Calculus II Notes

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1 Integration By Parts

Formula

$u \cdot v - \int v \cdot du$

Example

$$
\begin{gathered}
\int x^3 \cdot sin(x) \cdot dx \\
\begin{array}{l|l}
\text{u} & \text{dv} \\
x^3 & sin(x) \\
3x^2 & cos(x) \\
6x & -sin(x) \\
6 & -cos(x) \\
0 & sin(x)
\end{array} \\
x^3 \cdot cos(x) - \int 3x^2 \cdot cos(x) \cdot dx \\
x^3 \cdot cos(x) + 3x^2 \cdot sin(x) - \int -6x \cdot sin(x) \cdot dx \\
x^3 \cdot cos(x) + 3x^2 \cdot sin(x) + 6x \cdot cos(x) - \int 6 \cdot cos(x) \cdot dx \\
x^3 \cdot cos(x) + 3x^2 \cdot sin(x) + 6x \cdot cos(x) - 6 \cdot sin(x)
\end{gathered} \tag{1}
$$

2 Trigonometric Integrals and Substitutions

Trigonometric Identities

1. $sec(x) = \dfrac{1}{cos(x)}$
2. $csc(x) = \dfrac{1}{sec(x)}$
3. $cot(x) = \dfrac{1}{tan(x)}$
4. $sin^2(x) + cos^2(x) = 1$
5. $tan^2(x) + 1 = sec^2(x)$
6. Double Angles

(a) $sin^2(x) = \frac{1}{2} \cdot (1 - cos(2x))$

(b) $cos^2(x) = \frac{1}{2} \cdot (1 + cos(2x))$

7. $\frac{d}{dx} tan(x) = sec^2(x)$

8. $\frac{d}{dx} sec(x) = sec(x) \cdot tan(x)$

9. $\int sec(x) \cdot dx = ln|sec(x) + tan(x)| + C$

10. $\int tan(x) \cdot dx = -log(cos(x)) = ln|sec(x)| + C$

11. Substitutions

Integrand	Substitution	Boundaries	Trig Identity
$\sqrt{a^2 - x^2}$	$x = a \cdot sin(\Theta)$	$\frac{\Pi}{2} \leq \Theta \leq \frac{\Pi}{2}$	$sin^2(\Theta) + cos^2(\Theta) = 1$
$\sqrt{a^2 + x^2}$	$x = a \cdot tan(\Theta)$	$\frac{-\Pi}{2} < \Theta < \frac{\Pi}{2}$	$tan^2(x) + 1 = sec^2(x)$
$\sqrt{x^2 - a^2}$	$x = a \cdot sec(\Theta)$	$0 < \Theta < \frac{\Pi}{2}, \Pi < \Theta < \frac{3\Pi}{2}$	$tan^2(x) + 1 = sec^2(x)$

Examples

$$\begin{aligned}
\int \frac{x}{\sqrt{1 - x^2}} \cdot dx \\
x = sin(\Theta) \\
x^2 = sin^2(\Theta) \\
dx = cos(\Theta) \\
\int \frac{sin(\Theta)}{\sqrt{1 - sin^2(\Theta)}} \cdot d\Theta \\
\int \frac{sin(\Theta)}{\sqrt{cos^2(\Theta)}} \cdot d\Theta \\
\int \frac{sin(\Theta)}{cos(\Theta)} \cdot d\Theta \\
\int tan(\Theta) \cdot d\Theta \\
ln|sec(\Theta)| + C \\
ln|sec(arcsin(x))| + C
\end{aligned} \tag{2}$$

3 Partial Fraction Decomposition

This is meant to simplify integrals of rational functions.

Rational functions are ratios of polynomials in the form $\frac{P(x)}{Q(x)}$ while P(x) and Q(x) are arbitrary polynomials. PROPER iff (degree of Q(x) > degree of P(x))

Long Division

Suppose $Q(x) \leq P(x)$

After long division you will get $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$ while R(x) is the remainder (R(x) is ALWAYS less than Q(x)).

PFD: Replacing proper fractions by the sum of simpler fractions that we can integrate.

There are two ways to solve for A and B

1. Zero out one or the other (see ex.)
2. Expand and collect terms (see ex.)

Example

$$\begin{aligned}
\int \frac{x}{(x+1)(x-2)} \cdot dx \\
\frac{A}{x+1} + \frac{B}{x-2} \\
\frac{x}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \\
\frac{\cancel{(x+1)(x-2)} \cdot x}{\cancel{(x+1)(x-2)}} = \frac{\cancel{(x+1)}(x-2) \cdot A}{\cancel{x+1}} + \frac{(x+1)\cancel{(x-2)} \cdot B}{\cancel{x-2}} \\
x = A \cdot (x-2) + B \cdot (x+1) \\
x = 2 \therefore 2 = 3B \therefore B = \frac{2}{3}, A = \frac{1}{3} \\
or \\
x = Ax - 2A + Bx + B \\
1 = A + B \\
0 = B - 2A \\
B = 2A \\
A = \frac{1}{3}, B = \frac{2}{3}
\end{aligned} \tag{3}$$

Cases

1. For each 1^{st} order, non-repeated factor, you add to the PFD a term of the form $\dfrac{A}{ax+b}$

$$\frac{A_0}{a_0x+b_0} + \frac{A_1}{a_1x+b_1} + ... + \frac{A_n}{a_nx+b_n}$$

2. For each 1^{st} order factor $(ax+b)$ repeated n times, $[(ax+b)^n]$ add it to the PFD n times.

$$\frac{A_0}{a_0x+b_0} + \frac{A_1}{(a_1x+b_1)^2} + ... + \frac{A_n}{(a_nx+b_n)^n}$$

3. For each irreducible 2^{nd} order, non-repeated factor $[(ax^2+bx+c)$ for $(b^2-4ac) < 0]$ add it to the PFD one term.

$$\frac{Ax+B}{ax^2+bx+c}$$

4. For each irreducible 2^{nd} order, repeated factor $[(ax^2+bx+c)^n$ for $(b^2-4ac)<0]$ add it to the PFD n terms.

$$\frac{A_0x+B_0}{(ax^2+bx+c)^1}+\frac{A_1x+B_1}{(ax^2+bx+c)^2}+...+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

4 Sequences

A sequence is an ordered, infinite list of numbers.
$\lim_{n\to\infty} a_1, a_2, a_3, ..., a_n$
$\{a_n\}_{n=1}^{\infty}$ indicates a sequence.
We can think of a sequence as a function:
$n \in \mathbb{N}$ and $f(n) = a_n$
Two types of sequence definition

1. Linearly: $a_n = \dfrac{n}{n+1}$ so $a_1 = \dfrac{1}{2}, a_2 = \dfrac{2}{3}, etc.$

2. Recursively (Fibonacci): $\{f_n\}_1^{\infty} f_1 = 1, f_2 = 2, f_n = f_{n-1} + f_{n-2}$

A sequence can also be pictured by graphing.

Squeeze Theorem (Sammich Theorem)

Let $\{a_n\}_1^{\infty}, \{b_n\}_1^{\infty}, \{c_n\}_1^{\infty}$ and $a_n \leq b_n \leq c_n$
If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$

5 Series

A series is a sum of an infinite sequence of terms.
Let $\{a_n\}_{n=1}^{\infty}$, the series with these terms is $\sum_{n=1}^{\infty} a_n$
It is possible for a sum of an infinite number of terms to add up to a finite number. This is called a convergent series.

Consider:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_n &= a_1 + a_2 + ... + a_n \end{aligned} \tag{4}$$

s_n is called the sequence of partial sums ($\{s_n\}_{n=1}^{\infty}$) and the convergence of the series depends on its convergence.

If $\lim_{n\to\infty} s_n = L$ then it's convergent.
If $\lim_{n\to\infty} s_n = (+\infty, -\infty)$ then it's divergent.
If $\lim_{n\to\infty}$ does not exist, then the test is inconclusive.

Geometric Series

$\sum_{n=1}^{\infty} a \cdot r^{n-1}$ where $a \neq 0$ and $r =$ the ratio of the series
If $-1 < r < 1$ then the series is convergent to $\frac{a}{1-r}$.
If $r \geq 1$ then it is divergent.
If $r \leq -1$ then it is not regular (neither convergent or divergent).

Shifting range of series

Formula:

$$
\begin{aligned}
\sum_{n=x}^{\infty} a \cdot r^{n+y} \\
\sum_{n=x}^{\infty} a \cdot r^{n-x+y+x} \\
\sum_{n=x}^{\infty} a \cdot r^{n-x} \cdot r^{y+x} \\
\sum_{n=x}^{\infty} r^{n-1}(a(r^{y+x})) \\
\frac{a \cdot r^{y+x}}{1-r} \\
= \frac{a \cdot r^{y+x}}{1-r}
\end{aligned} \tag{5}
$$

Harmonic Series

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ is DIVERGENT
$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ is called the generalized harmonic series. It is convergent if $\alpha > 1$ and divergent if $\alpha \leq 1$.

6 Series Tests

Divergence Test (Test for un-convergence)

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$
If $\lim_{n\to\infty} a_n \neq 0$, then the series may or may not converge...

Integral Test

If $a_n = f(x)$ and the function is continuous, decreasing, and positive on $[1, +\infty)$, then the series is convergent iff the integral of the function is convergent. Iff $\int_1^{\infty} f(x) \cdot dx$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent and vice=versa with divergence.

Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with positive terms. If $a_n \leq b_n$ (for all n, or for all $n \geq N$) and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well. If $a_n \geq b_n$ (for all n, or for all $n \geq N$) and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = C, c \neq [0, \infty)$, then the two series are either both convergent or divergent.

Alternating Series Test (Leibniz' Test)

This ONLY applies to alternating series
$\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ where $a_n \geq 0$.
if $\lim_{n\to\infty} a_n = 0$ and a_n is decreasing for all n then the series is convergent.

Absolute Values Test

For any series $\sum_{n=1}^{\infty} a_n$ you must consider the absolute value series $\sum_{n=1}^{\infty} |a_n|$. If the series of absolute values is convergent, it is called absolutely convergent. Any series that is absolutely convergent is also convergent ($-|a_n| \leq a_n \leq |a_n|$). There exist many series that are convergent, but NOT absolutely convergent (these are called conditionally convergent). For example, an alternating harmonic series is conditionally convergent.

Ratio Test

$\sum_{n=1}^{\infty} a_n$
if:

$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$ then the series is absolutely convergent
$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L > 1$ the the series is not absolutely convergent
$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$ then the test is inconclusive

Root Test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} = L$$
If:

$L < 1$ then the series is absolutely convergent $L > 1$ then the series is not absolutely convergent
$L = 1$ then the test is inconclusive

7 Power Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

Representing Functions as Power Series

$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ where $|x| < 1$ (geometric series $a = 1$, ratio of$x$)

This is a power series centered at 0 with a radius of convergence of $R = 1$

If a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$ then the interval of convergence $|x-a| < R$

The function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable inside the interval of convergence.

$f'(x) = \sum_{n=0}^{\infty} c_n \cdot n \cdot (x-a)^{n-1}$ and $\int f(x) \cdot dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$

Examples

$$
\begin{aligned}
f(x) &= \frac{1}{1-x^2} \\
&\frac{1}{1-(-x^2)} \\
u &= (-x^2) \\
&\frac{1}{1-u} \\
&\sum_{n=0}^{\infty} u^n \\
&\sum_{n=0}^{\infty} (-x^2)^n \\
&\sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}
\end{aligned} \tag{6}
$$

$$\begin{aligned} f(x) &= \frac{1}{3+x} \\ &\frac{1}{3\cdot(1+\frac{x}{3})} \\ &\frac{1}{3}\cdot\frac{1}{1+\frac{x}{3}} \\ u &= \frac{-x}{3} \\ &\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n \\ &\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n\cdot x^n \\ &\sum_{n=0}^{\infty}\frac{1}{3}\cdot\frac{(-1)^n}{3^n}\cdot x^n \\ &\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}\cdot x^n \end{aligned} \tag{7}$$

Interval of convergence $= |x| < 3$

$$\begin{aligned} f(x) = \frac{1}{(1-x)^2} &\frac{1}{\to 1-2x-x^2} \\ &\frac{d}{dx}\frac{1}{1-x} \\ &\frac{d}{dx}\sum_{n=0}^{\infty}x^n, |x| < 1 \\ &\sum_{n=0}^{\infty}n\cdot x^{n-1} \end{aligned} \tag{8}$$

8 Taylor and MacLaurin Series

(Taylor series have arbitrary centers while MacLaurin are centered at 0)
Question: How do we know if a function has a power series representation? And for what values of x is it meaningful?

Assume: $\sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a| < R$

In other words: $f(x) = c_0 + c_1\cdot(x-a) + c_2\cdot(x-a)^2$

Evaluate c_n at $x = a$. $c_n = \frac{f^{(n)}(a)}{n!}$ while $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$

Theorem: If a function has a power series representation (or power series expansion) centered at a, i.e. $\sum_{n=0}^{\infty} c_n(x-a)^n, |x-a| < R$, then the coefficients are given by $c_n = \frac{f^{(n)}(a)}{n!}$.

These are all Taylor series centered at a. If $a = 0$, then it is called a MacLaurin series.

Need:

The function to be infinitely differentiable inside the interval $|x-a| < R$

Take partial sums in the power series ($T_n(x)$)

$T_n(x) = f(a) + f'(a)(x-a) + ... + \frac{f^{(n)}(a)(x-a)^n}{n!}$
$\lim\limits_{n\to\infty} T_n(x) = f(x)$
Consider $f(x) - T_n(x) = R_n(x)$ where $R_n(x)$ is the remainder of order n of the Taylor series.
$f(x) = \lim\limits_{n\to\infty} T_n(x)$ is equivalent to saying $\lim\limits_{n\to\infty} R_n(x) = 0$

Theorem: If $f(x) = T_n(x) + R_n(x)$ where $T_n(x)$ is a Taylor polynomial of degree n of $f(x)$ at a, and if $\lim\limits_{n\to\infty} R_n(x) = 0$ for all $|x-a| < R$, then $f(x) = \sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)(x-a)^n}{n!}$

Lagrange's Formula

The tricky bit is to show $\lim\limits_{n\to\infty} R_n(x) = 0$
In this case it is useful to consider special representations of remainder functions
Formula: If a function has at least $n+1$ derivatives in some interval I that contains the center, then there exists a number Z such that $x \leq Z \leq a$ and $R_n(x) = \frac{f^{(n+1)}(Z)(x-a)^{n+1}}{(n+1)!}$
If:

$x = 0$, then everything$= 0$
$x < 0$, then $x < Z < 0$
$x > 0$, then $0 < Z < x$

Application of Taylor Series

Given a function infinitely differentiable around $x = a$, to find its Taylor series centered at a:

1. Computer the Taylor coefficients $c_n = \frac{f^{(n)}(a)}{n!}$ and write down the corresponding Taylor series $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$
2. Find the radius of convergence and interval of convergence $|x-a| < R$
3. Apply Lagrange's formula for the remainder $R_n(x) = \frac{f^{(n+1)}(Z)(x-a)^{n+1}}{(n+1)!}$
4. $f(x) = \sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

Example

Find the MacLaurin series of $f(x) = e^x$ and its radius of convergence.

$$\boxed{1}$$

$$f^{(n)}(0) = e^0 = 1$$

$$c_n = \frac{1}{n!}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\boxed{2}$$

$$\text{Ratio Test of } \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{x^{\cancel{n+1}} \cdot \cancel{(n!)}}{(n+1)\cancel{!} \cdot \cancel{x^n}} \right| \tag{9}$$

$$\lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 \text{ regardless of x}$$

By the ratio test, the series is convergent for all $x \in \mathbb{R}$

The radius of convergence is $R = \infty$

$$\boxed{3}$$

$$0 < Z < x$$

$$R_n(x) = \frac{e^Z \cdot x^{n+1}}{(n+1)!}$$

if $x > 0$, then $0 < Z < x$ and by the Squeeze Theorem, it is 0

if $x < 0$, then $0 < Z < x$ and by the Squeeze Theorem, it is 0