Calculus 1 Workbook Solutions

Exponential growth and decay

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HALF LIFE

■ 1. Find the half-life of Tritium if its decay constant is 0.0562.

Solution:

Since we're calculating half-life, the exponential decay formula $y = y_0 e^{-rt}$ can be simplified to

$$\frac{1}{2} = e^{-rt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.0562t}$$

$$\ln\frac{1}{2} = \ln e^{-0.0562t}$$

$$-\ln 2 = -0.0562t$$

$$t = \frac{-\ln 2}{-0.0562} = \frac{\ln 2}{0.0562} \approx 12.33 \text{ years}$$

■ 2. Find the half-life of Cobalt-60 if its decay constant is 0.1315.

Solution:

Since we're calculating half-life, the exponential decay formula $y = y_0e^{-rt}$ can be simplified to

$$\frac{1}{2} = e^{-rt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.1315t}$$

$$\ln\frac{1}{2} = \ln e^{-0.1315t}$$

$$-\ln 2 = -0.1315t$$

$$t = \frac{-\ln 2}{-0.1315} = \frac{\ln 2}{0.1315} \approx 5.27 \text{ years}$$

■ 3. Find the half-life of Berkelium-97 if its decay constant is 0.000503.

Solution:

Since we're calculating half-life, the exponential decay formula $y = y_0e^{-rt}$ can be simplified to

$$\frac{1}{2} = e^{-rt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.000503t}$$

$$\ln\frac{1}{2} = \ln e^{-0.000503t}$$

$$-\ln 2 = -0.000503t$$

$$t = \frac{-\ln 2}{-0.000503} = \frac{\ln 2}{0.000503} \approx 1{,}378 \text{ years}$$

NEWTON'S LAW OF COOLING

1. A cup of coffee is 195° F when it's brewed. Room temperature is 74° F. If the coffee is 180° F after 5 minutes, to the nearest degree, how hot is the coffee after 25 minutes?

Solution:

Use the information given and the temperature after 5 minutes to solve for k in the Newton's Law of Cooling formula.

$$T - T_s = (T_0 - T_s)e^{-kt}$$

$$180 - 74 = (195 - 74)e^{-5k}$$

$$106 = 121e^{-5k}$$

$$\frac{106}{121} = e^{-5k}$$

$$\ln\frac{106}{121} = \ln e^{-5k}$$

$$\ln\frac{106}{121} = -5k$$

$$k = -\frac{1}{5}\ln\frac{106}{121} \approx 0.02647$$

Then use k to solve for T.

$$T - 74 = (195 - 74)e^{-0.02647(25)}$$

$$T - 74 = 121e^{-0.66175}$$

$$T - 74 = 62.42966$$

$$T = 136.42966$$

The coffee is approximately 136° F after 25 minutes.

■ 2. A boiled egg that's 99° C is placed in a pan of water that's 24° C. If the egg is 62° C after 5 minutes, how much longer, to the nearest minute, will it take the egg to reach 32° C.

Solution:

Use the information given and the temperature after 5 minutes to solve for k in the Newton's Law of Cooling formula.

$$T - T_s = (T_0 - T_s)e^{-kt}$$

$$62 - 24 = (99 - 24)e^{-5k}$$

$$38 = 75e^{-5k}$$

$$\frac{38}{75} = e^{-5k}$$

$$\ln\frac{38}{75} = \ln e^{-5k}$$

$$\ln \frac{38}{75} = -5k$$

$$k = -\frac{1}{5} \ln \frac{38}{75} \approx 0.13598$$

Then use k to solve for t.

$$32 - 24 = (62 - 24)e^{-0.13598t}$$

$$8 = 38e^{-0.13598t}$$

$$\frac{8}{38} = e^{-0.13598t}$$

$$\ln \frac{8}{38} = \ln e^{-0.13598t}$$

$$\ln \frac{8}{38} = -0.13598t$$

$$t = \frac{1}{-0.13598} \ln \frac{8}{38} \approx 11.4662$$

The egg will be 32° C after about 11 and a half more minutes.

■ 3. Suppose a cup of soup cooled from 200° F to 161° F in 10 minutes in a room whose temperature is 68° F. How much longer will it take for the soup to cool to 105° F?

Solution:

Use the information given and the temperature after 10 minutes to solve for k in the Newton's Law of Cooling formula.

$$T - T_s = (T_0 - T_s)e^{-kt}$$

$$161 - 68 = (200 - 68)e^{-10k}$$

$$93 = 132e^{-10k}$$

$$\frac{93}{132} = e^{-10k}$$

$$\ln \frac{93}{132} = \ln e^{-10k}$$

$$\ln \frac{93}{132} = -10k$$

$$k = -\frac{1}{10} \ln \frac{93}{132} \approx 0.03502$$

Then use k to solve for t.

$$105 - 68 = (161 - 68)e^{-0.03502t}$$

$$37 = 93e^{-0.03502t}$$

$$\frac{37}{93} = e^{-0.03502t}$$

$$\ln \frac{37}{93} = \ln e^{-0.03502t}$$

$$\ln \frac{37}{93} = -0.03502t$$

$$t = \frac{1}{-0.03502} \ln \frac{37}{93} \approx 26.3187$$

The egg will be 105° F after about 26 more minutes.

SALES DECLINE

1. Suppose a pizza company stops a special sale for their three-topping pizza. They will resume the sale if sales drop to 70% of the current sales level. If sales decline to 90% during the first week, when should the company expect to start the special sale again?

Solution:

Use the exponential function $FV = PVe^{-rt}$. Plug in what we know.

$$FV = PVe^{-rt}$$

$$90 = 100e^{-r(1)}$$

$$\frac{90}{100} = e^{-r}$$

$$\ln\frac{90}{100} = \ln e^{-r}$$

$$\ln 90 - \ln 100 = -r$$

$$r = \ln 100 - \ln 90 \approx 0.10536$$

Find t using a sales level of 70% and $r = 0.10536$.

$$70 = 100e^{-0.10536t}$$

$$\frac{70}{100} = e^{-0.10536t}$$

$$\ln\frac{70}{100} = \ln e^{-0.10536t}$$

$$\ln 70 - \ln 100 = -0.10536t$$

$$t = \frac{\ln 70 - \ln 100}{-0.10536} \approx 3.385$$

Since time t is in weeks, this means the company should expect to start the sale again in about 3 and a half weeks.

■ 2. Suppose a donut store experiments with raising the price of a dozen donuts to see if sales are affected. They'll resume the sale if sales drop to 80% of the current sales level. If sales decline to 90% after two weeks, when should the store change back to the original price?

Solution:

Use the exponential function $FV = PVe^{-rt}$. Plug in what we know.

$$FV = PVe^{-rt}$$

$$90 = 100e^{-r(2)}$$

$$\frac{90}{100} = e^{-2r}$$

$$\ln\frac{90}{100} = \ln e^{-2r}$$

$$\ln 90 - \ln 100 = -2r$$

$$r = \frac{\ln 90 - \ln 100}{-2} \approx 0.05268$$

Find t using a sales level of 80% and $r = 0.05268$.

$$80 = 100e^{-0.05268t}$$

$$\frac{80}{100} = e^{-0.05268t}$$

$$\ln \frac{80}{100} = \ln e^{-0.05268t}$$

$$\ln 80 - \ln 100 = -0.05268t$$

$$t = \frac{\ln 80 - \ln 100}{-0.05268} \approx 4.2358$$

Since time t is in weeks, this means the store should expect to change back to the original price in about 4 and a quarter weeks.

■ 3. Suppose a flower shop decides to stop ordering roses in the winter time to see if sales are affected. They will resume the sale if sales drop to 90% of the current sales level. If sales decline to 96% after three weeks, when should the shop begin ordering roses again?

Solution:

Use the exponential function $FV = PVe^{-rt}$. Plug in what we know.

$$FV = PVe^{-rt}$$

$$96 = 100e^{-r(3)}$$

$$\frac{96}{100} = e^{-3r}$$

$$\ln\frac{96}{100} = \ln e^{-3r}$$

$$\ln 96 - \ln 100 = -3r$$

$$r = \frac{\ln 96 - \ln 100}{-3} \approx 0.01361$$

Find t using a sales level of 90% and $r = 0.01361$.

$$90 = 100e^{-0.01361t}$$

$$\frac{90}{100} = e^{-0.01361t}$$

$$\ln\frac{90}{100} = \ln e^{-0.01361t}$$

$$\ln 90 - \ln 100 = -0.01361t$$

$$t = \frac{\ln 90 - \ln 100}{-0.01361} \approx 7.7414$$

Since time t is in weeks, this means the store should begin ordering roses again in about 7 and three-quarter weeks.

COMPOUNDING INTEREST

■ 1. Suppose you borrow $\$15,000$ with a single payment loan, payable in 2 years, with interest growing exponentially at 1.82% per month, compounded continuously. How much will it cost to pay off the loan after 2 years?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of months, we'll convert 2 years into 24 months for time t.

$$A(t) = Pe^{rt}$$

$$A(24) = 15{,}000e^{0.0182(24)}$$

$$A(24) = \$23{,}216.20$$

■ 2. Your parents deposit $\$5{,}000$ into a college savings account, with interest growing exponentially at 0.875% per quarter, compounded continuously. How much will be in the account after 18 years?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of quarters, we'll convert 18 years into 72 quarters for time t.

$$A(t) = Pe^{rt}$$

$$A(72) = 5{,}000e^{0.00875(72)}$$

$$A(72) = \$9{,}388.05$$

■ 3. Suppose you win $\$50{,}000$ in a contest and you decide to save it for your retirement. You deposit it into an annuity account that pays $2.4\,\%$ semi-annually, compounded continuously. How much will the account contain after 25 years, when you plan to retire?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of half-years, we'll convert 25 years into 50 half-years for time t.

$$A(t) = Pe^{rt}$$

$$A(50) = 50{,}000e^{0.024(50)}$$

$$A(50) = \$166{,}005.85$$

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