Completing the square

The zeroes of a single-variable polynomial are the values of that variable at which the polynomial is equal to 0. Completing the square is a method we can use to find the zeroes of a quadratic polynomial. Another way to say this is that completing the square is a method we can use to solve the corresponding quadratic equation (the equation that has the quadratic polynomial on one side and 0 on the other side).

The solutions of any polynomial equation are called the **roots** of that equation. So the zeroes of a quadratic polynomial are numerically equal to the roots of the corresponding quadratic equation.

Completing the square is a useful method when it's not possible to solve for the roots by factoring, because completing the square creates a trinomial that we can factor as the square of a binomial.

The formal way to write a quadratic polynomial is $ax^2 + bx + c$, where a is the coefficient of the x^2 term, b is the coefficient of the x term, and c is the constant term.

These are the steps we'll follow every time we want to complete the square in order to find the roots of a quadratic equation $ax^2 + bx + c = 0$.

Before we go through the steps, however, we'll first divide both sides of the equation by a (if $a \neq 1$), because it will be easier to solve the equation if the coefficient of the x^2 term is 1. If we have to do that division, we won't define b and c until after we do it. That is, b will be the coefficient of the **new** x term, and c will be the **new** constant term. So we'll actually be starting with an equation of the form $x^2 + bx + c = 0$.

1. Move the constant term to the right side of the equation by subtracting c from both sides.

$$x^2 + bx + c - c = 0 - c$$

$$x^2 + bx = - c$$

2. Find $(b/2)^2$. Take the coefficient of the x term, divide it by 2, and then square the result.

3. Add $(b/2)^2$ to both sides of the equation. Remember, what we do to one side of an equation, we have to do to the other side as well, to keep the equation balanced.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = - c + \left(\frac{b}{2}\right)^2$$

4. Factor the left side, which is now

$$x^2 + bx + \left(\frac{b}{2}\right)^2$$

To factor this quadratic polynomial, we have to find a pair of factors of $(b/2)^2$ whose sum is b. The only pair of factors with this property is $b/2$ and $b/2$. So the left side of the equation becomes

$$\left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right)$$

Notice that the trinomial

$$x^2 + bx + \left(\frac{b}{2}\right)^2$$

factors as the square of the binomial $x + (b/2)$:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right) = \left(x + \frac{b}{2}\right)^2$$

So the equation we have to solve is

$$\left(x + \frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2$$

5. Square root both sides of the equation. Remember that the right side will now include a ± sign.

$$\sqrt{\left(x + \frac{b}{2}\right)^2} = \sqrt{-c + \left(\frac{b}{2}\right)^2}$$

$$x + \frac{b}{2} = \pm\sqrt{-c + \left(\frac{b}{2}\right)^2}$$

6. Solve for x to get the roots of the original quadratic equation, by subtracting $b/2$ from both sides.

$$x = -\frac{b}{2} \pm \sqrt{-c + \left(\frac{b}{2}\right)^2}$$

Example

Solve for x by completing the square.

$$x^2 + 6x + 4 = 0$$

There is no pair of factors of 4 whose sum is 6, so we'll need to solve by completing the square. Start by moving the constant constant term, 4, to the right side of the equation by subtracting 4 from both sides.

$$x^2 + 6x + 4 - 4 = 0 - 4$$

$$x^2 + 6x = -4$$

Find $(b/2)^2$. In this case, $b = 6$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$$

Add 9 to both sides of the equation.

$$x^2 + 6x + 9 = -4 + 9$$

$$x^2 + 6x + 9 = 5$$

Factor $x^2 + 6x + 9$, and notice that it factors as the square of a binomial, $(x+3)^2$, so our equation becomes

$$(x+3)^2 = 5$$

Take the square root of each side of the equation.

$$\sqrt{(x+3)^2} = \sqrt{5}$$

Since $x+3$ could be either positive or negative (because $(x+3)^2$ is equal to 5, which is positive), we get

$$x+3=\pm\sqrt{5}$$

Solve for x by subtracting 3 from both sides. To avoid confusion, put the -3 in front of the $\pm\sqrt{5}$.

$$x+3-3=-3\pm\sqrt{5}$$

$$x=-3\pm\sqrt{5}$$

The roots of the original quadratic equation are

$x=-3+\sqrt{5}$ and $x=-3-\sqrt{5}$

Let's try another example of completing the square.

Example

Solve for the variable by completing the square.

$$v^3-5v^2+4v=0$$

In this case we could solve by factoring since we can first factor out a v and then factor the quadratic polynomial that remains.

$$v(v^2-5v+4)=0$$

$(-1)(-4) = 4$ and $-1 + (-4) = -5$, so $v^2 - 5v + 4$ can be factored as $(v-1)(v-4)$. Therefore, our original equation can be rewritten as

$$v(v-1)(v-4) = 0$$

One solution is $v = 0$, and we'll set each of the other two factors ($v - 1$ and $v - 4$) equal to 0 and solve each of the resulting equations for v.

$$v - 1 = 0$$

$$v - 1 + 1 = 0 + 1$$

$$v = 1$$

and

$$v - 4 = 0$$

$$v - 4 + 4 = 0 + 4$$

$$v = 4$$

The solutions are therefore

$v = 0$, $v = 1$, and $v = 4$

However, we were not asked to solve by factoring, so let's look at how this problem can be solved by completing the square. Start by factoring out a v so that we'll have a quadratic polynomial inside parentheses.

$$v(v^2 - 5v + 4) = 0$$

Again, one solution is $v = 0$, so now we'll find the other solutions, that is, the solutions of the equation

$$v^2 - 5v + 4 = 0$$

Now that we have a quadratic polynomial on the left side, we can start by moving the constant term, 4, to the right side of the equation by subtracting 4 from both sides.

$$v^2 - 5v + 4 - 4 = 0 - 4$$

$$v^2 - 5v = -4$$

Find $(b/2)^2$. In this case, $b = -5$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-5}{2}\right)^2 = \frac{25}{4}$$

Add $25/4$ to both sides of the equation.

$$v^2 - 5v + \frac{25}{4} = -4 + \frac{25}{4}$$

$$v^2 - 5v + \frac{25}{4} = \frac{9}{4}$$

We see that the trinomial on the left side,

$$v^2 - 5v + \frac{25}{4}$$

factors as the square of a binomial, $[v - (5/2)]^2$, so our equation becomes

$$\left(v-\frac{5}{2}\right)^2=\frac{9}{4}$$

Take the square root of each side of the equation.

$$\sqrt{\left(v-\frac{5}{2}\right)^2}=\sqrt{\frac{9}{4}}$$

So we get

$$v-\frac{5}{2}=\pm\frac{3}{2}$$

Solve for v by adding $5/2$ to both sides. To avoid confusion, put the $5/2$ in front of the $\pm(3/2)$.

$$v-\frac{5}{2}+\frac{5}{2}=\frac{5}{2}\pm\frac{3}{2}$$

$$v=\frac{5}{2}\pm\frac{3}{2}$$

If we solve this both ways (adding and subtracting), we get

$$v=\frac{5}{2}+\frac{3}{2}=\frac{8}{2}=4$$

$$v=\frac{5}{2}-\frac{3}{2}=\frac{2}{2}=1$$

So the solutions are

$v=0$, $v=1$, and $v=4$