Calculus 1 Workbook Solutions

Squeeze theorem

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SQUEEZE THEOREM

1. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) - 2$$

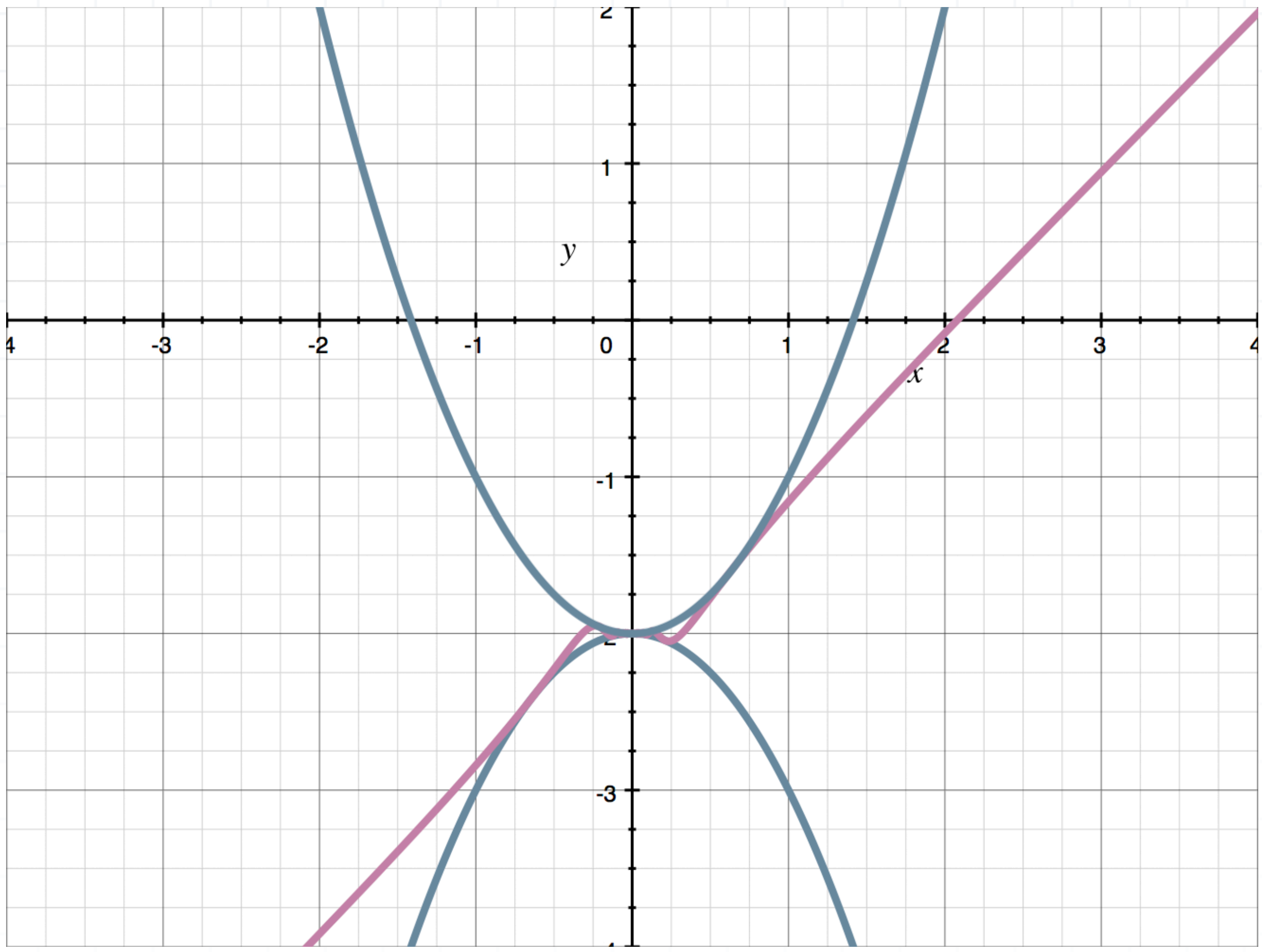
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -x^2 - 2$$

$$g(x) = x^2 \sin\left(\frac{1}{x}\right) - 2$$

$$h(x) = x^2 - 2$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x\to 0} f(x) \leq \lim_{x\to 0} g(x) \leq \lim_{x\to 0} h(x)$$

$$\lim_{x\to 0}(-x^2-2) \leq \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) - 2 \leq \lim_{x\to 0}(x^2-2)$$

We can evaluate the limits on the left and right sides.

$$-0^2 - 2 \leq \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) - 2 \leq 0^2 - 2$$

$$-2 \leq \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) - 2 \leq -2$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be -2.

2. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x\to\infty} \frac{3\sin x}{4x}$$

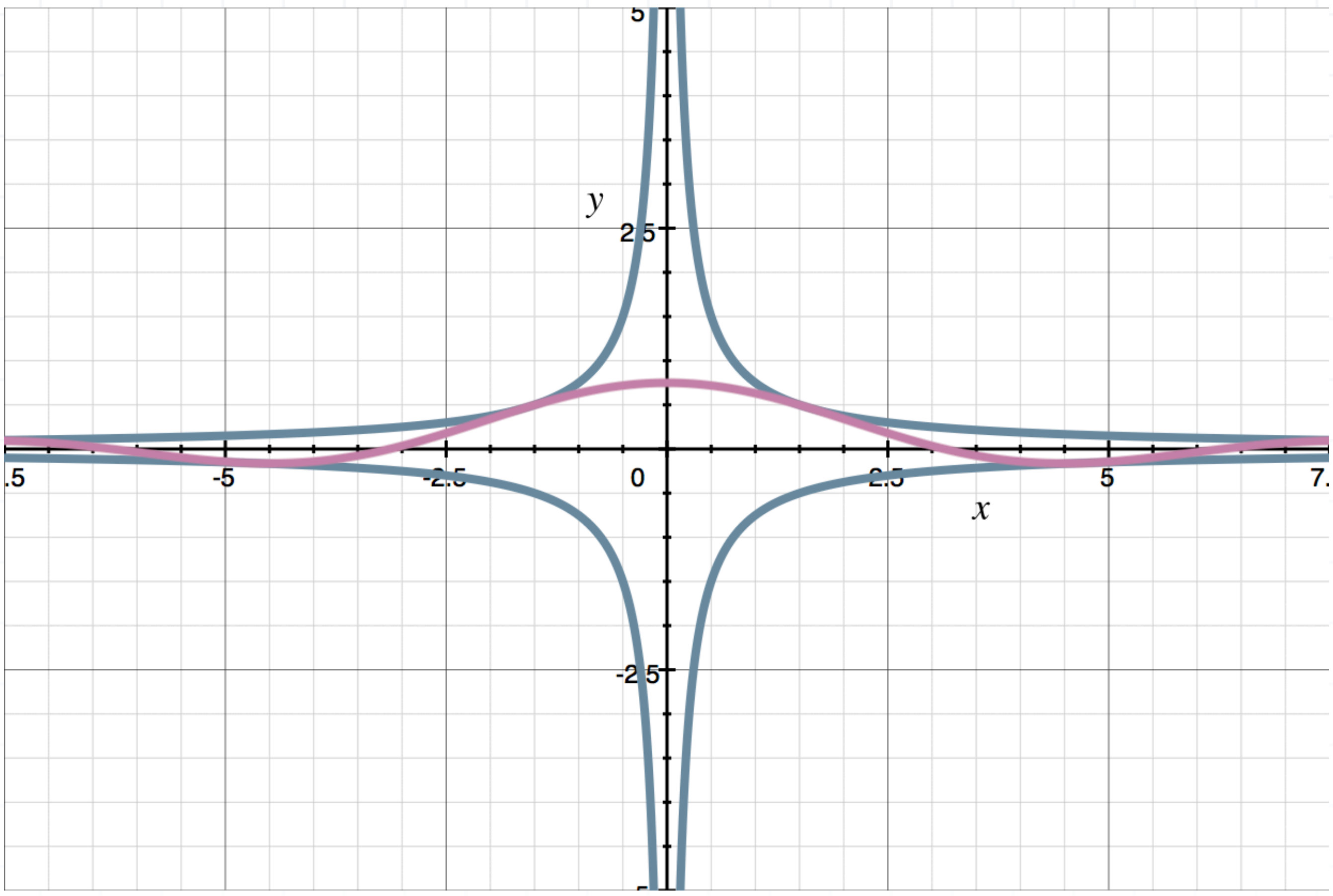
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -\frac{3}{4x}$$

$$g(x) = \frac{3\sin x}{4x}$$

$$h(x) = \frac{3}{4x}$$



Notice that $f(x) \le g(x) \le h(x)$. Therefore,

$$\lim_{x\to\infty} f(x) \le \lim_{x\to\infty} g(x) \le \lim_{x\to\infty} h(x)$$

$$\lim_{x\to\infty}\left(-\frac{3}{4x}\right) \le \lim_{x\to\infty}\frac{3\sin x}{4x} \le \lim_{x\to\infty}\left(\frac{3}{4x}\right)$$

We can evaluate the limits on the left and right sides.

$$0 \le \lim_{x\to\infty}\frac{3\sin x}{4x} \le 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

■ 3. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) + 1$$

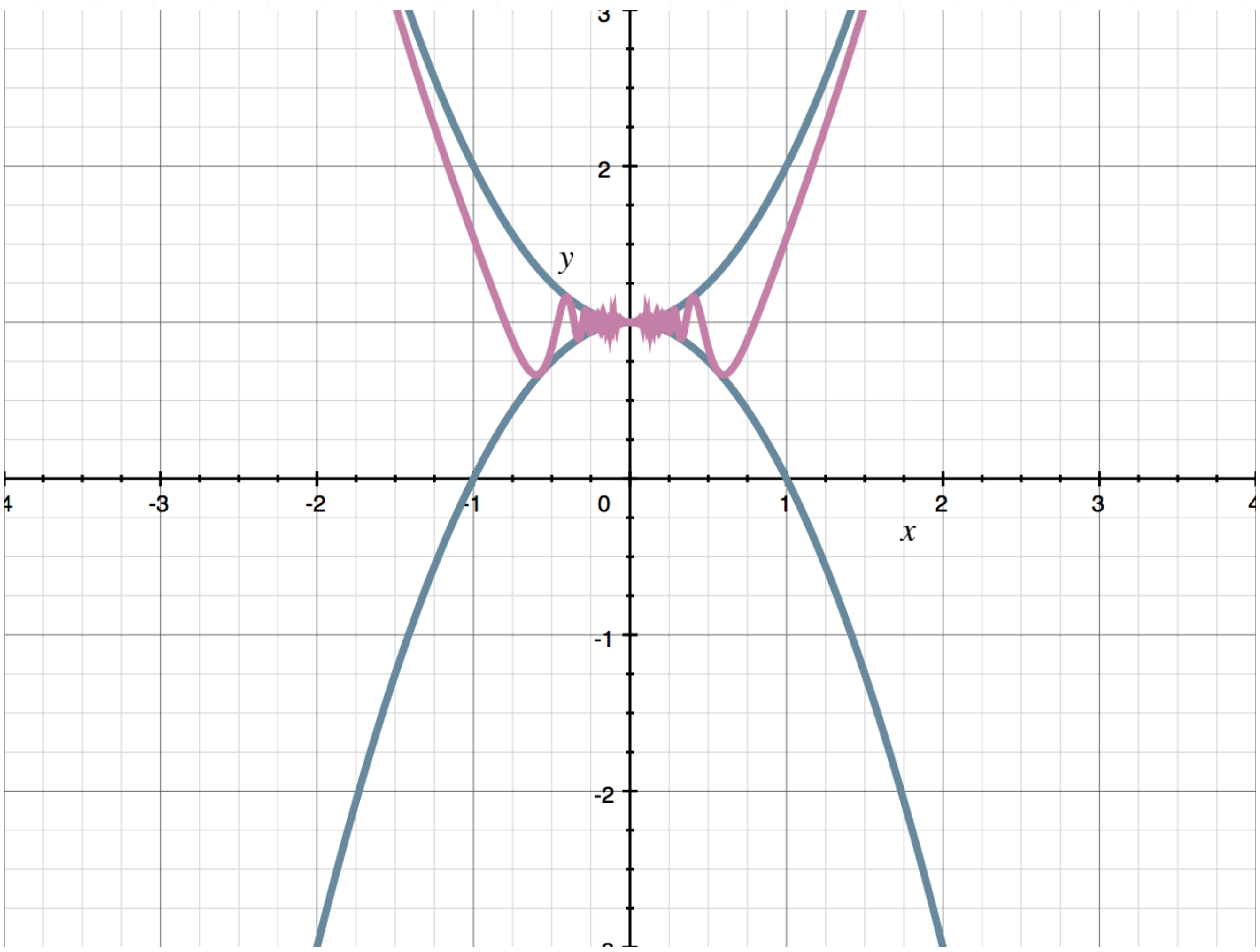
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -x^2 + 1$$

$$g(x) = x^2 \cos\left(\frac{1}{x^2}\right) + 1$$

$$h(x) = x^2 + 1$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x\to 0} f(x) \le \lim_{x\to 0} g(x) \le \lim_{x\to 0} h(x)$$

$$\lim_{x\to 0} -x^2+1 \le \lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right)+1 \le \lim_{x\to 0} x^2+1$$

We can evaluate the limits on the left and right sides.

$$-0^2+1 \le \lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right)+1 \le 0^2+1$$

$$1 \le \lim_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right)+1 \le 1$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 1.

■ 4. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x\to\infty} \frac{e^{-x}}{x}$$

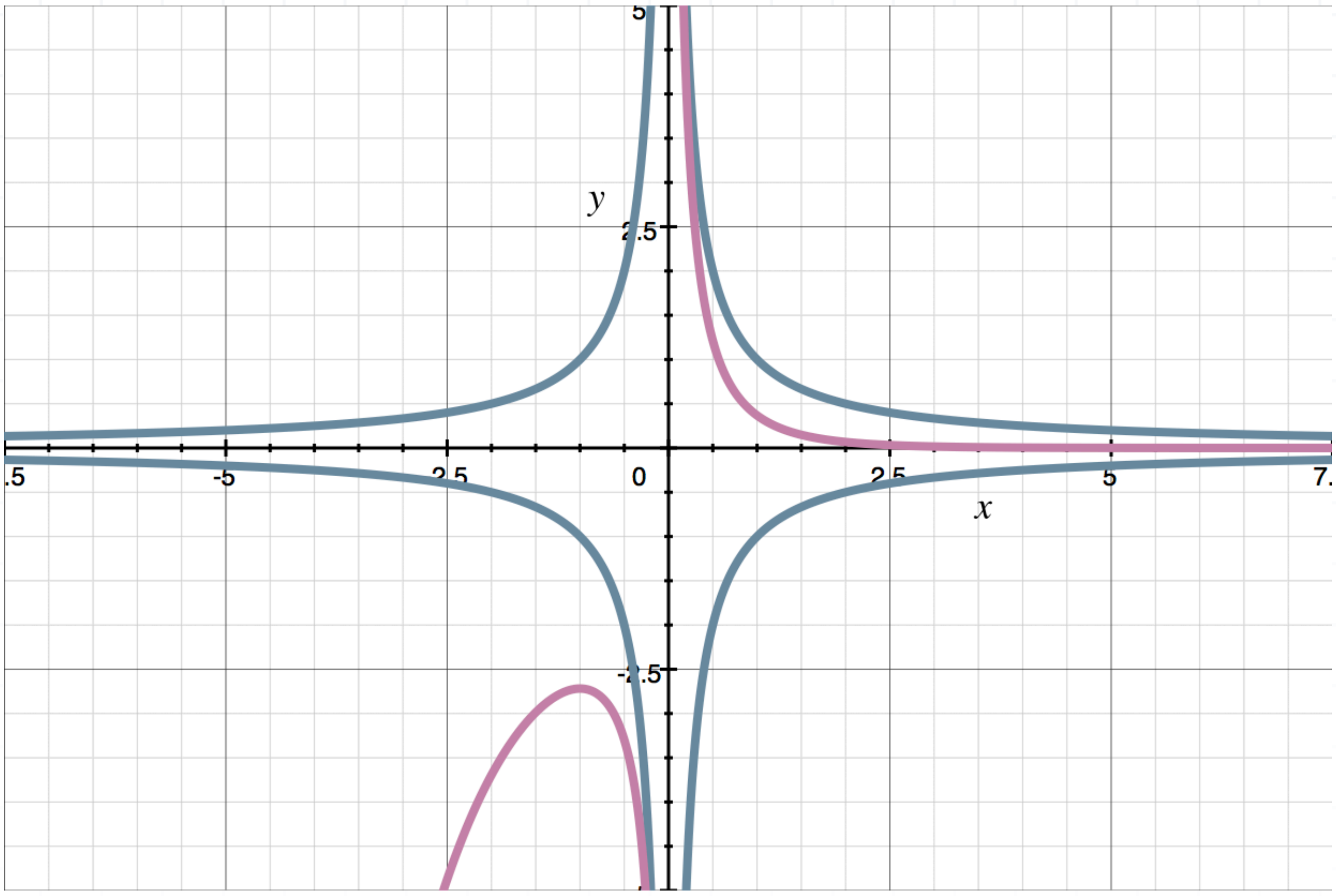
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -\frac{1}{x}$$

$$g(x) = \frac{e^{-x}}{x}$$

$$h(x) = \frac{1}{x}$$



Notice that $f(x) \le g(x) \le h(x)$. Therefore,

$$\lim_{x\to\infty} f(x) \le \lim_{x\to\infty} g(x) \le \lim_{x\to\infty} h(x)$$

$$\lim_{x\to\infty} \left(-\frac{1}{x}\right) \le \lim_{x\to\infty} \frac{e^{-x}}{x} \le \lim_{x\to\infty} \left(\frac{1}{x}\right)$$

We can evaluate the limits on the left and right sides.

$$0 \le \lim_{x\to\infty} \frac{e^{-x}}{x} \le 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

5. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

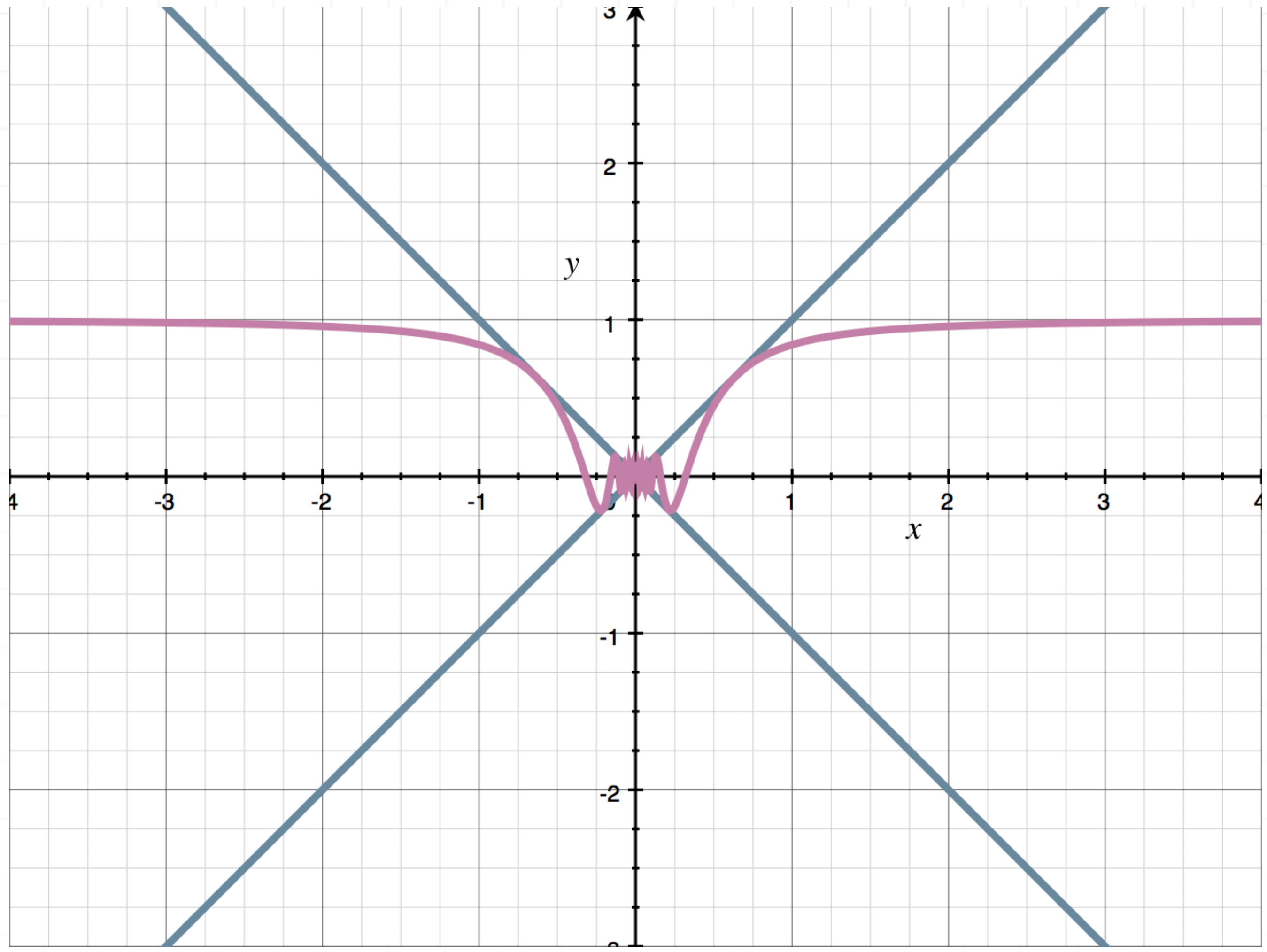
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -|x|$$

$$g(x) = \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$$h(x) = |x|$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x \to 0} f(x) \leq \lim_{x \to 0} g(x) \leq \lim_{x \to 0} h(x)$$

$$\lim_{x \to 0} -|x| \leq \lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \leq \lim_{x \to 0} |x|$$

We can evaluate the limits on the left and right sides.

$$-|0| \leq \lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \leq |0|$$

$$0 \leq \lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \leq 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

■ 6. Find $\lim_{x \to 4} f(x)$ if $x^2 + 1 \le f(x) \le 4x + 1$.

Solution:

Apply the limit to each part of the inequality.

$$x^2 + 1 \le f(x) \le 4x + 1$$

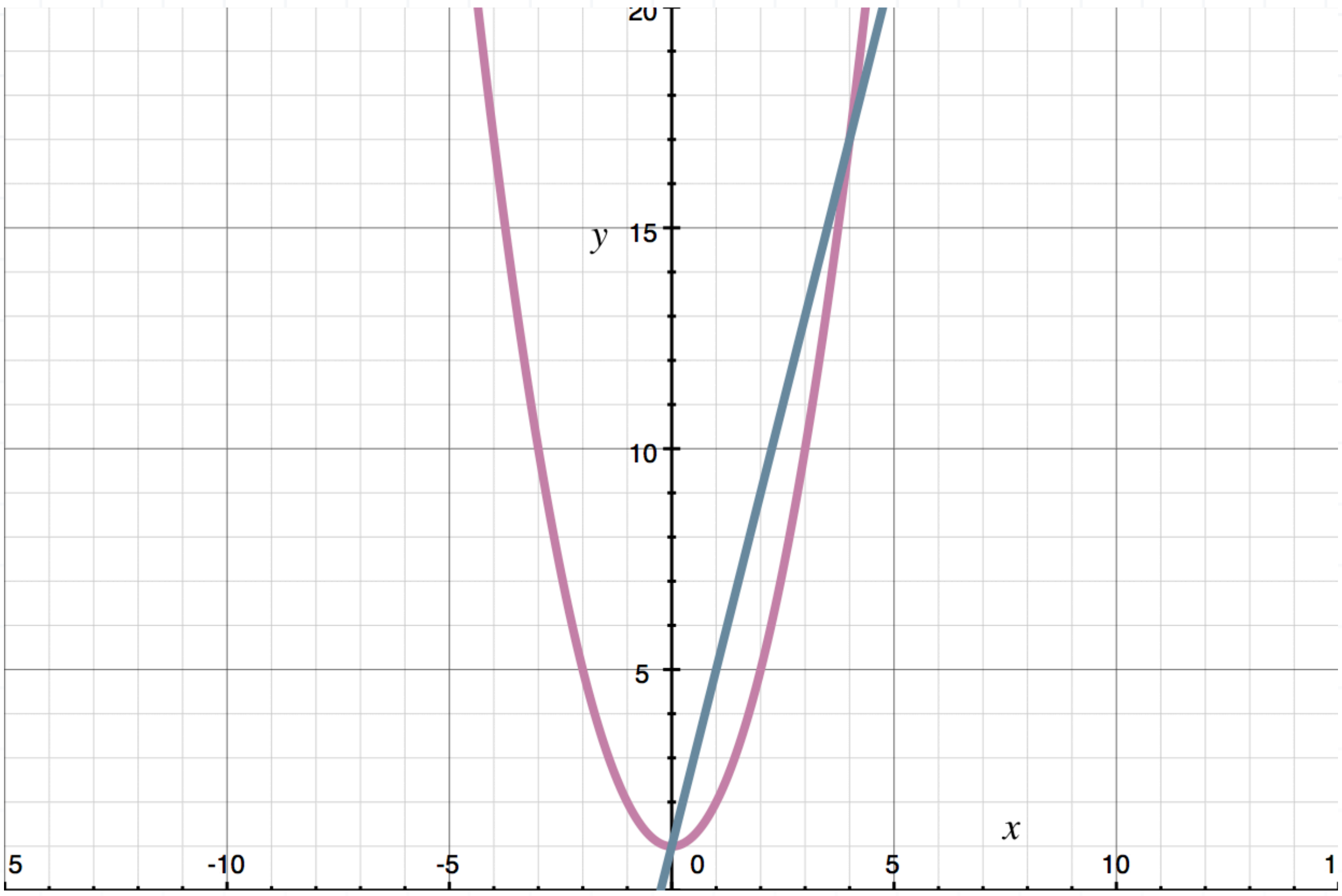
$$\lim_{x \to 4} x^2 + 1 \le \lim_{x \to 4} f(x) \le \lim_{x \to 4} 4x + 1$$

Evaluate the limits on the left and right sides using substitution.

$$4^2 + 1 \le \lim_{x \to 4} f(x) \le 4(4) + 1$$

$$17 \le \lim_{x \to 4} f(x) \le 17$$

Therefore, by the Squeeze Theorem, $\lim_{x \to 4} f(x) = 17$. The graph below shows the limit at the intersection point.



■ 7. Find $\lim_{x\to 3} g(x)$ if $x^2 - 7 \le g(x) \le \sqrt{13 - x^2}$.

Solution:

Apply the limit to each part of the inequality.

$$x^2 - 7 \le g(x) \le \sqrt{13 - x^2}$$

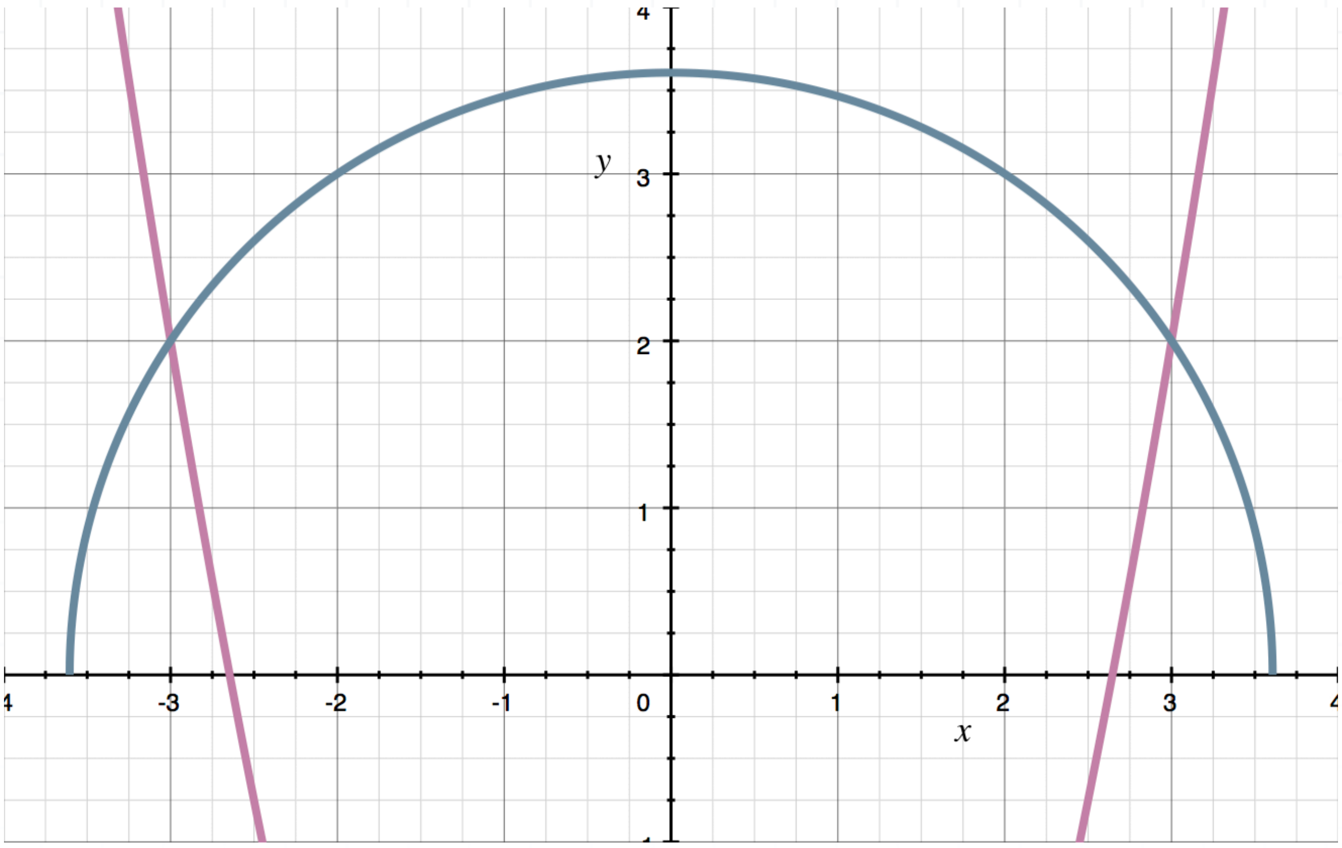
$$\lim_{x\to 3} x^2 - 7 \le \lim_{x\to 3} g(x) \le \lim_{x\to 3} \sqrt{13 - x^2}$$

Evaluate the limits on the left and right sides using substitution.

$$3^2 - 7 \le \lim_{x\to 3} g(x) \le \sqrt{13 - 3^2}$$

$$2 \le \lim_{x \to 3} g(x) \le 2$$

Therefore, by the Squeeze Theorem, $\lim_{x \to 3} g(x) = 2$. The graph below shows the limit at the intersection point.



■ 8. Find $\lim_{x \to 5} h(x)$ if $x^2 - 6x + 9 \le h(x) \le x - 1$.

Solution:

Apply the limit to each part of the inequality.

$$x^2 - 6x + 9 \le h(x) \le x - 1$$

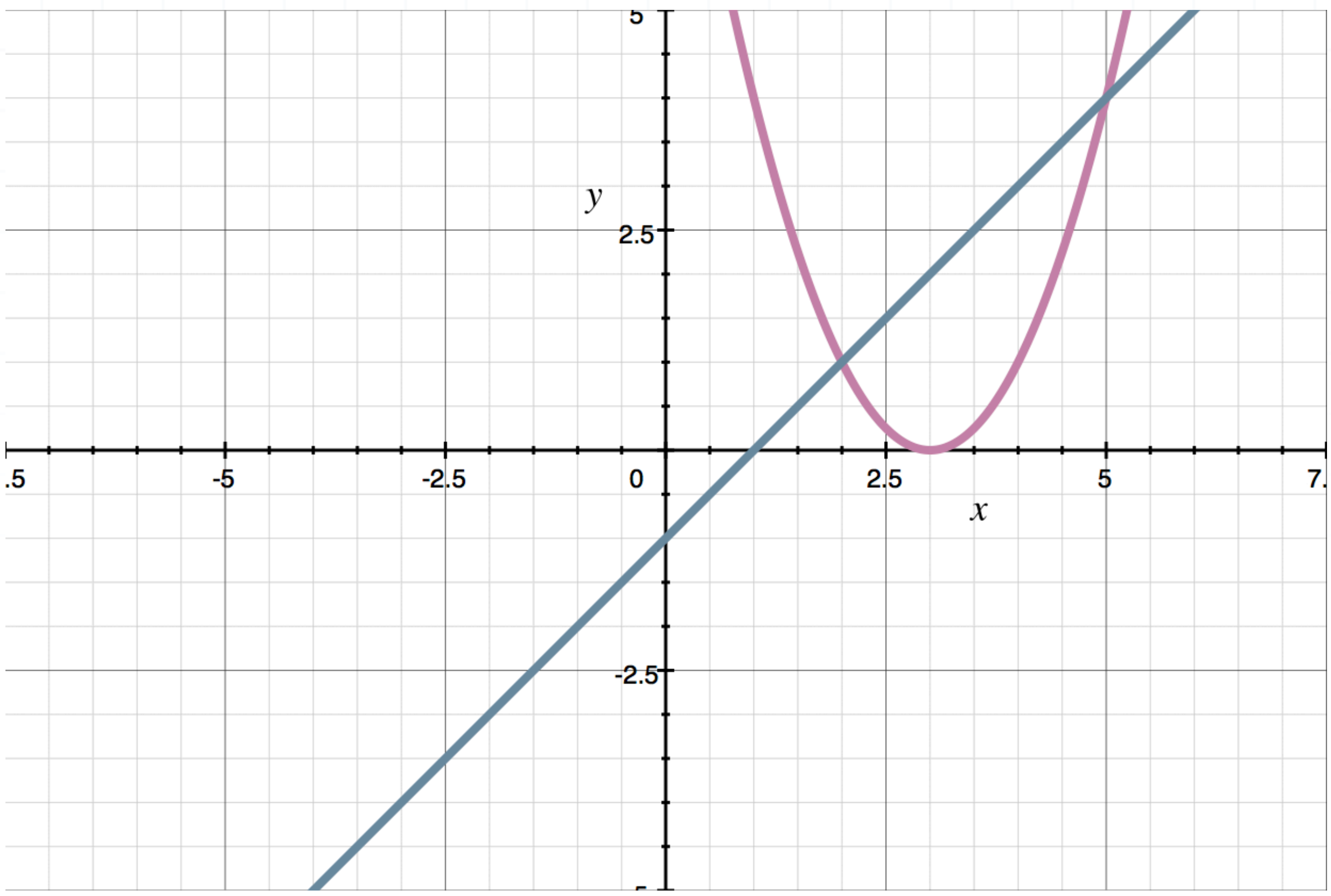
$$\lim_{x \to 5} x^2 - 6x + 9 \le \lim_{x \to 5} h(x) \le \lim_{x \to 5} x - 1$$

Evaluate the limits on the left and right sides using substitution.

$$5^2 - 6(5) + 9 \leq \lim_{x \to 5} h(x) \leq 5 - 1$$

$$4 \leq \lim_{x \to 5} h(x) \leq 4$$

Therefore, by the Squeeze Theorem, $\lim_{x \to 5} h(x) = 4$. The graph below shows the limit at the intersection point.



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