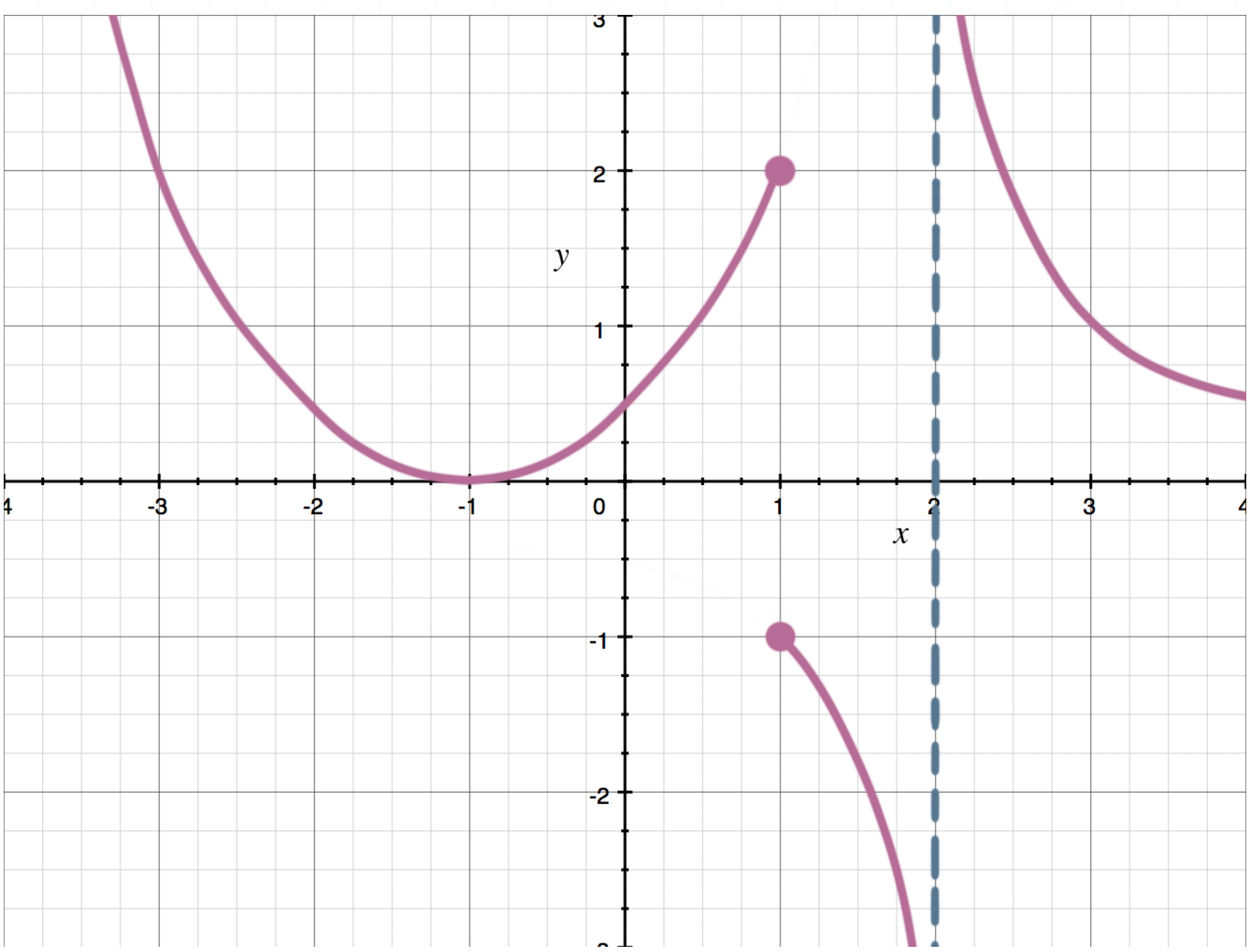
Point discontinuities

We already know that the general limit doesn't exist wherever the left- and right-hand limits are not equal.

The idea of continuity is exactly what it sounds like. If a function has continuity at a particular point, it means the function is continuous at that point, meaning that there are no holes, jumps, or asymptotes in the graph there.

For instance, in the graph,



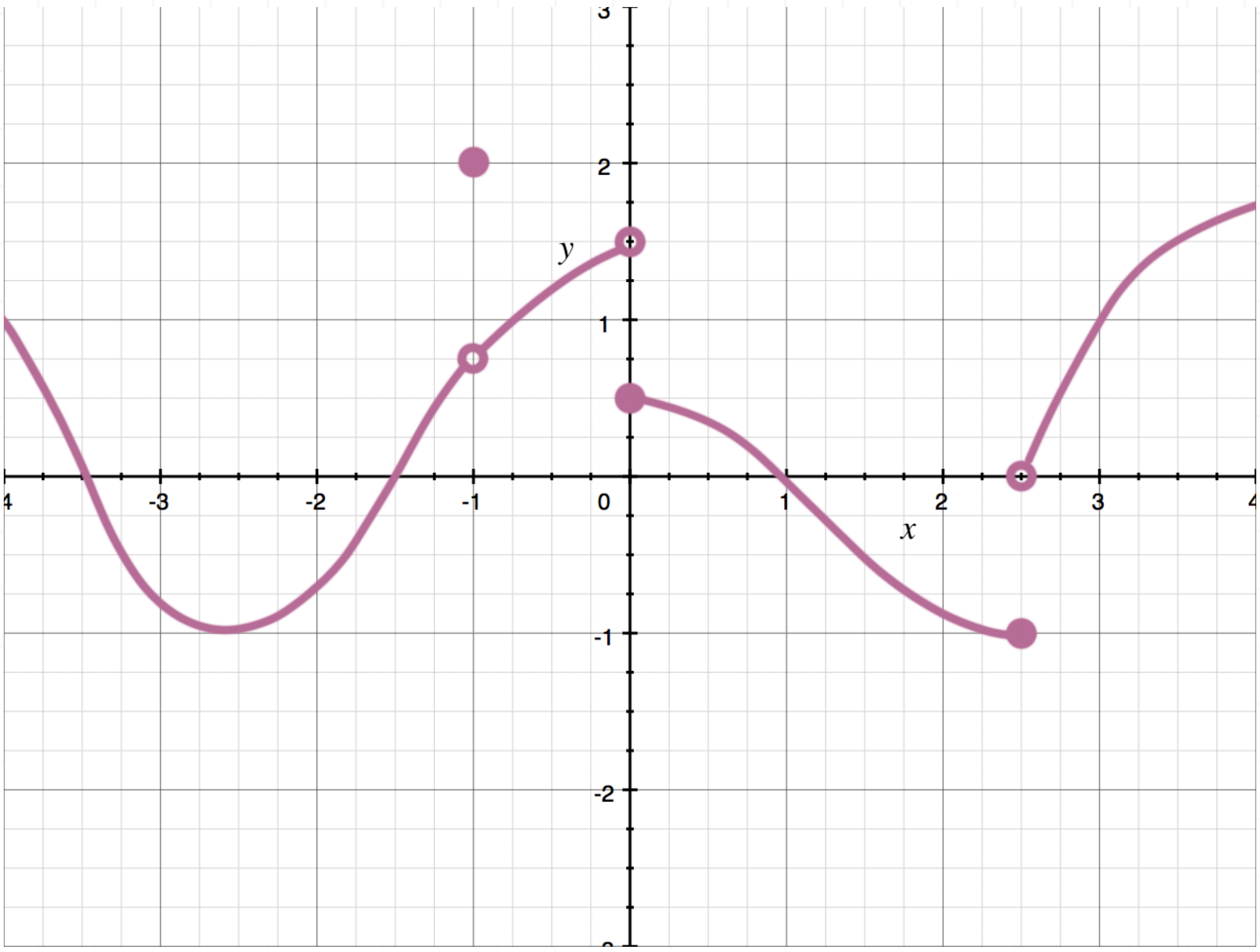
the function has a discontinuity at $x = 1$, because there's a jump there. The left piece of the graph has a value of 2 at $x = 1$, whereas the right piece of the graph has a value of -1 at $x = 1$. Because $2 \neq -1$, there's a jump in the graph at that point. The function also has a discontinuity at $x = 2$, because there's a vertical asymptote there. The function isn't continuous at the asymptote because the asymptote breaks the function into two pieces.

If we can draw the graph of the function without ever lifting our pencil off the paper as we sketch it out from left to right, then the function is continuous everywhere. At any point where we have to lift our pencil off the paper in order to continue sketching it, there must be a discontinuity at that point.

There are different types of discontinuities, all of which mean different things for the value of the limit at the discontinuity.

Point (removable) discontinuities

A **point discontinuity** exists wherever there's a hole in the graph at one specific point. In this graph,



there's a point discontinuity at $x = -1$, which is shown by the empty hole in the graph there. When there's a point discontinuity, the function will look continuous and smooth around the that point, but then have an empty hole in the graph at that exact spot.

We get this kind of discontinuity with rational functions (a rational function is a fraction in which the numerator and denominator are both polynomials). For the rational function

$$f(x) = \frac{x^2 + 11x + 28}{x + 4}$$

we can factor the numerator and then cancel common factors.

$$f(x) = \frac{(x+4)(x+7)}{x+4}$$

$$f(x) = \frac{x+4}{x+4}(x+7)$$

$$f(x) = 1(x+7)$$

$$f(x) = x+7$$

If we look at the original function, we'd say that there's a discontinuity at $x = -4$, because the denominator would be 0 at that point. But if we simplify the function down to $f(x) = x + 7$, then it's as if the discontinuity disappears. Because of the way that we can "take out" the point discontinuity by cancelling factors, we also refer to point discontinuities as **removable discontinuities**.

The function $f(x) = x + 7$ is perfectly defined at $x = -4$, even though the original function isn't defined there. It's this kind of situation that creates a point discontinuity.

Example

Find any point discontinuities in the graph of the function.

$$f(x) = \frac{x-2}{x^2+x-6}$$

If we factor the denominator of the function,

$$f(x) = \frac{x-2}{(x-2)(x+3)}$$

it looks as if there are discontinuities in the function at $x = 2$ and $x = -3$, because those values both make the denominator equal to 0. But we realize that we can cancel a factor of $x - 2$, leaving

$$f(x) = \frac{1}{x+3}$$

Therefore, we can say there's a point (removable) discontinuity at $x = 2$.

The general limit always exists at a point discontinuity, because the left- and right-hand limits both exist, and those one-sided limits are equivalent.