Calculus 1 Workbook Solutions

Inverse functions and logarithms

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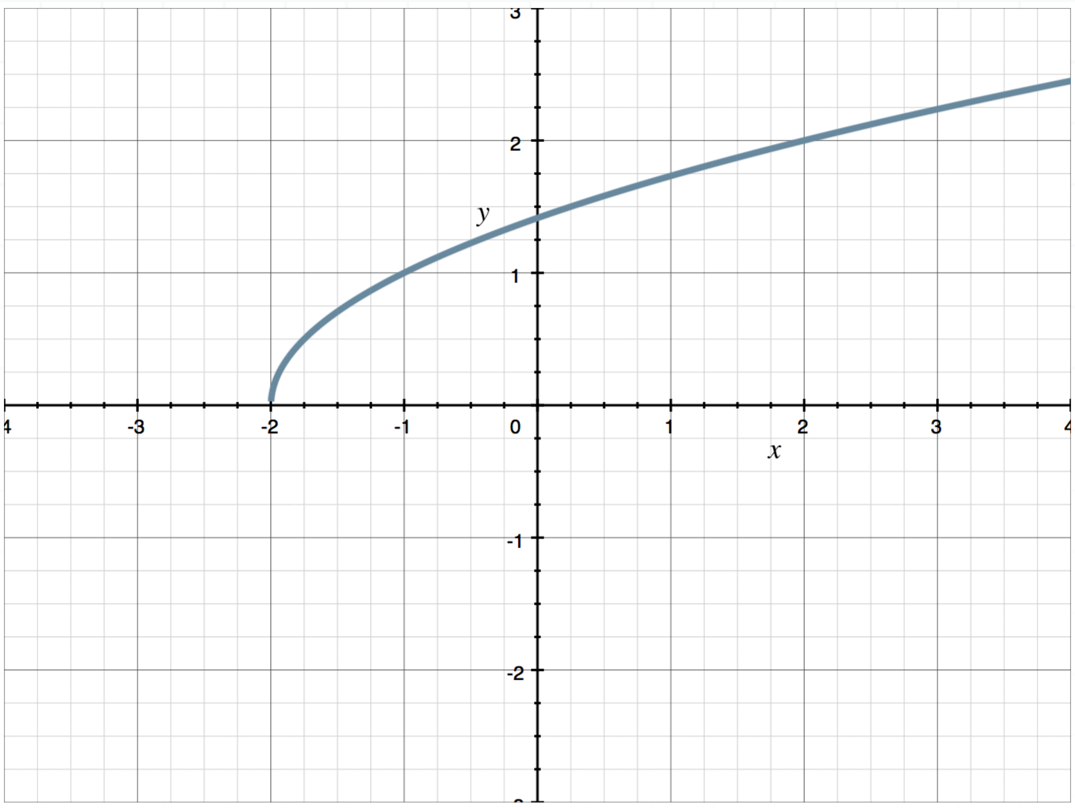
MATH

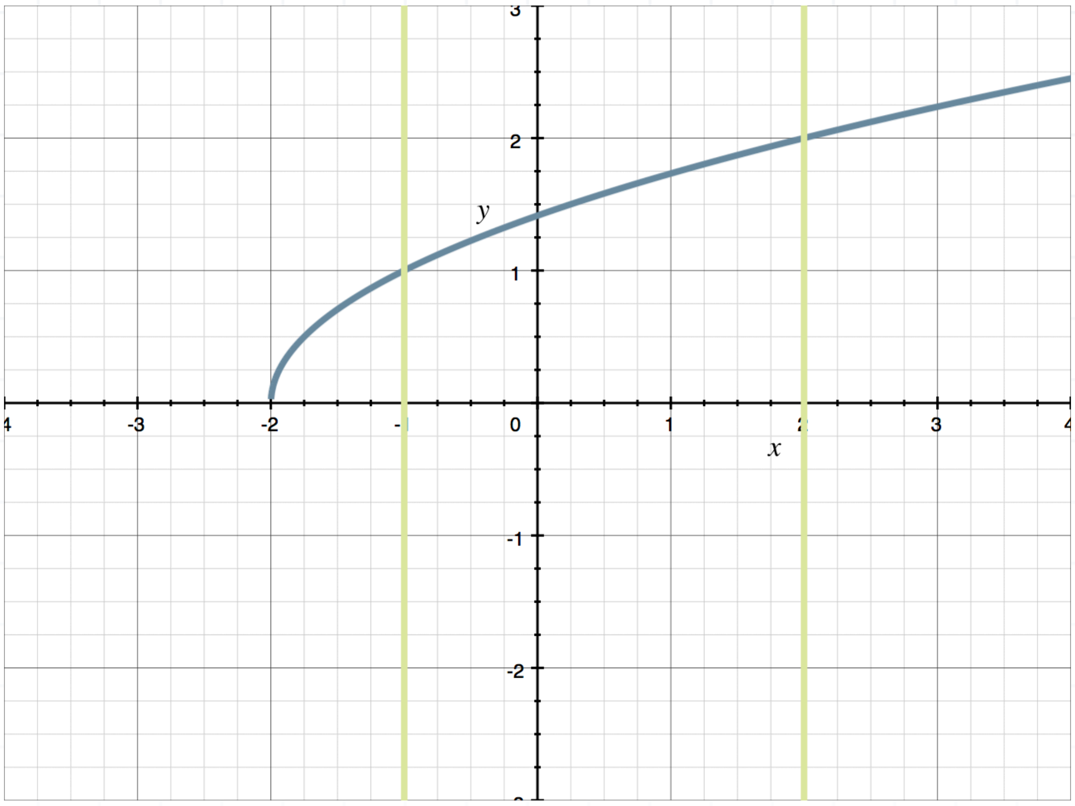
HORIZONTAL LINE TEST

1. Does the graph represent a one-to-one function?

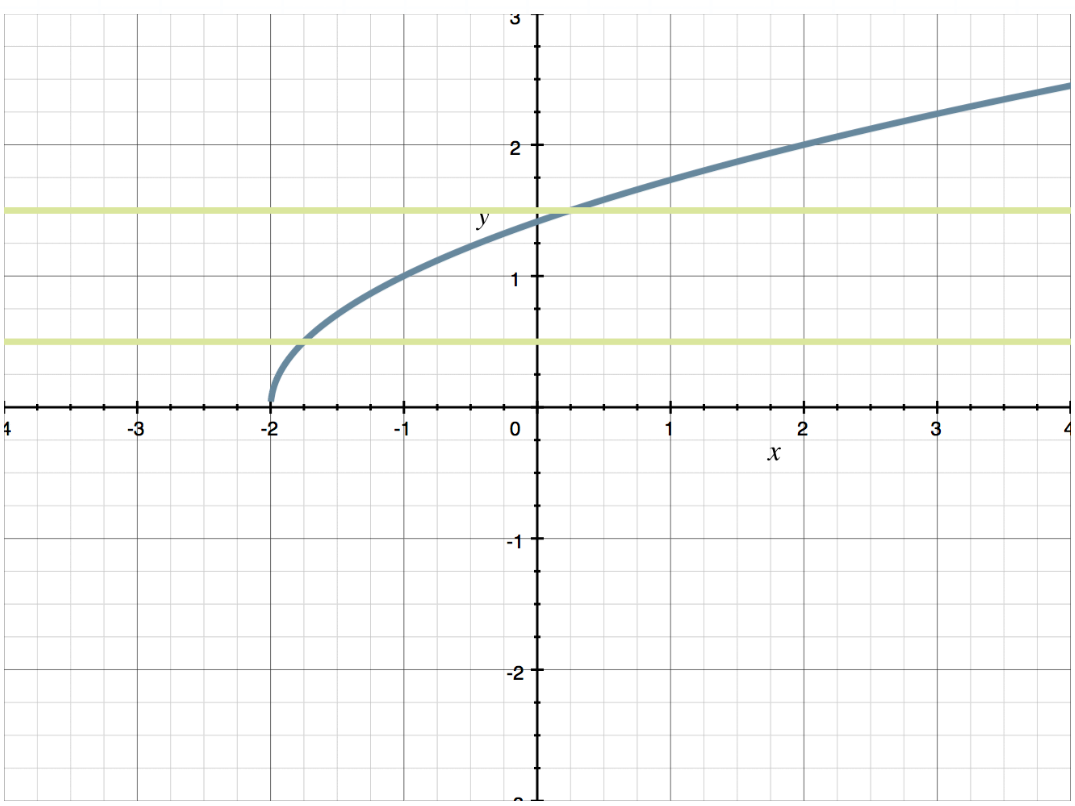


Solution:

First, see if the graph passes the Vertical Line Test. If any vertical line passes through the graph at two or more points, it will fail the Vertical Line Test and isn't a function.

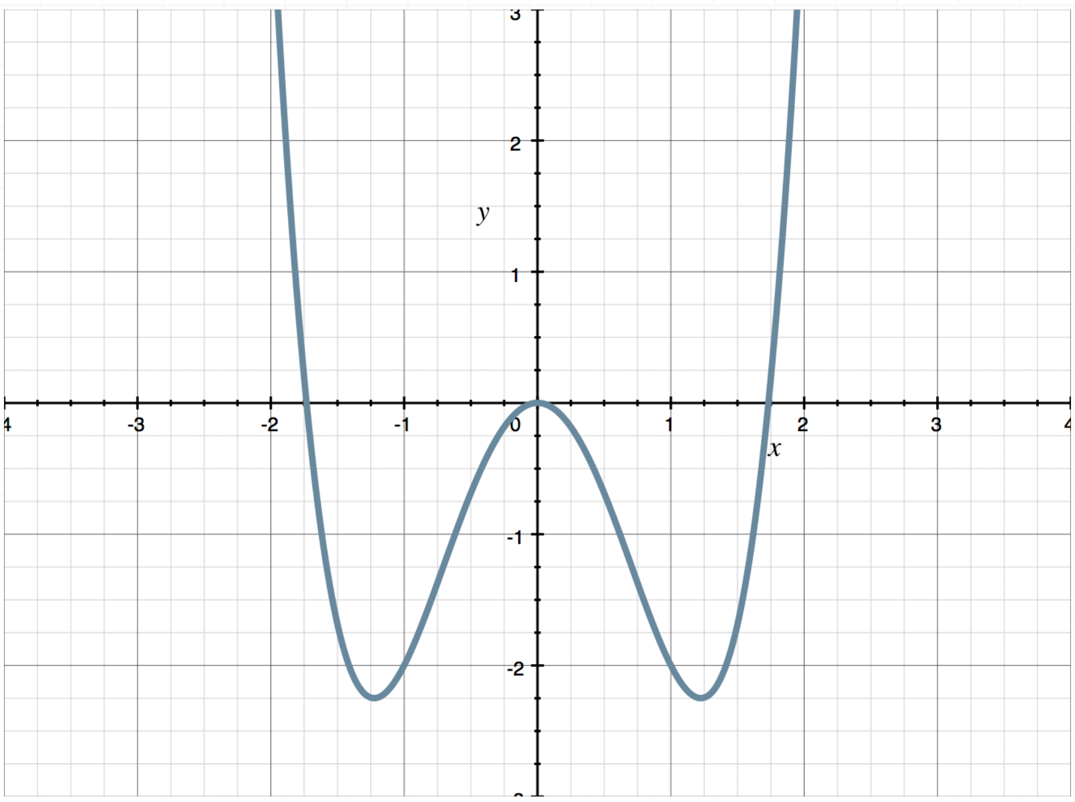


The graph passes the Vertical Line Test, which means that it represents a function. Next, see if the graph passes the Horizontal Line Test. If any horizontal line passes through the graph at two or more points, it will fail the Horizontal Line Test and isn’t a one-to-one function.



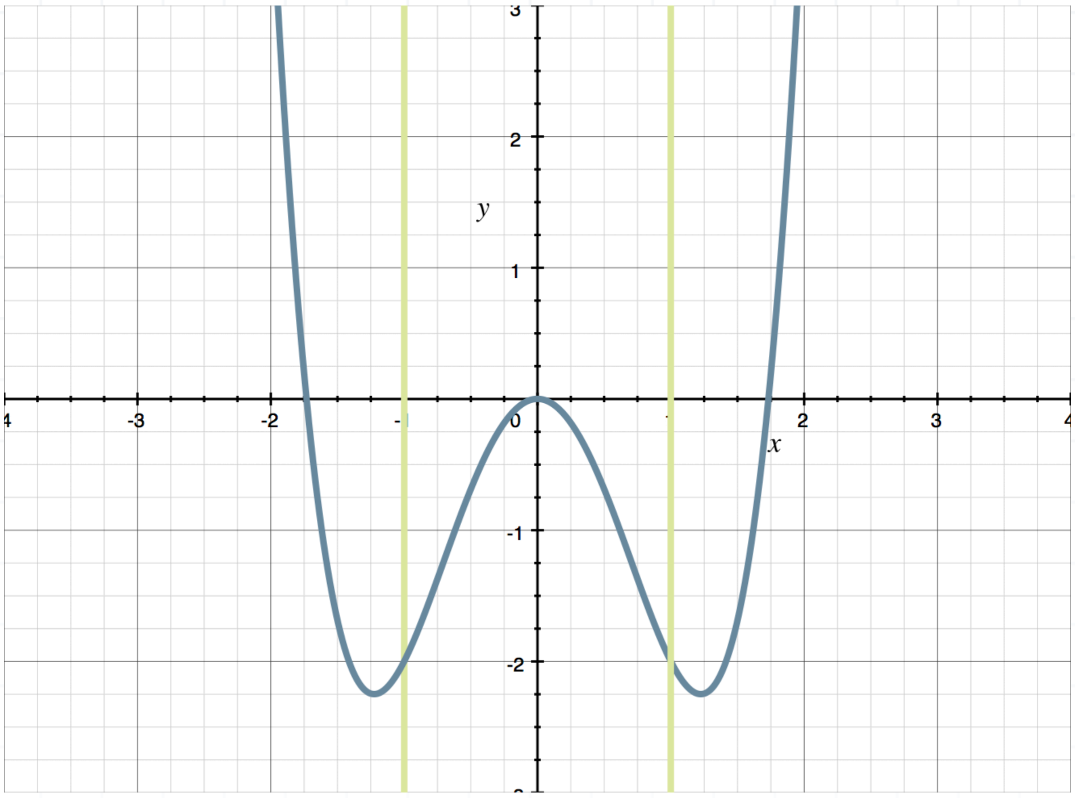
The graph passes the Horizontal Line Test, so it’s a one-to-one function.

2. Does the graph represent a one-to-one function?

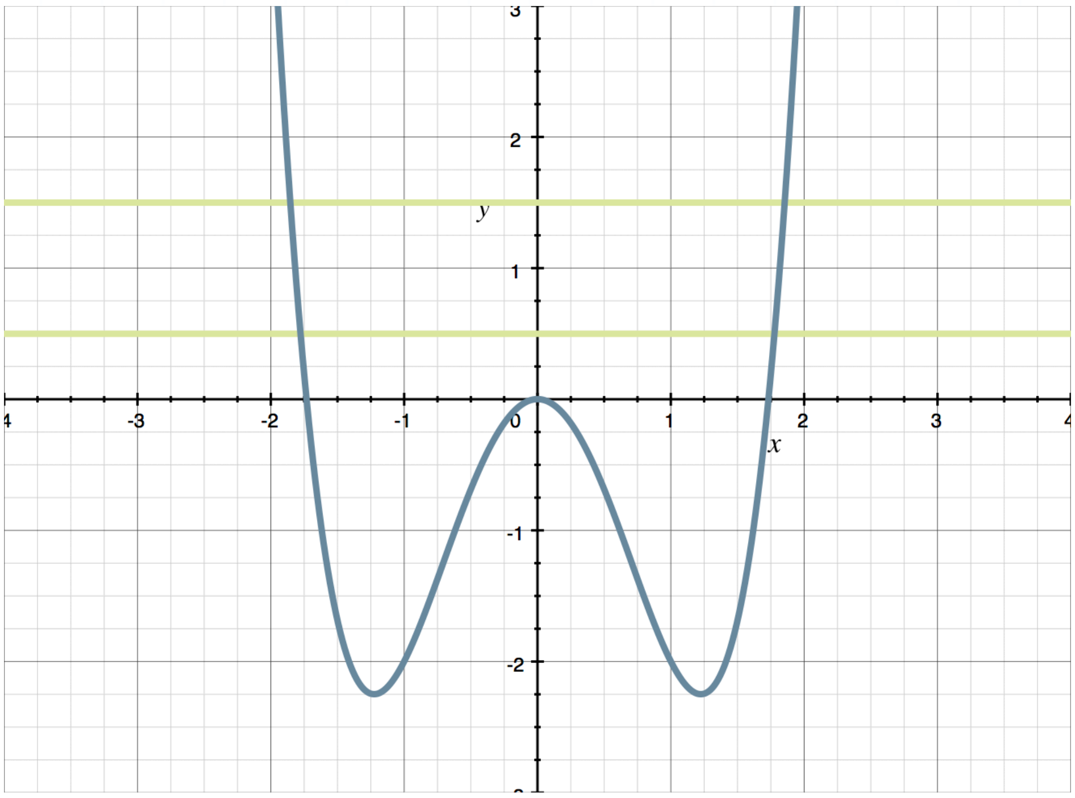


Solution:

First, see if the graph passes the Vertical Line Test. If any vertical line passes through the graph at two or more points, it will fail the Vertical Line Test and isn't a function.

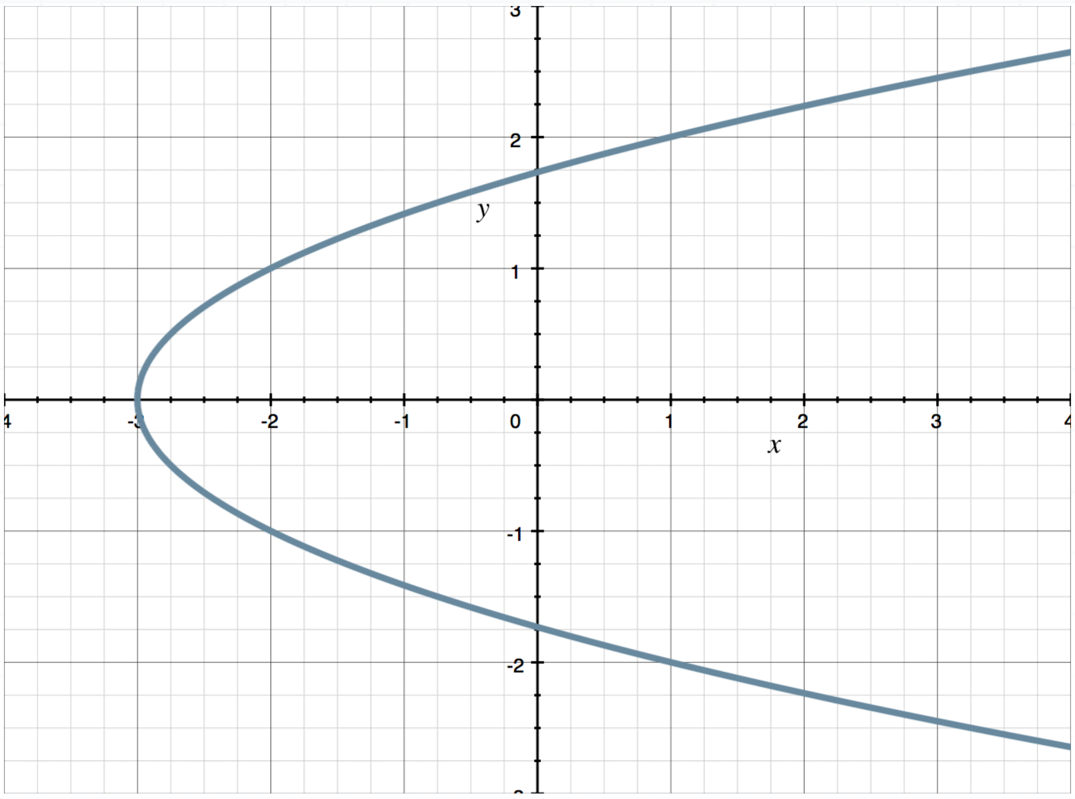


The graph passes the Vertical Line Test, which means that it represents a function. Next, see if the graph passes the Horizontal Line Test. If any horizontal line passes through the graph at two or more points, it will fail the Horizontal Line Test and isn't a one-to-one function.



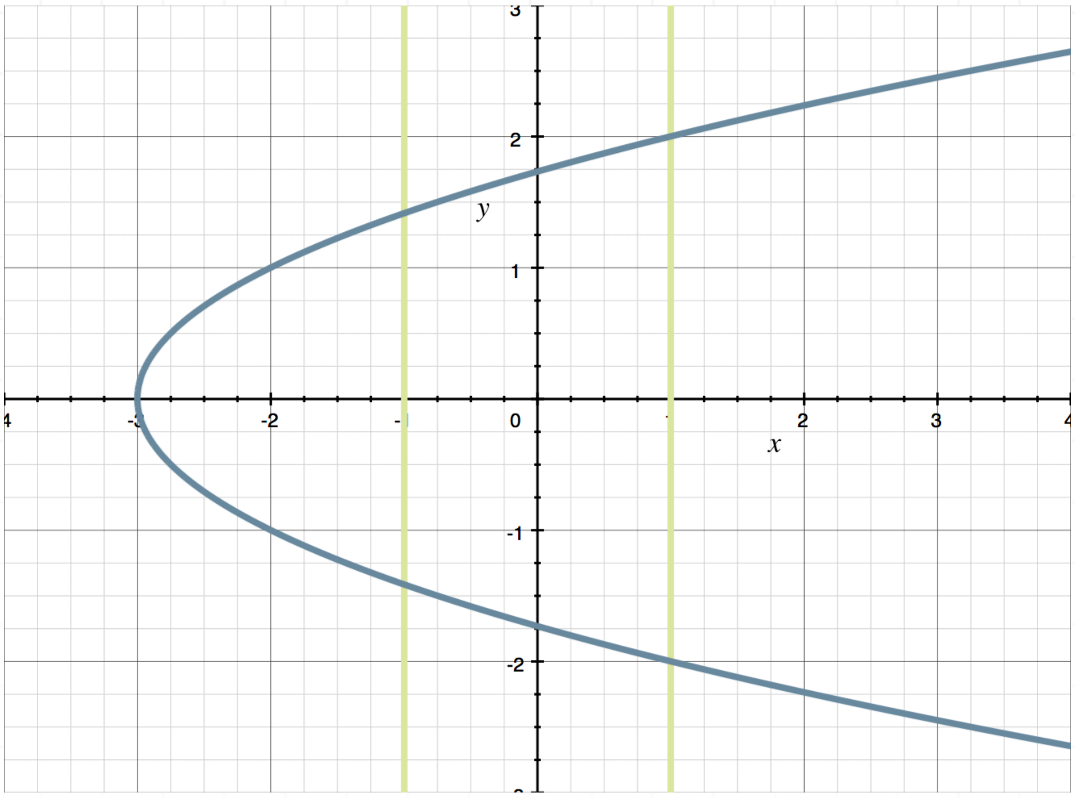
The graph does not pass the Horizontal Line Test, so it's still a function, just not a one-to-one function.

3. Does the graph represent a one-to-one function?



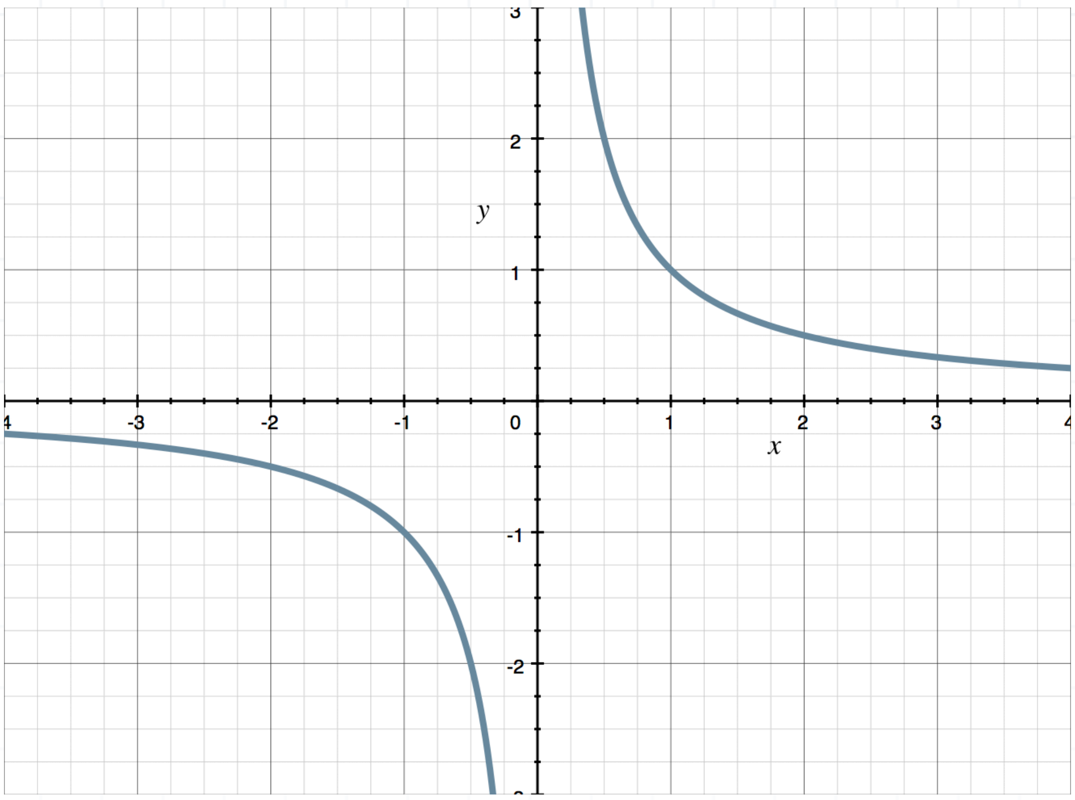
Solution:

First, see if the graph passes the Vertical Line Test. If any vertical line passes through the graph at two or more points, it will fail the Vertical Line Test and isn't a function.



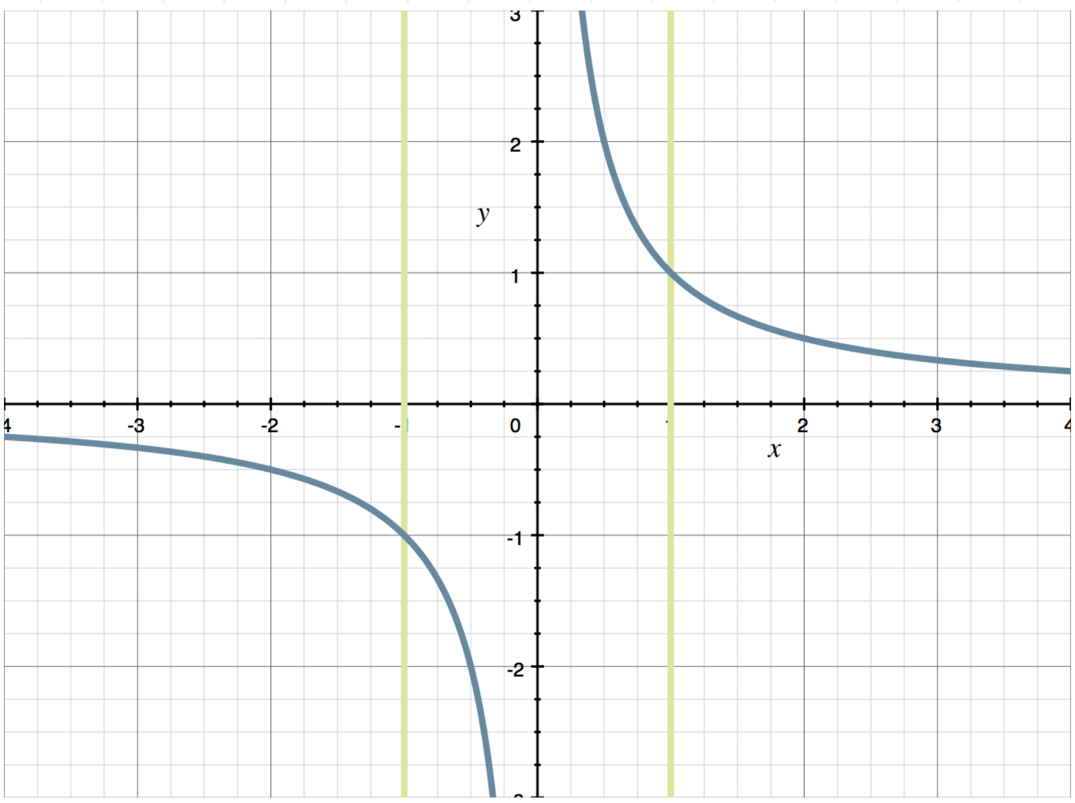
Clearly, a vertical line can cross the graph at two or more points. This means that the graph fails the Vertical Line Test and does not represent a function. Since the graph isn't a function, it can't be a one-to-one function, even if it passes the Horizontal Line Test.

4. Does the graph represent a one-to-one function?

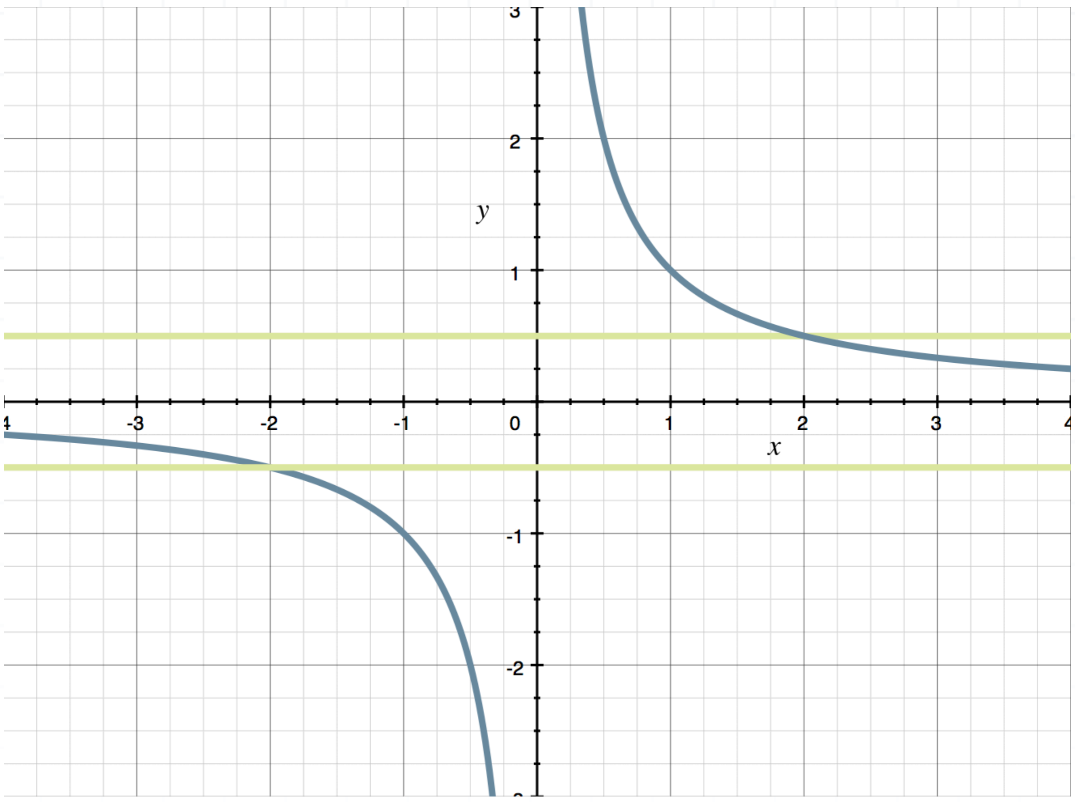


Solution:

First, see if the graph passes the Vertical Line Test. If any vertical line passes through the graph at two or more points, it will fail the Vertical Line Test and isn't a function.



The graph passes the Vertical Line Test, which means that it represents a function. Next, see if the graph passes the Horizontal Line Test. If any horizontal line passes through the graph at two or more points, it will fail the Horizontal Line Test and isn't a one-to-one function.



The graph passes the Horizontal Line Test, so it's a one-to-one function.

5. Show that the function is one-to-one by showing that $f(a) = f(b)$ leads to $a = b$.

$$f(x) = 3x - 4$$

Solution:

Start by replacing x with a and b.

$$3a-4$$

$$3b-4$$

Set these equal to one another, then simplify.

$$3a-4=3b-4$$

$$3a=3b$$

$$a=b$$

Since $a=b$, $f(x)$ is a one-to-one function.

■ 6. Show that the function is one-to-one by showing that $f(a)=f(b)$ leads to $a=b$.

$$f(x)=\frac{x+1}{x-5}$$

Solution:

Start by replacing x with a and b.

$$\frac{a+1}{a-5}$$

$$\frac{b+1}{b-5}$$

Set these equal to one another, then simplify.

$$\frac{a+1}{a-5} = \frac{b+1}{b-5}$$

$$(a+1)(b-5) = (b+1)(a-5)$$

$$ab - 5a + b - 5 = ab - 5b + a - 5$$

$$ab - 5a + b = ab - 5b + a$$

$$-5a + b = -5b + a$$

$$6b = 6a$$

$$b = a$$

Since $a = b$, $f(x)$ is a one-to-one function.

7. Show that the function is not one-to-one by showing that $f(a) = f(b)$ does not lead to $a = b$.

$$f(x) = x^2 - 6$$

Solution:

All we need is one counterexample to show that $f(a) = f(b)$ doesn't imply that $a = b$. Let's use $a = 1$ and $b = -1$.

$$f(a) = f(1) = 1^2 - 6 = 1 - 6 = -5$$

$$f(b) = f(-1) = (-1)^2 - 6 = 1 - 6 = -5$$

Since we get the same value for both functions, $f(a) = f(b)$, but we used different values to get the same answer, $a \neq b$, the function is not one-to-one.

8. Show that the function is not one-to-one by showing that $f(a) = f(b)$ does not lead to $a = b$.

$$f(x) = (x+3)(x-2)$$

Solution:

All we need is one counterexample to show that $f(a) = f(b)$ doesn't imply that $a = b$. Let's use $a = -3$ and $b = 2$.

$$f(a) = f(-3) = (-3+3)(-3-2) = 0(-5) = 0$$

$$f(b) = f(2) = (2+3)(2-2) = 5(0) = 0$$

Since we get the same value for both functions, $f(a) = f(b)$, but we used different values to get the same answer, $a \neq b$, the function is not one-to-one.

INVERSE FUNCTIONS

1. What is the inverse of the function?

$$f(x) = \frac{1}{2}x - 3$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{1}{2}x - 3$$

Switch x and y, then solve for y.

$$x = \frac{1}{2}y - 3$$

$$x + 3 = \frac{1}{2}y$$

$$2x + 6 = y$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = 2x + 6$$

2. What is the inverse of the function?

$$f(x) = -4x + 5$$

Solution:

Start by replacing $f(x)$ with y.

$$y = -4x + 5$$

Switch x and y, then solve for y.

$$x = -4y + 5$$

$$x - 5 = -4y$$

$$-\frac{1}{4}x + \frac{5}{4} = y$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{1}{4}x + \frac{5}{4}$$

■ 3. What is the inverse of the function?

$$f(x) = \frac{x}{x+2}$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{x}{x+2}$$

Switch x and y, then solve for y.

$$x = \frac{y}{y+2}$$

$$x(y+2) = y$$

$$xy + 2x = y$$

$$2x = y - xy$$

$$2x = y(1-x)$$

$$\frac{2x}{1-x} = y$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = \frac{2x}{1-x}$$

4. What is the inverse of the function?

$$f(x) = \frac{2x}{x-5}$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{2x}{x-5}$$

Switch x and y, then solve for y.

$$x = \frac{2y}{y-5}$$

$$x(y-5) = 2y$$

$$xy - 5x = 2y$$

$$-5x = 2y - xy$$

$$-5x = y(2-x)$$

$$-\frac{5x}{2-x} = y$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{5x}{2-x}$$

■ 5. What is the inverse of the function?

$$f(x) = \frac{1}{x} + 3$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{1}{x} + 3$$

Switch x and y, then solve for y.

$$x = \frac{1}{y} + 3$$

$$x - 3 = \frac{1}{y}$$

$$y(x - 3) = 1$$

$$y = \frac{1}{x - 3}$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = \frac{1}{x - 3}$$

■ 6. What is the inverse of the function?

$$f(x) = -\frac{3}{x - 2} - 4$$

Solution:

Start by replacing $f(x)$ with y.

$$y = -\frac{3}{x-2} - 4$$

Switch x and y, then solve for y.

$$x = -\frac{3}{y-2} - 4$$

$$x + 4 = -\frac{3}{y-2}$$

$$(y-2)(x+4) = -3$$

$$y - 2 = -\frac{3}{x+4}$$

$$y = -\frac{3}{x+4} + 2$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{3}{x+4} + 2$$

■ 7. What is the inverse of the function?

$$f(x) = \frac{x-2}{x+3}$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{x-2}{x+3}$$

Switch x and y, then solve for y.

$$x = \frac{y-2}{y+3}$$

$$x(y+3) = y-2$$

$$xy + 3x = y - 2$$

$$xy - y = -3x - 2$$

$$y(x-1) = -3x - 2$$

$$y(x-1) = -(3x+2)$$

$$y = -\frac{3x+2}{x-1}$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{3x+2}{x-1}$$

■ 8. What is the inverse of the function?

$$f(x) = \frac{5+x}{4-x}$$

Solution:

Start by replacing $f(x)$ with y.

$$y = \frac{5+x}{4-x}$$

Switch x and y, then solve for y.

$$x = \frac{5+y}{4-y}$$

$$x(4-y) = 5+y$$

$$4x - xy = 5+y$$

$$-y - xy = 5 - 4x$$

$$-y(1+x) = 5 - 4x$$

$$-y = \frac{5-4x}{1+x}$$

$$y = -\frac{5-4x}{1+x}$$

Replace y with $f^{-1}(x)$ to write the inverse function.

$$f^{-1}(x) = -\frac{5-4x}{1+x}$$

FINDING THE EQUATION OF A LINE FROM POINTS ON ITS INVERSE

■ 1. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(1) = -2$$

$$f^{-1}(-3) = -1$$

Solution:

$(1,-2)$ and $(-3,-1)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(-2,1)$$

$$(-1,-3)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-1}{-1-(-2)} = \frac{-4}{1} = -4$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(-2,1)$ and $(-1,-3)$. We'll use $(-2,1)$.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -4(x - (-2))$$

$$y - 1 = -4(x + 2)$$

$$y - 1 = -4x - 8$$

$$y = -4x - 7$$

So $f(x)$ is given by

$$f(x) = -4x - 7$$

2. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(0) = 3$$

$$f^{-1}(-2) = 1$$

Solution:

$(0,3)$ and $(-2,1)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(3,0)$$

$$(1, -2)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 0}{1 - 3} = \frac{-2}{-2} = 1$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(3,0)$ and $(1, -2)$. We'll use $(3,0)$.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 1(x - 3)$$

$$y = x - 3$$

So $f(x)$ is given by

$$f(x) = x - 3$$

3. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(2) = 5$$

$$f^{-1}(4) = 9$$

Solution:

$(2,5)$ and $(4,9)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(5,2)$$

$$(9,4)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{9 - 5} = \frac{2}{4} = \frac{1}{2}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(5,2)$ and $(9,4)$. We'll use $(5,2)$.

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{2}(x - 5)$$

$$y - 2 = \frac{1}{2}x - \frac{5}{2}$$

$$y = \frac{1}{2}x - \frac{5}{2} + \frac{4}{2}$$

$$y = \frac{1}{2}x - \frac{1}{2}$$

So $f(x)$ is given by

$$f(x) = \frac{1}{2}x - \frac{1}{2}$$

4. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(-3) = 2$$

$$f^{-1}(1) = 4$$

Solution:

$(-3,2)$ and $(1,4)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(2,-3)$$

$$(4,1)$$

Find the slope between these points to find the slope of $f(x)$.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-(-3)}{4-2}=\frac{4}{2}=2$$

Use point-slope form $y-y_1=m(x-x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(2,-3)$ and $(4,1)$. We'll use $(4,1)$.

$$y-y_1=m(x-x_1)$$

$$y-1=2(x-4)$$

$$y-1=2x-8$$

$$y=2x-7$$

So $f(x)$ is given by

$$f(x)=2x-7$$

■ 5. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(-4) = 7$$

$$f^{-1}(-1) = 14$$

Solution:

$(-4,7)$ and $(-1,14)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(7, -4)$$

$$(14, -1)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - (-4)}{14 - 7} = \frac{3}{7}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(7, -4)$ and $(14, -1)$. We'll use $(14, -1)$.

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = \frac{3}{7}(x - 14)$$

$$y + 1 = \frac{3}{7}x - 6$$

$$y = \frac{3}{7}x - 7$$

So $f(x)$ is given by

$$f(x) = \frac{3}{7}x - 7$$

■ 6. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(5) = -4$$

$$f^{-1}(10) = -12$$

Solution:

$(5, -4)$ and $(10, -12)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(-4,5)$$

$$(-12,10)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 5}{-12 - (-4)} = \frac{5}{-8} = -\frac{5}{8}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(-4,5)$ and $(-12,10)$. We'll use $(-4,5)$.

$$y - y_1 = m(x - x_1)$$

$$y - 5 = -\frac{5}{8}(x - (-4))$$

$$y - 5 = -\frac{5}{8}(x + 4)$$

$$y - 5 = -\frac{5}{8}x - \frac{5}{2}$$

$$y = -\frac{5}{8}x - \frac{5}{2} + \frac{10}{2}$$

$$y = -\frac{5}{8}x + \frac{5}{2}$$

So $f(x)$ is given by

$$f(x) = -\frac{5}{8}x + \frac{5}{2}$$

■ 7. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(-3) = -4$$

$$f^{-1}(3) = 12$$

Solution:

$(-3, -4)$ and $(3,12)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(-4, -3)$$

$$(12,3)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-3)}{12 - (-4)} = \frac{6}{16} = \frac{3}{8}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(-4, -3)$ and $(12,3)$. We'll use $(-4, -3)$.

$$y - y_1 = m(x - x_1)$$

$$y - (-3) = \frac{3}{8}(x - (-4))$$

$$y + 3 = \frac{3}{8}(x + 4)$$

$$y + 3 = \frac{3}{8}x + \frac{3}{2}$$

$$y = \frac{3}{8}x + \frac{3}{2} - \frac{6}{2}$$

$$y = \frac{3}{8}x - \frac{3}{2}$$

So $f(x)$ is given by

$$f(x) = \frac{3}{8}x - \frac{3}{2}$$

8. Find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(1) = 3$$

$$f^{-1}(2) = 6$$

Solution:

$(1,3)$ and $(2,6)$ are points on the inverse function $f^{-1}(x)$. Switch the x and y-values in those points in order to get points on $f(x)$.

$$(3,1)$$

$$(6,2)$$

Find the slope between these points to find the slope of $f(x)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{6-3} = \frac{1}{3}$$

Use point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the line by plugging in this slope, and either of the points $(3,1)$ and $(6,2)$. We'll use $(3,1)$.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{1}{3}(x - 3)$$

$$y - 1 = \frac{1}{3}x - 1$$

$$y = \frac{1}{3}x$$

So $f(x)$ is given by

$$f(x) = \frac{1}{3}x$$

LAWS OF LOGARITHMS

1. Write the expression as a single logarithm. Solve if possible.

$$\log_2 2 + \log_2 4$$

Solution:

Use the product rule

$$\log_a x + \log_a y = \log_a(xy)$$

to rewrite the expression as a single logarithm.

$$\log_2 2 + \log_2 4$$

$$\log_2 8$$

Simplify further by converting the logarithm into an exponential expression using the rule, if $\log_a x = y$ then $a^y = x$.

$$\log_2 8 = y$$

$$2^y = 8$$

$$y = 3$$

So the value of the logarithm is 3.

$$\log_2 2 + \log_2 4 = 3$$

2. Write the expression as a single logarithm. Solve if possible.

$$\log_3 216 - \log_3 24$$

Solution:

Use the quotient rule

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

to rewrite the expression as a single logarithm.

$$\log_3 216 - \log_3 24$$

$$\log_3 \frac{216}{24}$$

$$\log_3 9$$

Simplify further by converting the logarithm into an exponential expression using the rule, if $\log_a x = y$ then $a^y = x$.

$$\log_3 9 = y$$

$$3^y = 9$$

$$y = 2$$

So the value of the logarithm is 2.

$$\log_3 216 - \log_3 24 = 2$$

3. Write the expression as a single logarithm. Solve if possible.

$$\log_4 10 - 3\log_4 2$$

Solution:

Use the power rule

$$n\log_a x = \log_a x^n$$

to rewrite the expression.

$$\log_4 10 - 3\log_4 2$$

$$\log_4 10 - \log_4 2^3$$

$$\log_4 10 - \log_4 8$$

Use the quotient rule

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

to rewrite the expression as a single logarithm.

$$\log_4 10 - \log_4 8$$

$$\log_4 \frac{10}{8}$$

$$\log_4 \frac{5}{4}$$

■ 4. Write the expression as a single logarithm. Solve if possible.

$$2\log_7 4 + 3\log_7 5$$

Solution:

Use the power rule

$$n\log_a x = \log_a x^n$$

to rewrite the expression.

$$2\log_7 4 + 3\log_7 5$$

$$\log_7 4^2 + \log_7 5^3$$

$$\log_7 16 + \log_7 125$$

Use the product rule

$$\log_a x + \log_a y = \log_a(xy)$$

to rewrite the expression as a single logarithm.

$$\log_7 16 + \log_7 125$$

$$\log_7 2{,}000$$

5. Solve the equation.

$$\log_a 2 + \log_a 4 = \log_a(x+2)$$

Solution:

Use the product rule

$$\log_a x + \log_a y = \log_a(xy)$$

to rewrite the left side of the expression.

$$\log_a 2 + \log_a 4 = \log_a(x+2)$$

$$\log_a 8 = \log_a(x+2)$$

Since the logarithms are equal and have the same base, 8 must equal $x+2$.

$$x + 2 = 8$$

$$x = 6$$

6. Solve the equation.

$$\log_4(x+5) - \log_4(x-2) = \log_4 3$$

Solution:

Use the quotient rule

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

to rewrite the left side of the expression.

$$\log_4(x+5) - \log_4(x-2) = \log_4 3$$

$$\log_4 \left(\frac{x+5}{x-2}\right) = \log_4 3$$

Since the logarithms are equal and have the same base, $(x+5)/(x-2)$ must equal 3.

$$\frac{x+5}{x-2} = 3$$

$$x + 5 = 3(x-2)$$

$$x + 5 = 3x - 6$$

$$11 = 2x$$

$$x = \frac{11}{2}$$

■ 7. Solve the equation.

$$2\log_b x = \log_b 49$$

Solution:

Use the power rule

$$n\log_a x = \log_a x^n$$

to rewrite the left side of the expression.

$$2\log_b x = \log_b 49$$

$$\log_b x^2 = \log_b 49$$

Since the logarithms are equal and have the same base, x^2 must equal 49.

$$x^2 = 49$$

$$x = 7$$

8. Solve the equation.

$$\log_{12} x = \frac{3}{2}\log_{12} 16$$

Solution:

Use the power rule

$$n \log_a x = \log_a x^n$$

to rewrite the right side of the expression.

$$\log_{12} x = \frac{3}{2} \log_{12} 16$$

$$\log_{12} x = \log_{12} 16^{\frac{3}{2}}$$

$$\log_{12} x = \log_{12} 4^3$$

Since the logarithms are equal and have the same base, x must equal 4^3.

$$x = 4^3$$

$$x = 64$$

W W W . K R I S T A K I N G M A T H . C O M