Limits of composites

Think of a **composite function** as a "function of a function."

For instance, assume that $f(x) = x + 1$ and that $g(x) = x^2 - 4$. If we find the composite $f(g(x))$, it means we're plugging $g(x) = x^2 - 4$ into $f(x) = x + 1$. That means we replace every x in $f(x)$ with $x^2 - 4$.

$$f(x) = x + 1$$

$$f(g(x)) = x^2 - 4 + 1$$

$$f(g(x)) = x^2 - 3$$

Alternatively, we could find the composite $g(f(x))$, in which case, we'd be plugging $f(x) = x + 1$ into $g(x) = x^2 - 4$. That means we replace every x in $g(x)$ with $x + 1$.

$$g(x) = x^2 - 4$$

$$g(f(x)) = (x + 1)^2 - 4$$

$$g(f(x)) = (x + 1)(x + 1) - 4$$

$$g(f(x)) = x^2 + x + x + 1 - 4$$

$$g(f(x)) = x^2 + 2x - 3$$

To find the limit of a composite function, we'll find the composite first, and then take the limit of the composite.

Example

If $f(x) = x + 1$ and $g(x) = x^2 - 4$, find each limit.

$$\lim_{x \to -1} f(g(x))$$

$$\lim_{x \to -1} g(f(x))$$

First, find the composite $f(g(x))$.

$$f(x) = x + 1$$

$$f(g(x)) = x^2 - 4 + 1$$

$$f(g(x)) = x^2 - 3$$

Next, find the limit of $f(g(x))$.

$$\lim_{x \to -1} f(g(x))$$

$$\lim_{x \to -1} x^2 - 3$$

$$(-1)^2 - 3$$

$$-2$$

Now find the composite $g(f(x))$.

$$g(x) = x^2 - 4$$

$$g(f(x)) = (x + 1)^2 - 4$$

$$g(f(x)) = (x+1)(x+1) - 4$$

$$g(f(x)) = x^2 + x + x + 1 - 4$$

$$g(f(x)) = x^2 + 2x - 3$$

Next, find the limit of $g(f(x))$.

$$\lim_{x \to -1} g(f(x))$$

$$\lim_{x \to -1} x^2 + 2x - 3$$

$$(-1)^2 + 2(-1) - 3$$

$$1 - 2 - 3$$

$$-4$$

So the limits of the composite functions are

$$\lim_{x \to -1} f(g(x)) = -2$$

$$\lim_{x \to -1} g(f(x)) = -4$$