Quotient rule

In the same way that we use product rule to find the derivative of a product, we use quotient rule to find the derivative of a quotient.

Quotient rule

The **quotient rule** says that the derivative of

$$y=\frac{f(x)}{g(x)}$$

is given by

$$y'=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

Just like the other derivative rules so far (power rule and product rule), the derivative here is just a combination of the $f(x)$ and its derivative and $g(x)$ and its derivative.

But this time, $f(x)$ is the numerator of the original function, and $g(x)$ is the denominator of the original function.

Let's do an example where we use quotient rule to find the derivative.

Example

Use quotient rule to find the derivative.

$$y = \frac{x^4}{6x^2}$$

First, let's list out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = x^4$$

$$f'(x) = 4x^3$$

and

$$g(x) = 6x^2$$

$$g'(x) = 12x$$

Now we can plug these values directly into the quotient rule formula.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(4x^3)(6x^2) - (x^4)(12x)}{(6x^2)^2}$$

$$y' = \frac{24(x^3)(x^2) - 12(x^4)(x)}{(6x^2)(6x^2)}$$

$$y' = \frac{24x^5 - 12x^5}{36x^4}$$

$$y' = \frac{12x^5}{36x^4}$$

Simplify the function as much as possible by canceling common factors.

$$y' = \frac{x}{3}$$

$$y' = \frac{1}{3}x$$

We can verify the result from this last example by first simplifying the quotient,

$$y = \frac{x^4}{6x^2}$$

$$y = \frac{x^2}{6}$$

$$y = \frac{1}{6}x^2$$

and then applying power rule.

$$y' = \frac{2}{6}x^1$$

$$y' = \frac{1}{3}x$$

Reciprocal rule

When the numerator of the quotient we want to differentiate is a constant, the quotient rule formula gets simpler.

That's because we already know that the derivative of a constant is 0. So, assuming the numerator of the quotient is a constant,

$$y = \frac{a}{g(x)}$$

the derivative of the quotient would be

$$y' = \frac{(0)g(x) - ag'(x)}{[g(x)]^2}$$

$$y' = \frac{-ag'(x)}{[g(x)]^2}$$

This is the **reciprocal rule** formula. Again, it's just the simplified version of the quotient rule that applies only to quotients in which the numerator is a constant.

Because this rule comes directly from the quotient rule, there's really no reason to know the reciprocal rule (because we can always use the quotient rule, instead), other than to save ourselves a tiny bit of time when we're differentiating a quotient with a constant numerator.

That being said, let's still do an example where we apply the quotient rule directly.

Example

Use the reciprocal rule to find the derivative.

$$y = \frac{1}{2x+1} + \frac{5}{3x-1}$$

The reciprocal rule formula just needs the negative numerator, the denominator, and the denominator's derivative.

- For the first fraction, $a = 1$ so $-a = -1$, the denominator is $2x + 1$, and the denominator's derivative is 2.

- For the second fraction, $a = 5$ so $-a = -5$, the denominator is $3x - 1$, and the denominator's derivative is 3.

We'll set up the reciprocal rule formula for each fraction, and then plug in these values.

$$y' = \frac{-ag'(x)}{[g(x)]^2} + \frac{-ag'(x)}{[g(x)]^2}$$

$$y = \frac{-1(2)}{(2x+1)^2} + \frac{-5(3)}{(3x-1)^2}$$

$$y = -\frac{2}{(2x+1)^2} - \frac{15}{(3x-1)^2}$$