Problem Statement :3. Find the least positive integer n such that any set of n pairwise relatively prime integers greater than 1 and less than 2005 contains at least one prime number.
Solution :3. The example $2^2, 3^2, 5^2, \ldots, 43^2$, where we considered the squares of the first 14 prime numbers, shows that $n \geq 15$.Assume that there exist $a_1, a_2, \ldots, a_{16}$, pairwise relatively prime integers greater than 1 and less than 2005 , none of which is a prime. Let q_k be the least prime number in the factorization of $a_k, k = 1, 2, \ldots, 16$. Let q_i be the maximum of $q_1, q_2, \ldots, q_{15}$. Then $q_i \geq p_{16} = 47$. Because a_i is not a prime, $\frac{a_i}{q_i}$ is divisible by a prime number greater than or equal to q_i. Hence $a_i \geq q_i^2 = 47^2 > 2005$, a contradiction. We conclude that $n = 15$.
Topic :Methods of Proof
Book :Putnam and Beyond
Final Answer :

Problem Statement :8. Determine all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying

$$xf(y) + yf(x) = (x + y)f\left(x^2 + y^2\right)$$

for all positive integers x and y.
Solution :8. The constant function $f(x) = k$, where k is a positive integer, is the only possible solution. That any such function satisfies the given condition is easy to check.Now suppose there exists a nonconstant solution f. There must exist two positive integers a and b such that $f(a) < f(b)$. This implies that $(a + b)f(a) < af(b) + bf(a) < (a + b)f(b)$, which by the given condition is equivalent to $(a + b)f(a) < (a + b)f\left(a^2 + \ b^2\right) < (a + b)f(b)$. We can divide by $a + b > 0$ to find that $f(a) < f\left(a^2 + b^2\right) < f(b)$. Thus between any two different values of f we can insert another. But this cannot go on forever, since f takes only integer values. The contradiction shows that such a function cannot exist. Thus constant functions are the only solutions.(Canadian Mathematical Olympiad, 2002)
Topic :Methods of Proof
Book :Putnam and Beyond
Final Answer :

Problem Statement :64. Find all odd positive integers n greater than 1 such that for any coprime divisors a and b of n, the number $a+b-1$ is also a divisor of n.
Solution :64. We call a number good if it satisfies the given condition. It is not difficult to see that all powers of primes are good. Suppose n is a good number that has at least two distinct prime factors. Let $n = p^r s$, where p is the smallest prime dividing n and s is not divisible by p. Because n is good, $p + s - 1$ must divide n. For any prime q dividing $s, s < p + s - 1 < s + q$, so q does not divide $p + s - 1$. Therefore, the only prime factor of $p + s - 1$ is p. Then $s = p^c - p + 1$ for some integer $c > 1$. Because p^c must also divide $n, p^c + s - 1 = 2p^c - p$

divides n. Because $2p^{c-1}-1$ has no factors of p, it must divide s. But

$$\begin{aligned}\frac{p-1}{2}\left(2p^{c-1}-1\right) &= p^c - p^{c-1} - \frac{p-1}{2} < p^c - p + 1 < \frac{p+1}{2}\left(2p^{c-1}-1\right) \\ &= p^c + p^{c-1} - \frac{p+1}{2},\end{aligned}$$

a contradiction. It follows that the only good integers are the powers of primes.(Russian Mathematical Olympiad, 2001)
Topic :Methods of Proof
Book :Putnam and Beyond
Final Answer :

Problem Statement :67. An ordered triple of numbers is given. It is permitted to perform the following operation on the triple: to change two of them, say a and b, to $(a+b)/\sqrt{2}$ and $(a-b)/\sqrt{2}$. Is it possible to obtain the triple $(1, \sqrt{2}, 1+\sqrt{2})$ from the triple $(2, \sqrt{2}, 1/\sqrt{2})$ using this operation?
Solution :67. Because

$$a^2+b^2 = \left(\frac{a+b}{\sqrt{2}}\right)^2 + \left(\frac{a-b}{\sqrt{2}}\right)^2,$$

the sum of the squares of the numbers in a triple is invariant under the operation. The sum of squares of the first triple is $\frac{9}{2}$ and that of the second is $6+2\sqrt{2}$, so the first triple cannot be transformed into the second.(D. Fomin, S. Genkin, I. Itenberg, Mathematical Circles, AMS, 1996)
Topic :Methods of Proof
Book :Putnam and Beyond
Final Answer :

Problem Statement :69. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones by performing the operation finitely many times?
Solution :69. Let I be the sum of the number of stones and heaps. An easy check shows that the operation leaves I invariant. The initial value is 1002 . But a configuration with k heaps, each containing 3 stones, has $I=k+3k=4k$. This number cannot equal 1002 , since 1002 is not divisible by 4.(D. Fomin, S. Genkin, I. Itenberg, Mathematical Circles, AMS, 1996)
Topic :Methods of Proof
Book :Putnam and Beyond
Final Answer :

Problem Statement :74. The number $99\ldots99$ (having 1997 nines) is written on a blackboard. Each minute, one number written on the blackboard is factored

into two factors and erased, each factor is (independently) increased or decreased by 2 , and the resulting two numbers are written. Is it possible that at some point all of the numbers on the blackboard are equal to 9 ?
Solution :74. The answer is no. The essential observation is that

$$99 \ldots 99 \equiv 99 \equiv 3(\bmod 4).$$

When we write this number as a product of two factors, one of the factors is congruent to 1 and the other is congruent to 3 modulo 4 . Adding or subtracting a 2 from each factor produces numbers congruent to 3 , respectively, 1 modulo 4 . We deduce that what stays invariant in this process is the parity of the number of numbers on the blackboard that are congruent to 3 modulo 4 . Since initially this number is equal to 1 , there will always be at least one number that is congruent to 3 modulo 4 written on the blackboard. And this is not the case with the sequence of nines. This proves our claim.(St. Petersburg City Mathematical Olympiad, 1997)
Topic :Methods of Proof
Book :Putnam and Beyond
Final Answer :

Problem Statement :86. Factor $5^{1985}-1$ into a product of three integers, each of which is greater than 5^{100}.
Solution :86. We use the identity

$$a^5-1=(a-1)\left(a^4+a^3+a^2+a+1\right)$$

applied for $a=5^{397}$. The difficult part is to factor $a^4+a^3+a^2+a+1$. Note that

$$a^4+a^3+a^2+a+1=\left(a^2+3a+1\right)^2-5a(a+1)^2.$$

Hence

$$\begin{aligned} a^4+a^3+a^2+a+1 &= \left(a^2+3a+1\right)^2-5^{398}(a+1)^2 \\ &= \left(a^2+3a+1\right)^2-\left(5^{199}(a+1)\right)^2 \\ &= \left(a^2+3a+1+5^{199}(a+1)\right)\left(a^2+3a+1-5^{199}(a+1)\right). \end{aligned}$$

It is obvious that $a-1$ and $a^2+3a+1+5^{199}(a+1)$ are both greater than 5^{100}. As for the third factor, we have

$$a^2+3a+1-5^{199}(a+1)=a\left(a-5^{199}\right)+3a-5^{199}+1 \geq a+0+1 \geq 5^{100}.$$

Hence the conclusion.(proposed by Russia for the 26th International Mathematical Olympiad, 1985)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :90. Solve in real numbers the equation

$$\sqrt[3]{x-1}+\sqrt[3]{x}+\sqrt[3]{x+1}=0$$

Solution :90. First solution: Using the indentity

$$a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

applied to the (distinct) numbers $a=\sqrt[3]{x-1}, b=\sqrt[3]{x}$, and $c=\sqrt[3]{x+1}$, we transform the equation into the equivalent

$$(x-1)+x+(x+1)-3\sqrt[3]{(x-1)x(x+1)}=0.$$

We further change this into $x=\sqrt[3]{x^3-x}$. Raising both sides to the third power, we obtain $x^3=x^3-x$. We conclude that the equation has the unique solution $x=0$.Second solution: The function $f:\mathbb{R}\rightarrow\mathbb{R}, f(x)=\sqrt[3]{x-1}+\sqrt[3]{x}+\sqrt[3]{x+1}$ is strictly increasing, so the equation $f(x)=0$ has at most one solution. Since $x=0$ satisfies this equation, it is the unique solution.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :91. Find all triples (x, y, z) of positive integers such that

$$x^3+y^3+z^3-3xyz=p,$$

where p is a prime number greater than 3 .
Solution :91. The key observation is that the left-hand side of the equation can be factored as

$$(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)=p.$$

Since $x+y+z>1$ and p is prime, we must have $x+y+z=p$ and $x^2+y^2+z^2-xy-yz-zx=1$. The second equality can be written as $(x-y)^2+(y-z)^2+(z-x)^2=2$. Without loss of generality, we may assume that $x\geq y\geq z$. If $x>y>z$, then $x-y\geq 1$, $y-z\geq 1$, and $x-z\geq 2$, which would imply that $(x-y)^2+(y-z)^2+(z-x)^2\geq 6>2$.Therefore, either $x=y=z+1$ or $x-1=y=z$. According to whether the prime p is of the form $3k+1$ or $3k+2$, the solutions are $\left(\frac{p-1}{3}, \frac{p-1}{3}, \frac{p+2}{3}\right)$ and the corresponding permutations, or $\left(\frac{p-2}{3}, \frac{p+1}{3}, \frac{p+1}{3}\right)$ and the corresponding permutations.(T. Andreescu, D. Andrica, An Introduction to Diophantine Equations, GIL 2002)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :93. Find all triples (m, n, p) of positive integers such that $m+n+p=2002$ and the system of equations $\frac{x}{y}+\frac{y}{x}=m, \frac{y}{z}+\frac{z}{y}=n, \frac{z}{x}+\frac{x}{z}=p$

has at least one solution in nonzero real numbers.
Solution :93. This is a difficult exercise in completing squares. We have

$$mnp = 1 + \frac{x^2}{z^2} + \frac{z^2}{y^2} + \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} + 1$$

$$= \left(\frac{x}{y} + \frac{y}{x}\right)^2 + \left(\frac{y}{z} + \frac{z}{y}\right)^2 + \left(\frac{z}{x} + \frac{x}{z}\right)^2 - 4.$$

Hence

$$m^2 + n^2 + p^2 = mnp + 4.$$

Adding $2(mn + np + pm)$ to both sides yields

$$(m + n + p)^2 = mnp + 2(mn + np + pm) + 4.$$

Adding now $4(m + n + p) + 4$ to both sides gives

$$(m + n + p + 2)^2 = (m + 2)(n + 2)(p + 2).$$

It follows that

$$(m + 2)(n + 2)(p + 2) = 2004^2.$$

But $2004 = 2^2 \times 3 \times 167$, and a simple case analysis shows that the only possibilities are $(m+2, n+2, p+2) = (4, 1002, 1002), (1002, 4, 1002), (1002, 1002, 4)$. The desired triples are $(2, 1000, 1000), (1000, 2, 1000), (1000, 1000, 2)$.(proposed by T. Andreescu for the 43rd International Mathematical Olympiad, 2002)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :94. Find

$$\min_{a,b\in\mathbb{R}} \max\left(a^2 + b, b^2 + a\right).$$

Solution :94. Let $M(a, b) = \max\left(a^2 + b, b^2 + a\right)$. Then $M(a, b) \geq a^2 + b$ and $M(a, b) \geq b^2 + a$, so $2M(a, b) \geq a^2 + b + b^2 + a$. It follows that

$$2M(a, b) + \frac{1}{2} \geq \left(a + \frac{1}{2}\right)^2 + \left(b + \frac{1}{2}\right)^2 \geq 0,$$

hence $M(a, b) \geq -\frac{1}{4}$. We deduce that

$$\min_{a,b\in\mathbb{R}} M(a, b) = -\frac{1}{4},$$

which, in fact, is attained when $a = b = -\frac{1}{2}$.(T. Andreescu)
Topic :Algebra
Book :Putnam and Beyond

Final Answer :

Problem Statement :96. Find all positive integers n for which the equation

$$nx^4 + 4x + 3 = 0$$

has a real root.
Solution :96. Clearly, 0 is not a solution. Solving for n yields $\frac{-4x-3}{x^4} \geq 1$, which reduces to $x^4 + 4x + 3 \leq 0$. The last inequality can be written in its equivalent form,

$$\left(x^2 - 1\right)^2 + 2(x+1)^2 \leq 0,$$

whose only real solution is $x = -1$.Hence $n = 1$ is the unique solution, corresponding to $x = -1$.(T. Andreescu)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :97. Find all triples (x, y, z) of real numbers that are solutions to the system of equations

$$\frac{4x^2}{4x^2+1} = y,$$
$$\frac{4y^2}{4y^2+1} = z,$$
$$\frac{4z^2}{4z^2+1} = x.$$

Solution :97. If $x = 0$, then $y = 0$ and $z = 0$, yielding the triple $(x, y, z) = (0, 0, 0)$. If $x \neq 0$, then $y \neq 0$ and $z \neq 0$, so we can rewrite the equations of the system in the form

$$1 + \frac{1}{4x^2} = \frac{1}{y},$$
$$1 + \frac{1}{4y^2} = \frac{1}{z},$$
$$1 + \frac{1}{4z^2} = \frac{1}{x}.$$

Summing up the three equations leads to

$$\left(1 - \frac{1}{x} + \frac{1}{4x^2}\right) + \left(1 - \frac{1}{y} + \frac{1}{4y^2}\right) + \left(1 - \frac{1}{z} + \frac{1}{4z^2}\right) = 0.$$

This is equivalent to

$$\left(1 - \frac{1}{2x}\right)^2 + \left(1 - \frac{1}{2y}\right)^2 + \left(1 - \frac{1}{2z}\right)^2 = 0.$$

It follows that $\frac{1}{2x}=\frac{1}{2y}=\frac{1}{2z}=1$, yielding the triple $(x,y,z)=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. Both triples satisfy the equations of the system.(Canadian Mathematical Olympiad, 1996)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :98. Find the minimum of

$$\log_{x_1}\left(x_2-\frac{1}{4}\right)+\log_{x_2}\left(x_3-\frac{1}{4}\right)+\cdots+\log_{x_n}\left(x_1-\frac{1}{4}\right)$$

over all $x_1,x_2,\ldots,x_n\in\left(\frac{1}{4},1\right)$.
Solution :98. First, note that $\left(x-\frac{1}{2}\right)^2\geq 0$ implies $x-\frac{1}{4}\leq x^2$, for all real numbers x. Applying this and using the fact that the x_i 's are less than 1 , we find that

$$\log_{x_k}\left(x_{k+1}-\frac{1}{4}\right)\geq\log_{x_k}\left(x_{k+1}^2\right)=2\log_{x_k}x_{k+1}.$$

Therefore,

$$\sum_{k=1}^{n}\log_{x_k}\left(x_{k+1}-\frac{1}{4}\right)\geq 2\sum_{k=1}^{n}\log_{x_k}x_{k+1}\geq 2n\sqrt[n]{\frac{\ln x_2}{\ln x_1}\cdot\frac{\ln x_3}{\ln x_2}\cdots\frac{\ln x_n}{\ln x_1}}=2n.$$

So a good candidate for the minimum is $2n$, which is actually attained for $x_1=x_2=\cdots=x_n=\frac{1}{2}$.(Romanian Mathematical Olympiad, 1984, proposed by T. Andreescu)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :101. Find all pairs (x,y) of real numbers that are solutions to the system

$$x^4+2x^3-y=-\frac{1}{4}+\sqrt{3},$$
$$y^4+2y^3-x=-\frac{1}{4}-\sqrt{3}.$$

Solution :101. Adding up the two equations yields

$$\left(x^4+2x^3-x+\frac{1}{4}\right)+\left(y^4+2y^3-y+\frac{1}{4}\right)=0.$$

Here we recognize two perfect squares, and write this as

$$\left(x^2+x-\frac{1}{2}\right)^2+\left(y^2+y-\frac{1}{2}\right)^2=0.$$

Equality can hold only if $x^2+x-\frac{1}{2}=y^2+y-\frac{1}{2}=0$, which then gives $\{x,y\} \subset \left\{-\frac{1}{2}-\frac{\sqrt{3}}{2}, -\frac{1}{2}+\frac{\sqrt{3}}{2}\right\}$. Moreover, since $x \neq y, \{x,y\} = \left\{-\frac{1}{2}-\frac{\sqrt{3}}{2}, -\frac{1}{2}+\frac{\sqrt{3}}{2}\right\}$. A simple verification leads to $(x,y) = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}, -\frac{1}{2}-\frac{\sqrt{3}}{2}\right)$.(Mathematical Reflections, proposed by T. Andreescu)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :105. Let $a_1, a_2, \ldots, a_n$ be distinct real numbers. Find the maximum of

$$a_1 a_{\sigma(a)} + a_2 a_{\sigma(2)} + \cdots + a_n a_{\sigma(n)}$$

over all permutations of the set $\{1, 2, \ldots, n\}$.
Solution :105. Apply Cauchy-Schwarz:

$$\begin{aligned}\left(a_1 a_{\sigma(a)} + a_2 a_{\sigma(2)} + \cdots + a_n a_{\sigma(n)}\right)^2 &\leq \left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(n)}^2\right) \\ &= \left(a_1^2+a_2^2+\cdots+a_n^2\right)^2.\end{aligned}$$

The maximum is $a_1^2+a_2^2+\cdots+a_n^2$. The only permutation realizing it is the identity permutation.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :107. Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1+\cdots+k_n = 5n-4$ and

$$\frac{1}{k_1}+\cdots+\frac{1}{k_n}=1.$$

Solution :107. By the Cauchy-Schwarz inequality,

$$(k_1+\cdots+k_n)\left(\frac{1}{k_1}+\cdots+\frac{1}{k_n}\right) \geq n^2.$$

We must thus have $5n-4 \geq n^2$, so $n \leq 4$. Without loss of generality, we may suppose that $k_1 \leq \cdots \leq k_n$.If $n=1$, we must have $k_1=1$, which is a solution. Note that hereinafter we cannot have $k_1=1$If $n=2$, we have $(k_1,k_2) \in \{(2,4),(3,3)\}$, neither of which satisfies the relation from the statement.If $n=3$, we have $k_1+k_2+k_3=11$, so $2 \leq k_1 \leq 3$. Hence $(k_1,k_2,k_3) \in \{(2,2,7),(2,3,6),(2,4,5),(3,3,5),(3,4,4)\}$, and only $(2,3,6)$ works.If $n=4$, we must have equality in the Cauchy-Schwarz inequality, and this can happen only if $k_1=k_2=k_3=k_4=4$.Hence the solutions are $n=1$ and $k_1=1, n=3$, and (k_1,k_2,k_3) is a permutation of $(2,3,6)$, and $n=4$ and $(k_1,k_2,k_3,k_4)=(4,4,4,4)$.(66th W.L. Putnam Mathematical Competition, 2005, proposed by

T. Andreescu)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :125. Which number is larger,

$$\prod_{n=1}^{25}\left(1-\frac{n}{365}\right) \text{ or } \frac{1}{2}?$$

Solution :125. It is natural to try to simplify the product, and for this we make use of the AM-GM inequality:

$$\prod_{n=1}^{25}\left(1-\frac{n}{365}\right) \leq \left[\frac{1}{25}\sum_{n=1}^{25}\left(1-\frac{n}{365}\right)\right]^{25} = \left(\frac{352}{365}\right)^{25} = \left(1-\frac{13}{365}\right)^{25}.$$

We now use Newton's binomial formula to estimate this power. First, note that

$$\binom{25}{k}\left(\frac{13}{365}\right)^{k} \geq \binom{25}{k+1}\left(\frac{13}{365}\right)^{k+1},$$

since this reduces to

$$\frac{13}{365} \leq \frac{k+1}{25-k},$$

and the latter is always true for $1 \leq k \leq 24$. For this reason if we ignore the part of the binomial expansion beginning with the fourth term, we increase the value of the expression. In other words,

$$\left(1-\frac{13}{365}\right)^{25} \leq 1-\binom{25}{1}\frac{13}{365}+\binom{25}{2}\frac{13^2}{365^2} = 1-\frac{65}{73}+\frac{169\cdot 12}{63^2} < \frac{1}{2}.$$

We conclude that the second number is larger.(Soviet Union University Student Mathematical Olympiad, 1975)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :136. What is the maximal value of the expression $\sum_{i<j} x_i x_j$ if $x_1, x_2, \ldots, x_n$ are nonnegative integers whose sum is equal to m ?
Solution :136. There exist finitely many n-tuples of positive integers with the sum equal to m, so the expression from the statement has indeed a maximal value.We show that the maximum is not attained if two of the x_i 's differ by 2 or more. Without loss of generality, we may assume that $x_1 \leq x_2 - 2$. Increasing x_1 by 1 and decreasing x_2 by 1 yields

$$\begin{aligned}&\textstyle\sum_{2<i<j} x_i x_j + (x_1+1)\sum_{2<i} x_i + (x_2-1)\sum_{2<i} x_i + (x_1+1)(x_2-1)\\ &\quad = \textstyle\sum_{2<i<j} x_i x_j + x_1\sum_{2<i} x_i + x_2\sum_{2<i} x_i + x_1x_2 - x_1 + x_2 + 1.\end{aligned}$$

The sum increased by $x_2-x_1-1 \geq 1$, and hence the original sum was not maximal.This shows that the expression attains its maximum for a configuration in which the x_i 's differ from each other by at most 1 . If $\frac{m}{n}=r n+s$, with $0 \leq s<n$, then for this to happen $n-s$ of the x_i 's must be equal to $r+1$ and the remaining must be equal to r. This gives that the maximal value of the expression must be equal to

$$\frac{1}{2}(n-s)(n-s-1) r^2+s(n-s) r(r+1)+\frac{1}{2} s(s-1)(r+1)^2 .$$

(Mathematical Olympiad Summer Program 2002, communicated by Z. Sunik)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :137. Given the $n \times n$ array $\left(a_{i j}\right)_{i j}$ with $a_{i j}=i+j-1$, what is the smallest product of n elements of the array provided that no two lie on the same row or column?
Solution :137. There are finitely many such products, so a smallest product does exist. Examining the $2 \times 2,3 \times 3$, and 4×4 arrays, we conjecture that the smallest product is attained on the main diagonal and is $1 \cdot 3 \cdots 5 \cdots(2 n-1)$. To prove this, we show that if the permutation σ of $\{1,2, \ldots, n\}$ has an inversion, then $a_{1 \sigma(1)} a_{2 \sigma(2)} \cdots a_{n \sigma(n)}$ is not minimal.MATHPIX IMAGEFigure 62So assume that the inversion gives rise to the factors $i+(j+k)-1$ and $(i+m)+j-1$ in the product. Let us replace them with $i+j-1$ and $(i+m)+(j+k)-1$, as shown in Figure 62. The product of the first pair is

$$i^2+i k+i(j-1)+m i+m k+m(j-1)+(j-1) i+(j-1) k+(j-1)^2,$$

while the product of the second pair is

$$i^2+i m+i k+i(j-1)+(j-1) m+(j-1) k+(j-1)^2 .$$

We can see that the first of these expressions exceeds the second by $m k$. This proves that if the permutation has an inversion, then the product is not minimal. The only permutation without inversions is the identity permutation. By Sturm's principle, it is the permutation for which the minimum is attained. This minimum is $1 \cdot 3 \cdots 5 \cdots(2 n-1)$, as claimed.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :138. Given a positive integer n, find the minimum value of

$$\frac{x_1^3+x_2^3+\cdots+x_n^3}{x_1+x_2+\cdots+x_n}$$

subject to the condition that $x_1, x_2, \ldots, x_n$ be distinct positive integers.
Solution :138. Order the numbers $x_1<x_2<\cdots<x_n$ and call the expression from the statement $E\left(x_1, x_2, \ldots, x_n\right)$. Note that $E\left(x_1, x_2, \ldots, x_n\right)>\frac{x_n^2}{n}$,

which shows that as the variables tend to infinity, so does the expression. This means that the minimum exists. Assume that the minimum is attained at the point $(y_1, y_2, \ldots, y_n)$. If $y_n - y_1 > n$ then there exist indices i and $j, i < j$, such that $y_1, \ldots, y_i + 1, \ldots, y_j - 1, \ldots, y_n$ are still distinct integers. When substituting these numbers into E the denominator stays constant while the numerator changes by $3\left(y_j + y_i\right)\left(y_j - y_i - 1\right)$, a negative number, decreasing the value of the expression. This contradicts the minimality. We now look at the case with no gaps: $y_n - y_1 = n - 1$. Then there exists a such that $y_1 = a + 1, y_2 = a + 2, \ldots, y_n = a + n$. We have

$$\begin{aligned} E\left(y_1, \ldots, y_n\right) &= \frac{na^3 + 3\frac{n(n+1)}{2}a^2 + \frac{n(n+1)(2n+1)}{2}a + \frac{n^2(n+1)^2}{4}}{na + \frac{n(n+1)}{2}} \\ &= \frac{a^3 + \frac{3(n+1)}{2}a^2 + \frac{(n+1)(2n+1)}{2}a + \frac{n(n+1)^2}{4}}{a + \frac{n+1}{2}}. \end{aligned}$$

When $a = 0$ this is just $\frac{n(n+1)}{2}$. Subtracting this value from the above, we obtain

$$\frac{a^3 + \frac{3(n+1)}{2}a^2 + \left[\frac{(n+1)(2n+1)}{2} - \frac{n(n+1)}{2}\right]a}{a + \frac{n+1}{2}} > 0.$$

We deduce that $\frac{n(n+1)}{2}$ is a good candidate for the minimum.If $y_n - y_1 = n$, then there exist a and k such that $y_1 = a, \ldots, y_k = a+k-1, y_{k+1} = a+k+1, \ldots, y_n = a + n$. Then

$$\begin{aligned} E\left(y_1, \ldots, y_n\right) &= \frac{a^3 + \cdots + (a+k-1)^3 + (a+k+1)^3 + \cdots + (a+n)^3}{a + \cdots + (a+k-1) + (a+k+1) + \cdots + (a+n)} \\ &= \frac{\sum_{j=0}^{n}(a+j)^3 - (a+k)^3}{\sum_{j=0}^{n}(a+j) - (a+k)} \\ &= \frac{na^3 + 3\left[\frac{n(n+1)}{2} - k\right]a^2 + 3\left[\frac{n(n+1)(2n+1)}{6} - k^2\right]a + \left[\frac{n^2(n+1)^2}{4} - k^3\right]}{na + \frac{n(n+1)}{2} - k}. \end{aligned}$$

Subtracting $\frac{n(n+1)}{2}$ from this expression, we obtain

$$\frac{na^3 + 3\left[\frac{n(n+1)}{2} - k\right]a^2 + \left[\frac{n(n+1)(2n+1)}{2} - 3k^2 - \frac{n^2(n+1)}{2}\right]a - k^3 + \frac{n(n+1)}{2}k}{na + \frac{n(n+1)}{2} - k}.$$

The numerator is the smallest when $k = n$ and $a = 1$, in which case it is equal to 0 . Otherwise, it is strictly positive, proving that the minimum is not attained in that case. Therefore, the desired minimum is $\frac{n(n+1)}{2}$, attained only if $x_k = k, k = 1, 2, \ldots, n$.(American Mathematical Monthly, proposed by C. Popescu)

Topic :Algebra

Book :Putnam and Beyond

Final Answer :

Problem Statement :146. Find all polynomials satisfying the functional equation

$$(x+1)P(x) = (x-10)P(x+1).$$

Solution :146. The relation $(x+1)P(x) = (x-10)P(x+1)$ shows that $P(x)$ is divisible by $(x-10)$. Shifting the variable, we obtain the equivalent relation $xP(x-1) = (x-11)P(x)$, which shows that $P(x)$ is also divisible by x. Hence $P(x) = x(x-10)P_1(x)$ for some polynomial $P_1(x)$. Substituting in the original equation and canceling common factors, we find that $P_1(x)$ satisfies

$$xP_1(x) = (x-9)P_1(x+1).$$

Arguing as before, we find that $P_1(x) = (x-1)(x-9)P_2(x)$. Repeating the argument, we eventually find that $P(x) = x(x-1)(x-2)\cdots(x-10)Q(x)$, where $Q(x)$ satisfies $Q(x) = Q(x+1)$. It follows that $Q(x)$ is constant, and the solution to the problem is

$$P(x) = \mathrm{ax}(x-1)(x-2)\cdots(x-10),$$

where a is an arbitrary constant.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :148. Determine all polynomials $P(x)$ with real coefficients for which there exists a positive integer n such that for all x,

$$P\left(x+\frac{1}{n}\right) + P\left(x-\frac{1}{n}\right) = 2P(x).$$

Solution :148. First solution: Let m be the degree of $P(x)$, and write

$$P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0.$$

Using the binomial formula for $\left(x \pm \frac{1}{n}\right)^m$ and $\left(x \pm \frac{1}{n}\right)^{m-1}$ we transform the identity from the statement into

$$\begin{aligned} 2a_m x^m + 2a_{m-1}x^{m-1} + 2a_{m-2}x^{m-2} + a_m \tfrac{m(m-1)}{n^2}x^{m-2} + Q(x) \\ = 2a_m x^m + 2a_{m-1}x^{m-1} + 2a_{m-2}x^{m-2} + R(x), \end{aligned}$$

where Q and R are polynomials of degree at most $m-3$. If we identify the coefficients of the corresponding powers of x, we find that $a_m \frac{m(m-1)}{n^2} = 0$. But $a_m \neq 0$, being the leading coefficient of the polynomial; hence $m(m-1) = 0$. So either $m = 0$ or $m = 1$. One can check in an instant that all polynomials of degree 0 or 1 satisfy the required condition.Second solution: Fix a point x_0.

The graph of $P(x)$ has infinitely many points in common with the line that has slope

$$m=n\left(P\left(x_0+\frac{1}{n}\right)-P\left(x_0\right)\right)$$

and passes through the point $\left(x_0, P\left(x_0\right)\right)$. Therefore, the graph of $P(x)$ is a line, so the polynomial has degree 0 or 1 .Third solution: If there is such a polynomial of degree $m \geq 2$, differentiating the given relation $m-2$ times we find that there is a quadratic polynomial that satisfies the given relation. But then any point on its graph would be the vertex of the parabola, which of course is impossible. Hence only linear and constant polynomials satisfy the given relation.(Romanian Team Selection Test for the International Mathematical Olympiad, 1979, proposed by D. Buşneag)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :149. Find a polynomial with integer coefficients that has the zero $\sqrt{2}+\sqrt[3]{3}$.
Solution :149. Let $x=\sqrt{2}+\sqrt[3]{3}$. Then $\sqrt[3]{3}=x-\sqrt{2}$, which raised to the third power yields $3=x^3-3 \sqrt{2} x^2+6 x-2 \sqrt{2}$, or

$$x^3+6 x-3=\left(3 x^2+2\right) \sqrt{2}.$$

By squaring this equality we deduce that x satisfies the polynomial equation

$$x^6-6 x^4-6 x^3+12 x^2-36 x+1=0.$$

(Belgian Mathematical Olympiad, 1978, from a note by P. Radovici-Mărculescu)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :151. Let $P(x)$ be a polynomial of degree n. Knowing that

$$P(k)=\frac{k}{k+1}, \quad k=0,1, \ldots, n,$$

find $P(m)$ for $m>n$.
Solution :151. Because $P(0)=0$, there exists a polynomial $Q(x)$ such that $P(x)=x Q(x)$. Then

$$Q(k)=\frac{1}{k+1}, \quad k=1,2, \ldots, n.$$

Let $H(x)=(x+1) Q(x)-1$. The degree of $H(x)$ is n and $H(k)=0$ for $k=1,2, \ldots, n$. Hence

$$H(x)=(x+1) Q(x)-1=a_0(x-1)(x-2) \cdots(x-n).$$

In this equality $H(-1)=-1$ yields $a_0=\frac{(-1)^{n+1}}{(n+1)!}$. For $x=m, m>n$, which gives

$$Q(m)=\frac{(-1)^{n+1}(m-1)(m-2)\cdots(m-n)+1}{(n+1)!(m+1)}+\frac{1}{m+1},$$

and so

$$P(m)=\frac{(-1)^{m+1}m(m-1)\cdots(m-n)}{(n+1)!(m+1)}+\frac{m}{m+1}.$$

(D. Andrica, published in T. Andreescu, D. Andrica, 360 Problems for Mathematical Contests, GIL, 2003)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :154. Find the zeros of the polynomial

$$P(x)=x^4-6x^3+18x^2-30x+25$$

knowing that the sum of two of them is 4 .
Solution :154. Denote the zeros of $P(x)$ by x_1, x_2, x_3, x_4, such that $x_1+x_2=4$. The first Viète relation gives $x_1+x_2+x_3+x_4=6$; hence $x_3+x_4=2$. The second Viète relation can be written as

$$x_1x_2+x_3x_4+\left(x_1+x_2\right)\left(x_3+x_4\right)=18,$$

from which we deduce that $x_1x_2+x_3x_4=18-2\cdot 4=10$. This, combined with the fourth Viète relation $x_1x_2x_3x_4=25$, shows that the products x_1x_2 and x_3x_4 are roots of the quadratic equation $u^2-10u+25=0$. Hence $x_1x_2=x_3x_4=5$, and therefore x_1 and x_2 satisfy the quadratic equation $x^2-4x+5=0$, while x_3 and x_4 satisfy the quadratic equation $x^2-2x+5=0$. We conclude that the zeros of $P(x)$ are $2+i, 2-i, 1+2i, 1-2i$.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :156. Solve the system

$$\begin{aligned} x+y+z&=1, \\ xyz&=1, \end{aligned}$$

knowing that x, y, z are complex numbers of absolute value equal to 1 .
Solution :156. Taking the conjugate of the first equation, we obtain

$$\bar{x}+\bar{y}+\bar{z}=1,$$

and hence

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

Combining this with $xyz = 1$, we obtain

$$xy + yz + xz = 1.$$

Therefore, x, y, z are the roots of the polynomial equation

$$t^3 - t^2 + t - 1 = 0,$$

which are $1, i, -i$. Any permutation of these three complex numbers is a solution to the original system of equations.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :157. Find all real numbers r for which there is at least one triple (x, y, z) of nonzero real numbers such that

$$x^2y + y^2z + z^2x = xy^2 + yz^2 + zx^2 = rxyz.$$

Solution :157. Dividing by the nonzero xyz yields $\frac{x}{z} + \frac{y}{x} + \frac{z}{y} = \frac{y}{z} + \frac{z}{x} + \frac{x}{y} = r$. Let $a = \frac{x}{y}, b = \frac{y}{z}$, $c = \frac{z}{x}$. Then $abc = 1, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = r, a + b + c = r$. Hence

$$\begin{aligned} a + b + c &= r, \\ ab + bc + ca &= r, \\ abc &= 1. \end{aligned}$$

We deduce that a, b, c are the solutions of the polynomial equation $t^3 - rt^2 + rt - 1 = 0$. This equation can be written as

$$(t - 1)\left[t^2 - (r - 1)t + 1\right] = 0.$$

Since it has three real solutions, the discriminant of the quadratic must be positive. This means that $(r - 1)^2 - 4 \geq 0$, leading to $r \in (-\infty, -1] \cup [3, \infty)$. Conversely, all such r work.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :159. Find all polynomials whose coefficients are equal either to 1 or -1 and whose zeros are all real.
Solution :159. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$. Denote its zeros by $x_1, x_2, \ldots, x_n$. The first two of Viète's relations give

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= -\frac{a_{n-1}}{a_n}, \\ x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n &= \frac{a_{n-2}}{a_n} \end{aligned}$$

Combining them, we obtain

$$x_1^2 + x_2^2 + \cdots + x_n^2 = \left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right).$$

The only possibility is $x_1^2 + x_2^2 + \cdots + x_n^2 = 3$. Given that $x_1^2 x_2^2 \cdots x_n^2 = 1$, the AM-GM inequality yields

$$3 = x_1^2 + x_2^2 + \cdots + x_n^2 \geq n\sqrt[n]{x_1^2 x_2^2 \cdots x_n^2} = n.$$

Therefore, $n \leq 3$. Eliminating case by case, we find among linear polynomials $x+1$ and $x-1$, and among quadratic polynomials x^2+x-1 and x^2-x-1. As for the cubic polynomials, we should have equality in the AM-GM inequality. So all zeros should have the same absolute values. The polynomial should share a zero with its derivative. This is the case only for x^3+x^2-x-1 and x^3-x^2-x+1, which both satisfy the required property. Together with their negatives, these are all desired polynomials.(Indian Olympiad Training Program, 2005)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :161. The zeros of the polynomial $P(x) = x^3 - 10x + 11$ are u, v, and w. Determine the value of $\arctan u + \arctan v + \arctan w$.
Solution :161. First solution: Let $\alpha = \arctan u, \beta = \arctan v$, and $\arctan w$. We are required to determine the sum $\alpha + \beta + \gamma$. The addition formula for the tangent of three angles,

$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - (\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\alpha\tan\gamma)},$$

implies

$$\tan(\alpha+\beta+\gamma) = \frac{u+v+w-uvw}{1-(uv+vw+uv)}.$$

Using Viète's relations,

$$u+v+w=0, \quad uv+vw+uw=-10, \quad uvw=-11,$$

we further transform this into $\tan(\alpha+\beta+\gamma) = \frac{11}{1+10} = 1$. Therefore, $\alpha+\beta+\gamma = \frac{\pi}{4} + k\pi$, where k is an integer that remains to be determined.From Viète's relations we can see the product of the zeros of the polynomial is negative, so the number of negative zeros is odd. And since the sum of the zeros is 0 , two of them are positive and one is negative. Therefore, one of α, β, γ lies in the interval $\left(-\frac{\pi}{2}, 0\right)$ and two of them lie in $\left(0, \frac{\pi}{2}\right)$. Hence k must be equal to 0 , and $\arctan u +$ $\arctan v + \arctan w = \frac{\pi}{4}$.Second solution: Because

$$\operatorname{Im}\ln(1+ix) = \arctan x,$$

we see that

$$\arctan u + \arctan v + \arctan w = \operatorname{Im} \ln(iP(i)) = \operatorname{Im} \ln(11 + 11i)$$
$$= \arctan 1 = \frac{\pi}{4}.$$

(Kózépiskolai Matematikai Lapok (Mathematics Magazine for High Schools, Budapest), proposed by K. Bérczi).
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :163. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\begin{pmatrix} n \\ 1 \end{pmatrix}, a_{n-2} = \begin{pmatrix} n \\ 2 \end{pmatrix}$, and that all roots are real, find the remaining coefficients.
Solution :163. A good guess is that $P(x) = (x-1)^n$, and we want to show that this is the case. To this end, let $x_1, x_2, \ldots, x_n$ be the zeros of $P(x)$. Using Viète's relations, we can write

$$\sum_i (x_i - 1)^2 = \left(\sum_i x_i\right)^2 - 2\sum_{i<j} x_i x_j - 2\sum_i x_i + n$$
$$= n^2 - 2\frac{n(n-1)}{2} - 2n + n = 0.$$

This implies that all squares on the left are zero. So $x_1 = x_2 = \cdots = x_n = 1$, and $P(x) = (x-1)^n$, as expected.

1 (Gazeta Matematic ă (Mathematics Gazette, Bucharest))

Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :164. Determine the maximum value of λ such that whenever $P(x) = x^3 + ax^2 + bx + c$ is a cubic polynomial with all zeros real and nonnegative, then

$$P(x) \geq \lambda(x-a)^3$$

for all $x \geq 0$. Find the equality condition.
Solution :164. Let α, β, γ be the zeros of $P(x)$. Without loss of generality, we may assume that $0 \leq \alpha \leq \beta \leq \gamma$. Then

$$x - a = x + \alpha + \beta + \gamma \geq 0 \quad \text{and} \quad P(x) = (x-\alpha)(x-\beta)(x-\gamma).$$

If $0 \leq x \leq \alpha$, using the AM-GM inequality, we obtain

$$-P(x) = (\alpha - x)(\beta - x)(\gamma - x) \leq \frac{1}{27}(\alpha + \beta + \gamma - 3x)^3$$

$$\leq \frac{1}{27}(x + \alpha + \beta + \gamma)^3 = \frac{1}{27}(x - a)^3,$$

so that $P(x) \geq -\frac{1}{27}(x-a)^3$. Equality holds exactly when $\alpha - x = \beta - x = \gamma - x$ in the first inequality and $\alpha + \beta + \gamma - 3x = x + \alpha + \beta + \gamma$ in the second, that is, when $x = 0$ and $\alpha = \beta = \gamma$.If $\beta \leq x \leq \gamma$, then using again the AM-GM inequality, we obtain

$$-P(x) = (x - \alpha)(x - \beta)(\gamma - x) \leq \frac{1}{27}(x + \gamma - \alpha - \beta)^3$$
$$\leq \frac{1}{27}(x + \alpha + \beta + \gamma)^3 = \frac{1}{27}(x - a)^3,$$

so that again $P(x) \geq -\frac{1}{27}(x-a)^3$. Equality holds exactly when there is equality in both inequalities, that is, when $\alpha = \beta = 0$ and $\gamma = 2x$.Finally, when $\alpha < x < \beta$ or $x > \gamma$, then

$$P(x) > 0 \geq -\frac{1}{27}(x - a)^3.$$

Thus the desired constant is $\lambda = -\frac{1}{27}$, and the equality occurs when $\alpha = \beta = \gamma$ and $x = 0$, or when $\alpha = \beta = 0, \gamma$ is any nonnegative real, and $x = \frac{\gamma}{2}$.(Chinese Mathematical Olympiad, 1999)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :166. Find all polynomials $P(x)$ with integer coefficients satisfying $P\left(P'(x)\right) = P'(P(x))$ for all $x \in \mathbb{R}$.
Solution :166. Let us first consider the case $n \geq 2$. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$. Then

$$P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

Identifying the coefficients of $x^{n(n-1)}$ in the equality $P\left(P'(x)\right) = P'(P(x))$, we obtain

$$a_n^{n+1} \cdot n^n = a_n^n \cdot n.$$

This implies $a_n n^{n-1} = 1$, and so

$$a_n = \frac{1}{n^{n-1}}.$$

Since a_n is an integer, n must be equal to 1 , a contradiction. If $n = 1$, say $P(x) = ax + b$, then we should have $a^2 + b = a$, hence $b = a - a^2$. Thus the answer to the problem is the polynomials of the form $P(x) = ax^2 + a - a^2$.(Revista Matematică din Timişoara (Timişoara Mathematics Gazette), proposed by T.

Andreescu)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :167. Determine all polynomials $P(x)$ with real coefficients satisfying $(P(x))^n = P(x^n)$ for all $x \in \mathbb{R}$, where $n > 1$ is a fixed integer.
Solution :167. Let m be the degree of $P(x)$, so $P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$. If $P(x) = x^k Q(x)$, then

$$x^{kn} Q^n(x) = x^{kn} Q(x^n),$$

so

$$Q^n(x) = Q(x^n),$$

which means that $Q(x)$ satisfies the same relation.Thus we can assume that $P(0) \neq 0$. Substituting $x = 0$, we obtain $a_0^n = a_0$, and since a_0 is a nonzero real number, it must be equal to 1 if n is even, and to ± 1 if n is odd.Differentiating the relation from the statement, we obtain

$$nP^{n-1}(x)P'(x) = nP'(x^n)\, x^{n-1}.$$

For $x = 0$ we have $P'(0) = 0$; hence $a_1 = 0$. Differentiating the relation again and reasoning similarly, we obtain $a_2 = 0$, and then successively $a_3 = a_4 = \cdots = a_m = 0$. It follows that $P(x) = 1$ if n is even and $P(x) = \pm 1$ if n is odd.In general, the only solutions are $P(x) = x^m$ if n is even, and $P(x) = \pm x^m$ if n is odd, m being some nonnegative integer.(T. Andreescu)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :194. Let r be a positive real number such that $\sqrt[6]{r}+\frac{1}{\sqrt[6]{r}} = 6$. Find the maximum value of $\sqrt[4]{r} - \frac{1}{\sqrt[4]{r}}$
Solution :194. From the identity

$$x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right),$$

it follows that

$$\sqrt{r} + \frac{1}{\sqrt{r}} = 6^3 - 3 \times 6 = 198.$$

Hence

$$\left(\sqrt[4]{r} - \frac{1}{\sqrt[4]{r}}\right)^2 = 198 - 2,$$

and the maximum value of $\sqrt[4]{r} - \frac{1}{\sqrt[4]{r}}$ is 14 .(University of Wisconsin at Whitewater Math Meet, 2003, proposed by T. Andreescu)

Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :196. Find all quintuples (x, y, z, v, w) with $x, y, z, v, w \in [-2, 2]$ satisfying the system of equations

$$\begin{aligned} x + y + z + v + w &= 0, \\ x^3 + y^3 + z^3 + v^3 + w^3 &= 0, \\ x^5 + y^5 + z^5 + v^5 + w^5 &= -10. \end{aligned}$$

Solution :196. Because the five numbers lie in the interval $[-2, 2]$, we can find corresponding angles $t_1, t_2, t_3, t_4, t_5 \in [0, \pi]$ such that $x = 2\cos t_1, y = 2\cos t_2, z = 2\cos t_3, v = 2\cos t_4$, and $w = 2\cos t_5$. We would like to translate the third and fifth powers into trigonometric functions of multiples of the angles. For that we use the polynomials $\mathcal{T}_n(a)$. For example, $\mathcal{T}_5(a) = a^5 - 5a^3 + 5a$. This translates into the trigonometric identity $2\cos 5\theta = (2\cos\theta)^5 - 5(2\cos\theta)^3 + 5(2\cos\theta)$.Add to the third equation of the system the first multiplied by 5 and the second multiplied by -5, then use the above-mentioned trigonometric identity to obtain

$$2\cos 5t_1 + 2\cos 5t_2 + 2\cos 5t_3 + 2\cos 5t_4 + 2\cos 5t_5 = -10.$$

This can happen only if $\cos 5t_1 = \cos 5t_2 = \cos 5t_3 = \cos 5t_5 = \cos 5t_5 = -1$. Hence

$$t_1, t_2, t_3, t_4, t_5 \in \left\{ \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5} \right\}.$$

Using the fact that the roots of $x^5 = 1$, respectively, $x^{10} = 1$, add up to zero, we deduce that

$$\sum_{k=0}^{4} \cos \frac{2k\pi}{5} = 0 \quad \text{and} \quad \sum_{k=0}^{9} \cos \frac{k\pi}{5} = 0.$$

It follows that

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} + \cos\frac{5\pi}{5} + \cos\frac{7\pi}{5} + \cos\frac{9\pi}{5} = 0.$$

Since $\cos\frac{\pi}{5} = \cos\frac{9\pi}{5}$ and $\cos\frac{3\pi}{5} = \cos\frac{7\pi}{5}$, we find that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$. Also, it is not hard to see that the equation $\mathcal{T}_5(a) = -2$ has no rational solutions, which implies that $\cos\frac{\pi}{5}$ is irrational.The first equation of the system yields $\sum_{i=1}^{5} t_i = 0$, and the above considerations show that this can happen only when two of the t_i are equal to $\frac{\pi}{5}$, two are equal to $\frac{3\pi}{5}$, and one is equal to π. Let us show that in this situation the second equation is also satisfied. Using $\mathcal{T}_3(a) = a^3 - 3a$, we see that the first two equations are jointly equivalent to

$\sum_{k=1}^{5} \cos t_i = 0$ and $\sum_{k=1}^{5} \cos 3t_i = 0$. Thus we are left to check that this last equality is satisfied. We have

$$2\cos\frac{3\pi}{5} + 2\cos\frac{9\pi}{5} + \cos 3\pi = 2\cos\frac{3\pi}{5} + 2\cos\frac{\pi}{5} + \cos\pi = 0,$$

as desired. We conclude that up to permutations, the solution to the system is

$$\left(2\cos\frac{\pi}{5}, 2\cos\frac{\pi}{5}, 2\cos\frac{3\pi}{5}, 2\cos\frac{3\pi}{5}, 2\cos\pi\right).$$

(Romanian Mathematical Olympiad, 2002, proposed by T. Andreescu)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :200. Do there exist $n \times n$ matrices A and B such that $AB - BA = \mathcal{I}_n$?
Solution :200. The answer is negative. The trace of $AB - BA$ is zero, while the trace of $\mathcal{I}_n$ is n; the matrices cannot be equal.Remark. The equality cannot hold even for continuous linear transformations on an infinite-dimensional vector space. If P and Q are the linear maps that describe the momentum and the position in Heisenberg's matrix model of quantum mechanics, and if $\hbar$ is Planck's constant, then the equality $PQ - QP = \hbar\mathcal{I}$ is the mathematical expression of Heisenberg's uncertainty principle. We now see that the position and the momentum cannot be modeled using finite-dimensional matrices (not even infinite-dimensional continuous linear transformations). Note on the other hand that the matrices whose entries are residue classes in $\mathbb{Z}_4$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix},$$

satisfy $AB - BA = \mathcal{I}_4$.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :204. Compute the nth power of the $m \times m$ matrix

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{C}.$$

Solution :204. First solution: Computed by hand, the second, third, and fourth powers of $J_4(\lambda)$ are

$$\left(\begin{array}{cccc}\lambda^2 & 2\lambda & 1 & 0\\ 0 & \lambda^2 & 2\lambda & 1\\ 0 & 0 & \lambda^2 & 2\lambda\\ 0 & 0 & 0 & \lambda^2\end{array}\right),\quad \left(\begin{array}{cccc}\lambda^3 & 3\lambda^2 & 3\lambda & 1\\ 0 & \lambda^3 & 3\lambda^2 & 3\lambda\\ 0 & 0 & \lambda^3 & 3\lambda^2\\ 0 & 0 & 0 & \lambda^3\end{array}\right),\quad \left(\begin{array}{cccc}\lambda^4 & 4\lambda^3 & 6\lambda^2 & 4\lambda\\ 0 & \lambda^4 & 4\lambda^3 & 6\lambda^2\\ 0 & 0 & \lambda^4 & 4\lambda^3\\ 0 & 0 & 0 & \lambda^4\end{array}\right).$$

This suggest that in general, the ij th entry of $J_m(\lambda)^n$ is $(J_m(\lambda)^n)_{ij} = \left(\begin{array}{c} i \\ j-i \end{array}\right)\lambda^{n+i-j}$, with the convention $\left(\begin{array}{c} k \\ l \end{array}\right) = 0$ if $l < 0$. The proof by induction is based on the recursive formula for binomial coefficients. Indeed, from $J_m(\lambda)^{n+1} = J_m(\lambda)^n J_m(\lambda)$, we obtain

$$\begin{aligned}\left(J_m(\lambda)^{n+1}\right)_{ij} &= \lambda\left(J_m(\lambda)^n\right)_{ij} + \left(J_m(\lambda)^n\right)_{i,j-1}\\ &= \lambda\left(\begin{array}{c} n \\ j-i \end{array}\right)\lambda^{n+i-j} + \left(\begin{array}{c} n \\ j-1-i \end{array}\right)\lambda^{n+i-j+1} = \left(\begin{array}{c} n+1 \\ j-i \end{array}\right)\lambda^{n+1+i-j},\end{aligned}$$

which proves the claim.Second solution: Define S to be the $n \times n$ matrix with ones just above the diagonal and zeros elsewhere (usually called a shift matrix), and note that S^k has ones above the diagonal at distance k from it, and in particular $S^n = \mathcal{O}_n$. Hence

$$J_m(\lambda)^n = (\lambda\mathcal{I}_n + S)^n = \sum_{k=0}^{n-1}\left(\begin{array}{c} n \\ k \end{array}\right)\lambda^{n-k}S^k.$$

The conclusion follows.Remark. The matrix $J_m(\lambda)$ is called a Jordan block. It is part of the Jordan canonical form of a matrix. Specifically, given a square matrix A there exists an invertible matrix S such that $S^{-1}AS$ is a block diagonal matrix whose blocks are matrices of the form $J_{m_i}(\lambda_i)$. The numbers λ_i are the eigenvalues of A. As a consequence of this problem, we obtain a standard method for raising a matrix to the nth power. The idea is to write the matrix in the Jordan canonical form and then raise the blocks to the power.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :211. Compute the determinant of the $n \times n$ matrix $A = (a_{ij})_{ij}$, where

$$a_{ij} = \begin{cases}(-1)^{|i-j|} & \text{if } i \neq j,\\ 2 & \text{if } i = j.\end{cases}$$

Solution :211. By adding the second row to the first, the third row to the second, .., the nth row to the $(n-1)$ st, the determinant does not change. Hence

$$\det(A) = \begin{vmatrix} 2 & -1 & +1 & \cdots & \pm 1 & \mp 1 \\ -1 & 2 & -1 & \cdots & \mp 1 & \pm 1 \\ +1 & -1 & 2 & \cdots & \pm 1 \mp 1 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mp 1 \pm 1 & \mp 1 & \cdots & 2 & -1 & \\ \pm 1 & \mp 1 \pm 1 & \cdots & -1 & 2 & \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ \pm 1 \mp 1 \pm 1 & \mp 1 & \ldots & -1 & 2 & & \end{vmatrix}.$$

Now subtract the first column from the second, then subtract the resulting column from the third, and so on. This way we obtain

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \pm 1 & \mp 2 & \pm 3 & \cdots & -n+1 & n+1 \end{vmatrix} = n+1.$$

(9th International Mathematics Competition for University Students, 2002)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :223. Determine the matrix A knowing that its adjoint matrix (the one used in the computation of the inverse) is

$$A^* = \begin{pmatrix} m^2-1 & 1-m & 1-m \\ 1-m & m^2-1 & 1-m \\ 1-m & 1-m & m^2-1 \end{pmatrix}, \quad m \neq 1, -2$$

Solution :223. We know that $AA^* = A^*A = (\det A)\mathcal{I}_3$, so if A is invertible then so is A^*, and $A = \det A\,(A^*)^{-1}$. Also, $\det A \det A^* = (\det A)^3$; hence $\det A^* = (\det A)^2$. Therefore, $A = \pm\sqrt{\det A^*}\,(A^*)^{-1}$.Because

$$A^* = (1-m)\begin{pmatrix} -m-1 & 1 & 1 \\ 1 & -m-1 & 1 \\ 1 & 1 & -m-1 \end{pmatrix}$$

we have

$$\det A^* = (1-m)^3\left[-(m+1)^3 + 2 + 3(m+1)\right] = (1-m)^4(m+2)^2.$$

Using the formula with minors, we compute the inverse of the matrix

$$\begin{pmatrix} -m-1 & 1 & 1 \\ 1 & -m-1 & 1 \\ 1 & 1 & -m-1 \end{pmatrix}$$

to be

$$\frac{1}{(1-m)(m+2)^2}\begin{pmatrix} -m^2-m-2 & m+2 & m+2 \\ m+2 & -m^2-m-2 & m+2 \\ m+2 & m+2 & -m^2-m-2 \end{pmatrix}.$$

Then $(A^*)^{-1}$ is equal to this matrix divided by $(1-m)^3$. Consequently, the matrix we are looking for is

$$\begin{aligned} A &= \pm\sqrt{\det A^*}\,(A^*)^{-1} \\ &= \pm\frac{1}{(1-m)^2(m+2)}\begin{pmatrix} -m^2-m-2 & m+2 & m+2 \\ m+2 & -m^2-m-2 & m+2 \\ m+2 & m+2 & -m^2-m-2 \end{pmatrix}. \end{aligned}$$

(Romanian mathematics competition)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :231. Solve the system of linear equations

$$\begin{aligned} x_1+x_2+x_3&=0, \\ x_2+x_3+x_4&=0, \\ &\cdots \\ x_{99}+x_{100}+x_1&=0, \\ x_{100}+x_1+x_2&=0. \end{aligned}$$

Solution :231. Of course, one can prove that the coefficient matrix is nonsingular. But there is a slick solution. Add the equations and group the terms as

$$3\left(x_1+x_2+x_3\right)+3\left(x_4+x_5+x_6\right)+\cdots+3\left(x_{97}+x_{98}+x_{99}\right)+3x_{100}=0.$$

The terms in the parentheses are all zero; hence $x_{100}=0$. Taking cyclic permutations yields $x_1=x_2=\cdots=x_{100}=0$.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :232. Find the solutions x_1, x_2, x_3, x_4, x_5 to the system of equations

$$x_5+x_2=yx_1, \quad x_1+x_3=yx_2, \quad x_2+x_4=yx_3,$$

$$x_3+x_5=yx_1, \quad x_4+x_1=yx_5,$$

where y is a parameter.
Solution :232. If y is not an eigenvalue of the matrix

$$\left(\begin{array}{lllll}0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0\end{array}\right),$$

then the system has the unique solution $x_1=x_2=x_3=x_4=x_5=0$. Otherwise, the eigenvectors give rise to nontrivial solutions. Thus, we have to compute the determinant

$$\left|\begin{array}{ccccc}-y & 1 & 0 & 0 & 1 \\ 1 & -y & 1 & 0 & 0 \\ 0 & 1 & -y & 1 & 0 \\ 0 & 0 & 1 & -y & 1 \\ 1 & 0 & 0 & 1 & -y\end{array}\right| .$$

Adding all rows to the first and factoring $2-y$, we obtain

$$(2-y)\left|\begin{array}{ccccc}1 & 1 & 1 & 1 & 1 \\ 1 & -y & 1 & 0 & 0 \\ 0 & 1 & -y & 1 & 0 \\ 0 & 0 & 1 & -y & 1 \\ 1 & 0 & 0 & 1 & -y\end{array}\right| .$$

The determinant from this expression is computed using row-column operations as follows:

$$\left|\begin{array}{ccccc}1 & 1 & 1 & 1 & 1 \\ 1 & -y & 1 & 0 & 0 \\ 0 & 1 & -y & 1 & 0 \\ 0 & 0 & 1 & -y & 1 \\ 1 & 0 & 0 & 1 & -y\end{array}\right|=\left|\begin{array}{ccccc}1 & 0 & 0 & 0 & 0 \\ 1 & -y-1 & 0 & -1 & -1 \\ 0 & 1 & -y & 1 & 0 \\ 0 & 0 & 1 & -y & 1 \\ 1 & -1 & -1 & 0 & -y-1\end{array}\right|$$

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$$=\left|\begin{array}{cccc}-y-1 & 0 & 0 & -1 \\ 0 & -y & 1 & -1 \\ -1 & 1 & -y-1-1 & \\ -1 & 0 & 0 & -y\end{array}\right|,$$

which, after expanding with the rule of Laplace, becomes

$$-\left|\begin{array}{cc}-y & -1 \\ 1 & -y-1\end{array}\right| \cdot\left|\begin{array}{cc}-y-1 & -1 \\ -1 & -y\end{array}\right|=-\left(y^2+y-1\right)^2 .$$

Hence the original determinant is equal to $(y-2)\left(y^2+y-1\right)^2$. If $y=2$, the space of solutions is therefore one-dimensional, and it is easy to guess the solution $x_1 = x_2 = x_3 = x_4 = x_5 = \lambda, \lambda \in \mathbb{R}$.If $y = \frac{-1+\sqrt{5}}{2}$ or if $y = \frac{-1-\sqrt{5}}{2}$, the space of solutions is two-dimensional. In both cases, the minor

$$\begin{vmatrix} -y & 1 & 0 \\ 1 & -y & 1 \\ 0 & 1 & -y \end{vmatrix}$$

is nonzero, hence x_3, x_4, and x_5 can be computed in terms of x_1 and x_2. In this case the general solution is

$$(\lambda, \mu, -\lambda + y\mu, -y(\lambda + \mu), y\lambda - \mu), \quad \lambda, \mu \in \mathbb{R}.$$

Remark. The determinant of the system can also be computed using the formula for the determinant of a circulant matrix.(5th International Mathematical Olympiad, 1963, proposed by the Soviet Union)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :235. Let $P(x) = x^n + x^{n-1} + \cdots + x + 1$. Find the remainder obtained when $P\left(x^{n+1}\right)$ is divided by $P(x)$.
Solution :235. First solution: The zeros of $P(x)$ are $\epsilon, \epsilon^2, \ldots, \epsilon^n$, where ϵ is a primitive $(n+1)$ st root of unity. As such, the zeros of $P(x)$ are distinct. Let

$$P\left(x^{n+1}\right) = Q(x) \cdot P(x) + R(x),$$

where $R(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is the remainder. Replacing x successively by $\epsilon, \epsilon^2, \ldots, \epsilon^n$, we obtain

$$\begin{aligned} a_n \epsilon^{n-1} + \cdots + a_1 \epsilon + a_0 &= n+1, \\ a_n \left(\epsilon^2\right)^{n-1} + \cdots + a_1 \epsilon^2 + a_0 &= n+1, \\ &\cdots \\ a_n \left(\epsilon^n\right)^{n-1} + \cdots + a_1 \epsilon^n + a_0 &= n+1, \end{aligned}$$

or

$$\begin{aligned} [a_0 - (n+1)] + a_1 \epsilon + \cdots + a_{n-1} \epsilon^{n-1} &= 0, \\ [a_0 - (n+1)] + a_1 \left(\epsilon^2\right) + \cdots + a_{n-1} \left(\epsilon^2\right)^{n-1} &= 0, \\ &\cdots \\ [a_0 - (n+1)] + a_1 \left(\epsilon^n\right) + \cdots + a_{n-1} \left(\epsilon^n\right)^{n-1} &= 0. \end{aligned}$$

This can be interpreted as a homogeneous system in the unknowns $a_0 - (n+1)$, $a_1, a_2, \ldots, a_{n-1}$. The determinant of the coefficient matrix is Vandermonde, thus nonzero, and so the system has the unique solution $a_0 - (n+1) = a_1 = \cdots = a_{n-1} = 0$. We obtain $R(x) = n+1$.Second solution: Note that

$$x^{n+1} = (x-1)P(x) + 1;$$

hence

$$x^{k(n+1)} = (x-1)\left(x^{(k-1)(n+1)} + x^{(k-2)(n+1)} + \cdots + 1\right) P(x) + 1.$$

Thus the remainder of any polynomial $F\left(x^{n+1}\right)$ modulo $P(x)$ is $F(1)$. In our situation this is $n+1$, as seen above.(Gazeta Matematic ă (Mathematics Gazette, Bucharest), proposed by M. Diaconescu)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :236. Find all functions $f : \mathbb{R}\backslash\{-1, 1\} \to \mathbb{R}$ satisfying

$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) = x$$

for all $x \neq \pm 1$.
Solution :236. The function $\phi(t) = \frac{t-3}{t+1}$ has the property that $\phi \circ \phi \circ \phi$ equals the identity function. And $\phi(\phi(t)) = \frac{3+t}{1-t}$. Replace x in the original equation by $\phi(x)$ and $\phi(\phi(x))$ to obtain two more equations. The three equations form a linear system

$$\begin{aligned} f\left(\frac{x-3}{x+1}\right) + f\left(\frac{3+x}{1-x}\right) &= x, \\ f\left(\frac{3+x}{1-x}\right) + f(x) &= \frac{x-3}{x+1}, \\ f(x) + f\left(\frac{x-3}{x+1}\right) &= \frac{3+x}{1-x}, \end{aligned}$$

in the unknowns

$$f(x), \quad f\left(\frac{x-3}{x+1}\right), \quad f\left(\frac{3+x}{1-x}\right).$$

Solving, we find that

$$f(t) = \frac{4t}{1-t^2} - \frac{t}{2},$$

which is the unique solution to the functional equation.(Kvant (Quantum), also appeared at the Korean Mathematical Olympiad, 1999)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :237. Find all positive integer solutions (x, y, z, t) to the Diophantine equation

$$(x+y)(y+z)(z+x) = txyz$$

such that $\gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1$.
Solution :237. It is obvious that $\gcd(x, x+y) = \gcd(x, x+z) = 1$. So in the

equality from the statement, x divides $y+z$. Similarly, y divides $z+x$ and z divides $x+y$. It follows that there exist integers a, b, c with $abc=t$ and

$$
\begin{aligned}
& x+y=cz, \\
& y+z=ax, \\
& z+x=by .
\end{aligned}
$$

View this as a homogeneous system in the variables x, y, z. Because we assume that the system admits nonzero solutions, the determinant of the coefficient matrix is zero. Writing down this fact, we obtain a new Diophantine equation in the unknowns a, b, c :

$$
abc-a-b-c-2=0 .
$$

This can be solved by examining the following cases:1. $a=b=c$. Then $a=2$ and it follows that $x=y=z$, because these numbers are pairwise coprime. This means that $x=y=z=1$ and $t=8$. We have obtained the solution $(1,1,1,8)$.2. $a=b, a \neq c$. The equation becomes $a^2 c-2=2a+c$, which is equivalent to $c\left(a^2-1\right)=2(a+1)$, that is, $c(a-1)=2$. We either recover case 1 , or find the new solution $c=1, a=b=3$. This yields the solution to the original equation $(1,1,2,9)$3. $a>b>c$. In this case $abc-2=a+b+c<3a$. Therefore, $a(bc-3)<2$. It follows that $bc-3<2$, that is, $bc<5$. We have the following situations:(i) $b=2, c=1$, so $a=5$ and we obtain the solution $(1,2,3,10)$.(ii) $b=3, c=1$, so $a=3$ and we return to case 2 .(iii) $b=4, c=1$, so $3a=7$, which is impossible.In conclusion, we have obtained the solutions $(1,1,1,8),(1,1,2,9),(1,2,3,10)$, and those obtained by permutations of x, y, z.(Romanian Mathematical Olympiad, 1995)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :243. Let A be the $n \times n$ matrix whose i, j entry is $i+j$ for all $i, j=1,2, \ldots, n$. What is the rank of A ?
Solution :243. For $n=1$ the rank is 1 . Let us consider the case $n \geq 2$. Observe that the rank does not change under row/column operations. For $i=n, n-1, \ldots, 2$, subtract the $(i-1)$ st row from the i th. Then subtract the second row from all others. Explicitly, we obtain

$$
\begin{aligned}
& \operatorname{rank}\left(\begin{array}{cccc}
2 & 3 & \cdots & n+1 \\
3 & 4 & \cdots & n+2 \\
\vdots & \vdots & \ddots & \vdots \\
n+1 & n+2 & \cdots & 2n
\end{array}\right)=\operatorname{rank}\left(\begin{array}{cccc}
2 & 3 & \cdots & n+1 \\
1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{array}\right) \\
& =\operatorname{rank}\left(\begin{array}{cccc}
1 & 2 & \cdots & n \\
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{array}\right)=2 .
\end{aligned}
$$

(12th International Competition in Mathematics for University Students, 2005)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :244. For integers $n \geq 2$ and $0 \leq k \leq n-2$, compute the determinant

$$\left|\begin{array}{ccccc} 1^k & 2^k & 3^k & \cdots & n^k \\ 2^k & 3^k & 4^k & \cdots & (n+1)^k \\ 3^k & 4^k & 5^k & \cdots & (n+2)^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n^k & (n+1)^k & (n+2)^k & \cdots & (2n-1)^k \end{array}\right|.$$

Solution :244. The polynomials $P_j(x) = (x+j)^k, j = 0, 1, \ldots, n-1$, lie in the $(k+1)$-dimensional real vector space of polynomials of degree at most k. Because $k+1 < n$, they are linearly dependent. The columns consist of the evaluations of these polynomials at $1, 2, \ldots, n$, so the columns are linearly dependent. It follows that the determinant is zero.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :250. Consider the $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} = 1$ if $j - i \equiv 1 (\bmod n)$ and $a_{ij} = 0$ otherwise. For real numbers a and b find the eigenvalues of $aA + bA^t$.
Solution :250. First solution: The eigenvalues are the zeros of the polynomial $\det\left(\lambda \mathcal{I}_n - aA - bA^t\right)$. The matrix $\lambda \mathcal{I}_n - aA - bA^t$ is a circulant matrix, and the determinant of a circulant matrix was the subject of problem 211 in Section 2.3.2. According to that formula,

$$\det\left(\lambda \mathcal{I}_n - aA - bA^t\right) = (-1)^{n-1} \prod_{j=0}^{n-1} \left(\lambda \zeta^j - a\zeta^{2j} - b\right),$$

where $\zeta = e^{2\pi i/n}$ is a primitive nth root of unity. We find that the eigenvalues of $aA + bA^t$ are $a\zeta^j + b\zeta^{-j}, j = 0, 1, \ldots, n-1$.Second solution: Simply note that for $\zeta = e^{2\pi i/n}$ and $j = 0, 1, \ldots, n-1$, $\left(1, \zeta^j, \zeta^{2j}, \ldots, \zeta^{(n-1)j}\right)$ is an eigenvector with eigenvalue $a\zeta^j + b\zeta^{-j}$.
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :259. Find the 2×2 matrices with real entries that satisfy the equation

$$X^3 - 3X^2 = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}.$$

Solution :259. Rewriting the matrix equation as

$$X^2\left(X-3\mathcal{I}_2\right)=\left(\begin{array}{cc}-2 & -2 \\ -2 & -2\end{array}\right)$$

and taking determinants, we obtain that either $\det X=0$ or $\det\left(X-3\mathcal{I}_2\right)=0$. In the first case, the Cayley-Hamilton equation implies that $X^2=(\operatorname{tr} X)X$, and the equation takes the form

$$\left[(\operatorname{tr} X)^2-3 \operatorname{tr} X\right] X=\left(\begin{array}{cc}-2 & -2 \\ -2 & -2\end{array}\right).$$

Taking the trace of both sides, we find that the trace of X satisfies the cubic equation $t^3-3t^2+4=0$, with real roots $t=2$ and $t=-1$. In the case $\operatorname{tr} X=2$, the matrix equation is

$$-2X=\left(\begin{array}{cc}-2 & -2 \\ -2 & -2\end{array}\right)$$

with the solution

$$X=\left(\begin{array}{cc}1 & 1 \\ 1 & 1\end{array}\right).$$

When $\operatorname{tr} X=-1$, the matrix equation is

$$4X=\left(\begin{array}{cc}-2 & -2 \\ -2 & -2\end{array}\right)$$

with the solution

$$X=\left(\begin{array}{c}-\frac{1}{2}-\frac{1}{2} \\ -\frac{1}{2}-\frac{1}{2}\end{array}\right).$$

Let us now study the case $\det\left(X-3\mathcal{I}_2\right)=0$. One of the two eigenvalues of X is 3 . To determine the other eigenvalue, add $4\mathcal{I}_2$ to the equation from the statement. We obtain

$$X^3-3X^2+4\mathcal{I}_2=\left(X-2\mathcal{I}_2\right)\left(X+\mathcal{I}_2\right)=\left(\begin{array}{c}-2-2 \\ -2-2\end{array}\right).$$

Taking determinants we find that either $\det\left(X-2\mathcal{I}_2\right)=0$ or $\det\left(X+\mathcal{I}_2\right)=0$. So the second eigenvalue of X is either 2 or -1. In the first case, the Cayley-Hamilton equation for X is

$$X^2-5X+6\mathcal{I}_2=0,$$

which can be used to transform the original equation intoMATHPIX IMAGE-with the solution

$$X=\left(\begin{array}{cc}\frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2}\end{array}\right).$$

The case in which the second eigenvalue of X is -1 is treated similarly and yields the solution

$$X = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

(Romanian competition, 2004, proposed by A. Buju)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :265. Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable t that satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,$$
$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n,$$
$$\cdots$$
$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n,$$

for some constants $a_{ij} > 0$. Suppose that for all $i, x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly independent?
Solution :265. Let λ be the positive eigenvalue and $v = (v_1, v_2, \ldots, v_n)$ the corresponding eigenvector with positive entries of the transpose of the coefficient matrix. The function $y(t) = v_1x_1(t) + v_2x_2(t) + \cdots + v_nx_n(t)$ satisfies

$$\frac{dy}{dt} = \sum_{i,j} v_i a_{ij} x_j = \sum_j \lambda v_j x_j = \lambda y.$$

Therefore, $y(t) = e^{\lambda t}y_0$, for some vector y_0. Because

$$\lim_{t\to\infty} y(t) = \sum_i v_i \lim_{t\to\infty} x_i(t) = 0,$$

and $\lim_{t\to\infty} e^{\lambda t} = \infty$, it follows that y_0 is the zero vector. Hence

$$y(t) = v_1x_1(t) + v_2x_2(t) + \cdots + v_nx_n(t) = 0,$$

which shows that the functions $x_1, x_2, \ldots, x_n$ are necessarily linearly dependent.(56th W.L. Putnam Mathematical Competition, 1995)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :266. For a positive integer n and any real number c, define $(x_k)_{k\geq 0}$ recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and $k, 1 \leq k \leq n$.
Solution :266. We try some particular cases. For $n = 2$, we obtain $c = 1$ and the sequence 1 , 1 , or $n = 3, c = 2$ and the sequence $1, 2, 1$, and for $n = 4, c = 3$ and the sequence $1, 3, 3, 1$. We formulate the hypothesis that $c = n - 1$ and $x_k = \binom{n-1}{k-1}$.The condition $x_{n+1} = 0$ makes the recurrence relation from the statement into a linear system in the unknowns $(x_1, x_2, \ldots, x_n)$. More precisely, the solution is an eigenvector of the matrix $A = (a_{ij})_{ij}$ defined by

$$a_{ij} = \begin{cases} i & \text{if } j = i+1, \\ n-j & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix has nonnegative entries, so the Perron-Frobenius Theorem as stated here does not really apply. But let us first observe that A has an eigenvector with positive coordinates, namely $x_k = \binom{n-1}{k-1}$, $k = 1, 2, \ldots, n$, whose eigenvalue is $n - 1$. This follows by rewriting the combinatorial identity

$$\binom{n-1}{k} = \binom{n-2}{k} + \binom{n-2}{k-1}$$

as

$$\binom{n-1}{k} = \frac{k+1}{n-1}\binom{n-1}{k+1} + \frac{n-k}{n-1}\binom{n-1}{k-1}.$$

To be more explicit, this identity implies that for $c = n - 1$, the sequence $x_k = \binom{n-1}{k-1}$ satisfies the recurrence relation from the statement, and $x_{n+1} = 0$.Let us assume that $n - 1$ is not the largest value that c can take. For a larger value, consider an eigenvector v of A. Then $(A + \mathcal{I}_n)\, v = (c+1)v$, and $(A + \mathcal{I}_n)^n\, v = (c+1)^n v$. The matrix $(A + \mathcal{I}_n)^n$ has positive entries, and so by the Perron-Frobenius Theorem has a unique eigenvector with positive coordinates. We already found one such vector, that for which $x_k = \binom{n-1}{k-1}$. Its eigenvalue has the largest absolute value among all eigenvalues of $(A + \mathcal{I}_n)^n$, which means that $n^n > (c+1)^n$. This implies $n > c + 1$, contradicting our assumption. So $n - 1$ is the largest value c can take, and the sequence we found is the answer to the problem.(57th W.L. Putnam Mathematical Competition, 1997, solution by G. Kuperberg and published in K. Kedlaya, B. Poonen, R. Vakil, The William Lowell Putnam Mathematical Competition 1985-2000, MAA, 2002)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :267. With the aid of a calculator that can add, subtract, and determine the inverse of a nonzero number, find the product of two nonzero numbers using at most 20 operations.
Solution :267. Let us first show that if the two numbers are equal, then the product can be found in six steps. For $x \neq -1$, we compute (1) $x \rightarrow \frac{1}{x}$, (2)$x \rightarrow x+1$, (3)$x+1 \rightarrow \frac{1}{x+1}$, (4) $\frac{1}{x}, \frac{1}{x+1} \rightarrow \frac{1}{x}-\frac{1}{x+1}=\frac{1}{x^2+x}$, (5) $\frac{1}{x^2+x} \rightarrow x^2+x$, (6) $x^2+x, x \rightarrow x^2$. If $x=-1$, replace step (2) by $x \rightarrow x-1$ and make the subsequent modifications thereon.If the two numbers are distinct, say x and y, perform the following sequence of operations, where above each arrow we count the steps:

$$x, y \xrightarrow{1} x+y \xrightarrow{7} (x+y)^2,$$
$$x, y \xrightarrow{8} x-y \xrightarrow{14} (x-y)^2,$$
$$(x+y)^2, (x-y)^2 \xrightarrow{15} 4xy \xrightarrow{16} \frac{1}{4xy},$$
$$\frac{1}{4xy}, \frac{1}{4xy} \xrightarrow{17} \frac{1}{4xy}+\frac{1}{4xy}=\frac{2}{xy},$$
$$\frac{2}{4xy}, \frac{2}{4xy} \xrightarrow{18} \frac{2}{4xy}+\frac{2}{4xy}=\frac{4}{4xy}=\frac{1}{xy} \xrightarrow{19} xy.$$

So we are able to compute the product in just 19 steps.(Kvant (Quantum))
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :268. Invent a binary operation from which $+, -, \times$, and / can be derived.
Solution :268. Building on the previous problem, we see that it suffices to produce an operation o, from which the subtraction and reciprocal are derivable. A good choice is $\frac{1}{x-y}$. Indeed, $\frac{1}{x}=\frac{1}{x-0}$, and also $x-y=\frac{1}{(1/(x-y)-0)}$. Success!(D.J. Newman, A Problem Seminar, Springer-Verlag)
Topic :Algebra
Book :Putnam and Beyond
Final Answer :

Problem Statement :297. Find a formula for the general term of the sequence

$$1,2,2,3,3,3,4,4,4,4,5,5,5,5,5, \ldots$$

Solution :297. Examining the sequence, we see that the m th term of the sequence is equal to n exactly for those m that satisfy

$$\frac{n^2-n}{2}+1 \leq m \leq \frac{n^2+n}{2}.$$

So the sequence grows about as fast as the square root of twice the index. Let us rewrite the inequality as

$$n^2 - n + 2 \le 2m \le n^2 + n,$$

then try to solve for n. We can almost take the square root. And because m and n are integers, the inequality is equivalent to

$$n^2 - n + \frac{1}{4} < 2m < n^2 + n + \frac{1}{4}.$$

Here it was important that $n^2 - n$ is even. And now we can take the square root. We obtain

$$n - \frac{1}{2} < \sqrt{2m} < n + \frac{1}{2},$$

or

$$n < \sqrt{2m} + \frac{1}{2} < n + 1.$$

Now this happens if and only if $n = \lfloor \sqrt{2m} + \frac{1}{2} \rfloor$, which then gives the formula for the general term of the sequence

$$a_m = \left\lfloor \sqrt{2m} + \frac{1}{2} \right\rfloor, \quad m \ge 1.$$

(R. Graham, D. Knuth, O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, 2nd ed., Addison-Wesley, 1994)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :298. Find a formula in compact form for the general term of the sequence defined recursively by $x_1 = 1, x_n = x_{n-1} + n$ if n is odd, and $x_n = x_{n-1} + n - 1$ if n is even.
Solution :298. If we were given the recurrence relation $x_n = x_{n-1} + n$, for all n, the terms of the sequence would be the triangular numbers $T_n = \frac{n(n+1)}{2}$. If we were given the recurrence relation $x_n = x_{n-1} + n - 1$, the terms of the sequence would be $T_{n-1} + 1 = \frac{n^2-n+2}{2}$. In our case,

$$\frac{n^2 - n + 2}{2} \le x_n \le \frac{n^2 + n}{2}.$$

We expect $x_n = P(n)/2$ for some polynomial $P(n) = n^2 + an + b$; in fact, we should have $x_n = \lfloor P(n)/2 \rfloor$ because of the jumps. From here one can easily guess that $x_n = \left\lfloor \frac{n^2+1}{2} \right\rfloor$, and indeed

$$\left\lfloor \frac{n^2 + 1}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2 + 1}{2} + \frac{2(n-1) + 1}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2 + 1}{2} + \frac{1}{2} \right\rfloor + (n-1),$$

which is equal to $\left\lfloor \frac{(n-1)^2+1}{2} \right\rfloor + (n-1)$ if n is even, and to $\left\lfloor \frac{(n-1)^2+1}{2} \right\rfloor + n$ if n is odd.

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :300. The sequence $a_0, a_1, a_2, \ldots$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2}\left(a_{2m} + a_{2n}\right),$$

for all nonnegative integers m and n with $m \geq n$. If $a_1 = 1$, determine a_n.
Solution :300. The relations

$$a_m + a_m = \frac{1}{2}\left(a_{2m} + a_0\right) \quad \text{and} \quad a_{2m} + a_0 = \frac{1}{2}\left(a_{2m} + a_{2m}\right)$$

imply $a_{2m} = 4a_m$, as well as $a_0 = 0$. We compute $a_2 = 4, a_4 = 16$. Also, $a_1 + a_3 = \left(a_2 + a_4\right)/2 = 10$, so $a_3 = 9$. At this point we guess that $a_k = k^2$ for all $k \geq 1$.We prove our guess by induction on k. Suppose that $a_j = j^2$ for all $j < k$. The given equation with $m = k-1$ and $n = 1$ gives

$$\begin{aligned} a_n &= \frac{1}{2}\left(a_{2n-2} + a_2\right) - a_{n-2} = 2a_{n-1} + 2a_1 - a_{n-2} \\ &= 2\left(n^2 - 2n + 1\right) + 2 - \left(n^2 - 4n + 4\right) = n^2. \end{aligned}$$

This completes the proof.(Russian Mathematical Olympiad, 1995)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :303. Consider the sequences $\left(a_n\right)_n$ and $\left(b_n\right)_n$ defined by $a_1 = 3, b_1 = 100, a_{n+1} = 3^{a_n}$, $b_{n+1} = 100^{b_n}$. Find the smallest number m for which $b_m > a_{100}$.
Solution :303. We need to determine m such that $b_m > a_n > b_{m-1}$. It seems that the difficult part is to prove an inequality of the form $a_n > b_m$, which reduces to $3^{a_{n-1}} > 100^{b_{m-1}}$, or $a_{n-1} > \left(\log_3 100\right) b_{m-1}$. Iterating, we obtain $3^{a_{n-2}} > \left(\log_3 100\right) 100^{b_{m-2}}$, that is,

$$a_{n-2} > \log_3\left(\log_3 100\right) + \left(\left(\log_3 100\right) b_{m-2}.\right.$$

Seeing this we might suspect that an inequality of the form $a_n > u + vb_n$, holding for all n with some fixed u and v, might be useful in the solution. From such an inequality we would derive $a_{n+1} = 3^{a_n} > 3^u\left(3^v\right)^{b_m}$. If $3^v > 100$, then $a_{n+1} > 3^u b_{m+1}$, and if $3^u > u + v$, then we would obtain $a_{n+1} > u + vb_{m+1}$, the same inequality as the one we started with, but with $m+1$ and $n+1$ instead of m and n.The inequality $3^v > 100$ holds for $v = 5$, and $3^u > u + 5$ holds for $u = 2$. Thus $a_n > 2 + 5b_m$ implies $a_{n+1} > 2 + 5b_{m+1}$. We have $b_1 = 100, a_1 = 3, a_2 = 27, a_3 = 3^{27}$, and $2 + 5b_1 = 502 < 729 = 3^6$, so $a_3 > 2 + 5b_1$. We find that $a_n > 2 + 5b_{n-2}$ for all $n \geq 3$. In particular, $a_n \geq b_{n-2}$.On the other hand, $a_n < b_m$ implies $a_{n+1} = 3^{a_n} < 100^{b_m} < b_{m+1}$, which combined with $a_2 < b_1$

yields $a_n < b_{n-1}$ for all $n \geq 2$. Hence $b_{n-2} < a_n < b_{n-1}$, which implies that $m = n - 1$, and for $n = 100, m = 99$.(short list of the 21st International Mathematical Olympiad, 1979, proposed by Romania, solution by I. Cuculescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :305. Find the general term of the sequence given by $x_0 = 3, x_1 = 4$, and

$$(n+1)(n+2)x_n = 4(n+1)(n+3)x_{n-1} - 4(n+2)(n+3)x_{n-2}, \quad n \geq 2.$$

Solution :305. Divide through by the product $(n+1)(n+2)(n+3)$. The recurrence relation becomes

$$\frac{x_n}{n+3} = 4\frac{x_{n-1}}{n+2} + 4\frac{x_{n-2}}{n+1}.$$

The sequence $y_n = x_n/(n+3)$ satisfies the recurrence

$$y_n = 4y_{n-1} - 4y_{n-2}.$$

Its characteristic equation has the double root 2. Knowing that $y_0 = 1$ and $y_1 = 1$, we obtain $y_n = 2^n - n2^{n-1}$. It follows that the answer to the problem is

$$x_n = (n+3)2^n - n(n+3)2^{n-1}.$$

(D. Buşneag, I. Maftei, Teme pentru cercurile şi concursurile de matematic ă (Themes for mathematics circles and contests), Scrisul Românesc, Craiova)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :307. Define the sequence $(a_n)_n$ recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1 + 4a_n + \sqrt{1 + 24a_n}}{16}, \quad \text{for } n \geq 1$$

Find an explicit formula for a_n in terms of n.
Solution :307. A standard idea is to eliminate the square root. If we set $b_n = \sqrt{1 + 24a_n}$, then $b_n^2 = 1 + 24a_n$, and so

$$\begin{aligned} b_{n+1}^2 &= 1 + 24a_{n+1} = 1 + \frac{3}{2}\left(1 + 4a_n + \sqrt{1 + 24a_n}\right) \\ &= 1 + \frac{3}{2}\left(1 + \frac{1}{6}\left(b_n^2 - 1\right) + b_n\right) \\ &= \frac{1}{4}\left(b_n^2 + 6b_n + 9\right) = \left(\frac{b_n + 3}{2}\right)^2. \end{aligned}$$

Hence $b_{n+1} = \frac{1}{2}b_n + \frac{3}{2}$. This is an inhomogeneous first-order linear recursion. We can solve this by analogy with inhomogeneous linear first-order equations. Recall that if a, b are constants, then the equation $f'(x) = af(x) + b$ has the solution

$$f(x) = e^{ax} \int e^{-ax} b dx + ce^{ax}.$$

In our problem the general term should be

$$b_n = \frac{1}{2^{n+1}} + 3 \sum_{k=1}^{n} \frac{1}{2^k}, \quad n \geq 1.$$

Summing the geometric series, we obtain $b_n = 3 + \frac{1}{2^{n-2}}$, and the answer to our problem is

$$a_n = \frac{b_n^2 - 1}{24} = \frac{1}{3} + \frac{1}{2^n} + \frac{1}{3} \cdot \frac{1}{2^{2n-1}}.$$

(proposed by Germany for the 22nd International Mathematical Olympiad, 1981)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :310. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2001, \quad \text{for all } n \in \mathbb{N}.$$

Solution :310. We first try a function of the form $f(n) = n + a$. The relation from the statement yields $a = 667$, and hence $f(n) = n + 667$ is a solution. Let us show that this is the only solution.Fix some positive integer n and define $a_0 = n$, and $a_k = f(f(\cdots(f(n)\cdots))$, where the composition is taken k times, $k \geq 1$. The sequence $(a_k)_{k\geq 0}$ satisfies the inhomogeneous linear recurrence relation

$$a_{k+3} - 3a_{k+2} + 6a_{k+1} - 4a_k = 2001.$$

A particular solution is $a_k = 667\text{k}$. The characteristic equation of the homogeneous recurrence $a_{k+3} - 3a_{k+2} + 6a_{k+1} - 4a_k = 0$ is

$$\lambda^3 - 3\lambda^2 + 6\lambda - 4 = 0.$$

An easy check shows that $\lambda_1 = 1$ is a solution to this equation. Since $\lambda^3 - 3\lambda^2 + 6\lambda - 4 = (\lambda - 1)\left(\lambda^2 - 2\lambda + 4\right)$, the other two solutions are $\lambda_{2,3} = 1 \pm i\sqrt{3}$, that is, $\lambda_{2,3} = 2\left(\cos\frac{\pi}{3} \pm i \sin\frac{\pi}{3}\right)$. It follows that the formula for the general term of a sequence satisfying the recurrence relation is

$$a_k = c_1 + c_2 2^k \cos\frac{k\pi}{3} + c_3 2^k \sin\frac{k\pi}{3} + 667k, \quad k \geq 0,$$

with c_1, c_2, and c_3 some real constants.If $c_2 > 0$, then $a_{3(2m+1)}$ will be negative for large m, and if $c_2 < 0$, then a_{6m} will be negative for large m. Since $f(n)$

can take only positive values, this implies that $c_2 = 0$. A similar argument shows that $c_3 = 0$. It follows that $a_k = c_1 + 667k$. So the first term of the sequence determines all the others. Since $a_0 = n$, we have $c_1 = n$, and hence $a_k = n + 667k$, for all k. In particular, $a_1 = f(n) = n + 667$, and hence this is the only possible solution.(Mathematics Magazine, proposed by R. Gelca)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :313. Compute

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right|.$$

Solution :313. The function $|\sin x|$ is periodic with period π. Hence

$$\lim_{n\to\infty} \left|\sin\pi\sqrt{n^2+n+1}\right| = \lim_{n\to\infty}\left|\sin\pi\left(\sqrt{n^2+n+1}-n\right)\right|.$$

But

$$\lim_{n\to\infty}\left(\sqrt{n^2+n+1}-n\right) = \lim_{n\to\infty}\frac{n^2+n+1-n^2}{\sqrt{n^2+n+1}+n} = \frac{1}{2}.$$

It follows that the limit we are computing is equal to $\left|\sin\frac{\pi}{2}\right|$, which is 1 .
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :315. Let $(x_n)_n$ be a sequence of positive integers such that $x_{x_n} = n^4$ for all $n \geq 1$. Is it true that $\lim_{n\to\infty} x_n = \infty$?
Solution :315. Let us assume that the answer is negative. Then the sequence has a bounded subsequence $(x_{n_k})_k$. The set $\left\{x_{x_{n_k}} \mid k \in \mathbb{Z}\right\}$ is finite, since the indices x_{n_k} belong to a finite set. But $x_{x_{n_k}} = n_k^4$, and this takes infinitely many values for $k \geq 1$. We reached a contradiction that shows that our assumption was false. So the answer to the question is yes.(Romanian Mathematical Olympiad, 1978, proposed by S. Rădulescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :319. Let a be a positive real number and $(x_n)_{n\geq 1}$ a sequence of real numbers such that $x_1 = a$ and

$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} kx_k, \quad \text{for all } n \geq 1.$$

Find the limit of the sequence.
Solution :319. We will prove by induction on $n \geq 1$ that

$$x_{n+1} > \sum_{k=1}^{n} k x_k > a \cdot n!,$$

from which it will follow that the limit is ∞.For $n = 1$, we have $x_2 \geq 3x_1 > x_1 = a$. Now suppose that the claim holds for all values up through n. Then

$$x_{n+2} \geq (n+3)x_{n+1} - \sum_{k=1}^{n} k x_k = (n+1)x_{n+1} + 2x_{n+1} - \sum_{k=1}^{n} k x_k$$

$$> (n+1)x_{n+1} + 2\sum_{k=1}^{n} k x_k - \sum_{k=1}^{n} k x_k = \sum_{k=1}^{n+1} k x_k,$$

as desired. Furthermore, $x_1 > 0$ by definition and $x_2, x_3, \ldots, x_n$ are also positive by the induction hypothesis. Therefore, $x_{n+2} > (n+1)x_{n+1} > (n+1)(a \cdot n!) = a \cdot (n+1)!$. This completes the induction, proving the claim.(Romanian Team Selection Test for the International Mathematical Olympiad, 1999)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :321. Compute

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n^2}\right)^{\frac{k}{n^2}+1}.$$

Solution :321. We use the fact that

$$\lim_{x\to 0^+} x^x = 1.$$

As a consequence, we have

$$\lim_{x\to 0^+} \frac{x^{x+1}}{x} = 1.$$

For our problem, let $\epsilon > 0$ be a fixed small positive number. There exists $n(\epsilon)$ such that for any integer $n \geq n(\epsilon)$,

$$1 - \epsilon < \frac{\left(\frac{k}{n^2}\right)^{\frac{k}{n^2}+1}}{\frac{k}{n^2}} < 1 + \epsilon, \quad k = 1, 2, \ldots, n.$$

From this, using properties of ratios, we obtain

$$1 - \epsilon < \frac{\sum_{k=1}^{n} \left(\frac{k}{n^2}\right)^{\frac{k}{n^2}+1}}{\sum_{k=1}^{n} \frac{k}{n^2}} < 1 + \epsilon, \quad \text{for } n \geq n(\epsilon).$$

Knowing that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, this implies

$$(1-\epsilon)\frac{n+1}{2n} < \sum_{k=1}^{n} \left(\frac{k}{n^2}\right)^{\frac{k}{n^2}+1} < (1+\epsilon)\frac{n+1}{2n}, \quad \text{for } n \geq n(\epsilon).$$

It follows that

$$\lim_{n\to\infty} \sum_{k-1}^{n} \left(\frac{k}{n^2}\right)^{\frac{k}{n^2}+1} = \frac{1}{2}.$$

(D. Andrica)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :333. Compute

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$$

Solution :333. Define

$$x_n = \sqrt{1+\sqrt{1+\sqrt{1+\cdots+\sqrt{1}}}}, \quad n \geq 1,$$

where in this expression there are n square roots. Note that x_{n+1} is obtained from x_n by replacing $\sqrt{1}$ by $\sqrt{1+\sqrt{1}}$ at the far end. The square root function being increasing, the sequence $(x_n)_n$ is increasing. To prove that the sequence is bounded, we use the recurrence relation $x_{n+1} = \sqrt{1+x_n}, n \geq 1$. Then from $x_n < 2$, we obtain that $x_{n+1} = \sqrt{1+x_n} < \sqrt{1+2} < 2$, so inductively $x_n < 2$ for all n. Being bounded and monotonic, the sequence $(x_n)_n$ is convergent. Let L be its limit (which must be greater than 1). Passing to the limit in the recurrence relation, we obtain $L = \sqrt{1+L}$, or $L^2 - L - 1 = 0$. The only positive solution is the golden ratio $\frac{\sqrt{5}+1}{2}$, which is therefore the limit of the sequence.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :337. Compute up to two decimal places the number

$$\sqrt{1+2\sqrt{1+2\sqrt{1+\cdots+2\sqrt{1+2\sqrt{1969}}}}}$$

where the expression contains 1969 square roots.
Solution :337. Let

$$x_n = \sqrt{1+2\sqrt{1+2\sqrt{1+\cdots+2\sqrt{1+2\sqrt{1969}}}}}$$

with the expression containing n square root signs. Note that

$$x_1 - (1+\sqrt{2}) = \sqrt{1969} - (1+\sqrt{2}) < 50.$$

Also, since $\sqrt{1+2(1+\sqrt{2})} = 1+\sqrt{2}$, we have

$$x_{n+1} - (1+\sqrt{2}) = \sqrt{1+2x_n} - \sqrt{1+2(1+\sqrt{2})} = \frac{2\left(x_n - (1-\sqrt{2})\right)}{\sqrt{1+2x_n} + \sqrt{1+2(1+\sqrt{2})}}$$
$$< \frac{x_n - (1+\sqrt{2})}{1+\sqrt{2}}.$$

From here we deduce that

$$x_{1969} - (1+\sqrt{2}) < \frac{50}{(1+\sqrt{2})^{1968}} < 10^{-3},$$

and the approximation of x_{1969} with two decimal places coincides with that of $1+\sqrt{2} = 2.41$. This argument proves also that the limit of the sequence is $1+\sqrt{2}$.(St. Petersburg Mathematical Olympiad, 1969)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :338. Find the positive real solutions to the equation

$$\sqrt{x+2\sqrt{x+\cdots+2\sqrt{x+2\sqrt{3x}}}} = x.$$

Solution :338. Write the equation as

$$\sqrt{x+2\sqrt{x+\cdots+2\sqrt{x+2\sqrt{x+2x}}}} = x.$$

We can iterate this equality infinitely many times, always replacing the very last x by its value given by the left-hand side. We conclude that x should satisfy

$$\sqrt{x+2\sqrt{x+2\sqrt{x+2\cdots}}} = x,$$

provided that the expression on the left makes sense! Let us check that indeed the recursive sequence given by $x_0 = x$, and $x_{n+1} = \sqrt{x+2x_n}, n \geq 0$, converges for any solution x to the original equation. Squaring the equation, we find that $x < x^2$, hence $x > 1$. But then $x_{n+1} < x_n$, because it reduces to $x_n^2 - 2x_n + x > 0$. This is always true, since when viewed as a quadratic function in x_n, the left-hand side has negative discriminant. Our claim is proved, and we can now transform the equation, the one with infinitely many square roots, into the much simpler

$$x = \sqrt{x + 2x}.$$

This has the unique solution $x = 3$, which is also the unique solution to the equation from the statement, and this regardless of the number of radicals.(D.O. Shklyarski, N.N. Chentsov, I.M. Yaglom, Selected Problems and Theorems in Elementary Mathematics, Arithmetic and Algebra, Mir, Moscow)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :340. If $(u_n)_n$ is a sequence of positive real numbers and if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = u > 0$, then $\lim_{n\to\infty} \sqrt[n]{u_n} = u$
Solution :340. The solution is a direct application of the Cesàro-Stolz theorem. Indeed, if we let $a_n = \ln u_n$ and $b_n = n$, then

$$\ln \frac{u_{n+1}}{u_n} = \ln u_{n+1} - \ln u_n = \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

and

$$\ln \sqrt[n]{u_n} = \frac{1}{n} \ln u_n = \frac{a_n}{b_n}.$$

The conclusion follows.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :341. Let p be a real number, $p \neq 1$. Compute

$$\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}}$$

Solution :341. In view of the Cesàro-Stolz theorem, it suffices to prove the existence of and to compute the limit

$$\lim_{n\to\infty} \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}}.$$

We invert the fraction and compute instead

$$\lim_{n\to\infty} \frac{(n+1)^{p+1} - n^{p+1}}{(n+1)^p}.$$

Dividing both the numerator and denominator by $(n+1)^{p+1}$, we obtain

$$\lim_{n\to\infty} \frac{1 - \left(1 - \frac{1}{n+1}\right)^{p+1}}{\frac{1}{n+1}},$$

which, with the notation $h = \frac{1}{n+1}$ and $f(x) = (1-x)^{p+1}$, becomes

$$-\lim_{h \to 0} \frac{f(h) - f(0)}{h} = -f'(0) = p + 1.$$

We conclude that the required limit is $\frac{1}{p+1}$.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :342. Let $0 < x_0 < 1$ and $x_{n+1} = x_n - x_n^2$ for $n \geq 0$. Compute $\lim_{n\to\infty} nx_n$.
Solution :342. An inductive argument shows that $0 < x_n < 1$ for all n. Also, $x_{n+1} = x_n - x_n^2 < x_n$, so $(x_n)_n$ is decreasing. Being bounded and monotonic, the sequence converges; let x be its limit. Passing to the limit in the defining relation, we find that $x = x - x^2$, so $x = 0$.We now apply the Cesàro-Stolz theorem. We have

$$\begin{aligned} \lim_{n\to\infty} nx_n = \lim_{n\to\infty} \frac{n}{\frac{1}{x_n}} = \lim_{n\to\infty} \frac{n+1-n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{x_n - x_n^2} - \frac{1}{x_n}} \\ = \lim_{n\to\infty} \frac{x_n - x_n^2}{1 - (1 - x_n)} = \lim_{n\to\infty} (1 - x_n) = 1 \end{aligned}$$

and we are done.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :343. Let $x_0 \in [-1,1]$ and $x_{n+1} = x_n - \arcsin\left(\sin^2 x_n\right)$ for $n \geq 0$. Compute $\lim_{n\to\infty} \sqrt{n} x_n$
Solution :343. It is not difficult to see that $\lim_{n\to\infty} x_n = 0$. Because of this fact,

$$\lim_{n\to\infty} \frac{x_n}{\sin x_n} = 1.$$

If we are able to find the limit of

$$\frac{n}{\frac{1}{\sin^2 x_n}},$$

then this will equal the square of the limit under discussion. We use the Cesàro-Stolz theorem.Suppose $0 < x_0 \leq 1$ (the cases $x_0 < 0$ and $x_0 = 0$ being trivial; see above). If $0 < x_n \leq 1$, then $0 < \arcsin\left(\sin^2 x_n\right) < \arcsin\left(\sin x_n\right) = x_n$, so $0 < x_{n+1} < x_n$. It follows by induction on n that $x_n \in (0,1]$ for all n and x_n decreases to 0 . Rewriting the recurrence as $\sin x_{n+1} = \sin x_n \sqrt{1 - \sin^4 x_n} -$

$\sin^2 x_n \cos x_n$ gives

$$\begin{aligned}\frac{1}{\sin x_{n+1}} - \frac{1}{\sin x_n} &= \frac{\sin x_n - \sin x_{n+1}}{\sin x_n \sin x_{n+1}} \\ &= \frac{\sin x_n - \sin x_n \sqrt{1-\sin^4 x_n} + \sin^2 x_n \cos x_n}{\sin x_n \left(\sin x_n \sqrt{1-\sin^4 x_n} - \sin^2 x_n \cos x_n\right)} \\ &= \frac{1 - \sqrt{1-\sin^4 x_n} + \sin x_n \cos x_n}{\sin x_n \sqrt{1-\sin^4 x_n} - \sin^2 x_n \cos x_n} \\ &= \frac{\frac{\sin^4 x_n}{1+\sqrt{1-\sin^4 x_n}} + \sin x_n \cos x_n}{\sin x_n \sqrt{1-\sin^4 x_n} - \sin^2 x_n \cos x_n} \\ &= \frac{\frac{\sin^3 x_n}{1+\sqrt{1-\sin^4 x_n}} + \cos x_n}{\sqrt{1-\sin^4 x_n} - \sin x_n \cos x_n}.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \left(\frac{1}{\sin x_{n+1}} - \frac{1}{\sin x_n}\right) = 1$$

From the Cesàro-Stolz theorem it follows that $\lim_{n\to\infty} \frac{1}{n \sin x_n} = 1$, and so we have $\lim_{n\to\infty} n x_n = 1$.(Gazeta Matematica (Mathematics Gazette, Bucharest), 2002, proposed by T. Andreescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :344. For an arbitrary number $x_0 \in (0, \pi)$ define recursively the sequence $(x_n)_n$ by $x_{n+1} = \sin x_n, n \geq 0$. Compute $\lim_{n\to\infty} \sqrt{n} x_n$.
Solution :344. We compute the square of the reciprocal of the limit, namely $\lim_{n\to\infty} \frac{1}{n x_n^2}$. To this end, we apply the Cesàro-Stolz theorem to the sequences $a_n = \frac{1}{x_n^2}$ and $b_n = n$. First, note that $\lim_{n\to\infty} x_n = 0$. Indeed, in view of the inequality $0 < \sin x < x$ on $(0, \pi)$, the sequence is bounded and decreasing, and the limit L satisfies $L = \sin L$, so $L = 0$. We then have

$$\begin{aligned}\lim_{n\to\infty} \left(\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}\right) &= \lim_{n\to\infty} \left(\frac{1}{\sin^2 x_n} - \frac{1}{x_n^2}\right) = \lim_{n\to\infty} \frac{x_n^2 - \sin^2 x_n}{x_n^2 \sin^2 x_n} \\ &= \lim_{x_n\to 0} \frac{x_n^2 - \frac{1}{2}(1-\cos 2x_n)}{\frac{1}{2} x_n^2 (1-\cos 2x_n)} = \lim_{x_n \to 0} \frac{2x_n^2 - \left[\frac{(2x_n)^2}{2!} - \frac{(2x_n)^4}{4!} + \cdots\right]}{x_n^2 \left[\frac{(2x_n)^2}{2!} - \frac{(2x_n)^4}{4!} + \cdots\right]} \\ &= \frac{2^4/4!}{2^2/2!} = \frac{1}{3}.\end{aligned}$$

We conclude that the original limit is $\sqrt{3}$.(J. Dieudonné, Infinitesimal Calculus, Hermann, 1962, solution by Ch. Radoux)

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :350. Given a sequence $(a_n)_n$ such that for any $\gamma > 1$ the subsequence $a_{\lfloor \gamma^n \rfloor}$ converges to zero, does it follow that the sequence $(a_n)_n$ itself converges to zero?
Solution :350. The answer to the question is yes. We claim that for any sequence of positive integers n_k, there exists a number $\gamma > 1$ such that $\left(\lfloor \gamma^k \rfloor\right)_k$ and $(n_k)_k$ have infinitely many terms in common. We need the following lemma.Lemma. For any $\alpha, \beta, 1 < \alpha < \beta$, the set $\cup_{k=1}^{\infty}\left[\alpha^k, \beta^k - 1\right]$ contains some interval of the form (a, ∞)Proof. Observe that $(\beta/\alpha)^k \to \infty$ as $k \to \infty$. Hence for large $k, \alpha^{k+1} < \beta^k - 1$, and the lemma follows.Let us return to the problem and prove the claim. Fix the numbers α_1 and β_1, $1 < \alpha_1 < \beta_1$. Using the lemma we can find some k_1 such that the interval $\left[\alpha_1^{k_1}, \beta_1^{k_1} - 1\right]$ contains some terms of the sequence $(n_k)_k$. Choose one of these terms and call it t_1. Define

$$\alpha_2 = t_1^{1/k_1}, \quad \beta_2 = \left(t_1 + \frac{1}{2}\right)^{1/k_1}$$

Then $[\alpha_2, \beta_2] \subset [\alpha_1, \beta_1]$, and for any $x \in [\alpha_2, \beta_2], \lfloor x^{k_1} \rfloor = t_1$. Again by the lemma, there exists k_2 such that $\left[\alpha_2^{k_2}, \beta_2^{k_2} - 1\right]$ contains a term of $(n_k)_k$ different from n_1. Call this term t_2. Let

$$\alpha_3 = t_2^{1/k_2}, \quad \beta_3 = \left(t_2 + \frac{1}{2}\right)^{1/k_2}$$

As before, $[\alpha_3, \beta_3] \subset [\alpha_2, \beta_2]$ and $\lfloor x^{k_2} \rfloor = t_2$ for any $x \in [\alpha_3, \beta_3]$. Repeat the construction infinitely many times. By Cantor's nested intervals theorem, the intersection of the decreasing sequence of intervals $[\alpha_j, \beta_j], j = 1, 2, \ldots$, is nonempty. Let γ be an element of this intersection. Then $\lfloor \gamma^{k_j} \rfloor = t_j, j = 1, 2, \ldots$, which shows that the sequence $\left(\lfloor \gamma^j \rfloor\right)_j$ contains a subset of the sequence $(n_k)_k$. This proves the claim. To conclude the solution to the problem, assume that the sequence $(a_n)_n$ does not converge to 0 . Then it has some subsequence $(a_{n_k})_k$ that approaches a nonzero (finite or infinite) limit as $n \to \infty$. But we saw above that this subsequence has infinitely many terms in common with a sequence that converges to zero, namely with some $\left(a_{\lfloor \gamma^k \rfloor}\right)_k$. This is a contradiction. Hence the sequence $(a_n)_n$ converges to 0 .(Soviet Union University Student Mathematical Olympiad, 1975)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :353. For what positive x does the series

$$(x-1) + (\sqrt{x} - 1) + (\sqrt[3]{x} - 1) + \cdots + (\sqrt[n]{x} - 1) + \cdots$$

converge?
Solution :353. The series clearly converges for $x=1$. We will show that it does not converge for $x \neq 1$.The trick is to divide through by $x-1$ and compare to the harmonic series. By the mean value theorem applied to $f(t)=t^{1/n}$, for each n there exists c_n between x and 1 such that

$$\frac{\sqrt[n]{x}-1}{x-1}=\frac{1}{n} c^{\frac{1}{n}-1}.$$

It follows that

$$\frac{\sqrt[n]{x}-1}{x-1}>\frac{1}{n}(\max (1, x))^{\frac{1}{n}-1}>\frac{1}{n}(\max (1, x))^{-1}.$$

Summing, we obtain

$$\sum_{n=1}^{\infty} \frac{\sqrt[n]{x}-1}{x-1} \geq(\max (1, x))^{-1} \sum_{n=1}^{\infty} \frac{1}{n}=\infty,$$

which proves that the series diverges.(G.T. Gilbert, M.I. Krusemeyer, L.C. Larson, The Wohascum County Problem Book, MAA, 1996)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :357. Does the series $\sum_{n=1}^{\infty} \sin \pi \sqrt{n^2+1}$ converge?
Solution :357. We have

$$\sin \pi \sqrt{n^2+1}=(-1)^n \sin \pi\left(\sqrt{n^2+1}-n\right)=(-1)^n \sin \frac{\pi}{\sqrt{n^2+1}+n}.$$

Clearly, the sequence $x_n=\frac{\pi}{\sqrt{n^2+1}+n}$ lies entirely in the interval $\left(0, \frac{\pi}{2}\right)$, is decreasing, and converges to zero. It follows that $\sin x_n$ is positive, decreasing, and converges to zero. By Riemann's convergence criterion, $\sum_{k \geq 1}(-1)^n \sin x_n$, which is the series in question, is convergent.(Gh. Sireţchi, Calcul Diferential si Integral (Differential and Integral Calculus), Editura Ştiinţifică şi Enciclopedică, 1985)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :358. (a) Does there exist a pair of divergent series $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$, with $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, such that the series $\sum_n \min \left(a_n, b_n\right)$ is convergent?(b) Does the answer to this question change if we assume additionally that $b_n=\frac{1}{n}$, $n=1,2, \ldots$?
Solution :358. (a) We claim that the answer to the first question is yes. We construct the sequences $\left(a_n\right)_n$ and $\left(b_n\right)_n$ inductively, in a way inspired by the proof that the harmonic series diverges. At step 1 , let $a_1=1, b_1=\frac{1}{2}$. Then

at step 2 , let $a_2 = a_3 = \frac{1}{8}$ and $b_2 = b_3 = \frac{1}{2}$. In general, at step k we already know $a_1, a_2, \ldots, a_{n_k}$ and $b_1, b_2, \ldots, b_{n_k}$ for some integer n_k. We want to define the next terms. If k is even, and if

$$b_{n_k} = \frac{1}{2^{r_k}},$$

let

$$b_{n_k+1} = \cdots = b_{n_k+2^{r_k}} = \frac{1}{2^{r_k}}$$

and

$$a_{n_k+1} = \cdots = a_{n_k+2^{r_k}} = \frac{1}{2^k \cdot 2^{r_k}}.$$

If k is odd, we do precisely the same thing, with the roles of the sequences $(a_n)_n$ and $(b_n)_n$ exchanged. As such we have

$$\sum_n b_n \geq \sum_{k \text{ odd}} 2^{r_k} \frac{1}{2^{r_k}} = 1 + 1 + \cdots = \infty,$$
$$\sum_n a_n \geq \sum_{k \text{ even}} 2^{r_k} \frac{1}{2^{r_k}} = 1 + 1 + \cdots = \infty,$$

which shows that both series diverge. On the other hand, if we let $c_n = \min(a_n, b_n)$, then

$$\sum_n c_n = \sum_k 2^{r_k} \frac{1}{2^k 2^{r_k}} = \sum_k \frac{1}{2^k},$$

which converges to 1 . The example proves our claim.(b) The answer to the second question is no, meaning that the situation changes if we work with the harmonic series. Suppose there is a series $\sum_n a_n$ with the given property. If $c_n = \frac{1}{n}$ for only finitely many n 's, then for large $n, a_n = c_n$, meaning that both series diverge. Hence $c_n = \frac{1}{n}$ for infinitely many n. Let $(k_m)_m$ be a sequence of integers satisfying $k_{m+1} \geq 2k_m$ and $c_{k_m} = \frac{1}{k_m}$. Then

$$\sum_{k=k_m+1}^{k_{m+1}} c_k \geq (k_{m+1} - k_m) c_{k_{m+1}} = (k_{m+1} - k_m) \frac{1}{k_{m+1}} = \frac{1}{2}.$$

This shows that the series $\sum_n c_n$ diverges, a contradiction.(short list of the 44th International Mathematical Olympiad, 2003)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :360. Is the number

$$\sum_{n=1}^{\infty} \frac{1}{2^{n^2}}$$

rational?
Solution :360. The series is convergent because it is bounded from above by the geometric series with ratio $\frac{1}{2}$. Assume that its sum is a rational number $\frac{a}{b}$. Choose n such that $b < 2^n$. Then

$$\frac{a}{b} - \sum_{k=1}^{n} \frac{1}{2^{k^2}} = \sum_{k \geq n+1} \frac{1}{2^{k^2}}.$$

But the sum $\sum_{k=1}^{n} \frac{1}{2^{k^2}}$ is equal to $\frac{m}{2^{n^2}}$ for some integer n. Hence

$$\frac{a}{b} - \sum_{k=1}^{n} \frac{1}{2^{k^2}} = \frac{a}{b} - \frac{m}{2^{n^2}} > \frac{1}{2^{n^2}b} > \frac{1}{2^{n^2+n}} > \frac{1}{2^{(n+1)^2-1}} = \sum_{k \geq (n+1)^2} \frac{1}{2^k} > \sum_{k \geq n+1} \frac{1}{2^{k^2}},$$

a contradiction. This shows that the sum of the series is an irrational number.Remark. In fact, this number is transcendental.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :372. Evaluate in closed form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m!n!}{(m+n+2)!}.$$

Solution :372. Let us look at the summation over n first. Multiplying each term by $(m+n+2)-$ $(n+1)$ and dividing by $m+1$, we obtain

$$\frac{m!}{m+1} \sum_{n=0}^{\infty} \left(\frac{n!}{(m+n+1)!} - \frac{(n+1)!}{(m+n+2)!} \right)$$

This is a telescopic sum that adds up to

$$\frac{m!}{m+1} \cdot \frac{0!}{(m+1)!}.$$

Consequently, the expression we are computing is equal to

$$\sum_{m=0}^{\infty} \frac{1}{(m+1)^2} = \frac{\pi^2}{6}.$$

(Mathematical Mayhem, 1995)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :374. Evaluate in closed form

$$\sum_{k=0}^{n}(-1)^k(n-k)!(n+k)!$$

Solution :374. First solution: Let $S_n = \sum_{k=0}^{n}(-1)^k(n-k)!(n+k)\,!$. Reordering the terms of the sum, we have

$$\begin{aligned} S_n &= (-1)^n\sum_{k=0}^{n}(-1)^k k!(2n-k)! \\ &= (-1)^n\frac{1}{2}\left((-1)^n n!n! + \sum_{k=0}^{2n}(-1)^k k!(2n-k)!\right) \\ &= \frac{(n!)^2}{2} + (-1)^n\frac{T_n}{2}, \end{aligned}$$

where $T_n = \sum_{k=0}^{2n}(-1)^k k!(2n-k)\,!$. We now focus on the sum T_n. Observe that

$$\frac{T_n}{(2n)!} = \sum_{k=0}^{2n}\frac{(-1)^k}{\begin{pmatrix} 2n \\ k \end{pmatrix}}$$

and

$$\frac{1}{\begin{pmatrix} 2n \\ k \end{pmatrix}} = \frac{2n+1}{2(n+1)}\left[\frac{1}{\begin{pmatrix} 2n+1 \\ k \end{pmatrix}} + \frac{1}{\begin{pmatrix} 2n+1 \\ k+1 \end{pmatrix}}\right].$$

Hence

$$\frac{T_n}{(2n)!} = \frac{2n+1}{2(n+1)}\left[\frac{1}{\begin{pmatrix} 2n+1 \\ 0 \end{pmatrix}} + \frac{1}{\begin{pmatrix} 2n+1 \\ 1 \end{pmatrix}} - \frac{1}{\begin{pmatrix} 2n+1 \\ 1 \end{pmatrix}} - \frac{1}{\begin{pmatrix} 2n+1 \\ 2 \end{pmatrix}} + \cdots + \frac{1}{\begin{pmatrix} 2n+1 \\ 2n \end{pmatrix}} + \frac{1}{\begin{pmatrix} 2n+ \\ 2n+ \end{pmatrix}}\right.$$

This sum telescopes to

$$\frac{2n+1}{2(n+1)}\left[\frac{1}{\begin{pmatrix} 2n+1 \\ 0 \end{pmatrix}} + \frac{1}{\begin{pmatrix} 2n+1 \\ 2n+1 \end{pmatrix}}\right] = \frac{2n+1}{n+1}.$$

Thus $T_n = \frac{(2n+1)!}{n+1}$, and therefore

$$S_n = \frac{(n!)^2}{2} + (-1)^n\frac{(2n+1)!}{2(n+1)}.$$

Second solution: Multiply the k th term in S_n by $(n-k+1)+(n+k+1)$ and divide by $2(n+1)$ to obtain

$$S_n = \frac{1}{2(n+1)} \sum_{k=0}^{n} \left[(-1)^k (n-k+1)!(n+k)! + (-1)^k (n-k)!(n+k+1)!\right].$$

This telescopes to

$$\frac{1}{2(n+1)} \left[n!(n+1)! + (-1)^n (2n+1)!\right].$$

(T. Andreescu, second solution by R. Stong)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :377. Compute the product

$$\left(1-\frac{4}{1}\right)\left(1-\frac{4}{9}\right)\left(1-\frac{4}{25}\right)\cdots$$

Solution :377. For $N \geq 2$, define

$$a_N = \left(1-\frac{4}{1}\right)\left(1-\frac{4}{9}\right)\left(1-\frac{4}{25}\right)\cdots\left(1-\frac{4}{(2N-1)^2}\right).$$

The problem asks us to find $\lim_{N\to\infty} a_N$. The defining product for a_N telescopes as follows:

$$\begin{aligned} a_N &= \left[\left(1-\frac{2}{1}\right)\left(1+\frac{2}{1}\right)\right]\left[\left(1-\frac{2}{3}\right)\left(1+\frac{2}{3}\right)\right]\cdots\left[\left(1-\frac{2}{2N-1}\right)\left(1+\frac{2}{2N-1}\right)\right] \\ &= (-1\cdot 3)\left(\frac{1}{3}\cdot\frac{5}{3}\right)\left(\frac{3}{5}\cdot\frac{7}{5}\right)\cdots\left(\frac{2N-3}{2N-1}\cdot\frac{2N+1}{2N-1}\right) = -\frac{2N+1}{2N-1}. \end{aligned}$$

Hence the infinite product is equal to

$$\lim_{N\to\infty} a_N = -\lim_{N\to\infty}\frac{2N+1}{2N-1} = -1.$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :378. Let x be a positive number less than 1 . Compute the product

$$\prod_{n=0}^{\infty}\left(1+x^{2^n}\right).$$

Solution :378. Define the sequence $(a_N)_N$ by

$$a_N = \prod_{n=1}^{N} \left(1 + x^{2^n}\right).$$

Note that $(1-x)a_N$ telescopes as

$$\begin{aligned}
&(1-x)(1+x)\left(1+x^2\right)\left(1+x^4\right)\cdots\left(1+x^{2^N}\right) \\
&= \left(1-x^2\right)\left(1+x^2\right)\left(1+x^4\right)\cdots\left(1+x^{2^N}\right) \\
&= \left(1-x^4\right)\left(1+x^4\right)\cdots\left(1+x^{2^N}\right) \\
&= \cdots = \left(1-x^{2^{N+1}}\right).
\end{aligned}$$

Hence $(1-x)a_N \to 1$ as $N \to \infty$, and therefore

$$\prod_{n \geq 0} \left(1 + x^{2^n}\right) = \frac{1}{1-x}.$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :380. Find the real parameters m and n such that the graph of the function $f(x) = \sqrt[3]{8x^3 + mx^2} - nx$ has the horizontal asymptote $y = 1$.
Solution :380. We are supposed to find m and n such that

$$\lim_{x \to \infty} \sqrt[3]{8x^3 + mx^2} - nx = 1 \quad \text{or} \quad \lim_{x \to -\infty} \sqrt[3]{8x^3 + mx} - nx = 1.$$

We compute

$$\sqrt[3]{8x^3 + mx^2} - nx = \frac{\left(8 - n^3\right)x^3 + mx^2}{\sqrt[3]{\left(8x^3 + mx^2\right)^2} + nx\sqrt[3]{8x^3 + mx^2} + n^2x^2}.$$

For this to have a finite limit at either $+\infty$ or $-\infty, 8 - n^3$ must be equal to 0 (otherwise the highest degree of x in the numerator would be greater than the highest degree of x in the denominator). We have thus found that $n = 2$.Next, factor out and cancel an x^2 to obtain

$$f(x) = \frac{m}{\sqrt[3]{\left(8 + \frac{m}{x}\right)^2} + 2\sqrt[3]{8 + \frac{m}{x}} + 4}.$$

We see that $\lim_{x \to \infty} f(x) = \frac{m}{12}$. For this to be equal to $1, m$ must be equal to 12 . Hence the answer to the problem is $(m, n) = (12, 2)$.

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :381. Does

$$\lim_{x\to\pi/2}(\sin x)^{\frac{1}{\cos x}}$$

exist?
Solution :381. This is a limit of the form 1^∞. It can be computed as follows:

$$\begin{aligned}\lim_{x\to\pi/2}(\sin x)^{\frac{1}{\cos x}} &= \lim_{x\to\pi/2}(1+\sin x-1)^{\frac{1}{\sin x-1}\cdot\frac{\sin x-1}{\cos x}}\\ &= \left(\lim_{t\to 0}(1+t)^{1/t}\right)^{\lim_{x\to\pi/2}\frac{\sin x-1}{\cos x}} = \exp\left(\lim_{u\to 0}\frac{\cos u-1}{\sin u}\right)\\ &= \exp\left(\frac{\cos u-1}{u}\cdot\frac{u}{\sin u}\right) = e^{0.1} = e^0 = 1.\end{aligned}$$

The limit therefore exists.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :382. For two positive integers m and n, compute

$$\lim_{x\to 0}\frac{\sqrt[m]{\cos x}-\sqrt[n]{\cos x}}{x^2}.$$

Solution :382. Without loss of generality, we may assume that $m>n$. Write the limit as

$$\lim_{x\to 0}\frac{\sqrt[mn]{\cos^n x}-\sqrt[mn]{\cos^m x}}{x^2}.$$

Now we can multiply by the conjugate and obtain

$$\begin{aligned}&\lim_{x\to 0}\frac{\cos^n x-\cos^m x}{x^2\left(\sqrt[mn]{(\cos^n x)^{mn-1}}+\cdots+\sqrt[mn]{(\cos^m x)^{mn-1}}\right)}\\ &= \lim_{x\to 0}\frac{\cos^n x\,(1-\cos^{m-n}x)}{mnx^2} = \lim_{x\to 0}\frac{1-\cos^{m-n}x}{mnx^2}\\ &= \lim_{x\to 0}\frac{(1-\cos x)\left(1+\cos x+\cdots+\cos^{m-n-1}x\right)}{mnx^2}\\ &= \frac{m-n}{mn}\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{m-n}{2mn}.\end{aligned}$$

We are done.
Topic :Real Analysis

Book :Putnam and Beyond
Final Answer :

Problem Statement :383. Does there exist a nonconstant function $f:(1, \infty) \rightarrow \mathbb{R}$ satisfying the relation $f(x)=f\left(\frac{x^{2}+1}{2}\right)$ for all $x>1$ and such that $\lim_{x \rightarrow \infty} f(x)$ exists?
Solution :383. For $x>1$ define the sequence $\left(x_{n}\right)_{n \geq 0}$ by $x_{0}=x$ and $x_{n+1}=\frac{x_{n}^{2}+1}{2}, n \geq 0$. The sequence is increasing because of the AM-GM inequality. Hence it has a limit L, finite or infinite. Passing to the limit in the recurrence relation, we obtain $L=\frac{L^{2}+1}{2}$; hence either $L=1$ or $L=\infty$. Since the sequence is increasing, $L \geq x_{0}>1$, so $L=\infty$. We therefore have

$$f(x)=f\left(x_{0}\right)=f\left(x_{1}\right)=f\left(x_{2}\right)=\cdots=\lim_{n \rightarrow \infty} f\left(x_{n}\right)=\lim_{x \rightarrow \infty} f(x).$$

This implies that f is constant, which is ruled out by the hypothesis. So the answer to the question is negative.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :387. Does there exist a continuous function $f:[0,1] \rightarrow \mathbb{R}$ that assumes every element of its range an even (finite) number of times?
Solution :387. The answer is yes, there is a tooth function with this property. We construct f to have local maxima at $\frac{1}{2^{2n+1}}$ and local minima at 0 and $\frac{1}{2^{2n}}, n \geq 0$. The values of the function at the extrema are chosen to be $f(0)=f(1)=0, f\left(\frac{1}{2}\right)=\frac{1}{2}$, and $f\left(\frac{1}{2^{2n+1}}\right)=\frac{1}{2^{n}}$ and $f\left(\frac{1}{2^{2n}}\right)=\frac{1}{2^{n+1}}$ for $n \geq 1$. These are connected through segments. The graph from Figure 66 convinces the reader that f has the desired properties. dapest))(Kozépiskolai Matematikai Lapok (Mathematics Gazette for High Schools, Bu-
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :393. Give an example of a continuous function on an interval that is nowhere differentiable.
Solution :393. The first example of such a function was given by Weierstrass. The example we present here, of a function $f:[0,1] \rightarrow[0,1]$, was published by S. Marcus in the Mathematics Gazette, Bucharest.If $0 \leq x \leq 1$ and $x=0.a_{1} a_{2} a_{3} \ldots$ is the ternary expansion of x, we let the binary representation of $f(x)$ be $0.b_{1} b_{2} b_{3} \ldots$, where the binary digits $b_{1}, b_{2}, b_{3}, \ldots$ are uniquely determined by the conditions(i) $b_{1}=1$ if and only if $a_{1}=1$,(ii) $b_{n+1}=b_{n}$ if and only if $a_{n+1}=a_{n}, n \geq 1$.It is not hard to see that $f(x)$ does not depend on which ternary representation you choose for x. For example,

$$f(0.0222 \ldots)=0.0111 \cdots=0.1000 \cdots=f(0.1000 \ldots).$$

Let us prove first that the function is continuous. If x is a number that has a unique ternary expansion and $(x_n)_n$ is a sequence converging to x, then the first m digits of x_n become equal to the first m digits of x for n sufficiently large. It follows from the definition of f that the first m binary digits of $f(x_n)$ become equal to the first m binary digits of $f(x)$ for n sufficiently large. Hence $f(x_n)$ converges to $f(x)$, so f is continuous at x.If x is a number that has two possible ternary expansions, then in one expansion x has only finitely many nonzero digits $x = 0.a_1a_2\ldots a_k00\ldots$, with $a_k \neq 0$. The other expansion is $0.a_1a_2\ldots a_k'222\ldots$, with $a_k' = a_k - 1(= 0$ or 1). Given a sequence $(x_n)_n$ that converges to x, for sufficiently large n the first $k-1$ digits of x_n are equal to $a_1, a_2, \ldots, a_{k-1}$, while the next $m-k+1$ are either $a_k, 0, 0, \ldots, 0$, or $a_k', 2, 2, \ldots, 2$. If $f(x) = f(0.a_1a_2\ldots a_k00\ldots) = 0.b_1b_2b_3\ldots$, then for n sufficiently large, the first $k-1$ digits of $f(x_n)$ are $b_1, b_2, \ldots, b_{k-1}$, while the next $m-k+1$ are either $b_k, b_{k+1} = b_{k+2} = \cdots = b_m$ (the digits of $f(x)$) or $1-b_k, 1-b_{k+1} = \cdots = 1-b_m$. The two possible binary numbers are $0.b_1b_2\ldots b_{k-1}0111\ldots$ and $0.b_1b_2\ldots b_{k-1}1000\ldots$; they differ from $f(x)$ by at most $\frac{1}{2^{m+1}}$. We conclude again that as $n \to \infty, f(x_n) \to f(x)$. This proves the continuity of f.Let us show next that f does not have a finite derivative from the left at any point $x \in (0,1]$. For such x consider the ternary expansion $x = 0.a_1a_2a_3\ldots$ that has infinitely many nozero digits, and, applying the definition of f for this expansion, let $f(x) = 0.b_1b_2b_3\ldots$. Now consider an arbitrary positive number n, and let $k_n \geq n$ be such that $a_{k_n} \neq 0$. Construct a number $x' \in (0,1)$ whose first $k_n - 1$ digits are the same as those of x, whose k_n th digit is zero, and all of whose other digits are equal to 0 if $b_{k_n+1} = 1$ and to 1 if $b_{k_n+1} = 0$. Then

$$0 < x - x' < 2 \cdot 3^{-k_n} + 0.\underbrace{00\ldots0}_{k_n}22\ldots,0 = 3^{-k_n+1},$$

while in the first case,

$$|f(x) - f(x')| \geq 0.\underbrace{00\ldots0}_{k_n}b_{k_n+1} = 0.\underbrace{00\ldots0}_{k_n}1,$$

and in the second case,

$$|f(x) - f(x')| \geq 0.\underbrace{00\ldots0}_{k_n}11\ldots1 - 0.\underbrace{00\ldots0}_{k_n}0b_{k_n+2\ldots},$$

and these are both greater than or equal to 2^{-k_n-1}. Since $k_n \geq n$, we have $0 < x - x' < 3^{-n+1}$ and

$$\left|\frac{f(x) - f(x')}{x - x'}\right| > \frac{2^{-k_n-1}}{3^{-k_n+1}} = \frac{1}{6}\left(\frac{3}{2}\right)^{k_n} \geq \frac{1}{6}\left(\frac{3}{2}\right)^{n}.$$

Letting $n \to \infty$, we obtain

$$x' \to x, \quad \text{while} \left|\frac{f(x) - f(x')}{x - x'}\right| \to \infty.$$

This proves that f does not have a derivative on the left at x. The argument that f does not have a derivative on the right at x is similar and is left to the reader.Remark. S. Banach has shown that in some sense, there are far more continuous functions that are not differentiable at any point than continuous functions that are differentiable at least at some point.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :399. Let A and B be two cities connected by two different roads. Suppose that two cars can travel from A to B on different roads keeping a distance that does not exceed one mile between them. Is it possible for the cars to travel the first one from A to B and the second one from B to A in such a way that the distance between them is always greater than one mile?
Solution :399. Without loss of generality, we may assume that the cars traveled on one day from A to B keeping a distance of at most one mile between them, and on the next day they traveled in opposite directions in the same time interval, which we assume to be of length one unit of time.Since the first car travels in both days on the same road and in the same direction, it defines two parametrizations of that road. Composing the motions of both cars during the second day of travel with a homeomorphism (continuous bijection) of the time interval $[0,1]$, we can ensure that the motion of the first car yields the same parametrization of the road on both days. Let $f(t)$ be the distance from the second car to A when the first is at t on the first day, and $g(t)$ the distance from the second car to A when the first is at t on the second day. These two functions are continuous, so their difference is also continuous. But $f(0)-g(0)=-\operatorname{dist}(A,B)$, and $f(1)-g(1)=\operatorname{dist}(A,B)$, where $\operatorname{dist}(A,B)$ is the distance between the cities.The intermediate value property implies that there is a moment t for which $f(t)-\ g(t)=0$. At that moment the two cars are in the same position as they were the day before, so they are at distance at most one mile. Hence the answer to the problem is no.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :405. Find all positive real solutions to the equation $2^x=x^2$.
Solution :405. Taking the logarithm, transform the equation into the equivalent $x\ln 2=2\ln x$. Define the function $f:\mathbb{R}\rightarrow\mathbb{R}, f(x)=x\ln 2-2\ln x$. We are to find the zeros of f. Differentiating, we obtain

$$f'(x)=\ln 2-\frac{2}{x},$$

which is strictly increasing. The unique zero of the derivative is $\frac{2}{\ln 2}$, and so f' is negative for $x<2/\ln 2$ and positive for $x>\frac{2}{\ln 2}$. Note also that $\lim_{x\rightarrow 0} f(x)=\lim_{x\rightarrow\infty} f(x)=\infty$. There are two possibilities: either $f\left(\frac{2}{\ln 2}\right)>0$, in which

case the equation $f(x)=0$ has no solutions, or $f\left(\frac{2}{\ln 2}\right)<0$, in which case the equation $f(x)=0$ has exactly two solutions. The latter must be true, since $f(2)=f(4)=0$. Therefore, $x=2$ and $x=4$ are the only solutions to $f(x)=0$, and hence also to the original equation.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :407. Determine

$$\max_{z \in \mathbb{C},|z|=1}\left|z^{3}-z+2\right| .$$

Solution :407. Let $f: \mathbb{C} \rightarrow \mathbb{C}, f(z)=z^{3}-z+2$. We have to determine $\max_{|z|=1}|f(z)|^{2}$. For this, we switch to real coordinates. If $|z|=1$, then $z=x+i y$ with $y^{2}=1-x^{2},-1 \leq x \leq 1$. View the restriction of $|f(z)|^{2}$ to the unit circle as a function depending on the real variable x :

$$\begin{aligned}
|f(z)|^{2} & =\left|(x+i y)^{3}-(x+i y)+2\right|^{2} \\
& =\left|\left(x^{3}-3 x y^{2}-x+2\right)+i y\left(3 x^{2}-y^{2}-1\right)\right|^{2} \\
& =\left|\left(x^{3}-3 x\left(1-x^{2}\right)-x+2\right)+i y\left(3 x^{2}-\left(1-x^{2}\right)-1\right)\right|^{2} \\
& =\left(4 x^{3}-4 x+2\right)^{2}+\left(1-x^{2}\right)\left(4 x^{2}-2\right)^{2} \\
& =16 x^{3}-4 x^{2}-16 x+8 .
\end{aligned}$$

Call this last expression $g(x)$. Its maximum on $[-1,1]$ is either at a critical point or at an endpoint of the interval. The critical points are the roots of $g^{\prime}(x)=48 x^{2}-8 x-16=0$, namely, $x=\frac{2}{3}$ and $x=-\frac{1}{2}$. We compute $g(-1)=4, g\left(-\frac{1}{2}\right)=13, g\left(\frac{2}{3}\right)=\frac{8}{27}$, $g(1)=4$. The largest of them is 13 , which is therefore the answer to the problem. It is attained when $z=-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$(8th W.L. Putnam Mathematical Competition, 1947)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :408. Find the minimum of the function $f: \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x)=\frac{\left(x^{2}-x+1\right)^{3}}{x^{6}-x^{3}+1} .$$

Solution :408. After we bring the function into the form

$$f(x)=\frac{\left(x-1+\frac{1}{x}\right)^{3}}{x^{3}-1+\frac{1}{x^{3}}},$$

the substitution $x+\frac{1}{x}=s$ becomes natural. We are to find the minimum of the function

$$h(s)=\frac{(s-1)^3}{s^3-3s-1}=1+\frac{-3s^2+6s}{s^3-3s-1}$$

over the domain $(-\infty,-2]\cup[2,\infty)$. Setting the first derivative equal to zero yields the equation

$$3(s-1)\left(s^3-3s^2+2\right)=0.$$

The roots are $s=1$ (double root) and $s=1\pm\sqrt{3}$. Of these, only $s=1+\sqrt{3}$ lies in the domain of the function.We compute

$$\lim_{x\to\pm\infty} h(s)=1,\quad h(2)=1,\quad h(-2)=9,\quad h(1+\sqrt{3})=\frac{\sqrt{3}}{2+\sqrt{3}}.$$

Of these the last is the least. Hence the minimum of f is $\sqrt{3}/(2+\sqrt{3})$, which is attained when $x+\frac{1}{x}=1+\sqrt{3}$, that is, when $x=(1+\sqrt{3}\pm\sqrt[4]{12})/2$.(Mathematical Reflections, proposed by T. Andreescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :409. How many real solutions does the equation

$$\sin(\sin(\sin(\sin(\sin x))))=\frac{x}{3}$$

have?
Solution :409. Let $f(x)=\sin(\sin(\sin(\sin(\sin(x)))))$. The first solution is $x=0$. We have

$$\begin{aligned}f'(0)&=\cos 0\cos(\sin 0)\cos(\sin(\sin 0))\cos(\sin(\sin(\sin 0)))\cos(\sin(\sin(\sin(\sin 0))))\\&=1>\frac{1}{3}.\end{aligned}$$

Therefore, $f(x)>\frac{x}{3}$ in some neighborhood of 0 . On the other hand, $f(x)<1$, whereas $\frac{x}{3}$ is not bounded as $x\to\infty$. Therefore, $f\left(x_0\right)=\frac{x_0}{3}$ for some $x_0>0$. Because f is odd, $-x_0$ is also a solution. The second derivative of f is

$$\begin{aligned}&-\cos(\sin x)\cos(\sin(\sin x))\cos(\sin(\sin(\sin x)))\cos(\sin(\sin(\sin(\sin x))))\sin x\\&\quad-\cos^2 x\cos(\sin(\sin x))\cos(\sin(\sin(\sin x)))\cos(\sin(\sin(\sin(\sin x))))\sin(\sin x)\end{aligned}$$

$$\begin{aligned}&-\cos^2 x\cos^2(\sin x)\cos(\sin(\sin(\sin x)))\cos(\sin(\sin(\sin(\sin x))))\sin(\sin(\sin x))\\&-\cos^2 x\cos^2(\sin x)\cos^2(\sin(\sin x))\cos(\sin(\sin(\sin(\sin x))))\sin(\sin(\sin(\sin x)))\\&-\cos^2 x\cos^2(\sin x)\big)\cos^2(\sin(\sin x))\cos^2(\sin(\sin(\sin x)))\sin(\sin(\sin(\sin(\sin x))))\end{aligned}$$

which is clearly nonpositive for $0\le x\le 1$. This means that $f'(x)$ is monotonic. Therefore, $f'(x)$ has at most one root x' in $[0,+\infty)$. Then $f(x)$ is monotonic at $[0,x']$ and $[x',+\infty)$ and has at most two nonnegative roots. Because $f(x)$ is an

odd function, it also has at most two nonpositive roots. Therefore, $-x_0, 0, x_0$ are the only solutions.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :413. Let f and g be n-times continuously differentiable functions in a neighborhood of a point a, such that $f(a) = g(a) = \alpha, f'(a) = g'(a), \ldots, f^{(n-1)}(a) = g^{(n-1)}(a)$, and $f^{(n)}(a) \neq g^{(n)}(a)$. Find, in terms of α,

$$\lim_{x \to a} \frac{e^{f(x)} - e^{g(x)}}{f(x) - g(x)}.$$

Solution :413. Let us examine the function $F(x) = f(x) - g(x)$. Because $F^{(n)}(a) \neq 0$, we have $F^{(n)}(x) \neq 0$ for x in a neighborhood of a. Hence $F^{(n-1)}(x) \neq 0$ for $x \neq a$ and x in a neighborhood of a (otherwise, this would contradict Rolle's theorem). Then $F^{(n-2)}(x)$ is monotonic to the left, and to the right of a, and because $F^{(n-2)}(a) = 0, F^{(n-2)}(x) \neq 0$ for $x \neq a$ and x in a neighborhood of a. Inductively, we obtain $F'(x) \neq 0$ and $f(x) \neq 0$ in some neighborhood of a.The limit from the statement can be written as

$$\lim_{x \to a} e^{g(x)} \frac{e^{f(x)-g(x)} - 1}{f(x) - g(x)}.$$

We only have to compute the limit of the fraction, since $g(x)$ is a continuous function. We are in a $\frac{0}{0}$ situation, and can apply L'Hôpital's theorem:

$$\lim_{x \to a} \frac{e^{f(x)-g(x)} - 1}{f(x) - g(x)} = \lim_{x \to a} \frac{(f'(x) - g'(x))\, e^{f(x)-g(x)}}{f'(x) - g'(x)} = e^0 = 1.$$

Hence the limit from the statement is equal to $e^{g(a)} = e^{\alpha}$.(N. Georgescu-Roegen)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :423. Find all real solutions to the equation

$$6^x + 1 = 8^x - 27^{x-1}.$$

Solution :423. The equation is $a^3 + b^3 + c^3 = 3abc$, with $a = 2^x, b = -3^{x-1}$, and $c = -1$. Using the factorization

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)\left[(a - b)^2 + (b - c)^2 + (c - a)^2\right]$$

we find that $a + b + c = 0$ (the other factor cannot be zero since, for example, 2^x cannot equal -1). This yields the simpler equation

$$2^x = 3^{x-1} + 1.$$

Rewrite this as

$$3^{x-1}-2^{x-1}=2^{x-1}-1.$$

We immediately notice the solutions $x=1$ and $x=2$. Assume that another solution exists, and consider the function $f(t)=t^{x-1}$. Because $f(3)-f(2)=f(2)-f(1)$, by the mean value theorem there exist $t_1 \in (2,3)$ and $t_2 \in (1,2)$ such that $f'(t_1)=f'(t_2)$. This gives rise to the impossible equality $(x-1)t_1^{x-2}=(x-1)t_2^{x-2}$. We conclude that there are only two solutions: $x=1$ and $x=2$.(Mathematical Reflections, proposed by T. Andreescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :425. Let $x_1, x_2, \ldots, x_n$ be real numbers. Find the real numbers a that minimize the expression

$$|a-x_1|+|a-x_2|+\cdots+|a-x_n|.$$

Solution :425. Arrange the x_i 's in increasing order $x_1 \leq x_2 \leq \cdots \leq x_n$. The function

$$f(a)=|a-x_1|+|a-x_2|+\cdots+|a-x_n|$$

is convex, being the sum of convex functions. It is piecewise linear. The derivative at a point a, in a neighborhood of which f is linear, is equal to the difference between the number of x_i 's that are less than a and the number of x_i 's that are greater than a. The global minimum is attained where the derivative changes sign. For n odd, this happens precisely at $x_{\lfloor n/2\rfloor+1}$. If n is even, the minimum is achieved at any point of the interval $\left[x_{\lfloor n/2\rfloor}, x_{\lfloor n/2\rfloor+1}\right]$ at which the first derivative is zero and the function is constant.So the answer to the problem is $a=x_{\lfloor n/2\rfloor+1}$ if n is odd, and a is any number in the interval $\left[x_{\lfloor n/2\rfloor}, x_{\lfloor n/2\rfloor+1}\right]$ if n is even.Remark. The required number x is called the median of $x_1, x_2, \ldots, x_n$. In general, if the numbers $x \in \mathbb{R}$ occur with probability distribution $d\mu(x)$ then their median a minimizes

$$E(|x-a|)=\int_{-\infty}^{\infty}|x-a| d\mu(x).$$

The median is any number such that

$$\int_{-\infty}^{a} d\mu(x)=P(x \leq a) \geq \frac{1}{2}$$

and

$$\int_{a}^{\infty} d\mu(x)=P(x \geq a) \geq \frac{1}{2}.$$

In the particular case of our problem, the numbers $x_1, x_2, \ldots, x_n$ occur with equal probability, so the median lies in the middle.

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :434. Let α, β, and γ be three fixed positive numbers and $[a, b]$ a given interval. Find x, y, z in $[a, b]$ for which the expression

$$E(x, y, z)=\alpha(x-y)^2+\beta(y-z)^2+\gamma(z-x)^2$$

has maximal value.
Solution :434. We assume that $\alpha \leq \beta \leq \gamma$, the other cases being similar. The expression is a convex function in each of the variables, so it attains its maximum for some $x, y, z=a$ or b.Now let us fix three numbers $x, y, z \in[a, b]$, with $x \leq y \leq z$. We have

$$E(x, y, z)-E(x, z, y)=(\gamma-\alpha)\left((z-x)^2-(y-z)^2\right) \geq 0,$$

and hence $E(x, y, z) \geq E(x, z, y)$. Similarly, $E(x, y, z) \geq E(y, x, z)$ and $E(z, y, x) \geq E(y, z, x)$. So it suffices to consider the cases $x=a, z=b$ or $x=b$ and $z=a$. For these cases we have

$$E(a, a, b)=E(b, b, a)=(\beta+\gamma)(b-a)^2$$

and

$$E(a, b, b)=E(b, a, a)=(\alpha+\gamma)(b-a)^2 .$$

We deduce that the maximum of the expression under discussion is $(\beta+\gamma)(b-a)^2$, which is attained for $x=y=a, z=b$ and for $x=y=b, z=a$.(Revista Matematică din Timişoara (Timişoara Mathematics Gazette), proposed by D. Andrica and I. Raşa)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :444. Compute the integral

$$\int\left(1+2 x^2\right) e^{x^2} d x$$

Solution :444. Split the integral as

$$\int e^{x^2} d x+\int 2 x^2 e^{x^2} d x .$$

Denote the first integral by I_1. Then use integration by parts to transform the second integral as

$$\int 2 x^2 e^{x^2} d x=x e^{x^2}-\int e^{x^2} d x=x e^{x^2}-I_1 .$$

The integral from the statement is therefore equal to

$$I_1 + xe^{x^2} - I_1 = xe^{x^2} + C.$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :445. Compute
Solution :445. Adding and subtracting e^x in the numerator, we obtain

$$\begin{aligned}\int \frac{x+\sin x-\cos x-1}{x+e^x+\sin x}dx &= \int \frac{x+e^x+\sin x-1-e^x-\cos x}{x+e^x+\sin x}dx \\ &= \int \frac{x+e^x+\sin x}{x+e^x+\sin x}dx - \int \frac{1+e^x+\cos x}{x+e^x+\sin x}dx \\ &= x+\ln\left(x+e^x+\sin x\right)+C.\end{aligned}$$

(Romanian college entrance exam)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :446. Find

$$\int \frac{x+\sin x-\cos x-1}{x+e^x+\sin x}dx.$$

$$\int \left(x^6+x^3\right)\sqrt[3]{x^3+2}dx$$

Solution :446. The trick is to bring a factor of x inside the cube root:

$$\int \left(x^6+x^3\right)\sqrt[3]{x^3+2}dx = \int \left(x^5+x^2\right)\sqrt[3]{x^6+2x^3}dx.$$

The substitution $u = x^6 + 2x^3$ now yields the answer

$$\frac{1}{6}\left(x^6+2x^3\right)^{4/3}+C.$$

(G.T. Gilbert, M.I. Krusemeyer, L.C. Larson, The Wohascum County Problem Book, MAA, 1993)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :447. Compute the integral

$$\int \frac{x^2+1}{x^4-x^2+1} dx$$

Solution :447. We want to avoid the lengthy method of partial fraction decomposition. To this end, we rewrite the integral as

$$\int \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2-1+\frac{1}{x^2}\right)} dx=\int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}} dx.$$

With the substitution $x-\frac{1}{x}=t$ we have $\left(1+\frac{1}{x^2}\right) dx=dt$, and the integral takes the form

$$\int \frac{1}{t^2+1} dt=\arctan t+C$$

We deduce that the integral from the statement is equal to

$$\arctan \left(x-\frac{1}{x}\right)+C$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :448. Compute

$$\int \sqrt{\frac{e^x-1}{e^x+1}} dx, \quad x>0.$$

Solution :448. Substitute $u=\sqrt{\frac{e^x-1}{e^x+1}}, 0<u<1$. Then $x=\ln \left(1+u^2\right)-\ln \left(1-u^2\right)$, and $dx=\left(\frac{2u}{1+u^2}+\frac{2u}{1-u^2}\right) du$. The integral becomes

$$\begin{aligned}
\int u\left(\frac{2u}{u^2+1}+\frac{2u}{u^2-1}\right) du &= \int\left(4-\frac{2}{u^2+1}+\frac{2}{u^2-1}\right) du \\
&= 4u-2 \arctan u+\int\left(\frac{1}{u+1}+\frac{1}{1-u}\right) du \\
&= 4u-2 \arctan u+\ln (u+1)-\ln (u-1)+C.
\end{aligned}$$

In terms of x, this is equal to

$$4 \sqrt{\frac{e^x-1}{e^x+1}}-2 \arctan \sqrt{\frac{e^x-1}{e^x+1}}+\ln \left(\sqrt{\frac{e^x-1}{e^x+1}}+1\right)-\ln \left(\sqrt{\frac{e^x-1}{e^x+1}}-1\right)+C$$

Topic :Real Analysis

Book :Putnam and Beyond
Final Answer :

Problem Statement :449. Find the antiderivatives of the function $f:[0,2]\to\mathbb{R}$,

$$f(x)=\sqrt{x^3+2-2\sqrt{x^3+1}}+\sqrt{x^3+10-6\sqrt{x^3+1}}.$$

Solution :449. If we naively try the substitution $t=x^3+1$, we obtain

$$f(t)=\sqrt{t+1-2\sqrt{t}}+\sqrt{t+9-6\sqrt{t}}$$

Now we recognize the perfect squares, and we realize that

$$f(x)=\sqrt{\left(\sqrt{x^3+1}-1\right)^2}+\sqrt{\left(\sqrt{x^3+1}-3\right)^2}=\left|\sqrt{x^3+1}-1\right|+\left|\sqrt{x^3+1}-3\right|.$$

When $x\in[0,2], 1\leq\sqrt{x^3+1}\leq 3$. Therefore,

$$f(x)=\sqrt{x^3+1}-1+3-\sqrt{x^3+1}=2.$$

The antiderivatives of f are therefore the linear functions $f(x)=2x+C$, where C is a constant.(communicated by E. Craina)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :450. For a positive integer n, compute the integral

$$\int\frac{x^n}{1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}dx$$

Solution :450. Let $f_n=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$. Then $f'(x)=1+x+\cdots+\frac{x^{n-1}}{(n-1)!}$. The integral in the statement becomes

$$\begin{aligned}I_n&=\int\frac{n!\,(f_n(x)-f_n'(x))}{f_n(x)}dx=n!\int\left(1-\frac{f_n'(x)}{f_n(x)}\right)dx=n!x-n!\ln f_n(x)+C\\&=n!x-n!\ln\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)+C.\end{aligned}$$

(C. Mortici, Probleme Pregătitoare pentru Concursurile de Matematic ă (Training Problems for Mathematics Contests), GIL, 1999)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :451. Compute the integral

$$\int \frac{dx}{\left(1-x^2\right)\sqrt[4]{2x^2-1}}$$

Solution :451. The substitution is

$$u=\frac{x}{\sqrt[4]{2x^2-1}}$$

for which

$$du=\frac{x^2-1}{\left(2x^2-1\right)\sqrt[4]{2x^2-1}}dx$$

We can transform the integral as follows:

$$\begin{aligned}
\int \frac{2x^2-1}{-\left(x^2-1\right)^2}\cdot\frac{x^2-1}{\left(2x^2-1\right)\sqrt[4]{2x^2-1}}dx &= \int \frac{1}{\frac{-x^4+2x^2-1}{2x^2-1}}\cdot\frac{x^2-1}{\left(2x^2-1\right)\sqrt[4]{2x^2-1}}dx \\
&= \int \frac{1}{1-\frac{x^4}{2x^2-1}}\cdot\frac{x^2-1}{\left(2x^2-1\right)\sqrt[4]{2x^2-1}}dx \\
&= \int \frac{1}{1-u^4}du
\end{aligned}$$

This is computed using Jacobi's method for rational functions, giving the final answer to the problem

$$\frac{1}{4}\ln\frac{\sqrt[4]{2x^2-1}+x}{\sqrt[4]{2x^2-1}-x}-\frac{1}{2}\arctan\frac{\sqrt[4]{2x^2-1}}{x}+C$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :452. Compute

$$\int \frac{x^4+1}{x^6+1}dx.$$

Give the answer in the form $\alpha\arctan\frac{P(x)}{Q(x)}+C,\alpha\in\mathbb{Q}$, and $P(x),Q(x)\in\mathbb{Z}[x]$.
Solution :452. Of course, Jacobi's partial fraction decomposition method can be applied, but it is more laborious. However, in the process of applying it we factor the denominator as $x^6+1=\left(x^2+1\right)\left(x^4-x^2+1\right)$, and this expression can be related somehow to the numerator. Indeed, if we add and subtract an x^2 in the numerator, we obtain

$$\frac{x^4+1}{x^6+1}=\frac{x^4-x^2+1}{x^6+1}+\frac{x^2}{x^6+1}$$

Now integrate as follows:

$$\int \frac{x^4+1}{x^6+1}dx = \int \frac{x^4-x^2+1}{x^6+1}dx + \int \frac{x^2}{x^6+1}dx = \int \frac{1}{x^2+1}dx + \int \frac{1}{3}\frac{\left(x^3\right)'}{\left(x^3\right)^2+1}dx$$

$$= \arctan x + \frac{1}{3}\arctan x^3.$$

To write the answer in the required form we should have

$$3\arctan x + \arctan x^3 = \arctan \frac{P(x)}{Q(x)}.$$

Applying the tangent function to both sides, we deduce

$$\frac{\frac{3x-x^3}{1-3x^2}+x^3}{1-\frac{3x-x^3}{1-3x^2}\cdot x^3} = \tan\left(\arctan\frac{P(x)}{Q(x)}\right).$$

From here

$$\arctan\frac{P(x)}{Q(x)} = \arctan\frac{3x-3x^5}{1-3x^2-3x^4+x^6},$$

and hence $P(x) = 3x-3x^5, Q(x) = 1-3x^2-3x^4+x^6$. The final answer is

$$\frac{1}{3}\arctan\frac{3x-3x^5}{1-3x^2-3x^4+x^6}+C.$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :453. Compute the integral

$$\int_{-1}^{1}\frac{\sqrt[3]{x}}{\sqrt[3]{1-x}+\sqrt[3]{1+x}}dx$$

Solution :453. The function $f:[-1,1]\to\mathbb{R}$,

$$f(x) = \frac{\sqrt[3]{x}}{\sqrt[3]{1-x}+\sqrt[3]{1+x}},$$

is odd; therefore, the integral is zero.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :454. Compute

$$\int_0^{\pi}\frac{x\sin x}{1+\sin^2 x}dx.$$

Solution :454. We use the example from the introduction for the particular function $f(x)=\frac{x}{1+x^2}$ to transform the integral into

$$\pi\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\sin^2 x}dx.$$

This is the same as

$$\pi\int_0^{\frac{\pi}{2}}-\frac{d(\cos x)}{2-\cos^2 x},$$

which with the substitution $t=\cos x$ becomes

$$\pi\int_0^1\frac{1}{2-t^2}dt=\frac{\pi}{2\sqrt{2}}\ln\frac{\sqrt{2}+t}{\sqrt{2}-t}\bigg|_0^1=\frac{\pi}{2\sqrt{2}}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}.$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :455. Let a and b be positive real numbers. Compute

$$\int_a^b\frac{e^{\frac{x}{a}}-e^{\frac{b}{x}}}{x}dx.$$

Solution :455. Denote the value of the integral by I. With the substitution $t=\frac{ab}{x}$ we have

$$I=\int_a^b\frac{e^{\frac{b}{t}}-e^{\frac{t}{a}}}{\frac{ab}{t}}\cdot\frac{-ab}{t^2}dt=-\int_a^b\frac{e^{\frac{t}{a}}-e^{\frac{b}{t}}}{t}dt=-I.$$

Hence $I=0$.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :456. Compute the integral

$$I=\int_0^1\sqrt[3]{2x^3-3x^2-x+1}dx$$

Solution :456. The substitution $t=1-x$ yields

$$I=\int_0^1\sqrt[3]{2(1-t)^3-3(1-t)^2-(1-t)+1}dt=-\int_0^1\sqrt[3]{2t^3-3t^2-t+1}dt=-I.$$

Hence $I=0$.(Mathematical Reflections, proposed by T. Andreescu)
Topic :Real Analysis

Book :Putnam and Beyond
Final Answer :

Problem Statement :457. Compute the integral

$$\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} \quad (a>0).$$

Solution :457. Using the substitutions $x = a\sin t$, respectively, $x = a\cos t$, we find the integral to be equal to both the integral

$$L_1 = \int_0^{\pi/2} \frac{\sin t}{\sin t + \cos t} dt$$

and the integral

$$L_2 = \int_0^{\pi/2} \frac{\cos t}{\sin t + \cos t} dt.$$

Hence the desired integral is equal to

$$\frac{1}{2}(L_1 + L_2) = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 dt = \frac{\pi}{4}.$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :458. Compute the integral

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx$$

Solution :458. Denote the integral by I. With the substitution $t = \frac{\pi}{4} - x$ the integral becomes

$$I = \int_{\frac{\pi}{4}}^{0} \ln\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)(-1)dt = \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan t}{1+\tan t}\right)dt$$
$$= \int_0^{\frac{\pi}{4}} \ln\frac{2}{1+\tan t}dt = \frac{\pi}{4}\ln 2 - I.$$

Solving for I, we obtain $I = \frac{\pi}{8}\ln 2$.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :459. Find

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$$

Solution :459. With the substitution $\arctan x = t$ the integral takes the form

$$I = \int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt.$$

This we already computed in the previous problem. ("Happiness is longing for repetition," says M. Kundera.) So the answer to the problem is $\frac{\pi}{8}\ln 2$.(66th W.L. Putnam Mathematical Competition, 2005, proposed by T. Andreescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :460. Compute

$$\int_0^{\infty} \frac{\ln x}{x^2+a^2}dx,$$

where a is a positive constant.
Solution :460. The function $\ln x$ is integrable near zero, and the function under the integral sign is dominated by $x^{-3/2}$ near infinity; hence the improper integral converges. We first treat the case $a=1$. The substitution $x = 1/t$ yields

$$\int_0^{\infty} \frac{\ln x}{x^2+1}dx = \int_{\infty}^{0} \frac{\ln \frac{1}{t}}{\frac{1}{t^2}+1}\left(-\frac{1}{t^2}\right)dt = -\int_0^{\infty} \frac{\ln t}{t^2+1}dt$$

which is the same integral but with opposite sign. This shows that for $a=1$ the integral is equal to 0 . For general a we compute the integral using the substitution $x = a/t$ as follows

$$\begin{aligned}\int_0^{\infty} \frac{\ln x}{x^2+a^2}dx &= \int_{\infty}^{0} \frac{\ln a - \ln t}{\left(\frac{a}{t}\right)^2+a^2}\cdot\left(-\frac{a}{t^2}\right)dt = \frac{1}{a}\int_0^{\infty} \frac{\ln a - \ln t}{1+t^2}dt \\ &= \frac{\ln a}{a}\int_0^{\infty} \frac{dt}{t^2+1} - \frac{1}{a}\int_0^{\infty} \frac{\ln t}{t^2+1}dt = \frac{\pi \ln a}{2a}\end{aligned}$$

(P.N. de Souza, J.N. Silva, Berkeley Problems in Mathematics, Springer, 2004)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :461. Compute the integral

$$\int_0^{\frac{\pi}{2}} \frac{x\cos x - \sin x}{x^2+\sin^2 x}dx$$

Solution :461. The statement is misleading. There is nothing special about the limits of integration! The indefinite integral can be computed as follows:

$$\int \frac{x\cos x - \sin x}{x^2 + \sin^2 x} dx = \int \frac{\frac{\cos x}{x} - \frac{\sin x}{x^2}}{1 + \left(\frac{\sin x}{x}\right)^2} dx = \int \frac{1}{1 + \left(\frac{\sin x}{x}\right)^2} \left(\frac{\sin x}{x}\right)' dx$$
$$= \arctan\left(\frac{\sin x}{x}\right) + C.$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \frac{x\cos x - \sin x}{x^2 + \sin^2 x} dx = \arctan\frac{2}{\pi} - \frac{\pi}{4}.$$

(Z. Ahmed)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :462. Let α be a real number. Compute the integral

$$I(\alpha) = \int_{-1}^{1} \frac{\sin\alpha dx}{1 - 2x\cos\alpha + x^2}.$$

Solution :462. If α is a multiple of π, then $I(\alpha) = 0$. Otherwise, use the substitution $x = \cos\alpha + t\sin\alpha$. The indefinite integral becomes

$$\int \frac{\sin\alpha dx}{1 - 2x\cos\alpha + x^2} = \int \frac{dt}{1 + t^2} = \arctan t + C.$$

It follows that the definite integral $I(\alpha)$ has the value

$$\arctan\left(\frac{1 - \cos\alpha}{\sin\alpha}\right) - \arctan\left(\frac{-1 - \cos\alpha}{\sin\alpha}\right)$$

where the angles are to be taken between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But

$$\frac{1 - \cos\alpha}{\sin\alpha} \times \frac{-1 - \cos\alpha}{\sin\alpha} = -1.$$

Hence the difference between these angles is $\pm\frac{\pi}{2}$. Notice that the sign of the integral is the same as the sign of α. Hence $I(\alpha) = \frac{\pi}{2}$ if $\alpha \in (2k\pi, (2k+1)\pi)$ and $-\frac{\pi}{2}$ if $\alpha \in ((2k+1)\pi, (2k+2)\pi)$ for some integer k.Remark. This is an example of an integral with parameter that does not depend continuously on the parameter.(E. Goursat, A Course in Mathematical Analysis, Dover, NY, 1904)
Topic :Real Analysis
Book :Putnam and Beyond

Final Answer :

Problem Statement :465. Let $n \geq 0$ be an integer. Compute the integral

$$\int_0^\pi \frac{1-\cos nx}{1-\cos x} dx.$$

Solution :465. First, note that by L'Hôpital's theorem,

$$\lim_{x \to 0} \frac{1-\cos nx}{1-\cos x} = n^2,$$

which shows that the absolute value of the integrand is bounded as x approaches 0 , and hence the integral converges.Denote the integral by I_n. Then

$$\begin{aligned}
\frac{I_{n+1}+I_{n-1}}{2} &= \int_0^\pi \frac{2-\cos(n+1)x-\cos(n-1)x}{2(1-\cos x)} dx = \int_0^\pi \frac{1-\cos nx \cos x}{1-\cos x} dx \\
&= \int_0^\pi \frac{(1-\cos nx)+\cos nx(1-\cos x)}{1-\cos x} dx = I_n + \int_0^\pi \cos nx dx = I_n
\end{aligned}$$

Therefore,

$$I_n = \frac{1}{2}\left(I_{n+1}+I_{n-1}\right), \quad n \geq 1$$

This shows that $I_0, I_1, I_2, \ldots$ is an arithmetic sequence. From $I_0 = 0$ and $I_1 = \pi$ it follows that $I_n = n\pi, n \geq 1$.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :467. Compute

$$\int_{-\pi}^\pi \frac{\sin nx}{(1+2^x)\sin x} dx, \quad n \geq 0.$$

Solution :467. Denote the integral from the statement by $I_n, n \geq 0$. We have

$$I_n = \int_{-\pi}^0 \frac{\sin nx}{(1+2^x)\sin x} dx + \int_0^\pi \frac{\sin nx}{(1+2^x)\sin x} dx.$$

In the first integral change x to $-x$ to further obtain

$$\begin{aligned}
I_n &= \int_0^\pi \frac{\sin nx}{(1+2^{-x})\sin x} dx + \int_0^\pi \frac{\sin nx}{(1+2^x)\sin x} dx \\
&= \int_0^\pi \frac{2^x \sin nx}{(1+2^x)\sin x} dx + \int_0^\pi \frac{\sin nx}{(1+2^x)\sin x} dx \\
&= \int_0^\pi \frac{(1+2^x)\sin nx}{(1+2^x)\sin x} dx = \int_0^\pi \frac{\sin nx}{\sin x} dx.
\end{aligned}$$

And these integrals can be computed recursively. Indeed, for $n \geq 0$ we have

$$I_{n+2} - I_n = \int_0^\pi \frac{\sin(n+2)x - \sin nx}{\sin x} dx = 2\int_0^\pi \cos(n-1)x dx = 0,$$

a very simple recurrence. Hence for n even, $I_n = I_0 = 0$, and for n odd, $I_n = I_1 = \pi$.(3rd International Mathematics Competition for University Students, 1996)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :468. Compute

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \cdots + \frac{1}{\sqrt{4n^2-n^2}}\right].$$

Solution :468. We have

$$\begin{aligned} s_n &= \frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \cdots + \frac{1}{\sqrt{4n^2-n^2}} \\ &= \frac{1}{n}\left[\frac{1}{\sqrt{4-\left(\frac{1}{n}\right)^2}} + \frac{1}{\sqrt{4-\left(\frac{2}{n}\right)^2}} + \cdots + \frac{1}{\sqrt{4-\left(\frac{n}{n}\right)^2}}\right]. \end{aligned}$$

Hence s_n is the Riemann sum of the function $f : [0,1] \to \mathbb{R}, f(x) = \frac{1}{\sqrt{4-x^2}}$ associated to the subdivision $x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_n = \frac{n}{n} = 1$, with the intermediate points $\xi_i = \frac{i}{n} \in [x_i, x_{i+1}]$. The answer to the problem is therefore

$$\lim_{n\to\infty} s_n = \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \arcsin \frac{x}{2}\Big|_0^1 = \frac{\pi}{6}.$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :470. Compute

$$\lim_{n\to\infty}\left(\frac{2^{1/n}}{n+1} + \frac{2^{2/n}}{n+\frac{1}{2}} + \cdots + \frac{2^{n/n}}{n+\frac{1}{n}}\right).$$

Solution :470. We would like to recognize the general term of the sequence as being a Riemann sum. This, however, does not seem to happen, since we can only write

$$\sum_{i=1}^{n} \frac{2^{i/n}}{n+\frac{1}{i}} = \frac{1}{n}\sum_{i=1}^{n} \frac{2^{i/n}}{1+\frac{1}{ni}}.$$

But for $i \geq 2$,

$$2^{i/n} > \frac{2^{i/n}}{1+\frac{1}{ni}},$$

and, using the inequality $e^x > 1+x$,

$$\frac{2^{i/n}}{1+\frac{1}{ni}} = 2^{(i-1)/n}\frac{2^{1/n}}{1+\frac{1}{ni}} = 2^{(i-1)/n}\frac{e^{\ln 2/n}}{1+\frac{1}{ni}} > 2^{(i-1)/n}\frac{1+\frac{\ln 2}{n}}{1+\frac{1}{ni}} > 2^{(i-1)/n},$$

for $i \geq 2$. By the intermediate value property, for each $i \geq 2$ there exists $\xi_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$ such that

$$\frac{2^{i/n}}{1+\frac{1}{ni}} = 2^{\xi_i}.$$

Of course, the term corresponding to $i = 1$ can be neglected when n is large. Now we see that our limit is indeed the Riemann sum of the function 2^x integrated over the interval $[0,1]$. We obtain

$$\lim_{n\to\infty}\left(\frac{2^{1/n}}{n+1} + \frac{2^{2/n}}{n+\frac{1}{2}} + \cdots + \frac{2^{n/n}}{n+\frac{1}{n}}\right) = \int_0^1 2^x dx = \frac{1}{\ln 2}$$

(Soviet Union University Student Mathematical Olympiad, 1976)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :471. Compute the integral

$$\int_0^{\pi} \ln\left(1 - 2a\cos x + a^2\right) dx.$$

Solution :471. This is an example of an integral that is determined using Riemann sums. Divide the interval $[0,\pi]$ into n equal parts and consider the Riemann sum

$$\tfrac{\pi}{n}\left[\ln\left(a^2 - 2a\cos\tfrac{\pi}{n} + 1\right) + \ln\left(a^2 - 2a\cos\tfrac{2\pi}{n} + 1\right) + \cdots \right.$$
$$\left. + \ln\left(a^2 - 2a\cos\tfrac{(n-1)\pi}{n} + 1\right)\right].$$

This expression can be written as

$$\frac{\pi}{n}\ln\left(a^2 - 2a\cos\frac{\pi}{n} + 1\right)\left(a^2 - 2a\cos\frac{2\pi}{n} + 1\right)\ldots\left(a^2 - 2a\cos\frac{(n-1)\pi}{n} + 1\right).$$

The product inside the natural logarithm factors as

$$\prod_{k=1}^{n-1}\left[a - \left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right)\right]\left[a - \left(\cos\frac{k\pi}{n} - i\sin\frac{k\pi}{n}\right)\right].$$

These are exactly the factors in $a^{2n}-1$, except for $a-1$ and $a+1$. The Riemann sum is therefore equal to

$$\frac{\pi}{n}\ln\frac{a^{2n}-1}{a^2-1}.$$

We are left to compute the limit of this expression as n goes to infinity. If $a \leq 1$, this limit is equal to 0 . If $a > 1$, the limit is

$$\lim_{n\to\infty}\pi\ln\sqrt[n]{\frac{a^{2n}-1}{a^2-1}} = 2\pi\ln a.$$

(S.D. Poisson)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :472. Find all continuous functions $f:\mathbb{R}\to[1,\infty)$ for which there exist $a\in\mathbb{R}$ and k a positive integer such that

$$f(x)f(2x)\cdots f(nx)\leq an^k,$$

for every real number x and positive integer n.
Solution :472. The condition $f(x)f(2x)\cdots f(nx)\leq an^k$ can be written equivalently as

$$\sum_{j=1}^{n}\ln f(jx)\leq \ln a+k\ln n, \qquad \text{for all } x\in\mathbb{R}, n\geq 1.$$

Taking $\alpha > 0$ and $x=\frac{\alpha}{n}$, we obtain

$$\sum_{j=1}^{n}\ln f\left(\frac{j\alpha}{n}\right)\leq \ln a+k\ln n$$

or

$$\sum_{j=1}^{n}\frac{\alpha}{n}\ln f\left(\frac{j\alpha}{n}\right)\leq \frac{\alpha\ln a+k\alpha\ln n}{n}.$$

The left-hand side is a Riemann sum for the function $\ln f$ on the interval $[0,\alpha]$. Because f is continuous, so is $\ln f$, and thus $\ln f$ is integrable. Letting n tend to infinity, we obtain

$$\int_0^1 \ln f(x)dx \leq \lim_{n\to\infty}\frac{\alpha\ln a+k\alpha\ln n}{n}=0.$$

The fact that $f(x)\geq 1$ implies that $\ln f(x)\geq 0$ for all x. Hence $\ln f(x)=0$ for all $x\in[0,\alpha]$. Since α is an arbitrary positive number, $f(x)=1$ for all $x\geq 0$. A similar argument yields $f(x)=1$ for $x<0$. So there is only one such function, the constant function equal to 1 .(Romanian Mathematical Olympiad, 1999, proposed by R. Gologan)

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :473. Determine the continuous functions $f : [0, 1] \to \mathbb{R}$ that satisfy

$$\int_0^1 f(x)(x - f(x))dx = \frac{1}{12}.$$

Solution :473. The relation from the statement can be rewritten as

$$\int_0^1 \left(xf(x) - f^2(x)\right) dx = \int_0^1 \frac{x^2}{4} dx.$$

Moving everything to one side, we obtain

$$\int_0^1 \left(f^2(x) - xf(x) + \frac{x^2}{4}\right) dx = 0.$$

We now recognize a perfect square and write this as

$$\int_0^1 \left(f(x) - \frac{x}{2}\right)^2 dx = 0.$$

The integral of the nonnegative continuous function $\left(f(x) - \frac{x}{2}\right)^2$ is strictly positive, unless the function is identically equal to zero. It follows that the only function satisfying the condition from the statement is $f(x) = \frac{x}{2}, x \in [0, 1]$.(Revista de Matematica din Timişoara (Timişoara Mathematics Gazette), proposed by T. Andreescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :474. Let n be an odd integer greater than 1. Determine all continuous functions $f : [0, 1] \to \mathbb{R}$ such that

$$\int_0^1 \left(f\left(x^{\frac{1}{k}}\right)\right)^{n-k} dx = \frac{k}{n}, \quad k = 1, 2, \ldots, n-1.$$

Solution :474. Performing the substitution $x^{\frac{1}{k}} = t$, the given conditions become

$$\int_0^1 (f(t))^{n-k} t^{k-1} dt = \frac{1}{n}, \quad k = 1, 2, \ldots, n-1.$$

Observe that this equality also holds for $k = n$. With this in mind we write

$$\begin{aligned}\int_0^1 (f(t)-t)^{n-1}dt &= \int_0^1 \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (f(t))^{n-1-k} t^k dt \\ &= \int_0^1 \sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^{k-1} (f(t))^{n-k} t^{k-1} dt \\ &= \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \int_0^1 (f(t))^{n-k} t^{k-1} dt \\ &= \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \frac{1}{n} = \frac{1}{n}(1-1)^{n-1} = 0.\end{aligned}$$

Because $n-1$ is even, $(f(t)-t)^{n-1} \geq 0$. The integral of this function can be zero only if $f(t) - t = 0$ for all $t \in [0,1]$. Hence the only solution to the problem is $f : [0,1] \to \mathbb{R}$, $f(x) = x$.(Romanian Mathematical Olympiad, 2002, proposed by T. Andreescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :476. For each continuous function $f : [0,1] \to \mathbb{R}$, we define $I(f) = \int_0^1 x^2 f(x)dx$ and $J(f) = \int_0^1 x(f(x))^2 dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.
Solution :476. We change this into a minimum problem, and then relate the latter to an inequality of the form $x \geq 0$. Completing the square, we see that

$$x(f(x))^2 - x^2 f(x) = \sqrt{x} f(x))^2 - 2\sqrt{x} f(x) \frac{x^{\frac{3}{2}}}{2} = \left(\sqrt{x} f(x) - \frac{x^{\frac{3}{2}}}{2}\right)^2 - \frac{x^3}{4}$$

Hence, indeed,

$$J(f) - I(f) = \int_0^1 \left(\sqrt{x} f(x) - \frac{x^{\frac{3}{2}}}{2}\right)^2 dx - \int_0^1 \frac{x^3}{4} dx \geq -\frac{1}{16}$$

It follows that $I(f) - J(f) \leq \frac{1}{16}$ for all f. The equality holds, for example, for $f : [0,1] \to \mathbb{R}, f(x) = \frac{x}{2}$. We conclude that

$$\max_{f \in \mathcal{C}^0([0,1])} (I(f) - J(f)) = \frac{1}{16}.$$

(49th W.L. Putnam Mathematical Competition, 2006, proposed by T. Andreescu)
Topic :Real Analysis
Book :Putnam and Beyond

Final Answer :

Problem Statement :479. Find the maximal value of the ratio

$$\left(\int_0^3 f(x)dx\right)^3 / \int_0^3 f^3(x)dx,$$

as f ranges over all positive continuous functions on $[0,1]$.
Solution :479. By Hölder's inequality,

$$\int_0^3 f(x)\cdot 1dx \le \left(\int_0^3 |f(x)|^3dx\right)^{\frac{1}{3}} \left(\int_0^3 1^{\frac{3}{2}}dx\right)^{\frac{2}{3}} = 3^{\frac{2}{3}} \left(\int_0^3 |f(x)|^3dx\right)^{\frac{1}{3}}.$$

Raising everything to the third power, we obtain

$$\left(\int_0^3 f(x)dx\right)^3 / \int_0^3 f^3(x)dx \le 9.$$

To see that the maximum 9 can be achieved, choose f to be constant.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :487. Compute the ratio

$$\frac{1+\frac{\pi^4}{5!}+\frac{\pi^8}{9!}+\frac{\pi^{12}}{13!}+\cdots}{\frac{1}{3!}+\frac{\pi^4}{7!}+\frac{\pi^8}{11!}+\frac{\pi^{12}}{15!}+\cdots}.$$

Solution :487. Denote by p the numerator and by q the denominator of this fraction. Recall the Taylor series expansion of the sine function,

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots.$$

We recognize the denominators of these fractions inside the expression that we are computing, and now it is not hard to see that $p\pi - q\pi^3 = \sin\pi = 0$. Hence $p\pi = q\pi^3$, and the value of the expression from the statement is π^2.(Soviet Union University Student Mathematical Olympiad, 1975)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :489. Find a quadratic polynomial $P(x)$ with real coefficients such that

$$\left|P(x) + \frac{1}{x-4}\right| \le 0.01, \quad \text{for all } x \in [-1,1].$$

Solution :489. Consider the Taylor series expansion around 0 ,

$$\frac{1}{x-4} = -\frac{1}{4} - \frac{1}{16}x - \frac{1}{64}x^2 - \frac{1}{256}x^3 - \cdots .$$

A good guess is to truncate this at the third term and let

$$P(x) = \frac{1}{4} + \frac{1}{16}x + \frac{1}{64}x^2.$$

By the residue formula for Taylor series we have

$$\left|P(x) + \frac{1}{x-4}\right| = \frac{x^3}{256} + \frac{1}{(\xi-4)^4}x^5,$$

for some $\xi \in (0, x)$. Since $|x| \leq 1$ and also $|\xi| \leq 1$, we have $\frac{x^3}{256} \leq \frac{1}{256}$ and $x^4/(\xi-4)^5 \leq \frac{1}{243}$. An easy numerical computation shows that $\frac{1}{256} + \frac{1}{243} < \frac{1}{100}$, and we are done.(Romanian Team Selection Test for the International Mathematical Olympiad, 1979, proposed by O. Stănăşilă)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :490. Compute to three decimal places

$$\int_0^1 \cos\sqrt{x}dx.$$

Solution :490. The Taylor series expansion of $\cos\sqrt{x}$ around 0 is

$$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots .$$

Integrating term by term, we obtain

$$\int_0^1 \cos\sqrt{x}dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{(n+1)(2n)!}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{(n+1)(2n)!}.$$

Grouping consecutive terms we see that

$$\left(\frac{1}{5\cdot 8!} - \frac{1}{6\cdot 10!}\right) + \left(\frac{1}{7\cdot 12!} - \frac{1}{8\cdot 14!}\right) + \cdots < \frac{1}{2\cdot 10^4} + \frac{1}{2\cdot 10^5} + \frac{1}{2\cdot 10^6} + \cdots$$
$$< \frac{1}{10^4}.$$

Also, truncating to the fourth decimal place yields

$$0.7638 < 1 - \frac{1}{4} + \frac{1}{72} - \frac{1}{2880} < 0.7639.$$

We conclude that

$$\int_0^1 \cos\sqrt{x}dx \approx 0.763.$$

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :496. For a positive integer n find the Fourier series of the function

$$f(x) = \frac{\sin^2 nx}{\sin^2 x}.$$

Solution :496. We will use only trigonometric considerations, and compute no integrals. A first remark is that the function is even, so only terms involving cosines will appear. Using Euler's formula

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

we can transform the identity

$$\sum_{k=1}^{n} e^{2ikx} = \frac{e^{2i(n+1)x} - 1}{e^{2ix} - 1}$$

into the corresponding identities for the real and imaginary parts:

$$\cos 2x + \cos 4x + \cdots + \cos 2nx = \frac{\sin nx \cos(n+1)x}{\sin x}$$
$$\sin 2x + \sin 4x + \cdots + \sin 2nx = \frac{\sin nx \sin(n+1)x}{\sin x}$$

These two relate to our function as

$$\frac{\sin^2 nx}{\sin^2 x} = \left(\frac{\sin nx \cos(n+1)x}{\sin x}\right)^2 + \left(\frac{\sin nx \sin(n+1)x}{\sin x}\right)^2,$$

which allows us to write the function as an expression with no fractions:

$$f(x) = (\cos 2x + \cos 4x + \cdots + \cos 2nx)^2 + (\sin 2x + \sin 4x + \cdots + \sin 2nx)^2.$$

Expanding the squares, we obtain

$$\begin{aligned} f(x) &= n + \sum_{1\le l<k\le n} (2\sin 2lx \sin 2kx + 2\cos 2lx \cos 2kx) \\ &= n + 2\sum_{1\le l<k\le n} \cos 2(k-l)x = n + \sum_{m=1}^{n-1} 2(n-m)\cos 2mx. \end{aligned}$$

In conclusion, the nonzero Fourier coefficients of f are $a_0 = n$ and $a_{2m} = 2(n-m)$, $m = 1, 2, \ldots, n-1$.(D. Andrica)

Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :502. Find the global minimum of the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = x^4 + 6x^2y^2 + y^4 - \frac{9}{4}x - \frac{7}{4}y.$$

Solution :502. First, observe that if $|x|+|y| \to \infty$ then $f(x, y) \to \infty$, hence the function indeed has a global minimum. The critical points of f are solutions to the system of equations

$$\frac{\partial f}{\partial x}(x, y) = 4x^3 + 12xy^2 - \frac{9}{4} = 0,$$
$$\frac{\partial f}{\partial y}(x, y) = 12x^2y + 4y^3 - \frac{7}{4} = 0.$$

If we divide the two equations by 4 and then add, respectively, subtract them, we obtain $x^3 + 3x^2y + 3xy^2 + y^3 - 1 = 0$ and $x^3 - 3x^2y + 3xy^3 - y^3 = \frac{1}{8}$. Recognizing the perfect cubes, we write these as $(x+y)^3 = 1$ and $(x-y)^3 = \frac{1}{8}$, from which we obtain $x + y = 1$ and $x - y = \frac{1}{2}$. We find a unique critical point $x = \frac{3}{4}, y = \frac{1}{4}$. The minimum of f is attained at this point, and it is equal to $f\left(\frac{3}{4}, \frac{1}{4}\right) = -\frac{51}{32}$.(R. Gelca)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :503. Find the equation of the smallest sphere that is tangent to both of the lines (i) $x = t + 1, y = 2t + 4, z = -3t + 5$, and (ii) $x = 4t - 12, y = -t + 8, z = t + 17$.
Solution :503. The diameter of the sphere is the segment that realizes the minimal distance between the lines. So if $P(t + 1, 2t + 4, -3t + 5)$ and $Q(4s - 12, -t + 8, t + 17)$, we have to minimize the function

$$\begin{aligned}|PQ|^2 &= (s - 4t + 13)^2 + (2s + t - 4)^2 + (-3s - t - 12)^2 \\ &= 14s^2 + 2st + 18t^2 + 82s - 88t + 329.\end{aligned}$$

To minimize this function we set its partial derivatives equal to zero:

$$28s + 2t + 82 = 0,$$
$$2s + 36t - 88 = 0.$$

This system has the solution $t = -782/251, s = 657/251$. Substituting into the equation of the line, we deduce that the two endpoints of the diameter are $P\left(-\frac{531}{251}, -\frac{560}{251}, \frac{3601}{251}\right)$ and $Q\left(-\frac{384}{251}, \frac{1351}{251}, \frac{4924}{251}\right)$. The center of the sphere is $\frac{1}{502}(-915, 791, 8252)$, and the radius $\frac{147}{\sqrt{1004}}$. The equation of the sphere is

$$(502x + 915)^2 + (502y - 791)^2 + (502z - 8525)^2 = 251(147)^2.$$

(20th W.L. Putnam Competition, 1959)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :504. Determine the maximum and the minimum of $\cos A+\cos B+\cos C$ when A, B, and C are the angles of a triangle.
Solution :504. Writing $C=\pi-A-B$, the expression can be viewed as a function in the independent variables A and B, namely,

$$f(A, B)=\cos A+\cos B-\cos (A+B).$$

And because A and B are angles of a triangle, they are constrained to the domain $A, B>0, A+B<\pi$. We extend the function to the boundary of the domain, then study its extrema. The critical points satisfy the system of equations

$$\begin{aligned}
& \frac{\partial f}{\partial A}(A, B)=-\sin A+\sin (A+B)=0, \\
& \frac{\partial f}{\partial B}(A, B)=-\sin B+\sin (A+B)=0.
\end{aligned}$$

From here we obtain $\sin A=\sin B=\sin (A+B)$, which can happen only if $A=B=\frac{\pi}{3}$. This is the unique critical point, for which $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right)=\frac{3}{2}$. On the boundary, if $A=0$ or $B=0$, then $f(A, B)=1$. Same if $A+B=\pi$. We conclude that the maximum of $\cos A+\cos B+\cos C$ is $\frac{3}{2}$, attained for the equilateral triangle, while the minimum is 1 , which is attained only for a degenerate triangle in which two vertices coincide.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :511. Of all triangles circumscribed about a given circle, find the one with the smallest area.
Solution :511. Without loss of generality, we may assume that the circle has radius 1 . If a, b, c are the sides, and $S(a, b, c)$ the area, then (because of the formula $S=p r$, where p is the semiperimeter) the constraint reads $S=\frac{a+b+c}{2}$. We will maximize the function $f(a, b, c)=S(a, b, c)^{2}$ with the constraint $g(a, b, c)=S(a, b, c)^{2}-\left(\frac{a+b+c}{2}\right)^{2}=0$. Using Hero's formula, we can write

$$\begin{aligned}
f(a, b, c) & =\frac{a+b+c}{2} \cdot \frac{-a+b+c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2} \\
& =\frac{-a^{4}-b^{4}-c^{4}+2\left(a^{2} b^{2}+b^{2} c^{2}+a^{2} c^{2}\right)}{16} .
\end{aligned}$$

The method of Lagrange multipliers gives rise to the system of equations

$$
\begin{aligned}
(\lambda-1) \frac{-a^3+a\left(b^2+c^2\right)}{4} & =\frac{a+b+c}{2}, \\
(\lambda-1) \frac{-b^3+b\left(a^2+c^2\right)}{4} & =\frac{a+b+c}{2}, \\
(\lambda-1) \frac{-c^3+c\left(a^2+b^2\right)}{4} & =\frac{a+b+c}{2}, \\
g(a, b, c) & =0 .
\end{aligned}
$$

Because $a+b+c \neq 0, \lambda$ cannot be 1 , so this further gives

$$
-a^3+a\left(b^2+c^2\right)=-b^3+b\left(a^2+c^2\right)=-c^3+c\left(a^2+b^2\right) .
$$

The first equality can be written as $(b-a)\left(a^2+b^2-c^2\right)=0$. This can happen only if either $a=b$ or $a^2+b^2=c^2$, so either the triangle is isosceles, or it is right. Repeating this for all three pairs of sides we find that either $b=c$ or $b^2+c^2=a^2$, and also that either $a=c$ or $a^2+c^2=b^2$. Since at most one equality of the form $a^2+b^2=c^2$ can hold, we see that, in fact, all three sides must be equal. So the critical point given by the method of Lagrange multipliers is the equilateral triangle.Is this the global minimum? We just need to observe that as the triangle degenerates, the area becomes infinite. So the answer is yes, the equilateral triangle minimizes the area.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :514. Compute the integral $\iint_D x d x d y$, where

$$
D=\left\{(x, y) \in \mathbb{R}^2 \mid x \geq 0,1 \leq x y \leq 2,1 \leq \frac{y}{x} \leq 2\right\} .
$$

Solution :514. The domain is bounded by the hyperbolas $x y=1, x y=2$ and the lines $y=x$ and $y=2 x$. This domain can mapped into a rectangle by the transformation

$$
T: \quad u=x y, \quad v=\frac{y}{x} .
$$

Thus it is natural to consider the change of coordinates

$$
T^{-1}: \quad x=\sqrt{\frac{u}{v}}, \quad y=\sqrt{u v} .
$$

The domain becomes the rectangle $D^*=\left\{(u, v) \in \mathbb{R}^2 \mid 1 \leq u \leq 2,1 \leq v \leq 2\right\}$. The Jacobian of T^{-1} is $\frac{1}{2 v} \neq 0$. The integral becomes

$$
\int_1^2 \int_1^2 \sqrt{\frac{u}{v}} \frac{1}{2 v} d u d v=\frac{1}{2} \int_1^2 u^{1 / 2} d u \int_1^2 v^{-3 / 2} d v=\frac{1}{3}(5 \sqrt{2}-6) .
$$

(Gh. Bucur, E. Câmpu, S. Găină, Culegere de Probleme de Calcul Diferenţial şi Integral (Collection of Problems in Differential and Integral Calculus), Editura Tehnică, Bucharest, 1967)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :515. Find the integral of the function

$$f(x, y, z)=\frac{x^{4}+2 y^{4}}{x^{4}+4 y^{4}+z^{4}}$$

over the unit ball $B=\left\{(x, y, z) \mid x^{2}+y^{2}+z^{2} \leq 1\right\}$.
Solution :515. Denote the integral by I. The change of variable $(x, y, z) \rightarrow (z, y, x)$ transforms the integral into

$$\iiint_{B} \frac{z^{4}+2 y^{4}}{x^{4}+4 y^{4}+z^{4}} d x d y d z .$$

Hence

$$\begin{aligned}
2 I & =\iiint_{B} \frac{x^{4}+2 y^{4}}{x^{4}+4 y^{4}+z^{4}} d x d y d z+\iiint_{B} \frac{2 y^{4}+z^{4}}{x^{4}+4 y^{4}+z^{4}} d x d y d z \\
& =\iiint_{B} \frac{x^{4}+4 y^{4}+z^{4}}{x^{4}+4 y^{4}+z^{4}} d x d y d z=\frac{4 \pi}{3} .
\end{aligned}$$

It follows that $I=\frac{2 \pi}{3}$.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :516. Compute the integral

$$\iint_{D} \frac{d x d y}{\left(x^{2}+y^{2}\right)^{2}},$$

where D is the domain bounded by the circles

$$\begin{array}{ll}
x^{2}+y^{2}-2 x=0, & x^{2}+y^{2}-4 x=0, \\
x^{2}+y^{2}-2 y=0, & x^{2}+y^{2}-6 y=0 .
\end{array}$$

Solution :516. The domain D is depicted in Figure 71 . We transform it into the rectangle $D_{1}=\left[\frac{1}{4}, \frac{1}{2}\right] \times\left[\frac{1}{6}, \frac{1}{2}\right]$ by the change of coordinates

$$x=\frac{u}{u^{2}+v^{2}}, \quad y=\frac{v}{u^{2}+v^{2}} .$$

The Jacobian is MATHPIX IMAGEFigure 71

$$J=-\frac{1}{\left(u^{2}+v^{2}\right)^{2}} .$$

Therefore,

$$\iint_D \frac{dxdy}{(x^2+y^2)^2} = \iint_{D_1} dudv = \frac{1}{12}.$$

(D. Flondor, N. Donciu, Algebră şi Analiză Matematică (Algebra and Mathematical Analysis), Editura Didactică şi Pedagogică, Bucharest, 1965)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :517. Compute the integral

$$I = \iint_D |xy| dxdy,$$

where

$$D = \left\{ (x,y) \in \mathbb{R}^2 \mid x \geq 0, \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2 \leq \frac{x^2}{a^2} - \frac{y^2}{b^2} \right\}, \quad a, b > 0.$$

Solution :517. In the equation of the curve that bounds the domain

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2 = \frac{x^2}{a^2} - \frac{y^2}{b^2},$$

the expression on the left suggests the use of generalized polar coordinates, which are suited for elliptical domains. And indeed, if we set $x = ar\cos\theta$ and $y = br\sin\theta$, the equation of the curve becomes $r^4 = r^2\cos 2\theta$, or $r = \sqrt{\cos 2\theta}$. The condition $x \geq 0$ becomes $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and because $\cos 2\theta$ should be positive we should further have $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. Hence the domain of integration is

$$\left\{ (r,\theta); \quad 0 \leq r \leq \sqrt{\cos 2\theta}, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \right\}.$$

The Jacobian of the transformation is $J = abr$. Applying the formula for the change of variables, the integral becomes

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} a^2b^2r^3\cos\theta|\sin\theta| dr d\theta = \frac{a^2b^2}{4}\int_0^{\frac{\pi}{4}} \cos^2 2\theta \sin 2\theta d\theta = \frac{a^2b^2}{24}.$$

(Gh. Bucur, E. Câmpu, S. Găină, Culegere de Probleme de Calcul Diferenţial şi Integral (Collection of Problems in Differential and Integral Calculus), Editura Tehnică, Bucharest, 1967)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :519. Evaluate

$$\int_0^1 \int_0^1 \int_0^1 \left(1+u^2+v^2+w^2\right)^{-2} dudvdw$$

Solution :519. Call the integral I. By symmetry, we may compute it over the domain $\left\{(u,v,w) \in \mathbb{R}^3 \mid 0 \le v \le u \le 1\right\}$, then double the result. We substitute $u = r\cos\theta, v = r\sin\theta, w = \tan\phi$, taking into account that the limits of integration become $0 \le \theta, \phi \le \frac{\pi}{4}$, and $0 \le r \le \sec\theta$. We have

$$\begin{aligned}
I &= 2\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \frac{r\sec^2\phi}{\left(1+r^2\cos^2\theta + r^2\sin^2\theta + \tan^2\phi\right)^2} drd\theta d\phi \\
&= 2\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \frac{r\sec^2\phi}{\left(r^2+\sec^2\phi\right)^2} drd\theta d\phi \\
&= 2\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \sec^2\phi \left.\frac{-1}{2\left(r^2+\sec^2\phi\right)}\right|_{r=0}^{r=\sec\theta} d\theta d\phi \\
&= -\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{\sec^2\phi}{\sec^2\theta + \sec^2\phi} d\theta d\phi + \left(\frac{\pi}{4}\right)^2.
\end{aligned}$$

But notice that this is the same as

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \left(1 - \frac{\sec^2\phi}{\sec^2\theta+\sec^2\phi}\right) d\theta d\phi = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^2\theta+\sec^2\phi} d\theta d\phi.$$

If we exchange the roles of θ and ϕ in this last integral we see that

$$-\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{\sec^2\phi}{\sec^2\theta+\sec^2\phi} d\theta d\phi + \left(\frac{\pi}{4}\right)^2 = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{\sec^2\phi}{\sec^2\theta+\sec^2\phi} d\theta d\phi.$$

Hence

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{\sec^2\phi}{\sec^2\theta+\sec^2\phi} d\theta d\phi = \frac{\pi^2}{32}.$$

Consequently, the integral we are computing is equal to $\frac{\pi^2}{32}$.(American Mathematical Monthly, proposed by M. Hajja and P. Walker)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :525. Let $F(x) = \sum_{n=1}^{\infty} \frac{1}{x^2+n^4}, x \in \mathbb{R}$. Compute $\int_0^{\infty} F(t)dt$.
Solution :525. We can apply Tonelli's theorem to the function $f(x,n) = \frac{1}{x^2+n^4}$. Integrating term by term, we obtain

$$\int_0^x F(t)dt = \int_0^x \sum_{n=1}^{\infty} f(t,n)dt = \sum_{n=1}^{\infty} \int_0^x \frac{dt}{t^2+n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2} \arctan\frac{x}{n^2}.$$

This series is bounded from above by $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Hence the summation commutes with the limit as x tends to infinity. We have

$$\int_0^{\infty} F(t)dt = \lim_{x\to\infty} \int_0^x F(t)dt = \lim_{x\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^2} \arctan \frac{x}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \frac{\pi}{2}.$$

Using the identity $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$, we obtain

$$\int_0^{\infty} F(t)dt = \frac{\pi^3}{12}.$$

(Gh. Sireţchi, Calcul Diferenţial şi Integral (Differential and Integral Calculus), Editura Ştiinţifică şi Enciclopedică, Bucharest, 1985)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :527. Compute the flux of the vector field

$$\vec{F}(x,y,z) = x\left(e^{xy} - e^{zx}\right)\vec{i} + y\left(e^{yz} - e^{xy}\right)\vec{j} + z\left(e^{zx} - e^{yz}\right)\vec{k}$$

across the upper hemisphere of the unit sphere.
Solution :527. It can be checked that $\operatorname{div} \vec{F} = 0$ (in fact, $\vec{F}$ is the curl of the vector field $e^{yz}\vec{i} + e^{zx}\vec{j} + e^{xy}\vec{k}$). If S be the union of the upper hemisphere and the unit disk in the xy-plane, then by the divergence theorem $\iint_S \vec{F} \cdot \vec{n} dS = 0$. And on the unit disk $\vec{F} \cdot \vec{n} = 0$, which means that the flux across the unit disk is zero. It follows that the flux across the upper hemisphere is zero as well.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :528. Compute

$$\oint_C y^2dx + z^2dy + x^2dz,$$

where C is the Viviani curve, defined as the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ with the cylinder $x^2 + y^2 = ax$.
Solution :528. We simplify the computation using Stokes' theorem:

$$\oint_C y^2dx + z^2dy + x^2dz = -2\iint_S ydxdy + zdydz + xdzdx$$

where S is the portion of the sphere bounded by the Viviani curve. We have

$$-2\iint_S ydxdy + zdydz + xdzdx = -2\iint_S (z,x,y) \cdot \vec{n} d\sigma,$$

where (z, x, y) denotes the three-dimensional vector with coordinates z, x, and y, while $\vec{n}$ denotes the unit vector normal to the sphere at the point of coordinates (x, y, z). We parametrize the portion of the sphere in question by the coordinates (x, y), which range inside the circle $x^2+y^2-ax=0$. This circle is the projection of the Viviani curve onto the xy-plane.The unit vector normal to the sphere is

$$\vec{n}=\left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right)=\left(\frac{x}{a}, \frac{y}{a}, \frac{\sqrt{a^2-x^2-y^2}}{a}\right),$$

while the area element is

$$d\sigma=\frac{1}{\cos\alpha} dxdy,$$

α being the angle formed by the normal to the sphere with the xy-plane. It is easy to see that $\cos\alpha=\frac{z}{a}=\frac{\sqrt{a^2-x^2-y^2}}{a}$. Hence the integral is equal to

$$-2 \iint_D\left(z \frac{x}{a}+x \frac{y}{a}+y \frac{z}{a}\right) \frac{a}{z} dxdy=-2 \iint_D\left(x+y+\frac{xy}{\sqrt{a^2-x^2-y^2}}\right) dxdy$$

the domain of integration D being the disk $x^2+y^2-ax \leq 0$. Split the integral as

$$-2 \iint_D(x+y) dxdy-2 \iint_D \frac{xy}{\sqrt{a^2-x^2-y^2}} dxdy.$$

Because the domain of integration is symmetric with respect to the y-axis, the second double integral is zero. The first double integral can be computed using polar coordinates: $x=\frac{a}{2}+r\cos\theta, y=r\sin\theta, 0 \leq r \leq \frac{a}{2}, 0 \leq \theta \leq 2\pi$. Its value is $-\frac{\pi a^3}{4}$, which is the answer to the problem.(D. Flondor, N. Donciu, Algebră şi Analiză Matematică (Algebra and Mathematical Analysis), Editura Didactică şi Pedagogică, Bucharest, 1965)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :535. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$f\left(x^2-y^2\right)=(x-y)(f(x)+f(y)).$$

Solution :535. Plugging in $x=y$, we find that $f(0)=0$, and plugging in $x=-1, y=0$, we find that $f(1)=-f(-1)$. Also, plugging in $x=a, y=1$, and then $x=a, y=-1$, we obtain

$$\begin{aligned}
& f\left(a^2-1\right)=(a-1)(f(a)+f(1)), \\
& f\left(a^2-1\right)=(a+1)(f(a)-f(1)).
\end{aligned}$$

Equating the right-hand sides and solving for $f(a)$ gives $f(a)=f(1)a$ for all a.So any such function is linear. Conversely, a function of the form $f(x)=kx$

clearly satisfies the equation.(Korean Mathematical Olympiad, 2000)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :536. Find all complex-valued functions of a complex variable satisfying

$$f(z)+zf(1-z)=1+z, \quad \text{ for all } z.$$

Solution :536. Replace z by $1-z$ to obtain

$$f(1-z)+(1-z)f(z)=2-z.$$

Combine this with $f(z)+zf(1-z)=1+z$, and eliminate $f(1-z)$ to obtain

$$\left(1-z+z^{2}\right)f(z)=1-z+z^{2}.$$

Hence $f(z)=1$ for all z except maybe for $z=e^{\pm\pi i/3}$, when $1-z+z^{2}=0$. For $\alpha=e^{i\pi/3}, \bar{\alpha}=\alpha^{2}=1-\alpha$; hence $f(\alpha)+\alpha f(\bar{\alpha})=1+\alpha$. We therefore have only one constraint, namely $f(\bar{\alpha})=[1+\alpha-f(\alpha)]/\alpha=\bar{\alpha}+1-\bar{\alpha}f(\alpha)$. Hence the solution to the functional equation is of the form

$$\begin{gathered} f(z)=1 \quad \text{ for } z\neq e^{\pm i\pi/3}, \\ f\left(e^{i\pi/3}\right)=\beta \\ f\left(e^{-i\pi/3}\right)=\bar{\alpha}+1-\bar{\alpha}\beta \end{gathered}$$

where β is an arbitrary complex parameter.(20th W.L. Putnam Competition, 1959)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :537. Find all functions $f:\mathbb{R}\backslash\{1\}\rightarrow\mathbb{R}$, continuous at 0 , that satisfy

$$f(x)=f\left(\frac{x}{1-x}\right), \quad \text{ for } x\in\mathbb{R}\backslash\{1\}.$$

Solution :537. Successively, we obtain

$$f(-1)=f\left(-\frac{1}{2}\right)=f\left(-\frac{1}{3}\right)=\cdots=\lim_{n\rightarrow\infty}f\left(-\frac{1}{n}\right)=f(0).$$

Hence $f(x)=f(0)$ for $x\in\left\{0,-1,-\frac{1}{2},\ldots,-\frac{1}{n},\ldots\right\}$.If $x\neq 0,-1,\ldots,-\frac{1}{n},\ldots$, replacing x by $\frac{x}{1+x}$ in the functional equation, we obtain

$$f\left(\frac{x}{1+x}\right)=f\left(\frac{\frac{x}{1+x}}{1-\frac{x}{1+x}}\right)=f(x).$$

And this can be iterated to yield

$$f\left(\frac{x}{1+nx}\right) = f(x), \quad n = 1, 2, 3 \ldots$$

Because f is continuous at 0 it follows that

$$f(x) = \lim_{n\to\infty} f\left(\frac{x}{1+nx}\right) = f(0).$$

This shows that only constant functions satisfy the functional equation.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :538. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the inequality

$$f(x+y) + f(y+z) + f(z+x) \geq 3f(x+2y+3z)$$

for all $x, y, z \in \mathbb{R}$.
Solution :538. Plugging in $x = t, y = 0, z = 0$ gives

$$f(t) + f(0) + f(t) \geq 3f(t),$$

or $f(0) \geq f(t)$ for all real numbers t. Plugging in $x = \frac{t}{2}, y = \frac{t}{2}, z = -\frac{t}{2}$ gives

$$f(t) + f(0) + f(0) \geq 3f(0),$$

or $f(t) \geq f(0)$ for all real numbers t. Hence $f(t) = f(0)$ for all t, so f must be constant. Conversely, any constant function f clearly satisfies the given condition.(Russian Mathematical Olympiad, 2000)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :539. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2$ for all real numbers x ?
Solution :539. No! In fact, we will prove a more general result.Proposition. Let S be a set and $g : S \to S$ a function that has exactly two fixed points $\{a, b\}$ and such that $g \circ g$ has exactly four fixed points $\{a, b, c, d\}$. Then there is no function $f : S \to S$ such that $g = f \circ f$. Proof. Let $g(c) = y$. Then $c = g(g(c)) = g(y)$; hence $y = g(c) = g(g(y))$. Thus y is a fixed point of $g \circ g$. If $y = a$, then $a = g(a) = g(y) = c$, leading to a contradiction. Similarly, $y = b$ forces $c = b$. If $y = c$, then $c = g(y) = g(c)$, so c is a fixed point of g, again a contradiction. It follows that $y = d$, i.e., $g(c) = d$, and similarly $g(d) = c$.Suppose there is $f : S \to S$ such that $f \circ f = g$. Then $f \circ g = f \circ f \circ f = g \circ f$. Then $f(a) = f(g(a)) = g(f(a))$, so $f(a)$ is a fixed point of g. Examining case by case, we conclude that $f(\{a, b\}) \subset \{a, b\}$ and $f(\{a, b, c, d\}) \subset \{a, b, c, d\}$. Because

$f \circ f = g$, the inclusions are, in fact, equalities.Consider $f(c)$. If $f(c)=a$, then $f(a)=f(f(c))=g(c)=d$, a contradiction since $f(a)$ is in $\{a, b\}$. Similarly, we rule out $f(c)=b$. Of course, c is not a fixed point of f, since it is not a fixed point of g. We are left with the only possibility $f(c)=d$. But then $f(d)=f(f(c))=g(c)=d$, and this again cannot happen because d is not a fixed point of g. We conclude that such a function f cannot exist.In the particular case of our problem, $g(x)=x^{2}-2$ has the fixed points -1 and 2 , and $g(g(x))=\left(x^{2}-2\right)^{2}-2$ has the fixed points $-1,2, \frac{-1+\sqrt{5}}{2}$, and $\frac{-1-\sqrt{5}}{2}$. This completes the solution.(B.J. Venkatachala, Functional Equations: A Problem Solving Approach, Prism Books PVT Ltd., 2002)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :540. Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$f(x+y)=f(x) f(y)-c \sin x \sin y,$$

for all real numbers x and y, where c is a constant greater than 1 .
Solution :540. The standard approach is to substitute particular values for x and y. The solution found by the student S.P. Tungare does quite the opposite. It introduces an additional variable z. The solution proceeds as follows:

$$\begin{aligned}
& f(x+y+z) \\
& \quad=f(x) f(y+z)-c \sin x \sin (y+z) \\
& \quad=f(x)[f(y) f(z)-c \sin y \sin z]-c \sin x \sin y \cos z-c \sin x \cos y \sin z \\
& \quad=f(x) f(y) f(z)-c f(x) \sin y \sin z-c \sin x \sin y \cos z-c \sin x \cos y \sin z .
\end{aligned}$$

Because obviously $f(x+y+z)=f(y+x+z)$, it follows that we must have

$$\sin z[f(x) \sin y-f(y) \sin x]=\sin z[\cos x \sin y-\cos y \sin x] .$$

Substitute $z=\frac{\pi}{2}$ to obtain

$$f(x) \sin y-f(y) \sin x=\cos x \sin y-\cos y \sin x .$$

For $x=\pi$ and y not an integer multiple of π, we obtain $\sin y[f(\pi)+1]=0$, and hence $f(\pi)=-1$.Then, substituting in the original equation $x=y=\frac{\pi}{2}$ yields

$$f(\pi)=\left[f\left(\frac{\pi}{2}\right)\right]-c,$$

whence $f\left(\frac{\pi}{2}\right)= \pm \sqrt{c-1}$. Substituting in the original equation $y=\pi$ we also obtain $f(x+\pi)=-f(x)$. We then have

$$\begin{aligned}
-f(x)=f(x+\pi)=f\left(x+\frac{\pi}{2}\right) f\left(\frac{\pi}{2}\right)-c \cos x \\
=f\left(\frac{\pi}{2}\right)\left(f(x) f\left(\frac{\pi}{2}\right)-c \sin x\right)-c \cos x
\end{aligned}$$

whence

$$f(x)\left[\left(f\left(\frac{\pi}{2}\right)\right)^2-1\right]=cf\left(\frac{\pi}{2}\right)\sin x-c\cos x.$$

It follows that $f(x)=f\left(\frac{\pi}{2}\right)\sin x+\cos x$. We find that the functional equation has two solutions, namely,

$$f(x)=\sqrt{c-1}\sin x+\cos x \text{ and } f(x)=-\sqrt{c-1}\sin x+\cos x.$$

(Indian Team Selection Test for the International Mathematical Olympiad, 2004)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :542. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy the relation

$$3f(2x+1)=f(x)+5x, \quad \text{for all } x.$$

Solution :542. Substituting for f a linear function $ax+b$ and using the method of undetermined coefficients, we obtain $a=1, b=-\frac{3}{2}$, so $f(x)=x-\frac{3}{2}$ is a solution.Are there other solutions? Setting $g(x)=f(x)-\left(x-\frac{3}{2}\right)$, we obtain the simpler functional equation

$$3g(2x+1)=g(x), \quad \text{for all } x\in\mathbb{R}.$$

This can be rewritten as

$$g(x)=\frac{1}{3}g\left(\frac{x-1}{2}\right), \quad \text{for all } x\in\mathbb{R}$$

For $x=-1$ we have $g(-1)=\frac{1}{3}g(-1)$; hence $g(-1)=0$. In general, for an arbitrary x, define the recursive sequence $x_0=x, x_{n+1}=\frac{x_n-1}{2}$ for $n\geq 0$. It is not hard to see that this sequence is Cauchy, for example, because $|x_{m+n}-x_m|\leq\frac{1}{2^{m-2}}\max(1,|x|)$. This sequence is therefore convergent, and its limit L satisfies the equation $L=\frac{L-1}{2}$. It follows that $L=-1$. Using the functional equation, we obtain

$$g(x)=\frac{1}{3}g\left(x_1\right)=\frac{1}{9}g\left(x_2\right)=\cdots=\frac{1}{3^n}g\left(x_n\right)$$

Passing to the limit, we obtain $g(x)=0$. This shows that $f(x)=x-\frac{3}{2}$ is the unique solution to the functional equation.(B.J. Venkatachala, Functional Equations: A Problem Solving Approach, Prism Books PVT Ltd., 2002)
Topic :Real Analysis
Book :Putnam and Beyond

Final Answer :

Problem Statement :543. Find all functions $f:(0,\infty)\to(0,\infty)$ subject to the conditions(i) $f(f(f(x)))+2x=f(3x)$, for all $x>0$;(ii) $\lim_{x\to\infty}(f(x)-x)=0$.

Solution :543. We will first show that $f(x)\geq x$ for all x. From (i) we deduce that $f(3x)\geq 2x$, so $f(x)\geq\frac{2x}{3}$. Also, note that if there exists k such that $f(x)\geq kx$ for all x, then $f(x)\geq\frac{k^3+2}{3}x$ for all x as well. We can iterate and obtain $f(x)\geq k_n x$, where k_n are the terms of the recursive sequence defined by $k_1=\frac{2}{3}$, and $k_{n+1}=\frac{k_n^3+2}{3}$ for $k\geq 1$. Let us examine this sequence.By the AM-GM inequality,

$$k_{n+1}=\frac{k_n^3+1^3+1^3}{3}\geq k_n$$

so the sequence is increasing. Inductively we prove that $k_n<1$. Weierstrass' criterion implies that $(k_n)_n$ is convergent. Its limit L should satisfy the equation

$$L=\frac{L^3+2}{3}$$

which shows that L is a root of the polynomial equation $L^3-3L+2=0$. This equation has only one root in $[0,1]$, namely $L=1$. Hence $\lim_{n\to\infty}k_n=1$, and so $f(x)\geq x$ for all x.It follows immediately that $f(3x)\geq 2x+f(x)$ for all x. Iterating, we obtain that for all $n\geq 1$

$$f\left(3^n x\right)-f(x)\geq\left(3^n-1\right)x.$$

Therefore, $f(x)-x\leq f\left(3^n x\right)-3^n x$. If we let $n\to\infty$ and use (ii), we obtain $f(x)-x\leq 0$, that is, $f(x)\leq x$. We conclude that $f(x)=x$ for all $x>0$. Thus the identity function is the unique solution to the functional equation.(G. Dospinescu)

Topic :Real Analysis

Book :Putnam and Beyond

Final Answer :

Problem Statement :546. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x+y)=f(x)+f(y)+f(x)f(y),\quad \text{for all } x,y\in\mathbb{R}.$$

Solution :546. Adding 1 to both sides of the functional equation and factoring, we obtain

$$f(x+y)+1=(f(x)+1)(f(y)+1).$$

The continuous function $g(x)=f(x)+1$ satisfies the functional equation $g(x+y)=g(x)g(y)$, and we have seen in the previous problem that $g(x)=c^x$ for some nonnegative constant c. We conclude that $f(x)=c^x-1$ for all x.

Topic :Real Analysis

Book :Putnam and Beyond
Final Answer :

Problem Statement :547. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y)=\frac{f(x)+f(y)}{1+f(x)f(y)}, \quad \text{for all } x, y \in \mathbb{R}.$$

Solution :547. If there exists x_0 such that $f(x_0)=1$, then

$$f(x)=f(x_0+(x-x_0))=\frac{1+f(x-x_0)}{1+f(x-x_0)}=1.$$

In this case, f is identically equal to 1 . In a similar manner, we obtain the constant solution $f(x)\equiv -1$.Let us now assume that f is never equal to 1 or -1. Define $g:\mathbb{R}\to\mathbb{R}, g(x)=\frac{1+f(x)}{1-f(x)}$. To show that g is continuous, note that for all x,

$$f(x)=\frac{2f\left(\frac{x}{2}\right)}{1+f\left(\frac{x}{2}\right)}<1.$$

Now the continuity of g follows from that of f and of the function $h(t)=\frac{1+t}{1-t}$ on $(-\infty,1)$. Also,

$$\begin{aligned} g(x+y) &= \frac{1+f(x+y)}{1-f(x+y)}=\frac{f(x)f(y)+1+f(x)+f(y)}{f(x)f(y)+1-f(x)-f(y)} \\ &= \frac{1+f(x)}{1-f(x)}\cdot\frac{1+f(y)}{1-f(y)}=g(x)g(y) \end{aligned}$$

Hence g satisfies the functional equation $g(x+y)=g(x)g(y)$. As seen in problem 545, $g(x)=c^x$ for some $c>0$. We obtain $f(x)=\frac{c^x-1}{c^x+1}$. The solutions to the equation are therefore

$$f(x)=\frac{c^x-1}{c^x+1}, \quad f(x)=1, \quad f(x)=-1.$$

Remark. You might have recognized the formula for the hyperbolic tangent of the sum. This explains the choice of g, by expressing the exponential in terms of the hyperbolic tangent.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :548. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the condition

$$f(xy)=xf(y)+yf(x), \quad \text{for all } x, y \in \mathbb{R}.$$

Solution :548. Rewrite the functional equation as

$$\frac{f(xy)}{xy}=\frac{f(x)}{x}+\frac{f(y)}{y}.$$

It now becomes natural to let $g(x)=\frac{f(x)}{x}$, which satisfies the equation

$$g(xy)=g(x)+g(y).$$

The particular case $x=y$ yields $g(x)=\frac{1}{2}g\left(x^2\right)$, and hence $g(-x)=\frac{1}{2}g\left((-x)^2\right)=\frac{1}{2}g\left(x^2\right)=g(x)$. Thus we only need to consider the case $x>0$.Note that g is continuous on $(0,\infty)$. If we compose g with the continuous function $h:\mathbb{R}\rightarrow(0,\infty), h(x)=e^x$, we obtain a continuous function on $\mathbb{R}$ that satisfies Cauchy's equation. Hence $g\circ h$ is linear, which then implies $g(x)=\log_a x$ for some positive base a. It follows that $f(x)=x\log_a x$ for $x>0$ and $f(x)=x\log_a|x|$ if $x<0$.All that is missing is the value of f at 0 . This can be computed directly setting $x=y=0$, and it is seen to be 0 . We conclude that $f(x)=x\log_a|x|$ if $x\neq 0$, and $f(0)=0$, where a is some positive number. The fact that any such function is continuous at zero follows from

$$\lim_{x\rightarrow 0+} x\log_a x=0,$$

which can be proved by applying the L'Hôpital's theorem to the functions $\log_a x$ and $\frac{1}{x}$. This concludes the solution.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :549. Find the continuous functions $\phi, f, g, h:\mathbb{R}\rightarrow\mathbb{R}$ satisfying

$$\phi(x+y+z)=f(x)+g(y)+h(z),$$

for all real numbers x, y, z.
Solution :549. Setting $y=z=0$ yields $\phi(x)=f(x)+g(0)+h(0)$, and similarly $\phi(y)=g(y)+f(0)+h(0)$. Substituting these three relations in the original equation and letting $z=0$ gives rise to a functional equation for ϕ, namely

$$\phi(x+y)=\phi(x)+\phi(y)-(f(0)+g(0)+h(0)).$$

This should remind us of the Cauchy equation, which it becomes after changing the function ϕ to $\psi(x)=\phi(x)-(f(0)+g(0)+h(0))$. The relation $\psi(x+y)=\psi(x)+\psi(y)$ together with the continuity of ψ shows that $\psi(x)=cx$ for some constant c. We obtain the solution to the original equation

$$\phi(x)=cx+\alpha+\beta+\gamma, \quad f(x)=cx+\alpha, \quad g(x)=cx+\beta, \quad h(x)=cx+\gamma,$$

where α, β, γ are arbitrary real numbers.(Gazeta Matematică (Mathematics Gazette, Bucharest), proposed by M. Vlada)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :551. Do there exist continuous functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all $x \in \mathbb{R}$?
Solution :551. First solution: Assume that such functions do exist. Because $g \circ f$ is a bijection, f is one-to-one and g is onto. Since f is a one-to-one continuous function, it is monotonic, and because g is onto but $f \circ g$ is not, it follows that f maps $\mathbb{R}$ onto an interval I strictly included in $\mathbb{R}$. One of the endpoints of this interval is finite, call this endpoint a. Without loss of generality, we may assume that $I = (a, \infty)$. Then as $g \circ f$ is onto, $g(I) = \mathbb{R}$.MATHPIX IMAGE means that g oscillates in a neighborhood of infinity. But this is impossible because $f(g(x)) = x^2$ implies that g assumes each value at most twice. Hence the question has a negative answer; such functions do not exist.Second solution: Since $g \circ f$ is a bijection, f is one-to-one and g is onto. Note that $f(g(0)) = 0$. Since g is onto, we can choose a and b with $g(a) = g(0) - 1$ and $g(b) = g(0) + 1$. Then $f(g(a)) = a^2 > 0$ and $f(g(b)) = b^2 > 0$. Let $c = \min\left(a^2, b^2\right)/2 > 0$. The intermediate value property guarantees that there is an $x_0 \in (g(a), g(0))$ with $f\left(x_0\right) = c$ and an $x_1 \in (g(0), g(b))$ with $f\left(x_1\right) = c$. This contradicts the fact that f is one-to-one. Hence no such functions can exist.(R. Gelca, second solution by R. Stong)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :552. Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with the property that

$$f(f(x)) - 2f(x) + x = 0, \quad \text{for all } x \in \mathbb{R}.$$

Solution :552. The relation from the statement implies that f is injective, so it must be monotonic. Let us show that f is increasing. Assuming the existence of a decreasing solution f to the functional equation, we can find x_0 such that $f\left(x_0\right) \neq x_0$. Rewrite the functional equation as $f(f(x)) - f(x) = f(x) - x$. If $f\left(x_0\right) < x_0$, then $f\left(f\left(x_0\right)\right) < f\left(x_0\right)$, and if $f\left(x_0\right) > x_0$, then $f\left(f\left(x_0\right)\right) > f\left(x_0\right)$, which both contradict the fact that f is decreasing. Thus any function f that satisfies the given condition is increasing.Pick some $a > b$, and set $\Delta f(a) = f(a) - a$ and $\Delta f(b) = f(b) - b$. By adding a constant to f (which yields again a solution to the functional equation), we may assume that $\Delta f(a)$ and $\Delta f(b)$ are positive. Composing f with itself n times, we obtain $f^{(n)}(a) = a + n\Delta f(a)$ and $f^{(n)}(b) = b + n\Delta f(b)$. Recall that f is an increasing function, so $f^{(n)}$ is increasing, and hence $f^{(n)}(a) > f^{(n)}(b)$, for all n. This can happen only if $\Delta f(a) \geq \Delta f(b)$.On the other hand, there exists m such that $b + m\Delta f(b) = f^{(m)}(b) > a$, and the same argument shows that $\Delta f\left(f^{(m-1)}(b)\right) > \Delta f(a)$. But $\Delta f\left(f^{(m-1)}(b)\right) = \Delta f(b)$, so $\Delta f(b) \geq \Delta f(a)$. We conclude that $\Delta f(a) = \Delta f(b)$, and hence $\Delta f(a) = f(a) - a$ is independent of a. Therefore, $f(x) = x + c$, with $c \in \mathbb{R}$, and clearly any function of this type satisfies the equation from the statement.
Topic :Real Analysis

Book :Putnam and Beyond
Final Answer :

Problem Statement :553. A not uncommon mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for f and g on (a, b).
Solution :553. The answer is yes! We have to prove that for $f(x) = e^{x^2}$, the equation $f'g + fg' = f'g'$ has nontrivial solutions on some interval (a, b). Explicitly, this is the first-order linear equation in g,

$$(1 - 2x)e^{x^2} g' + 2xe^{x^2} g = 0.$$

Separating the variables, we obtain

$$\frac{g'}{g} = \frac{2x}{2x - 1} = 1 + \frac{1}{2x - 1},$$

which yields by integration $\ln g(x) = x + \frac{1}{2} \ln |2x - 1| + C$. We obtain the one-parameter family of solutions

$$g(x) = ae^x \sqrt{|2x - 1|}, \quad a \in \mathbb{R},$$

on any interval that does not contain $\frac{1}{2}$.(49th W.L. Putnam Mathematical Competition, 1988)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :554. Find the functions $f, g : \mathbb{R} \to \mathbb{R}$ with continuous derivatives satisfying

$$f^2 + g^2 = f'^2 + g'^2, \quad f + g = g' - f',$$

and such that the equation $f = g$ has two real solutions, the smaller of them being zero.
Solution :554. Rewrite the equation $f^2 + g^2 = f'^2 + g'^2$ as

$$(f + g)^2 + (f - g)^2 = (f' + g')^2 + (g' - f')^2.$$

This, combined with $f + g = g' - f'$, implies that $(f - g)^2 = (f' + g')^2$.Let x_0 be the second root of the equation $f(x) = g(x)$. On the intervals $I_1 = (-\infty, 0), I_2 = (0, x_0)$, and $I_3 = (x_0, \infty)$ the function $f - g$ is nonzero; hence so is $f' + g'$. These two functions maintain constant sign on the three intervals; hence $f - g = \epsilon_j (f' + g')$ on I_j, for some $\epsilon_j \in \{-1, 1\}, j = 1, 2, 3$. If on any of these intervals $f - g = f' + g'$, then since $f + g = g' - f'$ it follows that $f = g'$ on that interval, and so $g' + g = g' - g''$. This implies that g satisfies the equation $g'' + g = 0$, or that $g(x) = A \sin x + B \cos x$ on that interval. Also,

$f(x)=g'(x)=A\cos x-B\sin x$.If $f-g=-f'-g'$ on some interval, then using again $f+g=g'-f'$, we find that $g=g'$ on that interval. Hence $g(x)=C_1e^x$. From the fact that $f=-f'$, we obtain $f(x)=C_2e^{-x}$.Assuming that f and g are exponentials on the interval $(0,x_0)$, we deduce that $C_1=g(0)=f(0)=C_2$ and that $C_1e^{x_0}=g\left(x_0\right)=f\left(x_0\right)=C_2e^{-x}$. These two inequalities cannot hold simultaneously, unless f and g are identically zero, ruled out by the hypothesis of the problem. Therefore, $f(x)=A\cos x-B\sin x$ and $g(x)=A\sin x+B\cos x$ on $(0,x_0)$, and consequently $x_0=\pi$.On the intervals $(-\infty,0]$ and $[x_0,\infty)$ the functions f and g cannot be periodic, since then the equation $f=g$ would have infinitely many solutions. So on these intervals the functions are exponentials. Imposing differentiability at 0 and π, we obtain $B=A$, $C_1=A$ on I_1 and $C_1=-Ae^{-\pi}$ on I_3 and similarly $C_2=A$ on I_1 and $C_2=-Ae^{\pi}$ on I_3. Hence the answer to the problem is

$$f(x)=\begin{cases}Ae^{-x} & \text{for } x\in(-\infty,0],\\ A(\sin x+\cos x) & \text{for } x\in(0,\pi],\\ -Ae^{-x+\pi} & \text{for } x\in(\pi,\infty),\end{cases}$$

$$g(x)=\begin{cases}Ae^{x} & \text{for } x\in(-\infty,0],\\ A(\sin x-\cos x) & \text{for } x\in(0,\pi],\\ -Ae^{x-\pi} & \text{for } x\in(\pi,\infty),\end{cases}$$

where A is some nonzero constant.(Romanian Mathematical Olympiad, 1976, proposed by V. Matrosenco)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :556. Let A,B,C,D,m,n be real numbers with $AD-BC\neq 0$. Solve the differential equation

$$y\left(B+Cx^my^n\right)dx+x\left(A+Dx^my^n\right)dy=0.$$

Solution :556. The idea is to write the equation as

$$Bydx+Axdy+x^my^n(Dydx+Cxdy)=0,$$

then find an integrating factor that integrates simultaneously $Bydx+Axdy$ and $x^my^n(Dydx+Cxdy)$. An integrating factor of $Bydx+Axdy$ will be of the form $x^{-1}y^{-1}\phi_1\left(x^By^A\right)$, while an integrating factor of $x^my^n(Dydx+Cxdy)=Dx^my^{n+1}dx+Cx^{m+1}y^ndy$ will be of the form $x^{-m-1}y^{-n-1}\phi_2\left(x^Dy^C\right)$, where ϕ_1 and ϕ_2 are one-variable functions. To have the same integrating factor for both expressions, we should have

$$x^my^n\phi_1\left(x^By^A\right)=\phi_2\left(x^Dy^C\right).$$

It is natural to try power functions, say $\phi_1(t) = t^p$ and $\phi_2(t) = t^q$. The equality condition gives rise to the system

$$\begin{aligned} Ap - Cq &= -n, \\ Bp - Dq &= -m \end{aligned}$$

which according to the hypothesis can be solved for p and q. We find that

$$p = \frac{Bn - Am}{AD - BC}, \quad q = \frac{Dn - Cm}{AD - BC}.$$

Multiplying the equation by $x^{-1}y^{-1}\left(x^B y^A\right)^p = x^{-1-m}y^{-1-n}\left(x^D y^C\right)^q$ and integrating, we obtain

$$\frac{1}{p+1}\left(x^B y^A\right)^{p+1} + \frac{1}{q+1}\left(x^D y^C\right)^{q+1} = \text{ constant },$$

which gives the solution in implicit form.(M. Ghermănescu, Ecuaţii Diferenţiale (Differential Equations), Editura Didactică şi Pedagogică, Bucharest, 1963)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :557. Find all continuously differentiable functions $y : (0,\infty) \to (0,\infty)$ that are solutions to the initial value problem

$$y^{y'} = x, \quad y(1) = 1.$$

Solution :557. The differential equation can be rewritten as

$$e^{y' \ln y} = e^{\ln x}.$$

Because the exponential function is injective, this is equivalent to $y' \ln y = \ln x$. Integrating, we obtain the algebraic equation $y \ln y - y = x \ln x - x + C$, for some constant C. The initial condition yields $C = 0$. We are left with finding all differentiable functions y such that

$$y \ln y - y = x \ln x - x.$$

Let us focus on the function $f(t) = t \ln t - t$. Its derivative is $f'(t) = \ln t$, which is negative if $t < 1$ and positive if $t > 1$. The minimum of f is at $t = 1$, and is equal to -1. An easy application of L'Hôpital's rule shows that $\lim_{t\to 0} f(t) = 0$. It follows that the equation $f(t) = c$ fails to have a unique solution precisely when $c \in (0,1) \cup (1,e)$, in which case it has exactly two solutions.If we solve algebraically the equation $y \ln y - y = x \ln x - x$ on $(1,e)$, we obtain two possible continuous solutions, one that is greater than 1 and one that is less than 1 . The continuity of y at e rules out the second, so

on the interval $[1,\infty), y(x) = x$. On $(0,1)$ again we could have two solutions, $y_1(x) = x$, and some other function y_2 that is greater than 1 on this interval. Let us show that y_2 cannot be extended to a solution having continuous derivative at $x = 1$. On $(1,\infty), y_2(x) = x$, hence $\lim_{x\to 1^+} y_2'(x) = 1$. On $(0,1)$, as seen above, $y_2' \ln y_2 = \ln x$, so $y_2' = \ln x/\ln y_2 < 0$, since $x < 1$, and $y_2(x) > 1$. Hence $\lim_{x\to 1^-} y_2'(x) \leq 0$, contradicting the continuity of y_2' at $x = 1$. Hence the only solution to the problem is $y(x) = x$ for all $x \in (0,\infty)$.(R. Gelca)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :558. Find all differentiable functions $f : (0,\infty) \to (0,\infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)},$$

for all $x > 0$.
Solution :558. Define

$$g(x) = f(x)f'\left(\frac{a}{x}\right), \quad x \in (0,\infty).$$

We want to show that g is a constant function.Substituting $x \to \frac{a}{x}$ in the given condition yields

$$f\left(\frac{a}{x}\right)f'(x) = \frac{a}{x},$$

for all $x > 0$. We have

$$\begin{aligned} g'(x) &= f'(x)f\left(\frac{a}{x}\right) + f(x)f'\left(\frac{a}{x}\right)\left(-\frac{a}{x^2}\right) = f'(x)f\left(\frac{a}{x}\right) - \frac{a}{x^2}f\left(\frac{a}{x}\right)f(x) \\ &= \frac{a}{x} - \frac{a}{x} = 0, \end{aligned}$$

so g is identically equal to some positive constant b. Using the original equation we can write

$$b = g(x) = f(x)f\left(\frac{a}{x}\right) = f(x)\cdot\frac{a}{x}\cdot\frac{1}{f'(x)},$$

which gives

$$\frac{f'(x)}{f(x)} = \frac{a}{bx}.$$

Integrating both sides, we obtain $\ln f(x) = \frac{a}{b}\ln x + \ln c$, where $c > 0$. It follows that $f(x) = cx^{\frac{a}{b}}$, for all $x > 0$. Substituting back into the original equation yields

$$c\cdot\frac{a}{b}\cdot\frac{a^{\frac{a}{b}-1}}{x^{\frac{a}{b}-1}} = \frac{x}{cx^{\frac{a}{b}}}$$

which is equivalent to

$$c^2a^{\frac{a}{b}} = b$$

By eliminating c, we obtain the family of solutions

$$f_b(x) = \sqrt{b}\left(\frac{x}{\sqrt{a}}\right)^{\frac{a}{b}}, \quad b > 0$$

All such functions satisfy the given condition.(66th W.L. Putnam Mathematical Competition, 2005, proposed by T. Andreescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :560. Solve the differential equation

$$xy'' + 2y' + xy = 0.$$

Solution :560. The equation can be rewritten as

$$(xy)'' + (xy) = 0.$$

Solving, we find $xy = C_1 \sin x + C_2 \cos x$, and hence

$$y = C_1 \frac{\sin x}{x} + C_2 \frac{\cos x}{x}$$

on intervals that do not contain 0 .
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :561. Find all twice-differentiable functions defined on the entire real axis that satisfy $f'(x)f''(x) = 0$ for all x.
Solution :561. The function $f'(x)f''(x)$ is the derivative of $\frac{1}{2}\left(f'(x)\right)^2$. The equation is therefore equivalent to

$$\left(f'(x)\right)^2 = \text{ constant}$$

And because $f'(x)$ is continuous, $f'(x)$ itself must be constant, which means that $f(x)$ is linear. Clearly, all linear functions are solutions.
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :562. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x) + \int_0^x (x-t)f(t)dt = 1, \quad \text{for all } x \in \mathbb{R}.$$

Solution :562. The relation from the statement implies right away that f is differentiable. Differentiating

$$f(x) + x\int_0^x f(t)dt - \int_0^x tf(t)dt = 1,$$

we obtain

$$f'(x) + \int_0^x f(t)dt + xf(x) - xf(x) = 0,$$

that is, $f'(x) + \int_0^x f(t)dt = 0$. Again we conclude that f is twice differentiable, and so we can transform this equality into the differential equation $f'' + f = 0$. The general solution is $f(x) = A\cos x + B\sin x$. Substituting in the relation from the statement, we obtain $A = 1, B = 0$, that is, $f(x) = \cos x$.(E. Popa, Analiza Matematică, Culegere de Probleme (Mathematical Analysis, Collection of Problems), Editura GIL, 2005)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :563. Solve the differential equation

$$(x-1)y'' + (4x-5)y' + (4x-6)y = xe^{-2x}.$$

Solution :563. The equation is of Laplace type, but we can bypass the standard method once we make the following observation. The associated homogeneous equation can be written as

$$x\left(y'' + 4y' + 4y\right) - \left(y'' + 5y' + 6y\right) = 0,$$

and the equations $y'' + 4y' + 4y = 0$ and $y'' + 5y' + 6y = 0$ have the common solution $y(x) = e^{-2x}$. This will therefore be a solution to the homogeneous equation, as well. To find a solution to the inhomogeneous equation, we use the method of variation of the constant. Set $y(x) = C(x)e^{-2x}$. The equation becomes

$$(x-1)C'' - C' = x,$$

with the solution

$$C'(x) = \lambda(x-1) + (x-1)\ln|x-1| - 1.$$

Integrating, we obtain

$$C(x) = \frac{1}{2}(x-1)^2\ln|x-1| + \left(\frac{\lambda}{2} - \frac{1}{4}\right)(x-1)^2 - x + C_1.$$

If we set $c_2 = \frac{\lambda}{2} - \frac{1}{4}$, then the general solution to the equation is

$$y(x) = e^{-2x}\left[C_1 + C_2(x-1)^2 + \frac{1}{2}(x-1)^2\ln|x-1| - x\right].$$

(D. Flondor, N. Donciu, Algebră şi Analiză Matematică (Algebra and Mathematical Analysis), Editura Didactică şi Pedagogică, Bucharest, 1965)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :565. Find the one-to-one, twice-differentiable solutions y to the equation

$$\frac{d^2y}{dx^2} + \frac{d^2x}{dy^2} = 0.$$

Solution :565. We interpret the differential equation as being posed for a function y of x. In this perspective, we need to write $\frac{d^2x}{dy^2}$ in terms of the derivatives of y with respect to x. We have

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}},$$

and using this fact and the chain rule yields

$$\begin{aligned}\frac{d^2x}{dy^2} &= \frac{d}{dy}\left(\frac{1}{\frac{dy}{dx}}\right) = \frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right) \cdot \frac{dx}{dy} \\ &= -\frac{1}{\left(\frac{dy}{dx}\right)^2} \cdot \frac{d^2y}{dx^2} \cdot \frac{dx}{dy} = -\frac{1}{\left(\frac{dy}{dx}\right)^3} \cdot \frac{d^2y}{dx^2}.\end{aligned}$$

The equation from the statement takes the form

$$\frac{d^2y}{dx^2}\left(1 - \frac{1}{\left(\frac{dy}{dx}\right)^3}\right) = 0.$$

This splits into

$$\frac{d^2y}{dx^2} = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right)^3 = 1.$$

The first of these has the solutions $y = ax + b$, with $a \neq 0$, because y has to be one-toone, while the second reduces to $y' = 1$, whose family of solutions $y = x + c$ is included in the first. Hence the answer to the problem consists of the nonconstant linear functions.(M. Ghermănescu, Ecuaţii Diferenţiale (Differential Equations), Editura Didactică şi Pedagogică, Bucharest, 1963)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :570. Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) > 0$ and $f'(x) = f(f(x))$ for every $x \in \mathbb{R}$?

Solution :570. Assume that such a function exists. Because $f'(x) = f(f(x)) > 0$, the function is strictly increasing.The monotonicity and the positivity of f imply that $f(f(x)) > f(0)$ for all x. Thus $f(0)$ is a lower bound for $f'(x)$. Integrating the inequality $f(0) < f'(x)$ for $x < 0$, we obtain

$$f(x) < f(0) + f(0)x = (x+1)f(0).$$

But then for $x \leq -1$, we would have $f(x) \leq 0$, contradicting the hypothesis that $f(x) > 0$ for all x. We conclude that such a function does not exist.(9th International Mathematics Competition for University Students, 2002)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :571. Determine all n th-degree polynomials $P(x)$, with real zeros, for which the equality

$$\sum_{i=1}^{n} \frac{1}{P(x) - x_i} = \frac{n^2}{xP'(x)}$$

holds for all nonzero real numbers x for which $P'(x) \neq 0$, where $x_i, i = 1, 2, \ldots, n$, are the zeros of $P(x)$.
Solution :571. We use the separation of variables, writing the relation from the statement as

$$\sum_{i=1}^{n} \frac{P'(x)}{P(x) - x_i} = \frac{n^2}{x}.$$

Integrating, we obtain

$$\sum_{i=1}^{n} \ln |P(x) - x_i| = n^2 \ln C|x|$$

where C is some positive constant. After adding the logarithms on the left we have

$$\ln \prod_{i=1}^{n} |P(x) - x_i| = \ln C^{n^2} |x|^{n^2}$$

and so

$$\left| \prod_{i=1}^{n} (P(x) - x_i) \right| = k|x|^{n^2},$$

with $k = C^{n^2}$. Eliminating the absolute values, we obtain

$$P(P(x)) = \lambda x^{n^2}, \quad \lambda \in \mathbb{R}.$$

We end up with an algebraic equation. An easy induction can prove that the coefficient of the term of k th degree is 0 for $k < n$. Hence $P(x) = ax^n$, with a

some constant, are the only polynomials that satisfy the relation from the statement.(Revista Matematică din Timişoara (Timişoara Mathematics Gazette), proposed by T. Andreescu)
Topic :Real Analysis
Book :Putnam and Beyond
Final Answer :

Problem Statement :572. Let C be the class of all real-valued continuously differentiable functions f on the interval $[0,1]$ with $f(0)=0$ and $f(1)=1$. Determine

$$u=\inf_{f\in C}\int_0^1 |f'(x)-f(x)|\,dx.$$

2 Geometry and Trigonometry

Geometry is the oldest of the mathematical sciences. Its age-old theorems and the sharp logic of its proofs make you think of the words of Andrew Wiles, "Mathematics seems to have a permanence that nothing else has."This chapter is bound to take you away from the geometry of the ancients, with figures and pictorial intuition, and bring you to the science of numbers and equations that geometry has become today. In a dense exposition we have packed vectors and their applications, analytical geometry in the plane and in space, some applications of integral calculus to geometry, followed by a list of problems with Euclidean flavor but based on algebraic and combinatorial ideas. Special attention is given to conics and quadrics, for their study already contains the germs of differential and algebraic geometry.Four subsections are devoted to geometry's little sister, trigonometry. We insist on trigonometric identities, repeated in subsequent sections from different perspectives: Euler's formula, trigonometric substitutions, and telescopic summation and multiplication.Since geometry lies at the foundation of mathematics, its presence could already be felt in the sections on linear algebra and multivariable calculus. It will resurface again in the chapter on combinatorics.
Solution :572. The idea is to use an "integrating factor"' that transforms the quantity under the integral into the derivative of a function. We already encountered this situation in a previous problem, and should recognize that the integrating factor is e^{-x}. We can therefore write

$$\int_0^1 |f'(x)-f(x)|\,dx=\int_0^1 \left|f'(x)e^{-x}-f(x)e^{-x}\right|e^x dx=\int_0^1 \left|\left(f(x)e^{-x}\right)'\right|e^x dx$$
$$\geq \int_0^1 \left(f(x)e^{-x}\right)' \mid dx=f(1)e^{-1}-f(0)e^{-0}=\frac{1}{e}$$

We have found a lower bound. We will prove that it is the greatest lower bound. Define $f_a : [0,1] \to \mathbb{R}$

$$f_a(x) = \begin{cases} \frac{e^{a-1}}{a}x & \text{for } x \in [0,a] \\ e^{x-1} & \text{for } x \in [a,1] \end{cases}$$

The functions f_a are continuous but not differentiable at a, but we can smooth this "corner"' without altering too much the function or its derivative. Ignoring this problem, we can write

$$\int_0^1 |f_a'(x) - f_a(x)|\, dx = \int_0^a \left| \frac{e^{a-1}}{a} - \frac{e^{a-1}}{a}x \right| dx = \frac{e^{a-1}}{a}\left(a - \frac{a^2}{2}\right) = e^{a-1}\left(1 - \frac{a}{2}\right)$$

As $a \to 0$, this expression approaches $\frac{1}{e}$. This proves that $\frac{1}{e}$ is the desired greatest lower bound.(41st W.L. Putnam Mathematical Competition, 1980)

3 Geometry and Trigonometry

Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :575. Let $\vec{a}, \vec{b}, \vec{c}$ be vectors such that $\vec{b}$ and $\vec{c}$ are perpendicular, but $\vec{a}$ and $\vec{b}$ are not. Let m be a real number. Solve the system

$$\vec{x} \cdot \vec{a} = m,$$
$$\vec{x} \times \vec{b} = \vec{c}.$$

Solution :575. Multiply the second equation on the left by $\vec{a}$ to obtain

$$\vec{a} \times (\vec{x} \times \vec{b}) = \vec{a} \times \vec{c}.$$

Using the formula for the double cross-product, also known as the cab-bac formula, we transform this into

$$(\vec{a} \cdot \vec{b})\vec{x} - (\vec{a} \cdot \vec{x})\vec{b} = \vec{a} \times \vec{c}$$

Hence the solution to the equation is

$$\vec{x} = \frac{m}{\vec{a} \cdot \vec{b}}\vec{b} + \frac{1}{\vec{a} \cdot \vec{b}}\vec{a} \times \vec{c}.$$

(C. Coşniţă, I. Sager, I. Matei, I. Dragotă, Culegere de probleme de Geometrie Analitica (Collection of Problems in Analytical Geometry), Editura Didactică şi Pedagogică, Bucharest, 1963)
Topic :Geometry
Book :Putnam and Beyond

Final Answer :

Problem Statement :578. Find the vector-valued functions $\vec{u}$ (t) satisfying the differential equation

$$\vec{u} \times \vec{u}' = \vec{v},$$

where $\vec{v} = \vec{v}(t)$ is a twice-differentiable vector-valued function such that both $\vec{v}$ and $\vec{v}'$ are never zero or parallel.

Solution :578. Differentiating the equation from the statement, we obtain

$$\vec{u}' \times \vec{u}' + \vec{u} \times \vec{u}'' = \vec{u} \times \vec{u}'' = \vec{v}'$$

It follows that the vectors $\vec{u}$ and $\vec{v}'$ are perpendicular. But the original equation shows that $\vec{u}$ and $\vec{v}$ are also perpendicular, which means that $\vec{u}$ stays parallel to $\vec{v} \times \overrightarrow{v'}$. Then we can write $\vec{u} = f\vec{v} \times \vec{v}$ ' for some scalar function $f = f(t)$. The left-hand side of the original equation is therefore equal to

$$f\left(\vec{v} \times \vec{v}'\right) \times \left[f'\vec{v} \times \overrightarrow{v'} + f\vec{v}' \times \vec{v}' + f\vec{v} \times \vec{v}''\right]$$

$$= f^2\left(\vec{v} \times \vec{v}'\right) \times \left(\vec{v} \times \vec{v}''\right).$$

By the $cab - bac$ formula this is further equal to

$$f^2\left(\vec{v}'' \cdot \left(\vec{v} \times \vec{v}'\right)\vec{v} - \vec{v} \cdot \left(\vec{v} \times \vec{v}'\right)\vec{v}\right) = f^2\left(\left(\vec{v} \times \vec{v}'\right) \cdot \vec{v}''\right)\vec{v}.$$

The equation reduces therefore to

$$f^2\left(\left(\vec{v} \times \vec{v}'\right) \cdot \vec{v}''\right)\vec{v} = \vec{v}.$$

By hypothesis $\vec{v}$ is never equal to $\overrightarrow{0}$, so the above equality implies

$$f = \frac{1}{\sqrt{\left(\vec{v} \times \vec{v}'\right) \cdot \vec{v}''}}.$$

So the equation can be solved only if the frame $(\vec{v}, \vec{v}', \vec{v}'')$ consists of linearly independent vectors and is positively oriented and in that case the solution is

$$\vec{u} = \frac{1}{\sqrt{\operatorname{Vol}\left(\vec{v}, \vec{v}', \vec{v}''\right)}}\vec{v} \times \vec{v}',$$

where $\operatorname{Vol}(\vec{v}, \vec{v}', \vec{v}'')$ denotes the volume of the parallelepiped determined by the three vectors.(Revista Matematică din Timişoara (Timişoara Mathematics Gazette), proposed by M. Ghermănescu)

Topic :Geometry

Book :Putnam and Beyond

Final Answer :

Problem Statement :579. Does there exist a bijection f of (a) a plane with itself or (b) three-dimensional space with itself such that for any distinct points

A, B the lines AB and $f(A)f(B)$ are perpendicular?

Solution :579. (a) Yes: simply rotate the plane $90°$ about some axis perpendicular to it. For example, in the xy-plane we could map each point (x, y) to the point $(y, -x)$.(b) Suppose such a bijection existed. In vector notation, the given condition states that

$$(\vec{a} - \vec{b}) \cdot (f(\vec{a}) - f(\vec{b})) = 0$$

for any three-dimensional vectors $\vec{a}$ and $\vec{b}$.Assume without loss of generality that f maps the origin to itself; otherwise, $g(\vec{p}) = f(\vec{p}) - f(\overrightarrow{0})$ is still a bijection and still satisfies the above equation. Plugging $\vec{b} = (0, 0, 0)$ into the above equation, we obtain that $\vec{a} \cdot f(\vec{a}) = 0$ for all $\vec{a}$. The equation reduces to

$$\vec{a} \cdot f(\vec{b}) - \vec{b} \cdot f(\vec{a}) = 0$$

Given any vectors $\vec{a}, \vec{b}, \vec{c}$ and any real numbers m, n, we then have

$$m(\vec{a} \cdot f(\vec{b}) + \vec{b} \cdot f(\vec{a})) = 0,$$

$$n(\vec{a} \cdot f(\vec{c}) + \vec{c} \cdot f(\vec{a})) = 0,$$
$$a \cdot f(m\vec{b} + n\vec{c}) + (m\vec{b} + n\vec{c}) \cdot f(\vec{a}) = 0.$$

Adding the first two equations and subtracting the third gives

$$\vec{a} \cdot (mf(\vec{b}) + nf(\vec{c}) - f(m\vec{b} + n\vec{c})) = 0.$$

Because this is true for any vector $\vec{a}$, we must have

$$f(m\vec{b} + n\vec{c}) = mf(\vec{b}) + nf(\vec{c}).$$

Therefore, f is linear, and it is determined by the images of the unit vectors $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$. If

$$f(\vec{i}) = (a_1, a_2, a_3), \quad f(\vec{j}) = (b_1, b_2, b_3), \quad \text{and} \quad f(\vec{k}) = (c_1, c_2, c_3),$$

then for a vector $\vec{x}$ we have

$$f(\vec{x}) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \vec{x}.$$

Substituting in $f(\vec{a}) \cdot \vec{a} = 0$ successively $\vec{a} = \vec{i}, \vec{j}, \vec{k}$, we obtain $a_1 = b_2 = c_3 = 0$. Then substituting in $\vec{a} \cdot f(\vec{b}) + \vec{b} \cdot f(\vec{a}), (\vec{a}, \vec{b}) = (\vec{i}, \vec{j}), (\vec{j}, \vec{k}), (\vec{k}, \vec{i})$, we obtain $b_1 = -a_2, c_2 = -b_3, c_1 = -a_3$.Setting $k_1 = c_2, k_2 = -c_1$, and $k_3 = b_1$ yields

$$f\left(k_1\vec{i} + k_2\vec{j} + k_3\vec{k}\right) = k_1 f(\vec{i}) + k_2 f(\vec{j}) + k_3 f(\vec{k}) = \overrightarrow{0}.$$

Because f is injective and $f(\overrightarrow{0}) = \overrightarrow{0}$, this implies that $k_1 = k_2 = k_3 = 0$. Then $f(\vec{x}) = 0$ for all $\vec{x}$, contradicting the assumption that f was a surjection.

Therefore, our original assumption was false, and no such bijection exists.(Team Selection Test for the International Mathematical Olympiad, Belarus, 1999)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :583. Given a quadrilateral $ABCD$, consider the points A', B', C', D' on the half-lines (i.e., rays) $|AB|BC, , | \ CD$, and $| \ DA$, respectively, such that $AB = BA', BC = CB', CD = DC', DA = AD'$. Suppose now that we start with the quadrilateral $A'B'C'D'$. Using a straightedge and a compass only, reconstruct the quadrilateral $ABCD$
Solution :583. Set $\overrightarrow{v_1} = \overrightarrow{AB}, \overrightarrow{v_2} = \overrightarrow{BC}, \overrightarrow{v_3} = \overrightarrow{CD}, \overrightarrow{v_4} = \overrightarrow{DA}, \overrightarrow{u_1} = \overrightarrow{A'B'}, \overrightarrow{u_2} = \overrightarrow{B'C'}$, $\overrightarrow{u_3} = \overrightarrow{C'D'}, \overrightarrow{u_4} = \overrightarrow{D'A'}$. By examining Figure 72 we can write the system of equations

$$\begin{aligned} 2\overrightarrow{v_2} - \overrightarrow{v_1} &= \overrightarrow{u_1}, \\ 2\overrightarrow{v_3} - \overrightarrow{v_2} &= \overrightarrow{u_2}, \\ 2\overrightarrow{v_4} - \overrightarrow{v_3} &= \overrightarrow{u_3}, \\ 2\overrightarrow{v_1} - \overrightarrow{v_4} &= \overrightarrow{u_4}, \end{aligned}$$

in which the right-hand side is known. Solving, we obtain

$$\overrightarrow{v_1} = \frac{1}{15}\overrightarrow{u_1} + \frac{2}{15}\overrightarrow{u_2} + \frac{4}{15}\overrightarrow{u_3} + \frac{8}{15}\overrightarrow{u_4},$$

MATHPIX IMAGEFigure 72 and the analogous formulas for $\overrightarrow{v_2}, \overrightarrow{v_3}$, and $\overrightarrow{v_4}$. Since the rational multiple of a vector and the sum of two vectors can be constructed with straightedge and compass, we can construct the vectors $\overrightarrow{v_i}, i = 1, 2, 3, 4$. Then we take the vectors $\overrightarrow{A'B} = -\overrightarrow{v_1}, \overrightarrow{B'C} = -\overrightarrow{v_2}$, $\overrightarrow{C'D} = -\overrightarrow{v_3}$, and $\overrightarrow{D'A} = -\overrightarrow{v_4}$ from the points A', B', C', and D' to recover the vertices B, C, D, and A.Remark. Maybe we should elaborate more on how one effectively does these constructions. The sum of two vectors is obtained by constructing the parallelogram they form. Parallelograms can also be used to translate vectors. An integer multiple of a vector can be constructed by drawing its line of support and then measuring several lengths of the vector with the compass. This construction enables us to obtain segments divided into an arbitrary number of equal parts. In order to divide a given segment into equal parts, form a triangle with it and an already divided segment, then draw lines parallel to the third side and use Thales' theorem.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :591. Find the locus of points P in the interior of a triangle ABC such that the distances from P to the lines AB, BC, and CA are the side lengths of some triangle.
Solution :591. Denote by $\delta(P, MN)$ the distance from P to the line MN. The

problem asks for the locus of points P for which the inequalities

$$\delta(P, AB) < \delta(P, BC) + \delta(P, CA),$$
$$\delta(P, BC) < \delta(P, CA) + \delta(P, AB),$$
$$\delta(P, CA) < \delta(P, AB) + \delta(P, BC)$$

are simultaneously satisfied.Let us analyze the first inequality, written as $f(P) = \delta(P, BC) + \delta(P, CA) - \delta(P, AB) > 0$. As a function of the coordinates (x, y) of P, the distance from P to a line is of the form $mx+ny+p$. Combining three such functions, we see that $f(P) = f(x, y)$ is of the same form, $f(x, y) = \alpha x+\beta y+\gamma$. To solve the inequality $f(x, y) > 0$ it suffices to find the line $f(x, y) = 0$ and determine on which side of the line the function is positive. The line intersects the side BC where $\delta(P, CA) = \delta(P, AB)$, hence at the point E where the angle bisector from A intersects this side. It intersects side CA at the point F where the bisector from B intersects the side. Also, $f(x, y) > 0$ on side AB, hence on the same side of the line EF as the segment AB. Arguing similarly for the other two inequalities, we deduce that the locus is the interior of the triangle formed by the points where the angle bisectors meet the opposite sides.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :592. Let $A_1, A_2, \ldots, A_n$ be distinct points in the plane, and let m be the number of midpoints of all the segments they determine. What is the smallest value that m can have?
Solution :592. Consider an affine system of coordinates such that none of the segments determined by the n points is parallel to the x-axis. If the coordinates of the midpoints are (x_i, y_i), $i = 1, 2, \ldots, m$, then $x_i \neq x_j$ for $i \neq j$. Thus we have reduced the problem to the onedimensional situation. So let $A_1, A_2, \ldots, A_n$ lie on a line in this order. The midpoints of $A_1A_2, A_1A_3, \ldots, A_1A_n$ are all distinct and different from the (also distinct) midpoints of $A_2A_n, A_3A_n, \ldots, A_{n-1}A_n$. Hence there are at least $(n-1)+(n-2) = 2n-3$ midpoints. This bound can be achieved for $A_1, A_2, \ldots, A_n$ the points $1, 2, \ldots, n$ on the real axis.(Középiskolai Matematikai Lapok (Mathematics Magazine for High Schools, Budapest), proposed by M. Salát)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :594. In a planar Cartesian system of coordinates consider a fixed point $P(a, b)$ and a variable line through P. Let A be the intersection of the line with the x-axis. Connect A with the midpoint B of the segment OP (O being the origin), and through C, which is the point of intersection of this line with the y-axis, take the parallel to OP. This parallel intersects PA at M. Find the locus of M as the line varies.
Solution :594. We refer everything to Figure 74 . Let $A(c, 0), c$ being the

parameter that determines the variable line. Because B has the coordinates $\left(\frac{a}{2}, \frac{b}{2}\right)$, the line AB is given by the equation

$$y=\frac{b}{a-2c}x+\frac{bc}{2c-a}.$$

MATHPIX IMAGEFigure 74Hence C has coordinates $\left(0, \frac{bc}{2c-a}\right)$.The slope of the line CM is $\frac{b}{a}$, so the equation of this line is

$$y=\frac{b}{a}x+\frac{bc}{2c-a}.$$

Intersecting it with AP, whose equation is

$$y=\frac{b}{a-c}x+\frac{bc}{c-a}$$

we obtain M of coordinates $\left(\frac{ac}{2c-a}, \frac{2bc}{2c-a}\right)$. This point lies on the line $y=\frac{2b}{a}x$, so this line might be the locus.One should note, however, that $A=O$ yields an ambiguous construction, so the origin should be removed from the locus. On the other hand, any (x, y) on this line yields a point c, namely, $c=\frac{ax}{2x-a}$, except for $x=\frac{a}{2}$. Hence the locus consists of the line of slope $\frac{2b}{a}$ through the origin with two points removed.(A. Myller, Geometrie Analitică (Analytical Geometry), 3rd ed., Editura Didactică şi Pedagogică, Bucharest, 1972)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :596. Find all pairs of real numbers (p, q) such that the inequality

$$\left|\sqrt{1-x^2}-px-q\right| \leq \frac{\sqrt{2}-1}{2}$$

holds for every $x \in[0,1]$.
Solution :596. The inequality from the statement can be rewritten as

$$-\frac{\sqrt{2}-1}{2} \leq \sqrt{1-x^2}-(px+q) \leq \frac{\sqrt{2}-1}{2},$$

or

$$\sqrt{1-x^2}-\frac{\sqrt{2}-1}{2} \leq px+q \leq \sqrt{1-x^2}+\frac{\sqrt{2}-1}{2}.$$

Let us rephrase this in geometric terms. We are required to include a segment $y=px+q$, $0 \leq x \leq 1$, between two circular arcs.The arcs are parts of two circles of radius 1 and of centers $O_1\left(0, \frac{\sqrt{2}-1}{2}\right)$ and $\mathrm{O}_2\left(0,-\frac{\sqrt{2}-1}{2}\right)$. By examining Figure 76 we will conclude that there is just one such segment. On the first circle, consider the points $A\left(1, \frac{\sqrt{2}-1}{2}\right)$ and $B\left(0, \frac{\sqrt{2}+1}{2}\right)$. The distance from B to O_2

is $\sqrt{2}$, which is equal to the length of the segment AB. In the isosceles triangle BO_2 A, the altitudes from O_2 and A must be equal. The altitude from A is equal to the distance from A to the y-axis, hence is 1 . Thus the distance from O_2 to AB is 1 as well. This shows that the segment AB is tangent to the circle centered at O_2. This segment lies between the two arcs, and above the entire interval $[0,1]$. Being inscribed in one arc and tangent to the other, it is the only segment with this property.This answers the problem, by showing that the only possibility is $p=-1, q=\frac{\sqrt{2}+1}{2}$.MATHPIX IMAGEFigure 76(Romanian Team Selection Test for the International Mathematical Olympiad, 1983)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :600. On the sides of a convex quadrilateral $ABCD$ one draws outside the equilateral triangles ABM and CDP and inside the equilateral triangles BCN and ADQ. Describe the shape of the quadrilateral $MNPQ$.
Solution :600. With the convention that the lowercase letter denotes the complex coordinate of the point denoted by the same letter in uppercase, we translate the geometric conditions from the statement into the algebraic equations

$$\frac{m-a}{b-a}=\frac{n-c}{b-c}=\frac{p-c}{d-c}=\frac{q-a}{d-a}=\epsilon$$

where $\epsilon=\cos \frac{\pi}{3}+i \sin \frac{\pi}{3}$. Therefore,

$$\begin{aligned} & m=a+(b-a) \epsilon, n=c+(b-c) \epsilon, \\ & p=c+(d-c) \epsilon, q=a+(d-a) \epsilon . \end{aligned}$$

It is now easy to see that $\frac{1}{2}(m+p)=\frac{1}{2}(n+q)$, meaning that MP and NQ have the same midpoint. So either the four points are collinear, or they form a parallelogram.(short list of the 23rd International Mathematical Olympiad, 1982)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :602. Let $A_1 A_2 \ldots A_n$ be a regular polygon with circumradius equal to 1 . Find the maximum value of $\prod_{k=1}^n P A_k$ as P ranges over the circumcircle.
Solution :602. In the language of complex numbers we are required to find the maximum of $\prod_{k=1}^n\left|z-\epsilon^k\right|$ as z ranges over the unit disk, where $\epsilon=\cos \frac{2 \pi}{n}+i \sin \frac{2 \pi}{n}$. We have

$$\prod_{k=1}^n\left|z-\epsilon^k\right|=\left|\prod_{k=1}^n\left(z-\epsilon^k\right)\right|=\left|z^n-1\right| \leq\left|z^n\right|+1=2 .$$

The maximum is 2 , attained when z is an nth root of -1.(Romanian Mathematics Competition "Grigore Moisil," 1992, proposed by D. Andrica)

Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :604. Consider a circle of diameter AB and center O, and the tangent t at B. A variable tangent to the circle with contact point M intersects t at P. Find the locus of the point Q where the line OM intersects the parallel through P to the line AB.
Solution :604. First solution: We assume that the radius of the circle is equal to 1 . Set the origin at B with BA the positive x-semiaxis and t the y-axis (see Figure 78). If $\angle BOM=\theta$, then $BP=PM=\tan\frac{\theta}{2}$. In triangle $PQM, PQ=\tan\frac{\theta}{2}/\sin\theta$. So the coordinates of Q are

$$\left(\frac{\tan\frac{\theta}{2}}{\sin\theta}, \tan\frac{\theta}{2}\right)=\left(\frac{1}{1+\cos\theta}, \frac{\sin\theta}{1+\cos\theta}\right).$$

The x and y coordinates are related as follows:

$$\left(\frac{\sin\theta}{1+\cos\theta}\right)^2=\frac{1-\cos^2\theta}{(1+\cos\theta)^2}=\frac{1-\cos\theta}{1+\cos\theta}=2\frac{1}{1+\cos\theta}-1.$$

Hence the locus of Q is the parabola $y^2=2x-1$.MATHPIX IMAGEFigure 78Second solution: With $\angle BOM=\theta$ we have $\angle POM=\angle POB=\frac{\theta}{2}$. Since PQ is parallel to OB, it follows that $\angle OPQ=\frac{\theta}{2}$. So the triangle OPQ is isosceles, and therefore $QP=OQ$. We conclude that Q lies on the parabola of focus O and directrix t. A continuity argument shows that the locus is the entire parabola.(A. Myller, Geometrie Analitică (Analytical Geometry), 3rd ed., Editura Didactică şi Pedagogică, Bucharest, 1972, solutions found by the students from the Mathematical Olympiad Summer Program, 2004)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :608. Consider the parabola $y^2=4px$. Find the locus of the points such that the tangents to the parabola from those points make a constant angle ϕ.
Solution :608. The condition that a line through (x_0, y_0) be tangent to the parabola is that the system

$$\begin{gathered}y^2=4px,\\ y-y_0=m\left(x-x_0\right)\end{gathered}$$

have a unique solution. This means that the discriminant of the quadratic equation in x obtained by eliminating $y, \left(mx-mx_0+y_0\right)^2-4px=0$, is equal to zero. This translates into the condition

$$m^2x_0-my_0+p=0.$$

The slopes m of the two tangents are therefore the solutions to this quadratic equation. They satisfy

$$m_1 + m_2 = \frac{y_0}{x_0},$$
$$m_1 m_2 = \frac{p}{x_0}.$$

We also know that the angle between the tangents is ϕ. We distinguish two situations. First, if $\phi = 90^\circ$, then $m_1 m_2 = -1$. This implies $\frac{p}{x_0} = -1$, so the locus is the line $x = -p$, which is the directrix of the parabola.If $\phi \neq 90^\circ$, then

$$\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{m_1 - m_2}{1 + \frac{p}{x_0}}.$$

We thus have

$$m_1 + m_2 = \frac{y_0}{x_0},$$
$$m_1 - m_2 = \tan\phi + \frac{p}{x_0}\tan\phi.$$

We can compute $m_1 m_2$ by squaring the equations and then subtracting them, and we obtain

$$m_1 m_2 = \frac{y_0^2}{4x_0^2} - \left(1 + \frac{p}{x_0}\right)^2 \tan^2\phi.$$

This must equal $\frac{p}{x_0}$. We obtain the equation of the locus to be

$$-y^2 + (x+p)^2 \tan^2\phi + 4px = 0,$$

which is a hyperbola. One branch of the hyperbola contains the points from which the parabola is seen under the angle ϕ, and one branch contains the points from which the parabola is seen under an angle equal to the suplement of ϕ.(A. Myller, Geometrie Analitic ă (Analytical Geometry), 3rd ed., Editura Didactică şi Pedagogică, Bucharest, 1972)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :609. Let T_1, T_2, T_3 be points on a parabola, and t_1, t_2, t_3 the tangents to the parabola at these points. Compute the ratio of the area of triangle $T_1T_2T_3$ to the area of the triangle determined by the tangents.
Solution :609. Choose a Cartesian system of coordinates such that the equation of the parabola is $y^2 = 4px$. The coordinates of the three points are $T_i\left(4p\alpha_i^2, 4p\alpha_i\right)$, for appropriately chosen $\alpha_i, i = 1, 2, 3$. Recall that the equation of the tangent to the parabola at a point (x_0, y_0) is $yy_0 = 2p\,(x + x_0)$. In our situation the three tangents are given by

$$2\alpha_i y = x + 4p\alpha_i^2, \quad i = 1, 2, 3.$$

If P_{ij} is the intersection of t_i and t_j, then its coordinates are $(4p\alpha_i\alpha_j, 2p(\alpha_i+\alpha_j))$. The area of triangle $T_1T_2T_3$ is given by a Vandermonde determinant:

$$\pm\frac{1}{2}\begin{vmatrix} 4p\alpha_1^2 & 4p\alpha_1 & 1 \\ 4p\alpha_2^2 & 4p\alpha_2 & 1 \\ 4p\alpha_3^2 & 4p\alpha_3 & 1 \end{vmatrix} = \pm 8p^2 \begin{vmatrix} \alpha_1^2 & \alpha_1 & 1 \\ \alpha_2^2 & \alpha_2 & 1 \\ \alpha_3^2 & \alpha_3 & 1 \end{vmatrix} = 8p^2\left|(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)\right|.$$

The area of the triangle $P_{12}P_{23}P_{31}$ is given by

$$\begin{aligned} &\pm\frac{1}{2}\begin{vmatrix} 4p\alpha_1\alpha_2 & 2p(\alpha_1+\alpha_2) & 1 \\ 4p\alpha_2\alpha_3 & 2p(\alpha_2+\alpha_3) & 1 \\ 4p\alpha_3\alpha_1 & 2p(\alpha_3+\alpha_1) & 1 \end{vmatrix} \\ &= \pm 4p^2 \begin{vmatrix} \alpha_1\alpha_2 & (\alpha_1+\alpha_2) & 1 \\ \alpha_2\alpha_3 & (\alpha_2+\alpha_3) & 1 \\ \alpha_3\alpha_1 & (\alpha_3+\alpha_1) & 1 \end{vmatrix} = \pm 4p^2 \begin{vmatrix} (\alpha_1-\alpha_3)\alpha_2 & (\alpha_1-\alpha_3) & 0 \\ (\alpha_2-\alpha_1)\alpha_3 & (\alpha_2-\alpha_1) & 0 \\ \alpha_3\alpha_1 & (\alpha_3+\alpha_1) & 1 \end{vmatrix} \\ &= 4p^2\left|(\alpha_1-\alpha_3)(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)\right|. \end{aligned}$$

We conclude that the ratio of the two areas is 2 , regardless of the location of the three points or the shape of the parabola.(Gh. Călugăriţa, V. Mangu, Probleme de Matematică pentru Treapta I şi a II-a de Liceu (Mathematics Problems for High School), Editura Albatros, Bucharest, 1977)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :611. Find all regular polygons that can be inscribed in an ellipse with unequal semiaxes.
Solution :611. An equilateral triangle can be inscribed in any closed, non-self-intersecting curve, therefore also in an ellipse. The argument runs as follows. Choose a point A on the ellipse. Rotate the ellipse around A by 60°. The image of the ellipse through the rotation intersects the original ellipse once in A, so it should intersect it at least one more time. Let B an be intersection point different from A. Note that B is on both ellipses, and its preimage C through rotation is on the original ellipse. The triangle ABC is equilateral.A square can also be inscribed in the ellipse. It suffices to vary an inscribed rectangle with sides parallel to the axes of the ellipse and use the intermediate value property.Let us show that these are the only possibilities. Up to a translation, a rotation, and a dilation, the equation of the ellipse has the form

$$x^2+ay^2=b, \quad \text{with } a,b>0, a\neq 1.$$

Assume that a regular n-gon, $n\geq 5$, can be inscribed in the ellipse. Its vertices (x_i,y_i) satisfy the equation of the circumcircle:

$$x^2+y^2+cx+dy+e=0, \quad i=1,2,\ldots,n.$$

Writing the fact that the vertices also satisfy the equation of the ellipse and subtracting, we obtain $(1-a)y_i^2+cx_i+dy_i+(e+b)=0$. Hence

$$y_i^2=-\frac{c}{1-a}x_i-\frac{d}{1-a}y_i-\frac{e+b}{1-a}.$$

The number c cannot be 0 , for otherwise the quadratic equation would have two solutions y_i and each of these would yield two solutions x_i, so the polygon would have four or fewer sides, a contradiction. This means that the regular polygon is inscribed in a parabola. Change the coordinates so that the parabola has the standard equation $y^2 = 4px$. Let the new coordinates of the vertices be (ξ_i, η_i) and the new equation of the circumcircle be $x^2 + y^2 + c'x + d'y + e' = 0$. That the vertices belong to both the parabola and the circle translates to

$$\eta_i^2 = 4p\xi_i \quad \text{and} \quad \xi_i^2 + \eta_i^2 + c'\xi + d'\eta + e' = 0, \quad \text{for } i = 1, 2, \ldots, n.$$

So the η_i 's satisfy the fourth-degree equation

$$\frac{1}{16p^2}\eta_i^4 + \eta_i^2 + \frac{c'}{4p}\eta_i^2 + d'\eta_i + e' = 0.$$

This equation has at most four solutions, and each solution yields a unique x_i. So the regular polygon can have at most four vertices, a contradiction. We conclude that no regular polygon with five or more vertices can be inscribed in an ellipse that is not also a circle.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :614. Compute the integral

$$\int \frac{dx}{a + b\cos x + c\sin x},$$

where a, b, c are real numbers, not all equal to zero.
Solution :614. The interesting case occurs of course when b and c are not both equal to zero. Set $d = \sqrt{b^2 + c^2}$ and define the angle α by the conditions $\cos\alpha = \frac{b}{\sqrt{b^2+c^2}}$ and $\sin\alpha = \frac{c}{\sqrt{b^2+c^2}}$. The integral takes the form

$$\int \frac{dx}{a + d\cos(x - \alpha)},$$

which, with the substitution $u = x - \alpha$, becomes the simpler

$$\int \frac{du}{a + d\cos u}.$$

The substitution $t = \tan\frac{u}{2}$ changes this into

$$\frac{2}{a + d}\int \frac{dt}{1 + \frac{a-d}{a+d}t^2}.$$

If $a = d$ the answer to the problem is $\frac{1}{a}\tan\frac{x-\alpha}{2} + C$. If $\frac{a-d}{a+d} > 0$, the answer is

$$\frac{2}{\sqrt{a^2 - d^2}}\arctan\left(\sqrt{\frac{a-d}{a+d}}\tan\frac{x-\alpha}{2} + C\right),$$

while if $\frac{a-d}{a+d}<0$, the answer is

$$\frac{1}{\sqrt{d^{2}-a^{2}}} \ln \left|\frac{1+\sqrt{\frac{d-a}{d+a}} \tan \frac{x-\alpha}{2}}{1-\sqrt{\frac{d-a}{d+a}} \tan \frac{x-\alpha}{2}}\right|+C .$$

Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :616. Find the points where the tangent to the cardioid $r=1+\cos \theta$ is vertical.
Solution :616. We convert to Cartesian coordinates, obtaining the equation of the cardioid

$$\sqrt{x^{2}+y^{2}}=1+\frac{x}{\sqrt{x^{2}+y^{2}}},$$

or

$$x^{2}+y^{2}=\sqrt{x^{2}+y^{2}}+x .$$

By implicit differentiation, we obtain

$$2 x+2 y \frac{d y}{d x}=\left(x^{2}+y^{2}\right)^{-1 / 2}\left(x+y \frac{d y}{d x}\right)+1,$$

which yields

$$\frac{d y}{d x}=\frac{-2 x+x\left(x^{2}+y^{2}\right)^{-1 / 2}+1}{2 y-y\left(x^{2}+y^{2}\right)^{-1 / 2}} .$$

The points where the tangent is vertical are among those where the denominator cancels. Solving $2 y-y\left(x^{2}+y^{2}\right)^{-1 / 2}=0$, we obtain $y=0$ or $x^{2}+y^{2}=\frac{1}{4}$. Combining this MATHPIX IMAGEFigure 79with the equation of the cardioid, we find the possible answers to the problem as $(0,0),(2,0),\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$, and $\left(-\frac{1}{4},-\frac{\sqrt{3}}{4}\right)$. Of these the origin has to be ruled out, since there the cardioid has a corner, while the other three are indeed points where the tangent to the cardioid is vertical.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :617. Given a circle of diameter $A B$, a variable secant through A intersects the circle at C and the tangent through B at D. On the half-line $A C$ a point M is chosen such that $A M=C D$. Find the locus of M.
Solution :617. Let $A B=a$ and consider a system of polar coordinates with pole A and axis $A B$. The equation of the curve traced by M is obtained as

follows. We have $AM = r$, $AD = \frac{a}{\cos\theta}$, and $AC = a\cos\theta$. The equality $AM = AD - AC$ yields the equation

$$r = \frac{a}{\cos\theta} - a\cos\theta.$$

The equation of the locus is therefore $r = \frac{a\sin^2\theta}{\cos\theta}$. This curve is called the cisoid of Diocles (Figure 80).
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :618. Find the locus of the projection of a fixed point on a circle onto the tangents to the circle.
Solution :618. Let O be the center and a the radius of the circle, and let M be the point on the circle. Choose a system of polar coordinates with M the pole and MO the axis. For an arbitrary tangent, let I be its intersection with MO, T the tangency point, and P the projection of M onto the tangent. Then

$$OI = \frac{OT}{\cos\theta} = \frac{a}{\cos\theta}.$$

Hence

$$MP = r = (MO + OI)\cos\theta = \left(a + \frac{a}{\cos\theta}\right)\cos\theta.$$

We obtain $r = a(1+\cos\theta)$, which is the equation of a cardioid (Figure 80).
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :619. On a circle of center O consider a fixed point A and a variable point M. The circle of center A and radius AM intersects the line OM at L. Find the locus of L as M varies on the circle.
Solution :619. Working with polar coordinates we place the pole at O and axis OA. Denote by a the radius of the circle. We want to find the relation between the polar coordinates MATHPIX IMAGEFigure $80(r,\theta)$ of the point L. We have $AM = AL = 2a\sin\frac{\theta}{2}$. In the isosceles triangle LAM, $\angle LMA = \frac{\pi}{2} - \frac{\theta}{2}$; hence

$$LM = 2AM\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = 2\cdot 2a\sin\frac{\theta}{2}\cdot\sin\frac{\theta}{2} = 4a\sin^2\frac{\theta}{2}.$$

Substituting this in the relation $OL = OM - LM$, we obtain

$$r = a - 4a\sin^2\frac{\theta}{2} = a[1 - 2\cdot(1-\cos\theta)].$$

The equation of the locus is therefore

$$r = a(2\cos\theta - 1),$$

a curve known as Pascal's snail, or limaçon, whose shape is described in Figure 81.MATHPIX IMAGEFigure 81
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :620. The endpoints of a variable segment AB lie on two perpendicular lines that intersect at O. Find the locus of the projection of O onto AB, provided that the segment AB maintains a constant length.
Solution :620. As before, we work with polar coordinates, choosing O as the pole and OA as the axis. Denote by a the length of the segment AB and by $P(r, \theta)$ the projection of O onto this segment. Then $OA = \frac{r}{\cos\theta}$ and $OA = AB \sin\theta$, which yield the equation of the locus

$$r = a \sin\theta \cos\theta = \frac{a}{2} \sin 2\theta.$$

This is a four-leaf rose.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :621. From the center of a rectangular hyperbola a perpendicular is dropped to a variable tangent. Find the locus in polar coordinates of the foot of the perpendicular. (A hyperbola is called rectangular if its asymptotes are perpendicular.)
Solution :621. Choosing a Cartesian system of coordinates whose axes are the asymptotes, we can bring the equation of the hyperbola into the form $xy = a^2$. The equation of the tangent to the hyperbola at a point (x_0, y_0) is $x_0 y + y_0 x - 2a^2 = 0$. Since $a^2 = x_0 y_0$, the x and y intercepts of this line are $2x_0$ and $2y_0$, respectively.Let (r, θ) be the polar coordinates of the foot of the perpendicular from the origin to the tangent. In the right triangle determined by the center of the hyperbola and the two intercepts we have $2x_0 \cos\theta = r$ and $2y_0 \sin\theta = r$. Multiplying, we obtain the polar equation of the locus

$$r^2 = 2a^2 \sin 2\theta.$$

This is the lemniscate of Bernoulli, shown in Figure 82.(1st W.L. Putnam Mathematical Competition, 1938)MATHPIX IMAGEFigure 82
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :622. Find a transformation of the plane that maps the unit circle $x^2 + y^2 = 1$ into a cardioid. (Recall that the general equation of a cardioid is $r = 2a(1 + \cos\theta)$.)
Solution :622. The solution uses complex and polar coordinates. Our goal is

to map the circle onto a cardioid of the form

$$r = a(1 + \cos\theta), \quad a > 0.$$

Because this cardioid passes through the origin, it is natural to work with a circle that itself passes through the origin, for example $|z - 1| = 1$. If $\phi : \mathbb{C} \to \mathbb{C}$ maps this circle into the cardioid, then the equation of the cardioid will have the form

$$\left|\phi^{-1}(z) - 1\right| = 1.$$

So we want to bring the original equation of the cardioid into this form. First, we change it to

$$r = a \cdot 2\cos^2\frac{\theta}{2};$$

then we take the square root,

$$\sqrt{r} = \sqrt{2a}\cos\frac{\theta}{2}.$$

Multiplying by $\sqrt{r}$, we obtain

$$r = \sqrt{2a}\sqrt{r}\cos\frac{\theta}{2},$$

or

$$r - \sqrt{2a}\sqrt{r}\cos\frac{\theta}{2} = 0.$$

This should look like the equation of a circle. We modify the expression as follows:

$$\begin{aligned} r - \sqrt{2a}\sqrt{r}\cos\frac{\theta}{2} &= r\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) - \sqrt{2a}\sqrt{r}\cos\frac{\theta}{2} + 1 - 1 \\ &= \left(\sqrt{r}\cos\frac{\theta}{2}\right)^2 - \sqrt{2a}\sqrt{r}\cos\frac{\theta}{2} + 1 + \left(\sqrt{r}\sin\frac{\theta}{2}\right)^2 - 1. \end{aligned}$$

If we set $a = 2$, we have a perfect square, and the equation becomes

$$\left(\sqrt{r}\cos\frac{\theta}{2} - 1\right)^2 + \left(\sqrt{r}\sin\frac{\theta}{2}\right)^2 = 1,$$

which in complex coordinates reads $|\sqrt{z} - 1| = 1$. Of course, there is an ambiguity in taking the square root, but we are really interested in the transformation ϕ, not in ϕ^{-1}. Therefore, we can choose $\phi(z) = z^2$, which maps the circle $|z - 1| = 1$ into the cardioid $r = 2(1 + \cos\theta)$.Remark. Of greater practical importance is the Zhukovski transformation $z \to \frac{1}{2}\left(z + \frac{1}{z}\right)$, which maps the unit circle onto the profile of the airplane wing (the so-called aerofoil). Because the Zhukovski map preserves angles, it helps reduce the study of the air flow around an airplane wing to the much simpler study of the air flow around a circle.
Topic :Geometry

Book :Putnam and Beyond
Final Answer :

Problem Statement :625. What is the equation that describes the shape of a hanging flexible chain with ends supported at the same height and acted on by its own weight?
Solution :625. Let the equation of the curve be $y(x)$. Let $T(x)$ be the tension in the chain at the point $(x, y(x))$. The tension acts in the direction of the derivative $y'(x)$. Let $H(x)$ and $V(x)$ be, respectively, the horizontal and vertical components of the tension. Because the chain is in equilibrium, the horizontal component of the tension is constant at all points of the chain (just cut the chain mentally at two different points). Thus $H(x) = H$. The vertical component of the tension is then $V(x) = Hy'(x)$. On the other hand, for two infinitesimally close points, the difference in the vertical tension is given by $dV = \rho ds$, where ρ is the density of the chain and ds is the length of the arc between the two poins. Since $ds = \sqrt{1 + (y'(x))^2} dx$, it follows that y satisfies the differential equation

$$Hy'' = \rho\sqrt{1 + (y')^2}.$$

If we set $z(x) = y'(x)$, we obtain the separable first-order equation

$$Hz' = \rho\sqrt{1 + z^2}.$$

By integration, we obtain $z = \sinh\left(\frac{\rho}{H}x + C_1\right)$. The answer to the problem is therefore

$$y(x) = \frac{H}{\rho}\cosh\left(\frac{\rho}{H} + C_1\right) + C_2.$$

Remark. Galileo claimed that the curve was a parabola, but this was later proved to be false. The correct equation was derived by G.W. Leibniz, Ch. Huygens, and Johann Bernoulli. The curve is called a "catenary" and plays an important role in the theory of minimal surfaces.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :626. A cube is rotated about the main diagonal. What kind of surfaces do the edges describe?
Solution :626. An edge adjacent to the main diagonal describes a cone. For an edge not adjacent to the main diagonal, consider an orthogonal system of coordinates such that the rotation axis is the z-axis and, in its original position, the edge is parallel to the y-plane (Figure 83). In the appropriate scale, the line of support of the edge is $y = 1, z = \sqrt{3}x$.MATHPIX IMAGEFigure 83The locus of points on the surface of revolution is given in parametric form by

$$(x, y, z) = (t\cos\theta + \sin\theta, \cos\theta - t\sin\theta, \sqrt{3}t), \quad t \in \mathbb{R}, \quad \theta \in [0, 2\pi).$$

A glimpse at these formulas suggests the following computation:

$$
\begin{aligned}
& x^2+y^2-\frac{1}{3} z^2 \\
& \quad=t^2 \cos ^2 \theta+\sin ^2 \theta+2 t \sin \theta \cos \theta+\cos ^2 \theta+t^2 \sin ^2 \theta-2 t \cos \theta \sin \theta-t^2 \\
& \quad=t^2\left(\cos ^2 \theta+\sin ^2 \theta\right)+\cos ^2 \theta+\sin ^2 \theta-t^2=1 .
\end{aligned}
$$

The locus is therefore a hyperboloid of one sheet, $x^2+y^2-\frac{1}{3} z^3=1$.Remark. The fact that the hyperboloid of one sheet is a ruled surface makes it easy to build. It is a more resilient structure than the cylinder. This is why the cooling towers of power plants are built as hyperboloids of one sheet.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :629. Determine the radius of the largest circle that can lie on the ellipsoid

$$
\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad(a>b>c) .
$$

Solution :629. Figure 84 describes a generic ellipsoid. Since parallel cross-sections of the ellipsoid are always similar ellipses, any circular cross-section can be increased in size by taking a MATHPIX IMAGEFigure 84parallel cutting plane passing through the origin. Because of the condition $a>b>c$, a circular cross-section cannot lie in the $x y$-, $x z-$, or $y z$-plane. Looking at the intersection of the ellipsoid with the $y z$-plane, we see that some diameter of the circular cross-section is a diameter (segment passing through the center) of the ellipse $x=0, \frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Hence the radius of the circle is at most b. The same argument for the $x y$-plane shows that the radius is at least b, whence b is a good candidate for the maximum radius.To show that circular cross-sections of radius b actually exist, consider the intersection of the plane $\left(c \sqrt{a^2-b^2}\right) x=\left(a \sqrt{b^2-c^2}\right) z$ with the ellipsoid. We want to compute the distance from a point $\left(x_0, y_0, z_0\right)$ on this intersection to the origin. From the equation of the plane, we obtain by squaring

$$
x_0^2+z_0^2=b^2\left(\frac{x_0^2}{a^2}+\frac{z_0^2}{c^2}\right) .
$$

The equation of the ellipsoid gives

$$
y_0^2=b^2\left(1-\frac{x_0^2}{a^2}-\frac{z_0^2}{c^2}\right) .
$$

Adding these two, we obtain $x_0^2+y_0^2+z_0^2=1$; hence $\left(x_0, y_0, z_0\right)$ lies on the circle of radius 1 centered at the origin and contained in the plane $\left(c \sqrt{a^2-b^2}\right) x+\left(a \sqrt{b^2-c^2}\right) z=0$. This completes the proof.(31st W.L. Putnam Mathematical

Competition, 1970)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :635. Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two points is strictly greater than $\sqrt{2}$.
Solution :635. Consider the unit sphere in $\mathbb{R}^n$,

$$S^{n-1}=\left\{(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n \mid \sum_{k=1}^n x_k^2=1\right\}.$$

The distance between two points $X=(x_1,x_2,\ldots,x_n)$ and $Y=(y_1,y_2,\ldots,y_n)$ is given by

$$d(X,Y)=\left(\sum_{k=1}^n (x_k-y_k)^2\right)^{1/2}.$$

Note that $d(X,Y)>\sqrt{2}$ if and only if

$$d^2(X,Y)=\sum_{k=1}^n x_k^2+\sum_{k=1}^n y_k^2-2\sum_{k=1}^n x_k y_k>2.$$

Therefore, $d(X,Y)>\sqrt{2}$ implies $\sum_{k=1}^n x_k y_k<0$.Now let $A_1,A_2,\ldots,A_{m_n}$ be points satisfying the condition from the hypothesis, with m_n maximal. Using the symmetry of the sphere we may assume that $A_1=(-1,0,\ldots,0)$. Let $A_i=(x_1,x_2,\ldots,x_n)$ and $A_j=(y_1,y_2,\ldots,y_n)\,, i,j\geq 2$. Because $d\left(A_1,A_i\right)$ and $d\left(A_1,A_j\right)$ are both greater than $\sqrt{2}$, the above observation shows that x_1 and y_1 are positive.The condition $d\left(A_i,A_j\right)>\sqrt{2}$ implies $\sum_{k=1}^n x_k y_k<0$, and since $x_1 y_1$ is positive, it follows that

$$\sum_{k=2}^n x_k y_k<0.$$

This shows that if we normalize the last $n-1$ coordinates of the points A_i by

$$x_k'=\frac{x_k}{\sqrt{\sum_{k=1}^{n-1} x_k^2}},\quad k=1,2,\ldots,n-1,$$

we obtain the coordinates of point B_i in S^{n-2}, and the points $B_2,B_3,\ldots,B_n$ satisfy the condition from the statement of the problem for the unit sphere in $\mathbb{R}^{n-1}$.It follows that $m_n\leq 1+m_{n-1}$, and $m_1=2$ implies $m_n\leq n+1$. The example of the n-dimensional regular simplex inscribed in the unit sphere shows that $m_n=n+1$. To determine explicitly the coordinates of the vertices, we use the additional information that the distance from the center of the sphere to a hyperface of the n-dimensional simplex is $\frac{1}{n}$ and then find inductively

$$\begin{aligned}
A_1 &= (-1, 0, 0, 0, \ldots, 0, 0), \\
A_2 &= \left(\frac{1}{n}, -c_1, 0, 0, \ldots, 0, 0\right), \\
A_3 &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, -c_2, 0, \ldots, 0, 0\right), \\
A_4 &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, \frac{1}{n-2} \cdot c_2, c_3, \ldots, 0, 0\right), \\
&\ldots \\
A_{n-1} &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, \ldots, \frac{1}{3} \cdot c_{n-3}, -c_{n-2}, 0\right), \\
A_n &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, \ldots, \frac{1}{3} \cdot c_{n-3}, \frac{1}{2} \cdot c_{n-2}, -c_{n-1}\right) \\
A_{n+1} &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, \ldots, \frac{1}{3} \cdot c_{n-3}, \frac{1}{2} \cdot c_{n-2}, c_{n-1}\right)
\end{aligned}$$

where

$$c_k = \sqrt{\left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{n-k+1}\right)}, \quad k = 1, 2, \ldots, n-1$$

One computes that the distance between any two points is

$$\sqrt{2}\sqrt{1 + \frac{1}{n}} > \sqrt{2}$$

and the problem is solved.(8th International Mathematics Competition for University Students, 2001)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :641. In a triangle ABC for a variable point P on BC with $PB = x$ let $t(x)$ be the measure of $\angle PAB$. Compute

$$\int_0^a \cos t(x) dx$$

in terms of the sides and angles of triangle ABC.
Solution :641. The law of cosines in triangle APB gives

$$AP^2 = x^2 + c^2 - 2xc \cos B$$

and

$$x^2 = c^2 + AP^2 = x^2 + c^2 - 2xc \cos B - 2c\sqrt{x^2 + c^2 - 2xc \cos B} \cos t,$$

whence

$$\cos t = \frac{c - x\cos B}{\sqrt{x^2 + c^2 - 2xc\cos B}}$$

The integral from the statement is

$$\int_0^a \cos t(x)dx = \int_0^a \frac{c - x\cos B}{\sqrt{x^2 + c^2 - 2xc\cos B}}dx$$

Using the standard integration formulas

$$\int \frac{dx}{\sqrt{x^2 + \alpha x + \beta}} = \ln\left(2x + \alpha + 2\sqrt{x^2 + \alpha x + \beta}\right)$$

$$\int \frac{xdx}{\sqrt{x^2 + \alpha x + \beta}} = \sqrt{x^2 + \alpha x + \beta} - \frac{\alpha}{2}\ln\left(2x + \alpha + 2\sqrt{x^2 + \alpha x + \beta}\right)$$

we obtain

$$\begin{aligned}\int_0^a \cos t(x)dx = & c\sin^2 B\ln\left(2x + 2c\cos B + 2\sqrt{x^2 - 2cx\cos B + c^2}\right)\Big|_0^a \\ & - \cos B\sqrt{x^2 - 2cx\cos B + c^2}\Big|_0^a \\ = & c\sin^2 B\ln\frac{a - c\cos B + b}{c(1 - \cos B)} + \cos B(c - b)\end{aligned}$$

Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :642. Let $f : [0, a] \to \mathbb{R}$ be a continuous and increasing function such that $f(0) = 0$. Define by R the region bounded by $f(x)$ and the lines $x = a$ and $y = 0$. Now consider the solid of revolution obtained when R is rotated around the y-axis as a sort of dish. Determine f such that the volume of water the dish can hold is equal to the volume of the dish itself, this happening for all a.
Solution :642. It is equivalent to ask that the volume of the dish be half of that of the solid of revolution obtained by rotating the rectangle $0 \le x \le a$ and $0 \le y \le f(a)$. Specifically, this condition is

$$\int_0^a 2\pi x f(x)dx = \frac{1}{2}\pi a^2 f(a)$$

Because the left-hand side is differentiable with respect to a for all $a > 0$, the right-hand side is differentiable, too. Differentiating, we obtain

$$2\pi a f(a) = \pi a f(a) + \frac{1}{2}\pi a^2 f'(a).$$

This is a differential equation in f, which can be written as $f'(a)/f(a) = \frac{2}{a}$. Integrating, we obtain $\ln f(a) = 2\ln a$, or $f(a) = ca^2$ for some constant $c > 0$. This solves the problem.

4 (Math Horizons)

Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :659. Find the range of the function $f : \mathbb{R} \to \mathbb{R}, f(x) = (\sin x + 1)(\cos x + 1)$.
Solution :659. We have

$$f(x) = \sin x \cos x + \sin x + \cos x + 1 = \frac{1}{2}(\sin x + \cos x)^2 - \frac{1}{2} + \sin x + \cos x + 1$$
$$= \frac{1}{2}\left[(\sin x + \cos x)^2 + 2(\sin x + \cos x) + 1\right] = \frac{1}{2}[(\sin x + \cos x) + 1]^2.$$

This is a function of $y = \sin x + \cos x$, namely $f(y) = \frac{1}{2}(y+1)^2$. Note that

$$y = \cos\left(\frac{\pi}{2} - x\right) + \cos x = 2\cos\frac{\pi}{4}\cos\left(x - \frac{\pi}{4}\right) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right).$$

So y ranges between $-\sqrt{2}$ and $\sqrt{2}$. Hence $f(y)$ ranges between 0 and $\frac{1}{2}(\sqrt{2}+1)^2$.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :661. Compute the integral

$$\int \sqrt{\frac{1-x}{1+x}}dx, \quad x \in (-1, 1).$$

Solution :661. We would like to eliminate the square root, and for that reason we recall the trigonometric identity

$$\frac{1 - \sin t}{1 + \sin t} = \frac{\cos^2 t}{(1 + \sin t)^2}.$$

The proof of this identity is straightforward if we express the cosine in terms of the sine and then factor the numerator. Thus if we substitute $x = \sin t$, then $dx = \cos t dt$ and the integral becomes

$$\int \frac{\cos^2 t}{1 + \sin t}dt = \int 1 - \sin t dt = t + \cos t + C.$$

Since $t = \arcsin x$, this is equal to $\arcsin x + \sqrt{1 - x^2} + C$.(Romanian high school textbook)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :662. Find all integers k for which the two-variable function $f(x,y)=\cos(19x+99y)$ can be written as a polynomial in $\cos x, \cos y, \cos(x+ky)$.
Solution :662. We will prove that a function of the form $f(x,y)=\cos(ax+by), a, b$ integers, can be written as a polynomial in $\cos x, \cos y$, and $\cos(x+ky)$ if and only if b is divisible by k.For example, if $b=k$, then from

$$\cos(ax+ky)=2\cos x\cos((a\pm 1)x+ky)-\cos((a\pm 2)x+ky),$$

we obtain by induction on the absolute value of a that $\cos(ax+by)$ is a polynomial in $\cos x, \cos y, \cos(x+ky)$. In general, if $b=ck$, the identity

$$\cos(ax+cky)=2\cos y\cos(ax+(c\pm 1)ky)-\cos(ax+(c\pm 2)ky)$$

together with the fact that $\cos ax$ is a polynomial in $\cos x$ allows an inductive proof of the fact that $\cos(ax+by)$ can be written as a polynomial in $\cos x, \cos y$, and $\cos(x+ky)$ as well.For the converse, note that by using the product-to-sum formula we can write any polynomial in cosines as a linear combination of cosines. We will prove a more general statement, namely that if a linear combination of cosines is a polynomial in $\cos x, \cos y$, and $\cos(x+ky)$, then it is of the form

$$\sum_m\left[b_m\cos mx+\sum_{0\leq q<|p|}c_{m,p,q}(\cos(mx+(pk+q)y)+\cos(mx+(pk-q)y))\right].$$

This property is obviously true for polynomials of degree one, since any such polynomial is just a linear combination of the three functions. Also, any polynomial in $\cos x, \cos y, \cos(x+ky)$ can be obtained by adding polynomials of lower degrees, and eventually multiplying them by one of the three functions.Hence it suffices to show that the property is invariant under multiplication by $\cos x, \cos y$, and $\cos(x+ky)$. It can be verified that this follows from

$$\begin{aligned}2\cos(ax+by)\cos x&=\cos((a+1)x+by)+\cos((a-1)x+by)\\2\cos(ax+by)\cos y&=\cos(ax+(b+1)y)+\cos(ax+(b-1)y)\\2\cos(ax+by)\cos(x+ky)&=\cos((a+1)x+(b+k)y)+\cos(a-1)x+(b-k)y)\end{aligned}$$

So for $\cos(ax+by)$ to be a polynomial in $\cos x, \cos y$, and $\cos(x+ky)$, it must be such a sum with a single term. This can happen only if b is divisible by k.The answer to the problem is therefore $k=\pm 1, \pm 3, \pm 9, \pm 11, \pm 33, \pm 99$.(proposed by R. Gelca for the USA Mathematical Olympiad, 1999)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :668. Compute the sum

$$\binom{n}{1}\cos x+\binom{n}{2}\cos 2x+\cdots+\binom{n}{n}\cos nx.$$

Solution :668. Denote the sum in question by S_1 and let

$$S_2 = \binom{n}{1} \sin x + \binom{n}{2} \sin 2x + \cdots + \binom{n}{n} \sin nx.$$

Using Euler's formula, we can write

$$1 + S_1 + iS_2 = \binom{n}{0} + \binom{n}{1} e^{ix} + \binom{n}{2} e^{2ix} + \cdots + \binom{n}{n} e^{inx}.$$

By the multiplicative property of the exponential we see that this is equal to

$$\sum_{k=0}^{n} \binom{n}{k} \left(e^{ix}\right)^k = \left(1 + e^{ix}\right)^n = \left(2 \cos \frac{x}{2}\right)^n \left(e^{i\frac{x}{2}}\right)^n.$$

The sum in question is the real part of this expression less 1 , which is equal to

$$2^n \cos^n \frac{x}{2} \cos \frac{nx}{2} - 1.$$

Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :669. Find the Taylor series expansion at 0 of the function

$$f(x) = e^{x \cos \theta} \cos(x \sin \theta),$$

where θ is a parameter.
Solution :669. Combine $f(x)$ with the function $g(x) = e^{x \cos \theta} \sin(x \sin \theta)$ and write

$$\begin{aligned} f(x) + ig(x) &= e^{x \cos \theta}(\cos(x \sin \theta) + i \sin(x \sin \theta)) \\ &= e^{x \cos \theta} \cdot e^{ix \sin \theta} = e^{x(\cos \theta + i \sin \theta)}. \end{aligned}$$

Using the de Moivre formula we expand this in a Taylor series as

$$1 + \frac{x}{1!}(\cos \theta + i \sin \theta) + \frac{x^2}{2!}(\cos 2\theta + i \sin 2\theta) + \cdots + \frac{x^n}{n!}(\cos n\theta + i \sin n\theta) + \cdots .$$

Consequently, the Taylor expansion of $f(x)$ around 0 is the real part of this series, i.e.,

$$f(x) = 1 + \frac{\cos \theta}{1!} x + \frac{\cos 2\theta}{2!} x^2 + \cdots + \frac{\cos n\theta}{n!} x^n + \cdots .$$

Topic :Geometry
Book :Putnam and Beyond

Final Answer :

Problem Statement :672. Find $(\cos \alpha)(\cos 2\alpha)(\cos 3\alpha) \cdots (\cos 999\alpha)$ with $\alpha = \frac{2\pi}{1999}$.
Solution :672. More generally, for an odd integer n, let us compute

$$S = (\cos \alpha)(\cos 2\alpha) \cdots (\cos n\alpha)$$

with $\alpha = \frac{2\pi}{2n+1}$. We can let $\zeta = e^{i\alpha}$ and then $S = 2^{-n} \prod_{k=1}^{n} \left(\zeta^k + \zeta^{-k}\right)$. Since $\zeta^k + \zeta^{-k} = \zeta^{2n+1-k} + \zeta^{-(2n+1-k)}, k = 1, 2, \ldots, n$, we obtain

$$S^2 = 2^{-2n} \prod_{k=1}^{2n} \left(\zeta^k + \zeta^{-k}\right) = 2^{-2n} \times \prod_{k=1}^{2n} \zeta^{-k} \times \prod_{k=1}^{2n} \left(1 + \zeta^{2k}\right)$$

The first of the two products is just $\zeta^{-(1+2+\cdots+2n)}$. Because $1 + 2 + \cdots + 2n = n(2n+1)$, which is a multiple of $2n+1$, this product equals 1 .As for the product $\prod_{k=1}^{2n} \left(1 + \zeta^{2k}\right)$, note that it can be written as $\prod_{k=1}^{2n} \left(1 + \zeta^{k}\right)$, since the numbers ζ^{2k} range over the $(2n+1)$ st roots of unity other than 1 itself, taking each value exactly once. We compute this using the factorization

$$z^{n+1} - 1 = (z - 1) \prod_{k=1}^{2n} \left(z - \zeta^k\right)$$

Substituting $z = -1$ and dividing both sides by -2 gives $\prod_{k=1}^{2n} \left(-1 - \zeta^k\right) = 1$, so $\prod_{k=1}^{2n} \left(1 + \zeta^k\right) = 1$. Hence $S^2 = 2^{-2n}$, and so $S = \pm 2^{-n}$. We need to determine the sign.For $1 \leq k \leq n, \cos k\alpha < 0$ when $\frac{\pi}{2} < k\alpha < \pi$. The values of k for which this happens are $\left\lceil \frac{n+1}{2} \right\rceil$ through n. The number of such k is odd if $n \equiv 1$ or $2(\bmod 4)$, and even if $n \equiv 0$ or $3(\bmod 4)$. Hence

$$S = \begin{cases} +2^{-n} & \text{if } n \equiv 1 \text{ or } 2(\bmod 4) \\ -2^{-n} & \text{if } n \equiv 0 \text{ or } 3(\bmod 4) \end{cases}$$

Taking $n = 999 \equiv 3(\bmod 4)$, we obtain the answer to the problem, -2^{-999}.(proposed by J. Propp for the USA Mathematical Olympiad, 1999)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :676. Solve the equation $x^3 - 3x = \sqrt{x+2}$ in real numbers.
Solution :676. First, note that if $x > 2$, then $x^3 - 3x > 4x - 3x = x > \sqrt{x+2}$, so all solutions x should satisfy $-2 \leq x \leq 2$. Therefore, we can substitute $x = 2\cos a$ for some $a \in [0, \pi]$. Then the given equation becomes

$$2\cos 3a = \sqrt{2(1 + \cos a)} = 2\cos \frac{a}{2},$$

So

$$2 \sin \frac{7a}{4} \sin \frac{5a}{4} = 0$$

meaning that $a = 0, \frac{4\pi}{7}, \frac{4\pi}{5}$. It follows that the solutions to the original equation are $x = 2, 2\cos\frac{4\pi}{7}, -\frac{1}{2}(1+\sqrt{5})$.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :677. Find the maximum value of

$$S = (1-x_1)(1-y_1) + (1-x_2)(1-y_2)$$

if $x_1^2 + x_2^2 = y_1^2 + y_2^2 = c^2$, where c is some positive number.
Solution :677. The points (x_1, x_2) and (y_1, y_2) lie on the circle of radius c centered at the origin. Parametrizing the circle, we can write $(x_1, x_2) = (c\cos\phi, c\sin\phi)$ and $(y_1, y_2) = (c\cos\psi, c\sin\psi)$. Then

$$\begin{aligned} S &= 2 - c(\cos\phi + \sin\phi + \cos\psi + \sin\psi) + c^2(\cos\phi\cos\psi + \sin\phi\sin\psi) \\ &= 2 + c\sqrt{2}\left(-\sin\left(\phi + \frac{\pi}{4}\right) - \sin\left(\psi + \frac{\pi}{4}\right)\right) + c^2\cos(\phi - \psi) \end{aligned}$$

We can simultaneously increase each of $-\sin\left(\phi + \frac{\pi}{4}\right), -\sin\left(\psi + \frac{\pi}{4}\right)$, and $\cos(\phi - \psi)$ to 1 by choosing $\phi = \psi = \frac{5\pi}{4}$. Hence the maximum of S is $2 + 2c\sqrt{2} + c^2 = (c + \sqrt{2})^2$. (proposed by C. Rousseau for the USA Mathematical Olympiad, 2002)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :682. Solve the following system of equations in real numbers:

$$\begin{aligned} \frac{3x-y}{x-3y} &= x^2, \\ \frac{3y-z}{y-3z} &= y^2, \\ \frac{3z-x}{z-3x} &= z^2. \end{aligned}$$

Solution :682. From the first equation, it follows that if x is 0 , then so is y, making x^2 indeterminate; hence x, and similarly y and z, cannot be 0 . Solving the equations, respectively, for y, z, and x, we obtain the equivalent system

$$\begin{aligned} y &= \frac{3x - x^3}{1 - 3x^2}, \\ z &= \frac{3y - y^3}{1 - 3y^2}, \\ x &= \frac{3z - z^3}{1 - 3z^2}, \end{aligned}$$

where x, y, z are real numbers different from 0 .There exists a unique number u in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $x = \tan u$. Then

$$\begin{aligned} y &= \frac{3\tan u - \tan^3 u}{1 - 3\tan^2 u} = \tan 3u, \\ z &= \frac{3\tan 3u - \tan^3 3u}{1 - 3\tan^2 3u} = \tan 9u, \\ x &= \frac{3\tan 9u - \tan^3 9u}{1 - 3\tan^2 9u} = \tan 27u. \end{aligned}$$

The last equality yields $\tan u = \tan 27u$, so u and $27u$ differ by an integer multiple of π. Therefore, $u = \frac{k\pi}{26}$ for some k satisfying $-\frac{\pi}{2} < \frac{k\pi}{26} < \frac{\pi}{2}$. Besides, k must not be 0 , since $x \neq 0$. Hence the possible values of k are $\pm 1, \pm 2, \ldots, \pm 12$, each of them generating the corresponding triple

$$x = \tan \frac{k\pi}{26}, \quad y = \tan \frac{3k\pi}{26}, \quad z = \tan \frac{9k\pi}{26}.$$

It is immediately checked that all of these triples are solutions of the initial system.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :686. Compute the integral

$$\int \frac{dx}{x + \sqrt{x^2 - 1}}.$$

Solution :686. With the substitution $x = \cosh t$, the integral becomes

$$\begin{aligned} &\int \frac{1}{\sinh t + \cosh t} \sinh t dt \\ &= \int \frac{e^t - e^{-t}}{2e^t} dt = \frac{1}{2} \int \left(1 - e^{-2t}\right) dt = \frac{1}{2} t + \frac{e^{-2t}}{4} + C \\ &= \frac{1}{2} \ln\left(x + \sqrt{x^2 - 1}\right) + \frac{1}{4} \cdot \frac{1}{2x^2 - 1 + 2x\sqrt{x^2 - 1}} + C. \end{aligned}$$

Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :691. Obtain explicit values for the following series: (a) $\sum_{n=1}^{\infty} \arctan \frac{2}{n^2}$, (b) $\sum_{n=1}^{\infty} \arctan \frac{8n}{n^4 - 2n^2 + 5}$.
Solution :691. The formula

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

translates into

$$\arctan\frac{x-y}{1+xy}=\arctan x-\arctan y.$$

Applied to $x=n+1$ and $y=n-1$, it gives

$$\arctan\frac{2}{n^2}=\arctan\frac{(n+1)-(n-1)}{1+(n+1)(n-1)}=\arctan(n+1)-\arctan(n-1).$$

The sum in part (a) telescopes as follows:

$$\sum_{n=1}^{\infty}\arctan\frac{2}{n^2}=\lim_{N\to\infty}\sum_{n=1}^{N}\arctan\frac{2}{n^2}=\lim_{N\to\infty}\sum_{n=1}^{N}(\arctan(n+1)-\arctan(n-1))$$

$$=\lim_{N\to\infty}(\arctan(N+1)+\arctan N-\arctan 1-\arctan 0)$$
$$=\frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{4}=\frac{3\pi}{4}$$

The sum in part (b) is only slightly more complicated. In the above-mentioned formula for the difference of arctangents we have to substitute $x=\left(\frac{n+1}{\sqrt{2}}\right)^2$ and $y=\left(\frac{n-1}{\sqrt{2}}\right)^2$. This is because

$$\frac{8n}{n^4-2n^2+5}=\frac{8n}{4+\left(n^2-1\right)^2}=\frac{2\left[(n+1)^2-(n-1)^2\right]}{4-(n+1)^2(n-1)^2}=\frac{\left(\frac{n+1}{\sqrt{2}}\right)^2-\left(\frac{n-1}{\sqrt{2}}\right)^2}{1-\left(\frac{n+1}{\sqrt{2}}\right)^2\left(\frac{n-1}{\sqrt{2}}\right)^2}.$$

The sum telescopes as

$$\sum_{n=1}^{\infty}\arctan\frac{8n}{n^4-2n^2+5}$$
$$=\lim_{N\to\infty}\sum_{n=1}^{N}\arctan\frac{8n}{n^4-2n^2+5}=\lim_{N\to\infty}\sum_{n=1}^{N}\left[\arctan\left(\frac{n+1}{\sqrt{2}}\right)^2-\arctan\left(\frac{n-1}{\sqrt{2}}\right)^2\right]$$
$$=\lim_{N\to\infty}\left[\arctan\left(\frac{N+1}{\sqrt{2}}\right)^2+\arctan\left(\frac{N}{\sqrt{2}}\right)^2-\arctan 0-\arctan\frac{1}{2}\right]=\pi-\arctan\frac{1}{2}.$$

(American Mathematical Monthly, proposed by J. Anglesio)
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :693. In a circle of radius 1 a square is inscribed. A circle is inscribed in the square and then a regular octagon in the circle. The procedure continues, doubling each time the number of sides of the polygon. Find the limit of the lengths of the radii of the circles.

Solution :693. The radii of the circles satisfy the recurrence relation $R_1 = 1, R_{n+1} = R_n \cos \frac{\pi}{2^{n+1}}$. Hence

$$\lim_{n\to\infty} R_n = \prod_{n=1}^{\infty} \cos \frac{\pi}{2^n}.$$

The product can be made to telescope if we use the double-angle formula for sine written as $\cos x = \frac{\sin 2x}{2\sin x}$. We then have

$$\begin{aligned}\prod_{n=2}^{\infty} \cos \frac{\pi}{2^n} &= \lim_{N\to\infty} \prod_{n=2}^{N} \cos \frac{\pi}{2^n} = \lim_{N\to\infty} \prod_{n=2}^{N} \frac{1}{2} \cdot \frac{\sin \frac{\pi}{2^{n-1}}}{\sin \frac{\pi}{2^n}} \\ &= \lim_{N\to\infty} \frac{1}{2^N} \frac{\sin \frac{\pi}{2}}{\sin \frac{\pi}{2^N}} = \frac{2}{\pi} \lim_{N\to\infty} \frac{\frac{\pi}{2^N}}{\sin \frac{\pi}{2^N}} = \frac{2}{\pi}.\end{aligned}$$

Thus the answer to the problem is $\frac{2}{\pi}$.Remark. As a corollary, we obtain the formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots .$$

This formula is credited to F. Viète, although Archimedes already used this approximation of the circle by regular polygons to compute π.
Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :695. Evaluate the product

$$(1-\cot 1^\circ)(1-\cot 2^\circ)\cdots(1-\cot 44^\circ).$$

Solution :695. We have

$$\begin{aligned}&(1-\cot 1^\circ)(1-\cot 2^\circ)\cdots(1-\cot 44^\circ) \\ &\qquad = \left(1-\frac{\cos 1^\circ}{\sin 1^\circ}\right)\left(1-\frac{\cos 2^\circ}{\sin 2^\circ}\right)\cdots\left(1-\frac{\cos 44^\circ}{\sin 44^\circ}\right) \\ &\qquad = \frac{(\sin 1^\circ - \cos 1^\circ)(\sin 2^\circ - \cos 2^\circ)\cdots(\sin 44^\circ - \cos 44^\circ)}{\sin 1^\circ \sin 2^\circ \cdots \sin 44^\circ}\end{aligned}$$

Using the identity $\sin a - \cos a = \sqrt{2}\sin(a - 45^\circ)$ in the numerators, we transform this further into

$$\begin{aligned}&\frac{\sqrt{2}\sin(1^\circ - 45^\circ)\cdot\sqrt{2}\sin(2^\circ - 45^\circ)\cdots\sqrt{2}\sin(44^\circ - 45^\circ)}{\sin 1^\circ \sin 2^\circ \cdots \sin 44^\circ} \\ &= \frac{(\sqrt{2})^{44}(-1)^{44}\sin 44^\circ \sin 43^\circ \cdots \sin 1^\circ}{\sin 44^\circ \sin 43^\circ \cdots \sin 1^\circ}.\end{aligned}$$

After cancellations, we obtain 2^{22}.
Topic :Geometry

Book :Putnam and Beyond
Final Answer :

Problem Statement :696. Compute the product

$$\left(\sqrt{3}+\tan 1^\circ\right)\left(\sqrt{3}+\tan 2^\circ\right)\cdots\left(\sqrt{3}+\tan 29^\circ\right).$$

Solution :696. We can write

$$\begin{aligned}\sqrt{3}+\tan n^\circ &= \tan 60^\circ+\tan n^\circ = \frac{\sin 60^\circ}{\cos 60^\circ}+\frac{\sin n^\circ}{\cos n^\circ}\\ &= \frac{\sin\left(60^\circ+n^\circ\right)}{\cos 60^\circ\cos n^\circ} = 2\cdot\frac{\sin\left(60^\circ+n^\circ\right)}{\cos n^\circ} = 2\cdot\frac{\cos\left(30^\circ-n^\circ\right)}{\cos n^\circ}.\end{aligned}$$

And the product telescopes as follows:

$$\prod_{n=1}^{29}\left(\sqrt{3}+\tan n^\circ\right) = 2^{29}\prod_{n=1}^{29}\frac{\cos\left(30^\circ-n^\circ\right)}{\cos n^\circ} = 2^{29}\cdot\frac{\cos 29^\circ\cos 28^\circ\cdots\cos 1^\circ}{\cos 1^\circ\cos 2^\circ\cdots\cos 29^\circ} = 2^{29}.$$

5 (T. Andreescu)

Topic :Geometry
Book :Putnam and Beyond
Final Answer :

Problem Statement :699. Let k be a positive integer. The sequence $(a_n)_n$ is defined by $a_1=1$, and for $n\geq 2$, a_n is the nth positive integer greater than a_{n-1} that is congruent to n modulo k. Find a_n in closed form.
Solution :699. Because $a_{n-1}\equiv n-1(\bmod k)$, the first positive integer greater than a_{n-1} that is congruent to n modulo k must be $a_{n-1}+1$. The nth positive integer greater than a_{n-1} that is congruent to n modulo k is simply $(n-1)k$ more than the first positive integer greater than a_{n-1} that satisfies this condition. Therefore, $a_n=a_{n-1}+1+(n-1)k$. Solving this recurrence gives

$$a_n=n+\frac{(n-1)nk}{2}.$$

(Austrian Mathematical Olympiad, 1997)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :701. Find all functions $f:\mathbb{N}\to\mathbb{N}$ satisfying

$$f(n)+2f(f(n))=3n+5,\quad \text{for all } n\in\mathbb{N}.$$

Solution :701. From $f(1)+2f(f(1))=8$ we deduce that $f(1)$ is an even number between 1 and 6 , that is, $f(1)=2,4$, or 6 . If $f(1)=2$ then $2+2f(2)=8$, so $f(2)=3$. Continuing with $3+2f(3)=11$, we obtain $f(3)=4$, and formulate the conjecture that $f(n)=n+1$ for all $n \geq 1$. And indeed, in an inductive manner we see that $f(n)=n+1$ implies $n+1+2f(n+1)=3n+5$; hence $f(n+1)=n+2$.The case $f(1)=4$ gives $4+2f(4)=8$, so $f(4)=2$. But then $2+2f(f(4))=17$, which cannot hold for reasons of parity. Also, if $f(1)=6$, then $6+2f(6)=8$, so $f(6)=1$. This cannot happen, because $f(6)+2f(f(6))=1+2\cdot 6$, which does not equal $3\cdot 6+5$We conclude that $f(n)=n+1, n \geq 1$, is the unique solution to the functional equation.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :702. Find all functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ with the property that

$$2f(f(x))-3f(x)+x=0, \quad \text{for all } x \in \mathbb{Z}.$$

Solution :702. Let $g(x)=f(x)-x$. The given equation becomes $g(x)=2g(f(x))$. Iterating, we obtain that $g(x)=2^n f^{(n)}(x)$ for all $x \in \mathbb{Z}$, where $f^{(n)}(x)$ means f composed n times with itself. It follows that for every $x \in \mathbb{Z}, g(x)$ is divisible by all powers of 2 , so $g(x)=0$. Therefore, the only function satisfying the condition from the statement is $f(x)=x$ for all x.(Revista Matematică din Timişoara (Timişoara Mathematics Gazette), proposed by L. Funar)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :705. Determine all functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying

$$f\left(x^3+y^3+z^3\right)=(f(x))^3+(f(y))^3+(f(z))^3, \quad \text{for all } x, y, z \in \mathbb{Z}.$$

Solution :705. Setting $x=y=z=0$ we find that $f(0)=3(f(0))^3$. This cubic equation has the unique integer solution $f(0)=0$. Next, with $y=-x$ and $z=0$ we have $f(0)=(f(x))^3+(f(-x))^3+(f(0))^3$, which yields $f(-x)=-f(x)$ for all integers x; hence f is an odd function. Now set $x=1, y=z=0$ to obtain $f(1)=(f(1))^3+2(f(0))^3$; hence $f(1)=f(1)^3$. Therefore, $f(1) \in \{-1,0,1\}$. Continuing with $x=y=1$ and $z=0$ and $x=y=z=1$ we find that $f(2)=2(f(1))^3=2f(1)$ and $f(3)=3(f(1))^3=3f(1)$. We conjecture that $f(x)=xf(1)$ for all integers x. We will do this by strong induction on the absolute value of x, and for that we need the following lemma.Lemma. If x is an integer whose absolute value is greater than 3, then x^3 can be written as the sum of five cubes whose absolute values are less than x.Proof. We have

$$4^3=3^3+3^3+2^3+1^3+1^3, \quad 5^3=4^3+4^3+(-1)^3+(-1)^3+(-1)^3,$$
$$6^3=5^3+4^3+3^3+0^3+0^3, \quad 7^3=6^3+5^3+1^3+1^3+0^3$$

and if $x=2k+1$ with $k>3$, then

$$x^3=(2k+1)^3=(2k-1)^3+(k+4)^3+(4-k)^3+(-5)^3+(-1)^3.$$

In this last case it is not hard to see that $2k-1, k+4,|4-k|, 5$, and 1 are all less than $2k+1$. If $x>3$ is an arbitrary integer, then we write $x=my$, where y is 4,6 , or an odd number greater than 3 , and m is an integer. If we express $y^3=y_1^3+y_2^3+y_3^3+y_4^3+y_5^3$, then $x^3=\left(m y_1\right)^3+\left(m y_2\right)^3+\left(m y_3\right)^3+\left(m y_4\right)^3+\left(m y_5\right)^3$, and the lemma is proved.Returning to the problem, using the fact that f is odd and what we proved before, we see that $f(x)=x f(1)$ for $|x| \leq 3$. For $x>4$, suppose that $f(y)=y f(1)$ for all y with $|y|<|x|$. Using the lemma write $x^3=x_1^3+x_2^3+x_3^3+x_4^3+x_5^3$, where $\left|x_i\right|<|x|, i=1,2,3,4,5$. After writing

$$x^3+\left(-x_4\right)^3+\left(-x_5\right)^3=x_1^3+x_2^3+x_3^3,$$

we apply f to both sides and use the fact that f is odd and the condition from the statement to obtain

$$(f(x))^3-\left(f\left(x_4\right)\right)^3-\left(f\left(x_5\right)\right)^3=f\left(x_1\right)^3+f\left(x_2\right)^3+f\left(x_3\right)^3 .$$

The inductive hypothesis yields

$$(f(x))^3-\left(x_4 f(1)\right)^3-\left(x_5 f(1)\right)^3=\left(x_1 f(1)\right)^3+\left(x_2 f(1)\right)^3+\left(x_3 f(1)\right)^3 ;$$

hence

$$(f(x))^3=\left(x_1^3+x_2^3+x_3^3+x_4^3+x_5^3\right)(f(1))^3=x^3(f(1))^3 .$$

Hence $f(x)=x f(1)$, and the induction is complete. Therefore, the only answers to the problem are $f(x)=-x$ for all $x, f(x)=0$ for all x, and $f(x)=x$ for all x. That these satisfy the given equation is a straightforward exercise.(American Mathematical Monthly, proposed by T. Andreescu)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :712. Find all pairs of positive integers (a, b) with the property that $ab+a+b$ divides a^2+b^2+1.
Solution :712. The divisibility condition can be written as

$$k(a b+a+b)=a^2+b^2+1,$$

where k is a positive integer. The small values of k are easy to solve. For example, $k=1$ yields $ab+a+b=a^2+b^2+1$, which is equivalent to $(a-b)^2+(a-1)^2+(b-1)^2=0$, whose only solution is $a=b=1$. Also, for $k=2$ we have

$2ab+2a+2b=a^2+b^2+1$. This can be rewritten either as $4a=(b-a-1)^2$ or as $4b=(b-a+1)^2$, showing that both a and b are perfect squares. Assuming that $a \leq b$, we see that $(b-a-1)-(b-a+1)=2$, and hence a and b are consecutive squares. We obtain as an infinite family of solutions the pairs of consecutive perfect squares.Now let us examine the case $k \geq 3$. This is where we apply Fermat's infinite descent method. Again we assume that $a \leq b$. A standard approach is to interpret the divisibility condition as a quadratic equation in b :

$$b^2-k(a+1)b+\left(a^2-ka+1\right)=0.$$

Because one of the roots, namely b, is an integer, the other root must be an integer, too (the sum of the roots is $k(a+1)$). Thus we may substitute the pair (a, b) by the smaller pair (r, a), provided that $0<r<a$. Let us verify first that $0<r$. Assume the contrary. Since $br=a^2-ka+1$, we must have $a^2-ka+1 \leq 0$. The equality case is impossible, since $a(k-a)=1$ cannot hold if $k \geq 3$. If $a^2-ka+1<0$, the original divisibility condition implies $b(b-ak-k)=ak-a^2-1>0$, hence $b-ak-k>0$. But then $b(b-ak-k)>(ak+k) \cdot 1>ak-a^2-1$, a contradiction. This proves that r is positive. From the fact that $br=a^2-ka+1<a^2$ and $b \geq a$, it follows that $r<a$. Successively, we obtain the sequence of pairs of solutions to the original problem $(a_1, b_1)=(a, b),(a_2, b_2)=(r, a),(a_3, b_3), \ldots$, with $a_i \leq b_i$ and $a_1>a_2>a_3>\cdots, b_1>b_2>b_3>\cdots$, which of course is impossible. This shows that the ratio of a^2+b^2+1 to $ab+a+b$ cannot be greater than or equal to 3 , and so the answer to the problem consists of the pair $(1,1)$ together with all pairs of consecutive perfect squares.

6 (Mathematics Magazine)

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :715. For a positive integer n and a real number x, compute the sum

$$\sum_{0 \leq i<j \leq n}\left\lfloor\frac{x+i}{j}\right\rfloor .$$

Solution :715. Denote the sum in question by S_n. Observe that

$$\begin{aligned}
S_n-S_{n-1} & =\left\lfloor\frac{x}{n}\right\rfloor+\left\lfloor\frac{x+1}{n}\right\rfloor+\cdots+\left\lfloor\frac{x+n-1}{n}\right\rfloor \\
& =\left\lfloor\frac{x}{n}\right\rfloor+\left\lfloor\frac{x}{n}+\frac{1}{n}\right\rfloor+\cdots+\left\lfloor\frac{x}{n}+\frac{n-1}{n}\right\rfloor,
\end{aligned}$$

and, according to Hermite's identity,

$$S_n-S_{n-1}=\left\lfloor n \frac{x}{n}\right\rfloor=\lfloor x\rfloor .$$

Because $S_1 = \lfloor x \rfloor$, it follows that $S_n = n\lfloor x \rfloor$ for all $n \geq 1$.(S. Savchev, T. Andreescu, Mathematical Miniatures, MAA, 2002)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :717. Express $\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor$ in terms of n and $a = \lfloor \sqrt{n} \rfloor$.
Solution :717. We apply the identity proved in the introduction to the function $f : [1, n] \to [1, \sqrt{n}]$, $f(x) = \sqrt{x}$. Because $n\,(G_f) = \lfloor \sqrt{n} \rfloor$, the identity reads

$$\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor + \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \lfloor k^2 \rfloor - \lfloor \sqrt{n} \rfloor = n \lfloor \sqrt{n} \rfloor.$$

Hence the desired formula is

$$\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor = (n+1)a - \frac{a(a+1)(2a+1)}{6}.$$

(Korean Mathematical Olympiad, 1997)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :719. Find all pairs of real numbers (a, b) such that $a\lfloor bn \rfloor = b\lfloor an \rfloor$ for all positive integers n.
Solution :719. The property is clearly satisfied if $a = b$ or if $ab = 0$. Let us show that if neither of these is true, then a and b are integers.First, note that for an integer x, $\lfloor 2x \rfloor = 2\lfloor x \rfloor$ if $x - \lfloor x \rfloor \in \left[0, \frac{1}{2}\right)$ and $\lfloor 2x \rfloor = 2\lfloor x \rfloor + 1$ if $x - \lfloor x \rfloor \in \left[\frac{1}{2}, 1\right)$. Let us see which of the two holds for a and b. If $\lfloor 2a \rfloor = 2\lfloor a \rfloor + 1$, then

$$a\lfloor 2b \rfloor = b\lfloor 2a \rfloor = 2\lfloor a \rfloor b + b = 2a\lfloor b \rfloor + b.$$

This implies $\lfloor 2b \rfloor = 2\lfloor b \rfloor + \frac{b}{a}$, and so $\frac{b}{a}$ is either 0 or 1 , which contradicts our working hypothesis. Therefore, $\lfloor 2a \rfloor = 2\lfloor a \rfloor$ and also $\lfloor 2b \rfloor = 2\lfloor b \rfloor$. This means that the fractional parts of both a and b are less than $\frac{1}{2}$. With this as the base case, we will prove by induction that $\lfloor 2^m a \rfloor = 2^m \lfloor a \rfloor$ and $\lfloor 2^m b \rfloor = 2^m \lfloor b \rfloor$ for all $m \geq 1$.The inductive step works as follows. Assume that the property is true for m and let us prove it for $m+1$. If $\left\lfloor 2^{m+1} a \right\rfloor = 2\left\lfloor 2^m a \right\rfloor$, we are done. If $\left\lfloor 2^{m+1} a \right\rfloor = 2\left\lfloor 2^m a \right\rfloor + 1$, then

$$a\left\lfloor 2^{m+1} b \right\rfloor = b\left\lfloor 2^{m+1} a \right\rfloor = 2\left\lfloor 2^m a \right\rfloor b + b = 2^{m+1}\lfloor a \rfloor b + b = 2^{m+1} a\lfloor b \rfloor + 1.$$

As before, we deduce that $\left\lfloor 2^{m+1} b \right\rfloor = 2^{m+1}\lfloor b \rfloor + \frac{b}{a}$. Again this is an impossibility. Hence the only possibility is that $\left\lfloor 2^{m+1} a \right\rfloor = 2^{m+1}\lfloor a \rfloor$ and by a similar argument $\left\lfloor 2^{m+1} b \right\rfloor = 2^{m+1}\lfloor b \rfloor$. This completes the induction.From $\lfloor 2^m a \rfloor = 2^m \lfloor a \rfloor$ and $\lfloor 2^m b \rfloor = 2^m \lfloor b \rfloor$ we deduce that the fractional parts of a and b are less than

$\frac{1}{2^m}$. Taking $m \to \infty$, we conclude that a and b are integers.(short list of the 39th International Mathematical Olympiad, 1998)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :722. Does there exist a strictly increasing function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $f(1)=2$ and $f(f(n))=f(n)+n$ for all n ?
Solution :722. Let x_1 be the golden ratio, i.e., the (unique) positive root of the equation $x^2-x-1=0$. We claim that the following identity holds:

$$\left\lfloor x_1\left\lfloor x_1 n+\frac{1}{2}\right\rfloor+\frac{1}{2}\right\rfloor=\left\lfloor x_1+\frac{1}{2}\right\rfloor+n.$$

If this were so, then the function $f(n)=\left\lfloor x_1 n+\frac{1}{2}\right\rfloor$ would satisfy the functional equation. Also, since $\alpha=\frac{1+\sqrt{5}}{2}>1, f$ would be strictly increasing, and so it would provide an example of a function that satisfies the conditions from the statement.To prove the claim, we only need to show that

$$\left\lfloor\left(x_1-1\right)\left\lfloor x_1 n+\frac{1}{2}\right\rfloor+\frac{1}{2}\right\rfloor=n$$

We have

$$\begin{aligned}
\left\lfloor\left(x_1-1\right)\left\lfloor x_1 n+\frac{1}{2}\right\rfloor+\frac{1}{2}\right\rfloor & \leq\left\lfloor\left(x_1-1\right)\left(x_1 n+\frac{1}{2}\right)+\frac{1}{2}\right\rfloor \\
& =\left\lfloor x_1 n+n-x_1 n+\frac{x_1}{2}\right\rfloor=n
\end{aligned}$$

Also,

$$n=\left\lfloor n+\frac{2-x_1}{2}\right\rfloor \leq\left\lfloor\left(x_1-1\right)\left(x_1 n-\frac{1}{2}\right)+\frac{1}{2}\right\rfloor \leq\left\lfloor\left(x_1-1\right)\left\lfloor x_1 n+\frac{1}{2}\right\rfloor+\frac{1}{2}\right\rfloor .$$

This proves the claim and completes the solution.(34th International Mathematical Olympiad, 1993)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :724. Find the integers n for which $\left(n^3-3 n^2+4\right) /(2 n-1)$ is an integer.
Solution :724. If we multiplied the fraction by 8 , we would still get an integer. Note that

$$8 \frac{n^3-3 n^2+4}{2 n-1}=4 n^2-10 n-5+\frac{27}{2 n-1} .$$

Hence $2 n-1$ must divide 27 . This happens only when $2 n-1= \pm 1, \pm 3, \pm 9, \pm 27$, that is, when $n=-13,-4,-1,1,2,5,14$. An easy check shows that for each of

these numbers the original fraction is an integer.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :732. Determine the functions $f : \{0, 1, 2 \ldots\} \to \{0, 1, 2, \ldots\}$ satisfying(i) $f(2n+1)^2 - f(2n)^2 = 6f(n)+1$ and(ii) $f(2n) \geq f(n)$ for all $n \geq 0$.
Solution :732. Setting $n = 0$ in (i) gives

$$f(1)^2 = f(0)^2 + 6f(0) + 1 = (f(0) + 3)^2 - 8.$$

Hence

$$(f(0) + 3)^2 - f(1)^2 = (f(0) + 3 + f(1))(f(0) + 3 - f(1)) = 4 \times 2.$$

The only possibility is $f(0) + 3 + f(1) = 4$ and $f(0) + 3 - f(1) = 2$. It follows that $f(0) = 0$ and $f(1) = 1$.In general,

$$(f(2n + 1) - f(2n))(f(2n + 1) + f(2n)) = 6f(n) + 1.$$

We claim that $f(2n + 1) - f(2n) = 1$ and $f(2n + 1) + f(2n) = 6f(n) + 1$. To prove our claim, let $f(2n+1)-f(2n) = d$. Then $f(2n+1)+f(2n) = d+2f(2n)$. Multiplying, we obtain

$$6f(n) + 1 = d(d + 2f(2n)) \geq d(d + 2f(n)),$$

where the inequality follows from condition (ii). Moving everything to one side, we obtain the inequality

$$d^2 + (2d - 6)f(n) - 1 \leq 0,$$

which can hold only if $d \leq 3$. The cases $d = 2$ and $d = 3$ cannot hold, because d divides $6f(n) + 1$. Hence $d = 1$, and the claim is proved. From it we deduce that f is computed recursively by the rule

$$f(2n + 1) = 3f(n) + 1,$$
$$f(2n) = 3f(n).$$

At this moment it is not hard to guess the explicit formula for f; it associates to a number in binary representation the number with the same digits but read in ternary representation. For example, $f(5) = f\left(101_2\right) = 101_3 = 10$.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :735. Solve in positive integers the equation

$$x^{x+y} = y^{y-x}.$$

Solution :735. The numbers x and y have the same prime factors,

$$x=\prod_{i=1}^{k} p_i^{\alpha_i}, \quad y=\prod_{i=1}^{k} p_i^{\beta_i}.$$

The equality from the statement can be written as

$$\prod_{i=1}^{k} p_i^{\alpha_i(x+y)}=\prod_{i=1}^{k} p_i^{\beta_i(y-x)};$$

hence $\alpha_i(y+x)=\beta_i(y-x)$ for $i=1,2,\ldots,k$. From here we deduce that $\alpha_i<\beta_i$, $i=1,2,\ldots,k$, and therefore x divides y. Writing $y=zx$, the equation becomes $x^{x(z+1)}=(xz)^{x(z-1)}$, which implies $x^2=z^{z-1}$ and then $y^2=(xz)^2=z^{z+1}$. A power is a perfect square if either the base is itself a perfect square or if the exponent is even. For $z=t^2, t\geq 1$, we have $x=t^{t^2-1}, y=t^{t^2+1}$, which is one family of solutions. For $z-1=2s, s\geq 0$, we obtain the second family of solutions $x=(2s+1)^s, y=(2s+1)^{s+1}$.(Austrian-Polish Mathematics Competition, 1999, communicated by I. Cucurezeanu)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :737. Find all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle such that no two adjacent divisors are relatively prime.
Solution :737. The only numbers that do not have this property are the products of two distinct primes.Let n be the number in question. If $n=pq$ with p,q primes and $p\neq q$, then any cycle formed by p,q,pq will have p and q next to each other. This rules out numbers of this form.For any other number $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, with $k\geq 1, \alpha_i\geq 1$ for $i=1,2,\ldots,k$ and $\alpha_1+\alpha_2\geq 3$ if $k=2$, arrange the divisors of n around the circle according to the following algorithm. First, we place $p_1,p_2,\ldots,p_k$ arranged clockwise around the circle in increasing order of their indices. Second, we place p_ip_{i+1} between p_i and p_{i+1} for $i=1,\ldots,k-1$. (Note that the text has p_{i+i}, which is a typo and lets i go up to k, which is a problem if $k=2$, since p_1p_2 gets placed twice.) Third, we place n between p_k and p_1. Note that at this point every pair of consecutive numbers has a common factor and each prime p_i occurs as a common factor for some pair of adjacent numbers. Now for any remaining divisor of n we choose a prime p_i that divides it and place it between p_i and one of its neighbors.(USA Mathematical Olympiad, 2005, proposed by Z. Feng)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :738. Is it possible to place 1995 different positive integers around a circle so that for any two adjacent numbers, the ratio of the greater to the smaller is a prime?
Solution :738. The answer is negative. To motivate our claim, assume the contrary, and let $a_0, a_1, \ldots, a_{1995} = a_0$ be the integers. Then for $i = 1, 2, \ldots, 1995$, the ratio a_{k-1}/a_k is either a prime, or the reciprocal of a prime. Suppose the former happens m times and the latter $1995 - m$ times. The product of all these ratios is $a_0/a_{1995} = 1$, which means that the product of some m primes equals the product of some $1995 - m$ primes. This can happen only when the primes are the same (by unique factorization), and in particular they must be in the same number on both sides. But the equality $m = 1995 - m$ is impossible, since 1995 is odd, a contradiction. This proves our claim.(Russian Mathematical Olympiad, 1995)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :744. Find all positive integers n such that n ! ends in exactly 1000 zeros.
Solution :744. There are clearly more 2's than 5's in the prime factorization of n !, so it suffices to solve the equation

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots = 1000.$$

On the one hand,

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots < \frac{n}{5} + \frac{n}{5^2} + \frac{n}{5^3} + \cdots = \frac{n}{5} \cdot \frac{1}{1 - \frac{1}{5}} = \frac{n}{4},$$

and hence $n > 4000$. On the other hand, using the inequality $\lfloor a \rfloor > a - 1$, we have

$$\begin{aligned} 1000 &> \left(\frac{n}{5} - 1\right) + \left(\frac{n}{5^2} - 1\right) + \left(\frac{n}{5^3} - 1\right) + \left(\frac{n}{5^4} - 1\right) + \left(\frac{n}{5^5} - 1\right) \\ &= \frac{n}{5}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4}\right) - 5 = \frac{n}{5} \cdot \frac{1 - \left(\frac{1}{5}\right)^5}{1 - \frac{1}{5}} - 5, \end{aligned}$$

so

$$n < \frac{1005 \cdot 4 \cdot 3125}{3124} < 4022.$$

We have narrowed down our search to $\{4001, 4002, \ldots, 4021\}$. Checking each case with Polignac's formula, we find that the only solutions are $n = 4005, 4006, 4007,$ 4008, and 4009.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :752. Solve in positive integers the equation

$$2^{x} \cdot 3^{y}=1+5^{z}.$$

Solution :752. Reducing modulo 4 , the right-hand side of the equation becomes equal to 2 . So the left-hand side is not divisible by 4 , which means that $x=1$. If $y>1$, then reducing modulo 9 we find that z has to be divisible by 6 . A reduction modulo 6 makes the lefthand side 0 , while the right-hand side would be $1+(-1)^{z}=2$. This cannot happen. Therefore, $y=1$, and we obtain the unique solution $x=y=z=1$.(Matematika v Škole (Mathematics in Schools), 1979, proposed by I. Mihailov)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :753. Define the sequence $\left(a_{n}\right)_{n}$ recursively by $a_{1}=2, a_{2}=5$, and

$$a_{n+1}=\left(2-n^{2}\right) a_{n}+\left(2+n^{2}\right) a_{n-1} \text{ for } n \geq 2.$$

Do there exist indices p, q, r such that $a_{p} \cdot a_{q}=a_{r}$?
Solution :753. Note that a perfect square is congruent to 0 or to 1 modulo 3 . Using this fact we can easily prove by induction that $a_{n} \equiv 2(\bmod 3)$ for $n \geq 1$. Since $2 \cdot 2 \equiv 1(\bmod 3)$, the question has a negative answer.(Indian International Mathematical Olympiad Training Camp, 2005)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :759. Find all prime numbers p having the property that when divided by every prime number $q<p$ yield a remainder that is a squarefree integer.
Solution :759. Let S be the set of all primes with the desired property. We claim that $S=\{2,3,5,7,13\}$.It is easy to verify that these primes are indeed in S. So let us consider a prime p in $S, p>7$. Then $p-4$ can have no factor q larger than 4 , for otherwise $p-\left\lfloor\frac{p}{q}\right\rfloor q=4$. Since $p-4$ is odd, $p-4=3^{a}$ for some $a \geq 2$. For a similar reason, $p-8$ cannot have prime factors larger than 8 , and so $p-8=3^{a}-4=5^{b} 7^{c}$. Reducing the last equality modulo 24 , we find that a is even and b is odd.If $c \neq 0$, then $p-9=5^{b} 7^{c}-1=2^{d}$. Here we used the fact that $p-9$ has no prime factor exceeding 8 and is not divisible by 3,5 , or 7 . Reduction modulo 7 shows that the last equality is impossible, for the powers of 2 are 1,2 , and 4 modulo 7 . Hence $c=0$ and $3^{a}-4=5^{b}$, which, since $3^{a / 2}-2$ and $3^{a / 2}+2$ are relatively prime, gives $3^{a / 2}-2=1$ and $3^{a / 2}+2=5^{b}$. Thus $a=2, b=1$, and $p=13$. This proves the claim.(American Mathematical Monthly, 1987, proposed by M. Cipu and M. Deaconescu, solution by L. Jones)

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :768. Determine all integers a such that a^k+1 is divisible by 12321 for some appropriately chosen positive integer $k>1$.
Solution :768. We have $12321=(111)^2=3^2\times 37^2$. It becomes natural to work modulo 3 and modulo 37. By Fermat's little theorem,

$$a^2\equiv 1(\bmod 3),$$

and since we must have $a^k\equiv -1(\bmod 3)$, it follows that k is odd. Fermat's little theorem also gives

$$a^{36}\equiv 1(\bmod 37).$$

By hypothesis $a^k\equiv -1(\bmod 37)$. By the fundamental theorem of arithmetic there exist integers x and y such that $kx+36y=\operatorname{gcd}(k,36)$. Since the $\operatorname{gcd}(k,36)$ is odd, x is odd. We obtain that

$$a^{\operatorname{gcd}(k,36)}\equiv a^{kx+36y}\equiv(-1)\cdot 1=-1(\bmod 37).$$

Since $\operatorname{gcd}(k,36)$ can be 1,3 , or 9 , we see that a must satisfy $a\equiv -1, a^3\equiv -1$, or $a^9\equiv -1$ modulo 37 . Thus a is congruent to -1 modulo 3 and to $3,4,11,21,25,27,28,30$, or 36 modulo 37 . These residue classes modulo 37 are precisely those for which a is a perfect square but not a perfect fourth power. Note that if these conditions are satisfied, then $a^k\equiv -1(\bmod 3\times 37)$, for some odd integer k.How do the 3^2 and 37^2 come into the picture? The algebraic identity

$$x^n-y^n=(x-y)\left(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}\right)$$

shows that if $x\equiv y(\bmod n)$, then $x^n\equiv y^n\left(\bmod n^2\right)$. Indeed, modulo n, the factors on the right are 0 , respectively, nx^{n-1}, which is again 0 . We conclude that if a is a perfect square but not a fourth power modulo 37, and is -1 modulo 3 , then $a^k\equiv -1(\bmod 3\times 37)$ and $a^{k\times 3\times 37}\equiv -1\left(\bmod 3^2\times 37^2\right)$. The answer to the problem is the residue classes

$$11,41,62,65,77,95,101,104,110$$

modulo 111 .(Indian Team Selection Test for the International Mathematical Olympiad, 2004, proposed by S.A. Katre)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :769. For each positive integer n, find the greatest common divisor of $n!+1$ and $(n+1)!$.
Solution :769. If $n+1$ is composite, then each prime divisor of $(n+1)$! is

less than n, which also divides n !. Then it does not divide $n!+1$. In this case the greatest common divisor is 1 .If $n+1$ is prime, then by the same argument the greatest common divisor can only be a power of $n+1$. Wilson's theorem implies that $n+1$ divides $n!+1$. However, $(n+1)^2$ does not divide $(n+1)$!, and thus the greatest common divisor is $(n+1)$.(Irish Mathematical Olympiad, 1996)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :783. Devise a scheme by which a bank can transmit to its customers secure information over the Internet. Only the bank (and not the customers) is in the possession of the secret prime numbers p and q.
Solution :783. The customer picks a number k and transmits it securely to the bank using the algorithm described in the essay. Using the two large prime numbers p and q, the bank finds m such that $km \equiv 1(\bmod (p-1)(q-1))$. If α is the numerical information that the customer wants to receive, the bank computes $\alpha^m (\bmod n)$, then transmits the answer β to the customer. The customer computes $\beta^k (\bmod n)$. By Euler's theorem, this is α. Success!
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :784. A group of United Nations experts is investigating the nuclear program of a country. While they operate in that country, their findings should be handed over to the Ministry of Internal Affairs of the country, which verifies the document for leaks of classified information, then submits it to the United Nations. Devise a scheme by which the country can read the document but cannot modify its contents without destroying the information.
Solution :784. As before, let p and q be two large prime numbers known by the United Nations experts alone. Let also k be an arbitrary secret number picked by these experts with the property that $\gcd(k,(p-1)(q-1))=1$. The number $n=pq$ and the inverse m of k modulo $\phi(n)=(p-1)(q-1)$ are provided to both the country under investigation and to the United Nations.The numerical data α that comprises the findings of the team of experts is raised to the power k, then reduced modulo n. The answer β is handed over to the country. Computing β^m modulo n, the country can read the data. But it cannot encrypt fake data, since it does not know the number k.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :785. An old woman went to the market and a horse stepped on her basket and smashed her eggs. The rider offered to pay for the eggs and asked her how many there were. She did not remember the exact number, but when she had taken them two at a time there was one egg left,

and the same happened when she took three, four, five, and six at a time. But when she took them seven at a time, they came out even. What is the smallest number of eggs she could have had?
Solution :785. We are to find the smallest positive solution to the system of congruences

$$
\begin{aligned}
x &\equiv 1(\bmod 60), \\
x &\equiv 0(\bmod 7).
\end{aligned}
$$

The general solution is $7b_1 + 420t$, where b_1 is the inverse of 7 modulo 60 and t is an integer. Since b_1 is a solution to the Diophantine equation $7b_1 + 60y = 1$, we find it using Euclid's algorithm. Here is how to do it: $60 = 8 \cdot 7 + 4, 7 = 1 \cdot 4 + 3, 4 = 1 \cdot 3 + 1$. Then

$$
\begin{aligned}
1 &= 4 - 1 \cdot 3 = 4 - 1 \cdot (7 - 1 \cdot 4) = 2 \cdot 4 - 7 = 2 \cdot (60 - 8 \cdot 7) - 7 \\
&= 2 \cdot 60 - 17 \cdot 7
\end{aligned}
$$

Hence $b_1 = -17$, and the smallest positive number of the form $7b_1 + 420t$ is $-7 \cdot 17+$ $420 \cdot 1 = 301$.(Brahmagupta)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :790. Is there a sequence of positive integers in which every positive integer occurs exactly once and for every $k = 1, 2, 3, \ldots$ the sum of the first k terms is divisible by k ?
Solution :790. We construct such a sequence recursively. Suppose that $a_1, a_2, \ldots, a_m$ have been chosen. Set $s = a_1 + a_2 + \cdots + a_m$, and let n be the smallest positive integer that is not yet a term of the sequence. By the Chinese Remainder Theorem, there exists t such that $t \equiv -s(\bmod (m+1))$, and $t \equiv -s-n(\bmod (m+2))$. We can increase t by a suitably large multiple of $(m+1)(m+2)$ to ensure that it does not equal any of $a_1, a_2, \ldots, a_m$. Then $a_1, a_2, \ldots, a_m, t, n$ is also a sequence with the desired property. Indeed, $a_1 + a_2 + \cdots + a_m + t = s + t$ is divisible by $m + 1$ and $a_1 + \cdots + a_m + t + n = s + t + n$ is divisible by $m + 2$. Continue the construction inductively. Observe that the algorithm ensures that $1, \ldots, m$ all occur among the first $2m$ terms.(Russian Mathematical Olympiad, 1995)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :802. Find all positive integers x, y, z satisfying the equation $3^x + y^2 = 5^z$.
Solution :802. We guess immediately that $x = 2, y = 4$, and $z = 2$ is a solution because of the trigonometric triple $3, 4, 5$. This gives us a hint as to how to approach the problem. Checking parity, we see that y has to be even. A reduction modulo 4 shows that x must be even, while a reduction modulo 3 shows that z must be even. Letting $x = 2m$ and $z = 2n$, we obtain a Pythagorean equation

$$
(3^m)^2 + y^2 = (5^n)^2 .
$$

Because y is even, in the usual parametrization of the solution we should have $3^m = u^2 - v^2$ and $5^n = u^2 + v^2$. From $(u-v)(u+v) = 3^m$ we find that $u-v$ and $u+v$ are powers of 3. Unless $u-v$ is $1, u = (u-v+u+v)/2$ and $v = (u+v-u+v)/2$ are both divisible by 3 , which cannot happen because u^2+v^2 is a power of 5 . So $u-v=1$, $u+v=3^m$, and $u^2+v^2=5^n$. Eliminating the parameters u and v, we obtain the simpler equation

$$2 \cdot 5^n = 9^m + 1.$$

First, note that $n=1$ yields the solution mentioned in the beginning. If $n>1$, then looking at the equation modulo 25 , we see that m has to be an odd multiple of 5 , say $m = 5(2k+1)$. But then

$$2 \cdot 5^n = \left(9^5\right)^{2k+1} + 1 = \left(9^5+1\right)\left(\left(9^5\right)^{2k} - \left(9^5\right)^{2k-1} + \cdots + 1\right),$$

which implies that $2 \cdot 5^n$ is a multiple of $9^5 + 1 = 2 \cdot 5^2 \cdot 1181$. This is of course impossible; hence the equation does not have other solutions.(I. Cucurezeanu)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :804. Solve the following equation in positive integers:

$$x^2 + y^2 = 1997(x-y).$$

Solution :804. Here is how to transform the equation from the statement into a Pythagorean equation:

$$\begin{aligned} x^2 + y^2 &= 1997(x-y), \\ 2\left(x^2+y^2\right) &= 2 \cdot 1997(x-y), \\ (x+y)^2 + (x-y)^2 - 2 \cdot 1997(x-y) &= 0, \\ (x+y)^2 + (1997-x+y)^2 &= 1997^2. \end{aligned}$$

Because x and y are positive integers, $0 < x+y < 1997$, and for the same reason $0 < 1997 - x + y < 1997$. The problem reduces to solving the Pythagorean equation $a^2 + b^2 = 1997^2$ in positive integers. Since 1997 is prime, the greatest common divisor of a and b is 1 . Hence there exist coprime positive integers $u > v$ with the greatest common divisor equal to 1 such that

$$1997 = u^2 + v^2, \quad a = 2uv, \quad b = u^2 - v^2.$$

Because u is the larger of the two numbers, $\frac{1997}{2} < u^2 < 1997$; hence $33 \leq u \leq 44$. There are 12 cases to check. Our task is simplified if we look at the equality $1997 = u^2 + v^2$ and realize that neither u nor v can be divisible by 3 . Moreover, looking at the same equality modulo 5 , we find that u and v can only be 1 or -1

modulo 5 . We are left with the cases $m=34,41$, or 44 . The only solution is $(m,n)=(34,29)$. Solving $x+y=2\cdot 34\cdot 29$ and $1997-x+y=34^2-29^2$, we obtain $x=1827, y=145$. Solving $x+y=34^2-29^2$, $1997-x+y=2\cdot 34\cdot 29$, we obtain $(x,y)=(170,145)$. These are the two solutions to the equation.(Bulgarian Mathematical Olympiad, 1997)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :805. Find a solution to the Diophantine equation

$$x^2-\left(m^2+1\right)y^2=1,$$

where m is a positive integer.
Solution :805. One can verify that $x=2m^2+1$ and $y=2m$ is a solution. (Diophantus)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :809. Find the positive solutions to the Diophantine equation

$$(x+1)^3-x^3=y^2.$$

Solution :809. Expanding the cube, we obtain the equivalent equation $3x^2+3x+1=y^2$. After multiplying by 4 and completing the square, we obtain $(2y)^2-3(2x+1)^2=1$, a Pell equation, namely, $u^2-3v^2=1$ with u even and v odd. The solutions to this equation are generated by $u_n\pm v_n\sqrt{3}=(2\pm\sqrt{3})^n$, and the parity restriction shows that we must select every other solution. So the original equation has infinitely many solutions generated by

$$2y_n\pm\left(2x_n+1\right)\sqrt{3}=(2\pm\sqrt{3})(5\pm 4\sqrt{3})^n,$$

or, explicitly,

$$\begin{aligned}
x_n&=\frac{(2+\sqrt{3})(5+4\sqrt{3})^n-(2-\sqrt{3})(5-4\sqrt{3})^n-1}{2},\\
y_n&=\frac{(2+\sqrt{3})(5+4\sqrt{3})^n+(2-\sqrt{3})(5-4\sqrt{3})^n}{2}.
\end{aligned}$$

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :810. Find the positive integer solutions to the equation

$$(x-y)^5=x^3-y^3.$$

Solution :810. One family of solutions is of course $(n, n), n \in \mathbb{N}$. Let us see what other solutions the equation might have. Denote by t the greatest common divisor of x and y, and let $u = \frac{x}{t}, v = \frac{y}{t}$. The equation becomes $t^5(u-v)^5 = t^3\left(u^3 - v^3\right)$. Hence

$$t^2(u-v)^4 = \frac{u^3 - v^3}{u - v} = u^2 + uv + v^2 = (u-v)^2 + 3uv$$

or $(u-v)^2\left[t^2(u-v)^2 - 1\right] = 3uv$. It follows that $(u-v)^2$ divides $3uv$, and since u and v are relatively prime and $u > v$, this can happen only if $u - v = 1$. We obtain the equation $3v(v+1) = t^2 - 1$, which is the same as

$$(v+1)^3 - v^3 = t^2$$

This was solved in the previous problem. The solutions to the original equation are then given by $x = (v+1)t, y = vt$, for any solution (v, t) to this last equation.

7 (A. Rotkiewicz)

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :813. Find all integer solutions (x, y) to the equation

$$x^2 + 3xy + 4006(x+y) + 2003^2 = 0.$$

Solution :813. First solution: This solution is based on an idea that we already encountered in the section on factorizations and divisibility. Solving for y, we obtain

$$y = -\frac{x^2 + 4006x + 2003^2}{3x + 4006}.$$

To make the expression on the right easier to handle we multiply both sides by 9 and write

$$9y = -3x - 8012 - \frac{2003^2}{3x + 4006}.$$

If (x, y) is an integer solution to the given equation, then $3x + 4006$ divides 2003^2. Because 2003 is a prime number, we have $3x + 4006 \in \left\{\pm 1, \pm 2003, \pm 2003^2\right\}$. Working modulo 3 we see that of these six possibilities, only $1, -2003$, and 2003^2 yield integer solutions for x. We deduce that the equation from the statement has three solutions: $(-1334, -446224), (-2003, 0)$, and $(1336001, -446224)$.Second solution: Rewrite the equation as

$$(3x + 4006)(3x + 9y + 8012) = -2003^2.$$

This yields a linear system

$$3x + 4006 = d,$$
$$3x + 9y + 8012 = -\frac{2003^2}{d},$$

where d is a divisor of -2003^2. Since 2003 is prime, one has to check the cases $d = \pm 1, \pm 2003, \pm 2003^2$, which yield the above solutions.(American Mathematical Monthly, proposed by Wu Wei Chao)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :816. Find all nonnegative integers x, y, z, w satisfying

$$4^x + 4^y + 4^z = w^2.$$

Solution :816. If $x \leq y \leq z$, then since $4^x + 4^y + 4^z$ is a perfect square, it follows that the number $1 + 4^{y-x} + 4^{z-x}$ is also a perfect square. Then there exist an odd integer t and a positive integer m such that

$$1 + 4^{y-x} + 4^{z-x} = (1 + 2^m t)^2.$$

It follows that

$$4^{y-x}\left(1 + 4^{z-x}\right) = 2^{m+1} t \left(1 + 2^{m-1} t\right);$$

hence $m = 2y - 2x - 1$. From $1 + 4^{z-x} = t + 2^{m-1} t^2$, we obtain

$$t - 1 = 4^{y-x-1}\left(4^{z-2y+x+1} - t^2\right) = 4^{y-x-1}\left(2^{z-2y+x+1} + t\right)\left(2^{z-2y+x+1} - t\right).$$

Since $2^{z-2y+x+1} + t > t$, this equality can hold only if $t = 1$ and $z = 2y - x - 1$. The solutions are of the form $(x, y, 2y - x - 1)$ with x, y nonnegative integers.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :818. Find all positive integers x, y such that $7^x - 3^y = 4$.
Solution :818. Clearly, $y = 0$ does not yield a solution, while $x = y = 1$ is a solution. We show that there are no solutions with $y \geq 2$. Since in this case 7^x must give residue 4 when taken modulo 9 , it follows that $x \equiv 2(\bmod 4)$. In particular, we can write $x = 2n$, so that

$$3^y = 7^{2n} - 4 = (7^n + 2)(7^n - 2).$$

Both factors on the right must be powers of 3, but no two powers of 3 differ by 4 . Hence there are no solutions other than $x = y = 1$.(Indian Mathematical Olympiad, 1995)

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :819. Find all positive integers x satisfying

$$3^{2^{x!}} = 2^{3^{x!}} + 1.$$

Solution :819. First solution: One can see immediately that $x = 1$ is a solution. Assume that there exists a solution $x > 1$. Then x ! is even, so $3^{x!}$ has residue 1 modulo 4 . This implies that the last digit of the number $2^{3^{x!}}$ is 2 , so the last digit of $3^{2^{x!}} = 2^{3^{x!}} + 1$ is 3 . But this is impossible because the last digit of an even power of 3 is either 1 or 9 . Hence $x = 1$ is the only solution.Second solution: We will prove by induction the inequality

$$3^{2^{x!}} < 2^{3^{x!}},$$

for $x \geq 2$. The base case $x = 2$ runs as follows: $3^{2^2} = 3^4 = 81 < 512 = 2^9 = 2^{3^2}$. Assume now that $3^{2^{x!}} < 2^{3^{x!}}$ and let us show that $3^{2^{(x+1)!}} < 2^{3^{(x+1)!}}$.Raising the inequality $3^{2^{x!}} < 2^{3x^{x!}}$ to the power $2^{x! \cdot x}$, we obtain

$$\left(3^{2^{x!}}\right)^{2^{2!!x}} < \left(2^{3^{x!}}\right)^{2^{x! \cdot x}} < \left(2^{3^{x!}}\right)^{3^{x! \cdot x}}.$$

Therefore, $3^{2^{(x+1)!}} < 2^{3^{(x+1)!}}$, and the inequality is proved. The inequality we just proved shows that there are no solutions with $x \geq 2$. We are done.Remark. The proof by induction can be avoided if we perform some computations. Indeed, the inequality can be reduced to

$$3^{2^{x!}} < 2^{3^{x!}}$$

and then to

$$x! < \frac{\log \log 3 - \log \log 2}{\log 3 - \log 2} = 1.13588\ldots$$

(Romanian Mathematical Olympiad, 1985)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :820. Find all quadruples (u, v, x, y) of positive integers, where u and v are consecutive in some order, satisfying

$$u^x - v^y = 1.$$

8 Combinatorics and Probability

We conclude the book with combinatorics. First, we train combinatorial skills in set theory and geometry, with a glimpse at permutations. Then we turn to some specific techniques: generating functions, counting arguments, the inclusion-exclusion principle. A strong accent is placed on binomial coefficients.This is followed by probability, which, in fact, should be treated separately. But the level of this book restricts us to problems that use counting, classical schemes such as the Bernoulli and Poisson schemes and Bayes' theorem, recurrences, and some minor geometric considerations. It is only later in the development of mathematics that probability loses its combinatorial flavor and borrows the analytical tools of Lebesgue integration.

Solution :820. First solution: The solutions are

$$(v+1, v, 1, 1), \text{ for all } v; \quad (2,1,1,y), \text{ for all } y; \quad (2,3,2,1),(3,2,2,3).$$

To show that these are the only solutions, we consider first the simpler case $v=u+1$. Then $u^x-(u+1)^y=1$. Considering this equation modulo u, we obtain $-1 \equiv u^x-(u+1)^y=1(\bmod u)$. So $u=1$ or 2 . The case $u=1$ is clearly impossible, since then $v^y=0$, so we have $u=2, v=3$. We are left with the simpler equation $2^x-3^y=1$. Modulo 3 it follows that x is even, $x=2x'$. The equality $2^{2x'}-1=\left(2^{x'}-1\right)\left(2^{x'}+1\right)=3^y$ can hold only if $x'=1$ (the only consecutive powers of 3 that differ by 2 are 1 and 3). So $x=2, y=1$, and we obtain the solution $(2,3,2,1)$.Now suppose that $u=v+1$. If $v=1$, then $u=2, x=1$, and y is arbitrary. So we have found the solution $(2,1,2,y)$. If $v=2$, the equation reduces to $3^x-2^y=1$. If $y \geq 2$, then modulo 4 we obtain that x is even, $x=2x'$, and so $3^{2x'}-1=\left(3^{x'}-1\right)\left(3^{x'}+1\right)=2^y$. Two consecutive powers of 2 differ by 2 if they are 2 and 4 . We find that either $x=y=1$ or $x=2, y=3$. This gives the solutions $(2,1,1,1)$ and $(3,2,2,3)$.So let us assume $v \geq 3$. The case $y=1$ gives the solutions $(v+1, v, 1,1)$. If $y>1$, then v^2 divides v^y, so $1 \equiv(v+1)^x \equiv 0+\left(\begin{array}{l}x \\ 1\end{array}\right) v+1\left(\bmod v^2\right)$, and therefore v divides x. Considering the equation modulo $v+1$, we obtain $1 \equiv(v+1)^x-v^y \equiv -(-1)^y(\bmod (v+1))$. Since $v+1>2, 1 \not\equiv-1(\bmod (v+1))$, so y must be odd. Now if $x=1$, then $v^y=v$, so $v=1$, giving again the family of solutions $(v+1, v, 1,1)$. So assume $x>1$. Then $(v+1)^2$ divides $(v+1)^x$, so

$$\begin{aligned}
1 & \equiv(v+1)^x-v^y \equiv-(v+1-1)^y \\
& \equiv 0-\left(\begin{array}{l}y \\ 1\end{array}\right)(v+1)(-1)^{y-1}-(-1)^y \\
& \equiv-y(v+1)+1\left(\bmod (v+1)^2\right)
\end{aligned}$$

Hence $v+1$ divides y. Since y is odd, $v+1$ is odd and v is even. Since v divides x, x is also even. Because v is even and $v \geq 3$, it follows that $v \geq 4$. We will need the following result.Lemma. If a and q are odd, if $1 \leq m<t$, and if $a^{2^m q} \equiv 1\left(\bmod 2^t\right)$, then $a \equiv \pm 1\left(\bmod 2^{t-m}\right)$.Proof. First, let us prove

the property for $q = 1$. We will do it by induction on m. For $m = 1$ we have $a^2 = (a-1)(a+1)$, so one of the factors is divisible by 2^{t-1}. Assume that the property is true for $m-1$ and let us prove it for m. Factoring, we obtain $\left(a^{2^{m-1}}+1\right)\left(a^{2^{m-1}}-1\right)$. For $m \geq 2$, the first factor is 2 modulo 4 , hence $a^{2^{m-1}}$ is 1 modulo 2^{t-1}. From the induction hypothesis it follows that $a \equiv \pm 1 \,(\bmod 2^{t-m})$ (note that $t-m = (t-1)-(m-1)$). For arbitrary q, from what we have proved so far it follows that $a^q \equiv \pm 1 \,(\bmod 2^{t-m})$. Because $\phi\left(2^{t-m}\right) = 2^{t-m-1}$, by Euler's theorem $a^{2^{t-m-1}} \equiv 1 \,(\bmod 2^{t-m})$. Since q is odd, we can find a positive integer c such that $cq \equiv 1 \left(\bmod 2^{t-m-1}\right)$. Then $a \equiv a^{cq} \equiv (\pm 1)^c \equiv \pm 1 \,(\bmod 2^{t-m})$, and the lemma is proved.Let us return to the problem. Let $x = 2^m q$, where $m \geq 1$ and q is odd. Because $(v+1)^x - v^y = 1$, clearly $y \geq x$. We have shown that $v+1$ divides y, so $y \geq v+1$. Let us prove that $y \geq 2m+1$. Indeed, if $m \leq 2$ this holds since $y \geq v+1 \geq 5 \geq 2m+1$; otherwise, $y \geq x = 2^m q \geq 2^m \geq 2m+1$.Looking at the equation modulo 2^y, we have $(v+1)^{2^m q} \equiv 1 \,(\bmod 2^y)$, because 2^y divides v^y. By the lemma this implies that $v+1 \equiv \pm 1 \,(\bmod 2^{y-m})$. But $v+1 \equiv 1 \,(\bmod 2^{y-m})$ would imply that 2^{m+1} divides v, which is impossible since v divides x. Therefore, $v+1 \equiv -1 \,(\bmod 2^{y-m})$ and $v \equiv -2 \,(\bmod 2^{y-m})$. In particular, $v \geq 2^{y-m}-2$, so $y \geq 2^{y-m}-1$. But since $y \geq 2m+1$ and $y \geq 5$, it follows that $2^{y-m}-1 > y$, a contradiction. This shows that there are no other solutions.Second solution: Begin as before until we reduce to the case $u = v+1$ and $v \geq 3$. Then we use the following lemma.Lemma. Suppose $p^s \geq 3$ is a prime power, $r \geq 1$, and $a \equiv 1 \,(\bmod p^s)$, but not $\bmod p^{s+1}$. If $a^k \equiv 1 \,(\bmod p^{r+s})$, then p^r divides k.Proof. Write $a = 1 + cp^s + dp^{s+1}$, where $1 \leq c \leq p-1$. Then we compute $a^k \equiv 1 + kcp^s \left(\bmod p^{s+1}\right)$, and

$$a^p = 1 + cp^{s+1} + dp^{s+2} + \left(\begin{array}{c} p \\ 2 \end{array}\right) p^{2s}(c+dp) + \left(\begin{array}{c} p \\ 3 \end{array}\right) p^{3s}(c+dp)^3 + \cdots .$$

Since either $s \geq 2$ or p is odd, p^{s+2} divides $\left(\begin{array}{c} p \\ 2 \end{array}\right) p^{2s}$; hence the fourth term is zero $\bmod p^{s+2}$. Since $s+2 \leq 3s$, the latter terms are also zero $\bmod p^{s+2}$; hence $a^p \equiv 1 \left(\bmod p^{s+1}\right)$, but not $\bmod p^{s+2}$.We now proceed by induction on r. Since $r \geq 1$, the first equation above shows that p divides k, which is the base case. For the inductive step, we note that the second calculation above lets us apply the previous case to $(a^p)^{k/p}$.To use this lemma, let $p^s \geq 3$ be the highest power of the prime p that divides v. Then $u = v+1 \equiv 1 \,(\bmod p^s)$, but not $\bmod p^{s+1}$, and $u^x = v^y + 1 \equiv 1 \,(\bmod p^{sy})$. Hence by the lemma, $p^{s(y-1)}$ divides x, and in particular, $x \geq p^{s(y-1)} \geq 3^{y-1}$. Thus either $x > y$ or $y = 1$.Similarly, let $q^t \geq 3$ be the highest power of the prime q that divides u. Then $(-v) = 1-u \equiv 1 \,(\bmod q^t)$, but not $\bmod q^{t+1}$. Since $(-v)^y \equiv 1 \,(\bmod q^t)$ and $(-v)^y = (-1)^y - (-1)^y u^x \equiv (-1)^y \,(\bmod q^t)$, we see that y is even. Hence $(-v)^y = 1 - u^x \equiv 1 \,(\bmod q^{tx})$. Thus by the lemma, $q^{t(x-1)}$ divides y, and in particular, $y \geq q^{t(x-1)} \geq 3^{x-1}$, so either $y > x$ or $x = 1$. Combining these, we see that we must have either $x = 1$ or $y = 1$. Either of these implies the other and gives the solution $(v+1, v, 1, 1)$.Remark. Catalan conjectured in 1844 a more general fact, namely that the Diophantine equation $u^x - v^y = 1$ subject to the condition $x, y \geq 2$ has the unique solution $3^2 - 2^3 = 1$. This would mean that 8 and 9 are the only consecutive powers. Catalan's conjecture was proved by P. Mihăilescu in 2002.(Kvant (Quantum), first solution by R. Barton, second solution by R. Stong)

9 Combinatorics and Probability

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :821. Let A and B be two sets. Find all sets X with the property that

$$A \cap X = B \cap X = A \cap B,$$

$$A \cup B \cup X = A \cup B.$$

Solution :821. The relation from the statement implies

$$(A \cap X) \cup (B \cap X) = A \cap B.$$

Applying de Morgan's law, we obtain

$$(A \cup B) \cap X = A \cap B.$$

But the left-hand side is equal to $(A \cup B \cup X) \cap X$, and this is obviously equal to X. Hence $X = A \cap B$.(Russian Mathematics Competition, 1977)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :824. Let M be a subset of $\{1, 2, 3, \ldots, 15\}$ such that the product of any three distinct elements of M is not a square. Determine the maximum number of elements in M.
Solution :824. Note that the product of the three elements in each of the sets $\{1, 4, 9\}, \{2, 6, 12\}$, $\{3, 5, 15\}$, and $\{7, 8, 14\}$ is a square. Hence none of these sets is a subset of M. Because they are disjoint, it follows that M has at most $15 - 4 = 11$ elements.Since 10 is not an element of the aforementioned sets, if $10 \notin M$, then M has at most 10 elements. Suppose $10 \in M$. Then none of $\{2, 5\}, \{6, 15\}, \{1, 4, 9\}$, and $\{7, 8, 14\}$ is a subset of M. If $\{3, 12\} \not\subset M$, it follows again that M has at most 10 elements. If $\{3, 12\} \subset M$, then none of $\{1\}, \{4\}, \{9\}, \{2, 6\}, \{5, 15\}$, and $\{7, 8, 14\}$ is a subset of M, and then M has at most 9 elements. We conclude that M has at most 10 elements in any case.Finally, it is easy to verify that the subset

$$M = \{1, 4, 5, 6, 7, 10, 11, 12, 13, 14\}$$

has the desired property. Hence the maximum number of elements in M is 10 .(short list of the 35th International Mathematical Olympiad, 1994, proposed by Bulgaria)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :829. For each permutation $a_1, a_2, \ldots, a_{10}$ of the integers $1, 2, 3, \ldots, 10$, form the sum

$$|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|.$$

Find the average value of all such sums.
Solution :829. We solve the more general case of the permutations of the first $2n$ positive integers, $n \geq 1$. The average of the sum

$$\sum_{k=1}^{n} |a_{2k-1} - a_{2k}|$$

is just n times the average value of $|a_1 - a_2|$, because the average value of $|a_{2i-1} - a_{2i}|$ is the same for all $i = 1, 2, \ldots, n$. When $a_1 = k$, the average value of $|a_1 - a_2|$ is

$$\begin{aligned}
&\frac{(k-1)+(k-2)+\cdots+1+1+2+\cdots+(2n-k)}{2n-1} \\
&= \frac{1}{2n-1}\left[\frac{k(k-1)}{2} + \frac{(2n-k)(2n-k+1)}{2}\right] \\
&= \frac{k^2-(2n+1)k+n(2n+1)}{2n-1}.
\end{aligned}$$

It follows that the average value of the sum is

$$\begin{aligned}
&n \cdot \frac{1}{2n} \sum_{k=1}^{2n} \frac{k^2-(2n+1)k+n(2n+1)}{2n-1} \\
&\quad = \frac{1}{4n-2}\left[\frac{2n(2n+1)(4n+1)}{6} - (2n+1)\frac{2n(2n+1)}{2} + 2n^2(2n+1)\right] \\
&\quad = \frac{n(2n+1)}{3}.
\end{aligned}$$

For our problem $n = 5$ and the average of the sums is $\frac{55}{3}$.(American Invitational Mathematics Examination, 1996)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :830. Find the number of permutations $a_1, a_2, a_3, a_4, a_5, a_6$ of the numbers $1,2,3,4,5,6$ that can be transformed into $1,2,3,4,5,6$ through exactly four transpositions (and not fewer).
Solution :830. The condition from the statement implies that any such permutation has exactly two disjoint cycles, say $(a_{i_1}, \ldots, a_{i_r})$ and $\left(a_{i_{r+1}}, \ldots, a_{i_6}\right)$. This follows from the fact that in order to transform a cycle of length r into the identity $r-1$, transpositions are needed. Moreover, we can only have $r=5,4$, or 3 .When $r=5$, there are $\binom{6}{1}$ choices for the number that stays unpermuted. There are $(5-1)!$ possible cycles, so in this case we have $6 \times 4! = 144$ possibilities. When $r=4$, there are $\binom{6}{4}$ ways to split the numbers into the two cycles (two cycles are needed and not just one). One cycle is a transposition. There are $(4-1)! = 6$ choices for the other. Hence in this case the number is 90 . Note that here exactly four transpositions are needed.Finally, when $r=3$, then there are $\binom{6}{3} \times (3-1)! \times (3-1)! = 40$ cases. Therefore, the answer to the problem is $144+90+40=274$.(Korean Mathematical Olympiad, 1999)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :831. Let $f(n)$ be the number of permutations $a_1, a_2, \ldots, a_n$ of the integers $1,2,\ldots,n$ such that (i) $a_1 = 1$ and (ii) $|a_i - a_{i+1}| \leq 2, i = 1,2,\ldots,n-1$. Determine whether f (1996) is divisible by 3 .
Solution :831. We would like to find a recursive scheme for $f(n)$. Let us attempt the less ambitious goal of finding a recurrence relation for the number $g(n)$ of permutations of the desired form satisfying $a_n = n$. In that situation either $a_{n-1} = n-1$ or $a_{n-1} = n-2$, and in the latter case we necessarily have $a_{n-2} = n-1$ and $a_{n-3} = n-3$. We obtain the recurrence relation

$$g(n) = g(n-1) + g(n-3), \quad \text{for } n \geq 4.$$

In particular, the values of $g(n)$ modulo 3 are $1,1,1,2,0,1,0,0,\ldots$ repeating with period 8.Now let $h(n) = f(n) - g(n)$. We see that $h(n)$ counts permutations of the desired form with n occurring in the middle, sandwiched between $n-1$ and $n-2$. Removing n leaves an acceptable permutation, and any acceptable permutation on $n-1$ symbols can be so produced, except those ending in $n-4, n-2, n-3, n-1$. So for $h(n)$, we have the recurrence

$$h(n) = h(n-1) + g(n-1) - g(n-4) = h(n-1) + g(n-2), \quad \text{for } n \geq 5.$$

A routine check shows that $h(n)$ modulo 3 repeats with period 24.We find that $f(n)$ repeats with period equal to the least common multiple of 8 and 24 , which is 24 . Because $1996 \equiv 4(\bmod 24)$, we have $f(1996) \equiv f(4) = 4(\bmod 3)$. So f (1996) is not divisible by 3 .(Canadian Mathematical Olympiad, 1996)
Topic :Number Theory

Book :Putnam and Beyond
Final Answer :

Problem Statement :833. Let $a_1, a_2, \ldots, a_n$ be a permutation of the numbers $1, 2, \ldots, n$. We call a_i a large integer if $a_i > a_j$ for all $i < j < n$. Find the average number of large integers over all permutations of the first n positive integers.
Solution :833. Let $N(\sigma)$ be the number we are computing. Denote by $N_i(\sigma)$ the average number of large integers a_i. Taking into account the fact that after choosing the first $i-1$ numbers, the i th is completely determined by the condition of being large, for any choice of the first $i-1$ numbers there are $(n-i+1)$! choices for the last $n-i+1$, from which $(n-i)$! contain a large integer in the i th position. We deduce that $N_i(\sigma) = \frac{1}{n-i+1}$. The answer to the problem is therefore

$$N(\sigma) = \sum_{i=1}^{n} N_i(\sigma) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

(19th W.L. Putnam Mathematical Competition, 1958)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :834. Given some positive real numbers $a_1 < a_2 < \cdots < a_n$ find all permutations σ with the property that

$$a_1 a_{\sigma(1)} < a_2 a_{\sigma(2)} < \cdots < a_n a_{\sigma(n)}.$$

Solution :834. We will show that σ is the identity permutation. Assume the contrary and let $(i_1, i_2, \ldots, i_k)$ be a cycle, i.e., $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_k) = i_1$. We can assume that i_1 is the smallest of the i_j 's, $j = 1, 2, \ldots, k$. From the hypothesis,

$$a_{i_1} a_{i_2} = a_{i_1} a_{\sigma(i_1)} < a_{i_k} a_{\sigma(i_k)} = a_{i_k} a_{i_1},$$

so $a_{i_2} < a_{ik}$ and therefore $i_2 < i_k$. Similarly,

$$a_{i_2} a_{i_3} = a_{i_2} a_{\sigma(i_2)} < a_{i_k} a_{\sigma(i_k)} = a_{i_k} a_{i_1},$$

and since $a_{i_2} > a_{i_1}$ it follows that $a_{i_3} < a_{i_k}$, so $i_3 < i_k$. Inductively, we obtain that $i_j < i_k, j = 1, 2, \ldots, k-1$. But then

$$a_{i_{k-1}} a_{i_k} = a_{i_{k-1}} a_{\sigma(i_{k-1})} < a_{i_k} a_{\sigma(i_k)} = a_{i_k} a_{i_1},$$

hence $i_{k-1} < i_1$, a contradiction. This proves that σ is the identity permutation, and we are done.(C. Năstăsescu, C. Niţă, M. Brandiburu, D. Joiţa, Exerciţii şi Probleme de Algebră (Exercises and Problems in Algebra), Editura Didactică şi Pedagogică, Bucharest, 1983)

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :835. Determine the number of permutations $a_1, a_2, \ldots, a_{2004}$ of the numbers $1, 2, \ldots,$ 2004 for which

$$|a_1 - 1| = |a_2 - 2| = \cdots = |a_{2004} - 2004| > 0.$$

Solution :835. Let $S = \{1, 2, \ldots, 2004\}$. Write the permutation as a function $f : S \to S$, $f(n) = a_n, n = 1, 2, \ldots, 2004$. We start by noting three properties of f :(i) $f(i) \neq i$ for any i,(ii) $f(i) \neq f(j)$ if $i \neq j$,(iii) $f(i) = j$ implies $f(j) = i$. The first two properties are obvious, while the third requires a proof. Arguing by contradiction, let us assume that $f(i) = j$ but $f(j) \neq i$. We discuss first the case $j > i$. If we let $k = j - i$, then $f(i) = i + k$. Since $k = |f(i) - i| = |f(j) - j|$ and $f(j) \neq i$, it follows that $f(j) = i + 2k$, i.e., $f(i + k) = i + 2k$. The same reasoning yields $f(i + 2k) = i + k$ or $i + 3k$. Since we already have $f(i) = i + k$, the only possibility is $f(i + 2k) = i + 3k$. And the argument can be repeated to show that $f(i + nk) = i + (n + 1)k$ for all n. However, this then forces f to attain ever increasing values, which is impossible since its range is finite. A similar argument takes care of the case $j < i$. This proves (iii).The three properties show that f is an involution on S with no fixed points. Thus f partitions S into 1002 distinct pairs (i, j) with $i = f(j)$ and $j = f(i)$. Moreover, the absolute value of the difference of the elements in any pair is the same. If $f(1) - 1 = k$ then $f(2) = k + 1, \ldots, f(k) = 2k$, and since f is an involution, the values of f on $k + 1, k + 2, \ldots, 2k$ are already determined, namely $f(k + 1) = 1, f(k + 2) = 2, \ldots, f(2k) = k$. So the first block of $2k$ integers is invariant under f. Using similar reasoning, we obtain $f(2k + 1) = 3k + 1, f(2k + 2) = 3k + 2, \ldots, f(3k) = 4k, f(3k + 1) = 2k + 1, \ldots, f(4k) = 3k$. So the next block of $2k$ integers is invariant under f. Continuing this process, we see that f partitions S into blocks of $2k$ consecutive integers that are invariant under f. This can happen only if $2k$ divides 2004 , hence if k divides 1002. Furthermore, for each such k we can construct f following the recipe given above. Hence the number of such permutations equals the number of divisors of 1002 , which is 8.(Australian Mathematical Olympiad, 2004, solution by L. Field)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :836. Let n be an odd integer greater than 1 . Find the number of permutations σ of the set $\{1, 2, \ldots, n\}$ for which

$$|\sigma(1) - 1| + |\sigma(2) - 2| + \cdots + |\sigma(n) - n| = \frac{n^2 - 1}{2}.$$

Solution :836. Expanding $|\sigma(k)-k|$ as $\pm\sigma(k)\pm k$ and reordering, we see that

$$|\sigma(1)-1|+|\sigma(2)-2|+\cdots+|\sigma(n)-n| = \pm 1 \pm 1 \pm 2 \pm 2 \pm \cdots \pm n \pm n,$$

for some choices of signs. The maximum of $|\sigma(1)-1|+|\sigma(2)-2|+\cdots+|\sigma(n)-n|$ is reached by choosing the smaller of the numbers to be negative and the larger to be positive, and is therefore equal to

$$\begin{aligned} &2\left(-1-2-\cdots-\frac{n-1}{2}\right)-\frac{n+1}{2}+\frac{n+1}{2}+2\left(\frac{n+3}{2}+\cdots+n\right)\\ &\quad=-\left(1+\frac{n-1}{2}\right)\frac{n-1}{2}+\left(\frac{n+3}{2}+n\right)\frac{n-1}{2}=\frac{n^2-1}{2}. \end{aligned}$$

Therefore, in order to have $|\sigma(1)-1|+\cdots+|\sigma(n)-n| = \frac{n^2-1}{2}$, we must have

$$\left\{\sigma(1),\ldots,\sigma\left(\frac{n-1}{2}\right)\right\}\subset\left\{\frac{n+1}{2},\frac{n+3}{2},\ldots,n\right\}$$

and

$$\left\{\sigma\left(\frac{n+3}{2}\right),\sigma\left(\frac{n+5}{2}\right),\ldots,\sigma(n)\right\}\subset\left\{1,2,\ldots,\frac{n+1}{2}\right\}.$$

Let $\sigma\left(\frac{n+1}{2}\right)=k$. If $k\le\frac{n+1}{2}$, then

$$\left\{\sigma(1),\ldots,\sigma\left(\frac{n-1}{2}\right)\right\}=\left\{\frac{n+3}{2},\frac{n+5}{2},\ldots,n\right\}$$

and

$$\left\{\sigma\left(\frac{n+3}{2}\right),\sigma\left(\frac{n+5}{2}\right),\ldots,\sigma(n)\right\}=\left\{1,2,\ldots,\frac{n+1}{2}\right\}-\{k\}.$$

If $k\ge\frac{n+1}{2}$, then

$$\left\{\sigma(1),\ldots,\sigma\left(\frac{n-1}{2}\right)\right\}=\left\{\frac{n+1}{2},\frac{n+3}{2},\ldots,n\right\}-\{k\}$$

and

$$\left\{\sigma\left(\frac{n+3}{2}\right),\sigma\left(\frac{n+5}{2}\right),\ldots,\sigma(n)\right\}=\left\{1,2,\ldots,\frac{n-1}{2}\right\}.$$

For any value of k, there are $\left[\left(\frac{n-1}{2}\right)!\right]^2$ choices for the remaining values of σ, so there are

$$n\left[\left(\frac{n-1}{2}\right)!\right]^2$$

such permutations.

10 (T. Andreescu)

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :837. In how many regions do n great circles, any three nonintersecting, divide the surface of a sphere?
Solution :837. Let $f(n)$ be the desired number. We count immediately $f(1) = 2, f(2) = 4$. For the general case we argue inductively. Assume that we already have constructed n circles. When adding the $(n+1)$ st, it intersects the other circles in $2n$ points. Each of the $2n$ arcs determined by those points splits some region in two. This produces the recurrence relation $f(n+1) = f(n) + 2n$. Iterating, we obtain

$$f(n) = 2 + 2 + 4 + 6 + \cdots + 2(n-1) = n^2 - n + 2$$

(25th W.L. Putnam Mathematical Competition, 1965)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :838. In how many regions do n spheres divide the three-dimensional space if any two intersect along a circle, no three intersect along a circle, and no four intersect at one point?
Solution :838. Again we try to derive a recursive formula for the number $F(n)$ of regions. But this time counting the number of regions added by a new sphere is not easy at all. The previous problem comes in handy. The first n spheres determine on the $(n+1)$ st exactly $n^2 - n + 2$ regions. This is because the conditions from the statement give rise on the last sphere to a configuration of circles in which any two, but no three, intersect. And this is the only condition that we used in the solution to the previous problem. Each of the $n^2 - n + 2$ spherical regions divides some spatial region into two parts. This allows us to write the recursive formula

$$F(n+1) = F(n) + n^2 - n + 2, \quad F(1) = 2.$$

Iterating, we obtain

$$\begin{aligned} F(n) &= 2 + 4 + 8 + \cdots + \left[(n-1)^2 - (n-1) + 2\right] = \sum_{k=1}^{n-1} \left(k^2 - k + 2\right) \\ &= \frac{n^3 - 3n^2 + 8n}{3} \end{aligned}$$

Topic :Number Theory
Book :Putnam and Beyond

Final Answer :

Problem Statement :840. An equilateral triangle of side length n is drawn with sides along a triangular grid of side length 1 . What is the maximum number of grid segments on or inside the triangle that can be marked so that no three marked segments form a triangle?
Solution :840. The grid is made up of $\frac{n(n+1)}{2}$ small equilateral triangles of side length 1 . In each of these triangles, at most 2 segments can be marked, so we can mark at most $\frac{2}{3} \cdot \frac{3n(n+1)}{2}=n(n+1)$ segments in all. Every segment points in one of three directions, so we can achieve the maximum $n(n+1)$ by marking all the segments pointing in two of the three directions.(Russian Mathematical Olympiad, 1999)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :842. What is the largest number of internal right angles that an n-gon (convex or not, with non-self-intersecting boundary) can have?
Solution :842. We examine separately the cases $n=3,4,$ 5. A triangle can have at most one right angle, a quadrilateral four, and a pentagon three (if four angles of the pentagon were right, the fifth would have to be equal to 180°).Let us consider an n-gon with $n \geq 6$ having k internal right angles. Because the other $n-k$ angles are less than 360° and because the sum of all angles is $(n-2) \cdot 180^{\circ}$, the following inequality holds:

$$(n-k) \cdot 360^{\circ}+k \cdot 90^{\circ}>(n-2) \cdot 180^{\circ} .$$

This readily implies that $k<\frac{2n+4}{3}$, and since k and n are integers, $k \leq \left\lfloor\frac{2n}{3}\right\rfloor+1$.We will prove by induction on n that this upper bound can be reached. The base cases $n=6,7,8$ are shown in Figure 95 . MATHPIX IMAGEFigure 95We assume that the construction is done for n and prove that it can be done for $n+3$. For our method to work, we assume in addition that at least one internal angle is greater than 180°. This is the case with the polygons from Figure 95 . For the inductive step we replace the internal angle greater than 180° as shown in Figure 96. This increases the angles by 3 and the right angles by 2 . The new figure still has an internal angle greater than 180°, so the induction works. This construction proves that the bound can be reached.MATHPIX IMAGEFigure 96(short list of the 44th International Mathematical Olympiad, 2003)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :843. A circle of radius 1 rolls without slipping on the outside of a circle of radius $\sqrt{2}$. The contact point of the circles in the initial position is colored. Any time a point of one circle touches a colored point of

the other, it becomes itself colored. How many colored points will the moving circle have after 100 revolutions?
Solution :843. It seems that the situation is complicated by successive colorings. But it is not! Observe that each time the moving circle passes through the original position, a new point will be colored. But this point will color the same points on the fixed circle. In short, only the first colored point on one circle contributes to newly colored points on the other; all other colored points follow in its footsteps. So there will be as many colored points on the small circle as there are points of coordinate $2\pi k, k$ an integer, on the segment $[0, 200\pi\sqrt{2}]$. Their number is

$$\left\lfloor \frac{200\pi\sqrt{2}}{2\pi} \right\rfloor = \lfloor 100\sqrt{2} \rfloor = 141.$$

(Ukrainian Mathematical Olympiad)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :848. In the plane are given $n > 2$ points joined by segments, such that the interiors of any two segments are disjoint. Find the maximum possible number of such segments as a function of n.
Solution :848. For finding the upper bound we employ Euler's formula. View the configuration as a planar graph, and complete as many curved edges as possible, until a triangulation of the plane is obtained. If $V = n$ is the number of vertices, E the number of edges and F the number of faces (with the exterior counted among them), then $V - E + F = 2$, so $E - F = n + 2$. On the other hand, since every edge belongs to two faces and every face has three edges, $2E = 3F$. Solving, we obtain $E = 3n - 6$. Deleting the "alien" curved edges, we obtain the inequality $E \leq 3n - 6$. That the bound can be reached is demonstrated in Figure 98.MATHPIX IMAGEFigure 98(German Mathematical Olympiad, 1976)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :849. Three conflicting neighbors have three common wells. Can one draw nine paths connecting each of the neighbors to each of the wells such that no two paths intersect?
Solution :849. If this were possible, then the configuration would determine a planar graph with $V = 6$ vertices (the 3 neighbors and the 3 wells) and $E = 9$ edges (the paths). Each of its F faces would have 4 or more edges because there is no path between wells or between neighbors. So

$$F \leq \frac{2}{4}E = \frac{9}{2}.$$

On the other hand, by Euler's relation we have

$$F = 2 + E - V = 5.$$

We have reached a contradiction, which shows that the answer to the problem is negative.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :854. What is the largest number of vertices that a complete graph can have so that its edges can be colored by two colors in such a way that no monochromatic triangle is formed?
Solution :854. Figure 103 shows that this number is greater than or equal to 5.MATHPIX IMAGEFigure 103Let us show that any coloring by two colors of the edges of a complete graph with 6 vertices has a monochromatic triangle. Assume the contrary. By the pigeonhole principle, 3 of the 5 edges starting at some point have the same color (see Figure 104). Each pair of such edges forms a triangle with another edge. By hypothesis, this third edge must be of the other color. The three pairs produce three other edges that are of the same color and form a triangle. This contradicts our assumption. Hence any coloring of a complete graph with six vertices contains a monochromatic triangle. We conclude that $n = 5$.MATHPIX IMAGEFigure 104Remark. This shows that the Ramsey number $R(3, 3)$ is equal to 6 .
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :858. Let n be a positive integer satisfying the following property: If n dominoes are placed on a 6×6 chessboard with each domino covering exactly two unit squares, then one can always place one more domino on the board without moving any other dominoes. Determine the maximum value of n.
Solution :858. First solution: We will prove that the maximum value of n is 11 . Figure 105 describes an arrangement of 12 dominoes such that no additional domino can be placed on the board. Therefore, $n \leq 11$.MATHPIX IMAGE-Figure 105Let us show that for any arrangement of 11 dominoes on the board one can add one more domino. Arguing by contradiction, let us assume that there is a way of placing 11 dominoes on the board so that no more dominoes can be added. In this case there are $36 - 22 = 14$ squares not covered by dominoes.Denote by S_1 the upper 5×6 subboard, by S_2 the lower 1×6 subboard, and by S_3 the lower 5×6 subboard of the given chessboard as shown in Figure 106 .Because we cannot place another domino on the board, at least one of any two neighboring squares is covered by a domino. Hence there are at least three squares in S_2 that are covered by dominoes, and so in S_2 there are at most three uncovered squares. If A denotes the set of uncovered squares in S_1, then

$|A| \geq 14-3=11$. MATHPIX IMAGEFigure 106Let us also denote by B the set of dominoes that lie completely in S_3. We will construct a one-to-one map $f: A \rightarrow B$. First, note that directly below each square s in S_1 there is a square t of the chessboard (see Figure 107). If s is in A, then t must be covered by a domino d in B, since otherwise we could place a domino over s and t. We define $f(s)=d$. If f were not one-to-one, that is, if $f\left(s_1\right)=f\left(s_2\right)=d$, for some $s_1, s_2 \in A$, then d would cover squares directly below s_1 and s_2 as described in Figure 107. Then s_1 and s_2 must be neighbors, so a new domino can be placed to cover them. We conclude that f is one-to-one, and hence $|A| \leq|B|$. It follows that $|B| \geq 11$. But there are only 11 dominoes, so $|B|=11$. This means that all 11 dominoes lie completely in S_3 and the top row is not covered by any dominoes! We could then put three more dominoes there, contradicting our assumption on the maximality of the arrangement. Hence the assumption was wrong; one can always add a domino to an arrangement of 11 dominoes. The answer to the problem is therefore $n=11$. MATHPIX IMAGEFigure 107Second solution: Suppose we have an example with k dominoes to which no more can be added. Let X be the number of pairs of an uncovered square and a domino that covers an adjacent square. Let $m=36-2k$ be the number of uncovered squares, let m_∂ be the number of uncovered squares that touch the boundary (including corner squares), and m_c the number of uncovered corner squares. Since any neighbor of an uncovered square must be covered by some domino, we have $X=4m-m_\partial-m_c$. Similarly, let k_∂ be the number of dominoes that touch the boundary and k_c the number of dominoes that contain a corner square. A domino in the center of the board can have at most four unoccupied neighbors, for otherwise, we could place a new domino adjacent to it. Similarly, a domino that touches the boundary can have at most three unoccupied neighbors, and a domino that contains a corner square can have at most two unoccupied neighbors. Hence $X \leq 4k-k_\partial-k_c$. Also, note that $k_\partial \geq m_\partial$, since as we go around the boundary we can never encounter two unoccupied squares in a row, and $m_c+k_c \leq 4$, since there are only four corners. Thus $4m-m_\partial-m_c=X \leq 4k-k_\partial-k_c$ gives $4m-4 \leq 4k$; hence $35-2k \leq k$ and $3k \geq 35$. Thus k must be at least 12 . This argument also shows that on an $n \times n$ board, $3k^2 \geq n^2-1$.(T. Andreescu, Z. Feng, 102 Combinatorial Problems, Birkhäuser, 2000, second solution by R. Stong)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :860. Consider the triangular $n \times n$ matrix

$$
A=\left(\begin{array}{ccccc}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{array}\right) .
$$

Compute the matrix $A^k, k \geq 1$.
Solution :860. We computeMATHPIX IMAGEAlso,MATHPIX IMAGEIn general,MATHPIX IMAGEThis formula follows inductively from the combinatorial identity

$$\binom{m}{m}+\binom{m+1}{m}+\cdots+\binom{m+p}{m}=\binom{m+p+1}{m+1}$$

which holds for $m, p \geq 0$. This identity is quite straightforward and can be proved using Pascal's triangle as follows:

$$\begin{aligned}
\binom{m}{m}+\binom{m+1}{m}+\cdots+\binom{m+p}{m} & =\binom{m+1}{m+1}+\binom{m+1}{m}+\cdots+\binom{m+p}{m} \\
& =\binom{m+2}{m+1}+\binom{m+2}{m}+\cdots+\binom{m+p}{m} \\
& =\binom{m+3}{m+1}+\binom{m+3}{m}+\cdots+\binom{m+p}{m} \\
& =\cdots=\binom{m+p}{m+1}+\binom{m+p}{m}=\binom{m+p+1}{m+1} .
\end{aligned}$$

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :867. Find the general-term formula for the sequence $(y_n)_{n \geq 0}$ with $y_0=1$ and $y_n=a y_{n-1}+b^n$ for $n \geq 1$, where a and b are two fixed distinct real numbers.
Solution :867. Let $G(x)=\sum_n y_n x^n$ be the generating function of the sequence. It satisfies the functional equation

$$(1-a x) G(x)=1+b x+b x^2+\cdots=\frac{1}{1-b x}$$

We find that

$$G(x)=\frac{1}{(1-a x)(1-b x)}=\frac{A}{1-a x}+\frac{B}{1-b x}=\sum_n\left(A a^n+B b^n\right) x^n$$

for some A and B. It follows that $y_n=A a^n+B b^n$, and because $y_0=1$ and $y_1=a+b$, $A=\frac{a}{a-b}$ and $B=-\frac{b}{a-b}$. The general term of the sequence is therefore

$$\frac{1}{a-b}\left(a^{n+1}-b^{n+1}\right) .$$

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :868. Compute the sums

$$\sum_{k=1}^{n} k \binom{n}{k} \text{ and } \sum_{k=1}^{n} \frac{1}{k+1} \binom{n}{k}.$$

Solution :868. The first identity is obtained by differentiating $(x+1)^n = \sum_{k=1}^{n} \binom{n}{k} x^k$, then setting $x = 1$. The answer is $n2^{n-1}$. The second identity is obtained by integrating the same equality and then setting $x = 1$, in which case the answer is $\frac{2^{n+1}}{n+1}$.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :870. Compute the sum

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^m \binom{n}{m}.$$

Solution :870. The sum is equal to the coefficient of x^n in the expansion of

$$x^n(1-x)^n + x^{n-1}(1-x)^n + \cdots + x^{n-m}(1-x)^n.$$

This expression is equal to

$$x^{n-m} \cdot \frac{1-x^{m+1}}{1-x}(1-x)^n,$$

which can be written as $\left(x^{n-m} - x^{n+1}\right)(1-x)^{n-1}$. Hence the sum is equal to $(-1)^m \binom{n-1}{m}$ if $m < n$, and to 0 if $m = n$.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :871. Write in short form the sum

$$\binom{n}{k} + \binom{n+1}{k} + \binom{n+2}{k} + \cdots + \binom{n+m}{k}.$$

Solution :871. The sum from the statement is equal to the coefficient of x^k in the expansion of $(1+x)^n + (1+x)^{n+1} + \cdots + (1+x)^{n+m}$. This expression can be written in compact form as

$$\frac{1}{x}\left((1+x)^{n+m+1} - (1+x)^n\right).$$

We deduce that the sum is equal to $\left(\begin{array}{c} n+m+1 \\ k+1 \end{array}\right)-\left(\begin{array}{c} n \\ k+1 \end{array}\right)$ for $k<n$ and to $\left(\begin{array}{c} n+m+1 \\ n+1 \end{array}\right)$ for $k=n$.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :875. Let p be an odd prime number. Find the number of subsets of $\{1,2,\ldots,p\}$ with the sum of elements divisible by p.
Solution :875. The number of subsets with the sum of the elements equal to n is the coefficient of x^n in the product

$$G(x)=(1+x)\left(1+x^2\right)\cdots\left(1+x^p\right).$$

We are asked to compute the sum of the coefficients of x^n for n divisible by p. Call this number $s(p)$. There is no nice way of expanding the generating function; instead we compute $s(p)$ using particular values of G. It is natural to try p th roots of unity.The first observation is that if ξ is a p th root of unity, then $\sum_{k=1}^{p}\xi^p$ is zero except when $\xi=1$. Thus if we sum the values of G at the p th roots of unity, only those terms with exponent divisible by p will survive. To be precise, if ξ is a p th root of unity different from 1, then

$$\sum_{k=1}^{p} G\left(\xi^k\right)=ps(p).$$

We are left with the problem of computing $G\left(\xi^k\right), k=1,2,\ldots,p$. For $k=p$, this is just 2^p. For $k=1,2,\ldots,p-1$,

$$\begin{aligned} G\left(\xi^k\right) &= \prod_{j=1}^{p}\left(1+\xi^{kj}\right)=\prod_{j=1}^{p}\left(1+\xi^j\right)=(-1)^p\prod_{j=1}^{p}\left((-1)-\xi^j\right)=(-1)^p\left((-1)^p-1\right) \\ &= 2 \end{aligned}$$

We therefore have $ps(p)=2^p+2(p-1)=2^p+2p-2$. The answer to the problem is $s(p)=\frac{2^p-2}{p}+2$. The expression is an integer because of Fermat's little theorem.(T. Andreescu, Z. Feng, A Path to Combinatorics for Undergraduates, Birkhäuser 2004)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :878. Let $A_1,A_2,\ldots,A_n,\ldots$ and $B_1,B_2,\ldots,B_n,\ldots$ be sequences of sets defined by $A_1=\emptyset, B_1=\{0\}, A_{n+1}=\{x+1 \mid x\in B_n\}, B_{n+1}=(A_n\cup B_n)\setminus(A_n\cap B_n)$. Determine all positive integers n for which $B_n=\{0\}$.
Solution :878. We use the same generating functions as in the previous prob-

lem. So to the set A_n we associate the function

$$a_n x = \sum_{a=1}^{\infty} c_a x^a$$

with $c_a = 1$ if $a \in A_n$ and $c_a = 0$ if $a \notin A_n$. To B_n we associate the function $b_n(x)$ in a similar manner. These functions satisfy the recurrence $a_1(x) = 0, b_1(x) = 1$,

$$a_{n+1}(x) = xb_n(x)$$
$$b_{n+1} \equiv a_n(x) + b_n(x)(\text{mod}2)$$

From now on we understand all equalities modulo 2. Let us restrict our attention to the sequence of functions $b_n(x), n = 1, 2, \ldots$ It satisfies $b_1(x) = b_2(x) = 1$,

$$b_{n+1}(x) = b_n(x) + xb_{n-1}(x)$$

We solve this recurrence the way one usually solves second-order recurrences, namely by finding two linearly independent solutions $p_1(x)$ and $p_2(x)$ satisfying

$$p_i(x)^{n+1} = p_i(x)^n + xp_i(x)^{n-1}, \quad i = 1, 2.$$

Again the equality is to be understood modulo 2. The solutions $p_1(x)$ and $p_2(x)$ are formal power series whose coefficients are residue classes modulo 2 . They satisfy the "characteristic" equation

$$p(x)^2 = p(x) + x$$

which can be rewritten as

$$p(x)(p(x) + 1) = x$$

So $p_1(x)$ and $p_2(x)$ can be chosen as the factors of this product, and thus we may assume that $p_1(x) = xh(x)$ and $p_2(x) = 1 + p_1(x)$, where $h(x)$ is again a formal power series. Writing $p_1(x) = \sum \alpha_a x^a$ and substituting in the characteristic equation, we find that $\alpha_1 = 1, \alpha_{2k} = \alpha_k^2$, and $\alpha_{2k+1} = 0$ for $k > 1$. Therefore,

$$p_1(x) = \sum_{k=0}^{\infty} x^{2^k}$$

Since $p_1(x) + p_2(x) = p_1(x)^2 + p_2(x)^2 = 1$, it follows that in general,

$$b_n(x) = p_1(x)^n + p_2(x)^n = \left(\sum_{k=0}^{\infty} x^{2^k}\right)^n + \left(1 + \sum_{k=0}^{\infty} x^{2^k}\right)^n, \quad \text{for } n \geq 1.$$

We emphasize again that this is to be considered modulo 2. In order for $b_n(x)$ to be identically equal to 1 modulo 2 , we should have

$$\left(\left(\sum_{k=0}^{\infty} x^{2^k}\right) + 1\right)^n \equiv \left(\sum_{k=0}^{\infty} x^{2^k}\right)^n + 1(\text{mod}2).$$

This obviously happens if n is a power of 2 , since all binomial coefficients in the expansion are even.If n is not a power of 2 , say $n=2^i(2j+1), j\geq 1$, then the smallest m for which $\binom{n}{m}$ is odd is 2^j. The left-hand side will contain an x^{2^j} with coefficient equal to 1 , while the smallest nonzero power of x on the right is n. Hence in this case equality cannot hold.We conclude that $B_n=\{0\}$ if and only if n is a power of 2 .(Chinese Mathematical Olympiad)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :879. Find in closed form

$$1\cdot 2\binom{n}{2}+2\cdot 3\binom{n}{3}+\cdots+(n-1)\cdot n\binom{n}{n}.$$

Solution :879. We will count the number of committees that can be chosen from n people, each committee having a president and a vice-president.Choosing first a committee of k people, the president and the vice-president can then be elected in $k(k-1)$ ways. It is necessary that $k\geq 2$. The committees with president and vice-president can therefore be chosen in

$$1\cdot 2\binom{n}{2}+2\cdot 3\binom{n}{3}+\cdots+(n-1)\cdot n\binom{n}{n}$$

ways.But we can start by selecting first the president and the vice-president, and then adding the other members to the committee. From the n people, the president and the vice-president can be selected in $n(n-1)$ ways. The remaining members of the committee can be selected in 2^{n-2} ways, since they are some subset of the remaining $n-2$ people. We obtain

$$1\cdot 2\binom{n}{2}+2\cdot 3\binom{n}{3}+\cdots+(n-1)\cdot n\binom{n}{n}=n(n-1)2^{n-2}.$$

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :890. A set S containing four positive integers is called connected if for every $x\in S$ at least one of the numbers $x-1$ and $x+1$ belongs to S. Let C_n be the number of connected subsets of the set $\{1,2,\ldots,n\}$.(a) Evaluate C_7.(b) Find a general formula for C_n.
Solution :890. Let $a<b<c<d$ be the members of a connected set S. Because $a-1$ does not belong to the set, it follows that $a+1\in S$, hence $b=a+1$. Similarly, since $d+1\notin S$, we deduce that $d-1\in S$; hence $c=d-1$. Therefore, a connected set has the form $\{a,a+1,d-1,d\}$, with $d-a>2$.(a) There are

10 connected subsets of the set $\{1,2,3,4,5,6,7\}$, namely,

$$\{1,2,3,4\};\{1,2,4,5\};\{1,2,5,6\};\{1,2,6,7\},$$
$$\{2,3,4,5\};\{2,3,5,6\};\{2,3,6,7\}\{3,4,5,6\};\{2,4,6,7\};\text{ and }\{4,5,6,7\}.$$

(b) Call $D=d-a+1$ the diameter of the set $\{a,a+1,d-1,d\}$. Clearly, $D>3$ and $D\le n-1+1=n$. For $D=4$ there are $n-3$ connected sets, for $D=5$ there are $n-4$ connected sets, and so on. Adding up yields

$$C_n=1+2+3+\cdots+n-3=\frac{(n-3)(n-2)}{2},$$

which is the desired formula.(Romanian Mathematical Olympiad, 2006)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :893. Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A,B\}$ of subsets of S such that $A\cup B=S, A\cap B=\emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.
Solution :893. First, it is not hard to see that a configuration that maximizes the number of partitions should have no three collinear points. After examining several cases we guess that the maximal number of partitions is $\binom{n}{2}$. This is exactly the number of lines determined by two points, and we will use these lines to count the number of partitions. By pushing such a line slightly so that the two points lie on one of its sides or the other, we obtain a partition. Moreover, each partition can be obtained this way. There are $2\binom{n}{2}$ such lines, obtained by pushing the lines through the n points to one side or the other. However, each partition is counted at least twice this way, except for the partitions that come from the sides of the polygon that is the convex hull of the n points, but those can be paired with the partitions that cut out one vertex of the convex hull from the others. Hence we have at most $2\binom{n}{2}/2=\binom{n}{2}$ partitions.Equality is achieved when the points form a convex n-gon, in which case $\binom{n}{2}$ counts the pairs of sides that are intersected by the separating line.(67th W.L. Putnam Mathematical Competition, 2006)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :896. A sheet of paper in the shape of a square is cut by a line into two pieces. One of the pieces is cut again by a line, and so on. What

is the minimum number of cuts one should perform such that among the pieces one can find one hundred polygons with twenty sides.
Solution :896. At every cut the number of pieces grows by 1 , so after n cuts we will have $n+1$ pieces.Let us evaluate the total number of vertices of the polygons after n cuts. After each cut the number of vertices grows by 2 if the cut went through two vertices, by 3 if the cut went through a vertex and a side, or by 4 if the cut went through two sides. So after n cuts there are at most $4n+4$ vertices.Assume now that after N cuts we have obtained the one hundred polygons with 20 sides. Since altogether there are $N+1$ pieces, besides the one hundred polygons there are $N+1-100$ other pieces. Each of these other pieces has at least 3 vertices, so the total number of vertices is $100 \cdot 20+(N-99) \cdot 3$. This number does not exceed $4N+4$. Therefore,

$$4N+4 \geq 100 \cdot 20+(N-99) \cdot 3=3N+1703.$$

We deduce that $N \geq 1699$. We can obtain one hundred polygons with twenty sides by making 1699 cuts in the following way. First, cut the square into 100 rectangles (99 cuts needed). Each rectangle is then cut through 16 cuts into a polygon with twenty sides and some triangles. We have performed a total of $99+100 \cdot 16=1699$ cuts.(Kvant (Quantum), proposed by I. Bershtein)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :898. Let m, n, p, q, r, s be positive integers such that $p<r<m$ and $q<s<n$. In how many ways can one travel on a rectangular grid from $(0,0)$ to (m, n) such that at each step one of the coordinates increases by one unit and such that the path avoids the points (p, q) and (r, s) ?
Solution :898. First, let us forget about the constraint and count the number of paths from $(0,0)$ and (m, n) such that at each step one of the coordinates increases by 1 . There are a total of $m+n$ steps, out of which n go up. These n can be chosen in $\left(\begin{array}{c}m+n \\ n\end{array}\right)$ ways from the total of $m+n$. Therefore, the number of paths is $\left(\begin{array}{c}m+n \\ n\end{array}\right)$.How many of these go through (p, q) ? There are $\left(\begin{array}{c}p+q \\ q\end{array}\right)$ paths from $(0,0)$ to (p, q) and $\left(\begin{array}{c}m+n-p-q \\ n-q\end{array}\right)$ paths from (p, q) to (m, n). Hence

$$\left(\begin{array}{c}p+q \\ q\end{array}\right) \cdot\left(\begin{array}{c}m+n-p-q \\ n-q\end{array}\right)$$

of all the paths pass through (p, q). And, of course,

$$\left(\begin{array}{c}r+s \\ s\end{array}\right) \cdot\left(\begin{array}{c}m+n-r-s \\ n-s\end{array}\right)$$

paths pass through (r, s). To apply the inclusion-exclusion principle, we also need to count the number of paths that go simultaneously through (p, q) and (r, s). This number is

$$\binom{p+q}{q} \cdot \binom{r+s-p-q}{s-q} \cdot \binom{m+n-r-s}{n-s}.$$

Hence, by the inclusion-exclusion principle, the number of paths avoiding (p, q) and (r, s) is

$$\binom{m+n}{n} - \binom{p+q}{q} \cdot \binom{m+n-p-q}{n-q} - \binom{r+s}{s} \cdot \binom{m+n-r-s}{n-s} + \binom{p+q}{q} \cdot \binom{r+s-p-q}{s-q} \cdot \binom{m+n-r-s}{n-s}.$$

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :899. Let E be a set with n elements and F a set with p elements, $p \leq n$. How many surjective (i.e., onto) functions $f : E \to F$ are there?
Solution :899. Let $E = \{1, 2, \ldots, n\}$ and $F = \{1, 2, \ldots, p\}$. There are p^n functions from E to F. The number of surjective functions is $p^n - N$, where N is the number of functions that are not surjective. We compute N using the inclusion-exclusion principle.Define the sets

$$A_i = \{f : E \to F \mid i \notin f(E)\}.$$

Then

$$N = |\cup_{i=1}^p A_i| = \sum_i |A_i| - \sum_{i \neq j} |A_i \cap A_j| + \cdots + (-1)^{p-1} |\cap_{i=1}^p A_i|.$$

But A_i consists of the functions from E to $F \backslash \{i\}$; hence $|A_i| = (p-1)^n$. Similarly, for all $k, 2 \leq k \leq p-1, A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ is the set of functions from E to $F \backslash \{i_1, i_2, \ldots, i_k\}$; hence $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (p-k)^n$. Also, note that for a certain k, there are $\binom{p}{k}$ terms of the form $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$. It follows that

$$N = \binom{p}{1}(p-1)^n - \binom{p}{2}(p-2)^n + \cdots + (-1)^{p-1}\binom{p}{p-1}.$$

We conclude that the total number of surjections from E to F is

$$p^n - \binom{p}{1}(p-1)^n + \binom{p}{2}(p-2)^n - \cdots + (-1)^p \binom{p}{p-1}.$$

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :900. A permutation σ of a set S is called a derangement if it does not have fixed points, i.e., if $\sigma(x) \neq x$ for all $x \in S$. Find the number of derangements of the set $\{1,2,\ldots,n\}$.
Solution :900. We count instead the permutations that are not derangements. Denote by A_i the set of permutations σ with $\sigma(i)=i$. Because the elements in A_i have the value at i already prescribed, it follows that $|A_i|=(n-1)$!. And for the same reason, $|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}|=(n-k)$! for any distinct $i_1, i_2, \ldots, i_k, 1 \leq k \leq n$. Applying the inclusion-exclusion principle, we find that

$$|A_1 \cup A_2 \cup \cdots \cup A_n|=\binom{n}{1}(n-1)!-\binom{n}{2}(n-2)!+\cdots+(-1)^n\binom{n}{n}1!$$

The number of derangements is therefore $n!-|A_1 \cup A_2 \cup \cdots \cup A_n|$, which is

$$n!-\binom{n}{1}(n-1)!+\binom{n}{2}(n-2)!-\cdots+(-1)^n\binom{n}{n}0!$$

This number can also be written as

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\cdots+\frac{(-1)^n}{n!}\right]$$

This number is approximately equal to $\frac{n!}{e}$.
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :902. Let $m \geq 5$ and n be given positive integers, and suppose that $\mathcal{P}$ is a regular ($2n+1$) gon. Find the number of convex m-gons having at least one acute angle and having vertices exclusive among the vertices of $\mathcal{P}$.
Solution :902. If the m-gon has three acute angles, say at vertices A, B, C, then with a fourth vertex D they form a cyclic quadrilateral $ABCD$ that has three acute angles, which is impossible. Similarly, if the m-gon has two acute angles that do not share a side, say at vertices A and C, then they form with two other vertices B and D of the m-gon a cyclic quadrilateral $ABCD$ that has two opposite acute angles, which again is impossible. Therefore, the m-gon has either exactly one acute angle, or has two acute angles and they share a side.To count the number of such m-gons we employ the principle of inclusion and exclusion. Thus we first find the number of m-gons with at least one acute angle, then subtract the number of m-gons with two acute angles (which were counted twice the first time). If the acute angle of the m-gon is $A_k A_1 A_{k+r}$, the

condition that this angle is acute translates into $r \leq n$. The other vertices of the m-gon lie between A_k and A_{k+r}; hence $m-2 \leq r$, and these vertices can be chosen in $\binom{r-1}{m-3}$ ways. Note also that $1 \leq k \leq 2n-r$. Thus the number of m-gons with an acute angle at A_1 is

$$\sum_{r=m-2}^{n} \sum_{k=1}^{2n-r} \binom{r-1}{m-3} = 2n \sum_{m-2}^{n} \binom{r-1}{m-3} - \sum_{r=m-2}^{n} r \binom{r-1}{m-3}$$
$$= 2n \binom{n}{m-2} - (m-2) \binom{n+1}{m-1}$$

There are as many polygons with an acute angle at $A_2, A_3, \ldots, A_{2n+1}$.To count the number of m-gons with two acute angles, let us first assume that these acute angles are $A_s A_1 A_k$ and $A_1 A_k A_r$. The other vertices lie between A_r and A_s. We have the restrictions $2 \leq k \leq 2n-m+3, n+2 \leq r < s \leq k+n$ if $k \leq n$ and no restriction on r and s otherwise. The number of such m-gons is

$$\sum_{k=1}^{n} \binom{k-1}{m-2} + \sum_{k=n+1}^{2n+1-(m-2)} \binom{2n+1-k}{m-2} = \sum_{k=m-1}^{n} \binom{k-1}{m-2} + \sum_{s=m-2}^{n} \binom{s}{m-2}$$
$$= \binom{n+1}{m-1} + \binom{n}{m-1}$$

This number has to be multiplied by $2n+1$ to take into account that the first acute vertex can be at any other vertex of the regular n-gon.We conclude that the number of m-gons with at least one acute angle is

$$(2n+1)\left(2n \binom{n}{m-2} - (m-1)\binom{n+1}{m-1} - \binom{n}{m-1}\right).$$

Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :903. Let $S^1 = \{z \in \mathbb{C} || z \mid= 1\}$. For all functions $f : S^1 \to S^1$ set $f^1 = f$ and $f^{n+1} = f \circ f^n, n \geq 1$. Call $w \in S^1$ a periodic point of f of period n if $f^i(w) \neq w$ for $i = 1, \ldots, n-1$ and $f^n(w) = w$. If $f(z) = z^m, m$ a positive integer, find the number of periodic points of f of period 1989.
Solution :903. Denote by U_n the set of $z \in S^1$ such that $f^n(z) = z$. Because $f^n(z) = z^{m^n}, U_n$ is the set of the roots of unity of order $m^n - 1$. In our situation $n = 1989$, and we are looking for those elements of U_{1989} that do not have period less than 1989. The periods of the elements of U_{1989} are divisors of 1989. Note that $1989 = 3^2 \times 13 \times 17$. The elements we are looking for lie in the complement of $U_{1989/3} \cup U_{1989/13} \cup U_{1989/17}$. Using the inclusion-exclusion principle, we find that the answer to the problem is

$$|U_{1989}| - \left|U_{1989/3}\right| - \left|U_{1989/13}\right| - \left|U_{1989/17}\right| + \left|U_{1989/3} \cap U_{1898/13}\right| + \left|U_{1989/3} \cap U_{1989/17}\right|$$
$$+ \left|U_{1989/13} \cap U_{1989/17}\right| + \left|U_{1989/3} \cap U_{1989/13} \cap U_{1989/17}\right|,$$

i.e.,

$$|U_{1989}| - |U_{663}| - |U_{153}| - |U_{117}| + |U_{51}| + |U_{39}| + |U_9| - |U_3|.$$

This number is equal to

$$m^{1989} - m^{663} - m^{153} - m^{117} + m^{51} + m^{39} + m^9 - m^3,$$

since the -1 's in the formula for the cardinalities of the U_n 's cancel out.(Chinese Mathematical Olympiad, 1989)
Topic :Number Theory
Book :Putnam and Beyond
Final Answer :

Problem Statement :905. A $150 \times 324 \times 375$ rectangular solid is made by gluing together $1 \times 1 \times 1$ cubes. An internal diagonal of this solid passes through the interiors of how many of the $1 \times 1 \times 1$ cubes?
Solution :905. We solve the problem for the general case of a rectangular solid of width w, length l, and height h, where w, l, and h are positive integers. Orient the solid in space so that one vertex is at $O = (0,0,0)$ and the opposite vertex is at $A = (w, l, h)$. Then OA is the diagonal of the solid.The diagonal is transversal to the planes determined by the faces of the small cubes, so each time it meets a face, edge, or vertex, it leaves the interior of one cube and enters the interior of another. Counting by the number of interiors of small cubes that the diagonal leaves, we find that the number of interiors that OA intersects is equal to the number of points on OA having at least one integer coordinate. We count these points using the inclusion-exclusion principle. The first coordinate of the current point $P = (tw, tl, th), 0 < t \leq 1$, on the diagonal is a positive integer for exactly w values of t, namely, $t = \frac{1}{w}, \frac{2}{2}, \ldots, \frac{w}{w}$. The second coordinate is an integer for l values of t, and the third coordinate is an integer for h values of t. However, the sum $w + l + h$ doubly counts the points with two integer coordinates, and triply counts the points with three integer coordinates. The first two coordinates are integers precisely when $t = \frac{k}{\gcd(w,l)}$, for some integer $k, 1 \leq k \leq \gcd(w, l)$. Similarly, the second and third coordinates are positive integers for $\gcd(l, h)$, respectively, $\gcd(h, w)$ values of t, and all three coordinates are positive integers for $\gcd(w, l, h)$ values of t.The inclusion-exclusion principle shows that the diagonal passes through the interiors of

$$w + l + h - \gcd(w, l) - \gcd(l, h) - \gcd(h, w) + \gcd(w, l, h)$$

small cubes. For $w = 150, l = 324, h = 375$ this number is equal to 768 .(American Invitational Mathematics Examination, 1996)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :906. Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Find the probability that $\sqrt{2 + \sqrt{3}} \leq |v + w|$.

Solution :906. Because the 1997 roots of the equation are symmetrically distributed in the complex plane, there is no loss of generality to assume that $v = 1$. We are required to find the probability that

$$|1 + w|^2 = |(1 + \cos\theta) + i \sin\theta|^2 = 2 + 2\cos\theta \geq 2 + \sqrt{3}.$$

This is equivalent to $\cos\theta \geq \frac{1}{2}\sqrt{3}$, or $|\theta| \leq \frac{\pi}{6}$. Because $w \neq 1, \theta$ is of the form $\pm\frac{2k\pi}{1997}k$, $1 \leq k \leq \left\lfloor \frac{1997}{12} \right\rfloor$. There are $2 \cdot 166 = 332$ such angles, and hence the probability is $\frac{332}{1996} = \frac{83}{499} \approx 0.166$(American Invitational Mathematics Examination, 1997)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :907. Find the probability that in a group of n people there are two with the same birthday. Ignore leap years.
Solution :907. It is easier to compute the probability that no two people have the same birthday. Arrange the people in some order. The first is free to be born on any of the 365 days. But only 364 dates are available for the second, 363 for the third, and so on. The probability that no two people have the same birthday is therefore

$$\frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365 - n + 1}{365} = \frac{365!}{(365 - n)!365^n}.$$

And the probability that two people have the same birthday is

$$1 - \frac{365!}{(365 - n)!365^n}.$$

Remark. Starting with $n = 23$ the probability becomes greater than $\frac{1}{2}$, while when $n > 365$ the probability is clearly 1 by the pigeonhole principle.
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :908. A solitaire game is played as follows. Six distinct pairs of matched tiles are placed in a bag. The player randomly draws tiles one at a time from the bag and retains them, except that matching tiles are put aside as soon as they appear in the player's hand. The game ends if the player ever holds three tiles, no two of which match; otherwise, the drawing continues until the bag is empty. Find the probability that the bag will be emptied.
Solution :908. Denote by $P(n)$ the probability that a bag containing n distinct pairs of tiles will be emptied, $n \geq 2$. Then $P(n) = Q(n)P(n-1)$ where $Q(n)$ is the probability that two of the first three tiles selected make a pair. The latter

one is

$$Q(n) = \frac{\text{number of ways to select three tiles, two of which match}}{\text{number of ways to select three tiles}}$$
$$= \frac{n(2n-2)}{\begin{pmatrix} 2n \\ 3 \end{pmatrix}} = \frac{3}{2n-1}.$$

The recurrence
$$P(n) = \frac{3}{2n-1} P(n-1)$$
yields
$$P(n) = \frac{3^{n-2}}{(2n-1)(2n-3)\cdots 5} P(2).$$
Clearly, $P(2) = 1$, and hence the answer to the problem is
$$P(6) = \frac{3^4}{11 \cdot 9 \cdot 7 \cdot 5} = \frac{9}{385} \approx 0.023.$$
(American Invitational Mathematics Examination, 1994)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :909. An urn contains n balls numbered $1, 2, \ldots, n$. A person is told to choose a ball and then extract m balls among which is the chosen one. Suppose he makes two independent extractions, where in each case he chooses the remaining $m-1$ balls at random. What is the probability that the chosen ball can be determined?
Solution :909. Because there are two extractions each of with must contain a certain ball, the total number of cases is $\begin{pmatrix} n-1 \\ m-1 \end{pmatrix}^2$. The favorable cases are those for which the balls extracted the second time differ from those extracted first (except of course the chosen ball). For the first extraction there are $\begin{pmatrix} n-1 \\ m-1 \end{pmatrix}$ cases, while for the second there are $\begin{pmatrix} n-m \\ m-1 \end{pmatrix}$, giving a total number of cases $\begin{pmatrix} n-1 \\ m-1 \end{pmatrix}\begin{pmatrix} n-m \\ m-1 \end{pmatrix}$. Taking the ratio, we obtain the desired probability as
$$P = \frac{\begin{pmatrix} n-1 \\ m-1 \end{pmatrix}\begin{pmatrix} n-m \\ m-1 \end{pmatrix}}{\begin{pmatrix} n-1 \\ m-1 \end{pmatrix}} = \frac{\begin{pmatrix} n-m \\ m-1 \end{pmatrix}}{\begin{pmatrix} n-1 \\ m-1 \end{pmatrix}}.$$
(Gazeta Matematic ă (Mathematics Gazette, Bucharest), proposed by C. Marinescu)

Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :910. A bag contains 1993 red balls and 1993 black balls. We remove two balls at a time repeatedly and(i) discard them if they are of the same color,(ii) discard the black ball and return to the bag the red ball if they are of different colors.What is the probability that this process will terminate with one red ball in the bag?
Solution :910. First, observe that since at least one ball is removed during each stage, the process will eventually terminate, leaving no ball or one ball in the bag. Because red balls are removed 2 at a time and since we start with an odd number of red balls, the number of red balls in the bag at any time is odd. Hence the process will always leave red balls in the bag, and so it must terminate with exactly one red ball. The probability we are computing is therefore 1 .(Mathematics and Informatics Quarterly, proposed by D. Macks)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :911. The numbers $1,2,3,4,5,6,7$, and 8 are written on the faces of a regular octahedron so that each face contains a different number. Find the probability that no two consecutive numbers are written on faces that share an edge, where 8 and 1 are considered consecutive.
Solution :911. Consider the dual cube to the octahedron. The vertices A, B, C, D, E, F, G, H of this cube are the centers of the faces of the octahedron (here $ABCD$ is a face of the cube and $(A, G),(B, H),(C, E),(D, F)$ are pairs of diagonally opposite vertices). Each assignment of the numbers $1,2,3,4,5,6,7$, and 8 to the faces of the octahedron corresponds to a permutation of $ABCDEFGH$, and thus to an octagonal circuit of these vertices. The cube has 16 diagonal segments that join nonadjacent vertices. The problem requires us to count octagonal circuits that can be formed by eight of these diagonals.Six of these diagonals are edges of the tetrahedron $ACFH$, six are edges of the tetrahedron $DBEG$, and four are long diagonals, joining opposite vertices of the cube. Notice that each vertex belongs to exactly one long diagonal. It follows that an octagonal circuit must contain either 2 long diagonals separated by 3 tetrahedron edges (Figure 108a), or 4 long diagonals (Figure 108b) alternating with tetrahedron edges. MATHPIX IMAGEFigure 108When forming a (skew) octagon with 4 long diagonals, the four tetrahedron edges need to be disjoint; hence two are opposite edges of $ACFH$ and two are opposite edges of $DBEG$. For each of the three ways to choose a pair of opposite edges from the tetrahedron $ACFH$, there are two possible ways to choose a pair of opposite edges from tetrahedron $DBEG$. There are $3 \cdot 2=6$ octagons of this type, and for each of them, a circuit can start at 8 possible vertices and can be traced in two different ways, making a total of $6 \cdot 8 \cdot 2=96$ permutations.An octagon that contains exactly two long diagonals must also contain a three-edge path along the tetrahedron $ACFH$

and a three-edge path along tetrahedron the $DBEG$. A three-edge path along the tetrahedron the $ACFH$ can be chosen in $4! = 24$ ways. The corresponding three-edge path along the tetrahedron $DBEG$ has predetermined initial and terminal vertices; it thus can be chosen in only 2 ways. Since this counting method treats each path as different from its reverse, there are $8 \cdot 24 \cdot 2 = 384$ permutations of this type.In all, there are $96 + 384 = 480$ permutations that correspond to octagonal circuits formed exclusively from cube diagonals. The probability of randomly choosing such a permutation is $\frac{480}{8!} = \frac{1}{84}$.(American Invitational Mathematics Examination, 2001)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :912. What is the probability that a permutation of the first n positive integers has the numbers 1 and 2 within the same cycle.
Solution :912. The total number of permutations is of course n !. We will count instead the number of permutations for which 1 and 2 lie in different cycles.If the cycle that contains 1 has length k, we can choose the other $k-1$ elements in $\left(\begin{array}{c} n-2 \\ k-1 \end{array} \right)$ ways from the set $\{3, 4, \ldots, n\}$. There exist $(k-1)$! circular permutations of these elements, and $(n-k)$! permutations of the remaining $n-k$ elements. Hence the total number of permutations for which 1 and 2 belong to different cycles is equal to

$$\sum_{k=1}^{n-1} \left(\begin{array}{c} n-2 \\ k-1 \end{array} \right) (k-1)!(n-k)! = (n-2)! \sum_{k=1}^{n-1} (n-k) = (n-2)! \frac{n(n-1)}{2} = \frac{n!}{2}.$$

It follows that exactly half of all permutations contain 1 and 2 in different cycles, and so half contain 1 and 2 in the same cycle. The probability is $\frac{1}{2}$.(I. Tomescu Problems in Combinatorics, Wiley, 1985)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :913. An unbiased coin is tossed n times. Find a formula, in closed form, for the expected value of $|H-T|$, where H is the number of heads, and T is the number of tails.
Solution :913. There are $\left(\begin{array}{c} n \\ k \end{array} \right)$ ways in which exactly k tails appear, and in this case the difference is $n-2k$. Hence the expected value of $|H-T|$ is

$$\frac{1}{2^n} \sum_{k=0}^{n} |n-2k| \left(\begin{array}{c} n \\ k \end{array} \right).$$

Evaluate the sum as follows:

$$\begin{aligned}
\frac{1}{2^n}\sum_{m=0}^{n}|n-2m|\binom{n}{m} &= \frac{1}{2^{n-1}}\sum_{m=0}^{\lfloor n/2\rfloor}(n-2m)\binom{n}{m} \\
&= \frac{1}{2^{n-1}}\left(\sum_{m=0}^{\lfloor n/2\rfloor}(n-m)\binom{n}{m}-\sum_{m=0}^{\lfloor n/2\rfloor}m\binom{n}{m}\right) \\
&= \frac{1}{2^{n-1}}\left(\sum_{m=0}^{\lfloor n/2\rfloor}n\binom{n-1}{m}-\sum_{m=1}^{\lfloor n/2\rfloor}n\binom{n-1}{m-1}\right) \\
&= \frac{n}{2^{n-1}}\binom{n-1}{\lfloor \frac{n}{2}\rfloor}.
\end{aligned}$$

(35th W.L. Putnam Mathematical Competition, 1974)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :915. An exam consists of 3 problems selected randomly from a list of $2n$ problems, where n is an integer greater than 1 . For a student to pass, he needs to solve correctly at least two of the three problems. Knowing that a certain student knows how to solve exactly half of the $2n$ problems, find the probability that the student will pass the exam.
Solution :915. Denote by A_i the event "the student solves correctly exactly i of the three proposed problems, " $i=0,1,2,3$. The event A whose probability we are computing is

$$A=A_2\cup A_3,$$

and its probability is

$$P(A)=P(A_2)+P(A_3),$$

since A_2 and A_3 exclude each other.Because the student knows how to solve half of all the problems,

$$P(A_0)=P(A_3) \quad \text{and} \quad P(A_1)=P(A_2).$$

The equality

$$P(A_0)+P(A_1)+P(A_2)+P(A_3)=1$$

becomes

$$2\left[P(A_2)+P(A_3)\right]=1.$$

It follows that the probability we are computing is

$$P(A)=P(A_2)+P(A_3)=\frac{1}{2}.$$

(N. Negoescu, Probleme cu... Probleme (Problems with... Problems), Editura Facla, 1975)

Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :916. The probability that a woman has breast cancer is 1%. If a woman has breast cancer, the probability is 60% that she will have a positive mammogram. However, if a woman does not have breast cancer, the mammogram might still come out positive, with a probability of 7%. What is the probability for a woman with positive mammogram to actually have cancer?
Solution :916. For the solution we will use Bayes' theorem for conditional probabilities. We denote by $P(A)$ the probability that the event A holds, and by $P\left(\frac{B}{A}\right)$ the probability that the event B holds given that A in known to hold. Bayes' theorem states that

$$P(B/A) = \frac{P(B)}{P(A)} \cdot P(A/B).$$

For our problem A is the event that the mammogram is positive and B the event that the woman has breast cancer. Then $P(B) = 0.01$, while $P(A/B) = 0.60$. We compute $P(A)$ from the formula

$$P(A) = P(A/B)P(B) + P(A/\operatorname{not} B)P(\text{ not } B) = 0.6 \cdot 0.01 + 0.07 \cdot 0.99 = 0.0753.$$

The answer to the question is therefore

$$P(B/A) = \frac{0.01}{0.0753} \cdot 0.6 = 0.0795 \approx 0.08$$

The chance that the woman has breast cancer is only 8% !
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :917. Find the probability that in the process of repeatedly flipping a coin, one will encounter a run of 5 heads before one encounters a run of 2 tails.
Solution :917. We call a successful string a sequence of H 's and T 's in which $HHHHH$ appears before TT does. Each successful string must belong to one of the following three types:(i) those that begin with T, followed by a successful string that begins with H;(ii) those that begin with H, HH, HHH, or $HHHH$, followed by a successful string that begins with T;(iii) the string $HHHHH$.Let P_H denote the probability of obtaining a successful string that begins with H, and let P_T denote the probability of obtaining a successful string that begins with T. Then

$$\begin{aligned} P_T &= \frac{1}{2} P_H, \\ P_H &= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) P_T + \frac{1}{32}. \end{aligned}$$

Solving these equations simultaneously, we find that

$$P_H = \frac{1}{17} \quad \text{and} \quad P_T = \frac{1}{34}.$$

Hence the probability of obtaining five heads before obtaining two tails is $\frac{3}{34}$.(American Invitational Mathematics Examination, 1995)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :918. The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assumed to be independent; namely, we are given the following:(i) $P(x^\circ = 70^\circ) = a$, the probability that the temperature in Chicago is 70°,(ii) $P(y^\circ = 70^\circ) = b$, and(iii) $P(\max(x^\circ, y^\circ) = 70^\circ) = c$.Determine $P(\min(x^\circ, y^\circ) = 70^\circ)$ in terms of a, b, and c.
Solution :918. Let us denote the events $x = 70^\circ, y = 70^\circ, \max(x^\circ, y^\circ) = 70^\circ, \min(x^\circ, y^\circ) = 70^\circ$ by A, B, C, D, respectively. We see that $A \cup B = C \cup D$ and $A \cap B = C \cap D$. Hence$P(A) + P(B) = P(A \cup B) + P(A \cap B) = P(C \cup D) + P(C \cap D) = P(C) + P(D)$.Therefore, $P(D) = P(A) + P(B) - P(C)$, that is,

$$\begin{aligned} P(\min(x^\circ, y^\circ) = 70^\circ) &= P(x^\circ = 70^\circ) + P(y^\circ = 70^\circ) - P(\max(x^\circ, y^\circ) = 70^\circ) \\ &= a + b - c. \end{aligned}$$

(29th W.L. Putnam Mathematical Competition, 1968)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :919. An urn contains both black and white marbles. Each time you pick a marble you return it to the urn. Let p be the probability of drawing a white marble and $q = 1 - p$ the probability of drawing a black marble. Marbles are drawn until n black marbles have been drawn. If $n + x$ is the total number of draws, find the probability that $x = m$.
Solution :919. In order for n black marbles to show up in $n + x$ draws, two independent events should occur. First, in the initial $n + x - 1$ draws exactly $n - 1$ black marbles should be drawn. Second, in the $(n + x)$ th draw a black marble should be drawn. The probability of the second event is simply q. The probability of the first event is computed using the Bernoulli scheme; it is equal to

$$\binom{n+x-1}{x} p^x q^{n-1}.$$

The answer to the problem is therefore

$$\binom{n+m-1}{m} p^m q^{n-1} q = \binom{n+m-1}{m} p^m q^n.$$

(Romanian Mathematical Olympiad, 1971)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :920. Three independent students took an exam. The random variable X, representing the students who passed, has the distribution

$$\left(\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \frac{2}{5} & \frac{13}{30} & \frac{3}{20} & \frac{1}{60} \end{array}\right).$$

Find each student's probability of passing the exam.
Solution :920. First solution: Denote by p_1, p_2, p_3 the three probabilities. By hypothesis,

$$\begin{aligned} & P(X=0)=\prod_i\left(1-p_i\right)=1-\sum_i p_i+\sum_{i \neq j} p_i p_j-p_1 p_2 p_3=\frac{2}{5}, \\ & P(X=1)=\sum_{\{i, j, k\}=\{1,2,3\}} p_i\left(1-p_j\right)\left(1-p_k\right)=\sum_i p_i-2 \sum_{i \neq j} p_i p_j+3 p_1 p_2 p_3=\frac{13}{30}, \\ & P(X=2)=\sum_{\{i, j, k\}=\{1,2,3\}} p_i p_j\left(1-p_k\right)=\sum_{i \neq j} p_i p_j-3 p_1 p_2 p_3=\frac{3}{20}, \\ & P(X=3)=p_1 p_2 p_3=\frac{1}{60}. \end{aligned}$$

This is a linear system in the unknowns $\sum_i p_i, \sum_{i \neq j} p_i p_j$, and $p_1 p_2 p_3$ with the solution

$$\sum_i p_i=\frac{47}{60}, \quad \sum_{i \neq j} p_i p_j=\frac{1}{5}, \quad p_1 p_2 p_3=\frac{1}{60}.$$

It follows that p_1, p_2, p_3 are the three solutions to the equation

$$x^3-\frac{47}{60} x^2+\frac{1}{5} x-\frac{1}{60}=0.$$

Searching for solutions of the form $\frac{1}{q}$ with q dividing 60, we find the three probabilities to be equal to $\frac{1}{3}, \frac{1}{4}$, and $\frac{1}{5}$. Second solution: Using the Poisson scheme

$$\left(p_1 x+1-p_1\right)\left(p_2 x+1-p_2\right)\left(p_3 x+1-p_3\right)=\frac{2}{5}+\frac{13}{30} x+\frac{3}{20} x^2+\frac{1}{60} x^3,$$

we deduce that $1-\frac{1}{p_i}, i=1,2,3$, are the roots of $x^3+9 x^2+26 x+24=0$ and $p_1 p_2 p_3=\frac{1}{60}$. The three roots are $-2,-3,-4$, which again gives p_i 's equal to $\frac{1}{3}, \frac{1}{4}$, and $\frac{1}{5}$.(N. Negoescu, Probleme cu... Probleme (Problems with... Problems), Editura Facla, 1975)
Topic :Probability
Book :Putnam and Beyond

Final Answer :

Problem Statement :921. Given the independent events $A_1, A_2, \ldots, A_n$ with probabilities $p_1, p_2, \ldots, p_n$, find the probability that an odd number of these events occurs.
Solution :921. Set $q_i = 1 - p_i, i = 1, 2, \ldots, n$, and consider the generating function

$$Q(x) = \prod_{i=1}^{n}(p_i x + q_i) = Q_0 + Q_1 x + \cdots + Q_n x^n.$$

The probability for exactly k of the independent events $A_1, A_2, \ldots, A_n$ to occur is equal to the coefficient of x^k in $Q(x)$, hence to Q_k. The probability P for an odd number of events to occur is thus equal to $Q_1 + Q_3 + \cdots$. Let us compute this number in terms of $p_1, p_2, \ldots, p_n$.We have

$$Q(1) = Q_0 + Q_1 + \cdots + Q_n \quad \text{and} \quad Q(-1) = Q_0 - Q_1 + \cdots + (-1)^n Q_n.$$

Therefore,

$$P = \frac{Q(1) - Q(-1)}{2} = \frac{1}{2}\left(1 - \prod_{i=1}^{n}(1 - 2p_i)\right).$$

(Romanian Mathematical Olympiad, 1975)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :922. Out of every batch of 100 products of a factory, 5 are quality checked. If one sample does not pass the quality check, then the whole batch of one hundred will be rejected. What is the probability that a batch is rejected if it contains 5% faulty products.
Solution :922. It is easier to compute the probability of the contrary event, namely that the batch passes the quality check. Denote by A_i the probability that the i th checked product has the desired quality standard. We then have to compute $P\left(\cap_{i=1}^{5} A_i\right)$. The events are not independent, so we use the formula

$$\begin{aligned} P\left(\cap_{i=1}^{5} A_i\right) = & P\left(A_1\right) P\left(A_2 / A_1\right)\left(A_3 / A_1 \cap A_2\right) P\left(A_4 / A_1 \cap A_2 \cap A_3\right) \\ & \times P\left(A_5 / A_1 \cap A_2 \cap A_3 \cap A_4\right). \end{aligned}$$

We find successively $P\left(A_1\right) = \frac{95}{100}, P\left(A_2 / A_1\right) = \frac{94}{99}$ (because if A_1 occurs then we are left with 99 products out of which 94 are good), $P\left(A_3 / A_1 \cap A_2\right) = \frac{93}{98}, P\left(A_4 / A_1 \cap A_2 \cap A_3\right) = \frac{92}{97}, P\left(A_5 / A_1 \cap A_2 \cap A_3 \cap A_4\right) = \frac{91}{96}$. The answer to the problem is

$$1 - \frac{95}{100} \cdot \frac{94}{99} \cdot \frac{93}{98} \cdot \frac{92}{97} \cdot \frac{91}{96} \approx 0.230.$$

Topic :Probability
Book :Putnam and Beyond

Final Answer :

Problem Statement :923. There are two jet planes and a propeller plane at the small regional airport of Gauss City. A plane departs from Gauss City and arrives in Eulerville, where there were already five propeller planes and one jet plane. Later, a farmer sees a jet plane flying out of Eulerville. What is the probability that the plane that arrived from Gauss City was a propeller plane, provided that all events are equiprobable?
Solution :923. We apply Bayes' formula. Let B be the event that the plane flying out of Eulerville is a jet plane and A_1, respectively, A_2, the events that the plane flying between the two cities is a jet, respectively, a propeller plane. Then

$$P\left(A_1\right)=\frac{2}{3}, \quad P\left(A_2\right)=\frac{1}{3}, \quad P\left(B / A_1\right)=\frac{2}{7}, \quad P\left(B / A_2\right)=\frac{1}{7}.$$

Bayes formula gives

$$P\left(A_2 / B\right)=\frac{P\left(A_2\right) P\left(B / A_2\right)}{P\left(A_1\right) P\left(B / A_1\right)+P\left(A_2\right) P\left(B / A_2\right)}=\frac{\frac{1}{3} \cdot \frac{1}{7}}{\frac{2}{3} \cdot \frac{2}{7}+\frac{1}{3} \cdot \frac{1}{7}}=\frac{1}{5}.$$

Thus the answer to the problem is $\frac{1}{5}$.Remark. Without the farmer seeing the jet plane flying out of Eulerville, the probability would have been $\frac{1}{3}$. What you know affects your calculation of probabilities.
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :924. A coin is tossed n times. What is the probability that two heads will turn up in succession somewhere in the sequence?
Solution :924. We find instead the probability $P(n)$ for no consecutive heads to appear in n throws. We do this recursively. If the first throw is tails, which happens with probability $\frac{1}{2}$, then the probability for no consecutive heads to appear afterward is $P(n-1)$. If the first throw is heads, the second must be tails, and this configuration has probability $\frac{1}{4}$. The probability that no consecutive heads appear later is $P(n-2)$. We obtain the recurrence

$$P(n)=\frac{1}{2} P(n-1)+\frac{1}{4} P(n-2),$$

with $P(1)=1$, and $P(2)=\frac{3}{4}$. Make this relation more homogeneous by substituting $x_n=2^n P(n)$. We recognize the recurrence for the Fibonacci sequence $x_{n+1}=x_n+x_{n-1}$, with the remark that $x_1=F_3$ and $x_2=F_4$. It follows that $x_n=F_{n+2}, P(n)=\frac{F_{n+2}}{2^n}$, and the probability required by the problem is $P(n)=1-\frac{F_{n+2}}{2^n}$.(L.C. Larson, Problem-Solving Through Problems, Springer-Verlag, 1990)
Topic :Probability
Book :Putnam and Beyond

Final Answer :

Problem Statement :925. Two people, A and B, play a game in which the probability that A wins is p, the probability that B wins is q, and the probability of a draw is r. At the beginning, A has m dollars and B has n dollars. At the end of each game, the winner takes a dollar from the loser. If A and B agree to play until one of them loses all his/her money, what is the probability of A winning all the money?
Solution :925. Fix $N = m + n$, the total amount of money, and vary m. Denote by $P(m)$ the probability that A wins all the money when starting with m dollars. Clearly, $P(0) = 0$ and $P(N) = 1$. We want a recurrence relation for $P(m)$.Assume that A starts with k dollars. During the first game, A can win, lose, or the game can be a draw. If A wins this game, then the probability of winning all the money afterward is $P(k + 1)$. If A loses, the probability of winning in the end is $P(k - 1)$. Finally, if the first game is a draw, nothing changes, so the probability of A winning in the end remains equal to $P(k)$. These three situations occur with probabilities p, q, r, respectively; hence

$$P(k) = pP(k + 1) + qP(k - 1) + rP(k).$$

Taking into account that $p + q + r = 1$, we obtain the recurrence relation

$$pP(k + 1) - (p + q)P(k) + qP(k - 1) = 0.$$

The characteristic equation of this recurrence is $p\lambda^2 - (p + q)\lambda + q = 0$. There are two cases. The simpler is $p = q$. Then the equation has the double root $\lambda = 1$, in which case the general term is a linear function in k. Since $P(0) = 0$ and $P(N) = 1$, it follows that $P(m) = \frac{m}{N} = \frac{m}{n+m}$. If $p \neq q$, then the distinct roots of the equation are $\lambda_1 = 1$ and $\lambda_2 = \frac{q}{p}$, and the general term must be of the form $P(k) = c_1 + c_2 \left(\frac{q}{p}\right)^k$. Using the known values for $k = 0$ and N, we compute

$$c_1 = -c_2 = \frac{1}{1 - \left(\frac{q}{p}\right)^N}.$$

Hence the required probability is

$$\frac{m}{m+n} \text{ if } p = q \quad \text{ and } \quad \frac{1 - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^{m+n}} \text{ if } p \neq q.$$

(K.S. Williams, K. Hardy, The Red Book of Mathematical Problems, Dover, Mineola, NY, 1996)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :926. We play the coin tossing game in which if tosses match, I get both coins; if they differ, you get both. You have m coins, I have n. What is the expected length of the game (i.e., the number of tosses until one of us is wiped out)?
Solution :926. Seeking a recurrence relation, we denote by $E(m, n)$ this expected length. What happens, then, after one toss? Half the time you win, and then the parameters become $m+1, n-1$; the other half of the time you lose, and the parameters become $m-1, n+1$. Hence the recurrence

$$E(m, n)=1+\frac{1}{2} E(m-1, n+1)+\frac{1}{2} E(m+1, n-1),$$

the 1 indicating the first toss. Of course, this assumes $m, n>0$. The boundary conditions are that $E(0, n)=0$ and $E(m, 0)=0$, and these, together with the recurrence formula, do determine uniquely the function $E(m, n)$.View $E(m, n)$ as a function of one variable, say n, along the line $m+n=$ constant. Solving the inhomogeneous second-order recurrence, we obtain $E(m, n)=m n$. Alternately, the recursive formula says that the second difference is the constant (-2), and so $E(m, n)$ is a quadratic function. Vanishing at the endpoints forces it to be cmn, and direct evaluation shows that $c=1$.(D.J. Newman, A Problem Seminar, Springer-Verlag)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :927. What is the probability that the sum of two randomly chosen numbers in the interval $[0,1]$ does not exceed 1 and their product does not exceed $\frac{2}{9}$?
Solution :927. Let x and y be the two numbers. The set of all possible outcomes is the unit square

$$D=\{(x, y) \mid 0 \leq x \leq 1,0 \leq y \leq 1\}.$$

The favorable cases consist of the region

$$D_f=\left\{(x, y) \in D \mid x+y \leq 1, x y \leq \frac{2}{9}\right\}.$$

This is the set of points that lie below both the line $f(x)=1-x$ and the hyperbola $g(x)=\frac{2}{9 x}$. equal toThe required probability is $P=\frac{\operatorname{Area}\left(D_f\right)}{\operatorname{Area}(D)}$. The area of D is 1 . The area of D_f is

$$\int_0^1 \min (f(x), g(x)) d x.$$

The line and the hyperbola intersect at the points $\left(\frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}\right)$. Therefore,

$$\operatorname{Area}\left(D_f\right)=\int_0^{1 / 3}(1-x) d x+\int_{1 / 3}^{2 / 3} \frac{2}{9 x} d x+\int_{2 / 3}^1(1-x) d x=\frac{1}{3}+\frac{2}{9} \ln 2.$$

We conclude that $P=\frac{1}{3}+\frac{2}{9} \ln 2 \approx 0.487$.(C. Reischer, A. Sâmboan, Culegere de Probleme de Teoria Probabilitătilor şi Statistică Matematica (Collection of Problems of Probability Theory and Mathematical Statistics), Editura Didactică şi Pedagogică, Bucharest, 1972)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :928. Let α and β be given positive real numbers, with $\alpha<\beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α ?
Solution :928. The total region is a square of side β. The favorable region is the union of the two triangular regions shown in Figure 109, and hence the probability of a favorable outcome is

$$
\frac{(\beta-\alpha)^{2}}{\beta^{2}}=\left(1-\frac{\alpha}{\beta}\right)^{2} .
$$

MATHPIX IMAGEFigure 109(22nd W.L. Putnam Mathematical Competition, 1961)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :929. A husband and wife agree to meet at a street corner between 4 and 5 o'clock to go shopping together. The one who arrives first will await the other for 15 minutes, and then leave. What is the probability that the two meet within the given time interval, assuming that they can arrive at any time with the same probability?
Solution :929. Denote by x, respectively, y, the fraction of the hour when the husband, respectively, wife, arrive. The configuration space is the square

$$
D=\{(x, y) \mid 0 \leq x \leq 1,0 \leq y \leq 1\} .
$$

In order for the two people to meet, their arrival time must lie inside the region

$$
D_{f}=\left\{(x, y)||x-y \mid \leq \frac{1}{4}\right\} .
$$

The desired probability is the ratio of the area of this region to the area of the square.The complement of the region consists of two isosceles right triangles with legs equal to $\frac{3}{4}$, and hence of areas $\frac{1}{2}\left(\frac{3}{4}\right)^{2}$. We obtain for the desired probability

$$
1-2 \cdot \frac{1}{2} \cdot\left(\frac{3}{4}\right)^{2}=\frac{7}{16} \approx 0.44 .
$$

11 (B.V. Gnedenko)

Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :930. Two airplanes are supposed to park at the same gate of a concourse. The arrival times of the airplanes are independent and randomly distributed throughout the 24 hours of the day. What is the probability that both can park at the gate, provided that the first to arrive will stay for a period of two hours, while the second can wait behind it for a period of one hour?
Solution :930. The set of possible events is modeled by the square [0, 24] $\times[0,24]$. It is, however, better to identify the 0th and the 24th hours, thus obtaining a square with opposite sides identified, an object that in mathematics is called a torus (which is, in fact, the Cartesian product of two circles. The favorable region is outside a band of fixed thickness along the curve $x=y$ on the torus as depicted in Figure 110. On the square model this region is obtained by removing the points (x,y) with $|x-y| \leq 1$ together with those for which $|x-y-1| \leq 1$ and $|x-y+1| \leq 1$. The required probability is the ratio of the area of the favorable region to the area of the square, and is

$$P=\frac{24^2-2\cdot 24}{24^2}=\frac{11}{12} \approx 0.917.$$

MATHPIX IMAGEFigure 110
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :931. What is the probability that three points selected at random on a circle lie on a semicircle?
Solution :931. We assume that the circle of the problem is the unit circle centered at the origin O. The space of all possible choices of three points P_1, P_2, P_3 is the product of three circles; the volume of this space is $2\pi \times 2\pi \times 2\pi = 8\pi^3$. Let us first measure the volume of the configurations (P_1, P_2, P_3) such that the arc $P_1 \widehat{P_2} P_3$ is included in a semicircle and is oriented counterclockwise from P_1 to P_3. The condition that the arc is contained in a semicircle translates to $0 \leq \angle P_1 O P_2 \leq \pi$ and $0 \leq \angle P_2 O P_3 \leq \pi - \angle P_1 O P_2$ (see Figure 111). The point P_1 is chosen randomly on the circle, and for each P_1 the region of the angles θ_1 and θ_2 such that $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_1 \leq \pi - \theta_1$ is an isosceles right triangle with leg equal to π. Hence the region of points (P_1, P_2, P_3) subject to the above constraints has volume $2\pi \cdot \frac{1}{2}\pi^2 = \pi^3$. There are $3! = 6$ such regions and they are disjoint. Therefore, the volume of the favorable region is $6\pi^3$. The desired probability is therefore equal to $\frac{6\pi^3}{8\pi^3} = \frac{3}{4}$.MATHPIX IMAGEFigure 111
Topic :Probability
Book :Putnam and Beyond

Final Answer :

Problem Statement :932. Let $n \geq 4$ be given, and suppose that the points $P_1, P_2, \ldots, P_n$ are randomly chosen on a circle. Consider the convex n-gon whose vertices are these points. What is the probability that at least one of the vertex angles of this polygon is acute?
Solution :932. The angle at the vertex P_i is acute if and only if all other points lie on an open semicircle facing P_i. We first deduce from this that if there are any two acute angles at all, they must occur consecutively. Otherwise, the two arcs that these angles subtend would overlap and cover the whole circle, and the sum of the measures of the two angles would exceed 180°.So the polygon has either just one acute angle or two consecutive acute angles. In particular, taken in counterclockwise order, there exists exactly one pair of consecutive angles the second of which is acute and the first of which is not.We are left with the computation of the probability that for one of the points P_j, the angle at P_j is not acute, but the following angle is. This can be done using integrals. But there is a clever argument that reduces the geometric probability to a probability with a finite number of outcomes. The idea is to choose randomly $n-1$ pairs of antipodal points, and then among these to choose the vertices of the polygon. A polygon with one vertex at P_j and the other among these points has the desired property exactly when $n-2$ vertices lie on the semicircle to the clockwise side of P_j and one vertex on the opposite semicircle. Moreover, the points on the semicircle should include the counterclockwisemost to guarantee that the angle at P_j is not acute. Hence there are $n-2$ favorable choices of the total 2^{n-1} choices of points from the antipodal pairs. The probability for obtaining a polygon with the desired property is therefore $(n-2) 2^{-n+1}$.Integrating over all choices of pairs of antipodal points preserves the ratio. The events $j=1,2, \ldots, n$ are independent, so the probability has to be multiplied by n. The answer to the problem is therefore $n(n-2) 2^{-n+1}$.(66th W.L. Putnam Mathematical Competition, 2005, solution by C. Lin)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :933. Let C be the unit circle $x^2+y^2=1$. A point p is chosen randomly on the circumference of C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x-and y-axes with diagonal pq. What is the probability that no point of R lies outside of C ?
Solution :933. The pair (p, q) is chosen randomly from the three-dimensional domain $C \times$ int C, which has a total volume of $2 \pi^2$ (here int C denotes the interior of C). For a fixed p, the locus of points q for which R does not have points outside of C is the rectangle whose diagonal is the diameter through p and whose sides are parallel to the coordinate axes (Figure 112). If the coordinates of p are $(\cos \theta, \sin \theta)$, then the area of the rectangle is $2|\sin 2 \theta|$MATHPIX

IMAGEFigure 112The volume of the favorable region is therefore

$$V=\int_0^{2\pi} 2|\sin 2\theta| d\theta=4 \int_0^{\pi/2} 2 \sin 2\theta d\theta=8.$$

Hence the probability is

$$P=\frac{8}{2\pi^2}=\frac{4}{\pi^2} \approx 0.405.$$

(46th W.L. Putnam Mathematical Competition, 1985)
Topic :Probability
Book :Putnam and Beyond
Final Answer :

Problem Statement :934. If a needle of length 1 is dropped at random on a surface ruled with parallel lines at distance 2 apart, what is the probability that the needle will cross one of the lines?
Solution :934. Mark an endpoint of the needle. Translations parallel to the given (horizontal) lines can be ignored; thus we can assume that the marked endpoint of the needle always falls on the same vertical. Its position is determined by the variables (x, θ), where x is the distance to the line right above and θ the angle made with the horizontal (Figure 113).The pair (x, θ) is randomly chosen from the region $[0,2) \times[0,2\pi)$. The area of this region is 4π. The probability that the needle will cross the upper horizontal line is

$$\frac{1}{4\pi} \int_0^\pi \int_0^{\sin \theta} d x d \theta=\int_0^\pi \frac{\sin \theta}{4\pi} d \theta=\frac{1}{2\pi},$$

which is also equal to the probability that the needle will cross the lower horizontal line. The probability for the needle to cross either the upper or the lower horizontal line is therefore $\frac{1}{\pi}$. This gives an experimental way of approximating π.(G.-L. Leclerc, Comte de Buffon) MATHPIX IMAGEFigure 113
Topic :Probability
Book :Putnam and Beyond
Final Answer :