

# WHEN IS EVERY NON CENTRAL-UNIT A SUM OF TWO NILPOTENTS?

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ABSTRACT. A ring is said to satisfy the 2-nil-sum property if every non central-unit is the sum of two nilpotents. We prove that a ring satisfies the 2-nil-sum property iff it is either a simple ring with the 2-nil-sum property or a commutative local ring with nil Jacobson radical, and we provide an example of a simple rings with the 2-nil-sum property that is not commutative. Moreover, a simple right Goldie ring has the 2-nil-sum property iff it is a field.

## 1. INTRODUCTION

Throughout, rings are associative with identity. We start by recalling three special types of elements in a ring. An element in a ring is 2-good if it is a sum of two units (see [?]), is fine if it is a sum of a unit and a nilpotent (see [?]), and is 2-nilgood if it is a sum of two nilpotents (see [?]). By the terminology of Vámos [?], a ring is 2-good if each element is 2-good. The study of 2-good rings (also called rings with the 2-sum property in the literature), initiated by Wolfson [?] and Zelinsky [?], has attracted considerable interest (see, for example, [?], [?], [?], [?], [?] and the references there). A ring is called a fine ring if each nonzero element is fine (the zero element being fine implies that the ring is trivial). Fine rings were introduced and extensively investigated by Călugăreanu and Lam [?].

Motivated by the notions of 2-good rings and fine rings, we define a ring to satisfy the *2-nil-sum property* if every element that is not a central unit is 2-nilgood (a central unit in a ring being 2-nilgood implies that the ring is trivial). These rings belong to a larger class of rings for which every element is either a unit or a sum of two units. But, we will see that the 2-nil-sum property is very

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restrictive. For instance, a ring without nonzero nilpotents satisfies the 2-nil-sum property iff it is a field. The rings which is additively generated by nilpotents have been studied by several authors with connections to rings additively generated by commutators. We refer to [?] and [?] for more details. As far as we are aware, the 2-nil-sum property of rings has not been discussed in the literature. Here a study of this topic is conducted. The main results proved here are the following: a ring satisfies the 2-nil-sum property iff it is either a simple ring with the 2-nil-sum property or a commutative local ring with nil Jacobson radical (Theorem ??); there exists simple rings with the 2-nil-sum property that are not commutative (Example ??); a simple right Goldie ring has the 2-nil-sum property iff it is a field (Theorem ??).

For a ring  $R$ , we denote by  $C(R)$ ,  $J(R)$ ,  $U(R)$ , and  $\text{nil}(R)$  the center, the Jacobson radical, the group of units of  $R$ , and the set of nilpotents of  $R$ , respectively. We write  $M_n(R)$  for the ring of  $n \times n$  matrices over  $R$  whose identity is denoted by  $I_n$ . By a *non central-unit* we mean an element that is not a central unit. For  $i, j \in \{1, 2, \dots, n\}$ , we denote by  $E_{ij}$  the  $n \times n$  matrix whose  $(i, j)$ -entry is 1 and all other entries are 0.

## 2. BASIC PROPERTIES AND THE REDUCTION THEOREM

We start by recording some basic properties.

**Lemma 2.1.** *Let  $R$  be a ring with the 2-nil-sum property. The following hold:*

- (1)  $C(R) \setminus U(R) \subseteq \text{nil}(R)$ . Consequently,  $C(R)$  is a local ring with nil Jacobson radical.
- (2) For every proper ideal  $I$  of  $R$  we have  $I \subseteq C(R) \cap \text{nil}(R)$ .
- (3) All proper ideals of  $R$  are contained in  $J(R)$ .
- (4)  $J(R)$  is nil and  $J(R) \subseteq C(R)$ .
- (5) For any ideal  $I$  of  $R$ , central units of  $R/I$  can be lifted to central units of  $R$ .

*Proof.* (1) Assume that  $r$  is a central element that is not a unit. Then  $r = b_1 + b_2$ , where  $b_1, b_2 \in \text{nil}(R)$ . Since  $r$  is central,  $b_1$  and  $b_2$  commute. So  $r = b_1 + b_2 \in \text{nil}(R)$ .

(2) Assume that  $I \setminus C(R) \neq \emptyset$ . Let  $a \in I \setminus C(R)$ . Then  $1 + a$  is a sum of two nilpotents, so  $\bar{1} \in R/I$  is a sum of two nilpotents, a contradiction. Hence  $I \subseteq C(R) \setminus U(R)$ , so  $I \subseteq \text{nil}(R)$  by (1).

(3) and (4) are clear by (2).

(5) Let  $\bar{u} := u + I$  be a central unit of  $R/I$ . We claim that  $u \in R$  is a central unit. Otherwise,  $u$  is a sum of two nilpotents in  $R$ . Hence  $\bar{u}$  is a sum of two nilpotents in  $R/I$ , a contradiction.  $\square$

**Theorem 2.2.** (*Reduction Theorem*) *A ring  $R$  satisfies the 2-nil-sum property iff  $R$  is either a simple ring with the 2-nil-sum property or a commutative local ring with nil Jacobson radical.*

*Proof.* The sufficiency is clear. For the necessity, suppose that  $R$  satisfies the 2-nil-sum property but  $R$  is not simple. Then, by Lemma ??(3),  $J(R) \neq 0$ . Let  $0 \neq x \in J(R)$ . If  $a, b \in R$  then, noting  $J(R) \subseteq C(R)$ , we have

$$(ab)x = a(bx) = (bx)a = b(xa) = b(ax) = (ba)x,$$

hence  $(ab - ba)x = 0$ . It follows that  $ab - ba$  belongs to the ideal  $\text{Ann}(x) = \{r \in R \mid rx = 0\}$ . Since  $\text{Ann}(x) \neq R$ ,  $\text{Ann}(x) \subseteq J(R)$ . It follows that  $R/J(R)$  is a commutative ring with the 2-nil-sum property, and hence is local by Lemma 2.1(1). It follows that  $R/J(R)$  is a field. Thus, by Lemma ??(5), for any  $a \in R \setminus J(R)$ ,  $a = u + j$  where  $u$  is a central unit and  $j \in J(R)$ . It follows that  $a$  is central. Hence,  $R$  is commutative, and so  $R$  is a commutative local ring with  $J(R)$  nil.  $\square$

If  $R$  is a commutative ring with the 2-nil-sum property then every non central-unit  $a \in R$  is nilpotent, so it is a sum of the form  $a = b + c$  with  $b^1 = 0$  and  $c = a$  is nilpotent. Thus,  $R$  is a ring with the 2-nil-sum property of type  $(1, \infty)$

as defined below. The 2-nil-sum property is refined as follows: let  $p, q$  be integers with  $p \geq 1$  and  $q \geq 1$ . We say that a ring  $R$  has the 2-nil-sum property of type  $(p, q)$  if, for each non central-unit  $a$  in  $R$ ,  $a = b + c$  where  $b^p = c^q = 0$ . The ring  $R$  has the 2-nil-sum property of type  $(p, \infty)$  if, for each non central-unit  $a$  in  $R$ ,  $a = b + c$  where  $b^p = 0$  and  $c$  is nilpotent. Clearly, a ring  $R$  is a field iff  $R$  has the 2-nil-sum property of type  $(1, 1)$ .

**Proposition 2.3.** *Suppose that  $R$  has the 2-nil-sum property of type  $(2, \infty)$ . Then  $R$  is commutative.*

*Proof.* We observe that  $R$  has no non-trivial idempotents: if  $1 \neq e^2 = e \in R$  is a sum of a square-zero element and a nilpotent it follows by [?, Proposition 2] that  $e = 0$ .

Suppose that  $R$  is not commutative. Then  $R$  is a simple ring by Theorem 2.2, hence 0 is the only central nilpotent. But  $\text{nil}(R) \neq 0$ , hence there exists a square-zero element  $x$  that is not central. Since  $R$  is of type  $(2, \infty)$ , there exist  $y, z \in R$  such that

$$1 - x = y + z,$$

where  $y^2 = 0$  and  $z^n = 0$  for some  $n \geq 1$ . Then  $(1 - x - y)^n = 0$ . It follows that

$$1 = n_1(x + y) + n_2(xy + yx) + n_3(xyx + yxy) + n_4(xyxy + yxyx) + \cdots,$$

where  $n_1 = n$  and each  $n_i$  is an integer. Multiplying both sides of this equation from the left by  $x$  gives

$$x = n_1xy + n_2xyx + n_3xyxy + n_4xyxyx + \cdots$$

Multiplying both sides of this equality from the right by  $y$  gives

$$xy = n_2xyxy + n_4xyxyxy + \cdots = a(xy)^2 = (xy)^2a,$$

where  $a = n_2 + n_4xy + \cdots$ . It follows that  $a(xy)$  is an idempotent. Since  $x^2 = 0$ , we have  $a(xy) \neq 1$ . We obtain  $a(xy) = 0$ , hence  $xy = 0$ . It follows that  $x = 0$ , a contradiction.  $\square$

**Proposition 2.4.** *Suppose that  $R$  is of characteristic 0 and has the 2-nil-sum property of type  $(p, q)$  where  $p \leq 3$  and  $q \leq 5$ . Then  $R$  is commutative.*

*Proof.* We can assume that  $p = 3$  and  $q = 5$ . Suppose that  $R$  is not commutative. As in the proof of Proposition ??, there exists a square-zero element  $x$  that is not central. So, there exist  $y, z \in R$  such that  $1 - x = y + z$  and  $y^3 = z^5 = 0$ . This contradicts [?, Proposition 9].  $\square$

We close this section with the following

**Open Question.** If a ring  $R$  has the 2-nil-sum property of type  $(p, q)$  with  $p \in \mathbb{N}$  and  $q \in \mathbb{N} \cup \{\infty\}$ , is it commutative?

### 3. SIMPLE RINGS

**Example 3.1.** *There exists a simple ring with the 2-nil-sum property that is not commutative.*

*Proof.* Let  $\mathbb{F}_2$  be the field of 2 elements. For each integer  $n \geq 0$  we consider the diagonal morphism

$$\varphi_n : \mathbb{M}_{2^n}(\mathbb{F}_2) \rightarrow \mathbb{M}_{2^{n+1}}(\mathbb{F}_2), A \mapsto \text{diag}(A, A).$$

This is an inductive system, and we denote its colimit by  $R$ , while  $\varphi_n : \mathbb{M}_{2^n}(\mathbb{F}_2) \rightarrow R$  denote the canonical ring morphisms. Then  $R$  is a simple ring (see [?, Example 8.1]).

Let  $0 \neq r \in R$  a non central-unit of  $R$ . Then we can view it as an image  $r = \varphi_n(A)$  for some  $0 \neq A \in \mathbb{M}_{2^n}(\mathbb{F}_2)$ . Note that  $A \neq I_{2^n}$  since otherwise  $r$  is the identity of  $R$ . Observe that  $r = \varphi_{n+1}(\text{diag}(A, A))$ . From  $A \neq I_{2^n}$  it follows that  $\text{diag}(A, A)$  is not a scalar matrix. Since its trace is zero, there exist two nilpotent matrices  $N_1$  and  $N_2$  in  $\mathbb{M}_{2^{n+1}}(\mathbb{F}_2)$  such that  $\text{diag}(A, A) = N_1 + N_2$ , see [?, Proposition 3(ii)]. Then  $r = \varphi_{n+1}(N_1) + \varphi_{n+1}(N_2)$  is a sum of two nilpotents.  $\square$

Similar examples can be obtained by using other fields (of positive characteristic). We do not know other kind of examples of non-commutative rings with the

2-nil-sum property. In fact, we will prove that if a simple ring with the 2-nil-sum property satisfies some standard finiteness conditions then it is a field. A basic example is the following.

**Example 3.2.** Let  $F$  be a field and  $n \geq 2$  an integer. Since the traces of all nilpotent matrices in  $\mathbb{M}_n(F)$  are 0, it follows that  $\mathbb{M}_n(F)$  does not have the 2-nil-sum property. This can be easily extended to matrices over commutative rings (since in this case the trace of a nilpotent matrix has to be nilpotent).

Below we will study matrix rings over division rings. In this case, the above argument does not work because nilpotent matrices over division rings may have non-zero traces and, moreover, the trace is no longer invariant under similarity. For instance, if  $A = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ -\mathbf{j} & \mathbf{i} \end{pmatrix} \in \mathbb{M}_2(\mathbb{H})$ , where  $\mathbb{H}$  is the ring of real quaternions, then  $A^2 = 0$  and  $\begin{pmatrix} 1 & 0 \\ \mathbf{k} & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ -\mathbf{k} & 1 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{j} \\ 0 & 0 \end{pmatrix}$ .

We start with some elementary lemmas. We include the details of the proofs for the reader's convenience.

**Lemma 3.3.** *Let  $D$  be a division ring. If  $A \in \mathbb{M}_n(D)$  is nilpotent, then  $A^n = 0$ .*

*Proof.* We identify  $A$  with a linear transformation of the right vector space  $V := D^n$  over  $D$ . Then the chain

$$\text{Im}(A) \supset \text{Im}(A^2) \supset \text{Im}(A^3) \supset \cdots$$

should be strictly decreasing (otherwise,  $A$  is not nilpotent). Since  $\dim(V_D) = n$ , it follows that  $A^n = 0$ .  $\square$

**Lemma 3.4.** *Let  $R$  be a ring and  $n \geq 2$ . If  $A = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ 1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & a_{nn} \end{pmatrix} \in \mathbb{M}_n(R)$ ,*

*then*

$$A^k = \left( \begin{array}{cccc|c} & & & & * \\ \hline \sum_{i=1}^k a_{ii} & \cdots & \cdots & \cdots & * \\ 1 & \ddots & \ddots & \vdots & \\ & \ddots & \ddots & \vdots & * \\ & & 1 & \sum_{i=n-k+1}^n a_{ii} & \end{array} \right),$$

where the lower left block has size  $(n - k + 1) \times (n - k + 1)$ . So the  $(k, 1)$ -entry of  $A^k$  is  $a_{11} + \cdots + a_{kk}$  for  $k = 1, \dots, n$ ; in particular, the  $(n, 1)$ -entry of  $A^n$  is  $a_{11} + \cdots + a_{nn}$ .

*Proof.* The claim can be proved by induction on  $k$ . □

**Lemma 3.5.** *Let  $D$  be a division ring and  $n \geq 2$  an integer. If  $X = (x_{ij}) \in \mathbb{M}_n(D)$  is a matrix such that the  $(n - 1)$ -dimensional column vector  $\beta = \begin{bmatrix} x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}$  is not zero, then there exists an invertible matrix  $U = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix} \in \mathbb{M}_n(D)$  with  $V \in \mathbb{M}_{n-1}(D)$ , and  $1 < k \leq n$  such that*

$$UXU^{-1} = \left( \begin{array}{cccc|c} w_{11} & \cdots & \cdots & w_{1k} & \\ 1 & \ddots & \ddots & \vdots & \\ & \ddots & \ddots & \vdots & Y \\ & & 1 & w_{kk} & \\ \hline & & \mathbf{0} & & Z \end{array} \right).$$

*Proof.* Write  $X = \begin{pmatrix} x_{11} & Y_1 \\ W_1 & Z_1 \end{pmatrix}$  in block form where  $Z_1$  is an  $(n - 1) \times (n - 1)$  matrix. Since  $W_1 = \beta \neq 0$  there exists an invertible matrix  $V_1$  in  $\mathbb{M}_{n-1}(D)$

such that  $V_1 W_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ . Then  $U_1 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & V_1 \end{pmatrix}$  is invertible in  $\mathbb{M}_n(D)$  and

$$U_1 X U_1^{-1} = \begin{pmatrix} x_{11} & \gamma_1 V_1^{-1} \\ V_1 W_1 & V_1 X_1 V_1^{-1} \end{pmatrix}, \text{ so the first column of } U_1 X U_1^{-1} \text{ is } \begin{bmatrix} x_{11} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Write  $U_1 X U_1^{-1} = \left( \begin{array}{cc|c} y_{11} & y_{12} & Y_2 \\ 1 & y_{22} & \\ \hline W_2 & & Z_2 \end{array} \right)$ , where  $Z_2$  is an  $(n-2) \times (n-2)$  matrix and the first column of  $W_2$  is  $\mathbf{0}$ . If  $W_2 = \mathbf{0}$  we can simply take  $U = U_1$ . If  $W_2 \neq \mathbf{0}$ , there exists an invertible matrix  $V_2$  in  $\mathbb{M}_{n-2}(D)$  such that  $V_2 W_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$ .

Observe that  $U_2 = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & V_2 \end{pmatrix}$  is invertible in  $\mathbb{M}_n(D)$  and

$$U_2 U_1 X U_1^{-1} U_2^{-1} = \left( \begin{array}{ccc|c} z_{11} & z_{12} & z_{13} & Y_3 \\ 1 & z_{22} & z_{23} & \\ 0 & 1 & z_{33} & \\ \hline W_3 & & & Z_3 \end{array} \right).$$

The block consisting of the first two columns of  $U_2 U_1 X U_1^{-1} U_2^{-1}$  is of the form

$$\begin{bmatrix} z_{11} & z_{12} \\ 1 & z_{22} \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \text{ so the first two columns of } W_3 \text{ are zero. If } W_3 = \mathbf{0}, \text{ we can take}$$

$U = U_2 U_1$ . If  $W_3 \neq \mathbf{0}$ , we continue in a similar way.

This process has to stop after a finite number of steps. Therefore, there exists an integer  $1 < k \leq n$ , and a family of invertible matrices  $U_l = \begin{pmatrix} I_l & \mathbf{0} \\ \mathbf{0} & V_l \end{pmatrix}$ ,  $1 \leq l \leq k-1$ , such that



- $(U_l \cdots U_1)X(U_l \cdots U_1)^{-1} = \left( \begin{array}{ccccc|c} w_{11} & w_{12} & \cdots & w_{1l} & w_{1(l+1)} & \\ 1 & w_{22} & \cdots & w_{2l} & w_{2(l+1)} & \\ \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 & w_{(l+1)(l+1)} & \\ \hline & & & W_{l+1} & & \end{array} \middle| \begin{array}{c} X_{l+1} \\ \\ \\ Y_{l+1} \end{array} \right),$
- the first  $l$  columns of  $W_{l+1}$  are zero,
- $k = n$ , or  $k < n$  and  $W_k = \mathbf{0}$ .

Now take  $U = U_{k-1} \cdots U_1$ . By the constructions of  $U_1, \dots, U_{k-1}$ , one easily sees that  $U$  has the desired property.  $\square$

**Remark 3.6.** The above lemma will be applied for the case when  $X$  is nilpotent. We want to point out that it was proved in [?] that every nilpotent matrix is similar to a rational form (as in the commutative case). But we need the weaker property presented above since we have to use the form of the matrix  $U$ .

**Lemma 3.7.** *Let  $D$  be a division ring and  $A = (a_{ij}) \in \mathbb{M}_n(D)$  a matrix with the property that the  $k$ -th row of  $A$  is the only non-zero row of  $A$ . Then  $A$  is a sum of two nilpotents if and only if  $a_{kk} = 0$ .*

*Proof.* The sufficiency is obvious. For the necessity, let us observe that  $A$  is similar to  $A' = \begin{pmatrix} a & \alpha \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ , where  $a = a_{kk} \in D$  and  $\alpha$  is an  $(n-1)$ -dimensional row vector.

Hence  $A' = X + Y$ , where  $X, Y$  are nilpotents. Write  $X = \begin{pmatrix} x_{11} & \gamma \\ \beta & X_1 \end{pmatrix}$  in block form where  $X_1$  is an  $(n-1) \times (n-1)$  matrix. If  $\beta = 0$ , then  $a = x_{11} + (a - x_{11})$  is a sum of two nilpotents in  $D$ ; so  $a = 0$ . Thus, we can assume that  $\beta \neq 0$ . There exists an invertible matrix  $U = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix} \in \mathbb{M}_n(D)$  and an integer  $k > 1$  as in Lemma ?? such that

$$UXU^{-1} = \left( \begin{array}{cccc|c} w_{11} & \cdots & \cdots & w_{1k} & \\ 1 & \ddots & \ddots & \vdots & \\ & \ddots & \ddots & \vdots & \\ & & 1 & w_{kk} & \\ \hline & & \mathbf{0} & & \end{array} \middle| \begin{array}{c} T \\ \\ \\ Z \end{array} \right).$$

Note that  $UA'U^{-1} = \begin{pmatrix} a & (a_2 & \cdots & a_n) \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$  for some  $a_2, \dots, a_n$  in  $D$ , and that  $UXU^{-1}$  and  $UYU^{-1}$  are nilpotent matrices. It follows that the matrices

$$\begin{pmatrix} w_{11} & \cdots & \cdots & w_{1k} \\ 1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & w_{kk} \end{pmatrix} \text{ and } \begin{pmatrix} w_{11} - a & \cdots & \cdots & w_{1k} - a_k \\ 1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & w_{kk} \end{pmatrix}$$

are both nilpotent matrices. Using Lemma ?? and Lemma ??, we obtain the equalities  $w_{11} + \cdots + w_{kk} = 0 = w_{11} + \cdots + w_{kk} - a$ , and hence  $a = 0$ .  $\square$

**Theorem 3.8.** *Let  $R$  be a simple right Goldie ring with the 2-nil-sum property. Then  $R$  is a field.*

*Proof.* Note that we can view  $R$  as a right order of  $\mathbb{M}_n(D)$  where  $n \geq 1$  and  $D$  is a division ring. If  $n = 1$ , then  $R$  is a subring of a division ring. Since  $\text{nil}(R) = 0$ , the only non central-unit of  $R$  is 0, so  $R$  is a field.

We next show that  $n \geq 2$  yields a contradiction. Assume that  $n \geq 2$ . For all  $1 \leq k \leq n$ , we denote by  $\mathcal{A}_k$  the set of all non-zero matrices  $A = (a_{ij}) \in R$  with the property that the  $k$ -th row of  $A$  is the only non-zero row of  $A$ .

Since  $R$  is a right order of  $\mathbb{M}_n(D)$ , for  $1 \leq k \leq n$  we can write  $E_{kk} = U_k S_k^{-1}$  where  $U_k, S_k \in R$  and where  $U_k = (u_{ij}^{(k)})$  and  $S_k = (s_{ij}^{(k)})$ . It follows that  $U_k = E_{kk} S_k \in \mathcal{A}_k$ , so all sets  $\mathcal{A}_k$  are non-empty.

Moreover, if  $1 \leq k \leq n$ , we denote by  $i(k)$  the minimal integer with the property that there exists  $(a_{ij}) \in \mathcal{A}_k$  such that  $a_{kl} = 0$  for all  $l < i(k)$  and  $a_{ki(k)} \neq 0$ .

We show that if  $k < i(k)$  then  $i(k) < i(i(k))$ . In order to prove this, let  $A = (a_{ij}) \in \mathcal{A}_k$  be a matrix such that  $a_{kl} = 0$  for all  $l < i(k)$  and  $a := a_{ki(k)} \neq 0$ . For any  $B = (b_{ij}) \in \mathcal{A}_{i(k)}$ , the  $k$ -th row of  $AB$  is  $(ab_{i(k)1}, \dots, ab_{i(k)i(k)}, \dots, ab_{i(k)n})$  and this is the only possible non-zero row of  $AB$ . Since  $a \neq 0$  it follows, by the minimality of  $i(k)$ , that  $b_{i(k)1} = \cdots = b_{i(k)(i(k)-1)} = 0$ . Moreover, by Lemma ?? it follows that  $b_{i(k)i(k)} = 0$  as  $B \in \mathcal{A}_{i(k)}$ . Hence,  $i(k) < i(i(k))$ .

Now we close the proof by observing that  $1 < i(1)$ , hence we obtain a strictly increasing infinite sequence bounded above by  $n$ :  $1 < i(1) < i(i(1)) < \dots$ . This is a contradiction.  $\square$

**Remark 3.9.** By Theorem 3.8, for a division ring  $D$ , the ring  $\mathbb{M}_n(D)$  satisfies the 2-nil-sum property iff  $n = 1$  and  $D$  is a field. Note that there exists a division ring  $D$  such that all matrices in  $\mathbb{M}_3(D)$  are sums of three nilpotent matrices (see [?]).

We close the paper, with some applications of Theorem ?? and Theorem ?. A nonzero right ideal  $I$  of a ring  $R$  is called a uniform right ideal if the intersection of any two nonzero right ideals contained in  $I$  is nonzero. A ring  $R$  is semipotent if every nonzero right ideal not contained in  $J(R)$  contains a nonzero idempotent. A ring  $R$  is said to be of bounded index (of nilpotence) if there is a positive integer  $n$  such that  $a^n = 0$  for all nilpotents  $a$  of  $R$ .

**Corollary 3.10.** *A ring  $R$  with the 2-nil-sum property is commutative under any of the following additional assumptions:*

- (1)  *$R$  contains a uniform ideal;*
- (2)  *$R$  is a semipotent ring of bounded index;*
- (3)  *$R$  is a semilocal ring.*

*Proof.* By Theorem ??, suppose that  $R$  is a simple ring.

(1) It follows from [?] that  $R$  contains no infinite direct sums of right ideals (i.e.,  $R_R$  is of finite uniform dimension). Moreover, from the last remark of [?] it follows that the maximal right ring of quotients of  $R$  is simple artinian. Thus,  $R$  is right Goldie by [?, Proposition 13.41].

(2) If  $R$  is semipotent of bounded index, then  $R$  is simple artinian by [?, Corollary 6].

(3) If  $R$  is semilocal, then  $R$  is simple artinian.

So the claim follows from Theorem ?.  $\square$

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