MAT 105 - Summary of Lecture 16
Thursday, March 3, 2016

The probable is what usually happens. (Aristotle)

Working with normal random variables

If $X = N(0,1)$ is a standard normal, recall that we use the z-table to determine probabilities. Using $\phi(a)$ for the probability that X is less than a, the following properties hold.

(a) $P(X \leq a) = \phi(a) =$ area under the bell curve to the left of a

(b) $P(X > a) = 1 - P(X \leq a) = 1 - \phi(a) =$ area under the bell curve to the right of a

(c) $P(X \leq -a) = \phi(-a) = 1 - \phi(a) =$ area under the bell curve to the left of $-a$

(d) $P(X > a) = P(X \leq -a) = 1 - \phi(a)$ since the bell curve for standard normal is symmetric about the $y-$axis.

(e) $P(X > -a) = P(X < a) = \phi(a)$ which follows from (d).

(f) $P(a \leq X \leq b) = \phi(b) - \phi(a) =$ area under the bell curve between a and b

If $X = N(\mu, \sigma^2)$ is a normal random variable with mean μ and variance σ^2, in order to use the z-table we turn X into a standard normal by subtracting the mean and dividing by standard deviation. That is,

$$Z = \frac{X - \mu}{\sigma}$$

is a standard normal variable, for which we can use the z-table.

Examples:

1. Let $X = N(0,1)$. Find:

   (a) $P(X \leq 0.31) = \phi(0.31) = .6217$

   (b) $P(X = 2) = 0$ since there is zero area under the normal curve at $x = 2$.

   (c) $P(X > 1.27) = 1 - \phi(1.27) = 1 - .8980 = .1020$

   (d) $P(X \leq -2.33) = \phi(-2.33) = 1 - \phi(2.33) = 1 - .9901 = .0099$

2. Let $X = N(1,4)$ and denote the standard normal by Z. To use the $z-$table, we normalize X by subtracting the mean 1 and dividing the the standard deviation $\sigma = 2$. That is, $Z = \dfrac{X-1}{2}$.

   (a) $P(X \leq 1.32) = P\left(\frac{X-1}{2} \leq \frac{1.32-1}{2}\right) = P(Z \leq .16) = \phi(.16) = .5636$

   (b) $P(X = 1) = 0$ since there is zero area under the normal curve at $x = 1$.

   (c) $P(X > 1) = P\left(\frac{X-1}{2} > \frac{1-1}{2}\right) = P(Z > 0) = 1 - \phi(0) = 1 - .5 = .5$

   (d) $P(X < 0.68) = P\left(\frac{X-1}{2} < \frac{0.68-1}{2}\right) = P(Z < -.16) = 1 - \phi(.16) = .4364$

   (e) $P(X > 0.68) = P\left(\frac{X-1}{2} > \frac{0.68-1}{2}\right) = P(Z > -.16) = \phi(.16) = .5636$

Normal approximation to the binomial

We have seen that a binomial random variable $Binomial(n, p)$ can be approximated by a Poisson random variable $Poisson(np)$ as long as n is large, p is small and np is moderate in size. One can use a normal random variable to approximate $Binomial(n, p)$ as well, without any restriction to the mean np. However, keep in mind that the larger the n, the better the approximation! Why does it make sense to approximate using a normal? If we look at a histogram for binomials, we observe that its shape is roughly a bell curve.

The DeMoivre-Laplace Theorem: If X denotes the number of successes that occur in n trials, each resulting in success with probability p, then $X = Binomial(n, p)$ and $X \approx N(np, np(1-p))$. More formally, if $X = Binomial(n, p)$,

$$P\left(a \leq \frac{X - np}{\sqrt{np(1-p)}} \leq b\right) \approx \phi(b) - \phi(a).$$

Remark: When we approximate the binomial by a normal with mean μ and variance σ^2, we let $\mu = np$ and $\sigma^2 = np(1-p)$. That is, the mean and variance of the binomial match those of the approximating normal. This is just a special case of the Central Limit Theorem, discussed later in the course.

Examples:

1. Suppose we flip a coin 100 times. Let X count the number of heads. Then $X = Binomial(100, 0.5)$. We can use a normal with mean $\mu = np = 100(0.5) = 50$ and variance $\sigma^2 = np(1-p) = 100(0.5)(0.5) = 25$ to approximate X. Let the approximating normal be called Y.

   (a) Find $P(X \geq 55)$. The exact probability is

   $$P(X = 55) + P(X = 56) + \cdots + P(X = 100) = \binom{100}{55}(.5)^{55}(.5)^{45} + \binom{100}{56}(.5)^{56}(.5)^{44} + \cdots + \binom{100}{100}(.5)^{100}(.5)^{0},$$

   which takes long to compute. So instead, we approximate the probability with

   $$P(X \geq 55) \approx P(Y \geq 55) = P\left(\frac{Y - 50}{5} \geq \frac{55 - 50}{5}\right) = P(Z \geq 1) = 1 - \phi(1) = 1 - .8413 = .1587$$

   (b) Find $P(X = 22)$. The exact probability is

   $$P(X = 22) = \binom{100}{22}(.5)^{22}(.5)^{78}.$$

   To approximate by a normal random variable, we "trap" 22 between 21.5 and 22.5:

   $$P(21.5 \leq X \leq 22.5) = P\left(\frac{21.5 - 50}{5} \leq Z \leq \frac{22.5 - 50}{5}\right) = \phi(-5.7) - \phi(-5.5) = (1 - .9999) - (1 - .9999) \approx 0$$

2. The ideal size of a first year class at a particular college is 150 students. The college, knowing from experience that on average only 30 percent of those accepted for admission will actually attend, uses a policy of approving the applications of 450 students. Compute the probability that more than 150 first-year students attend this college.

   Let X count the number of accepted students who attend. Then $X = Binomial(450, .3) \approx N(135, 94.5)$. The exact probability follows the binomial

   $$P(X > 150) = P(X = 151) + \cdots + P(X = 450) = \binom{450}{151}(.3)^{151}(.7)^{299} + \cdots + \binom{450}{450}(.3)^{450}(.7)^{0}.$$

   Approximating by the normal with mean 135 and variance 94.5,

   $$P(X > 150) \approx P\left(Z > \frac{150 - 135}{\sqrt{94.5}}\right) = P(z > 1.54) = 1 - \phi(1.54) = 1 - .9382 = .0618$$