NE450 - Principles of nuclear engineering
Two group reactor equation

R. A. Borrelli

University of Idaho · Idaho Falls Center for Higher Education

Nuclear Engineering and Industrial Management Department

r.angelo.borrelli@gmail.com

2020.08.03

1 Equation of continuity

The equation of continuity for neutrons describes behavior in a reactor.

[rate of change of neutrons] = [production rate] - [absorption rate] - [leakage rate]

$$\frac{d}{dt}\int_V ndV = \int_V sdV - \int_V \Sigma_A \phi dV - \int_V \underline{J}\cdot\underline{n}dA \tag{1}$$

The terms are -

- $\int_V ndV$ - total number of neutrons
- $\frac{d}{dt}\int_V ndV$ - rate of change
- $\int_V sdV$ - production rate
- $\int_V \Sigma_A \phi dV$ - absorption rate
- $\int_A \underline{J}\cdot\underline{n}dA$ leakage rate

Applying some mathematical relationships -

$$\begin{aligned}\frac{d}{dt}\int_V ndV &= \int_V \frac{\partial n}{\partial t}dV \\ \int_A \underline{J}\cdot\underline{n}dA &= \int_V \nabla\underline{J}dV\end{aligned} \tag{2}$$

Substitute into eq. 1 -

$$\int_V \frac{\partial n}{\partial t}dV = \int_V sdV - \int_V \Sigma_A \phi dV - \int_V \nabla\underline{J}dV \tag{3}$$

Because the control volume is the same -

$$\frac{\partial n}{\partial t} = s - \Sigma_A\phi - \nabla J \tag{4}$$

Apply Fick's law -

$$\begin{aligned}\frac{\partial n}{\partial t} &= D\nabla^2\phi - \Sigma_A\phi + s \\ \frac{1}{v}\frac{\partial \phi}{\partial t} &= D\nabla^2\phi - \Sigma_A\phi + s\end{aligned} \tag{5}$$

Assume steady state -

$$D\nabla^2\phi - \Sigma_A\phi + s = 0 \tag{6}$$

2 Multigroup diffusion theory

The general multigroup diffusion equation based on eq. 6 for an arbitrary group g is -

$$D_g\nabla^2\phi_g - \Sigma_A^g\phi_g - \sum_{h=g+1}^{N}\Sigma_{g\rightarrow h}\phi_g + \sum_{h=1}^{g-1}\Sigma_{h\rightarrow g}\phi_h + s_g = 0 \tag{7}$$

In addition to absorption in group g, there is the loss due to scatter from group g to h but also a source terms due to scatter from h to g.

For 2 groups, then; i.e., a fast and thermal group, eq. 7 is -

$$\begin{aligned} &D_1 \nabla^2 \phi_1 - \Sigma_A^1 \phi_1 - \Sigma_{1\to 2}\phi_1 + \Sigma_{2\to 1}\phi_2 + s_1 = 0 \\ &D_T \nabla^2 \phi_T - \Sigma_A^T \phi_T - \Sigma_{T\to 1}\phi_T + s_T = 0 \end{aligned} \tag{8}$$

Neutrons will not scatter from the thermal group (T) to the fast group (1) -

$$\begin{aligned} &D_1 \nabla^2 \phi_1 - \Sigma_A^1 \phi_1 - \Sigma_{1\to 2}\phi_1 + s_1 = 0 \\ &D_T \nabla^2 \phi_T - \Sigma_A^T \phi_T + s_T = 0 \end{aligned} \tag{9}$$

3 Solution

3.1 Source

(1) Assume the bulk of fissions are induced by thermal neutrons.

$$s_1 = \eta_T f \epsilon \Sigma_A^T \phi_T = \frac{k_\infty}{p} \Sigma_A^T \phi_T \tag{10}$$

(2) Fast neutrons scattering into the thermal region must survive through the resonances.

$$s_T = p\Sigma_{1\to T}\phi_1 \tag{11}$$

3.2 Reactor equations

(1) Assume a one-dimensional, infinite slab.

(2) Compute the derivatives -

Let -

$$\frac{d^2\phi}{dx^2} \equiv \phi''$$

(3) Substitute the sources in eqns. 10 and 11 into eq. 9 -

$$D_1 \phi_1'' - \Sigma_A^1 \phi_1 - \Sigma_{1\to T}\phi_1 + \frac{k_\infty}{p}\Sigma_A^T \phi_T = 0 \tag{12}$$

$$D_T \phi_T'' - \Sigma_A^T \phi_T + p\Sigma_{1\to T}\phi_1 = 0 \tag{13}$$

(4) Then -

$$\phi_1'' - \frac{1}{K^2}\phi_1 + f\phi_T = 0 \tag{14}$$

$$\phi_T'' + g\phi_1 - \frac{1}{L^2}\phi_T = 0 \tag{15}$$

Where -

$$\begin{aligned} K^2 &= \frac{D_1}{\Sigma_{1\to T} + \Sigma_A^1} \\ L^2 &= \frac{D_T}{\Sigma_A^T} \\ f &= \frac{k\Sigma_A^T}{pD_1} \\ g &= \frac{p\Sigma_{1\to T}}{D_T} \end{aligned}$$

3.3 Eigenvalue approach

(1) Put in matrix form -

$$\underline{\phi}'' = \underline{\underline{A}} \cdot \underline{\phi} \tag{16}$$

$$\begin{bmatrix} \phi_1'' \\ \phi_T'' \end{bmatrix} = \begin{bmatrix} \frac{1}{K^2} & -f \\ -g & \frac{1}{L^2} \end{bmatrix} \cdot \begin{bmatrix} \phi_1 \\ \phi_T \end{bmatrix} \tag{17}$$

(2) Find eigenvalues -

$$\underline{\underline{A}} \cdot \underline{x} = \lambda^2 \underline{x}$$

$$(\underline{\underline{A}} - \lambda^2 \underline{\underline{I}})\underline{x} = 0$$

$$det(\underline{\underline{A}} - \lambda^2 \underline{\underline{I}}) = 0$$

(3) Applying eqns. 14 and 15 -

$$det \begin{bmatrix} \frac{1}{K^2} - \lambda^2 & -f \\ -g & \frac{1}{L^2} - \lambda^2 \end{bmatrix} = 0 \tag{18}$$

$$\frac{1}{K^2}(\frac{1}{L^2} - \lambda^2) + \lambda^4 - \lambda^2 \frac{1}{L^2} + fg = 0 \tag{19}$$

(4) Then -

$$K^2 L^2 \lambda^4 - (L^2 + K^2)\lambda^2 + fgK^2L^2 + 1 = 0 \tag{20}$$

Eq. 20 is quadratic in λ^2

(5) The general solution (p585) is -

$$\underline{\phi} = \sum_{j=1}^{n} (a_j \cos \omega_j t + b_j \frac{\sin \omega_j t}{\omega_j}) \underline{x}_j \tag{21}$$

Where the eigenvalues (λ) relate as -

$$\lambda_j = -\omega_j^2$$

Eq. 21 assumes all eigenvalues of $\underline{\underline{A}}$ are negative or zero.