MAT 105 - Summary of Lecture 5
Thursday, January 19, 2016

I will never believe that god plays dice with the universe. (Albert Einstein)

More examples on counting

MAT 105 - Group Work - Solution

1. Roll 3 six-sided dice, of different colors (R,G,B).
   (a) How many different outcomes are possible?

   $6 \times 6 \times 6 = 216$, by Multiplication Rule, since there are 6 possible outcomes for each die.

   (b) In how many ways can you get one 6, one 4 and one 1?

   We place 3 different numbers in 3 slots: 3 choices for the first value (6), then 2 choices left for second value (4) and the last value (1) gets placed in the remaining slot. Thus, there are $3 \times 2 \times 1 = 6$ ways to get this outcome. One can also use combinations or multinomial:

$$\binom{3}{1,1,1} = \frac{3!}{1! \times 1! \times 1!} = 6, \quad \text{or} \quad \binom{3}{1} \times \binom{2}{1} \times \binom{1}{1} = 6.$$

   (c) In how many ways can you get two 6's and one 1?

   We place the value 1 in one one of the 3 slots in 3 ways, the other two slots get filled with 6's. So, there are 3 ways to get this outcome. One can also use combinations or multinomial:

$$\binom{3}{1,2} = \frac{3!}{1! \times 2!} = 3, \quad \text{or} \quad \binom{3}{1} \times \binom{2}{2} = 3.$$

   (d) What are the probabilities for the events in (b) and (c) respectively?

   $P\{\text{event in (b)}\} = \dfrac{6}{216} = \dfrac{1}{36}$ and $P\{\text{event in (c)}\} = \dfrac{3}{216} = \dfrac{1}{72}$.

2. In the game of bridge, each of the 4 players are dealt 13 cards out of a deck of 52.
   (a) How many different hands (13 cards) could you get (order in a hand does not matter)?

   Out of the deck of 52 I choose 13 cards, hence $\binom{52}{13} = \dfrac{52!}{13! \times 39!} = 635,013,559,600$ outcomes.

   (b) In how many ways can you order your 13 cards, from left to right?

   Since no 2 cards are the same, there are $13! = 6,227,020,800$ ways to order my 13 cards.

(c) In how many ways can the dealer assign 13 cards each to players A, B, C, D.

There are $\binom{52}{13}$ hands for A, then, with 39 cards to choose from, there are $\binom{39}{13}$ possible hands for B, $\binom{26}{13}$ possible hands for C and D will get the leftovers $\binom{13}{13}$. The number of deals is:

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} = \frac{52!}{13! \times 39!} \times \frac{39!}{13! \times 26!} \times \frac{26!}{13! \times 13!} \times \frac{13!}{13! \times 0!} = \frac{52!}{(13!)^4} = 5.36 \times 10^{28}$$

(d) If you know you have exactly one Ace, exactly one Q and exactly two 5s, how many different hands can you get?

We choose 1 Ace out of 4, 1 Q out of 4 and 2 5s out of 4. There are 40 cards left, and we need 9 more cards for our hand, so we choose 9 out of 40. The number of possible hands is:

$$\binom{4}{1}\binom{4}{1}\binom{4}{2}\binom{40}{9} = \frac{4!}{1! \times 3!} \times \frac{4!}{1! \times 3!} \times \frac{4!}{2! \times 2!} \times \frac{40!}{9! \times 31!} = 26,250,132,480$$

3. In Poker, you are dealt 5 cards, out of a deck of 52.

(a) How many different hands (5 cards) could you get (order in a hand does not matter)?

Out of the deck of 52, I choose 5 cards, so I get $\binom{52}{5} = \frac{52!}{5! \times 47!} = 2,598,960$ possible hands.

(b) In how many ways can you get a royal flush? (5 cards of the same suit, in sequence AKQJ10)

There are 4 suits, and the sequence is set, so there are 4 ways to get a royal flush.

(c) In how many ways can you get 4 of a kind? (4 cards of same rank and a 5th card of another rank)

We pick the rank, from 13 possible ranks and then we get all 4 cards of that rank. This is done in 13 ways. The fifth card is chosen from the remaining 48 cards, so there are $13 \times 48 = 624$ four of a kind deals.

(d) In how many ways can you get a full house? (2 cards of one rank and 3 cards of a second rank.)

There are 13 ways to choose the rank of the pair, which we choose out of 4 cards. Then there are 12 ways left to pick the rank of the three cards, which we choose out of 4. The number of full houses is $13 \times \binom{4}{2} \times 12 \times \binom{4}{3} = 3,744$.

MAT 105 - Summary of Lecture 6
Thursday, January 21, 2016

I know too well that these arguments from probabilities are imposters, and unless great caution is observed in the use of them, they are apt to be deceptive. (Plato)

Additional examples on counting

1. A tourist wants to visit six of America's 59 national parks. In how many ways can she do this if the order of her visits is
   (a) important
   (b) not important

   Solution:
   (a) The tourist must pick 6 out of 59 National Parks and the order of travel matters, in
   $$\frac{59!}{53!} = 59 \times 58 \times 57 \times 56 \times 55 \times 54 = 32,441,381,280 \text{ ways.}$$
   (b) The tourist picks 6 out of 59 National Parks and order does not matter, so she has
   $$\binom{59}{6} = 45,057,474 \text{ ways.}$$

2. I want to watch a movie from my library. I have 10 comedies, 7 drama and 15 kids movies. How many choices of a movie do I have?

   Solution: There are $10 + 7 + 15 = 32$ choices (Sum Rule.)

3. A multiple choice test has 10 questions. There are 4 possible answers for each question. In how many ways can the student answer the questions on the test if
   (a) he answers all questions,
   (b) he can leave answers blank?

   Solution:
   (a) 4 choices for each question, so 4^{10} ways (Multiplication Rule).
   (b) 5 choices for each question, so 5^{10} ways (Multiplication Rule).

4. In how many different orders can 5 runners finish a race?

   Solution: We order them in $5! = 120$ ways.

5. (a) 20 coin flips reveal 10 heads (H) and 10 tails (T). In how many ways can you arrange them in a row if the H and T must alternate?
   (b) In how many ways can we arrange a group of 20 students (10 BAGD and 10 RTIS), if the degrees must alternate?

Solution:

(a) There are 2 ways to start: H or T, and then we alternate, and following outcomes are decided.

(b) There are 2 ways to start: BAGD or RTIS, and then we alternate. Then the order of the BAGD students can be decided in 10! ways and the order of the RTIS students can be decided in 10! ways, so there are $2 \times (10!) \times (10!)$ possible ways to place them.

6. A softball team has 13 players.

(a) How many ways are there to choose 10 players to take the field?

(b) How many ways are there to assign 10 (distinct) positions on the field by selecting from the 13 players?

(c) Of the 13 players, 3 are left-handed. In how many ways can you choose 10 players to take the field if at least one of these players must be left-handed?

Solution:

(a) We choose 10 players out of 13, order does not matter, in $C(13, 10) = \binom{13}{10} = 286$ ways.

(b) We choose 10 players out of 13, and order matters, in $P(13, 10) = \frac{13!}{3!}$ ways.

(c) We solve

$$\text{at least one left-handed} = \text{all ways} - \text{no left-handed} = C(13, 10) - C(10, 10) = 286 - 1 = 285.$$

Another way to solve is to consider all cases: 1L+9R, 2L+8R, 3L+7R, which amounts to

$$\binom{3}{1}\binom{10}{9} + \binom{3}{2}\binom{10}{8} + \binom{3}{3}\binom{10}{7} = 30 + 135 + 120 = 285 \text{ ways.}$$

7. Roll a six-sided die 5 times. In how many ways can you get one 1, two 2s, and two 5's, not necessarily in this order?

Solution: Imagine we have 5 spaces (rolls) to fill with one 1, two 2s, and two 5's:

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First we place the one (in 5 ways), then in the 4 remaining spaces we choose 2 spots to place the 2's (in $\binom{4}{2}$ ways) and then we place the 5's in the last 2 open spots. This can be done in $5 \times \binom{4}{2} = 5 \times 6 = 30$ ways. It can also be interpreted as the multinomial $\binom{5}{1, 2, 2} = \frac{5!}{1!2!2!} = 30$.