Calculus 1 Workbook Solutions

Derivative theorems

krista king MATH

MEAN VALUE THEOREM

1. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[1,5]$.

$$f(x) = x^3 - 9x^2 + 24x - 18$$

Solution:

First, $f(x)$ is continuous and differentiable on the interval $[1,5]$. The problem says to find c in the interval such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$

Find the values you need for the numerator.

$$f(5) = 5^3 - 9(5)^2 + 24(5) - 18 = 2$$

$$f(1) = 1^3 - 9(1)^2 + 24(1) - 18 = -2$$

Then

$$\frac{f(5) - f(1)}{5 - 1} = \frac{2 - (-2)}{4} = 1$$

Take the derivative $f'(x) = 3x^2 - 18x + 24$, then set $f'(x) = 1$ and solve for x.

$$3x^2 - 18x + 24 = 1$$

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$$3x^2 - 18x + 23 = 0$$

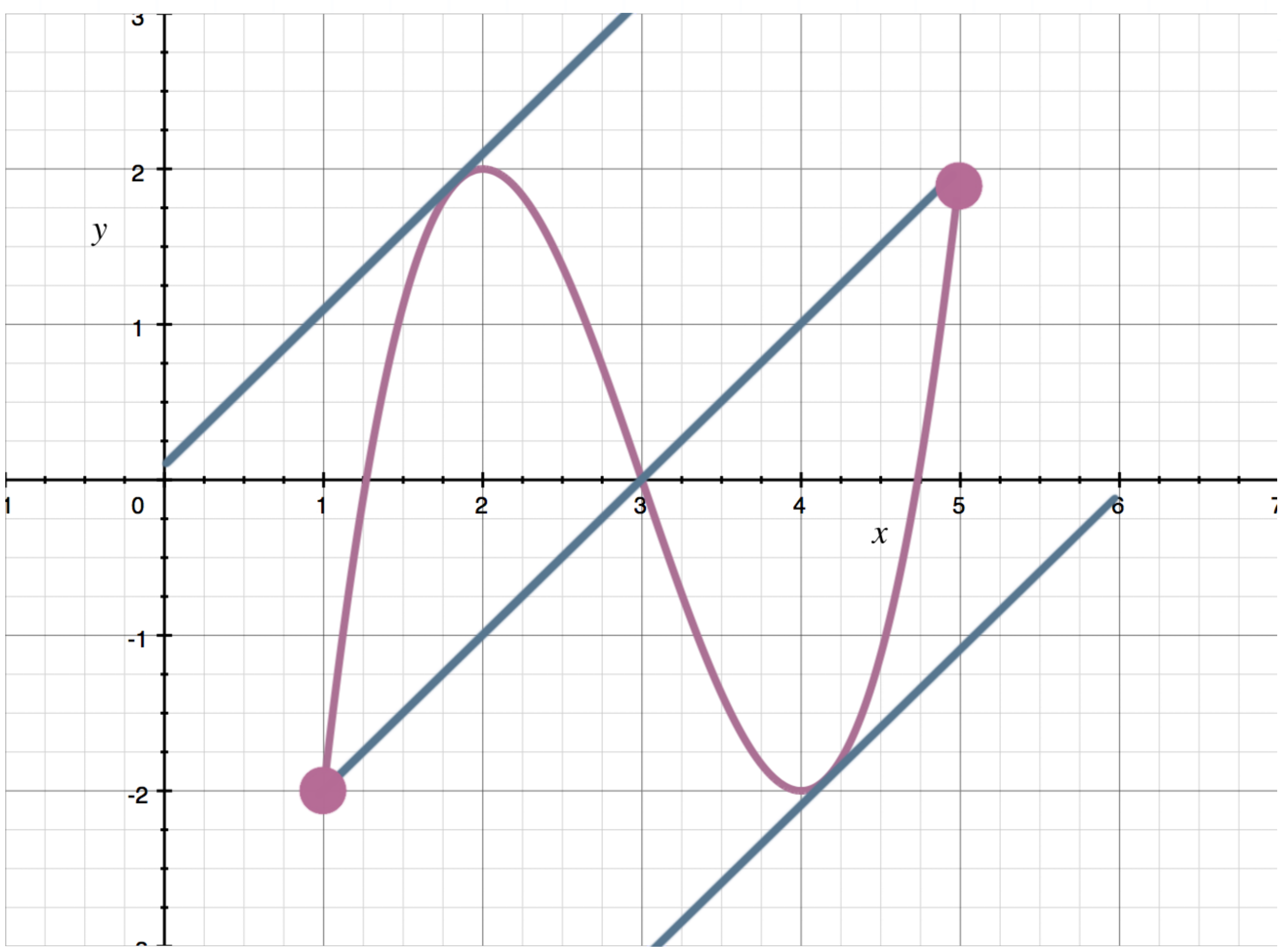
$$x = \frac{18 \pm \sqrt{18^2 - 4(3)(23)}}{2(3)} = \frac{18 \pm \sqrt{48}}{6} = \frac{18 \pm 4\sqrt{3}}{6} = \frac{9 \pm 2\sqrt{3}}{3}$$

Verify that the slope of the tangent line at these two x-values is 1.

$$f'\left(\frac{9 - 2\sqrt{3}}{3}\right) = 3\left(\frac{9 - 2\sqrt{3}}{3}\right)^2 - 18\left(\frac{9 - 2\sqrt{3}}{3}\right) + 24 = 1$$

$$f'\left(\frac{9 + 2\sqrt{3}}{3}\right) = 3\left(\frac{9 + 2\sqrt{3}}{3}\right)^2 - 18\left(\frac{9 + 2\sqrt{3}}{3}\right) + 24 = 1$$

Therefore, the values of c are $(9 \pm 2\sqrt{3})/3$. The figure illustrates how these two points satisfy the Mean Value Theorem.



■ 2. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[1,4]$.

$$g(x) = \frac{x^2 - 9}{3x}$$

Solution:

First, $g(x)$ is continuous and differentiable on the interval $[1,4]$. The problem says to find c in the interval such that

$$g'(c) = \frac{g(4) - g(1)}{4 - 1}$$

Find the values you need for the numerator.

$$g(4) = \frac{4^2 - 9}{3(4)} = \frac{16 - 9}{12} = \frac{7}{12}$$

$$g(1) = \frac{1^2 - 9}{3(1)} = \frac{1 - 9}{3} = -\frac{8}{3}$$

Then

$$\frac{g(4) - g(1)}{4 - 1} = \frac{\frac{7}{12} - \left(-\frac{8}{3}\right)}{3} = \frac{\frac{13}{4}}{3} = \frac{13}{4} \cdot \frac{1}{3} = \frac{13}{12}$$

Take the derivative,

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$$g'(x) = \frac{(3x)(2x) - (x^2 - 9)(3)}{(3x)^2} = \frac{6x^2 - 3x^2 + 27}{9x^2} = \frac{3x^2 + 27}{9x^2} = \frac{x^2 + 9}{3x^2}$$

then set $g'(x) = 13/12$ and solve for x.

$$\frac{x^2 + 9}{3x^2} = \frac{13}{12}$$

$$12x^2 + 108 = 39x^2$$

$$27x^2 = 108$$

$$x^2 = 4$$

$$x = \pm 2$$

Only $x = 2$ is in the given interval. Verify that the slope of the tangent line at this x-value is $13/12$.

$$g'(2) = \frac{2^2 + 9}{3(2)^2} = \frac{4 + 9}{3(4)} = \frac{13}{12}$$

Therefore, the value of c is 2. The figure illustrates how this point satisfies the Mean Value Theorem.

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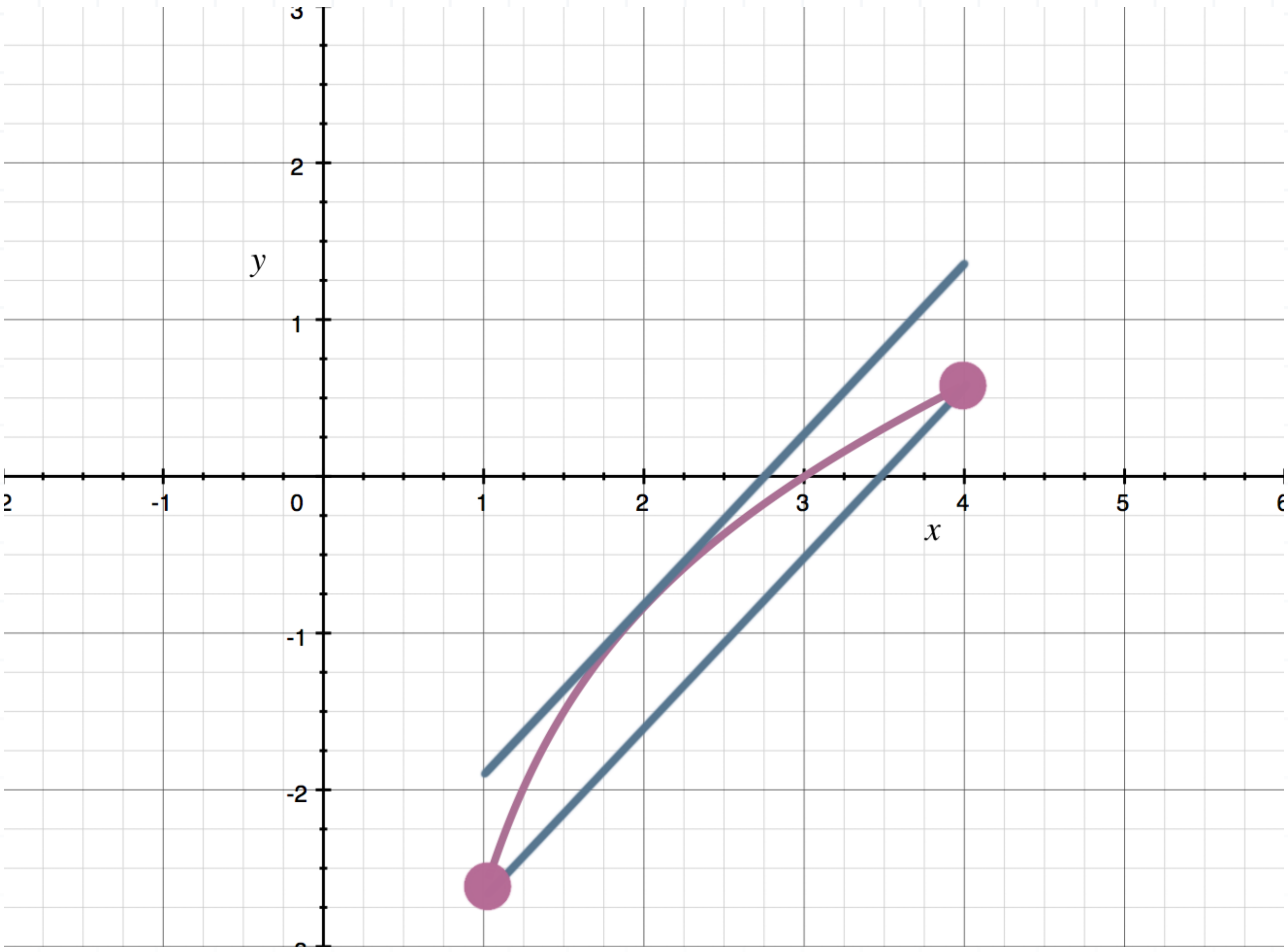

■ 3. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[0,5]$.

$$h(x) = -\sqrt{25 - 5x}$$

Solution:

First, $h(x)$ is continuous and differentiable on the interval $[0,5]$. The problem says to find c in the interval such that

$$h'(c) = \frac{h(5) - h(0)}{5 - 0}$$

Find the values you need for the numerator.

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$$h(5) = -\sqrt{25 - 5(5)} = -\sqrt{0} = 0$$

$$h(0) = -\sqrt{25 - 5(0)} = -\sqrt{25} = -5$$

Then

$$\frac{h(5) - h(0)}{5 - 0} = \frac{0 - (-5)}{5} = 1$$

Take the derivative,

$$h'(x) = -\frac{-5}{2\sqrt{25 - 5x}} = \frac{5}{2\sqrt{25 - 5x}}$$

then set $h'(x) = 1$ and solve for x.

$$\frac{5}{2\sqrt{25 - 5x}} = 1$$

$$5 = 2\sqrt{25 - 5x}$$

$$\frac{5}{2} = \sqrt{25 - 5x}$$

$$\frac{25}{4} = 25 - 5x$$

$$x = \left(\frac{25}{4} - 25\right) \div -5$$

$$x = \frac{15}{4}$$

Verify that the slope of the tangent line at this x-value is 1.

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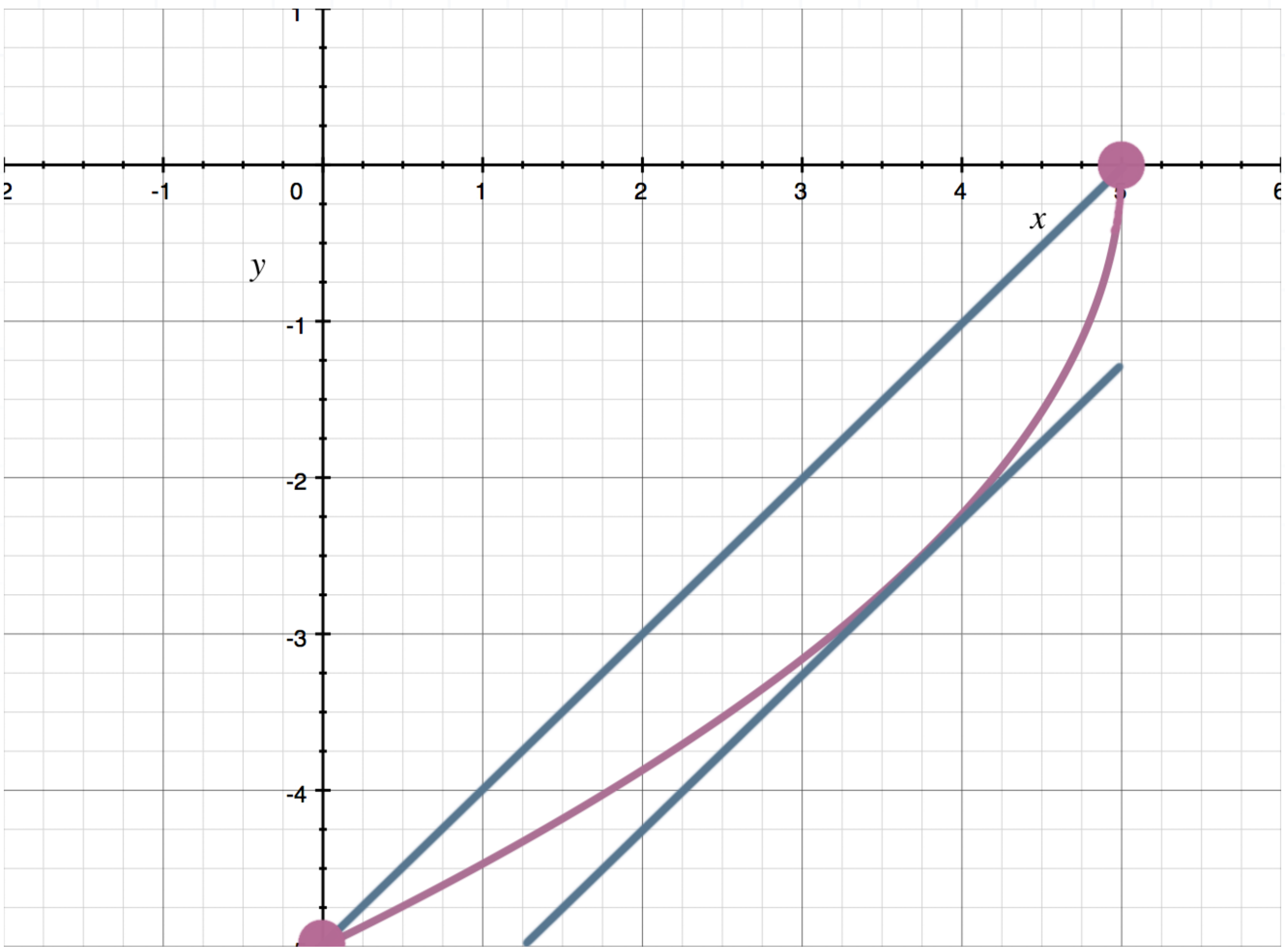
$$h'\left(\frac{15}{4}\right)=\frac{5}{2\sqrt{25-5\left(\frac{15}{4}\right)}}=\frac{5}{2\sqrt{\frac{25}{4}}}=\frac{5}{2\left(\frac{5}{2}\right)}=\frac{5}{5}=1$$

Therefore, the value of c is $15/4$. The figure illustrates how this point satisfies the Mean Value Theorem.



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ROLLE'S THEOREM

■ 1. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-1,2]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$f(x) = x^3 - 2x^2 - x - 3$$

Solution:

The function $f(x)$ is continuous and differentiable on the interval $[-1,2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-1,2]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(2) = f(-1)$.

$$f(2) = 2^3 - 2(2)^2 - 2 - 3 = -5$$

$$f(-1) = (-1)^3 - 2(-1)^2 - (-1) - 3 = -5$$

Thus, Rolle's Theorem applies. Next, find $f'(x) = 3x^2 - 4x - 1$ and set $f'(x) = 0$ and solve for x using the quadratic formula.

$$3x^2 - 4x - 1 = 0$$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-1)}}{2(3)} = \frac{4 \pm \sqrt{28}}{6} = \frac{4 \pm 2\sqrt{7}}{6} = \frac{2 \pm \sqrt{7}}{3}$$

Verify that the slope of the tangent line at these two x-values is 0.

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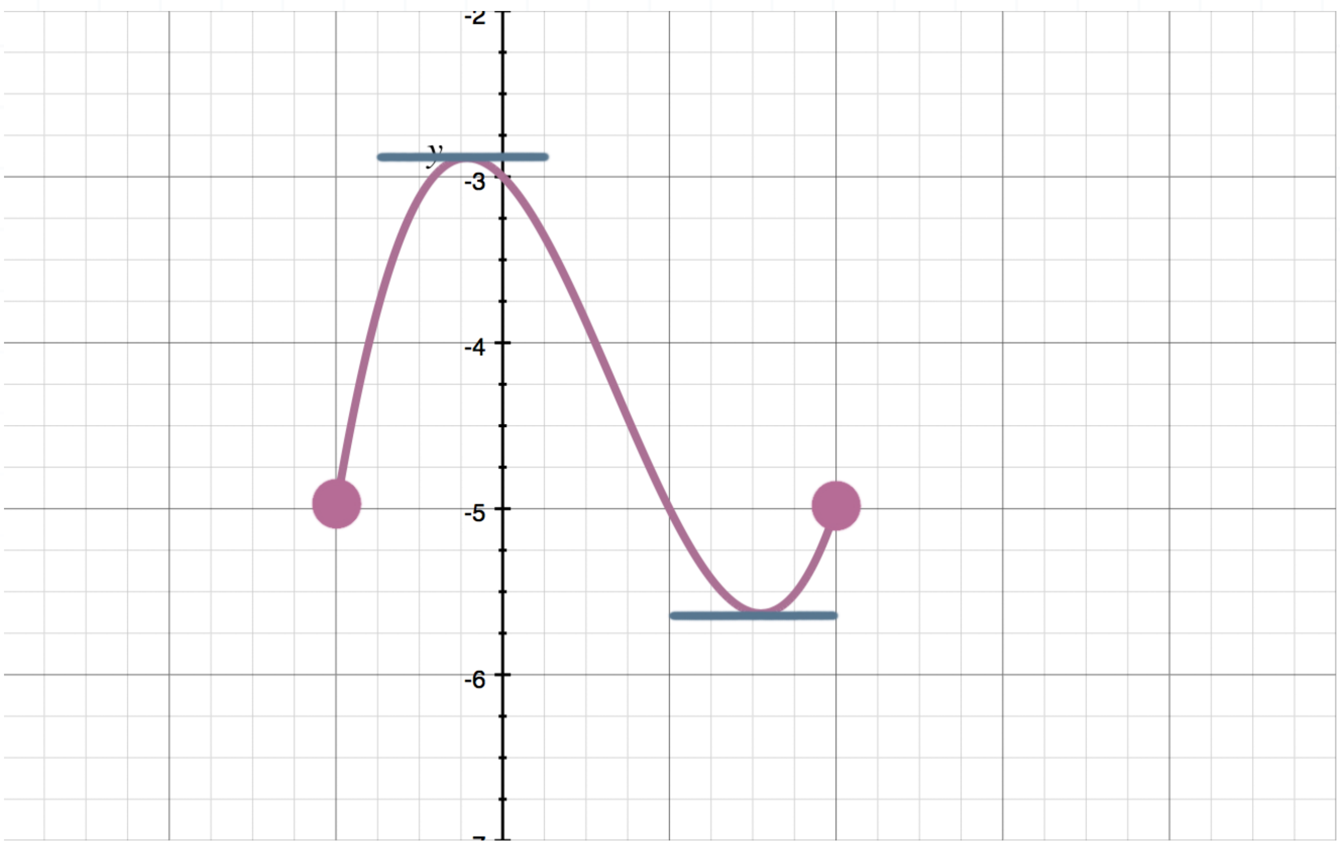
$$f'\left(\frac{2-\sqrt{7}}{3}\right)=3\left(\frac{2-\sqrt{7}}{3}\right)^2-4\left(\frac{2-\sqrt{7}}{3}\right)-1=0$$

$$f'\left(\frac{2+\sqrt{7}}{3}\right)=3\left(\frac{2+\sqrt{7}}{3}\right)^2-4\left(\frac{2+\sqrt{7}}{3}\right)-1=0$$

Therefore, the values of c such that $f'(c)=0$ are $(2\pm\sqrt{7})/3$. The figure illustrates how these two points satisfy Rolle's Theorem.



■ 2. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-3,5]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$g(x)=\frac{x^2-2x-15}{6-x}$$

Solution:

The function $g(x)$ is continuous and differentiable on the interval $[-3,5]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-3,5]$, such that $g'(c)=0$.

To use Rolle's Theorem, show that $g(5)=g(-3)$.

$$g(5)=\frac{5^2-2(5)-15}{6-5}=\frac{0}{1}=0$$

$$g(-3)=\frac{(-3)^2-2(-3)-15}{6-(-3)}=\frac{0}{9}=0$$

Thus, Rolle's Theorem applies. Next, find

$$g'(x)=\frac{(6-x)(2x-2)-(x^2-2x-15)(-1)}{(6-x)^2}=\frac{-x^2+12x-27}{(6-x)^2}$$

and set $g'(x)=0$ and solve for x using the quadratic formula.

$$-x^2+12x-27=0$$

$$-(x^2-12x+27)=0$$

$$-(x-3)(x-9)=0$$

$$x=3,\ 9$$

The value $x=9$ is outside of the given interval. Verify that the slope of the tangent line at $x=3$ is 0.

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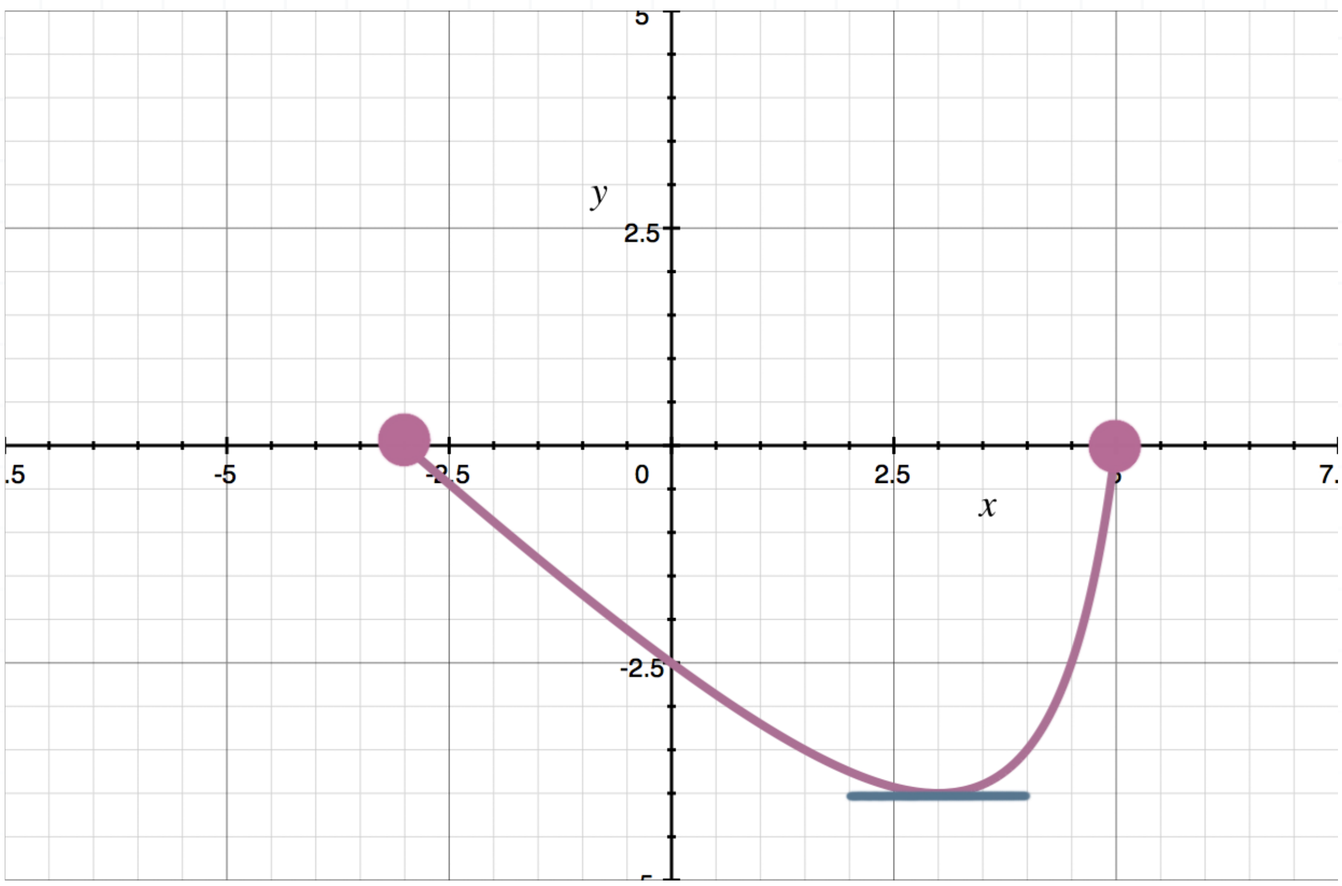
$$g'(3) = \frac{-3^2 + 12(3) - 27}{(6-3)^2} = \frac{0}{9} = 0$$

Therefore, the value of c such that $f'(c) = 0$ is 3. The figure illustrates how this point satisfies Rolle's Theorem.



■ 3. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-\pi/2,\pi/2]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$h(x) = \sin(2x)$$

Solution:

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The function $h(x)$ is continuous and differentiable on the interval $[-\pi/2,\pi/2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-\pi/2,\pi/2]$, such that $h'(c)=0$.

To use Rolle's Theorem, show that $h(\pi/2)=h(-\pi/2)$.

$$h\left(\frac{\pi}{2}\right)=\sin\left(2\cdot\frac{\pi}{2}\right)=\sin(\pi)=0$$

$$h\left(-\frac{\pi}{2}\right)=\sin\left(2\cdot-\frac{\pi}{2}\right)=\sin(-\pi)=0$$

Thus, Rolle's Theorem applies. Next, find $h'(x)=2\cos(2x)$ and set $h'(x)=0$ and solve for x.

$$2\cos(2x)=0$$

$$\cos(2x)=0$$

$$\arccos(0)=2x$$

$$2x=\pm\frac{\pi}{2}$$

$$x=\pm\frac{\pi}{4}$$

Verify that the slope of the tangent line at these two x-values is 0.

$$h'\left(-\frac{\pi}{4}\right)=2\cos\left(-\frac{\pi}{2}\right)=2\cdot 0=0$$

$$h'\left(\frac{\pi}{4}\right)=2\cos\left(\frac{\pi}{2}\right)=2\cdot 0=0$$

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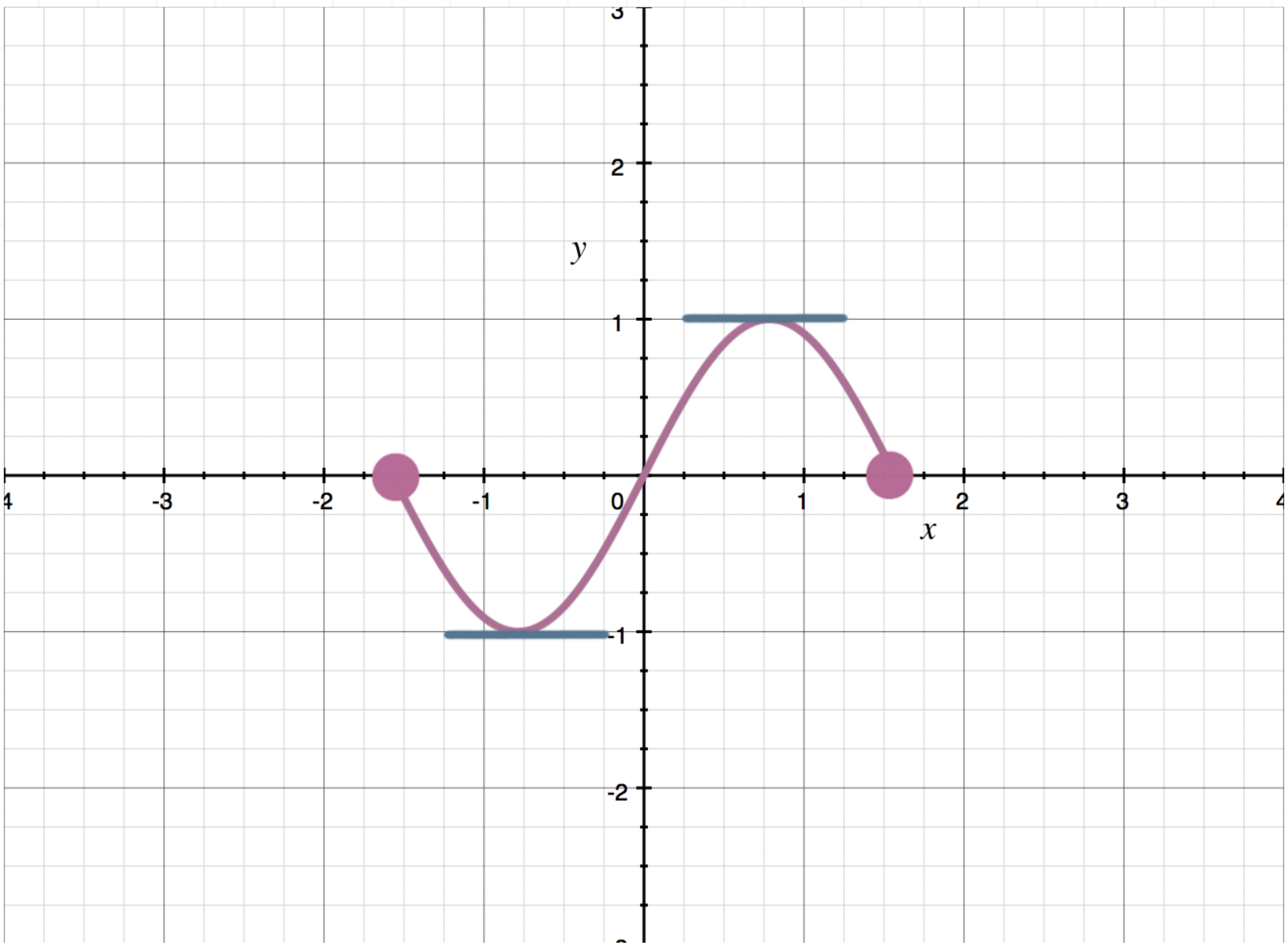
Therefore, the values of c such that $f'(c)=0$ are $\pm\pi/4$. The figure illustrates how these two points satisfy Rolle's Theorem.



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NEWTON'S METHOD

1. Use four iterations of Newton's method to approximate the root of $g(x) = x^3 - 12$ in the interval $[1,3]$. Give the answer to the nearest three decimal places.

Solution:

If $g(x) = x^3 - 12$ and $g'(x) = 3x^2$, and we start with an initial estimate of $x_0 = 1.5$, then $g(1.5) = -8.625$ and $g'(1.5) = 6.75$. Plug those values into the Newton's method formula.

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(n_n)}$$

$$x_1 = 1.5 - \frac{-8.625}{6.75} \approx 2.777$$

Next, $g(2.777) = 9.4155$ and $g'(2.777) = 23.1352$. So

$$x_2 = 2.777 - \frac{9.4155}{23.1352} = 2.3700$$

Next, $g(2.3700) = 1.3121$ and $g'(2.3700) = 16.851$. So

$$x_3 = 2.3700 - \frac{1.3121}{16.851} = 2.2921$$

Next, $g(2.2921) = 0.04206$ and $g'(2.2921) = 15.7612$. So

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$$x_4 = 2.2921 - \frac{0.04206}{15.7612} = 2.2894$$

■ 2. Use four iterations of Newton's method to approximate the root of $f(x) = x^4 - 15$ in the interval $[-2, -1]$. Give the answer to the nearest four decimal places.

Solution:

If $f(x) = x^4 - 14$ and $f'(x) = 4x^3$, and we start with an initial estimate of $x_0 = -1.5$, then $f(-1.5) = -9.8375$ and $f'(-1.5) = -13.5$. Plug those values into the Newton's method formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(n_n)}$$

$$x_1 = -1.5 - \frac{-9.9375}{-13.5} = -2.2361$$

Next, $f(-2.2361) = 10.0014$ and $f'(-2.2361) = -44.7233$. So

$$x_2 = -2.2361 - \frac{10.0014}{-44.7233} = -2.0125$$

Next, $f(-2.0125) = 1.4038$ and $f'(-2.0125) = -32.6038$. So

$$x_3 = -2.0125 - \frac{1.4038}{-32.6038} = -1.9694$$

Next, $f(-1.9694) = 0.0434$ and $f'(-1.9694) = -30.5536$. So

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$$x_4 = -1.9694 - \frac{0.0434}{-30.5536} = -1.9680$$

■ 3. Use four iterations of Newton's method to approximate the root of $h(x) = 3e^{x-3} - 4 + \sin x$ in the interval $[2,4]$. Give the answer to the nearest four decimal places.

Solution:

If $h(x) = 3e^{x-3} - 4 + \sin x$ and $h'(x) = 3e^{x-3} + \cos x$, and we start with an initial estimate of $x_0 = 3$, then $h(3) = -0.8589$ and $h'(3) = 2.0100$. Plug those values into the Newton's method formula.

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(n_n)}$$

$$x_1 = 3 - \frac{-0.8589}{2.0100} = 3.4273$$

Next, $h(3.4273) = 0.3175$ and $h'(3.4273) = 3.6399$. So

$$x_2 = 3.4273 - \frac{0.3175}{3.6399} = 3.3401$$

Next, $h(3.3401) = 0.0181$ and $h'(3.3401) = 3.2349$. So

$$x_3 = 3.3401 - \frac{0.0181}{3.2349} = 3.3345$$

Next, $h(3.3345) = 0.00001$ and $h'(3.3345) = 3.2103$. So

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$$x_4 = 3.3345 - \frac{0.00001}{3.2103} = 3.3345$$

L'HOSPITAL'S RULE

1. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to 0} \frac{2\sqrt{x+4} - 4 - \frac{1}{2}x}{x^2}$$

Solution:

Evaluating the limit as $x \to 0$ gives the indeterminate form $0/0$, so we'll use L'Hospital's rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to 0} \frac{\frac{1}{\sqrt{x+4}} - \frac{1}{2}}{2x}$$

But evaluating this $x \to 0$ still gives $0/0$, so we'll apply L'Hospital's rule again.

$$\lim_{x \to 0} \frac{-\frac{1}{2\sqrt{(x+4)^3}}}{2} = \lim_{x \to 0} -\frac{1}{4\sqrt{(x+4)^3}}$$

Then we can evaluate as $x \to 0$.

$$-\frac{1}{4\sqrt{(0+4)^3}} = -\frac{1}{4\sqrt{64}} = -\frac{1}{4(8)} = -\frac{1}{32}$$

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■ 2. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{3 + \tan x}$$

Solution:

Evaluating the limit as $x \to \pi/2$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to \frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x} \cdot \frac{\cos x}{1} = \lim_{x \to \frac{\pi}{2}} \sin x$$

Then we can evaluate as $x \to \pi/2$.

$$\sin \frac{\pi}{2} = 1$$

■ 3. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \infty} \frac{\ln x}{4\sqrt{x}}$$

Solution:

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Evaluating the limit as $x \to \infty$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{2}{\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{\sqrt{x}}{2} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}}$$

Then we can evaluate as $x \to \infty$.

$$\lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0$$

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