STABILITY OF CONSTANT STEADY STATES OF A CHEMOTAXIS MODEL

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ABSTRACT. The Cauchy problem for the parabolic–elliptic Keller–Segel system in the whole n-dimensional space is studied. For this model, every constant $A \in \mathbb{R}$ is a stationary solution. The main goal of this work is to show that $A < 1$ is a stable steady state while $A > 1$ is unstable. Uniformly local Lebesgue spaces are used in order to deal with solutions that do not decay at spatial variable on the unbounded domain.

1. INTRODUCTION

There are several mathematical works on the chemotaxis model introduced by Keller and Segel [?]. Here, we refer the reader only to the monograph [?] and the reviews [?, ?] for a discussion of those mathematical results as well as for additional references. In this paper, we consider the following minimal parabolic-elliptic Keller-Segel system

$$u_t - \Delta u + \nabla \cdot (u \nabla \psi) = 0, \quad -\Delta \psi + \psi = u \quad \text{for} \quad t > 0, \quad x \in \mathbb{R}^n,$$

where $u = u(t, x)$ denotes the density of cells and $\psi = \psi(t, x)$ is a concentration of chemoattractant. In these equations, all constant parameters are equal to one for simplicity of the exposition. System (??) was already studied in the whole space *e.g.* in the papers [?, ?, ?, ?, ?, ?, ?, ?], where several results either on a blow up or on a large time behavior of solutions have been obtained.

For each constant $A \in \mathbb{R}$, the couple $(u, \psi) = (A, A)$ is a stationary solution of system (??) and, since the domain is unbounded, it does not belong to any Lebesgue L^p-space with $p \in [1, \infty)$. Thus, in Theorem ?? and in Section ??, we develop a mathematical theory concerning local-in-time solutions to the initial value problem for system (??) in the uniformly local Lebesgue spaces $L^p_{\mathrm{uloc}}(\mathbb{R}^n)$. Then, we consider a constant stationary solution $(u, \psi) = (A, A)$ with $A \in [0, 1)$ and we show in Theorem ?? that a small L^p-perturbation of such an initial datum gives a global-in-time solution which converges toward (A, A) as $t \to \infty$. On the other hand, we prove in the Theorem ?? that the constant solution is unstable in the Lyapunov sense if $A > 1$.

A stability of constant solutions of chemotaxis models has been already studied in bounded domains. For example, the paper [?] describes dynamics near an unstable constant solution to the classical parabolic-parabolic Keller-Segel model in a bounded domain and obtained results are interpreted as an early pattern formation. Another

work [?] is devoted to the system

$$u_t - \Delta u + \nabla \cdot (u \nabla \psi) = 0, \quad -\Delta \psi + \mu = u, \quad \mu \equiv \frac{1}{|B_R(0)|} \int_{B_R(0)} u \, \mathrm{d}x,$$

in the ball of radius $R > 0$ with Neumann boundary condition. Here, constants are also stationary solutions and it is shown in the work [?] that there exists a critical number m_c such that at mass levels above m_c the constant steady states are extremely unstable and blow-up can occur. On the other hand, for $m < m_c$ there exist infinitely many radial solutions with a mass equal to m.

Notation. The usual norm of the Lebesgue space $L^p(\mathbb{R}^n)$ with respect to the spatial variable is denoted by $\|\cdot\|_p$ for all $p \in [1, \infty]$. In the following, we use also the uniformly local Lebesgue space $L^p_{\mathrm{uloc}}(\mathbb{R}^n)$ with the norm $\|\cdot\|_{p,\mathrm{uloc}}$ defined below by formula (??). Any other norm in a Banach space Y is denoted by $\|\cdot\|_Y$. The letter C corresponds to a generic constant (always independent of t and x) which may vary from line to line. We write $C = C(\alpha, \beta, \gamma, ...)$ when we want to emphasize the dependence of C on parameters $\alpha, \beta, \gamma, ...$. We use standard definition of Fourier transform $\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, \mathrm{d}x$.

2. RESULTS AND COMMENTS

Our goal is to study properties of solutions to the Cauchy problem for the simplified parabolic–elliptic Keller–Segel model of chemotaxis

$$\begin{cases} u_t - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0, \ x \in \mathbb{R}^n, \\ -\Delta \psi + \psi = u, & t > 0, \ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

with $n \geq 1$. We solve the second equation with respect to ψ to obtain $\psi = K * u$, where K is the Bessel function (see Lemma **??** below) which reduces problem (??) to the following one

$$\begin{cases} u_t - \Delta u + \nabla \cdot (u \nabla K * u) = 0, & t > 0, \ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

We begin by a result on an existence of local-in-time solutions to problem (??) in the uniformly local Lebesgue spaces

$$L^p_{\mathrm{uloc}}(\mathbb{R}^n) \equiv \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{p,\mathrm{uloc}} \equiv \sup_{x \in \mathbb{R}^n} \left(\int_{B_1(x)} |f(y)|^p \, \mathrm{d}y \right)^{1/p} < +\infty \right\}$$

for $p \in [1, \infty)$ and $L^\infty_{\mathrm{uloc}}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$.

Theorem 2.1. *For each p satisfying*

$$p \in [1, \infty] \quad if \quad n = 1,$$
$$p \in \left[\frac{3}{2}, \infty \right] \quad if \quad n = 2,$$
$$p \in \left(\frac{n}{2}, \infty \right] \quad if \quad n \geq 3,$$

and every $u_0 \in L^p_{\mathrm{uloc}}(\mathbb{R}^n)$, there exists $T > 0$ and a unique mild solution

$$u \in L^\infty \big([0, T); L^p_{\mathrm{uloc}}(\mathbb{R}^n) \big) \cap C \big((0, T); L^p_{\mathrm{uloc}}(\mathbb{R}^n) \big)$$

of problem (??). Moreover, if $u_0 \geq 0$, then $u(t, x) \geq 0$ almost everywhere in $[0, T) \times \mathbb{R}^n$.

The more-or-less standard proof of Theorem **??** is based on the Banach contraction principle applied to an integral representation of solutions to problem (**??**) (see Section **??** for more details).

Now, we formulate a simple consequence of Theorem **??** in the case when an initial condition is an L^p-perturbation of a constant $A \in \mathbb{R}$.

Corollary 2.2. *Let p satisfy conditions (**??**). For every $A \in \mathbb{R}$ and every $v_0 \in L^p(\mathbb{R}^n)$ there exists a unique local-in-time mild solution $u = u(t, x)$ of problem (**??**) (as stated in Theorem **??**) corresponding to the initial datum $u_0 = A + v_0 \in L^p_{\mathrm{uloc}}(\mathbb{R}^n)$. This solution satisfies $u - A \in C\big([0, T); L^p(\mathbb{R}^n)\big)$.*

This corollary is an immediate consequence of the uniqueness of solutions established in Theorem **??** combined with the uniqueness result of solutions to the perturbed problem considered in Proposition **??**, below.

Next, we show that one can construct global-in-time solutions around each constant solution $A \in [0, 1)$.

Theorem 2.3. *Let $A \in [0, 1)$. Assume that the exponent p satisfies conditions (**??**) and moreover $p \leq n$. Fix $q \in (n, 2p)$. There exists $\varepsilon > 0$ such that for every $v_0 \in L^p(\mathbb{R}^n)$ with $\|v_0\|_p < \varepsilon$, problem (**??**) with the initial condition $u_0 = A + v_0$ has a unique global-in-time mild solution $u(t, x)$ satisfying $u - A \in C\big([0, \infty); L^p(\mathbb{R}^n)\big)$ and*

$$\|u(t) - A\|_p + t^{\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}\|u(t) - A\|_q \leq C\|u_0 - A\|_p$$

for a constant $C > 0$ and all $t > 0$.

The smallness assumption imposed on initial conditions in Theorem **??** seems to be necessary. This is clear in the case $A = 0$, where sufficiently large initial data lead to solutions which blow-up in finite time, see *e.g.* [**?**, **?**, **?**, **?**] for blow-up results for solutions of system (**??**) considered in the whole space.

Next, we deal with $A > 1$ which appears to be the unstable constant stationary solution.

Theorem 2.4. *The constant stationary solution $A > 1$ of problem (**??**) is not stable in the Lyapunov sense under small perturbations from $L^p(\mathbb{R}^n)$ for each p satisfying condition (**??**) except $p = 1$ and $p = \infty$.*

In this theorem, we do not claim that solutions corresponding to L^p-perturbations of $A > 1$ are global-in-time. We show only that if they are global-in-time then they cannot be stable.

To conclude this section, we notice that a constant $A < 0$ is linearly stable which we comment in Remark **??**, below. The proof of nonlinear stability of this constant can be obtained by the method used in the proof of Theorem **??**. We add some comments on the linear stability of the constant solution $A = 1$ in Remark **??**, below.

3. Local-in-time solutions in uniformly local Lebesgue spaces

We find solutions to problem (**??**) via its formulation in the integral form

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u(s)\nabla K * u(s)\big)\, \mathrm{d}s$$

with the heat semigroup given by the formula

$$(e^{t\Delta}f)(x) \equiv (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)\, \mathrm{d}y.$$

This construction requires auxiliary results which we are going to gather and prove below. Then, Theorem **??** is proved at the end of this section. We begin by recalling properties of the heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ acting on the L^p_{uloc}-spaces.

Proposition 3.1 ([**?**, **?**]). *For all $1 \leq q \leq p \leq \infty$, $k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}^n_+$ there exist numbers $C = C(n, p, q, k, \alpha) > 0$ such that*

$$\|\partial_t^k \partial_x^\alpha e^{t\Delta} f\|_{p,\mathrm{uloc}} \leq C t^{-k-\frac{|\alpha|}{2}} \big(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\big) \|f\|_{q,\mathrm{uloc}}$$

for every $f \in L^p_{\mathrm{uloc}}(\mathbb{R}^n)$ and all $t > 0$. In particular, when $p = q$ and $1 \leq p \leq \infty$, it holds that

$$\|e^{t\Delta} f\|_{p,\mathrm{uloc}} \leq \|f\|_{p,\mathrm{uloc}} \qquad \text{for all} \quad t \geq 0.$$

Remark 3.2. Proposition **??** generalizes the following well-known estimates of the heat semigroup acting on the Lebesgue space $L^p(\mathbb{R}^n)$

$$\|\partial_t^k \partial_x^\alpha e^{t\Delta} f\|_p \leq C t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-k-\frac{|\alpha|}{2}} \|f\|_q \quad \text{for all} \quad t > 0.$$

Remark 3.3. It follows from estimates (**??**) applied with $k = 0$, $\alpha = 0$ and $p = q$ that $e^{t\Delta} f \in L^\infty\big((0, \infty); L^p_{\mathrm{uloc}}(\mathbb{R}^n)\big)$ for each $f \in L^p_{\mathrm{uloc}}(\mathbb{R}^n)$ and, in general, this mapping in not continuous in time. This continuity holds true in the smaller space

$$\mathcal{L}^p_{\mathrm{uloc}}(\mathbb{R}^n) = \overline{BUC(\mathbb{R}^n)}^{\|\cdot\|_{p,\mathrm{uloc}}},$$

where $BUC(\mathbb{R}^n)$ is a space of all bounded uniformly continuous functions on $\mathbb{R}^n$. In fact, the following statements are equivalent:

(1) $f \in \mathcal{L}^p_{\mathrm{uloc}}(\mathbb{R}^n)$.
(2) $\|f(\cdot + h) - f\|_{p,\mathrm{uloc}} \to 0$ as $|h| \to 0$.
(3) $\|e^{t\Delta} f - f\|_{p,\mathrm{uloc}} \to 0$ as $t \to 0$.

We refer the reader to [**?**, Proposition 2.2] for the proof of these properties.

Remark 3.4. There is an alternative definition of the norm in the uniformly local L^p-spaces

$$\|f\|_{p,\mathrm{uloc},\rho} \equiv \sup_{x\in\mathbb{R}^n} \Big(\int_{B_\rho(x)} |f(y)|^p \, \mathrm{d}y \Big)^{1/p}$$

for each $\rho > 0$. However, by a simple scaling property, we can show that all these norms are in fact equivalent.

Let us also recall properties of the Bessel kernel which are systematically used in this work.

Lemma 3.5. *Denote by $\psi \in \mathcal{S}'(\mathbb{R}^n)$ a solution of the equation $-\Delta\psi + \psi = u$ for some $u \in \mathcal{S}'(\mathbb{R}^n)$. The following statements hold true.*

(1) $\psi = K * u$, where $\widehat{K}(\xi) = \frac{1}{1+|\xi|^2}$.
(2) *For $n = 1$, $K(x) = \frac{1}{2}e^{-|x|}$.*
(3) *For $n \geq 2$, $K \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for each $p \in \big[1, \frac{n}{n-2}\big)$ and $\nabla K \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for each $q \in \big[1, \frac{n}{n-1}\big)$.*
(4) $|\nabla K(x)| = |K'(|x|)| = -K'(|x|)$ *is radially symmetric and decreasing in $|x|$.*

Sketch of the proof. Property 1 is well known. Item 2 can be obtained by a direct calculation. For the proofs of properties 3 and 4 we refer the reader to [**?**, Sec. 1.2.5.]. In particular, in order to show property 4, we recall that $K(x) = (2\pi)^{-\frac{n}{2}} |x|^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}(|x|)$,

where $K_\nu = K_\nu(r)$ $(r > 0, \nu > -\frac{1}{2})$ stands for the modified Bessel function of the third kind, that is

$$K_\nu(r) = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{2}}\Gamma(\nu + \frac{1}{2})} r^{-\frac{1}{2}} e^{-r} \int_0^\infty s^{\nu - \frac{1}{2}} \left(1 + \frac{s}{2r}\right)^{\nu - \frac{1}{2}} e^{-s} \, ds,$$

see *e.g.* [**?**, formula (1.2.25), p.12]. In particular, we observe that

$$K'(r) = -\frac{1}{4\pi} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} r^{-\frac{n-2}{2}} K_{\frac{n}{2}}(r).$$

Since $K_{\frac{n}{2}}(|x|) > 0$ and $|x|^{-\frac{n-2}{2}}$ decreases in $|x|$, the function $-|x|^{-\frac{n-2}{2}} K_{\frac{n}{2}}(|x|) < 0$ increases in $|x|$. Therefore $|\nabla K(x)| = |K'(|x|)| = -K'(|x|)$ is radially symmetric and decreasing in $|x|$. $\square$

Our goal is to estimate the nonlinear term on the right hand side of equation (**??**). We begin with the following elementary lemma.

Lemma 3.6. *For $f \in L^p_{\mathrm{uloc}}(\mathbb{R}^n)$ it holds that*

$$3^{-n} \sup_{x \in \mathbb{R}^n} \|f\|_{L^p(B_1(x))} \leq \sup_{k \in \mathbb{Z}^n} \|f\|_{L^p(Q_{\frac{1}{2}}(k))} \leq 2^n \sup_{x \in \mathbb{R}^n} \|f\|_{L^p(B_1(x))},$$

where $Q_r(k) \equiv \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \max_{1 \leq j \leq n} |x_j - k_j| \leq r\}$ for $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ and $r > 0$.

Proof. Obviously, it holds that $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^n} Q_{1/2}(k)$ with $|Q_{1/2}(k) \cap Q_{1/2}(\ell)| = 0$ for $k \neq \ell$. Thus, for $x \in \mathbb{R}^n$, there exists $k^0 \in \mathbb{Z}^n$ such that $x \in Q_{1/2}(k^0)$. Since $x \in Q_{1/2}(k^0)$, there exist lattice points $\{k^\ell\}_{\ell=1}^{3^n-1} \subset \mathbb{Z}^n$ such that $B_1(x) \subset \bigcup_{\ell=0}^{3^n-1} Q_{1/2}(k^\ell)$. Indeed, the set

$$P \equiv \left\{ k \in \mathbb{Z}^n : \begin{array}{c} \text{There exists } \ell \in \{1, \cdots, n\} \text{ and } \{m_j\}_{j=1}^\ell \subset \{1, \cdots, n\} \text{ such that} \\ m_j < m_{j+1}, e_{m_j} = (0, \cdots, \underbrace{1}_{m_j\text{-th}}, \cdots, 0) \text{ and } k = k^0 \pm e_{m_1} \pm \cdots \pm e_{m_\ell} \end{array} \right\},$$

consists of $(3^n - 1)$ points in $\mathbb{Z}^n$. Therefore the number of elements of the set $\{k^0\} \cup P$ is 3^n, and thus we have

$$\left(\int_{B_1(x)} |f(y)|^p \, dy\right)^{\frac{1}{p}} \leq \sum_{j=0}^{3^n-1} \left(\int_{Q_{\frac{1}{2}}(k^j)} |f(y)|^p \, dy\right)^{\frac{1}{p}} \leq 3^n \sup_{k \in \mathbb{Z}^n} \left(\int_{Q_{\frac{1}{2}}(k)} |f(y)|^p \, dy\right)^{\frac{1}{p}}.$$

Hence we obtain the first part of our claim. On the other hand, it holds that $Q_{1/2}(k) \subset \bigcup_{j=1}^{2^n} B_1(x^j)$, where $\{x^j\}_{j=1}^{2^n} \subset \mathbb{R}^n$ with $x_\ell^j = k_\ell \pm \frac{1}{2}$ for all $\ell = 1, \cdots, n$ and $j = 1, \cdots, 2^n$. Thus we observe

$$\left(\int_{Q_{\frac{1}{2}}(k)} |f(y)|^p \, dy\right)^{\frac{1}{p}} \leq \sum_{j=1}^{2^n} \left(\int_{B_1(x^j)} |f(y)|^p \, dy\right)^{\frac{1}{p}} \leq 2^n \sup_{x \in \mathbb{R}^n} \left(\int_{B_1(x)} |f(y)|^p \, dy\right)^{\frac{1}{p}}.$$

$\square$

Now, we prove a Young type inequality in the space $L^p_{\mathrm{uloc}}(\mathbb{R}^n)$.

Proposition 3.7. *Let $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ with $1 \leq q \leq p \leq \infty$ and $1 \leq r < \frac{n}{n-1}$. Then, there exists a number $C = C(n) > 0$ such that*

$$\|\nabla K * f\|_{p,\mathrm{uloc}} \leq C\big(\|\nabla K\|_1 + \|\nabla K\|_r\big)\|f\|_{q,\mathrm{uloc}}.$$

Proof. This lemma has been proved in [**?**, Theorem 3.1] in the case of an arbitrary bounded and integrable function K. Since the Bessel kernel is not bounded for $n \geq 2$, we revise those arguments in our setting. Using the decomposition $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^n} Q_{1/2}(k)$ and

$$\nabla K(x) = \sum_{k \in \mathbb{Z}^n} \chi_{Q_{\frac{1}{2}}(k)}(x) \nabla K(x), \quad f(x) = \sum_{k \in \mathbb{Z}^n} \chi_{Q_{\frac{1}{2}}(k)}(x) f(x) \qquad \text{a.e. } x \in \mathbb{R}^n,$$

we observe

$$\|\nabla K * f\|_{L^p(Q_{\frac{1}{2}}(k))} = \left\| \left(\sum_{k' \in \mathbb{Z}^n} \chi_{Q_{\frac{1}{2}}(k')} \nabla K \right) * \left(\sum_{k'' \in \mathbb{Z}^n} \chi_{Q_{\frac{1}{2}}(k'')} f \right) \right\|_{L^p(Q_{\frac{1}{2}}(k))}$$

$$= \left\| \sum_{k',k'' \in \mathbb{Z}^n} \left(\chi_{Q_{\frac{1}{2}}(k')} \nabla K \right) * \left(\chi_{Q_{\frac{1}{2}}(k'')} f \right) \right\|_{L^p(Q_{\frac{1}{2}}(k))}$$

$$\leq \sum_{k',k'' \in \mathbb{Z}^n} \left\| \left(\chi_{Q_{\frac{1}{2}}(k')} \nabla K \right) * \left(\chi_{Q_{\frac{1}{2}}(k'')} f \right) \right\|_{L^p(Q_{\frac{1}{2}}(k))}.$$

The well-known fact that $\operatorname{supp} f_1 * f_2 \subset \operatorname{supp} f_1 + \operatorname{supp} f_2$ implies that the support of the function $(\chi_{Q_{1/2}(k')} \nabla K) * (\chi_{Q_{1/2}(k'')} f)$ lies in the cube $Q_1(k' + k'')$. Thus for $1 \leq p, q, r \leq \infty$ with $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, the Young inequality shows that

$$\sum_{k',k'' \in \mathbb{Z}^n} \left\| \left(\chi_{Q_{\frac{1}{2}}(k')} \nabla K \right) * \left(\chi_{Q_{\frac{1}{2}}(k'')} f \right) \right\|_{L^p(Q_{\frac{1}{2}}(k) \cap Q_1(k'+k''))}$$

$$\leq \sum_{\substack{k',k'' \in \mathbb{Z}^n \\ \max_{1 \leq j \leq n} |k'_j + k''_j - k_j| \leq 1}} \left\| \left(\chi_{Q_{\frac{1}{2}}(k')} \nabla K \right) * \left(\chi_{Q_{\frac{1}{2}}(k'')} f \right) \right\|_{L^p(Q_{\frac{1}{2}}(k) \cap Q_1(k'+k''))}$$

$$\leq \sum_{\substack{k',k'' \in \mathbb{Z}^n \\ \max_{1 \leq j \leq n} |k'_j + k''_j - k_j| \leq 1}} \left\| \left(\chi_{Q_{\frac{1}{2}}(k')} \nabla K \right) * \left(\chi_{Q_{\frac{1}{2}}(k'')} f \right) \right\|_p$$

$$\leq \sum_{\substack{k',k'' \in \mathbb{Z}^n \\ \max_{1 \leq j \leq n} |k'_j + k''_j - k_j| \leq 1}} \|\chi_{Q_{\frac{1}{2}}(k')} \nabla K\|_r \|\chi_{Q_{\frac{1}{2}}(k'')} f\|_q.$$

The set $P(k) \equiv \{(k', k'') \in \mathbb{Z}^n \times \mathbb{Z}^n \mid \max_{1 \leq j \leq n} |k'_j + k''_j - k_j| \leq 1\}$ for $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ consists of 3^n elements. Therefore we obtain

$$\sum_{(k',k'') \in P(k)} \|\chi_{Q_{\frac{1}{2}}(k')} \nabla K\|_r \|\chi_{Q_{\frac{1}{2}}(k'')} f\|_q \leq 3^n \sup_{k'' \in \mathbb{Z}^n} \|f\|_{L^q(Q_{\frac{1}{2}}(k''))} \sum_{k' \in \mathbb{Z}^n} \|\nabla K\|_{L^r(Q_{\frac{1}{2}}(k'))}.$$

Due to the compactness of $Q_{1/2}(k)$ for $k \in \mathbb{Z}^n$, there exists $y_k \in Q_{1/2}(k)$ such that $|y_k| = \operatorname{dist}(Q_{1/2}(k), \{0\})$. If $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ satisfies $\max_{1 \leq j \leq n} |k_j| \geq 2$, it holds that $0 \notin Q_{1/2}(k)$. We notice that, for $k \in G \equiv \{\ell \in \mathbb{Z}^n \mid \max_{1 \leq j \leq n} |\ell_j| \geq 2\}$, the function $|\nabla K(x)|$ attains its maximum value at $x = y_k$ and we obtain $\|\nabla K\|_{L^r(Q_{1/2}(k))} \leq |\nabla K(y_k)|$.

Thus we have

$$\sum_{k \in \mathbb{Z}^n} \|\nabla K\|_{L^r(Q_{\frac{1}{2}}(k))} = \sum_{k \in G} \|\nabla K\|_{L^r(Q_{\frac{1}{2}}(k))} + \sum_{k \in G^c} \|\nabla K\|_{L^r(Q_{\frac{1}{2}}(k))}$$

$$\leq \sum_{k \in G} |\nabla K(y_k)| + \sum_{k \in G^c} \|\nabla K\|_{L^r(Q_{\frac{3}{2}}(0))}$$

$$= \sum_{k \in G} |\nabla K(y_k)| + 3^n \|\nabla K\|_{L^r(Q_{\frac{3}{2}}(0))}.$$

We define cubes corresponding to $y_k \in Q_{1/2}(k)$ for $k \in G$. Hence, there exists $\ell \in \mathbb{N}$ and $\{k^j\}_{j=1}^\ell \subset \mathbb{Z}^n$ such that $Q_{1/2}(k^j) \cap \{\lambda y_k : 0 < \lambda < 1\} \neq \emptyset$ for $j = 1, \cdots, \ell$. Here, we denote by $\{k^j\}_{j=1}^\ell$ the lattice points such that for $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{\ell-1} < \lambda_\ell = 1$, $Q_{1/2}(k^j)^\circ \cap \{\lambda y_k : 0 \leq \lambda \leq 1\} = \{\lambda y_k : \lambda_{j-1} < \lambda < \lambda_j\}$. We call such a cube the corresponding cube and we denote

$$P_k \equiv \{k' \in \mathbb{Z}^n \,|\, Q_{\frac{1}{2}}(k') \text{ is a corresponding cube such that } Q_{\frac{1}{2}}(k') = Q_{\frac{1}{2}}(k)\}$$

for $k \in G$. Since $|\nabla K(x)|$ decreases in $|x|$ by Lemma **??** and $x \in Q_{1/2}(k^j)$ satisfies $|x| \leq |y_k|$, it holds that

$$|\nabla K(y_k)| = \int_{Q_{\frac{1}{2}}(k^j)} |\nabla K(y_k)| \, \mathrm{d}x \leq \int_{Q_{\frac{1}{2}}(k^j)} |\nabla K(x)| \, \mathrm{d}x.$$

By [**?**, line 5 from above, p.384], for each fixed $k \in G$, the set P_k consists of at most 5^n points. Therefore,

$$\sum_{k \in G} |\nabla K(y_k)| + 3^n \|\nabla K\|_{L^r(Q_{\frac{3}{2}}(0))} \leq \sum_{k \in G} \sum_{k' \in P_k} \int_{Q_{\frac{1}{2}}(k')} |\nabla K(x)| \, \mathrm{d}x + 3^n \|\nabla K\|_r$$

$$= \sum_{k \in G} \sum_{k' \in P_k} \int_{Q_{\frac{1}{2}}(k)} |\nabla K(x)| \, \mathrm{d}x + 3^n \|\nabla K\|_r$$

$$\leq 5^n \sum_{k \in G} \int_{Q_{\frac{1}{2}}(k)} |\nabla K(x)| \, \mathrm{d}x + 3^n \|\nabla K\|_r$$

$$\leq 5^n \int_{\mathbb{R}^n} |\nabla K(x)| \, \mathrm{d}x + 3^n \|\nabla K\|_r.$$

Combining relations (**??**)-(**??**), we conclude

$$\|\nabla K * f\|_{L^p(Q_{\frac{1}{2}}(k))} \leq 3^n (5^n \|\nabla K\|_1 + 3^n \|\nabla K\|_r) \sup_{k' \in \mathbb{Z}^n} \|f\|_{L^q(Q_{\frac{1}{2}}(k'))}$$

and thus

$$\sup_{k \in \mathbb{Z}^n} \|\nabla K * f\|_{L^p(Q_{\frac{1}{2}}(k))} \leq (15^n \|\nabla K\|_1 + 9^n \|\nabla K\|_r) \sup_{k' \in \mathbb{Z}^n} \|f\|_{L^q(Q_{\frac{1}{2}}(k'))},$$

where $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ for $1 \leq r < \frac{n}{n-1}$. Thanks to Lemma **??**, we obtain

$$\|\nabla K * f\|_{p,\mathrm{uloc}} \leq (90^n \|\nabla K\|_1 + 54^n \|\nabla K\|_r)\|f\|_{q,\mathrm{uloc}},$$

which completes the proof. $\square$

Lemma 3.8. *For each p satisfying condition (**??**) there exists $k \in (n, \infty]$ and a number $C = C(p, k, n, \nabla K)$ such that*

$$\|\nabla e^{\tau \Delta} \cdot (u \nabla K * v)\|_{p,\mathrm{uloc}} \leq C\tau^{-\frac{1}{2}}(1 + \tau^{-\frac{n}{2}\frac{1}{k}})\|u\|_{p,\mathrm{uloc}}\|v\|_{p,\mathrm{uloc}}$$

for all $u, v \in L^p_{\mathrm{uloc}}(\mathbb{R}^n)$ and $\tau > 0$.

Proof. Using the heat semigroup estimates from Proposition **??** and the Hölder inequality (which also holds in L^p_{uloc}-norm) we have

$$\|\nabla e^{\tau\Delta} \cdot (u\nabla K * v)\|_{p,\mathrm{uloc}} \leq C\tau^{-\frac{1}{2}}(1 + \tau^{-\frac{n}{2}\frac{1}{k}})\|u\nabla K * v\|_{r,\mathrm{uloc}}$$
$$\leq C\tau^{-\frac{1}{2}}(1 + \tau^{-\frac{n}{2}\frac{1}{k}})\|u\|_{p,\mathrm{uloc}}\|\nabla K * v\|_{k,\mathrm{uloc}},$$

where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{k} \quad \text{and} \quad r \geq 1.$$

Moreover, applying Proposition **??**, we obtain

$$\|\nabla K * v\|_{k,\mathrm{uloc}} \leq C(\|\nabla K\|_1 + \|\nabla K\|_{q_1})\|v\|_{p,\mathrm{uloc}}$$

with

$$1 + \frac{1}{k} = \frac{1}{p} + \frac{1}{q_1} \quad \text{and} \quad q_1 \in \left[1, \frac{n}{n-1}\right).$$

Let us show that for every p satisfying condition (**??**) we can always choose $k \in (n, \infty]$ and $r \geq 1$ satisfying the conditions above. Indeed, if $n > 1$ then for every $k \in [p, 2p]$ we have

$$\frac{1}{q_1} = 1 + \frac{1}{k} - \frac{1}{p} \leq 1 \quad \text{and} \quad \frac{1}{q_1} = 1 + \frac{1}{k} - \frac{1}{p} \geq 1 - \frac{1}{2p} > 1 - \frac{1}{n}.$$

Analogously if $n = 1$ then the inequalities holds true for every $k \in [p, \infty]$. Next, the condition $r \geq 1$ is equivalent to $k \in [p/(p-1), \infty]$. For p satisfying condition (**??**) and $n > 1$ the intersection $[p, 2p] \cap [p/(p-1), \infty] \cap (n, \infty]$ is nonempty and hence the choice of k is always possible. Analogously, if $n = 1$ we choose arbitrary $k \in [p/(p-1), \infty] \cap (n, \infty]$. $\square$

We obtain a solution to integral equation (**??**) from the Banach fixed point theorem formulated in the following way.

Proposition 3.9. *Let X be a Banach space and let $Q = Q[\cdot, \cdot] : X \times X \to X$ be a bounded bilinear form with*

$$\|Q[u, v]\|_X \leq C_1\|u\|_X\|v\|_X$$

for some $C_1 > 0$ independent of $u, v \in X$. Assume that $\delta \in (0, 1/(4C_1))$. If $\|y_0\|_X \leq \delta$, then the equation

$$u = y_0 + Q[u, u]$$

has a solution with $\|u\|_X \leq 2\delta$. This solution is unique in the set $\{u \in X : \|u\|_X \leq 2\delta\}$ and stable in the following sense: if $y_0, \widetilde{y}_0 \in X$ satisfy $\|y_0\|_X \leq \delta$, $\|\widetilde{y}_0\|_X \leq \delta$ then for the corresponding solutions $u, \widetilde{u} \in X$ we have

$$\|u - \widetilde{u}\|_X \leq C_2\|y_0 - \widetilde{y}_0\|_X,$$

where $C_2 > 0$ is independent of u and $\widetilde{u}$.

*Proof of Theorem **??**.* For $T > 0$, we introduce $X_T \equiv L^\infty\big((0, T); L^p_{\mathrm{uloc}}(\mathbb{R}^n)\big)$ which is a Banach space with the norm $\|u\|_{X_T} \equiv \sup_{t \in (0, T)} \|u(t)\|_{p,\mathrm{uloc}}$. In order to apply Proposition **??**, it suffices to estimate the bilinear form

$$Q[u, v](t) = -\int_0^t \nabla e^{(t-s)\Delta} \cdot \big(u(s)\nabla K * v(s)\big) \, ds.$$

By Lemma **??**, for some $k > n$, we obtain

$$\|Q[u,v](t)\|_{p,\mathrm{uloc}} \leq \int_0^t \|\nabla e^{(t-s)\Delta} \cdot (u(s)\nabla K * v(s))\|_{p,\mathrm{uloc}}$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2}}(1 + (t-s)^{-\frac{n}{2}\frac{1}{k}})\|u(s)\|_{p,\mathrm{uloc}}\|v(s)\|_{p,\mathrm{uloc}}\,\mathrm{d}s$$

$$\leq C\|u\|_{X_T}\|v\|_{X_T} \int_0^t (t-s)^{-\frac{1}{2}}\big(1 + (t-s)^{-\frac{n}{2}\frac{1}{k}}\big)\,\mathrm{d}s$$

$$\leq C\big(t^{\frac{1}{2}} + t^{\frac{1}{2}-\frac{n}{2k}}\big)\|u\|_{X_T}\|v\|_{X_T}.$$

Therefore, we have

$$\|Q[u,v]\|_{X_T} \leq C\big(T^{\frac{1}{2}} + T^{\frac{1}{2}-\frac{n}{2k}}\big)\|u\|_{X_T}\|v\|_{X_T}.$$

Since inequality (**??**) in Proposition **??** provides $\|e^{t\Delta}u_0\|_{X_T} \leq \|u_0\|_{p,\mathrm{uloc}}$, we obtain a solution to the integral equation via Proposition **??** for sufficiently small $T > 0$. In order to show that $u \in C\big((0,T); L^p_{\mathrm{uloc}}(\mathbb{R}^n)\big)$ it suffices to follow the arguments from [**?**, p. 388]. This solution is unique by a usual reasoning.

To show that a solution is non-negative in the case of non-negative initial datum, we pass through an approximation process with a sequence of smooth solutions $\{u^\varepsilon\}_{\varepsilon>0}$ corresponding to the smooth, uniformly bounded non-negative initial conditions $u_0^\varepsilon(x) = (e^{\varepsilon\Delta}u_0)(x)$. Here, $u^\varepsilon \geq 0$ by the classical maximum principle, see *e.g.* [**?**, Theorem 9, p.43]. To complete this reasoning, we show that $\|u^\varepsilon(t) - u(t)\|_{p,\mathrm{uloc}} \to 0$ as $\varepsilon \to 0$ for each $t \in (0,T)$. Indeed, from inequality (**??**) in Proposition **??**, it holds that $\|u_0^\varepsilon\|_{p,\mathrm{uloc}} \leq \|u_0\|_{p,\mathrm{uloc}}$ for all $\varepsilon > 0$. Hence, by the above construction of a local-in-time solutions via Proposition **??**, there exists a constant $M > 0$ such that $\|u\|_{X_T} \leq M$ and $\|u^\varepsilon\|_{X_T} \leq M$. Now, computing the norm $\|u^\varepsilon(t) - u(t)\|_{p,\mathrm{uloc}}$ and using the integral representation (**??**) of these functions as well as a second inequality in estimate (**??**) we obtain

$$\|u^\varepsilon(t) - u(t)\|_{p,\mathrm{uloc}}$$
$$\leq \|e^{t\Delta}u_0^\varepsilon - e^{t\Delta}u_0\|_{p,\mathrm{uloc}}$$
$$+ C\big(\|u^\varepsilon\|_{X_T} + \|u\|_{X_T}\big) \int_0^t (t-s)^{-\frac{1}{2}}\big(1 + (t-s)^{-\frac{n}{2}\frac{1}{k}}\big)\|u^\varepsilon(s) - u(s)\|_{p,\mathrm{uloc}}\,\mathrm{d}s.$$

Applying the Volterra type inequality from [**?**, Ch. 9] we conclude that

$$\|u^\varepsilon(t) - u(t)\|_{p,\mathrm{uloc}} \leq \|e^{t\Delta}u_0^\varepsilon - e^{t\Delta}u_0\|_{p,\mathrm{uloc}} + C \int_0^t \kappa(t-s)\|e^{s\Delta}u_0^\varepsilon - e^{s\Delta}u_0\|_{p,\mathrm{uloc}}\,\mathrm{d}s,$$

where $\kappa = \kappa(\tau)$ is a suitable integrable kernel on $[0,t]$. Finally, by Remark **??** and the semigroup properties, we have

$$\|e^{t\Delta}u_0^\varepsilon - e^{t\Delta}u_0\|_{p,\mathrm{uloc}} = \|e^{\varepsilon\Delta}e^{t\Delta}u_0 - e^{t\Delta}u_0\|_{p,\mathrm{uloc}} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

$\square$

4. Linearized problem

4.1. **Preliminary properties.** The linearization procedure described below in Section **??** leads to the following linear problem

$$\begin{cases} v_t - \Delta v + A\Delta K * v = 0, & t > 0, \quad x \in \mathbb{R}^n \\ v(0, x) = v_0(x), \end{cases}$$

where $A \in \mathbb{R}$ is an arbitrary constant and the operator $\Delta - A\Delta K*$ can be expressed by the Fourier transform as follows

$$(\Delta\varphi - A\Delta K * \varphi)\widehat{}(\xi) = \left(-|\xi|^2 + A\frac{|\xi|^2}{1+|\xi|^2} \right)\widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n.$$

We begin by presenting preliminary properties of this operator.

Lemma 4.1. *There exists a constant $L > 0$ such that for each $p \in [1, \infty]$,*

$$\| - \Delta K * v\|_p \le L\|v\|_p \qquad for \ all \quad v \in L^p(\mathbb{R}^n).$$

This lemma is an immediate consequence of the fact that the operator $-\Delta K*$ can be represented by a convolution with a finite measure on $\mathbb{R}^n$. We skip the proof of this classical result from the harmonic analysis, see *e.g.* [**?**, Lemma 2.(i), p.133].

Lemma 4.2. *For each $A \in \mathbb{R}$, a closure in $L^p(\mathbb{R}^n)$ of the operator $\Delta - A\Delta K*$ generates an analytic semigroup $\{S_A(t)\}_{t \ge 0}$ on $L^p(\mathbb{R}^n)$ for every $p \in [1, \infty)$. This semigroup is defined by the Fourier transform by the formula*

$$\widehat{S_A(t)v_0}(\xi) = \widehat{\mu}_A(t, \xi)\widehat{v_0}(\xi),$$

where

$$\widehat{\mu}_A(t, \xi) = e^{-t\left(|\xi|^2 - A\frac{|\xi|^2}{1+|\xi|^2}\right)}.$$

Proof. It is well-known that Laplacian generates an analytic semigroup of linear operators on $L^p(\mathbb{R}^n)$ for every $p \in [1, \infty)$. A bounded perturbation of such an operator maintains the same property, see *e.g.* [**?**, Chapter III, Theorem 2.10]. The Fourier representation of this semigroup can be obtained by routine calculations. $\qquad\square$

Lemma 4.3. *Assume that $A \in \mathbb{R}$ and choose the constant L from inequality (**??**). Then for each $1 \le q \le p \le \infty$, there exists a constant $C = C(p, q, n) > 0$ such that*

$$\|S_A(t)v_0\|_p \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}e^{|A|Lt}\|v_0\|_q$$

and

$$\|\nabla S_A(t)v_0\|_p \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}e^{|A|Lt}\|v_0\|_q$$

for all $t > 0$ and $v_0 \in L^q(\mathbb{R}^n)$.

Proof. Here, we use the notation $S_A(t)v_0 = e^{t\Delta}(e^{-A\Delta K*}v_0)$. Using the L^p-L^q estimates of the heat semigroup (see Remark **??**) and Lemma **??**, we obtain

$$\|e^{t\Delta}(e^{-tA\Delta K*}v_0)\|_p \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|e^{-tA\Delta K*}v_0\|_q \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}e^{|A|Lt}\|v_0\|_q.$$

The proof for the second inequality is analogous. $\qquad\square$

4.2. **Decay estimates when $A < 1$.** The following theorem improves estimates from Lemma **??** in the case of $A \in [0, 1)$ and it plays a crucial role in the proof of stability of constant solutions to problem (**??**).

Theorem 4.4. *Assume that $A \in [0, 1)$. For all exponents satisfying $1 \le q \le p \le \infty$ there exist constants $C = C(p, q, n, A) > 0$ such that*

$$\|S_A(t)v_0\|_p \le Ct^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\|v_0\|_q$$

and

$$\|\nabla S_A(t)v_0\|_p \le Ct^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}}\|v_0\|_q$$

for all $t > 0$ and $v_0 \in L^q(\mathbb{R}^n)$.

The proof of this theorem is based on the following lemmas.

Lemma 4.5. *Let $\widehat{D^N v}(\xi) \equiv |\xi|^N \widehat{v}(\xi)$ for all $N \in \mathbb{R}$. For all $v \in \mathcal{S}(\mathbb{R}^n)$ and for every $N > \frac{n}{2}$, the following inequality holds*

$$\|v\|_1 \le C\|\widehat{v}\|_2^{1 - \frac{n}{2N}} \|D^N \widehat{v}\|_2^{\frac{n}{2N}},$$

with a constant $C = C(n, N) > 0$.

Proof. For $R = \left(\frac{2N-n}{n} \frac{\|D^N \widehat{v}\|_2}{\|\widehat{v}\|_2}\right)^{1/N}$, we obtain

$$\|v\|_1 = \int_{|x| \le R} |v(x)| \, dx + \int_{|x| > R} |v(x)| \, dx$$

$$\le \left(\int_{|x| \le R} dx\right)^{\frac{1}{2}} \left(\int_{|x| \le R} |v(x)|^2 \, dx\right)^{\frac{1}{2}} + \left(\int_{|x| > R} |x|^{-2N} \, dx\right)^{\frac{1}{2}} \left(\int_{|x| > R} |x|^{2N} |v(x)|^2 \, dx\right)^{\frac{1}{2}}$$

$$\le |\mathbb{S}^{n-1}|^{\frac{1}{2}} (R^{\frac{n}{2}} \|v\|_2 + R^{\frac{n}{2} - N} \|D^N \widehat{v}\|_2)$$

$$\le C\|\widehat{v}\|_2^{1 - \frac{n}{2N}} \|D^N \widehat{v}\|_2^{\frac{n}{2N}}.$$

$\square$

Lemma 4.6. *Assume that $A \in [0, 1)$. For the function $\widehat{\mu}_A$ defined by formula (**??**) and for every multi-index α with $|\alpha| = N$, there exists a constant $C = C(n, A, N) > 0$ such that*

$$\|\partial_\xi^\alpha \widehat{\mu}_A(t)\|_2^2 \le Ct^{N - \frac{n}{2}}, \quad \text{for all} \quad t \ge 1.$$

Proof. For $N = 0$, by the inequality $|\xi|^2/(1 + |\xi|^2) \le |\xi|^2$ ($\xi \in \mathbb{R}^n$), we obtain

$$\|\widehat{\mu}_A(t)\|_2^2 = \int_{\mathbb{R}^n} e^{-2t|\xi|^2 + 2At\frac{|\xi|^2}{1 + |\xi|^2}} \, d\xi \le \int_{\mathbb{R}^n} e^{-2t(1-A)|\xi|^2} \, d\xi = Ct^{-\frac{n}{2}}.$$

For $N \ge 1$, we introduce the C^∞-function $h(\xi) \equiv |\xi|^2 - A|\xi|^2/(1 + |\xi|^2)$ which satisfies estimates $|\partial_{\xi_j} h(\xi)| \le C|\xi|$ and $|\partial_\xi^\beta h(\xi)| \le C$ for every j, $1 \le j \le n$ and multi-index β with $|\beta| \ge 2$. We use the multivariate Faà di Bruno's formula (see *e.g.* [**?**])

$$\partial_\xi^\alpha e^{-th(\xi)} = e^{-th(\xi)} \sum_{k=1}^N (-t)^k H_k(\xi),$$

where $H_k(\xi)$ is a sum of products of k partial derivatives of the function $h(\xi)$ such that $|H_k(\xi)| \leq C(1 + |\xi|^k)$. We prove the following inequality by induction in $N \in \mathbb{N}$,

$$\left|\partial_\xi^\alpha e^{-th(\xi)}\right| \leq Ce^{-th(\xi)} \sum_{k - \frac{\ell}{2} \leq \frac{N}{2}} t^k\left(1 + |\xi|^\ell\right).$$

For $N = 1$, the inequality is obvious. We show the induction step for $N + 1$,

$$\left|\partial_{\xi_j}\partial_\xi^\alpha e^{-th(\xi)}\right| \leq t|\partial_{\xi_j}h(\xi)|\left|\partial_\xi^\alpha e^{-th(\xi)}\right| + \left|e^{-th(\xi)} \sum_{k=1}^N (-t)^k \partial_{\xi_j} H_k(\xi)\right|$$

$$\leq Ct|\xi|e^{-th(\xi)} \sum_{k - \frac{\ell}{2} \leq \frac{N}{2}} t^k\left(1 + |\xi|^\ell\right) + Ce^{-th(\xi)} \sum_{k - \frac{\ell}{2} \leq \frac{N}{2}} t^k\left(1 + |\xi|^\ell\right)$$

$$\leq Ce^{-th(\xi)} \sum_{k - \frac{\ell}{2} \leq \frac{N+1}{2}} t^k\left(1 + |\xi|^\ell\right)$$

which holds true because $|\partial_{\xi_j} H_k(\xi)| \leq C(1 + |\xi|^\ell)$ by the properties of the function $h(\xi)$ and $k + 1 - (\ell + 1)/2 \leq (N + 1)/2$.

Now we group coefficients t^k and $|\xi|^\ell$ in the following way, $t^k|\xi|^\ell = t^{k-\ell/2}|\sqrt{t}\xi|^\ell$ thus, by the assumption $t \geq 1$ and induction, we have $t^{k-\ell/2} \leq t^{N/2}$. We obtain an estimate

$$|\partial_\xi^\alpha \widehat{\mu}_A(t)| \leq Ct^{\frac{N}{2}} P(|\sqrt{t}\xi|)e^{-t|\xi|^2 + At\frac{|\xi|^2}{1+|\xi|^2}},$$

where $P(s)$ is a polynomial of degree N. By the same inequality as in the case $N = 0$ and properties of the exponential function,

$$|\partial_\xi^\alpha \widehat{\mu}_A(t)| \leq t^{\frac{N}{2}} P(|\sqrt{t}\xi|)e^{-t(1-A)|\xi|^2} \leq Ct^{\frac{N}{2}} e^{-\delta t|\xi|^2},$$

for some $\delta \in (0, 1 - A)$. Calculating the L^2-norm of both sides of inequality (**??**) we obtain the result. $\qquad\square$

Lemma 4.7. *Assume that $A \in [0, 1)$. For every $p \in [1, \infty]$ there exists a constant $C > 0$ such that*

$$\|S_A(t)v_0\|_p \leq C\|v_0\|_p,$$

for all $t > 0$ and all $v_0 \in L^p(\mathbb{R}^n)$.

Proof. For $t \in [0, 1]$, this is an immediate consequence of Lemma **??**. For $t \geq 1$, the function μ_A is from the Schwartz class in variable ξ. Thus, by the Young inequality,

$$\|S_A(t)v_0\|_p = \|\mu_A(t) * v_0\|_p \leq \|\mu_A(t)\|_1\|v_0\|_p.$$

In order to estimate $\|\mu_A(t)\|_1$, we recall a well known fact that both quantities $\|D^N v\|_2$ and $\sum_{|\alpha|=N} \|\partial_x^\alpha v\|_2$ are comparable for each $N \in \mathbb{N}$. Combining Lemma **??** and Lemma **??**, for $N > n/2$ and $N \in \mathbb{N}$, we obtain

$$\|\mu_A(t)\|_1 \leq C\|\widehat{\mu}_A(t)\|_2^{1-\frac{n}{2N}}\|D^N\widehat{\mu}_A(t)\|_2^{\frac{n}{2N}} \leq C(C_1 t^{-\frac{n}{4}})^{1-\frac{n}{2N}}(C_2 t^{\frac{N}{2}-\frac{n}{4}})^{\frac{n}{2N}} = C$$

for all $t \geq 0$. $\qquad\square$

*Proof of Theorem **??**.* We begin with inequality (**??**). Let us choose $\varepsilon \in (A, 1)$. By the standard heat semigroup estimates (see Remark **??**),

$$\|S_A(t)v_0\|_p = \|e^{(1-\varepsilon)t\Delta}(e^{\varepsilon t\Delta - tA\Delta(I-\Delta)^{-1}}v_0)\|_p \leq C_1 t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|e^{\varepsilon t\Delta - tA\Delta(I-\Delta)^{-1}}v_0\|_q.$$

Now we substitute $\tilde{t} = \varepsilon t$ to obtain $tA = \tilde{t}(A/\varepsilon) = \tilde{t}\tilde{A}$ and $0 \leq \tilde{A} < 1$. Thus, by Lemma **??**,

$$\|S_A(t)v_0\|_p \leq C_1 t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|e^{\tilde{t}\Delta - \tilde{t}\tilde{A}\Delta(I-\Delta)^{-1}}v_0\|_q \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|v_0\|_q.$$

We prove inequality (**??**) analogously using the formula

$$\nabla S_A(t)v_0 = \nabla e^{(1-\varepsilon)t\Delta}\big(e^{\varepsilon t\Delta - tA\Delta(I-\Delta)^{-1}}v_0\big).$$

$\square$

Remark 4.8. The L^q-L^p estimates (**??**)-(**??**) of the semigroup $\big\{S_A(t)\big\}_{t\geq 0}$ hold true for $A < 0$ as well. They can be proved by the same reasoning as above using the obvious inequality

$$e^{-t\left(|\xi|^2 - A\frac{|\xi|^2}{1+|\xi|^2}\right)} \leq e^{-t|\xi|^2} \quad \text{for each } A < 0.$$

Remark 4.9. For the completeness of this work, we notice that constant solution $A = 1$ is linearly stable in $L^2(\mathbb{R}^n)$. Indeed, since $e^{-t(|\xi|^2 - |\xi|^2/(1+|\xi|^2))} \leq 1$ for all $\xi \in \mathbb{R}^n$ and $t \geq 0$, by the Plancherel formula, we obtain $\|S_1(t)v_0\|_2 \leq \|v_0\|_2$ for all $v_0 \in L^2(\mathbb{R}^n)$. We skip a discussion of a stability of this constant solution for $p \neq 2$.

4.3. **Exponential growth when $A > 1$.** Next, we study an instability of solutions to linear problem (**??**).

Lemma 4.10. *Let $p \in (1,\infty)$. The closure in $L^p(\mathbb{R}^n)$ of the operator $\Delta - A\Delta K*$ has a real continuous spectrum $(-\infty, a]$, where $a = 0$ if $A \leq 1$ and $a = (\sqrt{A} - 1)^2$ if $A > 1$.*

Proof. The symbol $h(\xi) = |\xi|^2 - A|\xi|^2/(1+|\xi|^2)$ of the operator satisfies the estimates

$$|\partial_\xi^\alpha h(\xi)| \leq C_\alpha(1+|\xi|)^{2-|\alpha|} \quad \text{for all } \xi \in \mathbb{R}^n$$

and each multi-index $\alpha = (\alpha_1, ..., \alpha_n)$. Moreover $h(\xi)^{-1} = O(|\xi|^{-2})$ as $|\xi| \to \infty$. Now, it suffices to apply the result from [**?**, Theorem 2.1]. $\square$

Lemma 4.11. *Assume $A > 1$, $a = (\sqrt{A} - 1)^2$ and $1 < q \leq p < \infty$. For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, p, q, A) > 0$ such that*

$$\|S_A(t)v_0\|_p \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}e^{(a+\varepsilon)t}\|v_0\|_q$$

and

$$\|\nabla S_A(t)v_0\|_p \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}}e^{(a+\varepsilon)t}\|v_0\|_q$$

for all $v_0 \in L^q(\mathbb{R}^n)$ and $t \geq 0$.

Proof. Estimate (**??**) for $p = q$ is a direct consequence of Lemma **??** combined with estimates of strongly continuous semigroups, see *e.g.* [**?**, Ch.IV, Corollary 3.12]. In order to prove inequality (**??**) for $p = q$, notice that for every $\delta \in (0,1)$ we have

$$\nabla S_A(t)v_0 = \nabla\big(e^{\delta t\Delta}(e^{t((1-\delta)\Delta - A\Delta K*)})\big)v_0 = \nabla e^{\delta t\Delta}\big(e^{t(1-\delta)\left(\Delta - \frac{A}{1-\delta}\Delta K*\right)}v_0\big).$$

Applying the estimates of the heat semigroup from Remark **??** and inequality (**??**) with $\varepsilon/2$ to the operator $S_{\frac{A}{1-\delta}}(t)$ yields

$$\|\nabla S_A(t)v_0\|_p \leq C(\varepsilon/2)(\delta t)^{-1/2}e^{t(1-\delta)(a_\delta+\varepsilon/2)}\|v_0\|_p \quad \text{with} \quad a_\delta = \left(\sqrt{\frac{A}{1-\delta}} - 1\right)^2.$$

Choosing $\delta > 0$ sufficiently small to have $a_\delta \leq a + \varepsilon/2$.

For $q < p$ we proceed analogously using L^q-L^p estimates of the heat semigroup from Remark **??**. $\square$

Lemma 4.12. *Assume $A > 1$, $a = (\sqrt{A} - 1)^2$ and $p \in (1, \infty)$. For every $\gamma \in (0, 1]$ and every $T > 0$, there exists $v_0 \in L^p(\mathbb{R}^n)$ such that for each $t \in [0, T]$*

$$\|S_A(t)v_0 - e^{at}v_0\|_p \leq \gamma\|v_0\|_p \quad \text{and} \quad \|S_A(t)v_0\|_p \leq 2e^{at}\|v_0\|_p.$$

Proof. It follows from Lemma **??** that $a = (\sqrt{A} - 1)^2$ lies on the boundary of the spectrum of the operator $\Delta - A\Delta K*$ and such elements belong to the approximate point spectrum (see *e.g.* [**?**, Lemma 1]). Thus, for each $\varepsilon > 0$ there exists $v_\varepsilon \in L^p(\mathbb{R}^n)$ (in fact, v_ε belongs to the domain of the closure in $L^p(\mathbb{R}^n)$ of the operator $\Delta - A\Delta K*$) such that

$$\|\Delta v_\varepsilon - A\Delta K * v_\varepsilon - av_\varepsilon\|_p \leq \varepsilon\|v_\varepsilon\|_p.$$

Hence, by usual calculations for semigroups of linear operators (see *e.g.* [**?**, Ch.II, Sec. 3]) and by Lemma **??**, we obtain

$$\|S_A(t)v_\varepsilon - e^{at}v_\varepsilon\|_p = \left\|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}e^{ts(\Delta - A\Delta K*)}e^{t(1-s)a}v_\varepsilon \, \mathrm{d}s\right\|_p$$

$$\leq \int_0^1 \left\|e^{ts(\Delta - A\Delta K*)}e^{t(1-s)a}t(\Delta v_\varepsilon - A\Delta K * v_\varepsilon - av_\varepsilon)\right\|_p \mathrm{d}s$$

$$\leq Ct\|\Delta v_\varepsilon - A\Delta K * v_\varepsilon - av_\varepsilon\|_p \int_0^1 e^{|A|Lts}e^{t(1-s)a} \, \mathrm{d}s,$$

for some constant $C > 0$ from inequality (**??**) with $p = q$. Therefore, inequality (**??**) with $\varepsilon = \frac{\gamma}{CTe^{(|A|L+a)T}}$ provides the estimate

$$\|S_A(t)v_0 - e^{at}v_0\|_p \leq \frac{te^{(|A|L+a)t}}{Te^{(|A|L+a)T}}\gamma\|v_0\|_p \leq \gamma\|v_0\|_p.$$

Since $a > 0$, the second inequality in (**??**) can be obtained immediately from the first one by choosing $\gamma = 1$.   □

5. PERTURBATIONS OF CONSTANT SOLUTIONS

We study a solution $u = u(t, x)$ of problem (**??**) which is a perturbation of the constant stationary solution $A \in \mathbb{R}$. Thus, the function $v(t, x) = u(t, x) - A$ satisfies

$$\begin{cases} v_t - \Delta v + A\Delta K * v + \nabla \cdot (v\nabla K * v) = 0, & t > 0, \quad x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n. \end{cases}$$

Here, in fact, we consider a mild solution to this problem satisfying the integral equation

$$v(t) = S_A(t)v_0 - \int_0^t \nabla S_A(t - \tau) \cdot (v(\tau)\nabla K * v(\tau)) \, \mathrm{d}\tau,$$

where the semigroup $\{S_A(t)\}_{t\geq 0}$ has been studied in Section **??**.

5.1. **Local-in-time solutions.** We begin by a result on local-in-time solutions.

Proposition 5.1. *For each p satisfying condition (**??**) and every $v_0 \in L^p(\mathbb{R}^n)$, there exists $T > 0$ and a unique local-in-time mild solution to problem (**??**) in $C([0, T]; L^p(\mathbb{R}^n))$.*

For the proof of this proposition, one should follow the reasoning in the proof of Theorem **??**. In particular, a solution is obtained via Proposition **??** where the required estimate of the bilinear form

$$\widetilde{Q}[v, w](t) = -\int_0^t \nabla S_A(t - s) \cdot (v(s)\nabla K * w(s))(s) \, \mathrm{d}s$$

is a direct consequence of the following lemma.

Lemma 5.2. *For each p satisfying condition (??) there exist $k \in (n, \infty]$ and positive numbers $C_1 = C_1(p, k, n, \nabla K)$ and $C_2 = C_2(p, k, n, \nabla K)$ such that*

$$\|\nabla S_A(\tau) \cdot (v \nabla K * w)\|_p \leq C_2 \tau^{-\frac{n}{2}\frac{1}{k} - \frac{1}{2}} e^{C_1 \tau} \|v\|_p \|w\|_p \quad \text{for all} \quad v, w \in L^p(\mathbb{R}^n).$$

If $A < 1$ then $C_1 = 0$ and if $A \geq 1$ then C_1 can be an arbitrary constant satisfying $C_1 > a = (\sqrt{A} - 1)^2$.

We skip the proof of this lemma, because it is the same as the proof of Lemma **??**. In particular, it is based on the semigroup estimates from Lemma **??**.

5.2. **Global-in-time solutions for** $A \in [0, 1)$. The proof of Theorem **??** requires the following extension of Lemma **??**.

Lemma 5.3. *Assume that $A \in [0, 1)$. For*
- *each p satisfying condition (??),*
- *each $q > n$ if $n \geq 2$ and $q \geq 1$ if $n = 1$ satisfying $q \in [p, 2p]$,*
- *each $r \geq 1$ such that $r \in \left[\frac{q}{2}, p\right]$,*

there exists a constant $C > 0$ such that

$$\|\nabla S_A(\tau) \cdot (v \nabla K * w)\|_p \leq C \tau^{-\frac{n}{2}(\frac{1}{r} - \frac{1}{p}) - \frac{1}{2}} \|v\|_q \|w\|_q,$$

for all $v, w \in L^q(\mathbb{R}^n)$ and $\tau > 0$.

Proof. First, we use Theorem **??** and the Hölder inequality to estimate

$$\|\nabla S_A(\tau) \cdot (v \nabla K * w)\|_p \leq C \tau^{-\frac{n}{2}(\frac{1}{r} - \frac{1}{p}) - \frac{1}{2}} \|v \nabla K * w\|_r \leq C \tau^{-\frac{n}{2}(\frac{1}{r} - \frac{1}{p}) - \frac{1}{2}} \|v\|_q \|\nabla K * w\|_k$$

with $1 \leq r \leq p$ satisfying $\frac{1}{r} = \frac{1}{q} + \frac{1}{k}$. Next, we apply the Young inequality

$$\|\nabla K * w\|_k \leq C \|\nabla K\|_{q_1} \|w\|_q$$

with $\frac{1}{q_1} + \frac{1}{q} = 1 + \frac{1}{k}$. Let us show that $q_1 \in \left[1, \frac{n}{n-1}\right)$ in order to have $\nabla K \in L^{q_1}(\mathbb{R}^n)$. Indeed, by the assumption on p, q, r we have

$$1 - \frac{1}{q} \leq \frac{1}{q_1} = 1 + \frac{1}{r} - \frac{2}{q} \leq 1,$$

where, for $n \geq 2$, we use also the inequality $1 - \frac{1}{q} > 1 - \frac{1}{n}$. $\qquad\square$

Proof of Theorem **??**. It is sufficient to construct a global-in-time solution to problem (??) formulated in the mild form (??), because $u = A + v$ by the uniqueness of solutions from Proposition **??**. The solution is obtained via Proposition **??** applied to equation (??) in the Banach space

$$\mathcal{X} = C\big([0, \infty); L^p(\mathbb{R}^n)\big) \cap \left\{ v \in C\big((0, \infty); L^q(\mathbb{R}^n)\big) : \sup_{t > 0} t^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \|v(t)\|_q < \infty \right\}$$

with the norm $\|v\|_{\mathcal{X}} \equiv \sup_{t>0} \|v(t)\|_p + \sup_{t>0} t^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \|v(t)\|_q$.

First, for every $v_0 \in L^p(\mathbb{R}^n)$, it follows from Theorem **??** that

$$\|S_A(t) v_0\|_{\mathcal{X}} \leq C \|v_0\|_p$$

for all $t > 0$ and for some constant $C = C(p, q, r, n) > 0$.

Next, we estimate the bilinear form $\widetilde{Q}[u,v]$ given by formula (??) for all $v, w \in \mathcal{X}$. Here, for p satisfying condition (??) with $p \leq n$ and for $q \in (n, 2p]$, we choose one more exponent $r \in \left[\frac{q}{2}, p\right]$ and $r \geq 1$ such that

$$\frac{2}{q} + \frac{1}{n} - \frac{1}{p} \leq \frac{1}{r} \leq \frac{2}{q}.$$

By Lemma **??**, there exists $k > n$ such that

$$\|\widetilde{Q}[v,w](t)\|_p \leq C \int_0^t (t-s)^{-\frac{n}{2}\frac{1}{k}-\frac{1}{2}} \|v(s)\|_p \|w(s)\|_p \, ds \leq C t^{-\frac{n}{2k}+\frac{1}{2}} \|v\|_{\mathcal{X}} \|w\|_{\mathcal{X}}$$

with $-\frac{n}{2k} + \frac{1}{2} > 0$. Next, applying Lemma **??**, we obtain

$$\|\widetilde{Q}[v,w](t)\|_p \leq C \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}} \|v(s)\|_q \|w(s)\|_q \, ds$$

$$\leq C\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}} \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}} s^{-n(\frac{1}{p}-\frac{1}{q})} \, ds.$$

Notice that the integral on the right hand side is convergent because the exponents satisfy the inequalities

$$-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{1}{2} = -1+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right) > -1,$$

$$-n\left(\frac{1}{p}-\frac{1}{q}\right) > -n\left(\frac{2}{q}-\frac{1}{q}\right) = -\frac{n}{q} > -1$$

resulting immediately from the properties of the parameters p, q, r. Therefore,

$$\|\widetilde{Q}[v,w](t)\|_p \leq CB\left(1 - n\left(\frac{1}{p}-\frac{1}{q}\right), \frac{1}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)\right)$$

$$\times t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}-n(\frac{1}{p}-\frac{1}{q})+1}\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}},$$

where $B = B(x,y)$ denotes the beta function. Moreover, by the assumption on r, we have

$$-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)+1 = -\frac{n}{2r}-\frac{n}{2p}+\frac{n}{q}+\frac{1}{2}$$

$$\leq -\frac{n}{2}\left(\frac{2}{q}+\frac{1}{n}-\frac{1}{p}\right)-\frac{n}{2p}+\frac{n}{q}+\frac{1}{2} = 0.$$

Consequently, inequalities (**??**) and (**??**) provide the estimate

$$\sup_{t>0} \|\widetilde{Q}[v,w](t)\|_p \leq C \sup_{t>0}\left(\min\left\{t^{-\frac{n}{2k}+\frac{1}{2}}, \, t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{1}{2}-n(\frac{1}{p}-\frac{1}{q})+1}\right\}\right)\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}}$$

$$\leq C\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}}$$

with a positive constant $C > 0$.

We proceed with the L^q-component of the norm in $\mathcal{X}$ analogously. By Lemma **??**,

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\widetilde{Q}[v,w](t)\|_q \leq C t^{\frac{n}{2p}-\frac{n}{2}(\frac{1}{q}+\frac{1}{k})+\frac{1}{2}}\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}$$

with $\frac{n}{2p}-\frac{n}{2}\left(\frac{1}{q}+\frac{1}{k}\right)+\frac{1}{2} \geq 0$ and, by Lemma **??**,

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\widetilde{Q}[v,w](t)\|_q \leq C t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}} \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}} s^{-n(\frac{1}{p}-\frac{1}{q})} \, ds$$

$$= C t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+1}\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}}$$

with

$$-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+1\leq 0.$$

Therefore, there exists a constant $C > 0$ independent of $t > 0$ such that

$$\sup_{t>0} t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\widetilde{Q}[v,w](t)\|_q$$

$$\leq C\sup_{t>0}\left(\min\left\{t^{\frac{n}{2p}-\frac{n}{2}\left(\frac{1}{q}+\frac{1}{k}\right)+\frac{1}{2}},t^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+1}\right\}\right)\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}}$$

$$\leq C\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}}.$$

Finally, it follows from inequalities (**??**) and (**??**) that

$$\|\widetilde{Q}[v,w]\|_{\mathcal{X}}\leq\eta\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}}$$

for a positive number η independent of t, v and w. Hence, if $\|v_0\|_p$ is sufficiently small, by inequality (**??**) and Proposition **??**, there exists a mild solution of problem (**??**) in the space $\mathcal{X}$. This solution is unique by Proposition **??**. □

5.3. Instability for $A > 1$.
In this section, we prove that the constant solution $u = A$ of problem (**??**) is unstable in $L^p(\mathbb{R}^n)$ if $A > 1$. Here we apply the classical so-called linearization principle which was used *e.g.* in [**?**, **?**].

*Proof of Theorem **??**.* We begin with arbitrary $\delta \in (0,1)$ and arbitrary $v_0 \in L^p(\mathbb{R}^n)$ with $\|v_0\|_p = 1$ to be chosen later on. Under the assumpion on p, by Corollary **??**, there exists a unique local-in-time mild solution $v \in C\big([0,T_{\max}); L^p(\mathbb{R}^n)\big)$ to problem (**??**), with the initial datum δv_0. Suppose that this solution is global-in-time and for $a = (\sqrt{A}-1)^2$ define two numbers

$$T = \sup\left\{t:\ \|v(\tau)-S_A(\tau)\delta v_0\|_p \leq \frac{\delta}{2}e^{a\tau}\ \text{for all}\ \tau\in[0,t]\right\}$$

and $T' = \frac{1}{a}\log\left(\frac{2}{\delta}\right)$, hence $\delta e^{aT'} = 2$.

If either $T > T'$ or $T = \infty$, then the zero solution is unstable. Indeed, by Lemma **??**, for each $\gamma > 0$ we may choose $v_0 \in L^p(\mathbb{R}^n)$ with $\|v_0\|_p = 1$ such that

$$\left\|S_A(T')\delta v_0 - e^{aT'}\delta v_0\right\|_p \leq \gamma\delta\|v_0\|_p = \gamma\delta.$$

By the definition of T and by inequality (**??**), we obtain

$$\|v(T')\|_p \geq \|S_A(T')\delta v_0\|_p - \frac{\delta}{2}e^{aT'} \geq \|e^{aT'}\delta v_0\|_p - \gamma\delta - \frac{\delta}{2}e^{aT'} = \frac{\delta}{2}e^{aT'} - \gamma\delta \geq 1-\gamma.$$

In particular, $\|v(T')\|_p \geq \frac{1}{2}$ for $\gamma = \frac{1}{2}$.

Next, suppose that $T \leq T'$ and consider the mild representation of the solution of problem (**??**) with the initial condition δv_0

$$v(t)-S_A(t)\delta v_0 = \int_0^t \nabla S_A(t-\tau)\cdot\big(v(\tau)\nabla K * v(\tau)\big)\,\mathrm{d}\tau.$$

Lemma **??** with $C_1 = \frac{3}{2}a$, estimates **(??)** and definition of T in **(??)** lead to the inequality

$$\|v(t) - S_A(t)\delta v_0\|_p$$
$$\leq C \int_0^t (t-\tau)^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} e^{\frac{3}{2}a(t-\tau)} \|v(\tau)\|_p^2 \, d\tau$$
$$\leq C \int_0^t (t-\tau)^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} e^{\frac{3}{2}a(t-\tau)} \left(\|S_A(\tau)\delta v_0 - v(\tau)\|_p^2 + \|S_A(\tau)\delta v_0\|_p^2\right) d\tau$$
$$\leq C \int_0^t (t-\tau)^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} e^{\frac{3}{2}a(t-\tau)} \left(\frac{\delta^2}{4} e^{2a\tau} + 4\delta^2 e^{2a\tau}\right) d\tau$$
$$\leq C\delta^2 e^{2at}.$$

for all $t \in [0, T]$, where the last inequality is explained in Remark **??**, below. Thus, from the definition of the number T and from inequality **(??)** for $t = T$, we have the relations

$$\frac{\delta}{2} e^{aT} = \|v(T) - S_A(T)\delta v_0\|_p \leq C\delta^2 e^{2aT}$$

which imply the inequality $\frac{1}{2C} \leq \delta e^{aT}$. In particular, the number T^* defined by the equation $\delta e^{aT^*} = \frac{1}{2C}$ satisfies $T^* \leq T$. Hence, by inequality **(??)** with $t = T^*$ we have

$$\|v(T^*)\|_p \geq \|S_A(T^*)\delta v_0\|_p - \frac{1}{2}\delta e^{aT^*} = \|S_A(T^*)\delta v_0\|_p - \frac{1}{4C}.$$

Finally, we apply Lemma **??** with $\gamma = \frac{1}{4k_0 C} \leq 1$ for some fixed $k_0 \gg 1$ and T' (recall that $T^* \leq T \leq T'$) in order to obtain $v_0 \in L^p(\mathbb{R}^n)$ with $\|v_0\|_p = 1$ such that

$$\|v(T^*)\|_p \geq \|S_A(T^*)\delta v_0\|_p - \frac{1}{4C} \geq \delta e^{aT^*} - \gamma\delta - \frac{1}{4C} \geq \frac{1}{4C} - \gamma = \frac{1}{4C}\left(1 - \frac{1}{k_0}\right).$$

The proof of instability is completed because the right hand side is independent of δ. □

Remark 5.4. The last inequality in **(??)** follows from a direct calculation which we present for the reader convenience. For a fixed $\eta \in (0, T)$, we obtain

$$e^{\frac{3}{2}at} \int_0^t (t-\tau)^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} e^{\frac{1}{2}a\tau} \, d\tau$$
$$= e^{\frac{3}{2}at} \left(\int_0^{t-\eta} (t-\tau)^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} e^{\frac{1}{2}a\tau} \, d\tau + \int_{t-\eta}^t (t-\tau)^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} e^{\frac{1}{2}a\tau} \, d\tau\right)$$
$$\leq e^{\frac{3}{2}at} \left(\eta^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} e^{\frac{1}{2}a(t-1)} + e^{\frac{1}{2}at} \int_{t-\eta}^t (t-\tau)^{-\frac{n}{2}\frac{1}{q}-\frac{1}{2}} \, d\tau\right)$$
$$\leq C e^{2at}.$$

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