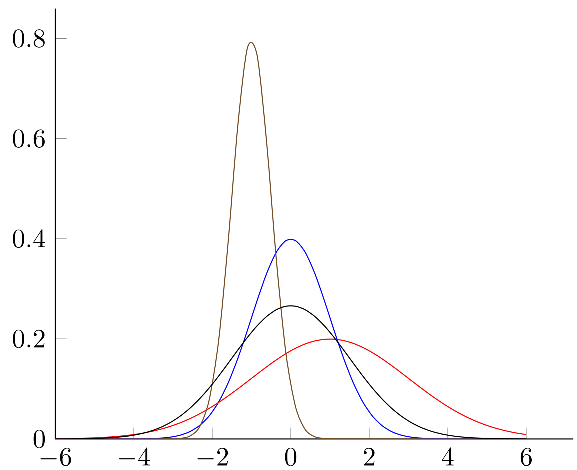
MAT 105 - Summary of Lecture 15
Tuesday, March 1, 2016

> *The 'Law of Frequency of Error' . . . reigns with serenity and in complete self-effacement amidst the wildest confusion. The huger the mob . . . the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand . . . an unsuspected and most beautiful form of regularity proves to have been latent all along.* (Francis Galton)

Normal random variables

Today we discussed the most important distribution in statistics and probability, the normal distribution.
Definition: A random variable is said to be **normally distributed** if its distribution (think of it as a histogram) has the shape of a bell curve (or normal curve). We write $X = N(\mu, \sigma^2)$ to say that X is a normal random variable with mean μ (read "mu") and variance σ^2 (read "sigma" squared).



The expectation and variance of X are $E[X] = \mu$ and $Var(X) = \sigma^2$. The four normal distributions above have various means and variances, however their shape is the same and the area under each curve is exactly 1.
Examples of normal random variables:

1. the weight and height a person;
2. velocity of a molecule in gas;
3. error in measuring a physical quantity (see quote at top of page);
4. grades in MAT 105;
5. gestation period of humans.

Note that this kind of random variable is not discrete as the set of all possible values includes intervals of real numbers. Such a random variable is said to be **continuous** and is described by a **probability density**, since for continuous random variables, $P(X = a) = 0$ for all possible values a. While we will not use the density of $X = N(\mu, \sigma^2)$ in this class, it might be interesting to know that it has the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty.$$

Then the probability

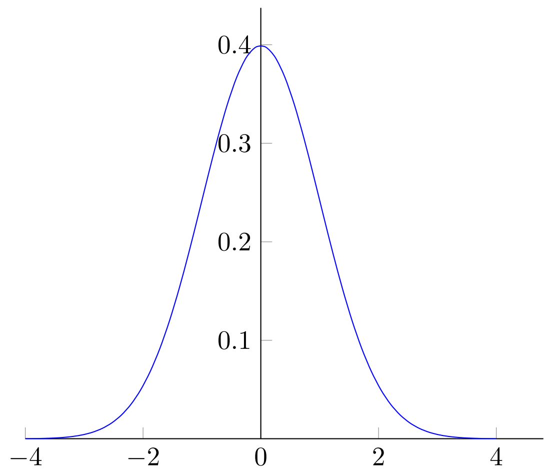
$$P(a \leq X \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx,$$

which does not have a nice (closed-form) antiderivative, as some of you who studied calculus might know. Therefore, we have to use a numerical approximation method to find these probabilities, which come to us in the form of the z-table, provided in class and available to download from Moodle.

It is important to note that the integral above gives the area under the curve $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, from $x = a$ to $x = b$.

Definition: A normal random variable with $\mu = 0$ and $\sigma = 1$ is called a **standard normal variable**. Its distribution is given by

$$\phi(a) = P(X \leq a) = \text{area under the curve to the left of } a.$$



Properties:

- The total area under the curve equals 1.
- $P(X \geq a) = 1 - P(X \leq a) = 1 - \phi(a)$
- The area under the curve to the left of $-a$ is equal to the area under the curve to the right of a, by symmetry with the y-axis, so
$$P(X \leq -a) = P(X \geq a) = 1 - \phi(a).$$
- $P(a \leq X \leq b) = \phi(b) - \phi(a)$.

Remark: Pay attention that the table of z-scores only applies to $N(0, 1)$! We will find probabilities associated to $N(\mu, \sigma^2)$ by **standardizing** or **normalizing** it. To read the table, we find $\phi(2.34) = .9904$ by looking for the entry in the row starting with 2.3 and the column with .04.

Examples: Let $X = N(0, 1)$

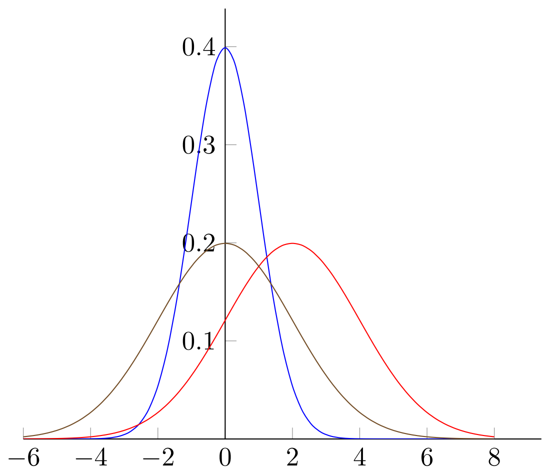
$$\begin{aligned}
P(X \leq 1) &= \phi(1.00) = .8413 \\
P(X \leq -2) &= \phi(-2) = 1 - \phi(2) = 1 - .9772 = .0128 \\
P(-2 \leq X \leq 1) &= \phi(1) - \phi(-2) = \phi(1) - (1 - \phi(2)) = .8413 - (1 - .9772) = .8185 \\
P(X \geq 0) &= 1 - P(X \leq 0) = 1 - \phi(0) = 1 - .5 = .5 \\
P(X \leq 1.23) &= \phi(1.23) = .8907
\end{aligned}$$

How does one find probabilities for $N(\mu, \sigma^2)$? We transform $X = N(\mu, \sigma^2)$ into a standard normal by subtracting the mean and dividing by the standard deviation. That is, let

$$z = \frac{X - \mu}{\sigma}.$$

While we will not verify that the resulting random variable is normal, and we'll assume that as a fact, let us verify at least that the mean and variance of z are 0 and 1. Using the properties of expectation and variance we have

$$\begin{aligned} E[X] &= E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}E[X-\mu] = \frac{E[X]-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0 \\ Var(X) &= Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}Var(X-\mu) = \frac{1}{\sigma^2}Var(X) = \frac{\sigma^2}{\sigma^2} = 1. \end{aligned}$$



Visually, standardizing $N(\mu, \sigma^2)$ works as follows. We start with the red curve, which here is $N(2, 4)$. Subtracting the mean, shifts to the brown curve, which is $N(0, 4)$; it has the same spread, but its mean is now at 0 (where the axis of symmetry is located). Dividing by the standard deviation changes the brown curve into the blue curve which is $N(0, 1)$; the brown and blue curves are both centered at 0 (so they have the same mean), but the blue curve has a smaller spread (and the peak is steeper).

Examples: Let $X = N(3, 4)$, so $\mu = 3$ and $\sigma = 2$

$$\begin{aligned} P(X \le 5) &= P\left(\frac{X-3}{2} \le \frac{5-3}{2}\right) = P(z \le 1) = \phi(1) = .8413 \\ P(X \ge 3) &= P\left(\frac{X-3}{2} \ge \frac{3-3}{2}\right) = P(z \ge 0) = 1 - \phi(0) = .5 \\ P(X \le 2.5) &= P\left(\frac{X-3}{2} \le \frac{2.5-3}{2}\right) = P(z \le -.25) = \phi(-.25) = 1 - \phi(.25) = 1 - .5987 = .4013. \end{aligned}$$