Calculus 1 Workbook Solutions

Continuity

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MATH

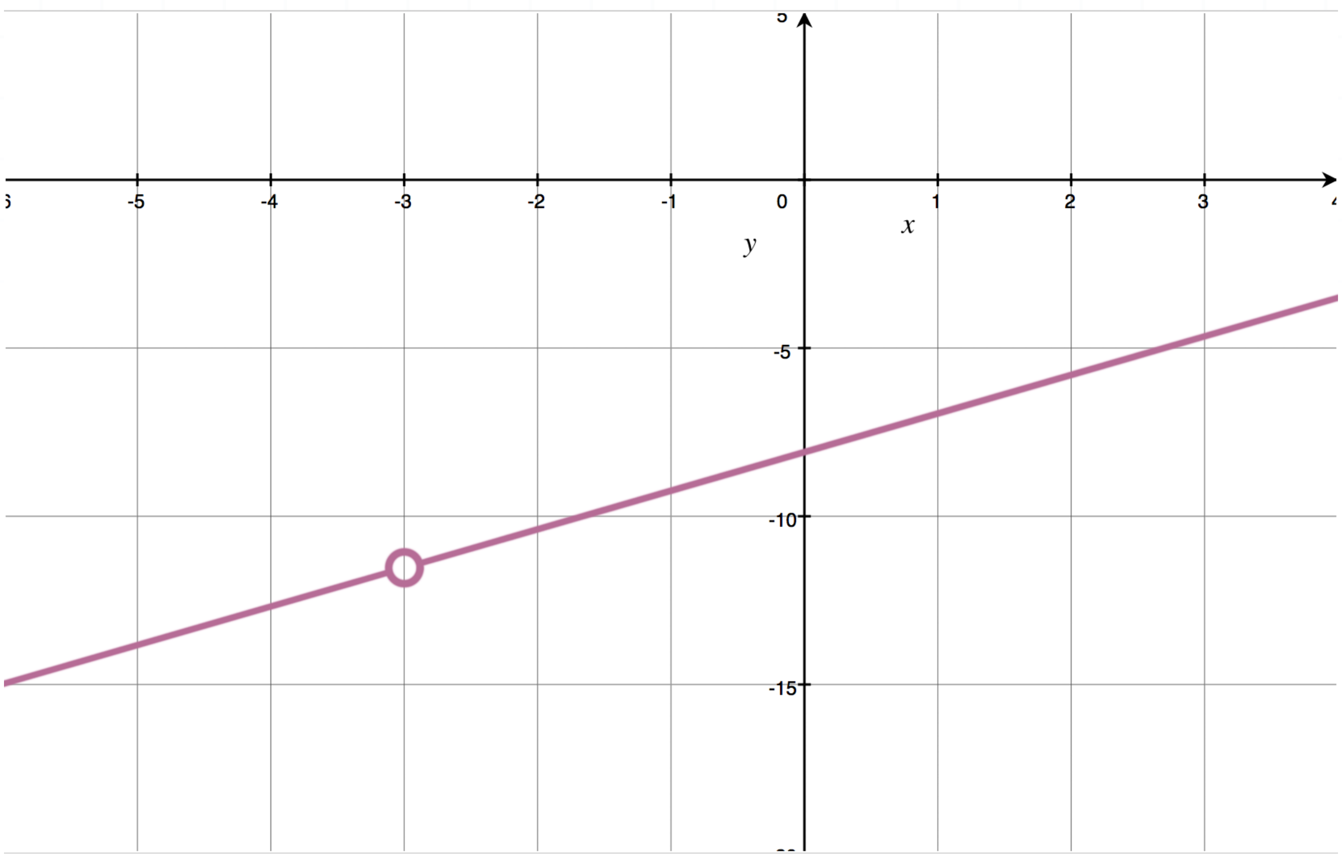
POINT DISCONTINUITIES

■ 1. Redefine the function as a continuous piecewise function.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$

Solution:

The function is discontinuous at $x = -3$.



Factor and reduce to remove the discontinuity.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$

$$f(x) = \frac{(x+3)(x-9)}{x+3}$$

$$f(x) = x - 9$$

Evaluate $f(x)$ at $x = -3$.

$$f(-3) = -3 - 9 = -12$$

Therefore, to make the function continuous, we have to redefine it as

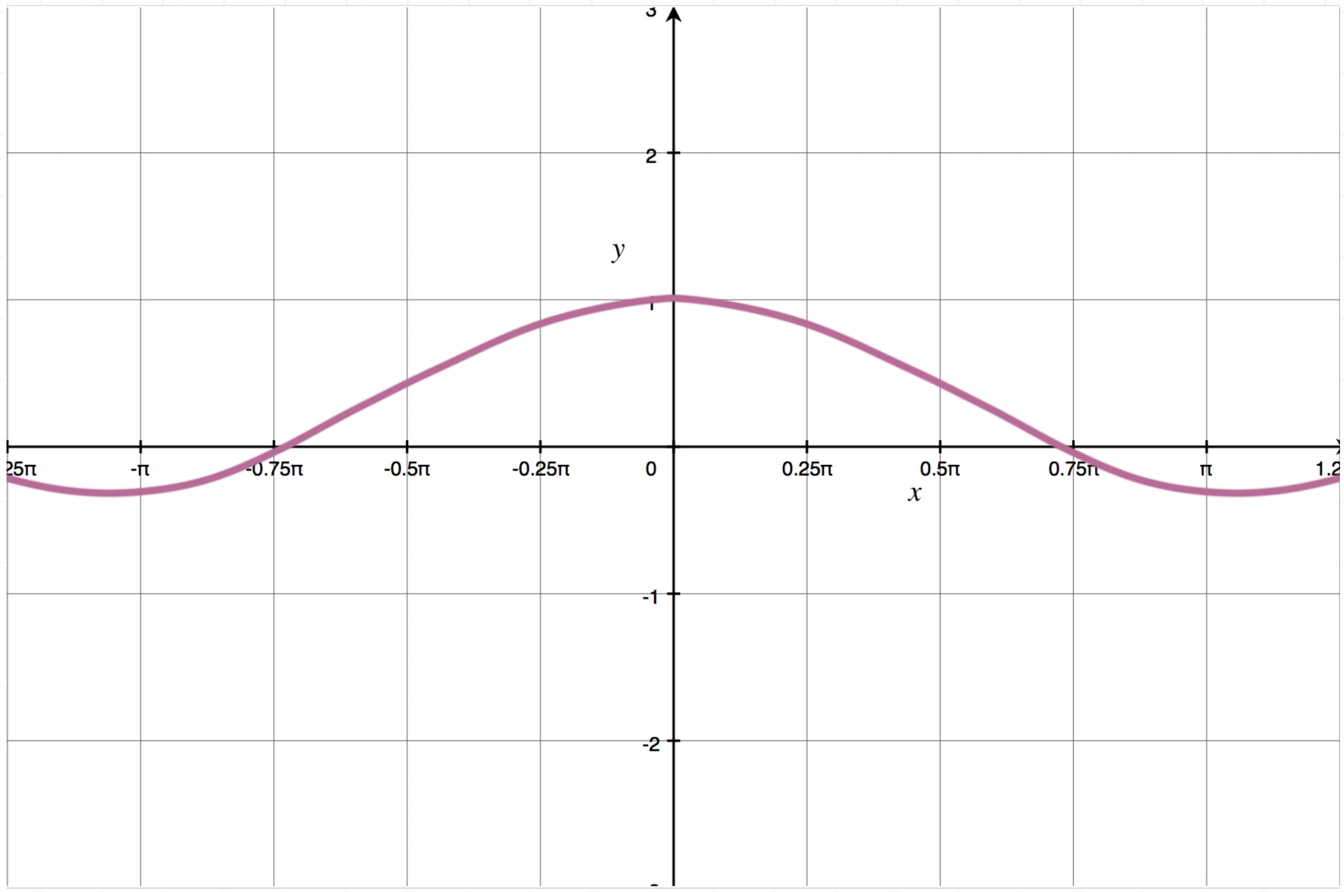
$$f(x) = \begin{cases} \frac{x^2 - 6x - 27}{x+3} & x \neq -3 \\ -12 & x = -3 \end{cases}$$

■ 2. Redefine the function as a continuous piecewise function.

$$g(x) = \frac{\sin x}{x}$$

Solution:

The function is discontinuous at $x = 0$, but is approaching a value of 1 from both sides.



Therefore, to make the function continuous, we have to redefine it as

$$g(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

3. What are the removable discontinuities of the function?

$$h(x) = \frac{x^4 - 5x^2 + 4}{x^2 - 1}$$

Solution:

Factor the function, then cancel common factors.

$$h(x) = \frac{x^4 - 5x^2 + 4}{x^2 - 1}$$

$$h(x) = \frac{(x+1)(x-1)(x+2)(x-2)}{(x+1)(x-1)}$$

$$h(x) = (x+2)(x-2)$$

The factors that were canceled are the ones that produced removable discontinuities. So the removable discontinuities are $x = -1, 1$.

■ 4. Identify the non-removable discontinuities of the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

Solution:

Factor the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

$$k(x) = \frac{(x+5)(x-5)(x+3)}{(x+4)(x-3)}$$

No factors can be canceled. Which means the function has discontinuities at $x = -4$ and $x = 3$, both of which are non-removable.

5. What is the set of removable discontinuities of the function?

$$j(\theta) = \frac{\cos^2\theta \cdot \sin^2\theta}{\tan^2\theta}$$

Solution:

We can rewrite the function as

$$j(\theta) = \frac{\cos^2\theta \cdot \sin^2\theta}{\tan^2\theta} = \frac{\cos^2\theta \cdot \sin^2\theta}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{\cos^2\theta \cdot \sin^2\theta \cdot \cos^2\theta}{\sin^2\theta} = \cos^4\theta$$

The removable discontinuities are the values of θ that make the sine function equal to 0, which are all the multiples of π.

$$\theta = \pm 0, \ \pm \pi, \ \pm 2\pi, \ \pm 3\pi, \ \pm 4\pi, \ \dots$$

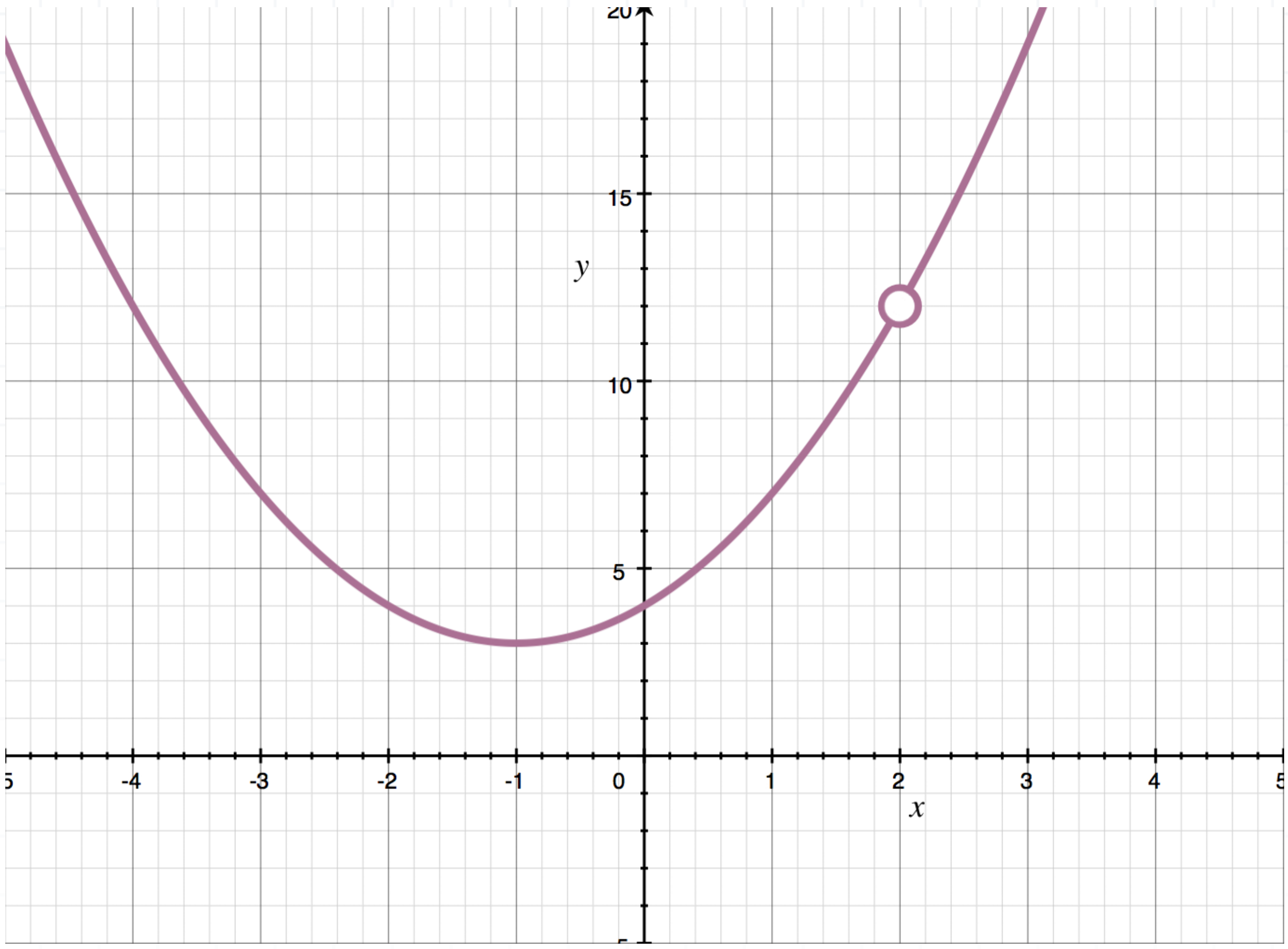
$\theta = n\pi$, where n is the set of all integers

6. Redefine the function as a continuous piecewise function.

$$g(x) = \frac{x^3 - 8}{x - 2}$$

Solution:

The function is discontinuous at $x = 2$.



Factor and reduce to remove the discontinuity.

$$g(x) = \frac{x^3 - 8}{x - 2}$$

$$g(x) = \frac{(x-2)(x^2 + 2x + 4)}{x - 2}$$

$$g(x) = x^2 + 2x + 4$$

Evaluate $g(x)$ at $x = 2$.

$$g(2) = 4 + 4 + 4 = 12$$

Therefore, to make the function continuous, we have to redefine it as

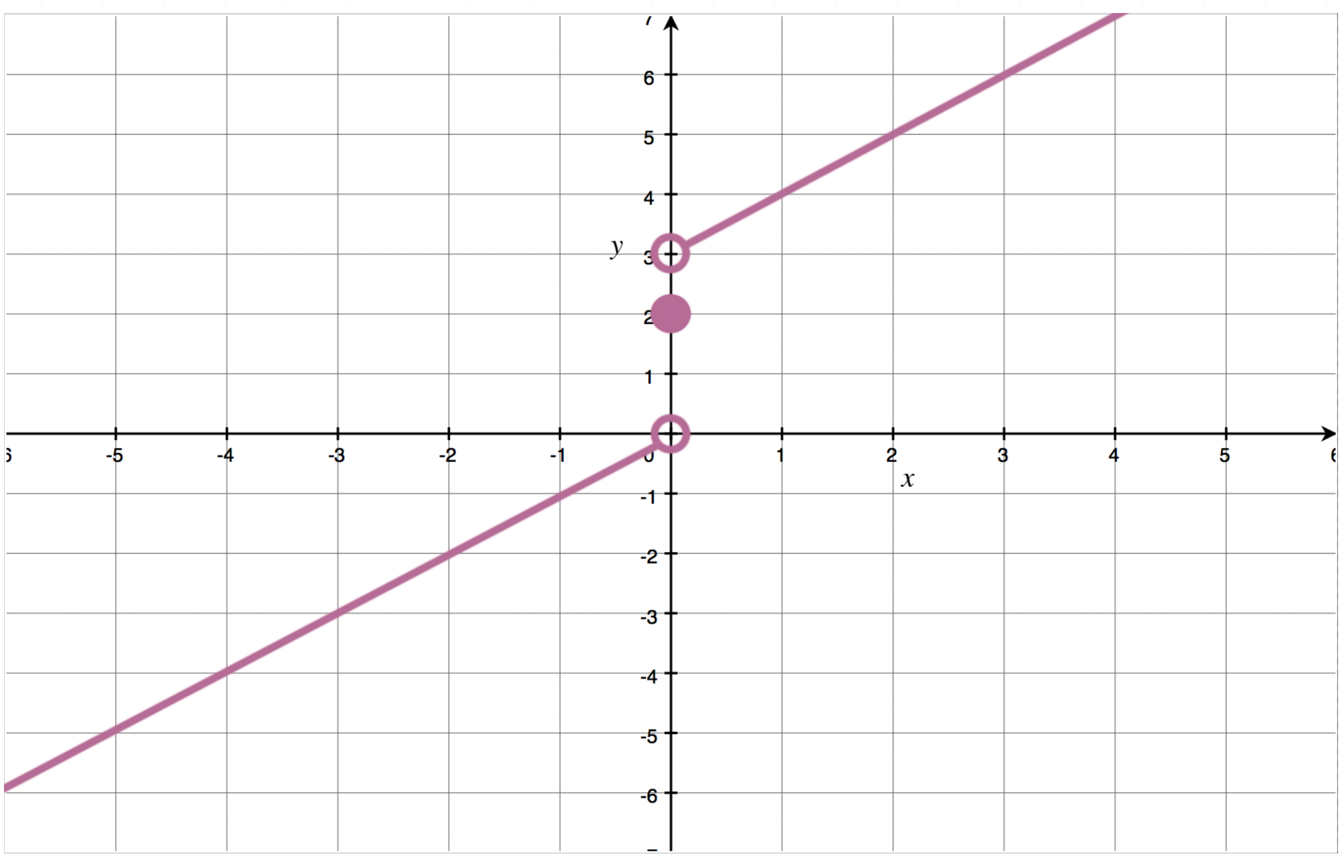
$$g(x) = \begin{cases} \frac{x^3 - 8}{x - 2} & x \neq 2 \\ 12 & x = 2 \end{cases}$$

■ 7. Identify the non-removable discontinuity in the function.

$$k(x) = \begin{cases} x & x < 0 \\ 2 & x = 0 \\ x + 3 & x > 0 \end{cases}$$

Solution:

The function $k(x)$ has a non-removable a discontinuity at $x = 0$ because the function has a jump discontinuity at $x = 0$, as shown in the graph below, and jump discontinuities are not removable.



■ 8. What is the removable discontinuity in the function?

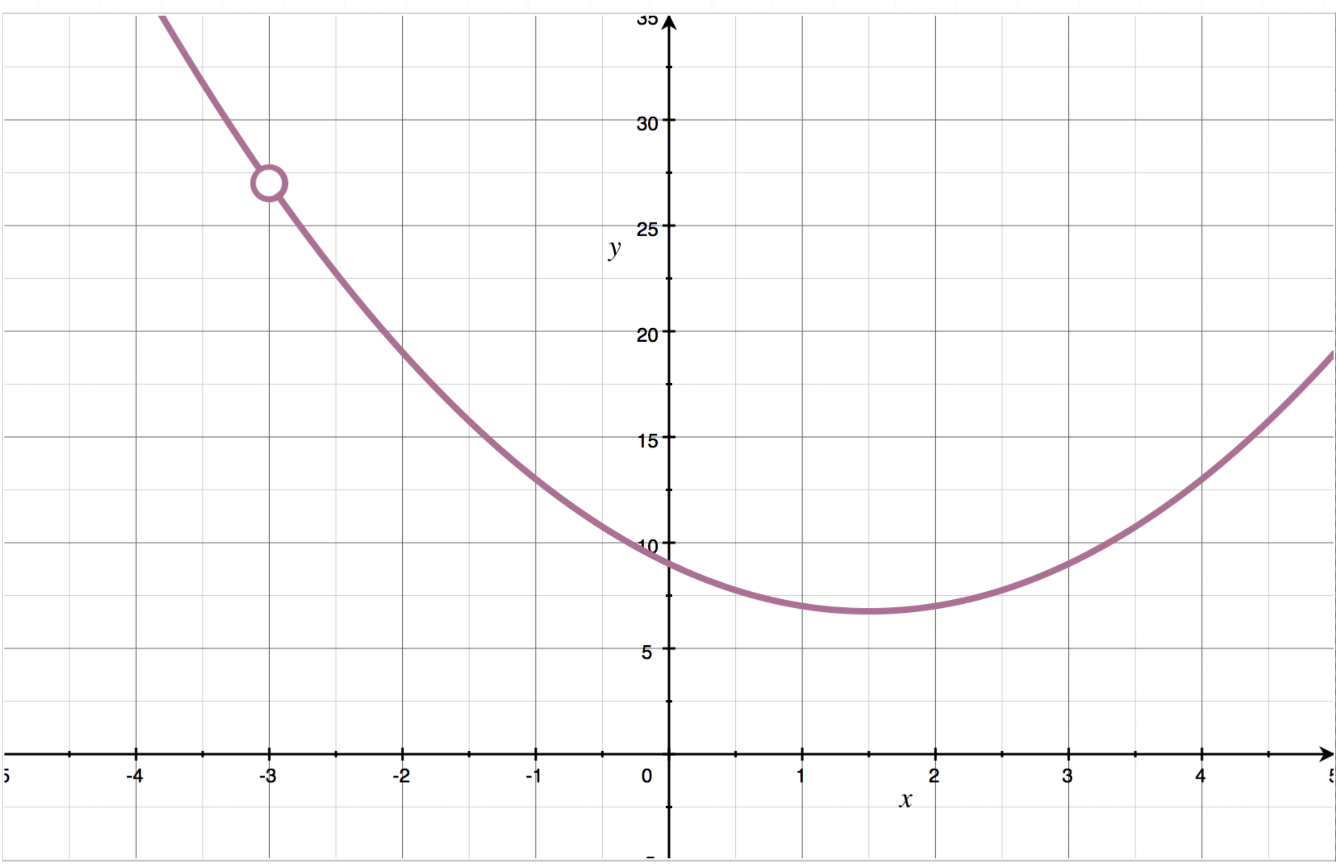
$$f(x) = \frac{x^3 + 27}{x + 3}$$

Solution:

If we factor the function, we can cancel a factor of $x = 3$.

$$f(x) = \frac{x^3 + 27}{x + 3} = \frac{(x + 3)(x^2 - 3x + 9)}{x + 3} = x^2 - 3x + 9$$

Therefore, the removable discontinuity is at $x = -3$.



9. Identify the removable discontinuities in the function.

$$k(x) = \frac{x^4 - 2x^3 - 16x^2 + 2x + 15}{x^2 - 2x - 15}$$

Solution:

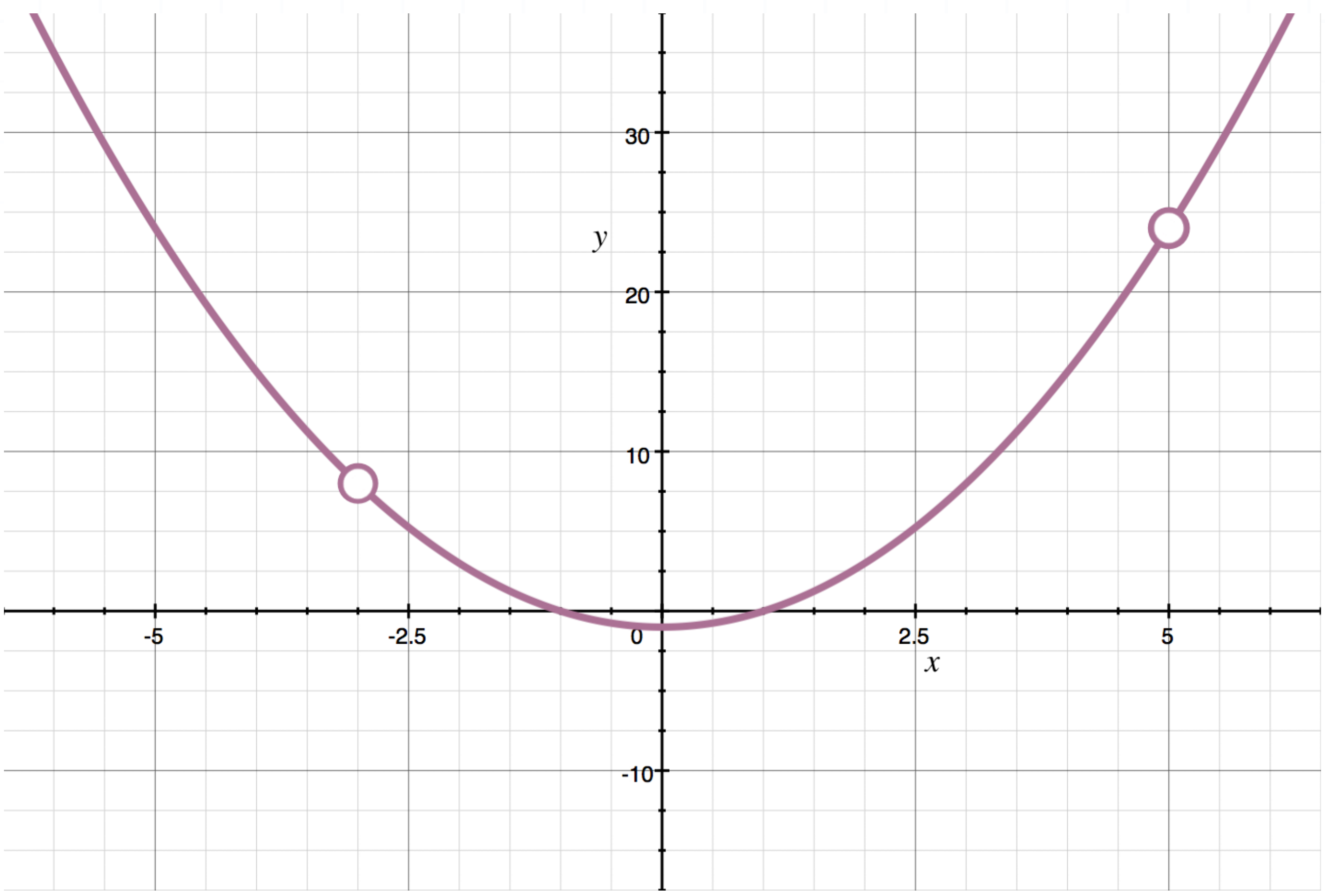
The function $k(x)$ has removable discontinuities at $x = -3$ and $x = 5$ because the function factors as

$$k(x) = \frac{(x+3)(x-5)(x+1)(x-1)}{(x+3)(x-5)}$$

and both factors from the denominator can be cancelled.

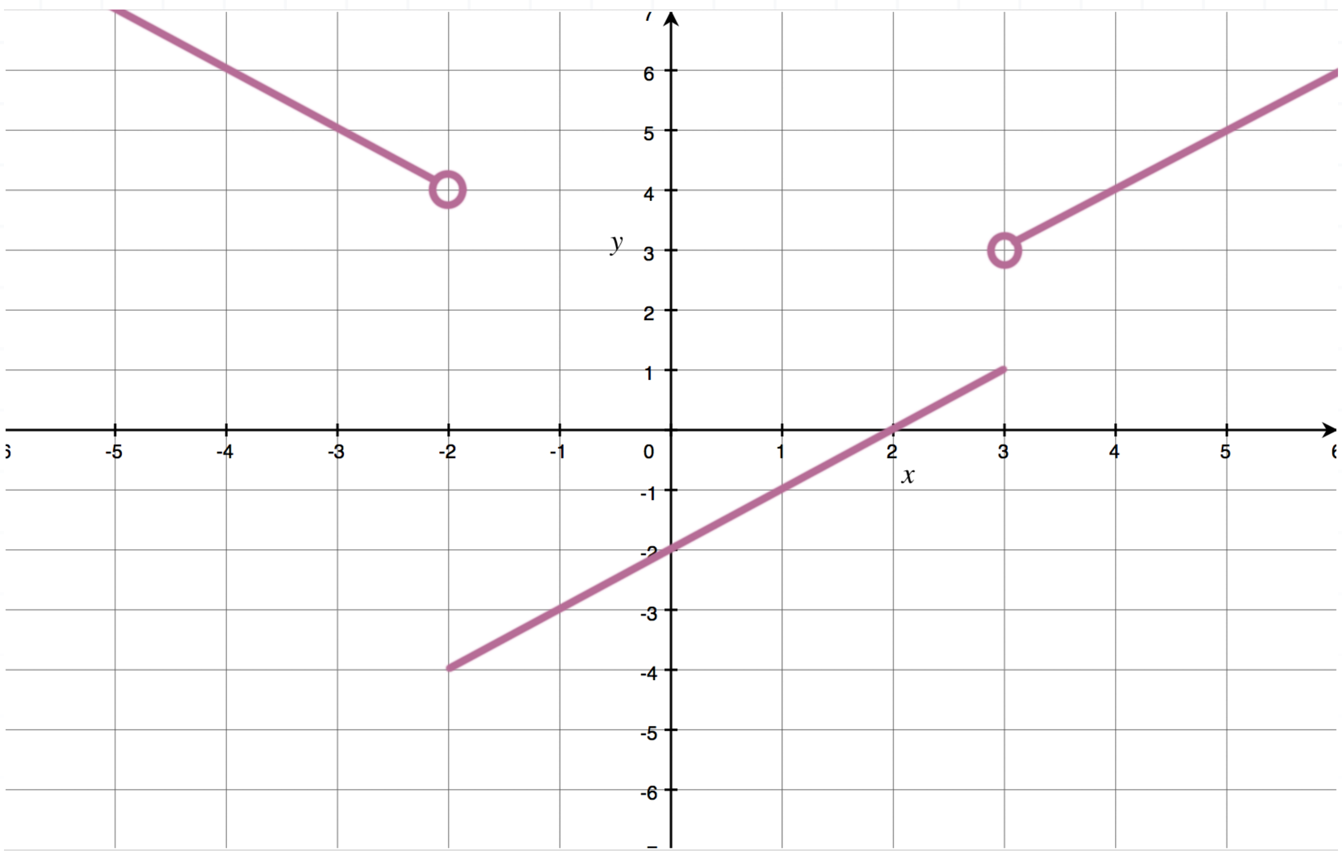
$$k(x) = (x+1)(x-1)$$

The graph is shown below.



JUMP DISCONTINUITIES

1. What are the x-values where the graph of $f(x)$, shown below, has jump discontinuities?



Solution:

The function $f(x)$ has jump discontinuities at $x = -2$ and $x = 3$ because the left- and right-hand limits aren't equal at $x = -2$

$$\lim_{x \to -2^-} f(x) = 4 \quad \neq \quad \lim_{x \to -2^+} f(x) = -2$$

and they aren't equal at $x = 3$.

$$\lim_{x \to 3^-} f(x) = 1 \quad \neq \quad \lim_{x \to 3^+} f(x) = 3$$

2. Where are the jump discontinuities in the graph of the function?

$$h(x)=\begin{cases}-\frac{1}{3}x^2+2 & x<0\\ 3 & 0\le x\le 1\\ \frac{1}{3}x^2+4 & x>1\end{cases}$$

Solution:

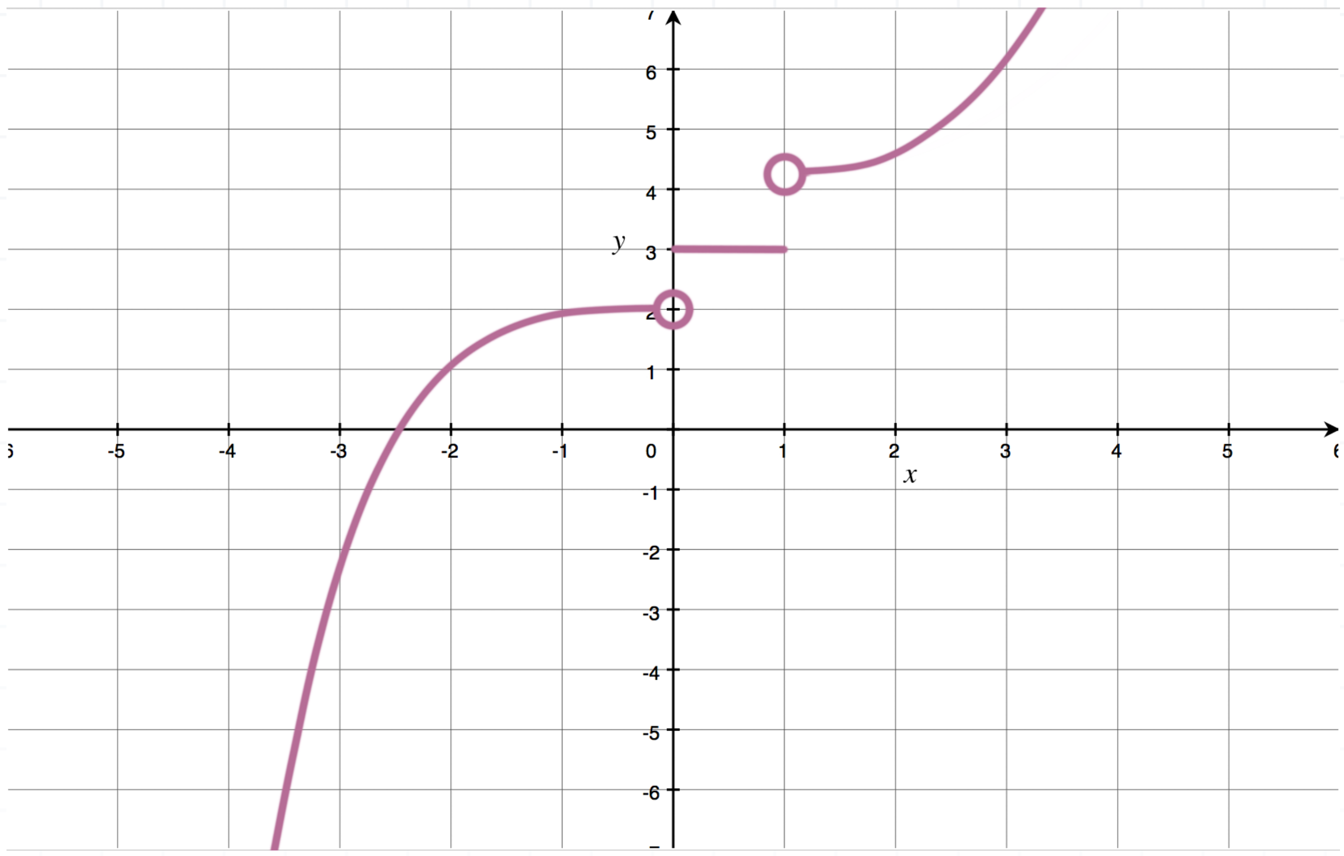
The function $h(x)$ has jump discontinuities at $x=0$ and $x=1$ because the left- and right-hand limits aren't equal at $x=0$,

$$\lim_{x\to 0^-} f(x)=2 \quad \neq \quad \lim_{x\to 0^+} f(x)=3$$

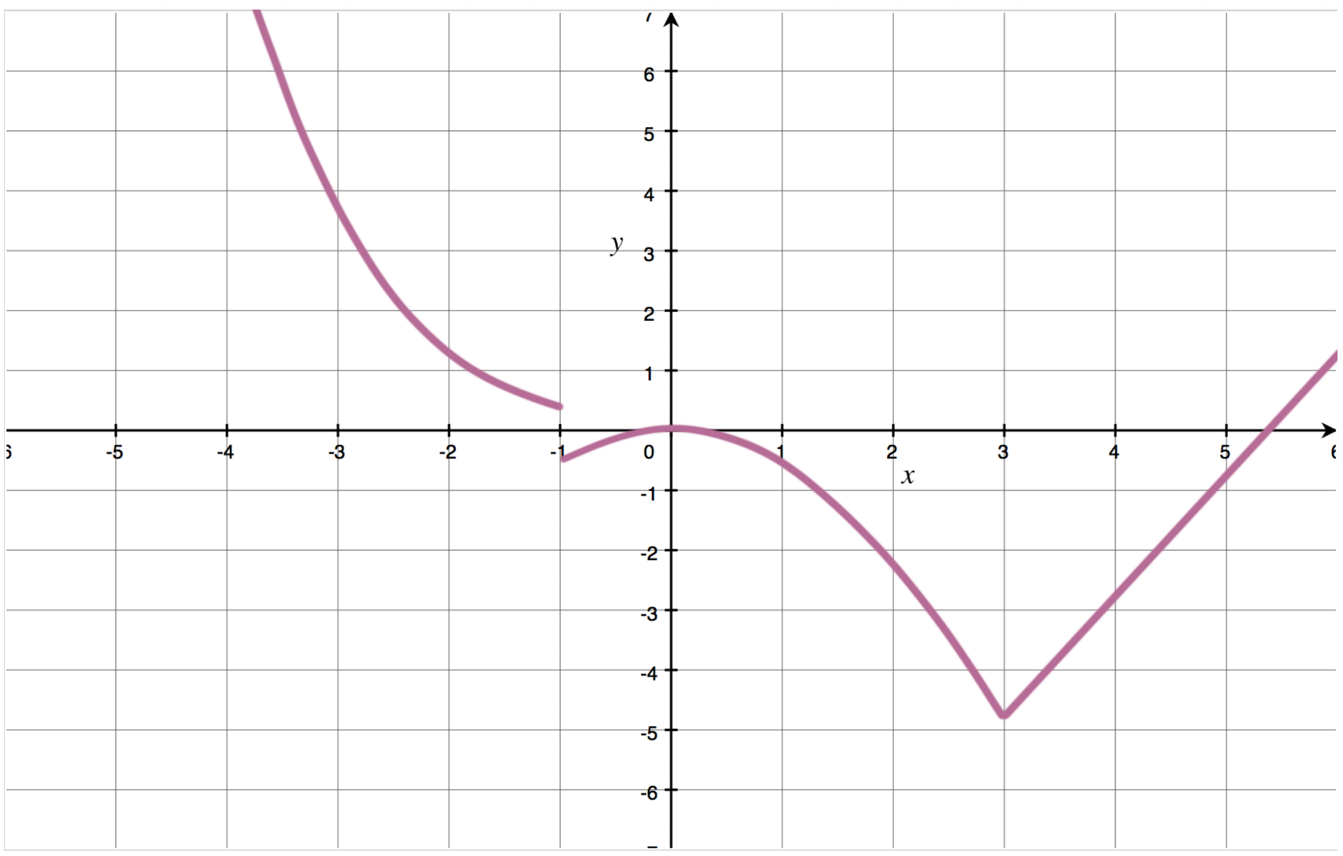
or at $x=1$.

$$\lim_{x\to 1^-} f(x)=3 \quad \neq \quad \lim_{x\to 1^+} f(x)=\frac{13}{3}$$

We can see the discontinuities in the function's graph, as well.



3. What are the x-values where the graph of $g(x)$ has jump discontinuities?

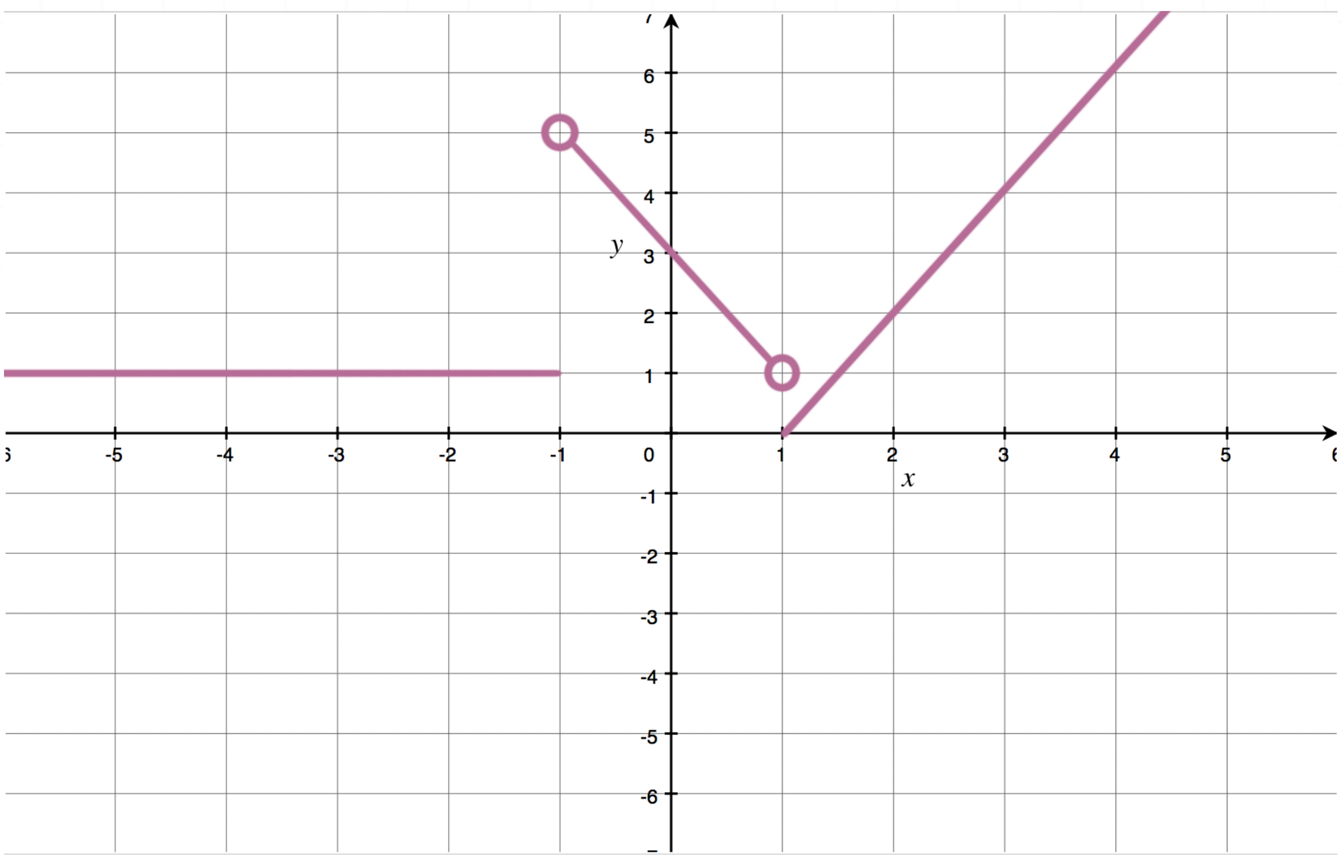


Solution:

The function $g(x)$ has a jump discontinuity at $x = -1$ because the left- and right-hand limits aren't equal there.

$$\lim_{x\to-1^-} f(x) = \frac{3}{4} \quad \neq \quad \lim_{x\to-1^+} f(x) = -\frac{2}{3}$$

■ 4. What are the x-values where the graph of $f(x)$, shown below, has jump discontinuities?



Solution:

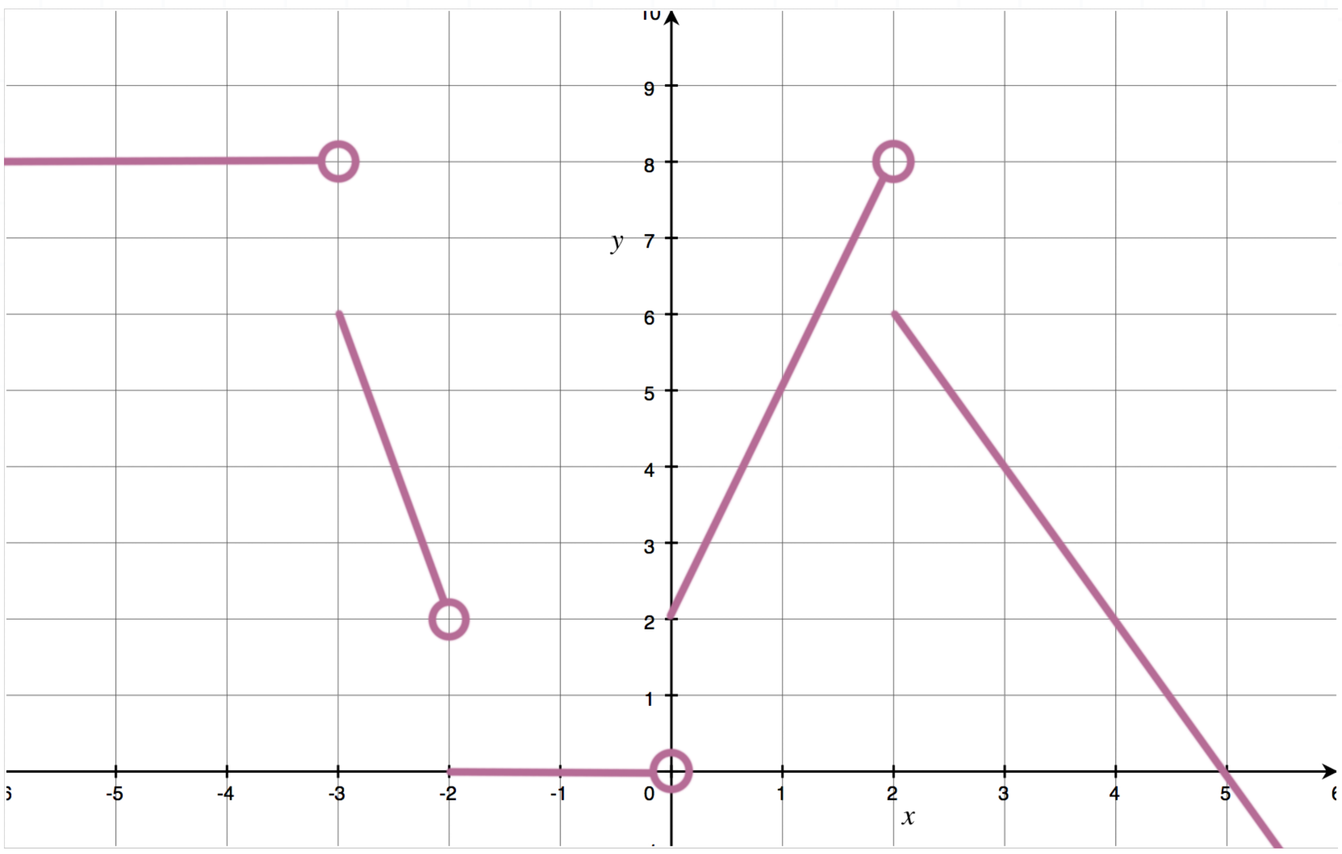
The function $f(x)$ has jump discontinuities at $x = -1$ and $x = 1$ because the left- and right-hand limits aren't equal at $x = -1$

$$\lim_{x \to -1^-} f(x) = 1 \quad \neq \quad \lim_{x \to -1^+} f(x) = 5$$

or at $x = 1$.

$$\lim_{x \to 1^-} f(x) = 1 \quad \neq \quad \lim_{x \to 1^+} f(x) = 0$$

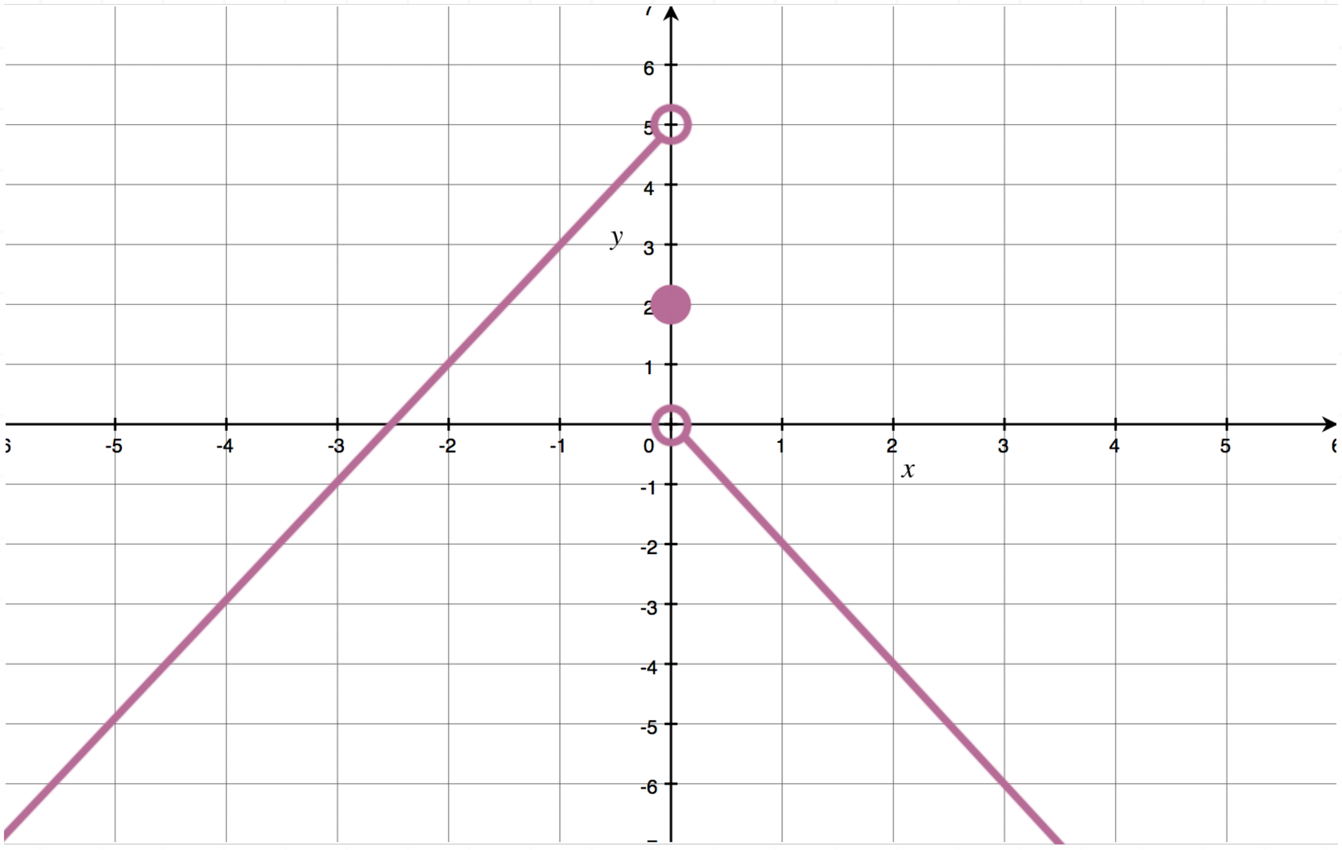
5. Where are the jump discontinuities in the graph of the function shown below?



Solution:

The function has jump discontinuities at $x = -3$, $x = -2$, $x = 0$, and $x = 2$, because at each x-value, the left- and right-hand limits aren't equal.

■ 6. What are the x-values where the graph of $h(x)$, shown below, has jump discontinuities?



Solution:

The function $h(x)$ has a jump discontinuity at $x = 0$ because the left- and right-hand limits aren't equal there.

$$\lim_{x\to 0^-} f(x) = 5 \quad \neq \quad \lim_{x\to 0^+} f(x) = 0$$

INFINITE DISCONTINUITIES

1. At what x-values does the function have infinite discontinuities?

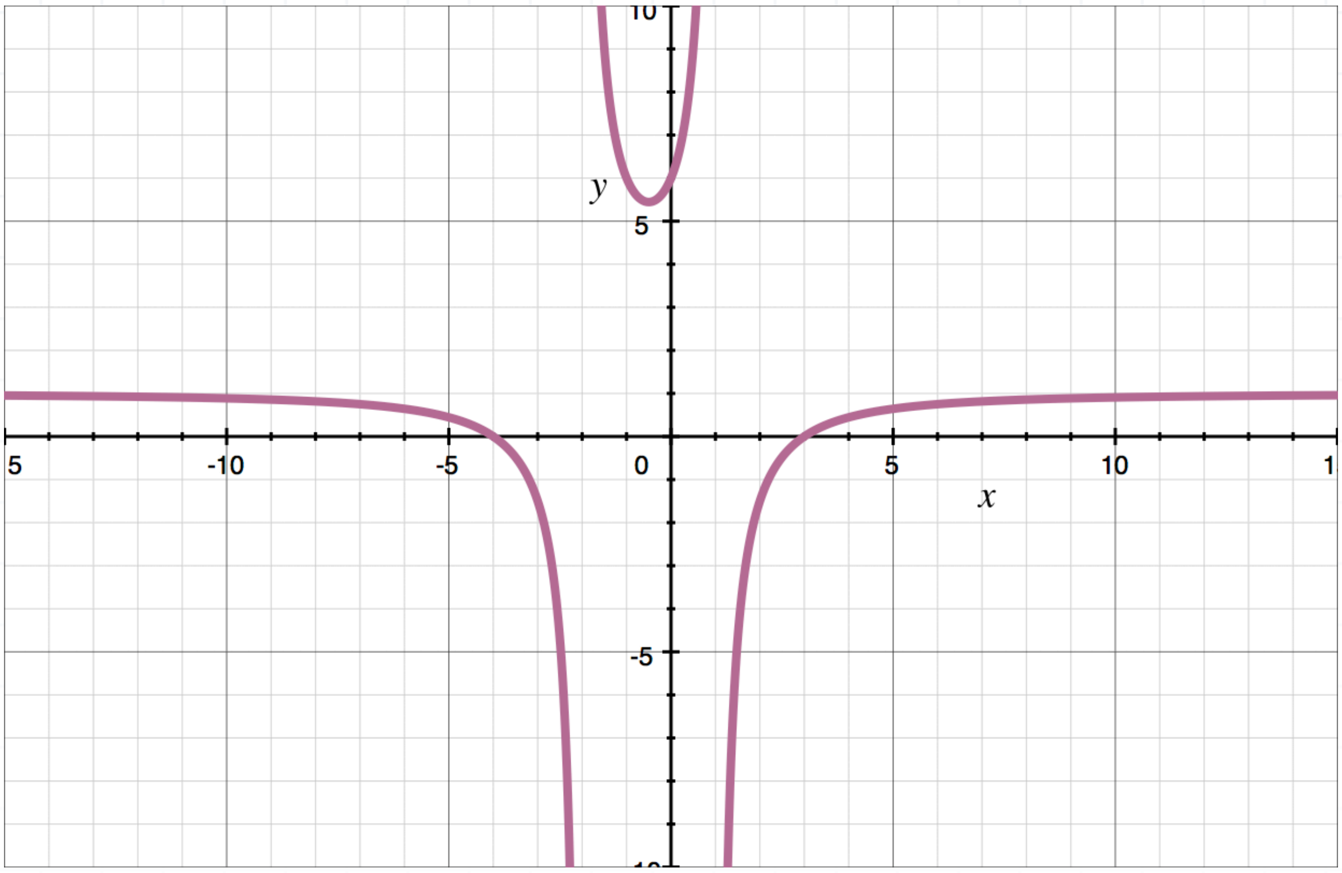
$$f(x) = \frac{x^2 + x - 12}{x^2 + x - 2}$$

Solution:

Factor the function.

$$f(x) = \frac{x^2 + x - 12}{x^2 + x - 2} = \frac{(x+4)(x-3)}{(x+2)(x-1)}$$

None of these factors cancel, which means that $x + 2 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -2$ and $x = 1$.



2. Where are the infinite discontinuities of the function?

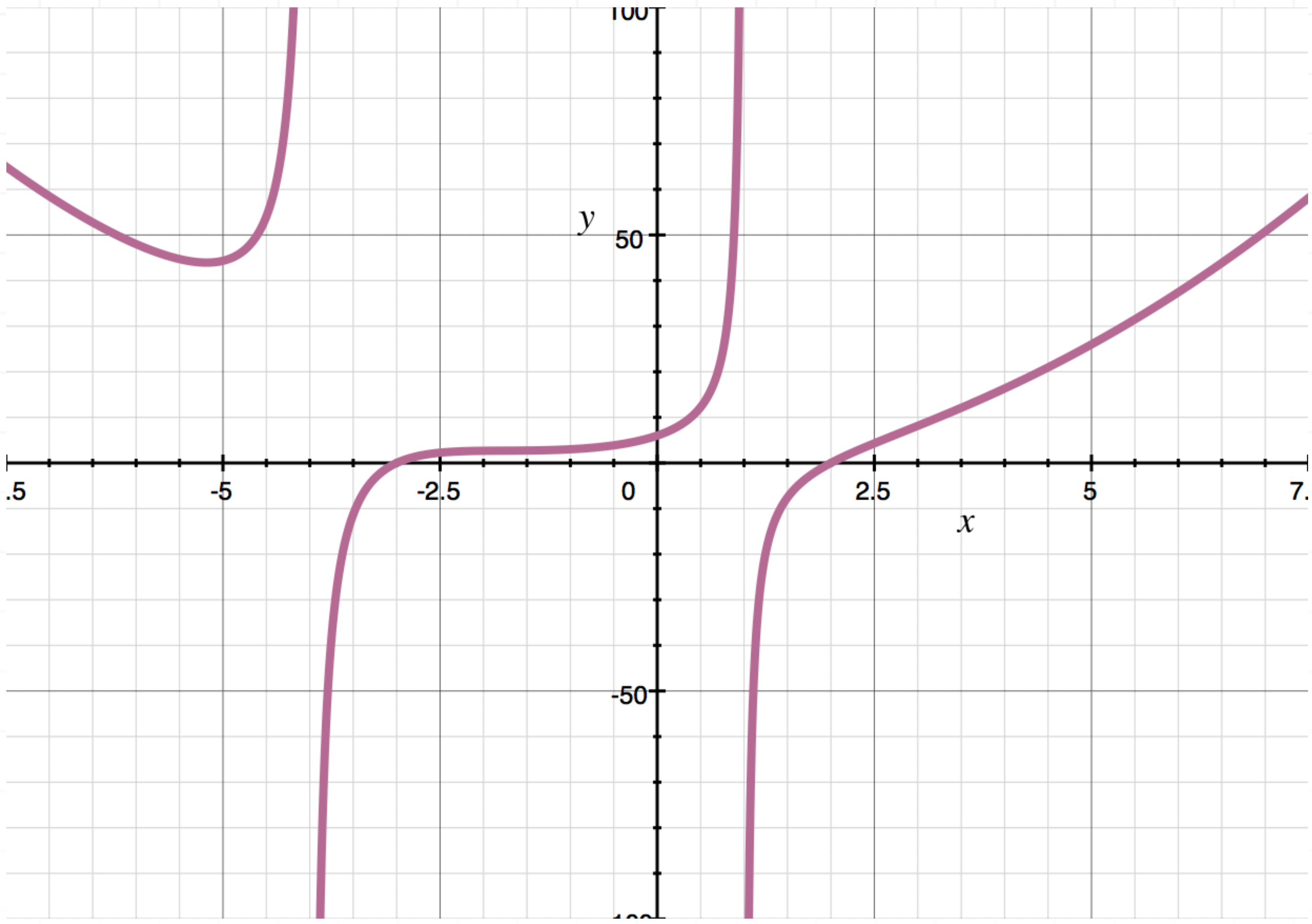
$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4}$$

Solution:

Factor the function.

$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4} = \frac{(x-2)(x^2+2x+4)(x+3)}{(x+4)(x-1)}$$

None of these factors cancel, which means that $x + 4 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -4$ and $x = 1$.



■ 3. At what x-values does the function have infinite discontinuities?

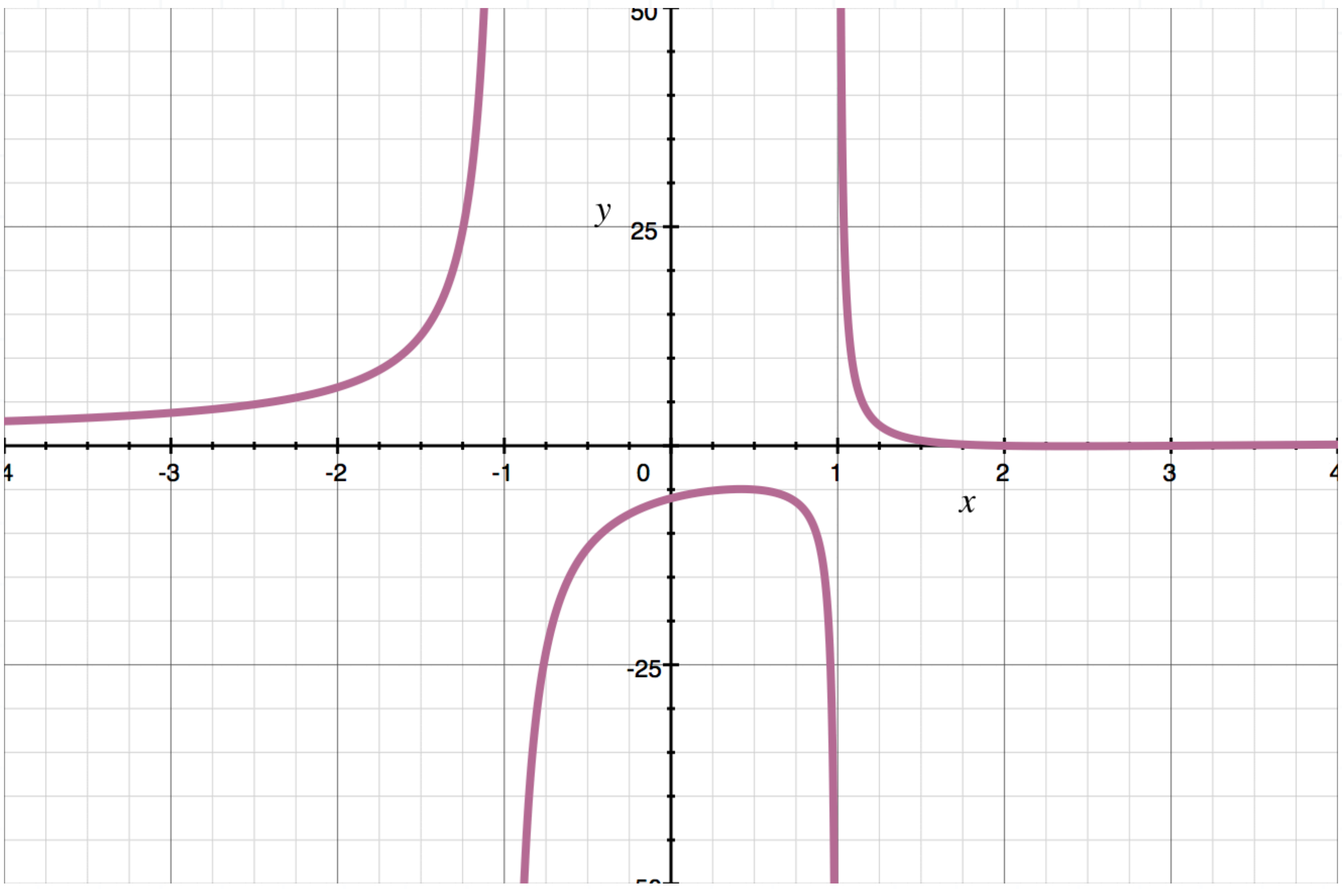
$$g(x) = \frac{x^2 - 5x + 6}{x^2 - 1}$$

Solution:

Factor the function.

$$g(x) = \frac{x^2 - 5x + 6}{x^2 - 1} = \frac{(x-3)(x-2)}{(x+1)(x-1)}$$

None of these factors cancel, which means that $x+1=0$ and $x-1=0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x=-1$ and $x=1$.



■ 4. Where are the infinite discontinuities of the function?

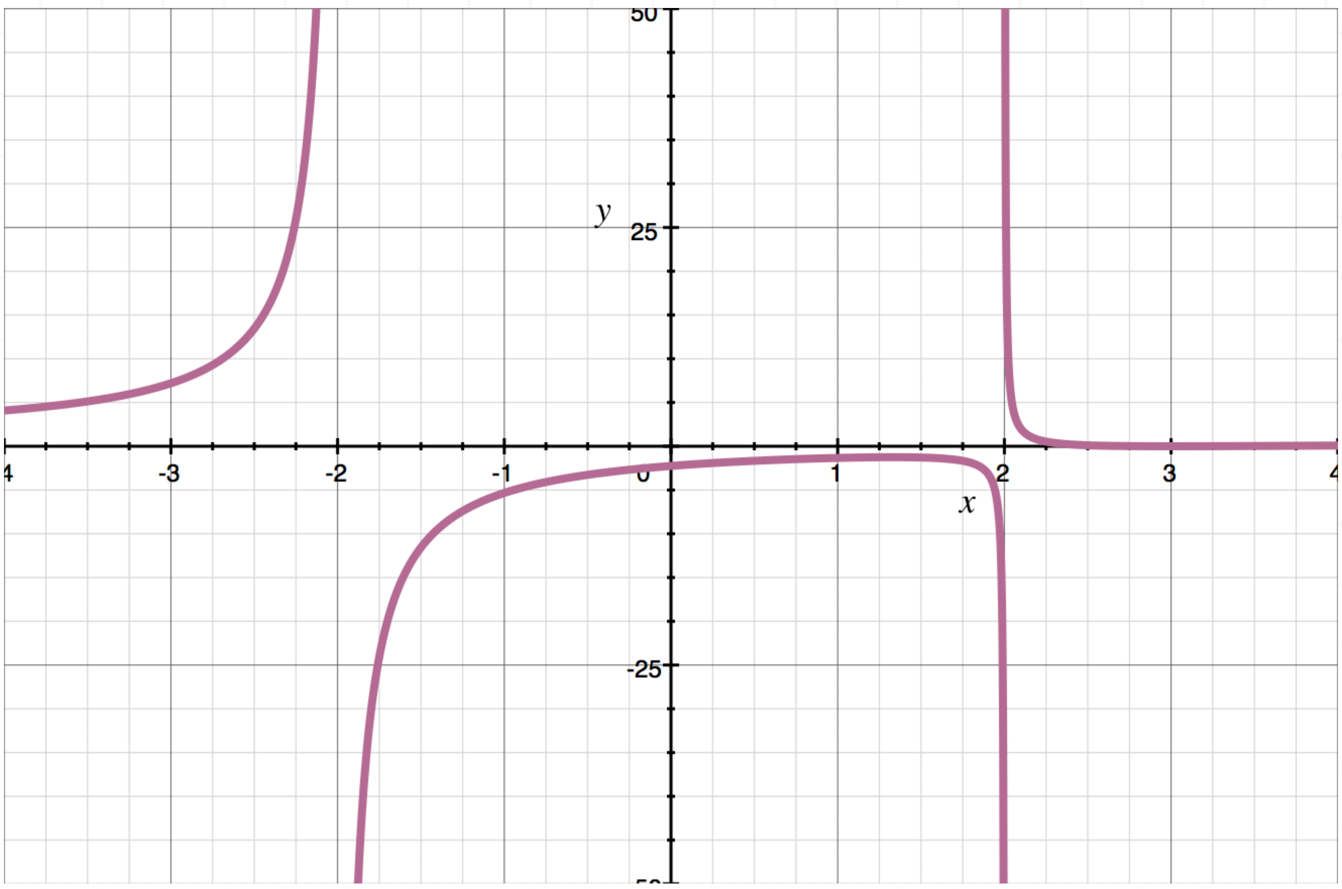
$$h(x)=\frac{x^2-6x+9}{x^2-4}$$

Solution:

Factor the function.

$$h(x)=\frac{x^2-6x+9}{x^2-4}=\frac{(x-3)^2}{(x+2)(x-2)}$$

None of these factors cancel, which means that $x+2=0$ and $x-2=0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x=-2$ and $x=2$.



■ 5. At what x-values does the function have infinite discontinuities?

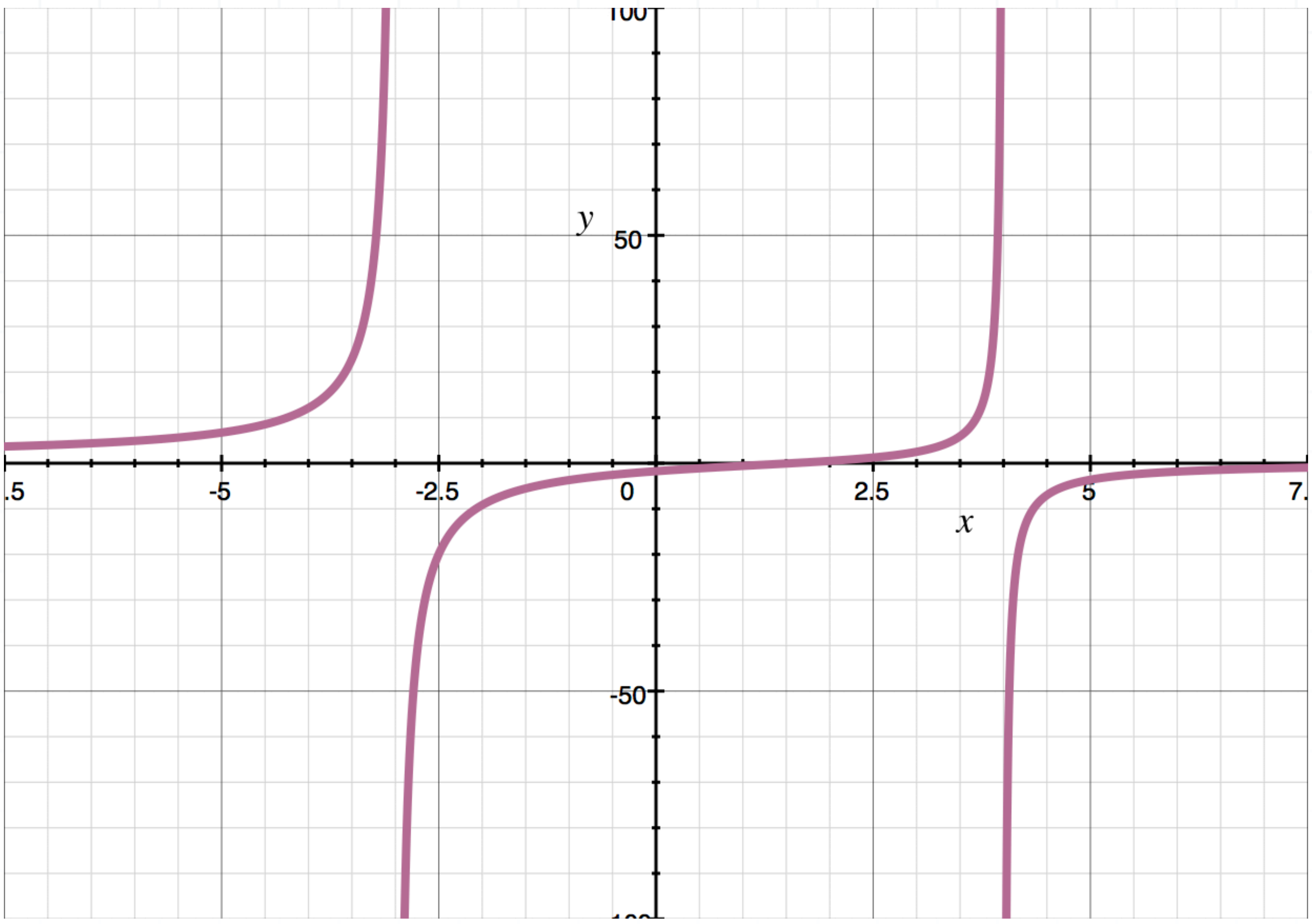
$$h(x)=\frac{x^2-15x+21}{x^2-x-12}$$

Solution:

Factor the function.

$$h(x)=\frac{x^2-15x+21}{x^2-x-12}=\frac{x^2-15x+21}{(x+3)(x-4)}$$

None of these factors cancel, which means that $x+3=0$ and $x-4=0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x=-3$ and $x=4$.



■ 6. Where are the infinite discontinuities of the function?

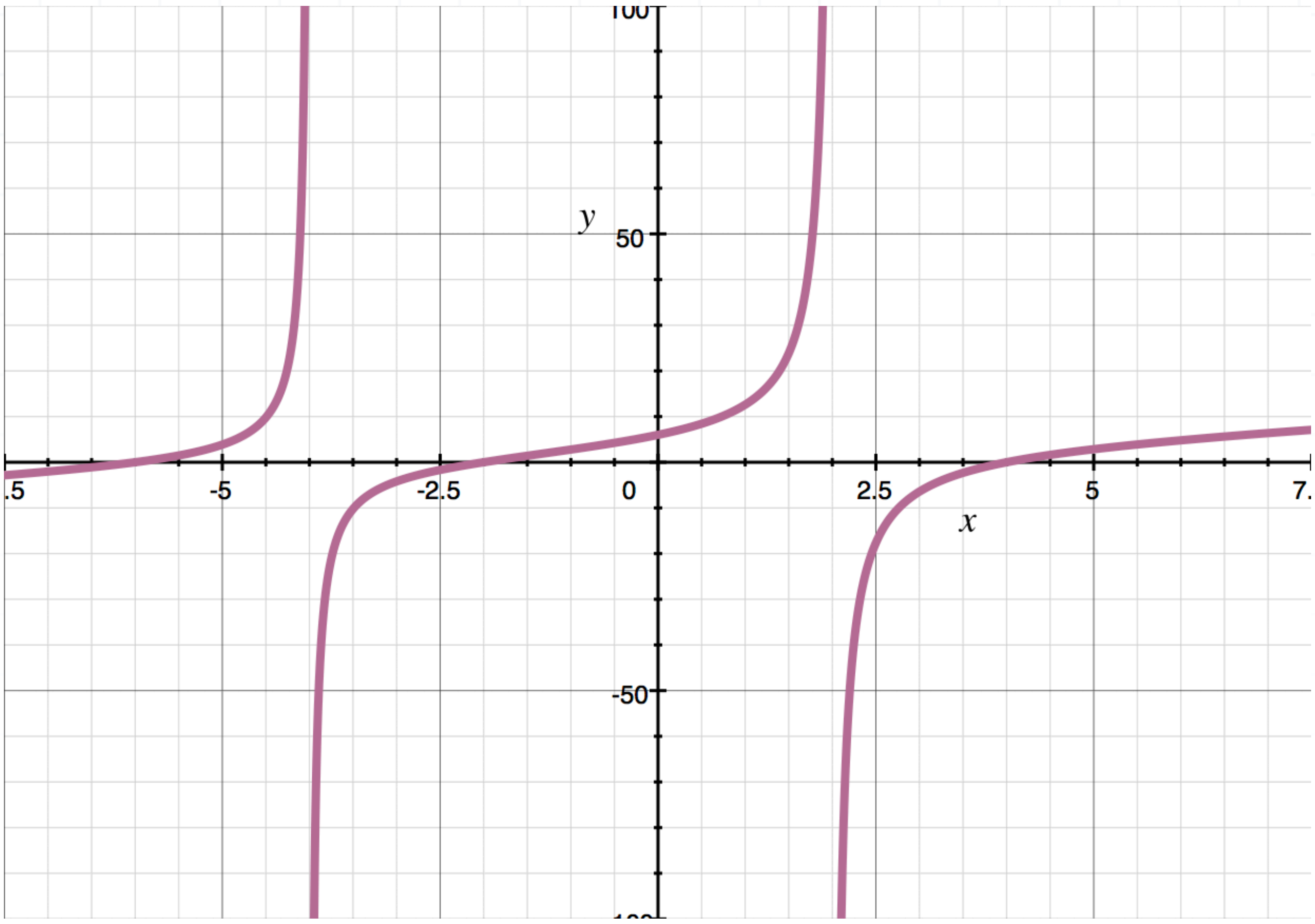
$$g(x)=\frac{x^3+4x^2-20x-48}{x^2+2x-8}$$

Solution:

Factor the function.

$$g(x) = \frac{x^3 + 4x^2 - 20x - 48}{x^2 + 2x - 8} = \frac{(x+2)(x-4)(x+6)}{(x+4)(x-2)}$$

None of these factors cancel, which means that $x + 4 = 0$ and $x - 2 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -4$ and $x = 2$.



ENDPOINT DISCONTINUITIES

1. What is the value of the limit on the interval $[0,3]$?

$$\lim_{x \to 3} -\sqrt{x+5}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = 3$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to 3^-} -\sqrt{x+5} = -2\sqrt{2} \quad \neq \quad \lim_{x \to 3^+} -\sqrt{x+5} = \text{DNE}$$

2. What is the value of the limit on the interval $[\pi,2\pi]$?

$$\lim_{x \to \pi} \sin x$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = \pi$. The left-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to \pi^+} \sin x = 0 \quad \neq \quad \lim_{x \to \pi^-} \sin x = \text{DNE}$$

■ 3. What is the value of the limit on the interval $(-\infty, 2]$.

$$\lim_{x \to 2} x^3 - x^2 + 4$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = 2$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to 2^-} x^3 - x^2 + 4 = 8 \quad \neq \quad \lim_{x \to 2^+} x^3 - x^2 + 4 = \text{DNE}$$

■ 4. What is the value of the limit on the interval $[4, \infty)$?

$$\lim_{x \to 4} -\frac{x + 7}{x^2 - 6x + 15}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = 4$. The left-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to4^+} -\frac{x+7}{x^2-6x+15} = -\frac{11}{7} \neq \lim_{x\to4^-} -\frac{x+7}{x^2-6x+15} = \text{DNE}$$

■ 5. What is the value of the limit on the interval $[-9/2,5/2]$?

$$\lim_{x\to\frac{5}{2}} \frac{x+3}{x^2+x+1}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x=5/2$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to\frac{5}{2}^-} \frac{x+3}{x^2+x+1} = \frac{22}{39} \neq \lim_{x\to\frac{5}{2}^+} \frac{x+3}{x^2+x+1} = \text{DNE}$$

■ 6. What is the value of the limit on the interval $(-2,2]$?

$$\lim_{x\to-2} \sqrt{2x+4}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = -2$. The left-hand limit does not exist, which means that the one-sided limits are not equal.

$$\lim_{x \to -2^+} \sqrt{2x+4} = 0 \quad \neq \quad \lim_{x \to -2^-} \sqrt{2x+4} = \text{DNE}$$

■ 7. What is the value of the limit on the interval $[-\pi, \pi]$?

$$\lim_{x \to \pi} -\frac{5 \cos x}{2}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = \pi$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to \pi^-} -\frac{\cos x}{2} = \frac{5}{2} \quad \neq \quad \lim_{x \to \pi^+} -\frac{\cos x}{2} = \text{DNE}$$

W W W . K R I S T A K I N G M A T H . C O M