Logarithmic derivatives

We briefly introduced the natural logarithm in the last lesson. Now we want to dig deeper into log and natural log functions and their derivatives.

The log functions and their derivatives are given as

Type	**Log function**	**Its derivative**
Logarithm	$y = \log_a x$	$y' = \dfrac{1}{x \ln a}$
Natural logarithm	$y = \ln x$	$y' = \dfrac{1}{x}$

There are two important things to say about these derivative formulas.

First, there's actually no difference between the derivative formulas for $y = \log_a x$ and $y = \ln x$. When we take the derivative of any $y = \log_a x$, we always need to include that $\ln a$ that's in the denominator of its derivative.

But the natural log function $y = \ln x$ is actually just the standard logarithm, but with a base of e instead of a. In other words, $y = \ln x$ is the same thing as $y = \log_e x$. With that in mind, we can say that the derivative of $y = \log_e x$ is

$$y' = \frac{1}{x \ln e}$$

And, like we learned in the last lesson, $\ln e = 1$, which means the derivative will simplify to just

$$y' = \frac{1}{x(1)}$$

$$y' = \frac{1}{x}$$

which is the derivative formula for $y = \ln x$.

So when the base of the logarithm is e, which is another way of saying that the logarithm $y = \log_a x$ becomes the natural logarithm $y = \ln x$, then the derivative formula simplifies from

$$y' = \frac{1}{x \ln a}$$

to

$$y' = \frac{1}{x}$$

Second, just like with exponential functions, we need to apply chain rule every time we take the derivative of a logarithmic function. With exponential functions, the “inside function” was the exponent. With logarithmic functions, the “inside function” is the argument of the log function. In both $y = \log_a x$ and $y = \ln x$, the argument is x, and the derivative of x is 1. So when we differentiate $y = \log_a x$ or $y = \ln x$, applying chain rule means we multiply by 1, which of course doesn’t change the value of the derivative.

But when the argument is anything other than x, the derivative of the argument will be something other than 1, which means that applying chain rule may change the value of the derivative. For instance, the derivative of $y = \ln(x^2)$ is $y' = 2x(1/(x^2)) = 2x/(x^2) = 2/x$. If we wanted to show chain rule as part of the logarithmic derivative formulas, we’d write them as

Type	Log function	Its derivative
Logarithm	$y = \log_a g(x)$	$y' = \frac{1}{g(x)\ln a} g'(x)$
Natural logarithm	$y = \ln g(x)$	$y' = \frac{1}{g(x)} g'(x)$

Let's try an example where we differentiate a function in the form $y = \log_a g(x)$.

Example

Find the derivative of the logarithmic function.

$$y = 6\log_8(5x^4)$$

Differentiate by applying the derivative formula for $y = \log_a g(x)$.

$$y' = \frac{1}{g(x)\ln a} g'(x)$$

$$y' = 6\left(\frac{1}{5x^4 \ln 8}\right)(20x^3)$$

$$y' = \frac{120x^3}{5x^4 \ln 8}$$

$$y' = \frac{24}{x \ln 8}$$

Now let's try an example with a natural log function.

Example

Find the derivative of the logarithmic function.

$$y = 5\ln(2x^3)$$

To find the derivative we need to apply the derivative formula for natural logs.

$$y' = \frac{1}{g(x)}g'(x)$$

$$y' = 5\left(\frac{1}{2x^3}\right)(6x^2)$$

$$y' = \frac{30x^2}{2x^3}$$

$$y' = \frac{15}{x}$$

Let's try one final example that's a little more complex.

Example

Find the derivative of the logarithmic function.

$$y = 9\ln(3x^7) + (4x^7)\log_3(8x^2) - 2x^{12}$$

We need to take the derivative one term at a time, applying the derivative formulas for the log and natural log. We'll also need to apply product rule to the second term.

$$y' = 9\left(\frac{1}{3x^7}\right)(21x^6) + \left[(28x^6)(\log_3(8x^2)) + (4x^7)\left(\frac{1}{8x^2\ln 3}\right)(16x)\right] - 24x^{11}$$

$$y' = \frac{189x^6}{3x^7} + 28x^6\log_3(8x^2) + \frac{64x^8}{8x^2\ln 3} - 24x^{11}$$

$$y' = \frac{63}{x} + 28x^6\log_3(8x^2) + \frac{8x^6}{\ln 3} - 24x^{11}$$