Squeeze theorem

The **squeeze theorem** allows us to find the limit of a function at a particular point, even when the function is undefined at that point. The way that we do it is by showing that our function can be "squeezed" between two other functions at the given point, and proving that the limits of these other functions are equal.

If we can show that two functions have the same value at a particular point, and we know that the original function has to run through the other two (be squeezed, or pinched, or sandwiched between them), then the original function can't take on any possible value other than the value of the other two.

Let's get a little more technical and take a look at the actual theorem.

We assume that our original function is $h(x)$, and that it's squeezed between two other functions, $f(x)$ and $g(x)$, so

$$f(x) \leq h(x) \leq g(x)$$

We also assume that the limits of our other two functions are equal as we approach the point c we're interested in, so

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$$

If we can show that both of the above statements are true, then we can say

$$L \leq \lim_{x \to c} h(x) \leq L$$

and we know that the original function $h(x)$ must have the same limit as the other two functions.

$$\lim_{x \to c} h(x) = L$$

We don't need to know what's actually happening to $h(x)$ at $x = c$. We're only concerned with the limit, so we just need to know what's happening *around* $x = c$.

Let's do an example where we use squeeze theorem to find the limit.

Example

Evaluate the limit.

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

If we try to evaluate the limit using substitution, we get a 0 value in the denominator of the fraction, which is undefined. So we can't use substitution, and there's nothing we can really do with factoring or conjugate method. So we'll try squeeze theorem.

We know that the sine function oscillates back and forth between -1 and 1, so no matter what value we use for x, we know that this inequality will always be true:

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

The goal now is to manipulate the inequality until the function in the center is identical to the original function. To get there, we'll multiply through by x^2.

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

We've now "squeezed" our original function between two other functions, $-x^2$ and x^2. Both of these squeezing functions have a well-defined limit as $x \to 0$.

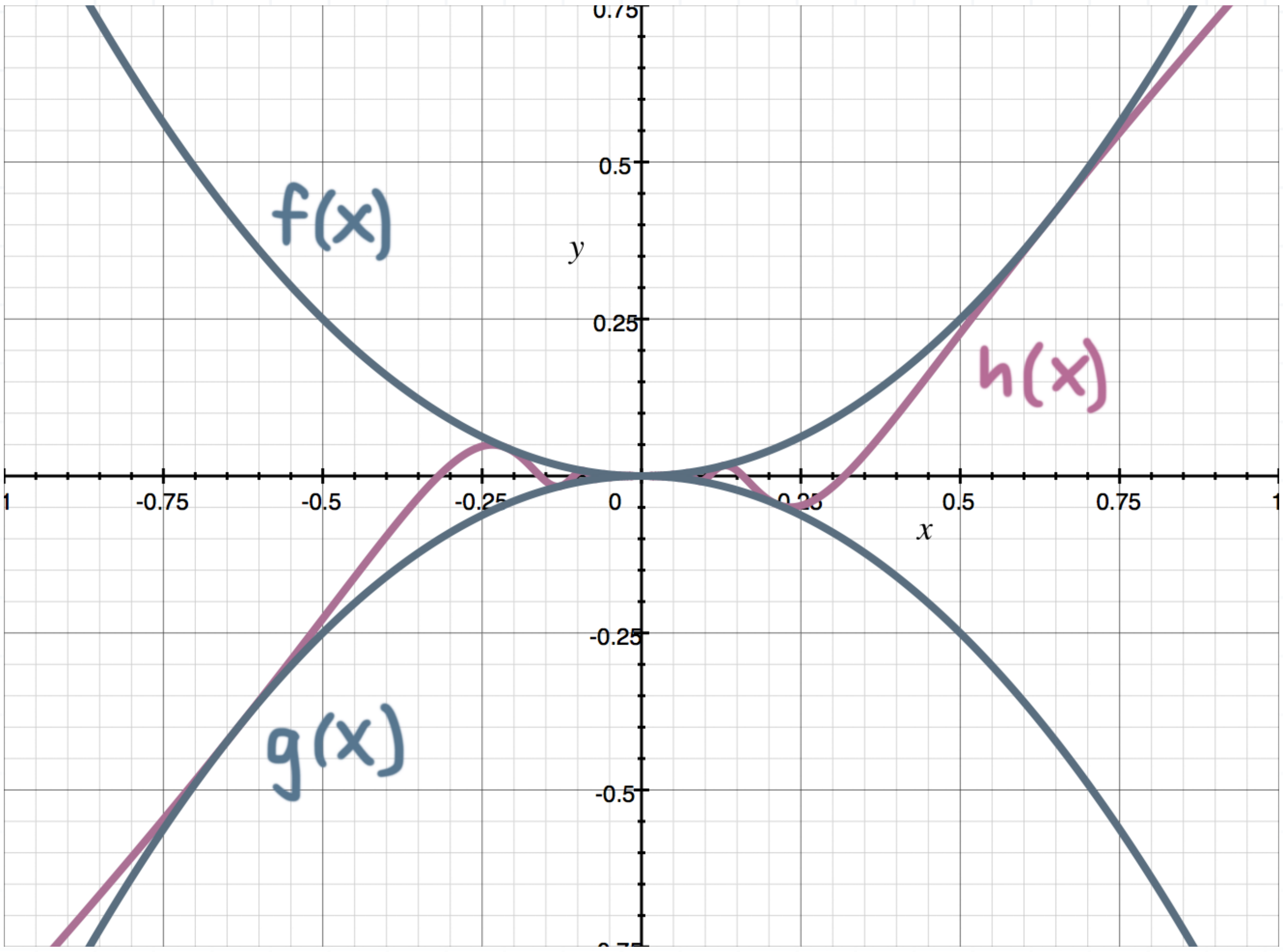
$$\lim_{x\to 0} -x^2 = -0^2 = 0$$

$$\lim_{x\to 0} x^2 = 0^2 = 0$$

Since these two functions have the same limit as $x \to 0$, and we know that the original function is squeezed between these other two, there's no possible value of the limit of the original function other than the value of the limit of the squeezing functions at the same point. Therefore, it must be true that

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

The graph below confirms that the three equations all exist as they approach $x = 0$ from both the left- and right-hand sides, and that they all have the same value at that point.


f(x)
g(x)
h(x)
y
x
0.75
0.5
0.25
-0.25
-0.5
-0.75
-0.5
-0.25
0
0.25
0.5
0.75
1