Inverse hyperbolic derivatives

Lastly, we want to be able to find the derivatives of the inverse hyperbolic functions.

Just like with the inverse trig functions, we can express inverse hyperbolic functions in a couple of different ways. For example, inverse hyperbolic sine can be written as $\sinh^{-1} x$, or as $\text{arcsinh}x$.

The derivatives of the inverse hyperbolic functions are given in the table.

Inverse function	**Derivative**	**Restriction**
$y = \sinh^{-1} x$	$y' = \dfrac{1}{\sqrt{x^2+1}}$	
$y = \cosh^{-1} x$	$y' = \dfrac{1}{\sqrt{x^2-1}}$	$x > 1$
$y = \tanh^{-1} x$	$y' = \dfrac{1}{1-x^2}$	$\|x\| < 1$
$y = \text{csch}^{-1} x$	$y' = -\dfrac{1}{\|x\|\sqrt{x^2+1}}$	$x \neq 0$
$y = \text{sech}^{-1} x$	$y' = -\dfrac{1}{x\sqrt{1-x^2}}$	$0 < x < 1$
$y = \coth^{-1} x$	$y' = \dfrac{1}{1-x^2}$	$\|x\| > 1$

As always, if the argument of the inverse hyperbolic function is anything other than x, then the derivative of the argument will be something other than 1, which means that applying the chain rule will actually have an affect on the value of the derivative.

So, if the argument of the hyperbolic function is something other than x, then the derivatives will be given by the formulas in this table:

Inverse function	**Derivative**	**Restrictions**
$y=\sinh^{-1}[g(x)]$	$y'=\dfrac{g'(x)}{\sqrt{[g(x)]^2+1}}$	
$y=\cosh^{-1}[g(x)]$	$y'=\dfrac{g'(x)}{\sqrt{[g(x)]^2-1}}$	$g(x)>1$
$y=\tanh^{-1}[g(x)]$	$y'=\dfrac{g'(x)}{1-[g(x)]^2}$	$\lvert g(x)\rvert<1$
$y=\operatorname{csch}^{-1}[g(x)]$	$y'=-\dfrac{g'(x)}{\lvert g(x)\rvert\sqrt{[g(x)]^2+1}}$	$g(x)\neq 0$
$y=\operatorname{sech}^{-1}[g(x)]$	$y'=-\dfrac{g'(x)}{g(x)\sqrt{1-[g(x)]^2}}$	$0<g(x)<1$
$y=\coth^{-1}[g(x)]$	$y'=\dfrac{g'(x)}{1-[g(x)]^2}$	$\lvert g(x)\rvert>1$

With these formulas in mind, let’s try an example where we find the derivative of an inverse hyperbolic function.

Example

Find the derivative of the inverse hyperbolic function.

$$y = -8\coth^{-1}(21x^3)$$

Apply the formula for the derivative of inverse hyperbolic cotangent, remembering to apply chain rule.

$$y' = \frac{g'(x)}{1 - [g(x)]^2}$$

$$y' = -8\left(\frac{1}{1 - (21x^3)^2}\right)(63x^2)$$

$$y' = -\frac{504x^2}{1 - 441x^6}$$

Now let's try an example with an inverse hyperbolic function occurring as part of a larger function.

Example

Find the derivative of the function.

$$y = 6x^{-4} - \cosh^{-1}(4x^7)$$

Take the derivative of one term at a time, applying the formula for the derivative of inverse hyperbolic cosine,

$$y' = \frac{g'(x)}{\sqrt{[g(x)]^2 - 1}}$$

and remembering to apply chain rule.

$$y' = -24x^{-5} - \left(\frac{1}{\sqrt{(4x^7)^2 - 1}}\right)(28x^6)$$

$$y' = -\frac{24}{x^5} - \frac{28x^6}{\sqrt{16x^{14} - 1}}$$

Let's try one more example that's a little more complex.

Example

Find the derivative of the function.

$$y = \text{sech}^{-1}(81x^4) - 5x^{-9}\sinh^{-1}(6x^7) + 103x^8$$

Apply the formulas for the derivative of inverse hyperbolic secant and inverse hyperbolic sine. We'll also need to use product rule for the second term.

$$y' = \left(-\frac{1}{81x^4\sqrt{1-(81x^4)^2}}\right)(324x^3)$$

$$-\left[(-45x^{-10})(\sinh^{-1}(6x^7)) + (5x^{-9})\left(\frac{1}{\sqrt{(6x^7)^2+1}}\right)(42x^6)\right] + 824x^7$$

$$y' = -\frac{324x^3}{81x^4\sqrt{1-(81x^4)^2}} - \left(-45x^{-10}\sinh^{-1}(6x^7) + \frac{210x^{-9}x^6}{\sqrt{(6x^7)^2+1}}\right) + 824x^7$$

$$y' = -\frac{324x^3}{81x^4\sqrt{1-(81x^4)^2}} + 45x^{-10}\sinh^{-1}(6x^7) - \frac{210x^{-9}x^6}{\sqrt{(6x^7)^2+1}} + 824x^7$$

$$y' = \frac{45\sinh^{-1}(6x^7)}{x^{10}} - \frac{4}{x\sqrt{1-6{,}561x^8}} - \frac{210}{x^3\sqrt{36x^{14}+1}} + 824x^7$$