Calculus 1 Workbook Solutions

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VERTICAL LINE TEST

1. Determine algebraically whether or not the equation represents a function.

$$(x-1)^2 + y = 3$$

Solution:

Notice that the equation can be rewritten as $y = -(x-1)^2 + 3$, which is just a transformation of $y = -x^2$. Therefore, the equation represents a function.

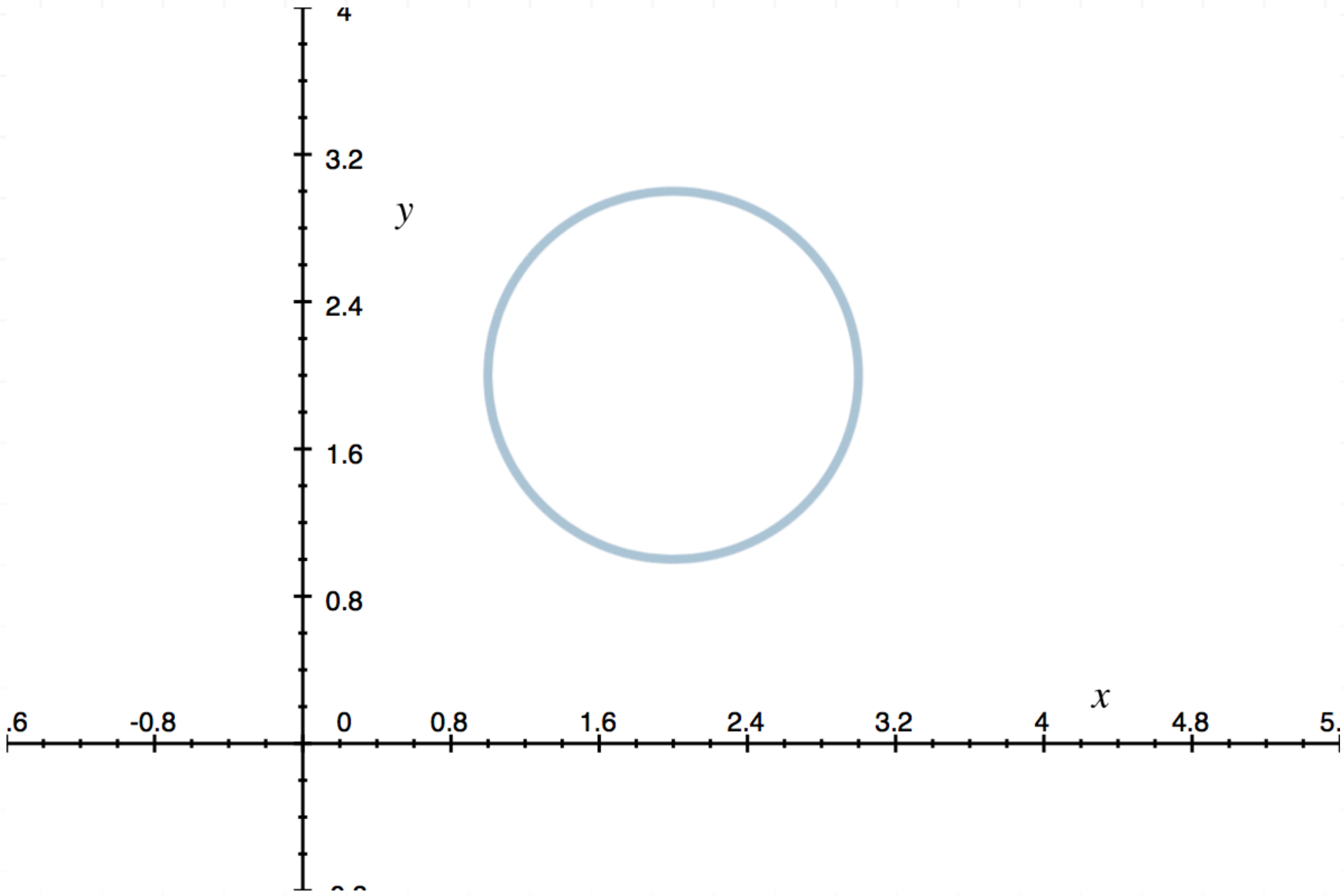
2. Fill in the blanks in the following statement using “equations,” and “functions.”

All __________ are __________.

Solution:

functions, equations

3. Use the Vertical Line Test to determine whether or not the graph is the graph of a function.



Solution:

The graph does not pass the Vertical Line Test, because any vertical line between the left edge of the circle and the right edge of the circle intersects the graph more than once. Therefore, the graph doesn't represent a function.

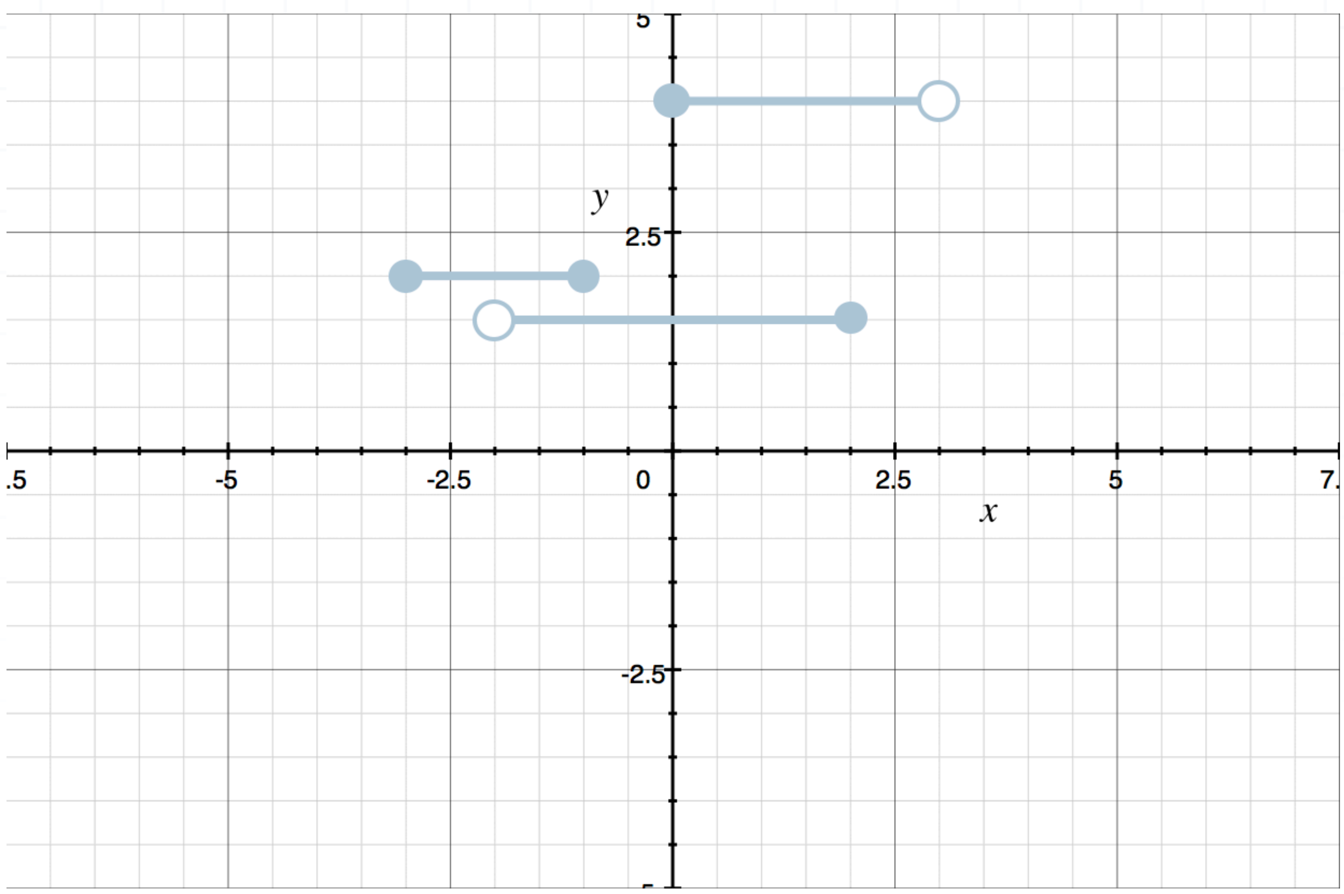
4. Determine algebraically whether or not the equation represents a function.

$$y^2 = x + 1$$

Solution:

Notice that for $x = 0$, $y^2 = 1$ which gives $y = -1, 1$. So for one input x, there are two outputs for y, so the equation does not represent a function.

5. Use the Vertical Line Test to determine whether or not the graph represents a function.



Solution:

There are different vertical lines that intersect the graph more than once. An example would be $x = 0$, which intersects the graph at $y = 3/2$ and $y = 4$. So by the Vertical Line Test, the graph is not a graph of a function.

6. Explain why the Vertical Line Test determines whether or not a graph represents a function.

Solution:

There are many correct answers. But they should all more or less say something like:

> "The Vertical Line Test can show whether or not a graph represents a function, because if any perfectly vertical line crosses the graph more than once, it proves that there are two output values of y for the one input value of x."

7. Fill in the blanks in the following statement using: equations, functions.

Not all __________ are __________.

Solution:

equations, functions

8. Determine algebraically whether or not the equation represents a function.

$$x^3 + y = 5$$

Solution:

Notice that this equation can be rewritten as $y = -x^3 + 5$, which is just a transformation of $y = -x^3$. Therefore, the equation represents a function.

DOMAIN AND RANGE

1. Find the domain of $f(x)$.

$$f(x)=\frac{3}{x(x+1)}+x^2$$

Solution:

In this function, the denominator cannot be equal to 0. The values of x that make the denominator 0 are $x=0$ and $x=-1$. So the domain of the function is all $x\neq 0,-1$, which we can write in interval notation as

$$(-\infty,-1)\cup(-1,0)\cup(0,\infty)$$

2. Find the domain and range of the given set.

$$(-1,-3),\quad (0,5),\quad (-3,6),\quad (0,-3)$$

Solution:

The domain is all the x-values and the range is all the y-values. Therefore the domain and range are

Domain: $-1,\ 0,\ -3$

Range: $-3, 5, 6$

■ 3. Find the domain and range of $g(x)$.

$$g(x) = \frac{\sqrt{x-2}}{3}$$

Solution:

In this function, the radicand (the expression under the square root) must be 0 or positive. So $x - 2 \geq 0$, which tells us that $x \geq 2$. Therefore the domain of the function in interval notation is $[2,\infty)$. Since the square root function cannot be negative, the range in interval notation is $[0,\infty)$.

■ 4. Find the domain and range of the function.

$$f(x) = \frac{2}{x} + 1$$

Solution:

In this function, the denominator cannot be 0, which means $x \neq 0$. Therefore the domain of the function in interval notation is

$$(-\infty,0) \cup (0,\infty)$$

Since the term $2/x$ is never 0, $f(x)$ can never be 1. Therefore the range of the function in interval notation is

$$(-\infty,1) \cup (1,\infty)$$

5. Give an example of a function that has a domain of $[1,\infty)$.

Solution:

There are many correct answers. The simplest one is

$$f(x) = \sqrt{x-1}$$

Note that since the 1 is included in the domain, the function $f(x) = \ln(x-1)$ would not work.

6. Find the domain and range of $f(x)$.

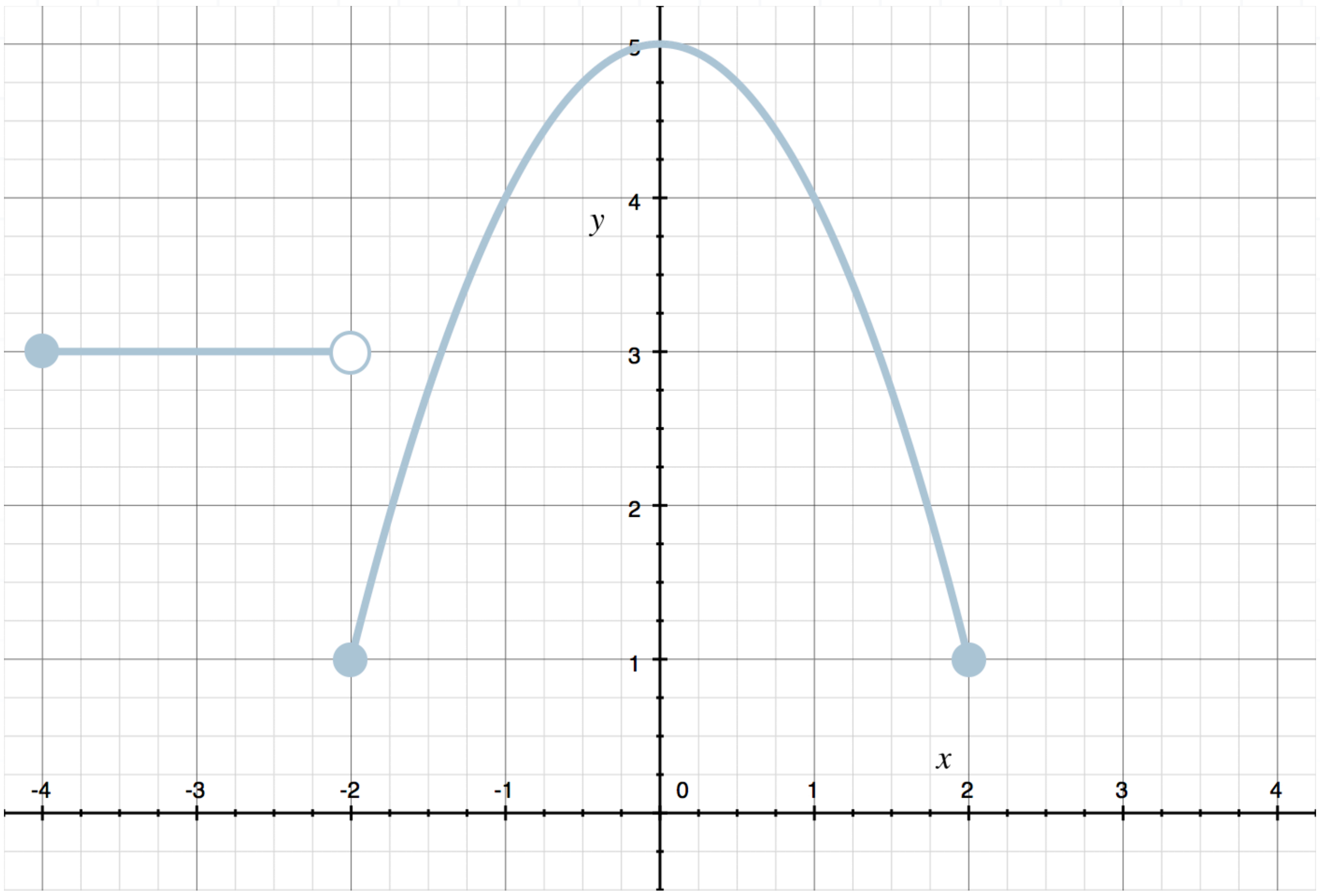
$$f(x) = \ln(x+3) + 5$$

Solution:

In this function, the input into $\ln$ must be positive. So $x + 3 > 0$, which tells us that $x > -3$. Therefore the domain in interval notation is $(-3,\infty)$. Since the range of $\ln(x)$ is $(-\infty, \infty)$, then the range of $f(x)$ is $(-\infty, \infty)$.

DOMAIN AND RANGE FROM A GRAPH

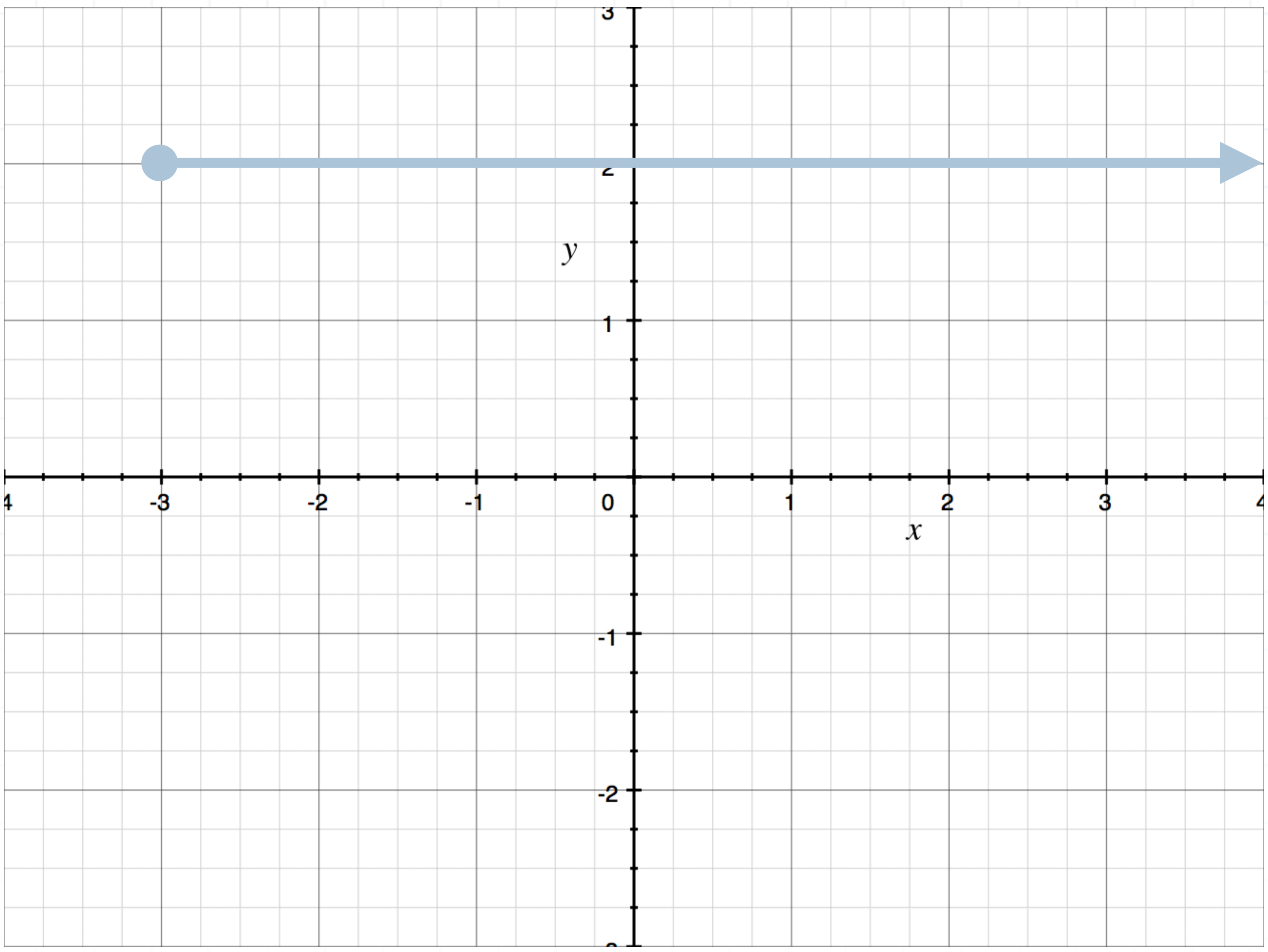
■ 1. What is the domain and range of the function? Assume the graph does not extend beyond the graph shown.



Solution:

Solution: The domain of the function given in the graph is determined by the x-values, which are defined on the interval $[-4,2]$. The range is determined by the y-values, which are defined on the interval $[1,5]$.

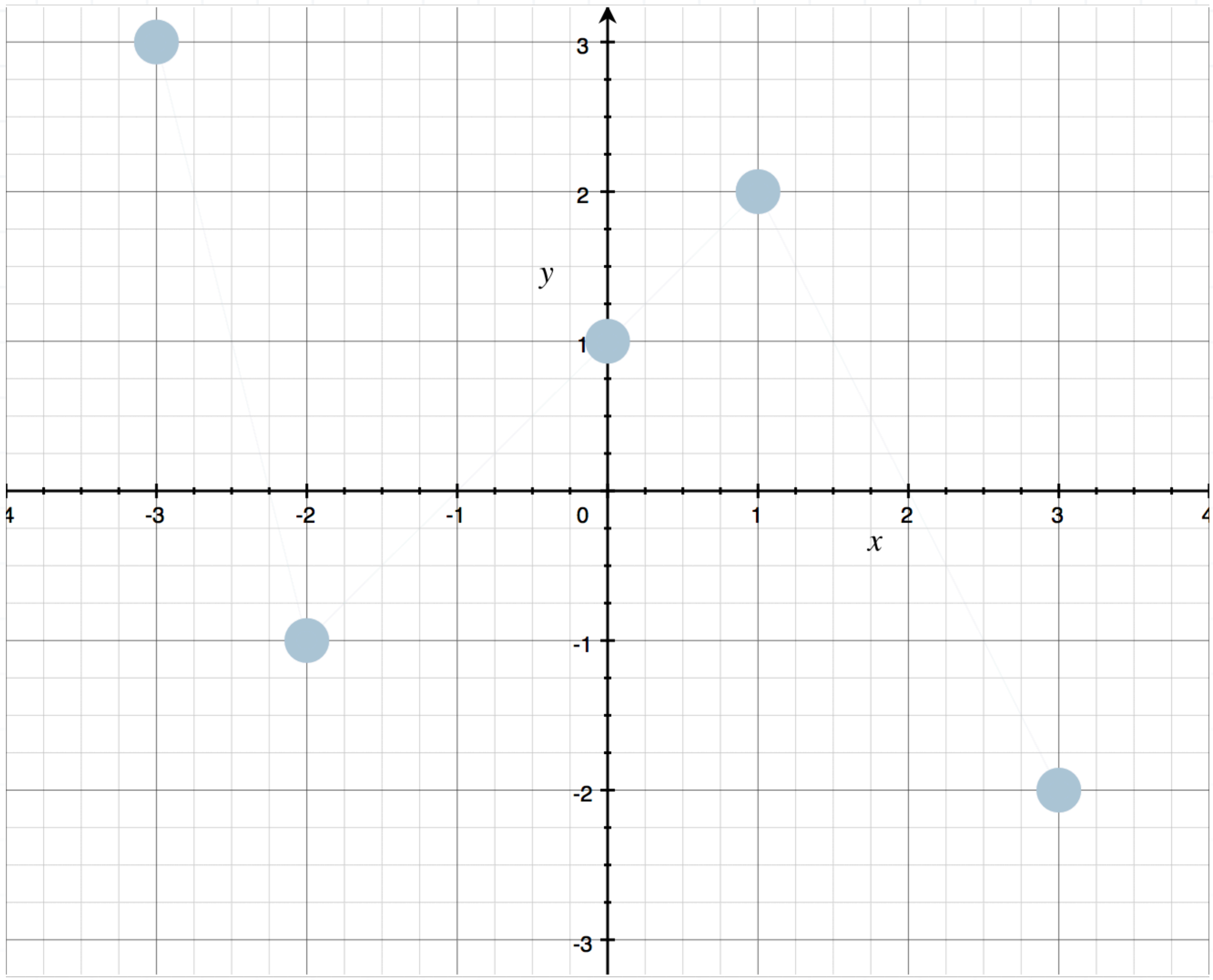
■ 2. What is the domain and range of the function?



Solution:

The domain of the function given in the graph is determined by the x-values, which are defined by the ray on the interval $[-3,\infty)$. The range is determined by the y-values, which is only $y=2$.

3. Determine the domain and range of the function.



Solution:

The domain of the function given in the graph is determined by the x-values, which are $\{-3,-2,0,1,3\}$. The range is determined by the y-values, which are $\{-2,-1,1,2,3\}$.

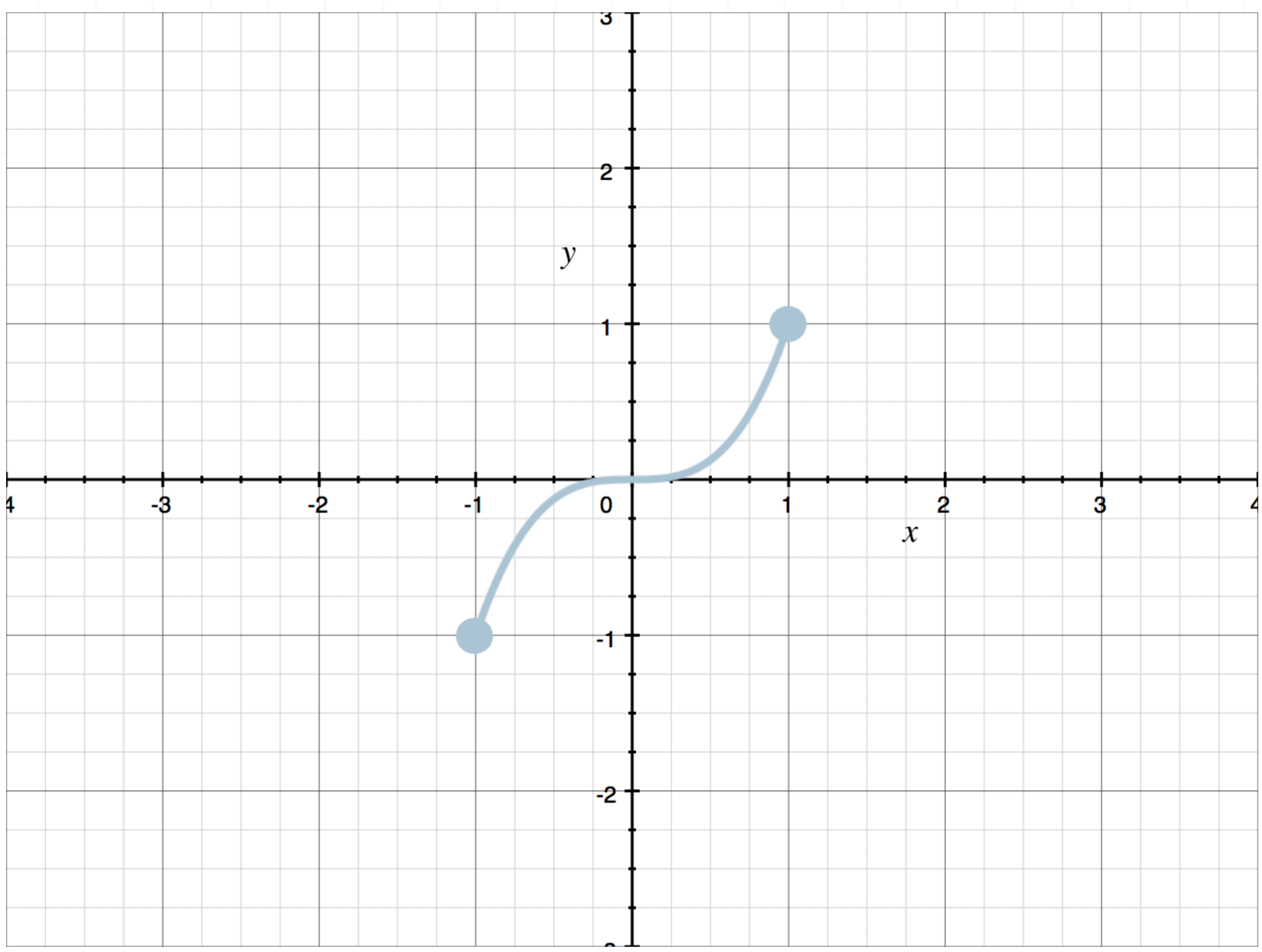
4. Fill in the blanks in the following description of the domain of a graph.

"The domain is all the values of the graph from __________ to __________."

Solution:

left, right

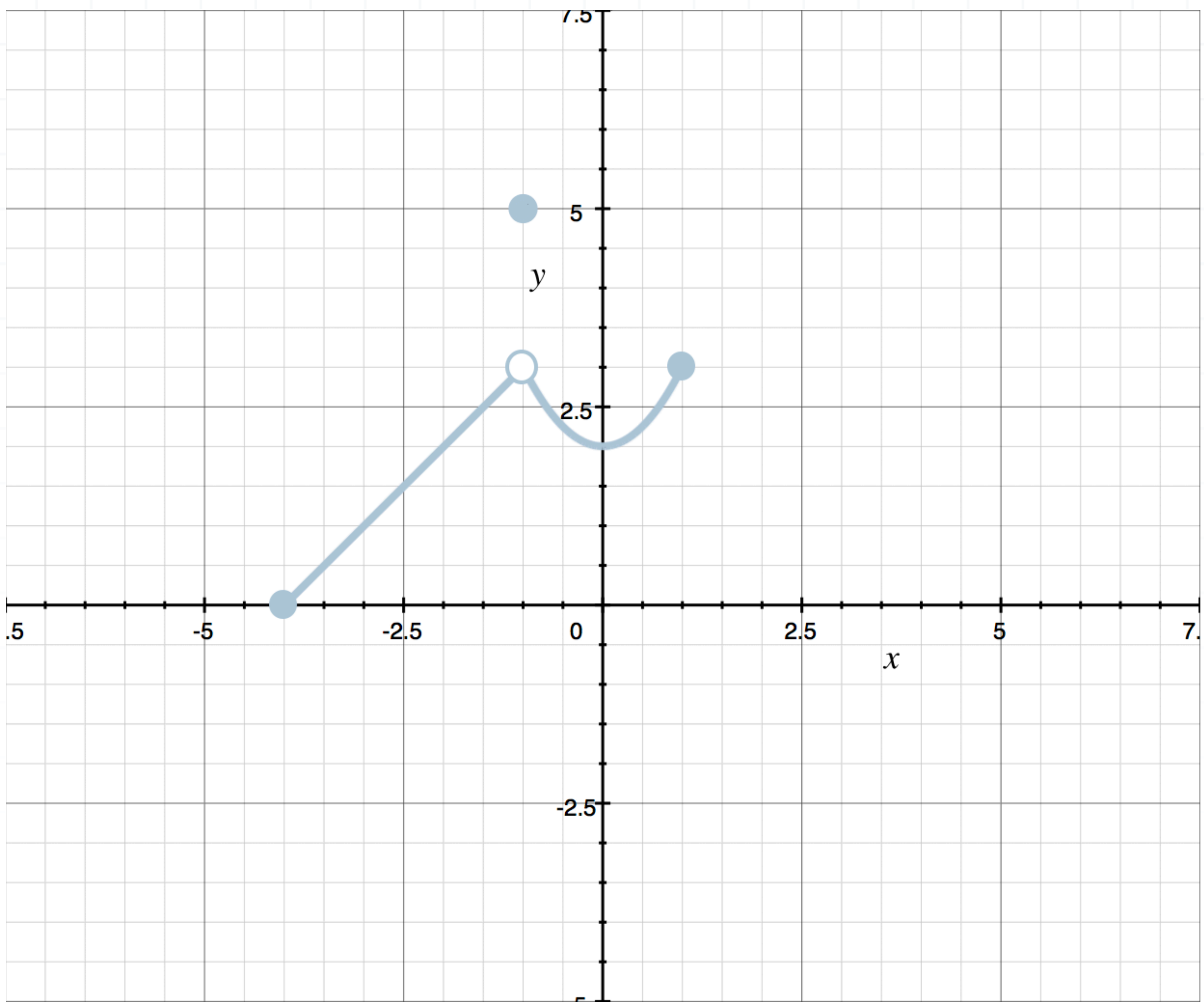
5. What is the domain and range of the function? Assume the graph does not extend beyond the graph shown.



Solution:

The domain of the function given in the graph is determined by the x-values, which are defined by the interval $[-1,1]$. The range is determined by the y-values, which are defined by the interval $[-1,1]$.

6. What is the domain and range of the function? Assume the graph does not extend beyond the graph shown.



Solution:

The domain of the function given in the graph is determined by the x-values, which are defined on the interval $[-4,1]$. The range is determined by the y-values, which are defined on the interval $[0,3]$ and at $y=5$.

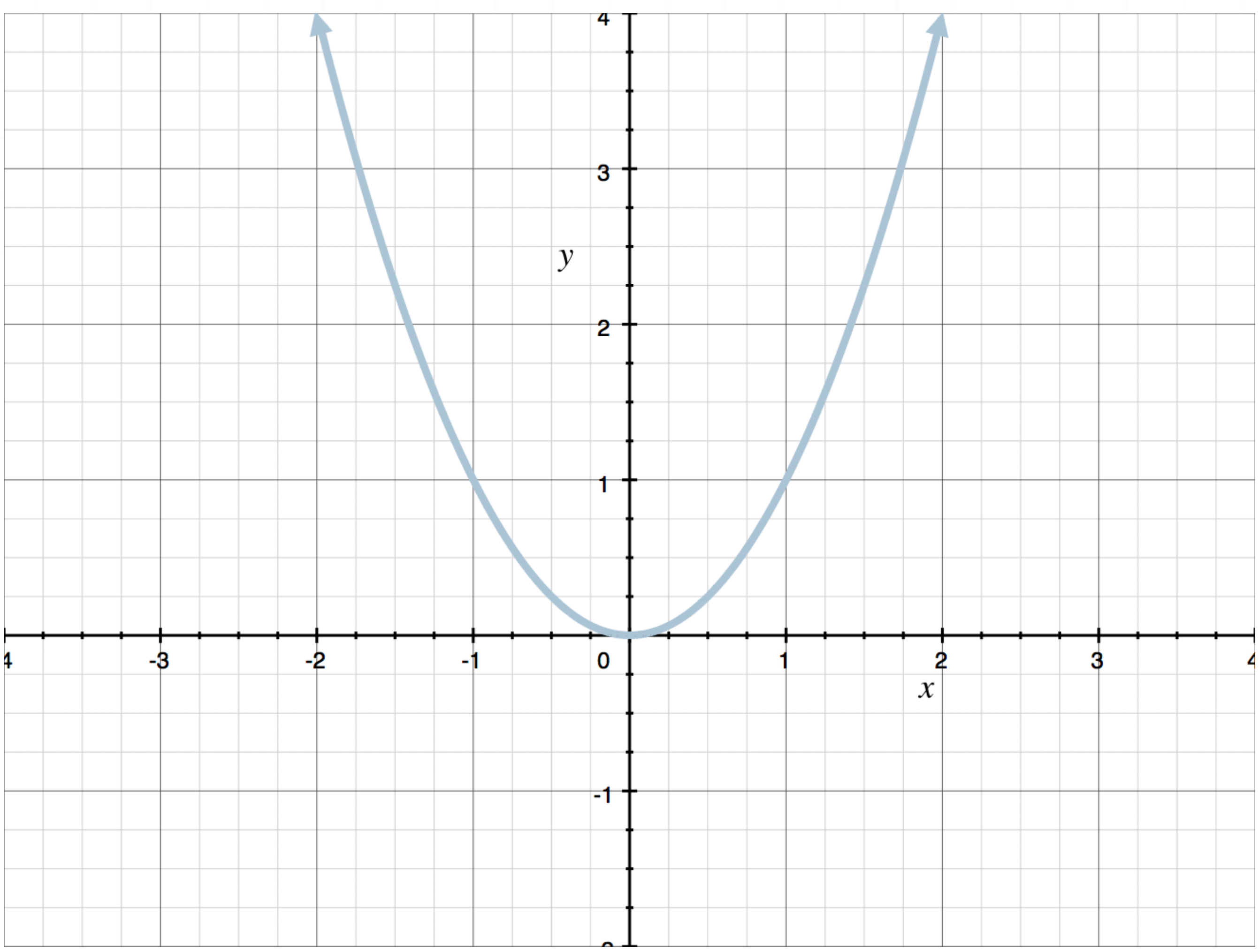
7. Fill in the blanks in the following description of the range of a graph.

"The range is all the values of the graph from __________ to __________."

Solution:

down, up

8. What is the domain and range of the function?



Solution:

The domain of the function given in the graph is determined by the x-values, which are defined on the interval $(-\infty, \infty)$. The range is determined by the y-values, which are defined on the interval $[0,\infty)$.

EVEN, ODD, OR NEITHER

1. Is the function even, odd, or neither?

$$f(x) = -x^5 + 2x^2 - 1$$

Solution:

Substitute $-x$ for x.

$$f(-x) = -(-x)^5 + 2(-x)^2 - 1$$

$$f(-x) = x^5 + 2x^2 - 1$$

Because $f(-x) \neq f(x)$, the function is not even. To see if it's odd, we check

$$-f(x) = -(-x^5 + 2x^2 - 1) = x^5 - 2x^2 + 1$$

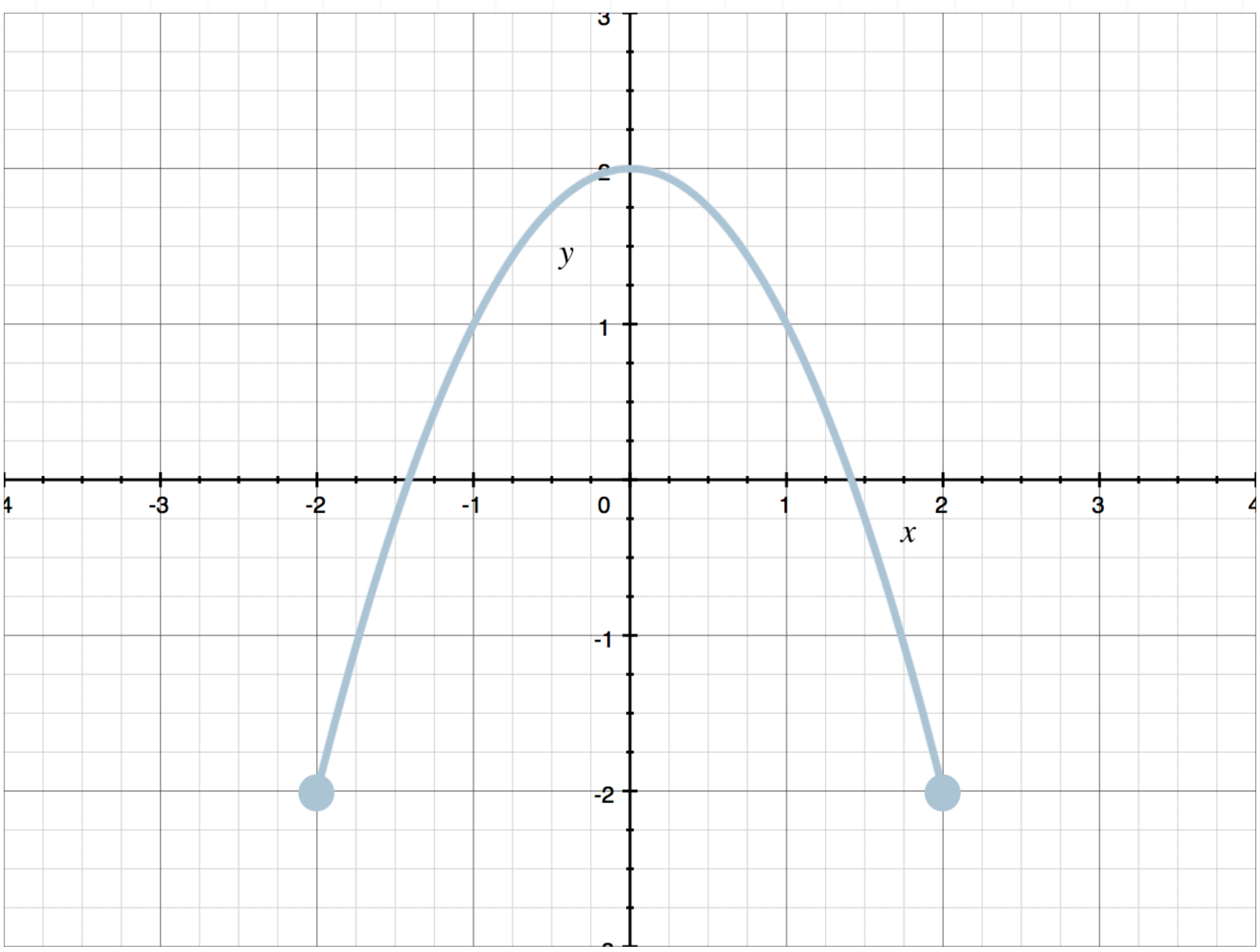
Because $f(-x) \neq -f(x)$, the function is not odd. Therefore, the function is neither even nor odd.

2. Describe the symmetry of an even function, and give an example of an even function.

Solution:

An even function is symmetric about the y-axis. There are many examples of even functions, one being $f(x) = x^2$.

3. Determine if the graph is the graph of a function that is even, odd, or neither.



Solution:

Notice that the graph is symmetric about the y-axis and therefore the graph is the graph of an even function.

4. Is the function even, odd, or neither?

$$g(x) = -3x^2 + 5x^6$$

Solution:

Substitute $-x$ for x.

$$g(-x) = -3(-x)^2 + 5(-x)^6$$

$$g(-x) = -3x^2 + 5x^6$$

Because $f(-x) = f(x)$, the function is even.

5. Show that the function is neither even nor odd.

$$f(x) = x^2 - 5x + 7$$

Solution:

Substitute $-x$ for x.

$$f(-x) = (-x)^2 - 5(-x) + 7$$

$$f(-x) = x^2 + 5x + 7$$

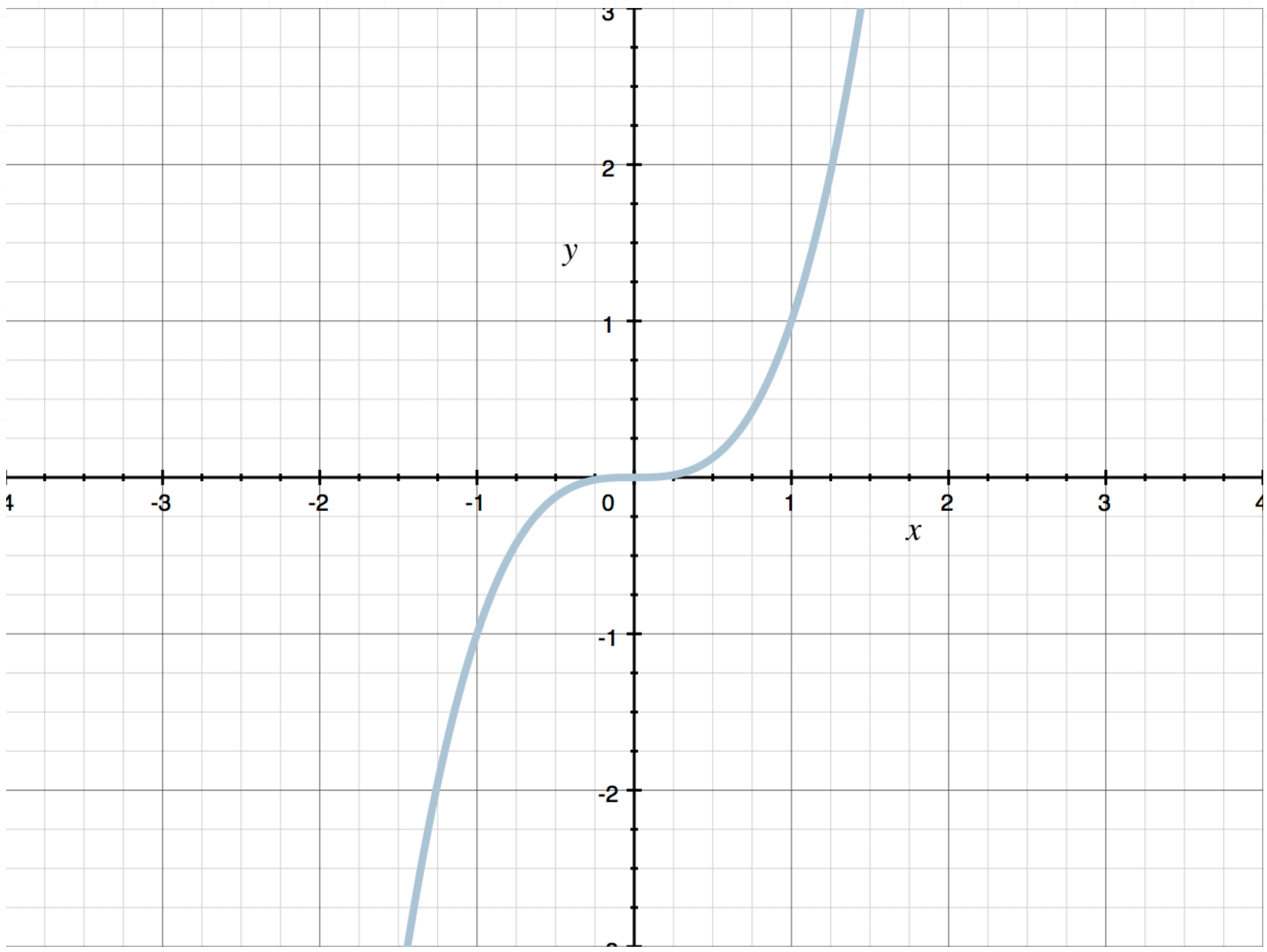
Because $f(-x) \neq f(x)$, the function is not even. To see if it's odd, we check

$$-f(x) = -(x^2 - 5x + 7)$$

$$-f(x) = -x^2 + 5x - 7$$

Because $f(-x) \neq -f(x)$, the function is not odd. Therefore, the function is neither even nor odd.

6. Determine if the graph is the graph of a function that is even, odd, or neither.



Solution:

Notice that the graph is symmetric about the origin, and therefore the graph is the graph of an odd function.

■ 7. Is the function even, odd, or neither?

$$h(x) = x^3 - 3x$$

Solution:

Substitute $-x$ for x.

$$h(-x) = (-x)^3 - 3(-x)$$

$$h(-x) = -x^3 + 3x$$

Because $f(-x) \neq f(x)$, the function is not even. To see if it's odd, we check

$$-f(x) = -(x^3 - 3x)$$

$$-f(x) = -x^3 + 3x$$

Because $f(-x) = -f(x)$, the function is odd.

■ 8. Describe the symmetry of an odd function, and give an example of an odd function.

Solution:

An odd function is symmetric about the origin. There are many correct examples of odd functions, one being $f(x) = x^3$.

EQUATION MODELING

1. A car and a truck were driven for a week. The car traveled 75 miles more than the truck. Each vehicle had different fuel mileage. Write an equation using t (where t is the number of miles the truck traveled) to calculate the number of gallons g, used during the week.

	Car	Truck
Mileage	28 mpg	14 mpg
Distance	c miles	t miles

Solution:

Write an expression in terms of t for the distance traveled by the car. The car traveled 75 more miles than the truck, so $c = t + 75$. To get the gallons used, divide the distance (in terms of t) by the mileage.

	Car	Truck
Mileage	28 mpg	14 mpg
Distance	t+75 miles	t miles
Gallons used g	(t+75)/28 gallons	t/14 gallons

To find the total gallons used, add the gallons the car used with the gallons the truck used.

$$g = \frac{t+75}{28} + \frac{t}{14}$$

$$g = \frac{t+75}{28} + \left(\frac{2}{2}\right)\frac{t}{14}$$

$$g = \frac{t+75}{28} + \frac{2t}{28}$$

$$g = \frac{t+75+2t}{28}$$

$$g = \frac{3t+75}{28} \text{ gallons}$$

■ 2. A motorcycle and a car were driven for a month. The motorcycle traveled 120 miles more than the car. Each vehicle had different fuel mileage. Write an equation using m (where m is the number of miles the motorcycle traveled) to calculate the number of gallons g, used during the month.

	Motorcycle	Car
Mileage	33 mpg	22 mpg
Distance	m miles	c miles

Solution:

Write an expression in terms of m for the distance traveled by the car. The car traveled 120 miles less than the motorcycle, so $c = m - 120$. To get the gallons used, divide the distance (in terms of m) by the mileage.

	Motorcycle	Car
Mileage	33 mpg	22 mpg
Distance	m miles	m-120 miles
Gallons used g	m/33 gallons	(m-120)/22 gallons

To find the total gallons used, add the gallons the motorcycle used with the gallons the car used.

$$g = \frac{m}{33} + \frac{m-120}{22}$$

$$g = \left(\frac{2}{2}\right)\frac{m}{33} + \left(\frac{3}{3}\right)\frac{m-120}{22}$$

$$g = \frac{2m}{66} + \frac{3m-360}{66}$$

$$g = \frac{2m+3m-360}{66}$$

$$g = \frac{5m-360}{66} \text{ gallons}$$

■ 3. A baseball is thrown at a speed of 21 ft/s straight down from a high platform. The distance it travels can be calculated using $D = 16t^2 + 21t$, where t is the amount of time in seconds that it's been falling. The average speed of any object can be calculated using $V = D/t$. Write an equation giving the time of the fall in terms of V.

Solution:

Plug $16t^2 + 21t$ into the $V = D/t$ for D.

$$V = \frac{D}{t}$$

$$V = \frac{16t^2 + 21t}{t}$$

$$V = 16t + 21$$

We were asked to find the equation for time in terms of V, so we need to solve for t.

$$V = 16t + 21$$

$$V - 21 = 16t$$

$$t = \frac{V - 21}{16}$$

■ 4. A rock is thrown at a speed of 8 ft/s straight down from a high platform. The distance it travels can be calculated using $D = 16t^2 + 8t$, where t is the amount of time in seconds that it's been falling. The average speed of any object can be calculated using $V = D/t$. Write an equation giving the time of the fall in terms of V.

Solution:

Plug $16t^2 + 8t$ into $V = D/t$ for D.

$$V = \frac{D}{t}$$

$$V = \frac{16t^2 + 8t}{t}$$

$$V = 16t + 8$$

We were asked to find the equation for time in terms of V, so we need to solve for t.

$$V = 16t + 8$$

$$V - 8 = 16t$$

$$t = \frac{V - 8}{16}$$

■ 5. Managers at a company are each paid $\$45{,}000$ in base salary. The company's owner wants to divide $\$162{,}000$ in annual bonus money evenly among the managers. Write an expression, in terms of the number of mangers m, that gives the amount a each manager earns per month.

Solution:

Find the monthly salary of a manager.

$$45{,}000 \div 12 = 3{,}750$$

Find the bonus money available each month to the group of all managers.

$$162{,}000 \div 12 = 13{,}500$$

This monthly bonus money needs to be divided evenly by the number of mangers m.

$$\frac{13{,}500}{m}$$

The total amount each manager earns monthly is the sum of their monthly salary and their monthly bonus money.

$$a = 3{,}750 + \frac{13{,}500}{m}$$

■ 6. Managers at a company are each paid $\$37{,}800$ in base salary. The company's owner wants to divide $\$102{,}000$ in annual bonus money evenly among the managers. Write an expression, in terms of the number of mangers m, that gives the amount a each manager earns per month.

Solution:

Find the monthly salary of a manager.

$$37{,}800 \div 12 = 3{,}150$$

Find the bonus money available each month to the group of all managers.

$$102{,}000 \div 12 = 8{,}500$$

This monthly bonus money needs to be divided evenly by the number of mangers m.

$$\frac{8{,}500}{m}$$

The total amount each manager earns monthly is the sum of their monthly salary and their monthly bonus money.

$$a = 3{,}150 + \frac{8{,}500}{m}$$

■ 7. The Jones and Anderson family go on vacation together with each family driving in their own car. The Anderson family travels 50 miles further than the Jones family. Each family averages 65 mph on the trip. Write an equation using D_a (where D_a is the total miles the Anderson family drove) to calculate the total time T both families spent driving to their destination.

	Jones	Anderson
Distance	D_j miles	D_a miles
Rate	65 mph	65 mph
Time	T_j hours	T_a hours

Solution:

The Jones family traveled 50 miles less than the Anderson family, so $D_j = D_a - 50$. We need to find the total time T, which is $T_j + T_a$. So we'll use the distance equation $D = RT$ and solve for T_j and T_a.

Time spent driving by the Jones family:

$$D_j = R_j T_j$$

$$D_a - 50 = 65T_j$$

$$T_j = \frac{D_a - 50}{65}$$

Time spent driving by the Anderson family:

$$D_a = R_a T_a$$

$$D_a = 65T_a$$

$$T_a = \frac{D_a}{65}$$

Total time spent driving:

$$T = T_j + T_a$$

$$T = \frac{D_a - 50}{65} + \frac{D_a}{65}$$

$$T = \frac{2D_a - 50}{65} \text{ hours}$$

8. The Frank and Harrington family go on vacation together with each family driving in their own car. The Frank family travels 120 miles less than the Harrington family. Each family averages 50 mph on the trip. Write an equation using D_f (where D_f is the total miles the Frank family drove) to calculate the total time T both families spent driving to their destination.

	Frank	Harrington
Distance	D_f miles	D_h miles
Rate	50 mph	50 mph
Time	T_f hours	T_h hours

Solution:

The Harrington family traveled 120 miles more than the Frank family, so $D_h = D_f + 120$. We need to find the total time T, which is $T_f + T_h$. So we'll use the distance equation $D = RT$ and solve for T_f and T_h.

Time spent driving by the Harrington family:

$$D_h = R_h T_h$$

$$D_f + 120 = 50T_h$$

$$T_h = \frac{D_f + 120}{50}$$

Time spent driving by the Frank family:

$$D_f = R_f T_f$$

$$D_f = 50T_f$$

$$T_f = \frac{D_f}{50}$$

Total time spent driving:

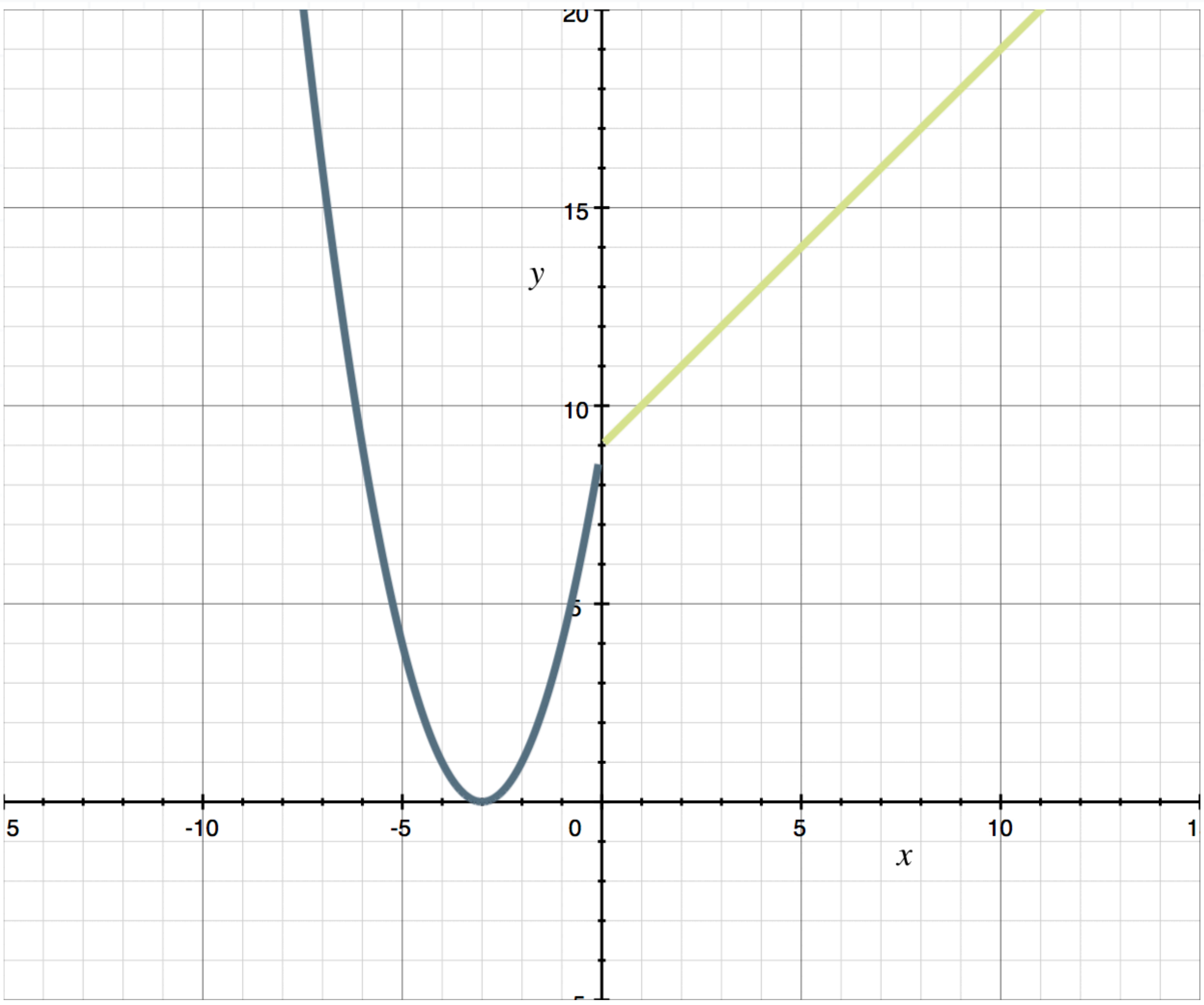
$$T = T_f + T_h$$

$$T = \frac{2D_f + 120}{50}$$

$$T = \frac{D_f + 60}{25} \text{ hours}$$

MODELING A PIECEWISE-DEFINED FUNCTION

1. Find the equation of the piecewise function.



Solution:

The dark blue parabola has a vertex at $(-3,0)$, so the equation is $f(x)=(x+3)^2$ from $-\infty$ to 0, or when $x \le 0$.

The green line has a slope of 1 and a y-intercept of 9, so the equation of the line is $f(x)=x+9$ from 0 to ∞, or when $x>0$.

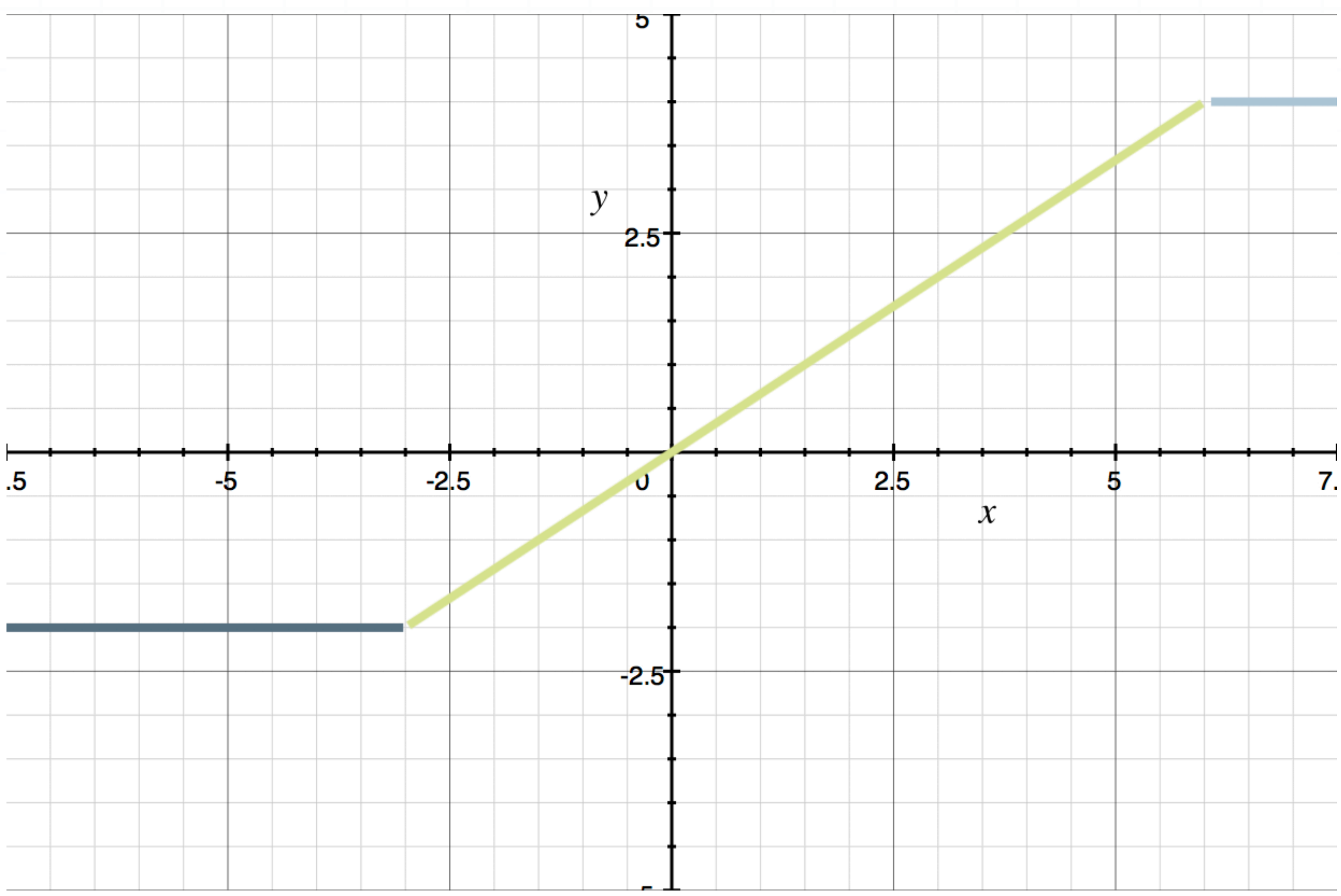
Putting these pieces together in a piecewise function gives

$$f(x) = \begin{cases} (x+3)^2 & x \le 0 \\ x+9 & x > 0 \end{cases}$$

We don't know if the "equal to" part of the equation is for the blue parabola or the green line, so we also could have written

$$f(x) = \begin{cases} (x+3)^2 & x < 0 \\ x+9 & x \ge 0 \end{cases}$$

■ 2. Find the equation of the piecewise function.



Solution:

The dark blue horizontal line is $f(x) = -2$, from $-\infty$ to -3, or when $x \leq 3$.

The green line has a slope of $2/3$ and a y-intercept of 0, so the equation of the line is $f(x) = (2/3)x$ from -3 to 6, or when $-3 < x < 6$.

The light blue horizontal line is at $f(x) = 4$, from 6 to ∞, or when $x \geq 6$.

Putting these pieces together in a piecewise function gives

$$f(x) = \begin{cases} -2 & x \leq -3 \\ \frac{2}{3}x & -3 < x < 6 \\ 4 & x \geq 6 \end{cases}$$

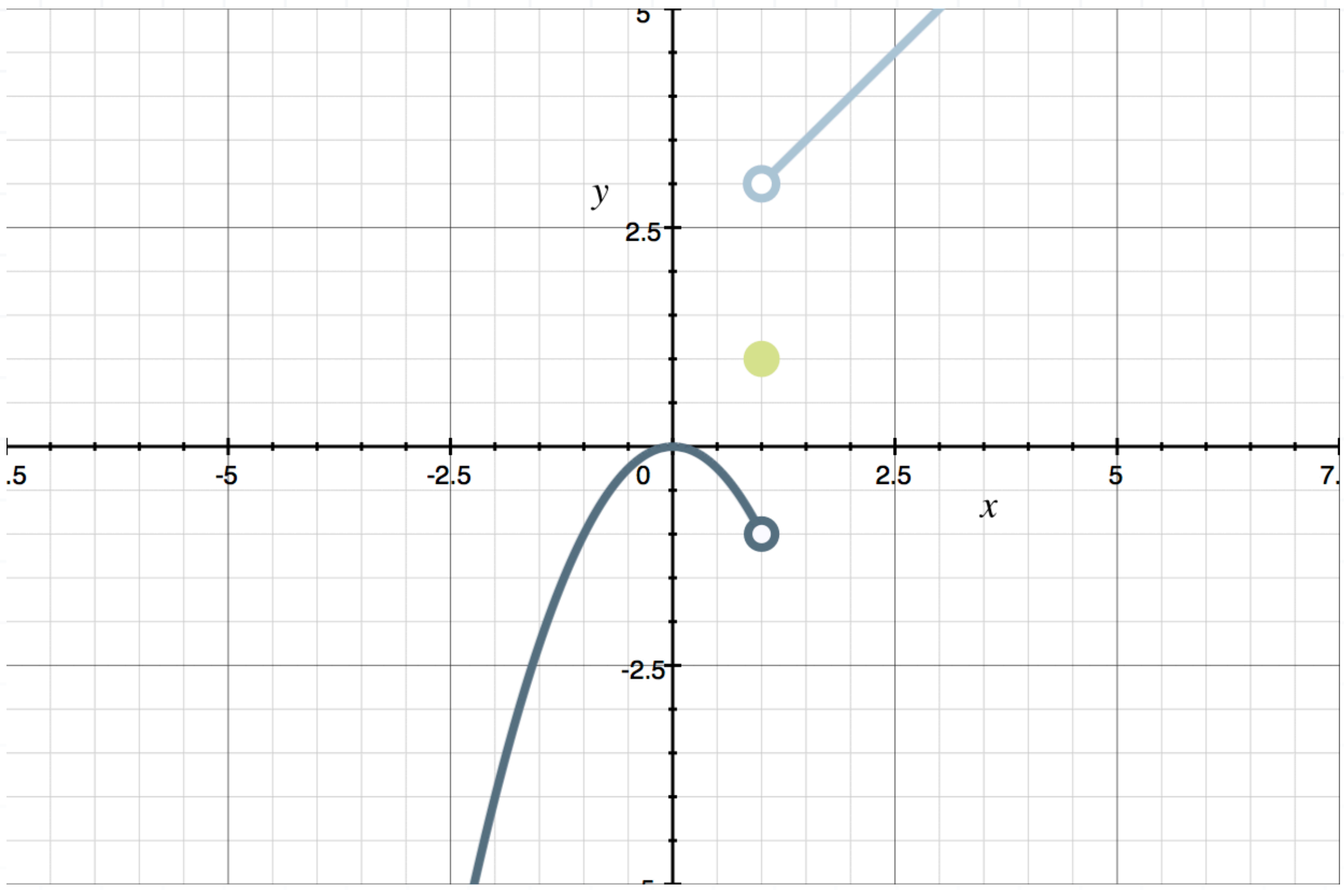
We don't which piece of the graph takes the "equal to" part of the equation, so we also could have written the answer as any of the following.

$$f(x) = \begin{cases} -2 & x < -3 \\ \frac{2}{3}x & -3 \leq x < 6 \\ 4 & x \geq 6 \end{cases}$$

$$f(x) = \begin{cases} -2 & x \leq -3 \\ \frac{2}{3}x & -3 < x \leq 6 \\ 4 & x > 6 \end{cases}$$

$$f(x) = \begin{cases} -2 & x < -3 \\ \frac{2}{3}x & -3 \leq x \leq 6 \\ 4 & x > 6 \end{cases}$$

3. Find the equation of the piecewise function.



Solution:

The dark blue parabola is $f(x) = -x^2$ from $-\infty$ to 1, or when $x < 1$. We know that it's strictly "less than" because of the hollow circle on the parabola at $x = 1$.

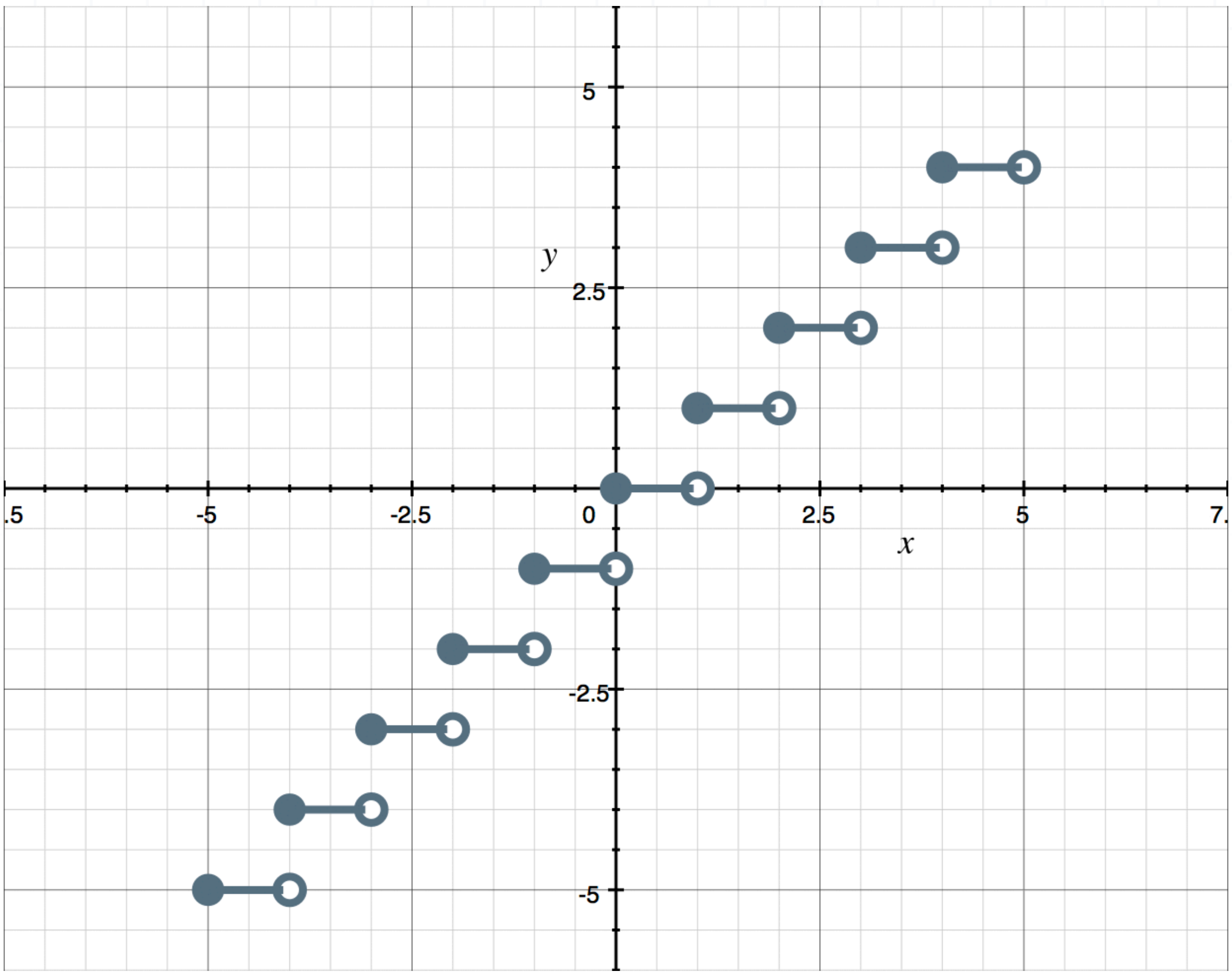
The solid green dot means that $f(x) = 1$ when $x = 1$.

The light blue line has a slope of 1 and would have a y-intercept of 2, so the equation of the line is $f(x) = x + 2$ from 1 to ∞, or when $x > 1$. We know that it's strictly "greater than" because of the hollow circle on the line at $x = 1$.

Putting these pieces together in a piecewise function gives

$$f(x) = \begin{cases} -x^2 & x < 1 \\ 1 & x = 1 \\ x + 2 & x > 1 \end{cases}$$

■ 4. Find the equation of the piecewise function.



Solution:

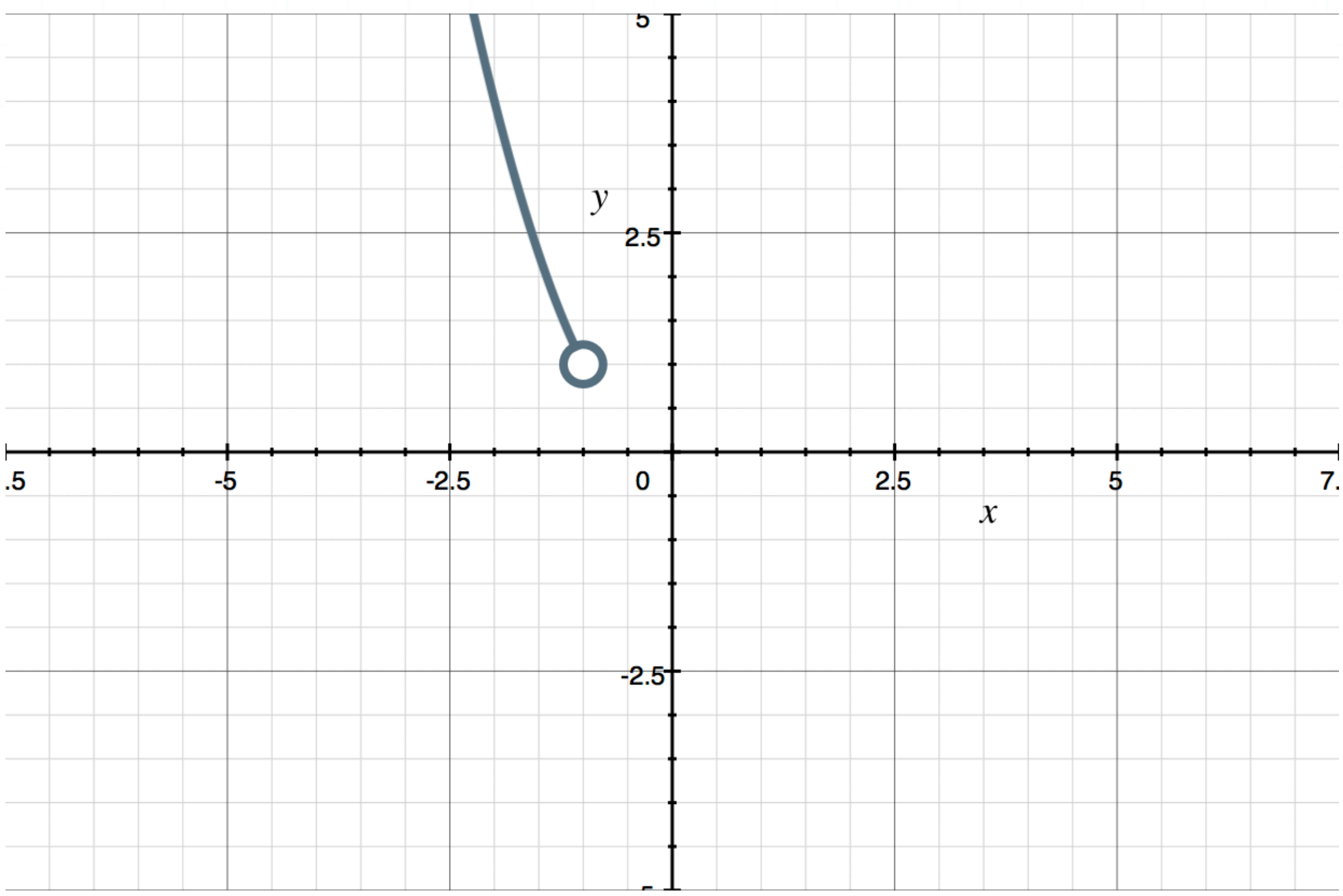
This is the "greatest integer function," a step or stair-case function that produces the greatest integer less than or equal to the number. It's denoted by $f(x) = [x]$.

5. Graph the piecewise function.

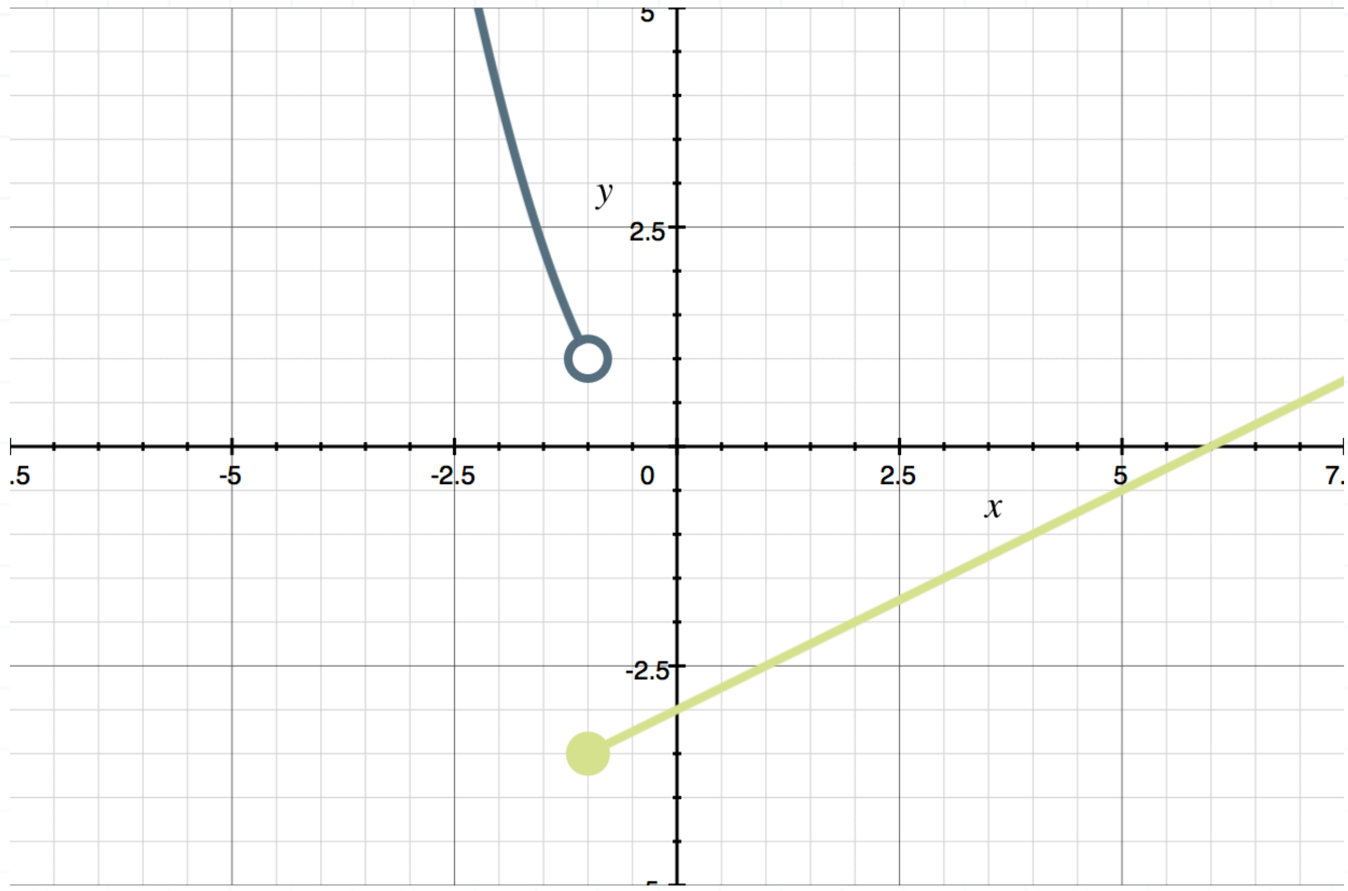
$$f(x) = \begin{cases} x^2 & x < -1 \\ \frac{1}{2}x - 3 & x \geq -1 \end{cases}$$

Solution:

First graph the parabola $f(x) = x^2$, but only when $x < -1$. This means that at $x = -1$ there will be an open circle.



Now graph the line $(1/2)x - 3$ when $x \geq -1$. This means that when $x = -1$ there will be a solid circle.

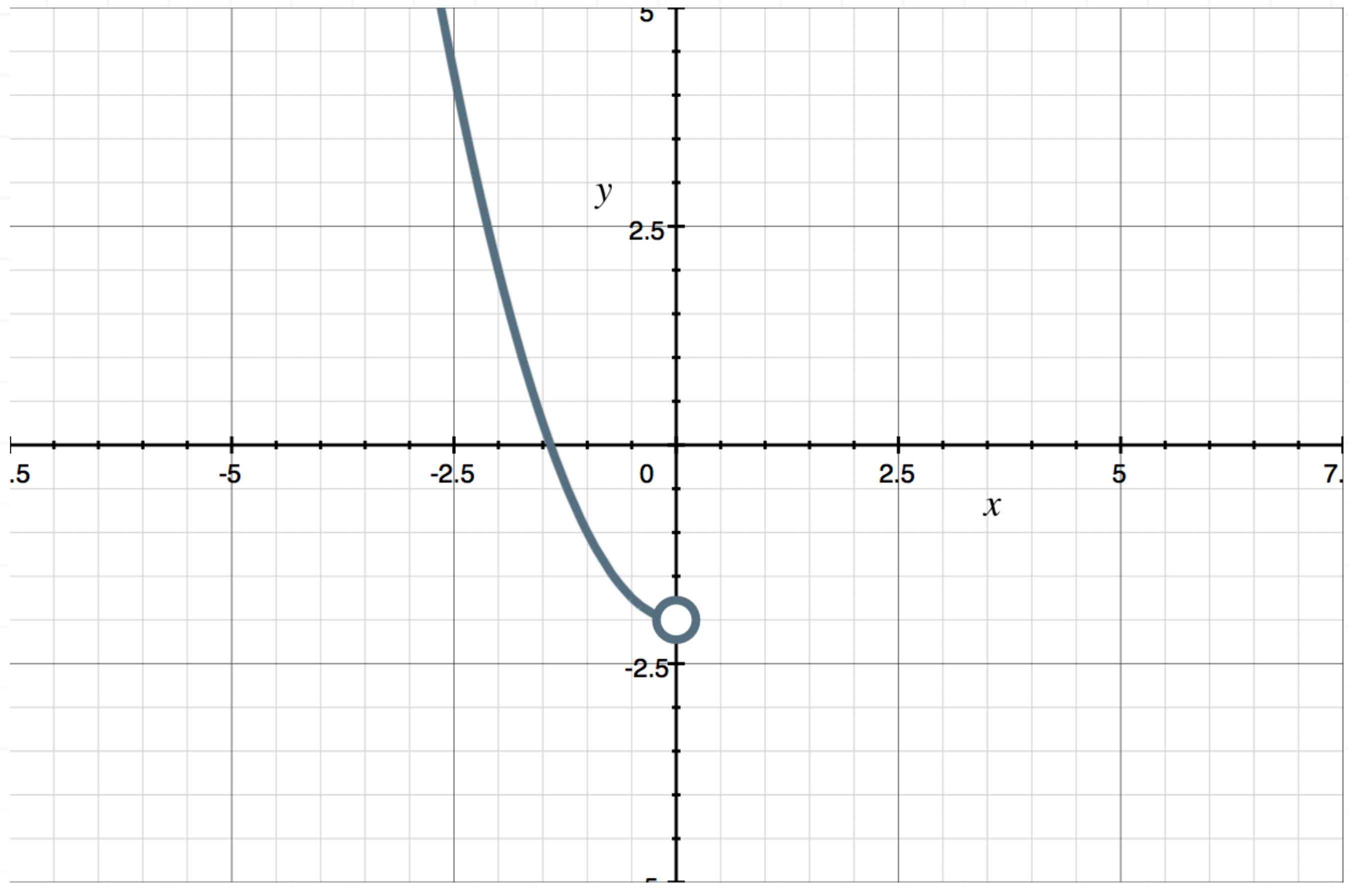


6. Graph the piecewise function.

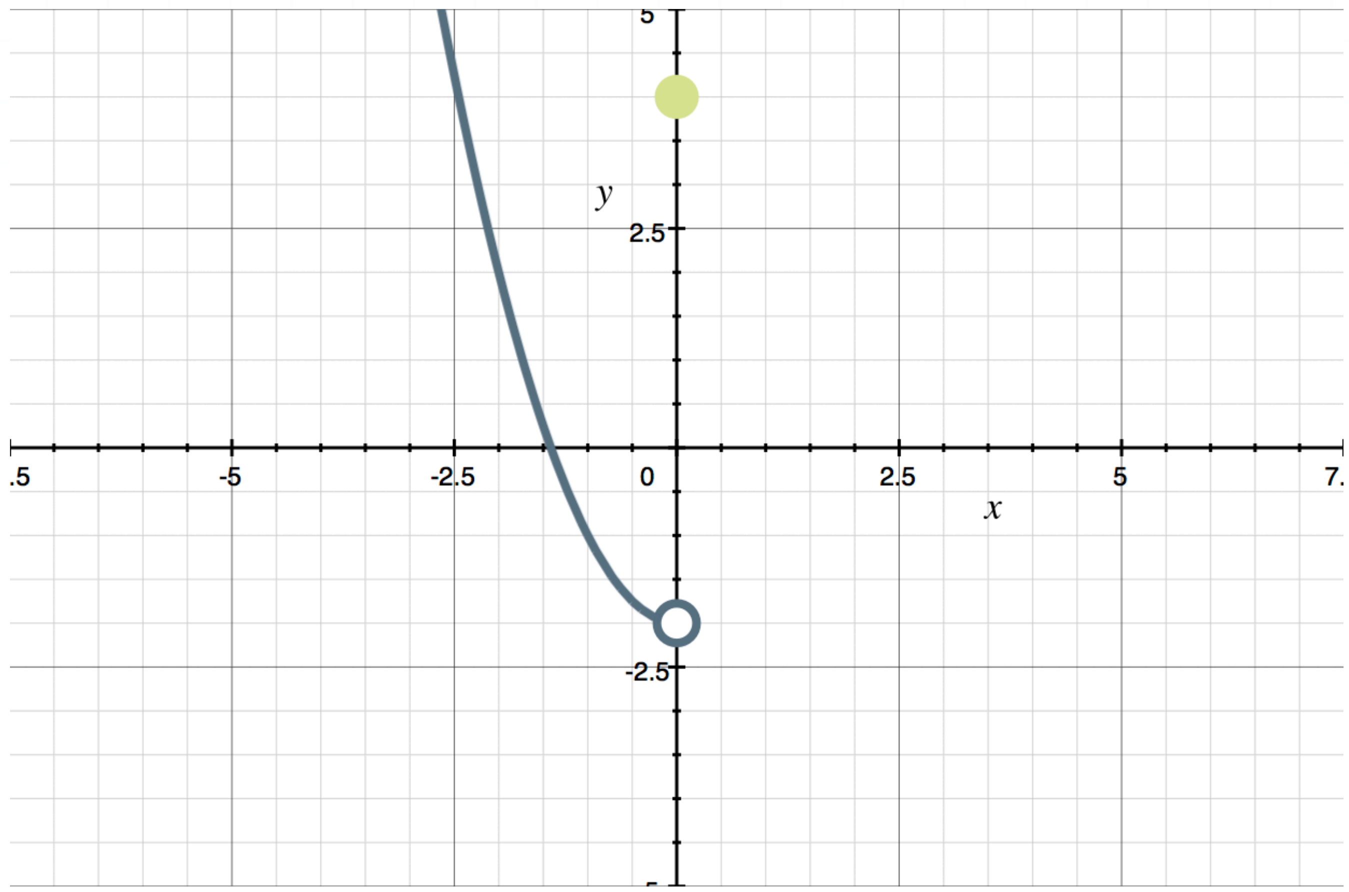
$$f(x) = \begin{cases} x^2 - 2 & x < 0 \\ 4 & x = 0 \\ -x^2 + 8 & x > 0 \end{cases}$$

Solution:

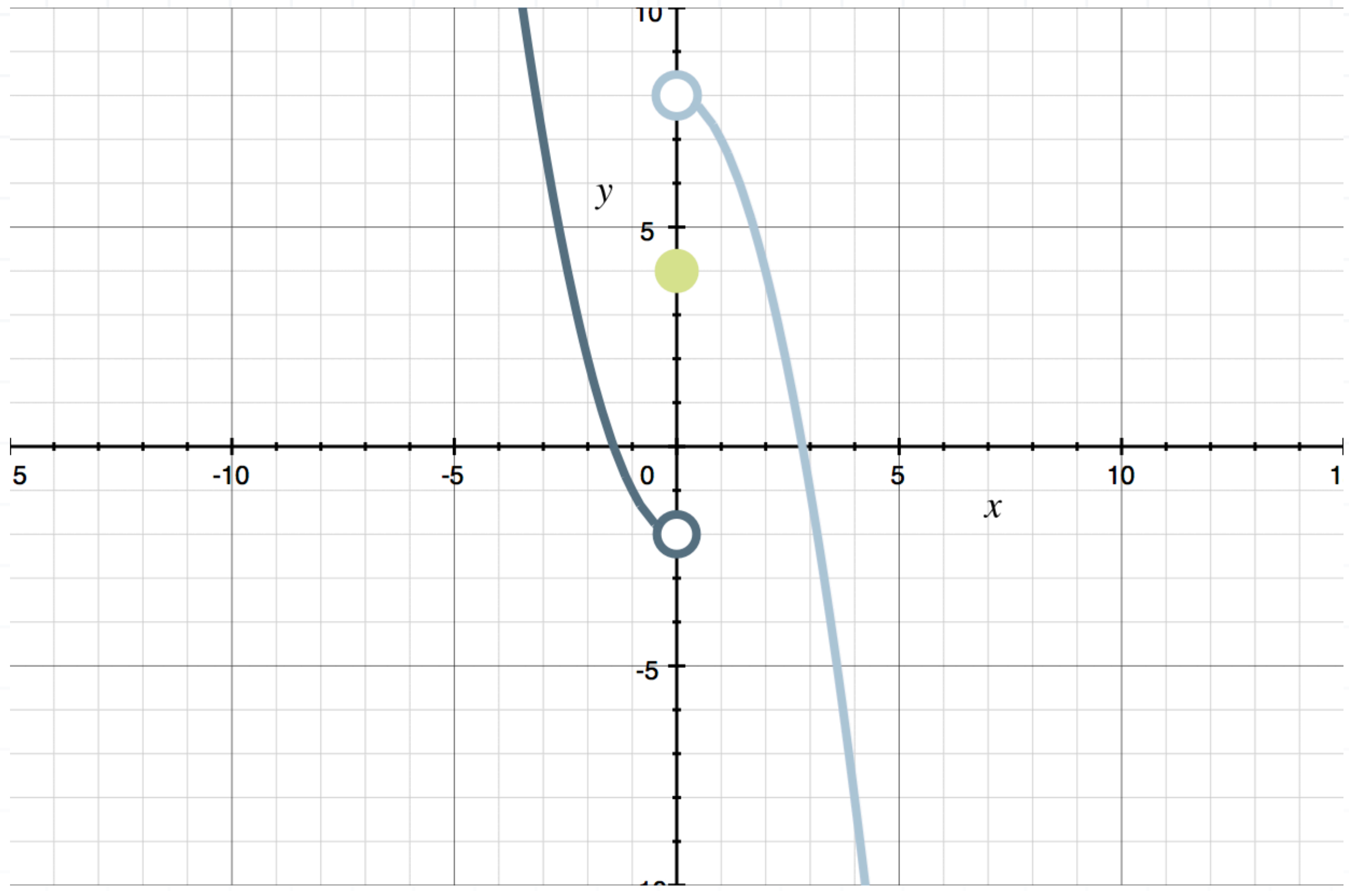
First graph the parabola $f(x) = x^2 - 2$, but only when $x < 0$. This means that at $x = 0$ there will be an open circle.



Graph the point $(0,4)$.



Now graph the parabola $f(x) = -x^2 + 8$ when $x > 0$. This means that at $x = 0$ there will be an open circle.

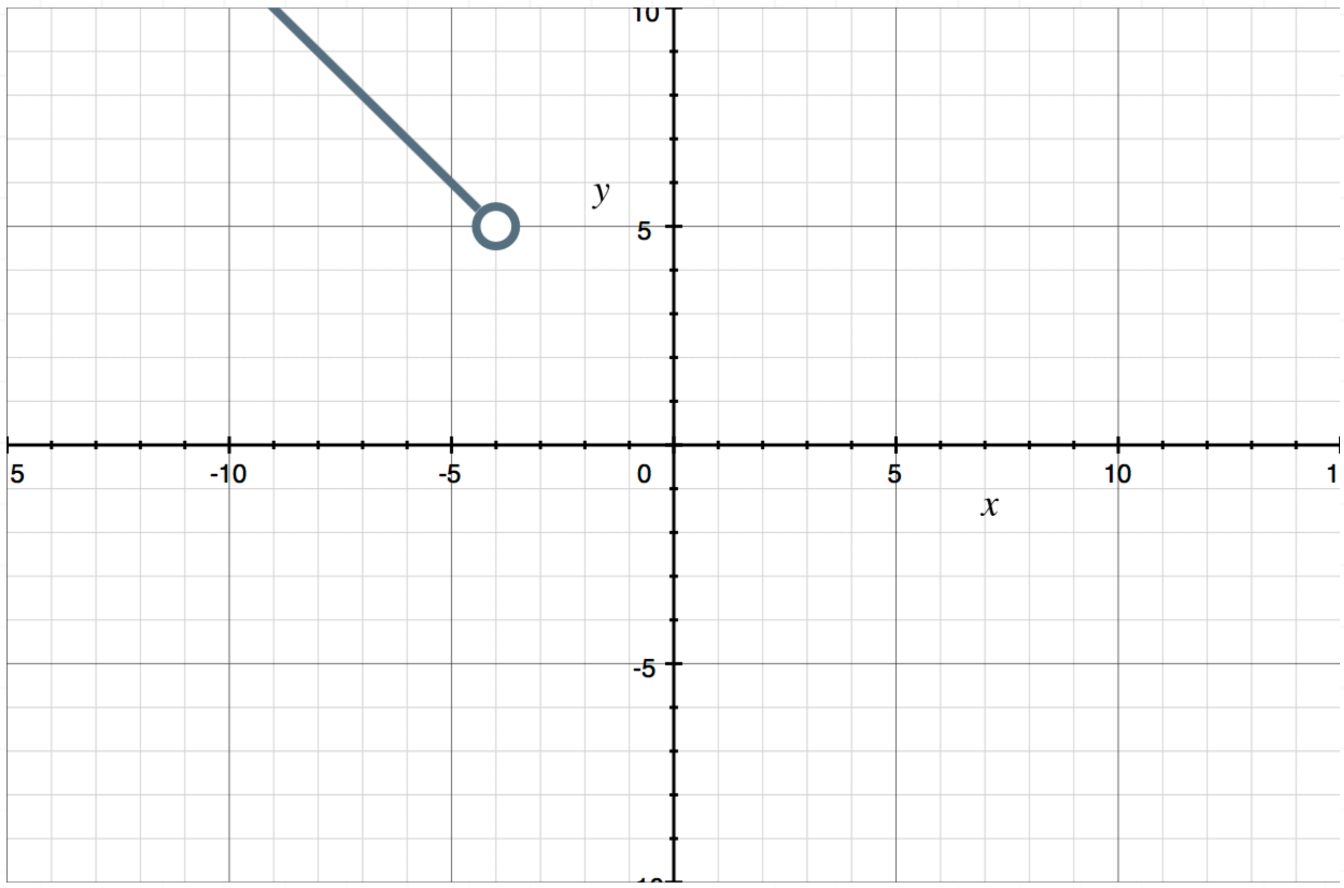


7. Graph the piecewise function.

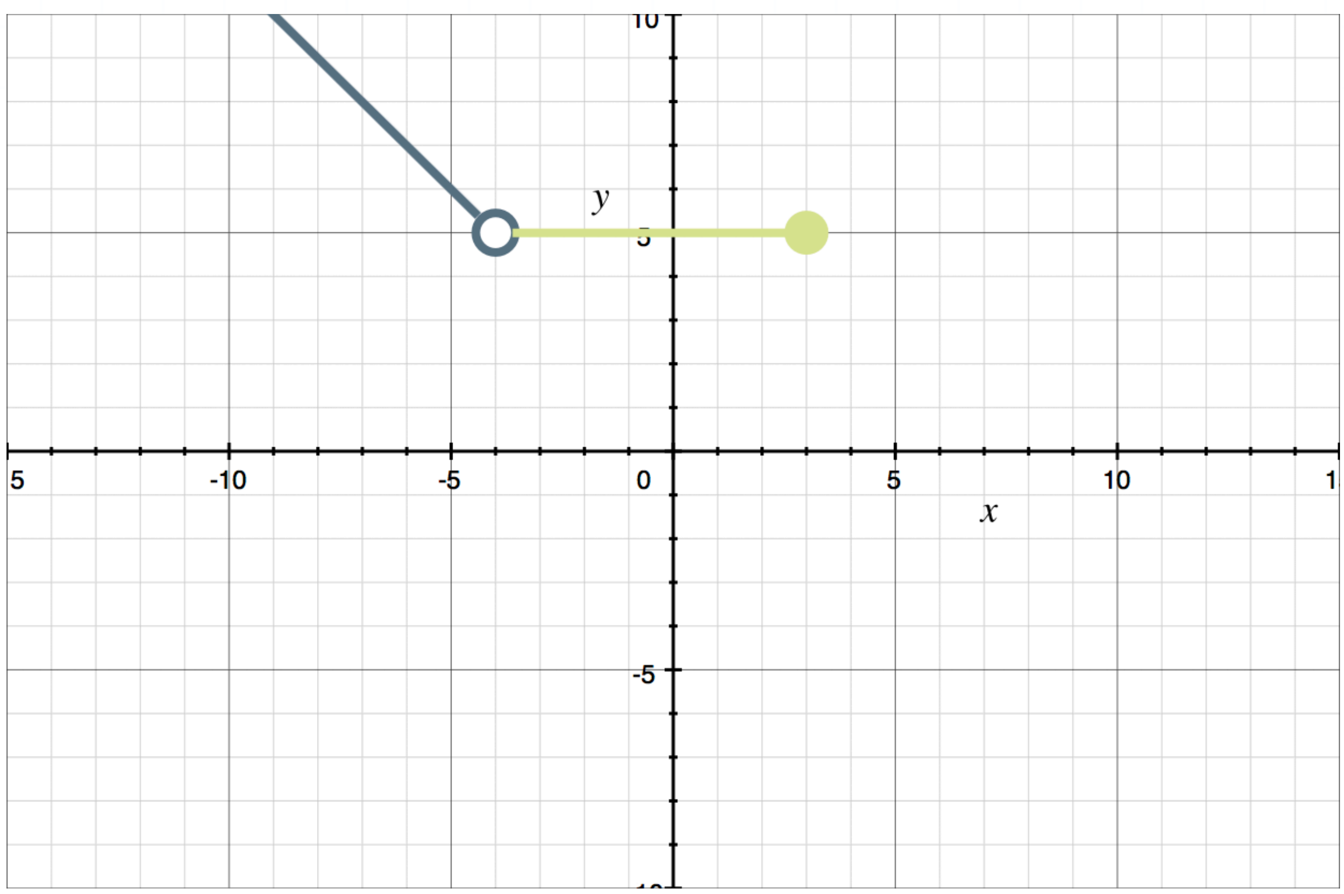
$$f(x) = \begin{cases} -x+1 & x < -4 \\ 5 & -4 < x \leq 3 \\ -2x+11 & x > 3 \end{cases}$$

Solution:

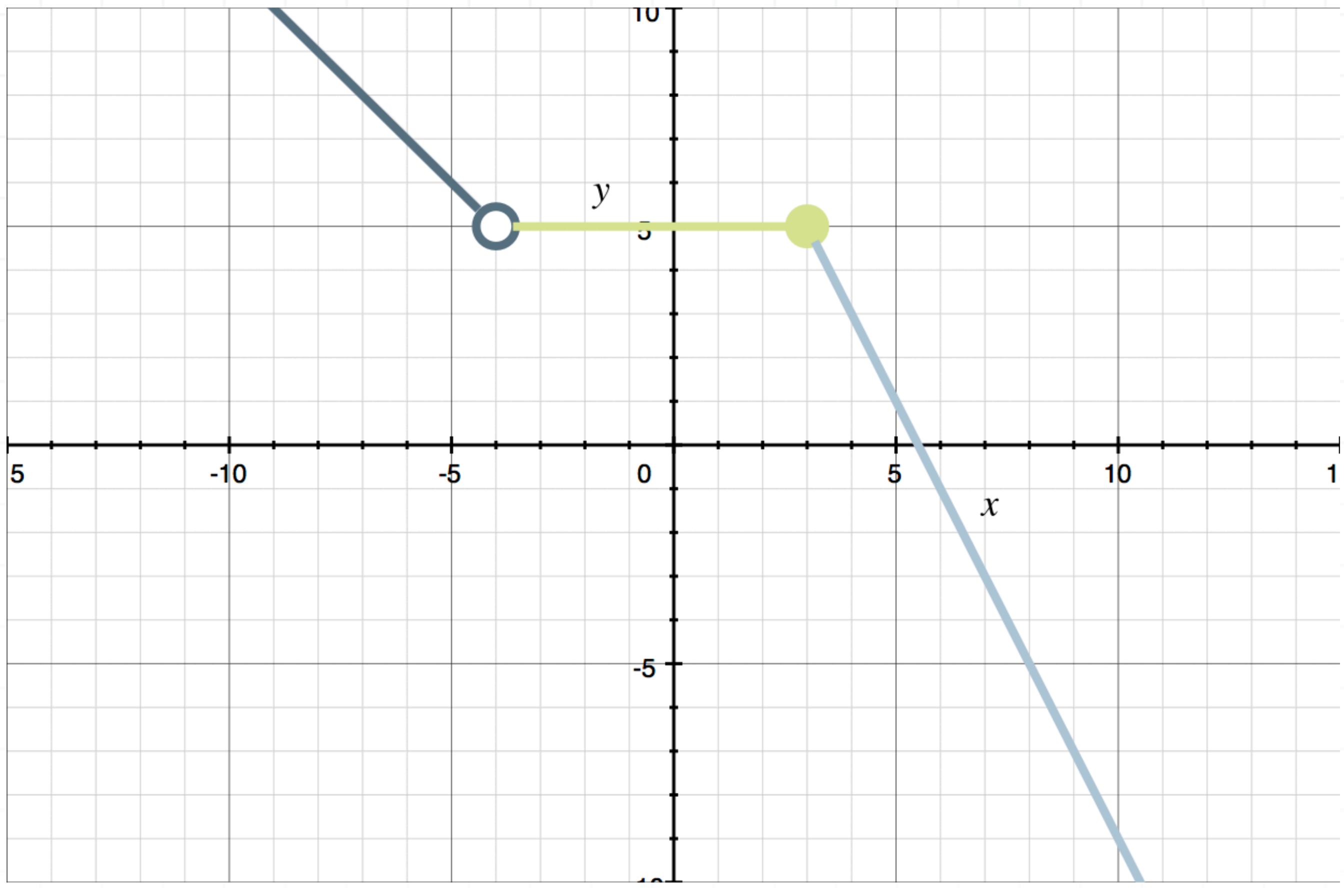
First graph the line $f(x) = -x + 1$, but only when $x < -4$. This means that at $x = -4$ there will be an open circle.



Graph the line $f(x) = 5$ when $-4 < x \leq 3$. This means that at $x = 3$ there will be solid circle.



Now graph the line $f(x) = -2x + 11$ when $x > 3$.

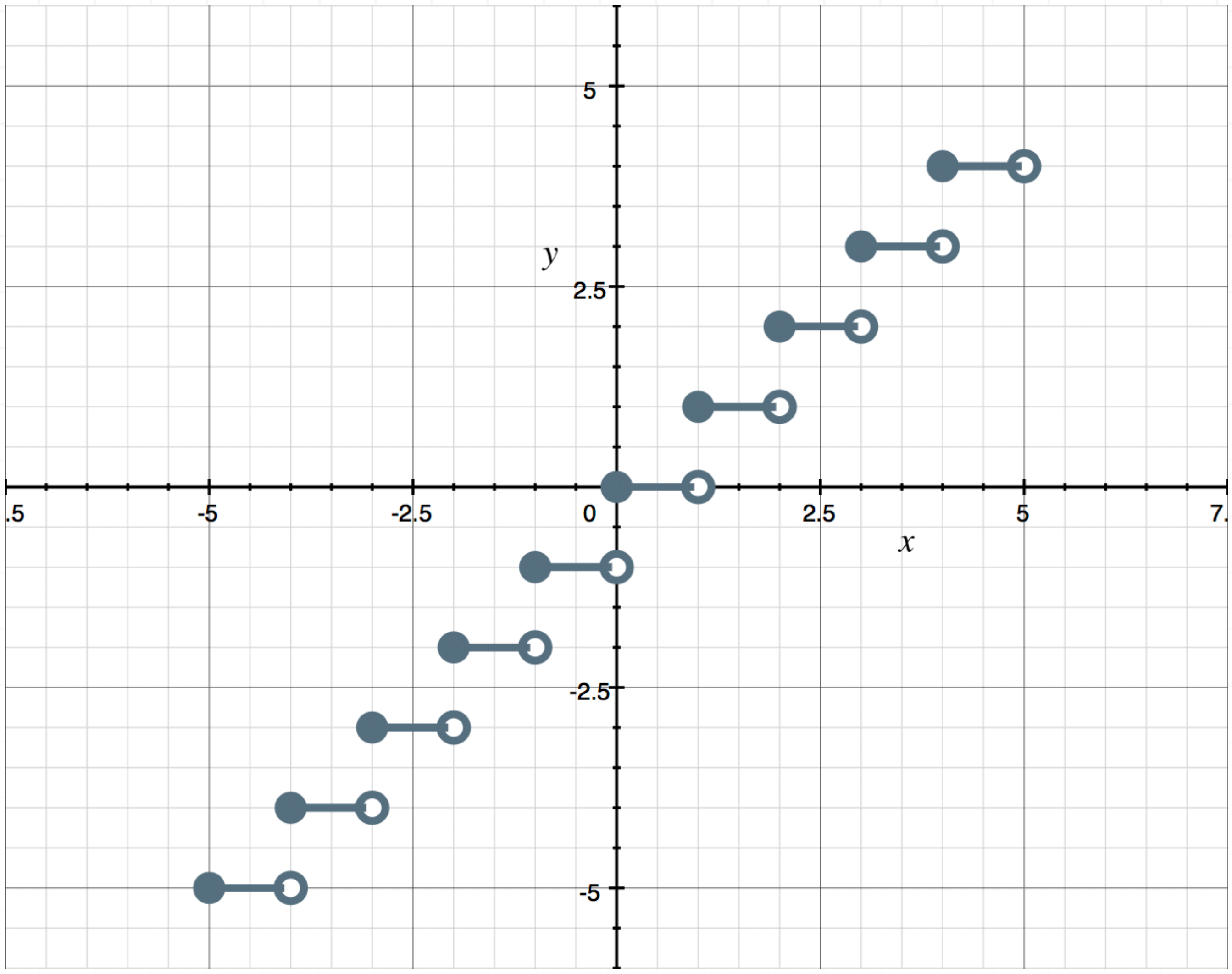


8. Graph the piecewise function.

$$f(x) = [x]$$

Solution:

This is the "greatest integer function," a step or staircase function that produces the greatest integer less than or equal to the number. This means there is a horizontal line between each interval that has a closed circle on the left and an open circle on the right.


5
2.5
-2.5
-5
.5
-5
-2.5
0
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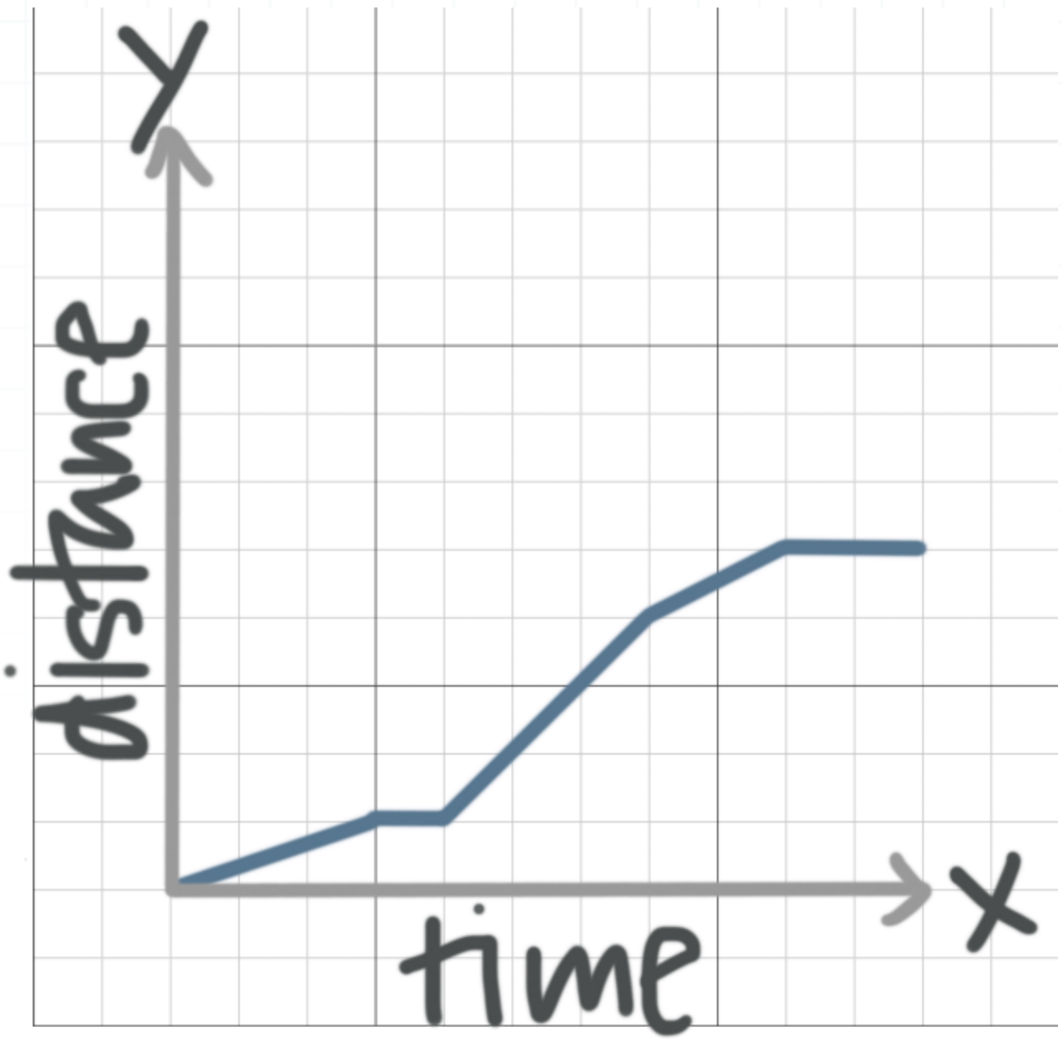
7.
y
x

SKETCHING GRAPHS FROM STORY PROBLEMS

1. Alex left in his car to visit his grandparents' house. The graph shows his distance from his house over time. Write a possible story to go along with the graph.

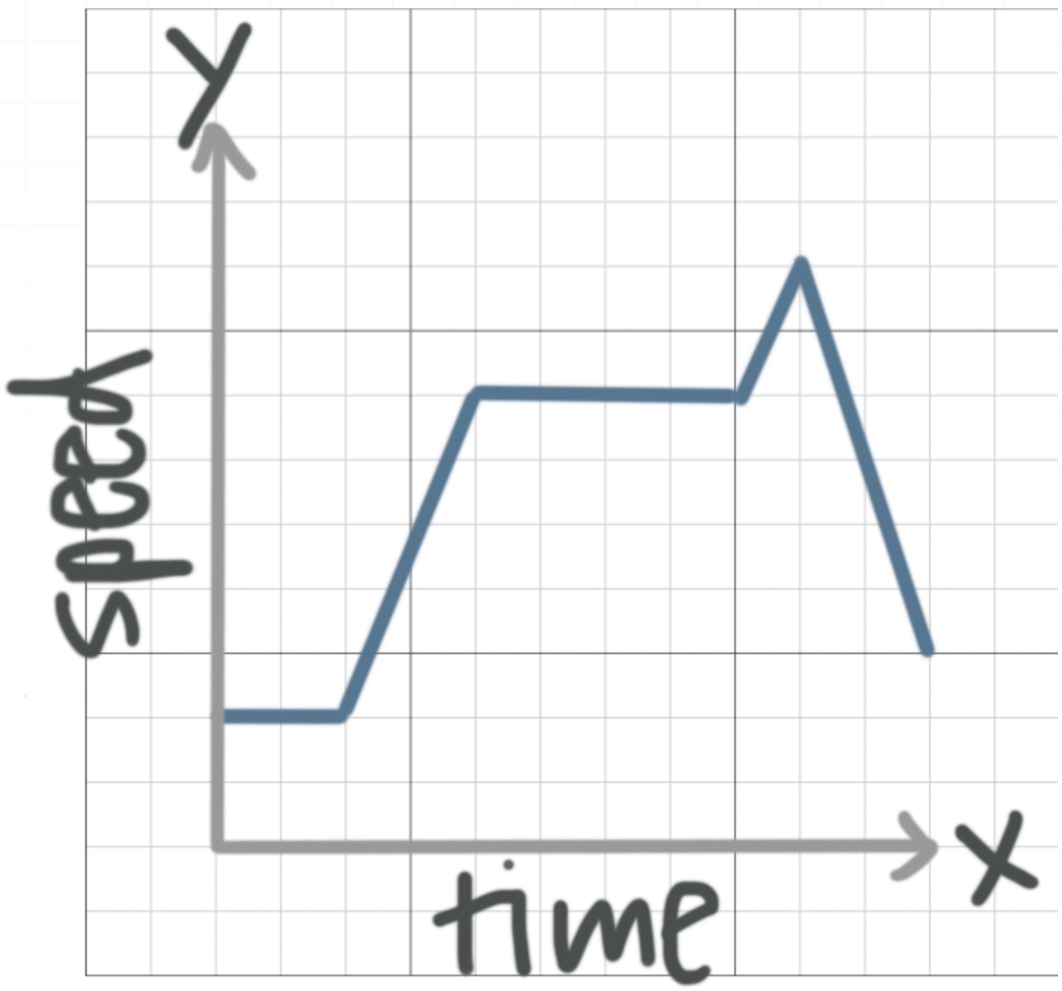


Solution:

Looking at the graph, there are two sections where the distance increases at a slower rate and one section where Alex's distance increases faster. There are also two horizontal lines, indicating stops. The last stop is when Alex arrives at his grandparents' house. So the story might be

"Alex is driving in town and then stops to get more gas. After stopping for gas he drives on the highway and his speed increases. When Alex gets off the highway he drives more slowly until he arrives at his grandparents' house, where he stops."

■ 2. A horse is practicing for a race. The graph shows the horse's speed over time. Write a possible story to go with the graph.

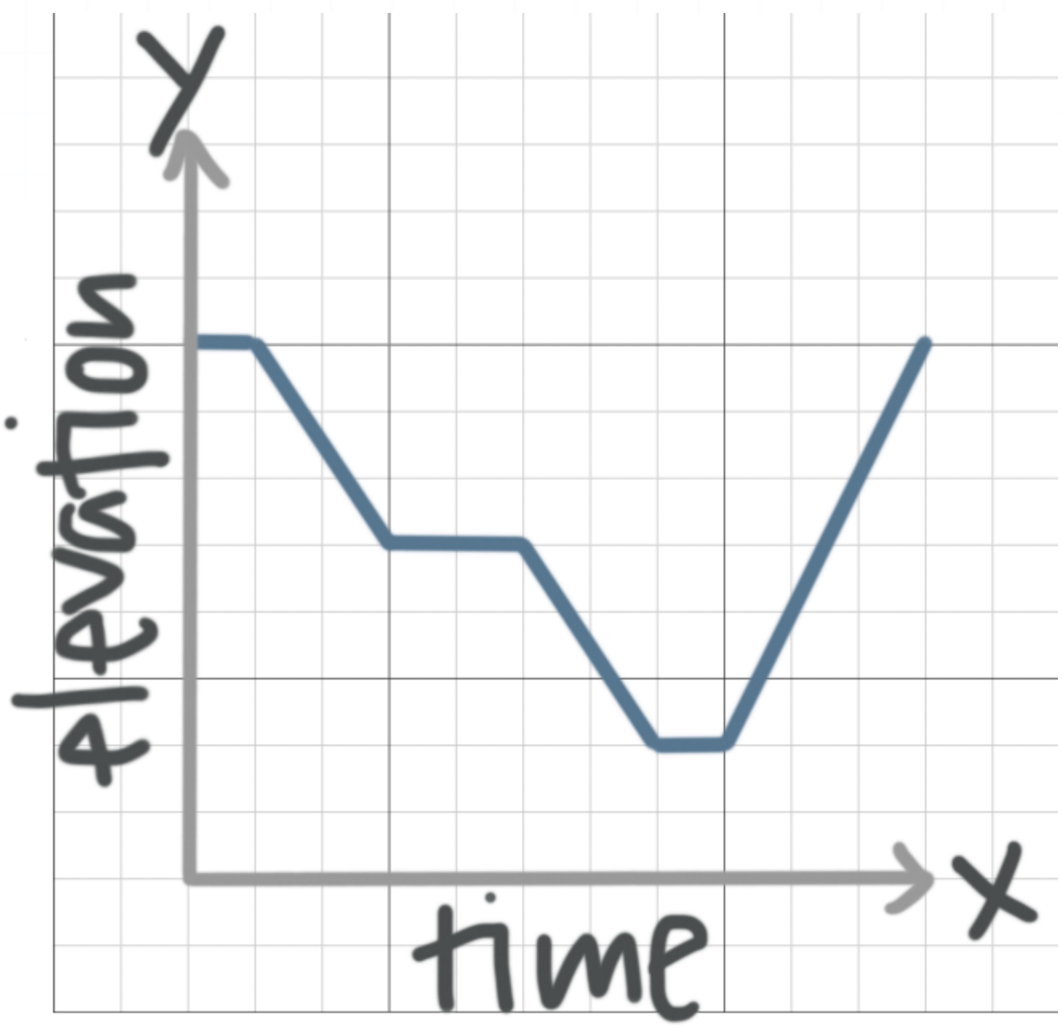


Solution:

The speed of the horse is slow at the beginning and then increases. In the middle of the graph, the horse maintains its speed. Then the horse increases speed for a short period of time before slowing quickly. So the story might be

> *"The horse walks to the beginning of the race track and then increases speed quickly. The horse runs at a constant speed in the middle of the track and then the rider encourages the horse to run even faster during the last stretch of the track. After the course the horse slows down rather quickly."*

■ 3. A scuba diver takes a dive to explore the ocean. The graph below shows the diver's elevation over time. Write a possible story to go with the graph.

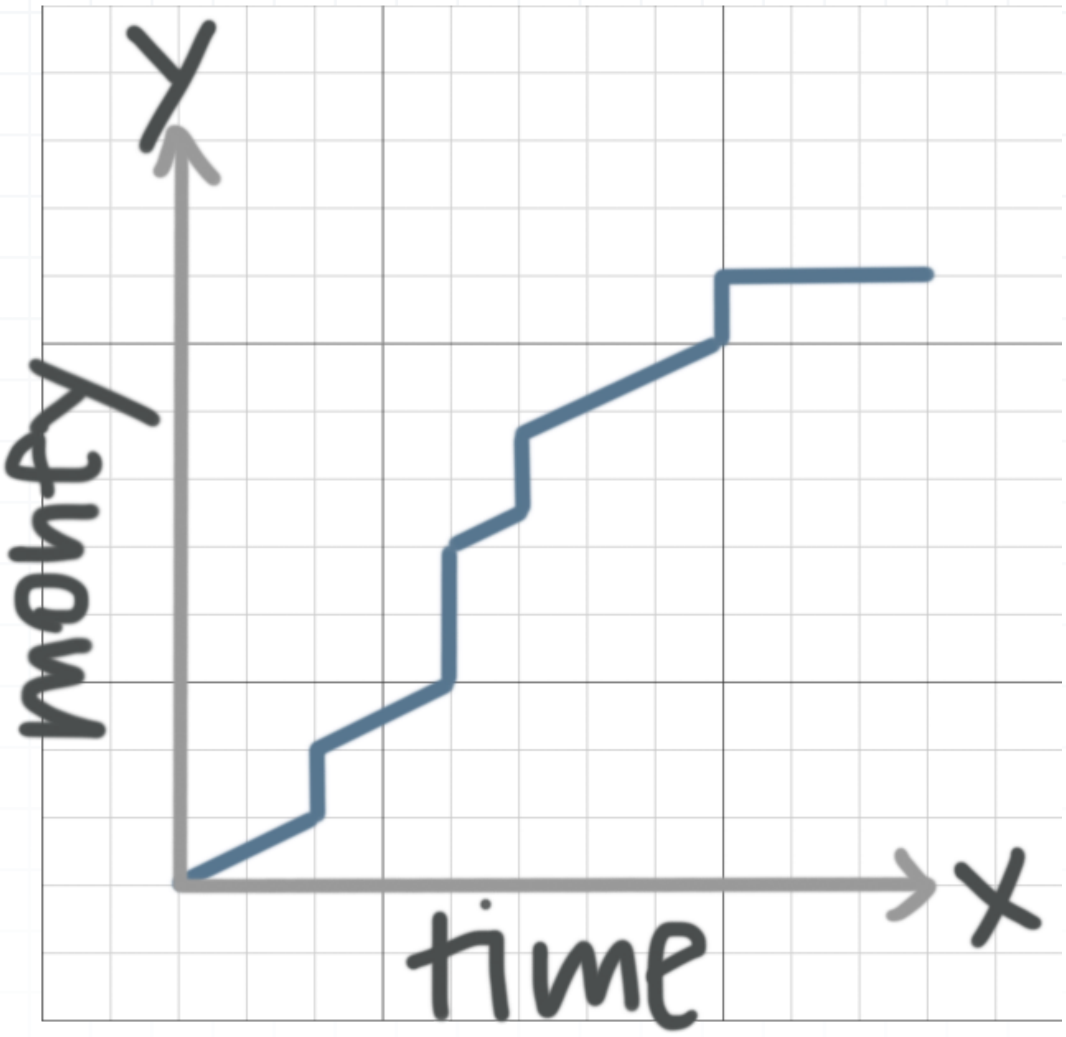


Solution:

The graph starts with a high elevation at the surface of the water, then the elevation decreases as the diver dives down. The elevation stays constant while the diver explores at a consistent depth. Next, the elevation decreases again indicating the diver dives to a deeper depth. The elevation remains constant for a shorter amount of time at the new depth. Finally the elevation increases back to the original elevation indicating that the diver is ascending back to the surface. So the story might be

> *"The scuba diver dives down and spends some time exploring at that depth. The diver then decides to dive deeper and spends a shorter amount of time exploring at the new depth. Finally the diver makes his way back up to the surface."*

■ 4. Janet delivers packages and get paid an hourly rate in addition to $\$1$ for every package she delivers. The graph shows Janet's pay over the course of the day. Write a possible story to go with the graph.



Solution:

When Janet is driving, her pay increases as a steady rate since she's paid hourly. The sudden increases in pay indicate times when Janet makes deliveries because she also gets paid per package delivered. The second delivery shows a sudden increase that's twice as high as the others, indicating that there were twice as many packages.

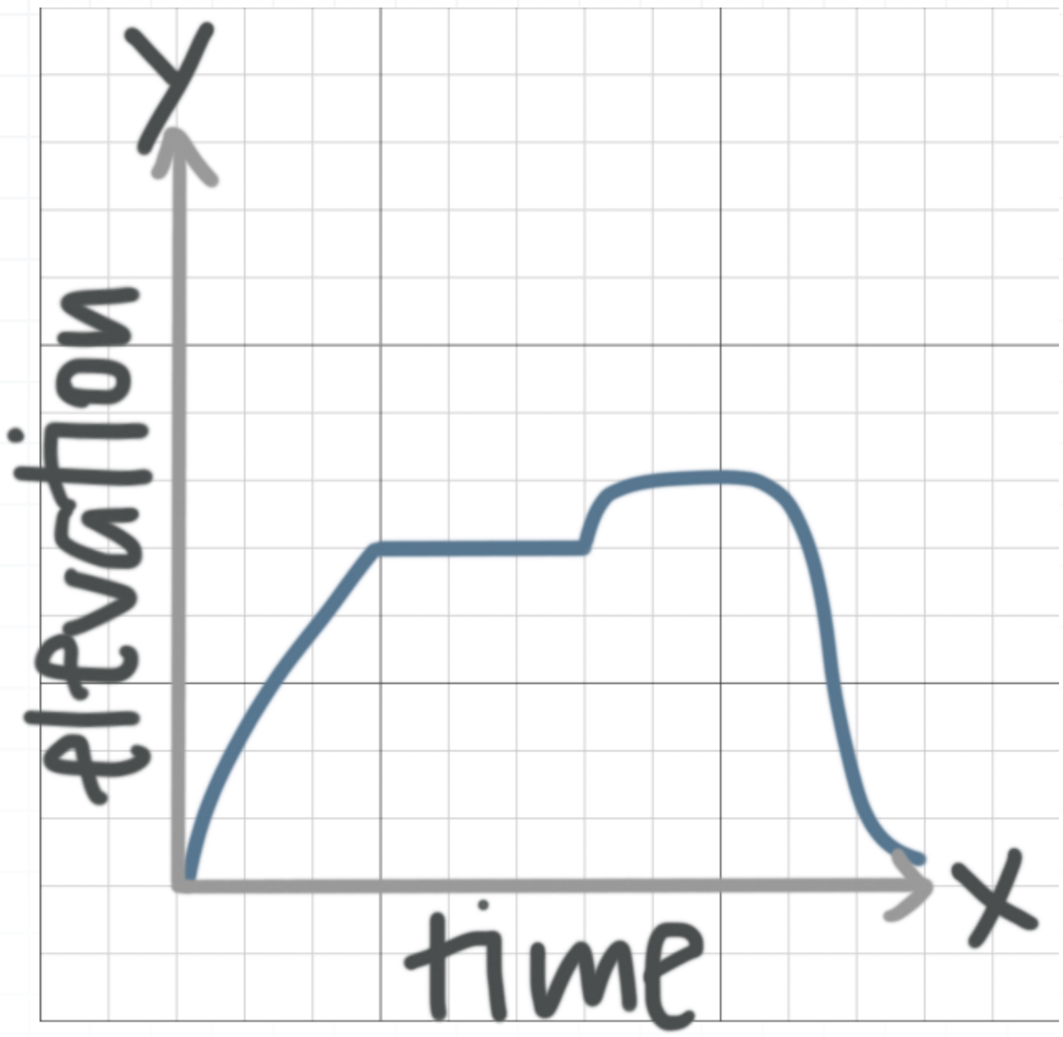
There are four sections that show the steady increase of pay when Janet is driving. The first two are the same length, the third is short, and the last is longer. At the end of the graph the amount of money becomes constant, indicating the end of Janet's shift, since she's no longer making money. So the story might be

"Janet delivers the same number of packages at her first, third, and last stop. At her second stop Janet delivers twice as many packages. Janet spends the same amount of time driving to her first and second stops. The third stop is closer, so Janet doesn't spend as much time driving, but she spends the most time driving to her final stop. After her final stop, Janet's shift is done for the day."

5. A plane takes off and then cruises at $30{,}000$ feet for several hours before rising in elevation to $35{,}000$ feet to avoid turbulence for the last few hours. The plane then reaches its destination and lands. Sketch a graph representing the situation.

Solution:

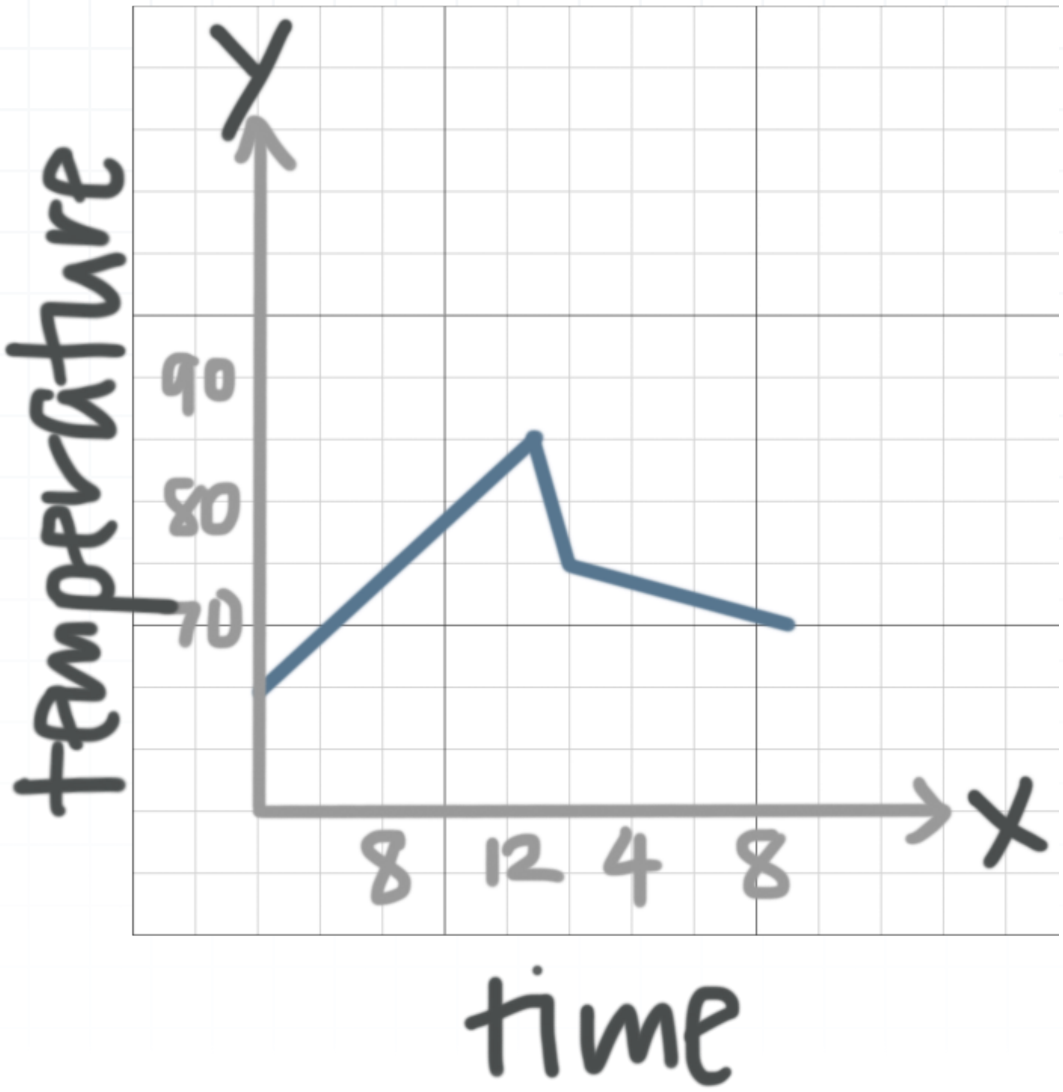
From left to right, elevation will rise dramatically and then level off. Then elevation will rise a little more to $35{,}000$ feet and level off at that elevation for a while before the elevation decreases for the landing.



6. The temperature throughout a summer day starts at 65° F at 6:00 a.m.. Over the next few hours the temperature rises steadily until it reaches 85° F at 1:00 p.m.. At 1:15 p.m., a rainstorm begins and cools the temperature down to 75° F. The temperature then steadily decreases until it reaches 70° F at 9:00 p.m.. Sketch a graph representing the situation.

Solution:

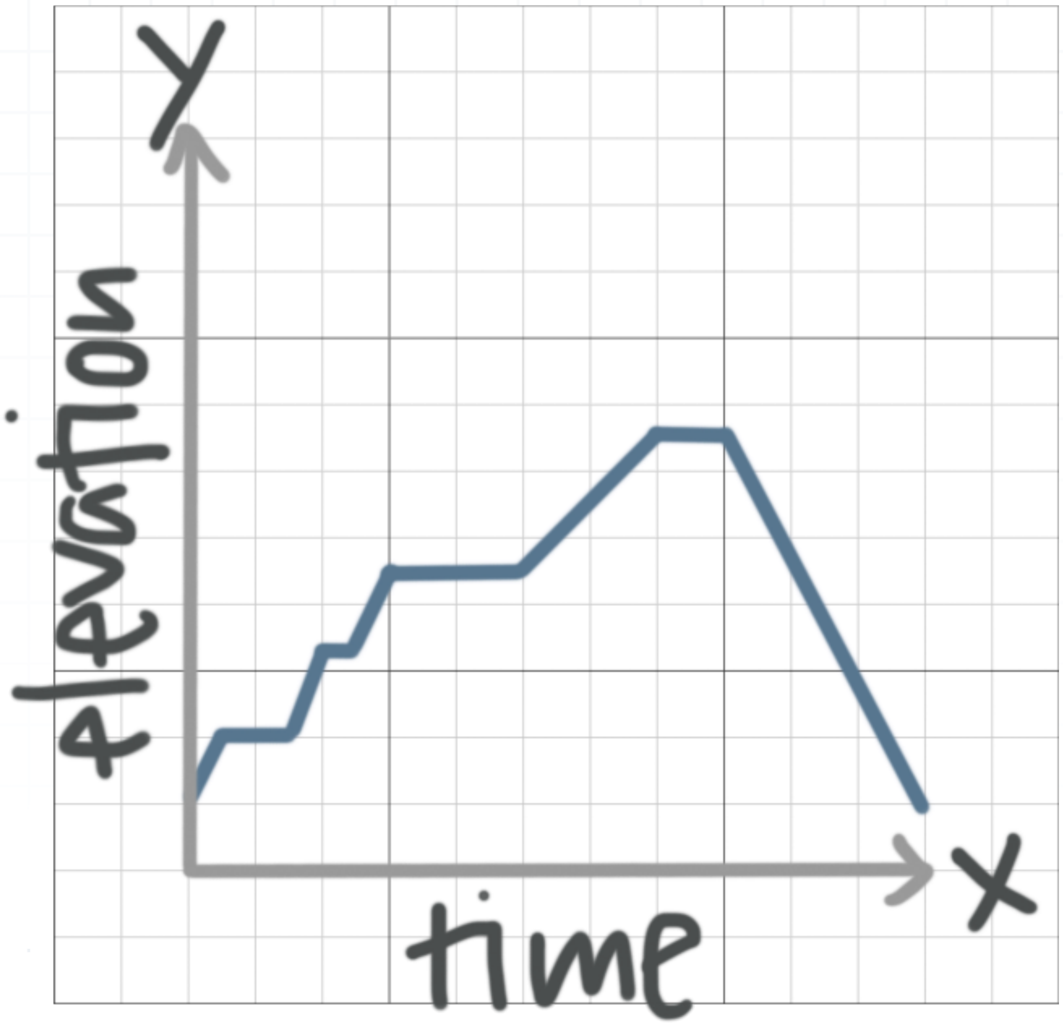
From left to right, the temperature will rise steadily until 1:00 p.m. from 65° to 85°. Then there will be a sharp decrease down to 75°, due to the rainstorm. After the rainstorm, the temperature decreases steadily until it reaches 70° at 9:00 p.m..



7. Brett goes for a hike up a mountain. He starts hiking up steadily for several hours with two stops for water. Then Brett stops for an hour to eat lunch and rest. He then continues up the mountain, summits, and spends a little time at the top of the mountain before climbing down. Sketch a graph representing Brett's elevation over time.

Solution:

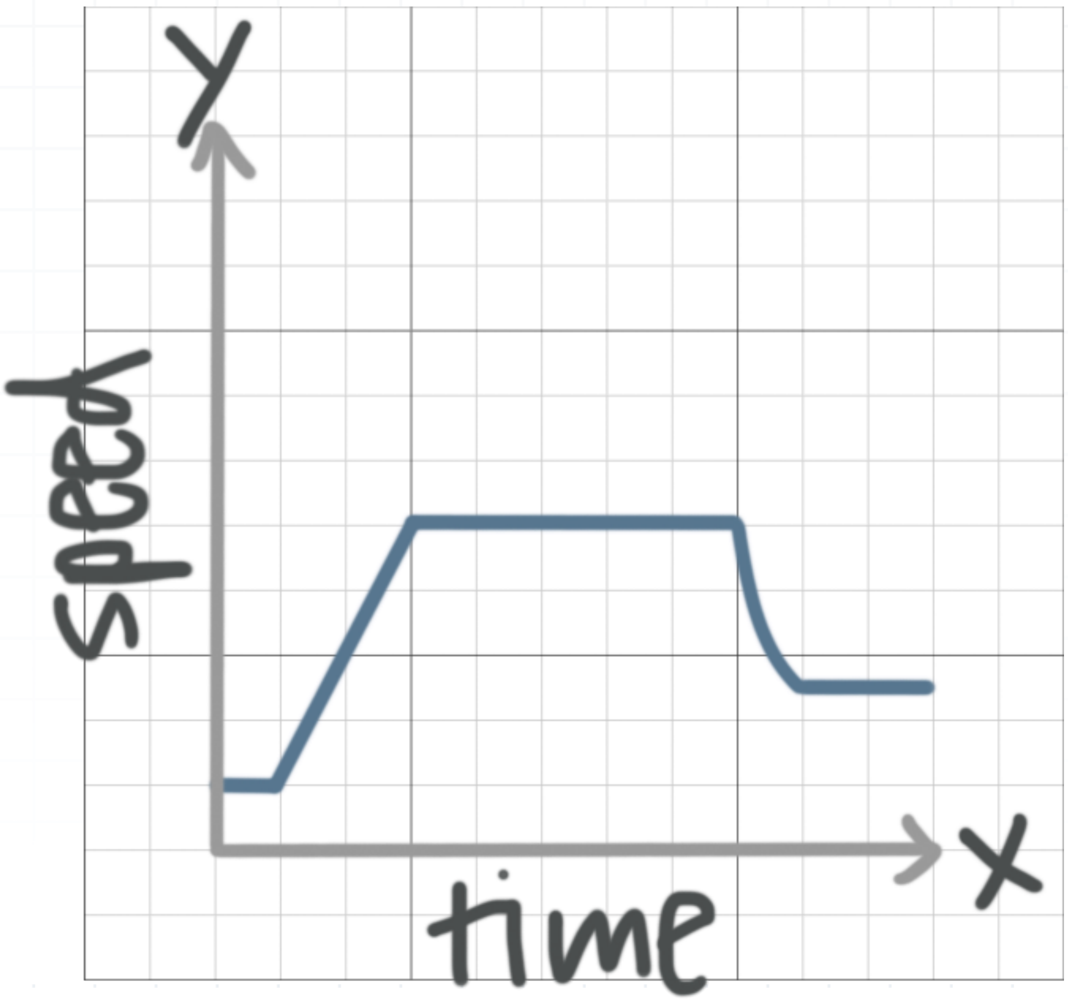
From left to right, Brett’s elevation will rise steadily with a couple of small stops for water before a longer stop for lunch. After lunch Brett’s elevation continues to increase until he reaches the top of the mountain. Then his elevation will stay steady as he spends some time at the top of the mountain, before decreasing as he descends down the mountain.



8. Heather went for a bike ride. She started at 12 mph to warm up, but quickly increased her speed to 20 mph and maintained that speed for most of the ride. Near the end of her bike ride, Heather decreased her speed to 15 mph until she reached her destination. Sketch a graph representing Heather’s speed over time.

Solution:

From left to right, Heather starts biking and maintains her slower speed for a short amount of time. Her speed increases and stays at 20 mph for most of the bike ride before decreasing at the end of the ride.



EQUATION OF A LINE IN POINT-SLOPE FORM

1. Find the equation of the line that passes through $(3,0)$ with slope -2.

Solution:

Using point-slope form, the equation of the line is

$$y - 0 = -2(x - 3)$$

$$y = -2x + 6$$

2. Name two (of four possible) pieces of information about a line that are required to write an equation of the line in point-slope form.

Solution:

Naming any two of the following is correct:

(1) A point

(2) Another point

(3) The slope

(4) The y-intercept

■ 3. Find the equation of the line that passes through the points $(-2,3)$ and $(2,-4)$.

Solution:

We first need to calculate the slope of the line as follows

$$m = \frac{-4-3}{2-(-2)}$$

$$m = \frac{-7}{4}$$

$$m = -\frac{7}{4}$$

Using point-slope form, the equation of the line is either of the following:

$$y - 3 = -\frac{7}{4}(x+2)$$

$$y + 4 = -\frac{7}{4}(x-2)$$

■ 4. Find the equation of the line that passes through $(-2,-5)$ with a slope 6.

Solution:

Using point-slope form, the equation of the line is

$$y + 5 = 6(x + 2)$$

$$y + 5 = 6x + 12$$

$$y = 6x + 7$$

■ 5. Identify the point (x_1, y_1) and slope m in the equation of the line.

$$y + 3 = \frac{1}{4}(x - 6)$$

Solution:

Using point-slope form, we can see that the point is $(6, -3)$ and the slope is $1/4$.

■ 6. Write the following equation in point-slope form.

$$y = -\frac{1}{2}x + 4$$

Solution:

Notice that the slope is given as $-1/2$ and the line passes through the point $(2,3)$, so we can write the equation of the line as

$$y - 3 = -\frac{1}{2}(x - 2)$$

7. Find the equation of the line that passes through the points $(5, -4)$ and $(6,0)$.

Solution:

We first need to calculate the slope of the line as

$$m = \frac{0 - (-4)}{6 - 5}$$

$$m = \frac{4}{1}$$

$$m = 4$$

Using point-slope form, the equation of the line is then either of the following:

$$y + 4 = 4(x - 5)$$

$$y = 4(x - 6)$$

EQUATION OF A LINE IN SLOPE-INTERCEPT FORM

■ 1. Find the equation of a line through the point $(0,5)$ with slope -2. Write the solution in slope-intercept form.

Solution:

Using slope-intercept form, the equation of the line is

$$y = -2x + 5$$

■ 2. Identify the y-intercept and slope m defining the line.

$$y = -\frac{1}{4}(x + 12)$$

Solution:

Notice that the slope of the line given is $-1/4$ and the y-intercept (when $x = 0$) is $(0, -3)$.

■ 3. Convert the following point-slope equation into a slope-intercept equation.

$$y - 3 = \frac{1}{3}(x - 6)$$

Solution:

Converting to slope-intercept form means that we need to solve for y, and simplify as much as we can.

$$y - 3 = \frac{1}{3}(x - 6)$$

$$y - 3 = \frac{1}{3}x - 2$$

$$y = \frac{1}{3}x - 2 + 3$$

$$y = \frac{1}{3}x + 1$$

■ 4. Find the equation of a line that passes through the points $(1, -1)$ and $(0,3)$. Write the solution in slope-intercept form.

Solution:

We first need to calculate the slope of the line as

$$m = \frac{3 - (-1)}{0 - 1}$$

$$m = \frac{4}{-1}$$

$$m = -4$$

Using slope-intercept form, noting that the y-intercept is 3, the equation of the line is

$$y = -4x + 3$$

5. Determine the y-intercept of a line with slope -3 that passes through the point $(1,1)$. Write your solution as a coordinate point.

Solution:

In point-slope form, the equation of the line is

$$y - 1 = -3(x - 1)$$

$$y = -3x + 3 + 1$$

$$y = -3x + 4$$

From the new form of the equation of the line, we can see that the y-intercept is $(0,4)$.

6. Name two (of four possible) pieces of information about a line that are required to write an equation of the line in point-slope form.

Solution:

Naming any two of the following is correct:

(1) A point

(2) Another point

(3) The slope

(4) The y-intercept

■ 7. Find the equation of a line that passes through the points $(-3,-2)$ and $(2,-4)$. Write the solution in slope-intercept form.

Solution:

We first need to calculate the slope of the line as

$$m=\frac{-4-(-2)}{2-(-3)}$$

$$m=\frac{-2}{5}$$

$$m=-\frac{2}{5}$$

Using point-slope form, the equation of the line is

$$y+2=-\frac{2}{5}(x+3)$$

$$y+2=-\frac{2}{5}x-\frac{6}{5}$$

$$y=-\frac{2}{5}x-\frac{6}{5}-\frac{10}{5}$$

$$y=-\frac{2}{5}x-\frac{16}{5}$$

GRAPHING PARABOLAS

1. Write the equation in vertex form.

$$y = x^2 + 8x + 5$$

Solution:

To convert to vertex form, we'll need to complete the square. Take the coefficient of 8 on the x term, divide it by 2, then square the result.

$$\frac{8}{2} = 4$$

$$(4)^2 = 16$$

We'll need to add and subtract 16 from the right side of the equation to keep it balanced.

$$y = (x^2 + 8x + 16) - 16 + 5$$

$$y = (x^2 + 8x + 16) - 11$$

Factor what's inside the parentheses in order to put the parabola in vertex form.

$$y = (x + 4)(x + 4) - 11$$

$$y = (x + 4)^2 - 11$$

2. Write the equation in vertex form.

$$y = -2x^2 + 24x - 68$$

Solution:

First factor out the coefficient on the x^2 term, which is -2.

$$y = -2x^2 + 24x - 68$$

$$y = -2(x^2 - 12x) - 68$$

To convert to vertex form, we'll need to complete the square. Take the coefficient of -12 on the x term, divide it by 2, then square the result.

$$\frac{-12}{2} = -6$$

$$(-6)^2 = 36$$

We'll need to add 36 on the inside of the parentheses. The -2 on the outside of the parentheses means that were really adding $-2(36) = -72$. Therefore, we'll also need to add 72 outside of the parentheses to keep the equation balanced.

$$y = -2(x^2 - 12x + 36) + 72 - 68$$

$$y = -2(x^2 - 12x + 36) + 4$$

Factor what's inside the parentheses in order to put the parabola in vertex form.

$$y = -2(x-6)(x-6)+4$$

$$y = -2(x-6)^2+4$$

■ 3. Find the vertex and axis of symmetry of $y = x^2 + 5x + 6$.

Solution:

Remember that the axis of symmetry is $x = -b/2a$ and that standard form for a parabola is $y = ax^2 + bx + c$. In this case, $a = 1$ and $b = 5$.

$$x = -\frac{5}{2(1)}$$

$$x = -\frac{5}{2}$$

To find the vertex, plug $x = -5/2$ into the equation of the parabola.

$$y = x^2 + 5x + 6$$

$$y = \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 6$$

$$y = \frac{25}{4} - \frac{25}{2} + 6$$

$$y = \frac{25}{4} - \frac{50}{4} + 6$$

$$y = -\frac{25}{4} + \frac{24}{4}$$

$$y = -\frac{1}{4}$$

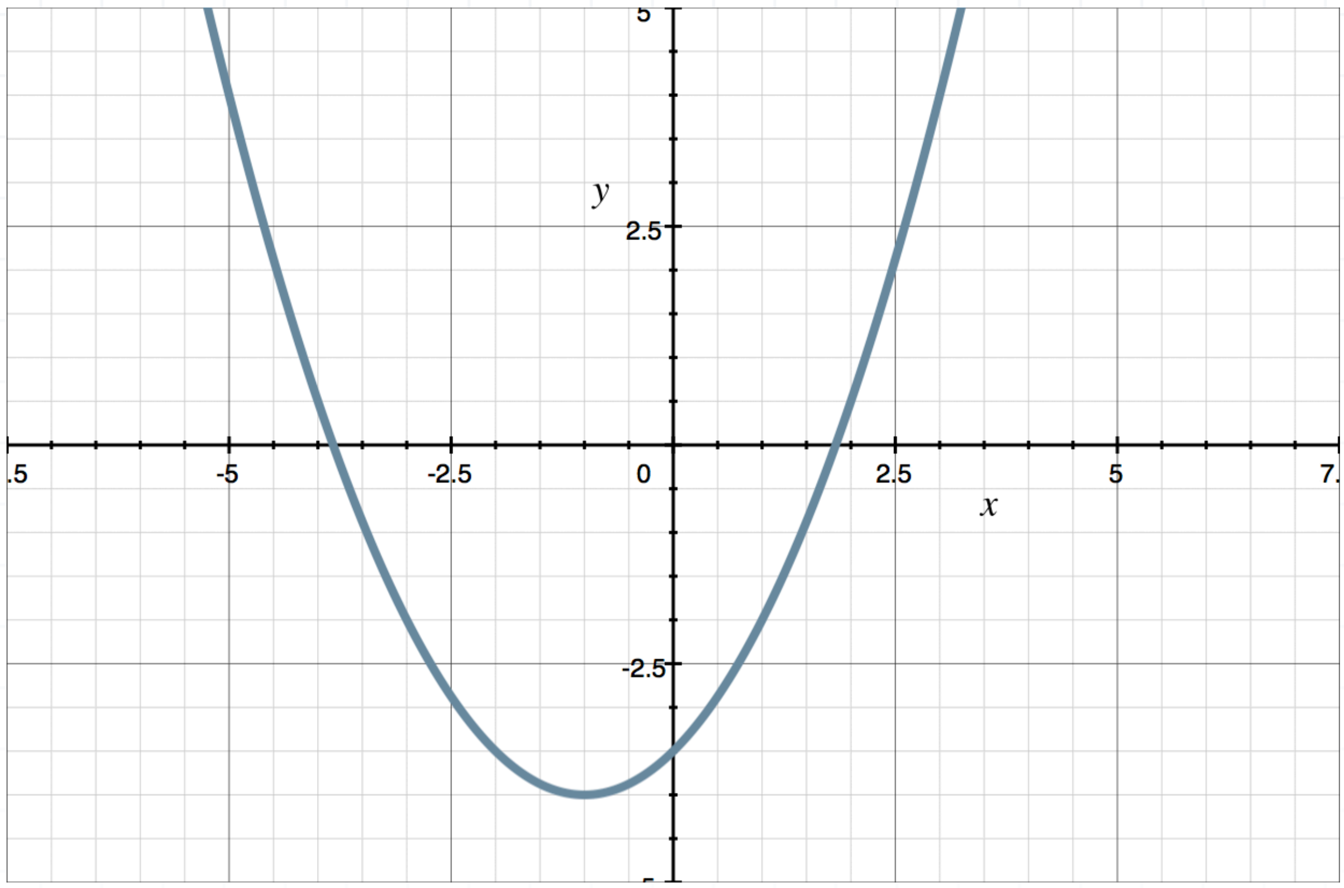
The vertex is therefore $(-5/2, -1/4)$ and the axis of symmetry is $x = -5/2$.

■ 4. Find the vertex and axis of symmetry of $y = 3(x + 2)^2 + 6$.

Solution:

Remember that vertex form is $a(x - h)^2 + k$ and that the vertex is (h, k). In this case, $h = -2$ and $k = 6$. So the vertex is $(-2,6)$. The axis of symmetry is $x = h$, so the axis of symmetry is $x = -2$.

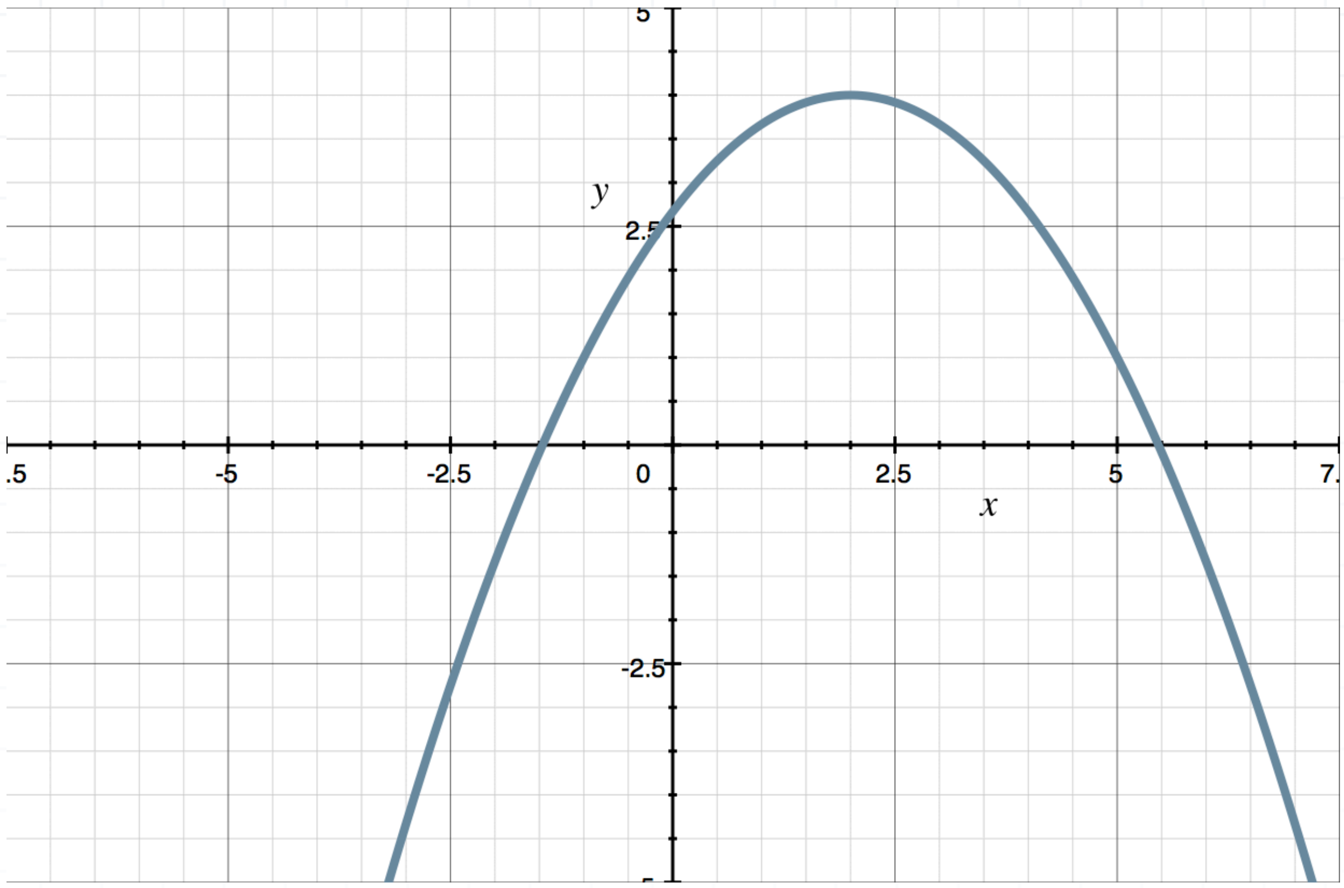
■ 5. Identify the vertex and axis of symmetry from the graph of the parabola.



Solution:

The vertex is the minimum point of the graph, $(-1,-4)$. The axis of symmetry is the line of symmetry, $x=-1$.

6. Using the graph below, find the equation of the parabola in standard form.



Solution:

First find the vertex of the graph. The vertex is the maximum point, which we can see from the graph is sitting at $(2,4)$. Write the equation of the parabola in vertex form and plug in $h = 2$ and $k = 4$.

$$y = a(x - h)^2 + k$$

$$y = a(x - 2)^2 + 4$$

To find a, we need to plug in another point from the parabola. We'll use $(-1,1)$, plugging $x = -1$ and $y = 1$ into the equation.

$$y = a(x - 2)^2 + 4$$

$$1 = a(-1 - 2)^2 + 4$$

$$1 = a(-3)^2 + 4$$

$$1 = 9a + 4$$

$$-3 = 9a$$

$$-\frac{1}{3} = a$$

Plug in $a = -1/3$ and expand to find standard form.

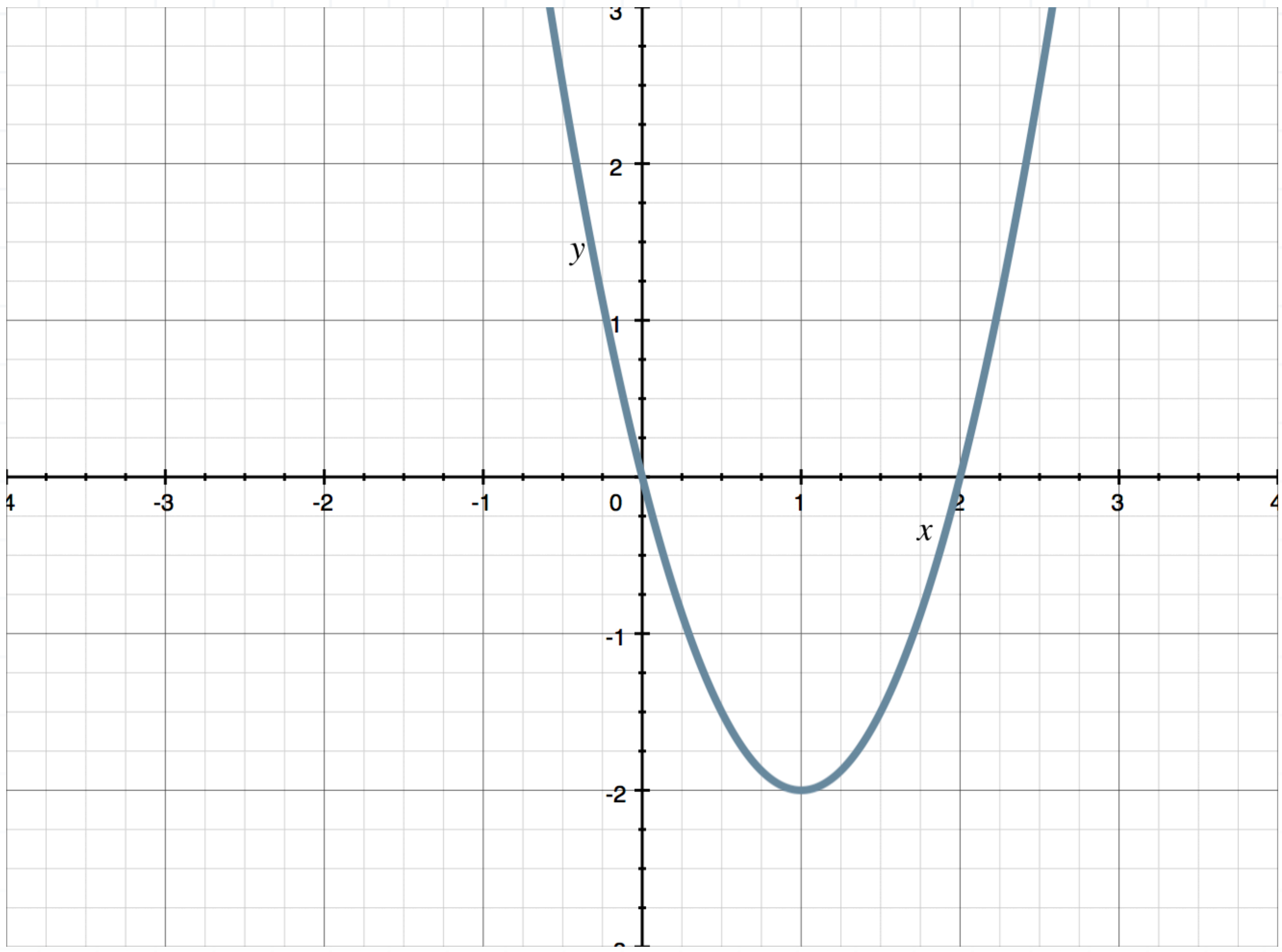
$$y = -\frac{1}{3}(x-2)^2 + 4$$

$$y = -\frac{1}{3}(x^2 - 4x + 4) + 4$$

$$y = -\frac{1}{3}x^2 + \frac{4}{3}x - \frac{4}{3} + 4$$

$$y = -\frac{1}{3}x^2 + \frac{4}{3}x + \frac{8}{3}$$

7. Using the graph, find the equation of the parabola in standard form.



Solution:

First, find the vertex of the graph. The vertex is the minimum point at $(1,-2)$. Write the equation of the parabola in vertex form and plug in $h=1$ and $k=-2$.

$$y=a(x-h)^2+k$$

$$y=a(x-1)^2+(-2)$$

$$y=a(x-1)^2-2$$

To find a, we need to plug in another point from the parabola. We'll use $(0,0)$, plugging $x=0$ and $y=0$ into the equation.

$$y = a(x-1)^2 - 2$$

$$0 = a(0-1)^2 - 2$$

$$0 = a(-1)^2 - 2$$

$$0 = a - 2$$

$$2 = a$$

Plug in $a = 2$ and expand to find standard form.

$$y = 2(x-1)^2 - 2$$

$$y = 2(x^2 - 2x + 1) - 2$$

$$y = 2x^2 - 4x + 2 - 2$$

$$y = 2x^2 - 4x$$

8. Complete the square to graph the parabola $y = x^2 + 6x + 5$.

Solution:

Complete the square to put the parabola in vertex form. Take the coefficient of 6 on x, divide it by 2, then square the result.

$$\frac{6}{2} = 3$$

$$3^2 = 9$$

We'll need to add and subtract 9 from the right side of the equation to keep it balanced.

$$y = (x^2 + 6x + 9) + 5 - 9$$

$$y = (x^2 + 6x + 9) - 4$$

Factor what's inside the parentheses.

$$y = (x + 3)(x + 3) - 4$$

$$y = (x + 3)^2 - 4$$

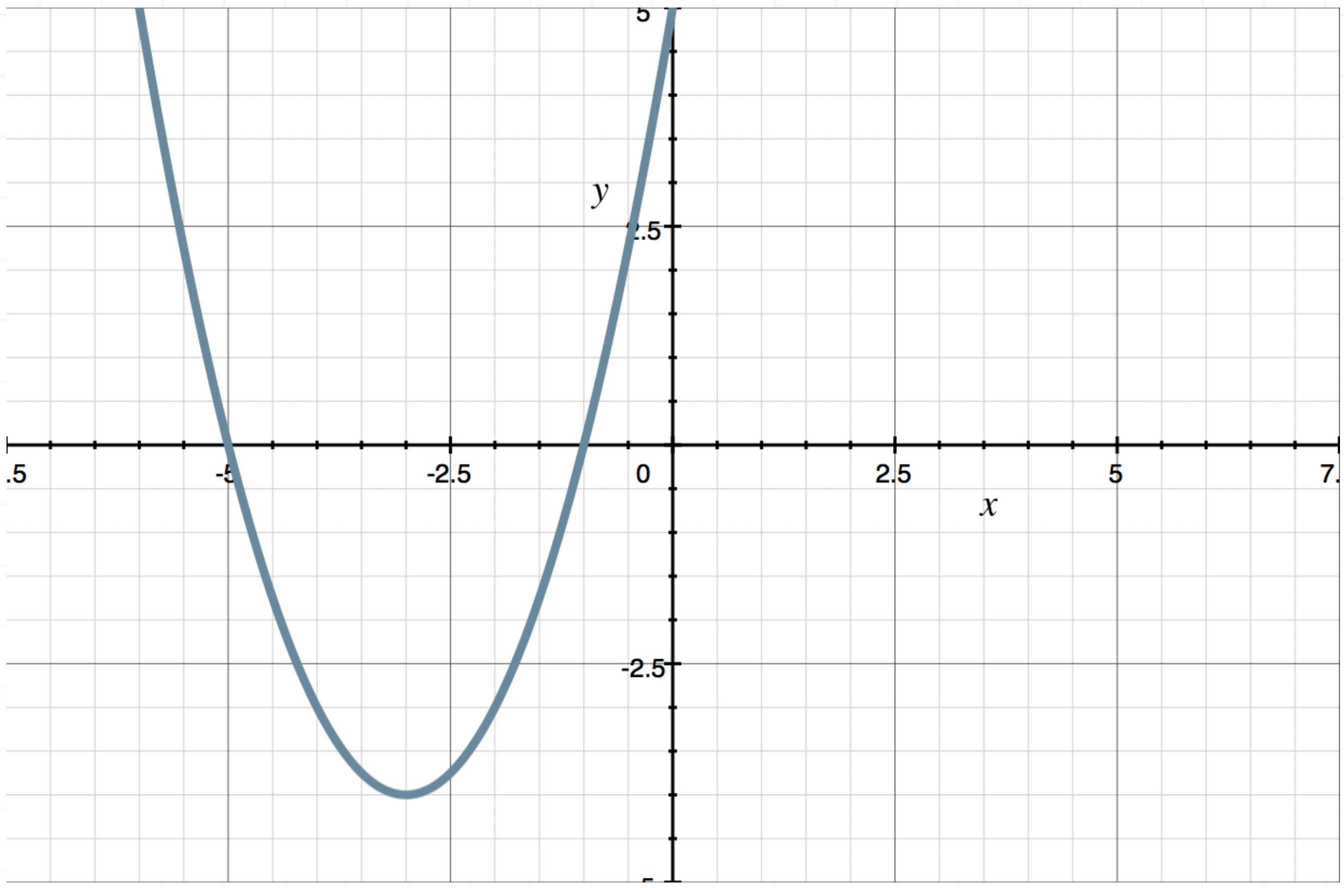
The vertex is $(-3, -4)$. We need to find at least one other point on the graph. In this case, we can find the zeros of the graph. Set the standard form of the parabola equal to 0, then factor and solve for x.

$$0 = x^2 + 6x + 5$$

$$0 = (x + 5)(x + 1)$$

$$x = -5, \ -1$$

Now we have three points to graph: $(-5,0)$, $(-1,0)$, and $(-3, -4)$. Graph the points, then connect them to sketch the graph of the parabola.



9. Complete the square to graph $y = -x^2 - 4x - 6$.

Solution:

We need to complete the square to put the parabola in vertex form, but first we'll need to factor out -1 from the first two terms so that the coefficient to the x^2 term is positive 1.

$$y = -1(x^2 + 4x) - 6$$

Take the coefficient of 4 on the x term, divide it by 2, then square the result.

$$\frac{4}{2} = 2$$

$$2^2 = 4$$

We'll need to add 4 on the inside of the parentheses. The -1 on the outside of the parentheses really means were adding $-1(4) = -4$ to the equation, so we'll need to add 4 outside of the parentheses to keep the equation balanced.

$$y = -(x^2 + 4x + 4) - 6 + 4$$

$$y = -(x^2 + 4x + 4) - 2$$

Factor what's inside the parentheses.

$$y = -(x + 2)(x + 2) - 2$$

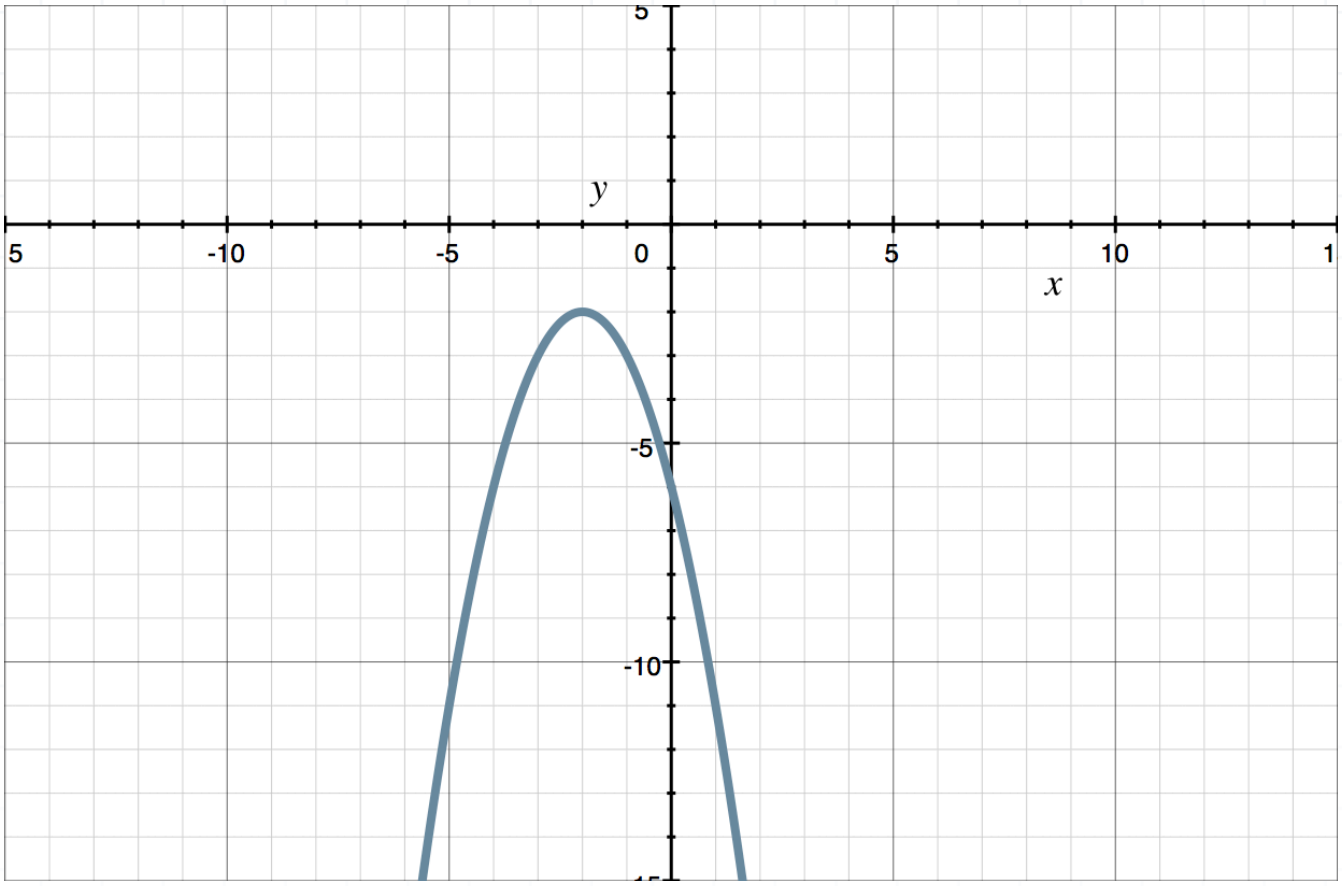
$$y = -(x + 2)^2 - 2$$

The vertex is $(-2, -2)$. We need to find at least one other point on the graph. In this case, let's plug $x = 0$ into the equation to find the corresponding y-value.

$$y = -0^2 - 4(0) - 6$$

$$y = -6$$

Now we have two points to graph: $(-2, -2)$ and $(0, -6)$. We can use the axis of symmetry to plot the third point. Graph the points, then connect them to sketch the parabola.


y
x
5
-10
-5
0
5
10
1
-5
-10

FINDING CENTER AND RADIUS OF A CIRCLE

1. Find the center and radius of the circle.

$$x^2 + y^2 - 2x - 3 = 0$$

Solution:

To change the equation into standard form, $(x - h)^2 + (y - k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$(x^2 - 2x) + y^2 = 3$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. Since there is no y term in this case, we'll just need to complete the square with respect to x. The coefficient of x is -2, so

$$\frac{-2}{2} = -1$$

$$(-1)^2 = 1$$

Add 1 to both sides of the equation, then factor and simplify.

$$(x^2 - 2x + 1) + y^2 = 3 + 1$$

$$(x - 1)^2 + y^2 = 4$$

The center of the circle (h, k) is at $(1,0)$, and the radius is $r = \sqrt{4} = 2$.

2. Find the center and radius of the circle.

$$x^2 + y^2 + 14x + 22y + 145 = 0$$

Solution:

To change the equation into standard form, $(x - h)^2 + (y - k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$(x^2 + 14x) + (y^2 + 22y) = -145$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is 14, so

$$\frac{14}{2} = 7$$

$$(7)^2 = 49$$

The coefficient of y is 22, so

$$\frac{22}{2} = 11$$

$$11^2 = 121$$

Add 49 and 121 to both sides of the equation, then factor and simplify.

$$(x^2 + 14x + 49) + (y^2 + 22y + 121) = -145 + 49 + 121$$

$$(x + 7)^2 + (y + 11)^2 = 25$$

The center of the circle (h, k) is at $(-7, -11)$, and the radius is $r = \sqrt{25} = 5$.

■ 3. Find the center and radius of the circle.

$$16x^2 + 16y^2 - 8x - 24y - 150 = 0$$

Solution:

To change the equation into standard form, $(x - h)^2 + (y - k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$16x^2 - 8x + 16y^2 - 24y = 150$$

Divide both sides of the equation by 16, so that the coefficients of x^2 and y^2 are both 1.

$$x^2 - \frac{1}{2}x + y^2 - \frac{3}{2}y = \frac{150}{16}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is $-1/2$, so

$$-\frac{1}{2}\cdot\frac{1}{2}=-\frac{1}{4}$$

$$\left(-\frac{1}{4}\right)^2=\frac{1}{16}$$

The coefficient of y is $-3/2$, so

$$-\frac{3}{2}\cdot\frac{1}{2}=-\frac{3}{4}$$

$$\left(-\frac{3}{4}\right)^2=\frac{9}{16}$$

Add $1/16$ and $9/16$ to both sides of the equation, then factor and simplify.

$$\left(x^2-\frac{1}{2}x+\frac{1}{16}\right)+\left(y^2-\frac{3}{2}y+\frac{9}{16}\right)=\frac{150}{16}+\frac{1}{16}+\frac{9}{16}$$

$$\left(x-\frac{1}{4}\right)^2+\left(y-\frac{3}{4}\right)^2=\frac{160}{16}$$

$$\left(x-\frac{1}{4}\right)^2+\left(y-\frac{3}{4}\right)^2=10$$

The center of the circle (h,k) is at $(1/4,3/4)$, and the radius is $r=\sqrt{10}$.

■ 4. Find the center and radius of the circle.

$$4x^2+4y^2+32x-4y+41=0$$

Solution:

To change the equation into standard form, $(x-h)^2+(y-k)^2=r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$4x^2+32x+4y^2-4y=-41$$

Divide both sides of the equation by 4, so that the coefficients of x^2 and y^2 are both 1.

$$x^2+8x+y^2-y=-\frac{41}{4}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is 8, so

$$\frac{8}{2}=4$$

$$4^2=16$$

The coefficient of y is -1, so

$$\frac{-1}{2}=-\frac{1}{2}$$

$$\left(-\frac{1}{2}\right)^2=\frac{1}{4}$$

Add 16 and $1/4$ to both sides of the equation, then factor and simplify.

$$\left(x^2+8x+16\right)+\left(y^2-y+\frac{1}{4}\right)=-\frac{41}{4}+16+\frac{1}{4}$$

$$(x+4)^2+\left(y-\frac{1}{2}\right)^2=-10+16$$

$$(x+4)^2+\left(y-\frac{1}{2}\right)^2=6$$

The center of the circle (h,k) is at $(-4,1/2)$, and the radius is $r=\sqrt{6}$.

■ 5. Find the center and radius of the circle.

$$9x^2+9y^2-30x-6y-118=0$$

Solution:

To change the equation into standard form, $(x-h)^2+(y-k)^2=r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$9x^2-30x+9y^2-6y=118$$

Divide both sides of the equation by 9, so that the coefficients of x^2 and y^2 are both 1.

$$x^2-\frac{10}{3}x+y^2-\frac{2}{3}y=\frac{118}{9}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is $-10/3$, so

$$-\frac{10}{3}\cdot\frac{1}{2}=-\frac{10}{6}$$

$$\left(-\frac{10}{6}\right)^2=\frac{100}{36}=\frac{25}{9}$$

The coefficient of y is $-2/3$, so

$$-\frac{2}{3}\cdot\frac{1}{2}=-\frac{1}{3}$$

$$\left(-\frac{1}{3}\right)^2=\frac{1}{9}$$

Add $25/9$ and $1/9$ to both sides of the equation, then factor and simplify.

$$\left(x^2-\frac{10}{3}x+\frac{25}{9}\right)+\left(y^2-\frac{2}{3}y+\frac{1}{9}\right)=\frac{118}{9}+\frac{25}{9}+\frac{1}{9}$$

$$\left(x-\frac{5}{3}\right)^2+\left(y-\frac{1}{3}\right)^2=16$$

The center of the circle (h,k) is at $(5/3,1/3)$, and the radius is $r=\sqrt{16}=4$.

■ 6. Find the center and radius of the circle.

$$x^2+y^2+4x-2y=0$$

Solution:

To change the equation into standard form, $(x-h)^2+(y-k)^2=r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2+4x+y^2-2y=0$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is 4, so

$$\frac{4}{2}=2$$

$$2^2=4$$

The coefficient of y is -2, so

$$\frac{-2}{2}=-1$$

$$(-1)^2=1$$

Add 4 and 1 to both sides of the equation, then factor and simplify.

$$(x^2+4x+4)+(y^2-2y+1)=0+4+1$$

$$(x+2)^2+(y-1)^2=5$$

The center of the circle (h,k) is at $(-2,1)$, and the radius is $r=\sqrt{5}$.

7. Find the center and radius of the circle.

$$x^2 + y^2 - 12x + 10y - 3 = 0$$

Solution:

To change the equation into standard form, $(x - h)^2 + (y - k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 - 12x + y^2 + 10y = 3$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is -12, so

$$\frac{-12}{2} = -6$$

$$(-6)^2 = 36$$

The coefficient of y is 10, so

$$\frac{10}{2} = 5$$

$$5^2 = 25$$

Add 36 and 25 to both sides of the equation, then factor and simplify.

$$(x^2 - 12x + 36) + (y^2 + 10y + 25) = 3 + 36 + 25$$

$$(x - 6)^2 + (y + 5)^2 = 64$$

The center of the circle (h, k) is at $(6, -5)$, and the radius is $r = \sqrt{64} = 8$.

■ 8. Find the center and radius of the circle.

$$x^2 + y^2 - \frac{1}{4} = 0$$

Solution:

To change the equation into standard form, $(x - h)^2 + (y - k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 + y^2 = \frac{1}{4}$$

The center of the circle (h, k) is at $(0,0)$, and the radius is $r = \sqrt{1/4} = 1/2$.

■ 9. Find the center and radius of the circle.

$$16x^2 + 16y^2 + 96x - 160y + 543 = 0$$

Solution:

To change the equation into standard form, $(x-h)^2+(y-k)^2=r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$16x^2+96x+16y^2-160y=-543$$

Divide both sides of the equation by 16, so that the coefficients of x^2 and y^2 are both 1.

$$x^2+6x+y^2-10y=-\frac{543}{16}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is 6, so

$$\frac{6}{2}=3$$

$$3^2=9$$

The coefficient of y is -10, so

$$\frac{-10}{2}=-5$$

$$(-5)^2=25$$

Add 9 and 25 to both sides of the equation, then factor and simplify.

$$(x^2+6x+9)+(y^2-10y+25)=-\frac{543}{16}+9+25$$

$$(x+3)^2 + (y-5)^2 = -\frac{543}{16} + \frac{144}{16} + \frac{400}{16}$$

$$(x+3)^2 + (y-5)^2 = \frac{1}{16}$$

The center of the circle (h, k) is at $(-3,5)$, and the radius is $r = \sqrt{1/16} = 1/4$.

■ 10. Find the center and radius of the circle.

$$9x^2 + 9y^2 - 72x + 12y - 77 = 0$$

Solution:

To change the equation into standard form, $(x-h)^2 + (y-k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$9x^2 - 72x + 9y^2 + 12y = 77$$

Divide both sides of the equation by 9, so that the coefficients of x^2 and y^2 are both 1.

$$x^2 - 8x + y^2 + \frac{4}{3}y = \frac{77}{9}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is -8, so

$$\frac{-8}{2} = -4$$

$$(-4)^2 = 16$$

The coefficient of y is $4/3$, so

$$\frac{4}{3} \cdot \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Add 16 and $4/9$ to both sides of the equation, then factor and simplify.

$$(x^2 - 8x + 16) + \left(y^2 + \frac{4}{3}y + \frac{4}{9}\right) = \frac{77}{9} + 16 + \frac{4}{9}$$

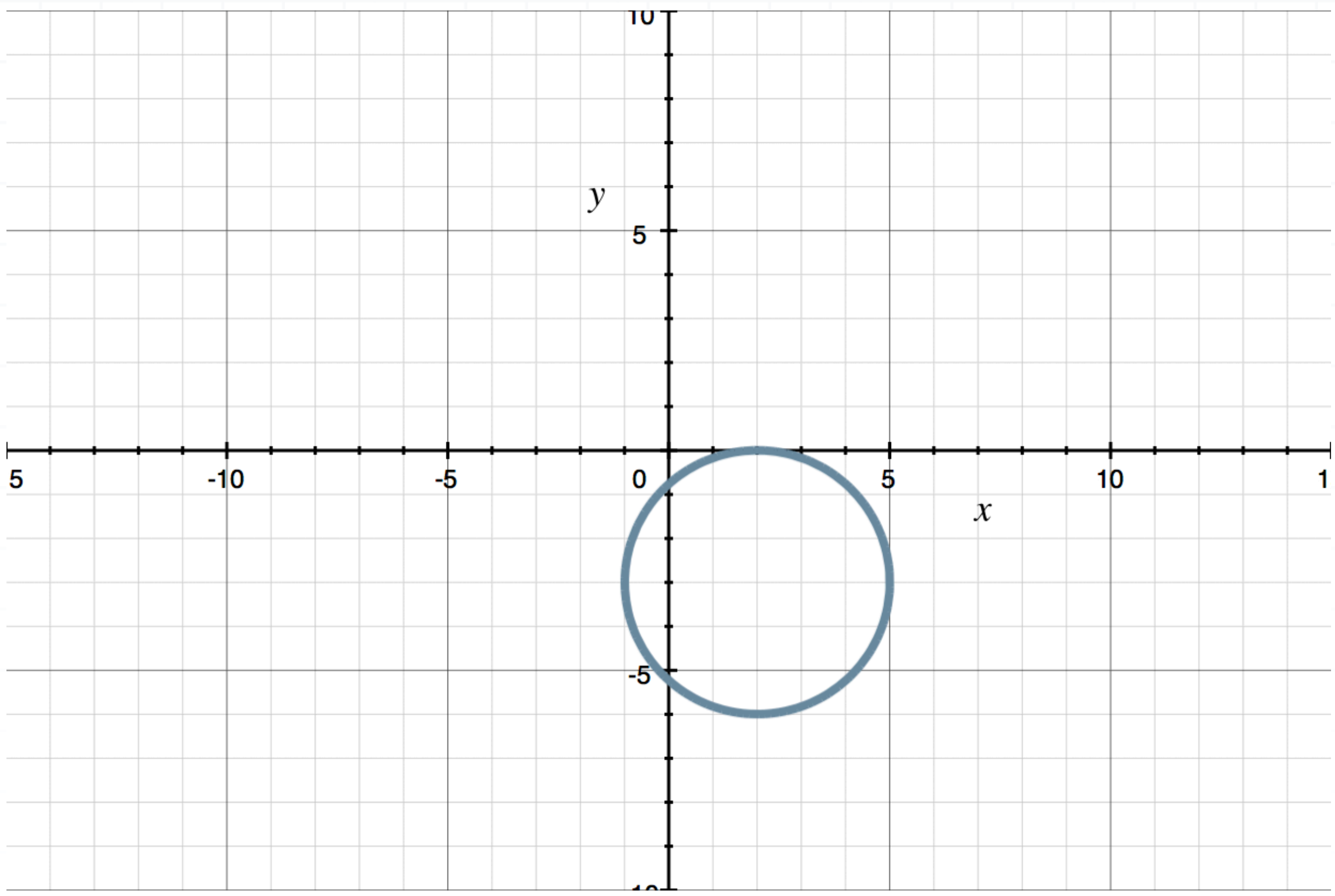
$$(x - 4)^2 + \left(y + \frac{2}{3}\right)^2 = 9 + 16$$

$$(x - 4)^2 + \left(y + \frac{2}{3}\right)^2 = 25$$

The center of the circle (h, k) is at $(4, -2/3)$, and the radius is $r = \sqrt{25} = 5$.

GRAPHING CIRCLES

1. Find the equation of the circle.



Solution:

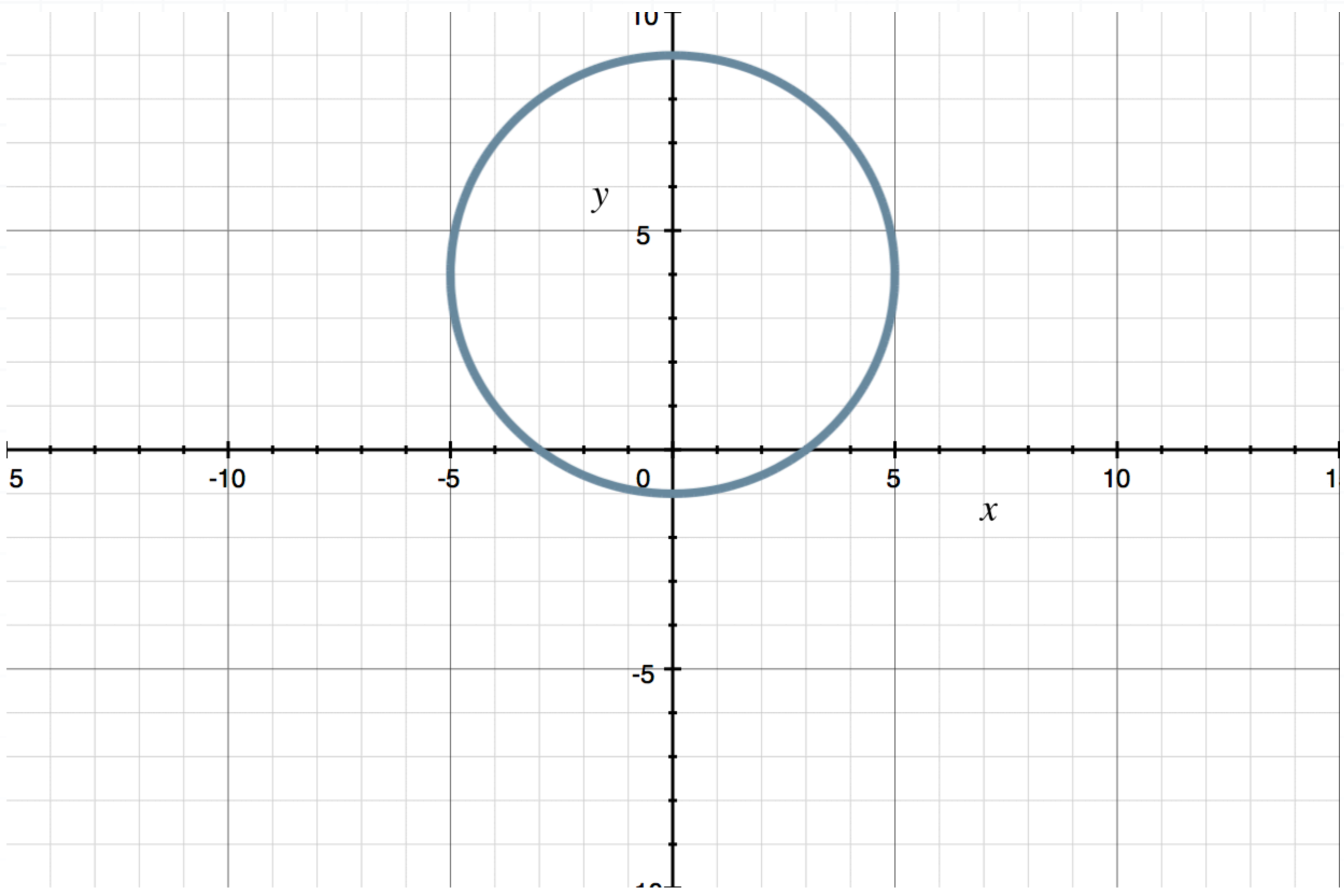
Visually, we can see that the center of the circle is at $(2,-3)$, so $h=2$ and $k=-3$. If we count from the center to a point on the circumference, we can see that the length of the radius is $r=3$. Plugging the center and radius into the standard equation of the circle gives

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-2)^2+(y-(-3))^2=3^2$$

$$(x-2)^2+(y+3)^2=9$$

■ 2. Find the equation of the circle.



Solution:

Visually, we can see that the center of the circle is at $(0,4)$, so $h=0$ and $k=4$. If we count from the center to a point on the circumference, we can see that the length of the radius is $r=5$. Plugging the center and radius into the standard equation of the circle gives

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-0)^2+(y-4)^2=5^2$$

$$x^2 + (y-4)^2 = 25$$

■ 3. Graph the circle $(x-1)^2 + (y-2)^2 = 4$.

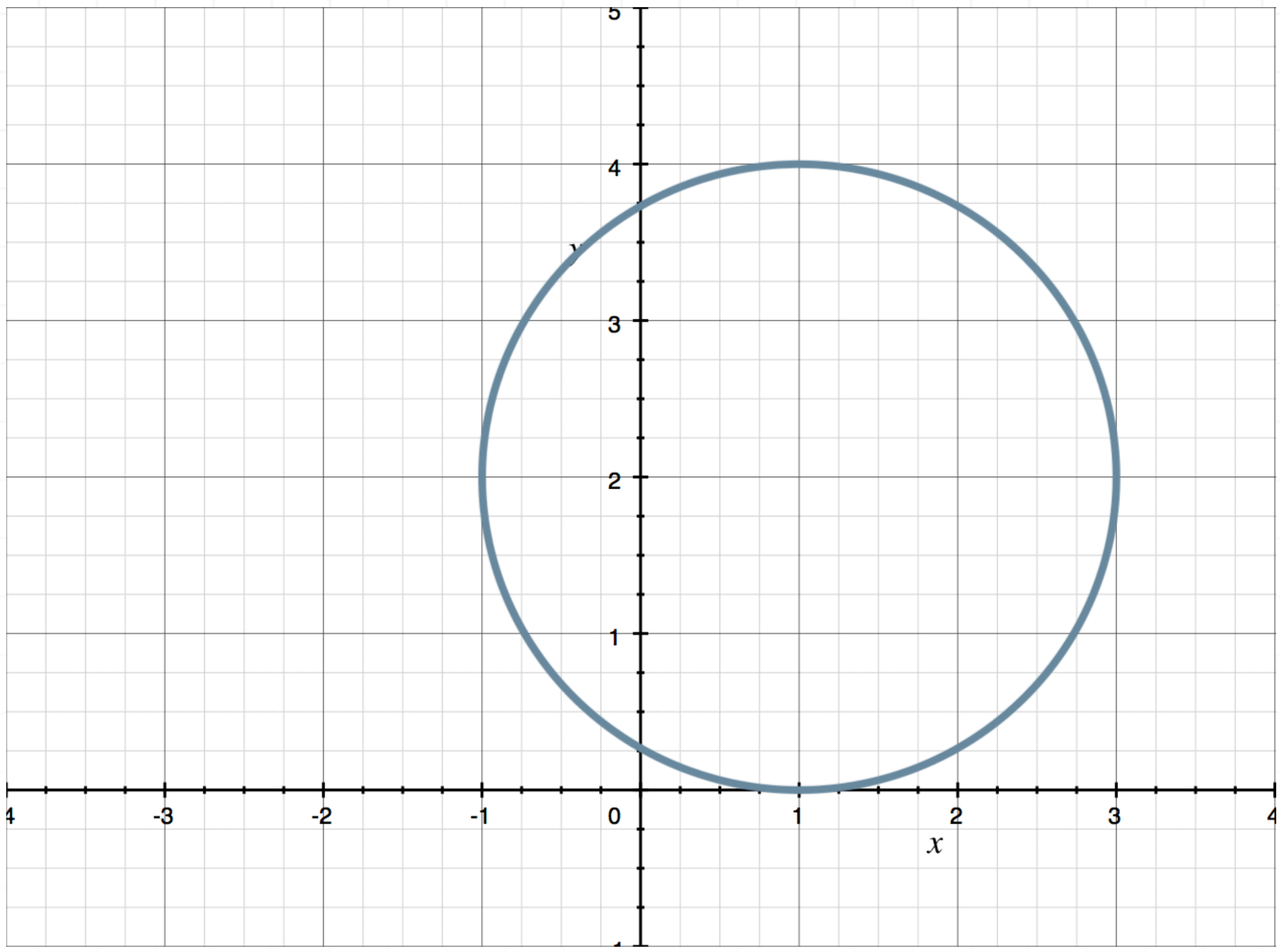
Solution:

We need to find the center and radius. The standard equation of the circle, is $(x-h)^2 + (y-k)^2 = r^2$, and if we match this up to the given equation,

$$(x-1)^2 + (y-2)^2 = 4$$

$$(x-1)^2 + (y-2)^2 = 2^2$$

we can say that the center is at $(h, k) = (1,2)$ and the radius is 2. Therefore, the graph of the circle is



4. Graph the circle $(x+3)^2 + (y-4)^2 = 25$.

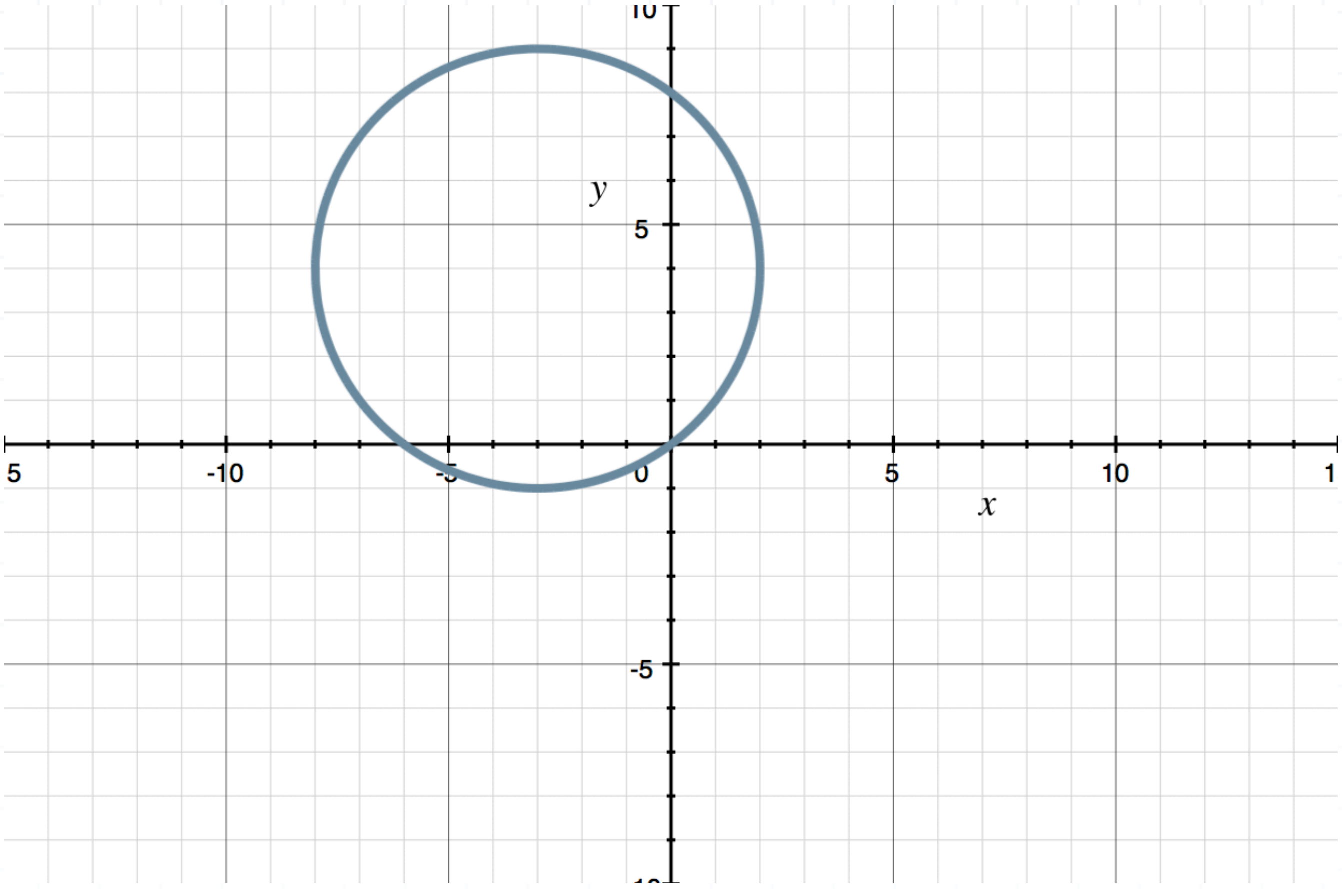
Solution:

We need to find the center and radius. The standard equation of the circle, is $(x-h)^2 + (y-k)^2 = r^2$, and if we match this up to the given equation,

$$(x+3)^2 + (y-4)^2 = 25$$

$$(x+3)^2 + (y-4)^2 = 5^2$$

we can say that the center is at $(h, k) = (-3,4)$ and the radius is 5. Therefore, the graph of the circle is



5. Graph the circle $x^2 + (y-3)^2 = 16$.

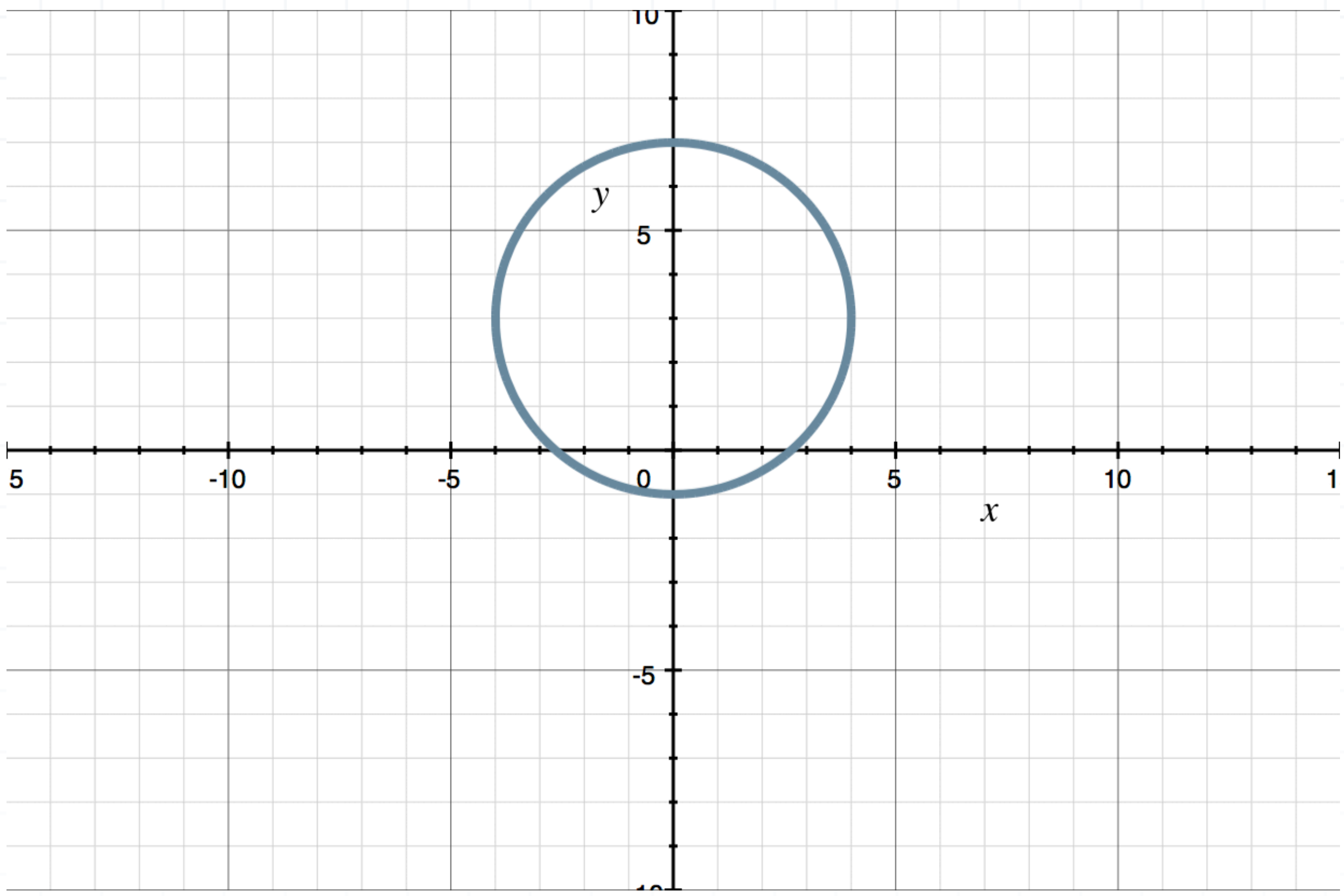
Solution:

We need to find the center and radius. The standard equation of the circle, is $(x-h)^2 + (y-k)^2 = r^2$, and if we match this up to the given equation,

$$x^2 + (y-3)^2 = 16$$

$$x^2 + (y-3)^2 = 4^2$$

we can say that the center is at $(h, k) = (0,3)$ and the radius is 4. Therefore, the graph of the circle is



6. Graph the circle $x^2 + y^2 + 2x + 2y - 14 = 0$.

Solution:

We need to find the center and radius of the circle by changing the equation of the circle into standard form, $(x - h)^2 + (y - k)^2 = r^2$, where h and k are the coordinates of the center and r is the radius.

Start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 + y^2 + 2x + 2y - 14 = 0$$

$$(x^2 + 2x) + (y^2 + 2y) = 14$$

To complete the square with respect to both x and y, take the coefficients on the x and y terms, divide them by 2, then square the results. The coefficient on x is 2, so

$$\frac{2}{2} = 1$$

$$(1)^2 = 1$$

The coefficient on y is 2, so

$$\frac{2}{2} = 1$$

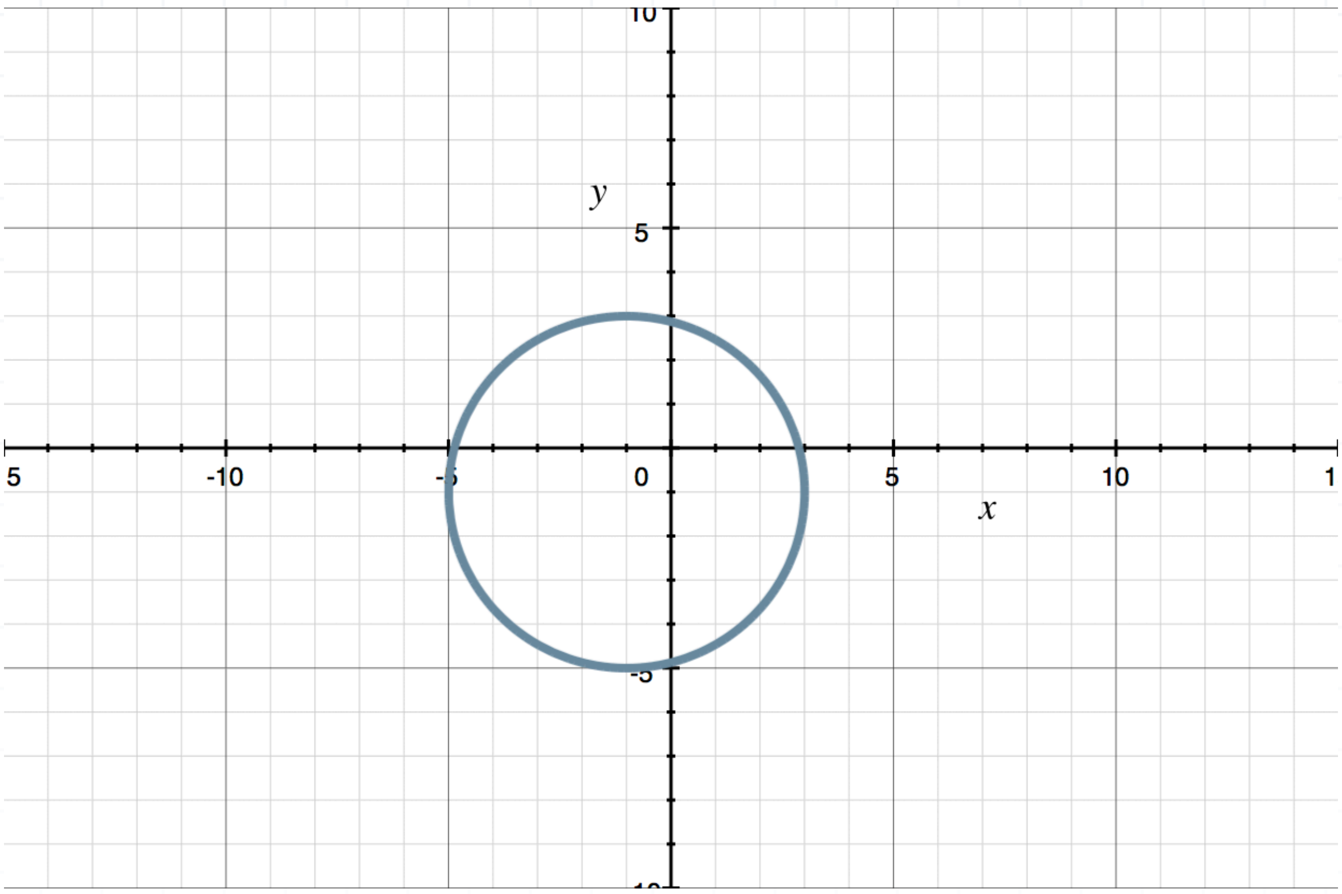
$$(1)^2 = 1$$

Add 1 and 1 to both sides of the equation. Then factor inside the parentheses and simplify the right side.

$$(x^2 + 2x + 1) + (y^2 + 2y + 1) = 14 + 1 + 1$$

$$(x + 1)^2 + (y + 1)^2 = 16$$

The center of the circle (h, k) is therefore at $(-1, -1)$ and the radius is $r = \sqrt{16} = 4$. So to graph the circle, plot the center point $(-1, -1)$, then move in any direction 4 units to get to a point on the edge of the circle.



7. Graph the circle $x^2 + y^2 - 8x - 4y + 11 = 0$.

Solution:

We need to find the center and radius of the circle by changing the equation of the circle into standard form, $(x-h)^2 + (y-k)^2 = r^2$, where h and k are the coordinates of the center and r is the radius.

Start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 + y^2 - 8x - 4y + 11 = 0$$

$$(x^2 - 8x) + (y^2 - 4y) = -11$$

To complete the square with respect to both x and y, take the coefficients on the x and y terms, divide them by 2, then square the results. The coefficient on x is -8, so

$$\frac{-8}{2} = -4$$

$$(-4)^2 = 16$$

The coefficient on y is -4, so

$$\frac{-4}{2} = -2$$

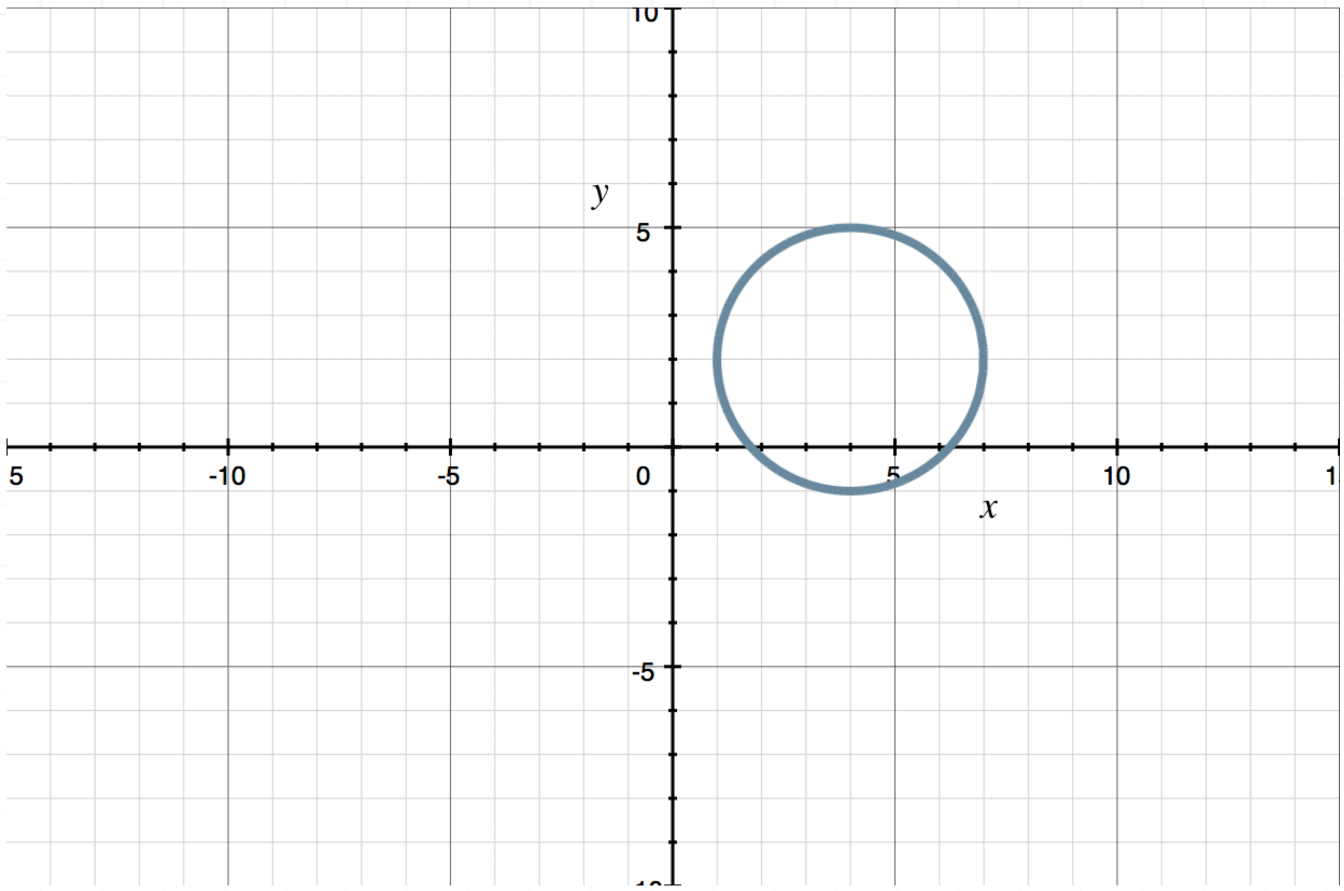
$$(-2)^2 = 4$$

Add 16 and 4 to both sides of the equation. Then factor inside the parentheses and simplify the right side.

$$(x^2 - 8x + 16) + (y^2 - 4y + 4) = -11 + 16 + 4$$

$$(x - 4)^2 + (y - 2)^2 = 9$$

The center of the circle (h, k) is therefore at $(4,2)$ and the radius is $r = \sqrt{9} = 3$. So to graph the circle, plot the center point $(4,2)$, then move in any direction 3 units to get to a point on the edge of the circle.



■ 8. Graph the circle $x^2 + y^2 + 6x - 8y - 11 = 0$.

Solution:

We need to find the center and radius of the circle by changing the equation of the circle into standard form, $(x-h)^2 + (y-k)^2 = r^2$, where h and k are the coordinates of the center and r is the radius.

Start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 + y^2 + 6x - 8y - 11 = 0$$

$$(x^2 + 6x) + (y^2 - 8y) = 11$$

To complete the square with respect to both x and y, take the coefficients on the x and y terms, divide them by 2, then square the results. The coefficient on x is 6, so

$$\frac{6}{2} = 3$$

$$(3)^2 = 9$$

The coefficient on y is -8, so

$$\frac{-8}{2} = -4$$

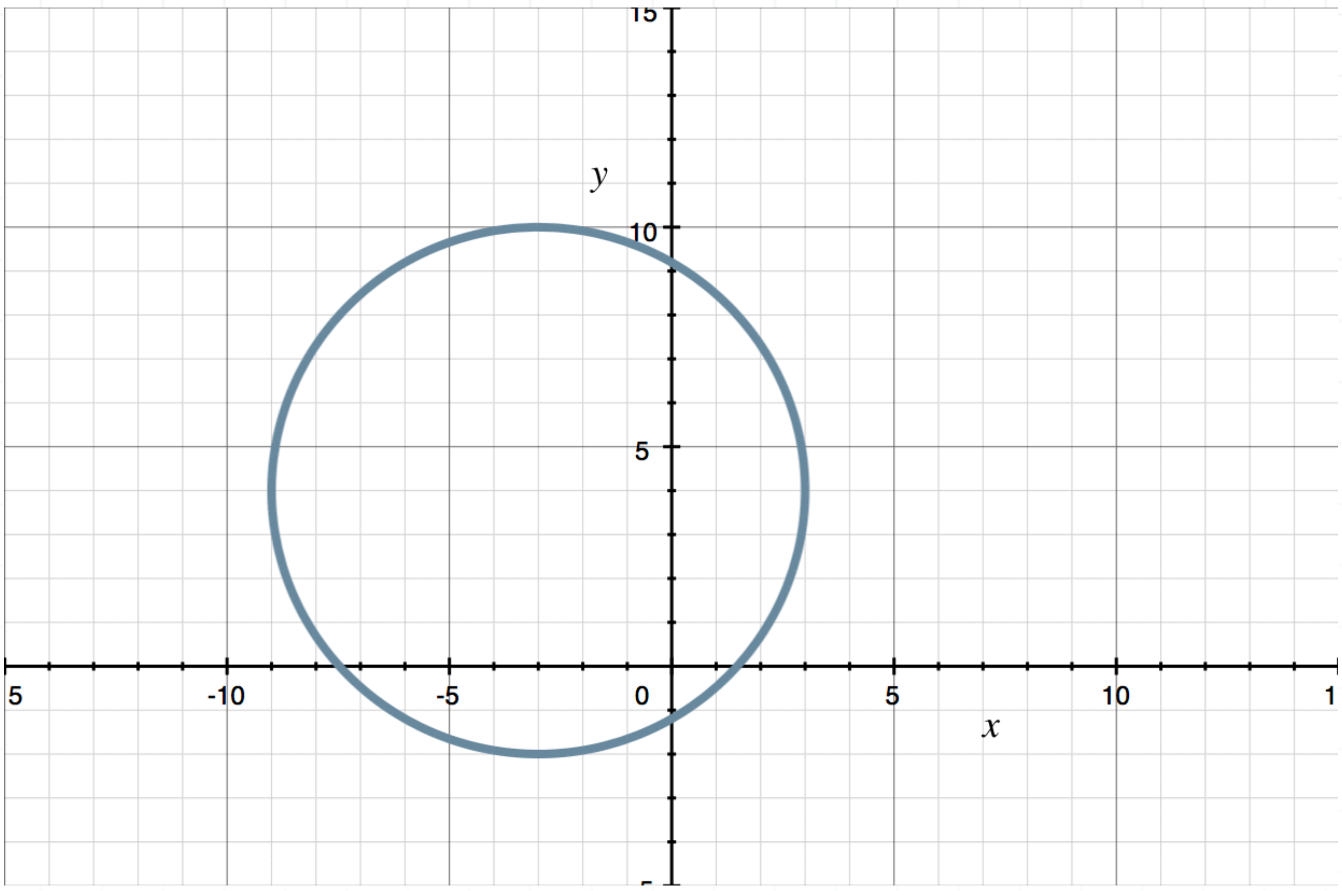
$$(-4)^2 = 16$$

Add 16 and 4 to both sides of the equation. Then factor inside the parentheses and simplify the right side.

$$(x^2 + 6x + 9) + (y^2 - 8y + 16) = 11 + 9 + 16$$

$$(x + 3)^2 + (y - 4)^2 = 36$$

The center of the circle (h, k) is therefore at $(-3,4)$ and the radius is $r = \sqrt{36} = 6$. So to graph the circle, plot the center point $(-3,4)$, then move in any direction 6 units to get to a point on the edge of the circle.


y
15
10
5
-5
5
-10
-5
0
5
10
1
x

COMBINATIONS OF FUNCTIONS

■ 1. Find $(f+g)(x)$.

$$f(x)=2x^2-x+5$$

$$g(x)=x^2+4x-7$$

Solution:

The combination $(f+g)(x)$ is the same as $f(x)+g(x)$, so we need to add the equations together.

$$(f+g)(x)=f(x)+g(x)$$

$$(f+g)(x)=2x^2-x+5+x^2+4x-7$$

Group like terms together and combine.

$$(f+g)(x)=2x^2+x^2-x+4x+5-7$$

$$(f+g)(x)=3x^2+3x-2$$

■ 2. Find $(f-g)(x)$.

$$f(x)=4x^2-2$$

$$g(x)=3x^2-5x$$

Solution:

The combination $(f-g)(x)$ is the same as $f(x)-g(x)$, so we need to find the difference.

$$(f-g)(x)=f(x)-g(x)$$

$$(f-g)(x)=4x^2-2-(3x^2-5x)$$

$$(f-g)(x)=4x^2-2-3x^2+5x$$

Group like terms together and combine.

$$(f-g)(x)=4x^2-3x^2+5x-2$$

$$(f-g)(x)=x^2+5x-2$$

■ 3. Find $(f-g)(x)$.

$$f(x)=x^2-3x+1$$

$$g(x)=2x-3$$

Solution:

The combination $(f-g)(x)$ is the same as $f(x)-g(x)$, so we need to find the difference.

$$(f-g)(x) = f(x) - g(x)$$

$$(f-g)(x) = x^2 - 3x + 1 - (2x - 3)$$

$$(f-g)(x) = x^2 - 3x + 1 - 2x + 3$$

Group like terms together and combine.

$$(f-g)(x) = x^2 - 3x - 2x + 1 + 3$$

$$(f-g)(x) = x^2 - 5x + 4$$

4. Find $(f \cdot g)(x)$.

$$f(x) = 2x - 3$$

$$g(x) = 3x^2 + 2$$

Solution:

The combination $(f \cdot g)(x)$ is the same as $f(x) \cdot g(x)$, so we need to find the product.

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$(f \cdot g)(x) = (2x - 3)(3x^2 + 2)$$

Multiply using the FOIL method.

$$(f \cdot g)(x) = 6x^3 + 4x - 9x^2 - 6$$

$$(f \cdot g)(x) = 6x^3 - 9x^2 + 4x - 6$$

5. Find $(f \cdot g)(x)$.

$$f(x) = x - 3$$

$$g(x) = x + 4$$

Solution:

The combination $(f \cdot g)(x)$ is the same as $f(x) \cdot g(x)$, so we need to find the product.

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$(f \cdot g)(x) = (x - 3)(x + 4)$$

Multiply using the FOIL method.

$$(f \cdot g)(x) = x^2 + 4x - 3x - 12$$

$$(f \cdot g)(x) = x^2 + x - 12$$

6. Find $(f \div g)(x)$.

$$f(x) = x^2 + 6x$$

$$g(x) = x$$

Solution:

The combination $(f \div g)(x)$ is the same as $f(x)/g(x)$, so we need to find the quotient.

$$(f \div g)(x) = \frac{x^2 + 6x}{x}$$

$$(f \div g)(x) = \frac{x(x + 6)}{x}$$

$$(f \div g)(x) = x + 6$$

■ 7. Find $(g \div f)(x)$.

$$f(x) = x^2 + 6x$$

$$g(x) = x$$

Solution:

The combination $(g \div f)(x)$ is the same as $g(x)/f(x)$, so we need to find the quotient.

$$(g \div f)(x) = \frac{x}{x^2 + 6x}$$

$$(g \div f)(x) = \frac{x}{x(x + 6)}$$

$$(g \div f)(x) = \frac{1}{x+6}$$

COMPOSITE FUNCTIONS

1. Find the composite function $(f \circ g)(x)$.

$$f(x) = \sqrt{2x - 1}$$

$$g(x) = 3x^2$$

Solution:

When we take the composite $(f \circ g)(x)$, we plug $g(x)$ into $f(x)$.

$$(f \circ g)(x) = f(g(x)) = \sqrt{2(3x^2) - 1}$$

$$(f \circ g)(x) = f(g(x)) = \sqrt{6x^2 - 1}$$

2. Find the composite function $(g \circ f)(x)$.

$$f(x) = \sqrt{2x - 1}$$

$$g(x) = 3x^2$$

Solution:

When we take the composite $(f \circ g)(x)$, we plug $f(x)$ into $g(x)$.

$$(g \circ f)(x) = g(f(x)) = 3(\sqrt{2x-1})^2$$

$$(g \circ f)(x) = g(f(x)) = 3(2x-1)$$

$$(g \circ f)(x) = g(f(x)) = 6x-3$$

■ 3. Find the composite function $f(g(x))$.

$$f(x) = x^2 - 4x + 3$$

$$g(x) = 2x + 1$$

Solution:

When we take the composite $f(g(x))$, we plug $g(x)$ into $f(x)$.

$$f(g(x)) = (2x+1)^2 - 4(2x+1) + 3$$

$$f(g(x)) = (2x+1)(2x+1) - 8x - 4 + 3$$

$$f(g(x)) = 4x^2 + 2x + 2x + 1 - 8x - 1$$

Group and combine like terms.

$$f(g(x)) = 4x^2 + 2x + 2x - 8x + 1 - 1$$

$$f(g(x)) = 4x^2 + 4x - 8x$$

$$f(g(x)) = 4x^2 - 4x$$

■ 4. Find the composite function $g(f(x))$.

$$f(x) = x^2 - 4x + 3$$

$$g(x) = 2x + 1$$

Solution:

When we take the composite $g(f(x))$, we plug $f(x)$ into $g(x)$.

$$g(f(x)) = 2(x^2 - 4x + 3) + 1$$

$$g(f(x)) = 2x^2 - 8x + 6 + 1$$

$$g(f(x)) = 2x^2 - 8x + 7$$

■ 5. Find the composite function $(g \circ h)(x)$.

$$g(x) = \frac{8}{x^3}$$

$$h(x) = \sqrt[3]{x + 4}$$

Solution:

When we take the composite $(g \circ h)(x)$, we plug $h(x)$ into $g(x)$.

$$(g \circ h)(x) = g(h(x)) = \frac{8}{(\sqrt[3]{x + 4})^3}$$

$$(g \circ h)(x) = g(h(x)) = \frac{8}{x+4}$$

6. Find the composite function $(h \circ g)(x)$.

$$g(x) = \frac{8}{x^3}$$

$$h(x) = \sqrt[3]{x+4}$$

Solution:

When we take the composite $(h \circ g)(x)$, we plug $g(x)$ into $h(x)$.

$$(h \circ g)(x) = h(g(x)) = \sqrt[3]{\frac{8}{x^3} + 4}$$

$$(h \circ g)(x) = h(g(x)) = \sqrt[3]{\frac{8}{x^3} + 4\frac{x^3}{x^3}}$$

$$(h \circ g)(x) = h(g(x)) = \sqrt[3]{\frac{8+4x^3}{x^3}}$$

Take the root of the numerator and denominator separately.

$$(h \circ g)(x) = h(g(x)) = \frac{\sqrt[3]{8+4x^3}}{\sqrt[3]{x^3}}$$

$$(h \circ g)(x) = h(g(x)) = \frac{\sqrt[3]{8+4x^3}}{x}$$

■ 7. Find the composite function $g(h(x))$.

$$g(x) = \frac{1}{x}$$

$$h(x) = 3x^2 - x$$

Solution:

When we take the composite $g(h(x))$, we plug $h(x)$ into $g(x)$.

$$g(h(x)) = \frac{1}{3x^2 - x}$$

■ 8. Find the composite function $h(g(x))$.

$$g(x) = \frac{1}{x}$$

$$h(x) = 3x^2 - x$$

Solution:

When we take the composite $h(g(x))$, we plug $g(x)$ into $h(x)$.

$$h(g(x)) = 3\left(\frac{1}{x}\right)^2 - \frac{1}{x}$$

$$h(g(x)) = \frac{3}{x^2} - \frac{1}{x}$$

Find a common denominator to combine the fractions.

$$h(g(x)) = \frac{3}{x^2} - \frac{1}{x}\left(\frac{x}{x}\right)$$

$$h(g(x)) = \frac{3}{x^2} - \frac{x}{x^2}$$

$$h(g(x)) = \frac{3-x}{x^2}$$

COMPOSITE FUNCTIONS, DOMAIN

1. What is the domain of $f \circ g$?

$$f(x) = x^2 - 2$$

$$g(x) = \sqrt{x+3}$$

Solution:

Find the domain of $g(x)$. Remember that you can't take the square root of negative numbers, since negative roots can't be defined by real numbers, so the expression inside the root must be positive or equal to 0.

$$x + 3 \geq 0$$

$$x \geq -3$$

Now find $f \circ g$.

$$f \circ g = (\sqrt{x+3})^2 - 2$$

$$f \circ g = x + 3 - 2$$

$$f \circ g = x + 1$$

We need to consider the domain of this composite function. The composite is a simple binomial with no domain restrictions. There's only the domain restriction that we found for $g(x)$, but all restrictions of $g(x)$

need to be included in the domain for $f(g(x))$. Therefore, the domain of $f(g(x))$ is $x \geq -3$.

■ 2. What is the domain of $f \circ g$?

$$f(x) = \frac{1}{x}$$

$$g(x) = x + 5$$

Solution:

Find the domain of $g(x)$. In this case, $g(x)$ is a simple binomial with no domain restrictions.

Now find $f \circ g$.

$$f \circ g = \frac{1}{x+5}$$

We need to consider the domain of this composite function. Remember that we can't divide by 0, which means the denominator of a fraction can never equal 0.

$$x + 5 \neq 0$$

$$x \neq -5$$

There were no restrictions on the domain of $g(x)$, so the domain of $f(g(x))$ is just $x \neq -5$.

3. What is the domain of $f \circ g$?

$$f(x) = \frac{2}{x-1}$$

$$g(x) = \sqrt{x-4}$$

Solution:

Find the domain of $g(x)$. Remember that you can't take the square root of negative numbers, since negative roots can't be defined by real numbers, so the expression inside the root must be positive or equal to 0.

$$x - 4 \geq 0$$

$$x \geq 4$$

Now find $f \circ g$.

$$f \circ g = \frac{2}{\sqrt{x-4} - 1}$$

We need to consider the domain of this composite function. Remember that we can't divide by 0, which means the denominator of a fraction can never equal 0.

$$\sqrt{x-4} - 1 \neq 0$$

$$\sqrt{x-4} \neq 1$$

$$x - 4 \neq 1$$

$$x \neq 5$$

The domain of the composite has to account for domain restrictions on $g(x)$ and the composite function $f \circ g$ itself, so the domain of $f(g(x))$ is $x \geq 4$, $x \neq 5$.

■ 4. What is the domain of $f \circ g$?

$$f(x) = \frac{1}{x} + 4$$

$$g(x) = \frac{3}{2x - 7}$$

Solution:

Find the domain of $g(x)$. Remember that we can't divide by 0, which means the denominator of a fraction can never equal 0.

$$2x - 7 \neq 0$$

$$2x \neq 7$$

$$x \neq \frac{7}{2}$$

Now find $f \circ g$.

$$f \circ g = \frac{1}{\frac{3}{2x-7}} + 4$$

$$f \circ g = \frac{2x-7}{3} + 4$$

$$f \circ g = \frac{2x-7}{3} + \frac{12}{3}$$

$$f \circ g = \frac{2x-7+12}{3}$$

$$f \circ g = \frac{2x+5}{3}$$

We need to consider the domain of this composite function. But there are no domain restrictions on $f \circ g$, so the only restriction we need to consider is the one on $g(x)$. The domain of $f(g(x))$ is $x \neq 7/2$.

■ 5. What is the domain of $f \circ g$?

$$f(x) = \frac{2}{x-3}$$

$$g(x) = \frac{4}{x+2}$$

Solution:

Find the domain of $g(x)$. Remember that we can't divide by 0, which means the denominator of a fraction can never equal 0.

$$x+2\neq 0$$

$$x\neq -2$$

Now find $f\circ g$.

$$f\circ g=\frac{2}{\frac{4}{x+2}-3}$$

$$f\circ g=\frac{2}{\frac{4}{x+2}-3\left(\frac{x+2}{x+2}\right)}$$

$$f\circ g=\frac{2}{\frac{4}{x+2}-\frac{3x+6}{x+2}}$$

Combine fractions within the denominator.

$$f\circ g=\frac{2}{\frac{4-3x-6}{x+2}}$$

$$f\circ g=\frac{2}{\frac{-3x-2}{x+2}}$$

$$f\circ g=\frac{2(x+2)}{-3x-2}$$

$$f\circ g=\frac{2x+4}{-3x-2}$$

$$f \circ g = -\frac{2x+4}{3x+2}$$

We need to consider the domain of this composite function. Remember that we can't divide by 0, which means the denominator of a fraction can never equal 0.

$$3x + 2 \neq 0$$

$$3x \neq -2$$

$$x \neq -\frac{2}{3}$$

Putting this restriction together with the one we found for $g(x)$, we can say that the domain of $f(g(x))$ is $x \neq -2, \ -2/3$.

■ 6. What is the domain of $f \circ g$?

$$f(x) = \frac{1}{x^2 - 3}$$

$$g(x) = \sqrt{x - 1}$$

Solution:

Find the domain of $g(x)$. Remember that you can't take the square root of negative numbers, since negative roots can't be defined by real numbers, so the expression inside the root must be positive or equal to 0.

$$x - 1 \geq 0$$

$$x \geq 1$$

Now find $f \circ g$.

$$f \circ g = \frac{1}{(\sqrt{x-1})^2 - 3}$$

$$f \circ g = \frac{1}{x - 1 - 3}$$

$$f \circ g = \frac{1}{x - 4}$$

We need to consider the domain of this composite function. Remember that we can't divide by 0, which means the denominator of a fraction can never equal 0.

$$x - 4 \neq 0$$

$$x \neq 4$$

Putting this restriction together with the one we found for $g(x)$, we can say that the domain of $f(g(x))$ is $x \geq 1$, $x \neq 4$.

7. What is the domain of $f \circ g$?

$$f(x) = 2x^2 - x + 1$$

$$g(x) = x - 3$$

Solution:

Find the domain of $g(x)$. In this case $g(x)$ is a simple binomial with no domain restrictions.

Now find $f \circ g$.

$$f \circ g = 2(x-3)^2 - (x-3) + 1$$

$$f \circ g = 2(x^2 - 6x + 9) - x + 3 + 1$$

$$f \circ g = 2x^2 - 12x + 18 - x + 4$$

$$f \circ g = 2x^2 - 13x + 22$$

We need to consider the domain of this composite function. But there are no domain restrictions on $f \circ g$. And, since there are also no domain restrictions for $g(x)$, the domain of $f(g(x))$ is all real numbers.

■ 8. What is the domain of $f \circ g$?

$$f(x) = x^2 + 4x - 10$$

$$g(x) = x + 6$$

Solution:

Find the domain of $g(x)$. In this case $g(x)$ is a simple binomial with no domain restrictions.

Now find $f \circ g$.

$$f \circ g = (x+6)^2 + 4(x+6) - 10$$

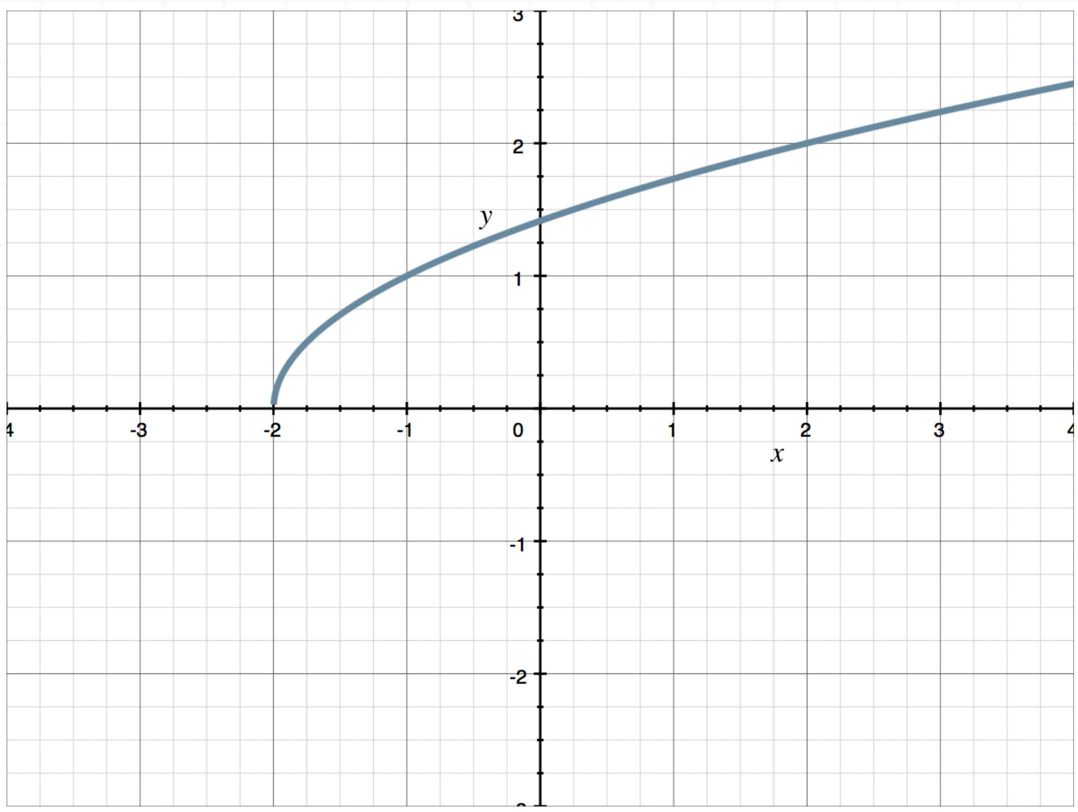
$$f \circ g = x^2 + 12x + 36 + 4x + 24 - 10$$

$$f \circ g = x^2 + 16x + 50$$

There are no domain restrictions for $f \circ g$. And, since there are also no domain restrictions for $g(x)$, the domain of $f(g(x))$ is all real numbers.

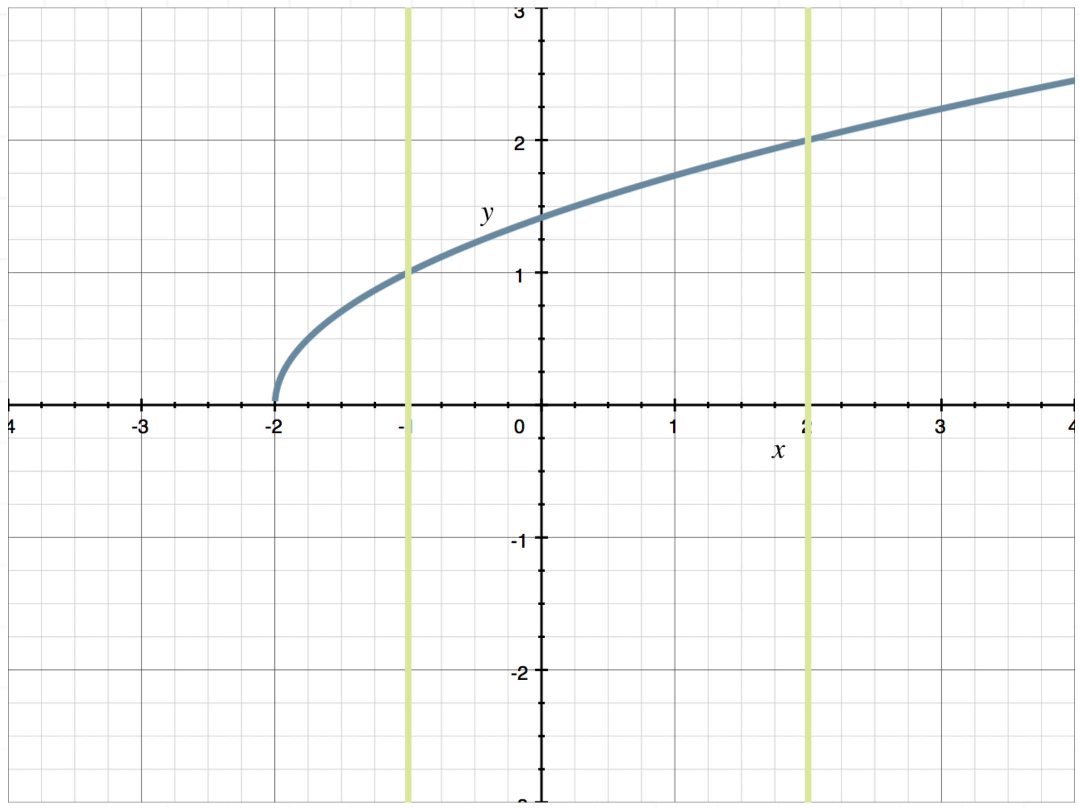
HORIZONTAL LINE TEST

1. Does the graph represent a one-to-one function?

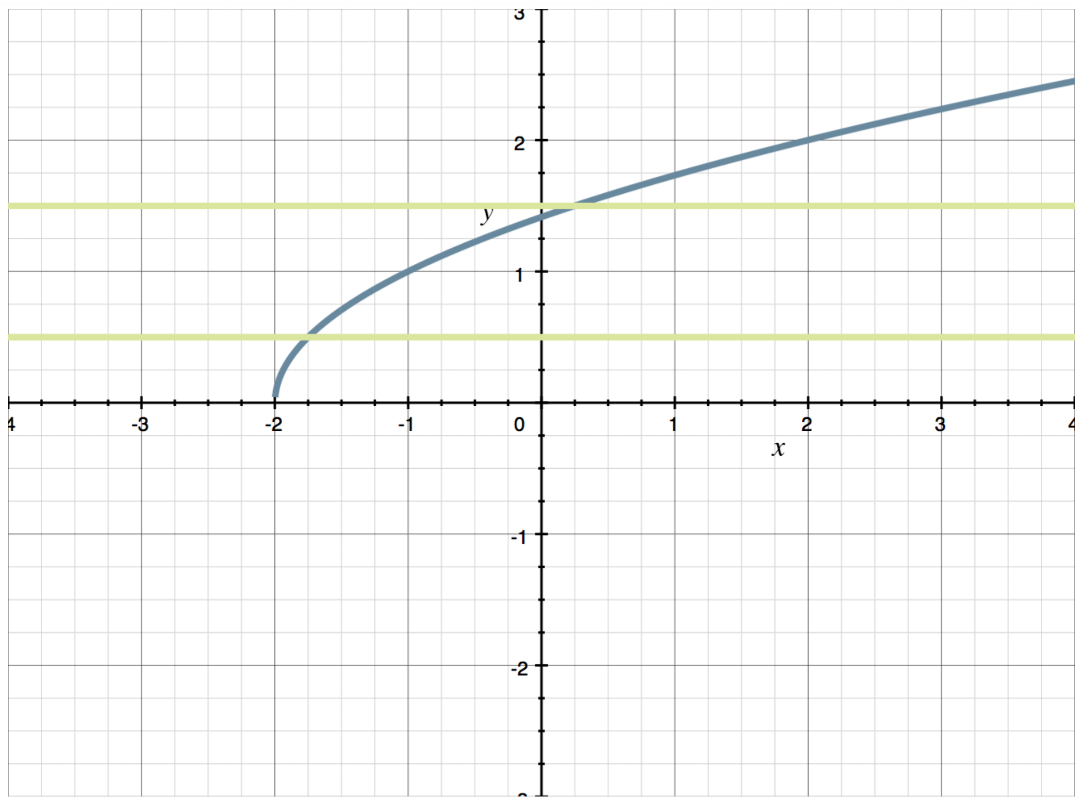


Solution:

First, see if the graph passes the Vertical Line Test. If any vertical line passes through the graph at two or more points, it will fail the Vertical Line Test and isn't a function.

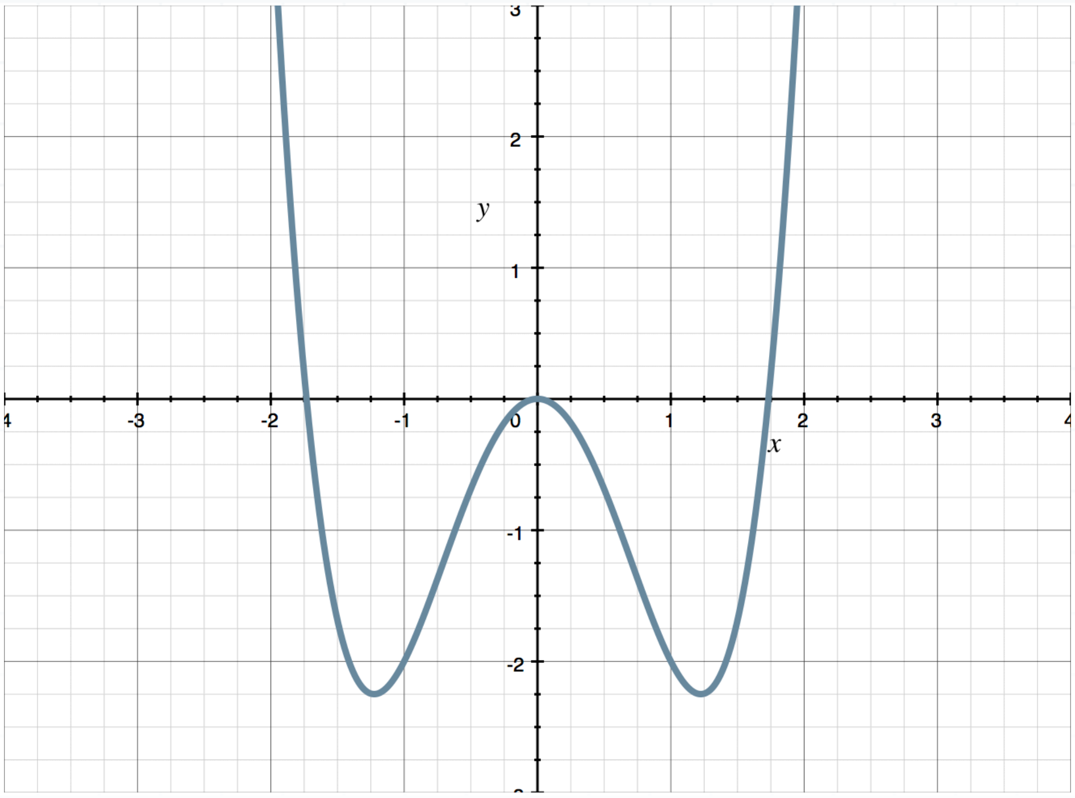


The graph passes the Vertical Line Test, which means that it represents a function. Next, see if the graph passes the Horizontal Line Test. If any horizontal line passes through the graph at two or more points, it will fail the Horizontal Line Test and isn't a one-to-one function.



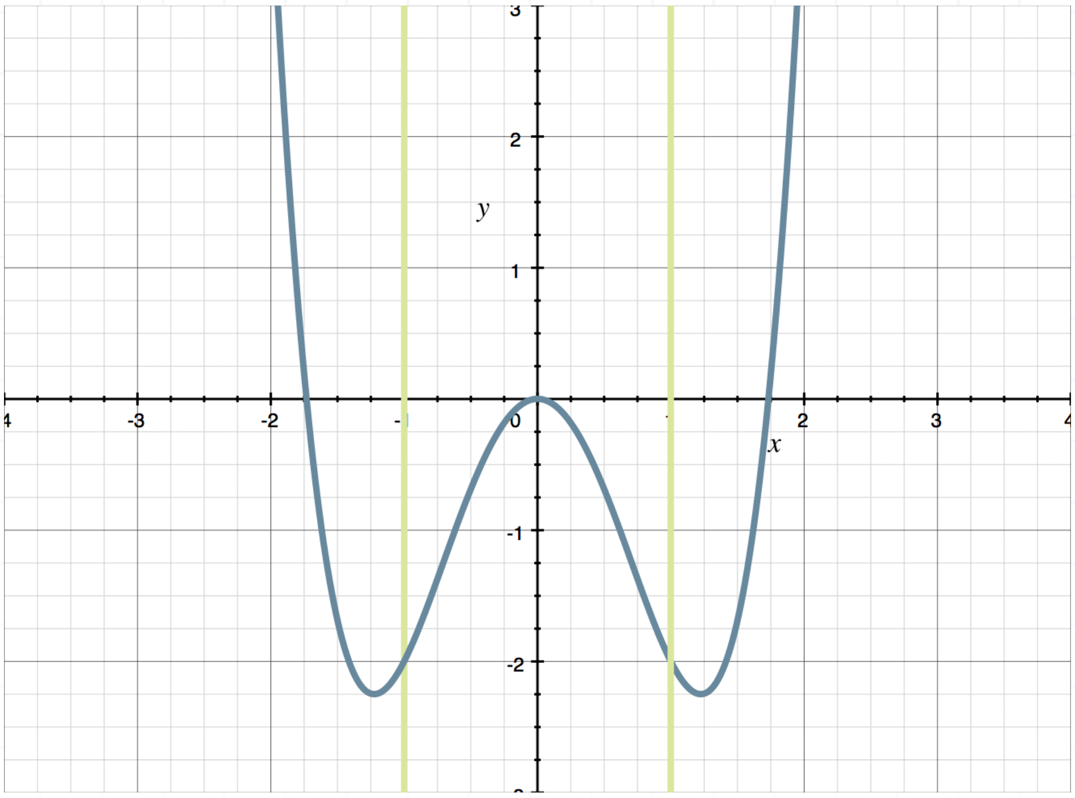
The graph passes the Horizontal Line Test, so it's a one-to-one function.

2. Does the graph represent a one-to-one function?

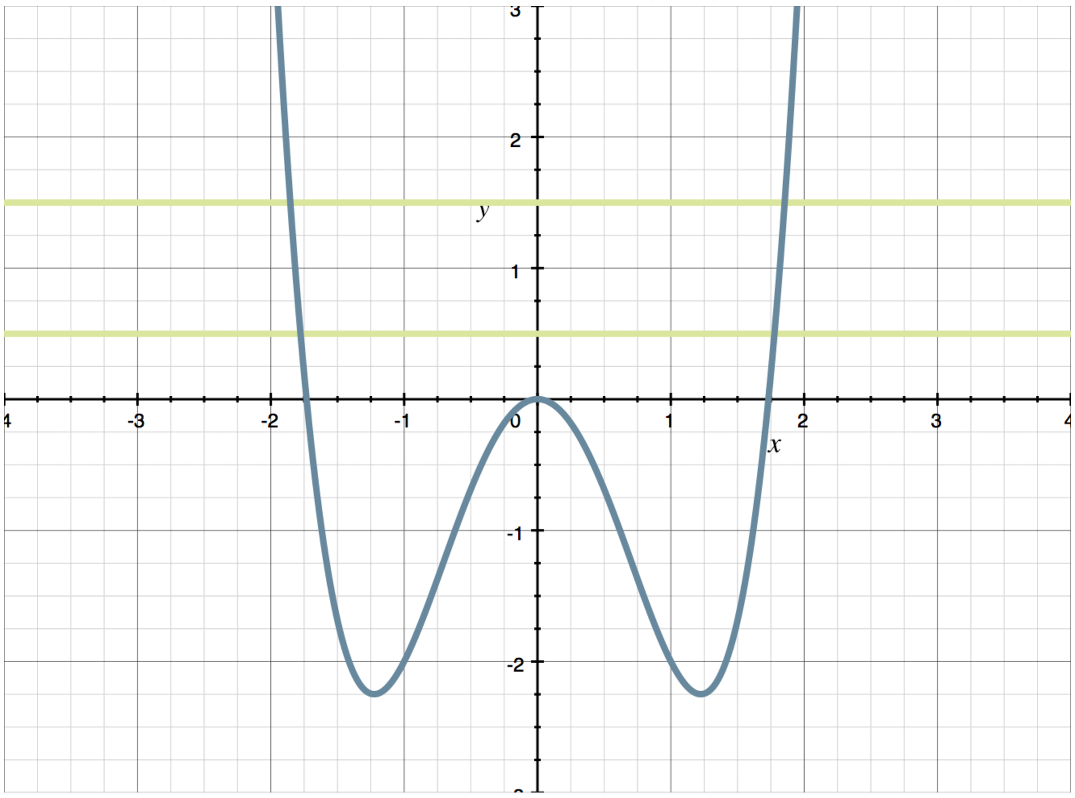


Solution:

First, see if the graph passes the Vertical Line Test. If any vertical line passes through the graph at two or more points, it will fail the Vertical Line Test and isn't a function.

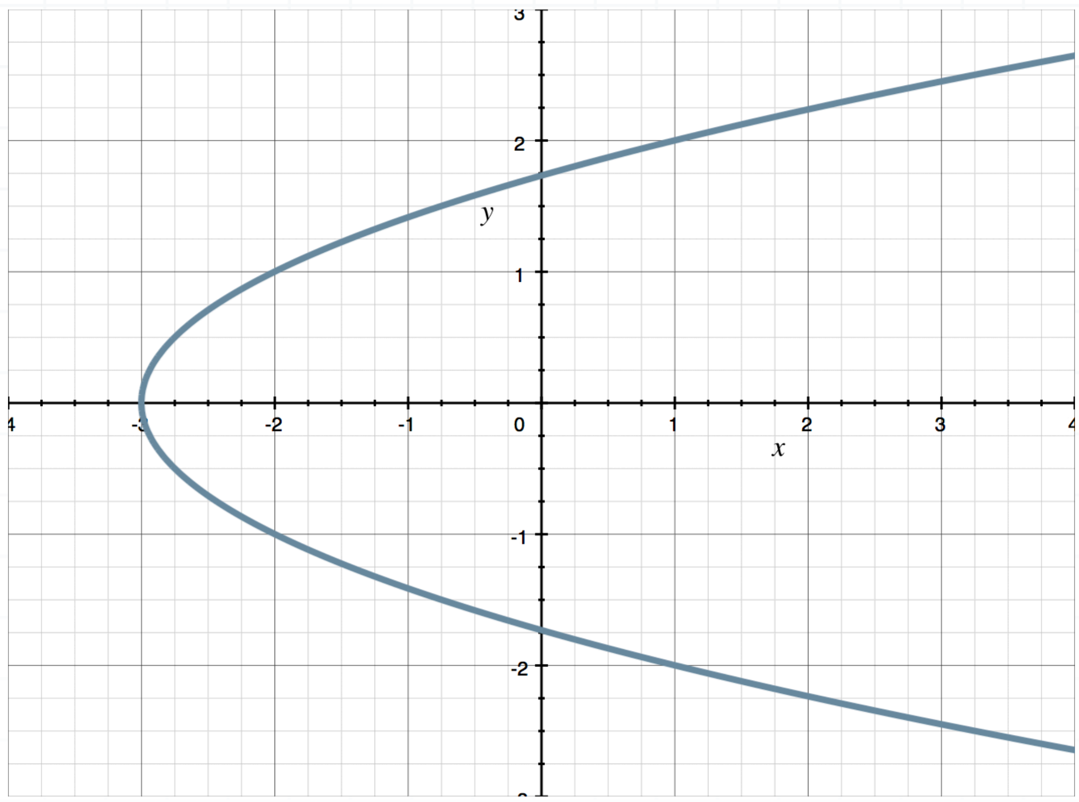


The graph passes the Vertical Line Test, which means that it represents a function. Next, see if the graph passes the Horizontal Line Test. If any horizontal line passes through the graph at two or more points, it will fail the Horizontal Line Test and isn't a one-to-one function.



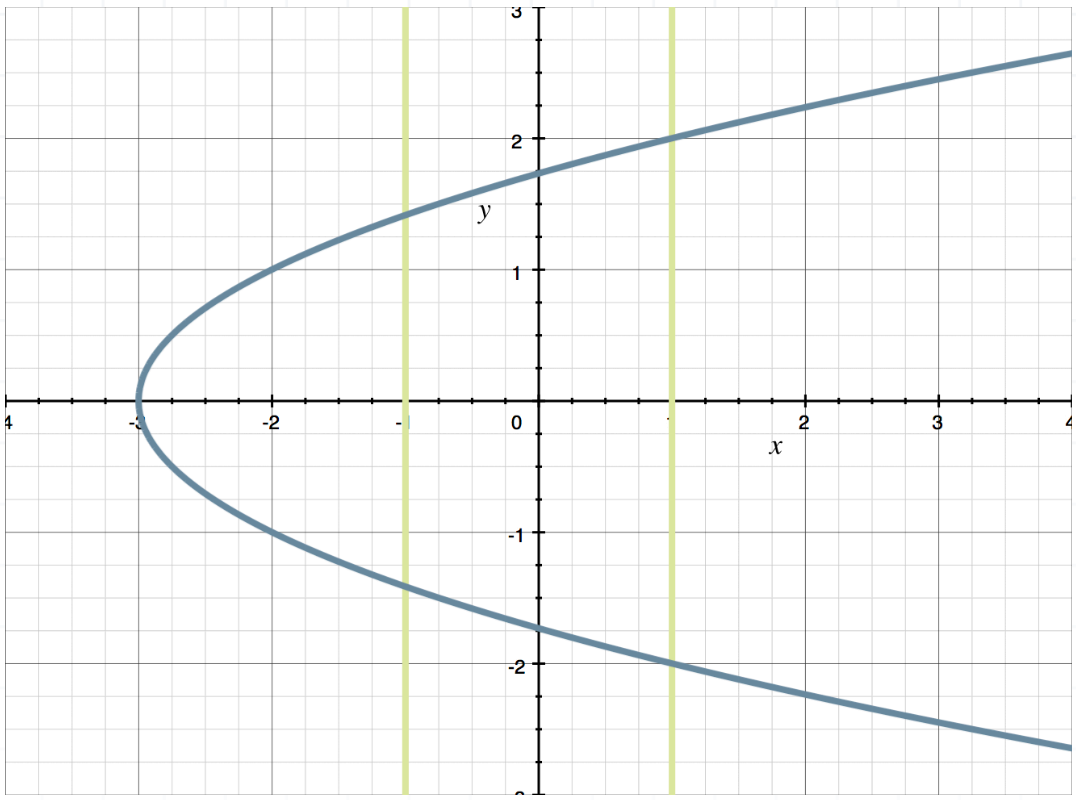
The graph does not pass the Horizontal Line Test, so it's still a function, just not a one-to-one function.

3. Does the graph represent a one-to-one function?



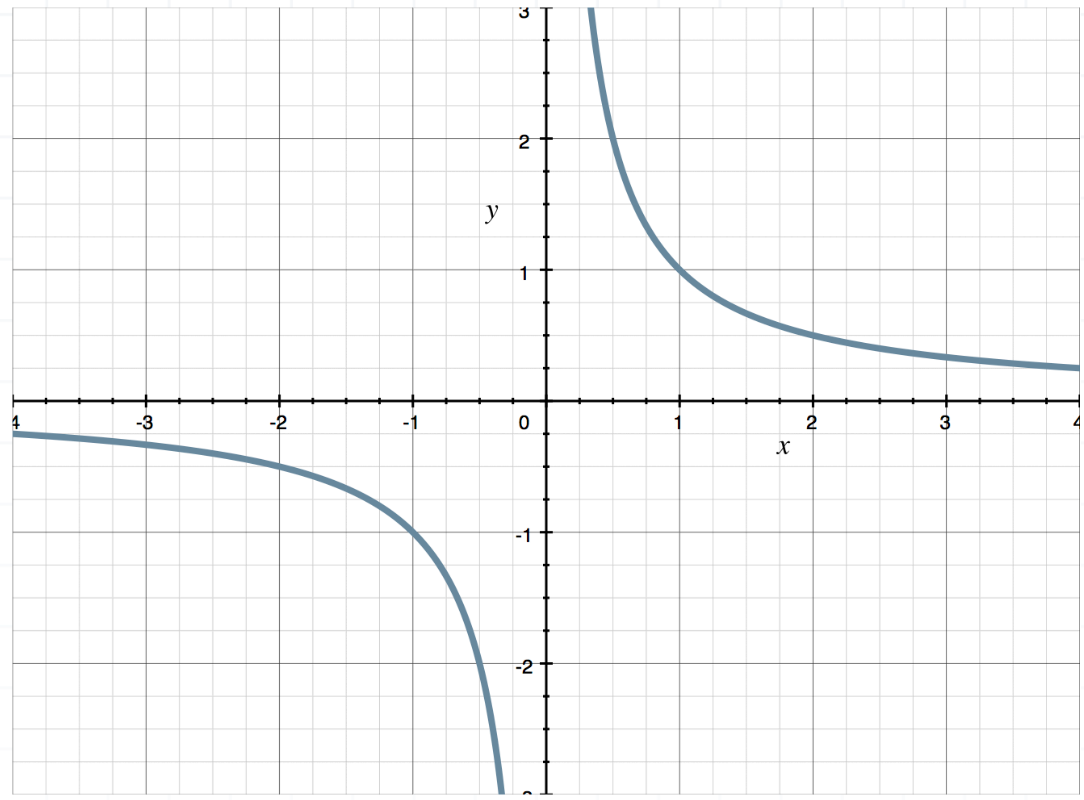
Solution:

First, see if the graph passes the Vertical Line Test. If any vertical line passes through the graph at two or more points, it will fail the Vertical Line Test and isn't a function.



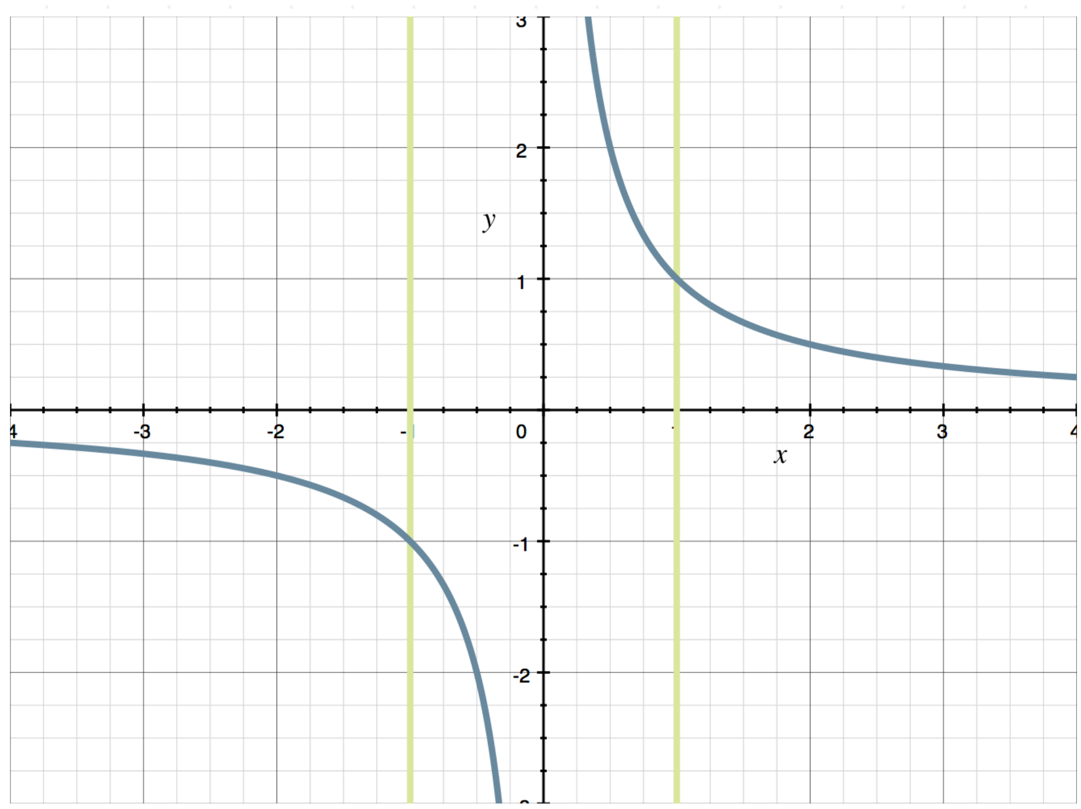
Clearly, a vertical line can cross the graph at two or more points. This means that the graph fails the Vertical Line Test and does not represent a function. Since the graph isn't a function, it can't be a one-to-one function, even if it passes the Horizontal Line Test.

■ 4. Does the graph represent a one-to-one function?

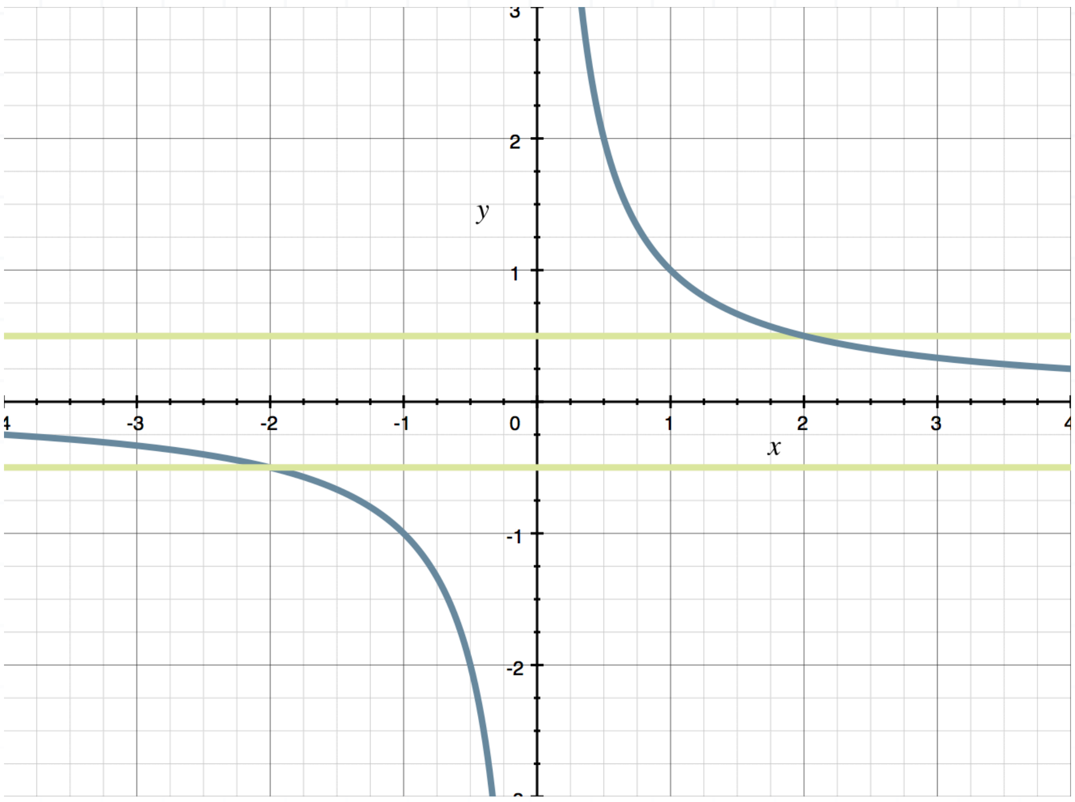


Solution:

First, see if the graph passes the Vertical Line Test. If any vertical line passes through the graph at two or more points, it will fail the Vertical Line Test and isn't a function.



The graph passes the Vertical Line Test, which means that it represents a function. Next, see if the graph passes the Horizontal Line Test. If any horizontal line passes through the graph at two or more points, it will fail the Horizontal Line Test and isn't a one-to-one function.



The graph passes the Horizontal Line Test, so it's a one-to-one function.

5. Show that the function is one-to-one by showing that $f(a)=f(b)$ leads to $a=b$.

$$f(x)=3x-4$$

Solution:

Start by replacing x with a and b.

$$3a - 4$$

$$3b - 4$$

Set these equal to one another, then simplify.

$$3a - 4 = 3b - 4$$

$$3a = 3b$$

$$a = b$$

Since $a = b$, $f(x)$ is a one-to-one function.

■ 6. Show that the function is one-to-one by showing that $f(a) = f(b)$ leads to $a = b$.

$$f(x) = \frac{x+1}{x-5}$$

Solution:

Start by replacing x with a and b.

$$\frac{a+1}{a-5}$$

$$\frac{b+1}{b-5}$$

Set these equal to one another, then simplify.

$$\frac{a+1}{a-5} = \frac{b+1}{b-5}$$

$$(a+1)(b-5) = (b+1)(a-5)$$

$$ab - 5a + b - 5 = ab - 5b + a - 5$$

$$ab - 5a + b = ab - 5b + a$$

$$-5a + b = -5b + a$$

$$6b = 6a$$

$$b = a$$

Since $a = b$, $f(x)$ is a one-to-one function.

■ 7. Show that the function is not one-to-one by showing that $f(a) = f(b)$ does not lead to $a = b$.

$$f(x) = x^2 - 6$$

Solution:

All we need is one counterexample to show that $f(a) = f(b)$ doesn't imply that $a = b$. Let's use $a = 1$ and $b = -1$.

$$f(a) = f(1) = 1^2 - 6 = 1 - 6 = -5$$

$$f(b) = f(-1) = (-1)^2 - 6 = 1 - 6 = -5$$

Since we get the same value for both functions, $f(a) = f(b)$, but we used different values to get the same answer, $a \neq b$, the function is not one-to-one.

8. Show that the function is not one-to-one by showing that $f(a) = f(b)$ does not lead to $a = b$.

$$f(x) = (x + 3)(x - 2)$$

Solution:

All we need is one counterexample to show that $f(a) = f(b)$ doesn't imply that $a = b$. Let's use $a = -3$ and $b = 2$.

$$f(a) = f(-3) = (-3 + 3)(-3 - 2) = 0(-5) = 0$$

$$f(b) = f(2) = (2 + 3)(2 - 2) = 5(0) = 0$$

Since we get the same value for both functions, $f(a) = f(b)$, but we used different values to get the same answer, $a \neq b$, the function is not one-to-one.

INVERSE FUNCTIONS

■ 1. What is the inverse of the function?

$$f(x) = \frac{1}{2}x - 3$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{1}{2}x - 3$$

Switch x and y, then solve for y.

$$x = \frac{1}{2}y - 3$$

$$x + 3 = \frac{1}{2}y$$

$$2x + 6 = y$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = 2x + 6$$

■ 2. What is the inverse of the function?

$$f(x) = -4x + 5$$

Solution:

Start by replacing $f(x)$ with y.

$$y = -4x + 5$$

Switch x and y, then solve for y.

$$x = -4y + 5$$

$$x - 5 = -4y$$

$$-\frac{1}{4}x + \frac{5}{4} = y$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{1}{4}x + \frac{5}{4}$$

3. What is the inverse of the function?

$$f(x) = \frac{x}{x+2}$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{x}{x+2}$$

Switch x and y, then solve for y.

$$x = \frac{y}{y+2}$$

$$x(y+2) = y$$

$$xy + 2x = y$$

$$2x = y - xy$$

$$2x = y(1-x)$$

$$\frac{2x}{1-x} = y$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = \frac{2x}{1-x}$$

■ 4. What is the inverse of the function?

$$f(x) = \frac{2x}{x-5}$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{2x}{x-5}$$

Switch x and y, then solve for y.

$$x = \frac{2y}{y-5}$$

$$x(y-5) = 2y$$

$$xy - 5x = 2y$$

$$-5x = 2y - xy$$

$$-5x = y(2-x)$$

$$-\frac{5x}{2-x} = y$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{5x}{2-x}$$

■ 5. What is the inverse of the function?

$$f(x) = \frac{1}{x} + 3$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{1}{x} + 3$$

Switch x and y, then solve for y.

$$x = \frac{1}{y} + 3$$

$$x - 3 = \frac{1}{y}$$

$$y(x - 3) = 1$$

$$y = \frac{1}{x - 3}$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = \frac{1}{x - 3}$$

■ 6. What is the inverse of the function?

$$f(x) = -\frac{3}{x - 2} - 4$$

Solution:

Start by replacing $f(x)$ with y.

$$y = -\frac{3}{x-2} - 4$$

Switch x and y, then solve for y.

$$x = -\frac{3}{y-2} - 4$$

$$x + 4 = -\frac{3}{y-2}$$

$$(y-2)(x+4) = -3$$

$$y - 2 = -\frac{3}{x+4}$$

$$y = -\frac{3}{x+4} + 2$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{3}{x+4} + 2$$

7. What is the inverse of the function?

$$f(x) = \frac{x-2}{x+3}$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{x-2}{x+3}$$

Switch x and y, then solve for y.

$$x = \frac{y-2}{y+3}$$

$$x(y+3) = y-2$$

$$xy + 3x = y-2$$

$$xy - y = -3x - 2$$

$$y(x-1) = -3x - 2$$

$$y(x-1) = -(3x+2)$$

$$y = -\frac{3x+2}{x-1}$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{3x+2}{x-1}$$

■ 8. What is the inverse of the function?

$$f(x) = \frac{5+x}{4-x}$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{5+x}{4-x}$$

Switch x and y, then solve for y.

$$x = \frac{5+y}{4-y}$$

$$x(4-y) = 5+y$$

$$4x - xy = 5+y$$

$$-y - xy = 5 - 4x$$

$$-y(1+x) = 5 - 4x$$

$$-y = \frac{5-4x}{1+x}$$

$$y = -\frac{5-4x}{1+x}$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{5-4x}{1+x}$$

FINDING THE EQUATION OF A LINE FROM POINTS ON ITS INVERSE

1. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(1) = -2$$

$$f^{-1}(-3) = -1$$

Solution:

$(1,-2)$ and $(-3,-1)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(-2,1)$$

$$(-1,-3)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-1}{-1-(-2)} = \frac{-4}{1} = -4$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(-2,1)$ and $(-1,-3)$. We'll use $(-2,1)$.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -4(x - (-2))$$

$$y - 1 = -4(x + 2)$$

$$y - 1 = -4x - 8$$

$$y = -4x - 7$$

So $f(x)$ is given by

$$f(x) = -4x - 7$$

■ 2. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(0) = 3$$

$$f^{-1}(-2) = 1$$

Solution:

$(0,3)$ and $(-2,1)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y -values in those points in order to get points on $f(x)$.

$$(3,0)$$

$$(1,-2)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 0}{1 - 3} = \frac{-2}{-2} = 1$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(3,0)$ and $(1,-2)$. We'll use $(3,0)$.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 1(x - 3)$$

$$y = x - 3$$

So $f(x)$ is given by

$$f(x) = x - 3$$

3. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(2) = 5$$

$$f^{-1}(4) = 9$$

Solution:

$(2,5)$ and $(4,9)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y -values in those points in order to get points on $f(x)$.

$$(5,2)$$

$$(9,4)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{9 - 5} = \frac{2}{4} = \frac{1}{2}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(5,2)$ and $(9,4)$. We'll use $(5,2)$.

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{2}(x - 5)$$

$$y - 2 = \frac{1}{2}x - \frac{5}{2}$$

$$y = \frac{1}{2}x - \frac{5}{2} + \frac{4}{2}$$

$$y = \frac{1}{2}x - \frac{1}{2}$$

So $f(x)$ is given by

$$f(x) = \frac{1}{2}x - \frac{1}{2}$$

■ 4. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(-3) = 2$$

$$f^{-1}(1) = 4$$

Solution:

$(-3,2)$ and $(1,4)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y -values in those points in order to get points on $f(x)$.

$$(2,-3)$$

$$(4,1)$$

Find the slope between these points to find the slope of $f(x)$.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-(-3)}{4-2}=\frac{4}{2}=2$$

Use point-slope form $y-y_1=m(x-x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(2,-3)$ and $(4,1)$. We'll use $(4,1)$.

$$y-y_1=m(x-x_1)$$

$$y-1=2(x-4)$$

$$y-1=2x-8$$

$$y=2x-7$$

So $f(x)$ is given by

$$f(x)=2x-7$$

■ 5. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(-4) = 7$$

$$f^{-1}(-1) = 14$$

Solution:

$(-4,7)$ and $(-1,14)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(7, -4)$$

$$(14, -1)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - (-4)}{14 - 7} = \frac{3}{7}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(7, -4)$ and $(14, -1)$. We'll use $(14, -1)$.

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = \frac{3}{7}(x - 14)$$

$$y + 1 = \frac{3}{7}x - 6$$

$$y = \frac{3}{7}x - 7$$

So $f(x)$ is given by

$$f(x) = \frac{3}{7}x - 7$$

6. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(5) = -4$$

$$f^{-1}(10) = -12$$

Solution:

$(5, -4)$ and $(10, -12)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(-4,5)$$

$$(-12,10)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 5}{-12 - (-4)} = \frac{5}{-8} = -\frac{5}{8}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(-4,5)$ and $(-12,10)$. We'll use $(-4,5)$.

$$y - y_1 = m(x - x_1)$$

$$y - 5 = -\frac{5}{8}(x - (-4))$$

$$y - 5 = -\frac{5}{8}(x + 4)$$

$$y - 5 = -\frac{5}{8}x - \frac{5}{2}$$

$$y = -\frac{5}{8}x - \frac{5}{2} + \frac{10}{2}$$

$$y = -\frac{5}{8}x + \frac{5}{2}$$

So $f(x)$ is given by

$$f(x) = -\frac{5}{8}x + \frac{5}{2}$$

■ 7. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(-3) = -4$$

$$f^{-1}(3) = 12$$

Solution:

$(-3, -4)$ and $(3,12)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$(-4, -3)$

$(12,3)$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-3)}{12 - (-4)} = \frac{6}{16} = \frac{3}{8}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(-4, -3)$ and $(12,3)$. We'll use $(-4, -3)$.

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = \frac{3}{8}(x - (-4))$$

$$y + 3 = \frac{3}{8}(x + 4)$$

$$y + 3 = \frac{3}{8}x + \frac{3}{2}$$

$$y = \frac{3}{8}x + \frac{3}{2} - \frac{6}{2}$$

$$y = \frac{3}{8}x - \frac{3}{2}$$

So $f(x)$ is given by

$$f(x) = \frac{3}{8}x - \frac{3}{2}$$

■ 8. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(1) = 3$$

$$f^{-1}(2) = 6$$

Solution:

$(1,3)$ and $(2,6)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y -values in those points in order to get points on $f(x)$.

$$(3,1)$$

$$(6,2)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{6-3} = \frac{1}{3}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(3,1)$ and $(6,2)$. We'll use $(3,1)$.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{1}{3}(x - 3)$$

$$y - 1 = \frac{1}{3}x - 1$$

$$y = \frac{1}{3}x$$

So $f(x)$ is given by

$$f(x) = \frac{1}{3}x$$

LAWS OF LOGARITHMS

1. Write the expression as a single logarithm. Solve if possible.

$$\log_2 2 + \log_2 4$$

Solution:

Use the product rule

$$\log_a x + \log_a y = \log_a(xy)$$

to rewrite the expression as a single logarithm.

$$\log_2 2 + \log_2 4$$

$$\log_2 8$$

Simplify further by converting the logarithm into an exponential expression using the rule, if $\log_a x = y$ then $a^y = x$.

$$\log_2 8 = y$$

$$2^y = 8$$

$$y = 3$$

So the value of the logarithm is 3.

$$\log_2 2 + \log_2 4 = 3$$

2. Write the expression as a single logarithm. Solve if possible.

$$\log_3 216 - \log_3 24$$

Solution:

Use the quotient rule

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

to rewrite the expression as a single logarithm.

$$\log_3 216 - \log_3 24$$

$$\log_3 \frac{216}{24}$$

$$\log_3 9$$

Simplify further by converting the logarithm into an exponential expression using the rule, if $\log_a x = y$ then $a^y = x$.

$$\log_3 9 = y$$

$$3^y = 9$$

$$y = 2$$

So the value of the logarithm is 2.

$$\log_3 216 - \log_3 24 = 2$$

■ 3. Write the expression as a single logarithm. Solve if possible.

$$\log_4 10 - 3\log_4 2$$

Solution:

Use the power rule

$$n\log_a x = \log_a x^n$$

to rewrite the expression.

$$\log_4 10 - 3\log_4 2$$

$$\log_4 10 - \log_4 2^3$$

$$\log_4 10 - \log_4 8$$

Use the quotient rule

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

to rewrite the expression as a single logarithm.

$$\log_4 10 - \log_4 8$$

$$\log_4 \frac{10}{8}$$

$$\log_4 \frac{5}{4}$$

■ 4. Write the expression as a single logarithm. Solve if possible.

$$2\log_7 4 + 3\log_7 5$$

Solution:

Use the power rule

$$n\log_a x = \log_a x^n$$

to rewrite the expression.

$$2\log_7 4 + 3\log_7 5$$

$$\log_7 4^2 + \log_7 5^3$$

$$\log_7 16 + \log_7 125$$

Use the product rule

$$\log_a x + \log_a y = \log_a(xy)$$

to rewrite the expression as a single logarithm.

$$\log_7 16 + \log_7 125$$

$$\log_7 2{,}000$$

5. Solve the equation.

$$\log_a 2 + \log_a 4 = \log_a(x+2)$$

Solution:

Use the product rule

$$\log_a x + \log_a y = \log_a(xy)$$

to rewrite the left side of the expression.

$$\log_a 2 + \log_a 4 = \log_a(x+2)$$

$$\log_a 8 = \log_a(x+2)$$

Since the logarithms are equal and have the same base, 8 must equal $x+2$.

$$x + 2 = 8$$

$$x = 6$$

6. Solve the equation.

$$\log_4(x+5) - \log_4(x-2) = \log_4 3$$

Solution:

Use the quotient rule

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

to rewrite the left side of the expression.

$$\log_4(x+5) - \log_4(x-2) = \log_4 3$$

$$\log_4 \left(\frac{x+5}{x-2}\right) = \log_4 3$$

Since the logarithms are equal and have the same base, $(x+5)/(x-2)$ must equal 3.

$$\frac{x+5}{x-2} = 3$$

$$x + 5 = 3(x-2)$$

$$x + 5 = 3x - 6$$

$$11 = 2x$$

$$x = \frac{11}{2}$$

7. Solve the equation.

$$2\log_b x = \log_b 49$$

Solution:

Use the power rule

$$n\log_a x = \log_a x^n$$

to rewrite the left side of the expression.

$$2\log_b x = \log_b 49$$

$$\log_b x^2 = \log_b 49$$

Since the logarithms are equal and have the same base, x^2 must equal 49.

$$x^2 = 49$$

$$x = 7$$

8. Solve the equation.

$$\log_{12} x = \frac{3}{2}\log_{12} 16$$

Solution:

Use the power rule

$$n \log_a x = \log_a x^n$$

to rewrite the right side of the expression.

$$\log_{12} x = \frac{3}{2} \log_{12} 16$$

$$\log_{12} x = \log_{12} 16^{\frac{3}{2}}$$

$$\log_{12} x = \log_{12} 4^3$$

Since the logarithms are equal and have the same base, x must equal 4^3.

$$x = 4^3$$

$$x = 64$$

QUADRATIC FORMULA

1. Solve for x using the quadratic formula.

$$4x^2 - 8x - 15 = 0$$

Solution:

The quadratic formula for the expression is

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(-15)}}{2(4)}$$

$$x = \frac{8 \pm \sqrt{64 + 240}}{8}$$

$$x = \frac{8 \pm \sqrt{304}}{8}$$

$$x = \frac{8 \pm 4\sqrt{19}}{8}$$

$$x = \frac{2 \pm \sqrt{19}}{2}$$

2. Write the quadratic formula for the following quadratic equation.

$$x^2 - 5x - 24 = 0$$

Solution:

The quadratic formula for the expression is

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-24)}}{2(1)}$$

We could continue to simplify to solve for the roots.

$$x = \frac{5 \pm \sqrt{25 + 96}}{2}$$

$$x = \frac{5 \pm \sqrt{121}}{2}$$

$$x = \frac{5 \pm 11}{2}$$

$$x = -3,\ 8$$

■ 3. What went wrong in the way the quadratic formula was applied?

$$3x^2 - 5x + 10 = 0$$

$$x = \frac{-5 \pm \sqrt{(-5)^2 - 4(3)(10)}}{2(3)}$$

Solution:

The $-b$ at the beginning of the quadratic formula is written as -5, but $b=-5$. Which means it should be written as $-(-5)$.

■ 4. Solve for z using the quadratic formula.

$$z^2=z+3$$

Solution:

Rewrite the expression as

$$z^2=z+3$$

$$z^2-z-3=0$$

Then the quadratic formula gives

$$z=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-3)}}{2(1)}$$

$$z=\frac{1\pm\sqrt{13}}{2}$$

5. Fill in the blank with the correct term if the quadratic formula below was built from the quadratic equation.

$$\_\_\_\_ x^2+3x-5=0$$

$$x=\frac{-3\pm\sqrt{(3)^2-4(-2)(-5)}}{2(-2)}$$

Solution:

The blank should be the term -2.

6. Simplify the expression.

$$\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(14)}}{2(1)}$$

Solution:

The expression is simplified as

$$\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(14)}}{2(1)}$$

$$\frac{8\pm\sqrt{64-56}}{2}$$

$$\frac{8 \pm \sqrt{8}}{2}$$

$$\frac{8 \pm 2\sqrt{2}}{2}$$

$$4 \pm \sqrt{2}$$

■ 7. What are two ways to solve a quadratic equation when you cannot easily factor?

Solution:

You can either use the method of completing the square or the quadratic formula.

■ 8. What went wrong if the quadratic formula below was built from the quadratic equation?

$$x^2 + 2x = 7$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(7)}}{2(1)}$$

Solution:

The expression was not written in the correct form before using the quadratic formula. It should be written as $x^2 + 2x - 7 = 0$, for which the quadratic formula would then be

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-7)}}{2(1)}$$

9. Solve for t using the quadratic formula.

$$4t^2 - 1 = -8t$$

Solution:

Rewrite the expression as

$$4t^2 - 1 = -8t$$

$$4t^2 + 8t - 1 = 0$$

Then the quadratic formula is

$$t = \frac{-(8) \pm \sqrt{(8)^2 - 4(4)(-1)}}{2(4)}$$

$$t = \frac{-8 \pm \sqrt{64 + 16}}{8}$$

$$t = \frac{-8 \pm 4\sqrt{5}}{8}$$

$$t = \frac{-2 \pm \sqrt{5}}{2}$$

COMPLETING THE SQUARE

1. Solve for x by completing the square.

$$x^2 - 6x + 5 = 0$$

Solution:

Completing the square gives

$$x^2 - 6x = -5$$

$$x^2 - 6x + 9 = -5 + 9$$

$$(x - 3)^2 = 4$$

$$x - 3 = \pm 2$$

$$x = 3 \pm 2$$

$$x = 1, 5$$

2. Fill in the blank with the correct term.

$$x^2 - \_\_\_\_ + \frac{9}{4} = -2 + \frac{9}{4}$$

Solution:

The blank should be the term $3x$.

3. Complete the square in the following expression, but do not solve.

$$3y^2 - 12y + 3 = 0$$

Solution:

To complete the square, we first write the expression as

$$3y^2 - 12y = -3$$

$$y^2 - 4y = -1$$

Now complete the square as

$$y^2 - 4y + 4 = -1 + 4$$

$$(y - 2)^2 = 3$$

4. Solve for a by completing the square.

$$2a^2 + 8a = -4$$

Solution:

Completing the square gives

$$a^2 + 4a = -2$$

$$a^2 + 4a + 4 = -2 + 4$$

$$(a + 2)^2 = 2$$

$$a + 2 = \pm\sqrt{2}$$

$$a = -2 \pm \sqrt{2}$$

■ 5. What is your first and second step in solving the problem by completing the square?

$$4x^2 - 16x + 28 = 0$$

Solution:

The first step is to move the 28 over to the other side. The second step is to divide everything by 4. These steps could be done in the opposite order, but they are the first two steps you must take before completing the square.

■ 6. Explain when and why completing the square is used for factoring.

Solution:

Completing the square is used when it's not possible to solve for the roots by factoring.

7. Solve for y by completing the square.

$$3y^2 + 9y = 3$$

Solution:

Completing the square gives

$$3y^2 + 9y = 3$$

$$y^2 + 3y = 1$$

$$y^2 + 3y + \frac{9}{4} = 1 + \frac{9}{4}$$

$$\left(y + \frac{3}{2}\right)^2 = \frac{13}{4}$$

$$y = -\frac{3}{2} \pm \frac{\sqrt{13}}{2}$$

$$y = -\frac{3 \pm \sqrt{13}}{2}$$

8. Fill in the blank with the correct term.

$$\_\_\_\_ - 4x = 6$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{22}{9}$$

Solution:

The blank should be filled in with $3x^2$. We can work backwards from the second equation.

$$\left(x - \frac{2}{3}\right)^2 = \frac{22}{9}$$

$$\left(x - \frac{2}{3}\right)\left(x - \frac{2}{3}\right) = \frac{22}{9}$$

$$x^2 - \frac{4}{3}x + \frac{4}{9} = \frac{22}{9}$$

$$3x^2 - 4x + \frac{4}{3} = \frac{22}{3}$$

$$3x^2 - 4x = \frac{22}{3} - \frac{4}{3}$$

$$3x^2 - 4x = \frac{18}{3}$$

$$3x^2 - 4x = 6$$

LONG DIVISION OF POLYNOMIALS

■ 1. Find the quotient.

$$\frac{x^2+2x-1}{x+3}$$

Solution:

Use long division to find the quotient.

$$
\begin{array}{r|l}
 & x-1+\frac{2}{x+3} \\
\hline
x+3 & x^2+2x-1 \\
 & -(x^2+3x) \\
\hline
 & \qquad -x-1 \\
 & \quad -(-x-3) \\
\hline
 & \qquad\qquad 2
\end{array}
$$

■ 2. Find the quotient.

$$\frac{2x^3-x^2-4x+5}{x-2}$$

Solution:

Use long division to find the quotient.

$$
\begin{array}{r}
2x^2+3x+2+\frac{9}{x-2} \\
x-2\,\overline{\big)\,2x^3-x^2-4x+5} \\
-(2x^3-4x^2) \\
\hline
3x^2-4x \\
-(3x^2-6x) \\
\hline
2x+5 \\
-(2x-4) \\
\hline
9
\end{array}
$$

3. Find the quotient.

$$\frac{2x^4+4x^3-x^2+5x-150}{x+4}$$

Solution:

Use long division to find the quotient.

$$
\begin{array}{r}
2x^3-4x^2+15x-55+\frac{70}{x+4} \\
x+4\;\overline{\big)\,2x^4+4x^3-x^2+5x-150} \\
-(2x^4+8x^3) \\
\hline
-4x^3-x^2 \\
-(-4x^3-16x^2) \\
\hline
15x^2+5x \\
-(15x^2+60x) \\
\hline
-55x-150 \\
-(-55x-220) \\
\hline
70
\end{array}
$$

■ 4. Find the quotient.

$$\frac{3x^3 - x^2 - 7x + 5}{x - 1}$$

Solution:

Use long division to find the quotient.

$$
\begin{array}{r}
3x^2+2x-5 \\
x-1 \overline{\big)\, 3x^3-x^2-7x+5} \\
-(3x^3-3x^2) \qquad\qquad \\
\hline
2x^2-7x \qquad \\
-(2x^2-2x) \qquad \\
\hline
-5x+5 \\
-(-5x+5) \\
\hline
0
\end{array}
$$

■ 5. Find the quotient.

$$\frac{-x^2+3x+15}{x+5}$$

Solution:

Use long division to find the quotient.

$$
\begin{array}{r|l}
 & -x+8-\dfrac{25}{x+5} \\ \hline
x+5 & -x^2+3x+15 \\
 & -(-x^2-5x) \\ \hline
 & \qquad 8x+15 \\
 & \quad -(8x+40) \\ \hline
 & \qquad\quad -25
\end{array}
$$

■ 6. Find the quotient.

$$\frac{x^4+x-3}{x-2}$$

Solution:

Use long division to find the quotient. Remember to represent the missing x^3 and x^2 terms.

$$
\begin{array}{r}
x^3+2x^2+4x+9+\dfrac{15}{x-2} \\
x-2\ \overline{\big)\ x^4+0x^3+0x^2+x-3} \\
-(x^4-2x^3) \qquad\qquad\qquad\quad \\
\hline
2x^3+0x^2 \qquad\qquad\quad \\
-(2x^3-4x^2) \qquad\qquad\quad \\
\hline
4x^2+x \qquad\quad \\
-(4x^2-8x) \qquad\quad \\
\hline
9x-3 \\
-(9x-18) \\
\hline
15
\end{array}
$$

7. Find the quotient.

$$\frac{x^3+6}{x+6}$$

Solution:

Use long division to find the quotient. Remember to represent the missing x^2 and x terms.

$$
\begin{array}{r}
x^2-6x+36-\frac{210}{x+6} \\
x+6\,\big)\overline{\,x^3+0x^2+0x+6\,} \\
-(x^3+6x^2) \\
\hline
-6x^2+0x \\
-(-6x^2-36x) \\
\hline
36x+6 \\
-(36x+216) \\
\hline
-210
\end{array}
$$

8. Find the quotient.

$$\frac{x^2+x}{x-3}$$

Solution:

Use long division to find the quotient. Remember to represent the missing constant term.

$$
\begin{array}{r|l}
 & x+4+\frac{12}{x-3} \\
\hline
x-3 & x^2+x+0 \\
 & -(x^2-3x) \\
\hline
 & \qquad 4x+0 \\
 & \qquad -(4x-12) \\
\hline
 & \qquad\qquad 12
\end{array}
$$

9. Find the quotient.

$$\frac{x^4-2x^2}{x-4}$$

Solution:

Use long division to find the quotient. Remember to represent the missing terms.

$$
\begin{array}{r|l}
 & x^3+4x^2+14x+56+\frac{224}{x-4} \\
\hline
x-4 & x^4+0x^3-2x^2+0x+0 \\
 & -(x^4-4x^3) \\
\hline
 & 4x^3-2x^2 \\
 & -(4x^3-16x^2) \\
\hline
 & 14x^2+0x \\
 & -(14x^2-56x) \\
\hline
 & 56x+0 \\
 & -(-56x-224) \\
\hline
 & 224
\end{array}
$$

■ 10. Find the quotient.

$$\frac{-2x^3+8x}{x+2}$$

Solution:

Use long division to find the quotient. Remember to represent the missing terms.

$$
\begin{array}{r|l}
 & -2x^2+4x \\
\hline
x+2 & -2x^3+0x^2+8x+0 \\
 & -(-2x^3-4x^2) \\
\hline
 & \qquad\qquad 4x^2+8x \\
 & \qquad\quad -(4x^2+8x) \\
\hline
 & \qquad\qquad\qquad 0
\end{array}
$$

THE UNIT CIRCLE

■ 1. What is the coordinate point associated with $\theta = 2\pi/3$ along the unit circle?

Solution:

Looking at the unit circle shows that the coordinate point associated with $\theta = 2\pi/3$ in the second quadrant is

$$\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$$

■ 2. The terminal side of the angle θ in $[0,2\pi)$ intersects the unit circle at the given point. Find the measure of θ in degrees.

$$\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$$

Solution:

Because both the x- and y-values in the coordinate point are negative, we know the angle lies in the third quadrant. Comparing the coordinate point to points along the unit circle in the third quadrant, we see that $\theta = 240^\circ$.

■ 3. Find $\sin\theta$ if $\theta \in [0,2\pi)$ and $\cos\theta = \sin\theta$.

Solution:

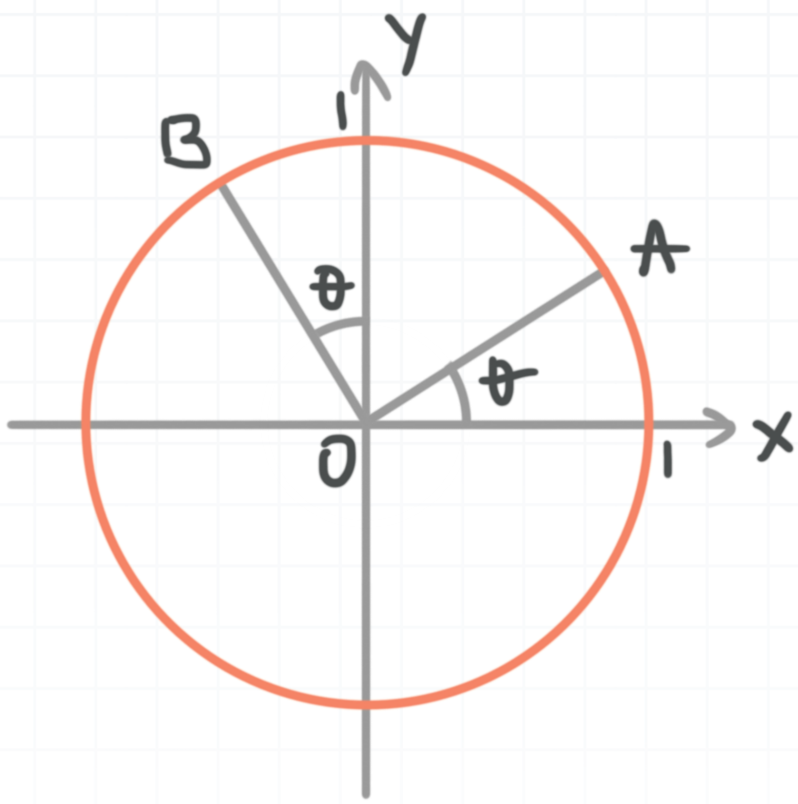
We know that $\sin\theta$ represents the y-coordinate, and $\cos\theta$ represents the x-coordinate. In the second and third quadrants, the signs of x and y are different, which means $\sin\theta$ and $\cos\theta$ cannot be equal.

If we look at the first and third quadrants of the unit circle, we can see that the x- and y-values are equal at $\theta = \pi/4$ and $\theta = 5\pi/4$.

At $\theta = \dfrac{\pi}{4}$, $\sin\theta = \dfrac{\sqrt{2}}{2}$

At $\theta = \dfrac{5\pi}{4}$, $\sin\theta = -\dfrac{\sqrt{2}}{2}$

■ 4. The points A and B lie on the unit circle in quadrants I and II respectively. The angle between OA and the positive x-axis is θ. The angle between OB and the positive y-axis is θ. Find the sine of $\angle AOB$.



Solution:

The angle between OB and the positive x-axis is $90^\circ + \theta$. So $\angle AOB$ is $(90^\circ + \theta) - \theta = 90^\circ$. Because we know $\sin 90^\circ = 1$,

$$\sin AOB = \sin 90^\circ = 1$$

■ 5. Evaluate the expression.

$$2 \csc \left(\frac{49\pi}{6} \right) - 3 \cos \left(\frac{13\pi}{3} \right) + \tan \left(\frac{25\pi}{4} \right)$$

Solution:

For each of the given angles, find a coterminal angle in $[0,2\pi)$.

$$\frac{49\pi}{6} = \frac{48\pi}{6} + \frac{\pi}{6} = 8\pi + \frac{\pi}{6}$$

$$\frac{13\pi}{3} = \frac{12\pi}{3} + \frac{\pi}{3} = 4\pi + \frac{\pi}{3}$$

$$\frac{25\pi}{4} = \frac{24\pi}{4} + \frac{\pi}{4} = 6\pi + \frac{\pi}{4}$$

Which means

$$\csc\left(\frac{49\pi}{6}\right) = \csc\left(\frac{\pi}{6}\right)$$

$$\cos\left(\frac{13\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$$

$$\tan\left(\frac{25\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right)$$

Rewrite the expression with these coterminal angles.

$$2\csc\left(\frac{\pi}{6}\right) - 3\cos\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right)$$

Apply reciprocal and quotient identities to put the expression in terms of only sine and cosine functions.

$$\frac{2}{\sin\left(\frac{\pi}{6}\right)} - 3\cos\left(\frac{\pi}{3}\right) + \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}$$

Plug in the known values from the unit circle.

$$\frac{2}{\frac{1}{2}} - 3\left(\frac{1}{2}\right) + \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$4 - \frac{3}{2} + 1$$

$$\frac{8}{2} - \frac{3}{2} + \frac{2}{2}$$

$$\frac{7}{2}$$

■ 6. Find the angle θ in the interval $[0,2\pi)$.

$$\sin\theta = \frac{1}{2} \text{ and } \cos\theta = -\frac{\sqrt{3}}{2}$$

Solution:

From the unit circle, we know the sine function takes the value $1/2$ at angles of $\pi/6$ and $5\pi/6$, so we need to evaluate cosine at both of these angles. We get

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ and } \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

Therefore, the matching angle is $\theta = 5\pi/6$.

IDEA OF THE LIMIT

1. The table below shows some values of a function $g(x)$. What does the table show for the value of $\lim_{x \to 4} g(x)$?

x	$g(x)$
3.9	1.9748
3.99	1.9975
3.999	1.9997
4.001	2.0002
4.01	2.0025
4.1	2.0248

Solution:

2

2. How would you express, mathematically, the limit of the function $f(x) = x^2 - x + 2$ as x approaches 3?

Solution:

$$\lim_{x \to 3} x^2 - x + 2$$

■ 3. How would you write the limit of $g(x)$ as x approaches ∞, using correct mathematical notation?

$$g(x) = \frac{5x^2 - 7}{3x^2 + 8}$$

Solution:

$$\lim_{x \to \infty} \frac{5x^2 - 7}{3x^2 + 8}$$

ONE-SIDED LIMITS

■ 1. Find the limit.

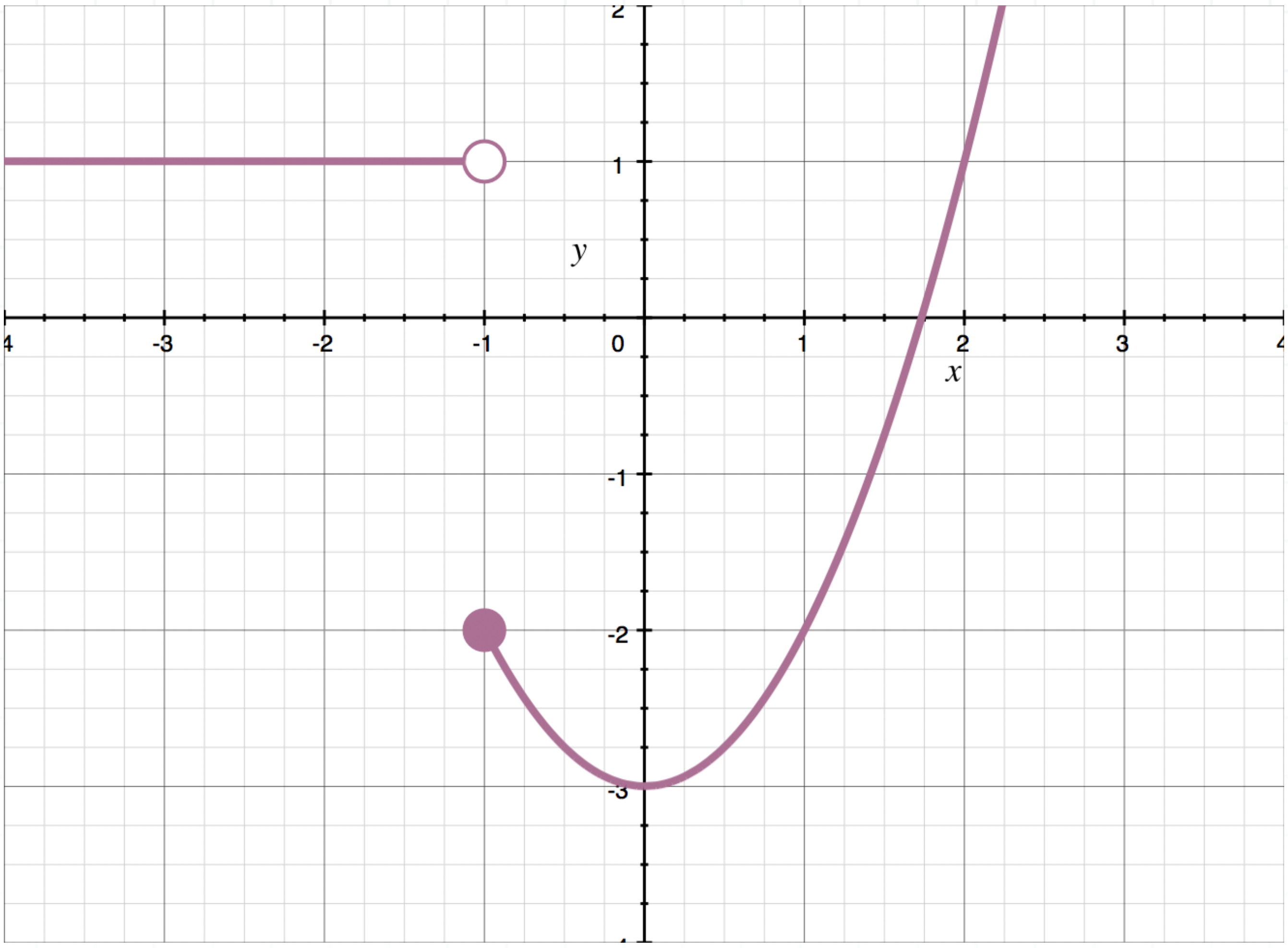
$$\lim_{x\to -7^+} x^2\sqrt{x+7}$$

Solution:

The value of the limit is 0.

x	-6.09	-6.9	-6.99	-6.999	-6.9999	-7
Value	35.38	15.056	4.886	1.5481	0.48999	0

■ 2. What does the graph of $f(x)$ say about the value of $\lim_{x\to -1^+} f(x)$?



Solution:

From the graph, the limit is

$$\lim_{x \to -1^+} f(x) = -2$$

■ 3. The table shows values of $k(x)$. What is $\lim_{x \to -5^-} k(x)$?

x	-5.1	-5.01	-5.0001	-5	-4.999	-4.99	-4.9
k(x)	-392.1	-3,812	-38,012	?	37,988	3,788	368.1

Solution:

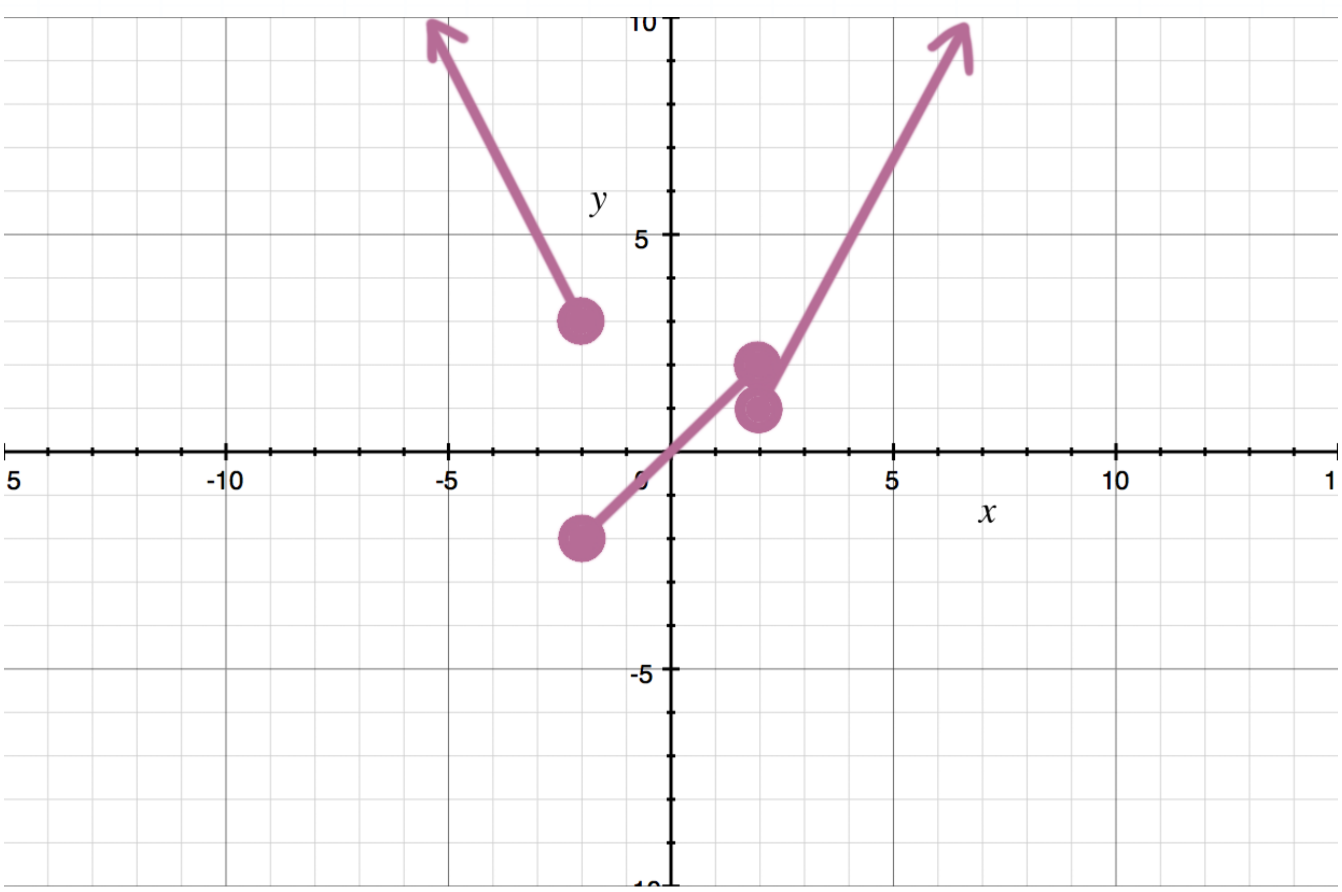
$$\lim_{x\to -5^-} k(x) = -\infty$$

■ 4. What is $\lim_{x\to -2^-} h(x)$?

$$h(x) = \begin{cases} -2x - 1 & x < -2 \\ x & -2 \le x < 2 \\ 2x - 3 & x \ge 2 \end{cases}$$

Solution:

The graph of $h(x)$ is



Based on the graph, the limit is 3. Or we could plug into the first piece of the function, which is the piece that approaches $x = -2$ from the left side.

$$\lim_{x \to -2^-} h(x) = \left[-2(-2) - 1\right] = 3$$

5. What is $\lim_{x \to 6^+} g(x)$?

$$g(x) = \frac{x^2 + x - 42}{x - 6}$$

Solution:

We could tell that the limit is 13 by making a table,

x	6	6.001	6.01	6.1
g(x)	?	13.001	13.01	13.1

Alternatively, we could have factored the numerator, canceled like terms, and then evaluated at the limit.

$$g(x) = \frac{x^2 + x - 42}{x - 6}$$

$$g(x) = \frac{(x + 7)(x - 6)}{x - 6}$$

$$g(x) = x + 7$$

Then the limit is

$$\lim_{x \to 6^+} x + 7$$

$$6 + 7$$

$$13$$

PROVING THAT THE LIMIT DOES NOT EXIST

1. Prove that the limit does not exist.

$$\lim_{x\to 0} \frac{-2|3x|}{3x}$$

Solution:

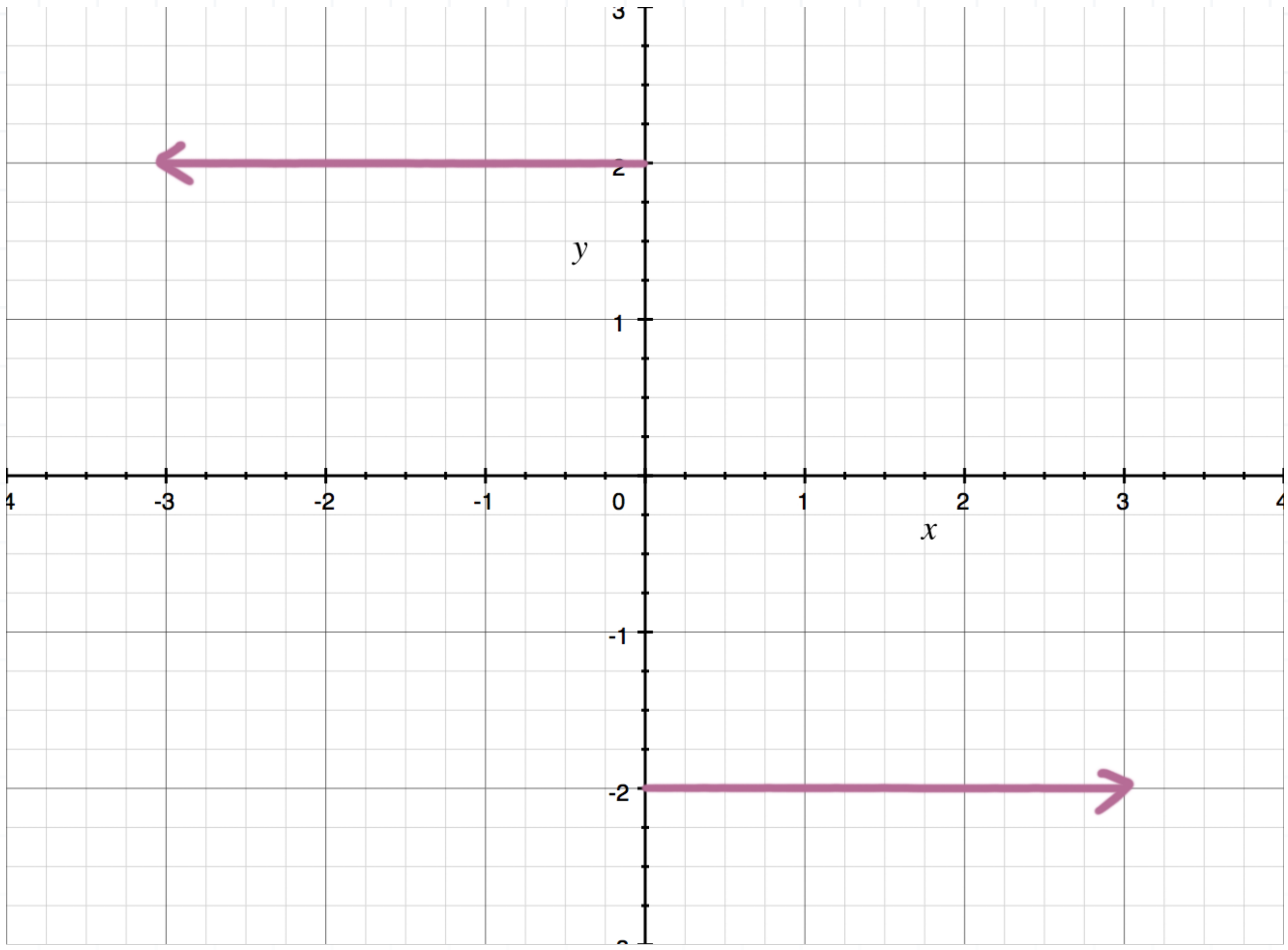
The left-hand limit is

$$\lim_{x\to 0^-} \frac{-2|3x|}{3x} = \frac{6x}{3x} = 2$$

The right-hand limit is

$$\lim_{x\to 0^+} \frac{-2|3x|}{3x} = \frac{-6x}{3x} = -2$$

Since the left- and right-hand limits aren't equal, the limit does not exist. The graph of the function would also prove that the limit doesn't exist.



2. Prove that the limit does not exist.

$$\lim_{x\to -5} \frac{x^2+7x+9}{x^2-25}$$

Solution:

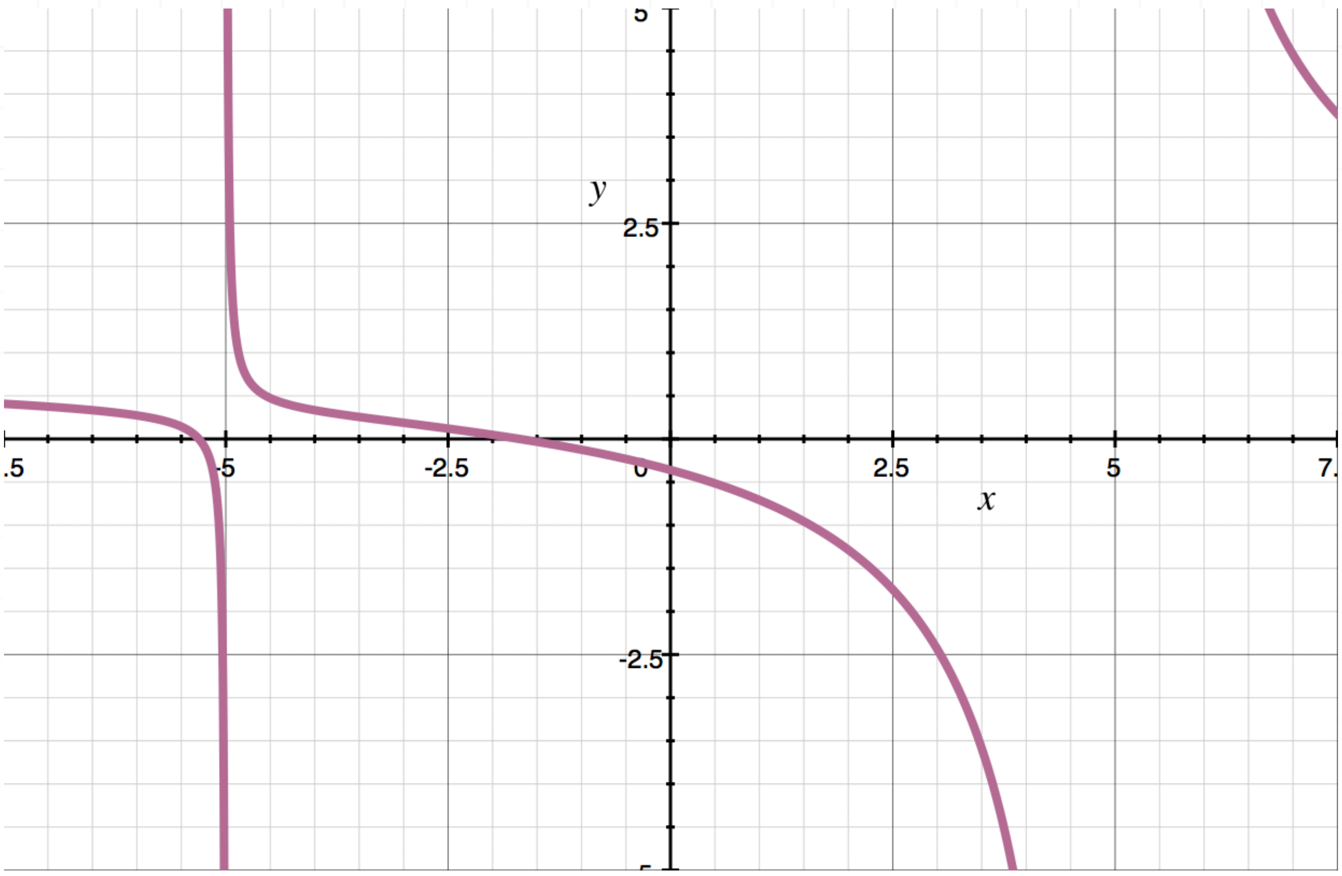
The left-hand limit is

$$\lim_{x\to -5^-} \frac{x^2+7x+9}{x^2-25} = -\infty$$

The right-hand limit is

$$\lim_{x \to -5^+} \frac{x^2 + 7x + 9}{x^2 - 25} = \infty$$

Since the left- and right-hand limits aren't equal, the limit does not exist. The graph of the function would also prove that the limit doesn't exist.



■ 3. Prove that $\lim_{x \to 1} f(x)$ does not exist.

$$f(x) = \begin{cases} -3x + 2 & x < 1 \\ 3x - 2 & x \geq 1 \end{cases}$$

Solution:

The left-hand limit is

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-3x+2) = \left[-3(1)+2\right] = -1$$

The right-hand limit is

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x-2) = \left[3(1)-2\right] = 1$$

Because the left- and right-hand limits aren't equal, the limit does not exist.

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

PRECISE DEFINITION OF THE LIMIT

■ 1. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x\to 4} 5x - 16 = 4$$

Solution:

If $0 < |x - 4| < \delta$, then $|(5x - 16) - 4| < \epsilon$. So,

$$|5x - 20| < \epsilon$$

$$|5(x - 4)| < \epsilon$$

$$|5| \cdot |x - 4| < \epsilon$$

$$5 \cdot |x - 4| < \epsilon$$

$$|x - 4| < \frac{\epsilon}{5}$$

Now if $|x - 4| < \epsilon/5$ and $0 < |x - 4| < \delta$, then if $\epsilon > 0$ then $\delta < \epsilon/5$. Therefore, the limit equation is true.

■ 2. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x\to -7} -2x + 15 = 29$$

Solution:

If $0 < |x - (-7)| < \delta$ then $|-2x + 15 - 29| < \epsilon$. Or we could rewrite this as $0 < |x + 7| < \delta$ and $|-2x - 14| < \epsilon$. So,

$$|(-2)(x + 7)| < \epsilon$$

$$|-2| \cdot |x + 7| < \epsilon$$

$$2 \cdot |x + 7| < \epsilon$$

$$|x + 7| < \frac{\epsilon}{2}$$

Now if $|x + 7| < \epsilon/2$ and $0 < |x + 7| < \delta$, then if $\epsilon > 0$ then $\delta < \epsilon/2$. Therefore, the limit equation is true.

■ 3. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x \to 16} \left(\frac{2}{5}x - \frac{17}{5} \right) = 3$$

Solution:

If $0 < |x - 16| < \delta$ then $\left| \big((2/5)x - (17/5)\big) - 3 \right| < \epsilon$. Or we could rewrite this as $0 < |x - 16| < \delta$ and

$$\left|\left(\frac{2}{5}x - \frac{17}{5}\right) - \frac{15}{5}\right| < \epsilon$$

$$\left|\frac{2}{5}x - \frac{32}{5}\right| < \epsilon$$

$$\left|\frac{2}{5}(x - 16)\right| < \epsilon$$

$$\left|\frac{2}{5}\right| |x - 16| < \epsilon$$

$$|x - 16| < \frac{5}{2}\epsilon$$

Now if $|x - 16| < (5/2)\epsilon$ and $0 < |x - 16| < \delta$, then if $\epsilon > 0$, then $\delta < (5/2)\epsilon$. Therefore, the limit equation is true.

■ 4. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x \to 7} \left(\frac{x^2 - 15x + 56}{x - 7}\right) = -1$$

Solution:

We'll apply the precise definition to the given limit.

If $0 < |x - 7| < \delta$, then $\left| \left(\frac{x^2 - 15x + 56}{x - 7} \right) - (-1) \right| < \epsilon$.

If $0 < |x - 7| < \delta$, then $\left| \left(\frac{x^2 - 15x + 56}{x - 7} \right) - \frac{-1(x - 7)}{x - 7} \right| < \epsilon$.

So,

$$\left| \left(\frac{x^2 - 15x + 56}{x - 7} \right) + \frac{x - 7}{x - 7} \right| < \epsilon$$

$$\left| \frac{x^2 - 14x + 49}{x - 7} \right| < \epsilon$$

$$\left| \frac{(x - 7)(x - 7)}{x - 7} \right| < \epsilon$$

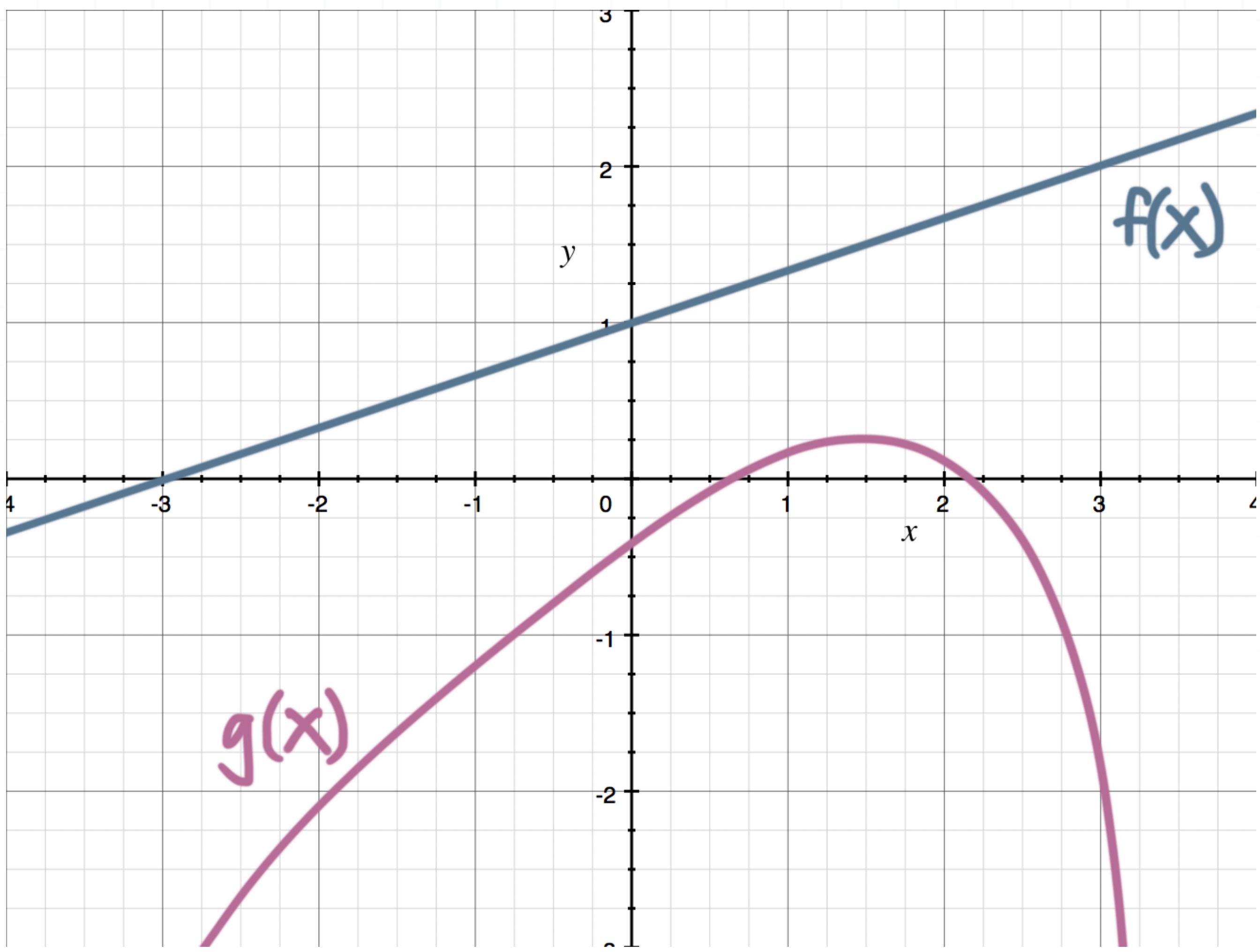
$$|x - 7| < \varepsilon$$

Now, if $|x - 7| < \epsilon$ and $0 < |x - 7| < \delta$, then if $\epsilon > 0$ and $\delta < \epsilon$. Therefore, the limit equation is true.

LIMITS OF COMBINATIONS

1. Use limit laws and the graph below to evaluate the limit.

$$\lim_{x \to 3} \left[4f(x) - 3g(x)\right]$$



Solution:

We can simplify the limit, and then evaluate both functions at $x = 3$.

$$\lim_{x \to 3} \left[4f(x) - 3g(x)\right]$$

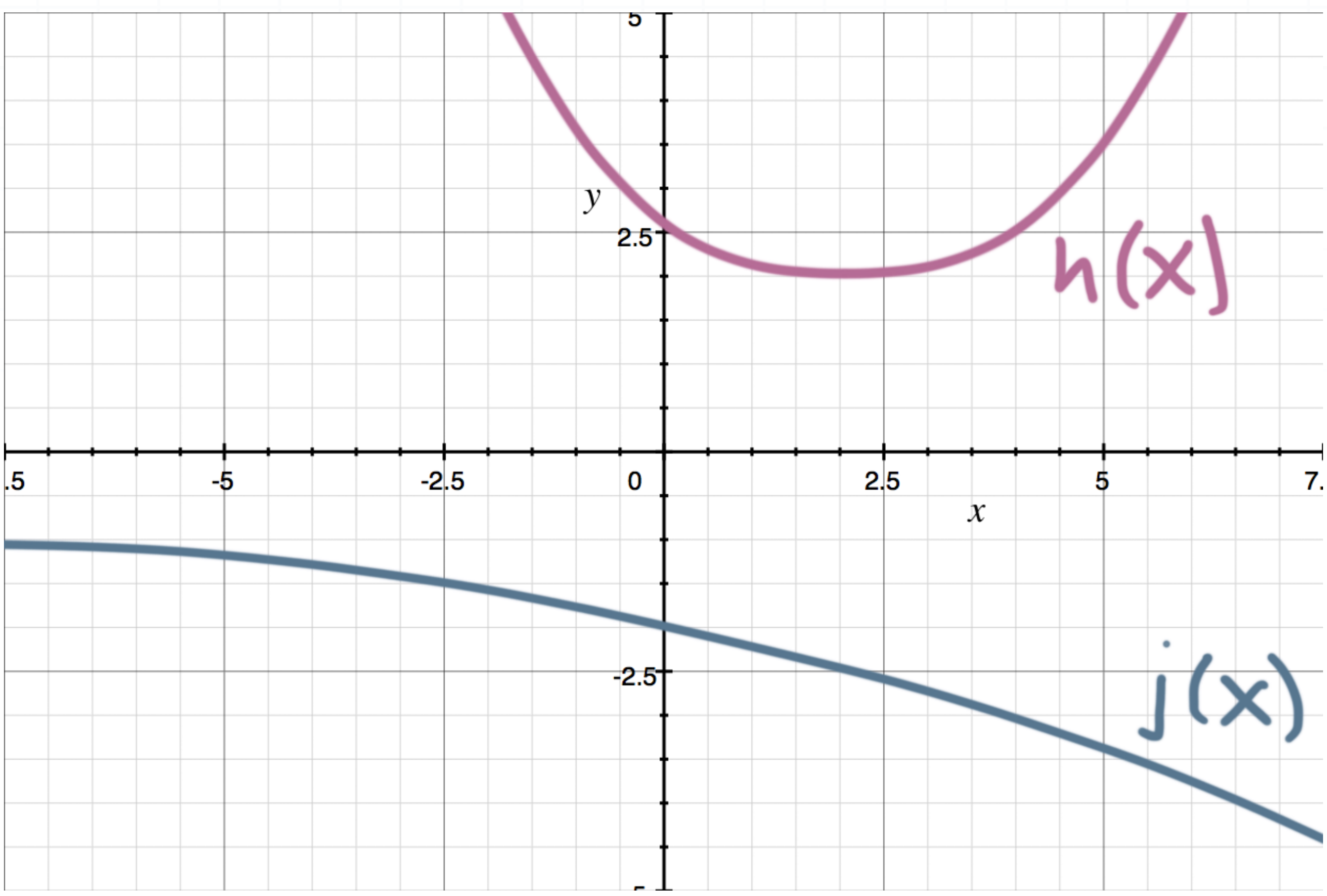
$$4 \lim_{x \to 3} f(x) - 3 \lim_{x \to 3} g(x)$$

$$4(2) - 3(-2)$$

$$14$$

2. Use limit laws and the graph below to evaluate the limit.

$$\lim_{x \to 4} \frac{h(x)}{j(x)}$$



Solution:

We can simplify the limit, and then evaluate both functions at $x = 4$.

$$\lim_{x \to 4} \frac{h(x)}{j(x)}$$

$$\frac{\lim_{x \to 4} h(x)}{\lim_{x \to 4} j(x)}$$

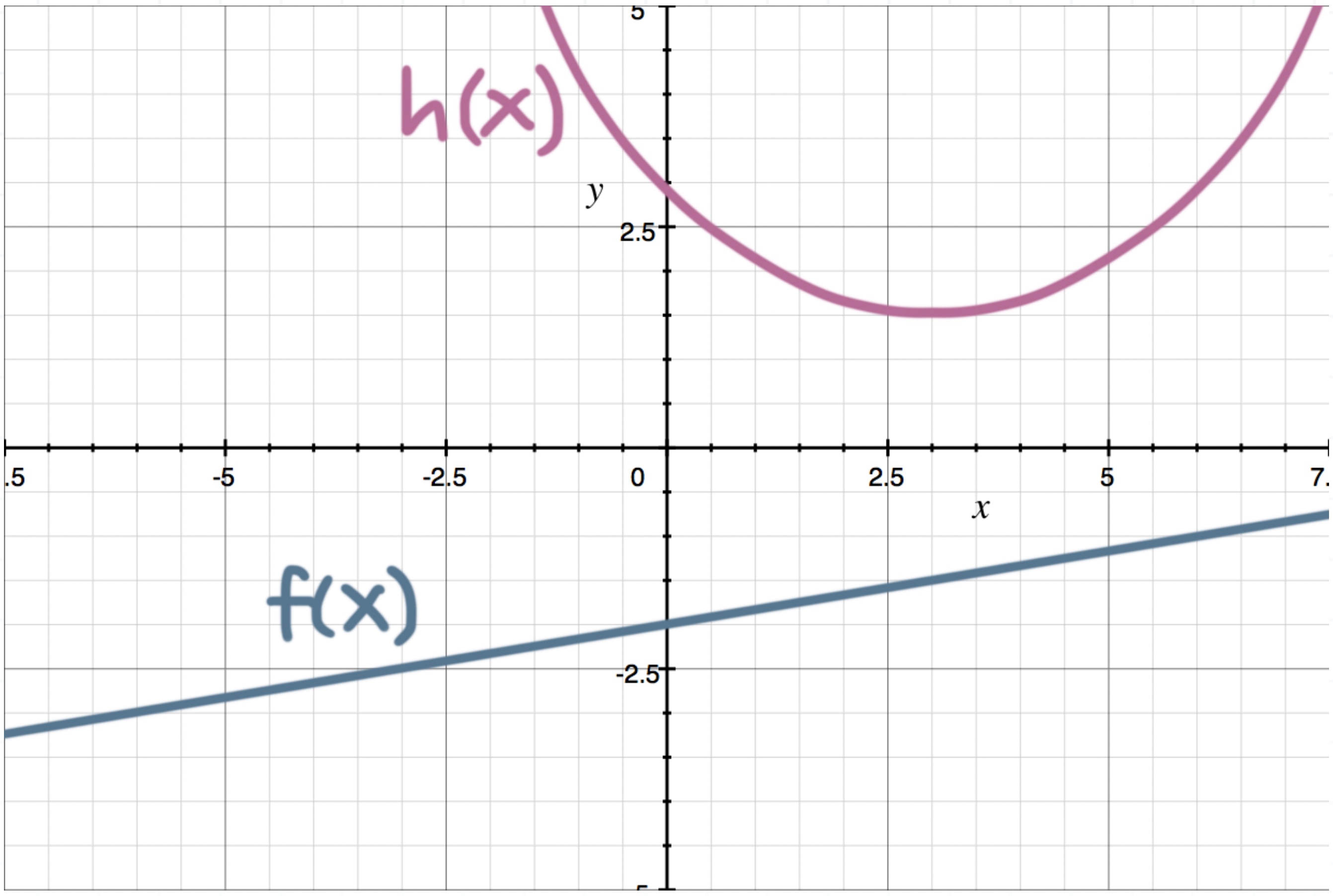
$$\frac{\frac{5}{2}}{-3}$$

$$\frac{5}{2} \cdot \frac{1}{-3}$$

$$-\frac{5}{6}$$

■ 3. Use limit laws and the graph below to evaluate the limit.

$$\lim_{x \to 0} \left[2f(x) \cdot 3h(x)\right]$$



Solution:

We can simplify the limit, and then evaluate both functions at $x = 0$.

$$\lim_{x \to 0} \left[2f(x) \cdot 3h(x)\right]$$

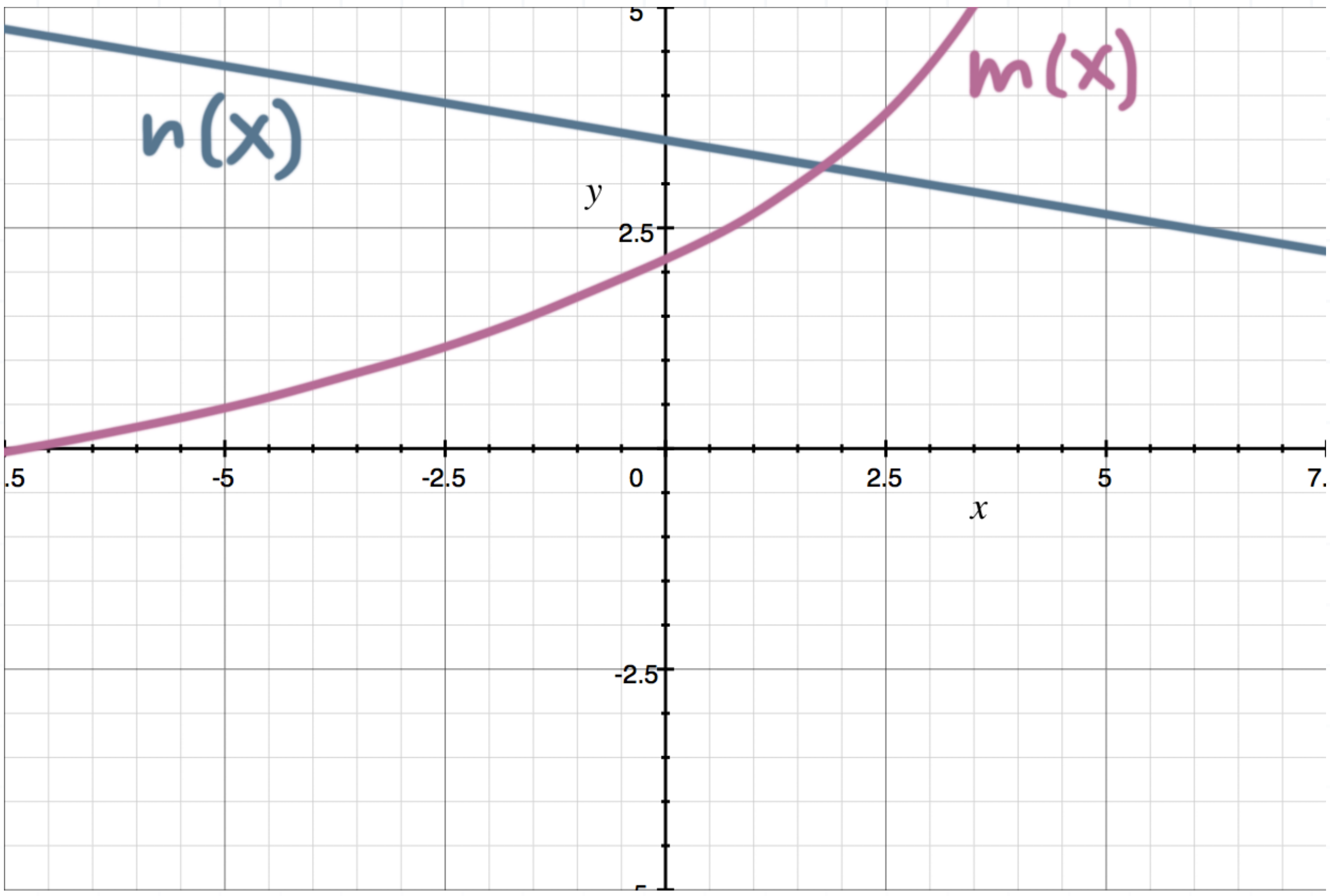
$$2 \lim_{x \to 0} f(x) \cdot 3 \lim_{x \to 0} h(x)$$

$$2(-2) \cdot 3(3)$$

$$-36$$

■ 4. Use limit laws and the graph below to evaluate the limit.

$$\lim_{x\to-3}\left[\frac{5m(x)}{n(x)}-\frac{4m(x)}{n(x)}\right]$$



Solution:

We can simplify the limit, and then evaluate both functions at $x=-3$.

$$\lim_{x\to-3}\left[\frac{5m(x)}{n(x)}-\frac{4m(x)}{n(x)}\right]$$

$$\frac{5\lim_{x\to-3}m(x)}{\lim_{x\to-3}n(x)}-\frac{4\lim_{x\to-3}m(x)}{\lim_{x\to-3}n(x)}$$

$$\frac{5(1)}{4}-\frac{4(1)}{4}$$

$$\frac{1}{4}$$

LIMITS OF COMPOSITES

■ 1. What is $\lim_{x\to 3} f(g(x))$ if $f(x) = 4x$ and $g(x) = 6x - 9$?

Solution:

If f is continuous at $x = 3$, then

$$\lim_{x\to 3} f(g(x)) = f\left(\lim_{x\to 3} g(x)\right)$$

$$\lim_{x\to 3} f(g(x)) = f\left(\lim_{x\to 3} 6x - 9\right)$$

$$\lim_{x\to 3} f(g(x)) = f(6(3) - 9) = f(9) = 4(9) = 36$$

■ 2. What is $\lim_{x\to -4} f(g(x))$ if $f(x) = 2x^2$ and $g(x) = 2x - 1$?

Solution:

If f is continuous at $x = -4$, then

$$\lim_{x\to -4} f(g(x)) = f\left(\lim_{x\to -4} g(x)\right)$$

$$\lim_{x \to -4} f(g(x)) = f\left(\lim_{x \to -4} 2x - 1\right)$$

$$\lim_{x \to -4} f(g(x)) = f(2(-4) - 1) = f(-9) = 2(-9)^2 = 162$$

■ 3. What is $\lim_{x \to \frac{\pi}{2}} f(g(x))$ if $f(x) = \sin x$ and $g(x) = x/2$?

Solution:

If f is continuous at $x = \pi/2$, then

$$\lim_{x \to \frac{\pi}{2}} f(g(x)) = f\left(\lim_{x \to \frac{\pi}{2}} g(x)\right)$$

$$\lim_{x \to \frac{\pi}{2}} f(g(x)) = f\left(\lim_{x \to \frac{\pi}{2}} \frac{x}{2}\right)$$

$$\lim_{x \to \frac{\pi}{2}} f(g(x)) = f\left(\frac{\frac{\pi}{2}}{2}\right) = f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

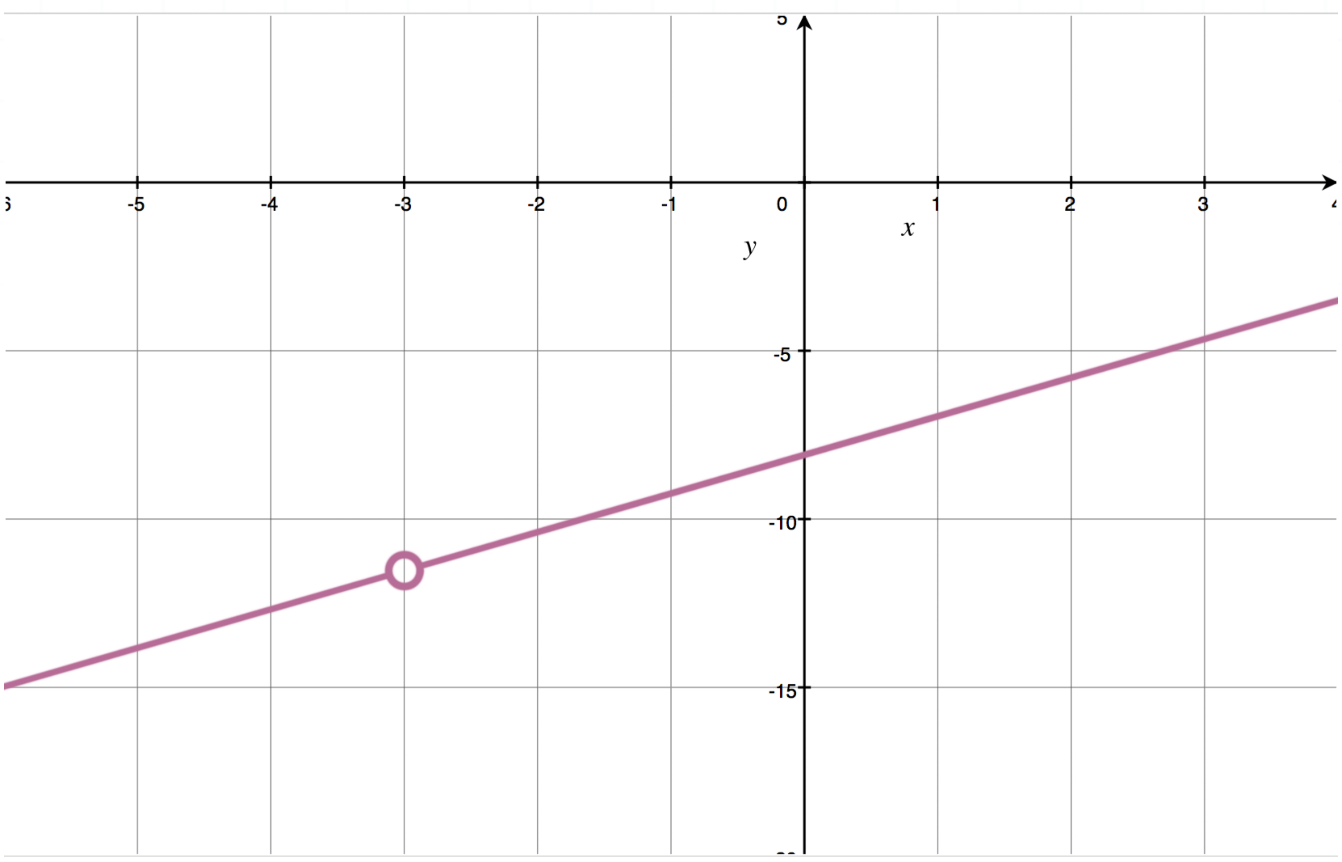
POINT DISCONTINUITIES

1. Redefine the function as a continuous piecewise function.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$

Solution:

The function is discontinuous at $x = -3$.



Factor and reduce to remove the discontinuity.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$

$$f(x) = \frac{(x+3)(x-9)}{x+3}$$

$$f(x) = x - 9$$

Evaluate $f(x)$ at $x = -3$.

$$f(-3) = -3 - 9 = -12$$

Therefore, to make the function continuous, we have to redefine it as

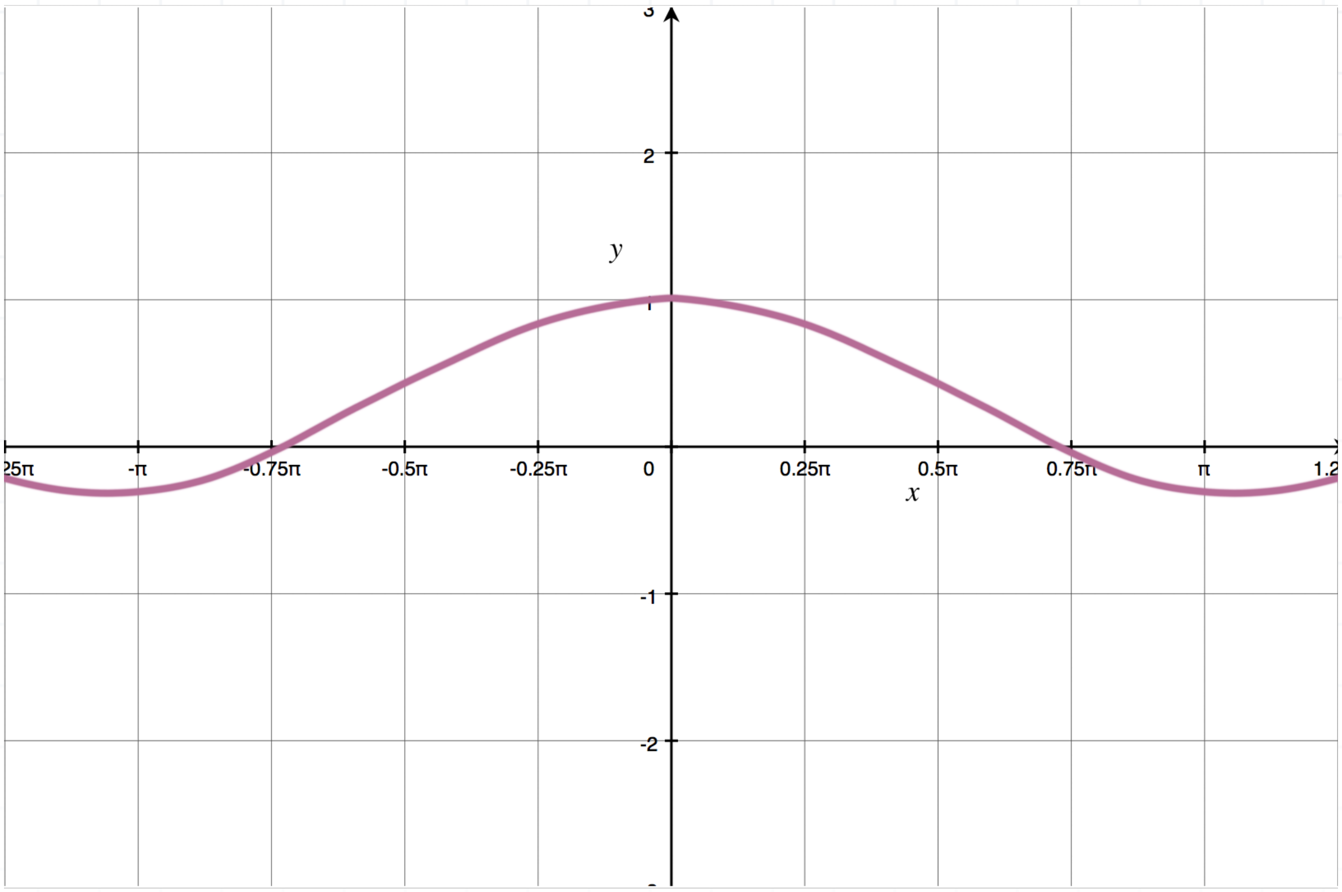
$$f(x) = \begin{cases} \frac{x^2 - 6x - 27}{x+3} & x \neq -3 \\ -12 & x = -3 \end{cases}$$

■ 2. Redefine the function as a continuous piecewise function.

$$g(x) = \frac{\sin x}{x}$$

Solution:

The function is discontinuous at $x = 0$, but is approaching a value of 1 from both sides.



Therefore, to make the function continuous, we have to redefine it as

$$g(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

■ 3. What are the removable discontinuities of the function?

$$h(x) = \frac{x^4 - 5x^2 + 4}{x^2 - 1}$$

Solution:

Factor the function, then cancel common factors.

$$h(x) = \frac{x^4 - 5x^2 + 4}{x^2 - 1}$$

$$h(x) = \frac{(x+1)(x-1)(x+2)(x-2)}{(x+1)(x-1)}$$

$$h(x) = (x+2)(x-2)$$

The factors that were canceled are the ones that produced removable discontinuities. So the removable discontinuities are $x = -1, 1$.

■ 4. Identify the non-removable discontinuities of the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

Solution:

Factor the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

$$k(x) = \frac{(x+5)(x-5)(x+3)}{(x+4)(x-3)}$$

No factors can be canceled. Which means the function has discontinuities at $x = -4$ and $x = 3$, both of which are non-removable.

■ 5. What is the set of removable discontinuities of the function?

$$j(\theta)=\frac{\cos^2\theta\cdot\sin^2\theta}{\tan^2\theta}$$

Solution:

We can rewrite the function as

$$j(\theta)=\frac{\cos^2\theta\cdot\sin^2\theta}{\tan^2\theta}=\frac{\cos^2\theta\cdot\sin^2\theta}{\frac{\sin^2\theta}{\cos^2\theta}}=\frac{\cos^2\theta\cdot\sin^2\theta\cdot\cos^2\theta}{\sin^2\theta}=\cos^4\theta$$

The removable discontinuities are the values of θ that make the sine function equal to 0, which are all the multiples of π.

$$\theta=\pm 0,\ \ \pm\pi,\ \ \pm 2\pi,\ \ \pm 3\pi,\ \ \pm 4\pi,\ \ldots$$

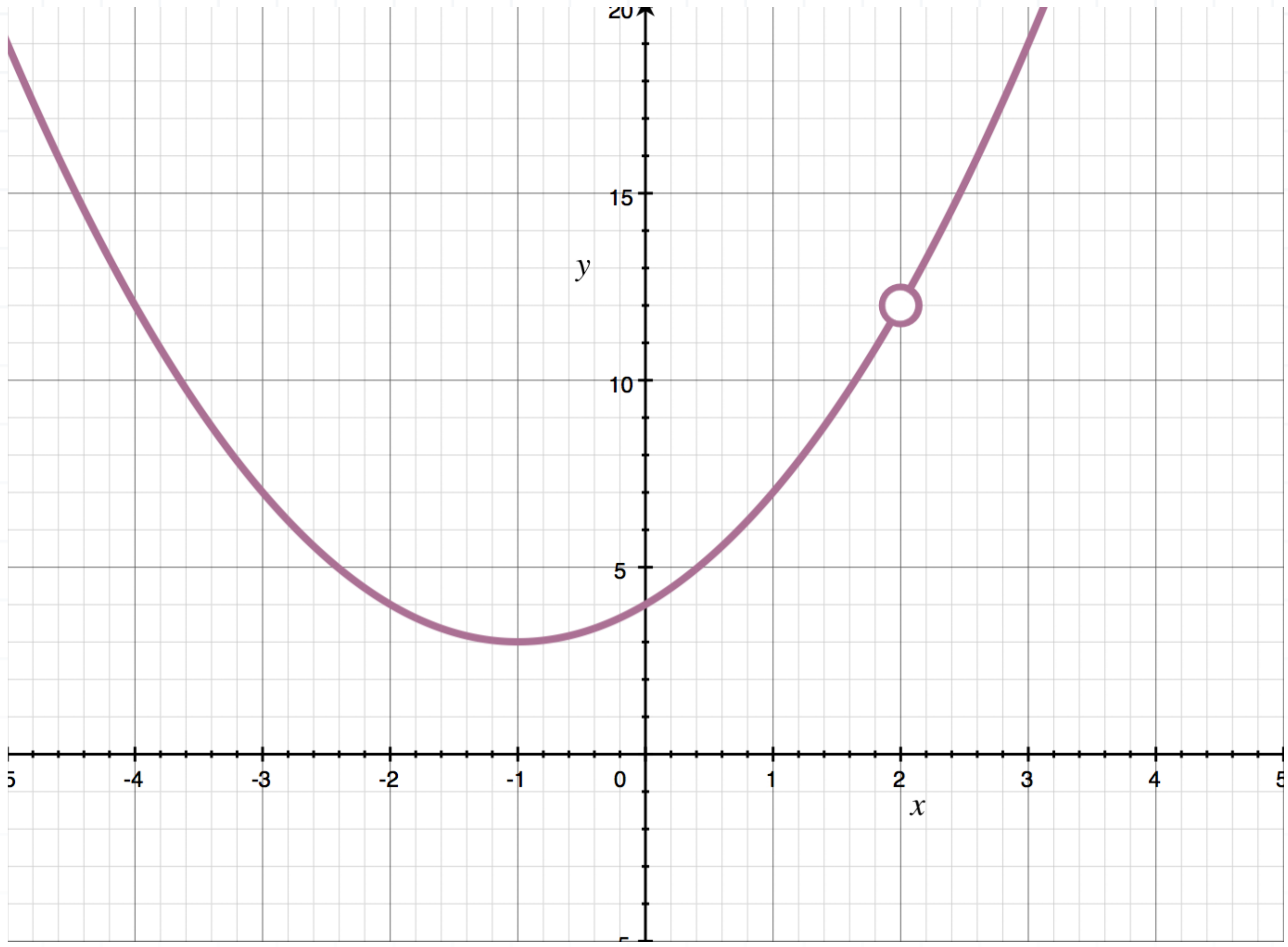
$\theta=n\pi$, where n is the set of all integers

■ 6. Redefine the function as a continuous piecewise function.

$$g(x)=\frac{x^3-8}{x-2}$$

Solution:

The function is discontinuous at $x=2$.



Factor and reduce to remove the discontinuity.

$$g(x) = \frac{x^3 - 8}{x - 2}$$

$$g(x) = \frac{(x-2)(x^2 + 2x + 4)}{x - 2}$$

$$g(x) = x^2 + 2x + 4$$

Evaluate $g(x)$ at $x = 2$.

$$g(2) = 4 + 4 + 4 = 12$$

Therefore, to make the function continuous, we have to redefine it as

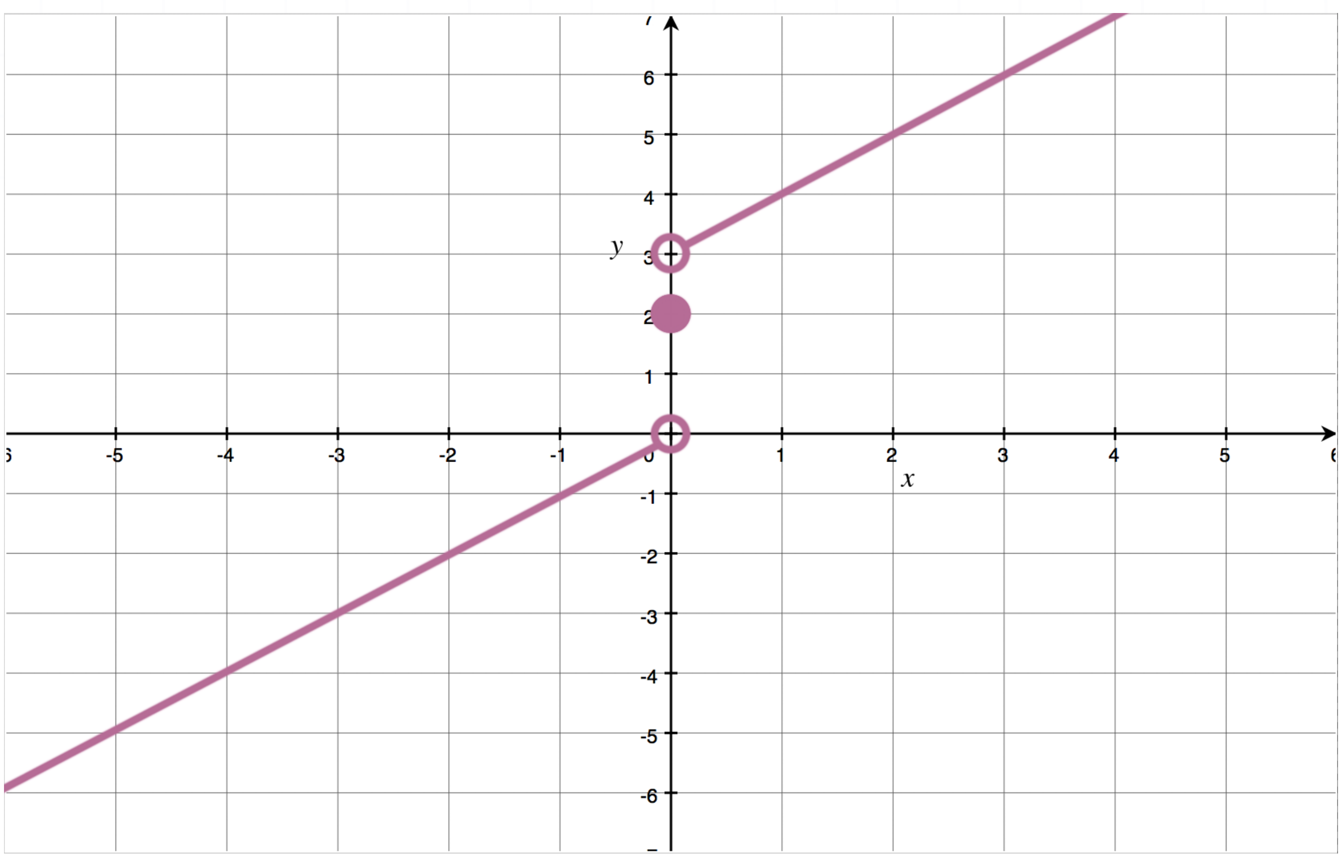
$$g(x) = \begin{cases} \frac{x^3 - 8}{x - 2} & x \neq 2 \\ 12 & x = 2 \end{cases}$$

■ 7. Identify the non-removable discontinuity in the function.

$$k(x) = \begin{cases} x & x < 0 \\ 2 & x = 0 \\ x + 3 & x > 0 \end{cases}$$

Solution:

The function $k(x)$ has a non-removable a discontinuity at $x = 0$ because the function has a jump discontinuity at $x = 0$, as shown in the graph below, and jump discontinuities are not removable.



■ 8. What is the removable discontinuity in the function?

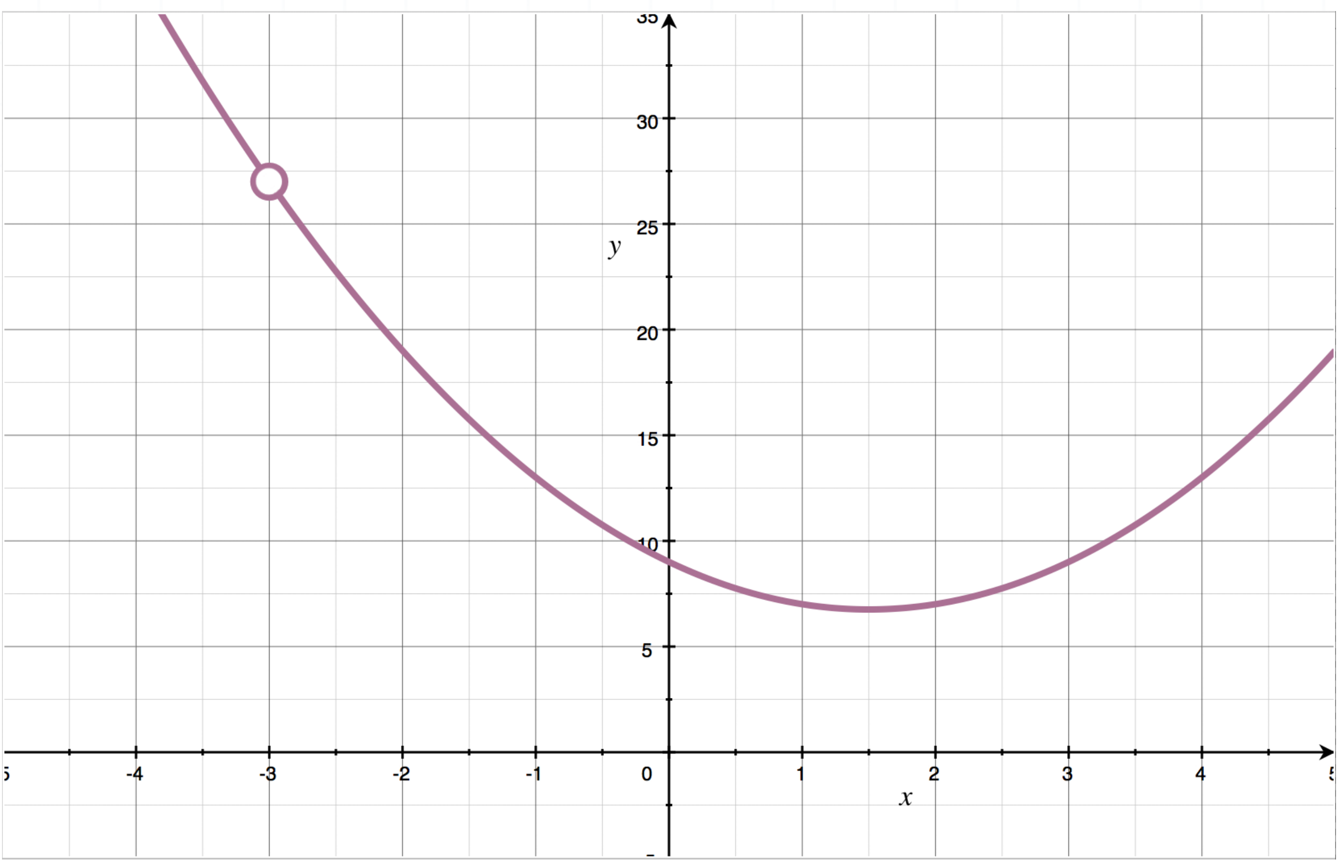
$$f(x) = \frac{x^3 + 27}{x + 3}$$

Solution:

If we factor the function, we can cancel a factor of $x = 3$.

$$f(x) = \frac{x^3 + 27}{x + 3} = \frac{(x + 3)(x^2 - 3x + 9)}{x + 3} = x^2 - 3x + 9$$

Therefore, the removable discontinuity is at $x = -3$.



■ 9. Identify the removable discontinuities in the function.

$$k(x) = \frac{x^4 - 2x^3 - 16x^2 + 2x + 15}{x^2 - 2x - 15}$$

Solution:

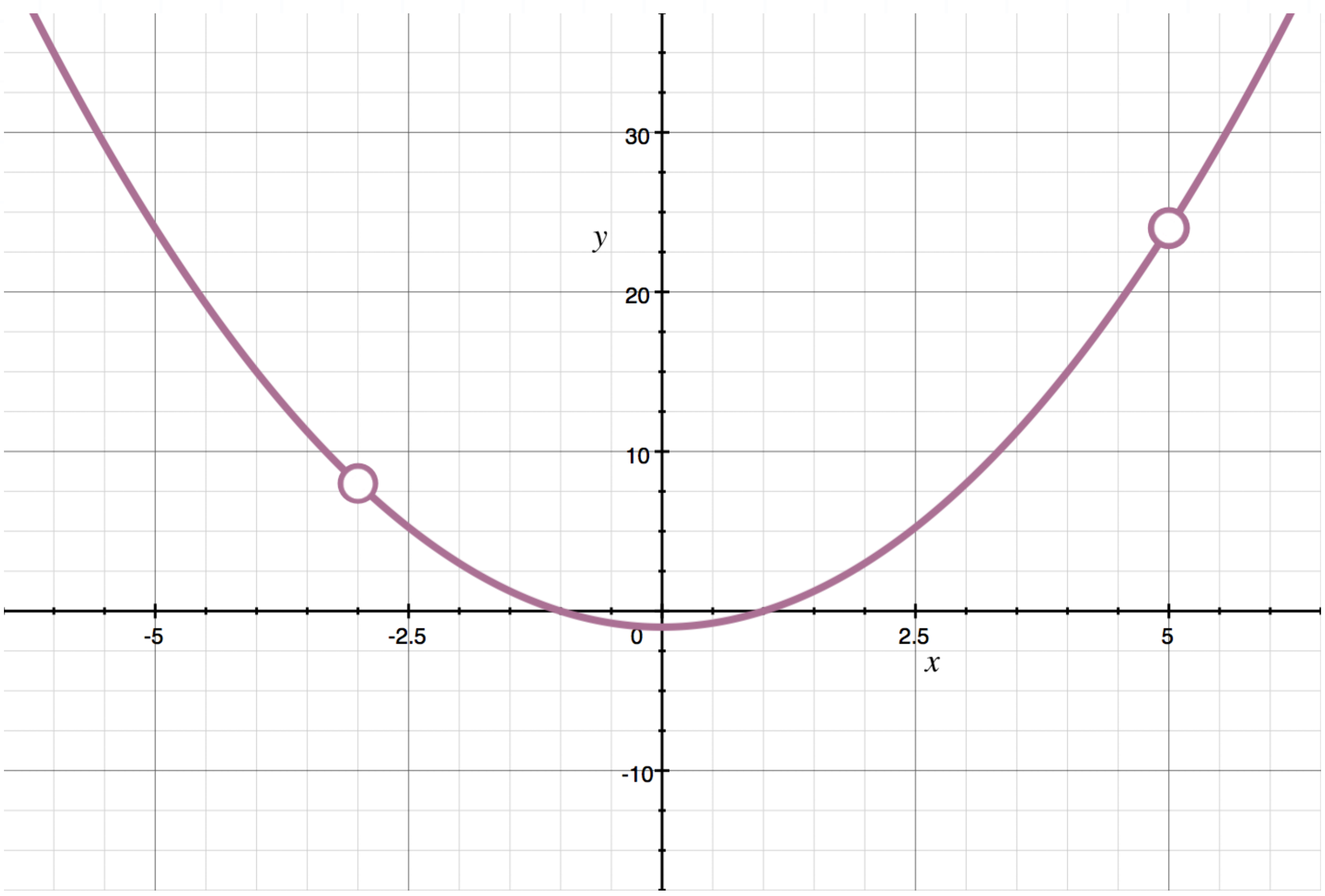
The function $k(x)$ has removable discontinuities at $x = -3$ and $x = 5$ because the function factors as

$$k(x) = \frac{(x+3)(x-5)(x+1)(x-1)}{(x+3)(x-5)}$$

and both factors from the denominator can be cancelled.

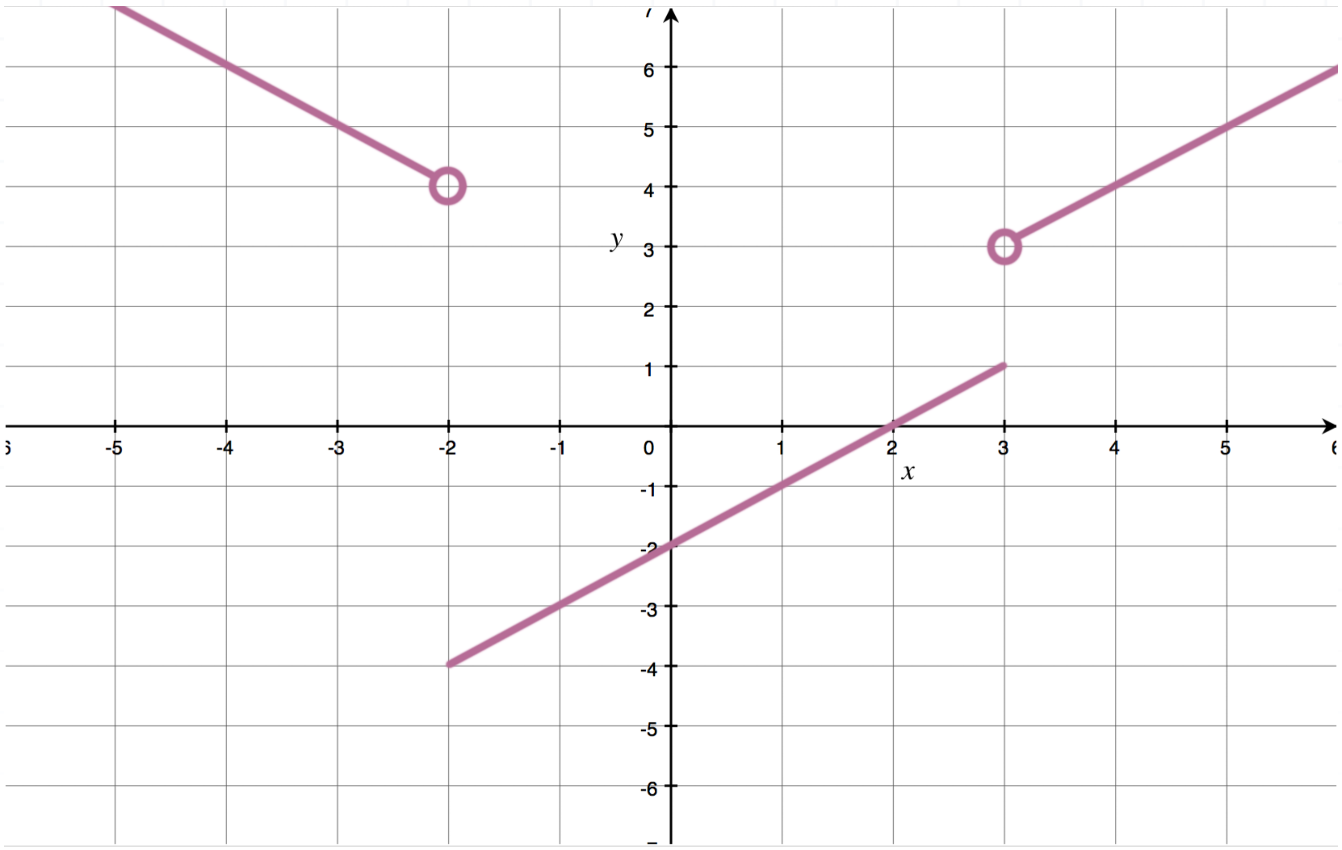
$$k(x) = (x+1)(x-1)$$

The graph is shown below.



JUMP DISCONTINUITIES

1. What are the x-values where the graph of $f(x)$, shown below, has jump discontinuities?



Solution:

The function $f(x)$ has jump discontinuities at $x = -2$ and $x = 3$ because the left- and right-hand limits aren't equal at $x = -2$

$$\lim_{x \to -2^-} f(x) = 4 \quad \neq \quad \lim_{x \to -2^+} f(x) = -2$$

and they aren't equal at $x = 3$.

$$\lim_{x \to 3^-} f(x) = 1 \quad \neq \quad \lim_{x \to 3^+} f(x) = 3$$

■ 2. Where are the jump discontinuities in the graph of the function?

$$h(x) = \begin{cases} -\frac{1}{3}x^2 + 2 & x < 0 \\ 3 & 0 \le x \le 1 \\ \frac{1}{3}x^2 + 4 & x > 1 \end{cases}$$

Solution:

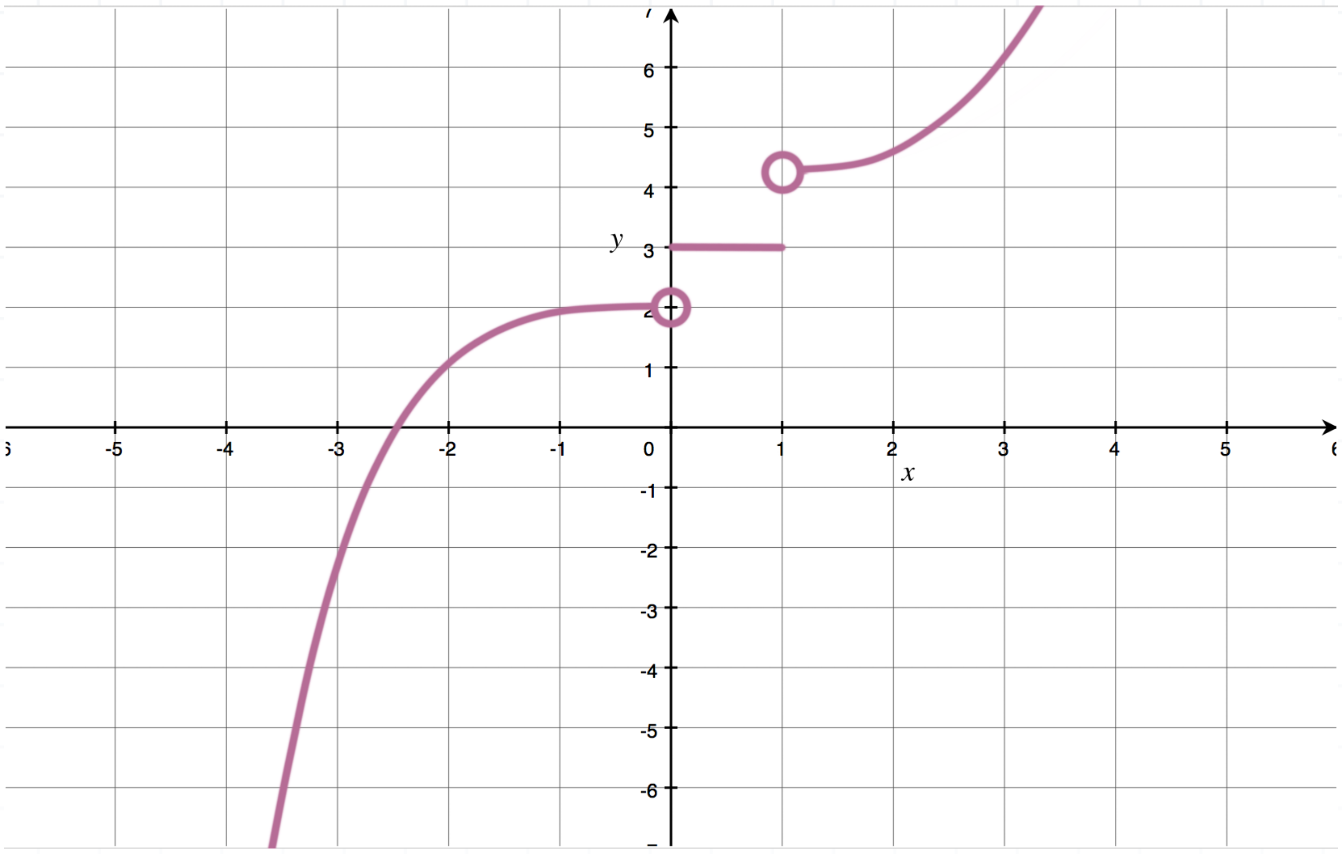
The function $h(x)$ has jump discontinuities at $x = 0$ and $x = 1$ because the left- and right-hand limits aren't equal at $x = 0$,

$$\lim_{x \to 0^-} f(x) = 2 \quad \neq \quad \lim_{x \to 0^+} f(x) = 3$$

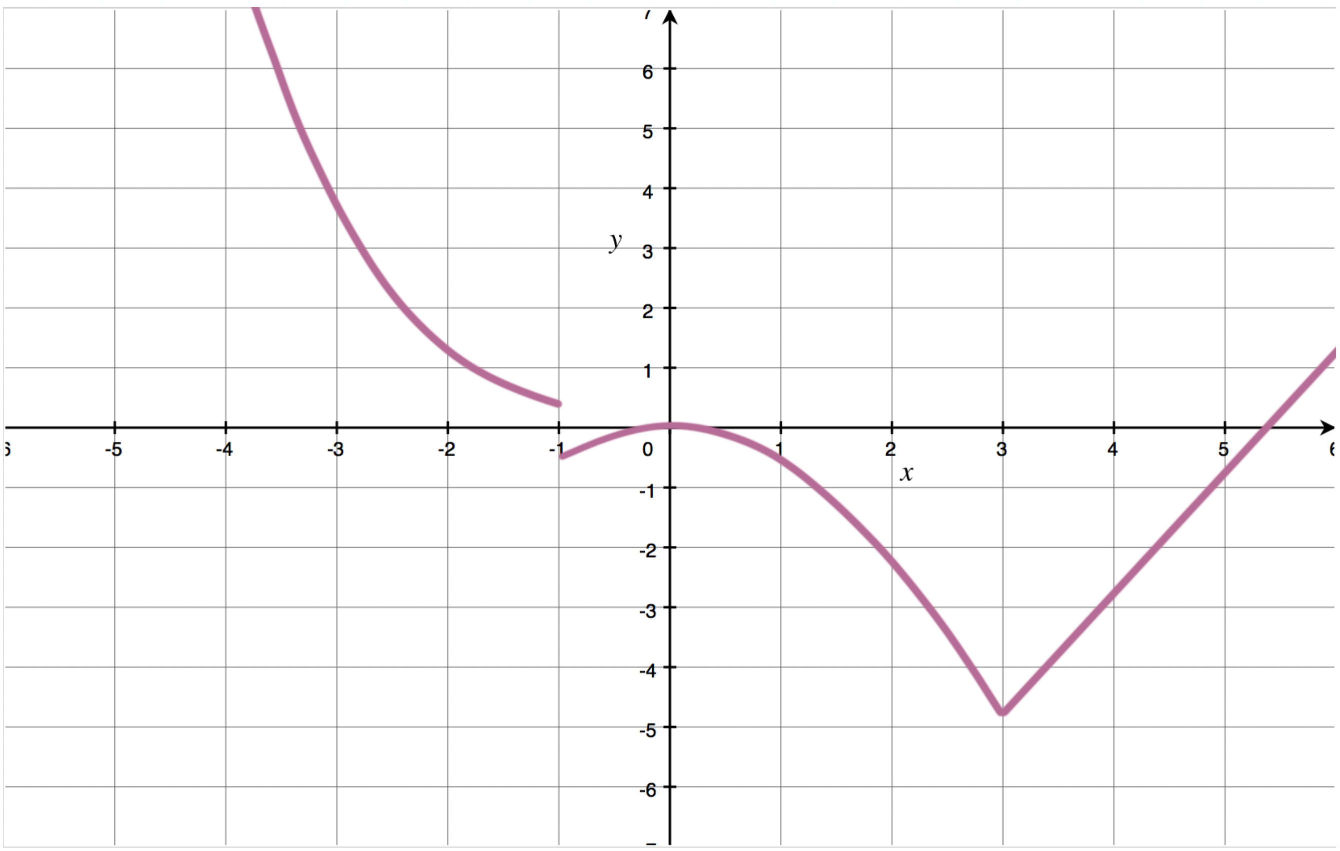
or at $x = 1$.

$$\lim_{x \to 1^-} f(x) = 3 \quad \neq \quad \lim_{x \to 1^+} f(x) = \frac{13}{3}$$

We can see the discontinuities in the function's graph, as well.



■ 3. What are the x-values where the graph of $g(x)$ has jump discontinuities?

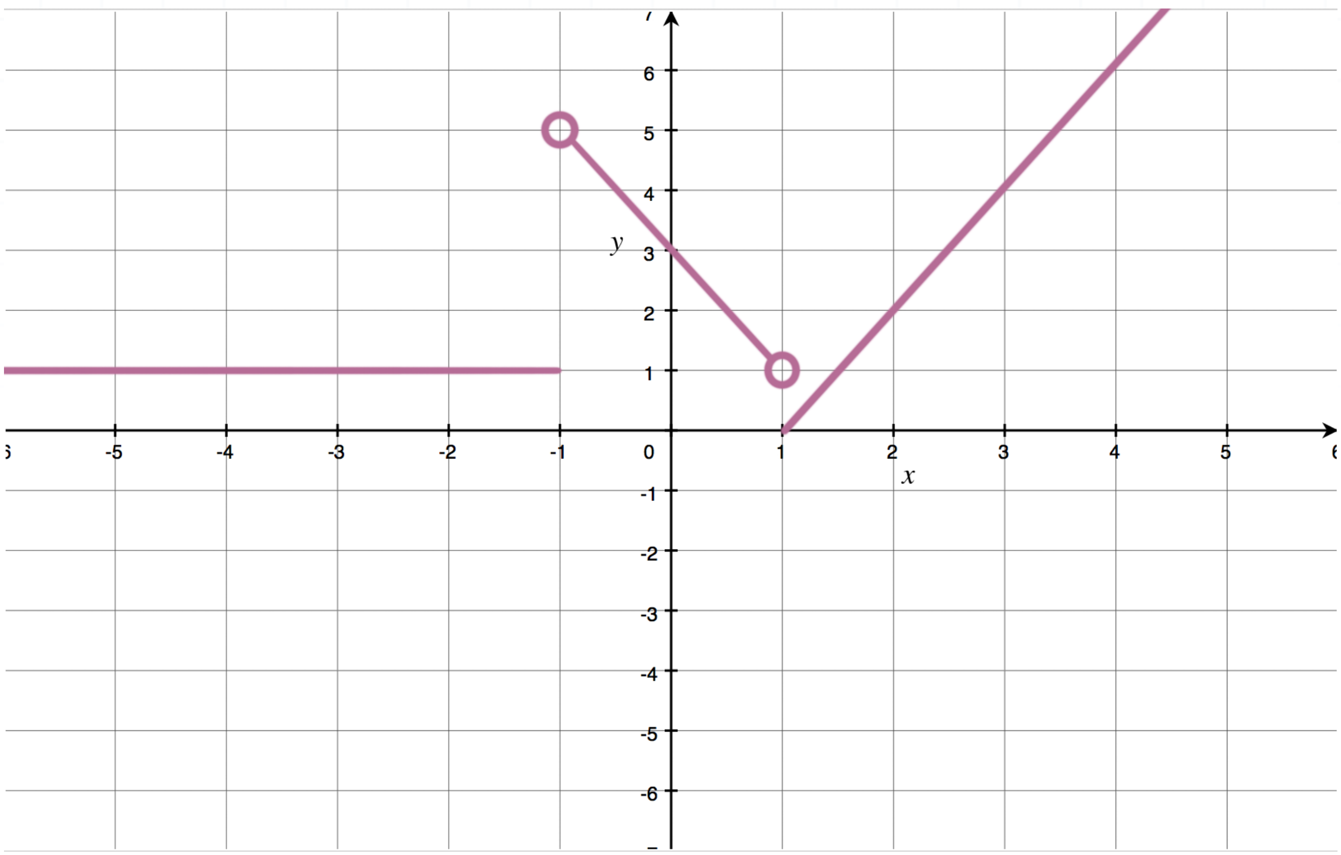


Solution:

The function $g(x)$ has a jump discontinuity at $x = -1$ because the left- and right-hand limits aren't equal there.

$$\lim_{x \to -1^-} f(x) = \frac{3}{4} \quad \neq \quad \lim_{x \to -1^+} f(x) = -\frac{2}{3}$$

■ 4. What are the x-values where the graph of $f(x)$, shown below, has jump discontinuities?



Solution:

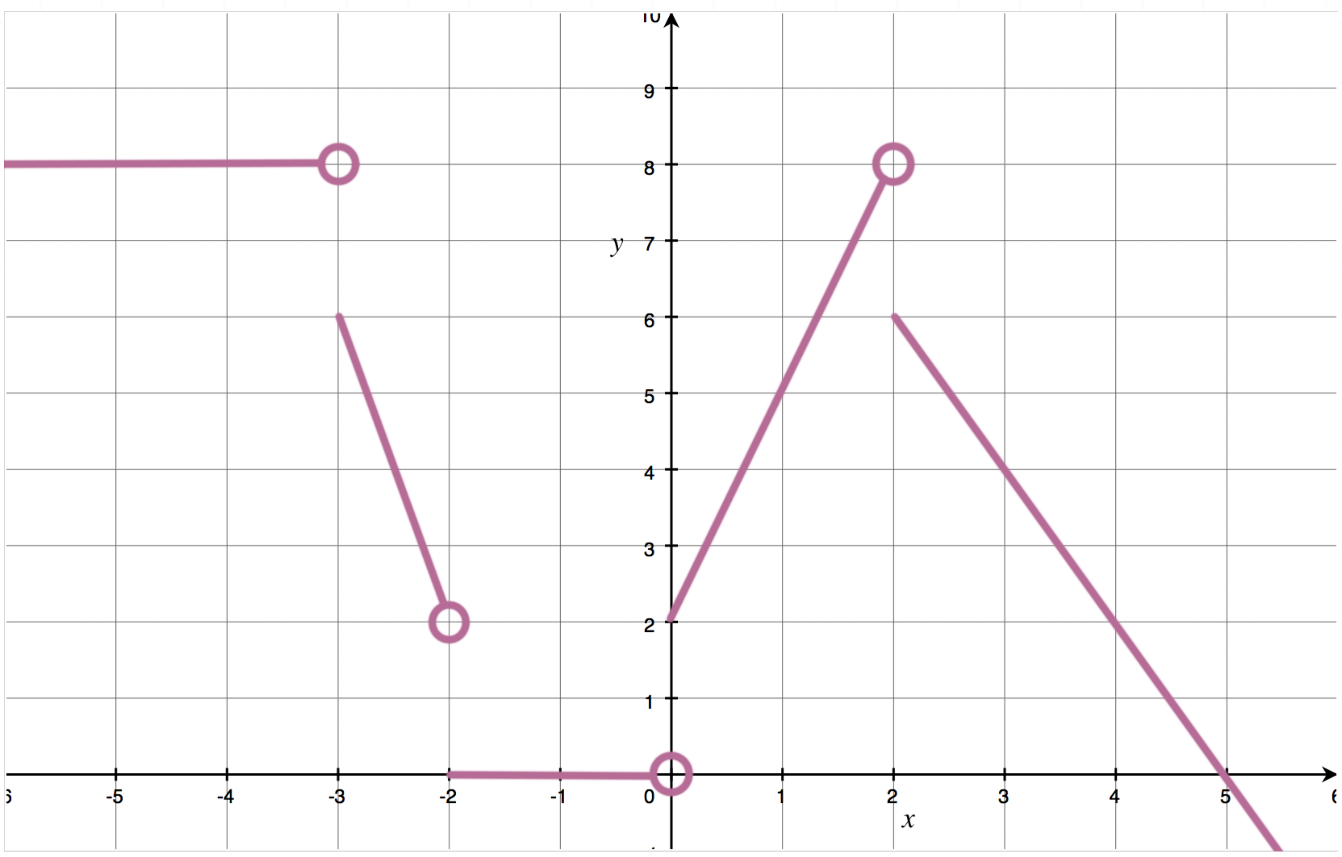
The function $f(x)$ has jump discontinuities at $x = -1$ and $x = 1$ because the left- and right-hand limits aren't equal at $x = -1$

$$\lim_{x\to -1^-} f(x) = 1 \quad \neq \quad \lim_{x\to -1^+} f(x) = 5$$

or at $x = 1$.

$$\lim_{x\to 1^-} f(x) = 1 \quad \neq \quad \lim_{x\to 1^+} f(x) = 0$$

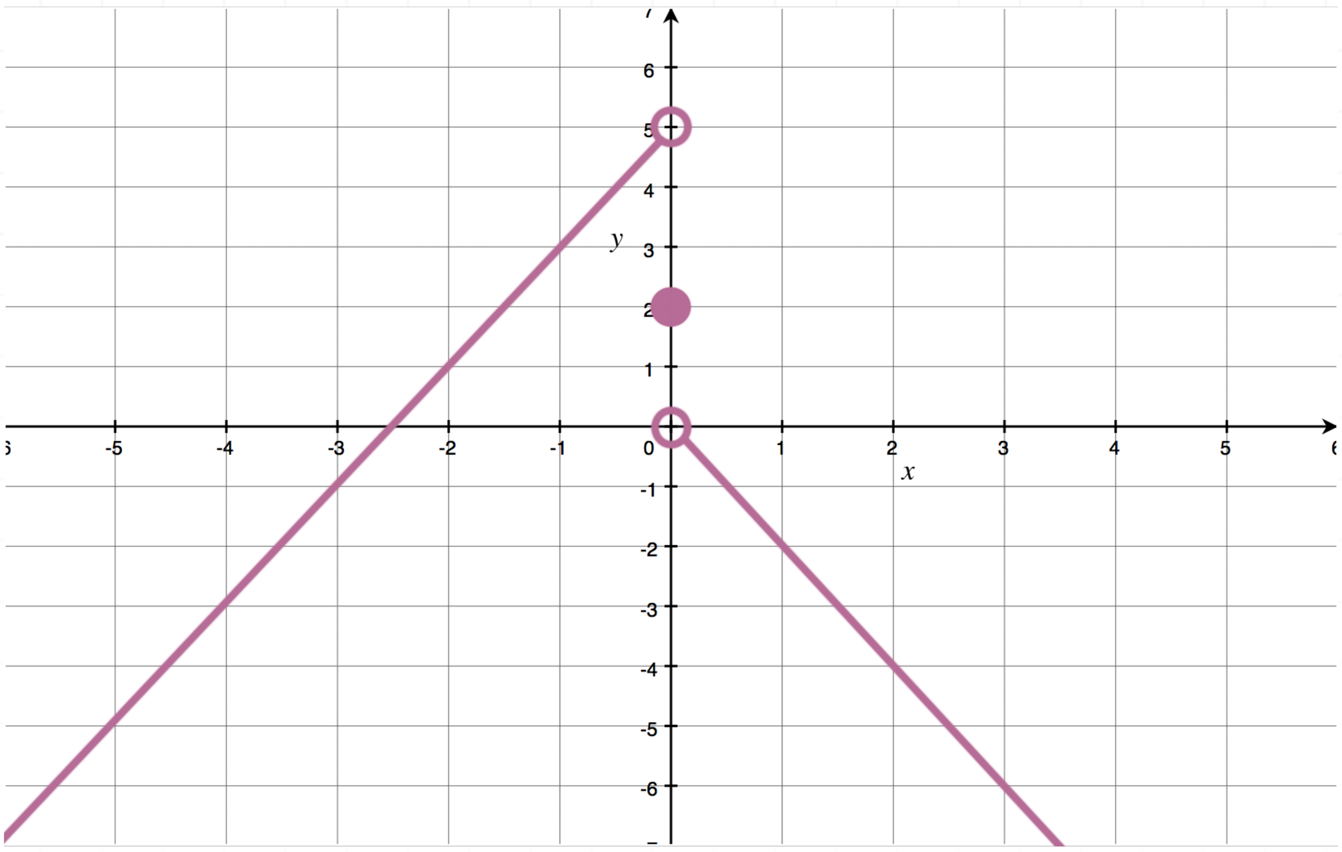
■ 5. Where are the jump discontinuities in the graph of the function shown below?



Solution:

The function has jump discontinuities at $x = -3$, $x = -2$, $x = 0$, and $x = 2$, because at each x-value, the left- and right-hand limits aren't equal.

6. What are the x-values where the graph of $h(x)$, shown below, has jump discontinuities?



Solution:

The function $h(x)$ has a jump discontinuity at $x = 0$ because the left- and right-hand limits aren't equal there.

$$\lim_{x \to 0^-} f(x) = 5 \quad \neq \quad \lim_{x \to 0^+} f(x) = 0$$

INFINITE DISCONTINUITIES

1. At what x-values does the function have infinite discontinuities?

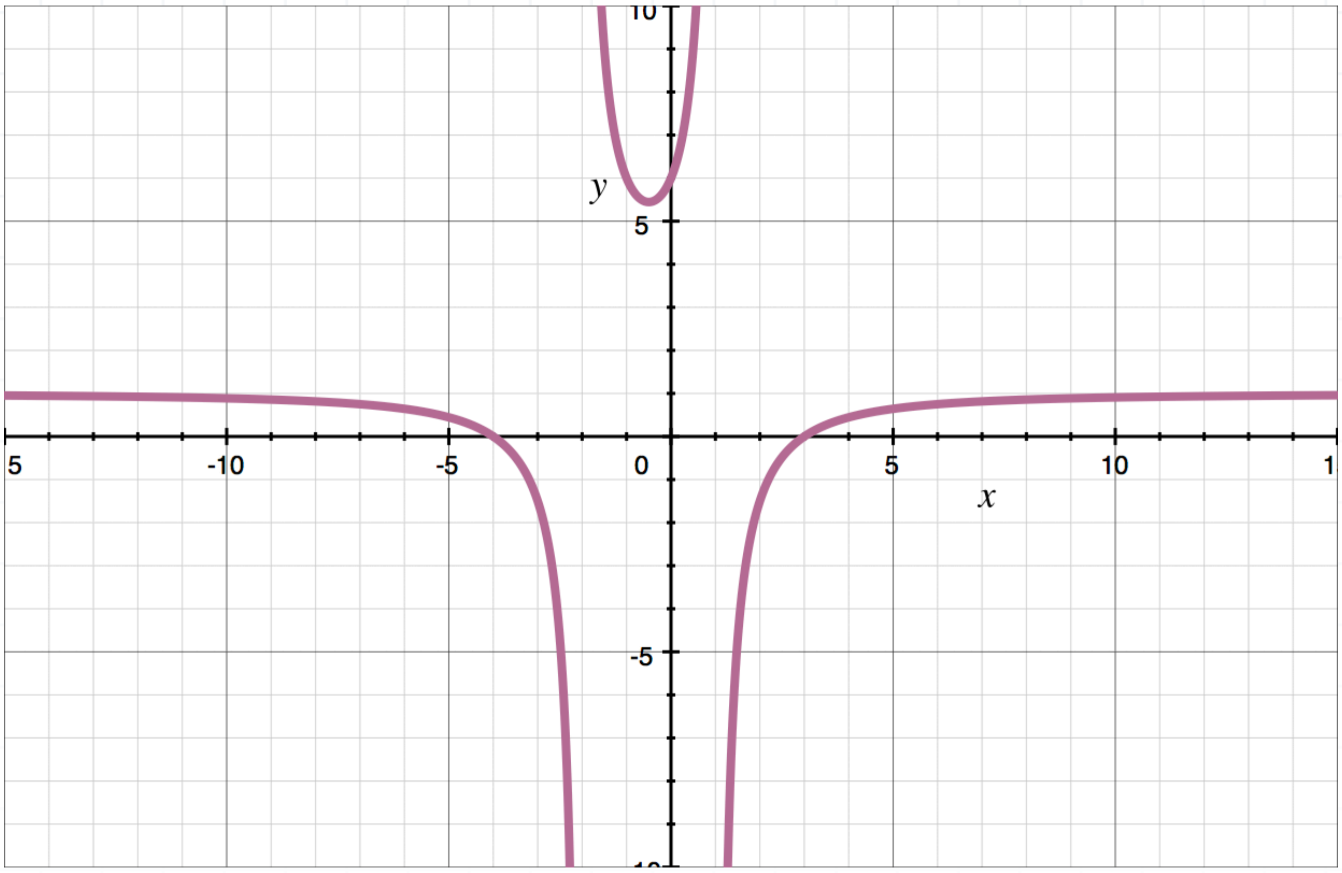
$$f(x) = \frac{x^2 + x - 12}{x^2 + x - 2}$$

Solution:

Factor the function.

$$f(x) = \frac{x^2 + x - 12}{x^2 + x - 2} = \frac{(x + 4)(x - 3)}{(x + 2)(x - 1)}$$

None of these factors cancel, which means that $x + 2 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -2$ and $x = 1$.



■ 2. Where are the infinite discontinuities of the function?

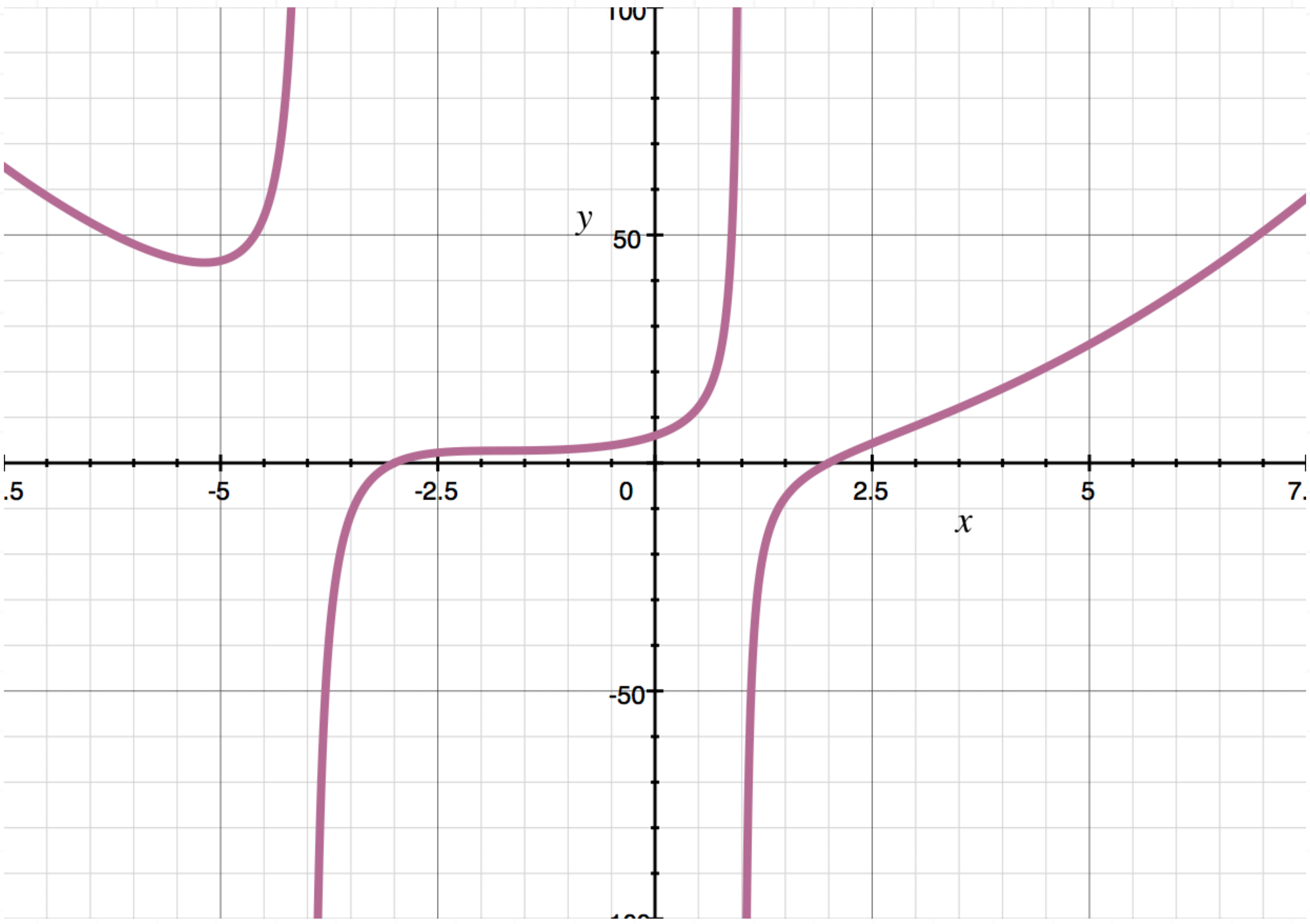
$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4}$$

Solution:

Factor the function.

$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4} = \frac{(x-2)(x^2+2x+4)(x+3)}{(x+4)(x-1)}$$

None of these factors cancel, which means that $x + 4 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -4$ and $x = 1$.



3. At what x-values does the function have infinite discontinuities?

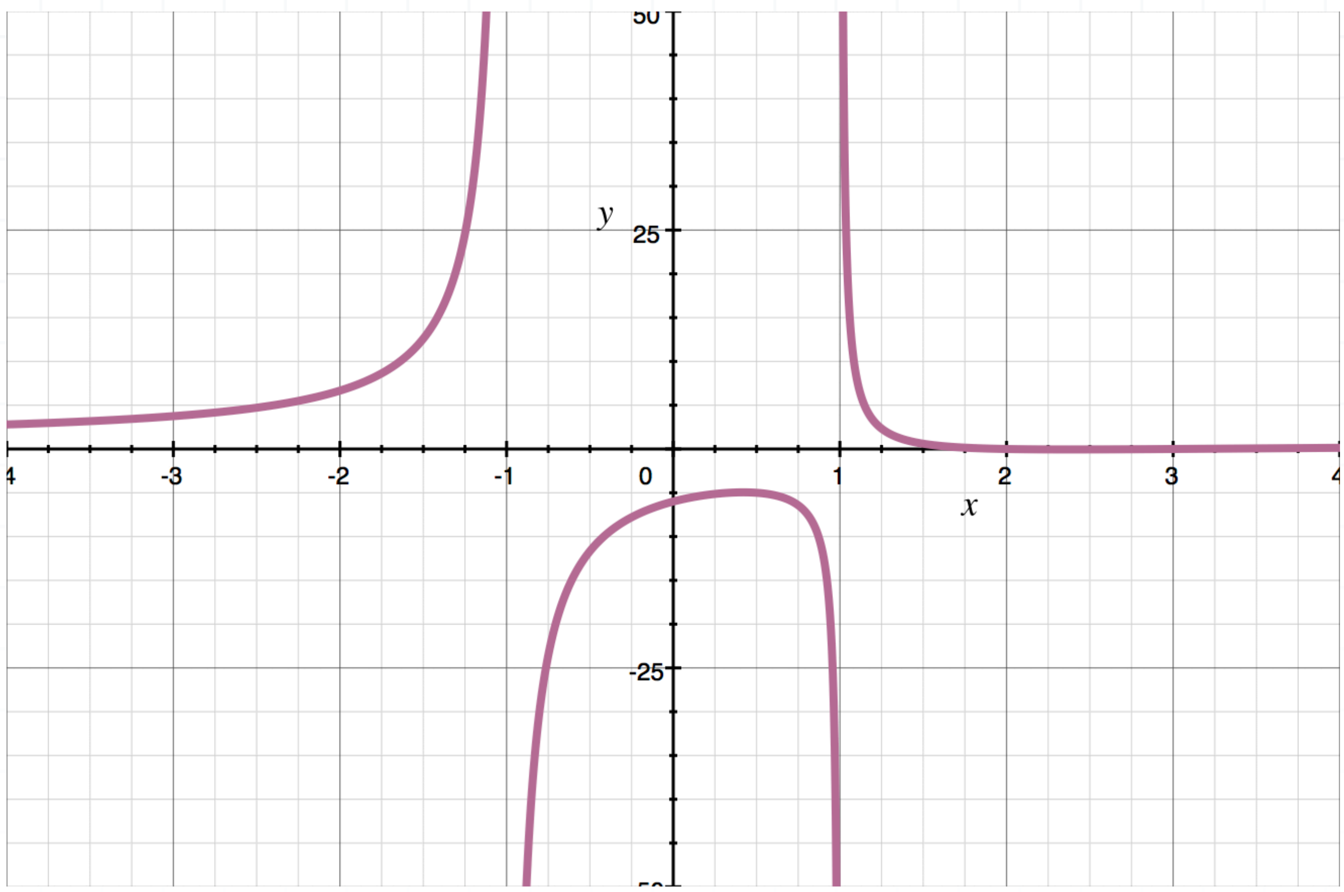
$$g(x) = \frac{x^2 - 5x + 6}{x^2 - 1}$$

Solution:

Factor the function.

$$g(x) = \frac{x^2 - 5x + 6}{x^2 - 1} = \frac{(x-3)(x-2)}{(x+1)(x-1)}$$

None of these factors cancel, which means that $x+1=0$ and $x-1=0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x=-1$ and $x=1$.



■ 4. Where are the infinite discontinuities of the function?

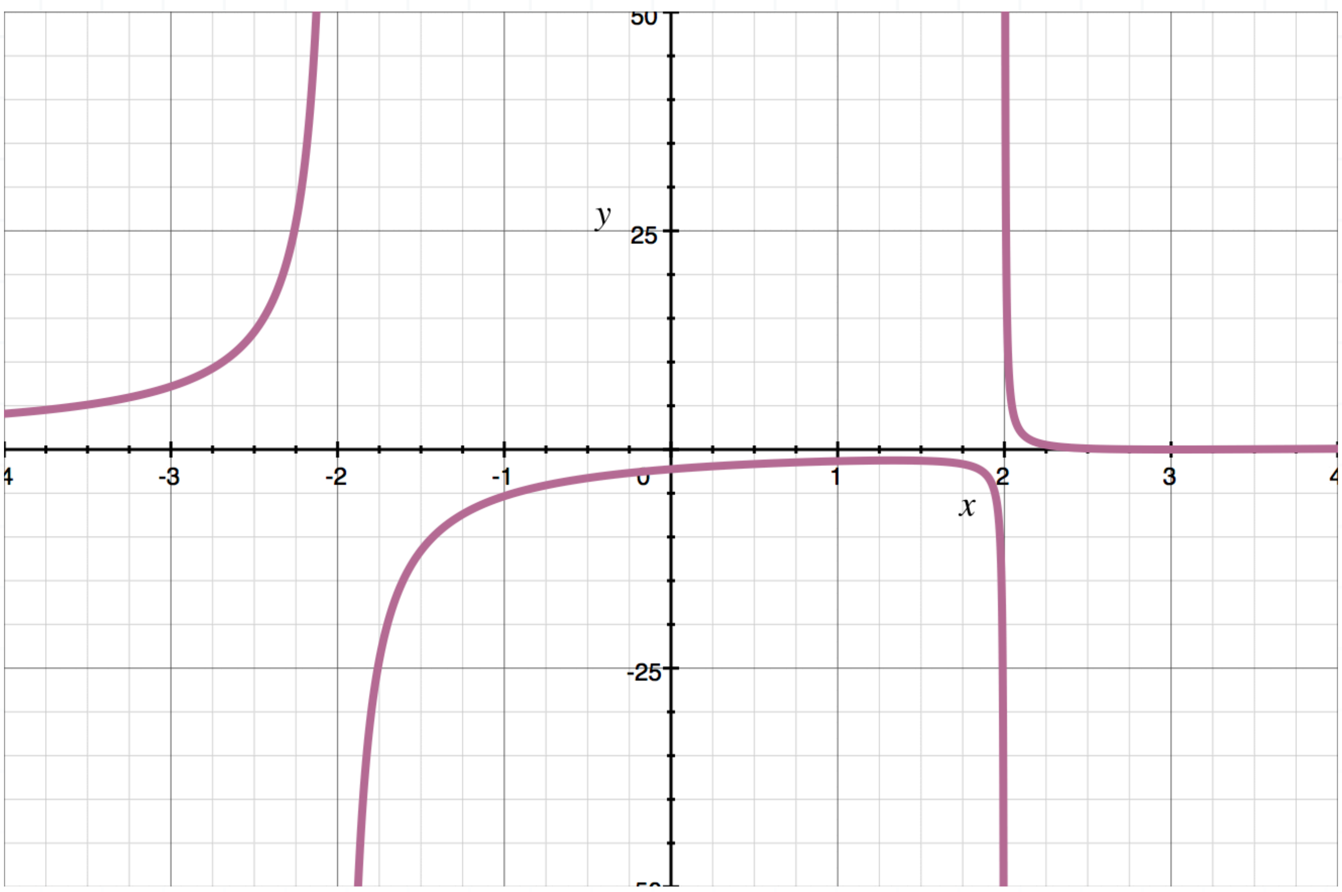
$$h(x)=\frac{x^2-6x+9}{x^2-4}$$

Solution:

Factor the function.

$$h(x)=\frac{x^2-6x+9}{x^2-4}=\frac{(x-3)^2}{(x+2)(x-2)}$$

None of these factors cancel, which means that $x+2=0$ and $x-2=0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x=-2$ and $x=2$.



■ 5. At what x-values does the function have infinite discontinuities?

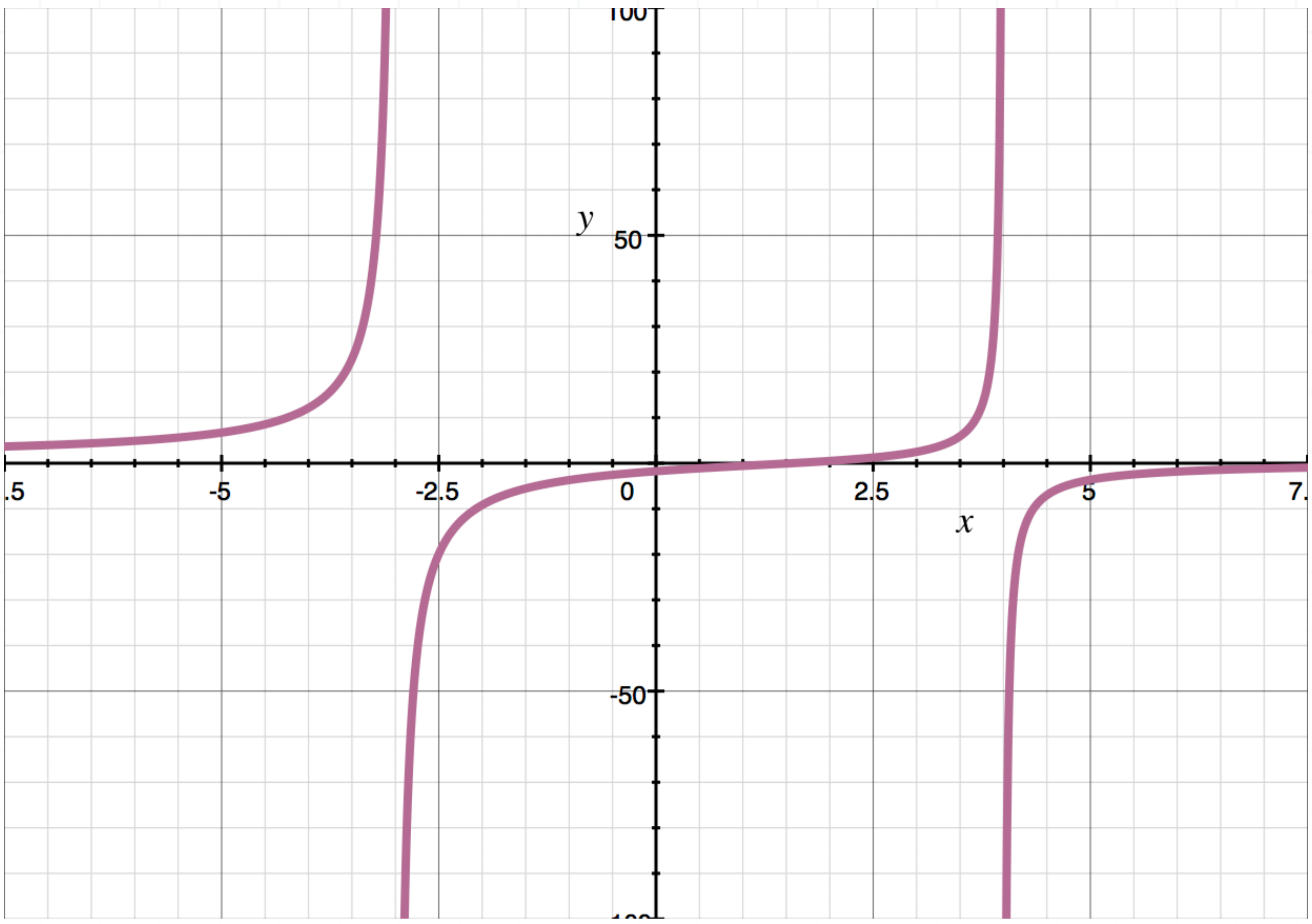
$$h(x)=\frac{x^2-15x+21}{x^2-x-12}$$

Solution:

Factor the function.

$$h(x)=\frac{x^2-15x+21}{x^2-x-12}=\frac{x^2-15x+21}{(x+3)(x-4)}$$

None of these factors cancel, which means that $x+3=0$ and $x-4=0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x=-3$ and $x=4$.



■ 6. Where are the infinite discontinuities of the function?

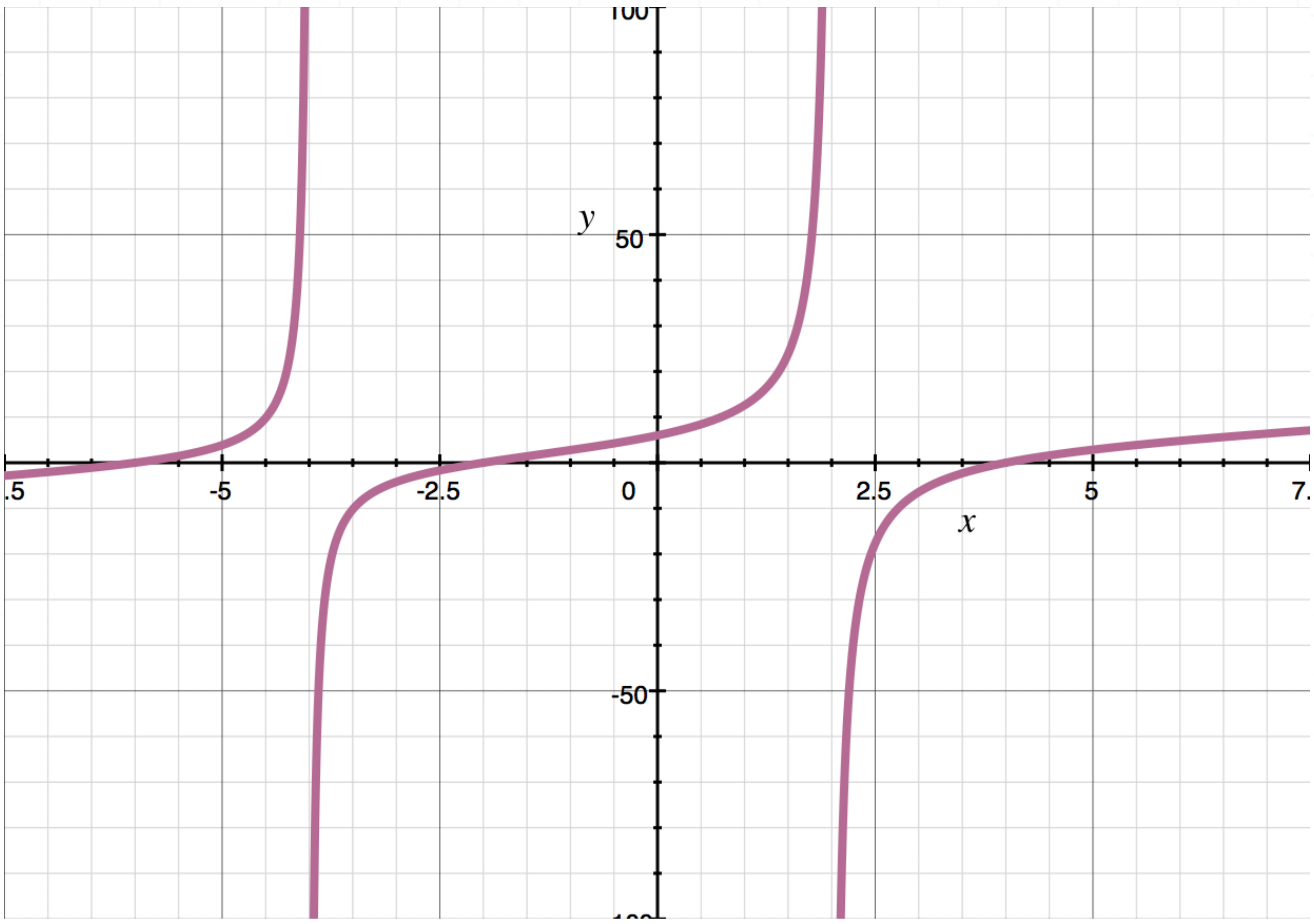
$$g(x)=\frac{x^3+4x^2-20x-48}{x^2+2x-8}$$

Solution:

Factor the function.

$$g(x) = \frac{x^3 + 4x^2 - 20x - 48}{x^2 + 2x - 8} = \frac{(x+2)(x-4)(x+6)}{(x+4)(x-2)}$$

None of these factors cancel, which means that $x + 4 = 0$ and $x - 2 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -4$ and $x = 2$.



ENDPOINT DISCONTINUITIES

1. What is the value of the limit on the interval $[0,3]$?

$$\lim_{x\to 3} -\sqrt{x+5}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x=3$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to 3^-} -\sqrt{x+5} = -2\sqrt{2} \quad \neq \quad \lim_{x\to 3^+} -\sqrt{x+5} = \text{DNE}$$

2. What is the value of the limit on the interval $[\pi,2\pi]$?

$$\lim_{x\to \pi} \sin x$$

Solution:

The limit does not exist because only the right-hand limit exists at $x=\pi$. The left-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to \pi^+} \sin x = 0 \quad \neq \quad \lim_{x \to \pi^-} \sin x = \text{DNE}$$

■ 3. What is the value of the limit on the interval $(-\infty,2]$.

$$\lim_{x \to 2} x^3 - x^2 + 4$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = 2$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to 2^-} x^3 - x^2 + 4 = 8 \quad \neq \quad \lim_{x \to 2^+} x^3 - x^2 + 4 = \text{DNE}$$

■ 4. What is the value of the limit on the interval $[4,\infty)$?

$$\lim_{x \to 4} -\frac{x+7}{x^2 - 6x + 15}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = 4$. The left-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to 4^+} -\frac{x+7}{x^2-6x+15} = -\frac{11}{7} \neq \lim_{x\to 4^-} -\frac{x+7}{x^2-6x+15} = \text{DNE}$$

■ 5. What is the value of the limit on the interval $[-9/2,5/2]$?

$$\lim_{x\to \frac{5}{2}} \frac{x+3}{x^2+x+1}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = 5/2$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to \frac{5}{2}^-} \frac{x+3}{x^2+x+1} = \frac{22}{39} \neq \lim_{x\to \frac{5}{2}^+} \frac{x+3}{x^2+x+1} = \text{DNE}$$

■ 6. What is the value of the limit on the interval $(-2,2]$?

$$\lim_{x\to -2} \sqrt{2x+4}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = -2$. The left-hand limit does not exist, which means that the one-sided limits are not equal.

$$\lim_{x\to-2^+}\sqrt{2x+4}=0 \quad \neq \quad \lim_{x\to-2^-}\sqrt{2x+4}=\text{DNE}$$

■ 7. What is the value of the limit on the interval $[-\pi, \pi]$?

$$\lim_{x\to\pi} -\frac{5\cos x}{2}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = \pi$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to\pi^-} -\frac{\cos x}{2}=\frac{5}{2} \quad \neq \quad \lim_{x\to\pi^+} -\frac{\cos x}{2}=\text{DNE}$$

INTERMEDIATE VALUE THEOREM WITH AN INTERVAL

1. The value $c = -1$ satisfies the conditions of the Intermediate Value Theorem for the function on the interval $[-3,5]$ because $f(c)$ equals what value?

$$f(x) = \frac{1}{4}(2x+5)(x-3)^2$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a,b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a,b]$. In this problem, $f(a) = f(-3) = -9$ and $f(b) = f(5) = 15$. Then,

$$f(c) = f(-1) = \frac{1}{4}(2(-1)+5)(-1-3)^2 = 12$$

The IVT requires that $f(a) \le f(c) \le f(b)$ and $-9 \le 12 \le 15$.

2. The value $c = 2$ does not satisfy the conditions of the Intermediate Value Theorem for $g(x) = 2x^2 - 11x + 4$ on the interval $[-2,8]$ because $g(c)$ equals what value?

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $g(a) = g(-2) = 34$ and $g(b) = g(8) = 44$. However, $g(c) = g(2) = 2(2)^2 - 11(2) + 4 = -10$. The IVT requires that $f(a) \le f(c) \le f(b)$, but -10 is not between 34 and 44.

■ 3. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[-3,3]$ if $h(x) = 3(x+1)^3$ and $h(c) = 24$?

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $h(a) = h(-3) = -24$ and $h(b) = h(3) = 192$. Thus, if $h(c) = 24$, the IVT requires that since $f(a) \le f(c) \le f(b)$, $a \le c \le b$. Thus, since $h(c) = 24$, we get

$$3(c+1)^3 = 24$$

$$(c+1)^3 = 8$$

$$c + 1 = 2$$

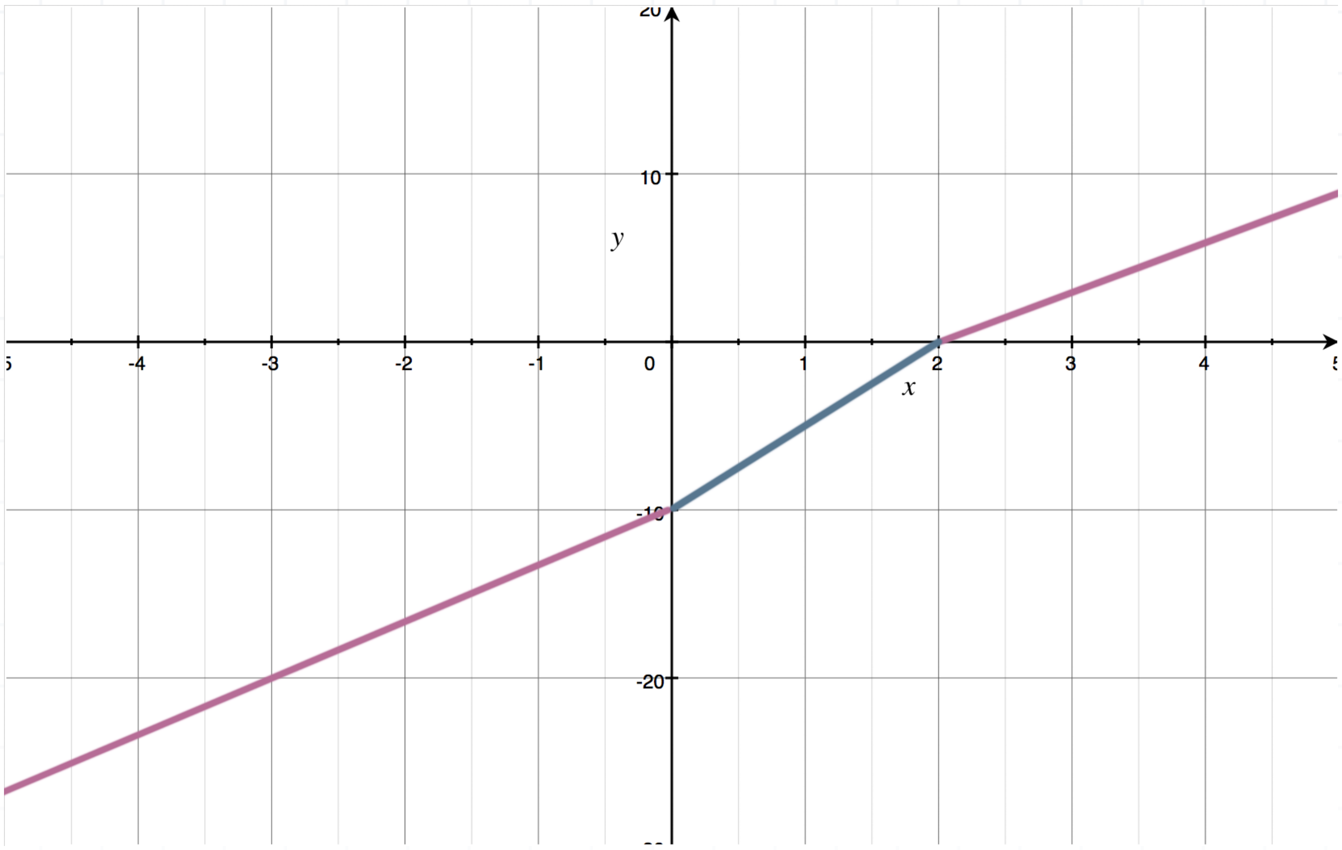
$$c = 1$$

■ 4. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[-5,6]$ if $f(c) = -6$ and

$$f(x) = \begin{cases} 3x - 10 & \text{if } x \le 0 \\ x^2 + 3x - 10 & \text{if } 0 < x < 2 \\ 3x - 6 & \text{if } x \ge 2 \end{cases}$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, first confirm that the function $f(x)$ is continuous on the interval $[-5,6]$ by evaluating the function on both sides of $x = 0$ and $x = 2$. The function is continuous, as shown in the graph below.



$f(a) = f(-5) = -25$ and $f(b) = f(6) = 12$. Thus, if $f(c) = -6$, the IVT requires that since $f(a) \le f(c) \le f(b)$, $a \le c \le b$. Thus, since $f(c) = -6$,

$$c^2 + 3c - 10 = -6$$

$$c^2 + 3c - 4 = 0$$

$$(c + 4)(c - 1) = 0$$

$$c = 1$$

when using $x^2 + 3x - 10$. Because $f(x)$ is defined piecewise, there can be other values of c that might satisfy the IVT, but there are no other values of c that satisfy the conditions.

■ 5. The value $c = 5$ satisfies the conditions of the Intermediate Value Theorem for the function on the interval $[3,9]$ because $g(c)$ equals what value?

$$g(x) = \frac{x^2 - 9}{x + 3}$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $g(a) = g(3) = 0$ and $g(b) = g(9) = 6$. Then,

$$g(c) = g(5) = \frac{5^2 - 9}{5 + 3} = 2$$

The IVT requires that $g(a) \leq g(c) \leq g(b)$ and $0 \leq 2 \leq 6$. Although $g(x)$, as defined, contains a discontinuity at $x = -3$, the function is continuous in the given interval, therefore satisfying the IVT.

■ 6. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[3,6]$ if c is a root of $h(x)$.

$$h(x) = \frac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

In this problem, $h(a) = h(3) = -12/5$ and $h(b) = h(6) = 9/8$. Thus, if $h(c)$ is a root of $h(x)$, then $h(c) = 0$. The IVT requires that since $f(a) \le f(c) \le f(b)$, $a \le c \le b$. Thus, since $h(c) = 0$, then

$$\frac{c^3 - 4c^2 - 11c + 30}{c^2 - 4} = 0$$

Solving this equation gives $c = 5$. Note that although $h(x)$, as defined, contains discontinuities at $x = -2$ and $x = 2$, the function is continuous in the given interval, therefore satisfying the IVT.

INTERMEDIATE VALUE THEOREM WITHOUT AN INTERVAL

■ 1. Use the Intermediate Value Theorem to prove that the equation $2e^x = 3\cos x$ has at least one positive solution. In what interval is that solution?

Solution:

Let $f(x) = 2e^x - 3\cos x$. The root of $f(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $f(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[0,1]$. Then,

$$f(0) = 2e^0 - 3\cos 0 = 2 - 3 = -1$$

$$f(1) = 2e^1 - 3\cos 1 = 2e - 1.6209$$

which is approximately 3.8157. Since the function's value changes sign in the interval $[0,1]$, and since $f(x)$ is continuous in the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 2. Use the Intermediate Value Theorem to prove that the equation $3\sin x + 7 = x^2 - 2x - 2$ has at least one positive solution. In what interval is that solution?

Solution:

Let $g(x) = 3\sin x + 7 - (x^2 - 2x - 2)$. The root of $g(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $g(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[3,4]$. Then,

$$g(3) = 3\sin(3) + 7 - (3^2 - 2(3) - 2) = 6.4234$$

$$g(4) = 3\sin(4) + 7 - (4^2 - 2(4) - 2) = -1.2704$$

Since the function's value changes sign in the interval $[3,4]$, and $g(x)$ is continuous on the interval, the function has a zero in that interval.

■ 3. Use the Intermediate Value Theorem to prove that the equation $x^6 - 9x^4 + 7 = x^5 - 8x^3 - 9$ has at least one positive solution. In what interval is that solution?

Solution:

Let $h(x) = (x^6 - 9x^4 + 7) - (x^5 - 8x^3 - 9)$. The root of $h(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $h(x)$ has a root in a certain closed interval if the function's value of changes sign in that closed interval.

Consider the interval $[1,2]$. Then,

$$h(1) = ((1)^6 - 9(1)^4 + 7) - ((1)^5 - 8(1)^3 - 9) = 15$$

$$h(2) = ((2)^6 - 9(2)^4 + 7) - ((2)^5 - 8(2)^3 - 9) = -32$$

Since the function's value changes sign in the interval $[1,2]$, and $h(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 4. Use the Intermediate Value Theorem to prove that the equation $4e^{x-3} = 2(x^3 - 5x + 9)$ has at least one negative solution. In what interval is that solution?

Solution:

Let $f(x) = 4e^{x-3} - 2(x^3 - 5x + 9)$. The root of $f(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $f(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[-3, -2]$. Then,

$$f(-3) = 4e^{-3-3} - 2((-3)^3 - 5(-3) + 9) = 6.0099$$

$$f(-2) = 4e^{-2-3} - 2((-2)^3 - 5(-2) + 9) = -21.97$$

Since the function's value changes sign in the interval $[-3,-2]$, and $f(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 5. Use the Intermediate Value Theorem to show that the equation has at least one positive solution. In what interval is that solution?

$$6e^{-x} = -\left(\frac{1}{5}x^2 - 4x + 9\right)$$

Solution:

Let

$$g(x) = 6e^{-x} + \frac{1}{5}x^2 - 4x + 9$$

The root of $g(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $g(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[2,3]$. Then,

$$g(2) = 6e^{-2} + \frac{1}{5}(2)^2 - 4(2) + 9 = 2.612$$

$$g(3) = 6e^{-3} + \frac{1}{5}(3)^2 - 4(3) + 9 = -0.9013$$

Since the function's value changes sign in the interval $[2,3]$, and $g(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 6. Use the Intermediate Value Theorem to show that the equation $2\sin(4x-1)=\cos(2x-3)$ has at least one negative solution. In what interval is that solution?

Solution:

Let $h(x)=2\sin(4x-1)-\cos(2x-3)$. The root of $h(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $h(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[-2,-1]$. Then,

$$h(-2)=2\sin(4(-2)-1)-\cos(2(-2)-3)=-1.578$$

$$h(-1)=2\sin(4(-1)-1)-\cos(2(-1)-3)=1.634$$

Since the function's value changes sign in the interval $[-2,-1]$, and $h(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

SOLVING WITH SUBSTITUTION

■ 1. What is the value of the limit?

$$\lim_{x\to 3} -x^4 + x^3 + 2x^2$$

Solution:

Use substitution and plug $x = 3$ into the function.

$$\lim_{x\to 3} -x^4 + x^3 + 2x^2$$

$$-(3)^4 + (3)^3 + 2(3)^2$$

$$-36$$

■ 2. What is the value of the limit?

$$\lim_{x\to 7} \frac{x^2 - 5}{x^2 + 5}$$

Solution:

Use substitution and plug $x = 7$ into the function.

$$\lim_{x \to 7} \frac{x^2 - 5}{x^2 + 5}$$

$$\frac{7^2 - 5}{7^2 + 5}$$

$$\frac{44}{54} = \frac{22}{27}$$

■ 3. What is the value of the limit.

$$\lim_{x \to -2} \frac{x^3 - 5x^2 + 4x - 6}{x^2 + 7x + 6}$$

Solution:

Use substitution and plug $x = -2$ into the function.

$$\lim_{x \to -2} \frac{x^3 - 5x^2 + 4x - 6}{x^2 + 7x + 6}$$

$$\frac{(-2)^3 - 5(-2)^2 + 4(-2) - 6}{(-2)^2 + 7(-2) + 6}$$

$$\frac{21}{2}$$

SOLVING WITH FACTORING

■ 1. What is the value of the limit?

$$\lim_{x \to -7} \frac{6x^3 + 42x^2}{2x^2 + 26x + 84}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to -7} \frac{6x^3 + 42x^2}{2x^2 + 26x + 84}$$

$$\lim_{x \to -7} \frac{6x^2(x + 7)}{2(x + 6)(x + 7)}$$

$$\lim_{x \to -7} \frac{6x^2}{2(x + 6)}$$

Now we can evaluate the limit at $x = -7$ using substitution.

$$\frac{6(-7)^2}{2\left(-7 + 6\right)}$$

$$-147$$

2. What is the value of the limit?

$$\lim_{x \to 10} \frac{3x^2 - 39x + 90}{x^2 - 3x - 70}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to 10} \frac{3x^2 - 39x + 90}{x^2 - 3x - 70}$$

$$\lim_{x \to 10} \frac{3(x - 10)(x - 3)}{(x - 10)(x + 7)}$$

$$\lim_{x \to 10} \frac{3(x - 3)}{x + 7}$$

Now we can evaluate the limit at $x = 10$ using substitution.

$$\frac{3(10 - 3)}{10 + 7}$$

$$\frac{21}{17}$$

3. What is the value of the limit?

$$\lim_{x \to -8} \frac{2x^2 + 10x - 48}{8x + 64}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to -8} \frac{2x^2 + 10x - 48}{8x + 64}$$

$$\lim_{x \to -8} \frac{2(x+8)(x-3)}{8(x+8)}$$

$$\lim_{x \to -8} \frac{x-3}{4}$$

Now we can evaluate the limit at $x = -8$ using substitution.

$$\frac{-8-3}{4}$$

$$-\frac{11}{4}$$

■ 4. What is the value of the limit?

$$\lim_{x \to 7} \frac{x^3 - x^2 - 42x}{2x^2 - 20x + 42}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to 7} \frac{x^3 - x^2 - 42x}{2x^2 - 20x + 42}$$

$$\lim_{x \to 7} \frac{x(x-7)(x+6)}{2(x-3)(x-7)}$$

$$\lim_{x \to 7} \frac{x(x+6)}{2(x-3)}$$

Now we can evaluate the limit at $x = 7$ using substitution.

$$\frac{7(7+6)}{2(7-3)}$$

$$\frac{91}{8}$$

■ 5. What is the value of the limit?

$$\lim_{x \to 8} \frac{x^2 + 2x - 80}{2x^3 - 24x^2 + 64x}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to 8} \frac{x^2 + 2x - 80}{2x^3 - 24x^2 + 64x}$$

$$\lim_{x \to 8} \frac{(x + 10)(x - 8)}{2x(x - 8)(x - 4)}$$

$$\lim_{x \to 8} \frac{x + 10}{2x(x - 4)}$$

Now we can evaluate the limit at $x = 8$ using substitution.

$$\frac{8 + 10}{2(8)(8 - 4)}$$

$$\frac{18}{64} = \frac{9}{32}$$

SOLVING WITH CONJUGATE METHOD

■ 1. What is the value of the limit?

$$\lim_{x \to 16} \frac{3(x-16)}{\sqrt{x}-4}$$

Solution:

Since the limit cannot be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x \to 16} \frac{3(x-16)(\sqrt{x}+4)}{(\sqrt{x}-4)(\sqrt{x}+4)}$$

$$\lim_{x \to 16} \frac{3(x-16)(\sqrt{x}+4)}{x-16}$$

$$\lim_{x \to 16} 3(\sqrt{x}+4)$$

Then use substitution to evaluate the limit.

$$3(\sqrt{16}+4)$$

$$3(4+4)$$

$$24$$

■ 2. What is the value of the limit?

$$\lim_{x\to 9}\frac{5(\sqrt{x}-3)}{x-9}$$

Solution:

Since the limit cannot be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x\to 9}\frac{5(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}$$

$$\lim_{x\to 9}\frac{5(x-9)}{(x-9)(\sqrt{x}+3)}$$

$$\lim_{x\to 9}\frac{5}{\sqrt{x}+3}$$

Then use substitution to evaluate the limit.

$$\frac{5}{\sqrt{9}+3}$$

$$\frac{5}{3+3}$$

$$\frac{5}{6}$$

■ 3. What is the value of the limit?

$$\lim_{x\to 25}\frac{2(x-25)}{\sqrt{x}-5}$$

Solution:

Since the limit cannot be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x\to 25}\frac{2(x-25)(\sqrt{x}+5)}{(\sqrt{x}-5)(\sqrt{x}+5)}$$

$$\lim_{x\to 25}\frac{2(x-25)(\sqrt{x}+5)}{x-25}$$

$$\lim_{x\to 25}2(\sqrt{x}+5)$$

Then use substitution to evaluate the limit.

$$2(\sqrt{25}+5)$$

$$2(5+5)$$

$$20$$

■ 4. What is the value of the limit?

$$\lim_{x \to 49} \frac{x - 49}{3(\sqrt{x} - 7)}$$

Solution:

Since the limit cannot be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x \to 49} \frac{(x - 49)(\sqrt{x} + 7)}{3(\sqrt{x} - 7)(\sqrt{x} + 7)}$$

$$\lim_{x \to 49} \frac{(x - 49)(\sqrt{x} + 7)}{3(x - 49)}$$

$$\lim_{x \to 49} \frac{\sqrt{x} + 7}{3}$$

Then use substitution to evaluate the limit.

$$\frac{\sqrt{49} + 7}{3}$$

$$\frac{7 + 7}{3}$$

$$\frac{14}{3}$$

■ 5. What is the value of the limit?

$$\lim_{x \to 1} \frac{8(x-1)}{3(\sqrt{x}-1)}$$

Solution:

Since the limit cannot be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x \to 1} \frac{8(x-1)(\sqrt{x}+1)}{3(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$\lim_{x \to 1} \frac{8(x-1)(\sqrt{x}+1)}{3(x-1)}$$

$$\lim_{x \to 1} \frac{8(\sqrt{x}+1)}{3}$$

Then use substitution to evaluate the limit.

$$\frac{8(\sqrt{1}+1)}{3}$$

$$\frac{8(1+1)}{3}$$

$$\frac{16}{3}$$

INFINITE LIMITS AND VERTICAL ASYMPTOTES

1. What is the value of the limit?

$$\lim_{x\to 2} \frac{x^2 - x - 6}{-3x^2 - 3x + 18}$$

Solution:

Factor to simplify the limit.

$$\lim_{x\to 2} \frac{x^2 - x - 6}{-3x^2 - 3x + 18}$$

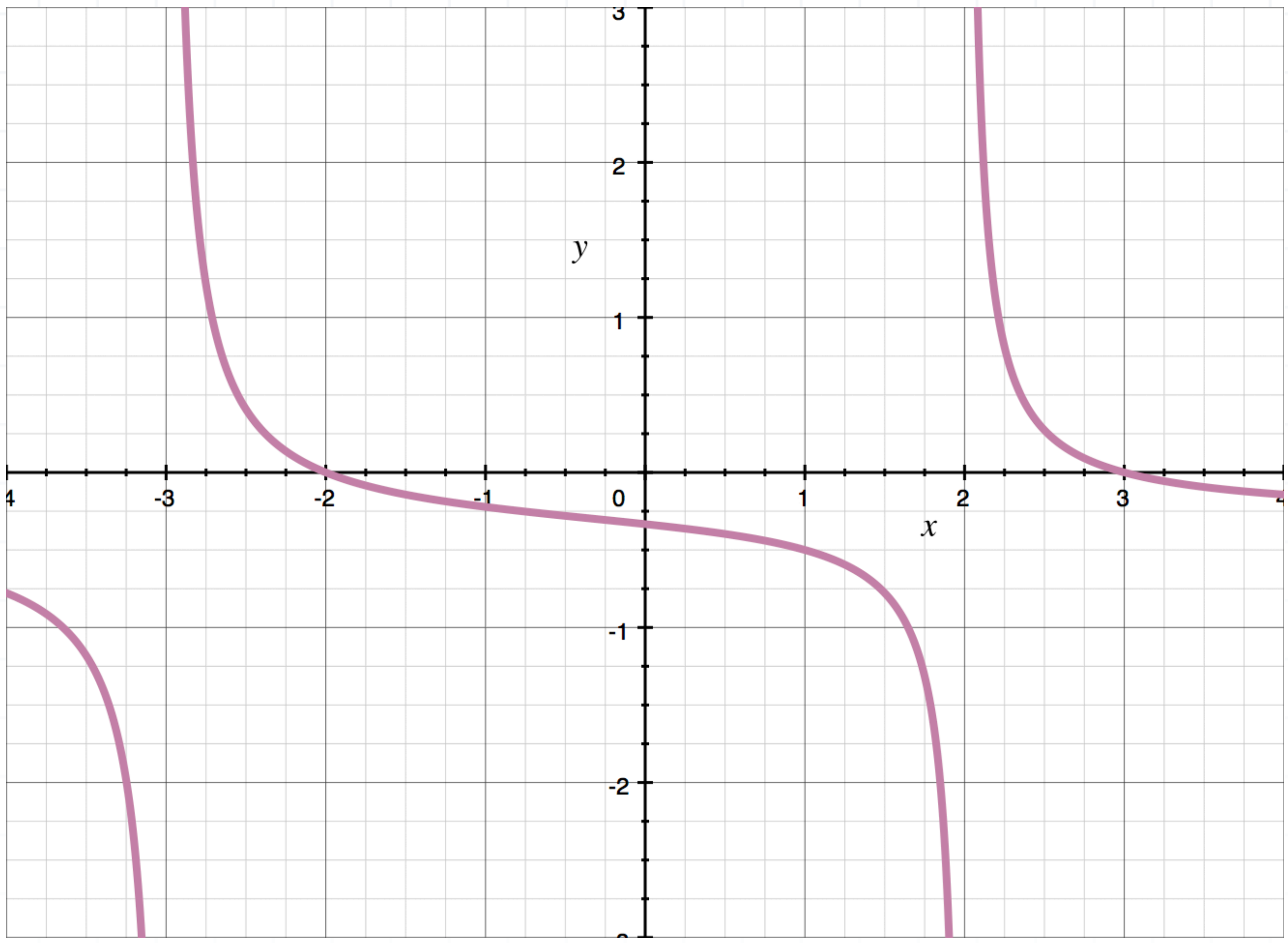
$$\lim_{x\to 2} \frac{(x-3)(x+2)}{-3(x+3)(x-2)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x\to 2^-} \frac{(x-3)(x+2)}{-3(x+3)(x-2)} = -\infty$$

$$\lim_{x\to 2^+} \frac{(x-3)(x+2)}{-3(x+3)(x-2)} = \infty$$

and they are not the same. Therefore, the limit does not exist (DNE). The graph is shown below.



■ 2. What is the value of the limit?

$$\lim_{x \to -1} \frac{x^2 + x - 6}{4x^2 + 16x + 12}$$

Solution:

Factor to simplify the limit.

$$\lim_{x \to -1} \frac{x^2 + x - 6}{4x^2 + 16x + 12}$$

$$\lim_{x \to -1} \frac{(x+3)(x-2)}{4(x+3)(x+1)}$$

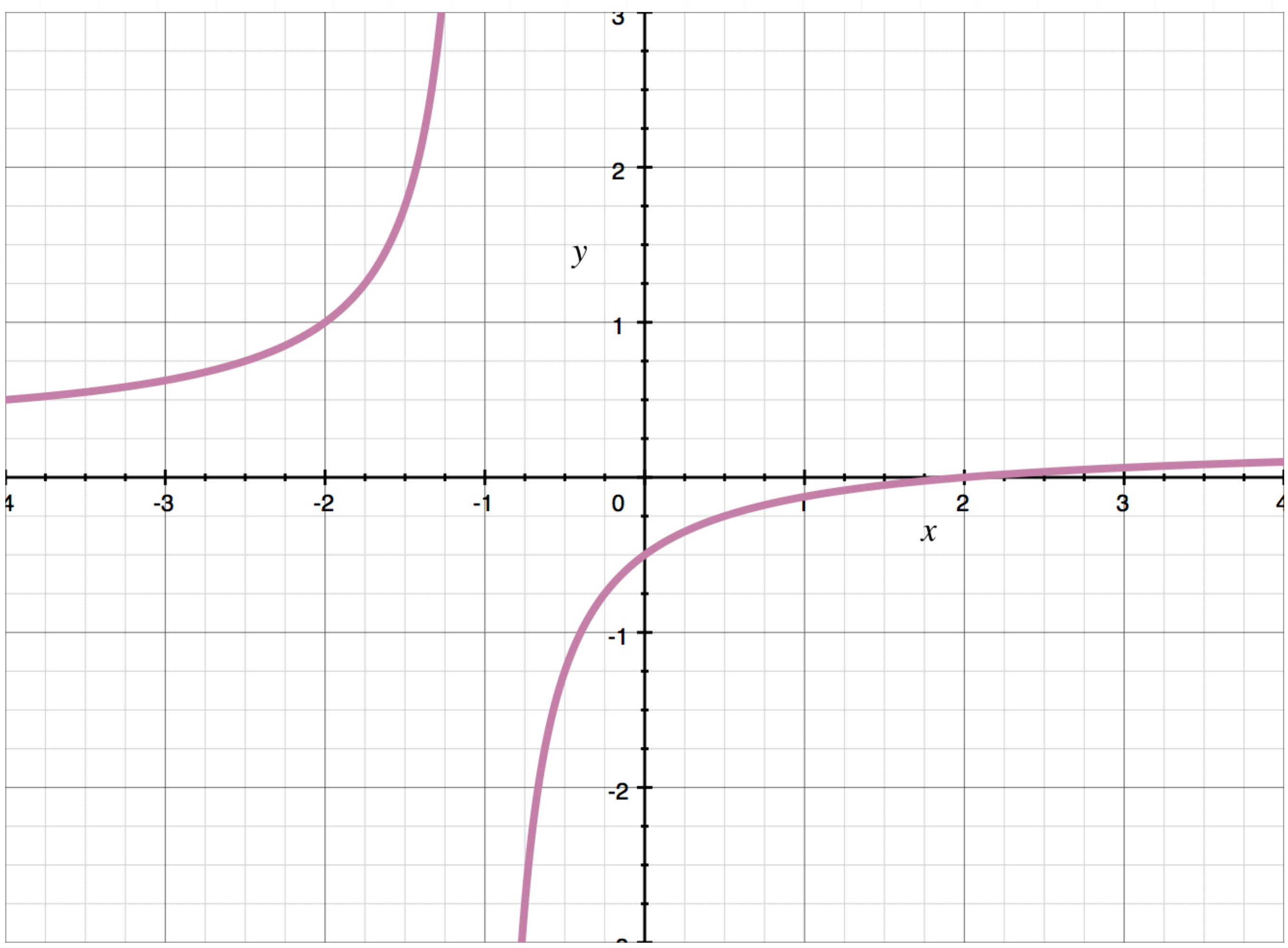
$$\lim_{x \to -1} \frac{x-2}{4(x+1)}$$

No other factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to -1^-} \frac{x-2}{4(x+1)} = \infty$$

$$\lim_{x \to -1^+} \frac{x-2}{4(x+1)} = -\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



■ 3. What is the value of the limit?

$$\lim_{x \to -4} \frac{x+5}{-4x-16}$$

Solution:

Factor to simplify the limit.

$$\lim_{x \to -4} \frac{x+5}{-4x-16}$$

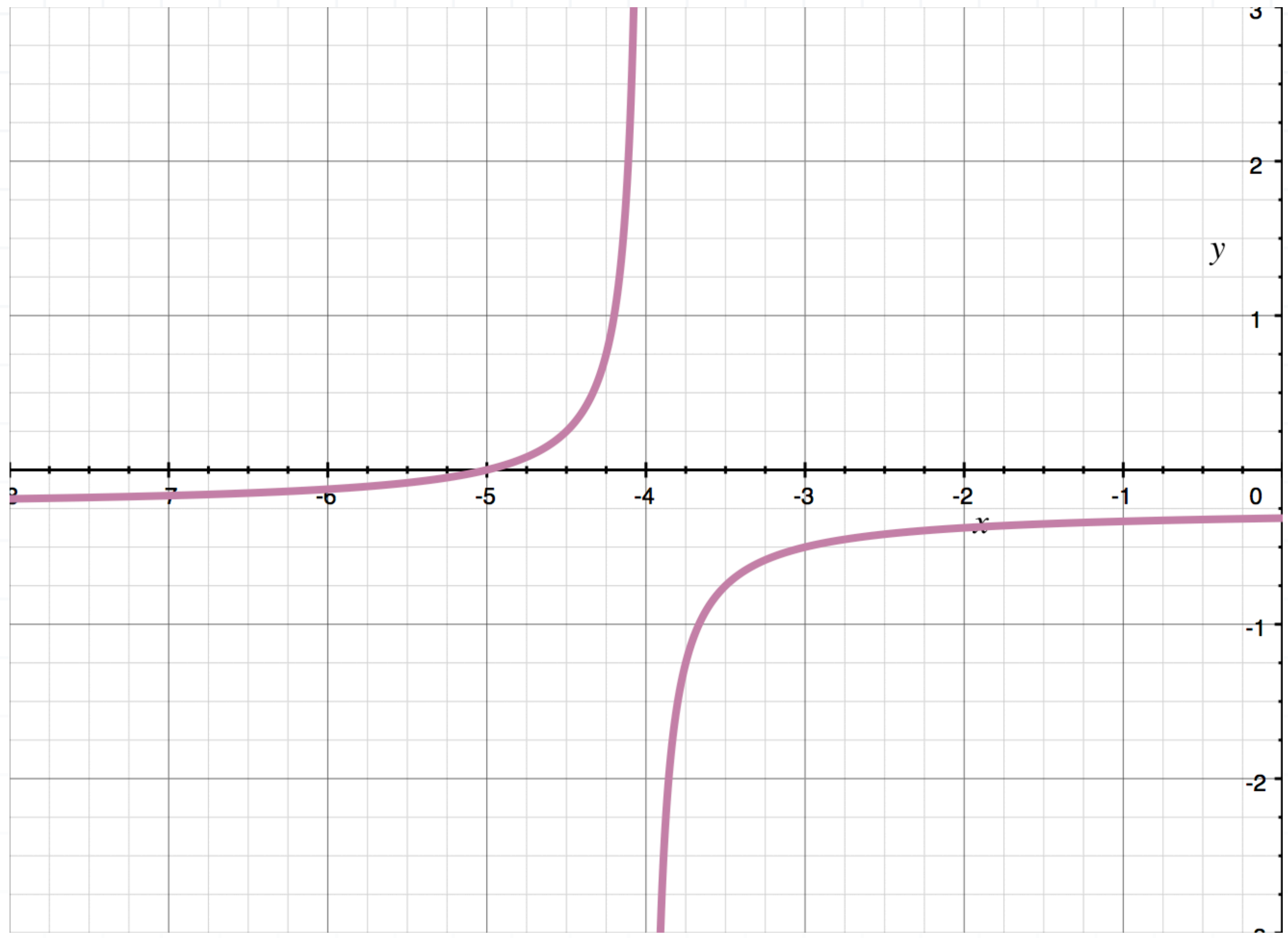
$$\lim_{x \to -4} \frac{x+5}{-4(x+4)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to -4^-} \frac{x+5}{-4(x+4)} = \infty$$

$$\lim_{x \to -4^+} \frac{x+5}{-4(x+4)} = -\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



■ 4. What is the value of the limit?

$$\lim_{x\to-1}\frac{x^2-9}{3x^2-6x-9}$$

Solution:

Factor to simplify the limit.

$$\lim_{x\to-1}\frac{x^2-9}{3x^2-6x-9}$$

$$\lim_{x\to-1}\frac{(x+3)(x-3)}{3(x-3)(x+1)}$$

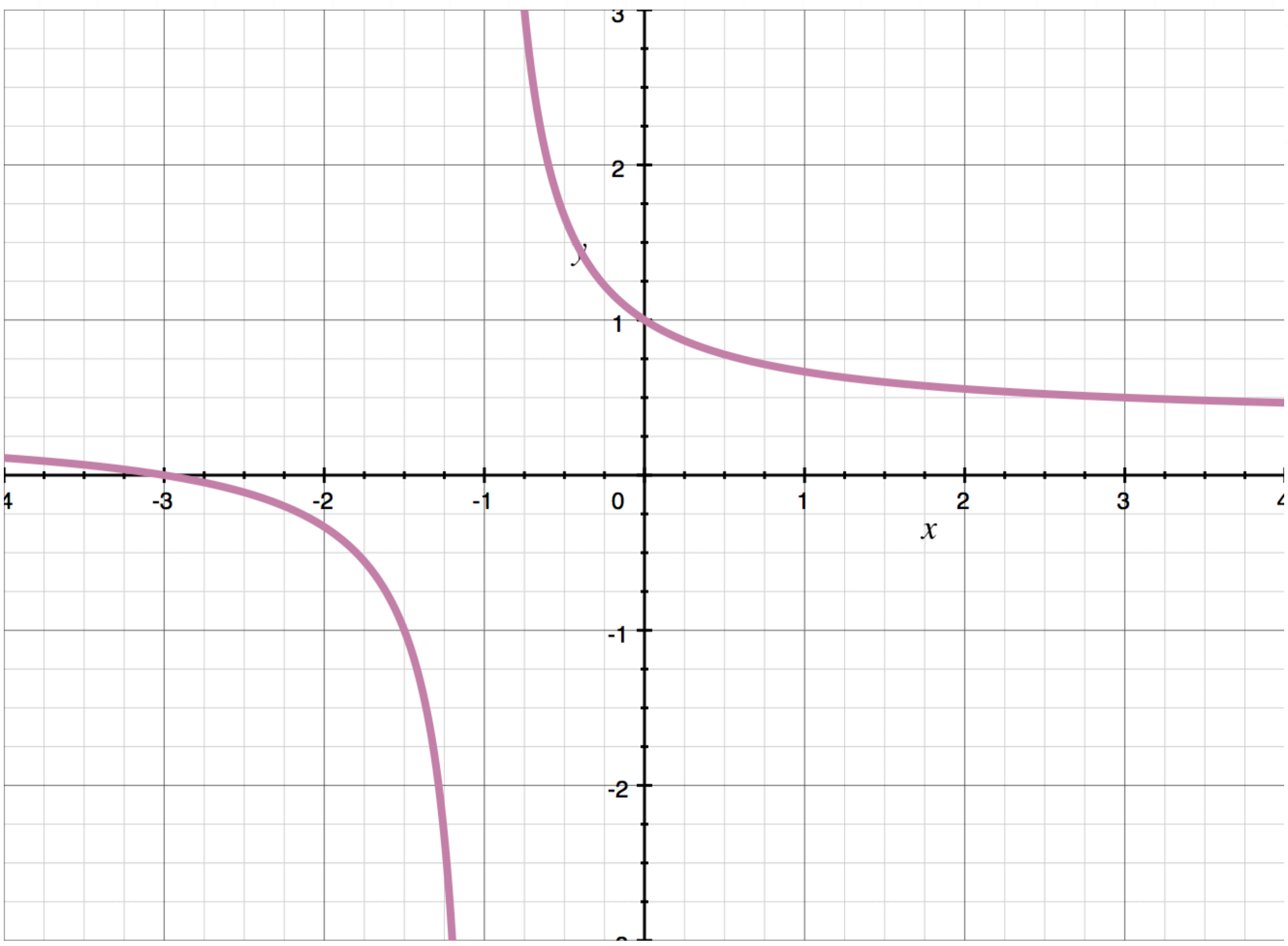
$$\lim_{x \to -1} \frac{x+3}{3(x+1)}$$

No other factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to -1^-} \frac{x+3}{3(x+1)} = -\infty$$

$$\lim_{x \to -1^+} \frac{x+3}{3(x+1)} = \infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



■ 5. What is the value of the limit?

$$\lim_{x\to 3} \frac{x^2 - 4x}{x^2 - 2x - 3}$$

Solution:

Factor to simplify the limit.

$$\lim_{x\to 3} \frac{x^2 - 4x}{x^2 - 2x - 3}$$

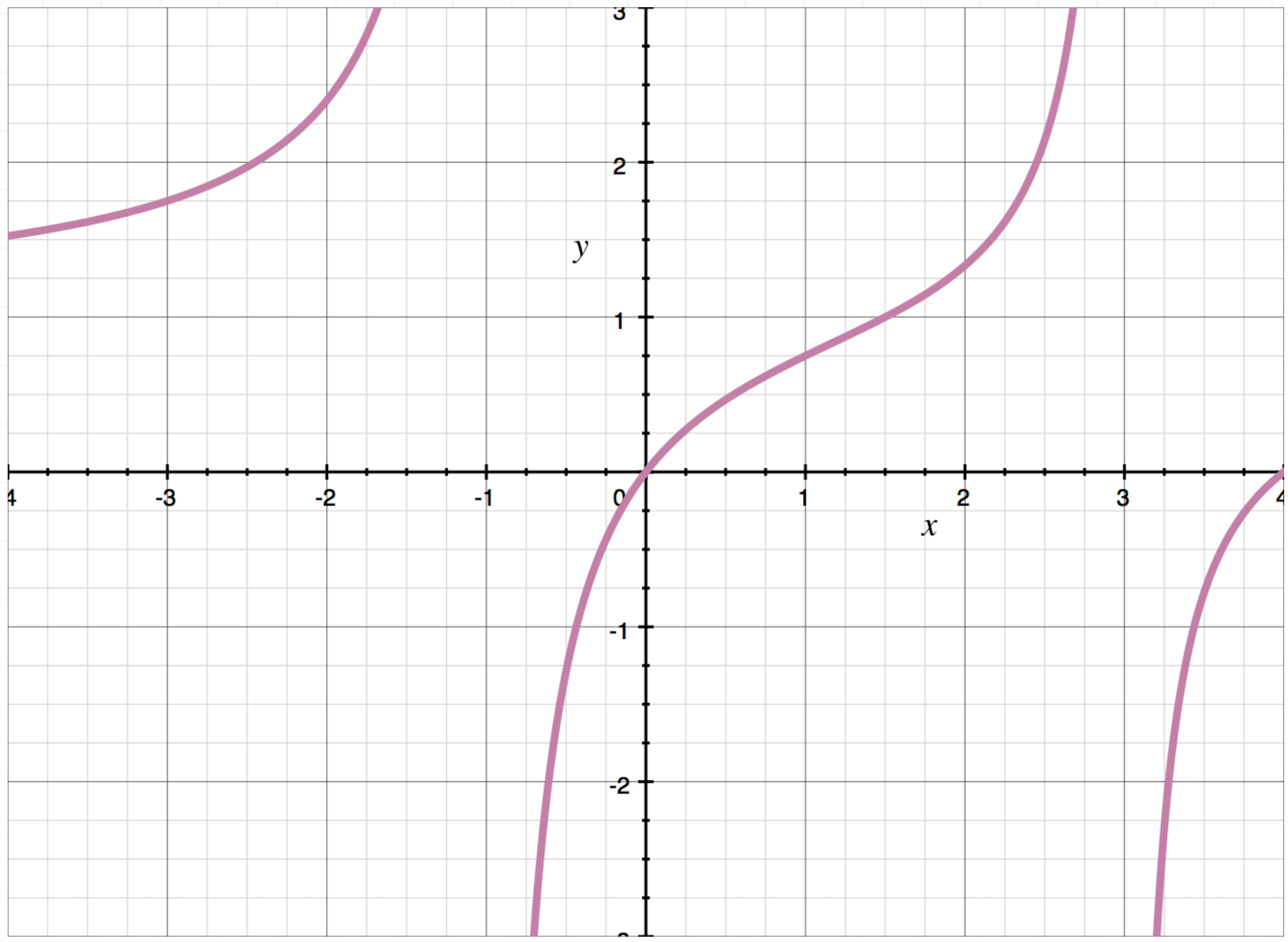
$$\lim_{x\to 3} \frac{x(x-4)}{(x-3)(x+1)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x\to 3^-} \frac{x(x-4)}{(x-3)(x+1)} = \infty$$

$$\lim_{x\to 3^+} \frac{x(x-4)}{(x-3)(x+1)} = -\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



6. What is the value of the limit?

$$\lim_{x \to -2} \frac{x^2 - 16}{-x^2 + x + 6}$$

Solution:

Factor to simplify the limit.

$$\lim_{x \to -2} \frac{x^2 - 16}{-x^2 + x + 6}$$

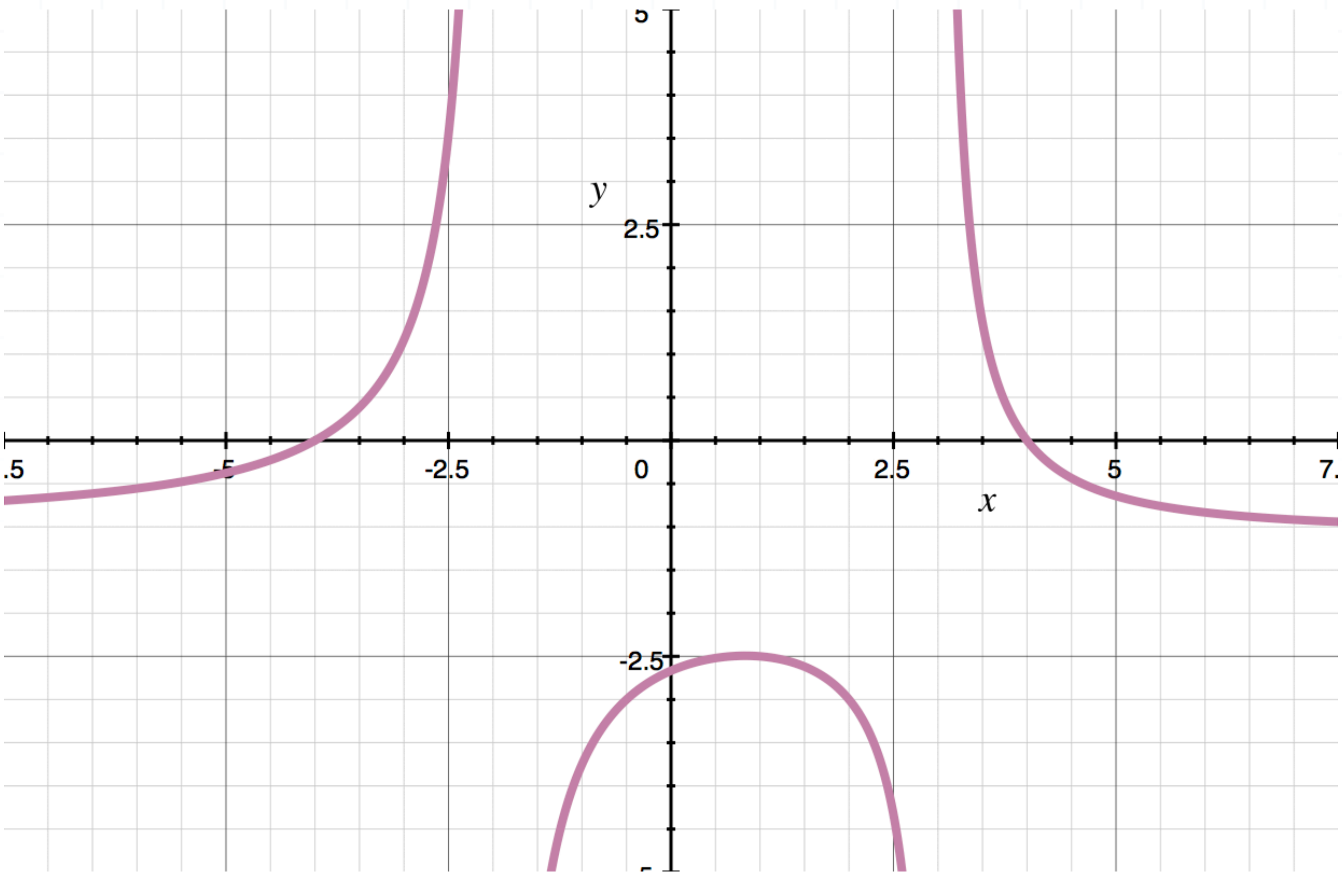
$$\lim_{x \to -2} \frac{(x+4)(x-4)}{-(x-3)(x+2)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x\to -2^-} \frac{(x+4)(x-4)}{-(x-3)(x+2)} = \infty$$

$$\lim_{x\to -2^+} \frac{(x+4)(x-4)}{-(x-3)(x+2)} = -\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

■ 1. What is the value of the limit?

$$\lim_{x\to\infty} \frac{3x^3 - 5x + 2}{9x^3 + 7x^2 - x}$$

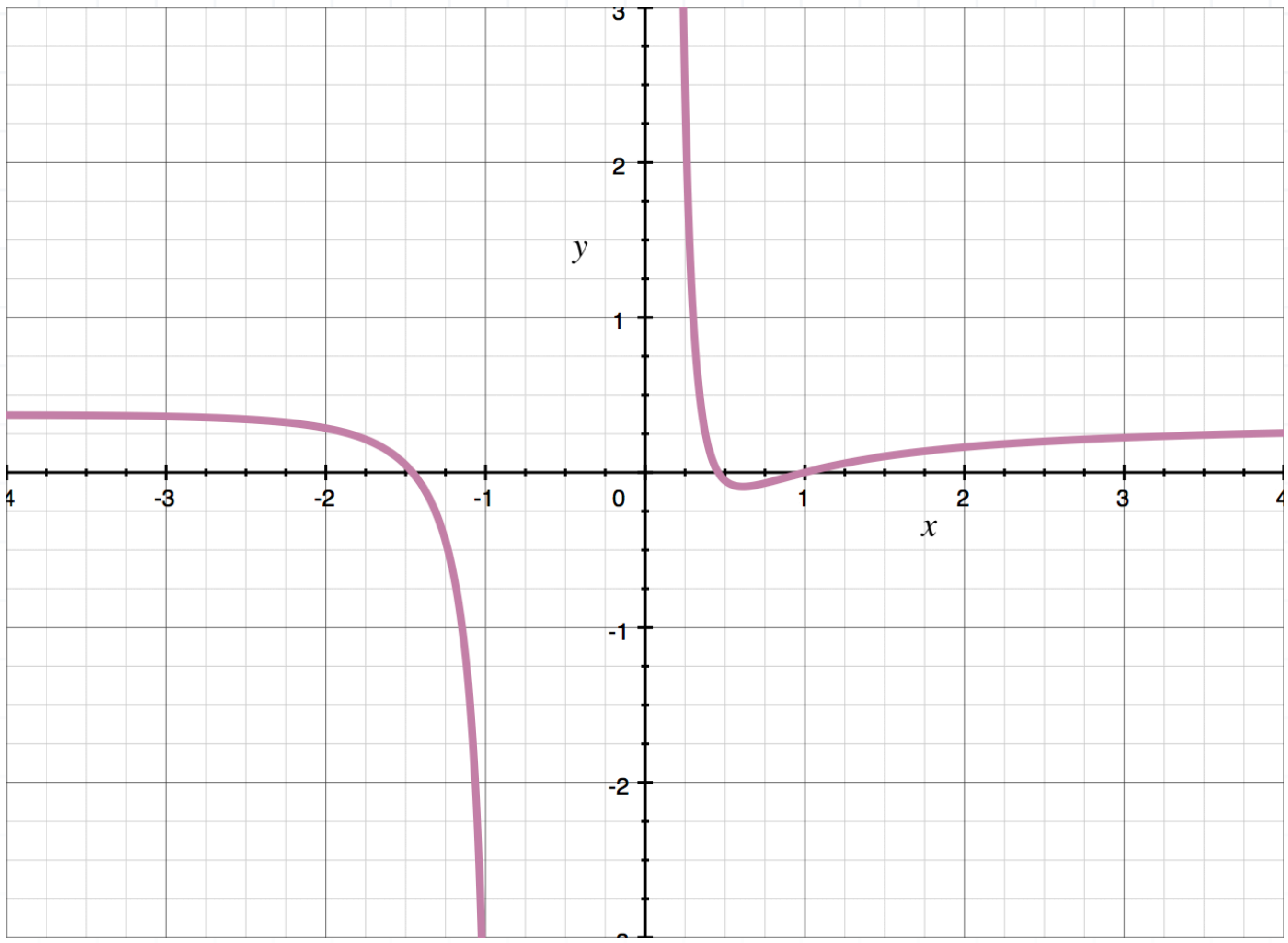
Solution:

Since this is a limit as $x \to \infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behaviors.

If the highest power in the numerator is the same as the highest power in the denominator, then the limit as $x \to \infty$ is the ratio of the leading coefficients.

$$\lim_{x\to\infty} \frac{3x^3 - 5x + 2}{9x^3 + 7x^2 - x} = \lim_{x\to\infty} \frac{3x^3}{9x^3} = \lim_{x\to\infty} \frac{3}{9} = \frac{3}{9} = \frac{1}{3}$$

The graph is shown below.



■ 2. What is the value of the limit?

$$\lim_{x \to -\infty} \frac{4x^2 + 3x + 5}{-2x^2 + x - 9}$$

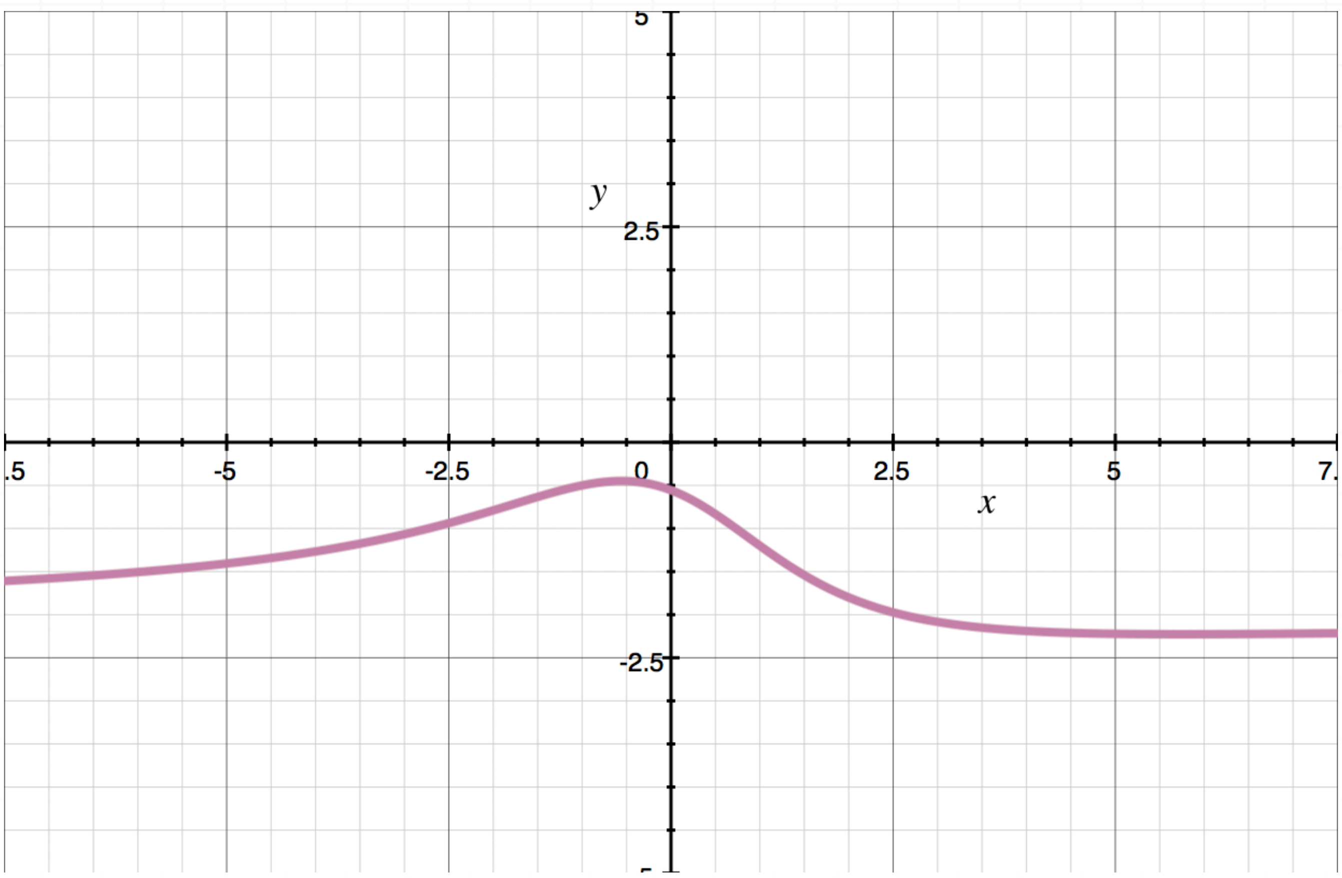
Solution:

Since this is a limit as $x \to -\infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behaviors.

If the highest power in the numerator is the same as the highest power in the denominator, then the limit as $x \to -\infty$ is the ratio of the leading coefficients.

$$\lim_{x \to -\infty} \frac{4x^2 + 3x + 5}{-2x^2 + x - 9} = \lim_{x \to -\infty} \frac{4x^2}{-2x^2} = \lim_{x \to -\infty} \frac{4}{-2} = -2$$

The graph is shown below.



3. What is the value of the limit?

$$\lim_{x \to \infty} \frac{x^3 + 6x^2 - 4x + 1}{x^3 + 9x + 8}$$

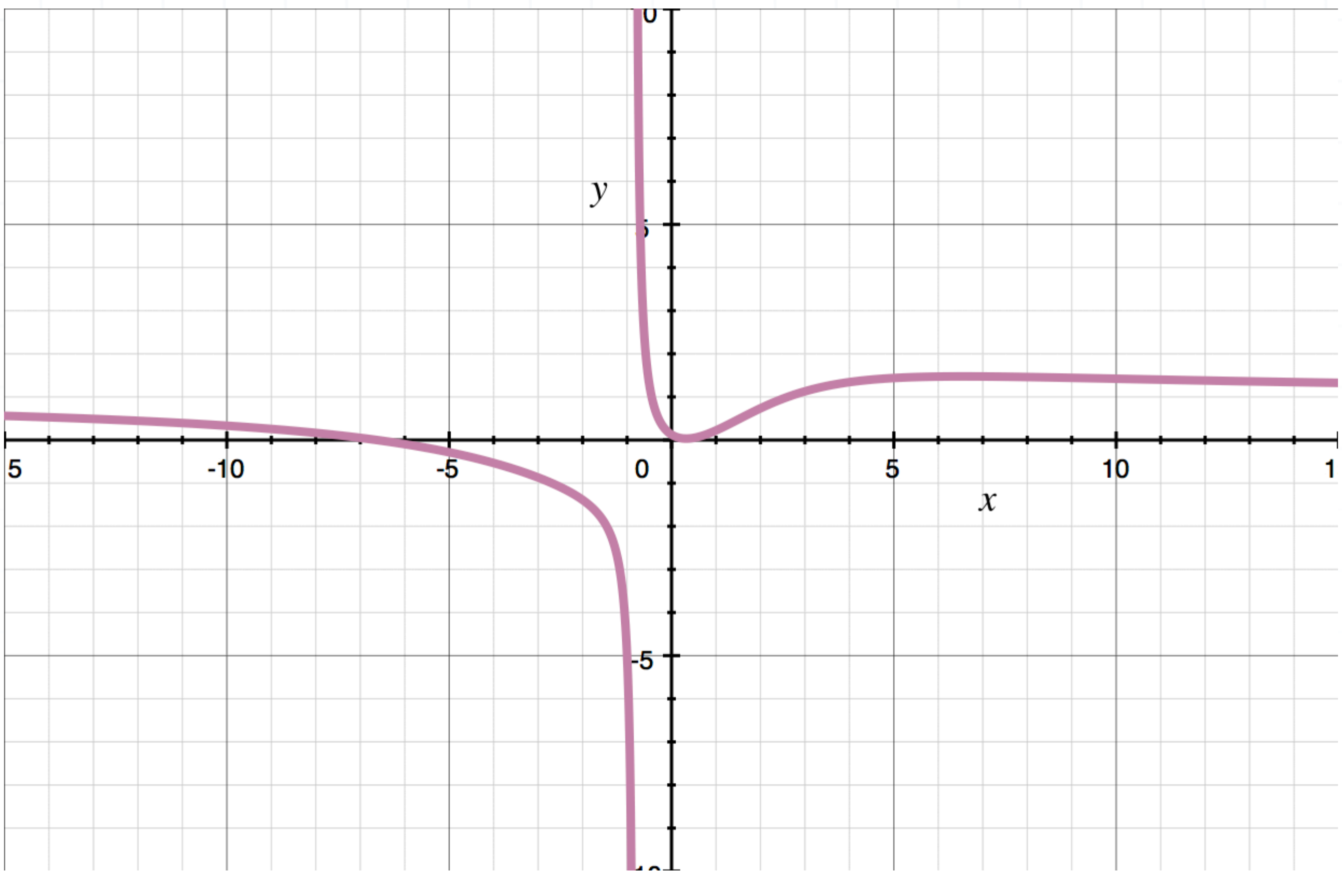
Solution:

Since this is a limit as $x \to \infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behaviors.

If the highest power in the numerator is the same as the highest power in the denominator, then the limit as $x \to \infty$ is the ratio of the leading coefficients.

$$\lim_{x\to\infty} \frac{x^3+6x^2-4x+1}{x^3+9x+8} = \lim_{x\to\infty} \frac{x^3}{x^3} = \lim_{x\to\infty} \frac{1}{1} = 1$$

The graph is shown below.



■ 4. What is the value of the limit?

$$\lim_{x\to\infty} \frac{3x^2+5x+8}{x^3-5x-9}$$

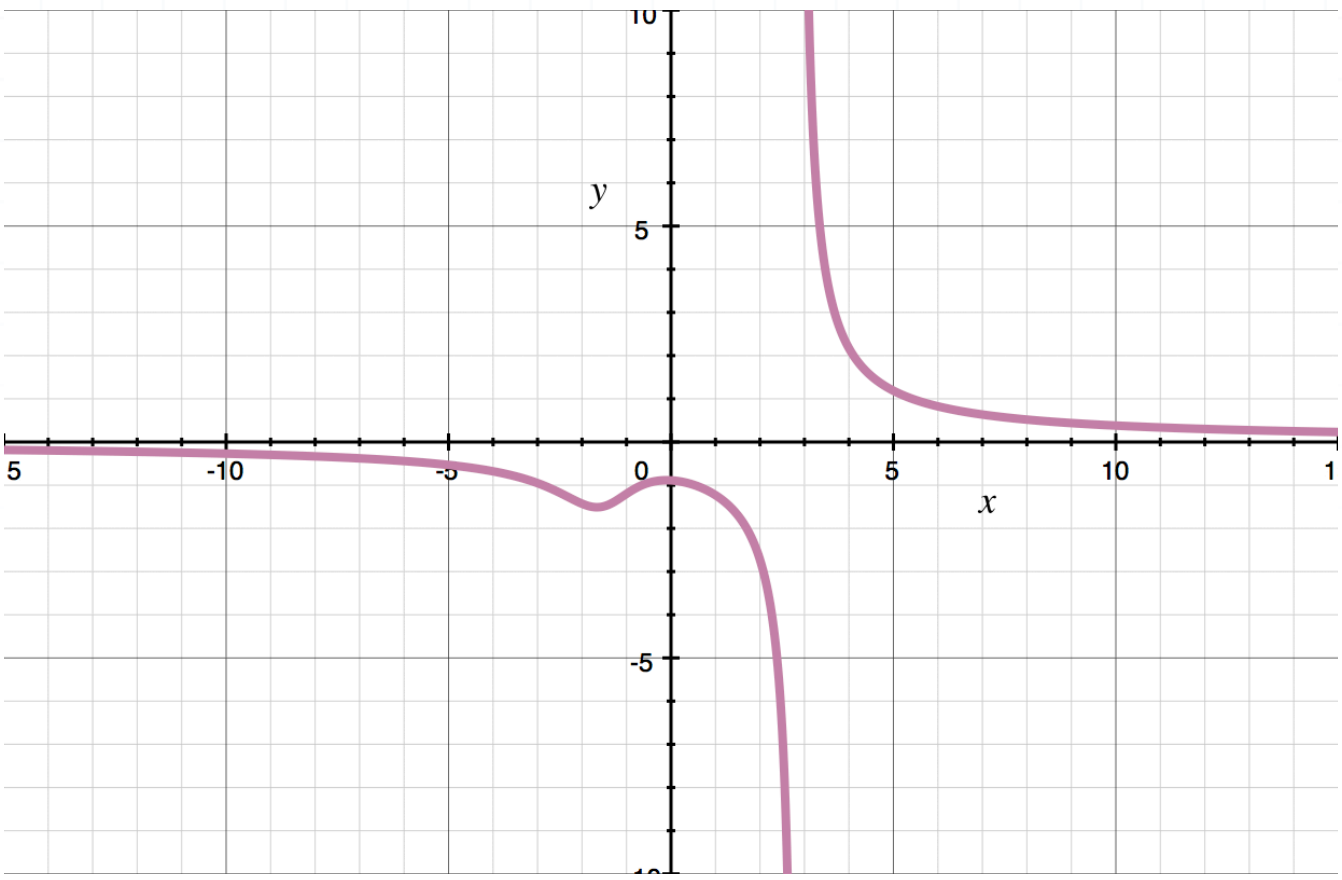
Solution:

Since this is a limit as $x \to \infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behaviors.

If the highest power in the numerator is smaller than the highest power in the denominator, then the limit as $x \to \infty$ is 0.

$$\lim_{x\to\infty} \frac{3x^2+5x+8}{x^3-5x-9} = \lim_{x\to\infty} \frac{3x^2}{x^3} = \lim_{x\to\infty} \frac{3}{x} = 0$$

The graph is shown below.



■ 5. What is the value of the limit?

$$\lim_{x\to-\infty} \frac{19x+21}{x^3+15x+11}$$

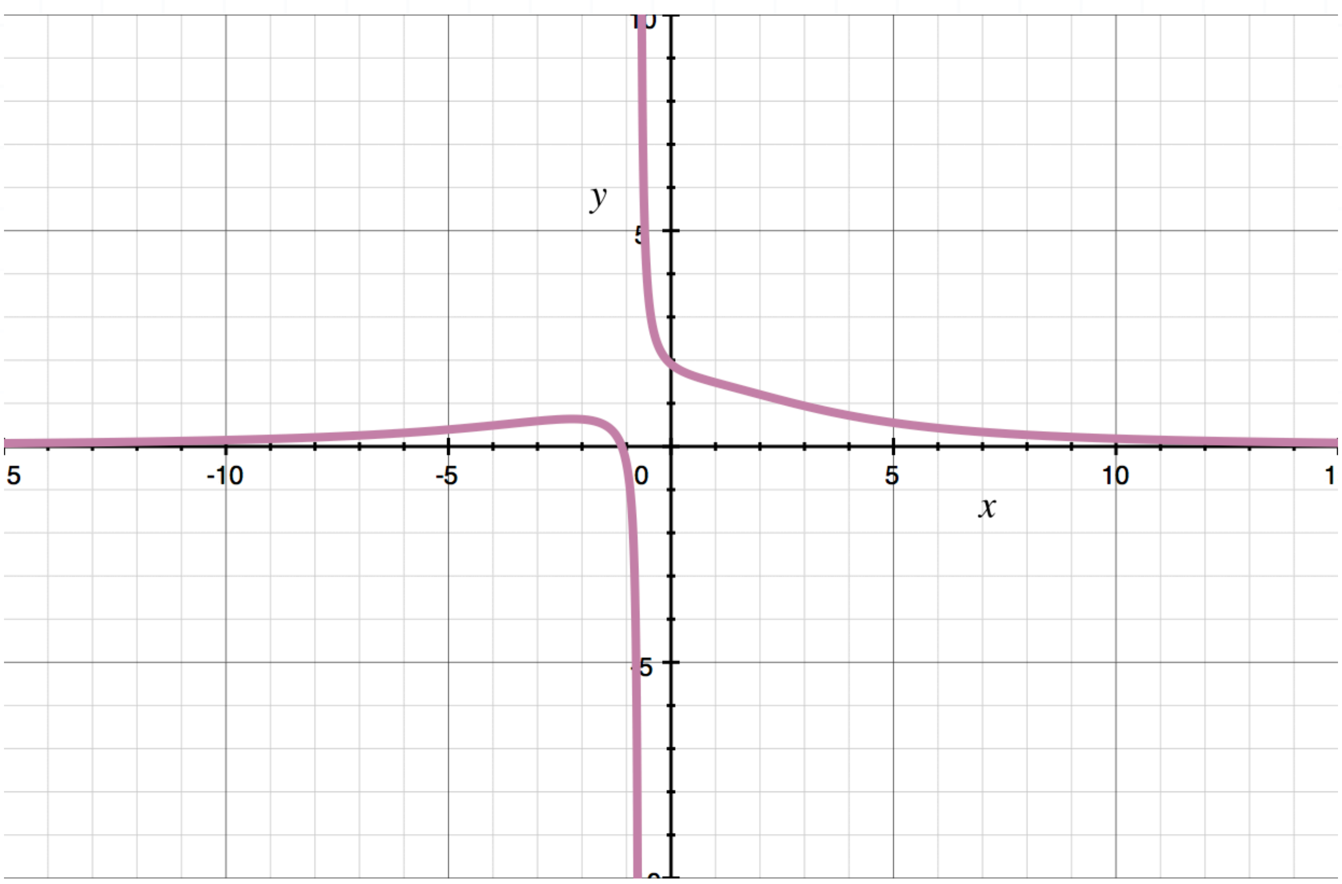
Solution:

Since this is a limit as $x \to -\infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behaviors.

If the highest power in the numerator is smaller than the highest power in the denominator, then the limit as $x \to -\infty$ is 0.

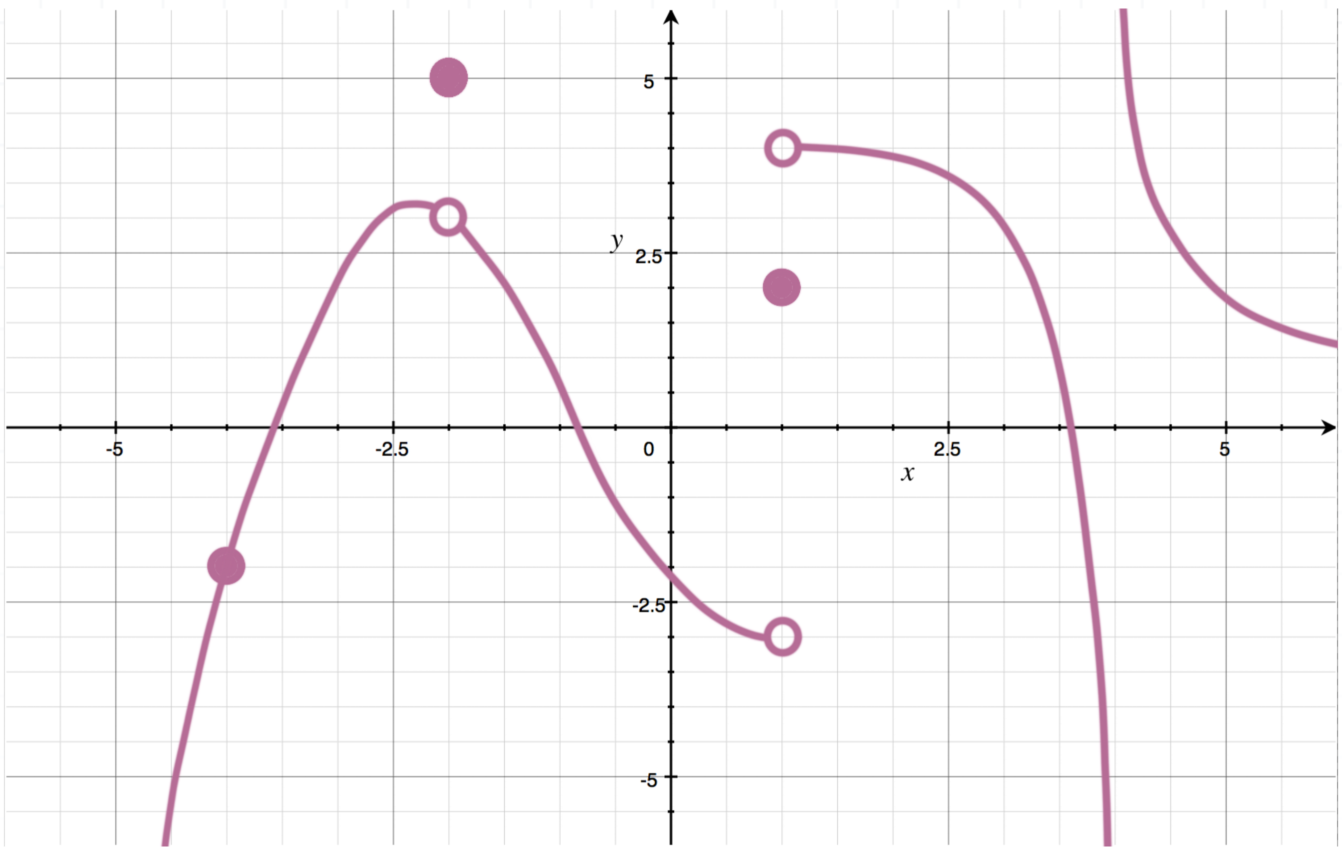
$$\lim_{x \to -\infty} \frac{19x + 21}{x^3 + 15x + 11} = \lim_{x \to -\infty} \frac{19x}{x^3} = \lim_{x \to -\infty} \frac{19}{x^2} = 0$$

The graph is shown below.



CRAZY GRAPHS

1. Use the graph to find the value of $\lim_{x\to 1} f(x)$.



Solution:

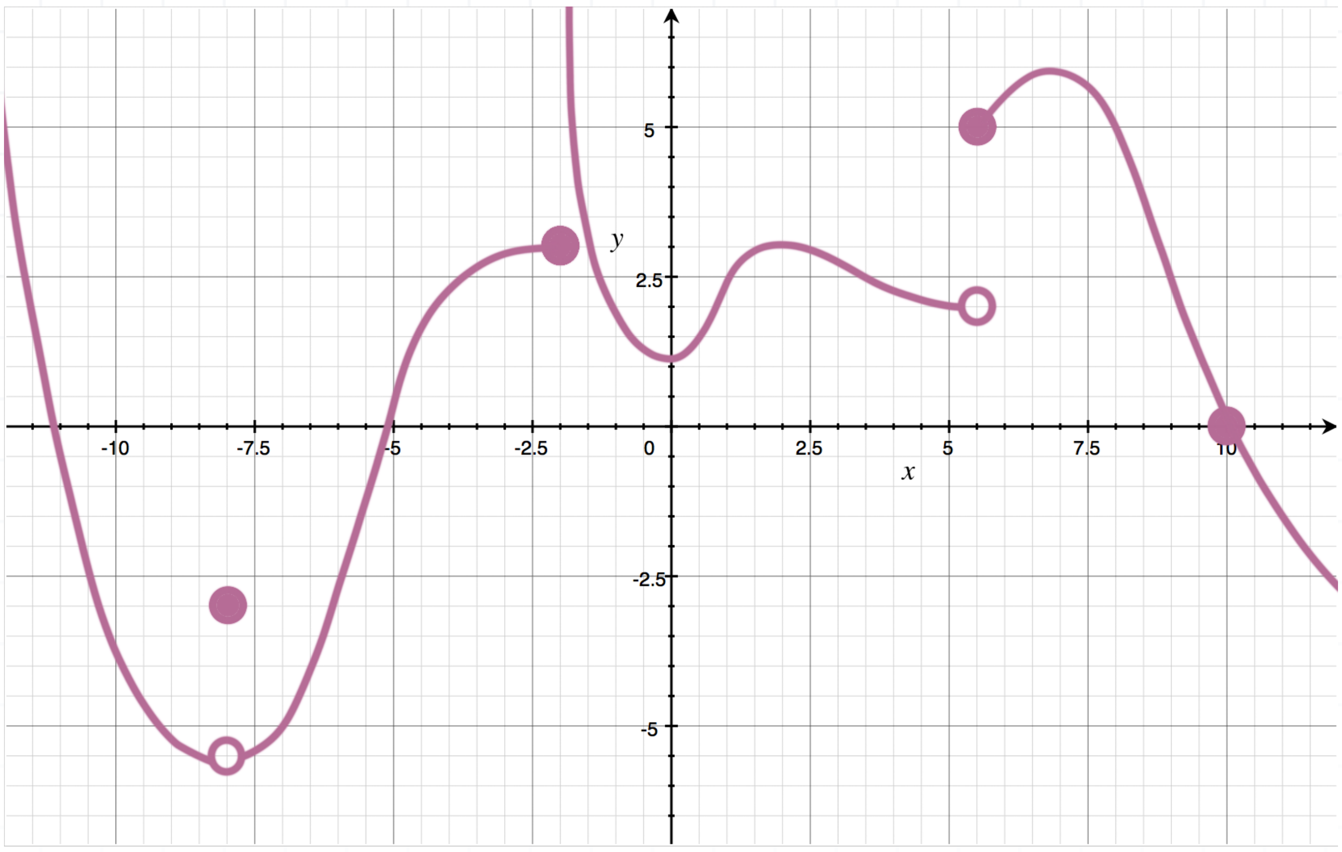
Notice in the graph that $f(x)$ has a jump discontinuity at $x = 1$.

$$\lim_{x\to 1^-} f(x) = -3$$

$$\lim_{x\to 1^+} f(x) = 4$$

Because these limits are unequal, the limit does not exist (DNE).

■ 2. Use the graph to find the value of $\lim_{x \to 5.5} g(x)$.



Solution:

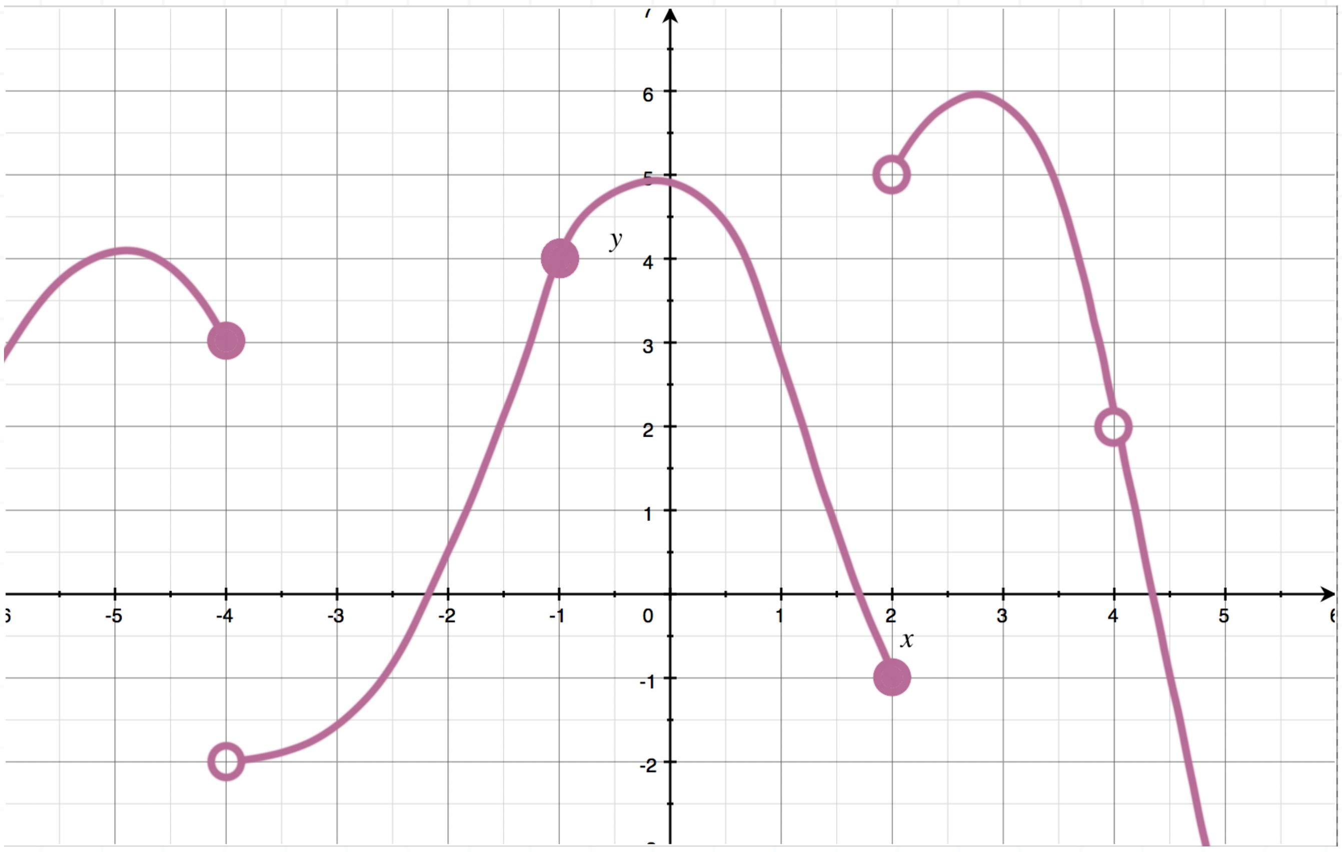
Notice in the graph that $g(x)$ has a jump discontinuity at $x = 5.5$.

$$\lim_{x \to 5.5^-} g(x) = 2$$

$$\lim_{x \to 5.5^+} g(x) = 5$$

Because these limits are unequal, the limit does not exist (DNE).

■ 3. Use the graph to find the value of $\lim_{x \to 4} h(x)$.



Solution:

Notice in the graph that $h(x)$ has a discontinuity at $x = 4$.

$$\lim_{x \to 4^-} h(x) = 2$$

$$\lim_{x \to 4^+} h(x) = 2$$

These limits are the same, which means

$$\lim_{x \to 4} h(x) = 2$$

TRIGONOMETRIC LIMITS

1. Find $\lim_{x\to\pi} f(x)$ if $f(x) = 3\cos x - 2$.

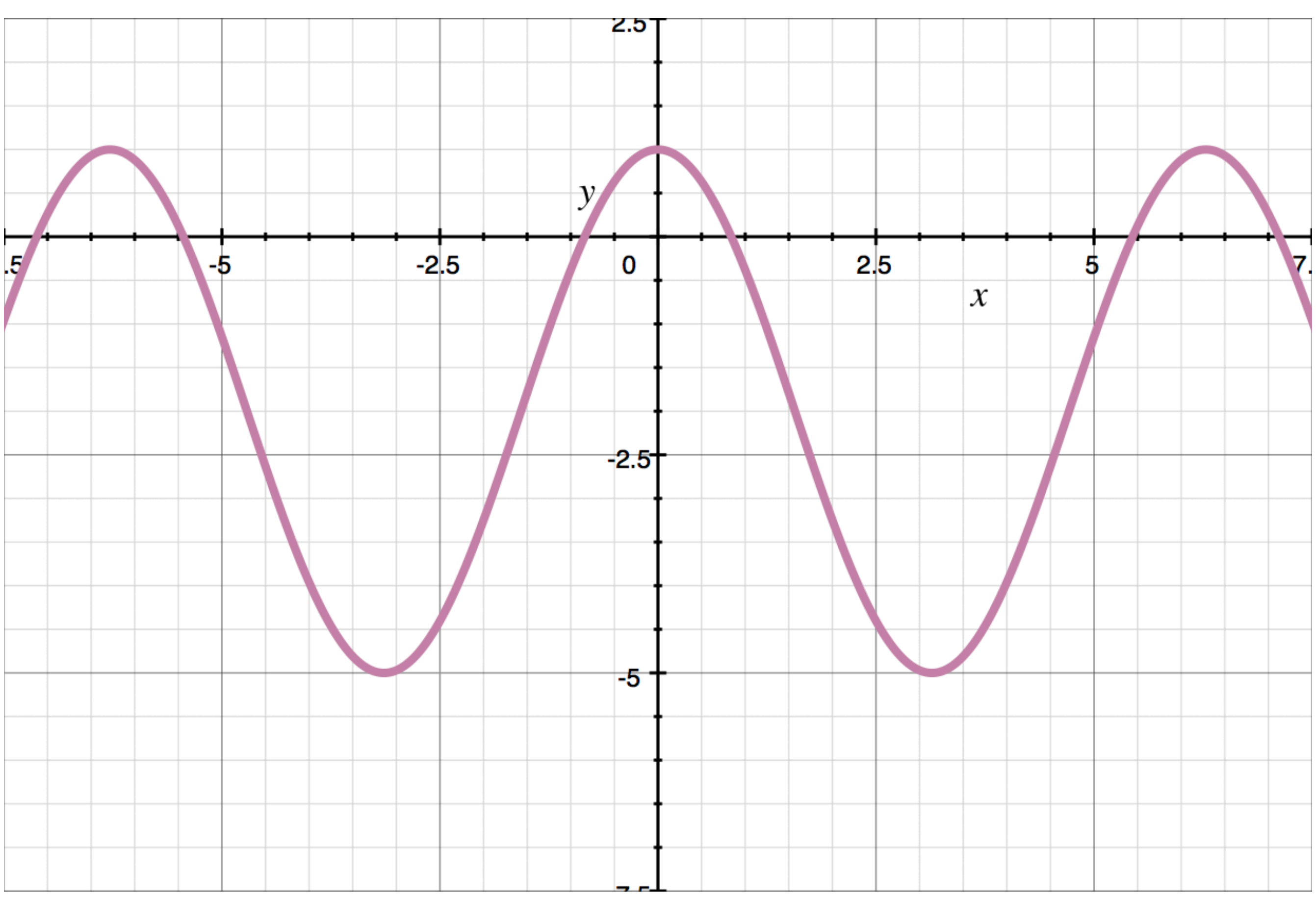
Solution:

The one-sided limits are

$$\lim_{x\to\pi^-} 3\cos x - 2 = -5$$

$$\lim_{x\to\pi^+} 3\cos x - 2 = -5$$

as shown in the graph below. Therefore, $\lim_{x\to\pi} f(x) = -5$.



■ 2. Find $\lim_{x\to\frac{3\pi}{2}} g(x)$ if $g(x) = 4\sin x + 1$.

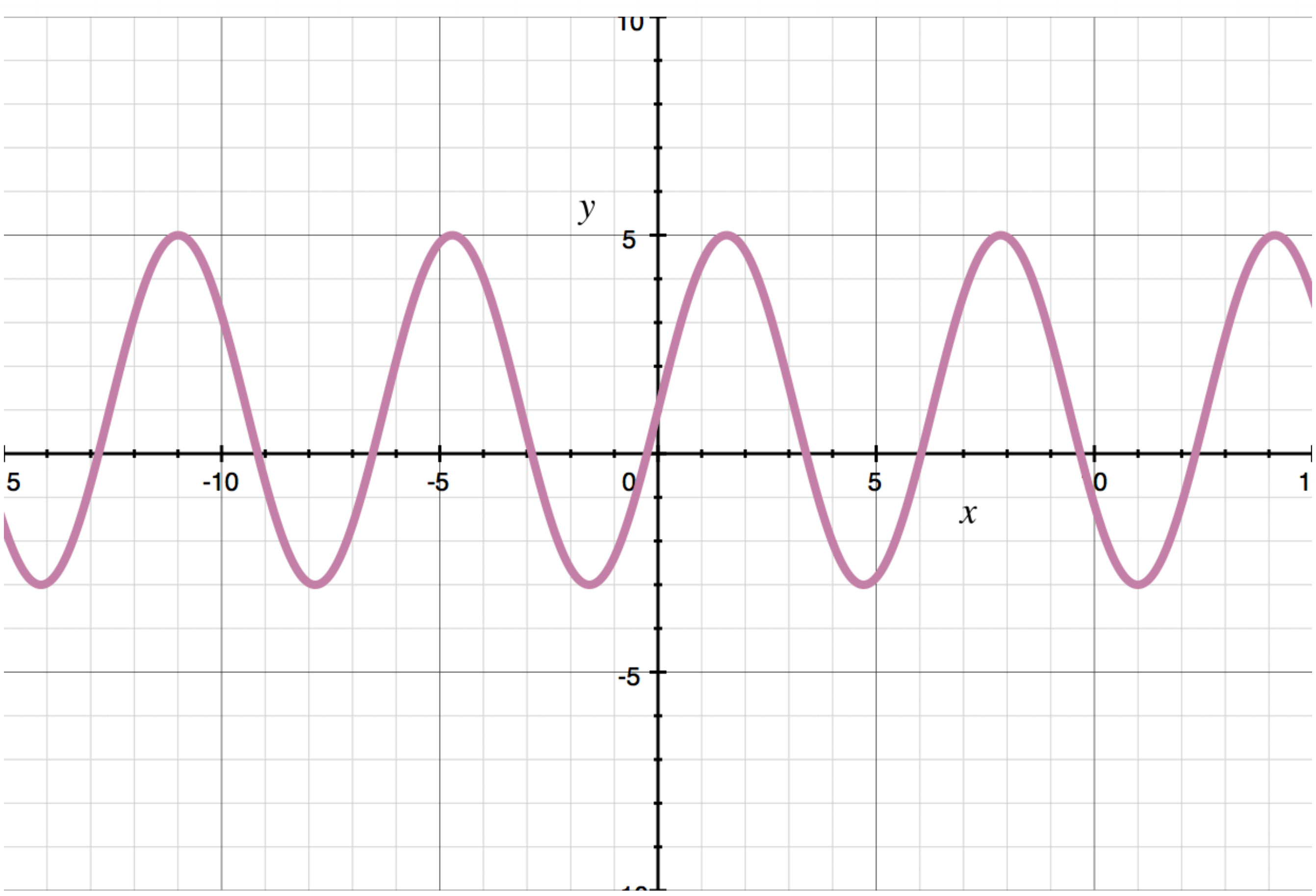
Solution:

The one-sided limits are

$$\lim_{x\to\frac{3\pi}{2}^-} 4\sin x + 1 = -3$$

$$\lim_{x\to\frac{3\pi}{2}^+} 4\sin x + 1 = -3$$

as shown in the graph below. Therefore, $\lim_{x\to\frac{3\pi}{2}} g(x) = -3$.



■ 3. Find $\lim_{x\to-\frac{3\pi}{2}} h(x)$ if $h(x) = \tan\left(\frac{x}{6}\right)$.

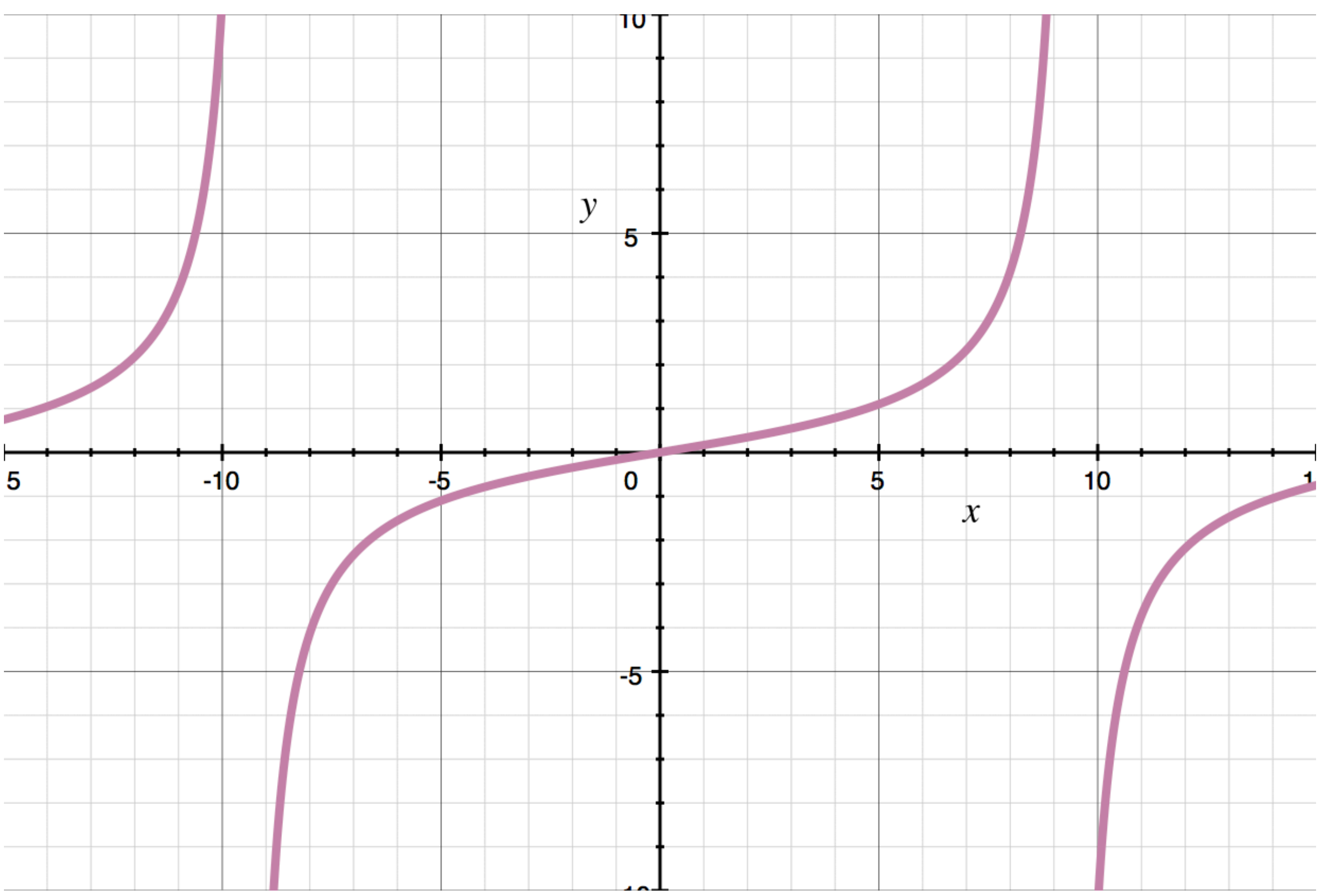
Solution:

The one-sided limits are

$$\lim_{x\to-\frac{3\pi}{2}^-} \tan\left(\frac{x}{6}\right) = -1$$

$$\lim_{x\to-\frac{3\pi}{2}^+} \tan\left(\frac{x}{6}\right) = -1$$

as shown in the graph below. Therefore, $\lim_{x\to-\frac{3\pi}{2}} h(x) = -1$.



MAKING THE FUNCTION CONTINUOUS

■ 1. What value of c makes the function $h(x)$ continuous if c is a constant?

$$h(x) = \begin{cases} x^2 & x \le 4 \\ 3x + c & x > 4 \end{cases}$$

Solution:

Since $h(x)$ is defined as a piecewise function, $x^2 = 3x + c$ at $x = 4$.

$$4^2 = 3(4) + c$$

$$16 = 12 + c$$

$$c = 4$$

■ 2. What value of c makes the function $f(x)$ continuous if c is a constant?

$$f(x) = \begin{cases} 5x - c & x \le 3 \\ 3x + 4 & x > 3 \end{cases}$$

Solution:

Since $f(x)$ is defined as a piecewise function, $5x - c = 3x + 4$ at $x = 3$.

$$5(3) - c = 3(3) + 4$$

$$15 - c = 9 + 4$$

$$c = 2$$

■ 3. What value of c makes the function $g(x)$ continuous if c is a constant?

$$g(x) = \begin{cases} x^2 - 4x + 8 & x \leq 2 \\ cx - 2 & x > 2 \end{cases}$$

Solution:

Since $g(x)$ is defined as a piecewise function, $x^2 - 4x + 8 = cx - 2$ at $x = 2$.

$$2^2 - 4(2) + 8 = c(2) - 2$$

$$4 - 8 + 8 = 2c - 2$$

$$6 = 2c$$

$$c = 3$$

■ 4. What value of c makes the function $f(x)$ continuous if c is a constant?

$$f(x) = \begin{cases} 2x^3 - 6x^2 + 8x + 3 & x \leq 1 \\ cx + 9 & x > 1 \end{cases}$$

Solution:

Since $f(x)$ is defined as a piecewise function, $2x^3 - 6x^2 + 8x + 3 = cx + 9$ at $x = 1$.

$$2(1)^3 - 6(1)^2 + 8(1) + 3 = c(1) + 9$$

$$2 - 6 + 8 + 3 = c + 9$$

$$7 = c + 9$$

$$c = -2$$

■ 5. What value of c makes the function $g(x)$ continuous if c is a constant?

$$g(x) = \begin{cases} \sqrt{x} + 18 & x \le 16 \\ x - 2c & x > 16 \end{cases}$$

Solution:

Since $g(x)$ is defined as a piecewise function, $\sqrt{x} = 18 = 4x - 2c$ at $x = 16$.

$$\sqrt{16} + 18 = 16 - 2c$$

$$4 + 18 = 16 - 2c$$

$$22 = 16 - 2c$$

$$6 = -2c$$

$$c = -3$$

■ 6. What value of c makes the function $h(x)$ continuous if c is a constant?

$$h(x) = \begin{cases} 11x - 9 & x \leq 3 \\ x^2 + 3x + c & x > 3 \end{cases}$$

Solution:

Since $h(x)$ is defined as a piecewise function, $11x - 9 = x^2 + 3x + c$ at $x = 3$.

$$11(3) - 9 = 3^2 + 3(3) + c$$

$$33 - 9 = 9 + 9 + c$$

$$24 = 18 + c$$

$$c = 6$$

SQUEEZE THEOREM

■ 1. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) - 2$$

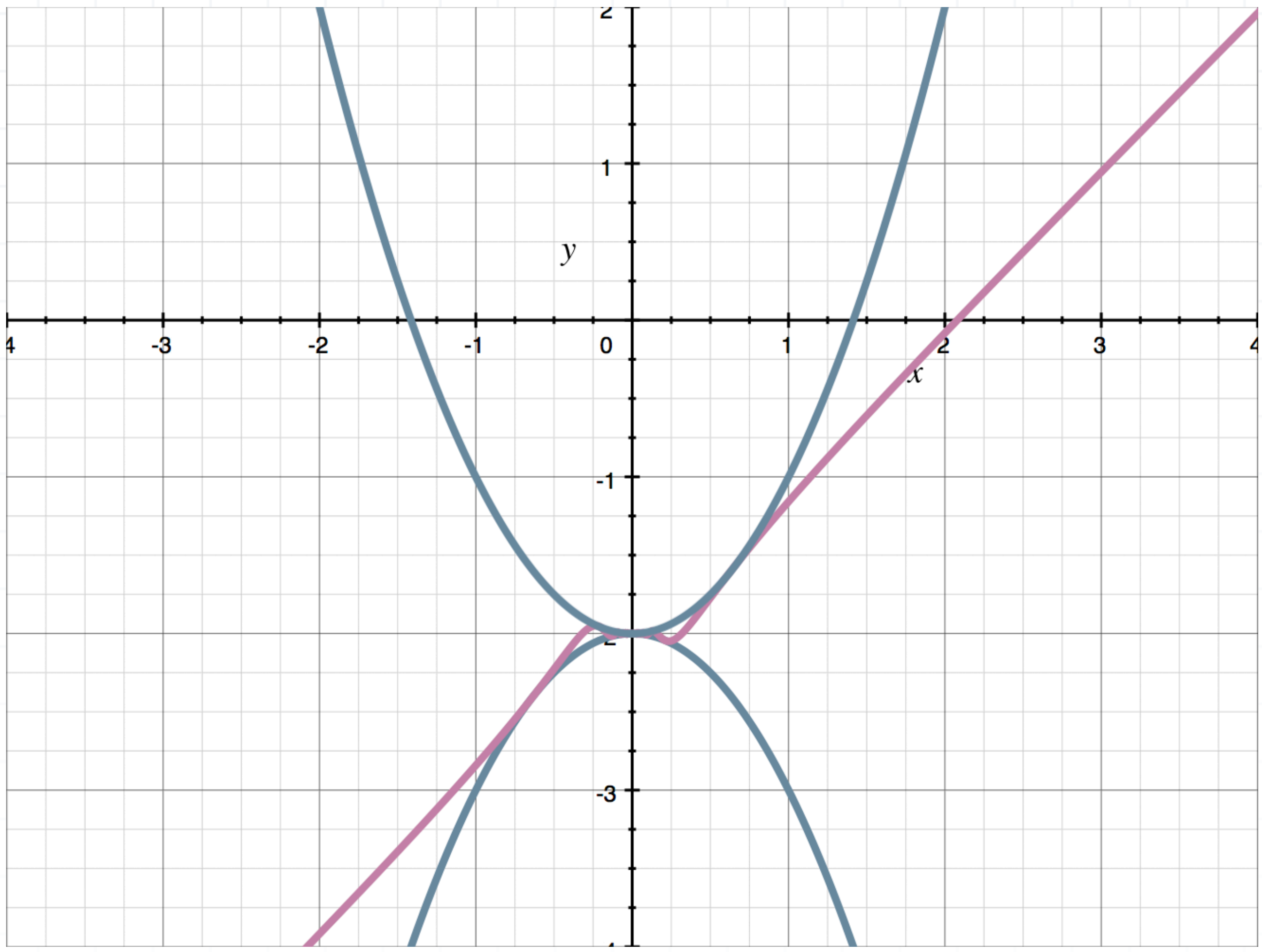
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -x^2 - 2$$

$$g(x) = x^2 \sin\left(\frac{1}{x}\right) - 2$$

$$h(x) = x^2 - 2$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x \to 0} f(x) \leq \lim_{x \to 0} g(x) \leq \lim_{x \to 0} h(x)$$

$$\lim_{x \to 0}(-x^2 - 2) \leq \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) - 2 \leq \lim_{x \to 0}(x^2 - 2)$$

We can evaluate the limits on the left and right sides.

$$-0^2 - 2 \leq \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) - 2 \leq 0^2 - 2$$

$$-2 \leq \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) - 2 \leq -2$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be -2.

2. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x\to\infty}\frac{3\sin x}{4x}$$

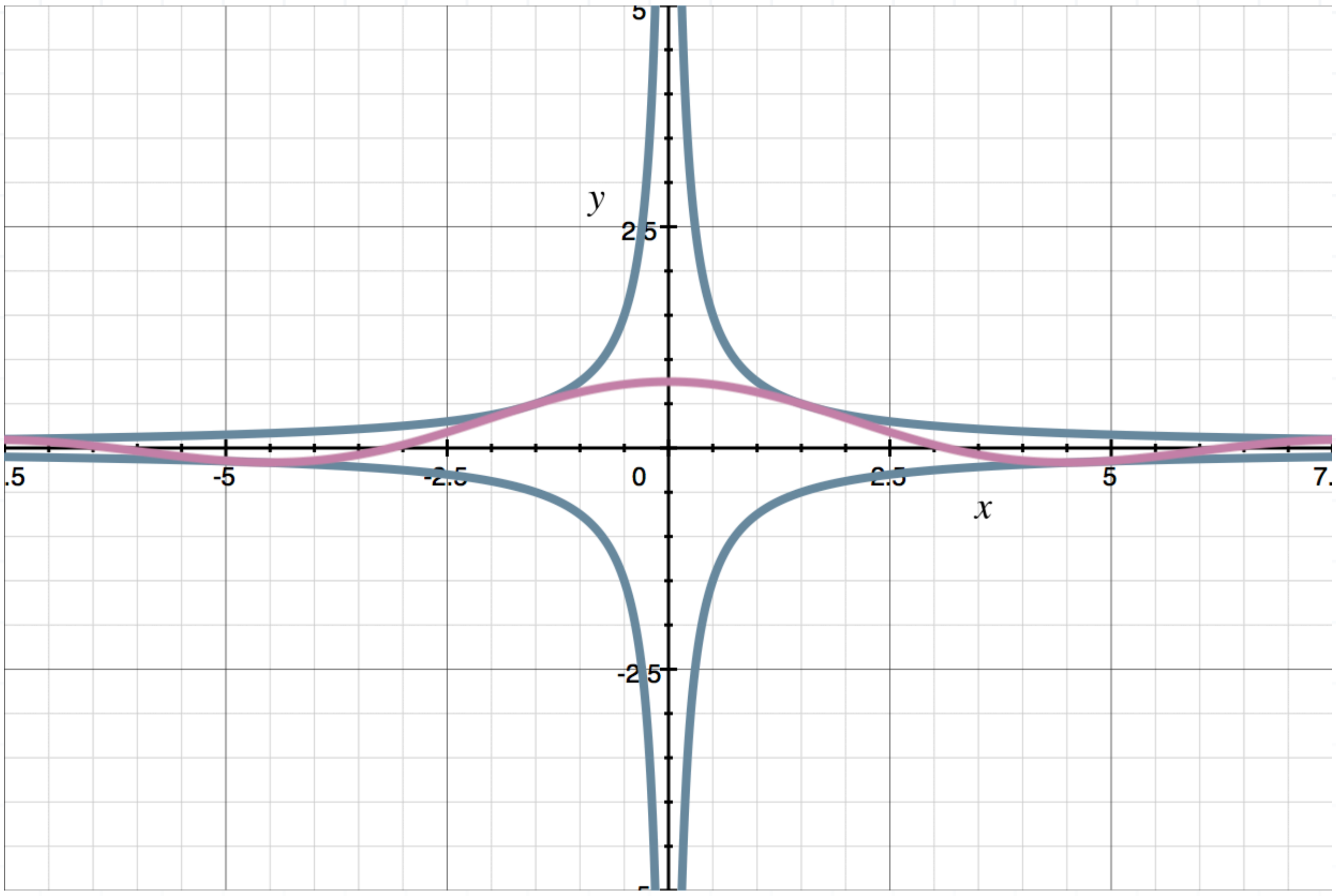
Solution:

Consider the graphs of the three functions shown below.

$$f(x)=-\frac{3}{4x}$$

$$g(x)=\frac{3\sin x}{4x}$$

$$h(x)=\frac{3}{4x}$$



Notice that $f(x) \le g(x) \le h(x)$. Therefore,

$$\lim_{x\to\infty} f(x) \le \lim_{x\to\infty} g(x) \le \lim_{x\to\infty} h(x)$$

$$\lim_{x\to\infty}\left(-\frac{3}{4x}\right) \le \lim_{x\to\infty}\frac{3\sin x}{4x} \le \lim_{x\to\infty}\left(\frac{3}{4x}\right)$$

We can evaluate the limits on the left and right sides.

$$0 \le \lim_{x\to\infty}\frac{3\sin x}{4x} \le 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

■ 3. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) + 1$$

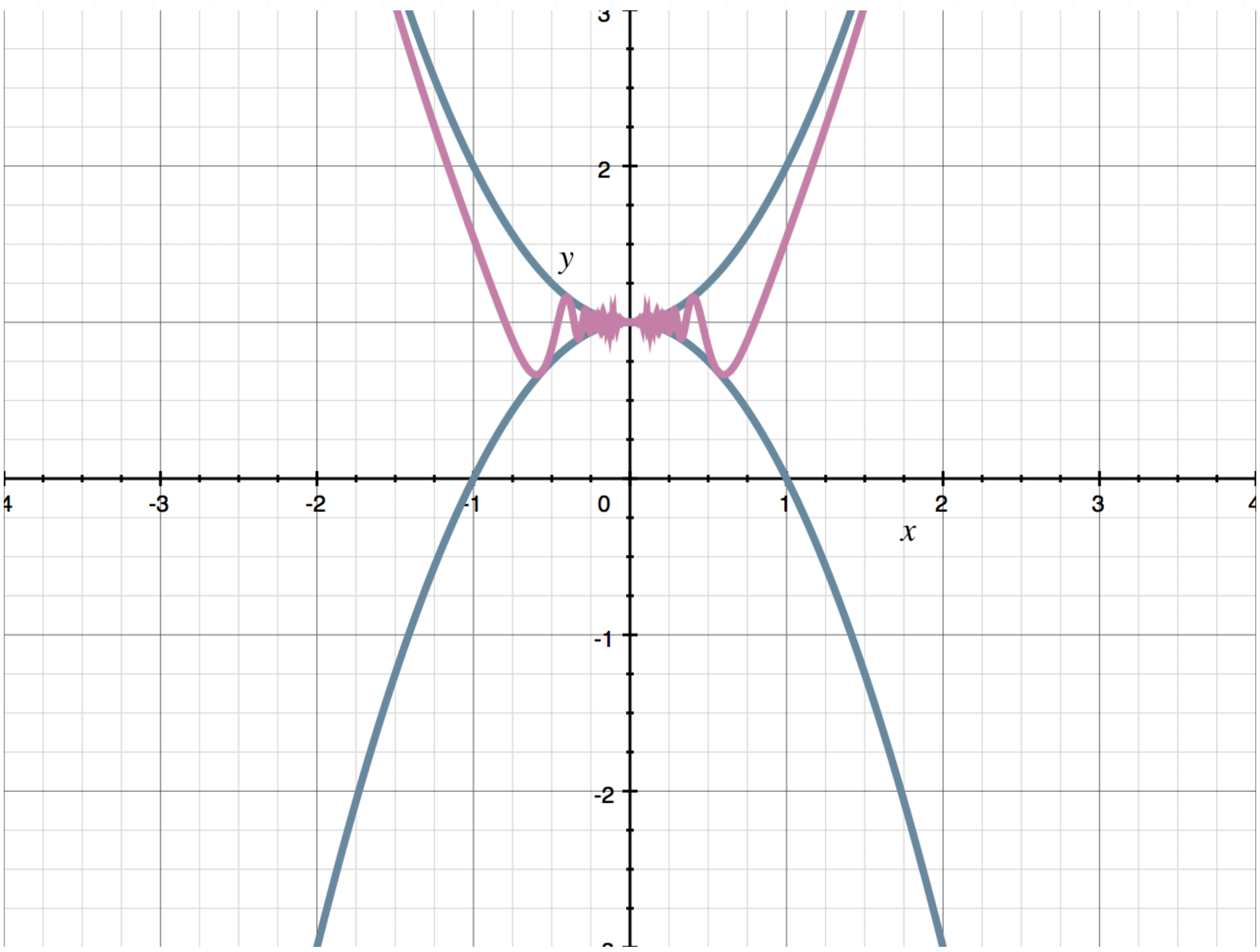
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -x^2 + 1$$

$$g(x) = x^2 \cos\left(\frac{1}{x^2}\right) + 1$$

$$h(x) = x^2 + 1$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x \to 0} f(x) \le \lim_{x \to 0} g(x) \le \lim_{x \to 0} h(x)$$

$$\lim_{x \to 0} -x^2 + 1 \le \lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) + 1 \le \lim_{x \to 0} x^2 + 1$$

We can evaluate the limits on the left and right sides.

$$-0^2 + 1 \le \lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) + 1 \le 0^2 + 1$$

$$1 \le \lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) + 1 \le 1$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 1.

■ 4. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{e^{-x}}{x}$$

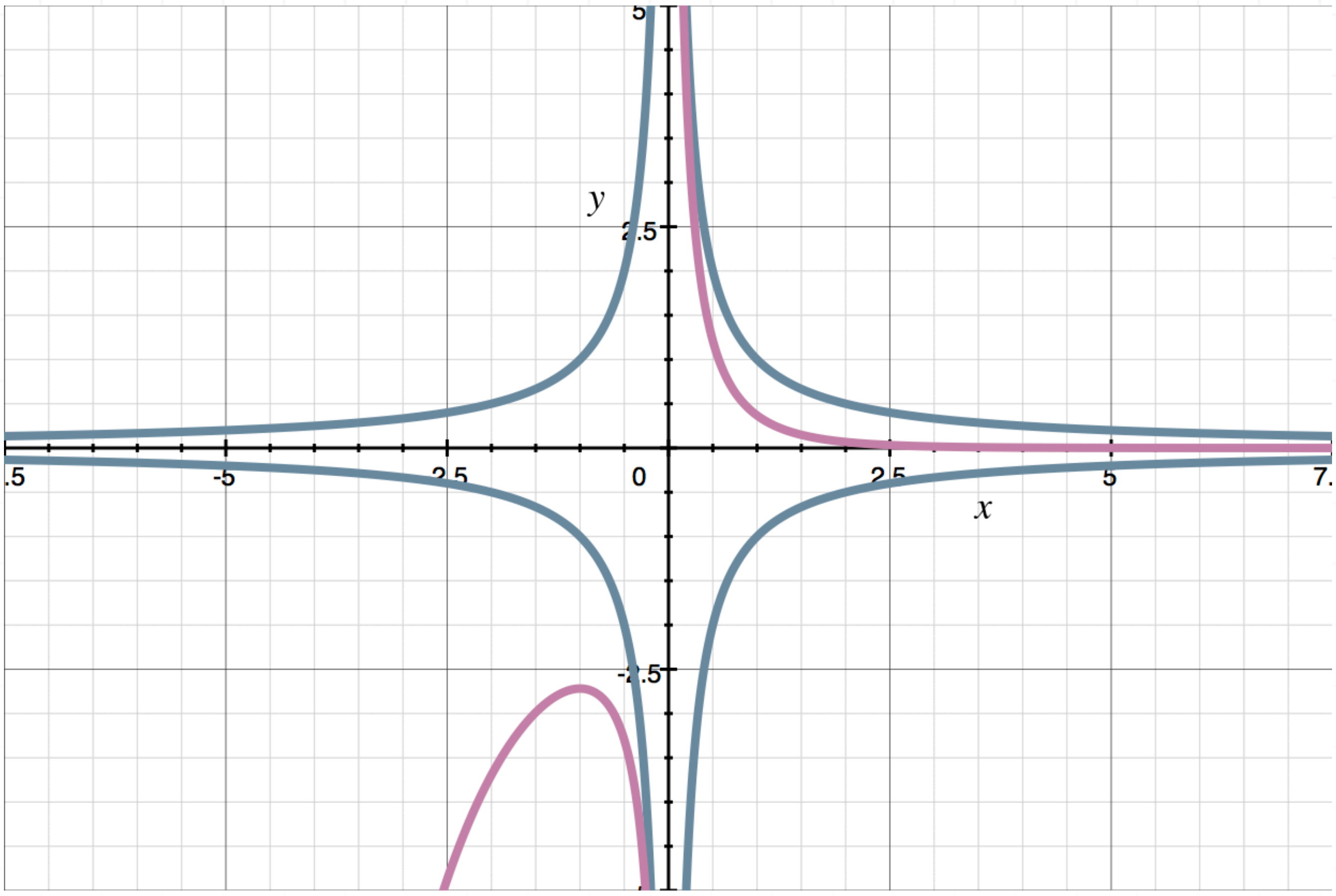
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -\frac{1}{x}$$

$$g(x) = \frac{e^{-x}}{x}$$

$$h(x) = \frac{1}{x}$$



Notice that $f(x) \le g(x) \le h(x)$. Therefore,

$$\lim_{x\to\infty} f(x) \le \lim_{x\to\infty} g(x) \le \lim_{x\to\infty} h(x)$$

$$\lim_{x\to\infty} \left(-\frac{1}{x}\right) \le \lim_{x\to\infty} \frac{e^{-x}}{x} \le \lim_{x\to\infty} \left(\frac{1}{x}\right)$$

We can evaluate the limits on the left and right sides.

$$0 \le \lim_{x\to\infty} \frac{e^{-x}}{x} \le 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

5. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

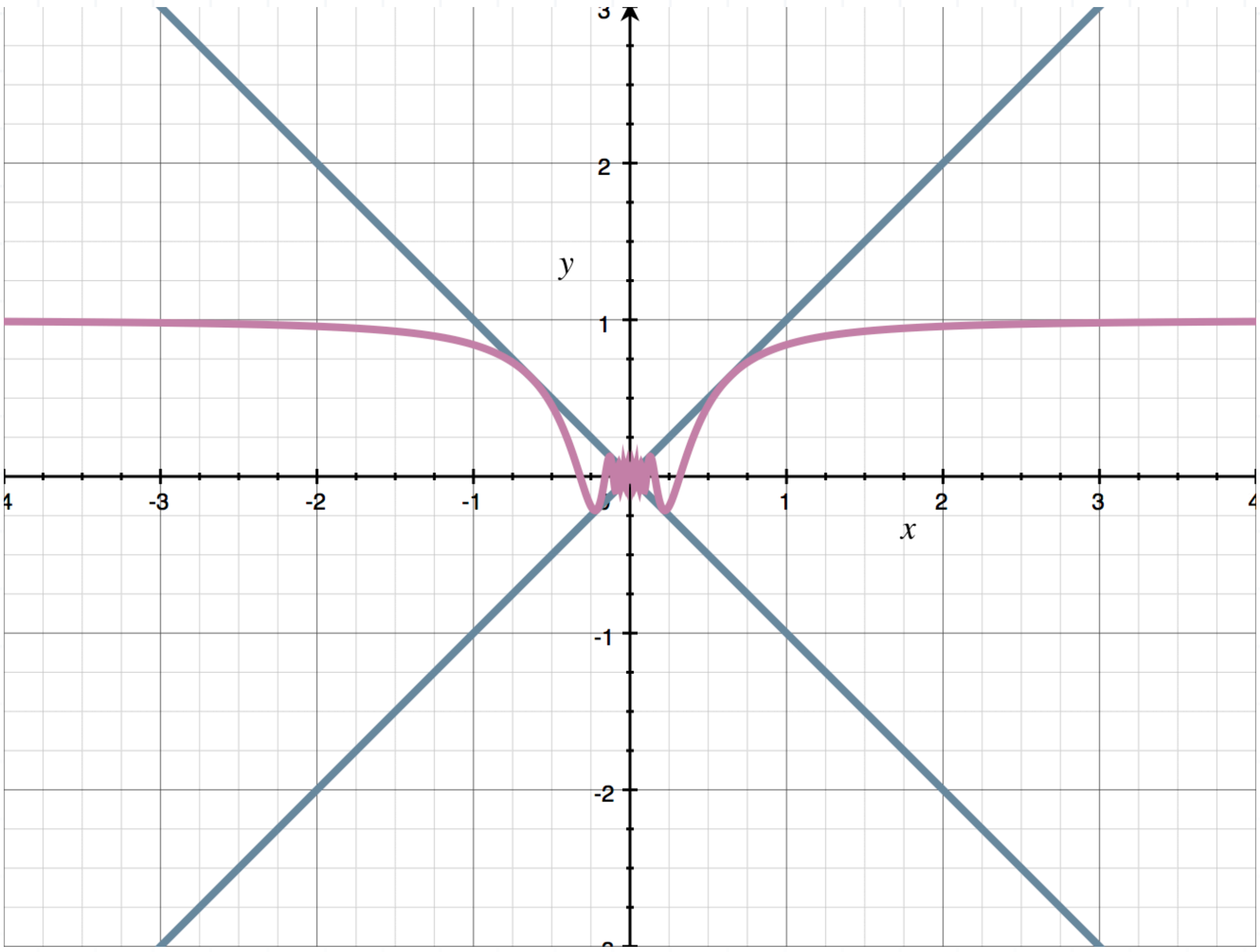
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -|x|$$

$$g(x) = \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$$h(x) = |x|$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x\to 0} f(x) \leq \lim_{x\to 0} g(x) \leq \lim_{x\to 0} h(x)$$

$$\lim_{x\to 0} -|x| \leq \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \leq \lim_{x\to 0} |x|$$

We can evaluate the limits on the left and right sides.

$$-|0| \leq \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \leq |0|$$

$$0 \leq \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \leq 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

■ 6. Find $\lim_{x\to 4} f(x)$ if $x^2 + 1 \le f(x) \le 4x + 1$.

Solution:

Apply the limit to each part of the inequality.

$$x^2 + 1 \le f(x) \le 4x + 1$$

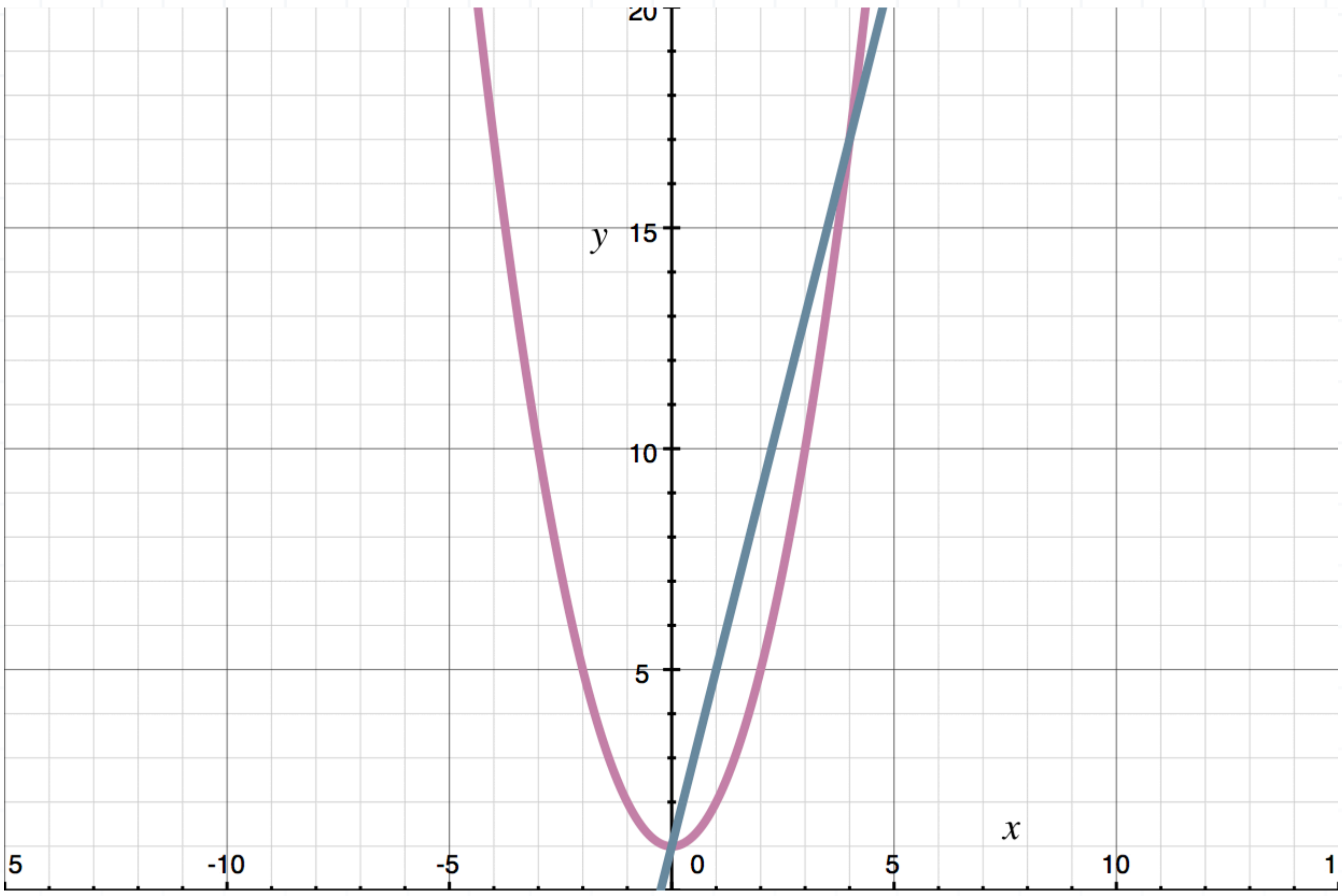
$$\lim_{x\to 4} x^2 + 1 \le \lim_{x\to 4} f(x) \le \lim_{x\to 4} 4x + 1$$

Evaluate the limits on the left and right sides using substitution.

$$4^2 + 1 \le \lim_{x\to 4} f(x) \le 4(4) + 1$$

$$17 \le \lim_{x\to 4} f(x) \le 17$$

Therefore, by the Squeeze Theorem, $\lim_{x\to 4} f(x) = 17$. The graph below shows the limit at the intersection point.



■ 7. Find $\lim_{x\to 3} g(x)$ if $x^2 - 7 \le g(x) \le \sqrt{13 - x^2}$.

Solution:

Apply the limit to each part of the inequality.

$$x^2 - 7 \le g(x) \le \sqrt{13 - x^2}$$

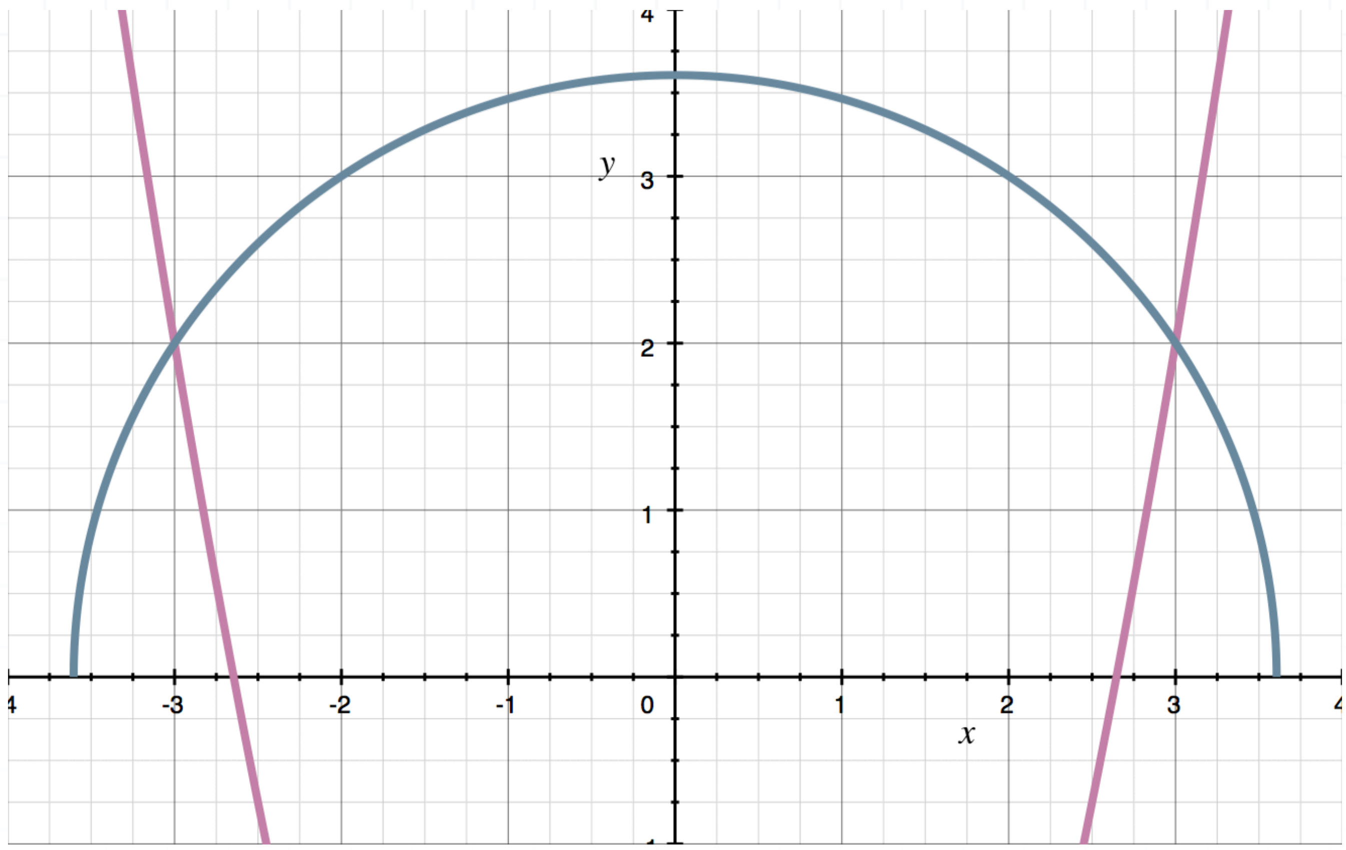
$$\lim_{x\to 3} x^2 - 7 \le \lim_{x\to 3} g(x) \le \lim_{x\to 3} \sqrt{13 - x^2}$$

Evaluate the limits on the left and right sides using substitution.

$$3^2 - 7 \le \lim_{x\to 3} g(x) \le \sqrt{13 - 3^2}$$

$$2 \leq \lim_{x \to 3} g(x) \leq 2$$

Therefore, by the Squeeze Theorem, $\lim_{x \to 3} g(x) = 2$. The graph below shows the limit at the intersection point.



■ 8. Find $\lim_{x \to 5} h(x)$ if $x^2 - 6x + 9 \leq h(x) \leq x - 1$.

Solution:

Apply the limit to each part of the inequality.

$$x^2 - 6x + 9 \leq h(x) \leq x - 1$$

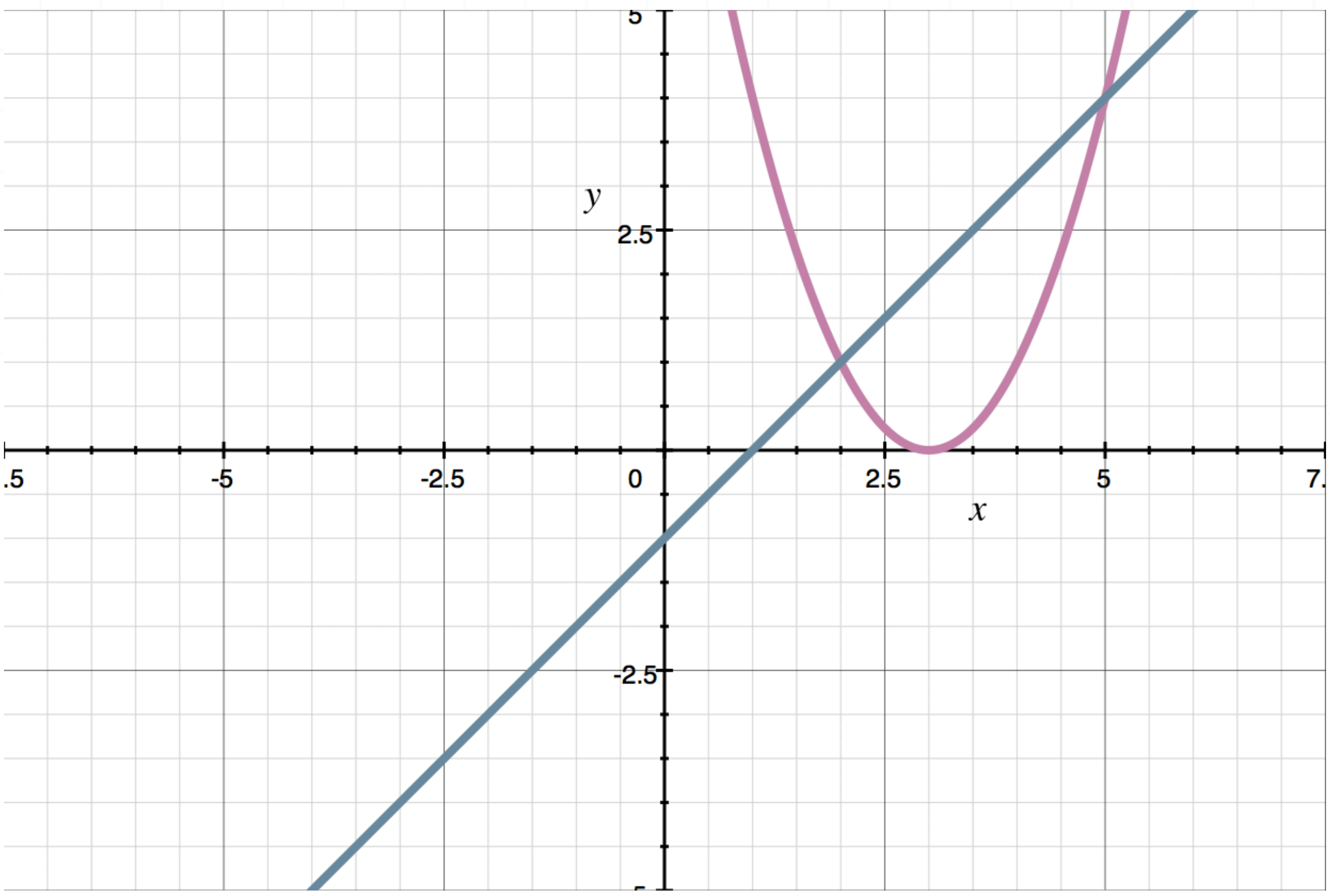
$$\lim_{x \to 5} x^2 - 6x + 9 \leq \lim_{x \to 5} h(x) \leq \lim_{x \to 5} x - 1$$

Evaluate the limits on the left and right sides using substitution.

$$5^2 - 6(5) + 9 \le \lim_{x\to 5} h(x) \le 5 - 1$$

$$4 \le \lim_{x\to 5} h(x) \le 4$$

Therefore, by the Squeeze Theorem, $\lim_{x\to 5} h(x) = 4$. The graph below shows the limit at the intersection point.



DEFINITION OF THE DERIVATIVE

■ 1. Use the definition of the derivative to find the derivative of $f(x) = 2x^2 + 2x - 12$ at $(4,28)$.

Solution:

The definition of the derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So at $(4,28)$, the derivative will be

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$f'(4) = \lim_{h \to 0} \frac{(2(4+h)^2 + 2(4+h) - 12) - (2(4)^2 + 2(4) - 12)}{h}$$

$$f'(4) = \lim_{h \to 0} \frac{(2(16 + 8h + h^2) + 8 + 2h - 12) - (32 + 8 - 12)}{h}$$

$$f'(4) = \lim_{h \to 0} \frac{32 + 16h + 2h^2 + 8 + 2h - 12 - 32 - 8 + 12}{h}$$

$$f'(4) = \lim_{h \to 0} \frac{18h + 2h^2}{h}$$

$$f'(4) = \lim_{h \to 0} 18 + 2h$$

Evaluate the limit to find the derivative at $(4,28)$.

$$f'(4) = 18 + 2(0)$$

$$f'(4) = 18$$

■ 2. Use the definition of the derivative to find the derivative of $g(x) = 3x^3 - 4x + 7$ at $(-2,-9)$.

Solution:

The definition of the derivative is

$$g'(x) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

So at $(-2,-9)$, the derivative will be

$$g'(-2) = \lim_{x \to -2} \frac{g(x) - g(-2)}{x - (-2)}$$

$$g'(-2) = \lim_{x \to -2} \frac{(3x^3 - 4x + 7) - (3(-2)^3 - 4(-2) + 7)}{x + 2}$$

$$g'(-2) = \lim_{x \to -2} \frac{(3x^3 - 4x + 7) - (-24 + 8 + 7)}{x + 2}$$

$$g'(-2) = \lim_{x \to -2} \frac{3x^3 - 4x + 16}{x + 2}$$

$$g'(-2) = \lim_{x \to -2} \frac{(x+2)(3x^2 - 6x + 8)}{x+2}$$

$$g'(-2) = \lim_{x \to -2} 3x^2 - 6x + 8$$

Evaluate the limit to find the derivative at $(-2, -9)$.

$$g'(-2) = 3(-2)^2 - 6(-2) + 8$$

$$g'(-2) = 32$$

■ 3. Use the definition of the derivative to find the derivative of $h(x) = 9x^2 - 7x - 4$ at $(2,18)$.

Solution:

The definition of the derivative is

$$h'(x) = \lim_{h \to 0} \frac{h(x+h) - h(x)}{h}$$

So at $(2,18)$, the derivative will be

$$h'(2) = \lim_{h \to 0} \frac{h(2+h) - h(2)}{h}$$

$$h'(2) = \lim_{h \to 0} \frac{(9(2+h)^2 - 7(2+h) - 4) - (9(2)^2 - 7(2) - 4)}{h}$$

$$h'(2) = \lim_{h \to 0} \frac{(9(4 + 4h + h^2) - 14 - 7h - 4) - (36 - 14 - 4)}{h}$$

$$h'(2) = \lim_{h \to 0} \frac{36 + 36h + 9h^2 - 14 - 7h - 4 - 36 + 14 + 4}{h}$$

$$h'(2) = \lim_{h \to 0} \frac{29h + 9h^2}{h}$$

$$h'(2) = \lim_{h \to 0} 29 + 9h$$

Evaluate the limit to find the derivative at $(2,18)$.

$$h'(2) = 29 + 9(0)$$

$$h'(2) = 29$$

■ 4. Use the definition of the derivative to find the derivative of $h(x) = 8x^2 - 19x + 15$ at $(2,9)$.

Solution:

The definition of the derivative is

$$h'(x) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

So at $(2,9)$, the derivative will be

$$h'(2)=\lim_{x\to 2}\frac{h(x)-h(2)}{x-2}$$

$$h'(2)=\lim_{x\to 2}\frac{(8x^2-19x+15)-(8(2)^2-19(2)+15)}{x-2}$$

$$h'(2)=\lim_{x\to 2}\frac{8x^2-19x+15-32+38-15}{x-2}$$

$$h'(2)=\lim_{x\to 2}\frac{8x^2-19x+6}{x+2}$$

$$h'(2)=\lim_{x\to 2}\frac{(8x-3)(x-2)}{x-2}$$

$$h'(2)=\lim_{x\to 2}8x-3$$

Evaluate the limit to find the derivative at $(2,9)$.

$$h'(2)=8(2)-3$$

$$h'(2)=13$$

POWER RULE

■ 1. Find the derivative of $f(x) = 7x^3 - 17x^2 + 51x - 25$ using the power rule.

Solution:

The power rule for a polynomial function $f(x)$ is

$$\frac{d}{dx}\left(ax^n\right) = anx^{n-1}$$

Differentiating

$$f(x) = 7x^3 - 17x^2 + 51x - 25$$

term by term gives

$$f'(x) = 7(3)x^{3-1} - 17(2)x^{2-1} + 51(1)x^{1-1} - 25(0)x^{0-1}$$

$$f'(x) = 21x^2 - 34x^1 + 51x^0 - 0x^{-1}$$

$$f'(x) = 21x^2 - 34x + 51(1) - 0$$

$$f'(x) = 21x^2 - 34x + 51$$

■ 2. Find the derivative of $g(x) = 2x^4 + 8x^3 + 6x^2 - 32x + 16$ using the power rule.

Solution:

The power rule for a polynomial function $f(x)$ is

$$\frac{d}{dx}\left(ax^n\right) = anx^{n-1}$$

Differentiating

$$g(x) = 2x^4 + 8x^3 + 6x^2 - 32x + 16$$

term by term gives

$$g'(x) = 2(4)x^{4-1} + 8(3)x^{3-1} + 6(2)x^{2-1} - 32(1)x^{1-1} + 16(0)x^{0-1}$$

$$g'(x) = 8x^3 + 24x^2 + 12x^1 - 32x^0 + 0x^{-1}$$

$$g'(x) = 8x^3 + 24x^2 + 12x - 32(1) + 0$$

$$g'(x) = 8x^3 + 24x^2 + 12x - 32$$

■ 3. Find the derivative of $h(x) = 22x^3 - 19x^2 + 13x - 17$ using the power rule.

Solution:

The power rule for a polynomial function $f(x)$ is

$$\frac{d}{dx}\left(ax^n\right) = anx^{n-1}$$

Differentiating

$$h(x) = 22x^3 - 19x^2 + 13x - 17$$

term by term gives

$$h'(x) = 22(3)x^{3-1} - 19(2)x^{2-1} + 13(1)x^{1-1} - 17(0)x^{0-1}$$

$$h'(x) = 66x^2 - 38x^1 + 13x^0 - 0x^{-1}$$

$$h'(x) = 66x^2 - 38x + 13(1) - 0$$

$$h'(x) = 66x^2 - 38x + 13$$

POWER RULE FOR NEGATIVE POWERS

■ 1. Find the derivative of the function using the power rule.

$$f(x) = \frac{7}{x^2} - \frac{5}{x^4} + \frac{2}{x}$$

Solution:

The power rule for a polynomial function is

$$\frac{d}{dx}\left(ax^n\right) = anx^{n-1}$$

Rearrange $f(x)$ to use the power rule.

$$f(x) = 7x^{-2} - 5x^{-4} + 2x^{-1}$$

Differentiating term by term gives

$$f'(x) = 7(-2)x^{-2-1} - 5(-4)x^{-4-1} + 2(-1)x^{-1-1}$$

$$f'(x) = -14x^{-3} + 20x^{-5} - 2x^{-2}$$

Move the variables back to the denominator to make positive exponents.

$$f'(x) = -\frac{14}{x^3} + \frac{20}{x^5} - \frac{2}{x^2}$$

■ 2. Find the derivative of the function using the power rule.

$$g(x) = \frac{1}{9x^4} + \frac{2}{3x^5} - \frac{1}{x}$$

Solution:

The power rule for a polynomial function is

$$\frac{d}{dx}\left(ax^n\right) = anx^{n-1}$$

Rearrange $g(x)$ to use the power rule.

$$g(x) = \frac{1}{9}x^{-4} + \frac{2}{3}x^{-5} - x^{-1}$$

Differentiating term by term gives

$$g'(x) = \frac{1}{9}(-4)x^{-4-1} + \frac{2}{3}(-5)x^{-5-1} - (-1)x^{-1-1}$$

$$g'(x) = -\frac{4}{9}x^{-5} - \frac{10}{3}x^{-6} + x^{-2}$$

Move the variables back to the denominator to make positive exponents.

$$g'(x) = -\frac{4}{9x^5} - \frac{10}{3x^6} + \frac{1}{x^2}$$

■ 3. Find the derivative of the function using the power rule.

$$h(x) = -\frac{7}{6x^6} - \frac{1}{4x^4} + \frac{9}{2x^2}$$

Solution:

The power rule for a polynomial function is

$$\frac{d}{dx}\left(ax^n\right) = anx^{n-1}$$

Rearrange $h(x)$ to use the power rule.

$$h(x) = -\frac{7}{6}x^{-6} - \frac{1}{4}x^{-4} + \frac{9}{2}x^{-2}$$

Differentiating term by term gives

$$h'(x) = -\frac{7}{6}(-6)x^{-6-1} - \frac{1}{4}(-4)x^{-4-1} + \frac{9}{2}(-2)x^{-2-1}$$

$$h'(x) = 7x^{-7} + x^{-5} - 9x^{-3}$$

Move the variables back to the denominator to make positive exponents.

$$h'(x) = \frac{7}{x^7} + \frac{1}{x^5} - \frac{9}{x^3}$$

POWER RULE FOR FRACTIONAL POWERS

■ 1. Find the derivative of the function using the power rule.

$$f(x) = 4x^{\frac{3}{2}} - 6x^{\frac{5}{3}}$$

Solution:

The power rule for a polynomial function is

$$\frac{d}{dx}\left(ax^n\right) = anx^{n-1}$$

Differentiating

$$f(x) = 4x^{\frac{3}{2}} - 6x^{\frac{5}{3}}$$

term by term gives

$$f'(x) = 4\left(\frac{3}{2}\right)x^{\frac{3}{2}-1} - 6\left(\frac{5}{3}\right)x^{\frac{5}{3}-1}$$

$$f'(x) = 6x^{\frac{3}{2}-\frac{2}{2}} - 10x^{\frac{5}{3}-\frac{3}{3}}$$

$$f'(x) = 6x^{\frac{1}{2}} - 10x^{\frac{2}{3}}$$

■ 2. Find the derivative of the function using the power rule.

$$g(x) = 6x^{\sqrt{3}} - 4x^{\sqrt{5}}$$

Solution:

The power rule for a polynomial function is

$$\frac{d}{dx}\left(ax^n\right) = anx^{n-1}$$

Differentiating

$$g(x) = 6x^{\sqrt{3}} - 4x^{\sqrt{5}}$$

term by term gives

$$g'(x) = 6\sqrt{3}x^{\sqrt{3}-1} - 4\sqrt{5}x^{\sqrt{5}-1}$$

■ 3. Find the derivative of the function using the power rule.

$$h(x) = \frac{1}{3}x^{\frac{6}{5}} + \frac{1}{4}x^{\frac{8}{3}} - \frac{1}{5}x^{\frac{5}{2}}$$

Solution:

The power rule for a polynomial function is

$$\frac{d}{dx}\left(ax^n\right) = anx^{n-1}$$

Differentiating

$$h(x) = \frac{1}{3}x^{\frac{6}{5}} + \frac{1}{4}x^{\frac{8}{3}} - \frac{1}{5}x^{\frac{5}{2}}$$

term by term gives

$$h'(x) = \frac{1}{3}\left(\frac{6}{5}\right)x^{\frac{6}{5}-1} + \frac{1}{4}\left(\frac{8}{3}\right)x^{\frac{8}{3}-1} - \frac{1}{5}\left(\frac{5}{2}\right)x^{\frac{5}{2}-1}$$

$$h'(x) = \frac{2}{5}x^{\frac{6}{5}-\frac{5}{5}} + \frac{2}{3}x^{\frac{8}{3}-\frac{3}{3}} - \frac{1}{2}x^{\frac{5}{2}-\frac{2}{2}}$$

$$h'(x) = \frac{2}{5}x^{\frac{1}{5}} + \frac{2}{3}x^{\frac{5}{3}} - \frac{1}{2}x^{\frac{3}{2}}$$

PRODUCT RULE, TWO FUNCTIONS

■ 1. Use the product rule to find the derivative of the function.

$$h(x) = (3x + 5)\ln(5x)$$

Solution:

The derivative of a function $h(x) = f(x) \cdot g(x)$ using the product rule is

$$h'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

Let

$$f(x) = 3x + 5$$

$$f'(x) = 3$$

$$g(x) = \ln(5x)$$

$$g'(x) = \frac{1}{5x}(5) = \frac{1}{x}$$

Then by product rule, the derivative is

$$h'(x) = (3x + 5) \cdot \frac{1}{x} + 3 \cdot \ln(5x)$$

$$h'(x) = \frac{3x + 5}{x} + 3\ln(5x)$$

■ 2. Use the product rule to find the derivative of the function.

$$h(x) = 8x^3e^{7x}$$

Solution:

The derivative of a function $h(x) = f(x) \cdot g(x)$ using the product rule is

$$h'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

Let

$$f(x) = 8x^3$$

$$f'(x) = 24x^2$$

$$g(x) = e^{7x}$$

$$g'(x) = 7e^{7x}$$

Then by product rule, the derivative is

$$h'(x) = 8x^3 \cdot 7e^{7x} + 24x^2 \cdot e^{7x}$$

$$h'(x) = 56x^3e^{7x} + 24x^2e^{7x}$$

■ 3. Use the product rule to find the derivative of the function.

$$h(x) = (5x^2 - x)(e^{4x} - 6)$$

Solution:

The derivative of a function $h(x) = f(x) \cdot g(x)$ using the product rule is

$$h'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

Let

$$f(x) = 5x^2 - x$$

$$f'(x) = 10x - 1$$

$$g(x) = e^{4x} - 6$$

$$g'(x) = 4e^{4x}$$

Then by product rule, the derivative is

$$h'(x) = (5x^2 - x) \cdot 4e^{4x} + (10x - 1) \cdot (e^{4x} - 6)$$

$$h'(x) = 20x^2e^{4x} - 4xe^{4x} + 10xe^{4x} - 60x - e^{4x} + 6$$

$$h'(x) = 20x^2e^{4x} + 6xe^{4x} - 60x - e^{4x} + 6$$

PRODUCT RULE, THREE OR MORE FUNCTIONS

■ 1. Use the product rule to find the derivative of the function.

$$y = 5x^4e^{3x}\cos(6x)$$

Solution:

The derivative of a function $y = f(x) \cdot g(x) \cdot h(x)$ using the product rule is

$$y' = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$$

Let

$$f(x) = 5x^4$$

$$f'(x) = 20x^3$$

$$g(x) = \cos(6x)$$

$$g'(x) = -6\sin(6x)$$

$$h(x) = e^{3x}$$

$$h'(x) = 3e^{3x}$$

Then by product rule, the derivative is

$$y' = (20x^3)(\cos(6x))(e^{3x}) + (5x^4)(-6\sin(6x)(e^{3x}) + (5x^4)(\cos(6x))(3e^{3x})$$

$$y' = 20x^3e^{3x}\cos(6x) - 30x^4e^{3x}\sin(6x) + 15x^4e^{3x}\cos(6x)$$

■ 2. Use the product rule to find the derivative of the function.

$$y = (-6x^2)(-2e^{5x})\tan(5x)$$

Solution:

The derivative of a function $y = f(x) \cdot g(x) \cdot h(x)$ using the product rule is

$$y' = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$$

Let

$$f(x) = -6x^2$$

$$f'(x) = -12x$$

$$g(x) = -2e^{5x}$$

$$g'(x) = -10e^{5x}$$

$$h(x) = \tan(5x)$$

$$h'(x) = 5\sec^2(5x)$$

Then by product rule, the derivative is

$$y' = (-12x)(-2e^{5x})(\tan(5x)) + (-6x^2)(-10e^{5x})(\tan(5x)) + (-6x^2)(-2e^{5x})(5\sec^2(5x))$$

$$y' = 24xe^{5x}\tan(5x) + 60x^2e^{5x}\tan(5x) + 60x^2e^{5x}\sec^2(5x)$$

■ 3. Use the product rule to find the derivative of the function.

$$y = (\sin(7x))(7e^{4x})(2x^6 + 1)$$

Solution:

The derivative of a function $y = f(x) \cdot g(x) \cdot h(x)$ using the product rule is

$$y' = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$$

Let

$$f(x) = \sin(7x)$$

$$f'(x) = 7\cos(7x)$$

$$g(x) = 7e^{4x}$$

$$g'(x) = 28e^{4x}$$

$$h(x) = 2x^6 + 1$$

$$h'(x) = 12x^5$$

Then by product rule, the derivative is

$$y' = (7\cos(7x))(7e^{4x})(2x^6 + 1) + (\sin(7x))(28e^{4x})(2x^6 + 1) + (\sin(7x))(7e^{4x})(12x^5)$$

$$y' = 49e^{4x}(2x^6 + 1)\cos(7x) + 28e^{4x}(2x^6 + 1)\sin(7x) + 84x^5e^{4x}\sin(7x)$$

■ 4. Use the product rule to find the derivative of the function.

$$y = (\cos(3x))(\sin(2x))(\tan(5x))(e^{2x})$$

Solution:

The derivative of a function $y = f(x) \cdot g(x) \cdot h(x) \cdot k(x)$ using the product rule is

$$y' = f'(x) \cdot g(x) \cdot h(x) \cdot k(x) + f(x) \cdot g'(x) \cdot h(x) \cdot k(x)$$

$$+f(x) \cdot g(x) \cdot h'(x) \cdot k(x) + f(x) \cdot g(x) \cdot h(x) \cdot k'(x)$$

Let

$$f(x) = \cos(3x)$$

$$f'(x) = -3\sin(3x)$$

$$g(x) = \sin(2x)$$

$$g'(x) = 2\cos(2x)$$

$$h(x) = \tan(5x)$$

$$h'(x) = 5\sec^2(5x)$$

$$k(x) = e^{2x}$$

$$k'(x) = 2e^{2x}$$

Then by product rule, the derivative is

$$y' = -3\sin(3x)\sin(2x)\tan(5x)e^{2x} + \cos(3x)(2\cos(2x))\tan(5x)e^{2x}$$

$$+\cos(3x)\sin(2x)(5\sec^2(5x))e^{2x} + \cos(3x)\sin(2x)\tan(5x)(2e^{2x})$$

$$y' = -3e^{2x}\sin(2x)\sin(3x)\tan(5x) + 2e^{2x}\cos(2x)\cos(3x)\tan(5x)$$

$$+5e^{2x}\sin(2x)\cos(3x)\sec^2(5x) + 2e^{2x}\sin(2x)\cos(3x)\tan(5x)$$

QUOTIENT RULE

■ 1. Use the quotient rule to find the derivative of the function.

$$h(x) = \frac{2x+6}{7x+5}$$

Solution:

The derivative of a function

$$h(x) = \frac{f(x)}{g(x)}$$

using the quotient rule is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Let

$$f(x) = 2x + 6$$

$$f'(x) = 2$$

$$g(x) = 7x + 5$$

$$g'(x) = 7$$

Then the derivative is

$$h'(x) = \frac{2 \cdot (7x + 5) - (2x + 6) \cdot 7}{(7x + 5)^2}$$

$$h'(x) = \frac{14x + 10 - 14x - 42}{(7x + 5)^2}$$

$$h'(x) = -\frac{32}{(7x + 5)^2}$$

■ 2. Use the quotient rule to find the derivative of the function.

$$h(x) = \frac{5x - 3}{4x - 9}$$

Solution:

The derivative of a function

$$h(x) = \frac{f(x)}{g(x)}$$

using the quotient rule is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Let

$$f(x) = 5x - 3$$

$$f'(x) = 5$$

$$g(x) = 4x - 9$$

$$g'(x) = 4$$

Then the derivative is

$$h'(x) = \frac{5 \cdot (4x - 9) - (5x - 3) \cdot 4}{(4x - 9)^2}$$

$$h'(x) = \frac{20x - 45 - 20x + 12}{(4x - 9)^2}$$

$$h'(x) = -\frac{33}{(4x - 9)^2}$$

■ 3. Use the quotient rule to find the derivative of the function.

$$h(x) = \frac{-8x}{5x + 2}$$

Solution:

The derivative of a function

$$h(x) = \frac{f(x)}{g(x)}$$

using the quotient rule is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Let

$$f(x) = -8x$$

$$f'(x) = -8$$

$$g(x) = 5x + 2$$

$$g'(x) = 5$$

Then the derivative is

$$h'(x) = \frac{-8 \cdot (5x+2) - (-8x) \cdot 5}{(5x+2)^2}$$

$$h'(x) = \frac{-40x - 16 + 40x}{(5x+2)^2}$$

$$h'(x) = -\frac{16}{(5x+2)^2}$$

■ 4. Use the quotient rule to find the derivative of the function.

$$h(x) = \frac{3x^2 + 12x}{e^x}$$

Solution:

The derivative of a function

$$h(x) = \frac{f(x)}{g(x)}$$

using the quotient rule is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Let

$$f(x) = 3x^2 + 12x$$

$$f'(x) = 6x + 12$$

$$g(x) = e^x$$

$$g'(x) = e^x$$

Then the derivative is

$$h'(x) = \frac{(6x + 12) \cdot e^x - (3x^2 + 12x) \cdot e^x}{(e^x)^2}$$

$$h'(x) = \frac{6xe^x + 12e^x - 3x^2e^x - 12xe^x}{e^{2x}}$$

$$h'(x) = \frac{-3x^2e^x - 6xe^x + 12e^x}{e^{2x}}$$

$$h'(x) = \frac{-3x^2 - 6x + 12}{e^x}$$

CHAIN RULE WITH POWER RULE

■ 1. Find $h'(x)$ if $h(x) = (3x^2 - 7)^4$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $3x^2 - 7$, and that the derivative of that inside function is $6x$. Therefore, the derivative is

$$h'(x) = 4(3x^2 - 7)^3(6x)$$

$$h'(x) = 24x(3x^2 - 7)^3$$

■ 2. Find $h'(x)$ if $h(x) = 2(5x^2 + 2x)^3$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $5x^2 + 2x$, and that the derivative of that inside function is $10x + 2$. Therefore, the derivative is

$$h'(x) = 6(5x^2 + 2x)^2(10x + 2)$$

$$h'(x) = 6(10x + 2)(5x^2 + 2x)^2$$

■ 3. Find $h'(x)$ if $h(x) = (2x^2 - 6x + 5)^7$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $2x^2 - 6x + 5$, and that the derivative of that inside function is $4x - 6$. Therefore, the derivative is

$$h'(x) = 7(2x^2 - 6x + 5)^6(4x - 6)$$

$$h'(x) = 7(4x - 6)(2x^2 - 6x + 5)^6$$

■ 4. Find $h'(x)$ if $h(x) = 2(x^3 + 4x^2 - 2x)^5$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $x^3 + 4x^2 - 2x$, and that the derivative of that inside function is $3x^2 + 8x - 2$. Therefore, the derivative is

$$h'(x) = 2(5)(x^3 + 4x^2 - 2x)^4(3x^2 + 8x - 2)$$

$$h'(x) = 10(3x^2 + 8x - 2)(x^3 + 4x^2 - 2x)^4$$

CHAIN RULE WITH PRODUCT RULE

■ 1. Find $y'(x)$ if $y(x) = (3x - 2)(5x^3)^5$.

Solution:

To find the derivative, we have to apply product rule.

$$y'(x) = \frac{d}{dx}(3x - 2) \cdot (5x^3)^5 + (3x - 2) \cdot \frac{d}{dx}(5x^3)^5$$

To find each derivative, we have to apply chain rule.

$$y'(x) = 3 \cdot (5x^3)^5 + (3x - 2) \cdot 5(5x^3)^4(15x^2)$$

$$y'(x) = 3(5x^3)^5 + 75x^2(3x - 2)(5x^3)^4$$

■ 2. Find $h'(x)$ if $h(x) = (x^2 - 5x)^2(2x^3 - 3x^2)^5$.

Solution:

To find the derivative, we have to apply product rule.

$$h'(x) = \frac{d}{dx}(x^2 - 5x)^2 \cdot (2x^3 - 3x^2)^5 + (x^2 - 5x)^2 \cdot \frac{d}{dx}(2x^3 - 3x^2)^5$$

To find each derivative, we have to apply chain rule.

$$h'(x) = 2(x^2 - 5x)(2x - 5) \cdot (2x^3 - 3x^2)^5 + (x^2 - 5x)^2 \cdot 5(2x^3 - 3x^2)^4(6x^2 - 6x)$$

$$h'(x) = 2(2x - 5)(x^2 - 5x)(2x^3 - 3x^2)^5 + 5(6x^2 - 6x)(x^2 - 5x)^2(2x^3 - 3x^2)^4$$

■ 3. Find $h'(x)$ if $h(x) = (x + 4)^5(3x - 2)^3$.

Solution:

To find the derivative, we have to apply product rule.

$$h'(x) = \frac{d}{dx}(x + 4)^5 \cdot (3x - 2)^3 + (x + 4)^5 \cdot \frac{d}{dx}(3x - 2)^3$$

To find each derivative, we have to apply chain rule.

$$h'(x) = 5(x + 4)^4(1) \cdot (3x - 2)^3 + (x + 4)^5 \cdot 3(3x - 2)^2(3)$$

$$h'(x) = 5(x + 4)^4(3x - 2)^3 + 9(x + 4)^5(3x - 2)^2$$

CHAIN RULE WITH QUOTIENT RULE

■ 1. Find $h'(x)$.

$$h(x)=\frac{(2x+1)^3}{(3x-2)^2}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x)=\frac{\frac{d}{dx}(2x+1)^3\cdot(3x-2)^2-(2x+1)^3\cdot\frac{d}{dx}(3x-2)^2}{((3x-2)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x)=\frac{3(2x+1)^2(2)\cdot(3x-2)^2-(2x+1)^3\cdot 2(3x-2)(3)}{((3x-2)^2)^2}$$

$$h'(x)=\frac{6(2x+1)^2(3x-2)^2-6(2x+1)^3(3x-2)}{(3x-2)^4}$$

$$h'(x)=\frac{6(2x+1)^2(3x-2)-6(2x+1)^3}{(3x-2)^3}$$

Factor the numerator.

$$h'(x)=\frac{6(2x+1)^2(3x-2-(2x+1))}{(3x-2)^3}$$

$$h'(x) = \frac{6(2x+1)^2(3x-2-2x-1)}{(3x-2)^3}$$

$$h'(x) = \frac{6(2x+1)^2(x-3)}{(3x-2)^3}$$

■ 2. Find $h'(x)$.

$$h(x) = \frac{(4x+5)^5}{(x+3)^2}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(4x+5)^5 \cdot (x+3)^2 - (4x+5)^5 \cdot \frac{d}{dx}(x+3)^2}{((x+3)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{5(4x+5)^4(4) \cdot (x+3)^2 - (4x+5)^5 \cdot 2(x+3)(1)}{(x+3)^4}$$

$$h'(x) = \frac{20(4x+5)^4(x+3)^2 - 2(4x+5)^5(x+3)}{(x+3)^4}$$

$$h'(x) = \frac{20(4x+5)^4(x+3) - 2(4x+5)^5}{(x+3)^3}$$

Factor the numerator.

$$h'(x) = \frac{2(4x+5)^4(10(x+3)-(4x+5))}{(x+3)^3}$$

$$h'(x) = \frac{2(4x+5)^4(10x+30-4x-5)}{(x+3)^3}$$

$$h'(x) = \frac{2(4x+5)^4(6x+25)}{(x+3)^3}$$

■ 3. Find $h'(x)$.

$$h(x) = \frac{(7x-4)^3}{(5x+3)^2}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(7x-4)^3 \cdot (5x+3)^2 - (7x-4)^3 \cdot \frac{d}{dx}(5x+3)^2}{((5x+3)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{3(7x-4)^2(7) \cdot (5x+3)^2 - (7x-4)^3 \cdot 2(5x+3)^1(5)}{(5x+3)^4}$$

$$h'(x) = \frac{21(7x-4)^2(5x+3)^2 - 10(7x-4)^3(5x+3)}{(5x+3)^4}$$

$$h'(x) = \frac{21(7x-4)^2(5x+3) - 10(7x-4)^3}{(5x+3)^3}$$

Factor the numerator.

$$h'(x) = \frac{(7x-4)^2(21(5x+3) - 10(7x-4))}{(5x+3)^3}$$

$$h'(x) = \frac{(7x-4)^2(105x+63-70x+40)}{(5x+3)^3}$$

$$h'(x) = \frac{(7x-4)^2(35x+103)}{(5x+3)^3}$$

■ 4. Find $h'(x)$.

$$h(x) = \frac{(6x-1)^4}{(8x+1)^2}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(6x-1)^4 \cdot (8x+1)^2 - (6x-1)^4 \cdot \frac{d}{dx}(8x+1)^2}{((8x+1)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{4(6x-1)^3(6) \cdot (8x+1)^2 - (6x-1)^4 \cdot 2(8x+1)(8)}{(8x+1)^4}$$

$$h'(x) = \frac{24(6x-1)^3(8x+1)^2 - 16(6x-1)^4(8x+1)}{(8x+1)^4}$$

$$h'(x) = \frac{24(6x-1)^3(8x+1) - 16(6x-1)^4}{(8x+1)^3}$$

Factor the numerator.

$$h'(x) = \frac{8(6x-1)^3(3(8x+1) - 2(6x-1))}{(8x+1)^3}$$

$$h'(x) = \frac{8(6x-1)^3(24x+3-12x+2)}{(8x+1)^3}$$

$$h'(x) = \frac{8(6x-1)^3(12x+5)}{(8x+1)^3}$$

TRIGONOMETRIC DERIVATIVES

■ 1. Find $f'(x)$ if $f(x) = 5x^7 + 8\sin(7x^7)$.

Solution:

Differentiate one term at a time, remembering to apply chain rule as you go.

$$f'(x) = 5(7)x^6 + 8\cos(7x^7)(49x^6)$$

$$f'(x) = 35x^6 + 392x^6\cos(7x^7)$$

$$f'(x) = 7x^6(5 + 56\cos(7x^7))$$

■ 2. Find $g'(x)$ if $g(x) = 3\sin(4x^3) - 4\cos(6x) + 3\sec(2x^4)$.

Solution:

Differentiate one term at a time, remembering to apply chain rule as you go.

$$g'(x) = 3\cos(4x^3)(12x^2) - 4(-\sin(6x))(6) + 3\sec(2x^4)\tan(2x^4)(8x^3)$$

$$g'(x) = 36x^2\cos(4x^3) + 24\sin(6x) + 24x^3\tan(2x^4)\sec(2x^4)$$

$$g'(x) = 12(3x^2\cos(4x^3) + 2\sin(6x) + 2x^3\tan(2x^4)\sec(2x^4))$$

■ 3. Find $h'(x)$ if $h(x) = 5\tan(4x^6) + 6\cot(6x^4)$.

Solution:

Differentiate one term at a time, remembering to apply chain rule as you go.

$$h'(x) = 5\sec^2(4x^6)(24x^5) + 6(-\csc^2(6x^4))(24x^3)$$

$$h'(x) = 120x^5\sec^2(4x^6) - 144x^3\csc^2(6x^4)$$

$$h'(x) = 24x^3(5x^2\sec^2(4x^6) - 6\csc^2(6x^4))$$

INVERSE TRIGONOMETRIC DERIVATIVES

■ 1. Find $f'(t)$.

$$f(t) = 4\sin^{-1}\left(\frac{t}{4}\right)$$

Solution:

The derivative of inverse sine is given by

$$\frac{d}{dt}a\sin^{-1}(y(t)) = a \cdot \frac{y'(t)}{\sqrt{1-\left[y(t)\right]^2}}$$

If $a = 4$ and $y(t) = t/4$, then $y'(t) = 1/4$. Then the derivative is

$$f'(t) = 4 \cdot \frac{\frac{1}{4}}{\sqrt{1-\left(\frac{t}{4}\right)^2}} = \frac{1}{\sqrt{\frac{16}{16}-\frac{t^2}{16}}} = \frac{1}{\sqrt{\frac{16-t^2}{16}}} = \frac{1}{\frac{\sqrt{16-t^2}}{4}} = \frac{4}{\sqrt{16-t^2}}$$

■ 2. Find $g'(t)$.

$$g(t) = -6\cos^{-1}(2t+3)$$

Solution:

The derivative of inverse cosine is given by

$$\frac{d}{dt}a\cos^{-1}(y(t)) = a \cdot -\frac{y'(t)}{\sqrt{1-\left[y(t)\right]^2}}$$

If $a = -6$ and $y(t) = 2t + 3$, then $y'(t) = 2$. Then the derivative is

$$g'(t) = -6 \cdot -\frac{2}{\sqrt{1-(2t+3)^2}} = \frac{12}{\sqrt{1-4t^2-12t-9}} = \frac{6}{\sqrt{-(t+1)(t+2)}}$$

■ 3. Find $h'(t)$.

$$h(t) = 3\tan^{-1}(6t^2)$$

Solution:

The derivative of inverse tangent is given by

$$\frac{d}{dt}a\tan^{-1}(y(t)) = a \cdot \frac{y'(t)}{1+\left[y(t)\right]^2}$$

If $a = 3$ and $y(t) = 6t^2$, then $y'(t) = 12t$. Then the derivative is

$$h'(t) = 3 \cdot \frac{12t}{1+(6t^2)^2} = \frac{36t}{1+36t^4}$$

HYPERBOLIC DERIVATIVES

■ 1. Find $f'(\theta)$ if $f(\theta) = 3\sinh(2\theta^2 - 5\theta + 2)$.

Solution:

The derivative of hyperbolic sine is given by

$$\frac{d}{d\theta}a\sinh(y(\theta)) = a \cdot \cosh(y(\theta)) \cdot y'(\theta)$$

If $a = 3$ and $y(\theta) = 2\theta^2 - 5\theta + 2$, then $y'(\theta) = 4\theta - 5$. Then the derivative is

$$f'(\theta) = 3\cosh(2\theta^2 - 5\theta + 2)(4\theta - 5)$$

$$f'(\theta) = 3(4\theta - 5)\cosh(2\theta^2 - 5\theta + 2)$$

■ 2. Find $g'(\theta)$ if $g(\theta) = 2\cosh(5\theta^{\frac{3}{2}} + 6\theta)$.

Solution:

The derivative of hyperbolic cosine is given by

$$\frac{d}{d\theta}a\cosh(y(\theta)) = a \cdot \sinh(y(\theta)) \cdot y'(\theta)$$

If $a = 2$ and $y(\theta) = 5\theta^{\frac{3}{2}} + 6\theta$, then $y'(\theta) = 5(3/2)\theta^{\frac{1}{2}} + 6$. Then the derivative is

$$g'(\theta) = 2\sinh(5\theta^{\frac{3}{2}} + 6\theta)\left(\frac{15}{2}\theta^{\frac{1}{2}} + 6\right)$$

$$g'(\theta) = (15\theta^{\frac{1}{2}} + 12)\sinh(5\theta^{\frac{3}{2}} + 6\theta)$$

$$g'(\theta) = 3(5\theta^{\frac{1}{2}} + 4)\sinh(5\theta^{\frac{3}{2}} + 6\theta)$$

■ 3. Find $h'(\theta)$ if $h(\theta) = 9\tanh\left(3\theta^2 - \theta^{\sqrt{3}}\right)$.

Solution:

The derivative of hyperbolic tangent is given by

$$\frac{d}{d\theta}a\tanh(y(\theta)) = a \cdot \text{sech}^2(y(\theta)) \cdot y'(\theta)$$

If $a = 9$ and $y(\theta) = 3\theta^2 - \theta^{\sqrt{3}}$, then $y'(\theta) = 6\theta - \sqrt{3} \cdot \theta^{\sqrt{3}-1}$. Then the derivative is

$$h'(\theta) = 9\left(6\theta - \sqrt{3} \cdot \theta^{\sqrt{3}-1}\right)\text{sech}^2\left(3\theta^2 - \theta^{\sqrt{3}}\right)$$

INVERSE HYPERBOLIC DERIVATIVES

■ 1. Find $f'(t)$ if $f(t) = 7\sinh^{-1}(5t^4)$.

Solution:

The derivative of inverse hyperbolic sine is given by

$$\frac{d}{dt}a\sinh^{-1}(y(t)) = a \cdot \frac{y'(t)}{\sqrt{\left[y(t)\right]^2 + 1}}$$

If $a = 7$ and $y(t) = 5t^4$, then $y'(t) = 20t^3$. Then the derivative is

$$f'(t) = 7 \cdot \frac{20t^3}{\sqrt{(5t^4)^2 + 1}} = \frac{140t^3}{\sqrt{25t^8 + 1}}$$

■ 2. Find $g'(t)$ if $g(t) = 4\cosh^{-1}(2t - 3)$.

Solution:

The derivative of inverse hyperbolic cosine is given by

$$\frac{d}{dt}a\cosh^{-1}(y(t)) = a \cdot \frac{y'(t)}{\sqrt{\left[y(t)\right]^2 - 1}}$$

If $a = 4$ and $y(t) = 2t - 3$, then $y'(t) = 2$. Then the derivative is

$$g'(t) = 4 \cdot \frac{2}{\sqrt{(2t-3)^2 - 1}}$$

$$g'(t) = \frac{8}{\sqrt{4t^2 - 12t + 9 - 1}}$$

$$g'(t) = \frac{8}{\sqrt{4t^2 - 12t + 8}}$$

$$g'(t) = \frac{8}{\sqrt{4(t-1)(t-2)}}$$

$$g'(t) = \frac{4}{\sqrt{(t-1)(t-2)}}$$

■ 3. Find $h'(t)$ if $h(t) = 9\tanh^{-1}(-7t + 2)$.

Solution:

The derivative of inverse hyperbolic tangent is given by

$$\frac{d}{dt} a \tanh^{-1}(y(t)) = a \cdot \frac{y'(t)}{1 - \left[y(t)\right]^2}$$

If $a = 9$ and $y(t) = -7t + 2$, then $y'(t) = -7$. Then the derivative is

$$h'(t) = 9 \cdot \frac{-7}{1 - (-7t + 2)^2}$$

$$h'(t) = -\frac{63}{1 - (49t^2 - 28t + 4)}$$

$$h'(t) = -\frac{63}{1 - 49t^2 + 28t - 4}$$

$$h'(t) = -\frac{63}{-49t^2 + 28t - 3}$$

$$h'(t) = \frac{63}{49t^2 - 28t + 3}$$

EXPONENTIAL DERIVATIVES

■ 1. Find $f'(x)$ if $f(x) = (x^3 - x)e^{2x}$.

Solution:

Use product rule to take the derivative.

$$f'(x) = \frac{d}{dx}(x^3 - x) \cdot e^{2x} + (x^3 - x) \cdot \frac{d}{dx}e^{2x}$$

$$f'(x) = (3x^2 - 1) \cdot e^{2x} + (x^3 - x) \cdot 2e^{2x}$$

Factor to simplify.

$$f'(x) = e^{2x}\left(3x^2 - 1 + 2x^3 - 2x\right)$$

$$f'(x) = e^{2x}\left(2x^3 + 3x^2 - 2x - 1\right)$$

■ 2. Find $g'(x)$ if $g(x) = 5x^2e^{2x^2} - 7x + 1$.

Solution:

Use product rule to take the derivative of the first term.

$$g'(x) = \frac{d}{dx}\left(5x^2\right) \cdot e^{2x^2} + 5x^2 \cdot \frac{d}{dx}\left(e^{2x^2}\right) - 7 + 0$$

$$g'(x) = 10x \cdot e^{2x^2} + 5x^2 \cdot e^{2x^2}(4x) - 7$$

$$g'(x) = 10xe^{2x^2} + 20x^3e^{2x^2} - 7$$

Factor to simplify.

$$g'(x) = 10xe^{2x^2}(1 + 2x^2) - 7$$

■ 3. Find $h'(x)$ if $h(x) = \sin(4x)e^{3x^2+4}$.

Solution:

Use product rule to take the derivative.

$$h'(x) = \frac{d}{dx}\sin(4x) \cdot e^{3x^2+4} + \sin(4x) \cdot \frac{d}{dx}e^{3x^2+4}$$

$$h'(x) = \cos(4x)(4) \cdot e^{3x^2+4} + \sin(4x) \cdot e^{3x^2+4}(6x)$$

$$h'(x) = 4e^{3x^2+4}\cos(4x) + 6xe^{3x^2+4}\sin(4x)$$

Factor to simplify.

$$h'(x) = 2e^{3x^2+4}\left(3x\sin(4x) + 2\cos(4x)\right)$$

LOGARITHMIC DERIVATIVES

■ 1. Find $f'(x)$.

$$f(x) = \ln(x^2 + 6x + 9)$$

Solution:

Take the derivative, remembering to apply chain rule.

$$f'(x) = \frac{1}{x^2 + 6x + 9} \cdot (2x + 6) = \frac{2x + 6}{x^2 + 6x + 9} = \frac{2(x + 3)}{(x + 3)(x + 3)} = \frac{2}{x + 3}$$

■ 2. Find $g'(x)$.

$$g(x) = \ln\sqrt{x^3 + x}$$

Solution:

Take the derivative, remembering to apply chain rule.

$$g'(x) = \frac{1}{\sqrt{x^3 + x}} \cdot \frac{1}{2}(x^3 + x)^{-\frac{1}{2}} \cdot (3x^2 + 1)$$

$$g'(x) = \frac{(3x^2 + 1)(x^3 + x)^{-\frac{1}{2}}}{2\sqrt{x^3 + x}}$$

$$g'(x) = \frac{3x^2 + 1}{2\sqrt{x^3 + x}\sqrt{x^3 + x}}$$

$$g'(x) = \frac{3x^2 + 1}{2(x^3 + x)}$$

$$g'(x) = \frac{3x^2 + 1}{2x(x^2 + 1)}$$

■ 3. Find $h'(x)$.

$$h(x) = \ln\left(\frac{x^3}{x^2 + 3}\right)$$

Solution:

Take the derivative, remembering to apply chain rule.

$$h'(x) = \frac{1}{\frac{x^3}{x^2 + 3}} \cdot \frac{3x^2(x^2 + 3) - x^3(2x)}{(x^2 + 3)^2}$$

$$h'(x) = \frac{x^2 + 3}{x^3} \cdot \frac{3x^2(x^2 + 3) - x^3(2x)}{(x^2 + 3)^2}$$

$$h'(x) = \frac{1}{x} \cdot \frac{3(x^2 + 3) - x(2x)}{x^2 + 3}$$

$$h'(x) = \frac{3x^2 + 9 - 2x^2}{x(x^2 + 3)}$$

$$h'(x) = \frac{x^2 + 9}{x(x^2 + 3)}$$

LOGARITHMIC DIFFERENTIATION

■ 1. Use logarithmic differentiation to find dy/dx.

$$y = x^4 e^x \sqrt{x}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln \left(x^4 e^x \sqrt{x} \right)$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln x^4 + \ln e^x + \ln \sqrt{x}$$

$$\ln y = \ln x^4 + x + \ln x^{\frac{1}{2}}$$

$$\ln y = 4 \ln x + x + \frac{1}{2} \ln x$$

Differentiate, remembering to apply chain rule, then solve for dy/dx.

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{4}{x} + 1 + \frac{1}{2x}$$

$$\frac{dy}{dx} = y \left(\frac{4}{x} + 1 + \frac{1}{2x} \right)$$

Substitute for y.

$$\frac{dy}{dx} = x^4e^x\sqrt{x}\left(\frac{4}{x} + 1 + \frac{1}{2x}\right)$$

You could leave the answer like this, or try to simplify.

$$\frac{dy}{dx} = x^4e^x\sqrt{x}\left(\frac{8}{2x} + \frac{2x}{2x} + \frac{1}{2x}\right)$$

$$\frac{dy}{dx} = x^4e^x\sqrt{x}\left(\frac{2x+9}{2x}\right)$$

$$\frac{dy}{dx} = \frac{x^3e^x\sqrt{x}(2x+9)}{2}$$

■ 2. Use logarithmic differentiation to find dy/dx.

$$y = 5x^4e^{3x}\sqrt[4]{x}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln\left(5x^4e^{3x}\sqrt[4]{x}\right)$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln 5x^4 + \ln e^{3x} + \ln\sqrt[4]{x}$$

$$\ln y = 4\ln 5x + 3x + \ln x^{\frac{1}{4}}$$

$$\ln y = 4\ln 5x + 3x + \frac{1}{4}\ln x$$

Differentiate, remembering to apply chain rule, then solve for dy/dx.

$$\frac{1}{y}\cdot\frac{dy}{dx} = 0 + (5)\frac{4}{5x} + 3 + \frac{1}{4x}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{4}{x} + 3 + \frac{1}{4x}$$

$$\frac{dy}{dx} = y\left(\frac{4}{x} + 3 + \frac{1}{4x}\right)$$

Substitute for y.

$$\frac{dy}{dx} = 5x^4e^{3x}\sqrt[4]{x}\left(\frac{4}{x} + 3 + \frac{1}{4x}\right)$$

You could leave the answer like this, or try to simplify.

$$\frac{dy}{dx} = 5x^4e^{3x}\sqrt[4]{x}\left(\frac{16}{4x} + \frac{12x}{4x} + \frac{1}{4x}\right)$$

$$\frac{dy}{dx} = 5x^4e^{3x}\sqrt[4]{x}\left(\frac{12x+17}{4x}\right)$$

$$\frac{dy}{dx} = \frac{5x^3e^{3x}\sqrt[4]{x}(12x+17)}{4}$$

■ 3. Use logarithmic differentiation to find dy/dx.

$$y = x^3 e^{2x} \sqrt{5x}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln \left(x^3 e^{2x} \sqrt{5x} \right)$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln x^3 + \ln e^{2x} + \ln \sqrt{5x}$$

$$\ln y = 3 \ln x + 2x + \ln (5x)^{\frac{1}{2}}$$

$$\ln y = 3 \ln x + 2x + \frac{1}{2} \ln (5x)$$

Differentiate, remembering to apply chain rule, then solve for dy/dx.

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{3}{x} + 2 + \frac{5}{2(5x)}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{3}{x} + 2 + \frac{1}{2x}$$

$$\frac{dy}{dx} = y \left(\frac{3}{x} + 2 + \frac{1}{2x} \right)$$

Substitute for y.

$$\frac{dy}{dx} = x^3 e^{2x} \sqrt{5x} \left(\frac{3}{x} + 2 + \frac{1}{2x} \right)$$

You could leave the answer like this, or try to simplify.

$$\frac{dy}{dx} = x^3 e^{2x} \sqrt{5x} \left(\frac{6}{2x} + \frac{4x}{2x} + \frac{1}{2x} \right)$$

$$\frac{dy}{dx} = x^3 e^{2x} \sqrt{5x} \left(\frac{4x+7}{2x} \right)$$

$$\frac{dy}{dx} = \frac{x^2 e^{2x} \sqrt{5x}(4x+7)}{2}$$

■ 4. Use logarithmic differentiation to find dy/dx.

$$y = \frac{(2e)^{\cos x}}{(3e)^{\sin x}}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln \left(\frac{(2e)^{\cos x}}{(3e)^{\sin x}} \right)$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln(2e)^{\cos x} - \ln(3e)^{\sin x}$$

$$\ln y = \cos x \ln(2e) - \sin x \ln(3e)$$

$$\ln y = (\cos x)(\ln 2 + \ln e) - (\sin x)(\ln 3 + \ln e)$$

$$\ln y = (\cos x)(\ln 2 + 1) - (\sin x)(\ln 3 + 1)$$

Differentiate, remembering to apply chain rule, then solve for dy/dx.

$$\frac{1}{y} \cdot \frac{dy}{dx} = (-\sin x)(\ln 2 + 1) + (\cos x)(0) - \left[(\cos x)(\ln 3 + 1) + (\sin x)(0)\right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = -(\ln 2 + 1)\sin x - (\ln 3 + 1)\cos x$$

$$\frac{dy}{dx} = -y\left[(\ln 2 + 1)\sin x + (\ln 3 + 1)\cos x\right]$$

Substitute for y.

$$\frac{dy}{dx} = -\frac{(2e)^{\cos x}}{(3e)^{\sin x}}\left[(\ln 2 + 1)\sin x + (\ln 3 + 1)\cos x\right]$$

■ 5. Use logarithmic differentiation to find dy/dx.

$$y = e^x(2e)^{\sin x}(3e)^{\cos x}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln\left(e^x(2e)^{\sin x}(3e)^{\cos x}\right)$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln e^x + \ln(2e)^{\sin x} + \ln(3e)^{\cos x}$$

$$\ln y = x + \sin x \ln(2e) + \cos x \ln(3e)$$

$$\ln y = x + \sin x(\ln 2 + \ln e) + \cos x(\ln 3 + \ln e)$$

$$\ln y = x + \sin x(\ln 2 + 1) + \cos x(\ln 3 + 1)$$

Differentiate, remembering to apply chain rule, then solve for dy/dx.

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + (\ln 2 + 1)\cos x - (\ln 3 + 1)\sin x$$

$$\frac{dy}{dx} = y\left[1 + (\ln 2 + 1)\cos x - (\ln 3 + 1)\sin x\right]$$

Substitute for y.

$$\frac{dy}{dx} = e^x(2e)^{\sin x}(3e)^{\cos x}\left[1 + (\ln 2 + 1)\cos x - (\ln 3 + 1)\sin x\right]$$

TANGENT LINES

1. Find the equation of the tangent line to the graph of the equation at $(1/2,\pi)$.

$$f(x)=4\arctan 2x$$

Solution:

The derivative of $\arctan x$ is given by

$$\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$$

So the derivative is

$$f'(x)=\frac{4}{1+(2x)^2}\cdot 2$$

$$f'(x)=\frac{8}{1+4x^2}$$

Evaluating the derivative at $(1/2,\pi)$, we get

$$f'\left(\frac{1}{2}\right)=\frac{8}{1+4\left(\frac{1}{2}\right)^2}=\frac{8}{1+1}=\frac{8}{2}=4$$

Now we can find the equation of the tangent line by plugging the slope $f'(1/2) = 4$ and the point $(1/2,\pi)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = f\left(\frac{1}{2}\right) + 4\left(x - \frac{1}{2}\right)$$

$$y = \pi + 4\left(x - \frac{1}{2}\right)$$

$$y = \pi + 4x - 2$$

$$y = 4x + \pi - 2$$

■ 2. Find the equation of the tangent line to the graph of the equation at $(-1, -9)$.

$$g(x) = x^3 - 2x^2 + x - 5$$

Solution:

The derivative is

$$g'(x) = 3x^2 - 4x + 1$$

Evaluating the derivative at $(-1, -9)$, we get

$$g'(-1) = 3(-1)^2 - 4(-1) + 1$$

$$g'(-1) = 3(1) + 4(1) + 1$$

$$g'(-1) = 3 + 4 + 1$$

$$g'(-1) = 8$$

Now we can find the equation of the tangent line by plugging the slope $g'(-1) = 8$ and the point $(-1, -9)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = g(-1) + 8(x - (-1))$$

$$y = -9 + 8(x + 1)$$

$$y = -9 + 8x + 8$$

$$y = 8x - 1$$

■ 3. Find the equation of the tangent line to the graph of the equation at $(0, -4)$.

$$h(x) = -4e^{-x} + 3x$$

Solution:

The derivative is

$$h'(x) = -4(-1)e^{-x} + 3$$

$$h'(x) = 4e^{-x} + 3$$

Evaluating the derivative at $(0, -4)$, we get

$$h'(0) = 4e^{-0} + 3$$

$$h'(0) = 4(1) + 3$$

$$h'(0) = 7$$

Now we can find the equation of the tangent line by plugging the slope $h'(0) = 7$ and the point $(0, -4)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = h(0) + 7(x - 0)$$

$$y = -4 + 7(x - 0)$$

$$y = -4 + 7x$$

$$y = 7x - 4$$

■ 4. Find the equation of the tangent line to the graph of the equation at $(1,1)$.

$$f(x) = -6x^4 + 4x^3 - 3x^2 + 5x + 1$$

Solution:

The derivative is

$$f'(x) = -24x^3 + 12x^2 - 6x + 5$$

Evaluating the derivative at $(1,1)$, we get

$$f'(1) = -24(1)^3 + 12(1)^2 - 6(1) + 5$$

$$f'(1) = -24 + 12 - 6 + 5$$

$$f'(1) = -13$$

Now we can find the equation of the tangent line by plugging the slope $f'(1) = -13$ and the point $(1,1)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = f(1) - 13(x - 1)$$

$$y = 1 - 13x + 13$$

$$y = -13x + 14$$

VALUE THAT MAKES TWO TANGENT LINES PARALLEL

■ 1. Find the value of a such that the tangent lines to $f(x)=2x^3+2$ at $x=a$ and $x=a+1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x=a$ and $x=a+1$. We know that

$$f(a)=2a^3+2$$

$$f'(x)=6x^2$$

$$f'(a)=6a^2$$

So the equation of the tangent line at $x=a$ is

$$y=f(a)+f'(a)(x-a)$$

$$y=2a^3+2+(6a^2)(x-a)$$

Now we'll do the same thing at $x=a+1$. We know that

$$f(a+1)=2(a+1)^3+2$$

$$f(a+1)=2a^3+6a^2+6a+4$$

$$f'(x)=6x^2$$

$$f'(a+1) = 6a^2 + 12a + 6$$

So the equation of the tangent line at $x = a + 1$ is

$$y = f(a+1) + f'(a+1)(x - (a+1))$$

$$y = 2a^3 + 6a^2 + 6a + 4 + (6a^2 + 12a + 6)(x - (a+1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$6a^2 = 6a^2 + 12a + 6$$

$$0 = 12a + 6$$

$$-12a = 6$$

$$a = -\frac{1}{2}$$

The slope of $f(x)$ at $x = -1/2$ is $3/2$, and the slope of $f(x)$ at $x = 1/2$ is $3/2$.

■ 2. Find the value of a such that the tangent lines to $g(x) = x^3 + x^2 + 7$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$g(a) = a^3 + a^2 + 7$$

$$g'(x) = 3x^2 + 2x$$

$$g'(a) = 3a^2 + 2a$$

So the equation of the tangent line at $x = a$ is

$$y = g(a) + g'(a)(x - a)$$

$$y = a^3 + a^2 + 7 + (3a^2 + 2a)(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$g(a + 1) = (a + 1)^3 + (a + 1)^2 + 7$$

$$g(a + 1) = a^3 + a^2 + 2a^2 + 2a + a + 1 + a^2 + a + a + 1 + 7$$

$$g(a + 1) = a^3 + 4a^2 + 5a + 9$$

$$g'(a) = 3a^2 + 2a$$

$$g'(a + 1) = 3a^2 + 8a + 5$$

So the equation of the tangent line at $x = a + 1$ is

$$y = g(a + 1) + g'(a + 1)(x - (a + 1))$$

$$y = a^3 + 4a^2 + 4a + 10 + (3a^2 + 8a + 5)(x - (a + 1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$3a^2 + 2a = 3a^2 + 8a + 5$$

$$-6a = 5$$

$$a = -\frac{5}{6}$$

The slope of $g(x)$ at $x = -5/6$ is $5/12$, and the slope of $g(x)$ at $x = 1/6$ is $5/12$.

■ 3. Find the value of a such that the tangent lines to $h(x) = \tan^{-1} x$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$h(a) = \tan^{-1} a$$

$$h'(x) = \frac{1}{1 + x^2}$$

$$h'(a) = \frac{1}{1 + a^2}$$

So the equation of the tangent line at $x = a$ is

$$y = h(a) + h'(a)(x - a)$$

$$y = \tan^{-1} a + \frac{1}{1 + a^2}(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$h(a+1) = \tan^{-1}(a+1)$$

$$h'(a) = \frac{1}{1+a^2}$$

$$h'(a+1) = \frac{1}{1+(a+1)^2}$$

$$h'(a+1) = \frac{1}{a^2+2a+2}$$

So the equation of the tangent line at $x = a+1$ is

$$y = h(a+1) + h'(a+1)(x-(a+1))$$

$$y = \tan^{-1}(a+1) + \frac{1}{a^2+2a+2}(x-(a+1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$\frac{1}{1+a^2} = \frac{1}{a^2+2a+2}$$

$$1+a^2 = a^2+2a+2$$

$$-1 = 2a$$

$$a = -\frac{1}{2}$$

The slope of $h(x)$ at $x = -1/2$ is $4/5$, and the slope of $h(x)$ at $x = 1/2$ is $4/5$.

■ 4. Find the value of a such that the tangent lines to $f(x) = 4x^3 - 6x + 7$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$f(a) = 4a^3 - 6a + 7$$

$$f'(x) = 12x^2 - 6$$

$$f'(a) = 12a^2 - 6$$

So the equation of the tangent line at $x = a$ is

$$y = f(a) + f'(a)(x - a)$$

$$y = 4a^3 - 6a + 7 + (12a^2 - 6)(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$f(a + 1) = 4(a + 1)^3 - 6(a + 1) + 7$$

$$f(a + 1) = 4a^3 + 12a^2 + 6a + 5$$

$$f'(x) = 12x^2 - 6$$

$$f'(a + 1) = 12a^2 + 24a + 6$$

So the equation of the tangent line at $x = a + 1$ is

$$y = f(a+1) + f'(a+1)(x-(a+1))$$

$$y = 4a^3 + 12a^2 + 6a + 5 + (12a^2 + 24a + 6)(x-(a+1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$12a^2 - 6 = 12a^2 + 24a + 6$$

$$-12 = 24a$$

$$a = -\frac{1}{2}$$

The slope of $f(x)$ at $x = -1/2$ is -3, and the slope of $f(x)$ at $x = 1/2$ is -3.

■ 5. Find the value of a such that the tangent lines to $g(x) = (x-2)^3 + x^2 + 3$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$g(a) = (a-2)^3 + a^2 + 3$$

$$g'(x) = 3(x-2)^2 + 2x$$

$$g'(a) = 3(a-2)^2 + 2a$$

So the equation of the tangent line at $x = a$ is

$$y = g(a) + g'(a)(x - a)$$

$$y = (a - 2)^3 + a^2 + 3 + (3(a - 2)^2 + 2a)(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$g(a + 1) = (a + 1 - 2)^3 + (a + 1)^2 + 3$$

$$g(a + 1) = (a - 1)^3 + (a + 1)^2 + 3$$

$$g'(a) = 3(a - 2)^2 + 2a$$

$$g'(a + 1) = 3(a + 1 - 2)^2 + 2(a + 1)$$

$$g'(a + 1) = 3(a - 1)^2 + 2(a + 1)$$

So the equation of the tangent line at $x = a + 1$ is

$$y = g(a + 1) + g'(a + 1)(x - (a + 1))$$

$$y = (a - 1)^3 + (a + 1)^2 + 3 + (3(a - 1)^2 + 2(a + 1))(x - (a + 1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$3(a - 2)^2 + 2a = 3(a - 1)^2 + 2(a + 1)$$

$$3a^2 - 10a + 12 = 3a^2 - 4a + 5$$

$$7 = 6a$$

$$a = \frac{7}{6}$$

The slope of $g(x)$ at $x = 7/6$ is $53/12$, and the slope of $g(x)$ at $x = 13/6$ is $53/12$.

■ 6. Find the approximate value of a, rounded to the nearest hundredth, such that the tangent lines to $h(x) = e^x - 3x^2$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We know that

$$h(a) = e^a - 3a^2$$

$$h'(x) = e^x - 6x$$

$$h'(a) = e^a - 6a$$

So the equation of the tangent line at $x = a$ is

$$y = h(a) + h'(a)(x - a)$$

$$y = e^a - 3a^2 + (e^a - 6a)(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$h(a + 1) = e^{a+1} - 3(a + 1)^2$$

$$h'(a) = e^a - 6a$$

$$h'(a + 1) = e^{a+1} - 6(a + 1)$$

So the equation of the tangent line at $x = a + 1$ is

$$y = h(a + 1) + h'(a + 1)(x - (a + 1))$$

$$y = e^{a+1} - 3(a + 1)^2 + e^{a+1} - 6(a + 1)(x - (a + 1))$$

For the two tangent lines to be parallel, set the slopes from the tangent lines equal to each other and solve for a.

$$e^a - 6a = e^{a+1} - 6(a + 1)$$

$$e^a - 6a = e^{a+1} - 6a - 6$$

$$e^a = e^{a+1} - 6$$

$$e^{a+1} - e^a - 6 = 0$$

$$e^a e^1 + (-e^a) - 6 = 0$$

$$(-e^a)e^1 - (-e^a) + 6 = 0$$

Substitute $x = - e^a$, then solve for x.

$$xe - x + 6 = 0$$

$$x(e - 1) + 6 = 0$$

$$x(e - 1) = - 6$$

$$x = -\frac{6}{e - 1}$$

Back-substitute.

$$-e^a = -\frac{6}{e-1}$$

$$e^a = \frac{6}{e-1}$$

Take the natural log of both sides.

$$\ln(e^a) = \ln\left(\frac{6}{e-1}\right)$$

$$a = \ln\left(\frac{6}{e-1}\right)$$

$$a \approx 1.25$$

The slope of $h(x)$ at $x = 1.25$ is -4, and the slope of $h(x)$ at $x = 2.25$ is -4.

VALUES THAT MAKE THE FUNCTION DIFFERENTIABLE

■ 1. What value of a and b will make the function differentiable?

$$f(x) = \begin{cases} x^2 & x \le 3 \\ ax - b & x > 3 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $f(x)$ continuous at $x = 3$,

$$x^2 = ax - b$$

$$3^2 = a(3) - b$$

$$9 = 3a - b$$

$$b = 3a - 9$$

If $f(x)$ is differentiable, then the derivatives of $f(x)$ at $x = 3$ must be equal to each other. So $2x = a$, and when $x = 3$, and $a = 6$. Therefore, $a = 6$ and

$$b = 3(6) - 9$$

$$b = 9$$

■ 2. What value of a and b will make the function differentiable?

$$g(x) = \begin{cases} ax + b & x \le -1 \\ bx^2 - 1 & x > -1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $g(x)$ continuous at $x = -1$,

$$ax + b = bx^2 - 1$$

$$a(-1) + b = b(-1)^2 - 1$$

$$-a + b = b - 1$$

$$a = 1$$

If $g(x)$ is differentiable, then the derivatives of $g(x)$ at $x = -1$ must be equal to each other. So $a = 2bx$ when $x = -1$, and $a = 1$.

$$1 = 2b(-1)$$

$$b = -\frac{1}{2}$$

Therefore, $a = 1$ and $b = -1/2$.

■ 3. What value of a and b will make the function differentiable?

$$h(x) = \begin{cases} ax^3 & x \le 2 \\ x^2 - b & x > 2 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $h(x)$ continuous at $x = 2$,

$$ax^3 = x^2 - b$$

$$a(2)^3 = (2)^2 - b$$

$$8a = 4 - b$$

$$8a - 4 = -b$$

$$b = 4 - 8a$$

If $h(x)$ is differentiable, then the derivatives of $h(x)$ at $x = 2$ must be equal to each other. So $3ax^2 = 2x$ when $x = 2$, and

$$3a(2)^2 = 2(2)$$

$$12a = 4$$

$$a = \frac{1}{3}$$

To get b, we'll plug in $a = 1/3$.

$$b = 4 - 8\left(\frac{1}{3}\right) = \frac{4}{3}$$

Therefore, $a = 1/3$ and $b = 4/3$.

■ 4. What value of a and b will make the function differentiable?

$$f(x) = \begin{cases} 3 - x & x \le 1 \\ ax^2 - bx & x > 1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $f(x)$ continuous at $x = 1$,

$$3 - x = ax^2 - bx$$

$$3 - (1) = a(1)^2 - b(1)$$

$$2 = a - b$$

$$a = 2 + b$$

If $f(x)$ is differentiable, then the derivatives of $f(x)$ at $x = 1$ must be equal to each other. So

$$-1 = 2ax - b$$

$$-1 = 2a(1) - b$$

$$-1 = 2a - b$$

$$-1 - 2a = -b$$

$$b = 2a + 1$$

Now, since $a = 2 + b$ and $b = 2a + 1$,

$$a = 2 + 2a + 1$$

$$-a = 3$$

$$a = -3$$

Then

$$b = 2a + 1$$

$$b = 2(-3) + 1$$

$$b = -5$$

The answer is $a = -3$ and $b = -5$.

■ 5. What value of a and b will make the function differentiable?

$$g(x) = \begin{cases} x^3 & x \leq 1 \\ a(x-2)^2 - b & x > 1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $g(x)$ continuous at $x = 1$,

$$x^3 = a(x-2)^2 - b$$

$$(1)^3 = a(1-2)^2 - b$$

$$1 = a - b$$

$$-b = 1 - a$$

$$b = a - 1$$

If $g(x)$ is differentiable, then the derivatives of $g(x)$ at $x = 1$ must be equal to each other. So

$$3x^2 = 2a(x-2)$$

$$3(1) = 2a(1-2)$$

$$3 = 2a(-1)$$

$$a = -\frac{3}{2}$$

So $a = -3/2$ and $b = -(3/2) - 1 = -(5/2)$.

■ 6. What value of a and b will make the function differentiable?

$$h(x) = \begin{cases} ax^2 + b & x \le 3 \\ bx + 4 & x > 3 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $h(x)$ continuous at $x = 3$,

$$ax^2 + b = bx + 4$$

$$a(3)^2 + b = b(3) + 4$$

$$9a + b = 3b + 4$$

$$9a = 2b + 4$$

$$a = \frac{2b + 4}{9}$$

If $h(x)$ is differentiable, then the derivatives of $h(x)$ at $x = 3$ must be equal to each other. So

$$2ax = b$$

$$2a(3) = b$$

$$b = 6a$$

Plugging $b = 6a$ into the equation for a gives

$$a = \frac{2(6a) + 4}{9}$$

$$9a = 12a + 4$$

$$-3a = 4$$

$$a = -\frac{4}{3}$$

Then, $b = 6(-4/3) = -8$. The answer is $a = -4/3$ and $b = -8$.

NORMAL LINES

■ 1. Find the equation of the normal line to the graph of $f(x) = 5x^4 + 3e^x$ at $(0,3)$.

Solution:

Begin by finding the tangent line at $(0,3)$, starting with taking the derivative. Then evaluate the derivative at $(0,3)$.

$$f'(x) = 20x^3 + 3e^x$$

$$f'(0) = 20(0)^3 + 3e^0$$

$$f'(0) = 0 + 3(1)$$

$$f'(0) = 3$$

With $f'(0) = 3$ and $(a, f(a)) = (0,3)$, the tangent line is

$$y = f(a) + f'(a)(x - a)$$

$$y = 3(x - 0) + 3$$

$$y = 3x + 3$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/3$, so the equation of the normal line is

$$y = 3 - \frac{1}{3}(x - 0)$$

$$y = -\frac{1}{3}x + 3$$

■ 2. Find the equation of the normal line to the graph of $g(x) = \ln e^{4x} + 2x^3$ at $(2,24)$.

Solution:

Begin by finding the tangent line at $(2,24)$, starting with taking the derivative. Then evaluate the derivative at $(2,24)$.

$$g'(x) = 4 + 6x^2$$

$$g'(2) = 4 + 6(2)^2$$

$$g'(2) = 4 + 24$$

$$g'(2) = 28$$

With $g'(2) = 28$ and $(a, g(a)) = (2,24)$, the tangent line is

$$y = g(a) + g'(a)(x - a)$$

$$y = 24 + 28(x - 2)$$

$$y = 28x - 32$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/28$, so the equation of the normal line is

$$y = 24 - \frac{1}{28}(x - 2)$$

$$y = -\frac{1}{28}x + \frac{337}{14}$$

■ 3. Find the equation of the normal line to the graph of $h(x) = 5\cos x + 5\sin x$ at $(\pi/2,5)$.

Solution:

Begin by finding the tangent line at $(\pi/2,5)$, starting with taking the derivative. Then evaluate the derivative at $(\pi/2,5)$.

$$h'(x) = -5\sin x + 5\cos x$$

$$h'\left(\frac{\pi}{2}\right) = -5\sin\left(\frac{\pi}{2}\right) + 5\cos\left(\frac{\pi}{2}\right)$$

$$h'\left(\frac{\pi}{2}\right) = -5(1) + 5(0)$$

$$h'\left(\frac{\pi}{2}\right) = -5$$

With $h'(\pi/2) = -5$ and $(a, h(a)) = (\pi/2,5)$, the tangent line is

$$y = h(a) + h'(a)(x - a)$$

$$y = 5 - 5\left(x - \frac{\pi}{2}\right)$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $1/5$, so the equation of the normal line is

$$y = 5 + \frac{1}{5}\left(x - \frac{\pi}{2}\right)$$

■ 4. Find the equation of the normal line to the graph of $f(x) = 7x^3 + 2x^2 - 5x + 9$ at $(2,63)$.

Solution:

Begin by finding the tangent line at $(2,63)$, starting with taking the derivative. Then evaluate the derivative at $(2,63)$.

$$f'(x) = 21x^2 + 4x - 5$$

$$f'(2) = 21(2)^2 + 4(2) - 5$$

$$f'(2) = 84 + 8 - 5$$

$$f'(2) = 87$$

With $f'(2) = 87$ and $(a, f(a)) = (2,63)$, the tangent line is

$$y = f(a) + f'(a)(x - a)$$

$$y = 63 + 87(x - 2)$$

$$y = 87x - 111$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $1/87$, so the equation of the normal line is

$$y = 63 - \frac{1}{87}(x - 2)$$

$$y = -\frac{1}{87}x + \frac{5{,}483}{87}$$

■ 5. Find the equation of the normal line to the graph of $g(x) = 5\sqrt{x^2 - 14x + 49}$ at $(2,25)$.

Solution:

Begin by finding the tangent line at $(2,25)$, starting with taking the derivative. Then evaluate the derivative at $(2,25)$.

$$g'(x) = \frac{5}{2\sqrt{x^2 - 14x + 49}} \cdot (2x - 14)$$

$$g'(x) = \frac{5x - 35}{\sqrt{x^2 - 14x + 49}}$$

$$g'(x) = \frac{5(x-7)}{|x-7|}$$

$$g'(2) = \frac{5(2-7)}{|2-7|}$$

$$g'(2) = \frac{-25}{5}$$

$$g'(2) = -5$$

With $g'(2) = -5$ and $(a, g(a)) = (2,25)$, the tangent line is

$$y = g(a) + g'(a)(x-a)$$

$$y = 25 - 5(x-2)$$

$$y = -5x + 35$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $1/5$, so the equation of the normal line is

$$y = 25 + \frac{1}{5}(x-2)$$

$$y = \frac{1}{5}x + \frac{123}{5}$$

AVERAGE RATE OF CHANGE

■ 1. Find the average rate of change of the function over the interval $[4,9]$.

$$f(x)=\frac{5\sqrt{x}-2}{3}$$

Solution:

In this question, $f(9)$ and $f(4)$ are

$$f(9)=\frac{5\sqrt{9}-2}{3}=\frac{5(3)-2}{3}=\frac{13}{3}$$

$$f(4)=\frac{5\sqrt{4}-2}{3}=\frac{5(2)-2}{3}=\frac{8}{3}$$

Therefore, average rate of change on $[a,b]=[4,9]$ is given by

$$\frac{f(b)-f(a)}{b-a}$$

$$\frac{\frac{13}{3}-\frac{8}{3}}{9-4}=\frac{\frac{5}{3}}{\frac{5}{1}}=\frac{5}{3}\cdot\frac{1}{5}=\frac{1}{3}$$

■ 2. Find the average rate of change of the function over the interval $[16,25]$.

$$g(x) = \frac{2x - 8}{\sqrt{x} - 2}$$

Solution:

In this question, $g(25)$ and $g(16)$ are

$$g(25) = \frac{2(25) - 8}{\sqrt{25} - 2} = \frac{42}{3} = 14$$

$$g(16) = \frac{2(16) - 8}{\sqrt{16} - 2} = \frac{24}{2} = 12$$

Therefore, average rate of change on $[a, b] = [16,25]$ is given by

$$\frac{g(b) - g(a)}{b - a}$$

$$\frac{14 - 12}{25 - 16} = \frac{2}{9}$$

■ 3. Find the average rate of change of the function over the interval $[0,4]$.

$$h(x) = \frac{x^3 - 8}{x^2 - 4x - 5}$$

Solution:

In this question, $g(4)$ and $g(0)$ are

$$h(4) = \frac{4^3 - 8}{4^2 - 4(4) - 5} = \frac{64 - 8}{16 - 16 - 5} = \frac{56}{-5} = -\frac{56}{5}$$

$$h(0) = \frac{0^3 - 8}{0^2 - 4(0) - 5} = \frac{0 - 8}{0 - 0 - 5} = \frac{-8}{-5} = \frac{8}{5}$$

Therefore, average rate of change on $[a, b] = [0,4]$ is given by

$$\frac{h(b) - h(a)}{b - a}$$

$$\frac{\frac{-56}{5} - \frac{8}{5}}{4 - 0} = \frac{-\frac{64}{5}}{\frac{4}{1}} = -\frac{64}{5} \cdot \frac{1}{4} = -\frac{16}{5}$$

IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find dy/dx at $(3,4)$ for the equation.

$$4x^3 - 3xy^2 + y^3 = 28$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$12x^2 - 3y^2 - 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(3y^2 - 6xy)\frac{dy}{dx} = 3y^2 - 12x^2$$

$$\frac{dy}{dx} = \frac{3y^2 - 12x^2}{3y^2 - 6xy}$$

Evaluate dy/dx at $(3,4)$.

$$\frac{dy}{dx}(3,4) = \frac{3(4)^2 - 12(3)^2}{3(4)^2 - 6(3)(4)} = \frac{48 - 108}{48 - 72} = \frac{5}{2}$$

■ 2. Use implicit differentiation to find dy/dx for the equation.

$$5x^3 + xy^2 = 4x^3y^3$$

Solution:

Rearrange the function. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$5x^3 + xy^2 = 4x^3y^3$$

$$xy^2 - 4x^3y^3 = -5x^3$$

Use implicit differentiation to take the derivative of both sides.

$$y^2 + 2xy\frac{dy}{dx} - 12x^2y^3 - 12x^3y^2\frac{dy}{dx} = -15x^2$$

$$2xy\frac{dy}{dx} - 12x^3y^2\frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$(2xy - 12x^3y^2)\frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$\frac{dy}{dx} = \frac{12x^2y^3 - 15x^2 - y^2}{2xy - 12x^3y^2}$$

■ 3. Use implicit differentiation to find dy/dx for the equation.

$$3x^2 = (3xy - 1)^2$$

Solution:

Rearrange the function. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$3x^2 = (3xy - 1)^2$$

$$3x^2 = 9x^2y^2 - 6xy + 1$$

Use implicit differentiation to take the derivative of both sides.

$$6x = 18xy^2 + 18x^2y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = 18x^2y\frac{dy}{dx} - 6x\frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = (18x^2y - 6x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x - 18xy^2 + 6y}{18x^2y - 6x}$$

$$\frac{dy}{dx} = \frac{x - 3xy^2 + y}{3x^2y - x}$$

EQUATION OF THE TANGENT LINE WITH IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find the equation of the tangent line to $5y^2 = 2x^3 - 5y + 6$ at $(3,3)$.

Solution:

Rearrange the function.

$$5y^2 = 2x^3 - 5y + 6$$

$$5y^2 + 5y = 2x^3 + 6$$

Use implicit differentiation to take the derivative of both sides.

$$10y\frac{dy}{dx} + 5\frac{dy}{dx} = 6x^2$$

$$(10y + 5)\frac{dy}{dx} = 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2}{10y + 5}$$

Evaluate dy/dx at $(3,3)$.

$$\frac{dy}{dx}(3,3) = \frac{6(3)^2}{10(3) + 5} = \frac{54}{35}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{54}{35}(x - 3)$$

$$y = \frac{54}{35}(x - 3) + 3$$

■ 2. Use implicit differentiation to find the equation of the tangent line to $5x^3 = -3xy + 4$ at $(2, -6)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$15x^2 = -3y - 3x\frac{dy}{dx}$$

$$15x^2 + 3y = -3x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{15x^2 + 3y}{-3x}$$

$$\frac{dy}{dx} = -\frac{5x^2 + y}{x}$$

Evaluate dy/dx at $(2, -6)$.

$$\frac{dy}{dx}(2, -6) = -\frac{5(2)^2 + (-6)}{2} = -\frac{20 - 6}{2} = -7$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 6 = -7(x - 2)$$

$$y = -7(x - 2) - 6$$

$$y = -7x + 8$$

■ 3. Use implicit differentiation to find the equation of the tangent line to $4y^2 + 8 = 3x^2$ at $(6, -5)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$8y\frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{8y} = \frac{3x}{4y}$$

Evaluate dy/dx at $(6, -5)$.

$$\frac{dy}{dx}(6, -5) = \frac{3(6)}{4(-5)} = -\frac{18}{20} = -\frac{9}{10}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 5 = -\frac{9}{10}(x - 6)$$

$$y = -\frac{9}{10}(x - 6) - 5$$

$$y = -\frac{9}{10}x + \frac{54}{10} - 5$$

$$y = -\frac{9}{10}x + \frac{27}{5} - \frac{25}{5}$$

$$y = -\frac{9}{10}x + \frac{2}{5}$$

SECOND DERIVATIVES WITH IMPLICIT DIFFERENTIATION

1. Use implicit differentiation to find d^2y/dx^2.

$$2x^3 = 2y^2 + 4$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$6x^2 = 4y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x^2}{4y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(4y)(12x) - (6x^2)\left(4 \cdot \frac{dy}{dx}\right)}{(4y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{48xy - 24x^2\frac{dy}{dx}}{16y^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = \frac{48xy - 24x^2\left(\frac{6x^2}{4y}\right)}{16y^2}$$

$$\frac{d^2y}{dx^2} = \frac{48xy - \frac{36x^4}{y}}{16y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

Multiply through the numerator and denominator by y to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = \frac{12xy^2 - 9x^4}{4y^3}$$

■ 2. Use implicit differentiation to find d^2y/dx^2.

$$4x^2 = 2y^3 + 4y - 2$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$8x = 6y^2\frac{dy}{dx} + 4\frac{dy}{dx}$$

$$8x = (6y^2 + 4)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{8x}{6y^2 + 4}$$

$$\frac{dy}{dx} = \frac{4x}{3y^2 + 2}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(4)(3y^2 + 2) - (4x)\left(6y \cdot \frac{dy}{dx}\right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy\frac{dy}{dx}}{(3y^2 + 2)^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy\left(\frac{4x}{3y^2 + 2}\right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - \frac{96x^2y}{3y^2 + 2}}{(3y^2 + 2)^2}$$

Multiply through the numerator and denominator by $3y^2 + 2$ to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = \frac{12y^2(3y^2 + 2) + 8(3y^2 + 2) - 96x^2y}{(3y^2 + 2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{(12y^2 + 8)(3y^2 + 2) - 96x^2y}{(3y^2 + 2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{4(3y^2 + 2)(3y^2 + 2) - 96x^2y}{(3y^2 + 2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{4(3y^2 + 2)^2 - 96x^2y}{(3y^2 + 2)^3}$$

■ 3. Use implicit differentiation to find d^2y/dx^2 at $(0,3)$.

$$3x^2 + 3y^2 = 27$$

Solution:

Rewrite the equation.

$$3x^2 + 3y^2 = 27$$

$$x^2 + y^2 = 9$$

Use implicit differentiation to take the derivative of both sides.

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = -\frac{(1)(y) - (x)(1)\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y + \frac{x^2}{y}}{y^2}$$

Multiply through the numerator and denominator by y to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = -\frac{y^2 + x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-x^2 - y^2}{y^3}$$

Evaluate the second derivative at $(0,3)$.

$$\frac{d^2y}{dx^2}(0,3) = \frac{-0^2 - 3^2}{3^3} = \frac{-9}{27} = -\frac{1}{3}$$

CRITICAL POINTS AND THE FIRST DERIVATIVE TEST

■ 1. Identify the critical point(s) of the function on the interval $[-3,2]$.

$$f(x) = x^{\frac{2}{3}}(x+2)$$

Solution:

Find $f'(x)$ and the x-values inside the given interval for which $f'(x) = 0$ or is undefined.

Rewrite the function.

$$f(x) = x^{\frac{2}{3}}(x+2)$$

$$f(x) = x^{\frac{5}{3}} + 2x^{\frac{2}{3}}$$

Find the derivative.

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} + 2 \cdot \frac{2}{3}x^{-\frac{1}{3}}$$

$$f'(x) = \frac{5}{3}\sqrt[3]{x^2} + \frac{4}{3\sqrt[3]{x}}$$

When $x = 0$, the denominator of the second fraction will be 0, which will make the derivative undefined. The derivative will also be equal to 0:

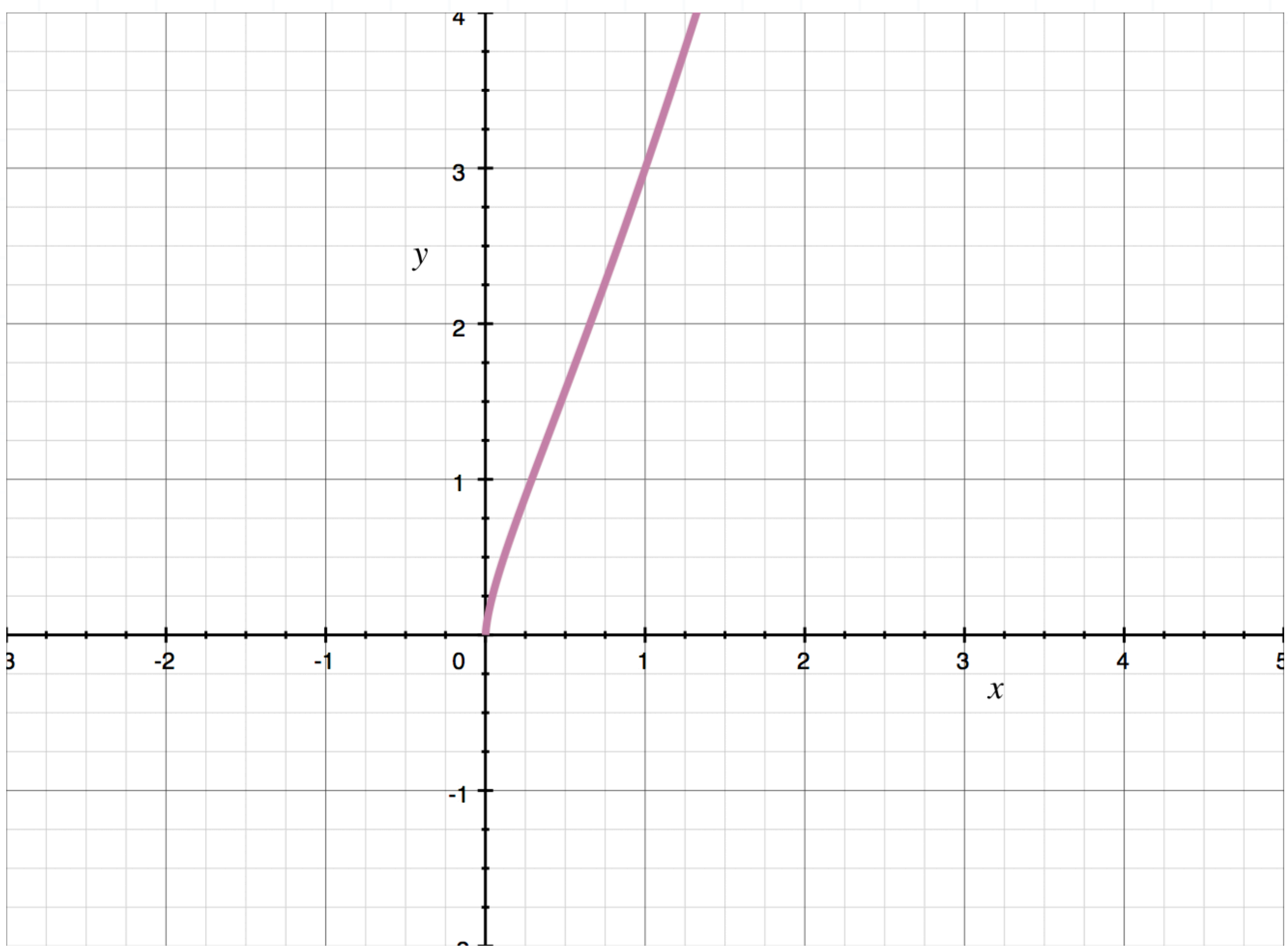
$$\frac{5}{3}\sqrt[3]{x^2} + \frac{4}{3\sqrt[3]{x}} = 0$$

$$\frac{5}{3}\sqrt[3]{x^2} = -\frac{4}{3\sqrt[3]{x}}$$

$$5x = -4$$

$$x = -\frac{4}{5}$$

The critical numbers are therefore $x = -4/5, 0$. The graph has a horizontal tangent at $x = -4/5$ and a cusp at $x = 0$.



■ 2. Identify the critical point(s) of the function on the interval $[-2,2]$.

$$g(x) = x\sqrt{4 - x^2}$$

Solution:

Find $g'(x)$ and the x-values inside the given interval for which $g'(x)=0$ or is undefined.

Find the derivative.

$$g'(x)=(1)\sqrt{4-x^2}+(x)\left(\frac{1}{2}\right)(4-x^2)^{-\frac{1}{2}}(-2x)$$

$$g'(x)=\sqrt{4-x^2}-x^2(4-x^2)^{-\frac{1}{2}}$$

$$g'(x)=\sqrt{4-x^2}-\frac{x^2}{\sqrt{4-x^2}}$$

When $x=\pm 2$, the denominator of the second fraction will be 0, which will make the derivative undefined. The derivative will also be equal to 0:

$$\sqrt{4-x^2}-\frac{x^2}{\sqrt{4-x^2}}=0$$

$$\sqrt{4-x^2}=\frac{x^2}{\sqrt{4-x^2}}$$

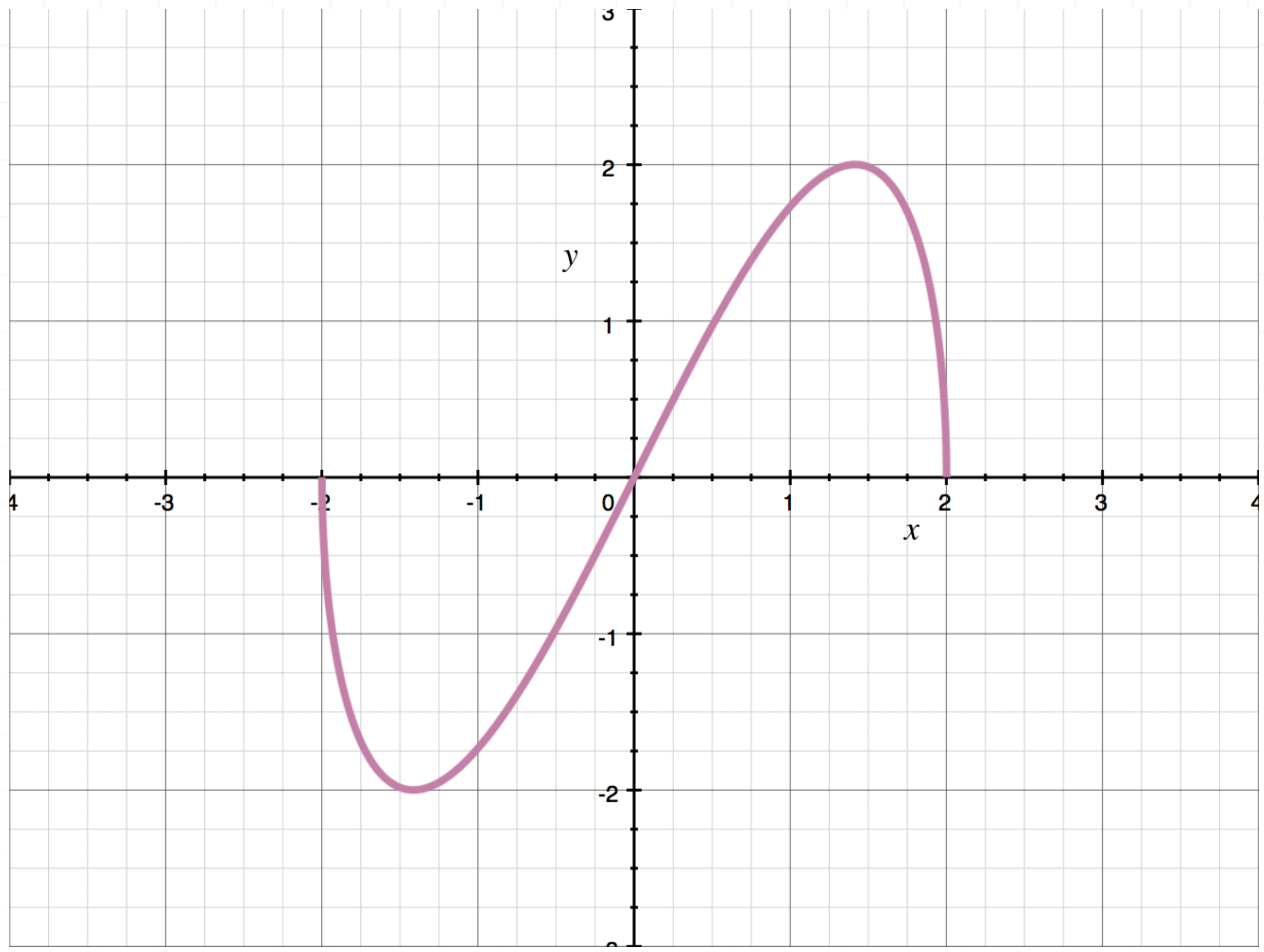
$$4-x^2=x^2$$

$$4=2x^2$$

$$2=x^2$$

$$x=\pm\sqrt{2}$$

The critical numbers are therefore $x = -\pm\sqrt{2},\ \pm 2$. The graph has horizontal tangents at $x = -\pm\sqrt{2}$ and endpoint discontinuities at $x = \pm 2$.



■ 3. Determine the intervals where the function is increasing and decreasing.

$$f(x) = \frac{5}{4}x^4 - 10x^2$$

Solution:

Find the derivative $f'(x) = 5x^3 - 20x$, then identify the critical points where $f'(x) = 0$.

$$5x^3 - 20x = 0$$

$$5x(x^2 - 4) = 0$$

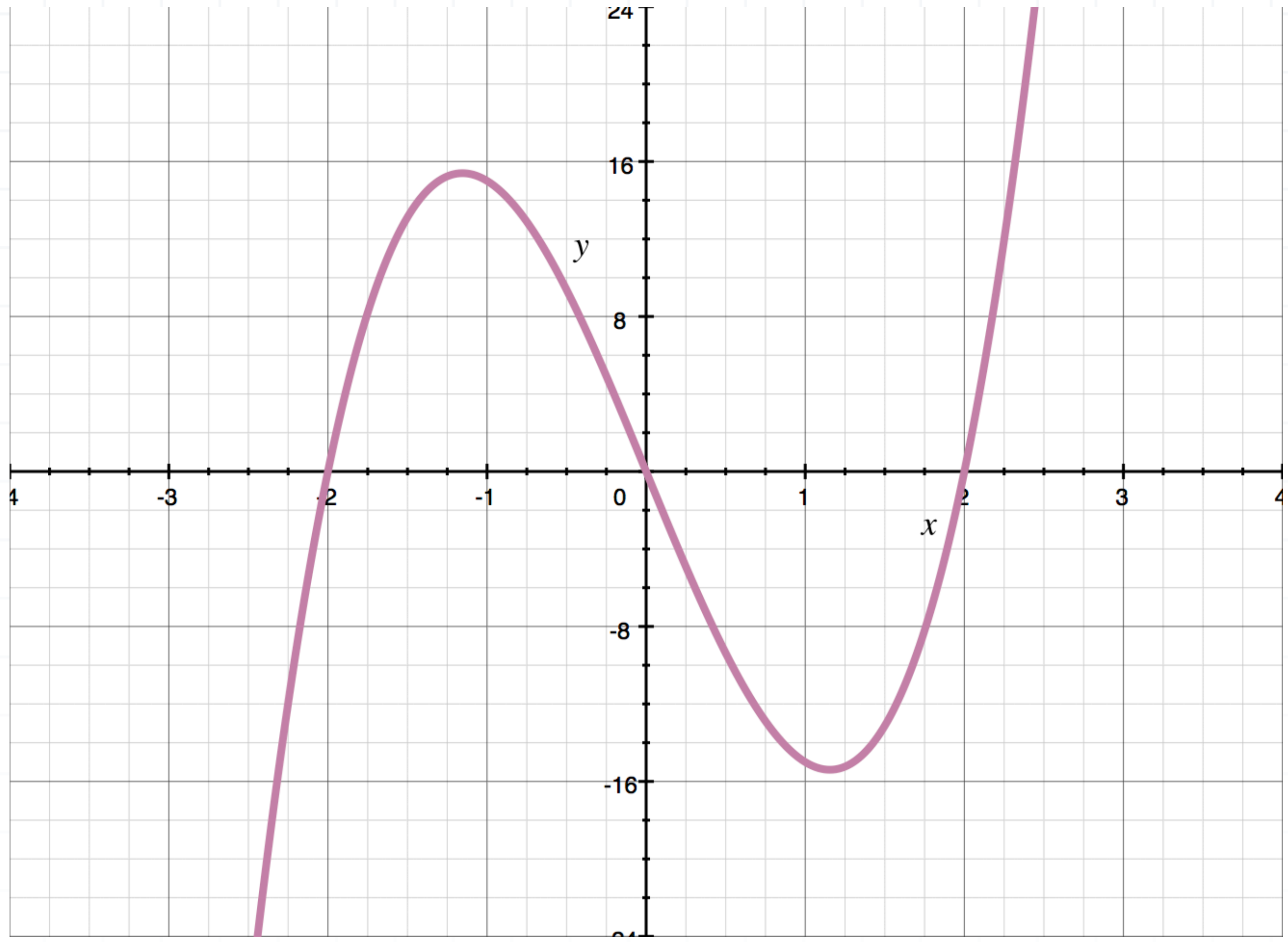
$$5x(x + 2)(x - 2) = 0$$

$$x = -2,\ 0,\ 2$$

Determine where $f'(x) > 0$ or $f'(x) < 0$ by selecting a value between each critical number.

Interval	x<-2	-2<x<0	0<x<2	x>2
x	-3	-1	1	3
f'(x)	<0	>0	<0	>0
f(x)	Decreasing	Increasing	Decreasing	Increasing

The graph of $f'(x)$ shows that $f'(x) < 0$ on $(-\infty, -2) \cup (0,2)$ and $f'(x) > 0$ on $(-2,0) \cup (2,\infty)$.



4. Determine the intervals where the function is increasing and decreasing.

$$g(x) = -x^3 + 2x^2 + 2$$

Solution:

Find the derivative $g'(x) = -3x^2 + 4x$, then identify the critical points where $g'(x) = 0$.

$$-3x^2 + 4x = 0$$

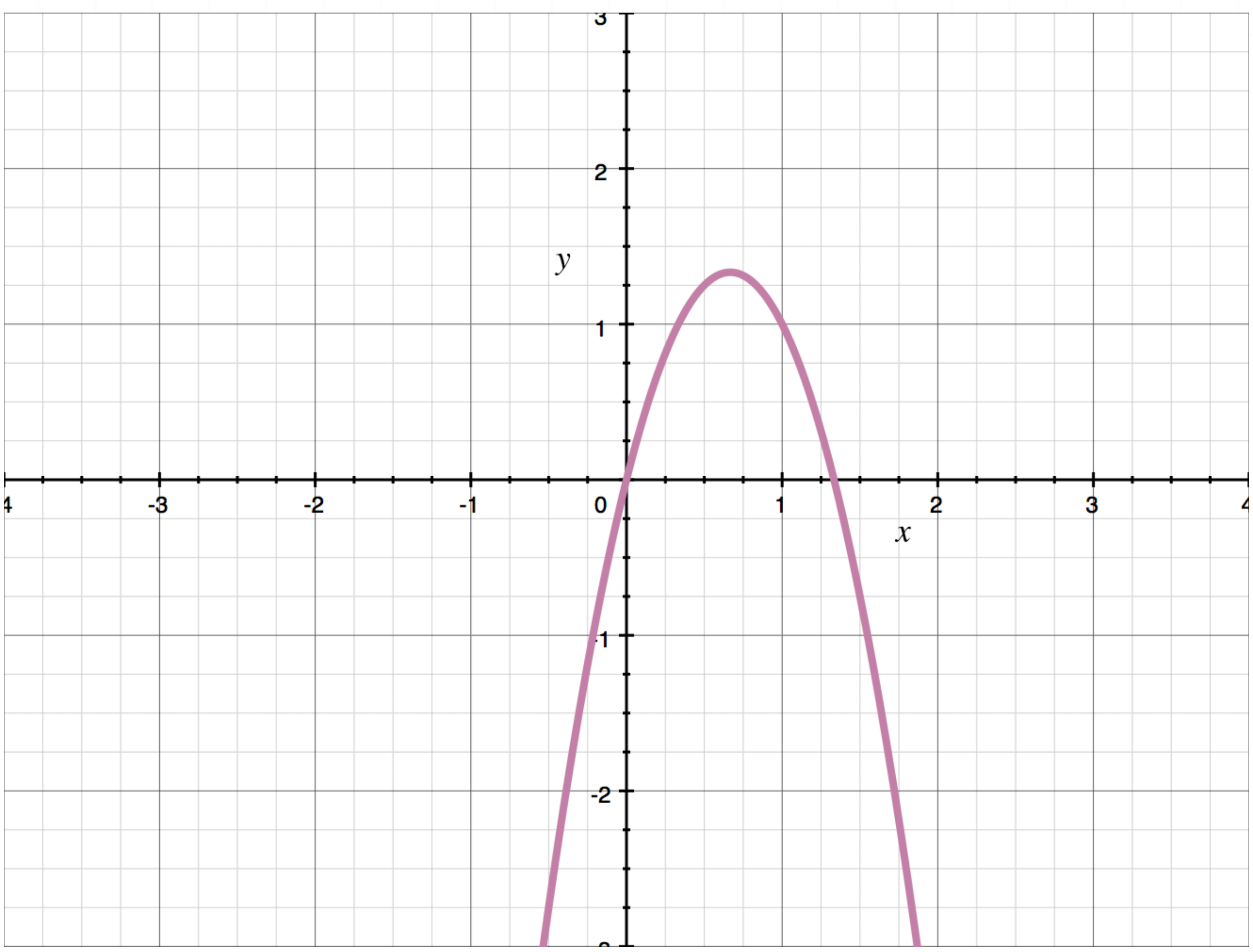
$$x(-3x + 4) = 0$$

$$x = 0, \frac{4}{3}$$

Determine where $g'(x) > 0$ or $g'(x) < 0$ by selecting a value between each critical number.

Interval	x<0	0<x<4/3	x>4/3
x	-1	1	2
g'(x)	<0	>0	<0
g(x)	Decreasing	Increasing	Decreasing

The graph of $g'(x)$ shows that $g'(x) < 0$ on $(-\infty,0) \cup (4/3,\infty)$ and $g'(x) > 0$ on $(0,4/3)$.



■ 5. Use the first derivative test to find the extrema of $f(x) = 4x^3 + 21x^2 + 36x - 5$.

Solution:

Find the derivative $f'(x) = 12x^2 + 42x + 36$. Set the derivative equal to 0 and solve for x.

$$12x^2 + 42x + 36 = 0$$

$$6(2x^2 + 7x + 6) = 0$$

$$6(x + 2)(2x + 3) = 0$$

$$x = -2, \ -\frac{3}{2}$$

Determine the sign of $f'(x)$ on each side of these critical numbers by selecting a value between each number.

Interval	x<-2	x=-2	-2<x<-3/2	x=-3/2	x>-3/2
x	-4	-2	-1.8	-3/2	0
f'(x)	+	0	-	0	+
f(x)	Increasing	Maximum	Decreasing	Minimum	Increasing

The table shows that $f(x)$ has a maximum at $x = -2$, so $f(-2) = -25$ and $(-2, -25)$ is a maximum point. The table also shows that $f(x)$ has a minimum at $x = -3/2$, so $f(-3/2) = -101/4$ and $(-3/2, -101/4)$ is a minimum point. The graph shows the extrema:

■ 6. Use the first derivative test to find the extrema of $g(x) = 2x^3 - 14x^2 + 22x + 3$.

Solution:

Find the derivative $g'(x) = 6x^2 - 28x + 22$. Set the derivative equal to 0 and solve for x.

$$6x^2 - 28x + 22 = 0$$

$$2(3x^2 - 14x + 11) = 0$$

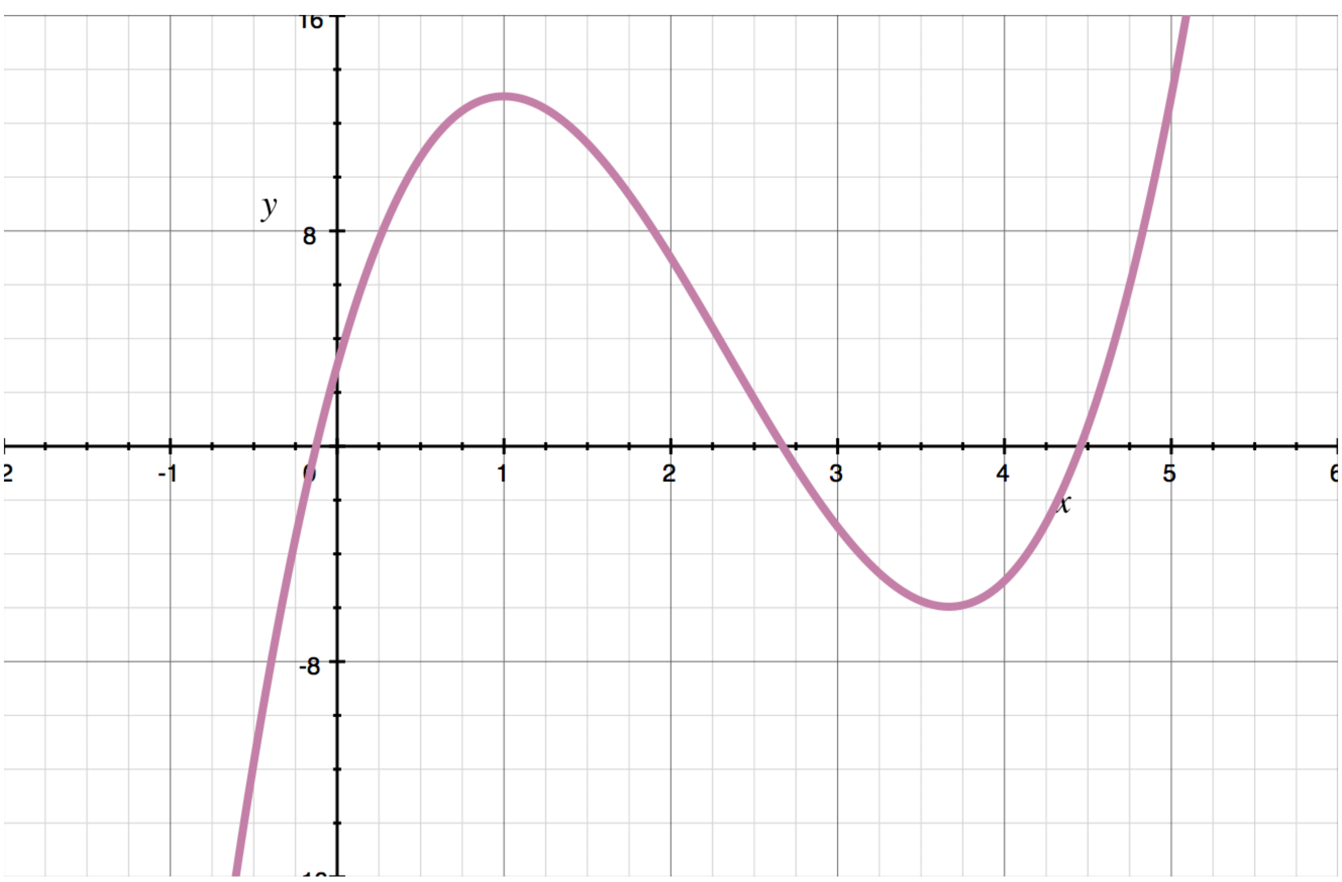
$$2(x-1)(3x-11)=0$$

$$x=1,\ \frac{11}{3}$$

Determine the sign of $g'(x)$ on each side of these critical numbers by selecting a value between each number.

Interval	x<1	x=1	1<x<11/3	x=11/3	x>11/3
x	-4	1	2	11/3	5
g'(x)	+	0	-	0	+
g(x)	Increasing	Maximum	Decreasing	Minimum	Increasing

The table shows that $g(x)$ has a maximum at $x=1$, so $g(1)=13$ and $(1,13)$ is a maximum point. The table also shows that $g(x)$ has a minimum at $x=11/3$, so $g(11/3)=-161/27$ and $(11/3,-161/27)$ is a minimum point. The graph shows the extrema:



INFLECTION POINTS AND THE SECOND DERIVATIVE TEST

1. For $f(x) = x^3 - 3x^2 + 5$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$f'(x) = 3x^2 - 6x$$

$$f''(x) = 6x - 6$$

The function has an inflection point when $f''(x) = 0$.

$$6x - 6 = 0$$

$$x - 1 = 0$$

$$x = 1$$

Check values around $x = 1$.

Interval	x<1	x=1	x>1
x	-1	1	2
f''(x)	-	0	+
Concavity	Down	Inflection	Up

The inflection point is at $x = 1$ and $f(1) = 3$, so the inflection point is $(1,3)$. The function is concave down on $(-\infty,1)$ and concave up on $(1,\infty)$.

■ 2. For $g(x) = -x^3 + 2x^2 + 3$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$g'(x) = -3x^2 + 4x$$

$$g''(x) = -6x + 4$$

The function has an inflection point when $g''(x) = 0$.

$$-6x + 4 = 0$$

$$-6x = -4$$

$$x = \frac{2}{3}$$

Check values around $x = 2/3$.

Interval	x<2/3	x=2/3	x>2/3
x	-1	2/3	1
g''(x)	+	0	-
Concavity	Up	Inflection	Down

The inflection point is at $x = 2/3$ and $g(2/3) = 97/27$, so the inflection point is $(2/3,97/27)$. The function is concave up on $(-\infty,2/3)$ and concave down on $(2/3,\infty)$.

■ 3. For $h(x) = x^4 + x^3 - 3x^2 + 2$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$h'(x) = 4x^3 + 3x^2 - 6x$$

$$h''(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x - 1)(x + 1)$$

The function has an inflection point when $h''(x) = 0$.

$$6(2x - 1)(x + 1) = 0$$

$$x = -1, \frac{1}{2}$$

Check values around these inflection points.

Interval	x<-1	x=-1	-1<x<1/2	x=1/2	x>1/2
x	-2	-1	0	1/2	1
h''(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

An inflection point is at $x = -1$ and $h(-1) = -1$, so an inflection point is $(-1, -1)$. Another inflection point is at $x = 1/2$ and $h(1/2) = 23/16$, so an inflection point is $(1/2, 23/16)$. The function is concave up on $(-\infty, -1) \cup (1/2, \infty)$ and concave down on $(-1, 1/2)$.

■ 4. Use the second derivative test to identify the extrema of $f(x) = x^3 - 12x - 2$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$f'(x) = 3x^2 - 12$$

$$f''(x) = 6x$$

The function has extrema when $f'(x) = 0$.

$$3x^2 - 12 = 0$$

$$(x+2)(x-2) = 0$$

$$x = -2, 2$$

Plug those values into the second derivative.

$$f''(-2) = 6(-2) = -12$$

$$f''(2) = 6(2) = 12$$

By the second derivative test, the function is concave down at $x = -2$. Since $f(-2) = 14$, $(-2,14)$ is a maximum. The function is concave up at $x = 2$. Since $f(2) = -18$, $(2, -18)$ is a minimum.

■ 5. Use the second derivative test to identify the extrema of $g(x) = -4x^3 + 12x^2 + 5$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$g'(x) = -12x^2 + 24x$$

$$g''(x) = -24x + 24$$

The function has extrema when $g'(x) = 0$.

$$-12x^2 + 24x = 0$$

$$x(x - 2) = 0$$

$$x = 0, 2$$

Plug those values into the second derivative.

$$g''(0) = -24(0) + 24 = 24$$

$$g''(2) = -24(2) + 24 = -24$$

By the second derivative test, the function is concave up at $x = 0$. Since $g(0) = 5$, $(0,5)$ is a minimum. The function is concave down at $x = 2$. Since $g(2) = 21$, $(2,21)$ is a maximum.

■ 6. Use the second derivative test to identify the extrema of $h(x) = 2x^4 - 4x^2 + 1$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$h'(x) = 8x^3 - 8x$$

$$h''(x) = 24x^2 - 8$$

The function has extrema when $h'(x) = 0$.

$$8x^3 - 8x = 0$$

$$x(x + 1)(x - 1) = 0$$

$$x = -1, 0, 1$$

Plug those values into the second derivative.

$$h''(-1) = 24(-1)^2 - 8 = 16$$

$$h''(0) = 24(0)^2 - 8 = -8$$

$$h''(1) = 24(1)^2 - 8 = 16$$

By the second derivative test, the function is concave up at $x = -1$. Since $h(-1) = -1$, $(-1, -1)$ is a minimum. The function is concave down at $x = 0$. Since $h(0) = 1$, $(0,1)$ is a maximum. The function is concave up at $x = 1$. Since $h(1) = -1$, $(1, -1)$ is a minimum.

INTERCEPTS AND VERTICAL ASYMPTOTES

■ 1. Find any vertical asymptote(s) of the function.

$$f(x) = \frac{-x^2 + 16x - 63}{x^2 - 2x - 35}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{-(x-7)(x-9)}{(x-7)(x+5)}$$

The denominator is equal to 0 if $x = 7$ or $x = -5$, which means the function has two discontinuities. However, the function simplifies to

$$f(x) = \frac{-(x-9)}{x+5}$$

$$f(x) = \frac{9-x}{x+5}$$

Therefore, the function has a removable discontinuity at $x = 7$ and a vertical asymptote at $x = -5$, which means the domain of the function is $(-\infty, -5) \cup (-5,7) \cup (7,\infty)$.

■ 2. Find any vertical asymptote(s) of the function.

$$g(x) = \frac{x^2 - 3x - 10}{x^2 + x - 2}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$g(x) = \frac{(x-5)(x+2)}{(x-1)(x+2)}$$

The denominator is equal to 0 if $x = -2$ or $x = 1$, which means the function has two discontinuities. However, the function simplifies to

$$g(x) = \frac{x-5}{x-1}$$

Therefore, the function has a removable discontinuity at $x = -2$ and a vertical asymptote at $x = 1$, which means the domain of the function is $(-\infty, -2) \cup (-2,1) \cup (1,\infty)$.

■ 3. Find any vertical asymptote(s) of the function.

$$h(x) = \frac{40 - 27x - 12x^2 - x^3}{9x^2 + 63x - 72}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$h(x) = \frac{-(x+8)(x+5)(x-1)}{9(x-1)(x+8)}$$

Cancel common factors from the numerator and denominator, then simplify.

$$h(x) = \frac{-(x+5)}{9}$$

$$h(x) = -\frac{x+5}{9}$$

There are no values of x that make this denominator 0, so the function has no vertical asymptotes. But it does have removable discontinuities for the factors we canceled, at $x = -8$ and $x = 1$.

HORIZONTAL AND SLANT ASYMPTOTES

■ 1. Find the horizontal asymptote(s) of the function.

$$f(x) = \frac{8x^4 - x^2 + 1}{4x^4 - 1}$$

Solution:

In a polynomial function, the term with the highest degree dominates the behavior of the function. So the behavior of $f(x)$ is dominated by the behavior of

$$\frac{8x^4}{4x^4}$$

Find the equation of the horizontal asymptote by taking the limit of the dominating function as $x \to \infty$.

$$\lim_{x \to \infty} \frac{8x^4}{4x^4} = \lim_{x \to \infty} \frac{8}{4} = 2$$

Therefore, the equation of the horizontal asymptote is $y = 2$.

■ 2. Find the horizontal asymptote(s) of the function.

$$g(x) = \frac{2x^2 - 5x + 12}{3x^2 - 11x - 4}$$

Solution:

In a polynomial function, the term with the highest degree dominates the behavior of the function. So the behavior of $g(x)$ is dominated by the behavior of

$$\frac{2x^2}{3x^2}$$

Find the equation of the horizontal asymptote by taking the limit of the dominating function as $x \to \infty$.

$$\lim_{x\to\infty} \frac{2x^2}{3x^2} = \lim_{x\to\infty} \frac{2}{3} = \frac{2}{3}$$

Therefore, the equation of the horizontal asymptote is $y = 2/3$.

■ 3. Find the horizontal asymptote(s) of the function.

$$h(x) = \frac{x^3 - x^2 + 6x - 1}{7x^4 - 1}$$

Solution:

In a polynomial function, the term with the highest degree dominates the behavior of the function. So the behavior of $g(x)$ is dominated by the behavior of

$$\frac{x^3}{7x^4}$$

Find the equation of the horizontal asymptote by taking the limit of the dominating function as $x \to \infty$.

$$\lim_{x \to \infty} \frac{x^3}{7x^4} = \lim_{x \to \infty} \frac{1}{7x} = 0$$

Therefore, the equation of the horizontal asymptote is $y = 0$.

■ 4. Find the slant asymptote of the function.

$$f(x) = \frac{3x^4 - x^3 + x^2 - 4}{x^3 - x^2 + 1}$$

Solution:

Use polynomial long division on the function to rewrite it as

$$3x + 2 + \frac{3x^2 - 3x - 6}{x^3 - x^2 + 1}$$

Then take the limit of the rational portion of the quotient as $x \to \infty$.

$$\lim_{x \to \infty} \frac{3x^2 - 3x - 6}{x^3 - x^2 + 1} = \lim_{x \to \infty} \frac{3x^2}{x^3} = \lim_{x \to \infty} \frac{3}{x} = 0$$

Therefore, the equation of the slant asymptote is $y = 3x + 2$.

■ 5. Find the slant asymptote of the function.

$$g(x) = \frac{8x^2 + 14x - 7}{4x - 1}$$

Solution:

Use polynomial long division on the function to rewrite it as

$$2x + 4 - \frac{3}{4x - 1}$$

Then take the limit of the rational portion of the quotient as $x \to \infty$.

$$\lim_{x \to \infty} -\frac{3}{4x - 1} = \lim_{x \to \infty} -\frac{3}{4x} = 0$$

Therefore, the equation of the slant asymptote is $y = 2x + 4$.

■ 6. Find the slant asymptote of the function.

$$h(x) = \frac{x^3 - 8}{x^2 - 5x + 6}$$

Solution:

Use polynomial long division on the function to rewrite it as

$$x + 5 + \frac{19x - 38}{x^2 - 5x + 6}$$

Then take the limit of the rational portion of the quotient as $x \to \infty$.

$$\lim_{x \to \infty} \frac{19x - 38}{x^2 - 5x + 6} = \lim_{x \to \infty} \frac{19x}{x^2} = \lim_{x \to \infty} \frac{19}{x} = 0$$

Therefore, the equation of the slant asymptote is $y = x + 5$.

SKETCHING GRAPHS

■ 1. Sketch the graph of the function.

$$f(x) = x^3 - 4x^2 + 8$$

Solution:

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = 3x^2 - 8x$$

$$3x^2 - 8x = 0$$

$$x(3x - 8) = 0$$

$$x = 0, \frac{8}{3}$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	x<0	x=0	0<x<8/3	x=8/3	x>8/3
x	-2	0	1	8/3	4
f'(x)	+	0	-	0	+
Direction	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $f(x)$

- increases on the interval $(-\infty, 0)$,

- has a local maximum at $x = 0$,

- decreases on the interval $(0,8/3)$,

- has a local minimum at $x = 8/3$, and then

- increases on the interval $(8/3,\infty)$.

Evaluate the function at the extrema.

$$f(0) = (0)^3 - 4(0)^2 + 8 = 8$$

$$f\left(\frac{8}{3}\right) = \left(\frac{8}{3}\right)^3 - 4\left(\frac{8}{3}\right)^2 + 8 = -\frac{40}{27}$$

There's a local maximum at $(0,8)$ and a local minimum at $(8/3, -40/27)$. Now use the second derivative to determine concavity.

$$f''(x) = 6x - 8$$

$$6x - 8 = 0$$

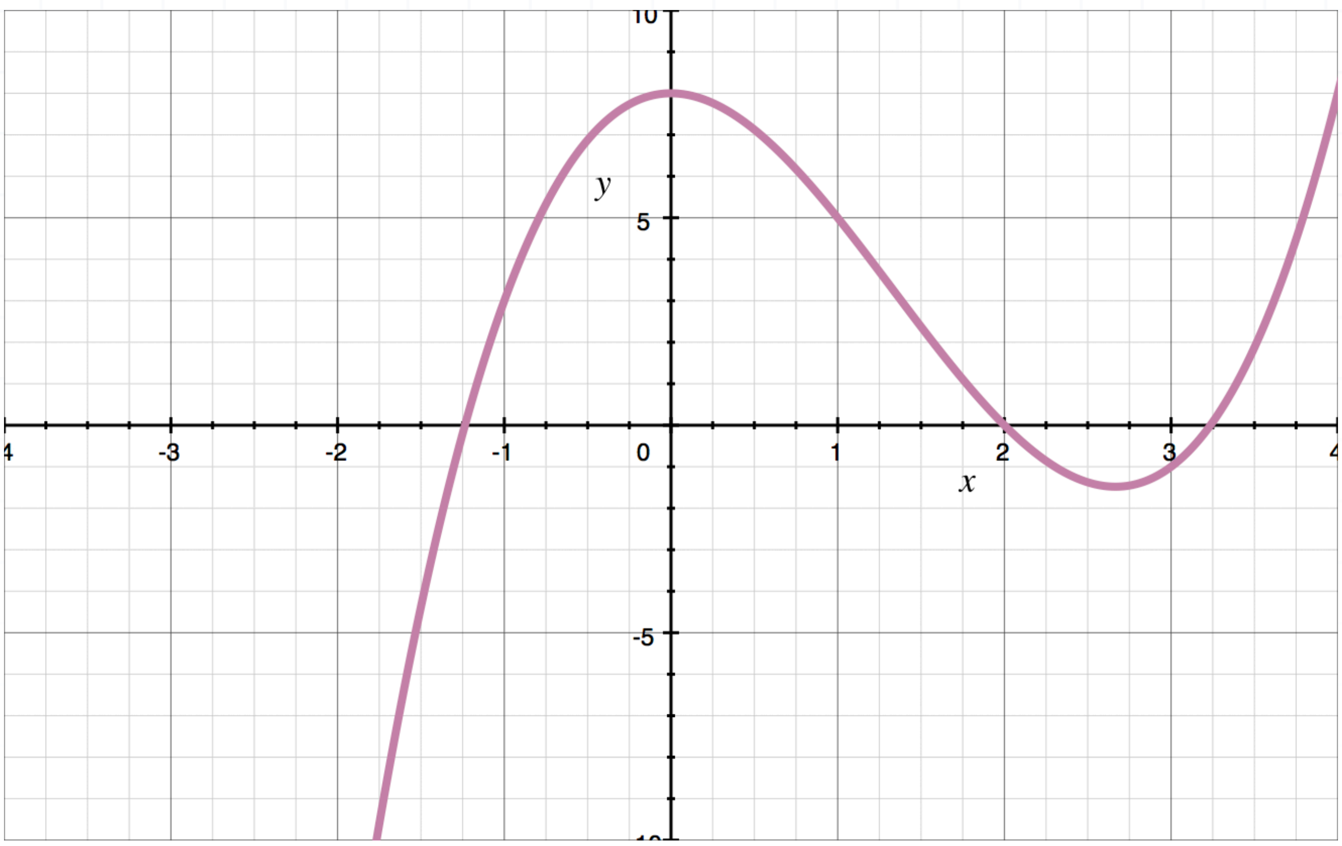
$$x = \frac{4}{3}$$

Test values around the inflection point $x = 4/3$.

Interval	x<4/3	x=4/3	x>4/3
x	0	4/3	3
f''(x)	-	0	+
Concavity	Down	Inflection	Up

We can see that $f(x)$ is concave down on the interval $(-\infty,4/3)$ and concave up on the interval $(4/3,\infty)$. Because $f(4/3)=88/27$, $f(x)$ has an inflection point at $(4/3,88/27)$. Since $f(x)$ is a polynomial function, its graph has no asymptotes.

Putting all this together, the graph is



■ 2. Sketch the graph of the function.

$$g(x)=\frac{1}{4}x^4-\frac{1}{3}x^3-3x^2+1$$

Solution:

Take the derivative, then set it equal to 0 to find critical points.

$$g'(x) = x^3 - x^2 - 6x$$

$$x^3 - x^2 - 6x = 0$$

$$x(x-3)(x+2) = 0$$

$$x = -2,\ 0,\ 3$$

Use the first derivative test to see where $g(x)$ is increasing and decreasing.

Interval	x<-2	x=-2	-2<x<0	x=0	0<x<3	x=3	x>3
x	-4	-2	-1	0	2	3	4
g'(x)	-	0	+	0	-	0	+
Direction	Decreasing	Minimum	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $g(x)$

- decreases on the interval $(-\infty, -2)$,
- has a local minimum at $x = -2$,
- increases on the interval $(-2,0)$,
- has a local maximum at $x = 0$
- decreases on the interval $(0,3)$
- has a local minimum at $x = 3$, and then
- increases on the interval $(3,\infty)$.

Evaluate the function at the extrema.

$$g(-2) = \frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 3(-2)^2 + 1 = -\frac{13}{3}$$

$$g(0) = \frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - 3(0)^2 + 1 = 1$$

$$g(3) = \frac{1}{4}(3)^4 - \frac{1}{3}(3)^3 - 3(3)^2 + 1 = -\frac{59}{4}$$

There's a local minimum at $(-2, -13/3)$, a local maximum at $(0,1)$, and a local minimum at $(3, -59/4)$. Now use the second derivative to determine concavity.

$$g''(x) = 3x^2 - 2x - 6$$

$$3x^2 - 2x - 6 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 72}}{6} = \frac{1 \pm \sqrt{19}}{3}$$

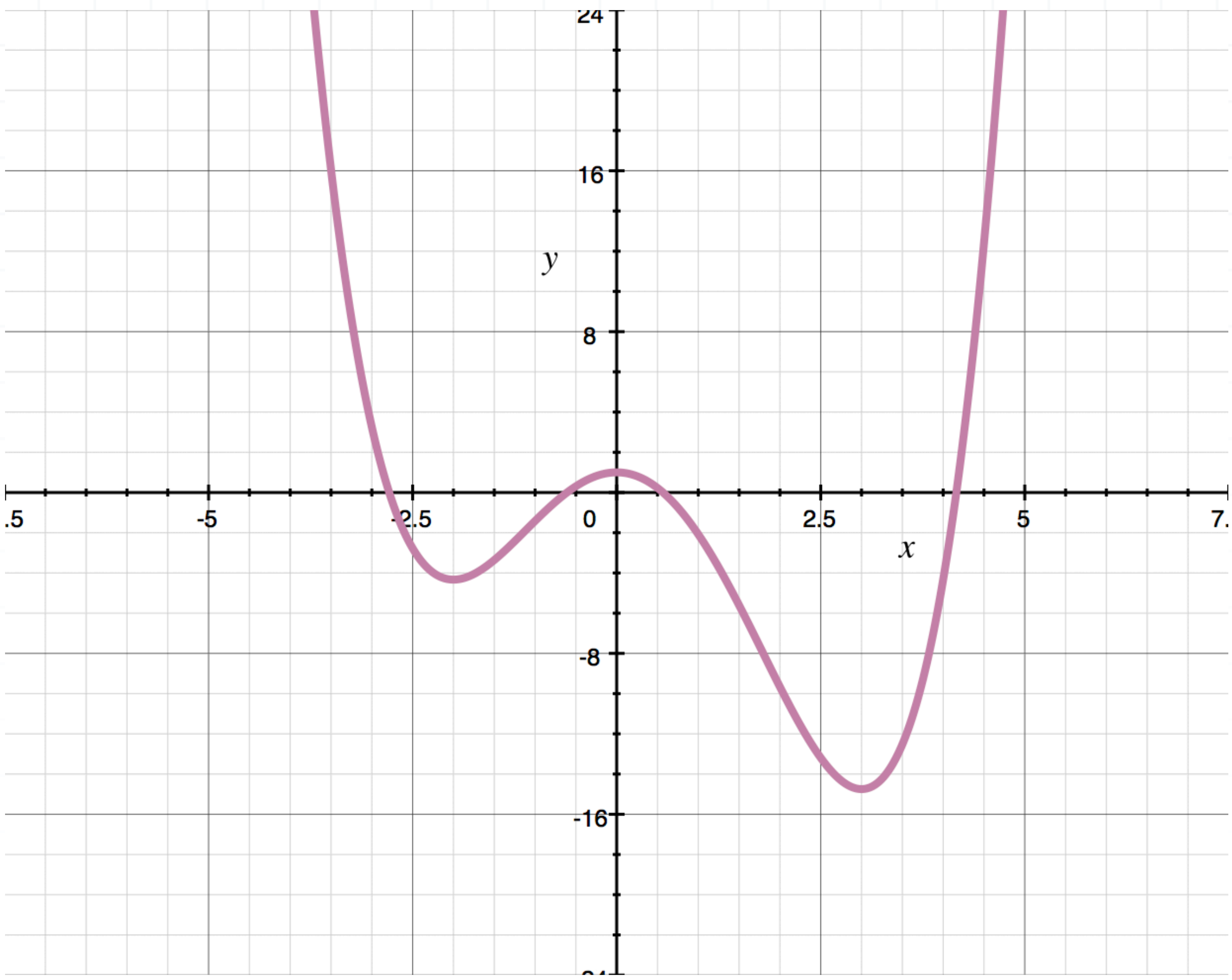
Test values around the inflection points using their approximate values.

Interval	x<-1.12	x=-1.12	-1.12<x<1.79	x=1.79	x>1.79
x	-2	-1.2	1	1.79	4
g''(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

We can see that $g(x)$ is concave up on the interval $(-\infty, (1 - \sqrt{19})/3)$, concave down on the interval $((1 - \sqrt{19})/3, (1 + \sqrt{19})/3)$, and concave up on the interval $((1 + \sqrt{19})/3, \infty)$. Because $g((1 - \sqrt{19})/3) \approx -1.9$, $g(x)$ has an inflection point at approximately $(-1.12, -1.9)$. Because $g((1 + \sqrt{19})/3) \approx -7.96$, $g(x)$ has an inflection point at approximately

$(1.79, -7.96)$. Since $g(x)$ is a polynomial function, its graph has no asymptotes.

Putting all this together, the graph is



■ 3. Sketch the graph of the function.

$$h(x) = \frac{x^2 + x - 6}{4x^2 + 16x + 12}$$

Solution:

Take the derivative, then set it equal to 0 to find critical points.

$$h'(x) = \frac{12(x^2+6x+9)}{(4x^2+16x+12)^2} = \frac{12(x+3)^2}{\left[4(x+1)(x+3)\right]^2} = \frac{12(x+3)^2}{16(x+1)^2(x+3)^2} = \frac{3}{4(x+1)^2}$$

There are no values for which $h'(x) = 0$, so there are no critical points. But $h'(x)$ is undefined when $x = -1$. Now use the second derivative to determine concavity.

$$h''(x) = -\frac{3}{2(x+1)^3}$$

There are no values for which $h''(x) = 0$, but $h''(x)$ is undefined when $x = -1$. Test values around the inflection point $x = -1$.

Interval	x<-1	x=-1	x>-1
x	-3	-1	3
h”(x)	+	DNE	-
Concavity	Up	Inflection	Down

We can see that $h(x)$ is concave up on the interval $(-\infty, -1)$ and concave down on the interval $(-1, \infty)$. Since $h(x)$ is a rational function, we need to look for asymptotes.

The behavior of the function is dominated by the highest degree terms in the numerator and denominator, which means the horizontal asymptote is

$$\lim_{x \to \pm\infty} \frac{x^2}{4x^2} = \lim_{x \to \pm\infty} \frac{1}{4} = \frac{1}{4}$$

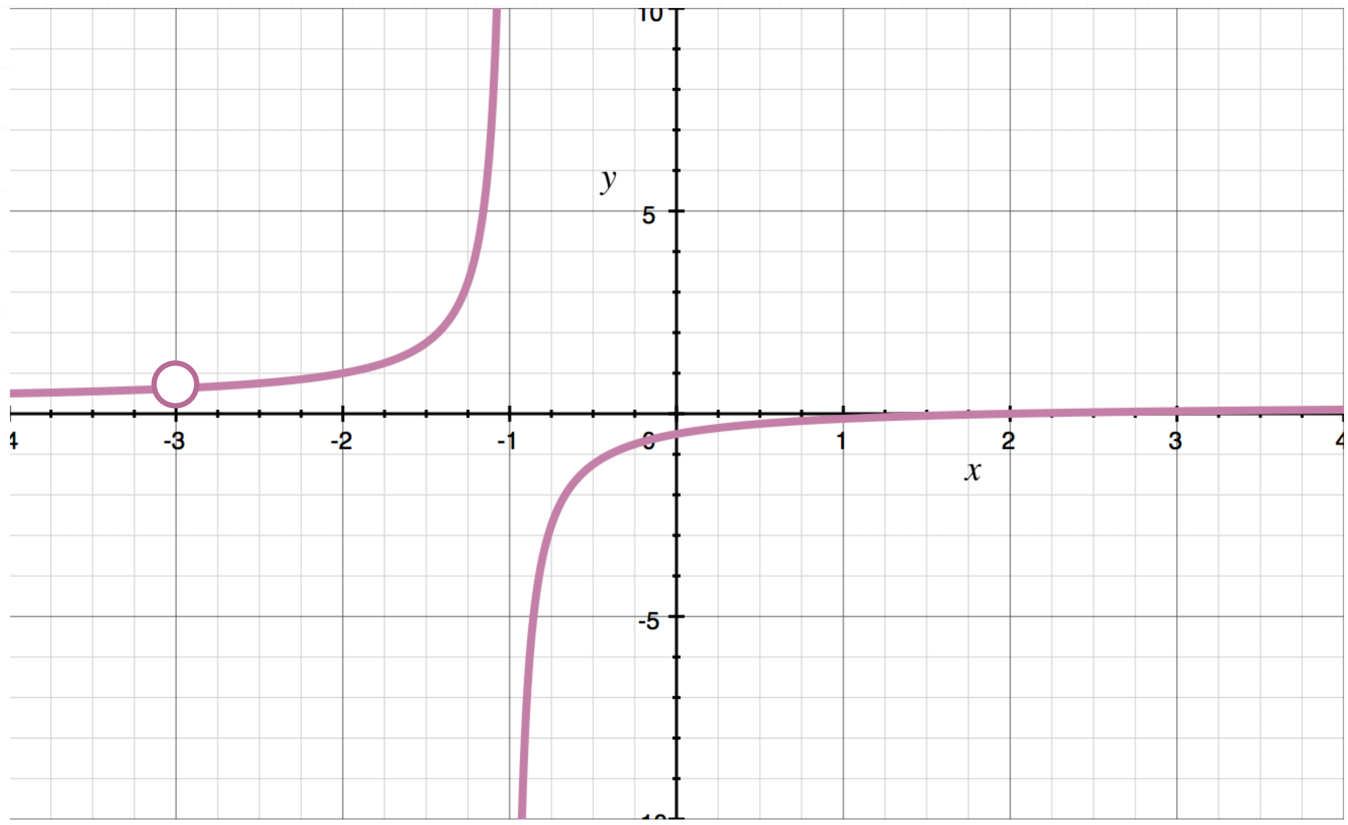
The denominator of $h(x)$ is 0 when $x = -1$, so the function has a vertical asymptote there. To determine the behavior of the function near $x = -1$, find

$$\lim_{x \to -1^-} h(x) = \infty$$

$$\lim_{x \to -1^+} h(x) = -\infty$$

Lastly, the function crosses the x-axis when the numerator equals 0, which occurs at $x = 2$, and the function crosses the y-axis when $x = 0$, which gives a y-intercept of $(0, -1/2)$.

Putting all this together, the graph is



EXTREMA ON A CLOSED INTERVAL

■ 1. Find the extrema of $f(x) = x^3 - 3x^2 + 5$ over the closed interval $[-3,4]$.

Solution:

Find the critical points of the function.

$$f'(x) = 3x^2 - 6x$$

$$3x^2 - 6x = 0$$

$$3x(x - 2) = 0$$

$$x = 0,\ 2$$

Evaluate the function at the endpoints and at the critical numbers.

For $x = -3$, $\quad (-3)^3 - 3(-3)^2 + 5 = -27 - 27 + 5 = -49$

For $x = 0$, $\quad (0)^3 - 3(0)^2 + 5 = 0 - 0 + 5 = 5$

For $x = 2$, $\quad (2)^3 - 3(2)^2 + 5 = 8 - 12 + 5 = 1$

For $x = 4$, $\quad (4)^3 - 3(4)^2 + 5 = 64 - 48 + 5 = 21$

The results show that $f(x)$ has a global minimum at $(-3,-49)$, a local maximum at $(0,5)$, a local minimum at $(2,1)$, and a global maximum at $(4,21)$.

■ 2. Find the extrema of $g(x)=\sqrt[3]{2x^2+3}$ over the closed interval $[-1,5]$.

Solution:

Find the critical points of the function.

$$g'(x)=\frac{1}{3}(2x^2+3)^{-\frac{2}{3}}(4x)=\frac{4x}{3\sqrt[3]{(2x^2+3)^2}}$$

$$4x=0$$

$$x=0$$

Evaluate the function at the endpoints and at the critical numbers.

For $x=-1$, $\sqrt[3]{2(-1)^2+3}=\sqrt[3]{5}\approx 1.71$

For $x=0$, $\sqrt[3]{2(0)^2+3}=\sqrt[3]{3}\approx 1.44$

For $x=5$, $\sqrt[3]{2(5)^2+3}=\sqrt[3]{53}\approx 3.76$

The results show that $g(x)$ has a global minimum at $(0,\sqrt[3]{3})$, and a global maximum at $(5,\sqrt[3]{53})$.

■ 3. Find the extrema of $h(x)=-4x^3+6x^2-3x-2$ over the closed interval $[-4,6]$.

Solution:

Find the critical points of the function.

$$h'(x) = -12x^2 + 12x - 3$$

$$-12x^2 + 12x - 3 = 0$$

$$-3(4x^2 - 4x + 1) = 0$$

$$-3(2x - 1)(2x - 1) = 0$$

$$x = 1/2$$

Evaluate the function at the endpoints and at the critical numbers.

For $x = -4$, $\quad -4(-4)^3 + 6(-4)^2 - 3(-4) - 2 = 362$

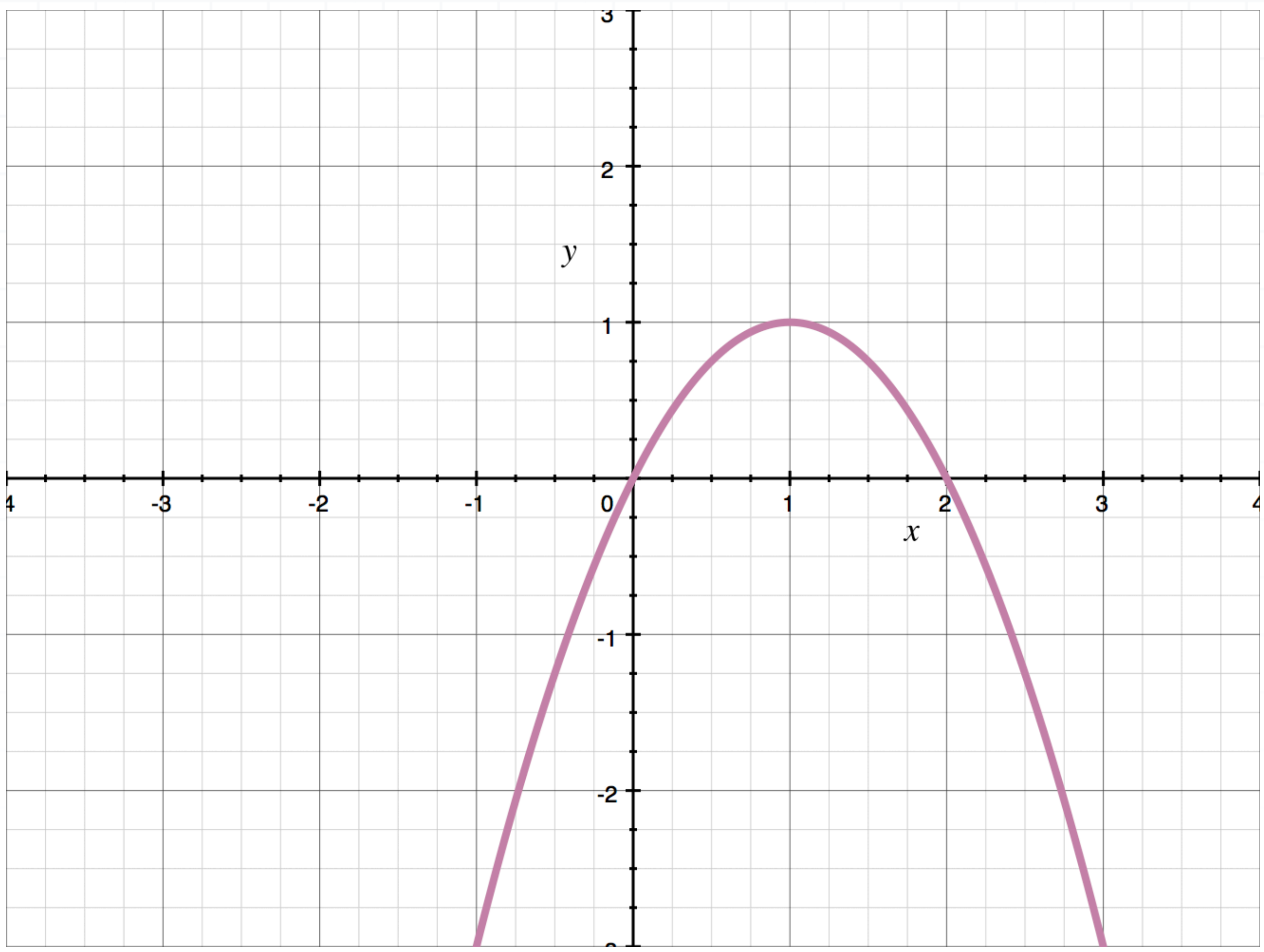
For $x = 1/2$, $\quad -4(1/2)^3 + 6(1/2)^2 - 3(1/2) - 2 = -5/2$

For $x = 6$, $\quad -4(6)^3 + 6(6)^2 - 3(6) - 2 = -668$

The results show that $h(x)$ has a global maximum at $(-4,362)$, a horizontal tangent line at $(1/2, -5/2)$, and a global minimum at $(6, -668)$.

SKETCHING F(X) FROM F'(X)

1. Sketch a possible graph of $f(x)$ given the graph below of $f'(x)$.



Solution:

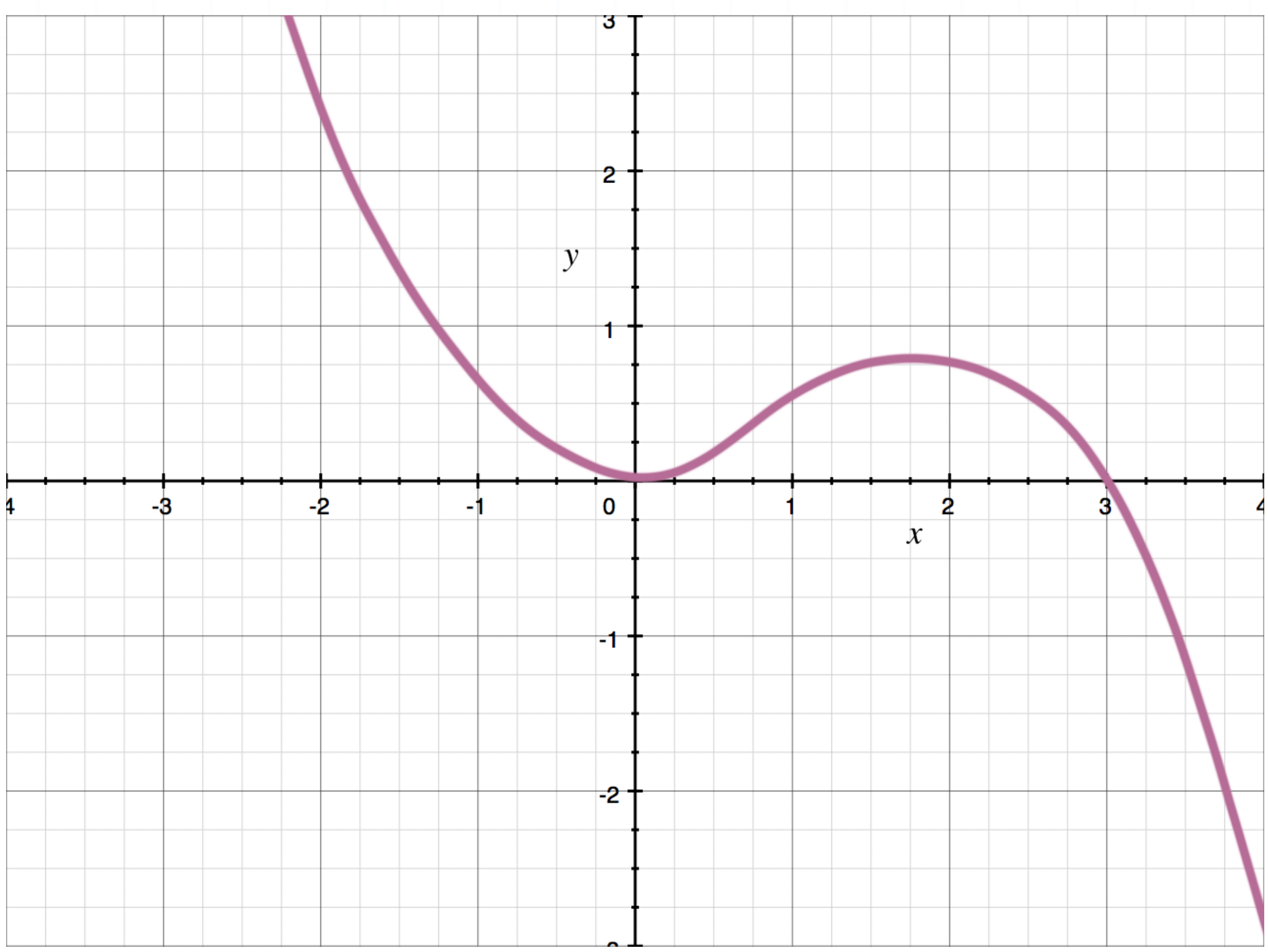
The graph of $f'(x)$ is below the x-axis on the intervals $(-\infty,0)$ and $(2,\infty)$, which means the function $f(x)$ has a negative slope and is decreasing on these intervals.

Additionally, the graph of $f'(x)$ is above the x-axis on the interval $(0,2)$, which means the function $f(x)$ has a positive slope and is increasing on this interval.

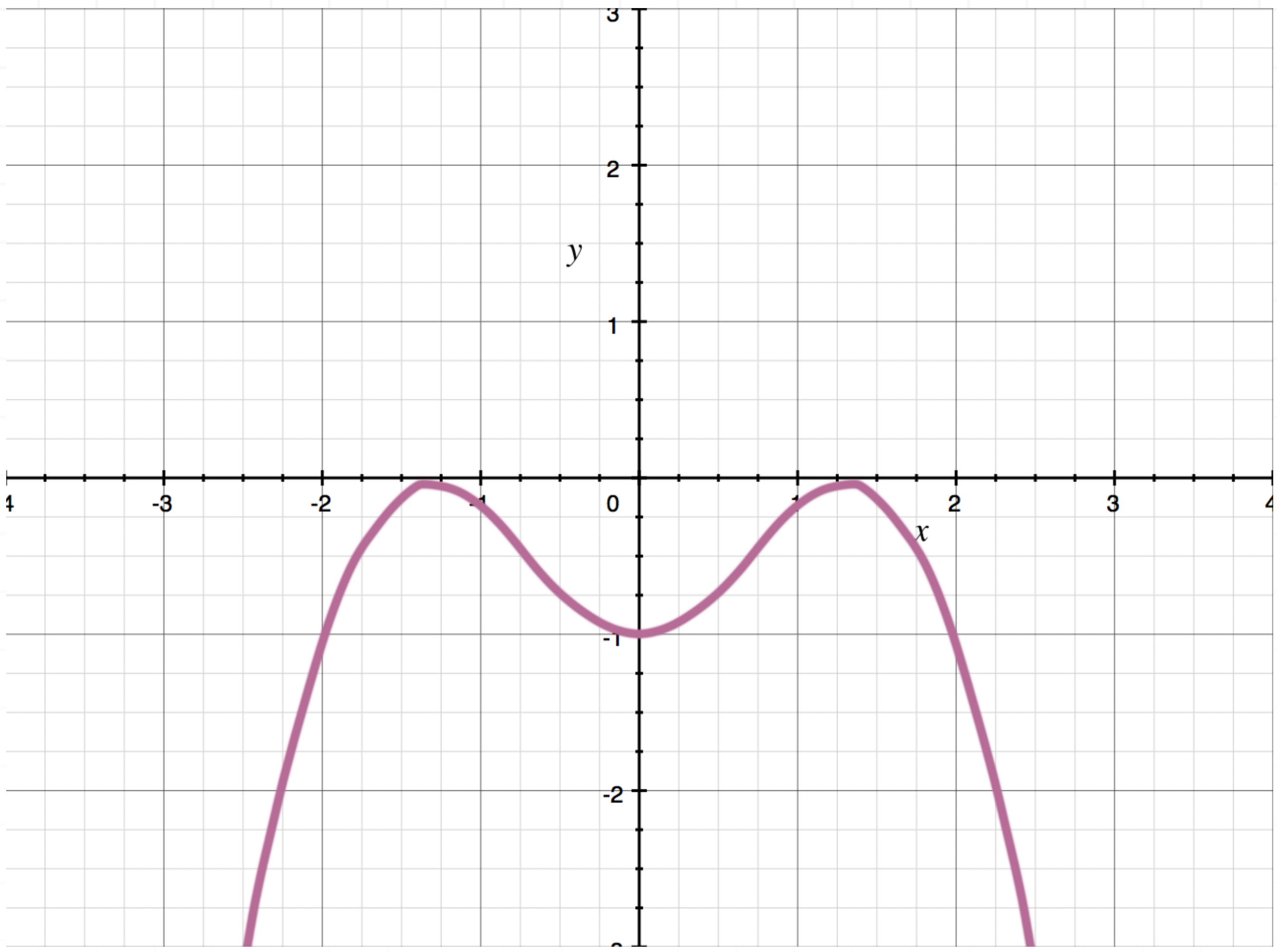
The graph of $f'(x)$ passes through the x-axis and changes sign from negative to positive at $x = 0$, which means that the graph of $f(x)$ has a minimum value at $x = 0$, and the graph of $f'(x)$ passes through the x-axis and changes sign from positive to negative at $x = 2$, which means that the graph of $f(x)$ has a maximum value at $x = 2$.

The graph of $f'(x)$ has a maximum value at $x = 1$, and its slope changes from positive to negative at that point. This means that the graph of $f(x)$ is concave up to the left of $x = 1$, has an inflection point at $x = 1$, and is concave down to the right of $x = 1$.

Putting these facts together, and based on the "assumption" that $f(x)$ contains the point $(0,0)$, this is a possible graph of $f(x)$:



■ 2. Sketch a possible graph of $g'(x)$ given the graph below of $g(x)$.



Solution:

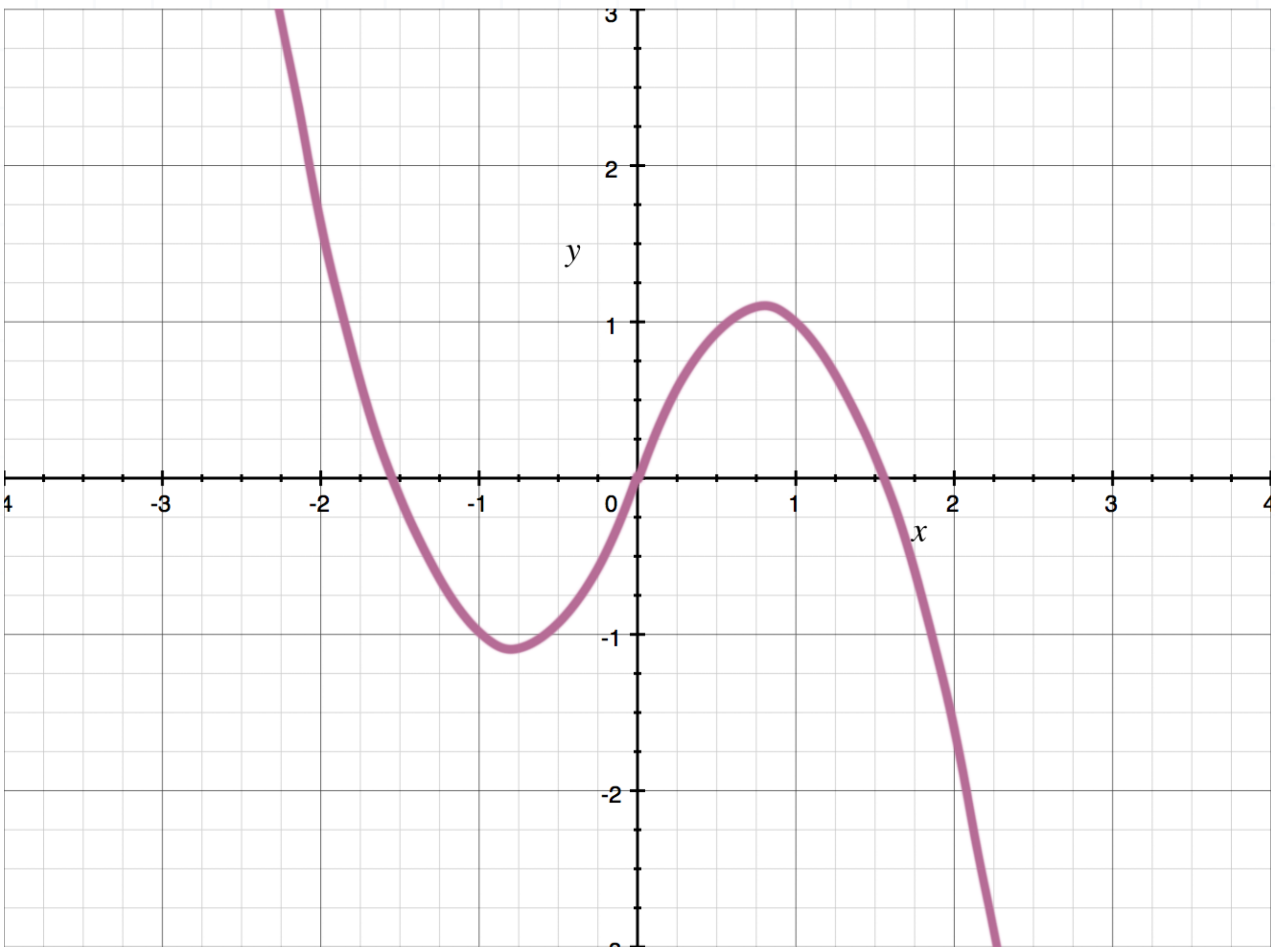
The graph of $g(x)$ has a positive slope on the intervals $(-\infty, -1.5)$ and $(0,1.5)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is above the x-axis on these intervals.

The graph of $g(x)$ has a negative slope on the intervals $(-1.5,0)$ and $(1.5,\infty)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is below the x-axis on these intervals.

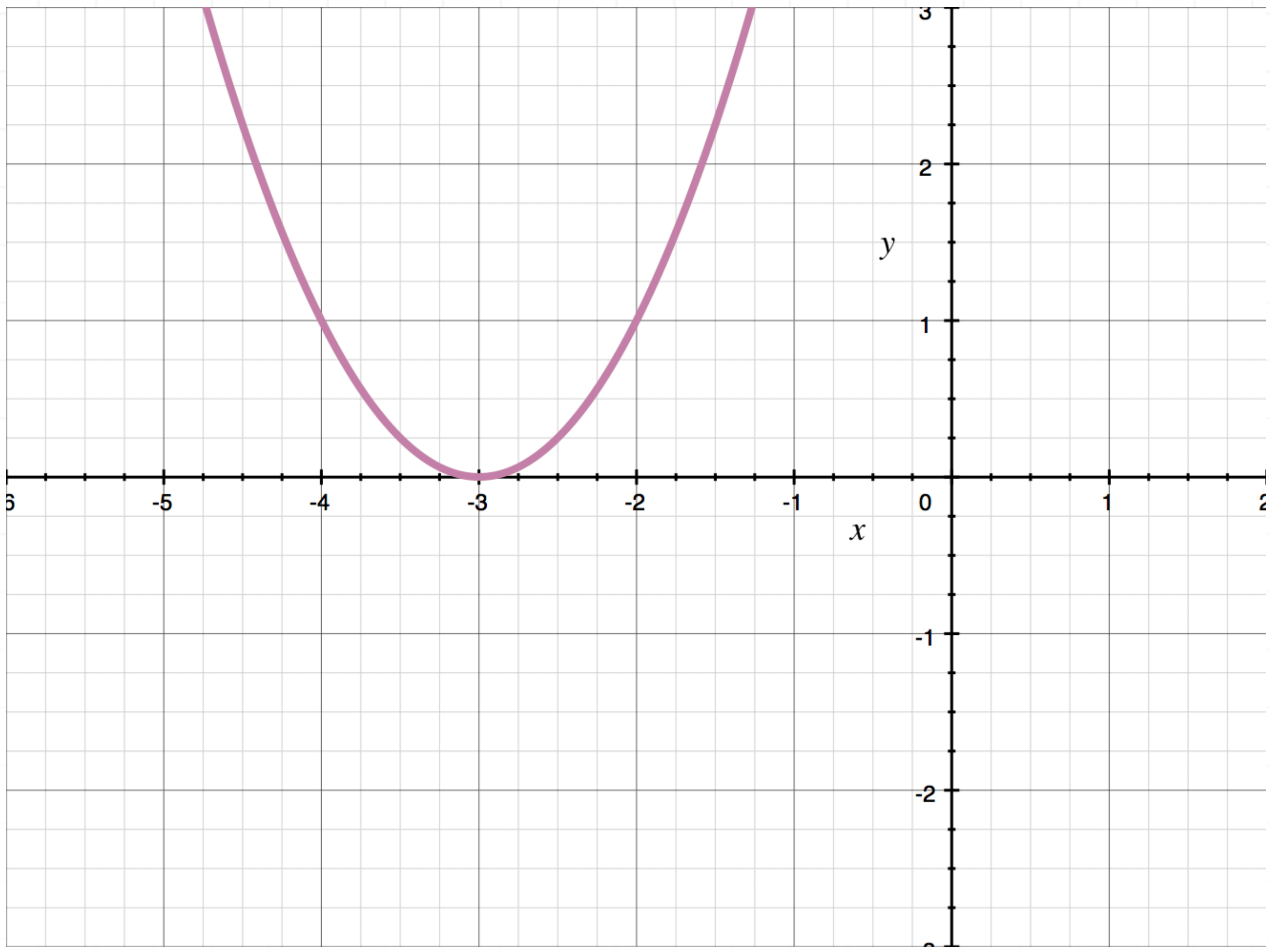
The graph of $g(x)$ has a maximum value at $x = -1.5$ and $x = 1.5$ and its slope is 0, so the graph of $g'(x)$ passes through the x-axis and changes sign from positive to negative at $x = -1.5$ and $x = 1.5$.

The graph of $g(x)$ has a minimum value at $x = 0$, and its slope changes from negative to positive at that point. This means that the graph of $g'(x)$ passes through the x-axis at $x = 0$, and changes from negative to positive.

It appears that the graph of $g(x)$ has an inflection point at $x = -0.75$ and $x = 0.75$, so the graph of $g'(x)$ has extrema at those points. Putting these facts together, this is a possible graph of $g'(x)$:



■ 3. Sketch a possible graph of $h(x)$ given the graph below of $h'(x)$.

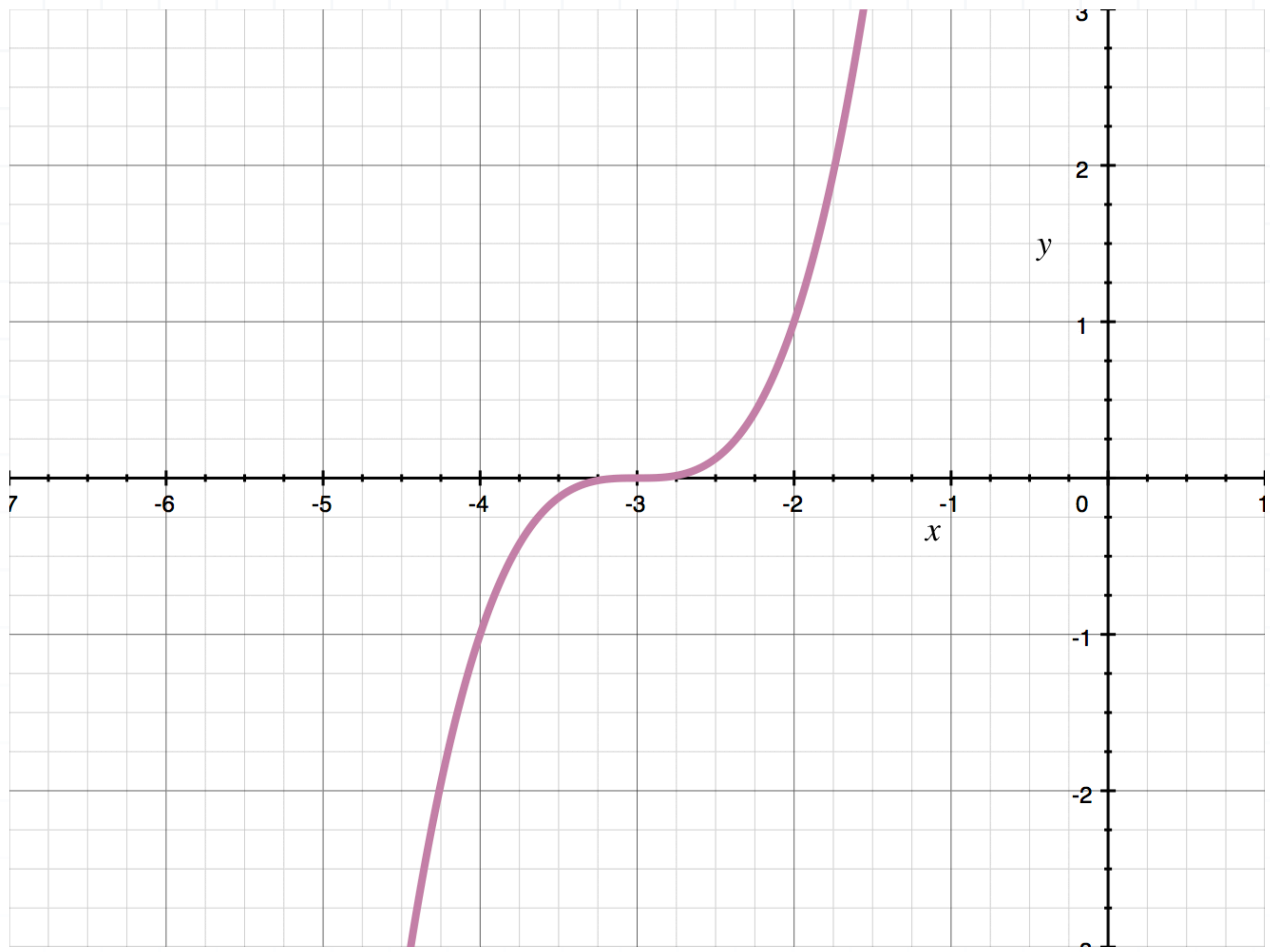


Solution:

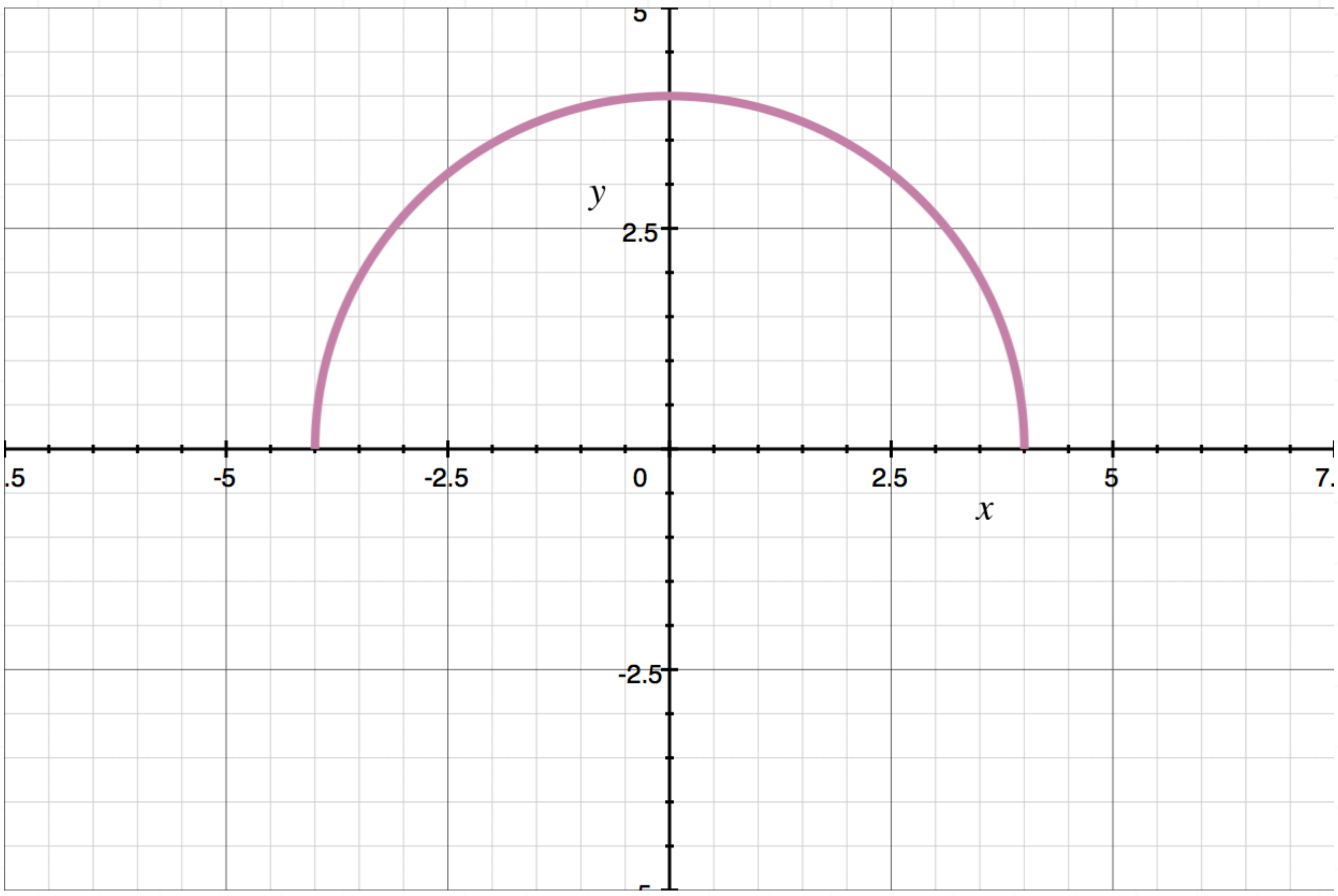
The graph of $h'(x)$ is above the x-axis on the intervals $(-\infty, -3)$ and $(-3,\infty)$, which means the function $h(x)$ has positive slopes and is increasing on these intervals. Since we're only excluding the single point $x = -3$, that means the function is essentially increasing everywhere.

The graph of $h'(x)$ is on the x-axis at $x = -3$, which means the function $h(x)$ has a horizontal tangent at $x = -3$ and is increasing on both sides of this point.

The graph of $h'(x)$ has a minimum value at $x = -3$, and its slope changes from positive to negative at that point. This means that the graph of $h(x)$ is concave down to the left of $x = -3$, has an inflection point at $x = -3$, and is concave up to the right of $x = -3$. Putting these facts together, this is a possible graph of $h(x)$:



■ 4. Sketch a possible graph of $f'(x)$ given the graph below of $f(x)$.



Solution:

The graph of $f(x)$ has a positive slope on the interval $(-4,0)$. Since $f'(x)$ is the derivative of $f(x)$, the graph of $f'(x)$ is above the x-axis on this interval.

The graph of $f(x)$ has a negative slope on the interval $(0,4)$. Since $f'(x)$ is the derivative of $f(x)$, the graph of $f'(x)$ is below the x-axis on this interval.

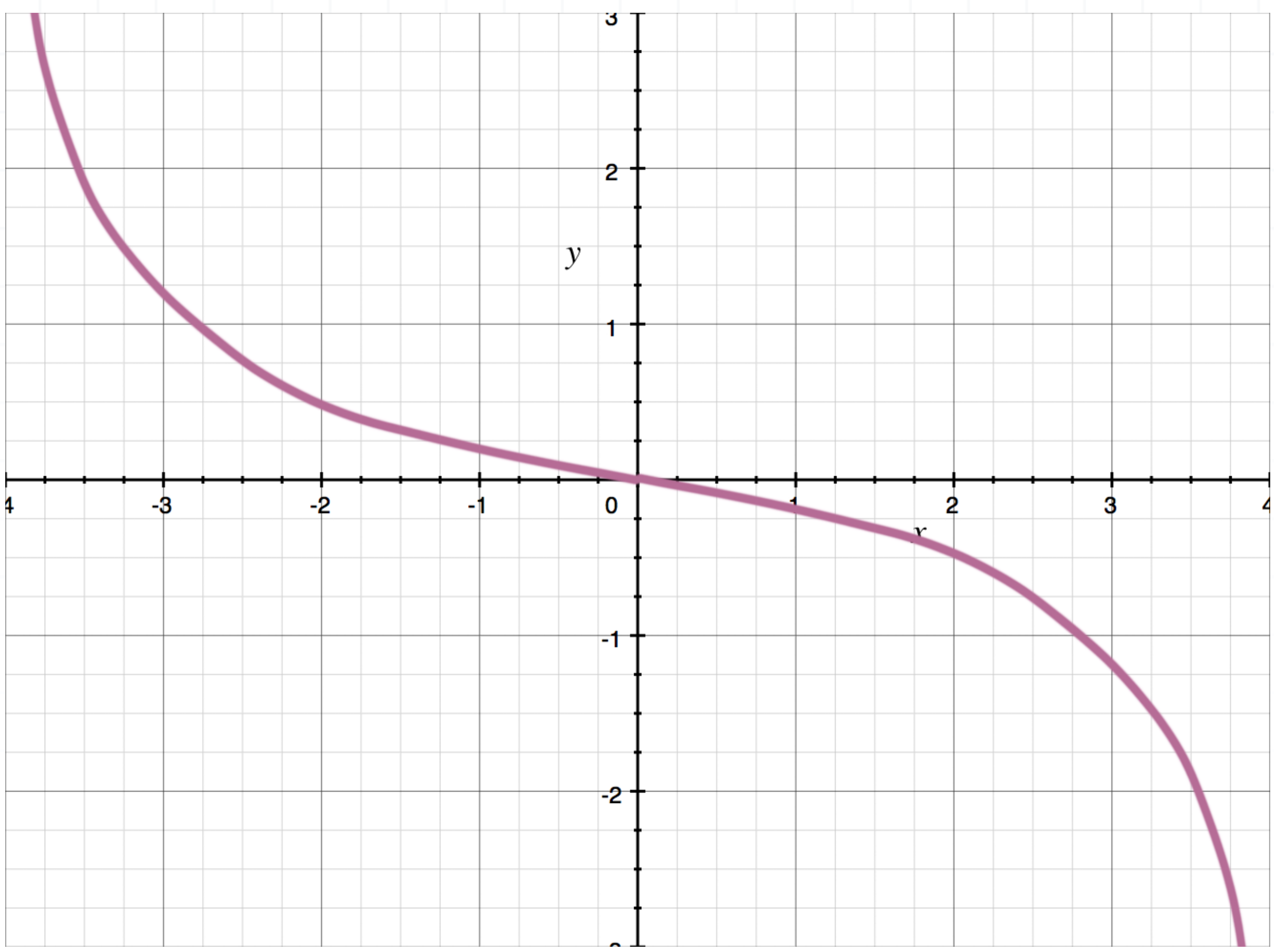
The graph of $f(x)$ has a maximum value at $x = 0$ and its slope is 0, so the graph of $f'(x)$ passes through the x-axis and changes sign from positive to negative at $x = 0$.

The graph of $f(x)$ has vertical tangents at $x = -4$ and $x = 4$. This means that the graph of $f'(x)$ has these limits:

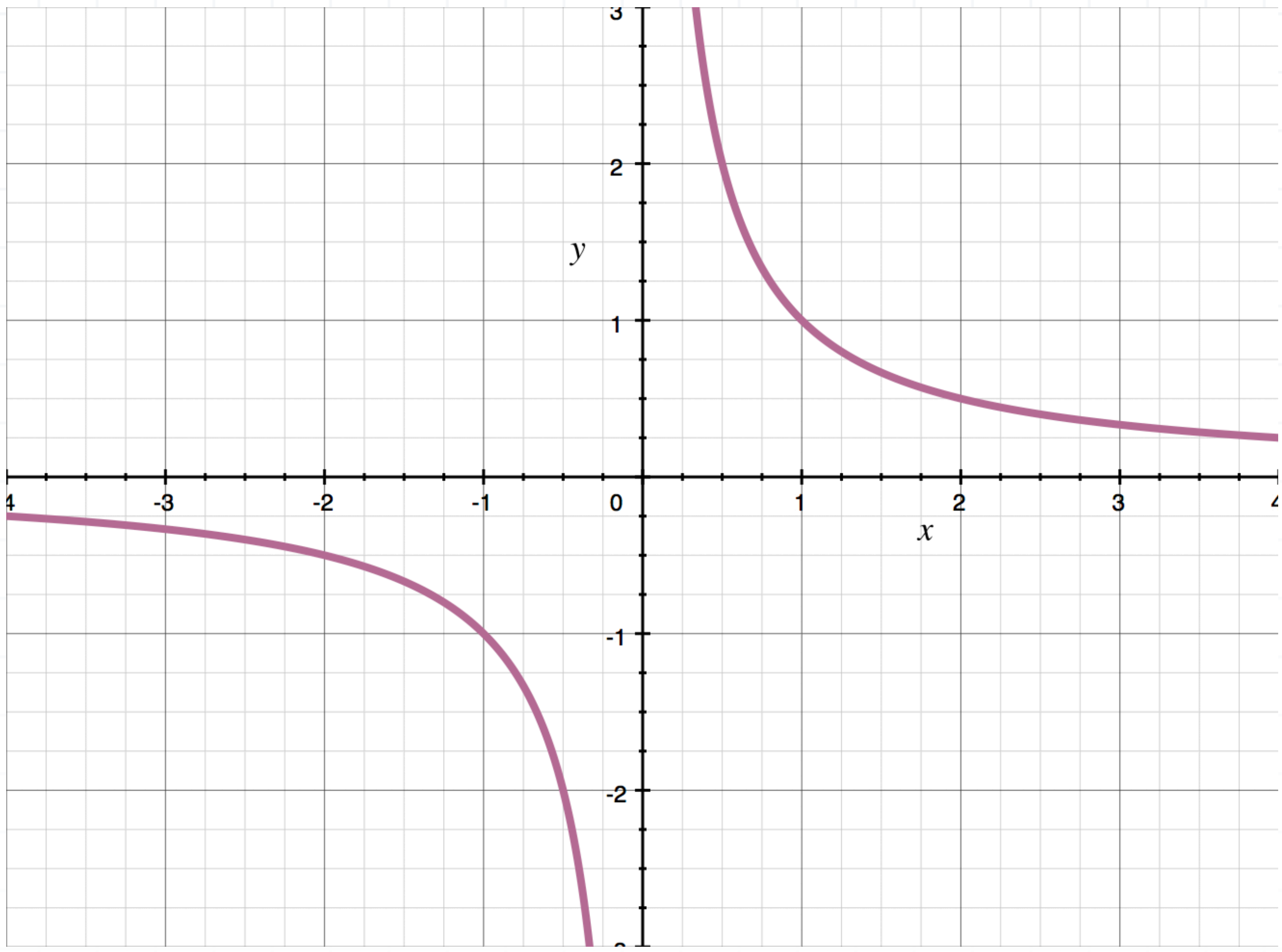
$$\lim_{x \to -4^+} f'(x) = \infty$$

$$\lim_{x \to 4^-} f'(x) = -\infty$$

The graph of $f(x)$ has no inflection points so the graph of $f'(x)$ has no extrema in the interval $(-4,4)$. Putting these facts together, this is a possible graph of $f'(x)$:



■ 5. Sketch a possible graph of $f(x)$ given the graph below of $f'(x)$.



Solution:

The graph of $f'(x)$ is below the x-axis on the interval $(-\infty,0)$, which means the function $f(x)$ has a negative slope and is decreasing on this interval.

The graph of $f'(x)$ is above the x-axis on the interval $(0,\infty)$, which means the function $f(x)$ has a positive slope and is increasing on this interval.

The graph of the $f'(x)$ does not pass through the x-axis, which means that the graph of $f(x)$ does not have any extrema.

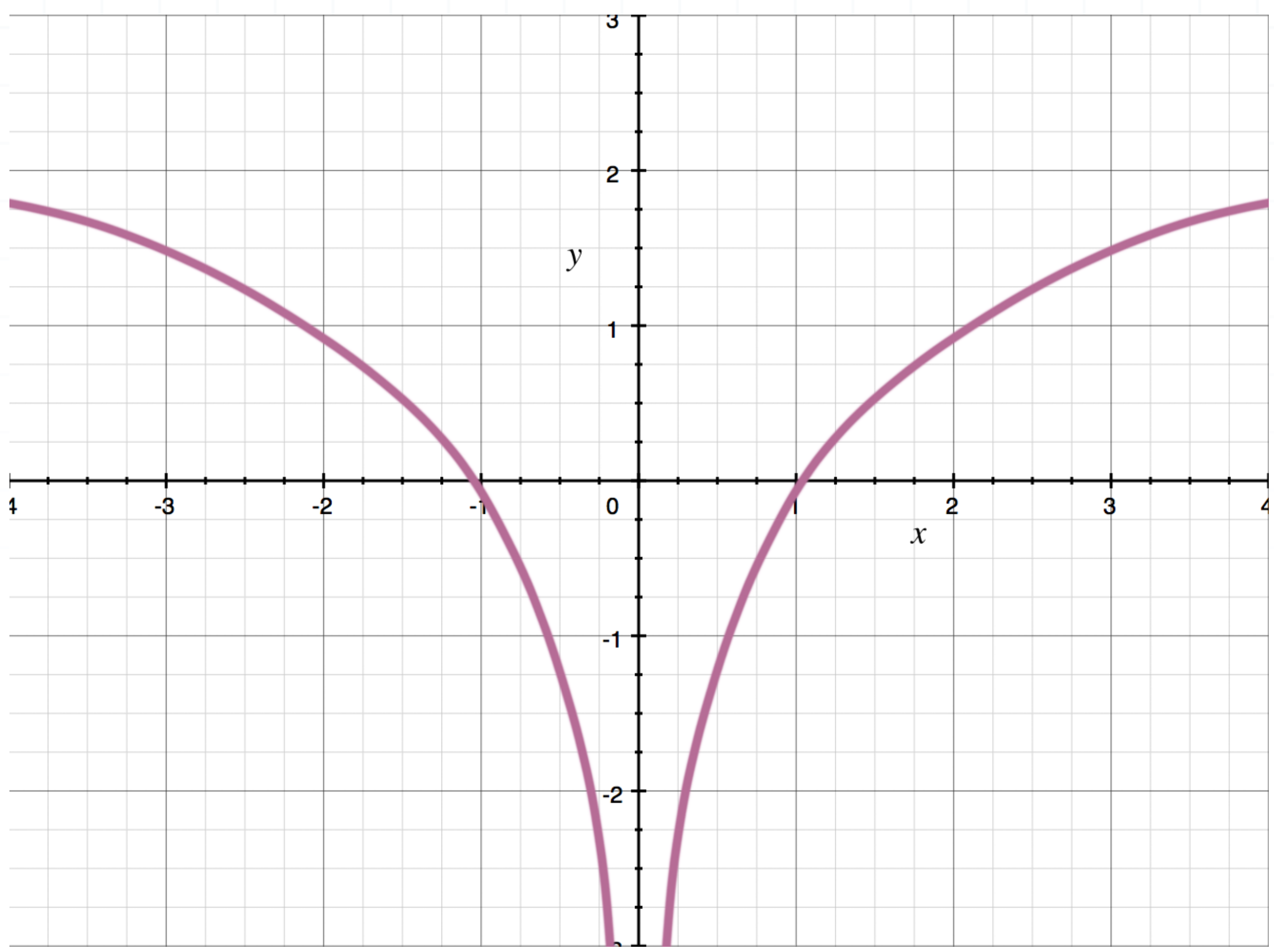
The slope of the graph of $f'(x)$ is negative on $(-\infty,0)$ and $(0,\infty)$. This means that the graph of $f(x)$ is concave down to the left and to the right of the y -axis.

The graph of $f'(x)$ has these limits:

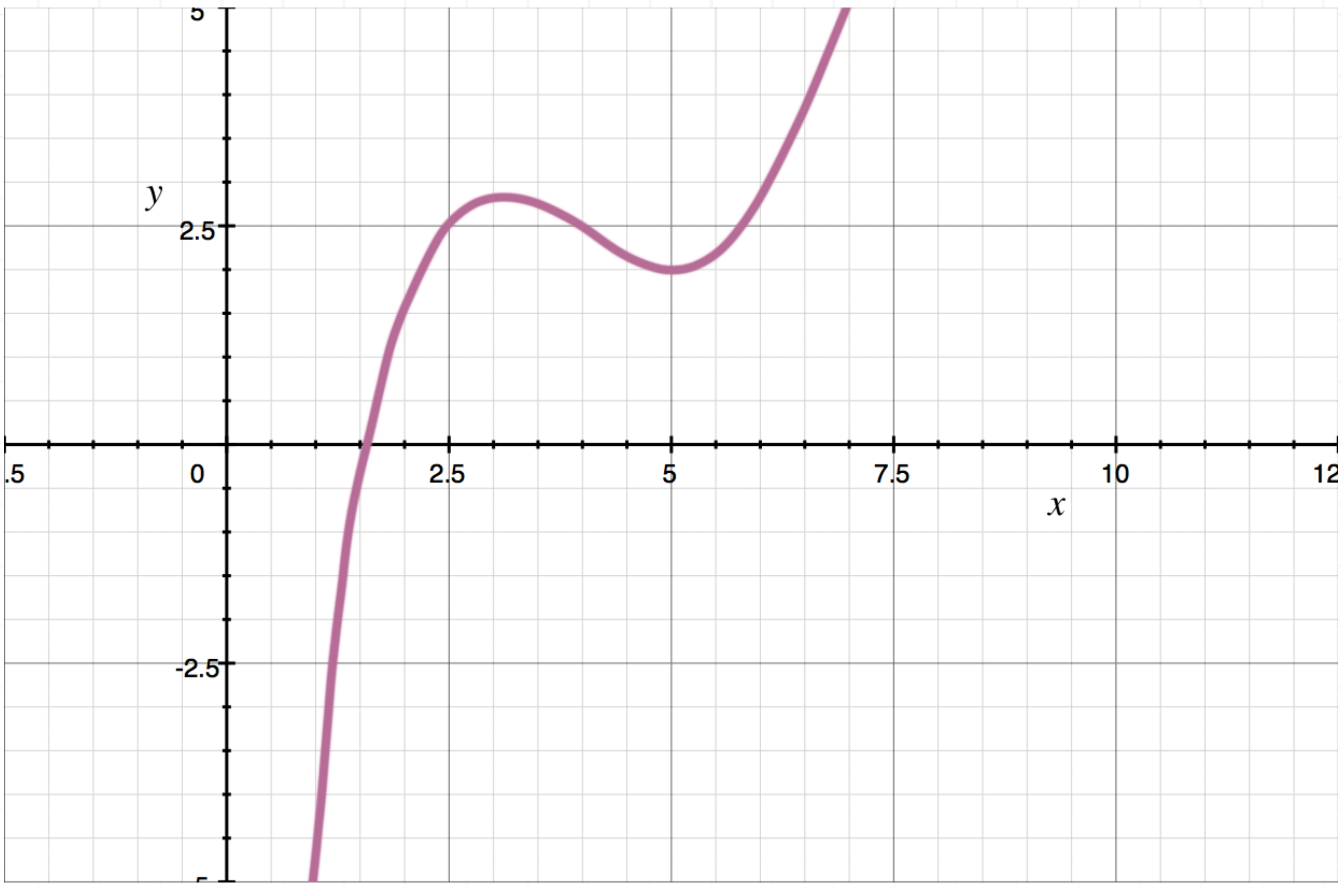
$$\lim_{x\to 0^-} f'(x) = -\infty$$

$$\lim_{x\to 0^+} f'(x) = \infty$$

This means the graph of $f(x)$ has an asymptote on the y-axis. Putting these facts together, this is a possible graph of $f(x)$:



6. Sketch a possible graph of $g'(x)$ given the graph below of $g(x)$.



Solution:

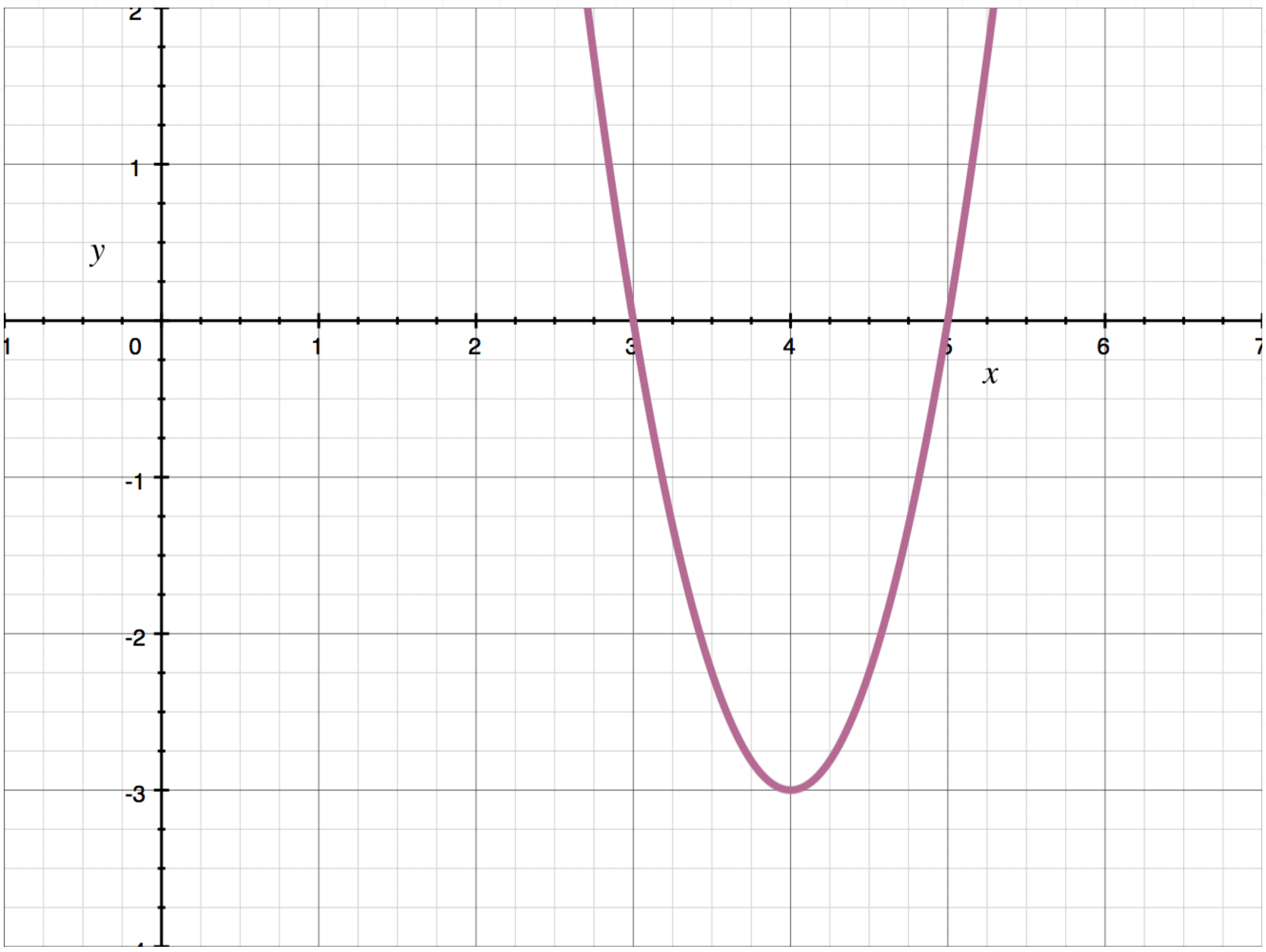
The graph of $g(x)$ has a positive slope on the intervals $(-\infty,3)$ and $(5,\infty)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is above the x-axis on these intervals.

The graph of $g(x)$ has a negative slope on the interval the $(3,5)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is below the x-axis on this interval.

The graph of $g(x)$ has a maximum value at $x=3$ and its slope is 0, so the graph of $g'(x)$ passes through the x-axis and changes sign from positive to negative at $x=3$.

The graph of $g(x)$ has a minimum value at $x=5$, and its slope changes from negative to positive at that point. This means that the graph of $g'(x)$ passes through the x-axis at $x=5$, and changes from negative to positive.

It appears that the graph of $g(x)$ has an inflection point at $x = 4$, so the graph of $g'(x)$ has extrema at $x = 4$. Putting these facts together, this is a possible graph of $g'(x)$:



LINEAR APPROXIMATION

1. Find the linearization of $f(x) = x^3 - 4x^2 + 2x - 3$ at $x = 3$ and use the linearization to approximate $f(3.02)$.

Solution:

The linearization formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 3$, so the linearization is $L(x) = f(3) + f'(3)(x - 3)$. Find the pieces that we need for the linearization formula.

$$f(3) = 3^3 - 4(3)^2 + 2(3) - 3 = -6$$

$$f'(x) = 3x^2 - 8x + 2$$

$$f'(3) = 3(3)^2 - 8(3) + 2 = 5$$

Plugging these pieces into the linearization gives

$$L(x) = -6 + 5(x - 3)$$

$$L(x) = -6 + 5x - 15$$

$$L(x) = 5x - 21$$

Use this equation to approximate $f(3.02)$.

$$f(3.02) = 5(3.02) - 21 = -5.9$$

■ 2. Find the linearization of $g(x)=\sqrt{8x-15}$ at $x=8$ and use the linearization to approximate $f(8.05)$.

Solution:

The linearization formula at $x=a$ is $L(x)=g(a)+g'(a)(x-a)$. In this problem, $a=8$, so the linearization is $L(x)=g(8)+g'(8)(x-8)$. Find the pieces that we need for the linearization formula.

$$g(8)=\sqrt{8(8)-15}=\sqrt{49}=7$$

$$g'(x)=\frac{8}{2\sqrt{8x-15}}=\frac{4}{\sqrt{8x-15}}$$

$$g'(8)=\frac{4}{\sqrt{8(8)-15}}=\frac{4}{\sqrt{49}}=\frac{4}{7}$$

Plugging these pieces into the linearization gives

$$L(x)=7+\frac{4}{7}(x-8)$$

$$L(x)=7+\frac{4}{7}x-\frac{32}{7}$$

$$L(x)=\frac{4}{7}x+\frac{17}{7}$$

Use this equation to approximate $g(8.05)$.

$$g(8.05)=\frac{4}{7}(8.05)+\frac{17}{7}=\frac{246}{35}\approx 7.029$$

■ 3. Find the linearization of $h(x)=2e^{x-4}+6$ at $x=5$ and use the linearization to approximate $h(5.1)$.

Solution:

The linearization formula at $x=a$ is $L(x)=h(a)+h'(a)(x-a)$. In this problem, $a=5$, so the linearization is $L(x)=h(5)+h'(5)(x-5)$. Find the pieces that we need for the linearization formula.

$$h(5)=2e^{5-4}+6=2e+6$$

$$h'(x)=2e^{x-4}$$

$$h'(5)=2e^{5-4}=2e$$

Plugging these pieces into the linearization gives

$$L(x)=2e+6+2e(x-5)$$

$$L(x)=2e+6+2ex-10e$$

$$L(x)=2ex-8e+6$$

Use this equation to approximate $h(5.1)$.

$$g(5.1)=2e(5.1)-8e+6=10.2e-8e+6=2.2e+6\approx 11.98$$

ESTIMATING A ROOT

1. Use linear approximation to estimate $\sqrt[5]{34}$.

Solution:

Let $f(x)=\sqrt[5]{x}$ and $a=32$. The linear approximation would be given by

$$L(x)=f(a)+f'(a)(x-a)$$

$$L(x)=f(32)+f'(32)(x-32)$$

Find the pieces needed for the formula.

$$f(32)=\sqrt[5]{32}=2$$

$$f(x)=\sqrt[5]{x}=x^{\frac{1}{5}}$$

$$f'(x)=\frac{1}{5}(x)^{-\frac{4}{5}}=\frac{1}{5\cdot\sqrt[5]{x^4}}$$

$$f'(32)=\frac{1}{5\cdot\sqrt[5]{32^4}}=\frac{1}{80}$$

Then the linear approximation is

$$L(x)=2+\frac{1}{80}(x-32)$$

$$L(x) = \frac{1}{80}x + \frac{8}{5}$$

Use this approximation to estimate $\sqrt[5]{34}$.

$$f(34) = \frac{1}{80}(34) + \frac{8}{5} = \frac{81}{40}$$

■ 2. Use linear approximation to estimate $\sqrt[8]{260}$.

Solution:

Let $f(x) = \sqrt[8]{x}$ and $a = 256$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(256) + f'(256)(x - 256)$$

Find the pieces needed for the formula.

$$f(256) = \sqrt[8]{256} = 2$$

$$f(x) = \sqrt[8]{x} = x^{\frac{1}{8}}$$

$$f'(x) = \frac{1}{8}(x)^{-\frac{7}{8}} = \frac{1}{8\sqrt[8]{x^7}}$$

$$f'(256) = \frac{1}{8\sqrt[8]{256^7}} = \frac{1}{1{,}024}$$

Then the linear approximation is

$$L(x) = 2 + \frac{1}{1{,}024}(x - 256)$$

$$L(x) = \frac{1}{1{,}024}x + \frac{7}{4}$$

Use this approximation to estimate $\sqrt[8]{260}$.

$$f(260) = \frac{1}{1{,}024}(260) + \frac{7}{4} = \frac{513}{256} \approx 2.0039$$

■ 3. Use linear approximation to estimate $\sqrt[4]{85}$.

Solution:

Let $f(x) = \sqrt[4]{x}$ and $a = 81$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(81) + f'(81)(x - 81)$$

Find the pieces needed for the formula.

$$f(81) = \sqrt[4]{81} = 3$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(81) = \frac{1}{4\sqrt[4]{81^3}} = \frac{1}{108}$$

Then the linear approximation is

$$L(x) = 3 + \frac{1}{108}(x - 81)$$

$$L(x) = \frac{1}{108}x + \frac{9}{4}$$

Use this approximation to estimate $\sqrt[4]{85}$.

$$f(85) = \frac{1}{108}(85) + \frac{9}{4} = \frac{82}{27} \approx 3.037$$

■ 4. Use linear approximation to estimate $\sqrt[4]{615}$.

Solution:

Let $f(x) = \sqrt[4]{x}$ and $a = 625$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(625) + f'(625)(x - 625)$$

Find the pieces needed for the formula.

$$f(625) = \sqrt[4]{625} = 5$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(625) = \frac{1}{4\sqrt[4]{625^3}} = \frac{1}{500}$$

Then the linear approximation is

$$L(x) = 5 + \frac{1}{500}(x - 625)$$

$$L(x) = \frac{1}{500}x + \frac{15}{4}$$

Use this approximation to estimate $\sqrt[4]{615}$.

$$f(615) = \frac{1}{500}(615) + \frac{15}{4} = \frac{249}{50} \approx 4.98$$

■ 5. Use linear approximation to estimate $\sqrt{95}$.

Solution:

Let $f(x) = \sqrt{x}$ and $a = 100$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(100) + f'(100)(x - 100)$$

Find the pieces needed for the formula.

$$f(100) = \sqrt{100} = 10$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

Then the linear approximation is

$$L(x) = 10 + \frac{1}{20}(x - 100)$$

$$L(x) = \frac{1}{20}x + 5$$

Use this approximation to estimate $\sqrt{95}$.

$$f(95) = \frac{1}{20}(95) + 5 = \frac{39}{4} \approx 9.75$$

■ 6. Use linear approximation to estimate $\sqrt[3]{700}$.

Solution:

Let $f(x) = \sqrt[3]{x}$ and $a = 729$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(729) + f'(729)(x - 729)$$

Find the pieces needed for the formula.

$$f(729) = \sqrt[3]{729} = 9.$$

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$f'(729) = \frac{1}{3\sqrt[3]{729^2}} = \frac{1}{243}$$

Then the linear approximation is

$$L(x) = 9 + \frac{1}{243}(x - 729)$$

$$L(x) = \frac{1}{243}x + 6$$

Use this approximation to estimate $\sqrt[3]{700}$.

$$f(700) = \frac{1}{243}(700) + 6 = \frac{2{,}158}{243} \approx 8.88$$

ABSOLUTE, RELATIVE, AND PERCENTAGE ERROR

■ 1. What is the absolute change of $f(x)$ from $x = \pi$ to $x = 2\pi$?

$$f(x) = 3x^2 - \cos\left(\frac{x}{2}\right)$$

Solution:

Absolute change is the difference between $f(2\pi)$ and $f(\pi)$.

$$f(2\pi) = 3(2\pi)^2 - \cos\left(\frac{2\pi}{2}\right) = 12\pi^2 - (-1) = 12\pi^2 + 1$$

$$f(\pi) = 3(\pi)^2 - \cos\left(\frac{\pi}{2}\right) = 3\pi^2 - (0) = 3\pi^2$$

The absolute change is the difference, or

$$f(2\pi) - f(\pi)$$

$$12\pi^2 + 1 - 3\pi^2$$

$$9\pi^2 + 1$$

■ 2. What is the relative change of $g(x)$ from $x = 2$ to $x = 3$?

$$g(x) = 2x^4 - 3x^2 - 5$$

Solution:

Relative change is the quotient of the difference between $g(3)$ and $g(2)$, and the value of the function at the left edge of the interval, $g(2)$.

$$g(3) = 2(3)^4 - 3(3)^2 - 5 = 2(81) - 3(9) - 5 = 130$$

$$g(2) = 2(2)^4 - 3(2)^2 - 5 = 2(16) - 3(4) - 5 = 15$$

The difference is $130 - 15 = 115$, so the relative change is

$$\frac{g(3) - g(2)}{g(2)} = \frac{115}{15} = \frac{23}{3} \approx 767\,\%$$

■ 3. What is the relative change of $h(x)$ from $x = 0$ to $x = \pi$?

$$h(x) = \tan x + 4x + 2$$

Solution:

Relative change is the quotient of the difference between $h(\pi)$ and $h(0)$, and the value of the function at the left edge of the interval, $h(0)$.

$$h(\pi) = \tan \pi + 4\pi + 2 = 0 + 4\pi + 2 = 4\pi + 2$$

$$h(0) = \tan 0 + 4(0) + 2 = 2$$

The difference is $4\pi + 2 - 2 = 4\pi$, so the relative change is

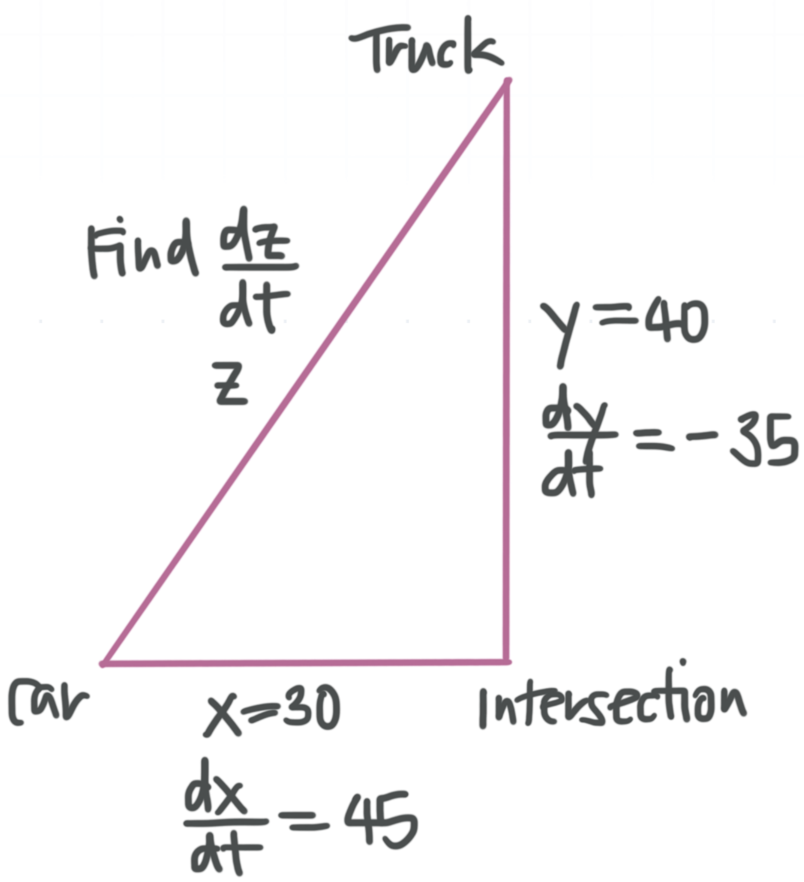
$$\frac{h(\pi) - h(0)}{h(0)} = \frac{4\pi}{2} = 2\pi \approx 628\,\%$$

RELATED RATES

■ 1. A truck is 40 miles north of an intersection, traveling toward the intersection at 35 mph. At the same time, another car is 30 miles west of the intersection, traveling away from the intersection at 45 mph. Is the distance between the vehicles increasing or decreasing at that moment? At what rate?

Solution:

Draw a diagram.



Use the Pythagorean theorem $x^2 + y^2 = z^2$, then differentiate with respect to time.

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}$$

$$x\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt}$$

Substitute what we know into the derivative, then solve for dz/dt.

$$30(45)+40(-35)=50\frac{dz}{dt}$$

$$\frac{dz}{dt}=\frac{30(45)+40(-35)}{50}$$

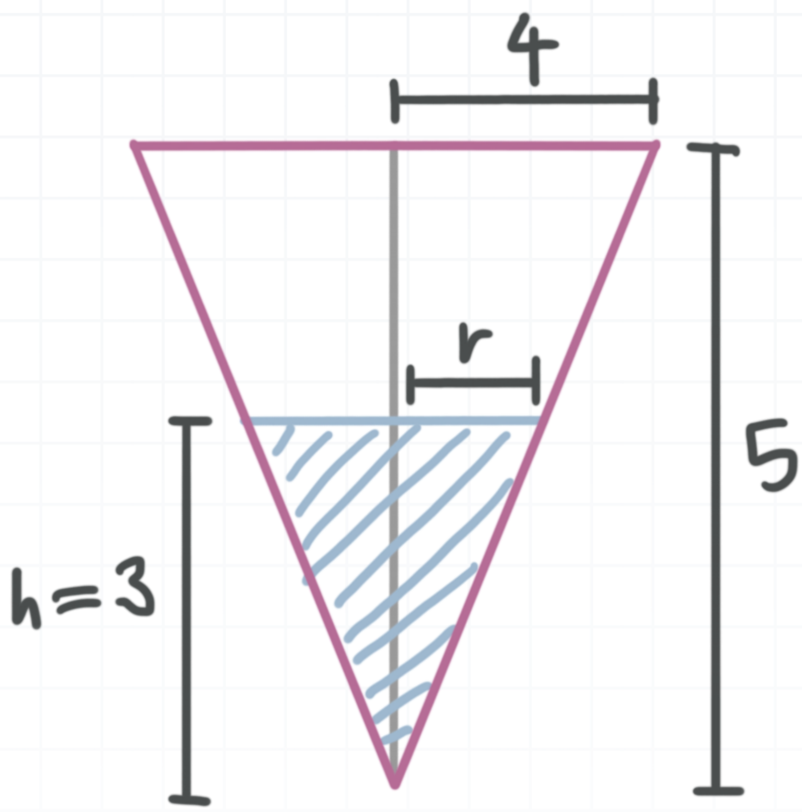
$$\frac{dz}{dt}=-\frac{50}{50}=-1$$

The distance between the two vehicles is decreasing at a rate of 1 mph.

■ 2. Water is flowing out of a cone-shaped tank at a rate of 6 cubic inches per second. If the cone has a height of 5 inches and a base radius of 4 inches, how fast is the water level falling when the water is 3 inches deep?

Solution:

Draw a diagram.



The volume of a cone is $V = (1/3)\pi r^2 h$. To find the base radius of the water, we'll use similar triangles.

$$\frac{r}{3} = \frac{4}{5}$$

$$r = \frac{12}{5}$$

The volume of the cone of water is

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{12}{5}\right)^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{144}{25}\right) h$$

$$V = \frac{48}{25}\pi h$$

Differentiate the volume equation with respect to t.

$$\frac{dV}{dt} = \frac{48}{25}\pi \cdot \frac{dh}{dt}$$

The problem states that $dV/dt = -6$. Substitute this into the derivative equation and solve for dh/dt.

$$-6 = \frac{48}{25}\pi \cdot \frac{dh}{dt}$$

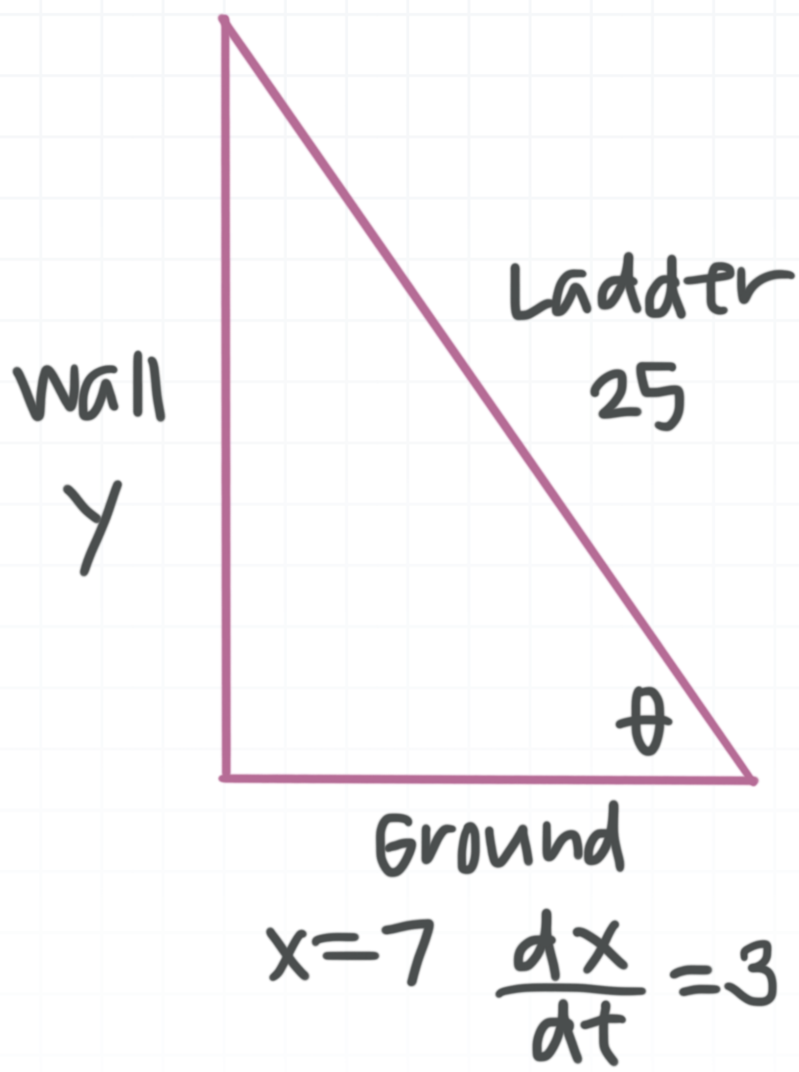
$$-6 \cdot \frac{25}{48\pi} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{25}{8\pi} \text{ inches per second}$$

■ 3. A ladder 25 feet long leans against a vertical wall of a building. If the bottom of the ladder is pulled away horizontally from the building at 3 feet per second, how fast is the angle formed by the ladder and the horizontal ground decreasing when the bottom of the ladder is 7 feet from the base of the wall?

Solution:

Draw a diagram.



Find y using the Pythagorean theorem.

$$y^2 + 7^2 = 25^2$$

$$y = 24$$

Use the cosine function, which gives the equation

$$\cos\theta = \frac{x}{25}$$

Differentiate with respect to t.

$$-\sin\theta\frac{d\theta}{dt} = \frac{1}{25}\frac{dx}{dt}$$

Substitute what we know.

$$-\frac{24}{25} \cdot \frac{d\theta}{dt} = \frac{1}{25} \cdot 3$$

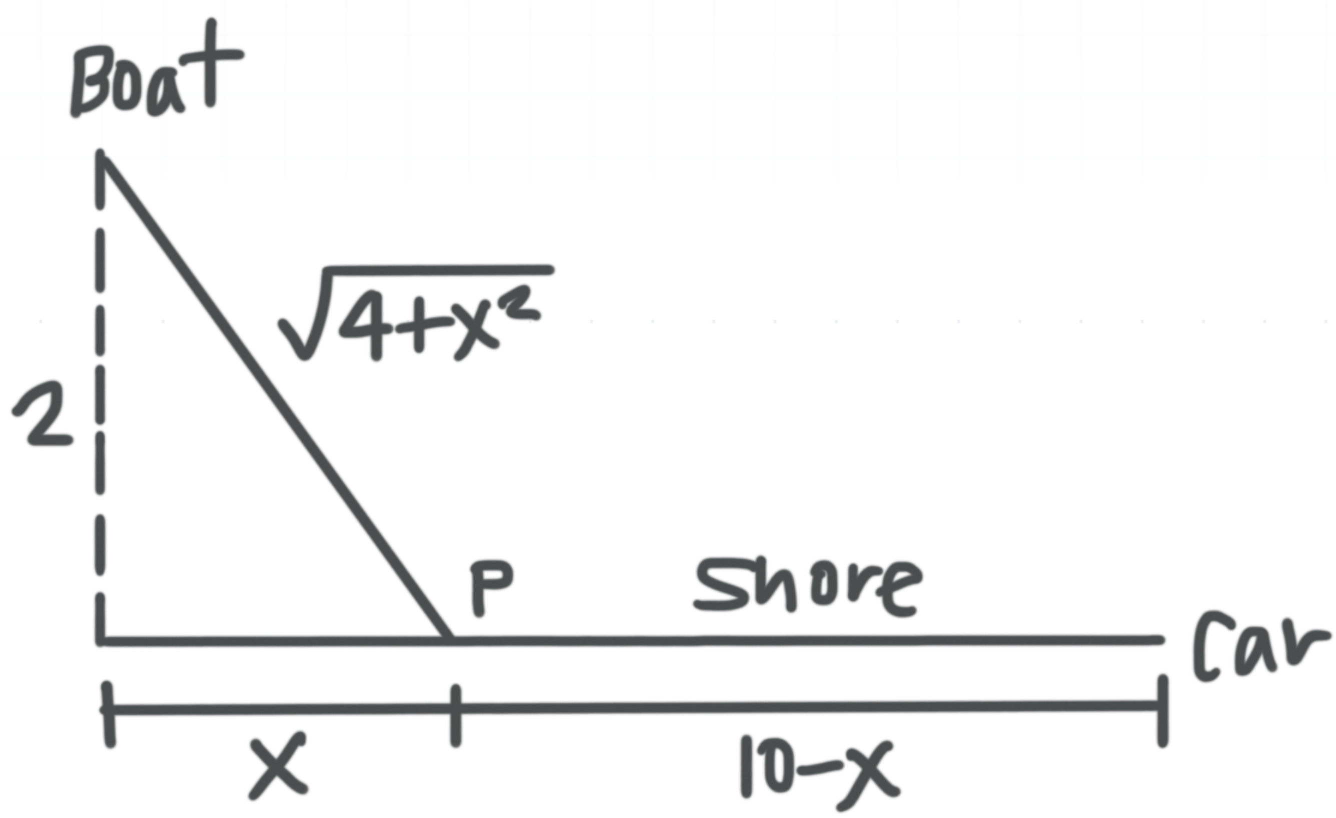
$$\frac{d\theta}{dt} = \frac{3}{25} \cdot -\frac{25}{24} = -\frac{3}{24} = -\frac{1}{8} \text{ feet per second}$$

APPLIED OPTIMIZATION

■ 1. A boater finds herself 2 miles from the nearest point to a straight shoreline, which is 10 miles down the shore from where she parked her car. She plans to row to shore and then walk to her car. If she can walk 4 miles per hour but only row 3 miles per hour, toward what point on the shore should she row in order to reach her car in in the least amount of time?

Solution:

Draw a diagram.



From the diagram, the distance to point P is $\sqrt{4+x^2}$ and the distance from point P to the car is $10-x$. So the total time to reach the car is

$$T = \frac{\sqrt{4+x^2}}{3} + \frac{10-x}{4} \text{ with } 0 \le x \le 10$$

Find the derivative of T.

$$dT = \frac{x}{3\sqrt{4+x^2}} - \frac{1}{4}$$

Set $dT = 0$ and solve for x.

$$\frac{x}{3\sqrt{4+x^2}} = \frac{1}{4}$$

$$\frac{4x}{3} = \sqrt{4+x^2}$$

$$\frac{16x^2}{9} = 4 + x^2$$

$$\frac{7}{9}x^2 = 4$$

$$x^2 = \frac{36}{7}$$

$$x = \frac{36}{\sqrt{7}} \approx 2.2678$$

If she rows to point P, where $x = 2.2678$ miles down the shoreline, it will take her

$$T = \frac{\sqrt{4+(2.2678)^2}}{3} + \frac{10-(2.2678)}{4} \approx 2.9409 \text{ hours}$$

If she rows directly to the shore, where $x = 0$, it will take her

$$T = \frac{2}{3} + \frac{10}{4} \approx 3.167 \text{ hours}$$

Find the distance directly to her car using the Pythagorean Theorem.

$$d^2 = 2^2 + 10^2 = 104$$

$$d = 2\sqrt{26} \text{ miles}$$

If she rows directly to her car, where $x = 10$,

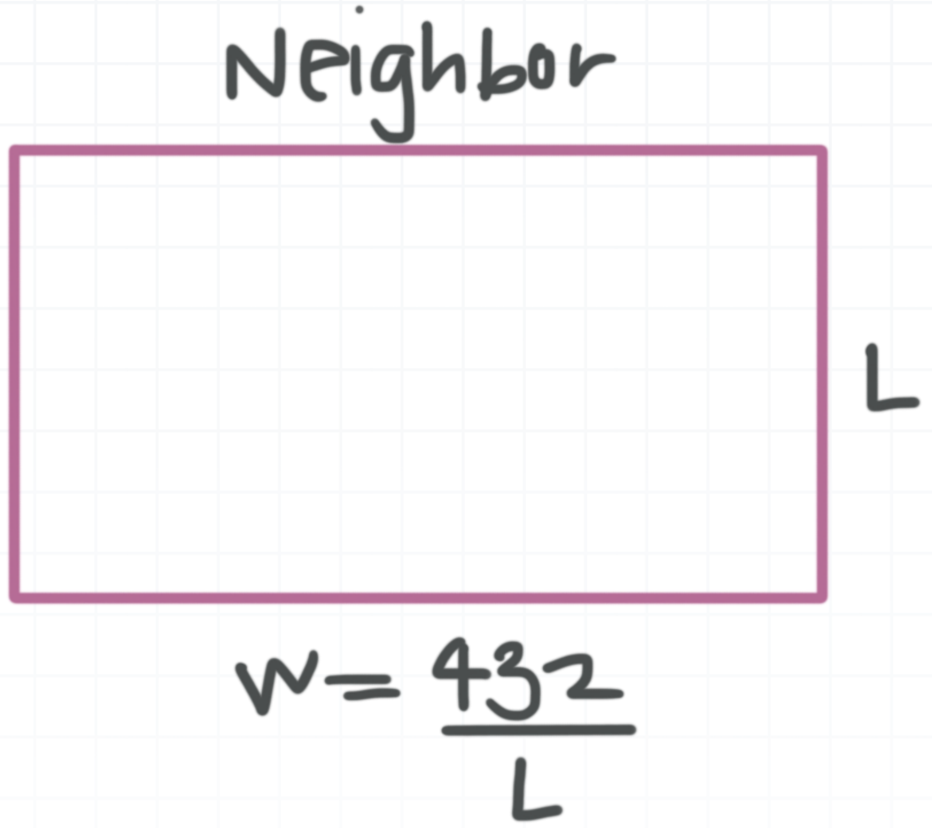
$$T = \frac{2\sqrt{26}}{3} = 3.399 \text{ hours}$$

She will minimize her time by rowing to a point that is 2.2678 miles down shore toward her car.

■ 2. Mr. Quizna wants to build in a completely fenced-in rectangular garden. The fence will be built so that one side is adjacent to his neighbor's property. The neighbor agrees to pay for half of that part of the fence because it borders his property. The garden will contain 432 square meters. What dimensions should Mr. Quizna select for his garden in order to minimize his cost?

Solution:

Draw a diagram.



The area is

$$A = L \cdot W$$

$$432 = L \cdot W$$

$$W = \frac{432}{L}$$

Let C be the total cost and y be the cost per meter. Then,

$$C = 2L \cdot y + \frac{432}{L} \cdot y + \frac{216}{L} \cdot y = 2yL + 648yL^{-1} = y(2L + 648L^{-1})$$

Take the derivative of the cost equation.

$$dC = y(2 - 648L^{-2})$$

Set the derivative equal to 0 and solve for L.

$$2 = \frac{648}{L^2}$$

$$2L^2 = 648$$

$$L^2 = 324$$

$$L = 18$$

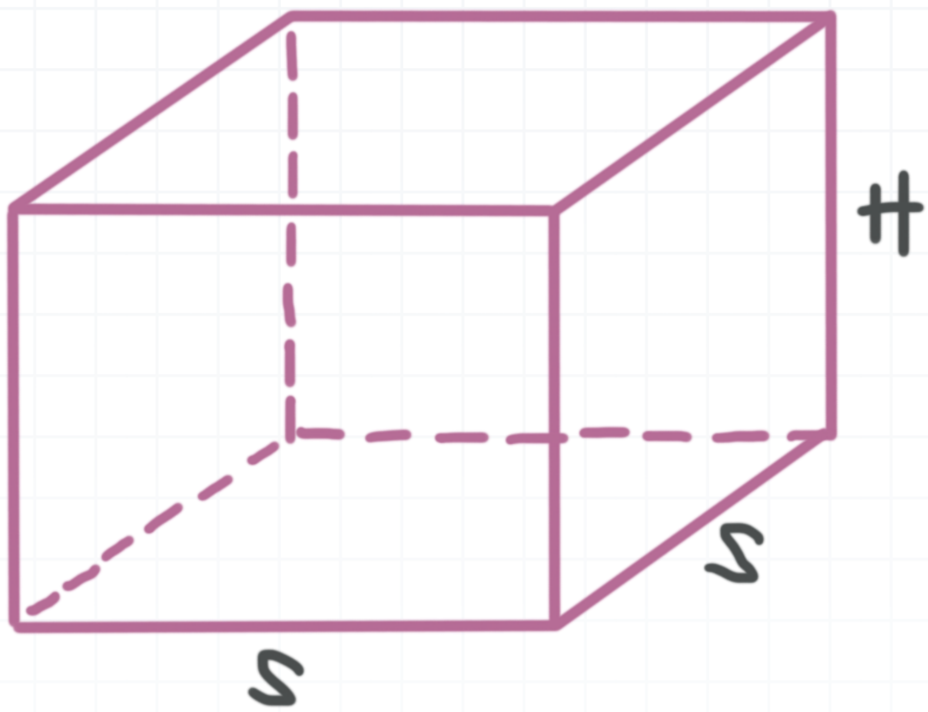
The length of the garden should be $L = 18$ meters and the width of the garden should be

$$W = \frac{432}{L} = \frac{432}{18} = 24 \text{ meters}$$

■ 3. A company is designing shipping crates and wants the volume of each crate to be 6 cubic feet, and each crate's base to be a square between 1.5 feet and 2.0 feet per side. The material for the bottom of the crate costs $\$5$ per square foot, the sides $\$3$ per square foot, and the top $\$1$ per square foot. What dimensions will minimize the cost of the shipping crates?

Solution:

Draw a diagram.



Based on the given information,

$$V = S \cdot S \cdot H$$

$$6 = S \cdot S \cdot H$$

$$H = \frac{6}{S^2}$$

The surface area of the bottom is S^2, the surface area of the top is S^2, and the surface area of the four sides is

$$4 \cdot S \cdot \frac{6}{S^2} = \frac{24}{S}$$

Create a cost function.

$$C = 5 \cdot S^2 + 1 \cdot S^2 + 3 \cdot \frac{24}{S} = 6S^2 + \frac{72}{S} = 6S^2 + 72S^{-1}$$

Differentiate the cost function.

$$dC = 12S - \frac{72}{S^2}$$

Set the derivative equal to 0 and solve for S.

$$12S = \frac{72}{S^2}$$

$$12S^3 = 72$$

$$S^3 = 6$$

$$S = \sqrt[3]{6}$$

$$S \approx 1.82$$

The dimensions that will give the minimum cost are $S = 1.82$ feet and $H = 1.81$ feet.

MEAN VALUE THEOREM

1. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[1,5]$.

$$f(x) = x^3 - 9x^2 + 24x - 18$$

Solution:

First, $f(x)$ is continuous and differentiable on the interval $[1,5]$. The problem says to find c in the interval such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$

Find the values you need for the numerator.

$$f(5) = 5^3 - 9(5)^2 + 24(5) - 18 = 2$$

$$f(1) = 1^3 - 9(1)^2 + 24(1) - 18 = -2$$

Then

$$\frac{f(5) - f(1)}{5 - 1} = \frac{2 - (-2)}{4} = 1$$

Take the derivative $f'(x) = 3x^2 - 18x + 24$, then set $f'(x) = 1$ and solve for x.

$$3x^2 - 18x + 24 = 1$$

$$3x^2 - 18x + 23 = 0$$

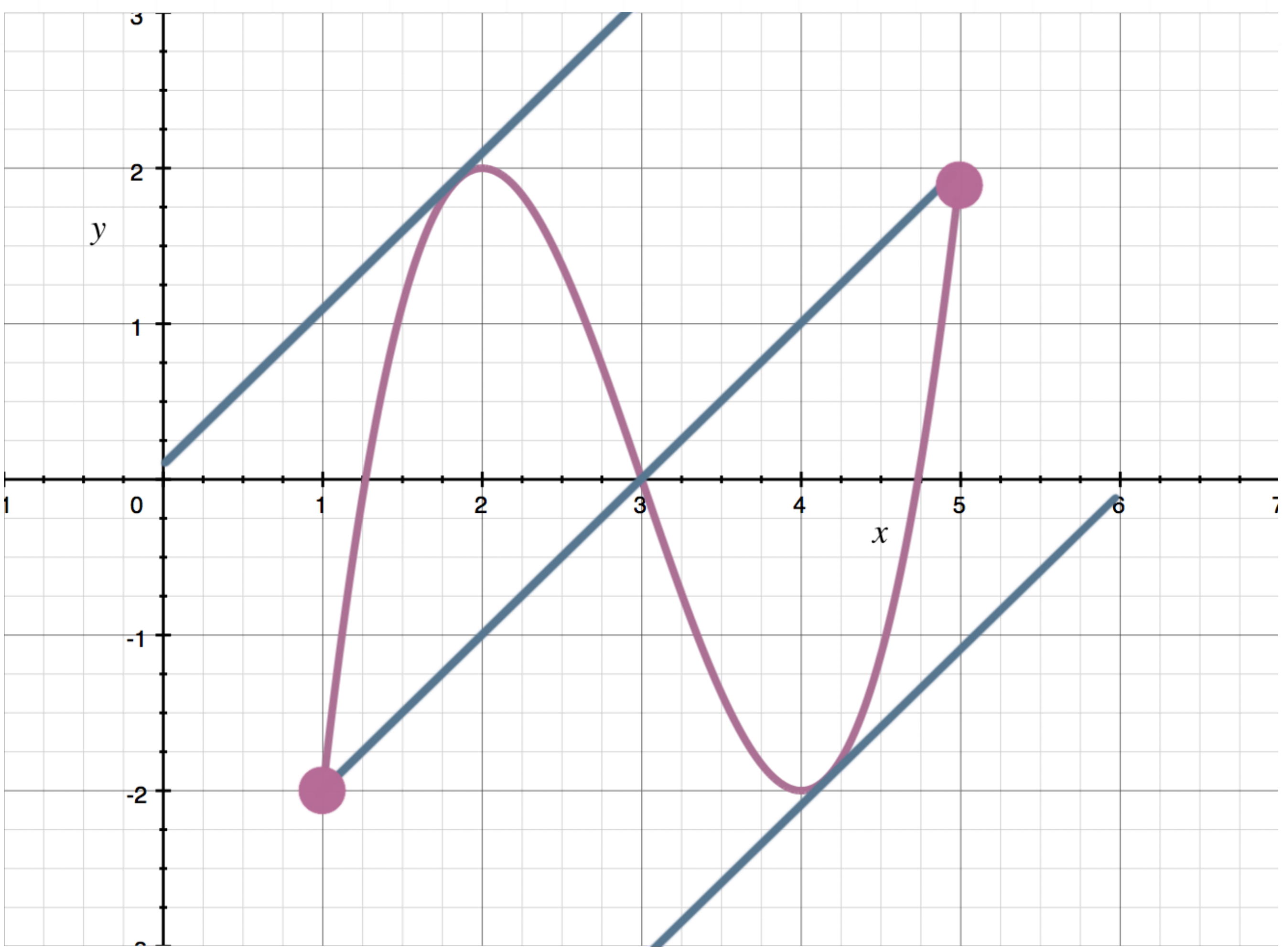
$$x=\frac{18\pm\sqrt{18^2-4(3)(23)}}{2(3)}=\frac{18\pm\sqrt{48}}{6}=\frac{18\pm4\sqrt{3}}{6}=\frac{9\pm2\sqrt{3}}{3}$$

Verify that the slope of the tangent line at these two x-values is 1.

$$f'\left(\frac{9-2\sqrt{3}}{3}\right)=3\left(\frac{9-2\sqrt{3}}{3}\right)^2-18\left(\frac{9-2\sqrt{3}}{3}\right)+24=1$$

$$f'\left(\frac{9+2\sqrt{3}}{3}\right)=3\left(\frac{9+2\sqrt{3}}{3}\right)^2-18\left(\frac{9+2\sqrt{3}}{3}\right)+24=1$$

Therefore, the values of c are $(9\pm2\sqrt{3})/3$. The figure illustrates how these two points satisfy the Mean Value Theorem.



■ 2. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[1,4]$.

$$g(x) = \frac{x^2 - 9}{3x}$$

Solution:

First, $g(x)$ is continuous and differentiable on the interval $[1,4]$. The problem says to find c in the interval such that

$$g'(c) = \frac{g(4) - g(1)}{4 - 1}$$

Find the values you need for the numerator.

$$g(4) = \frac{4^2 - 9}{3(4)} = \frac{16 - 9}{12} = \frac{7}{12}$$

$$g(1) = \frac{1^2 - 9}{3(1)} = \frac{1 - 9}{3} = -\frac{8}{3}$$

Then

$$\frac{g(4) - g(1)}{4 - 1} = \frac{\frac{7}{12} - \left(-\frac{8}{3}\right)}{3} = \frac{\frac{13}{4}}{3} = \frac{13}{4} \cdot \frac{1}{3} = \frac{13}{12}$$

Take the derivative,

$$g'(x) = \frac{(3x)(2x) - (x^2 - 9)(3)}{(3x)^2} = \frac{6x^2 - 3x^2 + 27}{9x^2} = \frac{3x^2 + 27}{9x^2} = \frac{x^2 + 9}{3x^2}$$

then set $g'(x) = 13/12$ and solve for x.

$$\frac{x^2 + 9}{3x^2} = \frac{13}{12}$$

$$12x^2 + 108 = 39x^2$$

$$27x^2 = 108$$

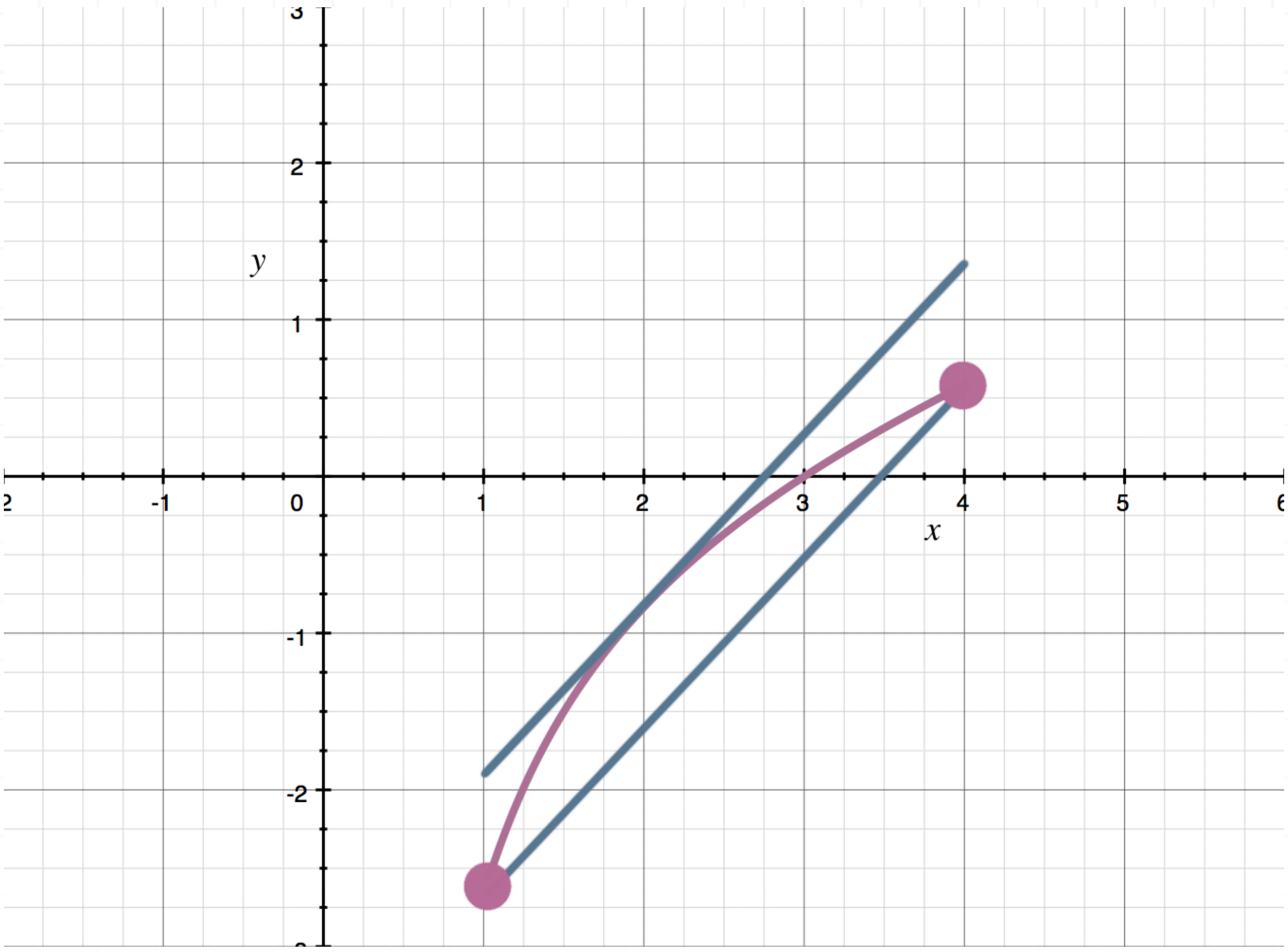
$$x^2 = 4$$

$$x = \pm 2$$

Only $x = 2$ is in the given interval. Verify that the slope of the tangent line at this x-value is $13/12$.

$$g'(2) = \frac{2^2 + 9}{3(2)^2} = \frac{4 + 9}{3(4)} = \frac{13}{12}$$

Therefore, the value of c is 2. The figure illustrates how this point satisfies the Mean Value Theorem.



■ 3. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[0,5]$.

$$h(x) = -\sqrt{25 - 5x}$$

Solution:

First, $h(x)$ is continuous and differentiable on the interval $[0,5]$. The problem says to find c in the interval such that

$$h'(c) = \frac{h(5) - h(0)}{5 - 0}$$

Find the values you need for the numerator.

$$h(5) = -\sqrt{25-5(5)} = -\sqrt{0} = 0$$

$$h(0) = -\sqrt{25-5(0)} = -\sqrt{25} = -5$$

Then

$$\frac{h(5)-h(0)}{5-0} = \frac{0-(-5)}{5} = 1$$

Take the derivative,

$$h'(x) = -\frac{-5}{2\sqrt{25-5x}} = \frac{5}{2\sqrt{25-5x}}$$

then set $h'(x) = 1$ and solve for x.

$$\frac{5}{2\sqrt{25-5x}} = 1$$

$$5 = 2\sqrt{25-5x}$$

$$\frac{5}{2} = \sqrt{25-5x}$$

$$\frac{25}{4} = 25-5x$$

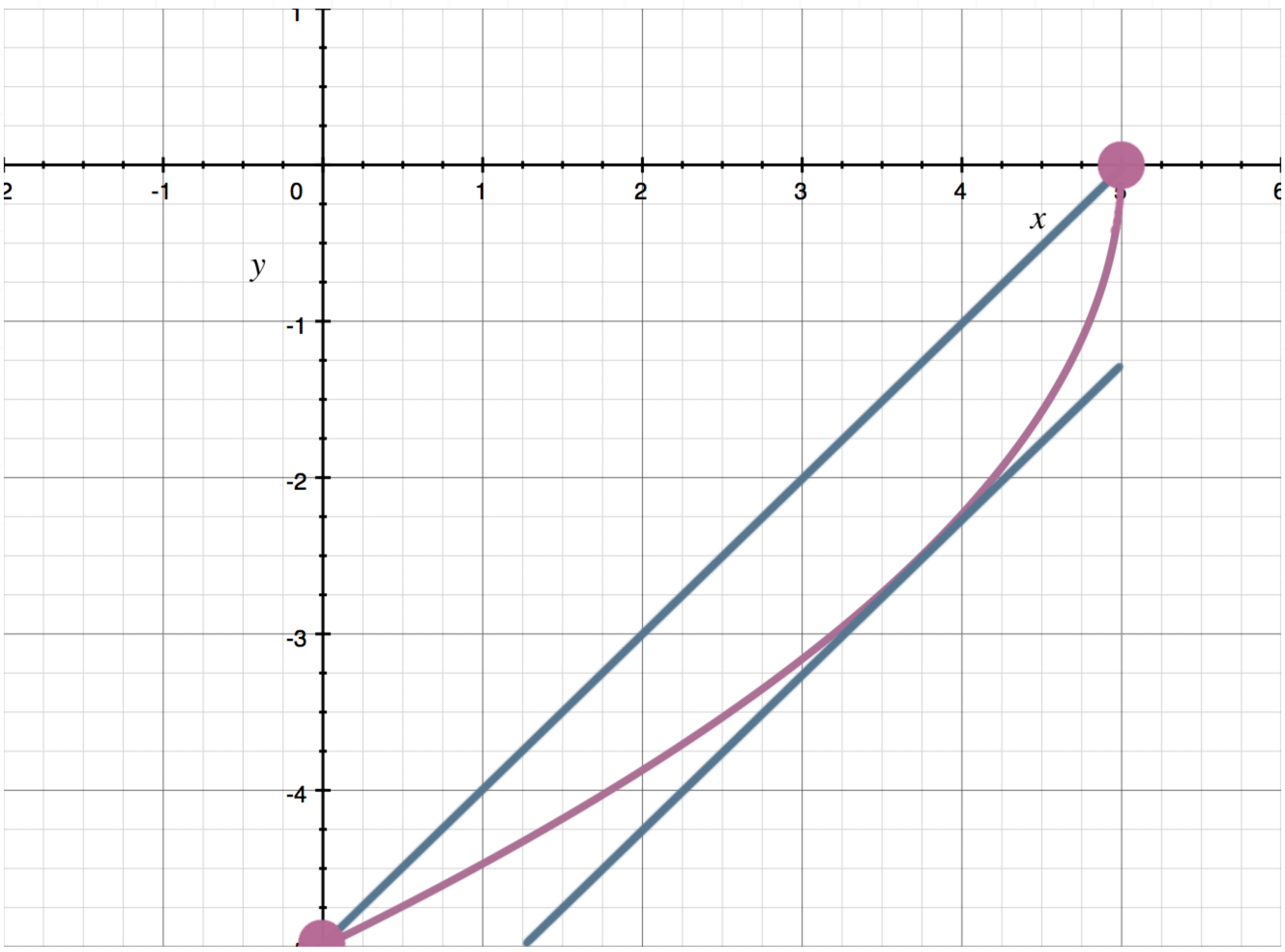
$$x = \left(\frac{25}{4}-25\right) \div -5$$

$$x = \frac{15}{4}$$

Verify that the slope of the tangent line at this x-value is 1.

$$h'\left(\frac{15}{4}\right) = \frac{5}{2\sqrt{25 - 5\left(\frac{15}{4}\right)}} = \frac{5}{2\sqrt{\frac{25}{4}}} = \frac{5}{2\left(\frac{5}{2}\right)} = \frac{5}{5} = 1$$

Therefore, the value of c is $15/4$. The figure illustrates how this point satisfies the Mean Value Theorem.



ROLLE'S THEOREM

1. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-1,2]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$f(x) = x^3 - 2x^2 - x - 3$$

Solution:

The function $f(x)$ is continuous and differentiable on the interval $[-1,2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-1,2]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(2) = f(-1)$.

$$f(2) = 2^3 - 2(2)^2 - 2 - 3 = -5$$

$$f(-1) = (-1)^3 - 2(-1)^2 - (-1) - 3 = -5$$

Thus, Rolle's Theorem applies. Next, find $f'(x) = 3x^2 - 4x - 1$ and set $f'(x) = 0$ and solve for x using the quadratic formula.

$$3x^2 - 4x - 1 = 0$$

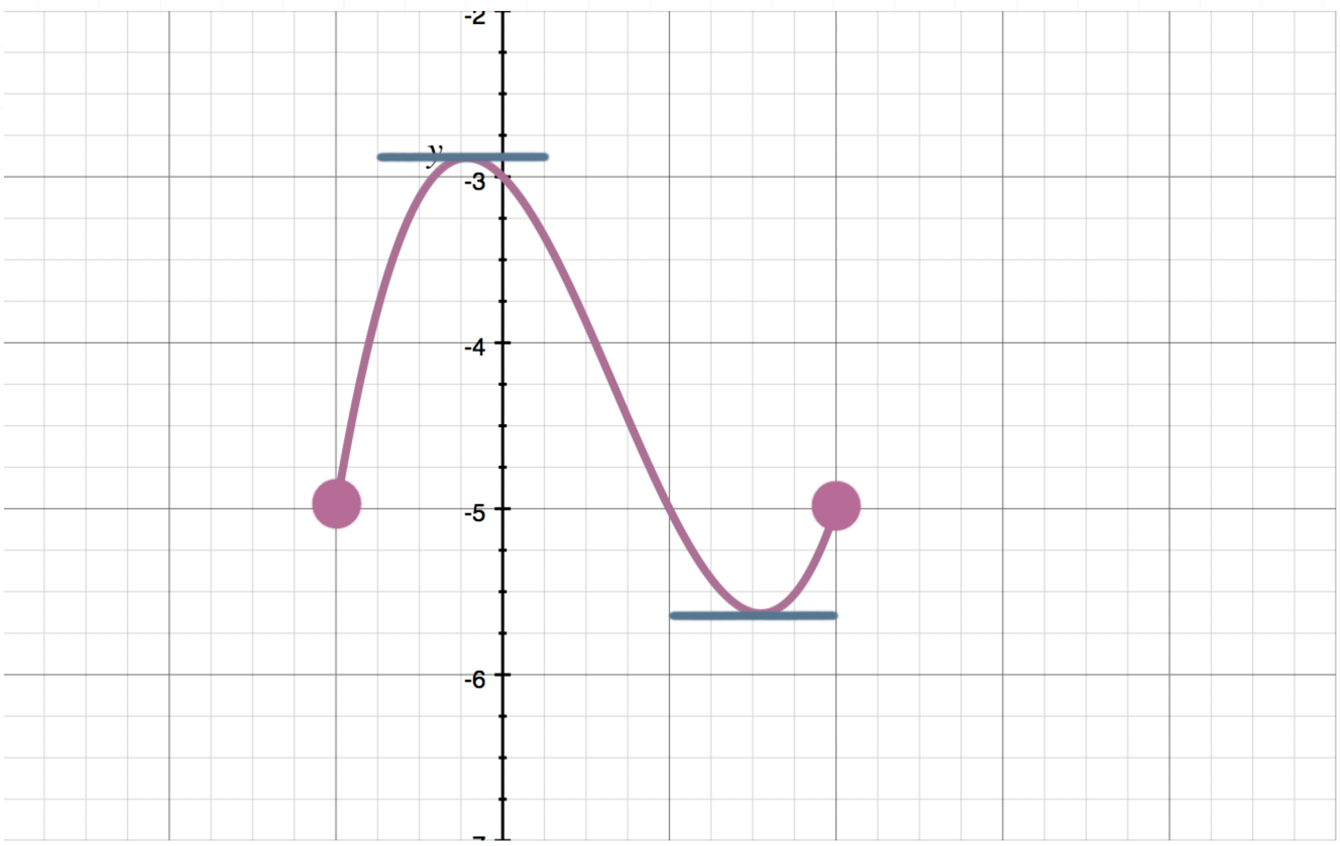
$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-1)}}{2(3)} = \frac{4 \pm \sqrt{28}}{6} = \frac{4 \pm 2\sqrt{7}}{6} = \frac{2 \pm \sqrt{7}}{3}$$

Verify that the slope of the tangent line at these two x-values is 0.

$$f'\left(\frac{2-\sqrt{7}}{3}\right)=3\left(\frac{2-\sqrt{7}}{3}\right)^2-4\left(\frac{2-\sqrt{7}}{3}\right)-1=0$$

$$f'\left(\frac{2+\sqrt{7}}{3}\right)=3\left(\frac{2+\sqrt{7}}{3}\right)^2-4\left(\frac{2+\sqrt{7}}{3}\right)-1=0$$

Therefore, the values of c such that $f'(c)=0$ are $(2\pm\sqrt{7})/3$. The figure illustrates how these two points satisfy Rolle's Theorem.



■ 2. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-3,5]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$g(x)=\frac{x^2-2x-15}{6-x}$$

Solution:

The function $g(x)$ is continuous and differentiable on the interval $[-3,5]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-3,5]$, such that $g'(c) = 0$.

To use Rolle's Theorem, show that $g(5) = g(-3)$.

$$g(5) = \frac{5^2 - 2(5) - 15}{6 - 5} = \frac{0}{1} = 0$$

$$g(-3) = \frac{(-3)^2 - 2(-3) - 15}{6 - (-3)} = \frac{0}{9} = 0$$

Thus, Rolle's Theorem applies. Next, find

$$g'(x) = \frac{(6 - x)(2x - 2) - (x^2 - 2x - 15)(-1)}{(6 - x)^2} = \frac{-x^2 + 12x - 27}{(6 - x)^2}$$

and set $g'(x) = 0$ and solve for x using the quadratic formula.

$$-x^2 + 12x - 27 = 0$$

$$-(x^2 - 12x + 27) = 0$$

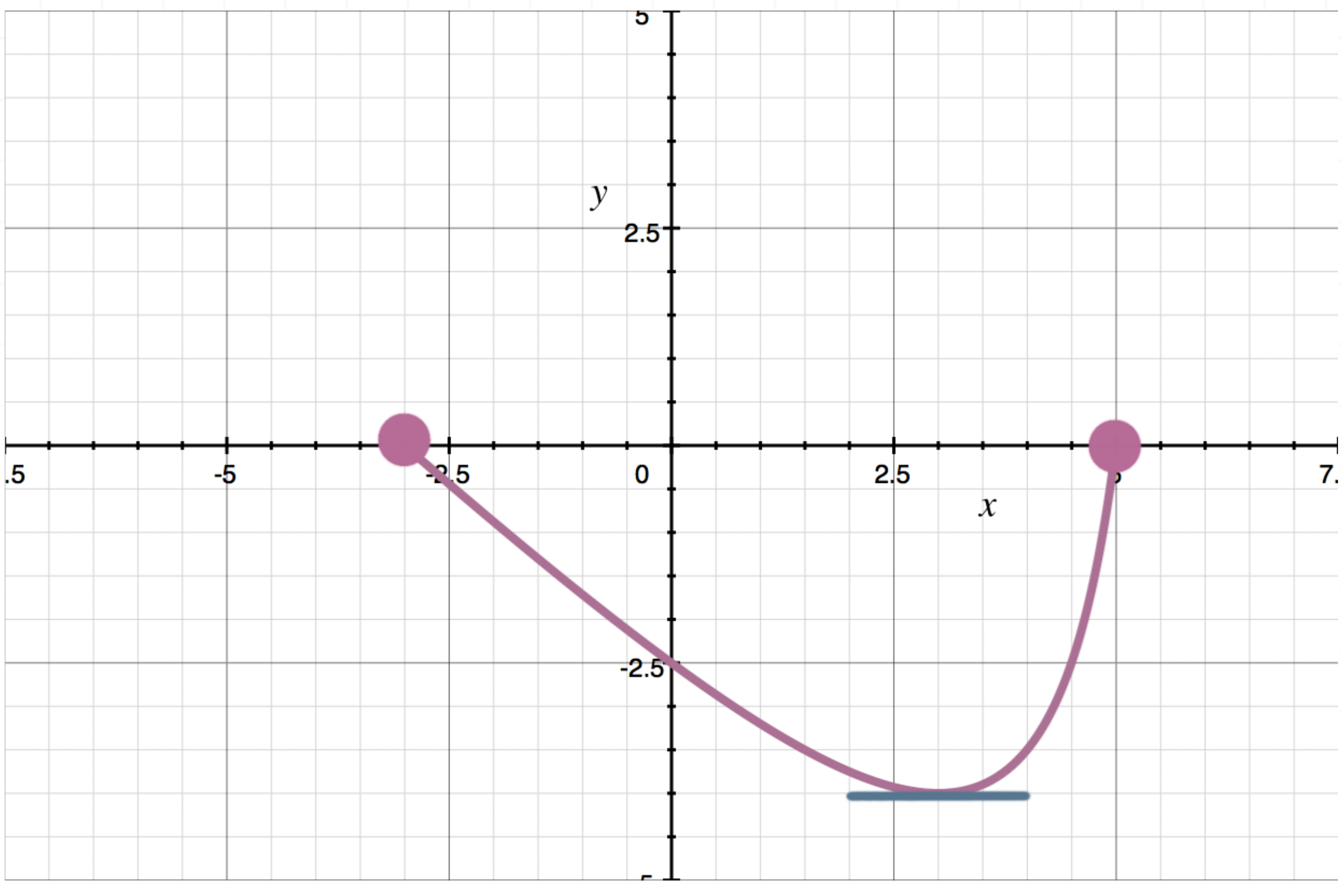
$$-(x - 3)(x - 9) = 0$$

$$x = 3,\ 9$$

The value $x = 9$ is outside of the given interval. Verify that the slope of the tangent line at $x = 3$ is 0.

$$g'(3) = \frac{-3^2 + 12(3) - 27}{(6-3)^2} = \frac{0}{9} = 0$$

Therefore, the value of c such that $f'(c) = 0$ is 3. The figure illustrates how this point satisfies Rolle's Theorem.



■ 3. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-\pi/2, \pi/2]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$h(x) = \sin(2x)$$

Solution:

The function $h(x)$ is continuous and differentiable on the interval $[-\pi/2,\pi/2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-\pi/2,\pi/2]$, such that $h'(c)=0$.

To use Rolle's Theorem, show that $h(\pi/2)=h(-\pi/2)$.

$$h\left(\frac{\pi}{2}\right)=\sin\left(2\cdot\frac{\pi}{2}\right)=\sin(\pi)=0$$

$$h\left(-\frac{\pi}{2}\right)=\sin\left(2\cdot-\frac{\pi}{2}\right)=\sin(-\pi)=0$$

Thus, Rolle's Theorem applies. Next, find $h'(x)=2\cos(2x)$ and set $h'(x)=0$ and solve for x.

$$2\cos(2x)=0$$

$$\cos(2x)=0$$

$$\arccos(0)=2x$$

$$2x=\pm\frac{\pi}{2}$$

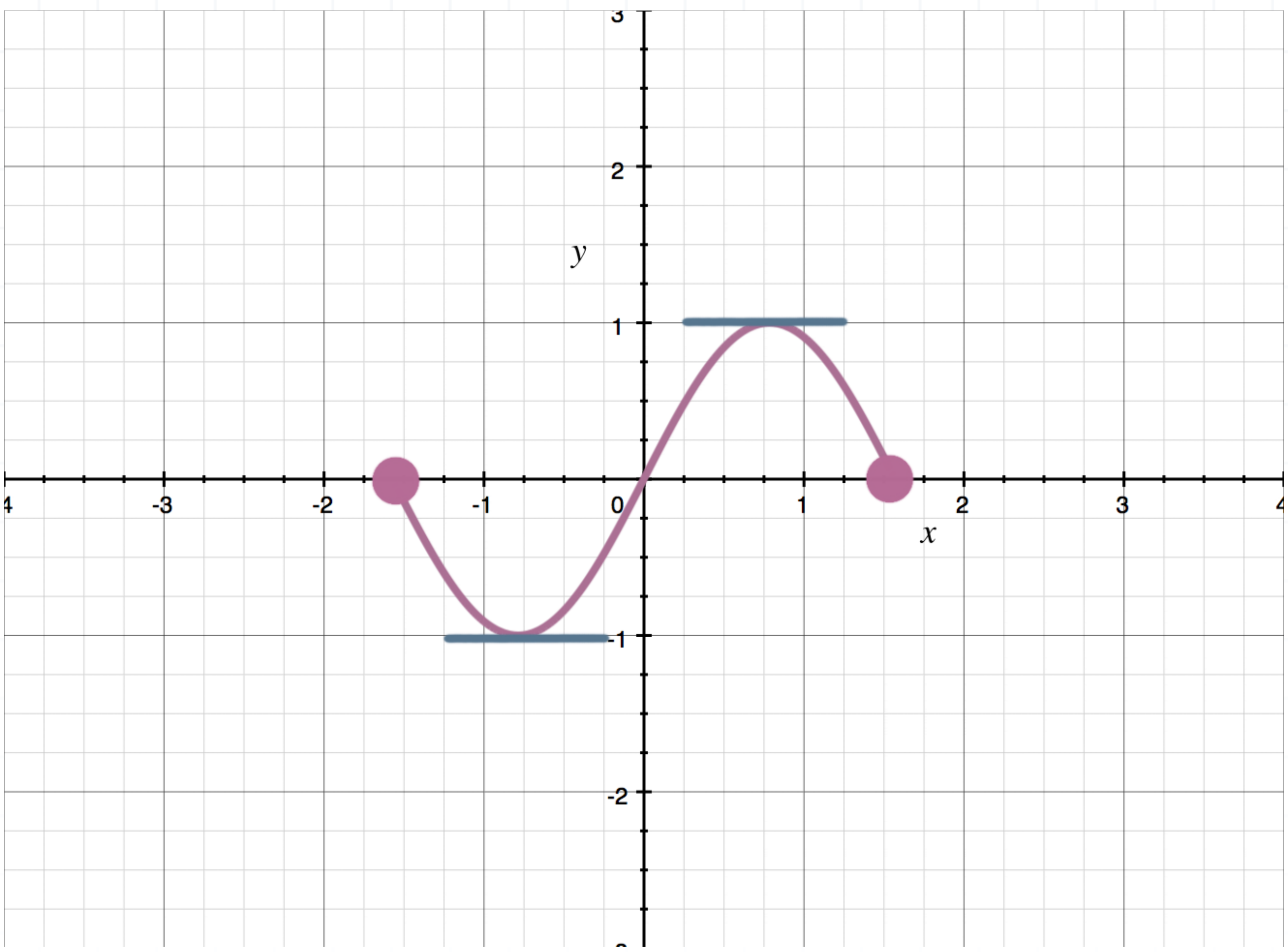
$$x=\pm\frac{\pi}{4}$$

Verify that the slope of the tangent line at these two x-values is 0.

$$h'\left(-\frac{\pi}{4}\right)=2\cos\left(-\frac{\pi}{2}\right)=2\cdot 0=0$$

$$h'\left(\frac{\pi}{4}\right)=2\cos\left(\frac{\pi}{2}\right)=2\cdot 0=0$$

Therefore, the values of c such that $f'(c) = 0$ are $\pm\pi/4$. The figure illustrates how these two points satisfy Rolle's Theorem.



NEWTON'S METHOD

■ 1. Use four iterations of Newton's method to approximate the root of $g(x) = x^3 - 12$ in the interval $[1,3]$. Give the answer to the nearest three decimal places.

Solution:

If $g(x) = x^3 - 12$ and $g'(x) = 3x^2$, and we start with an initial estimate of $x_0 = 1.5$, then $g(1.5) = -8.625$ and $g'(1.5) = 6.75$. Plug those values into the Newton's method formula.

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(n_n)}$$

$$x_1 = 1.5 - \frac{-8.625}{6.75} \approx 2.777$$

Next, $g(2.777) = 9.4155$ and $g'(2.777) = 23.1352$. So

$$x_2 = 2.777 - \frac{9.4155}{23.1352} = 2.3700$$

Next, $g(2.3700) = 1.3121$ and $g'(2.3700) = 16.851$. So

$$x_3 = 2.3700 - \frac{1.3121}{16.851} = 2.2921$$

Next, $g(2.2921) = 0.04206$ and $g'(2.2921) = 15.7612$. So

$$x_4 = 2.2921 - \frac{0.04206}{15.7612} = 2.2894$$

■ 2. Use four iterations of Newton's method to approximate the root of $f(x) = x^4 - 15$ in the interval $[-2, -1]$. Give the answer to the nearest four decimal places.

Solution:

If $f(x) = x^4 - 14$ and $f'(x) = 4x^3$, and we start with an initial estimate of $x_0 = -1.5$, then $f(-1.5) = -9.8375$ and $f'(-1.5) = -13.5$. Plug those values into the Newton's method formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(n_n)}$$

$$x_1 = -1.5 - \frac{-9.9375}{-13.5} = -2.2361$$

Next, $f(-2.2361) = 10.0014$ and $f'(-2.2361) = -44.7233$. So

$$x_2 = -2.2361 - \frac{10.0014}{-44.7233} = -2.0125$$

Next, $f(-2.0125) = 1.4038$ and $f'(-2.0125) = -32.6038$. So

$$x_3 = -2.0125 - \frac{1.4038}{-32.6038} = -1.9694$$

Next, $f(-1.9694) = 0.0434$ and $f'(-1.9694) = -30.5536$. So

$$x_4 = -1.9694 - \frac{0.0434}{-30.5536} = -1.9680$$

■ 3. Use four iterations of Newton's method to approximate the root of $h(x) = 3e^{x-3} - 4 + \sin x$ in the interval $[2,4]$. Give the answer to the nearest four decimal places.

Solution:

If $h(x) = 3e^{x-3} - 4 + \sin x$ and $h'(x) = 3e^{x-3} + \cos x$, and we start with an initial estimate of $x_0 = 3$, then $h(3) = -0.8589$ and $h'(3) = 2.0100$. Plug those values into the Newton's method formula.

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(n_n)}$$

$$x_1 = 3 - \frac{-0.8589}{2.0100} = 3.4273$$

Next, $h(3.4273) = 0.3175$ and $h'(3.4273) = 3.6399$. So

$$x_2 = 3.4273 - \frac{0.3175}{3.6399} = 3.3401$$

Next, $h(3.3401) = 0.0181$ and $h'(3.3401) = 3.2349$. So

$$x_3 = 3.3401 - \frac{0.0181}{3.2349} = 3.3345$$

Next, $h(3.3345) = 0.00001$ and $h'(3.3345) = 3.2103$. So

$$x_4 = 3.3345 - \frac{0.00001}{3.2103} = 3.3345$$

L'HOSPITAL'S RULE

1. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x\to 0}\frac{2\sqrt{x+4}-4-\frac{1}{2}x}{x^2}$$

Solution:

Evaluating the limit as $x \to 0$ gives the indeterminate form $0/0$, so we'll use L'Hospital's rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x\to 0}\frac{\frac{1}{\sqrt{x+4}}-\frac{1}{2}}{2x}$$

But evaluating this $x \to 0$ still gives $0/0$, so we'll apply L'Hospital's rule again.

$$\lim_{x\to 0}\frac{-\frac{1}{2\sqrt{(x+4)^3}}}{2}=\lim_{x\to 0}-\frac{1}{4\sqrt{(x+4)^3}}$$

Then we can evaluate as $x \to 0$.

$$-\frac{1}{4\sqrt{(0+4)^3}}=-\frac{1}{4\sqrt{64}}=-\frac{1}{4(8)}=-\frac{1}{32}$$

■ 2. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{3 + \tan x}$$

Solution:

Evaluating the limit as $x \to \pi/2$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to \frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x} \cdot \frac{\cos x}{1} = \lim_{x \to \frac{\pi}{2}} \sin x$$

Then we can evaluate as $x \to \pi/2$.

$$\sin \frac{\pi}{2} = 1$$

■ 3. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \infty} \frac{\ln x}{4\sqrt{x}}$$

Solution:

Evaluating the limit as $x \to \infty$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{2}{\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{\sqrt{x}}{2} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}}$$

Then we can evaluate as $x \to \infty$.

$$\lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0$$

POSITION, VELOCITY, AND ACCELERATION

1. Find the velocity $v(t)$, speed, and acceleration $a(t)$ at $t = 2$ of the position function.

$$s(t) = -\frac{t^3}{3} + t^2 + 3t - 1$$

Solution:

Velocity is given by the first derivative of the position function.

$$s'(t) = v(t) = -t^2 + 2t + 3$$

$$v(2) = -(2)^2 + 2(2) + 3$$

$$v(2) = 3$$

Acceleration is given by the second derivative of the position function.

$$s''(t) = v'(t) = a(t) = -2t + 2$$

$$a(2) = -2(2) + 2$$

$$a(2) = -2$$

Speed is the absolute value of velocity. So speed is

$$|v(2)| = |3| = 3$$

2. Find the velocity $v(t)$, speed, and acceleration $a(t)$ at $t = 1$ of the position function.

$$s(t) = \frac{t^2 - 3}{t^3}$$

Solution:

Velocity is given by the first derivative of the position function.

$$s'(t) = v(t) = \frac{(t^3)(2t) - (t^2 - 3)(3t^2)}{(t^3)^2} = \frac{2t^4 - 3t^4 + 9t^2}{t^6} = \frac{-t^4 + 9t^2}{t^6} = \frac{-t^2 + 9}{t^4}$$

$$v(1) = \frac{-1^2 + 9}{1^4} = -1 + 9$$

$$v(1) = 8$$

Acceleration is given by the second derivative of the position function.

$$s''(t) = v'(t) = a(t) = \frac{(-2t)(t^4) - (-t^2 + 9)(4t^3)}{(t^4)^2} = \frac{-2t^5 + 4t^5 - 36t^3}{t^8} = \frac{2t^2 - 36}{t^5}$$

$$a(1) = \frac{2(1)^2 - 36}{1^5} = 2 - 36$$

$$a(1) = -34$$

Speed is the absolute value of velocity. So speed is

$$|v(1)| = |8| = 8$$

■ 3. Find the velocity $v(t)$, speed, and acceleration $a(t)$ at $t = 4$ of the position function.

$$s(t) = \frac{t^2}{2t+4}$$

Solution:

Velocity is given by the first derivative of the position function.

$$s'(t) = v(t) = \frac{(2t)(2t+4) - (t^2)(2)}{(2t+4)^2} = \frac{4t^2 + 8t - 2t^2}{4t^2 + 16t + 16} = \frac{t^2 + 4t}{2t^2 + 8t + 8} = \frac{t(t+4)}{2(t+2)(t+2)}$$

$$v(4) = \frac{4(4+4)}{2(4+2)(4+2)} = \frac{4(8)}{2(6)(6)}$$

$$v(4) = \frac{32}{72} = \frac{4}{9}$$

Acceleration is given by the second derivative of the position function.

$$s''(t) = v'(t) = a(t) = \frac{(2t+4)(2t^2+8t+8) - (t^2+4t)(4t+8)}{(2t^2+8t+8)^2}$$

$$a(t) = \frac{16t+32}{(2t^2+8t+8)^2} = \frac{16(t+2)}{4(t^2+4t+4)^2} = \frac{4(t+2)}{(t+2)^4} = \frac{4}{(t+2)^3}$$

$$a(4) = \frac{4}{(4+2)^3} = \frac{4}{216}$$

$$a(4) = \frac{1}{54}$$

Speed is the absolute value of velocity. So speed is

$$|v(4)| = \left|\frac{4}{9}\right| = \frac{4}{9}$$

BALL THROWN UP FROM THE GROUND

1. A ball is thrown straight upward from the ground with an initial velocity of $v_0 = 86$ ft/sec. Assuming constant gravity, find the maximum height, in feet, that the ball attains, the time, in seconds, that it's in the air, as well as the ball's velocity, in ft/sec, when it hits the ground.

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = \frac{1}{2}gt^2 - v_0 t - y_0$$

$$h(t) = -\frac{1}{2}(32)t^2 + 86t + 0$$

$$h(t) = -16t^2 + 86t$$

When the ball is at its maximum height, velocity is 0, so find $h'(t)$ and set it equal to 0.

$$h'(t) = -32t + 86$$

$$-32t + 86 = 0$$

$$32t = 86$$

$$t = \frac{43}{16} \approx 2.69 \text{ seconds}$$

Next, find the maximum height.

$$h(t) = -16\left(\frac{43}{16}\right)^2 + 86\left(\frac{43}{16}\right) = \frac{1{,}849}{16} \approx 115.56 \text{ feet}$$

To find the time the ball stays in the air, set the height equal to 0 and solve for t.

$$h(t) = -16t^2 + 86t$$

$$-16t^2 + 86t = 0$$

$$t(43 - 8t) = 0$$

$$t = 0, \frac{43}{8} \approx 5.38 \text{ seconds}$$

Now, find the final velocity of the ball when it hits the ground. Substitute the time the ball lands into the velocity function.

$$h'\left(\frac{43}{8}\right) = -32\left(\frac{43}{8}\right) + 86 = -86 \text{ ft/sec}$$

■ 2. A ball is thrown straight upward from the top of a building, which is 56 feet above the ground, with an initial velocity of $v_0 = 48$ ft/sec. Assuming constant gravity, find the maximum height, in feet, that the ball attains, the time, in seconds, that it's in the air, as well as the ball's velocity, in ft/sec, when it hits the ground.

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = \frac{1}{2}gt^2 - v_0t - y_0$$

$$h(t) = -\frac{1}{2}(32)t^2 + 48t + 56$$

$$h(t) = -16t^2 + 48t + 56$$

When the ball is at its maximum height, velocity is 0, so find $h'(t)$ and set it equal to 0.

$$h'(t) = -32t + 48$$

$$-32t + 48 = 0$$

$$32t = 48$$

$$t = \frac{3}{2} = 1.5 \text{ seconds}$$

Next, find the maximum height.

$$h(t) = -16\left(\frac{3}{2}\right)^2 + 48\left(\frac{3}{2}\right) + 56 = 92 \text{ feet}$$

To find the time the ball stays in the air, set the height equal to 0 and solve for t.

$$-16t^2 + 48t + 56 = 0$$

$$2t^2 - 6t - 7 = 0$$

$$t = \frac{3 + \sqrt{23}}{2} \approx 3.90 \text{ seconds}$$

Now, find the final velocity of the ball when it hits the ground. Substitute the time the ball lands into the velocity function.

$$h'\left(\frac{3+\sqrt{23}}{2}\right) = -32\left(\frac{3+\sqrt{23}}{2}\right) + 48 \approx -76.73 \text{ ft/sec}$$

■ 3. A ball is thrown straight upward from a bridge, which is 24 meters above the water, with an initial velocity of $v_0 = 20$ m/sec. Assuming constant gravity, find the maximum height, in meters, that the ball attains, the time, in seconds, that it's in the air, as well as the ball's velocity, in m/sec, when it hits the water below.

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = \frac{1}{2}gt^2 - v_0t - y_0$$

$$h(t) = -\frac{1}{2}(9.8)t^2 + 20t + 24$$

$$h(t) = -4.9t^2 + 20t + 24$$

When the ball is at its maximum height, velocity is 0, so find $h'(t)$ and set it equal to 0.

$$h'(t) = -9.8t + 20$$

$$-9.8t + 20 = 0$$

$$9.8t = 20$$

$$t = \frac{20}{9.8} \approx 2.041 \text{ seconds}$$

Next, find the maximum height.

$$h(t) = -4.9\left(\frac{100}{49}\right)^2 + 20\left(\frac{100}{49}\right) + 24 \approx 44.41 \text{ meters}$$

To find the time the ball stays in the air, set the height equal to 0 and solve for t.

$$h(t) = -4.9t^2 + 20t + 24$$

$$-4.9t^2 + 20t + 24 = 0$$

$$t \approx 5.05 \text{ seconds}$$

Now, find the final velocity of the ball when it hits the water. Substitute the time the ball lands into the velocity function.

$$h'(5.05) = -9.8(5.05) + 20 \approx -29.5 \text{ m/sec}$$

COIN DROPPED FROM THE ROOF

■ 1. A rock is dropped from the top of an 800 foot tall cliff, with an initial velocity of $v_0 = 0$ ft/sec. Assuming constant gravity, when does the rock hit the ground, and what is its velocity when it hits the ground?

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = \frac{1}{2}gt^2 - v_0t - y_0$$

$$y(t) = \frac{1}{2}(32)t^2 - 0t - 800$$

$$y(t) = 16t^2 - 800$$

The rock hits the ground when its height is 0.

$$16t^2 - 800 = 0$$

$$16t^2 = 800$$

$$t^2 = 50$$

$$t = \sqrt{50} \approx 7 \text{ seconds}$$

To find the velocity of the rock when it hits the ground, find $y'(t)$ and evaluate it at the time the rock hits the ground.

$$y'(t) = 32t$$

$$y'(7.071) = 32(7.071) \approx 226.27 \text{ ft/sec}$$

However, since the rock is falling downward, the correct expression for the velocity is -226.27 ft/sec.

■ 2. A rock is tossed from the top of a 300 foot tall cliff, with an initial velocity of $v_0 = 15$ ft/sec. Assuming constant gravity, when does the rock hit the ground, and what is its velocity when it hits the ground?

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = \frac{1}{2}gt^2 - v_0t - y_0$$

$$y(t) = \frac{1}{2}(32)t^2 - 15t - 300$$

$$y(t) = 16t^2 - 15t - 300$$

The rock hits the ground when its height is 0.

$$16t^2 - 15t - 300 = 0$$

$$t = \frac{15 \pm \sqrt{15^2 - 4(16)(-300)}}{2(16)} = \frac{15 \pm 5\sqrt{777}}{32} \approx 4.8242 \text{ seconds}$$

To find the velocity of the rock when it hits the ground, find $y'(t)$ and evaluate it at the time the rock hits the ground.

$$y'(t) = 32t - 15$$

$$y'(4.8242) = 32(4.8242) - 15 \approx 139.37 \text{ ft/sec}$$

However, since the rock is falling downward, the correct expression for the velocity is -139.37 ft/sec.

■ 3. A coin is tossed downward from the top of a 36 meter tall building, with an initial velocity of $v_0 = 6$ m/sec. Assuming constant gravity, when does the rock hit the ground, and what is its velocity when it hits the ground?

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = \frac{1}{2}gt^2 - v_0t - y_0$$

$$y(t) = \frac{1}{2}(9.8)t^2 - 6t - 36$$

$$y(t) = 4.9t^2 - 6t - 36$$

The rock hits the ground when its height is 0.

$$4.9t^2 - 6t - 36 = 0$$

$$t = \frac{6 \pm \sqrt{6^2 - 4(4.9)(-36)}}{2(4.9)} = \frac{6 \pm \sqrt{741.6}}{9.8} \approx 3.391 \text{ seconds}$$

To find the velocity of the rock when it hits the ground, find $y'(t)$ and evaluate it at the time the rock hits the ground.

$$y'(t) = 9.8t - 6$$

$$y'(3.391) = 9.8(3.391) - 6 \approx 27.23 \text{ m/sec}$$

However, since the rock is falling downward, the correct expression for the velocity is -27.23 m/sec.

MARGINAL COST, REVENUE, AND PROFIT

■ 1. A company manufactures and sells basketballs for $\$9.50$ each. The company has a fixed cost of $\$395$ per week and a variable cost of $\$2.75$ per basketball. The company can make up to 300 basketballs per week. Find the marginal cost, marginal revenue, and marginal profit, if the company makes 150 basketballs.

Solution:

The cost function is $C(x) = 395 + 2.75x$, where x is the number of basketballs, so marginal cost is $C'(x) = 2.75$, and $C'(150) = \$2.75$.

The revenue function is $R(x) = 9.50x$, where x is the number of basketballs, so marginal revenue is $R'(x) = 9.50$, and $R'(150) = \$9.50$.

The profit function is

$$P(x) = R(x) - C(x)$$

$$P(x) = 9.50x - (395 + 2.75x)$$

$$P(x) = 6.75x - 395$$

Marginal profit is $P'(x) = 6.75$, and $P'(150) = \$6.75$.

2. A company manufactures and sells high end folding tables for $\$250$ each. The company has a fixed cost of $\$3{,}000$ per week and variable costs of $85x + 150\sqrt{x}$, where x is the number of tables manufactured. The company can make up to 200 tables per week. Find the marginal cost, marginal revenue, and marginal profit, if the company makes 64 tables.

Solution:

The cost function is $C(x) = 3{,}000 + 85x + 150\sqrt{x}$, where x is the number of folding tables, so marginal cost is $C'(x) = 85 + 75/\sqrt{x}$, and $C'(64) = 85 + 75/\sqrt{64} = 85 + 9.375 = \$94.375$.

The revenue function is $R(x) = 250x$, where x is the number of folding tables, so marginal revenue is $R'(x) = 250$, and $R'(64) = \$250$.

The profit function is

$$P(x) = R(x) - C(x)$$

$$P(x) = 250x - (3{,}000 + 85x + 150\sqrt{x})$$

$$P(x) = 165x - 150\sqrt{x} - 3{,}000$$

Marginal profit is $P'(x) = 165 - 75/\sqrt{x}$, and

$$P'(64) = 165 - \frac{75}{\sqrt{64}}$$

$$P'(64) = 165 - 9.375$$

$$P'(64) = \$155.63$$

3. A company manufactures and sells electric food mixers for $\$150$ each. The company has a fixed cost of $\$7,800$ per week and variable costs of $24x + 0.04x^2$, where x is the number of mixers manufactured. The company can make up to 200 mixers per week. Find the marginal cost, marginal revenue, and marginal profit, if the company makes 75 mixers.

Solution:

The cost function is $C(x) = 7,800 + 24x + 0.04x^2$, where x is the number of food mixers, so marginal cost is $C'(x) = 24 + 0.08x$, and $C'(75) = 24 + 0.08(75) = \$30$.

The revenue function is $R(x) = 150x$, where x is the number of food mixers, so marginal revenue is $R'(x) = 150$, and $R'(75) = \$150$.

The profit function is

$$P(x) = R(x) - C(x)$$

$$P(x) = 150x - (7,800 + 24x + 0.04x^2)$$

$$P(x) = 126x - 0.04x^2 - 7,800$$

Marginal profit is $P'(x) = 126 - 0.08x$, and

$$P'(75) = 126 - 0.08(75)$$

$$P'(75) = 126 - 6$$

$$P'(75) = \$120$$

HALF LIFE

■ 1. Find the half-life of Tritium if its decay constant is 0.0562.

Solution:

Since we're calculating half-life, the exponential decay formula $y = y_0 e^{-rt}$ can be simplified to

$$\frac{1}{2} = e^{-rt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.0562t}$$

$$\ln\frac{1}{2} = \ln e^{-0.0562t}$$

$$-\ln 2 = -0.0562t$$

$$t = \frac{-\ln 2}{-0.0562} = \frac{\ln 2}{0.0562} \approx 12.33 \text{ years}$$

■ 2. Find the half-life of Cobalt-60 if its decay constant is 0.1315.

Solution:

Since we're calculating half-life, the exponential decay formula $y = y_0e^{-rt}$ can be simplified to

$$\frac{1}{2} = e^{-rt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.1315t}$$

$$\ln\frac{1}{2} = \ln e^{-0.1315t}$$

$$-\ln 2 = -0.1315t$$

$$t = \frac{-\ln 2}{-0.1315} = \frac{\ln 2}{0.1315} \approx 5.27 \text{ years}$$

■ 3. Find the half-life of Berkelium-97 if its decay constant is 0.000503.

Solution:

Since we're calculating half-life, the exponential decay formula $y = y_0e^{-rt}$ can be simplified to

$$\frac{1}{2} = e^{-rt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.000503t}$$

$$\ln\frac{1}{2} = \ln e^{-0.000503t}$$

$$-\ln 2 = -0.000503t$$

$$t = \frac{-\ln 2}{-0.000503} = \frac{\ln 2}{0.000503} \approx 1{,}378 \text{ years}$$

NEWTON'S LAW OF COOLING

1. A cup of coffee is 195° F when it's brewed. Room temperature is 74° F. If the coffee is 180° F after 5 minutes, to the nearest degree, how hot is the coffee after 25 minutes?

Solution:

Use the information given and the temperature after 5 minutes to solve for k in the Newton's Law of Cooling formula.

$$T - T_s = (T_0 - T_s)e^{-kt}$$

$$180 - 74 = (195 - 74)e^{-5k}$$

$$106 = 121e^{-5k}$$

$$\frac{106}{121} = e^{-5k}$$

$$\ln\frac{106}{121} = \ln e^{-5k}$$

$$\ln\frac{106}{121} = -5k$$

$$k = -\frac{1}{5}\ln\frac{106}{121} \approx 0.02647$$

Then use k to solve for T.

$$T - 74 = (195 - 74)e^{-0.02647(25)}$$

$$T - 74 = 121e^{-0.66175}$$

$$T - 74 = 62.42966$$

$$T = 136.42966$$

The coffee is approximately 136° F after 25 minutes.

2. A boiled egg that's 99° C is placed in a pan of water that's 24° C. If the egg is 62° C after 5 minutes, how much longer, to the nearest minute, will it take the egg to reach 32° C.

Solution:

Use the information given and the temperature after 5 minutes to solve for k in the Newton's Law of Cooling formula.

$$T - T_s = (T_0 - T_s)e^{-kt}$$

$$62 - 24 = (99 - 24)e^{-5k}$$

$$38 = 75e^{-5k}$$

$$\frac{38}{75} = e^{-5k}$$

$$\ln \frac{38}{75} = \ln e^{-5k}$$

$$\ln \frac{38}{75} = -5k$$

$$k = -\frac{1}{5} \ln \frac{38}{75} \approx 0.13598$$

Then use k to solve for t.

$$32 - 24 = (62 - 24)e^{-0.13598t}$$

$$8 = 38e^{-0.13598t}$$

$$\frac{8}{38} = e^{-0.13598t}$$

$$\ln \frac{8}{38} = \ln e^{-0.13598t}$$

$$\ln \frac{8}{38} = -0.13598t$$

$$t = \frac{1}{-0.13598} \ln \frac{8}{38} \approx 11.4662$$

The egg will be 32° C after about 11 and a half more minutes.

■ 3. Suppose a cup of soup cooled from 200° F to 161° F in 10 minutes in a room whose temperature is 68° F. How much longer will it take for the soup to cool to 105° F?

Solution:

Use the information given and the temperature after 10 minutes to solve for k in the Newton's Law of Cooling formula.

$$T - T_s = (T_0 - T_s)e^{-kt}$$

$$161 - 68 = (200 - 68)e^{-10k}$$

$$93 = 132e^{-10k}$$

$$\frac{93}{132} = e^{-10k}$$

$$\ln\frac{93}{132} = \ln e^{-10k}$$

$$\ln\frac{93}{132} = -10k$$

$$k = -\frac{1}{10}\ln\frac{93}{132} \approx 0.03502$$

Then use k to solve for t.

$$105 - 68 = (161 - 68)e^{-0.03502t}$$

$$37 = 93e^{-0.03502t}$$

$$\frac{37}{93} = e^{-0.03502t}$$

$$\ln\frac{37}{93} = \ln e^{-0.03502t}$$

$$\ln\frac{37}{93} = -0.03502t$$

$$t = \frac{1}{-0.03502} \ln \frac{37}{93} \approx 26.3187$$

The egg will be 105° F after about 26 more minutes.

SALES DECLINE

■ 1. Suppose a pizza company stops a special sale for their three-topping pizza. They will resume the sale if sales drop to 70% of the current sales level. If sales decline to 90% during the first week, when should the company expect to start the special sale again?

Solution:

Use the exponential function $FV = PVe^{-rt}$. Plug in what we know.

$$FV = PVe^{-rt}$$

$$90 = 100e^{-r(1)}$$

$$\frac{90}{100} = e^{-r}$$

$$\ln \frac{90}{100} = \ln e^{-r}$$

$$\ln 90 - \ln 100 = -r$$

$$r = \ln 100 - \ln 90 \approx 0.10536$$

Find t using a sales level of 70% and $r = 0.10536$.

$$70 = 100e^{-0.10536t}$$

$$\frac{70}{100} = e^{-0.10536t}$$

$$\ln \frac{70}{100} = \ln e^{-0.10536t}$$

$$\ln 70 - \ln 100 = -0.10536t$$

$$t = \frac{\ln 70 - \ln 100}{-0.10536} \approx 3.385$$

Since time t is in weeks, this means the company should expect to start the sale again in about 3 and a half weeks.

■ 2. Suppose a donut store experiments with raising the price of a dozen donuts to see if sales are affected. They'll resume the sale if sales drop to 80% of the current sales level. If sales decline to 90% after two weeks, when should the store change back to the original price?

Solution:

Use the exponential function $FV = PVe^{-rt}$. Plug in what we know.

$$FV = PVe^{-rt}$$

$$90 = 100e^{-r(2)}$$

$$\frac{90}{100} = e^{-2r}$$

$$\ln \frac{90}{100} = \ln e^{-2r}$$

$$\ln 90 - \ln 100 = -2r$$

$$r = \frac{\ln 90 - \ln 100}{-2} \approx 0.05268$$

Find t using a sales level of 80% and $r = 0.05268$.

$$80 = 100e^{-0.05268t}$$

$$\frac{80}{100} = e^{-0.05268t}$$

$$\ln \frac{80}{100} = \ln e^{-0.05268t}$$

$$\ln 80 - \ln 100 = -0.05268t$$

$$t = \frac{\ln 80 - \ln 100}{-0.05268} \approx 4.2358$$

Since time t is in weeks, this means the store should expect to change back to the original price in about 4 and a quarter weeks.

■ 3. Suppose a flower shop decides to stop ordering roses in the winter time to see if sales are affected. They will resume the sale if sales drop to 90% of the current sales level. If sales decline to 96% after three weeks, when should the shop begin ordering roses again?

Solution:

Use the exponential function $FV = PVe^{-rt}$. Plug in what we know.

$$FV = PVe^{-rt}$$

$$96 = 100e^{-r(3)}$$

$$\frac{96}{100} = e^{-3r}$$

$$\ln \frac{96}{100} = \ln e^{-3r}$$

$$\ln 96 - \ln 100 = -3r$$

$$r = \frac{\ln 96 - \ln 100}{-3} \approx 0.01361$$

Find t using a sales level of 90% and $r = 0.01361$.

$$90 = 100e^{-0.01361t}$$

$$\frac{90}{100} = e^{-0.01361t}$$

$$\ln \frac{90}{100} = \ln e^{-0.01361t}$$

$$\ln 90 - \ln 100 = -0.01361t$$

$$t = \frac{\ln 90 - \ln 100}{-0.01361} \approx 7.7414$$

Since time t is in weeks, this means the store should begin ordering roses again in about 7 and three-quarter weeks.

COMPOUNDING INTEREST

■ 1. Suppose you borrow $\$15,000$ with a single payment loan, payable in 2 years, with interest growing exponentially at 1.82% per month, compounded continuously. How much will it cost to pay off the loan after 2 years?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of months, we'll convert 2 years into 24 months for time t.

$$A(t) = Pe^{rt}$$

$$A(24) = 15{,}000e^{0.0182(24)}$$

$$A(24) = \$23{,}216.20$$

■ 2. Your parents deposit $\$5,000$ into a college savings account, with interest growing exponentially at 0.875% per quarter, compounded continuously. How much will be in the account after 18 years?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of quarters, we'll convert 18 years into 72 quarters for time t.

$$A(t) = Pe^{rt}$$

$$A(72) = 5{,}000e^{0.00875(72)}$$

$$A(72) = \$9{,}388.05$$

■ 3. Suppose you win $\$50{,}000$ in a contest and you decide to save it for your retirement. You deposit it into an annuity account that pays $2.4\,\%$ semi-annually, compounded continuously. How much will the account contain after 25 years, when you plan to retire?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of half-years, we'll convert 25 years into 50 half-years for time t.

$$A(t) = Pe^{rt}$$

$$A(50) = 50{,}000e^{0.024(50)}$$

$$A(50) = \$166{,}005.85$$

W W W . K R I S T A K I N G M A T H . C O M