Precise definition of the limit

The precise definition of the limit is something we use as a proof for the existence of a limit.

The precise definition

Let's start by stating that $f(x)$ is a function on an open interval that contains $x = a$, but that the function doesn't necessarily exist at $x = a$. The **precise definition of the limit** of the function tells us that, at $x = a$, the limit is L,

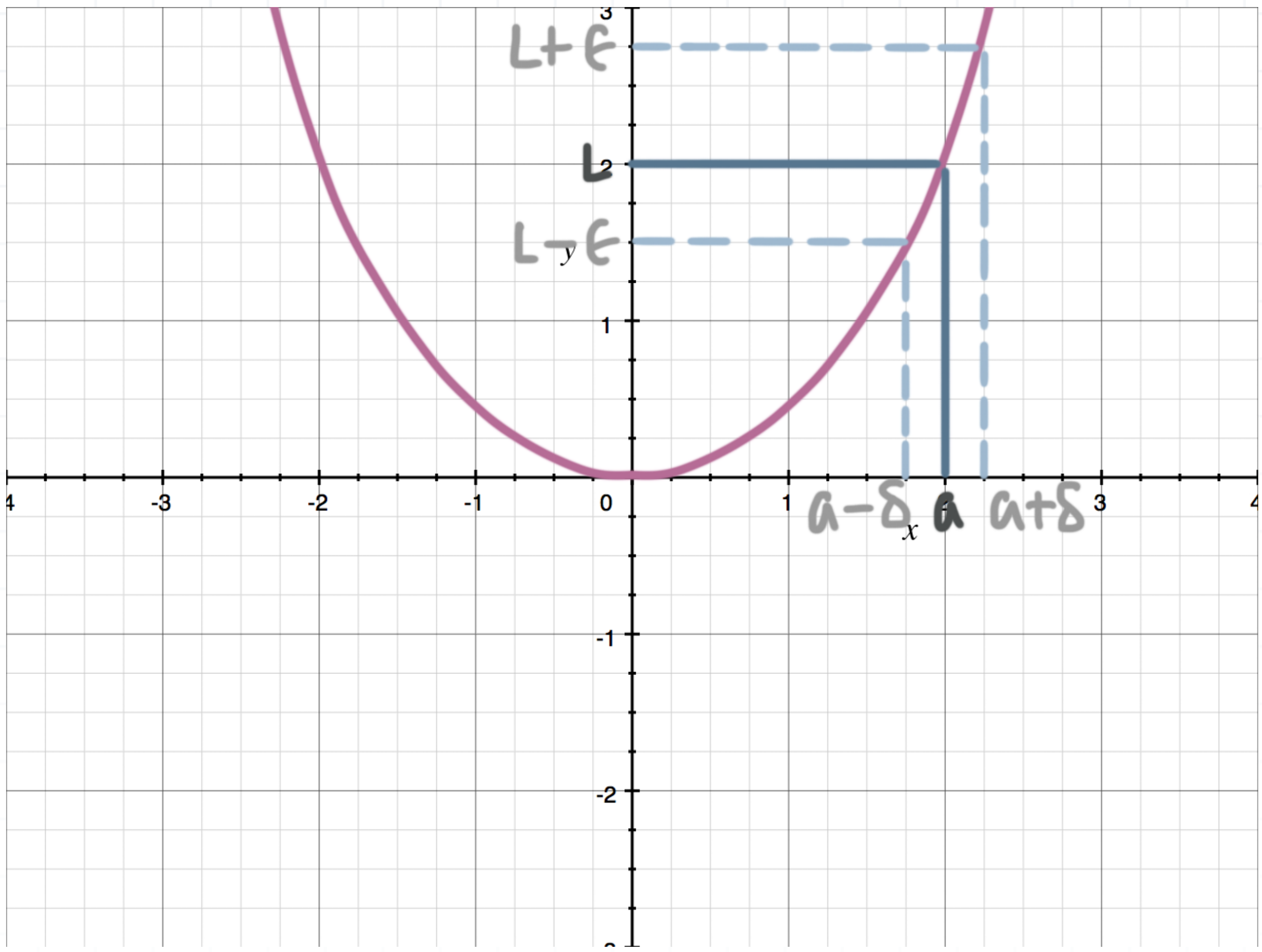
$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is some number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

What does all this mean? Well, since the open interval includes a but doesn't necessarily exist at a, we'll have to look at how the function behaves as it approaches a. L just represents the value of the limit.

When we're evaluating a limit, we're looking at the function as it approaches a specific point. In the graph,



that point is (a, L). The precise definition of the limit proves that the limit exists and is L, as long as any number we pick between $a - \delta$ and $a + \delta$ will always return a value between $L - \epsilon$ and $L + \epsilon$.

If this is true, then we know that if we pick a value that's closer and closer to a, the value we get back will be closer and closer to L. And that's the definition of of the limit, that, as we approach $x = a$, the value of the function gets closer to L.

Example

Using the precise definition of the limit, prove the following limit.

$$\lim_{x \to 4} 2x - 3 = 5$$

Substituting $2x-3$ for $f(x)$, 5 for L, and 4 for a into the definition, we get

$$|(2x-3)-5| < \epsilon \text{ whenever } 0 < |x-4| < \delta$$

If we simplify $|(2x-3)-5| < \epsilon$, we get

$$|2x-8| < \epsilon$$

$$2|x-4| < \epsilon$$

$$|x-4| < \frac{\epsilon}{2}$$

Notice now that the left side of this inequality looks just like the middle part of the inequality above that contains δ. When this happens, we set δ equal to the right-hand side of the last inequality, and we get

$$\delta = \frac{\epsilon}{2}$$

$$0 < |x-4| < \delta = \frac{\epsilon}{2}$$

Going back to the beginning,

$$|(2x-3)-5| = |2x-8|$$

$$|(2x-3)-5| = 2|x-4|$$

and using the assumption that $\delta = \epsilon/2$ and that $0 < |x-4| < \delta$, by substitution, we get

$$|(2x-3)-5| = 2\left|\frac{\epsilon}{2}\right|$$

$$|(2x-3)-5| = \epsilon$$

Since we started with $|(2x-3)-5| < \epsilon$ and ended with ϵ, we've shown that $\epsilon = \epsilon$ and that

$$|(2x-3)-5| < \epsilon \text{ whenever } 0 < |x-4| < \frac{\epsilon}{2}$$

Therefore,

$$\lim_{x \to 4} 2x - 3 = 5$$

Solving for delta

Sometimes we'll want to find δ, given other values in the precise definition of the limit. When this is the case, we'll follow a specific set of steps in order to find the value of δ.

Example

Find δ when $f(x) = x^2$, such that, if $|x-2| < \delta$ then $|x^2-4| < 0.5$.

We want to use the value for ϵ to determine the δ value by remembering that

$0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$

from the precise definition of the limit.

To solve for δ, we'll take the epsilon value $\epsilon = 0.5$ and the value of L to find the two y-values. This means we have $4 + 0.5 = 4.5$ and $4 - 0.5 = 3.5$. Then we can plug these values into the function to get the associated x-values.

$$4.5 = x^2$$

$$x = 2.12$$

and

$$3.5 = x^2$$

$$x = 1.87$$

We'll find $|x - a|$ with these two x-values and $a = 2$.

For $x = 2.12$: $|x - a| = |2.12 - 2| = |0.12| = 0.12$

For $x = 1.87$: $|x - a| = |1.87 - 2| = |-0.13| = 0.13$

If the two values are different, the smaller value will be the value we need to pick for δ. Which means that for the function $f(x) = x^2$, such that if $|x - 2| < \delta$ then $|x^2 - 4| < 0.5$, we know that $\delta = 0.12$.