A simple geometrical series that converges to π

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***Abstract*—This paper describes one of many possible ways to calculate the value of π with a geometrical series. This method will be particulary useful to calculate the value of π with iterative approach taking terms as per the required precision.**

I. Introduction

π comes flying all over high school mathematics. Beginner studensts often confuse the number π with 180^0 or $\frac{22}{7}$ which in fact is just a crude approximation of π. The approach taken in this paper will help beginners understand what actually is π.

II. Historical Note

Intuitively we can also guess that if we make bigger and bigger circle both diameter and circumference increase. It makes us wonder if there is a certain relation between the diameter and circumference of circle. Curious Babylonians in the early ages, out of similar curiosity, tried to measure the ratio of circumference to diameter. Unsurprisingly they found the ratio was close to 3. For various observations they found this ratio was always closer to 3, little greater than 3 and never larger than 4. They didn't precisely know the value of the ratio. So they named the ratio a magical symbol, which is π.

III. Main Idea

The main idea behind calculating the value of π is calculating the area of circle whose area is numerically equal to π. To formally calculate the value of π we begin with definition of π.

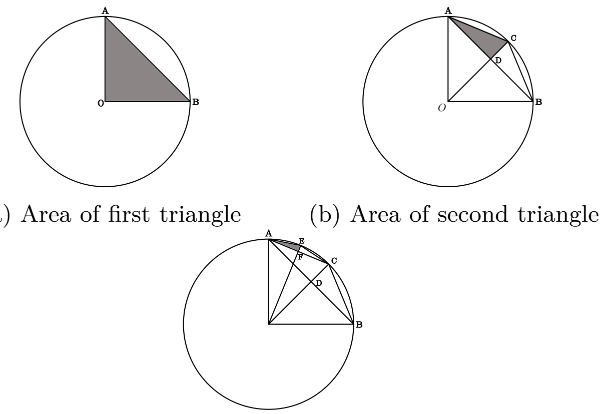
Definition 1. *π is the ratio of circumference to diameter of a circle. i.e., $\pi = \frac{C}{d}$, where C is the circumference, d is the diameter of circle*

Using this definition of circumference of circle, it is not hard to work out the area of circle. As it turns out the area of circle in terms of the named constant comes out to be [1]

$$A = \pi r^2$$

So far we don't know what is the value of number denoted by π. If we take a circle of unit radius, i.e., $r = 1$ then the area of circle turns, numerically, out to be π. We now calculate the area of unit circle geometrically. The area we calculate will, numerically, be the value of π which we all are looking for in this paper.

[1] The derivation of $A = \pi r^2$ is given in Appendix. A



(a) Area of first triangle (b) Area of second triangle

(c) Area of third triangle

Fig. 1: First two areas of the series in calculation of Area

IV. Area of unit circle

We use our knowledge of calculating the area of triangle to calculate the area of circle by dividing the circle . Lets divide the circle into four quarters. In Figure.(1a), we can approximate the area oc sector OAB (say S) with the area of right triangle ΔOAB. The are of ΔOAB is given simply by

$$\text{Area of } \Delta OAB\,(A_0) = \frac{1}{2} OA \times OB = \frac{1}{2}\text{units}$$

In first approximation the area of sector S is simply the area of triangle ΔOAB, i.e.,

$$S_0 = A_0 = \Delta OAB$$

But we know this is a very crude approximation for the area of sector. We can improve this approximation. For that let us construct a line OC through O and the midpoint D of AB meeting the perimeter of circle at C. The approximation to the area of sector OAB can now be improved by adding the area of two smaller triangles ΔDAC and ΔDBC which are equal in area (say A_1). The improved approximation to S can be

$$\begin{aligned} S_1 &= \Delta OAB + \Delta ADC + \Delta BDC \\ &= A_0 + 2A_1 \end{aligned}$$

Similarly, the approximation to the area can be improved by adding area of triangles in the gap left between the triangle and the circle further.

$$S_2 = A_0 + 2A_1 + 4A_2$$

Where A_2 is the area of triangle BDC.

Continuing this way, the approximation to area of sector S can be made as close to the actual area as we please by adding more and more triangles. And the series becomes

$$S_n = A_0 + 2A_1 + 4A_2 + 8A_3 + ... + 2^n A_n \quad (1)$$

And the exact area of sector OAB is the limit of the sum S_n as n tends to ∞. i.e.,

$$\text{Area of sector } OAB\,(S) = S_\infty = \sum_{n=0}^{\infty} 2^n A_n$$

V. Calculation

Since we have assumed a unit circle, the length of sides OA (p_0) and OB (b_0) corresponding to ΔOAB is unity. i.e.,

$$\begin{aligned} p_0(OA) &= 1 \\ b_0(OB) &= 1 \\ h_0(AB) &= \sqrt{p_0^2 + b_0^2} \end{aligned}$$

Thus the area of $\Delta OAB(A_0) = \frac{1}{2}p_0 \times b_0 = \frac{1}{2}$. In ΔACD in Figure.1b,

$$p_1(CD) = OC - CD = 1 - \sqrt{OB^2 - DB^2} \quad (2)$$

$$= 1 - \sqrt{1 - \left(\frac{AD}{2}\right)^2} = 1 - \sqrt{1 - \left(\frac{h_0}{2}\right)^2}$$

$$b_1(AD) = \frac{AB}{2} = \frac{h_0}{2} \qquad h_1(AC) = \sqrt{p_1^2 + b_1^2}$$

$$\text{Area of } \Delta ACD(A_1) = \frac{1}{2}p_1 \times b_1$$

Continuing this way, the general value of A_n can easily be calculated with the knowledge of p_n and b_n.

$$p_n = 1 - \sqrt{1 - \left(\frac{h_{n-1}}{2}\right)^2}$$

$$b_n = \frac{h_{n-1}}{2} \qquad h_n = \sqrt{p_n^2 + b_n^2}$$

$$A_n = \frac{1}{2}p_n \times b_n$$

Continuing this way we can calculate the value of every terms of Equation. (1) and hence find the value of series as we please.

VI. Numerical Value

We could write a simple python program to calculate the value of series in Equation. (1).

Listing 1: Python program to calculate the numerical value

```
#!/usr/bin/env python3

import math
def GeneratePlot():
    pn = 1
    bn = 1
    hn = math.sqrt(pn**2 + bn ** 2)
    s = 0
    for i in range(10):                    #iterate only
        10 times
        An = 0.5 * pn * bn                  # Area of nth
            trianlge
        bn = 0.5 * hn)
        pn = 1 - math.sqrt(1- bn ** 2)
        hn = math.sqrt(pn**2 + bn**2)
        s = s+( 2**i *An)
        res = 4*s
        print(res)

GeneratePlot()
```

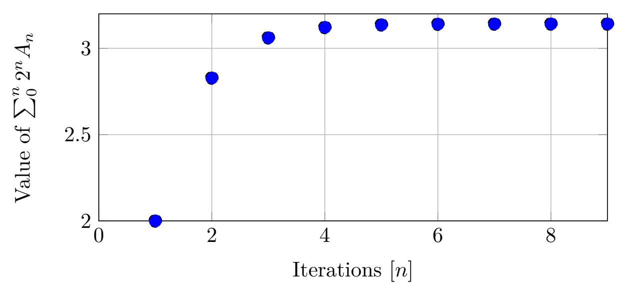


Fig. 2: Plot of values of series as a function of n

As in Figure.2 we see that the series converges rapidly to the actual value of π. In just 4 iterations we get the value with error of only 0.001%.

VII. Discussion

Value of π thus can be calculated with the beautiful series without much hassle, the geometric nature of its derivation can clearly be understood by high school level students without much difficulty. We hope this paper helps remove the common confusion among the beginners about what actually π is – a number whose value is between 3 and 4 in real line, precisely 3.141592653589 for 14 digit precision as we have calculated with the series proposed in this paper. The number π in essentially like any other number in real number system, the only difference being the location of it in real line.

VIII. Conclusion

We have constructed one series that is very intuitively easy to get to and converges very fast to the value of π. We have calculated the numerical value of the series which is 3.14159... [1].

Appendix

Let us take a circle of radius r. Suppose there is an infinitesimal ring at a distance $o < x < r$ from the center of the circle of width dx. The area of the infinitesimal ring can be calculated as

$$\mathrm{d}A = C\mathrm{d}x \tag{3}$$

Where C is the circumference of the infinitesimal ring of radius x. By definition of the circumference of the circle we get $C = 2\pi x$. Substituting the value of C in Equation. 3 we get

$$\begin{aligned} A = \int_0^r \mathrm{d}A &= \int_0^r C\mathrm{d}x \\ &= \int_0^r 2\pi x \mathrm{d}x \\ &= 2\pi \left[\frac{x^2}{2}\right]_0^r \\ &= \pi r^2 \end{aligned} \tag{4}$$

References

[1] H. Madhav and G. Prakash, The Delta Finders. The Delta Research Press (2016)