NE450 - Principles of nuclear engineering
Two group diffusion

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1 Equation of continuity

The equation of continuity for neutrons describes behavior in a reactor.

[rate of change of neutrons] = [production rate] - [absorption rate] - [leakage rate]

$$\frac{d}{dt}\int_V ndV = \int_V sdV - \int_V \Sigma_A \phi dV - \int_V \underline{J} \cdot \underline{n} dA \tag{1}$$

The terms are -

- $\int_V ndV$ - total number of neutrons
- $\frac{d}{dt}\int_V ndV$ - rate of change
- $\int_V sdV$ - production rate
- $\int_V \Sigma_A \phi dV$ - absorption rate
- $\int_A \underline{J} \cdot \underline{n} dA$ leakage rate

Applying some mathematical relationships -

$$\begin{aligned} \frac{d}{dt}\int_V ndV &= \int_V \frac{\partial n}{\partial t} dV \\ \int_A \underline{J} \cdot \underline{n} dA &= \int_V \nabla \underline{J} dV \end{aligned} \tag{2}$$

Substitute into eq. 1 -

$$\int_V \frac{\partial n}{\partial t} dV = \int_V sdV - \int_V \Sigma_A \phi dV - \int_V \nabla \underline{J} dV \tag{3}$$

Because the control volume is the same -

$$\frac{\partial n}{\partial t} = s - \Sigma_A \phi - \nabla J \tag{4}$$

Apply Fick's law -

$$\begin{aligned} \frac{\partial n}{\partial t} &= D\nabla^2 \phi - \Sigma_A \phi + s \\ \frac{1}{v}\frac{\partial \phi}{\partial t} &= D\nabla^2 \phi - \Sigma_A \phi + s \end{aligned} \tag{5}$$

Assume steady state -

$$D\nabla^2 \phi - \Sigma_A \phi + s = 0 \tag{6}$$

2 Multigroup diffusion theory

The general multigroup diffusion equation based on eq. 6 for an arbitrary group g is -

$$D_g \nabla^2 \phi_g - \Sigma_A^g \phi_g - \sum_{h=g+1}^{N} \Sigma_{g\to h}\phi_g + \sum_{h=1}^{g-1} \Sigma_{h\to g}\phi_h + s_g = 0 \tag{7}$$

In addition to absorption in group g, there is the loss due to scatter from group g to h but also a source terms due to scatter from h to g.

For 2 groups, then; i.e., a fast and thermal group, eq. 7 is -

$$\begin{aligned} D_1\nabla^2\phi_1 - \Sigma_A^1\phi_1 - \Sigma_{1\to 2}\phi_1 + \Sigma_{2\to 1}\phi_2 + s = 0 \\ D_2\nabla^2\phi_2 - \Sigma_A^2\phi_2 - \Sigma_{2\to 1}\phi_2 + \Sigma_{1\to 2}\phi_1 = 0 \end{aligned} \tag{8}$$

Neutrons will not scatter from the thermal group (2) to the fast group (1) -

$$\begin{aligned} D_1\nabla^2\phi_1 - \Sigma_A^1\phi_1 - \Sigma_{1\to 2}\phi_1 + s = 0 \\ D_2\nabla^2\phi_2 - \Sigma_A^2\phi_2 + \Sigma_{1\to 2}\phi_1 = 0 \end{aligned} \tag{9}$$

3 Solution

(1) Assume a one-dimensional, infinite slab.

(2) Compute the derivatives -

$$\frac{d^2\phi_1}{dx^2} - \frac{1}{K^2}\phi_1 + d = 0 \tag{10}$$

$$\frac{d^2\phi_2}{dx^2} - \frac{1}{L^2}\phi_2 + \Sigma_{1\to 2}\phi_1 = 0 \tag{11}$$

Where -

$$\begin{aligned} L^2 &= \frac{D_2}{\Sigma_A^2} \\ K^2 &= \frac{D_1}{\Sigma_A^1 + \Sigma_{1\to 2}} \\ d &= \frac{s}{D_1} \end{aligned}$$

Let -

$$\frac{d^2\phi}{dx^2} \equiv \phi''$$

3.1 Fast group

(1) Assume the solution is the sum of a homogeneous and a particular solution.

$$\phi_1(x) = \phi_H(x) + \phi_P(x) \tag{12}$$

(2) Making use of eq. 10 -

$$\frac{d^2\phi_H}{dx^2} - \frac{1}{K^2}\phi_H = 0 \tag{13}$$

(3) Assume the homogeneous solution -

$$Ae^{-\frac{x}{K}} + Ce^{\frac{x}{K}} \tag{14}$$

(4) Assume the particular solution is an arbitrary constant -

$$\phi_P = B \tag{15}$$

(5) Substitute eq. 15 in to eq. 10 -

$$0 - \frac{1}{K^2}B + d = 0 \tag{16}$$

(6) The particular solution is then -

$$\phi_P = dK^2 \tag{17}$$

(7) The fast flux is then -

$$\phi_1 = A_1e^{-\frac{x}{K}} + C_1e^{\frac{x}{K}} + dK^2 \tag{18}$$

3.2 Thermal flux

(1) Substitute eq. 18 into eq. 11, using the shorthand notation ($''$) for the second derivative -

$$\phi_2'' - \frac{1}{L^2}\phi_2 + \Sigma_1 A_1 e^{-\frac{x}{K}} + \Sigma_1 C_1 e^{\frac{x}{K}} + \Sigma_1 d K^2 = 0 \tag{19}$$

Where -

$$\Sigma_{1\to 2} \equiv \Sigma_1$$

(2) Following the same procedure, assume the solution is the sum of a homogeneous and a particular solution.

$$\phi_2 = \phi_H + \phi_P \tag{20}$$

(3) Making use of eq. 11 -

$$\phi_H'' - \frac{1}{L^2}\phi_H = 0 \tag{21}$$

(4) Assume the homogeneous solution -

$$\phi_H = A_2 e^{-\frac{x}{L}} + C_2 e^{\frac{x}{L}} \tag{22}$$

(5) Apply the following theorem for the particular solution -

(a) If $z_1(x)$ is a particular solution of -

$$y'' + p(x)y' + q(x)y = f(x)$$

(b) and $z_2(x)$ is a particular solution of -

$$y'' + p(x)y' + q(x)y = g(x)$$

(c) Then, for any number of forcing functions, $z = z_1(x) + z_2(x)$ is a particular solution of -

$$y'' + p(x)y' + q(x)y = f(x) + g(x)$$

(6) Therefore, because there are three forcing functions in eq. 19 -

$$\phi_P = \phi_F + \phi_G + \phi_I \tag{23}$$

(7) Based on the theorem -

$$\begin{aligned} &\phi_F'' - \frac{1}{L^2}\phi_F + \Sigma_1 A_1 e^{-\frac{x}{K}} \\ &\phi_G'' - \frac{1}{L^2}\phi_G + \Sigma_1 C_1 e^{\frac{x}{K}} \\ &\phi_I'' - \frac{1}{L^2}\phi_I + \Sigma_1 d K^2 \end{aligned} \tag{24}$$

The diffusion length (L^2) does not need to be split because the geometry is the same.

(8) Assume the correponding particular solutions for each forcing function -

$$\begin{aligned} \phi_F^P &= B_1 e^{-\frac{x}{K}} \\ \phi_G^P &= B_2 e^{\frac{x}{K}} \\ \phi_I^P &= B_3 \end{aligned} \tag{25}$$

(9) Compute the second derivative of each particular solution -

$$
\begin{aligned}
\phi_F'' &= \frac{B_1}{K^2} e^{-\frac{x}{K}} \\
\phi_G'' &= \frac{B_2}{K^2} e^{\frac{x}{K}} \\
\phi_H'' &= 0
\end{aligned}
\tag{26}
$$

(10) Substitute eq. 26 and 25 into eq. 24 -

$$
\begin{aligned}
&\frac{B_1}{K^2} e^{-\frac{x}{K}} - \frac{B_1}{L^2} e^{-\frac{x}{K}} + \Sigma_1 A_1 e^{-\frac{x}{K}} = 0 \\
&\frac{B_2}{K^2} e^{\frac{x}{K}} - \frac{B_2}{L^2} e^{\frac{x}{K}} + \Sigma_1 C_1 e^{\frac{x}{K}} = 0 \\
&0 - \frac{B_3}{L^2} + \Sigma_1 d K^2 = 0
\end{aligned}
\tag{27}
$$

(11) Simplfying -

$$
\begin{aligned}
&\frac{B_1}{K^2} - \frac{B_1}{L^2} + \Sigma_1 A_1 = 0 \\
&\frac{B_2}{K^2} - \frac{B_2}{L^2} + \Sigma_1 C_1 = 0 \\
&\frac{B_3}{L^2} - \Sigma_1 d K^2 = 0
\end{aligned}
\tag{28}
$$

(12) Solve for the constants -

$$
\begin{aligned}
B_1 &= \frac{\Sigma_1 A_1 K^2 L^2}{K^2 - L^2} \\
B_2 &= \frac{\Sigma_1 C_1 K^2 L^2}{K^2 - L^2} \\
B_3 &= \Sigma_1 d K^2 L^2
\end{aligned}
\tag{29}
$$

(13) Substitute the constants in eq. 29 into eq. 25 -

$$
\begin{aligned}
\phi_F^P &= \frac{\Sigma_1 A_1 K^2 L^2}{K^2 - L^2} e^{-\frac{x}{K}} \\
\phi_G^P &= \frac{\Sigma_1 C_1 K^2 L^2}{K^2 - L^2} e^{\frac{x}{K}} \\
\phi_I^P &= \Sigma_1 d K^2 L^2
\end{aligned}
\tag{30}
$$

(14) Finally, substitute the particular solutions in eq. 30 and the homogeneous solution in eq. 22 into eq. 20 to obtain the termal flux -

$$
\phi_2 = A_2 e^{-\frac{x}{L}} + C_2 e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{K^2 - L^2} e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{K^2 - L^2} e^{\frac{x}{K}} + \Sigma_1 d K^2 L^2
\tag{31}
$$

4 Solution check

4.1 Fast flux

(1) The fast flux is -

$$\phi_1 = A_1 e^{-\frac{x}{K}} + C_1 e^{\frac{x}{K}} + dK^2 \tag{32}$$

(2) And the solution must satisfy -

$$\frac{d^2\phi_1}{dx^2} - \frac{1}{K^2}\phi_1 + d = 0 \tag{33}$$

(3) Compute the second derivative of eq. 32 -

$$\phi_1'' = \frac{A_1}{K^2} e^{-\frac{x}{K}} + \frac{C_1}{K^2} e^{\frac{x}{K}} \tag{34}$$

(4) Substitute the result into eq. 33 and simplify -

$$(\frac{A_1}{K^2} e^{-\frac{x}{K}} + \frac{C_1}{K^2} e^{\frac{x}{K}}) - \frac{1}{K^2}[A_1 e^{-\frac{x}{K}} + C_1 e^{\frac{x}{K}} + dK^2] + d = 0$$
$$\frac{A_1}{K^2} e^{-\frac{x}{K}} + \frac{C_1}{K^2} e^{\frac{x}{K}} - \frac{A_1}{K^2} e^{-\frac{x}{K}} - \frac{C_1}{K^2} e^{\frac{x}{K}} - d + d = 0$$

Check satisfied.

4.2 Thermal flux

(1) The thermal flux is -

$$\phi_2 = A_2 e^{-\frac{x}{L}} + C_2 e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{K^2 - L^2} e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{K^2 - L^2} e^{\frac{x}{K}} + \Sigma_1 dK^2 L^2 \tag{35}$$

(2) And the solution must satisfy -

$$\frac{d^2\phi_2}{dx^2} - \frac{1}{L^2}\phi_2 + \Sigma_1\phi_1 = 0 \tag{36}$$

(3) Compute the second derivative of eq. 35 -

$$\phi_2'' = \frac{A_2}{L^2} e^{-\frac{x}{L}} + \frac{C_2}{L^2} e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)K^2} e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)K^2} e^{\frac{x}{K}} \tag{37}$$

(4) Substitute the result into eq. 36 and simplify -

$$\begin{aligned} 0 = & \\ & + [\frac{A_2}{L^2} e^{-\frac{x}{L}} + \frac{C_2}{L^2} e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)K^2} e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)K^2} e^{\frac{x}{K}}] \\ & - [\frac{A_2}{L^2} e^{-\frac{x}{L}} + \frac{C_2}{L^2} e^{\frac{x}{L}} + \frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)L^2} e^{-\frac{x}{K}} + \frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)L^2} e^{\frac{x}{K}} + \Sigma_1 dK^2] \\ & + [\Sigma_1 A_1 e^{-\frac{x}{K}} + \Sigma_1 C_1 e^{\frac{x}{K}} + \Sigma_1 dK^2] \end{aligned}$$

$$\begin{aligned} 0 = & \\ & + (\frac{A_2}{L^2} - \frac{A_2}{L^2}) e^{-\frac{x}{L}} + (\frac{C_2}{L^2} - \frac{C_2}{L^2}) e^{\frac{x}{L}} + (\Sigma_1 dK^2 - \Sigma_1 dK^2) \\ & + [\frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)K^2} - \frac{\Sigma_1 A_1 K^2 L^2}{(K^2 - L^2)L^2} + \Sigma_1 A_1] e^{-\frac{x}{K}} \\ & + [\frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)K^2} - \frac{\Sigma_1 C_1 K^2 L^2}{(K^2 - L^2)L^2} + \Sigma_1 C_1] e^{\frac{x}{K}} \end{aligned}$$

$$
\begin{aligned}
0 = \\
&+ (\frac{A_2}{L^2} - \frac{A_2}{L^2})e^{-\frac{x}{L}} + (\frac{C_2}{L^2} - \frac{C_2}{L^2})e^{\frac{x}{L}} + (\Sigma_1 dK^2 - \Sigma_1 dK^2) \\
&+ [\frac{\Sigma_1 A_1 L^2}{K^2 - L^2} - \frac{\Sigma_1 A_1 K^2}{K^2 - L^2} + \frac{\Sigma_1 A_1 (K^2 - L^2)}{K^2 - L^2}]e^{-\frac{x}{K}} \\
&+ [\frac{\Sigma_1 C_1 L^2}{K^2 - L^2} - \frac{\Sigma_1 C_1 K^2}{K^2 - L^2} + \frac{\Sigma_1 C_1 (K^2 - L^2)}{K^2 - L^2}]e^{\frac{x}{K}}
\end{aligned}
$$

Check satisfied.