Values that make the function differentiable

In this lesson, we want to look at another common derivative problem, similar to the way that we tackled parallel tangent lines in the last lesson.

In these kinds of problems, we're given a piecewise-defined function that includes some unknown constant(s), and asked to find the values of those constants that will force the differentiability of the piecewise function.

A function is differentiable at a particular point if it's continuous and smooth at that point.

1. A piecewise function will be continuous at its break point if the one-sided limits of the function's value at the break point are equal

2. A piecewise function will be smooth if the one-sided limits of the slopes (value of the derivative) of each piece at the break point are equal

So, in order to solve problems like these, we'll find the condition that forces the continuity of the piecewise function at its breakpoint, we'll find the condition that forces equal slopes of the piecewise function at its breakpoint, and then we'll use these conditions as a system of equations to solve for the unknown constant(s) that force the continuity and smoothness.

Let's work through an example so that we can actually break down the process we've just described.

Example

Find the values of a and b that make the piecewise-defined function differentiable.

$$f(x) = \begin{cases} 2x^2 - ax + b & x \le 1 \\ 3x^3 - x^2 - 6 & x > 1 \end{cases}$$

The break point of the function is at $x = 1$, because that's where the first piece of the function ends and the second piece takes over.

We'll work on continuity first by setting the one-sided limits at the break point $x = 1$ equal to one another.

$$\lim_{x \to 1^-} 2x^2 - ax + b = \lim_{x \to 1^+} 3x^3 - x^2 - 6$$

$$2(1)^2 - a(1) + b = 3(1)^3 - 1^2 - 6$$

$$2 - a + b = 3 - 1 - 6$$

$$2 - a + b = -4$$

$$-a + b = -6$$

Now we'll work on smoothness by setting the one-sided limits of the derivatives of each piece at the break point $x = 1$ equal to one another.

$$\lim_{x \to 1^-} 4x - a = \lim_{x \to 1^+} 9x^2 - 2x$$

$$4(1) - a = 9(1)^2 - 2(1)$$

$$4 - a = 9 - 2$$

$$4 - a = 7$$

$$-a = 3$$

$$a = -3$$

Pull together these two equations into a system of equations.

$$a = -3$$

$$-a + b = -6$$

We need to solve the system, which we can do by substituting the first equation $a = -3$ into the second equation.

$$-(-3) + b = -6$$

$$3 + b = -6$$

$$b = -9$$

Therefore, the values of the constants a and b that make $f(x)$ differentiable are $a = -3$ and $b = -9$.