Calculus 1 Workbook Solutions

Optimization and sketching graphs

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CRITICAL POINTS AND THE FIRST DERIVATIVE TEST

1. Identify the critical point(s) of the function on the interval $[-3,2]$.

$$f(x) = x^{\frac{2}{3}}(x+2)$$

Solution:

Find $f'(x)$ and the x-values inside the given interval for which $f'(x) = 0$ or is undefined.

Rewrite the function.

$$f(x) = x^{\frac{2}{3}}(x+2)$$

$$f(x) = x^{\frac{5}{3}} + 2x^{\frac{2}{3}}$$

Find the derivative.

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} + 2 \cdot \frac{2}{3}x^{-\frac{1}{3}}$$

$$f'(x) = \frac{5}{3}\sqrt[3]{x^2} + \frac{4}{3\sqrt[3]{x}}$$

When $x = 0$, the denominator of the second fraction will be 0, which will make the derivative undefined. The derivative will also be equal to 0:

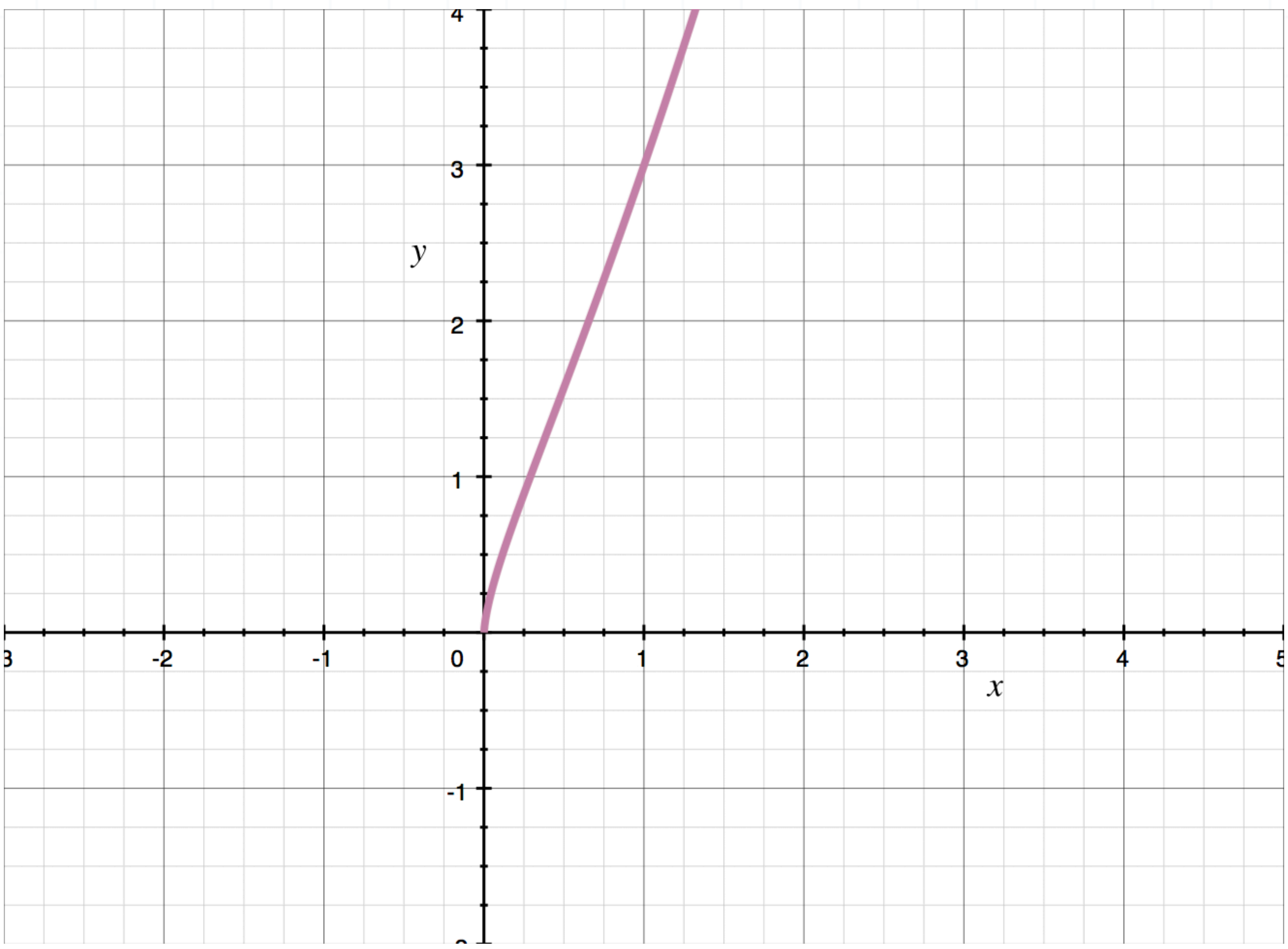
$$\frac{5}{3}\sqrt[3]{x^2} + \frac{4}{3\sqrt[3]{x}} = 0$$

$$\frac{5}{3}\sqrt[3]{x^2} = -\frac{4}{3\sqrt[3]{x}}$$

$$5x = -4$$

$$x = -\frac{4}{5}$$

The critical numbers are therefore $x = -4/5, 0$. The graph has a horizontal tangent at $x = -4/5$ and a cusp at $x = 0$.



■ 2. Identify the critical point(s) of the function on the interval $[-2,2]$.

$$g(x) = x\sqrt{4 - x^2}$$

Solution:

Find $g'(x)$ and the x-values inside the given interval for which $g'(x) = 0$ or is undefined.

Find the derivative.

$$g'(x) = (1)\sqrt{4 - x^2} + (x)\left(\frac{1}{2}\right)(4 - x^2)^{-\frac{1}{2}}(-2x)$$

$$g'(x) = \sqrt{4 - x^2} - x^2(4 - x^2)^{-\frac{1}{2}}$$

$$g'(x) = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}}$$

When $x = \pm 2$, the denominator of the second fraction will be 0, which will make the derivative undefined. The derivative will also be equal to 0:

$$\sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = 0$$

$$\sqrt{4 - x^2} = \frac{x^2}{\sqrt{4 - x^2}}$$

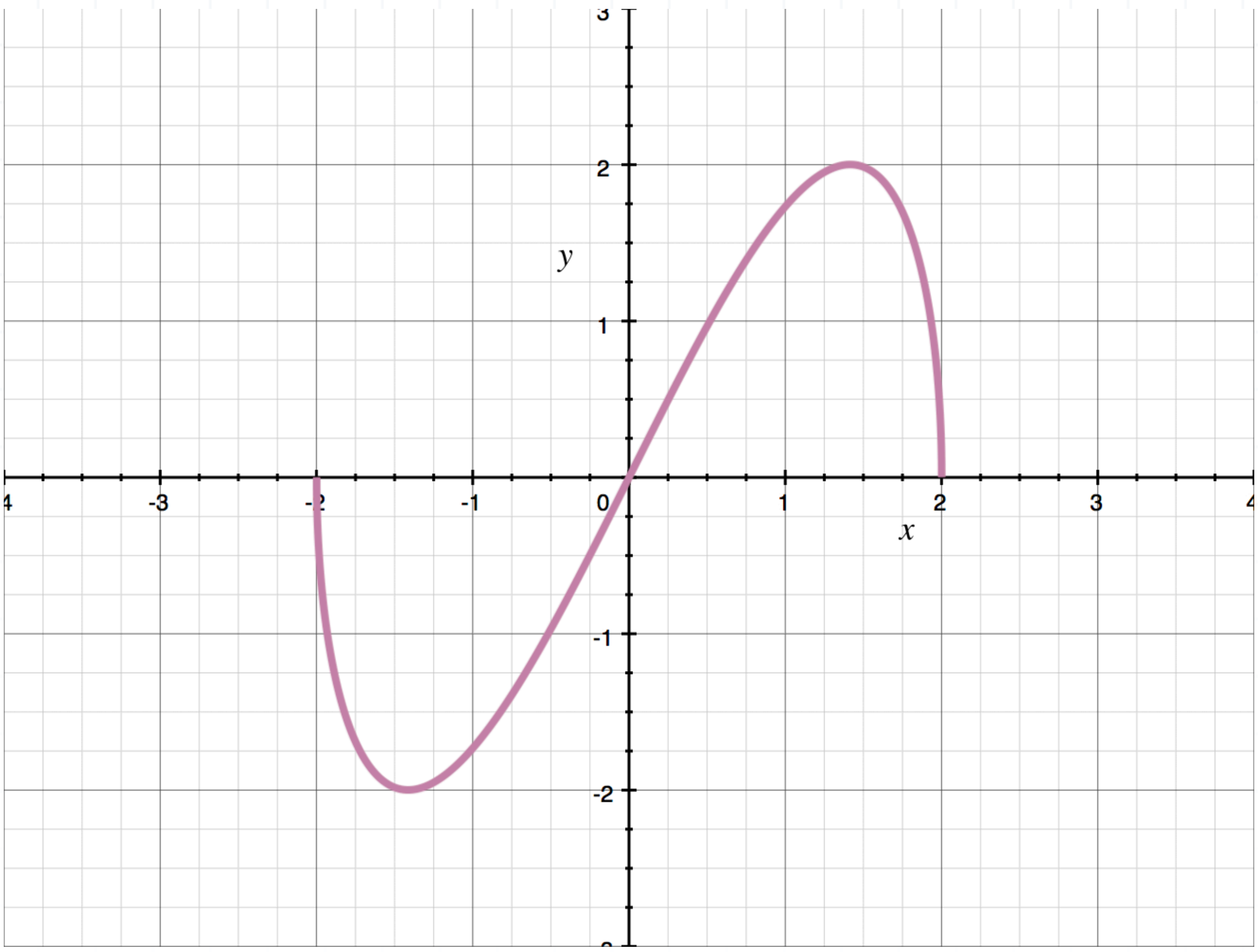
$$4 - x^2 = x^2$$

$$4 = 2x^2$$

$$2 = x^2$$

$$x = \pm\sqrt{2}$$

The critical numbers are therefore $x = - \pm \sqrt{2},\ \pm 2$. The graph has horizontal tangents at $x = - \pm \sqrt{2}$ and endpoint discontinuities at $x = \pm 2$.



■ 3. Determine the intervals where the function is increasing and decreasing.

$$f(x) = \frac{5}{4}x^4 - 10x^2$$

Solution:

Find the derivative $f'(x) = 5x^3 - 20x$, then identify the critical points where $f'(x) = 0$.

$$5x^3 - 20x = 0$$

$$5x(x^2 - 4) = 0$$

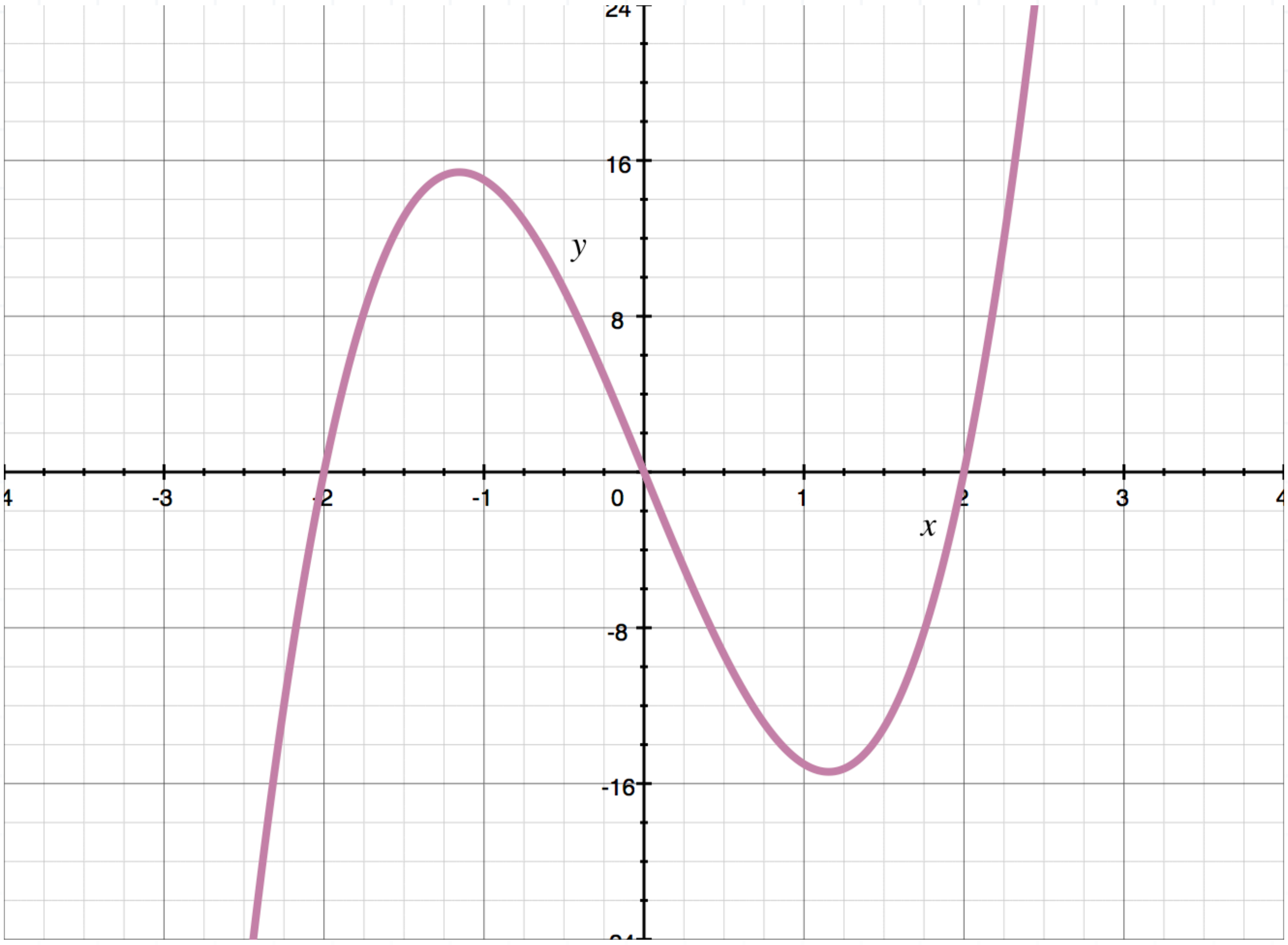
$$5x(x + 2)(x - 2) = 0$$

$$x = -2, 0, 2$$

Determine where $f'(x) > 0$ or $f'(x) < 0$ by selecting a value between each critical number.

Interval	x<-2	-2<x<0	0<x<2	x>2
x	-3	-1	1	3
f'(x)	<0	>0	<0	>0
f(x)	Decreasing	Increasing	Decreasing	Increasing

The graph of $f'(x)$ shows that $f'(x) < 0$ on $(-\infty, -2) \cup (0,2)$ and $f'(x) > 0$ on $(-2,0) \cup (2,\infty)$.



4. Determine the intervals where the function is increasing and decreasing.

$$g(x) = -x^3 + 2x^2 + 2$$

Solution:

Find the derivative $g'(x) = -3x^2 + 4x$, then identify the critical points where $g'(x) = 0$.

$$-3x^2 + 4x = 0$$

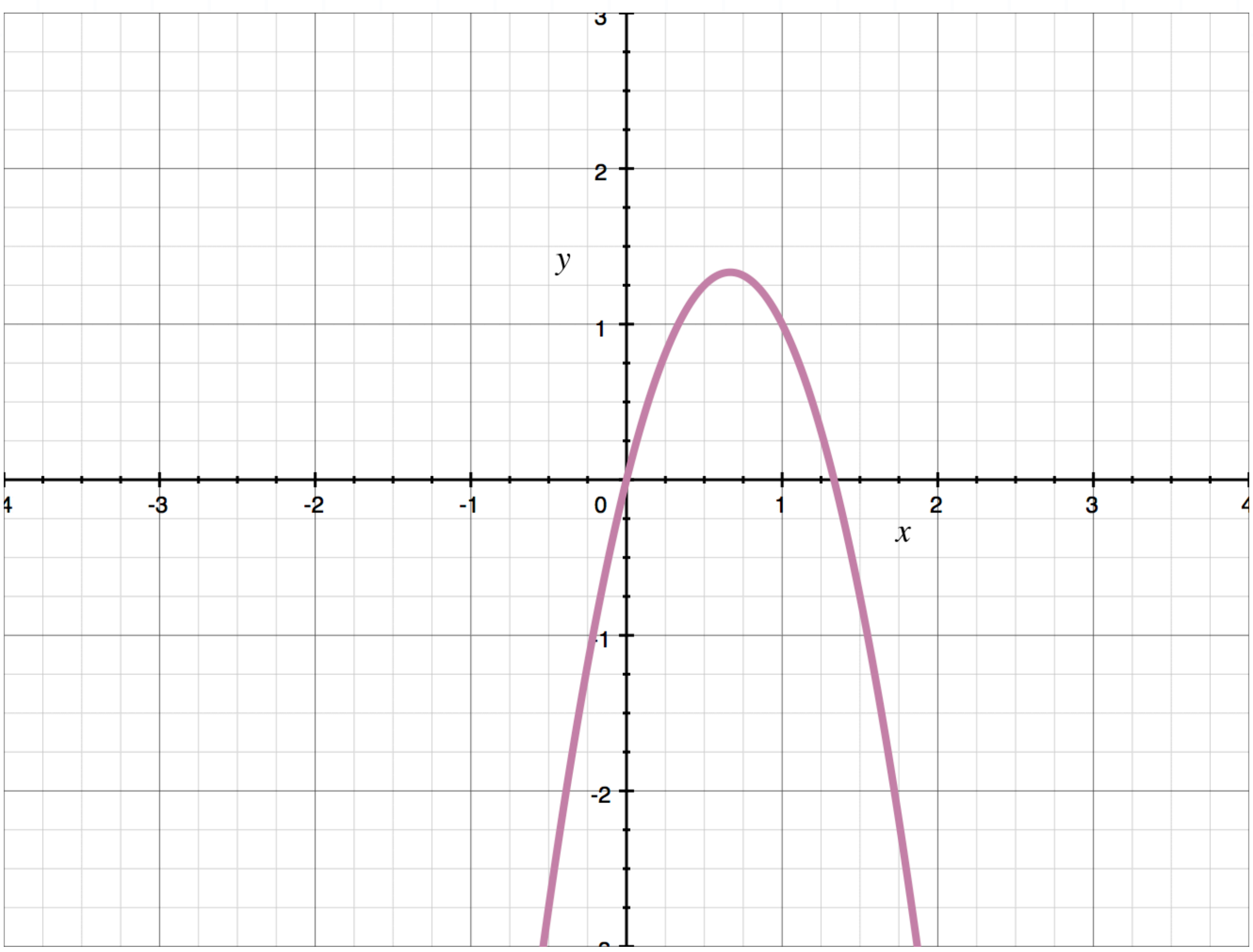
$$x(-3x + 4) = 0$$

$$x = 0, \frac{4}{3}$$

Determine where $g'(x) > 0$ or $g'(x) < 0$ by selecting a value between each critical number.

Interval	x<0	0<x<4/3	x>4/3
x	-1	1	2
g'(x)	<0	>0	<0
g(x)	Decreasing	Increasing	Decreasing

The graph of $g'(x)$ shows that $g'(x) < 0$ on $(-\infty,0) \cup (4/3,\infty)$ and $g'(x) > 0$ on $(0,4/3)$.



5. Use the first derivative test to find the extrema of $f(x) = 4x^3 + 21x^2 + 36x - 5$.

Solution:

Find the derivative $f'(x) = 12x^2 + 42x + 36$. Set the derivative equal to 0 and solve for x.

$$12x^2 + 42x + 36 = 0$$

$$6(2x^2 + 7x + 6) = 0$$

$$6(x + 2)(2x + 3) = 0$$

$$x = -2, \ -\frac{3}{2}$$

Determine the sign of $f'(x)$ on each side of these critical numbers by selecting a value between each number.

Interval	x<-2	x=-2	-2<x<-3/2	x=-3/2	x>-3/2
x	-4	-2	-1.8	-3/2	0
f'(x)	+	0	-	0	+
f(x)	Increasing	Maximum	Decreasing	Minimum	Increasing

The table shows that $f(x)$ has a maximum at $x = -2$, so $f(-2) = -25$ and $(-2, -25)$ is a maximum point. The table also shows that $f(x)$ has a minimum at $x = -3/2$, so $f(-3/2) = -101/4$ and $(-3/2, -101/4)$ is a minimum point. The graph shows the extrema:

■ 6. Use the first derivative test to find the extrema of $g(x) = 2x^3 - 14x^2 + 22x + 3$.

Solution:

Find the derivative $g'(x) = 6x^2 - 28x + 22$. Set the derivative equal to 0 and solve for x.

$$6x^2 - 28x + 22 = 0$$

$$2(3x^2 - 14x + 11) = 0$$

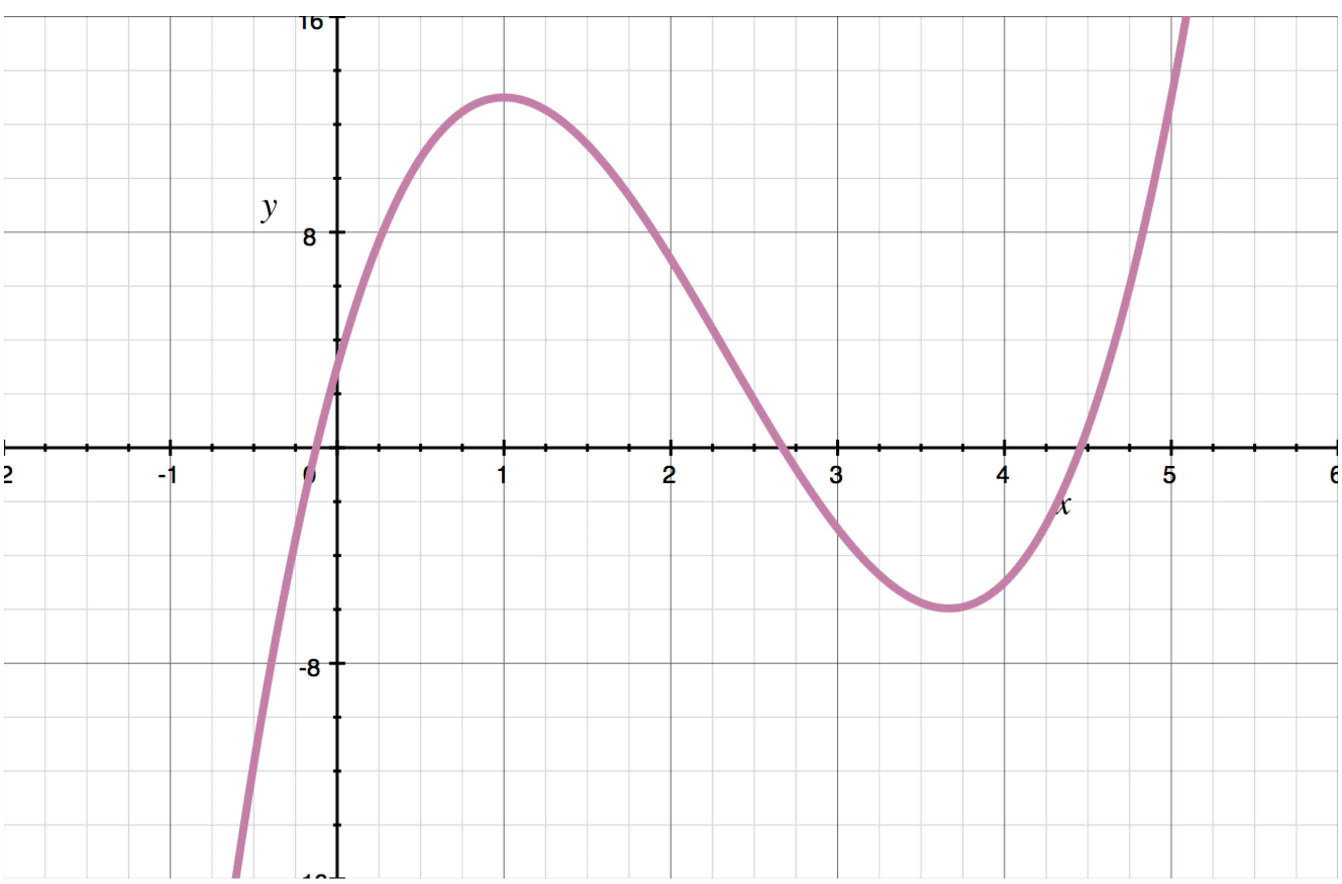
$$2(x-1)(3x-11)=0$$

$$x=1,\ \frac{11}{3}$$

Determine the sign of $g'(x)$ on each side of these critical numbers by selecting a value between each number.

Interval	x<1	x=1	1<x<11/3	x=11/3	x>11/3
x	-4	1	2	11/3	5
g'(x)	+	0	-	0	+
g(x)	Increasing	Maximum	Decreasing	Minimum	Increasing

The table shows that $g(x)$ has a maximum at $x=1$, so $g(1)=13$ and $(1,13)$ is a maximum point. The table also shows that $g(x)$ has a minimum at $x=11/3$, so $g(11/3)=-161/27$ and $(11/3,-161/27)$ is a minimum point. The graph shows the extrema:



INFLECTION POINTS AND THE SECOND DERIVATIVE TEST

■ 1. For $f(x) = x^3 - 3x^2 + 5$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$f'(x) = 3x^2 - 6x$$

$$f''(x) = 6x - 6$$

The function has an inflection point when $f''(x) = 0$.

$$6x - 6 = 0$$

$$x - 1 = 0$$

$$x = 1$$

Check values around $x = 1$.

Interval	x<1	x=1	x>1
x	-1	1	2
f''(x)	-	0	+
Concavity	Down	Inflection	Up

The inflection point is at $x = 1$ and $f(1) = 3$, so the inflection point is $(1,3)$. The function is concave down on $(-\infty,1)$ and concave up on $(1,\infty)$.

■ 2. For $g(x) = -x^3 + 2x^2 + 3$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$g'(x) = -3x^2 + 4x$$

$$g''(x) = -6x + 4$$

The function has an inflection point when $g''(x) = 0$.

$$-6x + 4 = 0$$

$$-6x = -4$$

$$x = \frac{2}{3}$$

Check values around $x = 2/3$.

Interval	x<2/3	x=2/3	x>2/3
x	-1	2/3	1
g''(x)	+	0	-
Concavity	Up	Inflection	Down

The inflection point is at $x = 2/3$ and $g(2/3) = 97/27$, so the inflection point is $(2/3,97/27)$. The function is concave up on $(-\infty,2/3)$ and concave down on $(2/3,\infty)$.

■ 3. For $h(x) = x^4 + x^3 - 3x^2 + 2$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$h'(x) = 4x^3 + 3x^2 - 6x$$

$$h''(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x - 1)(x + 1)$$

The function has an inflection point when $h''(x) = 0$.

$$6(2x - 1)(x + 1) = 0$$

$$x = -1, \frac{1}{2}$$

Check values around these inflection points.

Interval	x<-1	x=-1	-1<x<1/2	x=1/2	x>1/2
x	-2	-1	0	1/2	1
h”(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

An inflection point is at $x = -1$ and $h(-1) = -1$, so an inflection point is $(-1, -1)$. Another inflection point is at $x = 1/2$ and $h(1/2) = 23/16$, so an inflection point is $(1/2, 23/16)$. The function is concave up on $(-\infty, -1) \cup (1/2, \infty)$ and concave down on $(-1, 1/2)$.

■ 4. Use the second derivative test to identify the extrema of $f(x) = x^3 - 12x - 2$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$f'(x) = 3x^2 - 12$$

$$f''(x) = 6x$$

The function has extrema when $f'(x) = 0$.

$$3x^2 - 12 = 0$$

$$(x + 2)(x - 2) = 0$$

$$x = -2,\ 2$$

Plug those values into the second derivative.

$$f''(-2) = 6(-2) = -12$$

$$f''(2) = 6(2) = 12$$

By the second derivative test, the function is concave down at $x = -2$. Since $f(-2) = 14$, $(-2,14)$ is a maximum. The function is concave up at $x = 2$. Since $f(2) = -18$, $(2, -18)$ is a minimum.

■ 5. Use the second derivative test to identify the extrema of $g(x) = -4x^3 + 12x^2 + 5$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$g'(x) = -12x^2 + 24x$$

$$g''(x) = -24x + 24$$

The function has extrema when $g'(x) = 0$.

$$-12x^2 + 24x = 0$$

$$x(x - 2) = 0$$

$$x = 0, 2$$

Plug those values into the second derivative.

$$g''(0) = -24(0) + 24 = 24$$

$$g''(2) = -24(2) + 24 = -24$$

By the second derivative test, the function is concave up at $x = 0$. Since $g(0) = 5$, $(0,5)$ is a minimum. The function is concave down at $x = 2$. Since $g(2) = 21$, $(2,21)$ is a maximum.

■ 6. Use the second derivative test to identify the extrema of $h(x) = 2x^4 - 4x^2 + 1$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$h'(x) = 8x^3 - 8x$$

$$h''(x) = 24x^2 - 8$$

The function has extrema when $h'(x) = 0$.

$$8x^3 - 8x = 0$$

$$x(x + 1)(x - 1) = 0$$

$$x = -1,\ 0,\ 1$$

Plug those values into the second derivative.

$$h''(-1) = 24(-1)^2 - 8 = 16$$

$$h''(0) = 24(0)^2 - 8 = -8$$

$$h''(1) = 24(1)^2 - 8 = 16$$

By the second derivative test, the function is concave up at $x = -1$. Since $h(-1) = -1$, $(-1, -1)$ is a minimum. The function is concave down at $x = 0$. Since $h(0) = 1$, $(0,1)$ is a maximum. The function is concave up at $x = 1$. Since $h(1) = -1$, $(1, -1)$ is a minimum.

INTERCEPTS AND VERTICAL ASYMPTOTES

■ 1. Find any vertical asymptote(s) of the function.

$$f(x) = \frac{-x^2 + 16x - 63}{x^2 - 2x - 35}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{-(x-7)(x-9)}{(x-7)(x+5)}$$

The denominator is equal to 0 if $x = 7$ or $x = -5$, which means the function has two discontinuities. However, the function simplifies to

$$f(x) = \frac{-(x-9)}{x+5}$$

$$f(x) = \frac{9-x}{x+5}$$

Therefore, the function has a removable discontinuity at $x = 7$ and a vertical asymptote at $x = -5$, which means the domain of the function is $(-\infty, -5) \cup (-5,7) \cup (7,\infty)$.

■ 2. Find any vertical asymptote(s) of the function.

$$g(x) = \frac{x^2 - 3x - 10}{x^2 + x - 2}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$g(x) = \frac{(x-5)(x+2)}{(x-1)(x+2)}$$

The denominator is equal to 0 if $x = -2$ or $x = 1$, which means the function has two discontinuities. However, the function simplifies to

$$g(x) = \frac{x-5}{x-1}$$

Therefore, the function has a removable discontinuity at $x = -2$ and a vertical asymptote at $x = 1$, which means the domain of the function is $(-\infty, -2) \cup (-2,1) \cup (1,\infty)$.

■ 3. Find any vertical asymptote(s) of the function.

$$h(x) = \frac{40 - 27x - 12x^2 - x^3}{9x^2 + 63x - 72}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$h(x) = \frac{-(x+8)(x+5)(x-1)}{9(x-1)(x+8)}$$

Cancel common factors from the numerator and denominator, then simplify.

$$h(x) = \frac{-(x+5)}{9}$$

$$h(x) = -\frac{x+5}{9}$$

There are no values of x that make this denominator 0, so the function has no vertical asymptotes. But it does have removable discontinuities for the factors we canceled, at $x = -8$ and $x = 1$.

HORIZONTAL AND SLANT ASYMPTOTES

■ 1. Find the horizontal asymptote(s) of the function.

$$f(x) = \frac{8x^4 - x^2 + 1}{4x^4 - 1}$$

Solution:

In a polynomial function, the term with the highest degree dominates the behavior of the function. So the behavior of $f(x)$ is dominated by the behavior of

$$\frac{8x^4}{4x^4}$$

Find the equation of the horizontal asymptote by taking the limit of the dominating function as $x \to \infty$.

$$\lim_{x \to \infty} \frac{8x^4}{4x^4} = \lim_{x \to \infty} \frac{8}{4} = 2$$

Therefore, the equation of the horizontal asymptote is $y = 2$.

■ 2. Find the horizontal asymptote(s) of the function.

$$g(x) = \frac{2x^2 - 5x + 12}{3x^2 - 11x - 4}$$

Solution:

In a polynomial function, the term with the highest degree dominates the behavior of the function. So the behavior of $g(x)$ is dominated by the behavior of

$$\frac{2x^2}{3x^2}$$

Find the equation of the horizontal asymptote by taking the limit of the dominating function as $x \to \infty$.

$$\lim_{x\to\infty} \frac{2x^2}{3x^2} = \lim_{x\to\infty} \frac{2}{3} = \frac{2}{3}$$

Therefore, the equation of the horizontal asymptote is $y = 2/3$.

■ 3. Find the horizontal asymptote(s) of the function.

$$h(x) = \frac{x^3 - x^2 + 6x - 1}{7x^4 - 1}$$

Solution:

In a polynomial function, the term with the highest degree dominates the behavior of the function. So the behavior of $g(x)$ is dominated by the behavior of

$$\frac{x^3}{7x^4}$$

Find the equation of the horizontal asymptote by taking the limit of the dominating function as $x \to \infty$.

$$\lim_{x \to \infty} \frac{x^3}{7x^4} = \lim_{x \to \infty} \frac{1}{7x} = 0$$

Therefore, the equation of the horizontal asymptote is $y = 0$.

■ 4. Find the slant asymptote of the function.

$$f(x) = \frac{3x^4 - x^3 + x^2 - 4}{x^3 - x^2 + 1}$$

Solution:

Use polynomial long division on the function to rewrite it as

$$3x + 2 + \frac{3x^2 - 3x - 6}{x^3 - x^2 + 1}$$

Then take the limit of the rational portion of the quotient as $x \to \infty$.

$$\lim_{x \to \infty} \frac{3x^2 - 3x - 6}{x^3 - x^2 + 1} = \lim_{x \to \infty} \frac{3x^2}{x^3} = \lim_{x \to \infty} \frac{3}{x} = 0$$

Therefore, the equation of the slant asymptote is $y = 3x + 2$.

5. Find the slant asymptote of the function.

$$g(x) = \frac{8x^2 + 14x - 7}{4x - 1}$$

Solution:

Use polynomial long division on the function to rewrite it as

$$2x + 4 - \frac{3}{4x - 1}$$

Then take the limit of the rational portion of the quotient as $x \to \infty$.

$$\lim_{x \to \infty} -\frac{3}{4x - 1} = \lim_{x \to \infty} -\frac{3}{4x} = 0$$

Therefore, the equation of the slant asymptote is $y = 2x + 4$.

6. Find the slant asymptote of the function.

$$h(x) = \frac{x^3 - 8}{x^2 - 5x + 6}$$

Solution:

Use polynomial long division on the function to rewrite it as

$$x+5+\frac{19x-38}{x^2-5x+6}$$

Then take the limit of the rational portion of the quotient as $x \to \infty$.

$$\lim_{x\to\infty}\frac{19x-38}{x^2-5x+6}=\lim_{x\to\infty}\frac{19x}{x^2}=\lim_{x\to\infty}\frac{19}{x}=0$$

Therefore, the equation of the slant asymptote is $y=x+5$.

SKETCHING GRAPHS

■ 1. Sketch the graph of the function.

$$f(x) = x^3 - 4x^2 + 8$$

Solution:

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = 3x^2 - 8x$$

$$3x^2 - 8x = 0$$

$$x(3x - 8) = 0$$

$$x = 0, \frac{8}{3}$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	x<0	x=0	0<x<8/3	x=8/3	x>8/3
x	-2	0	1	8/3	4
f'(x)	+	0	-	0	+
Direction	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $f(x)$

- increases on the interval $(-\infty, 0)$,

- has a local maximum at $x = 0$,
- decreases on the interval $(0,8/3)$,
- has a local minimum at $x = 8/3$, and then
- increases on the interval $(8/3,\infty)$.

Evaluate the function at the extrema.

$$f(0) = (0)^3 - 4(0)^2 + 8 = 8$$

$$f\left(\frac{8}{3}\right) = \left(\frac{8}{3}\right)^3 - 4\left(\frac{8}{3}\right)^2 + 8 = -\frac{40}{27}$$

There's a local maximum at $(0,8)$ and a local minimum at $(8/3, -40/27)$. Now use the second derivative to determine concavity.

$$f''(x) = 6x - 8$$

$$6x - 8 = 0$$

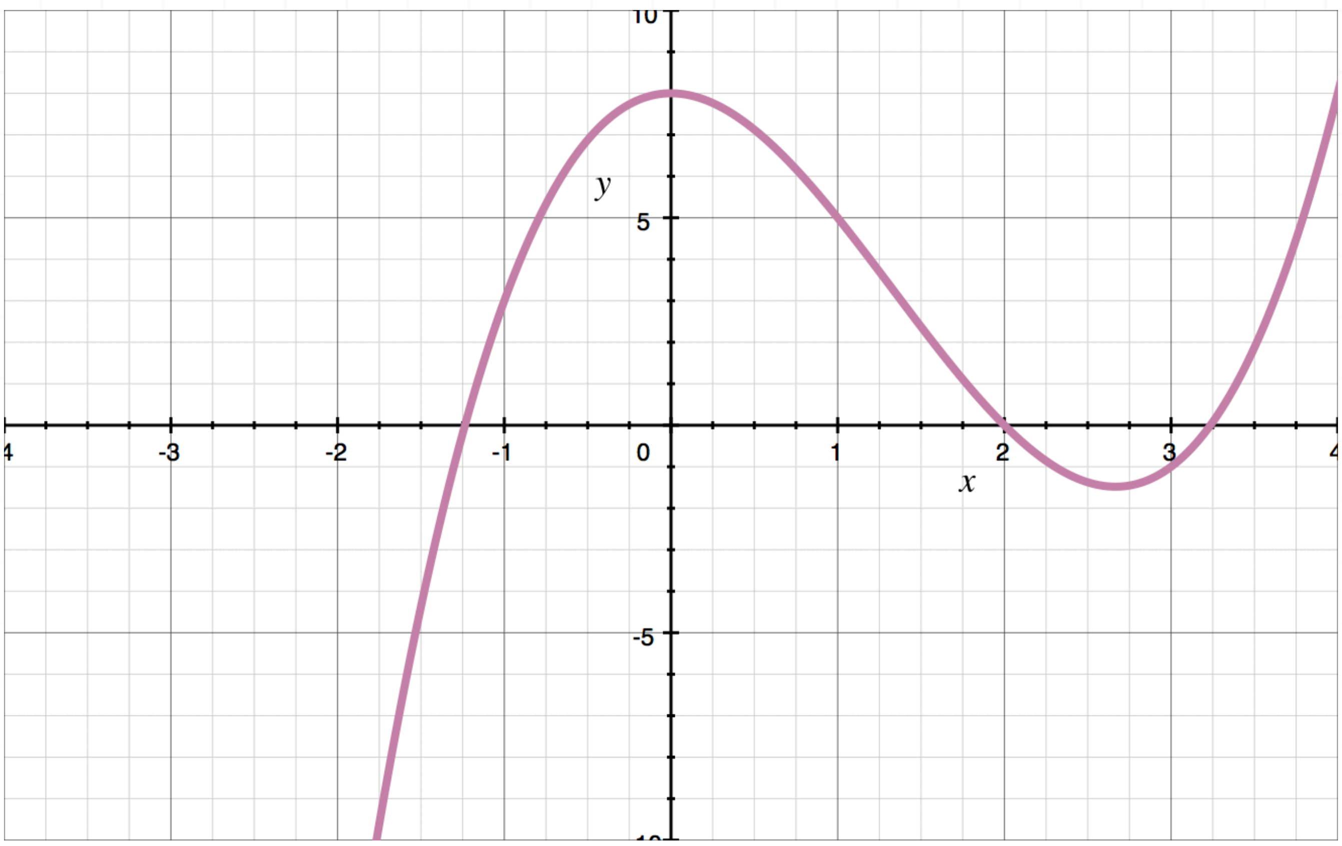
$$x = \frac{4}{3}$$

Test values around the inflection point $x = 4/3$.

Interval	x<4/3	x=4/3	x>4/3
x	0	4/3	3
f''(x)	-	0	+
Concavity	Down	Inflection	Up

We can see that $f(x)$ is concave down on the interval $(-\infty,4/3)$ and concave up on the interval $(4/3,\infty)$. Because $f(4/3)=88/27$, $f(x)$ has an inflection point at $(4/3,88/27)$. Since $f(x)$ is a polynomial function, its graph has no asymptotes.

Putting all this together, the graph is



■ 2. Sketch the graph of the function.

$$g(x)=\frac{1}{4}x^4-\frac{1}{3}x^3-3x^2+1$$

Solution:

Take the derivative, then set it equal to 0 to find critical points.

$$g'(x) = x^3 - x^2 - 6x$$

$$x^3 - x^2 - 6x = 0$$

$$x(x-3)(x+2) = 0$$

$$x = -2,\ 0,\ 3$$

Use the first derivative test to see where $g(x)$ is increasing and decreasing.

Interval	x<-2	x=-2	-2<x<0	x=0	0<x<3	x=3	x>3
x	-4	-2	-1	0	2	3	4
g'(x)	-	0	+	0	-	0	+
Direction	Decreasing	Minimum	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $g(x)$

- decreases on the interval $(-\infty, -2)$,
- has a local minimum at $x = -2$,
- increases on the interval $(-2,0)$,
- has a local maximum at $x = 0$
- decreases on the interval $(0,3)$
- has a local minimum at $x = 3$, and then
- increases on the interval $(3,\infty)$.

Evaluate the function at the extrema.

$$g(-2) = \frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 3(-2)^2 + 1 = -\frac{13}{3}$$

$$g(0) = \frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - 3(0)^2 + 1 = 1$$

$$g(3) = \frac{1}{4}(3)^4 - \frac{1}{3}(3)^3 - 3(3)^2 + 1 = -\frac{59}{4}$$

There's a local minimum at $(-2, -13/3)$, a local maximum at $(0,1)$, and a local minimum at $(3, -59/4)$. Now use the second derivative to determine concavity.

$$g''(x) = 3x^2 - 2x - 6$$

$$3x^2 - 2x - 6 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 72}}{6} = \frac{1 \pm \sqrt{19}}{3}$$

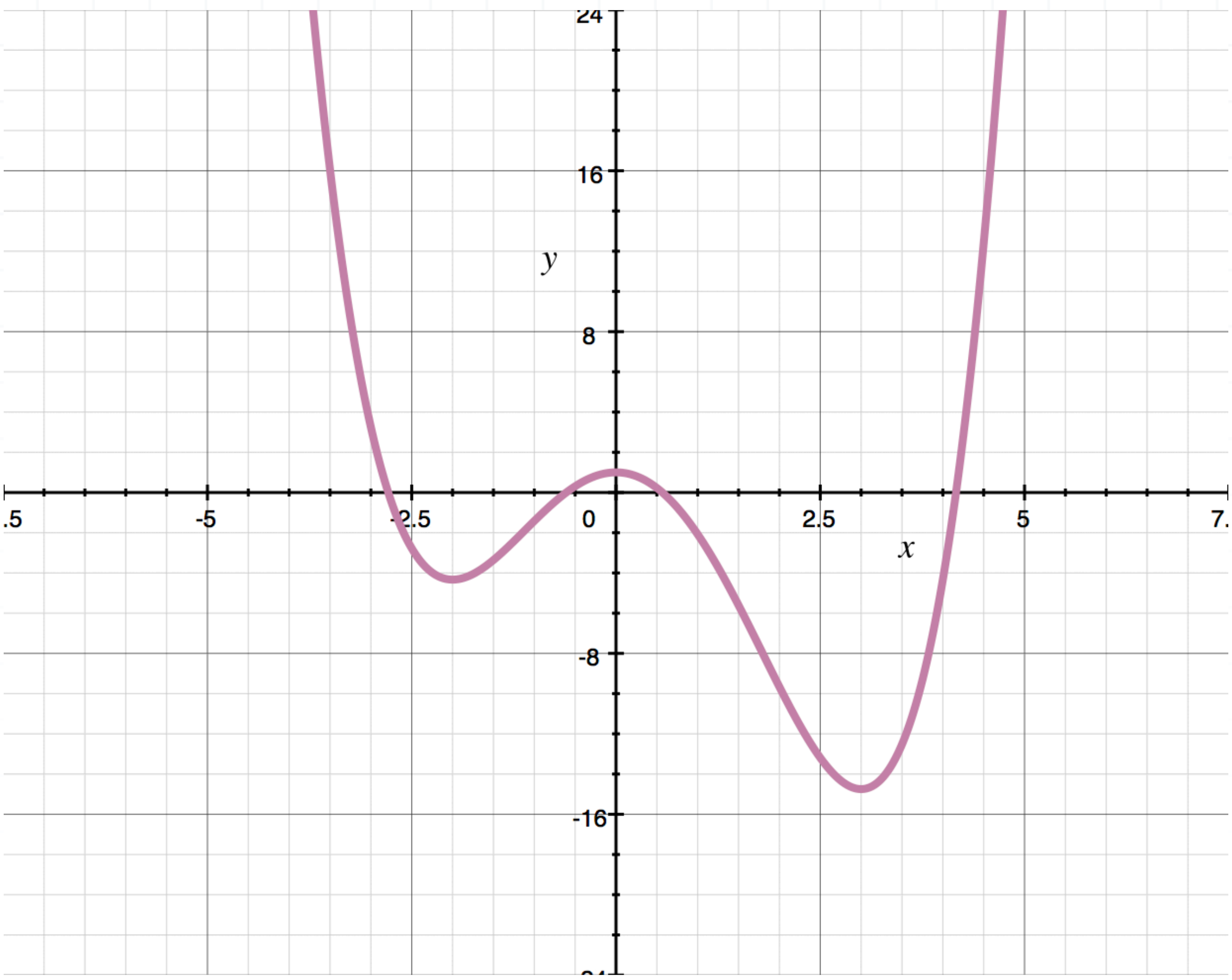
Test values around the inflection points using their approximate values.

Interval	x<-1.12	x=-1.12	-1.12<x<1.79	x=1.79	x>1.79
x	-2	-1.2	1	1.79	4
g''(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

We can see that $g(x)$ is concave up on the interval $(-\infty, (1 - \sqrt{19})/3)$, concave down on the interval $((1 - \sqrt{19})/3, (1 + \sqrt{19})/3)$, and concave up on the interval $((1 + \sqrt{19})/3, \infty)$. Because $g((1 - \sqrt{19})/3) \approx -1.9$, $g(x)$ has an inflection point at approximately $(-1.12, -1.9)$. Because $g((1 + \sqrt{19})/3) \approx -7.96$, $g(x)$ has an inflection point at approximately

$(1.79, -7.96)$. Since $g(x)$ is a polynomial function, its graph has no asymptotes.

Putting all this together, the graph is



3. Sketch the graph of the function.

$$h(x) = \frac{x^2 + x - 6}{4x^2 + 16x + 12}$$

Solution:

Take the derivative, then set it equal to 0 to find critical points.

$$h'(x) = \frac{12(x^2+6x+9)}{(4x^2+16x+12)^2} = \frac{12(x+3)^2}{\left[4(x+1)(x+3)\right]^2} = \frac{12(x+3)^2}{16(x+1)^2(x+3)^2} = \frac{3}{4(x+1)^2}$$

There are no values for which $h'(x) = 0$, so there are no critical points. But $h'(x)$ is undefined when $x = -1$. Now use the second derivative to determine concavity.

$$h''(x) = -\frac{3}{2(x+1)^3}$$

There are no values for which $h''(x) = 0$, but $h''(x)$ is undefined when $x = -1$. Test values around the inflection point $x = -1$.

Interval	x<-1	x=-1	x>-1
x	-3	-1	3
h"(x)	+	DNE	-
Concavity	Up	Inflection	Down

We can see that $h(x)$ is concave up on the interval $(-\infty, -1)$ and concave down on the interval $(-1,\infty)$. Since $h(x)$ is a rational function, we need to look for asymptotes.

The behavior of the function is dominated by the highest degree terms in the numerator and denominator, which means the horizontal asymptote is

$$\lim_{x\to\pm\infty} \frac{x^2}{4x^2} = \lim_{x\to\pm\infty} \frac{1}{4} = \frac{1}{4}$$

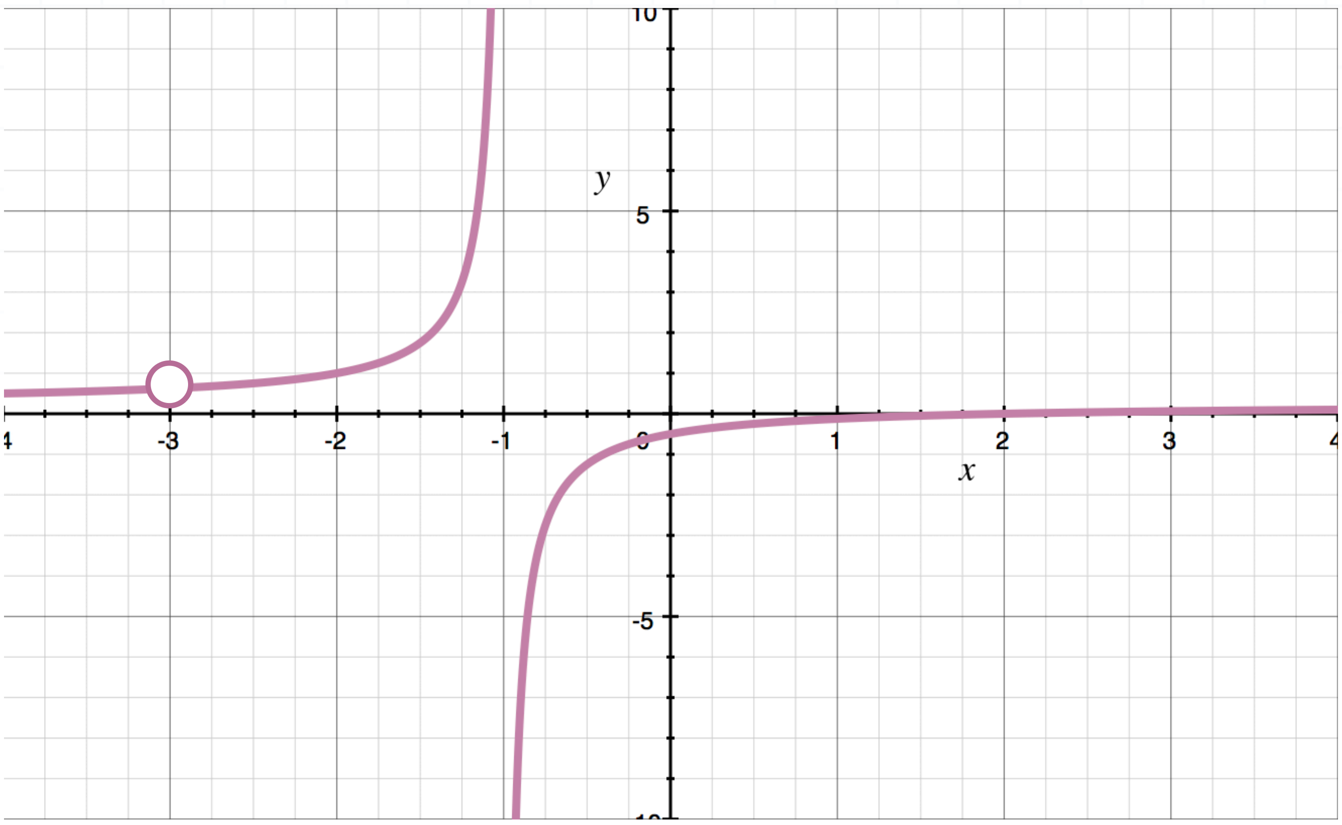
The denominator of $h(x)$ is 0 when $x = -1$, so the function has a vertical asymptote there. To determine the behavior of the function near $x = -1$, find

$$\lim_{x \to -1^-} h(x) = \infty$$

$$\lim_{x \to -1^+} h(x) = -\infty$$

Lastly, the function crosses the x-axis when the numerator equals 0, which occurs at $x = 2$, and the function crosses the y-axis when $x = 0$, which gives a y-intercept of $(0, -1/2)$.

Putting all this together, the graph is



EXTREMA ON A CLOSED INTERVAL

1. Find the extrema of $f(x) = x^3 - 3x^2 + 5$ over the closed interval $[-3,4]$.

Solution:

Find the critical points of the function.

$$f'(x) = 3x^2 - 6x$$

$$3x^2 - 6x = 0$$

$$3x(x - 2) = 0$$

$$x = 0, 2$$

Evaluate the function at the endpoints and at the critical numbers.

For $x = -3$, $\quad (-3)^3 - 3(-3)^2 + 5 = -27 - 27 + 5 = -49$

For $x = 0$, $\quad (0)^3 - 3(0)^2 + 5 = 0 - 0 + 5 = 5$

For $x = 2$, $\quad (2)^3 - 3(2)^2 + 5 = 8 - 12 + 5 = 1$

For $x = 4$, $\quad (4)^3 - 3(4)^2 + 5 = 64 - 48 + 5 = 21$

The results show that $f(x)$ has a global minimum at $(-3, -49)$, a local maximum at $(0,5)$, a local minimum at $(2,1)$, and a global maximum at $(4,21)$.

■ 2. Find the extrema of $g(x) = \sqrt[3]{2x^2 + 3}$ over the closed interval $[-1,5]$.

Solution:

Find the critical points of the function.

$$g'(x) = \frac{1}{3}(2x^2 + 3)^{-\frac{2}{3}}(4x) = \frac{4x}{3\sqrt[3]{(2x^2 + 3)^2}}$$

$$4x = 0$$

$$x = 0$$

Evaluate the function at the endpoints and at the critical numbers.

For $x = -1$, $\quad \sqrt[3]{2(-1)^2 + 3} = \sqrt[3]{5} \approx 1.71$

For $x = 0$, $\quad \sqrt[3]{2(0)^2 + 3} = \sqrt[3]{3} \approx 1.44$

For $x = 5$, $\quad \sqrt[3]{2(5)^2 + 3} = \sqrt[3]{53} \approx 3.76$

The results show that $g(x)$ has a global minimum at $(0,\sqrt[3]{3})$, and a global maximum at $(5,\sqrt[3]{53})$.

■ 3. Find the extrema of $h(x) = -4x^3 + 6x^2 - 3x - 2$ over the closed interval $[-4,6]$.

Solution:

Find the critical points of the function.

$$h'(x) = -12x^2 + 12x - 3$$

$$-12x^2 + 12x - 3 = 0$$

$$-3(4x^2 - 4x + 1) = 0$$

$$-3(2x - 1)(2x - 1) = 0$$

$$x = 1/2$$

Evaluate the function at the endpoints and at the critical numbers.

For $x = -4$, $\quad -4(-4)^3 + 6(-4)^2 - 3(-4) - 2 = 362$

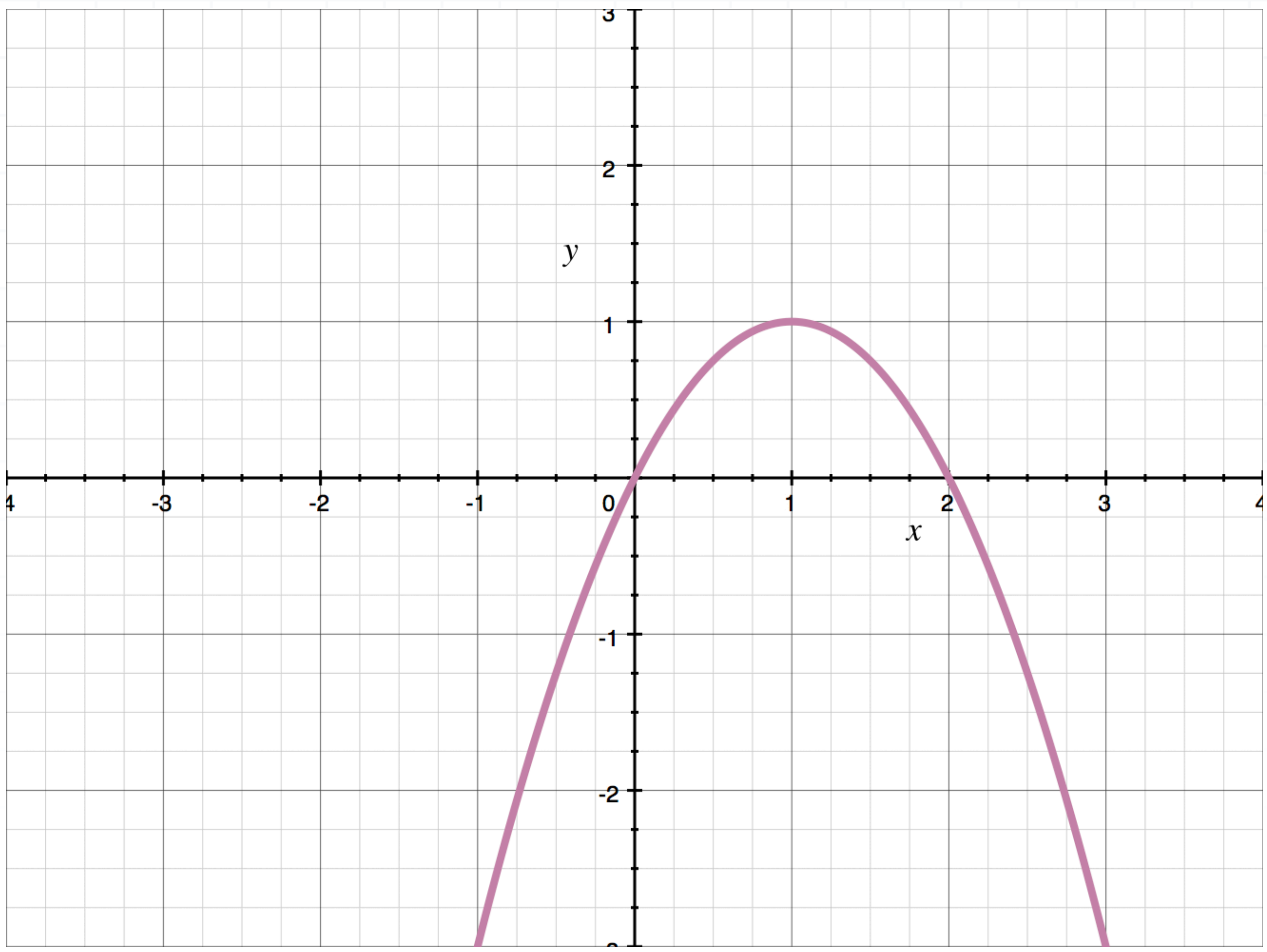
For $x = 1/2$, $\quad -4(1/2)^3 + 6(1/2)^2 - 3(1/2) - 2 = -5/2$

For $x = 6$, $\quad -4(6)^3 + 6(6)^2 - 3(6) - 2 = -668$

The results show that $h(x)$ has a global maximum at $(-4,362)$, a horizontal tangent line at $(1/2, -5/2)$, and a global minimum at $(6, -668)$.

SKETCHING F(X) FROM F'(X)

■ 1. Sketch a possible graph of $f(x)$ given the graph below of $f'(x)$.



Solution:

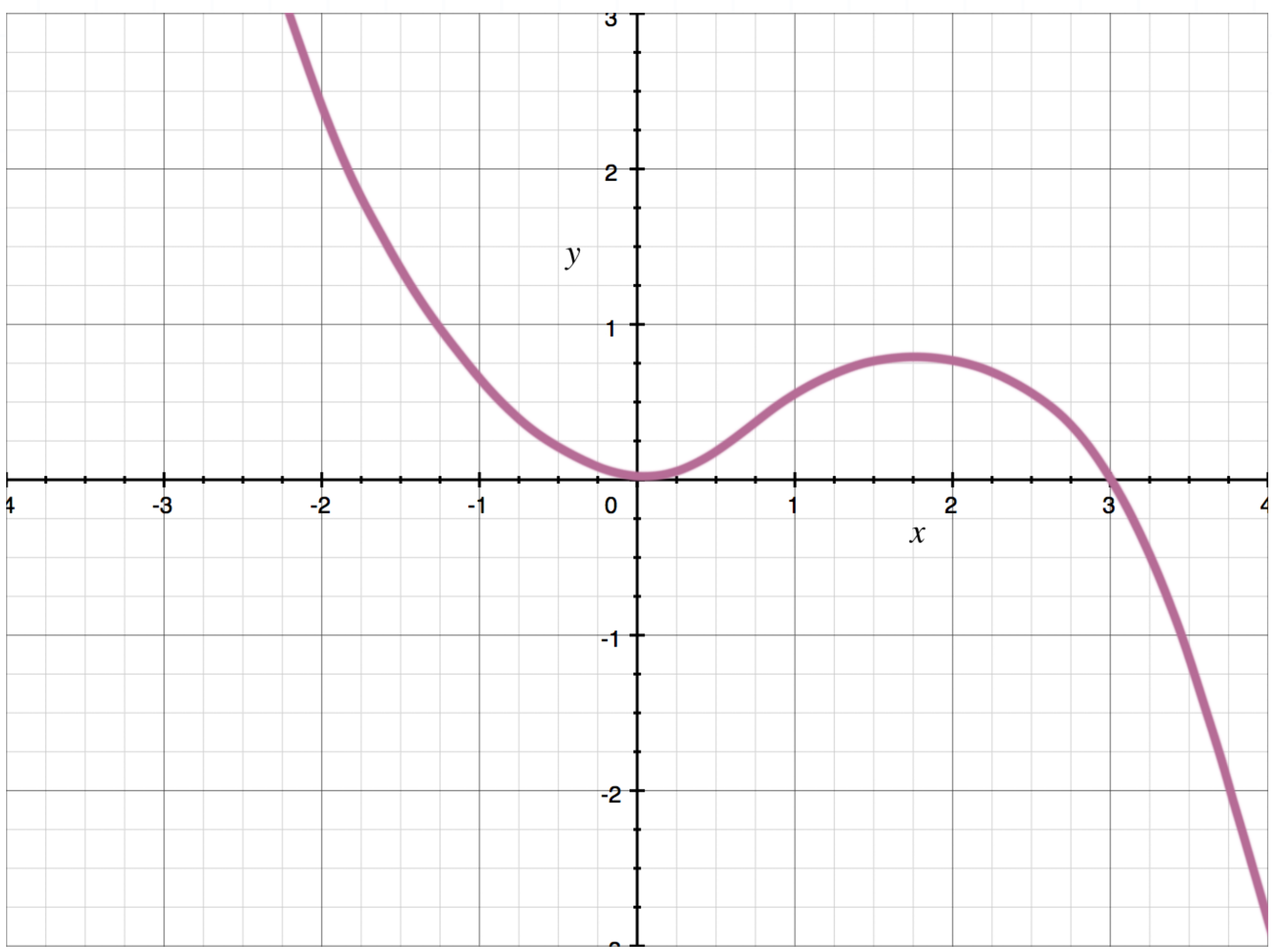
The graph of $f'(x)$ is below the x-axis on the intervals $(-\infty,0)$ and $(2,\infty)$, which means the function $f(x)$ has a negative slope and is decreasing on these intervals.

Additionally, the graph of $f'(x)$ is above the x-axis on the interval $(0,2)$, which means the function $f(x)$ has a positive slope and is increasing on this interval.

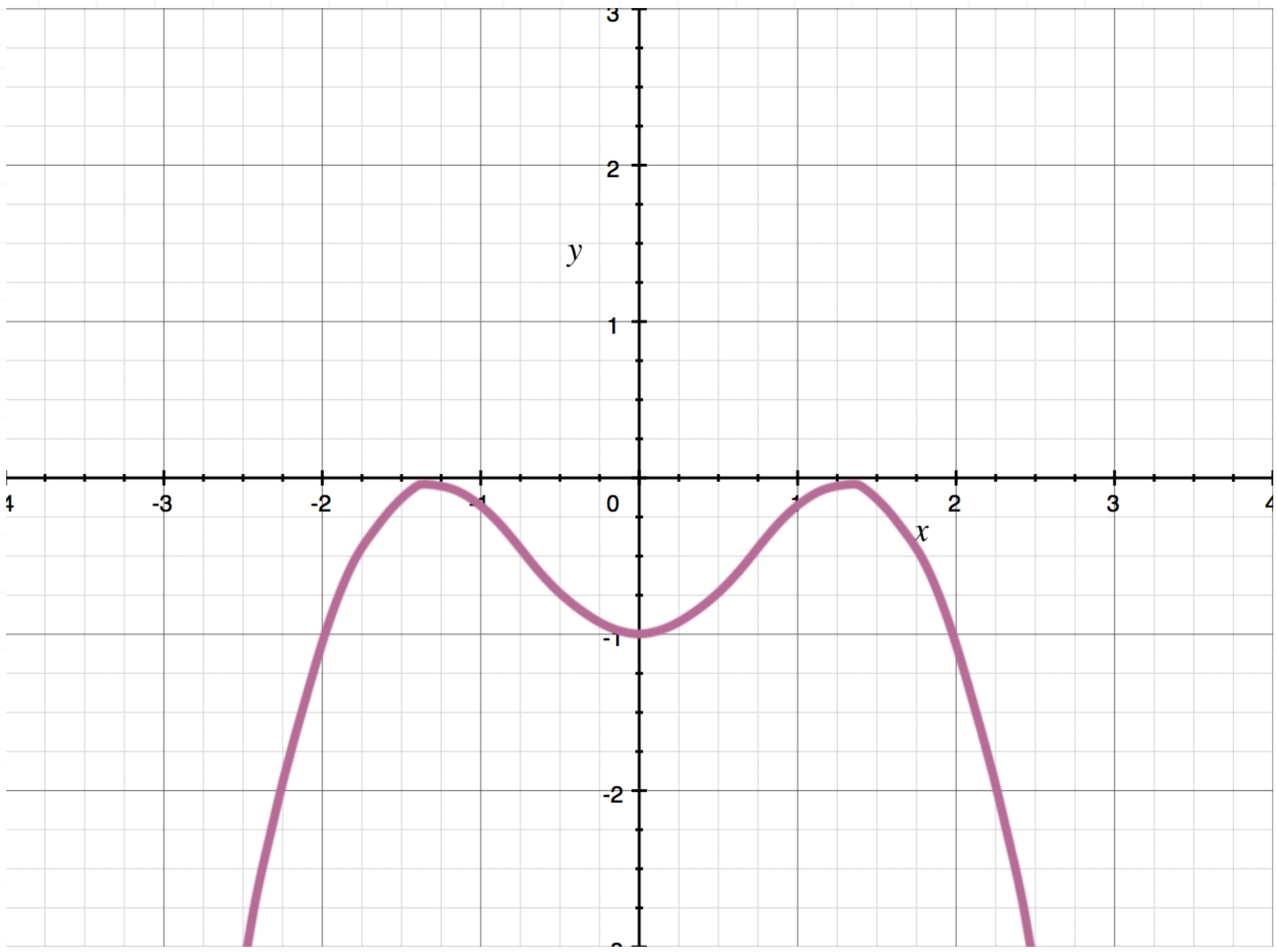
The graph of $f'(x)$ passes through the x-axis and changes sign from negative to positive at $x = 0$, which means that the graph of $f(x)$ has a minimum value at $x = 0$, and the graph of $f'(x)$ passes through the x-axis and changes sign from positive to negative at $x = 2$, which means that the graph of $f(x)$ has a maximum value at $x = 2$.

The graph of $f'(x)$ has a maximum value at $x = 1$, and its slope changes from positive to negative at that point. This means that the graph of $f(x)$ is concave up to the left of $x = 1$, has an inflection point at $x = 1$, and is concave down to the right of $x = 1$.

Putting these facts together, and based on the “assumption” that $f(x)$ contains the point $(0,0)$, this is a possible graph of $f(x)$:



■ 2. Sketch a possible graph of $g'(x)$ given the graph below of $g(x)$.



Solution:

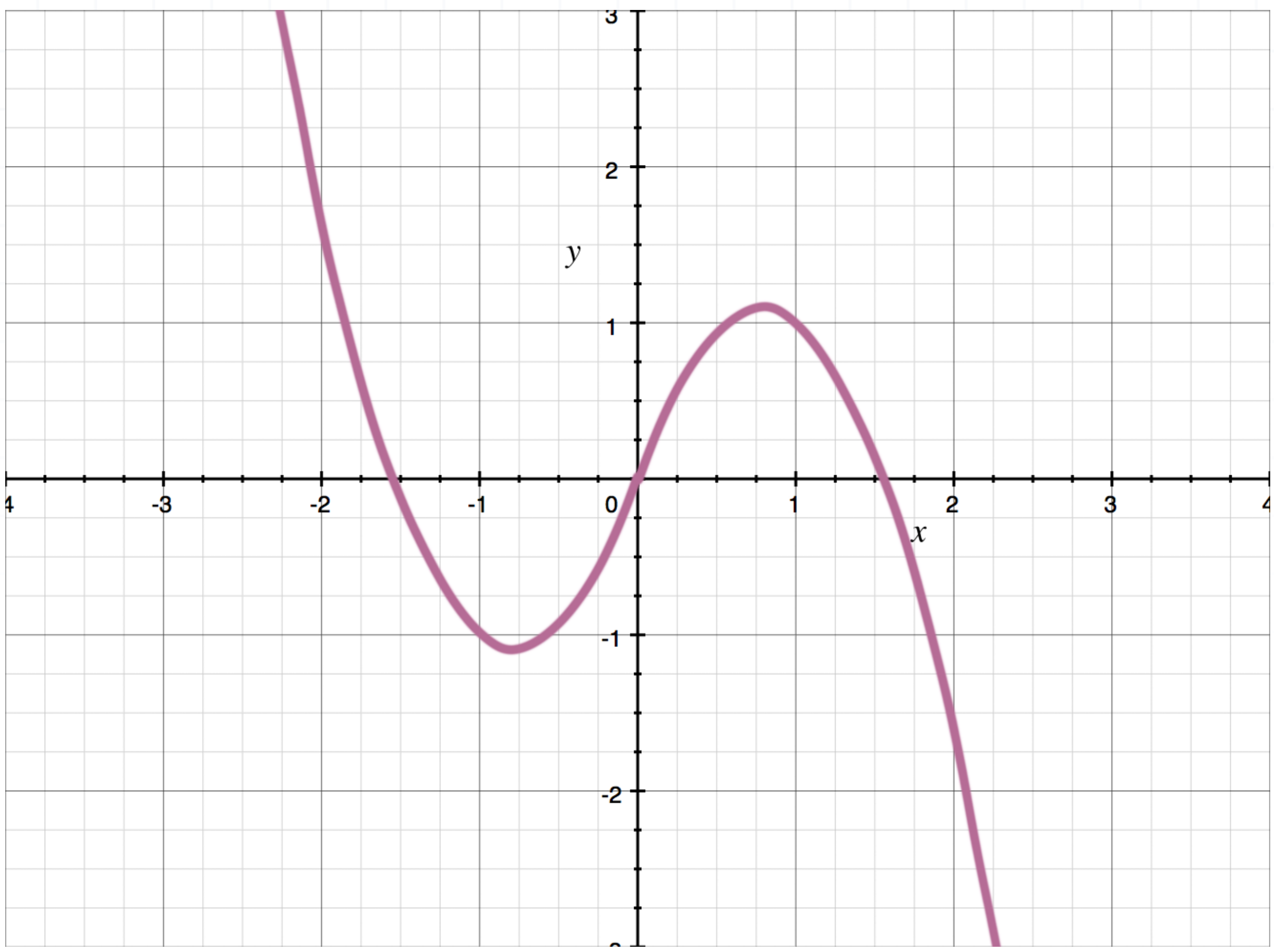
The graph of $g(x)$ has a positive slope on the intervals $(-\infty, -1.5)$ and $(0,1.5)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is above the x-axis on these intervals.

The graph of $g(x)$ has a negative slope on the intervals $(-1.5,0)$ and $(1.5,\infty)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is below the x-axis on these intervals.

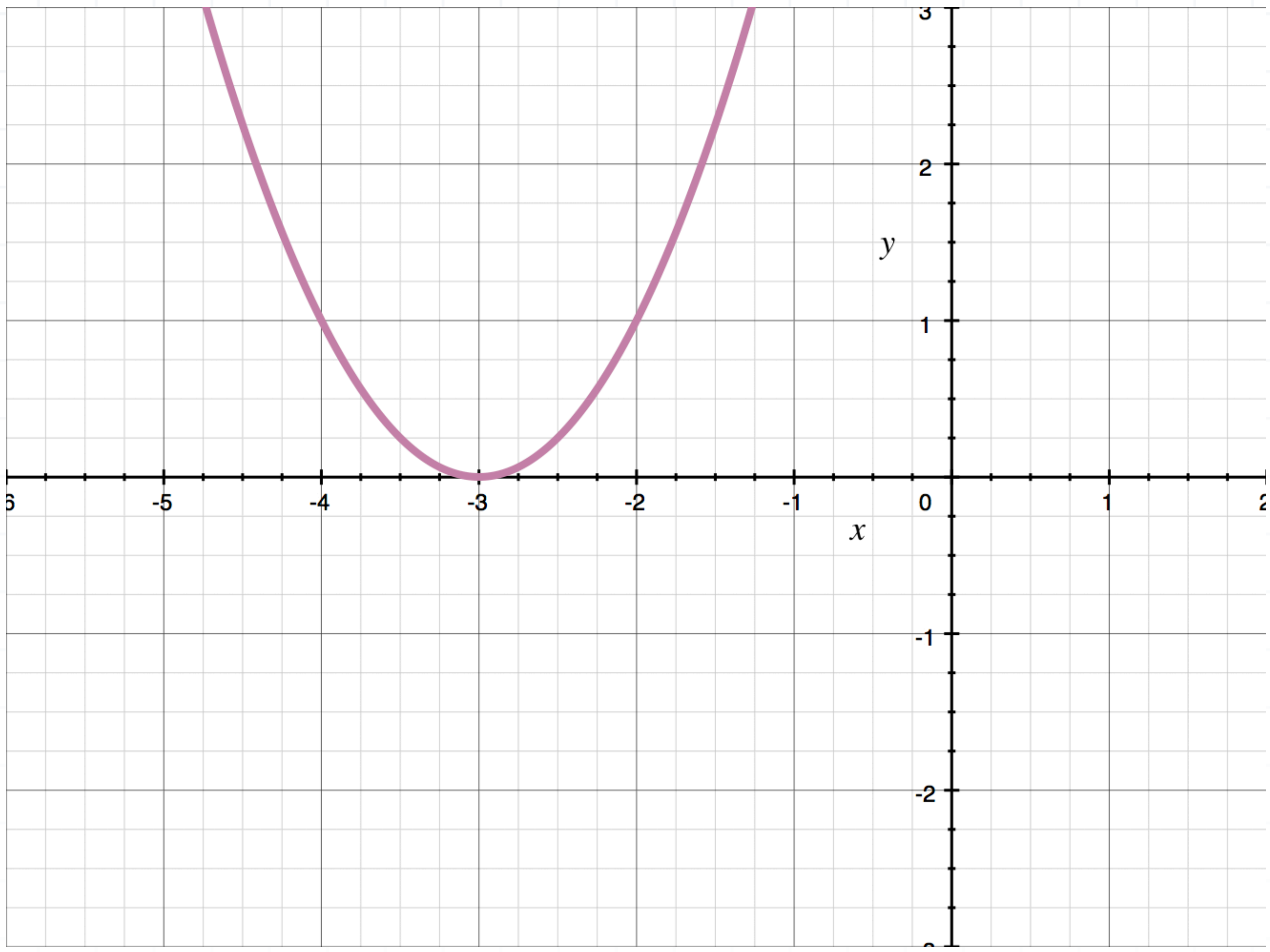
The graph of $g(x)$ has a maximum value at $x = -1.5$ and $x = 1.5$ and its slope is 0, so the graph of $g'(x)$ passes through the x-axis and changes sign from positive to negative at $x = -1.5$ and $x = 1.5$.

The graph of $g(x)$ has a minimum value at $x = 0$, and its slope changes from negative to positive at that point. This means that the graph of $g'(x)$ passes through the x-axis at $x = 0$, and changes from negative to positive.

It appears that the graph of $g(x)$ has an inflection point at $x = -0.75$ and $x = 0.75$, so the graph of $g'(x)$ has extrema at those points. Putting these facts together, this is a possible graph of $g'(x)$:



■ 3. Sketch a possible graph of $h(x)$ given the graph below of $h'(x)$.

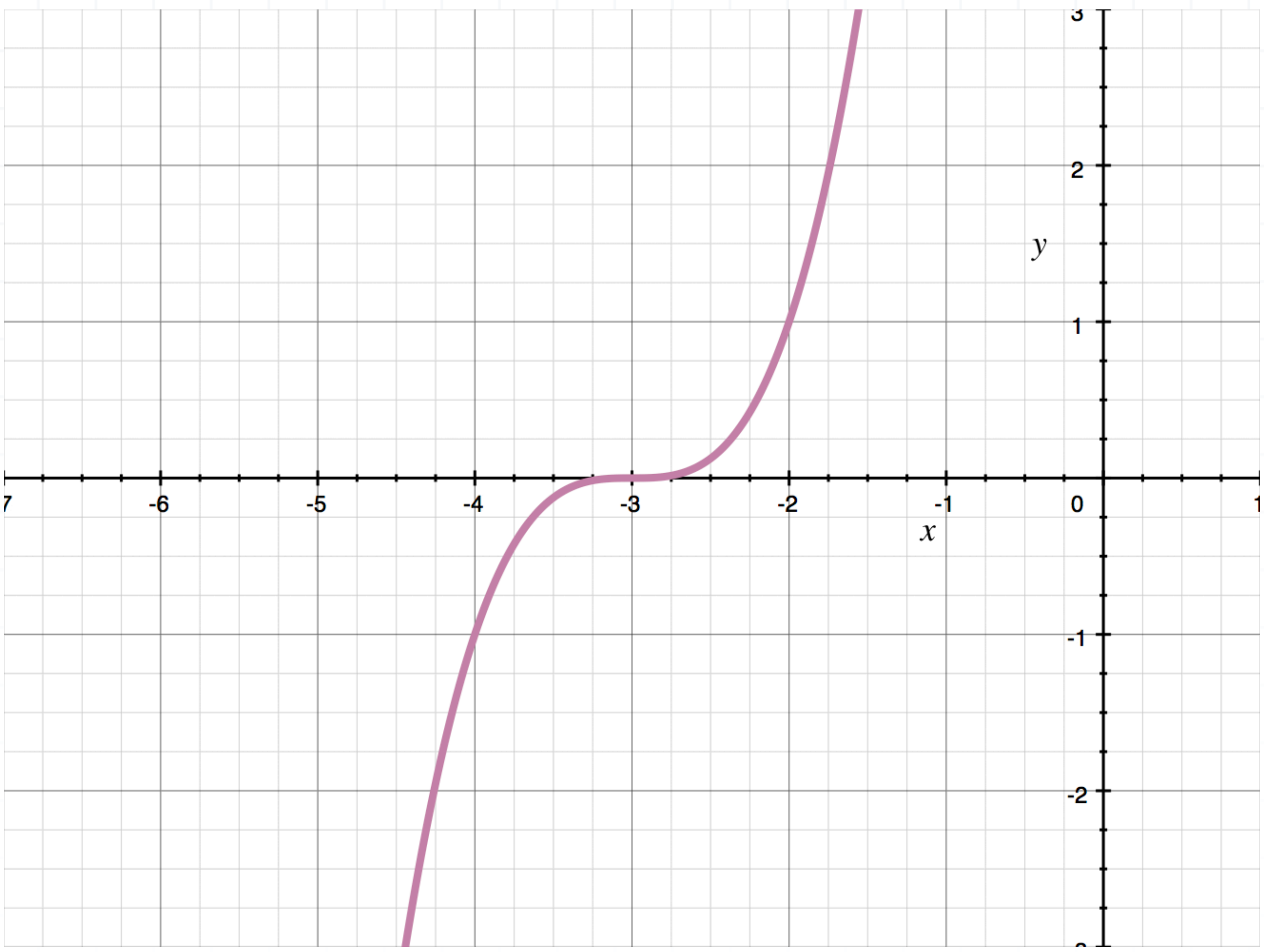


Solution:

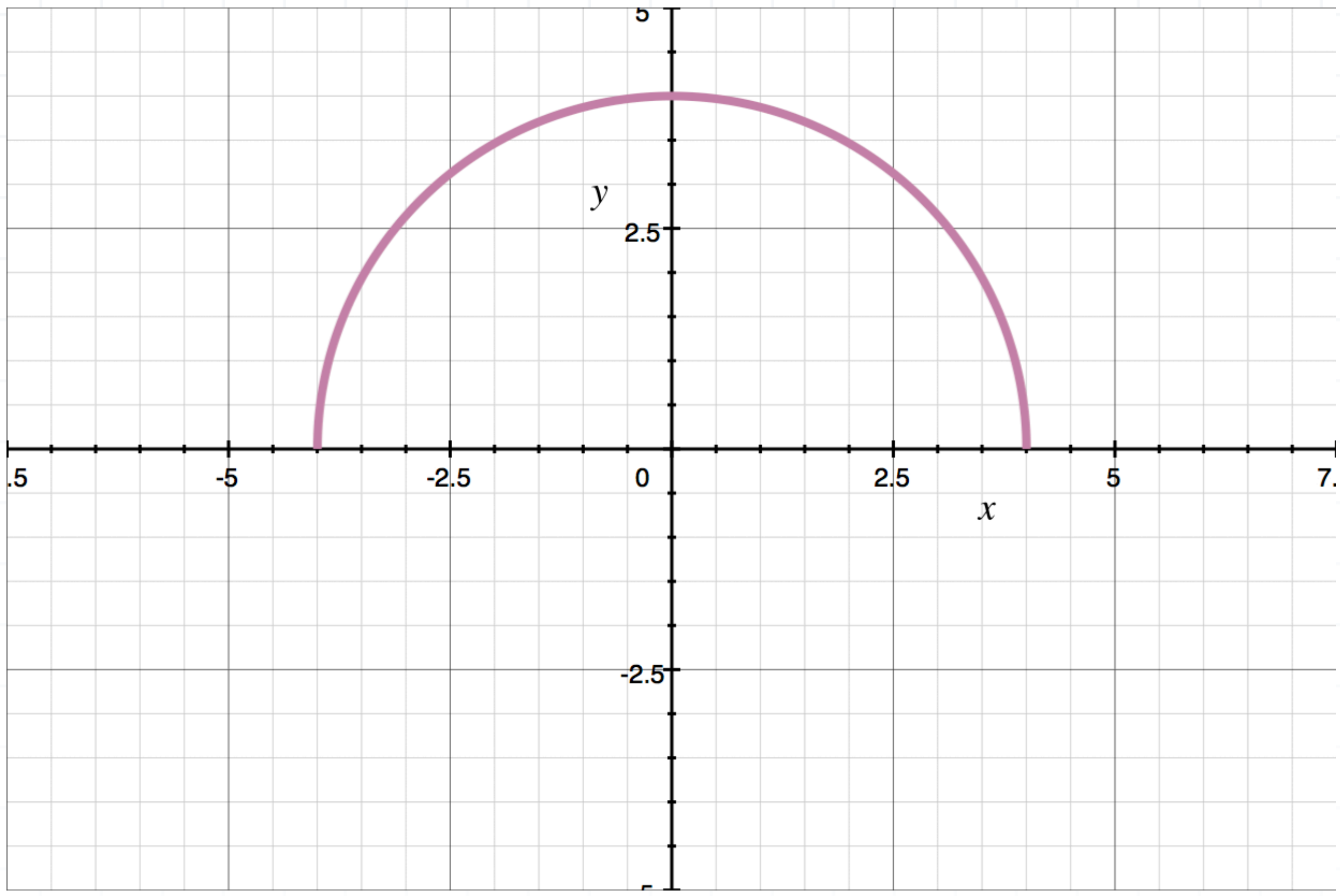
The graph of $h'(x)$ is above the x-axis on the intervals $(-\infty, -3)$ and $(-3,\infty)$, which means the function $h(x)$ has positive slopes and is increasing on these intervals. Since we're only excluding the single point $x = -3$, that means the function is essentially increasing everywhere.

The graph of $h'(x)$ is on the x-axis at $x = -3$, which means the function $h(x)$ has a horizontal tangent at $x = -3$ and is increasing on both sides of this point.

The graph of $h'(x)$ has a minimum value at $x = -3$, and its slope changes from positive to negative at that point. This means that the graph of $h(x)$ is concave down to the left of $x = -3$, has an inflection point at $x = -3$, and is concave up to the right of $x = -3$. Putting these facts together, this is a possible graph of $h(x)$:



■ 4. Sketch a possible graph of $f'(x)$ given the graph below of $f(x)$.



Solution:

The graph of $f(x)$ has a positive slope on the interval $(-4,0)$. Since $f'(x)$ is the derivative of $f(x)$, the graph of $f'(x)$ is above the x-axis on this interval.

The graph of $f(x)$ has a negative slope on the interval $(0,4)$. Since $f'(x)$ is the derivative of $f(x)$, the graph of $f'(x)$ is below the x-axis on this interval.

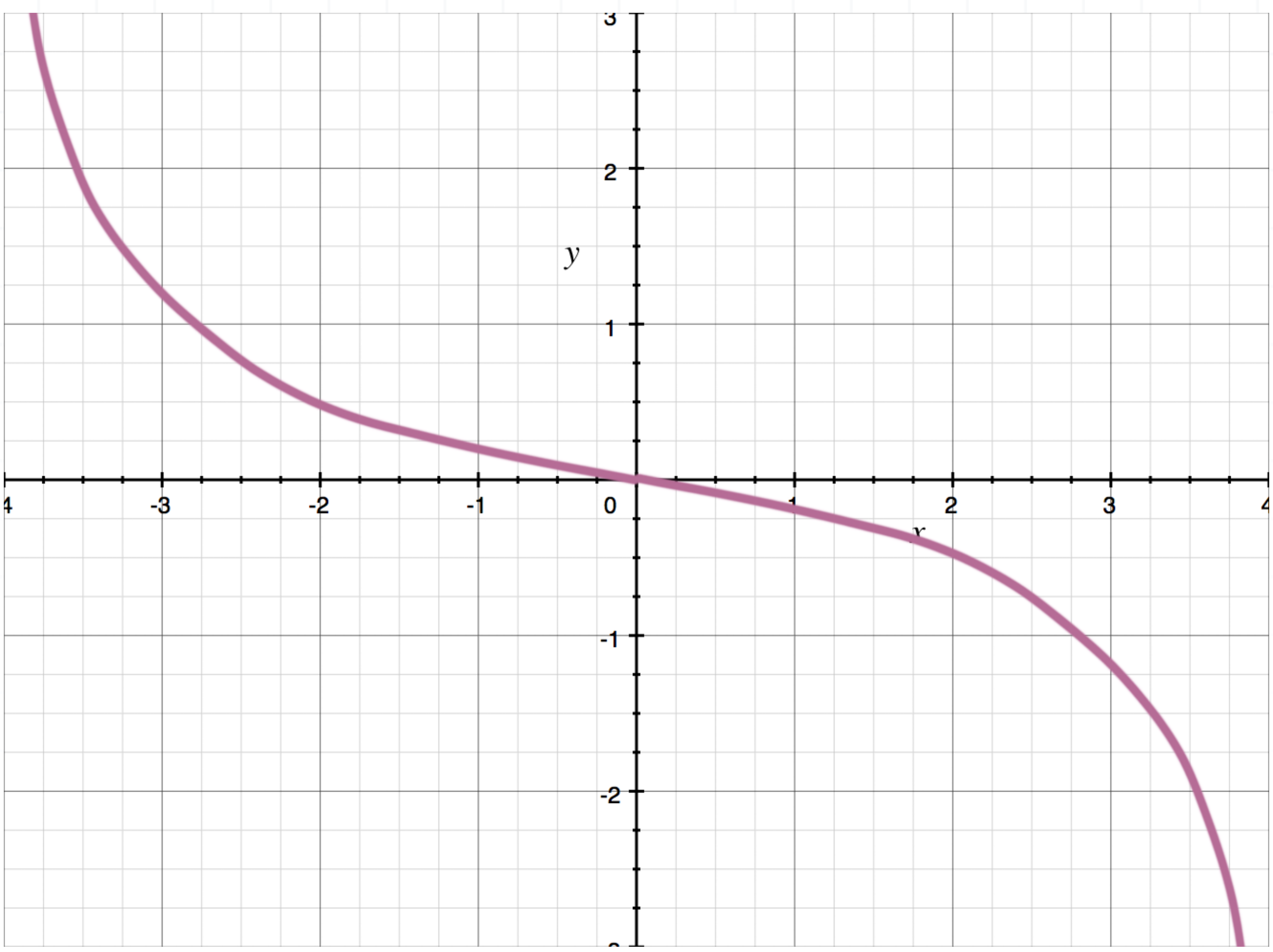
The graph of $f(x)$ has a maximum value at $x = 0$ and its slope is 0, so the graph of $f'(x)$ passes through the x-axis and changes sign from positive to negative at $x = 0$.

The graph of $f(x)$ has vertical tangents at $x = -4$ and $x = 4$. This means that the graph of $f'(x)$ has these limits:

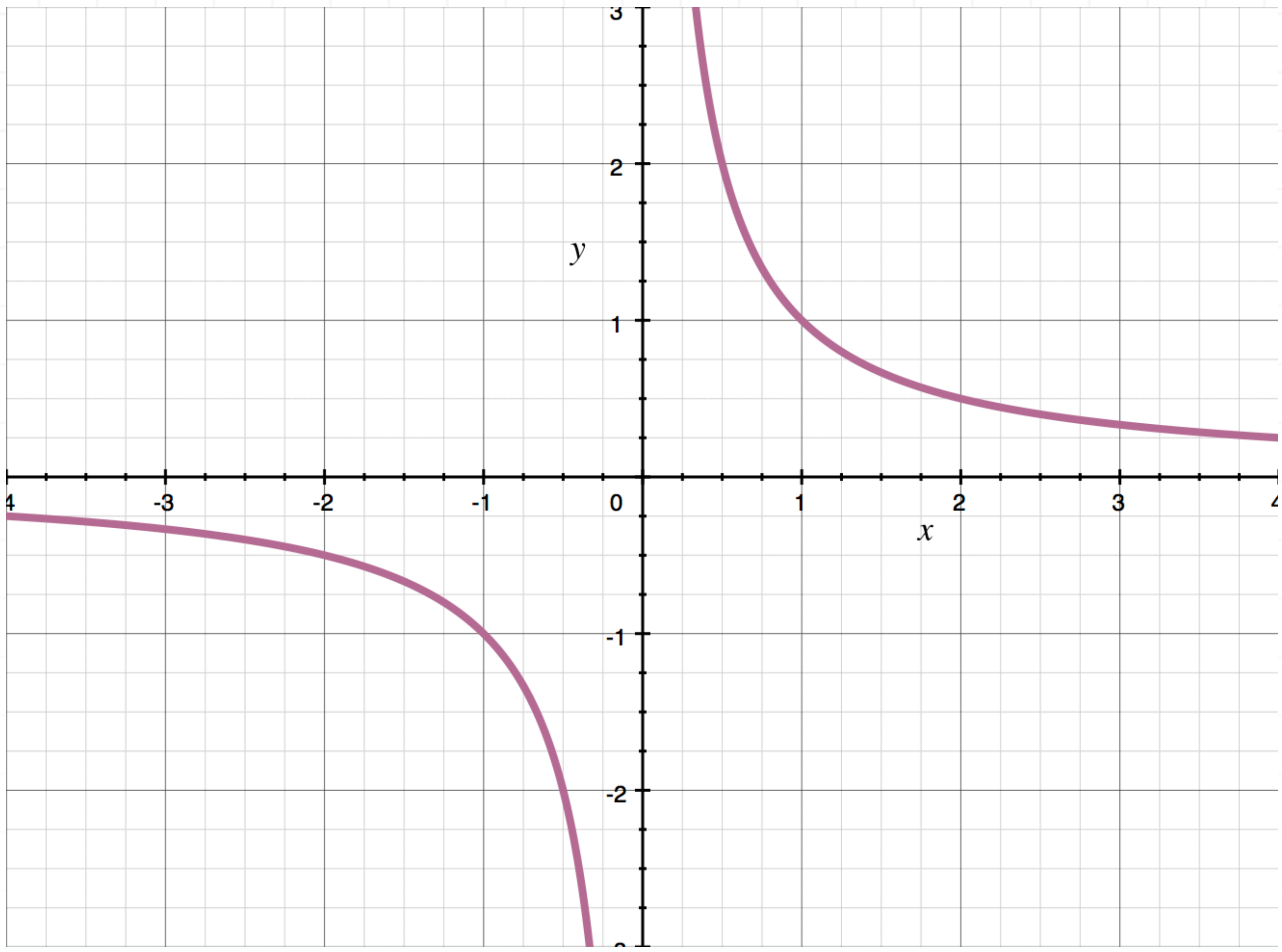
$$\lim_{x \to -4^+} f'(x) = \infty$$

$$\lim_{x\to 4^-} f'(x) = -\infty$$

The graph of $f(x)$ has no inflection points so the graph of $f'(x)$ has no extrema in the interval $(-4,4)$. Putting these facts together, this is a possible graph of $f'(x)$:



■ 5. Sketch a possible graph of $f(x)$ given the graph below of $f'(x)$.



Solution:

The graph of $f'(x)$ is below the x-axis on the interval $(-\infty,0)$, which means the function $f(x)$ has a negative slope and is decreasing on this interval.

The graph of $f'(x)$ is above the x-axis on the interval $(0,\infty)$, which means the function $f(x)$ has a positive slope and is increasing on this interval.

The graph of the $f'(x)$ does not pass through the x-axis, which means that the graph of $f(x)$ does not have any extrema.

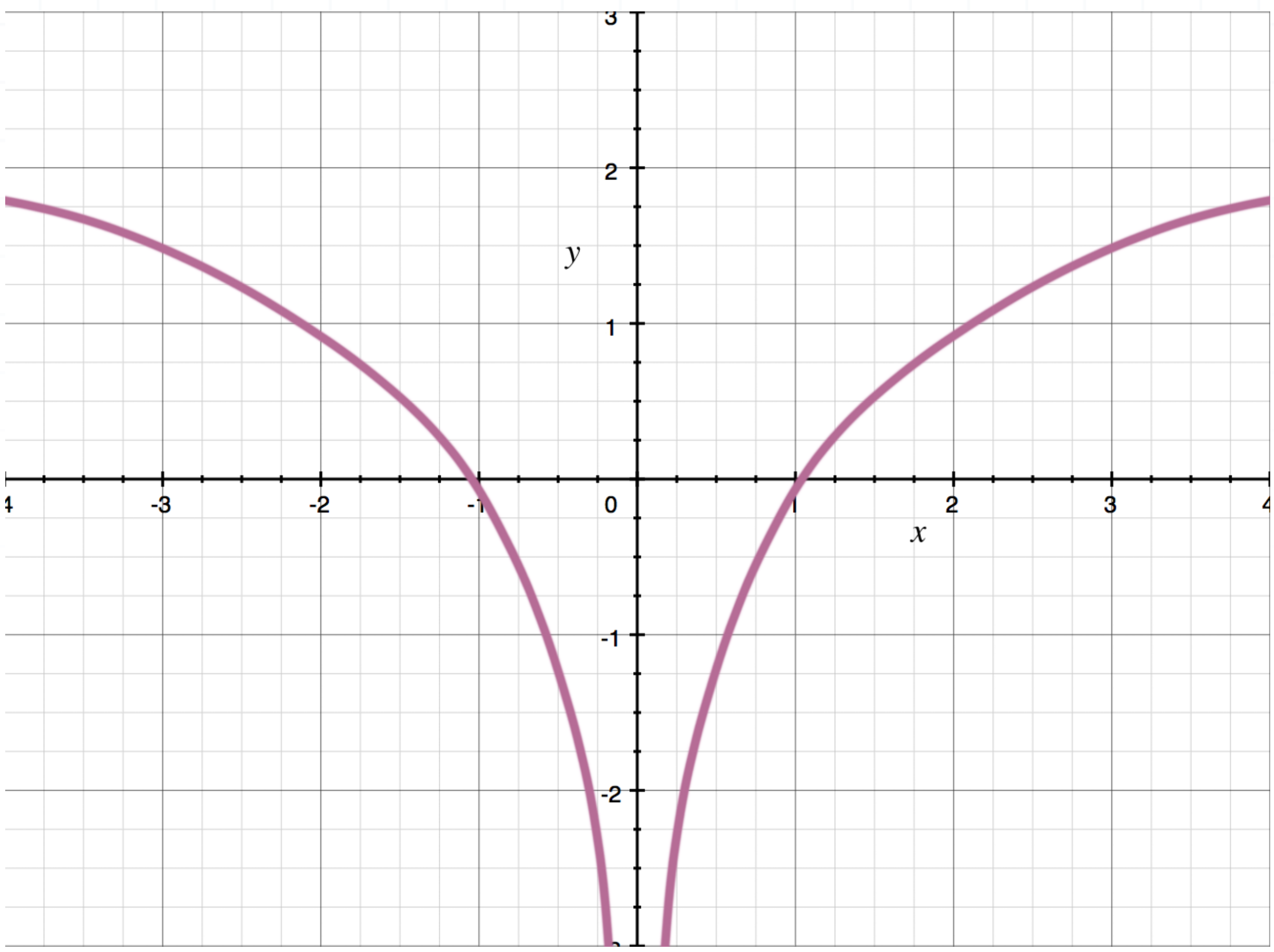
The slope of the graph of $f'(x)$ is negative on $(-\infty,0)$ and $(0,\infty)$. This means that the graph of $f(x)$ is concave down to the left and to the right of the y-axis.

The graph of $f'(x)$ has these limits:

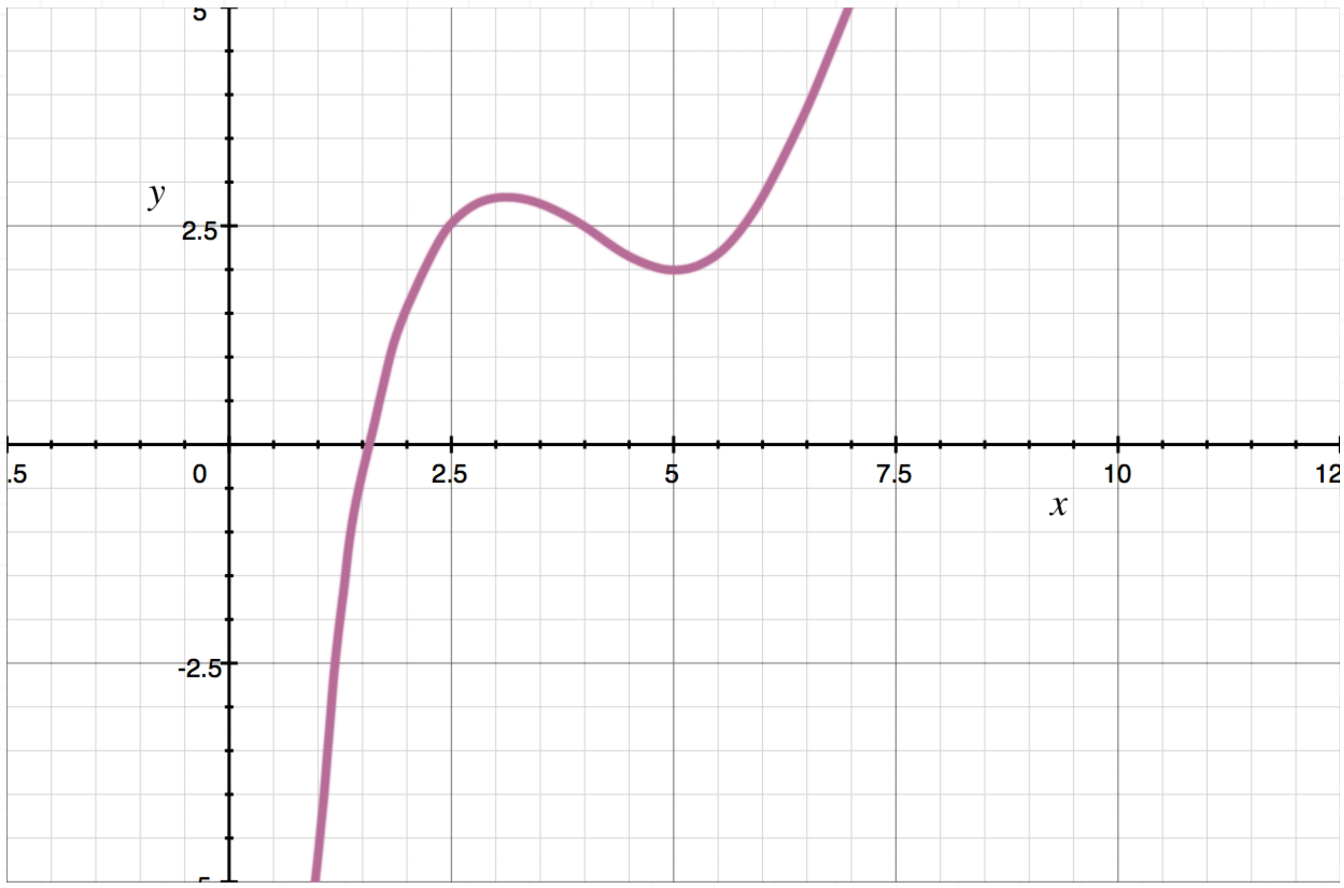
$$\lim_{x \to 0^-} f'(x) = -\infty$$

$$\lim_{x \to 0^+} f'(x) = \infty$$

This means the graph of $f(x)$ has an asymptote on the y-axis. Putting these facts together, this is a possible graph of $f(x)$:



■ 6. Sketch a possible graph of $g'(x)$ given the graph below of $g(x)$.



Solution:

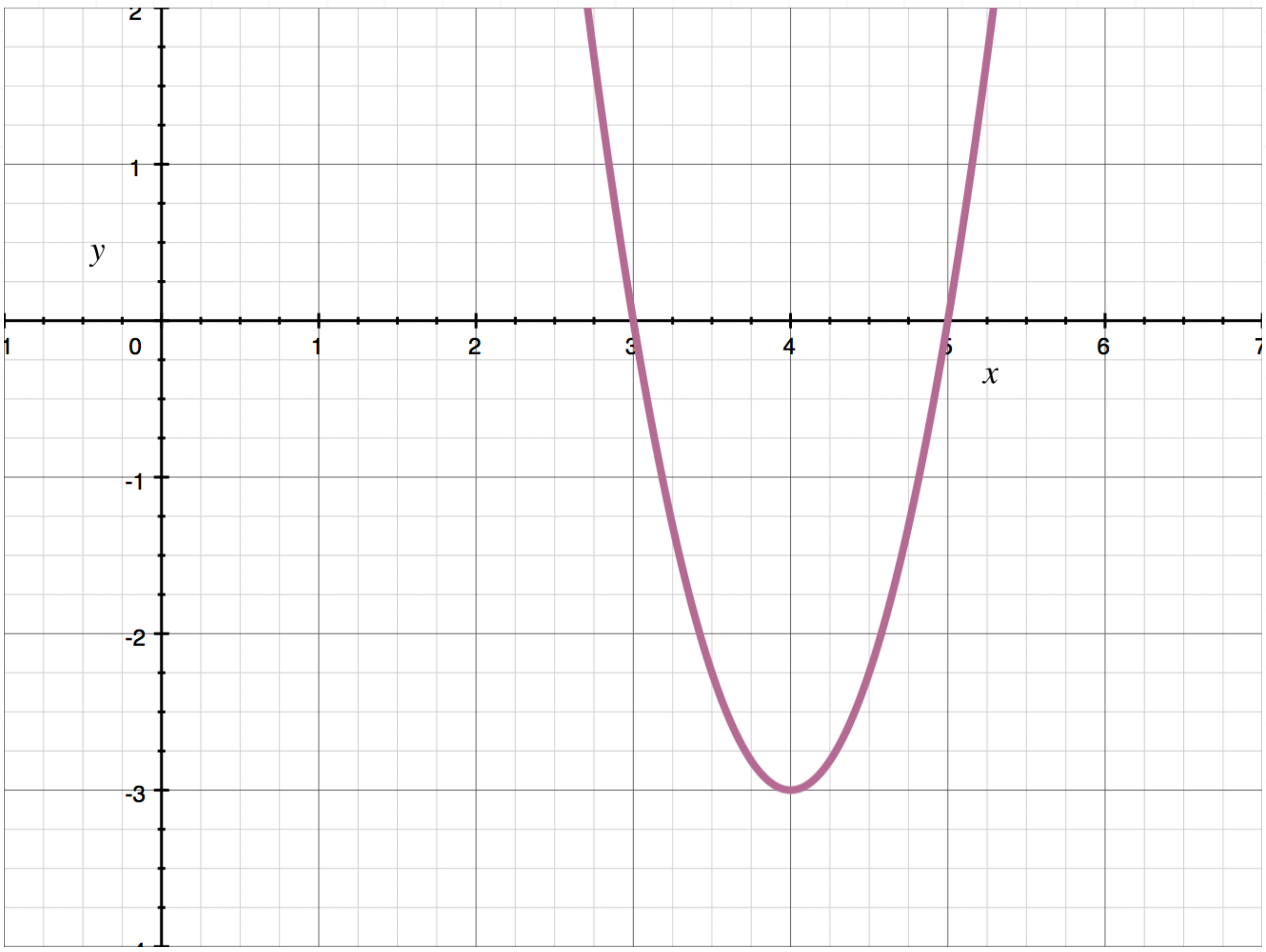
The graph of $g(x)$ has a positive slope on the intervals $(-\infty,3)$ and $(5,\infty)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is above the x-axis on these intervals.

The graph of $g(x)$ has a negative slope on the interval the $(3,5)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is below the x-axis on this interval.

The graph of $g(x)$ has a maximum value at $x = 3$ and its slope is 0, so the graph of $g'(x)$ passes through the x-axis and changes sign from positive to negative at $x = 3$.

The graph of $g(x)$ has a minimum value at $x = 5$, and its slope changes from negative to positive at that point. This means that the graph of $g'(x)$ passes through the x-axis at $x = 5$, and changes from negative to positive.

It appears that the graph of $g(x)$ has an inflection point at $x = 4$, so the graph of $g'(x)$ has extrema at $x = 4$. Putting these facts together, this is a possible graph of $g'(x)$:



W W W . K R I S T A K I N G M A T H . C O M