MAT 105 - Summary of Lecture 8
Tuesday, February 2, 2016

> *While the individual man is an unsolvable puzzle, in the aggregate he becomes a mathematical certainty. You can, for example, never foretell what any one man will do, but you can say with precision what an average number will be up to.* (Sherlock Holmes)

Recall that the expectation (also called the mean or the expected value) is simply a weighted average of all possible values of a random variable. More precisely,

$$\mu = E[X] = \sum_{\text{all } a} a \cdot P(X = a).$$

Similarly, we defined the variance of a random variable to be

$$Var(X) = \sum_{\text{all } a} (a - \mu)^2 \cdot P(X = a).$$

Example:

(a) We built a game and our data shows the user attacks 20% of the time, moves around 30% of the time, defends 40% of the time and does nothing (or something else) 10% of the time. For all these actions, the user expects to get points. Depending on feedback from the user, we can vary the amount of points we award for each of these actions, with the restriction they have to add up to 60. Here are some possibilities:

Action	Probabilities	Play 1	Play 2	Play 3	Play 4	Play 5
Attack	.2	24	12	60	20	15
Move	.3	18	18	0	20	15
Defend	.4	12	24	0	20	15
Nothing	.1	6	6	0	0	15
Expected points		15.6	18	12	18	15

Let X = points won. On average, the player wins $E[X]$ points per play.

- in Play 1, $E[X] = 24(.2) + 18(.3) + 12(.4) + 6(.1) = 15.6$;
- in Play 2, $E[X] = 12(.2) + 18(.3) + 24(.4) + 6(.1) = 18$;
- in Play 3, $E[X] = 60(.2) + 0(.3) + 0(.4) + 0(.1) = 12$;
- in Play 4, $E[X] = 20(.2) + 20(.3) + 20(.4) + 0(.1) = 18$;
- in Play 5, $E[X] = 15(.2) + 15(.3) + 15(.4) + 15(.1) = 15$.

(b) In the middle of a game, we are at a crossroads.

(a) If we turn right, we can find a gold coin worth 1000 points, with probability 1/100 of finding it.

(b) turn left, we can find 2 silver coins, each worth 50 points, with probability 1/10 of finding each.

(c) turn north, we can find 2 bronze coins, each worth 40 points, with probability 1/10 of finding each.

(d) turn south, we can find 2 copper coins, each worth 60 points, with probability 1/10 of finding each.

Which way should we go? We want to maximize the expected points, so we find expectation for each play. Let X denote the winnings.

(a) $X \in \{0, 1000\}$ with $P(X = 1000) = 1/100$. Then

$$E[X] = 1000 \times (1/100) + 0 \times (99/100) = 10.$$

(b) $X \in \{0, 50, 100\}$ with $P(X = 0) = 81/100$ since no coin was found; $P(X = 50) = 18/100$ since one coin was found and one was not; $P(X = 100) = 1/100$ with both coins found. Then

$$E[X] = 0 \times (81/100) + 50 \times (18/100) + 100 \times (1/100) = 0 + 9 + 1 = 10.$$

(c) $X \in \{0, 40, 80\}$ with $P(X = 0) = 81/100$ since no coin was found; $P(X = 40) = 18/100$ since one coin was found and one was not; $P(X = 80) = 1/100$ with both coins found. Then

$$E[X] = 0 \times (81/100) + 40 \times (18/100) + 80 \times (1/100) = 0 + 7.2 + .8 = 8$$

(d) $X \in \{0, 60, 120\}$ with $P(X = 0) = 81/100$ since no coin was found; $P(X = 60) = 18/100$ since one coin was found and one was not; $P(X = 120) = 1/100$ with both coins found. Then

$$E[X] = 0 \times (81/100) + 60 \times (18/100) + 120 \times (1/100) = 10.8 + 1.2 = 12$$

Note that the probabilities in (b) – (c) are the same, only the payout changes. That change in payout makes it worth picking the route going south, even if the change in payout is not significant! Also note that the expected points gained by going left or right is the same!

MAT 105 - Group Work - February 2, 2016

1. Roll 2D6. Let X count the number of 6's.
   (a) List all possible values of X.

   (b) Find the mean $E[X]$.

   (c) Find the variance $Var(X)$
2. Roll 2D4. Let X record the sum of the dice.
   (a) List all possible values of X.

   (b) Find the distribution of X. That is, find $P(X = k)$ for all possible values k you found in (a).

   (c) Find the mean $E[X]$.
3. In a popular gambling game, three dice are rolled. For a \$1 bet you win \$1 for each six that appears (plus your dollar back). If no six appears you lose your dollar. What is your expected value?
4. You want to invent a gambling game in which a person rolls 2D6 and is paid some money if the sum is 7 but otherwise they lose their money. How much should you pay them for winning a \$1 bet if you want this to be a fair game, that is, the net profit should have an expected value of 0?

Independence

Informally, two events are independent if the outcome of one has no influence on the outcome of the other. Furthermore if we know that A has occurred, it gives us no extra information on the occurrence of B. Some events can be easily be seen as independent or dependent. For example, if we flip two fair coins, the outcome of the first coin does not influence in any way the outcome of the second coin. However, if we remove cards from a deck of 52, one at a time, without replacement, the outcome of the second card will be influenced by what we removed from the deck before. Sometimes, we can not intuitively determine if two events are independent, in which case we use the following formal definition to decide.
Definition: We say the events A and B are **independent** if

$$P(A \cap B) = P(A)P(B).$$

If two events are *not independent* we call the **dependent.**
Remark: independence will play a big role in how we compute probabilities in multi-step experiments, such a flipping multiple coins, drawing cards without replacement from a deck etc.
Examples:

1. We roll two dice. Let A = first die lands on 6, B = second die lands on 6. Intuitively, the events are independent. Let us verify:

$$P(A) = 1/6, \;\; P(B) = 1/6, \;\;\; P(A \cap B) = 1/36 \;\; \Rightarrow \;\; P(A \cap B) = P(A)P(B).$$

2. Draw a card from a deck. Let A = the card is an ace, B = the card is a spade. Then $P(A) = 1/13$, $P(B) = 1/4$ and

$$P(A \cap B) = P(\text{the card is the ace of spade}) = 1/52.$$

   Thus, $P(A \cap B) = P(A)P(B)$, so A and B are independent.

3. We draw two cards from a deck of 52. Let B = second card is a spade.
   (a) Let A = the first card is an ace. Then $P(A) = 1/13$ and

$$\begin{aligned} P(B) &= P(\text{1st card} = \spadesuit, \text{2nd card} = \spadesuit) + P(\text{1st card} \neq \spadesuit, \text{2nd card} = \spadesuit) \\ &= \frac{13 \times 12}{52 \times 51} + \frac{39 \times 13}{52 \times 51} = \frac{1}{4}. \end{aligned}$$

   On the other hand,

$$\begin{aligned} P(A \cap B) &= P(\text{1st card} = \text{ace } \spadesuit, \text{2nd card} = \spadesuit) + P(\text{1st card} = \text{ace, not } \spadesuit, \text{2nd card} = \spadesuit) \\ &= \frac{1 \times 12}{52 \times 51} + \frac{3 \times 13}{52 \times 51} = \frac{1}{52}. \end{aligned}$$

   We check that $P(A \cap B) = P(A)P(B)$, so A and B are independent.

   (b) Let A = the first card is a spade. Then $P(A) = 1/4$ and $P(B) = 1/4$ from (a), but

$$P(A \cap B) = P(\text{1st card} = \spadesuit, \text{2nd card} = \spadesuit) = \frac{13 \times 12}{52 \times 51} = \frac{1}{17}.$$

   Thus, $P(A \cap B) \neq P(A)P(B)$ so A and B are dependent.