Implicit differentiation

Up to now, we've been differentiating functions defined for $f(x)$ in terms of x, or equations defined for y in terms of x. In other words, every equation we've differentiated has had the variables separated on either side of the equal sign.

For instance, the equation $y = 3x^2 + 2x + 1$ has the y variable on the left side, and the x variable on the right side. We don't have x and y variables mixed together on the left, and they aren't mixed together on the right, either.

Separable or not

We sometimes have equations where the variables are mixed together, but they can be easily separated. If we're given the equation $y - 2x = 3x^2 + 1$, we do have x and y variables mixed together on the left side, but we can easily separate the variables by simply adding $2x$ to both sides of the equation to get $y = 3x^2 + 2x + 1$.

Other times, we'll have the variables mixed together, and it's actually impossible to separate them. For instance, the equation $xy = 3(x - y)^2 + 2x + 1$ can't be rewritten with all the y variables on the left and all the x variables on the right. When the variables can't be separated, we can use implicit differentiation to find the function's derivative.

In other words, **implicit differentiation** allows us to take the derivative of a function that contains both x and y on the same side of the equation.

How to use implicit differentiation

When we use implicit differentiation, we have to treat y differently than we have in the past. With implicit differentiation, we treat y as a function and not just as a variable. We treat x just as we have before, as a variable, but we treat y as a function.

Practically, this means that each time we take the derivative of y, we multiply the result by the derivative of y. We can write the derivative of y as either y' or as dy/dx.

To use implicit differentiation, we'll follow these steps:

1. Differentiate both sides with respect to x.

2. Whenever we encounter y, we differentiate it like we would x, but we multiply that term by the derivative of y, which we write as y' or as dy/dx.

3. Move all terms involving dy/dx to the left side and everything else to the right.

4. Factor out dy/dx on the left and divide both sides by the other left-side factor so that dy/dx is the only thing remaining on the left.

Once we get dy/dx (or y') alone on the left, we've solved for the derivative of y', which was our goal when we started differentiating.

Let's walk through an example so that we can see how this set of steps gets us to the derivative.

Example

Use implicit differentiation to find the derivative.

$$x^3 + y^3 = 9xy$$

We'll differentiate both sides with respect to x. When we do, we'll treat x as we normally do, but we'll treat y as a function.

Working term-by-term, we take the derivative of x^3 like we normally would and we get $3x^2$. When we take the derivative of y^3, we get $3y^2$, but since we took the derivative of y, we have to multiply by dy/dx.

To take the derivative of the right side of the equation, we need to use product rule, since x and y are both variables and therefore need to be treated as separate functions. So we'll say that one function is $9x$ and that the other is y. The derivative of $9x$ will be 9, like normal. The derivative of y would be 1, but since we're taking the derivative of y, we have to multiply by dy/dx.

Therefore, after implicit differentiation, the derivative looks like this:

$$3x^2 + 3y^2\frac{dy}{dx} = (9)(y) + (9x)(1)\frac{dy}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 9y + 9x\frac{dy}{dx}$$

Move all terms that include dy/dx to the left side, and move everything else to the right side.

$$3y^2\frac{dy}{dx}-9x\frac{dy}{dx}=9y-3x^2$$

Factor out dy/dx on the left,

$$\frac{dy}{dx}(3y^2-9x)=9y-3x^2$$

and then divide both sides by $(3y^2-9x)$ in order to get dy/dx by itself.

$$\frac{dy}{dx}=\frac{9y-3x^2}{3y^2-9x}$$

$$\frac{dy}{dx}=\frac{3y-x^2}{y^2-3x}$$

We can leave the derivative this way, or we can replace dy/dx with y' if we prefer that notation.

$$y'=\frac{3y-x^2}{y^2-3x}$$

Notice how this answer looks similar to all the derivatives we've found before. It's a derivative equation that's solved for y' on the left side, which is exactly what we're used to seeing.

In the past, we were able to easily get the derivative in this form, because the equations we were differentiating were already solved for y. As we've seen from this example, when we get an equation that isn't already solved for y, we can still get to the derivative equation we want, we just have to use implicit differentiation to get there.