Calculus 1 Workbook Solutions

Linear approximation

krista king
MATH

LINEAR APPROXIMATION

1. Find the linearization of $f(x) = x^3 - 4x^2 + 2x - 3$ at $x = 3$ and use the linearization to approximate $f(3.02)$.

Solution:

The linearization formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 3$, so the linearization is $L(x) = f(3) + f'(3)(x - 3)$. Find the pieces that we need for the linearization formula.

$$f(3) = 3^3 - 4(3)^2 + 2(3) - 3 = -6$$

$$f'(x) = 3x^2 - 8x + 2$$

$$f'(3) = 3(3)^2 - 8(3) + 2 = 5$$

Plugging these pieces into the linearization gives

$$L(x) = -6 + 5(x - 3)$$

$$L(x) = -6 + 5x - 15$$

$$L(x) = 5x - 21$$

Use this equation to approximate $f(3.02)$.

$$f(3.02) = 5(3.02) - 21 = -5.9$$

■ 2. Find the linearization of $g(x)=\sqrt{8x-15}$ at $x=8$ and use the linearization to approximate $f(8.05)$.

Solution:

The linearization formula at $x=a$ is $L(x)=g(a)+g'(a)(x-a)$. In this problem, $a=8$, so the linearization is $L(x)=g(8)+g'(8)(x-8)$. Find the pieces that we need for the linearization formula.

$$g(8)=\sqrt{8(8)-15}=\sqrt{49}=7$$

$$g'(x)=\frac{8}{2\sqrt{8x-15}}=\frac{4}{\sqrt{8x-15}}$$

$$g'(8)=\frac{4}{\sqrt{8(8)-15}}=\frac{4}{\sqrt{49}}=\frac{4}{7}$$

Plugging these pieces into the linearization gives

$$L(x)=7+\frac{4}{7}(x-8)$$

$$L(x)=7+\frac{4}{7}x-\frac{32}{7}$$

$$L(x)=\frac{4}{7}x+\frac{17}{7}$$

Use this equation to approximate $g(8.05)$.

$$g(8.05)=\frac{4}{7}(8.05)+\frac{17}{7}=\frac{246}{35}\approx 7.029$$

■ 3. Find the linearization of $h(x) = 2e^{x-4} + 6$ at $x = 5$ and use the linearization to approximate $h(5.1)$.

Solution:

The linearization formula at $x = a$ is $L(x) = h(a) + h'(a)(x - a)$. In this problem, $a = 5$, so the linearization is $L(x) = h(5) + h'(5)(x - 5)$. Find the pieces that we need for the linearization formula.

$$h(5) = 2e^{5-4} + 6 = 2e + 6$$

$$h'(x) = 2e^{x-4}$$

$$h'(5) = 2e^{5-4} = 2e$$

Plugging these pieces into the linearization gives

$$L(x) = 2e + 6 + 2e(x - 5)$$

$$L(x) = 2e + 6 + 2ex - 10e$$

$$L(x) = 2ex - 8e + 6$$

Use this equation to approximate $h(5.1)$.

$$g(5.1) = 2e(5.1) - 8e + 6 = 10.2e - 8e + 6 = 2.2e + 6 \approx 11.98$$

ESTIMATING A ROOT

1. Use linear approximation to estimate $\sqrt[5]{34}$.

Solution:

Let $f(x)=\sqrt[5]{x}$ and $a=32$. The linear approximation would be given by

$$L(x)=f(a)+f'(a)(x-a)$$

$$L(x)=f(32)+f'(32)(x-32)$$

Find the pieces needed for the formula.

$$f(32)=\sqrt[5]{32}=2$$

$$f(x)=\sqrt[5]{x}=x^{\frac{1}{5}}$$

$$f'(x)=\frac{1}{5}(x)^{-\frac{4}{5}}=\frac{1}{5\cdot\sqrt[5]{x^4}}$$

$$f'(32)=\frac{1}{5\cdot\sqrt[5]{32^4}}=\frac{1}{80}$$

Then the linear approximation is

$$L(x)=2+\frac{1}{80}(x-32)$$

$$L(x) = \frac{1}{80}x + \frac{8}{5}$$

Use this approximation to estimate $\sqrt[5]{34}$.

$$f(34) = \frac{1}{80}(34) + \frac{8}{5} = \frac{81}{40}$$

■ 2. Use linear approximation to estimate $\sqrt[8]{260}$.

Solution:

Let $f(x) = \sqrt[8]{x}$ and $a = 256$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(256) + f'(256)(x - 256)$$

Find the pieces needed for the formula.

$$f(256) = \sqrt[8]{256} = 2$$

$$f(x) = \sqrt[8]{x} = x^{\frac{1}{8}}$$

$$f'(x) = \frac{1}{8}(x)^{-\frac{7}{8}} = \frac{1}{8\sqrt[8]{x^7}}$$

$$f'(256) = \frac{1}{8\sqrt[8]{256^7}} = \frac{1}{1{,}024}$$

Then the linear approximation is

$$L(x) = 2 + \frac{1}{1{,}024}(x - 256)$$

$$L(x) = \frac{1}{1{,}024}x + \frac{7}{4}$$

Use this approximation to estimate $\sqrt[8]{260}$.

$$f(260) = \frac{1}{1{,}024}(260) + \frac{7}{4} = \frac{513}{256} \approx 2.0039$$

■ 3. Use linear approximation to estimate $\sqrt[4]{85}$.

Solution:

Let $f(x) = \sqrt[4]{x}$ and $a = 81$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(81) + f'(81)(x - 81)$$

Find the pieces needed for the formula.

$$f(81) = \sqrt[4]{81} = 3$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(81) = \frac{1}{4\sqrt[4]{81^3}} = \frac{1}{108}$$

Then the linear approximation is

$$L(x) = 3 + \frac{1}{108}(x - 81)$$

$$L(x) = \frac{1}{108}x + \frac{9}{4}$$

Use this approximation to estimate $\sqrt[4]{85}$.

$$f(85) = \frac{1}{108}(85) + \frac{9}{4} = \frac{82}{27} \approx 3.037$$

■ 4. Use linear approximation to estimate $\sqrt[4]{615}$.

Solution:

Let $f(x) = \sqrt[4]{x}$ and $a = 625$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(625) + f'(625)(x - 625)$$

Find the pieces needed for the formula.

$$f(625) = \sqrt[4]{625} = 5$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(625) = \frac{1}{4\sqrt[4]{625^3}} = \frac{1}{500}$$

Then the linear approximation is

$$L(x) = 5 + \frac{1}{500}(x - 625)$$

$$L(x) = \frac{1}{500}x + \frac{15}{4}$$

Use this approximation to estimate $\sqrt[4]{615}$.

$$f(615) = \frac{1}{500}(615) + \frac{15}{4} = \frac{249}{50} \approx 4.98$$

■ 5. Use linear approximation to estimate $\sqrt{95}$.

Solution:

Let $f(x) = \sqrt{x}$ and $a = 100$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(100) + f'(100)(x - 100)$$

Find the pieces needed for the formula.

$$f(100) = \sqrt{100} = 10$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

Then the linear approximation is

$$L(x) = 10 + \frac{1}{20}(x - 100)$$

$$L(x) = \frac{1}{20}x + 5$$

Use this approximation to estimate $\sqrt{95}$.

$$f(95) = \frac{1}{20}(95) + 5 = \frac{39}{4} \approx 9.75$$

■ 6. Use linear approximation to estimate $\sqrt[3]{700}$.

Solution:

Let $f(x) = \sqrt[3]{x}$ and $a = 729$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(729) + f'(729)(x - 729)$$

Find the pieces needed for the formula.

$$f(729) = \sqrt[3]{729} = 9.$$

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$f'(729) = \frac{1}{3\sqrt[3]{729^2}} = \frac{1}{243}$$

Then the linear approximation is

$$L(x) = 9 + \frac{1}{243}(x - 729)$$

$$L(x) = \frac{1}{243}x + 6$$

Use this approximation to estimate $\sqrt[3]{700}$.

$$f(700) = \frac{1}{243}(700) + 6 = \frac{2{,}158}{243} \approx 8.88$$

ABSOLUTE, RELATIVE, AND PERCENTAGE ERROR

■ 1. What is the absolute change of $f(x)$ from $x=\pi$ to $x=2\pi$?

$$f(x)=3x^2-\cos\left(\frac{x}{2}\right)$$

Solution:

Absolute change is the difference between $f(2\pi)$ and $f(\pi)$.

$$f(2\pi)=3(2\pi)^2-\cos\left(\frac{2\pi}{2}\right)=12\pi^2-(-1)=12\pi^2+1$$

$$f(\pi)=3(\pi)^2-\cos\left(\frac{\pi}{2}\right)=3\pi^2-(0)=3\pi^2$$

The absolute change is the difference, or

$$f(2\pi)-f(\pi)$$

$$12\pi^2+1-3\pi^2$$

$$9\pi^2+1$$

■ 2. What is the relative change of $g(x)$ from $x=2$ to $x=3$?

$$g(x)=2x^4-3x^2-5$$

Solution:

Relative change is the quotient of the difference between $g(3)$ and $g(2)$, and the value of the function at the left edge of the interval, $g(2)$.

$$g(3) = 2(3)^4 - 3(3)^2 - 5 = 2(81) - 3(9) - 5 = 130$$

$$g(2) = 2(2)^4 - 3(2)^2 - 5 = 2(16) - 3(4) - 5 = 15$$

The difference is $130 - 15 = 115$, so the relative change is

$$\frac{g(3) - g(2)}{g(2)} = \frac{115}{15} = \frac{23}{3} \approx 767\,\%$$

■ 3. What is the relative change of $h(x)$ from $x = 0$ to $x = \pi$?

$$h(x) = \tan x + 4x + 2$$

Solution:

Relative change is the quotient of the difference between $h(\pi)$ and $h(0)$, and the value of the function at the left edge of the interval, $h(0)$.

$$h(\pi) = \tan \pi + 4\pi + 2 = 0 + 4\pi + 2 = 4\pi + 2$$

$$h(0) = \tan 0 + 4(0) + 2 = 2$$

The difference is $4\pi + 2 - 2 = 4\pi$, so the relative change is

$$\frac{h(\pi) - h(0)}{h(0)} = \frac{4\pi}{2} = 2\pi \approx 628\,\%$$

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