MAT 105 - Summary of Lecture 8
Thursday, January 28, 2016

> *The excitement that a gambler feels when making a bet is equal to the amount he might win times the probability of winning it.* (Blaise Pascal)

Random Variables

Recall that a **random variable** $X : S \to \mathbb{R}$ is a function that takes on values from the set of outcomes and maps into the reals.

Examples:

1. (a) You flip a coin. If the coin lands on Heads, you win \$5 and if the coin lands on Tails, you lose \$1. Let $X =$ the net amount of money won in this game. Then we define

$$X(H) = 5 \qquad\qquad X(T) = -1$$

We write $X \in \{-1, 5\}$ to mean that X can take on the values 5 or -1. We want to find probabilities for X taking on each value:

$$P(X = 5) = P(\text{Heads}) = 1/2, \qquad P(X = -1) = P(\text{Tails}) = 1/2.$$

(b) You roll 1D6. If the die lands on 6, you win \$5 and if it lands on a value other than 6, you lose \$1. Let $X =$ the net amount of money won in this game. Then we define

$$X(\text{land on } 6) = 5 \qquad\qquad X(\text{not land on } 6) = -1$$

Then $X \in \{-1, 5\}$ and

$$P(X = 5) = P(\text{lands on } 6) = 1/6, \qquad P(X = -1) = P(\text{not land on } 6) = 5/6.$$

Is it advantageous to play this game?

2. We play the following game. You pay \$3 at the start of the game. You flip two coins and for each Heads, you receive \$4. Let $X =$ the net amount of money won in this game. Then

$$\begin{aligned} X(HH) &= -3 + 4 + 4 = 5 \qquad & X(HT) &= -3 + 4 = 1 \\ X(TH) &= -3 + 4 = 1 & X(TT) &= -3 \end{aligned}$$

So $X \in \{-3, 1, 5\}$ and

$$P(X = -3) = P(TT) = \frac{1}{4}, \quad P(X = 1) = P(HT \text{ or } TH) = \frac{1}{2}, \quad P(X = 5) = P(HH) = \frac{1}{4}.$$

How about in this game, what are the expected winnings?

3. Roll 1D6. Let X =the number appearing on the die. Then $X \in \{1,2,3,4,5,6\}$ and its distribution is given by

$$P(X=1)=P(X=2)=P(X=3)=P(X=4)=P(X=5)=P(X=6)=1/6.$$

   Note that these probabilities add up to 1.

4. Let X denote the number of siblings for a DigiPen student. We survey a random sample of 40 students and get the following frequencies and then find the probability for each value:

$a=$ #sibilings	Frequency
0	8
1	17
2	11
3	3
4	1
	40

$\Rightarrow$

#sibilings $= a$	Frequency	$P(X=a)$
0	8	8/40
1	17	17/40
2	11	11/40
3	3	3/40
4	1	1/40
	40	1

   Again, note that the probabilities add up to 1.

5. We toss 3 coins. Let X denote the number of heads. Then $X \in \{0,1,2,3\}$ and

$$\begin{aligned} P(X=0) &= P(\text{all tails}) = 1/8 \\ P(X=1) &= P(\text{1 head, 2 tails}) = 3/8 \\ P(X=2) &= P(\text{2 heads, 1 tail}) = 3/8 \\ P(X=3) &= P(\text{all heads}) = 1/8 \end{aligned}$$

   What is the probability of at most 2 heads?

$$P(X \leq 2) = P(X=0)+P(X=1)+P(X=2) = \frac{1}{8}+\frac{3}{8}+\frac{3}{8} = \frac{7}{8}.$$

   Another way to find this probability is to observe that

$$P(X \leq 2) = 1 - P(X=3) = 1 - \frac{1}{8} = \frac{7}{8}.$$

Remarks:

- $\sum_{\text{all } a} P(X=a) = 1$, that is the sum of all probabilities add up to 1.
- If one draws a bar graph (or histogram) for $P(X=a)$, the graph will approximate the frequency histogram obtained by playing the game many times. (This is supported by the Law of Large Numbers, to be discussed later in the course.)

Expectation, variance, standard deviation

One question to consider is what is the payout for various games, or how many successes should one expect, on average, when playing a game? These quantities are given by the expectation of a random variable. Informally, think of expectation as a weighted average.

For example, one uses a weighted average when computing grades. Suppose your grades on exams etc. in this class are listed in the 3rd column in the following table (each score out of 100 possible). You want to compute your grade in the class, but each grade category counts as a different percentage (weight) in the grade. Here is your final grade in the course:

Criteria	%	Your score (100)	Points
Midterm	20	90	$(90) * (.2) = 18$
Final	30	80	$(80) * (.3) = 24$
Homework	20	100	$(100) * (.2) = 20$
Quiz	20	95	$(95) * (.2) = 19$
Group	10	100	$(100) * (.1) = 10$
Total	100		**91**

Formally, we define expectation (also known as mean or expected value) as follows.

Def: The **expectation** of a discrete random variable X (also called the mean or the expected value) is

$$\mu = E[X] = \sum_{\text{all } a} a \cdot P(X = a).$$

Remark: In a large number of independent observations of the random variable X, the average value of these observations will be approximately $E[X]$. More precisely, if X_1 is the 1st observation, X_2 is the 2nd observation, etc, X_n is the nth observation, then the Law of Large Numbers suggests

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \approx E[X].$$

To measure the spread of values away from the mean, we use the variance and standard deviation, defined similarly to the case of populations, but now the average is weighted, not just the arithmetic mean.

Def: The **variance of** X, denoted by $Var(X)$ or σ^2 is defined as

$$\sigma^2 = Var(X) = E[(X - E[X])^2] = \sum_{\text{all } a} (a - \mu)^2 P(X = a).$$

Note that this formula is the same as that for variance of populations, when we had n outcomes, labeled $x_1, x_2, \cdots, x_n$, each occurring with probability $1/n$:

$$Var(X)_{\text{population}} = \frac{(x_1 - \mu)^2 + \cdots + (x_n - \mu)^2}{n}.$$

Def: The **standard deviation of** X is then defined as the square root of variance $\sigma = \sqrt{Var(X)}$.

Properties: For any random variable X,

- the expectation, variance and standard deviation are constants, they are non-random!
- since the variance is an average of sums of squares, $Var(X) \geq 0$.
- if c is some constant, $E[c] = c$
- if c is some constant, $E[cX] = cE[X]$
- if c is some constant, $E[X + c] = E[X] + c$
- the variance can be expressed as

$$Var(X) = E[X^2] - (E[X])^2 = \left(\sum_{\text{all } a} a^2 P(X = a) \right) - \left(\sum_{\text{all } a} aP(X = a) \right)^2.$$

  This might be an easier formula to use in computations.
- if c is some constant, $Var(c) = 0$
- if c is some constant, $Var(X + c) = Var(X)$
- if c is some constant, $Var(cX) = c^2 X$

Examples: Expectation and standard deviation for all random variables in examples above:

1. (a) Get \$5 for heads and lose \$1 for tails. Then $P(X = 5) = 1/2$ and$P(X = -1) = 1/2$, so

$$\begin{aligned} E[X] &= (5)(1/2) + (-1)(1/2) = 4/2 = 2. \\ Var(X) &= (5-2)^2(1/2) + (-1-2)^2(1/2) = 9 \\ \sigma &= \sqrt{Var(X)} = 3 \end{aligned}$$

   This means that on average, we'd expect to win \$2 at this game.

   (b) Get \$5 for rolling a 6, and lose \$1 otherwise. Then $P(X = 5) = 1/6$ and$P(X = -1) = 5/6$, so

$$\begin{aligned} E[X] &= (5)(1/6) + (-1)(5/6) = 0. \\ Var(X) &= (5-0)^2(1/2) + (-1-0)^2(1/2) = 13 \\ \sigma &= \sqrt{Var(X)} = \sqrt{13} \end{aligned}$$

   This means that on average, we'd expect to win \$0 at this game. One can interpret this as being a "fair game".

2. You pay \$3 at the start of the game. You flip two coins and for each Heads, you receive \$4.

$$P(X = -3) = \frac{1}{4}, \qquad P(X = 1) = \frac{1}{2}, \qquad P(X = 5) = \frac{1}{4}.$$

$$\begin{aligned} E[X] &= (-3) \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 5 \cdot \frac{1}{4} = 1. \\ Var(X) &= (-3-1)^2 \cdot \frac{1}{4} + (1-1)^2 \cdot \frac{2}{4} + (5-1)^2 \cdot \frac{1}{4} = \frac{32}{4} = 8 \\ \sigma &= \sqrt{8} \end{aligned}$$

3. Roll 1D6. Let X =the number appearing on the die with

$$P(X=1)=P(X=2)=P(X=3)=P(X=4)=P(X=5)=P(X=6)=1/6.$$

Then

$$\begin{aligned}
E[X] &= 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6}=\frac{21}{6}=3.5 \\
Var(X) &= (1-3.5)^2\cdot\frac{1}{6}+(2-3.5)^2\cdot\frac{1}{6}+(3-3.5)^2\cdot\frac{1}{6}+(4-3.5)^2\cdot\frac{1}{6}+(5-3.5)^2\cdot\frac{1}{6}+(6-3.5)^2\cdot\frac{1}{6} \\
&= 35/12 \\
\sigma &= \sqrt{35/12}.
\end{aligned}$$

4.

#sibilings $= a$	Frequency	$P(X=a)$
0	8	8/40
1	17	17/40
2	11	11/40
3	3	3/40
4	1	1/40
	40	1

$$\begin{aligned}
E[X] &= 0\cdot\frac{8}{40}+1\cdot\frac{17}{40}+2\cdot\frac{11}{40}+3\cdot\frac{3}{40}+4\cdot\frac{1}{40}=\frac{52}{40}=1.3 \\
Var(X) &= (0-1.3)^2\cdot\frac{8}{40}+(1-1.3)^2\cdot\frac{17}{40}+(2-1.3)^2\cdot\frac{11}{40}+(3-1.3)^2\cdot\frac{3}{40}+(4-1.3)^2\cdot\frac{1}{40}=.91 \\
\sigma &= \sqrt{.91}=.954.
\end{aligned}$$

5. We toss 3 coins. Let X denote the number of heads.

$$\begin{aligned}
P(X=0) &= P(\text{all tails})=1/8 \\
P(X=1) &= P(\text{1 head, 2 tails})=3/8 \\
P(X=2) &= P(\text{2 heads, 1 tail})=3/8 \\
P(X=3) &= P(\text{all heads})=1/8
\end{aligned}$$

$$\begin{aligned}
E[X] &= 0\cdot\frac{1}{8}+1\cdot\frac{3}{8}+2\cdot\frac{3}{8}+3\cdot\frac{1}{8}=\frac{12}{8}=1.5 \\
Var(X) &= (0-1.5)^2\cdot\frac{1}{8}+(1-1.5)^2\cdot\frac{3}{8}+(2-1.5)^2\cdot\frac{3}{8}+(3-1.5)^2\cdot\frac{1}{8}=\frac{3}{4}=.75 \\
\sigma &= \sqrt{.75}=.866.
\end{aligned}$$

Lottery example

We pay \$1 for one ticket and to win we must pick the correct 6 (non-repeating) numbers out of 54.

(a) What is the probability we hold the winning ticket?

$$P(\text{win}) = \frac{\#\text{ ways to pick the correct 6 numbers}}{\#\text{ ways to pick any 6 numbers}} = \frac{\binom{6}{6}}{\binom{54}{6}} = \frac{1}{25827165} = 3.87 \times 10^{-8}.$$

(b) What is the expected profit of one ticket if the payout is 10 million?

Let X = net winnings from ticket. Then $X \in \{-1, 10^7\}$, so

$$E[X] = 10^7 \times P(X = 10^7) + (-1) \times P(X = -1) = 10^7 \times 3.87 \times 10^{-8} + (-1)(1 - 3.87 \times 10^{-8}) = -.613$$

(c) What is the expected profit from one ticket if the payout is 100 million?

$$E[X] = 10^8 \times P(X = 10^8) + (-1) \times P(X = -1) = 10^8 \times 3.87 \times 10^{-8} + (-1)(1 - 3.87 \times 10^{-8}) = 2.87$$

Note that the probabilities stay the same, only the payout changes!

(d) What is the probability that if we play 300 million times, we win at least once?

$$P(\text{at least one win}) = 1 - P(\text{no win}) = 1 - (1 - 3.87 \times 10^{-8})^{300{,}000{,}000} = .9999$$

(e) Suppose we play for the 300 millionth time. What is the probability we will win the lottery, if we lost all previous times?

$$P(\text{win on 300 millionth time}) = 3.87 \times 10^{-8}$$

Some may argue that because the probability of winning at least once in 300 million plays is roughly 1, we are "due for a win" on the 300 millionth play. But that is not the case! This is what we call **Gambler's Fallacy**. In fact, the probability of winning any play is the same, no matter what happened in the past. Note that the probability of losing the 300 millionth play is LARGE but the probability of losing all first 300 million plays is very SMALL.

(f) Would it be more advantageous to buy many tickets at once or to play the same number many times? Suppose we play \$10. Should we play them at once, or in different weeks?

$$P(\text{win by playing \$10 at once}) = \frac{10}{\binom{54}{6}} = 3.87 \times 10^{-7}.$$

$$\begin{aligned} P(\text{win by playing 10 weeks \$1}) &= 1 - P(\text{lose 10 times by playing \$1}) \\ &= 1 - P(\text{1st ticket loses}) \times \cdots \times P(\text{10th ticket loses}) \\ &= 1 - (1 - 3.87 \times 10^{-8})^{10} \approx 3.87 \times 10^{-7} \end{aligned}$$

The exact probability in playing 10 times the same numbers is slightly smaller than playing 10 numbers at once. A better illustration is to play \$10,000. where we can see that the probabilities differ (by a small margin still, but visible)

$$P(\text{win by playing \$10,000 at once}) = \frac{10{,}000}{\binom{54}{6}} = 3.87189 \times 10^{-4}.$$

$$P(\text{win by playing 10,000 weeks \$1}) = 1 - (1 - 3.87 \times 10^{-8})^{10{,}000} = 3.87114 \times 10^{-4}.$$