Topic: Precise definition of the limit

Question: Which of these is the precise definition of the limit?

Answer choices:

A Let f be a function defined on a closed interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < x - c < \delta$, then $f(x) - L < \epsilon$.

B Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

C Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $|f(x) - L| < \epsilon$, then $0 < |x - c| < \delta$.

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Solution: B

The correct statement of the precise definition of the limit is:

Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

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Topic: Precise definition of the limit

Question: Use the epsilon-delta definition to prove the value of the limit.

$$\lim_{x \to 9} \sqrt{x}$$

Answer choices:

A 1

B 2

C 3

D 4

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Solution: C

First we find by direct substitution that $\lim_{x \to 9} \sqrt{x} = \sqrt{9} = 3$.

To prove this, we need to show that, on some open interval surrounding $x = 9$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\sqrt{x} - 3| < \epsilon$ whenever $0 < |x - 9| < \delta$.

Let $\epsilon > 0$ and $0 < |x - 9| < \delta$. We need to find a δ (which will be in terms of ϵ) that will give $|\sqrt{x} - 3| < \epsilon$. To do this, we need to try to rewrite $|\sqrt{x} - 3|$ so that it involves $|x - 9|$ in some way.

We know that

$$|\sqrt{x} - 3| = |\sqrt{x} - 3| \cdot \frac{|\sqrt{x} + 3|}{|\sqrt{x} + 3|} = |x - 9| \cdot \frac{1}{|\sqrt{x} + 3|}$$

and we want to find a δ such that

$$|\sqrt{x} - 3| = |x - 9| \cdot \frac{1}{|\sqrt{x} + 3|} < \epsilon$$

Since we are assuming that $0 < |x - 9| < \delta$, this means that we need to find an upper bound for $1/|\sqrt{x} + 3|$ on some open interval around $x = 9$, say the interval $(4,16)$.

We choose this interval because the function is defined on the entire interval, as required by the definition of the limit, because the interval contains 9, and because it's easy to find the square roots of 4 and 16. We

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could have, of course, chosen any positive interval containing 9, as long as the function is defined on the entire interval, except possibly at 9 itself.

On the interval $(4,16)$, $1/|\sqrt{x}+3|$ is constantly decreasing (the denominator is always positive and increasing on $(4,16)$ and the numerator is a constant 1). Therefore, on the interval $(4,16)$, $1/|\sqrt{x}+3|$ is at its maximum at the left endpoint:

$$\frac{1}{|\sqrt{x}+3|} < \frac{1}{|\sqrt{4}+3|} = \frac{1}{5}$$

We can now use this to find the δ we are required to find for our proof to be complete.

We are assuming that $0 < |x-9| < \delta$ and we have found that, on the interval $(4,16)$, $1/(\sqrt{x}+3) < 1/5$. Therefore, since we want

$$|\sqrt{x}-3| = \frac{|\sqrt{x}-3||\sqrt{x}+3|}{|\sqrt{x}+3|} = |x-9| \cdot \frac{1}{|\sqrt{x}+3|} < \epsilon$$

we can choose δ to be the lesser of 5ϵ and 7.* By doing this, we see that whenever $0 < |x-9| < \delta = 5\epsilon$, we have

$$|\sqrt{x}-3| = \frac{|\sqrt{x}-3||\sqrt{x}+3|}{|\sqrt{x}+3|} = |x-9| \cdot \frac{1}{|\sqrt{x}+3|} < 5\epsilon\left(\frac{1}{5}\right) = \epsilon$$

This proves that $\lim_{x\to 0}\sqrt{x} = 3$.

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*In our proof, we have made the assumption that $0 < |x - 9| < \delta$. Since our choice of $\delta = 5\epsilon$ depends on x being in the interval $(4,16)$, this means that

$$4 < x < 16 \quad \to \quad -5 < x - 9 < 7 \quad \to \quad |x - 9| < 7$$

for this interval. Therefore, δ would have to be 7 if 5ϵ is greater than 7 in order for our assumption to remain true (a false assumption can't produce a true conclusion). This is why we choose δ to be the lesser of 5ϵ and 7.

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Topic: Precise definition of the limit

Question: True or false? The precise definition of the limit implies that picking a value of x inside the δ interval will return a resulting value in the ϵ interval.

Answer choices:

A True

B False

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Solution: A

According to the epsilon-delta definition of the limit, choosing a value for x between $x - \delta$ and $x + \delta$ will return a function value between $L - \epsilon$ and $L + \epsilon$.

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