Making the function continuous

In this lesson, we want to build on what we already know about discontinuities and one-sided limits.

Point discontinuities

Previously, we learned that a point discontinuity was a single pinpoint of discontinuity in the graph, at which the one-sided limits are equal, and the general limit therefore exists. But, even though the general limit exists, the function is still discontinuous at that point.

Now, we're interested in how we can plug that hole in the graph, by redefining the function in such a way that it becomes continuous at that point. To illustrate how easy this is, we'll take the function we used previously in the lesson on point discontinuities.

$$f(x) = \frac{x^2 + 11x + 28}{x + 4}$$

We factored the numerator in order to rewrite and simplify the function.

$$f(x) = \frac{(x+4)(x+7)}{x+4}$$

$$f(x) = \frac{x+4}{x+4}(x+7)$$

$$f(x) = 1(x+7)$$

$$f(x) = x + 7$$

Because we canceled the $x + 4$, we know the function has a point discontinuity at $x = -4$.

The value of the left- and right-hand limits, and therefore the value of the general limit, at $x = -4$, is

$$f(-4) = -4 + 7$$

$$f(-4) = 3$$

So, if we wanted to plug the hole in the graph and remove the point discontinuity, thereby making the function continuous at $x = -4$, we would redefine the function as

$$f(x) = \begin{cases} \frac{x^2 + 11x + 28}{x + 4} & x \neq -4 \\ 3 & x = -4 \end{cases}$$

Remember that a function written this way is called a piecewise-defined function, because different parts of the function are defined by different "pieces."

Defined this way, the first piece, the original function defines the graph everywhere, other than $x = -4$. But the second piece, $f(x) = 3$, steps in to define the function's value at $x = -4$. That second piece plugs the hole, removes the point discontinuity, and makes the function continuous everywhere.

Piecewise-defined functions

Sometimes we'll be given a piecewise-defined function, and asked to find the value of an unknown constant that will make the function continuous. For instance, consider this function, $f(x)$:

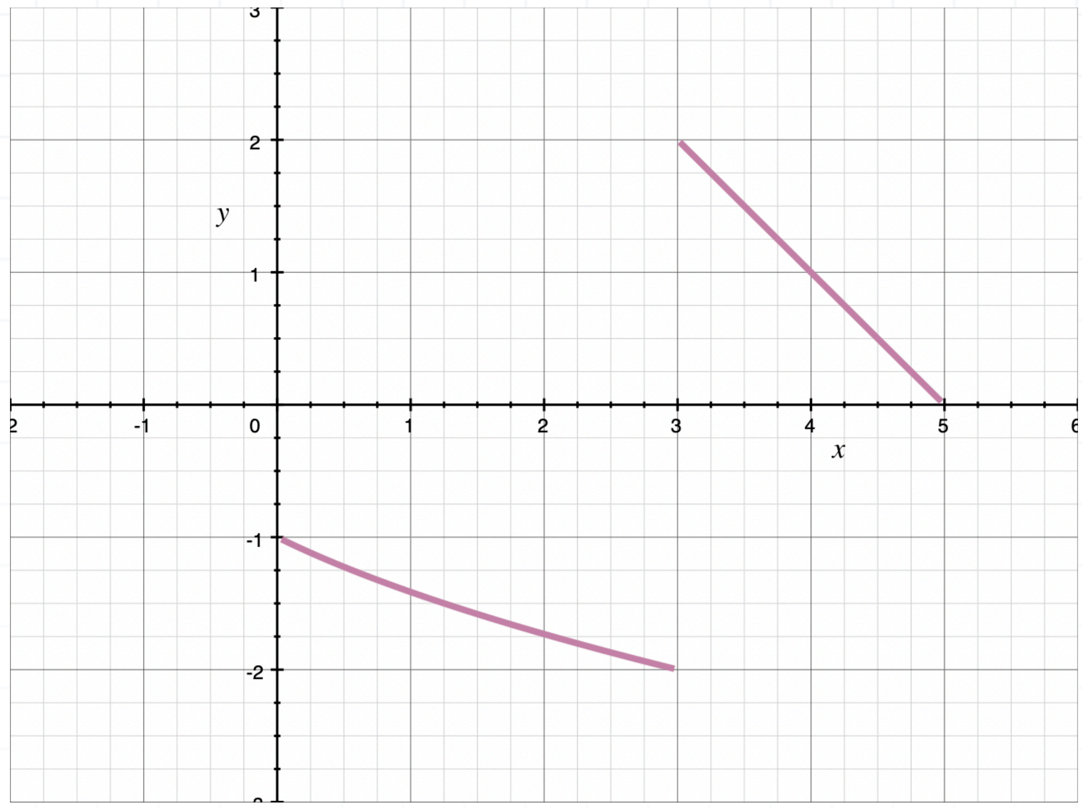
$$f(x) = \begin{cases} k\sqrt{x+1} & 0 \le x \le 3 \\ 5 - x & 3 < x \le 5 \end{cases}$$

It's a piecewise-defined function, where the first piece defines the function from 0 to 3 (including at 3), and the second piece defines the function for values above 3, until 5.

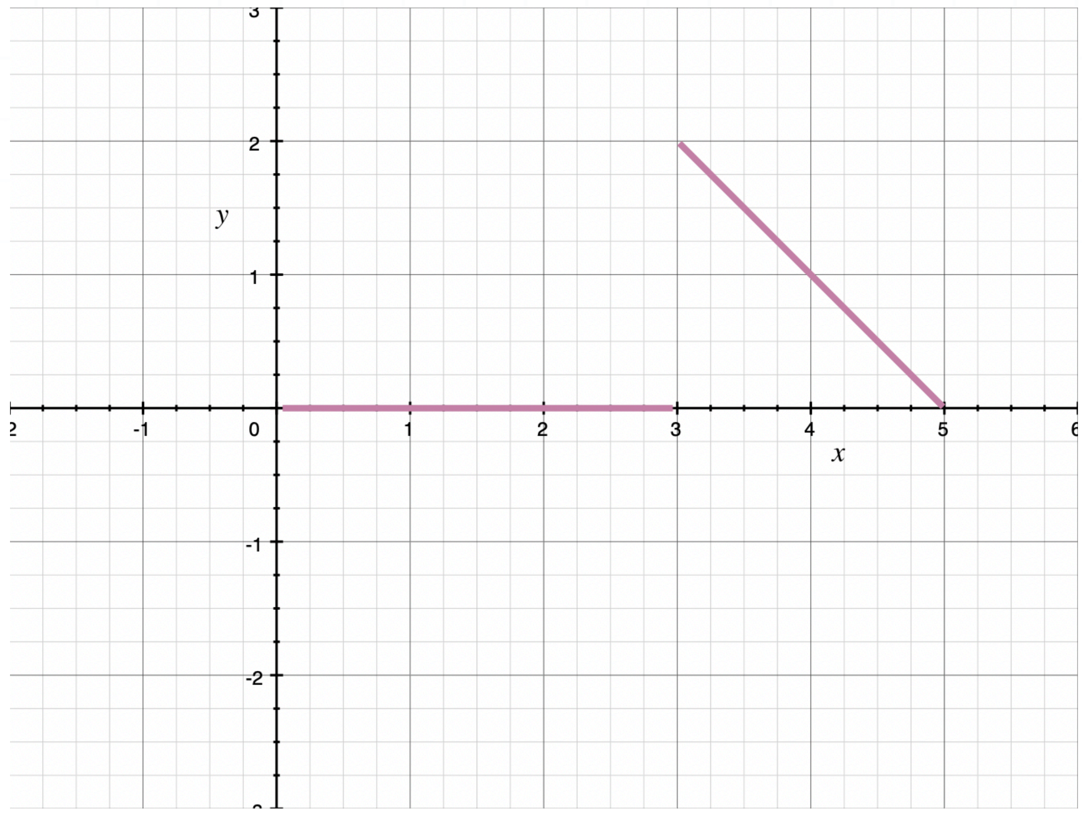
For a problem like this one, we need to find the value of k that makes $f(x)$ continuous. What that means is that we need to find the value of k that ensures that both pieces of the graph meet at the same value when $x = 3$. When $x = 3$, the first piece stops defining the function and the second piece takes over, so we can think of $x = 3$ as the "break point" between the pieces. If we can make the two pieces meet at the break point, then the function will be continuous.

So visualize this, here's what the graph of $f(x)$ looks like with some different values of k.

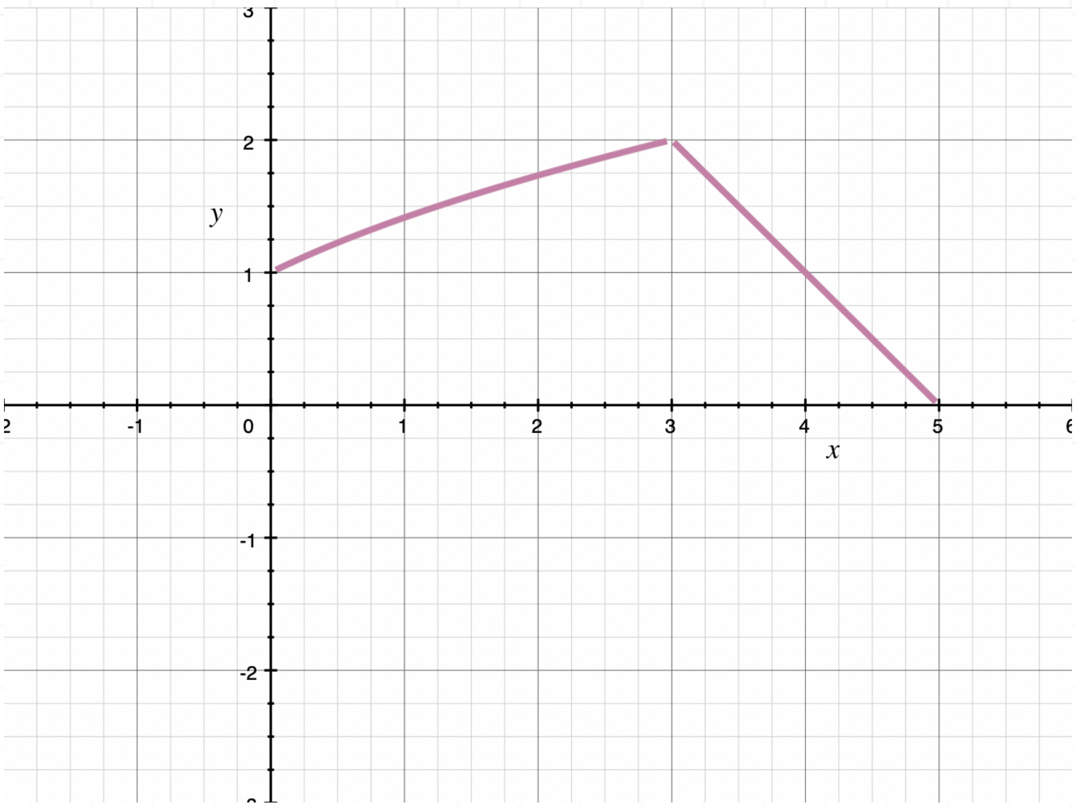
If $k = -1$, the graph of $f(x)$ is



If $k = 0$, the graph of $f(x)$ is



If $k = 1$, the graph of $f(x)$ is



So, graphically, we can see that $k = 1$ will be the value that makes the function continuous. But how do we solve for the value of k algebraically, so that we can avoid picking random values of the unknown constant and graphing the function with that value?

Well, we always want to start at the "break point" that we talked about earlier. For this function, the break point between the two pieces is at $x = 3$. When $x = 3$, we want the pieces to have equal value. So we set the pieces equal to one another, plug the break point $x = 3$ into the equation, and then solve the equation for the unknown k.

$$k\sqrt{x+1} = 5 - x$$

$$k\sqrt{3+1} = 5 - 3$$

$$k\sqrt{4} = 2$$

$$2k = 2$$

$$k = 1$$

So $k = 1$ is the value of the constant k that will force the continuity of the function. For any other value of k, we'll get a jump discontinuity at the break point $x = 3$, but $k = 1$ makes the two pieces meet at the same point, which makes the function continuous.