Limits at infinity and horizontal asymptotes

We've seen how to find the infinite limits around a vertical asymptote, but now we want to focus on finding the limits of a function at infinity. More specifically, we're interested in the limit of the function as $x \to -\infty$ and as $x \to \infty$.

If the function approaches a finite value as $x \to -\infty$ or $x \to \infty$, it means the function has a horizontal asymptote at that value. For instance, if the limit of the function as $x \to -\infty$ and as $x \to \infty$ is 0, then the function has a horizontal asymptote at $y = 0$.

Let's look at an example where we work through how to find the limits of a function at $\pm\infty$, and use those limits to draw a conclusion about any horizontal asymptote(s) of the function.

Example

Find the limits of the function as $x \to -\infty$ and $x \to \infty$, then say whether the function has a horizontal asymptote.

$$f(x) = \frac{1}{x - 3}$$

We can see right away that the vertical asymptotes exists at $x = 3$, since that's the value that makes the denominator 0.

To find any horizontal asymptotes, we'll look at the limits as $x \to -\infty$ and $x \to \infty$. Even though, technically, we can't plug $-\infty$ or ∞ into a function, we can imagine how the function behaves when we plug in infinitely large positive values or infinitely large negative values.

$$\lim_{x\to\infty}\frac{1}{x-3}=\frac{1}{\infty-3}=\frac{1}{\infty}=0$$

$$\lim_{x\to-\infty}\frac{1}{x-3}=\frac{1}{-\infty-3}=\frac{1}{-\infty}=0$$

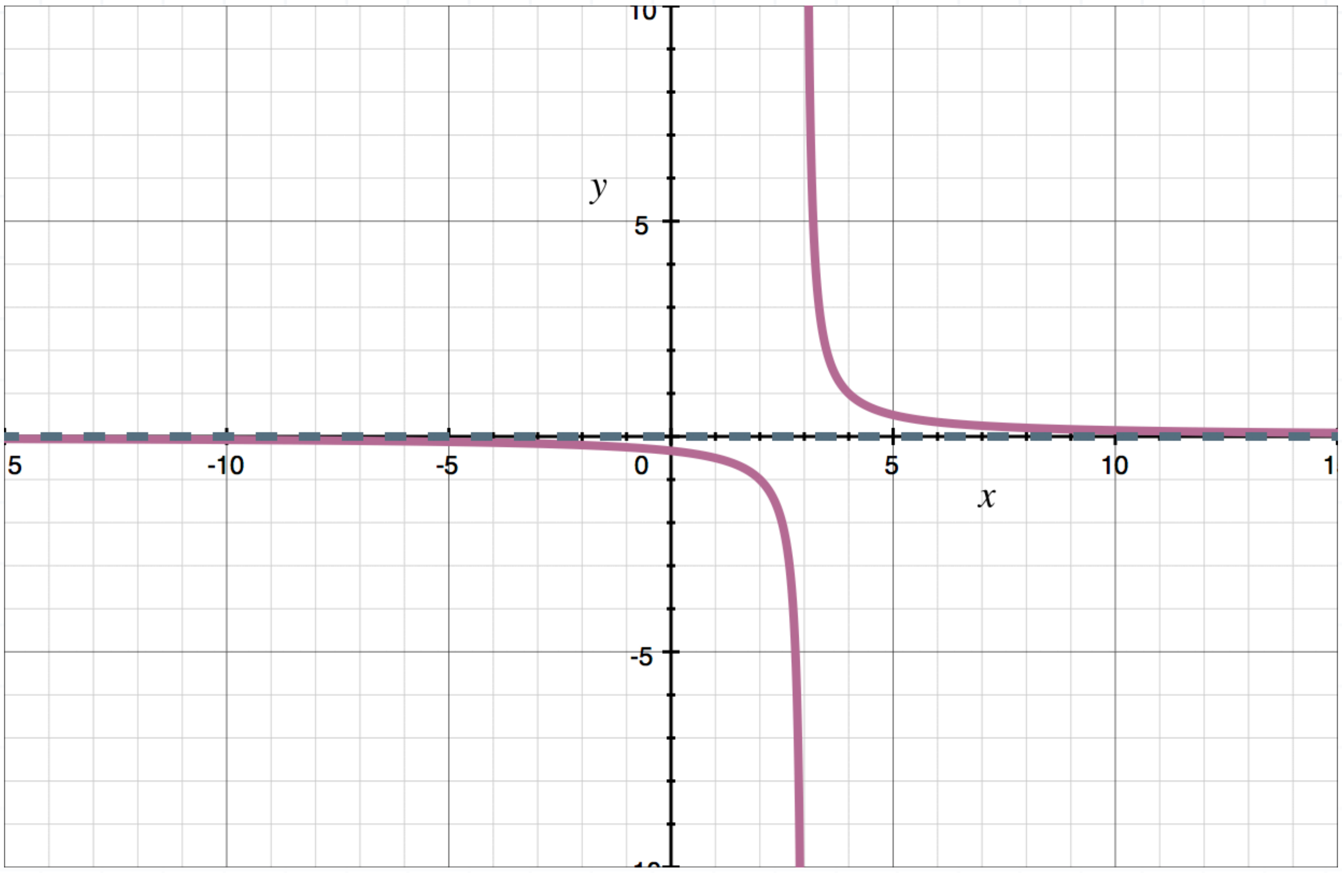
In both cases, as $x \to -\infty$ and $x \to \infty$, we have fractions with extremely large denominators, and numerators that are, in comparison, incredibly small. Any fraction with a constant value in the numerator and an infinitely large value in the denominator (whether that infinitely large value is positive or negative), will tend toward 0.

These results tell us that the limit of the function is 0 as $x \to -\infty$, and as $x \to \infty$.

$$\lim_{x\to\infty}\frac{1}{x-3}=0$$

$$\lim_{x\to-\infty}\frac{1}{x-3}=0$$

Therefore, the function has a horizontal asymptote at $y=0$. If we sketch the graph of the function, we can confirm our findings.



Rules for horizontal asymptotes

Given a rational function (a fraction in which the numerator and denominator are polynomials), we can determine the equation of the horizontal asymptote simply by comparing the numerator to the denominator.

We need to start by identifying the degree of the numerator and denominator. The **degree** is the exponent on the term with the largest exponent. For instance, $x^3 - 2x^2 + 6x + 1$ is a third-degree polynomial, because x^3 is the largest-degree term.

The largest-degree term isn't necessarily the first term. If the same polynomial is written as $6x - 2x^2 + x^3 + 1$, it's still a third-degree polynomial, because x^3 is the largest-degree term.

Once we've identified the degree of the numerator and denominator, then we compare them.

N < D: If the degree of the numerator is less than the degree of the denominator, then the horizontal asymptote is given by $y = 0$.

N > D: If the degree of the numerator is greater than the degree of the denominator, then the function doesn't have a horizontal asymptote.

N = D: If the degree of the numerator is equal to the degree of the denominator, then the horizontal asymptote is given by the ratio of the coefficients on the highest-degree terms.

Let's do an example where we see these rules in action.

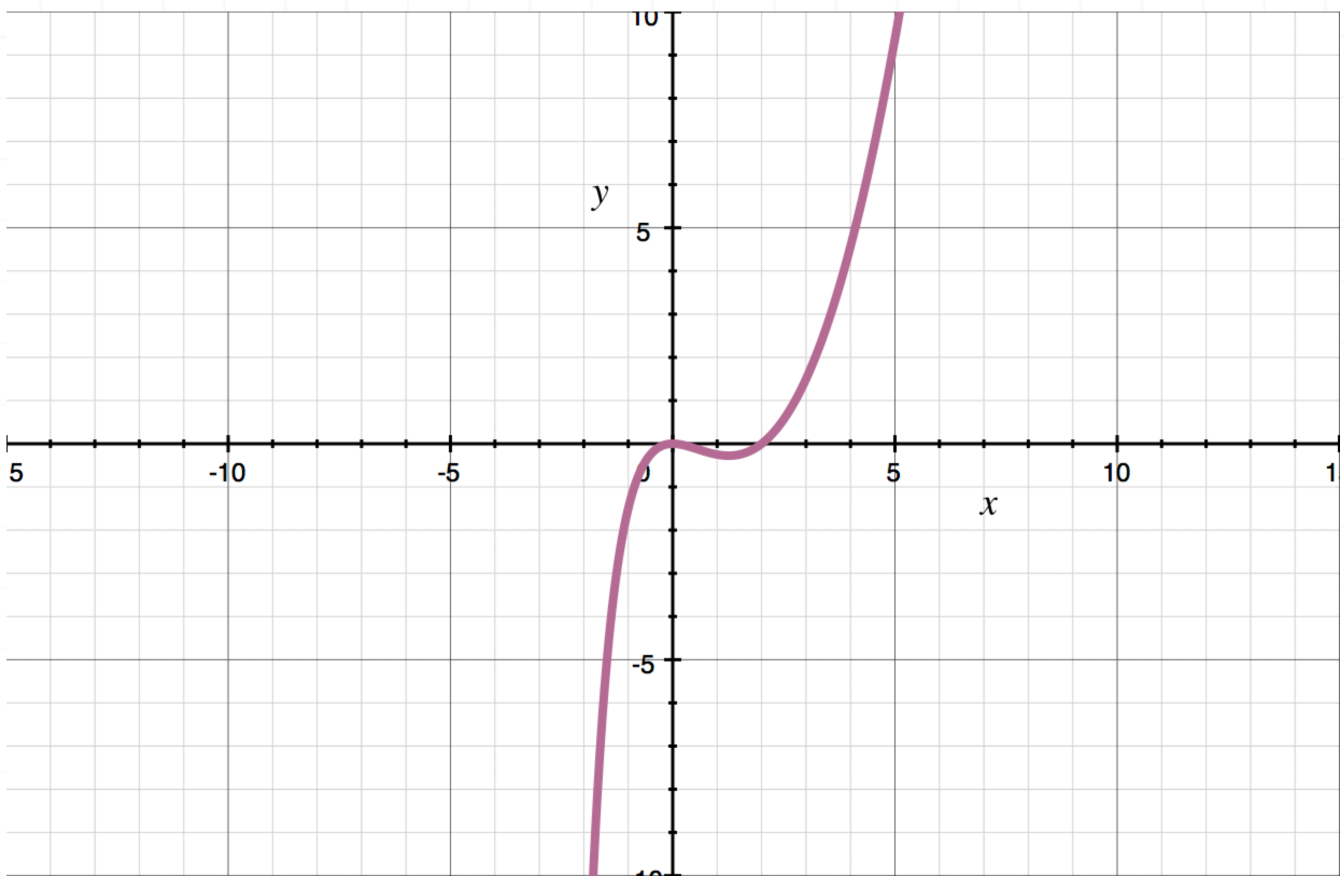
Example

Find any horizontal asymptote that exists for each function.

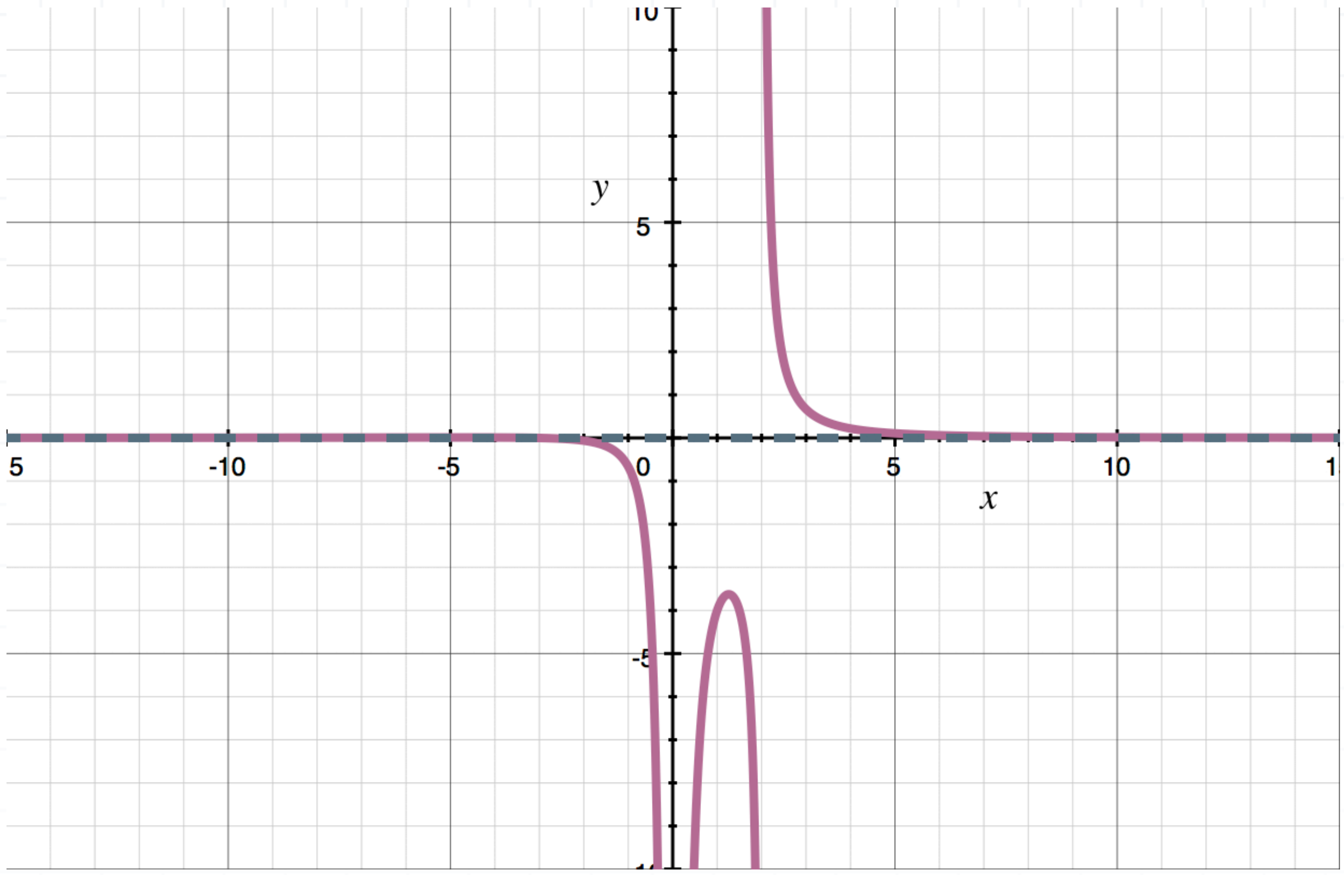
$$f(x) = \frac{x^3 - 2x^2}{x + 3} \qquad g(x) = \frac{x + 3}{x^3 - 2x^2} \qquad h(x) = \frac{x^3 + 3}{x^3 - 2x^2}$$

To determine any horizontal asymptotes that the functions may have, we need to compare the degree of the numerator to the degree of the denominator.

For the function $f(x)$, the degree of the numerator is 3, and the degree of the denominator is 1, so N > D, which means the function doesn't have a horizontal asymptote. The graph of $f(x)$ confirms this.



For the function $g(x)$, the degree of the numerator is 1, and the degree of the denominator is 3, so N < D, which means the function has a horizontal asymptote at $y = 0$. The graph of $g(x)$ confirms this.

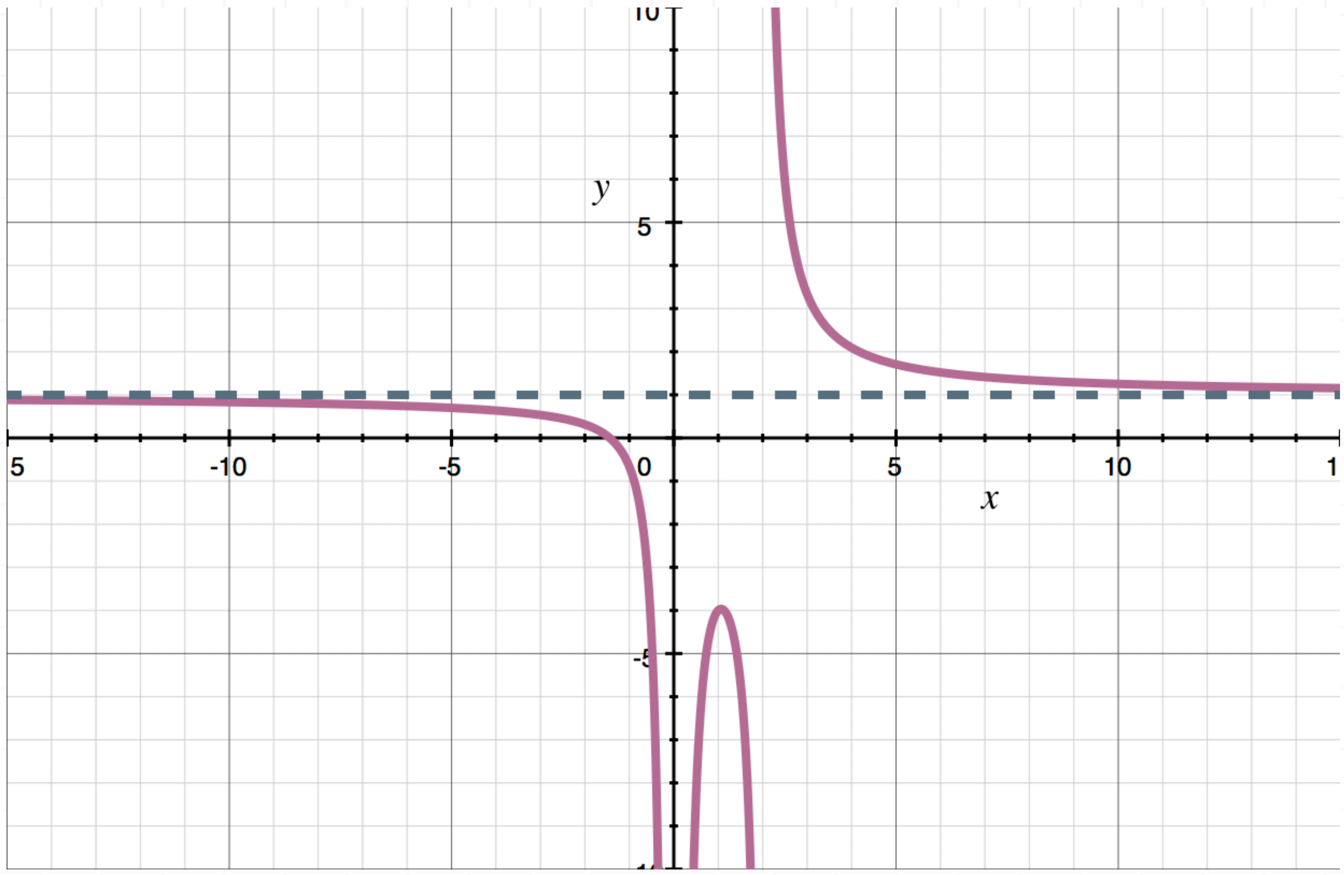


For the function $h(x)$, the degree of the numerator is 3, and the degree of the denominator is 3, so N = D, which means the equation of the horizontal asymptote is given by the ratio of the coefficients on the highest-degree terms. In $h(x)$, the highest-degree term in the numerator is x^3, and its coefficient is 1; the highest-degree term in the denominator is x^3, and its coefficient is 1. So the horizontal asymptote is

$$y = \frac{1}{1}$$

$$y = 1$$

The graph of $h(x)$ confirms this.



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y
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5
-10
-5
0
5
10
1
x
-5