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19TH EDITION

Sharon Weiner Green, M.A. and Ira K. Wolf, Ph.D.

- Fully updated to reflect the NEW GRE
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- Everything you need to know about the new numeric and multiple-answer question types
- Screen shots of the computer interface you will encounter on test day
- Brand new strategies for each of the new GRE question types

PART 4

QUANTITATIVE ABILITY: TACTICS, STRATEGIES, PRACTICE, AND REVIEW

Introduction to Part 4

Part 4 consists of five chapters. Chapter 7 presents several important strategies that can be used on any mathematics questions that appear on the GRE. In Chapters 8, 9, and 10 you will find tactics that are specific to one of the three different types of questions: discrete quantitative questions, quantitative comparison questions, and data interpretation questions, respectively. Chapter 11 contains a complete review of all the mathematics you need to know in order to do well on the GRE, as well as hundreds of sample problems patterned on actual test questions.

FIVE TYPES OF TACTICS

Five different types of tactics are discussed in this book.

1. In Chapters 1 and 2, you learned many basic tactics used by all good test-takers, such as read each question carefully, pace yourself, don't get bogged down on any one question, and never waste time reading the directions. You also learned the specific tactics required to excel on a computerized test. These tactics apply to both the verbal and quantitative sections of the GRE.
2. In Chapters 4 and 5 you learned the important tactics needed for handling each type of verbal question.
3. In Chapter 6 you learned the strategies for planning and writing the two essays that constitute the analytical writing section of the GRE.

4. In Chapters 7–10 you will find all of the tactics that apply to the quantitative sections of the GRE. Chapter 7 contains those techniques that can be applied to all types of mathematics questions; Chapters 8, 9, and 10 present specific strategies to deal with each of the three kinds of quantitative questions found on the GRE: discrete quantitative questions, quantitative comparison questions, and data interpretation questions.
5. In Chapter 11 you will learn or review all of the mathematics that is needed for the GRE, and you will master the specific tactics and key facts that apply to each of the different mathematical topics.

Using these tactics will enable you to answer more quickly many questions that you already know how to do. But the greatest value of these tactics is that they will allow you to correctly answer or make educated guesses on problems that *you do not know how to do.*

WHEN TO STUDY CHAPTER 11

How much time you initially devote to Chapter 11 should depend on how good your math reasoning skills are, how long it has been since you studied math, and how much of the math you learned in middle school and the first two years of high school you remember. If you think that your math skills are quite good, you can initially skip the instructional parts of Chapter 11. If, however, after doing the Model Tests in Part 5 of this book, you find that you made more than one or two mistakes on questions involving the same topic (averages, percents, geometry, etc.) or you spent too much time on them, you should then study the appropriate sections of Chapter 11. Even if your math skills are excellent, you should do the exercises in Chapter 11; they are a good source of additional GRE questions. If your math skills were never very good or if you feel they are rusty, it is advisable to review the material in Chapter 11, including working out the problems, *before* tackling the Model Tests.

AN IMPORTANT SYMBOL

Throughout the rest of this book, the symbol "$\Rightarrow$" is used to indicate that one step in the solution of a problem follows *immediately* from the preceding one, and no explanation is necessary. You should read

$3x = 12 \Rightarrow x = 4$

as $3x = 12$ *implies that* $x = 4$

or $3x = 12$, *which implies that* $x = 4$

or *since* $3x = 12$, *then* $x = 4$.

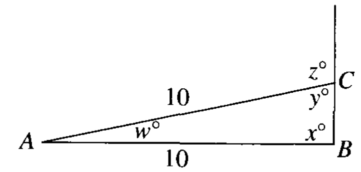
Here is a sample solution to the following problem using $\Rightarrow$:

What is the value of $2x^2 - 5$ when $x = -4$?

$$x = -4 \Rightarrow x^2 = (-4)^2 = 16 \Rightarrow 2x^2 = 2(16) = 32 \Rightarrow 2x^2 - 5 = 32 - 5 = \mathbf{27}$$

When the reason for a step is not obvious, $\Rightarrow$ is not used: rather, an explanation is given, often including a reference to a KEY FACT from Chapter 11. In many solutions, some steps are explained, while others are linked by the $\Rightarrow$ symbol, as in the following example.

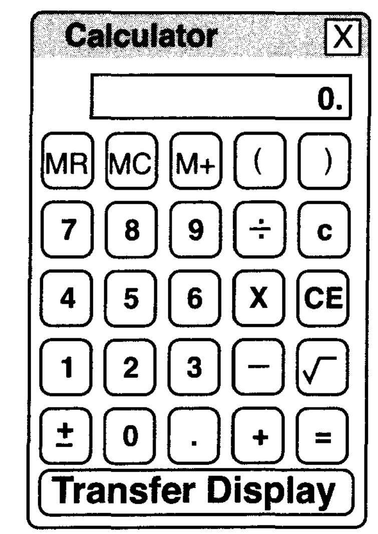
In the diagram below, if $w = 10$, what is the value of z?



- By KEY FACT J1, $w + x + y = 180$.
- Since $\triangle ABC$ is isosceles, $x = y$ (KEY FACT J5).
- Therefore, $w + 2y = 180 \Rightarrow 10 + 2y = 180 \Rightarrow 2y = 170 \Rightarrow y = 85$.
- Finally, since $y + z = 180$ (KEY FACT I3), $85 + z = 180 \Rightarrow z = \mathbf{95}$.

CALCULATORS ON THE GRE

You may *not* bring your own calculator to use when you take the GRE. However, starting in 2011, for the first time ever, you will have access to an onscreen calculator. While you are working on the math sections, one of the icons at the top of the screen will be a calculator icon. During the verbal and writing sections of the test, either that icon will be greyed out (meaning that you can't click on it) or it will simply not be there at all. During the math sections, however, you will be able to click on that icon at anytime; when you do, a calculator will instantly appear on the screen. Clicking the X in the upper-right-hand corner of the calculator will hide it.



Note that when the calculator appears on the screen, it may cover part of the question or the answer choices. If this occurs, just click on the top of the calculator and drag it to a convenient location. If you use the calculator to answer a question and then click NEXT to go to the next question, the calculator remains on the screen, exactly where it was, with the same numerical readout. This is actually a distraction. So, if you do use the calculator to answer a question, as soon as you have answered that question, click on the X to remove the calculator from the screen. Later, it takes only one click to get it back.

The onscreen calculator is a simple four-function calculator, with a square root key. It is not a graphing calculator; it is not a scientific calculator. The only operations you can perform with the onscreen calculator are adding, subtracting, multiplying, dividing, and taking square roots. Fortunately, these are the only operations you will ever need to answer any GRE question.

At the bottom of the onscreen calculator is a bar labeled TRANSFER DISPLAY. If you are using the calculator on a numeric entry question, and the result of your final calculation is the answer that you want to enter in the box, click on TRANSFER DISPLAY —the number currently displayed in the calculator's readout will instantly appear in the box under the question. This saves the few seconds that it would otherwise take to enter your answer; more important, it guarantees that you won't make an error typing in your answer.

Remember

Use your calculator only when you need to.

Just because you have a calculator at your disposal does not mean that you should use it very much. In fact, you shouldn't. The vast majority of questions that appear on the GRE do not require any calculations.

General Math Strategies

CHAPTER 7

In Chapters 8 and 9, you will learn tactics that are specifically applicable to discrete quantitative questions and quantitative comparison questions, respectively. In this chapter you will learn several important general math strategies that can be used on both of these types of questions.

The directions that appear on the screen at the beginning of the quantitative sections include the following cautionary information:

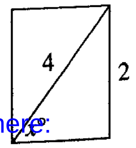
> Figures that accompany questions are intended to provide information useful in answering the questions.
>
> However, unless a note states that a figure is drawn to scale, you should solve these problems NOT by estimating sizes by sight or measurement, but by using your knowledge of mathematics.

Despite the fact that they are telling you that you cannot totally rely on *their* diagrams, if you learn how to draw diagrams accurately, *you can trust the ones you draw.* Knowing the best ways of handling diagrams on the GRE is critically important. Consequently, the first five tactics all deal with diagrams.

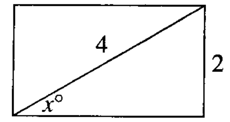
TACTIC 1.	**Draw a diagram.**
TACTIC 2.	**Trust a diagram that has been drawn to scale.**
TACTIC 3.	**Exaggerate or change a diagram.**
TACTIC 4.	**Add a line to a diagram.**
TACTIC 5.	**Subtract to find shaded regions.**

To implement these tactics, you need to be able to draw line segments and angles accurately, and you need to be able to look at segments and angles and accurately estimate their measures. Let's look at three variations of the same problem.

1. If the diagonal of a rectangle is twice as long as the shorter side, what is the degree measure of the angle it makes with the longer side?
2. In the rectangle below, what is the value of x?



3. In the rectangle below, what is the value of x?



For the moment, let's ignore the correct mathematical way of solving this problem. In the diagram in (3), the side labeled 2 appears to be half as long as the diagonal, which is labeled 4; consequently, you should assume that the diagram has been drawn to scale, and you should see that x is about 30, *certainly* between 25 and 35. In (1) you aren't given a diagram, and in (2) the diagram is useless because you can see that it has not been drawn to scale (the side labeled 2 is nearly as long as the diagonal, which is labeled 4). However, if while taking the GRE, you see a question such as (1) or (2), you should be able to quickly draw on your scrap paper a diagram that looks just like the one in (3), and then look at *your* diagram and see that the measure of x is just about 30. If the answer choices for these questions were

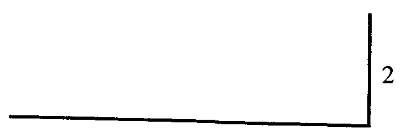
Ⓐ 15 Ⓑ 30 Ⓒ 45 Ⓓ 60 Ⓔ 75

you would, of course, choose **30**, **B**. If the choices were

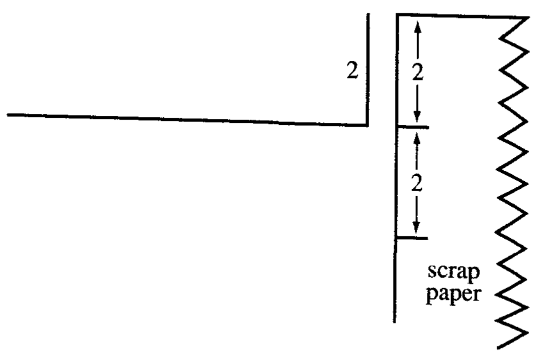
Ⓐ 20 Ⓑ 25 Ⓒ 30 Ⓓ 35 Ⓔ 40

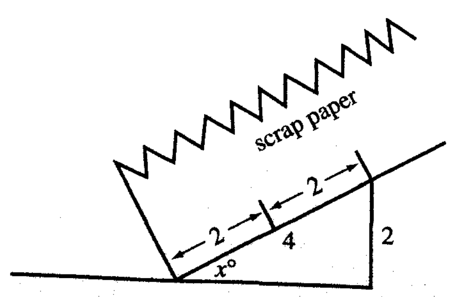
you might not be quite as confident, but you should still choose **30**, here **C**.

When you take the GRE, even though you are not allowed to have rulers or protractors, you should be able to draw your diagrams very accurately. For example, in (1) above, you should draw a horizontal line, and then, either freehand or by tracing the corner of a piece of scrap paper, draw a right angle on the line. The vertical line segment will be the width of the rectangle; label it 2.



Mark off that distance twice on a piece of scrap paper and use that to draw the diagonal.



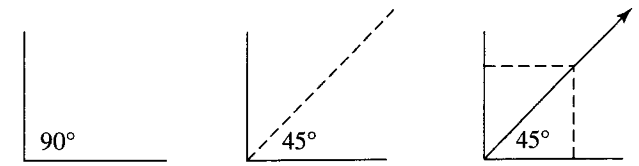


You should now have a diagram that is similar to that in (3), and you should be able to see that x is about 30.

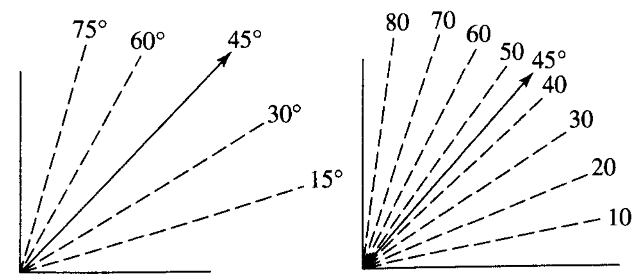
By the way, x is *exactly* 30. A right triangle in which one leg is half the hypotenuse must be a 30-60-90 triangle, and that leg is opposite the 30° angle [see KEY FACT J11].

Having drawn an accurate diagram, are you still unsure as to how you should know that the value of x is 30 just by looking at the diagram? You will now learn not only how to look at *any* angle and know its measure within 5 or 10 degrees, but how to draw any angle that accurately.

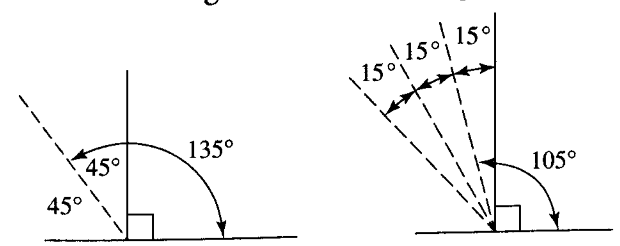
You should easily recognize a 90° angle and can probably draw one freehand; but you can always just trace the corner of a piece of scrap paper. To draw a 45° angle, just bisect a 90° angle. Again, you can probably do this freehand. If not, or to be even more accurate, draw a right angle, mark off the same distance on each side, draw a square, and then draw in the diagonal.



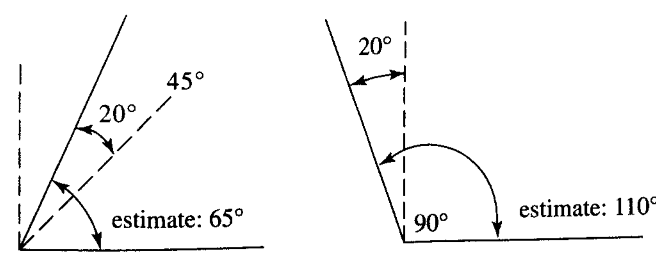
To draw other acute angles, just divide the two 45° angles in the above diagram with as many lines as necessary.

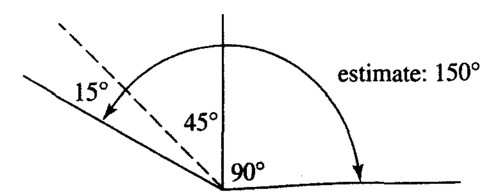


Finally, to draw an obtuse angle, add an acute angle to a right angle.

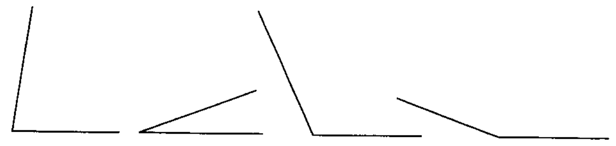


Now, to estimate the measure of a given angle, just draw in some lines.





To test yourself, find the measure of each angle shown. The answers are found below.



(a) (b) (c) (d)

Answers (a) 80° (b) 20° (c) 115° (d) 160°. Did you come within 10° on each one?

Testing Tactics

TACTIC

1

Draw a Diagram

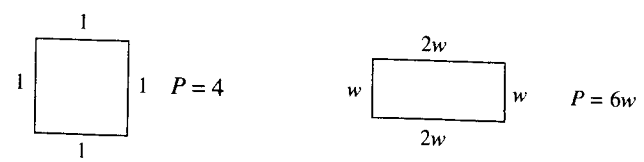
On *any* geometry question for which a figure is not provided, draw one (as accurately as possible) on your scrap paper — *never attempt a geometry problem without first drawing a diagram.*

EXAMPLE 1

What is the area of a rectangle whose length is twice its width and whose perimeter is equal to that of a square whose area is 1?

Ⓐ 1 Ⓑ 6 Ⓒ $\frac{2}{3}$ Ⓓ $\frac{4}{3}$ Ⓔ $\frac{8}{9}$

SOLUTION. Don't even think of answering this question until you have drawn a square and a rectangle and labeled each of them: each side of the square is 1, and if the width of the rectangle is w, its length (ℓ) is $2w$.



Now, write the required equation and solve it:

$$6w = 4 \Rightarrow w = \frac{4}{6} = \frac{2}{3} \Rightarrow 2w = \frac{4}{3}$$

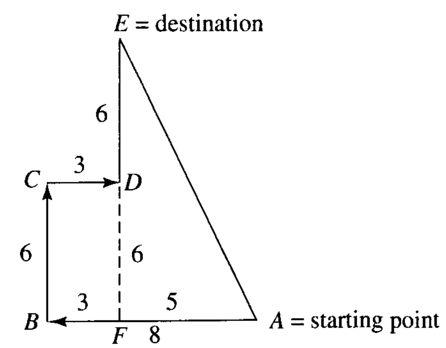
The area of the rectangle = $\ell w = \left(\frac{4}{3}\right)\left(\frac{2}{3}\right) = \frac{8}{9}$, **E.**

EXAMPLE 2

Betty drove 8 miles west, 6 miles north, 3 miles east, and 6 more miles north. How many miles was Betty from her starting place?

[ ] miles

SOLUTION. Draw a diagram showing Betty's route from A to B to C to D to E.



Now, extend line segment ED until it intersects AB at F. Then, AFE is a right triangle, whose legs are 5 and 12. The length of hypotenuse AE represents the distance from her starting point to her destination. Either recognize that $\triangle AFE$ is a 5-12-13 right triangle or use the Pythagorean theorem:

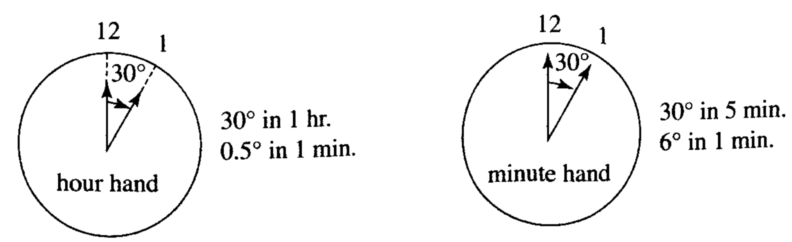
$$5^2 + 12^2 = (AE)^2 \Rightarrow (AE)^2 = 25 + 144 = 169 \Rightarrow AE = \mathbf{13}.$$

EXAMPLE 3

What is the difference in the degree measures of the angles formed by the hour hand and the minute hand of a clock at 12:35 and 12:36?

Ⓐ 1° Ⓑ 5° Ⓒ 5.5° Ⓓ 6° Ⓔ 30°

SOLUTION. Draw a simple picture of a clock. The hour hand makes a complete revolution, 360°, once every 12 hours. So, in 1 hour it goes through 360° ÷ 12 = 30°, and in one minute it advances through 30° ÷ 60 = 0.5°. The minute hand moves through 30° every 5 minutes or 6° per minute. So, in the minute from 12:35 to 12:36 (or any other minute), the *difference* between the hands increased by 6° – 0.5° = **5.5°**, **C.**



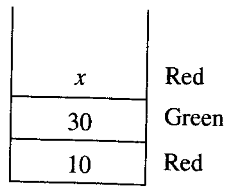
NOTE: It was not necessary, and would have been more time-consuming, to determine the angle between the hands at either 12:35 or 12:36. (See TACTIC 6: Don't do more than you have to.)

Drawings should not be limited to geometry questions; there are many other questions on which drawings will help.

EXAMPLE 4

A jar contains 10 red marbles and 30 green ones. How many red marbles must be added to the jar so that 60% of the marbles will be red?

SOLUTION. Let x represent the number of red marbles to be added, and draw a diagram and label it.



From the diagram it is clear that there are now $40 + x$ marbles in the jar, of which $10 + x$ are red. Since we want the fraction of red marbles to be 60%, we have $\frac{10+x}{40+x} = 60\% = \frac{60}{100} = \frac{3}{5}$. Cross-multiplying, we get:

$$5(10 + x) = 3(40 + x) \Rightarrow 50 + 5x = 120 + 3x \Rightarrow 2x = 70 \Rightarrow x = \mathbf{35}.$$

Of course, you could have set up the equation and solved it without the diagram, but the diagram makes the solution easier and you are less likely to make a careless mistake.

TACTIC 2 Trust a Diagram That Has Been Drawn to Scale

Whenever diagrams have been drawn to scale, they can be trusted. This means that you can look at the diagram and use your eyes to accurately estimate the sizes of angles and line segments. For example, in the first problem discussed at the beginning of this chapter, you could "see" that the measure of the angle was about 30°.

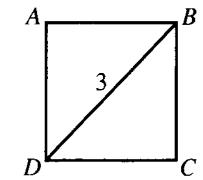
To take advantage of this situation:

- If a diagram is given that appears to be drawn to scale, trust it.
- If a diagram is given that has not been drawn to scale, try to draw it to scale on your scrap paper, and then trust it.
- When no diagram is provided, and you draw one on your scrap paper, try to draw it to scale.

In Example 5 below, we are told that $ABCD$ is a square and that diagonal BD is 3. In the diagram provided, quadrilateral $ABCD$ does indeed look like a square, and $BD = 3$ does not contradict any other information. We can, therefore, assume that the diagram has been drawn to scale.

EXAMPLE 5

In the figure at the right, diagonal BD of square $ABCD$ is 3. What is the perimeter of the square?



Ⓐ 4.5 Ⓑ 12 Ⓒ $3\sqrt{2}$ Ⓓ $6\sqrt{2}$ Ⓔ $12\sqrt{2}$

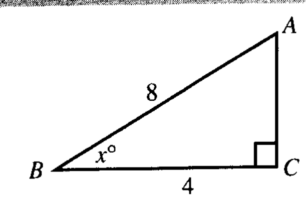
SOLUTION. Since this diagram has been drawn to scale, you can trust it. The sides of the square appear to be about two thirds as long as the diagonal, so assume that each side is about 2. Then the perimeter is about 8. Which of the choices is approximately 8? Certainly not A or B. Since $\sqrt{2} \approx 1.4$, Choices C, D, and E are approximately 4.2, 8.4, and 12.6, respectively. Clearly, the answer must be **D**.

Direct mathematical solution. Let s be a side of the square. Then since $\triangle BCD$ is a 45-45-90 right triangle, $s = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$, and the perimeter of the square is $4s = 4\left(\frac{3\sqrt{2}}{2}\right) = 6\sqrt{2}$.

Remember the goal of this book is to help you get credit for *all* the problems you know how to do, and, by using the TACTICS, to get credit for *many* that you don't know how to do. Example 5 is typical. Many students would miss this question. *You*, however, can now answer it correctly, even though you may not remember how to solve it directly.

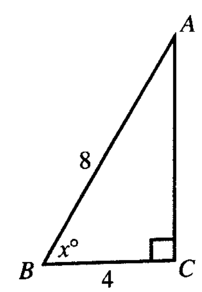
EXAMPLE 6

In $\triangle ABC$, what is the value of x?



Ⓐ 75 Ⓑ 60 Ⓒ 45 Ⓓ 30 Ⓔ 15

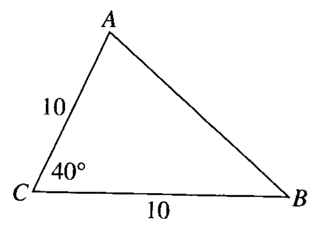
SOLUTION. If you don't see the correct mathematical solution, you should use TACTIC 2 and trust the diagram; but to do that you must be careful that when you copy it onto your scrap paper you *fix it*. What's wrong with the way it is drawn now? $AB = 8$ and $BC = 4$, but in the figure, AB and BC are almost the same length. Redraw it so that AB is *twice* as long as BC. Now, just look: x is about **60, B**.



In fact, x is exactly 60. If the hypotenuse of a right triangle is twice the length of one of the legs, then it's a 30-60-90 triangle, and the angle formed by the hypotenuse and that leg is 60° (see Section 11-J).

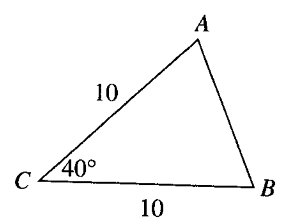
TACTIC 2 is equally effective on quantitative comparison questions that have diagrams. See pages 9–11 for directions on how to solve quantitative comparison questions.

EXAMPLE 7



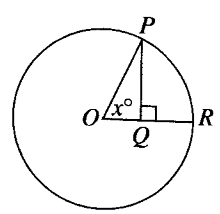
Quantity A	Quantity B
AB	10

SOLUTION. There are two things wrong with the given diagram: $\angle C$ is labeled 40°, but looks much more like 60° or 70°, and AC and BC are each labeled 10, but BC is drawn much longer. When you copy the diagram onto your scrap paper, be sure to correct these two mistakes: draw a triangle that has a 40° angle and two sides of the same length.



Now, it's clear: $AB < 10$. The answer is **B**.

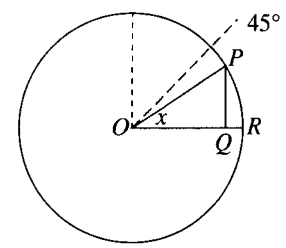
EXAMPLE 8



O is the center of the circle
$PQ = 6$
$OR = 12$

Quantity A	Quantity B
x	45

SOLUTION. In the diagram on page 154, the value of x is at least 60, so if the diagram has been drawn to scale, the answer would be **A**. If, on the other hand, the diagram has not been drawn to scale, we can't trust it. Which is it? The diagram is *not* drawn to scale — PQ is drawn almost as long as OR, even though OR is twice as long. Correct the diagram:

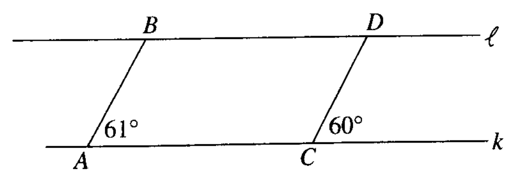


Now you can see that x is less than 45. The answer is **B**.

TACTIC 3 Exaggerate or Otherwise Change a Diagram

Sometimes it is appropriate to take a diagram that appears to be drawn to scale and intentionally exaggerate it. Why would we do this? Consider the following example.

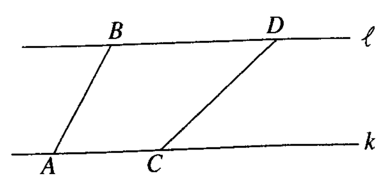
EXAMPLE 9



Line ℓ is parallel to line k.

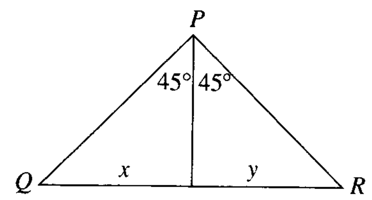
Quantity A	Quantity B
AB	CD

SOLUTION. In the diagram, which appears to be drawn correctly, AB and CD look as though they are the same length. However, there *might* be an imperceptible difference due to the fact that angle C is slightly smaller than angle A. So exaggerate the diagram: redraw it, making angle C *much* smaller than angle A. Now, it's clear: CD is longer. The answer is **B**.



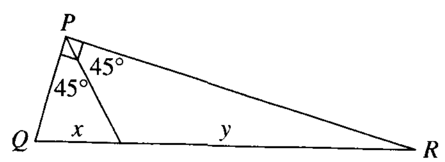
When you copy a diagram onto your scrap paper, you can change anything you like as long as your diagram is consistent with all the given data.

EXAMPLE 10



Quantity A	Quantity B
x	y

SOLUTION. You may redraw this diagram any way you like, as long as the two angles that are marked 45° remain 45°. If *PQ* and *PR* are equal, as they appear to be in the given diagram, then *x* would equal *y*. Since the given information doesn't state that *PQ* = *PR*, draw a diagram in which *PQ* and *PR* are clearly unequal. In the diagram below, *PR* is much longer than *PQ*, and *x* and *y* are clearly unequal. The answer is **D**.



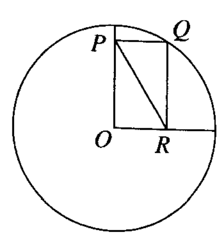
TACTIC 4

Add a Line to a Diagram

Occasionally, after staring at a diagram, you still have no idea how to solve the problem to which it applies. It looks as though not enough information has been given. When this happens, it often helps to draw another line in the diagram.

EXAMPLE 11

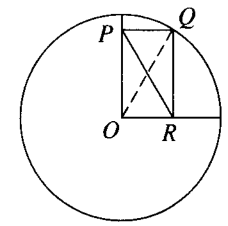
In the figure below, *Q* is a point on the circle whose center is *O* and whose radius is *r*, and *OPQR* is a rectangle. What is the length of diagonal *PR*?



Ⓐ r Ⓑ r^2 Ⓒ $\frac{r^2}{\pi}$ Ⓓ $\frac{r\sqrt{2}}{\pi}$

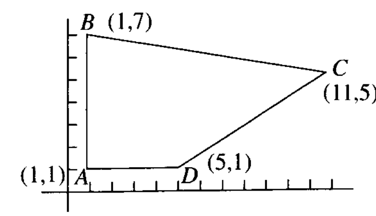
Ⓔ It cannot be determined from the information given.

SOLUTION. If after staring at the diagram and thinking about rectangles, circles, and the Pythagorean theorem, you're still lost, don't give up. Ask yourself, "Can I add another line to this diagram?" As soon as you think to draw in *OQ*, the other diagonal, the problem becomes easy: the two diagonals of a rectangle have the same length and, since *OQ* is a radius, it is equal to ***r***, **A**.

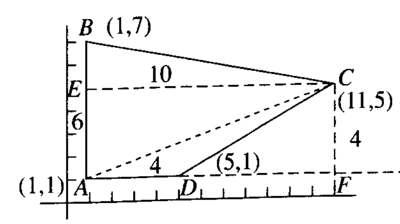


EXAMPLE 12

What is the area of quadrilateral *ABCD*?

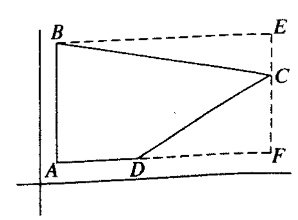


SOLUTION. Since the quadrilateral is irregular, there isn't a formula to find the area. However, if you draw in *AC*, you will divide *ABCD* into two triangles, each of whose areas can be determined.



If you then draw in the height of each triangle, you see that the area of $\triangle ACD$ is $\frac{1}{2}(4)(4) = 8$, and the area of $\triangle BAC$ is $\frac{1}{2}(6)(10) = 30$, so the area of *ABCD* is $30 + 8 =$ **38**.

Note that this problem could also have been solved by drawing in lines to create rectangle *ABEF*, and subtracting the areas of $\triangle BEC$ and $\triangle CFD$ from the area of the rectangle.



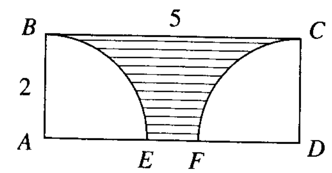
TACTIC

5 Subtract to Find Shaded Regions

Whenever part of a figure is shaded, the straightforward way to find the area of the shaded portion is to find the area of the entire figure and subtract from it the area of the unshaded region. Of course, if you are asked for the area of the unshaded region, you can, instead, subtract the shaded area from the total area. Occasionally, you may see an easy way to calculate the shaded area directly, but usually you should subtract.

EXAMPLE 13

In the figure below, *ABCD* is a rectangle, and *BE* and *CF* are arcs of circles centered at *A* and *D*. What is the area of the striped region?

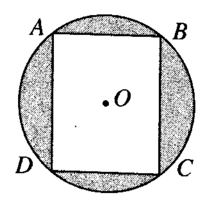


Ⓐ $10 - \pi$ Ⓑ $2(5 - \pi)$ Ⓒ $2(5 - 2\pi)$ Ⓓ $6 + 2\pi$ Ⓔ $5(2 - \pi)$

SOLUTION. The entire region is a 2×5 rectangle whose area is 10. Since the white region consists of two quarter-circles of radius 2, the total white area is that of a semicircle of radius 2: $\frac{1}{2}\pi(2)^2 = 2\pi$. Therefore, the area of the striped region is $10 - 2\pi = \mathbf{2(5 - \pi)}$, **B.**

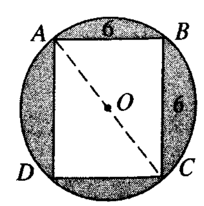
EXAMPLE 14

In the figure below, square *ABCD* is inscribed in circle *O*. If the perimeter of *ABCD* is 24, what is the area of the shaded region?



Ⓐ $18\pi - 36$ Ⓑ $18\pi - 24$ Ⓒ $12\pi - 36$ Ⓓ $9\pi - 36$ Ⓔ $9\pi - 24$

SOLUTION. Since the perimeter of square *ABCD* is 24, each of its sides is 6, and its area is $6^2 = 36$. Since diagonal *AC* is the hypotenuse of isosceles right triangle *ABC*, $AC = 6\sqrt{2}$. But *AC* is also a diameter of circle *O*, so the radius of the circle is $3\sqrt{2}$, and its area is $\pi(3\sqrt{2})^2 = 18\pi$. Finally, the area of the shaded region is $\mathbf{18\pi - 36}$, **A.**



TACTIC

6 Don't Do More Than You Have To

Very often a problem can be solved in more than one way. You should always try to do it in the easiest way possible. Consider the following examples.

EXAMPLE 15

If $5(3x - 7) = 20$, what is $3x - 8$?

Ⓐ $\frac{11}{3}$ Ⓑ 0 Ⓒ 3 Ⓓ 14 Ⓔ 19

It is not difficult to solve for x:

$$5(3x - 7) = 20 \Rightarrow 15x - 35 = 20 \Rightarrow 15x = 55 \Rightarrow x = \frac{55}{15} = \frac{11}{3}.$$

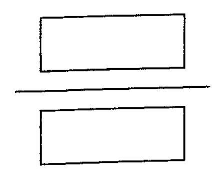
But it's too much work. Besides, once you find that $x = \frac{11}{3}$, you still have to multiply to get $3x$: $3\left(\frac{11}{3}\right) = 11$, and then subtract to get $3x - 8$: $11 - 8 = \mathbf{3}$.

SOLUTION. The key is to recognize that you don't need to find x. Finding $3x - 7$ is easy (just divide the original equation by 5), and $3x - 8$ is just 1 less:

$$5(3x - 7) = 20 \Rightarrow 3x - 7 = 4 \Rightarrow 3x - 8 = \mathbf{3}, \textbf{ C.}$$

EXAMPLE 16

If $7x + 3y = 17$ and $3x + 7y = 19$, what is the average (arithmetic mean) of x and y?



The obvious way to do this is to first find x and y by solving the two equations simultaneously and then to take their average. If you know how to do this, try it now, before reading further. If you worked carefully, you should have found that $x = \frac{31}{20}$ and $y = \frac{41}{20}$, and their average is $\frac{\frac{31}{20} + \frac{41}{20}}{2} = \frac{9}{5}$. Enter 9 as the numerator and 5 as the denominator.

This is not too difficult, but it is quite time-consuming, and questions on the GRE never require you to do that much work. Look for a shortcut. Is there a way to find the average without first finding x and y? Absolutely! Here's the best way to do this.

SOLUTION. Add the two equations:

$$\begin{array}{r} 7x + 3y = 17 \\ + \ 3x + 7y = 19 \\ \hline 10x + 10y = 36 \end{array}$$

Divide each side by 10: $x + y = 3.6$

Calculate the average: $\frac{x+y}{2} = \frac{3.6}{2} = \mathbf{1.8}$

Since this numeric entry question requires a fraction for the answer, note that $1.8 = 1\frac{8}{10} = \frac{18}{10}$. So enter 18 for the numerator and 10 for the denominator. Remember that you don't have to reduce fractions to lowest terms.

EXAMPLE 17

Benjamin worked from 9:47 A.M. until 12:11 P.M.
Jeremy worked from 9:11 A.M. until 12:47 P.M.

Quantity A	Quantity B
The number of minutes Benjamin worked	The number of minutes Jeremy worked

Do not spend any time calculating how many minutes either of them worked. You only need to know which column is greater, and since Jeremy started earlier and finished later, he clearly worked longer. The answer is **B**.

TACTIC 7 Pay Attention to Units

Often the answer to a question must be in units different from the data given in the question. As you read the question, write on your scratch paper exactly what you are being asked and circle it or put an asterisk next to it. Do they want hours or minutes or seconds, dollars or cents, feet or inches, meters or centimeters? On multiple-choice questions, an answer using the wrong units is almost always one of the choices.

EXAMPLE 18

Driving at 48 miles per hour, how many minutes will it take to drive 32 miles?

Ⓐ $\frac{2}{3}$ Ⓑ $\frac{3}{2}$ Ⓒ 40 Ⓓ 45 Ⓔ 2400

SOLUTION. This is a relatively easy question. Just be attentive. Divide the distance, 32, by the rate, 48: $\frac{32}{48} = \frac{2}{3}$, so it will take $\frac{2}{3}$ of an *hour* to drive 32 miles. Choice A is $\frac{2}{3}$, but that is not the correct answer, because you are asked how many *minutes* it will take. To convert hours to minutes, multiply by 60: it will take $\frac{2}{3}(60) =$ **40** minutes, **C**.

Note that you could have been asked how many *seconds* it would take, in which case the answer would be $40(60) = 2400$, Choice E.

EXAMPLE 19

At Nat's Nuts a $2\frac{1}{4}$-pound bag of pistachio nuts costs $6.00. At this rate, what is the cost in cents of a bag weighing 9 ounces?

Ⓐ 1.5 Ⓑ 24 Ⓒ 150 Ⓓ 1350 Ⓔ 2400

SOLUTION. This is a relatively simple ratio, but make sure you get the units right. To do this you need to know that there are 100 cents in a dollar and 16 ounces in a pound.

$$\frac{\text{price}}{\text{weight}} : \frac{6 \text{ dollars}}{2.25 \text{ pounds}} = \frac{600 \text{ cents}}{36 \text{ ounces}} = \frac{x \text{ cents}}{9 \text{ ounces}}.$$

Now cross-multiply and solve: $36x = 5400 \Rightarrow x =$ **150**, **C**.

TACTIC 8 Systematically Make Lists

When a question asks "how many," often the best strategy is to make a list of all the possibilities. If you do this it is important that you make the list in a *systematic* fashion so that you don't inadvertently leave something out. Usually, this means listing the possibilities in numerical or alphabetical order. Often, shortly after starting the list, you can see a pattern developing and you can figure out how many more entries there will be without writing them all down. Even if the question does not specifically ask "how many," you may need to count something to answer it; in this case, as well, the best plan may be to write out a list.

EXAMPLE 20

A palindrome is a number, such as 93539, that reads the same forward and backward. How many palindromes are there between 100 and 1,000?

SOLUTION. First, write down the numbers that begin and end in 1:

101, 111, 121, 131, 141, 151, 161, 171, 181, 191

Next write the numbers that begin and end in a 2:

202, 212, 222, 232, 242, 252, 262, 272, 282, 292

By now you should see the pattern: there are 10 numbers beginning with 1, 10 beginning with 2, and there will be 10 beginning with 3, 4, ..., 9 for a total of $9 \times 10 = \mathbf{90}$ palindromes.

EXAMPLE 21

The product of three positive integers is 300. If one of them is 5, what is the least possible value of the sum of the other two?

Ⓐ 16 Ⓑ 17 Ⓒ 19 Ⓓ 23 Ⓔ 32

SOLUTION. Since one of the integers is 5, the product of the other two is 60. Systematically, list all possible pairs, (a, b), of positive integers whose product is 60 and check their sums. First let $a = 1$, then 2, and so on.

a	b	$a + b$
1	60	61
2	30	32
3	20	23
4	15	19
5	12	17
6	10	16

The least possible sum is **16, A.**

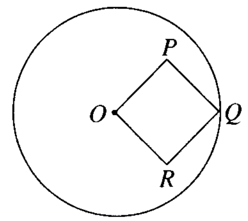
Practice Exercises

General Math Strategies

1. At Leo's Lumberyard, an 8-foot long wooden pole costs \$3.00. At this rate, what is the cost, in cents, of a pole that is 16 inches long?

Ⓐ 0.5
Ⓑ 48
Ⓒ 50
Ⓓ 64
Ⓔ 96

2. In the figure below, vertex Q of square $OPQR$ is on a circle with center O. If the area of the square is 8, what is the area of the circle?



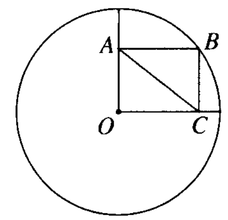
Ⓐ 8π
Ⓑ $8\pi\sqrt{2}$
Ⓒ 16π
Ⓓ 32π
Ⓔ 64π

3. In 1999, Diana read 10 English books and 7 French books. In 2000, she read twice as many French books as English books. If 60% of the books that she read during the two years were French, how many books did she read in 2000?

Ⓐ 16
Ⓑ 26
Ⓒ 32
Ⓓ 39
Ⓔ 48

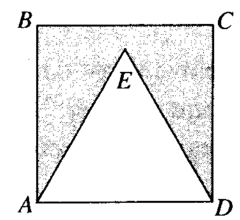
4. In writing all of the integers from 1 to 300, how many times is the digit 1 used?

5. In the figure below, if the radius of circle O is 10, what is the length of diagonal AC of rectangle $OABC$?



Ⓐ $\sqrt{2}$
Ⓑ $\sqrt{10}$
Ⓒ $5\sqrt{2}$
Ⓓ 10
Ⓔ $10\sqrt{2}$

6. In the figure below, $ABCD$ is a square and AED is an equilateral triangle. If $AB = 2$, what is the area of the shaded region?



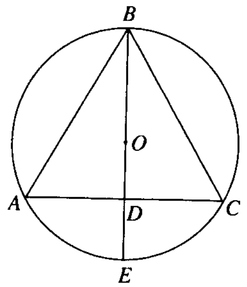
Ⓐ $\sqrt{3}$
Ⓑ 2
Ⓒ 3
Ⓓ $4 - 2\sqrt{3}$
Ⓔ $4 - \sqrt{3}$

7. If $5x + 13 = 31$, what is the value of $\sqrt{5x + 31}$?

Ⓐ $\sqrt{13}$
Ⓑ $\sqrt{\frac{173}{5}}$
Ⓒ 7
Ⓓ 13
Ⓔ 169

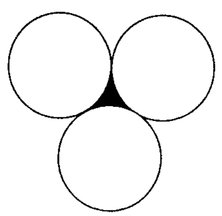
8. If $a + 2b = 14$ and $5a + 4b = 16$, what is the average (arithmetic mean) of a and b?

9. In the figure below, equilateral triangle ABC is inscribed in circle O, whose radius is 4. Altitude BD is extended until it intersects the circle at E. What is the length of DE?

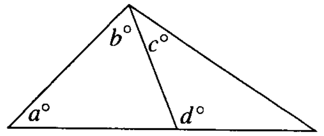


Ⓐ 1
Ⓑ $\sqrt{3}$
Ⓒ 2
Ⓓ $2\sqrt{3}$
Ⓔ $4\sqrt{3}$

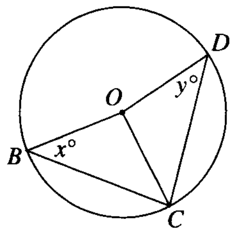
10. In the figure below, three circles of radius 1 are tangent to one another. What is the area of the shaded region between them?



Ⓐ $\frac{\pi}{2} - \sqrt{3}$
Ⓑ 1.5
Ⓒ $\pi - \sqrt{3}$
Ⓓ $\sqrt{3} - \frac{\pi}{2}$
Ⓔ $2 - \frac{\pi}{2}$



	Quantity A	Quantity B
11.	$a + b$	$c + d$



In circle O, $BC > CD$

	Quantity A	Quantity B
12.	x	y

	Quantity A	Quantity B
13.	The number of odd positive factors of 30	The number of even positive factors of 30

Questions 14–15 refer to the following definition.

$\{a, b\}$ represents the remainder when a is divided by b.

	Quantity A	Quantity B
14.	$\{10^3, 3\}$	$\{10^5, 5\}$

c and d are positive integers with $c < d$.

	Quantity A	Quantity B
15.	$\{c, d\}$	$\{d, c\}$

ANSWER KEY

1. **C**	6. **E**	11. **B**
2. **C**	7. **C**	12. **B**
3. **E**	8. **2.5**	13. **C**
4. **160**	9. **C**	14. **A**
5. **D**	10. **D**	15. **A**

ANSWER EXPLANATIONS

Two asterisks (**) indicate an alternative method of solving.

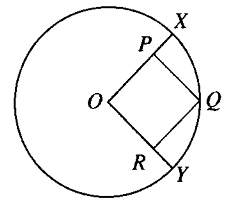
1. **(C)** This is a relatively simple ratio problem, but use TACTIC 7 and make sure you get the units right. To do this you need to know that there are 100 cents in a dollar and 12 inches in a foot.

$$\frac{\text{price}}{\text{weight}} : \frac{3 \text{ dollars}}{8 \text{ feet}} = \frac{300 \text{ cents}}{96 \text{ inches}} = \frac{x \text{ cents}}{16 \text{ inches}}.$$

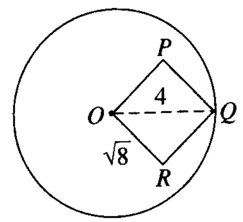
Now cross-multiply and solve:

$$96x = 4800 \Rightarrow x = 50.$$

2. **(C)** Use TACTICS 2 and 4. On your scrap paper, extend line segments OP and OR.

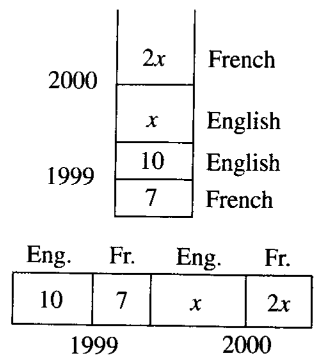


Square $OPQR$, whose area is 8, takes up most of quarter-circle OXY. So the area of the quarter-circle is certainly between 11 and 13. The area of the whole circle is 4 times as great: between 44 and 52. Check the five choices: they are approximately 25, 36, 50, 100, 200. The answer is clearly C.
**Another way to use TACTIC 4 is to draw in line segment OQ.



Since the area of the square is 8, each side is $\sqrt{8}$, and diagonal OQ is $\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$. But OQ is also a radius, so the area of the circle is $\pi(4)^2 = 16\pi$.

3. **(E)** Use TACTIC 1: draw a picture representing a pile of books or a bookshelf.



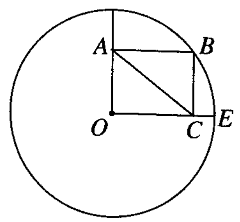
In the two years the number of French books Diana read was $7 + 2x$ and the total number of books was $17 + 3x$. Then 60% or $\frac{3}{5} = \frac{7+2x}{17+3x}$. To solve, cross-multiply:

$$5(7 + 2x) = 3(17 + 3x) \Rightarrow 35 + 10x = 51 + 9x \Rightarrow x = 16.$$

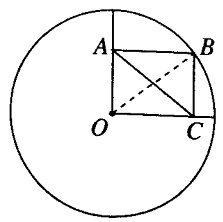
In 2000, Diana read 16 English books and 32 French books, a total of 48 books.

4. **(D)** Use TACTIC 8. Systematically list the numbers that contain the digit 1, writing as many as you need to see the pattern. Between 1 and 99 the digit 1 is used 10 times as the units digit (1, 11, 21, ... , 91) and 10 times as the tens digit (10, 11, 12, ... , 19) for a total of 20 times. From 200 to 299, there are 20 more (the same 20 preceded by a 2). From 100 to 199 there are 20 more plus 100 numbers where the digit 1 is used in the hundreds place. So the total is $20 + 20 + 20 + 100 = 160$.

5. **160** Use TACTIC 2. Trust the diagram: *AC*, which is clearly longer than *OC*, is approximately as long as radius *OE*.



Therefore, *AC* must be about 10. Check the choices. They are approximately 1.4, 3.1, 7, 10, and 14. The answer must be 10.
**The answer *is* 10. Use TACTIC 4: copy the diagram on your scrap paper and draw in diagonal *OB*.



Since the two diagonals of a rectangle are equal, and diagonal *OB* is a radius, $AC = OB = 10$.

6. **(E)** Use TACTIC 5: subtract to find the shaded area. The area of the square is 4.

The area of the equilateral triangle (see Section 11-J) is $\frac{2^2\sqrt{3}}{4} = \frac{4\sqrt{3}}{4} = \sqrt{3}$.

So the area of the shaded region is $4 - \sqrt{3}$.

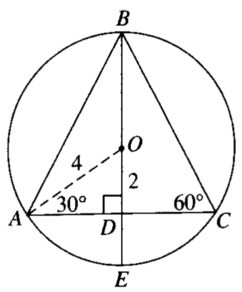
7. **(C)** Use TACTIC 6: don't do more than you have to. In particular, don't solve for *x*.

$$5x + 13 = 31 \Rightarrow 5x = 18 \Rightarrow 5x + 31 =$$
$$18 + 31 = 49 \Rightarrow \sqrt{5x+31} = \sqrt{49} = 7.$$

8. **2.5** Use TACTIC 6: don't do more than is necessary. We don't need to know the values of *a* and *b*, only their average. Adding the two equations, we get

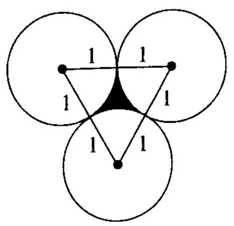
$$6a + 6b = 30 \Rightarrow a + b = 5 \Rightarrow \frac{a+b}{2} = \frac{5}{2} = 2.5.$$

9. **(C)** Use TACTIC 5: to get *DE*, subtract *OD* from radius *OE*, which is 4. Draw *AO* (TACTIC 4).



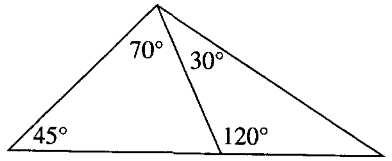
Since ΔAOD is a 30-60-90 right triangle, *OD* is 2 (one half of *OA*). So, $DE = 4 - 2 = 2$.

10. **(D)** Use TACTIC 4 and add some lines: connect the centers of the three circles to form an equilateral triangle whose sides are 2.



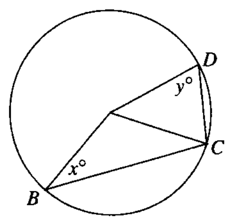
Now use TACTIC 5 and find the shaded area by subtracting the area of the three sectors from the area of the triangle. The area of the triangle is $\frac{2^2\sqrt{3}}{4} = \sqrt{3}$ (see Section 11-J).
Each sector is one sixth of a circle of radius 1. Together they form one half of such a circle, so their total area is $\frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$. Finally, subtract: the shaded area is $\sqrt{3} - \frac{\pi}{2}$.

11. **(B)** If you don't see how to answer this, use TACTIC 2: trust the diagram. Estimate the measure of each angle: for example, $a = 45$, $b = 70$, $c = 30$, and $d = 120$. So $c + d$ (150) is considerably greater than $a + b$ (115). Choose B.



**In fact, d by itself is equal to $a + b$ (an exterior angle of a triangle is equal to the sum of the opposite two interior angles). So $c + d > a + b$.

12. **(B)** From the figure, it appears that x and y are equal, or nearly so. However, the given information states that $BC > CD$, but this is not clear from the diagram. Use TACTIC 3: when you draw the figure on your scrap paper, exaggerate it. Draw it with BC much greater than CD. Now it is clear that y is greater.



13. **(C)** Use TACTIC 8. Systematically list all the factors of 30, either individually or in pairs: 1, 30; 2, 15; 3, 10; 5, 6. Of the 8 factors, 4 are even and 4 are odd.

14. **(A)** Quantity A: When 10^3 (1000) is divided by 3, the quotient is 333 and the remainder is 1. Quantity B: 10^5 is divisible by 5, so the remainder is 0.
Quantity A is greater.

15. **(A)** Quantity A: since $c < d$, the quotient when c is divided by d is 0, and the remainder is c. Quantity B: when d is divided by c the remainder must be less than c.
So Quantity A is greater.

CHAPTER 8

Discrete Quantitative Questions

About 20 of the 40 questions in the two math sections are what the ETS calls discrete quantitative questions. These questions are of three types:

- Multiple-choice questions
- Multiple-answer questions
- Numeric entry questions

Multiple-choice questions are just the standard multiple-choice questions that you are familiar with. Each one has five answer choices, exactly one of which is the correct answer. To get credit for a multiple-choice question you simply click on the oval in front of the one correct answer choice.

Multiple-answer questions are multiple-choice questions with a twist. These questions could have anywhere from 3 to 12 answer choices, any number of which could be correct, from just one to all of them. To alert you to the fact that there may be, and usually is, more than one correct answer, instead of an oval, a square appears in front of each answer choice. To get credit for a multiple-answer question, you must click on the square in front of each correct answer and leave blank the squares in front of each of the incorrect answers.

Numeric entry questions are the only questions on the test for which no answer choices are given. The answer to such a question may be a positive or negative integer, decimal, or fraction. To get credit for a numeric entry question you must use the keyboard to enter your answer into the box on the screen directly below the question. If in answering a question, you use the onscreen calculator and the digital readout is exactly the answer that you want to enter in the box, you can click on the calculator's TRANSFER DISPLAY bar and the readout will automatically appear in the box. Always enter the exact answer unless the question tells you to round your answer, in which case you must round it to the degree of accuracy asked for.

If the answer is to be entered as a fraction, there will be two boxes, and you are to enter the numerator in the upper box and the denominator in the lower box. Any answer equivalent to a correct answer earns full credit. If the correct answer to a question is 2.5, then 2.50 is equally acceptable, unless you were told to give the answer to the nearest tenth. Also, fractions do not have to be reduced: if the correct answer is $\frac{1}{2}$, then you would receive full credit for $\frac{3}{6}$ or $\frac{13}{26}$, or any other fraction equivalent to $\frac{1}{2}$.

TIP

When you take the GRE, dismiss the instructions for these questions instantly—do not spend even one second reading them—and certainly never accept their offer of clicking on "HELP" to return to them during the test.

TIP

On pages 11–12 you can see a worked out example of each of these three types of questions.

The majority of discrete quantitative questions are of the multiple-choice variety, and all of the tactics discussed in this chapter apply to them. Some of the tactics also apply to multiple-answer questions and numeric entry questions.

The important strategies you will learn in this chapter help you answer many questions on the GRE. However, as invaluable as these tactics are, use them only when you need them. *If you know how to do a problem and are confident that you can do it accurately and reasonably quickly, JUST DO IT!*

As we have done throughout this book, on multiple-choice questions we will continue to label the five answer choices A, B, C, D, and E and to refer to them as such. On multiple-answer questions, the choices will be consecutively labeled A, B, C, etc., using as many letters as there are answer choices. Of course, when you take the GRE, these letters will not appear—there will simply be a blank oval in front of each of the answer choices. When we refer to Choice C—as we do, for example, in TACTIC 1 (below)—we are simply referring to the third answer choice among the five presented.

Testing Tactics

TACTIC 1

Test the Choices, Starting with C

TACTIC 1, often called ***backsolving***, is useful when you are asked to solve for an unknown and you understand what needs to be done to answer the question, but you want to avoid doing the algebra. The idea is simple: test the various choices to see which one is correct.

NOTE: On the GRE the answers to virtually all numerical multiple-choice questions are listed in either increasing or decreasing order. Consequently, C is the middle value, and *in applying TACTIC 1, you should always start with C.* For example, assume that choices A, B, C, D, and E are given in increasing order. Try C. If it works, you've found the answer. If C doesn't work, you should know whether you need to test a larger number or a smaller one, and that permits you to eliminate two more choices. If C is too small, you need a larger number, and so A and B are out; if C is too large, eliminate D and E, which are even larger.

Examples 1 and 2 illustrate the proper use of TACTIC 1.

EXAMPLE 1

If the average (arithmetic mean) of 5, 6, 7, and w is 10, what is the value of w?

(A) 8 (B) 13 (C) 18 (D) 22 (E) 28

SOLUTION.

Use TACTIC 1. Test Choice C: $w = 18$.

- Is the average of 5, 6, 7, and 18 equal to 10?
- No: $\frac{5+6+7+18}{4} = \frac{36}{4} = 9$, which is *too small*.
- Eliminate C, and, since for the average to be 10, w must be *greater* than 18, eliminate A and B, as well.
- Try D: $w = 22$. Is the average of 5, 6, 7, and 22 equal to 10?
- Yes: $\frac{5+6+7+22}{4} = \frac{40}{4} = 10$. The answer is **D**.

Every problem that can be solved using TACTIC 1 can be solved directly, often in less time. So we stress: *if you are confident that you can solve a problem quickly and accurately, just do so.*

Here are two direct methods for solving Example 1, each of which is faster than backsolving. (See Section 11-E on averages.) If you know either method you should use it, and save TACTIC 1 for those problems that you can't easily solve directly.

DIRECT SOLUTION 1. If the average of four numbers is 10, their sum is 40. So, $5 + 6 + 7 + w = 40 \Rightarrow 18 + w = 40 \Rightarrow w = \mathbf{22}$.

DIRECT SOLUTION 2. Since 5 is *5 less than* 10, 6 is *4 less than* 10, and 7 is *3 less than* 10, to compensate, w must be $5 + 4 + 3 = 12$ more than 10.

So, $w = 10 + 12 = \mathbf{22}$.

EXAMPLE 2

Judy is now twice as old as Adam, but 6 years ago, she was 5 times as old as he was. How old is Judy now?

(A) 10 (B) 16 (C) 20 (D) 24 (E) 32

SOLUTION.

Use TACTIC 1: backsolve starting with C. If Judy is now 20, Adam is 10, and 6 years ago, they would have been 14 and 4. Since Judy would have been less than 5 times as old as Adam, eliminate C, D, and E, and try a smaller value. If Judy is now 16, Adam is 8; 6 years ago, they would have been 10 and 2. That's it; 10 *is* 5 times 2. The answer is **B**.

(See Section 11-H on word problems for the correct algebraic solution.)

Some tactics allow you to eliminate a few choices so you can make an educated guess. On those problems where it can be used, TACTIC 1 *always* gets you the right answer. The only reason not to use it on a particular problem is that you can *easily* solve the problem directly.

TIP

Don't start with C if some of the other choices are much easier to work with. If you start with B and it is too small, you may only get to eliminate two choices (A and B), instead of three, but it will save time if plugging in Choice C would be messy.

EXAMPLE 3

If $3x = 2(5 - 2x)$, then $x =$

Ⓐ $-\frac{10}{7}$ Ⓑ 0 Ⓒ $\frac{3}{7}$ Ⓓ 1 Ⓔ $\frac{10}{7}$

SOLUTION.

Since plugging in 0 is so much easier than plugging in $\frac{3}{7}$, start with B: then the left-hand side of the equation is 0 and the right-hand side is 10. The left-hand side is much too small. Eliminate A and B and try something bigger — D, of course; it will be much easier to deal with 1 than with $\frac{3}{7}$ or $\frac{10}{7}$. Now the left-hand side is 3 and the right-hand side is 6. We're closer, but not there. The answer must be **E**. Notice that we got the right answer without ever plugging in one of those unpleasant fractions. Are you uncomfortable choosing E without checking it? Don't be. If you *know* that the answer is greater than 1, and only one choice is greater than 1, that choice has to be right.

Again, we emphasize that, no matter what the choices are, you backsolve *only* if you can't easily do the algebra. Most students would probably do this problem directly:

$$3x = 2(5 - 2x) \Rightarrow 3x = 10 - 4x \Rightarrow 7x = 10 \Rightarrow x = \frac{10}{7}$$

and save backsolving for a harder problem. You have to determine which method is best for you.

TACTIC

2 Replace Variables with Numbers

Mastery of TACTIC 2 is critical for anyone developing good test-taking skills. This tactic can be used whenever the five choices involve the variables in the question. There are three steps:

1. Replace each letter with an easy-to-use number.
2. Solve the problem using those numbers.
3. Evaluate each of the five choices with the numbers you picked to see which choice is equal to the answer you obtained.

Examples 4 and 5 illustrate the proper use of TACTIC 2.

EXAMPLE 4

If a is equal to the sum of b and c, which of the following is equal to the difference of b and c?

Ⓐ $a - b - c$ Ⓑ $a - b + c$ Ⓒ $a - c$ Ⓓ $a - 2c$ Ⓔ $a - b - 2c$

SOLUTION.

- Pick three easy-to-use numbers which satisfy $a = b + c$: for example, $a = 5$, $b = 3$, $c = 2$.
- Then, solve the problem with these numbers: the difference of b and c is $3 - 2 = 1$.
- Finally, check each of the five choices to see which one is equal to 1:

Ⓐ Does $a - b - c = 1$?	NO. $5 - 3 - 2 = 0$
Ⓑ Does $a - b + c = 1$?	NO. $5 - 3 + 2 = 4$
Ⓒ Does $a - c = 1$?	NO. $5 - 2 = 3$
Ⓓ Does $a - 2c = 1$?	YES! $5 - 2(2) = 5 - 4 = 1$
Ⓔ Does $a - b - 2c = 1$?	NO. $5 - 3 - 2(2) = 2 - 4 = -2$

- The answer is **D**.

EXAMPLE 5

If the sum of five consecutive even integers is t, then, in terms of t, what is the greatest of these integers?

Ⓐ $\frac{t-20}{5}$ Ⓑ $\frac{t-10}{5}$ Ⓒ $\frac{t}{5}$ Ⓓ $\frac{t+10}{5}$ Ⓔ $\frac{t+20}{5}$

SOLUTION.

- Pick five easy-to-use consecutive even integers: say, 2, 4, 6, 8, 10. Then t, their sum, is 30.
- Solve the problem with these numbers: the greatest of these integers is 10.
- When $t = 30$, the five choices are $\frac{10}{5}, \frac{20}{5}, \frac{30}{5}, \frac{40}{5}, \frac{50}{5}$.
- Only $\frac{50}{5}$, Choice **E**, is equal to 10.

Of course, Examples 4 and 5 can be solved without using TACTIC 3 *if your algebra skills are good.* Here are the solutions.

SOLUTION 4. $a = b + c \Rightarrow b = a - c \Rightarrow b - c = (a - c) - c = a - 2c$.

SOLUTION 5. Let n, $n + 2$, $n + 4$, $n + 6$, and $n + 8$ be five consecutive even integers, and let t be their sum. Then,

$$t = n + (n + 2) + (n + 4) + (n + 6) + (n + 8) = 5n + 20$$

So, $n = \frac{t-20}{5} \Rightarrow n + 8 = \frac{t-20}{5} + 8 = \frac{t-20}{5} + \frac{40}{5} = \frac{t+20}{5}$.

The important point is that if you can't do the algebra, you can still use TACTIC 2 and *always* get the right answer. Of course, you should use TACTIC 2 even if you can do the algebra, if you think that by using this tactic you will solve the problem faster or will be less likely to make a mistake. This is a good example of what we mean when we say that with the proper use of these tactics, you can correctly answer many questions for which you may not know the correct mathematical solution.

TIP

Replace the letters with numbers that are easy to use, not necessarily ones that make sense. *It is perfectly OK to ignore reality.* A school can have 5 students, apples can cost 10 dollars each, trains can go 5 miles per hour or 1000 miles per hour — it doesn't matter.

Examples 6 and 7 are somewhat different. You are asked to reason through word problems involving only variables. Most students find problems like these mind-boggling. Here, the use of TACTIC 2 is essential. Without it, Example 6 is difficult and Example 7 is nearly impossible. This is not an easy tactic to master, but with practice you will catch on.

EXAMPLE 6

If a school cafeteria needs c cans of soup each week for each student, and if there are s students in the school, for how many weeks will x cans of soup last?

Ⓐ csx Ⓑ $\frac{xs}{c}$ Ⓒ $\frac{s}{cx}$ Ⓓ $\frac{x}{cs}$ Ⓔ $\frac{cx}{s}$

SOLUTION.

- Replace c, s, and x with three easy-to-use numbers. If a school cafeteria needs 2 cans of soup each week for each student, and if there are 5 students in the school, how many weeks will 20 cans of soup last?
- Since the cafeteria needs $2 \times 5 = 10$ cans of soup per week, 20 cans will last 2 weeks.
- Which of the choices equals 2 when $c = 2$, $s = 5$, and $x = 20$?
- $csx = 200$; $\frac{xs}{c} = 50$; $\frac{s}{cx} = \frac{1}{8}$; $\frac{x}{cs} = 2$; and $\frac{cx}{s} = 8$.

The answer is $\frac{x}{cs}$, **D.**

NOTE: You do not need to get the exact value of each choice. As soon as you see that a choice does not equal the value you are looking for, stop—eliminate that choice and move on. For example, in the preceding problem, it is clear that csx is much greater than 2, so eliminate it immediately; you do not need to multiply it out to determine that the value is 200.

CAUTION

In this type of problem it is *not* a good idea to replace any of the variables by 1. Since multiplying and dividing by 1 give the same result, you would not be able to distinguish between $\frac{cx}{s}$ and $\frac{x}{cs}$, both of which are equal to 4 when $c = 1$, $s = 5$, and $x = 20$. It is also not a good idea to use the same number for different variables: $\frac{cx}{s}$ and $\frac{xs}{c}$ are each equal to x when c and s are equal.

EXAMPLE 7

A vendor sells h hot dogs and s sodas. If a hot dog costs twice as much as a soda, and if the vendor takes in a total of d dollars, how many cents does a soda cost?

Ⓐ $\frac{100d}{s+2h}$ Ⓑ $\frac{s+2h}{100d}$ Ⓒ $\frac{d(s+2h)}{100}$ Ⓓ $100d(s+2h)$ Ⓔ $\frac{d}{100(s+2h)}$

SOLUTION.

- Replace h, s, and d with three easy-to-use numbers. Suppose a soda costs 50¢ and a hot dog $1.00. Then, if he sold 2 sodas and 3 hot dogs, he took in 4 dollars.
- Which of the choices equals 50 when $s = 2$, $h = 3$, and $d = 4$?
- Only $\frac{100d}{s+2h}$ **(A)**: $\frac{100(4)}{2+2(3)} = \frac{400}{8} = 50$.

Now, practice TACTIC 3 on the following problems.

EXAMPLE 8

Yann will be x years old y years from now. How old was he z years ago?

Ⓐ $x+y+z$ Ⓑ $x+y-z$ Ⓒ $x-y-z$ Ⓓ $y-x-z$ Ⓔ $z-y-x$

SOLUTION.

Assume that Yann will be 10 in 2 years. How old was he 3 years ago? If he will be 10 in 2 years, he is 8 now and 3 years ago he was 5. Which of the choices equals 5 when $x = 10$, $y = 2$, and $z = 3$? Only $x - y - z$, **C.**

EXAMPLE 9

Stan drove for h hours at a constant rate of r miles per hour. How many miles did he go during the final 20 minutes of his drive?

Ⓐ $20r$ Ⓑ $\frac{hr}{3}$ Ⓒ $3rh$ Ⓓ $\frac{hr}{20}$ Ⓔ $\frac{r}{3}$

SOLUTION.

If Stan drove at 60 miles per hour for 2 hours, how far did he go in the last 20 minutes? Since 20 minutes is $\frac{1}{3}$ of an hour, he went 20 ($\frac{1}{3}$ of 60) miles. Only $\frac{r}{3}$ is 20 when $r = 60$ and $h = 2$. Notice that h is irrelevant. Whether he had been driving for 2 hours or 20 hours, the distance he covered in the last 20 minutes would be the same.

TIP

In problems involving fractions, the best number to use is the least common denominator of all the fractions. In problems involving percents, the easiest number to use is 100. (See Sections 11-B and 11-C.)

TACTIC

3 Choose an Appropriate Number

TACTIC 3 is similar to TACTIC 2, in that we pick convenient numbers. However, here no variable is given in the problem. TACTIC 3 is especially useful in problems involving fractions, ratios, and percents.

EXAMPLE 10

At Madison High School each student studies exactly one foreign language. Three-fifths of the students take Spanish, and one-fourth of the remaining students take German. If all of the others take French, what percent of the students take French?

Ⓐ 10 Ⓑ 15 Ⓒ 20 Ⓓ 25 Ⓔ 30

SOLUTION.

The least common denominator of $\frac{3}{5}$ and $\frac{1}{4}$ is 20, so assume that there are 20 students at Madison High. (Remember the numbers don't have to be realistic.) The number of students taking Spanish is 12 $\left(\frac{3}{5}\text{ of }20\right)$. Of the remaining 8 students, 2 of them $\left(\frac{1}{4}\text{ of }8\right)$ take German. The other 6 take French. Finally, 6 is **30%** of 20. The answer is **E**.

EXAMPLE 11

From 1994 to 1995 the sales of a book decreased by 80%. If the sales in 1996 were the same as in 1994, by what percent did they increase from 1995 to 1996?

Ⓐ 80% Ⓑ 100% Ⓒ 120% Ⓓ 400% Ⓔ 500%

SOLUTION.

Since this problem involves percents, assume that 100 copies of the book were sold in 1994 (and 1996). Sales dropped by 80 (80% of 100) to 20 in 1995 and then increased by 80, from 20 back to 100, in 1996. The percent increase was

$$\frac{\text{the actual increase}}{\text{the original amount}} \times 100\% = \frac{80}{20} \times 100\% = \mathbf{400\%}, \textbf{ D}.$$

TACTIC

4 Eliminate Absurd Choices and Guess

When you have no idea how to solve a multiple-choice question, you can always make an educated guess—simply eliminate all the absurd choices and then guess from among the remaining ones.

During the course of a GRE, you will probably find at least a few multiple-choice questions that you don't know how to solve. Since you are not penalized for wrong answers, you are surely going to enter answers for them. But before taking a wild guess, take a moment to look at the answer choices. Often two or three of them are absurd. Eliminate those and then guess one of the others. Occasionally, four of the choices are absurd. When this occurs, your answer is no longer a guess.

What makes a choice absurd? Lots of things. Here are a few. Even if you don't know how to solve a problem you may realize that

- the answer must be positive, but some of the choices are negative;
- the answer must be even, but some of the choices are odd;
- the answer must be less than 100, but some choices exceed 100;
- a ratio must be less than 1, but some choices are greater than 1.

Let's look at several examples. In a few of them the information given is intentionally insufficient to solve the problem; but you will still be able to determine that some of the answers are absurd. In each case the "solution" will indicate which choices you should have eliminated. At that point you would simply guess. Remember, on the GRE when you guess, don't agonize. Just guess and move on.

EXAMPLE 12

A region inside a semicircle of radius *r* is shaded and you are asked for its area.

Ⓐ $\frac{1}{4}\pi r^2$ Ⓑ $\frac{1}{3}\pi r^2$ Ⓒ $\frac{1}{2}\pi r^2$ Ⓓ $\frac{2}{3}\pi r^2$ Ⓔ πr^2

SOLUTION.

You may have no idea how to find the area of the shaded region, but you should know that since the area of a circle is πr^2, the area of a semicircle is $\frac{1}{2}\pi r^2$. Therefore, the area of the shaded region must be *less* than $\frac{1}{2}\pi r^2$, so eliminate C, D, and E. On an actual GRE problem, you may be able to make an educated guess between A and B. If so, terrific; if not, just choose one or the other.

EXAMPLE 13

The average (arithmetic mean) of 5, 10, 15, and z is 20. What is z?

Ⓐ 0 Ⓑ 20 Ⓒ 25 Ⓓ 45 Ⓔ 50

SOLUTION.
If the average of four numbers is 20, and three of them are less than 20, the other one must be greater than 20. Eliminate A and B and guess. If you further realize that since 5 and 10 are a *lot less* than 20, z will probably be a *lot more* than 20; eliminate C, as well.

EXAMPLE 14

If 25% of 260 equals 6.5% of a, what is a?

Ⓐ 10 Ⓑ 65 Ⓒ 100 Ⓓ 130 Ⓔ 1000

SOLUTION.
Since 6.5% of a equals 25% of 260, which is surely greater than 6.5% of 260, a must be greater than 260. Eliminate A, B, C, and D. The answer *must* be **E**!

Example 14 illustrates an important point. *Even if you know how to solve a problem,* if you immediately see that four of the five choices are absurd, just pick the fifth choice and move on.

EXAMPLE 15

A jackpot of \$39,000 is to be divided in some ratio among three people. What is the value of the largest share?

Ⓐ \$23,400 Ⓑ \$19,500 Ⓒ \$11,700 Ⓓ \$7800 Ⓔ \$3900

SOLUTION.
If the prize were divided equally, each of the three shares would be worth \$13,000. If it is divided unequally, the largest share is surely worth *more than* \$13,000. Eliminate C, D, and E. In an actual question, you would be told what the ratio is, and that might enable you to eliminate A or B. If not, you just guess.

EXAMPLE 16

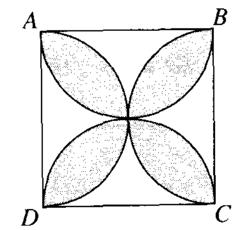
In a certain club, the ratio of the number of boys to girls is 5:3. What percent of the members of the club are girls?

Ⓐ 37.5% Ⓑ 50% Ⓒ 60% Ⓓ 62.5% Ⓔ 80%

SOLUTION.
Since there are 5 boys for every 3 girls, there are fewer girls than boys. Therefore, *fewer than half* (50%) of the members are girls. Eliminate B, C, D, and E. The answer is **A.**

EXAMPLE 17

In the figure below, four semicircles are drawn, each one centered at the midpoint of one of the sides of square $ABCD$. Each of the four shaded "petals" is the intersection of two of the semicircles. If $AB = 4$, what is the total area of the shaded region?



Ⓐ 8π Ⓑ $32 - 8\pi$ Ⓒ $16 - 8\pi$ Ⓓ $8\pi - 32$ Ⓔ $8\pi - 16$

SOLUTION.

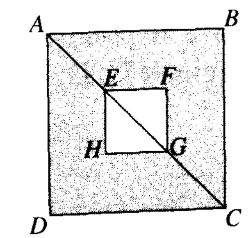
- Since $AB = 4$, the area of the square is 16, and so, obviously, the area of the shaded region must be much less.
- Check each choice. Since π is slightly more than 3 ($\pi \approx 3.14$), 8π is somewhat greater than 24, approximately 25.
- (A) $8\pi \approx 25$. More than the area of the whole square: way too big.
- (B) $32 - 8\pi \approx 32 - 25 = 7$.
- (C) $16 - 8\pi$ is negative.
- (D) $8\pi - 32$ is also negative.
- (E) $8\pi - 16 \approx 25 - 16 = 9$.

NOTE: Three of the choices are absurd: A is more than the area of the entire square and C and D are negative; they can be eliminated immediately. The answer must be B or E. If you think the shaded area takes up less than half of the square, guess B; if you think it takes up more than half of the square, guess E. (The answer is **E**).

Now use TACTIC 4 on each of the following problems. Even if you know how to solve them, don't. Practice this technique and see how many choices you can eliminate *without* actually solving.

EXAMPLE 18

In the figure at the right, diagonal EG of square $EFGH$ is $\frac{1}{2}$ of diagonal AC of the square $ABCD$. What is the ratio of the area of the shaded region to the area of $ABCD$?



Ⓐ $\sqrt{2}:1$ Ⓑ 3:4 Ⓒ $\sqrt{2}:2$ Ⓓ 1:2 Ⓔ $1:2\sqrt{2}$

SOLUTION.
Obviously, the shaded region is smaller than square *ABCD*, so the ratio must be less than 1. Eliminate A. Also, from the diagram, it is clear that the shaded region is more than half of square *ABCD*, so the ratio is greater than 0.5. Eliminate D and E. Since 3:4 = .75 and $\sqrt{2}:2 \approx .71$, B and C are too close to tell which is correct just by looking; so guess. The answer is **B**.

EXAMPLE 19

Shari receives a commission of 25¢ for every $20.00 worth of merchandise she sells. What percent is her commission?

Ⓐ $1\frac{1}{4}$% Ⓑ $2\frac{1}{2}$% Ⓒ 5% Ⓓ 25% Ⓔ 125%

SOLUTION.
Clearly, a commission of 25¢ on $20 is quite small. Eliminate D and E and guess one of the small percents. If you realize that 1% of $20 is 20¢, then you know the answer is a little more than 1%, and you should guess A (maybe B, but definitely not C). The answer is **A**.

EXAMPLE 20

From 1980 to 1990, Lior's weight increased by 25%. If his weight was *k* kilograms in 1990, what was it in 1980?

Ⓐ 1.75*k* Ⓑ 1.25*k* Ⓒ 1.20*k* Ⓓ .80*k* Ⓔ .75*k*

SOLUTION.
Since Lior's weight increased, his weight in 1980 was *less than k*. Eliminate A, B, and C and guess. The answer is **D**.

EXAMPLE 21

The average of 10 numbers is –10. If the sum of 6 of them is 100, what is the average of the other 4?

Ⓐ –100 Ⓑ –50 Ⓒ 0 Ⓓ 50 Ⓔ 100

SOLUTION.
Since the average of all 10 numbers is negative, so is their sum. But the sum of the first 6 is positive, so the sum (and the average) of the others must be negative. Eliminate C, D, and E. **B** is correct.

Practice Exercises

Discrete Quantitative Questions

1. Evan has 4 times as many books as David and 5 times as many as Jason. If Jason has more than 40 books, what is the least number of books that Evan could have?

Ⓐ 200
Ⓑ 205
Ⓒ 210
Ⓓ 220
Ⓔ 240

2. Judy plans to visit the National Gallery once each month in 2012 except in July and August when she plans to go three times each. A single admission costs $3.50, a pass valid for unlimited visits in any 3-month period can be purchased for $18, and an annual pass costs $60.00. What is the least amount, in dollars, that Judy can spend for her intended number of visits?

[ ] dollars

3. Alison is now three times as old as Jeremy, but 5 years ago, she was 5 times as old as he was. How old is Alison now?

Ⓐ 10
Ⓑ 12
Ⓒ 24
Ⓓ 30
Ⓔ 36

4. What is the largest prime factor of 255?

Ⓐ 5
Ⓑ 15
Ⓒ 17
Ⓓ 51
Ⓔ 255

5. If c is the product of a and b, which of the following is the quotient of a and b?

Ⓐ $\frac{b^2}{c}$

Ⓑ $\frac{c}{b^2}$

Ⓒ $\frac{b}{c^2}$

Ⓓ bc^2

Ⓔ b^2c

6. If w widgets cost c cents, how many widgets can you get for d dollars?

Ⓐ $\frac{100dw}{c}$

Ⓑ $\frac{dw}{100c}$

Ⓒ $100cdw$

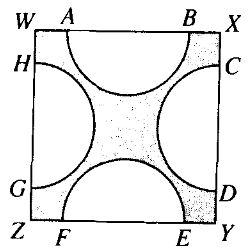
Ⓓ $\frac{dw}{c}$

Ⓔ cdw

7. If 120% of a is equal to 80% of b, which of the following is equal to $a + b$?

Ⓐ $1.5a$
Ⓑ $2a$
Ⓒ $2.5a$
Ⓓ $3a$
Ⓔ $5a$

8. In the figure below, $WXYZ$ is a square whose sides are 12. AB, CD, EF, and GH are each 8, and are the diameters of the four semicircles. What is the area of the shaded region?



Ⓐ $144 - 128\pi$
Ⓑ $144 - 64\pi$
Ⓒ $144 - 32\pi$
Ⓓ $144 - 16\pi$
Ⓔ 16π

9. If x and y are integers such that $x^3 = y^2$, which of the following could not be the value of y?
Indicate *all* such values.

[A] -1
[B] 1
[C] 8
[D] 12
[E] 16
[F] 27

10. What is a divided by $a\%$ of a?

Ⓐ $\frac{a}{100}$
Ⓑ $\frac{100}{a}$
Ⓒ $\frac{a^2}{100}$
Ⓓ $\frac{100}{a^2}$
Ⓔ $100a$

11. If an object is moving at a speed of 36 kilometers per hour, how many meters does it travel in one second?

meters

12. On a certain French-American committee, $\frac{2}{3}$ of the members are men, and $\frac{3}{8}$ of the men are Americans. If $\frac{3}{5}$ of the committee members are French, what fraction of the members are American women?

13. For what value of x is $8^{2x-4} = 16^x$?

Ⓐ 2
Ⓑ 3
Ⓒ 4
Ⓓ 6
Ⓔ 8

14. If $12a + 3b = 1$ and $7b - 2a = 9$, what is the average (arithmetic mean) of a and b?

Ⓐ 0.1
Ⓑ 0.5
Ⓒ 1
Ⓓ 2.5
Ⓔ 5

15. If $x\%$ of y is 10, what is y?

Ⓐ $\frac{10}{x}$
Ⓑ $\frac{100}{x}$
Ⓒ $\frac{1000}{x}$
Ⓓ $\frac{x}{100}$
Ⓔ $\frac{x}{10}$

ANSWER KEY

1. **D**	6. **A**	11. **10**	14. **B**
2. **49.50**	7. **C**	12. $\mathbf{\frac{3}{20}}$	15. **C**
3. **D**	8. **C**	13. **D**	
4. **C**	9. **D, E**		
5. **B**	10. **B**		

ANSWER EXPLANATIONS

Two asterisks (**) indicate an alternative method of solving.

1. **(D)** Test the answer choices starting with the smallest value. If Evan had 200 books, Jason would have 40. But Jason has more than 40, so 200 is too small. Trying 205 and 210, we see that neither is a multiple of 4, so David wouldn't have a whole number of books. Finally, 220 works. (So does 240, but we shouldn't even test it since we want the least value.)
   **Since Jason has at least 41 books, Evan has at least $41 \times 5 = 205$. But Evan's total must be a multiple of 4 and 5, hence of 20. The smallest multiple of 20 greater than 205 is 220.

2. **49.50** Judy intends to go to the Gallery 16 times during the year. Buying a single admission each time would cost $16 \times \$3.50 = \$56$, which is less than the annual pass. If she bought a 3-month pass for June, July, and August, she would pay \$18 plus \$31.50 for 9 single admissions ($9 \times \$3.50$), for a total expense of \$49.50, which is the least expensive option.

3. **(D)** Use TACTIC 1: backsolve starting with C. If Alison is now 24, Jeremy is 8, and 5 years ago, they would have been 19 and 3, which is more than 5 times as much. Eliminate A, B, and C, and try a bigger value. If Alison is now 30, Jeremy is 10, and 5 years ago, they would have been 25 and 5. That's it; 25 is 5 times 5.
   **If Jeremy is now x, Alison is $3x$, and 5 years ago they were $x - 5$ and $3x - 5$, respectively. Now, solve:
   $3x - 5 = 5(x - 5) \Rightarrow 3x - 5 = 5x - 25 \Rightarrow$
   $2x = 20 \Rightarrow x = 10 \Rightarrow 3x = 30.$

4. **(C)** Test the choices starting with C: 255 *is* divisible by 17 ($255 = 17 \times 15$), so this is a possible answer. Does 255 have a larger prime factor? Neither Choice D nor E is prime, so the answer must be Choice C.

5. **(B)** Use TACTIC 2. Pick simple values for a, b, and c. Let $a = 3$, $b = 2$, and $c = 6$. Then $a \div b = 3/2$. Without these values of a, b, and c, only B is equal to 3/2.

   ** $c = ab \Rightarrow a = \frac{c}{b} \Rightarrow a \div b = \frac{c}{b} \div b = \frac{c}{b} \cdot \frac{1}{b} = \frac{c}{b^2}.$

6. **(A)** Use TACTIC 2. If 2 widgets cost 10 cents, then widgets cost 5 cents each, and for 3 dollars, you can get 60. Which of the choices equals 60 when $w = 2$, $c = 10$, and $d = 3$? Only A.
   ** $\frac{\text{widgets}}{\text{cents}} = \frac{w}{c} = \frac{x}{100d} \Rightarrow x = \frac{100dw}{c}.$

7. **(C)** Since 120% of 80 = 80% of 120, let $a = 80$ and $b = 120$. Then $a + b = 200$, and $200 \div 80 = 2.5$.

8. **(C)** If you don't know how to solve this, you must use TACTIC 4 and guess after eliminating the absurd choices. Which choices are absurd? Certainly, A and B, both of which are negative. Also, since Choice D is about 94, which is much more than half the area of the square, it is much too big. Guess between Choice C (about 43) and Choice E (about 50). If you remember that the way to find shaded areas is to subtract, guess C.

   **The area of the square is $12^2 = 144$. The area of each semicircle is 8π, one-half the area of a circle of radius 4. So together the areas of the semicircles is 32π.

9. **(D)(E)** Test each choice until you find all the correct answers.

   (A) Could $y = -1$? Is there an integer x such that $x^3 = (-1)^2 = 1$? Yes, $x = 1$.
   (B) Similarly, if $y = 1$, $x = 1$.
   (C) Could $y = 8$? Is there an integer x such that $x^3 = (8)^2 = 64$? Yes, $x = 4$.
   (D) Could $y = 12$? Is there an integer such that $x^3 = 12^2 = 144$? No, $5^3 = 125$, which is too small, and $6^3 = 216$, which is too big.
   (E) Could $y = 16$? Is there an integer x such that $x^3 = 16^2 = 256$? No, $6^3 = 216$, which is too small; and $7^3 = 343$, which is too big.
   (F) Could $y = 27$? Is there an integer x such that $x^3 = 27^2 = 729$? Yes, $9^3 = 729$.
   The answer is D and E.

10. **(B)** $a \div (a\% \text{ of } a) = a \div \left(\frac{a}{100} \times a\right) = a \div \left(\frac{a^2}{100}\right) = a \times \frac{100}{a^2} = \frac{100}{a}$.

    **Use TACTICS 2 and 3: replace a by a number, and use 100 since the problem involves percents. $100 \div (100\% \text{ of } 100) = 100 \div 100 = 1$. Test each choice; which ones equal 1 when $a = 100$. Both A and B: $\frac{100}{100} = 1$. Eliminate Choices C, D, and E, and test A and B with another value for a. $50 \div (50\% \text{ of } 50) = 50 \div (25) = 2$. Now, only B works $\left(\frac{100}{50} = 2, \text{ whereas } \frac{50}{100} = \frac{1}{2}\right)$.

11. **10** Set up a ratio:

    $$\frac{\text{distance}}{\text{time}} = \frac{36 \text{ kilometers}}{1 \text{ hour}} = \frac{36{,}000 \text{ meters}}{60 \text{ minutes}} = \frac{36{,}000 \text{ meters}}{3600 \text{ seconds}} = 10 \text{ meters/second.}$$

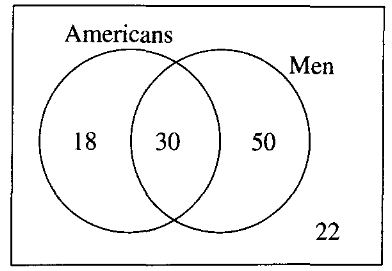
    **Use TACTIC 1: Test choices starting with C:

    100 meters/second = 6000 meters/minute = 360,000 meters/hour = 360 kilometers/hour.

    Not only is that too big, it is too big by a factor of 10. The answer is 10.

12. $\mathbf{\frac{3}{20}}$ Use TACTIC 3. The LCM of all the denominators is 120, so assume that the committee has 120 members. Then there are $\frac{2}{3} \times 120 = 80$ men and 40 women. Of the 80 men 30 $\left(\frac{3}{8} \times 80\right)$ are American. Since there are 72 $\left(\frac{3}{5} \times 120\right)$ French members, there are $120 - 72 = 48$ Americans, of whom 30 are men, so the other 18 are women. Finally, the fraction of American women is $\frac{18}{120} = \frac{3}{20}$.

    This is illustrated in the Venn diagram below.



13. **(D)** Use the laws of exponents to simplify the equation, and then solve it:
    $8^{2x-4} = 16^x \Rightarrow (2^3)^{2x-4} = (2^4)^x \Rightarrow 3(2x - 4) = 4x \Rightarrow 6x - 12 = 4x \Rightarrow$
    $2x = 12 \Rightarrow x = 6$.

14. **(B)** Add the two equations:

    $10a + 10b = 10 \Rightarrow a + b = 1 \Rightarrow \frac{a+b}{2} = \frac{1}{2}$.

    Do not waste time solving for a and b.

15. **(C)** Pick easy-to-use numbers. Since 100% of 10 is 10, let $x = 100$ and $y = 10$. When $x = 100$, Choices C and E are each 10. Eliminate Choices A, B, and D, and try some other numbers: 50% of 20 is 10. Of Choices C and E, only C = 20 when $x = 50$.

CHAPTER 9

Quantitative Comparison Questions

About 15 of the 40 questions on the two quantitative sections of the GRE are quantitative comparisons. Unless you took the SAT before 2005, it is very likely that you have never seen questions of this type and certainly never learned the correct strategies for answering them. Don't worry. In this chapter you will learn all of the necessary tactics. If you master them, you will quickly realize that quantitative comparisons are the easiest mathematics questions on the GRE and will wish that there were more than 15 of them.

Before the first quantitative comparison question appears on the screen, you will see these instructions.

Directions: In the following question, there are two quantities, labeled Quantity A and Quantity B. You are to compare those quantities, taking into consideration any additional information given and decide which of the following statements is true:

Quantity A is greater;
Quantity B is greater;
The two quantities are equal; or
It is impossible to determine which quantity is greater.

Note: The given information, if any, is centered above the two quantities. If a symbol appears more than once, it represents the same thing each time.

Before learning the different strategies for solving this type of question, let's clarify these instructions. In quantitative comparison questions there are two quantities, and it is your job to compare them. The correct answer to a quantitative comparison question is one of the four statements listed in the directions above. Of course, on the computer screen those choices will not be listed as A, B, C, and D. Rather, you will see an oval in front of each statement, and you will click on the oval in front of the statement you believe is true.

You should click on the oval in front of	if
Quantity A is greater.	Quantity A is greater *all the time, no matter what.*
Quantity B is greater.	Quantity B is greater *all the time, no matter what.*
The two quantities are equal.	The two quantities are equal *all the time, no matter what.*
It is impossible to determine which quantity is greater.	*The answer is not one of the first three choices.*

This means, for example, that *if you can find a single instance* when Quantity A is greater than Quantity B, then you can immediately eliminate two choices: the answer cannot be "Quantity B is greater," and the answer cannot be "The two quantities are equal." In order for the answer to be "Quantity B is greater," Quantity B would have to be greater *all the time*; but you know of one instance when it isn't. Similarly, since the quantities are not equal *all the time*, the answer can't be "The two quantities are equal." The correct answer, therefore, is either "Quantity A is greater" or "It is impossible to determine which quantity is greater." If it turns out that Quantity A *is* greater all the time, then that is the answer; if, however, you can find a single instance where Quantity A is not greater, the answer is "It is impossible to determine which quantity is greater."

By applying the tactics that you will learn in this chapter, you will probably be able to determine which of the choices is correct; if, however, after eliminating two of the choices, you still cannot determine which answer is correct, quickly guess between the two remaining choices and move on.

Before learning the most important tactics for handling quantitative comparison questions, let's look at two examples to illustrate the preceding instructions.

TIP

Right now, memorize the instructions for answering quantitative comparison questions. *When you take the GRE, dismiss the instructions for these questions immediately—do not spend even one second reading the directions (or looking at a sample problem).*

EXAMPLE 1

$1 < x < 3$

Quantity A	Quantity B
x^2	$2x$

- Quantity A is greater.
- Quantity B is greater.
- The two quantities are equal.
- It is impossible to determine which quantity is greater.

SOLUTION.

Throughout, x represents the same thing — a number between 1 and 3. If x is 2, then x^2 and $2x$ are each 4, and *in this case* the two quantities are equal. We can, therefore, eliminate the first two choices: neither Quantity A nor Quantity B is greater *all the time*. However, in order for the correct answer to be "The two quantities are equal," the quantities would have to be equal *all the time*. Are they? Note that although 2 is the only *integer* between 1 and 3, it is not the only *number* between 2 and 3: x could be 1.1 or 2.5 or any of infinitely many other numbers. And in those cases the quantities are not equal (for example, $2.5^2 = 6.25$, whereas $2(2.5) = 5$). The quantities are *not* always equal, and so the correct answer is the fourth choice: It is impossible to determine which quantity is greater.

EXAMPLE 2

p and q are primes
$p + q = 12$

Quantity A	Quantity B
p	8

- Quantity A is greater.
- Quantity B is greater.
- The two quantities are equal.
- It is impossible to determine which quantity is greater.

SOLUTION.

Since 5 and 7 are the only primes whose sum is 12, p could be 5 or 7. In either case, p is less than 8, and so Quantity B is greater, *all the time.* Note that although $1 + 11 = 12$, p cannot be 11, because 1 is not a prime [See Section 11-A].

NOTE: To simplify the discussion, throughout the rest of this chapter, in the explanations of the answers to all sample questions and in the Model Tests, the four answer choices will be referred to as A, B, C, and D, respectively. For example, we will write

The correct answer is **B.**

rather than

The correct answer is: Quantity B is greater.

Testing Tactics

TACTIC 1 Replace Variables with Numbers

Many problems that are hard to analyze because they contain variables become easy to solve when the variables are replaced by simple numbers.

TACTIC 1 is the most important tactic in this chapter. Using it properly will earn you more points on the quantitative comparison questions of the GRE than you can gain by applying any of the others. *Be sure to master it!*

Most quantitative comparison questions contain variables. When those variables are replaced by simple numbers such as 0 or 1, the quantities become much easier to compare.

The reason that TACTIC 1 is so important is that it *guarantees* that on any quantitative comparison question that involves variables, you will be able to immediately eliminate two of the four choices, and very often a third choice as well, leaving you with at least a 50% chance of guessing correctly, and often a certainty. Try the following example, and then read the explanation very carefully.

EXAMPLE 3

$a < b < c < d$

Quantity A	Quantity B
ab	cd

SOLUTION.

- Replace a, b, c, and d with easy-to-use numbers which satisfy the condition $a < b < c < d$: for example, $a = 1$, $b = 3$, $c = 6$, $d = 10$. [See the guidelines that follow to learn why 1, 2, 3, 4 is not a good choice.]
- Evaluate the two quantities: $ab = (1)(3) = 3$, and $cd = (6)(10) = 60$.
- So *in this case,* Quantity B is greater.
- Does that mean that B is the correct answer? Not necessarily. Quantity B *is* greater this time, but will it be greater ***every single time, no matter what?***
- What it does mean is that neither A nor C could possibly be the correct answer: Quantity A can't be greater ***every single time, no matter what*** because it isn't greater *this* time; and the quantities aren't equal ***every single time, no matter what*** because they aren't equal *this* time.

So in the few seconds that it took you to plug in 1, 3, 6, and 10 for a, b, c, and d, you were able to eliminate two of the four choices. You now know that the correct answer is either B or D, and if you could do nothing else, you would now guess with a 50% chance of being correct.

But, of course, *you will do something else.* You will try some other numbers. But *which* numbers? Since the first numbers you chose were positive, try some negative numbers this time.

- Let $a = -5$, $b = -3$, $c = -2$, and $d = -1$.
- Evaluate: $ab = (-5)(-3) = 15$ and $cd = (-2)(-1) = 2$.
- So *in this case,* Quantity A is greater.
- Quantity B is *not* greater all the time. B is *not* the correct answer.
- The correct answer is **D**: It is impossible to determine which quantity is greater.

NOTES:

1. If for your second substitution you had chosen 3, 7, 8, 10 or 2, 10, 20, 35 or *any* four positive numbers, Quantity B would have been bigger. No matter how many substitutions you made, Quantity B would have been bigger each time, and you would have incorrectly concluded that B was the answer. In fact, if the given condition had been $0 < a < b < c < d$, then B *would have been* the correct answer.
2. Therefore, knowing which numbers to plug in when you use TACTIC 1 is critical. As long as you comply with the conditions given in the question, you have complete freedom in choosing the numbers. Some choices, however, are much better than others.

Here are some guidelines for deciding which numbers to use when applying TACTIC 1.

1. **The very best numbers to use first are: 1, 0, and –1.**
2. **Often, fractions between 0 and 1 are useful.**
3. **Occasionally, "large" numbers such as 10 or 100 can be used.**
4. **If there is more than one letter, it is permissible to replace each with the same number.**
5. **Do not impose any conditions not specifically stated.** In particular, do not assume that variables must be integers. For example, 3 is not the only number that satisfies $2 < x < 4$ (2.1, 3.95, and π all work). The expression $a < b < c < d$ does not mean that a, b, c, d are *integers,* let alone *consecutive* integers (which is why we didn't choose 1, 2, 3, and 4 in Example 3), nor does it mean that any or all of them are *positive.*

When you replace the variables in a quantitative comparison question with numbers, remember:

If the value of Quantity A is ever greater:	eliminate B and C — the answer must be A or D.
If the value of Quantity B is ever greater:	eliminate A and C — the answer must be B or D.
If the two quantities are ever equal:	eliminate A and B — the answer must be C or D.

You have learned that, no matter how hard a quantitative comparison is, as soon as you replace the variables, two choices can *immediately* be eliminated; and if you can't decide between the other two, just guess. This guarantees that in addition to correctly answering all the questions that you know how to solve, you will be able to answer correctly at least half, and probably many more, of the questions that you don't know how to do.

Practice applying TACTIC 1 on these examples.

EXAMPLE 4

$m > 0$ and $m \neq 1$

Quantity A	Quantity B
m^2	m^3

SOLUTION.
Use TACTIC 1. Replace m with numbers satisfying $m > 0$ and $m \neq 1$.

	Quantity A	Quantity B	Compare	Eliminate
Let $m = 2$	$2^2 = 4$	$2^3 = 8$	B is greater	A and C
Let $m = \frac{1}{2}$	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	A is greater	B

The answer is **D**.

EXAMPLE 5

Quantity A	Quantity B
$13y$	$15y$

SOLUTION.
Use TACTIC 1. There are no restrictions on y, so use the best numbers: 1, 0, –1.

	Quantity A	Quantity B	Compare	Eliminate
Let $y = 1$	$13(1) = 13$	$15(1) = 15$	B is greater	A and C
Let $y = 0$	$13(0) = 0$	$15(0) = 0$	They're equal	B

The answer is **D**.

EXAMPLE 6

Quantity A	Quantity B
$w + 11$	$w - 11$

SOLUTION.
Use TACTIC 1. There are no restrictions on w, so use the best numbers: 1, 0, –1.

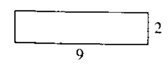
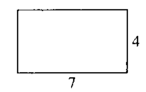
	Quantity A	Quantity B	Compare	Eliminate
Let $w = 1$	$1 + 11 = 12$	$1 - 11 = -10$	A is greater	B and C
Let $w = 0$	$0 + 11 = 11$	$0 - 11 = -11$	A is greater	
Let $w = -1$	$-1 + 11 = 10$	$-1 - 11 = -12$	A is greater	

Guess **A**. We let w be a positive number, a negative number, and 0. Each time, Quantity A was greater. That's not proof, but it justifies an educated guess. [The answer *is* A. Clearly, $11 > -11$, and if we add w to each side, we get: $w + 11 > w - 11$.]

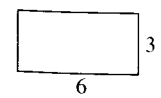
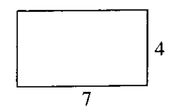
EXAMPLE 7

Quantity A	Quantity B
The perimeter of a rectangle whose area is 18	The perimeter of a rectangle whose area is 28

SOLUTION.
What's this question doing here? How can we use TACTIC 1? Where are the variables that we're supposed to replace? Well, each quantity is the perimeter of a rectangle, and the variables are the lengths and widths of these rectangles.

Quantity A	Quantity B	Compare	Eliminate
Choose a rectangle whose area is 18:  The perimeter here is $9 + 2 + 9 + 2 = \mathbf{22}$	Choose a rectangle whose area is 28:  The perimeter here is $7 + 4 + 7 + 4 = \mathbf{22}$	Quantities A and B are equal	A and B

Keep Quantity B, but take a different rectangle of area 18 when evaluating Quantity A:

 Perimeter $= 3 + 6 + 3 + 6 = \mathbf{18}$	 Perimeter $= \mathbf{22}$	B is greater	C

The answer is **D**.

EXAMPLE 8

$a = \frac{2}{3}t \quad b = \frac{5}{6}t \quad c = \frac{3}{5}b$

Quantity A	Quantity B
$3a$	$4c$

SOLUTION.
Use TACTIC 1. First, try the easiest number: let $t = 0$. Then a, b, and c are each 0, and *in this case*, the quantities are equal — they're both 0. Eliminate A and B. Now, try another number for t. The obvious choice is 1, but then a, b, and c will all be fractions. To avoid this, let $t = 6$. Then, $a = \frac{2}{3}(6) = 4$, $b = \frac{5}{6}(6) = 5$, and $c = \frac{3}{5}(5) = 3$. This time, $3a = 3(4) = \mathbf{12}$ and $4b = 4(3) = \mathbf{12}$. *Again, the two quantities are equal.* Choose **C**.

NOTE: You should consider answering this question directly (i.e., without plugging in numbers), *only if you are very comfortable with both fractions and elementary algebra.* Here's the solution:

$$c = \frac{3}{5}b = \frac{3}{5}\left(\frac{5}{6}t\right) = \frac{1}{2}t$$

Therefore, $2c = t$, and $4c = 2t$. Since $a = \frac{2}{3}t$, $3a = 2t$. So, $4c = 3a$. The answer is **C**.

TACTIC 2 Choose an Appropriate Number

This is just like TACTIC 1. We are replacing a variable with a number, but the variable isn't mentioned in the problem.

EXAMPLE 9

Every band member is either 15, 16, or 17 years old.
One third of the band members are 16, and
twice as many band members are 16 as 15.

Quantity A	Quantity B
The number of 17-year-old band members	The total number of 15- and 16-year-old band members

If the first sentence of Example 9 had been "There are n students in the school band, all of whom are 15, 16, or 17 years old," the problem would have been identical to this one. Using TACTIC 1, you could have replaced n with an easy-to-use number, such as 6, and solved: $\frac{1}{3}(6) = 2$ are 16 years old; 1 is 15, and the remaining 3 are 17. The answer is **C**.

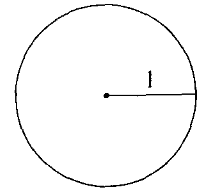
The point of TACTIC 2 is that you can plug in numbers even if there are no variables. As discussed in TACTIC 3, Chapter 8, this is especially useful on problems involving percents, in which case 100 is a good number, and problems involving fractions, in which case the LCD of the fractions is a good choice. However, the use of TACTIC 2 is not limited to these situations. Try using TACTIC 2 on the following three problems.

EXAMPLE 10

The perimeter of a square and the
circumference of a circle are equal.

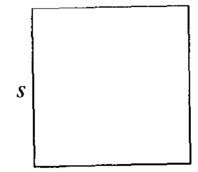
Quantity A	Quantity B
The area of the circle	The area of the square

SOLUTION.
First use TACTIC 1, Chapter 7: draw a diagram.



$C = 2\pi(1) = 2\pi$
$A = \pi(1)^2 = \pi \approx 3.14$

Then use TACTIC 2: choose an easy-to-use number. Let the radius of the circle be 1. Then its area is π. Let s be the side of the square:



$P = 4s$
$A = s^2$

$$4s = 2\pi \approx 6 \Rightarrow s \approx 1.5 \Rightarrow$$
$$\text{area of the square} \approx (1.5)^2 = 2.25$$

The answer is **A**.

EXAMPLE 11

Jen, Ken, and Len divided a cash prize.
Jen took 50% of the money and spent $\frac{3}{5}$ of what she took.
Ken took 40% of the money and spent $\frac{3}{4}$ of what he took.

Quantity A	Quantity B
The amount that Jen spent	The amount that Ken spent

SOLUTION.
Use TACTIC 2. Assume the prize was \$100. Then Jen took \$50 and spent $\frac{3}{5}(\$50) = \$30$. Ken took \$40 and spent $\frac{3}{4}(\$40) = \$30$. The answer is **C**.

EXAMPLE 12

Eliane types twice as fast as Delphine.
Delphine charges 50% more per page than Eliane.

Quantity A	Quantity B
Amount Eliane earns in 9 hours	Amount Delphine earns in 12 hours

SOLUTION.
Use TACTIC 2. Choose appropriate numbers. Assume Delphine can type 1 page per hour and Eliane can type 2. Assume Eliane charges \$1.00 per page and Delphine charges \$1.50. Then in 9 hours, Eliane types 18 pages, earning **\$18.00**. In 12 hours, Delphine types 12 pages, earning 12 × \$1.50 = **\$18.00**. The answer is **C**.

TACTIC 3 Make the Problem Easier: Do the Same Thing to Each Quantity

A quantitative comparison question can be treated as an equation or an inequality. Either:

Quantity A < Quantity B, or
Quantity A = Quantity B, or
Quantity A > Quantity B

In solving an equation or an inequality, you can always add the same thing to each side or subtract the same thing from each side. Similarly, in solving a quantitative comparison, you can always add the same thing to quantities A and B or subtract the same thing from quantities A and B. You can also multiply or divide each side of an equation or inequality by the same number, *but in the case of* *<u>inequalities</u>* *you can do this only if the number is positive.* Since you don't know whether the quantities are equal or unequal, you cannot multiply or divide by a variable *unless you know that it is positive.* If quantities A and B are both positive you may square them or take their square roots.

To illustrate the proper use of TACTIC 3, we will give alternative solutions to examples 4, 5, and 6, which we already solved using TACTIC 1.

EXAMPLE 4

$m > 0$ and $m \neq 1$

Quantity A	Quantity B
m^2	m^3

SOLUTION.

	Quantity A	Quantity B
Divide each quantity by m^2 (that's OK — m^2 is positive):	$\frac{m^2}{m^2} = 1$	$\frac{m^3}{m^2} = m$

This is a much easier comparison. Which is greater, m or 1? We don't know. We know $m > 0$ and $m \neq 1$, but it could be greater than 1 or less than 1. The answer is **D**.

EXAMPLE 5

Quantity A	Quantity B
$13y$	$15y$

SOLUTION.

	Quantity A	Quantity B
Subtract $13y$ from each quantity:	$13y - 13y = 0$	$15y - 13y = 2y$

Since there are no restrictions on y, $2y$ could be greater than, less than, or equal to 0. The answer is **D**.

EXAMPLE 6

Quantity A	Quantity B
$w + 11$	$w - 11$

SOLUTION.

	Quantity A	Quantity B
Subtract w from each quantity:	$(w + 11) - w = 11$	$(w - 11) - w = -11$

Clearly, 11 is greater than −11. Quantity **A** is greater.

Here are five more examples on which to practice TACTIC 3.

EXAMPLE 13

Quantity A	Quantity B
$\frac{1}{3} + \frac{1}{4} + \frac{1}{9}$	$\frac{1}{9} + \frac{1}{3} + \frac{1}{5}$

SOLUTION.

	Quantity A	Quantity B
Subtract $\frac{1}{3}$ and $\frac{1}{9}$ from each quantity:	$\cancel{\frac{1}{3}} + \frac{1}{4} + \cancel{\frac{1}{9}}$	$\cancel{\frac{1}{9}} + \cancel{\frac{1}{3}} + \frac{1}{5}$

Since $\frac{1}{4} > \frac{1}{5}$, the answer is **A**.

EXAMPLE 14

Quantity A	Quantity B
$(43 + 59)(17 - 6)$	$(43 + 59)(17 + 6)$

SOLUTION.

	Quantity A	Quantity B
Divide each quantity by (43 + 59):	$\cancel{(43+59)}(17-6)$	$\cancel{(43+59)}(17+6)$

Clearly, $(17 + 6) > (17 - 6)$. The answer is **B**.

EXAMPLE 15

Quantity A	Quantity B
$(43 - 59)(43 - 49)$	$(43 - 59)(43 + 49)$

SOLUTION.

> **CAUTION**
>
> $(43 - 59)$ is negative, and you may not divide the two quantities by a negative number.

The easiest alternative is to note that Quantity A, being the product of 2 negative numbers, is positive, whereas Quantity B, being the product of a negative number and a positive number, is negative, and so Quantity A is greater.

EXAMPLE 16

a is a negative number

Quantity A	Quantity B
a^2	$-a^2$

SOLUTION.

	Quantity A	Quantity B
Add a^2 to each quantity:	$a^2 + a^2 = 2a^2$	$-a^2 + a^2 = 0$

Since a is negative, $2a^2$ is positive. The answer is **A**.

EXAMPLE 17

Quantity A	Quantity B
$\dfrac{\sqrt{20}}{2}$	$\dfrac{5}{\sqrt{5}}$

SOLUTION.

	Quantity A	Quantity B
Square each quantity:	$\left(\dfrac{\sqrt{20}}{2}\right)^2 = \dfrac{20}{4} = 5$	$\left(\dfrac{5}{\sqrt{5}}\right)^2 = \dfrac{25}{5} = 5$

The answer is **C**.

TACTIC 4

Ask "Could They Be Equal?" and "Must They Be Equal?"

TACTIC 4 has many applications, but is most useful when one of the quantities contains a variable and the other contains a number. In this situation ask yourself, "Could they be equal?" If the answer is "yes," eliminate A and B, and then ask, "Must they be equal?" If the second answer is "yes," then C is correct; if the second answer is "no," then choose D. When the answer to "Could they be equal?" is "no," we usually know right away what the correct answer is. In both questions, "Could they be equal" and "Must they be equal," the word *they* refers, of course, to quantities A and B.

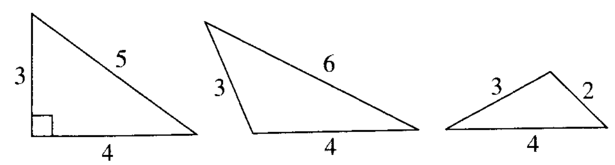
Let's look at a few examples.

EXAMPLE 18

The sides of a triangle are 3, 4, and x

Quantity A	Quantity B
x	5

SOLUTION.
Could they be equal? Could $x = 5$? Of course. That's the all-important 3-4-5 right triangle. Eliminate A and B. Must they be equal? Must $x = 5$? If you're not sure, try drawing an acute or an obtuse triangle. The answer is No. Actually, x can be any number satisfying: $1 < x < 7$. (See KEY FACT J12, the triangle inequality, and the figure below.) The answer is **D**.



EXAMPLE 19

$56 < 5c < 64$

Quantity A	Quantity B
c	12

SOLUTION.
Could they be equal? Could $c = 12$? If $c = 12$, then $5c = 60$, so, yes, they could be equal. Eliminate A and B. Must they be equal? Must $c = 12$? Could c be more or less than 12? BE CAREFUL: $5 \times 11 = 55$, which is too small; and $5 \times 13 = 65$, which is too big. Therefore, the only *integer* that c could be is 12; but *c doesn't have to be an integer*. The *only* restriction is that $56 < 5c < 64$. If $5c$ were 58 or 61.6 or 63, then c would not be 12. The answer is **D**.

EXAMPLE 20

School A has 100 teachers and School B has 200 teachers. Each school has more female teachers than male teachers.

Quantity A	Quantity B
The number of female teachers at School A	The number of female teachers at School B

SOLUTION.
Could they be equal? Could the number of female teachers be the same in both schools? No. More than half (i.e., more than 100) of School B's 200 teachers are female, but School A has only 100 teachers in all. The answer is **B**.

EXAMPLE 21

$(m + 1)(m + 2)(m + 3) = 720$

Quantity A	Quantity B
$m + 2$	10

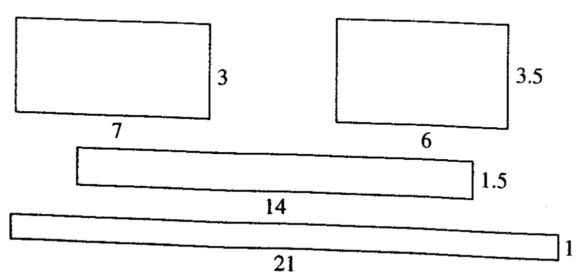
SOLUTION.
Could they be equal? Could $m + 2 = 10$? No, if $m + 2 = 10$, then $m + 1 = 9$ and $m + 3 = 11$, and $9 \times 10 \times 11 = 990$, which is too big. The answer is *not* C, and since $m + 2$ clearly has to be smaller than 10, the answer is **B**.

EXAMPLE 22

Quantity A	Quantity B
The perimeter of a rectangle whose area is 21	20

SOLUTION.
Could they be equal? Could a rectangle whose area is 21 have a perimeter of 20? Yes, if its length is 7 and its width is 3: $7 + 3 + 7 + 3 = 20$. Eliminate A and B. Must they be equal? If you're *sure* that there is no other rectangle with an area of 21, then choose C; if you're *not* sure, guess between C and D; if you *know* there are other rectangles of area 21, choose D.

There are other possibilities — lots of them; here are a 7×3 rectangle and a few other rectangles whose areas are 21:



TACTIC 5

Don't Calculate: Compare

Avoid unnecessary calculations. You don't have to determine the exact values of Quantity A and Quantity B; you just have to compare them.

TACTIC 5 is the special application of TACTIC 7, Chapter 10 (Don't do more than you have to) to quantitative comparison questions. Using TACTIC 5 allows you to solve many quantitative comparisons without doing tedious calculations, thereby saving you valuable test time that you can use on other questions. *Before you start calculating,* stop, look at the quantities, and ask yourself, "Can I easily and quickly determine which quantity is greater without doing *any* arithmetic?" Consider Examples 23 and 24, which look very similar, but really aren't.

EXAMPLE 23

Quantity A	Quantity B
37×43	30×53

EXAMPLE 24

Quantity A	Quantity B
37×43	39×47

Example 23 is very easy. Just multiply: $37 \times 43 = 1591$ and $30 \times 53 = 1590$. The answer is **A**.

Example 24 is even easier. *Don't* multiply. In less time than it takes to do the multiplications, even with the calculator, you can see that $37 < 39$ and $43 < 47$, so clearly $37 \times 43 < 39 \times 47$. The answer is **B**. *You don't get any extra credit for taking the time to determine the value of each product!*

Remember: do not start calculating immediately. Always take a second or two to glance at each quantity. In Example 23 it's not at all clear which product is larger, so you have to multiply. In Example 24, however, no calculations are necessary.

These are problems on which poor test-takers do a lot of arithmetic and good test-takers think! Practicing TACTIC 5 will help you become a good test-taker.

Now, test your understanding of TACTIC 5 by solving these problems.

EXAMPLE 25

Quantity A	Quantity B
The number of years from 1776 to 1929	The number of years from 1767 to 1992

EXAMPLE 26

Quantity A	Quantity B
$45^2 + 25^2$	$(45 + 25)^2$

EXAMPLE 27

Quantity A	Quantity B
$45(35 + 65)$	$45 \times 35 + 45 \times 65$

EXAMPLE 28

Marianne earned a 75 on each of her first three math tests and an 80 on her fourth and fifth tests.

Quantity A	Quantity B
Marianne's average after 4 tests	Marianne's average after 5 tests

SOLUTIONS 25–28

Performing the Indicated Calculations	Using TACTIC 5 to Avoid Doing the Calculations
25. Quantity A: $1929 - 1776 = 153$ Quantity B: $1992 - 1767 = 225$ The answer is **B**.	25. The subtraction is easy enough, but why do it? The dates in Quantity **B** start earlier and end later. Clearly, they span more years. You don't need to know how many years. The answer is **B**.
26. Quantity A: $45^2 + 25^2 = 2025 + 625 = 2650$ Quantity B: $(45 + 25)^2 = 70^2 = 4900$ The answer is **B**.	26. For *any positive numbers* a and b: $(a + b)^2 > a^2 + b^2$. You should do the calculations only if you don't know this fact. The answer is **B**.

Performing the Indicated Calculations	Using TACTIC 5 to Avoid Doing the Calculations
27. Quantity A: $45(35 + 65) = 45(100) = 4500$ Quantity B: $45 \times 35 + 45 \times 65 = 1575 + 2925 = 4500$ The answer is **C**.	27. This is just the distributive property (KEY FACT A20), which states that, for *any* numbers a, b, c: $a(b + c) = ab + ac$. The answer is **C**.
28. Quantity A: $\frac{75+75+75+80}{4} = \frac{305}{4} = 76.25$ Quantity B: $\frac{75+75+75+80+80}{5} = \frac{385}{5} = 77$ The answer is **B**.	28. Remember, you want to know which average is higher, *not* what the averages are. After 4 tests Marianne's average is clearly less than 80, so an 80 on the fifth test had to *raise* her average (KEY FACT E4). The answer is **B**.

Caution

TACTIC 5 is important, but *don't spend a lot of time looking for ways to avoid a simple calculation.*

TACTIC 6 Know When to Avoid Choice D

If Quantity A and Quantity B are both fixed numbers, the answer cannot be D.

Notice that D was not the correct answer to any of the six examples discussed under TACTIC 5. Those problems had no variables. The quantities were all specific numbers. In each of the next four examples, Quantity A and Quantity B are also fixed numbers. In each case, either the two numbers are equal or one is greater than the other. It can *always* be determined, and so D *cannot be the correct answer to any of these problems.* If, while taking the GRE, you find a problem of this type that you can't solve, just guess: A, B, or C. Now try these four examples.

EXAMPLE 29

Quantity A	Quantity B
The number of seconds in one day	The number of days in one century

EXAMPLE 30

Quantity A	Quantity B
The area of a square whose sides are 4	Twice the area of an equilateral triangle whose sides are 4

EXAMPLE 31

Three fair coins are flipped.

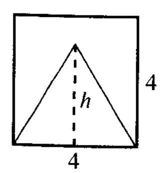
Quantity A	Quantity B
The probability of getting one head	The probability of getting two heads

EXAMPLE 32

Quantity A	Quantity B
The time it takes to drive 40 miles at 35 mph	The time it takes to drive 35 miles at 40 mph

Here's the important point to remember: don't choose D because *you* can't determine which quantity is bigger; choose D only if *nobody* could determine it. *You* may or may not know how to compute the number of seconds in a day, the area of an equilateral triangle, or a certain probability, but *these calculations can be made.*

SOLUTIONS 29–32

Direct Calculation	Solution Using Various TACTICS
29. Recall the facts you need and calculate. 60 seconds = 1 minute, 60 minutes = 1 hour, 24 hours = 1 day, 365 days = 1 year, and 100 years = 1 century. Quantity A: $60 \times 60 \times 24 = 86{,}400$ Quantity B: $365 \times 100 = 36{,}500$ Even if we throw in some days for leap years, the answer is clearly **A**.	29. The point of TACTIC 6 is that even if you have no idea how to calculate the number of seconds in a day, you can eliminate two choices. The answer *cannot* be D, and it would be an incredible coincidence if these two quantities were actually equal, so don't choose C. *Guess* between A and B.
30. Calculate both areas. (See KEY FACT J15 for the easy way to find the area of an equilateral triangle.) Quantity A: $A = s^2 = 4^2 = 16$ Quantity B: $A = \frac{s^2\sqrt{3}}{4} = \frac{4^2\sqrt{3}}{4} = 4\sqrt{3}$; and *twice A* is $8\sqrt{3}$. Since $\sqrt{3} \approx 1.7$, $8\sqrt{3} \approx 13.6$. The answer is **A**.	30. Use TACTIC 5: don't calculate—draw a diagram and then compare.  Since the height of the triangle is less than 4, its area is less than $\frac{1}{2}(4)(4) = 8$, and twice its area is less than 16, the area of the square. The answer is **A**. (If you don't see that, and just have to guess in order to move on, be sure not to guess D.)

Direct Calculation	Solution Using Various TACTICS
31. When a coin is flipped 3 times, there are 8 possible outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT. Of these, 3 have one head and 3 have two heads. Each probability is $\frac{3}{8}$. The answer is **C**.	31. Don't forget TACTIC 5. Even if you know how, you don't *have to* calculate the probabilities. When 3 coins are flipped, getting two heads means getting one tail. Therefore, the probability of two heads equals the probability of one tail, which by symmetry equals the probability of one head. The answer is **C**. (If you don't remember anything about probability, TACTIC 5 at least allows you to eliminate D before you guess.)
32. Since $d = rt$, $t = \frac{d}{r}$ [see Sect. 11-H]. Quantity A: $\frac{40}{35}$ hours—more than 1 hour. Quantity B: $\frac{35}{40}$ hours—less than 1 hour. The answer is **A**.	32. You *do* need to know these formulas, but *not* for this problem. At 35 mph it takes *more than an hour* to drive 40 miles. At 40 mph it takes *less than an hour* to drive 35 miles. Choose **A**.

Practice Exercises

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

	Quantity A	Quantity B
1.	197 + 398 + 586	203 + 405 + 607

$x > 0$

	Quantity A	Quantity B
2.	$10x$	$\frac{10}{x}$

	Quantity A	Quantity B
3.	The time that it takes to type 7 pages at a rate of 6 pages per hour	The time that it takes to type 6 pages at a rate of 7 pages per hour

$cd < 0$

	Quantity A	Quantity B
4.	$(c + d)^2$	$c^2 + d^2$

a, b, and c are the measures of the angles of isosceles triangle ABC.
x, y, and z are the measures of the angles of right triangle XYZ.

	Quantity A	Quantity B
5.	The average of a, b, and c	The average of x, y, and z

$b < 0$

	Quantity A	Quantity B
6.	$6b$	b^6

	Quantity A	Quantity B
7.	The area of a circle whose radius is 17	The area of a circle whose diameter is 35

Line k goes through (1,1) and (5,2).
Line m is perpendicular to k.

	Quantity A	Quantity B
8.	The slope of line k	The slope of line m

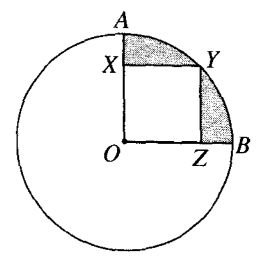
x is a positive integer

	Quantity A	Quantity B
9.	The number of multiples of 6 between 100 and $x + 100$	The number of multiples of 9 between 100 and $x + 100$

$x + y = 5$
$y - x = -5$

	Quantity A	Quantity B
10.	y	0

	Quantity A	Quantity B
11.	$\frac{7}{8}$	$\left(\frac{7}{8}\right)^5$



O is the center of the circle of radius 6. $OXYZ$ is a square.

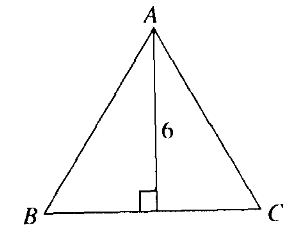
	Quantity A	Quantity B
12.	The area of the shaded region	12

The number of square inches in the surface area of a cube is equal to the number of cubic inches in its volume.

	Quantity A	Quantity B
13.	The length of an edge of the cube	6 inches

$1 < x < 4$

	Quantity A	Quantity B
14.	πx	x^2



$AB = AC$

	Quantity A	Quantity B
15.	The area of $\triangle ABC$	3

ANSWER KEY

1. **B**	4. **B**	7. **B**	10. **C**	13. **C**
2. **D**	5. **C**	8. **A**	11. **A**	14. **D**
3. **A**	6. **B**	9. **D**	12. **B**	15. **D**

ANSWER EXPLANATIONS

The direct mathematical solution to a problem is almost always the preferable one, so it is given first. It is often followed by one or more alternative solutions, indicated by a double asterisk (**), based on the various tactics discussed in this chapter. Occasionally, a solution based on one of the tactics is much easier than the straightforward one. In that case, it is given first.

1. **(B)** Using the onscreen calculator, this can easily be solved in 20 or 30 seconds by adding, but in only 5 seconds by thinking! Use TACTIC 5: don't calculate; compare. Each of the three numbers in Quantity B is greater than the corresponding numbers in Quantity A.

2. **(D)** Use TACTIC 1. When $x = 1$, the quantities are equal; when $x = 2$, they aren't.
**Use TACTIC 3

	Quantity A	Quantity B
	$10x$	$\frac{10}{x}$
Multiply each quantity by x (this is OK since $x > 0$):	$10x^2$	10
Divide each quantity by 10:	x^2	1

This is a much easier comparison. x^2 *could* equal 1, but doesn't have to. The answer is Choice D.

3. **(A)** You can easily calculate each of the times — divide 7 by 6 to evaluate Quantity A, and 6 by 7 in Quantity B. However, it is easier to just observe that Quantity A is more than one hour, whereas Quantity B is less than one hour.

4. **(B)** Use TACTIC 3

	Quantity A	Quantity B
Expand Quantity A:	$(c + d)^2 =$ $c^2 + 2cd + d^2$	$c^2 + d^2$
Subtract $c^2 + d^2$ from each quantity:	$2cd$	0

Since it is given that $cd < 0$, so is $2cd$.

**If you can't expand $(c + d)^2$, then use TACTIC 1. Replace c and d with numbers satisfying $cd < 0$.

	Quantity A	Quantity B	Compare	Eliminate
Let $c = 1$ and $d = -1$	$(1 + -1)^2 = 0$	$1^2 + (-1)^2 =$ $1 + 1 = 2$	B is greater	A and C
Let $c = 3$ and $d = -5$	$(3 + -5)^2 =$ $(-2)^2 = 4$	$3^2 + (-5)^2 =$ $9 + 25 = 34$	B is greater	

Both times Quantity B was greater: choose B.

5. **(C)** The average of 3 numbers is their sum divided by 3. Since in *any* triangle the sum of the measures of the 3 angles is 180°, the average in each quantity is equal to $180 \div 3 = 60$.

**Use TACTIC 1. Pick values for the measures of the angles. For example, in isosceles $\triangle ABC$ choose 70, 70, 40; in right $\triangle XYZ$, choose 30, 60, 90. Each average is 60. Choose C.

6. **(B)** Since $b < 0$, $6b$ is negative, whereas b^6 is positive.

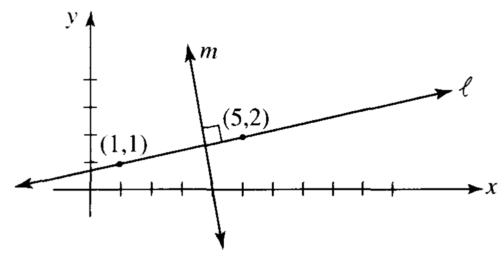
**Use TACTIC 1. Replace b with numbers satisfying $b < 0$.

	Quantity A	Quantity B	Compare	Eliminate
Let $b = -1$	$6(-1) = -6$	$(-1)^6 = 1$	B is greater	A and C
Let $b = -2$	$6(-2) = -12$	$(-2)^6 = 64$	B is greater	

Both times Quantity B was greater: choose B.

7. **(B)** Use TACTIC 5: don't calculate the two areas; compare them. The circle in Quantity A is the area of a circle whose radius is 17 and whose diameter is 34. Quantity B is the area of a circle whose diameter is 35, and so is clearly greater.

8. **(A)** Use TACTIC 5: don't calculate either slope. Quickly, make a rough sketch of line k, going through (1,1) and (5,2), and draw line m perpendicular to it.



Line k has a positive slope (it slopes upward), whereas line m has a negative slope (it slopes downward). Quantity A is greater.

[Note: The slope of k is $\frac{1}{4}$ and the slope of m is -4, but you don't need to calculate either one. See Section 11-N for all the facts you need to know about slopes.]

**If you don't know this fact about slopes, use TACTIC 6. The answer cannot be Choice D, and if two lines intersect, their slopes cannot be equal, so eliminate Choice C. Guess Choice A or B.

9. **(D)** Every sixth integer is a multiple of 6 and every ninth integer is a multiple of 9, so in a large interval there will be many more multiples of 6. But in a very small interval, there might be none or possibly just one of each.

**Use TACTIC 1. Let $x = 1$. Between 100 and 101 there are *no* multiples of 6 and *no* multiples of 9. Eliminate Choices A and B. Choose a large number for x: 100, for example. Between 100 and 200 there are many more multiples of 6 than there are multiples of 9. Eliminate Choice C.

10. **(C)** Add the equations.

$$\begin{array}{r} x + y = 5 \\ +\ y - x = -5 \\ \hline 2y = 0 \end{array}$$

Since $2y = 0$, $y = 0$.

**Use TACTIC 4. Could $y = 0$? In each equation, if $y = 0$, then $x = -5$. So, y can equal 0. Eliminate Choices A and B, and either guess between Choices C and D or continue. Must $y = 0$? Yes, when you have two linear equations in two variables, there is only one solution, so nothing else is possible.

11. **(A)** With a calculator, you can multiply $\frac{7}{8} \times \frac{7}{8} \times \frac{7}{8} \times \frac{7}{8} \times \frac{7}{8}$, but it is annoying and time-consuming. However, you can avoid the arithmetic, if you know KEY FACT A24:

If $0 < x < 1$ and $n > 1$, then $x^n < x$.

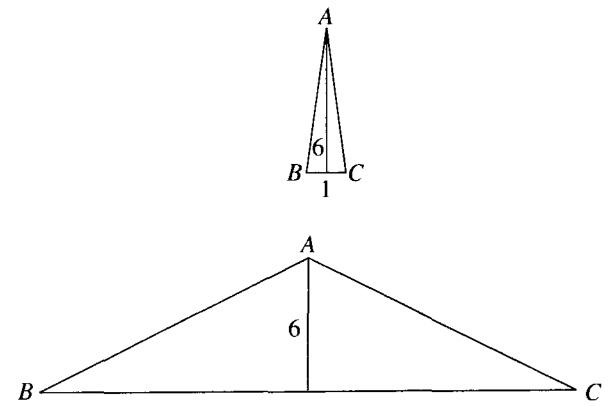
Since $\frac{7}{8} < 1$, then $\left(\frac{7}{8}\right)^5 < \frac{7}{8}$.

12. **(B)** The area of the shaded region is the area of quarter-circle AOB minus the area of the square. Since $r = OA = 6$, the area of the quarter-circle is $\frac{1}{4}\pi r^2 = \frac{1}{4}36\pi = 9\pi$. OY, the diagonal of the square, is 6 (since it is a radius of the circle), so OZ, the side of the square, is $\frac{6}{\sqrt{2}}$ [See KEY FACT J8]. So the area of the square is $\left(\frac{6}{\sqrt{2}}\right)^2 = \frac{36}{2} = 18$. Finally, the area of the shaded region is $9\pi - 18$, which is approximately 10.
   **The solution above requires several steps. [See Sections 11-J, K, L to review any of the facts used.] If you can't reason through this, you still should be able to answer this question correctly. Use TACTIC 6. The shaded region has a definite area, which is either 12, more than 12, or less than 12. Eliminate D. Also, the area of a curved region almost always involves π, so assume the area isn't exactly 12. Eliminate Choice C. You can now *guess* between Choices A and B, but if you trust the diagram and know a little bit you can improve your guess. If you know that the area of the circle is 36π, so that the quarter-circle is 9π or about 28, you can estimate the shaded region. It's well less than half of the quarter-circle, so less than 14 and probably less than 12. Guess Choice B.

13. **(C)** Use TACTIC 4. Could the edge be 6? Test it. If each edge is 6, the area of each face is $6 \times 6 = 36$, and since a cube has 6 faces, the total surface area is $6 \times 36 = 216$. The volume is $6^3 = 216$. So the quantities could be equal. Eliminate Choices A and B. If you have a sense that this is the only cube with this property, choose C. In fact, if you had no idea how to do this, you might use TACTIC 6, assume that there is only one way, eliminate Choice D, and then guess C. The direct solution is simple enough if you know the formulas. If e is the length of an edge of the cube, then the area is $6e^2$ and the volume is e^3: $6e^2 = e^3 \Rightarrow 6 = e$.

14. **(D)** There are several ways to answer this question. Use TACTIC 1: plug in a number for x. If $x = 2$, Quantity A is 2π, which is slightly more than 6, and Quantity B is $2^2 = 4$. Quantity A is greater: eliminate Choices B and C. Must Quantity A be greater? If the only other number you try is $x = 3$, you'll think so, because $3^2 = 9$, but $3\pi > 9$. But remember, x does not have to be an integer: $3.9^2 > 15$, whereas $3.9\pi < 4\pi$, which is a little over 12.
   **Use TACTIC 4. Could $\pi x = x^2$? Yes, if $x = \pi$. Must $x = \pi$? No.
   **Use TACTIC 3. Divide each quantity by x: Now Quantity A is π and Quantity B is x. Which is bigger, π or x? We cannot tell.

15. **(D)** Use TACTIC 4. Could the area of $\triangle ABC = 3$? Since the height is 6, the area would be 3 only if the base were 1: $\frac{1}{2}(1)(6) = 3$. Could $BC = 1$? Sure (see the figure). Must the base be 1? Of course not.



Data Interpretation Questions

CHAPTER 10

Three of the 20 questions in each quantitative section of the GRE are data interpretation questions. As their name suggests, these questions are always based on the information that is presented in some form of a graph or a chart. Occasionally, the data are presented in a chart or table, but much more often, they are presented graphically. The most common types of graphs are

- line graphs
- bar graphs
- circle graphs

In each section, the data interpretation questions are three consecutive questions, say questions 14, 15, and 16, all of which refer to the same set of graphs or charts.

When the first data interpretation question appears, either the graphs will be on the left-hand side of the screen, and the question will be on the right-hand side, or the graphs will be at the top of the screen and the question will be below them. It is possible, but unlikely, that you will have to scroll down in order to see all of the data. After you answer the first question, a second question will replace it on the right-hand side (or the bottom) of the screen; the graphs, of course, will still be on the screen for you to refer to.

The tactics discussed in this chapter can be applied to any type of data, no matter how they are displayed. In the practice exercises at the end of the chapter, there are data interpretation questions based on the types of graphs that normally appear on the GRE. Carefully, read through the answer explanations for each exercise, so that you learn the best way to handle each type of graph.

Infrequently, an easy data interpretation question will require only that you read the graph and find a numerical fact that is displayed. Usually, however, you will have to do some calculation on the data that you are analyzing. In harder questions, you may be given hypothetical situations and asked to make inferences based on the information provided in the given graphs.

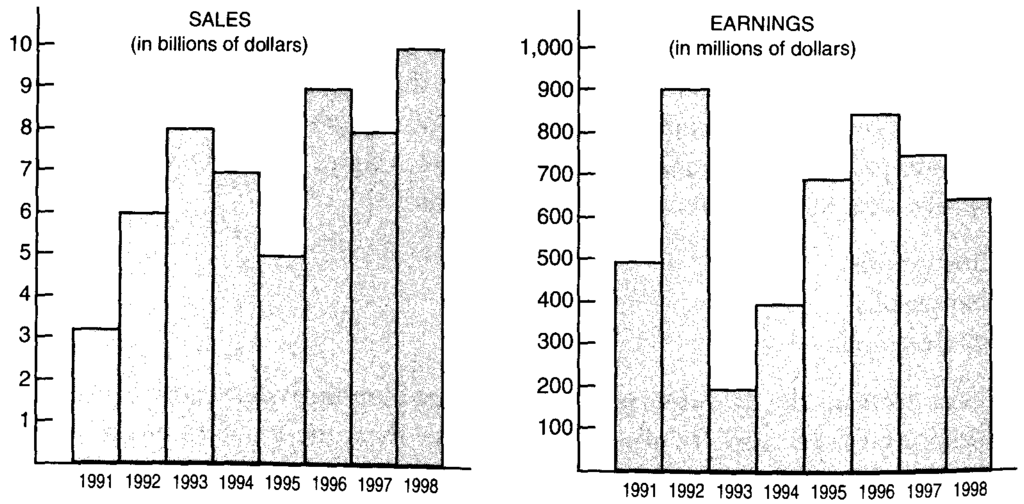
Most data interpretation questions are multiple-choice questions, but some could be multiple-answer or numeric entry questions. They are never quantitative comparisons.

Testing Tactics

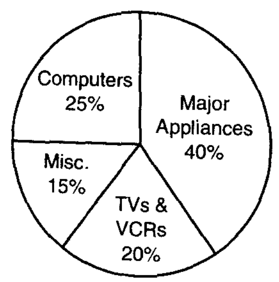
The four questions that follow will be used to illustrate the tactics that you should use in answering data interpretation questions. Remember, however, that on the GRE there will always be three questions that refer to a particular graph or set of graphs.

Questions 1–4 refer to the following graphs.

Sales and Earnings of XYZ Corporation 1991–1998



1998 Sales of XYZ Corporation by Category



1. What is the average (arithmetic mean) in billions of dollars of the sales of XYZ Corporation for the period 1991–1998?
   (A) 5.5 (B) 6.0 (C) 7.0 (D) 8.0 (E) 8.5

2. For which year was the percentage increase in earnings from the previous year the greatest?
   (A) 1992 (B) 1993 (C) 1994 (D) 1995 (E) 1996

3. Which of the following statements can be deduced from the data in the given charts and circle graph?

   Indicate *all* such statements.

   [A] Sales of major appliances in 1998 exceeded total sales in 1991.

   [B] Earnings for the year in which earnings were greatest were more than sales for the year in which sales were lowest.

   [C] If in 1998, the sales of major appliances had been 10% less, and the sales of computers had been 10% greater, the sales of major appliances would have been less than the sales of computers.

4. What was the ratio of earnings to sales in 1993?

   $\dfrac{\boxed{}}{\boxed{}}$

TACTIC 1 First Read the Titles

When the first data interpretation question appears on the screen, do not even read it! Before you attempt to answer a data interpretation question, take 15 to 30 seconds to study the graphs. Try to get a general idea about the information that is being displayed.

Observe that the bar graphs on which questions 1–4 are based present two different sets of data. The bar graph on the left-hand side provides information about the sales of XYZ Corporation, and the right-hand graph provides information about the corporation's earnings. Also, note that whereas sales are given in billions of dollars, earnings are given in millions of dollars. Finally, the circle graph gives a breakdown by category of the sales of XYZ Corporation for one particular year.

TACTIC 2 Don't Confuse Percents and Numbers

Many students make mistakes on data interpretation questions because they don't distinguish between absolute numbers and percents. Although few students would look at the circle graph shown and think that XYZ Corporation sold 25 computers in 1998, many would mistakenly think that it sold 15% more major appliances than computers.

The problem is particularly serious when the questions involve percent increases or percent decreases. In question 2 you are not asked for the year in which the increase in earnings from the previous year was the greatest. You are asked for the year in which the percent increase in earnings was the greatest. A quick glance at the right-hand graph reveals that the greatest increase occurred from 1991 to 1992 when earnings jumped by $400 million. However, when we solve this problem in the discussion of TACTIC 3, you will see that Choice A is not the correct answer.

NOTE: Since many data interpretation questions involve percents, you should carefully study Section 11-C, and be sure that you know all of the tactics for solving percent problems. In particular, always try to use the number 100 or 1000, since it is so easy to mentally calculate percents of powers of 10.

TACTIC 3 Whenever Possible, Estimate

Although you have access to the onscreen calculator, when you take the GRE, you will not be expected to do complicated or lengthy calculations. Often, thinking and using some common sense can save you considerable time. For example, it may seem that in order to get the correct answer to question 2, you have to calculate five different percents. In fact, you only need to do one calculation, and that one you can do in your head!

Just looking at the Earnings bar graph, it is clear that the only possible answers are 1992, 1994, and 1995, the three years in which there was a significant increase in earnings from the year before. From 1993 to 1994 expenditures doubled, from \$200 million to \$400 million — an increase of 100%. From 1991 to 1992 expenditures increased by \$400 million (from \$500 million to \$900 million), but that is less than a 100% increase (we don't care how much less). From 1994 to 1995 expenditures increased by \$300 million (from \$400 million to \$700 million); but again, this is less than a 100% increase. The answer is **C**.

TACTIC 4 Do Each Calculation Separately

As in all multiple-answer questions, question 3 requires you to determine which of the statements are true. The key is to work with the statements individually.

To determine whether or not statement A is true, look at both the Sales bar graph and the circle graph. In 1998, total sales were \$10 billion, and sales of major appliances accounted for 40% of the total: 40% of \$10 billion = \$4 billion. This exceeds the \$3 billion total sales figure for 1991, so statement A is true.

In 1992, the year in which earnings were greatest, earnings were \$900 million. In 1991, the year in which sales were lowest, sales were \$3 billion, which is much greater than \$900 million. Statement B is false.

In 1998, sales of major appliances were \$4 billion. If they had been 10% less, they would have been \$3.6 billion. That year, sales of computers were \$2.5 billion (25% of \$10 billion). If computer sales had increased by 10%, sales would have increased by \$0.25 billion to \$2.75 billion. Statement C is false.

The answer is **A**.

TACTIC 5 Use Only the Information Given

You must base your answer to each question only on the information in the given charts and graphs. It is unlikely that you have any preconceived notion as to the sales of XYZ Corporation, but you might think that you know the population of the United States for a particular year or the percent of women currently in the workplace. If your knowledge contradicts any of the data presented in the graphs, ignore what you know. First of all, you may be mistaken; but more important, the data may refer to a different, unspecified location or year. In any event, *always* base your answers on the given data.

TACTIC 6 Always Use the Proper Units

In answering question 4, observe that earnings are given in millions, while sales are in billions. If you answer too quickly, you might say that in 1993 earnings were 200 and sales were 8, and conclude that the desired ratio is $\frac{200}{8} = \frac{25}{1}$. You will avoid this mistake if you keep track of units: earnings were 200 *million* dollars, whereas sales were 8 *billion* dollars. The correct ratio is

$$\frac{2\cancel{00,000,000}}{8\cancel{,000,000,000}} = \frac{2}{80} = \frac{1}{40}.$$

Enter 1 in the box for the numerator and 40 in the box for the denominator.

TACTIC 7 Be Sure That Your Answer Is Reasonable

Before clicking on your answer, take a second to be sure that it is reasonable. For example, in question 4, Choices D and E are unreasonable. From the logic of the situation, you should realize that earnings can't exceed sales. The desired ratio, therefore, must be less than 1. If you use the wrong units (see TACTIC 6, above), your initial thought would be to choose D. By testing your answer for reasonableness, you will realize that you made a mistake.

Remember that if you don't know how to solve a problem, you should always guess. Before guessing, however, check to see if one or more of the choices are unreasonable. If so, eliminate them. For example, if you forget how to calculate a percent increase, you would have to guess at question 2. But before guessing wildly, you should at least eliminate Choice B, since from 1992 to 1993 earnings decreased.

TACTIC 8 Try to Visualize the Answer

Because graphs and tables present data in a form that enables you to readily see relationships and to make quick comparisons, you can often avoid doing any calculations. Whenever possible, use your eye instead of your computational skills.

For example, to answer question 1, rather than reading the sales figures in the bar graph on the left for each of the eight years, adding them, and then dividing by 8, visualize the situation. Where could you draw a horizontal line across the graph so that there would be the same amount of gray area above the line as white area below it? Imagine a horizontal line drawn through the 7 on the vertical axis. The portions of the bars above the line for 1993 and 1996–1998 are just about exactly the same size as the white areas below the line for 1991, 1992, and 1994. The answer is **C**.

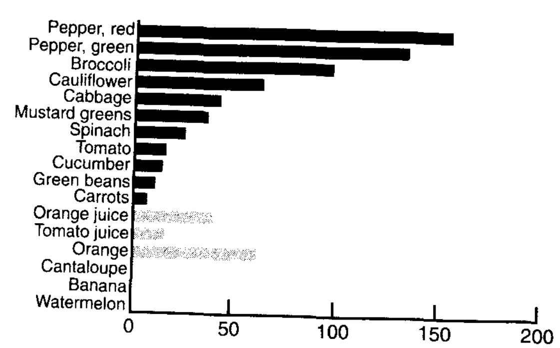
Practice Exercises

Data Interpretation Questions

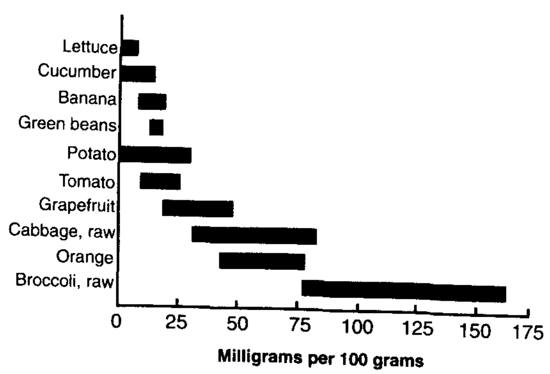
On the GRE there will typically be three questions based on any set of graphs. Accordingly, in each section of the model tests in this book, there are three data interpretation questions, each referring to the same set of graphs. However, to illustrate the variety of questions that can be asked, in this exercise set, for some of the graphs there are only two questions.

Questions 1–2 refer to the following graphs.

Vitamin C Content of Foods



Source: U.S. Department of Agriculture.



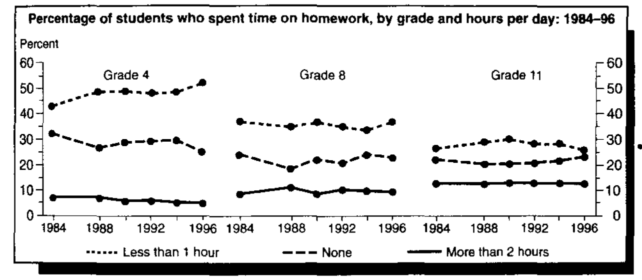
Source: U.S. Department of Agriculture.

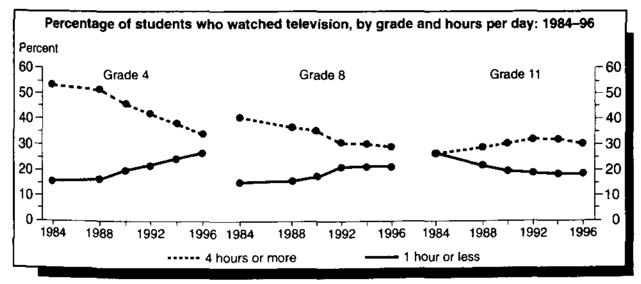
1. What is the ratio of the amount of Vitamin C in 500 grams of orange to the amount of Vitamin C in 500 grams of orange juice?
   Ⓐ 4:7
   Ⓑ 1:1
   Ⓒ 7:4
   Ⓓ 2:1
   Ⓔ 4:1

2. How many grams of tomato would you have to eat to be certain of getting more vitamin C than you would get by eating 100 grams of raw broccoli?
   Ⓐ 300
   Ⓑ 500
   Ⓒ 750
   Ⓓ 1200
   Ⓔ 1650

Questions 3–4 refer to the following graphs.

Percentage of students who reported spending time on homework and watching television





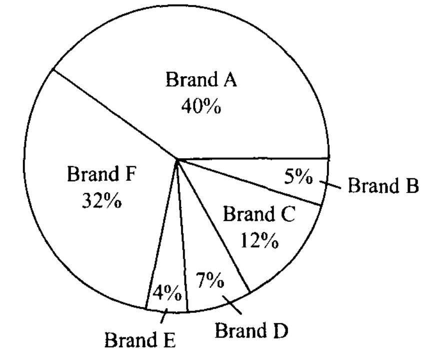
SOURCE: U.S. Department of Education.

3. In 1996, what percent of fourth-graders did between 1 and 2 hours of homework per day?
   Ⓐ 5%
   Ⓑ 15%
   Ⓒ 25%
   Ⓓ 40%
   Ⓔ 55%

4. If in 1984 there were 2,000,000 eleventh-graders, and if between 1984 and 1996 the number of eleventh-graders increased by 10%, then approximately how many more eleventh-graders watched 1 hour or less of television in 1996 than in 1984?
   Ⓐ 25,000
   Ⓑ 50,000
   Ⓒ 75,000
   Ⓓ 100,000
   Ⓔ 150,000

Questions 5–6 refer to the following graph.

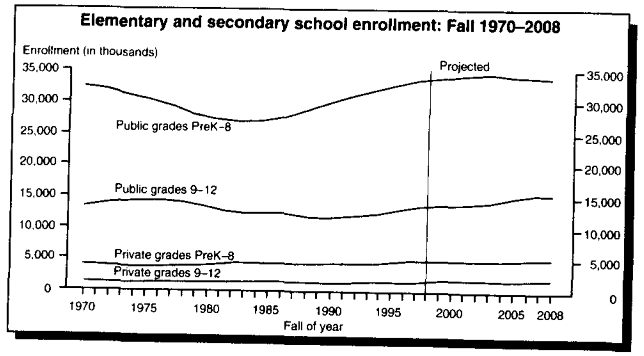
Total Sales of Coast Corporation in 2000: $1,000,000



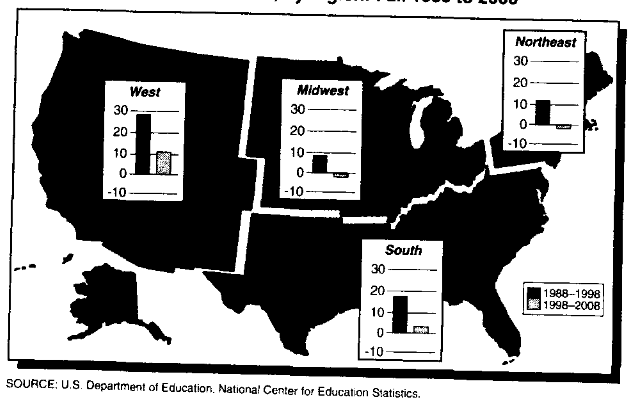
5. If the above circle graph were drawn to scale, then which of the following is closest to the difference in the degree measurements of the central angle of the sector representing Brand C and the central angle of the sector representing Brand D?
   Ⓐ 5°
   Ⓑ 12°
   Ⓒ 18°
   Ⓓ 25°
   Ⓔ 43°

6. The total sales of Coast Corporation in 2005 were 50% higher than in 2000. If the dollar value of the sales of Brand A was 25% higher in 2005 than in 2000, then the sales of Brand A accounted for what percentage of total sales in 2005?
   Ⓐ 20%
   Ⓑ 25%
   Ⓒ $33\frac{1}{3}$%
   Ⓓ 40%
   Ⓔ 50%

Questions 7–8 refer to the following graphs.



Projected percentage change in public elementary and secondary school enrollment, by region: Fall 1988 to 2008



SOURCE: U.S. Department of Education, National Center for Education Statistics.

7. To the nearest million, how many more students were enrolled in school — both public and private, preK–12 — in 1970 than in 1988?

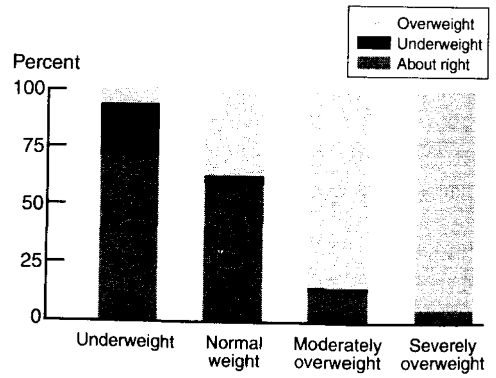
Ⓐ 3,000,000
Ⓑ 6,000,000
Ⓒ 10,000,000
Ⓓ 44,000,000
Ⓔ 51,000,000

8. In 1988 there were 40,000,000 public school students in the United States, of whom 22% lived in the West. Approximately, how many public school students are projected to be living in the West in 2008?

Ⓐ 9,000,000
Ⓑ 12,000,000
Ⓒ 15,000,000
Ⓓ 24,000,000
Ⓔ 66,000,000

Questions 9–10 refer to the following graph.

Perceptions of Body Weight Status



Actual weight status

Perceived compared with actual weight status of adult females.

Source: U.S. Department of Agriculture.

9. What percent of underweight adult females perceive themselves to be underweight?

[ ] %

10. The members of which of the four groups had the least accurate perception of their body weight?

Ⓐ Underweight
Ⓑ Normal weight
Ⓒ Moderately overweight
Ⓓ Severely overweight
Ⓔ It cannot be determined from the information given in the graph.

Questions 11–12 refer to the following table.

In 1979, residents of New York City paid both New York State and New York City tax. Residents of New York State who lived and worked outside of New York City paid only New York State tax.

Tax Rate Schedules for 1979

New York State: Taxable Income over	but not over	Amount of Tax					City of New York: Taxable Income over	but not over	Amount of Tax				
$ 0	$1,000			2%	of taxable income		$ 0	$1,000			0.9%	of taxable income	
1,000	3,000	$20	plus	3%	of excess over	$1,000	1,000	3,000	$ 9	plus	1.4%	of excess over	$1,000
3,000	5,000	80	plus	4%	of excess over	3,000	3,000	5,000	37	plus	1.8%	of excess over	3,000
5,000	7,000	160	plus	5%	of excess over	5,000	5,000	7,000	73	plus	2.0%	of excess over	5,000
7,000	9,000	260	plus	6%	of excess over	7,000	7,000	9,000	113	plus	2.3%	of excess over	7,000
9,000	11,000	380	plus	7%	of excess over	9,000	9,000	11,000	159	plus	2.5%	of excess over	9,000
11,000	13,000	520	plus	8%	of excess over	11,000	11,000	13,000	209	plus	2.7%	of excess over	11,000
13,000	15,000	680	plus	9%	of excess over	13,000	13,000	15,000	263	plus	2.9%	of excess over	13,000
15,000	17,000	860	plus	10%	of excess over	15,000	15,000	17,000	321	plus	3.1%	of excess over	15,000
17,000	19,000	1,060	plus	11%	of excess over	17,000	17,000	19,000	383	plus	3.3%	of excess over	17,000
19,000	21,000	1,280	plus	12%	of excess over	19,000	19,000	21,000	449	plus	3.5%	of excess over	19,000
21,000	23,000	1,520	plus	13%	of excess over	21,000	21,000	23,000	519	plus	3.8%	of excess over	21,000
23,000		1,780	plus	14%	of excess over	23,000	23,000	25,000	595	plus	4.0%	of excess over	23,000
							25,000		675	plus	4.3%	of excess over	25,000

11. In 1979 how much tax, in dollars, would a resident of New York State who lived and worked outside New York City have paid on a taxable income of $16,100?

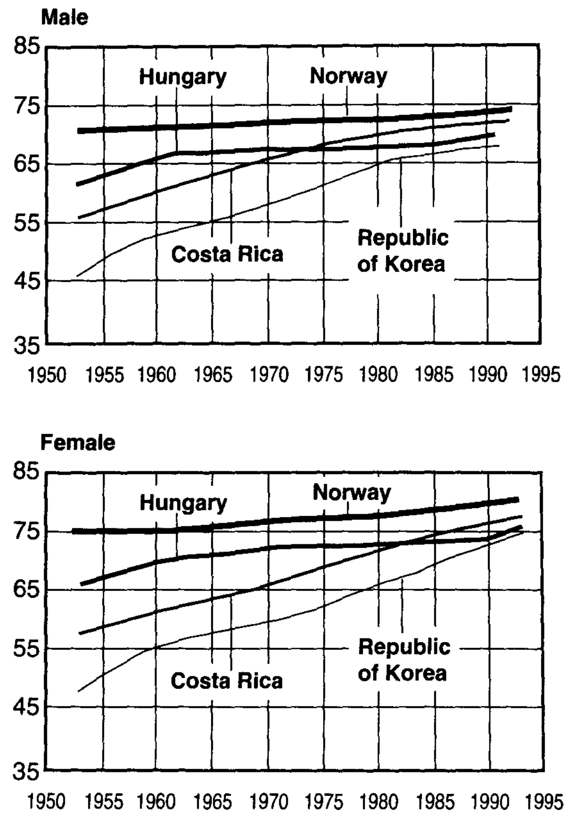
[ ] dollars

12. In 1979, how much more total tax would a resident of New York City who had a taxable income of $36,500 pay, compared to a resident of New York City who had a taxable income of $36,000?

Ⓐ $21.50
Ⓑ $43
Ⓒ $70
Ⓓ $91.50
Ⓔ $183

Questions 13–14 refer to the following tables.

Years of Life Expectancy at Birth
(Life expectancy in years)



Source: U.S. Bureau of the Census, Center for International Research.

13. For how many of the countries listed in the graphs is it true that the life expectancy of a female born in 1955 was higher than the life expectancy of a male born in 1990?

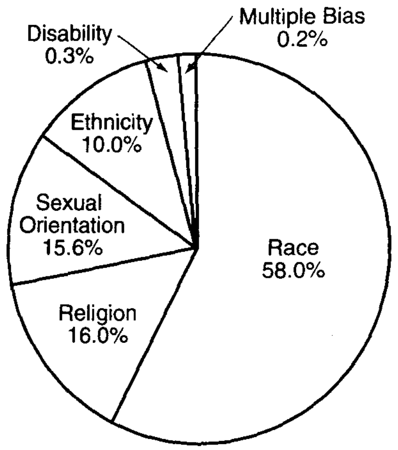
Ⓐ None
Ⓑ 1
Ⓒ 2
Ⓓ 3
Ⓔ 4

14. By sex and nationality, who had the greatest increase in life expectancy between 1955 and 1990?

Ⓐ A Korean female
Ⓑ A Korean male
Ⓒ A Costa Rican female
Ⓓ A Costa Rican male
Ⓔ A Norwegian female

Questions 15–16 refer to the following graph.

Bias-Motivated Offenses 1998
Percent Distribution



Source: U.S. Department of Justice, Federal Bureau of Investigation.

15. If in 1998 there were 10,000 bias-motivated offenses based on ethnicity, how many more offenses were based on religion than on sexual orientation?

Ⓐ 4
Ⓑ 40
Ⓒ 400
Ⓓ 4000
Ⓔ 40,000

16. If after further analysis it was determined that between 25% and 50% of the offenses included under Religion were, in fact, not bias-motivated, and those offenses were removed from the study, which of the following could be the percentage of bias-motivated offenses based on race?

Indicate *all* such percentages.

[A] 59%
[B] 60%
[C] 61%
[D] 62%
[E] 63%
[F] 64%
[G] 65%

ANSWER KEY

1. **C**	4. **E**	7. **B**	10. **A**	13. **B**	16. **C, D, E**
2. **E**	5. **C**	8. **B**	11. **970**	14. **A**	
3. **B**	6. **C**	9. **22**	12. **D**	15. **C**	

ANSWER EXPLANATIONS

1. **(C)** According to the graph on the left, there are approximately 70 milligrams of vitamin C in 100 grams of orange and 40 milligrams in the same amount of orange juice. This is a ratio of 70:40 = 7:4. Since the question refers to the same amount of orange and orange juice (500 grams), the ratio is unchanged.

2. **(E)** From the graph on the right, you can see that by eating 100 grams of raw broccoli, you could receive as much as 165 milligrams of vitamin C. Since 100 grams of tomato could have as little as 10 milligrams of vitamin C, you would have to eat 1650 grams of tomato to be sure of getting 165 milligrams of vitamin C.

3. **(B)** From the top graph, we see that among fourth-graders in 1996:

   25% did no homework;
   55% did less than 1 hour;
   5% did more than 2 hours.

   This accounts for 85% of the fourth-graders; the other 15% did between 1 and 2 hours of homework per day.

4. **(E)** In 1984, approximately 540,000 eleventh-graders watched television 1 hour or less per day (27% of 2,000,000). By 1996, the number of eleventh-graders had increased by 10% to 2,200,000, but the percent of them who watched television 1 hour or less per day decreased to about 18%: 18% of 2,200,000 is 396,000. This is a decrease of 144,000, or approximately 150,000.

5. **(C)** The central angle of the sector representing Brand C is 12% of 360°:

   $(0.12) \times 360° = 43.2°$

   The central angle of the sector representing Brand D is 7% of 360°.

   $(0.7) \times 360° = 25.2°$

   Finally, $43.2° \times 25.2° = 18°$

   **Note this can be done in one step by noticing that the percentage difference between Brands C and D is 5% and 5% of 360 is $(0.05) \times 380 = 18$.

6. **(C)** Since total sales in 2000 were $1,000,000, in 2005 sales were $1,500,000 (a 50% increase).

   In 2000, sales of Brand A were $400,000 (40% of $1,000,000).

   In 2005 sales of Brand A were $500,000 (25% or $\frac{1}{4}$ more than in 2000).

   Finally, $500,000 is $\frac{1}{3}$ or $33\frac{1}{3}$% of $1,500,000.

7. **(B)** Reading from the top graph, we get the following enrollment figures:

	1970	1988
Public PreK–8	33,000,000	28,000,000
Public 9–12	13,000,000	12,000,000
Private PreK–8	4,000,000	4,000,000
Private 9–12	1,000,000	1,000,000
Total	51,000,000	45,000,000

51,000,000 – 45,000,000 = 6,000,000.

8. **(B)** In 1988, 8,800,000 (22% of 40,000,000) students lived in the West. From 1988–1998 this figure increased by 27% — for simplicity use 25%: an additional 2,200,000 students; so the total was then 11,000,000. The projected increase from 1998–2008 is about 10%, so the number will grow by 1,100,000 to 12,100,000.

9. **22** The bar representing underweight adult females who perceive themselves to be underweight extends from about 70% to about 95%, a range of approximately 25%. Choice B is closest.

10. **(A)** Almost all overweight females correctly considered themselves to be overweight; and more than half of all females of normal weight correctly considered themselves "about right." But nearly 70% of underweight adult females inaccurately considered themselves "about right."

11. **970** Referring only to the New York State table, we see that the amount of tax on a taxable income between $15,000 and $17,000 was $860 plus 10% of the excess over $15,000. Therefore, the tax on $16,100 is $860 plus 10% of $1,100 = $860 + $110 = $970.

12. **(D)** According to the tables, each additional dollar of taxable income over $25,000 was subject to a New York State tax of 14% and a New York City tax of 4.3%, for a total tax of 18.3%. Therefore, an additional $500 in taxable income would have incurred an additional tax of 0.183×500 = $91.50.

13. **(B)** In Norway, the life expectancy of a female born in 1955 was 75 years, which is greater than the life expectancy of a male born in 1990. In Hungary, the life expectancy of a female born in 1955 was 66 years, whereas the life expectancy of a male born in 1990 was greater than 67. In the other two countries, the life expectancy of a female born in 1955 was less than 65 years, and the life expectancy of a male born in 1990 was greater than 65.

14. **(A)** The life expectancy of a Korean female born in 1955 was about 51 and in 1990 it was about 74, an increase of 23 years. This is greater than any other nationality and sex.

15. **(C)** Since there were 10,000 bias-motivated offenses based on ethnicity, and that represents 10% of the total, there were 100,000 bias-motivated offenses in total. Of these, 16,000 (16% of 100,000) were based on religion, and 15,600 (15.6% of 100,000) were based on sexual orientation. The difference is 400.

16. **(C)**, **(D)**, **(E)** Since this is a question about percentages, assume that the total number of bias-motivated offenses in 1998 was 100, of which 16 were based on religion and 58 were based on race.

- If 8 of the religion-based offenses (50% of 16) were deleted, then there would have been 92 offenses in all, of which 58 were based on race.

  $\frac{58}{92} = 0.6304 = 63.04\%$

- If 4 of the religion-based offenses (25% of 16) were deleted, then there would have been 96 offenses in all, of which 58 were based on race.

  $\frac{58}{96} = 0.6041 = 60.41\%$

Only choices C, D, and E lie between 60.41% and 63.04%.

Mathematics Review

CHAPTER 11

The mathematics questions on the GRE General Test require a working knowledge of mathematical principles, including an understanding of the fundamentals of algebra, plane geometry, and arithmetic, as well as the ability to translate problems into formulas and to interpret graphs. Very few questions require any math beyond what is typically taught in the first two years of high school, and even much of that is not tested. The following review covers those areas that you definitely need to know.

This chapter is divided into 15 sections, labeled 11-A through 11-O. For each question on the Diagnostic Test and the two Model Tests, the Answer Key indicates which section of Chapter 11 you should consult if you need help on a particular topic.

How much time you initially devote to reviewing mathematics should depend on your math skills. If you have always been a good math student and you have taken some math in college and remember most of your high school math, you can skip the instructional parts of this chapter for now. If while doing the Model Tests in Part 5 or on the accompanying CD-ROM, you find that you keep making mistakes on certain types of problems (averages, percents, circles, solid geometry, word problems, for example), or they take you too long, you should then study the appropriate sections here. Even if your math skills are excellent, and you don't need the review, you should complete the sample questions in those sections; they are an excellent source of additional GRE questions. If you know that your math skills are not very good and you have not done much math since high school, then it is advisable to review all of this material, including working out the problems, *before* tackling the model tests.

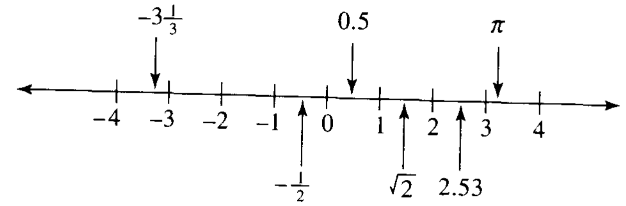
No matter how good you are in math, *you should carefully read and do the problems* in Chapters 7, 8, 9, and 10. For many of these problems, two solutions are given: the most direct mathematical solution and a second solution using one or more of the special tactics taught in these chapters.

Arithmetic

To do well on the GRE, you need to feel comfortable with most topics of basic arithmetic. In the first five sections of this chapter, we will review the basic arithmetic operations, signed numbers, fractions, decimals, ratios, percents, and averages. Since the GRE uses these concepts to test your reasoning skills, not your ability to perform tedious calculations, we will concentrate on the concepts and not on arithmetic drill. The solutions to more than one-third of the mathematics questions on the GRE depend on your knowing the key facts in these sections. Be sure to review them all.

11-A. BASIC ARITHMETIC CONCEPTS

Let's start by reviewing the most important sets of numbers and their properties. On the GRE the word *number* always means *real number*, a number that can be represented by a point on the number line.



Signed Numbers

The numbers to the right of 0 on the number line are called ***positive*** and those to the left of 0 are called ***negative***. Negative numbers must be written with a *negative sign* (−2); positive numbers can be written with a *plus sign* (+2) but are usually written without a sign (2). All numbers can be called ***signed numbers***.

KEY FACT A1

For any number *a*, exactly one of the following is true:

- ***a* is negative**
- **$a = 0$**
- ***a* is positive**

TIP

The absolute value of a number if *never* negative.

The ***absolute value*** of a number *a*, denoted $|a|$, is the distance between *a* and 0 on the number line. Since 3 is 3 units to the right of 0 on the number line and −3 is 3 units to the left of 0, both have an absolute value of 3:

- $|3| = 3$
- $|-3| = 3$

Two unequal numbers that have the same absolute value are called ***opposites***. So, 3 is the opposite of −3 and −3 is the opposite of 3.

KEY FACT A2

The only number that is equal to its opposite is 0.

EXAMPLE 1

$$a - b = -(a - b)$$

Quantity A	Quantity B
a	b

SOLUTION.
Since $-(a - b)$ is the opposite of $a - b$, $a - b = 0$, and so $a = b$. The answer is **C**.

In arithmetic we are basically concerned with the addition, subtraction, multiplication, and division of numbers. The third column of the following table gives the terms for the results of these operations.

Operation	Symbol	Result	Example
Addition	+	***Sum***	16 is the sum of 12 and 4 $16 = 12 + 4$
Subtraction	−	***Difference***	8 is the difference of 12 and 4 $8 = 12 - 4$
Multiplication*	×	***Product***	48 is the product of 12 and 4 $48 = 12 \times 4$
Division	÷	***Quotient***	3 is the quotient of 12 and 4 $3 = 12 \div 4$

*Multiplication can be indicated also by a dot, parentheses, or the juxtaposition of symbols without any sign: $2^2 \cdot 2^4$, $3(4)$, $3(x + 2)$, $3a$, $4abc$.

Given any two numbers *a* and *b*, we can *always* find their sum, difference, product, and quotient, except that we may *never divide by zero*.

- $0 \div 7 = 0$
- $7 \div 0$ is meaningless

EXAMPLE 2

What is the sum of the product and quotient of 8 and 8?

Ⓐ 16 Ⓑ 17 Ⓒ 63 Ⓓ 64 Ⓔ 65

SOLUTION.
Product: $8 \times 8 = 64$. Quotient: $8 \div 8 = 1$. Sum: $64 + 1 =$ **65** (**E**).

KEY FACT A3

- **The product of 0 and any number is 0. For any number *a*: $a \times 0 = 0$.**
- **Conversely, if the product of two numbers is 0, *at least one* of them must be 0:**

$$ab = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

EXAMPLE 3

Quantity A	Quantity B
The product of the integers from −7 to 2	The product of the integers from −2 to 7

SOLUTION.
Do not multiply. Each quantity is the product of 10 numbers, one of which is 0. So, by KEY FACT A3, each product is 0. The quantities are equal (**C**).

KEY FACT A4

The product and quotient of two positive numbers or two negative numbers are positive; the product and quotient of a positive number and a negative number are negative.

$\times$	+	−
+	+	−
−	−	+

$\div$	+	−
+	+	−
−	−	+

$6 \times 3 = 18$	$6 \times (-3) = -18$	$(-6) \times 3 = -18$	$(-6) \times (-3) = 18$
$6 \div 3 = 2$	$6 \div (-3) = -2$	$(-6) \div 3 = -2$	$(-6) \div (-3) = 2$

To determine whether a product of more than two numbers is positive or negative, count the number of negative factors.

KEY FACT A5

- **The product of an *even* number of negative factors is positive.**
- **The product of an *odd* number of negative factors is negative.**

EXAMPLE 4

Quantity A	Quantity B
$(-1)(2)(-3)(4)(-5)$	$(1)(-2)(3)(-4)(5)$

SOLUTION.
Don't waste time multiplying. Quantity A is negative since it has 3 negative factors, whereas Quantity B is positive since it has 2 negative factors. The answer is **B**.

KEY FACT A6

- **The *reciprocal* of any nonzero number a is $\frac{1}{a}$.**
- **The product of any number and its reciprocal is 1:**

$$a \times \left(\frac{1}{a}\right) = 1.$$

KEY FACT A7

- **The sum of two positive numbers is positive.**
- **The sum of two negative numbers is negative.**
- **To find the sum of a positive and a negative number, find the difference of their absolute values and use the sign of the number with the larger absolute value.**

$$6 + 2 = 8 \qquad (-6) + (-2) = -8$$

To calculate either $6 + (-2)$ or $(-6) + 2$, take the *difference*, $6 - 2 = 4$, and use the sign of the number whose absolute value is 6. So,

$$6 + (-2) = 4 \qquad (-6) + 2 = -4$$

KEY FACT A8

The sum of any number and its opposite is 0:

$$a + (-a) = 0.$$

Many of the properties of arithmetic depend on the relationship between subtraction and addition and between division and multiplication.

KEY FACT A9

- **Subtracting a number is the same as adding its opposite.**
- **Dividing by a number is the same as multiplying by its reciprocal.**

$$a - b = a + (-b) \qquad a \div b = a \times \left(\frac{1}{b}\right)$$

Many problems involving subtraction and division can be simplified by changing them to addition and multiplication problems, respectively.

KEY FACT A10

To subtract signed numbers, change the problem to an addition problem, by changing the sign of what is being subtracted, and use KEY FACT A7.

$$2 - 6 = 2 + (-6) = -4 \qquad 2 - (-6) = 2 + (6) = 8$$

$$(-2) - (-6) = (-2) + (6) = 4 \qquad (-2) - 6 = (-2) + (-6) = -8$$

In each case, the minus sign was changed to a plus sign, and either the 6 was changed to −6 or the −6 was changed to 6.

Integers

The ***integers*** are {... , –4, –3, –2, –1, 0, 1, 2, 3, 4, ...}.
The ***positive integers*** are {1, 2, 3, 4, 5, ...}.
The ***negative integers*** are {..., –5, –4, –3, –2, –1}.

TIP

0 is neither positive nor negative, but it *is* an integer.

There are five integers whose absolute value is less than 3—two negative integers (–2 and –1), two positive integers (1 and 2), and 0.

Consecutive integers are two or more integers written in sequence in which each integer is 1 more than the preceding integer. For example:

22, 23   6, 7, 8, 9   –2, –1, 0, 1 $n, n+1, n+2, n+3$

EXAMPLE 5

If the sum of three consecutive integers is less than 75, what is the greatest possible value of the smallest one?

Ⓐ 23 Ⓑ 24 Ⓒ 25 Ⓓ 26 Ⓔ 27

SOLUTION.
Let the numbers be n, $n + 1$, and $n + 2$. Then,

$$n + (n + 1) + (n + 2) = 3n + 3 \Rightarrow 3n + 3 < 75 \Rightarrow 3n < 72 \Rightarrow n < 24.$$

So, the most n can be is **23 (A)**.

CAUTION

Never assume that *number* means *integer*: 3 is not the only number between 2 and 4; there are infinitely many, including 2.5, 3.99, $\frac{10}{3}$, π, and $\sqrt{10}$.

EXAMPLE 6

If $2 < x < 4$ and $3 < y < 7$, what is the largest integer value of $x + y$?

SOLUTION.
If x and y are integers, the largest value is $3 + 6 = 9$. However, although $x + y$ is to be an integer, neither x nor y must be. If $x = 3.8$ and $y = 6.2$, then $x + y =$ **10**.

The sum, difference, and product of two integers are *always* integers; the quotient of two integers may be an integer, but it is not necessarily one. The quotient $23 \div 10$ can be expressed as $\frac{23}{10}$ or $2\frac{3}{10}$ or 2.3. If the quotient is to be an integer, we can say that the quotient is 2 and there is a ***remainder*** of 3. It depends upon our point of view. For example, if 23 dollars is to be divided among 10 people, each one will get \$2.30 (2.3 dollars); but if 23 books are to be divided among 10 people, each one will get 2 books and there will be 3 books left over (the remainder).

KEY FACT A11

If m and n are positive integers and if r is the remainder when n is divided by m, then n is r more than a multiple of m. That is, $n = mq + r$ where q is an integer and $0 \le r < m$.

EXAMPLE 7

How many positive integers less than 100 have a remainder of 3 when divided by 7?

SOLUTION.
To leave a remainder of 3 when divided by 7, an integer must be 3 more than a multiple of 7. For example, when 73 is divided by 7, the quotient is 10 and the remainder is 3: $73 = 10 \times 7 + 3$. So, just take the multiples of 7 and add 3. (*Don't forget that 0 is a multiple of 7.*)

$\underline{0} \times 7 + 3 = 3$; $\underline{1} \times 7 + 3 = 10$;
$\underline{2} \times 7 + 3 = 17$;   ... ;
$\underline{13} \times 7 + 3 = 94$

A total of **14** numbers.

Calculator Shortcut

The standard way to find quotients and remainders is to use long division; but on the GRE you *never* do long division: you use the onscreen calculator. To find the remainder when 100 is divided by 7, divide on your calculator: $100 \div 7 = 14.285714...$. This tells you that the quotient is 14. (Ignore everything to the right of the decimal point.) To find the remainder, multiply $14 \times 7 = 98$, and then subtract: $100 - 98 = 2$.

EXAMPLE 8

If today is Saturday, what day will it be in 500 days?

Ⓐ Friday Ⓑ Saturday Ⓒ Sunday Ⓓ Monday Ⓔ Tuesday

SOLUTION.
The days of the week form a repeating sequence. Seven days (1 week), 70 days (10 weeks), 700 days (100 weeks) from Saturday it is again Saturday. If 500 were a multiple of 7, then the answer would be Choice B, Saturday. Is it? With your calculator divide 500 by 7: $500 \div 7 = 71.428...$ So, 500 is not a multiple of 7; since $71 \times 7 = 497$, The quotient when 500 is divided by 7 is 71, and the remainder is 3. Therefore, 500 days is 3 days more than 71 complete weeks. 497 days from Saturday it will again be Saturday; three days later it will be **Tuesday**, **(E)**.

If a and b are integers, the following four terms are synonymous:

a is a ***divisor*** of b	a is a ***factor*** of b
b is ***divisible*** by a	b is a ***multiple*** of a

They all mean that when b is divided by a there is no remainder (or, more precisely, the remainder is 0). For example:

3 is a divisor of 12	3 is a factor of 12
12 is divisible by 3	12 is a multiple of 3

KEY FACT A12

Every integer has a finite set of factors (or divisors) and an infinite set of multiples.

The factors of 12: −12, −6, −4, −3, −2, −1, 1, 2, 3, 4, 6, 12
The multiples of 12: ... , −48, −36, −24, −12, 0, 12, 24, 36, 48, ...

The only positive divisor of 1 is 1. All other positive integers have at least 2 positive divisors: 1 and itself, and possibly many more. For example, 6 is divisible by 1 and 6, as well as 2 and 3, whereas 7 is divisible only by 1 and 7. Positive integers, such as 7, that have exactly 2 positive divisors are called ***prime numbers*** or ***primes***. The first ten primes are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29

TIP

1 is *not* a prime.

Memorize this list — it will come in handy.

Positive integers greater than 1 that are not prime are called ***composite numbers***. It follows from the definition that every composite number has at least three distinct positive divisors. The first ten composite numbers are

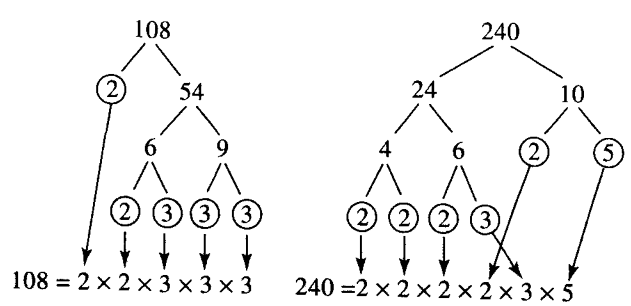
4, 6, 8, 9, 10, 12, 14, 15, 16, 18

KEY FACT A13

Every integer greater than 1 that is not a prime (i.e., every composite number) can be written as a product of primes.

To find the prime factorization of any integer, find any two factors; if they're both primes, you are done; if not, factor them. Continue until each factor has been written in terms of primes. A useful method is to make a *factor tree*.

For example, here are the prime factorizations of 108 and 240:



EXAMPLE 9

For any positive integer a, let ⌈a⌋ denote the smallest prime factor of a. Which of the following is equal to ⌈35⌋?

Ⓐ ⌈10⌋ Ⓑ ⌈15⌋ Ⓒ ⌈45⌋ Ⓓ ⌈55⌋ Ⓔ ⌈75⌋

SOLUTION.

Check the first few primes; 35 is not divisible by 2 or 3, but is divisible by 5, so 5 is the *smallest* prime factor of 35: ⌈35⌋ = 5. Now check the five choices: ⌈10⌋ = 2, and ⌈15⌋, ⌈45⌋, and ⌈75⌋ are all equal to 3. Only **⌈55⌋** = 5. The answer is **D**.

The ***least common multiple*** (**LCM**) of two or more integers is the smallest positive integer that is a multiple of each of them. For example, the LCM of 6 and 10 is 30. Infinitely many positive integers are multiples of both 6 and 10, including 60, 90, 180, 600, 6000, and 66,000,000, but 30 is the smallest one. The ***greatest common factor*** (**GCF**) or ***greatest common divisor*** (**GCD**) of two or more integers is the largest integer that is a factor of each of them. For example, the only positive integers that are factors of both 6 and 10 are 1 and 2, so the GCF of 6 and 10 is 2. For small numbers, you can often find their GCF and LCM by inspection. For larger numbers, KEY FACT A14 is very useful.

KEY FACT A14

The product of the GCF and LCM of two numbers is equal to the product of the two numbers.

TIP

It is usually easier to find the GCF than the LCM. For example, you might see immediately that the GCF of 36 and 48 is 12. You could then use KEY FACT A14 to find the LCM: since GCF × LCM = 36 × 48, then

$$\text{LCM} = \frac{\overset{3}{\cancel{36}} \times 48}{\underset{1}{\cancel{12}}} = 3 \times 48 = 144.$$

An easy way to find the GCF or LCM of two or more integers is to first get their prime factorizations.

- The GCF is the product of all the primes that appear in each factorization, using each prime the smallest number of times it appears in any of the factorizations.
- The LCM is the product of all the primes that appear in any of the factorizations, using each prime the largest number of times it appears in any of the factorizations.

For example, let's find the GCF and LCM of 108 and 240. As we saw:

$$108 = 2 \times 2 \times 3 \times 3 \times 3 \text{ and } 240 = 2 \times 2 \times 2 \times 2 \times 3 \times 5.$$

- **GCF.** The primes that appear in both factorizations are 2 and 3: 2 appears twice in the factorization of 108 and 4 times in the factorization of 240, so we take it twice; 3 appears 3 times in the factorization of 108, but only once in the factorization of 240, so we take it just once. The GCF = 2 × 2 × 3 = **12**.
- **LCM.** Take one of the factorizations and add to it any primes from the other that are not yet listed. So, start with 2 × 2 × 3 × 3 × 3 (108) and look at the primes from 240: there are four 2s; we already wrote two 2s, so we need two more; there is a 3 but we already have that; there is a 5, which we need. So, the LCM = (2 × 2 × 3 × 3 × 3) × (2 × 2 × 5) = 108 × 20 = **2,160**.

EXAMPLE 10

What is the smallest number that is divisible by both 34 and 35?

$\boxed{}$

SOLUTION. We are being asked for the LCM of 34 and 35. By KEY FACT A14, LCM = $\frac{34 \times 35}{\text{GCF}}$. But the GCF is 1 since no number greater than 1 divides evenly into both 34 and 35. So, the LCM is $34 \times 35 = $ **1,190**.

The ***even numbers*** are all the multiples of 2:

$$\{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$$

The ***odd numbers*** are the integers not divisible by 2:

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$$

TIP

The terms odd and even apply *only* to integers.

NOTE:

- Every integer (positive, negative, or 0) is either odd or even.
- 0 is an even integer; it is a multiple of 2. ($0 = 0 \times 2$)
- 0 is a multiple of *every* integer. ($0 = 0 \times n$)
- 2 is the only even prime number.

KEY FACT A15

The tables below summarize three important facts:

1. **If two integers are both even or both odd, their sum and difference are even.**
2. **If one integer is even and the other odd, their sum and difference are odd.**
3. **The product of two integers is even unless both of them are odd.**

+ and –	even	odd
even	even	odd
odd	odd	even

×	even	odd
even	even	even
odd	even	odd

Exponents and Roots

Repeated addition of the same number is indicated by multiplication:

$$17 + 17 + 17 + 17 + 17 + 17 + 17 = 7 \times 17$$

Repeated multiplication of the same number is indicated by an exponent:

$$17 \times 17 \times 17 \times 17 \times 17 \times 17 \times 17 = 17^7$$

In the expression 17^7, 17 is called the ***base*** and 7 is the ***exponent***.

At some time, you may have seen expressions such as 2^{-4}, $2^{\frac{1}{2}}$, or even $2^{\sqrt{2}}$. On the GRE, although the base, b, can be any number, the exponents you will see will almost always be positive integers.

KEY FACT A16

For any number b: $b^1 = b$, and $b^n = b \times b \times \cdots \times b$, where b is used as a factor n times.

(i) $2^5 \times 2^3 = (2 \times 2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2) = 2^8 = 2^{5+3}$

(ii) $\frac{2^5}{2^3} = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2 \times 2 = 2^2 = 2^{5-3}$

(iii) $(2^2)^3 = (2 \times 2)^3 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) = 2^6 = 2^{2 \times 3}$

(iv) $2^3 \times 7^3 = (2 \times 2 \times 2) \times (7 \times 7 \times 7) = (2 \times 7)(2 \times 7)(2 \times 7) = (2 \times 7)^3$

These four examples illustrate the following important ***laws of exponents*** given in KEY FACT A17.

KEY FACT A17

TIP

Memorize the laws of exponents. They come up often on the GRE.

For any numbers b and c and positive integers m and n:

(i) $b^m b^n = b^{m+n}$ **(ii)** $\frac{b^m}{b^n} = b^{m-n}$ **(iii)** $(b^m)^n = b^{mn}$ **(iv)** $b^m c^m = (bc)^m$

CAUTION

In (i) and (ii) the bases are the same and in (iv) the exponents are the same. None of these rules applies to expressions such as $7^5 \times 5^7$, in which both the bases and the exponents are different.

EXAMPLE 11

If $2^x = 32$, what is x^2?

Ⓐ 5 Ⓑ 10 Ⓒ 25 Ⓓ 100 Ⓔ 1024

SOLUTION.
To solve $2^x = 32$, just count (and keep track of) how many 2s you need to multiply to get 32: $2 \times 2 \times 2 \times 2 \times 2 = 32$, so $x = 5$ and $x^2 = $ **25 (C)**.

EXAMPLE 12

If $3^a \times 3^b = 3^{100}$, what is the average (arithmetic mean) of a and b?

$\boxed{}$

SOLUTION.

Since $3^a \times 3^b = 3^{a+b}$, we see that $a + b = 100 \Rightarrow \frac{a+b}{2} = $ **50.**

The next KEY FACT is an immediate consequence of KEY FACTS A4 and A5.

KEY FACT A18

For any positive integer n:

- **$0^n = 0$**
- **if a is positive, then a^n is positive**
- **if a is negative and n is even, then a^n is positive**
- **if a is negative and n is odd, then a^n is negative.**

EXAMPLE 13

Quantity A	Quantity B
$(-13)^{10}$	$(-13)^{25}$

SOLUTION.
Quantity A is positive and Quantity B is negative. So Quantity **A** is greater.

Squares and Square Roots

The exponent that appears most often on the GRE is 2. It is used to form the square of a number, as in πr^2 (the area of a circle), $a^2 + b^2 = c^2$ (the Pythagorean theorem), or $x^2 - y^2$ (the difference of two squares). Therefore, it is helpful to recognize the ***perfect squares***, numbers that are the squares of integers. The squares of the integers from 0 to 15 are as follows:

x	0	1	2	3	4	5	6	7
x^2	0	1	4	9	16	25	36	49

x	8	9	10	11	12	13	14	15
x^2	64	81	100	121	144	169	196	225

There are two numbers that satisfy the equation $x^2 = 9$: $x = 3$ and $x = -3$. The positive one, 3, is called the ***(principal) square root*** of 9 and is denoted by the symbol $\sqrt{9}$. Clearly, each perfect square has a square root: $\sqrt{0} = 0$, $\sqrt{36} = 6$, $\sqrt{81} = 9$, and $\sqrt{144} = 12$. But, it is an important fact that *every* positive number has a square root.

KEY FACT A19

For any positive number a, there is a positive number b that satisfies the equation $b^2 = a$. That number is called the square root of a and we write $b = \sqrt{a}$.

So, for any positive number a: $(\sqrt{a})^2 = \sqrt{a} \times \sqrt{a} = a$.

The only difference between $\sqrt{9}$ and $\sqrt{10}$ is that the first square root is an integer, while the second one isn't. Since 10 is a little more than 9, we should expect that $\sqrt{10}$ is a little more than $\sqrt{9} = 3$. In fact, $(3.1)^2 = 9.61$, which is close to 10, and $(3.16)^2 = 9.9856$, which is very close to 10. So, $\sqrt{10} \approx 3.16$. On the GRE you will *never* have to evaluate such a square root; if the solution to a problem involves a square root, that square root will be among the answer choices.

EXAMPLE 14

What is the circumference of a circle whose area is 10π?

Ⓐ 5π Ⓑ 10π Ⓒ $\pi\sqrt{10}$ Ⓓ $2\pi\sqrt{10}$ Ⓔ $\pi\sqrt{20}$

SOLUTION.
Since the area of a circle is given by the formula $A = \pi r^2$, we have

$$\pi r^2 = 10\pi \Rightarrow r^2 = 10 \Rightarrow r = \sqrt{10}.$$

The circumference is given by the formula $C = 2\pi r$, so $C = \mathbf{2\pi\sqrt{10}}$ **(D)**.

KEY FACT A20

For any positive numbers a and b:

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

CAUTION

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. For example:

$$5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

CAUTION

Although it is always true that $(\sqrt{a})^2 = a$, $\sqrt{a^2} = a$ is true *only if a* is positive:

$$\sqrt{(-5)^2} = \sqrt{25} = 5, \textit{ not } -5.$$

EXAMPLE 15

Quantity A	Quantity B
$\sqrt{x^{20}}$	$(x^5)^2$

SOLUTION.
Quantity A: Since $x^{10}x^{10} = x^{20}$, $\sqrt{x^{20}} = x^{10}$. Quantity B: $(x^5)^2 = x^{10}$. The quantities are equal **(C)**.

PEMDAS

When a calculation requires performing more than one operation, it is important to carry them out in the correct order. For decades students have memorized the sentence "Please Excuse My Dear Aunt Sally," or just the first letters, PEMDAS, to remember the proper order of operations. The letters stand for:

- Parentheses: first do whatever appears in parentheses, following PEMDAS within the parentheses if necessary.
- Exponents: next evaluate all terms with exponents.
- Multiplication and Division: then do all multiplications and divisions *in order from left to right* — *do not* multiply first and then divide.
- Addition and Subtraction: finally, do all additions and subtractions *in order from left to right* — do not add first and then subtract.

Here are some worked-out examples.

1. $12 + 3 \times 2 = 12 + 6 = 18$ [Multiply before you add.]
   $(12 + 3) \times 2 = 15 \times 2 = 30$ [First add in the parentheses.]
2. $12 \div 3 \times 2 = 4 \times 2 = 8$ [Just go from left to right.]
   $12 \div (3 \times 2) = 12 \div 6 = 2$ [First multiply inside the parentheses.]
3. $5 \times 2^3 = 5 \times 8 = 40$ [Do exponents first.]
   $(5 \times 2)^3 = 10^3 = 1000$ [First multiply inside the parentheses.]
4. $4 + 4 \div (2 + 6) = 4 + 4 \div 8 = 4 + .5 = 4.5$
   [First add in the parentheses, then divide, and finally add.]
5. $100 - 2^2(3 + 4 \times 5) = 100 - 2^2(23) = 100 - 4(23) = 100 - 92 = 8$
   [First evaluate what's inside the parentheses (using PEMDAS); then take the exponent; then multiply; and finally subtract.]

There is an important situation when you shouldn't start with what's in the parentheses. Consider the following two examples.

(i) What is the value of $7(100 - 1)$?

Using PEMDAS, you would write $7(100 - 1) = 7(99)$, and then multiply: $7 \times 99 = 693$. But you can do this even quicker in your head if you think of it this way: $7(100 - 1) = 700 - 7 = 693$.

(ii) What is the value of $(77 + 49) \div 7$?

If you followed the rules of PEMDAS, you would first add, $77 + 49 = 126$, and then divide, $126 \div 7 = 18$. This is definitely more difficult and time-consuming than mentally doing $\frac{77}{7} + \frac{49}{7} = 11 + 7 = 18$.

Both of these examples illustrate the very important distributive law.

Key Fact A21

The distributive law

For any real numbers a, b, and c:

- $a(b + c) = ab + ac$
- $a(b - c) = ab - ac$

and if $a \neq 0$

- $\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}$
- $\frac{b-c}{a} = \frac{b}{a} - \frac{c}{a}$

TIP

Many students who use the distributive law with multiplication forget about it with division. Don't you do that.

EXAMPLE 16

Quantity A	Quantity B
$5(a - 7)$	$5a - 7$

SOLUTION.
By the distributive law, Quantity A = $5a - 35$. The result of subtracting 35 from a number is *always less* than the result of subtracting 7 from that number. Quantity **B** is greater.

EXAMPLE 17

Quantity A	Quantity B
$\frac{50 + x}{5}$	$10 + x$

SOLUTION.

	Quantity A	Quantity B
By the distributive law:	$10 + \frac{x}{5}$	$10 + x$
Subtract 10 from each quantity:	$\frac{x}{5}$	x

The quantities are equal if $x = 0$, but not if $x = 1$.
The answer is **D**.

Inequalities

The number a is ***greater than*** the number b, denoted $a > b$, if a is to the right of b on the number line. Similarly, a is ***less than*** b, denoted $a < b$, if a is to the left of b on the number line. Therefore, if a is positive, $a > 0$, and if a is negative, $a < 0$. Clearly, if $a > b$, then $b < a$.

The following KEY FACT gives an important alternate way to describe greater than and less than.

KEY FACT A22

- **For any numbers a and b:**

  $a > b$ means that $a - b$ is positive.

- **For any numbers a and b:**

  $a < b$ means that $a - b$ is negative.

KEY FACT A23

- **For any numbers a and b, exactly one of the following is true:**

  $a > b$ or $a = b$ or $a < b$.

The symbol $\geq$ means ***greater than or equal to*** and the symbol $\leq$ means ***less than or equal to***. The statement "$x \geq 5$" means that x can be 5 or any number greater than 5; the statement "$x \leq 5$" means that x can be 5 or any number less than 5. The statement "$2 < x < 5$" is an abbreviation for the statement "$2 < x$ and $x < 5$." It means that x is a number between 2 and 5 (greater than 2 and less than 5).

Inequalities are very important on the GRE, especially on the quantitative comparison questions where you have to determine which of two quantities is the greater one. KEY FACTS A24 and A25 give some important facts about inequalities.

If the result of performing an arithmetic operation on an inequality is a new inequality in the same direction, we say that the inequality has been ***preserved***. If the result of performing an arithmetic operation on an inequality is a new inequality in the opposite direction, we say that the inequality has been ***reversed***.

KEY FACT A24

- **Adding a number to an inequality or subtracting a number from an inequality preserves it.**

  If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

  $3 < 7 \Rightarrow 3 + 100 < 7 + 100 \quad (103 < 107)$
  $3 < 7 \Rightarrow 3 - 100 < 7 - 100 \quad (-97 < -93)$

- **Adding inequalities in the same direction preserves them.**

  If $a < b$ and $c < d$, then $a + c < b + d$.

  $3 < 7$ and $5 < 10 \Rightarrow 3 + 5 < 7 + 10 \quad (8 < 17)$

- **Multiplying or dividing an inequality by a positive number preserves it.**

  If $a < b$, and c is positive, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

  $3 < 7 \Rightarrow 3 \times 100 < 7 \times 100 \quad (300 < 700)$
  $3 < 7 \Rightarrow 3 \div 100 < 7 \div 100 \quad \left(\frac{3}{100} < \frac{7}{100}\right)$

- **Multiplying or dividing an inequality by a negative number reverses it.**

  If $a < b$, and c is negative, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

  $3 < 7 \Rightarrow 3 \times (-100) > 7 \times (-100) \quad (-300 > -700)$
  $3 < 7 \Rightarrow 3 \div (-100) > 7 \div (-100) \quad \left(-\frac{3}{100} > -\frac{7}{100}\right)$

- **Taking negatives reverses an inequality.**

  If $a < b$, then $-a > -b$ and if $a > b$, then $-a < -b$.

  $3 < 7 \Rightarrow -3 > -7$ and $7 > 3 \Rightarrow -7 < -3$

- **If two numbers are each positive or negative, then taking reciprocals reverses an inequality.**

  If a and b are both positive or both negative and $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

  $3 < 7 \Rightarrow \frac{1}{3} > \frac{1}{7} \qquad -7 < -3 \Rightarrow -\frac{1}{7} > -\frac{1}{3}$

KEY FACT A25

Important inequalities for numbers between 0 and 1.

- **If $0 < x < 1$, and a is positive, then $xa < a$. For example: $.85 \times 19 < 19$.**
- **If $0 < x < 1$, and m and n are positive integers with $m > n$, then**

  $x^m < x^n < x$. For example, $\left(\frac{1}{2}\right)^5 < \left(\frac{1}{2}\right)^2 < \frac{1}{2}$.

TIP

Be sure you understand KEY FACT A24; it is very useful. Also, review the important properties listed in KEY FACTS A25 and A26. These properties come up often on the GRE.

- If $0 < x < 1$, then $\sqrt{x} > x$. For example, $\sqrt{\frac{3}{4}} > \frac{3}{4}$.
- If $0 < x < 1$, then $\frac{1}{x} > x$. In fact, $\frac{1}{x} > 1$. For example, $\frac{1}{0.2} > 1 > 0.2$.

KEY FACT A26

Properties of Zero

- **0 is the only number that is neither positive nor negative.**
- **0 is smaller than every positive number and greater than every negative number.**
- **0 is an even integer.**
- **0 is a multiple of every integer.**
- **For every number a: $a + 0 = a$ and $a - 0 = a$.**
- **For every number a: $a \times 0 = 0$.**
- **For every positive integer n: $0^n = 0$.**
- **For every number a (including 0): $a \div 0$ and $\frac{a}{0}$ are *meaningless symbols.* (They are *undefined.*)**
- **For every number a other than 0: $0 \div a = \frac{0}{a} = 0$.**
- **0 is the only number that is equal to its opposite: $0 = -0$.**
- **If the product of two or more numbers is 0, at least one of them is 0.**

Key Fact A27

Properties of 1

- **For any number a: $1 \times a = a$ and $\frac{a}{1} = a$.**
- **For any integer n: $1^n = 1$.**
- **1 is a divisor of every integer.**
- **1 is the smallest positive integer.**
- **1 is an odd integer.**
- **1 is *not* a prime.**

Practice Exercises—Basic Arithmetic

Discrete Quantitative Questions

1. For how many positive integers, a, is it true that $a^2 \leq 2a$?
   Ⓐ None
   Ⓑ 1
   Ⓒ 2
   Ⓓ 4
   Ⓔ More than 4

2. If $0 < a < b < 1$, which of the following statements are true?
   Indicate *all* such statements.
   [A] $a - b$ is negative
   [B] $\frac{1}{ab}$ is positive
   [C] $\frac{1}{b} - \frac{1}{a}$ is positive

3. If the product of 4 consecutive integers is equal to one of them, what is the largest possible value of one of the integers?
   [ ]

4. At 3:00 A.M. the temperature was 13° below zero. By noon it had risen to 32°. What was the average hourly increase in temperature?
   Ⓐ $\left(\frac{19}{9}\right)^{\circ}$
   Ⓑ $\left(\frac{19}{6}\right)^{\circ}$
   Ⓒ 5°
   Ⓓ 7.5°
   Ⓔ 45°

5. If a and b are negative, and c is positive, which of the following statements are true? Indicate *all* such statements.
   [A] $a - b < a - c$
   [B] If $a < b$, then $\frac{a}{c} < \frac{b}{c}$
   [C] $\frac{1}{b} < \frac{1}{c}$

6. If $-7 \leq x \leq 7$ and $0 \leq y \leq 12$, what is the greatest possible value of $y - x$?
   Ⓐ −19
   Ⓑ 5
   Ⓒ 7
   Ⓓ 17
   Ⓔ 19

7. If $(7^a)(7^b) = \frac{7^c}{7^d}$, what is d in terms of a, b, and c?
   Ⓐ $\frac{c}{ab}$
   Ⓑ $c - a - b$
   Ⓒ $a + b - c$
   Ⓓ $c - ab$
   Ⓔ $\frac{c}{a+b}$

8. If each of ★ and ❖ can be replaced by +, −, or ×, how many different values are there for the expression 2 ★ 2 ❖ 2?
   [ ]

9. A number is "terrific" if it is a multiple of 2 or 3. How many terrific numbers are there between –11 and 11?

Ⓐ 6
Ⓑ 7
Ⓒ 11
Ⓓ 15
Ⓔ 17

10. If $x \circledast y$ represents the number of integers greater than x and less than y, what is the value of $-\pi \circledast \sqrt{2}$?

Ⓐ 2
Ⓑ 3
Ⓒ 4
Ⓓ 5
Ⓔ 6

Questions 11 and 12 refer to the following definition.

For any positive integer n, $\tau(n)$ represents the number of positive divisors of n.

11. Which of the following statements are true? Indicate *all* such statements.

[A] $\tau(5) = \tau(7)$
[B] $\tau(5) \cdot \tau(7) = \tau(35)$
[C] $\tau(5) + \tau(7) = \tau(12)$

12. What is the value of $\tau(\tau(\tau(12)))$?

Ⓐ 1
Ⓑ 2
Ⓒ 3
Ⓓ 4
Ⓔ 6

13. If p and q are primes greater than 2, which of the following statements must be true? Indicate *all* such statements.

[A] $p + q$ is even
[B] pq is odd
[C] $p^2 - q^2$ is even

14. If $0 < x < 1$, which of the following lists the numbers in increasing order?

Ⓐ $\sqrt{x}, x, x^2$
Ⓑ $x^2, x, \sqrt{x}$
Ⓒ $x^2, \sqrt{x}, x$
Ⓓ $x, x^2, \sqrt{x}$
Ⓔ $x, \sqrt{x}, x^2$

15. Which of the following is equal to $(7^8 \times 7^9)^{10}$?

Ⓐ 7^{27}
Ⓑ 7^{82}
Ⓒ 7^{170}
Ⓓ 49^{170}
Ⓔ 49^{720}

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

	Quantity A	Quantity B
16.	The product of the odd integers between –8 and 8	The product of the even integers between –9 and 9

a and b are nonzero integers

	Quantity A	Quantity B
17.	$a + b$	ab

	Quantity A	Quantity B
18.	The remainder when a positive integer is divided by 7	7

	Quantity A	Quantity B
19.	$24 \div 6 \times 4$	12

	Quantity A	Quantity B
20.	$\frac{2x - 17}{2}$	$x - 17$

n is an integer greater than 1 that leaves a remainder of 1 when it is divided by 2, 3, 4, 5, and 6

	Quantity A	Quantity B
21.	n	60

	Quantity A	Quantity B
22.	The number of primes that are divisible by 2	The number of primes that are divisible by 3

n is a positive integer

	Quantity A	Quantity B
23.	The number of different prime factors of n	The number of different prime factors of n^2

	Quantity A	Quantity B
24.	The number of even positive factors of 30	The number of odd positive factors of 30

n is a positive integer

	Quantity A	Quantity B
25.	$(-10)^n$	$(-10)^{n+1}$

ANSWER KEY

1. **C**	6. **E**	11. **A, B**	16. **A**	21. **A**
2. **A, B**	7. **B**	12. **C**	17. **D**	22. **C**
3. **3**	8. **4**	13. **A, B, C**	18. **B**	23. **C**
4. **C**	9. **D**	14. **B**	19. **A**	24. **C**
5. **B, C**	10. **D**	15. **C**	20. **A**	25. **D**

ANSWER EXPLANATIONS

1. **(C)** Since a is positive, we can divide both sides of the given inequality by a: $a^2 \le 2a \Rightarrow a \le 2 \Rightarrow a = 1$ or 2.

2. **(A)(B)** Since $a < b$, $a - b$ is negative (A is true). Since a and b are positive, so is their product, ab; and the reciprocal of a positive number is positive (B is true).
$\frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab}$, and we have just seen that the numerator is negative and the denominator positive; so the value of the fraction is negative (C is false).

3. **3** If all four integers were negative, their product would be positive, and so could not equal one of them. If all of the integers were positive, their product would be much greater than any of them (even $1 \times 2 \times 3 \times 4 = 24$). So, the integers must include 0, in which case their product *is* 0. The largest set of four consecutive integers that includes 0 is 0, 1, 2, 3.

4. **(C)** In the 9 hours from 3:00 to 12:00, the temperature rose $32 - (-13) = 32 + 13 = 45$ degrees. So, the average hourly increase was $45° \div 9 = 5°$.

5. **(B)(C)** Since b is negative and c is positive, $b < c \Rightarrow -b > -c \Rightarrow a - b > a - c$ (A is false). Since c is positive, dividing by c preserves the inequality. (B is true.) Since b is negative, $\frac{1}{b}$ is negative, and so is less than $\frac{1}{c}$, which is positive (C is true).

6. **(E)** To make $y - x$ as large as possible, let y be as big as possible (12), and subtract the smallest amount possible ($x = -7$): $12 - (-7) = 19$.

7. **(B)** $(7^a)(7^b) = 7^{a+b}$, and $\frac{7^c}{7^d} = 7^{c-d}$. Therefore,
$a + b = c - d \Rightarrow a + b + d = c \Rightarrow d = c - a - b$

8. **4** Just list the 9 possible outcomes of replacing ★ and ❖ by +, –, and ×, and see that there are 4 different values: –2, 2, 6, 8.

$2 + 2 + 2 = 6$	$2 - 2 - 2 = -2$	$2 \times 2 \times 2 = 8$
$2 + 2 - 2 = 2$	$2 - 2 \times 2 = -2$	$2 \times 2 + 2 = 6$
$2 + 2 \times 2 = 6$	$2 - 2 + 2 = 2$	$2 \times 2 - 2 = 2$

9. **(D)** There are 15 "terrific" numbers: 2, 3, 4, 6, 8, 9, 10, their opposites, and 0.

10. **(D)** There are 5 integers (1, 0, –1, –2, –3) that are greater than –3.14 ($-\pi$) and less than 1.41 $(\sqrt{2})$.

11. **(A)(B)** Since 5 and 7 have two positive factors each, $\tau(5) = \tau(7)$. (A is true.) Since 35 has 4 divisors (1, 5, 7, and 35) and $\tau(5) \cdot \tau(7) = 2 \times 2 = 4$. (B is true.) Since the positive divisors of 12 are 1, 2, 3, 4, 6, and 12, $\tau(12)$ is 6, which is *not* equal to 2 + 2. (C is false.)

12. **(C)** $\tau(\tau(\tau(12))) = \tau(\tau(6)) = \tau(4) = 3$

13. **(A)(B)(C)** All primes greater than 2 are odd, so p and q are odd, and $p + q$, the sum of two odd numbers, is even (A is true). The product of two odd numbers is odd (B is true). Since p and q are odd, so are their squares, and so the difference of their squares is even (C is true).

14. **(B)** For any number, x, between 0 and 1: $x^2 < x$ and $x < \sqrt{x}$.

15. **(C)** First, multiply inside the parentheses: $7^8 \times 7^9 = 7^{17}$; then, raise to the 10th power: $(7^{17})^{10} = 7^{170}$.

16. **(A)** Since Quantity A has 4 negative factors (–7, –5, –3, –1), it is positive. Quantity B also has 4 negative factors, but be careful—it also has the factor 0, and so Quantity B is 0.

17. **(D)** If a and b are each 1, then $a + b = 2$, and $ab = 1$; so, Quantity A is greater. But, if a and b are each 3, $a + b = 6$, and $ab = 9$, then Quantity B is greater.

18. **(B)** The remainder is *always* less than the divisor.

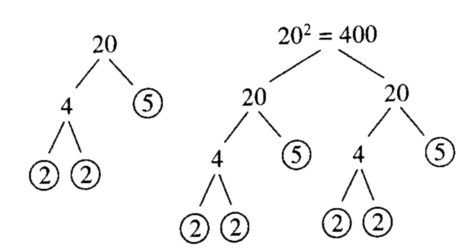
19. **(A)** According to PEMDAS, you divide and multiply from left to right (do *not* do the multiplication first): $24 \div 6 \times 4 = 4 \times 4 = 16$.

20. **(A)** By the distributive law, $\frac{2x-17}{2} = \frac{2x}{2} - \frac{17}{2} = x - 8.5$, which is greater than $x - 17$ (the larger the number you subtract, the smaller the difference.)

21. **(A)** The LCM of 2, 3, 4, 5, 6 is 60; and all multiples of 60 are divisible by each of them. So, n could be 61 or 1 more than any multiple of 60.

22. **(C)** The only prime divisible by 2 is 2, and the only prime divisible by 3 is 3. Quantity A and Quantity B are each 1.

23. **(C)** If you make a factor tree for n^2, the first branches would be n and n. Now, when you factor each n, you get exactly the same prime factors. (See the example below.)



24. **(C)** Just list the factors of 30: 1, 2, 3, 5, 6, 10, 15, 30. Four of them are odd and four are even.

25. **(D)** If n is even, then $n + 1$ is odd, and consequently $(-10)^n$ is positive, whereas $(-10)^{n+1}$ is negative. If n is odd, exactly the opposite is true.

11-B. FRACTIONS AND DECIMALS

Several questions on the GRE involve fractions or decimals. The KEY FACTS in this section cover all of the important facts you need to know for the GRE.

When a whole is *divided* into n equal parts, each part is called *one-nth* of the whole, written $\frac{1}{n}$. For example, if a pizza is cut (*divided*) into 8 equal slices, each slice is one-eighth $\left(\frac{1}{8}\right)$ of the pizza; a day is *divided* into 24 equal hours, so an hour is one-twenty-fourth $\left(\frac{1}{24}\right)$ of a day; and an inch is one-twelfth $\left(\frac{1}{12}\right)$ of a foot.

- If Donna slept for 5 hours, she slept for five-twenty-fourths $\left(\frac{5}{24}\right)$ of a day.
- If Taryn bought 8 slices of pizza, she bought eight-eighths $\left(\frac{8}{8}\right)$ of a pie.
- If Aviva's shelf is 30 inches long, it measures thirty-twelfths $\left(\frac{30}{12}\right)$ of a foot.

Numbers such as $\frac{5}{24}$, $\frac{8}{8}$, and $\frac{30}{12}$, in which one integer is written over a second integer, are called ***fractions***. The center line is called the fraction bar. The number above the bar is called the ***numerator***, and the number below the bar is called the ***denominator***.

> **CAUTION**
> The denominator of a fraction can *never* be 0.

- A fraction, such as $\frac{5}{24}$, in which the numerator is less than the denominator, is called a ***proper fraction***. Its value is less than 1.
- A fraction, such as $\frac{30}{12}$, in which the numerator is more than the denominator, is called an ***improper fraction***. Its value is greater than 1.
- A fraction, such as $\frac{8}{8}$, in which the numerator and denominator are the same, is also ***improper***, but it is equal to 1.

It is useful to think of the fraction bar as a symbol for division. If three pizzas are divided equally among eight people, each person gets $\frac{3}{8}$ of a pizza. If you actually divide 3 by 8, you get that $\frac{3}{8} = 0.375$.

KEY FACT B1

Every fraction, proper or improper, can be expressed in decimal form (or as a whole number) by dividing the numerator by the denominator.

$\frac{3}{10} = 0.3 \quad \frac{3}{4} = 0.75 \quad \frac{5}{8} = 0.625 \quad \frac{3}{16} = 0.1875$

$\frac{8}{8} = 1 \quad \frac{11}{8} = 1.375 \quad \frac{48}{16} = 3 \quad \frac{100}{8} = 12.5$

Note that any number beginning with a decimal point can be written with a 0 to the left of the decimal point. In fact, some calculators will express $3 \div 8$ as .375, whereas others will print 0.375.

Unlike the examples above, when most fractions are converted to decimals, the division does not terminate after 2 or 3 or 4 decimal places; rather it goes on forever with some set of digits repeating itself.

$\frac{2}{3} = 0.666666... \quad \frac{3}{11} = 0.272727... \quad \frac{5}{12} = 0.416666... \quad -\frac{17}{15} = -1.133333...$

A convenient way to represent repeating decimals is to place a bar over the digits that repeat. For example, the decimal equivalent of the four fractions, above, could be written as follows:

$$\frac{2}{3} = 0.\overline{6} \quad \frac{3}{11} = 0.\overline{27} \quad \frac{5}{12} = 0.41\overline{6} \quad -\frac{17}{15} = -1.1\overline{3}$$

A ***rational number*** is any number that can be expressed as a fraction, $\frac{a}{b}$, where a and b are integers. For example, 2, −2.2, and $2\frac{1}{2}$ are all rational since they can be expressed as $\frac{2}{1}$, $\frac{-22}{10}$, and $\frac{5}{2}$, respectively. When written as decimals, all rational numbers either terminate or repeat. Numbers such as $\sqrt{2}$ and π that cannot be expressed as fractions whose numerators and denominators are integers are called ***irrational numbers***. The decimal expansions of irrational numbers neither terminate nor repeat. For example, $\sqrt{2} = 1.414213562.....$ and $\pi = 3.141592654....$

Comparing Fractions and Decimals

KEY FACT B2

To compare two positive decimals, follow these rules.

- **Whichever number has the greater number to the left of the decimal point is greater: since 11 > 9, 11.001 > 9.896; since 1 > 0, 1.234 > 0.8; and since 3 > −3, 3.01 > −3.95. (Recall that if a decimal is written without a number to the left of the decimal point, you may assume that a 0 is there. So, 1.234 > 0.8.)**

- **If the numbers to the left of the decimal point are equal (or if there are no numbers to the left of the decimal point), proceed as follows:**
  1. **If the numbers do not have the same number of digits to the right of the decimal point, add zeros to the end of the shorter one.**
  2. **Now, compare the numbers *ignoring* the decimal point.**

For example, to compare 1.83 and 1.823, add a 0 to the end of 1.83, forming 1.830. Now compare them, *thinking of them as whole numbers*: since, 1830 > 1823, then 1.830 > 1.823.

EXAMPLE 1

Quantity A	Quantity B
.2139	.239

SOLUTION.
Do not think that Quantity A is greater because 2139 > 239. Be sure to add a 0 to the end of 0.239 (forming 0.2390) before comparing. Now, since 2390 > 2139, Quantity **B** is greater.

KEY FACT B3

There are two methods of comparing positive fractions:

1. **Convert them to decimals (by dividing), and use KEY FACT B2.**
2. **Cross-multiply.**

For example, to compare $\frac{1}{3}$ and $\frac{3}{8}$, we have two choices.

1. Write $\frac{1}{3}$ = .3333... and $\frac{3}{8}$ = .375. Since .375 > .333, then $\frac{3}{8} > \frac{1}{3}$.

2. Cross-multiply: $\frac{1}{3} \times \frac{3}{8}$. Since $3 \times 3 > 8 \times 1$, then $\frac{3}{8} > \frac{1}{3}$.

KEY FACT B4

When comparing positive fractions, there are three situations in which it is easier just to look at the fractions, and not use either method in KEY FACT B3.

1. **If the fractions have the same denominator, the fraction with the larger numerator is greater. Just as $9 is more than $7, and 9 books are more than 7 books, 9 fortieths are more than 7 fortieths: $\frac{9}{40} > \frac{7}{40}$.**
2. **If the fractions have the same numerator, the fraction with the smaller denominator is greater.**

If you divide a cake into 5 equal pieces, each piece is larger than the pieces you would get if you had divided the cake into 10 equal pieces: $\frac{1}{5} > \frac{1}{10}$, and similarly $\frac{3}{5} > \frac{3}{10}$.

3. **Sometimes the fractions are so familiar or easy to work with, you just know the answer. For example, $\frac{3}{4} > \frac{1}{5}$ and $\frac{11}{20} > \frac{1}{2}$ $\left(\text{since } \frac{10}{20} = \frac{1}{2}\right)$.**

KEY FACTS B2, B3, and B4 apply to *positive* decimals and fractions.

KEY FACT B5

- **Clearly, any positive number is greater than any negative number:**

$$\frac{1}{2} > -\frac{1}{5} \quad \text{and} \quad 0.123 > -2.56$$

- **For negative decimals and fractions, use KEY FACT A24, which states that if $a > b$, then $-a < -b$:**

$$\frac{1}{2} > \frac{1}{5} \Rightarrow -\frac{1}{2} < -\frac{1}{5} \quad \text{and} \quad 0.83 > 0.829 \Rightarrow -0.83 < -0.829$$

EXAMPLE 2

Which of the following lists the fractions $\frac{2}{3}, \frac{5}{8}$, and $\frac{13}{20}$ in order from least to greatest?

Ⓐ $\frac{2}{3}, \frac{5}{8}, \frac{13}{20}$ Ⓑ $\frac{5}{8}, \frac{2}{3}, \frac{13}{20}$ Ⓒ $\frac{5}{8}, \frac{13}{20}, \frac{2}{3}$ Ⓓ $\frac{13}{20}, \frac{5}{8}, \frac{2}{3}$ Ⓔ $\frac{13}{20}, \frac{2}{3}, \frac{5}{8}$

SOLUTION.
Use your calculator to quickly convert each to a decimal, writing down the first few decimal places: $\frac{2}{3} = 0.666$, $\frac{5}{8} = 0.625$, and $\frac{13}{20} = 0.65$. It is now easy to order the decimals: 0.625 < 0.650 < 0.666. The answer is **C**.

ALTERNATIVE SOLUTION.
Cross-multiply.

- $\frac{2}{3} > \frac{5}{8}$ since $8 \times 2 > 3 \times 5$.
- $\frac{13}{20} > \frac{5}{8}$ since $8 \times 13 > 20 \times 5$.
- $\frac{2}{3} > \frac{13}{20}$ since $20 \times 2 > 3 \times 13$.

EXAMPLE 3

$0 < x < y$

Quantity A	Quantity B
$\frac{1}{x} - \frac{1}{y}$	0

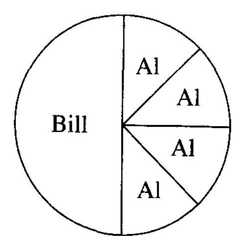
SOLUTION.

By KEY FACT B4, $x < y \Rightarrow \frac{1}{x} > \frac{1}{y}$, and so by KEY FACT A22, $\frac{1}{x} - \frac{1}{y}$ is positive. Quantity **A** is greater.

Equivalent Fractions

If Bill and Al shared a pizza, and Bill ate $\frac{1}{2}$ the pizza and Al ate $\frac{4}{8}$ of it, they had exactly the same amount.

We express this idea by saying that $\frac{1}{2}$ and $\frac{4}{8}$ are ***equivalent fractions***: they have the exact same value.



NOTE: If you multiply both the numerator and denominator of $\frac{1}{2}$ by 4 you get $\frac{4}{8}$; and if you divide both the numerator and denominator of $\frac{4}{8}$ by 4 you get $\frac{1}{2}$. This illustrates the next KEY FACT.

KEY FACT B6

Two fractions are equivalent if multiplying or dividing both the numerator and denominator of the first one *by the same number* gives the second one.

Consider the following two cases.

1. When the numerator and denominator of $\frac{3}{8}$ are each multiplied by 15, the products are $3 \times 15 = 45$ and $8 \times 15 = 120$. Therefore, $\frac{3}{8}$ and $\frac{45}{120}$ are equivalent fractions.

2. $\frac{2}{3}$ and $\frac{28}{45}$ are not equivalent fractions because 2 must be multiplied by 14 to get 28, but 3 must be multiplied by 15 to get 45.

KEY FACT B7

To determine if two fractions are equivalent, cross-multiply. The fractions are equivalent if and only if the two products are equal.

For example, since $120 \times 3 = 8 \times 45$, then $\frac{3}{8}$ and $\frac{45}{120}$ are equivalent.

Since $45 \times 2 \neq 3 \times 28$, then $\frac{2}{3}$ and $\frac{28}{45}$ are not equivalent fractions.

A fraction is in ***lowest terms*** if no positive integer greater than 1 is a factor of both the numerator and denominator. For example, $\frac{9}{20}$ is in lowest terms, since no integer greater than 1 is a factor of both 9 and 20; but $\frac{9}{24}$ is not in lowest terms, since 3 is a factor of both 9 and 24.

KEY FACT B8

Every fraction can be *reduced* to lowest terms by dividing the numerator and the denominator by their greatest common factor (GCF). If the GCF is 1, the fraction is already in lowest terms.

For any positive integer n: $n!$, read *n factorial*, is the product of all the integers from 1 to n, inclusive.

EXAMPLE 4

What is the value of $\frac{6!}{8!}$?

Ⓐ $\frac{1}{56}$ Ⓑ $\frac{1}{48}$ Ⓒ $\frac{1}{8}$ Ⓓ $\frac{1}{4}$ Ⓔ $\frac{3}{4}$

SOLUTION.

Even with a calculator, you do not want to calculate 6! ($1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$) and 8! ($1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 40{,}320$) and then take the time to reduce $\frac{720}{40{,}320}$. Here's the easy solution:

$$\frac{6!}{8!} = \frac{\overset{1}{\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}}{8 \times 7 \times \underset{1}{\cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}} = \frac{1}{8 \times 7} = \frac{1}{56}$$

Arithmetic Operations with Decimals

Arithmetic operations with decimals should be done on your calculator, unless they are so easy that you can do them in your head.

Multiplying and dividing by powers of 10 is particularly easy and does not require a calculator: they can be accomplished just by moving the decimal point.

KEY FACT B9

To multiply any decimal or whole number by a power of 10, move the decimal point as many places to the *right* as there are 0s in the power of 10, filling in with 0s, if necessary.

$1.35 \times 10 = 13.5$ (1) $\quad 1.35 \times 100 = 135$ (2)

$1.35 \times 1000 = 1350$ (3)

$23 \times 10 = 230$ (1) $\quad 23 \times 100 = 2300$ (2)

$23 \times 1{,}000{,}000 = 23{,}000{,}000$ (6)

KEY FACT B10

To divide any decimal or whole number by a power of 10, move the decimal point as many places to the *left* as there are 0s in the power of 10, filling in with 0s, if necessary.

$67.8 \div 10 = 6.78$ (1) $\quad 67.8 \div 100 = 0.678$ (2)

$67.8 \div 1000 = 0.0678$ (3)

$14 \div 10 = 1.4$ (1) $\quad 14 \div 100 = 0.14$ (2)

$14 \div 1{,}000{,}000 = 0.000014$ (6)

EXAMPLE 5

Quantity A	Quantity B
3.75×10^4	$37{,}500{,}000 \div 10^3$

SOLUTION.

To evaluate Quantity A, move the decimal point 4 places to the right: **37,500**. To evaluate Quantity B, move the decimal point 3 places to the left: **37,500**. The answer is **C**.

Arithmetic Operations with Fractions

KEY FACT B11

To multiply two fractions, multiply their numerators and multiply their denominators:

$$\frac{3}{5} \times \frac{4}{7} = \frac{3\times 4}{5\times 7} = \frac{12}{35} \qquad \frac{3}{5} \times \frac{\pi}{2} = \frac{3\times \pi}{5\times 2} = \frac{3\pi}{10}$$

KEY FACT B12

To multiply a fraction by any other number, write that number as a fraction whose denominator is 1:

$$\frac{3}{5} \times 7 = \frac{3}{5} \times \frac{7}{1} = \frac{21}{5} \qquad \frac{3}{5} \times \pi = \frac{3}{5} \times \frac{\pi}{1} = \frac{3\pi}{5}$$

TACTIC B1

Before multiplying fractions, reduce. You may reduce by dividing any numerator and any denominator by a common factor.

EXAMPLE 6

Express the product, $\frac{3}{4} \times \frac{8}{9} \times \frac{15}{16}$, in lowest terms.

$$\frac{\boxed{}}{\boxed{}}$$

SOLUTION.

You could use your calculator to multiply the numerators and denominators: $\frac{360}{576}$.

It is better, however, to use TACTIC B1 and reduce first:

$$\frac{\overset{1}{\cancel{3}}}{4} \times \frac{\overset{1}{\cancel{8}}}{\underset{\underset{1}{\cancel{3}}}{\cancel{9}}} \times \frac{\overset{5}{\cancel{15}}}{\underset{2}{\cancel{16}}} = \frac{1\times 1\times 5}{4\times 1\times 2} = \frac{5}{8}$$

TACTIC

B2

When a problem requires you to find a fraction of a number, multiply.

EXAMPLE 7

If $\frac{4}{7}$ of the 350 sophomores at Monroe High School are girls, and $\frac{7}{8}$ of them play on a team, how many sophomore girls do not play on a team?

$\boxed{}$

SOLUTION.

There are $\frac{4}{7} \times 350 = 200$ sophomore girls.

Of these, $\frac{7}{8} \times 200 = 175$ play on a team. So, $200 - 175 = \mathbf{25}$ do not play on a team.

The ***reciprocal*** of any nonzero number x is that number y such that $xy = 1$. Since $x\left(\frac{1}{x}\right) = 1$, then $\frac{1}{x}$ is the reciprocal of x. Similarly, the reciprocal of the fraction $\frac{a}{b}$ is the fraction $\frac{b}{a}$, since $\frac{a}{b} \cdot \frac{b}{a} = 1$.

KEY FACT B13

To divide any number by a fraction, multiply that number by the reciprocal of the fraction.

$$20 \div \frac{2}{3} = \frac{20}{1} \times \frac{3}{2} = 30 \qquad \frac{3}{5} \div \frac{2}{3} = \frac{3}{5} \times \frac{3}{2} = \frac{9}{10}$$

$$\sqrt{2} \div \frac{2}{3} = \frac{\sqrt{2}}{1} \times \frac{3}{2} = \frac{3\sqrt{2}}{2} \qquad \frac{\pi}{5} \div \frac{2}{3} = \frac{\pi}{5} \times \frac{3}{2} = \frac{3\pi}{10}$$

EXAMPLE 8

In the meat department of a supermarket, 100 pounds of chopped meat was divided into packages, each of which weighed $\frac{4}{7}$ of a pound. How many packages were there?

$\boxed{}$

SOLUTION.

$$100 \div \frac{4}{7} = \frac{100}{1} \times \frac{7}{4} = \mathbf{175}$$

KEY FACT B14

- **To add or subtract fractions with the same denominator, add or subtract the numerators and keep the denominator:**

$$\frac{4}{9} + \frac{1}{9} = \frac{5}{9} \quad \textbf{and} \quad \frac{4}{9} - \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$$

- **To add or subtract fractions with different denominators, first rewrite the fractions as equivalent fractions with the same denominators:**

$$\frac{1}{6} + \frac{3}{4} = \frac{2}{12} + \frac{9}{12} = \frac{11}{12}$$

NOTE: The *easiest* common denominator to find is the product of the denominators ($6 \times 4 = 24$, in this example), but the best denominator to use is the ***least common denominator***, which is the least common multiple (LCM) of the denominators (12, in this case). Using the least common denominator minimizes the amount of reducing that is necessary to express the answer in lowest terms.

KEY FACT B15

If $\frac{a}{b}$ is the fraction of a whole that satisfies some property, then $1 - \frac{a}{b}$ is the fraction of that whole that does not satisfy it.

EXAMPLE 9

In a jar, $\frac{1}{2}$ of the marbles are red, $\frac{1}{4}$ are white, and $\frac{1}{5}$ are blue. What fraction of the marbles are neither red, white, nor blue?

$$\frac{\boxed{}}{\boxed{}}$$

SOLUTION.
The red, white, and blue marbles constitute

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{5}=\frac{10}{20}+\frac{5}{20}+\frac{4}{20}=\frac{19}{20}$$

of the total, so $1-\frac{19}{20}=\frac{20}{20}-\frac{19}{20}=\frac{1}{20}$ of the marbles are neither red, white, nor blue.

Alternatively, you could convert the fractions to decimals and use your calculator.

$$0.5+0.25+0.2=0.95$$

$$1-0.95=0.05=\frac{5}{100}$$

Remember, on the GRE you do not have to reduce fractions, so $\frac{5}{100}$ is an acceptable answer.

EXAMPLE 10

Lindsay ate $\frac{1}{3}$ of a cake and Emily ate $\frac{1}{4}$ of it. What fraction of the cake was still uneaten?

$$\frac{\boxed{}}{\boxed{}}$$

SOLUTION.
$\frac{1}{3}+\frac{1}{4}=\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$ of the cake was eaten, and $1-\frac{7}{12}=\frac{5}{12}$ was uneaten.

EXAMPLE 11

Lindsay ate $\frac{1}{3}$ of a cake and Emily ate $\frac{1}{4}$ of what was left. What fraction of the cake was still uneaten?

$$\frac{\boxed{}}{\boxed{}}$$

CAUTION: Be sure to read questions carefully. In Example 10, Emily ate $\frac{1}{4}$ of the cake. In Example 11, however, she only ate $\frac{1}{4}$ of the $\frac{2}{3}$ that was left after Lindsay had her piece: she ate $\frac{1}{4}\times\frac{2}{3}=\frac{1}{6}$ of the cake.

SOLUTION.
$\frac{1}{3}+\frac{1}{6}=\frac{2}{6}+\frac{1}{6}=\frac{1}{2}$ of the cake was eaten, and the other $\frac{1}{2}$ was uneaten.

Arithmetic Operations with Mixed Numbers

A ***mixed number*** is a number such as $3\frac{1}{2}$, which consists of an integer followed by a fraction. It is an abbreviation for the *sum* of the number and the fraction; so, $3\frac{1}{2}$ is an abbreviation for $3+\frac{1}{2}$. Every mixed number can be written as an improper fraction, and every improper fraction can be written as a mixed number:

$$3\frac{1}{2}=3+\frac{1}{2}=\frac{3}{1}+\frac{1}{2}=\frac{6}{2}+\frac{1}{2}=\frac{7}{2} \quad \text{and} \quad \frac{7}{2}=\frac{6}{2}+\frac{1}{2}=3+\frac{1}{2}=3\frac{1}{2}$$

On the GRE you should perform all arithmetic operations on mixed numbers in one of the following two ways:

- Change the mixed numbers to improper fractions and use the rules you already know for performing arithmetic operations on fractions.
- Change the mixed numbers to decimals and perform the arithmetic on your calculator.

CAUTION

A common mistake for many students is to think that

$$3\times 5\frac{1}{2} \text{ is } 15\frac{1}{2}\text{— it isn't!}$$

If you need to multiply $3\times 5\frac{1}{2}$, use one of the two methods mentioned above.

- $3\times 5\frac{1}{2}=3\left(5+\frac{1}{2}\right)=15+\frac{3}{2}=15+1\frac{1}{2}=16\frac{1}{2}$
- $3\times 5\frac{1}{2}=3\times 5.5=16.5=16\frac{1}{2}$

Complex Fractions

A ***complex fraction*** is a fraction, such as $\dfrac{1+\frac{1}{6}}{2-\frac{3}{4}}$, which has one or more fractions in its numerator or denominator or both.

KEY FACT B16

There are two ways to simplify a complex fraction:

- **Multiply *every* term in the numerator and denominator by the least common multiple of all the denominators that appear in the fraction.**
- **Simplify the numerator and the denominator, and then divide.**

To simplify $\dfrac{1+\frac{1}{6}}{2-\frac{3}{4}}$

- either multiply each term by 12, the LCM of 6 and 4:

$$\frac{12(1)+12\left(\frac{1}{6}\right)}{12(2)-12\left(\frac{3}{4}\right)}=\frac{12+2}{24-9}=\frac{14}{15}\text{, or}$$

- simplify the numerator and denominator:

$$\frac{1+\frac{1}{6}}{2-\frac{3}{4}}=\frac{\frac{7}{6}}{\frac{5}{4}}=\frac{7}{6}\times\frac{4}{5}=\frac{14}{15}.$$

Practice Exercises—Fractions and Decimals

Discrete Quantitative Questions

1. A biology class has 12 boys and 18 girls. What fraction of the class are boys?

$$\frac{\boxed{}}{\boxed{}}$$

2. For how many integers, *a*, between 30 and 40 is it true that $\frac{5}{a}$, $\frac{8}{a}$, and $\frac{13}{a}$ are all in lowest terms?

Ⓐ 1
Ⓑ 2
Ⓒ 3
Ⓓ 4
Ⓔ 5

3. What fractional part of a week is 98 hours?

$$\frac{\boxed{}}{\boxed{}}$$

4. What is the value of the product $\frac{5}{5}\times\frac{5}{10}\times\frac{5}{15}\times\frac{5}{20}\times\frac{5}{25}$?

Ⓐ $\frac{1}{120}$
Ⓑ $\frac{1}{60}$
Ⓒ $\frac{1}{30}$
Ⓓ $\frac{5}{30}$
Ⓔ $\frac{1}{2}$

5. If $\frac{3}{11}$ of a number is 22, what is $\frac{6}{11}$ of that number?

Ⓐ 6
Ⓑ 11
Ⓒ 12
Ⓓ 33
Ⓔ 44

6. Jason won some goldfish at the state fair. During the first week, $\frac{1}{5}$ of them died, and during the second week, $\frac{3}{8}$ of those still alive at the end of the first week died. What fraction of the original goldfish were still alive after two weeks?

Ⓐ $\frac{3}{10}$
Ⓑ $\frac{17}{40}$
Ⓒ $\frac{1}{2}$
Ⓓ $\frac{23}{40}$
Ⓔ $\frac{7}{10}$

7. $\frac{5}{8}$ of 24 is equal to $\frac{15}{7}$ of what number?

Ⓐ 7
Ⓑ 8
Ⓒ 15
Ⓓ $\frac{7}{225}$
Ⓔ $\frac{225}{7}$

8. If $7a = 3$ and $3b = 7$, what is the value of $\frac{a}{b}$?

Ⓐ $\frac{9}{49}$

Ⓑ $\frac{3}{7}$

Ⓒ 1

Ⓓ $\frac{7}{3}$

Ⓔ $\frac{49}{9}$

9. What is the value of $\dfrac{\frac{7}{9}\times\frac{7}{9}}{\frac{7}{9}+\frac{7}{9}+\frac{7}{9}}$?

Ⓐ $\frac{7}{27}$

Ⓑ $\frac{2}{3}$

Ⓒ $\frac{7}{9}$

Ⓓ $\frac{9}{7}$

Ⓔ $\frac{3}{2}$

10. Which of the following expressions are greater than x when $x = \frac{9}{11}$?

Indicate *all* such expressions.

[A] $\frac{1}{x}$

[B] $\frac{x+1}{x}$

[C] $\frac{x+1}{x-1}$

11. One day at Lincoln High School, $\frac{1}{12}$ of the students were absent, and $\frac{1}{5}$ of those present went on a field trip. If the number of students staying in school that day was 704, how many students are enrolled at Lincoln High?

12. If $a = 0.87$, which of the following expressions are less than a?

Indicate *all* such expressions.

[A] $\sqrt{a}$

[B] a^2

[C] $\frac{1}{a}$

13. For what value of x is $\frac{(34.56)(7.89)}{x} = (.3456)(78.9)$?

Ⓐ .001
Ⓑ .01
Ⓒ .1
Ⓓ 10
Ⓔ 100

14. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, and C is the set consisting of all the fractions whose numerators are in A and whose denominators are in B, what is the product of all of the numbers in C?

Ⓐ $\frac{1}{64}$

Ⓑ $\frac{1}{48}$

Ⓒ $\frac{1}{24}$

Ⓓ $\frac{1}{12}$

Ⓔ $\frac{1}{2}$

15. For the final step in a calculation, Ezra accidentally divided by 1000 instead of multiplying by 1000. What should he do to his incorrect answer to correct it?

Ⓐ Multiply it by 1000.
Ⓑ Multiply it by 100,000.
Ⓒ Multiply it by 1,000,000.
Ⓓ Square it.
Ⓔ Double it.

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

	Quantity A	Quantity B
16.	$\frac{5}{13}$ of 47	$\frac{47}{13}$ of 5

$x = -\frac{2}{3}$ and $y = \frac{3}{5}$

	Quantity A	Quantity B
17.	xy	$\frac{x}{y}$

	Quantity A	Quantity B
18.	$\dfrac{15}{\frac{1}{15}}$	1

Judy needed 8 pounds of chicken. At the supermarket, the only packages available weighed $\frac{3}{4}$ of a pound each.

	Quantity A	Quantity B
19.	The number of packages Judy needed to buy	11

	Quantity A	Quantity B
20.	$\frac{11}{12}$ or $\frac{13}{14}$	$\frac{14}{15}$

$a\nabla b = \frac{a}{b} + \frac{b}{a}$

	Quantity A	Quantity B
21.	$3\nabla 4$	$\frac{1}{2}\nabla\frac{2}{3}$

	Quantity A	Quantity B
22.	$\frac{100}{2^{100}}$	$\frac{100}{3^{100}}$

	Quantity A	Quantity B
23.	$\left(-\frac{1}{2}\right)\left(-\frac{3}{4}\right)\left(-\frac{5}{6}\right)\left(-\frac{7}{8}\right)$	$\left(-\frac{3}{7}\right)\left(-\frac{5}{9}\right)\left(-\frac{7}{11}\right)$

$a = \frac{1}{2}$ and $b = \frac{1}{3}$

	Quantity A	Quantity B
24.	$\frac{a}{b}$	$\frac{b}{a}$

	Quantity A	Quantity B
25.	$\left(\frac{3}{11}\right)^2$	$\sqrt{\frac{3}{11}}$

ANSWER KEY

1. $\frac{2}{5}$	4. **A**	9. **A**	14. **A**	19. **C**	24. **A**
	5. **E**	10. **A, B**	15. **C**	20. **B**	25. **B**
2. **C**	6. **C**	11. **960**	16. **C**	21. **C**	
	7. **A**	12. **B**	17. **A**	22. **A**	
3. $\frac{7}{12}$	8. **A**	13. **D**	18. **A**	23. **A**	

Answer Explanations

1. $\frac{2}{5}$ The class has 30 students, of whom 12 are boys. So, the boys make up $\frac{12}{30} = \frac{5}{2}$ of the class.

2. **(C)** If a is even, then $\frac{8}{a}$ is *not* in lowest terms, since both a and 8 are divisible by 2. Therefore, the only possibilities are 31, 33, 35, 37, and 39; but $\frac{5}{35} = \frac{1}{7}$ and $\frac{13}{39} = \frac{1}{3}$, so only 3 integers—31, 33, and 37—satisfy the given condition.

3. $\frac{7}{12}$ There are 24 hours in a day and 7 days in a week, so there are $24 \times 7 = 168$ hours in a week: $\frac{98}{168} = \frac{7}{12}$.

4. **(A)** Reduce each fraction and multiply:

$$1 \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} = \frac{1}{120}.$$

5. **(E)** Don't bother writing an equation for this one; just think. We know that $\frac{3}{11}$ of the number is 22, and $\frac{6}{11}$ of a number is twice as much as $\frac{3}{11}$ of it: $2 \times 22 = 44$.

6. **(C)** The *algebra* way is to let x = the number of goldfish Jason won. During the first week $\frac{1}{5}x$ died, so $\frac{4}{5}x$ were still alive. During week two, $\frac{3}{8}$ of them died and $\frac{5}{8}$ of them survived:

$$\left(\frac{\cancel{5}^{1}}{\cancel{8}_{2}}\right)\left(\frac{\cancel{4}^{1}}{\cancel{5}_{1}}x\right) = \frac{1}{2}x.$$

On the GRE, the best way is to assume that the original number of goldfish was 40, the LCM of the denominators (see TACTIC 3, Chapter 9). Then, 8 died the first week $\left(\frac{1}{5} \text{ of } 40\right)$, and 12 of the 32 survivors $\left(\frac{3}{8} \text{ of } 32\right)$ died the second week. In all, 8 + 12 = 20 died; the other 20 $\left(\frac{1}{2}\right.$ the original number$\left.\right)$ were still alive.

7. **(A)** If x is the number, then $\frac{15}{7}x = \frac{5}{\cancel{8}_1} \times \cancel{24}^3 = 15$. So, $\frac{15}{7}x = 15$, which means (dividing by 15) that $\frac{1}{7}x = 1$, and so $x = 7$.

8. **(A)** $7a = 3$ and $3b = 7 \Rightarrow a = \frac{3}{7}$ and $b = \frac{7}{3} \Rightarrow \frac{a}{b} = \frac{3}{7} \div \frac{7}{3} = \frac{3}{7} \times \frac{3}{7} = \frac{9}{49}$.

9. **(A)** Don't start by doing the arithmetic. This is just $\frac{(a)(a)}{a+a+a} = \frac{(a)(\cancel{a})}{3\cancel{a}} = \frac{a}{3}$.

Now, replacing a with $\frac{7}{9}$ gives $\frac{7}{9} \div 3 = \frac{7}{9} \times \frac{1}{3} = \frac{7}{27}$.

10. **(A)(B)** The reciprocal of a positive number less than 1 is greater than 1 (A is true). $\frac{x+1}{x} = 1 + \frac{1}{x}$, which is greater than 1 (B is true). Since $\frac{9}{11} + 1$ is positive and $\frac{9}{11} - 1$ is negative, when $x = \frac{9}{11}$, $\frac{x+1}{x-1} < 0$ and, therefore, less than x (C is false).

11. **960** If s is the number of students enrolled, $\frac{1}{12}s$ is the number who were absent, and $\frac{11}{12}s$ is the number who were present. Since $\frac{1}{5}$ of them went on a field trip, $\frac{4}{5}$ of them stayed in school. Therefore,

$$704 = \frac{\cancel{4}^1}{5} \times \frac{11}{\cancel{12}_3}s = \frac{11}{15}s \Rightarrow$$

$$s = 704 \div \frac{11}{15} = 704 \times \frac{15}{11} = 960.$$

12. **(B)** Since $a < 1$, $\sqrt{a} > a$ (A is false). Since $a < 1$, $a^2 < a$ (B is true). The reciprocal of a positive number less than 1 is greater than 1 (C is false).

13. **(D)** There are two easy ways to do this. The first is to see that (34.56)(7.89) has 4 decimal places, whereas (.3456)(78.9) has 5, so the numerator has to be divided by 10. The second is to round off and calculate mentally: since $30 \times 8 = 240$, and $.3 \times 80 = 24$, we must divide by 10.

14. **(A)** Nine fractions are formed:

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{2}, \frac{3}{3}, \frac{3}{4}$$

Note that although some of these fractions are equivalent, we do have nine distinct fractions.

When you multiply, the three 2s and the three 3s in the numerators cancel with the three 2s and three 3s in the denominators. So, the numerator is 1 and the denominator is $4 \times 4 \times 4 = 64$.

15. **(C)** Multiplying Ezra's incorrect answer by 1000 would undo the final division he made. At that point he should have multiplied by 1000. So, to correct his error, he should multiply again by 1000. In all, Ezra should multiply his incorrect answer by $1000 \times 1000 = 1{,}000{,}000$.

16. **(C)** Each quantity equals $\dfrac{5 \times 47}{3}$.

17. **(A)** Quantity A: $-\dfrac{2}{3} \times \dfrac{3}{5} = -\dfrac{2}{5}$.

Quantity B: $-\dfrac{2}{3} \div \dfrac{3}{5} = -\dfrac{2}{3} \times \dfrac{5}{3} = -\dfrac{10}{9}$.

Finally, $\dfrac{10}{9} > \dfrac{2}{5} \Rightarrow -\dfrac{10}{9} < -\dfrac{2}{5}$.

18. **(A)** Quantity A: $\dfrac{15}{\frac{1}{15}} = 15 \times 15 = 225$.

19. **(C)** $8 \div \dfrac{3}{4} = 8 \times \dfrac{4}{3} = \dfrac{32}{3} = 10\dfrac{2}{3}$. Since 10 packages wouldn't be enough, she had to buy 11. (10 packages would weigh only $7\frac{1}{2}$ pounds.)

20. **(B)** You don't need to multiply on this one: since $\dfrac{11}{12} < 1$, $\dfrac{11}{12}$ of $\dfrac{13}{14}$ is less than $\dfrac{13}{14}$, which is already less than $\dfrac{14}{15}$.

21. **(C)** Quantity B is the sum of 2 complex fractions:

$$\frac{\frac{1}{2}}{\frac{2}{3}} + \frac{\frac{2}{3}}{\frac{1}{2}}$$

Simplifying each complex fraction, by multiplying numerator and denominator by 6, or treating these as the quotient of 2 fractions, we get $\dfrac{3}{4} + \dfrac{4}{3}$, which is exactly the value of Quantity A.

22. **(A)** When two fractions have the same numerator, the one with the smaller denominator is bigger, and $2^{100} < 3^{100}$.

23. **(A)** Since Quantity A is the product of 4 negative numbers, it is positive, and so is greater than Quantity B, which, being the product of 3 negative numbers, is negative.

24. **(A)** Quantity A: $\dfrac{1}{2} \div \dfrac{1}{3} = \dfrac{1}{2} \times \dfrac{3}{1} = \dfrac{3}{2}$.

Since Quantity B is the reciprocal of Quantity A, Quantity B = $\dfrac{2}{3}$.

25. **(B)** If $0 < x < 1$, then $x^2 < x < \sqrt{x}$. In this question, $x = \dfrac{3}{11}$.

11-C. PERCENTS

The word ***percent*** means hundredth. We use the symbol "%" to express the word "percent." For example, "17 percent" means "17 hundredths," and can be written with a % symbol, as a fraction, or as a decimal:

$$17\% = \frac{17}{100} = 0.17.$$

TIP

A percent is just a fraction whose denominator is 100.

KEY FACT C1

- **To convert a percent to a decimal, drop the % symbol and move the decimal point two places to the left, adding 0s if necessary. (Remember that we assume that there is a decimal point to the right of any whole number.)**
- **To convert a percent to a fraction, drop the % symbol, write the number over 100, and reduce.**

$25\% = 0.25 = \frac{25}{100} = \frac{1}{4}$ $\quad 100\% = 1.00 = \frac{100}{100}$ $\quad 12.5\% = 0.125 = \frac{12.5}{100} = \frac{125}{1000} = \frac{1}{8}$

$1\% = 0.01 = \frac{1}{100}$ $\quad \frac{1}{2}\% = 0.5\% = 0.005 = \frac{.5}{100} = \frac{1}{200}$ $\quad 250\% = 2.50 = \frac{250}{100} = \frac{5}{2}$

KEY FACT C2

- **To convert a decimal to a percent, move the decimal point two places to the right, adding 0s if necessary, and add the % symbol.**
- **To convert a fraction to a percent, first convert the fraction to a decimal, then convert the decimal to a percent, as indicated above.**

$0.375 = 37.5\%$ $\quad 0.3 = 30\%$ $\quad 1.25 = 125\%$ $\quad 10 = 1000\%$

$\frac{3}{4} = 0.75 = 75\%$ $\quad \frac{1}{3} = 0.33333\ldots = 33.333\ldots\% = 33\frac{1}{3}\%$ $\quad \frac{1}{5} = 0.2 = 20\%$

You should be familiar with the following basic conversions:

$\frac{1}{2} = 50\%$ $\quad \frac{1}{10} = 10\%$ $\quad \frac{6}{10} = \frac{3}{5} = 60\%$

$\frac{1}{3} = 33\frac{1}{3}\%$ $\quad \frac{2}{10} = \frac{1}{5} = 20\%$ $\quad \frac{7}{10} = 70\%$

$\frac{2}{3} = 66\frac{2}{3}\%$ $\quad \frac{3}{10} = 30\%$ $\quad \frac{8}{10} = \frac{4}{5} = 80\%$

$\frac{1}{4} = 25\%$ $\quad \frac{4}{10} = \frac{2}{5} = 40\%$ $\quad \frac{9}{10} = 90\%$

$\frac{3}{4} = 75\%$ $\quad \frac{5}{10} = \frac{1}{2} = 50\%$ $\quad \frac{10}{10} = 1 = 100\%$

Knowing these conversions can help solve many problems more quickly. For example, the fastest way to find 25% of 32 is not to multiply 32 by 0.25; rather, it is to know that $25\% = \frac{1}{4}$, and that $\frac{1}{4}$ of 32 is 8.

Many questions involving percents can actually be answered more quickly in your head than by using paper and pencil. Since $10\% = \frac{1}{10}$, to take 10% of a number, just divide by 10 by moving the decimal point one place to the left: 10% of 60 is 6. Also, since 5% is half of 10%, then 5% of 60 is 3 (half of 6); and since 30% is 3 times 10%, then 30% of 60 is 18 (3×6).

Practice doing this, because improving your ability to do mental math will add valuable points to your score on the GRE.

CAUTION

Do not confuse 0.5 and 0.5%.
Just as 5 is 100 times 5%, 0.5 is 100 times 0.5% = 0.005.

CAUTION

Although 35% can be written as $\frac{35}{100}$ or 0.35, x% can *only* be written as $\frac{x}{100}$.

Solving Percent Problems

Consider the following three questions:

(i) What is 45% of 200?
(ii) 90 is 45% of what number?
(iii) 90 is what percent of 200?

The arithmetic needed to answer each of these questions is very easy, but unless you set a question up properly, you won't know whether you should multiply or divide. In each case, there is one unknown, which we will call x. Now just translate each sentence, replacing "is" by "=" and the unknown by x.

(i) $x = 45\%$ of $200 \Rightarrow x = .45 \times 200 = 90$
(ii) $90 = 45\%$ of $x \Rightarrow 90 = .45x \Rightarrow x = 90 \div .45 = 200$
(iii) $90 = x\%$ of $200 \Rightarrow 90 = \frac{x}{\cancel{100}_{1}}(\cancel{200}^{2}) \Rightarrow 90 = 2x \Rightarrow x = 45$

EXAMPLE 1

Charlie gave 20% of his baseball cards to Kenne and 15% to Paulie. If he still had 520 cards, how many did he have originally?

SOLUTION.
Originally, Charlie had 100% of the cards (all of them). Since he gave away 35% of them, he has 100% – 35% = 65% of them left. So, 520 is 65% of what number?

$$520 = .65x \Rightarrow x = 520 \div .65 = \mathbf{800}.$$

EXAMPLE 2

After Ruth gave 110 baseball cards to Alison and 75 to Susanna, she still had 315 left. What percent of her cards did Ruth give away?

Ⓐ 25% Ⓑ $33\frac{1}{3}$% Ⓒ 37% Ⓓ 40% Ⓔ 50%

SOLUTION. Ruth gave away a total of 185 cards and had 315 left. Therefore, she started with 185 + 315 = 500 cards. So, 185 is what percent of 500?

$$185 = \frac{x}{\cancel{100}_1}(\cancel{500}^5) \Rightarrow 5x = 185 \Rightarrow x = 185 \div 5 = \mathbf{37}$$

Ruth gave away 37% of her cards, **(C)**.

Since percent means hundredth, the easiest number to use in any percent problem is 100:

$$a\% \text{ of } 100 = \frac{a}{\cancel{100}_1} \times \cancel{100}^1 = a.$$

KEY FACT C3

For any positive number a: a% of 100 is a.

For example: 91.2% of 100 is 91.2; 300% of 100 is 300; and $\frac{1}{2}$% of 100 is $\frac{1}{2}$.

TACTIC

C1

In any problem involving percents, use the number 100. (It doesn't matter whether or not 100 is a realistic number—a country can have a population of 100; an apple can cost \$100; a man can run 100 miles per hour.)

EXAMPLE 3

In 1985 the populations of town A and town B were the same. From 1985 to 1995 the population of town A increased by 60% while the population of town B decreased by 60%. In 1995, the population of town B was what percent of the population of town A?

Ⓐ 25% Ⓑ 36% Ⓒ 40% Ⓓ 60% Ⓔ 120%

SOLUTION.
On the GRE, do not waste time with a nice algebraic solution. Simply, assume that in 1985 the population of each town was 100. Then, since 60% of 100 is 60, in 1995, the populations were 100 + 60 = 160 and 100 – 60 = 40. So, in 1995, town B's population was $\frac{40}{160} = \frac{1}{4}$ = **25%** of town A's **(A)**.

Since a% of b is $\frac{a}{100} \times b = \frac{ab}{100}$, and b% of a is $\frac{b}{100} \times a = \frac{ba}{100}$, we have the result shown in KEY FACT C4.

KEY FACT C4

For any positive numbers a and b: a% of b = b% of a.

KEY FACT C4 often comes up on the GRE in quantitative comparison questions: Which is greater, 13% of 87 or 87% of 13? Don't multiply — they're equal.

Percent Increase and Decrease

KEY FACT C5

- **The *percent increase* of a quantity is**

$$\frac{\textbf{actual increase}}{\textbf{original amount}} \times \mathbf{100\%}.$$

- **The *percent decrease* of a quantity is**

$$\frac{\textbf{actual decrease}}{\textbf{original amount}} \times \mathbf{100\%}.$$

For example:

- If the price of a lamp goes from \$80 to \$100, the actual increase is \$20, and the percent increase is $\frac{20}{80} \times 100\% = \frac{1}{4} \times 100\% = 25\%$.
- If a \$100 lamp is on sale for \$80, the actual decrease in price is \$20, and the percent decrease is $\frac{20}{100} \times 100\% = 20\%$.

Notice that the percent increase in going from 80 to 100 is not the same as the percent decrease in going from 100 to 80.

KEY FACT C6

If $a < b$, the percent increase in going from a to b is *always* greater than the percent decrease in going from b to a.

KEY FACT C7

- **To increase a number by $k\%$, multiply it by $(1 + k\%)$.**
- **To decrease a number by $k\%$, multiply it by $(1 - k\%)$.**

For example:

- The value of a \$1600 investment after a 25% increase is $\$1600(1 + 25\%) = \$1600(1.25) = \$2000$.
- If the investment then loses 25% of its value, it is worth $\$2000(1 - 25\%) = \$2000(.75) = \$1500$.

Note that, after a 25% increase followed by a 25% decrease, the value is \$1500, \$100 *less* than the original amount.

KEY FACT C8

An increase of $k\%$ followed by a decrease of $k\%$ is equal to a decrease of $k\%$ followed by an increase of $k\%$, and is *always* less than the original value. The original value is never regained.

EXAMPLE 4

Store B always sells CDs at 60% off the list price. Store A sells its CDs at 40% off the list price, but often runs a special sale during which it reduces its prices by 20%.

Quantity A	Quantity B
The price of a CD when it is on sale at store A	The price of the same CD at store B

SOLUTION.
Assume the list price of the CD is \$100. Store B always sells the CD for \$40 (\$60 off the list price). Store A normally sells the CD for \$60 (\$40 off the list price), but on sale reduces its price by 20%. Since 20% of 60 is 12, the sale price is \$48 (\$60 – \$12). The price is greater at Store A.

Notice that a decrease of 40% followed by a decrease of 20% is not the same as a single decrease of 60%; it is less. In fact, a decrease of 40% followed by a decrease of 30% wouldn't even be as much as a single decrease of 60%.

KEY FACT C9

- **A decrease of $a\%$ followed by a decrease of $b\%$ *always* results in a smaller decrease than a single decrease of $(a + b)\%$.**
- **An increase of $a\%$ followed by an increase of $b\%$ *always* results in a larger increase than a single increase of $(a + b)\%$.**
- **An increase (or decrease) of $a\%$ followed by another increase (or decrease) of $a\%$ is *never* the same as a single increase (or decrease) of $2a\%$.**

EXAMPLE 5

Sally and Heidi were both hired in January at the same salary. Sally got two 40% raises, one in July and another in November. Heidi got one 90% raise in October.

Quantity A	Quantity B
Sally's salary at the end of the year	Heidi's salary at the end of the year

SOLUTION.
Since this is a percent problem, assume their salaries were \$100. Quantity A: Sally's salary rose to $100(1.40) = 140$, and then to $140(1.40) = \$196$. Quantity B: Heidi's salary rose to $100(1.90) = \$190$. Quantity **A** is greater.

EXAMPLE 6

In January, the value of a stock increased by 25%, and in February, it decreased by 20%. How did the value of the stock at the end of February compare with its value at the beginning of January?

Ⓐ It was less.
Ⓑ It was the same.
Ⓒ It was 5% greater.
Ⓓ It was more than 5% greater.
Ⓔ It cannot be determined from the information given.

SOLUTION. Assume that at the beginning of January the stock was worth \$100. Then at the end of January it was worth \$125. Since 20% of 125 is 25, during February its value decreased from \$125 to \$100. The answer is **B**.

KEY FACT C10

- **If a number is the result of increasing another number by $k\%$, to find the original number, divide by $(1 + k\%)$.**
- **If a number is the result of decreasing another number by $k\%$, to find the original number, divide it by $(1 - k\%)$.**

For example, if the population of a town in 1990 was 3000, and this represents an increase of 20% since 1980, to find the population in 1980, divide 3000 by (1 + 20%): 3000 ÷ 1.20 = 2500.

EXAMPLE 7

From 1989 to 1990, the number of applicants to a college increased 15% to 5060. How many applicants were there in 1989?

SOLUTION.
The number of applicants in 1989 was 5060 ÷ 1.15 = **4400**.

CAUTION

Percents over 100%, which come up most often on questions involving percent increases, are often confusing for students. First of all, be sure you understand that 100% of a number is that number, 200% of a number is 2 times the number, and 1000% of a number is 10 times the number. If the value of an investment goes from $1000 to $5000, it is now worth 5 times, or 500%, as much as it was originally; but there has only been a *400%* increase in value:

$$\frac{\text{actual increase}}{\text{original amount}} \times 100\% = \frac{4000}{1000} \times 100\% = 4 \times 100\% = 400\%.$$

EXAMPLE 8

The population of a country doubled every 10 years from 1960 to 1990. What was the percent increase in population during this time?

Ⓐ 200% Ⓑ 300% Ⓒ 700% Ⓓ 800% Ⓔ 1000%

SOLUTION.
The population doubled three times (once from 1960 to 1970, again from 1970 to 1980, and a third time from 1980 to 1990). Assume that the population was originally 100. Then it increased from 100 to 200 to 400 to 800. So the population in 1990 was 8 times the population in 1960, but this was an increase of 700 people, or **700%** (**C**).

Practice Exercises — Percents

Discrete Quantitative Questions

1. If 25 students took an exam and 4 of them failed, what percent of them passed?

Ⓐ 4%
Ⓑ 21%
Ⓒ 42%
Ⓓ 84%
Ⓔ 96%

2. Amanda bought a $60 sweater on sale at 5% off. How much did she pay, including 5% sales tax?

Ⓐ $54.15
Ⓑ $57.00
Ⓒ $57.75
Ⓓ $59.85
Ⓔ $60.00

3. What is 10% of 20% of 30%?

Ⓐ 0.006%
Ⓑ 0.6%
Ⓒ 6%
Ⓓ 60%
Ⓔ 6000%

4. If c is a positive number, 500% of c is what percent of $500c$?

Ⓐ 0.01
Ⓑ 0.1
Ⓒ 1
Ⓓ 10
Ⓔ 100

5. What percent of 50 is b?

Ⓐ $\frac{b}{50}$

Ⓑ $\frac{b}{2}$

Ⓒ $\frac{50}{b}$

Ⓓ $\frac{2}{b}$

Ⓔ $2b$

6. 8 is $\frac{1}{3}$% of what number?

7. During his second week on the job, Mario earned $110. This represented a 25% increase over his earnings of the previous week. How much did he earn during his first week of work?

Ⓐ $82.50
Ⓑ $85.00
Ⓒ $88.00
Ⓓ $137.50
Ⓔ $146.67

8. At Bernie's Bargain Basement everything is sold for 20% less than the price marked. If Bernie buys radios for $80, what price should he mark them if he wants to make a 20% profit on his cost?

Ⓐ $96
Ⓑ $100
Ⓒ $112
Ⓓ $120
Ⓔ $125

9. Mrs. Fisher usually deposits the same amount of money each month into a vacation fund. This year she decided not to make any contributions during November and December. To make the same annual contribution that she had originally planned, by what percent should she increase her monthly deposits from January through October?

Ⓐ $16\frac{2}{3}\%$
Ⓑ 20%
Ⓒ 25%
Ⓓ $33\frac{1}{3}\%$
Ⓔ It cannot be determined from the information given.

10. The price of a loaf of bread was increased by 20%. How many loaves can be purchased for the amount of money that used to buy 300 loaves?

Ⓐ 240
Ⓑ 250
Ⓒ 280
Ⓓ 320
Ⓔ 360

11. If 1 micron = 10,000 angstroms, then 100 angstroms is what percent of 10 microns?

Ⓐ 0.0001%
Ⓑ 0.001%
Ⓒ 0.01%
Ⓓ 0.1%
Ⓔ 1%

12. There are twice as many girls as boys in an English class. If 30% of the girls and 45% of the boys have already handed in their book reports, what percent of the students have not yet handed in their reports?

[ ] %

13. An art dealer bought a Ming vase for \$1000 and later sold it for \$10,000. By what percent did the value of the vase increase?

Ⓐ 10%
Ⓑ 90%
Ⓒ 100%
Ⓓ 900%
Ⓔ 1000%

14. During a sale a clerk was putting a new price tag on each item. On one jacket, he accidentally raised the price by 15% instead of lowering the price by 15%. As a result the price on the tag was \$45 too high. What was the original price of the jacket?

[ ] dollars

15. On a test consisting of 80 questions, Eve answered 75% of the first 60 questions correctly. What percent of the other 20 questions does she need to answer correctly for her grade on the entire exam to be 80%?

Ⓐ 85%
Ⓑ 87.5%
Ⓒ 90%
Ⓓ 95%
Ⓔ 100%

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

	Quantity A	Quantity B
16.	400% of 3	300% of 4

$n\%$ of 25 is 50

	Quantity A	Quantity B
17.	50% of n	75

	Quantity A	Quantity B
18.	The price of a television when it is on sale at 25% off	The price of that television when it's on sale at \$25 off

The price of cellular phone 1 is 20% more than the price of cellular phone 2.

	Quantity A	Quantity B
19.	The price of cellular phone 1 when it is on sale at 20% off	The price of cellular phone 2

	Quantity A	Quantity B
20.	$\frac{2}{3}\%$ of $\frac{3}{4}$	$\frac{3}{4}\%$ of $\frac{2}{3}$

	Quantity A	Quantity B
21.	$a\%$ of $\frac{1}{b}$	$b\%$ of $\frac{1}{a}$

Bank A pays 5% interest on its savings accounts. Bank B pays 4% interest on its savings accounts.

	Quantity A	Quantity B
22.	Percent by which bank B would have to raise its interest rate to match bank A	20%

A solution that is 20% sugar is made sweeter by doubling the amount of sugar.

	Quantity A	Quantity B
23.	The percent of sugar in the new solution	40%

b is an integer greater than 1, and b equals $n\%$ of b^2

	Quantity A	Quantity B
24.	n	50

After Ali gave Lior 50% of her money, she had 20% as much as he did.

	Quantity A	Quantity B
25.	75% of the amount Lior had originally	150% of the amount Ali had originally

ANSWER KEY

1. **D**	6. **2400**	11. **D**	16. **C**	21. **D**
2. **D**	7. **C**	12. **65**	17. **A**	22. **A**
3. **B**	8. **D**	13. **D**	18. **D**	23. **B**
4. **C**	9. **B**	14. **150**	19. **B**	24. **D**
5. **E**	10. **B**	15. **D**	20. **C**	25. **C**

Answer Explanations

1. **(D)** If 4 students failed, then the other 25 – 4 = 21 students passed, and $\frac{21}{25} = 0.84 = 84\%$.

2. **(D)** Since 5% of 60 is 3, Amanda saved \$3, and thus paid \$57 for the sweater. She then had to pay 5% sales tax on the \$57: $.05 \times 57 = 2.85$, so the total cost was \$57 + \$2.85 = \$59.85.

3. **(B)** 10% of 20% of 30% = $.10 \times .20 \times .30 = .006 = .6\%$.

4. **(C)** 500% of $c = 5c$, which is 1% of $500c$.

5. **(E)** $b = \frac{x}{\cancel{100}_2}(\cancel{50}^1) \Rightarrow b = \frac{x}{2} \Rightarrow x = 2b.$

6. **2400** $8 = \frac{\frac{1}{3}}{100}x = \frac{1}{300}x \Rightarrow x = 8 \times 300 = 2400.$

7. **(C)** To find Mario's earnings during his first week, divide his earnings from the second week by 1.25: $110 \div 1.25 = 88$.

8. **(D)** Since 20% of 80 is 16, Bernie wants to get \$96 for each radio he sells. What price should the radios be marked so that after a 20% discount, the customer will pay \$96? If x represents the marked price, then $.80x = 96 \Rightarrow x = 96 \div .80 = 120$.

9. **(B)** Assume that Mrs. Fisher usually contributed \$100 each month, for an annual total of \$1200. Having decided not to contribute for 2 months, the \$1200 will have to be paid in 10 monthly deposits of \$120 each. This is an increase of \$20, and a percent increase of
$$\frac{\text{actual increase}}{\text{original amount}} \times 100\% = \frac{20}{100} \times 100\% = 20\%.$$

10. **(B)** Assume that a loaf of bread used to cost \$1 and that now it costs \$1.20 (20% more). Then 300 loaves of bread used to cost \$300. How many loaves costing \$1.20 each can be bought for \$300? $300 \div 1.20 = 250$.

11. **(D)** 1 micron = 10,000 angstroms ⇒ 10 microns = 100,000 angstroms; dividing both sides by 1000, we get 100 angstroms = $\frac{1}{1000}$ (10 microns); and $\frac{1}{1000} = .001 = 0.1\%$.

12. **65** Assume that there are 100 boys and 200 girls in the class. Then, 45 boys and 60 girls have handed in their reports. So 105 students have handed them in, and 300 – 105 = 195 have not handed them in. What percent of 300 is 195?
$$\frac{195}{300} = .65 = 65\%.$$

13. **(D)** The increase in the value of the vase was \$9,000. So the percent increase is $\frac{\text{actual increase}}{\text{original cost}} \times 100\% = \frac{9000}{1000} = 9 = 900\%$.

14. **150** If p represents the original price, the jacket was priced at $1.15p$ instead of $.85p$. Since this was a \$45 difference, $45 = 1.15p - .85p = .30p \Rightarrow p = 45 \div .30 = \$150$.

15. **(D)** To earn a grade of 80% on the entire exam, Eve needs to correctly answer 64 questions (80% of 80). So far, she has answered 45 questions correctly (75% of 60). Therefore, on the last 20 questions she needs 64 – 45 = 19 correct answers; and $\frac{19}{20} = 95\%$.

16. **(C)** Quantity A: 400% of 3 = $4 \times 3 = 12$.
Quantity B: 300% of 4 = $3 \times 4 = 12$.

17. **(A)** Since n% of 25 is 50, then 25% of n is also 50, and 50% of n is twice as much: 100. If you don't see that, just solve for n:
$$\frac{n}{\cancel{100}_4} \times \cancel{25}^1 = 50 \Rightarrow \frac{n}{4} = 50 \Rightarrow n = 200 \text{ and } 50\% \text{ of } n = 100.$$

18. **(D)** A 25% discount on a \$10 television is much less than \$25, whereas a 25% discount on a \$1000 television is much more than \$25. (They would be equal only if the regular price of the television were \$100.)

19. **(B)** Assume that the list price of cellular phone 2 is \$100; then the list price of cellular phone 1 is \$120, and on sale at 20% off it costs \$24 less: \$96.

20. **(C)** For *any* numbers a and b: a% of b is equal to b% of a.

21. **(D)**

Quantity A	Quantity B
$a\%$ of $\frac{1}{b}$	$b\%$ of $\frac{1}{a}$
$\frac{a}{100} \times \frac{1}{b} = \frac{a}{100b}$	$\frac{b}{100} \times \frac{1}{a} = \frac{b}{100a}$

Multiply by 100:

$\frac{a}{b}$	$\frac{b}{a}$

The quantities are equal if a and b are equal, and unequal otherwise.

22. **(A)** Bank B would have to increase its rate from 4% to 5%, an actual increase of 1%. This represents a percent increase of $\frac{1\%}{4\%} \times 100\% = 25\%$.

23. **(B)** Assume a vat contains 100 ounces of a solution, of which 20% or 20 ounces is sugar (the remaining 80 ounces being water). If the amount of sugar is doubled, there would be 40 ounces of sugar and 80 ounces of water. The sugar will then comprise $\frac{40}{120} = \frac{1}{3} = 33\frac{1}{3}\%$ of the solution.

24. **(D)** If $b = 2$, then $b^2 = 4$, and $2 = 50\%$ of 4; in this case, the quantities are equal. If $b = 4$, $b^2 = 16$, and 4 is not 50% of 16; in this case, the quantities are not equal.

25. **(C)** Avoid the algebra and just assume Ali started with \$100. After giving Lior \$50, she had \$50 left, which was 20% or one-fifth of what he had. So, Lior had $5 \times \$50 = \$250$, which means that originally he had \$200.
Quantity A: 75% of \$200 = \$150.
Quantity B: 150% of \$100 = \$150.
The quantities are equal.

11-D. RATIOS AND PROPORTIONS

A ***ratio*** is a fraction that compares two quantities that are measured in the same units. The first quantity is the numerator and the second quantity is the denominator.

For example, if there are 4 boys and 16 girls on the debate team, we say that the ratio of the number of boys to the number of girls on the team is 4 to 16, or $\frac{4}{16}$. This is often written 4:16. Since a ratio is just a fraction, it can be reduced or converted to a decimal or a percent. The following are all different ways to express the same ratio:

4 to 16 4:16 $\frac{4}{16}$ 2 to 8 2:8 $\frac{2}{8}$ 1 to 4 1:4 $\frac{1}{4}$ 0.25 25%

CAUTION

Saying that the ratio of boys to girls on the team is 1:4 does *not* mean that $\frac{1}{4}$ of the team members are boys. It means that for each boy on the team there are 4 girls; so for every 5 members of the team, there are 4 girls and 1 boy. Boys, therefore, make up $\frac{1}{5}$ of the team, and girls $\frac{4}{5}$.

KEY FACT D1

If a set of objects is divided into two groups in the ratio of $a:b$, then the first group contains $\frac{a}{a+b}$ of the objects and the second group contains $\frac{b}{a+b}$ of the objects.

EXAMPLE 1

Last year, the ratio of the number of tennis matches that Central College's women's team won to the number of matches they lost was 7:3. What percent of their matches did the team win?

$\boxed{}$ %

SOLUTION.

The team won $\frac{7}{7+3} = \frac{7}{10} = $ **70%** of their matches.

EXAMPLE 2

If 45% of the students at a college are male, what is the ratio of male students to female students?

TIP

In problems involving percents the best number to use is 100.

SOLUTION.
Assume that there are 100 students. Then 45 of them are male, and 100 – 45 = 55 of them are female. So, the ratio of males to females is $\frac{45}{55} = \frac{9}{11}$.

If we know how many boys and girls there are in a club, then, clearly, we know not only the ratio of boys to girls, but several other ratios too. For example, if the club has 7 boys and 3 girls: the ratio of boys to girls is $\frac{7}{3}$, the ratio of girls to boys is $\frac{3}{7}$, the ratio of boys to members is $\frac{7}{10}$, the ratio of members to girls is $\frac{10}{3}$, and so on.

However, if we know a ratio, we *cannot* determine how many objects there are. For example, if a jar contains only red and blue marbles, and if the ratio of red marbles to blue marbles is 3:5, there *may* be 3 red marbles and 5 blue marbles, but *not necessarily*. There may be 300 red marbles and 500 blue ones, since the ratio 300:500 reduces to 3:5. In the same way, all of the following are possibilities for the distribution of marbles.

Red	6	12	33	51	150	3000	**$3x$**
Blue	10	20	55	85	250	5000	**$5x$**

The important thing to observe is that the number of red marbles can be *any* multiple of 3, as long as the number of blue marbles is the *same* multiple of 5.

KEY FACT D2

If two numbers are in the ratio of $a{:}b$, then for some number x, the first number is ax and the second number is bx. If the ratio is in lowest terms, and if the quantities must be integers, then x is also an integer.

TACTIC D1

In any ratio problem, write the letter x after each number and use some given information to solve for x.

EXAMPLE 3

If the ratio of men to women in a particular dormitory is 5:3, which of the following could not be the number of residents in the dormitory?

Ⓐ 24 Ⓑ 40 Ⓒ 96 Ⓓ 150 Ⓔ 224

SOLUTION.
If $5x$ and $3x$ are the number of men and women in the dormitory, respectively, then the number of residents in the dormitory is $5x + 3x = 8x$. So, the number of students must be a multiple of 8. Of the five choices, only **150 (D)** is not divisible by 8.

NOTE: Assume that the ratio of the number of pounds of cole slaw to the number of pounds of potato salad consumed in the dormitory's cafeteria was 5:3. Then, it is possible that a total of exactly 150 pounds was eaten: 93.75 pounds of cole slaw and 56.25 pounds of potato salad. In Example 3, 150 wasn't possible because there had to be a *whole* number of men and women.

EXAMPLE 4

The measures of the two acute angles in a right triangle are in the ratio of 5:13. What is the measure of the larger angle?

Ⓐ 25° Ⓑ 45° Ⓒ 60° Ⓓ 65° Ⓔ 75°

SOLUTION.
Let the measure of the smaller angle be $5x$ and the measure of the larger angle be $13x$. Since the sum of the measures of the two acute angles of a right triangle is 90° (KEY FACT J1), $5x + 13x = 90 \Rightarrow 18x = 90 \Rightarrow x = 5$.

Therefore, the measure of the larger angle is $13 \times 5 =$ **65° (D)**.

Ratios can be extended to three or four or more terms. For example, we can say that the ratio of freshmen to sophomores to juniors to seniors in a college marching band is 6:8:5:8, which means that for every 6 freshmen in the band there are 8 sophomores, 5 juniors, and 8 seniors.

TIP

TACTIC D1 applies to extended ratios, as well.

EXAMPLE 5

The concession stand at Cinema City sells popcorn in three sizes: large, super, and jumbo. One day, Cinema City sold 240 bags of popcorn, and the ratio of large to super to jumbo was 8:17:15. How many super bags of popcorn were sold that day?

SOLUTION.
Let $8x$, $17x$, and $15x$ be the number of large, super, and jumbo bags of popcorn sold, respectively. Then $8x + 17x + 15x = 240 \Rightarrow 40x = 240 \Rightarrow x = 6$.
The number of super bags sold was $17 \times 6 =$ **102**.

KEY FACT D3

KEY FACT D1 applies to extended ratios, as well. If a set of objects is divided into 3 groups in the ratio $a{:}b{:}c$, then the first group contains $\frac{a}{a+b+c}$ of the objects, the second $\frac{b}{a+b+c}$, and the third $\frac{c}{a+b+c}$.

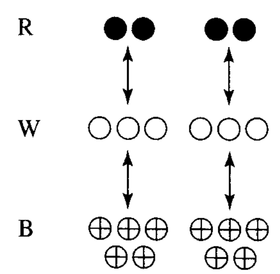
EXAMPLE 6

If the ratio of large to super to jumbo bags of popcorn sold at Cinema City was 8:17:15, what percent of the bags sold were super?

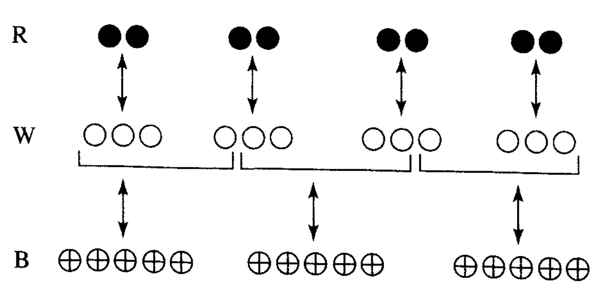
Ⓐ 20% Ⓑ 25% Ⓒ $33\frac{1}{3}$% Ⓓ 37.5% Ⓔ 42.5%

SOLUTION.
Super bags made up $\frac{17}{8+17+15} = \frac{17}{40} =$ **42.5%** of the total (**E**).

A jar contains a number of red (R), white (W), and blue (B) marbles. Suppose that R:W = 2:3 and W:B = 3:5. Then, for every 2 red marbles, there are 3 white ones, and for those 3 white ones, there are 5 blue ones. So, R:B = 2:5, and we can form the extended ratio R:W:B = 2:3:5.



If the ratios were R:W = 2:3 and W:B = 4:5, however, we wouldn't be able to combine them as easily. From the diagram below, you see that for every 8 reds there are 15 blues, so R:B = 8:15.



To see this without drawing a picture, we write the ratios as fractions: $\frac{R}{W} = \frac{2}{3}$ and $\frac{W}{B} = \frac{4}{5}$. Then, we multiply the fractions:

$$\frac{R}{\cancel{W}} \times \frac{\cancel{W}}{B} = \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}, \quad \text{so} \quad \frac{R}{B} = \frac{8}{15}.$$

Not only does this give us R:B = 8:15, but also, if we multiply both W numbers, $3 \times 4 = 12$, we can write the extended ratio: R:W:B = 8:12:15.

EXAMPLE 7

Jar A and jar B each have 70 marbles,
all of which are red, white, or blue.
In jar A, R:W = 2:3 and W:B = 3:5.
In jar B, R:W = 2:3 and W:B = 4:5.

Quantity A	Quantity B
The number of white marbles in jar A	The number of white marbles in jar B

SOLUTION.
From the discussion immediately preceding this example, in jar A the extended ratio R:W:B is 2:3:5, which implies that the white marbles constitute $\frac{3}{2+3+5} = \frac{3}{10}$ of the total: $\frac{3}{10} \times 70 =$ **21**.

In jar B the extended ratio R:W:B is 8:12:15, so the white marbles are $\frac{12}{8+12+15} = \frac{12}{35}$ of the total: $\frac{12}{35} \times 70 =$ **24**. The answer is **B**.

A ***proportion*** is an equation that states that two ratios are equivalent. Since ratios are just fractions, any equation such as $\frac{4}{6} = \frac{10}{15}$ in which each side is a single fraction is a proportion. Usually the proportions you encounter on the GRE involve one or more variables.

TACTIC

D2

Solve proportions by cross-multiplying: if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

Setting up a proportion is a common way of solving a problem on the GRE.

EXAMPLE 8

If $\frac{3}{7} = \frac{x}{84}$, what is the value of x?

[ ]

SOLUTION.
Cross-multiply: $3(84) = 7x \Rightarrow 252 = 7x \Rightarrow x = \mathbf{36}$.

EXAMPLE 9

If $\frac{x+2}{17} = \frac{x}{16}$, what is the value of $\frac{x+6}{19}$?

Ⓐ $\frac{1}{2}$ Ⓑ 1 Ⓒ $\frac{3}{2}$ Ⓓ 2 Ⓔ 3

SOLUTION.
Cross-multiply: $16(x + 2) = 17x \Rightarrow 16x + 32 = 17x \Rightarrow x = 32$.

So, $\frac{x+6}{19} = \frac{32+6}{19} = \frac{38}{19} = \mathbf{2}$ **(D)**.

EXAMPLE 10

A state law requires that on any field trip the ratio of the number of chaperones to the number of students must be at least 1:12. If 100 students are going on a field trip, what is the minimum number of chaperones required?

Ⓐ 6 Ⓑ 8 Ⓒ $8\frac{1}{3}$ Ⓓ 9 Ⓔ 12

SOLUTION.
Let x represent the number of chaperones required, and set up a proportion: $\frac{\text{number of chaperones}}{\text{number of students}} = \frac{1}{12} = \frac{x}{100}$. Cross-multiply: $100 = 12x \Rightarrow x = 8\frac{1}{3}$. This, of course, is *not* the answer since, clearly, the number of chaperones must be a whole number. Since x is greater than 8, 8 chaperones would not be enough. The answer is **9 (D)**.

A ***rate*** is a fraction that compares two quantities measured in different units. The word "per" often appears in rate problems: miles per hour, dollars per week, cents per ounce, students per classroom, and so on.

TIP

A rate can always be written as a fraction.

TACTIC

D3

Set up rate problems just like ratio problems. Solve the proportions by cross-multiplying.

EXAMPLE 11

Brigitte solved 24 math problems in 15 minutes. At this rate, how many problems can she solve in 40 minutes?

Ⓐ 25 Ⓑ 40 Ⓒ 48 Ⓓ 60 Ⓔ 64

SOLUTION.
Handle this rate problem exactly like a ratio problem. Set up a proportion and cross-multiply:

$$\frac{\text{problems}}{\text{minutes}} = \frac{24}{15} = \frac{x}{40} \Rightarrow 15x = 40 \times 24 = 960 \Rightarrow x = \mathbf{64}\ \mathbf{(E)}.$$

When the denominator in the given rate is 1 unit (1 minute, 1 mile, 1 dollar), the problem can be solved by a single division or multiplication. Consider Examples 12 and 13.

EXAMPLE 12

If Stefano types at the rate of 35 words per minute, how long will it take him to type 987 words?

[ ]

SOLUTION.
Set up a proportion and cross-multiply:

$$\frac{\text{words typed}}{\text{minutes}} = \frac{35}{1} = \frac{987}{x} \Rightarrow 35x = 987 \Rightarrow x = \frac{987}{35} = \mathbf{28.2} \text{ minutes.}$$

EXAMPLE 13

If Mario types at the rate of 35 words per minute, how many words can he type in 85 minutes?

[ ]

SOLUTION.
Set up a proportion and cross-multiply:

$$\frac{\text{words typed}}{\text{minutes}} = \frac{35}{1} = \frac{x}{85} \Rightarrow x = 35 \times 85 = \mathbf{2975} \text{ words.}$$

Notice that in Example 12, all we did was divide 987 by 35, and in Example 13, we multiplied 35 by 85. If you realize that, you don't have to introduce x and set up a proportion. You must know, however, whether to multiply or divide. If you're not absolutely positive which is correct, write the proportion; then you can't go wrong.

CAUTION

In rate problems it is essential that the units in both fractions be the same.

EXAMPLE 14

If 3 apples cost 50¢, how many apples can you buy for \$20?

Ⓐ 20 Ⓑ 60 Ⓒ 120 Ⓓ 600 Ⓔ 2000

SOLUTION.
We have to set up a proportion, but it is *not* $\frac{3}{50} = \frac{x}{20}$. In the first fraction, the denominator represents *cents*, whereas in the second fraction, the denominator represents *dollars*. The units must be the same. We can change 50 cents to 0.5 dollar or we can change 20 dollars to 2000 cents:

$$\frac{3}{50} = \frac{x}{2000} \Rightarrow 50x = 6000 \Rightarrow x = \mathbf{120} \text{ apples } \mathbf{(C)}.$$

On the GRE, some rate problems involve only variables. They are handled in exactly the same way.

EXAMPLE 15

If a apples cost c cents, how many apples can be bought for d dollars?

Ⓐ $100acd$ Ⓑ $\frac{100d}{ac}$ Ⓒ $\frac{ad}{100c}$ Ⓓ $\frac{c}{100ad}$ Ⓔ $\frac{100ad}{c}$

SOLUTION.
First change d dollars to $100d$ cents, and set up a proportion: $\frac{\text{apples}}{\text{cents}} = \frac{a}{c} = \frac{x}{100d}$.

Now cross-multiply: $100ad = cx \Rightarrow x = \frac{\mathbf{100ad}}{\mathbf{c}}$ **(E)**.

Most students find problems such as Example 15 very difficult. If you get stuck on such a problem, use TACTIC 2, Chapter 8, which gives another strategy for handling these problems.

Notice that in rate problems, as one quantity increases or decreases, so does the other. If you are driving at 45 miles per hour, the more hours you drive, the further you go; if you drive fewer miles, it takes less time. If chopped meat cost \$3.00 per pound, the less you spend, the fewer pounds you get; the more meat you buy, the more it costs.

In some problems, however, as one quantity increases, the other decreases. These *cannot* be solved by setting up a proportion. Consider the following two examples, which look similar but must be handled differently.

EXAMPLE 16

A hospital needs 150 pills to treat 6 patients for a week. How many pills does it need to treat 10 patients for a week?

SOLUTION.
Example 16 is a standard rate problem. The more patients there are, the more pills are needed.

The *ratio* or *quotient* remains constant: $\frac{150}{6} = \frac{x}{10} \Rightarrow 6x = 1500 \Rightarrow x = \mathbf{250}$.

EXAMPLE 17

A hospital has enough pills on hand to treat 10 patients for 14 days. How long will the pills last if there are 35 patients?

SOLUTION.
In Example 17, the situation is different. With more patients, the supply of pills will last for a shorter period of time; if there were fewer patients, the supply would last longer. It is not the ratio that remains constant, it is the *product*.

There are enough pills to last for $10 \times 14 = 140$ patient-days:

$$\frac{140 \text{ patient-days}}{10 \text{ patients}} = 14 \text{ days} \qquad \frac{140 \text{ patient-days}}{35 \text{ patients}} = \mathbf{4} \text{ days}$$

$$\frac{140 \text{ patient-days}}{70 \text{ patients}} = 2 \text{ days} \qquad \frac{140 \text{ patient-days}}{1 \text{ patient}} = 140 \text{ days}$$

There are many mathematical situations in which one quantity increases as another decreases, but their product is not constant. Those types of problems, however, do not appear on the GRE.

TACTIC
D4

If one quantity increases as a second quantity decreases, multiply them; their product will be a constant.

EXAMPLE 18

If 15 workers can pave a certain number of driveways in 24 days, how many days will 40 workers take, working at the same rate, to do the same job?

Ⓐ 6 Ⓑ 9 Ⓒ 15 Ⓓ 24 Ⓔ 40

SOLUTION.
Clearly, the more workers there are, the less time it will take, so use TACTIC D4: multiply. The job takes $15 \times 24 = 360$ worker-days:

$$\frac{360 \text{ worker-days}}{40 \text{ workers}} = \mathbf{9} \text{ days } (\mathbf{B}).$$

Note that it doesn't matter how many driveways have to be paved, as long as the 15 workers and the 40 workers are doing the same job. Even if the question had said, "15 workers can pave 18 driveways in 24 days," the number 18 would not have entered into the solution. This number would be important only if the second group of workers was going to pave a different number of driveways.

EXAMPLE 19

If 15 workers can pave 18 driveways in 24 days, how many days would it take 40 workers to pave 22 driveways?

Ⓐ 6 Ⓑ 9 Ⓒ 11 Ⓓ 15 Ⓔ 18

SOLUTION.
This question is similar to Example 18, except that now the jobs that the two groups of workers are doing are different. The solution, however, starts out exactly the same way. Just as in Example 18, 40 workers can do in 9 days the *same* job that 15 workers can do in 24 days. Since that job is to pave 18 driveways, 40 workers can pave $18 \div 9 = 2$ driveways every day. So, it will take **11** days for them to pave 22 driveways (**C**).

Practice Exercises—Ratios and Proportions

Discrete Quantitative Questions

1. If $\frac{3}{4}$ of the employees in a supermarket are not college graduates, what is the ratio of the number of college graduates to those who are not college graduates?
   Ⓐ 1:3
   Ⓑ 3:7
   Ⓒ 3:4
   Ⓓ 4:3
   Ⓔ 3:1

2. If $\frac{a}{9} = \frac{10}{2a}$, what is the value of a^2?
   Ⓐ $3\sqrt{6}$
   Ⓑ $3\sqrt{5}$
   Ⓒ $9\sqrt{6}$
   Ⓓ 45
   Ⓔ 90

3. If 80% of the applicants to a program were rejected, what is the ratio of the number accepted to the number rejected?

   $\frac{\square}{\square}$

4. Scott can read 50 pages per hour. At this rate, how many pages can he read in 50 minutes?
   Ⓐ 25
   Ⓑ $41\frac{2}{3}$
   Ⓒ $45\frac{1}{2}$
   Ⓓ 48
   Ⓔ 60

5. If all the members of a team are juniors or seniors, and if the ratio of juniors to seniors on the team is 3:5, what percent of the team members are seniors?
   Ⓐ 37.5%
   Ⓑ 40%
   Ⓒ 60%
   Ⓓ 62.5%
   Ⓔ It cannot be determined from the information given.

6. The measures of the three angles in a triangle are in the ratio of 1:1:2. Which of the following statements must be true?

   Indicate *all* such statements.

   [A] The triangle is isosceles.
   [B] The triangle is a right triangle.
   [C] The triangle is equilateral.

7. What is the ratio of the circumference of a circle to its radius?
   Ⓐ 1
   Ⓑ $\frac{\pi}{2}$
   Ⓒ $\sqrt{\pi}$
   Ⓓ π
   Ⓔ 2π

8. The ratio of the number of freshmen to sophomores to juniors to seniors on a college basketball team is 4:7:6:8. What percent of the team are sophomores?
   Ⓐ 16%
   Ⓑ 24%
   Ⓒ 25%
   Ⓓ 28%
   Ⓔ 32%

9. At Central State College the ratio of the number of students taking Spanish to the number taking French is 7:2. If 140 students are taking French, how many are taking Spanish?

students

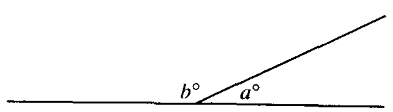
10. If $a:b = 3:5$ and $a:c = 5:7$, what is the value of $b:c$?

Ⓐ 3:7
Ⓑ 21:35
Ⓒ 21:25
Ⓓ 25:21
Ⓔ 7:3

11. If x is a positive number and $\frac{x}{3} = \frac{12}{x}$, then $x =$

Ⓐ 3
Ⓑ 4
Ⓒ 6
Ⓓ 12
Ⓔ 36

12. In the diagram below, $b:a = 7:2$. What is $b - a$?



Ⓐ 20
Ⓑ 70
Ⓒ 100
Ⓓ 110
Ⓔ 160

13. A snail can move i inches in m minutes. At this rate, how many feet can it move in h hours?

Ⓐ $\frac{5hi}{m}$
Ⓑ $\frac{60hi}{m}$
Ⓒ $\frac{hi}{12m}$
Ⓓ $\frac{5m}{hi}$
Ⓔ $5him$

14. Gilda can grade t tests in $\frac{1}{x}$ hours. At this rate, how many tests can she grade in x hours?

Ⓐ tx
Ⓑ tx^2
Ⓒ $\frac{1}{t}$
Ⓓ $\frac{x}{t}$
Ⓔ $\frac{1}{tx}$

15. A club had 3 boys and 5 girls. During a membership drive the same number of boys and girls joined the club. How many members does the club have now if the ratio of boys to girls is 3:4?

Ⓐ 12
Ⓑ 14
Ⓒ 16
Ⓓ 21
Ⓔ 28

16. If $\frac{3x-1}{25} = \frac{x+5}{11}$, what is the value of x?

Ⓐ $\frac{3}{4}$
Ⓑ 3
Ⓒ 7
Ⓓ 17
Ⓔ 136

17. If 4 boys can shovel a driveway in 2 hours, how many minutes will it take 5 boys to do the job?

Ⓐ 60
Ⓑ 72
Ⓒ 96
Ⓓ 120
Ⓔ 150

18. If 500 pounds of mush will feed 20 pigs for a week, for how many days will 200 pounds of mush feed 14 pigs?

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

The ratio of red to blue marbles in a jar was 3:5. The same number of red and blue marbles were added to the jar.

	Quantity A	Quantity B
19.	The ratio of red to blue marbles now	3:5

Three associates agreed to split the \$3000 profit of an investment in the ratio of 2:5:8.

	Quantity A	Quantity B
20.	The difference between the largest and the smallest share	\$1200

The ratio of the number of boys to girls in the chess club is 5:2. The ratio of the number of boys to girls in the glee club is 11:4.

	Quantity A	Quantity B
21.	The number of boys in the chess club	The number of boys in the glee club

Sally invited the same number of boys and girls to her party. Everyone who was invited came, but 5 additional boys showed up. This caused the ratio of girls to boys at the party to be 4:5.

	Quantity A	Quantity B
22.	The number of people she invited to her party	40

A large jar is full of marbles. When a single marble is drawn at random from the jar, the probability that it is red is $\frac{3}{7}$.

	Quantity A	Quantity B
23.	The ratio of the number of red marbles to non-red marbles in the jar	$\frac{1}{2}$

$3a = 2b$ and $3b = 5c$

	Quantity A	Quantity B
24.	The ratio of a to c	1

The radius of circle II is 3 times the radius of circle I

	Quantity A	Quantity B
25.	$\frac{\text{area of circle II}}{\text{area of circle I}}$	3π

ANSWER KEY

1. **A**	6. **A, B**	12. **C**	18. **4**	24. **A**
2. **D**	7. **E**	13. **A**	19. **A**	25. **B**
3. $\frac{1}{4}$	8. **D**	14. **B**	20. **C**	
	9. **490**	15. **B**	21. **D**	
4. **B**	10. **D**	16. **D**	22. **C**	
5. **D**	11. **C**	17. **C**	23. **A**	

Answer Explanations

1. **(A)** Of every 4 employees, 3 are not college graduates, and 1 is a college graduate. So the ratio of graduates to nongraduates is 1:3.

2. **(D)** Cross-multiplying, we get: $2a^2 = 90 \Rightarrow a^2 = 45$.

3. $\mathbf{\frac{1}{4}}$ If 80% were rejected, 20% were accepted, and the ratio of accepted to rejected is 20:80 = 1:4.

4. **(B)** Set up a proportion: $\frac{50 \text{ pages}}{1 \text{ hour}} = \frac{50 \text{ pages}}{60 \text{ minutes}} = \frac{x \text{ pages}}{50 \text{ minutes}}$,

   and cross-multiply: $50 \times 50 = 60x \Rightarrow 2500 = 60x \Rightarrow x = 41\frac{2}{3}$.

5. **(D)** Out of every 8 team members, 3 are juniors and 5 are seniors. Seniors, therefore, make up $\frac{5}{8} = 62.5\%$ of the team.

6. **(A)(B)** It is worth remembering that if the ratio of the measures of the angles of a triangle is 1:1:2, the angles are 45-45-90 (see Section 11-J). Otherwise, the first step is to write $x + x + 2x = 180 \Rightarrow 4x = 180 \Rightarrow x = 45$.

   Since two of the angles have the same measure, the triangle is isosceles, and since one of the angles measures 90°, it is a right triangle. I and II are true, and, of course, III is false.

7. **(E)** By definition, π is the ratio of the circumference to the diameter of a circle (see Section 11-L). Therefore, $\pi = \frac{C}{d} = \frac{C}{2r} \Rightarrow 2\pi = \frac{C}{r}$.

8. **(D)** The *fraction* of the team that is sophomores is $\frac{7}{4+7+6+8} = \frac{7}{25}$, and $\frac{7}{25} \times 100\% = 28\%$.

9. **490** Let the number of students taking Spanish be $7x$, and the number taking French be $2x$. Then, $2x = 140 \Rightarrow x = 70 \Rightarrow 7x = 490$.

10. **(D)** Since $\frac{a}{b} = \frac{3}{5}, \frac{b}{a} = \frac{5}{3}$. So, $b{:}c = \frac{b}{c} = \frac{b}{\cancel{a}_1} \times \frac{\cancel{a}^1}{c} = \frac{5}{3} \times \frac{5}{7} = \frac{25}{21} = 25{:}21$.

    Alternatively, we could write equivalent ratios with the same value for a:

    $a{:}b = 3{:}5 = 15{:}25$ and $a{:}c = 5{:}7 = 15{:}21$.

    So, when $a = 15$, $b = 25$, and $c = 21$.

11. **(C)** To solve a proportion, cross-multiply: $\frac{x}{3} = \frac{12}{x} \Rightarrow x^2 = 36 \Rightarrow x = 6$.

12. **(C)** Let $b = 7x$ and $a = 2x$. Then, $7x + 2x = 180 \Rightarrow 9x = 180 \Rightarrow x = 20 \Rightarrow b = 140$ and $a = 40 \Rightarrow b - a = 140 - 40 = 100$.

13. **(A)** Set up the proportion, keeping track of

    units: $\frac{x \text{ feet}}{h \text{ hours}} = \frac{{}^{1}\cancel{12}x \text{ inches}}{{}_{5}\cancel{60}h \text{ minutes}} = \frac{i \text{ inches}}{m \text{ minutes}} \Rightarrow \frac{x}{5h} = \frac{i}{m} \Rightarrow x = \frac{5hi}{m}$.

14. **(B)** Gilda grades at the rate of $\frac{t \text{ tests}}{\frac{1}{x}\text{hours}} = \frac{tx \text{ tests}}{1 \text{ hour}}$.

    Since she can grade tx tests each hour, in x hours she can grade $x(tx) = tx^2$ tests.

15. **(B)** Suppose that x boys and x girls joined the club. Then, the new ratio of boys to girls would be $(3 + x){:}(5 + x)$, which we are told is 3:4.

    So, $\frac{3+x}{5+x} = \frac{3}{4} \Rightarrow 4(3 + x) = 3(5 + x) \Rightarrow 12 + 4x = 15 + 3x \Rightarrow x = 3$.

    Therefore, 3 boys and 3 girls joined the other 3 boys and 5 girls: a total of 14.

16. **(D)** Cross-multiplying, we get:

    $11(3x - 1) = 25(x + 5) \Rightarrow 33x - 11 = 25x + 125 \Rightarrow 8x = 136 \Rightarrow x = 17$.

17. **(C)** Since 4 boys can shovel the driveway in 2 hours, or $2 \times 60 = 120$ minutes, the job takes $4 \times 120 = 480$ boy-minutes; and so 5 boys would need $\frac{480 \text{ boy-minutes}}{5 \text{ boys}} = 96$ minutes.

18. **4** Since 500 pounds will last for 20 pig-weeks = 140 pig-days, 200 pounds will last for $\frac{2}{5} \times 140$ pig-days = 56 pig-days, and $\frac{56 \text{ pig-days}}{14 \text{ pigs}} = 4$ days.

19. **(A)** Assume that to start there were $3x$ red marbles and $5x$ blue ones and that y of each color were added.

	Quantity A	Quantity B
	$\frac{3x+y}{5x+y}$	$\frac{3}{5}$
Cross-multiply:	$5(3x + y)$	$3(5x + y)$
Distribute:	$15x + 5y$	$15x + 3y$
Subtract $15x$:	$5y$	$3y$

Since y is positive, Quantity A is greater.

20. **(C)** The shares are $2x$, $5x$, and $8x$, and their sum is 3000:
$2x + 5x + 8x = 3000 \Rightarrow 15x = 3000 \Rightarrow x = 200$, and so $8x - 2x = 6x = 1200$.

21. **(D)** Ratios alone can't answer the question, "How many?" There could be 5 boys in the chess club or 500. We can't tell.

22. **(C)** Assume that Sally invited x boys and x girls. When she wound up with x girls and $x + 5$ boys, the girl:boy ratio was 4:5. So,
$\frac{x}{x+5} = \frac{4}{5} \Rightarrow 5x = 4x + 20 \Rightarrow x = 20$
Sally invited 40 people (20 boys and 20 girls).

23. **(A)** If the probability of drawing a red marble is $\frac{3}{7}$, 3 out of every 7 marbles are red, and 4 out of every 7 are non-red. So the ratio of red:non-red = 3:4, which is greater than $\frac{1}{2}$.

24. **(A)** Multiplying the first equation by 3 and the second by 2 to get the same coefficient of b, we have: $9a = 6b$ and $6b = 10c$. So, $9a = 10c$ and $\frac{a}{c} = \frac{10}{9}$.

25. **(B)** Assume the radius of circle I is 1 and the radius of circle II is 3. Then the areas are π and 9π, respectively. So, the area of circle II is 9 times the area of circle I, and $3\pi > 9$.

11-E. AVERAGES

The ***average*** of a set of n numbers is the sum of those numbers divided by n.

$$\text{average} = \frac{\text{sum of the } n \text{ numbers}}{n} \quad \text{or simply} \quad A = \frac{\text{sum}}{n}.$$

If the weights of three children are 80, 90, and 76 pounds, respectively, to calculate the average weight of the children, you would add the three weights and divide by 3:

$$\frac{80 + 90 + 76}{3} = \frac{246}{3} = 82$$

The technical name for this type of average is "arithmetic mean," and on the GRE those words always appear in parentheses—for example, "What is the average (arithmetic mean) of 80, 90, and 76?"

Usually, on the GRE, you are not asked to find an average; rather, you are given the average of a set of numbers and asked for some other information. The key to solving all of these problems is to first find the sum of the numbers. Since $A = \frac{\text{sum}}{n}$, multiplying both sides by n yields the equation: sum = nA.

TACTIC

E1

If you know the average, A, of a set of n numbers, multiply A by n to get their sum.

EXAMPLE 1

One day a supermarket received a delivery of 25 frozen turkeys. If the average (arithmetic mean) weight of a turkey was 14.2 pounds, what was the total weight, in pounds, of all the turkeys?

SOLUTION.
Use TACTIC E1: $25 \times 14.2 =$ **355**.

NOTE: We do not know how much any individual turkey weighed nor how many turkeys weighed more or less than 14.2 pounds. All we know is their total weight.

EXAMPLE 2

Sheila took five chemistry tests during the semester and the average (arithmetic mean) of her test scores was 85. If her average after the first three tests was 83, what was the average of her fourth and fifth tests?

Ⓐ 83 Ⓑ 85 Ⓒ 87 Ⓓ 88 Ⓔ 90

SOLUTION.
- Use TACTIC E1: On her five tests, Sheila earned $5 \times 85 = 425$ points.
- Use TACTIC E1 again: On her first three tests she earned $3 \times 83 = 249$ points.

- Subtract: On her last two tests Sheila earned 425 – 249 = 176 points.
- Calculate her average on her last two tests: $\frac{176}{2} = \mathbf{88}$ **(D)**.

NOTE: We cannot determine Sheila's grade on even one of the tests.

KEY FACT E1

- **If all the numbers in a set are the same, then that number is the average.**
- **If the numbers in a set are not all the same, then the average must be greater than the smallest number and less than the largest number. Equivalently, at least one of the numbers is less than the average and at least one is greater.**

If Jessica's test grades are 85, 85, 85, and 85, her average is 85. If Gary's test grades are 76, 83, 88, and 88, his average must be greater than 76 and less than 88. What can we conclude if, after taking five tests, Kristen's average is 90? We know that she earned exactly $5 \times 90 = 450$ points, and that either she got a 90 on every test or at least one grade was less than 90 and at least one was over 90. Here are a few of the thousands of possibilities for Kristen's grades:

(a) 90, 90, 90, 90, 90 (b) 80, 90, 90, 90, 100 (c) 83, 84, 87, 97, 99
(d) 77, 88, 93, 95, 97 (e) 50, 100, 100, 100, 100

In (b), 80, the one grade below 90, is *10 points below*, and 100, the one grade above 90, is *10 points above*. In (c), 83 is 7 points below 90, 84 is 6 points below 90, and 87 is 3 points below 90, for a total of 7 + 6 + 3 = *16 points below 90*; 97 is 7 points above 90, and 99 is 9 points above 90, for a total of 7 + 9 = *16 points above 90*.

These differences from the average are called ***deviations***, and the situation in these examples is not a coincidence.

KEY FACT E2

The total deviation below the average is equal to the total deviation above the average.

EXAMPLE 3

If the average (arithmetic mean) of 25, 31, and x is 37, what is the value of x?

SOLUTION 1.
Use KEY FACT E2. Since 25 is 12 less than 37 and 31 is 6 less than 37, the total deviation below the average is 12 + 6 = 18. Therefore, the total deviation above must also be 18. So, $x = 37 + 18 = \mathbf{55}$.

SOLUTION 2.
Use TACTIC E1. Since the average of the three numbers is 37, the sum of the 3 numbers is $3 \times 37 = 111$. Then,

$$25 + 31 + x = 111 \Rightarrow 56 + x = 111 \Rightarrow x = \mathbf{55}.$$

KEY FACT E3

Assume that the average of a set of numbers is A. If a number x is added to the set and a new average is calculated, then the new average will be less than, equal to, or greater than A, depending on whether x is less than, equal to, or greater than A, respectively.

EXAMPLE 4

Quantity A	Quantity B
The average (arithmetic mean) of the integers from 0 to 12	The average (arithmetic mean) of the integers from 1 to 12

TIP

Remember TACTIC 5 from Chapter 9. We don't have to *calculate* the averages, we just have to *compare* them.

SOLUTION 1.
Quantity B is the average of the integers from 1 to 12, which is surely greater than 1. Quantity A is the average of those same 12 numbers and 0. Since the extra number, 0, is less than Quantity B, Quantity A must be *less* [KEY FACT E3]. The answer is **B**.

SOLUTION 2.

Clearly the sum of the 13 integers from 0 to 12 is the same as the sum of the 12 integers from 1 to 12. Since that sum is positive, dividing by 13 yields a smaller quotient than dividing by 12 [KEY FACT B4].

Although in solving Example 4 we didn't calculate the averages, we could have:

$0 + 1 + 2 + 3 + 4 + 5 + \mathbf{6} + 7 + 8 + 9 + 10 + 11 + 12 = 78$ and $\frac{78}{13} = 6$.

$1 + 2 + 3 + 4 + 5 + \mathbf{6} + \mathbf{7} + 8 + 9 + 10 + 11 + 12 = 78$ and $\frac{78}{12} = 6.5$.

Notice that the average of the 13 *consecutive* integers 0, 1,...,12 is the *middle integer*, **6**, and the average of the 12 *consecutive* integers 1, 2,...,12 is the *average of the two middle integers*, **6** and **7**. This is a special case of KEY FACT E4.

KEY FACT E4

Whenever n numbers form an arithmetic sequence (one in which the difference between any two consecutive terms is the same): (i) if n is odd, the average of the numbers is the middle term in the sequence and (ii) if n is even, the average of the numbers is the average of the two middle terms, which is the same as the average of the first and last terms.

For example, in the arithmetic sequence 6, 9, 12, 15, 18, the average is the middle number, **12**; in the sequence 10, 20, 30, 40, 50, 60, the average is **35**, the average of the two middle numbers — 30 and 40. Note that 35 is also the average of the first and last terms—10 and 60.

EXAMPLE 5

On Thursday, 20 of the 25 students in a chemistry class took a test and their average was 80. On Friday, the other 5 students took the test, and their average was 90. What was the average (arithmetic mean) for the entire class?

[ ]

SOLUTION.
The class average is calculated by dividing the sum of all 25 test grades by 25.

- The first 20 students earned a total of: $20 \times 80 = 1600$ points
- The other 5 students earned a total of: $5 \times 90 = 450$ points
- Add: altogether the class earned: $1600 + 450 = 2050$ points
- Calculate the class average: $\frac{2050}{25} = \mathbf{82}$.

Notice that the answer to Example 5 is *not* 85, which is the average of 80 and 90. This is because the averages of 80 and 90 were earned by different numbers of students, and so the two averages had to be given different weights in the calculation. For this reason, this is called a ***weighted average.***

KEY FACT E5

To calculate the weighted average of a set of numbers, multiply each number in the set by the number of times it appears, add all the products, and divide by the total number of numbers in the set.

So, the solution to Example 5 should look like this:

$$\frac{20(80)+5(90)}{25} = \frac{1600+450}{25} = \frac{2050}{25} = 82$$

Problems involving *average speed* will be discussed in Section 11-H, but we mention them briefly here because they are closely related to problems on weighted averages.

TIP

Without doing any calculations, you should immediately realize that since the grade of 80 is being given more weight than the grade of 90, the average will be closer to 80 than to 90 — certainly *less than 85*.

EXAMPLE 6

For the first 3 hours of his trip, Justin drove at 50 miles per hour. Then, due to construction delays, he drove at only 40 miles per hour for the next 2 hours. What was his average speed, in miles per hour, for the entire trip?

Ⓐ 40 Ⓑ 43 Ⓒ 46 Ⓓ 48 Ⓔ 50

SOLUTION.
This is just a weighted average:

$$\frac{3(50)+2(40)}{5} = \frac{150+80}{5} = \frac{230}{5} = \mathbf{46}.$$

Note that in the fractions above, the numerator is the total distance traveled and the denominator the total time the trip took. This is *always* the way to find an average speed. Consider the following slight variation on Example 6.

EXAMPLE 6A

For the first 100 miles of his trip, Justin drove at 50 miles per hour, and then due to construction delays, he drove at only 40 miles per hour for the next 120 miles. What was his average speed, in miles per hour, for the entire trip?

[ ] miles per hour

SOLUTION.
This is not a *weighted* average. Here we immediately know the total distance traveled, 220 miles. To get the total time the trip took, we find the time for each portion and add: the first 100 miles took $100 \div 50 = 2$ hours, and the next 120 miles took $120 \div 40 = 3$ hours. So the average speed was $\frac{220}{5} = \mathbf{44}$ miles per hour.

Notice that in Example 6, since Justin spent more time traveling at 50 miles per hour than at 40 miles per hour, his average speed was closer to 50; in Example 6a, he spent more time driving at 40 miles per hour than at 50 miles per hour, so his average speed was closer to 40.

Two other terms that are associated with averages are ***median*** and ***mode***. In a set of n numbers that are arranged in increasing order, the ***median*** is the middle number (if n is odd), or the average of the two middle numbers (if n is even). The ***mode*** is the number in the set that occurs most often.

EXAMPLE 7

During a 10-day period, Jorge received the following number of phone calls each day: 2, 3, 9, 3, 5, 7, 7, 10, 7, 6. What is the average (arithmetic mean) of the median and mode of this set of data?

Ⓐ 6 Ⓑ 6.25 Ⓒ 6.5 Ⓓ 6.75 Ⓔ 7

SOLUTION.
The first step is to write the data in increasing order:

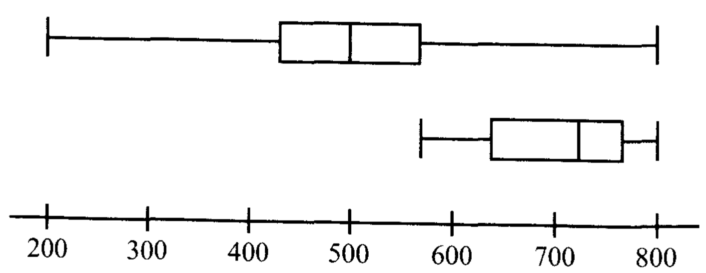
2, 3, 3, 5, 6, 7, 7, 7, 9, 10.

- The median is 6.5, the average of the middle two numbers.
- The mode is 7, the number that appears more times than any other.
- The average of the median and the mode is $\frac{6.5+7}{2} = \mathbf{6.75}$ **(D)**.

The median is actually a special case of a measure called a ***percentile***. In the same way that the median divides a set of data into two roughly equal groups, percentiles divide a set of data into 100 roughly equal groups. P_{63}, the 63rd percentile, for example, is a number with the property that 63% of the data in the group is less than or equal to that number and the rest of the data is greater than that number. Clearly, percentiles are mainly used for large groups of data—it doesn't make much sense to talk about the 63rd percentile of a set of data with 5 or 10 or 20 numbers in it. When you receive your GRE scores in the mail, you will receive a percentile ranking for each of your scores. If you are told that your Verbal score is at the 63rd percentile, that means that your score was higher than the scores of approximately 63% of all GRE test takers (and, therefore, that your score was lower than those of approximately 37% of GRE test takers).

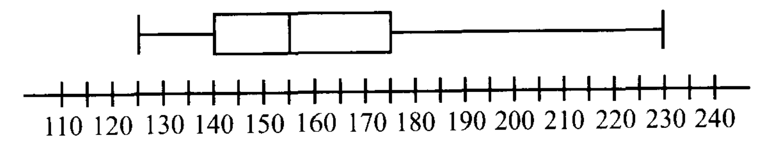
From the definition of percentile, it follows that the median is exactly the same as the 50th percentile. Another term that is often used in analyzing data is ***quartile***. There are three quartiles, Q_1, Q_2, and Q_3, which divide a set of data into four roughly equal groups. Q_1, Q_2, and Q_3 are called the first, second, and third quartiles and are equal to P_{25}, P_{50}, and P_{75}, respectively. So, if M represents the median, then $M = Q_2 = P_{50}$. A measure that is sometimes used to show how spread out the numbers in a set of data are is the ***interquartile range***, which is defined as the difference between the first and third quartiles: $Q_3 - Q_1$.

The interquartile range shows where the middle half of all the data lies. The interquartile range can be graphically illustrated in a diagram called a ***boxplot***. A boxplot extends from the smallest number in the set of data (S) to the largest number in the set of data (L) and has a box representing the interquartile range. The box, which begins and ends at the first and third quartiles, also shows the location of the median (Q_2). The box may be symmetric about the median, but does not need to be, as is illustrated in the two boxplots, below. The upper boxplot shows the distribution of math SAT scores for all students who took the SAT in 2010, while the lower boxplot shows the distribution of math scores for the students at a very selective college.



EXAMPLE 8

Twelve hundred 18-year-old boys were weighed, and their weights, in pounds, are summarized in the following boxplot.



If the 91st percentile of the weights is 200 pounds, approximately how many of the students weigh less than 140 pounds or more than 200 pounds?

Ⓐ 220 Ⓑ 280 Ⓒ 350 Ⓓ 410 Ⓔ 470

SOLUTION.

From the boxplot, we see that the first quartile is 140. So, approximately 25% of the boys weigh less than 140. And since the 91st percentile is 200, approximately 9% of the boys weigh more than 200. So 25% + 9% = 34% of the 1,200 boys fall within the range we are considering.

Finally, 34% of 1,200 = 0.34 × 1200 = 408, or approximately **410 (D)**.

Practice Exercises—Averages

Discrete Quantitative Questions

1. Michael's average (arithmetic mean) on 4 tests is 80. What does he need on his fifth test to raise his average to 84?
   - (A) 82
   - (B) 84
   - (C) 92
   - (D) 96
   - (E) 100

2. Maryline's average (arithmetic mean) on 4 tests is 80. Assuming she can earn no more than 100 on any test, what is the least she can earn on her fifth test and still have a chance for an 85 average after seven tests?
   - (A) 60
   - (B) 70
   - (C) 75
   - (D) 80
   - (E) 85

3. Sandrine's average (arithmetic mean) on 4 tests is 80. Which of the following <u>cannot</u> be the number of tests on which she earned exactly 80 points?
   - (A) 0
   - (B) 1
   - (C) 2
   - (D) 3
   - (E) 4

4. What is the average (arithmetic mean) of the positive integers from 1 to 100, inclusive?
   - (A) 49
   - (B) 49.5
   - (C) 50
   - (D) 50.5
   - (E) 51

5. If $10a + 10b = 35$, what is the average (arithmetic mean) of a and b?

6. If $x + y = 6$, $y + z = 7$, and $z + x = 9$, what is the average (arithmetic mean) of x, y, and z?
   - (A) $\frac{11}{3}$
   - (B) $\frac{11}{2}$
   - (C) $\frac{22}{3}$
   - (D) 11
   - (E) 22

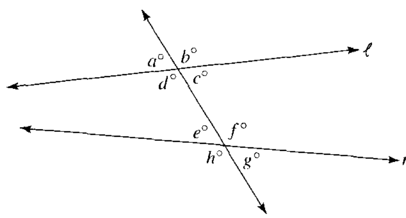
7. If the average (arithmetic mean) of 5, 6, 7, and w is 8, what is the value of w?
   - (A) 8
   - (B) 12
   - (C) 14
   - (D) 16
   - (E) 24

8. What is the average (arithmetic mean) in degrees of the measures of the five angles in a pentagon?

   degrees

9. If $a + b = 3(c + d)$, which of the following is the average (arithmetic mean) of a, b, c, and d?
   - (A) $\frac{c+d}{4}$
   - (B) $\frac{3(c+d)}{8}$
   - (C) $\frac{c+d}{2}$
   - (D) $\frac{3(c+d)}{4}$
   - (E) $c + d$

10. In the diagram below, lines ℓ and m are *not* parallel.



If A represents the average (arithmetic mean) of the degree measures of all eight angles, which of the following is true?
   - (A) $A = 45$
   - (B) $45 < A < 90$
   - (C) $A = 90$
   - (D) $90 < A < 180$
   - (E) $A = 180$

11. What is the average (arithmetic mean) of 2^{10} and 2^{20}?
   - (A) 2^{15}
   - (B) $2^5 + 2^{10}$
   - (C) $2^9 + 2^{19}$
   - (D) 2^{29}
   - (E) 30

12. Let M be the median and m the mode of the following set of numbers: 10, 70, 20, 40, 70, 90. What is the average (arithmetic mean) of M and m?
   - (A) 50
   - (B) 55
   - (C) 60
   - (D) 62.5
   - (E) 65

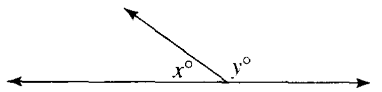
Quantitative Comparison Questions

- (A) Quantity A is greater.
- (B) Quantity B is greater.
- (C) Quantities A and B are equal.
- (D) It is impossible to determine which quantity is greater.

	Quantity A	Quantity B
13.	The average (arithmetic mean) of the measures of the three angles of an equilateral triangle	The average (arithmetic mean) of the measures of the three angles of a right triangle

10 students took a test and the average grade was 80. No one scored exactly 80.

	Quantity A	Quantity B
14.	The number of grades over 80	5



	Quantity A	Quantity B
15.	The average (arithmetic mean) of $2x$ and $2y$	180

There are the same number of boys and girls in a club.
The average weight of the boys is 150 pounds.
The average weight of the girls is 110 pounds.

	Quantity A	Quantity B
16.	The number of boys weighing over 150	The number of girls weighing over 110

The average (arithmetic mean) of 22, 38, x, and y is 15.
$x > 0$

	Quantity A	Quantity B
17.	y	0

	Quantity A	Quantity B
18.	The average (arithmetic mean) of the even numbers between 1 and 11	The average (arithmetic mean) of the odd numbers between 2 and 12

	Quantity A	Quantity B
19.	The average (arithmetic mean) of 17, 217, 417	The average (arithmetic mean) of 0, 17, 217, 417

$y > 0$

	Quantity A	Quantity B
20.	The average (arithmetic mean) of x and y	The average (arithmetic mean) of x, y, and $2y$

ANSWER KEY

1. **E**	5. **1.75**	9. **E**	13. **C**	17. **B**
2. **C**	6. **A**	10. **C**	14. **D**	18. **B**
3. **D**	7. **C**	11. **C**	15. **C**	19. **A**
4. **D**	8. **108**	12. **D**	16. **D**	20. **D**

Answer Explanations

1. **(E)** Use TACTIC E1. For Michael's average on five tests to be an 84, he needs a total of $5 \times 84 = 420$ points. So far, he has earned $4 \times 80 = 320$ points. Therefore, he needs 100 points more.

2. **(C)** Use TACTIC E1. So far, Maryline has earned 320 points. She can survive a low grade on test five if she gets the maximum possible on both the sixth and seventh tests. So, assume she gets two 100s. Then her total for tests 1, 2, 3, 4, 6, and 7 would be 520. For her seven-test average to be 85, she needs a total of $7 \times 85 = 595$ points. Therefore, she needs at least $595 - 520 = 75$ points.

3. **(D)** Since Sandrine's 4-test average is 80, she earned a total of $4 \times 80 = 320$ points. Could Sandrine have earned a total of 320 points with:
   0 grades of 80? Easily; for example, 20, 100, 100, 100 or 60, 70, 90, 100.
   1 grade of 80? Lots of ways; 80, 40, 100, 100, for instance.
   2 grades of 80? Yes; 80, 80, 60, 100.
   4 grades of 80? Sure: 80, 80, 80, 80.
   3 grades of 80? NO! $80 + 80 + 80 + x = 320 \Rightarrow x = 80$, as well.

4. **(D)** Clearly, the sequence of integers from 1 to 100 has 100 terms, and so by KEY FACT E4, we know that the average of all the numbers is the average of the two middle ones: 50 and 51. The average, therefore, is 50.5.

5. **1.75** Since $10a + 10b = 35$, dividing both sides of the equation by 10, we get that $a + b = 3.5$. Therefore, the average of a and b is $3.5 \div 2 = 1.75$.

6. **(A)** Whenever a question involves three equations, add them:
$$\begin{array}{rr} & x + y = 6 \\ & y + z = 7 \\ + & z + x = 9 \\ \hline & 2x + 2y + 2z = 22 \end{array}$$
   Divide by 2: $x + y + z = 11$
   The average of x, y, and z is $\frac{x+y+z}{3} = \frac{11}{3}$.

7. **(C)** Use TACTIC E1: the sum of the 4 numbers is 4 times their average:
   $5 + 6 + 7 + w = 4 \times 8 = 32 \Rightarrow 18 + w = 32 \Rightarrow w = 14$.

8. **108** The average of the measures of the five angles is the sum of their measures divided by 5. The sum is $(5 - 2) \times 180 = 3 \times 180 = 540$ (see Section 11-K). So, the average is $540 \div 5 = 108$.

9. **(E)** Calculate the average:
$$\frac{a+b+c+d}{4} = \frac{3(c+d)+c+d}{4} = \frac{3c+3d+c+d}{4} = \frac{4c+4d}{4} = c + d$$

10. **(C)** Since $a + b + c + d = 360$, and $e + f + g + h = 360$ (see Section 11-I), the sum of the measures of all 8 angles is $360 + 360 = 720$, and their average is $720 \div 8 = 90$.

11. **(C)** The average of 2^{10} and 2^{20} is $\frac{2^{10}+2^{20}}{2} = \frac{2^{10}}{2} + \frac{2^{20}}{2} = 2^9 + 2^{19}$.

12. **(D)** Arrange the numbers in increasing order: 10, 20, 40, 70, 70, 90. M, the median, is the average of the middle two numbers: $\frac{40+70}{2} = 55$; the mode, m, is 70, the number that appears most frequently. The average of M and m, therefore, is the average of 55 and 70, which is 62.5.

13. **(C)** In *any* triangle, the sum of the measures of the three angles is 180°, and the average of their measures is $180 \div 3 = 60$.

14. **(D)** From KEY FACT E1, we know only that *at least one grade was above 80.* In fact, there may have been only one (9 grades of 79 and 1 grade of 89, for example). But there could have been five or even nine (for example, 9 grades of 85 and 1 grade of 35).
    Alternative solution. The ten students scored exactly 800 points. Ask, "Could they be equal?" Could there be exactly five grades above 80? Sure, five grades of 100 for 500 points and five grades of 60 for 300 points. Must they be equal? No, eight grades of 100 and two grades of 0 also total 800.

15. **(C)** The average of $2x$ and $2y$ is $\frac{2x+2y}{2} = x + y$, which equals 180.

16. **(D)** It is possible that no boy weighs over 150 (if every single boy weighs exactly 150); on the other hand, it is possible that almost every boy weighs over 150. The same is true for the girls.

17. **(B)** Use TACTIC E1: $22 + 38 + x + y = 4(15) = 60 \Rightarrow 60 + x + y = 60 \Rightarrow x + y = 0$.
    Since it is given that x is positive, y must be negative.

18. **(B)** Don't calculate the averages. Quantity A is the average of 2, 4, 6, 8, and10. Quantity B is the average of 3, 5, 7, 9, and 11. Since each of the five numbers from Quantity A is less than the corresponding number from Quantity B, Quantity A must be less than Quantity B.

19. **(A)** You don't have to calculate the averages. Quantity A is clearly positive, and by KEY FACT E3, adding 0 to the set of numbers being averaged must lower the average.

20. **(D)** Use KEY FACT E3: If $x < y$, then the average of x and y is less than y, and surely less than $2y$. So, $2y$ has to raise the average. On the other hand, if x is much larger than y, then $2y$ would lower the average.

Algebra

For the GRE you need to know only a small portion of the algebra normally taught in a high school elementary algebra course and none of the material taught in an intermediate or advanced algebra course. Sections 11-F, 11-G, and 11-H review only those topics that you absolutely need for the GRE.

11-F. POLYNOMIALS

Even though the terms *monomial, binomial, trinomial,* and *polynomial* are not used on the GRE, you must be able to work with simple polynomials, and the use of these terms will make it easier for us to discuss the important concepts.

A ***monomial*** is any number or variable or product of numbers and variables. Each of the following is a monomial:

$$3 \quad -4 \quad x \quad y \quad 3x \quad -4xyz \quad 5x^3 \quad 1.5xy^2 \quad a^3b^4$$

The number that appears in front of the variables in a monomial is called the *coefficient.* The coefficient of $5x^3$ is 5. If there is no number, the coefficient is 1 or -1, because x means $1x$ and $-ab^2$ means $-1ab^2$.

On the GRE, you could be asked to evaluate a monomial for specific values of the variables.

EXAMPLE 1

What is the value of $-3a^2b$ when $a = -4$ and $b = 0.5$?

Ⓐ −72 Ⓑ −24 Ⓒ 24 Ⓓ 48 Ⓔ 72

SOLUTION.
Rewrite the expression, replacing the letters a and b with the numbers -4 and 0.5, respectively. Make sure to write each number in parentheses. Then evaluate: $-3(-4)^2(0.5) = -3(16)(0.5) = \mathbf{-24}$ **(B)**.

CAUTION

Be sure you follow PEMDAS (see Section 11-A): handle exponents before the other operations. In Example 1, you *cannot* multiply −4 by −3, get 12, and then square the 12; you must first square −4.

A ***polynomial*** is a monomial or the sum of two or more monomials. Each monomial that makes up the polynomial is called a ***term*** of the polynomial. Each of the following is a polynomial:

$$2x^2 \quad 2x^2+3 \quad 3x^2-7 \quad x^2+5x-1 \quad a^2b+b^2a \quad x^2-y^2 \quad w^2-2w+1$$

The first polynomial in the above list is a monomial; the second, third, fifth, and sixth polynomials are called ***binomials***, because each has two terms; the fourth and seventh polynomials are called ***trinomials***, because each has three terms. Two terms are called ***like terms*** if they have exactly the same variables and exponents; they can differ only in their coefficients: $5a^2b$ and $-3a^2b$ are like terms, whereas a^2b and b^2a are not.

The polynomial $3x^2 + 4x + 5x + 2x^2 + x - 7$ has 6 terms, but some of them are like terms and can be combined:

$$3x^2 + 2x^2 = 5x^2 \quad \text{and} \quad 4x + 5x + x = 10x.$$

So, the original polynomial is equivalent to the trinomial $5x^2 + 10x - 7$.

KEY FACT F1

The only terms of a polynomial that can be combined are like terms.

KEY FACT F2

To add two polynomials, put a plus sign between them, erase the parentheses, and combine like terms.

TIP

To add, subtract, multiply, and divide polynomials, use the usual laws of arithmetic. To avoid careless errors, before performing any arithmetic operations, write each polynomial in parentheses.

EXAMPLE 2

What is the sum of $5x^2 + 10x - 7$ and $3x^2 - 4x + 2$?

SOLUTION.
$(5x^2 + 10x - 7) + (3x^2 - 4x + 2)$
$= 5x^2 + 10x - 7 + 3x^2 - 4x + 2$
$= (5x^2 + 3x^2) + (10x - 4x) + (-7 + 2)$
$= \mathbf{8x^2 + 6x - 5}$.

KEY FACT F3

To subtract two polynomials, change the minus sign between them to a plus sign and change the sign of every term in the second parentheses. Then just use KEY FACT F2 to add them: erase the parentheses and then combine like terms.

CAUTION

Make sure you get the order right in a subtraction problem.

EXAMPLE 3

Subtract $3x^2 - 4x + 2$ from $5x^2 + 10x - 7$.

SOLUTION.
Be careful. Start with the second polynomial and subtract the first:

$(5x^2 + 10x - 7) - (3x^2 - 4x + 2) = (5x^2 + 10x - 7) + (-3x^2 + 4x - 2) = \mathbf{2x^2 + 14x - 9}$.

EXAMPLE 4

What is the average (arithmetic mean) of $5x^2 + 10x - 7$, $3x^2 - 4x + 2$, and $4x^2 + 2$?

SOLUTION.
As in any average problem, add and divide:

$$(5x^2 + 10x - 7) + (3x^2 - 4x + 2) + (4x^2 + 2) = 12x^2 + 6x - 3,$$

and by the distributive law (KEY FACT A21), $\dfrac{12x^2 + 6x - 3}{3} = \mathbf{4x^2 + 2x - 1}$.

KEY FACT F4

To multiply monomials, first multiply their coefficients, and then multiply their variables (letter by letter), by adding the exponents (see Section 11-A).

EXAMPLE 5

What is the product of $3xy^2z^3$ and $-2x^2y^2$?

SOLUTION.
$(3xy^2z^3)(-2x^2y^2) = 3(-2)(x)(x^2)(y^2)(y^2)(z^3) = \mathbf{-6x^3y^4z^3}$.

All other polynomials are multiplied by using the distributive law.

KEY FACT F5

To multiply a monomial by a polynomial, just multiply each term of the polynomial by the monomial.

EXAMPLE 6

What is the product of $2a$ and $3a^2 - 6ab + b^2$?

SOLUTION.

$2a(3a^2 - 6ab + b^2) = \mathbf{6a^3 - 12a^2b + 2ab^2}$.

On the GRE, the only other polynomials that you could be asked to multiply are two binomials.

KEY FACT F6

To multiply two binomials, use the so-called FOIL method, which is really nothing more than the distributive law: Multiply each term in the first parentheses by each term in the second parentheses and simplify by combining terms, if possible.

$$(2x - 7)(3x + 2) = \underbrace{(2x)(3x)}_{\text{First terms}} + \underbrace{(2x)(2)}_{\text{Outer terms}} + \underbrace{(-7)(3x)}_{\text{Inner terms}} + \underbrace{(-7)(2)}_{\text{Last terms}} =$$

$$6x^2 + 4x - 21x - 14 = 6x^2 - 17x - 14$$

EXAMPLE 7

What is the value of $(x - 2)(x + 3) - (x - 4)(x + 5)$?

$\boxed{}$

SOLUTION.
First, multiply both pairs of binomials:

$$(x - 2)(x + 3) = x^2 + 3x - 2x - 6 = x^2 + x - 6$$
$$(x - 4)(x + 5) = x^2 + 5x - 4x - 20 = x^2 + x - 20$$

Now, subtract:

$$(x^2 + x - 6) - (x^2 + x - 20) = x^2 + x - 6 - x^2 - x + 20 = \mathbf{14}.$$

KEY FACT F7

The three most important binomial products on the GRE are these:

- $(x - y)(x + y) = x^2 + xy - yx - y^2 = x^2 - y^2$
- $(x - y)^2 = (x - y)(x - y) = x^2 - xy - yx + y^2 = x^2 - 2xy + y^2$
- $(x + y)^2 = (x + y)(x + y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$

TIP

If you memorize these, you won't have to multiply them out each time you need them.

EXAMPLE 8

If $a - b = 7$ and $a + b = 13$, what is the value of $a^2 - b^2$?

SOLUTION.
In Section 11-G, we will review how to solve such a pair of equations; but even if you know how, *you should not do it here.* You do not need to know the values of a and b to answer this question. The moment you see $a^2 - b^2$, you should think $(a - b)(a + b)$. Then:

$$a^2 - b^2 = (a - b)(a + b) = (7)(13) = \mathbf{91}.$$

EXAMPLE 9

If $x^2 + y^2 = 36$ and $(x + y)^2 = 64$, what is the value of xy?

SOLUTION.

$$64 = (x + y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 + 2xy = 36 + 2xy.$$

Therefore, $2xy = 64 - 36 = 28 \Rightarrow xy = \mathbf{14}$.

On the GRE, the only division of polynomials you might have to do is to divide a polynomial by a monomial. You will *not* have to do long division of polynomials.

KEY FACT F8

To divide a polynomial by a monomial, use the distributive law. Then simplify each term by reducing the fraction formed by the coefficients to lowest terms and applying the laws of exponents.

EXAMPLE 10

What is the quotient when $32a^2b + 12ab^3c$ is divided by $8ab$?

SOLUTION.
By the distributive law, $\dfrac{32a^2b + 12ab^3c}{8ab} = \dfrac{32a^2b}{8ab} + \dfrac{12ab^3c}{8ab}$.

Now reduce each fraction: $\mathbf{4a + \frac{3}{2}b^2c}$.

On the GRE, the most important way to use the three formulas in KEY FACT F7 is to recognize them in reverse. In other words, whenever you see $x^2 - y^2$, you should realize that it can be rewritten as $(x - y)(x + y)$. This process, which is the reverse of multiplication, is called ***factoring***.

EXAMPLE 11

Quantity A	Quantity B
The value of $x^2 + 4x + 4$ when $x = 95.9$	The value of $x^2 - 4x + 4$ when $x = 99.5$

SOLUTION.
Obviously, you don't want to plug in 95.9 and 99.5 (remember that the GRE *never* requires you to do tedious arithmetic). Recognize that $x^2 + 4x + 4$ is equal to $(x + 2)^2$ and that $x^2 - 4x + 4$ is equal to $(x - 2)^2$. So, Quantity A is just $(95.9 + 2)^2 = 97.9^2$, whereas Quantity B is $(99.5 - 2)^2 = 97.5^2$. Quantity **A** is greater.

EXAMPLE 12

What is the value of $(1{,}000{,}001)^2 - (999{,}999)^2$?

SOLUTION.
Do not even consider squaring 999,999. You know that there has to be an easier way to do this. In fact, if you stop to think, you can get the right answer in a few seconds. This is just $a^2 - b^2$ where $a = 1{,}000{,}001$ and $b = 999{,}999$, so change it to $(a - b)(a + b)$:

$$(1{,}000{,}001)^2 - (999{,}999)^2 = (1{,}000{,}001 - 999{,}999)(1{,}000{,}001 + 999{,}999) = (2)(2{,}000{,}000) = \mathbf{4{,}000{,}000}.$$

Although the coefficients of any of the terms in a polynomial can be fractions, as in $\frac{2}{3}x^2 - \frac{1}{2}x$, the variable itself cannot be in the denominator. An expression such as $\frac{3+x}{x^2}$, which does have a variable in the denominator, is called an ***algebraic fraction***. Fortunately, you should have no trouble with algebraic fractions, since they are handled just like regular fractions. The rules that you reviewed in Section 11-B for adding, subtracting, multiplying, and dividing fractions apply to algebraic fractions, as well.

EXAMPLE 13

What is the sum of the reciprocals of x^2 and y^2?

SOLUTION.

To add $\frac{1}{x^2} + \frac{1}{y^2}$, you need a common denominator, which is x^2y^2.

Multiply the numerator and denominator of $\frac{1}{x^2}$ by y^2 and the numerator and denominator of $\frac{1}{y^2}$ by x^2, and then add:

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{y^2}{x^2y^2} + \frac{x^2}{x^2y^2} = \mathbf{\frac{x^2+y^2}{x^2y^2}}.$$

Often, the way to simplify algebraic fractions is to factor the numerator or the denominator or both. Consider the following example, which is harder than anything you will see on the GRE, but still quite manageable.

EXAMPLE 14

What is the value of $\frac{4x^3 - x}{(2x+1)(6x-3)}$ when $x = 9999$?

$\boxed{}$

SOLUTION.

Don't use FOIL to multiply the denominator. That's going the wrong way. We want to simplify this fraction by factoring everything we can. First factor an x out of the numerator and notice that what's left is the difference of two squares, which can be factored. Then factor out the 3 in the second factor in the denominator:

$$\frac{4x^3 - x}{(2x+1)(6x-3)} = \frac{x(4x^2-1)}{(2x+1)3(2x-1)} = \frac{x\cancel{(2x-1)}\cancel{(2x+1)}}{3\cancel{(2x+1)}\cancel{(2x-1)}} = \frac{x}{3}.$$

So, instead of plugging 9999 into the original expression, plug it into $\frac{x}{3}$: $9999 \div 3 =$ **3333**.

Practice Exercises—Polynomials

Discrete Quantitative Questions

1. What is the value of $\frac{a^2 - b^2}{a - b}$ when $a = 117$ and $b = 118$?

$\boxed{}$

2. If $a^2 - b^2 = 21$ and $a^2 + b^2 = 29$, which of the following could be the value of ab? Indicate *all* possible values.

[A] -10
[B] $5\sqrt{2}$
[C] 10

3. What is the average (arithmetic mean) of $x^2 + 2x - 3$, $3x^2 - 2x - 3$, and $30 - 4x^2$?

(A) $\frac{8x^2 + 4x + 24}{3}$
(B) $\frac{8x^2 + 24}{3}$
(C) $\frac{24 - 4x}{3}$
(D) -12
(E) 8

4. What is the value of $x^2 + 12x + 36$ when $x = 994$?

(A) 11,928
(B) 98,836
(C) 100,000
(D) 988,036
(E) 1,000,000

5. If $c^2 + d^2 = 4$ and $(c - d)^2 = 2$, what is the value of cd?

(A) 1
(B) $\sqrt{2}$
(C) 2
(D) 3
(E) 4

6. What is the value of $(2x + 3)(x + 6) - (2x - 5)(x + 10)$?

(A) 32
(B) 16
(C) 68
(D) $4x^2 + 30x + 68$
(E) $4x^2 + 30x - 32$

7. If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ and $ab = c$, what is the average of a and b?

(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) $\frac{c}{2}$
(E) $\frac{a+b}{2c}$

8. If $x^2 - y^2 = 28$ and $x - y = 8$, what is the average of x and y?

(A) 1.75
(B) 3.5
(C) 7
(D) 8
(E) 10

9. Which of the following is equal to

$$\left(\frac{1}{a}+a\right)^2 - \left(\frac{1}{a}-a\right)^2?$$

Ⓐ 0
Ⓑ 4
Ⓒ $\frac{1}{a^2} - a^2$
Ⓓ $\frac{2}{a^2} - 2a^2$
Ⓔ $\frac{1}{a^2} - 4 - a^2$

10. If $\left(\frac{1}{a}+a\right)^2 = 100$, what is the value of $\frac{1}{a^2} + a^2$?

Ⓐ 10
Ⓑ 64
Ⓒ 98
Ⓓ 100
Ⓔ 102

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

$n < 0$

	Quantity A	Quantity B
11.	$-2n^2$	$(-2n)^2$

$d < c$

	Quantity A	Quantity B
12.	$(c-d)(c+d)$	$(c-d)(c-d)$

$x = -3$ and $y = 2$

	Quantity A	Quantity B
13.	$-x^2y^3$	0

	Quantity A	Quantity B
14.	$(r+s)(r-s)$	$r(s+r) - s(r+s)$

	Quantity A	Quantity B
15.	$\frac{5x^2-20}{x-2}$	$4x+8$

ANSWER KEY

1. **235**	4. **E**	7. **B**	10. **C**	13. **B**
2. **A, C**	5. **A**	8. **A**	11. **B**	14. **C**
3. **E**	6. **C**	9. **B**	12. **D**	15. **D**

Answer Explanations

1. **235** $\frac{a^2-b^2}{a-b} = \frac{\cancel{(a-b)}(a+b)}{\cancel{a-b}} = a + b = 117 + 118 = 235.$

2. **(A)(C)** Adding the two equations, we get that $2a^2 = 50 \Rightarrow a^2 = 25 \Rightarrow b^2 = 4$. So, $a = 5$ or -5 and $b = 2$ or -2. The only possibilities for their product are 10 and -10. (Only A and C are true.)

3. **(E)** To find the average, take the sum of the three polynomials and then divide by 3. Their sum is $(x^2 + 2x - 3) + (3x^2 - 2x - 3) + (30 - 4x^2) = 24$, and $24 \div 3 = 8$.

4. **(E)** You can avoid messy, time-consuming arithmetic if you recognize that $x^2 + 12x + 36 = (x + 6)^2$. The value is $(994 + 6)^2 = 1000^2 = 1{,}000{,}000$.

5. **(A)** Start by squaring $c - d$: $2 = (c - d)^2 = c^2 - 2cd + d^2 = c^2 + d^2 - 2cd = 4 - 2cd$.
So, $2 = 4 - 2cd \Rightarrow 2cd = 2 \Rightarrow cd = 1$.

6. **(C)** First multiply out both pairs of binomials: $(2x + 3)(x + 6) = 2x^2 + 15x + 18$ and $(2x - 5)(x + 10) = 2x^2 + 15x - 50$.
Now subtract: $(2x^2 + 15x + 18) - (2x^2 + 15x - 50) = 18 - (-50) = 68$.

7. **(B)** $\frac{1}{c} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{a+b}{c} \Rightarrow 1 = a + b \Rightarrow \frac{a+b}{2} = \frac{1}{2}$.

8. **(A)** $x^2 - y^2 = (x - y)(x + y) \Rightarrow 28 = 8(x + y) \Rightarrow x + y = 28 \div 8 = 3.5$.
Finally, the average of x and y is $\frac{x+y}{2} = \frac{3.5}{2} = 1.75$.

9. **(B)** Expand each square: $\left(\frac{1}{a}+a\right)^2 = \frac{1}{a^2} + 2\left(\frac{1}{a}\right)(a) + a^2 = \frac{1}{a^2} + 2 + a^2$.

Similarly, $\left(\frac{1}{a}-a\right)^2 = \frac{1}{a^2} - 2 + a^2$.

Subtract: $\left(\frac{1}{a^2}+2+a^2\right) - \left(\frac{1}{a^2}-2+a^2\right) = 4$.

10. **(C)** $100 = \left(\frac{1}{a}+a\right)^2 = \frac{1}{a^2} + 2 + a^2 \Rightarrow \frac{1}{a^2} + a^2 = 98$.

11. **(B)** Since n is negative, n^2 is positive, and so $-2n^2$ is negative. Therefore, Quantity A is negative, whereas Quantity B is positive.

	Quantity A	Quantity B
12. **(D)** $c > d \Rightarrow c - d$ is positive,		
so divide each side by $c - d$:	$c + d$	$c - d$
Subtract c from each quantity:	d	$-d$

If $d = 0$ the quantities are equal; if $d = 1$, they aren't.

13. **(B)** Quantity A: $-(-3)^2 2^3 = -(9)(8) = -72$.

14. **(C)** Quantity B: $r(s + r) - s(r + s) = rs + r^2 - sr - s^2 = r^2 - s^2$
Quantity A: $(r + s)(r - s) = r^2 - s^2$.

15. **(D)** Quantity A: $\dfrac{5x^2 - 20}{x - 2} = \dfrac{5(x^2 - 4)}{x - 2} = \dfrac{5\cancel{(x - 2)}(x + 2)}{\cancel{x - 2}} = 5(x + 2)$.

Quantity B: $4x + 8 = 4(x + 2)$. If $x = -2$, both quantities are 0; for any other value of x the quantities are unequal.

11-G. SOLVING EQUATIONS AND INEQUALITIES

The basic principle that you must adhere to in solving any *equation* is that you can manipulate it in any way, as long as *you do the same thing to both sides.* For example, you may always add the same number to each side; subtract the same number from each side; multiply or divide each side by the same number (except 0); square each side; take the square root of each side (if the quantities are positive); or take the reciprocal of each side. These comments apply to inequalities, as well, except you must be very careful, because some procedures, such as multiplying or dividing by a negative number and taking reciprocals, reverse inequalities (see Section 11-A).

Most of the equations and inequalities that you will have to solve on the GRE have only one variable and no exponents. The following simple six-step method can be used on all of them.

EXAMPLE 1

If $\frac{1}{2}x + 3(x - 2) = 2(x + 1) + 1$, what is the value of x?

SOLUTION.
Follow the steps outlined in the following table.

Step	What to Do	Example 1
1	Get rid of fractions and decimals by multiplying both sides by the Lowest Common Denominator (LCD).	Multiply each term by 2: $x + 6(x - 2) = 4(x + 1) + 2$.
2	Get rid of all parentheses by using the distributive law.	$x + 6x - 12 = 4x + 4 + 2$.
3	Combine like terms on each side.	$7x - 12 = 4x + 6$.
4	By adding or subtracting, get all the variables on one side.	Subtract $4x$ from each side: $3x - 12 = 6$.
5	By adding or subtracting, get all the plain numbers on the other side.	Add 12 to each side: $3x = 18$.
6	Divide both sides by the coefficient of the variable.*	Divide both sides by 3: $x =$ **6**.

*Note: If you start with an inequality and in Step 6 you divide by a negative number, remember to reverse the inequality (see KEY FACT A24).

Example 1 is actually harder than any equation on the GRE, because it required all six steps. On the GRE that never happens. Think of the six steps as a list of questions that must be answered. Ask if each step is necessary. If it isn't, move on to the next one; if it is, do it.

Let's look at Example 2, which does not require all six steps.

EXAMPLE 2

For what real number n is it true that $3(n - 20) = n$?

☐

SOLUTION. Do whichever of the six steps are necessary.

Step	Question	Yes/No	What to Do
1	Are there any fractions or decimals?	No	
2	Are there any parentheses?	Yes	Get rid of them: $3n - 60 = n$.
3	Are there any like terms to combine?	No	
4	Are there variables on both sides?	Yes	Subtract n from each side: $2n - 60 = 0$.
5	Is there a plain number on the same side as the variable?	Yes	Add 60 to each side: $2n = 60$.
6	Does the variable have a coefficient?	Yes	Divide both sides by 2: $n = \mathbf{30}$.

TACTIC

G1

Memorize the six steps *in order* and use this method whenever you have to solve this type of equation or inequality.

EXAMPLE 3

Three brothers divided a prize as follows. The oldest received $\frac{2}{5}$ of it, the middle brother received $\frac{1}{3}$ of it, and the youngest received the remaining $120. What was the value of the prize?

☐

SOLUTION.
If x represents the value of the prize, then $\frac{2}{5}x + \frac{1}{3}x + 120 = x$.
Solve this equation using the six-step method.

Step	Question	Yes/No	What to Do
1	Are there any fractions or decimals?	Yes	To get rid of them, multiply by 15. $15\left(\frac{2}{5}x\right) + 15\left(\frac{1}{3}x\right) +$ $15(120) = 15(x)$ $6x + 5x + 1800 = 15x$
2	Are there any parentheses?	No	
3	Are there any like terms to combine?	Yes	Combine them: $11x + 1800 = 15x$.
4	Are there variables on both sides?	Yes	Subtract $11x$ from each side: $1800 = 4x$.
5	Is there a plain number on the same side as the variable?	No	
6	Does the variable have a coefficient?	Yes	Divide both sides by 4: $x = \mathbf{450}$.

Sometimes on the GRE, you are given an equation with several variables and asked to solve for one of them in terms of the others.

TIP

In applying the six-step method, you shouldn't actually write out the table, as we did in Examples 1–4, since it would be too time consuming. Instead, use the method as a guideline and mentally go through each step, doing whichever ones are required.

TACTIC

G2

When you have to solve for one variable in terms of the others, treat all of the others as if they were numbers, and apply the six-step method.

EXAMPLE 4

If $a = 3b - c$, what is the value of b in terms of a and c?

SOLUTION.
To solve for b, treat a and c as numbers and use the six-step method with b as the variable.

Step	Question	Yes/No	What to Do
1	Are there any fractions or decimals?	No	
2	Are there any parentheses?	No	
3	Are there any like terms to combine?	No	
4	Are there variables on both sides?	No	Remember: the only variable is b.
5	Is there a plain number on the same side as the variable?	Yes	Remember: we're considering c as a number, and it is on the same side as b, the variable. Add c to both sides: $a + c = 3b$.
6	Does the variable have a coefficient?	Yes	Divide both sides by 3: $b = \mathbf{\frac{a+c}{3}}$.

Sometimes when solving equations, you may see a shortcut. For example, to solve $7(w - 3) = 42$, it saves time to start by dividing both sides by 7, getting $w - 3 = 6$, rather than by using the distributive law to eliminate the parentheses. Similarly, if you have to solve a proportion such as $\frac{x}{7} = \frac{3}{5}$, it is easier to cross-multiply, getting $5x = 21$, than to multiply both sides by 35 to get rid of the fractions (although that's exactly what cross-multiplying accomplishes). Other shortcuts will be illustrated in the problems at the end of the section. If you spot such a shortcut, use it; but if you don't, be assured that the six-step method *always* works.

EXAMPLE 5

If $x - 4 = 11$, what is the value of $x - 8$?

Ⓐ −15 Ⓑ −7 Ⓒ −1 Ⓓ 7 Ⓔ 15

SOLUTION.
Going immediately to Step 5, add 4 to each side: $x = 15$. But this is *not* the answer. You need the value not of x, but of $x - 8$: $15 - 8 = 7$ **(D)**.

As in Example 5, on the GRE you are often asked to solve for something other than the simple variable. In Example 5, you could have been asked for the value of x^2 or $x + 4$ or $(x - 4)^2$, and so on.

TACTIC
G3

As you read each question on the GRE, on your scrap paper write down whatever you are looking for, and circle it. This way you will always be sure that you are answering the question that is asked.

TIP
Very often, solving the equation is not the quickest way to answer the question. Consider Example 6.

EXAMPLE 6

If $2x - 5 = 98$, what is the value of $2x + 5$?

SOLUTION.
The first thing you should do is write $2x + 5$ on your paper and circle it. The fact that you are asked for the value of something other than x should alert you to look at the question carefully to see if there is a shortcut.

- The best approach here is to observe that $2x + 5$ is 10 more than $2x - 5$, so the answer is **108** (10 more than 98).
- Next best would be to do only one step of the six-step method, add 5 to both sides: $2x = 103$. Now, add 5 to both sides: $2x + 5 = 103 + 5 = 108$.
- The *worst* method would be to divide $2x = 103$ by 2, get $x = 51.5$, and then use that to calculate $2x + 5$.

EXAMPLE 7

If w is an integer, and the average (arithmetic mean) of 3, 4, and w is less than 10, what is the greatest possible value of w?

Ⓐ 9 Ⓑ 10 Ⓒ 17 Ⓓ 22 Ⓔ 23

SOLUTION.
Set up the inequality: $\frac{3+4+w}{3} < 10$. Do Step 1 (get rid of fractions by multiplying by 3): $3 + 4 + w < 30$. Do Step 3 (combine like terms): $7 + w < 30$. Finally, do Step 5 (subtract 7 from each side): $w < 23$. Since w is an integer, the most it can be is **22 (D)**.

The six-step method also works when there are variables in denominators.

EXAMPLE 8

For what value of x is $\frac{4}{x} + \frac{3}{5} = \frac{10}{x}$?

Ⓐ 5 Ⓑ 10 Ⓒ 20 Ⓓ 30 Ⓔ 50

SOLUTION. Multiply each side by the LCD, $5x$:

$$5\cancel{x}\left(\frac{4}{\cancel{x}}\right) + \cancel{5}x\left(\frac{3}{\cancel{5}}\right) = 5\cancel{x}\left(\frac{10}{\cancel{x}}\right) \Rightarrow 20 + 3x = 50.$$

Now solve normally: $20 + 3x = 50 \Rightarrow 3x = 30$ and so $x =$ **10 (B)**.

EXAMPLE 9

If x is positive, and $y = 5x^2 + 3$, which of the following is an expression for x in terms of y?

Ⓐ $\sqrt{\frac{y}{5} - 3}$ Ⓑ $\sqrt{\frac{y-3}{5}}$ Ⓒ $\frac{\sqrt{y-3}}{5}$ Ⓓ $\frac{\sqrt{y}-3}{5}$ Ⓔ $\frac{\sqrt{y}-\sqrt{3}}{5}$

SOLUTION.
The six-step method works when there are no exponents. However, we can treat x^2 as a single variable, and use the method as far as possible:

$$y = 5x^2 + 3 \Rightarrow y - 3 = 5x^2 \Rightarrow \frac{y-3}{5} = x^2.$$

Now take the square root of each side; since x is positive, the only solution is $x = \sqrt{\frac{y-3}{5}}$ **(B)**.

CAUTION

Doing the same thing to each *side* of an equation does *not* mean doing the same thing to each *term* of the equation. Study Examples 10 and 11 carefully.

NOTE

You *cannot* just take the reciprocal of each term; the answer is *not* $a = b + c$. Here are two solutions.

EXAMPLE 10

If $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, what is a in terms of b and c?

SOLUTION 1.
First add the fractions on the right hand side:

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c} = \frac{b+c}{bc}.$$

Now, take the reciprocal of each *side*: $a = \frac{bc}{b+c}$.

SOLUTION 2.
Use the six-step method. Multiply each term by abc, the LCD:

$$\cancel{a}bc\left(\frac{1}{\cancel{a}}\right) = a\cancel{b}c\left(\frac{1}{\cancel{b}}\right) + ab\cancel{c}\left(\frac{1}{\cancel{c}}\right) \Rightarrow bc = ac + ab = a(c + b) \Rightarrow a = \frac{bc}{c+b}.$$

EXAMPLE 11

If $a > 0$ and $a^2 + b^2 = c^2$, what is a in terms of b and c?

SOLUTION. $a^2 + b^2 = c^2 \Rightarrow a^2 = c^2 - b^2$. Be careful: you *cannot* now take the square root of each *term* and write, $a = c - b$. Rather, you must take the square root of each *side*: $a = \sqrt{a^2} = \sqrt{c^2 - b^2}$.

There are a few other types of equations that you could have to solve on the GRE. Fortunately, they are quite easy. You probably will not have to solve a quadratic equation. However, if you do, you will *not* need the quadratic formula, and you will not have to factor a trinomial. Here are two examples.

EXAMPLE 12

If x is a positive number and $x^2 + 64 = 100$, what is the value of x?

Ⓐ 6 Ⓑ 12 Ⓒ 13 Ⓓ 14 Ⓔ 36

SOLUTION. When there is an x^2-term, but no x-term, we just have to take a square root:

$$x^2 + 64 = 100 \Rightarrow x^2 = 36 \Rightarrow x = \sqrt{36} = \mathbf{6}\ \mathbf{(A)}.$$

EXAMPLE 13

What is the largest value of x that satisfies the equation $2x^2 - 3x = 0$?

Ⓐ 0 Ⓑ 1.5 Ⓒ 2 Ⓓ 2.5 Ⓔ 3

SOLUTION.
When an equation has an x^2-term and an x-term but no constant term, the way to solve it is to factor out the x and to use the fact that if the product of two numbers is 0, one of them must be 0 (KEY FACT A3):

$$\begin{aligned} 2x^2 - 3x = 0 &\Rightarrow x(2x - 3) = 0 \\ x = 0 &\text{ or } 2x - 3 = 0 \\ x = 0 &\text{ or } 2x = 3 \\ x = 0 &\text{ or } x = 1.5. \end{aligned}$$

The largest value is **1.5 (B)**.

In another type of equation that occasionally appears on the GRE, the variable is in the exponent. These equations are particularly easy and are basically solved by inspection.

EXAMPLE 14

If $2^{x+3} = 32$, what is the value of 3^{x+2}?

Ⓐ 5 Ⓑ 9 Ⓒ 27 Ⓓ 81 Ⓔ 125

SOLUTION.
How many 2s do you have to multiply together to get 32? If you don't know that it's 5, just multiply and keep track. Count the 2s on your fingers as you say to yourself, "2 times 2 is 4, times 2 is 8, times 2 is 16, times 2 is 32." Then

$$2^{x+3} = 32 = 2^5 \Rightarrow x + 3 = 5 \Rightarrow x = 2.$$

Therefore, $x + 2 = 4$, and $3^{x+2} = 3^4 = 3 \times 3 \times 3 \times 3 = \mathbf{81}$ **(D)**.

Occasionally, both sides of an equation have variables in the exponents. In that case, it is necessary to write both exponentials with the same base.

EXAMPLE 15

If $4^{w+3} = 8^{w-1}$, what is the value of w?

Ⓐ 0 Ⓑ 1 Ⓒ 2 Ⓓ 3 Ⓔ 9

SOLUTION.
Since it is necessary to have the same base on each side of the equation, write $4 = 2^2$ and $8 = 2^3$. Then

$$4^{w+3} = (2^2)^{w+3} = 2^{2(w+3)} = 2^{2w+6} \quad \text{and} \quad 8^{w-1} = (2^3)^{w-1} = 2^{3(w-1)} = 2^{3w-3}.$$

So, $2^{2w+6} = 2^{3w-3} \Rightarrow 2w + 6 = 3w - 3 \Rightarrow w = \mathbf{9}$ **(E)**.

Systems of Linear Equations

The equations $x + y = 10$ and $x - y = 2$ each have lots of solutions (infinitely many, in fact). Some of them are given in the tables below.

$x + y = 10$

x	5	**6**	4	1	1.2	10	20
y	5	**4**	6	9	8.8	0	−10
$x+y$	10	10	10	10	10	10	10

$x - y = 2$

x	5	**6**	2	0	2.5	19	40
y	3	**4**	0	−2	.5	17	38
$x-y$	2	2	2	2	2	2	2

However, only one pair of numbers, $\boldsymbol{x = 6}$ and $\boldsymbol{y = 4}$, satisfy both equations simultaneously: $6 + 4 = 10$ and $6 - 4 = 2$. This then is the only solution of the ***system of equations***: $\begin{Bmatrix} x + y = 10 \\ x - y = 2 \end{Bmatrix}$.

A system of equations is a set of two or more equations involving two or more variables. To solve such a system, you must find values for each of the variables that will make each equation true. In an algebra course you learn several ways to solve systems of equations. On the GRE, the most useful way to solve them is to add or subtract (usually add) the equations. After demonstrating this method, we will show in Example 19 one other way to handle some systems of equations.

TACTIC

G4

To solve a system of equations, add or subtract them. If there are more than two equations, add them.

EXAMPLE 16

$x + y = 10$
$x - y = 2$

Quantity A	Quantity B
x	y

SOLUTION.
Add the two equations:

$$\begin{array}{rl} x + y &= 10 \\ + \; x - y &= \;\; 2 \\ \hline 2x \quad &= 12 \\ x &= \;\; 6 \end{array}$$

Replacing x with 6 in $x + y = 10$ yields $y = 4$. So, Quantity **A** is greater.

TIP

On the GRE, most problems involving systems of equations do not require you to solve the system. They usually ask for something other than the values of each variable. Read the questions very carefully, circle what you need, and do no more than is required.

EXAMPLE 17

If $3a + 5b = 10$ and $5a + 3b = 30$, what is the average (arithmetic mean) of a and b?

☐

SOLUTION.
Add the two equations:

$$\begin{array}{rl} 3a + 5b &= 10 \\ + \; 5a + 3b &= 30 \\ \hline 8a + 8b &= 40 \end{array}$$

Divide both sides by 8: $a + b = 5$

The average of a and b is: $\dfrac{a+b}{2} = \dfrac{5}{2} = \mathbf{2.5}$

NOTE: It is not only unnecessary to first solve for a and b ($a = 7.5$ and $b = -2.5$.), but, because that procedure is so much more time-consuming, it would be foolish to do so.

EXAMPLE 18

$7a - 3b = 200$
$7a + 3b = 100$

Quantity A	Quantity B
a	b

TIP

Remember TACTIC 5, Chapter 9. On quantitative comparison questions, you don't need to know the value of the quantity in each column; you only need to know which one is greater.

SOLUTION.
Don't actually solve the system. Add the equations:

$$14a = 300 \Rightarrow 7a = 150.$$

So, replacing $7a$ with 150 in the second equation, we get $150 + 3b = 100$; so $3b$, and hence b, must be negative, whereas a is positive. Therefore, $a > b$, and Quantity **A** is greater.

Occasionally on the GRE, it is as easy, or easier, to solve the system by substitution.

TACTIC

G5

If one of the equations in a system of equations consists of a single variable equal to some expression, substitute that expression for the variable in the other equation.

EXAMPLE 19

$$x + y = 10$$
$$y = x - 2$$

Quantity A	Quantity B
x	y

SOLUTION.
Since the second equation states that a single variable (y), is equal to some expression ($x - 2$), substitute that expression for y in the first equation: $x + y = 10$ becomes $x + (x - 2) = 10$. Then, $2x - 2 = 10$, $2x = 12$, and $x = 6$. As always, to find the value of the other variable (y), plug the value of x into one of the two original equations: $y = 6 - 2 = 4$. Quantity **A** is greater.

Practice Exercises — Equations/Inequalities

Discrete Quantitative Questions

1. If $4x + 12 = 36$, what is the value of $x + 3$?

Ⓐ 3
Ⓑ 6
Ⓒ 9
Ⓓ 12
Ⓔ 18

2. If $7x + 10 = 44$, what is the value of $7x - 10$?

Ⓐ $-6\frac{6}{7}$
Ⓑ $4\frac{6}{7}$
Ⓒ $14\frac{6}{7}$
Ⓓ 24
Ⓔ 34

3. If $4x + 13 = 7 - 2x$, what is the value of x?

Ⓐ $-\frac{10}{3}$
Ⓑ -3
Ⓒ -1
Ⓓ 1
Ⓔ $\frac{10}{3}$

4. If $x - 4 = 9$, what is the value of $x^2 - 4$?

[ ]

5. If $ax - b = c - dx$, what is the value of x in terms of a, b, c, and d?

Ⓐ $\frac{b+c}{a+d}$
Ⓑ $\frac{c-b}{a-d}$
Ⓒ $\frac{b+c-d}{a}$
Ⓓ $\frac{c-b}{a+d}$
Ⓔ $\frac{c}{b} - \frac{d}{a}$

6. If $\frac{1}{3}x + \frac{1}{6}x + \frac{1}{9}x = 33$, what is the value of x?

Ⓐ 3
Ⓑ 18
Ⓒ 27
Ⓓ 54
Ⓔ 72

7. If $3x - 4 = 11$, what is the value of $(3x - 4)^2$?

Ⓐ 22
Ⓑ 36
Ⓒ 116
Ⓓ 121
Ⓔ 256

8. If $64^{12} = 2^{a-3}$, what is the value of a?

Ⓐ 9
Ⓑ 15
Ⓒ 69
Ⓓ 72
Ⓔ 75

9. If the average (arithmetic mean) of $3a$ and $4b$ is less than 50, and a is twice b, what is the largest possible integer value of a?
Ⓐ 9
Ⓑ 10
Ⓒ 11
Ⓓ 19
Ⓔ 20

10. If $\frac{1}{a-b} = 5$, then $a =$
Ⓐ $b + 5$
Ⓑ $b - 5$
Ⓒ $b + \frac{1}{5}$
Ⓓ $b - \frac{1}{5}$
Ⓔ $\frac{1-5b}{5}$

11. If $x = 3a + 7$ and $y = 9a^2$, what is y in terms of x?
Ⓐ $(x-7)^2$
Ⓑ $3(x-7)^2$
Ⓒ $\frac{(x-7)^2}{3}$
Ⓓ $\frac{(x+7)^2}{3}$
Ⓔ $(x+7)^2$

12. If $4y - 3x = 5$, what is the smallest integer value of x for which $y > 100$?

☐

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

$a + b = 13$
$a - b = 13$

	Quantity A	Quantity B
13.	b	13

$\frac{2^{a-1}}{2^{b+1}} = 8$

	Quantity A	Quantity B
14.	a	b

$4x^2 = 3x$

	Quantity A	Quantity B
15.	x	1

$a + b = 1$
$b + c = 2$
$c + a = 3$

	Quantity A	Quantity B
16.	The average (arithmetic mean) of a, b, and c	1

$3x - 4y = 5$
$y = 2x$

	Quantity A	Quantity B
17.	x	y

$\frac{x}{2} - 2 > \frac{x}{3}$

	Quantity A	Quantity B
18.	x	12

$3r - 5s = 17$
$2r - 6s = 7$

	Quantity A	Quantity B
19.	The average (arithmetic mean) of r and s	10

$\frac{1}{c} = 1 + \frac{1}{d}$
c and d are positive

	Quantity A	Quantity B
20.	c	d

ANSWER KEY

1. **C**	5. **A**	9. **D**	13. **B**	17. **A**
2. **D**	6. **D**	10. **C**	14. **A**	18. **A**
3. **C**	7. **D**	11. **A**	15. **B**	19. **B**
4. **165**	8. **E**	12. **132**	16. **C**	20. **B**

Answer Explanations

1. **(C)** The easiest method is to recognize that $x + 3$ is $\frac{1}{4}$ of $4x + 12$ and, therefore, equals $\frac{1}{4}$ of 36, which is 9. If you don't see that, solve normally: $4x + 12 = 36 \Rightarrow 4x = 24 \Rightarrow x = 6$ and so $x + 3 = 9$.

2. **(D)** Subtracting 20 from each side of $7x + 10 = 44$ gives $7x - 10 = 24$. If you don't see that, subtract 10 from each side, getting $7x = 34$. Then subtract 10 to get $7x - 10 = 24$. The worst alternative is to divide both sides of $7x = 34$ by 7 to get $x = \frac{34}{7}$; then you have to multiply by 7 to get back to 34, and then subtract 10.

3. **(C)** Add $2x$ to each side: $6x + 13 = 7$. Subtract 13 from each side: $6x = -6$. Divide by 6: $x = -1$.

4. **165** $x - 4 = 9 \Rightarrow x = 13 \Rightarrow x^2 = 169$ and so $x^2 - 4 = 165$.

5. **(A)** Treat a, b, c, and d as constants, and use the six-step method to solve for x:
$ax - b = c - dx \Rightarrow ax - b + dx = c \Rightarrow ax + dx = c + b \Rightarrow x(a + d) = b + c \Rightarrow$
$x = \frac{b+c}{a+d}$.

6. **(D)** Multiply both sides by 18, the LCD:

$$18\left(\frac{1}{3}x + \frac{1}{6}x + \frac{1}{9}x\right) = 18(33) \Rightarrow 6x + 3x + 2x = 594 \Rightarrow 11x = 594 \Rightarrow x = 54.$$

It's actually easier not to multiply out 18×33; leave it in that form, and then divide by 11: $\frac{18 \times \cancel{33}^{3}}{\cancel{11}_{1}} = 3 \times 18 = 54.$

7. **(D)** Be alert. Since you are given the value of $3x - 4$, and want the value of $(3x - 4)^2$, just square both sides: $11^2 = 121$. If you don't see that, you'll waste time solving $3x - 4 = 11(x = 5)$, only to use that value to calculate that $3x - 4$ is equal to 11, which you aready knew.

8. **(E)** $2^{a-3} = 64^{12} = (2^6)^{12} = 2^{72} \Rightarrow a - 3 = 72$, and so $a = 75$.

9. **(D)** Since $a = 2b$, $2a = 4b$. Therefore, the average of $3a$ and $4b$ is the average of $3a$ and $2a$, which is $2.5a$. Therefore, $2.5a < 50 \Rightarrow a < 20$. So the largest *integer* value of a is 19.

10. **(C)** Taking the reciprocal of each side, we get $a - b = \frac{1}{5}$. So $a = b + \frac{1}{5}$.

11. **(A)** $x = 3a + 7 \Rightarrow x - 7 = 3a \Rightarrow a = \frac{x-7}{3}$.

Therefore, $y = 9a^2 = 9\left(\frac{x-7}{3}\right)^2 = 9\frac{(x-7)^2}{3^2} = (x-7)^2$.

12. **132** Solving for y yields $y = \frac{5+3x}{4}$.

Then, since $y > 100$: $\frac{5+3x}{4} > 100 \Rightarrow 5 + 3x > 400 \Rightarrow 3x > 395 \Rightarrow$

$x > 131.666$.

The smallest integer value of x is 132.

13. **(B)** Adding the two equations, we get that $2a = 26$. Therefore, $a = 13$ and $b = 0$.

14. **(A)** Express each side of $\frac{2^{a-1}}{2^{b+1}} = 8$ as a power of 2:

$8 = 2^3$ and $\frac{2^{a-1}}{2^{b+1}} = 2^{(a-1)-(b+1)} = 2^{a-b-2}$.

Therefore, $a - b - 2 = 3 \Rightarrow a = b + 5$, and so a is greater.

15. **(B)** $4x^2 = 3x \Rightarrow 4x^2 - 3x = 0 \Rightarrow x(4x - 3) = 0$.

So,

$$\begin{aligned} x = 0 \quad &\text{or} \quad 4x - 3 = 0 \Rightarrow \\ x = 0 \quad &\text{or} \quad 4x = 3 \Rightarrow \\ x = 0 \quad &\text{or} \quad x = \frac{3}{4}. \end{aligned}$$

There are two possible values of x, both of which are less than 1.

16. **(C)** When we add all three equations, we get

$2a + 2b + 2c = 6 \Rightarrow a + b + c = 3$, and so $\frac{a+b+c}{3} = 1$.

17. **(A)** Use substitution. Replace y in the first equation with $2x$:

$3x - 4(2x) = 5 \Rightarrow 3x - 8x = 5 \Rightarrow -5x = 5 \Rightarrow x = -1 \Rightarrow y = -2$.

18. **(A)** Multiply both sides by 6, the LCD:

$$6\left(\frac{x}{2} - 2\right) > 6\left(\frac{x}{3}\right) \Rightarrow 3x - 12 > 2x \Rightarrow -12 > -x \Rightarrow x > 12.$$

19. **(B)** The first thing to try is to add the equations. That yields $5r - 11s = 24$, which does not appear to be useful. So now try to subtract the equations. That yields $r + s = 10$.

So the average of r and s is $\frac{r+s}{2} = \frac{10}{2} = 5$.

20. **(B)** Multiply both sides of the given equation by cd, the LCD of the fractions:

$$cd\left(\frac{1}{c}\right) = cd\left(1 + \frac{1}{d}\right) \Rightarrow d = cd + c = c(d + 1) \Rightarrow c = \frac{d}{d+1}.$$

Since d is positive, $d + 1 > 1$, and so $\frac{d}{d+1} < d$.

So $c < d$.

11-H. WORD PROBLEMS

On a typical GRE you will see several word problems, covering almost every math topic for which you are responsible. In this chapter you have already seen word problems on consecutive integers in Section A; fractions and percents in Sections B and C; ratios and rates in Section D; and averages in Section E. Later in this chapter you will see word problems involving probability, circles, triangles, and other geometric figures. A few of these problems can be solved with just arithmetic, but most of them require basic algebra.

To solve word problems algebraically, you must treat algebra as a foreign language and learn to translate "word for word" from English into algebra, just as you would from English into French or Spanish or any other language. When translating into algebra, we use some letter (often x) to represent the unknown quantity we are trying to determine. It is this translation process that causes difficulty for some students. Once translated, solving is easy using the techniques we have already reviewed. Consider the following pairs of typical GRE questions. The first ones in each pair (1A and 2A) would be considered easy, whereas the second ones (1B and 2B) would be considered harder.

EXAMPLE 1A

What is 4% of 4% of 40,000?

EXAMPLE 1B

In a lottery, 4% of the tickets printed can be redeemed for prizes, and 4% of those tickets have values in excess of $100. If the state prints 40,000 tickets, how many of them can be redeemed for more than $100?

EXAMPLE 2A

If $x + 7 = 2(x - 8)$, what is the value of x?

EXAMPLE 2B

In 7 years Erin will be twice as old as she was 8 years ago. How old is Erin now?

Once you translate the words into arithmetic expressions or algebraic equations, Examples 1A and 1B and 2A and 2B are identical. The problem that many students have is doing the translation. It really isn't very difficult, and we'll show you how. First, though, look over the following English to algebra "dictionary."

English Words	Mathematical Meaning	Symbol
Is, was, will be, had, has, will have, is equal to, is the same as	Equals	$=$
Plus, more than, sum, increased by, added to, exceeds, received, got, older than, farther than, greater than	Addition	$+$
Minus, fewer, less than, difference, decreased by, subtracted from, younger than, gave, lost	Subtraction	$-$
Times, of, product, multiplied by	Multiplication	$\times$
Divided by, quotient, per, for	Division	$\div$, $\frac{a}{b}$
More than, greater than	Inequality	$>$
At least	Inequality	$\geq$
Fewer than, less than	Inequality	$<$
At most	Inequality	$\leq$
What, how many, etc.	Unknown quantity	x (or some other variable)

Let's use our dictionary to translate some phrases and sentences.

1. The sum of 5 and some number is 13. $5 + x = 13$
2. John was 2 years younger than Sam. $J = S - 2$
3. Bill has at most $100. $B \leq 100$
4. The product of 2 and a number exceeds that number by 5 (is 5 more than). $2N = N + 5$

In translating statements, you first must decide what quantity the variable will represent. Often it's obvious. Other times there is more than one possibility.

Let's translate and solve the two questions from the beginning of this section, and then we'll look at a few new ones.

TIP

In all word problems on the GRE, remember to write down and circle what you are looking for. Don't answer the wrong question!

EXAMPLE 1B

In a lottery, 4% of the tickets printed can be redeemed for prizes, and 4% of those tickets have values in excess of $100. If the state prints 40,000 tickets, how many of them can be redeemed for more than $100?

SOLUTION.
Let x = the number of tickets worth more than $100. Then

$$x = 4\% \text{ of } 4\% \text{ of } 40{,}000 = .04 \times .04 \times 40{,}000 = \mathbf{64},$$

which is also the solution to Example 1a.

EXAMPLE 2B

In 7 years Erin will be twice as old as she was 8 years ago. How old is Erin now?

TIP

In problems involving ages, remember that "years ago" means you need to subtract, and "years from now" means you need to add.

SOLUTION.
Let x = Erin's age now. Then 8 years ago she was $x - 8$, and 7 years from now she will be $x + 7$. So,

$$x + 7 = 2(x - 8) \Rightarrow x + 7 = 2x - 16 \Rightarrow 7 = x - 16 \Rightarrow x = \mathbf{23},$$

which is also the solution to Example 2a.

Most algebraic word problems on the GRE are not too difficult, and if you can do the algebra, that's usually the best way. But if, after studying this section, you still get stuck on a question during the test, don't despair. Use the tactics that you learned in Chapter 8, especially TACTIC 1—backsolving.

Age Problems

EXAMPLE 3

In 1980, Judy was 3 times as old as Adam, but in 1984 she was only twice as old as he was. How old was Adam in 1990?

Ⓐ 4 Ⓑ 8 Ⓒ 12 Ⓓ 14 Ⓔ 16

TIP

It is often very useful to organize the data from a word problem in a table.

SOLUTION.
Let x be Adam's age in 1980 and fill in the table below.

Year	Judy	Adam
1980	$3x$	x
1984	$3x + 4$	$x + 4$

Now translate: Judy's age in 1984 was twice Adam's age in 1984:

$$3x + 4 = 2(x + 4) = 2x + 8$$

$$3x + 4 = 2x + 8 \Rightarrow x + 4 = 8, \text{ and so } x = 4.$$

So, Adam was 4 in 1980. However, 4 is *not* the answer to this question. Did you remember to circle what you're looking for? The question *could have* asked for Adam's age in 1980 (Choice A) or 1984 (Choice B) or Judy's age in any year whatsoever (Choice C is 1980 and Choice E is 1984); but it didn't. It asked for *Adam's age in 1990.* Since he was 4 in 1980, then 10 years later, in 1990, he was **14 (D)**.

Distance Problems

Distance problems all depend on three variations of the same formula:

$$\textbf{distance} = \textbf{rate} \times \textbf{time} \qquad \textbf{rate} = \frac{\textbf{distance}}{\textbf{time}} \qquad \textbf{time} = \frac{\textbf{distance}}{\textbf{rate}}$$

These are usually abbreviated, $\boldsymbol{d = rt}$, $\boldsymbol{r} = \frac{d}{t}$, and $\boldsymbol{t} = \frac{d}{r}$.

EXAMPLE 4

How much longer, in *seconds*, is required to drive 1 mile at 40 miles per hour than at 60 miles per hour?

seconds

SOLUTION.
The time to drive 1 mile at 40 miles per hour is given by

$$t = \frac{1}{40} \text{ hour} = \frac{1}{\cancel{40}_2} \times \cancel{60}^{3} \text{ minutes} = 1\frac{1}{2} \text{ minutes}.$$

The time to drive 1 mile at 60 miles per hour is given by $t = \frac{1}{60}$ hour = 1 minute.

The difference is $\frac{1}{2}$ minute = **30** seconds.

Note that this solution used the time formula given, but required only arithmetic, not algebra. Example 5 requires an algebraic solution.

EXAMPLE 5

Avi drove from his home to college at 60 miles per hour. Returning over the same route, there was a lot of traffic, and he was only able to drive at 40 miles per hour. If the return trip took 1 hour longer, how many miles did he drive each way?

Ⓐ 2 Ⓑ 3 Ⓒ 5 Ⓓ 120 Ⓔ 240

SOLUTION.
Let x = the number of hours Avi took going to college and make a table.

	rate	time	distance
Going	60	x	$60x$
Returning	40	$x + 1$	$40(x + 1)$

Since he drove the same distance going and returning,

$$60x = 40(x + 1) \Rightarrow 60x = 40x + 40 \Rightarrow 20x = 40, \text{ and so } x = 2.$$

Now be sure to answer the correct question. When $x = 2$, Choices A, B, and C are the time in hours that it took going, returning, and round-trip; Choices D and E are the distances each way and round-trip. You could have been asked for any of the five. If you circled what you're looking for, you won't make a careless mistake. Avi drove **120** miles each way, and so the correct answer is **D**.

The d in $d = rt$ stands for "distance," but it could really be any type of work that is performed at a certain rate, r, for a certain amount of time, t. Example 5 need not be about distance. Instead of driving 120 miles at 60 miles per hour for 2 hours, Avi could have read 120 pages at a rate of 60 pages per hour for 2 hours; or planted 120 flowers at the rate of 60 flowers per hour for 2 hours; or typed 120 words at a rate of 60 words per minute for 2 minutes.

Examples 6 and 7 illustrate two additional word problems of the type that you might find on the GRE.

EXAMPLE 6

Lindsay is trying to collect all the cards in a special commemorative set of baseball cards. She currently has exactly $\frac{1}{4}$ of the cards in that set. When she gets 10 more cards, she will then have $\frac{1}{3}$ of the cards. How many cards are in the set?

Ⓐ 30 Ⓑ 60 Ⓒ 120 Ⓓ 180 Ⓔ 240

SOLUTION.
Let x be the number cards in the set. First, translate this problem from English into algebra: $\frac{1}{4}x + 10 = \frac{1}{3}x$. Now, use the six-step method of Section 11-G to solve the equation. Multiply by 12 to get, $3x + 120 = 4x$, and then subtract $3x$ from each side: $x = \mathbf{120}$ **(C)**.

EXAMPLE 7

Jen, Ken, and Len have a total of \$390. Jen has 5 times as much as Len, and Ken has $\frac{3}{4}$ as much as Jen. How much money does Ken have?

Ⓐ \$40 Ⓑ \$78 Ⓒ \$150 Ⓓ \$195 Ⓔ \$200

Suppose, for example, that in this problem you let x represent the amount of money that Ken has. Then since Ken has $\frac{3}{4}$ as much as Jen, Jen has $\frac{4}{3}$ as much as Ken: $\frac{4}{3}x$; and Jen would have $\frac{1}{5}$ of that: $\left(\frac{1}{5}\right)\left(\frac{4}{3}x\right)$. It is much easier here to let x represent the amount of money Len has.

TIP

You often have a choice as to what to let the variable represent. Don't necessarily let it represent what you're looking for; rather, choose what will make the problem easiest to solve.

SOLUTION.
Let x represent the amount of money Len has. Then $5x$ is the amount that Jen has, and $\frac{3}{4}(5x)$ is the amount that Ken has. Since the total amount of money is \$390, $x + 5x + \frac{15}{4}x = 390$.

Multiply by 4 to get rid of the fraction: $4x + 20x + 15x = 1560$.
Combine like terms and then divide: $39x = 1560 \Rightarrow x = 40$.

So Len has \$40, Jen has 5×40 = \$200, and Ken has $\frac{3}{4}(200)$ = **\$150 (C)**.

Practice Exercises — Word Problems

Discrete Quantitative Questions

1. Howard has three times as much money as Ronald. If Howard gives Ronald $50, Ronald will then have three times as much money as Howard. How much money, in dollars, do the two of them have together?

[ ] dollars

2. In the afternoon, Beth read 100 pages at the rate of 60 pages per hour; in the evening, when she was tired, she read another 100 pages at the rate of 40 pages per hour. What was her average rate of reading for the day?

(A) 45
(B) 48
(C) 50
(D) 52
(E) 55

3. If the sum of five consecutive integers is S, what is the largest of those integers in terms of S?

(A) $\frac{S-10}{5}$
(B) $\frac{S+4}{4}$
(C) $\frac{S+5}{4}$
(D) $\frac{S-5}{2}$
(E) $\frac{S+10}{5}$

4. As a fund-raiser, the school band was selling two types of candy: lollipops for 40 cents each and chocolate bars for 75 cents each. On Monday, they sold 150 candies and raised 74 dollars. How many lollipops did they sell?

(A) 75
(B) 90
(C) 96
(D) 110
(E) 120

5. A jar contains only red, white, and blue marbles. The number of red marbles is $\frac{4}{5}$ the number of white ones, and the number of white ones is $\frac{3}{4}$ the number of blue ones. If there are 470 marbles in all, how many of them are blue?

(A) 120
(B) 135
(C) 150
(D) 184
(E) 200

6. The number of shells in Judy's collection is 80% of the number in Justin's collection. If Justin has 80 more shells than Judy, how many shells do they have altogether?

[ ] shells

7. What is the greater of two numbers whose product is 900, if the sum of the two numbers exceeds their difference by 30?

(A) 15
(B) 60
(C) 75
(D) 90
(E) 100

8. On a certain project the only grades awarded were 80 and 100. If 10 students completed the project and the average of their grades was 94, how many earned 100?

(A) 2
(B) 3
(C) 5
(D) 7
(E) 8

9. If $\frac{1}{2}x$ years ago Adam was 12, and $\frac{1}{2}x$ years from now he will be $2x$ years old, how old will he be $3x$ years from now?

(A) 18
(B) 24
(C) 30
(D) 54
(E) It cannot be determined from the information given.

10. Since 1950, when Barry was discharged from the army, he has gained 2 pounds every year. In 1980 he was 40% heavier than in 1950. What percent of his 1995 weight was his 1980 weight?

(A) 80
(B) 85
(C) 87.5
(D) 90
(E) 95

Quantitative Comparison Questions

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) Quantities A and B are equal.
(D) It is impossible to determine which quantity is greater.

Lindsay is twice as old as she was 10 years ago.
Kimberly is half as old as she will be in 10 years.

	Quantity A	Quantity B
11.	Lindsay's age now	Kimberly's age now

Boris spent $\frac{1}{4}$ of his take-home pay on Saturday and $\frac{1}{3}$ of what was left on Sunday. The rest he put in his savings account.

	Quantity A	Quantity B
12.	The amount of his take-home pay that he spent	The amount of his take-home pay that he saved

In 8 years, Tiffany will be 3 times as old as she is now.

	Quantity A	Quantity B
13.	The number of years until Tiffany will be 6 times as old as she is now	16

Rachel put exactly 50 cents worth of postage on an envelope using only 4-cent stamps and 7-cent stamps.

	Quantity A	Quantity B
14.	The number of 4-cent stamps she used	The number of 7-cent stamps she used

Car A and Car B leave from the same spot at the same time.
Car A travels due north at 40 mph.
Car B travels due east at 30 mph.

	Quantity A	Quantity B
15.	Distance from Car A to Car B 9 hours after they left	450 miles

ANSWER KEY

1. **100**	4. **D**	7. **B**	10. **C**	13. **A**
2. **B**	5. **E**	8. **D**	11. **A**	14. **D**
3. **E**	6. **720**	9. **D**	12. **C**	15. **C**

Answer Explanations

1. **100**

	Ronald	Howard
At the beginning	x	$3x$
After the gift	$x + 50$	$3x - 50$

After the gift, Ronald will have 3 times as much money as Howard:
$x + 50 = 3(3x - 50) \Rightarrow x + 50 = 9x - 150 \Rightarrow 8x = 200$, and so $x = 25$.
So Ronald has \$25 and Howard has \$75, for a total of \$100.

2. **(B)** Beth's average rate of reading is determined by dividing the total number of pages she read (200) by the total amount of time she spent reading. In the afternoon she read for $\frac{100}{60} = \frac{5}{3}$ hours, and in the evening for $\frac{100}{40} = \frac{5}{2}$ hours,

for a total time of $\frac{5}{3} + \frac{5}{2} = \frac{10}{6} + \frac{15}{6} = \frac{25}{6}$ hours. So, her average rate was

$200 \div \frac{25}{6} = 200 \times \frac{6}{25} = 48$ pages per hour.

3. **(E)** Let the 5 consecutive integers be n, $n + 1$, $n + 2$, $n + 3$, $n + 4$. Then,
$S = n + n + 1 + n + 2 + n + 3 + n + 4 = 5n + 10 \Rightarrow 5n = S - 10 \Rightarrow n = \frac{S-10}{5}$.
Choice A, therefore, is the *smallest* of the integers; the *largest* is
$n + 4 = \frac{S-10}{5} + 4 = \frac{S-10}{5} + \frac{20}{5} = \frac{S+10}{5}$.

4. **(D)** Let x represent the number of chocolate bars sold; then $150 - x$ is the number of lollipops sold. We must use the same units, so we could write 75 cents as .75 dollars or 74 dollars as 7400 cents. Let's avoid the decimals: x chocolates sold for $75x$ cents and $(150 - x)$ lollipops sold for $40(150 - x)$ cents. So,
$7400 = 75x + 40(150 - x) = 75x + 6000 - 40x = 6000 + 35x \Rightarrow$
$1400 = 35x \Rightarrow x = 40$ and $150 - 40 = 110$.

5. **(E)** If b is the number of blue marbles, then there are $\frac{3}{4}b$ white ones, and
$\frac{4}{5}\left(\frac{3}{4}b\right) = \frac{3}{5}b$ red ones.

Therefore, $470 = b + \frac{3}{4}b + \frac{3}{5}b = b\left(1 + \frac{3}{4} + \frac{3}{5}\right) = \frac{47}{20}b$.

So, $b = 470 \div \frac{47}{20} = \overset{10}{\cancel{470}} \times \frac{20}{\underset{1}{\cancel{47}}} = 200$.

6. **720** If x is the number of shells in Justin's collection, then Judy has $.80x$. Since Justin has 80 more shells than Judy,
$x = .80x + 80 \Rightarrow .20x = 80 \Rightarrow x = 80 \div .20 = 400$.
So Justin has 400 and Judy has 320: a total of 720.

7. **(B)** If x represents the greater and y the smaller of the two numbers, then $(x + y) = 30 + (x - y) \Rightarrow y = 30 - y \Rightarrow 2y = 30$, and so $y = 15$. Since $xy = 900$, $x = 900 \div 15 = 60$.

8. **(D)** If x represents the number of students earning 100, then $10 - x$ is the number of students earning 80. So
$94 = \frac{100x + 80(10 - x)}{10} \Rightarrow 94 = \frac{100x + 800 - 80x}{10} = \frac{20x + 800}{10} \Rightarrow$

$94 \times 10 = 940 = 20x + 800 \Rightarrow 140 = 20x$, and $x = 7$.

9. **(D)** Since $\frac{1}{2}x$ years ago, Adam was 12, he is now $12 + \frac{1}{2}x$. So $\frac{1}{2}x$ years from

now, he will be $12 + \frac{1}{2}x + \frac{1}{2}x = 12 + x$. But, we are told that at that time

he will be $2x$ years old. So, $12 + x = 2x \Rightarrow x = 12$.

Thus, he is now $12 + 6 = 18$, and $3x$ or 36 years from now he will be $18 + 36 = 54$.

10. **(C)** Let x be Barry's weight in 1950. By 1980, he had gained 60 pounds (2 pounds per year for 30 years) and was 40% heavier: $60 = .40x \Rightarrow$ $x = 60 \div .4 = 150$. So in 1980, he weighed 210. Fifteen years later, in 1995,

he weighed 240: $\frac{210}{240} = \frac{7}{8} = 87.5\%$.

11. **(A)** You can do the simple algebra, but you might realize that if in the past 10 years Lindsay's age doubled, she was 10 and is now 20. Similarly, Kimberly is now 10 and in 10 years will be 20.

Here is the algebra: if x represents Lindsay's age now,
$x = 2(x - 10) \Rightarrow x = 2x - 20 \Rightarrow x = 20$.
Similarly, Kimberly is now 10 and will be 20 in 10 years.

12. **(C)** Let x represent the amount of Boris's take-home pay. On Saturday, he

spent $\frac{1}{4}x$ and still had $\frac{3}{4}x$; but on Sunday, he spent $\frac{1}{3}$ of that:

$\frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{4}x\right) = \frac{1}{4}x$. Therefore, he spent $\frac{1}{4}$ of his take-home pay each day.

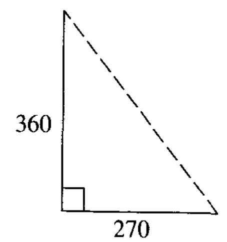
So, he spent $\frac{1}{2}$ of his pay and saved $\frac{1}{2}$ of his pay.

13. **(A)** If x represents Tiffany's age now, then in 8 years she will be $x + 8$, and so $x + 8 = 3x \Rightarrow 8 = 2x \Rightarrow x = 4$.

    Tiffany will be 6 times as old 20 years from now, when she will be 24.

14. **(D)** If x and y represent the number of 4-cent stamps and 7-cent stamps that Rachel used, respectively, then $4x + 7y = 50$. This equation has infinitely many solutions but only 2 in which x and y are both positive integers: $y = 2$ and $x = 9$ or $y = 6$ and $x = 2$.

15. **(C)** Draw a diagram. In 9 hours Car A drove 360 miles north and Car B drove 270 miles east. These are the legs of a right triangle, whose hypotenuse is the distance between them. Use the Pythagorean theorem if you don't recognize that this is just a $3x$–$4x$–$5x$ right triangle: the legs are 90×3 and 90×4, and the hypotenuse is $90 \times 5 = 450$.



Geometry

Although about 30% of the math questions on the GRE have to do with geometry, there are only a relatively small number of facts you need to know — far less than you would learn in a geometry course — and, of course, there are no proofs. In the next six sections we will review all of the geometry that you need to know to do well on the GRE. We will present the material exactly as it appears on the GRE, using the same vocabulary and notation, which might be slightly different from the terminology you learned in your high school math classes. The numerous examples in the next six sections will show you exactly how these topics are treated on the GRE.

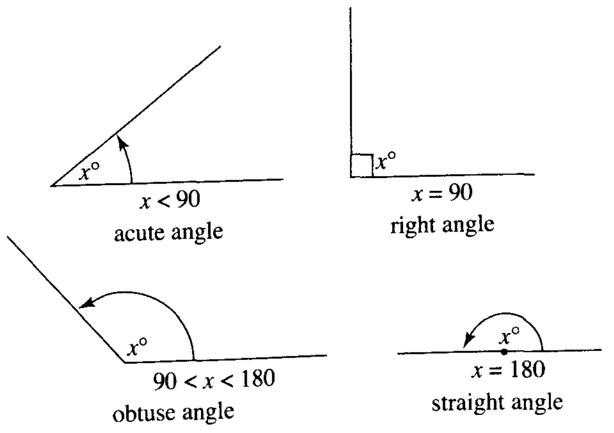
11-I. LINES AND ANGLES

An ***angle*** is formed by the intersection of two line segments, rays, or lines. The point of intersection is called the ***vertex***. On the GRE, angles are always measured in degrees.

KEY FACT I1

Angles are classified according to their degree measures.

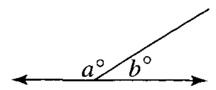
- **An acute angle measures less than 90°.**
- **A right angle measures 90°.**
- **An obtuse angle measures more than 90° but less than 180°.**
- **A straight angle measures 180°.**



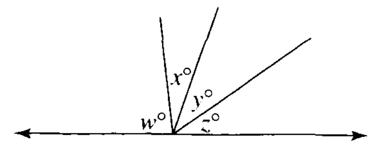
NOTE: The small square in the second angle in the figure above is *always* used to mean that the angle is a right angle. On the GRE, if an angle has a square in it, it must measure exactly 90°, *whether or not you think that the figure has been drawn to scale.*

KEY FACT I2

If two or more angles form a straight angle, the sum of their measures is 180°.



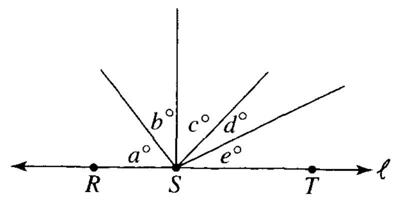
$a + b = 180$



$w + x + y + z = 180$

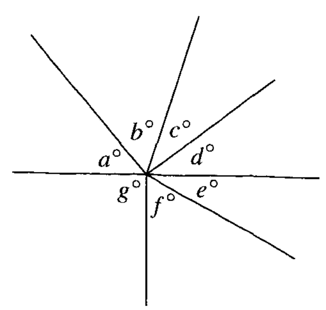
EXAMPLE 1

In the figure below, *R*, *S*, and *T* are all on line ℓ. What is the average of *a*, *b*, *c*, *d*, and *e*?

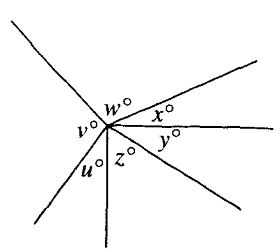


SOLUTION.
Since $\angle RST$ is a straight angle, by KEY FACT I2, the sum of *a*, *b*, *c*, *d*, and *e* is 180, and so their average is $\frac{180}{5} = \mathbf{36}$.

In the figure below, since $a + b + c + d = 180$ and $e + f + g = 180$, $a + b + c + d + e + f + g = 180 + 180 = 360$.



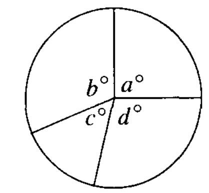
It is also true that $u + v + w + x + y + z = 360$, even though none of the angles forms a straight angle.



KEY FACT I3

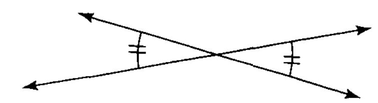
The sum of all the measures of all the angles around a point is 360°.

NOTE: This fact is particularly important when the point is the center of a circle, as we shall see in Section 11-L.



$a + b + c + d = 360$

When two lines intersect, four angles are formed. The two angles in each pair of opposite angles are called ***vertical angles.***



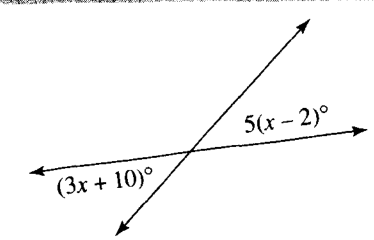
KEY FACT I4

Vertical angles have equal measures.

EXAMPLE 2

In the figure at the right, what is the value of *x*?

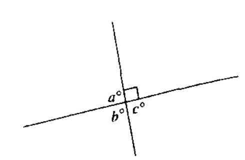
Ⓐ 6 Ⓑ 8 Ⓒ 10 Ⓓ 20 Ⓔ 40



SOLUTION.
Since the measures of vertical angles are equal, $3x + 10 = 5(x - 2) \Rightarrow 3x + 10 = 5x - 10 \Rightarrow 3x + 20 = 5x \Rightarrow 20 = 2x \Rightarrow x = \mathbf{10}$ **(C)**.

KEY FACT I5

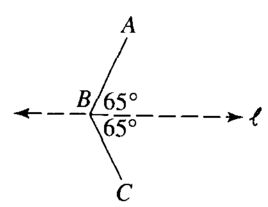
If one of the angles formed by the intersection of two lines (or line segments) is a right angle, then all four angles are right angles.

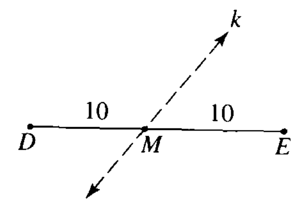


$a = b = c = 90$

Two lines that intersect to form right angles are called ***perpendicular.***

In the figures below, line ℓ divides $\angle ABC$ into two equal parts, and line k divides line segment DE into two equal parts. Line ℓ is said to ***bisect*** the angle, and line k ***bisects*** the line segment. Point M is called the ***midpoint*** of segment DE.

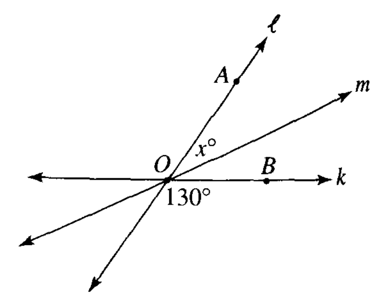




EXAMPLE 3

In the figure at the right, lines k, ℓ, and m intersect at O. If line m bisects $\angle AOB$, what is the value of x?

Ⓐ 25 Ⓑ 35 Ⓒ 45 Ⓓ 50 Ⓔ 60

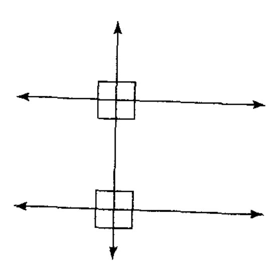


SOLUTION.
$m\angle AOB + 130 = 180 \Rightarrow m\angle AOB = 50$; and since m bisects $\angle AOB$, $x =$ **25 (A)**.

Two lines that never intersect are said to be parallel. Consequently, parallel lines form no angles. However, if a third line, called a ***transversal***, intersects a pair of parallel lines, eight angles are formed, and the relationships among these angles are very important.

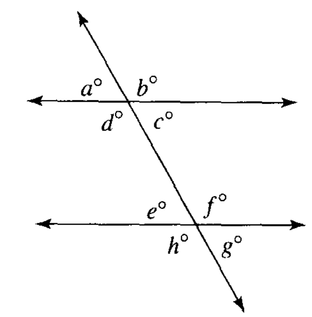
KEY FACT I6

If a pair of parallel lines is cut by a transversal that is perpendicular to the parallel lines, all eight angles are right angles.



KEY FACT I7

If a pair of parallel lines is cut by a transversal that is not perpendicular to the parallel lines,



- **Four of the angles are acute and four are obtuse;**
- **The four acute angles are equal: $a = c = e = g$;**
- **The four obtuse angles are equal: $b = d = f = h$;**
- **The sum of any acute angle and any obtuse angle is 180°: for example, $d + e = 180$, $c + f = 180$, $b + g = 180$, ...**

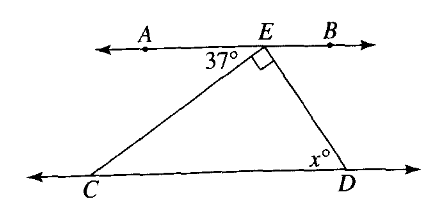
KEY FACT I8

If a pair of lines that are not parallel is cut by a transversal, *none* of the properties listed in KEY FACT I7 is true.

You must know KEY FACT I7 — virtually every GRE has at least one question based on it. However, you do *not* need to know the special terms you learned in high school for these pairs of angles; those terms are not used on the GRE.

EXAMPLE 4

In the figure below, AB is parallel to CD. What is the value of x?



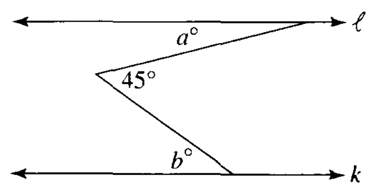
SOLUTION.
Let y be the measure of $\angle BED$. Then by KEY FACT I2:

$$37 + 90 + y = 180 \Rightarrow 127 + y = 180 \Rightarrow y = 53.$$

Since AB is parallel to CD, by KEY FACT I7, $x = y \Rightarrow x =$ **53**.

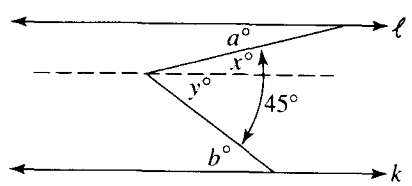
EXAMPLE 5

In the figure below, lines ℓ and k are parallel. What is the value of $a + b$?

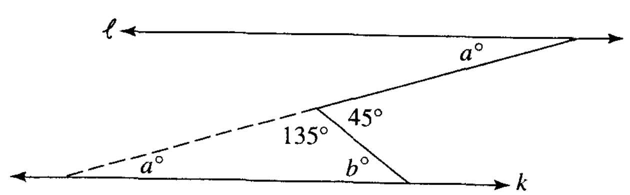


Ⓐ 45 Ⓑ 60 Ⓒ 75 Ⓓ 90 Ⓔ 135

SOLUTION.
It is impossible to determine the value of either a or b. We can, however, find the value of $a + b$. We draw a line through the vertex of the angle parallel to ℓ and k. Then, looking at the top two lines, we see that $a = x$, and looking at the bottom two lines, we see that $b = y$. So, $a + b = x + y =$ **45 (A)**.



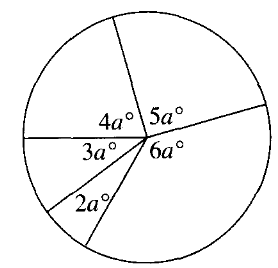
Alternative solution. Draw a different line and use a Key Fact from Section 11-J on triangles. Extend one of the line segments to form a triangle. Since ℓ and k are parallel, the measure of the third angle in the triangle equals a. Now, use the fact that the sum of the measures of the three angles in a triangle is 180° or, even easier, that the given 45° angle is an external angle of the triangle, and so is equal to the sum of a and b.



Practice Exercises — Lines and Angles

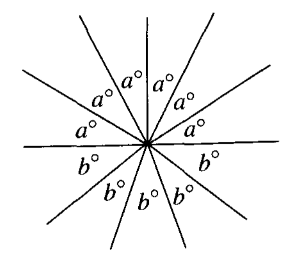
Discrete Quantitative Questions

1. In the figure below, what is the average (arithmetic mean) of the measures of the five angles?



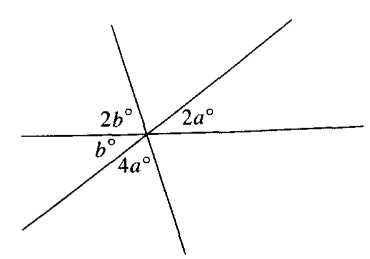
Ⓐ 36
Ⓑ 45
Ⓒ 60
Ⓓ 72
Ⓔ 90

2. In the figure below, what is the value of $\dfrac{b+a}{b-a}$?

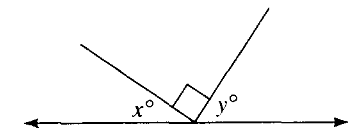


Ⓐ 1
Ⓑ 10
Ⓒ 11
Ⓓ 30
Ⓔ 36

3. In the figure below, what is the value of b?



4. In the figure below, what is the value of x if $y{:}x = 3{:}2$?



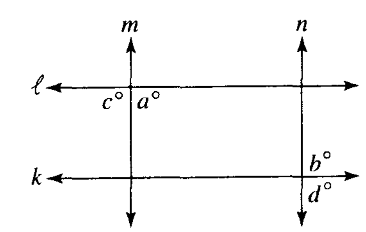
Ⓐ 18
Ⓑ 27
Ⓒ 36
Ⓓ 45
Ⓔ 54

5. What is the measure, in degrees, of the angle formed by the minute and hour hands of a clock at 1:50?

degrees

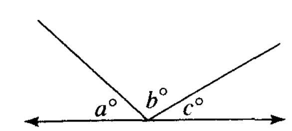
6. Concerning the figure below, if $a = b$, which of the following statements must be true?

Indicate *all* such statements.



A $c = d$
B ℓ and k are parallel
C m and ℓ are perpendicular

7. In the figure below, $a{:}b = 3{:}5$ and $c{:}b = 2{:}1$. What is the measure of the largest angle?

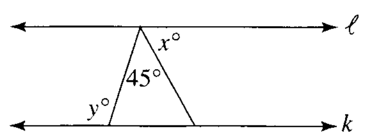


Ⓐ 30
Ⓑ 45
Ⓒ 50
Ⓓ 90
Ⓔ 100

8. A, B, and C are points on a line with B between A and C. Let M and N be the midpoints of AB and BC, respectively. If $AB:BC = 3:1$, what is $MN:BC$?

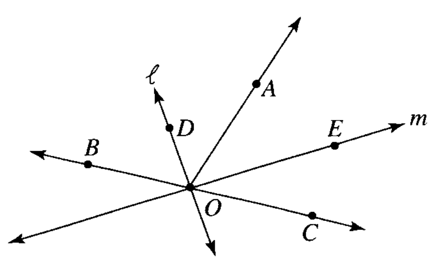
Ⓐ 1:2
Ⓑ 2:3
Ⓒ 1:1
Ⓓ 3:2
Ⓔ 2:1

9. In the figure below, lines k and ℓ are parallel. What is the value of $y - x$?



Ⓐ 15
Ⓑ 30
Ⓒ 45
Ⓓ 60
Ⓔ 75

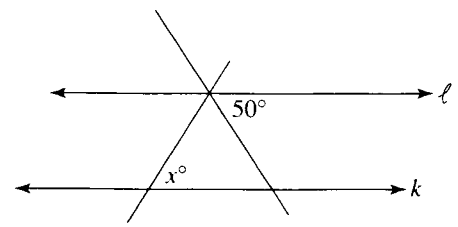
10. In the figure below, line m bisects $\angle AOC$ and line ℓ bisects $\angle AOB$. What is the measure of $\angle DOE$?



Ⓐ 75
Ⓑ 90
Ⓒ 100
Ⓓ 105
Ⓔ 120

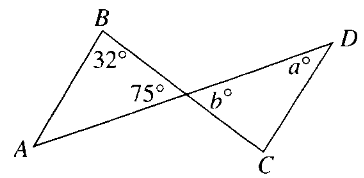
Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.



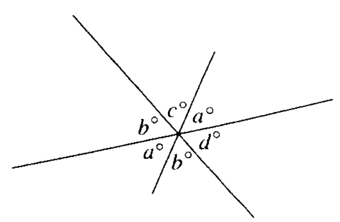
ℓ is parallel to k.

	Quantity A	Quantity B
11.	x	50

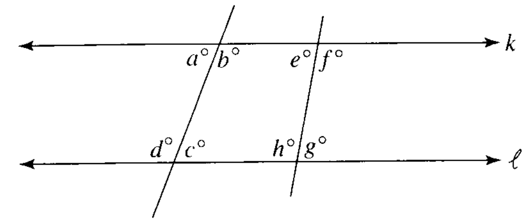


AB is parallel to CD.

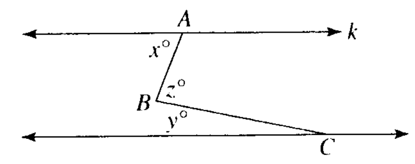
	Quantity A	Quantity B
12.	a	b



	Quantity A	Quantity B
13.	$a + b + c + d$	$2a + 2b$



	Quantity A	Quantity B
14.	$a + b + c + d$	$e + f + g + h$



k and ℓ are parallel.

	Quantity A	Quantity B
15.	z	$x + y$

ANSWER KEY

1. **D**	4. **C**	7. **E**	10. **B**	13. **D**
2. **C**	5. **115**	8. **E**	11. **D**	14. **C**
3. **36**	6. **A**	9. **C**	12. **B**	15. **C**

Answer Explanations

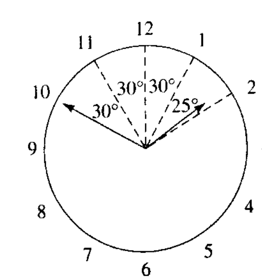
1. **(D)** The markings in the five angles are irrelevant. The sum of the measures of the five angles is 360°, and $360 \div 5 = 72$. If you calculated the measure of each angle you should have gotten 36, 54, 72, 90, and 108; but you would have wasted time.

2. **(C)** From the diagram, we see that $6a = 180$, which implies that $a = 30$, and that $5b = 180$, which implies that $b = 36$. So, $\frac{b+a}{b-a} = \frac{36+30}{36-30} = \frac{66}{6} = 11$.

3. **36** Since vertical angles are equal, the two unmarked angles are $2b$ and $4a$. Since the sum of all six angles is 360°,
$360 = 4a + 2b + 2a + 4a + 2b + b = 10a + 5b$.
However, since vertical angles are equal, $b = 2a \Rightarrow 5b = 10a$. Hence,
$360 = 10a + 5b = 10a + 10a = 20a$, so $a = 18$ and $b = 36$.

4. **(C)** Since $x + y + 90 = 180$, $x + y = 90$. Also, since $y{:}x = 3{:}2$, $y = 3t$ and $x = 2t$. Therefore, $3t + 2t = 90 \Rightarrow 5t = 90$. So $t = 18$, and $x = 2(18) = 36$.

5. **115** For problems such as this, always draw a diagram. The measure of each of the 12 central angles from one number to the next on the clock is 30°. At 1:50 the minute hand is pointing at 10, and the hour hand has gone $\frac{50}{60} = \frac{5}{6}$ of the way from 1 to 2. So from 10 to 1 on the clock is 90°, and from 1 to the hour hand is $\frac{5}{6}(30°) = 25°$, for a total of $90° + 25° = 115°$.



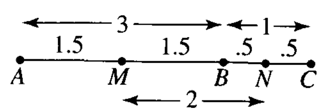
6. **(A)** No conclusions can be made about the lines; they could form any angles whatsoever. (B and C are both false.) Since $a = b$,

$$c = 180 - a = 180 - b = d.$$

(A is true.)

7. **(E)** Since $a{:}b = 3{:}5$, then $a = 3x$ and $b = 5x$. $c{:}b = c{:}5x = 2{:}1 \Rightarrow c = 10x$. Then, $3x + 5x + 10x = 180 \Rightarrow 18x = 180$. So, $x = 10$ and $c = 10x = 100$.

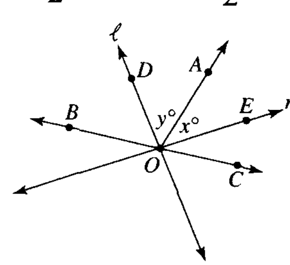
8. **(E)** If a diagram is not provided on a geometry question, draw one on your scrap paper. From the figure below, you can see that $MN{:}BC = 2{:}1$.



9. **(C)** Since the lines are parallel, the angle marked y and the sum of the angles marked x and 45 are equal: $y = x + 45 \Rightarrow y - x = 45$.

10. **(B)** Let $x = \frac{1}{2}m\angle AOC$, and $y = \frac{1}{2}m\angle AOB$.

Then, $x + y = \frac{1}{2}m\angle AOC + \frac{1}{2}m\angle AOB = \frac{1}{2}(180) = 90$.



11. **(D)** No conclusion can be made: x could equal 50 or be more or less.

12. **(B)** Since $m\angle A + 32 + 75 = 180$, $m\angle A = 73$; and since AB is parallel to CD, $a = 73$, whereas, because vertical angles are equal, $b = 75$.

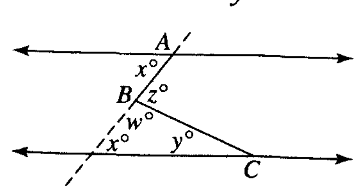
13. **(D)**

	Quantity A	Quantity B
	$a + b + c + d$	$2a + 2b$
Subtract a and b from each quantity:	$c + d$	$a + b$
Since $b = d$, subtract them:	c	a

There is no way to determine whether a is less than, greater than, or equal to c.

14. **(C)** Whether the lines are parallel or not, $a + b = c + d = e + f = g + h = 180$. Each quantity is equal to 360.

15. **(C)** Extend line segment AB to form a transversal. Since $w + z = 180$ and $w + (x + y) = 180$, it follows that $z = x + y$.



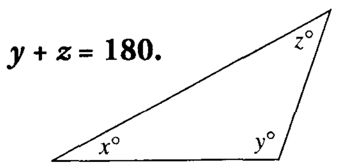
11-J. TRIANGLES

More geometry questions on the GRE pertain to triangles than to any other topic. To answer them, there are several important facts that you need to know about the angles and sides of triangles. The KEY FACTS in this section are extremely useful. Read them carefully, a few times if necessary, and *make sure you learn them all.*

KEY FACT J1

In any triangle, the sum of the measures of the three angles is 180°:

$$x + y + z = 180.$$



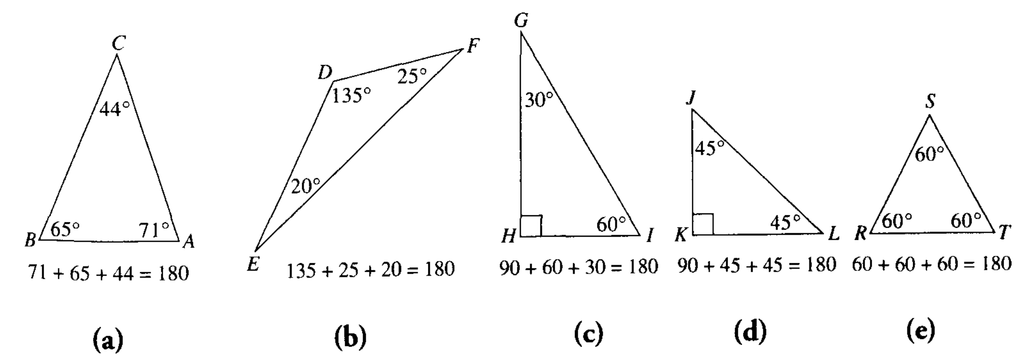
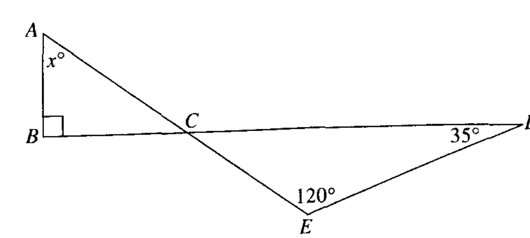


FIGURE 1

Figure 1 (a–e) illustrates KEY FACT J1 for five different triangles, which will be discussed below.

EXAMPLE 1

In the figure below, what is the value of x?



Ⓐ 25 Ⓑ 35 Ⓒ 45 Ⓓ 55 Ⓔ 65

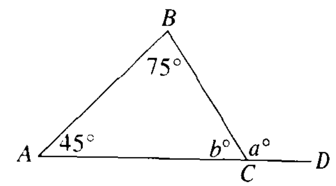
SOLUTION.
Use KEY FACT J1 twice: first, for $\triangle CDE$ and then for $\triangle ABC$.

- $m\angle DCE + 120 + 35 = 180 \Rightarrow m\angle DCE + 155 = 180 \Rightarrow m\angle DCE = 25$.
- Since vertical angles are equal, $m\angle ACB = 25$ (see KEY FACT I6).
- $x + 90 + 25 = 180 \Rightarrow x + 115 = 180$, and so $x =$ **65 (E)**.

EXAMPLE 2

In the figure at the right, what is the value of a?

Ⓐ 45 Ⓑ 60 Ⓒ 75 Ⓓ 120 Ⓔ 135



SOLUTION.
First find the value of b: $180 = 45 + 75 + b = 120 + b \Rightarrow b = 60$.

Then, $a + b = 180 \Rightarrow a = 180 - b = 180 - 60 =$ **120 (D)**.

In Example 2, $\angle BCD$, which is formed by one side of $\triangle ABC$ and the extension of another side, is called an ***exterior angle***. Note that to find a we did not have to first find b; we could have just added the other two angles: $a = 75 + 45 = 120$. This is a useful fact to remember.

KEY FACT J2

The measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles.

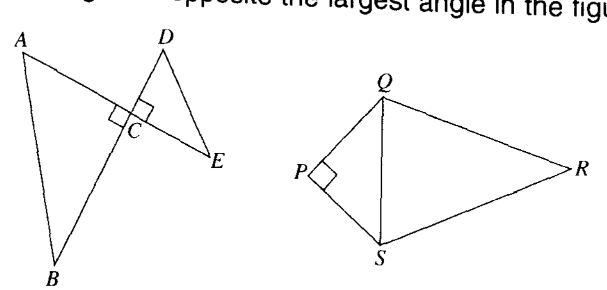
KEY FACT J3

In any triangle:

- **the longest side is opposite the largest angle;**
- **the shortest side is opposite the smallest angle;**
- **sides with the same length are opposite angles with the same measure.**

CAUTION

In KEY FACT J3 the condition "in any triangle" is crucial. If the angles are not in the same triangle, none of the conclusions hold. For example, in the figures below, AB and DE are *not* equal even though they are each opposite a 90° angle, and QS is not the longest side in the figure, even though it is opposite the largest angle in the figure.



Consider triangles ABC, JKL, and RST in Figure 1 on the previous page.

- In $\triangle ABC$: BC is the longest side since it is opposite angle A, the largest angle (71°). Similarly, AB is the shortest side since it is opposite angle C, the smallest angle (44°). So $AB < AC < BC$.
- In $\triangle JKL$: angles J and L have the same measure (45°), so $JK = KL$.
- In $\triangle RST$: since all three angles have the same measure (60°), all three sides have the same length: $RS = ST = TR$.

EXAMPLE 3

Which of the following statements concerning the length of side YZ is true?

Indicate *all* such statements.

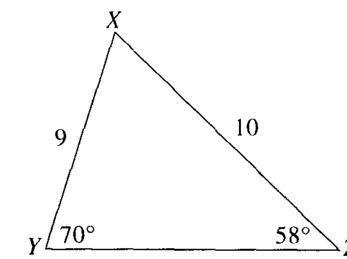
[A] $YZ < 9$

[B] $YZ = 9$

[C] $9 < YZ < 10$

[D] $YZ = 10$

[E] $YZ > 10$



SOLUTION.
Since the five answer choices are mutually exclusive, only one of them can be true.

- By KEY FACT J1, $m\angle X + 70 + 58 = 180 \Rightarrow m\angle X = 52$.
- So, X is the smallest angle.
- Therefore, by KEY FACT J3, YZ is the shortest side. So $\boldsymbol{YZ < 9}$ **(A)**.

Classification of Triangles

Name	Lengths of the Sides	Measures of the Angles	Examples from Figure 1
scalene	all 3 different	all 3 different	*ABC, DEF, GHI*
isosceles	2 the same	2 the same	*JKL*
equilateral	all 3 the same	all 3 the same	*RST*

Acute triangles are triangles such as ABC and RST, in which all three angles are acute. An acute triangle could be scalene, isosceles, or equilateral.

Obtuse triangles are triangles such as DEF, in which one angle is obtuse and two are acute. An obtuse triangle could be scalene or isosceles.

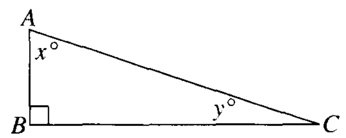
Right triangles are triangles such as GHI and JKL, which have one right angle and two acute ones. A right triangle could be scalene or isosceles. The side opposite the 90° angle is called the ***hypotenuse***, and by KEY FACT J3, it is the longest side. The other two sides are called the ***legs***.

If x and y are the measures of the acute angles of a right triangle, then by KEY FACT J1, $90 + x + y = 180 \Rightarrow x + y = 90$.

KEY FACT J4

In any right triangle, the sum of the measures of the two acute angles is 90°.

EXAMPLE 4



Quantity A	Quantity B
The average of x and y	45

TIP

The Pythagorean theorem is probably the most important theorem you need to know. Be sure to review all of its uses.

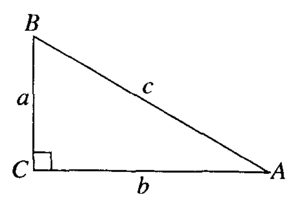
SOLUTION.
Since the diagram indicates that $\triangle ABC$ is a right triangle, then, by KEY FACT J1, $x + y = 90$. So the average of x and $y = \frac{x+y}{2} = \frac{90}{2} = 45$.

The quantities are equal (**C**).

The most important facts concerning right triangles are the **Pythagorean theorem** and its converse, which are given in KEY FACT J5 and repeated as the first line of KEY FACT J6.

KEY FACT J5

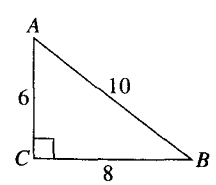
Let a, b, and c be the sides of $\triangle ABC$, with $a \le b \le c$. If $\triangle ABC$ is a right triangle, $a^2 + b^2 = c^2$; and if $a^2 + b^2 = c^2$, then $\triangle ABC$ is a right triangle.



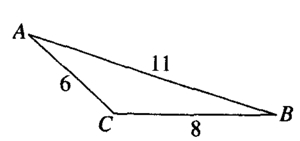
KEY FACT J6

Let a, b, and c be the sides of $\triangle ABC$, with $a \le b \le c$.

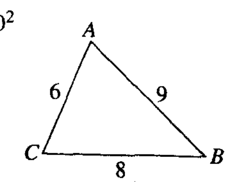
- **$a^2 + b^2 = c^2$ if and only if angle C is a right angle. ($\triangle ABC$ is a right triangle.)**
- **$a^2 + b^2 < c^2$ if and only if angle C is obtuse. ($\triangle ABC$ is an obtuse triangle.)**
- **$a^2 + b^2 > c^2$ if and only if angle C is acute. ($\triangle ABC$ is an acute triangle.)**



$6^2 + 8^2 = 10^2$



$6^2 + 8^2 < 11^2$



$6^2 + 8^2 > 9^2$

EXAMPLE 5

Which of the following triples are *not* the sides of a right triangle?
Indicate *all* such triples.

[A] 3, 4, 5

[B] 1, 1, $\sqrt{3}$

[C] 1, $\sqrt{3}$, 2

[D] $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$

[E] 30, 40, 50

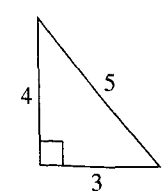
SOLUTION.
Just check each choice.

(A) $3^2 + 4^2 = 9 + 16 = 25 = 5^2$	These *are* the sides of a right triangle.
(B) $1^2 + 1^2 = 1 + 1 = 2 \ne (\sqrt{3})^2$	These *are not* the sides of a right triangle.
(C) $1^2 + (\sqrt{3})^2 = 1 + 3 = 4 = 2^2$	These *are* the sides of a right triangle.
(D) $(\sqrt{3})^2 + (\sqrt{4})^2 = 3 + 4 = 7 \ne (\sqrt{5})^2$	These *are not* the sides of a right triangle.
(E) $30^2 + 40^2 = 900 + 1600 = 2500 = 50^2$	These *are* the sides of a right triangle.

The answer is **B** and **D**.

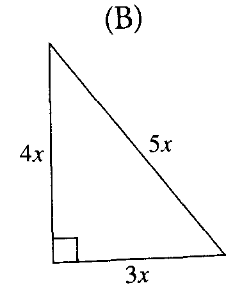
Below are the right triangles that appear most often on the GRE. You should recognize them immediately whenever they come up in questions. Carefully study each one, and memorize KEY FACTS J7–J11.

(A)



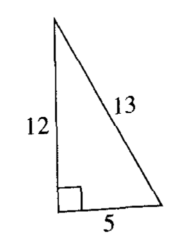
3, 4, 5

(B)



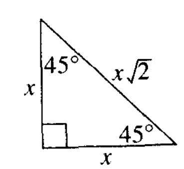
$3x$, $4x$, $5x$

(C)



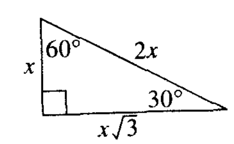
5, 12, 13

(D)



x, x, $x\sqrt{2}$

(E)



x, $x\sqrt{3}$, $2x$

On the GRE, the most common right triangles whose sides are integers are the 3-4-5 triangle (A) and its multiples (B).

KEY FACT J7

For any positive number x, there is a right triangle whose sides are $3x$, $4x$, $5x$.

For example:

$x = 1$	3, 4, 5	$x = 5$	15, 20, 25
$x = 2$	6, 8, 10	$x = 10$	30, 40, 50
$x = 3$	9, 12, 15	$x = 50$	150, 200, 250
$x = 4$	12, 16, 20	$x = 100$	300, 400, 500

TIP

KEY FACT J7 applies even if x is not an integer. For example:

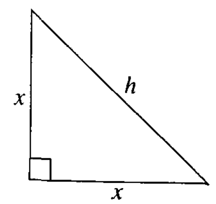
$x = .5$ 1.5, 2, 2.5

$x = \pi$ 3π, 4π, 5π

NOTE: The only other right triangle with integer sides that you should recognize immediately is the one whose sides are 5, 12, 13 (C).

Let x = length of each leg, and h = length of the hypotenuse, of an isosceles right triangle (D). By the Pythagorean theorem (KEY FACT J5), $x^2 + x^2 = h^2$.

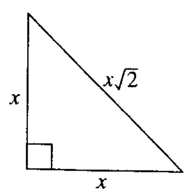
So, $2x^2 = h^2$, and $h = \sqrt{2x^2} = x\sqrt{2}$.

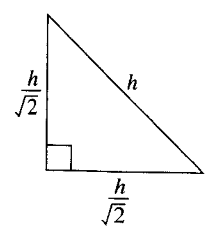


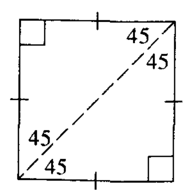
KEY FACT J8

In a 45-45-90 right triangle, the sides are x, x, and $x\sqrt{2}$. So,

- **by multiplying the length of a leg by $\sqrt{2}$, you get the hypotenuse.**
- **by dividing the hypotenuse by $\sqrt{2}$, you get the length of each leg.**







KEY FACT J9

The diagonal of a square divides the square into two isosceles right triangles.

The last important right triangle is the one whose angles measure 30°, 60°, and 90°. (E)

KEY FACT J10

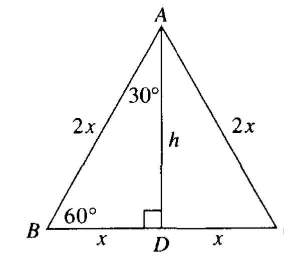
An altitude divides an equilateral triangle into two 30-60-90 right triangles.

Let $2x$ be the length of each side of equilateral $\triangle ABC$ in which altitude AD is drawn. Then $\triangle ABD$ is a 30-60-90 right triangle, and its sides are x, $2x$, and h.

By the Pythagorean theorem,

$x^2 + h^2 = (2x)^2 = 4x^2$.

So $h^2 = 3x^2$, and $h = \sqrt{3x^2} = x\sqrt{3}$.



KEY FACT J11

In a 30-60-90 right triangle the sides are x, $x\sqrt{3}$, and $2x$.

If you know the length of the shorter leg (x),

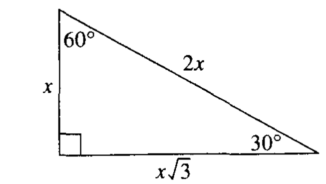
- multiply it by $\sqrt{3}$ to get the longer leg, and
- multiply it by 2 to get the hypotenuse.

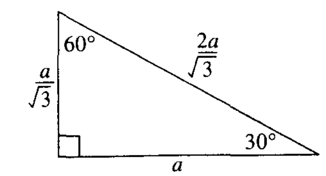
If you know the length of the longer leg (a),

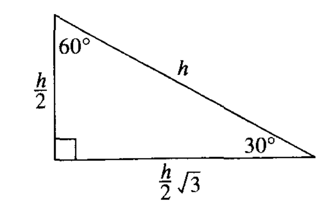
- divide it by $\sqrt{3}$ to get the shorter leg, and
- multiply the shorter leg by 2 to get the hypotenuse.

If you know the length of the hypotenuse (h),

- divide it by 2 to get the shorter leg, and
- multiply the shorter leg by $\sqrt{3}$ to get the longer leg.







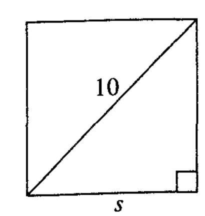
EXAMPLE 6

What is the area of a square whose diagonal is 10?

SOLUTION.

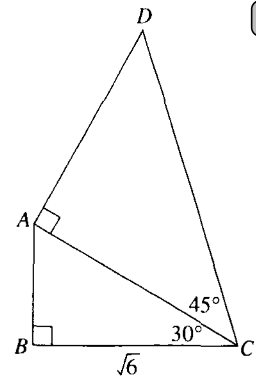
Draw a diagonal in a square of side s, creating a 45-45-90 right triangle. By KEY FACT J8:

$s = \frac{10}{\sqrt{2}}$ and $A = s^2 = \left(\frac{10}{\sqrt{2}}\right)^2 = \frac{100}{2} = \mathbf{50}$.



EXAMPLE 7

In the diagram at the right, if $BC = \sqrt{6}$, what is the value of CD?



Ⓐ $2\sqrt{2}$

Ⓑ $4\sqrt{2}$

Ⓒ $2\sqrt{3}$

Ⓓ $2\sqrt{6}$

Ⓔ 4

SOLUTION.
Since $\triangle ABC$ and $\triangle DAC$ are 30-60-90 and 45-45-90 right triangles, respectively, use KEY FACTS J11 and J8.

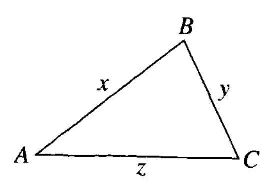
- Divide the longer leg, BC, by $\sqrt{3}$ to get the shorter leg, AB: $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$.
- Multiply AB by 2 to get the hypotenuse: $AC = 2\sqrt{2}$.
- Since AC is also a leg of isosceles right $\triangle DAC$, to get hypotenuse CD, multiply AC by $\sqrt{2}$: $CD = 2\sqrt{2} \times \sqrt{2} = 2 \times 2 = \mathbf{4}$ **(E)**.

Key Fact J12

Triangle Inequality

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

The best way to remember this is to see that $x + y$, the length of the path from A to C through B, is greater than z, the length of the direct path from A to C.



$x + y > z$

NOTE: If you subtract x from each side of $x + y > z$, you see that $z - x < y$.

KEY FACT J13

The difference of the lengths of any two sides of a triangle is less than the length of the third side.

EXAMPLE 8

If the lengths of two of the sides of a triangle are 6 and 7, which of the following could be the length of the third side?

Indicate *all* possible lengths.

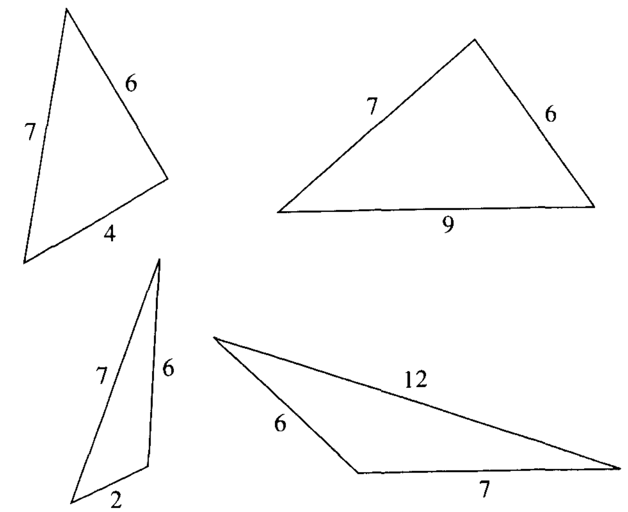
[A] 1	[C] π	[E] 12	[G] 15
[B] 2	[D] 7	[F] 13	

SOLUTION.
Use KEY FACTS J12 and J13.

- The third side must be *less* than 6 + 7 = 13. (Eliminate F and G.)
- The third side must be *greater* than 7 – 6 = 1. (Eliminate A.)
- *Any* number between 1 and 13 could be the length of the third side.

The answer is **B, C, D, E.**

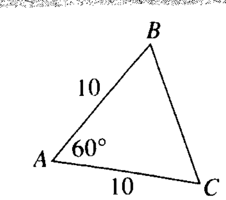
The following diagram illustrates several triangles, two of whose sides have lengths of 6 and 7.



On the GRE, two other terms that appear regularly in triangle problems are ***perimeter*** and ***area*** (see Section 11-K).

EXAMPLE 9

In the figure at the right, what is the perimeter of $\triangle ABC$?



Ⓐ $20 + 10\sqrt{2}$

Ⓑ $20 + 10\sqrt{3}$

Ⓒ 25

Ⓓ 30

Ⓔ 40

SOLUTION.
First, use KEY FACTS J3 and J1 to find the measures of the angles.

- Since $AB = AC$, $m\angle B = m\angle C$. Represent each of them by x.
- Then, $x + x + 60 = 180 \Rightarrow 2x = 120 \Rightarrow x = 60$.
- Since the measure of each angle of $\triangle ABC$ is 60, the triangle is equilateral.
- So $BC = 10$, and the perimeter is $10 + 10 + 10 =$ **30 (D)**.

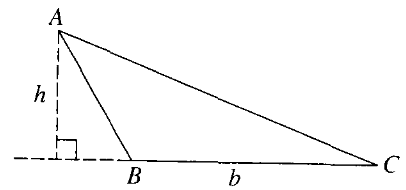
KEY FACT J14

The area of a triangle is given by $A = \frac{1}{2}bh$, where b is the base and h is the height.

NOTE:

1. *Any* side of the triangle can be taken as the ***base.***
2. The ***height*** or ***altitude*** is a line segment drawn to the base or, if necessary, to an extension of the base from the opposite vertex.
3. In a right triangle, either leg can be the base and the other the height.
4. The height may be outside the triangle. [See the figure below.]

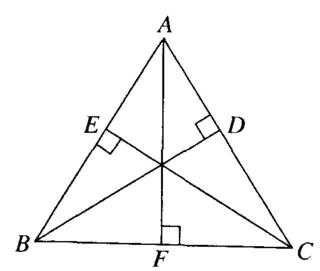
TIP
If one endpoint of the base of a triangle is the vertex of an obtuse angle, then the height drawn to that base will be outside the triangle.



Note: $\triangle ABC$ is obtuse.

In the figure below:

- If AC is the base, BD is the height.
- If AB is the base, CE is the height.
- If BC is the base, AF is the height.

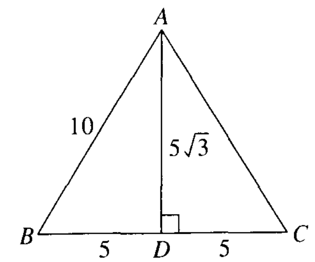


EXAMPLE 10

What is the area of an equilateral triangle whose sides are 10?

Ⓐ 30 Ⓑ $25\sqrt{3}$ Ⓒ 50 Ⓓ $50\sqrt{3}$ Ⓔ 100

SOLUTION.
Draw an equilateral triangle and one of its altitudes.



- By KEY FACT J10, $\triangle ABD$ is a 30-60-90 right triangle.
- By KEY FACT J11, $BD = 5$ and $AD = 5\sqrt{3}$.
- The area of $\triangle ABC = \frac{1}{2}(10)(5\sqrt{3}) =$ **$25\sqrt{3}$ (B)**.

Replacing 10 by s in Example 10 yields a very useful result.

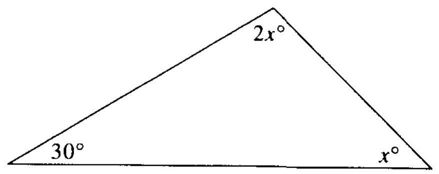
KEY FACT J15

If A represents the area of an equilateral triangle with side s, then $A = \frac{s^2\sqrt{3}}{4}$.

TIP
Learn this formula for the area of an equilateral triangle. It can save you time.

Practice Exercises — Triangles

Discrete Quantitative Questions



1. In the triangle above, what is the value of x?
   (A) 20
   (B) 30
   (C) 40
   (D) 50
   (E) 60

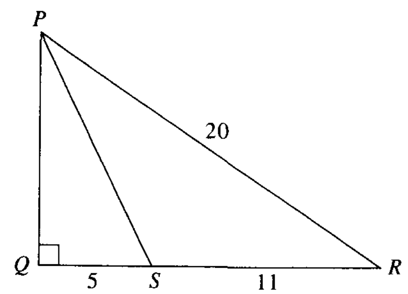
2. If the difference between the measures of the two smaller angles of a right triangle is 8°, what is the measure, in degrees, of the smallest angle?
   (A) 37
   (B) 41
   (C) 42
   (D) 49
   (E) 53

3. What is the area of an equilateral triangle whose altitude is 6?
   (A) 18
   (B) $12\sqrt{3}$
   (C) $18\sqrt{3}$
   (D) 36
   (E) $24\sqrt{3}$

4. Two sides of a right triangle are 12 and 13. Which of the following *could be* the length of the third side?

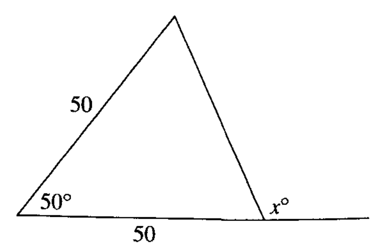
   Indicate *all* possible lengths.

   [A] 2
   [B] 5
   [C] $\sqrt{31}$
   [D] 11
   [E] $\sqrt{313}$

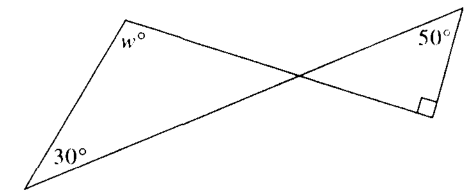


5. What is the value of PS in the triangle above?
   (A) $5\sqrt{2}$
   (B) 10
   (C) 11
   (D) 13
   (E) $12\sqrt{2}$

6. If the measures of the angles of a triangle are in the ratio of 1:2:3, and if the length of the smallest side of the triangle is 10, what is the length of the longest side?
   (A) $10\sqrt{2}$
   (B) $10\sqrt{3}$
   (C) 15
   (D) 20
   (E) 30

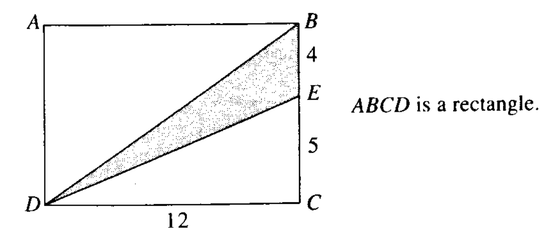


7. What is the value of x in the figure above?



8. In the figure above, what is the value of w?

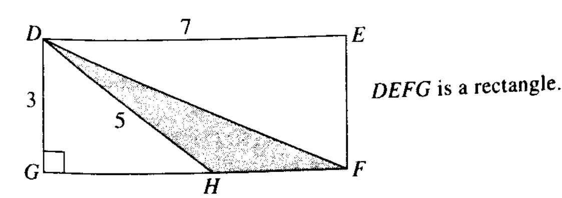
Questions 9–10 refer to the following figure.



9. What is the area of $\triangle BED$?
   (A) 12
   (B) 24
   (C) 36
   (D) 48
   (E) 60

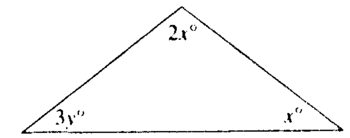
10. What is the perimeter of $\triangle BED$?
   (A) $19 + 5\sqrt{2}$
   (B) 28
   (C) $17 + \sqrt{185}$
   (D) 32
   (E) 36

Questions 11–12 refer to the following figure.



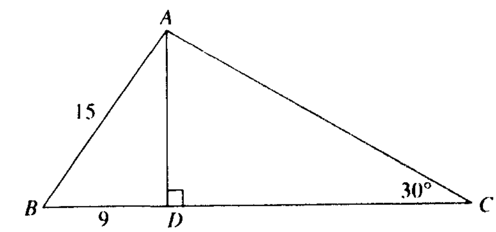
11. What is the area of $\triangle DFH$?
   (A) 3
   (B) 4.5
   (C) 6
   (D) 7.5
   (E) 10

12. What is the perimeter of $\triangle DFH$?
   (A) $8 + \sqrt{41}$
   (B) $8 + \sqrt{58}$
   (C) 16
   (D) 17
   (E) 18



13. Which of the following expresses a true relationship between x and y in the figure above?
   (A) $y = 60 - x$
   (B) $y = x$
   (C) $x + y = 90$
   (D) $y = 180 - 3x$
   (E) $x = 90 - 3y$

Questions 14–15 refer to the following figure.



14. What is the perimeter of $\triangle ABC$?
   (A) 48
   (B) $48 + 12\sqrt{2}$
   (C) $48 + 12\sqrt{3}$
   (D) 60
   (E) $60 + 6\sqrt{3}$

15. What is the area of $\triangle ABC$?
   (A) 108
   (B) $54 + 72\sqrt{2}$
   (C) $54 + 72\sqrt{3}$
   (D) 198
   (E) 216

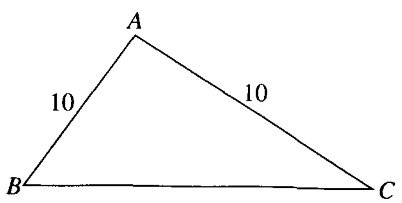
Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

The lengths of two sides of a triangle are 7 and 11.

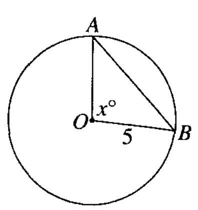
	Quantity A	Quantity B
16.	The length of the third side	4

	Quantity A	Quantity B
17.	The ratio of the length of a diagonal to the length of a side of a square	$\sqrt{2}$



	Quantity A	Quantity B
18.	The perimeter of $\triangle ABC$	30

Questions 19–20 refer to the following figure.



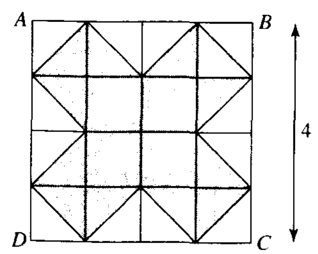
$90 < x$

	Quantity A	Quantity B
19.	The length of AB	7

	Quantity A	Quantity B
20.	The perimeter of $\triangle AOB$	20

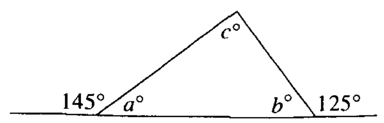
	Quantity A	Quantity B
21.	The area of an equilateral triangle whose sides are 10	The area of an equilateral triangle whose altitude is 10

Questions 22–23 refer to the following figure in which the horizontal and vertical lines divide square $ABCD$ into 16 smaller squares.

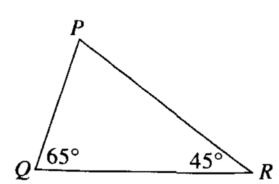


	Quantity A	Quantity B
22.	The perimeter of the shaded region	The perimeter of the square

	Quantity A	Quantity B
23.	The area of the shaded region	The area of the white region



	Quantity A	Quantity B
24.	$a + b$	c



	Quantity A	Quantity B
25.	PR	QR

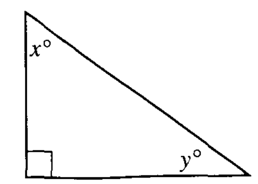
ANSWER KEY

1. **D**	6. **D**	11. **B**	16. **A**	21. **B**
2. **B**	7. **115**	12. **B**	17. **C**	22. **A**
3. **B**	8. **110**	13. **A**	18. **D**	23. **A**
4. **B, E**	9. **B**	14. **C**	19. **A**	24. **C**
5. **D**	10. **D**	15. **C**	20. **B**	25. **B**

Answer Explanations

1. **(D)** $x + 2x + 30 = 180 \Rightarrow 3x + 30 = 180 \Rightarrow 3x = 150 \Rightarrow x = 50.$

2. **(B)** Draw a diagram and label it.

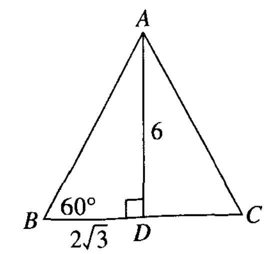


Then write the equations: $x + y = 90$ and $x - y = 8$.

Add the equations:
$$\begin{array}{r} x + y = 90 \\ \underline{+\ x - y = \ \ 8} \\ 2x = 98 \end{array}$$

So $x = 49$ and $y = 90 - 49 = 41$.

3. **(B)** Draw altitude AD in equilateral $\triangle ABC$.



By KEY FACT J11, $BD = \dfrac{6}{\sqrt{3}} = \dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$, and BD is one half the base.

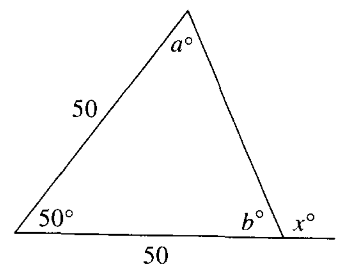
So, the area is $2\sqrt{3} \times 6 = 12\sqrt{3}$.

4. **(B)(E)** If the triangle were not required to be a right triangle, by KEY FACTS J11 and J12 *any* number greater than 1 and less than 25 could be the length of the third side, and the answer would be A, B, C, D, E. But for a right triangle, there are only *two* possibilities:
   - If 13 is the hypotenuse, then the legs are 12 and 5. (B is true.) (If you didn't recognize the 5-12-13 triangle, use the Pythagorean theorem: $12^2 + x^2 = 13^2$, and solve.)
   - If 12 and 13 are the two legs, then use the Pythagorean theorem to find the hypotenuse: $12^2 + 13^2 = c^2 \Rightarrow c^2 = 144 + 169 = 313 \Rightarrow c = \sqrt{313}$. (E is true.)

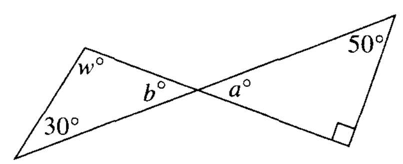
5. **(D)** Use the Pythagorean theorem twice, unless you recognize the common right triangles in this figure (*which you should*). Since $PR = 20$ and $QR = 16$, $\triangle PQR$ is a $3x$-$4x$-$5x$ right triangle with $x = 4$. So $PQ = 12$, and $\triangle PQS$ is a right triangle whose legs are 5 and 12. The hypotenuse, PS, therefore, is 13.
6. **(D)** If the measures of the angles are in the ratio of 1:2:3,

   $x + 2x + 3x = 180 \Rightarrow 6x = 180 \Rightarrow x = 30$.

   So the triangle is a 30-60-90 right triangle, and the sides are a, $2a$, and $a\sqrt{3}$. Since $a = 10$, then $2a$, the length of the longest side, is 20.
7. **115** Label the other angles in the triangle.



   $50 + a + b = 180 \Rightarrow a + b = 130$, and since the triangle is isosceles, $a = b$. Therefore, a and b are each 65, and $x = 180 - 65 = 115$.
8. **110** Here, $50 + 90 + a = 180 \Rightarrow a = 40$, and since vertical angles are equal, $b = 40$. Then, $40 + 30 + w = 180 \Rightarrow w = 110$.

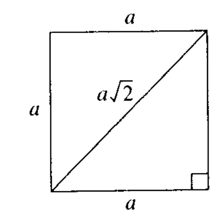


9. **(B)** You *could* calculate the area of the rectangle and subtract the area of the two white right triangles, but you shouldn't. It is easier to solve this problem if you realize that the shaded area is a triangle whose base is 4 and whose height is 12. The area is $\frac{1}{2}(4)(12) = 24$.
10. **(D)** Since both BD and ED are the hypotenuses of right triangles, their lengths can be calculated by the Pythagorean theorem, but these are triangles you should recognize: the sides of $\triangle DCE$ are 5-12-13, and those of $\triangle BAD$ are 9-12-15 ($3x$-$4x$-$5x$, with $x = 3$). So the perimeter of $\triangle BED$ is $4 + 13 + 15 = 32$.
11. **(B)** Since $\triangle DGH$ is a right triangle whose hypotenuse is 5 and one of whose legs is 3, the other leg, GH, is 4. Since $GF = DE$ is 7, HF is 3. Now, $\triangle DFH$ has a base of 3 (HF) and a height of 3 (DG), and its area is $\frac{1}{2}(3)(3) = 4.5$.

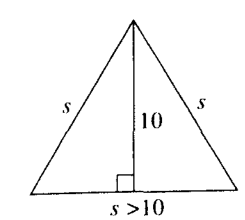
12. **(B)** In $\triangle DFH$, we already have that $DH = 5$ and $HF = 3$; we need only find DF, which is the hypotenuse of $\triangle DEF$. By the Pythagorean theorem,

    $(DF)^2 = 3^2 + 7^2 = 9 + 49 = 58 \Rightarrow DF = \sqrt{58}$.

    So the perimeter is $3 + 5 + \sqrt{58} = 8 + \sqrt{58}$.
13. **(A)** $x + 2x + 3y = 180 \Rightarrow 3x + 3y = 180$. So $x + y = 60$, and $y = 60 - x$.
14. **(C)** $\triangle ABD$ is a right triangle whose hypotenuse is 15 and one of whose legs is 9, so this is a $3x$-$4x$-$5x$ triangle with $x = 3$; so $AD = 12$. Now $\triangle ADC$ is a 30-60-90 triangle, whose shorter leg is 12. Hypotenuse AC is 24, and leg CD is $12\sqrt{3}$. So the perimeter is $24 + 15 + 9 + 12\sqrt{3} = 48 + 12\sqrt{3}$.
15. **(C)** From the solution to 14, we have the base $(9 + 12\sqrt{3})$ and the height (12) of $\triangle ABC$. Then, the area is $\frac{1}{2}(12)(9 + 12\sqrt{3}) = 54 + 72\sqrt{3}$.
16. **(A)** Any side of a triangle must be greater than the difference of the other two sides (KEY FACT J13), so the third side is greater than $11 - 7 = 4$.
17. **(C)** Draw a diagram. A diagonal of a square is the hypotenuse of each of the two 45-45-90 right triangles formed. The ratio of the length of the hypotenuse to the length of the leg in such a triangle is $\sqrt{2}:1$, so the quantities are equal.



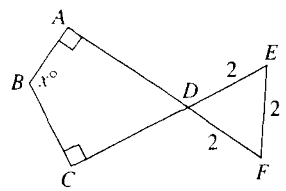
18. **(D)** BC can be any positive number less than 20 (by KEY FACTS J12 and J13, $BC > 10 - 10 = 0$ and $BC < 10 + 10 = 20$). So the perimeter can be *any* number greater than 20 and less than 40.
19. **(A)** Since OA and OB are radii, they are each equal to 5. With no restrictions on x, AB could be any positive number less than 10, and the bigger x is, the bigger AB is. If x were 90, AB would be $5\sqrt{2}$, but we are told that $x > 90$, so $AB > 5\sqrt{2} > 7$.
20. **(B)** Since AB must be less than 10, the perimeter is *less* than 20.
21. **(B)** Don't calculate either area. The length of a side of an equilateral triangle is *greater* than the length of an altitude. So Quantity B is larger since it is the area of a triangle whose sides are greater.



EXAMPLE 1

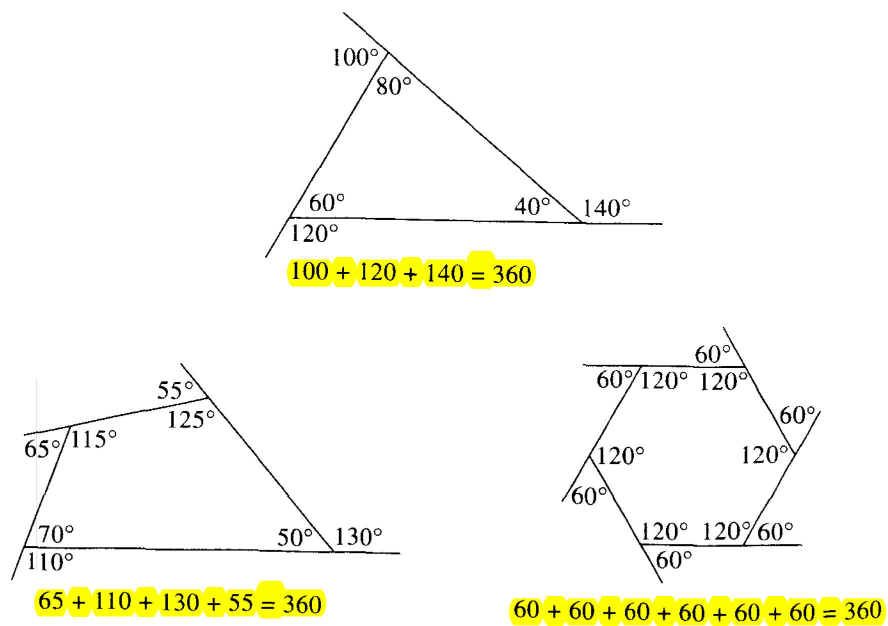
In the figure at the right, what is the value of x?

Ⓐ 60 Ⓑ 90 Ⓒ 100 Ⓓ 120 Ⓔ 150



SOLUTION.
Since $\triangle DEF$ is equilateral, all of its angles measure 60°; also, since the two angles at vertex D are vertical angles, their measures are equal. Therefore, the measure of $\angle D$ in quadrilateral $ABCD$ is 60°. Finally, since the sum of the measures of all four angles of $ABCD$ is 360°, $60 + 90 + 90 + x = 360 \Rightarrow 240 + x = 360 \Rightarrow x =$ **120 (D)**.

In the polygons in the figure that follows, one exterior angle has been drawn at each vertex. Surprisingly, if you add the measures of all of the exterior angles in any of the polygons, the sums are equal.



KEY FACT K3

In any polygon, the sum of the exterior angles, taking one at each vertex, is 360°.

A ***regular polygon*** is a polygon in which all of the sides are the same length and each angle has the same measure. KEY FACT K4 follows immediately from this definition and from KEY FACTS K2 and K3.

KEY FACT K4

In any regular polygon the measure of each interior angle is $\frac{(n-2)\times 180°}{n}$ and the measure of each exterior angle is $\frac{360°}{n}$.

EXAMPLE 2

What is the measure, in degrees, of each interior angle in a regular decagon?

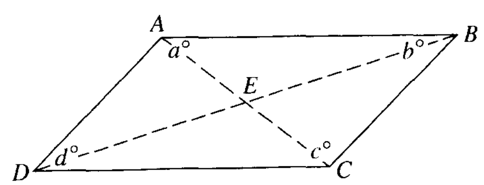
[ ] degrees

SOLUTION 1.
The measure of each of the 10 interior angles is

$$\frac{(10-2)\times 180°}{10} = \frac{8\times 180°}{10} = \frac{1440°}{10} = \mathbf{144°}.$$

SOLUTION 2.
The measure of each of the 10 exterior angles is 36° (360° ÷ 10). Therefore, the measure of each interior angle is 180° − 36° = **144°**.

A ***parallelogram*** is a quadrilateral in which both pairs of opposite sides are parallel.

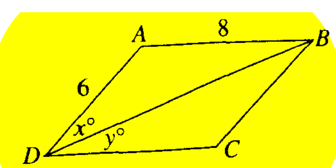


KEY FACT K5

Parallelograms have the following properties:

- **Opposite sides are equal: $AB = CD$ and $AD = BC$.**
- **Opposite angles are equal: $a = c$ and $b = d$.**
- **Consecutive angles add up to 180°: $a + b = 180$, $b + c = 180$, $c + d = 180$, and $a + d = 180$.**
- **The two diagonals bisect each other: $AE = EC$ and $BE = ED$.**
- **A diagonal divides the parallelogram into two triangles that have the exact same size and shape. (The triangles are congruent.)**

EXAMPLE 3

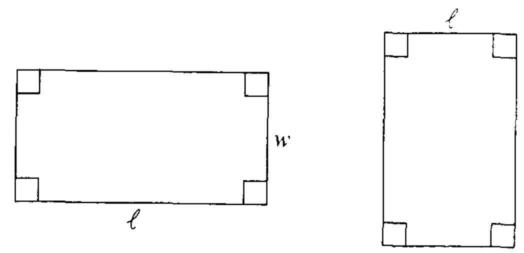


$ABCD$ is a parallelogram.

Quantity A	Quantity B
x	y

SOLUTION.
In $\triangle ABD$ the larger angle is opposite the larger side (KEY FACT J2); so $x > m\angle ABD$. However, since AB and CD are parallel sides cut by transversal BD, $y = m\angle ABD$. Therefore, $x > y$. Quantity **A** is greater.

A ***rectangle*** is a parallelogram in which all four angles are right angles. Two adjacent sides of a rectangle are usually called the ***length*** (ℓ) and the ***width*** (w). Note in the right-hand figure that the length is not necessarily greater than the width.

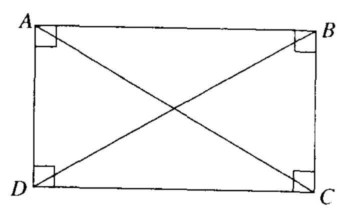


KEY FACT K6

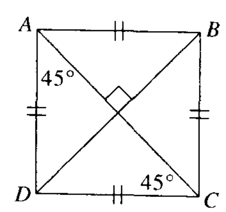
TIP
A rectangle is a parallelogram.

Since a rectangle is a parallelogram, all of the properties listed in KEY FACT K5 hold for rectangles. In addition:

- **The measure of each angle in a rectangle is 90°.**
- **The diagonals of a rectangle have the same length: $AC = BD$.**



A ***square*** is a rectangle in which all four sides have the same length.



KEY FACT K7

TIP
A square is a rectangle and, hence, a parallelogram.

Since a square is a rectangle, all of the properties listed in KEY FACTS K5 and K6 hold for squares. In addition:

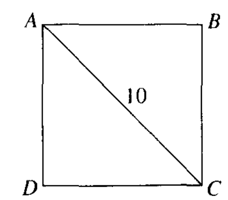
- **All four sides have the same length.**
- **Each diagonal divides the square into two 45-45-90 right triangles.**
- **The diagonals are perpendicular to each other: $AC \perp BD$.**

EXAMPLE 4

What is the length of each side of a square if its diagonals are 10?

Ⓐ 5 Ⓑ 7 Ⓒ $5\sqrt{2}$ Ⓓ $10\sqrt{2}$ Ⓔ $10\sqrt{3}$

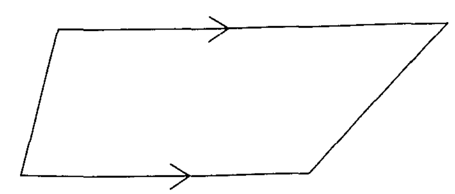
SOLUTION.
Draw a diagram.



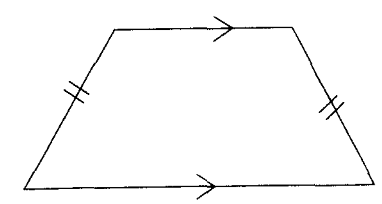
In square $ABCD$, diagonal AC is the hypotenuse of $\triangle ABC$ a 45-45-90 right triangle, and side AB is a leg of that triangle. By KEY FACT J7,

$$AB = \frac{AC}{\sqrt{2}} = \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} \text{ (C).}$$

A ***trapezoid*** is a quadrilateral in which one pair of sides is parallel and the other pair of sides is not parallel. The parallel sides are called the ***bases*** of the trapezoid. The two bases are never equal. In general, the two nonparallel sides are not equal; if they are the trapezoid is called an ***isosceles trapezoid***.

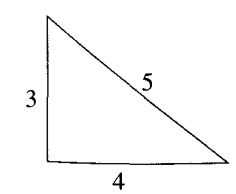


Trapezoid

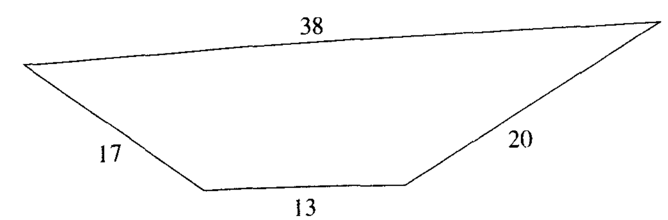


Isoceles trapezoid

The ***perimeter*** (P) of any polygon is the sum of the lengths of all of its sides.



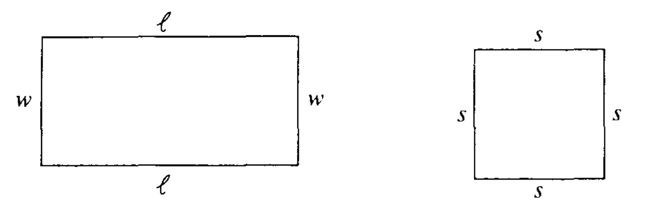
$P = 3 + 4 + 5 = 12$



$P = 13 + 17 + 20 + 38 = 88$

KEY FACT K8

In a rectangle, $P = 2(\ell + w)$; in a square, $P = 4s$.



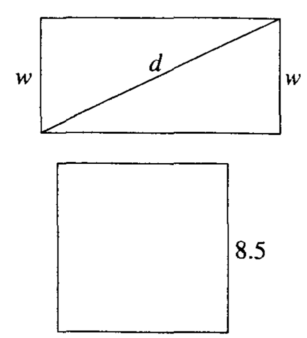
$P = \ell + w + \ell + w = 2(\ell + w)$ $P = s + s + s + s = 4s$

EXAMPLE

The length of a rectangle is 7 more than its width. If the perimeter of the rectangle is the same as the perimeter of a square of side 8.5, what is the length of a diagonal of the rectangle?

SOLUTION.

Don't do anything until you have drawn a diagram.



Since the perimeter of the square $= 4 \times 8.5 = 34$, the perimeter of the rectangle is also 34: $2(\ell + w) = 34 \Rightarrow \ell + w = 17$. Replacing ℓ by $w + 7$, we get:

$$w + 7 + w = 17 \Rightarrow 2w + 7 = 17 \Rightarrow 2w = 10 \Rightarrow w = 5$$

Then $\ell = 5 + 7 = 12$. Finally, realize that the diagonal is the hypotenuse of a 5-12-13 triangle, or use the Pythagorean theorem:

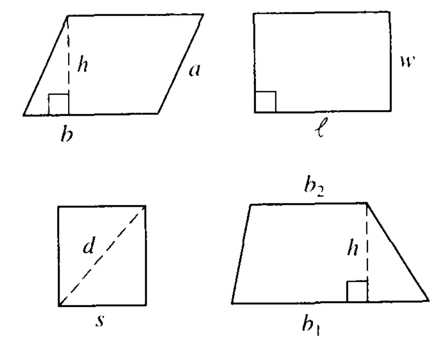
$$d^2 = 5^2 + 12^2 = 25 + 144 = 169 \Rightarrow d = \mathbf{13}.$$

In Section 11-J we reviewed the formula for the area of a triangle. The only other figures for which you need to know area formulas are the parallelogram, rectangle, square, and trapezoid.

KEY FACT K9

Here are the area formulas you need to know:

- **For a parallelogram: $A = bh$.**
- **For a rectangle: $A = \ell w$.**
- **For a square: $A = s^2$ or $A = \frac{1}{2}d^2$.**
- **For a trapezoid: $A = \frac{1}{2}(b_1 + b_2)h$.**

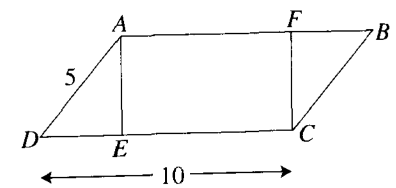


TIP

Be sure to learn the alternative formula for the area of a square: $A = 1/2d^2$, where d is the length of a diagonal.

EXAMPLE 6

In the figure below, the area of parallelogram $ABCD$ is 40. What is the area of rectangle $AFCE$?



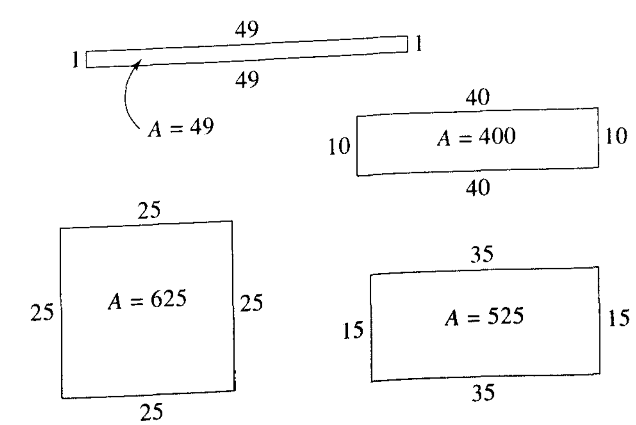
Ⓐ 20 Ⓑ 24 Ⓒ 28 Ⓓ 32 Ⓔ 36

SOLUTION.

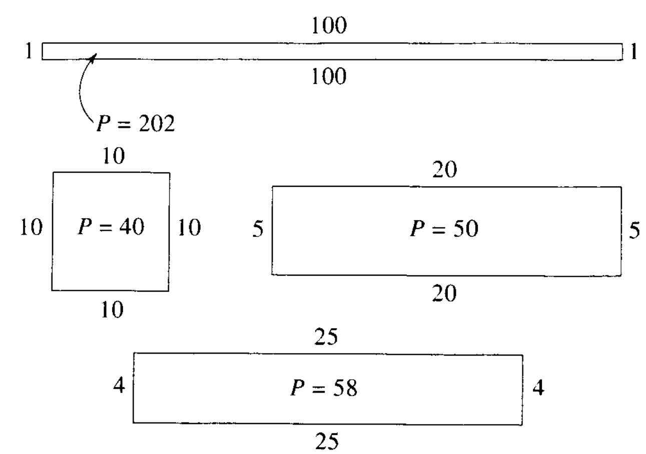
Since the base, CD, is 10 and the area is 40, the height, AE, must be 4. Then $\triangle AED$ must be a 3-4-5 right triangle with $DE = 3$, which implies that $EC = 7$. So the area of the rectangle is $7 \times 4 = \mathbf{28}$ **(C)**.

Two rectangles with the same perimeter can have different areas, and two rectangles with the same area can have different perimeters. These facts are a common source of questions on the GRE.

RECTANGLES WHOSE PERIMETERS ARE 100



RECTANGLES WHOSE AREAS ARE 100



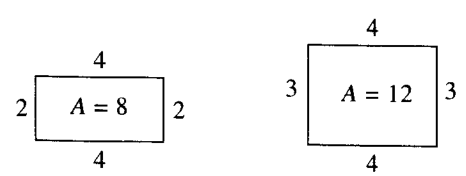
KEY FACT K10

For a given perimeter, the rectangle with the largest area is a square. For a given area, the rectangle with the smallest perimeter is a square.

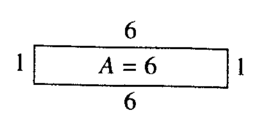
EXAMPLE 7

Quantity A	Quantity B
The area of a rectangle whose perimeter is 12	The area of a rectangle whose perimeter is 14

SOLUTION.
Draw any rectangles whose perimeters are 12 and 14 and compute their areas. As drawn below, Quantity A = 8 and Quantity B = 12.



This time Quantity B is greater. Is it always? Draw a different rectangle whose perimeter is 14.



The one drawn here has an area of 6. Now Quantity B isn't greater. The answer is **D**.

EXAMPLE 8

Quantity A	Quantity B
The area of a rectangle whose perimeter is 12	10

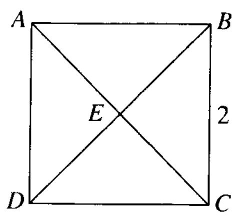
SOLUTION.
There are many rectangles of different areas whose perimeters are 12. But the largest area is 9, when the rectangle is a 3 × 3 square. Quantity **B** is greater.

Practice Exercises — Quadrilaterals

Discrete Quantitative Questions

1. If the length of a rectangle is 4 times its width, and if its area is 144, what is its perimeter?

Questions 2–3 refer to the diagram below in which the diagonals of square *ABCD* intersect at *E*.



2. What is the area of $\triangle DEC$?

Ⓐ $\frac{1}{2}$
Ⓑ 1
Ⓒ $\sqrt{2}$
Ⓓ 2
Ⓔ $2\sqrt{2}$

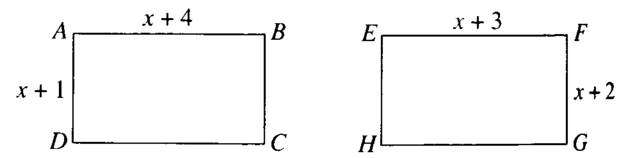
3. What is the perimeter of $\triangle DEC$?

Ⓐ $1 + \sqrt{2}$
Ⓑ $2 + \sqrt{2}$
Ⓒ 4
Ⓓ $2 + 2\sqrt{2}$
Ⓔ 6

4. If the angles of a five-sided polygon are in the ratio of 2:3:3:5:5, what is the measure of the smallest angle?

Ⓐ 20
Ⓑ 40
Ⓒ 60
Ⓓ 80
Ⓔ 90

5. If in the figures below, the area of rectangle *ABCD* is 100, what is the area of rectangle *EFGH*?



Ⓐ 98
Ⓑ 100
Ⓒ 102
Ⓓ 104
Ⓔ 106

Questions 6–7 refer to a rectangle in which the length of each diagonal is 12, and one of the angles formed by the diagonal and a side measures 30°.

6. What is the area of the rectangle?

Ⓐ 18
Ⓑ 72
Ⓒ $18\sqrt{3}$
Ⓓ $36\sqrt{3}$
Ⓔ $36\sqrt{2}$

7. What is the perimeter of the rectangle?

Ⓐ 18
Ⓑ 24
Ⓒ $12 + 12\sqrt{3}$
Ⓓ $18 + 6\sqrt{3}$
Ⓔ $24\sqrt{2}$

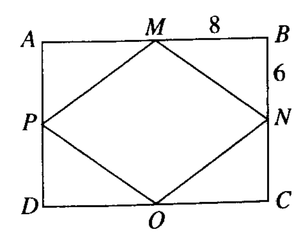
8. How many sides does a polygon have if the measure of each interior angle is 8 times the measure of each exterior angle?

Ⓐ 8
Ⓑ 9
Ⓒ 10
Ⓓ 12
Ⓔ 18

9. The length of a rectangle is 5 more than the side of a square, and the width of the rectangle is 5 less than the side of the square. If the area of the square is 45, what is the area of the rectangle?

Ⓐ 20
Ⓑ 25
Ⓒ 45
Ⓓ 50
Ⓔ 70

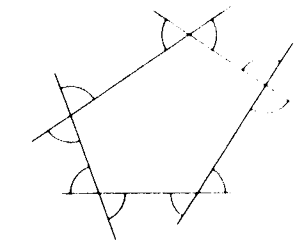
Questions 10–11 refer to the following figure, in which *M*, *N*, *O*, and *P* are the midpoints of the sides of rectangle *ABCD*.



10. What is the perimeter of quadrilateral *MNOP*?

Ⓐ 24
Ⓑ 32
Ⓒ 40
Ⓓ 48
Ⓔ 60

11. What is the area of quadrilateral *MNOP*?



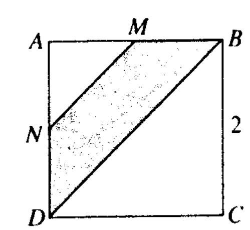
12. In the figure above, what is the sum of the measures of all of the marked angles?

Ⓐ 360
Ⓑ 540
Ⓒ 720
Ⓓ 900
Ⓔ 1080

13. In quadrilateral *WXYZ*, the measure of angle *Z* is 10 more than twice the average of the measures of the other three angles. What is the measure of angle *Z*?

Ⓐ 100
Ⓑ 105
Ⓒ 120
Ⓓ 135
Ⓔ 150

Questions 14–15 refer to the following figure, in which *M* and *N* are the midpoints of two of the sides of square *ABCD*.



14. What is the perimeter of the shaded region?

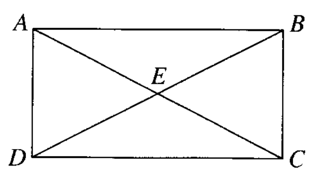
Ⓐ 3
Ⓑ $2 + 3\sqrt{2}$
Ⓒ $3 + 2\sqrt{2}$
Ⓓ 5
Ⓔ 8

15. What is the area of the shaded region?

Ⓐ 1.5
Ⓑ 1.75
Ⓒ 3
Ⓓ $2\sqrt{2}$
Ⓔ $3\sqrt{2}$

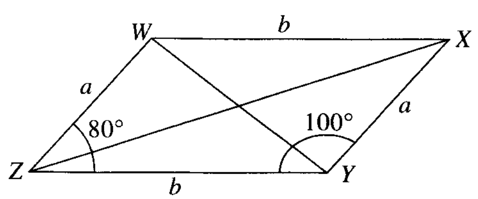
Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.



ABCD is a rectangle.

	Quantity A	Quantity B
16.	The area of $\triangle AED$	The area of $\triangle EDC$

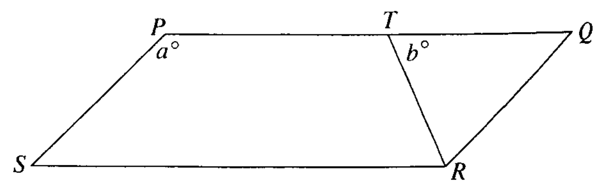


WXYZ is a parallelogram.

	Quantity A	Quantity B
17.	Diagonal *WY*	Diagonal *XZ*

	Quantity A	Quantity B
18.	The perimeter of a 30-60-90 right triangle whose shorter leg is $2x$	The perimeter of an octagon, each of whose sides is x

	Quantity A	Quantity B
19.	The perimeter of a rectangle whose area is 50	28



In parallelogram *PQRS*, *TR* bisects $\angle QRS$.

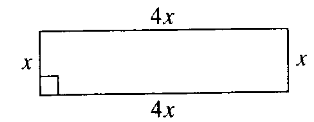
	Quantity A	Quantity B
20.	a	$2b$

ANSWER KEY

1. **60**	5. **C**	9. **A**	13. **150**	17. **B**
2. **B**	6. **D**	10. **C**	14. **B**	18. **A**
3. **D**	7. **C**	11. **96**	15. **A**	19. **A**
4. **C**	8. **E**	12. **C**	16. **C**	20. **C**

Answer Explanations

1. **60** Draw a diagram and label it.



Since the area is 144, then $144 = (4x)(x) = 4x^2 \Rightarrow x^2 = 36 \Rightarrow x = 6$.
So the width is 6, the length is 24, and the perimeter is 60.

2. **(B)** The area of the square is $2^2 = 4$, and each triangle is one-fourth of the square. So the area of $\triangle DEC$ is 1.

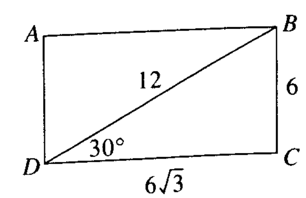
3. **(D)** $\triangle DEC$ is a 45-45-90 right triangle whose hypotenuse, *DC*, is 2.

Therefore, each of the legs is $\frac{2}{\sqrt{2}} = \sqrt{2}$. So the perimeter is $2 + 2\sqrt{2}$.

4. **(C)** The sum of the angles of a five-sided polygon is $(5 - 2) \times 180 = 3 \times 180 = 540$. Therefore, $540 = 2x + 3x + 3x + 5x + 5x = 18x$.
So, $x = 540 \div 18 = 30$.
The measure of the smallest angle is $2x = 2 \times 30 = 60$.

5. **(C)** The area of rectangle *ABCD* is $(x + 1)(x + 4) = x^2 + 5x + 4$.
The area of rectangle *EFGH* is $(x + 2)(x + 3) = x^2 + 5x + 6$, which is exactly 2 more than the area of rectangle *ABCD*: $100 + 2 = 102$.

6. **(D)** Draw a picture and label it.



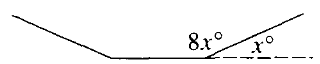
Since $\triangle BCD$ is a 30-60-90 right triangle, *BC* is 6 (half the hypotenuse) and *CD* is $6\sqrt{3}$.

So the area is $\ell w = 6(6\sqrt{3}) = 36\sqrt{3}$.

7. **(C)** The perimeter of the rectangle is $2(\ell + w) = 2(6 + 6\sqrt{3}) = 12 + 12\sqrt{3}$.

8. **(E)** The sum of the degree measures of an interior and exterior angle is 180, so $180 = 8x + x = 9x \Rightarrow x = 20$.

   Since the sum of the measures of all the exterior angles is 360, there are $360 \div 20 = 18$ angles and 18 sides.



9. **(A)** Let x represent the side of the square. Then the dimensions of the rectangle are $(x + 5)$ and $(x - 5)$, and its area is $(x + 5)(x - 5) = x^2 - 25$. Since the area of the square is 45, $x^2 = 45 \Rightarrow x^2 - 25 = 45 - 25 = 20$.

10. **(C)** Each triangle surrounding quadrilateral $MNOP$ is a 6-8-10 right triangle. So each side of $MNOP$ is 10, and its perimeter is 40.

11. **96** The area of each of the triangles is $\frac{1}{2}(6)(8) = 24$, so together the four triangles have an area of 96. The area of the rectangle is $16 \times 12 = 192$. Therefore, the area of quadrilateral $MNOP$ is $192 - 96 = 96$.
    Note: Joining the midpoints of the four sides of *any* quadrilateral creates a parallelogram whose area is one-half the area of the original quadrilateral.

12. **(C)** Each of the 10 marked angles is an exterior angle of the pentagon. If we take one angle at each vertex, the sum of those five angles is 360; the sum of the other five is also 360: $360 + 360 = 720$.

13. **150** Let W, X, Y, and Z represent the measures of the four angles. Since $W + X + Y + Z = 360$, $W + X + Y = 360 - Z$. Also,

$$Z = 10 + 2\left(\frac{W + X + Y}{3}\right) = 10 + 2\left(\frac{360 - Z}{3}\right).$$

$$\text{So } Z = 10 + \frac{2}{3}(360) - \frac{2}{3}Z = 10 + 240 - \frac{2}{3}Z \Rightarrow \frac{5}{3}Z = 250 \Rightarrow Z = 150.$$

14. **(B)** Since M and N are midpoints of sides of length 2, AM, MB, AN, and ND are all 1. $MN = \sqrt{2}$, since it's the hypotenuse of an isosceles right triangle whose legs are 1; and $BD = 2\sqrt{2}$, since it's the hypotenuse of an isosceles right triangle whose legs are 2. So the perimeter of the shaded region is $1 + \sqrt{2} + 1 + 2\sqrt{2} = 2 + 3\sqrt{2}$.

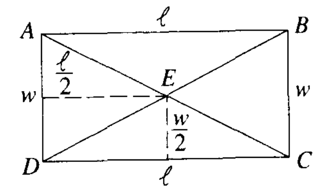
15. **(A)** The area of $\triangle ABD = \frac{1}{2}(2)(2) = 2$, and the area of $\triangle AMN$ is $\frac{1}{2}(1)(1) = 0.5$.

    So the area of the shaded region is $2 - 0.5 = 1.5$.

16. **(C)** The area of $\triangle AED$ is $\frac{1}{2}w\left(\frac{\ell}{2}\right) = \frac{\ell w}{4}$.

    The area of $\triangle EDC$ is $\frac{1}{2}\ell\left(\frac{w}{2}\right) = \frac{\ell w}{4}$.

    Note: Each of the four small triangles has the same area.

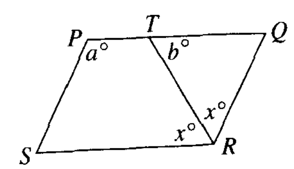


17. **(B)** By KEY FACT J5, since $\angle Z$ is acute and $\angle Y$ is obtuse, $(WY)^2 < a^2 + b^2$, whereas $(XZ)^2 > a^2 + b^2$.

18. **(A)** Since an octagon has eight sides, Quantity B is $8x$.
    Quantity A: By KEY FACT J10, the hypotenuse of the triangle is $4x$, and the longer leg is $2x\sqrt{3}$. So the perimeter is $2x + 4x + 2x\sqrt{3}$. Since $\sqrt{3} > 1$, then $2x + 4x + 2x\sqrt{3} > 2x + 4x + 2x = 8x$.

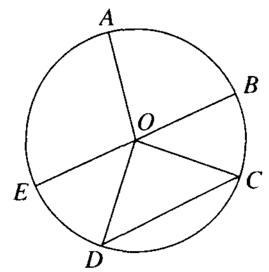
19. **(A)** The perimeter of a rectangle of area 50 can be as large as we like, but the least it can be is when the rectangle is a square. In that case, each side is $\sqrt{50}$, which is greater than 7, and so the perimeter is greater than 28.

20. **(C)** TR is a transversal cutting the parallel sides PQ and RS. So $b = x$ and $2b = 2x$. But since the opposite angles of a parallelogram are equal, $a = 2x$. So $a = 2b$.



11-L. CIRCLES

A ***circle*** consists of all the points that are the same distance from one fixed point called the ***center***. That distance is called the ***radius*** of the circle. The figure below is a circle of radius 1 unit whose center is at the point O. A, B, C, D, and E, which are each 1 unit from O, are all points on circle O. The word *radius* is also used to represent any of the line segments joining the center and a point on the circle. The plural of ***radius*** is ***radii***. In circle O, below, OA, OB, OC, OD, and OE are all radii. If a circle has radius r, each of the radii is r units long.

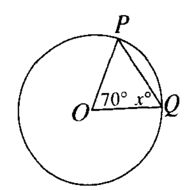


KEY FACT L1

Any triangle, such as $\triangle COD$ in the figure above, formed by connecting the endpoints of two radii, is isosceles.

EXAMPLE 1

If P and Q are points on circle O, what is the value of x?



SOLUTION.

Since $\triangle POQ$ is isosceles, angles P and Q have the same measure. Then, $70 + x + x = 180 \Rightarrow 2x = 110 \Rightarrow x = 55$.

A line segment, such as CD in circle O at the beginning of this section, both of whose endpoints are on a circle is called a ***chord***. A chord such as BE, which passes through the center of the circle, is called a ***diameter***. Since BE is the sum of two radii, OB and OE, it is twice as long as a radius.

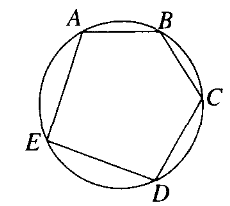
KEY FACT L2

If d is the diameter and r the radius of a circle, $d = 2r$.

KEY FACT L3

A diameter is the longest chord that can be drawn in a circle.

EXAMPLE 2



The radius of the circle is 0.1.

Quantity A	Quantity B
$AB + BC + CD + DE + EA$	1

SOLUTION.

Since the radius of the circle is 0.1, the diameter is 0.2. Therefore, the length of each of the five sides of pentagon $ABCDE$ is less than 0.2, and the sum of their lengths is less than $5 \times 0.2 = 1$. The answer is **B**.

The total length around a circle, from A to B to C to D to E and back to A, is called the ***circumference*** of the circle. In every circle the ratio of the circumference to the diameter is exactly the same and is denoted by the symbol π (the Greek letter "pi").

KEY FACT L4

- $\pi = \dfrac{\text{circumference}}{\text{diameter}} = \dfrac{C}{d}$
- $C = \pi d$
- $C = 2\pi r$

KEY FACT L5

The value of π is approximately 3.14.

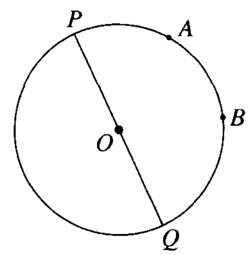
On GRE questions that involve circles, you are almost always expected to leave your answer in terms of π. So *don't* multiply by 3.14 until the final step, and then only if you have to. If you are ever stuck on a problem whose answers involve π, use your calculator to evaluate the answer or to test the answers. For example, assume that you think that an answer is about 50, and the answer choices are 4π, 6π, 12π, 16π, and 24π. Since π is slightly greater than 3, these choices are a little greater than 12, 18, 36, 48, and 72. The answer must be 16π. (To the nearest hundredth, 16π is actually 50.27, but approximating it by 48 was close enough.)

EXAMPLE 3

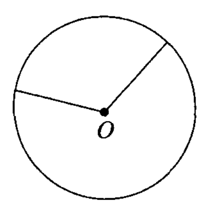
Quantity A	Quantity B
The circumference of a circle whose diameter is 12	The perimeter of a square whose side is 12

SOLUTION.
Quantity A: $C = \pi d = \pi(12)$. Quantity B: $P = 4s = 4(12)$.
Since $4 > \pi$, Quantity **B** is greater. (Note: $12\pi = 12(3.14) = 37.68$, but *you should not have wasted any time calculating this.*)

An ***arc*** consists of two points on a circle and all the points between them. On the GRE, *arc AB* always refers to the smaller arc joining *A* and *B*.

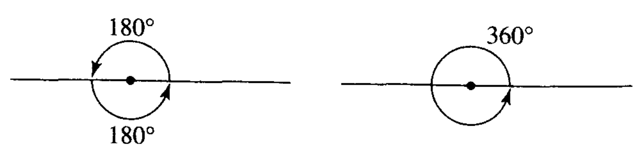


If we wanted to refer to the large arc going from *A* to *B* through *P* and *Q*, we would refer to it as arc *APB* or arc *AQB*. If two points, such as *P* and *Q* in circle *O*, are the endpoints of a diameter, they divide the circle into two arcs called ***semicircles***.
An angle whose vertex is at the center of a circle is called a ***central angle***.



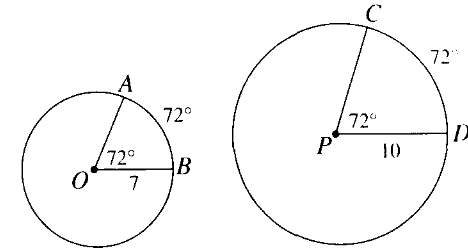
KEY FACT L6

The degree measure of a complete circle is 360°.



KEY FACT L7

The degree measure of an arc equals the degree measure of the central angle that intercepts it.



CAUTION

Degree measure is *not* a measure of length. In the circles above, arc *AB* and arc *CD* each measure 72°, even though arc *CD* is much longer.

How long *is* arc *CD*? Since the radius of Circle *P* is 10, its diameter is 20, and its circumference is 20π. Since there are 360° in a circle, arc *CD* is $\frac{72}{360}$, or $\frac{1}{5}$, of the circumference: $\frac{1}{5}(20\pi) = 4\pi$.

KEY FACT L8

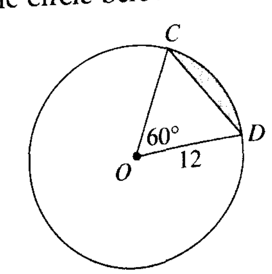
The formula for the area of a circle of radius r is $A = \pi r^2$.

The area of Circle *P*, below KEY FACT L7, is $\pi(10)^2 = 100\pi$ square units. The area of sector *CPD* is $\frac{1}{5}$ of the area of the circle: $\frac{1}{5}(100\pi) = 20\pi$.

KEY FACT L9

If an arc measures $x°$, the length of the arc is $\frac{x}{360}(2\pi r)$, and the area of the sector formed by the arc and 2 radii is $\frac{x}{360}(\pi r^2)$.

Examples 4 and 5 refer to the circle below.



EXAMPLE 4

What is the area of the shaded region?

Ⓐ $144\pi - 144\sqrt{3}$

Ⓑ $144\pi - 36\sqrt{3}$

Ⓒ $144 - 72\sqrt{3}$

Ⓓ $24\pi - 36\sqrt{3}$

Ⓔ $24\pi - 72\sqrt{3}$

SOLUTION.
The area of the shaded region is equal to the area of sector COD minus the area of $\triangle COD$. The area of the circle is $\pi(12)^2 = 144\pi$.

- Since $\frac{60}{360} = \frac{1}{6}$, the area of sector COD is $\frac{1}{6}(144\pi) = 24\pi$.
- Since $m\angle O = 60°$, $m\angle C + \angle D = 120°$ and since $\triangle COD$ is isosceles, $m\angle C = m\angle D$. So, they each measure 60°, and the triangle is equilateral.
- By KEY FACT J15, area of $\triangle COD = \frac{12^2\sqrt{3}}{4} = \frac{144\sqrt{3}}{4} = 36\sqrt{3}$. So the area of the shaded region is $\mathbf{24\pi - 36\sqrt{3}}$ **(D)**.

EXAMPLE 5

What is the perimeter of the shaded region?

Ⓐ $12 + 4\pi$

Ⓑ $12 + 12\pi$

Ⓒ $12 + 24\pi$

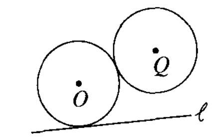
Ⓓ $12\sqrt{2} + 4\pi$

Ⓔ $12\sqrt{2} + 24\pi$

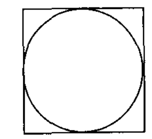
SOLUTION.
Since $\triangle COD$ is equilateral, $CD = 12$. Since the circumference of the circle is $2\pi(12) = 24\pi$, arc $CD = \frac{1}{6}(24\pi) = 4\pi$. So the perimeter is $\mathbf{12 + 4\pi}$ **(A)**.

Suppose that in Example 5 you see that $CD = 12$, but you don't remember how to find the length of arc CD. From the diagram, it is clear that it is slightly longer than CD, say 13. So you know that the perimeter is *about* 25. Now, mentally, using 3 for π, or with your calculator, using 3.14 for π, approximate the value of each of the choices and see which one is closest to 25. Only Choice A is even close.

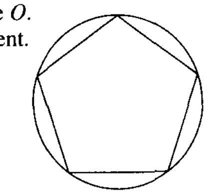
A line and a circle or two circles are ***tangent*** if they have only one point of intersection. A circle is ***inscribed*** in a triangle or square if it is tangent to each side. A polygon is ***inscribed*** in a circle if each vertex is on the circle.



Line ℓ is tangent to circle O.
Circles O and Q are tangent.



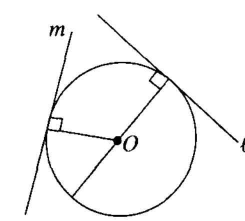
The circle is inscribed in the square.



The pentagon is inscribed in the circle.

KEY FACT L10

If a line is tangent to a circle, a radius (or diameter) drawn to the point where the tangent touches the circle is perpendicular to the tangent line.



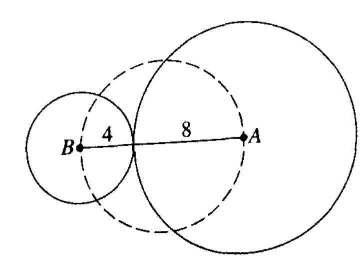
Lines ℓ and m are tangent to circle O.

EXAMPLE 6

A is the center of a circle whose radius is 8, and B is the center of a circle whose diameter is 8. If these two circles are tangent to one another, what is the area of the circle whose diameter is AB?

Ⓐ 12π Ⓑ 36π Ⓒ 64π Ⓓ 144π Ⓔ 256π

SOLUTION.
Draw a diagram.



Since the diameter, AB, of the dotted circle is 12, its radius is 6, and its area is $\pi(6)^2 = \mathbf{36\pi}$ **(B)**.

Practice Exercises — Circles

Discrete Quantitative Questions

1. What is the circumference of a circle whose area is 100π?

Ⓐ 10
Ⓑ 20
Ⓒ 10π
Ⓓ 20π
Ⓔ 25π

2. What is the area of a circle whose circumference is π?

Ⓐ $\frac{\pi}{4}$
Ⓑ $\frac{\pi}{2}$
Ⓒ π
Ⓓ 2π
Ⓔ 4π

3. What is the area of a circle that is inscribed in a square of area 2?

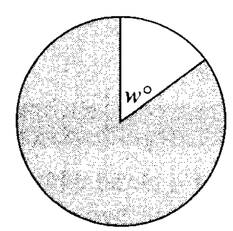
Ⓐ $\frac{\pi}{4}$
Ⓑ $\frac{\pi}{2}$
Ⓒ π
Ⓓ $\pi\sqrt{2}$
Ⓔ 2π

4. A square of area 2 is inscribed in a circle. What is the area of the circle?

Ⓐ $\frac{\pi}{4}$
Ⓑ $\frac{\pi}{2}$
Ⓒ π
Ⓓ $\pi\sqrt{2}$
Ⓔ 2π

5. A 5×12 rectangle is inscribed in a circle. What is the radius of the circle?

6. If, in the figure below, the area of the shaded sector is 85% of the area of the entire circle, what is the value of w?

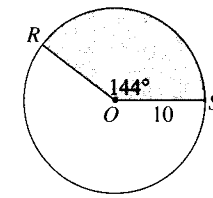


Ⓐ 15
Ⓑ 30
Ⓒ 45
Ⓓ 54
Ⓔ 60

7. The circumference of a circle is $a\pi$ units, and the area of the circle is $b\pi$ square units. If $a = b$, what is the radius of the circle?

Ⓐ 1
Ⓑ 2
Ⓒ 3
Ⓓ π
Ⓔ 2π

Questions 8–9 refer to the following figure.

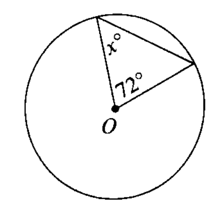


8. What is the length of arc RS?

Ⓐ 8
Ⓑ 20
Ⓒ 8π
Ⓓ 20π
Ⓔ 40π

9. What is the area of the shaded sector?

Ⓐ 8
Ⓑ 20
Ⓒ 8π
Ⓓ 20π
Ⓔ 40π



10. In the figure above, what is the value of x?

11. If A is the area and C the circumference of a circle, which of the following is an expression for A in terms of C?

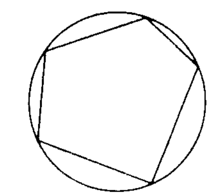
Ⓐ $\frac{C^2}{4\pi}$
Ⓑ $\frac{C^2}{4\pi^2}$
Ⓒ $2C$
Ⓓ $2C^2\sqrt{\pi}$
Ⓔ $\frac{C^2\sqrt{\pi}}{4}$

12. What is the area of a circle whose radius is the diagonal of a square whose area is 4?

Ⓐ 2π
Ⓑ $2\pi\sqrt{2}$
Ⓒ 4π
Ⓓ 8π
Ⓔ 16π

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.



	Quantity A	Quantity B
13.	The perimeter of the pentagon	The circumference of the circle

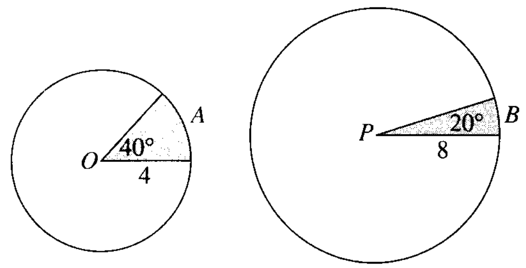
The circumference of a circle is C inches. The area of the same circle is A square inches.

	Quantity A	Quantity B
14.	$\frac{C}{A}$	$\frac{A}{C}$

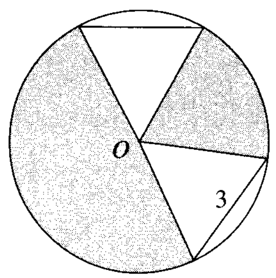
	Quantity A	Quantity B
15.	The area of a circle of radius 2	The area of a semicircle of radius 3

C is the circumference of a circle of radius r

	Quantity A	Quantity B
16.	$\frac{C}{r}$	6



	Quantity A	Quantity B
17.	The area of sector A	The area of sector B



Each of the triangles is equilateral.

	Quantity A	Quantity B
18.	The area of the shaded region	6π

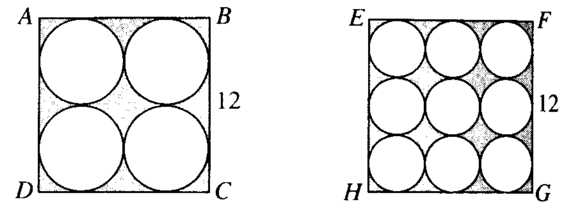


Figure 1 Figure 2

$ABCD$ and $EFGH$ are squares, and all the circles are tangent to one another and to the sides of the squares.

	Quantity A	Quantity B
19.	The area of the shaded region in Figure 1	The area of the shaded region in Figure 2

A square and a circle have equal areas.

	Quantity A	Quantity B
20.	The perimeter of the square	The circumference of the circle

ANSWER KEY

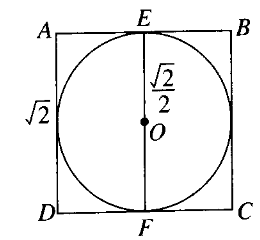
1. **D**	5. **6.5**	9. **E**	13. **B**	17. **B**
2. **A**	6. **D**	10. **54**	14. **D**	18. **C**
3. **B**	7. **B**	11. **A**	15. **B**	19. **C**
4. **C**	8. **C**	12. **D**	16. **A**	20. **A**

Answer Explanations

1. **(D)** $A = \pi r^2 = 100\pi \Rightarrow r^2 = 100 \Rightarrow r = 10 \Rightarrow C = 2\pi r = 2\pi(10) = 20\pi$.

2. **(A)** $C = 2\pi r = \pi \Rightarrow 2r = 1 \Rightarrow r = \frac{1}{2} \Rightarrow A = \pi r^2 = \pi\left(\frac{1}{2}\right)^2 = \frac{1}{4}\pi = \frac{\pi}{4}$.

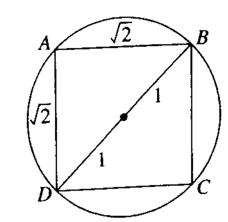
3. **(B)** Draw a diagram.



Since the area of square $ABCD$ is 2, $AD = \sqrt{2}$.

Then diameter $EF = \sqrt{2}$ and radius $OE = \frac{\sqrt{2}}{2}$. Then the area of the circle is $\pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4}\pi = \frac{\pi}{2}$

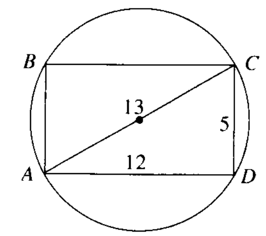
4. **(C)** Draw a diagram.



Since the area of square $ABCD$ is 2, $AD = \sqrt{2}$. Then, since $\triangle ABD$ is an isosceles right triangle, diagonal $BD = \sqrt{2} \times \sqrt{2} = 2$.

But BD is also a diameter of the circle. So the diameter is 2 and the radius is 1. Therefore, the area is $\pi(1)^2 = \pi$.

5. **6.5** Draw a diagram.



By the Pythagorean theorem (or by recognizing a 5-12-13 triangle), we see that diagonal AC is 13. But AC is also a diameter of the circle, so the diameter is 13 and the radius is 6.5.

6. **(D)** Since the shaded area is 85% of the circle, the white area is 15% of the circle. So, w is 15% of 360°: $0.15 \times 360 = 54$.

7. **(B)** Since $C = a\pi = b\pi = A$, we have $2\pi r = \pi r^2 \Rightarrow 2r = r^2 \Rightarrow r = 2$.

8. **(C)** The length of arc RS is $\frac{144}{360}$ of the circumference:

$$\left(\frac{144}{360}\right)2\pi(10) = \left(\frac{2}{5}\right)20\pi = 8\pi.$$

9. **(E)** The area of the shaded sector is $\left(\frac{144}{360}\right)$ of the area of the circle:

$$\left(\frac{144}{360}\right)\pi(10)^2 = \left(\frac{2}{5}\right)100\pi = 40\pi.$$

10. **54** Since two of the sides are radii of the circles, the triangle is isosceles. So the unmarked angle is also x:

$$180 = 72 + 2x \Rightarrow 2x = 108 \Rightarrow x = 54.$$

11. **(A)** $C = 2\pi r \Rightarrow r = \frac{C}{2\pi} \Rightarrow A = \pi\left(\frac{C}{2\pi}\right)^2 = \pi\left(\frac{C^2}{4\pi^2}\right) = \frac{C^2}{4\pi}$

12. **(D)** If the area of the square is 4, each side is 2, and the length of a diagonal is $2\sqrt{2}$. The area of a circle whose radius is $2\sqrt{2}$ is $\pi(2\sqrt{2})^2 = 8\pi$.

13. **(B)** There's nothing to calculate here. Each arc of the circle is clearly longer than the corresponding chord, which is a side of the pentagon. So the circumference, which is the sum of all the arcs, is greater than the perimeter, which is the sum of all the chords.

14. **(D)** Quantity A: $\frac{C}{A} = \frac{2\pi r}{\pi r^2} = \frac{2}{r}$.

Quantity B: $\frac{A}{C} = \frac{r}{2}$.

If $r = 2$, the quantities are equal; otherwise, they're not.

15. **(B)** Quantity A: $A = \pi(2)^2 = 4\pi$.

Quantity B: The area of a semicircle of radius 3 is $\frac{1}{2}\pi(3)^2 = \frac{1}{2}(9\pi) = 4.5\pi$.

16. **(A)** By KEY FACT L4, $\pi = \frac{C}{d} = \frac{C}{2r} \Rightarrow \frac{C}{r} = 2\pi$, which is greater than 6.

17. **(B)** The area of sector A is $\frac{40}{360}(16\pi) = \frac{16\pi}{9}$.

The area of sector B is $\frac{20}{360}(64\pi) = \frac{64\pi}{18} = \frac{32\pi}{9}$.

So sector B is twice as big as sector A.

18. **(C)** Since the triangles are equilateral, the two white central angles each measure 60°, and their sum is 120°. So the white area is $\frac{120}{360} = \frac{1}{3}$ of the circle, and the shaded area is $\frac{2}{3}$ of the circle. The area of the circle is $\pi(3)^2 = 9\pi$, so the shaded area is $\frac{2}{3}(9\pi) = 6\pi$.

19. **(C)** In Figure 1, since the radius of each circle is 3, the area of each circle is 9π, and the total area of the 4 circles is 36π. In Figure 2, the radius of each circle is 2, and so the area of each circle is 4π, and the total area of the 9 circles is 36π. In the two figures, the white areas are equal, as are the shaded areas.

20. **(A)** Let A represent the area of the square and the circle.

Quantity A: $A = s^2 \Rightarrow s = \sqrt{A} \Rightarrow P = 4\sqrt{A}$.

Quantity B: $A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}} = \frac{\sqrt{A}}{\sqrt{\pi}}$.

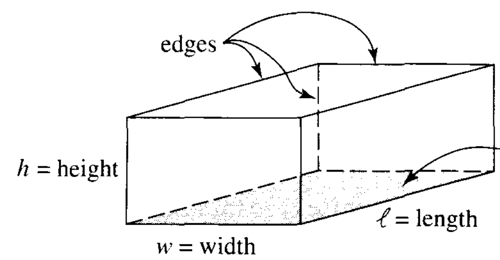
So $C = 2\pi\left(\frac{\sqrt{A}}{\sqrt{\pi}}\right) = 2\sqrt{\pi}\sqrt{A}$.

Since $\pi \approx 3.14$, $\sqrt{\pi} \approx 1.77 \Rightarrow 2\sqrt{\pi} \approx 3.54$. Quantity A is $4\sqrt{A}$; Quantity B is $3.54\sqrt{A}$. Quantity A is greater.

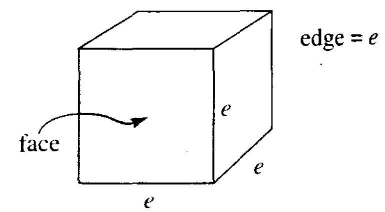
11-M. SOLID GEOMETRY

There are very few solid geometry questions on the GRE, and they cover only a few elementary topics. Basically, all you need to know are the formulas for the volume and surface areas of rectangular solids (including cubes) and cylinders.

A ***rectangular solid*** or ***box*** is a solid formed by six rectangles, called ***faces***. The sides of the rectangles are called ***edges***. As shown in the diagram below (left), the edges are called the ***length***, ***width***, and ***height***. A ***cube*** is a rectangular solid in which the length, width, and height are equal; so all the edges are the same length.



RECTANGULAR SOLID



CUBE

The ***volume*** of a solid is the amount of space it takes up and is measured in ***cubic units***. One cubic unit is the amount of space occupied by a cube all of whose edges are one unit long. In the figure above (right), if each edge of the cube is 1 inch long, then the area of each face is 1 square inch, and the volume of the cube is 1 cubic inch.

KEY FACT M1

- **The formula for the volume of a rectangular solid is $V = \ell wh$.**
- **The formula for the volume of a cube is $V = e \cdot e \cdot e = e^3$.**

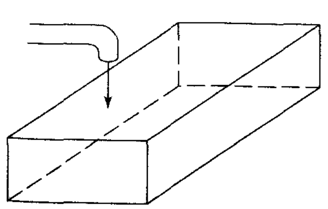
EXAMPLE 1

The base of a rectangular tank is 12 feet long and 8 feet wide; the height of the tank is 30 inches. If water is pouring into the tank at the rate of 2 cubic feet per second, how many minutes will be required to fill the tank?

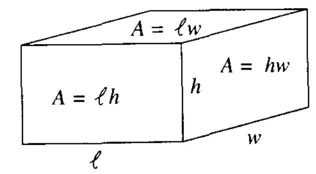
Ⓐ 1 Ⓑ 2 Ⓒ 10 Ⓓ 120 Ⓔ 240

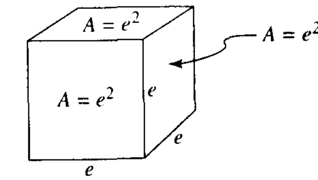
SOLUTION.

Draw a diagram. In order to express all of the dimensions of the tank in the same units, convert 30 inches to 2.5 feet. Then the volume of the tank is $12 \times 8 \times 2.5 = 240$ cubic feet. At 2 cubic feet per second, it will take $240 \div 2 = 120$ seconds = **2** minutes to fill the tank (**B**).



The ***surface area*** of a rectangular solid is the sum of the areas of the six faces. Since the top and bottom faces are equal, the front and back faces are equal, and the left and right faces are equal, we can calculate the area of one from each pair and then double the sum. In a cube, each of the six faces has the same area.





KEY FACT M2

- **The formula for the surface area of a rectangular solid is $A = 2(\ell w + \ell h + wh)$.**
- **The formula for the surface area of a cube is $A = 6e^2$.**

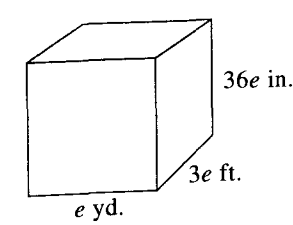
EXAMPLE 2

The volume of a cube is v cubic *yards*, and its surface area is a square *feet*. If $v = a$, what is the length in *inches* of each edge?

inches

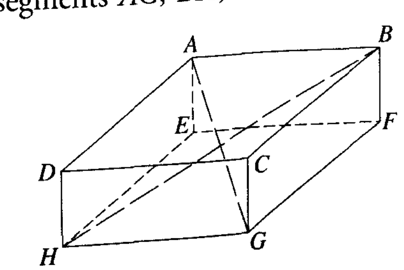
SOLUTION.

Draw a diagram.



If e is the length of the edge in yards, $3e$ is the length in feet, and $36e$ the length in inches. Therefore, $v = e^3$ and $a = 6(3e)^2 = 6(9e^2) = 54e^2$. Since $v = a$, $e^3 = 54e^2$, and $e = 54$. So the length of each edge is $36(54) =$ **1,944** inches.

A ***diagonal*** of a box is a line segment joining a vertex on the top of the box to the opposite vertex on the bottom. A box has 4 diagonals, all the same length. In the box below they are line segments AG, BH, CE, and DF.



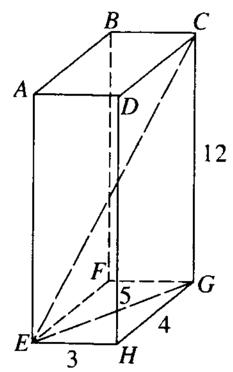
KEY FACT M3

A diagonal of a box is the longest line segment that can be drawn between two points on the box.

KEY FACT M4

If the dimensions of a box are ℓ, w, and h, and if d is the length of a diagonal, then $d^2 = \ell^2 + w^2 + h^2$ and $d = \sqrt{\ell^2 + w^2 + h^2}$.

For example, in the box below: $d^2 = 3^2 + 4^2 + 12^2 = 9 + 16 + 144 = 169 \Rightarrow d = 13$.



This formula is really just an extended Pythagorean theorem. EG is the diagonal of rectangular base $EFGH$. Since the sides of the base are 3 and 4, EG is 5. Now, $\triangle CGE$ is a right triangle whose legs are 12 and 5, so diagonal CE is 13.

EXAMPLE 3

What is the length of a diagonal of a cube whose edges are 1?

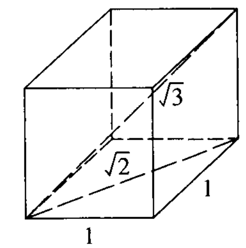
Ⓐ 1 Ⓑ 2 Ⓒ 3 Ⓓ $\sqrt{2}$ Ⓔ $\sqrt{3}$

SOLUTION.
Use the formula:

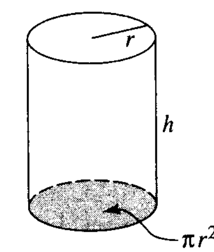
$$d^2 = 1^2 + 1^2 + 1^2 = 3 \Rightarrow d = \sqrt{3} \text{ (E).}$$

Without the formula you would draw a diagram and label it. Since the base is a 1×1 square, its diagonal is $\sqrt{2}$. Then the diagonal of the cube is the hypotenuse of a right triangle whose legs are 1 and $\sqrt{2}$, so

$$d^2 = 1^2 + (\sqrt{2})^2 = 1 + 2 = 3, \text{ and } d = \sqrt{3}.$$



A ***cylinder*** is similar to a rectangular solid except that the base is a circle instead of a rectangle. The volume of a cylinder is the area of its circular base (πr^2) times its height (h). The surface area of a cylinder depends on whether you are envisioning a tube, such as a straw, without a top or bottom, or a can, which has both a top and a bottom.



KEY FACT M5

- **The formula for the volume, V, of a cylinder whose circular base has radius r and whose height is h is $V = \pi r^2 h$.**
- **The surface area, A, of the side of the cylinder is the circumference of the circular base times the height: $A = 2\pi rh$.**
- **The area of the top and bottom are each πr^2, so the total area of a can is $2\pi rh + 2\pi r^2$.**

EXAMPLE 4

The radius of cylinder II equals the height of cylinder I.
The height of cylinder II equals the radius of cylinder I.

Quantity A	Quantity B
The volume of cylinder I	The volume of cylinder II

SOLUTION.
Let r and h be the radius and height, respectively, of cylinder I. Then

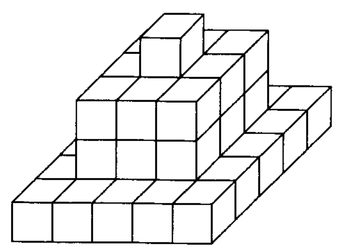
	Quantity A	Quantity B
	$\pi r^2 h$	$\pi h^2 r$
Divide each quantity by πrh:	r	h

Either r or h could be greater, or the two could be equal. The answer is **D**.

These are the only formulas you need to know. Any other solid geometry questions that might appear on the GRE would require you to visualize a situation and reason it out, rather than to apply a formula.

EXAMPLE 5

How many small cubes are needed to construct the tower in the figure below?



SOLUTION.
You need to "see" the answer. The top level consists of 1 cube, the second and third levels consist of 9 cubes each, and the bottom layer consists of 25 cubes. The total is 1 + 9 + 9 + 25 = **44**.

Practice Exercises — Solid Geometry

Discrete Quantitative Questions

1. The sum of the lengths of all the edges of a cube is 6 centimeters. What is the volume, in cubic centimeters, of the cube?
   - Ⓐ $\frac{1}{8}$
   - Ⓑ $\frac{1}{4}$
   - Ⓒ $\frac{1}{2}$
   - Ⓓ 1
   - Ⓔ 8

2. What is the volume of a cube whose surface area is 150?

3. What is the surface area of a cube whose volume is 64?
   - Ⓐ 16
   - Ⓑ 64
   - Ⓒ 96
   - Ⓓ 128
   - Ⓔ 384

4. What is the number of cubic inches in one cubic foot?
   - Ⓐ 12
   - Ⓑ 24
   - Ⓒ 144
   - Ⓓ 684
   - Ⓔ 1728

5. A solid metal cube of edge 3 feet is placed in a rectangular tank whose length, width, and height are 3, 4, and 5 feet, respectively. What is the volume, in cubic feet, of water that the tank can now hold?
   - Ⓐ 20
   - Ⓑ 27
   - Ⓒ 33
   - Ⓓ 48
   - Ⓔ 60

6. A 5-foot-long cylindrical pipe has an inner diameter of 6 feet and an outer diameter of 8 feet. If the total surface area (inside and out, including the ends) is $k\pi$, what is the value of k?
   - Ⓐ 7
   - Ⓑ 40
   - Ⓒ 48
   - Ⓓ 70
   - Ⓔ 84

7. The height, h, of a cylinder is equal to the edge of a cube. If the cylinder and cube have the same volume, what is the radius of the cylinder?
   - Ⓐ $\frac{h}{\sqrt{\pi}}$
   - Ⓑ $h\sqrt{\pi}$
   - Ⓒ $\frac{\sqrt{\pi}}{h}$
   - Ⓓ $\frac{h^2}{\pi}$
   - Ⓔ πh^2

8. A rectangular tank has a base that is 10 centimeters by 5 centimeters and a height of 20 centimeters. If the tank is half full of water, by how many centimeters will the water level rise if 325 cubic centimeters of water are poured into the tank?

Ⓐ 3.25
Ⓑ 6.5
Ⓒ 16.25
Ⓓ 32.5
Ⓔ 65

9. If the height of a cylinder is 4 times its circumference, what is the volume of the cylinder in terms of its circumference, C?

Ⓐ $\dfrac{C^3}{\pi}$

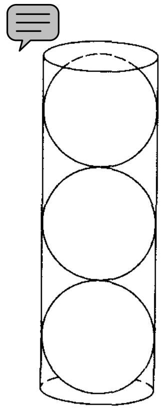
Ⓑ $\dfrac{2C^3}{\pi}$

Ⓒ $\dfrac{2C^2}{\pi^2}$

Ⓓ $\dfrac{\pi C^2}{4}$

Ⓔ $4\pi C^3$

10. Three identical balls fit snugly into a cylindrical can: the radius of the spheres equals the radius of the can, and the balls just touch the bottom and the top of the can. If the formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, what fraction of the volume of the can is taken up by the balls?



Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

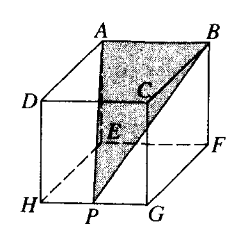
Jack and Jill each roll a sheet of 9×12 paper to form a cylinder. Jack tapes the two 9-inch edges together. Jill tapes the two 12-inch edges together.

	Quantity A	Quantity B
11.	The volume of Jack's cylinder	The volume of Jill's cylinder

	Quantity A	Quantity B
12.	The volume of a cube whose edges are 6	The volume of a box whose dimensions are 5, 6, and 7

A is the surface area of a rectangular box in square units.
V is the volume of the same box in cubic units.

	Quantity A	Quantity B
13.	A	V



P is a point on edge GH of cube $ABCDEFGH$. Each edge of the cube is 1.

	Quantity A	Quantity B
14.	The area of $\triangle ABP$	1

	Quantity A	Quantity B
15.	The volume of a sphere whose radius is 1	The volume of a cube whose edge is 1

ANSWER KEY

1. **A**	4. **E**	7. **A**	10. $\dfrac{2}{3}$	12. **A**	15. **A**
2. **125**	5. **C**	8. **B**		13. **D**	
3. **C**	6. **E**	9. **A**	11. **A**	14. **B**	

Answer Explanations

1. **(A)** Since a cube has 12 edges, we have $12e = 6 \Rightarrow e = \frac{1}{2}$.

   Therefore, $V = e^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

2. **125** Since the surface area is 150, each of the 6 faces is a square whose area is $150 \div 6 = 25$. So the edges are all 5, and the volume is $5^3 = 125$.

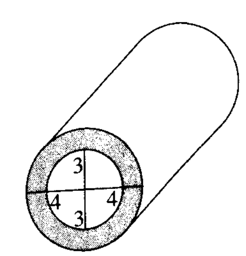
3. **(C)** Since the volume of the cube is 64, we have $e^3 = 64 \Rightarrow e = 4$. The surface area is $6e^2 = 6 \times 16 = 96$.

4. **(E)** The volume of a cube whose edges are 1 foot can be expressed in either of two ways:

   $(1\text{ foot})^3 = 1$ cubic foot or
   $(12\text{ inches})^3 = 1728$ cubic inches.

5. **(C)** The volume of the tank is $3 \times 4 \times 5 = 60$ cubic units, but the solid cube is taking up $3^3 = 27$ cubic units. Therefore, the tank can hold $60 - 27 = 33$ cubic units of water.

6. **(E)** Draw a diagram and label it.



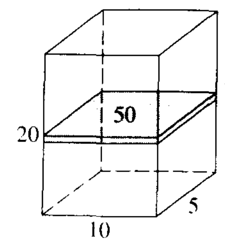
Since the surface of a cylinder is given by $A = 2\pi rh$, the area of the exterior is $2\pi(4)(5) = 40\pi$, and the area of the interior is $2\pi(3)(5) = 30\pi$. The area of *each* shaded end is the area of the outer circle minus the area of the inner circle: $16\pi - 9\pi = 7\pi$, so the total surface area is

$$40\pi + 30\pi + 7\pi + 7\pi = 84\pi \Rightarrow k = 84.$$

7. **(A)** Since the volumes are equal, $\pi r^2 h = e^3 = h^3$.

   Therefore, $\pi r^2 = h^2 \Rightarrow r^2 = \dfrac{h^2}{\pi} \Rightarrow r = \dfrac{h}{\sqrt{\pi}}$.

8. **(B)** Draw a diagram.



   Since the area of the base is $5 \times 10 = 50$ square centimeters, each 1 centimeter of depth has a volume of 50 cubic centimeters. Therefore, 325 cubic centimeters will raise the water level $325 \div 50 = 6.5$ centimeters.

   (Note that we didn't use the fact that the tank was half full, except to be sure that the tank didn't overflow. Since the tank was half full, the water was 10 centimeters deep, and the water level could rise by 6.5 centimeters. Had the tank been three-fourths full, the water would have been 15 centimeters deep and the extra water would have caused the level to rise 5 centimeters, filling the tank; the rest of the water would have spilled out.)

9. **(A)** Since $V = \pi r^2 h$, we need to express r and h in terms of C. It is given that $h = 4C$ and since $C = 2\pi r$, then $r = \dfrac{C}{2\pi}$. Therefore,

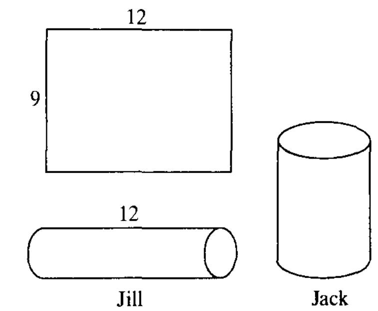
$$V = \pi\left(\frac{C}{2\pi}\right)^2(4C) = \pi\left(\frac{C^2}{4\pi^2}\right)(4C) = \frac{C^3}{\pi}.$$

10. $\dfrac{\mathbf{2}}{\mathbf{3}}$ To avoid using r, assume that the radii of the spheres and the can are 1.

   Then the volume of each ball is $\frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$, and the total volume of the 3 balls is $3\left(\frac{4}{3}\pi\right) = 4\pi$.

   Since the height of the can is 6 (the diameter of each sphere is 2), the volume of the can is $\pi(1)^2(6) = 6\pi$. So the balls take up $\dfrac{4\pi}{6\pi} = \dfrac{2}{3}$ of the can.

11. **(A)** Drawing a diagram makes it easier to visualize the problem. The volume of a cylinder is $\pi r^2 h$. In each case, we know the height but have to determine the radius in order to calculate the volume.



   Jack's cylinder has a circumference of 12:

$$2\pi r = 12 \Rightarrow r = \frac{12}{2\pi} = \frac{6}{\pi} \Rightarrow V = \pi\left(\frac{6}{\pi}\right)^2(9) = \pi\left(\frac{36}{\pi^2}\right)(9) = \frac{324}{\pi}.$$

   Jill's cylinder has a circumference of 9:

$$2\pi r = 9 \Rightarrow r = \frac{9}{2\pi} \Rightarrow V = \pi\left(\frac{9}{2\pi}\right)^2(12) = \pi\left(\frac{81}{4\pi^2}\right)(12) = \frac{243}{\pi}.$$

12. **(A)** Quantity A: $V = 6^3 = 216$. Quantity B: $V = 5 \times 6 \times 7 = 210$.

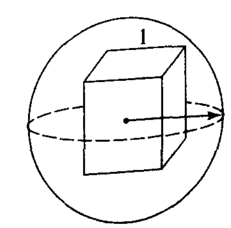
13. **(D)** There is no relationship between the two quantities. If the box is a cube of edge 1, $A = 6$ and $V = 1$. If the box is a cube of edge 10, $A = 600$ and $V = 1000$.

14. **(B)** The base, AB, of $\triangle ABP$ is 1. Since the diagonal is the longest line segment in the cube, the height, h, of the triangle is definitely less than the diagonal, which is $\sqrt{1^2+1^2+1^2} = \sqrt{3}$.

   So the area of the triangle is less than $\frac{1}{2}(1)\sqrt{3} \approx .87$, which is less than 1.

   (You could also have just calculated the area: $h = BG = \sqrt{2}$, so the area is $\frac{1}{2}\sqrt{2} \approx .71$.)

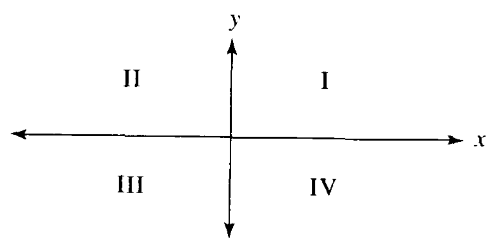
15. **(A)** You probably don't know how to find the volume of a sphere; fortunately, you don't need to. You should be able to visualize that the sphere is *much* bigger than the cube. (In fact, it is more than 4 times the size.)



11-N. COORDINATE GEOMETRY

The GRE has very few questions on coordinate geometry. Most often they deal with the coordinates of points and occasionally with the slope of a line. You will *never* have to draw a graph.

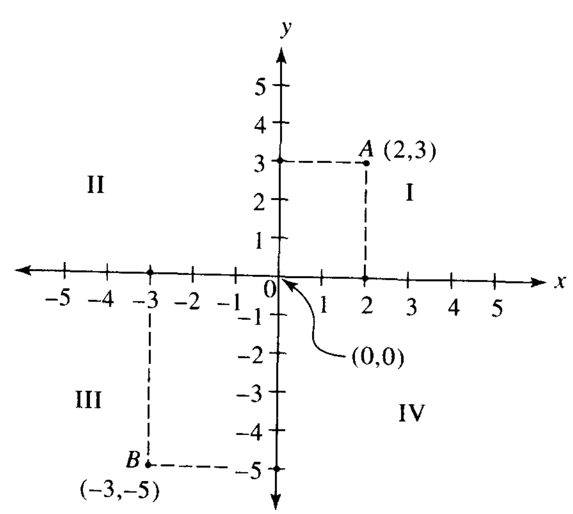
The coordinate plane is formed by two perpendicular number lines called the ***x-axis*** and ***y-axis***, which intersect at the ***origin***. The axes divide the plane into four ***quadrants***, labeled I, II, III, and IV.



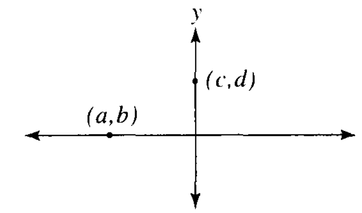
Each point in the plane is assigned two numbers, an ***x-coordinate*** and a ***y-coordinate***, which are written as an ordered pair, ***(x, y)***.

- Points to the right of the y-axis have positive x-coordinates, and those to the left have negative x-coordinates.
- Points above the x-axis have positive y-coordinates, and those below it have negative y-coordinates.
- If a point is on the x-axis, its y-coordinate is 0.
- If a point is on the y-axis, its x-coordinate is 0.

For example, point A in the following figure is labeled (2, 3), since it is 2 units to the right of the y-axis and 3 units above the x-axis. Similarly, $B(-3, -5)$ is in Quadrant III, 3 units to the left of the y-axis and 5 units below the x-axis.



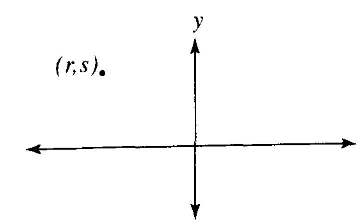
EXAMPLE 1



Quantity A	Quantity B
b	c

SOLUTION.
Since (a, b) lies on the x-axis, $b = 0$. Since (c, d) lies on the y-axis, $c = 0$. The answer is **C.**

EXAMPLE 2



Quantity A	Quantity B
r	s

SOLUTION.
Since (r, s) is in Quadrant II, r is negative and s is positive. The answer is **B.**

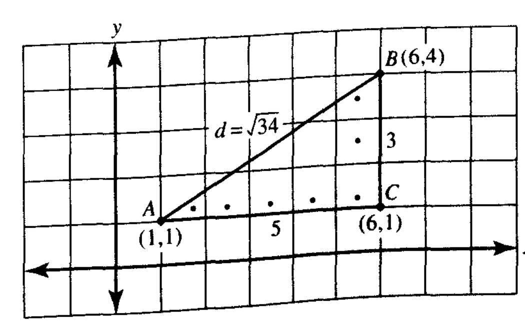
Often a question requires you to calculate the distance between two points. This is easiest when the points lie on the same horizontal or vertical line.

TIP

If the points have been plotted on a graph, you can find the distance between them by counting boxes.

KEY FACT N1

- **All the points on a horizontal line have the same y-coordinate. To find the distance between them, subtract their x-coordinates.**
- **All the points on a vertical line have the same x-coordinate. To find the distance between them, subtract their y-coordinates.**

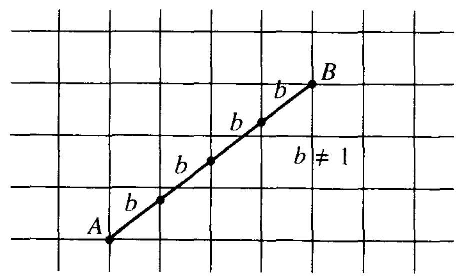


The distance from A to C is $6 - 1 = 5$. The distance from B to C is $4 - 1 = 3$.

It is a little more difficult to find the distance between two points that are not on the same horizontal or vertical line. In this case, use the Pythagorean theorem. For example, in the previous figure, if d represents the distance from A to B, $d^2 = 5^2 + 3^2 = 25 + 9 = 34$, and so $d = \sqrt{34}$.

CAUTION

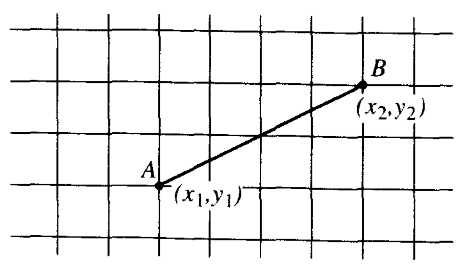
You *cannot* count boxes unless the points are on the same horizontal or vertical line. The distance between A and B is 5, not 4.

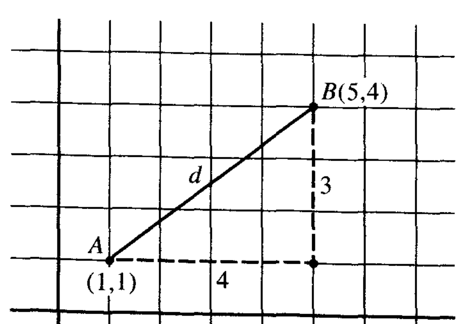


KEY FACT N2

The distance, d, between two points, $A(x_1, y_1)$ and $B(x_2, y_2)$, can be calculated using the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

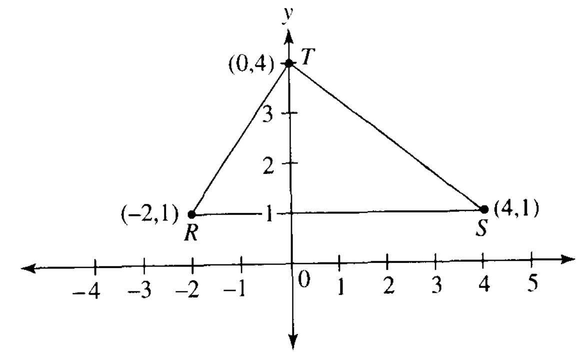




TIP

The "distance formula" is nothing more than the Pythagorean theorem. If you ever forget the formula, and you need the distance between two points that do not lie on the same horizontal or vertical line, do as follows: create a right triangle by drawing a horizontal line through one of the points and a vertical line through the other, and then use the Pythagorean theorem.

Examples 3–4 refer to the triangle in the following figure.



EXAMPLE 3

What is the area of $\triangle RST$?

SOLUTION.

$R(-2, 1)$ and $S(4, 1)$ lie on the same horizontal line, so $RS = 4 - (-2) = 6$. Let that be the base of the triangle. Then the height is the distance along the vertical line from T to RS: $4 - 1 = 3$. The area is $\frac{1}{2}(6)(3) = \mathbf{9}$.

EXAMPLE 4

What is the perimeter of $\triangle RST$?

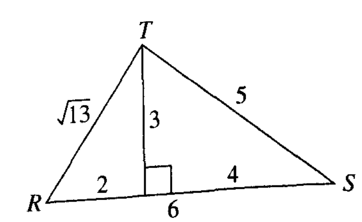
Ⓐ 13 Ⓑ 14 Ⓒ 16 Ⓓ $11 + \sqrt{13}$ Ⓔ $11 + \sqrt{61}$

SOLUTION.

The perimeter is $RS + ST + RT$. From the solution to Example 3, you know that $RS = 6$. Also, $ST = 5$, since it is the hypotenuse of a 3-4-5 right triangle. To calculate RT, either use the distance formula:

$$RT = \sqrt{(-2-0)^2 + (1-4)^2} = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$$

or the Pythagorean theorem: $RT^2 = 2^2 + 3^2 = 4 + 9 = 13 \Rightarrow RT = \sqrt{13}$.



So the perimeter is: $6 + 5 + \sqrt{13} = \mathbf{11 + \sqrt{13}}$ **(D)**.

The ***slope*** of a line is a number that indicates how steep the line is.

KEY FACT N3

- **Vertical lines *do not have slopes.***
- **To find the slope of any other line proceed as follows:**
  1. **Choose any two points A(x_1, y_1) and B(x_2, y_2) on the line.**
  2. **Take the differences of the *y*-coordinates, $y_2 - y_1$, and the *x*-coordinates, $x_2 - x_1$.**
  3. **Divide: slope = $\dfrac{y_2 - y_1}{x_2 - x_1}$.**

We will illustrate the next KEY FACT by using this formula to calculate the slopes of *RS*, *RT*, and *ST* from Example 3: *R*(–2, 1), *S*(4, 1), *T*(0, 4).

Key Fact N4

- **The slope of any horizontal line is 0: slope of *RS* = $\dfrac{1-1}{4-(-2)} = \dfrac{0}{6} = 0$**
- **The slope of any line that goes up as you move from left to right is positive:**

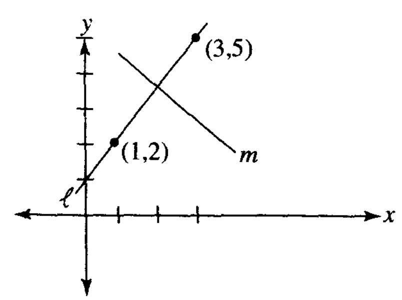
  slope of *RT* = $\dfrac{4-1}{0-(-2)} = \dfrac{3}{2}$
- **The slope of any line that goes down as you move from left to right is negative: slope of *ST* = $\dfrac{1-4}{4-0} = \dfrac{-3}{4} = -\dfrac{3}{4}$**

EXAMPLE 5

Line ℓ passes through (1, 2) and (3, 5)
Line *m* is perpendicular to ℓ

Quantity A	Quantity B
The slope of ℓ	The slope of *m*

SOLUTION.
First, make a quick sketch.



Do not use the formula to calculate the slope of ℓ. Simply notice that ℓ slopes upward, so its slope is positive, whereas *m* slopes downward, so its slope is negative. Quantity A is greater.

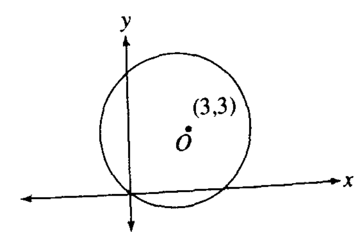
Practice Exercises — Coordinate Geometry

Discrete Quantitative Questions

1. What is the slope of the line that passes through points (0, –2) and (3, 0)?

2. If the coordinates of $\triangle RST$ are *R*(0, 0), *S*(7, 0), and *T*(2, 5), what is the sum of the slopes of the three sides of the triangle?
   Ⓐ –1.5
   Ⓑ 0
   Ⓒ 1.5
   Ⓓ 2.5
   Ⓔ 3.5

3. If *A*(–1, 1) and *B*(3, –1) are the endpoints of one side of square *ABCD*, what is the area of the square?
   Ⓐ 12
   Ⓑ 16
   Ⓒ 20
   Ⓓ 25
   Ⓔ 36



4. If the area of circle *O* above is $k\pi$, what is the value of *k*?
   Ⓐ 3
   Ⓑ 6
   Ⓒ 9
   Ⓓ 18
   Ⓔ 27

5. If *P*(2, 1) and *Q*(8, 1) are two of the vertices of a rectangle, which of the following could *not* be another of the vertices?
   Ⓐ (2, 8)
   Ⓑ (8, 2)
   Ⓒ (2, –8)
   Ⓓ (–2, 8)
   Ⓔ (8, 8)

6. A circle whose center is at (6, 8) passes through the origin. Which of the following points is *not* on the circle?
   Ⓐ (12, 0)
   Ⓑ (6, –2)
   Ⓒ (16, 8)
   Ⓓ (–2, 12)
   Ⓔ (–4, 8)

Questions 7–8 concern parallelogram *JKLM*, whose coordinates are *J*(–5, 2), *K*(–2, 6), *L*(5, 6), *M*(2, 2).

7. What is the area of parallelogram *JKLM*?
   Ⓐ 35
   Ⓑ 28
   Ⓒ 24
   Ⓓ 20
   Ⓔ 12

8. What is the perimeter of parallelogram *JKLM*?

9. If (a, b) and $\left(\frac{1}{a}, b\right)$ are two distinct points, what is the slope of the line that passes through them?

Ⓐ 0

Ⓑ $\frac{1}{b}$

Ⓒ $\frac{1-a^2}{a}$

Ⓓ $\frac{a^2-1}{a}$

Ⓔ undefined

10. If $c \neq 0$ and the slope of the line passing through $(-c, c)$ and $(3c, a)$ is 1, which of the following is an expression for a in terms of c?

Ⓐ $-3c$

Ⓑ $-\frac{c}{3}$

Ⓒ $2c$

Ⓓ $3c$

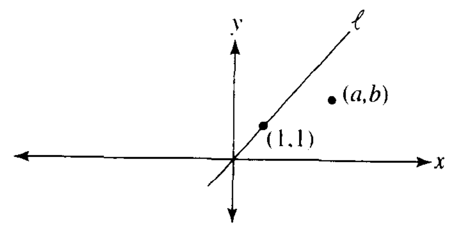
Ⓔ $5c$

Quantitative Comparison Questions

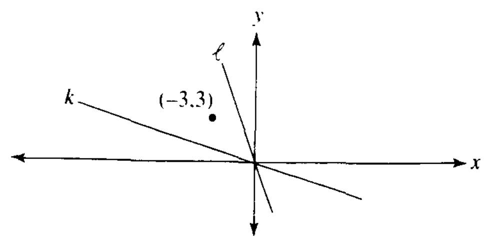
Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

m is the slope of one of the diagonals of a square.

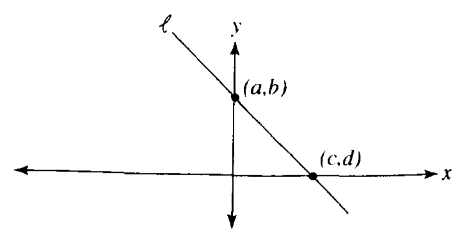
	Quantity A	Quantity B
11.	m^2	1



	Quantity A	Quantity B
12.	$a - b$	0



	Quantity A	Quantity B
13.	The slope of line k	The slope of line ℓ



The slope of line ℓ is -0.8.

	Quantity A	Quantity B
14.	c	b

The distance from $(b, 5)$ to $(c, -3)$ is 10.
$b < c$

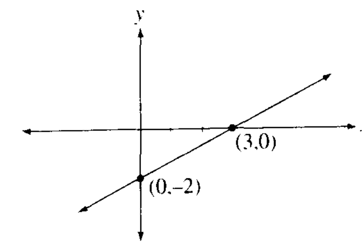
	Quantity A	Quantity B
15.	$c - b$	6

ANSWER KEY

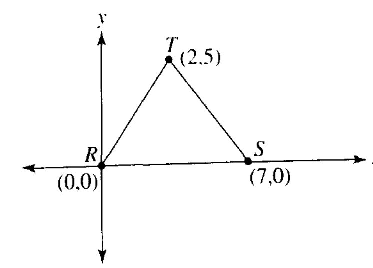
1. $\frac{2}{3}$	3. **C**	6. **D**	9. **A**	12. **A**	15. **C**
2. **C**	4. **D**	7. **B**	10. **E**	13. **A**	
	5. **D**	8. **24**	11. **D**	14. **A**	

Answer Explanations

1. $\frac{2}{3}$ If you sketch the line, you see immediately that the slope of the line is positive. Without even knowing the slope formula, therefore, you can eliminate Choices A, B, and C. To determine the actual slope, use the formula: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-2)}{3-0} = \frac{2}{3}$.

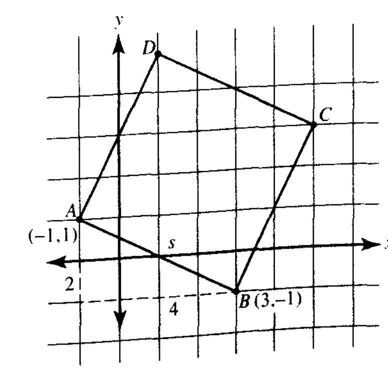


2. **(C)** Sketch the triangle, and then calculate the slopes.



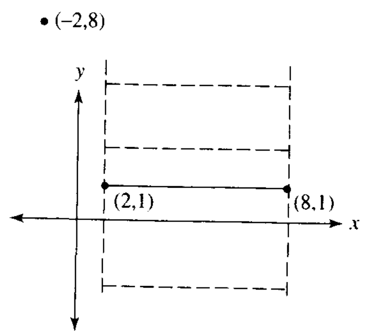
Since RS is horizontal, its slope is 0.
The slope of $RT = \frac{5-0}{2-0} = 2.5$. The slope of $ST = \frac{5-0}{2-7} = \frac{5}{-5} = -1$.
Now add: $0 + 2.5 + (-1) = 1.5$

3. **(C)** Draw a diagram and label it. The area of square $ABCD$ is s^2, where $s = AB$, the length of a side. By the Pythagorean theorem:
$$s^2 = 2^2 + 4^2 = 4 + 16 = 20.$$

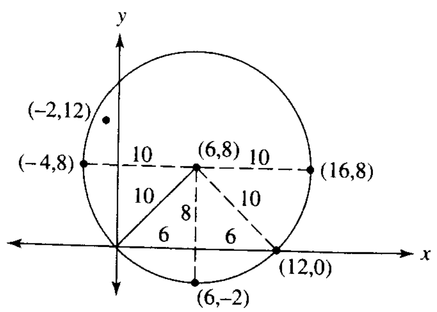


4. **(D)** Since the line segment joining (3, 3) and (0, 0) is a radius of the circle, $r^2 = 3^2 + 3^2 = 18$. Therefore, area $= \pi r^2 = 18\pi \Rightarrow k = 18$. Note that you do not actually have to find that the value of r is $3\sqrt{2}$.

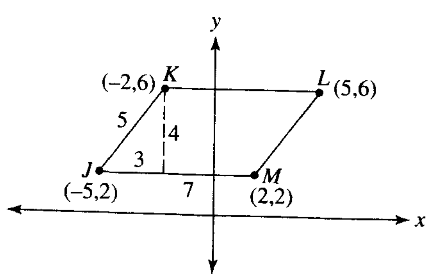
5. **(D)** Draw a diagram. Any point whose x-coordinate is 2 or 8 could be another vertex. Of the choices, only (–2, 8) is *not* possible.



6. **(D)** Draw a diagram. The radius of the circle is 10 (since it's the hypotenuse of a 6-8-10 right triangle). Which of the choices are 10 units from (6, 8)? First, check the easy ones: (–4, 8) and (16, 8) are 10 units to the left and right of (6, 8), and (6, –2) is 10 units below. What remains is to check (12, 0), which works, and (–2, 12), which doesn't.



Here is the diagram for solutions 7 and 8.



7. **(B)** The base is 7 and the height is 4. So, the area is $7 \times 4 = 28$.

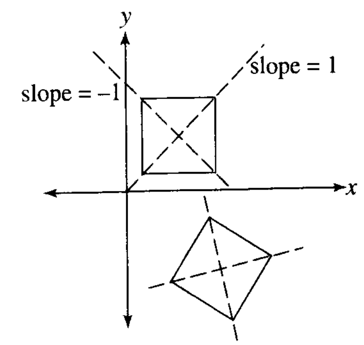
8. **24** Sides JM and KL are each 7, and sides JK and LM are each the hypotenuse of a 3-4-5 right triangle, so they are 5. The perimeter is $2(7 + 5) = 24$.

9. **(A)** The formula for the slope is $\dfrac{y_2 - y_1}{x_2 - x_1}$, but before using it, look. Since the y-coordinates are equal and the x-coordinates are not equal, the numerator is 0 and the denominator is not 0. So the value of the fraction is 0.

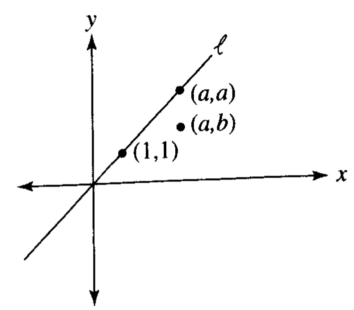
10. **(E)** The slope is equal to $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{a - c}{3c - (-c)} = \dfrac{a - c}{4c} = 1$.

    So $a - c = 4c$ and $a = 5c$.

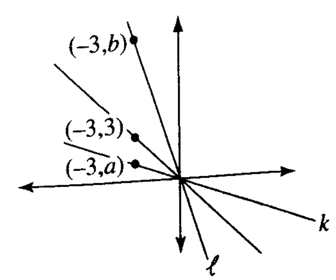
11. **(D)** If the sides of the square are horizontal and vertical, then m is 1 or –1, and m^2 is 1. But the square could be positioned any place, and the slope of a diagonal could be any number.



12. **(A)** Line ℓ, which goes through (0, 0) and (1, 1), also goes through (a, a), and since (a, b) is below (a, a), $b < a$. Therefore, $a - b$ is positive. Quantity A is greater.



13. **(A)** The line going through (–3, 3) and (0, 0) has slope –1. Since ℓ is steeper, its slope is a number such as –2 or –3; since k is less steep, its slope is a number such as –0.5 or –0.3. Therefore, the slope of k is greater.

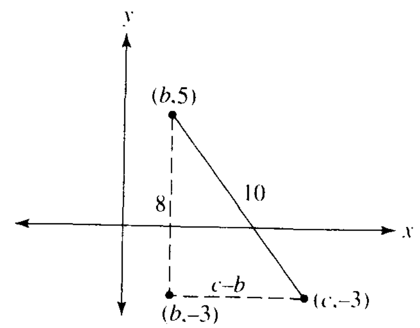


14. **(A)** Since (a, b) is on the y-axis, $a = 0$; and since (c, d) is on the x-axis, $d = 0$.

Then by the slope formula, $-0.8 = \frac{0-b}{c-0} = -\frac{b}{c} \Rightarrow b = 0.8c$.

Since b and c are both positive, $b < c$.

15. **(C)** Draw a diagram.



Since the distance between the two points is 10, by the distance formula:

$10 = \sqrt{(c-b)^2 + (-3-5)^2} = \sqrt{(c-b)^2 + (-8)^2} = \sqrt{(c-b)^2 + 64}$.

Squaring both sides gives $100 = (c - b)^2 + 64 \Rightarrow (c - b)^2 = 36$. So $c - b = 6$.

Notice that the equation $100 = (c - b)^2 + 64$ is exactly what you could have gotten immediately by using the Pythagorean theorem: $10^2 = (c - b)^2 + 8^2$.

11-O. COUNTING AND PROBABILITY

Some questions on the GRE begin, "How many ... ." In these problems you are being asked to count something: how many apples can she buy, how many dollars did he spend, how many pages did she read, how many numbers satisfy a certain property, or how many ways are there to complete a particular task. Sometimes these problems can be handled by simple arithmetic. Other times it helps to use TACTIC 8 from Chapter 7 and systematically make a list. Occasionally it helps to know the counting principle and other strategies that we will review in this section.

COUNTING

USING ARITHMETIC TO COUNT

The following three examples require only arithmetic. But be careful; they are not the same.

EXAMPLE 1

Brian bought some apples. If he entered the store with \$113 and left with \$109, how much, in dollars, did the apples cost?

EXAMPLE 2

Scott was selling tickets for the school play. One day he sold tickets numbered 109 through 113. How many tickets did he sell that day?

EXAMPLE 3

Brian is the 109th person in a line and Scott is the 113th person. How many people are there between Brian and Scott?

SOLUTIONS 1–3.

- It may seem that each of these examples requires a simple subtraction: $113 - 109 = 4$. In Example 1, Brian did spend **\$4** on apples; in Example 2, however, Scott sold **5** tickets; and in Example 3, only **3** people are on line between Brian and Scott!

- Assume that Brian went into the store with 113 one-dollar bills, numbered 1 through 113; he spent the 4 dollars numbered 113, 112, 111, and 110, and still had the dollars numbered 1 through 109; Scott sold the 5 tickets numbered 109, 110, 111, 112, and 113; and between Brian and Scott the 110th, 111th, and 112th persons — 3 people — were on line.

In Example 1, you just needed to subtract: 113 – 109 = 4. In Example 2, you need to subtract *and then add 1*: 113 – 109 + 1 = 4 + 1 = 5. And in Example 3, you need to subtract and then *subtract 1 more*: 113 – 109 – 1 = 3. Although Example 1 is too easy for the GRE, questions such as Examples 2 and 3 do appear, because they're not as obvious and they require that little extra thought. *When do you have to add or subtract 1?*

The issue is whether or not the first and last numbers are included. In Example 1, Brian spent dollar number 113, but he still had dollar number 109 when he left the store. In Example 2, Scott sold both ticket number 109 and ticket 113. In Example 3, neither Scott (the 113th person) nor Brian (the 109th person) was to be counted.

KEY FACT O1

To count how many integers there are between two integers, follow these rules:

- **If exactly one of the endpoints is included, subtract.**
- **If both endpoints are included, subtract and add 1.**
- **If neither endpoint is included, subtract and subtract 1 more.**

EXAMPLE 4

From 1:09 to 1:13, Adam read pages 109 through 113 in his English book. What was his rate of reading, in pages per minute?

$$\frac{\boxed{}}{\boxed{}}$$

SOLUTION.

Since Adam read both pages 109 and 113, he read 113 – 109 + 1 = 5 pages. He started reading during the minute that started at 1:09 (and ended at 1:10). Since he stopped reading at 1:13, he did not read during the minute that began at 1:13 (and ended at 1:14). So he read for 1:13 – 1:09 = 4 minutes. He read at the rate of $\frac{5}{4}$ pages per minute.

SYSTEMATICALLY MAKING A LIST

TACTIC O1

When a question asks "How many...?" and the numbers in the problem are small, just systematically list all of the possibilities.

Proper use of TACTIC O1 eliminates the risk of making an error in arithmetic. In Example 4, rather than even thinking about whether or not to add 1 or subtract 1 after subtracting the number of pages, you could have just quickly jotted down the numbers of the pages Adam read (109, 110, 111, 112, 113), and then counted them.

EXAMPLE 5

Ariel has 4 paintings in the basement. She is going to bring up 2 of them and hang 1 in her den and 1 in her bedroom. In how many ways can she choose which paintings go in each room?

$\boxed{}$

SOLUTION.

Label the paintings 1, 2, 3, and 4, write B for bedroom and D for den, and make a list.

B-D	B-D	B-D	B-D
1-2	2-2	3-1	4-1
1-3	2-3	3-2	4-2
1-4	2-4	3-4	4-3

There are **12** ways to choose which paintings go in each room.

In Example 5, making a list was feasible, but if Ariel had 10 paintings and needed to hang 4 of them, it would be impossible to list all the different ways of hanging them. In such cases, we need the ***counting principle.***

THE COUNTING PRINCIPLE

KEY FACT O2

If two jobs need to be completed and there are *m* ways to do the first job and *n* ways to do the second job, then there are $m \times n$ ways to do one job followed by the other. This principle can be extended to any number of jobs.

In Example 5, the first job was to pick 1 of the 4 paintings and hang it in the bedroom. That could be done in 4 ways. The second job was to pick a second painting to hang in the den. That job could be accomplished by choosing any of the remaining 3 paintings. So there are 4 × 3 = **12** ways to hang 2 of the paintings.

Now, assume that there are 10 paintings to be hung in 4 rooms. The first job is to choose one of the 10 paintings for the bedroom. The second job is to choose one of the 9 remaining paintings to hang in the den. The third job is to choose one of the 8 remaining paintings for the living room. Finally, the fourth job is to pick one of the 7 remaining paintings for the dining room. These 4 jobs can be completed in $10 \times 9 \times 8 \times 7 =$ **5040** ways.

EXAMPLE 6

How many integers are there between 100 and 1000 all of whose digits are odd?

SOLUTION.
We're looking for three-digit numbers, such as 135, 711, 353, and 999, in which all three digits are odd. Note that we are *not* required to use three different digits. Although you certainly wouldn't want to list all of them, you could count them by listing some of them and seeing if a pattern develops.

- In the 100s there are 5 numbers that begin with 11: 111, 113, 115, 117, 119.
- Similarly, there are 5 numbers that begin with 13: 131, 133, 135, 137, 139.
- There are 5 that begin with 15; 5 that begin with 17; and 5 that begin with 19.
- A total of $5 \times 5 = 25$ in the 100s.
- In the same way there are 25 in the 300s, 25 in the 500s, 25 in the 700s, and 25 in the 900s, for a grand total of $5 \times 25 =$ **125**.

You can actually do this in less time than it takes to read this paragraph.

The best way to solve Example 6, however, is to use the counting principle. Think of writing a three-digit number as three jobs that need to be done. The first job is to select one of the five odd digits and use it as the digit in the hundreds place. The second job is to select one of the five odd digits to be the digit that goes in the tens place. Finally, the third job is to select one of the five odd digits to be the digit in the units place. Each of these jobs can be done in 5 ways. So the total number of ways is $5 \times 5 \times 5 =$ **125**.

EXAMPLE 7

How many different arrangements are there of the letters *A*, *B*, *C*, and *D*?

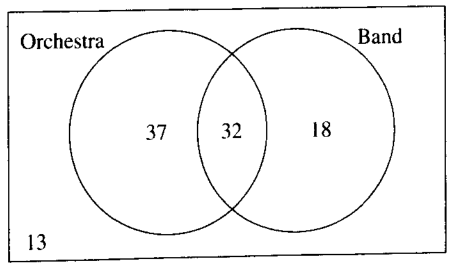
Ⓐ 4 Ⓑ 6 Ⓒ 8 Ⓓ 12 Ⓔ 24

Since from the choices given, we know that the answer is a relatively small number, we could just use TACTIC O1 and systematically list them: *ABCD*, *ABDC*, *ACBD*, ... However, this method would not be suitable if you had to arrange as few as 5 or 6 letters and would be essentially impossible if you had to arrange 10 or 20 letters.

SOLUTION.
Think of the act of arranging the four letters as four jobs that need to be done, and use the counting principle. The first job is to choose one of the four letters to write in the first position; there are 4 ways to complete that job. The second job is to choose one of the remaining three letters to write in the second position; there are 3 ways to complete that job. The third job is to choose one of the two remaining letters to write in the third position; there are 2 ways to complete that job. Finally, the fourth job is to choose the only remaining letter and to write it: $4 \times 3 \times 2 \times 1 =$ **24**.

VENN DIAGRAMS

A ***Venn diagram*** is a figure with two or three overlapping circles, usually enclosed in a rectangle, which is used to solve certain counting problems. To illustrate this, assume that a school has 100 seniors. The following Venn diagram, which divides the rectangle into four regions, shows the distribution of those students in the band and the orchestra.



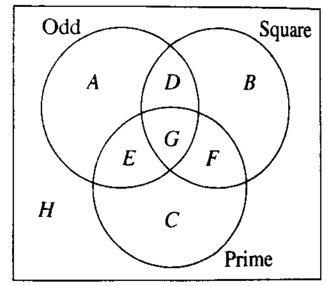
The 32 written in the part of the diagram where the two circles overlap represents the 32 seniors who are in both band and orchestra. The 18 written in the circle on the right represents the 18 seniors who are in band but not in orchestra, while the 37 written in the left circle represents the 37 seniors who are in orchestra but not in band. Finally, the 13 written in the rectangle outside of the circles represents the 13 seniors who are in neither band nor orchestra. The numbers in all four regions must add up to the total number of seniors: $32 + 18 + 37 + 13 = 100$. Note that there are 50 seniors in the band — 32 who are also in the orchestra and 18 who are not in the orchestra. Similarly, there are $32 + 37 = 69$ seniors in the orchestra. Be careful: the 50 names on the band roster and the 69 names on the orchestra roster add up to 119 names — more than the number of seniors. That's because 32 names are on both lists and so have been counted twice. The number of seniors who are in band or orchestra is only $119 - 32 = 87$. Those 87 together with the 13 seniors who are in neither make up the total of 100.

On the GRE, Venn diagrams are used in two ways. It is possible to be given a Venn diagram and asked a question about it, as in Example 8. More often, you will come across a problem, such as Example 9, that you will be able to solve more easily if you think to draw a Venn diagram.

EXAMPLE 8

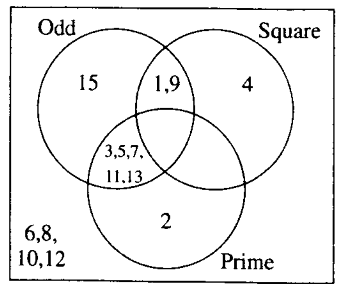
If the integers from 1 through 15 are each placed in the diagram at the right, which regions are empty?

Indicate *all* such regions.



[A] A
[B] B
[C] C
[D] D
[E] E
[F] F
[G] G
[H] H

SOLUTION. The easiest way is just to put each of the numbers from 1 through 15 in the appropriate region.

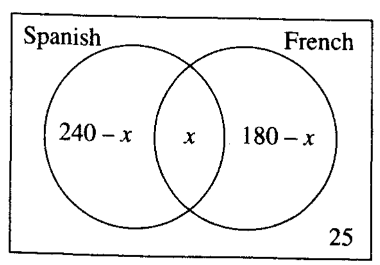


The empty regions are **F and G.**

EXAMPLE 9

Of the 410 students at H. S. Truman High School, 240 study Spanish and 180 study French. If 25 students study neither language, how many study both?

SOLUTION.
Draw a Venn diagram.



Let x represent the number of students who study both languages, and write x in the part of the diagram where the two circles overlap. Then the number who study only Spanish is $240 - x$, and the number who study only French is $180 - x$. The number who study at least one of the languages is $410 - 25 = 385$, so we have

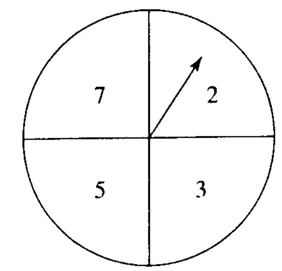
$$385 = (240 - x) + x + (180 - x) = 420 - x \Rightarrow x = 420 - 385 = \mathbf{35}$$

students who study both.

NOTE: No problem *requires* the use of a Venn diagram. On some problems you might even find it easier not to use one. In Example 9, you could have reasoned that if there were 410 students in the school and 25 didn't study either language, then there were $410 - 25 = 385$ students who studied at least one language. There are 240 names on the Spanish class lists and 180 on the French class lists, a total of $240 + 180 = 420$ names. But those 420 names belong to only 385 students. It must be that $420 - 385 = 35$ names were repeated. In other words, 35 students are in both French and Spanish classes.

PROBABILITY

The ***probability*** that an ***event*** will occur is a number between 0 and 1, usually written as a fraction, which indicates how likely it is that the event will happen. For example, if you spin the spinner in the diagram, there are 4 possible outcomes. It is equally likely that the spinner will stop in any of the 4 regions. There is 1 chance in 4 that it will stop in the region marked 2. So we say that the probability of spinning a 2 is one-fourth and write $P(2) = \frac{1}{4}$. Since 2 is the only even number on the spinner we could also say $P(\text{even}) = \frac{1}{4}$. There are 3 chances in 4 that the spinner will land in a region with an odd number in it, so $P(\text{odd}) = \frac{3}{4}$.



KEY FACT O3

If *E* is any event, the probability that *E* will occur is given by

$$P(E) = \frac{\textbf{number of favorable outcomes}}{\textbf{total number of possible outcomes}},$$

assuming that the possible outcomes are all equally likely.

In the preceding example, each of the 4 regions is the same size, so it is equally likely that the spinner will land on the 2, 3, 5, or 7. Therefore,

$$P(\text{odd}) = \frac{\text{number of ways of getting an odd number}}{\text{total number of possible outcomes}} = \frac{3}{4}$$

Note that the probability of *not* getting an odd number is 1 minus the probability of getting an odd number: $1 - \frac{3}{4} = \frac{1}{4}$. Let's look at some other probabilities associated with spinning this spinner once.

$$P(\text{number} > 10) = \frac{\text{number of ways of getting a number} > 10}{\text{total number of possible outcomes}} = \frac{0}{4} = 0.$$

$$P(\text{prime number}) = \frac{\text{number of ways of getting a prime number}}{\text{total number of possible outcomes}} = \frac{4}{4} = 1.$$

$$P(\text{number} < 4) = \frac{\text{number of ways of getting a number} < 4}{\text{total number of possible outcomes}} = \frac{2}{4} = \frac{1}{2}.$$

KEY FACT O4

Let E be an event, and $P(E)$ the probability it will occur.

- If E is ***impossible*** (such as getting a number greater than 10), $\boldsymbol{P(E) = 0}$.
- If it is ***certain*** that E will occur (such as getting a prime number), $\boldsymbol{P(E) = 1}$.
- In all cases $\boldsymbol{0 \leq P(E) \leq 1}$.
- The probability that event E will not occur is $\boldsymbol{1 - P(E)}$.
- If 2 or more events are mutually exclusive and constitute all the outcomes, the sum of their probabilities is 1.

  [For example, $P(\text{even}) + P(\text{odd}) = \frac{1}{4} + \frac{3}{4} = 1$.]
- The more likely it is that an event will occur, the higher its probability (the closer to 1 it is); the less likely it is that an event will occur, the lower its probability (the closer to 0 it is).

Even though probability is defined as a fraction, we can write probabilities as decimals or percents, as well.

Instead of writing $P(E) = \frac{1}{2}$, we can write $P(E) = .50$ or $P(E) = 50\%$.

EXAMPLE 10

An integer between 100 and 999, inclusive, is chosen at random. What is the probability that all the digits of the number are odd?

SOLUTION.
By KEY FACT O1, since both endpoints are included, there are $999 - 100 + 1 = 900$ integers between 100 and 999. In Example 6, we saw that there are 125 three-digit numbers all of whose digits are odd. So the probability is

$$\frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}} = \frac{\mathbf{125}}{\mathbf{900}} = \frac{\mathbf{5}}{\mathbf{36}}.$$

NOTE: On a numeric entry question it is ***not*** necessary to reduce fractions, so $\frac{125}{900}$ is perfectly acceptable.

KEY FACT O5

If an experiment is done two (or more) times, the probability that first one event will occur and then a second event will occur is the product of the probabilities.

EXAMPLE 11

A fair coin is flipped three times. What is the probability that the coin lands heads each time?

SOLUTION.
When a fair coin is flipped:

$$P(\text{head}) = \frac{1}{2} \text{ and } P(\text{tail}) = \frac{1}{2}.$$

By KEY FACT O5, $P(3 \text{ heads}) =$

$$P(\text{head 1st time}) \times P(\text{head 2nd time}) \times P(\text{head 3rd time}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{\mathbf{1}}{\mathbf{8}}.$$

Another way to handle problems such as Example 11 is to make a list of all the possible outcomes. For example, if a coin is tossed three times, the possible outcomes are

head, head, head	head, head, tail
head, tail, head	head, tail, tail
tail, head, head	tail, head, tail
tail, tail, head	tail, tail, tail

On the GRE, of course, if you choose to list the outcomes on your scrap paper, you should abbreviate and just write HHH, HHT, and so on. In any event, there are eight possible outcomes, and only one of them (HHH) is favorable. So the probability is $\frac{\mathbf{1}}{\mathbf{8}}$.

EXAMPLE 12

Three fair coins are flipped.

Quantity A	Quantity B
The probability of getting more heads than tails	The probability of getting more tails than heads

SOLUTION.
From the list of the 8 possible outcomes mentioned, you can see that in 4 of them (HHH, HHT, HTH, THH) there are more heads than tails, and that in 4 of them (TTT, TTH, THT, HTT) there are more tails than heads. Each probability is $\frac{4}{8}$. The answer is **C**.

In Example 12, it wasn't even necessary to calculate the two probabilities. Since heads and tails are equally likely, when several coins are flipped, it is just as likely to have more heads as it is to have more tails. This is typical of quantitative comparison questions on probability; you usually can tell which of the two probabilities is greater without having to calculate either one. This is another instance where you can use TACTIC 5 from Chapter 10: don't calculate, compare.

EXAMPLE 13

The numbers from 1 to 1000 are each written on a slip of paper and placed in a box. Then 1 slip is removed.

Quantity A	Quantity B
The probability that the number drawn is a multiple of 5	The probability that the number drawn is a multiple of 7

SOLUTION.
Since there are many more multiples of 5 than there are of 7, it is more likely that a multiple of 5 will be drawn. Quantity **A** is greater.

Practice Exercises — Counting and Probability

Discrete Quantitative Questions

1. Alyssa completed exercises 6–20 on her math review sheet in 30 minutes. At this rate, how long, in minutes, will it take her to complete exercises 29–57?

Ⓐ 56
Ⓑ 57
Ⓒ 58
Ⓓ 60
Ⓔ 65

2. A diner serves a lunch special, consisting of soup or salad, a sandwich, coffee or tea, and a dessert. If the menu lists 3 soups, 2 salads, 7 sandwiches, and 8 desserts, how many different lunches can you choose? (*Note*: Two lunches are different if they differ in any aspect.)

Ⓐ 22
Ⓑ 280
Ⓒ 336
Ⓓ 560
Ⓔ 672

3. Dwight Eisenhower was born on October 14, 1890 and died on March 28, 1969. What was his age, in years, at the time of his death?

Ⓐ 77
Ⓑ 78
Ⓒ 79
Ⓓ 80
Ⓔ 81

4. How many four-digit numbers have only even digits?

[ ]

5. There are 27 students on the college debate team. What is the probability that at least 3 of them have their birthdays in the same month?

Ⓐ 0
Ⓑ $\frac{3}{27}$
Ⓒ $\frac{3}{12}$
Ⓓ $\frac{1}{2}$
Ⓔ 1

6. Let A be the set of primes less than 6, and B be the set of positive odd numbers less than 6. How many different sums of the form $a + b$ are possible, if a is in A and b is in B?

Ⓐ 6
Ⓑ 7
Ⓒ 8
Ⓓ 9
Ⓔ 10

7. There are 100 people on a line. Aviva is the 37th person and Naomi is the 67th person. If a person on line is chosen at random, what is the probability that the person is standing between Aviva and Naomi?

$\frac{[\ ]}{[\ ]}$

8. A jar has 5 marbles, 1 of each of the colors red, white, blue, green, and yellow. If 4 marbles are removed from the jar, what is the probability that the yellow one was removed?

Ⓐ $\frac{1}{20}$

Ⓑ $\frac{1}{5}$

Ⓒ $\frac{1}{4}$

Ⓓ $\frac{4}{5}$

Ⓔ $\frac{5}{4}$

9. Josh works on the second floor of a building. There are 10 doors to the building and 8 staircases from the first to the second floor. Josh decided that each day he would enter by one door and leave by a different one, and go up one staircase and down another. How many days could Josh do this before he had to repeat a path he had previously taken?

Ⓐ 80
Ⓑ 640
Ⓒ 800
Ⓓ 5040
Ⓔ 6400

10. A jar contains 20 marbles: 4 red, 6 white, and 10 blue. If you remove marbles one at a time, randomly, what is the minimum number that must be removed to be certain that you have at least 2 marbles of each color?

Ⓐ 6
Ⓑ 10
Ⓒ 12
Ⓓ 16
Ⓔ 18

11. At the audition for the school play, n people tried out. If k people went before Judy, who went before Liz, and m people went after Liz, how many people tried out between Judy and Liz?

Ⓐ $n - m - k - 2$
Ⓑ $n - m - k - 1$
Ⓒ $n - m - k$
Ⓓ $n - m - k + 1$
Ⓔ $n - m - k + 2$

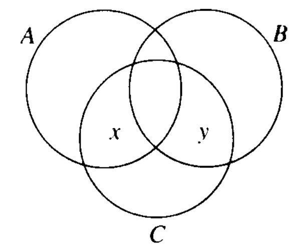
12. In a group of 100 students, more students are on the fencing team than are members of the French club. If 70 are in the club and 20 are neither on the team nor in the club, what is the minimum number of students who could be both on the team and in the club?

Ⓐ 10
Ⓑ 49
Ⓒ 50
Ⓓ 60
Ⓔ 61

13. In a singles tennis tournament that has 125 entrants, a player is eliminated whenever he loses a match. How many matches are played in the entire tournament?

Questions 14–15 refer to the following diagram.

A is the set of positive integers less than 20; B is the set of positive integers that contain the digit 7; and C is the set of primes.



14. How many numbers are in the region labeled x?

Ⓐ 4
Ⓑ 5
Ⓒ 6
Ⓓ 7
Ⓔ 8

15. What is the sum of all the numbers less than 50 that are in the region labeled y?

Quantitative Comparison Questions

Ⓐ Quantity A is greater.
Ⓑ Quantity B is greater.
Ⓒ Quantities A and B are equal.
Ⓓ It is impossible to determine which quantity is greater.

	Quantity A	Quantity B
16.	The probability of getting no heads when a fair coin is flipped 7 times	The probability of getting 7 heads when a fair coin is flipped 7 times

A jar contains 4 marbles: 2 red and 2 white. 2 marbles are chosen at random.

	Quantity A	Quantity B
17.	The probability that the marbles chosen are the same color	The probability that the marbles chosen are different colors

	Quantity A	Quantity B
18.	The number of ways to assign a number from 1 to 5 to each of 4 people	The number of ways to assign a number from 1 to 5 to each of 5 people

	Quantity A	Quantity B
19.	The probability that 2 people chosen at random were born on the same day of the week	The probability that 2 people chosen at random were born in the same month

	Quantity A	Quantity B
20.	The probability a number chosen at random from the primes between 100 and 199 is odd.	.99

ANSWER KEY

1. **C**	5. **E**	8. **D**	12. **E**	16. **C**	20. **A**
2. **D**	6. **B**	9. **D**	13. **124**	17. **B**	
3. **B**	7. $\frac{\mathbf{29}}{\mathbf{100}}$	10. **E**	14. **C**	18. **C**	
4. **500**		11. **A**	15. **84**	19. **A**	

Answer Explanations

1. **(C)** Alyssa completed 20 – 6 + 1 = 15 exercises in 30 minutes, which is a rate of 1 exercise every 2 minutes. Therefore, to complete 57 – 29 + 1 = 29 exercises would take her 58 minutes.

2. **(D)** You can choose your soup or salad in any of 5 ways, your beverage in any of 2 ways, your sandwich in 7 ways, and your dessert in 8 ways. The counting principle says to multiply: 5 × 2 × 7 × 8 = 560. (Note that if you got soup *and* a salad, then instead of 5 choices for the first course there would have been 2 × 3 = 6 choices for the first two courses.)

3. **(B)** His last birthday was in October 1968, when he turned 78: 1968 – 1890 = 78.

4. **500** The easiest way to solve this problem is to use the counting principle. The first digit can be chosen in any of 4 ways (2, 4, 6, 8), whereas the second, third, and fourth digits can be chosen in any of 5 ways (0, 2, 4, 6, 8). Therefore, the total number of four-digit numbers all of whose digits are even is 4 × 5 × 5 × 5 = 500.

5. **(E)** If there were no month in which at least 3 students had a birthday, then each month would have the birthdays of at most 2 students. But that's not possible. Even if there were 2 birthdays in January, 2 in February, ..., and 2 in December, that would account for only 24 students. It is guaranteed that with more than 24 students, at least one month will have 3 or more birthdays. The probability is 1.

6. **(B)** A = {2, 3, 5} and B = {1, 3, 5}. Any of the 3 numbers in A could be added to any of the 3 numbers in B, so there are 9 sums that could be formed. However, there could be some duplication. List the sums systematically; first add 1 to each number in A, then 3, and then 5: 3, 4, 6; 5, 6, 10; 7, 8, 10. There are 7 different sums.

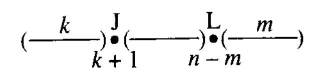
7. $\frac{\mathbf{29}}{\mathbf{100}}$ There are 67 – 37 – 1 = 29 people between Aviva and Naomi, so, the probability that one of them is chosen is $\frac{29}{100}$.

8. **(D)** It is equally likely that any one of the 5 marbles will be the one that is not removed. So, the probability that the yellow one is left is $\frac{1}{5}$ and the probability that it is removed is $\frac{4}{5}$.

9. **(D)** This is the counting principle at work. Each day Josh has four jobs to do: choose 1 of the 10 doors to enter and 1 of the 9 other doors to exit; choose 1 of the 8 staircases to go up and 1 of the other 7 to come down. This can be done in 10 × 9 × 8 × 7 = 5040 ways. So on each of 5040 days Josh could choose a different path.

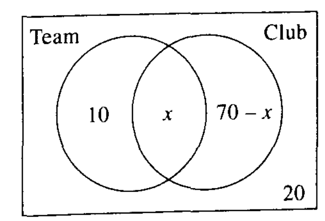
10. **(E)** In a problem like this the easiest thing to do is to see what could go wrong in your attempt to get 2 marbles of each color. If you were really unlucky, you might remove 10 blue ones in a row, followed by all 6 white ones. At that point you would have 16 marbles, and you still wouldn't have even 1 red one. The next 2 marbles, however, must both be red. The answer is 18.

11. **(A)** It may help to draw a line and label it:



Since k people went before Judy, she was number k + 1 to try out; and since m people went after Liz, she was number n – m to try out. So the number of people to try out between them was (n – m) – (k + 1) – 1 = n – m – k – 2.

12. **(E)** Draw a Venn diagram, letting x be the number of students who are on the team and in the club.



Of the 100 students, 70 are in the club, so 30 are not in the club. Of these 30, 20 are also not on the team, so 10 are on the team but not in the club. Since more students are on the team than in the club, 10 + x > 70 ⇒ x > 60. Since x must be an integer, the least it can be is 61.

13. **124** You could try to break it down by saying that first 124 of the 125 players would be paired off and play 62 matches. The 62 losers would be eliminated and there would still be 63 people left, the 62 winners and the 1 person who didn't play yet. Then continue until only 1 person was left. This is too time-consuming. An easier way is to observe that the winner never loses and the other 124 players each lose once. Since each match has exactly one loser, there must be 124 matches.

14. **(C)** The region labeled x contains all of the primes less than 20 that do *not* contain the digit 7. They are 2, 3, 5, 11, 13, 19.

15. **84** Region y consists of primes that contain the digit 7 and are greater than 20. There are two of them that are less than 50: 37 and 47. Their sum is 84.

16. **(C)** Don't calculate the probabilities. The probability of no heads is equal to the probability of no tails; but no tails means all heads.

17. **(B)** The simplest solution is to notice that whatever color the first marble is, there is only 1 more marble of that color, but there are 2 of the other color, so it is twice as likely that the marbles will be of different colors.

18. **(C)** By the counting principle, Quantity A is $5 \cdot 4 \cdot 3 \cdot 2$ and Quantity B is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Clearly, the quantities are equal.

19. **(A)** Quantity A: the probability is $\frac{1}{7}$. Quantity B: the probability is $\frac{1}{12}$.

20. **(A)** Every prime between 100 and 199 is odd (the only even prime is 2). So Quantity A is 1, which is greater than .99.

PART 5

MODEL TESTS