Logarithmic differentiation

Some problems are easiest to solve using logarithmic differentiation. **Logarithmic differentiation** is a problem-solving method in which we start by applying the natural log function to both sides of the equation.

Once we've taken the natural log of both sides, then we differentiate both sides of the function, and work toward rewriting the equation so that it's solved for y'.

Oftentimes, we'll utilize laws of logarithms in order to simplify one or both sides of the equation. As a reminder, these are the laws of logs we'll want to use:

Laws of logs	**Laws of natural logs**
$\log_a a^x = x$	$\ln(e^x) = x$
$a^{\log_a x} = x$	$e^{\ln x} = x$
$\log_a x^r = r \log_a x$	$\ln(x^a) = a \ln x$
$\log_a(xy) = \log_a x + \log_a y$	$\ln(xy) = \ln x + \ln y$
$\log_a \frac{x}{y} = \log_a x - \log_a y$	$\ln \frac{x}{y} = \ln x - \ln y$

The easiest way to illustrate this method is to work through an example.

Example

Use logarithmic differentiation to find the derivative of the function.

$$y = \frac{(\ln x)^x}{2^{3x+1}}$$

To start, we'll apply the natural log to both sides of the equation.

$$\ln y = \ln\left(\frac{(\ln x)^x}{2^{3x+1}}\right)$$

Use laws of logs to rewrite the right-hand side.

$$\ln y = \ln((\ln x)^x) - \ln(2^{3x+1})$$

$$\ln y = x\ln(\ln x) - (3x + 1)\ln 2$$

$$\ln y = x\ln(\ln x) - 3x\ln 2 - \ln 2$$

Now we'll take the derivative of both sides. We'll need to use product rule to differentiate $x\ln(\ln x)$.

$$\frac{1}{y}y' = \left[(1)(\ln(\ln x)) + (x)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)\right] - 3\ln 2 - 0$$

$$\frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x} - 3\ln 2$$

$$\frac{1}{y}y' = \ln(\ln x) + \frac{1}{\ln x} - \ln(2^3)$$

We want to solve for y', so we'll multiply both sides by y in order to get y' by itself.

$$y' = y\left[\ln(\ln x) + \frac{1}{\ln x} - \ln 8\right]$$

Now we'll use the original equation to substitute for y.

$$y' = \frac{(\ln x)^x}{2^{3x+1}}\left[\ln(\ln x) + \frac{1}{\ln x} - \ln 8\right]$$

Since these are a little tricky, let's do one more example.

Example

Use logarithmic differentiation to find the derivative of the function.

$$y = x^{(x^{(x^4)})}$$

To start, we'll apply the natural log to both sides of the equation.

$$\ln y = \ln(x^{(x^{(x^4)})})$$

Use laws of logs to rewrite the right-hand side.

$$\ln y = (x^{(x^4)})\ln x$$

Apply the natural log to both sides again.

$$\ln(\ln y) = \ln((x^{(x^4)})\ln x)$$

Use laws of logs to rewrite the right-hand side.

$$\ln(\ln y) = \ln(x^{(x^4)}) + \ln(\ln x)$$

$$\ln(\ln y) = x^4 \ln x + \ln(\ln x)$$

Now we'll take the derivative of both sides. We'll need to use product rule to differentiate $x^4 \ln x$.

$$\left(\frac{1}{\ln y}\right)\left(\frac{1}{y}\right)(y') = \left[(x^4)\left(\frac{1}{x}\right) + (4x^3)(\ln x)\right] + \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)$$

$$\frac{1}{y \ln y} y' = x^3 + 4x^3 \ln x + \frac{1}{x \ln x}$$

We want to solve for y', so we'll multiply both sides by $y \ln y$ in order to get y' by itself.

$$y' = y \ln y \left(x^3 + 4x^3 \ln x + \frac{1}{x \ln x}\right)$$

Now we'll use the original equation to substitute for y.

$$y' = x^{(x^{(x^4)})} \ln(x^{(x^{(x^4)})}) \left(x^3 + 4x^3 \ln x + \frac{1}{x \ln x}\right)$$