Problem Statement :Let G be a finite group, let V be a representation of G on a finitedimensional vector space over $\mathbb{C}$, and let $W \subset V$ be a subrepresentation. Show that there is a subrepresentation $W' \subset V$ such that $V = W \oplus W'$

Solution :Choose any complementary subspace $U \subset V$ with $V = W \oplus U$, and let

$$\pi : V \longrightarrow W$$

be the corresponding projection onto the first component. Define a new linear map $\pi' : V \longrightarrow W$ by averaging π over the group G - that is,

$$\pi'(v) := \frac{1}{|G|} \sum_{g \in G} g\pi\left(g^{-1} v\right)$$

This is a G-equivariant map such that, for any $w \in W$,

$$\pi'(w) = w$$

Its kernel $W' := \ker \pi'$ is G-invariant, of complementary dimension to W, and has the property that $W \cap W' = 0$. Therefore

$$V = W \oplus W'$$

.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Consider the varieties in the affine plane $\mathbb{A}^2_{\mathbb{C}}$ with coordinates (x, y) defined by the following polynomials:
1. $X_1 = V\left(x^2 - 1\right)$
2. $X_2 = V\left(x^2 - y\right)$
3. $X_3 = V\left(x^2 - y^2\right)$
4. $X_4 = V\left(x^2 - y^3\right)$
5. $X_5 = V\left(x^2 - y^4\right)$. Prove that no two of the varieties X_i are isomorphic. (Note: we are not adopting the convention that varieties are assumed irreducible.)

Solution :First off, the varieties X_2 and X_4 are irreducible, whereas the other three are reducible. Since X_2 is nonsingular and X_4 is singular, they are not isomorphic to each other.
Among the varieties X_1, X_3 and X_5, the first is nonsingular whereas the other two are singular. And finally, in the case of X_3, the intersection of the two irreducible components is transverse, while in X_5 the two irreducible components are tangent at their point of intersection.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Let D^n be a closed disc in $\mathbb{R}^n$ and $S^{n-1} = \partial D^n$ its boundary. For any topological space X and map $\alpha : S^{n-1} \rightarrow X$, we define the space Y obtained from X by attaching an n-cell via the map α to be the quotient of the disjoint union $D^n \sqcup X$ by the equivalence relation generated by $p \sim \alpha(p)$ for all $p \in \partial D^n$. Assuming that the Betti numbers of X are finite, show that one of the two following statements holds:
1. the nth Betti number of Y is 1 greater than the nth Betti number of X, and all other Betti numbers are equal; or
2. the $(n-1)$ st Betti number of Y is 1 less than the $(n-1)$ st Betti number of X, and all other Betti numbers are equal.

Solution :We consider the covering of Y by the two open sets U and V, where $U = Y \backslash \{0\}$ is the complement in Y of the image of the origin $0 \in D^n$, and V is the image in Y of the open disc $D^n \backslash S^{n-1}$. Here V is contractible, so its reduced homology is 0 , and V may be retracted back to X, so its reduced homology is the same as that of X. Finally, the intersection $U \cap V$ has the homotopy type of S^{n-1}, so its reduced homology is $\mathbb{Z}$ in degree $n-1$ and 0 otherwise. The relevant part of the Mayer-Vietoris sequence is thus

$$0 \rightarrow H_n(X) \rightarrow H_n(Y) \rightarrow H_{n-1}\left(S^{n-1}\right) \cong \mathbb{Z} \rightarrow^{\alpha_*} H_{n-1}(X) \rightarrow H_{n-1}(Y) \rightarrow 0$$

If the rank of the map α_* is zero-that is, if the image in $H_{n-1}(X)$ of the fundamental class of S^{n-1} is torsion-then the first statement holds; if the rank of α_* is 1 , the second holds.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Evaluate the series

$$\sum_{n=-\infty}^{\infty} \frac{n^2+n+1}{n^4+1}$$

by integrating $R(z) \cot \pi z$ for some appropriate rational function $R(z)$ over the boundary of the square $C_n \subset \mathbb{C}$ whose four vertices are $\left(n+\frac{1}{2}\right)(\pm 1 \pm i)$ and then letting $n \rightarrow \infty$.

Solution :Since

$$\cot \pi z = i \frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}} = i \frac{e^{-\pi y} e^{i\pi x}+e^{\pi y} e^{-i\pi x}}{e^{-\pi y} e^{i\pi x}-e^{\pi y} e^{-i\pi x}},$$

by looking at $y \rightarrow \infty$ and $y \rightarrow -\infty$ separately, we conclude from

$$\cot \pi(z+2) = \cot \pi z$$

that $\cot \pi z$ is uniformly bounded on C_n (independent of n). Let

$$f(z) = \frac{z^2+z+1}{z^4+1}\pi \cot \pi z.$$

From

$$\lim_{n\to\infty} \sup_{z\in C_n} \frac{z^2+z+1}{z^4+1} = O\left(\frac{1}{n^2}\right)$$

and the length of C_n of order $O(n)$, it follows that

$$\lim_{n\to\infty} \int_{C_n} f(z)dz = 0$$

and the sum of the residues of $f(z)$ on $\mathbb{C}$ vanishes. The poles of f are all simple poles and are at $z \in \mathbb{Z}$ and the four roots $e^{\frac{ik\pi}{4}}$ $(k = 1, 3, 5, 7)$ of $z^4 + 1 = 0$. The residue at $z = n$ is $\frac{n^2+n+1}{n^4+1}$ and the residue at $e^{\frac{ik\pi}{4}}$ is

$$\left(\frac{z^2+z+1}{4z^3}\pi \cot \pi z\right)_{z=e^{\frac{ik\pi}{4}}}.$$

The sum of the four residues at $e^{\frac{ik\pi}{4}}$ $(k = 1, 3, 5, 7)$ is

$$-\frac{i\pi}{\sqrt{2}}\left(\cot\left(\pi e^{\frac{i\pi}{4}}\right) + \cot\left(\pi e^{\frac{3i\pi}{4}}\right)\right).$$

Thus,

$$\sum_{n=-\infty}^{\infty} \frac{n^2+n+1}{n^4+1} = \frac{i\pi}{\sqrt{2}}\left(\cot\left(\pi e^{\frac{i\pi}{4}}\right) + \cot\left(\pi e^{\frac{3i\pi}{4}}\right)\right).$$

Book Title :Harvard Math Qualifying Exams

Problem Statement :Let $c > 0$. Consider the catenary C defined by

$$x = c\cosh\left(\frac{z}{c}\right)$$

in the xz-plane. Let S be the catenoid in the xyz-space obtained by rotating the catenary C with respect to the z-axis. Use θ, z as coordinates for S, where θ is from the polar coordinates (r, θ) of the xy-plane. In terms of (θ, z), write down the first and second fundamental forms of S and the mean curvature and Gaussian curvature of S.

Solution :The parametric equations for S are

$$x = c\cosh\left(\frac{z}{c}\right)\cos\theta$$
$$y = c\cosh\left(\frac{z}{c}\right)\sin\theta$$
$$z = z.$$

The first fundamental form $I = Ed\theta^2 + 2Fd\theta dz + Gdz^2$ is

$$\begin{aligned}
ds^2 =& dx^2 + dy^2 + dz^2 \\
=& \left(-c\cosh\left(\frac{z}{c}\right)\sin\theta d\theta + \sinh\left(\frac{z}{c}\right)\cos\theta dz\right)^2 \\
&+ \left(c\cosh\left(\frac{z}{c}\right)\cos\theta d\theta + \sinh\left(\frac{z}{c}\right)\sin\theta dz\right)^2 + dz^2 \\
=& c^2\cosh^2\left(\frac{z}{c}\right)d\theta^2 + \cosh^2\left(\frac{z}{c}\right)dz^2
\end{aligned}$$

with

$$\begin{aligned}
E &= c^2\cosh^2\left(\frac{z}{c}\right), \\
F &= 0, \\
G &= \cosh^2\left(\frac{z}{c}\right).
\end{aligned}$$

To compute the unit normal vector $\vec{n}$, we compute the partial derivatives of the radius vector $\vec{r}$ with respect θ and z,

$$\begin{aligned}
\vec{r}_\theta &= \left(-c\cosh\left(\frac{z}{c}\right)\sin\theta, c\cosh\left(\frac{z}{c}\right)\cos\theta, 0\right) \\
\vec{r}_z &= \left(\sinh\left(\frac{z}{c}\right)\cos\theta, \sinh\left(\frac{z}{c}\right)\sin\theta, 1\right),
\end{aligned}$$

to form

$$\vec{r}_\theta \times \vec{r}_z = \left(c\cosh\left(\frac{z}{c}\right)\cos\theta, c\cosh\left(\frac{z}{c}\right)\sin\theta, -c\sinh\left(\frac{z}{c}\right)\cosh\left(\frac{z}{c}\right)\right).$$

The length of $\vec{r}_\theta \times \vec{r}_z$ is equal to $\sqrt{EG - F^2} = c\cosh^2\left(\frac{z}{c}\right)$ so that

$$\vec{n} = \left(\cosh\left(\frac{z}{c}\right)^{-1}\cos\theta, \cosh\left(\frac{z}{c}\right)^{-1}\sin\theta, -\sinh\left(\frac{z}{c}\right)\cosh\left(\frac{z}{c}\right)^{-1}\right).$$

To obtain the coefficients L, M, N of the second fundamental form $II = Ldz^2 + 2Mdzd\theta + Nd\theta^2$, we compute the partial derivatives of the radius vector $\vec{r}$,

$$\begin{aligned}
\vec{r}_{\theta\theta} &= \left(-c\cosh\left(\frac{z}{c}\right)\cos\theta, -c\cosh\left(\frac{z}{c}\right)\sin\theta, 0\right), \\
\vec{r}_{\theta z} &= \left(-\sinh\left(\frac{z}{c}\right)\sin\theta, \sinh\left(\frac{z}{c}\right)\cos\theta, 0\right), \\
\vec{r}_{zz} &= \left(\frac{1}{c}\cosh\left(\frac{z}{c}\right)\cos\theta, \frac{1}{c}\cosh\left(\frac{z}{c}\right)\sin\theta, 0\right).
\end{aligned}$$

The coefficients L, M, N of the second fundamental form are given by

$$\begin{aligned}
L &= \vec{n}\cdot\vec{r}_{\theta\theta} = -c, \\
M &= \vec{n}\cdot r_{\theta z} = 0, \\
N &= \vec{n}\cdot\vec{r}_{zz} = \frac{1}{c}.
\end{aligned}$$

The mean curvature of S is

$$\frac{1}{2}\frac{LG-2MF+NE}{EG-F^2}=\frac{1}{2}\frac{(-c)\cosh^2\left(\frac{z}{c}\right)+\frac{1}{c}c^2\cosh^2\left(\frac{z}{c}\right)}{c^2\cosh^2\left(\frac{z}{c}\right)\cosh^2\left(\frac{z}{c}\right)}=0.$$

The Gaussian curvature of S is

$$\frac{LN-M^2}{EG-F^2}=\frac{(-c)\frac{1}{c}}{c^2\cosh^2\left(\frac{z}{c}\right)\cosh^2\left(\frac{z}{c}\right)}=\frac{-1}{c^2\cosh^4\left(\frac{z}{c}\right)}.$$

Book Title :Harvard Math Qualifying Exams

Problem Statement :Suppose $f:[-1,1]\to\mathbf{R}$ is a continuous function such that

$$\int_{-1}^{1}x^{2n}f(x)dx=0$$

for each $n=0,1,2,3,\ldots$. Prove that f is an odd function (i.e., that $f(-x)=-f(x)$ for all $x\in[-1,1]$).

Solution :Let $g:[-1,1]\to\mathbf{R}$ be the continuous function defined by $g(x)=f(x)+f(-x)$. We prove that g is the zero function, which is equivalent to the desired $f(-x)=-f(x)$
First note that $\int_{-1}^{1}x^m g(x)dx=0$ for each $m=0,1,2,3,\ldots$; this is automatic for m odd, and follows from the hypothesis for m even. By linearity it follows that $\int_{-1}^{1}P(x)g(x)dx=0$ for all polynomials P. By the Weierstrass approximation theorem there exists a sequence $\{P_k\}_{k=1}^{\infty}$ of polynomials such that $P_k(x)\to g(x)$ uniformly for all $x\in[-1,1]$. Since g is bounded (continuous function on a compact set), it follows that

$$\int_{-1}^{1}g(x)^2dx=\lim_{k\to\infty}\int_{-1}^{1}P(x)g(x)dx.$$

Hence $\int_{-1}^{1}g(x)^2dx=0$. Since g is continuous and real valued, it thus vanishes identically, and we are done.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Let X and Y be compact, connected, oriented n-manifolds, and $f:X\to Y$ a continuous map. Define the degree of the map f.

Solution :By hypothesis we have $H_n(X)=H_n(Y)\cong\mathbb{Z}$, where we choose the identification so that the generator $1\in\mathbb{Z}$ corresponds to the fundamental class given by the orientation. The map $f_* : H_n(X) \to H_n(Y)$ is then simply multiplication by an integer d; the degree of the map is defined to be this integer d.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, and let $r_i : S^n \to S^n$ be the reflection in the i th axis; that is, the map

Solution :The reflection r_i is an orientation-reversing automorphism of S^n, so its degree is -1.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, and let $a : S^n \to S^n$ be the antipodal map sending x to $-x$. What is the degree of a ?

Solution :We note that a is the composition of the $n+1$ reflections $r_0, \ldots, r_n$, so its degree is $(-1)^{n+1}$.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Suppose that $f : \{z : 0 < |z| < 1\} \to \mathbb{C}$ is holomorphic and $|f(z)| \leq A|z|^{-3/2}$ for some constant A. Prove that there is a complex constant α such that $g(z) := f(z) - \alpha z^{-1}$ can be extended to a holomorphic function on $\{z : |z| < 1\}$

Solution :For $0 < a < |z| < b < 1$, we can write

$$2\pi i f(z) = \int_{|w|=b} \frac{f(w)}{z-w} \mathrm{d}w - \int_{|w|=a} \frac{f(w)}{z-w} \mathrm{d}w$$

Notice that

$$\int_{|w|=a} \frac{f(w)}{z-w} \mathrm{d}w = \frac{1}{z} \int_{|w|=a} f(w) \mathrm{d}w - \frac{1}{z} \int_{|w|=a} O(w/z) f(w) \mathrm{d}w$$

By assumption, the last term can be estimated by

$$\frac{1}{|z|} \int_{|w|=a} O(w/z) |f(w)| |\mathrm{d}w| \leq \frac{A}{|z|^2} \sqrt{a}$$

As $a \to 0$, the last term vanishes. Thus we have

$$2\pi i f(z) + \frac{c}{z} = \int_{|w|=b} \frac{f(w)}{z-w} \mathrm{d}w, \quad c = \int_{|w|=a} f(w) \mathrm{d}w$$

Notice that c is independent of the choice of a. The right hand side defines a holomorphic function near $z=0$.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Which of the following smooth manifolds:

1. S^2

2. $\mathbb{RP}^2$ and

3. $S^1 \times S^1$

admit a closed, non-exact differential 1-form? In each case, either argue why such form does not exist or give an example.

Solution :By deRham's theorem, if M is a manifold, then the real-valued singular cohomology groups $H^*(M, \mathbb{R})$ are isomorphic to the cohomology of the complex of differential forms. Thus, it follows that M admits a closed, non-exact differential 1 -form if and only if $H^1(M, \mathbb{R}) eq 0$.
Since $H^1\left(S^2, \mathbb{R}\right)=0$ and $H^1\left(\mathbb{RP}^2, \mathbb{R}\right)=0$, these are no such forms in these cases.
In the last case, we have $H^1\left(S^1 \times S^1, \mathbb{R}\right) \simeq \mathbb{R} \oplus \mathbb{R}$, so such forms exist. As an explicit example, let us choose a diffeomorphism

$$(\theta, \rho): S^1 \times S^1 \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$

Then, θ can be considered as a real-valued function, well-defined up to adding an integral constant, so that $d\theta$ is a well-defined differential 1-form on $S^1 \times S^1$. By definition, $d\theta$ is locally a differential of a real-valued function, so it is closed. On the other hand, it is not exact, as its integral around the loop corresponding to $\mathbb{R}/\mathbb{Z} \times \{e\}$ is not zero.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Let $\mathbf{T}$ be the torus $(\mathbf{R}/\mathbf{Z})^2$, and let $a: \mathbf{T} \rightarrow \mathbf{R}$ be any continuous function. Prove that the $\mathbf{R}$-vector space of solutions of the partial differential equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=a f$$

in functions $f: \mathbf{T} \rightarrow \mathbf{R}$ is finite dimensional.

Solution :Call that vector space V, and write the differential equation as

$(1-a) f=(1-\Delta) f$ where Δ is the Laplacian $\partial^2 / \partial x^2+\partial^2 / \partial y^2$. Let A be the operator $(1-\Delta)^{-1}$ on $L^2(\mathbf{T})$, which is compact because it is diagonalized by the Fourier basis with eigenvalues $\left(1+4 \pi^2\left(m^2+n^2\right)\right)^{-1}$, only finitely many of which are outside $(0, \epsilon)$ for any $\epsilon>0$. Then V is the fixed subspace of $A(1-a)$, which is also compact (composition of the compact operator A with the bounded operator $1-a$). Hence V is finite dimensional (for example, because the closure of its unit ball is compact),

Book Title :Harvard Math Qualifying Exams

Problem Statement :Consider the polynomial $f(x)=x^4+1$. Is there any prime $p>2$ such that f is irreducible over the finite field of order p?

Solution :Let $\alpha \in \mathbb{C}$ be a root of f. Then the full set of roots of f is given by

$$\{\pm \alpha, \pm i \alpha\} .$$

Since $\alpha^2= \pm i$, it follows that $\mathbb{Q}[\alpha]$ is the splitting field of f over $\mathbb{Q}$, and we have

$$|G|=[\mathbb{Q}[\alpha]: \mathbb{Q}]=4 .$$

On the other hand, note that the Galois group G acts transitively on the roots of f, so it contains elements σ and τ such that

$$\sigma(\alpha)=-\alpha \quad \text { and } \quad \tau(\alpha)=\alpha^3 .$$

Then

$$\begin{aligned}
& \sigma^2(\alpha)=\alpha, \text { and } \\
& \tau^2(\alpha)=\alpha^9=(-1)^2 \cdot \alpha=\alpha .
\end{aligned}$$

Since G is a group of order 4 which contains two elements of order 2 , it must be isomorphic to $\mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 2 \mathbb{Z}$. Finally, the arguments above show that, if $\mathbb{F}$ is a field of characteristic not equal to 2 or 3 over which f is irreducible, the Galois group of f over $\mathbb{F}$ is isomorphic to $\mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 2 \mathbb{Z}$. However, the Galois group of any finite extension of $\mathbb{F}_p$ is cyclic. Therefore, f cannot be irreducible over $\mathbb{F}_p$. Alternatively, we could argue that the cyclic group $\mathbb{F}_{p^2}^{\times}$has order p^2-1, which is always congruent to $0(\bmod 8)$. This implies that there is an element $\alpha \in \mathbb{F}_{p^2}^{\times}$ of order 8 . Then $\alpha^4=-1$, so α is a root of the polynomial $f(x) \in \mathbb{F}_p[x]$. Suppose that $f(x)$ is irreducible over $\mathbb{F}_p$. Then $\mathbb{F}_p(\alpha)$ is the splitting field of f over $\mathbb{F}_p$ and it has degree 4 . We get

$$2=\left[\mathbb{F}_{p^2}: \mathbb{F}_p\right]=\left[\mathbb{F}_{p^2}: \mathbb{F}_p(\alpha)\right]\left[\mathbb{F}_p(\alpha): \mathbb{F}_p\right]=\left[\mathbb{F}_{p^2}: \mathbb{F}_p(\alpha)\right] \cdot 4$$

- a contradiction!

Book Title :Harvard Math Qualifying Exams

Problem Statement :Let $C \subset \mathbb{P}^3$ be a smooth, irreducible, nondegenerate curve of degree 4. Show that the genus of C cannot be greater than 1.

Solution :There are many ways to do this. Probably the simplest would be to argue that for a general point $p \in C$, the projection map $\pi_p : C \to \mathbb{P}^2$ maps C birationally onto a plane cubic curve, which will either have genus 1 (if it's smooth) or 0 (if it's singular).

Book Title :Harvard Math Qualifying Exams

Problem Statement :Let a_{ij} for $1 \leq i \leq n-1$ and $1 \leq j \leq n$ be real constants. For $1 \leq i \leq n-1$ consider the vector field

$$X_i = (\underbrace{0, \cdots, 0, 1, 0 \cdots, 0}_{1 \text{ in } i^{\text{th}} \text{ position}}, \sum_{j=1}^{n} a_{ij} x_j)$$

on $\mathbb{R}^n$ (with coordinates $x_1, \cdots, x_n$). Let Π be the distribution of the tangent subspace of dimension $n-1$ in $\mathbb{R}^n$ spanned by $X_1, \cdots, X_{n-1}$. Determine the necessary and sufficient condition for Π to be integrable. Express the condition in terms of symmetry properties of the $(n-1) \times (n-1)$ matrix $(a_{ij})_{1 \leq i,j \leq n-1}$ and the relation among the ratios $\frac{a_{ik}}{a_{jk}}$ for $1 \leq i < j \leq n-1$ and $1 \leq k \leq n$.

Solution :Write

$$X_i = \frac{\partial}{\partial x_i} + \left(\sum_{j=1}^{n} a_{ij} x_j \right) \frac{\partial}{\partial x_n}$$

for $1 \leq i \leq n-1$. By Frobenius theorem, integrability of Π is equivalent to $[X_i, X_j]$ being spanned by $X_1, \cdots, X_{n-1}$ for $1 \leq i < j \leq n-1$. Since

$$[X_i, X_j] = \left(a_{ji} + \left(\sum_{k=1}^{n} a_{ik} x_k \right) a_{jn} - a_{ij} - \left(\sum_{k=1}^{n} a_{jk} x_k \right) a_{in} \right) \frac{\partial}{\partial x_n}$$

has zero coefficients for $\frac{\partial}{\partial x_k}$ for $1 \leq k \leq n-1$, the integrability condition can be rewritten as the vanishing of $[X_i, X_j]$ for $1 \leq i < j \leq n-1$, which means

$$a_{ji} + \left(\sum_{k=1}^{n} a_{ik} x_k \right) a_{jn} = a_{ij} + \left(\sum_{k=1}^{n} a_{jk} x_k \right) a_{in}.$$

Equating the coefficients, we obtain $a_{ji} = a_{ij}$ and $a_{ik} a_{jn} = a_{jk} a_{in}$ for $1 \leq i < j \leq n-1$ and $1 \leq k \leq n$. The necessary and sufficient condition is that the $(n-1) \times (n-1)$ matrix $(a_{ij})_{1 \leq i,j \leq n-1}$ is symmetric and for $1 \leq i < j \leq n-1$

the n ratios $\frac{a_{ik}}{a_{jk}}$ for $1 \leq k \leq n$ are equal in the sense of equality after crossmultiplication

Book Title :Harvard Math Qualifying Exams

Problem Statement :Suppose U and V are two random variables. We say that U and V are uncorrelated if $\mathrm{Cov}(U, V) = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V] = 0$. Suppose X and Y are distributed by the following bivariate normal distribution with density

$$f(x, y) = \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}},$$

where $0 < \rho < 1$ is a parameter. Let $U = X + aY$ and $V = X + bY$ with $a, beq0$. Find the condition that $\mathrm{Cov}(U, V) = 0$. In this case, prove that U and V are independent (you cannot just cite a theorem).

Solution :Define the matrix

$$A^{-1} = \frac{1}{(1-\rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

and denote the column vectors $\mathbf{x} = (x, y)^t$ and $\mathbf{s} = (s, t)^t$. Then the characteristic function

$$\phi(s, t) = \mathbb{E}e^{isX + itY} = \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} \int e^{i\mathbf{s}\cdot\mathbf{x}} e^{-\mathbf{x}^t A^{-1}\mathbf{x}/2} \, \mathrm{d}x \, \mathrm{d}y = e^{-\mathbf{s}^t A\mathbf{s}/2}$$

This gives $\mathrm{Cov}(X, X) = \mathrm{Cov}(Y, Y) = 1$ and $\mathrm{Cov}(X, Y) = \rho$. Now the characteristic function of U, V can be computed from $\phi(s, t)$, i.e.,

$$\mathbb{E}e^{i\alpha U + i\beta V} = \phi(\alpha + \beta, a\alpha + b\beta)$$

The condition $\mathrm{Cov}(U, V) = 0$ will imply that $\mathbb{E}e^{i\alpha U + i\beta V} = \mathbb{E}e^{i\alpha U}\mathbb{E}e^{i\beta V}$ and hence U, V are independent.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Suppose R is a commutative ring with unit, I an ideal in R, and M a finitely-generated R-module. If $IM = M$, prove that there exists $r \in R$ such that $r - 1 \in I$ and $rM = 0$.

Solution :Let M be generated by $x_1, \ldots, x_n$. Then IM consists of module elements of the form $\sum_{j=1}^n a_j x_j$ with each $a_j \in I$. Thus $M = IM$ means that each x_i can be written as $\sum_{j=1}^n a_{ij} x_j$ for some $a_{ij} \in I$. Let A be the $n \times n$

matrix (a_{ij}), and $\vec{x}$ the column vector (x_i); then we have $(\mathbf{1} - A)\vec{x} = 0$. Multiplying from the left by $\operatorname{adj} A$, we deduce that $\det(\mathbf{1} - A) \cdot \vec{x} = 0$, and thus that $\det(\mathbf{1} - A) \cdot M = 0$. But the ring element $\det(\mathbf{1} - A)$ is in $1 + I$ because $1 - A \equiv \mathbf{1} \bmod I$.

Book Title :Harvard Math Qualifying Exams

Problem Statement :Let $\mathbb{P}^{n^2-1}$ be the variety of nonzero $n \times n$ complex matrices modulo scalars. Consider the set

$$X := \left\{ [A] \in \mathbb{P}^{n^2-1} \mid A \text{ is nilpotent} \right\}.$$

Show that X is irreducible, and find its dimension.

Solution :Let $\mathcal{F}$ be the variety of complete flags in $\mathbb{C}^n$ - that is, let $\operatorname{Gr}(k, n)$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ and let

$$\mathcal{F} := \{V_\bullet = (V_0, V_1, \ldots, V_n) \mid V_k \in \operatorname{Gr}(k, n) \text{ and } V_k \subset V_{k+1}\}.$$

Note that

$$\dim \mathcal{F} = \frac{n(n-1)}{2}.$$

Define an incidence variety

$$\Lambda := \{(A, V_\bullet) \in X \times \mathcal{F} \mid A \cdot V_\bullet \subset V_\bullet\}$$

which consists of pairs of a nilpotent element A and a flag $V_\bullet$ such that A preserves V. The fiber over the standard flag $E_\bullet$ defined by

$$E_k = \{(x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{C}^n\}$$

consists exactly of the upper-triangular nilpotent matrices. Since any complete flag is conjugate to the standard flag, it follows that Λ fibers over $\mathcal{F}$ with fiber the projective space of dimension

$$\frac{n(n-1)}{2} - 1.$$

Therefore Λ is irreducible of dimension $n^2 - n - 1$.
The projection onto the first component

$$\pi : \Lambda \longrightarrow X$$

is surjective, because any nilpotent matrix is conjugate to an upper-triangular one and therefore stabilizes at least one flag. This implies that X is irreducible. Moreover, recall that any nilpotent matrix of rank $n - 1$ is conjugate to the maximal nilpotent Jordan block, which stabilizes only the standard flag $E_.$. Therefore π is generically one-to-one, and it follows that

$$\dim X = n^2 - n - 1.$$

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Problem Statement :Let M be a connected closed 4-manifold such that $\pi_1(M)$ is perfect; that is, does not have any non-trivial abelian quotients. Determine the possible cohomology groups $H^*(M, \mathbb{Z})$.

Solution :We first claim that M is orientable. Let $p : \widetilde{M} \to M$ be the orientation cover of M. As M is connected, this is completely classified as a covering by any fibre $p^{-1}(m)$ together with the action of $\pi_1(M, m)$.

As the orientation covering is 2-fold, this is the same as a homomorphism $\pi_1(M, m) \to \Sigma_2 \simeq \mathbb{Z}/2$. It this was non-trivial, then it would be surjective, which is impossible since $\pi_1(M)$ is assumed to be perfect. Thus, we deduce that the orientation covering is the trivial 2 -fold covering, so that M is orientable.

We will now determine the possible homology groups. Orientability tells us that $H^4(M, \mathbb{Z}) \simeq \mathbb{Z}$ and similarly $H_4(M, \mathbb{Z}) \simeq \mathbb{Z}$.

By Hurewicz theorem, we have $H_1(M, \mathbb{Z}) \simeq \pi_1(M)^{ab}$. By the perfectness assumption, the latter vanishes, and hence so does the former.

By universal coefficient theorem combined with vanishing of H_1, we deduce that

$$H^2(M, \mathbb{Z}) \simeq \operatorname{Hom}\left(H_2(M, \mathbb{Z}), \mathbb{Z}\right).$$

The latter group is torsion-free, and we deduce that $H^2(M, \mathbb{Z})$ is a torsion free abelian group, hence finite free rank as M is compact. The Poincare duality isomorphism $H_2(M, \mathbb{Z}) \simeq H^2(M, \mathbb{Z})$ allows us to deduce that the second homology group is also free of finite rank.

Similarly, we have Poincare isomorphism $H_3(M, \mathbb{Z}) \simeq H^1(M, \mathbb{Z})$ and a universal coefficient isomorphism $H^1(M, \mathbb{Z}) \simeq \operatorname{Hom}\left(H_1(M, \mathbb{Z}), \mathbb{Z}\right)$. The last group vanishes, and we deduce the same is true for the third homology groups. These shows that the possible homology groups of M are respectively

$$\mathbb{Z}, 0, A, 0, \mathbb{Z}$$

where A is free of finite rank. All of these can be realized by a connected sum of complex projective planes.

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Problem Statement :Let $a < b$ and $f(z)$ be a continuous function on the closed strip $\{a \leq x \leq b\}$ which is holomorphic on its interior $\{a < x < b\}$, where $z = x + iy$, such that $|f(z)| = O\left(e^{\varepsilon|y|}\right)$ on $\{a \leq x \leq b\}$ for every $\varepsilon > 0$ as $|y| \to \infty$. If $|f(z)| \leq M$ on the boundary $\{x = a \text{ or } x = b\}$ of the strip $\{a \leq x \leq b\}$ and on the interval $[a, b]$ for some positive number M, prove that $|f(z)| \leq M$ on the entire closed strip $\{a \leq x \leq b\}$.

Solution :Fix arbitrarily $\varepsilon > 0$. Let $C_\varepsilon > 0$ such that $|f(x + iy)| \leq C_\varepsilon e^{\frac{\varepsilon}{2}|y|}$ on $\{a \leq x \leq b\}$ for any $y \in \mathbb{R}$. Since

$$|g_\varepsilon(a + iy)| = e^{-\varepsilon y}|f(a + iy)| \leq e^{-\varepsilon y} C_\varepsilon e^{\frac{\varepsilon}{2} y} \leq M$$

and

$$|g_\varepsilon(b + iy)| = e^{-\varepsilon y}|f(b + iy)| \leq e^{-\varepsilon y} C_\varepsilon e^{\frac{\varepsilon}{2} y} \leq M$$

when $y \geq T_\varepsilon$ for some sufficiently large positive number T_ε. By the maximum modulus principle applied to $g_\varepsilon(z)$ on the rectangle with vertices

$$a, b, b + iT, a + iT$$

when $T \geq T_\varepsilon$, we conclude that $|g_\varepsilon(z)| \leq M$ on the half strip

$$\{a \leq x \leq b\} \cap \{y \geq 0\}.$$

Passing to limit as $\varepsilon \to 0^+$, we obtain $|f(z)| \leq M$ on the half strip

$$\{a \leq x \leq b\} \cap \{y \geq 0\}.$$

Repeat the same argument with $g_\varepsilon(z)$ replaced by $h_\varepsilon(z)$ and with the condition $y \geq T_\varepsilon$ replaced by $y \leq -S_\varepsilon$ for some sufficiently large positive number S_ε. Analogously we get the conclusion that $|f(z)| \leq M$ on the half strip

$$\{a \leq x \leq b\} \cap \{y \leq 0\}.$$

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