Calculus 1 Quizzes

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Topic: Vertical Line Test

Question: If a perfectly straight vertical line crosses a graph at more than one point, the graph fails the Vertical Line Test.

Answer choices:

A True

B False

Solution: A

A graph passes the Vertical Line Test if it’s impossible to draw a perfectly straight vertical line that crosses the graph more than once.

If you can draw a perfectly straight vertical line that crosses the graph more than once, then the graph fails the Vertical Line Test, and the graph does not represent a function.

Topic: Vertical Line Test

Question: Which of the following will never pass the Vertical Line Test and therefore can never represent a function?

Answer choices:

A A horizontal line

B A graph that consists of six points all of which have different x-coordinates

C A “slanted” line (neither vertical nor horizontal)

D A circle

Solution: D

A graph fails the Vertical Line Test when you can draw a vertical line that crosses the graph more than once. Since you'll always be able to draw a vertical line that crosses the graph of a circle more than once, a circle will always fail the Vertical Line Test, and therefore can never represent a function.

Topic: Vertical Line Test

Question: Does the graph of the equation pass the Vertical Line Test?

$$2x^2 + 2y^2 = 18$$

Answer choices:

A Yes, because $2x^2 + 2y^2 = 18$ is a function.

B No, because $2x^2 + 2y^2 = 18$ is a function.

C No, because $2x^2 + 2y^2 = 18$ is not a function.

D Yes, because $2x^2 + 2y^2 = 18$ is not a function.

Solution: C

In order for a graph to pass the Vertical Line Test, it must be the graph of a function, because only functions pass the Vertical Line Test. The test is simply that a vertical line drawn at any point in the graph must only pass through it once.

The equation $2x^2 + 2y^2 = 18$ is a circle. The graph of a circle allows for a vertical line to pass through it twice at many points, which means it automatically fails the Vertical Line Test. This means that a circle is not a function.

We could also solve the equation algebraically to prove that it doesn't represent a function.

$$2x^2 + 2y^2 = 18$$

$$\frac{2x^2}{2} + \frac{2y^2}{2} = \frac{18}{2}$$

$$x^2 + y^2 = 9$$

$$x^2 - x^2 + y^2 = 9 - x^2$$

$$y^2 = 9 - x^2$$

$$\sqrt{y^2} = \sqrt{9 - x^2}$$

$$y = \pm\sqrt{9 - x^2}$$

Now that we have the equation in this form, we can find values of x that return multiple y-values. For instance, at $x = 1$,

$$y = \pm\sqrt{9 - 1^2}$$

$$y = \pm\sqrt{8}$$

$$y = \pm 2\sqrt{2}$$

Because the equation takes on the values $y = -2\sqrt{2}$ and $y = 2\sqrt{2}$ at the single value $x = 1$, we know the equation doesn't represent a function.

Topic: Domain and range

Question: What are the domain and range of the function?

$$(3,4), (4,1), (5,2), (7,1)$$

Answer choices:

A The domain is $3, 4, 5, 7$. The range is $1, 2, 4$.

B The domain is $3, 7$. The range is $1, 4$.

C The domain is $3, 4, 5, 7$. The range is $1, 2$.

D None of these

Solution: A

Remember that the coordinates of points in the Cartesian coordinate system are given in the form (x, y).

Since the domain of a function is all of the x-values, we can see that the domain of this function is

$$3,\ 4,\ 5,\ 7$$

The range of a function is all of the y-values, so we can see that the range of this function is

$$4,\ 1,\ 2,\ 1$$

We don't need to include the same value more than once, so we'll list 1 only once, and rearrange the numbers so that they are in ascending order. The range is

$$1,\ 2,\ 4$$

Topic: Domain and range

Question: What are the domain and range of the function?

$$y = \frac{2}{x}$$

Answer choices:

A	Domain: all real numbers except 2	Range: all real numbers except 2
B	Domain: all real numbers except 0	Range: all real numbers except 0
C	Domain: all real numbers except 0	Range: all real numbers except 2
D	Domain: all real numbers except 2	Range: all real numbers except 0

Solution: B

The domain of a function is all of the x-values for which the function is defined. The range of a function is all of the y-values that correspond to the x-values in the domain. To solve for the domain of a function, we look for any places where the function is not defined. For example, this can happen if there's a variable in the denominator of the function or if a radical has a negative number under it.

The function

$$y = \frac{2}{x}$$

is undefined for $x = 0$, because division by 0 is undefined. However, this function is defined for all other values of x, so its domain consists of all real numbers except 0.

To solve for the range of the function, we need to look for the y-values that correspond to numbers in the domain and for those that don't.

For every nonzero real number y, there's some nonzero real number x such that

$$y = \frac{2}{x}$$

To see this, multiply both sides of this equation by x/y.

$$y\left(\frac{x}{y}\right) = \left(\frac{2}{x}\right)\left(\frac{x}{y}\right)$$

$$x = \frac{2}{y}$$

So for any nonzero real number y, we divide 2 by y to get a nonzero real number x for which $y = 2/x$.

However, there's no nonzero real number x such that

$$0 = \frac{2}{x}$$

To see this, multiply both sides of this equation by x.

$$0(x) = \left(\frac{2}{x}\right)(x)$$

$$0 = 2$$

This gives us the false equation $0 = 2$.

Combining these results, we find that the range of this function is all real numbers except 0.

Topic: Domain and range

Question: What is the domain of the function?

$$f(x) = \sqrt{4x^3}$$

Answer choices:

A The domain is all values of x that make $4x^3$ positive

B The domain is all values of x that make $4x^3$ negative

C The domain is all values of x that make $4x^3$ either 0 or positive

D The domain is all values of x that make $4x^3$ either 0 or negative

Solution: C

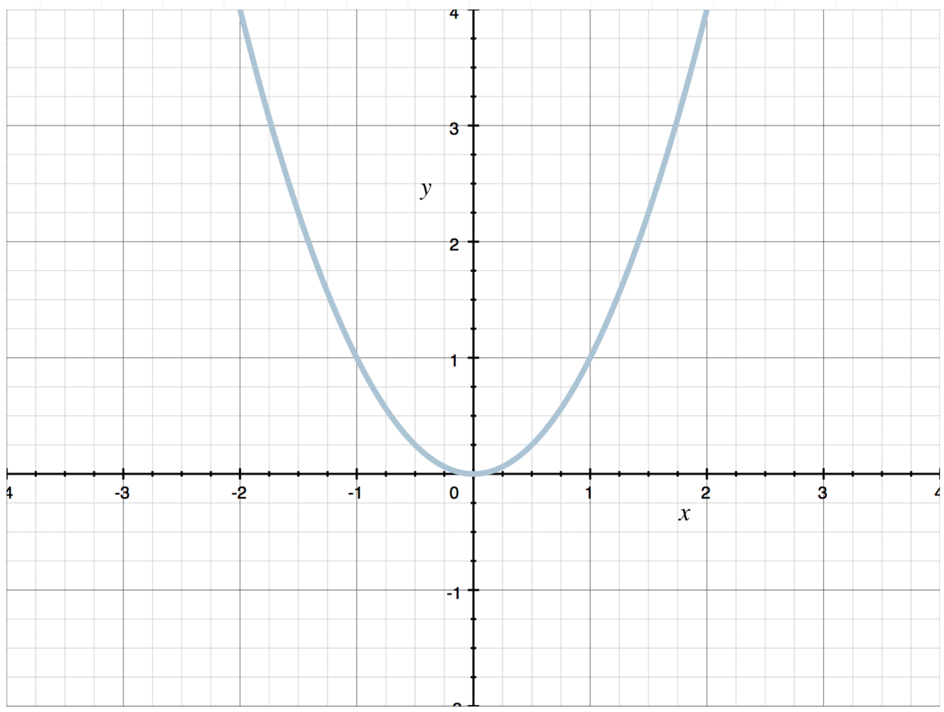
When we're dealing in real numbers, we can only take the square root of 0, or of positive values.

In other words, we won't be able to find the square root of $4x^3$ unless the value of $4x^3$ is positive, or equal to 0.

Therefore, any values of x that make $4x^3$ equivalent to 0, or equivalent to any positive value, will be included in the domain of the function.

Topic: Domain and range from a graph

Question: What are the domain and range of the function? Assume the graph does not extend beyond what's shown in the graph.



Answer choices:

A The domain is any value of x between -2 and 2, and the range is any value of y between -3 and 0.

B The domain is any value of x between 0 and 3, and the range is any value of y between -2 and 2.

C The domain is any value of x between -3 and 0, and the range is any value of y between -2 and 2.

D The domain is any value of x between -2 and 2, and the range is any value of y between 0 and 4.

Solution: D

To find the domain of the function, start by looking at the leftmost point of the graph. That point is at

$$x = -2$$

Then the graph continues with no breaks until it ends at

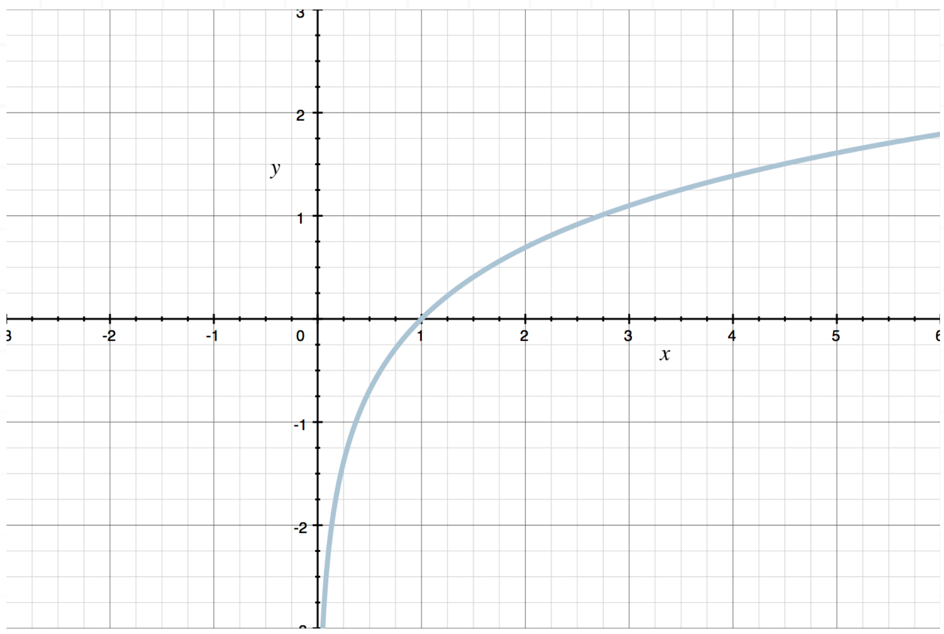
$$x = 2$$

This means that the domain of the function is any value of x between -2 and 2.

To find the range of the function, start by looking at the lowest point of the graph. The y-coordinate of the lowest point is 0. The graph has two highest points, and the y-coordinate of each of the highest points is 4. There are no breaks in the graph going from the lowest point to either of the highest points. This means that the range of the function is any value of y between 0 and 4.

Topic: Domain and range from a graph

Question: What are the domain and range of the function? Assume the graph does not extend beyond what's shown in the graph.



Answer choices:

A The domain is any value of x between -3 and 0, and the range is any value of y between $1/4$ and 1.

B The domain is any value of x between $1/4$ and 1, and the range is any value of y between 0 and 3.

C The domain is any value of x between 0 and 6, and the range is any value of y between -3 and $7/4$.

D The domain is any value of x between 0 and 3, and the range is any value of y between $1/4$ and 1.

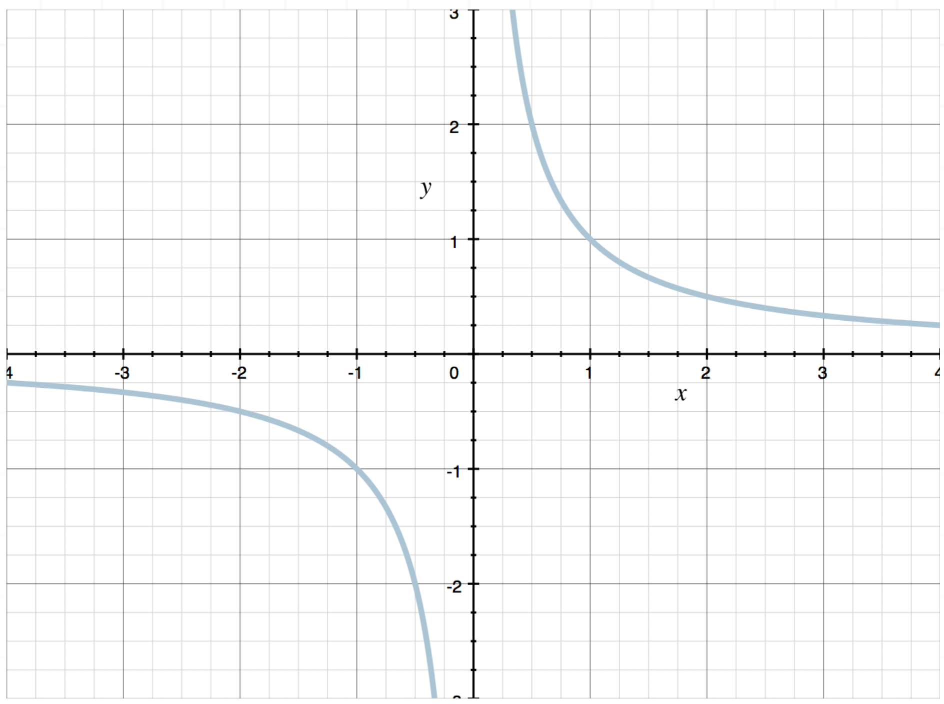
Solution: C

To find the domain of the function, start by looking at the leftmost point of the graph. The x-coordinate of the leftmost point is about 0. Then the graph continues with no breaks until it ends at $x = 6$. This means that the domain of the function is any value of x between 0 and 6.

To find the range of the function, start by looking at the lowest point of the graph. The y-coordinate of the lowest point is -3. Then the graph continues with no breaks until it ends at about $y = 7/4$. This means that the range of the function is any value of y between -3 and $7/4$.

Topic: Domain and range from a graph

Question: What are the domain and range of the function? Assume the graph does not extend beyond what's shown in the graph.



Answer choices:

A The domain is any value of x between -4 and $-1/4$, or between $1/4$ and 4, and the range is any value of y between -3 and $-1/4$, or between $1/4$ and 3.

B The domain is any value of x between -4 and $-1/4$, or between $1/4$ and 4, and the range is any value of y between -3 and $1/4$, or between $1/4$ and 3.

C The domain is any value of x between -4 and $1/4$, or between $1/4$ and 4, and the range is any value of y between -3 and $-1/4$, or between $1/4$ and 3.

D The domain is any value of x between -4 and $1/4$, or between $1/4$ and 4, and the range is any value of y between -3 and $1/4$, or between $1/4$ and 3.

Solution: A

To solve for the domain of the function on the graph, look at the graph from left to right. The first x-value that exists for the function is at $x = -4$, then there's a break at $x = -1/4$, and the function picks up again at $x = 1/4$ and continues smoothly until it ends at $x = 4$. This means the domain of the function is any value of x between -4 and $-1/4$, or between $1/4$ and 4.

To solve for the range of the function on the graph, look at the graph from bottom to top. The first y-value that exists for the function is at $y = -3$, then there's a break at $y = -1/4$, and the function picks up again at $y = 1/4$ and continues smoothly until it ends at $y = 3$. This means the range of the function is any value of y between -3 and $-1/4$, or between $1/4$ and 3.

Topic: Even, odd, or neither

Question: Is the function even, odd, or neither?

$$f(x) = 2x^3 - x^7$$

Answer choices:

A Even

B Odd

C Neither

Solution: B

A function is even if $f(x) = f(-x)$, odd if $f(-x) = -f(x)$, and neither if $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. So to classify the given function, we'll substitute $-x$ into the expression for the function and then simplify the result.

Given

$$f(x) = 2x^3 - x^7$$

we get

$$f(-x) = 2(-x)^3 - (-x)^7$$

$$f(-x) = 2(-1x)^3 - (-1x)^7$$

$$f(-x) = 2(-1)^3(x^3) - (-1)^7(x^7)$$

$$f(-x) = 2(-1)x^3 - (-1)x^7$$

$$f(-x) = -2x^3 + x^7$$

$$f(-x) = -(2x^3 - x^7)$$

This function is odd, because $f(-x) = -f(x)$.

Topic: Even, odd, or neither

Question: Is the function even, odd, or neither?

$$f(x) = 5x^2 - 2x^3$$

Answer choices:

A Even

B Odd

C Neither

Solution: C

A function is even if $f(x) = f(-x)$, odd if $f(-x) = -f(x)$, and neither if $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. So to classify the given function, we'll substitute $-x$ into the expression for the function and then simplify the result.

Given

$$f(x) = 5x^2 - 2x^3$$

we get

$$f(-x) = 5(-x)^2 - 2(-x)^3$$

$$f(-x) = 5(-1x)^2 - 2(-1x)^3$$

$$f(-x) = 5(-1)^2(x^2) - 2(-1)^3(x^3)$$

$$f(-x) = 5(1)x^2 - 2(-1)x^3$$

$$f(-x) = 5x^2 + 2x^3$$

This function is neither even nor odd, because $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$.

Topic: Even, odd, or neither

Question: Is the function even, odd, or neither?

$$f(x) = -x^4 - 6x^2$$

Answer choices:

A Even

B Odd

C Neither

Solution: A

A function is even if $f(x) = f(-x)$, odd if $f(-x) = -f(x)$, and neither if $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. So to classify the given function, we'll substitute $-x$ into the expression for the function and then simplify the result.

Given

$$f(x) = -x^4 - 6x^2$$

we get

$$f(-x) = -(-x)^4 - 6(-x)^2$$

$$f(-x) = -(-1x)^4 - 6(-1x)^2$$

$$f(-x) = -(-1)^4(x^4) - 6(-1)^2(x^2)$$

$$f(-x) = -(1)x^4 - 6(1)x^2$$

$$f(-x) = -x^4 - 6x^2$$

This function is even, because $f(x) = f(-x)$.

Topic: Equation modeling

Question: A car and a truck were driven for a week. The car traveled 400 miles more than the truck. The fuel mileage for the car was 27 miles per gallon (mpg), and the fuel mileage for the truck was 18 mpg. Write an equation which gives the total amount of fuel, g (in gallons), that was used by the two vehicles that week in terms of c, the distance (in miles) traveled by the car.

	Car	Truck
Mileage	27 mpg	18 mpg
Distance	c miles	t miles

Answer choices:

A $\quad g = \dfrac{5c - 1{,}200}{54}$

B $\quad g = \dfrac{5c + 1{,}200}{54}$

C $\quad g = \dfrac{c - 1{,}200}{18}$

D $\quad g = \dfrac{c + 400}{27}$

Solution: A

Write an equation that gives the distance t (in miles) traveled by the truck in terms of c. We know that $c = t + 400$, so $t = c - 400$.

To get the amount of fuel used by each vehicle, divide its distance by its fuel mileage.

We'll replace "t miles" in the table with "$c - 400$ miles." Also, we'll add a row to the table, for the amounts of fuel used.

	Car	Truck
Mileage	27 mpg	18 mpg
Distance	c miles	t miles
Fuel used	c/27 gallons	(c-400)/18 gallons

We can write:

$$g = \frac{c}{27} + \frac{c-400}{18}$$

$$g = \left(\frac{2}{2}\right)\frac{c}{27} + \left(\frac{3}{3}\right)\frac{c-400}{18}$$

$$g = \frac{2c}{54} + \frac{3c-1{,}200}{54}$$

$$g = \frac{2c+3c-1{,}200}{54}$$

$$g = \frac{5c-1{,}200}{54}$$

Topic: Equation modeling

Question: The average speed of any falling object between time 0 (the time at which it starts falling) and time t is given by the ratio of the distance d through which it has fallen to the elapsed time (t), $V = D/t$. A ball is thrown at a speed of 12 ft/s straight downward from a tall cliff. The distance through which it has fallen is $D = 16t^2 + 12t$, where t is the time (in seconds) that it's been falling. Write an equation that gives t in terms of V.

Answer choices:

A $\quad t = \dfrac{V + 12}{16}$

B $\quad t = \dfrac{V}{28}$

C $\quad t = \dfrac{V - 12}{16}$

D $\quad t = \dfrac{V}{4}$

Solution: C

Start with $V = D/t$, and substitute $16t^2 + 12t$ for D.

$$V = \frac{16t^2 + 12t}{t}$$

Simplify, and then solve for t.

$$V = 16t + 12$$

$$t = \frac{V - 12}{16}$$

Topic: Equation modeling

Question: At a movie theater, prices are $\$4$ for children and $\$8$ for adults. If 800 people came to see a movie, write an equation that gives the number of children, c, in terms of T, the total amount of money taken in.

Answer choices:

A $\quad c = \frac{T}{4} - 1{,}600$

B $\quad c = 1{,}600 - 4T$

C $\quad c = 6{,}400 + 8T$

D $\quad c = 1{,}600 - \frac{T}{4}$

Solution: D

Let c be the number of children and a the number of adults. The total money taken in is

$$T = 4c + 8a$$

We also know that the total number of people who came to see the movie is 800, that is, $c + a = 800$. So $a = 800 - c$. Substituting $800 - c$ for a in the equation $T = 4c + 8a$ gives

$$T = 4c + 8(800 - c)$$

$$T = 4c + 6{,}400 - 8c$$

$$T = -4c + 6{,}400$$

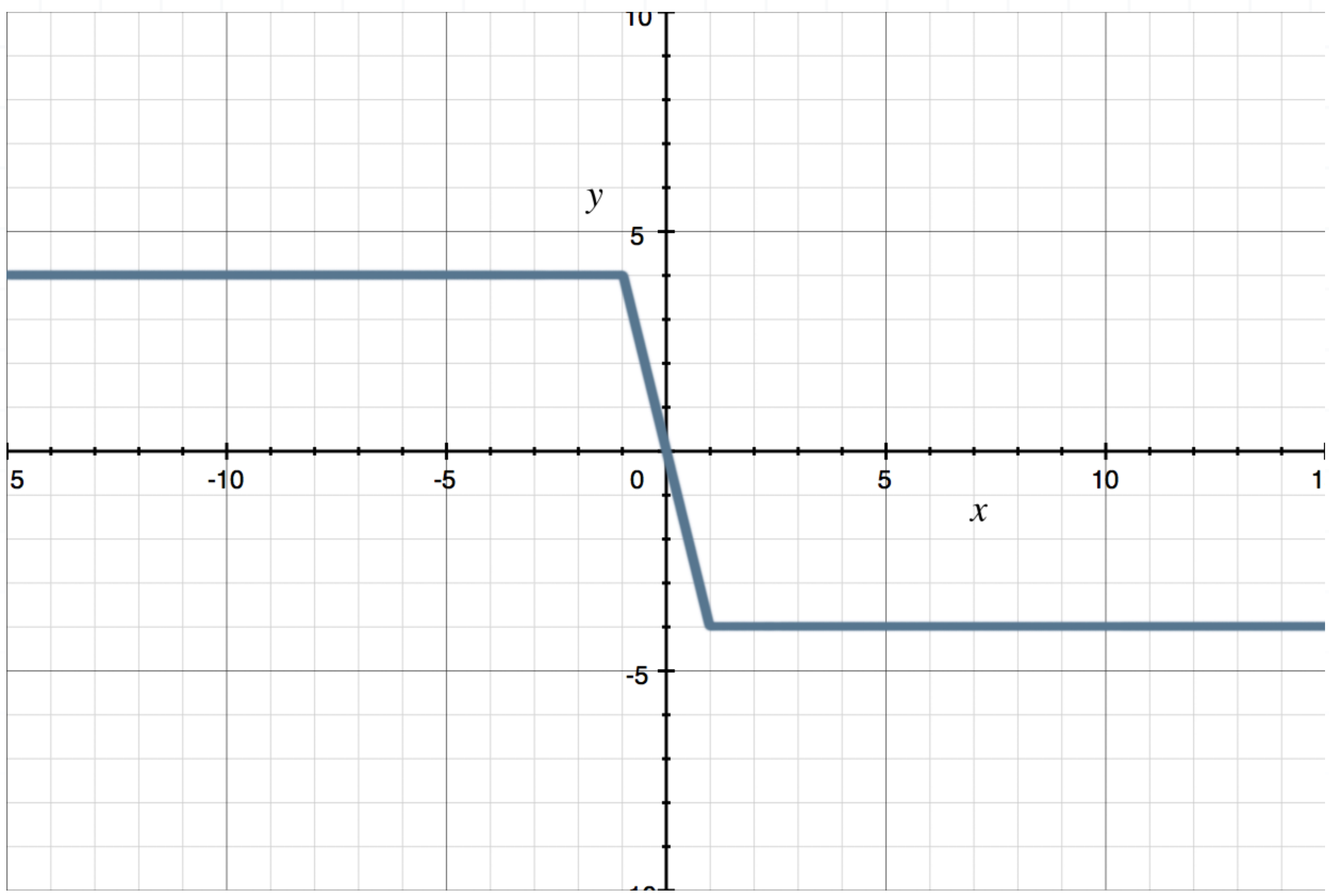
Now solve for c.

$$T + 4c = 6{,}400$$

$$4c = 6{,}400 - T$$

Topic: Modeling a piecewise-defined function

Question: What is the definition of the piecewise function shown in the graph?



Answer choices:

A $$f(x) = \begin{cases} -4 & \text{if} \quad x \le -1 \\ -4x & \text{if} \quad -1 < x \le 1 \\ 4 & \text{if} \quad 1 < x \end{cases}$$

B $$f(x) = \begin{cases} 4 & \text{if} \quad x \le -1 \\ 4x & \text{if} \quad 1 < x \le -1 \\ -4 & \text{if} \quad 1 < x \end{cases}$$

C $$f(x) = \begin{cases} 4 & \text{if} \quad x \le -1 \\ -4x & \text{if} \quad -1 < x \le 1 \\ -4 & \text{if} \quad 1 < x \end{cases}$$

D $$f(x) = \begin{cases} 4 & \text{if} \quad x < -1 \\ -4x & \text{if} \quad -1 < x \le 1 \\ -4 & \text{if} \quad 1 < x \end{cases}$$

Solution: C

Going from left to right, the first part of the graph is part of the line $y = 4$ and it goes from the left, to $x = -1$. For this piece, we write 4 for the function and $x \leq -1$ for its domain.

The second part of the graph is part of the line that has a slope of -4 and a y-intercept of 0, so the equation of this line is $y = -4x$. Remember: the slope-intercept of the equation of a line is $y = mx + b$.) To see how to get the slope, notice that $(-1,4)$ and $(0,0)$ are points on this line, so its slope is

$$m = \frac{0-4}{0-(-1)} = \frac{-4}{1} = -4$$

This part of the graph goes from $x = -1$ to $x = 1$. So for the second piece, we write $-4x$ for the function and $-1 < x \leq 1$ for its domain. We can't include $x = -1$ in the domain of this piece, because we included $x = -1$ in the domain of the first piece.

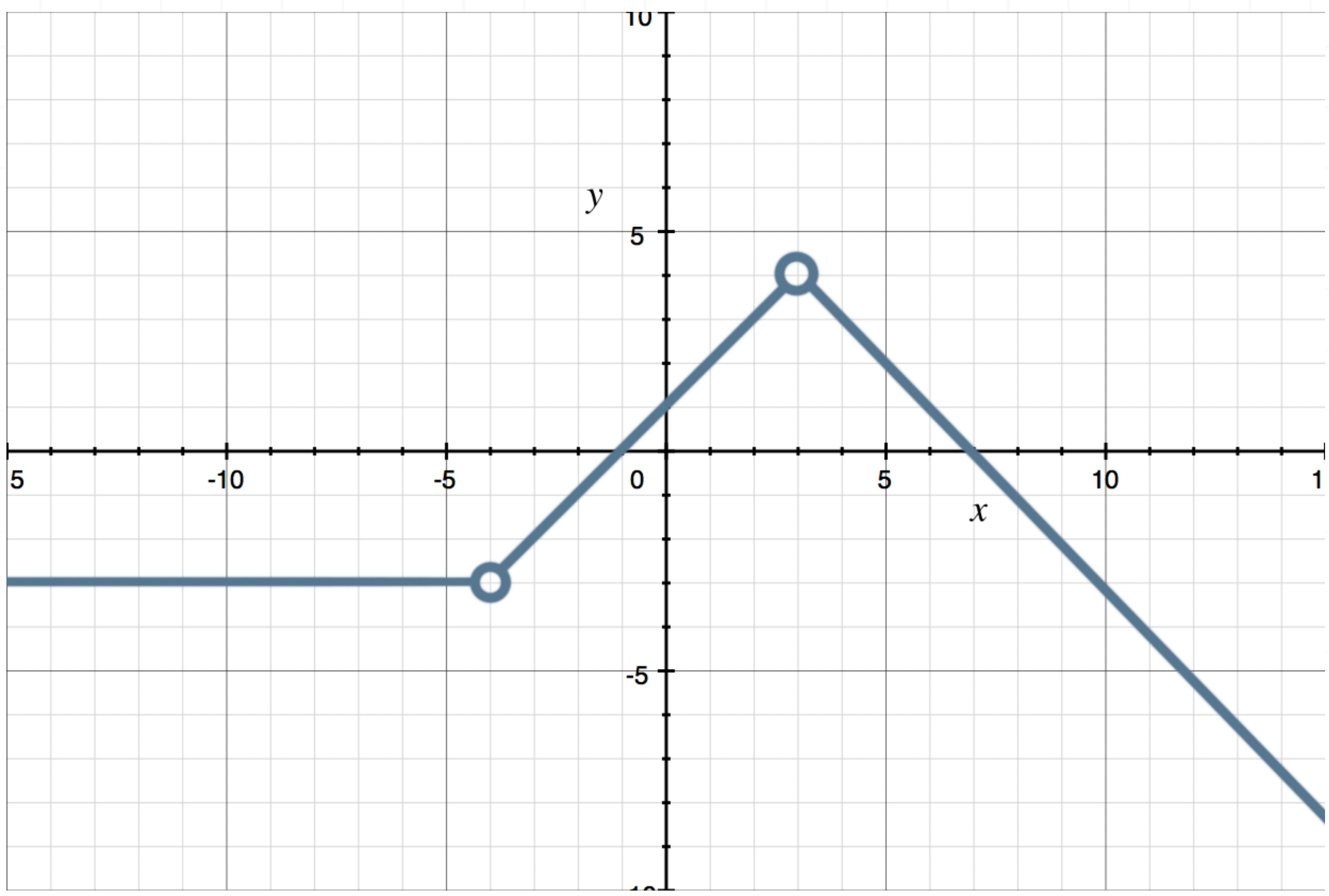
The third part of the graph is part of the line $y = -4$, and it goes from $x = 1$ to the right. For this piece, we write -4 for the function and $1 < x$ for its domain. We can't include $x = 1$ in the domain of this piece, because we included $x = 1$ in the domain of the second piece.

When we put the pieces together, we get this function:

$$f(x) = \begin{cases} 4 & \text{if} \quad x \leq -1 \\ -4x & \text{if} \quad -1 < x \leq 1 \\ -4 & \text{if} \quad 1 < x \end{cases}$$

Topic: Modeling a piecewise-defined function

Question: What is the definition of the piecewise function shown in the graph?



Answer choices:

A $f(x) = \begin{cases} -4 & \text{if} \quad x < -3 \\ x+1 & \text{if} \quad -4 < x < 3 \\ -x+7 & \text{if} \quad x > 3 \end{cases}$

B $f(x) = \begin{cases} -3 & \text{if} \quad x < -4 \\ x+1 & \text{if} \quad -4 < x < 3 \\ -x+7 & \text{if} \quad x > 3 \end{cases}$

C $f(x) = \begin{cases} -3 & \text{if} \quad x \le -4 \\ x+1 & \text{if} \quad -4 < x < 3 \\ -x+7 & \text{if} \quad x \ge 3 \end{cases}$

D $f(x) = \begin{cases} -3 & \text{if} \quad x < -4 \\ x-1 & \text{if} \quad -4 < x < 3 \\ x+7 & \text{if} \quad x > 3 \end{cases}$

Solution: B

Going from left to right, the first part of the graph is part of the line $y = -3$, and it includes all values of x in the interval $x < -4$ (but not $x = -4$, because there's an open circle on the graph at $x = -4$). For this piece, we write -3 for the function and $x < -4$ for its domain.

The second part of the graph is (part of) the line that has a slope of 1, so the equation of this line is $y = x + 1$. To see how to get the slope, notice that $(-4, -3)$ and $(0,1)$ are points on this line, so its slope is

$$m = \frac{1 - (-3)}{0 - (-4)} = \frac{4}{4} = 1$$

This piece goes from $x = -4$ to $x = 3$, but neither -4 nor 3 is in its domain (or in the domain of this entire piecewise function), because there's an open circle at each of those two values of x on the graph. So for this piece, we write $x + 1$ for the function and $-4 < x < 3$ for its domain.

The graph of the third part is part of the line that has a slope of -1 and a y-intercept of 7, so the equation of this line is $y = -x + 7$. To see this, we'll first compute the slope from the points $(3,4)$ and $(5,2)$, both of which are on this line. Then we'll use the slope and the point $(3,4)$ to get the point-slope form of the equation of the line (and then use that to get the slope-intercept form). The slope is

Combining the three pieces, we get this function:

$$f(x) = \begin{cases} -3 & \text{if} \quad x < -4 \\ x + 1 & \text{if} \quad -4 < x < 3 \\ -x + 7 & \text{if} \quad x > 3 \end{cases}$$

Topic: Modeling a piecewise-defined function

Question: For the given function, evaluate $f(-4)+f(8)+f(3)$.

$$f(x) = \begin{cases} -\frac{1}{2}x - 3 & \text{if} \quad -6 \le x \le 2 \\ -4 & \text{if} \quad 2 < x \le 7 \\ 3x - 25 & \text{if} \quad 7 < x \le 9 \end{cases}$$

Answer choices:

A -10

B -6

C -2

D 2

Solution: B

Given the piecewise function

$$f(x) = \begin{cases} -\frac{1}{2}x - 3 & \text{if} \quad -6 \le x \le 2 \\ -4 & \text{if} \quad 2 < x \le 7 \\ 3x - 25 & \text{if} \quad 7 < x \le 9 \end{cases}$$

First, evaluate $f(-4)$. Notice that -4 is in the interval $-6 \le x \le 2$, so we use the function for the first piece.

$$f(x) = -\frac{1}{2}x - 3$$

$$f(-4) = -\frac{1}{2}(-4) - 3 = 2 - 3 = -1$$

Next, evaluate $f(8)$. Notice that 8 is in the interval $7 < x \le 9$, so we use the function for the third piece.

$$f(x) = 3x - 25$$

$$f(8) = 3(8) - 25 = 24 - 25 = -1$$

Now, evaluate $f(3)$. Notice that 3 is in the interval $2 < x \le 7$, so we use the function for the second piece.

$$f(x) = -4$$

$$f(3) = -4$$

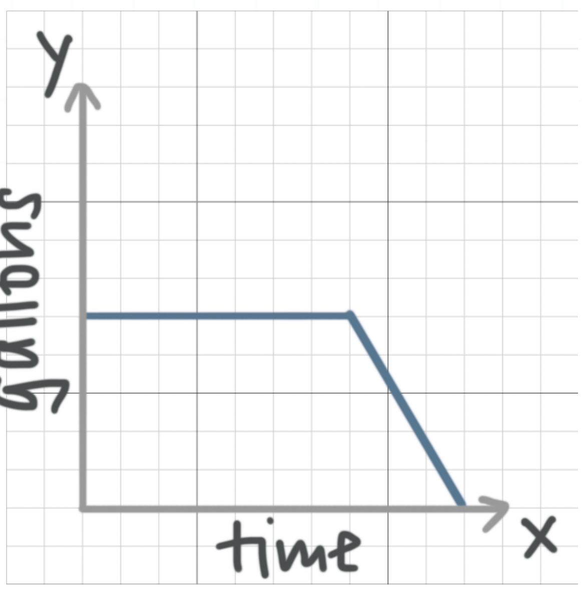
Finally, compute the sum of the three values.

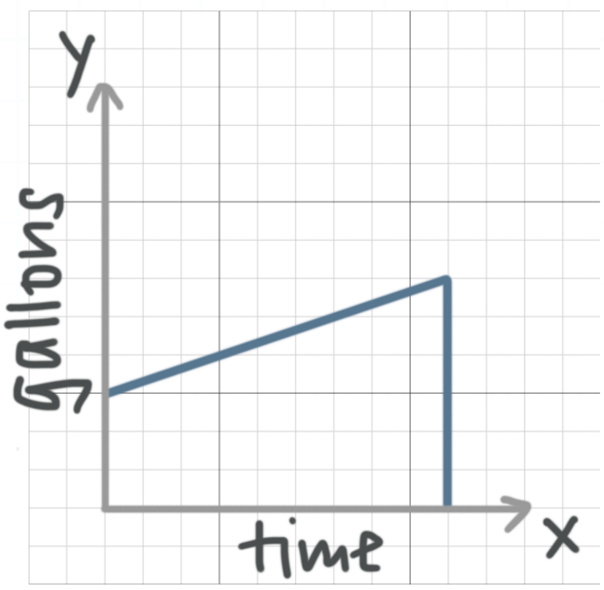
$$f(-4) + f(8) + f(3) = -1 + (-1) + (-4) = -6$$

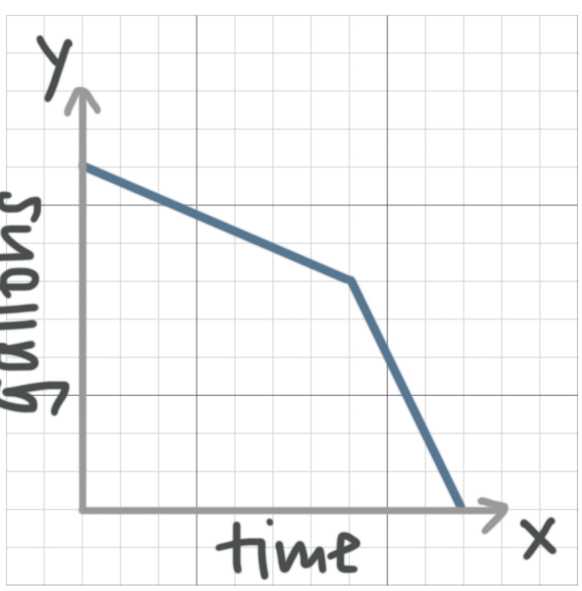
Topic: Sketching graphs from story problems

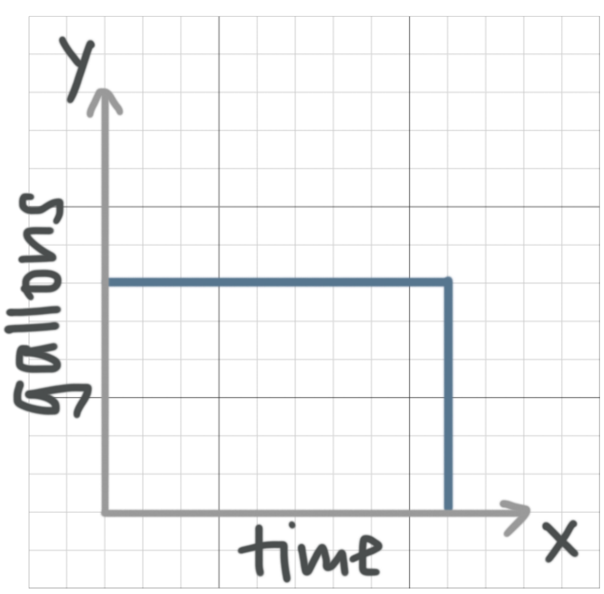
Question: A car with a full tank of gas starts on a long trip. The gas is being used up at a constant rate until a stone comes flying by and knocks a small hole in the bottom of the tank. The tank is now losing gas faster than before and eventually runs dry. Which graph best depicts the number of gallons of gas in the tank during this trip as a function of time?

Answer choices:

A 

B 

C 

D 

Solution: C

At first, the amount of gas in the tank is going down slowly, so the graph should have a mild negative slope.

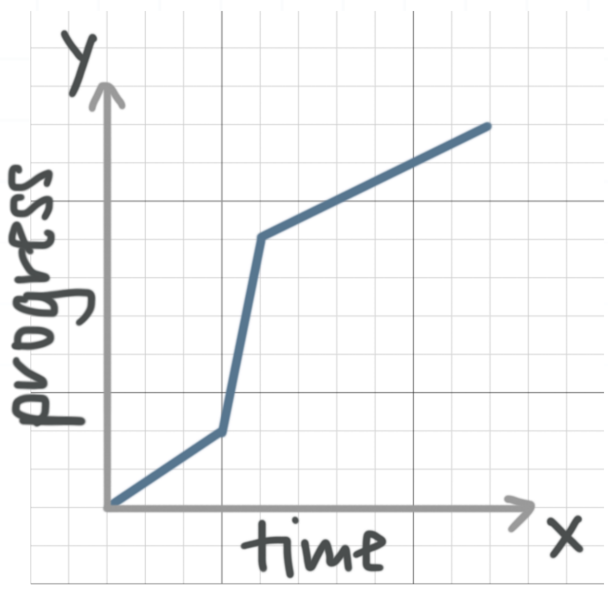
After the stone knocks a hole in the tank, the amount of gas in the tank goes down faster, and the graph will show a steeper negative slope, eventually reaching 0 gallons.

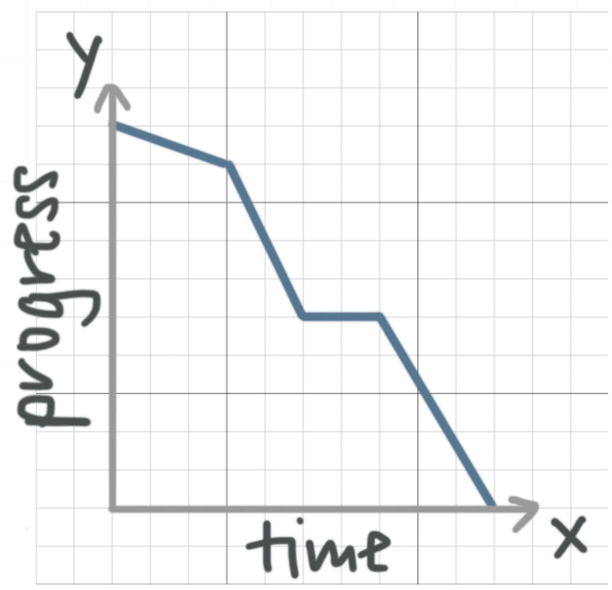
The graph in answer choice C best fits this description.

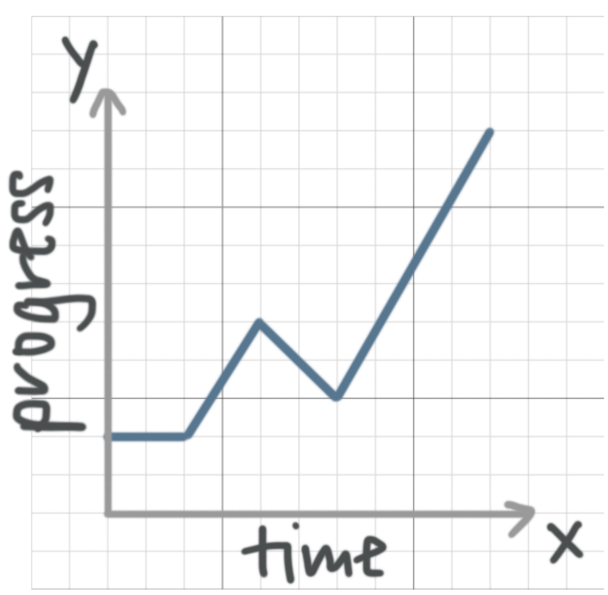
Topic: Sketching graphs from story problems

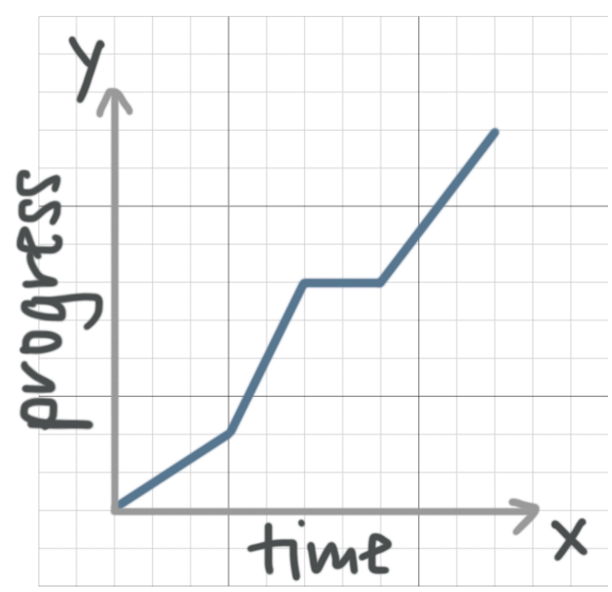
Question: A man starts to paint a large room. After an hour or so, he is joined by two other painters. An hour or so later, they all take a break for lunch. After lunch, they work for another couple of hours and finish the room. Which graph of painting progress vs. time best fits this description?

Answer choices:

A


B


C


D


Solution: D

The graph must start at the None level. This immediately rules out answer choices B and C. Now consider the rate at which the painting is done in each part of the job.

1) One man working: The room is slowly getting painted, so the graph will have a positive slope. This would also rule out answer choices B and C.

2) Three men working: The room is getting painted faster now, so the graph will have a positive slope much steeper than in the first part. This would also rule out answer choice B.

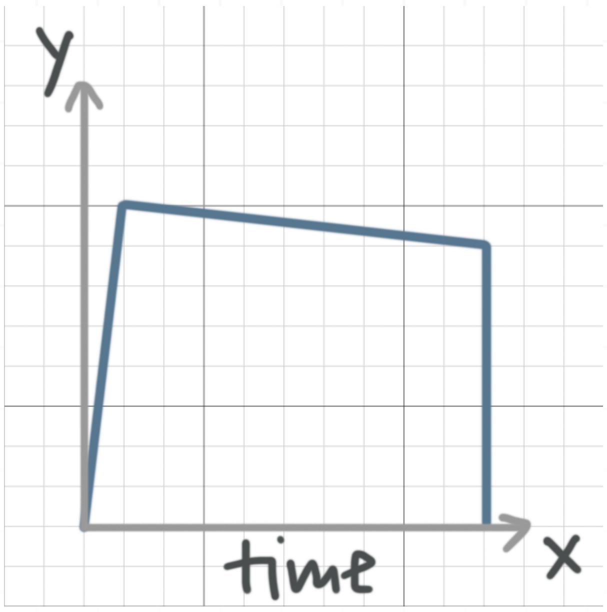
3) Lunch time: The clock is ticking, but no painting is being done, so the graph will be horizontal here. This rules out answer choice A and would also rule out answer choice C.

4) Three men working again: The slope here should be the same as in the second part and will end at the Done level.

The graph in answer choice D satisfies all four criteria.

Topic: Sketching graphs from story problems

Question: Which of the following descriptions would best be represented by the graph?



Answer choices:

A The variable represented on the vertical axis is the depth of water in a bathtub. At time 0 the water is turned on and fills the tub at a normal rate. After the bathing is over, the tub suddenly cracks wide open and all the water runs out on the floor.

B The variable represented on the vertical axis is height above ground level. At time 0 a ball is dropped from a tall building, picking up speed all the way down. It bounces off the ground once and is caught after the bounce.

C The variable represented on the vertical axis is water temperature. There is an overnight rain. At time 0 (the next morning), the sun starts to warm a large puddle. After warming, it stays at a constant temperature until evening. Then it slowly cools back down.

D The variable represented on the vertical axis is the speed of an arrow. At time 0 the bow releases the arrow, causing it to go from stationary to a high speed very quickly. The arrow travels through the air for a time and then hits a tree and stops.

Solution: D

An important clue here is that the value of the variable represented on the vertical axis increases very suddenly (steep positive slope) at the at the start. This fact alone rules out answer choices A, B, and C. In answer choice A, water is flowing into the bathtub, but at a normal rate. In answer choice B, the ball is initially falling, so its height above ground level is initially decreasing (not increasing). In answer choice C, the temperature of the water in the puddle is initially increasing, but at a low rate.

That leaves only answer choice D. The almost-horizontal section of the graph in answer choice D represents the arrow speeding through the air at an almost constant speed, and the almost-vertical section at the end represents the arrow suddenly going to a speed of 0.

Topic: Equation of a line in point-slope form

Question: Find the equation of the line.

$$m = -\frac{2}{3}$$

$$(-7,2)$$

Answer choices:

A $\quad y + 2 = \frac{2}{3}(x - 7)$

B $\quad y - 2 = \frac{2}{3}(x + 7)$

C $\quad y + 2 = -\frac{2}{3}(x - 7)$

D $\quad y - 2 = -\frac{2}{3}(x + 7)$

Solution: D

When we're given a point and the slope, we can use the point-slope form of the equation of a line, which is

$$y - y_1 = m(x - x_1)$$

where m is the slope and (x_1, y_1) is a point on the line.

We'll first plug in the slope and the coordinates of the point we've been given, and then simplify the equation by solving for y.

$$y - 2 = -\frac{2}{3}(x - (-7))$$

$$y - 2 = -\frac{2}{3}(x + 7)$$

Topic: Equation of a line in point-slope form

Question: Find the equation, in point-slope form, of the line that passes through $(2,3)$ and $(4,11)$. Use $(2,3)$ for (x_1, y_1).

Answer choices:

A $y - 3 = 4(x - 2)$

B $y - 3 = 8(x - 2)$

C $y + 3 = 4(x + 2)$

D $y - 3 = 4(x + 2)$

Solution: A

First, find the slope of the line using the given points.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11 - 3}{4 - 2} = \frac{8}{2} = 4$$

Next, substitute $m = 4$ and the coordinates of the point $(2,3)$ into the formula $y - y_1 = m(x - x_1)$.

$$y - 3 = 4(x - 2)$$

Topic: Equation of a line in point-slope form

Question: Find the equation, in point-slope form, of the line that passes through $(-3,0)$ and rises 2 in a run of 4.

Answer choices:

A $\quad y = \frac{1}{2}(x - 3)$

B $\quad y = 2(x - 3)$

C $\quad y + 3 = \frac{1}{2}x$

D $\quad y = \frac{1}{2}(x + 3)$

Solution: D

First, find the slope of the line using the rise and run.

$$m = \frac{2}{4} = \frac{1}{2}$$

Next, use $y - y_1 = m(x - x_1)$. Use $(-3,0)$ as (x_1, y_1), and get

$$y - 0 = \frac{1}{2}(x - (-3))$$

$$y = \frac{1}{2}(x + 3)$$

Topic: Equation of a line in slope-intercept form

Question: Find the equation, in slope-intercept form, of the line that passes through $(0, -2)$ and has a slope of $1/2$.

Answer choices:

A $\quad y = \frac{1}{2}(x - 2)$

B $\quad y = \frac{1}{2}x + 2$

C $\quad y = \frac{1}{2}x - 2$

D $\quad y = \frac{1}{2}x - 1$

Solution: C

First, use $m = 1/2$ and $(x_1, y_1) = (0, -2)$ in the equation $y - y_1 = m(x - x_1)$.

$$y - (-2) = \frac{1}{2}(x - 0)$$

$$y + 2 = \frac{1}{2}x$$

Now subtract 2 from both sides to get this equation into slope-intercept form.

$$y = \frac{1}{2}x - 2$$

Topic: Equation of a line in slope-intercept form

Question: Find the equation, in slope-intercept form, of the line that passes through $(-3,-2)$ and $(3,-4)$.

Answer choices:

A $\quad y=-3x-3$

B $\quad y=-\frac{1}{3}x-3$

C $\quad y=-3x-1$

D $\quad y=-\frac{1}{3}x-1$

Solution: B

First, find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-4 - (-2)}{3 - (-3)} = \frac{-2}{6} = -\frac{1}{3}$$

Next, use the equation $y - y_1 = m(x - x_1)$. m will be $-1/3$. Using the point $(-3, -2)$ for (x_1, y_1), we get

$$y - (-2) = -\frac{1}{3}(x - (-3))$$

$$y + 2 = -\frac{1}{3}(x + 3)$$

$$y + 2 = -\frac{1}{3}x - 1$$

Finally, subtract 2 from both sides to get this equation into slope-intercept form.

$$y = -\frac{1}{3}x - 3$$

Topic: Equation of a line in slope-intercept form

Question: Find the equation, in slope-intercept form, of the line that passes through $(2,5)$ and is parallel to $y = 3 - 2x$.

Answer choices:

A $\quad y = -2x - 12$

B $\quad y = 2x + 2$

C $\quad y = 2x + 1$

D $\quad y = -2x + 9$

Solution: D

First, rewrite the given equation in slope-intercept form.

$$y = -2x + 3$$

Remembering $y = mx + b$, we can see that the slope of the given equation is -2. For the two lines to be parallel, the equation we're looking for must also have a slope of -2.

Next, use the equation $y - y_1 = m(x - x_1)$. Use -2 for m and $(2,5)$ for (x_1, y_1).

$$y - 5 = -2(x - 2)$$

$$y - 5 = -2x + 4$$

Add 5 to both sides to put the equation in slope-intercept form.

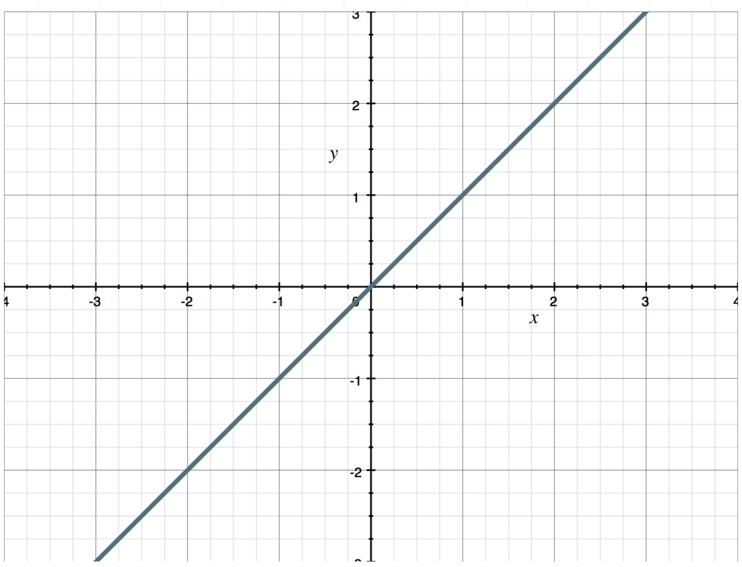
$$y = -2x + 9$$

Topic: Graphing parabolas

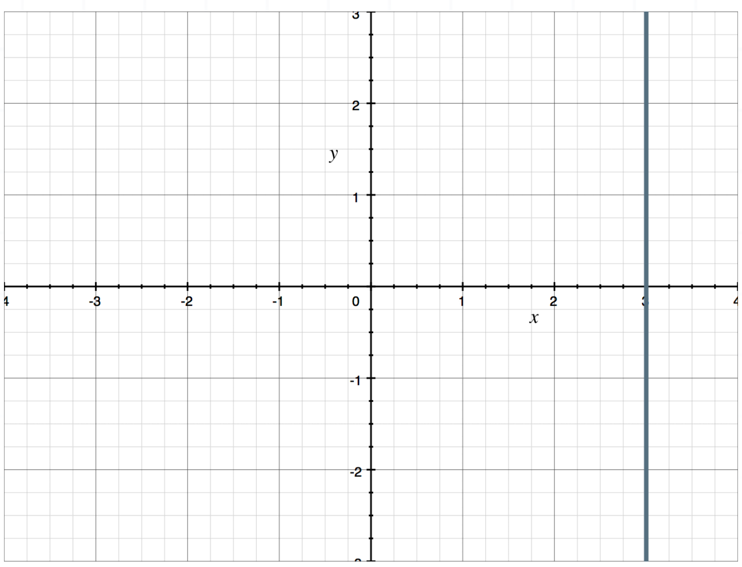
Question: Which graph represents a non-linear function?

Answer choices:

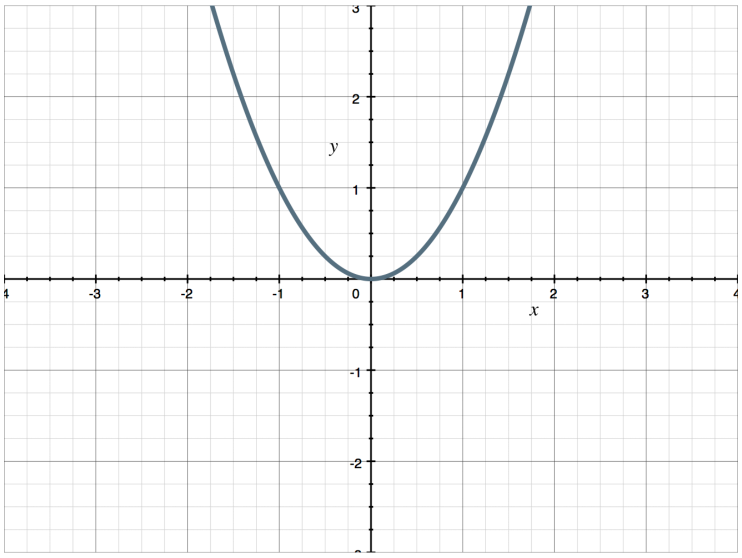
A $y = x$



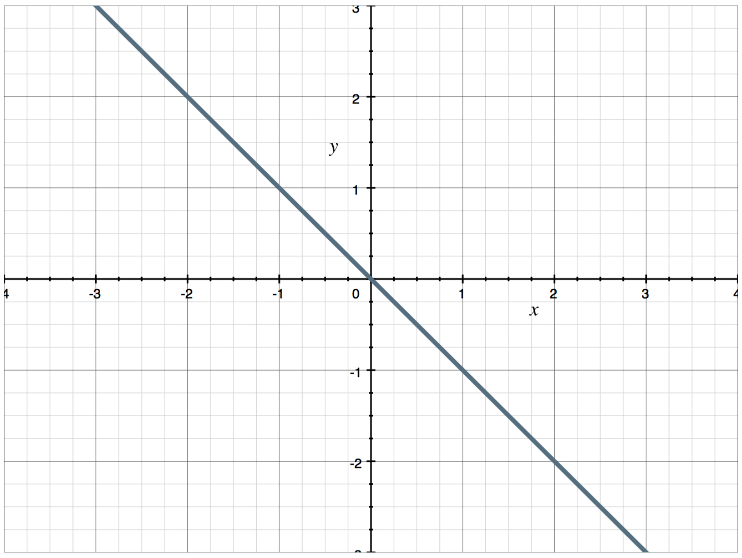
B $x = 3$



C $y = x^2$



D $y = -x$



Solution: C

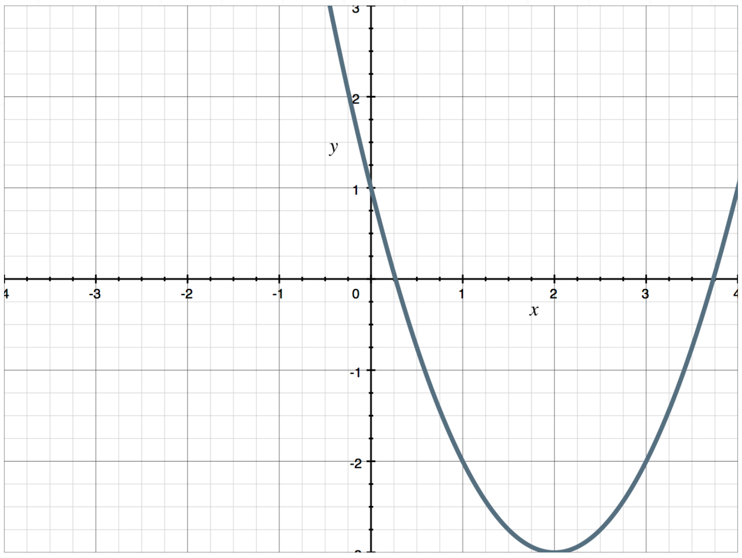
The graph in answer choice C is the only graph that isn't a line, which means it's the only one that represents a non-linear function.

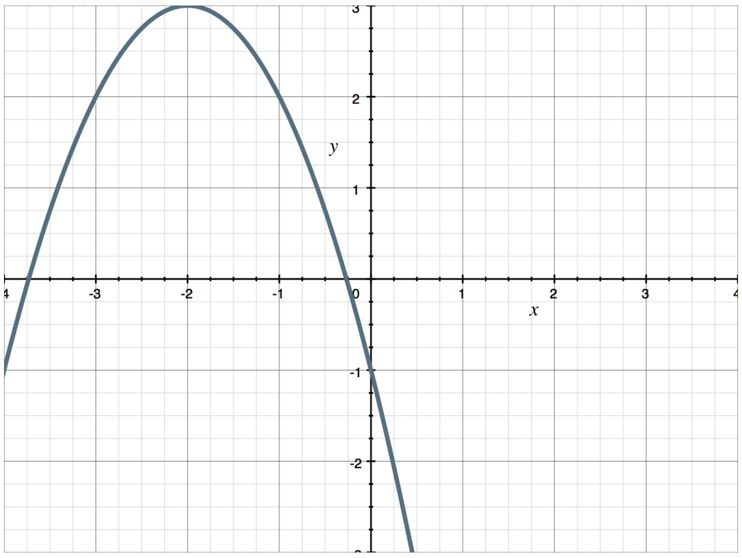
Topic: Graphing parabolas

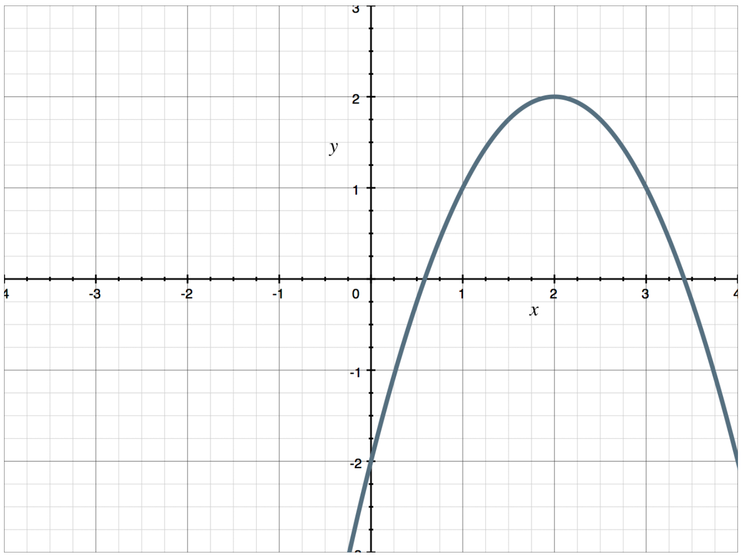
Question: Which graph represents the function?

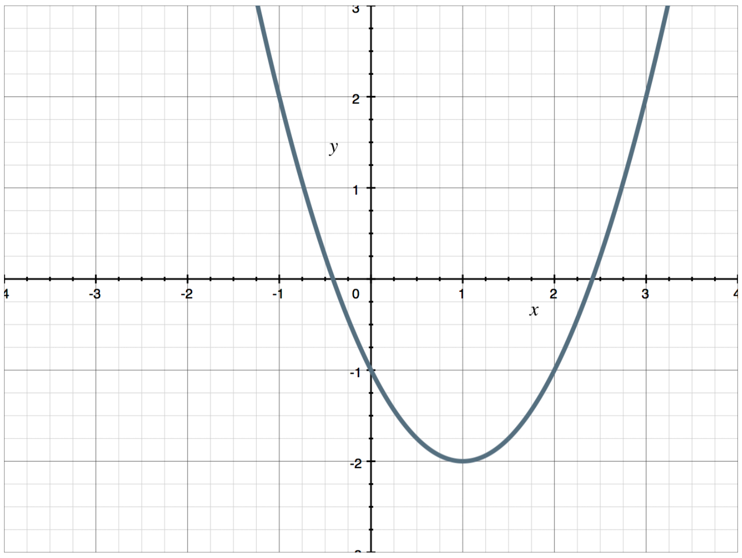
$$y = -x^2 - 4x - 1$$

Answer choices:

A 

B 

C 

D 

Solution: B

We'll first convert the equation of the parabola to vertex form,

$$y = a(x-h)^2 + k$$

where (h, k) are the coordinates of the vertex of the parabola.

We'll do this by completing the square.

Before we complete the square, we'll factor a -1 out of the expression on the right-hand side of the given equation, because it's easier to deal with a polynomial in which the leading term has a positive coefficient.

$$y = -x^2 - 4x - 1$$

$$y = -(x^2 + 4x + 1)$$

To complete the square, we need to find the number d that satisfies the equation

$$x^2 + 4x + d^2 = (x+d)^2$$

That is, we need to find the number d for which

$$x^2 + 4x + d^2 = x^2 + 2dx + d^2$$

This means that the coefficient of the x term of the expression inside the parentheses must be equal to $2d$. That coefficient is 4, so we'll set $2d$ equal to 4 and solve for d.

$$2d = 4 \quad \to \quad d = 2$$

To keep our equation balanced, we need to add and subtract d^2 (4) inside the parentheses, and then regroup and simplify.

$$y = -(x^2 + 4x + 1)$$

$$y = -(x^2 + 4x + 4 - 4 + 1)$$

$$y = -\left[(x^2 + 4x + 4) - 4 + 1\right]$$

$$y = -\left[(x^2 + 4x + 4) - 3\right]$$

$$y = -(x^2 + 4x + 4) + 3$$

Finally, we'll factor the expression that's now inside the parentheses $(x^2 + 4x + 4)$. By construction ("completing the square"), that expression factors as $(x + d)^2$.

$$x^2 + 4x + 4 = (x + d)^2$$

$$x^2 + 4x + 4 = (x + 2)^2$$

Therefore, the vertex form of the equation of the parabola is

$$y = -(x + 2)^2 + 3$$

Now that we've got the equation of the parabola in vertex form, we can identify its characteristics.

1. The negative sign in front of the parentheses indicates that the parabola opens downwards.

2. The coordinates of the vertex (in this case the point at the top of the parabola) are $(h, k) = (-2,3)$.

3. The y-coordinate of the y-intercept, which is the point of the parabola whose x-coordinate is 0, is found by substituting 0 for x in the equation of the parabola.

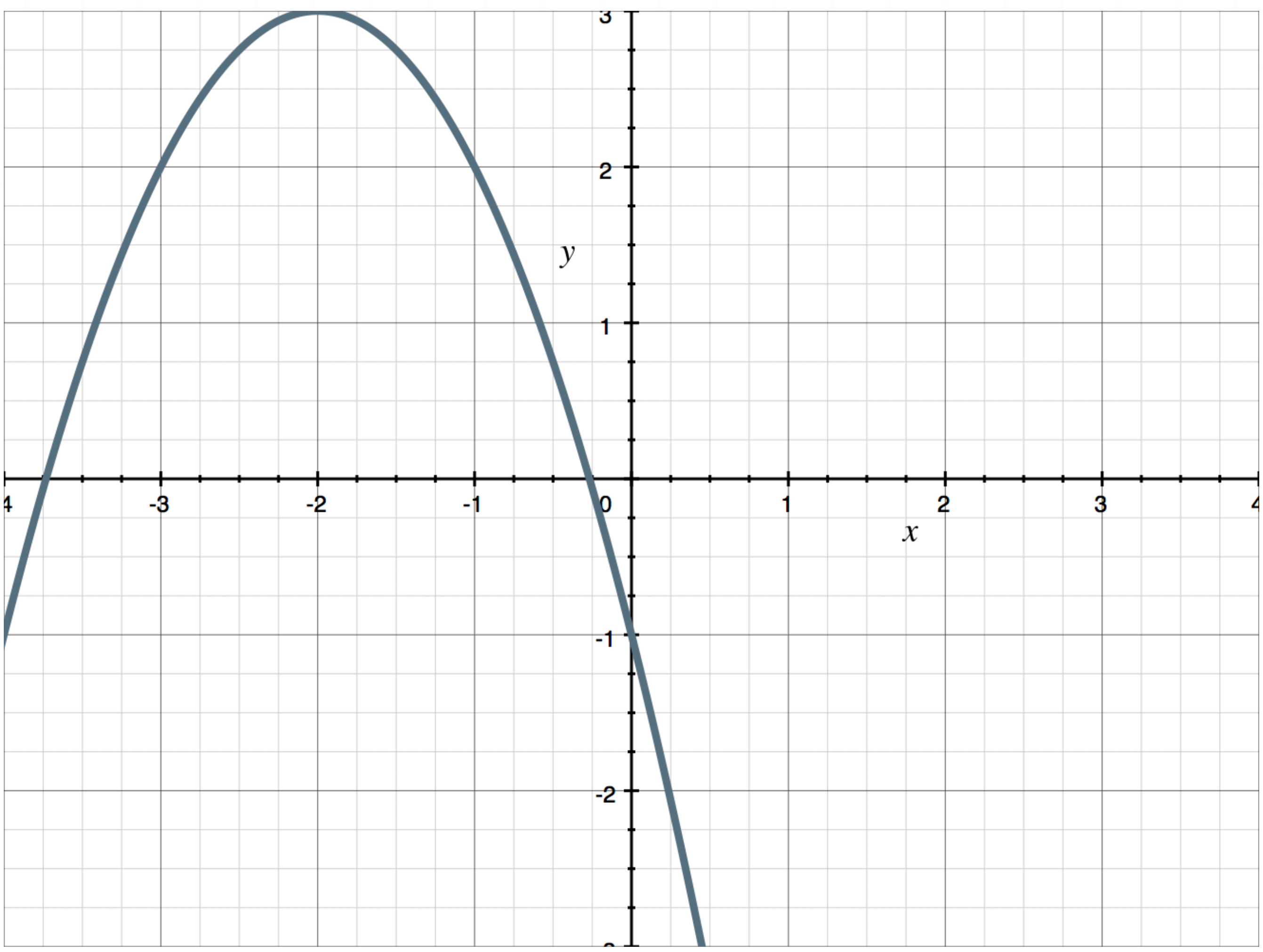
$$y = -(x+2)^2 + 3$$

$$y = -(0+2)^2 + 3$$

$$y = -2^2 + 3$$

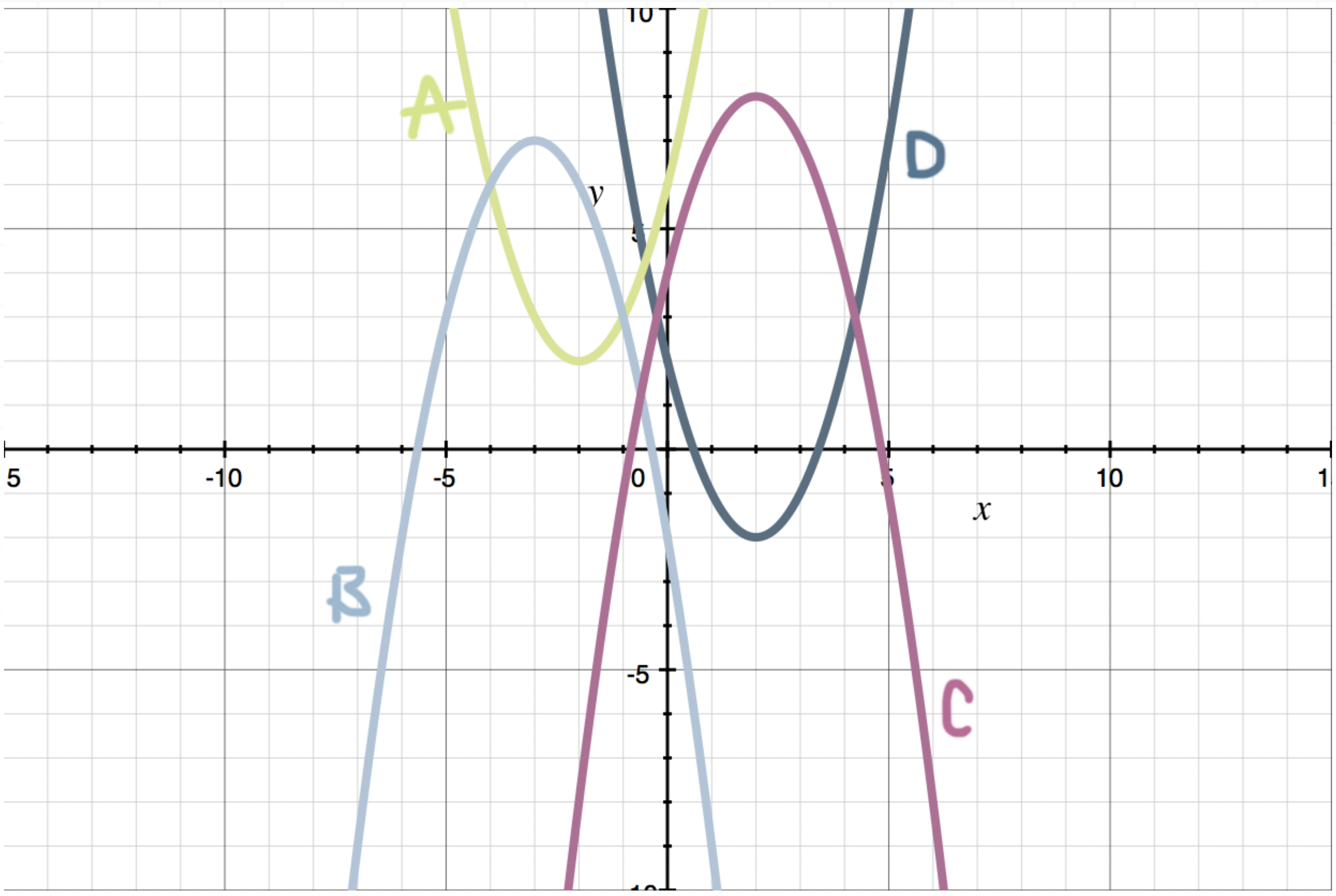
$$y = -4 + 3$$

$$y = -1$$



Topic: Graphing parabolas

Question: Which parabola is the graph of $y = x^2 - 4x + 2$?



Answer choices:

A A

B B

C C

D D

Solution: D

The easiest way to do this is to find the y-intercept. Remember, the y-intercept is the point of the parabola whose x-coordinate is 0. We can substitute 0 for x in the given equation to find the y-coordinate of the y-intercept.

$$y = x^2 - 4x + 2$$

$$y = 0^2 - 4(0) + 2$$

$$y = 2$$

The y-intercept is $(0,2)$.

Although the graphs are a little crowded there, you can see that the graph in answer choice D is the only one that passes through $(0,2)$.

Another method of doing the problem is to find the vertex by matching the given equation to the standard form of the equation of a parabola, $y = ax^2 + bx + c$, and then finding the coordinates of the vertex. We know that the x-coordinate of the vertex is $-b/(2a)$. For this parabola, $a = 1$ and $b = -4$, so

$$-\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2$$

The y-coordinate of the vertex can be found by substituting 2 for x in the equation of the parabola.

$$y = x^2 - 4x + 2$$

$$y = (2^2) - 4(2) + 2$$

$$y = 4 - 8 + 2$$

$$y = -2$$

Therefore, the coordinates of the vertex are $(2, -2)$. The graph in answer choice D is the only one whose vertex is at the point $(2, -2)$.

Topic: Center and radius of a circle

Question: Find the center and radius of the circle.

$$4x^2 + 4y^2 + 4x - 12y + 1 = 0$$

Answer choices:

A Center at $\left(\frac{1}{2}, \frac{3}{2}\right)$ Radius of $\frac{3}{2}$

B Center at $\left(-\frac{1}{2}, \frac{3}{2}\right)$ Radius of $\frac{3}{2}$

C Center at $\left(\frac{1}{2}, \frac{3}{2}\right)$ Radius of $\frac{1}{2}$

D Center at $\left(-\frac{1}{2}, \frac{3}{2}\right)$ Radius of $\frac{1}{2}$

Solution: B

In order to find the center and radius, we need to convert the equation of the circle to standard form, $(x-h)^2+(y-k)^2=r^2$, where h and k are the coordinates of the center and r is the radius.

We'll begin by grouping the x terms separately from the y terms, and moving the constant term to the right side of the equation.

$$4x^2+4y^2+4x-12y+1=0$$

$$4x^2+4y^2+4x-12y=-1$$

In standard from, the coefficients of the x^2 term and the y^2 term must be equal to 1. Since the coefficient of each of those terms is now 4, we'll factor out a 4 on the left side of the equation and then divide both sides by 4.

$$4(x^2+y^2+x-3y)=-1$$

$$x^2+y^2+x-3y=-\frac{1}{4}$$

In order to get this equation into standard form, we need to complete the square on both x and y.

$$(x^2+x)+(y^2-3y)=-\frac{1}{4}$$

To complete the square on x, we need to find the number a that satisfies the equation

$$x^2+x+a^2=(x+a)^2$$

That is, we need to find the number a for which

$$x^2+x+a^2=x^2+2ax+a^2$$

This means that the coefficient of the x term of the expression inside the first set of parentheses must be equal to $2a$. That coefficient is 1, so we'll set $2a$ to 1 and solve for a.

$$2a=1 \quad \to \quad a=\frac{1}{2}$$

To keep our equation balanced, we need to add and subtract a^2 $(1/4)$ inside that set of parentheses and then regroup.

$$(x^2+x)+(y^2-3y)=-\frac{1}{4}$$

$$\left(x^2+x+\frac{1}{4}-\frac{1}{4}\right)+(y^2-3y)=-\frac{1}{4}$$

$$\left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}+(y^2-3y)=-\frac{1}{4}$$

To complete the square on y, we need to find the number b that satisfies the equation

$$y^2-3y+b^2=(y+b)^2$$

That is, we need to find the number b for which

$$y^2-3y+b^2=y^2+2by+b^2$$

This means that the coefficient of the y term of the expression inside the second set of parentheses must be equal to $2b$. That coefficient is -3, so we'll set $2b$ to -3 and solve for b.

$$2b = -3 \quad \to \quad b = -\frac{3}{2}$$

To keep our equation balanced, we need to add and subtract b^2 $(9/4)$ inside that set of parentheses and then regroup.

$$\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} + (y^2 - 3y) = -\frac{1}{4}$$

$$\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} + \left(y^2 - 3y + \frac{9}{4} - \frac{9}{4}\right) = -\frac{1}{4}$$

$$\left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} + \left(y^2 - 3y + \frac{9}{4}\right) - \frac{9}{4} = -\frac{1}{4}$$

Moving the $-1/4$ and $-9/4$ to the right side, we have

$$\left(x^2 + x + \frac{1}{4}\right) + \left(y^2 - 3y + \frac{9}{4}\right) = -\frac{1}{4} + \frac{1}{4} + \frac{9}{4}$$

Factoring the expressions in parentheses and simplifying the right side, we get

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$$

If you think of $x + (1/2)$ and $9/4$ as $x - (-1/2)$ and $(3/2)^2$, respectively, you'll see that the center of the circle is at

$$(h, k) = \left(-\frac{1}{2}, \frac{3}{2}\right)$$

and its radius is

$$r = \frac{3}{2}$$

Topic: Center and radius of a circle

Question: Find the center and radius of the given circle.

$$x^2 + y^2 - 6y = 5$$

Answer choices:

A Center is $(0,3)$. Radius is $\sqrt{14}$.

B Center is $(0,-3)$. Radius is $\sqrt{14}$.

C Center is $(0,3)$. Radius is 14.

D Center is $(0,-3)$. Radius is 14.

Solution: A

x^2 is already a perfect square, so we'll complete the square on y.

$$x^2 + (y^2 - 6y) = 5$$

To do that, we need to find the number a that satisfies the equation

$$y^2 - 6y + a^2 = (y + a)^2$$

That is, we need to find the number a for which

$$y^2 - 6y + a^2 = y^2 + 2ay + a^2$$

This means that the coefficient of the y term of the expression inside the parentheses must be equal to $2a$. That coefficient is -6, so we'll set $2a$ equal to -6 and solve for a.

$$2a = -6 \quad \to \quad a = -3$$

To keep our equation balanced, we need to add and subtract a^2 (9) inside the parentheses and then regroup.

$$x^2 + (y^2 - 6y) = 5$$

$$x^2 + (y^2 - 6y + 9 - 9) = 5$$

$$x^2 + (y^2 - 6y + 9) - 9 = 5$$

Moving the -9 to the right side, we have

$$x^2 + (y^2 - 6y + 9) = 5 + 9$$

Factoring the expression in parentheses and simplifying the right side, we get

$$x^2 + (y - 3)^2 = 14$$

If you think of x and 14 as $x - 0$ and $(\sqrt{14})^2$, respectively, you'll see that the center of the circle is at $(h, k) = (0,3)$ and the radius is $\sqrt{14}$.

Topic: Center and radius of a circle

Question: Find the center and radius of the given circle.

$$x^2 + y^2 + 10x - 4y + 13 = 0$$

Answer choices:

A Center is $(-5,2)$. Radius is 16.

B Center is $(5, -2)$. Radius is 4.

C Center is $(-5,2)$. Radius is 4.

D Center is $(5, -2)$. Radius is 16.

Solution: C

Starting with

$$x^2 + y^2 + 10x - 4y + 13 = 0$$

we'll group the terms in x separately from the terms in y, and subtract 13 from both sides.

$$x^2 + 10x + y^2 - 4y = -13$$

We need to complete the square on both x and y.

$$(x^2 + 10x) + (y^2 - 4y) = -13$$

To complete the square on x, we need to find the number a that satisfies the equation

$$x^2 + 10x + a^2 = (x + 1)^2$$

That is, we need to find the number a for which

$$x^2 + 10x + a^2 = s^2 + 2ax + a^2$$

This means that the coefficient of the x term of the expression inside the first set of parentheses must be equal to $2a$. That coefficient is 10, so we'll set $2a$ equal to 10 and solve for a.

$$2a = 10 \quad \to \quad z = 5$$

To keep our equation balanced, we need to add and subtract a^2 (25) inside that set of parentheses and then regroup.

$$(x^2 + 10x) + (y^2 - 4y) = -13$$

$$(x^2 + 10x + 25 - 25) + (y^2 - 4y) = -13$$

$$(x^2 + 10x + 25) - 25 + (y^2 - 4y) = -13$$

To complete the square on y, we need to find the number b that satisfies the equation

$$y^2 - 4y + b^2 = (y + b)^2$$

That is, we need to find the number b for which

$$y^2 - 4y + b^2 = y^2 + 2by + b^2$$

This means that the coefficient of the y term of the expression inside the second set of parentheses must be equal to $2b$. That coefficient is -4, so we'll set $2b$ equal to -4 and solve for b.

$$2b = -4 \quad \to \quad b = -2$$

To keep our equation balanced, we need to add and subtract b^2 (4) inside that set of parentheses and then regroup.

$$(x^2 + 10x + 25) - 25 + (y^2 - 4y) = -13$$

$$(x^2 + 10x + 25) - 25 + (y^2 - 4y + 4 - 4) = -13$$

$$(x^2 + 10x + 25) - 25 + (y^2 - 4y + 4) - 4 = -13$$

Moving the -25 and -4 to the right side, we have

$$(x^2 + 10x + 25) + (y^2 - 4y + 4) = -13 + 25 + 4$$

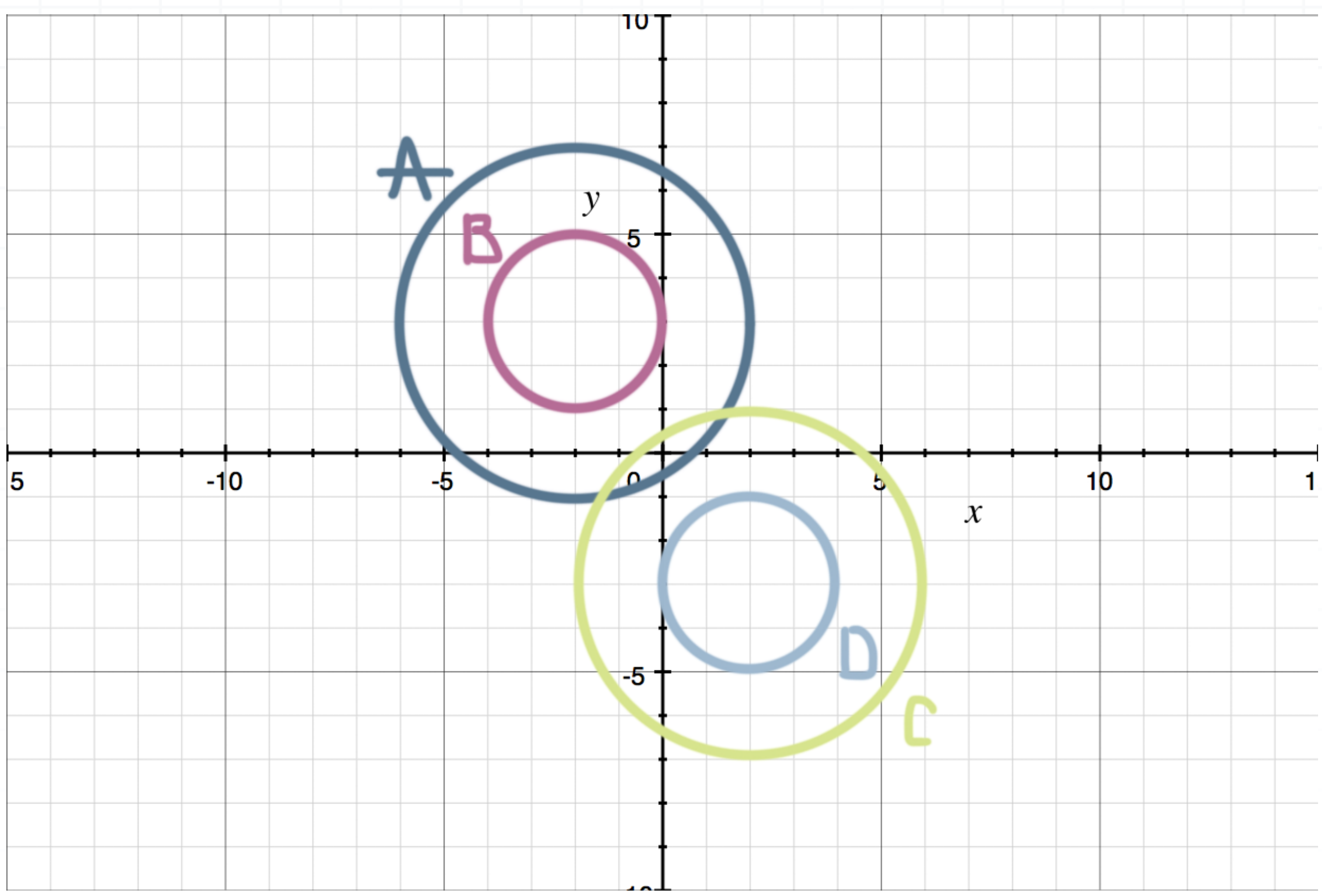
Factoring the expressions in parentheses and simplifying the right side, we get

$$(x+5)^2 + (y-2)^2 = 16$$

If you think of $x+5$ and 16 as $x-(-5)$ and 4^2, respectively, you'll see that the center of the circle is at $(h,k)=(-5,2)$ and the radius is 4.

Topic: Graphing circles

Question: Which circle is the graph of $(x-2)^2+(y+3)^2=4$?



Answer choices:

A A

B B

C C

D D

Solution: D

Given

$$(x-2)^2+(y+3)^2=4$$

we can put this in the form $(x-h)^2+(y-k)^2=r^2$ by rewriting it.

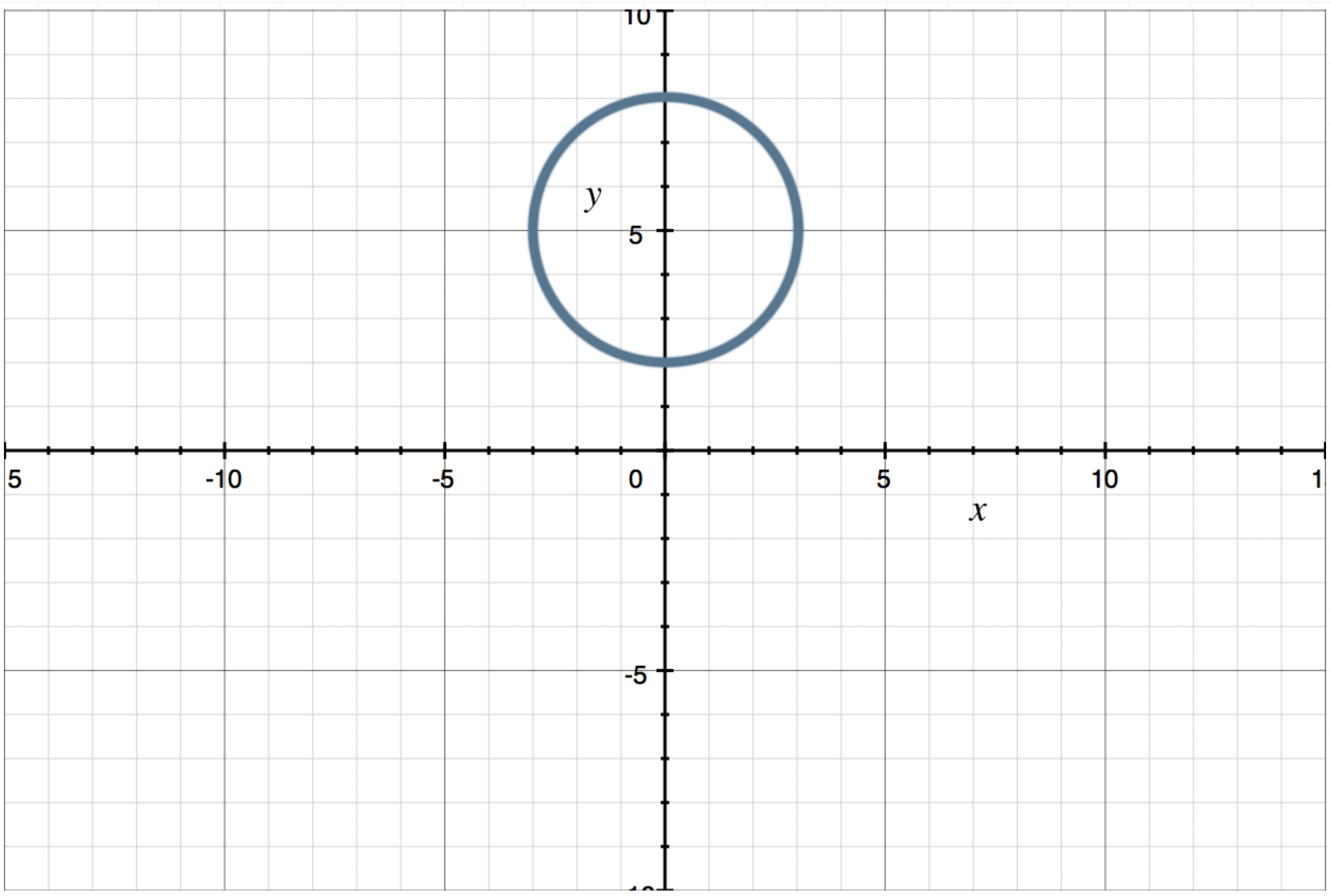
$$(x-2)^2+\left[y-(-3)\right]^2=2^2$$

We can see that $h=2$, $k=-3$, and $r=2$.

The circle with center at $(2,-3)$ and radius 2 is D.

Topic: Graphing circles

Question: What is the equation of the given circle?



Answer choices:

A $x^2 - 10x + y^2 + 16 = 0$

B $x^2 + y^2 + 10y + 16 = 0$

C $x^2 + y^2 - 10y + 16 = 0$

D $x^2 + 10x + y^2 + 16 = 0$

Solution: C

The points $(3,5)$ and $(-3,5)$ are at opposite ends of a diameter of this circle. They both have a y-coordinate of 5, so the center of the circle also has a y-coordinate of 5 and it lies halfway between them.

The distance between the points $(3,5)$ and $(-3,5)$ is the difference in their x-coordinates, which is $3-(-3)=6$, so the center of the circle is at a distance of 3 units from both of those points. Therefore, the center of the circle is at the point $(0,5)$, and it has a radius of 3. That tells us that $h=0$, $k=5$, and $r=3$.

Substitute the values of h, k, and r into the equation $(x-h)^2+(y-k)^2=r^2$, then expand and simplify.

$$(x-0)^2+(y-5)^2=3^2$$

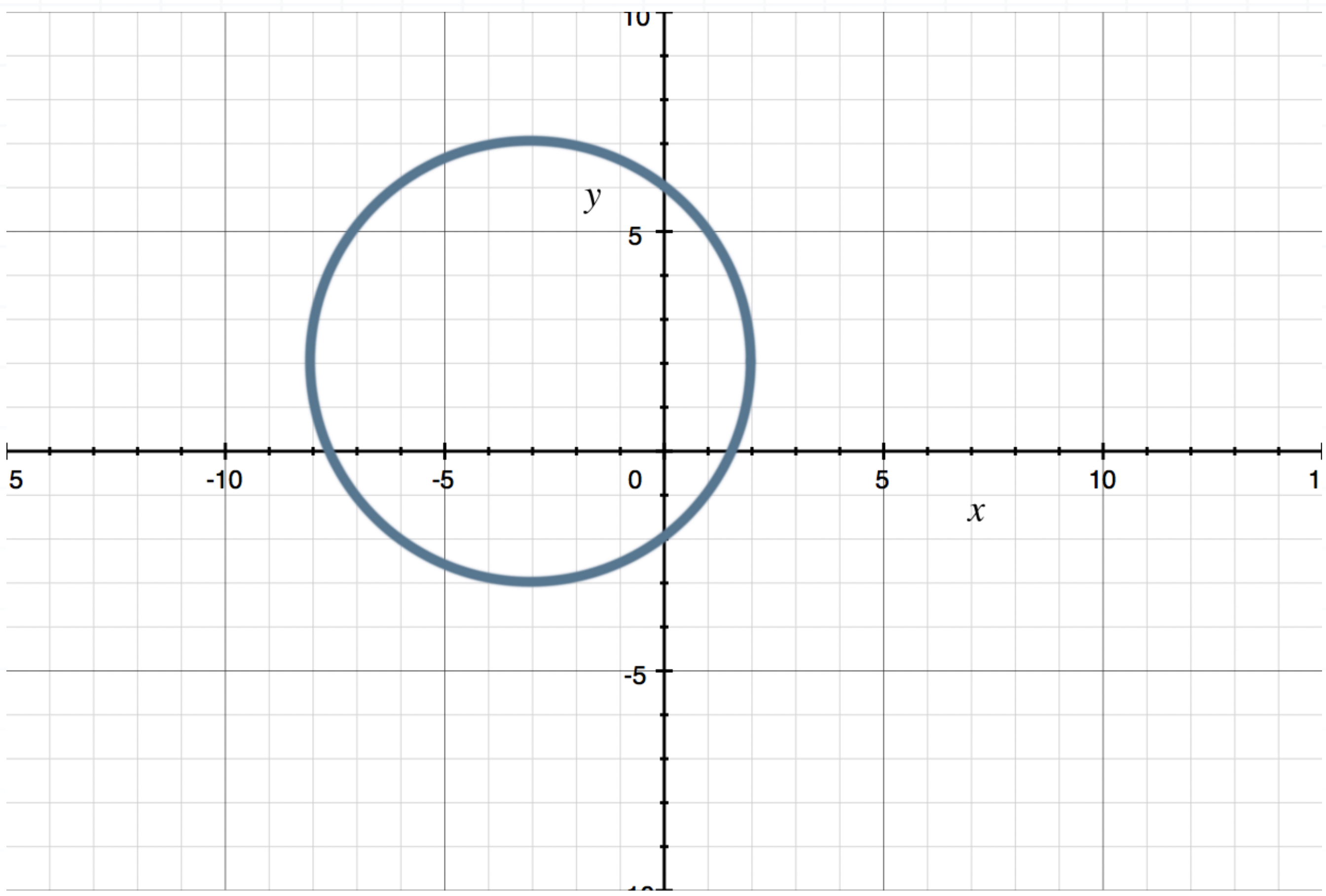
$$x^2+y^2-10y+25=9$$

$$x^2+y^2-10y+16=0$$

This matches answer choice C.

Topic: Graphing circles

Question: What is the equation of the given circle?



Answer choices:

A $x^2 + 6x + y^2 - 4y = 12$

B $x^2 - 6x + y^2 + 4y = 12$

C $x^2 + 6x + y^2 - 4y = -12$

D $x^2 + 6x + y^2 - 4y = 25$

Solution: A

The points $(-3,7)$ and $(-3,-3)$ are at opposite ends of a diameter of this circle. They both have an x-coordinate of -3, so the center of the circle also has an x-coordinate of -3 and it lies halfway between them.

The distance between the points $(-3,7)$ and $(-3,-3)$ is the difference in their y-coordinates, which is $7-(-3)=10$, so the center of the circle is at a distance of 5 units from both of those points. Therefore, the center of the circle is at the point $(-3,2)$, and it has a radius of 5. That tells us that $h=-3$, $k=2$, and $r=5$.

Substitute the values of h, k, and r into the equation $(x-h)^2+(y-k)^2=r^2$, then expand and simplify.

$$\left[x-(-3)\right]^2+(y-2)^2=5^2$$

$$(x+3)^2+(y-2)^2=5^2$$

$$x^2+6x+9+y^2-4y+4=25$$

$$x^2+6x+y^2-4y=12$$

This matches answer choice A.

Topic: Combinations of functions

Question: Find $(f \cdot g)(x)$.

$$f(x) = 2x^2 + 5$$

$$g(x) = x + 2$$

Answer choices:

A $2x^3 + 5x^2 + 4x + 10$

B $2x^3 + 3x^2 + 3x + 10$

C $2x^3 + 4x^2 + 5x + 10$

D $2x^3 + 10x^2 + 10x + 10$

Solution: C

The combination $(f \cdot g)(x)$ is the same as the product $f(x) \cdot g(x)$. Therefore,

$$(f \cdot g)(x) = \left(2x^2 + 5\right)(x + 2)$$

We can find this product using the FOIL method.

$$(f \cdot g)(x) = 2x^3 + 4x^2 + 5x + 10\text{v}$$

Topic: Combinations of functions

Question: Find $(f-g)(x)$.

$$f(x) = 2x^2 + 6x - 3$$

$$g(x) = 3x^2 - 5x - 2$$

Answer choices:

A $\quad -x^2 + 11x - 1$

B $\quad x^2 + x - 5$

C $\quad -x^2 + 11x - 5$

D $\quad -x^2 + x - 1$

Solution: A

The combination $(f-g)(x)$ is the same as the difference $f(x)-g(x)$. Therefore,

$$(f-g)(x) = (2x^2+6x-3)-(3x^2-5x-2)$$

$$(f-g)(x) = 2x^2+6x-3-3x^2-(-5x)-(-2)$$

$$(f-g)(x) = 2x^2+6x-3-3x^2+5x+2$$

$$(f-g)(x) = -x^2+11x-1$$

Topic: Combinations of functions

Question: The domain of $(f/g)(x)$ is all real numbers, except what?

$$f(x) = x^2 - 9$$

$$g(x) = 2x - 6$$

Answer choices:

A 6

B 3

C 0

D -3

Solution: B

The function

$$\left(\frac{f}{g}\right)(x)$$

is the same as the quotient

$$\frac{f(x)}{g(x)}$$

The domain of

$$\frac{x^2-9}{2x-6}$$

is all real numbers except those that make the denominator 0.

$$2x-6=0 \quad \to \quad 2x=6 \quad \to \quad x=3$$

So the only real number that isn't in the domain is 3.

Topic: Composite functions

Question: Find the composite function.

$$g(f(x))$$

$$f(x) = \frac{1}{x^2}$$

$$g(x) = \sqrt{x - 3}$$

Answer choices:

A $\quad g(f(x)) = \dfrac{1}{x - 3}$

B $\quad g(f(x)) = \sqrt{\dfrac{1}{x^2} - 3}$

C $\quad g(f(x)) = \sqrt{\dfrac{1}{(x - 3)^2}}$

D $\quad g(f(x)) = \dfrac{1}{\sqrt{x - 3}}$

Solution: B

To find the composite function $g(f(x))$, we plug $f(x)$ into $g(x)$, which means that we take the algebraic expression for $f(x)$ and substitute it for x in the algebraic expression for $g(x)$.

$$g(f(x)) = \sqrt{\frac{1}{x^2} - 3}$$

Topic: Composite functions

Question: Find $g(h(x))$.

$$g(x) = x^2 - x - 4$$

$$h(x) = x\sqrt{2} + 1$$

Answer choices:

A $\sqrt{2}(x^2 - x - 4) + 1$

B $2x^2 + x\sqrt{2} - 4$

C $\sqrt{2}x^2 - 2x + 3$

D $2x^2 + 3x\sqrt{2} + 5$

Solution: B

To find $g(h(x))$, we have to plug $h(x)$ into $g(x)$. Given

$$g(x) = x^2 - x - 4$$

$$h(x) = x\sqrt{2} + 1$$

we get

$$g(h(x)) = \left(x\sqrt{2} + 1\right)^2 - \left(x\sqrt{2} + 1\right) - 4$$

$$g(h(x)) = 2x^2 + 2x\sqrt{2} + 1 - x\sqrt{2} - 1 - 4$$

$$g(h(x)) = 2x^2 + x\sqrt{2} - 4$$

Topic: Composite functions

Question: Find $f(g(x)) - g(f(x))$.

$$f(x) = x^2 - 2x$$

$$g(x) = 3x + 1$$

Answer choices:

A $6x^2 - 6x + 2$

B $6x^2 + 6x + 2$

C $6x^2 - 6x - 2$

D $6x^2 + 6x - 2$

Solution: D

To find $f(g(x))$, we have to plug $g(x)$ into $f(x)$, and to find $g(f(x))$, we have to plug $f(x)$ into $g(x)$. Given

$$f(x) = x^2 - 2x$$

$$g(x) = 3x + 1$$

we get

$$f(g(x)) = (3x + 1)^2 - 2(3x + 1)$$

$$f(g(x)) = 9x^2 + 6x + 1 - 6x - 2$$

$$f(g(x)) = 9x^2 - 1$$

and

$$g(f(x)) = 3(x^2 - 2x) + 1$$

$$g(f(x)) = 3x^2 - 6x + 1$$

Therefore, the function $f(g(x)) - g(f(x))$, which is the difference of the composite functions $f(g(x))$ and $g(f(x))$, is

$$f(g(x)) - g(f(x)) = (9x^2 - 1) - (3x^2 - 6x + 1)$$

$$f(g(x)) - g(f(x)) = 9x^2 - 1 - 3x^2 + 6x - 1$$

$$f(g(x)) - g(f(x)) = 6x^2 + 6x - 2$$

Topic: Composite functions, domain

Question: What is the domain of $f \circ g$?

$$f(x) = x^2 - 5$$

$$g(x) = \sqrt{x+4}$$

Answer choices:

A $\quad x \leq -4$

B $\quad x > -4$

C $\quad x \geq -4$

D $\quad x < -4$

Solution: C

First, find the domain of $g(x)$. The expression $\sqrt{x+4}$ is undefined if the radicand is negative. For example, if $x = -5$, then $x+4$ is -1. In general, if x is any number less than -4, then $x+4$ is negative. However, -4 itself is okay, because $\sqrt{-4+4=0}$.

Therefore, the domain of $g(x)$ is all reals x such that $x \geq -4$.

The algebraic expression for the composite function is

$$f(g(x)) = \left(\sqrt{x+4}\right)^2 - 5$$

$$f(g(x)) = (x+4) - 5$$

$$f(g(x)) = x - 1$$

For this simple binomial ($x-1$), no real numbers are excluded, so its domain is all reals. But because the domain of $g(x)$ excludes all $x < -4$, those values of x also have to be excluded from the domain of the composite function $f(g(x))$.

That means the domain of $f(g(x))$ is $x \geq -4$.

Topic: Composite functions, domain

Question: What is the domain of $f \circ g$?

$$f(x) = \frac{1}{x+3}$$

$$g(x) = \frac{x}{x-2}$$

Answer choices:

A $\quad x \neq 2, \ -3$

B $\quad x \neq \frac{3}{2}, 2$

C $\quad x \neq -2, 3$

D $\quad x \neq -\frac{3}{2}, 2$

Solution: B

First, find the domain of $g(x)$. The expression $x/(x-2)$ is undefined if the denominator is 0. That means $x=2$ isn't in the domain of $g(x)$. Therefore, the domain of $g(x)$ is all reals x such that $x \neq 2$.

The algebraic expression for the composite function is

$$f(g(x)) = \frac{1}{\left(\frac{x}{x-2}\right)+3}$$

$$f(g(x)) = \frac{1}{\left(\frac{x}{x-2}\right)+3\left(\frac{x-2}{x-2}\right)}$$

$$f(g(x)) = \frac{1}{\left(\frac{x+3x-6}{x-2}\right)}$$

$$f(g(x)) = \frac{1}{\left(\frac{4x-6}{x-2}\right)}$$

$$f(g(x)) = \frac{x-2}{4x-6}$$

$$f(g(x)) = \frac{x-2}{2(2x-3)}$$

For this rational function ($(x-2)/[2(2x-3)]$), any numbers that make the denominator 0 are excluded from the domain.

$$2(2x-3)=0 \quad \to \quad 2x-3=0 \quad \to \quad 2x=3 \quad \to \quad x=\frac{3}{2}$$

Putting both exclusions together, the domain of the composite is all real numbers except $3/2$ and 2, so

$$f(g(x)) = \frac{x-2}{2(2x-3)}, x \neq \frac{3}{2}, 2$$

Topic: Composite functions, domain

Question: What is the domain of $f \circ g$?

$$f(x) = \sqrt{x - 1}$$

$$g(x) = \frac{1}{x - 1}$$

Answer choices:

A $\quad 1 < x \le 2$

B $\quad 1 \le x \le 2$

C $\quad 1 < x < 2$

D $\quad 1 \le x < 2$

Solution: A

First, find the domain of $g(x)$. The expression $1/(x-1)$ is undefined if the denominator is 0. That means $x=1$ isn't in the domain of $g(x)$. Therefore, the domain of $g(x)$ is all reals x such that $x \neq 1$.

The algebraic expression for the composite function is

$$f(g(x)) = \sqrt{\frac{1}{x-1} - 1}$$

$$f(g(x)) = \sqrt{\frac{1-(x-1)}{x-1}}$$

$$f(g(x)) = \sqrt{\frac{2-x}{x-1}}$$

For the rational function under the radical sign ($(2-x)/(x-1)$), any numbers that make the denominator 0 are excluded from the domain.

$$x - 1 = 0 \quad \to \quad x = 1$$

And any time that rational function is negative, the values of x that make it negative will be excluded from the domain. A rational function is negative when either the numerator is negative and the denominator is positive, or vice versa. The numerator is negative when $2-x<0$.

$$2 - x < 0 \quad \to \quad -x < -2 \quad \to \quad x > 2$$

The denominator is positive when $x-1>0$.

$$x - 1 > 0 \quad \to \quad x > 1$$

The values of x where the numerator is negative and the denominator is positive are the values of x such that $x > 2$ and $x > 1$. Notice that $x > 2$ and $x > 1$ if and only if $x > 2$.

The denominator is negative when $x - 1 < 0$.

$$x - 1 < 0 \quad \to \quad x < 1$$

The numerator is positive when $2 - x > 0$.

$$2 - x > 0 \quad \to \quad -x > -2 \quad \to \quad x < 2$$

The values of x where the denominator is negative and the numerator is positive are the values of x such that $x < 1$ and $x < 2$. Notice that $x < 1$ and $x < 2$ if and only if $x < 1$.

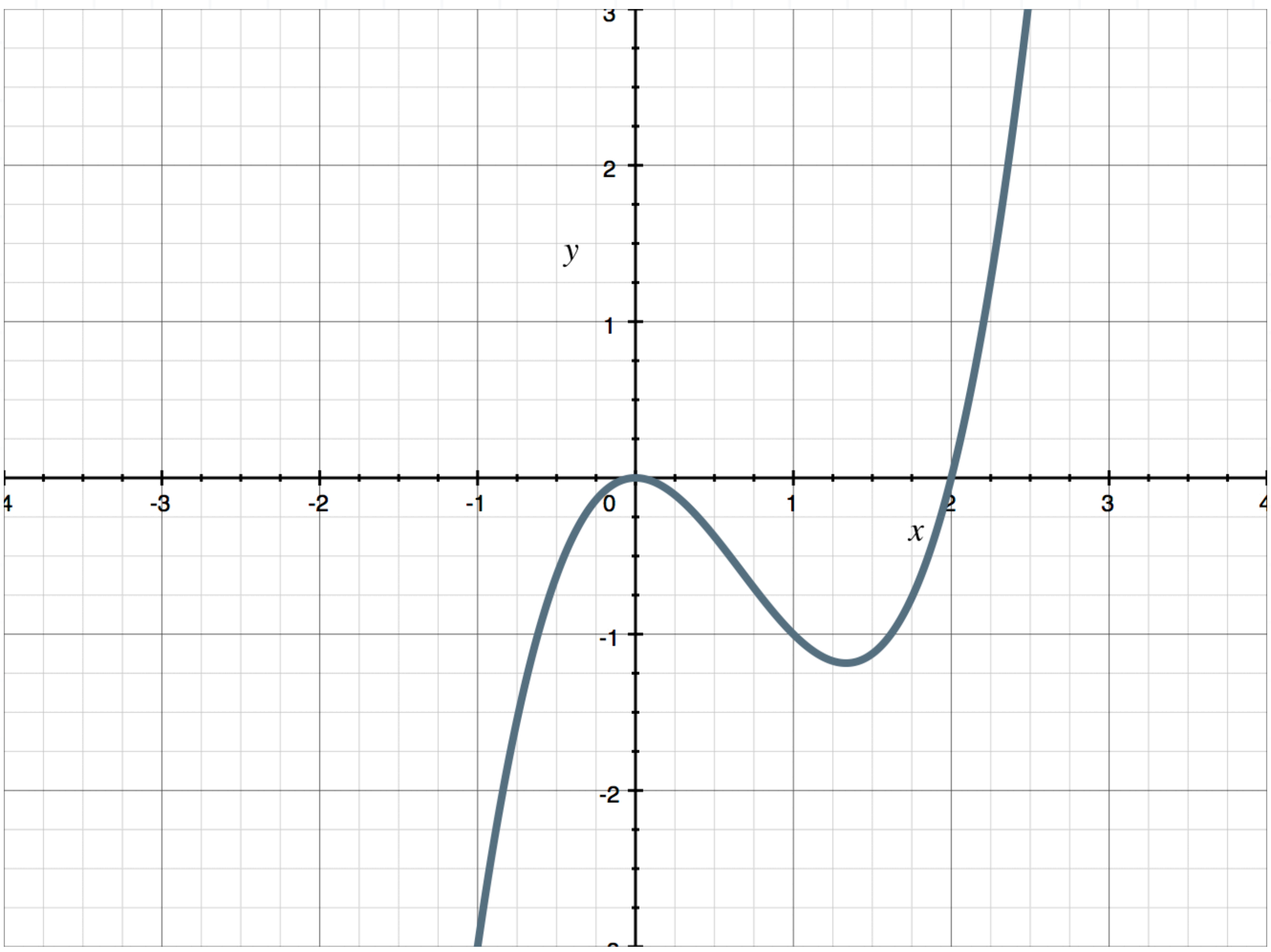
Therefore, the radicand is negative on the intervals $x > 2$ and $x < 1$, so the real numbers x in those intervals are excluded from the domain of this composite function.

We found earlier that $x = 1$ is excluded from the domain of this composite function, so its domain is the set of all real numbers x such that

$$1 < x \leq 2$$

Topic: One-to-one functions and the horizontal line test

Question: This is a graph of $y = x^3 - 2x^2$. Which of these describes the relation it represents?



Answer choices:

A Not a function

B A function, but not one-to-one

C A one-to-one function

D An exponential function

Solution: B

A look at the graph tells us that there’s no vertical line that intersects the graph at more than one point, so the relation is a function.

On the other hand, it's easy to find a horizontal line that intersects the graph at more than one point (the x-axis is one example), which indicates that the function is not one-to-one.

The given graph doesn’t represent a quadratic function, because the graph of a quadratic function is a parabola (not the type of curve given here), and the function represented by the graph ($x^3 - 2x^2$) is a cubic polynomial (not a quadratic polynomial). Therefore the only answer choice that works is B.

Topic: One-to-one functions and the horizontal line test

Question: Which function is not one-to-one?

Hint: Picture the graphs of these functions if you can, or use the method $f(a) = f(b)$ implies $a = b$ to determine which function isn't one-to-one.

Answer choices:

A $j(x) = \sqrt{x+2}$

B $g(x) = 7x - 2$

C $h(x) = (x-2)(x+3)$

D $f(x) = \dfrac{x+7}{x+5}$

Solution: C

Answer choice A: $j(x)=\sqrt{x+2}$

Use the method $j(a)=j(b)$ implies $a=b$.

$$\sqrt{a+2}=\sqrt{b+2}$$

$$a+2=b+2$$

$$a=b$$

That worked, so $j(x)$ is a one-to-one function.

Answer choice B: $g(x)=7x-2$

This is a linear function whose graph is a line that has a slope of 7, so so any horizontal or vertical line will intersect it at only one point. Therefore, $g(x)$ is a one-to-one function.

Answer choice C: $h(x)=(x-2)(x+3)$

This can be written as $h(x)=x^2+x-6$, a quadratic function. The graph will be a parabola that opens upward. There are horizontal lines that will intersect the parabola at two points, so $h(x)$ is not one-to-one.

Answer choice D: $f(x)=(x+7)/(x+5)$

Use the method $f(a)=f(b)$ implies $a=b$.

$$\frac{a+7}{a+5}=\frac{b+7}{b+5}$$

$$(a+7)(b+5)=(b+7)(a+5)$$

$$ab+5a+7b+35=ab+5b+7a+35$$

$$5a+7b=5b+7a$$

$$5a-7a=5b-7b$$

$$-2a=-2b$$

$$a=b$$

That worked, so $f(x)$ is a one-to-one function.

Topic: One-to-one functions and the horizontal line test

Question: Which function below is one-to-one?

Answer choices:

A $f(x) = 5x(x - 3)$

B $g(x) = \dfrac{2 - x^3}{4}$

C $h(x) = x^4$

D $j(x) = x^2 - 4$

Solution: B

Choices A and D are both quadratic functions, and their graphs are parabolas that open upward. For each of those functions, there are horizontal lines that will intersect its graph at two points, so neither function is one-to-one.

Answer choice B: $g(x) = \left(2 - x^3\right)/4$

Use the method $f(a) = f(b)$ implies $a = b$.

$$\frac{2 - a^3}{4} = \frac{2 - b^3}{4}$$

$$2 - a^3 = 2 - b^3$$

$$-a^3 = -b^3$$

$$a^3 = b^3$$

If the cubes of two real numbers are equal, then the numbers themselves are equal, so

$$a = b$$

That worked, so $g(x)$ is one-to-one.

Answer choice C: $h(x) = x^4$

Use the method $h(a) = h(b)$ implies $a = b$.

$$a^4 = b^4$$

$$(a^2)^2 = (b^2)^2$$

Since a and b are both real numbers, we know that a^2 and b^2 are both nonnegative real numbers. Also, if the squares of two nonnegative real numbers are equal, then the numbers themselves are equal, so

$$a^2 = b^2$$

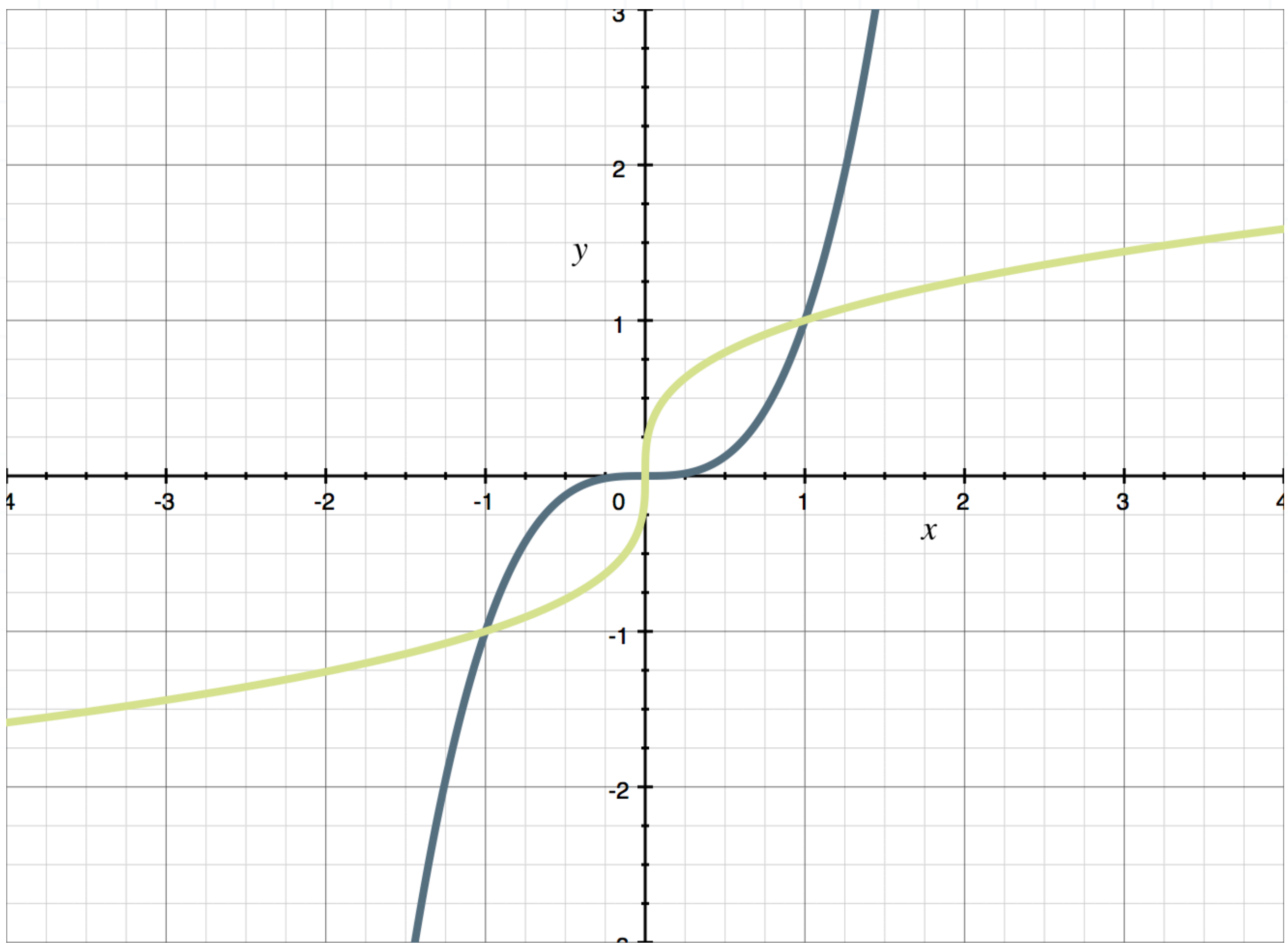
From this it follows that if $a = 0$ then $b = 0$, and vice versa, so $a = b$. If a and b are nonzero real numbers, then

$$a^2 = b^2 \text{ if and only if } |a| = |b|$$

This solution allows a to be negative while b is positive, or vice versa, so it doesn't follow that $a = b$. Therefore, the function $h(x)$ isn't one-to-one.

Topic: Inverse functions

Question: The blue curve is the graph of a function, and the green curve is the graph of its inverse. Which pair of functions do the graphs represent?



Answer choices:

A $\quad f(x)=x^3 \quad g(x)=-\sqrt[3]{x}$

B $\quad f(x)=x^3 \quad g(x)=\sqrt[3]{x}$

C $\quad f(x)=-x^3 \quad g(x)=-\sqrt[3]{x}$

D $\quad f(x)=-x^3 \quad g(x)=\sqrt[3]{x}$

Solution: B

The point $(1,1)$ is common to the two graphs, so let's input 1 into both functions in each pair and see if it returns 1 for both of them.

A	$f(1) = 1^3 = 1$	$g(1) = -\sqrt[3]{1} = -1$
B	$f(1) = 1^3 = 1$	$g(1) = \sqrt[3]{1} = 1$
C	$f(1) = -(1^3) = -1$	$g(1) = -\sqrt[3]{1} = -1$
D	$f(1) = -(1^3) = -1$	$g(1) = \sqrt[3]{1} = 1$

Look at answer choice B. Evaluating $f(1)$ returns a value of 1. Likewise, evaluating $g(1)$ also returns a value of 1. This tells us that $(1,1)$ is a point of the graphs of the functions $f(x)$ and $g(x)$ that are defined in answer choice B.

Topic: Inverse functions

Question: Which of these functions is the inverse of the given function?

$$f(x) = \frac{1}{x} - 2$$

Answer choices:

A $\quad f^{-1}(x) = 2 - \frac{1}{x}$

B $\quad f^{-1}(x) = \frac{x+1}{2}$

C $\quad f^{-1}(x) = \frac{1}{x+2}$

D $\quad f^{-1}(x) = \frac{x}{2} + 1$

Solution: C

To find the inverse of

$$f(x) = \frac{1}{x} - 2$$

first replace $f(x)$ with y.

$$y = \frac{1}{x} - 2$$

Next, solve for x.

$$y + 2 = \frac{1}{x}$$

$$x(y + 2) = 1$$

$$x = \frac{1}{y + 2}$$

Now switch x with y.

$$y = \frac{1}{x + 2}$$

Finally, write the inverse function by replacing y with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{1}{x + 2}$$

Topic: Inverse functions

Question: Which of these is the inverse of the given function?

$$f(x) = 3x^3 - 4$$

Answer choices:

A $\quad f^{-1}(x) = \sqrt[3]{\frac{3x}{4}}$

B $\quad f^{-1}(x) = \frac{\sqrt[3]{x}}{3} + 4$

C $\quad f^{-1}(x) = \sqrt[3]{3x + 4}$

D $\quad f^{-1}(x) = \sqrt[3]{\frac{x + 4}{3}}$

Solution: D

To find the inverse of $f(x) = 3x^3 - 4$, first replace $f(x)$ with y.

$$y = 3x^3 - 4$$

$$y + 4 = 3x^3$$

$$\frac{y+4}{3} = x^3$$

$$\sqrt[3]{\frac{y+4}{3}} = x$$

Now switch x with y.

$$\sqrt[3]{\frac{x+4}{3}} = y$$

Finally, write the inverse function by replacing y with $f^{-1}(x)$ (and then turning the equation around so that $f^{-1}(x)$ is on the left side).

$$f^{-1}(x) = \sqrt[3]{\frac{x+4}{3}}$$

Topic: Finding a function from its inverse

Question: Use the given information to find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(4) = 1$$

$$f^{-1}(-2) = -2$$

Answer choices:

A $\quad f(x) = 2x + 6$

B $\quad f(x) = 2x - 2$

C $\quad f(x) = 2x - 6$

D $\quad f(x) = 2x + 2$

Solution: D

Use the points $(4,1)$ and $(-2,-2)$ to find the slope of the line that represents $f^{-1}(x)$.

$$m = \frac{-2-1}{-2-4} = \frac{-3}{-6} = \frac{1}{2}$$

Use one of the two points and $y - y_1 = m(x - x_1)$ to find the equation of that line.

We'll use the point $(4,1)$.

$$y - 1 = \frac{1}{2}(x - 4)$$

$$y - 1 = \frac{1}{2}x - 2$$

$$y = \frac{1}{2}x - 1$$

Switch x with y, and then solve for y to get the equation of the line that represents $f(x)$.

$$x = \frac{1}{2}y - 1$$

$$x + 1 = \frac{1}{2}y$$

$$2(x + 1) = y$$

$$2x + 2 = y$$

Now replace y with $f(x)$.

$$f(x) = 2x + 2$$

Topic: Finding a function from its inverse

Question: Use the given information to find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(2) = 5$$

$$f^{-1}(1) = -1$$

Answer choices:

A $\quad f(x) = \frac{1}{6}x + \frac{7}{6}$

B $\quad f(x) = \frac{1}{4}x + \frac{3}{4}$

C $\quad f(x) = \frac{1}{6}x - \frac{7}{6}$

D $\quad f(x) = \frac{1}{4}x - \frac{3}{4}$

Solution: A

Use the points $(2,5)$ and $(1, -1)$ to find the slope of the line that represents $f^{-1}(x)$.

$$m = \frac{5 - (-1)}{2 - 1} = 6$$

Use one of the two points and $y - y_1 = m(x - x_1)$ to find the equation of that line.

We'll use the point $(2,5)$.

$$y - 5 = 6(x - 2)$$

$$y - 5 = 6x - 12$$

$$y = 6x - 7$$

Switch x with y, and then solve for y to get the equation of the line that represents $f(x)$.

$$x = 6y - 7$$

$$x + 7 = 6y$$

$$\frac{x + 7}{6} = y$$

$$\frac{1}{6}x + \frac{7}{6} = y$$

Now replace y with $f(x)$.

$$f(x) = \frac{1}{6}x + \frac{7}{6}$$

Topic: Finding a function from its inverse

Question: Use the given information to find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(3) = 4$$

$$f^{-1}(-6) = -2$$

Answer choices:

A $\quad f(x) = \dfrac{2}{3}x - 3$

B $\quad f(x) = \dfrac{3}{2}x - 3$

C $\quad f(x) = \dfrac{2}{3}x + 3$

D $\quad f(x) = \dfrac{3}{2}x + 3$

Solution: B

Use the points $(3,4)$ and $(-6,-2)$ to find the slope of the line that represents $f^{-1}(x)$.

$$m = \frac{4-(-2)}{3-(-6)} = \frac{6}{9} = \frac{2}{3}$$

Use one of the two points and $y - y_1 = m(x - x_1)$ to find the equation of that line.

We'll use the point $(3,4)$.

$$y - 4 = \frac{2}{3}(x-3)$$

$$y - 4 = \frac{2}{3}x - 2$$

$$y = \frac{2}{3}x + 2$$

Switch x with y, and then solve for y to get the equation of the line that represents $f(x)$.

$$x = \frac{2}{3}y + 2$$

$$x - 2 = \frac{2}{3}y$$

$$\frac{3}{2}(x-2) = \frac{3}{2}\left(\frac{2}{3}y\right)$$

$$\frac{3}{2}x - 3 = y$$

Now replace y with $f(x)$.

$$f(x) = \frac{3}{2}x - 3$$

Topic: Laws of logarithms

Question: Write the expression as a rational number if possible, or if not, as a single logarithm.

$$\log_3 54 - \log_3 2$$

Answer choices:

A $-\log_3 9$

B 9

C $\log_3 27$

D 3

Solution: D

First, use the rule

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

to rewrite the given expression.

$$\log_3 54 - \log_3 2$$

$$\log_3 \frac{54}{2}$$

$$\log_3 27$$

To simplify further, use this rule:

If $\log_a y = x$, then $a^x = y$.

If we let $x = \log_3 27$, then $3^x = 27$. Therefore, $x = 3$, because $27 = 3 \cdot 3 \cdot 3 = 3^3$.

Topic: Laws of logarithms

Question: Which expression is equal to 1?

Answer choices:

A $\log_5 20 - \log_5 10$

B $\log_3 18 - \log_3 6$

C $\log_2 8 - \log_2 7$

D $\log_8 128 - \log_8 2$

Solution: B

Use these two rules to evaluate each expression.

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

If $\log_a y = x$, then $a^x = y$.

Applying the first rule to the answer choices gives

A $\qquad \log_5 20 - \log_5 10 = \log_5 \frac{20}{10} = \log_5 2$

B $\qquad \log_3 18 - \log_3 6 = \log_3 \frac{18}{6} = \log_3 3 = 1$

C $\qquad \log_2 8 - \log_2 7 = \log_2 \frac{8}{7}$

D $\qquad \log_8 128 - \log_8 2 = \log_8 \frac{128}{2} = \log_8 64 = 2$

Now we'll show that the values of the expressions we found for answer choices A and C are not equal to 1.

For answer choice A, let $x = \log_5 2$. By the second rule given above, $5^x = 2$. We know that $5^1 = 5$, so $5^x \neq 5^1$. Since $5^x \neq 5^1$, this tells us that $x \neq 1$. So $\log_5 2 \neq 1$.

For answer choice C, we can use similar reasoning. Let $x = \log_2(8/7)$. By the second rule given above, $2^x = 8/7$. We know that $2^1 = 2$, so $2^x \neq 2^1$. Therefore, $x \neq 1$, and $\log_2(8/7) \neq 1$.

Topic: Laws of logarithms

Question: Write the expression as a rational number if possible, or if not, as a single logarithm.

$$\log_2 \frac{1}{4} + \log_2 16$$

Answer choices:

A $\quad 2$

B $\quad 4$

C $\quad \log_2 8$

D $\quad \log_2 64$

Solution: A

Use these two rules to evaluate the expression.

$$\log_a x + \log_a y = \log_a xy$$

If $\log_a y = x$, then $a^x = y$.

Applying the first rule to the given expression gives

$$\log_2 \frac{1}{4} + \log_2 16$$

$$\log_2 \left(\frac{1}{4} \cdot 16 \right)$$

$$\log_2 4$$

It's probably obvious from this that $\log_2 4 = 2$, but if not, use the second rule above. If we let $x = \log_2 4$, then

$$2^x = 4$$

$$x = 2$$

Topic: Quadratic formula

Question: Use the quadratic formula to solve for the variable.

$$3x^2 + 2x - 1 = 0$$

Answer choices:

A $\quad x = \frac{1}{3},\ 1$

B $\quad x = -1,\ \frac{1}{3}$

C $\quad x = -\frac{1}{3},\ 1$

D $\quad x = 2,\ 3$

Solution: B

We can factor this quadratic equation directly.

$$3x^2 + 2x - 1 = 0$$

$$(3x - 1)(x + 1) = 0$$

So the solutions are found by setting the individual factors ($3x - 1$ and $x + 1$) to 0 and solving each of the resulting equations for x.

$$3x - 1 = 0$$

$$3x = 1$$

$$x = \frac{1}{3}$$

and

$$x + 1 = 0$$

$$x = -1$$

But since we've been asked to use the quadratic formula to find the solutions, we'll solve it that way. If $ax^2 + bx + c = 0$, then the quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If we compare the standard form of a quadratic equation to the equation we've been given in this problem, we see that

$$a = 3$$

$$b = 2$$

$$c = -1$$

Plugging these numbers into the quadratic formula, we get

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-1)}}{2(3)}$$

$$x = \frac{-2 \pm \sqrt{4 + 12}}{6}$$

$$x = \frac{-2 \pm \sqrt{16}}{6}$$

$$x = \frac{-2 \pm 4}{6}$$

So the solutions are

$$x = \frac{-2 + 4}{6} = \frac{2}{6} = \frac{1}{3}$$

and

$$x = \frac{-2 - 4}{6} = \frac{-6}{6} = -1$$

Topic: Quadratic formula

Question: Find the solution(s) to the polynomial equation.

$$2x^2 - 7x - 3 = 0$$

Answer choices:

A $\quad x = \dfrac{7 + \sqrt{73}}{4}$ and $x = \dfrac{7 - \sqrt{73}}{4}$

B $\quad x = \dfrac{7 + \sqrt{73}}{2}$ and $x = \dfrac{-7 + \sqrt{73}}{2}$

C $\quad x = \dfrac{-7 + \sqrt{73}}{4}$ and $x = \dfrac{-7 - \sqrt{73}}{4}$

D $\quad x = \dfrac{7 + \sqrt{73}}{2}$ and $x = \dfrac{7 - \sqrt{73}}{2}$

Solution: A

Since the polynomial can't be factored, we have to use the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to find the solutions, or roots, of the given polynomial equation. Remember that, in order to use the quadratic formula, we need our polynomial equation to be in the form

$$ax^2 + bx + c = 0$$

Our function is already in this form, so we'll first identify a, b, and c, and then plug them into the quadratic formula.

Matching up $2x^2 - 7x - 3 = 0$ with $ax^2 + bx + c = 0$, we see that

$$a = 2$$

$$b = -7$$

$$c = -3$$

Plugging these numbers into the quadratic formula, we get

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{7 \pm \sqrt{49 + 24}}{4}$$

$$x = \frac{7 \pm \sqrt{73}}{4}$$

Therefore, the roots of our equation are

$$x = \frac{7 + \sqrt{73}}{4}$$

and

$$x = \frac{7 - \sqrt{73}}{4}$$

Topic: Quadratic formula

Question: Find the roots of the equation using the quadratic formula.

$$3x^2 + 10x + 5 = 0$$

Answer choices:

A $\quad x = \dfrac{5 \pm \sqrt{10}}{3}$

B $\quad x = \dfrac{-5 \pm \sqrt{10}}{3}$

C $\quad x = \dfrac{-5 \pm \sqrt{10}}{6}$

D $\quad x = \dfrac{5 \pm \sqrt{10}}{6}$

Solution: B

Since the polynomial can't be factored, we have to use the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to find the solutions, or roots, of the function. Remember that, in order to use the quadratic formula, we need our polynomial function in the form

$$ax^2 + bx + c = 0$$

Our function is already in this form, so we'll match it up to the form above to identify a, b and c before we plug them into the quadratic formula.

Matching up $3x^2 + 10x + 5 = 0$ with $ax^2 + bx + c = 0$, we see that

$$a = 3$$

$$b = 10$$

$$c = 5$$

Plugging these values into the quadratic formula, we get

$$x = \frac{-10 \pm \sqrt{10^2 - 4(3)(5)}}{2(3)}$$

$$x = \frac{-10 \pm \sqrt{100 - 60}}{6}$$

$$x = \frac{-10 \pm \sqrt{40}}{6}$$

$$x = \frac{-10 \pm 2\sqrt{10}}{6}$$

$$x = \frac{-5 \pm \sqrt{10}}{3}$$

Topic: Completing the square

Question: Complete the square to solve for the variable.

$$x^2 + 4x + 2 = 0$$

Answer choices:

A $\quad x = -2 \pm \sqrt{2}$

B $\quad x = 2 \pm \sqrt{2}$

C $\quad x = -2 \pm \sqrt{3}$

D $\quad x = 2 \pm \sqrt{3}$

Solution: A

We have a quadratic polynomial of the form $x^2 + bx + c$ (with $b = 4$ and $c = 2$) on the left side of the given equation.

First, we'll subtract c (which is 2) from both sides of the equation.

$$x^2 + 4x + 2 - 2 = 0 - 2$$

$$x^2 + 4x = -2$$

Next, we'll find $(b/2)^2$. Here, $b = 4$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 2^2 = 4$$

This is the number we have to add to both sides of the equation $x^2 + 4x = -2$ in order to complete the square.

$$x^2 + 4x + 4 = -2 + 4$$

Now we can factor the left-hand side as the square of a binomial.

$$(x + 2)(x + 2) = 2$$

$$(x + 2)^2 = 2$$

$$\sqrt{(x + 2)^2} = \sqrt{2}$$

$$x + 2 = \pm\sqrt{2}$$

To solve this equation for x, we subtract 2 from both sides.

$$x + 2 - 2 = -2 \pm \sqrt{2}$$

$$x = -2 \pm \sqrt{2}$$

We can't reduce $\sqrt{2}$ at all, so the roots of our equation are

$$x = -2 + \sqrt{2}$$

and

$$x = -2 - \sqrt{2}$$

Topic: Completing the square

Question: Complete the square to solve for the variable.

$$u^2 - 4u + 3 = 0$$

Answer choices:

A $u = -1, \ -3$

B $u = 1, \ -3$

C $u = 1, \ 3$

D $u = -1, \ 3$

Solution: C

Find $(b/2)^2$, where b is the coefficient of the u term. Here, $b=-4$.

$$\left(\frac{b}{2}\right)^2=\left(\frac{-4}{2}\right)^2=(-2)^2=4$$

This is the number we have to add to both sides of the equation in order to complete the square.

$$u^2-4u+4+3=0+4$$

$$u^2-4u+4+3=4$$

$$u^2-4u+4=1$$

Factor the left-hand side as the square of a binomial.

$$(u-2)^2=1$$

$$u-2=\pm\sqrt{1}$$

$$u=2\pm 1$$

$$u=1,\ 3$$

Topic: Completing the square

Question: Complete the square to solve for the variable.

$$x^2 - 5x - 4 = 0$$

Answer choices:

A $\quad x = \dfrac{5 \pm \sqrt{73}}{2}$

B $\quad x = \dfrac{-5 \pm \sqrt{73}}{2}$

C $\quad x = \dfrac{5 \pm \sqrt{41}}{2}$

D $\quad x = \dfrac{-5 \pm \sqrt{41}}{2}$

Solution: C

To complete the square, we'll add $(b/2)^2$ to both sides of the equation, where b is equal to -5, the coefficient on the first-degree term. We'll also move the other constant term to the right side of the equation.

$$x^2 - 5x + \left(-\frac{5}{2}\right)^2 = 4 + \left(-\frac{5}{2}\right)^2$$

$$x^2 - 5x + \frac{25}{4} = \frac{16}{4} + \frac{25}{4}$$

$$x^2 - 5x + \frac{25}{4} = \frac{41}{4}$$

We'll factor the left-hand side and solve for x.

$$\left(x - \frac{5}{2}\right)^2 = \frac{41}{4}$$

$$x - \frac{5}{2} = \pm\sqrt{\frac{41}{4}}$$

$$x = \frac{5}{2} \pm \frac{\sqrt{41}}{2}$$

$$x = \frac{5 \pm \sqrt{41}}{2}$$

Topic: Long division of polynomials

Question: Find the quotient.

$$\frac{x^2-26}{x-5}$$

Answer choices:

A $\quad x+5+\dfrac{1}{x-5}$

B $\quad x+5-\dfrac{1}{x+5}$

C $\quad x+5+\dfrac{1}{x+5}$

D $\quad x+5-\dfrac{1}{x-5}$

Solution: D

Using long division,

$$
\begin{array}{rl}
 & x+5-\dfrac{1}{x-5} \\
x-5 & \overline{)\,x^2+0x-26} \\
 & -(x^2-5x) \\ \hline
 & \qquad 5x-26 \\
 & \qquad -(5x-25) \\ \hline
 & \qquad\qquad -1
\end{array}
$$

the quotient is

$$x+5-\frac{1}{x-5}$$

Topic: Long division of polynomials

Question: Find the quotient.

$$\frac{12x^3 - 11x^2 + 9x + 18}{4x + 3}$$

Answer choices:

A $3x^2 - 5x + 6$

B $3x^2 + 5x + 6$

C $3x^2 + 5x - 6$

D $3x^2 - 5x - 6$

Solution: A

Using long division,

$$
\begin{array}{r|l}
 & 3x^2-5x+6 \\
\hline
4x+3 & 12x^3-11x^2+9x+18 \\
 & -(12x^3+9x^2) \\
\hline
 & \quad -20x^2+9x \\
 & \quad -(-20x^2-15x) \\
\hline
 & \qquad 24x+18 \\
 & \qquad -(24x+18) \\
\hline
 & \qquad\qquad 0
\end{array}
$$

the quotient is

$$3x^2-5x+6$$

Topic: Long division of polynomials

Question: Find the quotient.

$$\frac{x^3+7x^2+14x+3}{x+2}$$

Answer choices:

A $\quad x^2+5x-4-\dfrac{5}{x+2}$

B $\quad x^2+5x-4+\dfrac{11}{x+2}$

C $\quad x^2+5x+4-\dfrac{5}{x+2}$

D $\quad x^2+9x+8-\dfrac{5}{x+2}$

Solution: C

Using long division,

$$
\begin{array}{r|l}
 & x^2+5x+4-\frac{5}{x+2} \\
\hline
x+2 & x^3+7x^2+14x+3 \\
 & -(x^3+2x^2) \\
\cline{2-2}
 & \quad 5x^2+14x \\
 & \quad -(5x^2+10x) \\
\cline{2-2}
 & \qquad 4x+3 \\
 & \qquad -(4x+8) \\
\cline{2-2}
 & \qquad\quad -5
\end{array}
$$

the quotient is

$$x^2+5x+4-\frac{5}{x+2}$$

Topic: The unit circle

Question: What is the coordinate point associated with $\theta = 30^\circ$ along the unit circle?

Answer choices:

A $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

B $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

C $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

D $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Solution: A

Looking at the unit circle shows that the coordinate point associated with $\theta = 30^\circ$ in the first quadrant is

$$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$$

Topic: The unit circle

Question: At which angles are sine and cosine equivalent?

Answer choices:

A $\frac{\pi}{4}$

B $\frac{\pi}{4}, \frac{3\pi}{4}$

C $\frac{\pi}{4}, \frac{5\pi}{4}$

D $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Solution: C

The sign of cosine and sine is the same in the first and third quadrants, so we need to find the angles in these quadrants. From the unit circle, we see that sine and cosine are the same at $\theta = \pi/4$ and $\theta = 5\pi/4$.

Topic: The unit circle

Question: Use the unit circle to find the value of tangent of the angle $\theta = 5\pi/3$.

Answer choices:

A -1

B $\frac{\sqrt{3}}{3}$

C $\sqrt{3}$

D $-\sqrt{3}$

Solution: D

Looking at the unit circle, we know that sine of $\theta = 5\pi/3$ is the y-value of the coordinate point at that angle, and that cosine of $\theta = 5\pi/3$ is the x-value of the coordinate point at that angle.

$$\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$$

Use the quotient identity for tangent to find the value of tangent.

$$\tan\left(\frac{5\pi}{3}\right) = \frac{\sin\left(\frac{5\pi}{3}\right)}{\cos\left(\frac{5\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

Topic: Idea of the limit

Question: What statement is being made by the limit equation?

$$\lim_{x \to 3} x^2 - 1 = 8$$

Answer choices:

A The limit as x approaches 8 of the function $f(x) = x^2 - 1$ is 3.

B The limit as x approaches 3 of the function $f(x) = x^2 - 1$ is not 8.

C The limit as x approaches 8 of the function $f(x) = x^2 - 1$ is not 3.

D The limit as x approaches 3 of the function $f(x) = x^2 - 1$ is 8.

Solution: D

Break down the limit

$$\lim_{x\to 3} x^2 - 1 = 8$$

into its component parts:

- x approaches 3
- the function is $f(x) = x^2 - 1$
- the value of the limit is 8

Putting these pieces together gives a full statement of the limit:

"The limit as x approaches 3 of the function $f(x) = x^2 - 1$ is equal to 8."

Topic: Idea of the limit

Question: Use limit notation to write the limit of the function $f(x)$ as x approaches 3.

$$f(x) = \frac{x-6}{x}$$

Answer choices:

A $\lim_{x \to -3} f(x) = \frac{x-6}{x}$

B $\lim_{x \to 3} f(x) = \frac{x-6}{x}$

C $\lim_{x \to 3} \frac{x-6}{x}$

D $\lim_{x \to -3} \frac{x-6}{x}$

Solution: C

When a is the value that x approaches, and $f(x)$ is the given function, the limit is written as

$$\lim_{x\to a} f(x)$$

In this case, x approaches 3, so $a = 3$, and the function is

$$f(x) = \frac{x-6}{x}$$

So we'd write the limit as

$$\lim_{x\to 3} \frac{x-6}{x}$$

Topic: Idea of the limit

Question: Evaluate the limit.

$$\lim_{x \to 3} \frac{x-6}{x}$$

Answer choices:

A -3

B 3

C -1

D 1

Solution: C

To evaluate the limit,

$$\lim_{x\to 3}\frac{x-6}{x}$$

plug the value that's being approached into the function, then simplify the answer.

$$\frac{3-6}{3}$$

$$\frac{-3}{3}$$

$$-1$$

Topic: One-sided limits

Question: Find the left-hand limit.

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2}$$

Answer choices:

A -1

B 1

C -2

D 2

Solution: A

If we try substitution to evaluate the limit, we get the undefined value $0/0$. Instead, let's try substituting a value to the left of $x = 2$ that's very close to $x = 2$, like $x = 1.9999$.

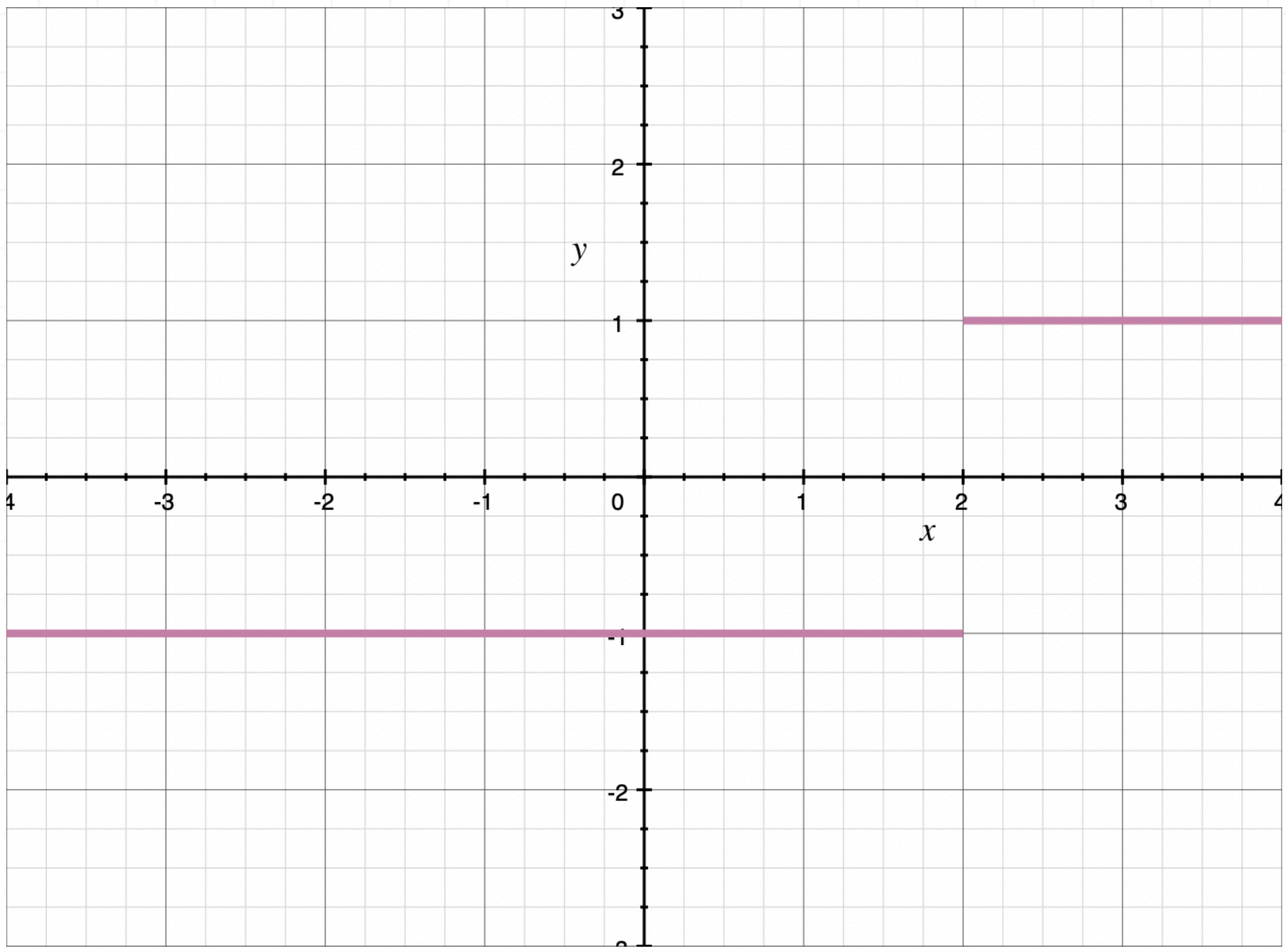
$$\frac{|1.9999 - 2|}{1.9999 - 2}$$

$$\frac{|-0.0001|}{-0.0001}$$

$$\frac{0.0001}{-0.0001}$$

$$-1$$

As we approach $x = 2$ from the left, the function is a constant -1 (the numerator is always positive and the denominator is always negative). The graph of the function confirms this value for the left-hand limit.



3
2
y
1
-4
-3
-2
-1
0
1
2
3
4
x
-1
-2

Topic: One-sided limits

Question: Find the right-hand limit.

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2}$$

Answer choices:

A -1

B 1

C -2

D 2

Solution: B

If we try substitution to evaluate the limit, we get the undefined value $0/0$. Instead, let's try substituting a value to the right of $x = 2$ that's very close to $x = 2$, like $x = 2.0001$.

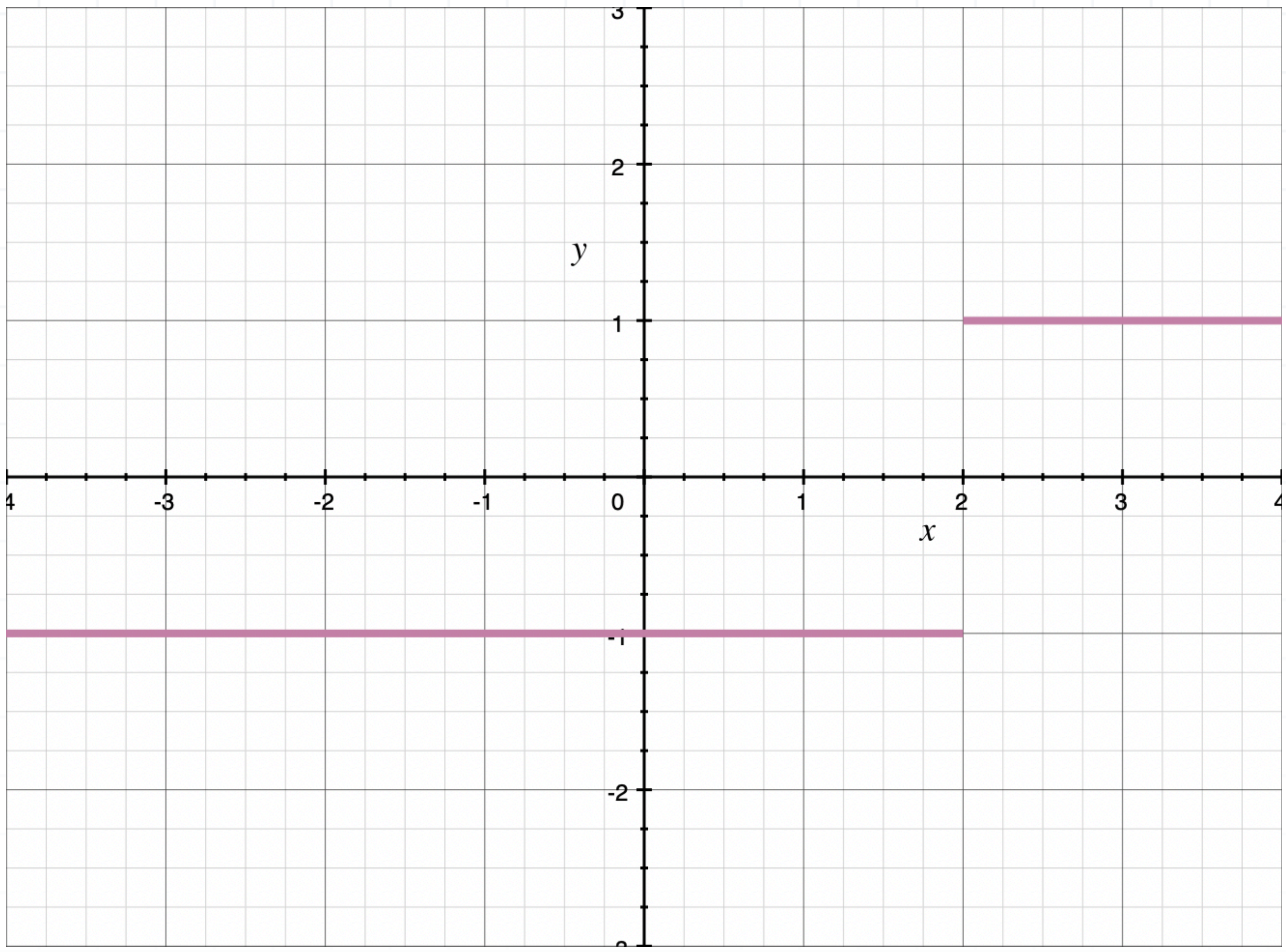
$$\frac{|2.0001 - 2|}{2.0001 - 2}$$

$$\frac{|0.0001|}{0.0001}$$

$$\frac{0.0001}{0.0001}$$

$$1$$

As we approach $x = 2$ from the right, the function is a constant 1 (the numerator is always positive and the denominator is always positive). The graph of the function confirms this value for the right-hand limit.



3
2
y
1
-4
-3
-2
-1
0
1
2
3
4
x
-1
-2
-3

Topic: One-sided limits

Question: Find the limit.

$$\lim_{x \to 2} \frac{|x-2|}{x-2}$$

Answer choices:

A -1

B 1

C -2

D Does not exist (DNE)

Solution: D

We can see the left-hand limit of the function at $x = 2$ if we try substituting $x = 1.9999$.

$$\frac{|1.9999 - 2|}{1.9999 - 2}$$

$$\frac{|-0.0001|}{-0.0001}$$

$$\frac{0.0001}{-0.0001}$$

$$-1$$

We can see the right-hand limit of the function at $x = 2$ if we try substituting $x = 2.0001$.

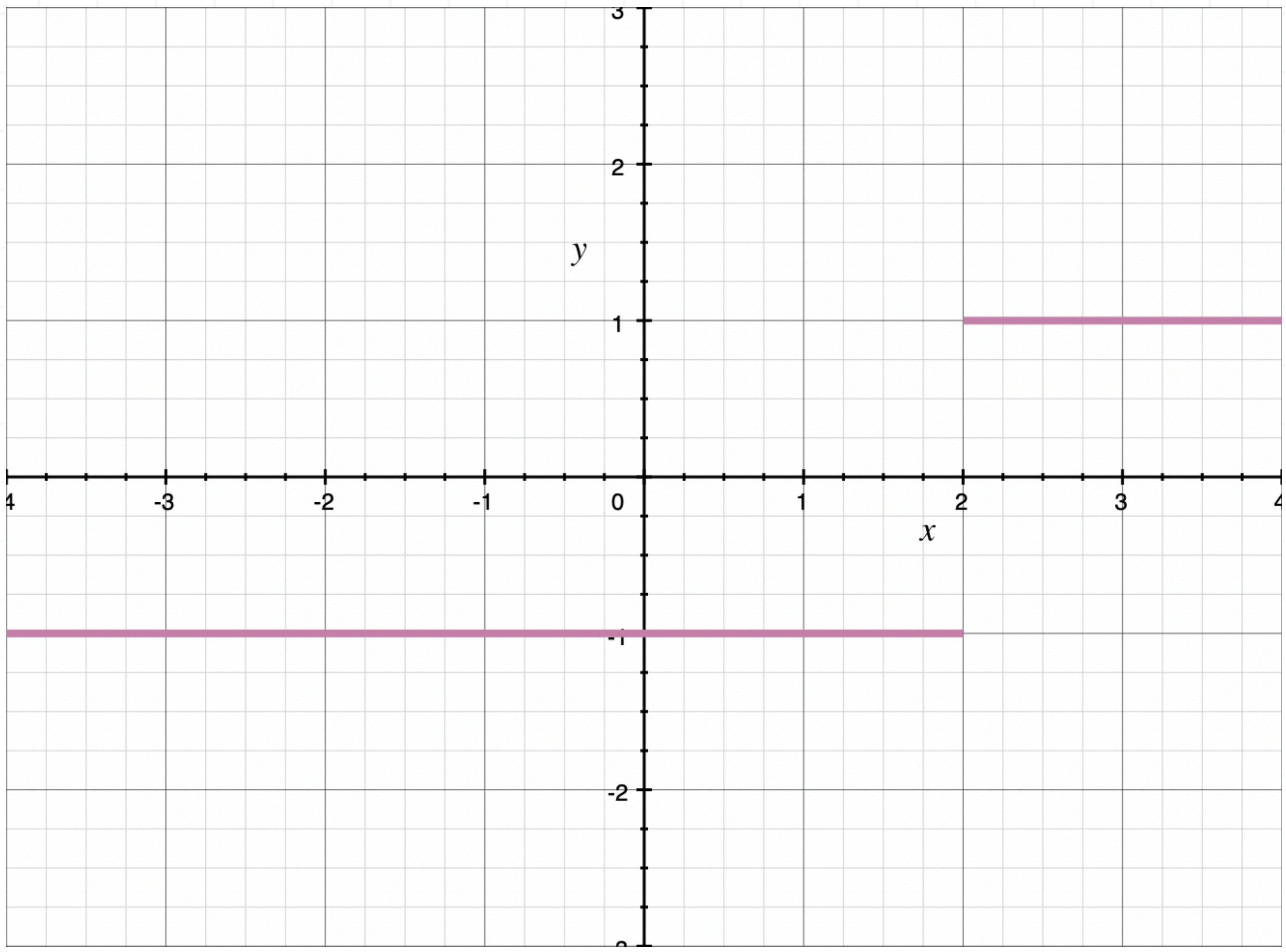
$$\frac{|2.0001 - 2|}{2.0001 - 2}$$

$$\frac{|0.0001|}{0.0001}$$

$$\frac{0.0001}{0.0001}$$

$$1$$

The graph of the function confirms these one-sided limits.



Because the one-sided limits aren't equivalent, the general limit of the function doesn't exist at $x = 2$.

Topic: Proving that the limit does not exist

Question: How do we know that the limit does not exist?

$$\lim_{x \to 0} \frac{1}{x}$$

Answer choices:

A The left-hand limit is 0.

B The right-hand limit does not equal the left-hand limit.

C The right-hand limit is equal to the left-hand limit.

D The right-hand limit exists.

Solution: B

We know that the general limit only exists if both the left- and right-hand limits exist. We can't find the limit using substitution, so we'll pick values close to $x = 0$, but on either side of it, to get an idea of what the one-sided limits are doing.

$$f(-0.0001) = \frac{1}{-0.0001} = -10{,}000$$

$$f(0.0001) = \frac{1}{0.0001} = 10{,}000$$

These values tell us that the left-hand limit is $-\infty$, and the right-hand limit is ∞. Because the one-sided limits aren't equal, the general limit does not exist.

Topic: Proving that the limit does not exist

Question: How do we know that the limit does not exist?

$$\lim_{x \to 0} \frac{\sqrt{x^2}}{x}$$

Answer choices:

A The right-hand limit is 1.

B The right-hand limit is equal to the left-hand limit.

C The right-hand limit does not equal the left-hand limit.

D The left-hand limit does exist.

Solution: C

We know that the general limit only exists if both the left- and right-hand limits exist. We can't find the limit using substitution, so we'll pick values close to $x = 0$, but on either side of it, to get an idea of what the one-sided limits are doing.

$$f(-0.0001) = \frac{\sqrt{(-0.0001)^2}}{-0.0001} = \frac{0.0001}{-0.0001} = -1$$

$$f(0.0001) = \frac{\sqrt{(0.0001)^2}}{0.0001} = \frac{0.0001}{0.0001} = 1$$

These values tell us that the left-hand limit is -1, and the right-hand limit is 1. Because the one-sided limits aren't equal, the general limit does not exist.

Topic: Proving that the limit does not exist

Question: How do we know that the limit does not exist?

$$\lim_{x\to 1} \ln(x-1)$$

Answer choices:

A The right-hand limit does not exist.

B The right-hand limit is equal to the left-hand limit.

C The left-hand limit is approaching $-\infty$.

D The left-hand limit does not exist.

Solution: D

We know that the general limit only exists if both the left- and right-hand limits exist. We can't find the limit using substitution (because the natural log function is undefined when the argument is 0, and substituting $x = 1$ makes the argument 0), so we'll pick values close to $x = 1$, but on either side of it, to get an idea of what the one-sided limits are doing.

$$f(0.9999) = \ln(0.9999 - 1) = \ln(-0.0001) = \text{DNE}$$

$$f(1.0001) = \ln(1.0001 - 1) = \ln(0.0001) = -\infty$$

The natural log function isn't defined in real numbers for negative arguments. So, because we end up with $\ln(-0.0001)$ when we investigate the left-hand limit, we can say that the left-hand limit does not exist (DNE).

This fact alone tells us that the general limit does not exist, since both of the one-sided limits must exist in order for the general limit to be defined.

Topic: Precise definition of the limit

Question: Which of these is the precise definition of the limit?

Answer choices:

A Let f be a function defined on a closed interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < x - c < \delta$, then $f(x) - L < \epsilon$.

B Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

C Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $|f(x) - L| < \epsilon$, then $0 < |x - c| < \delta$.

Solution: B

The correct statement of the precise definition of the limit is:

Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Topic: Precise definition of the limit

Question: Use the epsilon-delta definition to prove the value of the limit.

$$\lim_{x \to 9} \sqrt{x}$$

Answer choices:

A 1

B 2

C 3

D 4

Solution: C

First we find by direct substitution that $\lim_{x \to 9} \sqrt{x} = \sqrt{9} = 3$.

To prove this, we need to show that, on some open interval surrounding $x = 9$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\sqrt{x} - 3| < \epsilon$ whenever $0 < |x - 9| < \delta$.

Let $\epsilon > 0$ and $0 < |x - 9| < \delta$. We need to find a δ (which will be in terms of ϵ) that will give $|\sqrt{x} - 3| < \epsilon$. To do this, we need to try to rewrite $|\sqrt{x} - 3|$ so that it involves $|x - 9|$ in some way.

We know that

$$|\sqrt{x} - 3| = |\sqrt{x} - 3| \cdot \frac{|\sqrt{x} + 3|}{|\sqrt{x} + 3|} = |x - 9| \cdot \frac{1}{|\sqrt{x} + 3|}$$

and we want to find a δ such that

$$|\sqrt{x} - 3| = |x - 9| \cdot \frac{1}{|\sqrt{x} + 3|} < \epsilon$$

Since we are assuming that $0 < |x - 9| < \delta$, this means that we need to find an upper bound for $1/|\sqrt{x} + 3|$ on some open interval around $x = 9$, say the interval $(4,16)$.

We choose this interval because the function is defined on the entire interval, as required by the definition of the limit, because the interval contains 9, and because it's easy to find the square roots of 4 and 16. We could have, of course, chosen any positive interval containing 9, as long as the function is defined on the entire interval, except possibly at 9 itself.

On the interval $(4,16)$, $1/|\sqrt{x}+3|$ is constantly decreasing (the denominator is always positive and increasing on $(4,16)$ and the numerator is a constant 1). Therefore, on the interval $(4,16)$, $1/|\sqrt{x}+3|$ is at its maximum at the left endpoint:

$$\frac{1}{|\sqrt{x}+3|} < \frac{1}{|\sqrt{4}+3|} = \frac{1}{5}$$

We can now use this to find the δ we are required to find for our proof to be complete.

We are assuming that $0 < |x-9| < \delta$ and we have found that, on the interval $(4,16)$, $1/(\sqrt{x}+3) < 1/5$. Therefore, since we want

$$|\sqrt{x}-3| = \frac{|\sqrt{x}-3||\sqrt{x}+3|}{|\sqrt{x}+3|} = |x-9| \cdot \frac{1}{|\sqrt{x}+3|} < \epsilon$$

we can choose δ to be the lesser of 5ϵ and 7.* By doing this, we see that whenever $0 < |x-9| < \delta = 5\epsilon$, we have

$$|\sqrt{x}-3| = \frac{|\sqrt{x}-3||\sqrt{x}+3|}{|\sqrt{x}+3|} = |x-9| \cdot \frac{1}{|\sqrt{x}+3|} < 5\epsilon\left(\frac{1}{5}\right) = \epsilon$$

This proves that $\lim_{x\to 0}\sqrt{x} = 3$.

*In our proof, we have made the assumption that $0 < |x - 9| < \delta$. Since our choice of $\delta = 5\epsilon$ depends on x being in the interval $(4,16)$, this means that

$$4 < x < 16 \quad \to \quad -5 < x - 9 < 7 \quad \to \quad |x - 9| < 7$$

for this interval. Therefore, δ would have to be 7 if 5ϵ is greater than 7 in order for our assumption to remain true (a false assumption can't produce a true conclusion). This is why we choose δ to be the lesser of 5ϵ and 7.

Topic: Precise definition of the limit

Question: True or false? The precise definition of the limit implies that picking a value of x inside the δ interval will return a resulting value in the ϵ interval.

Answer choices:

A True

B False

Solution: A

According to the epsilon-delta definition of the limit, choosing a value for x between $x-\delta$ and $x+\delta$ will return a function value between $L-\epsilon$ and $L+\epsilon$.

Topic: Limits of combinations

Question: If $f(x) = x - 5$ and $g(x) = 3$, evaluate the limit.

$$\lim_{x \to 2} f(x) - g(x)$$

Answer choices:

A $\lim_{x \to 2} f(x) - g(x) = 6$

B $\lim_{x \to 2} f(x) - g(x) = -6$

C $\lim_{x \to 2} f(x) - g(x) = 0$

D $\lim_{x \to 2} f(x) - g(x) = \infty$

Solution: B

We'll start by distributing the limit across the combination.

$$\lim_{x \to 2} f(x) - g(x)$$

$$\lim_{x \to 2} f(x) - \lim_{x \to 2} g(x)$$

$$\lim_{x \to 2} (x - 5) - \lim_{x \to 2} (3)$$

Now we'll substitute the value we're approaching into each function.

$$(2 - 5) - (3)$$

$$-3 - 3$$

$$-6$$

Topic: Limits of combinations

Question: If $f(x) = x^3$ and $g(x) = 2 - x^2$, evaluate the limit.

$$\lim_{x \to 3} 2f(x)g(x)$$

Answer choices:

A $\lim_{x \to 3} 2f(x)g(x) = 189$

B $\lim_{x \to 3} 2f(x)g(x) = -189$

C $\lim_{x \to 3} 2f(x)g(x) = 378$

D $\lim_{x \to 3} 2f(x)g(x) = -378$

Solution: D

We'll start by distributing the limit across the combination.

$$\lim_{x \to 3} 2f(x)g(x)$$

$$\lim_{x \to 3} 2f(x) \lim_{x \to 3} g(x)$$

$$2 \lim_{x \to 3} f(x) \lim_{x \to 3} g(x)$$

$$2 \lim_{x \to 3} (x^3) \lim_{x \to 3} (2 - x^2)$$

Now we'll substitute the value we're approaching into each function.

$$2(3^3)(2 - 3^2)$$

$$2(27)(-7)$$

$$-378$$

Topic: Limits of combinations

Question: If $f(x) = x^2 + 2x + 1$ and $g(x) = x - 1$, evaluate the limit.

$$\lim_{x \to -1} \frac{f(x)}{4g(x)}$$

Answer choices:

A $\lim_{x \to -1} \frac{f(x)}{4g(x)} = -\infty$

B $\lim_{x \to -1} \frac{f(x)}{4g(x)} = \infty$

C $\lim_{x \to -1} \frac{f(x)}{4g(x)} = 0$

D The limit does not exist (DNE)

Solution: C

We'll start by plugging $f(x)$ and $g(x)$ into the limit.

$$\lim_{x\to-1}\frac{f(x)}{4g(x)}$$

$$\lim_{x\to-1}\frac{x^2+2x+1}{4(x-1)}$$

Now we'll substitute the value we're approaching into the function.

$$\frac{(-1)^2+2(-1)+1}{4(-1-1)}$$

$$\frac{1-2+1}{4(-2)}$$

$$\frac{0}{-8}$$

$$0$$

Topic: Limits of composites

Question: If $f(x) = x^3$ and $g(x) = x^2 + 3$, evaluate the limit.

$$\lim_{x \to 5} f(g(x))$$

Answer choices:

A $\lim_{x \to 5} f(g(x)) = 21,952$

B $\lim_{x \to 5} f(g(x)) = 81$

C $\lim_{x \to 5} f(g(x)) = 15,628$

D $\lim_{x \to 5} f(g(x)) = 253$

Solution: A

First find the composite $f(g(x))$, when $f(x) = x^3$ and $g(x) = x^2 + 3$.

$$f(x) = x^3$$

$$f(g(x)) = (x^2 + 3)^3$$

Then find the limit of the composite function.

$$\lim_{x \to 5} f(g(x))$$

$$\lim_{x \to 5} (x^2 + 3)^3$$

$$(5^2 + 3)^3$$

$$28^3$$

$$21{,}952$$

Topic: Limits of composites

Question: If $f(x) = \cos x$ and $g(x) = x + 4$, evaluate the limit.

$$\lim_{x \to -4} f(g(x))$$

Answer choices:

A $\lim_{x \to -4} f(g(x)) = -1$

B $\lim_{x \to -4} f(g(x)) = 0$

C $\lim_{x \to -4} f(g(x)) = 1$

D The limits does not exist (DNE)

Solution: C

First find the composite $f(g(x))$, when $f(x)=\cos x$ and $g(x)=x+4$.

$$f(x)=\cos x$$

$$f(g(x))=\cos(x+4)$$

Then find the limit of the composite function.

$$\lim_{x\to -4} f(g(x))$$

$$\lim_{x\to -4} \cos(x+4)$$

$$\cos(-4+4)$$

$$\cos(0)$$

$$1$$

Topic: Limits of composites

Question: If $f(x) = x^2 - 2x - 4$ and $g(x) = 5x - 5$, evaluate the limit.

$$\lim_{x \to 3} f(g(x))$$

Answer choices:

A $\lim_{x \to 3} f(g(x)) = 0$

B $\lim_{x \to 3} f(g(x)) = 76$

C $\lim_{x \to 3} f(g(x)) = 10$

D $\lim_{x \to 3} f(g(x)) = \infty$

Solution: B

First find the composite $f(g(x))$, when $f(x) = x^2 - 2x - 4$ and $g(x) = 5x - 5$.

$$f(x) = x^2 - 2x - 4$$

$$f(g(x)) = (5x - 5)^2 - 2(5x - 5) - 4$$

$$f(g(x)) = (5x - 5)(5x - 5) - 2(5x - 5) - 4$$

$$f(g(x)) = 25x^2 - 25x - 25x + 25 - 2(5x - 5) - 4$$

$$f(g(x)) = 25x^2 - 25x - 25x + 25 - 10x + 10 - 4$$

$$f(g(x)) = 25x^2 - 60x + 31$$

Then find the limit of the composite function.

$$\lim_{x \to 3} f(g(x))$$

$$\lim_{x \to 3} 25x^2 - 60x + 31$$

$$25(3)^2 - 60(3) + 31$$

$$25(9) - 180 + 31$$

$$225 - 180 + 31$$

$$76$$

Topic: Point discontinuities

Question: Find any point discontinuities of the function.

$$f(x) = \frac{x-5}{x^2-25}$$

Answer choices:

A $x = 0$

B $x = -5$

C $x = 5$

D $x = 25$

Solution: C

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{x-5}{x^2-25}$$

$$f(x) = \frac{x-5}{(x+5)(x-5)}$$

The factor $x-5$ can be canceled from the numerator and denominator.

$$f(x) = \frac{1}{x+5}$$

Because $x=5$ is a value that *would have* made the denominator 0, but we canceled it out when we canceled $x-5$, we know that the function has a point discontinuity at $x=5$.

Topic: Point discontinuities

Question: Find any point discontinuities of the function.

$$f(x) = \frac{x-1}{x^2+x-2}$$

Answer choices:

A $\quad x = 1$

B $\quad x = -1$

C $\quad x = 2$

D $\quad x = -2$

Solution: A

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{x-1}{x^2+x-2}$$

$$f(x) = \frac{x-1}{(x+2)(x-1)}$$

The factor $x-1$ can be canceled from the numerator and denominator.

$$f(x) = \frac{1}{x+2}$$

Because $x=1$ is a value that *would have* made the denominator 0, but we canceled it out when we canceled $x-1$, we know that the function has a point discontinuity at $x=1$.

Topic: Point discontinuities

Question: Find any point discontinuities of the function.

$$f(x) = \frac{x^2 + 2x - 3}{x^2 + x - 2}$$

Answer choices:

A $\quad x = -3$

B $\quad x = -2$

C $\quad x = -1$

D $\quad x = 1$

Solution: D

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{x^2 + 2x - 3}{x^2 + x - 2}$$

$$f(x) = \frac{(x + 3)(x - 1)}{(x + 2)(x - 1)}$$

The factor $x - 1$ can be canceled from the numerator and denominator.

$$f(x) = \frac{x + 3}{x + 2}$$

Because $x = 1$ is a value that *would have* made the denominator 0, but we canceled it out when we canceled $x - 1$, we know that the function has a point discontinuity at $x = 1$.

Topic: Jump discontinuities

Question: Which of the following statements is true?

Answer choices:

A A jump discontinuity occurs when the right- and left-hand limits both exist, but aren't equal.

B A jump discontinuity occurs when the right- and left-hand limits are not equal, and only one exists.

C A jump discontinuity occurs when the right- and left-hand limits do not exist.

D A jump discontinuity occurs when the right- and left-hand limits both exist, and they're equal.

Solution: A

For a jump discontinuity to occur, the left- and right-hand limits must both exist, but they must not be equal. In other words, the left-hand limit will exist, and the right-hand limit will exist, but the left- and right-hand limits will have different values.

Topic: Jump discontinuities

Question: Choose the correct description of the jump discontinuity that exists in the function.

$$f(x) = \frac{x}{|x|}$$

Answer choices:

A The function has a jump discontinuity at $x = 1$.

B The function has a jump discontinuity at $x = -1$.

C The function has a jump discontinuity at $x = \infty$.

D The function has a jump discontinuity at $x = 0$.

Solution: D

To answer this question, we could investigate the limit at each of the values given in the answer choices.

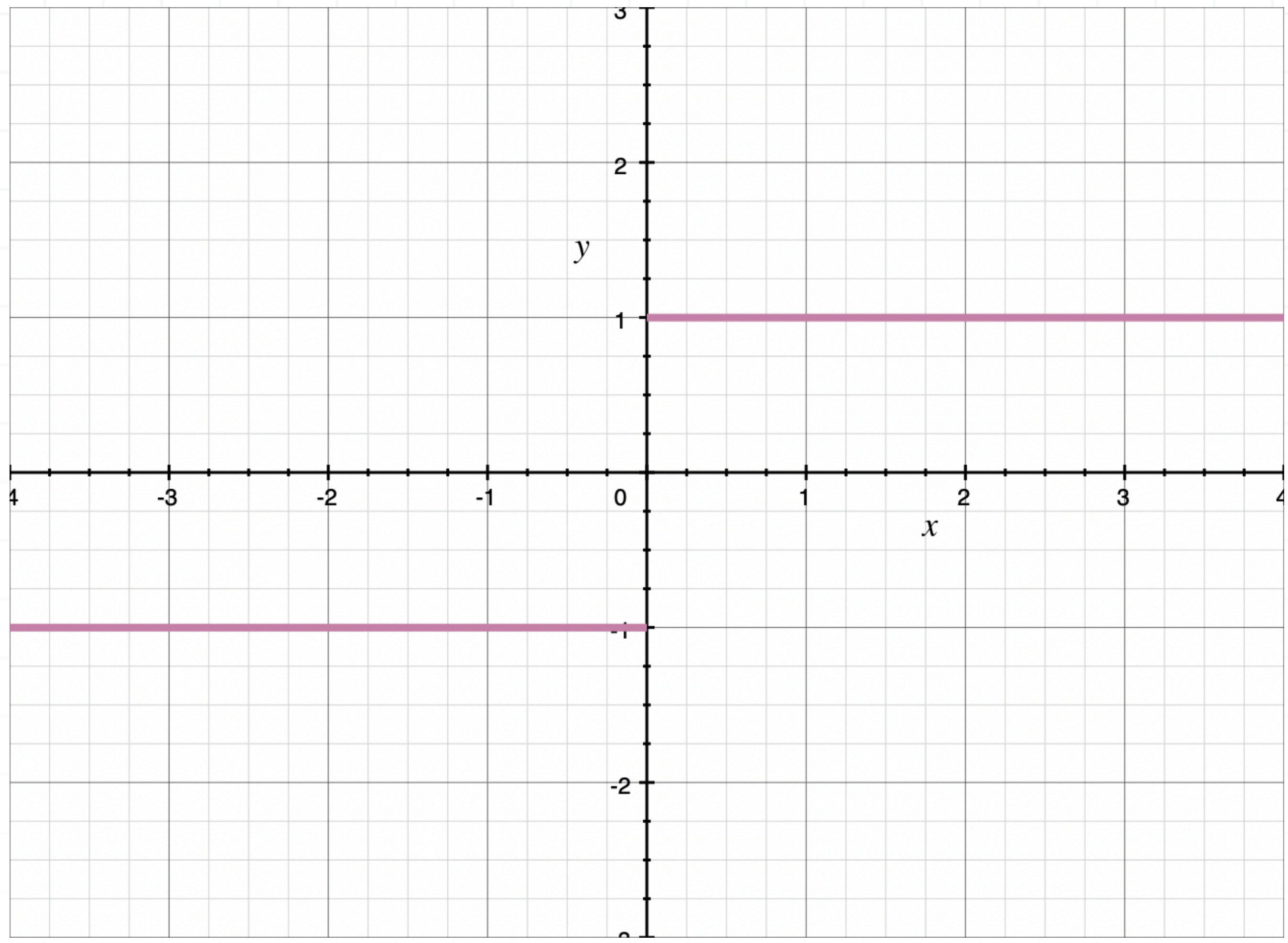
However, if we consider the answer choices briefly, or if we investigate answer choice D, we see that the function is undefined at $x = 0$. So $x = 0$ is an interesting place to start.

We'll look at what the function is doing on either side of $x = 0$.

$$f(-0.0001) = \frac{-0.0001}{|-0.0001|} = \frac{-0.0001}{0.0001} = -1$$

$$f(0.0001) = \frac{0.0001}{|0.0001|} = \frac{0.0001}{0.0001} = 1$$

What we see when we substitute values on either side of $x = 0$ is that, no matter which values we pick, any value to the left of $x = 0$ will return a value of -1, and any value to the right of $x = 0$ will return a value of 1.



Therefore, the jump discontinuity occurs at $x = 0$.

Topic: Jump discontinuities

Question: Choose the correct description of the jump discontinuity.

$$f(x) = \frac{x-1}{|x-1|}$$

Answer choices:

A The function has a jump discontinuity at $x = -1$.

B The function has a jump discontinuity at $x = 1$.

C The function has a jump discontinuity at $x = \infty$.

D The function has a jump discontinuity at $x = 0$.

Solution: B

To answer this question, we could investigate the limit at each of the values given in the answer choices.

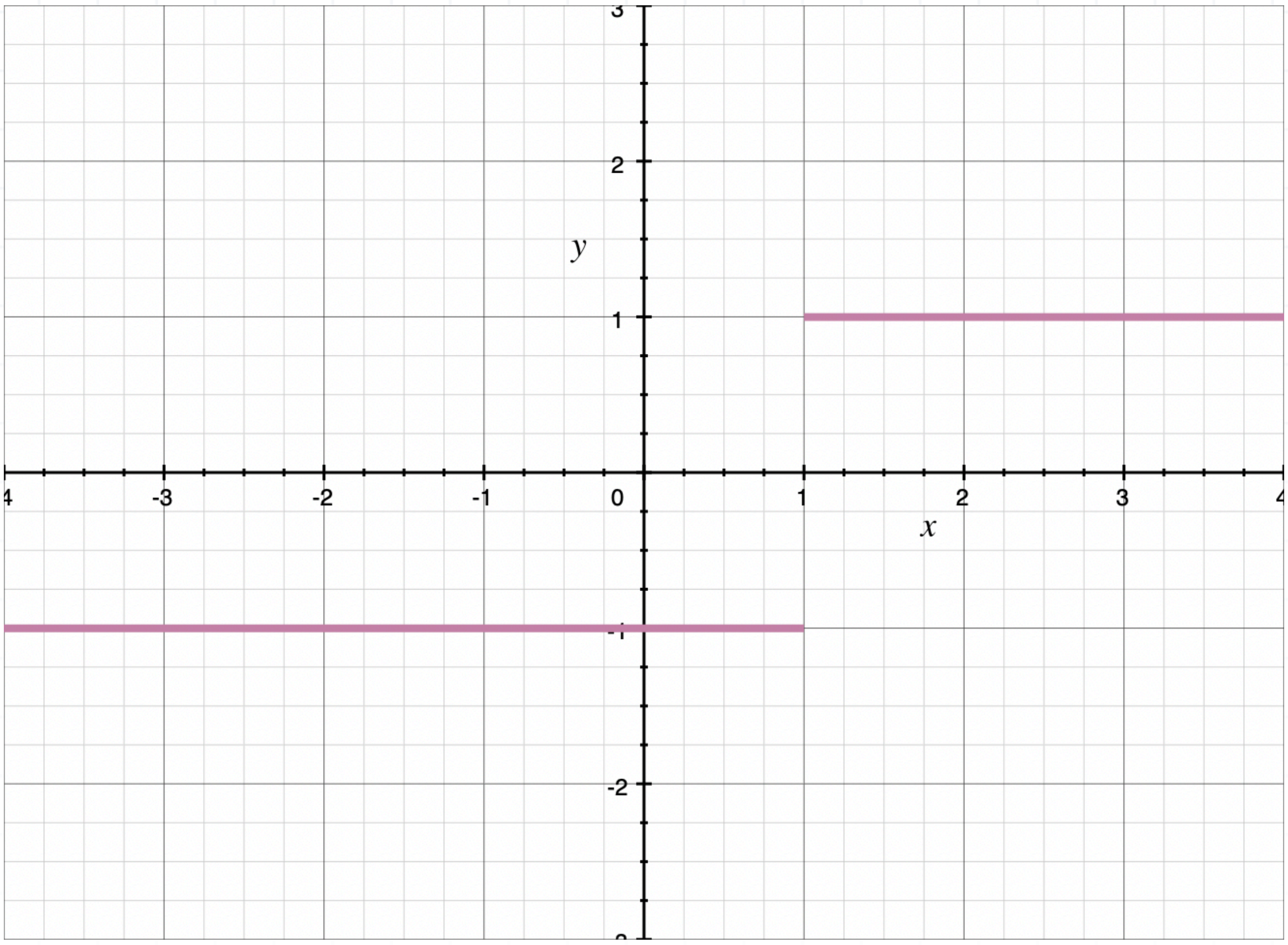
However, if we consider the answer choices briefly, or if we investigate answer choice B, we see that the function is undefined at $x = 1$. So $x = 1$ is an interesting place to start.

We'll look at what the function is doing on either side of $x = 1$.

$$f(0.9999) = \frac{0.9999 - 1}{|0.9999 - 1|} = \frac{-0.0001}{|-0.0001|} = \frac{-0.0001}{0.0001} = -1$$

$$f(1.0001) = \frac{1.0001 - 1}{|1.0001 - 1|} = \frac{0.0001}{|0.0001|} = \frac{0.0001}{0.0001} = 1$$

What we see when we substitute values on either side of $x = 1$ is that, no matter which values we pick, any value to the left of $x = 1$ will return a value of -1, and any value to the right of $x = 1$ will return a value of 1.



Therefore, the jump discontinuity occurs at $x = 1$.

Topic: Infinite discontinuities

Question: Choose the correct statement.

Answer choices:

A An infinite discontinuity exists where the right- and left-hand limits both approach ∞, or both approach $-\infty$.

B An infinite discontinuity exists where the right-hand limit approaches $-\infty$ while the left-hand limit approaches ∞.

C An infinite discontinuity exists where the right-hand limit approaches ∞ while the left-hand limit approaches $-\infty$.

D All of the above are true.

Solution: D

In answer choices A, B, and C, both the left-hand limit and right-hand limit are tending toward an infinite value, whether that value is $-\infty$ or ∞.

Wherever both the left- and right-hand limit are approaching an infinite value, the function has a vertical asymptote, and therefore an infinite discontinuity.

Topic: Infinite discontinuities

Question: Choose the correct description of the infinite discontinuity.

$$f(x) = \frac{1}{x}$$

Answer choices:

A The function has an infinite discontinuity at $x = 1$.

B The function has an infinite discontinuity at $x = -\infty$.

C The function has an infinite discontinuity at $x = 0$.

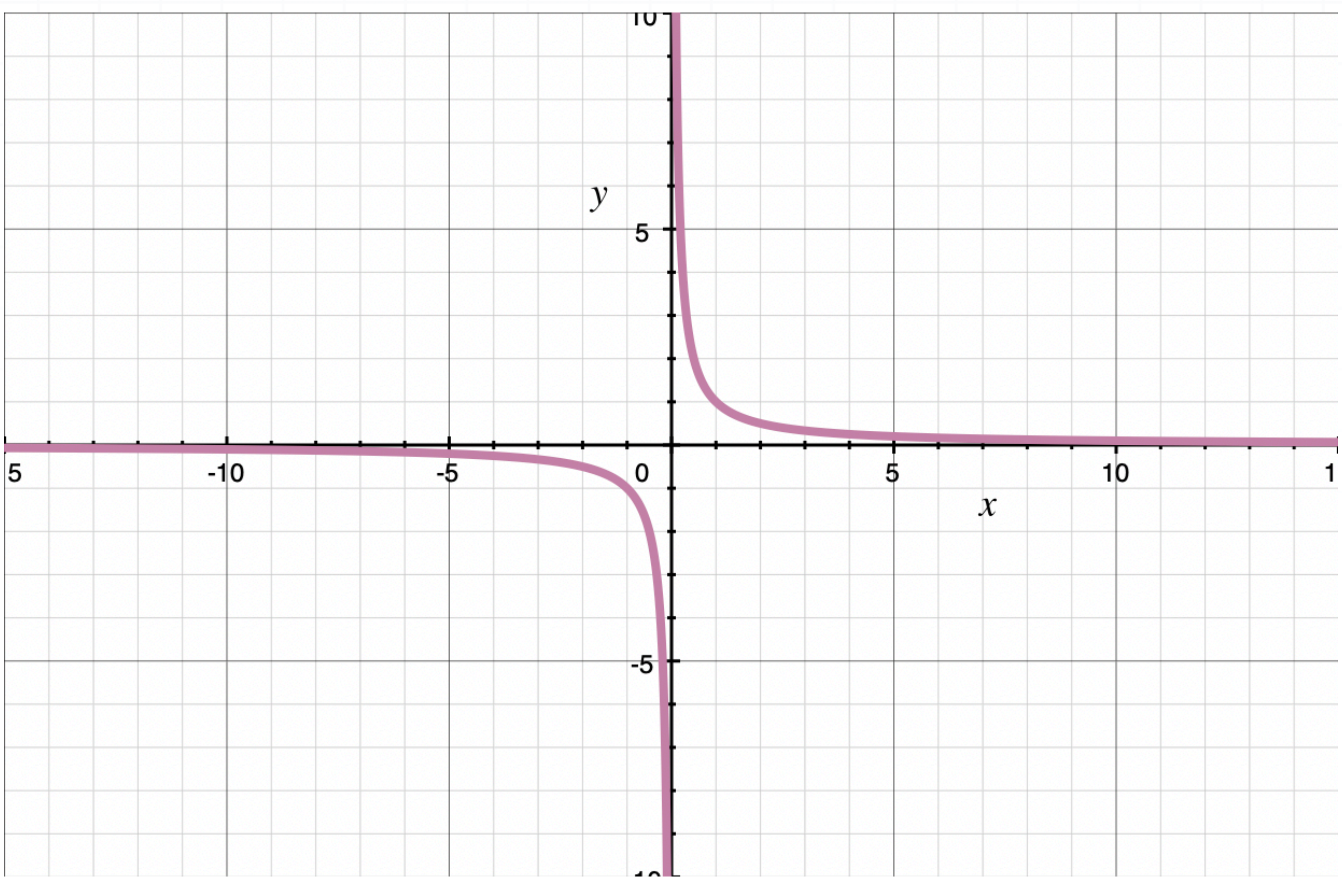
D The function has an infinite discontinuity at $x = \infty$.

Solution: C

The denominator of the function will be 0 when $x = 0$. That discontinuity can't be removed by canceling factors from the function, so $x = 0$ doesn't represent a point discontinuity.

Which means the function has a vertical asymptote, and therefore an infinite discontinuity, at $x = 0$.

The graph of the function confirms the discontinuity.



Topic: Infinite discontinuities

Question: Choose the correct description of the infinite discontinuity.

$$f(x) = \frac{x}{x-4}$$

Answer choices:

A The function has an infinite discontinuity at $x = 4$.

B The function has an infinite discontinuity at $x = -\infty$.

C The function has an infinite discontinuity at $x = -4$.

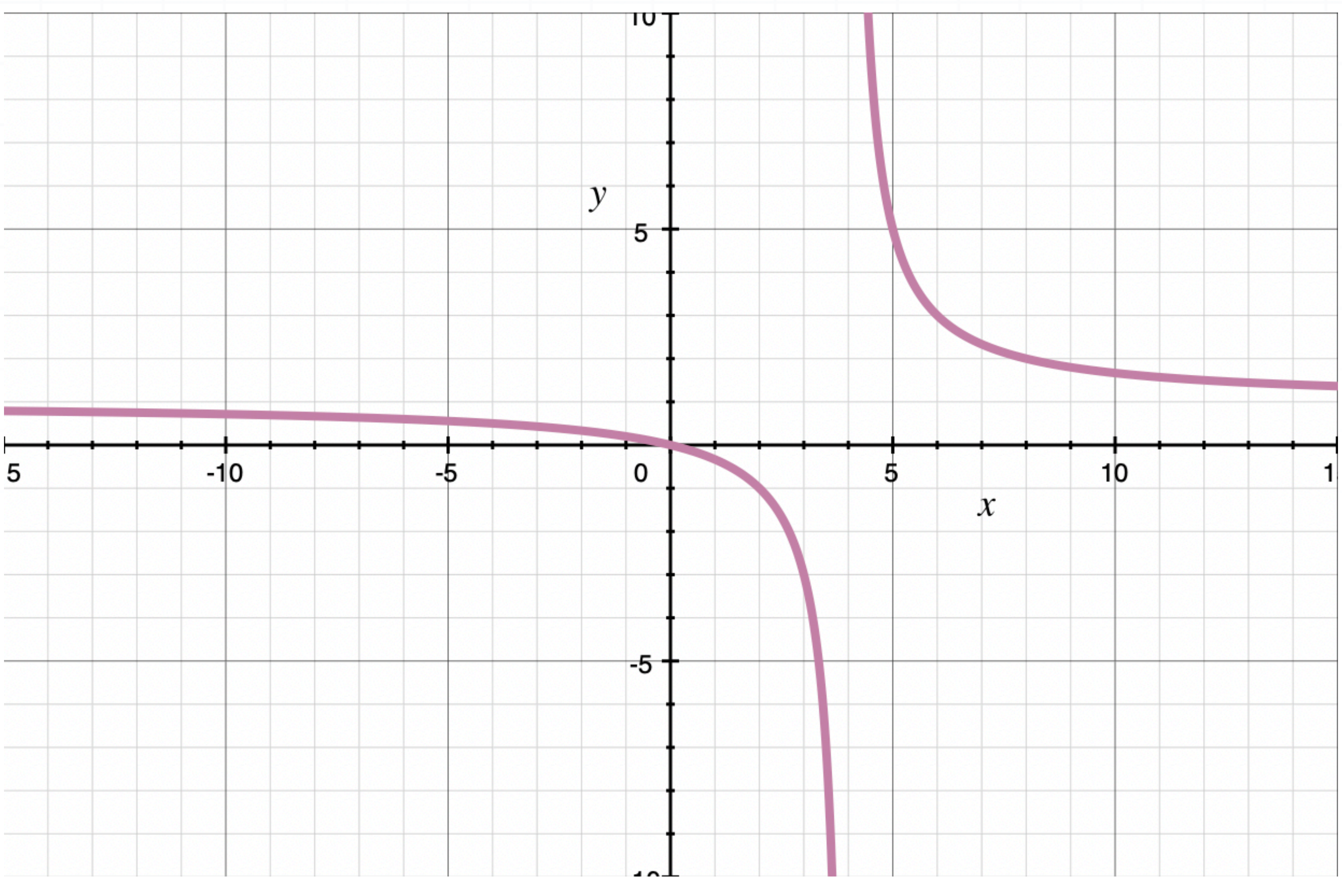
D The function has an infinite discontinuity at $x = \infty$.

Solution: A

The denominator of the function will be 0 when $x = 4$. That discontinuity can't be removed by canceling factors from the function, so $x = 4$ doesn't represent a point discontinuity.

Which means the function has a vertical asymptote, and therefore an infinite discontinuity, at $x = 4$.

The graph of the function confirms the discontinuity.



Topic: Endpoint discontinuities

Question: Which of the following statements is true?

Answer choices:

A The endpoint of an interval is discontinuous because one of the one-sided limits will be 0.

B The endpoint of an interval is discontinuous because one of the one-sided limits will be ∞.

C The endpoint of an interval is discontinuous because one of the one-sided limits will not exist.

D The endpoint of an interval is discontinuous because both of the one-sided limits will not exist.

Solution: C

The endpoint of an interval is discontinuous because one of the one-sided limits does not exist.

Because the function stops at an endpoint, either the left-hand limit will exist while the right-hand limit does not, or the right-hand limit will exist while the left-hand limit does not.

Topic: Endpoint discontinuities

Question: If the function $f(x) = x^2$ is only defined on $[1,4]$, and does not extend beyond that interval, what are the discontinuities of the function?

Answer choices:

A Endpoint discontinuities at $x = 0,\ 4$.

B A jump discontinuity at $x = 0$.

C Endpoint discontinuities at $x = 1,\ 4$ and a jump discontinuity at $x = 0$.

D Endpoint discontinuities at $x = 1,\ 4$.

Solution: D

The endpoints of an interval are discontinuous for a function because one of the one-sided limits will not exist at each endpoint.

The function $f(x) = x^2$ is a continuous function, but the interval $[1,4]$ means that there will be endpoint discontinuities at $x = 1$ and $x = 4$.

At $x = 1$, only the right-hand limit exists. The left-hand limit would be outside the function's domain. By the definition of continuity (that the left-hand limit exists, the right-hand limit exists, and the left- and right-hand limits are equal), that means the function isn't continuous at $x = 1$, so there's an endpoint discontinuity there.

At $x = 4$, only the left-hand limit exists. The right-hand limit is outside the function's domain. By the definition of continuity, that means the function isn't continuous at $x = 4$, so there's an endpoint discontinuity there.

Topic: Endpoint discontinuities

Question: What are the discontinuities of the function on the interval $[2,5]$?

$$f(x) = \sqrt{x}$$

Answer choices:

A Endpoint discontinuities at $x = 2$ and $x = 5$ and when $x \geq 0$.

B Endpoint discontinuities at $x = 2$ and $x = 5$.

C Endpoint discontinuities at $x = 2$ and $x = 5$ and when $x \leq 0$.

D Endpoint discontinuities at $x = 0$ and $x = 5$.

Solution: B

The function $f(x) = \sqrt{x}$ is a continuous function when $x \geq 0$ but the interval $[2,5]$ means that there will be endpoint discontinuities at the points $x = 2$ and $x = 5$.

An endpoint discontinuity exists at $x = 2$ because the left-hand limit doesn't exist there, and an endpoint discontinuity exists at $x = 5$ because the right-hand limit doesn't exist there.

Topic: Intermediate value theorem with an interval

Question: Which statement is true?

Answer choices:

A The IVT only applies to discontinuous functions.

B The IVT only applies when there’s no interval.

C The IVT only applies to open intervals.

D The IVT only applies to closed intervals.

Solution: D

The Intermediate Value Theorem states that for a function on a closed interval $[a, b]$ where the function is continuous on the interval, a point c exists on the interval where $f(c) = k$.

$f(a) < k < f(b)$ and $a < c < b$

Topic: Intermediate value theorem with an interval

Question: Does the Intermediate Value Theorem prove that the function $f(x) = x^2 + 2x - 35$ has a root on the interval $[0,10]$?

Answer choices:

A The IVT shows that the function has a root on the open interval.

B The IVT shows that the function has a root on the closed interval.

C The IVT shows that there's not root on the closed interval.

D The IVT is inconclusive.

Solution: B

Evaluate the function at both endpoints of the interval $(0,10)$.

$$f(0) = 0^2 + 2(0) - 35$$

$$f(0) = -35$$

and

$$f(10) = 10^2 + 2(10) - 35$$

$$f(10) = 85$$

Because the function is below the x-axis at the left edge of the interval, and above the x-axis at the right edge of the interval, we can say $f(a) < 0 < f(b)$, or more specifically, $-35 < 0 < 85$.

Therefore, by the intermediate value theorem, it must be true that the function has a root on the interval $[0,10]$.

Topic: Intermediate value theorem with an interval

Question: Is there a root for the function $f(x) = x^2 - 4$ on the interval $(1,6)$?

Answer choices:

A Yes, there's a root at $(0,4)$.

B Yes, there's a root at $(0, -4)$.

C Yes, there's a root at $(2,0)$.

D Yes, there's a root at $(-2,0)$.

Solution: C

Evaluate the function at both endpoints of the interval $(1,6)$.

$$f(1) = 1^2 - 4$$

$$f(1) = 1 - 4$$

$$f(1) = -3$$

and

$$f(6) = 6^2 - 4$$

$$f(6) = 36 - 4$$

$$f(6) = 32$$

The IVT confirms that the function has a root on the interval, because the function's value crosses from below the x-axis to above the x-axis at some point within that interval.

To find the root, which is the point where the graph of the function crosses the x-axis, we'll set the function equal to 0.

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

Therefore, the root in the interval $(1,6)$ is at $x = 2$, and that coordinate point is $(2,0)$.

Topic: Intermediate value theorem without an interval

Question: If we're trying to use the Intermediate Value Theorem to prove the existence of a root for the function, but no interval is given, what should we do?

Answer choices:

A We should give up, because there's no way to use the IVT if no interval is given.

B We should give up, because by definition, the function has no root if no interval is given.

C We should try to find our own interval, and the only way to do this is to guess random intervals.

D We should try to find our own interval, and to do this, we can try to consider what we might already know about the function's values, we can look at the graph of the function, or otherwise try to be strategic about how to narrow down a useful interval.

Solution: D

When no interval is given to us in which we should look for the root, it still may be possible to use the Intermediate Value Theorem to prove the existence of a root.

But we have to find an interval first. In order to do that, we can employ different techniques, and get creative in order to narrow down what interval we might be able to use.

For instance, graphing the function might show us the approximate location of the root, and we can pick values for the interval that are on either side of the root.

Or, if we know something about the shape of the graph of the function, and we can use that information to narrow down an interval, we can take that approach as well.

Topic: Intermediate value theorem without an interval

Question: There are no real roots for the function $f(x) = \sin x$.

Answer choices:

A True

B False

Solution: B

No interval is given, but the sine function oscillates back and forth between $[-1,1]$.

If it oscillates back and forth from -1 below the x-axis, to 1 above the x-axis, and because $f(x)=\sin x$ is a continuous function, it must cross the x-axis at some point, which means it has at least one real root somewhere in its domain.

Topic: Intermediate value theorem without an interval

Question: Find an interval for the function $f(x) = \cos x$ on which the function has a real root.

Answer choices:

A $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

B $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

C $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

D $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Solution: D

We could answer this question by evaluating the function $f(x) = \cos x$ at the endpoints of each interval given in the answer choices.

If we did that, only answer choices C and D give a negative value at one edge of the interval and a positive value at the other edge of the interval. From there, only answer choice D is a closed interval, so answer choice D can be the only correct answer.

Alternatively, we could have looked at the graph of $f(x) = \cos x$ and investigated its value at the endpoints of each of the intervals. We'd see that the graph was above the x-axis everywhere from $-\pi/4$ to $\pi/4$, thereby eliminating answer choices A and B, allowing us to conclude that D must be the correct choice, since C is not a closed interval.

Topic: Solving with substitution

Question: Use substitution to evaluate the limit.

$$\lim_{x \to 5} \frac{x^2 + 2x + 1}{x + 5}$$

Answer choices:

A 12

B 6

C 1.6

D 3.6

Solution: D

Substitute $x = 5$ into the function to evaluate the limit.

$$f(x) = \frac{x^2 + 2x + 1}{x + 5}$$

$$f(5) = \frac{5^2 + 2(5) + 1}{5 + 5}$$

$$f(5) = \frac{36}{10}$$

$$f(5) = 3.6$$

Topic: Solving with substitution

Question: Use substitution to evaluate the limit.

$$\lim_{x \to 6} x^3 + 6 - 3x$$

Answer choices:

A $\quad 204$

B $\quad 198$

C $\quad 240$

D $\quad 234$

Solution: A

Substitute $x = 6$ into the function to evaluate the limit.

$$f(x) = x^3 + 6 - 3x$$

$$f(6) = 6^3 + 6 - 3(6)$$

$$f(6) = 216 + 6 - 18$$

$$f(6) = 204$$

Topic: Solving with substitution

Question: Use substitution to evaluate the limit.

$$\lim_{x\to 0} \frac{-1}{3(x+3)}$$

Answer choices:

A $-\frac{1}{9}$

B $\frac{1}{9}$

C $-\frac{1}{6}$

D $\frac{1}{6}$

Solution: A

Substitute $x = 0$ into the function to evaluate the limit.

$$f(x) = \frac{-1}{3(x+3)}$$

$$f(0) = \frac{-1}{3(0+3)}$$

$$f(0) = \frac{-1}{3(3)}$$

$$f(0) = -\frac{1}{9}$$

Topic: Solving with factoring

Question: Use factoring to find the limit.

$$\lim_{t \to -1} \frac{(t+1)(t^2 - t + 1)}{t+1}$$

Answer choices:

A 0

B 3

C -1

D ∞

Solution: B

The numerator and denominator share a common factor of $t+1$, which can be canceled from the function.

$$\lim_{t \to -1} \frac{(t+1)(t^2-t+1)}{t+1}$$

$$\lim_{t \to -1} t^2 - t + 1$$

Now use substitution to evaluate the limit.

$$(-1)^2 - (-1) + 1$$

$$1 + 1 + 1$$

$$3$$

Topic: Solving with factoring

Question: Use factoring to find the limit.

$$\lim_{x \to 2} \frac{x^2 - 2x}{x - 2}$$

Answer choices:

A 4

B -4

C 2

D -2

Solution: C

Factor the numerator and denominator as completely as possible.

$$\lim_{x \to 2} \frac{x^2 - 2x}{x - 2}$$

$$\lim_{x \to 2} \frac{x(x - 2)}{x - 2}$$

Cancel the common factor of $x - 2$.

$$\lim_{x \to 2} x$$

Then use direct substitution to evaluate the limit.

$$2$$

Topic: Solving with factoring

Question: Use factoring to find the limit.

$$\lim_{x \to 3} \frac{x^2 - 7x + 12}{x^2 - 9}$$

Answer choices:

A $\frac{1}{3}$

B $-\frac{1}{3}$

C $\frac{1}{6}$

D $-\frac{1}{6}$

Solution: D

Factor the numerator and denominator as completely as possible.

$$\lim_{x\to 3} \frac{x^2 - 7x + 12}{x^2 - 9}$$

$$\lim_{x\to 3} \frac{(x-4)(x-3)}{(x+3)(x-3)}$$

Cancel the common factor of $x - 3$.

$$\lim_{x\to 3} \frac{x-4}{x+3}$$

Then use direct substitution to evaluate the limit.

$$\frac{3-4}{3+3}$$

$$-\frac{1}{6}$$

Topic: Solving with conjugate method

Question: Use conjugate method to find the limit.

$$\lim_{m \to 0} \frac{\sqrt{m+4}-2}{m}$$

Answer choices:

A 0

B 4

C $\frac{1}{4}$

D ∞

Solution: C

The conjugate of $\sqrt{m+4}-2$ is $\sqrt{m+4}+2$. Multiply both the numerator and denominator by this conjugate.

$$\lim_{m\to 0}\frac{\sqrt{m+4}-2}{m}\left(\frac{\sqrt{m+4}+2}{\sqrt{m+4}+2}\right)$$

$$\lim_{m\to 0}\frac{m+4+2\sqrt{m+4}-2\sqrt{m+4}-4}{m(\sqrt{m+4}+2)}$$

$$\lim_{m\to 0}\frac{m+4-4}{m(\sqrt{m+4}+2)}$$

$$\lim_{m\to 0}\frac{m}{m(\sqrt{m+4}+2)}$$

Cancel the common factor of m from both the numerator and denominator.

$$\lim_{m\to 0}\frac{1}{\sqrt{m+4}+2}$$

Now use substitution to evaluate the limit.

$$\frac{1}{\sqrt{0+4}+2}$$

$$\frac{1}{2+2}$$

$$\frac{1}{4}$$

Topic: Solving with conjugate method

Question: Use conjugate method to find the limit.

$$\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3}$$

Answer choices:

A 6

B 3

C 9

D 0

Solution: A

The conjugate of $\sqrt{x}-3$ is $\sqrt{x}+3$. Multiply both the numerator and denominator by this conjugate.

$$\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right)$$

$$\lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$\lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{x+3\sqrt{x}-3\sqrt{x}-9}$$

$$\lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{x-9}$$

Cancel the common factor of $x-9$ from both the numerator and denominator.

$$\lim_{x\to 9}\sqrt{x}+3$$

Now use substitution to evaluate the limit.

$$\sqrt{9}+3$$

$$3+3$$

$$6$$

Topic: Solving with conjugate method

Question: Use conjugate method to find the limit.

$$\lim_{x \to 16} \frac{16 - x}{4 - \sqrt{x}}$$

Answer choices:

A 4

B 8

C 0

D 16

Solution: B

The conjugate of $4-\sqrt{x}$ is $4+\sqrt{x}$. Multiply both the numerator and denominator by this conjugate.

$$\lim_{x\to 16}\frac{16-x}{4-\sqrt{x}}\left(\frac{4+\sqrt{x}}{4+\sqrt{x}}\right)$$

$$\lim_{x\to 16}\frac{(16-x)(4+\sqrt{x})}{(4-\sqrt{x})(4+\sqrt{x})}$$

$$\lim_{x\to 16}\frac{(16-x)(4+\sqrt{x})}{16+4\sqrt{x}-4\sqrt{x}-x}$$

$$\lim_{x\to 16}\frac{(16-x)(4+\sqrt{x})}{16-x}$$

Cancel the common factor of $16-x$ from both the numerator and denominator.

$$\lim_{x\to 16} 4+\sqrt{x}$$

Now use substitution to evaluate the limit.

$$4+\sqrt{16}$$

$$4+4$$

$$8$$

Topic: Infinite limits and vertical asymptotes

Question: Evaluate the limit.

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

Answer choices:

A 0

B 1

C ∞

D $-\infty$

Solution: C

We can't use substitution to evaluate the limit, because we get a 0 in the denominator and therefore an undefined value.

$$\frac{1}{(1-1)^2}$$

$$\frac{1}{0^2}$$

$$\frac{1}{0}$$

So we'll test values close to, and on either side of, $x = 1$ to see how the function is behaving close to that point.

$$f(0.9999) = \frac{1}{(0.9999-1)^2} = \frac{1}{(-0.0001)^2} = \frac{1}{0.00000001} = 100{,}000{,}000$$

$$f(1.0001) = \frac{1}{(1.0001-1)^2} = \frac{1}{0.0001^2} = \frac{1}{0.00000001} = 100{,}000{,}000$$

The function is approaching ∞ on both sides of $x = 1$, so we can say that the value of the limit is ∞.

$$\lim_{x\to1}\frac{1}{(x-1)^2} = \infty$$

Topic: Infinite limits and vertical asymptotes

Question: Evaluate the limit.

$$\lim_{x \to \frac{\pi}{2}} \tan x$$

Answer choices:

A $\lim_{x \to \frac{\pi}{2}^-} \tan x = -\infty$ $\lim_{x \to \frac{\pi}{2}^+} \tan x = \infty$

B $\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$ $\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$

C $\lim_{x \to \frac{\pi}{2}^-} \tan x = -\infty$ $\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$

D $\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$ $\lim_{x \to \frac{\pi}{2}^+} \tan x = \infty$

Solution: B

We can't use substitution to evaluate the limit, because we get a 0 in the denominator and therefore an undefined value.

$$\tan\frac{\pi}{2}$$

$$\frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}$$

$$\frac{1}{0}$$

So we'll test values close to, and on either side of, $x=\pi/2$ to see how the function is behaving close to that point.

$$f\left(\frac{49\pi}{100}\right)=\tan\frac{49\pi}{100}\approx 31.82$$

$$f\left(\frac{51\pi}{100}\right)=\tan\frac{51\pi}{100}\approx -31.82$$

The function is approaching ∞ to the left of $x=\pi/2$ and $-\infty$ to the right of $x=\pi/2$, so the general limit does not exist. But the one-sided limits are

$$\lim_{x\to\frac{\pi}{2}^-}\tan x=\infty$$

$$\lim_{x\to\frac{\pi}{2}^+}\tan x=-\infty$$

Topic: Infinite limits and vertical asymptotes

Question: Evaluate the limit.

$$\lim_{x \to \pi} \cot x$$

Answer choices:

A 0

B 1

C ∞

D Does not exist (DNE)

Solution: D

We can't use substitution to evaluate the limit, because we get a 0 in the denominator and therefore an undefined value.

$$\cot \pi$$

$$\frac{\cos \pi}{\sin \pi}$$

$$\frac{-1}{0}$$

So we'll test values close to, and on either side of, $x = \pi$ to see how the function is behaving close to that point.

$$f\left(\frac{99\pi}{100}\right) = \cot \frac{99\pi}{100} \approx -31.82$$

$$f\left(\frac{101\pi}{100}\right) = \cot \frac{101\pi}{100} \approx 31.82$$

The function is approaching $-\infty$ to the left of $x = \pi$ and ∞ to the right of $x = \pi$, so the general limit does not exist.

Topic: Limits at infinity and horizontal asymptotes

Question: Evaluate the limit.

$$\lim_{x\to\infty} \frac{6}{4x^2}$$

Answer choices:

A $\frac{3}{2}$

B 0

C ∞

D 1

Solution: B

To find the limit as $x \to \infty$, we'll look at the highest-degree terms in both the numerator and denominator.

The highest-degree term in the numerator is 6, which has a degree of 0. The highest-degree term in the denominator is $4x^2$, which has a degree of 2. Therefore,

N < D: $0 < 2$

When the degree of the numerator is less that the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

So, as $x \to \infty$, the limit is 0.

Topic: Limits at infinity and horizontal asymptotes

Question: Evaluate the limit.

$$\lim_{x \to \infty} \frac{6x^3 + 2x^2 - x + 1}{8x^3 - 1}$$

Answer choices:

A $\quad -1$

B $\quad \infty$

C $\quad \frac{3}{4}$

D $\quad \frac{3}{8}$

Solution: C

To find the limit as $x \to \infty$, we'll look at the highest-degree terms in both the numerator and denominator.

The highest-degree term in the numerator is $6x^3$, which has a degree of 3. The highest-degree term in the denominator is $8x^3$, which has a degree of 3. Therefore,

N = D: $3 = 3$

When the degree of the numerator is equal to the degree of the denominator, the function has a horizontal asymptote given by the ratio of the coefficients on those highest-degree terms.

$$\frac{6x^3 + 2x^2 - x + 1}{8x^3 - 1}$$

$$\frac{6x^3}{8x^3}$$

$$\frac{6}{8}$$

$$\frac{3}{4}$$

So, as $x \to \infty$, the limit is $3/4$.

Topic: Limits at infinity and horizontal asymptotes

Question: Evaluate the limit.

$$\lim_{x\to\infty} \frac{x^2+1}{5x^4+2x^2}$$

Answer choices:

A $\sqrt{5}$

B $\frac{\sqrt{5}}{5}$

C ∞

D 0

Solution: D

To find the limit as $x \to \infty$, we'll look at the highest-degree terms in both the numerator and denominator.

The highest-degree term in the numerator is x^2, which has a degree of 2. The highest-degree term in the denominator is $5x^4$, which has a degree of 4. Therefore,

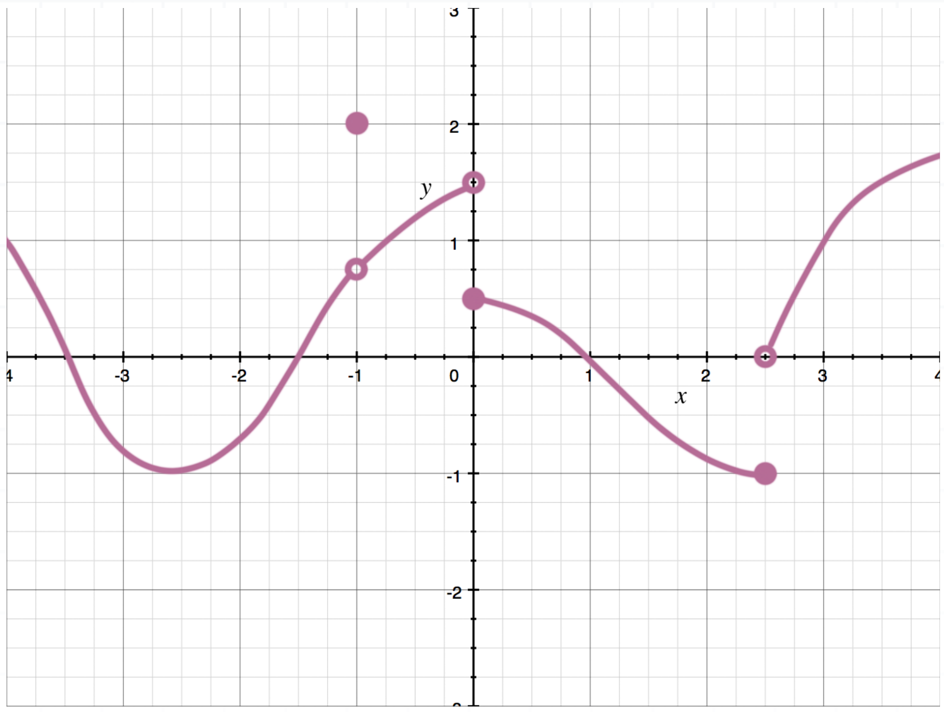
N < D: $2 < 4$

When the degree of the numerator is less that the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

So, as $x \to \infty$, the limit is 0.

Topic: Crazy graphs

Question: Use the graph to find the function's limit as $x \to -1^+$.



Answer choices:

A $\lim_{x \to -1^+} f(x) = 0$

B $\lim_{x \to -1^+} f(x) = 2$

C $\lim_{x \to -1^+} f(x) = \frac{3}{4}$

D $\lim_{x \to -1^+} f(x) = \frac{1}{2}$

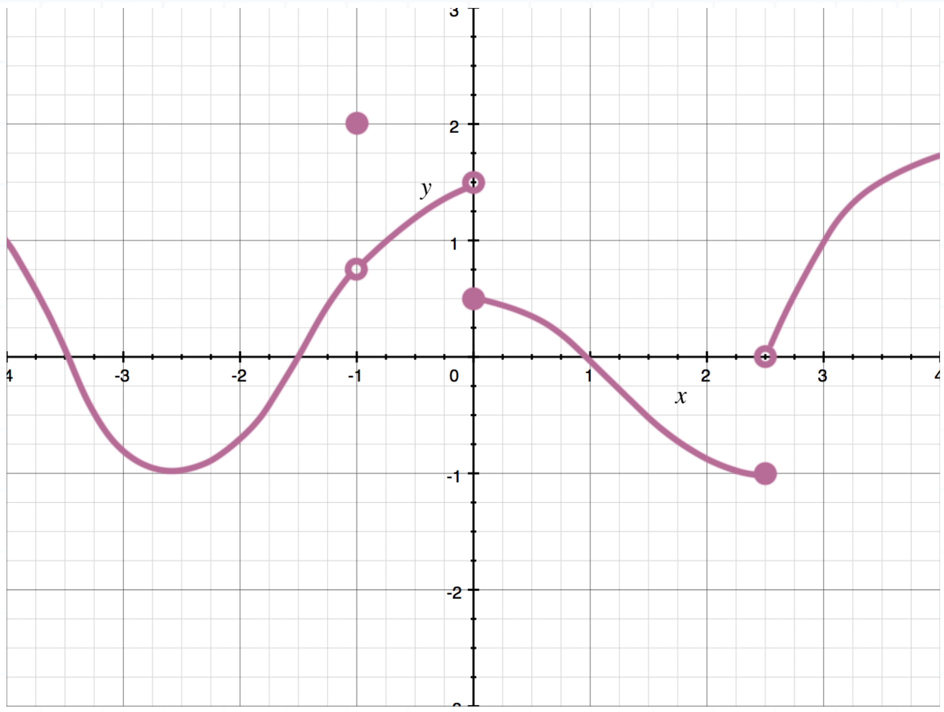
Solution: C

Using the graph, we'll look at the limit as x gets close to -1 from the right side. We can see that

$$\lim_{x \to -1^+} f(x) = \frac{3}{4}$$

Topic: Crazy graphs

Question: Use the graph to find the function's limit as $x \to 0^-$ and $x \to 0^+$.



Answer choices:

A $\lim_{x\to 0^-} f(x) = -\frac{1}{2}$ $\lim_{x\to 0^+} f(x) = -\frac{3}{2}$

B $\lim_{x\to 0^-} f(x) = -\frac{3}{2}$ $\lim_{x\to 0^+} f(x) = -\frac{1}{2}$

C $\lim_{x\to 0^-} f(x) = \frac{1}{2}$ $\lim_{x\to 0^+} f(x) = \frac{3}{2}$

D $\lim_{x\to 0^-} f(x) = \frac{3}{2}$ $\lim_{x\to 0^+} f(x) = \frac{1}{2}$

Solution: D

Using the graph, we'll look at the limit as x gets close to 0 from the left side. We can see that

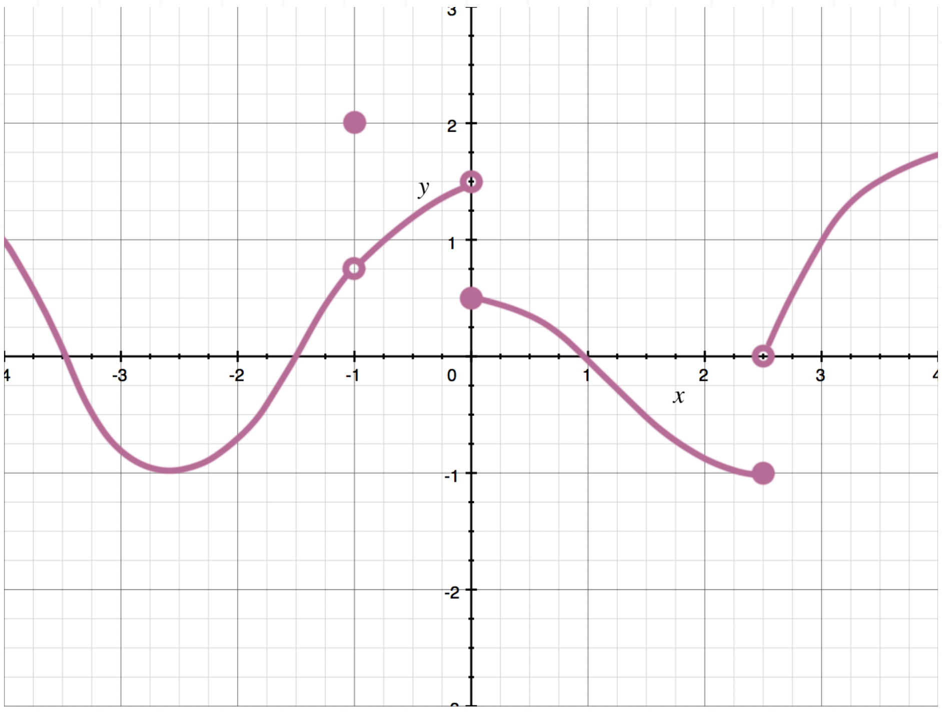
$$\lim_{x \to 0^-} f(x) = \frac{3}{2}$$

And as x gets close to 0 from the right side, we can see that

$$\lim_{x \to 0^+} f(x) = \frac{1}{2}$$

Topic: Crazy graphs

Question: Use the graph to find the function's limit as $x \to (5/2)^-$ and $x \to (5/2)^+$.



Answer choices:

A $\quad \lim_{x \to \frac{5}{2}^-} f(x) = -1 \qquad \lim_{x \to \frac{5}{2}^+} f(x) = 0$

B $\quad \lim_{x \to \frac{5}{2}^-} f(x) = 0 \qquad \lim_{x \to \frac{5}{2}^+} f(x) = -1$

C $\quad \lim_{x \to \frac{5}{2}^-} f(x) = 1 \qquad \lim_{x \to \frac{5}{2}^+} f(x) = 0$

D $\quad \lim_{x \to \frac{5}{2}^-} f(x) = 0 \qquad \lim_{x \to \frac{5}{2}^+} f(x) = 1$

Solution: A

Using the graph, we'll look at the limit as x gets close to $5/2$ from the left side. We can see that

$$\lim_{x \to \frac{5}{2}^-} f(x) = -1$$

And as x gets close to $5/2$ from the right side, we can see that

$$\lim_{x \to \frac{5}{2}^+} f(x) = 0$$

Topic: Trigonometric limits

Question: Evaluate the limit.

$$\lim_{x \to 0} \frac{\cos x \sin x}{x}$$

Answer choices:

A 0

B -1

C 1

D Does not exist (DNE)

Solution: C

If we use direct substitution to evaluate the limit, we get the undefined value $0/0$.

$$\frac{\cos(0)\sin(0)}{0}$$

$$\frac{1(0)}{0}$$

$$\frac{0}{0}$$

But if we rewrite the limit as

$$\lim_{x \to 0} \cos x \frac{\sin x}{x}$$

then we see that we have the product of two of the three key trig limit formulas,

$$\lim_{x \to 0} \cos x = 1 \text{ and } \lim_{x \to 0} \frac{\sin x}{x} = 1$$

So we can evaluate the limit using these formulas.

$$\lim_{x \to 0} \cos x \cdot \lim_{x \to 0} \frac{\sin x}{x}$$

$$1 \cdot 1$$

$$1$$

Topic: Trigonometric limits

Question: Use a reciprocal identity to move the function toward one of the key trig limits, and then evaluate the limit.

$$\lim_{x \to 0} \frac{7}{x \csc x}$$

Answer choices:

A $\quad 0$

B $\quad 7$

C $\quad -7$

D $\quad \infty$

Solution: B

Rewrite the function as using the reciprocal identity that relates $\sin x$ and $\csc x$.

$$\lim_{x\to 0} \frac{7}{x\csc x}$$

$$\lim_{x\to 0} \frac{7}{\frac{x}{\sin x}}$$

$$\lim_{x\to 0} \frac{7\sin x}{x}$$

We know the value of the trig limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Therefore,

$$7(1)$$

$$7$$

Topic: Trigonometric limits

Question: Use conjugate method, then the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, to evaluate the limit.

$$\lim_{h \to 0} \frac{\cos h - 1}{h}$$

Answer choices:

A 0

B 1

C -1

D ∞

Solution: A

If we use direct substitution to evaluate the limit, we get the undefined value $0/0$.

$$\frac{\cos(0) - 1}{0}$$

$$\frac{1 - 1}{0}$$

$$\frac{0}{0}$$

But we've been asked to start with conjugate method, anyway. We'll multiply both the numerator and denominator of the function by the conjugate of $\cos h - 1$.

$$\lim_{h \to 0} \frac{\cos h - 1}{h} \left(\frac{\cos h + 1}{\cos h + 1} \right)$$

$$\lim_{h \to 0} \frac{\cos^2 h + \cos h - \cosh h - 1}{h(\cos h + 1)}$$

$$\lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

If we factor out a negative sign, we can rewrite the limit as

$$\lim_{h \to 0} -\frac{1 - \cos^2 h}{h(\cos h + 1)}$$

We were told in the question to use the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, which we can rewrite.

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

Now that the right side of this trigonometric identity matches the numerator of the function, we can make a substitution.

$$\lim_{h \to 0} -\frac{\sin^2 h}{h(\cos h + 1)}$$

Now we'll rewrite the limit

$$\lim_{h \to 0} -\sin h \frac{\sin h}{h} \left(\frac{1}{\cos h + 1} \right)$$

One of the three key trig limits is

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

which means we can simplify the limit to

$$\lim_{h \to 0} -\sin h \left(\frac{1}{\cos h + 1} \right)$$

$$\lim_{h \to 0} -\frac{\sin h}{\cos h + 1}$$

Now we can use substitution to evaluate the limit.

$$-\frac{\sin(0)}{\cos(0) + 1}$$

$$-\frac{0}{1 + 1}$$

$$-\frac{0}{2}$$

$$0$$

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 1/2$. If it's discontinuous, identify the type of discontinuity and redefine the function to make it continuous.

$$f(x) = \begin{cases} |2x - 1| & x < \frac{1}{2} \\ \frac{3}{4} & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

Answer choices:

A The function is continuous at $x = 1/2$.

B The function is discontinuous at $x = 1/2$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 1/2$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} |2x - 1| & x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

D The function is discontinuous at $x = 1/2$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

Solution: D

In order for the function to be continuous at $x = 1/2$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 1/2$.

We already know that the value of the function at $x = 1/2$ is $3/4$, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to $3/4$. If they exist but aren't equal to $3/4$, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x = 1/2$.

To look at the left-hand limit, we'll use the first "piece" of our piecewise-defined function, because it defines the function to the left of $x = 1/2$ (the domain of that piece is $x < 1/2$).

$$\lim_{x \to (1/2)^-} |2x - 1|$$

Since the domain of $|2x - 1|$ is $x < 1/2$, we know that no matter what value in the domain we plug in for, we're always going to get a negative value for $2x - 1$. That means we can take away the absolute value bars as long as we put a negative sign in front of $2x - 1$.

$$\lim_{x \to (1/2)^-} -(2x - 1)$$

$$\lim_{x \to (1/2)^-} 1 - 2x$$

$$1 - 2\left(\frac{1}{2}\right)$$

$$1 - 1$$

$$0$$

We know now that the left-hand limit exists, and that it's equal to 0. Let's look at the right-hand limit by using the third "piece" of the piecewise-defined function, since it defines the function to the right of $x = 1/2$ (the domain of that piece is $x > 1/2$).

$$\lim_{x\to(1/2)^+} \frac{2x-1}{2}$$

$$\frac{2\left(\frac{1}{2}\right)-1}{2}$$

$$\frac{1-1}{2}$$

$$0$$

We know now that the right-hand limit exists, and that it's equal to 0.

Since the left-hand limit exists at $x = 1/2$, the right-hand limit exists at $x = 1/2$, and the left- and right-hand limits are equal and therefore the general limit exists at $x = 1/2$, but the general limit at $x = 1/2$ isn't equal to the value of the function at $x = 1/2$, we have a removable discontinuity and we need to redefine the function in order to make it continuous at $x = 1/2$.

So we just redefine the value of the function at $x = 1/2$ to be equal to the general limit at $x = 1/2$ that we found earlier by taking the left- and right-hand limits at $x = 1/2$.

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 0$. If it's discontinuous, identify the type of discontinuity and redefine the function to make it continuous.

$$f(x) = \begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x > 0 \\ -2 & x = 0 \\ \dfrac{x}{x^2+2x} & x < 0 \end{cases}$$

Answer choices:

A The function is continuous at $x = 0$.

B The function is discontinuous at $x = 0$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 0$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x > 0 \\ \dfrac{1}{2} & x = 0 \\ \dfrac{x}{x^2+2x} & x < 0 \end{cases}$$

D The function is discontinuous at $x = 0$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \frac{\sqrt{4x+4}-2}{2x} & x > 0 \\ 0 & x = 0 \\ \frac{x}{x^2+2x} & x < 0 \end{cases}$$

Solution: C

In order for the function to be continuous at $x = 0$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 0$.

We already know that the value of the function at $x = 0$ is -2, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to -2. If they exist but aren't equal to -2, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x = 0$.

To look at the left-hand limit, we'll use the third "piece" of the piecewise-defined function, because it defines the function to the left of $x = 0$ (the domain of that piece is $x < 0$).

$$\lim_{x \to 0^-} \frac{x}{x^2 + 2x}$$

$$\lim_{x \to 0^-} \frac{x}{x(x + 2)}$$

$$\lim_{x \to 0^-} \frac{1}{x + 2}$$

$$\frac{1}{0 + 2}$$

$$\frac{1}{2}$$

We know now that the left-hand limit exists, and that it's equal to $1/2$. Let's look at the right-hand limit by using the first "piece" of our piecewise-defined function, since it defines the function to the right of $x = 0$ (the domain of that piece is $x > 0$).

$$\lim_{x \to 0^+} \frac{\sqrt{4x + 4} - 2}{2x}$$

$$\lim_{x \to 0^+} \frac{\sqrt{4x + 4} - 2}{2x} \left(\frac{\sqrt{4x + 4} + 2}{\sqrt{4x + 4} + 2} \right)$$

$$\lim_{x \to 0^+} \frac{4x + 4 - 4}{2x(\sqrt{4x + 4} + 2)}$$

$$\lim_{x \to 0^+} \frac{2}{\sqrt{4x+4}+2}$$

$$\lim_{x \to 0^+} \frac{2}{\sqrt{4(x+1)}+2}$$

$$\lim_{x \to 0^+} \frac{1}{\sqrt{x+1}+1}$$

$$\frac{1}{\sqrt{0+1}+1}$$

$$\frac{1}{1+1}$$

$$\frac{1}{2}$$

We know now that the right-hand limit exists, and that it's equal to $1/2$.

Since the left-hand limit exists at $x = 0$, the right-hand limit exists at $x = 0$, and the left- and right-hand limits are equal and therefore the general limit exists at $x = 0$, but the general limit at $x = 0$ isn't equal to the value of the function at $x = 0$, we have a removable discontinuity and we need to redefine the function in order to make it continuous at $x = 0$.

So we just redefine the value of the function at $x = 0$ to be equal to the general limit at $x = 0$ that we found earlier by taking the left- and right-hand limits at $x = 0$.

$$f(x) = \begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x > 0 \\ \dfrac{1}{2} & x = 0 \\ \dfrac{x}{x^2+2x} & x < 0 \end{cases}$$

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 0$. If it's discontinuous, identify the type of discontinuity and solve for the value of k that makes it continuous.

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ k - \frac{1}{2} & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

Answer choices:

A The function is continuous at $x = 0$.

B The function is discontinuous at $x = 0$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 0$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ \frac{1}{2} & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

D The function is discontinuous at $x = 0$ and the discontinuity is removable. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ -\frac{1}{2} & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

Solution: B

In order for the function to be continuous at $x = 0$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 0$.

We already know that the value of the function at $x = 0$ is $k - 1/2$, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're equal to one another. If they are, then we'll set the value of the general limit equal to $k - 1/2$ to solve for k.

To look at the left-hand limit, we'll use the third "piece" of the piecewise-defined function, because it defines the function to the left of $x = 0$ (the domain of that piece is $x < 0$).

$$\lim_{x \to 0^-} \frac{4 - x}{x^2 - 2x - 8}$$

$$\lim_{x \to 0^-} \frac{4 - x}{(x - 4)(x + 2)}$$

$$\lim_{x \to 0^-} -\frac{x - 4}{(x - 4)(x + 2)}$$

We can cancel the $x - 4$ from the numerator and denominator to simplify the function. Keep in mind that this tells us we have a removable discontinuity at $x = 4$. That means that the function isn't continuous everywhere, but for the purposes of this problem, we really only care about continuity at $x = 0$, so we can move on.

$$\lim_{x \to 0^-} -\frac{1}{x + 2}$$

$$-\frac{1}{0 + 2}$$

$$-\frac{1}{2}$$

We know now that the left-hand limit exists, and that it's equal to $-1/2$. Let's look at the right-hand limit by using the first "piece" of the piecewise-defined function, since it defines the function to the right of $x = 0$ (the domain of that piece is $x > 0$). Substitute using a Pythagorean identity.

$$\lim_{x \to 0^+} \frac{\sqrt{\cos x} - 1}{\sin^2 x}$$

$$\lim_{x \to 0^+} \frac{\sqrt{\cos x} - 1}{1 - \cos^2 x}$$

Factor the denominator in order to simplify the fraction and then evaluate the limit.

$$\lim_{x \to 0^+} \frac{\sqrt{\cos x} - 1}{(1 + \cos x)(1 - \cos x)}$$

$$\lim_{x \to 0^+} \frac{\sqrt{\cos x} - 1}{(1 + \cos x)\left[(1 + \sqrt{\cos x})(1 - \sqrt{\cos x})\right]}$$

$$\lim_{x \to 0^+} -\frac{1 - \sqrt{\cos x}}{(1 + \cos x)(1 + \sqrt{\cos x})(1 - \sqrt{\cos x})}$$

$$\lim_{x \to 0^+} -\frac{1}{(1 + \cos x)(1 + \sqrt{\cos x})}$$

$$-\frac{1}{(1 + \cos(0))(1 + \sqrt{\cos(0)})}$$

$$-\frac{1}{(1 + 1)(1 + 1)}$$

$$-\frac{1}{4}$$

We know now that the right-hand limit exists, and that it’s equal to $-1/4$.

Since the left-hand limit exists at $x = 0$, the right-hand limit exists at $x = 0$, but the left- and right-hand limits are not equal to another, that means the general limit does not exist at $x = 0$. Furthermore, because the one-sided limits are unequal, it means the discontinuity is non-removable.

Topic: Squeeze theorem

Question: Use squeeze theorem to evaluate the limit.

$$\lim_{x \to 0} x^2 \cos x$$

Answer choices:

A ∞

B -1

C 0

D 1

Solution: C

We know the value of the cosine function oscillates back and forth between -1 and 1, so we'll start with

$$-1 \le \cos x \le 1$$

Multiply through the inequality by x^2 to get the function at the center of the inequality to match the one we were given.

$$-x^2 \le x^2 \cos x \le x^2$$

Apply the limit throughout the inequality.

$$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2 \cos x \le \lim_{x\to 0} x^2$$

$$-0^2 \le \lim_{x\to 0} x^2 \cos x \le 0^2$$

$$0 \le \lim_{x\to 0} x^2 \cos x \le 0$$

We've squeezed the limit we wanted to find in the inequality, so we can say the value of the limit is

$$\lim_{x\to 0} x^2 \cos x = 0$$

Topic: Squeeze theorem

Question: Use squeeze theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{\sin(6x)}{x^2}$$

Answer choices:

A -1

B 0

C ∞

D 1

Solution: B

We know the value of the sine function oscillates back and forth between -1 and 1, so we'll start with

$$-1 \leq \sin(6x) \leq 1$$

Divide through the inequality by x^2 to get the function at the center of the inequality to match the one we were given.

$$-\frac{1}{x^2} \leq \frac{\sin(6x)}{x^2} \leq \frac{1}{x^2}$$

Apply the limit throughout the inequality.

$$\lim_{x\to\infty} -\frac{1}{x^2} \leq \lim_{x\to\infty} \frac{\sin(6x)}{x^2} \leq \lim_{x\to\infty} \frac{1}{x^2}$$

$$-\frac{1}{\infty^2} \leq \lim_{x\to\infty} \frac{\sin(6x)}{x^2} \leq \frac{1}{\infty^2}$$

$$0 \leq \lim_{x\to\infty} \frac{\sin(6x)}{x^2} \leq 0$$

We've squeezed the limit we wanted to find in the inequality, so we can say the value of the limit is

$$\lim_{x\to\infty} \frac{\sin(6x)}{x^2} = 0$$

Topic: Squeeze theorem

Question: Use squeeze theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5}$$

Answer choices:

A $\frac{1}{3}$

B ∞

C 0

D 3

Solution: A

We know the value of the sine function oscillates back and forth between -1 and 1, so we'll start with

$$-1 \le \sin(4x) \le 1$$

Add $2x^3$ to each part of the inequality.

$$2x^3 - 1 \le 2x^3 + \sin(4x) \le 2x^3 + 1$$

Divide through the inequality by $6x^3 + 5$ to get the function at the center of the inequality to match the one we were given.

$$\frac{2x^3 - 1}{6x^3 + 5} \le \frac{2x^3 + \sin(4x)}{6x^3 + 5} \le \frac{2x^3 + 1}{6x^3 + 5}$$

Apply the limit throughout the inequality.

$$\lim_{x \to \infty} \frac{2x^3 - 1}{6x^3 + 5} \le \lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5} \le \lim_{x \to \infty} \frac{2x^3 + 1}{6x^3 + 5}$$

$$\frac{2}{6} \le \lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5} \le \frac{2}{6}$$

$$\frac{1}{3} \le \lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5} \le \frac{1}{3}$$

We've squeezed the limit we wanted to find in the inequality, so we can say the value of the limit is

$$\lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5} = \frac{1}{3}$$

Topic: Squeeze theorem, limit of an inequality

Question: If $x^2 \leq f(x) \leq x^3 - 6x$, find the limit of $f(x)$ as $x \to 3$.

Answer choices:

A $\lim_{x \to 3} f(x) = 9$

B $\lim_{x \to 3} f(x) = 3$

C $\lim_{x \to 3} f(x) = 0$

D $\lim_{x \to 3} f(x) =$ DNE

Solution: A

Apply the limit throughout the inequality.

$$x^2 \le f(x) \le x^3 - 6x$$

$$\lim_{x \to 3} x^2 \le \lim_{x \to 3} f(x) \le \lim_{x \to 3} x^3 - 6x$$

Use substitution to evaluate the limits.

$$3^2 \le \lim_{x \to 3} f(x) \le 3^3 - 6(3)$$

$$9 \le \lim_{x \to 3} f(x) \le 27 - 18$$

$$9 \le \lim_{x \to 3} f(x) \le 9$$

We've squeezed the limit we wanted to find in the inequality, so we can say the value of the limit is

$$\lim_{x \to 3} f(x) = 9$$

Topic: Squeeze theorem, limit of an inequality

Question: If $x^2 - x + 3 \le f(x) \le x^2 - 2x - 1$, find the limit of $f(x)$ as $x \to -4$.

Answer choices:

A $\lim_{x \to -4} f(x) = 15$

B $\lim_{x \to -4} f(x) = 23$

C $\lim_{x \to -4} f(x) = 7$

D $\lim_{x \to -4} f(x) = \text{DNE}$

Solution: B

Apply the limit throughout the inequality.

$$x^2 - x + 3 \le f(x) \le x^2 - 2x - 1$$

$$\lim_{x \to -4} x^2 - x + 3 \le \lim_{x \to -4} f(x) \le \lim_{x \to -4} x^2 - 2x - 1$$

Use substitution to evaluate the limits.

$$(-4)^2 - (-4) + 3 \le \lim_{x \to -4} f(x) \le (-4)^2 - 2(-4) - 1$$

$$16 + 4 + 3 \le \lim_{x \to -4} f(x) \le 16 + 8 - 1$$

$$23 \le \lim_{x \to -4} f(x) \le 23$$

We've squeezed the limit we wanted to find in the inequality, so we can say the value of the limit is

$$\lim_{x \to -4} f(x) = 23$$

Topic: Squeeze theorem, limit of an inequality

Question: If $2x^2 - 4x - 5 \le f(x) \le x^2 + 2x - 10$, find the limit of $f(x)$ as $x \to 5$.

Answer choices:

A $\lim_{x \to 5} f(x) = \text{DNE}$

B $\lim_{x \to 5} f(x) = 5$

C $\lim_{x \to 5} f(x) = 25$

D $\lim_{x \to 5} f(x) = 50$

Solution: C

Apply the limit throughout the inequality.

$$2x^2 - 4x - 5 \le f(x) \le x^2 + 2x - 10$$

$$\lim_{x\to 5} 2x^2 - 4x - 5 \le \lim_{x\to 5} f(x) \le \lim_{x\to 5} x^2 + 2x - 10$$

Use substitution to evaluate the limits.

$$2(5)^2 - 4(5) - 5 \le \lim_{x\to 5} f(x) \le 5^2 + 2(5) - 10$$

$$50 - 20 - 5 \le \lim_{x\to 5} f(x) \le 25 + 10 - 10$$

$$25 \le \lim_{x\to 5} f(x) \le 25$$

We've squeezed the limit we wanted to find in the inequality, so we can say the value of the limit is

$$\lim_{x\to 5} f(x) = 25$$

Topic: Definition of the derivative

Question: Use the definition of the derivative to find the derivative of the function.

$$f(x) = x^3 - 2x$$

Answer choices:

A $f'(x) = 2x^2 - 2$

B $f'(x) = 3x^2 - 2$

C $f'(x) = 3x^2 + 2$

D $f'(x) = x^2 - 1$

Solution: B

After replacing x with $(x+\Delta x)$ in $f(x)$,

$$f(x)=(x+\Delta x)^3-2(x+\Delta x)$$

we'll substitute for $f(c+\Delta x)$.

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-2(x+\Delta x)-f(c)}{\Delta x}$$

Then plug $f(x)$ into the definition for $f(c)$.

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-2(x+\Delta x)-(x^3-2x)}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{x^3+x^2\Delta x+2x^2\Delta x+2x\Delta x^2+x\Delta x^2+\Delta x^3-2x-2\Delta x-x^3+2x}{\Delta x}$$

Collect like terms,

$$f'(x)=\lim_{\Delta x\to 0}\frac{x^2\Delta x+2x^2\Delta x+2x\Delta x^2+x\Delta x^2+\Delta x^3-2x-2\Delta x+2x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{x^2\Delta x+2x^2\Delta x+2x\Delta x^2+x\Delta x^2+\Delta x^3-2\Delta x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{3x^2\Delta x+3x\Delta x^2+\Delta x^3-2\Delta x}{\Delta x}$$

then factor Δx out of the numerator and cancel out that common factor from the numerator and denominator.

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta x(3x^2+3x\Delta x+\Delta x^2-2)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 - 2$$

Now we evaluate the limit using substitution, which means we'll substitute $\Delta x = 0$.

$$f'(x) = 3x^2 + 3x(0) + 0^2 - 2$$

$$f'(x) = 3x^2 - 2$$

Topic: Definition of the derivative

Question: Use the definition of the derivative to find the derivative of the function.

$$f(x) = x^2$$

Answer choices:

A $\quad f'(x) = 0$

B $\quad f'(x) = 2$

C $\quad f'(x) = 2x$

D $\quad f'(x) = x^2 + 2x$

Solution: C

After replacing x with $(x + \Delta x)$ in $f(x)$,

$$f(x) = (x + \Delta x)^2$$

we'll substitute for $f(c + \Delta x)$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - f(c)}{\Delta x}$$

Then plug $f(x)$ into the definition for $f(c)$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x^2 + x\Delta x + x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

Collect like terms,

$$f'(x) = \lim_{\Delta x \to 0} \frac{x\Delta x + x\Delta x + \Delta x^2}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

then factor Δx out of the numerator and cancel out that common factor from the numerator and denominator.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} 2x + \Delta x$$

Now we evaluate the limit using substitution, which means we'll substitute $\Delta x = 0$.

$$f'(x) = 2x + 0$$

$$f'(x) = 2x$$

Topic: Definition of the derivative

Question: Use the definition of the derivative to find the derivative of the function.

$$f(x) = 2 - x^2 + x$$

Answer choices:

A $f'(x) = 2$

B $f'(x) = 2x$

C $f'(x) = -2x$

D $f'(x) = -2x + 1$

Solution: D

After replacing x with $(x+\Delta x)$ in $f(x)$,

$$f(x)=2-(x+\Delta x)^2+(x+\Delta x)$$

we'll substitute for $f(c+\Delta x)$.

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-(x+\Delta x)^2+(x+\Delta x)-f(c)}{\Delta x}$$

Then plug $f(x)$ into the definition for $f(c)$.

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-(x+\Delta x)^2+(x+\Delta x)-(2-x^2+x)}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-(x^2+x\Delta x+x\Delta x+\Delta x^2)+x+\Delta x-2+x^2-x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-x^2-x\Delta x-x\Delta x-\Delta x^2+x+\Delta x-2+x^2-x}{\Delta x}$$

Collect like terms,

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-x\Delta x-x\Delta x-\Delta x^2+x+\Delta x-2-x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{-x\Delta x-x\Delta x-\Delta x^2+x+\Delta x-x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{-x\Delta x-x\Delta x-\Delta x^2+\Delta x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{-2x\Delta x-\Delta x^2+\Delta x}{\Delta x}$$

then factor Δx out of the numerator and cancel out that common factor from the numerator and denominator.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x(-2x - \Delta x + 1)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} -2x - \Delta x + 1$$

Now we evaluate the limit using substitution, which means we'll substitute $\Delta x = 0$.

$$f'(x) = -2x - 0 + 1$$

$$f'(x) = -2x + 1$$

Topic: Power rule

Question: Find the derivative.

$$y = -3x^3$$

Answer choices:

A $\quad y' = -9x^4$

B $\quad y' = 9x^2$

C $\quad y' = -9x^2$

D $\quad y' = -6x^2$

Solution: C

Apply power rule to differentiate the equation.

$$y' = -3(3)x^{3-1}$$

$$y' = -9x^2$$

Topic: Power rule

Question: Find the derivative.

$$y = 5x^5 - 4x^2$$

Answer choices:

A $\quad y' = 5x^4 - 4x$

B $\quad y' = 25x^4 - 8x$

C $\quad y' = 25x^5 - 8x^2$

D $\quad y' = 25x^3 - 8$

Solution: B

Apply power rule to differentiate the equation, one term at a time.

$$y' = 5(5)x^{5-1} - 4(2)x^{2-1}$$

$$y' = 25x^4 - 8x^1$$

$$y' = 25x^4 - 8x$$

Topic: Power rule

Question: Find the derivative.

$$y = 3x^7 - 9x^2 + 21$$

Answer choices:

A $\quad y' = 21x^{-6} - 18x$

B $\quad y' = 3x(7x^5 - 6x)$

C $\quad y' = 21x^8 - 18x^2$

D $\quad y' = 21x^6 - 18x$

Solution: D

Apply power rule to differentiate the equation, one term at a time.

$$y' = 3(7)x^{7-1} - 9(2)x^{2-1} + 0$$

$$y' = 21x^6 - 18x^1$$

$$y' = 21x^6 - 18x$$

Topic: Power rule for negative powers

Question: Differentiate the function.

$$f(x) = x^{-2}$$

Answer choices:

A $f'(x) = -3x^{-2}$

B $f'(x) = -2x^{-3}$

C $f'(x) = -2x^{-2}$

D $f'(x) = -2x^{-1}$

Solution: B

Apply power rule to differentiate the function.

$$f'(x) = -2x^{-2-1}$$

$$f'(x) = -2x^{-3}$$

Topic: Power rule for negative powers

Question: Differentiate the function.

$$f(x) = 3x^{-5} - 4$$

Answer choices:

A $f'(x) = -12x^{-6}$

B $f'(x) = -15x^{-4}$

C $f'(x) = -5x^{-6}$

D $f'(x) = -15x^{-6}$

Solution: D

Apply power rule to differentiate the equation, one term at a time.

$$f'(x) = 3(-5)x^{-5-1} - 0$$

$$f'(x) = -15x^{-6}$$

Topic: Power rule for negative powers

Question: Differentiate the function.

$$f(x) = \frac{4}{x^2} - \frac{6}{x} + 2$$

Answer choices:

A $f'(x) = -8x^{-3} + 6x^{-2}$

B $f'(x) = -8x^{-4} + 6x^{-3}$

C $f'(x) = -2x^{-3} + x^{-2}$

D $f'(x) = -8x^{-2} + 6x^{-1}$

Solution: A

First, rewrite the fractions by changing the signs of the exponents.

$$f(x) = 4x^{-2} - 6x^{-1} + 2$$

Apply power rule to differentiate the equation, one term at a time.

$$f'(x) = 4(-2)x^{-2-1} - 6(-1)x^{-1-1} + 0$$

$$f'(x) = -8x^{-3} + 6x^{-2}$$

Topic: Power rule for fractional powers

Question: Differentiate the function.

$$f(x) = \sqrt{x}$$

Answer choices:

A $\quad f'(x) = -\frac{1}{2}\sqrt{x}$

B $\quad f'(x) = -\frac{1}{2\sqrt{x}}$

C $\quad f'(x) = \frac{1}{2\sqrt{x}}$

D $\quad f'(x) = \frac{1}{2}\sqrt{x}$

Solution: C

Rewrite the function by converting the radical into a fractional exponent.

$$f(x) = x^{\frac{1}{2}}$$

Apply power rule to differentiate the function.

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

Because the original function was given in terms of a root, rewrite this answer with a root instead of a fractional exponent.

$$f'(x) = \frac{1}{2x^{\frac{1}{2}}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Topic: Power rule for fractional powers

Question: Differentiate the function.

$$f(x) = \frac{4}{\sqrt{x}}$$

Answer choices:

A $\quad f'(x) = -\frac{2}{\sqrt{x^3}}$

B $\quad f'(x) = -\frac{2}{\sqrt{x}}$

C $\quad f'(x) = -2\sqrt{x^3}$

D $\quad f'(x) = -2\sqrt{x}$

Solution: A

Rewrite the function by converting the radical into a fractional exponent.

$$f(x) = \frac{4}{x^{\frac{1}{2}}}$$

$$f(x) = 4x^{-\frac{1}{2}}$$

Apply power rule to differentiate the function.

$$f'(x) = 4\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$

$$f'(x) = -\frac{4}{2}x^{-\frac{1}{2}-\frac{2}{2}}$$

$$f'(x) = -2x^{-\frac{3}{2}}$$

Because the original function was given in terms of a root, rewrite this answer with a root instead of a fractional exponent.

$$f'(x) = -\frac{2}{x^{\frac{3}{2}}}$$

$$f'(x) = -\frac{2}{\sqrt{x^3}}$$

Topic: Power rule for fractional powers

Question: Differentiate the function.

$$f(x) = \frac{5}{\sqrt[3]{x^2}}$$

Answer choices:

A $f'(x) = -\frac{10}{3}\sqrt[3]{x^5}$

B $f'(x) = -\frac{10}{3\sqrt[3]{x^5}}$

C $f'(x) = -\frac{10}{3\sqrt[3]{x}}$

D $f'(x) = -\frac{10}{3}\sqrt[3]{x}$

Solution: B

Rewrite the function by converting the radical into a fractional exponent.

$$f(x) = \frac{5}{x^{\frac{2}{3}}}$$

$$f(x) = 5x^{-\frac{2}{3}}$$

Apply power rule to differentiate the function.

$$f'(x) = 5\left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1}$$

$$f'(x) = -\frac{10}{3}x^{-\frac{2}{3}-\frac{3}{3}}$$

$$f'(x) = -\frac{10}{3}x^{-\frac{5}{3}}$$

Because the original function was given in terms of a root, rewrite this answer with a root instead of a fractional exponent.

$$f'(x) = -\frac{10}{3x^{\frac{5}{3}}}$$

$$f'(x) = -\frac{10}{3\sqrt[3]{x^5}}$$

Topic: Product rule, two functions

Question: Find the derivative.

$$y = (x^2 + 2)(x^3 + 1)$$

Answer choices:

A $\quad y' = 5x^3 + 6x + 2$

B $\quad y' = x^4 + 12x^2 + 2x$

C $\quad y' = 5x^4 + 6x^2 + 2x$

D $\quad y' = 5x^4 - 6x^2 + 2x$

Solution: C

Let $f(x) = x^2 + 2$ and $g(x) = x^3 + 1$, and then apply product rule.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (x^2 + 2)(3x^2) + (2x)(x^3 + 1)$$

Expand the derivative, then collect like terms.

$$y' = 3x^2(x^2) + 3x^2(2) + 2x(x^3) + 2x(1)$$

$$y' = 3x^4 + 6x^2 + 2x^4 + 2x$$

$$y' = 5x^4 + 6x^2 + 2x$$

Topic: Product rule, two functions

Question: Find the derivative.

$$y = (3x^2 + 2x)(x^4 - 3x + 1)$$

Answer choices:

A $\quad y' = 18x^4 + 10x^3 - 27x^2 - 6x + 2$

B $\quad y' = 10x^5 + 27x^4 - 27x^2 - 6x + 2$

C $\quad y' = 18x^5 + 10x^4 + 27x^2 - 6x + 2$

D $\quad y' = 18x^5 + 10x^4 - 27x^2 - 6x + 2$

Solution: D

Let $f(x) = 3x^2 + 2x$ and $g(x) = x^4 - 3x + 1$, and then apply product rule.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (3x^2 + 2x)(4x^3 - 3) + (6x + 2)(x^4 - 3x + 1)$$

Expand the derivative, then collect like terms.

$$y' = 3x^2(4x^3 - 3) + 2x(4x^3 - 3) + 6x(x^4 - 3x + 1) + 2(x^4 - 3x + 1)$$

$$y' = 3x^2(4x^3) - 3x^2(3) + 2x(4x^3) - 2x(3)$$

$$+6x(x^4) - 6x(3x) + 6x(1) + 2(x^4) - 2(3x) + 2(1)$$

$$y' = 12x^5 - 9x^2 + 8x^4 - 6x + 6x^5 - 18x^2 + 6x + 2x^4 - 6x + 2$$

$$y' = 12x^5 + 6x^5 + 8x^4 + 2x^4 - 9x^2 - 18x^2 - 6x + 6x - 6x + 2$$

$$y' = 18x^5 + 10x^4 - 27x^2 - 6x + 2$$

Topic: Product rule, two functions

Question: Find the derivative.

$$h(x) = (3x^2 - 7)(x^2 - 4x + 3)$$

Answer choices:

A $h'(x) = 6x^3 - 36x^2 + 4x + 28$

B $h'(x) = 12x^3 - 36x^2 + 32x + 28$

C $h'(x) = 12x^3 - 12x^2 + 4x + 28$

D $h'(x) = 12x^3 - 36x^2 + 4x + 28$

Solution: D

Let $f(x) = 3x^2 - 7$ and $g(x) = x^2 - 4x + 3$, and then apply product rule.

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(x) = (3x^2 - 7)(2x - 4) + (6x)(x^2 - 4x + 3)$$

Expand the derivative, then collect like terms.

$$h'(x) = (3x^2 - 7)(2x) - (3x^2 - 7)(4) + (6x)(x^2) - (6x)(4x) + (6x)(3)$$

$$h'(x) = 2x(3x^2) - 2x(7) - 4(3x^2) - 4(-7) + (6x)(x^2) - (6x)(4x) + (6x)(3)$$

$$h'(x) = 6x^3 - 14x - 12x^2 + 28 + 6x^3 - 24x^2 + 18x$$

$$h'(x) = 6x^3 + 6x^3 - 12x^2 - 24x^2 - 14x + 18x + 28$$

$$h'(x) = 12x^3 - 36x^2 + 4x + 28$$

Topic: Product rule, three or more functions

Question: Use the product rule to find the derivative of the function.

$$f(x) = (3x^2)(x)(-2x^4)$$

Answer choices:

A $f'(x) = 6x^7$

B $f'(x) = 42x^6$

C $f'(x) = -6x^7$

D $f'(x) = -42x^6$

Solution: D

Let $r(x) = 3x^2$, $s(x) = x$, and $t(x) = -2x^4$. Find each of their derivatives.

$$r(x) = 3x^2$$

$$r'(x) = 6x$$

and

$$s(x) = x$$

$$s'(x) = 1$$

and

$$t(x) = -2x^4$$

$$t'(x) = -8x^3$$

Apply product rule.

$$f'(x) = r'(x)s(x)t(x) + r(x)s'(x)t(x) + r(x)s(x)t'(x)$$

$$f'(x) = (6x)(x)(-2x^4) + (3x^2)(1)(-2x^4) + (3x^2)(x)(-8x^3)$$

Expand the derivative, then collect like terms.

$$f'(x) = (6x^2)(-2x^4) + (3x^2)(-2x^4) + (3x^3)(-8x^3)$$

$$f'(x) = -12x^6 - 6x^6 - 24x^6$$

$$f'(x) = -42x^6$$

Topic: Product rule, three or more functions

Question: Use the product rule to find the derivative of the function.

$$f(x) = (x+3)(2x^2-5)(-x^3+2)$$

Answer choices:

A $\quad f'(x) = -12x^3$

B $\quad f'(x) = -12x^5 - 30x^4 + 20x^3 + 57x^2 + 24x - 10$

C $\quad f'(x) = -22x^4 - 36x^3 + 15x^2 + 8x$

D $\quad f'(x) = -2x^5 + 2x^3 - x^2 + 12x - 10$

Solution: B

Let $r(x) = x + 3$, $s(x) = 2x^2 - 5$, and $t(x) = -x^3 + 2$. Find each of their derivatives.

$$r(x) = x + 3$$

$$r'(x) = 1$$

and

$$s(x) = 2x^2 - 5$$

$$s'(x) = 4x$$

and

$$t(x) = -x^3 + 2$$

$$t'(x) = -3x^2$$

Apply product rule.

$$f'(x) = r'(x)s(x)t(x) + r(x)s'(x)t(x) + r(x)s(x)t'(x)$$

$$f'(x) = (1)(2x^2 - 5)(-x^3 + 2) + (x + 3)(4x)(-x^3 + 2) + (x + 3)(2x^2 - 5)(-3x^2)$$

Expand the derivative, then collect like terms.

$$f'(x) = -2x^5 + 4x^2 + 5x^3 - 10$$

$$+(4x^2 + 12x)(-x^3 + 2) + (2x^3 - 5x + 6x^2 - 15)(-3x^2)$$

$$f'(x) = -2x^5 + 5x^3 + 4x^2 - 10$$

$$+(-4x^5+8x^2-12x^4+24x)+(-6x^5+15x^3-18x^4+45x^2)$$

$$f'(x)=-2x^5+5x^3+4x^2-10-4x^5+8x^2-12x^4+24x-6x^5+15x^3-18x^4+45x^2$$

$$f'(x)=-2x^5-4x^5-6x^5-12x^4-18x^4+5x^3+15x^3+4x^2+8x^2+45x^2+24x-10$$

$$f'(x)=-12x^5-30x^4+20x^3+57x^2+24x-10$$

Topic: Product rule, three or more functions

Question: Use the product rule to find the derivative of the function.

$$y = (3x^5 - 4)^3$$

Answer choices:

A $y' = 45x^4(3x^5 - 4)^2$

B $y' = (15x^4)^3$

C $y' = 3(15x^4)^2(3x^5 - 4)$

D $y' = 3(15x^4)^3$

Solution: A

Let $r(x) = 3x^5 - 4$, $s(x) = 3x^5 - 4$, and $t(x) = 3x^5 - 4$. Find each of their derivatives.

$$r(x) = 3x^5 - 4$$

$$r'(x) = 15x^4$$

and

$$s(x) = 3x^5 - 4$$

$$s'(x) = 15x^4$$

and

$$t(x) = 3x^5 - 4$$

$$t'(x) = 15x^4$$

Apply product rule.

$$f'(x) = (15x^4)(3x^5 - 4)(3x^5 - 4) + (3x^5 - 4)(15x^4)(3x^5 - 4) + (3x^5 - 4)(3x^5 - 4)(15x^4)$$

$$f'(x) = 3(15x^4)(3x^5 - 4)(3x^5 - 4)$$

$$f'(x) = 45x^4(3x^5 - 4)(3x^5 - 4)$$

$$f'(x) = 45x^4(3x^5 - 4)^2$$

Topic: Quotient rule

Question: Find the derivative.

$$y = \frac{2x^2 - 1}{3x + 5}$$

Answer choices:

A $\quad y' = \dfrac{6x^2 - 20x + 3}{(3x + 5)^2}$

B $\quad y' = \dfrac{6x^2 + 20x + 3}{3x + 5}$

C $\quad y' = \dfrac{6x^2 + 20x + 3}{(3x + 5)^2}$

D $\quad y' = \dfrac{6x^2 + 10x + 3}{(3x + 5)^2}$

Solution: C

Apply the quotient rule to find the derivative.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(4x)(3x+5) - (2x^2-1)(3)}{(3x+5)^2}$$

Simplify the derivative.

$$y' = \frac{(12x^2+20x) - (6x^2-3)}{(3x+5)^2}$$

$$y' = \frac{12x^2+20x-6x^2+3}{(3x+5)^2}$$

$$y' = \frac{6x^2+20x+3}{(3x+5)^2}$$

Topic: Quotient rule

Question: Find the derivative.

$$y = \frac{x^2 - x + 1}{x^2 + 1}$$

Answer choices:

A $\quad y' = \dfrac{x^2 - 1}{(x^2 + 1)^2}$

B $\quad y' = \dfrac{x - 1}{(x^2 + 1)^2}$

C $\quad y' = \dfrac{x^2 - 1}{x^2 + 1}$

D $\quad y' = \dfrac{x^2}{(x^2 + 1)^2}$

Solution: A

Apply the quotient rule to find the derivative.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(2x-1)(x^2+1) - (x^2-x+1)(2x)}{(x^2+1)^2}$$

Simplify the derivative.

$$y' = \frac{(2x^3+2x-x^2-1) - (2x^3-2x^2+2x)}{(x^2+1)^2}$$

$$y' = \frac{2x^3+2x-x^2-1-2x^3+2x^2-2x}{(x^2+1)^2}$$

$$y' = \frac{x^2-1}{(x^2+1)^2}$$

Topic: Quotient rule

Question: Find the derivative.

$$y = \frac{1}{3x^2 + 1}$$

Answer choices:

A $\quad y' = -\frac{6x}{(3x^2 + 1)^2}$

B $\quad y' = \frac{6x}{(3x^2 + 1)^2}$

C $\quad y' = -\frac{6x}{3x^2 + 1}$

D $\quad y' = -\frac{6}{(3x^2 + 1)^2}$

Solution: A

Apply the quotient rule to find the derivative.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(0)(3x^2 + 1) - (1)(6x)}{(3x^2 + 1)^2}$$

Simplify the derivative.

$$y' = \frac{-6x}{(3x^2 + 1)^2}$$

$$y' = -\frac{6x}{(3x^2 + 1)^2}$$

Topic: Chain rule with power rule

Question: Apply power rule and chain rule to find the derivative.

$$y = (5x^2 + 2x - 8)^5$$

Answer choices:

A $\quad y' = (5x + 2)(5x^2 + 2x - 8)^4$

B $\quad y' = (50x + 10)(5x^2 + 2x - 8)^5$

C $\quad y' = (50x + 10)(5x^2 + 2x - 8)^4$

D $\quad y' = (50x - 10)(5x^2 + 2x - 8)^4$

Solution: C

Use substitution with $u = 5x^2 + 2x - 8$ and $u' = 10x + 2$, and rewrite the function with the substitution.

$$y = u^5$$

Then the derivative is

$$y' = 5u^4u'$$

Back-substitute.

$$y' = 5(5x^2 + 2x - 8)^4(10x + 2)$$

$$y' = (50x + 10)(5x^2 + 2x - 8)^4$$

Topic: Chain rule with power rule

Question: Apply power rule and chain rule to find the derivative.

$$f(x) = 8(6x^2 + 2)^4$$

Answer choices:

A $\quad f'(x) = 384x(6x^2 + 2)^3$

B $\quad f'(x) = 384(6x^2 + 2)^3$

C $\quad f'(x) = 32x(6x^2 + 2)^3$

D $\quad f'(x) = 32(6x^2 + 2)^3$

Solution: A

Use substitution with $u = 6x^2 + 2$ and $u' = 12x$, and rewrite the function with the substitution.

$$f(x) = 8u^4$$

Then the derivative is

$$f'(x) = 32u^3u'$$

Back-substitute.

$$f'(x) = 32(6x^2 + 2)^3(12x)$$

$$f'(x) = 384x(6x^2 + 2)^3$$

Topic: Chain rule with power rule

Question: Apply power rule and chain rule to find the derivative.

$$f(y) = (y^3 + 1)^{25}$$

Answer choices:

A $f'(y) = (3y^2)^{25}$

B $f'(y) = 25(y^3 + 1)^{24}$

C $f'(y) = 75y^2(y^3 + 1)^{24}$

D $f'(y) = 25(3y^2)^{24}$

Solution: C

Use substitution with $u = y^3 + 1$ and $u' = 3y^2$, and rewrite the function with the substitution.

$$f(y) = u^{25}$$

Then the derivative is

$$f'(y) = 25u^{24}u'$$

Back-substitute.

$$f'(y) = 25(y^3 + 1)^{24}(3y^2)$$

$$f'(y) = 75y^2(y^3 + 1)^{24}$$

Topic: Chain rule with product rule

Question: Apply product rule and chain rule to find the derivative.

$$y = (4x-7)^2(2x+3)$$

Answer choices:

A $\quad y' = (4x-7)(12x+5)$

B $\quad y' = 2(4x-7)(2x+3)$

C $\quad y' = 2(4x-7)(12x+5)$

D $\quad y' = 2(4x-7)^3(12x+5)$

Solution: C

Set $f(x) = (4x - 7)^2$ and $g(x) = 2x + 3$. Then

$$f(x) = (4x - 7)^2$$

$$f'(x) = 8(4x - 7)$$

and

$$g(x) = 2x + 3$$

$$g'(x) = 2$$

Now we can apply product rule.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = ((4x - 7)^2)(2) + (8(4x - 7))(2x + 3)$$

The two terms $2(4x - 7)^2$ and $8(4x - 7)(2x + 3)$ share a common factor of $2(4x - 7)$, so factor that out.

$$y' = 2(4x - 7)\left[(4x - 7) + 4(2x + 3)\right]$$

$$y' = 2(4x - 7)(4x - 7 + 8x + 12)$$

$$y' = 2(4x - 7)(12x + 5)$$

Topic: Chain rule with product rule

Question: Apply product rule and chain rule to find the derivative.

$$y = 2x^2(-5x^2)^3$$

Answer choices:

A $y' = -200x^7$

B $y' = -200x^8$

C $y' = -2{,}000x^7$

D $y' = -2{,}000x^8$

Solution: C

Set $f(x)=2x^2$ and $g(x)=(-5x^2)^3$. Then

$$f(x)=2x^2$$

$$f'(x)=4x$$

and

$$g(x)=(-5x^2)^3$$

$$g'(x)=-30x(-5x^2)^2$$

Now we can apply product rule.

$$y'=f(x)g'(x)+f'(x)g(x)$$

$$y'=(2x^2)(-30x(-5x^2)^2)+(4x)((-5x^2)^3)$$

$$y'=-60x^3(-5x^2)^2+4x(-5x^2)^3$$

The two terms $-60x^3(-5x^2)^2$ and $4x(-5x^2)^3$ share a common factor of $4x(-5x^2)^2$, so factor that out.

$$y'=4x(-5x^2)^2\left[-15x^2+(-5x^2)\right]$$

$$y'=4x(-5x^2)^2(-15x^2-5x^2)$$

$$y'=4x(-5x^2)^2(-20x^2)$$

$$y'=4x(25x^4)(-20x^2)$$

$$y'=-2{,}000x^7$$

Topic: Chain rule with product rule

Question: Apply product rule and chain rule to find the derivative.

$$y = (9x)(2x^3)(-3x^2)$$

Answer choices:

A $y' = -92x^4$

B $y' = -92x^5$

C $y' = -324x^4$

D $y' = -324x^5$

Solution: D

Set $f(x) = 9x$, $g(x) = 2x^3$, and $h(x) = -3x^2$. Then

$$f(x) = 9x$$

$$f'(x) = 9$$

and

$$g(x) = 2x^3$$

$$g'(x) = 6x^2$$

and

$$h(x) = -3x^2$$

$$h'(x) = -6x$$

Now we can apply product rule.

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$y' = (9)(2x^3)(-3x^2) + (9x)(6x^2)(-3x^2) + (9x)(2x^3)(-6x)$$

Simplify the derivative.

$$y' = 18x^3(-3x^2) + 54x^3(-3x^2) + 18x^4(-6x)$$

$$y' = -54x^5 - 162x^5 - 108x^5$$

$$y' = -324x^5$$

Topic: Chain rule with quotient rule

Question: Apply quotient rule and chain rule to find the derivative.

$$y = \frac{(2x^2 + 1)^3}{x}$$

Answer choices:

A $\quad y' = 12x(2x^2 + 1)^2$

B $\quad y' = \dfrac{(2x^2 + 1)^2(10x^2 - 1)}{x^2}$

C $\quad y' = \dfrac{1}{12x(2x^2 + 1)^2}$

D $\quad y' = \dfrac{1 - 10x^2}{(2x^2 + 1)^4}$

Solution: B

List out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = (2x^2 + 1)^3$$

$$f'(x) = 12x(2x^2 + 1)^2$$

and

$$g(x) = x$$

$$g'(x) = 1$$

Now we can plug these values directly into the quotient rule formula.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(12x(2x^2 + 1)^2)(x) - ((2x^2 + 1)^3)(1)}{x^2}$$

$$y' = \frac{12x^2(2x^2 + 1)^2 - (2x^2 + 1)^3}{x^2}$$

Within the numerator, we have a common factor of $(2x^2 + 1)^2$, so factor that out.

$$y' = \frac{(2x^2 + 1)^2(12x^2 - (2x^2 + 1))}{x^2}$$

$$y' = \frac{(2x^2 + 1)^2(12x^2 - 2x^2 - 1)}{x^2}$$

$$y' = \frac{(2x^2+1)^2(10x^2-1)}{x^2}$$

Topic: Chain rule with quotient rule

Question: Apply quotient rule and chain rule to find the derivative.

$$y = \frac{4x}{(x^2 - 1)^3}$$

Answer choices:

A $\quad y' = -\frac{4x}{(x - 1)^2}$

B $\quad y' = -\frac{4x}{(x^2 - 1)^4}$

C $\quad y' = -\frac{4(5x^2 + 1)}{(x^2 - 1)^4}$

D $\quad y' = -\frac{8x}{(x - 1)^2}$

Solution: C

List out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = 4x$$

$$f'(x) = 4$$

and

$$g(x) = (x^2 - 1)^3$$

$$g'(x) = 6x(x^2 - 1)^2$$

Now we can plug these values directly into the quotient rule formula.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(4)((x^2 - 1)^3) - (4x)(6x(x^2 - 1)^2)}{[(x^2 - 1)^3]^2}$$

$$y' = \frac{4(x^2 - 1)^3 - 24x^2(x^2 - 1)^2}{(x^2 - 1)^6}$$

Within the fraction, we have a common factor of $(x^2 - 1)^2$, so cancel that out.

$$y' = \frac{4(x^2 - 1) - 24x^2}{(x^2 - 1)^4}$$

$$y' = \frac{4x^2 - 4 - 24x^2}{(x^2 - 1)^4}$$

$$y' = \frac{-4 - 20x^2}{(x^2 - 1)^4}$$

$$y' = -\frac{4(5x^2 + 1)}{(x^2 - 1)^4}$$

Topic: Chain rule with quotient rule

Question: Apply quotient rule and chain rule to find the derivative.

$$y = \frac{(3x^2+4)^2}{(4-2x)^4}$$

Answer choices:

A $\quad y' = -\frac{16(3x^2+4)}{(4-2x)^4}$

B $\quad y' = -\frac{16(3x^2+4)(3x+2)}{(4-2x)^5}$

C $\quad y' = \frac{16(3x^2+4)}{(4-2x)^4}$

D $\quad y' = \frac{16(3x^2+4)(3x+2)}{(4-2x)^5}$

Solution: D

List out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = (3x^2 + 4)^2$$

$$f'(x) = 12x(3x^2 + 4)$$

and

$$g(x) = (4 - 2x)^4$$

$$g'(x) = -8(4 - 2x)^3$$

Now we can plug these values directly into the quotient rule formula.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(12x(3x^2 + 4))((4 - 2x)^4) - ((3x^2 + 4)^2)(-8(4 - 2x)^3)}{[(4 - 2x)^4]^2}$$

$$y' = \frac{12x(3x^2 + 4)(4 - 2x)^4 + 8(3x^2 + 4)^2(4 - 2x)^3}{(4 - 2x)^8}$$

Within the fraction, we have a common factor of $(4 - 2x)^3$, so cancel that out.

$$y' = \frac{12x(3x^2 + 4)(4 - 2x) + 8(3x^2 + 4)^2}{(4 - 2x)^5}$$

Within the numerator, we have a common factor of $4(3x^2 + 4)$, so factor that out.

$$y' = \frac{4(3x^2+4)\left[3x(4-2x)+2(3x^2+4)\right]}{(4-2x)^5}$$

$$y' = \frac{4(3x^2+4)\left[(12x-6x^2)+(6x^2+8)\right]}{(4-2x)^5}$$

$$y' = \frac{4(3x^2+4)(12x-6x^2+6x^2+8)}{(4-2x)^5}$$

$$y' = \frac{4(3x^2+4)(12x+8)}{(4-2x)^5}$$

Factor out another 4 from the $12x+8$.

$$y' = \frac{16(3x^2+4)(3x+2)}{(4-2x)^5}$$

Topic: Trigonometric derivatives

Question: Find the derivative of the trigonometric function.

$$y = \sin(3x^2 + 11x)$$

Answer choices:

A $\quad y' = -(6x + 11)\cos(3x^2 + 11x)$

B $\quad y' = -(6x + 11)\sin(3x^2 + 11x)$

C $\quad y' = (6x + 11)\cos(3x^2 + 11x)$

D $\quad y' = (6x + 11)\sin(3x^2 + 11x)$

Solution: C

Set $u = 3x^2 + 11x$ and $u' = 6x + 11$. Then $y = \sin u$, and the derivative is

$$y' = \cos u \cdot u'$$

$$y' = \cos(3x^2 + 11x) \cdot (6x + 11)$$

$$y' = (6x + 11)\cos(3x^2 + 11x)$$

Topic: Trigonometric derivatives

Question: Find the derivative of the trigonometric function.

$$y = 2\sin x\csc(2x)$$

Answer choices:

A $y' = -4\sin x\csc(2x)\cot(2x) + 2\cos x\csc(2x)$

B $y' = -2\sin x\csc(2x)\cot(2x) + \cos x\csc(2x)$

C $y' = \csc(2x)\cot(2x) + \cos x\csc(2x)$

D $y' = \csc(2x)\cot(2x)$

Solution: A

Use the product rule with

$$f(x) = 2\sin x$$

$$f'(x) = 2\cos x$$

and

$$g(x) = \csc(2x)$$

$$g'(x) = -2\csc(2x)\cot(2x)$$

Then the derivative is

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (2\sin x)(-2\csc(2x)\cot(2x)) + (2\cos x)(\csc(2x))$$

$$y' = -4\sin x\csc(2x)\cot(2x) + 2\cos x\csc(2x)$$

Topic: Trigonometric derivatives

Question: Find the derivative of the trigonometric function.

$$y = \cot^5(7x)$$

Answer choices:

A $\quad y' = -35\csc^2(7x)\cot^4(7x)$

B $\quad y' = -35\csc^2(7x)\cot^2(7x)$

C $\quad y' = -35\csc^4(7x)\cot^2(7x)$

D $\quad y' = -35\csc^4(7x)\cot^4(7x)$

Solution: A

Rewrite the trigonometric function.

$$y = (\cot(7x))^5$$

Set $u = \cot(7x)$ and $u' = -7\csc^2(7x)$. Then $y = u^5$, and the derivative is

$$y' = 5u^4u'$$

$$y' = 5(\cot(7x))^4(-7\csc^2(7x))$$

$$y' = -35\csc^2(7x)\cot^4(7x)$$

Topic: Inverse trigonometric derivatives

Question: Find the derivative of the inverse trig function.

$$f(x) = \tan^{-1}(x^2 - 1)$$

Answer choices:

A $\quad f'(x) = \dfrac{2x}{x^4 - 2x^2 + 2}$

B $\quad f'(x) = \dfrac{1}{x^4 - 2x^2 + 2}$

C $\quad f'(x) = \dfrac{2x}{1 + 4x^2}$

D $\quad f'(x) = \dfrac{1}{1 + 4x^2}$

Solution: A

Apply the formula for the derivative of inverse tangent, with $g(x) = x^2 - 1$, in order to differentiate the function.

$$y' = \frac{g'(x)}{1 + [g(x)]^2}$$

$$y' = \frac{2x}{1 + (x^2 - 1)^2}$$

Simplify the derivative.

$$y' = \frac{2x}{1 + (x^4 - 2x^2 + 1)}$$

$$y' = \frac{2x}{x^4 - 2x^2 + 2}$$

Topic: Inverse trigonometric derivatives

Question: Find the derivative of the inverse trig function.

$$y = \frac{1}{\cos^{-1} x}$$

Answer choices:

A $\quad y' = \dfrac{1}{1 - x^2}$

B $\quad y' = \dfrac{1}{(\cos^{-1} x)^2 \sqrt{1 - x^2}}$

C $\quad y' = -\sqrt{1 - x^2}$

D $\quad y' = \dfrac{\sqrt{1 - x^2}}{(\cos^{-1} x)^2}$

Solution: B

Rewrite the function.

$$y = \frac{1}{\arccos x}$$

$$y = (\arccos x)^{-1}$$

Use substitution with $u = \arccos x$ and

$$u' = -\frac{1}{\sqrt{1 - x^2}}$$

Then the function is

$$y = u^{-1}$$

The derivative is

$$y' = -u^{-2}u'$$

$$y' = -(\arccos x)^{-2}\left(-\frac{1}{\sqrt{1 - x^2}}\right)$$

$$y' = \frac{1}{(\arccos x)^2}\left(\frac{1}{\sqrt{1 - x^2}}\right)$$

$$y' = \frac{1}{(\cos^{-1} x)^2\sqrt{1 - x^2}}$$

Topic: Inverse trigonometric derivatives

Question: Find the derivative of the inverse trig function.

$$y = x \sin^{-1} \sqrt{x}$$

Answer choices:

A $\quad y' = x \sin^{-1} \sqrt{x} + \dfrac{1}{\sqrt{1 - x}}$

B $\quad y' = \sin^{-1} \sqrt{x} + \dfrac{x}{\sqrt{1 - x}}$

C $\quad y' = x \sin^{-1} \sqrt{x} + \dfrac{1}{2\sqrt{x(1 - x)}}$

D $\quad y' = \dfrac{\sqrt{x}}{2\sqrt{1 - x}} + \sin^{-1} \sqrt{x}$

Solution: D

We need to apply product rule, with

$$f(x) = x$$

$$f'(x) = 1$$

and

$$g(x) = \sin^{-1}\sqrt{x}$$

$$g'(x) = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

Take the derivative using product rule.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (x)\left[\frac{1}{\sqrt{1-(\sqrt{x})^2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\right] + (1)(\sin^{-1}\sqrt{x})$$

$$y' = \frac{1}{2}x\left(\frac{1}{x^{\frac{1}{2}}\sqrt{1-x}}\right) + \sin^{-1}\sqrt{x}$$

$$y' = \frac{x}{2\sqrt{x}\sqrt{1-x}} + \sin^{-1}\sqrt{x}$$

$$y' = \frac{\sqrt{x}\sqrt{x}}{2\sqrt{x}\sqrt{1-x}} + \sin^{-1}\sqrt{x}$$

$$y' = \frac{\sqrt{x}}{2\sqrt{1-x}} + \sin^{-1}\sqrt{x}$$

Topic: Hyperbolic derivatives

Question: Find the derivative of the hyperbolic function.

$$y = \cosh(x^2 - 1)$$

Answer choices:

A $y' = -2x\sinh(x^2 - 1)$

B $y' = -2x\cosh(x^2 - 1)$

C $y' = 2x\sinh(x^2 - 1)$

D $y' = 2x\cosh(x^2 - 1)$

Solution: C

Make a substitution, letting $u = x^2 - 1$ and $u' = 2x$. Then $y = \cosh u$, and the derivative is

$$y' = \sinh u \cdot u'$$

$$y' = \sinh(x^2 - 1) \cdot (2x)$$

$$y' = 2x \sinh(x^2 - 1)$$

Topic: Hyperbolic derivatives

Question: Find the derivative of the hyperbolic function.

$$y = x^2 \sinh x$$

Answer choices:

A $y' = x \cosh x + 2 \sinh x$

B $y' = x^2 \cosh x + 2x \sinh x$

C $y' = x \cosh x - 2 \sinh x$

D $y' = x^2 \cosh x - 2x \sinh x$

Solution: B

Use the product rule with

$$f(x) = x^2$$

$$f'(x) = 2x$$

and

$$g(x) = \sinh x$$

$$g'(x) = \cosh x$$

Then the derivative is

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (x^2)(\cosh x) + (2x)(\sinh x)$$

$$y' = x^2 \cosh x + 2x \sinh x$$

Topic: Hyperbolic derivatives

Question: Find the derivative of the hyperbolic function.

$$f(x) = \sinh(\cosh(2x))$$

Answer choices:

A $f'(x) = 2\sinh(2x)\sinh(\sinh(2x))$

B $f'(x) = 2\sinh(2x)\cosh(\sinh(2x))$

C $f'(x) = 2\sinh(2x)\cosh(\cosh(2x))$

D $f'(x) = 2\cosh(2x)\cosh(\cosh(2x))$

Solution: C

Use a substitution with $u = \cosh(2x)$ and $u' = 2\sinh(2x)$. Then the function can be rewritten as

$$f(x) = \sinh u$$

and the derivative is

$$f'(x) = \cosh u \cdot u'$$

$$f'(x) = \cosh(\cosh(2x)) \cdot (2\sinh(2x))$$

$$f'(x) = 2\sinh(2x)\cosh(\cosh(2x))$$

Topic: Inverse hyperbolic derivatives

Question: Find the derivative of the inverse hyperbolic function.

$$y = \cosh^{-1}(x^3)$$

Answer choices:

A $\quad y' = \dfrac{3x^2}{\sqrt{x^6 - 1}}$ $\quad$ with $x^3 < 1$

B $\quad y' = \dfrac{3x^2}{\sqrt{x^6 + 1}}$ $\quad$ with $x^3 < 1$

C $\quad y' = \dfrac{3x^2}{\sqrt{x^6 + 1}}$ $\quad$ with $x^3 > 1$

D $\quad y' = \dfrac{3x^2}{\sqrt{x^6 - 1}}$ $\quad$ with $x^3 > 1$

Solution: D

Apply the formula for the derivative of inverse hyperbolic cosine, with $g(x) = x^3$ and $g'(x) = 3x^2$.

$$y' = \frac{g'(x)}{\sqrt{[g(x)]^2 - 1}} \qquad \text{with } x^3 > 1$$

$$y' = \frac{3x^2}{\sqrt{(x^3)^2 - 1}} \qquad \text{with } x^3 > 1$$

$$y' = \frac{3x^2}{\sqrt{x^6 - 1}} \qquad \text{with } x^3 > 1$$

Topic: Inverse hyperbolic derivatives

Question: Find the derivative of the inverse hyperbolic function.

$$y = \tanh^{-1}(2x^5 - 1)$$

Answer choices:

A $\quad y' = \dfrac{5}{2x + 2x^6}$ $\quad$ with $|2x^5 - 1| < 1$

B $\quad y' = \dfrac{1}{2x - 2x^6}$ $\quad$ with $|2x^5 - 1| < 1$

C $\quad y' = \dfrac{5}{x - x^6}$ $\quad$ with $|2x^5 - 1| < 1$

D $\quad y' = \dfrac{5}{2x - 2x^6}$ $\quad$ with $|2x^5 - 1| < 1$

Solution: D

Apply the formula for the derivative of inverse hyperbolic tangent, with $g(x) = 2x^5 - 1$ and $g'(x) = 10x^4$.

$$y' = \frac{g'(x)}{1 - [g(x)]^2} \quad \text{with } |g(x)| < 1$$

$$y' = \frac{10x^4}{1 - (2x^5 - 1)^2} \quad \text{with } |2x^5 - 1| < 1$$

Simplify the derivative.

$$y' = \frac{10x^4}{1 - (4x^{10} - 4x^5 + 1)} \quad \text{with } |2x^5 - 1| < 1$$

$$y' = \frac{10x^4}{1 - 4x^{10} + 4x^5 - 1} \quad \text{with } |2x^5 - 1| < 1$$

$$y' = \frac{10x^4}{-4x^{10} + 4x^5} \quad \text{with } |2x^5 - 1| < 1$$

$$y' = \frac{5}{-2x^6 + 2x} \quad \text{with } |2x^5 - 1| < 1$$

$$y' = \frac{5}{2x - 2x^6} \quad \text{with } |2x^5 - 1| < 1$$

Topic: Inverse hyperbolic derivatives

Question: Find the derivative of the inverse hyperbolic function.

$$f(x) = (\sinh^{-1}(2x^3))^4$$

Answer choices:

A $$f'(x) = \frac{24x^2(\sinh^{-1}(2x^3))^3}{\sqrt{x^2+1}}$$

B $$f'(x) = \frac{24x^2(\sinh^{-1}(2x^3))^3}{\sqrt{4x^6+1}}$$

C $$f'(x) = \frac{6x^2(\sinh^{-1}(2x^3))^3}{\sqrt{x^2+1}}$$

D $$f'(x) = \frac{6x^2(\sinh^{-1}(2x^3))^3}{\sqrt{4x^6+1}}$$

Solution: B

Use a substitution with $u = \sinh^{-1}(2x^3)$ and

$$u' = \frac{6x^2}{\sqrt{(2x^3)^2 + 1}}$$

$$u' = \frac{6x^2}{\sqrt{4x^6 + 1}}$$

Then the function is

$$f(x) = u^4$$

and the derivative is

$$f'(x) = 4u^3 \cdot u'$$

$$f'(x) = 4(\sinh^{-1}(2x^3))^3 \cdot \frac{6x^2}{\sqrt{4x^6 + 1}}$$

$$f'(x) = \frac{24x^2(\sinh^{-1}(2x^3))^3}{\sqrt{4x^6 + 1}}$$

Topic: Exponential derivatives

Question: Find the derivative of the exponential function.

$$y = e^{2x+1}$$

Answer choices:

A $y' = e^{2x+1}$

B $y' = 2e^{2x+1}$

C $y' = 2e^{3x}$

D $y' = (2x+1)e^{2x+1}$

Solution: B

Make a substitution, letting $u = 2x + 1$ and $u' = 2$. Then the function is

$$y = e^u$$

and the derivative is

$$y' = e^u \cdot u'$$

$$y' = e^{2x+1} \cdot 2$$

$$y' = 2e^{2x+1}$$

Topic: Exponential derivatives

Question: Find the derivative of the exponential function.

$$y = e^{\sqrt{x+1}}$$

Answer choices:

A $\quad y' = \dfrac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$

B $\quad y' = \dfrac{e^{\sqrt{x}}}{2\sqrt{x+1}}$

C $\quad y' = \dfrac{e^{\sqrt{x+1}}}{\sqrt{x+1}}$

D $\quad y' = e^{\sqrt{x+1}}$

Solution: A

Make a substitution, letting $u = \sqrt{x+1}$ and

$$u' = \frac{1}{2\sqrt{x+1}}$$

Then the function is

$$y = e^u$$

and the derivative is

$$y' = e^u \cdot u'$$

$$y' = e^{\sqrt{x+1}} \cdot \frac{1}{2\sqrt{x+1}}$$

$$y' = \frac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$$

Topic: Exponential derivatives

Question: Find the derivative of the exponential function.

$$y = 4xe^{5x^2-2}$$

Answer choices:

A $\quad y' = 4e^{5x^2-2}(5x + 1)$

B $\quad y' = 4e^{5x^2-2}(5x^2 + 1)$

C $\quad y' = 4e^{5x^2-2}(10x^2 + 1)$

D $\quad y' = 4e^{5x^2-2}(10x + 1)$

Solution: C

We'll apply product rule with

$$f(x) = 4x$$

$$f'(x) = 4$$

and

$$g(x) = e^{5x^2-2}$$

$$g'(x) = 10xe^{5x^2-2}$$

Then the derivative is

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (4x)(10xe^{5x^2-2}) + (4)(e^{5x^2-2})$$

$$y' = 40x^2e^{5x^2-2} + 4e^{5x^2-2}$$

The terms in the denominator share a common factor of $4e^{5x^2-2}$, so factor that out.

$$y' = 4e^{5x^2-2}(10x^2 + 1)$$

Topic: Logarithmic derivatives

Question: Find the derivative of the logarithmic function.

$$y = \ln(x^2 - 5x)$$

Answer choices:

A $\quad y' = \dfrac{2x + 5}{x^2 + 5x}$

B $\quad y' = \dfrac{2x - 5}{x^2 - 5}$

C $\quad y' = \dfrac{5 - 2x}{x^2 - 5x}$

D $\quad y' = \dfrac{2x - 5}{x^2 - 5x}$

Solution: D

Let $u = x^2 - 5x$ and $u' = 2x - 5$. Then the function is

$$y = \ln u$$

and the derivative is

$$y' = \frac{1}{u} \cdot u'$$

$$y' = \frac{1}{x^2 - 5x} \cdot (2x - 5)$$

$$y' = \frac{2x - 5}{x^2 - 5x}$$

Topic: Logarithmic derivatives

Question: Find the derivative of the logarithmic function.

$$y = \ln \sqrt[3]{2x^3 - 5}$$

Answer choices:

A $\quad y' = \dfrac{2x}{2x^3 - 5}$

B $\quad y' = \dfrac{x^2}{2x^3 - 5}$

C $\quad y' = \dfrac{2x^2}{2x^3 + 5}$

D $\quad y' = \dfrac{2x^2}{2x^3 - 5}$

Solution: D

Let $u=\sqrt[3]{2x^3-5}$ and

$$u'=\frac{1}{3}(2x^3-5)^{-\frac{2}{3}}(6x^2)$$

$$u'=2x^2(2x^3-5)^{-\frac{2}{3}}$$

Then the function is

$$y=\ln u$$

and the derivative is

$$y'=\frac{1}{u}\cdot u'$$

$$y'=\frac{1}{\sqrt[3]{2x^3-5}}\cdot 2x^2(2x^3-5)^{-\frac{2}{3}}$$

$$y'=\frac{2x^2}{(2x^3-5)^{\frac{2}{3}}\sqrt[3]{2x^3-5}}$$

$$y'=\frac{2x^2}{(2x^3-5)^{\frac{2}{3}}(2x^3-5)^{\frac{1}{3}}}$$

$$y'=\frac{2x^2}{(2x^3-5)^1}$$

$$y'=\frac{2x^2}{2x^3-5}$$

Topic: Logarithmic derivatives

Question: Find the derivative of the logarithmic function.

$$f(x) = 4 \ln x$$

Answer choices:

A $f'(x) = \dfrac{4}{x}$

B $f'(x) = 4x$

C $f'(x) = 4$

D $f'(x) = x$

Solution: A

The derivative is

$$f'(x) = 4\left(\frac{1}{x}\right)$$

$$f'(x) = \frac{4}{x}$$

Topic: Logarithmic differentiation

Question: Use logarithmic differentiation to find the derivative.

$$y = 3^{5x}$$

Answer choices:

A $\quad y' = 15^{5x}(\ln 3)$

B $\quad y' = 3^{4x}(5 \ln 3)$

C $\quad y' = 3^{5x}(5 \ln 3)$

D $\quad y' = 3^{5x}(\ln 15)$

Solution: C

Apply the natural log to both sides of the equation.

$$y = 3^{5x}$$

$$\ln y = \ln(3^{5x})$$

Use laws of logs to rewrite the equation.

$$\ln y = 5x \ln 3$$

Take the derivative of both sides of the equation, using product rule for the right side and remembering to multiply by y' when we take the derivative of y.

$$\frac{1}{y}y' = (5)(\ln 3) + (5x)(0)$$

$$\frac{1}{y}y' = 5 \ln 3$$

Solve for y' in terms of x by isolating y' and substituting for y.

$$y' = 5y \ln 3$$

$$y' = 5(3^{5x})\ln 3$$

$$y' = 3^{5x}(5 \ln 3)$$

Topic: Logarithmic differentiation

Question: Use logarithmic differentiation to find the derivative.

$$y = 3\ln(5x)$$

Answer choices:

A $\quad y' = \dfrac{5}{x}$

B $\quad y' = \dfrac{3}{x}$

C $\quad y' = \dfrac{3}{5x}$

D $\quad y' = \dfrac{15}{x}$

Solution: B

Apply the natural log to both sides of the equation.

$$y = 3\ln(5x)$$

$$\ln y = \ln(3\ln(5x))$$

Use laws of logs to rewrite the equation.

$$\ln y = \ln 3 + \ln(\ln(5x))$$

Take the derivative of both sides of the equation, remembering to multiply by y' when we take the derivative of y.

$$\frac{1}{y}y' = 0 + \frac{1}{\ln(5x)}\left(\frac{1}{5x}(5)\right)$$

$$\frac{1}{y}y' = \frac{1}{x\ln(5x)}$$

Solve for y' in terms of x by isolating y' and substituting for y.

$$y' = \frac{y}{x\ln(5x)}$$

$$y' = \frac{3\ln(5x)}{x\ln(5x)}$$

$$y' = \frac{3}{x}$$

Topic: Logarithmic differentiation

Question: Use logarithmic differentiation to find the derivative.

$$y = \ln(6x^2 - 3x + 9)^{4x}$$

Answer choices:

A $\quad y' = 4\ln(6x^2 - 3x + 9)$

B $\quad y' = \dfrac{4x(4x - 1)}{2x^2 - x + 3}$

C $\quad y' = 4\ln(6x^2 - 3x + 9) + \dfrac{4x(4x - 1)}{2x^2 - x + 3}$

D $\quad y' = 4\ln(6x^2 - 3x + 9) + \dfrac{1}{6x^2 - 3x + 9}$

Solution: C

Apply the natural log to both sides of the equation.

$$y = \ln(6x^2 - 3x + 9)^{4x}$$

$$\ln y = \ln(\ln(6x^2 - 3x + 9)^{4x})$$

Use laws of logs to rewrite the equation.

$$\ln y = \ln(4x\ln(6x^2 - 3x + 9))$$

$$\ln y = \ln(4x) + \ln(\ln(6x^2 - 3x + 9))$$

Take the derivative of both sides of the equation, remembering to multiply by y' when we take the derivative of y.

$$\frac{1}{y}y' = \frac{1}{4x}(4) + \frac{1}{\ln(6x^2 - 3x + 9)}\left(\frac{1}{6x^2 - 3x + 9}(12x - 3)\right)$$

$$\frac{1}{y}y' = \frac{1}{x} + \frac{1}{\ln(6x^2 - 3x + 9)}\left(\frac{12x - 3}{6x^2 - 3x + 9}\right)$$

$$\frac{1}{y}y' = \frac{1}{x} + \frac{12x - 3}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}$$

Solve for y' in terms of x by isolating y' and substituting for y.

$$y' = y\left[\frac{1}{x} + \frac{12x - 3}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}\right]$$

$$y' = \ln(6x^2 - 3x + 9)^{4x}\left[\frac{1}{x} + \frac{12x - 3}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}\right]$$

$$y' = \frac{\ln(6x^2 - 3x + 9)^{4x}}{x} + \frac{(12x - 3)\ln(6x^2 - 3x + 9)^{4x}}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}$$

Use laws of logarithms to simplify the derivative.

$$y' = \frac{4x\ln(6x^2 - 3x + 9)}{x} + \frac{4x(12x - 3)\ln(6x^2 - 3x + 9)}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}$$

$$y' = 4\ln(6x^2 - 3x + 9) + \frac{4x(12x - 3)}{6x^2 - 3x + 9}$$

$$y' = 4\ln(6x^2 - 3x + 9) + \frac{12x(4x - 1)}{3(2x^2 - x + 3)}$$

$$y' = 4\ln(6x^2 - 3x + 9) + \frac{4x(4x - 1)}{2x^2 - x + 3}$$

Topic: Tangent lines

Question: Find the equation of the tangent line to the function at $(1, -2)$.

$$y = 3x^2 - 6x + 1$$

Answer choices:

A $y = -2$

B $x + y = -2$

C $y = 2$

D $x - y = 2$

Solution: A

Take the derivative of the function.

$$y' = 6x - 6$$

Find the slope of the tangent line at $(1, -2)$ by evaluating the derivative at that point.

$$y' = 6(1) - 6$$

$$y' = 0$$

The slope of the tangent line at $(1, -2)$ is $m = 0$, so plugging this slope and the point of tangency into the point-slope formula for the equation of a line gives the equation of the tangent line:

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = 0(x - 1)$$

$$y + 2 = 0$$

$$y = -2$$

Topic: Tangent lines

Question: Find the equation of the tangent line to the function at $(1,1/2)$.

$$y = \frac{1}{x^2 + 1}$$

Answer choices:

A $\quad y = -\frac{1}{2}x + 1$

B $\quad y = x - 1$

C $\quad y = -2x + 2$

D $\quad y = \frac{1}{2}x - 1$

Solution: A

Use quotient rule to take the derivative of the function.

$$y' = \frac{(0)(x^2+1) - (1)(2x)}{(x^2+1)^2}$$

$$y' = \frac{0 - 2x}{(x^2+1)^2}$$

$$y' = -\frac{2x}{(x^2+1)^2}$$

Find the slope of the tangent line at $(1,1/2)$ by evaluating the derivative at that point.

$$y' = -\frac{2(1)}{(1^2+1)^2}$$

$$y' = -\frac{1}{2}$$

The slope of the tangent line at $(1,1/2)$ is $m = -1/2$, so plugging this slope and the point of tangency into the point-slope formula for the equation of a line gives the equation of the tangent line:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{2} = -\frac{1}{2}(x - 1)$$

$$y - \frac{1}{2} = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{1}{2} + \frac{1}{2}$$

$$y = -\frac{1}{2}x + 1$$

Topic: Tangent lines

Question: Where on the interval $-1 \leq x \leq 1$ does the function have horizontal tangent lines?

$$f(x) = x^3 - x - 3$$

Answer choices:

A At $x = 0$

B At $x = \pm \frac{\sqrt{3}}{3}$

C At $x = \pm \sqrt{3}$

D At $x = \pm 3$

Solution: B

Take the derivative of the function.

$$f'(x) = 3x^2 - 1$$

Horizontal tangent lines exist when $f'(x) = 0$, so we'll set the derivative equal to 0.

$$3x^2 - 1 = 0$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{1}}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \pm\frac{\sqrt{3}}{3}$$

On the interval $-1 \le x \le 1$, the function has two horizontal tangent lines, located at

$$x = \pm\frac{\sqrt{3}}{3}$$

Topic: Value that makes two tangent lines parallel

Question: What is the value of a such that the tangent lines of $f(x)$ at $x = a$ and $x = a + 1$ are parallel?

$$f(x) = x^3 + x^2 + x - 1$$

Answer choices:

A $a = -\frac{1}{2}$

B $a = \frac{1}{2}$

C $a = -\frac{5}{6}$

D $a = \frac{5}{6}$

Solution: C

Start by finding the derivative of $f(x)$.

$$f'(x) = 3x^2 + 2x + 1$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$f'(a) = 3a^2 + 2a + 1$$

$$f'(a + 1) = 3(a + 1)^2 + 2(a + 1) + 1$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3a^2 + 2a + 1 = 3(a + 1)^2 + 2(a + 1) + 1$$

$$3a^2 + 2a + 1 = 3(a^2 + 2a + 1) + 2a + 2 + 1$$

$$3a^2 + 2a + 1 = 3a^2 + 6a + 3 + 2a + 2 + 1$$

Collect like terms and solve for a.

$$2a + 1 = 6a + 3 + 2a + 2 + 1$$

$$1 = 6a + 3 + 2 + 1$$

$$0 = 6a + 3 + 2$$

$$0 = 6a + 5$$

$$6a = -5$$

$$a = -\frac{5}{6}$$

Topic: Value that makes two tangent lines parallel

Question: What is the value of a such that the tangent lines of $f(x)$ at $x = a$ and $x = a + 1$ are parallel?

$$f(x) = 2x^3 - x^2 + x + 12$$

Answer choices:

A $\quad a = \frac{1}{3}$

B $\quad a = -\frac{1}{3}$

C $\quad a = \frac{1}{2}$

D $\quad a = -\frac{1}{2}$

Solution: B

Start by finding the derivative of $f(x)$.

$$f'(x) = 6x^2 - 2x + 1$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$f'(a) = 6a^2 - 2a + 1$$

$$f'(a + 1) = 6(a + 1)^2 - 2(a + 1) + 1$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$6a^2 - 2a + 1 = 6(a + 1)^2 - 2(a + 1) + 1$$

$$6a^2 - 2a + 1 = 6(a^2 + 2a + 1) - 2a - 2 + 1$$

$$6a^2 - 2a + 1 = 6a^2 + 12a + 6 - 2a - 2 + 1$$

Collect like terms and solve for a.

$$-2a + 1 = 12a + 6 - 2a - 2 + 1$$

$$1 = 12a + 6 - 2 + 1$$

$$0 = 12a + 6 - 2$$

$$0 = 12a + 4$$

$$12a = -4$$

$$a = -\frac{1}{3}$$

Topic: Value that makes two tangent lines parallel

Question: What is the value of a such that the tangent lines of $f(x)$ at $x = a$ and $x = a + 2$ are parallel?

$$f(x) = x^3 + 3x^2 - x - 5$$

Answer choices:

A $\quad a = \dfrac{1}{2}$

B $\quad a = -\dfrac{1}{2}$

C $\quad a = 2$

D $\quad a = -2$

Solution: D

Start by finding the derivative of $f(x)$.

$$f'(x) = 3x^2 + 6x - 1$$

Now we'll plug both $x = a$ and $x = a + 2$ into the derivative.

$$f'(a) = 3a^2 + 6a - 1$$

$$f'(a + 2) = 3(a + 2)^2 + 6(a + 2) - 1$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3a^2 + 6a - 1 = 3(a + 2)^2 + 6(a + 2) - 1$$

$$3a^2 + 6a - 1 = 3(a^2 + 4a + 4) + 6a + 12 - 1$$

$$3a^2 + 6a - 1 = 3a^2 + 12a + 12 + 6a + 12 - 1$$

Collect like terms and solve for a.

$$6a - 1 = 12a + 12 + 6a + 12 - 1$$

$$-1 = 12a + 12 + 12 - 1$$

$$0 = 12a + 12 + 12$$

$$0 = 12a + 24$$

$$12a = -24$$

$$a = -2$$

Topic: Values that make the function differentiable

Question: Which values of a and b would make the function differentiable?

$$f(x) = \begin{cases} 3x^2 & x > 1 \\ bx^2 - a & x \leq 1 \end{cases}$$

Answer choices:

A	$a = 3$	and	$b = 0$
B	$a = 0$	and	$b = 0$
C	$a = 0$	and	$b = 3$
D	$a = 3$	and	$b = 3$

Solution: C

The break point of the function is at $x=1$, because that's where the first piece of the function ends and the second piece takes over.

We'll work on continuity first by setting the one-sided limits at the break point $x=1$ equal to one another.

$$\lim_{x\to1^+} 3x^2 = \lim_{x\to1^-} bx^2 - a$$

$$3(1)^2 = b(1)^2 - a$$

$$3 = b - a$$

$$b - a = 3$$

Now we'll work on smoothness by setting the one-sided limits of the derivatives of each piece at the break point $x=1$ equal to one another.

$$\lim_{x\to1^+} 6x = \lim_{x\to1^-} 2bx$$

$$6(1) = 2b(1)$$

$$6 = 2b$$

$$b = 3$$

Pull together these two equations into a system of equations.

$$b = 3$$

$$b - a = 3$$

We need to solve the system, which we can do by substituting the first equation $b = 3$ into the second equation.

$$3 - a = 3$$

$$-a = 0$$

$$a = 0$$

Therefore, the values of the constants a and b that make $f(x)$ differentiable are $a = 0$ and $b = 3$.

Topic: Values that make the function differentiable

Question: Which values of a and b would make the function differentiable?

$$f(x) = \begin{cases} x^2 - 5 & x > 3 \\ 4x^2 - 2ax + b & x \leq 3 \end{cases}$$

Answer choices:

A	$a = 22$	and	$b = 9$
B	$a = 22$	and	$b = 22$
C	$a = 9$	and	$b = 9$
D	$a = 9$	and	$b = 22$

Solution: D

The break point of the function is at $x = 3$, because that's where the first piece of the function ends and the second piece takes over.

We'll work on continuity first by setting the one-sided limits at the break point $x = 3$ equal to one another.

$$\lim_{x\to 3^+} x^2 - 5 = \lim_{x\to 3^-} 4x^2 - 2ax + b$$

$$3^2 - 5 = 4(3)^2 - 2a(3) + b$$

$$9 - 5 = 4(9) - 6a + b$$

$$4 = 36 - 6a + b$$

$$-6a + b = -32$$

$$6a - b = 32$$

Now we'll work on smoothness by setting the one-sided limits of the derivatives of each piece at the break point $x = 3$ equal to one another.

$$\lim_{x\to 3^+} 2x = \lim_{x\to 3^-} 8x - 2a$$

$$2(3) = 8(3) - 2a$$

$$6 = 24 - 2a$$

$$-18 = -2a$$

$$a = 9$$

Pull together these two equations into a system of equations.

$$a = 9$$

$$6a - b = 32$$

We need to solve the system, which we can do by substituting the first equation $a = 9$ into the second equation.

$$6(9) - b = 32$$

$$54 - b = 32$$

$$-b = -22$$

$$b = 22$$

Therefore, the values of the constants a and b that make $f(x)$ differentiable are $a = 9$ and $b = 22$.

Topic: Values that make the function differentiable

Question: Which values of a and b would make the function differentiable?

$$f(x) = \begin{cases} ax^2 + 10 & x \leq 2 \\ x^2 - 6x + b & x > 2 \end{cases}$$

Answer choices:

A $\quad a = \frac{1}{2} \quad$ and $\quad b = 16$

B $\quad a = -\frac{1}{2} \quad$ and $\quad b = -16$

C $\quad a = \frac{1}{2} \quad$ and $\quad b = -16$

D $\quad a = -\frac{1}{2} \quad$ and $\quad b = 16$

Solution: D

The break point of the function is at $x = 2$, because that's where the first piece of the function ends and the second piece takes over.

We'll work on continuity first by setting the one-sided limits at the break point $x = 2$ equal to one another.

$$\lim_{x \to 2^-} ax^2 + 10 = \lim_{x \to 2^+} x^2 - 6x + b$$

$$a(2)^2 + 10 = 2^2 - 6(2) + b$$

$$4a + 10 = 4 - 12 + b$$

$$4a - b = -18$$

Now we'll work on smoothness by setting the one-sided limits of the derivatives of each piece at the break point $x = 2$ equal to one another.

$$\lim_{x \to 2^-} 2ax = \lim_{x \to 2^+} 2x - 6$$

$$2a(2) = 2(2) - 6$$

$$4a = 4 - 6$$

$$4a = -2$$

$$a = -\frac{1}{2}$$

Pull together these two equations into a system of equations.

$$a = -\frac{1}{2}$$

$$4a - b = -18$$

We need to solve the system, which we can do by substituting the first equation $a = -1/2$ into the second equation.

$$4\left(-\frac{1}{2}\right) - b = -18$$

$$-2 - b = -18$$

$$-b = -16$$

$$b = 16$$

Therefore, the values of the constants a and b that make $f(x)$ differentiable are $a = -1/2$ and $b = 16$.

Topic: Normal lines

Question: Find the equation of the normal line to the function at $(1,2)$.

$$f(x) = 2x^4$$

Answer choices:

A $y = 8x - 6$

B $y = -\frac{1}{8}x - \frac{17}{8}$

C $y = -\frac{1}{8}x + \frac{17}{8}$

D $y = 8x - 10$

Solution: C

Take the derivative of the function,

$$f'(x) = 8x^3$$

and then evaluate it at $(1,2)$.

$$f'(1) = 8(1)^3$$

$$f'(1) = 8$$

This is the slope of the tangent line at $(1,2)$. Since $m = 8$, we'll take the negative reciprocal to find n, the slope of the normal line.

$$n = -\frac{1}{8}$$

We'll plug $n = -1/8$ and the point $(1,2)$ into the point-slope formula for the equation of the line. Once we simplify, we'll have the equation of the normal line to the function at $(1,2)$.

$$y - y_1 = n(x - x_1)$$

$$y - 2 = -\frac{1}{8}(x - 1)$$

$$y - 2 = -\frac{1}{8}x + \frac{1}{8}$$

$$y = -\frac{1}{8}x + \frac{1}{8} + \frac{16}{8}$$

$$y = -\frac{1}{8}x + \frac{17}{8}$$

Topic: Normal lines

Question: Find the equation of the normal line to the function at $(3,6)$.

$$f(x) = x\sqrt{x+1}$$

Answer choices:

A $\quad y = -\frac{4}{11}x + \frac{78}{11}$

B $\quad y = \frac{11}{4}x - \frac{57}{4}$

C $\quad y = \frac{11}{4}x - \frac{9}{4}$

D $\quad y = -\frac{4}{11}x - \frac{54}{11}$

Solution: A

Take the derivative of the function,

$$f'(x) = (1)(\sqrt{x+1}) + (x)\left(\frac{1}{2}(x+1)^{-\frac{1}{2}}\right)$$

$$f'(x) = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}$$

and then evaluate it at $(3,6)$.

$$f'(3) = \sqrt{3+1} + \frac{3}{2\sqrt{3+1}}$$

$$f'(3) = 2 + \frac{3}{2(2)}$$

$$f'(3) = \frac{8}{4} + \frac{3}{4}$$

$$f'(3) = \frac{11}{4}$$

This is the slope of the tangent line at $(3,6)$. Since $m = 11/4$, we'll take the negative reciprocal to find n, the slope of the normal line.

$$n = -\frac{4}{11}$$

We'll plug $n = -4/11$ and the point $(3,6)$ into the point-slope formula for the equation of the line. Once we simplify, we'll have the equation of the normal line to the function at $(3,6)$.

$$y - y_1 = n(x - x_1)$$

$$y - 6 = -\frac{4}{11}(x - 3)$$

$$y - 6 = -\frac{4}{11}x + \frac{12}{11}$$

$$y = -\frac{4}{11}x + \frac{12}{11} + \frac{66}{11}$$

$$y = -\frac{4}{11}x + \frac{78}{11}$$

Topic: Normal lines

Question: Find the equation of the normal line to the function at $(2,2)$.

$$f(x) = \frac{2x^2}{x+2}$$

Answer choices:

A $\quad y = -\frac{2}{3}x - \frac{2}{3}$

B $\quad y = \frac{3}{2}x - 1$

C $\quad y = \frac{3}{2}x - 5$

D $\quad y = -\frac{2}{3}x + \frac{10}{3}$

Solution: D

Take the derivative of the function,

$$f'(x) = \frac{(4x)(x+2) - (2x^2)(1)}{(x+2)^2}$$

$$f'(x) = \frac{4x^2 + 8x - 2x^2}{(x+2)^2}$$

$$f'(x) = \frac{2x^2 + 8x}{(x+2)^2}$$

and then evaluate it at $(2,2)$.

$$f'(2) = \frac{2(2)^2 + 8(2)}{(2+2)^2}$$

$$f'(2) = \frac{2(4) + 16}{4^2}$$

$$f'(2) = \frac{8 + 16}{16}$$

$$f'(2) = \frac{3}{2}$$

This is the slope of the tangent line at $(2,2)$. Since $m = 3/2$, we'll take the negative reciprocal to find n, the slope of the normal line.

$$n = -\frac{2}{3}$$

We'll plug $n = -2/3$ and the point $(2,2)$ into the point-slope formula for the equation of the line. Once we simplify, we'll have the equation of the normal line to the function at $(2,2)$.

$$y - y_1 = n(x - x_1)$$

$$y - 2 = -\frac{2}{3}(x - 2)$$

$$y - 2 = -\frac{2}{3}x + \frac{4}{3}$$

$$y = -\frac{2}{3}x + \frac{4}{3} + \frac{6}{3}$$

$$y = -\frac{2}{3}x + \frac{10}{3}$$

Topic: Average rate of change

Question: Find the average rate of change of the function on $[0,2]$.

$$f(x) = x^2$$

Answer choices:

A $\frac{\Delta f}{\Delta x} = 0$

B $\frac{\Delta f}{\Delta x} = \frac{1}{2}$

C $\frac{\Delta f}{\Delta x} = 1$

D $\frac{\Delta f}{\Delta x} = 2$

Solution: D

From the interval, we know $x_1 = 0$ and $x_2 = 2$. We'll find $f(x_1)$ and $f(x_2)$ by plugging these values into the function. We get

$$f(0) = 0^2$$

$$f(0) = 0$$

and

$$f(2) = 2^2$$

$$f(2) = 4$$

Now we can plug the values we've found into the formula for average rate fo change.

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(2) - f(0)}{2 - 0}$$

$$\frac{\Delta f}{\Delta x} = \frac{4 - 0}{2}$$

$$\frac{\Delta f}{\Delta x} = \frac{4}{2}$$

$$\frac{\Delta f}{\Delta x} = 2$$

Topic: Average rate of change

Question: Find the average rate of change of the function on $[1,4]$.

$$f(x) = \frac{2x}{3x^2 - 1}$$

Answer choices:

A $\quad \frac{\Delta f}{\Delta x} = -\frac{47}{13}$

B $\quad \frac{\Delta f}{\Delta x} = -\frac{13}{47}$

C $\quad \frac{\Delta f}{\Delta x} = \frac{47}{13}$

D $\quad \frac{\Delta f}{\Delta x} = \frac{13}{47}$

Solution: B

From the interval, we know $x_1 = 1$ and $x_2 = 4$. We'll find $f(x_1)$ and $f(x_2)$ by plugging these values into the function. We get

$$f(1) = \frac{2(1)}{3(1)^2 - 1}$$

$$f(1) = \frac{2}{2}$$

$$f(1) = 1$$

and

$$f(4) = \frac{2(4)}{3(4)^2 - 1}$$

$$f(4) = \frac{8}{48 - 1}$$

$$f(4) = \frac{8}{47}$$

Now we can plug the values we've found into the formula for average rate fo change.

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(4) - f(1)}{4 - 1}$$

$$\frac{\Delta f}{\Delta x} = \frac{\frac{8}{47} - 1}{3}$$

$$\frac{\Delta f}{\Delta x} = \frac{\frac{8}{47} - \frac{47}{47}}{3}$$

$$\frac{\Delta f}{\Delta x} = \frac{-\frac{39}{47}}{3}$$

$$\frac{\Delta f}{\Delta x} = -\frac{39}{47} \cdot \frac{1}{3}$$

$$\frac{\Delta f}{\Delta x} = -\frac{13}{47}$$

Topic: Average rate of change

Question: Find the average rate of change of the function on $[2,3]$.

$$f(x) = 6e^x - 4\sqrt{x^3}$$

Answer choices:

A $\quad \dfrac{\Delta f}{\Delta x} = 6e^2(e-1) - 4(3\sqrt{3} - 2\sqrt{2})$

B $\quad \dfrac{\Delta f}{\Delta x} = e^2(6e+1) + 3\sqrt{3} + 8\sqrt{2}$

C $\quad \dfrac{\Delta f}{\Delta x} = 6e^2(e+1) + 4(3\sqrt{3} + 2\sqrt{2})$

D $\quad \dfrac{\Delta f}{\Delta x} = e^2(6e-1) - 3\sqrt{3} + 8\sqrt{2}$

Solution: A

From the interval, we know $x_1 = 2$ and $x_2 = 3$. We'll find $f(x_1)$ and $f(x_2)$ by plugging these values into the function. We get

$$f(3) = 6e^3 - 4\sqrt{3^3}$$

$$f(3) = 6e^3 - 4\sqrt{27}$$

$$f(3) = 6e^3 - 12\sqrt{3}$$

and

$$f(2) = 6e^2 - 4\sqrt{2^3}$$

$$f(2) = 6e^2 - 4\sqrt{8}$$

$$f(2) = 6e^2 - 8\sqrt{2}$$

Now we can plug the values we've found into the formula for average rate fo change.

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2}$$

$$\frac{\Delta f}{\Delta x} = \frac{6e^3 - 12\sqrt{3} - (6e^2 - 8\sqrt{2})}{3 - 2}$$

$$\frac{\Delta f}{\Delta x} = \frac{6e^3 - 12\sqrt{3} - 6e^2 + 8\sqrt{2}}{1}$$

$$\frac{\Delta f}{\Delta x} = 6e^3 - 12\sqrt{3} - 6e^2 + 8\sqrt{2}$$

$$\frac{\Delta f}{\Delta x} = 6e^2(e - 1) - 4(3\sqrt{3} - 2\sqrt{2})$$

Topic: Implicit differentiation

Question: Using implicit differentiation, we...

Answer choices:

A treat x as a variable and y as a function

B treat x as a variable and y as a variable

C treat x as a function and y as a function

D treat x as a function and y as a variable

Solution: A

When we use implicit differentiation, it's important to remember that we can't treat y as a variable, the same way we would if we were differentiating "normally."

In contrast, we have to treat y as a function of y in terms of x, and therefore apply chain rule whenever we take the derivative of y, which means we multiply by y'.

Topic: Implicit differentiation

Question: Every time we take the derivative of y, implicit differentiation requires us to multiply by which of the following?

Answer choices:

A 1

B y

C 0

D y'

Solution: D

We have to treat y as a function of y in terms of x, and therefore apply chain rule whenever we take the derivative of y, which means we multiply by y' every time we differentiate y.

Topic: Implicit differentiation

Question: Use implicit differentiation to find the derivative.

$$x^2 - y^2 = 9$$

Answer choices:

A $\quad y' = -\frac{x}{y}$

B $\quad y' = -\frac{y}{x}$

C $\quad y' = \frac{x}{y}$

D $\quad y' = \frac{y}{x}$

Solution: C

Using implicit differentiation to take the derivative of both sides of the equation gives

$$2x - 2yy' = 0$$

$$-2yy' = -2x$$

$$yy' = x$$

$$y' = \frac{x}{y}$$

Topic: Equation of the tangent line with implicit differentiation

Question: Find the equation of the tangent line to the curve at $(3,2)$.

$$\frac{x^2}{9}+\frac{y^2}{4}=2$$

Answer choices:

A $y=-\frac{2}{3}x+4$

B $y=-\frac{2}{3}x-4$

C $y=\frac{2}{3}x+4$

D $y=\frac{2}{3}x-4$

Solution: A

Using implicit differentiation to find the derivative of the curve, we get

$$\frac{2x}{9}+\frac{2y}{4}y'=0$$

Simplify and solve for y'.

$$\frac{2y}{4}y'=-\frac{2x}{9}$$

$$y'=-\frac{2x(4)}{9(2y)}$$

$$y'=-\frac{8x}{18y}$$

$$y'=-\frac{4x}{9y}$$

Evaluate the derivative at $(3,2)$ to find the slope of the tangent line.

$$m=-\frac{4(3)}{9(2)}$$

$$m=-\frac{12}{18}$$

$$m=-\frac{2}{3}$$

Plug the slope $m=-2/3$ and the point of tangency $(3,2)$ into the point-slope formula for the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{2}{3}(x - 3)$$

$$y - 2 = -\frac{2}{3}x + 2$$

$$y = -\frac{2}{3}x + 4$$

Topic: Equation of the tangent line with implicit differentiation

Question: Find the equation of the tangent line to the curve at $(0,1)$.

$$5x^2 + y^2 + 4xy = 1$$

Answer choices:

A $\quad y = -2x - 1$

B $\quad y = -2x + 1$

C $\quad y = 2x - 1$

D $\quad y = 2x + 1$

Solution: B

Using implicit differentiation to find the derivative of the curve, we get

$$10x + 2yy' + [(4)(y) + (4x)(1)(y')] = 0$$

$$10x + 2yy' + 4y + 4xy' = 0$$

Simplify and solve for y'.

$$2yy' + 4xy' = -10x - 4y$$

$$y'(2y + 4x) = -10x - 4y$$

$$y' = -\frac{10x + 4y}{2y + 4x}$$

Evaluate the derivative at $(0,1)$ to find the slope of the tangent line.

$$m = -\frac{10(0) + 4(1)}{2(1) + 4(0)}$$

$$m = -\frac{4}{2}$$

$$m = -2$$

Plug the slope $m = -2$ and the point of tangency $(0,1)$ into the point-slope formula for the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2(x - 0)$$

$$y - 1 = -2x$$

$$y = -2x + 1$$

Topic: Equation of the tangent line with implicit differentiation

Question: Find the equation of the tangent line to the curve at $(1,4)$.

$$8 + x^2y^2 - 5xy = 4$$

Answer choices:

A $y = 4x - 8$

B $y = 4x + 8$

C $y = -4x - 8$

D $y = -4x + 8$

Solution: D

Using implicit differentiation to find the derivative of the curve, we get

$$0 + [(2x)(y^2) + (x^2)(2yy')] - [(5)(y) + (5x)(1)(y')] = 0$$

$$2xy^2 + 2x^2yy' - 5y - 5xy' = 0$$

Simplify and solve for y'.

$$2x^2yy' - 5xy' = 5y - 2xy^2$$

$$y'(2x^2y - 5x) = 5y - 2xy^2$$

$$y' = \frac{5y - 2xy^2}{2x^2y - 5x}$$

Evaluate the derivative at $(1,4)$ to find the slope of the tangent line.

$$m = \frac{5(4) - 2(1)(4)^2}{2(1)^2(4) - 5(1)}$$

$$m = \frac{20 - 32}{8 - 5}$$

$$m = \frac{-12}{3}$$

$$m = -4$$

Plug the slope $m = -4$ and the point of tangency $(1,4)$ into the point-slope formula for the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -4(x - 1)$$

$$y - 4 = -4x + 4$$

$$y = -4x + 8$$

Topic: Second derivatives with implicit differentiation

Question: Use implicit differentiation to find the second derivative.

$$x^3 + y^3 = 9$$

Answer choices:

A $\quad y'' = -\dfrac{2x(y^3 - x^3)}{y^6}$

B $\quad y'' = -\dfrac{2x(x^3 + y^3)}{y^5}$

C $\quad y'' = -\dfrac{2x(y^3 - x^3)}{y^5}$

D $\quad y'' = -\dfrac{2x(x^3 + y^3)}{y^6}$

Solution: B

The first derivative is

$$3x^2 + 3y^2y' = 0$$

Solve for y'.

$$3y^2y' = -3x^2$$

$$y' = \frac{-3x^2}{3y^2}$$

$$y' = \frac{-x^2}{y^2}$$

Use quotient rule to find the second derivative.

$$y'' = \frac{(-2x)(y^2) - (-x^2)(2yy')}{(y^2)^2}$$

$$y'' = \frac{-2xy^2 + 2x^2yy'}{y^4}$$

$$y'' = \frac{2xy(-y + xy')}{y^4}$$

$$y'' = \frac{2x(-y + xy')}{y^3}$$

Substitute for the first derivative.

$$y'' = \frac{2x\left[-y + x\left(\frac{-x^2}{y^2}\right)\right]}{y^3}$$

$$y'' = \frac{2x\left(-y - \frac{x^3}{y^2}\right)}{y^3}$$

Find a common denominator within the numerator, then combine the fractions in the numerator into one fraction.

$$y'' = \frac{2x\left(\frac{-y^3 - x^3}{y^2}\right)}{y^3}$$

$$y'' = 2x\left(\frac{-y^3 - x^3}{y^5}\right)$$

$$y'' = -\frac{2x(x^3 + y^3)}{y^5}$$

Topic: Second derivatives with implicit differentiation

Question: Use implicit differentiation to find the second derivative.

$$x^2 + 2xy + y^3 = 25$$

Answer choices:

A $\quad y'' = \frac{8x^2 - 24x^2y + 16xy - 48xy^2 - 18y^4}{(2x + 3y^2)^3}$

B $\quad y'' = -\frac{8x^2 - 24x^2y + 16xy - 48xy^2 - 18y^4}{(2x + 3y^2)^3}$

C $\quad y'' = -\frac{24x^2 + 18y^4}{(2x + 3y^2)^3}$

D $\quad y'' = \frac{24x^2 + 18y^4}{(2x + 3y^2)^3}$

Solution: A

The first derivative is

$$2x + [(2)(y) + (2x)(1)(y')] + 3y^2y' = 0$$

$$2x + 2y + 2xy' + 3y^2y' = 0$$

Solve for y'.

$$2xy' + 3y^2y' = -2x - 2y$$

$$y'(2x + 3y^2) = -2x - 2y$$

$$y' = \frac{-2x - 2y}{2x + 3y^2}$$

Use quotient rule to find the second derivative.

$$y'' = \frac{(-2 - 2(1)(y'))(2x + 3y^2) - (-2x - 2y)(2 + 6yy')}{(2x + 3y^2)^2}$$

$$y'' = \frac{(-2 - 2y')(2x + 3y^2) + (2x + 2y)(2 + 6yy')}{(2x + 3y^2)^2}$$

$$y'' = \frac{-4x - 6y^2 - 4xy' - 6y^2y' + 4x + 12xyy' + 4y + 12y^2y'}{(2x + 3y^2)^2}$$

Collect like terms, then factor out y'.

$$y'' = \frac{-4xy' + 12xyy' + 6y^2y' + 4y - 6y^2}{(2x + 3y^2)^2}$$

$$y'' = \frac{y'(-4x + 12xy + 6y^2) + 4y - 6y^2}{(2x + 3y^2)^2}$$

Substitute for the first derivative.

$$y'' = \frac{\frac{-2x - 2y}{2x + 3y^2}(-4x + 12xy + 6y^2) + 4y - 6y^2}{(2x + 3y^2)^2}$$

$$y'' = \frac{\frac{(-2x - 2y)(-4x + 12xy + 6y^2)}{2x + 3y^2} + \frac{(4y - 6y^2)(2x + 3y^2)}{2x + 3y^2}}{(2x + 3y^2)^2}$$

$$y'' = \frac{\frac{(-2x - 2y)(-4x + 12xy + 6y^2) + (4y - 6y^2)(2x + 3y^2)}{2x + 3y^2}}{(2x + 3y^2)^2}$$

$$y'' = \frac{(-2x - 2y)(-4x + 12xy + 6y^2) + (4y - 6y^2)(2x + 3y^2)}{(2x + 3y^2)^3}$$

Expand the numerator, then simplify.

$$y'' = \frac{(8x^2 - 24x^2y - 12xy^2 + 8xy - 24xy^2 - 12y^3) + (8xy + 12y^3 - 12xy^2 - 18y^4)}{(2x + 3y^2)^3}$$

$$y'' = \frac{8x^2 - 24x^2y - 12xy^2 + 8xy - 24xy^2 - 12y^3 + 8xy + 12y^3 - 12xy^2 - 18y^4}{(2x + 3y^2)^3}$$

$$y'' = \frac{8x^2 - 24x^2y + 16xy - 48xy^2 - 18y^4}{(2x + 3y^2)^3}$$

Topic: Second derivatives with implicit differentiation

Question: Use implicit differentiation to find the second derivative.

$$x^2y^2 + 3xy = 100$$

Answer choices:

A $\quad y'' = -\frac{y}{x^2}$

B $\quad y'' = \frac{y}{x^2}$

C $\quad y'' = -\frac{2y}{x^2}$

D $\quad y'' = \frac{2y}{x^2}$

Solution: D

The first derivative is

$$[(2x)(y^2) + (x^2)(2y)(y')] + [(3)(y) + (3x)(1)(y')] = 0$$

$$2xy^2 + 2x^2yy' + 3y + 3xy' = 0$$

Solve for y'.

$$2x^2yy' + 3xy' = -2xy^2 - 3y$$

$$y'(2x^2y + 3x) = -2xy^2 - 3y$$

$$y' = \frac{-2xy^2 - 3y}{2x^2y + 3x}$$

Factor and simplify the first derivative.

$$y' = \frac{-y(2xy + 3)}{x(2xy + 3)}$$

$$y' = -\frac{y}{x}$$

Use the quotient rule to find the second derivative.

$$y'' = -\frac{(y')(x) - (y)(1)}{x^2}$$

$$y'' = -\frac{xy' - y}{x^2}$$

Substitute for the first derivative.

$$y'' = -\frac{x\left(-\frac{y}{x}\right) - y}{x^2}$$

$$y'' = -\frac{-y - y}{x^2}$$

$$y'' = \frac{y + y}{x^2}$$

$$y'' = \frac{2y}{x^2}$$

Topic: Critical points and the first derivative test

Question: Find the critical point of the function.

$$f(x) = x^2 - 10x + 2$$

Answer choices:

A $x = \frac{1}{5}$

B $x = 5$

C $x = -5$

D $x = -\frac{1}{5}$

Solution: B

Take the derivative of the function.

$$f(x) = x^2 - 10x + 2$$

$$f'(x) = 2x - 10$$

Set the derivative equal to 0 and solve for x.

$$2x - 10 = 0$$

$$2x = 10$$

$$x = 5$$

The function has one potential critical point at $x = 5$.

Topic: Critical points and the first derivative test

Question: Where is the function increasing and decreasing?

$$f(x) = x^2$$

Answer choices:

A Increasing on $x < 1$ and decreasing on $x > 1$

B Increasing on $x < 0$ and decreasing on $x > 0$

C Increasing on $x > 0$ and decreasing on $x < 0$

D Increasing on $x > 1$ and decreasing on $x < 1$

Solution: C

Find the derivative.

$$f(x) = x^2$$

$$f'(x) = 2x$$

Set the derivative equal to 0 and solve for x.

$$0 = 2x$$

$$x = 0$$

Investigate the critical point $x = 0$ by testing $x = -1$ and $x = 1$ in the first derivative.

$$f'(-1) = 2(-1)$$

$$f'(-1) = -2$$

and

$$f'(1) = 2(1)$$

$$f'(1) = 2$$

On the left side of $x = 0$ the derivative is negative so the function is decreasing. On the right side of $x = 0$ the derivative is positive so the function is increasing.

Topic: Critical points and the first derivative test

Question: Where is the function increasing and decreasing?

$$f(x) = x^4 - 4x^3 + 4x^2 - 7$$

Answer choices:

A Decreasing on $x < 0$ and $1/2 < x < 3/2$, increasing on $0 < x < 1/2$ and $x > 3/2$

B Decreasing on $0 < x < 1/2$ and $x > 3/2$, increasing on $x < 0$ and $1/2 < x < 3/2$

C Decreasing on $x < 0$ and $1 < x < 2$, increasing on $0 < x < 1$ and $x > 2$

D Decreasing on $0 < x < 1$ and $x > 2$, increasing on $x < 0$ and $1 < x < 2$

Solution: C

Take the first derivative of the function.

$$f(x) = x^4 - 4x^3 + 4x^2 - 7$$

$$f'(x) = 4x^3 - 12x^2 + 8x$$

$$f'(x) = 4x(x^2 - 3x + 2)$$

$$f'(x) = 4x(x - 2)(x - 1)$$

Set the derivative equal to 0 and solve for x.

$$4x(x - 2)(x - 1) = 0$$

$$x = 0,\ 1,\ 2$$

Investigate each interval by evaluating the first derivative at $x = -1$, $x = 1/2$, $x = 3/2$, and $x = 3$.

$$f'(-1) = 4(-1)^3 - 12(-1)^2 + 8(-1)$$

$$f'(-1) = -4 - 12 - 8$$

$$f'(-1) = -24$$

and

$$f'\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right)$$

$$f'\left(\frac{1}{2}\right) = 4\left(\frac{1}{8}\right) - 12\left(\frac{1}{4}\right) + 4$$

$$f'\left(\frac{1}{2}\right)=\frac{1}{2}-3+4$$

$$f'\left(\frac{1}{2}\right)=\frac{3}{2}$$

and

$$f'\left(\frac{3}{2}\right)=4\left(\frac{3}{2}\right)^3-12\left(\frac{3}{2}\right)^2+8\left(\frac{3}{2}\right)$$

$$f'\left(\frac{3}{2}\right)=4\left(\frac{27}{8}\right)-12\left(\frac{9}{4}\right)+12$$

$$f'\left(\frac{3}{2}\right)=\frac{27}{2}-27+12$$

$$f'\left(\frac{3}{2}\right)=-\frac{3}{2}$$

and

$$f'(3)=4(3)^3-12(3)^2+8(3)$$

$$f'(3)=4(27)-12(9)+24$$

$$f'(3)=108-108+24$$

$$f'(3)=24$$

To the left of $x=0$ the derivative is negative so the function is decreasing. Between $x=0$ and $x=1$, the derivative is positive so the function is increasing. Between $x=1$ and $x=2$, the derivative is negative so the function is decreasing. To the right of $x = 2$ the derivative is positive so the function is increasing.

The function $f(x) = x^4 - 4x^3 + 4x^2 - 7$ is decreasing when $x < 0$, increasing between $x = 0$ and $x = 1$, decreasing between $x = 1$ and $x = 2$, and increasing when $x > 2$.

Topic: Inflection points and the second derivative test

Question: Find the function's inflection points.

$$f(x) = x^4 - 6x^2 - 3x + 2$$

Answer choices:

A $(1,0)$ and $(-1,-6)$

B $(-1,0)$ and $(1,6)$

C $(-1,0)$ and $(1,-6)$

D $(-1,-6)$ and $(0,1)$

Solution: C

Find the second derivative of the function.

$$f(x) = x^4 - 6x^2 - 3x + 2$$

$$f'(x) = 4x^3 - 12x - 3$$

$$f''(x) = 12x^2 - 12$$

Set the second derivative equal to 0 and solve for x.

$$12x^2 - 12 = 0$$

$$12x^2 = 12$$

$$x^2 = 1$$

$$x = \pm 1$$

There are two possible inflection points at $x = -1$ and $x = 1$. Investigate $x = -1$ by testing $x = -2$ and $x = 0$ in the second derivative.

$$f''(-2) = 12(-2)^2 - 12$$

$$f''(-2) = 36$$

and

$$f''(0) = 12(0)^2 - 12$$

$$f''(0) = -12$$

Since $f''(-2) = 36 > 0$, the function is concave up to the left of $x = -1$, and since $f''(0) = -12 < 0$, the function is concave down to the right of $x = -1$.

Because the function changes concavity at $x = -1$, there's an inflection point there. We'll get the y-coordinate of the inflection point by substituting $x = -1$ into $f(x)$.

$$f(-1) = (-1)^4 - 6(-1)^2 - 3(-1) + 2$$

$$f(-1) = 0$$

The function has an inflection point at $(-1,0)$.

Investigate $x = 1$ by testing $x = 0$ and $x = 2$ into the second derivative.

$$f''(0) = 12(0)^2 - 12$$

$$f''(0) = -12$$

and

$$f''(2) = 12(2)^2 - 12$$

$$f''(2) = 36$$

Since $f''(0) = -12 < 0$, the function is concave down to the left of $x = 1$, and since $f''(2) = 36 > 0$, the function is concave up to the right of $x = 1$.

Because the function changes concavity at $x = 1$, there's an inflection point there. We'll get the y-coordinate of the inflection point by substituting $x = 1$ into $f(x)$.

$$f(1) = (1)^4 - 6(1)^2 - 3(1) + 2$$

$$f(1) = -6$$

The function has a second inflection point at $(1, -6)$.

Topic: Inflection points and the second derivative test

Question: Use the Second Derivative Test to classify the critical points at $x = 0$ and $x = 2$.

$$f''(x) = -6x + 6$$

Answer choices:

A Relative minimum at $x = 0$; Relative maximum at $x = 2$

B Relative minimum at $x = 2$; Relative maximum at $x = 0$

C Relative minima at $x = 0$ and $x = 2$

D Relative maxima at $x = 0$ and $x = 2$

Solution: A

The second derivative is positive at $x = 0$,

$$f''(0) = -6(0) + 6 = 6 > 0$$

so the function is concave up at that critical point, which means there's a relative minimum there.

The second derivative is negative at $x = 2$,

$$f''(2) = -6(2) + 6 = -6 < 0$$

so the function is concave down at that critical point, which means there's a relative maximum there.

Topic: Inflection points and the second derivative test

Question: Use the second derivative test to find the function's extrema?

$$f(x) = x^2 + x + 4$$

Answer choices:

A The function has a local minimum at $x = -1/2$.

B The function has a local maximum at $x = -1/2$.

C The function has a local minimum at $x = 1/2$.

D The function has a local maximum at $x = 1/2$.

Solution: A

Take the first derivative.

$$f(x) = x^2 + x + 4$$

$$f'(x) = 2x + 1$$

Set the derivative equal to 0 and solve for x.

$$2x + 1 = 0$$

$$2x = -1$$

$$x = -\frac{1}{2}$$

There's one critical point at $x = -1/2$. Take the second derivative.

$$f'(x) = 2x + 1$$

$$f''(x) = 2$$

Substitute the critical point $x = -1/2$ into the second derivative.

$$f''\left(-\frac{1}{2}\right) = 2$$

Because the second derivative is positive at the critical point, it means there's a local minimum at $x = -1/2$.

Topic: Intercepts and vertical asymptotes

Question: Find the function’s vertical asymptote.

$$f(x) = \frac{1}{x^2}$$

Answer choices:

A The function has a vertical asymptote at $x = 1$

B The function has a vertical asymptote at $x = 0$

C The function has a vertical asymptote at $x = \infty$

D The function has a vertical asymptote at $x = -1$

Solution: B

Set the function's denominator equal to 0.

$$x^2 = 0$$

$$x = 0$$

This is the value that makes the denominator 0, so the function has a vertical asymptote at $x = 0$.

Topic: Intercepts and vertical asymptotes

Question: Find the function's vertical asymptotes.

$$f(x) = \frac{x-2}{x^2-3}$$

Answer choices:

A The function has vertical asymptotes at $x = -2$ and $x = 2$

B The function has vertical asymptotes at $x = -3$ and $x = 3$

C The function has vertical asymptotes at $x = -\sqrt{3}$ and $x = \sqrt{3}$

D The function has vertical asymptotes at $x = -\sqrt{2}$ and $x = \sqrt{2}$

Solution: C

Set the function's denominator equal to 0.

$$x^2 - 3 = 0$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

These are the values that make the denominator 0, so the function has vertical asymptotes at $x = \pm\sqrt{3}$.

Topic: Intercepts and vertical asymptotes

Question: Find the function's vertical asymptotes.

$$f(x) = \frac{2x}{x^2 - 4x + 3}$$

Answer choices:

A The function has vertical asymptotes at $x = 1$ and $x = -3$

B The function has vertical asymptotes at $x = -1$ and $x = -3$

C The function has vertical asymptotes at $x = -1$ and $x = 3$

D The function has vertical asymptotes at $x = 1$ and $x = 3$

Solution: D

Set the function's denominator equal to 0.

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0$$

$x = 3$ or $x = 1$

These are the values that make the denominator 0, so the function has vertical asymptotes at $x = 1$ and $x = 3$.

Topic: Horizontal and slant asymptotes

Question: Find the function's horizontal asymptote(s).

$$f(x) = \frac{4x^3 - 2x^2 + 1}{2x^3 - 3x}$$

Answer choices:

A $y = 0$

B $y = -3$

C $y = 2$

D $y = \pm 2$

Solution: C

To find the horizontal asymptote, take the limit of the function when $x \to \infty$.

$$\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 1}{2x^3 - 3x}$$

$$\lim_{x\to\infty} \frac{4x^3 - 2x^2 + 1}{2x^3 - 3x} \left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}} \right)$$

$$\lim_{x\to\infty} \frac{\frac{4x^3 - 2x^2 + 1}{x^3}}{\frac{2x^3 - 3x}{x^3}}$$

$$\lim_{x\to\infty} \frac{4 - \frac{2}{x} + \frac{1}{x^3}}{2 - \frac{3}{x^2}}$$

Evaluate at the limit.

$$\frac{4 - 0 + 0}{2 - 0}$$

$$\frac{4}{2}$$

$$2$$

So $y = 2$ is the horizontal asymptote.

Topic: Horizontal and slant asymptotes

Question: Find the function's horizontal asymptote(s).

$$y = \frac{x^5 - x + 6}{x^7 - x^4 + 3x^2 - 1}$$

Answer choices:

A The function has a horizontal asymptote at $y = 1$

B The function has a horizontal asymptote at $y = 5/7$

C The function has a horizontal asymptote at $y = 0$

D The function has no horizontal asymptote

Solution: C

The x^5 term is the highest-degree term in the numerator, and the x^7 term is the highest-degree term in the denominator.

Because the degree of the numerator is less than the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

Topic: Horizontal and slant asymptotes

Question: Find the function's slant asymptote(s).

$$f(x) = \frac{x^2 - x + 3}{x + 1}$$

Answer choices:

A The function has a slant asymptote at $y = x + 2 + \dfrac{5}{x + 1}$

B The function has a slant asymptote at $y = x - 2 + \dfrac{5}{x + 1}$

C The function has a slant asymptote at $y = x - 2$

D The function has a slant asymptote at $y = x + 2$

Solution: C

We want to do polynomial long division with the function, which we set up as

$$x+1\,\overline{\big)\,x^2-x+3}$$

If we work through this division, we end up with

$$f(x)=x-2+\frac{5}{x+1}$$

The slant asymptote is what we get when we remove the remainder from this rewritten function. If we remove the remainder, we get

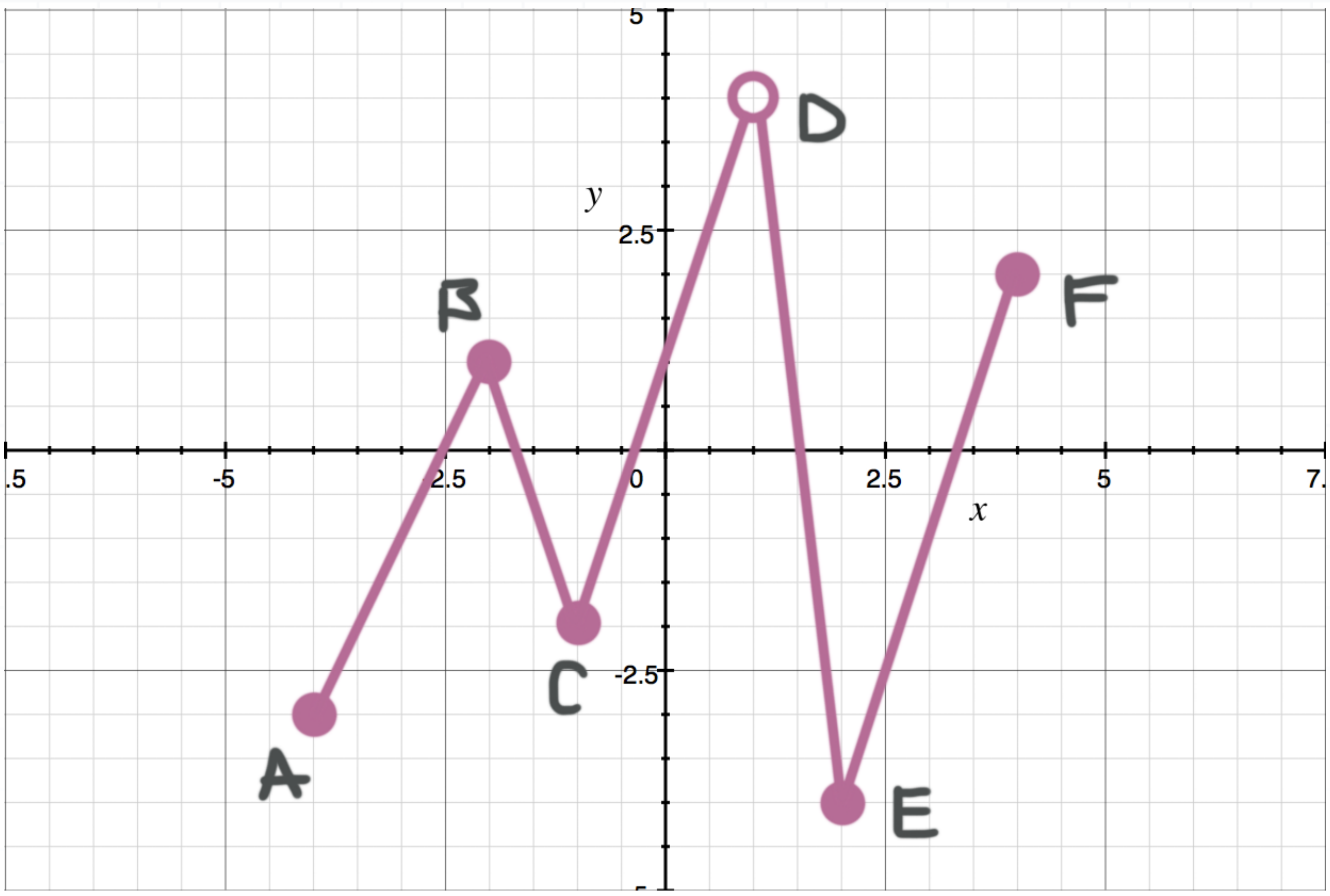
$$f(x)=x-2$$

So the equation of the slant asymptote is

$$y=x-2$$

Topic: Sketching graphs

Question: Classify the point B.



Answer choices:

A Absolute maximum

B Absolute minimum

C Local maximum

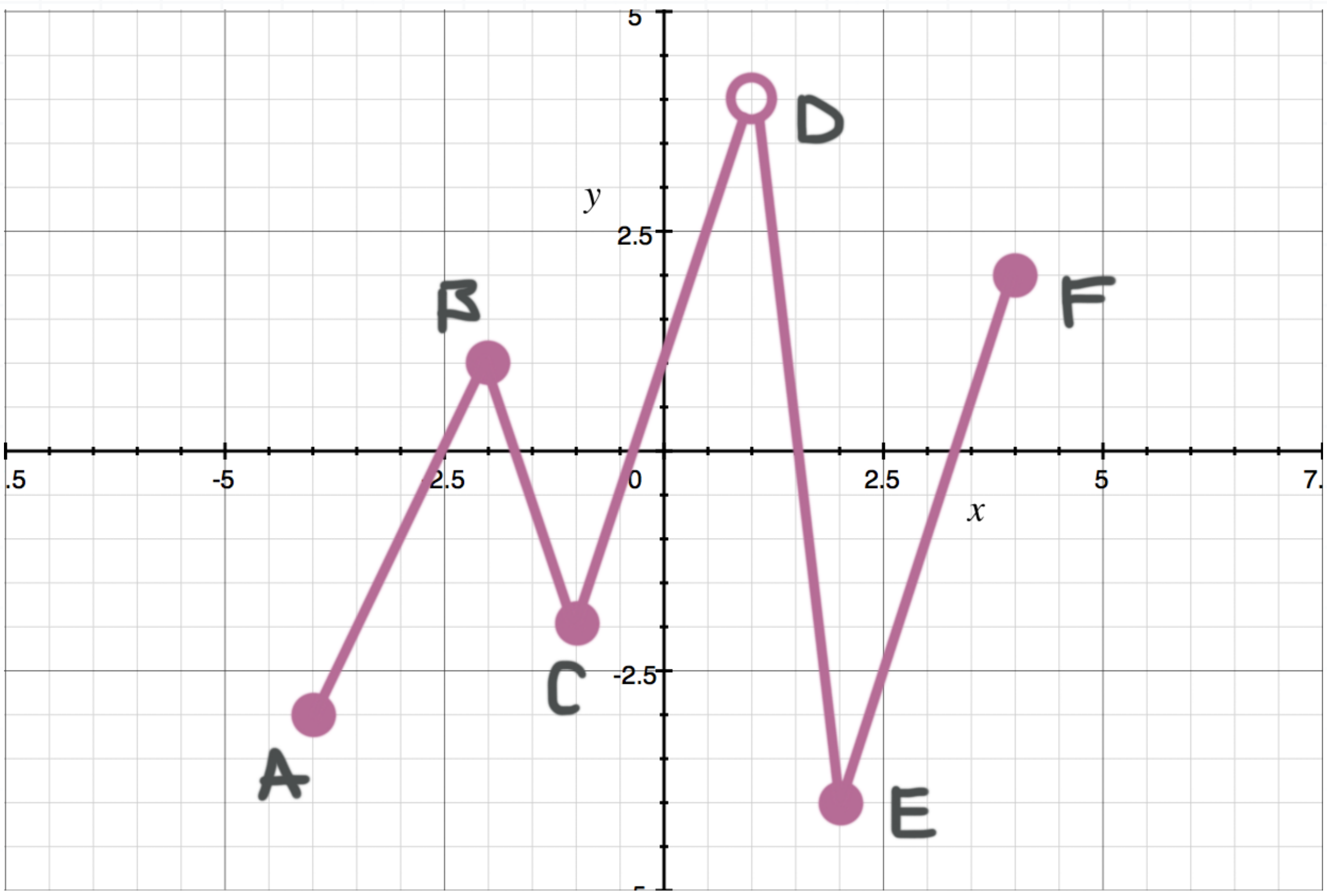
D Local minimum

Solution: C

The point B on the graph is a relative maximum, because B is the highest point in that area of the graph, but not the highest point on the entire domain of the graph.

Topic: Sketching graphs

Question: Classify the point C.



Answer choices:

A Absolute maximum

B Absolute minimum

C Local maximum

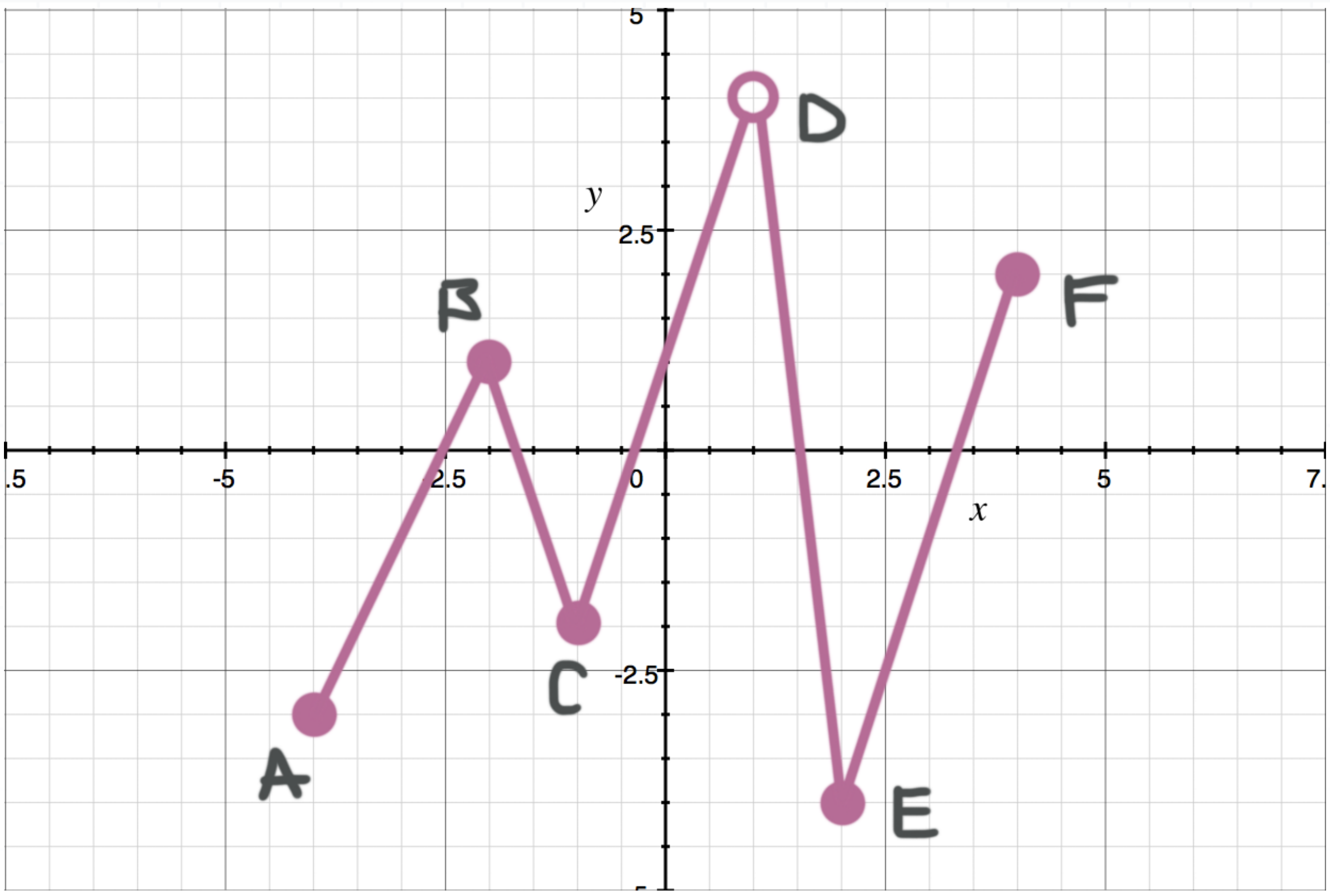
D Local minimum

Solution: D

The point C on the graph is a relative minimum because C is the lowest point in that area of the graph, but not the lowest point on the entire domain of the graph.

Topic: Sketching graphs

Question: Classify the point E.



Answer choices:

A Absolute maximum

B Absolute minimum

C Local maximum

D Local minimum

Solution: B

The point E on the graph is an absolute minimum because E is the lowest point on the entire domain of the graph.

Topic: Extrema on a closed interval

Question: Find the absolute extrema of the function on the interval $[-2,1]$.

$$f(x) = x^3 - 2x$$

Answer choices:

A	Minimum at $(1,1)$	Maximum at $(-2,10)$
B	Minimum at $(-2,-4)$	Maximum at $(-0.82,1.09)$
C	Minima at $(\pm 0.82,0)$	Maximum at $(-2,10)$
D	Minimum at $(-0.82,-2.18)$	Maximum at $(0.82,2.18)$

Solution: B

Find the first derivative,

$$f'(x) = 3x^2 - 2$$

then set it equal to 0 and solve for x.

$$3x^2 - 2 = 0$$

$$3x^2 = 2$$

$$x^2 = \frac{2}{3}$$

$$x = \pm\sqrt{\frac{2}{3}}$$

Absolute extrema could occur at these critical points and/or at the endpoints of the interval. So we'll find the value of $f(x)$ at each of these points.

At $x = -2$,

$$f(-2) = (-2)^3 - 2(-2)$$

$$f(-2) = -8 + 4$$

$$f(-2) = -4$$

At $x = -\sqrt{2/3} \approx -0.82$,

$$f\left(-\sqrt{\frac{2}{3}}\right)=\left(-\sqrt{\frac{2}{3}}\right)^3-2\left(-\sqrt{\frac{2}{3}}\right)$$

$$f\left(-\sqrt{\frac{2}{3}}\right)=-\frac{2\sqrt{2}}{3\sqrt{3}}+\frac{2\sqrt{2}}{\sqrt{3}}$$

$$f\left(-\sqrt{\frac{2}{3}}\right)=-\frac{2\sqrt{2}}{3\sqrt{3}}+\frac{6\sqrt{2}}{3\sqrt{3}}$$

$$f\left(-\sqrt{\frac{2}{3}}\right)=\frac{4\sqrt{2}}{3\sqrt{3}}$$

$$f\left(-\sqrt{\frac{2}{3}}\right)\approx 1.09$$

At $x=\sqrt{2/3}\approx 0.82$,

$$f\left(\sqrt{\frac{2}{3}}\right)=\left(\sqrt{\frac{2}{3}}\right)^3-2\left(\sqrt{\frac{2}{3}}\right)$$

$$f\left(\sqrt{\frac{2}{3}}\right)=\frac{2\sqrt{2}}{3\sqrt{3}}-\frac{2\sqrt{2}}{\sqrt{3}}$$

$$f\left(\sqrt{\frac{2}{3}}\right)=\frac{2\sqrt{2}}{3\sqrt{3}}-\frac{6\sqrt{2}}{3\sqrt{3}}$$

$$f\left(\sqrt{\frac{2}{3}}\right)=-\frac{4\sqrt{2}}{3\sqrt{3}}$$

$$f\left(\sqrt{\frac{2}{3}}\right) \approx -1.09$$

At $x = 1$,

$$f(1) = 1^3 - 2(1)$$

$$f(1) = 1 - 2$$

$$f(1) = -1$$

If we rank these points from least to greatest in terms of the function's value, we get

$$(-2, -4)$$

$$(0.82, -1.09)$$

$$(1, -1)$$

$$(-0.82, 1.09)$$

So on the interval $[-2, 1]$, the function has an absolute minimum at $(-2, -4)$ and an absolute maximum at $(-0.82, 1.09)$.

Topic: Extrema on a closed interval

Question: Find the absolute extrema of the function on the interval $[0,3]$.

$$f(x) = x^2 - 4x$$

Answer choices:

A Global minimum at $(3, -3)$; Global maximum at $(2, -4)$

B Global maximum at $(2, -4)$; Global maximum at $(3, -3)$

C Global minimum at $(0,0)$; Global maximum at $(2, -4)$

D Global minimum at $(2, -4)$; Global maximum at $(0,0)$

Solution: D

Find the first derivative,

$$f'(x) = 2x - 4$$

$$f'(x) = 2(x - 2)$$

then set it equal to 0 and solve for x.

$$2(x - 2) = 0$$

$$x - 2 = 0$$

$$x = 2$$

Absolute extrema could occur at this critical point and/or at the endpoints of the interval. So we'll find the value of $f(x)$ at each of these points.

At $x = 0$,

$$f(0) = 0^2 - 4(0)$$

$$f(0) = 0$$

At $x = 2$,

$$f(2) = 2^2 - 4(2)$$

$$f(2) = 4 - 8$$

$$f(2) = -4$$

At $x = 3$,

$$f(3) = 3^2 - 4(3)$$

$$f(3) = 9 - 12$$

$$f(3) = -3$$

If we rank these points from least to greatest in terms of the function's value, we get

$$(2, -4)$$

$$(3, -3)$$

$$(0,0)$$

So on the interval $[0,3]$, the function has an absolute minimum at $(2, -4)$ and an absolute maximum at $(0,0)$.

Topic: Extrema on a closed interval

Question: Find the absolute extrema of the function on the interval $[0,2]$.

$$f(x) = x^3 - 3x$$

Answer choices:

A Global minimum at $(1, -2)$; Global maximum at $(2,2)$

B Global minimum at $(2,2)$; Global maximum at $(1, -2)$

C Global minimum at $(-1,2)$; Global maximum at $(2,2)$

D Global minimum at $(1, -2)$; Global maxima at $(-1,2)$ and $(2,2)$

Solution: A

Find the first derivative,

$$f'(x) = 3x^2 - 3$$

$$f'(x) = 3(x^2 - 1)$$

$$f'(x) = 3(x + 1)(x - 1)$$

then set it equal to 0 and solve for x.

$$3(x + 1)(x - 1) = 0$$

$$x = -1,\ 1$$

The critical point $x = -1$ is outside the interval $[0,2]$, so we'll ignore it. Then we can say that absolute extrema could occur at just $x = 1$ and/or at the endpoints of the interval. So we'll find the value of $f(x)$ at each of these points.

At $x = 0$,

$$f(0) = 0^3 - 3(0)$$

$$f(0) = 0 - 0$$

$$f(0) = 0$$

At $x = 1$,

$$f(1) = 1^3 - 3(1)$$

$$f(1) = 1 - 3$$

$$f(1) = -2$$

At $x = 2$,

$$f(2) = 2^3 - 3(2)$$

$$f(2) = 8 - 6$$

$$f(2) = 2$$

If we rank these points from least to greatest in terms of the function's value, we get

$$(1, -2)$$

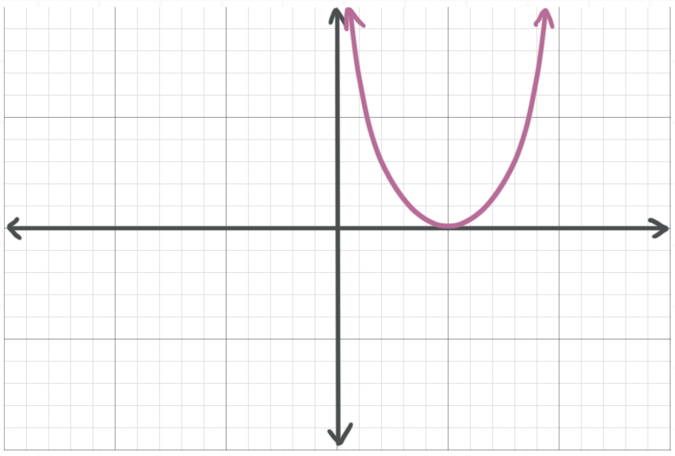
$$(0,0)$$

$$(2,2)$$

So on the interval $[0,2]$, the function has an absolute minimum at $(1, -2)$ and an absolute maximum at $(2,2)$.

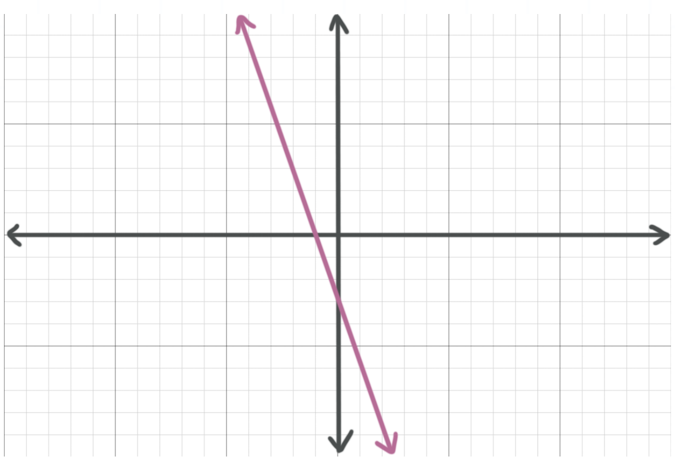
Topic: Sketching f(x) from f'(x)

Question: Given the graph of $f(x)$, which is a possible graph of $f'(x)$?

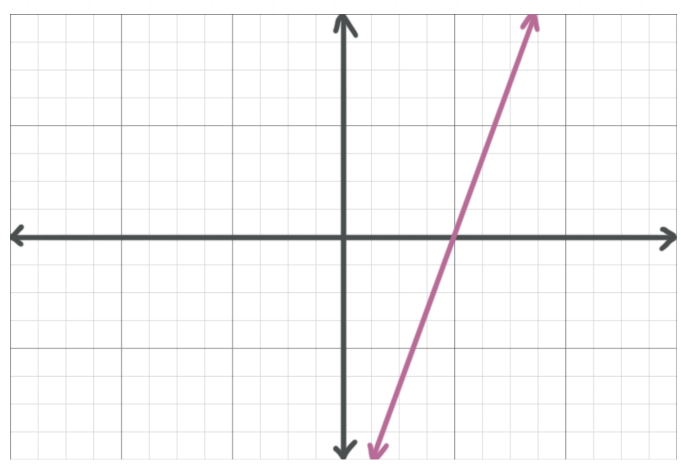


Answer choices:

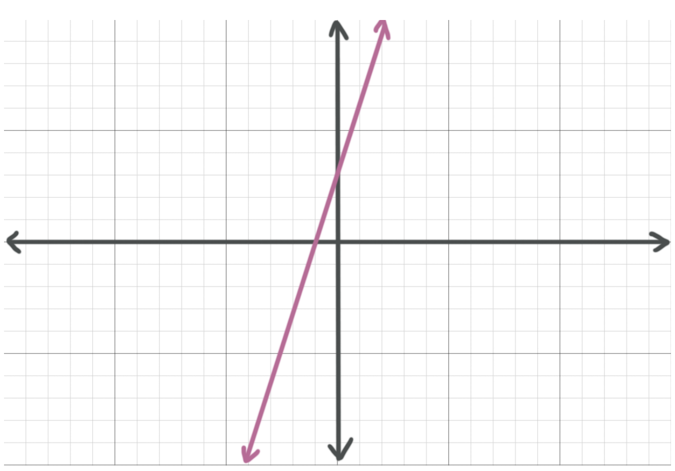
A



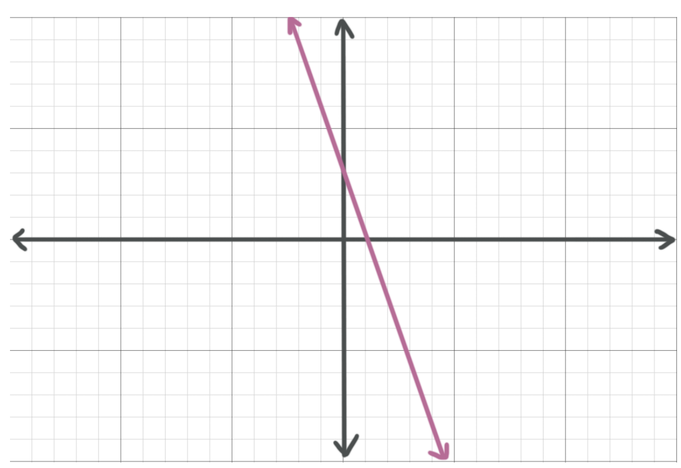
B



C



D

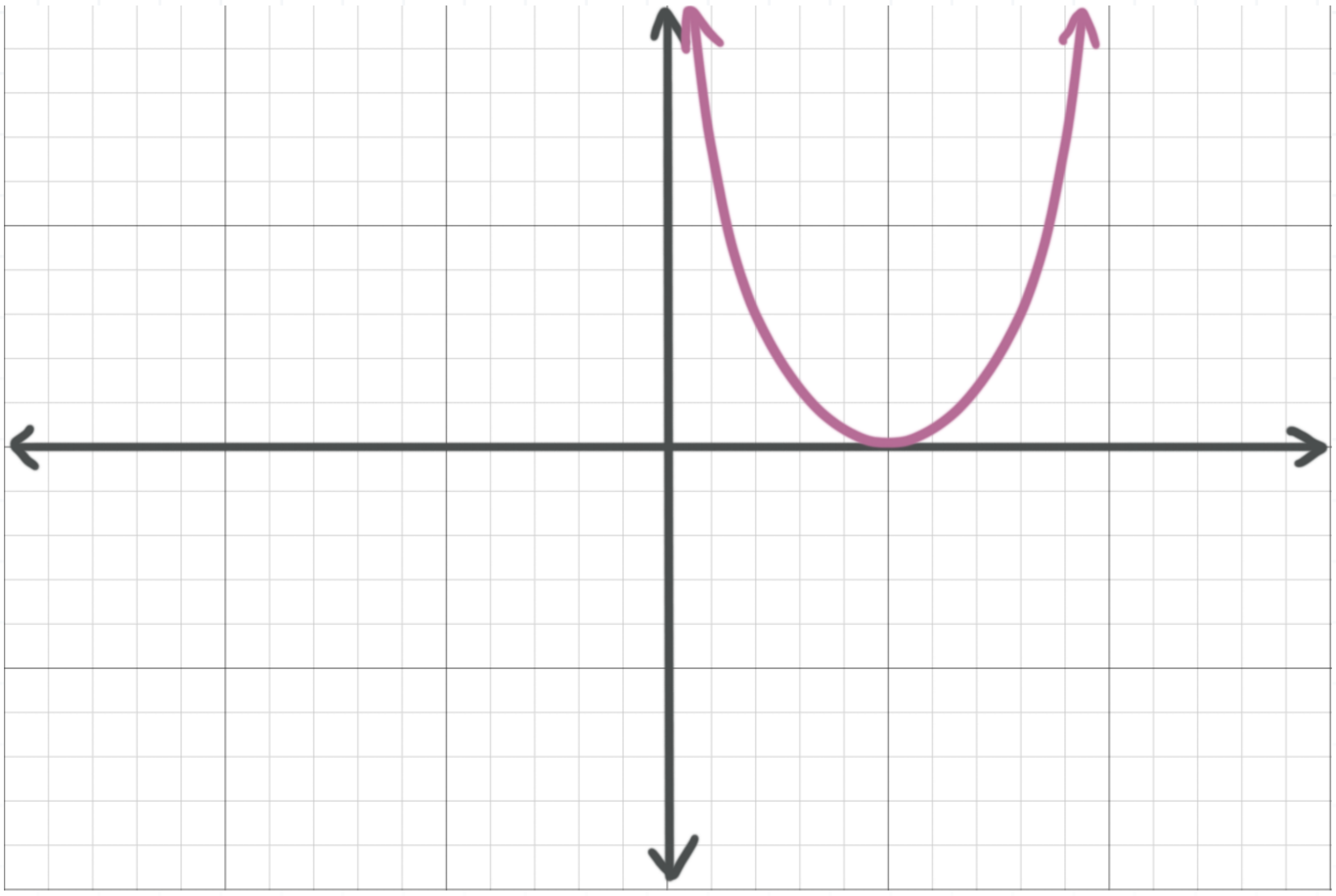


Solution: B

To take a sketch of $f(x)$ and translate it into a sketch of a possible $f'(x)$, we'll use the chart that compares the graphs of those two functions.

$f(x)$	$f'(x)$
Critical point	0 **(x-intercept)**
Increasing	**Positive (above the x-axis)**
Decreasing	**Negative (below the x-axis)**
Inflection point	**Critical point**
Concave up	**Increasing**
Concave down	**Decreasing**

If we consider the graph of $f(x)$ we've been given,



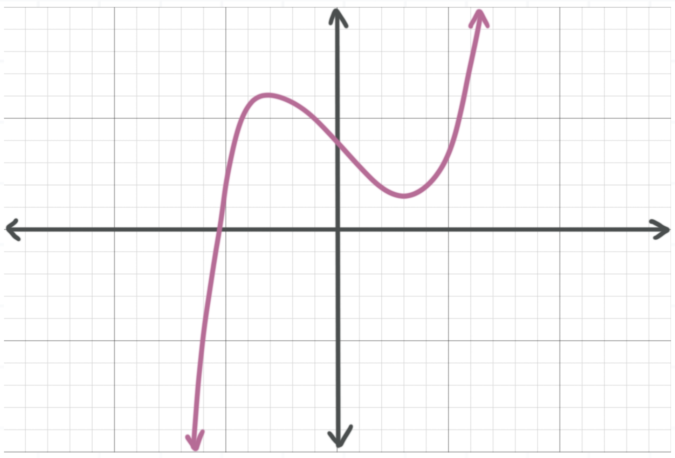
we can see right away that it's concave up everywhere, which means the graph of $f'(x)$ will be increasing everywhere, which eliminates answer choices A and D.

We can also see that the graph of $f(x)$ is decreasing to the left of what looks like $x = 5$, and then increasing to the right of that point. If that's the case, then the graph of $f'(x)$ should be negative (below the x-axis) to the left of $x = 5$ and positive (above the x-axis) to the right of $x = 5$.

That matches answer choice B, and not answer choice C.

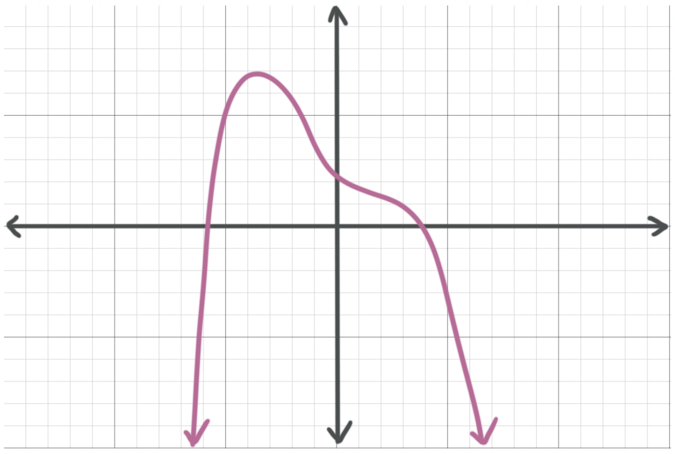
Topic: Sketching f(x) from f'(x)

Question: Given the graph of $f'(x)$, which is a possible graph of $f(x)$?

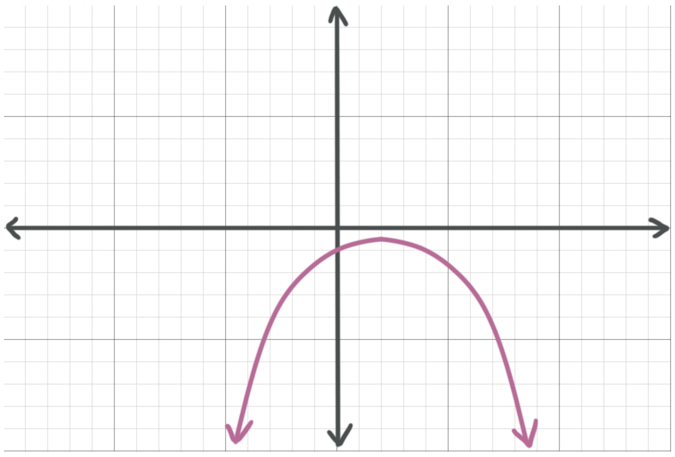


Answer choices:

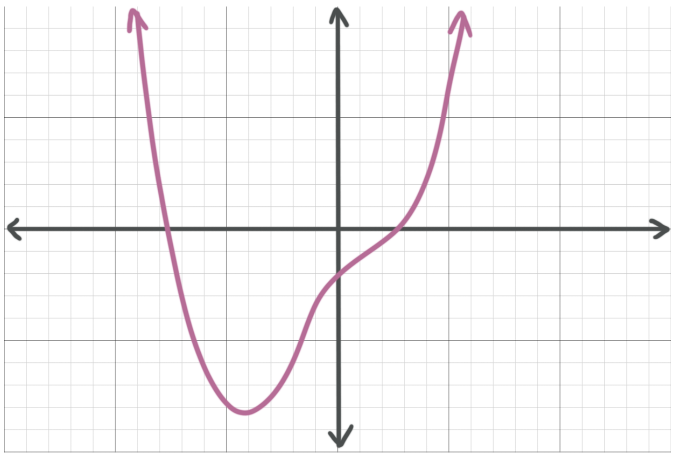
A



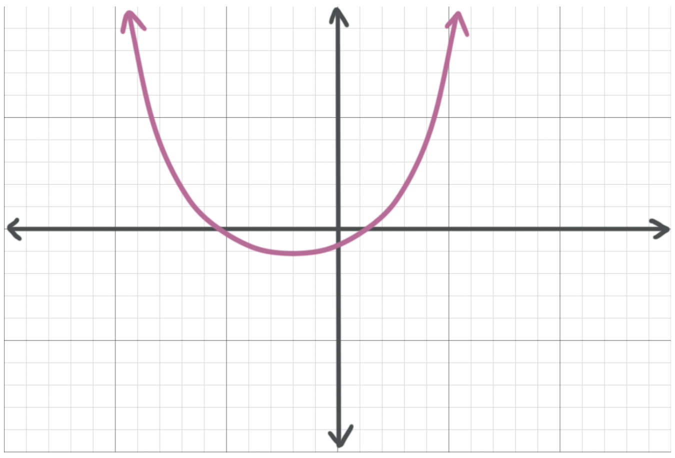
B



C



D

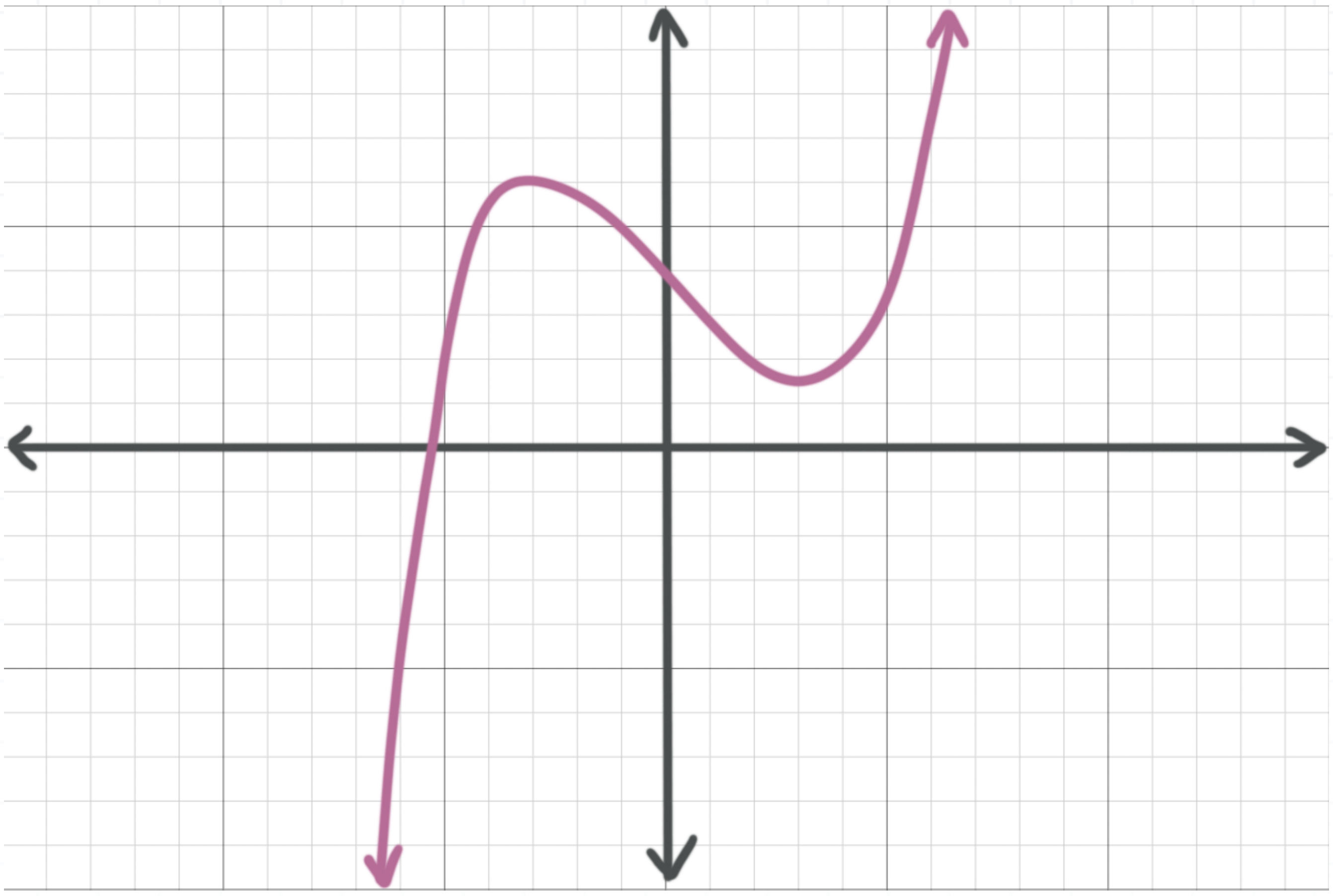


Solution: C

To take a sketch of $f'(x)$ and translate it into a sketch of a possible $f(x)$, we'll use the chart that compares the graphs of those two functions.

$f(x)$	$f'(x)$
Critical point	0 **(x-intercept)**
Increasing	**Positive (above the x-axis)**
Decreasing	**Negative (below the x-axis)**
Inflection point	**Critical point**
Concave up	**Increasing**
Concave down	**Decreasing**

If we consider the graph of $f'(x)$ we've been given,

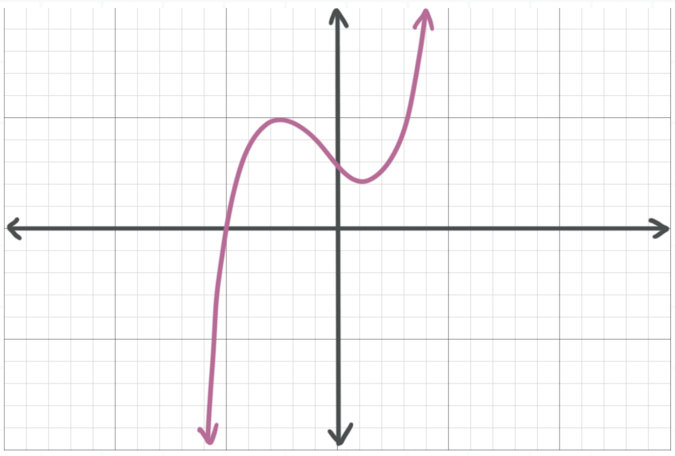


we can see right away that, as we move from left to right, it's increasing, then decreasing, then increasing again. Which means that the graph of $f(x)$, as we move from left to right, must be concave up, then concave down, then concave up again.

The only graph matching that concavity pattern is the graph in answer choice C.

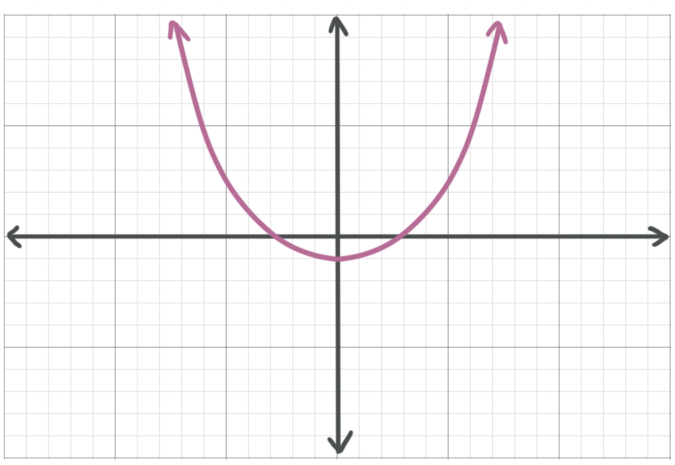
Topic: Sketching f(x) from f'(x)

Question: Given the graph of $f(x)$, which is a possible graph of $f'(x)$?

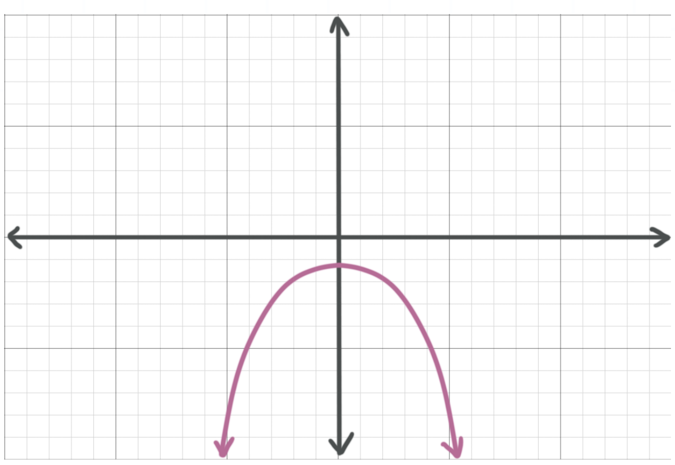


Answer choices:

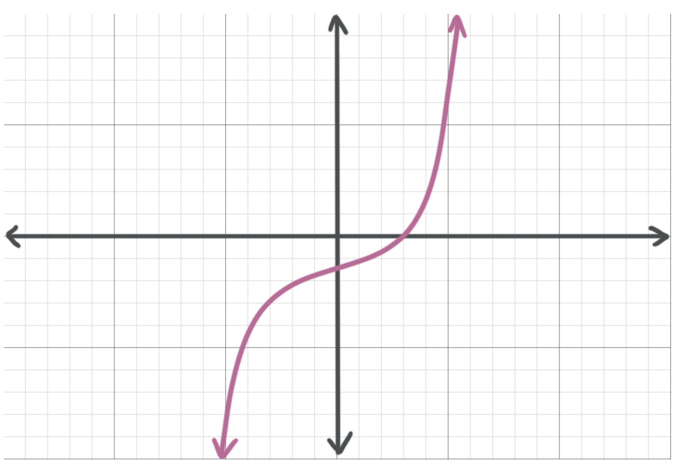
A



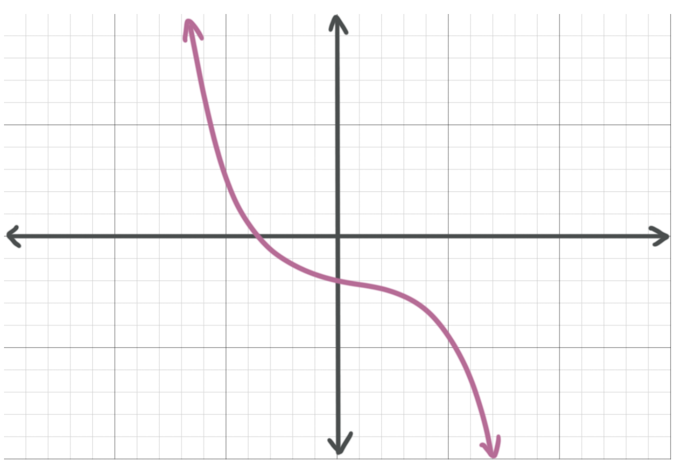
B



C



D

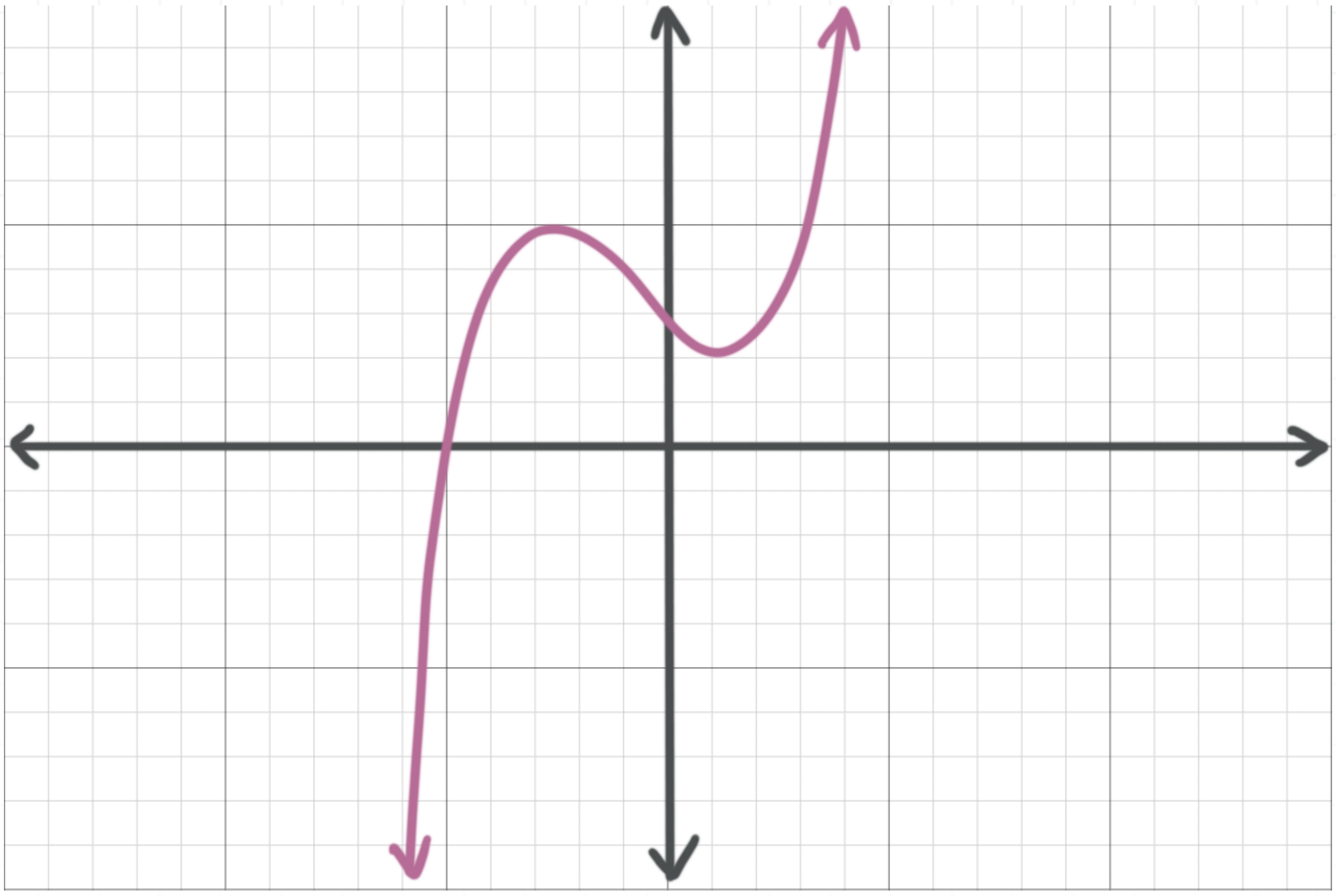


Solution: A

To take a sketch of $f(x)$ and translate it into a sketch of a possible $f'(x)$, we'll use the chart that compares the graphs of those two functions.

$f(x)$	$f'(x)$
Critical point	**0 (x-intercept)**
Increasing	**Positive (above the x-axis)**
Decreasing	**Negative (below the x-axis)**
Inflection point	**Critical point**
Concave up	**Increasing**
Concave down	**Decreasing**

If we consider the graph of $f(x)$ we've been given,



we can see right away that, as we move from left to right, it's concave down and then concave up, which means that, as we move from left to right, the graph of $f'(x)$ will be decreasing and then increasing.

The only graph matching that direction pattern is the graph in answer choice A.

Topic: Linear approximation

Question: Find the linear approximation of the function at $a = 1$.

$$f(x) = x^2$$

Answer choices:

A $L(x) = 2x - 1$

B $L(x) = -2x - 1$

C $L(x) = 2x + 1$

D $L(x) = 1 - 2x$

Solution: A

Take the derivative.

$$f(x) = x^2$$

$$f'(x) = 2x$$

Evaluate the original function at $a = 1$.

$$f(1) = 1^2$$

$$f(1) = 1$$

Evaluate the derivative at $a = 1$.

$$f'(1) = 2(1)$$

$$f'(1) = 2$$

Substitute all of these pieces into the linear approximation formula.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + 2(x - 1)$$

$$L(x) = 1 + 2x - 2$$

$$L(x) = 2x - 1$$

Topic: Linear approximation

Question: Find the linear approximation of the function at $a = 0$.

$$f(x) = 3x^2 - 6x + 1$$

Answer choices:

A $L(x) = 1 + 6x$

B $L(x) = -1 - 6x$

C $L(x) = 1 - 6x$

D $L(x) = -1 + 6x$

Solution: C

Take the derivative.

$$f(x) = 3x^2 - 6x + 1$$

$$f'(x) = 6x - 6$$

Evaluate the original function at $a = 0$.

$$f(0) = 3(0)^2 - 6(0) + 1$$

$$f(0) = 1$$

Evaluate the derivative at $a = 0$.

$$f'(0) = 6(0) - 6$$

$$f'(0) = -6$$

Substitute all of these pieces into the linear approximation formula.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + (-6)(x - 0)$$

$$L(x) = 1 - 6x$$

Topic: Linear approximation

Question: Find the linear approximation of the function at $a = 2$.

$$f(x) = (x + 4)^2$$

Answer choices:

A $L(x) = 1 + x$

B $L(x) = 12 + 12x$

C $L(x) = -12 - 12x$

D $L(x) = 1 - x$

Solution: B

Take the derivative.

$$f(x) = (x+4)^2$$

$$f'(x) = 2(x+4)(1)$$

$$f'(x) = 2x+8$$

Evaluate the original function at $a = 2$.

$$f(2) = (2+4)^2$$

$$f(2) = 36$$

Evaluate the derivative at $a = 2$.

$$f'(2) = 2(2)+8$$

$$f'(2) = 12$$

Substitute all of these pieces into the linear approximation formula.

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = 36 + 12(x-2)$$

$$L(x) = 36 + 12x - 24$$

$$L(x) = 12 + 12x$$

Topic: Estimating a root

Question: Use linear approximation to estimate $\sqrt[3]{9}$.

Answer choices:

A $\frac{41}{12}$

B $\frac{23}{12}$

C $\frac{7}{12}$

D $\frac{25}{12}$

Solution: D

We don't know the value of $\sqrt[3]{9}$, but we know that $\sqrt[3]{8} = 2$. So instead of trying to calculate $\sqrt[3]{9}$ directly, let's use the function $f(x) = \sqrt[3]{x}$.

Differentiate the function

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$

and then evaluate it at $a = 8$.

$$f'(8) = \frac{1}{3(8)^{\frac{2}{3}}}$$

$$f'(8) = \frac{1}{3(8^{\frac{1}{3}})^2}$$

$$f'(8) = \frac{1}{3(2)^2}$$

$$f'(8) = \frac{1}{3(4)}$$

$$f'(8) = \frac{1}{12}$$

So along the function $f(x) = \sqrt[3]{x}$, we have the point of tangency $(8,2)$ and the slope $m = 1/12$. Substitute these into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 2 + \frac{1}{12}(x - 8)$$

$$L(x) = 2 + \frac{1}{12}x - \frac{8}{12}$$

$$L(x) = \frac{1}{12}x - \frac{8}{12} + \frac{24}{12}$$

$$L(x) = \frac{1}{12}x + \frac{16}{12}$$

Now that we have the linear approximation equation, we can use it to estimate $\sqrt[3]{9}$. Substitute $x = 9$.

$$L(9) = \frac{1}{12}(9) + \frac{16}{12}$$

$$L(9) = \frac{9}{12} + \frac{16}{12}$$

$$L(9) = \frac{25}{12}$$

Topic: Estimating a root

Question: Use linear approximation to estimate $\sqrt[3]{29}$.

Answer choices:

A $\frac{83}{27}$

B $\frac{25}{27}$

C $\frac{137}{27}$

D $\frac{79}{27}$

Solution: A

We don't know the value of $\sqrt[3]{29}$, but we know that $\sqrt[3]{27}=3$. So instead of trying to calculate $\sqrt[3]{29}$ directly, let's use the function $f(x)=\sqrt[3]{x}$.

Differentiate the function

$$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(x)=\frac{1}{3x^{\frac{2}{3}}}$$

and then evaluate it at $a=27$.

$$f'(27)=\frac{1}{3(27)^{\frac{2}{3}}}$$

$$f'(27)=\frac{1}{3(27^{\frac{1}{3}})^2}$$

$$f'(27)=\frac{1}{3(3)^2}$$

$$f'(27)=\frac{1}{3(9)}$$

$$f'(27)=\frac{1}{27}$$

So along the function $f(x)=\sqrt[3]{x}$, we have the point of tangency $(27,3)$ and the slope $m=1/27$. Substitute these into the linear approximation equation.

$$L(x)=f(a)+f'(a)(x-a)$$

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

$$L(x) = 3 + \frac{1}{27}x - 1$$

$$L(x) = \frac{1}{27}x + 2$$

Now that we have the linear approximation equation, we can use it to estimate $\sqrt[3]{29}$. Substitute $x = 29$.

$$L(29) = \frac{1}{27}(29) + 2$$

$$L(29) = \frac{29}{27} + \frac{54}{27}$$

$$L(29) = \frac{83}{27}$$

Topic: Estimating a root

Question: Use linear approximation to estimate $\sqrt[4]{79}$.

Answer choices:

A $\quad \dfrac{322}{54}$

B $\quad \dfrac{164}{54}$

C $\quad \dfrac{161}{54}$

D $\quad \dfrac{82}{54}$

Solution: C

We don't know the value of $\sqrt[4]{79}$, but we know that $\sqrt[4]{81}=3$. So instead of trying to calculate $\sqrt[4]{79}$ directly, let's use the function $f(x)=\sqrt[4]{x}$.

Differentiate the function

$$f'(x)=\frac{1}{4}x^{-\frac{3}{4}}$$

$$f'(x)=\frac{1}{4x^{\frac{3}{4}}}$$

and then evaluate it at $a=81$.

$$f'(81)=\frac{1}{4(81)^{\frac{3}{4}}}$$

$$f'(81)=\frac{1}{4(81^{\frac{1}{4}})^3}$$

$$f'(81)=\frac{1}{4(3)^3}$$

$$f'(81)=\frac{1}{4(27)}$$

$$f'(81)=\frac{1}{108}$$

So along the function $f(x)=\sqrt[4]{x}$, we have the point of tangency $(81,3)$ and the slope $m=1/108$. Substitute these into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 3 + \frac{1}{108}(x - 81)$$

$$L(x) = 3 + \frac{1}{108}x - \frac{81}{108}$$

$$L(x) = \frac{1}{108}x - \frac{81}{108} + \frac{324}{108}$$

$$L(x) = \frac{1}{108}x + \frac{243}{108}$$

Now that we have the linear approximation equation, we can use it to estimate $\sqrt[4]{79}$. Substitute $x = 79$.

$$L(79) = \frac{1}{108}(79) + \frac{243}{108}$$

$$L(79) = \frac{79}{108} + \frac{243}{108}$$

$$L(79) = \frac{322}{108}$$

$$L(79) = \frac{161}{54}$$

Topic: Absolute, relative, and percentage error

Question: Use a linear approximation to estimate the value of $\sqrt{99}$, then find the absolute error of the estimate.

Answer choices:

A $E_A(100) \approx 0.0050$

B $E_A(100) \approx 0.0001$

C $E_A(99) \approx 0.0050$

D $E_A(99) \approx 0.0001$

Solution: D

The root $\sqrt{99}$ is very close to $\sqrt{100}$, which we already know is 10. So instead of thinking specifically about $\sqrt{99}$, let's think about $\sqrt{x}$ and use the function $f(x)=\sqrt{x}$.

Differentiate the function,

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$$

$$f'(x)=\frac{1}{2x^{\frac{1}{2}}}$$

$$f'(x)=\frac{1}{2\sqrt{x}}$$

then evaluate the derivative at $x=100$.

$$f'(100)=\frac{1}{2\sqrt{100}}$$

$$f'(100)=\frac{1}{2(10)}$$

$$f'(100)=\frac{1}{20}$$

So the linear approximation line intersects $f(x)=\sqrt{x}$ at the point of tangency $(100,10)$, and has a slope of $m=1/20$. Substitute these values into the linear approximation equation.

$$L(x)=f(a)+f'(a)(x-a)$$

$$L(x) = 10 + \frac{1}{20}(x - 100)$$

$$L(x) = 10 + \frac{1}{20}x - \frac{100}{20}$$

$$L(x) = \frac{1}{20}x - \frac{100}{20} + \frac{200}{20}$$

$$L(x) = \frac{1}{20}x + \frac{100}{20}$$

Then the linear approximation at $x = 99$ is

$$L(99) = \frac{1}{20}(99) + \frac{100}{20}$$

$$L(99) = \frac{99}{20} + \frac{100}{20}$$

$$L(99) = \frac{199}{20}$$

$$L(99) = 9.95$$

In comparison, the actual value of $\sqrt{99}$ is

$$f(x) = \sqrt{x}$$

$$f(99) = \sqrt{99}$$

$$f(99) \approx 9.9499$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(99) = |f(99) - L(99)|$$

$$E_A(99) \approx |9.9499 - 9.95|$$

$$E_A(99) \approx |-0.0001|$$

$$E_A(99) \approx 0.0001$$

Topic: Absolute, relative, and percentage error

Question: If the absolute error of a linear approximation estimate of $\sqrt{99}$ is $E_A(99) = 0.0001$, then find the relative error of the estimate.

Answer choices:

A $\quad E_R(99) \approx 0.00005001$

B $\quad E_R(99) \approx 0.00001005$

C $\quad E_R(100) \approx 0.00005001$

D $\quad E_R(100) \approx 0.00001005$

Solution: B

The actual value of $\sqrt{99}$ is

$$f(x) = \sqrt{x}$$

$$f(99) = \sqrt{99}$$

$$f(99) \approx 9.9499$$

and we've been told that the absolute error of the estimate is

$$E_A(99) = 0.0001$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(99) = \frac{E_A(99)}{f(99)}$$

$$E_R(99) \approx \frac{0.0001}{9.9499}$$

$$E_R(99) \approx 0.00001005$$

Topic: Absolute, relative, and percentage error

Question: If the absolute error of a linear approximation estimate of $\sqrt{99}$ is $E_A(99) = 0.0001$ and the relative error is $E_R(99) \approx 0.00001005$, then find the percentage error of the estimate.

Answer choices:

A $E_P(100) \approx 0.001005$

B $E_P(100) \approx 0.00001005$

C $E_P(99) \approx 0.001005$

D $E_P(99) \approx 0.00001005$

Solution: C

The actual value of $\sqrt{99}$ is

$$f(x) = \sqrt{x}$$

$$f(99) = \sqrt{99}$$

$$f(99) \approx 9.9499$$

and we've been told that the absolute error of the estimate is $E_A(99) = 0.0001$ and that the relative error of the estimate is $E_R(99) \approx 0.00001005$.

The percentage error is

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(99) = 100\% \cdot E_R(99)$$

$$E_P(99) \approx 100\% \cdot 0.00001005$$

$$E_P(99) \approx 0.001005$$

Topic: Radius of the balloon

Question: Air is being pumped into a spherical ball at a rate of 2 cm^3/s. How fast is the length of the radius increasing when $r = 10$ cm?

Answer choices:

A $\dfrac{1}{200\pi}$ cm/s

B 400π cm/s

C 200π cm/s

D $\dfrac{1}{400\pi}$ cm/s

Solution: A

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

From the question, we know that $dV/dt = 2$ and that $r = 10$, so we'll plug those in.

$$2 = 4\pi(10)^2\frac{dr}{dt}$$

$$2 = 400\pi\frac{dr}{dt}$$

Solve for dr/dt, which is the rate we were asked to find.

$$\frac{dr}{dt} = \frac{2}{400\pi}$$

$$\frac{dr}{dt} = \frac{1}{200\pi}$$

Topic: Radius of the balloon

Question: Air is being pumped into a spherical balloon at a rate of 100 cm^3/s. How fast is the radius increasing when $r = 10$ cm?

Answer choices:

A $\frac{1}{4}$ cm/s

B 4 cm/s

C $\frac{1}{4\pi}$ cm/s

D 4π cm/s

Solution: C

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

From the question, we know that $dV/dt = 100$ and that $r = 10$, so we'll plug those in.

$$100 = 4\pi(10)^2\frac{dr}{dt}$$

$$100 = 400\pi\frac{dr}{dt}$$

Solve for dr/dt, which is the rate we were asked to find.

$$\frac{dr}{dt} = \frac{100}{400\pi}$$

$$\frac{dr}{dt} = \frac{1}{4\pi}$$

Topic: Radius of the balloon

Question: Air is being sucked out of a spherical balloon so that its volume is decreasing by 250 cm^3/s. How fast is the radius decreasing when the radius is 5 cm?

Answer choices:

A $\frac{5}{2\pi}$ cm/s

B $-\frac{5}{2\pi}$ cm/s

C $\frac{2}{5\pi}$ cm/s

D $-\frac{2}{5\pi}$ cm/s

Solution: B

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

From the question, we know that $dV/dt = -250$ and that $r = 5$, so we'll plug those in.

$$-250 = 4\pi(5)^2\frac{dr}{dt}$$

$$-250 = 100\pi\frac{dr}{dt}$$

Solve for dr/dt, which is the rate we were asked to find.

$$\frac{dr}{dt} = -\frac{250}{100\pi}$$

$$\frac{dr}{dt} = -\frac{5}{2\pi}$$

Topic: Price of the product

Question: An item is currently selling for $\$50$/unit. The quantity supplied is decreasing by 10 units/week. At what rate is the price of the item changing?

$$q = 4{,}000e^{-0.01p}$$

Answer choices:

A $\$2.43$ per week

B $\$1.86$ per week

C $\$6.59$ per week

D $\$0.41$ per week

Solution: D

Use implicit differentiation to take the derivative of both sides of the quantity equation.

$$q = 4{,}000e^{-0.01p}$$

$$\frac{dq}{dt} = -40e^{-0.01p}\frac{dp}{dt}$$

From the question, we know that $p = 50$ and $dq/dt = -10$, so we'll plug those in.

$$-10 = -40e^{-0.01(50)}\frac{dp}{dt}$$

$$-10 = -40e^{-0.50}\frac{dp}{dt}$$

Solve for dp/dt, which is the rate we were asked to find.

$$\frac{dp}{dt} = \frac{-10}{-40e^{-0.50}}$$

$$\frac{dp}{dt} = \frac{1}{4e^{-0.50}}$$

$$\frac{dp}{dt} = \frac{e^{0.50}}{4}$$

$$\frac{dp}{dt} \approx \$0.41$$

Topic: Price of the product

Question: An item is currently selling for $\$100$/unit. The quantity supplied is decreasing by 20 units/week. At what rate is the price of the item changing?

$$q = 4{,}000e^{-0.01p}$$

Answer choices:

A $\$0.18$ per week

B $\$0.74$ per week

C $\$1.36$ per week

D $\$5.44$ per week

Solution: C

Use implicit differentiation to take the derivative of both sides of the quantity equation.

$$q = 4{,}000e^{-0.01p}$$

$$\frac{dq}{dt} = -40e^{-0.01p}\frac{dp}{dt}$$

From the question, we know that $p = 100$ and $dq/dt = -20$, so we'll plug those in.

$$-20 = -40e^{-0.01(100)}\frac{dp}{dt}$$

$$-20 = -40e^{-1}\frac{dp}{dt}$$

Solve for dp/dt, which is the rate we were asked to find.

$$\frac{dp}{dt} = \frac{-20}{-40e^{-1}}$$

$$\frac{dp}{dt} = \frac{1}{2e^{-1}}$$

$$\frac{dp}{dt} = \frac{e}{2}$$

$$\frac{dp}{dt} \approx \$1.36$$

Topic: Price of the product

Question: An item is currently selling for $\$150$/unit. The quantity supplied is decreasing by 25 units/week. At what rate is the price of the item changing?

$$q = 4{,}000e^{-0.01p}$$

Answer choices:

A $\$1.03$ per week

B $\$2.80$ per week

C $\$2.64$ per week

D $\$0.97$ per week

Solution: B

Use implicit differentiation to take the derivative of both sides of the quantity equation.

$$q = 4{,}000e^{-0.01p}$$

$$\frac{dq}{dt} = -40e^{-0.01p}\frac{dp}{dt}$$

From the question, we know that $p = 150$ and $dq/dt = -25$, so we'll plug those in.

$$-25 = -40e^{-0.01(150)}\frac{dp}{dt}$$

$$-25 = -40e^{-1.50}\frac{dp}{dt}$$

Solve for dp/dt, which is the rate we were asked to find.

$$\frac{dp}{dt} = \frac{-25}{-40e^{-1.50}}$$

$$\frac{dp}{dt} = \frac{5}{8e^{-1.50}}$$

$$\frac{dp}{dt} = \frac{5e^{1.50}}{8}$$

$$\frac{dp}{dt} \approx \$2.80$$

Topic: Water level in the tank

Question: The water level in an inverted, cone-shaped funnel is decreasing at a rate of 0.5 m/s. How fast is the water volume decreasing when the top surface of the water has radius $r = 2$?

Answer choices:

A $-\frac{3}{2}\pi$ m^3/s

B $-\frac{4}{3}\pi$ m^3/s

C $-\frac{2}{3}\pi$ m^3/s

D $-\frac{1}{3}\pi$ m^3/s

Solution: C

The formula for the volume of a cone is

$$V = \frac{1}{3}\pi r^2 h$$

From the question, we know that $r = 2$, so plug that in.

$$V = \frac{1}{3}\pi(2)^2 h$$

$$V = \frac{4}{3}\pi h$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = \frac{4}{3}\pi(1)\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\frac{dh}{dt}$$

From the question, we know that $dh/dt = -1/2$, so make that substitution.

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(-\frac{1}{2}\right)$$

$$\frac{dV}{dt} = -\frac{2}{3}\pi$$

Topic: Water level in the tank

Question: Water is being pumped from a cylindrical tank with a radius of 3 ft at a rate of 18 cubic feet per minute. How fast is the water level falling when the water is 2 ft deep?

Answer choices:

A $\quad -\frac{2}{\pi}$ ft/min

B $\quad -2$ ft/min

C $\quad -\pi$ ft/min

D $\quad -\frac{\pi}{2}$ ft/min

Solution: A

The formula for the volume of a cylinder is

$$V = \pi r^2 h$$

From the question, we know that $r = 3$, so plug that in.

$$V = \pi(3)^2 h$$

$$V = 9\pi h$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = 9\pi(1)\frac{dh}{dt}$$

$$\frac{dV}{dt} = 9\pi\frac{dh}{dt}$$

From the question, we know that $dV/dt = -18$, so make that substitution.

$$-18 = 9\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{18}{9\pi}$$

$$\frac{dh}{dt} = -\frac{2}{\pi}$$

Topic: Water level in the tank

Question: Water is being pumped from a cylindrical tank with a radius of 2 ft at a rate of 10 cubic feet per minute. How fast is the water level falling when the water is 5 ft deep?

Answer choices:

A $-\frac{2\pi}{5}$ ft/min

B $-\frac{5}{2\pi}$ ft/min

C $-\frac{3}{2\pi}$ ft/min

D $-\frac{2\pi}{3}$ ft/min

Solution: B

The formula for the volume of a cylinder is

$$V = \pi r^2 h$$

From the question, we know that $r = 2$, so plug that in.

$$V = \pi(2)^2 h$$

$$V = 4\pi h$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = 4\pi(1)\frac{dh}{dt}$$

$$\frac{dV}{dt} = 4\pi\frac{dh}{dt}$$

From the question, we know that $dV/dt = -10$, so make that substitution.

$$-10 = 4\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{10}{4\pi}$$

$$\frac{dh}{dt} = -\frac{5}{2\pi}$$

Topic: Observer and the airplane

Question: An airplane is flying horizontally at 720 miles/hr, 3 miles above the ground, when it passes an observer on the ground. How fast is the distance between the person and the plane increasing 10 seconds later?

Answer choices:

A About 400 miles/hr

B About 500 miles/hr

C About 600 miles/hr

D About 700 miles/hr

Solution: A

The horizontal path of the plane, the vertical distance between the observer and that path, and the diagonal connecting the observer to the plane's location, together form a right triangle.

So we'll use the Pythagorean theorem, which relates the three side lengths of a right triangle.

$$a^2 + b^2 = c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

We'll call a the vertical distance, b the horizontal distance, and c the diagonal distance. We know from the question that $a = 3$. And because a stays constant, $da/dt = 0$.

$$2(3)(0) + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

Convert $t = 10$ seconds to hours,

$$x \text{ hours} = 10 \text{ seconds} \times \frac{1 \text{ minute}}{60 \text{ seconds}} \times \frac{1 \text{ hour}}{60 \text{ minutes}}$$

$$x \text{ hours} = 0.00278 \text{ hours}$$

then use it to find the horizontal distance b.

$$b \text{ miles} = 0.00278 \text{ hours} \times \frac{720 \text{ miles}}{\text{hour}}$$

$$b \text{ miles} \approx 2 \text{ miles}$$

With $a = 3$ and $b \approx 2$, we can find c.

$$a^2 + b^2 = c^2$$

$$3^2 + 2^2 = c^2$$

$$9 + 4 = c^2$$

$$13 = c^2$$

$$c \approx 3.61$$

Substitute $b = 2$ and $c = 3.61$, along with $db/dt = 720$, into the derivative. This will give us the rate at which the distance between the observer and the airplane is increasing.

$$2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2(2)(720) = 2(3.61)\ \frac{dc}{dt}$$

$$\frac{dc}{dt} \approx 399$$

Topic: Observer and the airplane

Question: An airplane is flying horizontally at 540 miles/hr, 4 miles above the ground, when it passes an observer on the ground. How fast is the distance between the person and the plane increasing 20 seconds later?

Answer choices:

A About 424 miles/hr

B About 324 miles/hr

C About 224 miles/hr

D About 124 miles/hr

Solution: B

The horizontal path of the plane, the vertical distance between the observer and that path, and the diagonal connecting the observer to the plane's location, together form a right triangle.

So we'll use the Pythagorean theorem, which relates the three side lengths of a right triangle.

$$a^2 + b^2 = c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

We'll call a the vertical distance, b the horizontal distance, and c the diagonal distance. We know from the question that $a = 4$. And because a stays constant, $da/dt = 0$.

$$2(4)(0) + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

Convert $t = 20$ seconds to hours,

$$x \text{ hours} = 20 \text{ seconds} \times \frac{1 \text{ minute}}{60 \text{ seconds}} \times \frac{1 \text{ hour}}{60 \text{ minutes}}$$

$$x \text{ hours} = 0.0056 \text{ hours}$$

then use it to find the horizontal distance b.

$$b \text{ miles} = 0.0056 \text{ hours} \times \frac{540 \text{ miles}}{\text{hour}}$$

$$b \text{ miles} \approx 3 \text{ miles}$$

With $a = 4$ and $b \approx 3$, we can find c.

$$a^2 + b^2 = c^2$$

$$4^2 + 3^2 = c^2$$

$$16 + 9 = c^2$$

$$25 = c^2$$

$$c = 5$$

Substitute $b = 3$ and $c = 5$, along with $db/dt = 540$, into the derivative. This will give us the rate at which the distance between the observer and the airplane is increasing.

$$2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2(3)(540) = 2(5)\ \frac{dc}{dt}$$

$$\frac{dc}{dt} = 324$$

Topic: Observer and the airplane

Question: An airplane is flying horizontally at 360 miles/hr, 5 miles above the ground, when it passes an observer on the ground. How fast is the distance between the person and the plane increasing 40 seconds later?

Answer choices:

A 125 miles/hr

B 225 miles/hr

C 325 miles/hr

D 425 miles/hr

Solution: B

The horizontal path of the plane, the vertical distance between the observer and that path, and the diagonal connecting the observer to the plane's location, together form a right triangle.

So we'll use the Pythagorean theorem, which relates the three side lengths of a right triangle.

$$a^2 + b^2 = c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

We'll call a the vertical distance, b the horizontal distance, and c the diagonal distance. We know from the question that $a = 5$. And because a stays constant, $da/dt = 0$.

$$2(5)(0) + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

Convert $t = 40$ seconds to hours,

$$x \text{ hours} = 40 \text{ seconds} \times \frac{1 \text{ minute}}{60 \text{ seconds}} \times \frac{1 \text{ hour}}{60 \text{ minutes}}$$

$$x \text{ hours} = 0.0111 \text{ hours}$$

then use it to find the horizontal distance b.

$$b \text{ miles} = 0.0111 \text{ hours} \times \frac{360 \text{ miles}}{\text{hour}}$$

$$b \text{ miles} \approx 4 \text{ miles}$$

With $a = 5$ and $b \approx 4$, we can find c.

$$a^2 + b^2 = c^2$$

$$5^2 + 4^2 = c^2$$

$$25 + 16 = c^2$$

$$41 = c^2$$

$$c \approx 6.4$$

Substitute $b = 4$ and $c = 6.4$, along with $db/dt = 360$, into the derivative. This will give us the rate at which the distance between the observer and the airplane is increasing.

$$2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2(4)(360) = 2(6.4)\ \frac{dc}{dt}$$

$$\frac{dc}{dt} = 225$$

Topic: Ladder sliding down the wall

Question: A gardener's shovel is 1 m long and leaning against a fence, sliding down the fence at a rate of 0.25 m/s. When the top of the shovel is 0.5 m off the ground, at what rate is the bottom of the shovel sliding along the ground away from the fence?

Answer choices:

A $\quad \dfrac{3\sqrt{3}}{4}$ m/s

B $\quad \dfrac{4\sqrt{3}}{3}$ m/s

C $\quad \dfrac{4}{3}$ m/s

D $\quad \dfrac{\sqrt{3}}{12}$ m/s

Solution: D

The ground, the fence, and the shovel form a right triangle, so we'll use the Pythagorean Theorem as the equation that relates the side lengths.

$$a^2 + b^2 = c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

From the question we know that the length of the shovel is $c = 1$, and that the length of the shovel doesn't change, so $dc/dt = 0$.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2(1)(0)$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$a\frac{da}{dt} + b\frac{db}{dt} = 0$$

If we say that the vertical fence is side b, and that the horizontal ground is side a, then the question tells us that $b = 1/2$ and that $db/dt = -1/4$.

$$a\frac{da}{dt} + \frac{1}{2}\left(-\frac{1}{4}\right) = 0$$

$$a\frac{da}{dt} - \frac{1}{8} = 0$$

Find the value of a when $b = 1/2$ and $c = 1$.

$$a^2 + b^2 = c^2$$

$$a^2 + \left(\frac{1}{2}\right)^2 = 1^2$$

$$a^2 + \frac{1}{4} = 1$$

$$a^2 = \frac{3}{4}$$

$$a = \frac{\sqrt{3}}{2}$$

We're asked to solve for da/dt, so we'll plug in this value of a that we've found and then solve the equation for da/dt.

$$\left(\frac{\sqrt{3}}{2}\right)\frac{da}{dt} - \frac{1}{8} = 0$$

$$\left(\frac{\sqrt{3}}{2}\right)\frac{da}{dt} = \frac{1}{8}$$

$$\frac{da}{dt} = \frac{2}{8\sqrt{3}}$$

$$\frac{da}{dt} = \frac{1}{4\sqrt{3}}$$

Rationalize the denominator.

$$\frac{1}{4\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$\frac{\sqrt{3}}{4(3)}$$

$$\frac{\sqrt{3}}{12}$$

Topic: Ladder sliding down the wall

Question: A 5 foot ladder is sliding down a vertical wall while its bottom slides away from the wall at 3 ft/s. How fast is the top moving when the top is 4 feet off the ground?

Answer choices:

A $\quad -\frac{9}{4}$ ft/s

B $\quad -\frac{4}{9}$ ft/s

C $\quad -\frac{3}{2}$ ft/s

D $\quad -\frac{2}{3}$ ft/s

Solution: A

The ground, the wall, and the ladder form a right triangle, so we'll use the Pythagorean Theorem as the equation that relates the side lengths.

$$a^2 + b^2 = c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

From the question we know that the length of the ladder is $c = 5$, and that the length of the ladder doesn't change, so $dc/dt = 0$.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2(5)(0)$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$a\frac{da}{dt} + b\frac{db}{dt} = 0$$

If we say that the vertical wall is side b, and that the horizontal ground is side a, then the question tells us that $b = 4$ and that $da/dt = 3$.

$$a(3) + 4\frac{db}{dt} = 0$$

$$3a + 4\frac{db}{dt} = 0$$

Find the value of a when $b = 4$ and $c = 5$.

$$a^2 + b^2 = c^2$$

$$a^2 + 4^2 = 5^2$$

$$a^2 + 16 = 25$$

$$a^2 = 9$$

$$a = 3$$

We're asked to solve for db/dt, so we'll plug in this value of a that we've found and then solve the equation for db/dt.

$$3(3) + 4\frac{db}{dt} = 0$$

$$9 + 4\frac{db}{dt} = 0$$

$$4\frac{db}{dt} = -9$$

$$\frac{db}{dt} = -\frac{9}{4}$$

Topic: Ladder sliding down the wall

Question: A 13 foot ladder is sliding down a vertical wall while its bottom slides away from the wall at 9 ft/s. How fast is the top moving when the top is 5 feet off the ground?

Answer choices:

A -2.16 ft/s

B -21.6 ft/s

C -216 ft/s

D $-2{,}160$ ft/s

Solution: B

The ground, the wall, and the ladder form a right triangle, so we'll use the Pythagorean Theorem as the equation that relates the side lengths.

$$a^2 + b^2 = c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

From the question we know that the length of the ladder is $c = 13$, and that the length of the ladder doesn't change, so $dc/dt = 0$.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2(13)(0)$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$a\frac{da}{dt} + b\frac{db}{dt} = 0$$

If we say that the vertical wall is side b, and that the horizontal ground is side a, then the question tells us that $b = 5$ and that $da/dt = 9$.

$$a(9) + 5\frac{db}{dt} = 0$$

$$9a + 5\frac{db}{dt} = 0$$

Find the value of a when $b = 5$ and $c = 13$.

$$a^2 + b^2 = c^2$$

$$a^2 + 5^2 = 13^2$$

$$a^2 + 25 = 169$$

$$a^2 = 144$$

$$a = 12$$

We're asked to solve for db/dt, so we'll plug in this value of a that we've found and then solve the equation for db/dt.

$$9(12) + 5\frac{db}{dt} = 0$$

$$108 + 5\frac{db}{dt} = 0$$

$$5\frac{db}{dt} = -108$$

$$\frac{db}{dt} = -\frac{108}{5}$$

$$\frac{db}{dt} = -21.6$$

Topic: Applied optimization

Question: A rancher plans to build a fenced rectangular area adjacent to an existing stone wall. He wants the fence to enclose $160,000$ square meters for his horses, but he's low on fencing. Which dimensions require the least amount of fencing?

Answer choices:

A 400×400

B 283×565

C $4,000 \times 8,000$

D $100 \times 1,600$

Solution: B

Since one side of the enclosure doesn't need any fencing, the amount of fencing needed to enclose the area is

$$P = 2x + y$$

where x is the length of a side adjacent to the wall and y is the length of the side opposite the wall. We can also write an equation for area in terms of x and y, and then solve it for y.

$$A = xy$$

$$160{,}000 = xy$$

$$y = \frac{160{,}000}{x}$$

Substitute this value into the perimeter equation.

$$P = 2x + \frac{160{,}000}{x}$$

Take the derivative,

$$P' = 2 - \frac{160{,}000}{x^2}$$

then set it equal to 0 to find critical points.

$$2 - \frac{160{,}000}{x^2} = 0$$

$$2 = \frac{160{,}000}{x^2}$$

$$2x^2 = 160{,}000$$

$$x^2 = 80{,}000$$

$$x = \sqrt{80{,}000}$$

$$x \approx 283$$

Use the first derivative test with test values of 280 and 290 to confirm that $x \approx 283$ represents a minimum.

$$P'(280) = 2 - \frac{160{,}000}{280^2}$$

$$P'(280) \approx -0.04$$

and

$$P'(290) = 2 - \frac{160{,}000}{290^2}$$

$$P'(290) \approx 0.10$$

Because we get a negative value to the left of the critical point and a positive value to the right, $x \approx 283$ represents a minimum. The associated value for y is

$$y = \frac{160{,}000}{283}$$

$$y \approx 565$$

Topic: Applied optimization

Question: With $1{,}000$ m of new fencing material, you need to enclose a rectangular yard and maximize its area. What dimensions should you use?

Answer choices:

A 250×250

B 250×750

C 500×500

D 125×375

Solution: A

If we use l for length and w for width, we can say the perimeter of the yard is

$$P = 2l + 2w$$

$$1{,}000 = 2l + 2w$$

and that the area is

$$A = lw$$

We want to maximize area, so we'll solve the perimeter equation for width.

$$2w = 1{,}000 - 2l$$

$$w = 500 - l$$

Substitute this value into the area equation.

$$A = l(500 - l)$$

$$A = 500l - l^2$$

Take the derivative, then set it equal to 0 to find critical points.

$$A' = 500 - 2l$$

$$500 - 2l = 0$$

$$2l = 500$$

$$l = 250$$

Use the first derivative test with test values of $l=240$ and $l=260$ to make sure that $l=250$ represents a maximum.

$$A'(240)=500-2(240)$$

$$A'(240)=500-480$$

$$A'(240)=20$$

and

$$A'(260)=500-2(260)$$

$$A'(260)=500-520$$

$$A'(260)=-20$$

Since we got a positive value to the left of the critical point and a negative value to the right of it, we know $l=250$ represents a maximum. The corresponding width is

$$w=500-250$$

$$w=250$$

Topic: Applied optimization

Question: You want to construct a box with a square bottom and you only have 36 m^2 of material. Assuming you use all of the material, what is the maximum volume of the box?

Answer choices:

A $12\sqrt{6}$ m^3

B 36 m^3

C 18 m^3

D $6\sqrt{6}$ m^3

Solution: D

The volume of a box is always given by $V = lwh$, but since we've been told that the box has a square base, we know that $l = w$, so we can simplify the volume equation to $V = l^2h$.

The surface area of a box is always given by $A = 2lw + 2lh + 2wh$. But since $l = w$, we can simplify this as

$$A = 2ll + 2lh + 2lh$$

$$A = 2l^2 + 4lh$$

We know that total surface area is 36, so

$$36 = 2l^2 + 4lh$$

$$18 = l^2 + 2lh$$

Solve this area equation for height h.

$$2lh = 18 - l^2$$

$$h = \frac{18 - l^2}{2l}$$

Substitute this value into the volume equation.

$$V = l^2h$$

$$V = l^2\left(\frac{18 - l^2}{2l}\right)$$

$$V = \frac{18l - l^3}{2}$$

$$V = 9l - \frac{1}{2}l^3$$

Take the derivative,

$$V' = 9 - \frac{3}{2}l^2$$

then set it equal to 0 to find critical points.

$$9 - \frac{3}{2}l^2 = 0$$

$$\frac{3}{2}l^2 = 9$$

$$l^2 = 6$$

$$l = \pm\sqrt{6}$$

It's nonsensical to have a negative length, so the only critical point is $l = \sqrt{6}$. Use the first derivative test with test values of $l = 2$ and $l = 3$ to confirm that the critical point represents a maximum.

$$V'(2) = 9 - \frac{3}{2}(2)^2$$

$$V'(2) = 9 - 6$$

$$V'(2) = 3$$

and

$$V'(3) = 9 - \frac{3}{2}(3)^2$$

$$V'(3) = \frac{18}{2} - \frac{27}{2}$$

$$V'(3) = -\frac{9}{2}$$

Since we get a positive value to the left of the critical point and a negative value to the right of it, the function has a maximum at the critical point.

We found the length l associated with the critical point, but we were asked for the maximum volume, so now we just need to find the volume that corresponds with this length, which we'll do by plugging $l = \sqrt{6}$ into $V = 9l - (1/2)l^3$.

$$V = 9\sqrt{6} - \frac{1}{2}(\sqrt{6})^3$$

$$V = 9\sqrt{6} - \frac{6}{2}\sqrt{6}$$

$$V = 9\sqrt{6} - 3\sqrt{6}$$

$$V = 6\sqrt{6}$$

The maximum volume of the box is $6\sqrt{6}$ m^3.

Topic: Mean Value Theorem

Question: Which is the correct statement of the Mean Value Theorem?

Answer choices:

A	If f is continuous and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

B	If f is continuous on the closed interval $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

C	If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

D	If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Solution: D

The Mean Value Theorem states:

> If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Topic: Mean Value Theorem

Question: Which statement is true?

Answer choices:

A The Mean Value Theorem applies to functions that are continuous and differentiable on a given interval (a, b) and states that there will be a point c in the interval such that $f(b) - f(a) = f'(c)(a - b)$.

B The Mean Value Theorem applies to functions that are discontinuous and differentiable on a given interval $[a, b]$ and states that there will be a point c in the interval such that $f(b) - f(a) = f'(c)(b - a)$.

C The Mean Value Theorem applies to functions that are continuous and differentiable on a given interval $[a, b]$ and states that there will be a point c in the interval such that $f(b) - f(a) = f'(c)(b - a)$.

D The Mean Value Theorem applies to functions that are discontinuous and differentiable on a given interval $[a, b]$ and states that there will be a point c in the interval such that $f(b) - f(a) = f'(c)(a - b)$.

Solution: C

The Mean Value Theorem applies to functions that are continuous and differentiable on a given interval $[a, b]$ and states that there will be a point c in the interval such that

$$f(b) - f(a) = f'(c)(b - a)$$

Topic: Mean Value Theorem

Question: Two police officers are sitting separately along a highway, 3 miles apart. A car passes the first officer at 60 mph, and passes the second offer two minutes later at 58 mph. Can the officers prove that the car was speeding (going faster than 65 mph) at some point between them?

Answer choices:

A Yes, they can prove the car was speeding

B No, they can't prove the car was speeding

C It's impossible to say one way or the other

Solution: A

The officers can use the Mean Value Theorem to prove that the car was traveling faster than the 65 mph speed limit at some point between them.

Let time $t = 0$ and position $s = 0$ when the car passes the first officer, and let $t = 2/60 = 1/30$ hr and $s(1/30) = 3$ miles. Then the average speed of the car over the 3 miles (or 2 minutes) is

$$v_{avg} = \frac{s\left(\frac{1}{30}\right) - s(0)}{\frac{1}{30} - 0}$$

$$v_{avg} = \frac{3 - 0}{\frac{1}{30}}$$

$$v_{avg} = 3 \cdot \frac{30}{1}$$

$$v_{avg} = 90 \text{ mph}$$

By the Mean Value Theorem, the car must have been traveling at 90 mph at some point along the 3-mile stretch.

Topic: Rolle's Theorem

Question: Which of these is not part of Rolle's Theorem?

Answer choices:

A The function $f(x)$ must be continuous on the closed interval $[a, b]$.

B That the function $f(x)$ meets the condition $f(a) = f(b)$ for the closed interval $[a, b]$.

C The function $f(x)$ can be integrated over the open interval (a, b).

D The function $f(x)$ is differentiable on the open interval (a, b).

Solution: C

Rolle's Theorem requires three conditions be met in order for its conclusion to be true:

- The function $f(x)$ must be continuous on the closed interval $[a, b]$
- The function $f(x)$ must be differentiable on the open interval (a, b)
- The function $f(x)$ meets the condition $f(a) = f(b)$ for the interval $[a, b]$

If these three conditions are met, Rolle's Theorem states that there must exist a point c within the interval (a, b) where $f'(c) = 0$.

Topic: Rolle's Theorem

Question: Does the function meet the criteria of Rolle's Theorem on the interval $[0,1]$?

$$f(x) = x^2 - x + 6$$

Answer choices:

A Yes, it's continuous and differentiable over the interval, and $f(0) = f(1)$.

B Yes, it's continuous and $f(0) \neq f(1)$.

C No, it's not differentiable over the interval.

D No, it's discontinuous, and $f(0) \neq f(1)$.

Solution: A

Rolle's Theorem requires three conditions be met in order for its conclusion to be true:

- The function $f(x)$ must be continuous on the closed interval $[a,b]$
- The function $f(x)$ is differentiable on the open interval (a,b)
- The function $f(x)$ meets the condition $f(a)=f(b)$ for the interval $[a,b]$

If these three conditions are met, Rolle's Theorem states that there must exist a point c within the interval (a,b) where $f'(c)=0$.

Since this is a polynomial function, and we know that polynomial functions are continuous for all real numbers, we know that the function is continuous on the interval $[0,1]$.

Taking the function's derivative,

$$f'(x)=2x-1$$

we can see that the function is differentiable.

Confirm that $f(0)=f(1)$.

$$f(0)=0^2-0+6$$

$$f(0)=6$$

and

$$f(1) = 1^2 - 1 + 6$$

$$f(1) = 6$$

Since $f(0) = 6 = f(1)$, we've confirmed that this function over the given interval meets all three conditions of Rolle's Theorem.

Topic: Rolle's Theorem

Question: Use Rolle's Theorem to find the point in the interval $[0,4]$ where the function has a horizontal tangent line.

$$f(x) = -x^2 + 4x + 16$$

Answer choices:

A $(0,4)$

B $(-2,20)$

C $(0,-4)$

D $(2,20)$

Solution: D

Since this is a polynomial function, and we know that polynomial functions are continuous for all real numbers, we know that the function is continuous on the interval $[0,4]$.

Taking the derivative,

$$f'(x) = -2x + 4$$

we can see that the function is differentiable. And

$$f(0) = -0^2 + 4(0) + 16$$

$$f(0) = 16$$

and

$$f(4) = -4^2 + 4(4) + 16$$

$$f(4) = 16$$

Since $f(0) = 16 = f(4)$, we've confirmed that the function over the given interval meets all three conditions of Rolle's Theorem.

Now we can find the point c by solving the equation $f'(c) = 0$.

$$-2x + 4 = 0$$

$$2x = 4$$

$$x = 2$$

To find the coordinate point associated with $c = 2$, we'll plug it back into the original function.

$$f(2) = -2^2 + 4(2) + 16$$

$$f(2) = -4 + 8 + 16$$

$$f(2) = 20$$

The conclusion of Rolle's Theorem tells us that the function has a horizontal tangent line at $(2,20)$ inside the interval $[0,4]$.

Topic: Newton's Method

Question: Use Newton’s Method to approximate to three decimal places the root of the function on the interval $[1/2,2]$.

$$f(x) = x^2 - 1$$

Answer choices:

A $x = 1.500$

B $x = 1.000$

C $x = 2.000$

D $x = 0.500$

Solution: B

Take the derivative of the function.

$$f(x_n) = x_n^2 - 1$$

$$f'(x_n) = 2x_n$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 1}{2x_n}$$

Let's start with the left endpoint of the interval, $x_n = 1/2$, and work our problem with the number of decimals we were asked for.

$$x_0 = 0.500$$

$$x_1 = 0.500 - \frac{0.500^2 - 1}{2(0.500)} = 1.250$$

$$x_2 = 1.250 - \frac{1.250^2 - 1}{2(1.250)} = 1.025$$

$$x_3 = 1.025 - \frac{1.025^2 - 1}{2(1.025)} = 1.000$$

$$x_4 = 1.000 - \frac{1.000^2 - 1}{2(1.000)} = 1.000$$

Since these last two approximations were identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 1.000$.

Topic: Newton's Method

Question: Use Newton's Method to approximate to three decimal places the root of the function on the interval $[1,2]$.

$$f(x) = 2x^2 - 3$$

Answer choices:

A $x = 1.525$

B $x = 1.255$

C $x = 1.522$

D $x = 1.225$

Solution: D

Take the derivative of the function.

$$f(x_n) = 2x_n^2 - 3$$

$$f'(x_n) = 4x_n$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2x_n^2 - 3}{4x_n}$$

Let's start with the left endpoint of the interval, $x_n = 1$, and work our problem with the number of decimals we were asked for.

$$x_0 = 1.000$$

$$x_1 = 1.000 - \frac{2(1.000)^2 - 3}{4(1.000)} = 1.250$$

$$x_2 = 1.250 - \frac{2(1.250)^2 - 3}{4(1.250)} = 1.225$$

$$x_3 = 1.225 - \frac{2(1.225)^2 - 3}{4(1.225)} = 1.225$$

Since these last two approximations were identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 1.225$.

Topic: Newton's Method

Question: Use Newton's Method to approximate to three decimal places the root of the function on the interval $[3,4]$.

$$f(x) = x^2 - 3x - 1$$

Answer choices:

A $x = 3.303$

B $x = 3.322$

C $x = 3.032$

D $x = 3.332$

Solution: A

Take the derivative of the function.

$$f(x_n) = x_n^2 - 3x_n - 1$$

$$f'(x_n) = 2x_n - 3$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 3x_n - 1}{2x_n - 3}$$

Let's start with the left endpoint of the interval, $x_n = 3$, and work our problem with the number of decimals we were asked for.

$$x_0 = 3.000$$

$$x_1 = 3.000 - \frac{(3.000)^2 - 3(3.000) - 1}{2(3.000) - 3} = 3.333$$

$$x_2 = 3.333 - \frac{(3.333)^2 - 3(3.333) - 1}{2(3.333) - 3} = 3.303$$

$$x_3 = 3.303 - \frac{(3.303)^2 - 3(3.303) - 1}{2(3.303) - 3} = 3.303$$

Since these last two approximations were identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 3.303$.

Topic: L'Hospital's Rule

Question: Use L'Hospital's rule to evaluate the limit.

$$\lim_{x \to \infty} \frac{e^{2x}}{x^3}$$

Answer choices:

A 0

B 1

C ∞

D $-\infty$

Solution: C

Since we get the indeterminate form ∞/∞ with direct substitution, we apply L'Hospital's rule until we get a determinate form.

$$\lim_{x\to\infty}\frac{e^{2x}}{x^3}$$

We get an indeterminate form when we evaluate the limit.

$$\lim_{x\to\infty}\frac{2e^{2x}}{3x^2}$$

We get an indeterminate form when we evaluate the limit.

$$\lim_{x\to\infty}\frac{4e^{2x}}{6x}$$

We get an indeterminate form when we evaluate the limit.

$$\lim_{x\to\infty}\frac{8e^{2x}}{6}$$

We get ∞ when we evaluate the limit.

Since the last application of the rule allowed us to evaluate the limit by direct substitution without giving us an indeterminate form, we've found that the limit is ∞.

Topic: L'Hospital's Rule

Question: Use L'Hospital's rule to evaluate the limit.

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

Answer choices:

A 0

B 1

C ∞

D $-\infty$

Solution: B

Since we get the indeterminate form $0/0$ with direct substitution, but we can't eliminate the zero in the denominator by factoring, we apply L'Hospital's rule until we get a determinate form.

$$\lim_{x\to 1} \frac{\ln x}{x-1}$$

We get an indeterminate form when we evaluate the limit.

$$\lim_{x\to 1} \frac{\frac{1}{x}}{1}$$

We get $1/1 = 1$ when we evaluate the limit.

Since the last application of the rule allowed us to evaluate the limit by direct substitution without giving us an indeterminate form, we've found that the limit is 1.

Topic: L'Hospital's Rule

Question: Use L’Hospital’s rule to evaluate the limit.

$$\lim_{x \to 4} (5 - x)^{\frac{1}{4 - x}}$$

Answer choices:

A 0

B 1

C ∞

D e

Solution: D

If we try substitution to evaluate at $x = 4$, we get an indeterminate form.

$$(5-4)^{\frac{1}{4-4}}$$

$$1^{\infty}$$

Because we get an indeterminate form, we want to use L'Hospital's Rule. But before we do, we need to get the fraction by itself. So we'll set the limit equal to y,

$$y = \lim_{x \to 4}(5-x)^{\frac{1}{4-x}}$$

and then take the natural log of both sides.

$$\ln y = \lim_{x \to 4} \ln\left((5-x)^{\frac{1}{4-x}}\right)$$

$$\ln y = \lim_{x \to 4} \frac{1}{4-x} \ln(5-x)$$

$$\ln y = \lim_{x \to 4} \frac{\ln(5-x)}{4-x}$$

With the limit rewritten, we'll apply L'Hospital's rule to the fraction.

$$\ln y = \lim_{x \to 4} \frac{\frac{1}{5-x}(-1)}{-1}$$

$$\ln y = \lim_{x \to 4} \frac{-\frac{1}{5-x}}{-1}$$

$$\ln y = \lim_{x \to 4} \frac{1}{5 - x}$$

Evaluate the limit,

$$\ln y = \frac{1}{5 - 4}$$

$$\ln y = \frac{1}{1}$$

$$\ln y = 1$$

then raise both sides to the base e to solve for y.

$$e^{\ln y} = e^1$$

$$y = e$$

Remember earlier that we set the limit equal to y,

$$y = \lim_{x \to 4}(5 - x)^{\frac{1}{4 - x}}$$

so because we now have two values both equal to y, we can set those values equal to each other.

$$\lim_{x \to 4}(5 - x)^{\frac{1}{4 - x}} = e$$

Topic: Position, velocity, and acceleration

Question: Find the velocity function.

$$x(t) = 4t^2 - 6t + 2$$

Answer choices:

A $v(t) = 8t - 4$

B $v(t) = 8t + 6$

C $v(t) = 4t - 6$

D $v(t) = 8t - 6$

Solution: D

Take the derivative of the position function.

$$x(t) = 4t^2 - 6t + 2$$

$$x'(t) = 8t - 6$$

Velocity is the derivative of position.

$$v(t) = 8t - 6$$

Topic: Position, velocity, and acceleration

Question: Find the position of a car when its velocity is zero.

$$x(t) = 4t^2 - 8t + 10$$

Answer choices:

A $x = 6$

B $x = 7$

C $x = 10$

D $x = 0$

Solution: A

Take the derivative of the position function.

$$x(t) = 4t^2 - 8t + 10$$

$$x'(t) = 8t - 8$$

Velocity is the derivative of position.

$$v(t) = 8t - 8$$

We need to find position when velocity is 0.

$$8t - 8 = 0$$

$$8t = 8$$

$$t = 1$$

Velocity is 0 when $t = 1$. To find position at the same time, substitute $t = 1$ into the position function.

$$x(1) = 4(1)^2 - 8(1) + 10$$

$$x(1) = 4 - 8 + 10$$

$$x(1) = 6$$

Topic: Position, velocity, and acceleration

Question: Use the position function to find the velocity of a rocket at $t = 4$.

$$x(t) = 6t^3 - t^2 + 3t - 9$$

Answer choices:

A $v(4) = 238$

B $v(4) = 371$

C $v(4) = 283$

D $v(4) = 317$

Solution: C

Take the derivative of the position function.

$$x(t) = 6t^3 - t^2 + 3t - 9$$

$$x'(t) = 18t^2 - 2t + 3$$

Velocity is the derivative of position.

$$v(t) = 18t^2 - 2t + 3$$

We need to find velocity when $t = 4$, so we'll plug $t = 4$ into the velocity function we just found.

$$v(4) = 18(4)^2 - 2(4) + 3$$

$$v(4) = 288 - 8 + 3$$

$$v(4) = 283$$

Topic: Ball thrown up from the ground

Question: A ball's thrown straight up from the ground with initial velocity $v_0 = 64$ ft/s. What is the ball's maximum height, and what is its velocity when it hits the ground?

Answer choices:

A Maximum height is 32 ft; Velocity is -32 ft/s

B Maximum height is 32 ft; Velocity is -64 ft/s

C Maximum height is 64 ft; Velocity is -64 ft/s

D Maximum height is 64 ft; Velocity is -32 ft/s

Solution: C

Substitute $g = 32$ ft/s², $v_0 = 64$ ft/s, and $y_0 = 0$ into the vertical motion formula.

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$y(t) = -\frac{1}{2}(32)t^2 + (64)t + 0$$

$$y(t) = -16t^2 + 64t$$

To find velocity when the ball hits the ground, set the position function equal to 0, since height is 0 when the ball hits the ground.

$$-16t^2 + 64t = 0$$

$$-16t(t-4) = 0$$

$$t = 0, 4$$

We know that the height is 0 when the ball is initially thrown up from the ground at $t = 0$, which means it hits the ground again when $t = 4$.

To find velocity when the ball hits the ground at $t = 4$, we need to find the velocity function by taking the derivative of the position function.

$$y'(t) = -32t + 64$$

$$v(t) = -32t + 64$$

Substitute $t = 4$ to find velocity when the ball hits the ground.

$$v(4) = -32(4) + 64$$

$$v(4) = -128 + 64$$

$$v(4) = -64$$

The ball's velocity when it hits the ground is -64 ft/s.

The ball reaches its maximum height when $v(t) = 0$, so set the velocity function equal to 0.

$$-32t + 64 = 0$$

$$32t = 64$$

$$t = 2$$

The ball reaches maximum height at $t = 2$, so substitute $t = 2$ into the position function.

$$y(2) = -16(2)^2 + 64(2)$$

$$y(2) = -64 + 128$$

$$y(2) = 64$$

The ball's maximum height is 64 ft.

Topic: Ball thrown up from the ground

Question: A ball's thrown straight up from the ground with initial velocity $v_0 = 128$ ft/s. What is the ball's maximum height, and what is its velocity when it hits the ground?

Answer choices:

A Maximum height is 256 ft; Velocity is -256 ft/s

B Maximum height is 128 ft; Velocity is -128 ft/s

C Maximum height is 128 ft; Velocity is -256 ft/s

D Maximum height is 256 ft; Velocity is -128 ft/s

Solution: D

Substitute $g = 32$ ft/s², $v_0 = 128$ ft/s, and $y_0 = 0$ into the vertical motion formula.

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$y(t) = -\frac{1}{2}(32)t^2 + (128)t + 0$$

$$y(t) = -16t^2 + 128t$$

To find velocity when the ball hits the ground, set the position function equal to 0, since height is 0 when the ball hits the ground.

$$-16t^2 + 128t = 0$$

$$-16t(t - 8) = 0$$

$$t = 0,\ 8$$

We know that the height is 0 when the ball is initially thrown up from the ground at $t = 0$, which means it hits the ground again when $t = 8$.

To find velocity when the ball hits the ground at $t = 8$, we need to find the velocity function by taking the derivative of the position function.

$$y'(t) = -32t + 128$$

$$v(t) = -32t + 128$$

Substitute $t = 8$ to find velocity when the ball hits the ground.

$$v(8) = -32(8) + 128$$

$$v(8) = -256 + 128$$

$$v(8) = -128$$

The ball’s velocity when it hits the ground is -128 ft/s.

The ball reaches its maximum height when $v(t) = 0$, so set the velocity function equal to 0.

$$-32t + 128 = 0$$

$$32t = 128$$

$$t = 4$$

The ball reaches maximum height at $t = 4$, so substitute $t = 4$ into the position function.

$$y(4) = -16(4)^2 + 128(4)$$

$$y(4) = -256 + 512$$

$$y(4) = 256$$

The ball’s maximum height is 256 ft.

Topic: Ball thrown up from the ground

Question: An apple is thrown straight up from the ground with an initial velocity of 100 m/s. Assuming constant gravity, find the apple's maximum height.

$$s(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

Answer choices:

A $\quad 520.1$ m

B $\quad 512.0$ m

C $\quad 51.02$ m

D $\quad 510.2$ m

Solution: D

Substitute $g = 9.8$ m/s², $v_0 = 100$ m/s, and $y_0 = 0$ into the vertical motion formula.

$$s(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$s(t) = -\frac{1}{2}(9.8)t^2 + 100t + 0$$

$$s(t) = -4.9t^2 + 100t$$

Take the derivative of the position function.

$$s'(t) = -9.8t + 100$$

$$v(t) = -9.8t + 100$$

The apple reaches its maximum height when $v(t) = 0$, so set the velocity function equal to 0.

$$-9.8t + 100 = 0$$

$$9.8t = 100$$

$$t = 10.2$$

The apple reaches maximum height at $t = 10.2$, so substitute $t = 10.2$ into the position function.

$$s(10.2) = -4.9(10.2)^2 + 100(10.2)$$

$$s(10.2) \approx -509.8 + 1{,}020$$

$$s(10.2) \approx 510.2$$

The apple's maximum height is about 510.2 m.

Topic: Coin dropped from the roof

Question: A pumpkin is dropped from the top of a building and falls 5 m to the ground. Given the position function of the pumpkin, find instantaneous velocity at $t = 3$ seconds.

$$s(t) = -6t^2 + 3t - 5$$

Answer choices:

A -40 m/s

B -39 m/s

C -33 m/s

D -50 m/s

Solution: C

Take the derivative of the position function to get the velocity function.

$$s(t) = -6t^2 + 3t - 5$$

$$s'(t) = -12t + 3$$

$$v(t) = -12t + 3$$

Substitute $t = 3$ to find instantaneous velocity at that time.

$$v(3) = -12(3) + 3$$

$$v(3) = -33$$

The instantaneous velocity at $t = 3$ is -33 m/s. Because the velocity is negative, it means that the pumpkin is falling toward the ground.

Topic: Coin dropped from the roof

Question: A baseball is dropped from the top of a bridge that's 8 m high. Find its average velocity during the first 4 seconds.

$$s(t) = -8t^2 - 4t - 8$$

Answer choices:

A -36 m/s

B -68 m/s

C -86 m/s

D -23 m/s

Solution: A

Substitute $t_1 = 0$ and $t_2 = 4$ into the formula for average velocity.

$$v_{avg} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$v_{avg} = \frac{s(4) - s(0)}{4 - 0}$$

$$v_{avg} = \frac{s(4) - s(0)}{4}$$

Find $s(0)$ and $s(4)$.

$$s(0) = -8(0)^2 - 4(0) - 8$$

$$s(0) = -8$$

and

$$s(4) = -8(4)^2 - 4(4) - 8$$

$$s(4) = -152$$

Substitute these values into the average velocity equation.

$$v_{avg} = \frac{-152 - (-8)}{4}$$

$$v_{avg} = \frac{-144}{4}$$

$$v_{avg} = -36 \text{ m/s}$$

Topic: Coin dropped from the roof

Question: A coin is dropped from the roof of a 400 ft building with an initial velocity of -64 ft/s. When does it hit the ground and what is the velocity at that time?

Answer choices:

A The coin hits the ground after 7.78 s at -108.48 ft/s

B The coin hits the ground after 3.39 s at -108.48 ft/s

C The coin hits the ground after 3.39 s at -172.48 ft/s

D The coin hits the ground after 7.78 s at -172.48 ft/s

Solution: C

Substitute $g = 32$ ft/s², $v_0 = -64$ ft/s, and $y_0 = 400$ into the vertical motion formula.

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$y(t) = -\frac{1}{2}(32)t^2 + (-64)t + 400$$

$$y(t) = -16t^2 - 64t + 400$$

$$y(t) = -16(t^2 + 4t - 25)$$

To find velocity when the coin hits the ground, set the position function equal to 0, since height is 0 when the coin hits the ground.

$$-16(t^2 + 4t - 25) = 0$$

$$t^2 + 4t - 25 = 0$$

Use the quadratic formula to find the roots of the function.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-4 \pm \sqrt{4^2 - 4(1)(-25)}}{2(1)}$$

$$t = \frac{-4 \pm \sqrt{16 + 100}}{2}$$

$$t = \frac{-4 \pm 2\sqrt{29}}{2}$$

$$t = -2 \pm \sqrt{29}$$

$$t \approx -7.39, 3.39$$

A negative value for time is nonsensical, which means the coin hits the ground when $t \approx 3.39$.

To find velocity when the coin hits the ground at $t \approx 3.39$, we need to find the velocity function by taking the derivative of the position function.

$$y'(t) = -32t - 64$$

$$v(t) = -32t - 64$$

Substitute $t \approx 3.39$ to find velocity when the ball hits the ground.

$$v(3.39) = -32(3.39) - 64$$

$$v(3.39) = -108.48 - 64$$

$$v(3.39) = -172.48$$

The coin's velocity when it hits the ground is -172.48 ft/s.

Topic: Marginal cost, revenue, and profit

Question: The cost function C models the weekly expenses of a balloon company. What is the company's weekly marginal cost?

$$C(x) = 1.5x + 300$$

Answer choices:

A $\$1.00$

B $\$1.50$

C $\$1.05$

D $\$300$

Solution: B

We can find marginal cost by taking the derivative of the cost formula.

$$C(x) = 1.5x + 300$$

$$C'(x) = 1.5$$

The balloon company’s weekly marginal cost is $\$1.50$.

Topic: Marginal cost, revenue, and profit

Question: If a candy company's weekly revenue is modeled by R, how many units should they sell in order to maximize weekly revenue?

$$R(x) = -0.52x^2 + 12x$$

Answer choices:

A 12

B 44

C 23

D 32

Solution: A

To find the marginal revenue function R', take the derivative of R.

$$R(x) = -0.52x^2 + 12x$$

$$R'(x) = -1.04x + 12$$

Set marginal revenue equal to 0, then solve for x.

$$-1.04x + 12 = 0$$

$$1.04x = 12$$

$$x = 11.5$$

Since we can't sell a partial unit, we'll round to $x = 12$. The candy company needs to sell 12 units in order to maximize weekly revenue.

Topic: Marginal cost, revenue, and profit

Question: The cell phone store has monthly costs described by $C(x) = 22.5x + 675$ and monthly revenue described by $R(x) = 0.89x^2 - 22x$. What's their marginal profit if they sell $1{,}000$ units this month?

Answer choices:

A $\$1{,}753.50$

B $\$844{,}825.00$

C $\$1{,}735.50$

D $\$846{,}175.00$

Solution: C

Create a profit equation by subtracting costs from revenue.

$$P(x) = 0.89x^2 - 22x - (22.5x + 675)$$

$$P(x) = 0.89x^2 - 22x - 22.5x - 675$$

$$P(x) = 0.89x^2 - 44.5x - 675$$

To find the marginal profit function, take the derivative of the profit function.

$$P'(x) = 1.78x - 44.5$$

The marginal profit when the stores sells $1{,}000$ units is therefore

$$P'(1{,}000) = 1.78(1{,}000) - 44.5$$

$$P'(1{,}000) = 1{,}735.50$$

Topic: Half life

Question: Americium-241 has a half-life of 432 years. Find the decay constant.

Answer choices:

A 0.160

B 0.0160

C 0.00160

D 0.000160

Solution: C

The half life equation is

$$\frac{1}{2}=e^{kt}$$

Solve this for the decay constant k.

$$\ln\frac{1}{2}=\ln e^{kt}$$

$$\ln\frac{1}{2}=kt$$

$$k=\frac{\ln\frac{1}{2}}{t}$$

Use laws of logarithms to rewrite the log.

$$k=\frac{\ln 1-\ln 2}{t}$$

$$k=\frac{0-\ln 2}{t}$$

$$k=-\frac{\ln 2}{t}$$

Because k is a constant, we can absorb the negative sign into it.

$$k=\frac{\ln 2}{t}$$

Substitute $t=432$.

$$k = \frac{\ln 2}{432}$$

$$k \approx 0.00160$$

Topic: Half life

Question: Carbon-19 has a decay constant of 0.000121. Find its half life.

Answer choices:

A $5,782$ years

B $5,728$ years

C $5,278$ years

D $5,827$ years

Solution: B

The half life equation is

$$\frac{1}{2} = e^{kt}$$

Solve this for time t.

$$\ln \frac{1}{2} = \ln e^{kt}$$

$$\ln \frac{1}{2} = kt$$

$$t = \frac{1}{k} \ln \frac{1}{2}$$

Use laws of logarithms to rewrite the log.

$$t = \frac{1}{k}(\ln 1 - \ln 2)$$

$$t = \frac{1}{k}(0 - \ln 2)$$

$$t = -\frac{\ln 2}{k}$$

Because k is a constant, we can absorb the negative sign into it.

$$t = \frac{\ln 2}{k}$$

Substitute the decay constant $k = 0.000121$.

$$t = \frac{\ln 2}{0.000121}$$

$$t \approx 5{,}728$$

Topic: Half life

Question: Plutonium-239 has a half-life of $24{,}110$ years. Find the decay constant.

Answer choices:

A 0.0000287

B 0.000287

C 0.000000287

D 0.00000287

Solution: A

The half life equation is

$$\frac{1}{2} = e^{kt}$$

Solve this for the decay constant k.

$$\ln\frac{1}{2} = \ln e^{kt}$$

$$\ln\frac{1}{2} = kt$$

$$k = \frac{\ln\frac{1}{2}}{t}$$

Use laws of logarithms to rewrite the log.

$$k = \frac{\ln 1 - \ln 2}{t}$$

$$k = \frac{0 - \ln 2}{t}$$

$$k = -\frac{\ln 2}{t}$$

Because k is a constant, we can absorb the negative sign into it.

$$k = \frac{\ln 2}{t}$$

Substitute $t = 24{,}110$.

$$k = \frac{\ln 2}{24{,}110}$$

$$k \approx 0.0000287$$

Topic: Newton's Law of Cooling

Question: The function T models the temperature (in Celsius) of a cooling object. What is the starting temperature of the object?

$$T(t) = 14e^{-5t}$$

Answer choices:

A $5^\circ\ C$

B $14^\circ\ C$

C $8^\circ\ C$

D $7^\circ\ C$

Solution: B

The starting temperature of the object is given by T when $t=0$. So substitute $t=0$ into the temperature function.

$$T(t)=14e^{-5t}$$

$$T(t)=14e^{-5(0)}$$

$$T(t)=14e^{0}$$

$$T(t)=14(1)$$

$$T(t)=14$$

Topic: Newton's Law of Cooling

Question: The function T models the temperature (in Celsius) of a cooling object. What is the approximate temperature of the object after 1 hour?

$$T(t) = 8e^{-t}$$

Answer choices:

A $0^\circ\ C$

B $1^\circ\ C$

C $8^\circ\ C$

D $3^\circ\ C$

Solution: D

The temperature of the object after 1 hour is given by T when $t = 1$. So substitute $t = 1$ into the temperature function.

$$T(t) = 8e^{-t}$$

$$T(t) = 8e^{-1}$$

$$T(t) \approx 8(0.37)$$

$$T(t) \approx 3$$

Topic: Newton's Law of Cooling

Question: The function T models the temperature (in Celsius) of a cooling object. How many hours does it take to cool the object to 100°?

$$T(t) = 124e^{-0.6t}$$

Answer choices:

A 0.35 hours

B 0.70 hours

C 3.5 hours

D 7.0 hours

Solution: A

The time it takes to cool the object to 100° is given by T when $T = 100$. So substitute $T = 100$ into the temperature function.

$$T(t) = 124e^{-0.6t}$$

$$100 = 124e^{-0.6t}$$

$$0.81 = e^{-0.6t}$$

Apply the natural logarithm to both sides.

$$\ln 0.81 = \ln e^{-0.6t}$$

$$-0.21 = -0.6t$$

$$t = \frac{-0.21}{-0.6}$$

$$t = 0.35$$

Topic: Sales decline

Question: A t-shirt company noticed that their inventory of blue shirts was declining exponentially at a rate of 23% per year. They currently have 300 blue shirts in stock and they don't plan to purchase any more. How many blue shirts will they have in stock in 3 years?

Answer choices:

A 250

B 200

C 150

D 100

Solution: C

Both the interest rate and time have units in years, so with matching units we can plug directly into the exponential decay formula to find the number of blue shirts the company will have in stock in 3 years.

$$S(t) = S_0 e^{-\lambda t}$$

$$S(t) = 300e^{-0.23(3)}$$

$$S(t) = 300e^{-0.69}$$

$$S(t) \approx 150$$

Topic: Sales decline

Question: A pet store noticed that sales of generic cat food was declining at an exponential rate of 8% per year. If they currently sell 600 bags of generic cat food in one year, how many years will it take before they're only selling 100 bags annually?

Answer choices:

A 22.4 years

B 24.4 years

C 24.2 years

D 42.4 years

Solution: A

Both the interest rate and time have units in years, so with matching units we can plug directly into the exponential decay formula to find the number of years until sales reach the level of 100 bags per year.

$$S(t) = S_0 e^{-\lambda t}$$

$$100 = 600e^{(-0.08)t}$$

$$\frac{1}{6} = e^{-0.08t}$$

Apply the natural logarithm to both sides.

$$\ln \frac{1}{6} = \ln(e^{-0.08t})$$

$$\ln \frac{1}{6} = -0.08t$$

$$t = \frac{\ln \frac{1}{6}}{-0.08}$$

$$t \approx 22.4$$

Topic: Sales decline

Question: Pixie stick sales are declining at a candy store. Two years ago, the store sold 450 pixie sticks, but this year they're only selling 150. Assuming that sales have declined exponentially, what's been the annual rate of decline?

Answer choices:

A $5\,\%$

B $5.50\,\%$

C $0.55\,\%$

D $55\,\%$

Solution: D

Both the interest rate and time have units in years, so with matching units we can plug directly into the exponential decay formula to find the rate of decline.

$$S(t) = S_0 e^{-\lambda t}$$

$$150 = 450e^{-\lambda(2)}$$

$$\frac{1}{3} = e^{-2\lambda}$$

Apply the natural logarithm to both sides.

$$\ln\frac{1}{3} = \ln(e^{-2\lambda})$$

$$\ln\frac{1}{3} = -2\lambda$$

$$\lambda = \frac{\ln\frac{1}{3}}{-2}$$

$$\lambda = 0.55$$

Topic: Compounding interest

Question: Four years ago, you owed $\$10$ on your credit card. Since then, you haven't made any payments. If the card carries an annual interest rate of 22%, compounded continuously, how much do you owe on the credit card today?

Answer choices:

A $\$21.14$

B $\$24.11$

C $\$21.41$

D $\$14.11$

Solution: B

Both the interest rate and time have units in years, so with matching units we can plug directly into the exponential growth formula to find the amount you currently owe on the credit card.

$$A(t) = A_0e^{rt}$$

$$A(t) = 10e^{(0.22)(4)}$$

$$A(t) = 24.11$$

Topic: Compounding interest

Question: You made an initial investment of $\$1{,}000$ at an annual interest rate of 4.5%, compounded continuously. If the investment is currently worth $\$5{,}632$, how many years have you had the investment?

Answer choices:

A 384 years

B $3{,}840$ years

C 38.4 years

D 3.84 years

Solution: C

Both the interest rate and time have units in years, so with matching units we can plug directly into the exponential growth formula to find the number of years we've held the investment.

$$A(t) = A_0 e^{rt}$$

$$5{,}632 = 1{,}000e^{(0.045)t}$$

$$5.632 = e^{0.045t}$$

Apply the natural logarithm to both sides.

$$\ln 5.632 = \ln(e^{0.045t})$$

$$\ln 5.632 = 0.045t$$

$$t = \frac{\ln 5.632}{0.045}$$

$$t \approx 38.4$$

Topic: Compounding interest

Question: Twenty years ago, you purchased $\$5{,}000$ worth of stock. This stock paid a continuously compounded, annual interest rate, and today your shares are worth $\$45{,}000$. What was the interest rate?

Answer choices:

A $111\ \%$

B $0.11\ \%$

C $1.1\ \%$

D $11\ \%$

Solution: D

Both the interest rate and time have units in years, so with matching units we can plug directly into the exponential growth formula to find the interest rate that the stock earned.

$$A(t) = A_0 e^{rt}$$

$$45{,}000 = 5{,}000 e^{r(20)}$$

$$9 = e^{20r}$$

Apply the natural logarithm to both sides.

$$\ln 9 = \ln(e^{20r})$$

$$\ln 9 = 20r$$

$$r = \frac{\ln 9}{20}$$

$$r \approx 0.11$$

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