MAT 105 - Summary of Lecture 17
Tuesday, March 8, 2016

> *And sometimes the stupidest man – by some instinct of nature per se and by no previous instruction (this is truly amazing) – knows for sure that the more observations of this sort that are taken, the less the danger will be of straying from the mark.* (James Bernoulli, 1713)

Limit Laws

Let $X_1, X_2, \dots$ be independent random variables with the same distribution (think same experiment, repeated over and over, with X_1 being the first outcome, X_2 the second etc.) with mean $E[X_1] = E[X_2] = \cdots = \mu$ and variance $Var(X_1) = Var(X_2) = \cdots = \sigma^2$. We say the random variables $X_1, X_2, \dots$ are **i.i.d.**, short for **independent identically distributed**. Let

$$S_n = X_1 + X_2 + \cdots + X_n.$$

Then,

$$E[S_n] = E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n] = n\mu,$$
$$Var(S_n) = Var(X_1 + X_2 + \cdots + X_n) = Var(X_1) + Var(X_2) + \cdots + Var(X_n) = n\sigma^2,$$
$$StDev(S_n) = \sqrt{Var(S_n)} = \sqrt{n}\,\sigma.$$

Remarks:

- It is true that for any random variables, even *dependent* random variables,

$$E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

- In general, the variance of the sum is not the sum of variances. It is equal to

$$Var(X_1 + X_2 + \cdots + X_n) = Var(X_1) + Var(X_2) + \cdots + Var(X_n) + \text{(Covariance Term)}.$$

  The covariance term measures the correlation between the random variables. While we will not use or further discuss covariance in this course, if X and Y are random variables, their covariance is defined as

$$Cov(X, Y) = E[X * Y] - E[X] * E[Y].$$

  More importantly, there are 3 cases to consider

  (a) if X and Y are independent, then $E[X * Y] = E[X] * E[Y]$, so $Cov(X, Y) = 0$ which intuitively makes sense, since there is no correlation between X and Y.

  (b) if $Cov(X, Y) < 0$, we say they are *negatively correlated*, which means that if X tends to be large, Y tends to be small and vice versa.

  (c) if $Cov(X, Y) > 0$, we say they are *positively correlated*, which means that if X tends to be large, Y also tends to be large and vice versa.

  For example, one can have X measure the time a person spends playing a game per week, and Y the amount of money they spend per week on the game. Are X and Y correlated? Does one who spends more time on the game also spend more money? That would mean the two variables are positively correlated and convincing one to spend more time on the game (by making certain adjustments to your game) would bring you more revenue.

For us, we will be mainly concerned with independent trials, in which case the covariance term is zero.

Law of Large Numbers

We have seen that an intuitive way to view the probability of a certain outcome is as the frequency with which that outcome occurs in the long run, when the experiment is repeated a large number of times. We have also defined probability mathematically as a value of a distribution function for the random variable representing the experiment. The Law of Large Numbers shows that this mathematical model is consistent with the frequency interpretation of probability. This theorem is sometimes called the law of averages.

Theorem (Laws of Large Numbers): If $X_1, X_2, \dots$ are independent trials of an experiment, each trial having the same distribution with expectation μ (finite) and variance σ^2, and we let $S_n = X_1 + X_2 + \dots + X_n$, then $\dfrac{S_n}{n}$ converges to μ

$$\frac{S_n}{n} \to \mu \text{ as } n \to \infty,$$

that is,

$$\frac{S_n}{n} \approx \mu \text{ for } n \text{ very large.}$$

Depending on how the convergence above is described, we have different versions of the law (Weak Law of Large Numbers, Strong Law of Large Numbers), but we are not interested in their differences in this class.

Examples:

1. Flip a coin many times. We let

$$\begin{aligned} X_1 &= 1 \text{ if the 1st flip lands on Heads and 0 otherwise} \\ X_2 &= 1 \text{ if the 2nd flip lands on Heads and 0 otherwise} \\ X_3 &= 1 \text{ if the 3rd flip lands on Heads and 0 otherwise...} \end{aligned}$$

   Since $X_1, X_2, \dots$ all are $Bernoulli(1/2)$, their expectation is 1/2 and their variance is 1/4.

   Let S_{400} denote the number of Heads in 400 flips, so $S_{400} = X_1 + X_2 + \dots + X_{400}$. The Law of Large Numbers guarantees that

$$\frac{S_{400}}{400} \approx \frac{1}{2}.$$

2. Roll a fair six-sided die many times and let

$$\begin{aligned} X_1 &= \text{number in the 1st roll} \\ X_2 &= \text{number in the 2nd roll} \\ X_3 &= \text{number in the 3rd roll...} \end{aligned}$$

   Since $X_1, X_2, \dots$ all have the same distribution, $P(X = k) = 1/6$ for all $1 \le k \le 6$, we find the expectation and variance: $E[X] = 3.5$ and $Var(X) = \dfrac{35}{12}$.

   Let S_n denote the sum of nD6, so $S_n = X_1 + X_2 + \dots + X_n$. The Law of Large Numbers guarantees that if n is large, then

$$\frac{S_n}{n} \approx 3.5.$$

We will see next lecture that, even more striking, the sum of a large number of random variables (independent and with the same distribution) can be modeled by the normal distribution. Let us think about this for a moment: no matter what distribution we have for $X_1, X_2, \dots$, their sum has the **normal** distribution! That is indeed surprising.

NAMES:

MAT 105 - Group Work
March 8, 2016

1. If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$, find
   (a) $P(2 < X < 5)$

   (b) $P(0 < X)$

   (c) $P(|X - 3| > 6)$

2. Find the value a such that the standard normal random variable $Z = N(0, 1)$ satisfies:
   (a) $P(Z \leq a) = 0.7413$

   (b) $P(-a \leq Z \leq a) = 0.3716$

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3. A snow-boarder has a difficult trick with a 10% of success. She tries the trick 25 times and wants to know the probability she will get exactly 2 successes. Compute

   (a) the exact answer

   (b) the Poisson approximation

   (c) the normal approximation