Calculus 1 Workbook Solutions

Implicit differentiation

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MATH

IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find dy/dx at $(3,4)$ for the equation.

$$4x^3 - 3xy^2 + y^3 = 28$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$12x^2 - 3y^2 - 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(3y^2 - 6xy)\frac{dy}{dx} = 3y^2 - 12x^2$$

$$\frac{dy}{dx} = \frac{3y^2 - 12x^2}{3y^2 - 6xy}$$

Evaluate dy/dx at $(3,4)$.

$$\frac{dy}{dx}(3,4) = \frac{3(4)^2 - 12(3)^2}{3(4)^2 - 6(3)(4)} = \frac{48 - 108}{48 - 72} = \frac{5}{2}$$

■ 2. Use implicit differentiation to find dy/dx for the equation.

$$5x^3 + xy^2 = 4x^3y^3$$

Solution:

Rearrange the function. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$5x^3 + xy^2 = 4x^3y^3$$

$$xy^2 - 4x^3y^3 = -5x^3$$

Use implicit differentiation to take the derivative of both sides.

$$y^2 + 2xy\frac{dy}{dx} - 12x^2y^3 - 12x^3y^2\frac{dy}{dx} = -15x^2$$

$$2xy\frac{dy}{dx} - 12x^3y^2\frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$(2xy - 12x^3y^2)\frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$\frac{dy}{dx} = \frac{12x^2y^3 - 15x^2 - y^2}{2xy - 12x^3y^2}$$

■ 3. Use implicit differentiation to find dy/dx for the equation.

$$3x^2 = (3xy - 1)^2$$

Solution:

Rearrange the function. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$3x^2 = (3xy - 1)^2$$

$$3x^2 = 9x^2y^2 - 6xy + 1$$

Use implicit differentiation to take the derivative of both sides.

$$6x = 18xy^2 + 18x^2y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = 18x^2y\frac{dy}{dx} - 6x\frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = (18x^2y - 6x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x - 18xy^2 + 6y}{18x^2y - 6x}$$

$$\frac{dy}{dx} = \frac{x - 3xy^2 + y}{3x^2y - x}$$

EQUATION OF THE TANGENT LINE WITH IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find the equation of the tangent line to $5y^2 = 2x^3 - 5y + 6$ at $(3,3)$.

Solution:

Rearrange the function.

$$5y^2 = 2x^3 - 5y + 6$$

$$5y^2 + 5y = 2x^3 + 6$$

Use implicit differentiation to take the derivative of both sides.

$$10y\frac{dy}{dx} + 5\frac{dy}{dx} = 6x^2$$

$$(10y + 5)\frac{dy}{dx} = 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2}{10y + 5}$$

Evaluate dy/dx at $(3,3)$.

$$\frac{dy}{dx}(3,3) = \frac{6(3)^2}{10(3) + 5} = \frac{54}{35}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{54}{35}(x - 3)$$

$$y = \frac{54}{35}(x - 3) + 3$$

2. Use implicit differentiation to find the equation of the tangent line to $5x^3 = -3xy + 4$ at $(2, -6)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$15x^2 = -3y - 3x\frac{dy}{dx}$$

$$15x^2 + 3y = -3x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{15x^2 + 3y}{-3x}$$

$$\frac{dy}{dx} = -\frac{5x^2 + y}{x}$$

Evaluate dy/dx at $(2, -6)$.

$$\frac{dy}{dx}(2, -6) = -\frac{5(2)^2 + (-6)}{2} = -\frac{20 - 6}{2} = -7$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 6 = -7(x - 2)$$

$$y = -7(x - 2) - 6$$

$$y = -7x + 8$$

■ 3. Use implicit differentiation to find the equation of the tangent line to $4y^2 + 8 = 3x^2$ at $(6, -5)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$8y\frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{8y} = \frac{3x}{4y}$$

Evaluate dy/dx at $(6, -5)$.

$$\frac{dy}{dx}(6, -5) = \frac{3(6)}{4(-5)} = -\frac{18}{20} = -\frac{9}{10}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 5 = -\frac{9}{10}(x - 6)$$

$$y = -\frac{9}{10}(x - 6) - 5$$

$$y = -\frac{9}{10}x + \frac{54}{10} - 5$$

$$y = -\frac{9}{10}x + \frac{27}{5} - \frac{25}{5}$$

$$y = -\frac{9}{10}x + \frac{2}{5}$$

SECOND DERIVATIVES WITH IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find d^2y/dx^2.

$$2x^3 = 2y^2 + 4$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$6x^2 = 4y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x^2}{4y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(4y)(12x) - (6x^2)\left(4 \cdot \frac{dy}{dx}\right)}{(4y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{48xy - 24x^2\frac{dy}{dx}}{16y^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = \frac{48xy - 24x^2\left(\frac{6x^2}{4y}\right)}{16y^2}$$

$$\frac{d^2y}{dx^2} = \frac{48xy - \frac{36x^4}{y}}{16y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

Multiply through the numerator and denominator by y to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = \frac{12xy^2 - 9x^4}{4y^3}$$

■ 2. Use implicit differentiation to find d^2y/dx^2.

$$4x^2 = 2y^3 + 4y - 2$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$8x = 6y^2\frac{dy}{dx} + 4\frac{dy}{dx}$$

$$8x = (6y^2 + 4)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{8x}{6y^2 + 4}$$

$$\frac{dy}{dx} = \frac{4x}{3y^2 + 2}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(4)(3y^2 + 2) - (4x)\left(6y \cdot \frac{dy}{dx}\right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy\frac{dy}{dx}}{(3y^2 + 2)^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy\left(\frac{4x}{3y^2 + 2}\right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - \frac{96x^2y}{3y^2 + 2}}{(3y^2 + 2)^2}$$

Multiply through the numerator and denominator by $3y^2 + 2$ to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = \frac{12y^2(3y^2 + 2) + 8(3y^2 + 2) - 96x^2y}{(3y^2 + 2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{(12y^2 + 8)(3y^2 + 2) - 96x^2y}{(3y^2 + 2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{4(3y^2 + 2)(3y^2 + 2) - 96x^2y}{(3y^2 + 2)^3}$$

$$\frac{d^2y}{dx^2} = \frac{4(3y^2 + 2)^2 - 96x^2y}{(3y^2 + 2)^3}$$

■ 3. Use implicit differentiation to find d^2y/dx^2 at $(0,3)$.

$$3x^2 + 3y^2 = 27$$

Solution:

Rewrite the equation.

$$3x^2 + 3y^2 = 27$$

$$x^2 + y^2 = 9$$

Use implicit differentiation to take the derivative of both sides.

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = -\frac{(1)(y) - (x)(1)\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y + \frac{x^2}{y}}{y^2}$$

Multiply through the numerator and denominator by y to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = -\frac{y^2 + x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-x^2 - y^2}{y^3}$$

Evaluate the second derivative at $(0,3)$.

$$\frac{d^2y}{dx^2}(0,3) = \frac{-0^2 - 3^2}{3^3} = \frac{-9}{27} = -\frac{1}{3}$$

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