Product rule, three or more functions

As we learned in the last lesson, the product rule is what we use to take the derivative of a product. So we already know that the product rule works well for differentiating the product of two functions.

Product rule with three functions

In this lesson, we want to show that the product rule can be extended to more than two functions. For instance, given the function

$$y = f(x)g(x)h(x)$$

then the product rule tells us that the derivative is

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

Notice how each term in the derivative is just the derivative of one of the functions from the original equation, multiplied by the rest of the functions from the original equation. And in the product rule formula, we include every possible combination.

Extending the product rule

That being said, we can use this same method to extend the product rule to as many terms as we have multiplied together in the original function.

So for an equation that's the product of four functions, $y = ABCD$, the derivative will include four terms: each possible combination.

$$y' = A'BCD + AB'CD + ABC'D + ABCD'$$

Let's do an example where we apply the product rule to the product of three functions.

Example

Use product rule to find the derivative.

$$y = (4x^6)(-2x)(-x^3)$$

First, let's list out each of the functions in the product, as well as their derivatives.

$$f(x) = 4x^6$$

$$f'(x) = 24x^5$$

and

$$g(x) = -2x$$

$$g'(x) = -2$$

and

$$h(x) = -x^3$$

$$h'(x) = -3x^2$$

Now we can plug everything we've found directly into the product rule formula.

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$y' = (24x^5)(-2x)(-x^3) + (4x^6)(-2)(-x^3) + (4x^6)(-2x)(-3x^2)$$

$$y' = 48(x^5)(x)(x^3) + 8(x^6)(x^3) + 24(x^6)(x)(x^2)$$

$$y' = 48x^9 + 8x^9 + 24x^9$$

$$y' = 80x^9$$

We can confirm this answer if we instead start the problem by multiplying out the functions.

$$y = (4x^6)(-2x)(-x^3)$$

$$y = 8(x^6)(x)(x^3)$$

$$y = 8x^{10}$$

Once we've simplified the function this way, we can take the derivative using power rule.

$$y' = 8(10)x^{10-1}$$

$$y' = 80x^9$$

We get the same answer both ways, so we know that we applied the product rule correctly.

Let's do another example with more functions.

Example

Use product rule to find the derivative.

$$y = (8x^{12})\left(\frac{6}{7}x^2\right)(x)(-3)(2x^3)$$

Let's apply product rule.

$$y' = (96x^{11})\left(\frac{6}{7}x^2\right)(x)(-3)(2x^3)$$

$$+(8x^{12})\left(\frac{12}{7}x\right)(x)(-3)(2x^3)$$

$$+(8x^{12})\left(\frac{6}{7}x^2\right)(1)(-3)(2x^3)$$

$$+(8x^{12})\left(\frac{6}{7}x^2\right)(x)(0)(2x^3)$$

$$+(8x^{12})\left(\frac{6}{7}x^2\right)(x)(-3)(6x^2)$$

$$y' = -\frac{3{,}456}{7}(x^{11})(x^2)(x)(x^3)$$

$$-\frac{576}{7}(x^{12})(x)(x)(x^3)$$

$$-\frac{288}{7}(x^{12})(x^2)(x^3)$$

$$+0$$

$$-\frac{864}{7}(x^{12})(x^2)(x)(x^2)$$

$$y' = -\frac{3{,}456}{7}x^{17} - \frac{576}{7}x^{17} - \frac{288}{7}x^{17} - \frac{864}{7}x^{17}$$

$$y' = -\frac{5{,}184}{7}x^{17}$$

We can confirm this result by first multiplying out the functions,

$$y = (8x^{12})\left(\frac{6}{7}x^2\right)(x)(-3)(2x^3)$$

$$y = -\frac{288}{7}(x^{12})(x^2)(x)(x^3)$$

$$y = -\frac{288}{7}x^{18}$$

and then applying power rule, instead.

$$y = -\frac{288}{7}(18)x^{18-1}$$

$$y = -\frac{5{,}184}{7}x^{17}$$