Calculus 1 Workbook Solutions

Graphing functions

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MATH

EQUATION MODELING

■ 1. A car and a truck were driven for a week. The car traveled 75 miles more than the truck. Each vehicle had different fuel mileage. Write an equation using t (where t is the number of miles the truck traveled) to calculate the number of gallons g, used during the week.

	Car	Truck
Mileage	28 mpg	14 mpg
Distance	c miles	t miles

Solution:

Write an expression in terms of t for the distance traveled by the car. The car traveled 75 more miles than the truck, so $c = t + 75$. To get the gallons used, divide the distance (in terms of t) by the mileage.

	Car	Truck
Mileage	28 mpg	14 mpg
Distance	t+75 miles	t miles
Gallons used g	(t+75)/28 gallons	t/14 gallons

To find the total gallons used, add the gallons the car used with the gallons the truck used.

$$g = \frac{t + 75}{28} + \frac{t}{14}$$

$$g = \frac{t + 75}{28} + \left(\frac{2}{2}\right)\frac{t}{14}$$

$$g = \frac{t + 75}{28} + \frac{2t}{28}$$

$$g = \frac{t + 75 + 2t}{28}$$

$$g = \frac{3t + 75}{28} \text{ gallons}$$

■ 2. A motorcycle and a car were driven for a month. The motorcycle traveled 120 miles more than the car. Each vehicle had different fuel mileage. Write an equation using m (where m is the number of miles the motorcycle traveled) to calculate the number of gallons g, used during the month.

	Motorcycle	Car
Mileage	33 mpg	22 mpg
Distance	m miles	c miles

Solution:

Write an expression in terms of m for the distance traveled by the car. The car traveled 120 miles less than the motorcycle, so $c = m - 120$. To get the gallons used, divide the distance (in terms of m) by the mileage.

	Motorcycle	Car
Mileage	33 mpg	22 mpg
Distance	m miles	m-120 miles
Gallons used g	m/33 gallons	(m-120)/22 gallons

To find the total gallons used, add the gallons the motorcycle used with the gallons the car used.

$$g = \frac{m}{33} + \frac{m - 120}{22}$$

$$g = \left(\frac{2}{2}\right)\frac{m}{33} + \left(\frac{3}{3}\right)\frac{m - 120}{22}$$

$$g = \frac{2m}{66} + \frac{3m - 360}{66}$$

$$g = \frac{2m + 3m - 360}{66}$$

$$g = \frac{5m - 360}{66} \text{ gallons}$$

■ 3. A baseball is thrown at a speed of 21 ft/s straight down from a high platform. The distance it travels can be calculated using $D = 16t^2 + 21t$, where t is the amount of time in seconds that it's been falling. The average speed of any object can be calculated using $V = D/t$. Write an equation giving the time of the fall in terms of V.

Solution:

Plug $16t^2 + 21t$ into the $V = D/t$ for D.

$$V = \frac{D}{t}$$

$$V = \frac{16t^2 + 21t}{t}$$

$$V = 16t + 21$$

We were asked to find the equation for time in terms of V, so we need to solve for t.

$$V = 16t + 21$$

$$V - 21 = 16t$$

$$t = \frac{V - 21}{16}$$

■ 4. A rock is thrown at a speed of 8 ft/s straight down from a high platform. The distance it travels can be calculated using $D = 16t^2 + 8t$, where t is the amount of time in seconds that it's been falling. The average speed of any object can be calculated using $V = D/t$. Write an equation giving the time of the fall in terms of V.

Solution:

Plug $16t^2 + 8t$ into $V = D/t$ for D.

$$V = \frac{D}{t}$$

$$V = \frac{16t^2 + 8t}{t}$$

$$V = 16t + 8$$

We were asked to find the equation for time in terms of V, so we need to solve for t.

$$V = 16t + 8$$

$$V - 8 = 16t$$

$$t = \frac{V - 8}{16}$$

■ 5. Managers at a company are each paid $\$45,000$ in base salary. The company's owner wants to divide $\$162,000$ in annual bonus money evenly among the managers. Write an expression, in terms of the number of mangers m, that gives the amount a each manager earns per month.

Solution:

Find the monthly salary of a manager.

$$45{,}000 \div 12 = 3{,}750$$

Find the bonus money available each month to the group of all managers.

$$162{,}000 \div 12 = 13{,}500$$

This monthly bonus money needs to be divided evenly by the number of mangers m.

$$\frac{13{,}500}{m}$$

The total amount each manager earns monthly is the sum of their monthly salary and their monthly bonus money.

$$a = 3{,}750 + \frac{13{,}500}{m}$$

■ 6. Managers at a company are each paid $\$37{,}800$ in base salary. The company's owner wants to divide $\$102{,}000$ in annual bonus money evenly among the managers. Write an expression, in terms of the number of mangers m, that gives the amount a each manager earns per month.

Solution:

Find the monthly salary of a manager.

$$37{,}800 \div 12 = 3{,}150$$

Find the bonus money available each month to the group of all managers.

$$102{,}000 \div 12 = 8{,}500$$

This monthly bonus money needs to be divided evenly by the number of mangers m.

$$\frac{8{,}500}{m}$$

The total amount each manager earns monthly is the sum of their monthly salary and their monthly bonus money.

$$a = 3{,}150 + \frac{8{,}500}{m}$$

7. The Jones and Anderson family go on vacation together with each family driving in their own car. The Anderson family travels 50 miles further than the Jones family. Each family averages 65 mph on the trip. Write an equation using D_a (where D_a is the total miles the Anderson family drove) to calculate the total time T both families spent driving to their destination.

	Jones	Anderson
Distance	D_j miles	D_a miles
Rate	65 mph	65 mph
Time	T_j hours	T_a hours

Solution:

The Jones family traveled 50 miles less than the Anderson family, so $D_j = D_a - 50$. We need to find the total time T, which is $T_j + T_a$. So we'll use the distance equation $D = RT$ and solve for T_j and T_a.

Time spent driving by the Jones family:

$$D_j = R_j T_j$$

$$D_a - 50 = 65T_j$$

$$T_j = \frac{D_a - 50}{65}$$

Time spent driving by the Anderson family:

$$D_a = R_a T_a$$

$$D_a = 65T_a$$

$$T_a = \frac{D_a}{65}$$

Total time spent driving:

$$T = T_j + T_a$$

$$T = \frac{D_a - 50}{65} + \frac{D_a}{65}$$

$$T = \frac{2D_a - 50}{65} \text{ hours}$$

8. The Frank and Harrington family go on vacation together with each family driving in their own car. The Frank family travels 120 miles less than the Harrington family. Each family averages 50 mph on the trip. Write an equation using D_f (where D_f is the total miles the Frank family drove) to calculate the total time T both families spent driving to their destination.

	Frank	Harrington
Distance	D_f miles	D_h miles
Rate	50 mph	50 mph
Time	T_f hours	T_h hours

Solution:

The Harrington family traveled 120 miles more than the Frank family, so $D_h = D_f + 120$. We need to find the total time T, which is $T_f + T_h$. So we'll use the distance equation $D = RT$ and solve for T_f and T_h.

Time spent driving by the Harrington family:

$$D_h = R_h T_h$$

$$D_f + 120 = 50T_h$$

$$T_h = \frac{D_f + 120}{50}$$

Time spent driving by the Frank family:

$$D_f = R_f T_f$$

$$D_f = 50T_f$$

$$T_f = \frac{D_f}{50}$$

Total time spent driving:

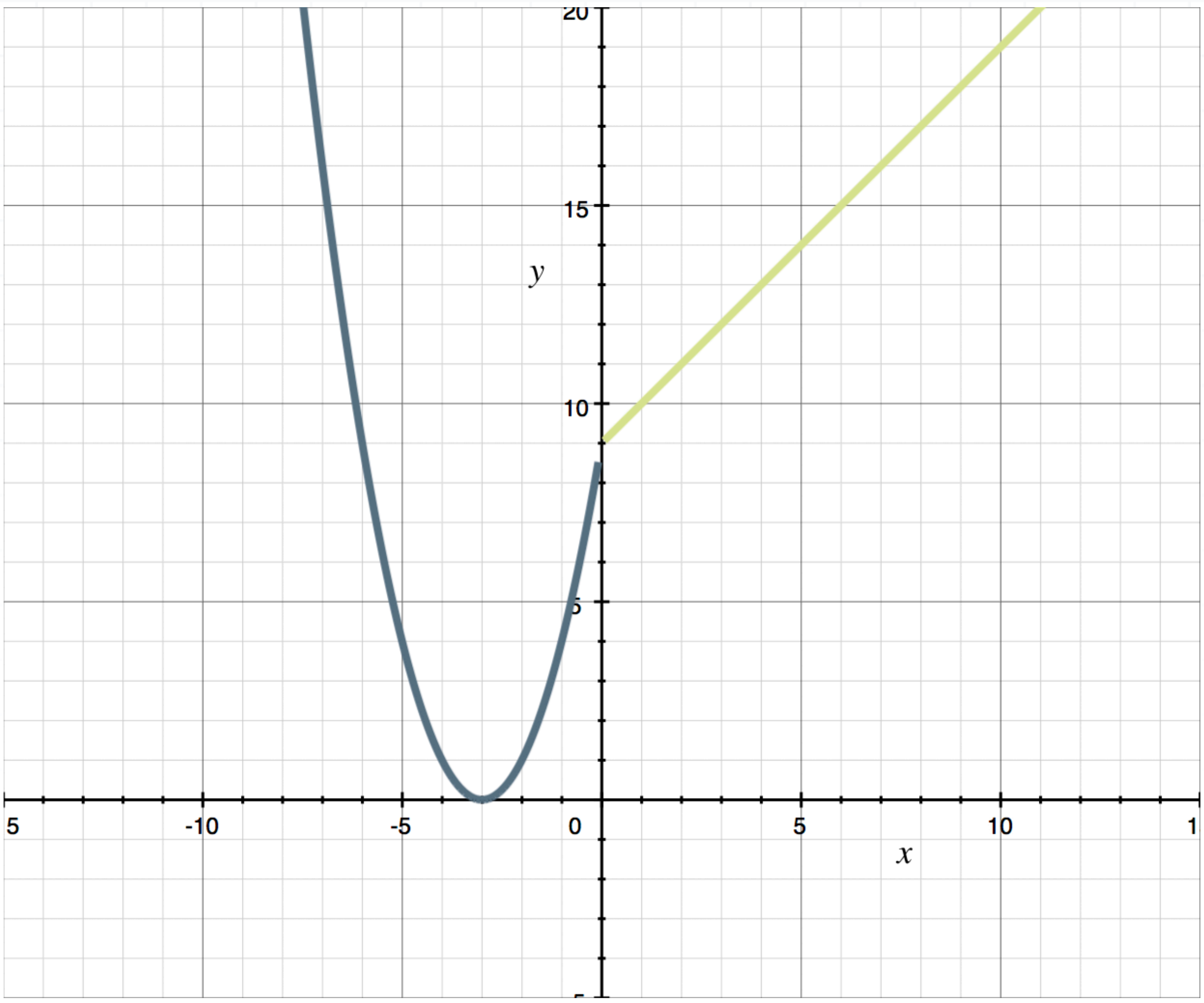
$$T = T_f + T_h$$

$$T = \frac{2D_f + 120}{50}$$

$$T = \frac{D_f + 60}{25} \text{ hours}$$

MODELING A PIECEWISE-DEFINED FUNCTION

1. Find the equation of the piecewise function.



Solution:

The dark blue parabola has a vertex at $(-3,0)$, so the equation is $f(x) = (x+3)^2$ from $-\infty$ to 0, or when $x \leq 0$.

The green line has a slope of 1 and a y-intercept of 9, so the equation of the line is $f(x) = x + 9$ from 0 to ∞, or when $x > 0$.

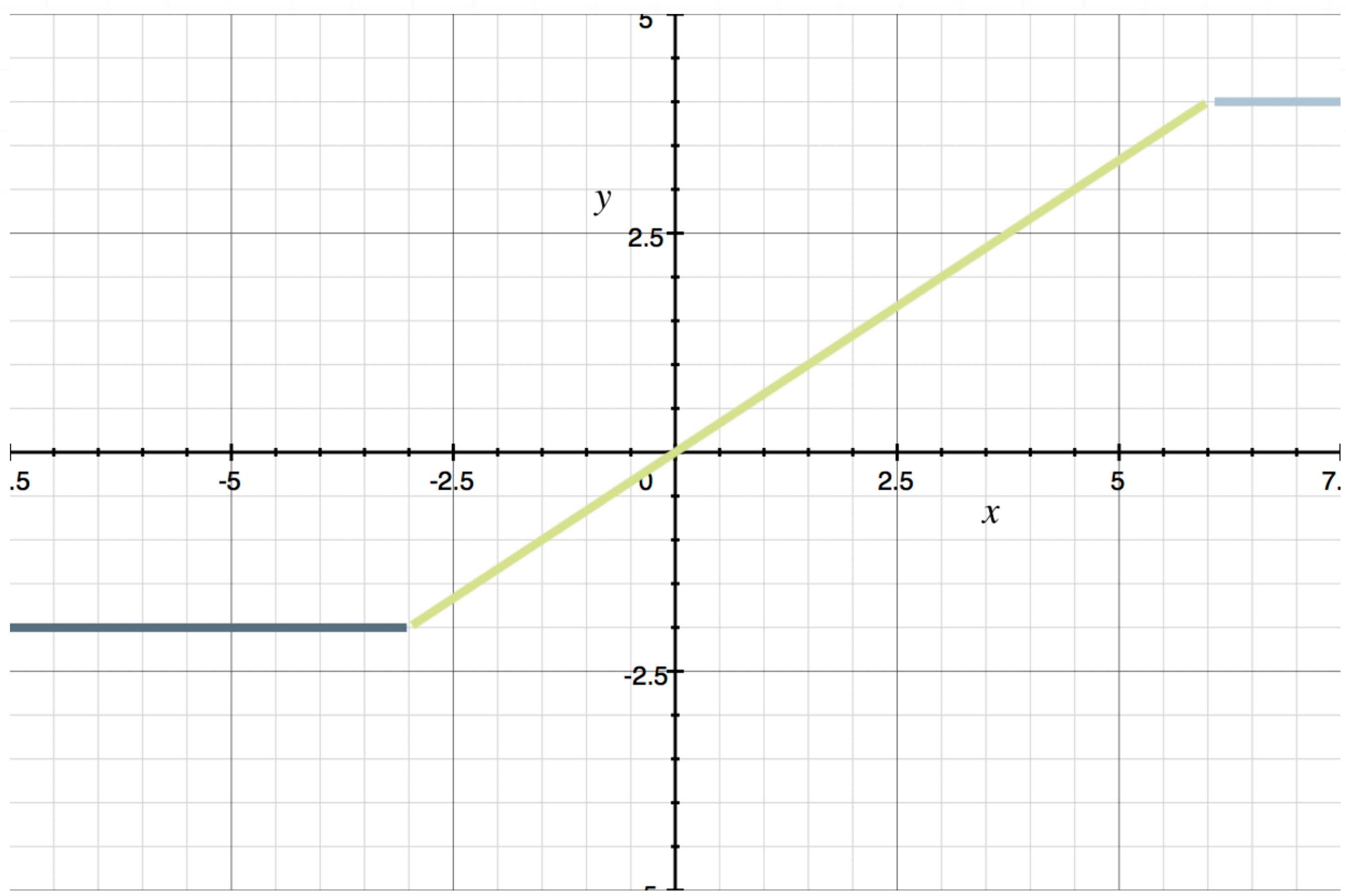
Putting these pieces together in a piecewise function gives

$$f(x) = \begin{cases} (x+3)^2 & x \le 0 \\ x+9 & x > 0 \end{cases}$$

We don't know if the "equal to" part of the equation is for the blue parabola or the green line, so we also could have written

$$f(x) = \begin{cases} (x+3)^2 & x < 0 \\ x+9 & x \ge 0 \end{cases}$$

■ 2. Find the equation of the piecewise function.



Solution:

The dark blue horizontal line is $f(x) = -2$, from $-\infty$ to -3, or when $x \leq 3$.

The green line has a slope of $2/3$ and a y-intercept of 0, so the equation of the line is $f(x) = (2/3)x$ from -3 to 6, or when $-3 < x < 6$.

The light blue horizontal line is at $f(x) = 4$, from 6 to ∞, or when $x \geq 6$.

Putting these pieces together in a piecewise function gives

$$f(x) = \begin{cases} -2 & x \leq -3 \\ \frac{2}{3}x & -3 < x < 6 \\ 4 & x \geq 6 \end{cases}$$

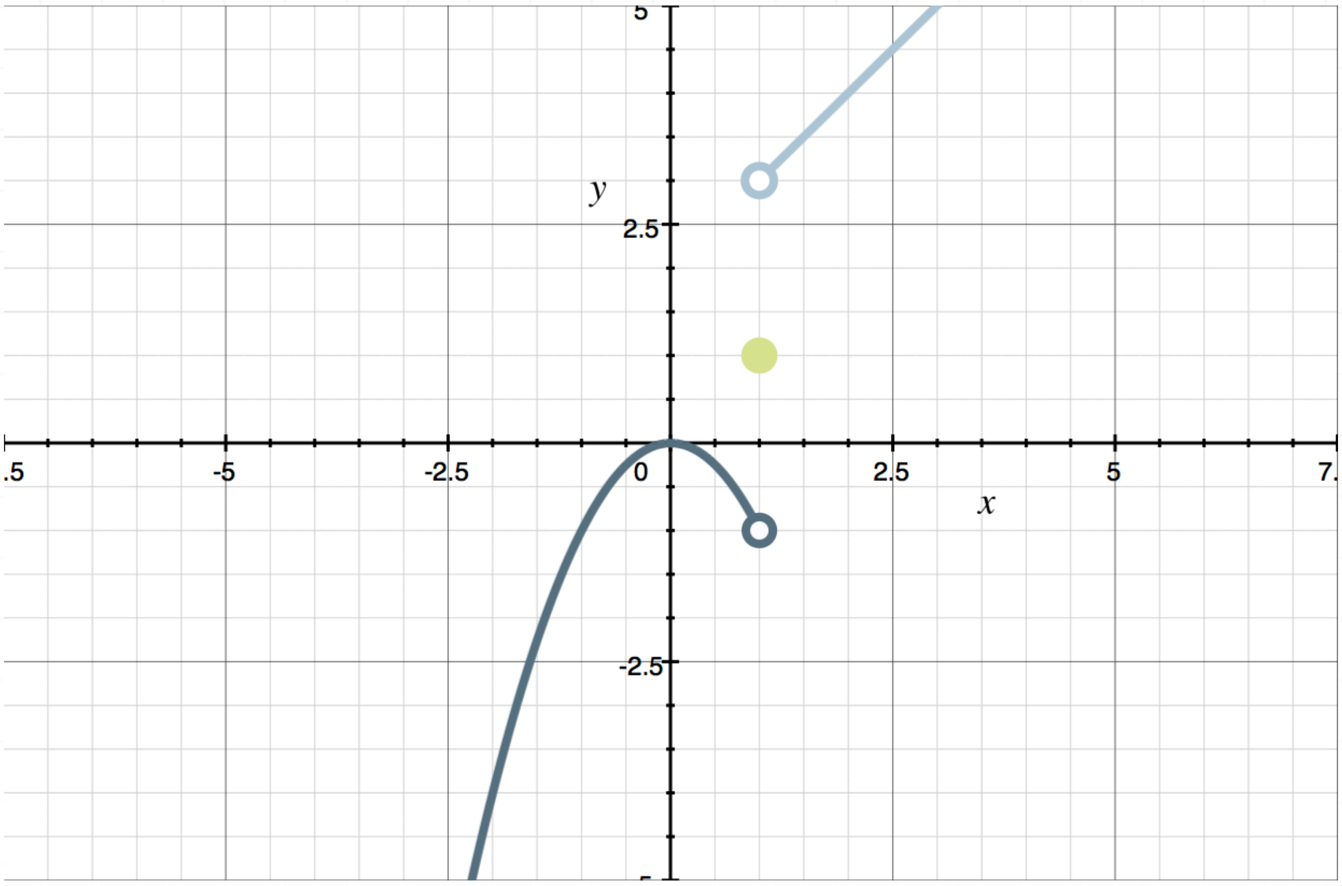
We don't which piece of the graph takes the "equal to" part of the equation, so we also could have written the answer as any of the following.

$$f(x) = \begin{cases} -2 & x < -3 \\ \frac{2}{3}x & -3 \leq x < 6 \\ 4 & x \geq 6 \end{cases}$$

$$f(x) = \begin{cases} -2 & x \leq -3 \\ \frac{2}{3}x & -3 < x \leq 6 \\ 4 & x > 6 \end{cases}$$

$$f(x) = \begin{cases} -2 & x < -3 \\ \frac{2}{3}x & -3 \leq x \leq 6 \\ 4 & x > 6 \end{cases}$$

3. Find the equation of the piecewise function.



Solution:

The dark blue parabola is $f(x) = -x^2$ from $-\infty$ to 1, or when $x < 1$. We know that it's strictly "less than" because of the hollow circle on the parabola at $x = 1$.

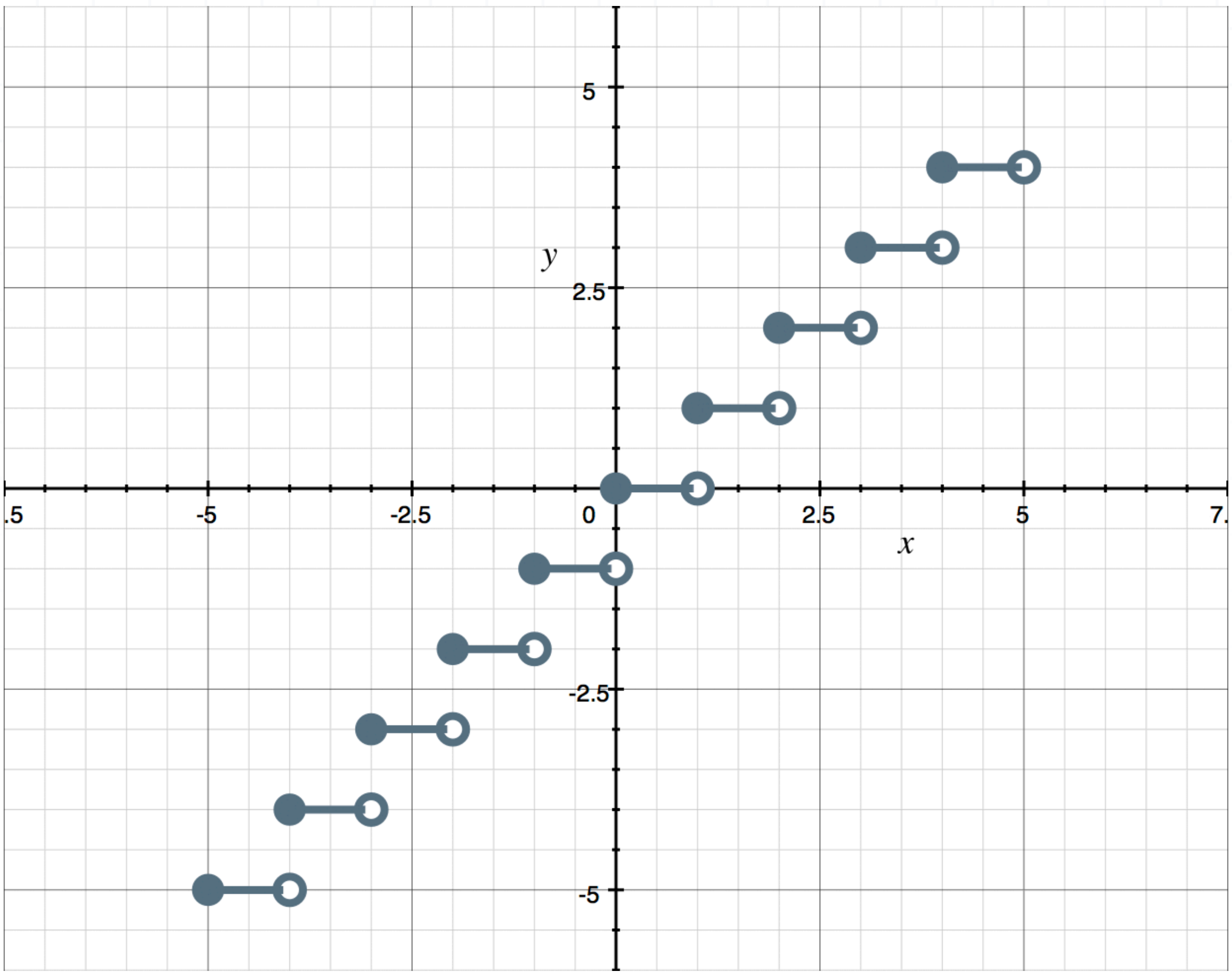
The solid green dot means that $f(x) = 1$ when $x = 1$.

The light blue line has a slope of 1 and would have a y-intercept of 2, so the equation of the line is $f(x) = x + 2$ from 1 to ∞, or when $x > 1$. We know that it's strictly "greater than" because of the hollow circle on the line at $x = 1$.

Putting these pieces together in a piecewise function gives

$$f(x) = \begin{cases} -x^2 & x < 1 \\ 1 & x = 1 \\ x + 2 & x > 1 \end{cases}$$

4. Find the equation of the piecewise function.



Solution:

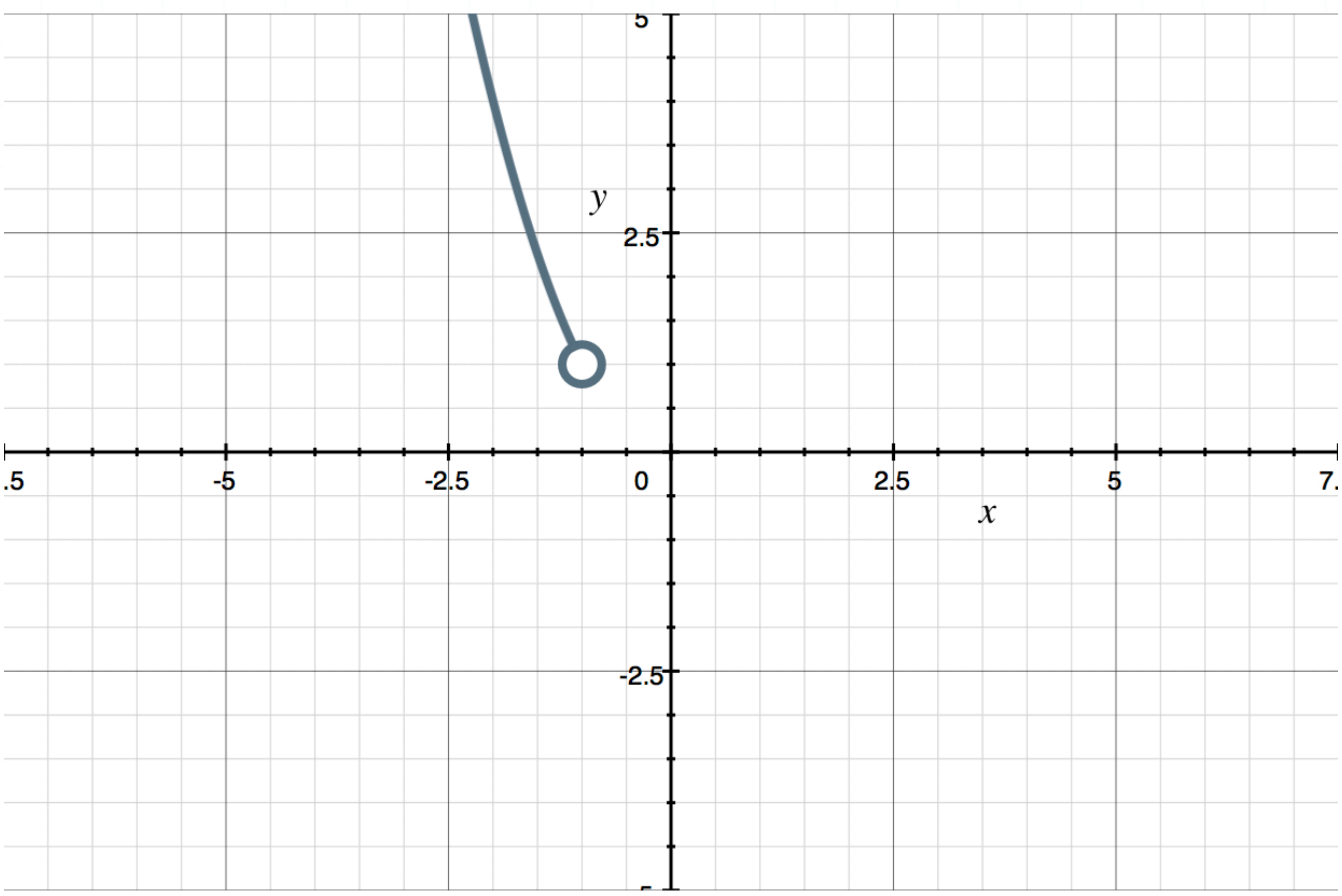
This is the "greatest integer function," a step or stair-case function that produces the greatest integer less than or equal to the number. It's denoted by $f(x) = [x]$.

5. Graph the piecewise function.

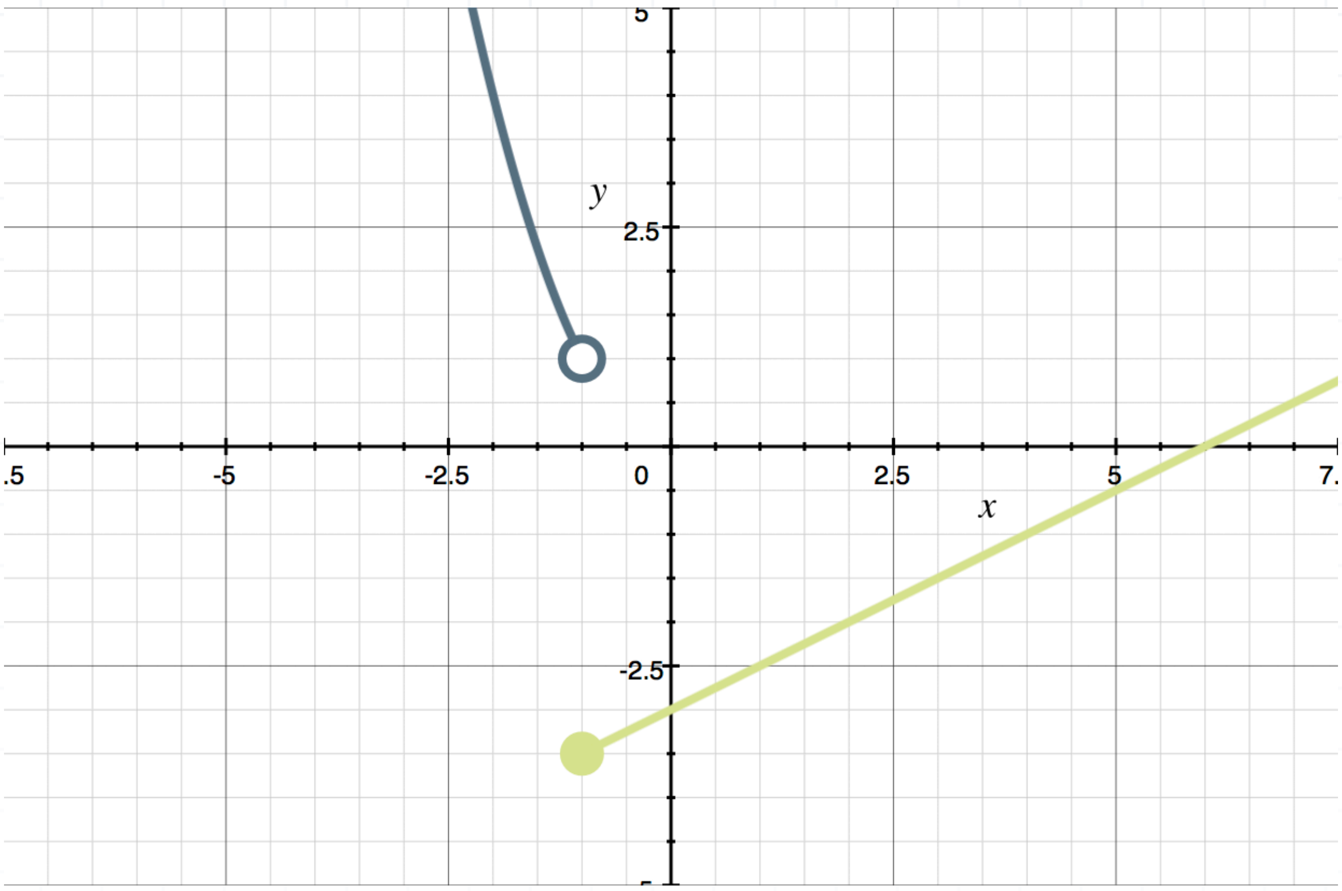
$$f(x) = \begin{cases} x^2 & x < -1 \\ \frac{1}{2}x - 3 & x \geq -1 \end{cases}$$

Solution:

First graph the parabola $f(x) = x^2$, but only when $x < -1$. This means that at $x = -1$ there will be an open circle.



Now graph the line $(1/2)x - 3$ when $x \geq -1$. This means that when $x = -1$ there will be a solid circle.

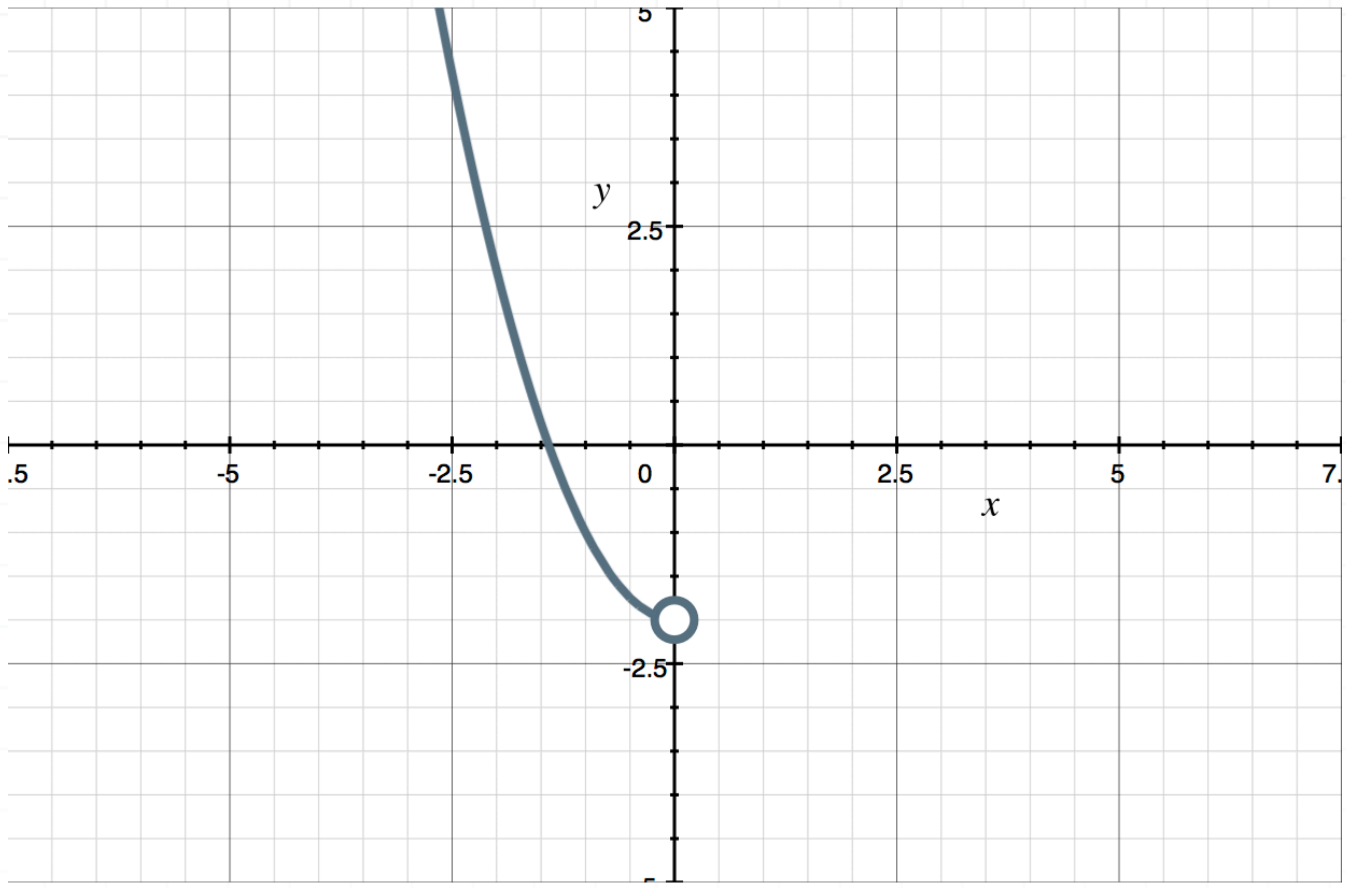


6. Graph the piecewise function.

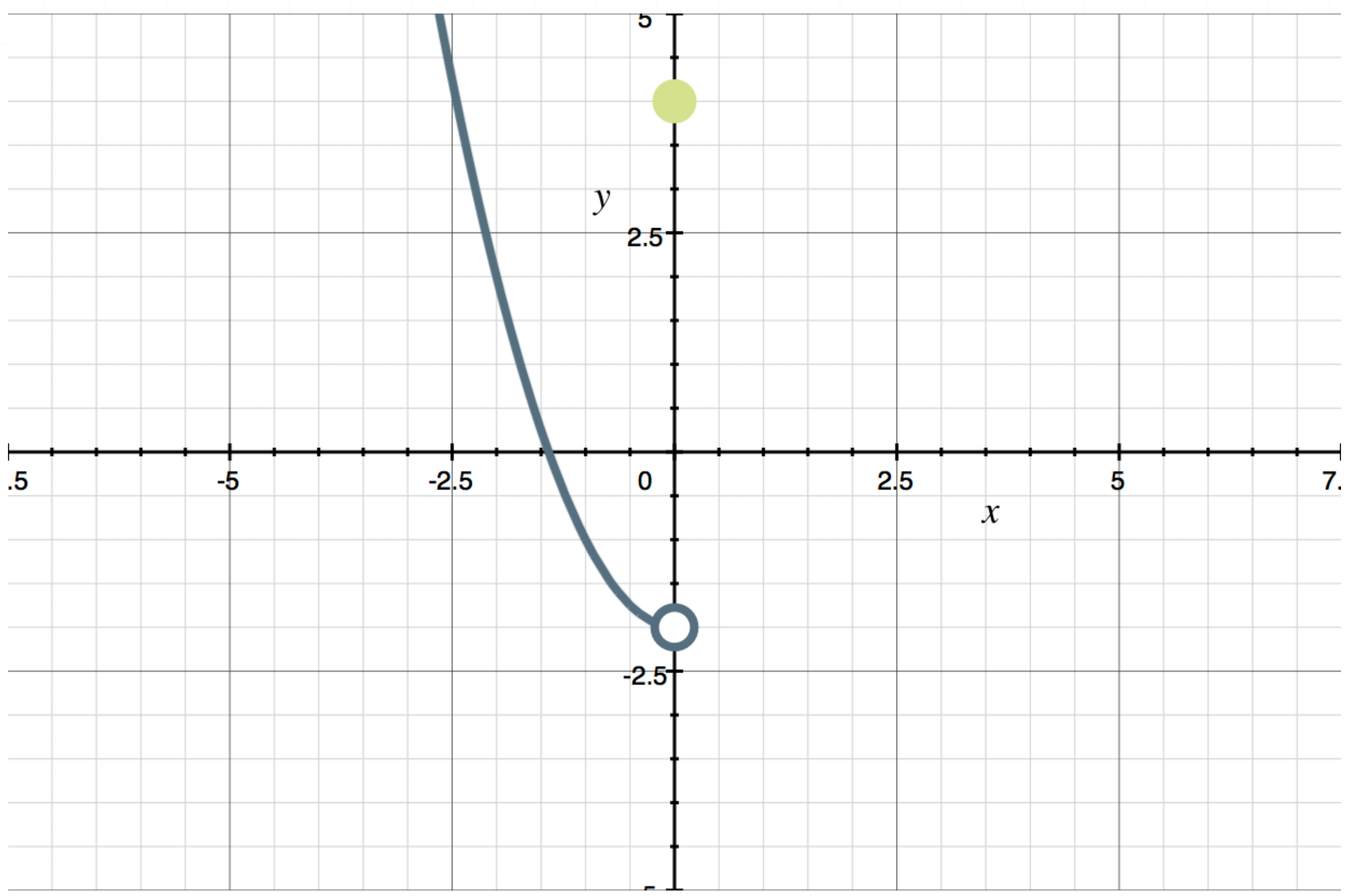
$$f(x) = \begin{cases} x^2 - 2 & x < 0 \\ 4 & x = 0 \\ -x^2 + 8 & x > 0 \end{cases}$$

Solution:

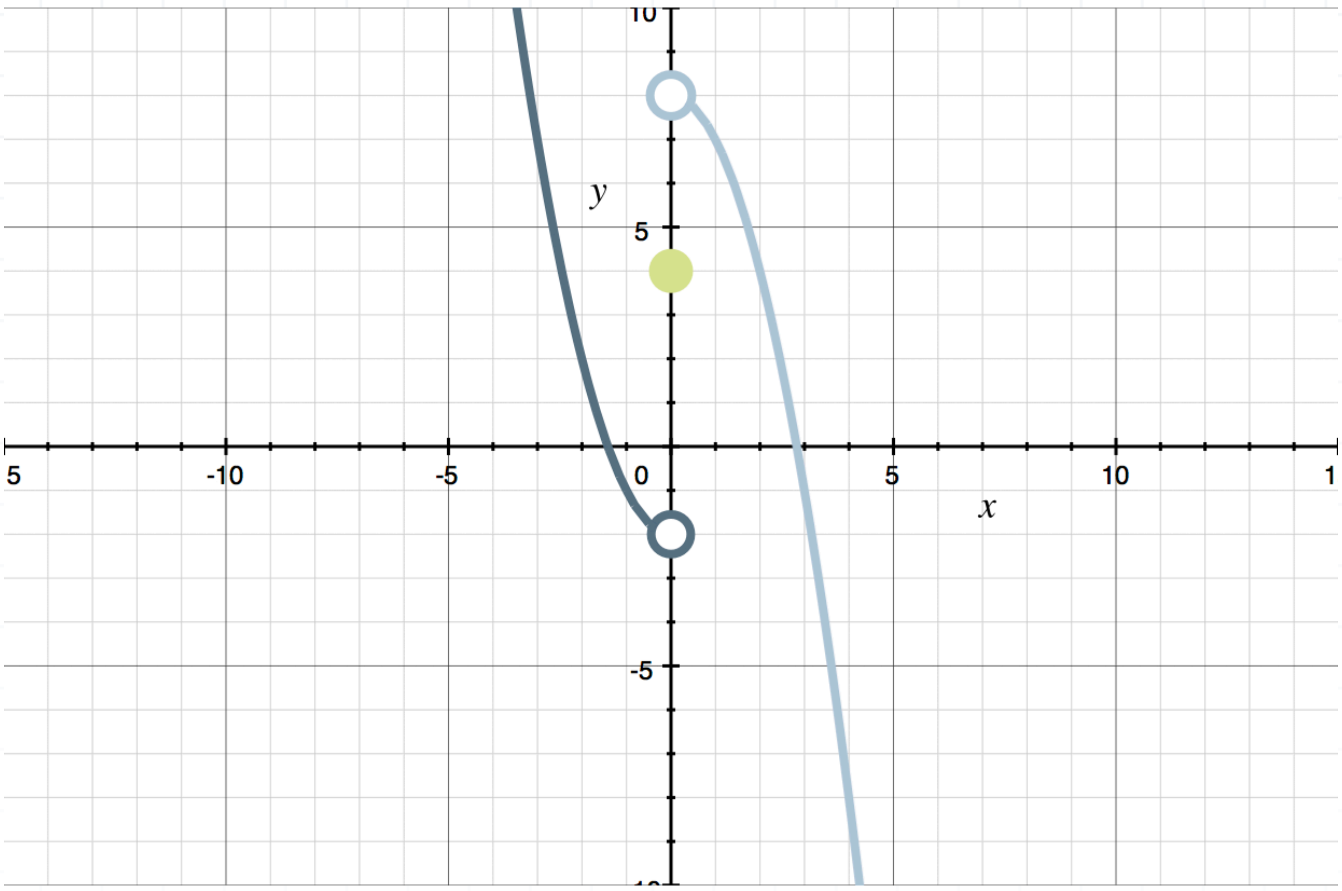
First graph the parabola $f(x) = x^2 - 2$, but only when $x < 0$. This means that at $x = 0$ there will be an open circle.



Graph the point $(0,4)$.



Now graph the parabola $f(x) = -x^2 + 8$ when $x > 0$. This means that at $x = 0$ there will be an open circle.

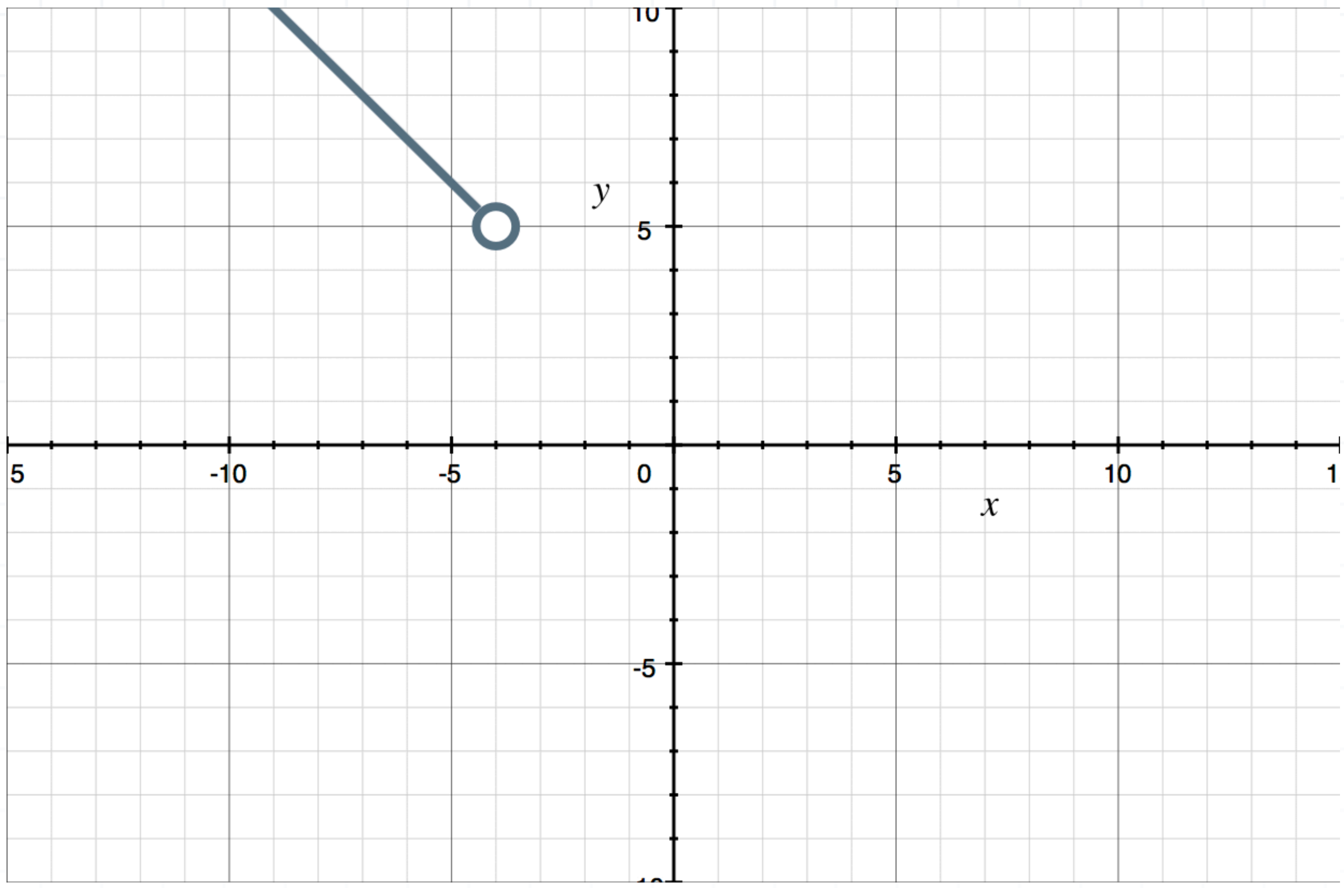


7. Graph the piecewise function.

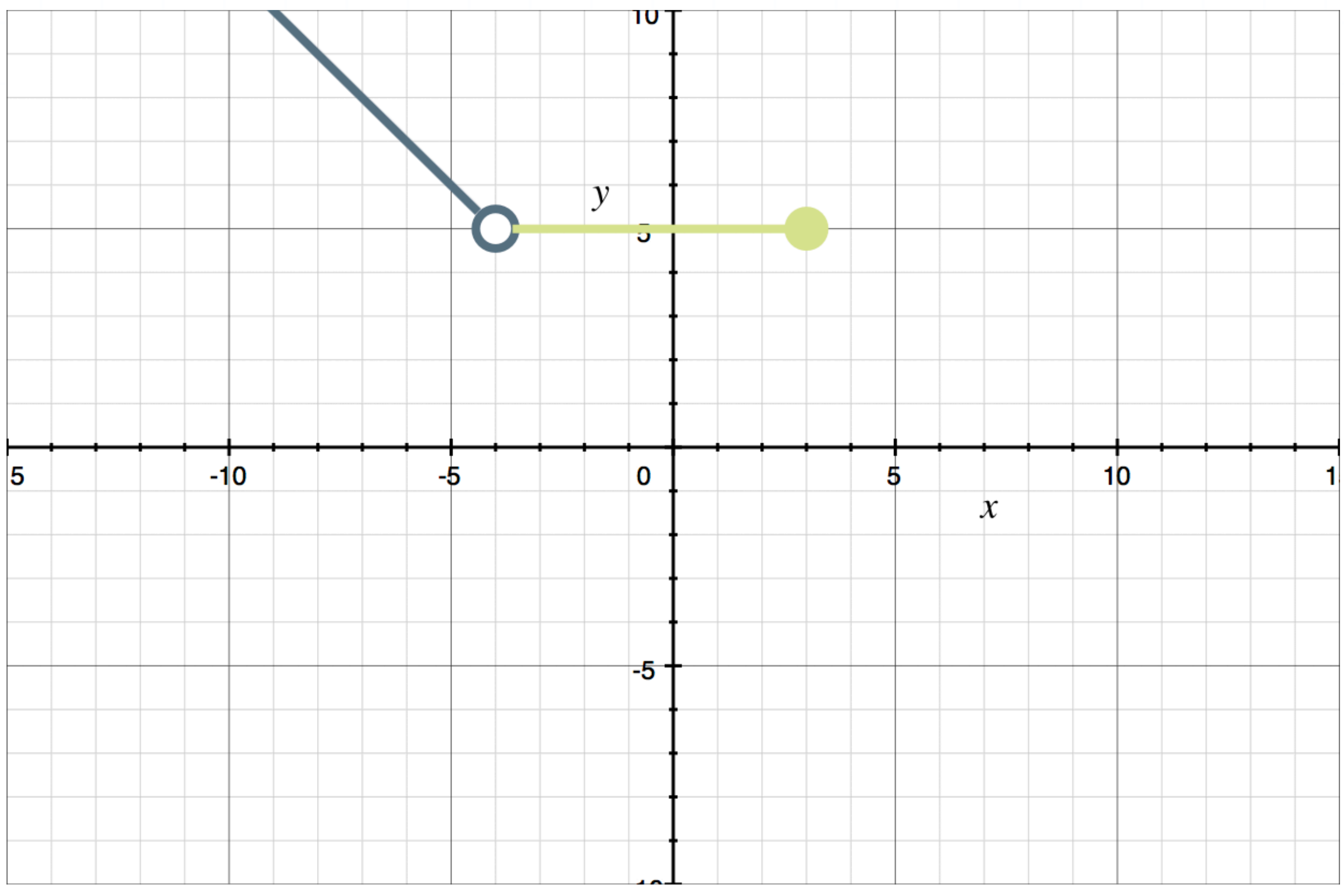
$$f(x) = \begin{cases} -x+1 & x < -4 \\ 5 & -4 < x \leq 3 \\ -2x+11 & x > 3 \end{cases}$$

Solution:

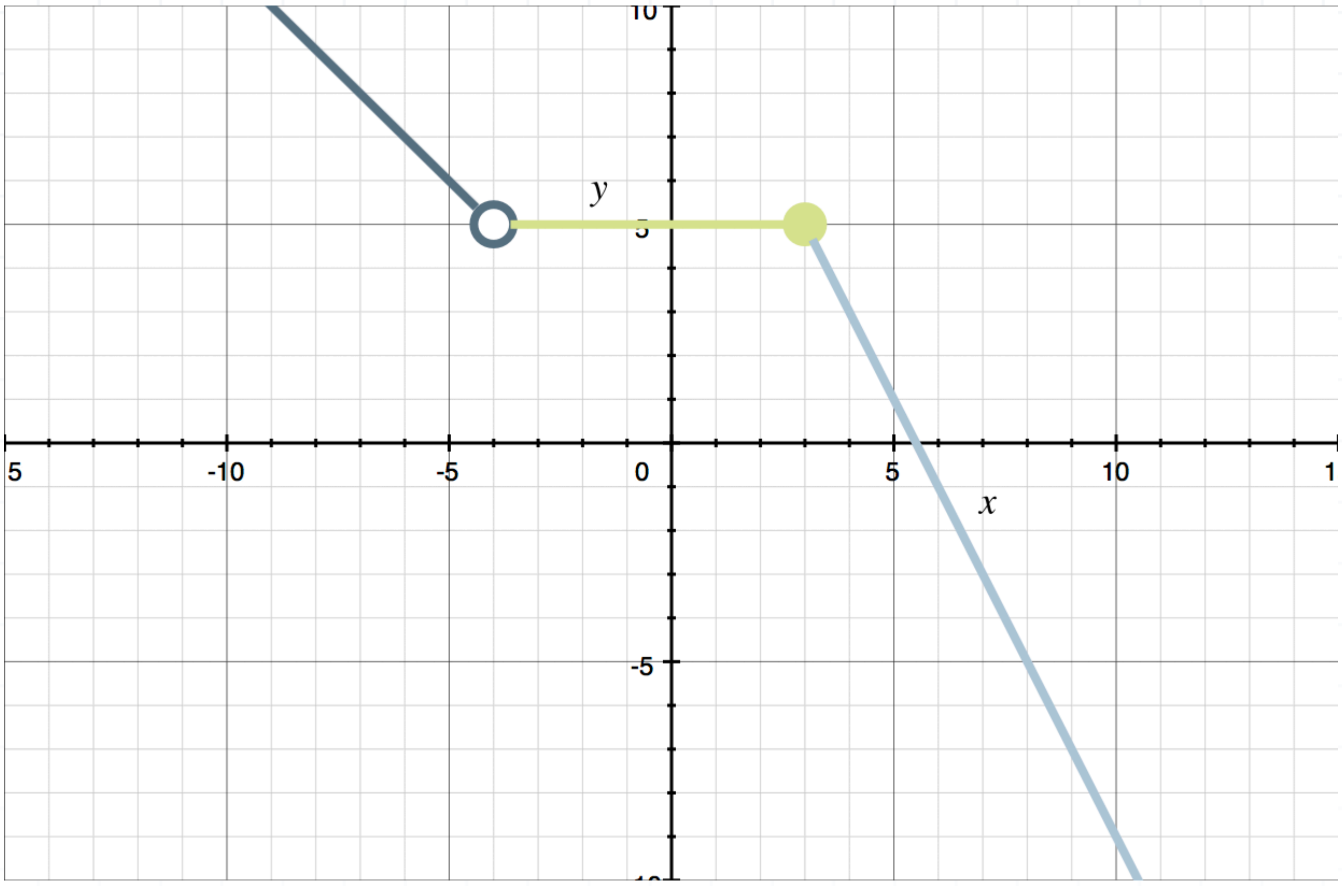
First graph the line $f(x) = -x + 1$, but only when $x < -4$. This means that at $x = -4$ there will be an open circle.



Graph the line $f(x) = 5$ when $-4 < x \leq 3$. This means that at $x = 3$ there will be solid circle.



Now graph the line $f(x) = -2x + 11$ when $x > 3$.

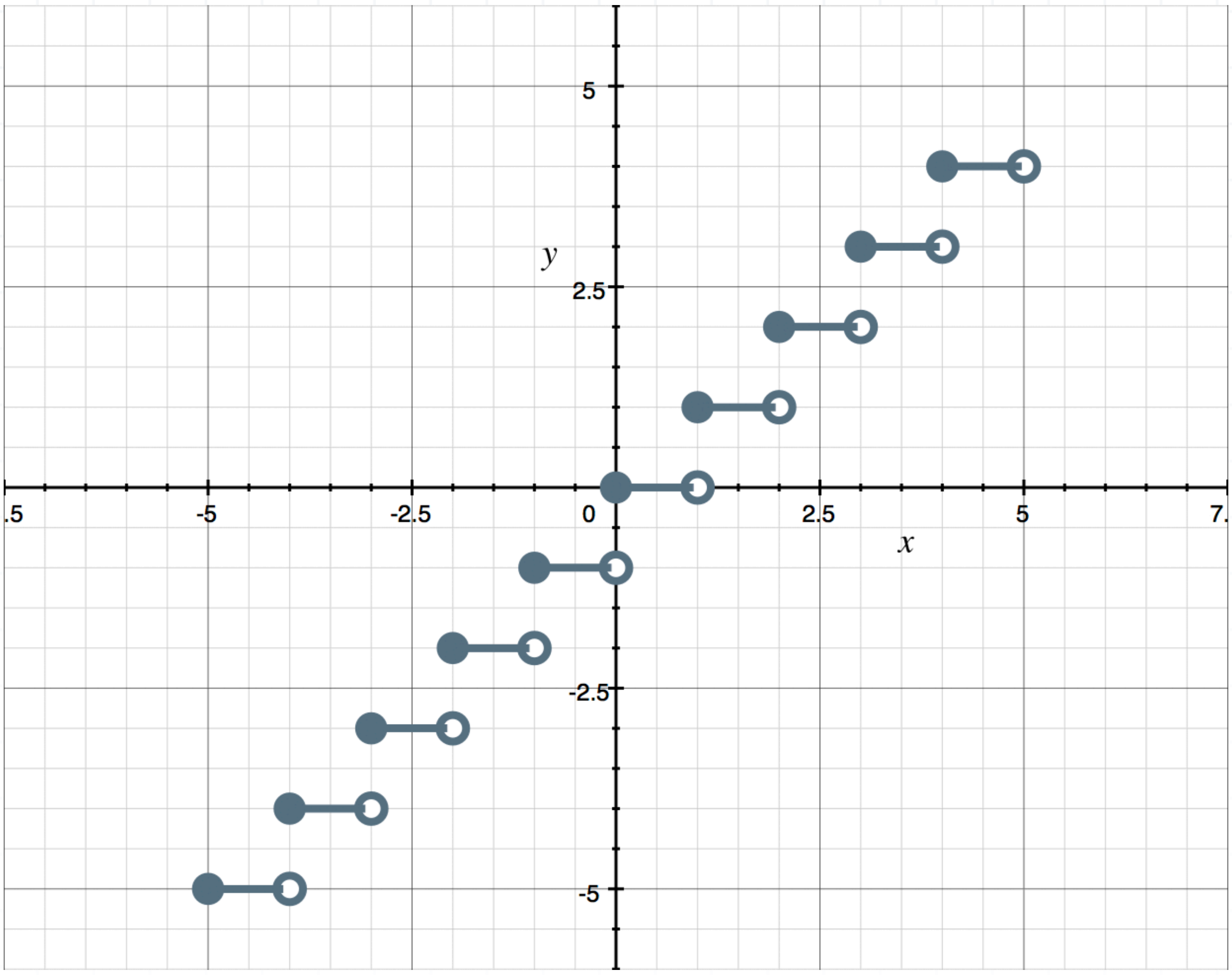


8. Graph the piecewise function.

$$f(x) = [x]$$

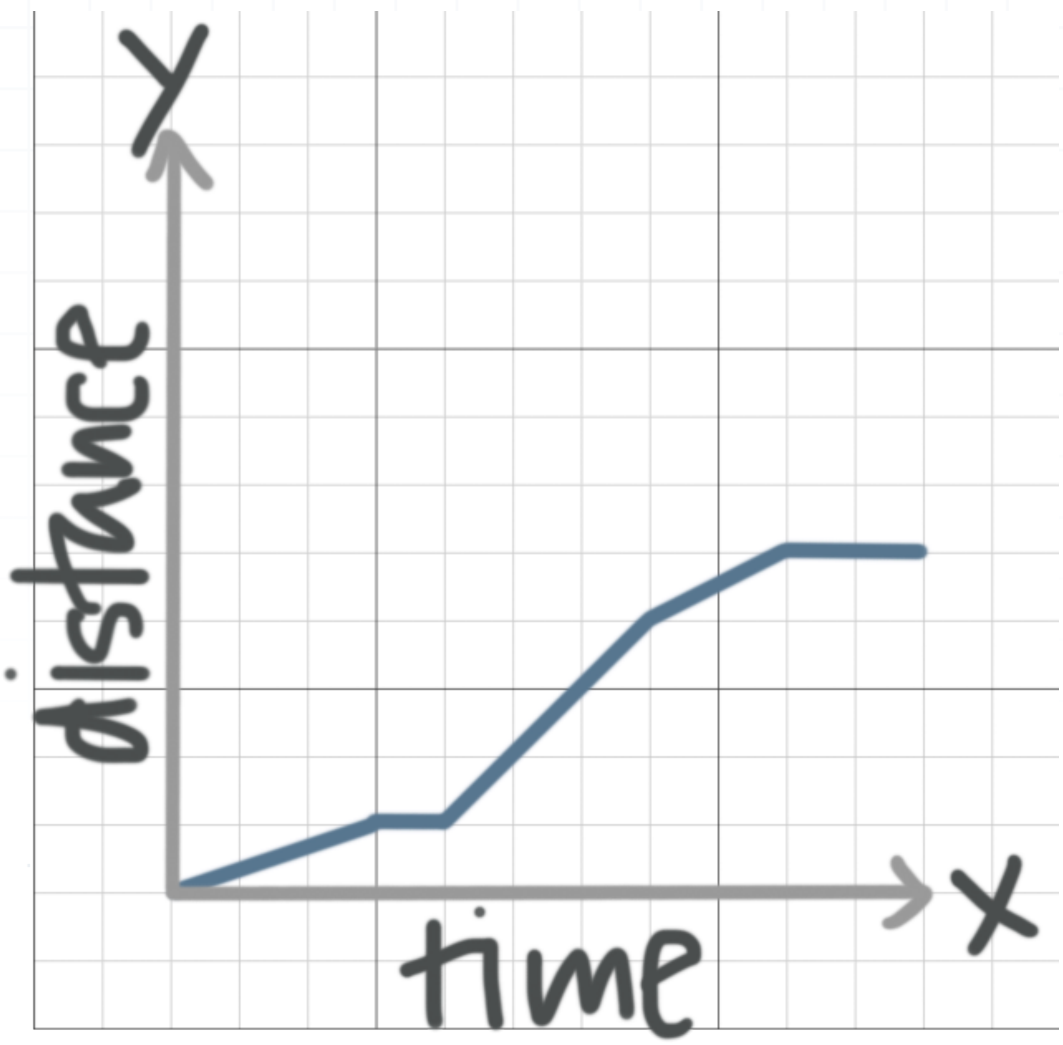
Solution:

This is the "greatest integer function," a step or staircase function that produces the greatest integer less than or equal to the number. This means there is a horizontal line between each interval that has a closed circle on the left and an open circle on the right.


5
y
2.5
.5
-5
-2.5
0
2.5
5
7.
x
-2.5
-5

SKETCHING GRAPHS FROM STORY PROBLEMS

1. Alex left in his car to visit his grandparents' house. The graph shows his distance from his house over time. Write a possible story to go along with the graph.

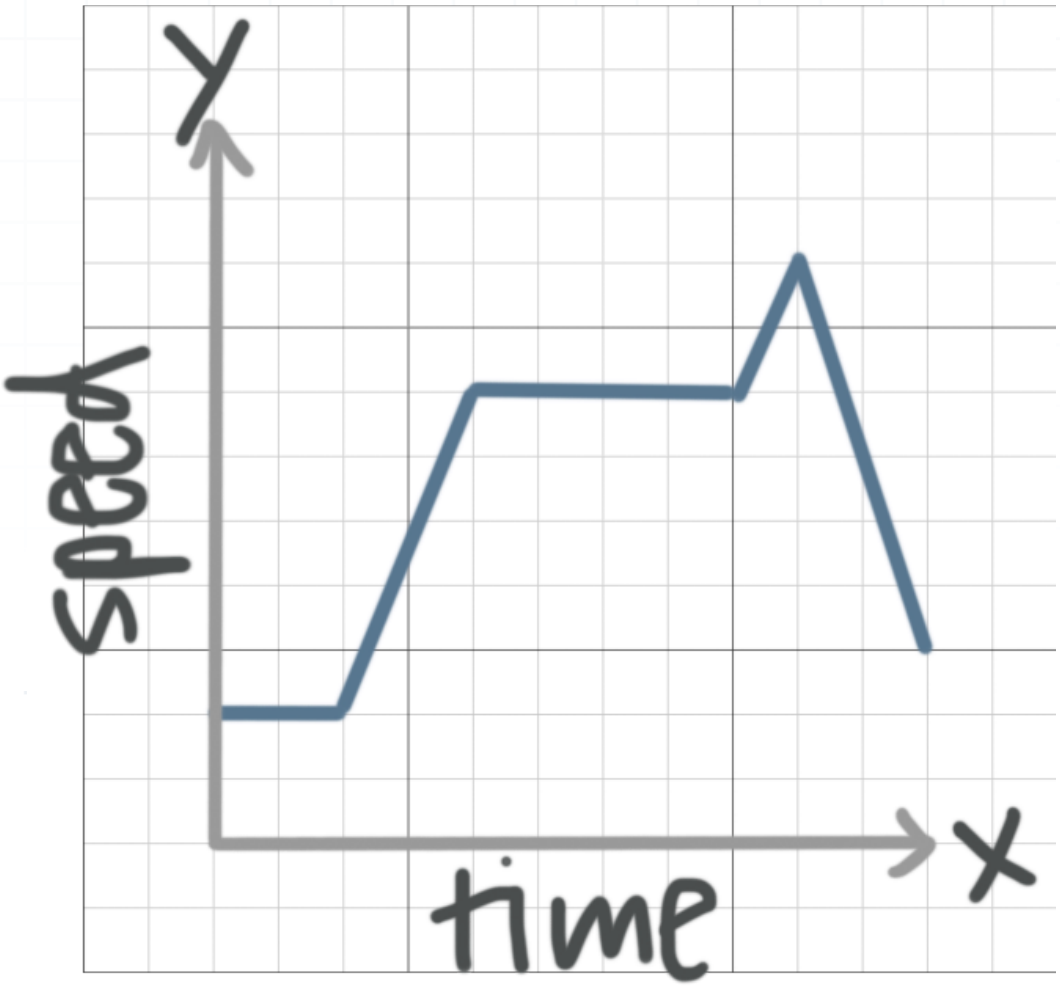


Solution:

Looking at the graph, there are two sections where the distance increases at a slower rate and one section where Alex's distance increases faster. There are also two horizontal lines, indicating stops. The last stop is when Alex arrives at his grandparents' house. So the story might be

"Alex is driving in town and then stops to get more gas. After stopping for gas he drives on the highway and his speed increases. When Alex gets off the highway he drives more slowly until he arrives at his grandparents' house, where he stops."

2. A horse is practicing for a race. The graph shows the horse's speed over time. Write a possible story to go with the graph.

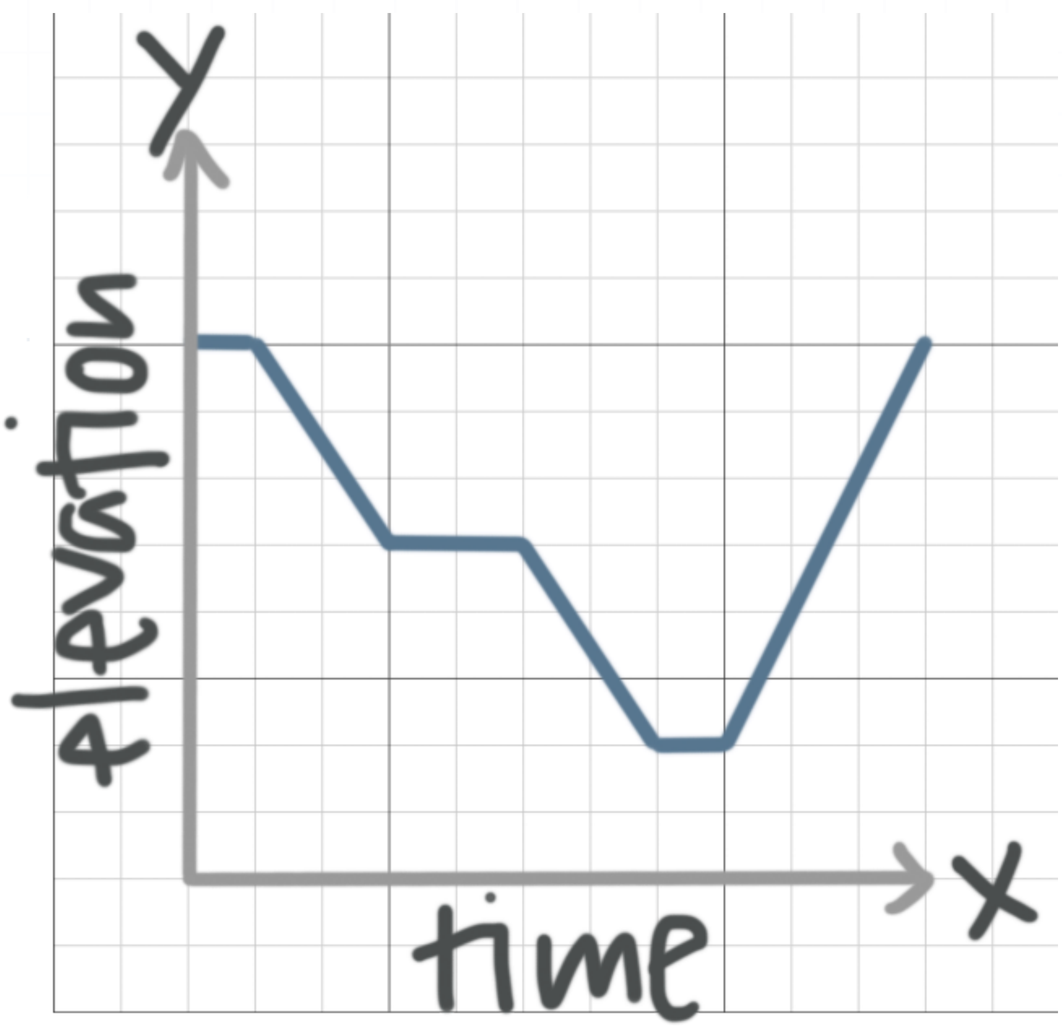


Solution:

The speed of the horse is slow at the beginning and then increases. In the middle of the graph, the horse maintains its speed. Then the horse increases speed for a short period of time before slowing quickly. So the story might be

> *"The horse walks to the beginning of the race track and then increases speed quickly. The horse runs at a constant speed in the middle of the track and then the rider encourages the horse to run even faster during the last stretch of the track. After the course the horse slows down rather quickly."*

3. A scuba diver takes a dive to explore the ocean. The graph below shows the diver's elevation over time. Write a possible story to go with the graph.

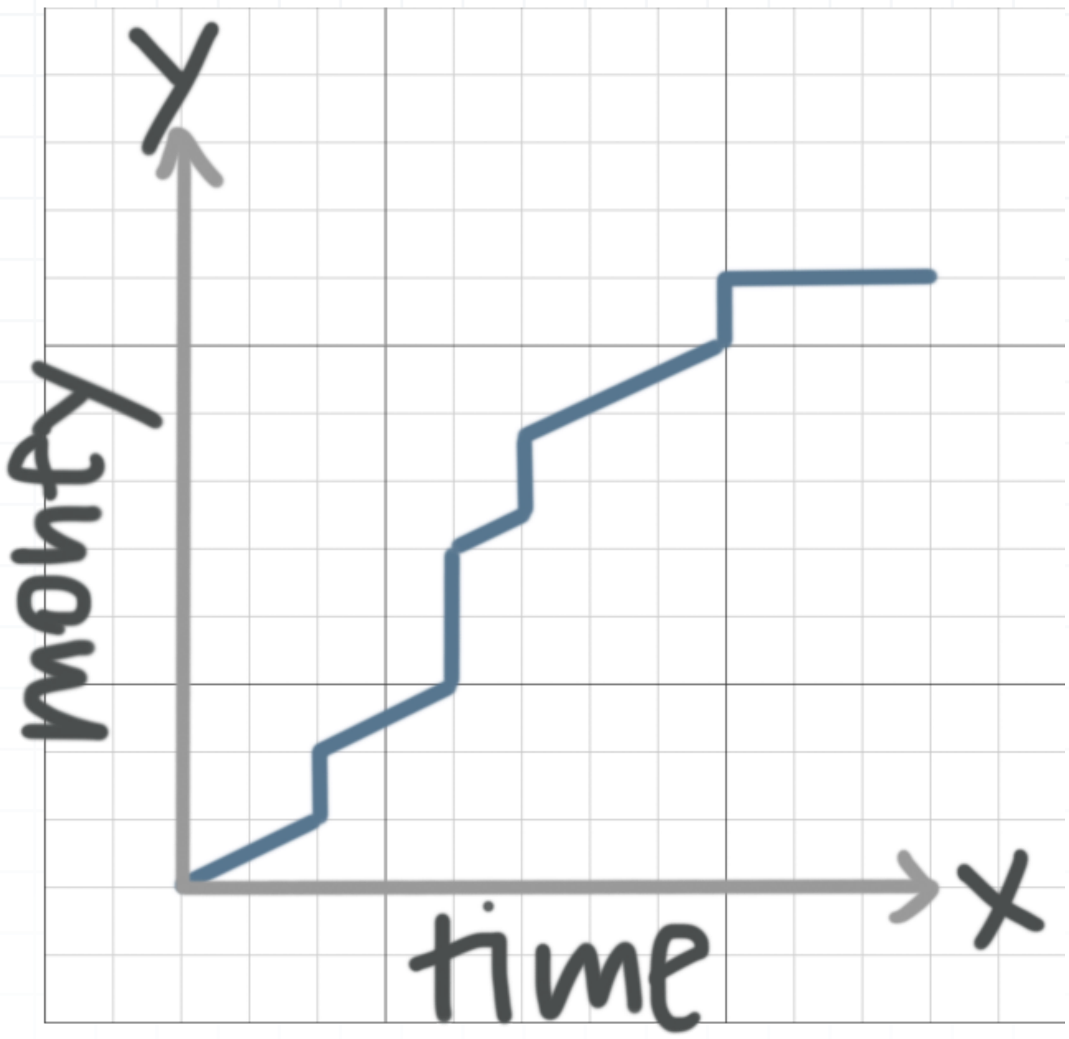


Solution:

The graph starts with a high elevation at the surface of the water, then the elevation decreases as the diver dives down. The elevation stays constant while the diver explores at a consistent depth. Next, the elevation decreases again indicating the diver dives to a deeper depth. The elevation remains constant for a shorter amount of time at the new depth. Finally the elevation increases back to the original elevation indicating that the diver is ascending back to the surface. So the story might be

> *"The scuba diver dives down and spends some time exploring at that depth. The diver then decides to dive deeper and spends a shorter amount of time exploring at the new depth. Finally the diver makes his way back up to the surface."*

■ 4. Janet delivers packages and get paid an hourly rate in addition to $\$1$ for every package she delivers. The graph shows Janet's pay over the course of the day. Write a possible story to go with the graph.



Solution:

When Janet is driving, her pay increases as a steady rate since she's paid hourly. The sudden increases in pay indicate times when Janet makes deliveries because she also gets paid per package delivered. The second delivery shows a sudden increase that's twice as high as the others, indicating that there were twice as many packages.

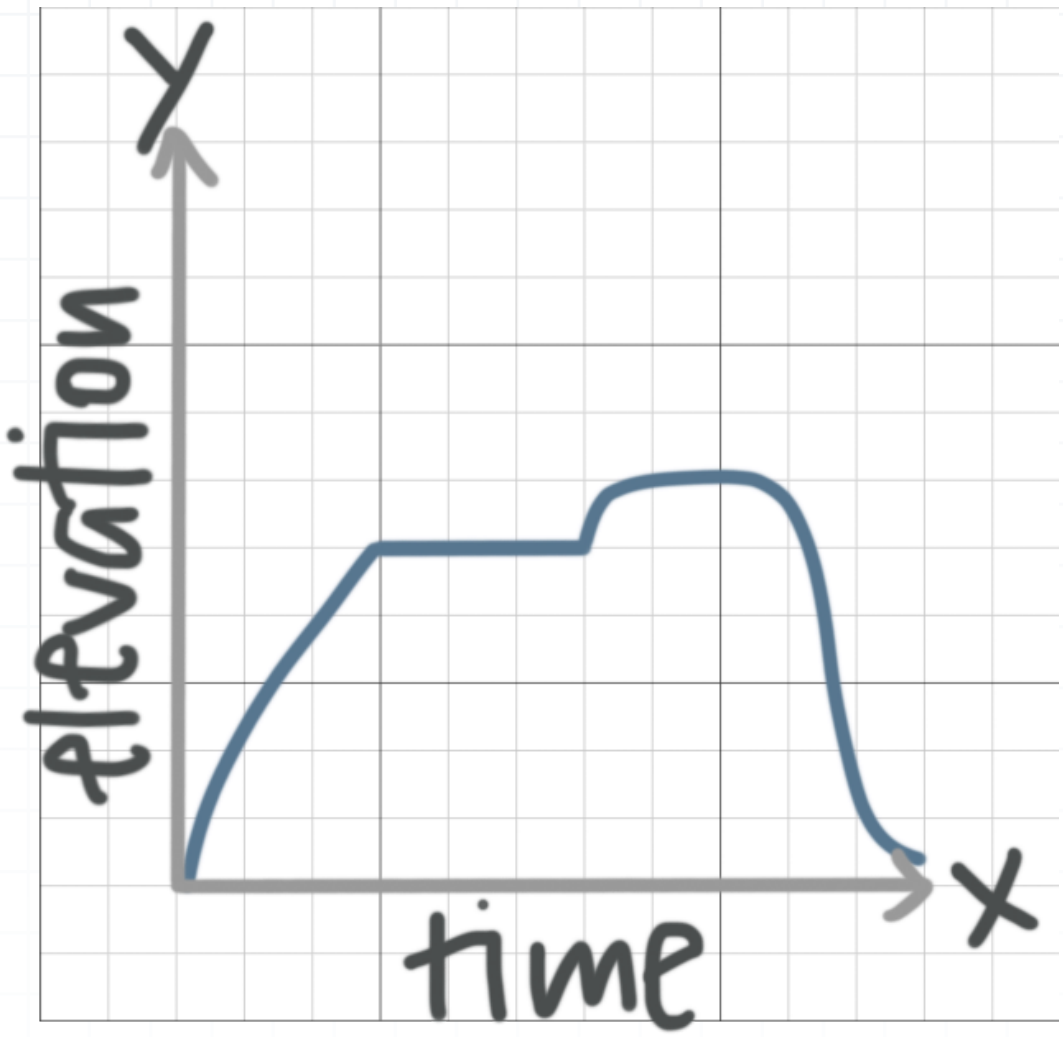
There are four sections that show the steady increase of pay when Janet is driving. The first two are the same length, the third is short, and the last is longer. At the end of the graph the amount of money becomes constant, indicating the end of Janet's shift, since she's no longer making money. So the story might be

"Janet delivers the same number of packages at her first, third, and last stop. At her second stop Janet delivers twice as many packages. Janet spends the same amount of time driving to her first and second stops. The third stop is closer, so Janet doesn't spend as much time driving, but she spends the most time driving to her final stop. After her final stop, Janet's shift is done for the day."

■ 5. A plane takes off and then cruises at $30{,}000$ feet for several hours before rising in elevation to $35{,}000$ feet to avoid turbulence for the last few hours. The plane then reaches its destination and lands. Sketch a graph representing the situation.

Solution:

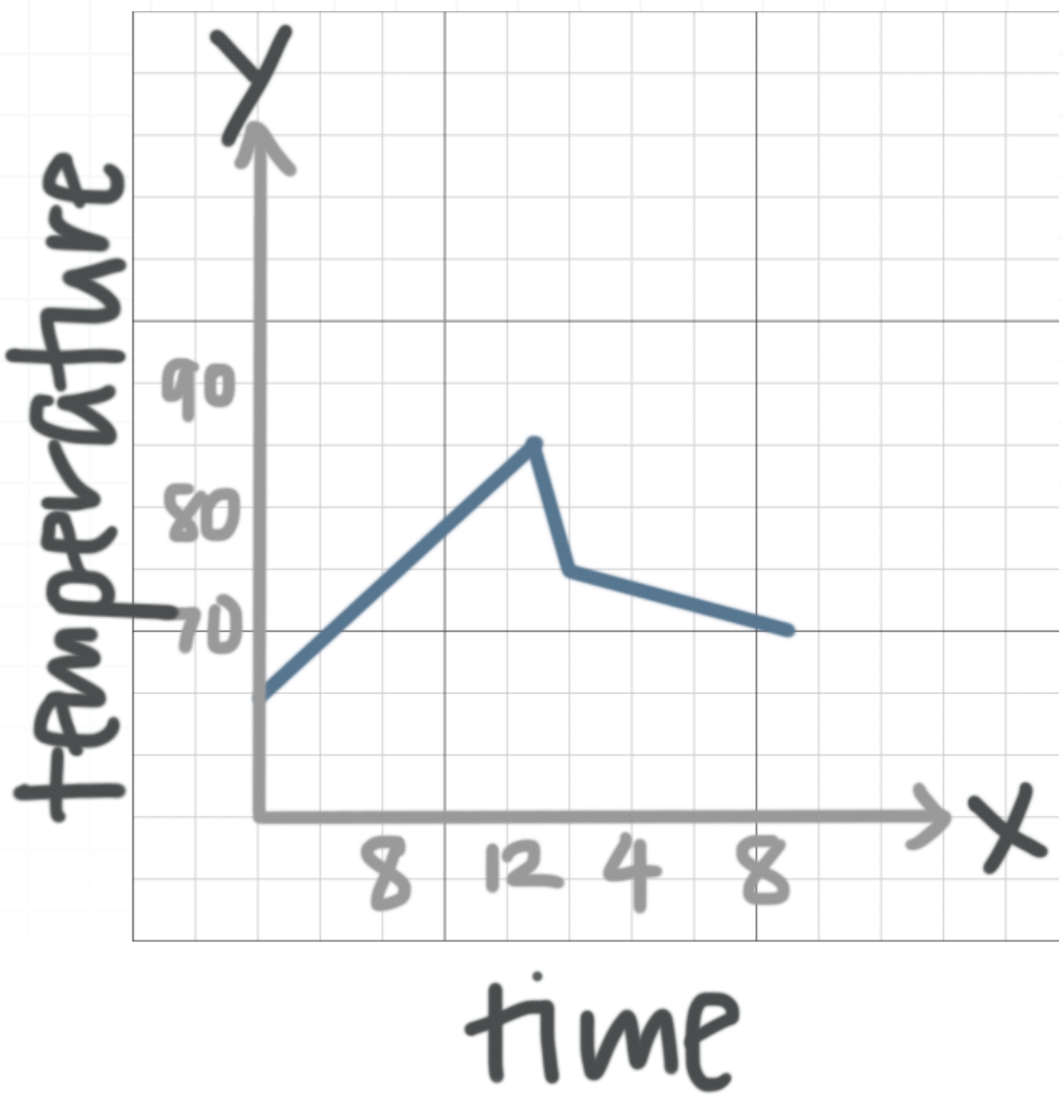
From left to right, elevation will rise dramatically and then level off. Then elevation will rise a little more to $35{,}000$ feet and level off at that elevation for a while before the elevation decreases for the landing.



6. The temperature throughout a summer day starts at 65° F at 6:00 a.m.. Over the next few hours the temperature rises steadily until it reaches 85° F at 1:00 p.m.. At 1:15 p.m., a rainstorm begins and cools the temperature down to 75° F. The temperature then steadily decreases until it reaches 70° F at 9:00 p.m.. Sketch a graph representing the situation.

Solution:

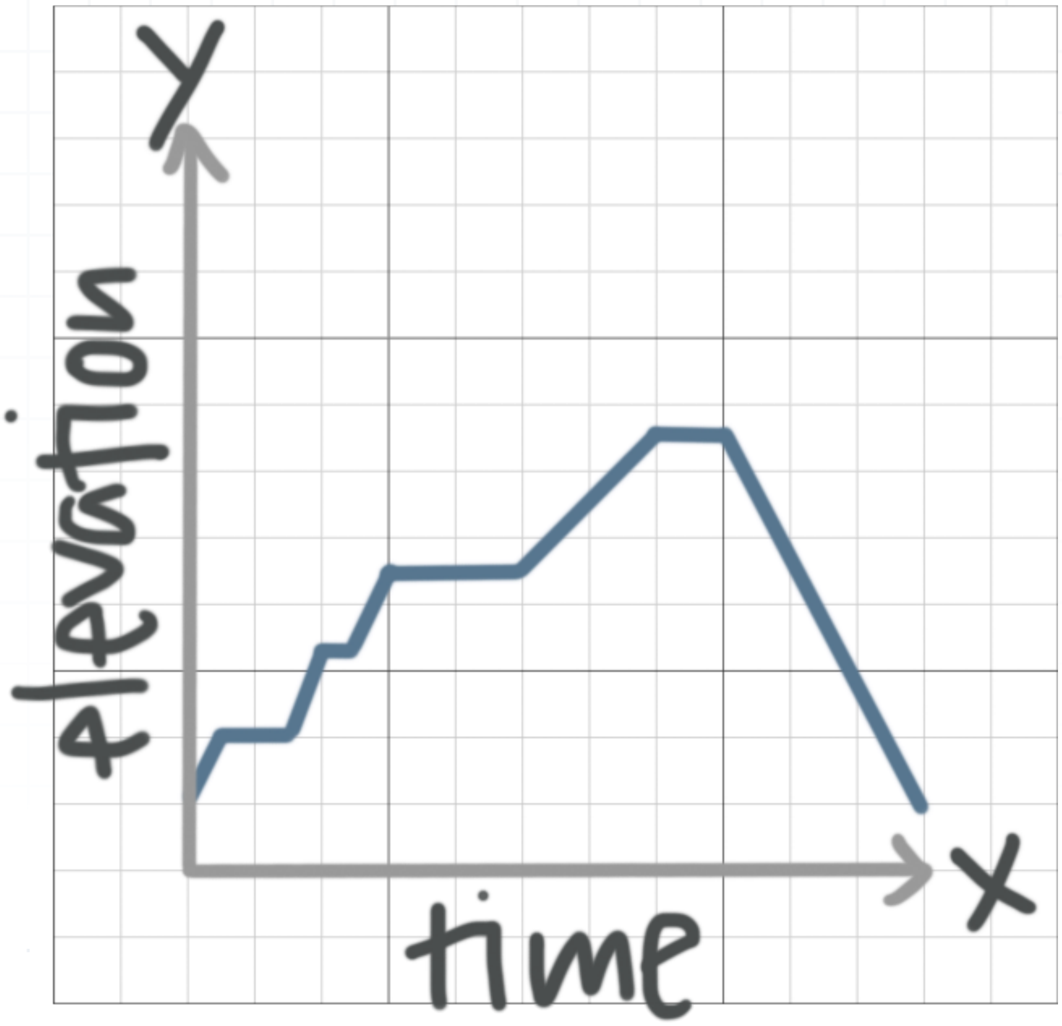
From left to right, the temperature will rise steadily until 1:00 p.m. from 65° to 85°. Then there will be a sharp decrease down to 75°, due to the rainstorm. After the rainstorm, the temperature decreases steadily until it reaches 70° at 9:00 p.m..



7. Brett goes for a hike up a mountain. He starts hiking up steadily for several hours with two stops for water. Then Brett stops for an hour to eat lunch and rest. He then continues up the mountain, summits, and spends a little time at the top of the mountain before climbing down. Sketch a graph representing Brett's elevation over time.

Solution:

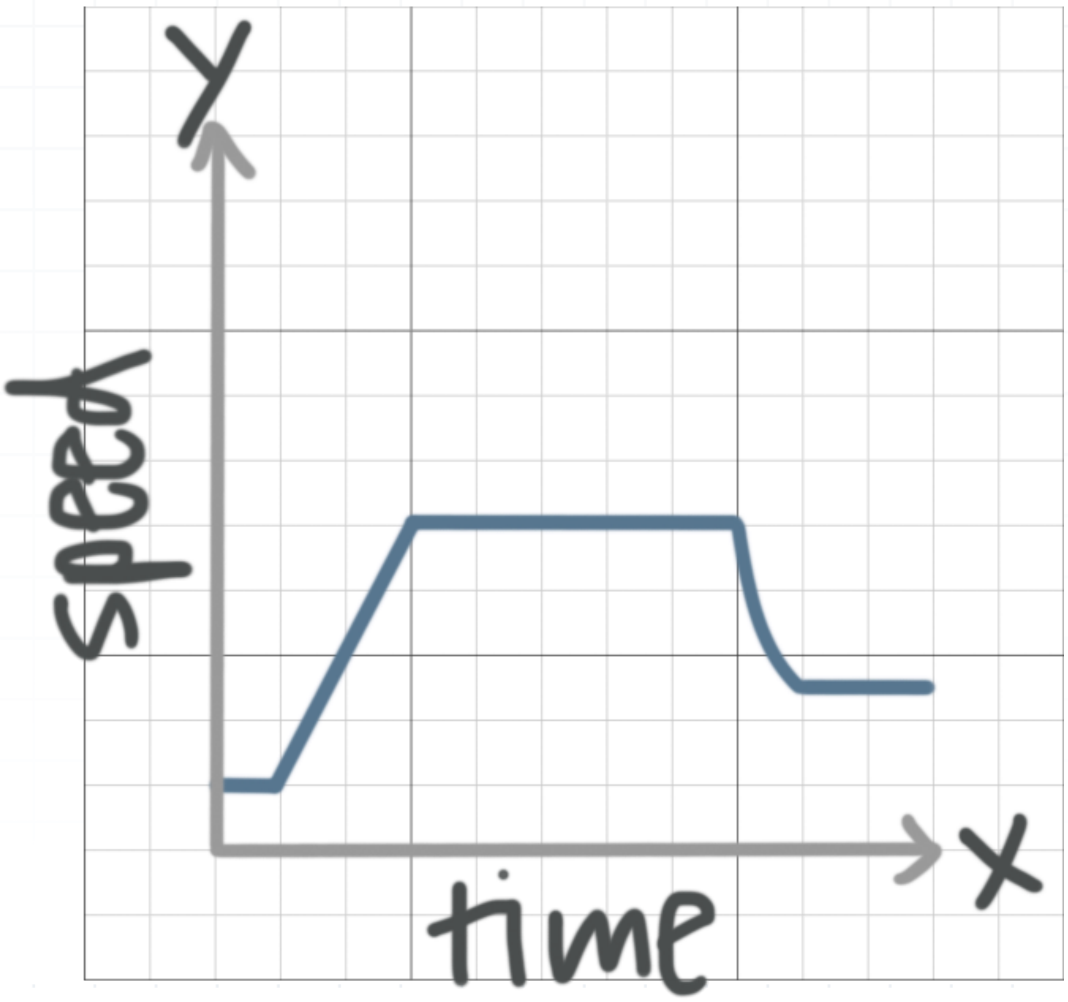
From left to right, Brett's elevation will rise steadily with a couple of small stops for water before a longer stop for lunch. After lunch Brett's elevation continues to increase until he reaches the top of the mountain. Then his elevation will stay steady as he spends some time at the top of the mountain, before decreasing as he descends down the mountain.



■ 8. Heather went for a bike ride. She started at 12 mph to warm up, but quickly increased her speed to 20 mph and maintained that speed for most of the ride. Near the end of her bike ride, Heather decreased her speed to 15 mph until she reached her destination. Sketch a graph representing Heather's speed over time.

Solution:

From left to right, Heather starts biking and maintains her slower speed for a short amount of time. Her speed increases and stays at 20 mph for most of the bike ride before decreasing at the end of the ride.



EQUATION OF A LINE IN POINT-SLOPE FORM

1. Find the equation of the line that passes through $(3,0)$ with slope -2.

Solution:

Using point-slope form, the equation of the line is

$$y - 0 = -2(x - 3)$$

$$y = -2x + 6$$

2. Name two (of four possible) pieces of information about a line that are required to write an equation of the line in point-slope form.

Solution:

Naming any two of the following is correct:

(1) A point

(2) Another point

(3) The slope

(4) The y-intercept

■ 3. Find the equation of the line that passes through the points $(-2,3)$ and $(2,-4)$.

Solution:

We first need to calculate the slope of the line as follows

$$m = \frac{-4-3}{2-(-2)}$$

$$m = \frac{-7}{4}$$

$$m = -\frac{7}{4}$$

Using point-slope form, the equation of the line is either of the following:

$$y - 3 = -\frac{7}{4}(x+2)$$

$$y + 4 = -\frac{7}{4}(x-2)$$

■ 4. Find the equation of the line that passes through $(-2,-5)$ with a slope 6.

Solution:

Using point-slope form, the equation of the line is

$$y + 5 = 6(x + 2)$$

$$y + 5 = 6x + 12$$

$$y = 6x + 7$$

5. Identify the point (x_1, y_1) and slope m in the equation of the line.

$$y + 3 = \frac{1}{4}(x - 6)$$

Solution:

Using point-slope form, we can see that the point is $(6, -3)$ and the slope is $1/4$.

6. Write the following equation in point-slope form.

$$y = -\frac{1}{2}x + 4$$

Solution:

Notice that the slope is given as $-1/2$ and the line passes through the point $(2,3)$, so we can write the equation of the line as

$$y-3=-\frac{1}{2}(x-2)$$

7. Find the equation of the line that passes through the points $(5,-4)$ and $(6,0)$.

Solution:

We first need to calculate the slope of the line as

$$m=\frac{0-(-4)}{6-5}$$

$$m=\frac{4}{1}$$

$$m=4$$

Using point-slope form, the equation of the line is then either of the following:

$$y+4=4(x-5)$$

$$y=4(x-6)$$

EQUATION OF A LINE IN SLOPE-INTERCEPT FORM

■ 1. Find the equation of a line through the point $(0,5)$ with slope -2. Write the solution in slope-intercept form.

Solution:

Using slope-intercept form, the equation of the line is

$$y = -2x + 5$$

■ 2. Identify the y-intercept and slope m defining the line.

$$y = -\frac{1}{4}(x + 12)$$

Solution:

Notice that the slope of the line given is $-1/4$ and the y-intercept (when $x = 0$) is $(0, -3)$.

■ 3. Convert the following point-slope equation into a slope-intercept equation.

$$y - 3 = \frac{1}{3}(x - 6)$$

Solution:

Converting to slope-intercept form means that we need to solve for y, and simplify as much as we can.

$$y - 3 = \frac{1}{3}(x - 6)$$

$$y - 3 = \frac{1}{3}x - 2$$

$$y = \frac{1}{3}x - 2 + 3$$

$$y = \frac{1}{3}x + 1$$

4. Find the equation of a line that passes through the points $(1, -1)$ and $(0,3)$. Write the solution in slope-intercept form.

Solution:

We first need to calculate the slope of the line as

$$m = \frac{3 - (-1)}{0 - 1}$$

$$m = \frac{4}{-1}$$

$$m = -4$$

Using slope-intercept form, noting that the y-intercept is 3, the equation of the line is

$$y = -4x + 3$$

■ 5. Determine the y-intercept of a line with slope -3 that passes through the point $(1,1)$. Write your solution as a coordinate point.

Solution:

In point-slope form, the equation of the line is

$$y - 1 = -3(x - 1)$$

$$y = -3x + 3 + 1$$

$$y = -3x + 4$$

From the new form of the equation of the line, we can see that the y-intercept is $(0,4)$.

■ 6. Name two (of four possible) pieces of information about a line that are required to write an equation of the line in point-slope form.

Solution:

Naming any two of the following is correct:

(1) A point

(2) Another point

(3) The slope

(4) The y-intercept

7. Find the equation of a line that passes through the points $(-3,-2)$ and $(2,-4)$. Write the solution in slope-intercept form.

Solution:

We first need to calculate the slope of the line as

$$m=\frac{-4-(-2)}{2-(-3)}$$

$$m=\frac{-2}{5}$$

$$m=-\frac{2}{5}$$

Using point-slope form, the equation of the line is

$$y + 2 = -\frac{2}{5}(x + 3)$$

$$y + 2 = -\frac{2}{5}x - \frac{6}{5}$$

$$y = -\frac{2}{5}x - \frac{6}{5} - \frac{10}{5}$$

$$y = -\frac{2}{5}x - \frac{16}{5}$$

GRAPHING PARABOLAS

1. Write the equation in vertex form.

$$y = x^2 + 8x + 5$$

Solution:

To convert to vertex form, we'll need to complete the square. Take the coefficient of 8 on the x term, divide it by 2, then square the result.

$$\frac{8}{2} = 4$$

$$(4)^2 = 16$$

We'll need to add and subtract 16 from the right side of the equation to keep it balanced.

$$y = (x^2 + 8x + 16) - 16 + 5$$

$$y = (x^2 + 8x + 16) - 11$$

Factor what's inside the parentheses in order to put the parabola in vertex form.

$$y = (x + 4)(x + 4) - 11$$

$$y = (x + 4)^2 - 11$$

2. Write the equation in vertex form.

$$y = -2x^2 + 24x - 68$$

Solution:

First factor out the coefficient on the x^2 term, which is -2.

$$y = -2x^2 + 24x - 68$$

$$y = -2(x^2 - 12x) - 68$$

To convert to vertex form, we'll need to complete the square. Take the coefficient of -12 on the x term, divide it by 2, then square the result.

$$\frac{-12}{2} = -6$$

$$(-6)^2 = 36$$

We'll need to add 36 on the inside of the parentheses. The -2 on the outside of the parentheses means that were really adding $-2(36) = -72$. Therefore, we'll also need to add 72 outside of the parentheses to keep the equation balanced.

$$y = -2(x^2 - 12x + 36) + 72 - 68$$

$$y = -2(x^2 - 12x + 36) + 4$$

Factor what's inside the parentheses in order to put the parabola in vertex form.

$$y = -2(x-6)(x-6)+4$$

$$y = -2(x-6)^2+4$$

■ 3. Find the vertex and axis of symmetry of $y = x^2 + 5x + 6$.

Solution:

Remember that the axis of symmetry is $x = -b/2a$ and that standard form for a parabola is $y = ax^2 + bx + c$. In this case, $a = 1$ and $b = 5$.

$$x = -\frac{5}{2(1)}$$

$$x = -\frac{5}{2}$$

To find the vertex, plug $x = -5/2$ into the equation of the parabola.

$$y = x^2 + 5x + 6$$

$$y = \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 6$$

$$y = \frac{25}{4} - \frac{25}{2} + 6$$

$$y = \frac{25}{4} - \frac{50}{4} + 6$$

$$y = -\frac{25}{4} + \frac{24}{4}$$

$$y = -\frac{1}{4}$$

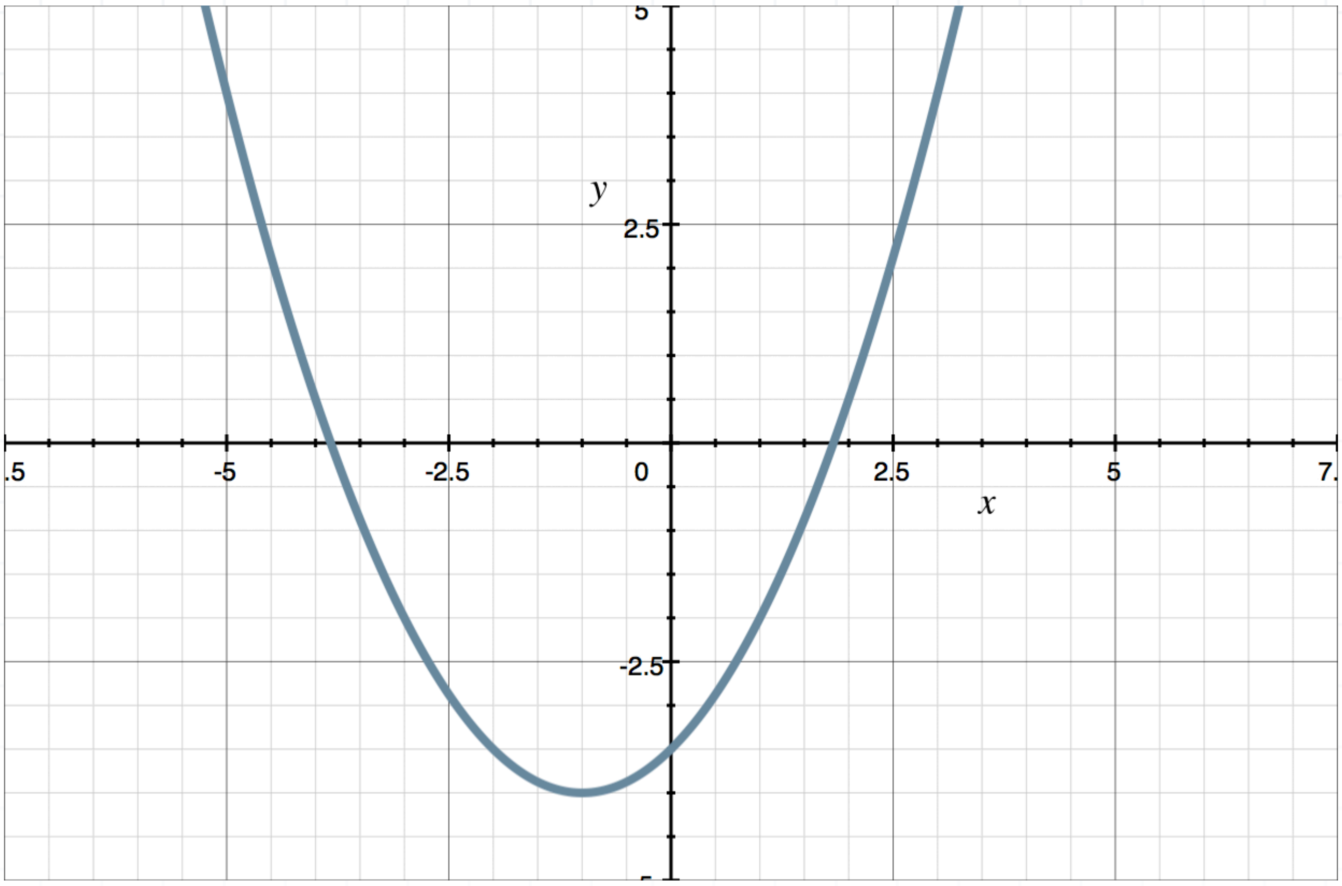
The vertex is therefore $(-5/2, -1/4)$ and the axis of symmetry is $x = -5/2$.

■ 4. Find the vertex and axis of symmetry of $y = 3(x+2)^2 + 6$.

Solution:

Remember that vertex form is $a(x-h)^2 + k$ and that the vertex is (h, k). In this case, $h = -2$ and $k = 6$. So the vertex is $(-2,6)$. The axis of symmetry is $x = h$, so the axis of symmetry is $x = -2$.

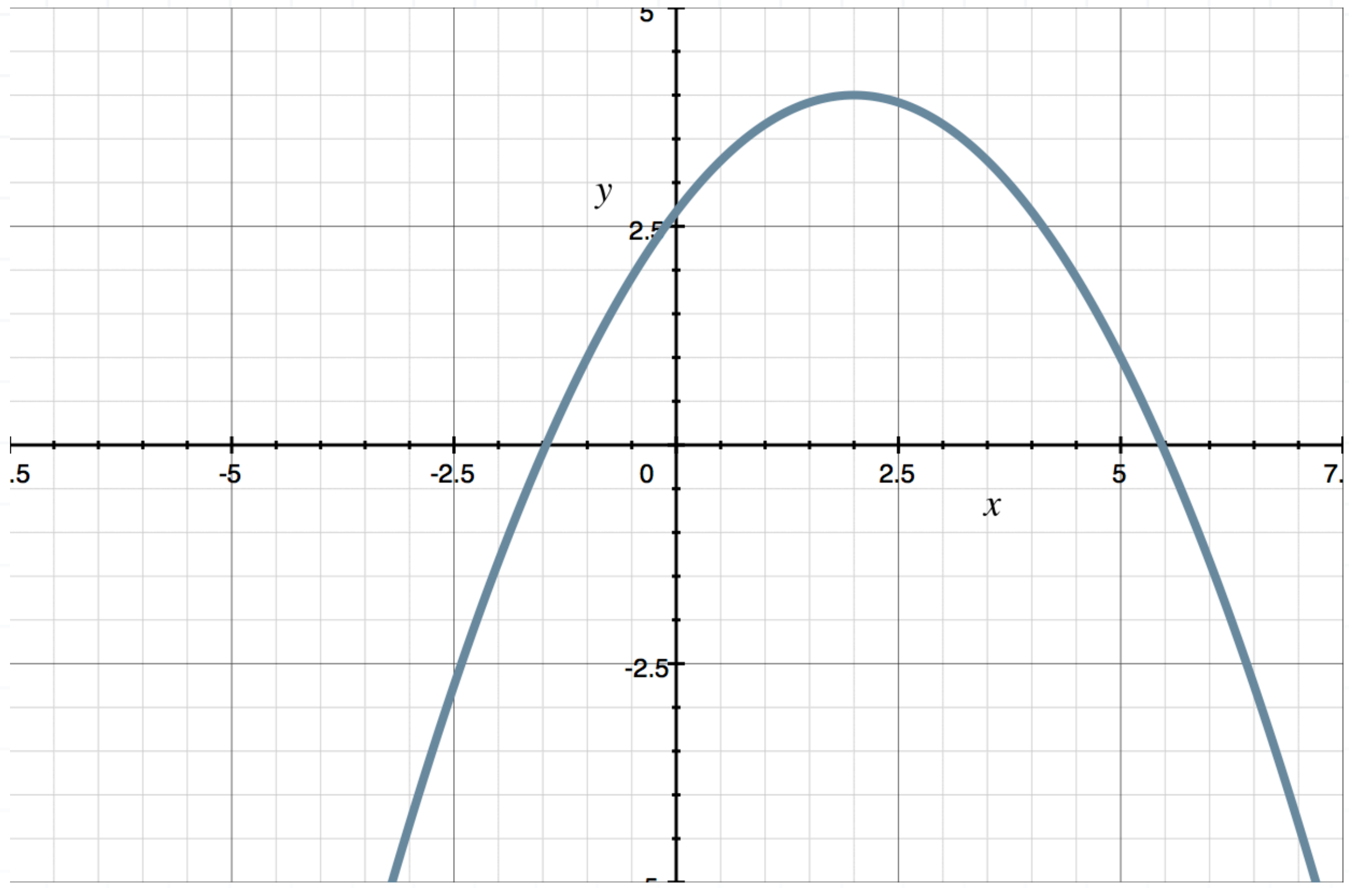
■ 5. Identify the vertex and axis of symmetry from the graph of the parabola.



Solution:

The vertex is the minimum point of the graph, $(-1,-4)$. The axis of symmetry is the line of symmetry, $x=-1$.

6. Using the graph below, find the equation of the parabola in standard form.



Solution:

First find the vertex of the graph. The vertex is the maximum point, which we can see from the graph is sitting at $(2,4)$. Write the equation of the parabola in vertex form and plug in $h=2$ and $k=4$.

$$y=a(x-h)^2+k$$

$$y=a(x-2)^2+4$$

To find a, we need to plug in another point from the parabola. We'll use $(-1,1)$, plugging $x=-1$ and $y=1$ into the equation.

$$y=a(x-2)^2+4$$

$$1=a(-1-2)^2+4$$

$$1 = a(-3)^2 + 4$$

$$1 = 9a + 4$$

$$-3 = 9a$$

$$-\frac{1}{3} = a$$

Plug in $a = -1/3$ and expand to find standard form.

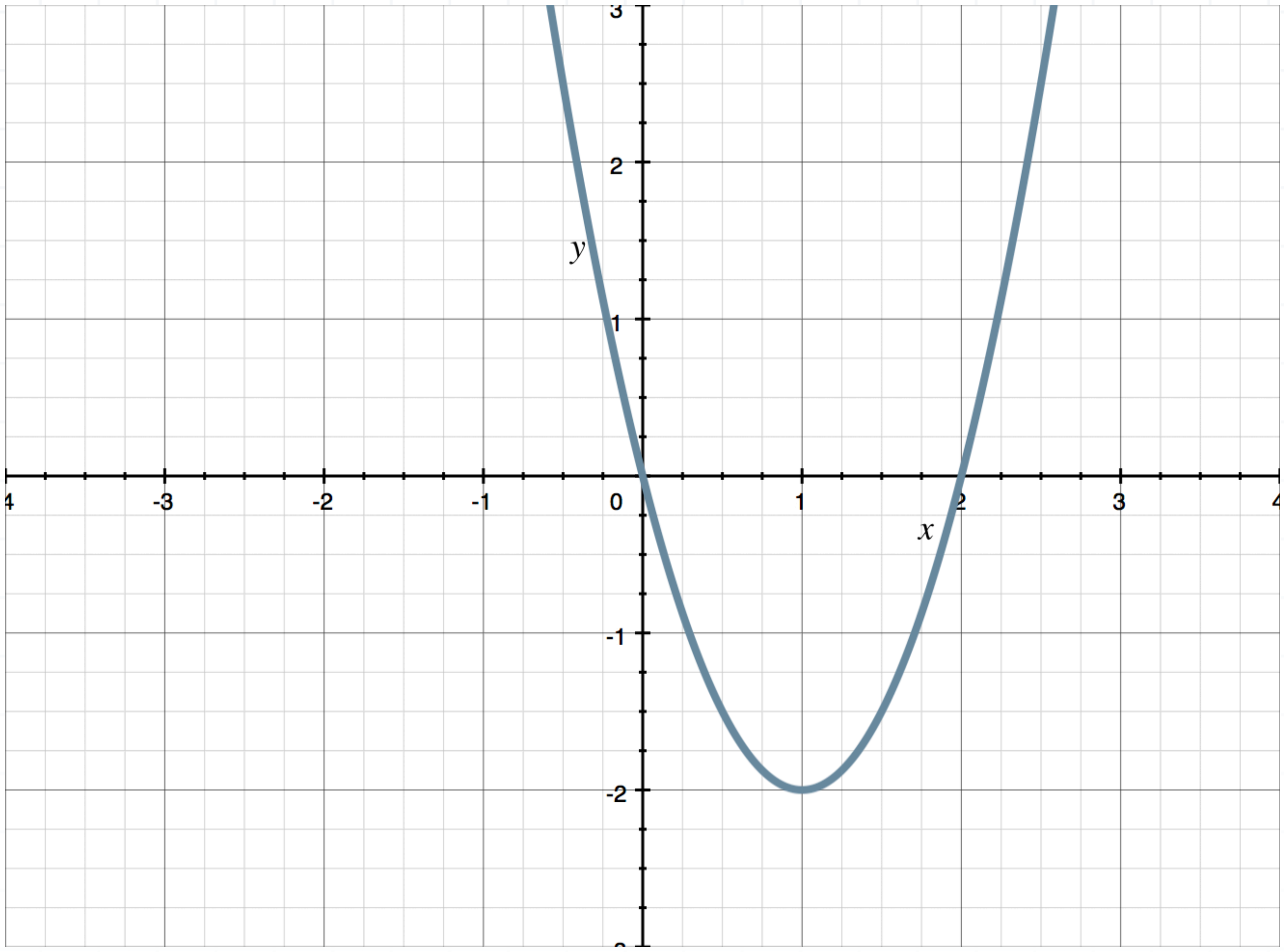
$$y = -\frac{1}{3}(x-2)^2 + 4$$

$$y = -\frac{1}{3}(x^2 - 4x + 4) + 4$$

$$y = -\frac{1}{3}x^2 + \frac{4}{3}x - \frac{4}{3} + 4$$

$$y = -\frac{1}{3}x^2 + \frac{4}{3}x + \frac{8}{3}$$

■ 7. Using the graph, find the equation of the parabola in standard form.



Solution:

First, find the vertex of the graph. The vertex is the minimum point at $(1,-2)$. Write the equation of the parabola in vertex form and plug in $h=1$ and $k=-2$.

$$y=a(x-h)^2+k$$

$$y=a(x-1)^2+(-2)$$

$$y=a(x-1)^2-2$$

To find a, we need to plug in another point from the parabola. We'll use $(0,0)$, plugging $x=0$ and $y=0$ into the equation.

$$y = a(x-1)^2 - 2$$

$$0 = a(0-1)^2 - 2$$

$$0 = a(-1)^2 - 2$$

$$0 = a - 2$$

$$2 = a$$

Plug in $a = 2$ and expand to find standard form.

$$y = 2(x-1)^2 - 2$$

$$y = 2(x^2 - 2x + 1) - 2$$

$$y = 2x^2 - 4x + 2 - 2$$

$$y = 2x^2 - 4x$$

■ 8. Complete the square to graph the parabola $y = x^2 + 6x + 5$.

Solution:

Complete the square to put the parabola in vertex form. Take the coefficient of 6 on x, divide it by 2, then square the result.

$$\frac{6}{2} = 3$$

$$3^2 = 9$$

We'll need to add and subtract 9 from the right side of the equation to keep it balanced.

$$y = (x^2 + 6x + 9) + 5 - 9$$

$$y = (x^2 + 6x + 9) - 4$$

Factor what's inside the parentheses.

$$y = (x + 3)(x + 3) - 4$$

$$y = (x + 3)^2 - 4$$

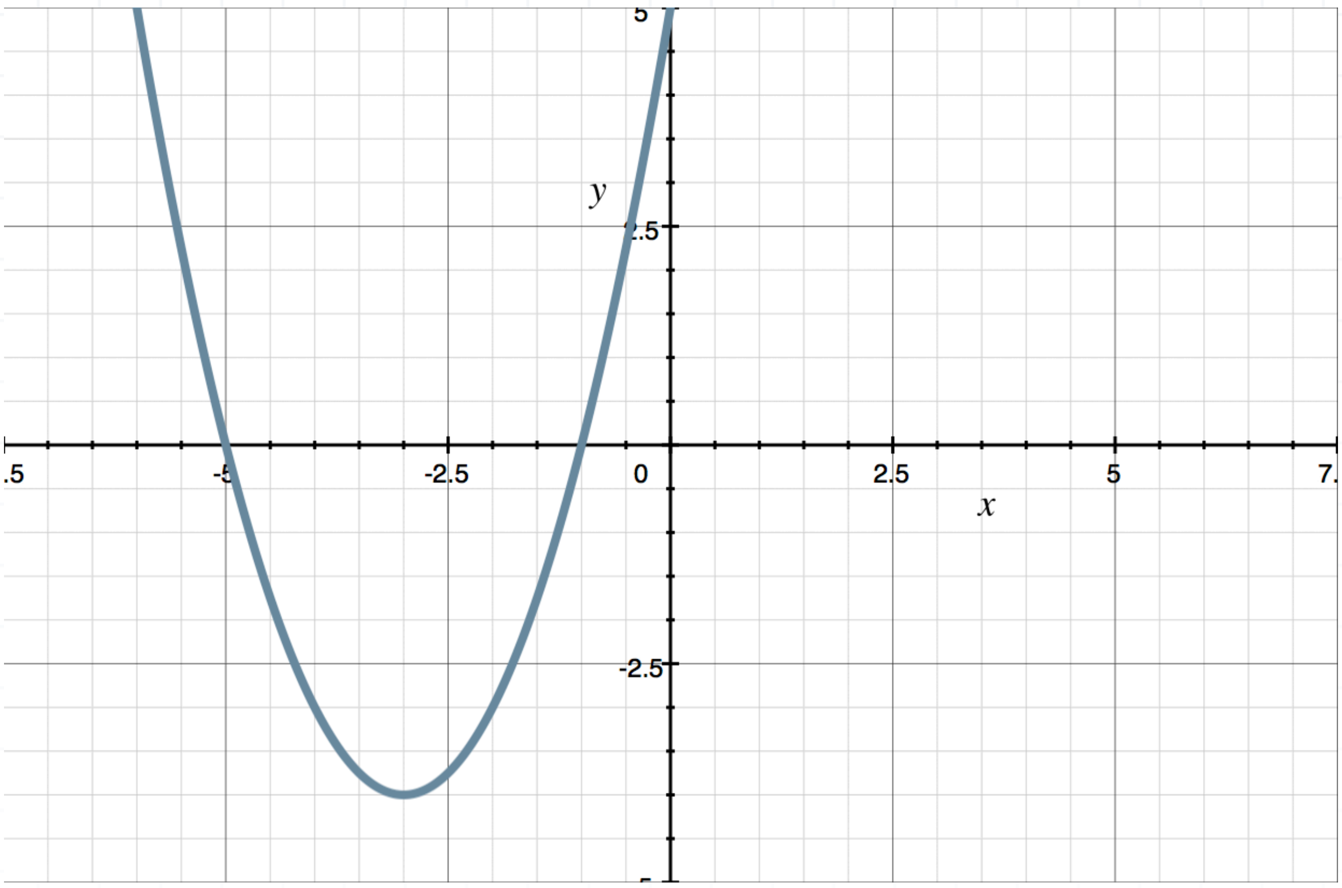
The vertex is $(-3, -4)$. We need to find at least one other point on the graph. In this case, we can find the zeros of the graph. Set the standard form of the parabola equal to 0, then factor and solve for x.

$$0 = x^2 + 6x + 5$$

$$0 = (x + 5)(x + 1)$$

$$x = -5, \ -1$$

Now we have three points to graph: $(-5,0)$, $(-1,0)$, and $(-3, -4)$. Graph the points, then connect them to sketch the graph of the parabola.



9. Complete the square to graph $y = -x^2 - 4x - 6$.

Solution:

We need to complete the square to put the parabola in vertex form, but first we'll need to factor out -1 from the first two terms so that the coefficient to the x^2 term is positive 1.

$$y = -1(x^2 + 4x) - 6$$

Take the coefficient of 4 on the x term, divide it by 2, then square the result.

$$\frac{4}{2} = 2$$

$$2^2 = 4$$

We'll need to add 4 on the inside of the parentheses. The -1 on the outside of the parentheses really means were adding $-1(4) = -4$ to the equation, so we'll need to add 4 outside of the parentheses to keep the equation balanced.

$$y = -(x^2 + 4x + 4) - 6 + 4$$

$$y = -(x^2 + 4x + 4) - 2$$

Factor what's inside the parentheses.

$$y = -(x + 2)(x + 2) - 2$$

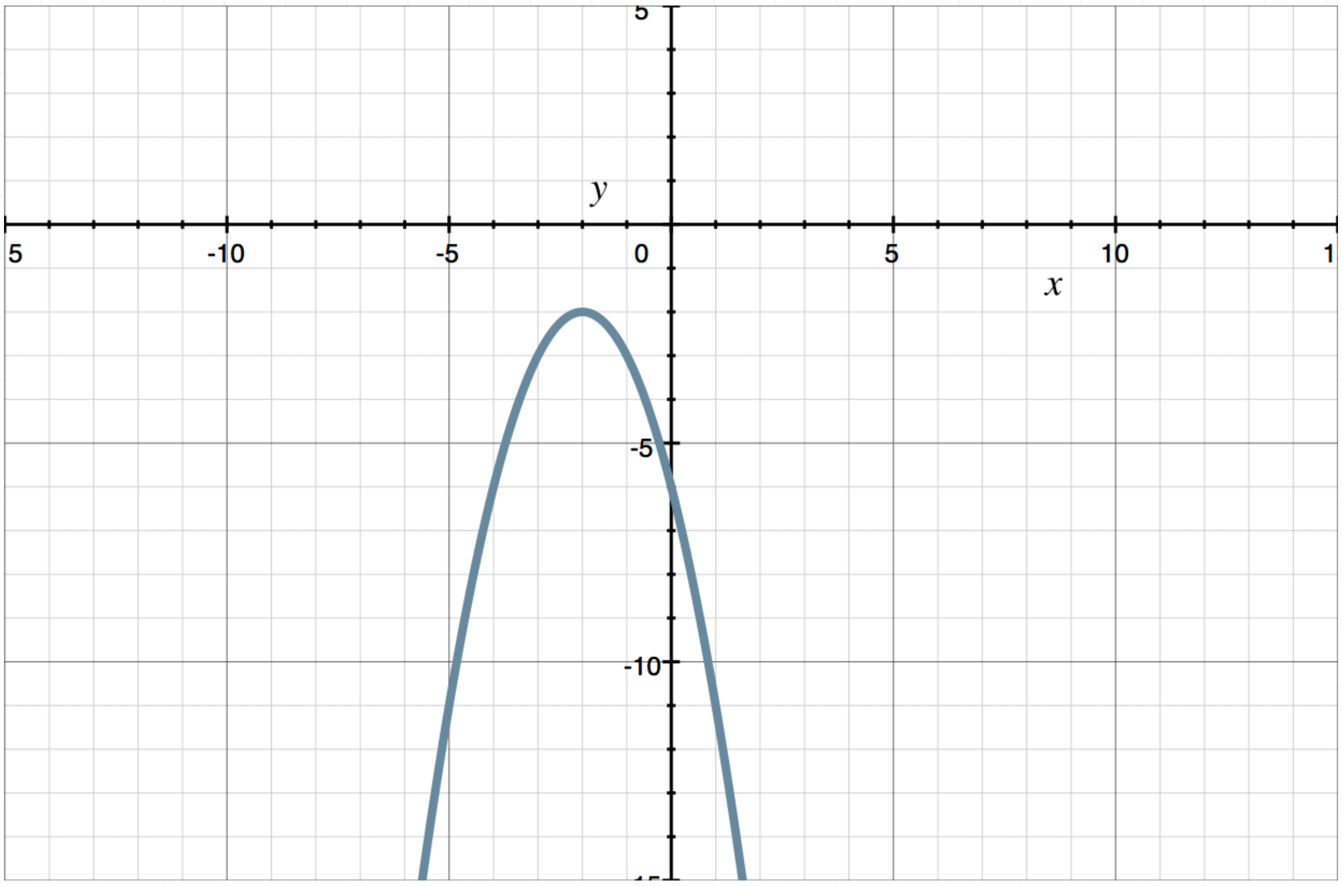
$$y = -(x + 2)^2 - 2$$

The vertex is $(-2, -2)$. We need to find at least one other point on the graph. In this case, let's plug $x = 0$ into the equation to find the corresponding y-value.

$$y = -0^2 - 4(0) - 6$$

$$y = -6$$

Now we have two points to graph: $(-2, -2)$ and $(0, -6)$. We can use the axis of symmetry to plot the third point. Graph the points, then connect them to sketch the parabola.


y
x
5
-10
-5
0
5
10
1
5
-5
-10

FINDING CENTER AND RADIUS OF A CIRCLE

1. Find the center and radius of the circle.

$$x^2+y^2-2x-3=0$$

Solution:

To change the equation into standard form, $(x-h)^2+(y-k)^2=r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$(x^2-2x)+y^2=3$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. Since there is no y term in this case, we'll just need to complete the square with respect to x. The coefficient of x is -2, so

$$\frac{-2}{2}=-1$$

$$(-1)^2=1$$

Add 1 to both sides of the equation, then factor and simplify.

$$(x^2-2x+1)+y^2=3+1$$

$$(x-1)^2+y^2=4$$

The center of the circle (h, k) is at $(1,0)$, and the radius is $r = \sqrt{4} = 2$.

2. Find the center and radius of the circle.

$$x^2 + y^2 + 14x + 22y + 145 = 0$$

Solution:

To change the equation into standard form, $(x - h)^2 + (y - k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$(x^2 + 14x) + (y^2 + 22y) = -145$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is 14, so

$$\frac{14}{2} = 7$$

$$(7)^2 = 49$$

The coefficient of y is 22, so

$$\frac{22}{2} = 11$$

$$11^2 = 121$$

Add 49 and 121 to both sides of the equation, then factor and simplify.

$$(x^2 + 14x + 49) + (y^2 + 22y + 121) = -145 + 49 + 121$$

$$(x + 7)^2 + (y + 11)^2 = 25$$

The center of the circle (h, k) is at $(-7, -11)$, and the radius is $r = \sqrt{25} = 5$.

■ 3. Find the center and radius of the circle.

$$16x^2 + 16y^2 - 8x - 24y - 150 = 0$$

Solution:

To change the equation into standard form, $(x - h)^2 + (y - k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$16x^2 - 8x + 16y^2 - 24y = 150$$

Divide both sides of the equation by 16, so that the coefficients of x^2 and y^2 are both 1.

$$x^2 - \frac{1}{2}x + y^2 - \frac{3}{2}y = \frac{150}{16}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is $-1/2$, so

$$-\frac{1}{2}\cdot\frac{1}{2}=-\frac{1}{4}$$

$$\left(-\frac{1}{4}\right)^2=\frac{1}{16}$$

The coefficient of y is $-3/2$, so

$$-\frac{3}{2}\cdot\frac{1}{2}=-\frac{3}{4}$$

$$\left(-\frac{3}{4}\right)^2=\frac{9}{16}$$

Add $1/16$ and $9/16$ to both sides of the equation, then factor and simplify.

$$\left(x^2-\frac{1}{2}x+\frac{1}{16}\right)+\left(y^2-\frac{3}{2}y+\frac{9}{16}\right)=\frac{150}{16}+\frac{1}{16}+\frac{9}{16}$$

$$\left(x-\frac{1}{4}\right)^2+\left(y-\frac{3}{4}\right)^2=\frac{160}{16}$$

$$\left(x-\frac{1}{4}\right)^2+\left(y-\frac{3}{4}\right)^2=10$$

The center of the circle (h,k) is at $(1/4,3/4)$, and the radius is $r=\sqrt{10}$.

■ 4. Find the center and radius of the circle.

$$4x^2+4y^2+32x-4y+41=0$$

Solution:

To change the equation into standard form, $(x-h)^2+(y-k)^2=r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$4x^2+32x+4y^2-4y=-41$$

Divide both sides of the equation by 4, so that the coefficients of x^2 and y^2 are both 1.

$$x^2+8x+y^2-y=-\frac{41}{4}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is 8, so

$$\frac{8}{2}=4$$

$$4^2=16$$

The coefficient of y is -1, so

$$\frac{-1}{2}=-\frac{1}{2}$$

$$\left(-\frac{1}{2}\right)^2=\frac{1}{4}$$

Add 16 and $1/4$ to both sides of the equation, then factor and simplify.

$$\left(x^2+8x+16\right)+\left(y^2-y+\frac{1}{4}\right)=-\frac{41}{4}+16+\frac{1}{4}$$

$$(x+4)^2+\left(y-\frac{1}{2}\right)^2=-10+16$$

$$(x+4)^2+\left(y-\frac{1}{2}\right)^2=6$$

The center of the circle (h,k) is at $(-4,1/2)$, and the radius is $r=\sqrt{6}$.

■ 5. Find the center and radius of the circle.

$$9x^2+9y^2-30x-6y-118=0$$

Solution:

To change the equation into standard form, $(x-h)^2+(y-k)^2=r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$9x^2-30x+9y^2-6y=118$$

Divide both sides of the equation by 9, so that the coefficients of x^2 and y^2 are both 1.

$$x^2-\frac{10}{3}x+y^2-\frac{2}{3}y=\frac{118}{9}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is $-10/3$, so

$$-\frac{10}{3}\cdot\frac{1}{2}=-\frac{10}{6}$$

$$\left(-\frac{10}{6}\right)^2=\frac{100}{36}=\frac{25}{9}$$

The coefficient of y is $-2/3$, so

$$-\frac{2}{3}\cdot\frac{1}{2}=-\frac{1}{3}$$

$$\left(-\frac{1}{3}\right)^2=\frac{1}{9}$$

Add $25/9$ and $1/9$ to both sides of the equation, then factor and simplify.

$$\left(x^2-\frac{10}{3}x+\frac{25}{9}\right)+\left(y^2-\frac{2}{3}y+\frac{1}{9}\right)=\frac{118}{9}+\frac{25}{9}+\frac{1}{9}$$

$$\left(x-\frac{5}{3}\right)^2+\left(y-\frac{1}{3}\right)^2=16$$

The center of the circle (h,k) is at $(5/3,1/3)$, and the radius is $r=\sqrt{16}=4$.

■ 6. Find the center and radius of the circle.

$$x^2+y^2+4x-2y=0$$

Solution:

To change the equation into standard form, $(x-h)^2+(y-k)^2=r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2+4x+y^2-2y=0$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is 4, so

$$\frac{4}{2}=2$$

$$2^2=4$$

The coefficient of y is -2, so

$$\frac{-2}{2}=-1$$

$$(-1)^2=1$$

Add 4 and 1 to both sides of the equation, then factor and simplify.

$$(x^2+4x+4)+(y^2-2y+1)=0+4+1$$

$$(x+2)^2+(y-1)^2=5$$

The center of the circle (h,k) is at $(-2,1)$, and the radius is $r=\sqrt{5}$.

7. Find the center and radius of the circle.

$$x^2 + y^2 - 12x + 10y - 3 = 0$$

Solution:

To change the equation into standard form, $(x-h)^2 + (y-k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 - 12x + y^2 + 10y = 3$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is -12, so

$$\frac{-12}{2} = -6$$

$$(-6)^2 = 36$$

The coefficient of y is 10, so

$$\frac{10}{2} = 5$$

$$5^2 = 25$$

Add 36 and 25 to both sides of the equation, then factor and simplify.

$$(x^2 - 12x + 36) + (y^2 + 10y + 25) = 3 + 36 + 25$$

$$(x - 6)^2 + (y + 5)^2 = 64$$

The center of the circle (h, k) is at $(6, -5)$, and the radius is $r = \sqrt{64} = 8$.

■ 8. Find the center and radius of the circle.

$$x^2 + y^2 - \frac{1}{4} = 0$$

Solution:

To change the equation into standard form, $(x - h)^2 + (y - k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 + y^2 = \frac{1}{4}$$

The center of the circle (h, k) is at $(0,0)$, and the radius is $r = \sqrt{1/4} = 1/2$.

■ 9. Find the center and radius of the circle.

$$16x^2 + 16y^2 + 96x - 160y + 543 = 0$$

Solution:

To change the equation into standard form, $(x-h)^2 + (y-k)^2 = r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$16x^2 + 96x + 16y^2 - 160y = -543$$

Divide both sides of the equation by 16, so that the coefficients of x^2 and y^2 are both 1.

$$x^2 + 6x + y^2 - 10y = -\frac{543}{16}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is 6, so

$$\frac{6}{2} = 3$$

$$3^2 = 9$$

The coefficient of y is -10, so

$$\frac{-10}{2} = -5$$

$$(-5)^2 = 25$$

Add 9 and 25 to both sides of the equation, then factor and simplify.

$$(x^2 + 6x + 9) + (y^2 - 10y + 25) = -\frac{543}{16} + 9 + 25$$

$$(x+3)^2+(y-5)^2=-\frac{543}{16}+\frac{144}{16}+\frac{400}{16}$$

$$(x+3)^2+(y-5)^2=\frac{1}{16}$$

The center of the circle (h,k) is at $(-3,5)$, and the radius is $r=\sqrt{1/16}=1/4$.

10. Find the center and radius of the circle.

$$9x^2+9y^2-72x+12y-77=0$$

Solution:

To change the equation into standard form, $(x-h)^2+(y-k)^2=r^2$, start by grouping x and y terms together and moving the constant to the right side of the equation.

$$9x^2-72x+9y^2+12y=77$$

Divide both sides of the equation by 9, so that the coefficients of x^2 and y^2 are both 1.

$$x^2-8x+y^2+\frac{4}{3}y=\frac{77}{9}$$

To complete the square with respect to both x and y, take the coefficients of the x and y terms, divide by 2, then square the results. The coefficient of x is -8, so

$$\frac{-8}{2} = -4$$

$$(-4)^2 = 16$$

The coefficient of y is $4/3$, so

$$\frac{4}{3} \cdot \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Add 16 and $4/9$ to both sides of the equation, then factor and simplify.

$$(x^2 - 8x + 16) + \left(y^2 + \frac{4}{3}y + \frac{4}{9}\right) = \frac{77}{9} + 16 + \frac{4}{9}$$

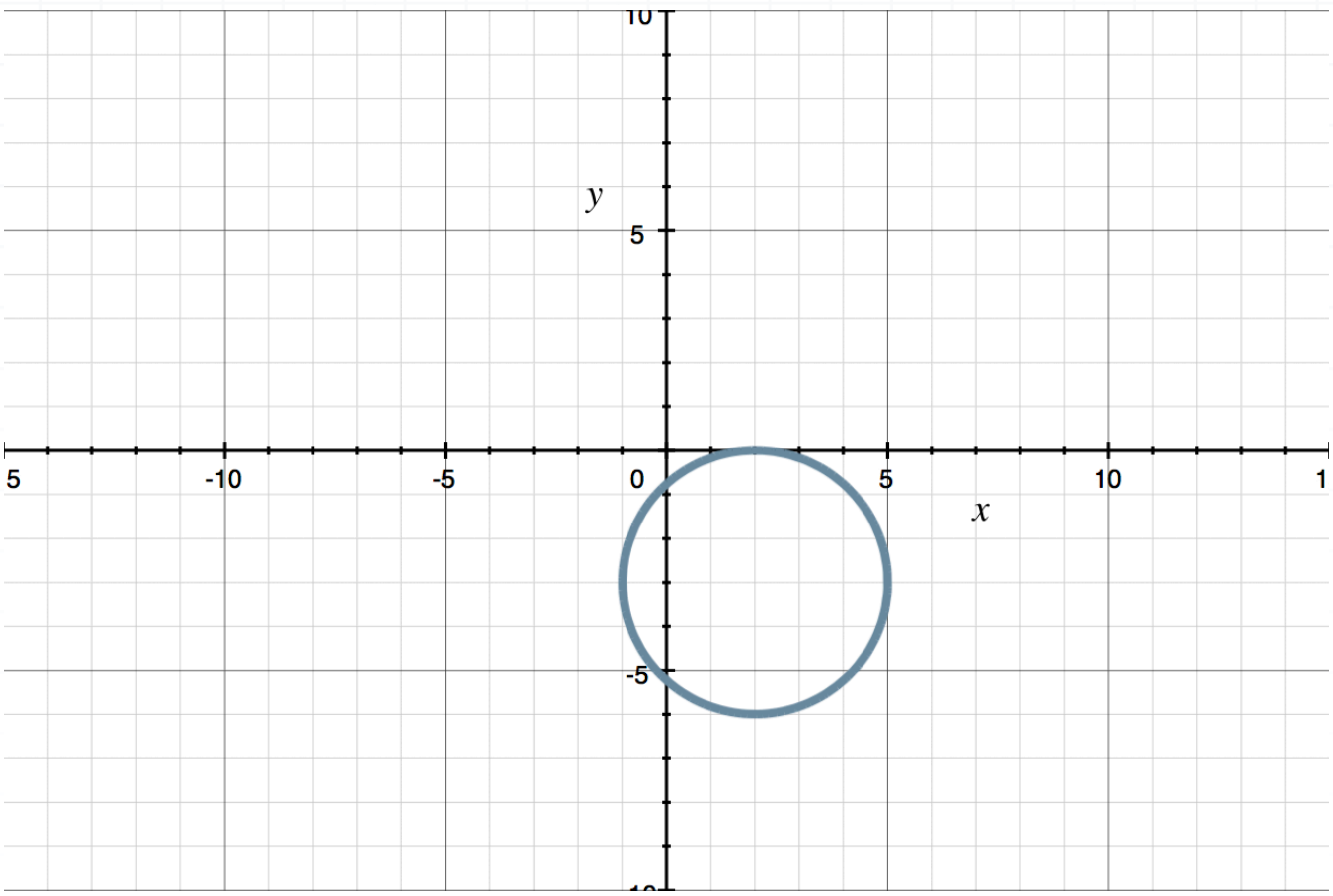
$$(x-4)^2 + \left(y + \frac{2}{3}\right)^2 = 9 + 16$$

$$(x-4)^2 + \left(y + \frac{2}{3}\right)^2 = 25$$

The center of the circle (h, k) is at $(4, -2/3)$, and the radius is $r = \sqrt{25} = 5$.

GRAPHING CIRCLES

1. Find the equation of the circle.



Solution:

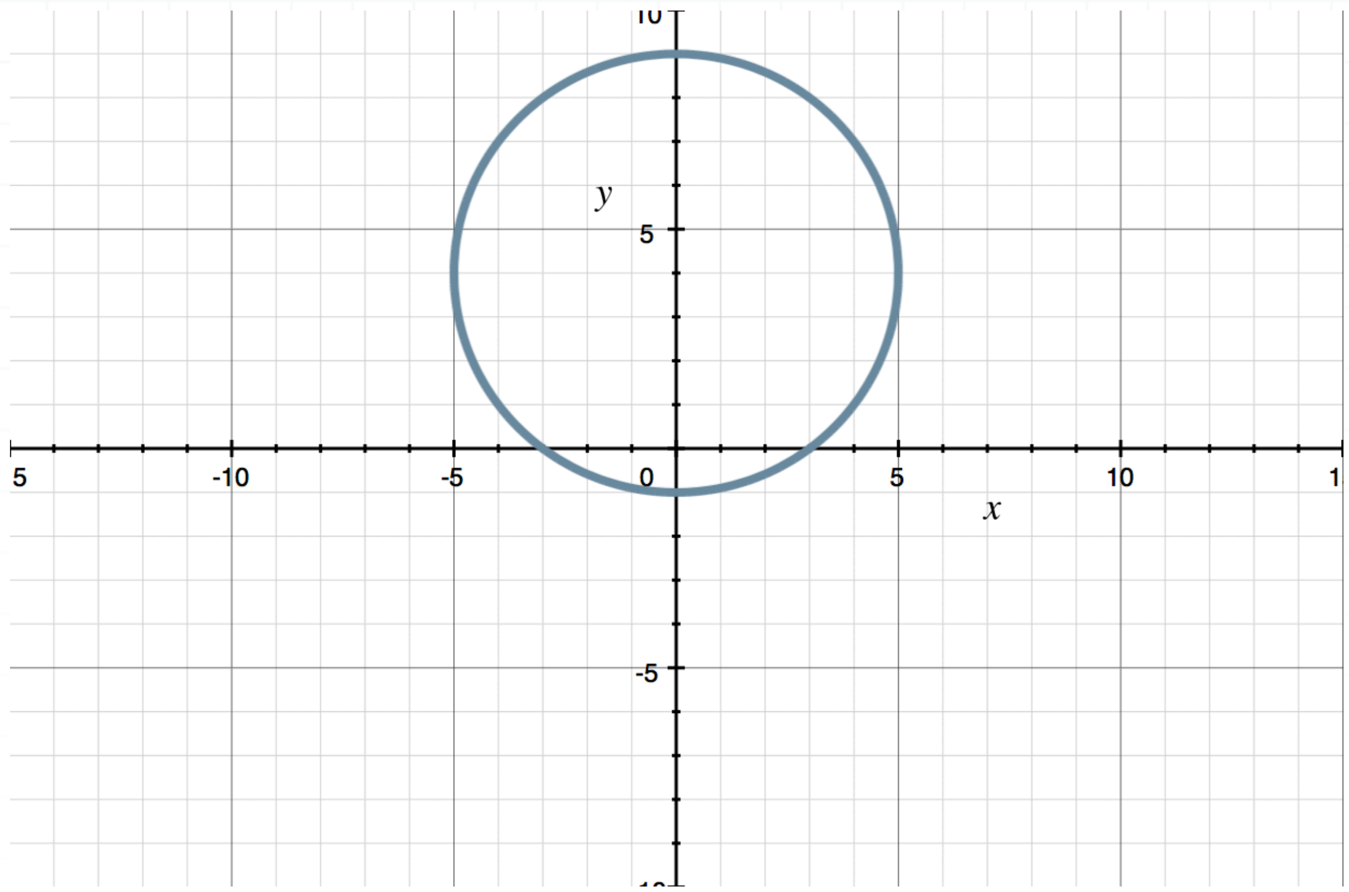
Visually, we can see that the center of the circle is at $(2,-3)$, so $h=2$ and $k=-3$. If we count from the center to a point on the circumference, we can see that the length of the radius is $r=3$. Plugging the center and radius into the standard equation of the circle gives

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-2)^2+(y-(-3))^2=3^2$$

$$(x-2)^2 + (y+3)^2 = 9$$

■ 2. Find the equation of the circle.



Solution:

Visually, we can see that the center of the circle is at $(0,4)$, so $h = 0$ and $k = 4$. If we count from the center to a point on the circumference, we can see that the length of the radius is $r = 5$. Plugging the center and radius into the standard equation of the circle gives

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-0)^2 + (y-4)^2 = 5^2$$

$$x^2 + (y-4)^2 = 25$$

■ 3. Graph the circle $(x-1)^2 + (y-2)^2 = 4$.

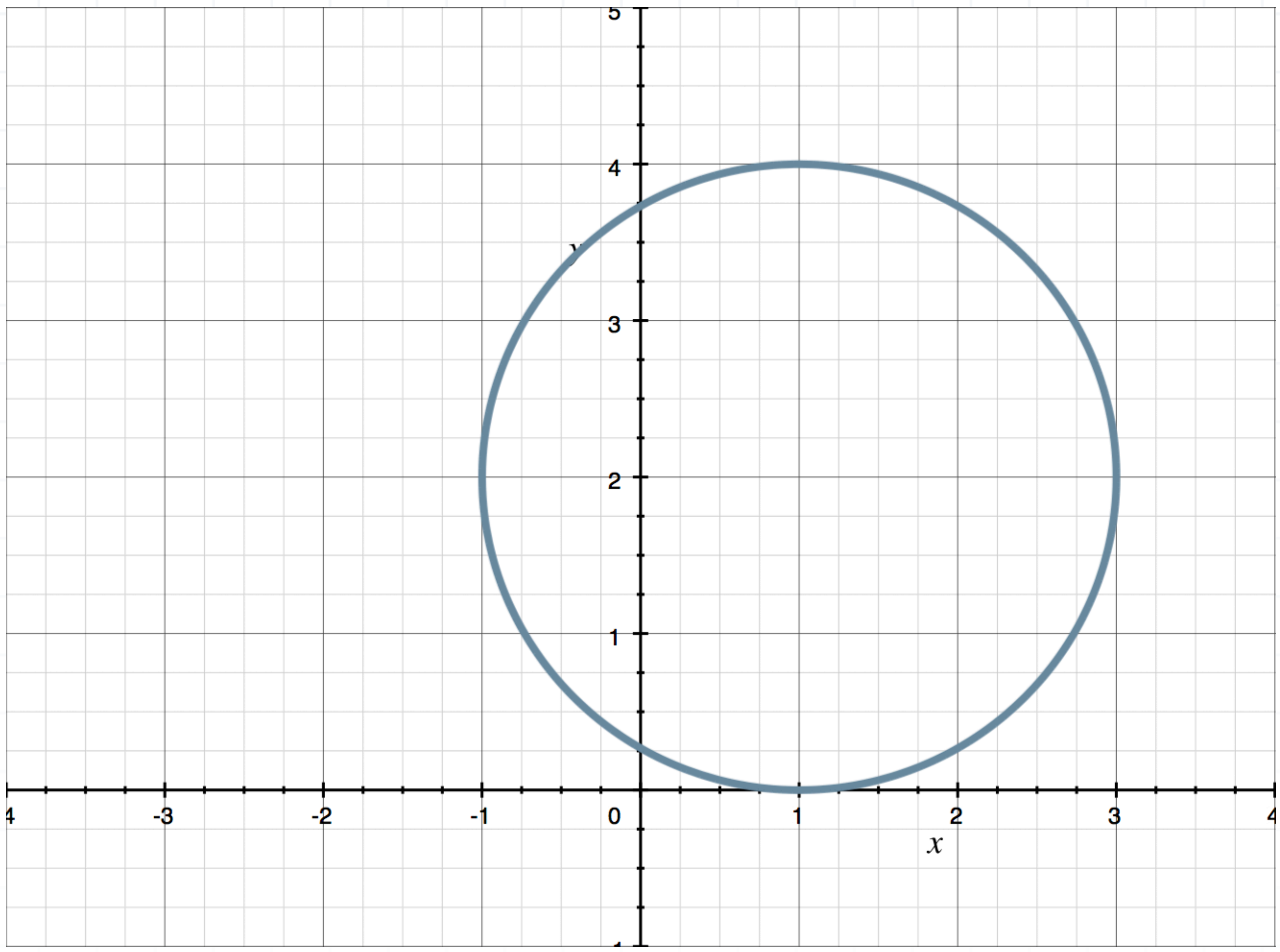
Solution:

We need to find the center and radius. The standard equation of the circle, is $(x-h)^2 + (y-k)^2 = r^2$, and if we match this up to the given equation,

$$(x-1)^2 + (y-2)^2 = 4$$

$$(x-1)^2 + (y-2)^2 = 2^2$$

we can say that the center is at $(h, k) = (1,2)$ and the radius is 2. Therefore, the graph of the circle is



4. Graph the circle $(x+3)^2 + (y-4)^2 = 25$.

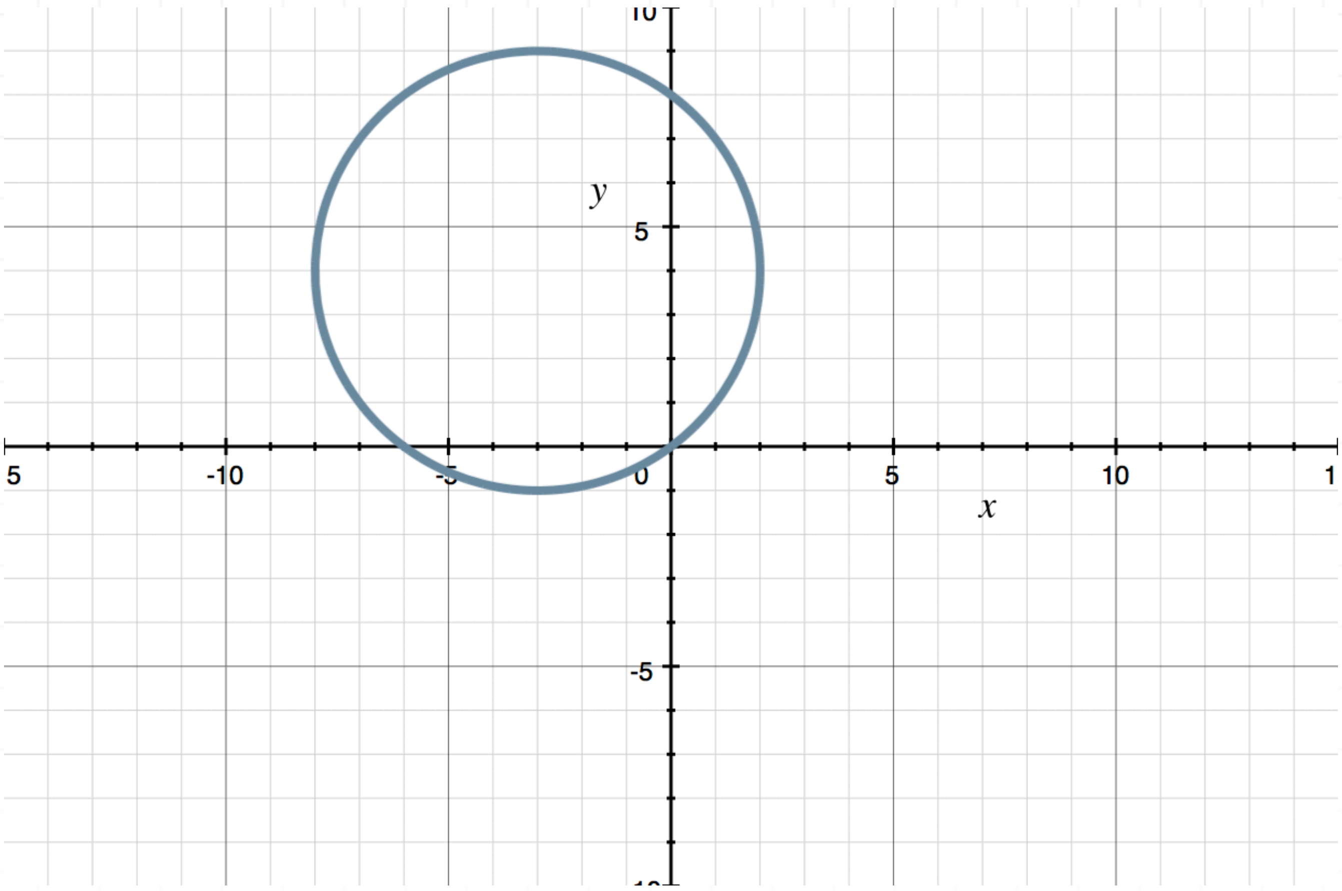
Solution:

We need to find the center and radius. The standard equation of the circle, is $(x-h)^2 + (y-k)^2 = r^2$, and if we match this up to the given equation,

$$(x+3)^2 + (y-4)^2 = 25$$

$$(x+3)^2 + (y-4)^2 = 5^2$$

we can say that the center is at $(h, k) = (-3,4)$ and the radius is 5. Therefore, the graph of the circle is



5. Graph the circle $x^2 + (y-3)^2 = 16$.

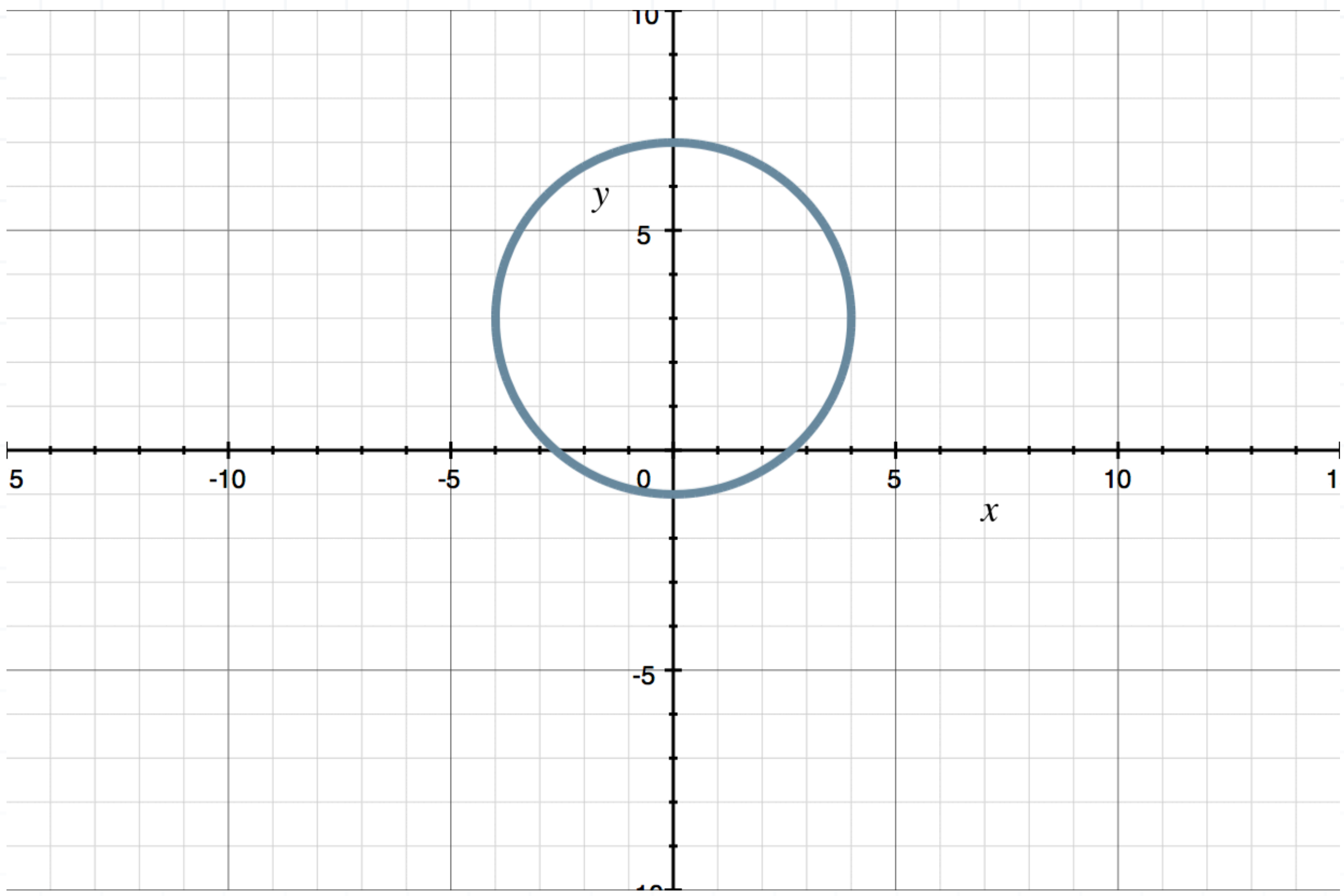
Solution:

We need to find the center and radius. The standard equation of the circle, is $(x-h)^2 + (y-k)^2 = r^2$, and if we match this up to the given equation,

$$x^2 + (y-3)^2 = 16$$

$$x^2 + (y-3)^2 = 4^2$$

we can say that the center is at $(h,k) = (0,3)$ and the radius is 4. Therefore, the graph of the circle is



6. Graph the circle $x^2 + y^2 + 2x + 2y - 14 = 0$.

Solution:

We need to find the center and radius of the circle by changing the equation of the circle into standard form, $(x - h)^2 + (y - k)^2 = r^2$, where h and k are the coordinates of the center and r is the radius.

Start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 + y^2 + 2x + 2y - 14 = 0$$

$$(x^2 + 2x) + (y^2 + 2y) = 14$$

To complete the square with respect to both x and y, take the coefficients on the x and y terms, divide them by 2, then square the results. The coefficient on x is 2, so

$$\frac{2}{2} = 1$$

$$(1)^2 = 1$$

The coefficient on y is 2, so

$$\frac{2}{2} = 1$$

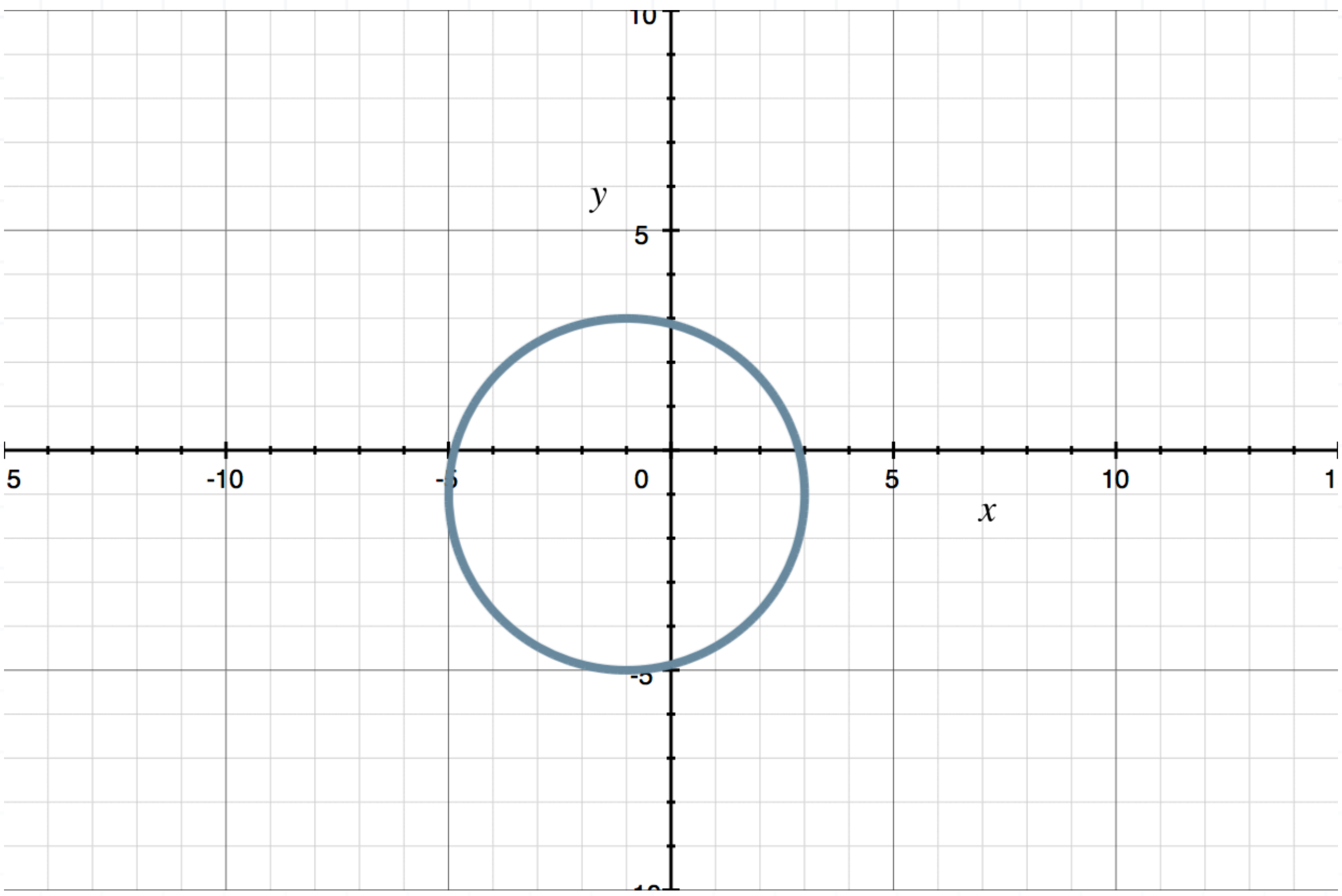
$$(1)^2 = 1$$

Add 1 and 1 to both sides of the equation. Then factor inside the parentheses and simplify the right side.

$$(x^2 + 2x + 1) + (y^2 + 2y + 1) = 14 + 1 + 1$$

$$(x + 1)^2 + (y + 1)^2 = 16$$

The center of the circle (h, k) is therefore at $(-1, -1)$ and the radius is $r = \sqrt{16} = 4$. So to graph the circle, plot the center point $(-1, -1)$, then move in any direction 4 units to get to a point on the edge of the circle.



7. Graph the circle $x^2 + y^2 - 8x - 4y + 11 = 0$.

Solution:

We need to find the center and radius of the circle by changing the equation of the circle into standard form, $(x - h)^2 + (y - k)^2 = r^2$, where h and k are the coordinates of the center and r is the radius.

Start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 + y^2 - 8x - 4y + 11 = 0$$

$$(x^2 - 8x) + (y^2 - 4y) = -11$$

To complete the square with respect to both x and y, take the coefficients on the x and y terms, divide them by 2, then square the results. The coefficient on x is -8, so

$$\frac{-8}{2} = -4$$

$$(-4)^2 = 16$$

The coefficient on y is -4, so

$$\frac{-4}{2} = -2$$

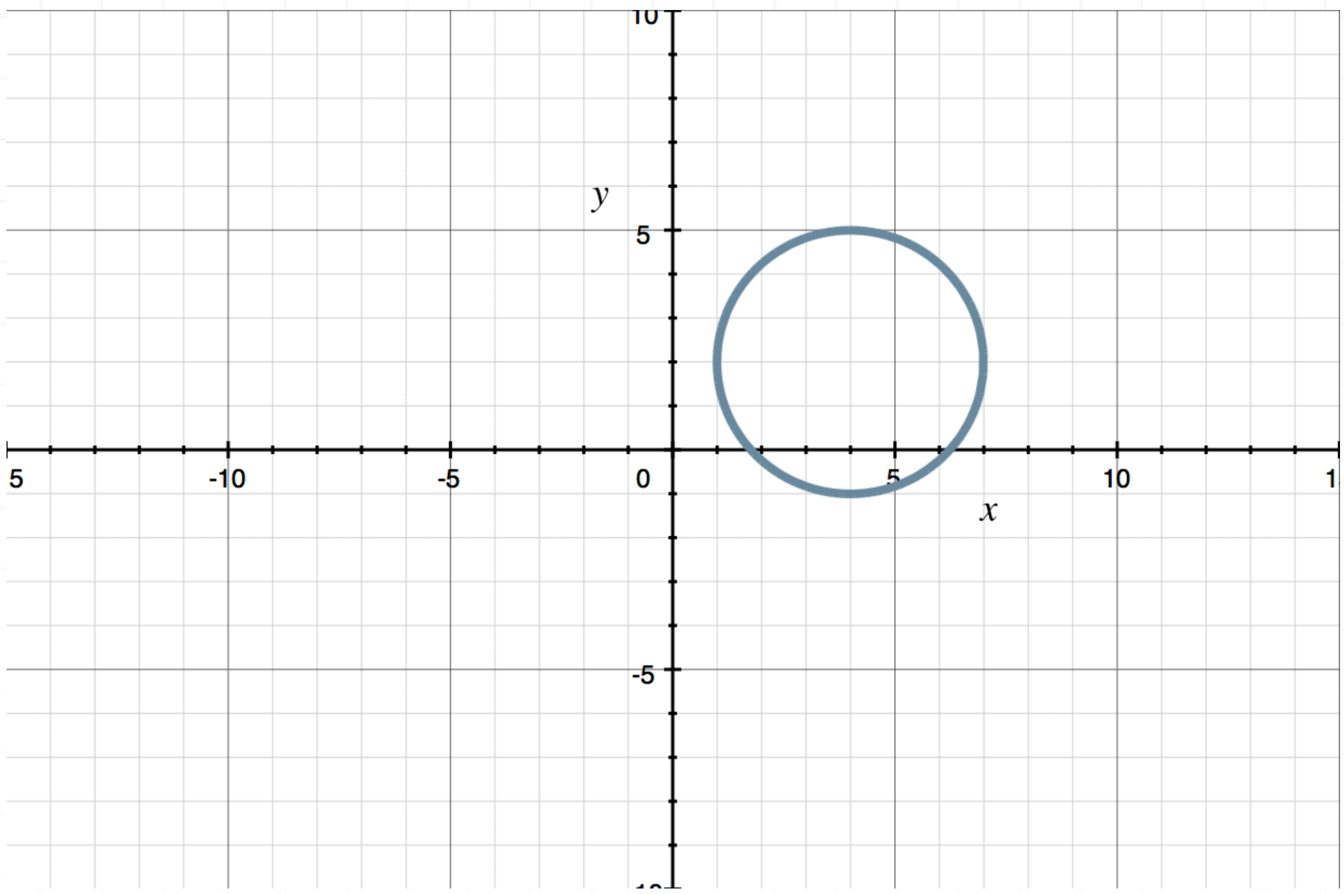
$$(-2)^2 = 4$$

Add 16 and 4 to both sides of the equation. Then factor inside the parentheses and simplify the right side.

$$(x^2 - 8x + 16) + (y^2 - 4y + 4) = -11 + 16 + 4$$

$$(x - 4)^2 + (y - 2)^2 = 9$$

The center of the circle (h, k) is therefore at $(4,2)$ and the radius is $r = \sqrt{9} = 3$. So to graph the circle, plot the center point $(4,2)$, then move in any direction 3 units to get to a point on the edge of the circle.



8. Graph the circle $x^2 + y^2 + 6x - 8y - 11 = 0$.

Solution:

We need to find the center and radius of the circle by changing the equation of the circle into standard form, $(x - h)^2 + (y - k)^2 = r^2$, where h and k are the coordinates of the center and r is the radius.

Start by grouping x and y terms together and moving the constant to the right side of the equation.

$$x^2 + y^2 + 6x - 8y - 11 = 0$$

$$(x^2 + 6x) + (y^2 - 8y) = 11$$

To complete the square with respect to both x and y, take the coefficients on the x and y terms, divide them by 2, then square the results. The coefficient on x is 6, so

$$\frac{6}{2} = 3$$

$$(3)^2 = 9$$

The coefficient on y is -8, so

$$\frac{-8}{2} = -4$$

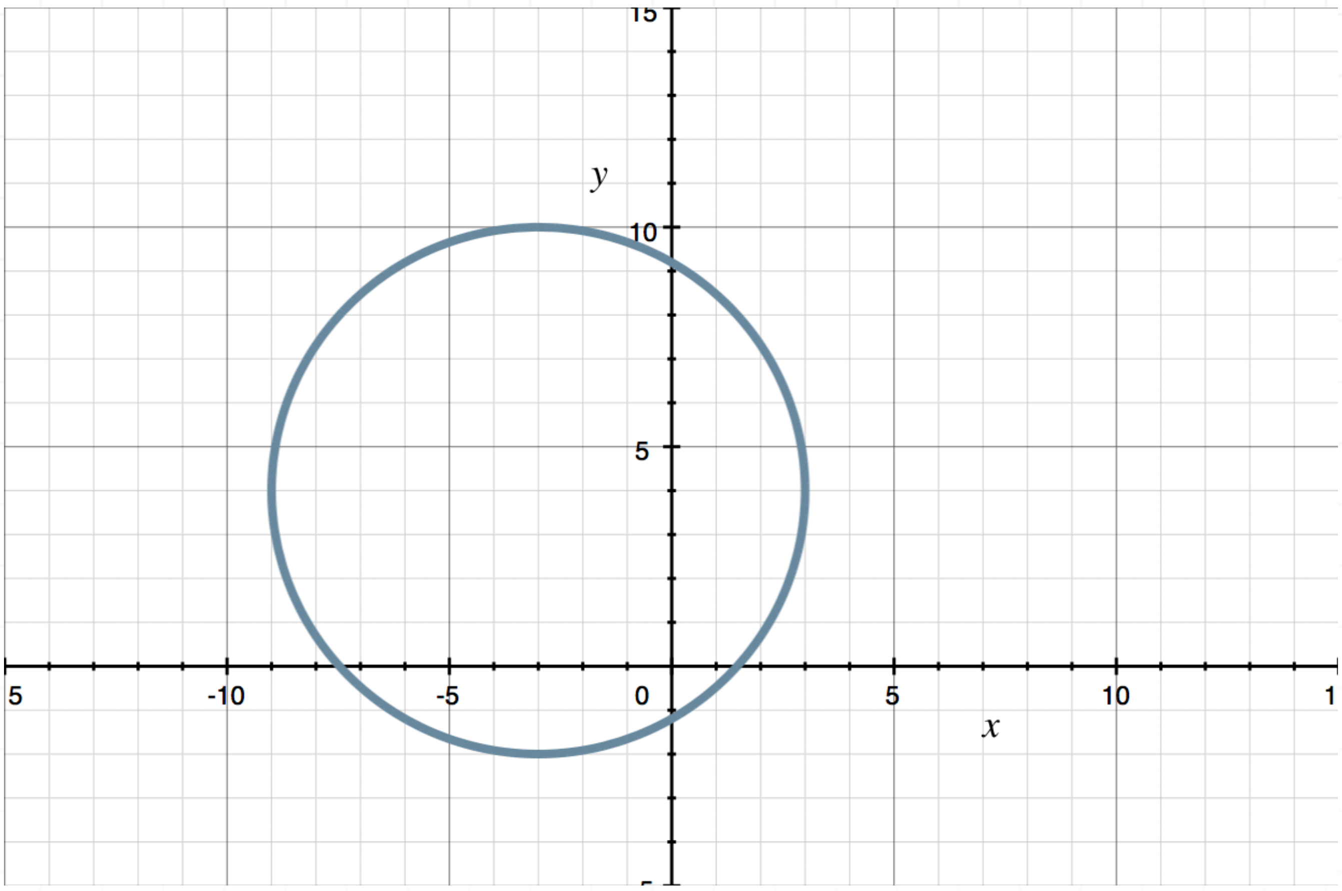
$$(-4)^2 = 16$$

Add 16 and 4 to both sides of the equation. Then factor inside the parentheses and simplify the right side.

$$(x^2 + 6x + 9) + (y^2 - 8y + 16) = 11 + 9 + 16$$

$$(x + 3)^2 + (y - 4)^2 = 36$$

The center of the circle (h, k) is therefore at $(-3,4)$ and the radius is $r = \sqrt{36} = 6$. So to graph the circle, plot the center point $(-3,4)$, then move in any direction 6 units to get to a point on the edge of the circle.



15
y
10
5
5
-10
-5
0
5
10
1
x

W W W . K R I S T A K I N G M A T H . C O M