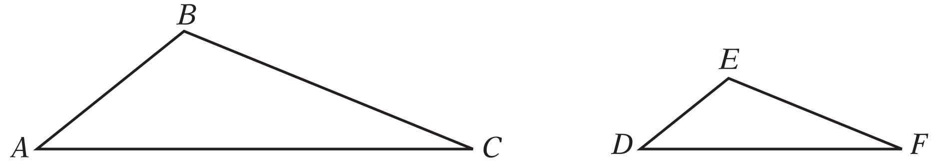
Two triangles that have the same shape and size are called **congruent triangles**. More precisely, two triangles are congruent if their vertices can be matched up so that the corresponding angles and the corresponding sides are congruent.

The following three propositions can be used to determine whether two triangles are congruent by comparing only some of their sides and angles.

- If the three sides of one triangle are congruent to the three sides of another triangle, then the triangles are congruent.
- If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.
- If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

Two triangles that have the same shape but not necessarily the same size are called **similar triangles**. More precisely, two triangles are similar if their vertices can be matched up so that the corresponding angles are congruent or, equivalently, the lengths of corresponding sides have the same ratio, called the scale factor of similarity. For example, all 30°- 60°- 90° right triangles, discussed above, are similar triangles, though they may differ in size.

When we say that triangles *ABC* and *DEF* are similar, it is assumed that angles *A* and *D* are congruent, angles *B* and *E* are congruent, and angles *C* and *F* are congruent, as shown in the figure below. In other words, the order of the letters indicates the correspondences.

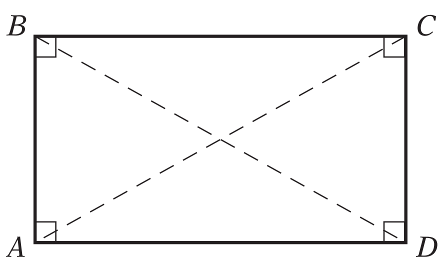


Since triangles *ABC* and *DEF* are similar, we have $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$. By cross multiplication, we can obtain other proportions, such as $\frac{AB}{BC} = \frac{DE}{EF}$.

3.4 Quadrilaterals

Every quadrilateral has four sides and four interior angles. The measures of the interior angles add up to 360°. The following are special quadrilaterals.

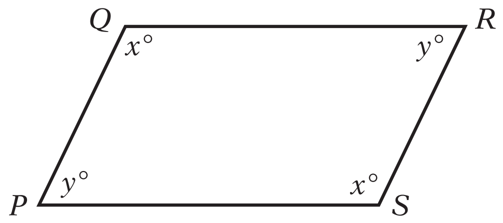
- A quadrilateral with four right angles is called a **rectangle**. Opposite sides of a rectangle are parallel and congruent, and the two diagonals are also congruent.



$AB \parallel CD$ and $AD \parallel BC$
$AB = CD$ and $AD = BC$
$AC = BD$

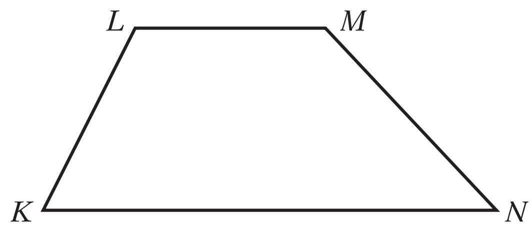
- A rectangle with four congruent sides is called a **square**.

- A quadrilateral in which both pairs of opposite sides are parallel is called a **parallelogram**. In a parallelogram, opposite sides are congruent and opposite angles are congruent.



$PQ \parallel SR$ and $PS \parallel QR$
$PQ = SR$ and $PS = QR$

- A quadrilateral in which two opposite sides are parallel is called a **trapezoid**.

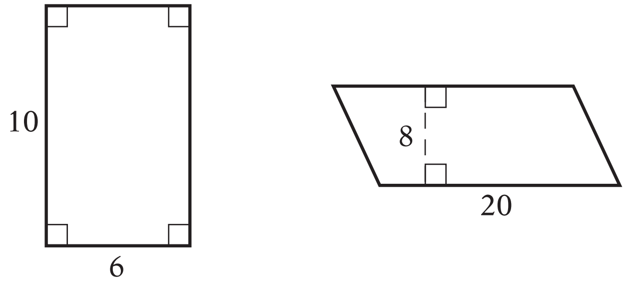


$KN \parallel LM$

For all parallelograms, including rectangles and squares, the **area** A equals the product of the length of a base b and the corresponding height h; that is,

$$A = bh$$

Any side can be used as a base. The height corresponding to the base is the perpendicular line segment from any point of a base to the opposite side (or to an extension of that side). Below are examples of finding the areas of a rectangle and a parallelogram.



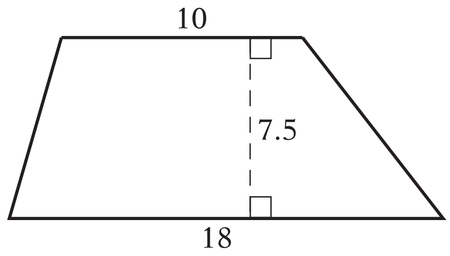
$A = (6)(10) = 60$ $\qquad$ $A = (20)(8) = 160$

The **area** A of a trapezoid equals half the product of the sum of the lengths of the two parallel sides b_1 and b_2 and the corresponding height h; that is,

$$A = \frac{1}{2}(b_1 + b_2)(h)$$

For example, for the trapezoid below with bases of length 10 and 18 and a height of 7.5, the area is

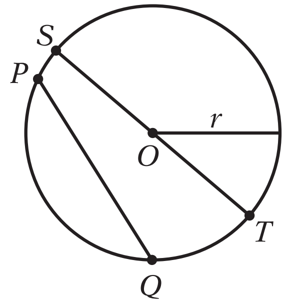
$$A = \frac{1}{2}(10 + 18)(7.5) = 105$$



3.5 Circles

Given a point O in a plane and a positive number r, the set of points in the plane that are a distance of r units from O is called a **circle**. The point O is called the **center** of the circle and the distance r is called the **radius** of the circle. The **diameter** of the circle is twice the radius. Two circles with equal radii are called **congruent circles**.

Any line segment joining two points on the circle is called a **chord**. The terms "radius" and "diameter" can also refer to line segments: A **radius** is any line segment joining a point on the circle and the center of the circle, and a **diameter** is any chord that passes through the center of the circle. In the figure below, O is the center of the circle, r is the radius, PQ is a chord, and ST is a diameter.



The distance around a circle is called the **circumference** of the circle, which is analogous to the perimeter of a polygon. The ratio of the circumference C to the diameter d is the same for all circles and is denoted by the Greek letter π; that is,

$$\frac{C}{d} = \pi$$

The value of π is approximately 3.14 and can also be approximated by the fraction $\frac{22}{7}$.

If r is the radius of a circle, then $\frac{C}{d} = \frac{C}{2r} = \pi$, and so the circumference is related to the radius by the equation

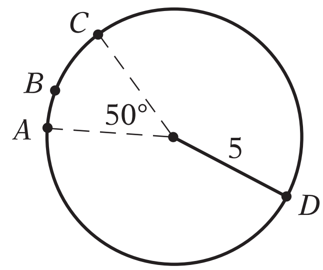
$$C = 2\pi r$$

For example, if a circle has a radius of 5.2, then its circumference is

$$(2)(\pi)(5.2) = (10.4)(\pi) \approx (10.4)(3.14)$$

which is approximately 32.7.

Given any two points on a circle, an **arc** is the part of the circle containing the two points and all the points between them. Two points on a circle are always the endpoints of two arcs. It is customary to identify an arc by three points to avoid ambiguity. In the figure below, arc ABC is the shorter arc between A and C, and arc ADC is the longer arc between A and C.



A **central angle** of a circle is an angle with its vertex at the center of the circle. The **measure of an arc** is the measure of its central angle, which is the angle formed by two radii that connect the center of the circle to the two endpoints of the arc. An entire circle is considered to be an arc with measure 360°. In the figure above, the measure of arc ABC is 50° and the measure of arc ADC is 310°.

To find the **length of an arc** of a circle, note that the ratio of the length of an arc to the circumference is equal to the ratio of the degree measure of the arc to 360°. The circumference of the circle above is 10π. Therefore,

$$\frac{\text{length of arc } ABC}{10\pi} = \frac{50}{360}$$

$$\text{length of arc } ABC = \left(\frac{50}{360}\right)(10\pi) = \frac{25\pi}{18} \approx \frac{(25)(3.14)}{18} \approx 4.4$$

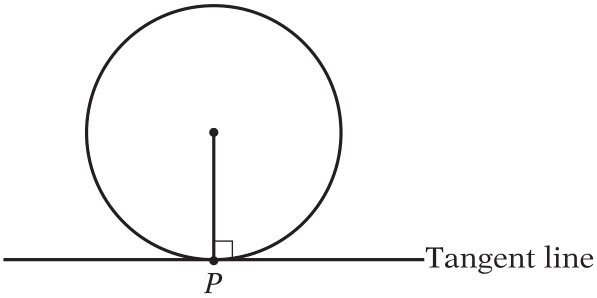
The **area** of a circle with radius r is equal to πr^2. For example, the area of the circle above with radius 5 is $\pi(5)^2 = 25\pi$.

A **sector** of a circle is a region bounded by an arc of the circle and two radii. In the circle above, the region bounded by arc ABC and the two dashed radii is a sector with central angle 50°. Just as in the case of the length of an arc, the ratio of the area of a sector of a circle to the area of the entire circle is equal to the ratio of the degree measure of its arc to 360°. Therefore, if S represents the area of the sector with central angle 50°, then

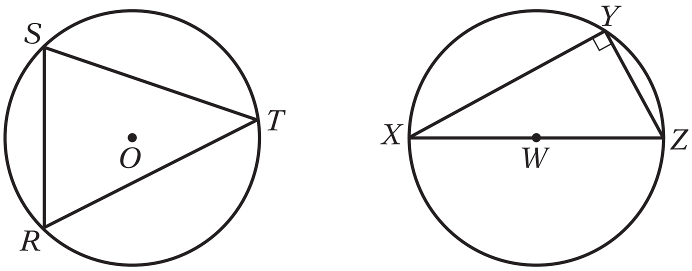
$$\frac{S}{25\pi} = \frac{50}{360}$$

$$S = \left(\frac{50}{360}\right)(25\pi) = \frac{125\pi}{36} \approx \frac{(125)(3.14)}{36} \approx 10.9$$

A **tangent** to a circle is a line that intersects the circle at exactly one point, called the **point of tangency**, denoted by P in the figure at the top of the following page. If a line is tangent to a circle, then a radius drawn to the point of tangency is perpendicular to the tangent line. The converse is also true; that is, if a line is perpendicular to a radius at its endpoint on the circle, then the line is a tangent to the circle at that endpoint.

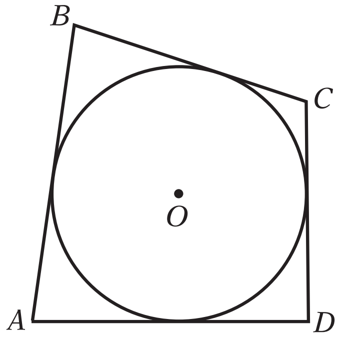


A polygon is **inscribed** in a circle if all its vertices lie on the circle, or equivalently, the circle is **circumscribed** about the polygon. Triangles RST and XYZ below are inscribed in the circles with centers O and W, respectively.

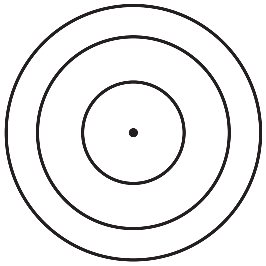


If one side of an inscribed triangle is a diameter of the circle, as in triangle XYZ above, then the triangle is a right triangle. Conversely, if an inscribed triangle is a right triangle, then one of its sides is a diameter of the circle.

A polygon is circumscribed about a circle if each side of the polygon is tangent to the circle, or equivalently, the circle is inscribed in the polygon. In the figure below, quadrilateral $ABCD$ is circumscribed about the circle with center O.



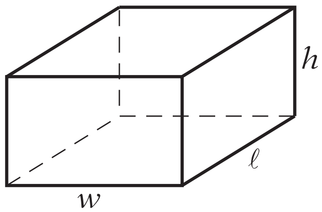
Two or more circles with the same center are called **concentric circles**, as shown in the figure below.



3.6 Three-dimensional Figures

Basic three-dimensional figures include rectangular solids, cubes, cylinders, spheres, pyramids, and cones. In this section, we look at some properties of rectangular solids and right circular cylinders.

A **rectangular solid** has six rectangular surfaces called **faces**, as shown in the figure below. Adjacent faces are perpendicular to each other. Each line segment that is the intersection of two faces is called an **edge**, and each point at which the edges intersect is called a **vertex**. There are 12 edges and 8 vertices. The dimensions of a rectangular solid are the length ℓ, the width w, and the height h.



A rectangular solid with six square faces is called a **cube**, in which case $\ell = w = h$.

The **volume** V of a rectangular solid is the product of its three dimensions, or

$$V = \ell wh$$

The **surface area** A of a rectangular solid is the sum of the areas of the six faces, or

$$A = 2(\ell w + \ell h + wh)$$

For example, if a rectangular solid has length 8.5, width 5, and height 10, then its volume is

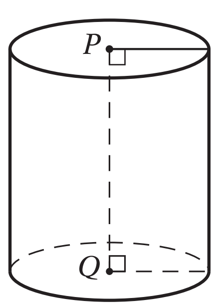
$$V = (8.5)(5)(10) = 425$$

and its surface area is

$$A = 2((8.5)(5) + (8.5)(10) + (5)(10)) = 355$$

A **circular cylinder** consists of two bases that are congruent circles and a **lateral surface** made of all line segments that join points on the two circles and that are parallel to the line segment joining the centers of the two circles. The latter line segment is called the **axis** of the cylinder. A **right circular cylinder** is a circular cylinder whose axis is perpendicular to its bases.

The right circular cylinder shown in the figure below has circular bases with centers P and Q. Line segment PQ is the axis of the cylinder and is perpendicular to both bases. The length of PQ is called the height of the cylinder.



The **volume** V of a right circular cylinder that has height h and a base with radius r is the product of the height and the area of the base, or

$$V = \pi r^2 h$$

The **surface area** A of a right circular cylinder is the sum of the areas of the two bases and the lateral area, or

$$A = 2(\pi r^2) + 2\pi r h$$

For example, if a right circular cylinder has height 6.5 and a base with radius 3, then its volume is

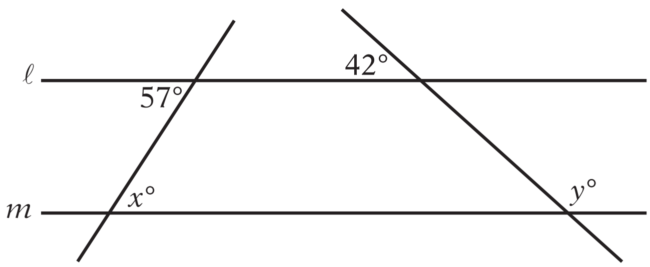
$$V = \pi(3)^2(6.5) = 58.5\pi$$

and its surface area is

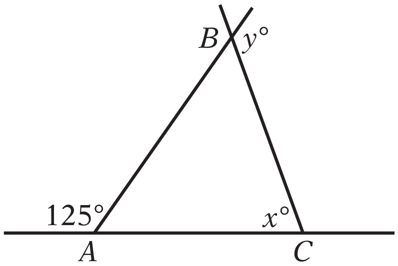
$$A = (2)(\pi)(3)^2 + (2)(\pi)(3)(6.5) = 57\pi$$

GEOMETRY EXERCISES

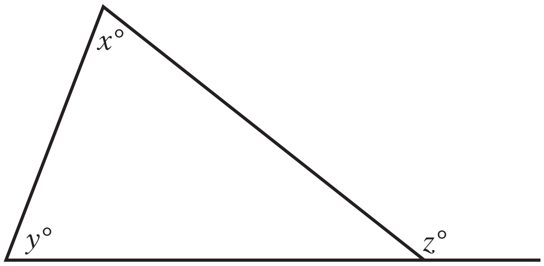
1. Lines ℓ and m below are parallel. Find the values of x and y.



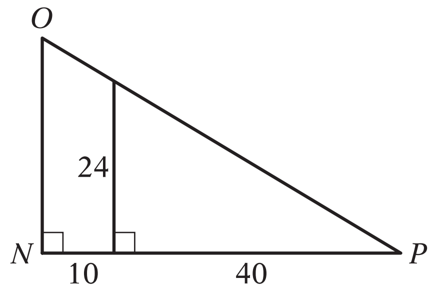
2. In the figure below, $AC = BC$. Find the values of x and y.



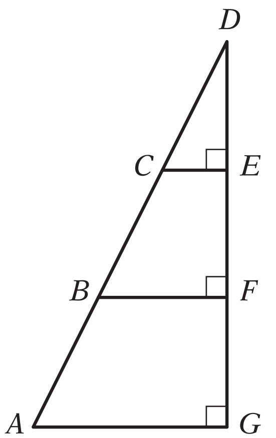
3. In the figure below, what is the relationship between x, y, and z?



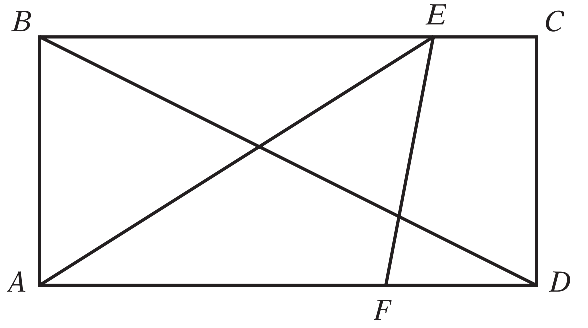
4. What is the sum of the measures of the interior angles of a decagon (10-sided polygon)?
5. If the decagon in exercise 4 is regular, what is the measure of each interior angle?
6. The lengths of two sides of an isosceles triangle are 15 and 22, respectively. What are the possible values of the perimeter?
7. Triangles PQR and XYZ are similar. If $PQ = 6$, $PR = 4$, and $XY = 9$, what is the length of side XZ?
8. What are the lengths of sides NO and OP in triangle NOP below?



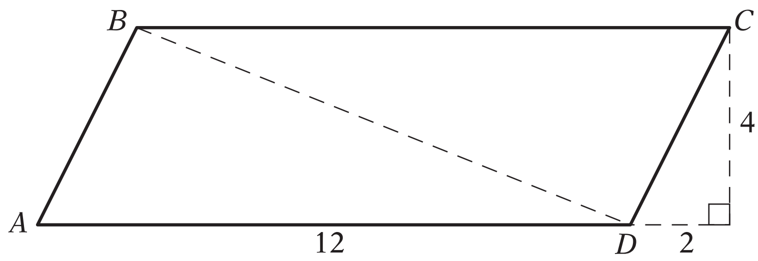
9. In the figure below, $AB = BC = CD$. If the area of triangle CDE is 42, what is the area of triangle ADG?



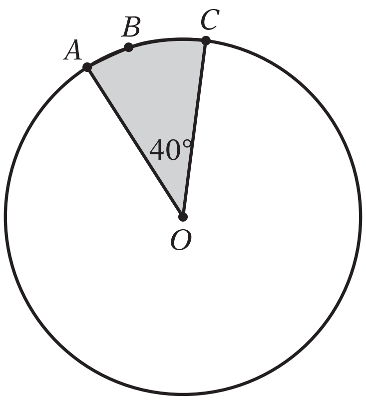
10. In rectangle $ABCD$ below, $AB = 5$, $AF = 7$, and $FD = 3$. Find the following.
(a) Area of $ABCD$
(b) Area of triangle AEF
(c) Length of BD
(d) Perimeter of $ABCD$



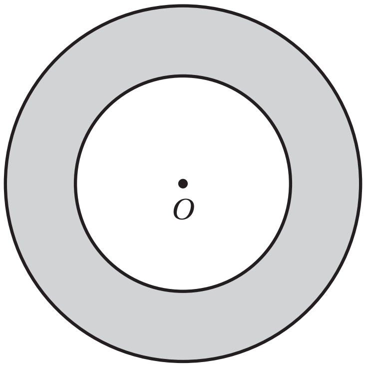
11. In parallelogram *ABCD* below, find the following.
    (a) Area of *ABCD*
    (b) Perimeter of *ABCD*
    (c) Length of diagonal *BD*



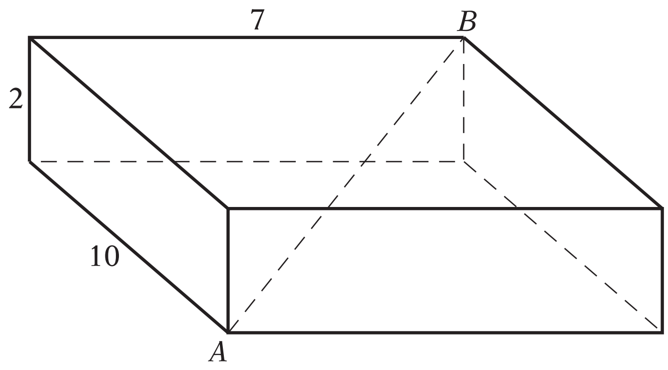
12. The circle with center *O* below has radius 4. Find the following.
    (a) Circumference of the circle
    (b) Length of arc *ABC*
    (c) Area of the shaded region



13. The figure below shows two concentric circles, each with center *O*. Given that the larger circle has radius 12 and the smaller circle has radius 7, find the following.
    (a) Circumference of the larger circle
    (b) Area of the smaller circle
    (c) Area of the shaded region



14. For the rectangular solid below, find the following.
(a) Surface area of the solid
(b) Length of diagonal AB



ANSWERS TO GEOMETRY EXERCISES

1. $x = 57$ and $y = 138$

2. $x = 70$ and $y = 125$

3. $z = x + y$

4. $1{,}440°$

5. $144°$

6. 52 and 59

7. 6

8. $NO = 30$ and $OP = 10\sqrt{34}$

9. 378

10. (a) 50
(b) 17.5
(c) $5\sqrt{5}$
(d) 30

11. (a) 48
(b) $24 + 4\sqrt{5}$
(c) $2\sqrt{29}$

12. (a) 8π
(b) $\frac{8\pi}{9}$
(c) $\frac{16\pi}{9}$

13. (a) 24π
(b) 49π
(c) 95π

14. (a) 208
(b) $3\sqrt{17}$

4. DATA ANALYSIS

The goal of data analysis is to understand data well enough to describe past and present trends, predict future events with some certainty, and thereby make better decisions. In this limited review of data analysis, we begin with tools for describing data; follow with tools for understanding counting and probability; review the concepts of distributions of data, random variables, and probability distributions; and end with examples of interpreting data.

4.1 Graphical Methods for Describing Data

Data can be organized and summarized using a variety of methods. Tables are commonly used, and there are many graphical and numerical methods as well. The appropriate type of representation for a collection of data depends in part on the nature of the data, such as whether the data are numerical or nonnumerical. In this section, we review some common graphical methods for describing and summarizing data.

Variables play a major role in algebra because a variable serves as a convenient name for many values at once, and it also can represent a particular value in a given problem to solve. In data analysis, variables also play an important role but with a somewhat different meaning. In data analysis, a **variable** is any characteristic that can vary for the population of individuals or objects being analyzed. For example, both gender and age represent variables among people.

Data are collected from a population after observing either a single variable or observing more than one variable simultaneously. The distribution of a variable, or **distribution of data**, indicates the values of the variable and how frequently the values are observed in the data.

Frequency Distributions

The **frequency**, or **count**, of a particular category or numerical value is the number of times that the category or value appears in the data. A **frequency distribution** is a table or graph that presents the categories or numerical values along with their associated frequencies. The **relative frequency** of a category or a numerical value is the associated frequency divided by the total number of data. Relative frequencies may be expressed in terms of percents, fractions, or decimals. A **relative frequency distribution** is a table or graph that presents the relative frequencies of the categories or numerical values.

Example 4.1.1: A survey was taken to find the number of children in each of 25 families. A list of the values collected in the survey follows.

1 2 0 4 1 3 3 1 2 0 4 5 2 3 2 3 2 4 1 2 3 0 2 3 1

Here are the resulting frequency and relative frequency distributions of the data.

Frequency Distribution

Number of Children	Frequency
0	3
1	5
2	7
3	6
4	3
5	1
Total	25

Relative Frequency Distribution

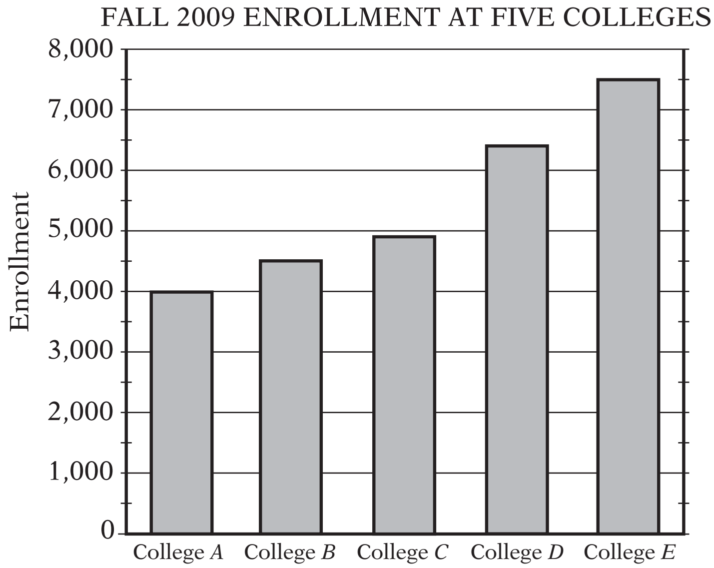
Number of Children	Relative Frequency
0	12%
1	20%
2	28%
3	24%
4	12%
5	4%
Total	100%

Note that the total for the relative frequencies is 100%. If decimals were used instead of percents, the total would be 1. The sum of the relative frequencies in a relative frequency distribution is always 1.

Bar Graphs

A commonly used graphical display for representing frequencies, or counts, is a **bar graph**, or bar chart. In a bar graph, rectangular bars are used to represent the categories of the data, and the height of each bar is proportional to the corresponding frequency or relative frequency. All of the bars are drawn with the same width, and the bars can be presented either vertically or horizontally. Bar graphs enable comparisons across several categories, making it easy to identify frequently and infrequently occurring categories.

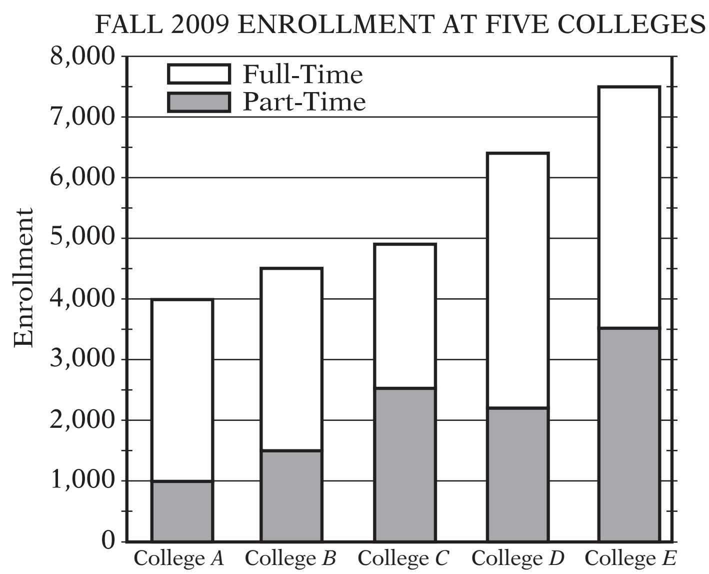
Example 4.1.2:



From the graph, we can conclude that the college with the greatest fall 2009 enrollment was College E, and the college with the least enrollment was College A. Also, we can estimate that the enrollment for College D was about 6,400.

A **segmented bar graph** is used to show how different subgroups or subcategories contribute to an entire group or category. In a segmented bar graph, each bar represents a category that consists of more than one subcategory. Each bar is divided into segments that represent the different subcategories. The height of each segment is proportional to the frequency or relative frequency of the subcategory that the segment represents.

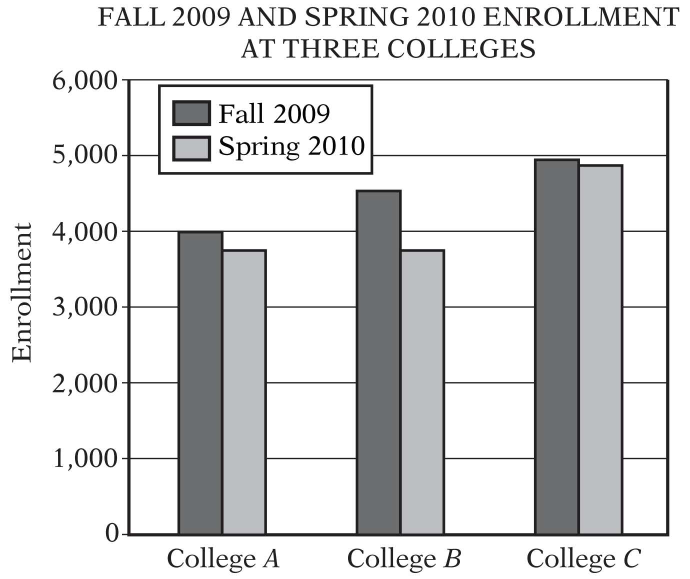
Example 4.1.3:



Different values can be estimated from the segmented bar graph above. For example, for College D, the total enrollment was approximately 6,400 students, the part-time enrollment was approximately 2,200, and the full-time enrollment was approximately 6,400 − 2,200, or 4,200 students.

Bar graphs can also be used to compare different groups using the same categories.

Example 4.1.4:



Observe that for all three colleges, the fall 2009 enrollment was greater than the spring 2010 enrollment. Also, the greatest decrease in the enrollment from fall 2009 to spring 2010 occurred at College *B*.

Although bar graphs are commonly used to compare frequencies, as in the examples above, they are sometimes used to compare numerical data that could be displayed in a table, such as temperatures, dollar amounts, percents, heights, and weights. Also, the categories sometimes are numerical in nature, such as years or other time intervals.

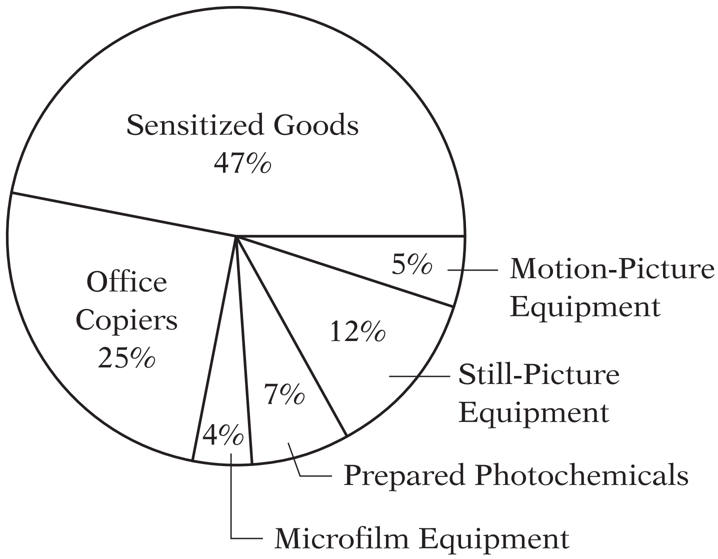
Circle Graphs

Circle graphs, often called pie charts, are used to represent data with a relatively small number of categories. They illustrate how a whole is separated into parts. The area of the circle graph representing each category is proportional to the part of the whole that the category represents.

Example 4.1.5:

UNITED STATES PRODUCTION OF PHOTOGRAPHIC
EQUIPMENT AND SUPPLIES IN 1971

Total: $3,980 million



The graph shows that of all United States production of photographic equipment and supplies in 1971, Sensitized Goods was the category with the greatest dollar value.

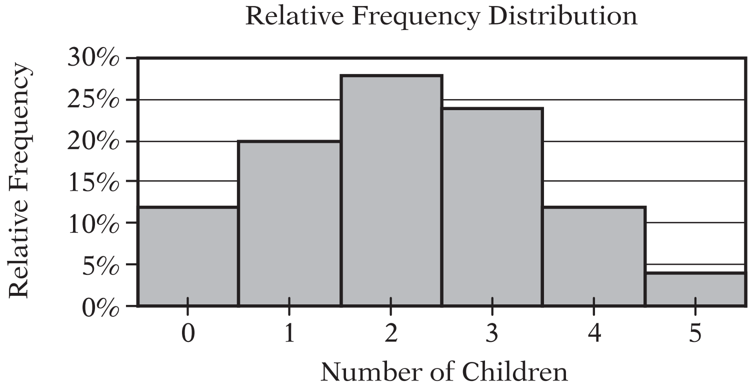
Each part of a circle graph is called a **sector**. Because the area of each sector is proportional to the percent of the whole that the sector represents, the measure of the central angle of a sector is proportional to the percent of 360 degrees that the sector represents. For example, the measure of the central angle of the sector representing the category Prepared Photochemicals is 7 percent of 360 degrees, or 25.2 degrees.

Histograms

When a list of data is large and contains many different values of a numerical variable, it is useful to organize it by grouping the values into intervals, often called classes. To do this, divide the entire interval of values into smaller intervals of equal length and then count the values that fall into each interval. In this way, each interval has a frequency and a relative frequency. The intervals and their frequencies (or relative frequencies) are often displayed in a **histogram**. Histograms are graphs of frequency distributions that are similar to bar graphs, but they have a number line for the horizontal axis. Also, in a histogram, there are no regular spaces between the bars. Any spaces between bars in a histogram indicate that there are no data in the intervals represented by the spaces.

Example 4.5.1 in section 4.5 illustrates a histogram with 50 bars. Numerical variables with just a few values can also be displayed using histograms, where the frequency or relative frequency of each value is represented by a bar centered over the value, as in the histogram in the following example.

Example 4.1.6: The relative frequency distribution in example 4.1.1 can be displayed as a histogram.



Histograms are useful for identifying the general shape of a distribution of data. Also evident are the "center" and degree of "spread" of the distribution, as well as high-frequency and low-frequency intervals. From the histogram above, you can see that the distribution is shaped like a mound with one peak; that is, the data are frequent in the middle and sparse at both ends. The central values are 2 and 3, and the distribution is close to being symmetric about those values. Because the bars all have the same width, the area of each bar is proportional to the amount of data that the bar represents. Thus, the areas of the bars indicate where the data are concentrated and where they are not.

Finally, note that because each bar has a width of 1, the sum of the areas of the bars equals the sum of the relative frequencies, which is 100% or 1, depending on whether percents or decimals are used. This fact is central to the discussion of probability distributions in section 4.5.

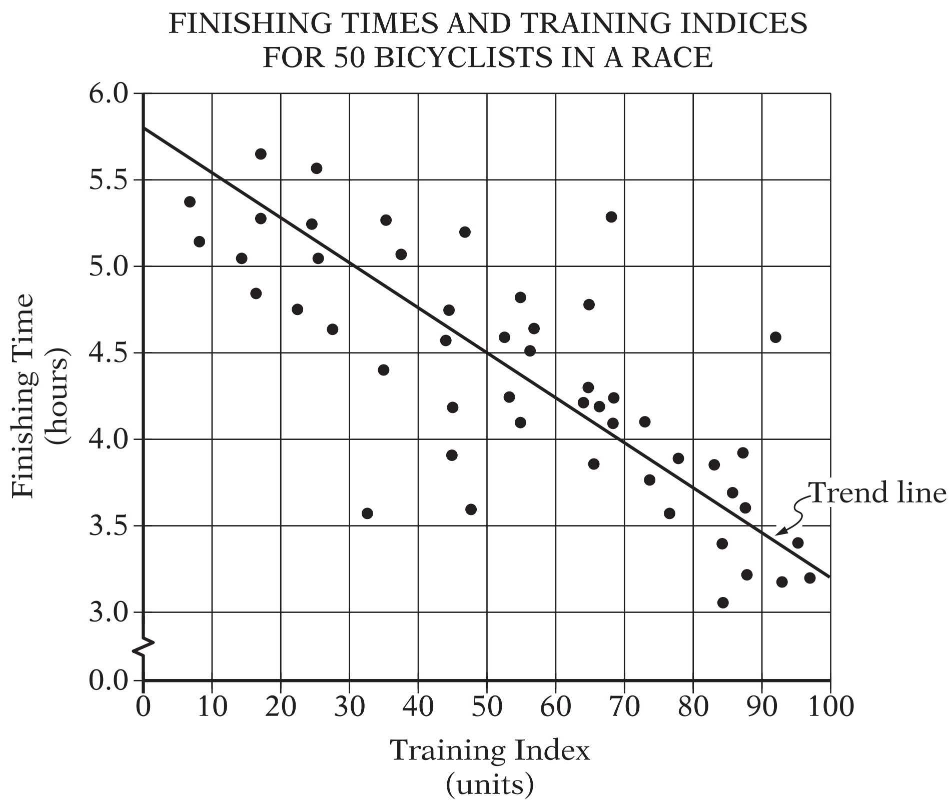
Scatterplots

All examples used thus far have involved data resulting from a single characteristic or variable. These types of data are referred to as **univariate**, that is, data observed for one variable. Sometimes data are collected to study two different variables in the same population of individuals or objects. Such data are called **bivariate** data. We might want to study the variables separately or investigate a relationship between the two variables. If the variables were to be analyzed separately, each of the graphical methods for univariate numerical data presented above could be applied.

To show the relationship between two numerical variables, the most useful type of graph is a **scatterplot**. In a scatterplot, the values of one variable appear on the horizontal axis of a rectangular coordinate system and the values of the other variable appear on the vertical axis. For each individual or object in the data, an ordered pair of numbers is collected, one number for each variable, and the pair is represented by a point in the coordinate system.

A scatterplot makes it possible to observe an overall pattern, or **trend**, in the relationship between the two variables. Also, the strength of the trend as well as striking deviations from the trend are evident. In many cases, a line or a curve that best represents the trend is also displayed in the graph and is used to make predictions about the population.

Example 4.1.7: A bicycle trainer studied 50 bicyclists to examine how the finishing time for a certain bicycle race was related to the amount of physical training in the three months before the race. To measure the amount of training, the trainer developed a training index, measured in "units" and based on the intensity of each bicyclist's training. The data and the trend of the data, represented by a line, are displayed in the scatterplot below.



In addition to the given trend line, you can see how scattered or close the data are to the trend line; or to put it another way, you can see how well the trend line fits the data. You can also see that the finishing times generally decrease as the training indices increase and that three or four data are relatively far from the trend.

Several types of predictions can be based on the trend line. For example, it can be predicted, based on the trend line, that a bicyclist with a training index of 70 units would finish the race in approximately 4 hours. This value is obtained by noting that the vertical line at the training index of 70 units intersects the trend line very close to 4 hours.

Another prediction based on the trend line is the number of minutes that a bicyclist can expect to lower his or her finishing time for each increase of 10 training index units. This prediction is basically the ratio of the change in finishing time to the change in training index, or the slope of the trend line. Note that the slope is negative. To estimate the slope, estimate the coordinates of any two points on the line—for instance, the points at the extreme left and right ends of the line: $(0, 5.8)$ and $(100, 3.2)$. The slope is

$$\frac{3.2 - 5.8}{100 - 0} = \frac{-2.6}{100} = -0.026$$

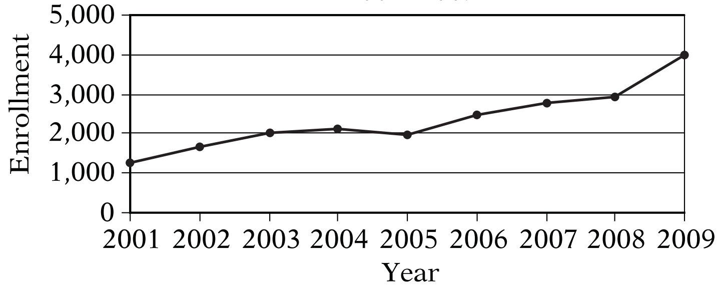
which is measured in hours per unit. The slope can be interpreted as follows: the finishing time is predicted to decrease 0.026 hours for every unit by which the training index increases. Since we want to know how much the finishing time decreases for an increase of *10 units*, we multiply the rate by 10 to get 0.26 hour per 10 units. To compute the decrease in *minutes* per 10 units, we multiply 0.26 by 60 to get approximately 16 minutes. Based on the trend line, the bicyclist can expect to decrease the finishing time by 16 minutes for every increase of 10 training index units.

Time Plots

Sometimes data are collected in order to observe changes in a variable over time. For example, sales for a department store may be collected monthly or yearly. A **time plot** (sometimes called a time series) is a graphical display useful for showing changes in data collected at regular intervals of time. A time plot of a variable plots each observation corresponding to the time at which it was measured. A time plot uses a coordinate plane similar to a scatterplot, but the time is always on the horizontal axis, and the variable measured is always on the vertical axis. Additionally, consecutive observations are connected by a line segment to emphasize increases and decreases over time.

Example 4.1.8:

FALL ENROLLMENT FOR COLLEGE *A*
2001–2009



You can observe from the graph that the greatest increase in fall enrollment between consecutive years occurred from 2008 to 2009. One way to determine this is by noting that the slope of the line segment joining the values for 2008 and 2009 is greater than the slopes of the line segments joining all other consecutive years, because the time intervals are regular.

Although time plots are commonly used to compare frequencies, as in the example above, they can be used to compare any numerical data as the data change over time, such as temperatures, dollar amounts, percents, heights, and weights.

4.2 Numerical Methods for Describing Data

Data can be described numerically by various **statistics**, or **statistical measures**. These statistical measures are often grouped in three categories: measures of central tendency, measures of position, and measures of dispersion.

Measures of Central Tendency

Measures of **central tendency** indicate the "center" of the data along the number line and are usually reported as values that represent the data. There are three common measures of central tendency: (i) the **arithmetic mean**—usually called the **average** or simply the **mean**, (ii) the **median**, and (iii) the **mode**.

To calculate the **mean** of n numbers, take the sum of the n numbers and divide it by n.

Example 4.2.1: For the five numbers 6, 4, 7, 10, and 4, the mean is

$$\frac{6 + 4 + 7 + 10 + 4}{5} = \frac{31}{5} = 6.2$$

When several values are repeated in a list, it is helpful to think of the mean of the numbers as a **weighted mean** of only those values in the list that are *different*.

Example 4.2.2: Consider the following list of 16 numbers.

2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9

There are only 6 different values in the list: 2, 4, 5, 7, 8, and 9. The mean of the numbers in the list can be computed as

$$\frac{1(2) + 2(4) + 1(5) + 6(7) + 2(8) + 4(9)}{1 + 2 + 1 + 6 + 2 + 4} = \frac{109}{16} = 6.8125$$

The number of times a value appears in the list, or the frequency, is called the **weight** of that value. So the mean of the 16 numbers is the weighted mean of the values 2, 4, 5, 7, 8, and 9, where the respective weights are 1, 2, 1, 6, 2, and 4. Note that the sum of the weights is the number of numbers in the list, 16.

The mean can be affected by just a few values that lie far above or below the rest of the data, because these values contribute directly to the sum of the data and therefore to the mean. By contrast, the **median** is a measure of central tendency that is fairly unaffected by unusually high or low values relative to the rest of the data.

To calculate the median of n numbers, first order the numbers from least to greatest. If n is odd, then the median is the middle number in the ordered list of numbers. If n is even, then there are *two* middle numbers, and the median is the average of these two numbers.

Example 4.2.3: The five numbers in example 4.2.1 listed in increasing order are 4, 4, 6, 7, 10, so the median is 6, the middle number. Note that if the number 10 in the list is replaced by the number 24, the mean increases from 6.2 to

$$\frac{4 + 4 + 6 + 7 + 24}{5} = \frac{45}{5} = 9$$

but the median remains equal to 6. This example shows how the median is relatively unaffected by an unusually large value.

The median, as the "middle value" of an ordered list of numbers, divides the list into roughly two equal parts. However, if the median is equal to one of the data values and it is repeated in the list, then the numbers of data above and below the median may be rather different. See example 4.2.2, where the median is 7, but four of the data are less than 7 and six of the data are greater than 7.

The **mode** of a list of numbers is the number that occurs most frequently in the list.

Example 4.2.4: The mode of the numbers in the list 1, 3, 6, 4, 3, 5 is 3. A list of numbers may have more than one mode. For example, the list 1, 2, 3, 3, 3, 5, 7, 10, 10, 10, 20 has two modes, 3 and 10.

Measures of Position

The three most basic **positions**, or locations, in a list of data ordered from least to greatest are the beginning, the end, and the middle. It is useful here to label these as L for the least, G for the greatest, and M for the median. Aside from these, the most common measures of position are **quartiles** and **percentiles**. Like the median M, quartiles and percentiles are numbers that divide the data into roughly equal groups after the data have been ordered from the least value L to the greatest value G. There are three quartile numbers that divide the data into four roughly equal groups, and there are 99 percentile numbers that divide the data into 100 roughly equal groups. As with the mean and median, the quartiles and percentiles may or may not themselves be values in the data.

The **first quartile** Q_1, the **second quartile** Q_2 (which is simply the median M), and the **third quartile** Q_3 divide a group of data into four roughly equal groups as follows. After the data are listed in increasing order, the first group consists of the data from L to Q_1, the second group is from Q_1 to M, the third group is from M to Q_3, and the fourth group is from Q_3 to G. Because the number of data in a list may not be divisible by 4, there are various rules to determine the exact values of Q_1 and Q_3 and some statisticians use different rules, but in all cases $Q_2 = M$. We use perhaps the most common rule, in which $Q_2 = M$ divides the data into two equal parts—the lesser numbers and the greater numbers—and then Q_1 is the median of the lesser numbers and Q_3 is the median of the greater numbers.

Example 4.2.5: To find the quartiles for the list of 16 numbers 2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9 (already listed in order), first divide the data into two groups of 8 numbers each. The first group is 2, 4, 4, 5, 7, 7, 7, 7 and the second group is 7, 7, 8, 8, 9, 9, 9, 9, so that the second quartile, or median, is $Q_2 = M = 7$. To find the other quartiles, you can take each of the two smaller groups and find *its* median: the first quartile is $Q_1 = 6$ (the average of 5 and 7) and the third quartile is $Q_3 = 8.5$ (the average of 8 and 9).

In example 4.2.5, note that the number 4 is in the lowest 25 percent of the distribution of data. There are different ways to describe this. We can say that 4 is *below* the first quartile, that is, below Q_1; we can also say that 4 is *in* the first quartile. The phrase "*in* a quartile" refers to being *in* one of the four groups determined by Q_1, Q_2, and Q_3.

Percentiles are mostly used for very large lists of numerical data ordered from least to greatest. Instead of dividing the data into four groups, the 99 percentiles P_1, $P_2, P_3, \ldots, P_{99}$ divide the data into 100 groups. Consequently, $Q_1 = P_{25}$, $M = Q_2 = P_{50}$, and $Q_3 = P_{75}$. Because the number of data in a list may not be divisible by 100, statisticians apply various rules to determine values of percentiles.

Measures of Dispersion

Measures of **dispersion** indicate the degree of "spread" of the data. The most common statistics used as measures of dispersion are the range, the interquartile range, and the standard deviation. These statistics measure the spread of the data in different ways.

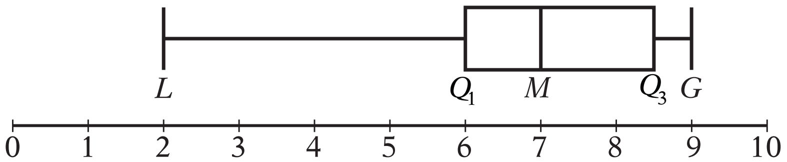
The **range** of the numbers in a group of data is the difference between the greatest number G in the data and the least number L in the data, that is, $G - L$. For example, given the list 11, 10, 5, 13, 21, the range of the numbers is $21 - 5 = 16$.

The simplicity of the range is useful in that it reflects that maximum spread of the data. However, sometimes a data value is so unusually small or so unusually large in comparison with the rest of the data that it is viewed with suspicion when the data are analyzed—the value could be erroneous or accidental in nature. Such data are called **outliers** because they lie so far out that in most cases, they are ignored when analyzing the data. Unfortunately, the range is directly affected by outliers.

A measure of dispersion that is not affected by outliers is the **interquartile range**. It is defined as the difference between the third quartile and the first quartile, that is, $Q_3 - Q_1$. Thus, the interquartile range measures the spread of the middle half of the data.

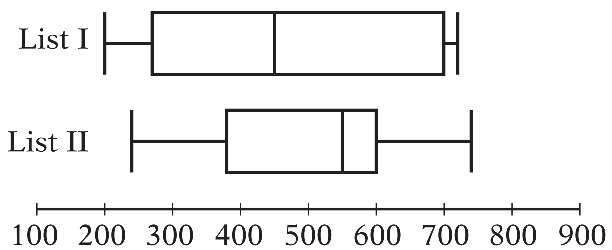
Example 4.2.6: In the list of 16 numbers 2, 4, 4, 5, 7, 7, 7, 7, 7, 7, 8, 8, 9, 9, 9, 9, the range is $9 - 2 = 7$, the first quartile is $Q_1 = 6$, and the third quartile is $Q_3 = 8.5$. So the interquartile range for the numbers in this list is $8.5 - 6 = 2.5$.

One way to summarize a group of numerical data and to illustrate its center and spread is to use the five numbers L, Q_1, Q_2, Q_3, and G. These five numbers can be plotted along a number line to show where the four quartile groups lie. Such plots are called **boxplots** or **box-and-whisker plots**, because a box is used to identify each of the two middle quartile groups of data, and "whiskers" extend outward from the boxes to the least and greatest values. The following graph shows the boxplot for the list of 16 numbers in example 4.2.6.



There are a few variations in the way boxplots are drawn—the position of the ends of the boxes can vary slightly, and some boxplots identify outliers with certain symbols—but all boxplots show the center of the data at the median and illustrate the spread of the data in each of the four quartile groups. As such, boxplots are useful for comparing sets of data side by side.

Example 4.2.7: Two large lists of numerical data, list I and list II, are summarized by the following boxplots.



Based on the boxplots, several different comparisons of the two lists can be made. First, the median of list II, which is approximately 550, is greater than the median of list I, which is approximately 450. Second, the two measures of spread, range and interquartile range, are greater for list I than for list II. For list I, these measures are approximately 520 and 430, respectively; and for list II, they are approximately 500 and 220, respectively.

Unlike the range and the interquartile range, the **standard deviation** is a measure of spread that depends on each number in the list. Using the mean as the center of the data, the standard deviation takes into account how much each value differs from the mean and then takes a type of average of these differences. As a result, the more the data are spread away from the mean, the greater the standard deviation; and the more the data are clustered around the mean, the lesser the standard deviation.

The standard deviation of a group of n numerical data is computed by (1) calculating the mean of the n values, (2) finding the difference between the mean and each of the n values, (3) squaring each of the differences, (4) finding the average of the n squared differences, and (5) taking the nonnegative square root of the average squared difference.

Example 4.2.8: For the five data 0, 7, 8, 10, and 10, the standard deviation can be computed as follows. First, the mean of the data is 7, and the squared differences from the mean are

$$(7-0)^2, (7-7)^2, (7-8)^2, (7-10)^2, (7-10)^2$$

or 49, 0, 1, 9, 9. The average of the five squared differences is $\frac{68}{5}$, or 13.6, and the positive square root of 13.6 is approximately 3.7.

Note on terminology: The term "standard deviation" defined above is slightly different from another measure of dispersion, the **sample standard deviation**. The latter term is qualified with the word "sample" and is computed by dividing the sum of the squared differences by $n - 1$ instead of n. The sample standard deviation is only slightly different from the standard deviation but is preferred for technical reasons for a sample of data that is taken from a larger population of data. Sometimes the standard deviation is called the **population standard deviation** to help distinguish it from the sample standard deviation.

Example 4.2.9: Six hundred applicants for several post office jobs were rated on a scale from 1 to 50 points. The ratings had a mean of 32.5 points and a standard deviation of 7.1 points. How many standard deviations above or below the mean is a rating of 48 points? A rating of 30 points? A rating of 20 points?

Solution: Let d be the standard deviation, so $d = 7.1$ points. Note that 1 standard deviation above the mean is

$$32.5 + d = 32.5 + 7.1 = 39.6$$

and 2 standard deviations above the mean is

$$32.5 + 2d = 32.5 + 2(7.1) = 46.7$$

So 48 is a little more than 2 standard deviations above the mean. Since 48 is actually 15.5 points above the mean, the number of standard deviations that 48 is above the mean is $\frac{15.5}{7.1} \approx 2.2$. Thus, to answer the question, we first found the difference from the mean and then we divided by the standard deviation. The number of standard deviations that a rating of 30 is away from the mean is

$$\frac{30 - 32.5}{7.1} = \frac{-2.5}{7.1} \approx -0.4$$

where the negative sign indicates that the rating is 0.4 standard deviation *below* the mean.

The number of standard deviations that a rating of 20 is away from the mean is

$$\frac{20 - 32.5}{7.1} = \frac{-12.5}{7.1} \approx -1.8$$

where the negative sign indicates that the rating is 1.8 standard deviations *below* the mean.

To summarize:

- 48 points is 15.5 points above the mean, or approximately 2.2 standard deviations above the mean.
- 30 points is 2.5 points below the mean, or approximately 0.4 standard deviation below the mean.
- 20 points is 12.5 points below the mean, or approximately 1.8 standard deviations below the mean.

One more instance, which may seem trivial, is important to note:

- 32.5 points is 0 points from the mean, or 0 standard deviations from the mean.

Example 4.2.9 shows that for a group of data, each value can be located with respect to the mean by using the standard deviation as a ruler. The process of subtracting the mean from each value and then dividing the result by the standard deviation is called **standardization**. Standardization is a useful tool because for each data value, it provides a measure of position relative to the rest of the data independently of the variable for which the data was collected and the units of the variable.

Note that the standardized values 2.2, −0.4, and −1.8 from example 4.2.9 are all between −3 and 3; that is, the corresponding ratings 48, 30, and 20 are all within 3 standard deviations above or below the mean. This is not surprising, based on the following fact about the standard deviation.

*In **any group of data**, most of the data are within about **3** standard deviations above or below the mean.*

Thus, when *any group of data* are standardized, most of the data are transformed to an interval on the number line centered about 0 and extending from about −3 to 3. The mean is always transformed to 0.

4.3 Counting Methods

Uncertainty is part of the process of making decisions and predicting outcomes. Uncertainty is addressed with the ideas and methods of probability theory. Since elementary probability requires an understanding of counting methods, we now turn to a discussion of counting objects in a systematic way before reviewing probability.

When a set of objects is small, it is easy to list the objects and count them one by one. When the set is too large to count that way, and when the objects are related in a patterned or systematic way, there are some useful techniques for counting the objects without actually listing them.

Sets and Lists

The term **set** has been used informally in this review to mean a collection of objects that have some property, whether it is the collection of all positive integers, all points in a circular region, or all students in a school that have studied French. The objects of a set are called **members** or **elements**. Some sets are **finite**, which means that their members can be completely counted. Finite sets can, in principle, have all of their members listed, using curly brackets, such as the set of even digits $\{0, 2, 4, 6, 8\}$. Sets that are not finite are called **infinite** sets, such as the set of all integers. A set that has no members is called the **empty set** and is denoted by the symbol $\varnothing$. A set with one or more members is called **nonempty**. If A and B are sets and all of the members of A are also members of B, then A is a **subset** of B. For example, $\{2, 8\}$ is a subset of $\{0, 2, 4, 6, 8\}$. Also, by convention, $\varnothing$ is a subset of every set.

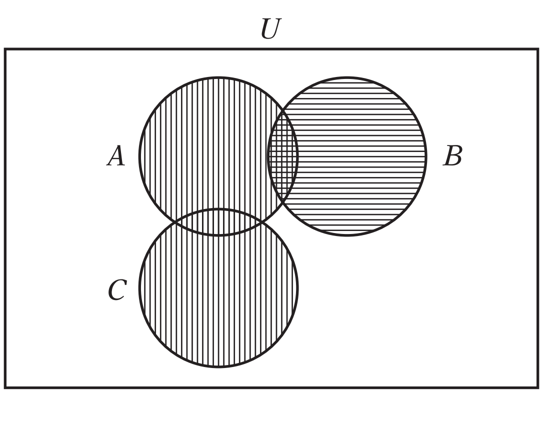
A **list** is like a finite set, having members that can all be listed, but with two differences. In a list, the members are ordered; that is, rearranging the members of a list makes it a different list. Thus, the terms “first element,” “second element,” etc., make sense in a list. Also, elements can be repeated in a list and the repetitions matter. For example, the lists 1, 2, 3, 2 and 1, 2, 2, 3 are different lists, each with four elements, and they are both different from the list 1, 2, 3, which has three elements.

In contrast to a list, when the elements of a set are given, repetitions are not counted as additional elements and the order of the elements does not matter. For example, the sets $\{1, 2, 3, 2\}$ and $\{3, 1, 2\}$ are the same set, which has three elements. For any finite set S, the number of elements of S is denoted by $|S|$. Thus, if $S = \{6.2, -9, \pi, 0.01, 0\}$, then $|S| = 5$. Also, $|\varnothing| = 0$.

Sets can be formed from other sets. If S and T are sets, then the **intersection** of S and T is the set of all elements that are in both S and T and is denoted by $S \cap T$. The **union** of S and T is the set of all elements that are in S or T, or both, and is denoted by $S \cup T$. If sets S and T have no elements in common, they are called **disjoint** or **mutually exclusive**.

A useful way to represent two or three sets and their possible intersections and unions is a **Venn diagram**. In a Venn diagram, sets are represented by circular regions that overlap if they have elements in common but do not overlap if they are disjoint. Sometimes the circular regions are drawn inside a rectangular region, which represents a **universal set**, of which all other sets involved are subsets.

Example 4.3.1: The sets A, B, and C are represented in the Venn diagram below, where U represents a universal set.



The regions with vertical stripes represent the set $A \cup C$. The regions with horizontal stripes represent the set B. The region with both kinds of stripes represents the set $A \cap B$. The sets B and C are mutually exclusive, often written $B \cap C = \varnothing$.

The example above can be used to illustrate an elementary counting principle involving intersecting sets, called the **inclusion-exclusion principle** for two sets. This principle relates the numbers of elements in the union and intersection of two finite sets: The number of elements in the union of two sets equals the sum of their individual numbers of elements minus the number of elements in their intersection. If the sets in example 4.3.1 are finite, then we have for the union of A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Because $A \cap B$ is a subset of both A and B, the subtraction is necessary to avoid counting the elements in $A \cap B$ twice. For the union of B and C, we have

$$|B \cup C| = |B| + |C|$$

because $B \cap C = \varnothing$.

Multiplication Principle

Suppose there are two choices to be made sequentially and that the second choice is independent of the first choice. Suppose also that there are k different possibilities for the first choice and m different possibilities for the second choice. The **multiplication principle** states that under those conditions, there are km different possibilities for the pair of choices.

For example, suppose that a meal is to be ordered from a restaurant menu and that the meal consists of one entrée and one dessert. If there are 5 entrées and 3 desserts on the menu, then there are $(5)(3) = 15$ different meals that can be ordered from the menu.

The multiplication principle applies in more complicated situations as well. If there are more than two independent choices to be made, then the number of different possible outcomes of all of the choices is the product of the numbers of possibilities for each choice.

Example 4.3.2: Suppose that a computer password consists of four characters such that the first character is one of the 10 digits from 0 to 9 and each of the next 3 characters is any one of the uppercase letters from the 26 letters of the English alphabet. How many different passwords are possible?

Solution: The description of the password allows repetitions of letters. Thus, there are 10 possible choices for the first character in the password and 26 possible choices for each of the next 3 characters in the password. Therefore, applying the multiplication principle, the number of possible passwords is $(10)(26)(26)(26) = 175{,}760$.

Note that if repetitions of letters are *not* allowed in the password, then the choices are not all independent, but a modification of the multiplication principle can still be applied. There are 10 possible choices for the first character in the password, 26 possible choices for the second character, 25 for the third character because the first letter cannot be repeated, and 24 for the fourth character because the first two letters cannot be repeated. Therefore, the number of possible passwords is $(10)(26)(25)(24) = 156{,}000$.

Example 4.3.3: Each time a coin is tossed, there are 2 possible outcomes—either it lands heads up or it lands tails up. Using this fact and the multiplication principle, you can conclude that if a coin is tossed 8 times, there are $(2)(2)(2)(2)(2)(2)(2)(2) = 2^8 = 256$ possible outcomes.

Permutations and Factorials

Suppose you want to determine the number of different ways the 3 letters A, B, and C can be placed in order from 1st to 3rd. The following is a list of all the possible orders in which the letters can be placed.

ABC   ACB   BAC   BCA   CAB   CBA

There are 6 possible orders for the 3 letters.

Now suppose you want to determine the number of different ways the 4 letters A, B, C, and D can be placed in order from 1st to 4th. Listing all of the orders for 4 letters is time-consuming, so it would be useful to be able to count the possible orders without listing them.

To order the 4 letters, one of the 4 letters must be placed first, one of the remaining 3 letters must be placed second, one of the remaining 2 letters must be placed third, and the last remaining letter must be placed fourth. Therefore, applying the multiplication principle, there are (4)(3)(2)(1), or 24, ways to order the 4 letters.

More generally, suppose n objects are to be ordered from 1st to nth, and we want to count the number of ways the objects can be ordered. There are n choices for the first object, $n-1$ choices for the second object, $n-2$ choices for the third object, and so on, until there is only 1 choice for the nth object. Thus, applying the multiplication principle, the number of ways to order the n objects is equal to the product

$$n(n-1)(n-2)\cdots(3)(2)(1)$$

Each order is called a **permutation**, and the product above is called the number of permutations of n objects.

Because products of the form $n(n-1)(n-2)\cdots(3)(2)(1)$ occur frequently when counting objects, a special symbol $n!$, called ***n* factorial**, is used to denote this product.

For example,

$$\begin{aligned}1! &= 1\\ 2! &= (2)(1) = 2\\ 3! &= (3)(2)(1) = 6\\ 4! &= (4)(3)(2)(1) = 24\end{aligned}$$

As a special definition, $0! = 1$.

Note that $n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)!$ and so on.

Example 4.3.4: Suppose that 10 students are going on a bus trip, and each of the students will be assigned to one of the 10 available seats. Then the number of possible different seating arrangements of the students on the bus is

$$10! = (10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 3{,}628{,}800$$

Now suppose you want to determine the number of ways in which you can select 3 of the 5 letters A, B, C, D, and E and place them in order from 1st to 3rd. Reasoning as in the preceding examples, you find that there are (5)(4)(3), or 60, ways to select and order them.

More generally, suppose that k objects will be selected from a set of n objects, where $k \le n$, and the k objects will be placed in order from 1st to kth. Then there are n choices for the first object, $n-1$ choices for the second object, $n-2$ choices for the third object, and so on, until there are $n-k+1$ choices for the kth object. Thus, applying the multiplication principle, the number of ways to select and order k objects from a set of n objects is $n(n-1)(n-2)\cdots(n-k+1)$. It is useful to note that

$$\begin{aligned} n(n-1)(n-2)\cdots(n-k+1) &= n(n-1)(n-2)\cdots(n-k+1)\frac{(n-k)!}{(n-k)!} \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

This expression represents the number of **permutations of n objects taken k at a time**, that is, the number of ways to select and order k objects out of n objects.

Example 4.3.5: How many different five-digit positive integers can be formed using the digits 1, 2, 3, 4, 5, 6, and 7 if none of the digits can occur more than once in the integer?

Solution: This example asks how many ways there are to order 5 integers chosen from a set of 7 integers. According to the counting principle above, there are $(7)(6)(5)(4)(3) = 2{,}520$ ways to do this. Note that this is equal to $\frac{7!}{(7-5)!} = \frac{(7)(6)(5)(4)(3)(2!)}{2!} = (7)(6)(5)(4)(3)$.

Combinations

Given the five letters A, B, C, D, and E, suppose that you want to determine the number of ways in which you can select 3 of the 5 letters, but unlike before, you do not want to count different orders for the 3 letters. The following is a list of all of the ways in which 3 of the 5 letters can be selected without regard to the order of the letters.

ABC ABD ABE ACD ACE ADE BCD BCE BDE CDE

There are 10 ways of selecting the 3 letters without order. There is a relationship between selecting with order and selecting without order.

The number of ways to select 3 of the 5 letters without order, which is 10, *multiplied by* the number of ways to order the 3 letters, which is 3!, or 6, *is equal to* the number of ways to select 3 of the 5 letters and order them, which is $\frac{5!}{2!} = 60$. In short,

$$\begin{aligned} &(\text{number of ways to select without order}) \times (\text{number of ways to order}) \\ &\qquad = (\text{number of ways to select with order}) \end{aligned}$$

This relationship can also be described as follows.

$$\begin{aligned} (\text{number of ways to select without order}) &= \frac{(\text{number of ways to select with order})}{(\text{number of ways to order})} \\ &= \frac{\frac{5!}{2!}}{3!} = \frac{5!}{3!2!} = 10 \end{aligned}$$

More generally, suppose that k objects will be chosen from a set of n objects, where $k \le n$, but that the k objects will *not* be put in order. The number of ways in which this can be done is called the number of **combinations of n objects taken k at a time** and is given by the formula $\frac{n!}{k!(n-k)!}$.

Another way to refer to the number of combinations of n objects taken k at a time is ***n* choose *k***, and two notations commonly used to denote this number are ${}_nC_k$ and $\binom{n}{k}$.

Example 4.3.6: Suppose you want to select a 3-person committee from a group of 9 students. How many ways are there to do this?

Solution: Since the 3 students on the committee are not ordered, you can use the formula for the combination of 9 objects taken 3 at a time, or "9 choose 3":

$$\frac{9!}{3!(9-3)!} = \frac{9!}{3!6!} = \frac{(9)(8)(7)}{(3)(2)(1)} = 84$$

Using the terminology of sets, given a set S consisting of n elements, n choose k is simply the number of subsets of S that consist of k elements. The formula $\frac{n!}{k!(n-k)!}$ also holds when $k = 0$ and $k = n$.

- n choose 0 is $\frac{n!}{0!n!} = 1$, which corresponds to the fact that there is only one subset of S with 0 elements, namely the empty set.
- n choose n is $\frac{n!}{n!0!} = 1$, since there is only one subset of S with n elements, namely the set S itself.

Finally, note that n choose k is always equal to n choose $n - k$, because

$$\frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!}$$

4.4 Probability

Probability is a way of describing uncertainty in numerical terms. In this section we review some of the terminology used in elementary probability theory.

A **probability experiment**, also called a **random experiment**, is an experiment for which the result, or **outcome**, is uncertain. We assume that all of the possible outcomes of an experiment are known before the experiment is performed, but which outcome will actually occur is unknown. The set of all possible outcomes of a random experiment is called the **sample space**, and any particular set of outcomes is called an **event**. For example, consider a cube with faces numbered 1 to 6, called a 6-sided die. Rolling the die once is an experiment in which there are 6 possible outcomes—either 1, 2, 3, 4, 5, or 6 will appear on the top face. The sample space for this experiment is the set of numbers 1, 2, 3, 4, 5, and 6. Two examples of events for this experiment are (i) rolling the number 4, which has only one outcome, and (ii) rolling an odd number, which has three outcomes.

The **probability** of an event is a number from 0 to 1, inclusive, that indicates the likelihood that the event occurs when the experiment is performed. The greater the number, the more likely the event.

Example 4.4.1: Consider the following experiment. A box contains 15 pieces of paper, each of which has the name of one of the 15 students in a class consisting of 7 male and 8 female students, all with different names. The instructor will shake the box for a while and then, without looking, choose a piece of paper at random and read the name. Here the sample space is the set of 15 names. The assumption of **random selection** means that each of the names is **equally likely** to be selected. If this assumption is made, then the probability that any one particular name is selected is equal to $\frac{1}{15}$. For any event E, the probability that E occurs, denoted by $P(E)$, is defined by the ratio

$$P(E) = \frac{\text{the number of names in the event } E}{15}$$

If M is the event that the student selected is male, then $P(M) = \frac{7}{15}$.

In general, for a random experiment with a finite number of possible outcomes, if each outcome is equally likely to occur, then the probability that an event E occurs is defined by the ratio

$$P(E) = \frac{\text{the number of outcomes in the event } E}{\text{the number of possible outcomes in the experiment}}$$

In the case of rolling a 6-sided die, if the die is "fair," then the 6 outcomes are equally likely. So the probability of rolling a 4 is $\frac{1}{6}$, and the probability of rolling an odd number—rolling a 1, 3, or 5—can be calculated as $\frac{3}{6} = \frac{1}{2}$.

The following are general facts about probability.

- If an event E is certain to occur, then $P(E) = 1$.
- If an event E is certain *not* to occur, then $P(E) = 0$.
- If an event E is possible but not certain to occur, then $0 < P(E) < 1$.
- The probability that an event E will not occur is equal to $1 - P(E)$.
- If E is an event, then the probability of E is the sum of the probabilities of the outcomes in E.
- The sum of the probabilities of all possible outcomes of an experiment is 1.

If E and F are two events of an experiment, we consider two other events related to E and F.

- The event that both E and F occur, that is, all outcomes in the set $E \cap F$.
- The event that E or F, or both, occur, that is, all outcomes in the set $E \cup F$.

Events that cannot occur at the same time are said to be **mutually exclusive**. For example, if a 6-sided die is rolled once, the event of rolling an odd number and the event of rolling an even number are mutually exclusive. But rolling a 4 and rolling an even number are not mutually exclusive, since 4 is an outcome that is common to both events.

For events E and F, we have the following rules.

- $P(E \text{ or } F\text{, or both, occur}) = P(E) + P(F) - P(\text{both } E \text{ and } F \text{ occur})$, which is the inclusion-exclusion principle applied to probability.
- If E and F are mutually exclusive, then $P(\text{both } E \text{ and } F \text{ occur}) = 0$, and therefore, $P(E \text{ or } F\text{, or both, occur}) = P(E) + P(F)$.
- E and F are said to be **independent** if the occurrence of either event does not affect the occurrence of the other. If two events E and F are independent, then $P(\text{both } E \text{ and } F \text{ occur}) = P(E)P(F)$. For example, if a fair 6-sided die is rolled twice, the event E of rolling a 3 on the first roll and the event F of rolling a 3 on the second roll are independent, and the probability of rolling a 3 on both rolls is $P(E)P(F) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$. In this example, the experiment is actually

"rolling the die twice," and each outcome is an ordered pair of results like "4 on the first roll and 1 on the second roll." But event E restricts only the first roll—to a 3—having no effect on the second roll; similarly, event F restricts only the second roll—to a 3—having no effect on the first roll.

Note that if $P(E) \neq 0$ and $P(F) \neq 0$, then events E and F cannot be both mutually exclusive and independent. For if E and F are independent, then $P(\text{both } E \text{ and } F \text{ occur}) = P(E)P(F) \neq 0$; but if E and F are mutually exclusive, then $P(\text{both } E \text{ and } F \text{ occur}) = 0$.

It is common to use the shorter notation "E and F" instead of "both E and F occur" and use "E or F" instead of "E or F, or both, occur." With this notation, we have the following rules.

- $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$
- $P(E \text{ or } F) = P(E) + P(F)$ if E and F are mutually exclusive.
- $P(E \text{ and } F) = P(E)P(F)$ if E and F are independent.

Example 4.4.2: If a fair 6-sided die is rolled once, let E be the event of rolling a 3 and let F be the event of rolling an odd number. These events are *not* independent. This is because rolling a 3 makes certain that the event of rolling an odd number occurs. Note that $P(E \text{ and } F) \neq P(E)P(F)$, since

$$P(E \text{ and } F) = P(E) = \frac{1}{6} \quad \text{and} \quad P(E)P(F) = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \frac{1}{12}$$

Example 4.4.3: A 12-sided die, with faces numbered 1 to 12, is to be rolled once, and each of the 12 possible outcomes is equally likely to occur. The probability of rolling a 4 is $\frac{1}{12}$, so the probability of rolling a number that is *not* a 4 is $1 - \frac{1}{12} = \frac{11}{12}$. The probability of rolling a number that is either a multiple of 5—a 5 or a 10—or an odd number—a 1, 3, 5, 7, 9, or 11—is equal to

$$\begin{aligned} P(\text{multiple of 5}) + P(\text{odd}) - P(\text{multiple of 5 and odd}) &= \frac{2}{12} + \frac{6}{12} - \frac{1}{12} \\ &= \frac{7}{12} \end{aligned}$$

Another way to calculate this probability is to notice that rolling a number that is either a multiple of 5 (5 or 10) or an odd number (1, 3, 5, 7, 9, or 11) is the same as rolling one of 1, 3, 5, 7, 9, 10, and 11, which are 7 equally likely outcomes. So by using the ratio formula to calculate the probability, the required probability is $\frac{7}{12}$.

Example 4.4.4: Consider an experiment with events A, B, and C for which $P(A) = 0.23$, $P(B) = 0.40$, and $P(C) = 0.85$. Suppose that events A and B are mutually exclusive and events B and C are independent. What are the probabilities $P(A \text{ or } B)$ and $P(B \text{ or } C)$?

Solution: Since A and B are mutually exclusive,

$$P(A \text{ or } B) = P(A) + P(B) = 0.23 + 0.40 = 0.63$$

Since B and C are independent, $P(B \text{ and } C) = P(B)P(C)$. So,

$$P(B \text{ or } C) = P(B) + P(C) - P(B \text{ and } C) = P(B) + P(C) - P(B)P(C)$$

Therefore,

$$\begin{aligned} P(B \text{ or } C) &= 0.40 + 0.85 - (0.40)(0.85) = 1.25 - 0.34 \\ &= 0.91 \end{aligned}$$

Example 4.4.5: Suppose that there is a 6-sided die that is weighted in such a way that each time the die is rolled, the probabilities of rolling any of the numbers from 1 to 5 are all equal, but the probability of rolling a 6 is twice the probability of rolling a 1. When you roll the die once, the 6 outcomes are *not equally likely*. What are the probabilities of the 6 outcomes?

Solution: Using the notation $P(1)$ for the probability of rolling a 1, let $p = P(1)$. Then each of the probabilities of rolling a 2, 3, 4, or 5 is equal to p, and the probability of rolling a 6 is $2p$. Therefore, since the sum of the probabilities of all possible outcomes is 1, it follows that

$$\begin{aligned} 1 &= P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = p + p + p + p + p + 2p \\ &= 7p \end{aligned}$$

So the probability of rolling each of the numbers from 1 to 5 is $p = \frac{1}{7}$, and the probability of rolling a 6 is $\frac{2}{7}$.

Example 4.4.6: Suppose that you roll the weighted 6-sided die from example 4.4.5 twice. What is the probability that the first roll will be an odd number and the second roll will be an even number?

Solution: To calculate the probability that the first roll will be odd and the second roll will be even, note that these two events are independent. To calculate the probability that both occur, you must multiply the probabilities of the two independent events. First compute the individual probabilities.

$$P(\text{odd}) = P(1) + P(3) + P(5) = \frac{3}{7}$$

$$P(\text{even}) = P(2) + P(4) + P(6) = \frac{4}{7}$$

Then, $P(\text{first roll is odd and second roll is even}) = P(\text{odd})P(\text{even}) = \left(\frac{3}{7}\right)\left(\frac{4}{7}\right) = \frac{12}{49}$.

Two events that happen sequentially are not always independent. The occurrence of one event may affect the occurrence of a following event. In that case, the probability that *both* events happen is equal to the probability that the first event happens multiplied by the probability that *given that the first event has already happened*, the second event happens as well.

Example 4.4.7: A box contains 5 orange disks, 4 red disks, and 1 blue disk. You are to select two disks at random and without replacement from the box. What is the probability that the first disk you select will be red and the second disk you select will be orange?

Solution: To solve, you need to calculate the following two probabilities and then multiply them

- The probability that the first disk selected from the box will be red
- The probability that the second disk selected from the box will be orange, given that the first disk selected from the box is red

The probability that the first disk you select will be red is $\frac{4}{10} = \frac{2}{5}$. If the first disk you select is red, there will be 5 orange disks, 3 red disks, and 1 blue disk left in the box, for a total of 9 disks. Therefore, the probability that the second disk you select will be orange, given that the first disk you selected is red, is $\frac{5}{9}$. Multiply the two probabilities to get $\left(\frac{2}{5}\right)\left(\frac{5}{9}\right) = \frac{2}{9}$.

4.5 Distributions of Data, Random Variables, and Probability Distributions

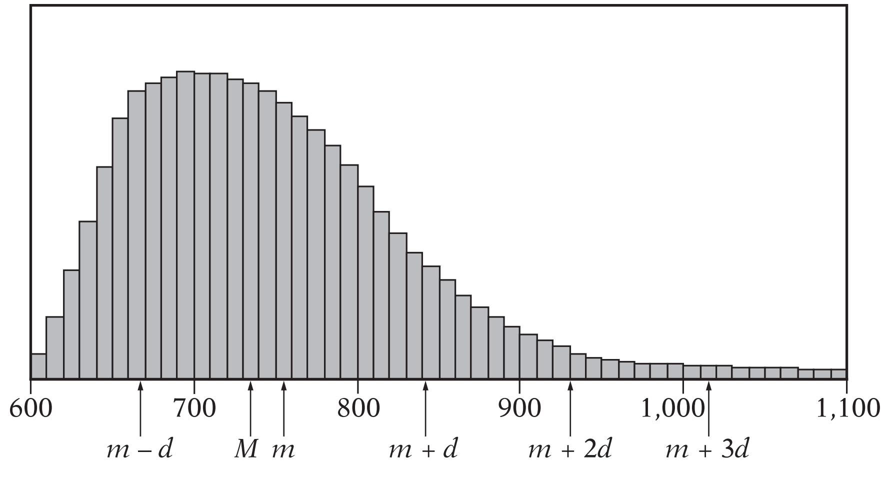
In data analysis, variables whose values depend on chance play an important role in linking distributions of data to probability distributions. Such variables are called random variables. We begin with a review of distributions of data.

Distributions of Data

Recall that relative frequency distributions given in a table or histogram are a common way to show how numerical data are distributed. In a histogram, the areas of the bars indicate where the data are concentrated. The histogram in example 4.1.6 illustrates a small group of data, with only 6 possible values and only 25 data altogether. Many groups of data are much larger than 25 and have many more than 6 possible values, which are often measurements of quantities like length, money, and time.

Example 4.5.1: The lifetimes of 800 electric devices were measured. Because the lifetimes had many different values, the measurements were grouped into 50 intervals, or **classes**, of 10 hours each: 601–610 hours, 611–620 hours, . . . , 1,091–1,100 hours. The resulting relative frequency distribution, as a histogram, has 50 thin bars and many different bar heights, as shown below.

Relative Frequency Distribution for Lifetimes of 800 Electric Devices



Note that the tops of the bars of the histogram have a relatively smooth appearance and begin to look like a curve. In general, histograms that represent very large data sets with many classes appear to have smooth shapes. Consequently, the distribution can be modeled by a smooth curve that is close to the tops of the bars. Such a model retains the shape of the distribution but is independent of classes.

Recall from example 4.1.6 that the sum of the areas of the bars of a relative frequency histogram is 1. Although the units on the horizontal axis of a histogram vary from one data set to another, the vertical scale can be adjusted (stretched or shrunk) so that the sum of the areas of the bars is 1. With this vertical scale adjustment, the area under the curve that models the distribution is also 1. This model curve is called a **distribution curve**, but it has other names as well, including **density curve** and **frequency curve**.

The purpose of the distribution curve is to give a good illustration of a large distribution of numerical data that doesn't depend on specific classes. To achieve this, the main property of a distribution curve is that the area under the curve in any vertical slice, just like a histogram bar, represents the proportion of the data that lies in the corresponding interval on the horizontal axis, which is at the base of the slice.

Before leaving this histogram, note that the mean m and the median M of the data are marked on the horizontal axis. Also, several standard deviations above and below the mean are marked, where d is the standard deviation of the data. The standard deviation marks show how most of the data are within about 3 standard deviations above or below the mean (that is, between the numbers $m - 3d$ (not shown) and $m + 3d$).

Finally, regarding the mean and the median, recall that the median splits the data into a lower half and an upper half, so that the sum of the areas of the bars to the left of M is the same as the sum of the areas to the right. On the other hand, m takes into account the exact value of each of the data, not just whether a value is high or low. The nature of the mean is such that if an imaginary fulcrum were placed somewhere under the horizontal axis in order to balance the distribution perfectly, the balancing position would be exactly at m. That is why m is somewhat to the right of M. The balance point at m takes into account *how high* the few very high values are (to the far right), while M just counts them as "high." To summarize, the median is the "halving point" and the mean is the "balance point."

Random Variables

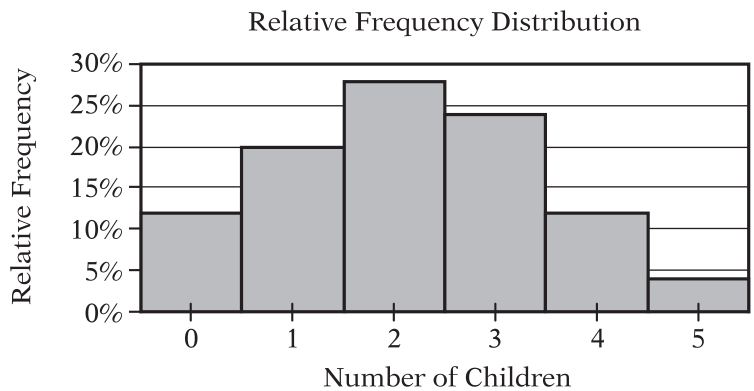
When analyzing data, it is common to choose a value of the data at random and consider that choice as a random experiment, as introduced in section 4.4. Then, the probabilities of events involving the randomly chosen value may be determined. Given a distribution of data, a variable, say X, may be used to represent a randomly chosen value from the distribution. Such a variable X is an example of a **random variable**, which is a variable whose value is a numerical outcome of a random experiment.

Example 4.5.2: In the data from example 4.1.1 consisting of numbers of children, let X represent the number of children in a randomly chosen family among the 25 families. What is the probability that $X = 3$? That $X > 3$? That X is less than the mean of the distribution?

Solution: For convenience, here is the frequency distribution of the data.

Number of Children	Frequency
0	3
1	5
2	7
3	6
4	3
5	1
Total	25

Since there are 6 families with 3 children and each of the 25 families is equally likely to be chosen, the probability that a family with 3 children will be chosen is $\frac{6}{25}$. That is, $X = 3$ is an event, and its probability is $P(X = 3) = \frac{6}{25}$, or 0.24. It is common to use the shorter notation $P(3)$ instead of $P(X = 3)$, so you could write $P(3) = 0.24$. Note that in the histogram, shown below, the area of the bar corresponding to $X = 3$ as a proportion of the combined areas of all of the bars is equal to this probability. This indicates how probability is related to area in a histogram for a relative frequency distribution.



As for the event $X > 3$, it is the same as the event "$X = 4$ or $X = 5$". Because $X = 4$ and $X = 5$ are mutually exclusive events, we can use the rules of probability from section 4.4.

$$P(X > 3) = P(4) + P(5) = \frac{3}{25} + \frac{1}{25} = 0.12 + 0.04 = 0.16$$

For the last question, first compute the mean of the distribution.

$$\frac{0(3) + 1(5) + 2(7) + 3(6) + 4(3) + 5(1)}{25} = \frac{54}{25} = 2.16$$

Then,

$$P(X < 2.16) = P(0) + P(1) + P(2) = \frac{3}{25} + \frac{5}{25} + \frac{7}{25} = \frac{15}{25} = 0.6$$

A table showing all 6 possible values of X and their probabilities is called the **probability distribution** of the random variable X.

Probability Distribution of the Random Variable X

X	$P(X)$
0	0.12
1	0.20
2	0.28
3	0.24
4	0.12
5	0.04

Note that the probabilities are simply the relative frequencies of the 6 possible values expressed as decimals instead of percents. The following statement indicates a fundamental link between data distributions and probability distributions.

For a random variable that represents a randomly chosen value from a distribution of data, the probability distribution of the random variable is the same as the relative frequency distribution of the data.

Because the probability distribution and the relative frequency distribution are essentially the same, the probability distribution can be represented by a histogram. Also, all of the descriptive statistics—such as mean, median, and standard deviation—that apply to the distribution of data also apply to the probability distribution. For example, we say that the probability distribution above has a mean of 2.16, a median of 2, and a standard deviation of about 1.3, since the 25 data values have these statistics, as you can check.

These statistics are similarly defined for the random variable X above. Thus, we would say that the **mean of the random variable *X*** is 2.16. Another name for the mean of a random variable is **expected value**. So we would also say that the expected value of X is 2.16. Note that the mean of X can be expressed in terms of probabilities as follows.

$$\begin{aligned}\frac{0(3)+1(5)+2(7)+3(6)+4(3)+5(1)}{25} &= 0\left(\frac{3}{25}\right)+1\left(\frac{5}{25}\right)+2\left(\frac{7}{25}\right)+3\left(\frac{6}{25}\right)+4\left(\frac{3}{25}\right)+5\left(\frac{1}{25}\right)\\ &= 0P(0)+1P(1)+2P(2)+3P(3)+4P(4)+5P(5)\end{aligned}$$

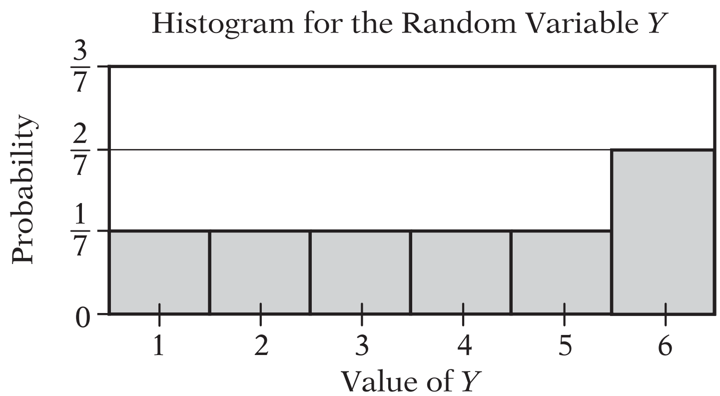
which is the sum of the products $XP(X)$, that is, the sum of each value of X multiplied by its corresponding probability $P(X)$.

The preceding example involves a common type of random variable—one that represents a randomly chosen value from a distribution of data. However, the concept of a random variable is more general. A random variable can be any quantity whose value is the result of a random experiment. The possible values of the random variable are the same as the outcomes of the experiment. So any random experiment with numerical outcomes naturally has a random variable associated with it, as in the following example.

Example 4.5.3: Let Y represent the outcome of the experiment in example 4.4.5 of rolling a weighted 6-sided die. Then Y is a random variable with 6 possible values, the numbers 1 through 6. Each value of Y has a probability, which is listed in the probability distribution of the random variable Y and is shown in a histogram for Y.

Probability Distribution of the Random Variable Y

Y	1	2	3	4	5	6
$P(Y)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$



The mean, or expected value, of Y can be computed as

$$\begin{aligned} P(1) + 2P(2) + 3P(3) + 4P(4) + 5P(5) + 6P(6) &= \left(\frac{1}{7}\right) + 2\left(\frac{1}{7}\right) + 3\left(\frac{1}{7}\right) + 4\left(\frac{1}{7}\right) + 5\left(\frac{1}{7}\right) + 6\left(\frac{2}{7}\right) \\ &= \frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{4}{7} + \frac{5}{7} + \frac{12}{7} \\ &= \frac{27}{7} \approx 3.86 \end{aligned}$$

Both of the random variables X and Y above are examples of **discrete random variables** because their values consist of discrete points on a number line.

A basic fact about probability from section 4.4 is that the sum of the probabilities of all possible outcomes of an experiment is 1, which can be confirmed by adding all of the probabilities in each of the probability distributions for the random variables X and Y above. Also, the sum of the areas of the bars in a histogram for the probability distribution of a random variable is 1. This fact is related to a fundamental link between the areas of the bars of a histogram and the probabilities of a discrete random variable.

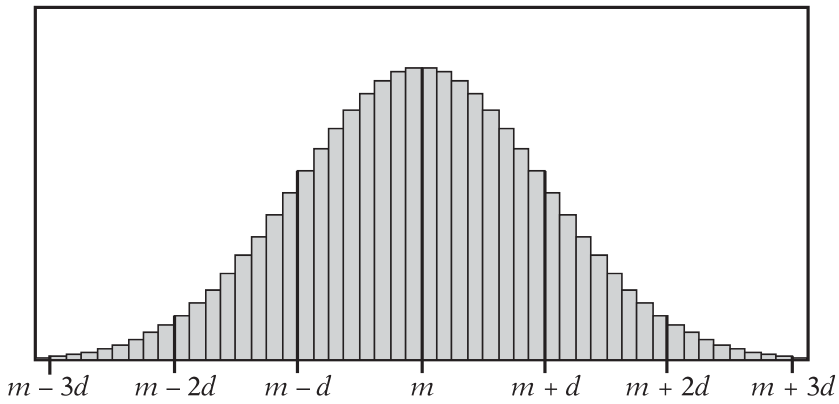
In the histogram for a random variable, the area of each bar is proportional to the probability represented by the bar.

If the die in example 4.4.5 were a fair die instead of weighted, then the probability of each of the outcomes would be $\frac{1}{6}$, and consequently, each of the bars in the histogram would have the same height. Such a flat histogram indicates a **uniform distribution**, since the probability is distributed uniformly over all possible outcomes.

The Normal Distribution

Many natural processes yield data that have a relative frequency distribution shaped somewhat like a bell, as in the distribution below with mean m and standard deviation d.

Approximately Normal Relative Frequency Distribution



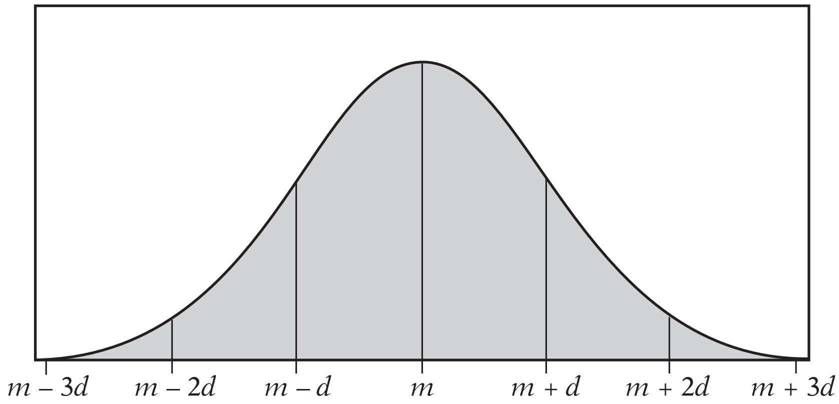
Such data are said to be **approximately normally distributed** and have the following properties.

- The mean, median, and mode are all nearly equal.
- The data are grouped fairly symmetrically about the mean.
- About two-thirds of the data are within 1 standard deviation of the mean.
- Almost all of the data are within 2 standard deviations of the mean.

As stated above, you can always associate a random variable X with a distribution of data by letting X be a randomly chosen value from the distribution. If X is such a random variable for the distribution above, we say that X is approximately normally distributed.

As described in example 4.5.1, relative frequency distributions are often approximated using a smooth curve—a distribution curve or density curve—for the tops of the bars in the histogram. The region below such a curve represents a distribution, called a **continuous probability distribution**. There are many different continuous probability distributions, but the most important one is the **normal distribution**, which has a bell-shaped curve like the one shown in the figure below.

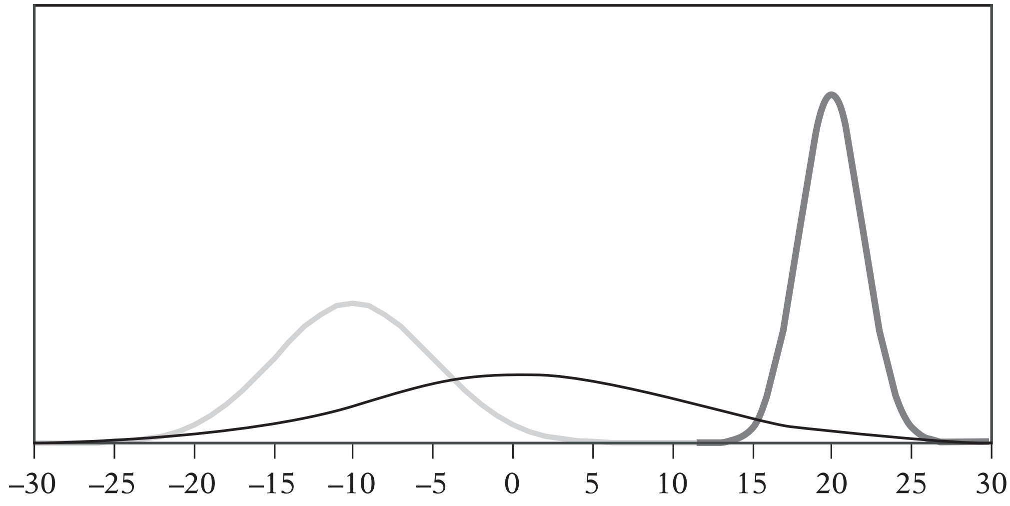
Normal Distribution



Just as a data distribution has a mean and standard deviation, the normal probability distribution has a mean and standard deviation. Also, the properties listed above for the approximately normal distribution of data hold for the normal distribution, except that the mean, median, and mode are exactly the same and the distribution is perfectly symmetric about the mean.

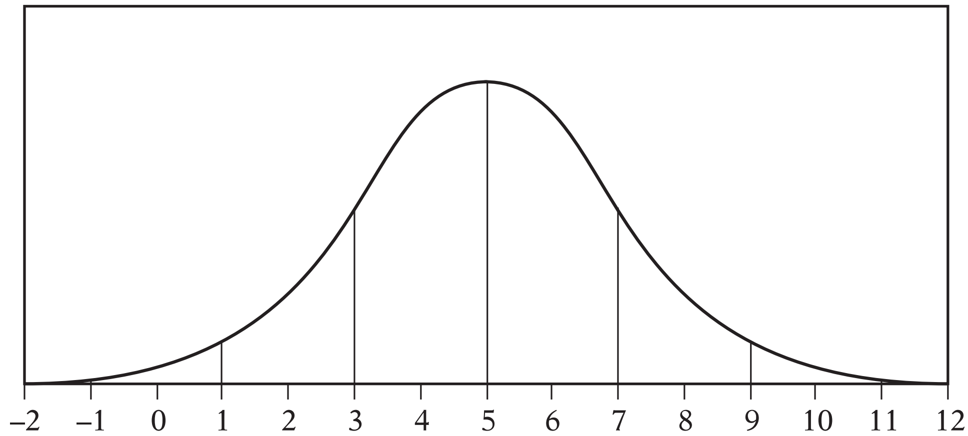
A normal distribution, though always shaped like a bell, can be centered around any mean and can be spread out to a greater or lesser degree, depending on the standard deviation. Below are three normal distributions that have different centers and spreads. From left to right, the means of the three distributions are -10, 1, and 20; and the standard deviations are 5, 10, and 2.

Three Normal Distributions



As mentioned earlier, areas of the bars in a histogram for a discrete random variable correspond to probabilities for the values of the random variable; the sum of the areas is 1 and the sum of the probabilities is 1. This is also true for a continuous probability distribution: the area of the region under the curve is 1, and the areas of vertical slices of the region—similar to the bars of a histogram—are equal to probabilities of a random variable associated with the distribution. Such a random variable is called a **continuous random variable**, and it plays the same role as a random variable that represents a randomly chosen value from a distribution of data. The main difference is that we seldom consider the event in which a continuous random variable is equal to a single value like $X = 3$; rather, we consider events that are described by intervals of values such as $1 < X < 3$ and $X > 10$. Such events correspond to vertical slices under a continuous probability distribution, and the areas of the vertical slices are the probabilities of the corresponding events. (Consequently, the probability of an event such as $X = 3$ would correspond to the area of a line segment, which is 0.)

Example 4.5.4: If W is a random variable that is normally distributed with a mean of 5 and a standard deviation of 2, what is $P(W > 5)$? Approximately what is $P(3 < W < 7)$? Which of the four numbers 0.5, 0.1, 0.05, or 0.01 is the best estimate of $P(W < -1)$?



Solution: Since the mean of the distribution is 5, and the distribution is symmetric about the mean, the event $W > 5$ corresponds to exactly half of the area under the normal distribution. So $P(W > 5) = \frac{1}{2}$.

For the event $3 < W < 7$, note that since the standard deviation of the distribution is 2, the values 3 and 7 are one standard deviation below and above the mean, respectively. Since about two-thirds of the area is within one standard deviation of the mean, $P(3 < W < 7)$ is approximately $\frac{2}{3}$.

For the event $W < -1$, note that -1 is 3 standard deviations below the mean. Since the graph makes it fairly clear that the area of the region under the normal curve to the left of -1 is much less than 5 percent of all of the area, the best of the four estimates given for $P(W < -1)$ is 0.01.

The **standard normal distribution** is a normal distribution with a mean of 0 and standard deviation equal to 1. To transform a normal distribution with a mean of m and a standard deviation of d to a standard normal distribution, you standardize the values (as explained below example 4.2.9); that is, you subtract m from any observed value of the normal distribution and then divide the result by d.

Very precise values for probabilities associated with normal distributions can be computed using calculators, computers, or statistical tables for the standard normal distribution. For example, more precise values for $P(3 < W < 7)$ and $P(W < -1)$ are 0.683 and 0.0013. Such calculations are beyond the scope of this review.

4.6 Data Interpretation Examples

Example 4.6.1:

DISTRIBUTION OF CUSTOMER COMPLAINTS
RECEIVED BY AIRLINE *P*, 2003 AND 2004

Category	2003	2004
Flight problem	20.0%	22.1%
Baggage	18.3	21.8
Customer service	13.1	11.3
Oversales of seats	10.5	11.8
Refund problem	10.1	8.1
Fare	6.4	6.0
Reservation and ticketing	5.8	5.6
Tours	3.3	2.3
Smoking	3.2	2.9
Advertising	1.2	1.1
Credit	1.0	0.8
Special passenger accommodation	0.9	0.9
Other	6.2	5.3
Total	100.0%	100.0%
Total number of complaints	22,998	13,278

(a) Approximately how many complaints concerning credit were received by Airline *P* in 2003?

(b) By approximately what percent did the total number of complaints decrease from 2003 to 2004?

(c) Based on the information in the table, which of the following statements are true?

I. In each of the years 2003 and 2004, complaints about flight problems, baggage, and customer service together accounted for more than 50 percent of all customer complaints received by Airline *P*.

II. The number of special passenger accommodation complaints was unchanged from 2003 to 2004.

III. From 2003 to 2004, the number of flight problem complaints increased by more than 2 percent.

Solutions:

(a) According to the table, in 2003, 1 percent of the total number of complaints concerned credit. Therefore, the number of complaints concerning credit is equal to 1 percent of 22,998. By converting 1 percent to its decimal equivalent, you obtain that the number of complaints in 2003 is equal to (0.01)(22,998), or about 230.

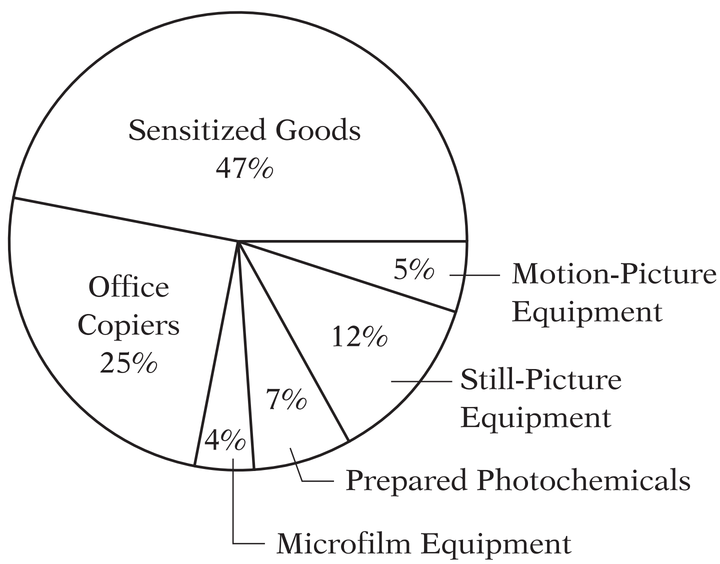
(b) The decrease in the total number of complaints from 2003 to 2004 was 22,998 − 13,278, or 9,720. Therefore, the percent decrease was $\left(\frac{9{,}720}{22{,}998}\right)(100\%)$, or approximately 42 percent.

(c) Since 20.0 + 18.3 + 13.1 and 22.1 + 21.8 + 11.3 are both greater than 50, statement I is true. For statement II, the *percent* of special passenger accommodation complaints *did* remain the same from 2003 to 2004, but the *number* of such complaints decreased because the total number of complaints decreased. Thus, statement II is false. For statement III, the *percents* shown in the table for flight problems do in fact increase by more than 2 percentage points, but the bases of the percents are different. The total number of complaints in 2004 was much lower than the total number of complaints in 2003, and clearly 20 percent of 22,998 is greater than 22.1 percent of 13,278. So, the number of flight problem complaints actually decreased from 2003 to 2004, and statement III is false.

Example 4.6.2:

UNITED STATES PRODUCTION OF PHOTOGRAPHIC
EQUIPMENT AND SUPPLIES IN 1971

Total: $3,980 million



(a) Approximately what was the ratio of the value of sensitized goods to the value of still-picture equipment produced in 1971 in the United States?

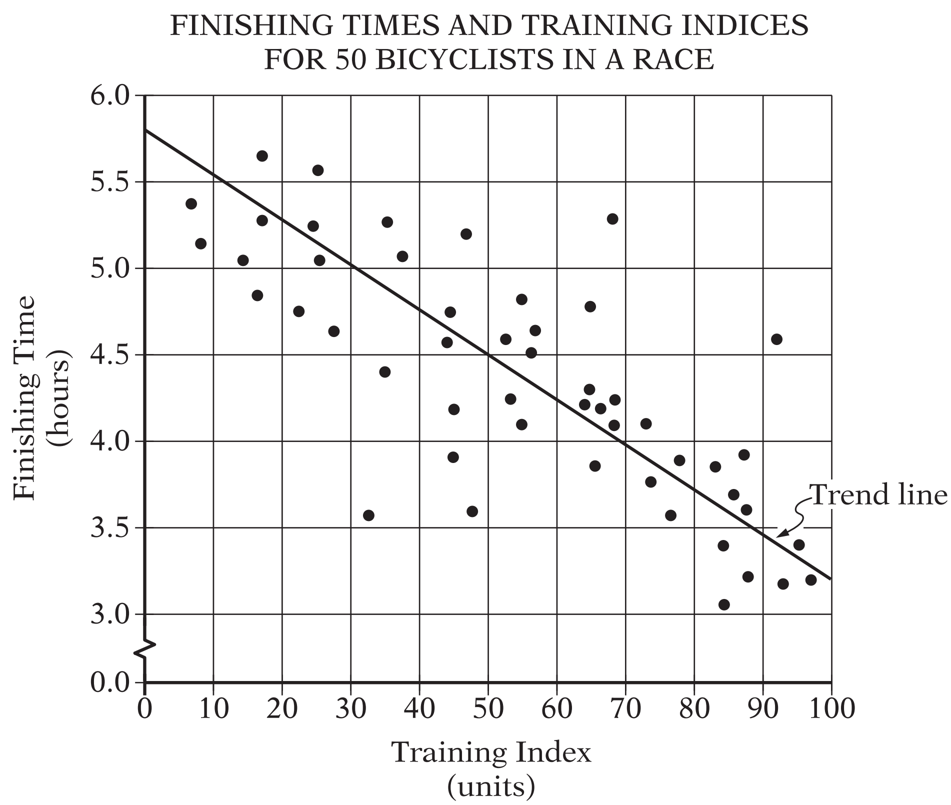
(b) If the value of office copiers produced in 1971 was 30 percent greater than the corresponding value in 1970, what was the value of office copiers produced in 1970 ?

Solutions:

(a) The ratio of the value of sensitized goods to the value of still-picture equipment is equal to the ratio of the corresponding percents shown because the percents have the same base, which is the total value. Therefore, the ratio is 47 to 12, or approximately 4 to 1.

(b) The value of office copiers produced in 1971 was 0.25 times \$3,980 million, or \$995 million. Therefore, if the corresponding value in 1970 was x million dollars, then $1.3x = 995$ million. Solving for x yields $x = \frac{995}{1.3} \approx 765$, so the value of office copiers produced in 1970 was approximately \$765 million.

Example 4.6.3:



A bicycle trainer studied 50 bicyclists to examine how the finishing time for a certain bicycle race was related to the amount of physical training in the three months before the race. To measure the amount of training, the trainer developed a training index, measured in "units" and based on the intensity of each bicyclist's training. The data and the trend of the data, represented by a line, are displayed in the scatterplot above.

(a) How many of the 50 bicyclists had both a training index less than 50 units and a finishing time less than 4.5 hours?

(b) What percent of the 10 fastest bicyclists in the race had a training index less than 90 units?

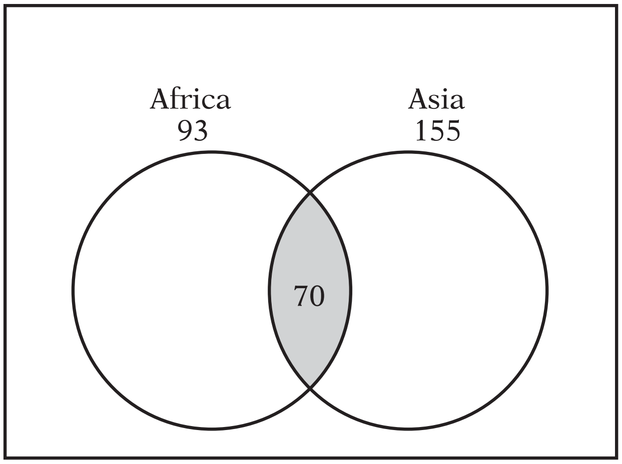
Solutions:

(a) The number of bicyclists who had both a training index less than 50 units and a finishing time less than 4.5 hours is equal to the number of points on the graph to the left of 50 and below 4.5. Since there are five data points that are both to the left of 50 units and below 4.5 hours, the correct answer is five.

(b) The 10 lowest data points represent the 10 fastest bicyclists. Of these 10 data points, 3 points are to the right of 90 units, so the number of points to the left of 90 units is 7, which represents 70 percent of the 10 fastest bicyclists.

Example 4.6.4:

TRAVELERS SURVEYED: 250



In a survey of 250 European travelers, 93 have traveled to Africa, 155 have traveled to Asia, and of these two groups, 70 have traveled to both continents, as illustrated in the Venn diagram above.

(a) How many of the travelers surveyed have traveled to Africa but not to Asia?

(b) How many of the travelers surveyed have traveled to at least one of the two continents of Africa and Asia?

(c) How many of the travelers surveyed have traveled neither to Africa nor to Asia?

Solutions: In the Venn diagram, the rectangular region represents the set of all travelers surveyed; the two circular regions represent the two sets of travelers to Africa and Asia, respectively; and the shaded region represents the subset of those who have traveled to both continents.

(a) The set described here is represented by *the part of the left circle that is not shaded*. This description suggests that the answer can be found by taking the shaded part away from the first circle—in effect, subtracting the 70 from the 93, to get 23 travelers who have traveled to Africa but not to Asia.

(b) The set described here is represented by that part of the rectangle that is *in at least one of the two circles*. This description suggests adding the two numbers 93 and 155. But the 70 travelers who have traveled to both continents would be counted twice in the sum 93 + 155. To correct the double counting, subtract 70 from the sum so that these 70 travelers are counted only once:

$$93 + 155 - 70 = 178$$

(c) The set described here is represented by the part of the rectangle that is *not in either circle*. Let N be the number of these travelers. Note that the entire rectangular region has two main nonoverlapping parts: the part *outside* the circles and the part *inside* the circles. The first part represents N travelers and the second part represents $93 + 155 - 70 = 178$ travelers (from question (b)). Therefore,

$$250 = N + 178$$

and solving for N yields

$$N = 250 - 178 = 72$$

DATA ANALYSIS EXERCISES

1. The daily temperatures, in degrees Fahrenheit, for 10 days in May were 61, 62, 65, 65, 65, 68, 74, 74, 75, and 77.
   (a) Find the mean, median, mode, and range of the temperatures.
   (b) If each day had been 7 degrees warmer, what would have been the mean, median, mode, and range of those 10 temperatures?

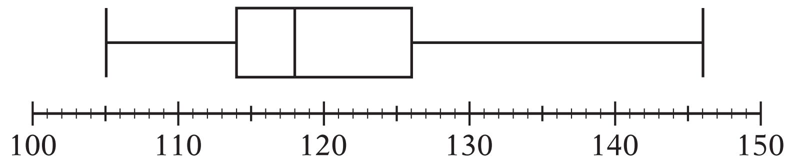
2. The numbers of passengers on 9 airline flights were 22, 33, 21, 28, 22, 31, 44, 50, and 19. The standard deviation of these 9 numbers is approximately equal to 10.2.
   (a) Find the mean, median, mode, range, and interquartile range of the 9 numbers.
   (b) If each flight had had 3 times as many passengers, what would have been the mean, median, mode, range, interquartile range, and standard deviation of the 9 numbers?
   (c) If each flight had had 2 fewer passengers, what would have been the interquartile range and standard deviation of the 9 numbers?

3. A group of 20 values has a mean of 85 and a median of 80. A different group of 30 values has a mean of 75 and a median of 72.
   (a) What is the mean of the 50 values?
   (b) What is the median of the 50 values?

4. Find the mean and median of the values of the random variable X, whose relative frequency distribution is given in the table below.

X	Relative Frequency
0	0.18
1	0.33
2	0.10
3	0.06
4	0.33

5. Eight hundred insects were weighed, and the resulting measurements, in milligrams, are summarized in the boxplot below.



(a) What are the range, the three quartiles, and the interquartile range of the measurements?
(b) If the 80th percentile of the measurements is 130 milligrams, about how many measurements are between 126 milligrams and 130 milligrams?

6. In how many different ways can the letters in the word STUDY be ordered?

7. Martha invited 4 friends to go with her to the movies. There are 120 different ways in which they can sit together in a row of 5 seats, one person per seat. In how many of those ways is Martha sitting in the middle seat?

8. How many 3-digit positive integers are odd and do not contain the digit 5 ?

9. From a box of 10 lightbulbs, you are to remove 4. How many different sets of 4 lightbulbs could you remove?

10. A talent contest has 8 contestants. Judges must award prizes for first, second, and third places, with no ties.
(a) In how many different ways can the judges award the 3 prizes?
(b) How many different groups of 3 people can get prizes?

11. If an integer is randomly selected from all positive 2-digit integers, what is the probability that the integer chosen has
(a) a 4 in the tens place?
(b) at least one 4 in the tens place or the units place?
(c) no 4 in either place?

12. In a box of 10 electrical parts, 2 are defective.
(a) If you choose one part at random from the box, what is the probability that it is not defective?
(b) If you choose two parts at random from the box, without replacement, what is the probability that both are defective?

13. The table below shows the distribution of a group of 40 college students by gender and class.

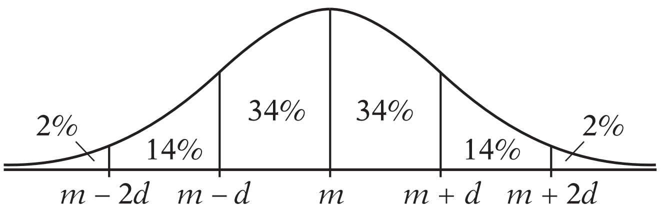
	Sophomores	Juniors	Seniors
Males	6	10	2
Females	10	9	3

If one student is randomly selected from this group, find the probability that the student chosen is
(a) not a junior
(b) a female or a sophomore
(c) a male sophomore or a female senior

14. Let A, B, C, and D be events for which $P(A \text{ or } B) = 0.6$, $P(A) = 0.2$, $P(C \text{ or } D) = 0.6$, and $P(C) = 0.5$. The events A and B are mutually exclusive, and the events C and D are independent.
(a) Find $P(B)$
(b) Find $P(D)$

15. Lin and Mark each attempt independently to decode a message. If the probability that Lin will decode the message is 0.80 and the probability that Mark will decode the message is 0.70, find the probability that
(a) both will decode the message
(b) at least one of them will decode the message
(c) neither of them will decode the message

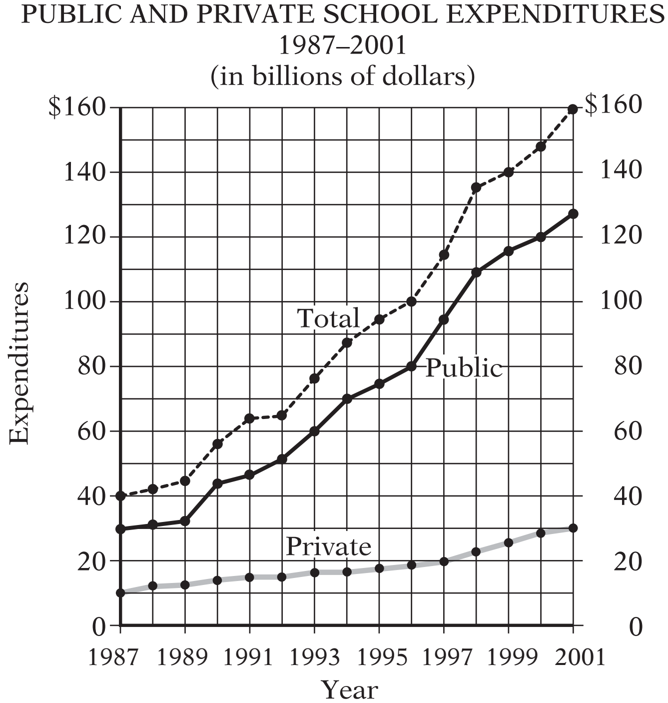
16.



The figure above shows a normal distribution with mean m and standard deviation d, including approximate percents of the distribution corresponding to the six regions shown.

Suppose the heights of a population of 3,000 adult penguins are approximately normally distributed with a mean of 65 centimeters and a standard deviation of 5 centimeters.
(a) Approximately how many of the adult penguins are between 65 centimeters and 75 centimeters tall?
(b) If an adult penguin is chosen at random from the population, approximately what is the probability that the penguin's height will be less than 60 centimeters? Give your answer to the nearest 0.05.

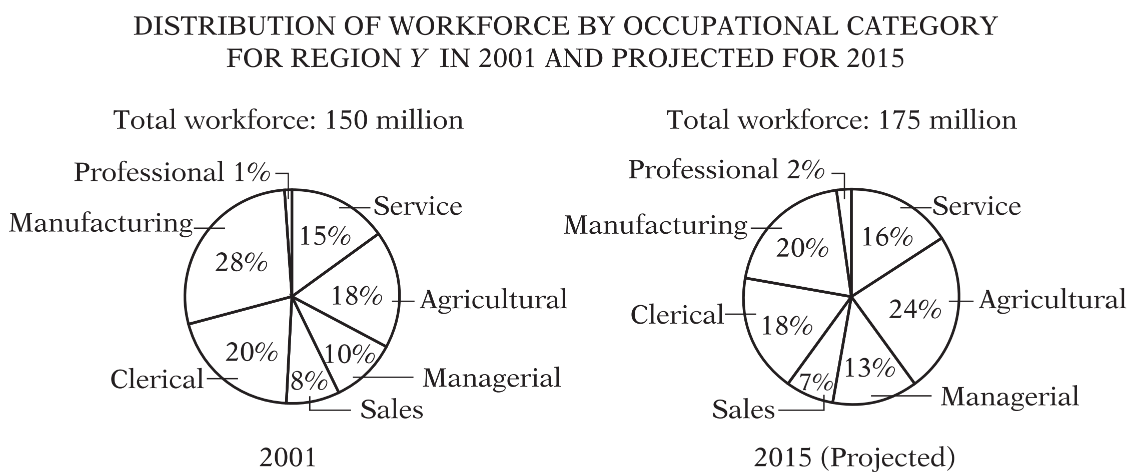
17.



(a) For which year did total expenditures increase the most from the year before?

(b) For 2001, private school expenditures were approximately what percent of total expenditures?

18.



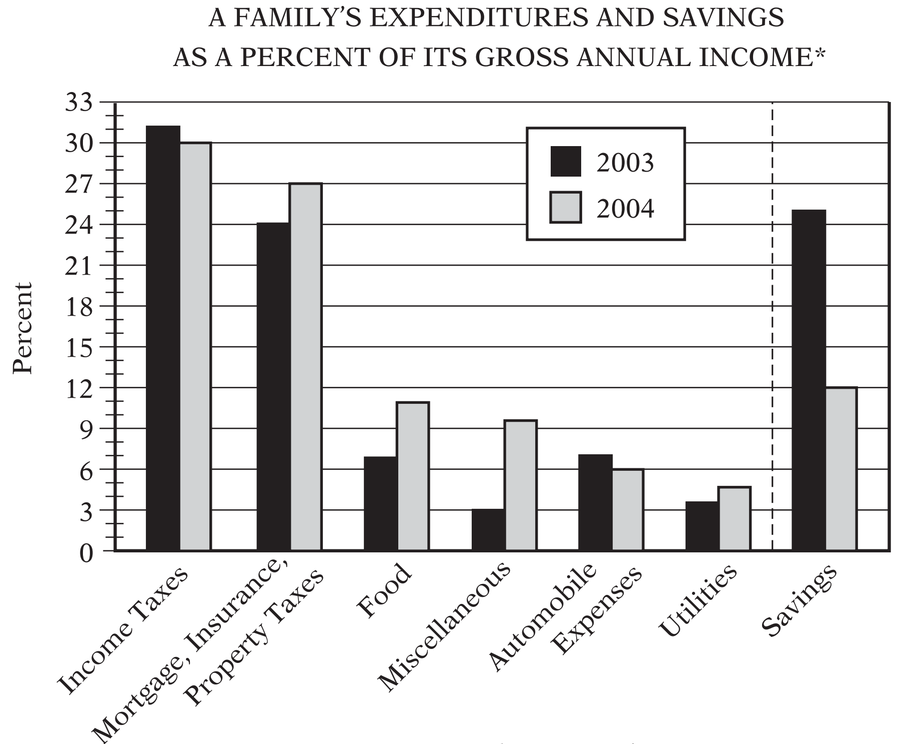
(a) In 2001, how many categories each comprised more than 25 million workers?

(b) What is the ratio of the number of workers in the Agricultural category in 2001 to the projected number of such workers in 2015 ?

(c) From 2001 to 2015, there is a projected increase in the number of workers in which of the following categories?

I. Sales II. Service III. Clerical

19.



*2003 Gross annual income: \$50,000
2004 Gross annual income: \$45,000

(a) In 2003 the family used a total of 49 percent of its gross annual income for two of the categories listed. What was the total amount of the family's income used for those same categories in 2004?

(b) Of the seven categories listed, which category of expenditure had the greatest percent increase from 2003 to 2004 ?

ANSWERS TO DATA ANALYSIS EXERCISES

1. In degrees Fahrenheit, the statistics are
   (a) mean = 68.6, median = 66.5, mode = 65, range = 16
   (b) mean = 75.6, median = 73.5, mode = 72, range = 16

2. (a) mean = 30, median = 28, mode = 22, range = 31, interquartile range = 17
   (b) mean = 90, median = 84, mode = 66, range = 93, interquartile range = 51,
   standard deviation = $3\sqrt{\frac{940}{9}} \approx 30.7$
   (c) interquartile range = 17, standard deviation ≈ 10.2

3. (a) mean = 79
   (b) The median cannot be determined from the information given.

4. mean = 2.03, median = 1

5. (a) range = 41, $Q_1 = 114$, $Q_2 = 118$, $Q_3 = 126$, interquartile range = 12
   (b) 40 measurements

6. 5! = 120
7. 24
8. 288
9. 210
10. (a) 336 (b) 56
11. (a) $\frac{1}{9}$ (b) $\frac{1}{5}$ (c) $\frac{4}{5}$
12. (a) $\frac{4}{5}$ (b) $\frac{1}{45}$
13. (a) $\frac{21}{40}$ (b) $\frac{7}{10}$ (c) $\frac{9}{40}$
14. (a) 0.4 (b) 0.2
15. (a) 0.56 (b) 0.94 (c) 0.06
16. (a) 1,440 (b) 0.15
17. (a) 1998 (b) 19%
18. (a) Three (b) 9 to 14, or $\frac{9}{14}$ (c) I, II, and III
19. (a) $17,550 (b) Miscellaneous