MAT 105 - Summary of Lecture 10
Thursday, February 4, 2016

I know too well that these arguments from probabilities are imposters, and unless great caution is observed in the use of them, they are apt to be deceptive. (Plato)

More on independence

Recall that two events are independent if the outcome of one has no influence on the outcome of the other. Furthermore if we know that A has occurred, it gives us no extra information on the occurrence of B. Formally, independence is defined as follows.

Definition: We say the events A and B are **independent** if

$$P(A \cap B) = P(A)P(B).$$

If two events are *not independent* we call the **dependent.**

Example: We roll two dice. Let $A =$ first die lands on 6. Then $P(A) = 1/6$.
(a) Let $B =$ the sum of the dice is 7. Then $B = \{16, 25, 34, 43, 52, 61\}$ and $A \cap B = \{61\}$

$$P(A) = 1/6, \;\; P(B) = 6/36, \;\; P(A \cap B) = 1/36 \;\; \Rightarrow \;\; P(A \cap B) = P(A)P(B).$$

So A and B are independent.

(b) Let $B =$ the sum of the dice is 9. Then $B = \{36, 45, 54, 63\}$ and $A \cap B = \{63\}$

$$P(A) = 1/6, \;\; P(B) = 4/36, \;\; P(A \cap B) = 1/36 \;\; \Rightarrow \;\; P(A \cap B) \neq P(A)P(B).$$

So A and B are dependent in this case.

We can define independence for more than 2 sets in the following way.

Definition: If $A_1, A_2, \cdots, A_n$ are n events, they are independent if

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2) \cdots P(A_n).$$

The events are pairwise independent if $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all pairs $i \neq j$.

Remark: If a set of events are independent, then they must also be pairwise independent. There could be events that are pairwise independent but not independent.

Examples:

1. Flip 20 coins. Let A_k denote the event the kth coin land on heads. These events are independent because

$$P(A_1) = \cdots = P(A_{20}) = \frac{1}{2}, \;\; P(A_1 \cap A_2 \cap \cdots \cap A_{20}) = \frac{1}{2^{20}} \;\; \Rightarrow P(A_1 \cap \cdots \cap A_{20}) = P(A_1)P(A_2) \cdots P(A_{20}).$$

2. Let A=Al and Bob share a birthday, B=Bob and Cate share a birthday, and C=Cate and Al share a birthday. Then

$$P(A) = P(B) = P(C) = \frac{365}{365^2} = \frac{1}{365}$$

since there are 365 possible days for the common birthday and the number of ways for two people to be born is 365×365 (365 days possible for each). Now

$$P(A \cap B \cap C) = P(A \cap B) = P(B \cap C) = P(C \cap A) = \frac{365}{365^3} = \frac{1}{365^2}.$$

This follows from the fact that $A \cap B \cap C = A \cap B = B \cap C = C \cap A$ all mean "Al, Bob and Cate share a birthday". Therefore, these events are pairwise independent because

$$P(A \cap B) = P(A)P(B), \;\; P(A \cap C) = P(A)P(C), \;\; P(B \cap C) = P(B)P(C),$$

but they are not independent because $P(A \cap B \cap C) \neq P(A)P(B)P(B)$.

MAT 105 - Group Work - Solution

1. Roll 2D6. Let X count the number of 6's.
   (a) List all possible values of X.
   (b) Find the mean $E[X]$.
   (c) Find the variance $Var(X)$

   Solution:

   (a) $X \in \{0, 1, 2\}$

$$\begin{aligned} P(X = 0) &= P(\text{no } 6) = \frac{25}{36} \\ P(X = 1) &= P(\text{one } 6) = \frac{10}{36} \\ P(X = 2) &= P(\text{two } 6) = \frac{1}{36}. \end{aligned}$$

   Note that these probabilities add up to 1, as they should!

   (b) $E[X] = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 2 \cdot P(X = 2) = 0 \cdot \frac{25}{36} + 1 \cdot \frac{10}{36} + 2 \cdot \frac{1}{36} = \frac{12}{36} = \frac{1}{3}.$

   (c)

$$\begin{aligned} Var(X) &= \left(0 - \frac{1}{3}\right)^2 \cdot P(X = 0) + \left(1 - \frac{1}{3}\right)^2 \cdot P(X = 1) + \left(2 - \frac{1}{3}\right)^2 \cdot P(X = 2) \\ &= \frac{1}{9} \cdot \frac{25}{36} + \frac{4}{9} \cdot \frac{10}{36} + \frac{25}{9} \cdot \frac{1}{36} = \frac{5}{18} \end{aligned}$$

2. Roll 2D4. Let X record the sum of the dice.
   (a) List all possible values of X.
   (b) Find the distribution of X. That is, find $P(X = k)$ for all possible values k you found in (a).
   (c) Find the mean $E[X]$.

Solution:

(a) $X \in \{2, 3, 4, 5, 6, 7, 8\}$

(b)

$$\begin{aligned}
P(X = 2) &= P(11) = \frac{1}{16} \\
P(X = 3) &= P(12 \text{ or } 21) = \frac{2}{16} \\
P(X = 4) &= P(13 \text{ or } 22 \text{ or } 31) = \frac{3}{16} \\
P(X = 5) &= P(14 \text{ or } 23 \text{ or } 32 \text{ or } 41) = \frac{4}{16} \\
P(X = 6) &= P(24 \text{ or } 33 \text{ or } 42) = \frac{3}{16} \\
P(X = 7) &= P(34 \text{ or } 43) = \frac{2}{16} \\
P(X = 8) &= P(44) = \frac{1}{16}.
\end{aligned}$$

Note that these probabilities add up to 1.

(c) $E[X] = 2 \cdot \frac{1}{16} + 3 \cdot \frac{2}{16} + 4 \cdot \frac{3}{16} + 5 \cdot \frac{4}{16} + 6 \cdot \frac{3}{16} + 7 \cdot \frac{2}{16} + 8 \cdot \frac{1}{16} = \frac{80}{16} = 5.$

3. In a popular gambling game, three dice are rolled. For a \$1 bet you win \$1 for each six that appears (plus your dollar back). If no six appears you lose your dollar. What is your expected value?

Solution: Let X stand for the net winnings. Then $X \in \{-1, 1, 2, 3\}$ and

$$\begin{aligned}
P(X = -1) &= P(\text{no } 6) = \frac{5 \times 5 \times 5}{6 \times 6 \times 6} \\
P(X = 1) &= P(\text{one } 6) = \frac{3 \times 5 \times 5}{6 \times 6 \times 6} \\
P(X = 2) &= P(\text{two } 6) = \frac{3 \times 5}{6 \times 6 \times 6} \\
P(X = 3) &= P(\text{three } 6) = \frac{1}{6 \times 6 \times 6}
\end{aligned}$$

Then we compute the expectation to be

$$E[X] = (-1) \cdot \frac{125}{216} + 1 \cdot \frac{75}{216} + 2 \cdot \frac{15}{216} + 3 \cdot \frac{1}{216} = -\frac{17}{216}$$

4. You want to invent a gambling game in which a person rolls 2D6 and is paid some money if the sum is 7 but otherwise they lose their money. How much should you pay them for winning a \$1 bet if you want this to be a fair game, that is, the net profit should have an expected value of 0?

Solution: Let X be the net winnings. Then

$$P(X = -1) = P(\text{sum is not } 7) = 1 - P(\text{sum is } 7) = 1 - P(16,\ 25,\ 34,\ 43,\ 52, \text{ or } 61) = 1 - \frac{6}{36} = \frac{5}{6}.$$

Let us denote the desired payout by w. Then we want

$$0 = E[X] = (-1) \cdot \frac{5}{6} + w \cdot \left(1 - \frac{5}{6}\right) \quad \Rightarrow \quad w = \frac{5/6}{1/6} = 5.$$

Therefore, to make this a fair game, we need a payout of \$5.