Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 1/2$. If it's discontinuous, identify the type of discontinuity and redefine the function to make it continuous.

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ \frac{3}{4} & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

Answer choices:

A The function is continuous at $x = 1/2$.

B The function is discontinuous at $x = 1/2$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 1/2$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

D The function is discontinuous at $x = 1/2$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

Solution: D

In order for the function to be continuous at $x = 1/2$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 1/2$.

We already know that the value of the function at $x = 1/2$ is $3/4$, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to $3/4$. If they exist but aren't equal to $3/4$, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x = 1/2$.

To look at the left-hand limit, we'll use the first "piece" of our piecewise-defined function, because it defines the function to the left of $x = 1/2$ (the domain of that piece is $x < 1/2$).

$$\lim_{x \to (1/2)^-} |2x - 1|$$

Since the domain of $|2x - 1|$ is $x < 1/2$, we know that no matter what value in the domain we plug in for, we're always going to get a negative value for $2x - 1$. That means we can take away the absolute value bars as long as we put a negative sign in front of $2x - 1$.

$$\lim_{x \to (1/2)^-} -(2x - 1)$$

$$\lim_{x \to (1/2)^-} 1 - 2x$$

$$1 - 2\left(\frac{1}{2}\right)$$

$$1 - 1$$

$$0$$

We know now that the left-hand limit exists, and that it's equal to 0. Let's look at the right-hand limit by using the third "piece" of the piecewise-defined function, since it defines the function to the right of $x = 1/2$ (the domain of that piece is $x > 1/2$).

$$\lim_{x \to (1/2)^+} \frac{2x-1}{2}$$

$$\frac{2\left(\frac{1}{2}\right)-1}{2}$$

$$\frac{1-1}{2}$$

$$0$$

We know now that the right-hand limit exists, and that it's equal to 0.

Since the left-hand limit exists at $x = 1/2$, the right-hand limit exists at $x = 1/2$, and the left- and right-hand limits are equal and therefore the general limit exists at $x = 1/2$, but the general limit at $x = 1/2$ isn't equal to the value of the function at $x = 1/2$, we have a removable discontinuity and we need to redefine the function in order to make it continuous at $x = 1/2$.

So we just redefine the value of the function at $x = 1/2$ to be equal to the general limit at $x = 1/2$ that we found earlier by taking the left- and right-hand limits at $x = 1/2$.

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 0$. If it's discontinuous, identify the type of discontinuity and redefine the function to make it continuous.

$$f(x) = \begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x > 0 \\ -2 & x = 0 \\ \dfrac{x}{x^2+2x} & x < 0 \end{cases}$$

Answer choices:

A The function is continuous at $x = 0$.

B The function is discontinuous at $x = 0$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 0$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x > 0 \\ \dfrac{1}{2} & x = 0 \\ \dfrac{x}{x^2+2x} & x < 0 \end{cases}$$

D The function is discontinuous at $x=0$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x)=\begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x>0 \\ 0 & x=0 \\ \dfrac{x}{x^2+2x} & x<0 \end{cases}$$

Solution: C

In order for the function to be continuous at $x=0$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x=0$.

We already know that the value of the function at $x=0$ is -2, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to -2. If they exist but aren't equal to -2, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x=0$.

To look at the left-hand limit, we'll use the third "piece" of the piecewise-defined function, because it defines the function to the left of $x = 0$ (the domain of that piece is $x < 0$).

$$\lim_{x\to 0^-} \frac{x}{x^2 + 2x}$$

$$\lim_{x\to 0^-} \frac{x}{x(x+2)}$$

$$\lim_{x\to 0^-} \frac{1}{x+2}$$

$$\frac{1}{0+2}$$

$$\frac{1}{2}$$

We know now that the left-hand limit exists, and that it's equal to $1/2$. Let's look at the right-hand limit by using the first "piece" of our piecewise-defined function, since it defines the function to the right of $x = 0$ (the domain of that piece is $x > 0$).

$$\lim_{x\to 0^+} \frac{\sqrt{4x+4} - 2}{2x}$$

$$\lim_{x\to 0^+} \frac{\sqrt{4x+4} - 2}{2x} \left(\frac{\sqrt{4x+4} + 2}{\sqrt{4x+4} + 2} \right)$$

$$\lim_{x\to 0^+} \frac{4x + 4 - 4}{2x(\sqrt{4x+4} + 2)}$$

$$\lim_{x\to 0^+} \frac{2}{\sqrt{4x+4}+2}$$

$$\lim_{x\to 0^+} \frac{2}{\sqrt{4(x+1)}+2}$$

$$\lim_{x\to 0^+} \frac{1}{\sqrt{x+1}+1}$$

$$\frac{1}{\sqrt{0+1}+1}$$

$$\frac{1}{1+1}$$

$$\frac{1}{2}$$

We know now that the right-hand limit exists, and that it's equal to $1/2$.

Since the left-hand limit exists at $x=0$, the right-hand limit exists at $x=0$, and the left- and right-hand limits are equal and therefore the general limit exists at $x=0$, but the general limit at $x=0$ isn't equal to the value of the function at $x=0$, we have a removable discontinuity and we need to redefine the function in order to make it continuous at $x=0$.

So we just redefine the value of the function at $x=0$ to be equal to the general limit at $x=0$ that we found earlier by taking the left- and right-hand limits at $x=0$.

$$f(x) = \begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x > 0 \\ \dfrac{1}{2} & x = 0 \\ \dfrac{x}{x^2+2x} & x < 0 \end{cases}$$

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 0$. If it's discontinuous, identify the type of discontinuity and solve for the value of k that makes it continuous.

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ k - \frac{1}{2} & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

Answer choices:

A The function is continuous at $x = 0$.

B The function is discontinuous at $x = 0$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 0$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ \frac{1}{2} & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

D The function is discontinuous at $x = 0$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ -\dfrac{1}{2} & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

Solution: B

In order for the function to be continuous at $x = 0$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 0$.

We already know that the value of the function at $x = 0$ is $k - 1/2$, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're equal to one another. If they are, then we'll set the value of the general limit equal to $k - 1/2$ to solve for k.

To look at the left-hand limit, we'll use the third "piece" of the piecewise-defined function, because it defines the function to the left of $x = 0$ (the domain of that piece is $x < 0$).

$$\lim_{x \to 0^-} \frac{4 - x}{x^2 - 2x - 8}$$

$$\lim_{x \to 0^-} \frac{4 - x}{(x - 4)(x + 2)}$$

$$\lim_{x \to 0^-} -\frac{x - 4}{(x - 4)(x + 2)}$$

We can cancel the $x - 4$ from the numerator and denominator to simplify the function. Keep in mind that this tells us we have a removable discontinuity at $x = 4$. That means that the function isn't continuous everywhere, but for the purposes of this problem, we really only care about continuity at $x = 0$, so we can move on.

$$\lim_{x \to 0^-} -\frac{1}{x + 2}$$

$$-\frac{1}{0 + 2}$$

$$-\frac{1}{2}$$

We know now that the left-hand limit exists, and that it's equal to $-1/2$. Let's look at the right-hand limit by using the first "piece" of the piecewise-defined function, since it defines the function to the right of $x = 0$ (the domain of that piece is $x > 0$). Substitute using a Pythagorean identity.

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x} - 1}{\sin^2 x}$$

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x} - 1}{1 - \cos^2 x}$$

Factor the denominator in order to simplify the fraction and then evaluate the limit.

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x} - 1}{(1 + \cos x)(1 - \cos x)}$$

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x} - 1}{(1 + \cos x)\left[(1 + \sqrt{\cos x})(1 - \sqrt{\cos x})\right]}$$

$$\lim_{x\to 0^+} -\frac{1 - \sqrt{\cos x}}{(1 + \cos x)(1 + \sqrt{\cos x})(1 - \sqrt{\cos x})}$$

$$\lim_{x\to 0^+} -\frac{1}{(1 + \cos x)(1 + \sqrt{\cos x})}$$

$$-\frac{1}{(1 + \cos(0))(1 + \sqrt{\cos(0)})}$$

$$-\frac{1}{(1 + 1)(1 + 1)}$$

$$-\frac{1}{4}$$

We know now that the right-hand limit exists, and that it's equal to $-1/4$.

Since the left-hand limit exists at $x = 0$, the right-hand limit exists at $x = 0$, but the left- and right-hand limits are not equal to another, that means the general limit does not exist at $x = 0$. Furthermore, because the one-sided limits are unequal, it means the discontinuity is non-removable.