TANGENT ∞-CATEGORIES AND GOODWILLIE CALCULUS

KRISTINE BAUER, MATTHEW BURKE, AND MICHAEL CHING

ABSTRACT. We make precise the analogy between Goodwillie's calculus of functors in homotopy theory and the differential calculus of smooth manifolds by introducing a higher-categorical framework of which both theories are examples. That framework is an extension to ∞-categories of the *tangent categories* of Cockett and Cruttwell (introduced originally by Rosický). A tangent structure on an ∞-category $\mathbb{X}$ consists of an endofunctor on $\mathbb{X}$, which plays the role of the tangent bundle construction, together with various natural transformations that mimic structure possessed by the ordinary tangent bundles of smooth manifolds and that satisfy similar conditions.

The tangent bundle functor in Goodwillie calculus is Lurie's tangent bundle for ∞-categories, introduced to generalize the cotangent complexes of André, Quillen and Illusie. We show that Lurie's construction admits the additional structure maps and satisfies the conditions needed to form a tangent ∞-category, which we refer to as the *Goodwillie tangent structure* on the ∞-category of ∞-categories.

Cockett and Cruttwell (and others) have started to develop various aspects of differential geometry in the abstract context of tangent categories, and we begin to apply those ideas to Goodwillie calculus. For example, we show that the role of Euclidean spaces in the calculus of manifolds is played in Goodwillie calculus by the stable ∞-categories. We also show that Goodwillie's n-excisive functors are the direct analogues of *n-jets* of smooth maps between manifolds; to state that connection precisely, we develop a notion of tangent $(\infty, 2)$-category and show that Goodwillie calculus is best understood in that context.

1

Contents

INTRODUCTION

Goodwillie's calculus of functors, developed in the series of papers [**?**, **?**, **?**], provides a systematic way to apply ideas from ordinary calculus to homotopy theory. For example, central to this functor calculus is the 'Taylor tower', an analogue of the Taylor series, which comprises a sequence of 'polynomial' approximations to a functor between categories of topological spaces or spectra.

The purpose of this paper is to explore a slightly different analogy, also proposed by Goodwillie, where we view the categories of topological spaces or spectra (or, to be more precise, the corresponding ∞-categories) as analogues of smooth manifolds, and the functors between those categories as playing the role of smooth maps. Our main goal is to make this analogy precise by introducing a common framework within which both theories exist as examples. That framework is the theory of *tangent categories*[1] of Cockett and Cruttwell [**?**], extended to ∞-categories.

A central object in the calculus of manifolds is the tangent bundle construction: associated to each smooth manifold M is another smooth manifold TM along with a projection $p_M : TM \to M$ as well as various other structure maps that, among other things, make p_M into a vector bundle for each M. The first step in merging functor calculus with differential geometry is to describe the 'tangent bundle' for an ∞-category, such as that of spaces or spectra.

Fortunately for us, the analogue of the tangent bundle construction in homotopy theory is well-known. In the generality we want in this paper, that construction is given by Lurie in [**?**, 7.3.1], but the ideas go back at least to work of André [**?**] and Quillen [**?**] on cohomology theories for commutative rings, and Illusie [**?**] on cotangent complexes in algebraic geometry.

For an object X in an ordinary category $\mathcal{C}$, we can define the 'tangent space' to $\mathcal{C}$ at X to be the category of abelian group objects in the slice category $\mathcal{C}_{/X}$ of objects over X. These abelian group objects are also called *Beck modules* after their introduction by Beck in his 1967 Ph.D. thesis [**?**], and they were used by Quillen [**?**] as the coefficients for his definition of cohomology in an arbitrary model category.

---

[1]The phrase 'tangent category' is unfortunately used in the literature to describe two different objects, both of which feature extensively in this paper. We follow Cockett and Cruttwell's terminology, and use 'tangent category' to refer to the broader notion: a category whose objects admit tangent bundles. The notion referred to by, for example, Harpaz-Nuiten-Prasma in the title of [**?**] will for us be called the 'tangent bundle' on a category (or ∞-category).

Quillen's approach was refined by Basterra and Mandell [**?**], building on previous work of Basterra [**?**] on ring spectra, to what is now known as *topological* Quillen homology. In that refinement the abelian group objects are replaced by cohomology theories, or spectra[2] in the sense of stable homotopy theory. In particular, for an object X in a suitably nice ∞-category $\mathcal{C}$, our *tangent space* to $\mathcal{C}$ at X:

$$(0.1) \qquad\qquad T_X\mathcal{C} := \mathcal{S}p(\mathcal{C}_{/X})$$

is the ∞-category of spectra in the corresponding slice ∞-category of objects over X. This ∞-category $T_X\mathcal{C}$ is the 'stabilization' of $\mathcal{C}_{/X}$: its best approximation by an ∞-category that is stable.[3]

For example, when $\mathcal{C}$ is the ∞-category of topological spaces, $T_X\mathcal{C}$ can be identified with the ∞-category of spectra parameterized[4] over the topological space X. Basterra and Mandell [**?**] prove that when $\mathcal{C}$ is the ∞-category of commutative ring spectra, then $T_R\mathcal{C}$ is the ∞-category of R-module spectra. Various other calculations of these tangent spaces have been done recently in a series of papers by Harpaz, Nuiten and Prasma: [**?**] (for algebras over operads of spectra), [**?**] (for ∞-categories themselves) and [**?**] (for $(\infty, 2)$-categories).

Lurie's construction [**?**, 7.3.1.10] bundles all of these individual tangent spaces together into a 'tangent bundle'. So for each suitably nice ∞-category $\mathcal{C}$ we have a functor

$$p_{\mathcal{C}} : T\mathcal{C} \to \mathcal{C}$$

whose fibre over X is precisely $T_X\mathcal{C}$. One of the aims of this paper is to give substance to the claim that this $p_{\mathcal{C}}$ is analogous to the ordinary tangent bundle $p_M : TM \to M$ for a smooth manifold M, by presenting both as examples of the notion of 'tangent category'.

Tangent categories. The task of providing a categorical framework for the tangent bundle construction on manifolds was first carried out by Rosický in [**?**]. In that work, he describes various structure that can be built on the tangent bundle functor

$$T : \mathbb{M}\mathrm{fld} \to \mathbb{M}\mathrm{fld}$$

where $\mathbb{M}\mathrm{fld}$ denotes the category of smooth manifolds and smooth maps. Of course there is a natural transformation $p : T \to \mathrm{Id}$ that provides the tangent bundle projection maps. There are also natural transformations $0 : \mathrm{Id} \to T$,

---

[2]See [**?**] for a survey of spectra in this sense.

[3]An ∞-category is *stable* when it admits a null object, it has finite limits and colimits, and when pushout and pullback squares coincide; see [**?**, 1.1] for an extended introduction.

[4]See [**?**] for parameterized spectra.

given by the zero section for each tangent bundle, and $+ : T \times_{\mathrm{Id}} T \to T$, capturing the additive structure of those vector bundles.

Rosický's work was largely unused until resurrected by Cockett and Cruttwell [?] in 2014 in order to describe connections between calculus on manifolds and structures appearing in logic and computer science, such as the differential λ-calculus [?]. Cockett and Cruttwell define a tangent structure on a category $\mathbb{X}$ to consist of an endofunctor $T : \mathbb{X} \to \mathbb{X}$ together with five natural transformations:

- the *projection* $p : T \to \mathrm{Id}$
- the *zero section* $0 : \mathrm{Id} \to T$
- the *addition* $+ : T \times_{\mathrm{Id}} T \to T$
- the *flip* $c : T^2 \to T^2$
- the *vertical lift* $\ell : T \to T^2$

for which a large collection of diagrams are required to commute. As indicated above, the first three of these natural transformations make TM into a bundle of commutative monoids over M, for any object $M \in \mathbb{X}$.[5] The maps c and ℓ express aspects of the 'double' tangent bundle $T^2M = T(TM)$ that are inspired by the case of smooth manifolds: its symmetry in the two tangent directions, and a canonical way of 'lifting' tangent vectors to the double tangent bundle.

In addition to the required commutative diagrams, there is one crucial additional condition, referred to by Cockett and Cruttwell [?, 2.1] as the 'universality of the vertical lift'. This axiom states that certain squares of the form

(0.2)
$$\begin{array}{ccc} TM \times_M TM & \longrightarrow & T^2M \\ \downarrow & & \downarrow {\scriptstyle T(p)} \\ M & \xrightarrow{\ 0\ } & TM \end{array}$$

are required to be pullback diagrams in $\mathbb{X}$. (See ?? for a precise statement.) When $\mathbb{X} = \mathbb{M}\mathrm{fld}$ and M is a smooth manifold, this vertical lift axiom reduces to a collection of diffeomorphisms

$$T(T_xM) \cong T_xM \times T_xM$$

---

[5] In Rosický's original work, the addition map was required to have fibrewise negatives so that TM is a bundle of abelian groups. Cockett and Cruttwell relaxed that conditions, and we will take full advantage of this relaxation since Lurie's tangent bundle does not admit negatives.

varying smoothly with $x \in M$.

Thus the vertical lift axiom tells us a familiar fact: that the tangent bundle of a tangent space $T_x M$ (indeed of any Euclidean space) is trivial. Cockett and Cruttwell's insight was that this axiom is also the key to translating many of the constructions of ordinary differential geometry into the context of abstract tangent categories. Since the publication of [**?**] a small industry has developed around this task, providing, for example:

- a *Lie bracket for vector fields* (described by Rosický in his original work [**?**]);
- an analogue for *smooth vector bundles*: the 'differential bundles' introduced in [**?**];
- notions of *connection*, *torsion* and *curvature*, by Cockett and Cruttwell [**?**], also studied by Lucyshyn-Wright [**?**];
- analogues of *affine spaces*, developed by Blute, Cruttwell and Lucyshyn-Wright [**?**]; and
- *differential forms* and *cohomology*, studied by Cruttwell and Lucyshyn-Wright [**?**].

In Section **??** of this paper we add to this list the notion of 'n-jet' of smooth maps in order to make precise the connection with Goodwillie's n-excisive functors. We do not address any of these other topics here, though we expect each of them has an analogue for tangent ∞-categories that might be worth studying. Note, however, that some of these concepts, such as the Lie bracket, depend on the existence of additive inverses in the tangent bundle, and hence would not be applicable in the Goodwillie tangent structure where those inverses do not exist.

Tangent ∞-categories. Our paper is split into two parts, and the first part focuses on extending the theory of tangent categories to the ∞-categorical context. The principal goal of this part is Definition **??** which defines a notion of tangent structure on an ∞-category $\mathbb{X}$ and which recovers Cockett and Cruttwell's definition when $\mathbb{X}$ is an ordinary category.

The main challenge in making this definition is that the large array of commutative diagrams listed by Cockett and Cruttwell in their definition of tangent category would require an even larger array of higher cohering homotopies if translated directly to the ∞-categorical framework. Fortunately, work of Leung [**?**] has provided a more conceptual definition of tangent category based on a category of 'Weil-algebras' already used in algebraic geometry and synthetic differential geometry.

In Definition **??** we describe a symmetric monoidal category $\mathbb{W}\mathrm{eil}$ whose objects are certain augmented commutative semi-rings of the form

$$\mathbb{N}[x_1, \ldots, x_n]/(x_i x_j)$$

where the relations are given by some set of quadratic monomials that includes the squares x_i^2. The monoidal structure on $\mathbb{W}\mathrm{eil}$ is given by tensor product. These objects form only a subset of the more general Weil-algebras introduced by Weil in [**?**] to study tangent vectors on manifolds in terms of infinitesimals. Note that the appearance of semi-rings in this definition, rather than algebras over $\mathbb{Z}$ or a field, corresponds to the fact that the additive structure in our tangent bundles is not required to admit fibrewise negation.

Leung's main result [**?**, 14.1] is that the structure of a tangent category $\mathbb{X}$ is precisely captured by a strong monoidal functor

$$(0.3) \qquad\qquad\qquad T^{\otimes} : \mathbb{W}\mathrm{eil} \to \mathrm{End}(\mathbb{X})$$

from $\mathbb{W}\mathrm{eil}$ to the category of endofunctors on $\mathbb{X}$ under composition, or equivalently, to an *action* of $\mathbb{W}\mathrm{eil}$ on $\mathbb{X}$. This action is subject to the additional condition that certain pullbacks be preserved by T; in particular, this condition provides the 'universality of vertical lift' axiom referred to in (**??**).

The simplest non-trivial example of a Weil-algebra is the 'dual numbers' object $W = \mathbb{N}[x]/(x^2)$ which, under Leung's formulation, corresponds to the tangent bundle functor $T : \mathbb{X} \to \mathbb{X}$. Evaluating the functor $T^{\otimes}$ on morphisms in $\mathbb{W}\mathrm{eil}$, i.e. the homomorphisms between Weil-algebras, provides the various natural transformations that make up the tangent structure.

Our Definition **??** of tangent structure on an ∞-category $\mathbb{X}$ is a direct generalization to ∞-categories of that suggested by Leung's theorem; it is a functor of monoidal ∞-categories of the form (**??**) that preserves those same pullbacks (though now with pullback understood in the ∞-categorical sense). Under this definition any ordinary tangent category, such as $\mathbb{M}\mathrm{fld}$ with its usual tangent bundle construction, is also a tangent ∞-category.

There are, however, tangent ∞-categories that do not arise from an ordinary tangent category. One example is given by the 'derived smooth manifolds' of [**?**]. There, Spivak defines an ∞-category $\mathbb{DM}\mathrm{fld}$ that contains $\mathbb{M}\mathrm{fld}$ as a full subcategory, but which admits all pullbacks, not only those along transverse pairs of smooth maps. We show in Proposition **??** that the tangent structure on $\mathbb{M}\mathrm{fld}$ extends naturally to $\mathbb{DM}\mathrm{fld}$ using a universal property described by Carchedi and Steffens [**?**].

Tangent $(\infty, 2)$-categories and other objects. The definition of tangent structure on an ∞-category can easily be extended to a wide range of other types of objects, and we examine this generalization in Section **??**. Let $\mathbb{X}$ be an object in an $(\infty, 2)$-category $\mathbf{C}$ which, for the purposes of this introduction, one may view simply as a category enriched in ∞-categories. Then $\mathbb{X}$ admits a monoidal ∞-category $\mathrm{End}_{\mathbf{C}}(\mathbb{X})$ of endomorphisms. We thus define (in **??**) a tangent structure on the object $\mathbb{X}$ to be a monoidal functor

$$T : \mathbb{W}\mathrm{eil} \to \mathrm{End}_{\mathbf{C}}(\mathbb{X})$$

that preserves the appropriate pullbacks. This definition recovers our notion of tangent ∞-category (and hence of tangent category) when $\mathbf{C}$ is the $(\infty, 2)$-category of ∞-categories. Taking $\mathbf{C}$ to be a suitable $(\infty, 2)$-category of $(\infty, 2)$-categories, we also obtain a notion of tangent $(\infty, 2)$-category.

The Goodwillie tangent structure on $\mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$. In the second part of this paper, we construct a specific tangent ∞-category for which the tangent bundle functor is equivalent to that defined by Lurie, and which encodes the theory of Goodwillie calculus. The existence of this tangent structure (which we refer to as the *Goodwillie tangent structure*) justifies the analogy between functor calculus and the calculus of smooth manifolds.

The underlying ∞-category for this tangent structure is $\mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$: a subcategory of Lurie's ∞-category of ∞-categories [**?**, 3.0.0.1]. The objects in $\mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$ are those ∞-categories $\mathcal{C}$ that are *differentiable* in the sense introduced in [**?**, 6.1.1.6]: those $\mathcal{C}$ that admit finite limits and sequential colimits, which commute. This condition is satisfied by many ∞-categories of interest including any compactly generated ∞-category, such as that of topological spaces, and any ∞-topos. The morphisms in the ∞-category $\mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$ are those functors between differentiable ∞-categories that preserve sequential colimits.

The tangent bundle construction from [**?**, 7.3.1.10] can be described explicitly as the functor $T : \mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}} \to \mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$ given by

$$T\mathcal{C} := \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C})$$

where $\mathcal{S}_{\mathrm{fin},*}$ denotes the ∞-category of finite pointed spaces, and $\mathrm{Exc}(-, -)$ denotes the ∞-category of functors that are *excisive* in the sense of Goodwillie (i.e. map pushouts in $\mathcal{S}_{\mathrm{fin},*}$ to pullbacks in $\mathcal{C}$). With this definition in mind, the Goodwillie tangent structure on $\mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$ consists of the following natural transformations:

- the *projection* map $p : T\mathcal{C} \to \mathcal{C}$ is evaluation at the null object:

$$L \mapsto L(*);$$

- the *zero section* $0 : \mathcal{C} \to T\mathcal{C}$ maps an object X of $\mathcal{C}$ to the constant functor with value X;
- *addition* $+ : T\mathcal{C} \times_{\mathcal{C}} T\mathcal{C} \to T\mathcal{C}$ is the fibrewise product:

$$(L_1, L_2) \mapsto L_1(-) \times_{L_1(*) = L_2(*)} L_2(-);$$

- identifying $T^2\mathcal{C}$ with the ∞-category of functors $\mathcal{S}_{\mathrm{fin},*} \times \mathcal{S}_{\mathrm{fin},*} \to \mathcal{C}$ that are excisive in each variable individually; the *flip* $c : T^2\mathcal{C} \to T^2\mathcal{C}$ is the symmetry in those two variables:

$$L \mapsto [(X, Y) \mapsto L(Y, X)];$$

- the *vertical lift* map $\ell : T\mathcal{C} \to T^2\mathcal{C}$ is precomposition with the smash product:

$$L \mapsto [(X, Y) \mapsto L(X \wedge Y)].$$

A complete definition of the Goodwillie tangent structure requires the construction of a monoidal functor

$$T : \mathbb{W}\mathrm{eil} \to \mathrm{End}(\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}).$$

For a Weil-algebra of the form

$$A = \mathbb{N}[x_1, \ldots, x_n]/(x_i x_j)$$

we define the corresponding endofunctor $T^A : \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ by

$$T^A(\mathcal{C}) := \mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}),$$

the full subcategory consisting of those functors $\mathcal{S}_{\mathrm{fin},*}^n \to \mathcal{C}$ that satisfy the property of being 'A-excisive' (see Definition **??**). When $A = \mathbb{N}[x]/(x^2)$, A-excisive is excisive, and we recover Lurie's definition of the tangent bundle $T\mathcal{C}$. We define the functor T on a morphism ϕ of Weil-algebras via precomposition with a certain functor

$$\tilde{\phi} : \mathcal{S}_{\mathrm{fin},*}^{n'} \to \mathcal{S}_{\mathrm{fin},*}^n$$

whose construction mimics the algebra homomorphism ϕ; see Definition **??** for details.

In Section **??** we prove the key homotopy-theoretic results needed to check that the definitions above do indeed form a tangent structure. Principal among those is the vertical lift axiom analogous to that in (**??**) above; this axiom is verified in Proposition **??**, and involves in detail the classification of multilinear functors in Goodwillie calculus, and splitting results for functors with values in a stable ∞-category. It is there that lies the technical heart of the construction of the Goodwillie tangent structure.

The full definition of T as a monoidal functor between monoidal ∞-categories is rather involved and relies on a model for $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$ based on 'relative' ∞-categories; see [**?**]. The specifics of this construction are in Section **??**. The reader interested in understanding the basic idea of our construction, rather than the intricate details, should focus on Section **??** where the most important of the underlying definitions are provided.

Tangent functors, differential objects and jets. Much of this paper is concerned with the definition of tangent ∞-category and the construction of the Goodwillie tangent structure on the ∞-category $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$. However, we also begin the task of extending the current tangent category literature to the ∞-categorical context. In this paper, we focus on three specific aspects of that theory: functors between tangent categories, differential objects, and jets.

As with any categorical structure, it is crucial to identify the appropriate morphisms. Cockett and Cruttwell introduced in [**?**, 2.7] two notions of 'morphism of tangent structure' ('lax' and 'strong'), and in Section **??** we extend those notions to tangent ∞-categories. Briefly, a tangent functor between tangent ∞-categories is a functor that commutes (up to higher coherent equivalences, in the strong case, or up to coherent natural transformations, in the lax case) with the corresponding actions of $\mathbb{W}\mathrm{eil}$. We give an explicit model for these higher coherences in Definition **??** that, using work of Garner [**?**], reduces to Cockett and Cruttwell's definition in the case of ordinary categories.

Differential objects were introduced by Cockett and Cruttwell [**?**, 4.8] in order to axiomatize the role of Euclidean spaces in the theory of smooth manifolds: these are the objects that play the role of tangent *spaces*. In Section **??** we describe our extension of the theory of differential objects to tangent ∞-categories. Our description provides a new perspective on differential objects even in the setting of ordinary tangent categories; see Proposition **??**.

Another role for differential objects is in making the connection between tangent categories and the 'cartesian differential categories' of Blute, Cockett and Seely [**?**]. Roughly speaking, a cartesian differential category is a tangent category in which every object has a canonical differential structure; for example, the subcategory of $\mathbb{M}\mathrm{fld}$ whose objects are only the Euclidean spaces $\mathbb{R}^n$. We show that a similar relationship (Theorem **??**) also holds in the tangent ∞-category setting after passage to homotopy categories.

In Section **??** we analyze the notion of differential object in the specific context of the Goodwillie tangent structure on $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$. It is not hard to see that the differential objects in $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$ are precisely the *stable* ∞-categories. This

result is not at all surprising; the role of stabilization is built into our tangent structure via the tangent spaces described in (**??**). It does, however, confirm Goodwillie's intuition that categories of spectra should be viewed, from the point of view of calculus, as analogues of Euclidean spaces.

We also deduce the existence of a cartesian differential category whose objects are the stable ∞-categories and whose morphisms are natural equivalence classes of functors. This result extends work of the first author and Johnson, Osborne, Riehl, and Tebbe [**?**] which describes a similar construction for (chain complexes of) abelian categories in the context of Johnson and McCarthy's 'abelian functor calculus' variant of Goodwillie's theory [**?**]. In fact, the paper [**?**] provided much of the inspiration for our development of tangent ∞-categories and for the construction of the Goodwillie tangent structure.

In Section **??** we turn to the notion of 'n-jet' of a morphism which does not appear explicitly in the tangent category literature, though it has been studied in the context of synthetic differential geometry, e.g. see [**?**, 2.7]. In our case, the importance of n-jets is that they correspond in the Goodwillie tangent structure on $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ to the n-excisive functors, i.e. Goodwillie's analogues of degree n polynomials.

In an arbitrary tangent ∞-category $\mathbb{X}$, we can say that two morphisms $F, G : \mathcal{C} \to \mathcal{D}$ determine the same n-jet at a (generalized) point $x \in \mathcal{C}$ if they induce equivalent maps on the n-fold tangent spaces $T_x^n \mathcal{C}$ at x. For smooth manifolds, this definition recovers the ordinary notion of n-jet: the equivalence class of smooth maps that agree to order n in a neighbourhood of the point x. For ∞-categories, we prove an analogous result (Theorem **??**) saying that a natural transformation $\alpha : F \to G$ between two functors $\mathcal{C} \to \mathcal{D}$ induces an equivalence $P_n^x F \xrightarrow{\sim} P_n^x G$ between Goodwillie's n-excisive approximations at x if and only if α induces an equivalence on $T_x^n \mathcal{C}$.

The significance of the previous result is that it shows that the notion of n-excisive equivalence, and hence n-excisive functor, can be recovered directly from the Goodwillie tangent structure. To make proper sense of this claim though, we need to be able to talk about non-invertible natural transformations between functors of ∞-categories. This observation reveals that Goodwillie calculus is better understood in the context of tangent structures on an $(\infty, 2)$-category.

In Theorem **??** we show that there is an $(\infty, 2)$-category $\mathbb{C}\mathrm{AT}_\infty^{\mathrm{diff}}$ of differentiable ∞-categories which admits a Goodwillie tangent structure extending that on $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$. This tangent $(\infty, 2)$-category completely encodes Goodwillie calculus and the notion of Taylor tower.

Connections and conjectures. There are other topics and questions that we would like to have addressed in this paper. One such question is the extent to which the Goodwillie tangent structure on $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ is unique. We believe that it is indeed the unique (up to contractible choice) tangent ∞-category which extends Lurie's tangent bundle construction, but we do not try to give a proof of that conjecture here.

Another topic concerns what are known as 'representable' tangent categories. It was observed by Rosický that a model for synthetic differential geometry gives rise to a tangent category in which the tangent bundle functor is represented by an object with so-called 'infinitesimal' structure. (See [**?**, 5.6].) We expect it to be relatively easy to extend the definition of representable tangent structure to ∞-categories, and also to prove that the Goodwillie tangent structure is *not* representable in this sense. (In fact, Lurie's tangent bundle functor by itself is not representable.)

However, if we restrict the Goodwillie tangent structure to the subcategory $\mathbb{T}\mathrm{opos}_\infty^{op} \subseteq \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ consisting of ∞-toposes (in the sense of Lurie [**?**, 6.1.0.4]) and the left exact colimit-preserving functors, then we conjecture that this restricted tangent structure is *dual*, in the sense of [**?**, 5.17], to a representable tangent structure on $\mathbb{T}\mathrm{opos}_\infty$ (i.e. the ∞-category of ∞-toposes and geometric morphisms). The representing object for that tangent structure would be the ∞-topos $T(\mathcal{S}) = \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{S})$, whose objects are parameterized spectra over arbitrary topological spaces, with infinitesimal structure arising directly from the Goodwillie tangent structure. We wonder if such structure is related to work of Anel, Biedermann, Finster and Joyal [**?**] on Goodwillie calculus for ∞-toposes.

Heuts [**?**, 1.7] has developed a notion of Goodwillie tower for (pointed compactly-generated) ∞-*categories*, instead of functors, that provides each such ∞-category $\mathcal{C}$ with a sequence of approximations $\mathcal{P}_n\mathcal{C}$. The first approximation $\mathcal{P}_1\mathcal{C}$ is equal to the stabilization $\mathcal{S}p(\mathcal{C})$ hence equal to the tangent space $T_*\mathcal{C}$ to $\mathcal{C}$ at the null object. It would be interesting to see if Heuts's higher approximations also admit a description in terms of the Goodwillie tangent structure.

There are two possible directions for generalization that seem worthy of exploration. One is to look for different ∞-toposes that might also represent, in a similar sense, tangent structures related to functor calculus. In particular, one might imagine that the 'manifold calculus' of Goodwillie and Weiss [**?**, **?**], or the 'orthogonal calculus' of Weiss [**?**] might be fit into a similar framework by considering, respectively, the ∞-topos of space-valued sheaves on a

manifold, or that of degree 1 functors $\mathcal{J} \to \mathcal{S}$ where $\mathcal{J}$ is Weiss's category of finite-dimensional inner-product spaces and isometric embeddings.

A second direction would be to replace $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ in the Goodwillie tangent structure with a different ∞-cosmos in the sense of Riehl and Verity [**?**, 1.2]. An arbitrary ∞-cosmos $\mathbb{K}$ is cotensored over quasi-categories, which allows one to make sense of the tangent bundle construction $T(\mathcal{C}) := \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C})$ for any object $\mathcal{C}$ in $\mathbb{K}$. It would be interesting to explore under what conditions on $\mathbb{K}$ this construction extends to a full tangent structure.

Finally, we might speculate on how the Goodwillie tangent structure fits into the much bigger programme of 'higher differential geometry' developed by Schreiber [**?**, 4.1], or into the framework of homotopy type theory [**?**], though we don't have anything concrete to say about these possible connections.

Background on ∞-categories; notation and conventions. This paper is written largely in the language of ∞-categories, as developed by Lurie in the books [**?**] and [**?**], based on the quasi-categories of Boardman and Vogt [**?**]. However, for much of the paper, details of that theory are not particularly important, and other models for $(\infty, 1)$-categories could easily be used instead, especially if the reader is interested only in the main ideas of this work rather than the technical details.

All the basic concepts needed to define tangent ∞-categories, and to describe the underlying data of the Goodwillie tangent structure, will be familiar to readers versed in ordinary category theory. Those ideas include limits and colimits (in particular, pullbacks and pushouts), adjunctions, monoidal structures, and functor categories. To follow the details of our constructions, however, the reader will need a close acquaintance with simplicial sets and, especially, their relationship to categories via the nerve construction.

Throughout this paper, we take the perspective that categories, ∞-categories, and indeed $(\infty, 2)$-categories, are all really the same sort of thing, namely simplicial sets (sometimes with additional data). We typically do not distinguish notationally between any of these types of object. In particular, we identify a category $\mathbb{X}$ with its nerve, an ∞-category. In a few places we will consider simplicially-enriched categories, which we also usually identify with simplicial sets via the simplicial (or homotopy coherent) nerve [**?**, 1.1.5.5].

One exception to this convention is the category of simplicial sets itself which we denote as Set_Δ, and some related categories (such as 'marked' or 'scaled' simplicial sets) which we introduce in the course of this paper. We will make

some use of the various model structures on these categories, including the Quillen and Joyal model structures on Set_Δ. There are ∞-categories associated to these model categories, but we will use separate notation to denote those ∞-categories when we need to use them.

A functor $F : \mathbb{X} \to \mathbb{Y}$ between two ∞-categories (or $(\infty, 2)$-categories) is then simply a map of simplicial sets (possibly required to respect the additional data). When $\mathbb{X}$ is an ordinary category, such F can be viewed as a 'homotopy coherent' diagram in the ∞-category $\mathbb{Y}$. If $\mathbb{Y}$ is also an ordinary category, then F is an ordinary functor from $\mathbb{X}$ to $\mathbb{Y}$.

For simplicial sets A, B, we write $\mathrm{Fun}(A, B)$ for the simplicial set whose n-simplexes are the simplicial maps $\Delta^n \times A \to B$, i.e. the ordinary simplicial mapping object. When B is an ∞-category, so is $\mathrm{Fun}(A, B)$, and in that case we refer to $\mathrm{Fun}(A, B)$ is the ∞-*category of functors from A to B*.

One of the most confusing aspects of our theory is that ∞-categories (and $(\infty, 2)$-categories) play several different roles in this paper at different 'levels':

(1) we can define **tangent objects in** any (perhaps very large) $(\infty, 2)$-category $\mathbf{C}$ (Definition **??**);
(2) taking $\mathbf{C}$ to be the $(\infty, 2)$-category of (large) ∞-categories $\mathbf{Cat}_\infty$, we get a notion of **tangent structure on** a specific ∞-category $\mathbb{C} \in \mathbf{C}$ (Definition **??**);
(3) taking $\mathbb{C}$ to be an ∞-category of (smaller) ∞-categories, such as $\mathbb{Cat}_\infty^{\mathrm{diff}}$, we get the **tangent bundle of** a specific ∞-category $\mathcal{C} \in \mathbb{C}$ (Definition **??**).

We can also take $\mathcal{C}$ in (3) to be an ∞-category $\mathcal{Cat}_\infty$ of (even smaller) ∞-categories, in which case we would also have the **tangent space at** an ∞-category Y, i.e. $T_Y \mathcal{C}$. These tangent spaces are the subject of [**?**].

We distinguish between these three uses for ∞- and $(\infty, 2)$-categories by applying the fonts $\mathbf{C}, \mathbb{C}, \mathcal{C}$ as indicated in the list above. In particular, we use these different fonts to signify the size restrictions that are implicit in our hierarchy. To be precise we assume, where necessary, the existence of inaccessible cardinals that determine the different 'sizes' of the ∞-categories described in (1), (2) and (3) above; see also [**?**, 1.2.15] for a discussion of this foundational issue. Beyond the requirement to keep these three levels separate, size issues do not play any significant role in this paper.

One final (and important) comment on notation, especially for readers familiar with the papers of Cockett and Cruttwell: in this paper we use 'algebraic' order

for writing composition, as opposed to the diagrammatic order employed in many papers in the tangent category literature. So, for morphisms $f : A \to B$ and $g : B \to C$, we write gf for the composite morphism $A \to C$. Because of this choice, some of the expressions we use look different to those appearing in a corresponding place in [?].

Part 1. Tangent structures on ∞-categories

The goal of this first part of the paper is to extend the notion of tangent category of Cockett and Cruttwell to an ∞-categorical context, and to begin the study of these 'tangent ∞-categories'.

We refer the reader to the paper of Cockett and Cruttwell [?] for a detailed introduction to the theory of tangent categories. The definition that we use in this paper, however, is based on an alternative characterization of tangent structure, due to Leung [?, 14.1]. That characterization is in terms of a monoidal category of 'Weil-algebras', and in Section ?? we recall that category and the corresponding notion of tangent structure on a category.

In Section ?? we make the generalization to ∞-categories, giving our definition of tangent ∞-category along with some simple examples. We start to develop the theory of tangent ∞-categories in Section ?? with a definition of tangent functor, the appropriate notion of morphism between tangent ∞-categories. In Section ?? we consider the notion of 'differential object' in a tangent structure, also due to Cockett and Cruttwell, and extend that notion to tangent ∞-categories.

In the last topic in this part, Section ??, we extend the definition of tangent structure to $(\infty, 2)$-categories and other kinds of objects.

1. The monoidal category of Weil-algebras

We use the term $\mathbb{N}$-*algebra* to refer to what is also called a 'commutative semi-ring', or 'commutative rig', that is, a commutative ring without the requirement for additive inverses. In particular $\mathbb{N} = \{0, 1, 2, \dots\}$ itself is an $\mathbb{N}$-algebra.

Definition 1.1. Let $\mathbb{W}$eil be the following category:

- an object of $\mathbb{W}$eil is an augmented commutative $\mathbb{N}$-algebra of the form

$$A = \mathbb{N}[x_1, \dots, x_n]/(x_i x_j \mid i \sim j)$$

  where n is a nonnegative integer, and $\sim$ is an equivalence relation on $\{1, \dots, n\}$ that corresponds to a block partition[6] $n = n_1 + \dots + n_r$ of n

---

[6]A *block partition* is one in which each equivalence class is a set of consecutive integers; for example $\{\{1, 2, 3\}, \{4, 5\}\}$. Such a partition is determined by the ordered sequence of sizes of those equivalence classes, i.e. the partition $5 = 3 + 2$ in this example.

into an ordered sum of positive integers; the augmentation $\epsilon : A \to \mathbb{N}$ is (necessarily) the algebra map given by $x_i \mapsto 0$ for $i = 1, \dots, n$;
- a morphism $A \to A'$ in $\mathbb{W}$eil is a map of augmented commutative $\mathbb{N}$-algebras, i.e. a function that preserves 0, 1, addition and multiplication, and which commutes with the augmentations;
- the identity morphism 1_A is the identity algebra map on A;
- composition in $\mathbb{W}$eil is composition of algebra maps.

Our formal definition is restricted to generators labelled $x_1, \dots, x_n$ for some nonnegative integer n, but it will be convenient, especially when n is small, to use labels such as x, y, z or a, b instead.

We refer to an object of $\mathbb{W}$eil as a *Weil-algebra*, and to the morphisms as *morphisms of Weil-algebras*.

Remark 1.2. Allowing all equivalence relations on $\{1, \dots, n\}$ in the definition, rather than restricting to the block partitions, does not change the category $\mathbb{W}$eil up to equivalence. However, the restriction will be convenient in some of our later constructions.

Remark 1.3. In [?], the category we are calling $\mathbb{W}$eil is denoted $\mathbb{N}\text{-}\mathbf{Weil_1}$, while Leung uses an undecorated $\mathbf{Weil}$ to denote a larger category of augmented commutative k-algebras (over an unspecified field k).

Examples 1.4. Here are some examples of objects and morphisms in the category $\mathbb{W}$eil:

(1) The unique Weil-algebra with 0 generators is $\mathbb{N}$ itself. This object is both initial and terminal in $\mathbb{W}$eil.
(2) The unique Weil-algebra with 1 generator will be denoted
$$W = \mathbb{N}[x]/(x^2).$$
(3) A morphism $\phi : W \to W$ is uniquely determined by the element $\phi(x) \in W$, which, so that ϕ commutes with the augmentation, must be of the form kx for some $k \in \mathbb{N}$. In other words, ϕ is given by
$$\phi(a + bx) = a + bkx.$$
(4) There are two Weil-algebras with 2 generators:
$$W \otimes W := \mathbb{N}[x, y]/(x^2, y^2), \quad W^2 := \mathbb{N}[x, y]/(x^2, xy, y^2)$$
corresponding, respectively, to the discrete and indiscrete partitions of $\{1, 2\}$.

Remark 1.5. A morphism $A \to A'$ in $\mathbb{W}$eil is determined by the images $\phi(x_i)$ of each of the generators x_i of A. Each $\phi(x_i)$ is a sum of monomials in

$\mathbb{N}[x_1, \ldots, x_{n'}]/(x_i x_j \mid i \sim' j)$, and we may assume that none of those monomials has a factor of the form $x_i x_j$ for $i \sim' j$. This assumption uniquely determines a set of monomials whose sum is $\phi(x_i)$. For example, a morphism

$$\mathbb{N}[x, y]/(x^2, xy, y^2) \to \mathbb{N}[x, y]/(x^2, y^2)$$

might be determined by setting

$$\phi(x) = x + x + xy + y, \quad \phi(y) = xy + xy + xy.$$

Not all choices of sums of monomials determine a morphism: in the above case we must check that $\phi(x)\phi(y) = \phi(xy) = \phi(0) = 0$ which is the case here.

We have the following description of coproducts in $\mathbb{W}$eil.

Lemma 1.6. *The category* $\mathbb{W}$eil *has finite coproducts given by the tensor product of* $\mathbb{N}$-*algebras. A specific choice of the coproduct is given by concatenating lists of generators. Thus for* $A = \mathbb{N}[x_1, \ldots, x_n]/(x_i x_j \mid i \sim j)$ *and* $A' = \mathbb{N}[x_1, \ldots, x_{n'}]/(x_i x_j \mid i \sim' j)$, *we define*

$$A \otimes A' := \mathbb{N}[x_1, \ldots, x_{n+n'}]/(x_i x_j \mid i \sim'' j)$$

where $i \sim'' j$ *if and only if:*

- $i, j \leq n$ *and* $i \sim j$*; or*
- $i, j \geq n + 1$ *and* $(i - n) \sim' (j - n)$.

Definition 1.7. Let $\otimes$ be the strict monoidal product on $\mathbb{W}$eil given by the coproduct described in Lemma **??**, with unit object $\mathbb{N}$. For example, we have

$$W \otimes \cdots \otimes W = \mathbb{N}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2),$$

the coproduct of n copies of W, i.e. the object of $\mathbb{W}$eil corresponding to the discrete equivalence relation on $\{1, \ldots, n\}$.

Leung proves in [**?**, 8.5] that the only non-trivial finite products in $\mathbb{W}$eil are the powers of W.

Definition 1.8. We write

$$W^n = W \times \cdots \times W := \mathbb{N}[x_1, \ldots, x_n]/(x_i x_j)_{i,j=1}^n$$

for the product in $\mathbb{W}$eil of n copies of W, i.e. the object of $\mathbb{W}$eil corresponding to the indiscrete equivalence relation on $\{1, \ldots, n\}$.

Remark 1.9. Each object in $\mathbb{W}$eil is uniquely of the form $W^{n_1} \otimes \cdots \otimes W^{n_r}$ for a (possibly empty) ordered sequence of positive integers $(n_1, \ldots, n_r)$.

There are certain pullback squares in $\mathbb{W}$eil that play a crucial role in the definition of tangent structure.

Lemma 1.10 (Leung, [**?**, 3.14]). *For arbitrary $A \in \mathbb{W}\text{eil}$ and $m, n \geq 0$, there is a pullback square in $\mathbb{W}\text{eil}$ of the form*

(1.11)

$$
\begin{array}{ccc}
A \otimes W^{m+n} & \longrightarrow & A \otimes W^m \\
\downarrow & & \downarrow \\
A \otimes W^n & \longrightarrow & A
\end{array}
$$

in which the horizontal and vertical maps are induced by the augmentations $W^n \to \mathbb{N}$ and $W^m \to \mathbb{N}$ respectively. We refer to these diagrams as the foundational pullbacks in $\mathbb{W}\text{eil}$.

Lemma 1.12. *There is a pullback square in $\mathbb{W}\text{eil}$ of the form*

(1.13)

$$
\begin{array}{ccc}
W^2 & \xrightarrow{\ \mu\ } & W \otimes W \\
{\scriptstyle \epsilon}\downarrow & & \downarrow {\scriptstyle 1_W \otimes \epsilon} \\
\mathbb{N} & \xrightarrow{\ \eta\ } & W
\end{array}
$$

where $\mu : \mathbb{N}[x, y]/(x^2, xy, y^2) \to \mathbb{N}[a, b]/(a^2, b^2)$ is given by

$$
\mu(x) = ab, \quad \mu(y) = b.
$$

We refer to this square as the vertical lift pullback in $\mathbb{W}\text{eil}$.

Proof. A cone over the maps $1_W \otimes \epsilon$ and η consists of a morphism

$$
\phi : \mathbb{N}[z_1, \ldots, z_k]/(z_i z_j \mid i \sim j) \to \mathbb{N}[a, b]/(a^2, b^2)
$$

such that each $\phi(z_i)$ is a sum of monomials ab and b (but not a). The corresponding lift

$$
\tilde{\phi} : \mathbb{N}[z_1, \ldots, z_k]/(z_i z_j \mid i \sim j) \to \mathbb{N}[x, y]/(x^2, xy, y^2)
$$

is given by replacing ab with x and b with y in the formula for each $\phi(z_i)$. $\quad\square$

We can now give Leung's characterization of a tangent structure on a category $\mathbb{X}$, which we will use as our definition.

Definition 1.14 (Leung, [**?**, 14.1]). Let $\mathbb{X}$ be a category, and let $\text{End}(\mathbb{X})$ be the strict monoidal category of endofunctors $\mathbb{X} \to \mathbb{X}$ under composition. A *tangent structure* on $\mathbb{X}$ is a strict monoidal functor

$$
T : (\mathbb{W}\text{eil}, \otimes, \mathbb{N}) \to (\text{End}(\mathbb{X}), \circ, \text{Id})
$$

for which the underlying functor $T : \mathbb{Weil} \to \mathrm{End}(\mathbb{X})$ preserves the foundational and vertical lift pullbacks (**??**, **??**). We will refer to these pullback diagrams collectively as the *tangent pullbacks* in $\mathbb{Weil}$.

A *tangent category* $(\mathbb{X}, T)$ consists of a category $\mathbb{X}$ and a tangent structure T on $\mathbb{X}$.

Remarks 1.15. There are two minor differences between our definition of tangent structure and Leung's formulation:

(1) The statement of Leung's theorem [**?**, 14.1] is that tangent structures correspond to *strong* monoidal functors, yet the proof therein actually constructs a strict monoidal functor T for each tangent structure. We will see in the next section that the monoidal category $\mathbb{Weil}$ satisfies a cofibrancy condition (Lemma **??**) which implies that any strong monoidal functor $\mathbb{Weil} \to \mathrm{End}(\mathbb{X})$ is equivalent to a strict monoidal functor (an observation also made by Garner [**?**, Thm. 7]).
(2) Leung uses a certain equalizer in place of the vertical lift pullback (**??**). Cockett and Cruttwell demonstrated the equivalence of these two approaches to the definition of tangent structure in [**?**, 2.12], and focused on the pullback condition in their later work, see [**?**, 2.1].

Notation 1.16. A strict monoidal functor $T : \mathbb{Weil} \to \mathrm{End}(\mathbb{X})$ can be described equivalently via an action map

$$\mathbb{Weil} \times \mathbb{X} \to \mathbb{X}$$

that makes $\mathbb{X}$ into a $\mathbb{Weil}$-*actegory*. We typically denote this action map also by T and move freely between the two descriptions.

For each Weil-algebra A, a tangent structure on $\mathbb{X}$ provides for an endofunctor on $\mathbb{X}$ which we denote $T^A : \mathbb{X} \to \mathbb{X}$. In particular, when $A = W = \mathbb{N}[x]/(x^2)$, we have an endofunctor $T^W : \mathbb{X} \to \mathbb{X}$ which we refer to as the *tangent bundle functor* of the tangent structure. We usually overload the notation still further and denote T^W also by T.

We now recall how Definition **??** reduces to Cockett and Cruttwell's original definition of tangent category [**?**, 2.1].

Remark 1.17. Let $T : \mathbb{Weil} \to \mathrm{End}(\mathbb{X})$ be a tangent structure on a category $\mathbb{X}$. The functors $T^A : \mathbb{X} \to \mathbb{X}$ for Weil-algebras A are determined by the single functor $T = T^W : \mathbb{X} \to \mathbb{X}$ corresponding to the Weil-algebra $W = \mathbb{N}[x]/(x^2)$:

- since T is (strict) monoidal, we have, for the unit object $\mathbb{N}$ of the monoidal structure on $\mathbb{Weil}$:

$$T^{\mathbb{N}} = I$$

  the identity functor on $\mathbb{X}$;
- since the tangent structure is required to preserve the foundational pullbacks, we have, for any positive integers n:

$$T_n := T^{W^n} \cong T \times_I \cdots \times_I T;$$

  the wide pullback of n copies of the 'projection' map $p : T \to I$ corresponding to the augmentation $\epsilon : W \to \mathbb{N}$;
- since any Weil-algebra $A \in \mathbb{Weil}$ is of the form $W^{n_1} \otimes \cdots \otimes W^{n_r}$, the monoidal condition then implies

$$T^A = T_{n_1} \cdots T_{n_r}.$$

The main content of Leung's result is that the values of a tangent structure T on morphisms in $\mathbb{Weil}$ are determined by those values on five specific morphisms which correspond to the five natural transformations appearing in Cockett and Cruttwell's definition:

- corresponding to the augmentation $\epsilon : W \to \mathbb{N}$ is the *projection*,

$$p_T : T \to I;$$

- corresponding to the unit map $\eta : \mathbb{N} \to W$ is the *zero section*

$$0_T : I \to T;$$

- corresponding to the map

$$\phi : \mathbb{N}[x,y]/(x^2, xy, y^2) \to \mathbb{N}[z]/(z^2); \quad x \mapsto z; y \mapsto z,$$

  is the *addition*

$$+_T : T_2 \to T;$$

- corresponding to the symmetry map

$$\sigma : \mathbb{N}[x,y]/(x^2, y^2) \to \mathbb{N}[x,y]/(x^2, y^2); \quad x \mapsto y; y \mapsto x,$$

  is the *flip*

$$c_T : T^2 \to T^2;$$

- corresponding to the map

$$\delta : \mathbb{N}[z]/(z^2) \to \mathbb{N}[x,y]/(x^2, y^2); \quad z \mapsto xy,$$

  is the *vertical lift*

$$\ell_T : T \to T^2.$$

Finally, the requirement that a tangent structure $T : \mathbb{Weil} \to \text{End}(\mathbb{X})$ preserve the vertical lift pullback (**??**) corresponds to the condition that Cockett and Cruttwell refer to in [**?**, 2.1] as the 'Universality of the Vertical Lift', i.e. that for all $M \in \mathbb{X}$ there is a pullback square (in $\mathbb{X}$) of the form

(1.18)
$$
\begin{array}{ccc}
TM \times_M TM & \longrightarrow & T(TM) \\
\downarrow & & \downarrow {\scriptstyle T(p)} \\
M & \xrightarrow{\ \ 0\ \ } & TM.
\end{array}
$$

Examples 1.19. Here are some of the standard examples of tangent categories. More examples, and more details, appear in the papers [**?**, **?**].

(1) Let $\mathbb{X} = \mathbb{Mfld}$, the category of finite-dimensional smooth manifolds and smooth maps, and let $T : \mathbb{Mfld} \to \mathbb{Mfld}$ be the ordinary tangent bundle functor. Then there is a tangent structure on $\mathbb{Mfld}$ with tangent bundle functor T and projection map given by the usual bundle projections $TM \to M$. The zero and addition maps come from the vector bundle structure on TM, and the flip and vertical lift can be defined directly in terms of tangent vectors [**?**, 2.2(i)].

(2) The category $\mathbb{Sch}$ of schemes has a tangent structure in which the tangent bundle functor $T : \mathbb{Sch} \to \mathbb{Sch}$ on a scheme X is the vector bundle associated to the $\mathcal{O}_X$-module of Kähler differentials of X (over $\text{Spec } \mathbb{Z}$):
$$ T(X) = \text{Spec Sym } \Omega_{X/\mathbb{Z}}. $$
See [**?**, Ex. 2(iii)] for further details.

(3) The category $\mathbb{CRing}$ of commutative rings (with identity) has a tangent structure given by $T^A(R) := A \otimes R$, i.e. with tangent bundle functor $T(R) = R[x]/(x^2)$.

(4) The category $\mathbb{X}$ of 'infinitesimally linear' objects in a model of synthetic differential geometry (SDG) has a tangent structure whose tangent bundle functor $T : \mathbb{X} \to \mathbb{X}$ is given by the exponential $T(C) = C^D$ where D is an 'object of infinitesimals' in $\mathbb{X}$. See [**?**, 5.1] for more details.

(5) Let $\mathbb{X}$ be any category. Then there is a trivial tangent structure on $\mathbb{X}$ in which $T^A : \mathbb{X} \to \mathbb{X}$ is the identity functor for every Weil-algebra A..

(6) Let $\mathbb{X}$ be a tangent category with tangent bundle functor T, and let $\mathbb{I}$ be a category. Then the functor category $\text{Fun}(\mathbb{I}, \mathbb{X})$ has a tangent structure whose tangent bundle functor is composition with T.

2. TANGENT ∞-CATEGORIES

We now turn to ∞-categories. The goal of this section is to extend Definition **??** to tangent structures on an ∞-category. We start by giving our definition in a 'model-independent' language that does not assume knowledge of any particular model for ∞-categories.

That definition requires very little change from **??**, but it does rely on a notion of *monoidal* ∞-category. We will give an explicit definition of monoidal ∞-category in the context of quasi-categories in Definition **??**. For now, we need the following two examples:

- any ordinary monoidal category, such as $(\mathbb{Weil}, \otimes, \mathbb{N})$, determines a monoidal ∞-category which we will denote $\mathbb{Weil}^{\otimes}$;
- for any ∞-category $\mathbb{X}$, there is a monoidal ∞-category $\mathrm{End}(\mathbb{X})^{\circ}$ whose underlying ∞-category is $\mathrm{End}(\mathbb{X}) = \mathrm{Fun}(\mathbb{X}, \mathbb{X})$, the ∞-category of endofunctors of $\mathbb{X}$, with monoidal structure given by composition.

Explicit models for these two examples are given in **??** and **??**.

To state our definition of tangent ∞-category, we also need the notion of monoidal functor between monoidal ∞-categories which we define in **??**. With that, we make the following definition.

Definition 2.1. Let $\mathbb{X}$ be an ∞-category. A *tangent structure* on $\mathbb{X}$ is a monoidal functor
$$T : \mathbb{Weil}^{\otimes} \to \mathrm{End}(\mathbb{X})^{\circ}$$
for which the underlying functor $T : \mathbb{Weil} \to \mathrm{End}(\mathbb{X})$ preserves the tangent pullbacks (**??**, **??**). By a 'pullback' in $\mathrm{End}(\mathbb{X})$ we mean in the ∞-categorical sense [**?**, 1.2.13.4].

Remark 2.2. Pullbacks in a functor ∞-category such as $\mathrm{End}(\mathbb{X})$ are calculated pointwise [**?**, 5.1.2.3], so the condition in **??** can be expressed by saying that for each tangent pullback diagram in $\mathbb{Weil}$, and each object $C \in \mathbb{X}$, there is a certain pullback square in the ∞-category $\mathbb{X}$. In particular, a tangent structure on $\mathbb{X}$ determines pullbacks in $\mathbb{X}$ of the form

$$
\begin{CD}
TC \times_C TC @>>> T^2C \\
@VVV @VVV \\
C @>>> TC
\end{CD}
$$

for each $C \in \mathbb{X}$.

Remark 2.3. Much of Remark **??** extends to the ∞-categorical case. The value of a tangent structure $T : \mathbb{Weil}^{\otimes} \to \operatorname{End}(\mathbb{X})^{\circ}$ on any Weil-algebra A is determined, up to equivalence, by the tangent bundle functor $T^W : \mathbb{X} \to \mathbb{X}$ which we usually denote also by T. A tangent structure on $\mathbb{X}$ also entails the five natural transformations $p, 0, +, c, \ell$ described in **??**. However, in place of the strictly commutative diagrams in Cockett and Cruttwell's definition of tangent category [**?**, 2.1], a tangent ∞-category includes higher-level coherence data that establishes the commutativity of those diagrams up to homotopy.

Example 2.4. Let $\mathbb{X}$ be an ordinary category. In Lemma **??** we prove that a tangent structure on $\mathbb{X}$, in the sense of Definition **??**, determines a tangent structure on the ∞-category corresponding to $\mathbb{X}$, in the sense of Definition **??**.

Warning 2.5. Every ∞-category $\mathbb{X}$ has an associated homotopy category $h\mathbb{X}$. A tangent structure T on $\mathbb{X}$ determines a monoidal functor on the level of homotopy categories

$$hT : \mathbb{Weil} \times h\mathbb{X} \to h\mathbb{X}.$$

However, hT is typically *not* a tangent structure on $h\mathbb{X}$ since pullbacks in an ∞-category $\mathbb{X}$ do not often represent pullbacks in $h\mathbb{X}$.

Our next goal is to give a more precise version of Definition **??** by choosing a specific model for ∞-categories and, in particular, their monoidal counterparts. We focus on the theory of *quasi-categories* introduced by Boardman and Vogt in [**?**], but developed considerably in recent years by Joyal ([**?**, **?**]) and then Lurie (see, e.g. [**?**]).

Monoidal quasi-categories and monoidal functors. A *quasi-category* is a simplicial set that satisfies the 'inner horn condition' of [**?**, 1.1.2.4]. The nerve of an ordinary category is a quasi-category and we will not usually distinguish between a category and its nerve. For example, the nerve of the category $\mathbb{Weil}$ is a quasi-category which we will also denote $\mathbb{Weil}$.

One of the appealing aspects of quasi-categories from our perspective is that they admit a robust theory of monoidal ∞-categories. There are various ways to implement these 'monoidal quasi-categories'. We follow the approach outlined by Lurie in [**?**, 4.1.8.7], which allows a particularly simple definition.

Definition 2.6. A *monoidal quasi-category* is a simplicial monoid $\mathbb{M}^{\otimes}$ for which the underlying simplicial set $\mathbb{M}$ is a quasi-category. We will often drop the superscript $\otimes$ and simply refer to 'the monoidal quasi-category $\mathbb{M}$' with the monoid structure understood.

Remark 2.7. The definition of monoidal quasi-category is analogous to that of *strict* monoidal category in that the monoidal structure is required to be strictly associative. In other models of monoidal ∞-category, such as Lurie's ∞-operads [**?**, 2.1.2.13], non-trivial associativity isomorphisms (and higher coherences) are built in to the definition. However, as in the case of ordinary monoidal categories, every monoidal ∞-category is equivalent to one that is strictly associative; that claim is explained in [**?**, 4.1.8.7]. As the following two examples show, the monoidal ∞-categories that are relevant to describing tangent structures already come equipped with a strictly associative monoidal structure.

Example 2.8. The nerve of a strict monoidal category is a monoidal quasi-category. In particular, the strict monoidal category $(\mathbb{Weil}, \otimes, \mathbb{N})$ determines a monoidal quasi-category which we denote $\mathbb{Weil}^{\otimes}$, or usually simply as $\mathbb{Weil}$.

Example 2.9. For any quasi-category $\mathbb{X}$, the functor quasi-category

$$\mathrm{End}(\mathbb{X}) := \mathrm{Fun}(\mathbb{X}, \mathbb{X}),$$

given by the ordinary simplicial mapping space for the simplicial set $\mathbb{X}$, is a monoidal quasi-category, which we denote $\mathrm{End}(\mathbb{X})^{\circ}$, with monoidal structure given by composition of functors.

Another convenient aspect of our definition of monoidal quasi-categories is that *strict* monoidal functors between them are familiar and easy to describe.

Definition 2.10. A *strict monoidal functor* $\mathbb{M}^{\otimes} \to \mathbb{N}^{\otimes}$ between monoidal quasi-categories is a map of simplicial monoids $\mathbb{M}^{\otimes} \to \mathbb{N}^{\otimes}$, i.e. a map of simplicial sets $F : \mathbb{M} \to \mathbb{N}$ that strictly commutes with the monoid structures.

We can now provide a more concrete version of Definition **??**.

Definition 2.11. Let $\mathbb{X}$ be a quasi-category. Then a *tangent structure* on $\mathbb{X}$ is a strict monoidal functor

$$T : \mathbb{Weil}^{\otimes} \to \mathrm{End}(\mathbb{X})^{\circ}$$

for which the underlying map of quasi-categories $T : \mathbb{Weil} \to \mathrm{End}(\mathbb{X})$ preserves the tangent pullbacks.

We will refer to the pair $(\mathbb{X}, T)$ as a *tangent ∞-category*, or *tangent quasi-category* if we wish to emphasize that we are using the quasi-categorical model for ∞-categories. For brevity we often leave T understood and refer to 'the tangent ∞-category $\mathbb{X}$'.

Remark 2.12. As with ordinary tangent categories, a tangent structure on a quasi-category $\mathbb{X}$ can also be described via an *action map*, i.e. a map of simplicial sets

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{X} \to \mathbb{X}$$

which forms an action of the simplicial monoid $\mathbb{W}\mathrm{eil}$ on the simplicial set $\mathbb{X}$. We will also say that $\mathbb{X}$ is a $\mathbb{W}\mathrm{eil}$-*module*. Thus a tangent ∞-category can equivalently be defined as a $\mathbb{W}\mathrm{eil}$-module for which the $\mathbb{W}\mathrm{eil}$-action preserves tangent pullbacks.

Lemma 2.13. *Let $\mathbb{X}$ be an ordinary category. Then a tangent structure on (the nerve of) $\mathbb{X}$, in the sense of Definition ??, is the same thing as a tangent structure on the category $\mathbb{X}$ in the sense of Definition ??.*

Proof. The monoidal quasi-category $\mathrm{End}(\mathbb{X})^\circ$ can be identified with (the nerve of) the ordinary strict monoidal category $\mathrm{End}(\mathbb{X})$, and a map of simplicial monoids $\mathbb{W}\mathrm{eil}^\otimes \to \mathrm{End}(\mathbb{X})^\circ$ is then the nerve of an ordinary strict monoidal functor. Finally, we note that a diagram in $\mathbb{X}$ is a pullback square in the ∞-categorical sense if and only if it is a pullback square in the ordinary categorical sense. $\qquad\square$

Perhaps it is surprising that our definition of tangent structure involves *strict* monoidal functors instead of a weaker notion more akin to that of *strong* monoidal functor. Our justification for this claim is a cofibrancy property of the monoidal quasi-category $\mathbb{W}\mathrm{eil}^\otimes$ which implies that every 'strong' monoidal functor with source $\mathbb{W}\mathrm{eil}^\otimes$ is equivalent, in a suitable sense, to a strict monoidal functor. To describe that property, we now develop in more detail the model for monoidal ∞-categories and their monoidal functors that is outlined by Lurie in [**?**, 4.1.8.7].

Underlying that model is a model structure on the category of 'marked' simplicial sets which forms a particularly nice foundation for the theory of ∞-categories.

Definition 2.14. A *marked simplicial set* consist of a pair (S, E) where S is a simplicial set and $E \subseteq S_1$ is a subset of the set of edges in E that contains all degenerate edges. We refer to E as a *marking* of S, and an element of E as a *marked edge*. Let Set_Δ^+ be the category of marked simplicial sets, with morphisms given by the maps of simplicial sets that preserve marked edges.

The category Set_Δ^+ admits a simplicial model structure [**?**, 3.1.3.7], which we refer to as the *marked model structure*, in which the cofibrations are the monomorphisms, and the fibrant objects are those marked simplicial sets

(S, E) for which S is a quasi-category and E is precisely the set of equivalences in S. The marked model structure on Set_Δ^+ is Quillen equivalent to the Joyal model structure on Set_Δ by [**?**, 3.1.5] and so provides an alternative model for the homotopy theory of ∞-categories.

We will frequently treat a quasi-category S as a marked simplicial set without comment. Whenever we do so, we mean that the set E of marked edges is the set of equivalences in S unless stated otherwise. This choice is called the *natural marking* on S.

Definition 2.15. A *marked simplicial monoid* is a simplicial monoid together with a marking of its underlying simplicial set that is preserved by the monoid multiplication map. Let Mon_Δ^+ denote the category whose objects are the marked simplicial monoids and whose morphisms are the maps of simplicial monoids that preserve the marking.

Proposition 2.16. *There is a model structure on Mon_Δ^+ in which a morphism of marked simplicial monoids is a weak equivalence (or fibration) if and only if its underlying morphism of marked simplicial sets is a weak equivalence (or, respectively, a fibration) in the marked model structure.*

Proof. The existence of this model structure is an application of [**?**, 4.1(3)]. See also [**?**, 4.1.8.3]. □

It follows from [**?**, 4.1.8.4] that the model structure of Proposition **??** models the homotopy theory of monoidal ∞-categories. The fibrant objects in that model structure are the marked simplicial monoids for which the underlying marked simplicial set is fibrant, i.e. the monoidal quasi-categories with the equivalences marked.

The appropriate notion of (strong but not strict) monoidal functor between monoidal quasi-categories is that determined by this model structure. Thus a monoidal functor from $\mathbb{M}^\otimes$ to $\mathbb{N}^\otimes$ is given by a map of marked simplicial monoids from a cofibrant replacement of $\mathbb{M}^\otimes$ to (the already fibrant object) $\mathbb{N}^\otimes$. The fact that strict monoidal functors suffice to describe tangent structures on a monoidal quasi-category is then a consequence of the following calculation.

Lemma 2.17. *The simplicial monoid $\mathbb{Weil}^\otimes$ (with only the identity morphisms in $\mathbb{Weil}$ as the marked edges) is cofibrant in the model structure of Proposition **??**.*

Proof. The category Mon_Δ of (non-marked) simplicial monoids also admits a model structure in which the weak equivalences and fibrations are those maps

whose underlying maps of simplicial sets are weak equivalences and fibrations in the Quillen model structure on simplicial sets. The cofibrant objects in this model structure are identified in [**?**, 7.6]; they are the retracts of the simplicial monoids which are free in each dimension, and for which the degeneracy maps take generators to generators. The simplicial monoid $\mathbb{W}$eil has this property: the set of vertices is freely generated under $\otimes$ by generators of the form W^n (see Remark **??**), and any edge in $\mathbb{W}$eil can be written uniquely as a $\otimes$-product of edges that cannot be decomposed further. So $\mathbb{W}$eil is cofibrant in Mon_Δ.

Let $(-)^\flat : \mathsf{Set}_\Delta \to \mathsf{Set}_\Delta^+$ be the 'minimal' marking functor, i.e. the functor which takes a simplicial set X to the marked simplicial set $X^\flat$ which has only degenerate edges marked. The forgetful functor $U : \mathsf{Set}_\Delta^+ \to \mathsf{Set}_\Delta$ is right adjoint to $(-)^\flat$, and both U and $(-)^\flat$ are monoidal functors with respect to the cartesian product. It follows that $(-)^\flat$ and U induce an adjoint pair of functors, which we denote with the same names, on the corresponding categories of monoids

$$\mathsf{Mon}_\Delta \xrightarrow[\;\;U\;\;]{\;\;(-)^\flat\;\;} \mathsf{Mon}_\Delta^+.$$

We claim that the left adjoint $(-)^\flat : \mathsf{Mon}_\Delta \to \mathsf{Mon}_\Delta^+$ preserves cofibrations, and it is sufficient to show that the right adjoint $U : \mathsf{Mon}_\Delta^+ \to \mathsf{Mon}_\Delta$ preserves acyclic fibrations. Since those acyclic fibrations are detected in both cases in the underlying categories of (marked or not) simplicial sets, it is sufficient to check that $U : \mathsf{Set}_\Delta^+ \to \mathsf{Set}_\Delta$ preserves acyclic fibrations (where Set_Δ has the Quillen model structure). This claim follows from the fact that the minimal marking functor $(-)^\flat : \mathsf{Set}_\Delta \to \mathsf{Set}_\Delta^+$ preserves cofibrations (which are monomorphisms in both categories).

Since $\mathbb{W}$eil is cofibrant in Mon_Δ, we deduce that $\mathbb{W}$eil$^\flat$ is cofibrant in Mon_Δ^+, as claimed. $\square$

It follows from Lemma **??** that any morphism in the homotopy category of monoidal quasi-categories, from $\mathbb{W}$eil to $\mathrm{End}(\mathbb{X})$, can be represented by a strict map of marked simplicial monoids of the form

$$\mathbb{W}\mathrm{eil}^\flat \to \mathrm{End}(\mathbb{X})$$

or equivalently by a map of (non-marked) simplicial monoids $\mathbb{W}$eil $\to \mathrm{End}(\mathbb{X})$. This observation confirms that the strict monoidal functors appearing in Definition **??** provide an accurate model for the monoidal functors between monoidal ∞-categories in Definition **??**.

Examples of tangent ∞-categories. Recall that a tangent structure on an ordinary category $\mathbb{X}$ determines a tangent structure on $\mathbb{X}$ viewed as an ∞-category. However, there are also tangent structures on ∞-categories that do not arise from ordinary categories. We start with some simple examples and constructions.

Example 2.18. Let $\mathbb{X}$ be an arbitrary ∞-category. Then there is a tangent structure $I : \mathbb{Weil}^\otimes \to \mathrm{End}(\mathbb{X})^\circ$ given by the constant map to the identity functor on $\mathbb{X}$, the *trivial tangent structure* on $\mathbb{X}$.

Example 2.19. Let $(\mathbb{X}, T)$ be a tangent ∞-category, and let S be any simplicial set. We give the ∞-category $\mathrm{Fun}(S, \mathbb{X})$, of S-indexed diagrams in $\mathbb{X}$, a tangent structure by extending T to a map

$$\mathrm{Fun}(S, T) : \mathbb{Weil} \times \mathrm{Fun}(S, \mathbb{X}) \to \mathrm{Fun}(S, \mathbb{X}),$$

which is a tangent structure because pullbacks in the diagram ∞-category $\mathrm{Fun}(S, \mathbb{X})$ are determined objectwise.

Lemma 2.20. *Let $i : \mathbb{X} \xrightarrow{\sim} \mathbb{Y}$ be an equivalence of ∞-categories, and let $T : \mathbb{X} \to \mathbb{X}$ be the tangent bundle functor for a tangent structure on $\mathbb{X}$. Then there is a tangent structure on $\mathbb{Y}$ whose underlying tangent bundle functor $\mathbb{Y} \to \mathbb{Y}$ is equivalent to iTi^{-1}.*

Proof. We show that i induces an equivalence in the homotopy category of monoidal ∞-categories

$$\mathrm{End}(\mathbb{X})^\circ \simeq \mathrm{End}(\mathbb{Y})^\circ$$

whose underlying functor is $i(-)i^{-1}$. This claim follows from the methods of [**?**, 4.7.1] in which a universal property is established for a monoidal ∞-category of endomorphisms such as $\mathrm{End}(\mathbb{X})^\circ$. That universal property says that $\mathrm{End}(\mathbb{X})^\circ$ is the terminal object in the ∞-category $\mathbb{Cat}_\infty[\mathbb{X}]$ in which an object is a pair $(\mathbb{F}, \eta)$ consisting of an ∞-category $\mathbb{F}$ and a functor $\eta : \mathbb{F} \to \mathrm{End}(\mathbb{X})$. Any such terminal object has a unique associative algebra structure in $\mathbb{Cat}_\infty[\mathbb{X}]$ which induces the structure of an associative algebra in $\mathbb{Cat}_\infty$, i.e. a monoidal ∞-category. (To be precise, this claim is an application of [**?**, 4.7.1.40] with $\mathcal{C} = \mathcal{M} = \mathbb{Cat}_\infty$, the ∞-category of ∞-categories, and $M = \mathbb{X}$.)

Now note that we have an equivalence of ∞-categories

$$\mathbb{Cat}_\infty[\mathbb{X}] \xrightarrow{\sim} \mathbb{Cat}_\infty[\mathbb{Y}]; \quad (\mathbb{F}, \eta) \mapsto (\mathbb{F}, i(-)i^{-1} \circ \eta)$$

which therefore preserves the terminal objects and their algebra structures. In particular, $(\mathrm{End}(\mathbb{X}), i(-)i^{-1})$ is a terminal object in $\mathbb{Cat}_\infty[\mathbb{Y}]$, so is equivalent, as an associative algebra, to $(\mathrm{End}(\mathbb{Y}), \mathrm{id})$, yielding the desired monoidal equivalence.

Composing the given tangent structure map $T : \mathbb{Weil}^{\otimes} \to \mathrm{End}(\mathbb{X})^{\circ}$ with the equivalence constructed above determines the necessary tangent structure on the ∞-category $\mathbb{Y}$. $\hfill\square$

We conclude this section with a tangent ∞-category that extends the standard tangent structure on $\mathbb{Mfld}$, the category of smooth manifolds and smooth maps.

Spivak introduced in [**?**] an ∞-category $\mathbb{DMfld}$ of *derived manifolds*, constructed to allow for pullbacks along non-transverse pairs of smooth maps. We use a version of $\mathbb{DMfld}$ that is characterized by the following universal property described by Carchedi and Steffens [**?**]; see there for more details.

Definition 2.21. Let $\mathbb{DMfld}$ denote an idempotent-complete ∞-category with finite limits that admits a functor $i : \mathbb{Mfld} \to \mathbb{DMfld}$ with the following universal property: for any other idempotent-complete ∞-category with finite limits $\mathbb{C}$, the functor i induces an equivalence

$$i^* : \mathrm{Fun}^{\mathrm{lex}}(\mathbb{DMfld}, \mathbb{C}) \xrightarrow{\sim} \mathrm{Fun}^{\pitchfork}(\mathbb{Mfld}, \mathbb{C})$$

between the ∞-categories of finite-limit-preserving functors (on the left-hand side) and functors that preserve the *transverse pullbacks* and terminal object of $\mathbb{Mfld}$ (on the right-hand side). In particular, i preserves those transverse pullbacks and terminal object.

An explicit model for $\mathbb{DMfld}$ is given by the opposite of the ∞-category of (homotopically finitely presented) simplicial C^{∞}-rings [**?**, 5.4].

Lemma 2.22. *There is a monoidal functor*

$$\tilde{\bullet} : \mathrm{Fun}^{\pitchfork}(\mathbb{Mfld}, \mathbb{Mfld}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathbb{DMfld}, \mathbb{DMfld})$$

*induced by i along with the inverse to the equivalence of Definition **??**.*

Proof. A formal argument can be made using ideas from [**?**, 4.7.1] in a similar manner to the proof of Lemma **??**. Here we sketch an informal approach. Given $F, G : \mathbb{Mfld} \to \mathbb{Mfld}$ that preserve the transverse pullbacks and terminal object, the universal property on $\mathbb{DMfld}$ implies that iF and iG factor uniquely (up to contractible choice) as

$$iF \simeq \tilde{F}i, \quad iG \simeq \tilde{G}i$$

for $\tilde{F}, \tilde{G} : \mathbb{DMfld} \to \mathbb{DMfld}$. We then have

$$iFG \simeq \tilde{F}iG \simeq \tilde{F}\tilde{G}i$$

which implies that there is a canonical equivalence $\widetilde{FG} \simeq \tilde{F}\tilde{G}$. $\hfill\square$

Proposition 2.23. *The standard tangent structure on* $\mathbb{M}$fld *induces, by composition with* $\tilde{\bullet}$, *a tangent structure on the* ∞-*category* $\mathbb{DM}$fld.

Proof. First we argue that the tangent structure map

$$T : \mathbb{W}\text{eil} \to \text{Fun}(\mathbb{M}\text{fld}, \mathbb{M}\text{fld})$$

factors via $\text{Fun}^{\pitchfork}(\mathbb{M}\text{fld}, \mathbb{M}\text{fld})$. It is sufficient to note that the tangent bundle functor $T : \mathbb{M}\text{fld} \to \mathbb{M}\text{fld}$ preserves the transverse pullbacks and terminal object, and this claim follows directly from the definition of transversality.

Composing T with the monoidal functor $\tilde{\bullet}$ described in Lemma **??**, we obtain a monoidal functor

$$\tilde{T} : \mathbb{W}\text{eil} \to \text{Fun}^{\text{lex}}(\mathbb{DM}\text{fld}, \mathbb{DM}\text{fld}).$$

Since the target of $\tilde{T}$ is a full subcategory of $\text{End}(\mathbb{DM}\text{fld})$, it is now sufficient to show that $\tilde{T}$ preserves the tangent pullbacks. Equivalently, we must show that the composite

$$\mathbb{W}\text{eil} \xrightarrow{T} \text{Fun}^{\pitchfork}(\mathbb{M}\text{fld}, \mathbb{M}\text{fld}) \xrightarrow{i_*} \text{Fun}^{\pitchfork}(\mathbb{M}\text{fld}, \mathbb{DM}\text{fld})$$

preserves those pullbacks. The first map does (as T is a tangent structure on $\mathbb{M}$fld), and the second map does too (as each tangent pullback is transverse by [**?**, Ex. 4.4(ii)], and i preserves transverse pullbacks). ☐

3. Tangent functors

We now turn to morphisms between tangent structures on ∞-categories. In the context of ordinary tangent categories, these are described by Cockett and Cruttwell in [**?**, 2.7]. They define a 'strong' morphism between tangent categories to be a functor on the underlying categories that commutes, up to natural isomorphism, with the tangent structures. They also have a weaker notion involving a natural transformation that is not necessarily invertible.

To apply the Cockett-Cruttwell definition to tangent ∞-categories, we need to express it in terms of the action by Weil-algebras. This approach is described by Garner in [**?**, Thm. 9] where he notes that a (strong) morphism of tangent categories can, equivalently, be given via a map of 'Weil-actegories'. In our language of 'modules' over the monoidal ∞-category $\mathbb{W}$eil (see Remark **??**) we would say that a (strong) tangent functor between tangent ∞-categories $\mathbb{X}$ and $\mathbb{Y}$ is simply a map of $\mathbb{W}$eil-modules $\mathbb{X} \to \mathbb{Y}$.

However, a map of modules in the ∞-categorical context involves a lot of higher coherence data. To give a precise description of this data, we introduce a

suitable model category of $\mathbb{W}$eil-modules. Throughout this section we consider $\mathbb{W}$eil as a marked simplicial monoid with its minimal marking, denoted $\mathbb{W}\text{eil}^\flat$ in the proof of Lemma **??**.

Definition 3.1. A *marked* $\mathbb{W}$eil-*module* consists of a marked simplicial set together with a (strict) action of the marked simplicial monoid $\mathbb{W}$eil. Let $\mathsf{Mod}^+_{\mathbb{W}\text{eil}}$ denote the category of marked $\mathbb{W}$eil-modules with morphisms given by those maps of marked simplicial sets that commute (strictly) with the action maps.

Proposition 3.2. *There is a simplicial model structure on $\mathsf{Mod}^+_{\mathbb{W}\text{eil}}$ in which the weak equivalences and fibrations are those maps of marked $\mathbb{W}$eil-modules for which the underlying map of marked simplicial sets is a weak equivalence (respectively, a fibration) in the marked model structure on Set^+_Δ.*

Proof. The model structure is an application of [**?**, 4.1] to the marked model structure on marked simplicial sets. The simplicial structure on $\mathsf{Mod}^+_{\mathbb{W}\text{eil}}$ is determined by the simplicial enrichment for Set^+_Δ.   □

We use the model structure of Proposition **??** as our foundation for the appropriate notion of functor between tangent quasi-categories. A tangent quasi-category $\mathbb{Y}$ naturally determines a fibrant marked $\mathbb{W}$eil-module (with the equivalences in $\mathbb{Y}$ as the marked edges). The appropriate notion of (strong) tangent functor $\mathbb{X} \to \mathbb{Y}$ between tangent quasi-categories is then given by a strict map of marked $\mathbb{W}$eil-modules $\mathbb{X}' \to \mathbb{Y}$ where $\mathbb{X}'$ denotes a cofibrant replacement for $\mathbb{X}$ in $\mathsf{Mod}^+_{\mathbb{W}\text{eil}}$. To construct this cofibrant replacement, we employ a standard bar resolution using the free-forgetful adjunction between marked $\mathbb{W}$eil-modules and marked simplicial sets.

Let $\theta : \mathsf{Mod}^+_{\mathbb{W}\text{eil}} \to \mathsf{Set}^+_\Delta$ denote the forgetful functor, and its left adjoint $L : \mathsf{Set}^+_\Delta \to \mathsf{Mod}^+_{\mathbb{W}\text{eil}}$ defined by $L(X) := \mathbb{W}\text{eil} \times X$ be the corresponding free functor. Associated to the monad θL is a bar construction for marked $\mathbb{W}$eil-modules defined by May in [**?**, 9.6].

Definition 3.3. Let $\mathbb{M}$ be a marked $\mathbb{W}$eil-module. The *simplicial bar construction* on $\mathbb{M}$ is the simplicial object in the category of marked $\mathbb{W}$eil-modules

$$B_\bullet\mathbb{M} = B_\bullet(L, \theta L, \theta\mathbb{M})$$

with terms

$$B_k\mathbb{M} := (L\theta)^{k+1}\mathbb{M} = \mathbb{W}\text{eil}^{k+1} \times \mathbb{M}$$

and with face and degeneracy maps determined by the counit and unit maps for the adjunction (L, θ), i.e. by the multiplication and unit maps for the monoid $\mathbb{W}$eil, and the action of $\mathbb{W}$eil on $\mathbb{M}$.

Let $B\mathbb{M}$ denote the 'realization' of the simplicial object $B_\bullet\mathbb{M}$ in the sense of [?, 18.6.2]. This is the marked Weil-module given by the coend

$$B\mathbb{M} := \int_\Delta \Delta^\bullet \times B_\bullet\mathbb{M}$$

and we will refer to $B\mathbb{M}$ as the *bar resolution* of $\mathbb{M}$. We can identify $\mathbb{M}$ with a marked simplicial subset of $B\mathbb{M}$ via the unit map for the monoid Weil, but note that the inclusion $\mathbb{M} \subseteq B\mathbb{M}$ does not respect the Weil-actions.

It follows from [?, 9.6] that the construction in Definition **??** determines a functor $B : \mathsf{Mod}^+_{\mathbb{W}\mathrm{eil}} \to \mathsf{Mod}^+_{\mathbb{W}\mathrm{eil}}$.

Remark 3.4. According to [?, 15.11.6], the underlying simplicial set of the realization $B\mathbb{M}$ is given by the diagonal of the bisimplicial set $B_\bullet\mathbb{M}$. Thus a vertex in $B\mathbb{M}$ is a pair (A, M) consisting of a Weil-algebra A and object $M \in \mathbb{M}$, and an edge in $B\mathbb{M}$ from (A, M) to (A', M') consists of a triple (ϕ_0, ϕ_1, γ) where

$$\phi_0 : A \to A'_0, \quad \phi_1 : A_1 \to A'_1$$

are Weil-algebra morphisms,

$$\gamma : M_2 \to M'$$

is a morphism in $\mathbb{M}$, and $A'_0 \otimes A'_1 = A'$ and $A_1 \cdot M_1 = M$ (writing $\cdot$ for the action of Weil on $\mathbb{M}$). The edge (ϕ_0, ϕ_1, γ) is marked in the marked Weil-module $B\mathbb{M}$ if each of its components is marked; i.e. ϕ_0, ϕ_1 are identity morphisms in Weil, and γ is marked in $\mathbb{M}$.

The counit of the adjunction (L, θ) induces an augmentation for the simplicial object $B_\bullet\mathbb{M}$ over $\mathbb{M}$, and hence a map of simplicial marked Weil-modules

$$\epsilon_\bullet : B_\bullet\mathbb{M} \to \mathbb{M}_\bullet$$

where $\mathbb{M}_\bullet$ denotes the constant simplicial object with value $\mathbb{M}$. Taking realizations we get a map of marked Weil-modules

$$\epsilon : B\mathbb{M} \to \mathbb{M}.$$

Lemma 3.5. *The map $\epsilon : B\mathbb{M} \to \mathbb{M}$ is a functorial cofibrant replacement for the marked Weil-module $\mathbb{M}$ in the model structure of Proposition* **??**.

Proof. We have to show that ϵ is a weak equivalence, and that $B\mathbb{M}$ is cofibrant in $\mathsf{Mod}^+_{\mathbb{W}\mathrm{eil}}$. First, by [?, 9.8], the map of simplicial marked Weil-modules

$$\theta_\bullet\epsilon_\bullet : \theta_\bullet B_\bullet\mathbb{M} = B_\bullet(\theta L, \theta L, \theta\mathbb{M}) \to (\theta\mathbb{M})_\bullet,$$

given by applying θ levelwise to the map $\epsilon_\bullet$, is a simplicial homotopy equivalence in the category of simplicial marked simplicial sets. Taking realizations,

we deduce that $\theta\epsilon : \theta B\mathbb{M} \to \theta\mathbb{M}$ is a simplicial homotopy equivalence in the simplicial model category Set_Δ^+. By [**?**, 9.5.16], it follows that $\theta\epsilon$ is a weak equivalence in Set_Δ^+, and hence that ϵ is a weak equivalence in $\mathsf{Mod}_{\mathbb{W}\mathrm{eil}}^+$.

To show that $B\mathbb{M}$ is cofibrant in $\mathsf{Mod}_{\mathbb{W}\mathrm{eil}}^+$, it is sufficient, by [**?**, 18.6.7], to show that $B_\bullet\mathbb{M}$ is a Reedy cofibrant simplicial object, i.e. that each of the latching maps

$$L_k B_\bullet\mathbb{M} = \mathrm{colim}_{[k]\twoheadrightarrow[i]} B_i\mathbb{M} \to B_k\mathbb{M}$$

is a cofibration in $\mathsf{Mod}_{\mathbb{W}\mathrm{eil}}^+$. The morphisms in this colimit diagram are induced by the unit map for the monoid $\mathbb{W}\mathrm{eil}$, and the latching map is given by applying the free functor L to the inclusion into $\mathbb{W}\mathrm{eil}^k \times \mathbb{M}$ of the union of the marked simplicial subsets $\mathbb{W}\mathrm{eil}^i \times \{\mathbb{N}\} \times \mathbb{W}\mathrm{eil}^{k-1-i} \times \mathbb{M}$ for $i = 0, \ldots, k-1$. The functor L is left adjoint to the right Quillen functor θ so is left Quillen and preserves cofibrations. Thus the latching map above is a cofibration as required. $\square$

We can now introduce a precise notion of tangent functor between tangent quasi-categories.

Definition 3.6. Let $\mathbb{X}$ and $\mathbb{Y}$ be tangent quasi-categories. A *lax tangent functor* $\mathcal{F} : \mathbb{X} \to \mathbb{Y}$ is a map of simplicial sets

$$\mathcal{F} : B\mathbb{X} \to \mathbb{Y}$$

that commutes with the $\mathbb{W}\mathrm{eil}$-actions. We say that $\mathcal{F}$ is a *strong* tangent functor (or simply a *tangent functor*) if $\mathcal{F}$ takes marked edges in $B\mathbb{X}$ to equivalences in $\mathbb{Y}$, that is, if $\mathcal{F}$ is a map of marked $\mathbb{W}\mathrm{eil}$-modules.

The *underlying functor* of a tangent functor $\mathcal{F}$ is given by restricting $\mathcal{F}$ to the simplicial subset $\mathbb{X} \subseteq B\mathbb{X}$. We use the same notation $\mathcal{F}$ for this underlying functor.

Example 3.7. For any tangent quasi-category $\mathbb{X}$, the canonical map $\epsilon : B\mathbb{X} \to \mathbb{X}$ is a tangent functor from $\mathbb{X}$ to itself, which we refer to as the *identity tangent functor on* $\mathbb{X}$. The underlying functor for ϵ is the identity functor on $\mathbb{X}$.

Definition 3.8. We write

$$\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y}) := \mathrm{Fun}_{\mathbb{W}\mathrm{eil}}(B\mathbb{X}, \mathbb{Y}) \subseteq \mathrm{Fun}(B\mathbb{X}, \mathbb{Y})$$

for the simplicial subset in which the n-simplexes are those maps of simplicial sets

$$B\mathbb{X} \to \mathrm{Fun}(\Delta^n, \mathbb{Y})$$

that commute with the $\mathbb{W}\mathrm{eil}$-actions, i.e. the strong tangent functors from $\mathbb{X}$ to $\mathrm{Fun}(\Delta^n, \mathbb{Y})$ equipped with the tangent structure of Example **??**. We refer to $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y})$ as the ∞-*category of tangent functors from* $\mathbb{X} \to \mathbb{Y}$. That terminology is justified by the following result.

Proposition 3.9. *Let $\mathbb{X}$ and $\mathbb{Y}$ be tangent quasi-categories. The simplicial set*

$$\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y})$$

is a quasi-category.

Proof. We claim that the model structure on marked $\mathbb{W}$eil-modules of Proposition **??** is a $\mathsf{Set}_{\Delta}^{+}$-model category in the sense of [**?**, 4.2.18]; the pushout-product axiom can be checked by noting that the (acyclic) cofibrations in $\mathsf{Mod}_{\mathbb{W}\mathrm{eil}}^{+}$ are generated by the free marked $\mathbb{W}$eil-module maps on (acyclic) cofibrations in $\mathsf{Set}_{\Delta}^{+}$. Since $B\mathbb{X}$ and $\mathbb{Y}$ are cofibrant and fibrant marked $\mathbb{W}$eil-modules respectively, it follows that $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y}) = \mathrm{Fun}_{\mathbb{W}\mathrm{eil}}(B\mathbb{X}, \mathbb{Y})$ is fibrant in $\mathsf{Set}_{\Delta}^{+}$. $\square$

We now wish to define composition for tangent functors. It seems unlikely that there is a strictly associative composition operation, in part because quasi-categories themselves do not have a canonical composition for morphisms. In Definition **??** we describe an $(\infty, 2)$-category of tangent ∞-categories that more fully captures the composition of tangent functors, but for now we describe an ad hoc but explicit composition operation for use in constructing differential objects in the next section.

Definition 3.10. Given tangent ∞-categories $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ we define a composition operation

$$\circ : \mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y}) \times \mathrm{Fun}^{\mathrm{tan}}(\mathbb{Y}, \mathbb{Z}) \to \mathrm{Fun}^{\mathrm{tan}}(B\mathbb{X}, \mathbb{Z})$$

which, on n-simplexes, takes the tangent functors $\sigma_0 : B\mathbb{X} \to \mathrm{Fun}(\Delta^n, \mathbb{Y})$ and $\sigma_1 : B\mathbb{Y} \to \mathrm{Fun}(\Delta^n, \mathbb{Z})$ to the tangent functor $BB\mathbb{X} \to \mathrm{Fun}(\Delta^n, \mathbb{Z})$ given by the composite

$$BB\mathbb{X} \xrightarrow{B\sigma_0} B\,\mathrm{Fun}(\Delta^n, \mathbb{Y})$$
$$\xrightarrow{} \mathrm{Fun}(\Delta^n, B\mathbb{Y})$$
$$\xrightarrow{\mathrm{Fun}(\Delta^n, \sigma_1)} \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\Delta^n, \mathbb{Z}))$$
$$\xrightarrow{\cong} \mathrm{Fun}(\Delta^n \times \Delta^n, \mathbb{Z})$$
$$\xrightarrow{\delta^*} \mathrm{Fun}(\Delta^n, \mathbb{Z})$$

where the second map comes from the simplicial structure on the bar resolution functor B, and δ^* is restriction along the diagonal.

The canonical equivalence of marked $\mathbb{W}$eil-modules $\epsilon : B\mathbb{X} \xrightarrow{\sim} \mathbb{X}$ determines an equivalence of ∞-categories

$$\epsilon^* : \mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{tan}}(B\mathbb{X}, \mathbb{Z}).$$

Composing the map $\circ$ with a fixed inverse to ϵ^* then determines a choice of composition operation

$$(3.11) \qquad \mathrm{Fun}^{\tan}(\mathbb{X}, \mathbb{Y}) \times \mathrm{Fun}^{\tan}(\mathbb{Y}, \mathbb{Z}) \to \mathrm{Fun}^{\tan}(\mathbb{X}, \mathbb{Z}).$$

The goal of the remainder of this section is to connect Definition **??** to the Cockett-Cruttwell notion of tangent functor. To do that, it is convenient to reinterpret our definition slightly. First note that the marked $\mathbb{W}$eil-modules $B\mathbb{W}$eil (given by applying Definition **??** to the action of $\mathbb{W}$eil on itself) can be viewed as a marked $\mathbb{W}$eil-*bimodule* with right action determined by the action of $\mathbb{W}$eil on the rightmost copy of itself in the simplicial bar construction. For two (left) $\mathbb{W}$eil-modules $\mathbb{X}, \mathbb{Y}$, the functor quasi-category $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ is also a $\mathbb{W}$eil-bimodule with right action determined by the module structure on $\mathbb{X}$ and left action by the module structure on $\mathbb{Y}$.

Lemma 3.12. *Let $\mathbb{X}$ and $\mathbb{Y}$ be tangent quasi-categories. There is a one-to-one correspondence between maps of marked $\mathbb{W}$eil-modules $\mathcal{F} : B\mathbb{X} \to \mathbb{Y}$ and maps of marked $\mathbb{W}$eil-bimodules*

$$\underline{\mathcal{F}} : B\mathbb{W}\mathrm{eil} \to \mathrm{Fun}(\mathbb{X}, \mathbb{Y})$$

where the marked edges in the quasi-category $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ are the equivalences.

Proof. This claim is based on an isomorphism of marked $\mathbb{W}$eil-modules

$$B\mathbb{X} \cong B\mathbb{W}\mathrm{eil} \times_{\mathbb{W}\mathrm{eil}} \mathbb{X}$$

where the right-hand side denotes a coend over the the actions of $\mathbb{W}$eil on $B\mathbb{W}$eil (on the right) and $\mathbb{X}$ (on the left). That isomorphism in turn comes from isomorphisms at each simplicial level

$$\mathbb{W}\mathrm{eil}^{k+1} \times \mathbb{X} \cong \mathbb{W}\mathrm{eil}^{k+2} \times_{\mathbb{W}\mathrm{eil}} \mathbb{X}$$

by taking the diagonal of these bisimplicial sets. $\square$

Remark 3.13. We can use the formulation suggested by Lemma **??** to describe more explicitly the structure of a tangent functor $\mathcal{F}$ between tangent quasi-categories $(\mathbb{X}, T)$ and $(\mathbb{Y}, U)$. Applying $\underline{\mathcal{F}}$ to the object $(\mathbb{N}, \mathbb{N})$ of $B\mathbb{W}$eil yields the underlying functor $\mathcal{F} : \mathbb{X} \to \mathbb{Y}$. Applying $\underline{\mathcal{F}}$ to another object (A, A') of $B\mathbb{W}$eil we obtain another functor which, since $\underline{\mathcal{F}}$ is a $\mathbb{W}$eil-bimodule map, is necessarily given by the composite

$$U^A \mathcal{F} T^{A'} : \mathbb{X} \to \mathbb{Y}.$$

The edges in the simplicial set $B\mathbb{W}$eil determine natural maps between these functors. In particular, applying $\underline{\mathcal{F}}$ to the morphism $e_A : (\mathbb{N}, A) \to (A, \mathbb{N})$

given by the triple of Weil-algebra identity maps $(1_{\mathbb{N}}, 1_A, 1_{\mathbb{N}})$ yields a natural transformation

$$\alpha^A : \mathcal{F}T^A \to U^A\mathcal{F}.$$

The edge e_A is marked in $B\mathbb{W}$eil, so if $\mathcal{F}$ is a strong tangent functor, the natural transformation α^A is required to be an equivalence in $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$. These equivalences witness the requirement that the underlying functor $\mathcal{F}$ commute with the tangent structures on $\mathbb{X}$ and $\mathbb{Y}$, and the required higher coherences are indexed by the structure of the simplicial set $B\mathbb{W}$eil.

The description given in Remark **??** highlights the connection between our notion of tangent functor and the Cockett and Cruttwell's morphisms of tangent structure. Recall that a *strong morphism*, in [**?**, 2.7], between tangent categories $(\mathbb{X}, T)$ and $(\mathbb{Y}, U)$ consists of a functor $\mathcal{F} : \mathbb{X} \to \mathbb{Y}$, which preserves the tangent pullbacks and commutes with the tangent structure maps T and U, up to a natural isomorphism $\alpha : \mathcal{F}T \cong U\mathcal{F}$. A *tangent transformation* [**?**, Def. 5] between strong tangent functors $(\mathcal{F}, \alpha)$ and $(\mathcal{F}', \alpha')$ is a natural transformation $\psi : \mathcal{F} \to \mathcal{F}'$ such that $(U\psi)\alpha = \alpha'(\psi T)$. The strong tangent morphisms from $(\mathbb{X}, T)$ to $(\mathbb{Y}, U)$, together with these tangent transfomations, form a category which we can now relate to our previous definitions.

Proposition 3.14. *Let $(\mathbb{X}, T)$ and $(\mathbb{Y}, U)$ be tangent categories, viewed as tangent quasi-categories via the identification of Lemma **??**. Then $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{X}, \mathbb{Y})$ is a category, and is equivalent to the category whose objects are the strong morphisms of tangent categories and whose morphisms are the tangent transformations.*

Proof. Garner's proof of [**?**, Thm. 9] determines a correspondence between Cockett and Cruttwell's strong morphisms of tangent structure and maps of Weil-actegories from $\mathbb{X}$ to $\mathbb{Y}$. Such a map consists of a functor $\mathcal{F} : \mathbb{X} \to \mathbb{Y}$ and natural isomorphisms

$$\alpha^A : \mathcal{F}T^A \xrightarrow{\cong} U^A\mathcal{F},$$

for $A \in \mathbb{W}$eil, such that the following diagrams in $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ commute: for a Weil-algebra morphism $\phi : A \to A'$

$$
\begin{array}{ccc}
\mathcal{F}T^A & \xrightarrow{\alpha^A} & U^A\mathcal{F} \\
{\scriptstyle \mathcal{F}T^\phi}\downarrow & & \downarrow{\scriptstyle U^\phi F} \\
\mathcal{F}T^{A'} & \xrightarrow{\alpha^{A'}} & U^{A'}\mathcal{F}
\end{array}
$$

where we will denote the diagonal composite by α^ϕ, and for a pair of Weil-algebras A, A':

$$
\begin{array}{ccccc}
\mathcal{F}T^A T^{A'} & \xrightarrow{\alpha^A T^{A'}} & U^A \mathcal{F}T^{A'} & \xrightarrow{U^A \alpha^{A'}} & U^A U^{A'} \mathcal{F} \\
\| & & & & \| \\
\mathcal{F}T^{A \otimes A'} & & \xrightarrow{\quad \alpha^{A \otimes A'} \quad} & & U^{A \otimes A'} \mathcal{F}
\end{array}
$$

and a unit condition

$$
\begin{array}{ccc}
 & \mathcal{F} & \\
 & \diagup \quad \| & \\
\mathcal{F}T^{\mathbb{N}} & \xrightarrow{\quad \alpha^{\mathbb{N}} \quad} & U^{\mathbb{N}} \mathcal{F}
\end{array}
$$

Such information can be identified with part of the data of a $\mathbb{W}$eil-bimodule map $\underline{\mathcal{F}} : B\mathbb{W}\mathrm{eil} \to \mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ as in Remark **??**. The first diagram above corresponds to a diagram in the simplicial set $B\mathbb{W}\mathrm{eil}$, and the remaining diagrams follow from the condition that $\underline{\mathcal{F}}$ is a map of bimodules. It remains to show that these partial data extend uniquely to a full bimodule map $\underline{\mathcal{F}}$.

Such a map $\underline{\mathcal{F}}$ is determined by the bimodule condition to be given on vertices by

$$\underline{\mathcal{F}}(A_0, A_1) := U^{A_0} \mathcal{F}T^{A_1}$$

and on the edge $(\sigma_0, \sigma_1, \sigma_2) : (A_0^0, A_1^0 \otimes A_2^0) \to (A_0^1 \otimes A_1^1, A_2^1)$ by

$$U^{A_0^0} \mathcal{F}T^{A_1^0 \otimes A_2^0} \xrightarrow{U^{\sigma_0} \alpha^{\sigma_1} T^{\sigma_2}} U^{A_0^1 \otimes A_1^1} \mathcal{F}T^{A_2^1}.$$

A map of simplicial sets *into* a category is uniquely determined by where it sends vertices and edges, so such an extension $\underline{\mathcal{F}}$ is unique if it exists. To see that it does exist, we can deduce an explicit formula as follows.

An n-simplex σ in $B\mathbb{W}\mathrm{eil}$ consists of a sequence $(\sigma_0, \ldots, \sigma_{n+1})$ of n-simplexes in $\mathbb{W}\mathrm{eil}$, each of which we will write as a sequence of Weil-algebras morphisms:

$$\sigma_i : A_i^0 \xrightarrow{\sigma_i^1} A_i^1 \xrightarrow{\sigma_i^2} \ldots \xrightarrow{\sigma_i^n} A_i^n.$$

Given the data of a $\mathbb{W}$eil-actegory map, we define $\underline{\mathcal{F}}(\sigma)$ to be the n-simplex in the category $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ given by the sequence

$$U^{A_0^0 \otimes \cdots \otimes A_n^0} \mathcal{F}T^{A_{n+1}^0} \to U^{A_0^1 \otimes \cdots \otimes A_{n-1}^1} \mathcal{F}T^{A_n^1 \otimes A_{n+1}^1} \cdots \to U^{A_0^n} \mathcal{F}T^{A_1^n \otimes \cdots \otimes A_{n+1}^n}$$

consisting of the natural transformations

$$U^{\sigma_0^i \otimes \cdots \otimes \sigma_{n-i}^i} \alpha^{\sigma_{n-i+1}^i} T^{\sigma_{n-i+2}^i \otimes \cdots \otimes \sigma_{n+1}^i}$$

for $i = 1, \ldots, n$.

We can use the commutative diagrams laid out above to show that $\underline{\mathcal{F}}$ commutes with the face and degeneracy maps, so determines a map of simplicial sets $B\mathbb{W}\mathrm{eil} \to \mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ as desired. We can also check that $\underline{\mathcal{F}}$ is a map of $\mathbb{W}\mathrm{eil}$-bimodules. Finally, the marked edges in $B\mathbb{W}\mathrm{eil}$ are those of the form $(1_{A_0}, 1_{A_1}, 1_{A_2})$, and these are sent by $\underline{\mathcal{F}}$ to equivalences in $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ because α^{A_1} is an isomorphism. Therefore $\underline{\mathcal{F}}$ is a map of marked $\mathbb{W}\mathrm{eil}$-bimodules and corresponds by Lemma **??** to the desired tangent functor $\mathbb{X} \to \mathbb{Y}$. $\square$

Remark 3.15. We can extend many of the results of this section to lax tangent functors by ignoring the requirement that markings are preserved. In particular, a lax tangent functor between tangent categories, in the sense described in Definition **??**, determines a (not necessarily strong) 'morphism of tangent structure' in the sense of Cockett and Cruttwell [**?**, 2.7].

4. Differential objects in cartesian tangent ∞-categories

Having established the basic definitions of tangent ∞-categories and tangent functors, we now start to generalize the broader theory developed by Cockett and Cruttwell to the ∞-categorical setting. In this section, we consider the notion of 'differential object' introduced in [**?**, Def. 4.8] (and strengthened slightly in [**?**, 3.1]) to describe the connection between tangent categories and cartesian differential categories.

Roughly speaking, the differential objects in a tangent category $\mathbb{X}$ are the tangent *spaces*, that is the fibres T_xM of the tangent bundle projections over (generalized) points $x : * \to M$ of objects in $\mathbb{X}$. In the tangent category of smooth manifolds these differential objects are the Euclidean vector spaces $\mathbb{R}^n$; in the tangent category of schemes they are the affine spaces $\mathbb{A}^n = \mathrm{Spec}\,\mathbb{Z}[x_1, \ldots, x_n]$.

Our first goal in this section is to generalize Cockett and Cruttwell's definition of differential object to tangent ∞-categories. We do that by describing a tangent category $\mathbb{N}^\bullet$ which *represents* differential objects in the following sense. We show in **??** that differential objects in a tangent category $\mathbb{X}$ correspond to (product-preserving) tangent functors $\mathbb{N}^\bullet \to \mathbb{X}$. We use that characterization as the basis for our definition (**??**) of differential object in a tangent ∞-category.

We then prove Proposition **??**, our analogue of [**?**, 4.15], which provides a canonical structure of a differential object on each tangent space in a tangent ∞-category. In Corollary **??** we note that conversely any object that admits a differential structure is equivalent to a tangent space.

Finally in this section we turn to the connection between differential objects and the cartesian differential categories of Blute, Cockett and Seely [**?**]. In **??**, we prove a generalization of [**?**, 4.11] by showing that from every (cartesian) tangent ∞-category $\mathbb{X}$ we can construct a cartesian differential category whose objects are the differential objects of $\mathbb{X}$, and whose morphisms come from those in the homotopy category of $\mathbb{X}$.

We start by recalling Cockett and Cruttwell's definition of differential object, which they give in the context of tangent categories which are *cartesian* in the following sense.

Definition 4.1. A tangent category $(\mathbb{X}, T)$ is *cartesian* if $\mathbb{X}$ admits finite products (including a terminal object which we denote $*$), and the tangent bundle functor $T : \mathbb{X} \to \mathbb{X}$ preserves those products.

A *differential object* in a cartesian tangent category $(\mathbb{X}, T)$ consists of

- an object D in $\mathbb{X}$ (the *underlying object*),
- morphisms $\sigma : D \times D \to D$ and $\zeta : * \to D$ that provide D with the structure of a commutative monoid in $\mathbb{X}$, and
- a morphism $\hat{p} : T(D) \to D$,

such that

- the map $\langle p, \hat{p} \rangle : T(D) \to D \times D$ is an isomorphism, and
- the five diagrams listed in [**?**, 3.1] commute.

A *morphism of differential objects* is a morphism of underlying objects that commutes with the differential structure maps σ, ζ and $\hat{p}$. We denote by $\mathbb{D}\text{iff}(\mathbb{X})$ the category of differential objects in $\mathbb{X}$ and their morphisms. (Warning: this notation conflicts with that used by Cockett and Cruttwell in [**?**, 4.11] where they consider *all* morphisms between underlying objects, not only those that commute with the structure maps. We will write $\widehat{\mathbb{D}\text{iff}}(\mathbb{X})$ when we need to refer to that larger category.)

Our first goal is to produce a characterization of differential objects that is more amenable to translation into the context of tangent ∞-categories.

Definition 4.2. Let $\mathbb{N}^\bullet$ denote the following category:

- an object of $\mathbb{N}^\bullet$ is a free finitely-generated module over the semi-ring $\mathbb{N}$ (i.e. a commutative monoid) together with a chosen (unordered) basis; we can identify an object with $\mathbb{N}^J$ for some finite set J;

- a morphism $M \to M'$ in $\mathbb{N}^\bullet$ is an $\mathbb{N}$-linear map, i.e. $f : M \to M'$ that satisfies $f(0_M) = 0_{M'}$ and $f(x+y) = f(x) + f(y)$ for all $x, y \in M$.

The category $\mathbb{N}^\bullet$ is precisely the Lawvere theory for commutative monoids, so that commutative monoids in a category $\mathbb{X}$ correspond to product-preserving functors $\mathbb{N}^\bullet \to \mathbb{X}$.

We define a tangent structure on the category $\mathbb{N}^\bullet$ by the map

$$\mathbb{W}\mathrm{eil} \times \mathbb{N}^\bullet \to \mathbb{N}^\bullet; \quad (A, M) \mapsto A \otimes M.$$

The chosen basis for the tensor product $A \otimes M$ is

$$\{x_{i_1} \cdots x_{i_r} \otimes m_j\}$$

where $\{m_1, \ldots, m_k\}$ is the chosen basis of M, and $x_{i_1} \cdots x_{i_r}$ are the nonzero monomials in the Weil-algebra A.

Lemma 4.3. *The map $\mathbb{W}\mathrm{eil} \times \mathbb{N}^\bullet \to \mathbb{N}^\bullet$ determines a cartesian tangent structure on the category $\mathbb{N}^\bullet$ with tangent bundle functor*

$$T_{\mathbb{N}^\bullet} : \mathbb{N}^\bullet \to \mathbb{N}^\bullet; \quad M \mapsto W \otimes M = M\{1, x\}.$$

Proof. To make the tensor product into a strict action of $\mathbb{W}\mathrm{eil}$ on the category $\mathbb{N}^\bullet$, we identify $\mathbb{N} \otimes M$ with M, and $A' \otimes (A \otimes M)$ with $(A' \otimes A) \otimes M$. Notice that each nonzero monomial in $A' \otimes A$ can be identified uniquely with a product of a nonzero monomial in A' and a nonzero monomial in A.

Each of the tangent pullbacks in $\mathbb{W}\mathrm{eil}$ (**??,??**) is a pullback in the underlying category of $\mathbb{N}$-modules, and the tensor product of free $\mathbb{N}$-modules preserves those pullbacks. Therefore the structure map described above makes $\mathbb{N}^\bullet$ into a tangent category.

Finally, we note that $\mathbb{N}^\bullet$ has finite products, and these are preserved by the tangent bundle functor $T_{\mathbb{N}^\bullet}$. Therefore the tangent structure is cartesian. $\square$

Lemma 4.4. *The commutative monoid structure provided by addition and zero, and the projection map*

$$\hat{p} : T_{\mathbb{N}^\bullet}(\mathbb{N}) = \mathbb{N}\{1, x\} \to \mathbb{N}; \quad 1 \mapsto 0, \quad x \mapsto 1,$$

make $\mathbb{N}$ into a differential object in the tangent category $\mathbb{N}^\bullet$.

Proof. The projection $p : T_{\mathbb{N}^\bullet}(\mathbb{N}) \to \mathbb{N}$ is given by $p(x) = 0$, $p(1) = 1$, so it is immediately clear that $\langle p, \hat{p} \rangle : T_{\mathbb{N}^\bullet}(\mathbb{N}) \to \mathbb{N} \times \mathbb{N}$ is an isomorphism. The remaining conditions are that certain diagrams in the category $\mathbb{N}^\bullet$ must commute, and these are readily checked by looking at basis elements. $\square$

Differential objects are represented. The first main result of this section is that differential objects in any cartesian tangent category $\mathbb{X}$ can be detected by tangent functors $\mathbb{N}^\bullet \to \mathbb{X}$ in the sense of Definition **??**. Recall from Proposition **??** that these tangent functors, i.e. the objects of $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{N}^\bullet, \mathbb{X})$, can be identified with the (strong) morphisms of tangent structure defined by Cockett and Cruttwell.

Proposition 4.5. *Let $(\mathbb{X}, T)$ be a cartesian tangent category, and let*

$$\mathrm{Fun}^{\mathrm{tan},\times}(\mathbb{N}^\bullet, \mathbb{X}) \subseteq \mathrm{Fun}^{\mathrm{tan}}(\mathbb{N}^\bullet, \mathbb{X})$$

be the full subcategory on those tangent functors which preserve finite products. Then there is an equivalence of categories

$$e : \mathrm{Fun}^{\mathrm{tan},\times}(\mathbb{N}^\bullet, \mathbb{X}) \xrightarrow{\sim} \mathbb{D}\mathrm{iff}(\mathbb{X})$$

*which maps a product-preserving tangent functor $\mathcal{D} : \mathbb{N}^\bullet \to \mathbb{X}$ to the object $\mathcal{D}(\mathbb{N})$ together with a differential structure induced from that on $\mathbb{N}$ in Lemma **??**.*

Proof. Any product-preserving tangent functor preserves differential objects, and the components of a tangent transformation are morphisms that commute with the differential structure maps σ, ζ and $\hat{p}$. We therefore have a functor e as claimed.

To see that e is fully faithful, take product-preserving tangent functors $(\mathcal{D}, \alpha)$ and $(\mathcal{D}', \alpha') : \mathbb{N}^\bullet \to \mathbb{X}$. Recall from the discussion before Proposition **??** that a tangent transformation $\beta : (\mathcal{D}, \alpha) \to (\mathcal{D}', \alpha')$ is a natural transformation $\beta : \mathcal{D} \to \mathcal{D}'$ such that the following diagram commutes

$$(4.6)$$

$$
\begin{array}{ccc}
\mathcal{D}T_{\mathbb{N}^\bullet} & \xrightarrow{\ \alpha\ } & T\mathcal{D} \\
{\scriptstyle \beta T_{\mathbb{N}^\bullet}}\Big\downarrow & & \Big\downarrow{\scriptstyle T\beta} \\
\mathcal{D}'T_{\mathbb{N}^\bullet} & \xrightarrow{\ \alpha'\ } & T\mathcal{D}'
\end{array}
$$

where T denotes the tangent bundle functor on $\mathbb{X}$.

Since all the functors in (**??**) preserve finite products, that diagram commutes if and only if it does so on its $\mathbb{N}$-component, and the natural transformation β is uniquely determined by the component $\beta_{\mathbb{N}} : \mathcal{D}(\mathbb{N}) \to \mathcal{D}'(\mathbb{N})$, which must be a map of commutative monoids in $\mathbb{X}$.

Therefore, the natural transformation β is a tangent transformation (i.e. the diagram above commutes) if and only if $\beta_{\mathbb{N}}$ commutes with the differential

structure map $\hat{p}$. Thus e determines a bijection between tangent transformations $(\mathcal{D}, \alpha) \to (\mathcal{D}', \alpha')$ and morphisms of differential objects $\mathcal{D}(\mathbb{N}) \to \mathcal{D}'(\mathbb{N})$.

To see that e is essentially surjective, consider a differential object $(D, \sigma, \zeta, \hat{p})$ in $\mathbb{X}$. The commutative monoid (D, σ, ζ) determines (uniquely up to isomorphism) a product-preserving functor $\mathcal{D} : \mathbb{N}^\bullet \to \mathbb{X}$ with $\mathcal{D}(\mathbb{N}^J) \cong D^J$.

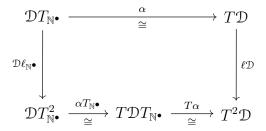To make $\mathcal{D}$ into a tangent functor, we have to define a natural isomorphism

$$\alpha : \mathcal{D}T_{\mathbb{N}^\bullet} \to T\mathcal{D}$$

that commutes with the tangent structure maps. We define the component

$$\alpha_\mathbb{N} : \mathcal{D}T_{\mathbb{N}^\bullet}(\mathbb{N}) = \mathcal{D}(W) \cong D^{\{1,x\}} \xrightarrow{\sim} T(D) = T\mathcal{D}(\mathbb{N})$$

of α to be the inverse of the isomorphism $\langle p, \hat{p} \rangle$ of Definition **??**. Other components are determined by the naturality requirement and the fact that all functors involved preserve finite products. Thus we obtain the natural isomorphism α.

To see that $(\mathcal{D}, \alpha)$ is a tangent functor, we check the conditions of [**?**, 2.7]. We will write out the proof for the commutative diagram involving the vertical lift ℓ; the other conditions are much easier to verify. We must show that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{D}T_{\mathbb{N}^\bullet} & \xrightarrow[\cong]{\alpha} & T\mathcal{D} \\
{\scriptstyle \mathcal{D}\ell_{\mathbb{N}^\bullet}} \downarrow & & \downarrow {\scriptstyle \ell\mathcal{D}} \\
\mathcal{D}T_{\mathbb{N}^\bullet}^2 \xrightarrow[\cong]{\alpha T_{\mathbb{N}^\bullet}} & T\mathcal{D}T_{\mathbb{N}^\bullet} \xrightarrow[\cong]{T\alpha} & T^2\mathcal{D}
\end{array}
$$

Since all functors in this diagram preserve finite products, it is sufficient to look at the diagram of components at the object $\mathbb{N} \in \mathbb{N}^\bullet$. This diagram in $\mathbb{X}$ takes the form

$$
\begin{array}{ccc}
D^2 & \xleftarrow[\cong]{\langle p, \hat{p} \rangle} & T(D) \\
{\scriptstyle \langle \pi_1, \zeta!, \zeta!, \pi_2 \rangle} \downarrow & & \downarrow {\scriptstyle \ell} \\
D^4 \xleftarrow[\cong]{\langle p\pi_1, \hat{p}\pi_1, p\pi_2, \hat{p}\pi_2 \rangle} & T(D)^2 \xleftarrow[\cong]{\langle T(p), T(\hat{p}) \rangle} & T^2(D)
\end{array}
$$

where we have identified the terms D^2, $T(D)^2$ and D^4 by choosing an order on the basis elements of W (we use $\{1, x\}$) and $W \otimes W$ (we use $\{1, x_1, x_2, x_1x_2\}$), and we are writing ! for the unique map to the terminal object. We check that this diagram commutes by looking at each of the four components in turn:

$$pT(p)\ell = p0p = p, \quad \hat{p}T(p)\ell = \hat{p}0p = \zeta!, \quad pT(\hat{p})\ell = \hat{p}p_T\ell = \zeta!, \quad \hat{p}T(\hat{p})\ell = \hat{p}.$$

These equations follow from the definition of differential object in [**?**, 3.1]. In particular, the last equation is precisely the 'extra' axiom that was added to the definition of differential object in [**?**].

The remaining four diagrams in [**?**, 2.7] are verified in a similar manner. To see that $(\mathcal{D}, \alpha)$ is a strong morphism of tangent categories, we also need to show that the functor $\mathcal{D} : \mathbb{N}^J \mapsto D^J$ preserves the tangent pullbacks in $\mathbb{N}^\bullet$. Each of those pullbacks can be explicitly described using basis elements, and its preservation follows from the form of $\mathcal{D}$.

Finally we check that the differential structure on $e(\mathcal{D}) = \mathcal{D}(\mathbb{N}) = D$ induced by that on $\mathbb{N}$ in Lemma **??** agrees with the original structure $(D, \sigma, \zeta, \hat{p})$. For the commutative monoid structure this is a standard fact about models for the Lawvere theory $\mathbb{N}^\bullet$. For the projection map $\hat{p}$ we have to check that the map

$$T(D) = T\mathcal{D}(\mathbb{N}) \xrightarrow{\alpha_{\mathbb{N}}^{-1}} \mathcal{D}T_{\mathbb{N}^\bullet}(\mathbb{N}) = \mathcal{D}(W) \xrightarrow{\mathcal{D}(\hat{p}_{\mathbb{N}})} \mathcal{D}(\mathbb{N}) = D$$

agrees with $\hat{p}$, which it does. Thus e is essentially surjective and is an equivalence of categories as claimed. $\square$

We now turn to tangent ∞-categories. Motivated by **??**, we make the following definitions.

Definition 4.7. We say that a tangent ∞-category $\mathbb{X}$ is *cartesian* if $\mathbb{X}$ admits finite products which are preserved by the tangent bundle functor $T : \mathbb{X} \to \mathbb{X}$. A *differential object* in a cartesian tangent ∞-category $\mathbb{X}$ is a (strong) tangent functor

$$\mathcal{D} : \mathbb{N}^\bullet \to \mathbb{X}$$

whose underlying functor preserves finite products. The *underlying object* of such $\mathcal{D}$ is the object $D := \mathcal{D}(\mathbb{N}) \in \mathbb{X}$. We will also say that $\mathcal{D}$ is a *differential structure* on the object D.

We write $\mathbb{D}\mathrm{iff}(\mathbb{X}) := \mathrm{Fun}^{\mathrm{tan}, \times}(\mathbb{N}^\bullet, \mathbb{X})$ for the ∞-*category of differential objects in* $\mathbb{X}$, the full subcategory of $\mathrm{Fun}^{\mathrm{tan}}(\mathbb{N}^\bullet, \mathbb{X})$ whose objects are those tangent functors that preserve finite products.

Differential structure on tangent spaces. The next main goal of this section is to identify differential objects in a tangent ∞-category with the 'tangent spaces'. This part of the paper parallels [**?**, Sec. 4.4].

Definition 4.8. Let $(\mathbb{X}, T)$ be a cartesian tangent ∞-category. For a pointed object in $\mathbb{X}$, that is a morphism $x : * \to C$ in $\mathbb{X}$ where $*$ denotes a terminal object, the *tangent space to C at x* is the pullback

$$
\begin{array}{ccc}
T_x C & \longrightarrow & T(C) \\
\downarrow & & \downarrow {\scriptstyle p_C} \\
* & \xrightarrow{\ x\ } & C
\end{array}
$$

if that pullback exists and is preserved by each $T^A : \mathbb{X} \to \mathbb{X}$.

We wish to construct a differential object in $\mathbb{X}$ whose underlying object is the tangent space $T_x C$, i.e. a product-preserving tangent functor $\mathcal{T}_x C : \mathbb{N}^\bullet \to \mathbb{X}$ with $\mathbb{N} \mapsto T_x C$. In the next paragraph we will describe that tangent functor informally, before turning to a more precise construction in Definition **??** below.

Definition 4.9. Let $x : * \to C$ be a pointed object in a cartesian tangent ∞-category $\mathbb{X}$, such that the tangent space $T_x C$ exists (and is preserved by each $T^A : \mathbb{X} \to \mathbb{X}$). We define a functor

$$\mathcal{T}_x C : \mathbb{N}^\bullet \to \mathbb{X}$$

by the pullbacks

$$
\mathcal{T}_x C(M) := \lim \left(
\begin{array}{ccc}
 & & T^{W^M}(C) \\
 & & \downarrow \\
* & \xrightarrow{\ x\ } & C
\end{array}
\right)
$$

where W^M is the Weil-algebra given by the square-zero extension $\mathbb{N} \oplus M$ of the $\mathbb{N}$-module M.

More precisely, we define W^M by fixing an order $(m_1, \ldots, m_k)$ for the chosen basis of M and setting $W^M := W^k$. Via these orderings of the chosen bases, a morphism $\mu : M \to M'$ induces a Weil-algebra morphism $W^\mu : W^M \to W^{M'}$, and hence induces a map $\mathcal{T}_x C(M) \to \mathcal{T}_x C(M')$.

Note that acccording to this notation we have $W^{\mathbb{N}} = W$, and so $T^{W^{\mathbb{N}}} = T^W$. Therefore the underlying object of the proposed differential object $\mathfrak{T}_x C : \mathbb{N}^{\bullet} \to \mathbb{X}$ is indeed the tangent space $T_x C$.

Lemma 4.10. *The pullbacks required by Definition ?? exist, and the resulting functor $\mathfrak{T}_x C$ preserves finite products.*

Proof. The ∞-category $\mathbb{X}$ admits finite products, and so it is sufficient to show that $(T_x C)^k$ provides the desired pullback in the case that $M = \mathbb{N}^k$. Since the tangent structure on $\mathbb{X}$ preserves foundational pullbacks, we have

$$T^{W^k}(C) \simeq T(C) \times_C \cdots \times_C T(C)$$

and so we define

$$(T_x C)^k \to T^{W^k}(C)$$

via the maps

$$(T_x C)^k \xrightarrow{\pi_i} T_x C \to T(C)$$

for $i = 1, \ldots, k$. To see that the resulting square

(4.11)
$$\begin{array}{ccc} (T_x C)^k & \longrightarrow & T^{W^k}(C) \\ \downarrow & & \downarrow \\ * & \xrightarrow{x} & C \end{array}$$

is a pullback in $\mathbb{X}$, we observe that this square is the pullback power of k copies of the map of pullback squares

$$\begin{array}{ccc} T_x C & \longrightarrow & T(C) \\ \downarrow & & \downarrow \\ * & \xrightarrow{x} & C \end{array} \quad \to \quad \begin{array}{ccc} * & \xrightarrow{x} & C \\ \downarrow & & \downarrow \\ * & \xrightarrow{x} & C \end{array}$$

induced by the projection $p_C : T(C) \to C$, and so (??) is itself a pullback. $\square$

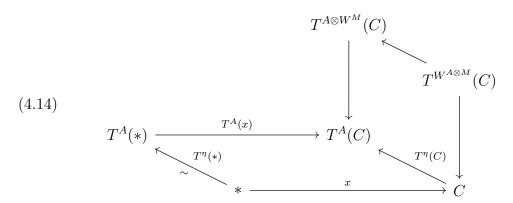**Definition 4.12.** For a Weil-algebra A, we now construct a natural equivalence

$$\alpha_A : T^A(\mathfrak{T}_x C) \xrightarrow{\sim} (\mathfrak{T}_x C)T^A_{\mathbb{N}^{\bullet}}.$$

which forms part of the tangent structure on the functor $\mathcal{T}_x C$. The M-component of α is given by the composite of two equivalences

$$
(4.13) \quad T^A \lim \left(\begin{array}{c} T^{W^M}(C) \\ \downarrow \\ * \xrightarrow{\ x\ } C \end{array} \right) \xrightarrow{\ \sim\ } \lim \left(\begin{array}{c} T^A T^{W^M}(C) \\ \downarrow \\ T^A(*) \xrightarrow{\ T^A(x)\ } T^A(C) \end{array} \right)
$$

$$
\xleftarrow{\ \sim\ } \lim \left(\begin{array}{c} T^{W^{A \otimes M}}(C) \\ \downarrow \\ * \xrightarrow{\ x\ } C \end{array} \right)
$$

where the first map is the canonical map $T^A \lim \to \lim T^A$, and the second map (note the backwards direction, and that $T^A T^{W^M} = T^{A \otimes W^M}$ since the tangent structure is a strict monoidal functor) is induced by the following diagram in $\mathbb{X}$:

$$
(4.14)
$$

The bottom-left square in this diagram is a naturality square for the natural transformation $T^\eta : I \to T^A$ associated to the unit map $\eta : \mathbb{N} \to A$, and the top-right square is determined by the following diagram of Weil-algebras, applied via the tangent structure on $\mathbb{X}$ to the object C:

$$
(4.15) \quad
\begin{array}{ccc}
W^{A \otimes M} & \xrightarrow{\ \theta\ } & A \otimes W^M \\
\downarrow & & \downarrow \\
\mathbb{N} & \xrightarrow{\ \eta\ } & A
\end{array}
$$

In this diagram, the map θ sends the generator $x_{i_1} \cdots x_{i_k} \otimes m_j$ in $W^{A \otimes M}$ to the element of the tensor product $A \otimes W^M$ given by the same expression. Note that the square (??) reduces to the vertical lift pullback of (??) in the case $A = W$ and $M = \mathbb{N}$.

In order to complete the construction of the natural equivalence α, we prove the following result.

Lemma 4.16. *The two maps in (??) are equivalences.*

Proof. For the first map we have to show that for each Weil-algebra A, the functor T^A preserves the pullback square (??). Since $\mathbb{X}$ is a cartesian tangent ∞-category, T^A commutes with finite products, so we have

$$T^A((T_xC)^k) \simeq (T^A T_x C)^k.$$

Since a tangent structure preserves foundational pullbacks, we have

$$T^A T^{W^k}(C) \simeq T^A TC \times_{T^A C} \cdots \times_{T^A C} T^A TC.$$

Therefore, as in the proof of Lemma **??**, we can identify the square obtained by applying T^A to (??) with the wide pullback of k copies of the map of pullback squares

$$
\begin{array}{ccc}
T^A T_x C & \longrightarrow & T^A TC \\
\downarrow & & \downarrow \\
* & \xrightarrow{\ x\ } & T^A C
\end{array}
\qquad \rightarrow \qquad
\begin{array}{ccc}
* & \xrightarrow{\ x\ } & T^A C \\
\downarrow & & \downarrow \\
* & \xrightarrow{\ x\ } & T^A C
\end{array}
$$

which is therefore also a pullback.

For the second map in (??) we have to show that the diagram (??) induces an equivalence on pullbacks. Equivalently, it is sufficient to show that the top-right square in that diagram is a pullback in $\mathbb{X}$. We do this by arguing that, for each $A \in \mathbb{W}\mathrm{eil}$ and $M \in \mathbb{N}^\bullet$, the square (??) is a pullback in $\mathbb{W}\mathrm{eil}$ that is preserved by the tangent structure map $T : \mathbb{W}\mathrm{eil} \to \mathrm{End}(\mathbb{X})$.
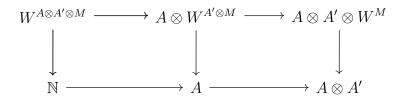
As indicated above, when $A = W$ and $M = \mathbb{N}$, the diagram (??) is simply the vertical lift pullback (??) which of course is preserved by any tangent structure. We build up the general case inductively from this observation.

We start with the case $A = W^n$ and $M = \mathbb{N}$ for arbitrary $n \geq 0$. The diagram (??) is then the pullback power of n copies of the corresponding diagram for $A = W$ and $M = \mathbb{N}$, over the diagram for $A = \mathbb{N}$ and $M = \mathbb{N}$ (which is trivially a pullback preserved by T). That claim relies on various examples of

the foundational pullbacks in $\mathbb{W}$eil (**??**). Thus (**??**) is a pullback in that case, and since a tangent structure T preserves all of these foundational pullbacks, we deduce that it also preserves the pullback (**??**).

Next we turn to the case $A = W^n$ and $M = \mathbb{N}^k$ for arbitrary $n, k \geq 0$. In this case (**??**) is the pullback power of k copies of the case $A = W^n$ and $M = \mathbb{N}$ over the case $A = W^n$ and $M = 0$ (which again is trivially a pullback preserved by T). Once again that claim depends on the foundational pullbacks, and we again deduce that (**??**) is a pullback in $\mathbb{W}$eil that is preserved by T.

Finally, we deduce the general case by showing that if T preserves (**??**) for Weil-algebras A and A' (and arbitrary M), then it also does so for $A \otimes A'$. To see that, consider the diagram

$$
\begin{array}{ccccc}
W^{A \otimes A' \otimes M} & \longrightarrow & A \otimes W^{A' \otimes M} & \longrightarrow & A \otimes A' \otimes W^M \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{N} & \longrightarrow & A & \longrightarrow & A \otimes A'
\end{array}
$$

Each of these individual squares is a pullback preserved by T, so the composite square is too. $\qquad\square$

Definitions **??** and **??** contain the basic underlying data of a tangent functor $\mathcal{T}_x C : \mathbb{N}^\bullet \to \mathbb{X}$, but to give a more precise construction we have to, according to Definition **??**, produce a map of simplicial sets $\mathcal{T}_x C : B\mathbb{N}^\bullet \to \mathbb{X}$. We turn now to this construction, which we also make functorial in the pointed object $x : * \to C$.

Definition 4.17. Let $\mathbb{X}$ be a cartesian tangent ∞-category, and let

$$\mathbb{X}^T_* \subseteq \mathrm{Fun}(\Delta^1, \mathbb{X})$$

denote the full subcategory of the arrow ∞-category $\mathrm{Fun}(\Delta^1, \mathbb{X})$ whose objects are the (functors corresponding to) morphisms in $\mathbb{X}$ of the form

$$x : * \to C$$

whose source is a terminal object of $\mathbb{X}$, and for which the tangent space $T_x C$ exists in the sense of Definition **??**.

Our goal is to construct a map of ∞-categories

$$\mathcal{T} : \mathbb{X}^T_* \to \mathbb{D}\mathrm{iff}(\mathbb{X}); \quad (* \xrightarrow{x} C) \mapsto \mathcal{T}_x C$$

where $\mathcal{T}_x C$ is a differential object in $\mathbb{X}$ whose underlying object is equivalent to the tangent space $T_x C$. The crucial underlying construction for building $\mathcal{T}$ is a map of $\mathbb{W}$eil-modules

$$t : B\mathbb{N}^{\bullet} \to \mathrm{Fun}(\Delta^1, \mathbb{W}\mathrm{eil})$$

where $\mathbb{W}$eil acts on $B\mathbb{N}^{\bullet}$ as in Definition **??** and objectwise on the arrow ∞-category $\mathrm{Fun}(\Delta^1, \mathbb{W}\mathrm{eil})$ as in Example **??**.

Definition 4.18. We define t on vertices by

$$t(A, M) := (A \otimes W^M \xrightarrow{A \otimes \epsilon} A).$$

An n-simplex σ in $B\mathbb{N}^{\bullet}$ consists of a sequence

$$(\sigma_0, \ldots, \sigma_n, \tau)$$

where

$$\sigma_i : A_0^i \to A_1^i \to \cdots \to A_n^i$$

is an n-simplex in $\mathbb{W}$eil, for $i = 0, \ldots, n$, and

$$\tau : M_0 \to M_1 \to \cdots \to M_n$$

is an n-simplex in $\mathbb{N}^{\bullet}$. We then define $t(\sigma)$ to be the n-simplex in $\mathrm{Fun}(\Delta^1, \mathbb{W}\mathrm{eil})$ given by the diagram

$$
\begin{array}{ccc}
A_0^0 \otimes W^{A_0^1 \otimes A_0^2 \otimes \cdots \otimes A_0^n \otimes M_0} & \longrightarrow & A_0^0 \\
\downarrow & & \downarrow \\
A_1^0 \otimes A_1^1 \otimes W^{A_1^2 \otimes \cdots \otimes A_1^n \otimes M_1} & \longrightarrow & A_1^0 \otimes A_1^1 \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
A_n^0 \otimes \cdots \otimes A_n^n \otimes W^{M_n} & \longrightarrow & A_n^0 \otimes \cdots \otimes A_n^n
\end{array}
$$

where each left-hand vertical map is built from a map of the form θ as constructed for the diagram (**??**), together with the relevant terms in the n-simplexes $\sigma_0, \ldots, \sigma_n, \tau$. Each right-hand vertical map is similarly built from the unit map for the $\mathbb{W}$eil-algebra A_i^i, and each horizontal map is given by the augmentation map.

The reader may verify that this construction does define a map of simplicial sets
$$t : B\mathbb{N}^\bullet \to \mathrm{Fun}(\Delta^1, \mathbb{W}\mathrm{eil})$$
which commutes with the $\mathbb{W}\mathrm{eil}$-actions.

Definition 4.19. Let $\lrcorner \subseteq \Delta^1 \times \Delta^1$ be the 'punctured square', i.e. the indexing diagram for a cospan (two morphisms with a common target). We define a map
$$\mathcal{U} : \mathbb{X}^T_* \to \mathrm{Fun}^{\mathrm{tan}}(\mathbb{N}^\bullet, \mathrm{Fun}(\lrcorner, \mathbb{X}))$$
where the diagram category $\mathrm{Fun}(\lrcorner, \mathbb{X})$ is given the tangent structure of Example **??**. We define $\mathcal{U}$ as adjunct to the composite map

$$B\mathbb{N}^\bullet \times \mathbb{X}^T_* \xrightarrow{t \times \iota} \mathrm{Fun}(\Delta^1, \mathbb{W}\mathrm{eil}) \times \mathrm{Fun}(\Delta^1, \mathbb{X})$$

(4.20)
$$\xrightarrow{\times} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathbb{W}\mathrm{eil} \times \mathbb{X})$$

$$\longrightarrow \mathrm{Fun}(\lrcorner, \mathbb{X}).$$

The final map is induced by the restriction along the inclusion $\lrcorner \subseteq \Delta^1 \times \Delta^1$, and by the tangent structure map $T : \mathbb{W}\mathrm{eil} \times \mathbb{X} \to \mathbb{X}$. Notice that the underlying functor of $\mathcal{U}(x)$ consists of the diagrams whose pullbacks were used to define $\mathcal{T}_x C$ in Definition **??**.

Lemma 4.21. *The map $\mathcal{U}$ described in Definition **??** takes values in the ∞-category of tangent functors $\mathbb{N}^\bullet \to \mathrm{Fun}(\lrcorner, \mathbb{X})$.*

Proof. We have to check that for each $x : * \to C$ in $\mathbb{X}^T_*$, the induced map $\mathcal{U}(x) : B\mathbb{N}^\bullet \to \mathrm{Fun}(\lrcorner, \mathbb{X})$ is a map of marked $\mathbb{W}\mathrm{eil}$-modules. This map commutes with the $\mathbb{W}\mathrm{eil}$-actions because that is true of the map $t : B\mathbb{N}^\bullet \to \mathrm{Fun}(\Delta^1, \mathbb{W}\mathrm{eil})$ of Definition **??**. It preserves the markings in $B\mathbb{N}^\bullet$ by the first part of Lemma **??**. $\qquad\square$

We now wish to take the pullback of each cospan in the image of $\mathcal{U}(x)$ to obtain the desired differential object $\mathcal{T}_x C$.

Definition 4.22. Let $\mathbb{Y} \subseteq \mathrm{Fun}(\lrcorner, \mathbb{X})$ be the full subcategory generated by the cospans in the image of the composite map (**??**), i.e. all those of the form
$$T^A(*) \xrightarrow{T^A(x)} T^A(C) \longleftarrow T^A T^{W^M}(C)$$
for $x : * \to C$ in $\mathbb{X}^T_*$, $A \in \mathbb{W}\mathrm{eil}$ and $M \in \mathbb{N}^\bullet$.

The definition of $\mathbb{X}^T_*$, together with the second part of Lemma **??**, implies that each cospan in $\mathbb{Y}$ has a pullback in $\mathbb{X}$ that is preserved by each T^A. This

preservation of pullbacks is reflected in the existence of a (strong) tangent functor

$$\underline{\lim} : \mathbb{Y} \to \mathbb{X}$$

which computes the pullback of each cospan in the subcategory $\mathbb{Y}$. The construction of such a functor can be found in Definition **??** at the end of this section.

As described in Definition **??**, composition with the tangent functor $\underline{\lim}$ determines (non-canonically) a functor

$$\mathrm{Fun}^{\mathrm{tan}}(\mathbb{N}^\bullet, \mathbb{Y}) \to \mathrm{Fun}^{\mathrm{tan}}(\mathbb{N}^\bullet, \mathbb{X})$$

and combining this functor with $\mathcal{U}$, we obtain the desired map

$$\mathfrak{T} : \mathbb{X}_*^T \to \mathrm{Fun}^{\mathrm{tan}}(\mathbb{N}^\bullet, \mathbb{X}).$$

Proposition 4.23. *Let $\mathbb{X}$ be a cartesian tangent ∞-category. Then the construction of Definition **??** determines a functor*

$$\mathfrak{T} : \mathbb{X}_*^T \to \mathbb{D}\mathrm{iff}(\mathbb{X})$$

that sends a pointed object $x : * \to C$ in $\mathbb{X}$ to a differential object $\mathfrak{T}_x C$ in $\mathbb{X}$ whose underlying object is the tangent space $T_x C$.

Proof. It remains to check that the functor $\mathfrak{T}$ constructed above actually takes values in the ∞-category of differential objects in $\mathbb{X}$, i.e. that each underlying functor preserves finite products. But this claim is proved in Lemma **??**. $\square$

Corollary 4.24. *Let $\mathbb{X}$ be a cartesian tangent ∞-category. Then an object D in $\mathbb{X}$ admits a differential structure if and only if D is equivalent to some tangent space $T_x C$.*

Proof. The if direction follows from Proposition **??**. To see the only if direction, it is sufficient to note that if D admits a differential structure, then we have a diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\langle \zeta, 1 \rangle} & D \times D \\
\sim \downarrow & & \sim \downarrow {\scriptstyle \alpha_{\mathbb{N}}} \\
T_\zeta D & \longrightarrow & T(D) \\
\downarrow & & \downarrow {\scriptstyle p} \\
* & \xrightarrow{\zeta} & D
\end{array}
$$

and the top-left map is an equivalence because the bottom square and composite squares are both pullbacks in $\mathbb{X}$. $\square$

Remark 4.25. Proposition **??** implies that a morphism $f : C \to D$ in $\mathbb{X}$ induces a map
$$T_x f : T_x C \to T_{f(x)} D$$
that preserves the differential structures on these tangent spaces. This map $T_x f$ is the analogue of the ordinary derivative of a map f at a point x and preservation of the differential structure is the analogue of this derivative being a linear map in a setting where there is no precise version of the vector space structure on an ordinary tangent space.

Tangent ∞-categories and cartesian differential categories. One of the motivations for Cockett and Cruttwell to study differential objects was to make a connection between cartesian tangent categories and cartesian differential categories. Roughly speaking, they show that for a cartesian tangent category $\mathbb{X}$ in which every object has a canonical differential structure there is a corresponding cartesian differential structure on $\mathbb{X}$, and that every cartesian differential category arises in this way. We provide a generalization of that result to tangent ∞-categories.

Definition 4.26. Let $\mathbb{X}$ be a cartesian tangent ∞-category. We define a category $\widehat{h\mathbb{D}\mathrm{iff}}(\mathbb{X})$ with objects the differential objects of $\mathbb{X}$, and with morphisms from $\mathcal{D}$ to $\mathcal{D}'$ given by morphisms in the homotopy category of $\mathbb{X}$ between the underlying objects $\mathcal{D}(\mathbb{N})$ and $\mathcal{D}'(\mathbb{N})$. This construction is not the homotopy category of $\mathbb{D}\mathrm{iff}(\mathbb{X})$ because we are now, like Cockett and Cruttwell, including *all* morphisms between the underlying objects, not only those that commute with the differential structures.

The following result generalizes [**?**, 4.11]. See [**?**, 2.1.1] for the definition of cartesian differential structure on a category.

Theorem 4.27. *Let $\mathbb{X}$ be a cartesian tangent ∞-category. There is a cartesian differential structure on $\widehat{h\mathbb{D}\mathrm{iff}}(\mathbb{X})$ in which the monoid structure on an object is that inherited from its differential structure, and the derivative of a morphism $f : A \to B$ is given by the composite*
$$\nabla(f) : A \times A \xrightarrow[\sim]{\langle p_A, \hat{p}_A \rangle^{-1}} TA \xrightarrow{T(f)} TB \xrightarrow{\hat{p}_B} B.$$
where $\hat{p}_A$ and $\hat{p}_B$ are determined by the differential structures on A and B respectively.

Proof. Let $\mathbb{X}_d \subseteq \mathbb{X}$ be the full subcategory of $\mathbb{X}$ whose objects are those that admit a differential structure. We claim that $\mathbb{X}_d$ is a tangent subcategory, i.e.

is closed under the action of $\mathbb{W}$eil. To see this claim, we note that, if D admits a differential structure, $T^A(D) \simeq D^{m(A)}$ where $m(A)$ denotes the set of nonzero monomials in the Weil-algebra A. A finite product of differential objects has a canonical differential structure, so $\mathbb{X}_d$ is closed under finite products, and it follows that $\mathbb{X}_d$ is a cartesian tangent subcategory of $\mathbb{X}$.

We now claim that, unusually, the homotopy category $h\mathbb{X}_d$ inherits a tangent structure from that on $\mathbb{X}_d$. The $\mathbb{W}$eil-action on $\mathbb{X}_d$ passes to the homotopy category so the key is to show that each of the foundational and vertical lift pullbacks in $\mathbb{X}_d$ is also a pullback in $h\mathbb{X}_d$.

First consider an object $D \in \mathbb{X}_d$, a Weil-algebra A and positive integers k, l. We have to show that

$$
\begin{array}{ccc}
T^{A \otimes W^{k+l}}(D) & \longrightarrow & T^{A \otimes W^l}(D) \\
\downarrow & & \downarrow \\
T^{A \otimes W^k}(D) & \longrightarrow & T^A(D)
\end{array}
$$

is a pullback in $h\mathbb{X}_d$. Recall that a choice of differential structure on D determines equivalences $T^A(D) \simeq D^{m(A)}$ where $m(A)$ denotes the set of nonzero monomials in A, so we can replace this square by a corresponding diagram

(4.28)
$$
\begin{array}{ccc}
D^{m(A \otimes W^{k+l})} & \longrightarrow & D^{m(A \otimes W^l)} \\
\downarrow & & \downarrow \\
D^{m(A \otimes W^k)} & \longrightarrow & D^{m(A)}
\end{array}
$$

which is determined by a pushout of sets of the form

$$
\begin{array}{ccc}
m(A) & \longrightarrow & m(A \otimes W^l) \\
\downarrow & & \downarrow \\
m(A \otimes W^l) & \longrightarrow & m(A \otimes W^{k+l})
\end{array}
$$

where each map is an appropriate inclusion. It follows that, in any category with finite products and for any object D, the square (**??**) is a pullback.

For the vertical lift axiom, we have to consider the following diagram in $h\mathbb{X}_d$

$$
\begin{array}{ccc}
D^3 & \xrightarrow{\langle \pi_1, \pi_2, \zeta!, \pi_3 \rangle} & D^4 \\
\downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \langle \pi_3, \pi_4 \rangle} \\
D & \xrightarrow{\langle \zeta!, 1 \rangle} & D^2
\end{array}
$$

where $\zeta : * \to D$ is the zero map for the differential structure on D. A diagram of this form is a pullback in any category (regardless of the map ζ).

We have completed the check that $h\mathbb{X}_d$ has a cartesian tangent structure, and we now apply [**?**, 4.11] to this structure. This provides a cartesian differential structure on the category $\widehat{\mathbb{D}\text{iff}}(h\mathbb{X}_d)$ whose objects are the differential objects in $h\mathbb{X}_d$, and whose morphisms are morphisms in $h\mathbb{X}_d$ between the underlying objects.

We complete the proof of our theorem by constructing an embedding of the category $\widehat{h\mathbb{D}\text{iff}}(\mathbb{X})$ into $\widehat{\mathbb{D}\text{iff}}(h\mathbb{X}_d)$. Given a differential object in $\mathbb{X}$ with underlying object D, we obtain a differential structure on D in $h\mathbb{X}_d$ in the same manner as in the proof of Proposition **??**. Since morphisms in each case are homotopy classes of maps in $\mathbb{X}$, we see that this construction determines a fully faithful functor

$$U : \widehat{h\mathbb{D}\text{iff}}(\mathbb{X}) \to \widehat{\mathbb{D}\text{iff}}(h\mathbb{X}_d).$$

Since $\widehat{h\mathbb{D}\text{iff}}(\mathbb{X})$ is closed under finite products, we obtain the desired cartesian differential structure by restriction along U. The formula for the derivative $\nabla(f)$ follows from that in $\widehat{\mathbb{D}\text{iff}}(h\mathbb{X}_d)$. $\qquad\square$

The one outstanding task in this section is to construct the pullback tangent functor $\varprojlim : \mathbb{Y} \to \mathbb{X}$ used in Definition **??**. Recall that $\mathbb{X}$ is a cartesian tangent ∞-category and $\mathbb{Y} \subseteq \text{Fun}(\lrcorner, \mathbb{X})$ is a full subcategory of cospans each of which admits a pullback in $\mathbb{X}$ which is preserved by the tangent structure maps T^A.

Definition 4.29. By definition of $\mathbb{Y}$ we can choose a functor

$$\lim : \mathbb{Y} \to \mathbb{X}$$

that calculates the pullback of each object in $\mathbb{Y}$. We first define a $\mathbb{W}$eil-module map

$$\varprojlim_0 : B_0 \mathbb{Y} = \mathbb{W}\text{eil} \times \mathbb{Y} \to \mathbb{X}$$

by freely extending $\lim$, i.e.

$$\varprojlim_0(A, D) := T^A(\lim D).$$

The universal property of pullbacks then ensures that there is a functor

$$\mathbb{Weil} \times \mathbb{Y} \to \mathrm{Fun}(\Delta^1, \mathbb{X})$$

given by

$$(A, D) \mapsto [T^A(\lim D) \to \lim T^A D]$$

and our hypothesis on $\mathbb{Y}$ is that this map is a natural equivalence. We can therefore choose a natural inverse, and extend freely, to obtain a $\mathbb{Weil}$-module map

$$\underline{\lim}_1 : B_1 \mathbb{Y} = \mathbb{Weil} \times \mathbb{Weil} \times \mathbb{Y} \to \mathrm{Fun}(\Delta^1, \mathbb{X})$$

given by equivalences

$$\underline{\lim}_1(A_1, A_0, D) = [T^{A_1} \lim T^{A_0} D \xrightarrow{\sim} T^{A_1} T^{A_0} \lim D].$$

Similarly, the universal property of pullbacks in an ∞-category ensures that there is a functor

$$\mathbb{Weil} \times \mathbb{Weil} \times \mathbb{Y} \to \mathrm{Fun}(\Delta^2, \mathbb{X})$$

consisting of 2-simplexes in $\mathbb{X}$ of the form

$$
\begin{array}{ccc}
T^{A_1} T^{A_0} \lim D & \xrightarrow{\quad\sim\quad} & \lim T^{A_1} T^{A_0} D \\
 & \searrow{\scriptstyle\sim} \qquad \nearrow{\scriptstyle\sim} & \\
 & T^{A_1} \lim T^{A_0} D &
\end{array}
$$

whose edges are the functors whose inverses were used to construct $\underline{\lim}_1$. We have already chosen inverses for the edges of these 2-simplexes, and we can use the fact that $\mathrm{Fun}(\mathbb{Weil}^2 \times \mathbb{Y}, \mathbb{X})$ is a quasi-category to 'fill in' a corresponding 'inverse' for the 2-simplexes themselves. Extending freely, we obtain a $\mathbb{Weil}$-module map

$$\underline{\lim}_2 : B_2 \mathbb{Y} = \mathbb{Weil} \times \mathbb{Weil}^2 \times \mathbb{Y} \to \mathrm{Fun}(\Delta^2, \mathbb{X})$$

that is compatible with $\underline{\lim}_1$ via the face maps $B_2 \mathbb{Y} \to B_1 \mathbb{Y}$, and coface maps $\Delta^1 \to \Delta^2$.

Inductively, using the universal property of the ∞-categorical pullback, we construct a sequence of functors

$$\underline{\lim}_n : B_n \mathbb{Y} \to \mathrm{Fun}(\Delta^n, \mathbb{X})$$

which are compatible with the face maps on each side, i.e. we have a map of semi-simplicial objects

$$\underline{\lim}_\bullet : B_\bullet \mathbb{Y} \to \mathrm{Fun}(\Delta^\bullet, \mathbb{X}).$$

This map induces a map

$$\underline{\lim}' : B' \mathbb{Y} \to \mathbb{X}$$

where $B'\mathbb{Y}$ denotes the so-called 'fat' geometric realization [**?**, 8.5.12] of the simplicial object $B_\bullet\mathbb{Y}$. Since that fat realization is equivalent to the ordinary realization $B\mathbb{Y}$, and $B\mathbb{Y}$ is cofibrant as a marked $\mathbb{W}$eil-module, we can lift $\underline{\lim}'$ to the desired map of $\mathbb{W}$eil-modules

$$\underline{\lim} : B\mathbb{Y} \to \mathbb{X}$$

whose underlying functor is lim.

5. Tangent structures in and on an $(\infty, 2)$-category

The goal of this section is to generalize our notion of 'tangent ∞-category' to $(\infty, 2)$-categories in two different ways. Firstly, we will observe that one can easily extend Definition **??** to a notion of *tangent object* **in** any $(\infty, 2)$-category, and that this context provides the natural setting for such a notion. Secondly, we apply that general notion to give a definition of *tangent $(\infty, 2)$-category*, i.e. a tangent structure **on** an $(\infty, 2)$-category, or tangent object in the $(\infty, 2)$-category of $(\infty, 2)$-categories.

Models for $(\infty, 2)$-categories. The basic intuition for an $(\infty, 2)$-category is that it is a category enriched in $(\infty, 1)$-categories, i.e. in ∞-categories. The most convenient model in many cases is to take this intuition as a definition.

Definition 5.1. Let qCat be the category of quasi-categories with monoidal structure given by cartesian product. A qCat-*category* **C** is a category enriched in qCat. In particular, for any object $\mathbb{X} \in \mathbf{C}$, the enrichment provides a monoidal quasi-category

$$\mathrm{End}_\mathbf{C}(\mathbb{X}) := \mathrm{Hom}_\mathbf{C}(\mathbb{X}, \mathbb{X})$$

which can be used (see Definition **??**) to define a notion of tangent structure on the object $\mathbb{X}$.

Example 5.2. Let $\mathbf{Cat}_\infty$ be the qCat-category consisting of qCat itself with self-enrichment given by its closed monoidal structure, i.e. with mapping objects

$$\mathrm{Hom}_{\mathbf{Cat}_\infty}(\mathbb{X}, \mathbb{Y}) = \mathrm{Fun}(\mathbb{X}, \mathbb{Y}).$$

We refer to $\mathbf{Cat}_\infty$ as the $(\infty, 2)$-*category of ∞-categories*.

The qCat-categories are the fibrant objects in a model structure on *marked simplicial categories*, i.e. categories enriched in the category Set_Δ^+ of marked simplicial sets. That model structure is constructed in [**?**, A.3.2] and generalizes the Bergner model structure [**?**] on simplicial categories. In [**?**], Lurie

shows that this model structure is Quillen equivalent to other common models for $(\infty, 2)$-categories.

From our perspective, the biggest drawback of using qCat-categories to model $(\infty, 2)$-categories is that they do not accurately describe the 'mapping $(\infty, 2)$-category' between two $(\infty, 2)$-categories. Given qCat-categories $\mathbf{C}, \mathbf{D}$, one can define a new qCat-category $\mathrm{Fun}_{\mathsf{qCat}}(\mathbf{C}, \mathbf{D})$ whose objects are the qCat-enriched functors $\mathbf{C} \to \mathbf{D}$, yet this construction does not typically describe the correct 'mapping $(\infty, 2)$-category', even when $\mathbf{C}$ is cofibrant.[7]
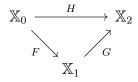
It is therefore convenient to introduce a second model for $(\infty, 2)$-categories for which these mapping $(\infty, 2)$-categories are easier to describe; we will use Lurie's ∞-*bicategories* from [?, 4.1] based on his notion of *scaled simplicial set* which we now recall.

Definition 5.3. A *scaled simplicial set* (X, S) consists of a simplicial set X and a subset $S \subseteq X_2$ of the set of 2-simplexes in X, such that S contains all the degenerate 2-simplexes. We refer to the 2-simplexes in S as being *thin*. A *scaled morphism* $(X, S) \to (X', S')$ is a map of simplicial sets $f : X \to X'$ such that $f(S) \subseteq S'$. Let $\mathsf{Set}^{\mathsf{sc}}_{\Delta}$ denote the category of scaled simplicial sets and scaled morphisms.

There is a model structure on $\mathsf{Set}^{\mathsf{sc}}_{\Delta}$ described in [?, 4.2.7] which we take as our model for the ∞-category of $(\infty, 2)$-categories and refer to as the *scaled model structure*. The cofibrations in $\mathsf{Set}^{\mathsf{sc}}_{\Delta}$ are the monomorphisms, so every object is cofibrant. An ∞-*bicategory* is a fibrant object in this model structure.

Example 5.4. The Duskin nerve [?] of an ordinary bicategory can be given the structure of an ∞-bicategory; for example, see [?, 3.16].

Example 5.5. Given a qCat-category $\mathbf{C}$, we define a scaled simplicial set, called the *scaled nerve* of $\mathbf{C}$, as follows. The underlying simplicial set of the scaled nerve is the simplicial nerve of the simplicially-enriched category $\mathbf{C}$; see [?, 1.1.5.5]. A 2-simplex in that simplicial nerve is given by a (not-necessarily-commutative) diagram in $\mathbf{C}$ of the form

$$
\begin{array}{ccc}
\mathbb{X}_0 & \xrightarrow{\ H\ } & \mathbb{X}_2 \\
& {\scriptstyle F} \searrow \quad \nearrow {\scriptstyle G} & \\
& \mathbb{X}_1 &
\end{array}
$$

together with a morphism $\alpha : H \to GF$ in the quasi-category $\mathrm{Hom}_{\mathbf{C}}(\mathbb{X}_0, \mathbb{X}_2)$. We designate such a 2-simplex as *thin* if the corresponding morphism α is an

---

[7]One explanation for this deficit is that the model structure on marked simplicial categories is not compatible with the cartesian monoidal structure.

equivalence. This choice of thin 2-simplexes gives the simplicial nerve of $\mathbf{C}$ a scaling making it an ∞-bicategory by [**?**, 4.2.7].

Lurie proves in [**?**, 4.2.7] that the scaled nerve is the right adjoint in a Quillen equivalence between the scaled model structure on $\mathsf{Set}^{\mathsf{sc}}_\Delta$ and the model structure on marked simplicial categories described above. In particular every ∞-bicategory $\mathbf{C}$ is weakly equivalent to the scaled nerve of a qCat-category. When working with an arbitrary ∞-bicategory, we will usually assume an implicit choice of such a qCat-category, and hence of the corresponding mapping ∞-categories $\mathrm{Hom}_{\mathbf{C}}(\mathbb{X}, \mathbb{Y})$ for objects $\mathbb{X}, \mathbb{Y} \in \mathbf{C}$. Similarly, we will usually not distinguish notationally between the qCat-category $\mathbf{C}$ and the ∞-bicategory given by its scaled nerve, extending our convention for nerves of categories.

Example 5.6. Let $\mathbf{C}$ be a quasi-category. The *maximal scaling* on $\mathbf{C}$ is that in which every 2-simplex is thin. This scaling makes $\mathbf{C}$ into an ∞-bicategory by [**?**, 4.1.2] and the fact, proved by Gagna, Harpaz and Lanari in [**?**, 5.1], that any weak ∞-bicategory is also an ∞-bicategory. In this way we treat a quasi-category as a special kind of ∞-bicategory.

Conversely, any ∞-bicategory has an *underlying* quasi-category.

Definition 5.7. Let $\mathbf{C}$ be an ∞-bicategory, and denote by $\mathbf{C}^{\simeq}$ the simplicial subset of $\mathbf{C}$ consisting of those simplexes for which all of the 2-dimensional faces are thin. Then $\mathbf{C}^{\simeq}$ is a quasi-category; this follows from [**?**, 4.1.3] and the fact that any scaled anodyne map is a trivial cofibration in $\mathsf{Set}^{\mathsf{sc}}_\Delta$ (so that any ∞-bicategory is also a 'weak' ∞-bicategory).

We now describe the internal mapping objects for ∞-bicategories.

Definition 5.8. Let $\mathbf{B}$ and $\mathbf{C}$ be ∞-bicategories. The ∞-*bicategory of functors* $\mathbf{B} \to \mathbf{C}$ is the scaled simplicial set

$$\mathrm{Fun}_{(\infty,2)}(\mathbf{B}, \mathbf{C})$$

whose underlying simplicial set is the maximal simplicial subset of $\mathrm{Fun}(\mathbf{B}, \mathbf{C})$ with vertices those maps $\mathbf{B} \to \mathbf{C}$ that are scaled morphisms. The scaling is that in which a 2-simplex $\Delta^2 \times \mathbf{B} \to \mathbf{C}$ is thin if it is a scaled morphism (where Δ^2 has the maximal scaling).

Lemma 5.9. *Let* $\mathbf{B}, \mathbf{C}$ *be* ∞-*bicategories. Then the scaled simplicial set* $\mathrm{Fun}_{(\infty,2)}(\mathbf{B}, \mathbf{C})$ *is an* ∞-*bicategory.*

Proof. Devalapurkar proves in [**?**, 2.1] that $\mathrm{Fun}_{(\infty,2)}(-, -)$ makes $\mathsf{Set}^{\mathsf{sc}}_\Delta$ into a cartesian closed model category. Since any object in $\mathsf{Set}^{\mathsf{sc}}_\Delta$ is cofibrant, and $\mathbf{B}$ is fibrant, it follows that $\mathrm{Fun}_{(\infty,2)}(\mathbf{B}, \mathbf{C})$ is fibrant. $\square$

Tangent objects in an $(\infty, 2)$**-category.** We now turn to tangent structures, and our first goal is to describe what is meant by a tangent structure on an object *in* an $(\infty, 2)$-category.

Let $\mathbf{C}$ be a qCat-category. Then any object $\mathbb{X} \in \mathbf{C}$ has a monoidal quasi-category of 'endofunctors'

$$\mathrm{End}_{\mathbf{C}}(\mathbb{X}) := \mathrm{Hom}_{\mathbf{C}}(\mathbb{X}, \mathbb{X})$$

with monoidal structure given by composition, and we therefore have the following simple generalization of Definition **??**.

Definition 5.10. Let $\mathbf{C}$ be a qCat-category, and let $\mathbb{X} \in \mathbf{C}$ be an object. A *tangent structure* on $\mathbb{X}$ is a strict monoidal functor

$$T : \mathbb{W}\mathrm{eil}^{\otimes} \to \mathrm{End}_{\mathbf{C}}(\mathbb{X})^{\circ}$$

for which the underlying functor of ∞-categories $\mathbb{W}\mathrm{eil} \to \mathrm{End}_{\mathbf{C}}(\mathbb{X})$ preserves the tangent pullbacks.

We can phrase Definition **??** in terms of functors of $(\infty, 2)$-categories as follows. The monoidal quasi-category $\mathbb{W}\mathrm{eil}$ determines a qCat-category $\mathbf{Weil}$ that has a single object $\bullet$, with $\mathrm{Hom}_{\mathbf{Weil}}(\bullet, \bullet) := \mathbb{W}\mathrm{eil}$. If we consider the identity morphisms in $\mathbb{W}\mathrm{eil}$ as marked, then it follows from Lemma **??** that $\mathbf{Weil}$ is cofibrant in the model structure on marked simplicial categories. We make the following definition.

Definition 5.11. Let $\mathbf{C}$ be a qCat-category. A *tangent object* in $\mathbf{C}$ is a qCat-enriched functor

$$\mathbf{T} : \mathbf{Weil} \to \mathbf{C}$$

for which the induced map on endomorphism ∞-categories preserves the tangent pullbacks.

Example 5.12. Let $\mathbf{Cat}_{\infty}$ denote the $(\infty, 2)$-category of ∞-categories of Example **??**. Then a tangent object in $\mathbf{Cat}_{\infty}$ is a tangent ∞-category in the sense of Definition **??**.

Morphisms between tangent objects $\mathbb{X}$ and $\mathbb{Y}$ in an $(\infty, 2)$-category $\mathbf{C}$ can be defined via a generalization of the characterization of Lemma **??**, with $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ replaced by the mapping object $\mathrm{Hom}_{\mathbf{C}}(\mathbb{X}, \mathbb{Y})$.

However, we can now go further and define an entire $(\infty, 2)$-category of tangent objects in $\mathbf{C}$. The following definition can be viewed as the culmination of the first half of this paper. Recall that we treat the qCat-category $\mathbf{Weil}$ as an ∞-bicategory via the scaled nerve.

Definition 5.13. Let $\mathbf{C}$ be an ∞-bicategory. The *∞-bicategory of tangent objects in* $\mathbf{C}$ is the full $(\infty, 2)$-subcategory (i.e. the maximal scaled simplicial subset)

$$\mathbf{Tan}(\mathbf{C}) \subseteq \mathrm{Fun}_{(\infty,2)}(\mathbf{Weil}, \mathbf{C})$$

whose vertices are those functors $\mathbf{T} : \mathbf{Weil} \to \mathbf{C}$ for which the induced functor on mapping ∞-categories $\mathbb{W}\mathrm{eil} \to \mathrm{End}_{\mathbf{C}}(\mathbb{X})$ preserves the tangent pullbacks.

Remark 5.14. The mapping ∞-categories in $\mathbf{Tan}(\mathbf{C})$ are generalizations of the ∞-categories of strong tangent functors in Definition **??** and reduce, up to equivalence, to those ∞-categories when $\mathbf{C} = \mathbf{Cat}_\infty$. Using work of Johnson-Freyd and Scheimbauer [**?**] on lax natural transformations for (∞, n)-categories, we can also extend $\mathbf{Tan}(\mathbf{C})$ to an ∞-bicategory of tangent objects and their *lax* tangent morphisms, though we will not pursue that extension here.

Tangent $(\infty, 2)$-categories. The preceding theory defines a notion of tangent structure *in* an $(\infty, 2)$-category, but it also determines a notion of tangent structure *on* an $(\infty, 2)$-category, by applying Definition **??** in the case that $\mathbf{C} = \mathbf{Cat}_{(\infty,2)}$ is a suitable $(\infty, 2)$-category of $(\infty, 2)$-categories.

Definition 5.15. Let $\mathbf{Cat}_{(\infty,2)}$ be the qCat-category whose objects are the ∞-bicategories, and whose mapping objects are the underlying ∞-categories

$$\mathrm{Hom}_{\mathbf{Cat}_{(\infty,2)}}(\mathbb{X}, \mathbb{Y}) := \mathrm{Fun}_{(\infty,2)}(\mathbb{X}, \mathbb{Y})^\simeq$$

of the ∞-bicategories of functors $\mathbb{X} \to \mathbb{Y}$. (See Definitions **??** and **??**.)

An ∞-bicategory $\mathbb{X}$ has an endomorphism ∞-category

$$\mathrm{End}_{(\infty,2)}(\mathbb{X}) := \mathrm{Fun}_{(\infty,2)}(\mathbb{X}, \mathbb{X})^\simeq$$

which is a monoidal quasi-category under composition. Definition **??** then gives us the following notion.

Definition 5.16. Let $\mathbb{X}$ be an ∞-bicategory. A *tangent structure* on $\mathbb{X}$ is a strict monoidal functor

$$T : \mathbb{W}\mathrm{eil} \to \mathrm{End}_{(\infty,2)}(\mathbb{X})$$

for which the underlying functor $\mathbb{W}\mathrm{eil} \to \mathrm{End}_{(\infty,2)}(\mathbb{X})$ preserves tangent pullbacks. A *tangent ∞-bicategory* consists of an ∞-bicategory $\mathbb{X}$ and a tangent structure T on $\mathbb{X}$.

The theory of tangent ∞-bicategories subsumes that of tangent ∞-categories: recall that a quasi-category can be viewed as an ∞-bicategory in which all 2-simplexes are thin.

Lemma 5.17. *Let $\mathbb{X}$ be a quasi-category. Then a tangent structure on $\mathbb{X}$ (in the sense of Definition* **??***) is the same thing as a tangent structure on the corresponding ∞-bicategory $\mathbb{X}$ (in the sense of Definition* **??***).*

Proof. When $\mathbb{X}$ is a quasi-category, there is an isomorphism of monoidal quasi-categories

$$\mathrm{End}_{(\infty,2)}(\mathbb{X}) = \mathrm{Fun}_{(\infty,2)}(\mathbb{X},\mathbb{X})^{\simeq} = \mathrm{Fun}(\mathbb{X},\mathbb{X}) = \mathrm{End}(\mathbb{X}).$$

$\square$

Remark 5.18. As with tangent ∞-categories, we can also view a tangent structure on an ∞-bicategory $\mathbb{X}$ via an 'action' map:
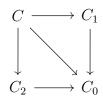
$$T : \mathbb{W}\mathrm{eil} \times \mathbb{X} \to \mathbb{X}$$

which is an action of the scaled simplicial monoid $\mathbb{W}\mathrm{eil}$ (where every 2-simplex is thin) on the scaled simplicial set $\mathbb{X}$.

Unlike the situation for a tangent structure on an ∞-category, however, we do not know a simple characterization of when an action map of the form T corresponds to a map of quasi-categories $\mathbb{W}\mathrm{eil} \to \mathrm{End}_{(\infty,2)}(\mathbb{X})$ which preserves the tangent pullbacks. The issue is that we don't know in general how to identify pullbacks in an ∞-category of the form $\mathrm{Fun}_{(\infty,2)}(\mathbb{X},\mathbb{Y})^{\simeq}$. Nonetheless we can provide a sufficient criterion for a diagram in this ∞-category to be a pullback, and we will use this criterion in Section **??** to verify that the Goodwillie tangent structure extends to an ∞-bicategory.

Our criterion is based on the following notion of pullback in an $(\infty,2)$-category. Recall that we define mapping ∞-categories $\mathrm{Hom}_{\mathbb{X}}(C, D)$ by choosing a qCat-category whose scaled nerve is equivalent to $\mathbb{X}$.

Definition 5.19. Let $\mathbb{X}$ be an ∞-bicategory, and consider a commutative diagram (i.e. a pair of thin 2-simplexes with a common edge):

$$
\begin{array}{ccc}
C & \longrightarrow & C_1 \\
\downarrow & \searrow & \downarrow \\
C_2 & \longrightarrow & C_0
\end{array}
$$

in $\mathbb{X}$. We say that this diagram is a *homotopy 2-pullback* if, for every $D \in \mathbb{X}$, the induced diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{X}}(D, C) & \longrightarrow & \mathrm{Hom}_{\mathbb{X}}(D, C_1) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbb{X}}(D, C_2) & \longrightarrow & \mathrm{Hom}_{\mathbb{X}}(D, C_0)
\end{array}
$$

is a homotopy pullback of ∞-categories (that is, a homotopy pullback in the Joyal model structure on simplicial sets).
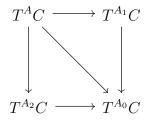
Proposition 5.20. *Let $\mathbb{X}$ be an ∞-bicategory, and let*

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{X} \to \mathbb{X}$$

be an action of the simplicial monoid $\mathbb{W}\mathrm{eil}$ (with every 2-simplex considered thin) on the scaled simplicial set $\mathbb{X}$. Suppose that for each of the tangent pullback squares in $\mathbb{W}\mathrm{eil}$

$$
\begin{array}{ccc}
A & \longrightarrow & A_1 \\
\downarrow & \searrow & \downarrow \\
A_2 & \longrightarrow & A_0
\end{array}
$$

and each object $C \in \mathbb{X}$, the resulting diagram

$$
\begin{array}{ccc}
T^A C & \longrightarrow & T^{A_1} C \\
\downarrow & \searrow & \downarrow \\
T^{A_2} C & \longrightarrow & T^{A_0} C
\end{array}
$$

is a homotopy 2-pullback in $\mathbb{X}$. Then T defines a tangent structure on $\mathbb{X}$.

Proof. We need to show that each diagram

$$
\begin{array}{ccc}
T^A & \longrightarrow & T^{A_1} \\
\downarrow & \searrow & \downarrow \\
T^{A_2} & \longrightarrow & T^{A_0}
\end{array}
$$

determines a pullback in the ∞-category $\mathrm{End}_{(\infty,2)}(\mathbb{X}) = \mathrm{Fun}_{(\infty,2)}(\mathbb{X}, \mathbb{X})^{\simeq}$. That claim follows from the following more general lemma. $\qquad\square$

Lemma 5.21. *Let* $\mathbb{X}, \mathbb{Y}$ *be* ∞*-bicategories, and let* $D : \square \times \mathbb{X} \to \mathbb{Y}$ *be a scaled morphism (where every 2-simplex in* $\square := \Delta^1 \times \Delta^1$ *is considered thin) such that for each* $x \in \mathbb{X}$*, the resulting diagram* $D(-, x) : \square \to \mathbb{Y}$ *is a homotopy 2-pullback in* $\mathbb{Y}$*. Then* D *corresponds to a pullback square in the* ∞*-category* $\mathrm{Fun}_{(\infty,2)}(\mathbb{X}, \mathbb{Y})^{\simeq}$*.*

Proof. To get a handle on the ∞-category $\mathrm{Fun}_{(\infty,2)}(\mathbb{X}, \mathbb{Y})^{\simeq}$ we translate this problem into the context of qCat-categories and use the following characterization by Lurie [**?**, A.3.4] for the (derived) internal mapping object for qCat-categories.

Without loss of generality, we may assume that $\mathbb{X}$ and $\mathbb{Y}$ are (the scaled nerves of) qCat-categories. It is then possible to form the qCat-category

$$\mathrm{Fun}_{\mathsf{qCat}}(\mathbb{X}, \mathbb{Y})$$

of qCat-enriched functors $\mathbb{X} \to \mathbb{Y}$, but typically this qCat-category does *not* model the $(\infty, 2)$-category $\mathrm{Fun}_{(\infty,2)}(\mathbb{X}, \mathbb{Y})$. The problem, in some sense, is that there is no notion of when a functor $\mathbb{X} \to \mathbb{Y}$ is 'cofibrant'. We rectify this concern by embedding $\mathbb{Y}$ into a suitable model category via its Yoneda map.

The enriched Yoneda embedding for $\mathbb{Y}$ determines a fully faithful map of qCat-categories

$$\mathrm{Fun}_{\mathsf{qCat}}(\mathbb{X}, \mathbb{Y}) \to \mathrm{Fun}_{\mathsf{qCat}}(\mathbb{X} \times \mathbb{Y}^{op}, \mathsf{qCat}).$$

The right-hand side here can be identified with the full subcategory of fibrant objects in the category

$$\mathrm{Fun}_{\mathsf{Set}_{\Delta}^+}(\mathbb{X} \times \mathbb{Y}^{op}, \mathsf{Set}_{\Delta}^+)$$

of Set_{Δ}^+-enriched functors $\mathbb{X} \times \mathbb{Y}^{op} \to \mathsf{Set}_{\Delta}^+$ equipped with the projective model structure in which an Set_{Δ}^+-enriched natural transformation is a weak equivalence and/or fibration if and only each of its components is a weak equivalence and/or fibration in the marked model structure on Set_{Δ}^+.

It follows from [**?**, A.3.4.14] that the correct internal mapping object for qCat-categories is given by the full subcategory of $\mathrm{Fun}_{\mathsf{qCat}}(\mathbb{X} \times \mathbb{Y}^{op}, \mathsf{qCat})$ whose objects are functors $F : \mathbb{X} \times \mathbb{Y}^{op} \to \mathsf{qCat}$ which are also cofibrant in the model structure described above, and which have the property that, for each $x \in \mathbb{X}$, the qCat-enriched functor $F(x, -) : \mathbb{Y}^{op} \to \mathsf{qCat}$ is in the essential image of the Yoneda embedding. We denote that full subcategory by

$$\widetilde{\mathrm{Fun}}_{\mathsf{qCat}}(\mathbb{X}, \mathbb{Y}).$$

By [**?**, A.3.4.14], the diagram D corresponds to a homotopy coherent square $\bar{D}$ in the qCat-category $\widetilde{\mathrm{Fun}}_{\mathsf{qCat}}(\mathbb{X}, \mathbb{Y})$, and the hypothesis on D implies that for each $x \in \mathbb{X}$, the resulting square $\bar{D}(x, -) : \mathbb{Y}^{op} \to \mathsf{qCat}$ has the property that for each $y \in \mathbb{Y}$, the diagram $\bar{D}(x, y)$ is a homotopy pullback of quasi-categories, i.e. a homotopy pullback in the marked model structure on Set^+_Δ. Since fibrations and weak equivalences are detected objectwise in the projective model structure on $\mathrm{Fun}_{\mathsf{Set}^+_\Delta}(\mathbb{X} \times \mathbb{Y}^{op}, \mathsf{Set}^+_\Delta)$, this condition implies that $\bar{D}$ is a homotopy pullback square of fibrant-cofibrant objects in that model structure, and hence determines a pullback in the corresponding ∞-category. Since $\mathrm{Fun}_{(\infty,2)}(\mathbb{X}, \mathbb{Y})^\simeq$ is equivalent to a full subcategory of that ∞-category, it follows that D is a pullback there too. $\square$

Corollary 5.22. *Let $(\mathbb{X}, T)$ be a tangent ∞-bicategory that satisfies the condition of Proposition* **??**. *Then T restricts to a tangent structure on the underlying quasi-category $\mathbb{X}^\simeq$.*

Proof. Applying $(-)^\simeq$ to the map of ∞-bicategories

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{X} \to \mathbb{X}$$

we get the desired action map

$$T^\simeq : \mathbb{W}\mathrm{eil} \times \mathbb{X}^\simeq \to \mathbb{X}^\simeq.$$

A homotopy pullback of quasi-categories (in the Joyal model structure) determines a homotopy pullback of maximal Kan-subcomplexes (in the Quillen model structure) and so a homotopy 2-pullback in $\mathbb{X}$ determines a pullback in the ∞-category $\mathbb{X}^\simeq$. Thus the condition of Proposition **??** verifies the necessary tangent pullbacks in $\mathbb{X}^\simeq$. $\square$

Part 2. A tangent ∞-category of ∞-categories

Up to this point we have been developing the general theory of tangent ∞-categories, but now we start to look at the specific tangent structure which motivated this work. The construction and study of this structure, which we term the *Goodwillie tangent structure*, occupies the remainder of this paper. We start with a brief review of Goodwillie's calculus of functors, ideas from which will permeate the definitions and proofs to come. Most of the following section is based on Lurie's development [**?**, Ch. 6] of Goodwillie's ideas in the general context of ∞-categories.

6. Goodwillie calculus and the tangent bundle functor

The central notion in Goodwillie calculus is the *Taylor tower* of a functor; that is, a sequence of approximations to the functor that plays the role of the Taylor series in ordinary calculus. To describe the Taylor tower, we recall Goodwillie's analogues of polynomials in the context of functors. We need the following preliminary definition.

Definition 6.1. An *n-cube* in an ∞-category $\mathcal{C}$ is a diagram $\mathcal{X}$ in $\mathcal{C}$ indexed by the poset $\mathcal{P}[n]$ of subsets of $[n] = \{1, \ldots, n\}$. Such a cube is *strongly cocartesian* if each 2-dimensional face is a pushout, and is *cartesian* if the cube as a whole is a limit diagram, i.e. the map

$$\mathcal{X}(\emptyset) \to \mathrm{holim}_{\emptyset \neq S \subseteq [n]} \mathcal{X}(S)$$

is an equivalence in $\mathcal{C}$.

Definition 6.2 (Goodwillie [**?**, 3.1])**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We say that F is *n-excisive* if it maps each strongly cocartesian $(n + 1)$-cube in $\mathcal{C}$ to a cartesian cube in $\mathcal{D}$. In particular, F is 1-excisive (or, simply, *excisive*) if it maps pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$, and F is 0-excisive if and only if it is constant (up to equivalence). We write

$$\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

for the full subcategory of the functor ∞-category whose objects are the n-excisive functors.

Remark 6.3. Motivation for this definition comes from algebraic topology; the property of being excisive is closely related to the excision property in the Eilenberg-Steenrod axioms for homology, and excisive functors on the ∞-category of pointed topological spaces are closely related to homology theories.

One of Goodwillie's key constructions is of a universal n-excisive approximation for functors between suitable ∞-categories. In order to state a condition for this approximation to exist, we introduce the following definition from Lurie [**?**, 6.1.1.6].

Definition 6.4. An ∞-category $\mathcal{C}$ is *differentiable* if it admits finite limits and sequential colimits (i.e. colimits along countable sequences of composable morphisms), and those limits and colimits commute.

Remark 6.5. We caution the reader to distinguish carefully between the words 'differentiable', as applied to an ∞-category in the above definition, and 'differential', which we introduced in Section **??** to refer to the structure inherent on a tangent space in any tangent ∞-category. In Section **??** we make that distinction particularly challenging by considering differential objects in the ∞-category of differentiable ∞-categories, or 'differential differentiable ∞-categories' if you prefer.

The following result is due to Lurie [**?**, 6.1.1.10] in this generality, but its proof is based on Goodwillie's original construction from [**?**, Sec. 1].

Proposition 6.6. *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories such that $\mathcal{C}$ has finite colimits and a terminal object, and $\mathcal{D}$ is differentiable. Then the inclusion*

$$\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

admits a left adjoint P_n which preserves finite limits.

Proposition **??** implies that a functor $F : \mathcal{C} \to \mathcal{D}$ admits a universal n-excisive approximation map $p_n : F \to P_n F$, with the property that any map $F \to G$, where G is n-excisive, factors uniquely (i.e. up to contractible choice) as $F \to P_n F \to G$.

Definition 6.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories that satisfy the conditions in Proposition **??**. The *Taylor tower* of F is the sequence of natural transformations

$$F \to \cdots \to P_n F \to P_{n-1} F \to \cdots \to P_1 F \to P_0 F = F(*)$$

determined by the universal property of each $P_n F$ and the observation that an n-excisive functor is also $(n+1)$-excisive, for each n.

We need the following generalization of [**?**, 3.1].

Lemma 6.8. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors between ∞-categories that satisfy the conditions of Proposition **??**, and suppose that F preserves the*

terminal object in $\mathcal{C}$*. Then the map*

$$P_n(GF) \to P_n((P_nG)F)$$

induced by $p_n : G \to P_nG$*, is an equivalence.*

Proof. Following Goodwillie [?], Lurie defines [?, 6.1.1.27]

$$P_nF := \text{colim}_k\, T_n^k F$$

where

$$T_nF(X) \simeq \lim_{\emptyset \neq S \subseteq [n]} F(C_S(X))$$

and $C_S(X) \simeq \text{hocofib}(\bigvee_S X \to X)$ is a model for the 'join' of the object $X \in \mathcal{C}$ with a finite set S. Since F preserves the terminal object, we get canonical maps

$$C_S(F(X)) \to F(C_S(X))$$

and hence

$$T_n(G)F \to T_n(GF)$$

and therefore also a map (commuting with the canonical maps from GF) of the form

$$P_n(G)F \to P_n(GF).$$

Since $P_n(GF)$ is n-excisive, this must factor via a map

$$P_n((P_nG)F) \to P_n(GF)$$

which we claim to be an inverse to the map in the statement of the Lemma. It follows from the universal property of P_n that the composite

$$P_n(GF) \to P_n((P_nG)F) \to P_n(GF)$$

is an equivalence. In the following diagram

$$
\begin{array}{ccccc}
P_n(GF) & \longrightarrow & P_n((P_nG)F) & \longrightarrow & P_n(GF) \\
\downarrow & & \downarrow & & \downarrow \\
P_n((P_nG)F) & \longrightarrow & P_nP_n((P_nG)F) & \longrightarrow & P_n((P_nG)F)
\end{array}
$$

the horizontal composites are equivalences, and the middle vertical map is an equivalence. It follows that the end map is a retract of an equivalence, hence is an equivalence. $\square$

A significant role in our later constructions is played by a multivariable version of Goodwillie's calculus, so we describe that briefly too. More details are in [?, 6.1.3].

Definition 6.9. Let $F : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ be a functor between ∞-categories such that each of $\mathcal{C}_1, \mathcal{C}_2$ admits finite colimits and a terminal object, and $\mathcal{D}$ is differentiable. We say that F is n_1-*excisive in its first variable* (and similarly for its second variable) if, for all $X_2 \in \mathcal{C}_2$, the functor $F(-, X_2) : \mathcal{C}_1 \to \mathcal{D}$ is n_1-excisive. We say that F is (n_1, n_2)-*excisive* if it is n_i-excisive in its i-th variable, for each i. We write

$$\mathrm{Exc}^{n_1, n_2}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$$

for the full subcategory on the (n_1, n_2)-excisive functors. The inclusion of this subcategory has a left adjoint P_{n_1, n_2} given by applying the functor P_{n_i} of Proposition **??** to each variable in turn, keeping the other variable constant.

We now begin our description of the Goodwillie tangent structure. First we introduce the underlying ∞-category for this structure.

Definition 6.10. Let $\mathbb{C}\mathrm{at}_\infty$ be Lurie's model [**?**, 3.0.0.1] for the ∞-category of ∞-categories[8], and let $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \subseteq \mathbb{C}\mathrm{at}_\infty$ be the subcategory whose objects are the differentiable ∞-categories and whose morphisms are the functors that preserve sequential colimits.

The tangent bundle functor for the Goodwillie tangent structure is an endofunctor $T : \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ defined by Lurie in [**?**, 7.3.1.10]. That definition, and much of the rest of this paper, depends heavily on the ∞-category $\mathcal{S}_{\mathrm{fin},*}$ of 'finite pointed spaces', so let us be entirely explicit about that object.

Definition 6.11. We say that a simplicial set is *finite* if it is homotopy-equivalent to the singular simplicial set on a finite CW-complex. Let $\mathcal{S}_{\mathrm{fin},*}$ denote the simplicial nerve of the simplicial category in which an object is a pointed Kan complex (X, x_0) with X finite, with enrichment given by the pointed mapping spaces

$$\mathrm{Hom}_{\mathcal{S}_{\mathrm{fin},*}}((X, x_0), (Y, y_0)) \subseteq \mathrm{Hom}(X, Y)$$

whose vertices are the basepoint-preserving maps $X \to Y$. Since these mapping spaces are Kan complexes, $\mathcal{S}_{\mathrm{fin},*}$ is a quasi-category by [**?**, 1.1.5.10].

Explicitly, an object of $\mathcal{S}_{\mathrm{fin},*}$ is a finite pointed Kan complex, and a morphism is a basepoint-preserving map. A 2-simplex in $\mathcal{S}_{\mathrm{fin},*}$ consists of a diagram of

---

[8]Recall that we are not being explicit about the various size restrictions on our ∞-categories, but for this definition to make sense, we of course require the objects in the ∞-category $\mathbb{C}\mathrm{at}_\infty$ to be restricted to be smaller than $\mathbb{C}\mathrm{at}_\infty$ itself.

basepoint-preserving maps

$$(X, x_0) \xrightarrow{\quad h \quad} (Z, z_0)$$

with f down to (Y, y_0) and g up to (Z, z_0)

together with a basepoint-preserving simplicial homotopy $h \simeq gf$. Higher simplexes in $\mathcal{S}_{\mathrm{fin},*}$ involve more complicated diagrams of basepoint-preserving maps and higher-dimension homotopies.

Definition 6.12 ([**?**, 7.3.1.10]). Let $\mathcal{C}$ be a differentiable ∞-category. The *tangent bundle* on $\mathcal{C}$ is the ∞-category

$$T(\mathcal{C}) := \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C})$$

of excisive functors $\mathcal{S}_{\mathrm{fin},*} \to \mathcal{C}$ (i.e. those that map pushout squares in $\mathcal{S}_{\mathrm{fin},*}$ to pullback squares in $\mathcal{C}$).

The tangent bundle on $\mathcal{C}$ is equipped with a projection map

$$p_{\mathcal{C}} : T(\mathcal{C}) \to \mathcal{C}; \quad L \mapsto L(*)$$

given by evaluating an excisive functor L at the one-point space $*$.

Remark 6.13. Lurie actually makes Definition **??** for a slightly different class of ∞-categories: those that are *presentable* in the sense of [**?**, 5.5.0.1]. There is a significant overlap between the presentable and differentiable ∞-categories including any compactly-generated ∞-category and any ∞-topos; see [**?**, 6.1.1].

It seems very likely that most of the rest of this paper could be made with the ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ replaced by an ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{pres}}$ of presentable ∞-categories in which the morphisms are those functors that preserve all *filtered* colimits, not merely the sequential colimits. For example, Proposition **??** holds when $\mathcal{D}$ is a presentable ∞-category, and [**?**, 7.3.1.14] implies that Definition **??** allows for T to be extended to a functor $T : \mathbb{C}\mathrm{at}_\infty^{\mathrm{pres}} \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{pres}}$. However, much of our argument is based on results from [**?**, Sec. 6] which is written in the context of differentiable ∞-categories, so we follow that lead.

Definition 6.14. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between differentiable ∞-categories. We define a functor

$$T(F) : T(\mathcal{C}) \to T(\mathcal{D})$$

by the formula

$$L \mapsto P_1(FL).$$

It follows from Lemma **??** below that the constructions of Definitions **??** and **??** can be made into a functor

$$T : \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$$

so that the projection maps $p_{\mathcal{C}}$ form a natural transformation p from T to the identity functor I.

The following result is the main theorem of this paper.

Theorem 6.15. *The tangent bundle functor* $T : \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ *and the projection map* $p : T \to I$ *extend to a tangent structure (in the sense of Definition* **??***) on the* ∞*-category* $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$*. We refer to this structure as the* Goodwillie tangent structure.

We believe that the tangent structure described in Theorem **??** is *unique* (i.e. that the space of tangent structures that extend T and p is contractible) though we will not prove that claim here.

The proof of Theorem **??** occupies the next two sections, concluding with the proof of Theorem **??**. In Section **??** we introduce the basic constructions and definitions which underlie the tangent structure; that is, we describe how to construct, from a Weil-algebra A and a differentiable ∞-category $\mathcal{C}$, a new ∞-category $T^A(\mathcal{C})$, and how these constructions interact with morphisms of Weil-algebras and (sequential-colimit-preserving) functors of ∞-categories.

In Section **??** we turn those constructions into an actual tangent structure on the ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$. It turns out to be more convenient to define that tangent structure on an ∞-category $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ that is equivalent to $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, and whose definition is based on the notion of *relative* ∞-category which we describe in Definition **??**. Having established a tangent structure on $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, we transfer that structure to $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ using Lemma **??**.

In Section **??** we begin the study of the Goodwillie tangent structure by identifying its differential objects in the sense of Definition **??**. Those differential objects turn out to be precisely the stable ∞-categories. That fact is of no surprise given that the tangent bundle construction **??** is formed precisely so that its tangent spaces are the stable ∞-categories $\mathcal{S}p(\mathcal{C}_{/X})$. Nonetheless this observation is a check that our tangent structure is acting as intended.

The characterization of differential objects as stable ∞-categories also confirms the intuition, promoted by Goodwillie, that in the analogy between functor calculus and the ordinary calculus of manifolds one should view the category of spectra as playing the role of Euclidean space.

Further developing that analogy, in Section **??** we characterize the *n-excisive* functors for $n > 1$, as corresponding to a notion of '*n*-jet' of a smooth map between manifolds. The construction of a Taylor tower itself does not precisely fit into the framework of tangent ∞-categories because it involves non-invertible natural transformations. We therefore show in Section **??** that the Goodwillie tangent structure on $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ extends to a tangent structure, in the sense of Definition **??**, on an ∞-bicategory $\mathbb{C}\mathrm{AT}_\infty^{\mathrm{diff}}$ whose underlying ∞-category is $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$. That tangent ∞-bicategory is the natural setting for Goodwillie calculus.

7. The Goodwillie tangent structure: underlying data

Our goal in this section is to introduce the basic data of the Goodwillie tangent structure on the ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ without paying attention to the higher coherence information needed to obtain an actual tangent ∞-category. Thus we define the tangent structure on objects and morphisms in $\mathbb{W}\mathrm{eil}$ and $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, and prove basic lemmas concerning functoriality and the preservation of pullbacks, including the crucial vertical lift axiom (Proposition **??**).

Tangent structure on objects. We start by defining $T^A(\mathcal{C})$ for a Weil-algebra A and a differentiable ∞-category $\mathcal{C}$. When $A = W$, this definition reduces precisely to Definition **??**.

Definition 7.1. Let $A = W^{n_1} \otimes \cdots \otimes W^{n_r}$ be an object in $\mathbb{W}\mathrm{eil}$ with $n = n_1 + \cdots + n_r$ generators. We write

$$\mathcal{S}_{\mathrm{fin},*}^n := \mathcal{S}_{\mathrm{fin},*} \times \cdots \times \mathcal{S}_{\mathrm{fin},*}$$

for the product of n copies of the ∞-category $\mathcal{S}_{\mathrm{fin},*}$ (with $\mathcal{S}_{\mathrm{fin},*}^0 = *$). Let

$$(7.2) \qquad T^A(\mathcal{C}) = \mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}) := \mathrm{Exc}^{1,\ldots,1}(\mathcal{S}_{\mathrm{fin},*}^{n_1} \times \cdots \times \mathcal{S}_{\mathrm{fin},*}^{n_r}, \mathcal{C})$$

be the full subcategory of the functor ∞-category $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ that consists of those functors

$$L : \mathcal{S}_{\mathrm{fin},*}^{n_1} \times \cdots \times \mathcal{S}_{\mathrm{fin},*}^{n_r} \to \mathcal{C}$$

that are excisive (i.e. take pushouts to pullbacks) in each of their r variables individually; see Definition **??**. We say that a functor L with this property is *A-excisive*.

It is crucial in Definition **??** that we view L as a functor of r variables, each of the form $\mathcal{S}_{\mathrm{fin},*}^{n_j}$, according to the description of the Weil-algebra A as a tensor product of terms of the form W^{n_j}.

Examples 7.3. We can identify some particular examples of $T^A(\mathcal{C})$ to show that we are on the right track to define a tangent structure.

(1) For $A = \mathbb{N}$, the unit object for the monoidal structure on $\mathbb{W}$eil, we get the identity functor:
$$T^{\mathbb{N}}(\mathcal{C}) = \mathrm{Fun}(\mathcal{S}^0_{\mathrm{fin},*}, \mathcal{C}) \cong \mathcal{C}.$$

(2) For $A = W = \mathbb{N}[x]/(x^2)$, we get the tangent bundle functor from Definition **??**:
$$T^W(\mathcal{C}) = T(\mathcal{C}) = \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}).$$

(3) For $A = W \otimes W = \mathbb{N}[x, y]/(x^2, y^2)$, we get the ∞-category of functors $\mathcal{S}^2_{\mathrm{fin},*} \to \mathcal{C}$ that are excisive *in each variable individually*. In the notation of [**?**, 6.1.3.1] we can write
$$T^{W \otimes W}(\mathcal{C}) = \mathrm{Exc}^{1,1}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C}),$$
and we have $T^{W \otimes W}(\mathcal{C}) \simeq T^W(T^W(\mathcal{C}))$ as required in a tangent structure; see [**?**, 6.1.3.3].

(4) For $A = W^2 = \mathbb{N}[x, y]/(x^2, xy, y^2)$, we get the ∞-category of functors $\mathcal{S}^2_{\mathrm{fin},*} \to \mathcal{C}$ that are excisive when viewed as a functor of *one* variable:
$$T^{W^2}(\mathcal{C}) = \mathrm{Exc}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C}).$$
For this definition to satisfy the pullback conditions in a tangent structure, we need to have an equivalence of ∞-categories
$$\mathrm{Exc}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C}) \simeq \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}),$$
and indeed there is such an equivalence under which an excisive functor $L : \mathcal{S}^2_{\mathrm{fin},*} \to \mathcal{C}$ corresponds to the pair of excisive functors
$$(L(*, -), L(-, *)),$$
and the pair (L_1, L_2) with $L_1(*) = L_2(*)$ corresponds to the excisive functor given by the fibre product
$$(X, Y) \mapsto L_1(X) \times_{L_1(*) = L_2(*)} L_2(Y).$$
That claim is proved in the next Lemma.

Lemma 7.4. *Let $\mathcal{S}_1, \mathcal{S}_2$ be ∞-categories each with finite colimits and a terminal object $*$, and let $\mathcal{C}$ be a differentiable ∞-category. Then a functor*
$$L : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{C}$$
is excisive (as a functor of one variable) if and only if

(1) *L is excisive in each variable individually; and*
(2) *The morphisms $X_1 \to *$ and $X_2 \to *$ determine equivalences*

$$L(X_1, X_2) \simeq L(X_1, *) \times_{L(*,*)} L(*, X_2)$$

for all $X_1 \in \mathcal{S}_1$, $X_2 \in \mathcal{S}_2$.

Proof. Key to this result is the fact that a pushout square in the ∞-category $\mathcal{S}_1 \times \mathcal{S}_2$ is a square for which each component is a pushout in its respective $\mathcal{S}_i$.

Suppose that L is excisive. Applying that condition to a pushout square in $\mathcal{S}_1 \times \mathcal{S}_2$ of the form

$$
\begin{array}{ccc}
(X_0, Y) & \longrightarrow & (X_1, Y) \\
\downarrow & & \downarrow \\
(X_2, Y) & \longrightarrow & (X_{12}, Y)
\end{array}
$$

consisting of an arbitrary pushout in $\mathcal{S}_1$ and a fixed object in $\mathcal{S}_2$, we deduce that L is excisive in its $\mathcal{S}_1$ variable. Similarly for its $\mathcal{S}_2$ variable.

Applying the condition that L is excisive to a pushout square in $\mathcal{S}_1 \times \mathcal{S}_2$ of the form

$$
\begin{array}{ccc}
(X_1, X_2) & \longrightarrow & (X_1, *) \\
\downarrow & & \downarrow \\
(*, X_2) & \longrightarrow & (*, *)
\end{array}
$$

we deduce condition (2).

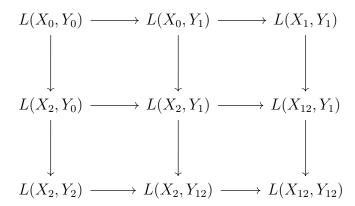Conversely, suppose that L satisfies conditions (1) and (2), and consider a diagram

$$
\begin{array}{ccc}
(X_0, Y_0) & \longrightarrow & (X_1, Y_1) \\
\downarrow & & \downarrow \\
(X_2, Y_2) & \longrightarrow & (X_{12}, Y_{12})
\end{array}
$$

that is a pushout in each component.

Consider the following diagram in $\mathcal{C}$:

$$
\begin{array}{ccccc}
L(X_0, Y_0) & \longrightarrow & L(X_0, Y_1) & \longrightarrow & L(X_1, Y_1) \\
\downarrow & & \downarrow & & \downarrow \\
L(X_2, Y_0) & \longrightarrow & L(X_2, Y_1) & \longrightarrow & L(X_{12}, Y_1) \\
\downarrow & & \downarrow & & \downarrow \\
L(X_2, Y_2) & \longrightarrow & L(X_2, Y_{12}) & \longrightarrow & L(X_{12}, Y_{12})
\end{array}
$$

The top-right and bottom-left squares are pullbacks because L is excisive in each variable individually, so it is sufficient to show that the top-left and bottom-right squares are also pullbacks, since then the whole square is a pullback by standard pasting properties of pullbacks (see [**?**, 4.4.2.1]).

For the top-left square (the bottom-right is similar), consider the diagram

$$
\begin{array}{ccccc}
L(X_0, Y_0) & \longrightarrow & L(X_0, Y_1) & \longrightarrow & L(X_0, *) \\
\downarrow & & \downarrow & & \downarrow \\
L(X_2, Y_0) & \longrightarrow & L(X_2, Y_1) & \longrightarrow & L(X_2, *) \\
\downarrow & & \downarrow & & \downarrow \\
L(*, Y_0) & \longrightarrow & L(*, Y_1) & \longrightarrow & L(*, *)
\end{array}
$$

Condition (2) implies that the bottom-right square, the bottom half, right-hand half, and entire square are all pullbacks. From that it follows by a succession of applications of [**?**, 4.4.2.1] that each individual square is a pullback too, including the top-left as required. $\qquad\square$

A crucial role in the following sections will be played by the following universal A-excisive approximation.

Proposition 7.5. *Let A be a Weil-algebra with n generators, and let $\mathcal{C}$ be a differentiable ∞-category. Then there is a functor*

$$
P_A : \operatorname{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C}) \to \operatorname{Exc}^A(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C})
$$

that is left adjoint to the inclusion and preserves finite limits. Moreover
$T^A(\mathcal{C}) = \mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ *is a differentiable ∞-category with finite limits and sequential colimits all calculated objectwise in $\mathcal{C}$. We write*

$$p_A : F \to P_A(F)$$

for the corresponding universal P_A-approximation map.

Proof. The first part is an example of the multivariable excisive approximation construction described in Definition **??**. It follows from the definition of excisive, and the fact that $\mathcal{C}$ is differentiable, that the subcategory $\mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ of $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ is closed under finite limits and sequential colimits which are computed objectwise in $\mathcal{C}$, and hence commute. Therefore $\mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ is differentiable. $\square$

The following property of P_A is used in the proof of Lemma **??**.

Lemma 7.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between differentiable ∞-categories that preserves both finite limits and sequential colimits. Then the following diagram commutes (up to natural equivalence)*

$$
\begin{array}{ccc}
\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}) & \xrightarrow{\ P_A\ } & \mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}) \\
\downarrow{\scriptstyle F_*} & & \downarrow{\scriptstyle F_*} \\
\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{D}) & \xrightarrow{\ P_A\ } & \mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{D})
\end{array}
$$

where each functor F_* denotes post-composition with F.

Proof. This claim is a multivariable version of [**?**, 6.1.1.32] and follows from the explicit construction of the excisive approximation, as in the proof of Lemma **??**. $\square$

Tangent structure on morphisms in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$. We now turn to the action of the tangent structure functors T^A on morphisms in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, i.e. functors $F : \mathcal{C} \to \mathcal{D}$ between differentiable ∞-categories that preserve sequential colimits. This action is the obvious extension of that described in **??** for the tangent bundle functor T.

Definition 7.7. Let A be a Weil-algebra, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor between differentiable ∞-categories that preserves sequential colimits. We define

$$T^A(F) : T^A(\mathcal{C}) \to T^A(\mathcal{D})$$

to be the composite

$$\mathrm{Exc}^A(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C}) \xrightarrow{F_*} \mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{D}) \xrightarrow{P_A} \mathrm{Exc}^A(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{D}).$$

That is, we have

(7.8) $$T^A(F)(L) := P_A(FL)$$

for an A-excisive functor $L : \mathcal{S}^n_{\mathrm{fin},*} \to \mathcal{C}$.

Lemma 7.9. *Let A be a Weil-algebra, and let $F : \mathcal{C} \to \mathcal{D}$ be a sequential-colimit-preserving functor between differential ∞-categories. Then $T^A(F)$ also preserves sequential colimits.*

Proof. Each of the functors F_*, P_A in Definition **??** preserves sequential colimits which, in all cases, are computed pointwise in the ∞-categories $\mathcal{C}$ and $\mathcal{D}$. $\square$

We now check that Definition **??** makes T^A into a functor $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty \to \mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$, at least up to higher equivalence.

Lemma 7.10. *Let A be a Weil-algebra. Then:*

(1) *for the identity functor $I_\mathcal{C}$ on a differential ∞-category $\mathcal{C}$, there is a natural equivalence*

$$I_{T^A(\mathcal{C})} \xrightarrow{\sim} T^A(I_\mathcal{C})$$

given by the maps $p_A : L \xrightarrow{\sim} P_A(L)$ for A-excisive $L : \mathcal{S}^n_{\mathrm{fin},*} \to \mathcal{C}$;

(2) *for sequential-colimit-preserving functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ between differential ∞-categories, there is a natural equivalence*

$$T^A(GF) \xrightarrow{\sim} T^A(G)T^A(F)$$

which comprises natural equivalences

$$P_A(GFL) \xrightarrow{\sim} P_A(GP_A(FL))$$

induced by the A-excisive approximation map $p_A : FL \to P_A(FL)$ for $L \in \mathrm{Exc}^A(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C})$.

Proof. Part (1) is a standard property of excisive approximation. Part (2) is more substantial. When $A = W$, so that $P_A = P_1$, this result is proved by Lurie in [**?**, 7.3.1.14]. (That result is in the context of presentable ∞-categories and functors that preserve all filtered colimits, but the proof works equally well for differentiable ∞-categories and functors that only preserve sequential colimits. Fundamentally this result relies on the Klein-Rognes [**?**] Chain Rule as extended to ∞-categories by Lurie in [**?**, 6.2.1.24].)

In particular, Lurie's argument shows that when the functor $G : \mathcal{D} \to \mathcal{E}$ between differentiable ∞-categories preserves sequential colimits, we have an equivalence

$$P_1(GH) \to P_1(GP_1(H))$$

for any $H : \mathcal{S}_{\text{fin},*} \to \mathcal{D}$. That argument extends to the case of $H : \mathcal{S}_{\text{fin},*}^n \to \mathcal{D}$, simply by replacing $\mathcal{S}_{\text{fin},*}$ with $\mathcal{S}_{\text{fin},*}^n$ and the null object $*$ with $(*, \ldots, *)$. This observation provides the desired result when $A = W^n$.

Now consider an arbitrary Weil-algebra $A = W^{n_1} \otimes \cdots \otimes W^{n_r}$. Then we have

$$P_A = P_1^{(r)} \cdots P_1^{(1)}$$

where $P_1^{(j)}$ is the ordinary excisive approximation applied to a functor

$$\mathcal{S}_{\text{fin},*}^{n_1} \times \cdots \times \mathcal{S}_{\text{fin},*}^{n_r} \to \mathcal{D}$$

in its $\mathcal{S}_{\text{fin},*}^{n_j}$ variable. For any $H : \mathcal{S}_{\text{fin},*}^n \to \mathcal{D}$, the map $p_A : H \to P_A(H)$ factors as a composite

$$H \xrightarrow{\ p_1^{(1)}\ } P_1^{(1)} H \xrightarrow{\ p_1^{(2)}\ } P_1^{(2)} P_1^{(1)} H \xrightarrow{\ p_1^{(3)}\ } \ldots \xrightarrow{\ p_1^{(r)}\ } P_A H$$

of excisive approximations in each variable separately. Applying G to the j-th map yields a $P_1^{(j)}$-equivalence by the extended version of Lurie's argument, and hence a P_A-equivalence as required. Thus we have an equivalence

(7.11) $$P_A(GH) \xrightarrow{\ \sim\ } P_A(GP_A(H)).$$

Taking H to be FL yields the desired result. $\square$

Tangent structure on morphisms in $\mathbb{W}$**eil.** We next address functoriality of our tentative tangent structure in the $\mathbb{W}$eil variable. Let $\phi : A \to A'$ be a morphism in $\mathbb{W}$eil, and let $\mathcal{C}$ be a differentiable ∞-category. We construct a (sequential-colimit-preserving) functor

$$T^\phi(\mathcal{C}) : T^A(\mathcal{C}) \to T^{A'}(\mathcal{C})$$

as follows. Roughly speaking, T^ϕ is the map

$$\text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) \to \text{Exc}^{A'}(\mathcal{S}_{\text{fin},*}^{n'}, \mathcal{C})$$

given by precomposition with a suitable functor

$$\tilde{\phi} : \mathcal{S}_{\text{fin},*}^{n'} \to \mathcal{S}_{\text{fin},*}^n$$

whose definition mirrors the Weil-algebra morphism ϕ.

Definition 7.12. Let $\phi : A \to A'$ be a morphism between Weil-algebras with n and n' generators respectively. We know that ϕ is determined by the values

$$\phi(x_i) \in A' = \mathbb{N}[y_1, \ldots, y_{n'}]/(y_i y_j \mid i \sim' j)$$

for $i = 1, \ldots, n$, and that each such value can be written (uniquely up to reordering) as a sum of nontrivial monomials in A'. In other words, for each generator x_i of A we can write

(7.13)
$$\phi(x_i) = \sum_{j=1}^{r_i} y_{ij1} \cdots y_{ijl}$$

where each y_{ijk} is one of the generators of A'. We define a functor

$$\tilde{\phi} : \mathcal{S}_{\text{fin},*}^{n'} \to \mathcal{S}_{\text{fin},*}^{n}$$

by setting

$$\tilde{\phi}(Y_1, \ldots, Y_{n'}) := (X_1, \ldots, X_n)$$

where

$$X_i := \bigvee_{j=1}^{r_i} Y_{ij1} \wedge \ldots \wedge Y_{ijl}$$

is the wedge sum of smash products described by the same pattern as $\phi(x_i)$ in (**??**). If $\phi(x_i) = 0$, then we take $X_i := *$.

Definition 7.14. Let $\phi : A \to A'$ be a morphism between Weil-algebras, and let $\mathcal{C}$ be a differential ∞-category. Then we define

$$T^\phi(\mathcal{C}) : T^A(\mathcal{C}) \to T^{A'}(\mathcal{C}); \quad L \mapsto P_{A'}(L\tilde{\phi}),$$

that is, $T^\phi(\mathcal{C})$ is the composite

$$\text{Exc}^A(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) \subseteq \text{Fun}(\mathcal{S}_{\text{fin},*}^n, \mathcal{C}) \xrightarrow{\tilde{\phi}^*} \text{Fun}(\mathcal{S}_{\text{fin},*}^{n'}, \mathcal{C}) \xrightarrow{P_{A'}} \text{Exc}^{A'}(\mathcal{S}_{\text{fin},*}^{n'}, \mathcal{C})$$

where $\tilde{\phi}^*$ denotes precomposition with $\tilde{\phi}$.

Examples 7.15. Using Definition **??**, we can now work out how the fundamental natural transformations from Cockett and Cruttwell's definition of tangent structure [**?**, 2.3] manifest in our case.

(1) Let $\epsilon : W \to \mathbb{N}$ be the augmentation. Then $\tilde{\epsilon} : * \to \mathcal{S}_{\text{fin},*}$ is the functor that picks out the null object $*$, and so the *projection*

$$p := T^\epsilon : T(\mathcal{C}) \to \mathcal{C}$$

can be identified with the evaluation map

$$\text{Exc}(\mathcal{S}_{\text{fin},*}, \mathcal{C}) \to \mathcal{C}; \quad L \mapsto L(*),$$

as in Definition **??**.

(2) Let $\eta : \mathbb{N} \to W$ be the unit map. Then $\tilde{\eta} : \mathcal{S}_{\mathrm{fin},*} \to *$ is, of course, the constant map, and so the *zero section*

$$0 := T^\eta : \mathcal{C} \to T(\mathcal{C})$$

can be identified with the map

$$\mathcal{C} \to \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}); \quad C \mapsto \mathrm{const}_C$$

that picks out the constant functors.

(3) Let $\phi : W^2 \to W$ be the addition map given by $x \mapsto x$ and $y \mapsto x$. Then $\tilde{\phi} : \mathcal{S}_{\mathrm{fin},*} \to \mathcal{S}_{\mathrm{fin},*}^2$ is the diagonal $X \mapsto (X, X)$, and so the *addition*

$$+ := T^\phi : T^{W^2}(\mathcal{C}) \to T(\mathcal{C})$$

is given by the map

$$\mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}^2, \mathcal{C}) \to \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}); \quad L \mapsto (X \mapsto L(X, X)).$$

Under the equivalence $T^{W^2}(\mathcal{C}) \simeq T(\mathcal{C}) \times_{\mathcal{C}} T(\mathcal{C})$ described in **??**(**??**), we can identify $+$ with the fibrewise product map

$$\mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \to \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C});$$
$$(L_1, L_2) \mapsto L_1(-) \times_{L_1(*)=L_2(*)} L_2(-).$$

(4) Let $\sigma : W \otimes W \to W \otimes W$ be the symmetry map. Then $\tilde{\sigma} : \mathcal{S}_{\mathrm{fin},*}^2 \to \mathcal{S}_{\mathrm{fin},*}^2$ is given by $(X, Y) \mapsto (Y, X)$, and the *flip*

$$c := T^\sigma : T^2(\mathcal{C}) \to T^2(\mathcal{C})$$

is the symmetry map

$$\mathrm{Exc}^{1,1}(\mathcal{S}_{\mathrm{fin},*}^2, \mathcal{C}) \to \mathrm{Exc}^{1,1}(\mathcal{S}_{\mathrm{fin},*}^2, \mathcal{C}); \quad L \mapsto [(X, Y) \mapsto L(Y, X)].$$

(5) Let $\delta : W \to W \otimes W$ be the map $x \mapsto xy$. Then $\tilde{\delta} : \mathcal{S}_{\mathrm{fin},*}^2 \to \mathcal{S}_{\mathrm{fin},*}$ is the smash product $(X, Y) \mapsto X \wedge Y$, and the *vertical lift*

$$\ell := T^\delta : T(\mathcal{C}) \to T^2(\mathcal{C})$$

is the map

$$\mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \to \mathrm{Exc}^{1,1}(\mathcal{S}_{\mathrm{fin},*}^2, \mathcal{C}); \quad L \mapsto [(X, Y) \mapsto L(X \wedge Y)].$$

Remark 7.16. In each of the cases in Examples **??**, we did not need to apply $P_{A'}$ at the end because precomposition with $\tilde{\phi}$ preserved the excisiveness property. However, this is not always the case; for example, if $\phi : W \otimes W \to W$ is the fold map given by $x_1, x_2 \mapsto x$, then $\tilde{\phi}(X) = (X, X)$, but the functor $X \mapsto L(X, X)$ is not typically excisive when L is excisive in each variable individually.

We should check that T^ϕ is a morphism in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, and it turns out to have a stronger property too.

Lemma 7.17. *The functor $T^\phi : T^A(\mathcal{C}) \to T^{A'}(\mathcal{C})$ preserves finite limits and sequential colimits.*

Proof. This result follows immediately from Definition **??**, since $P_{A'}$ preserves these (co)limits which in each case are calculated objectwise in $\mathcal{C}$.. $\qquad\square$

We now check functoriality in the $\mathbb{W}$eil factor, at least up to higher equivalence.

Lemma 7.18. *Let $\mathcal{C}$ be a differential ∞-category. Then:*

(1) *for the identity morphism 1_A on a Weil-algebra A we have $\tilde{1}_A = I_{\mathcal{S}^n_{\mathrm{fin},*}}$, the identity functor on $\mathcal{S}^n_{\mathrm{fin},*}$, and so there is a natural equivalence*

$$I_{T^A(\mathcal{C})} \xrightarrow{\ \sim\ } T^{1_A}(\mathcal{C})$$

given by the maps $p_A : L \xrightarrow{\ \sim\ } P_A(L)$ for A-excisive $L : \mathcal{S}^n_{\mathrm{fin},*} \to \mathcal{C}$;

(2) *for morphisms $\phi_1 : A \to A'$ and $\phi_2 : A' \to A''$, there is a natural equivalence*

$$T^{\phi_2\phi_1}(\mathcal{C}) \xrightarrow{\ \sim\ } T^{\phi_2}(\mathcal{C})T^{\phi_1}(\mathcal{C})$$

given by a composite of two maps of the form

$$P_{A''}(-\widetilde{\phi_2\phi_1}) \xrightarrow{\ (i)\ } P_{A''}(-\tilde{\phi}_1\tilde{\phi}_2) \xrightarrow{\ (ii)\ } P_{A''}(P_{A'}(-\tilde{\phi}_1)\tilde{\phi}_2)$$

*induced (i) by a natural map $\alpha : \widetilde{\phi_2\phi_1} \to \tilde{\phi}_1\tilde{\phi}_2$, described in Definition **??** below, and (ii) by the universal A'-excisive approximation map $p_{A'}$.*

Definition 7.19. Suppose the Weil-algebras A, A', A'' are given by

$$A = \mathbb{N}[x_1, \ldots, x_n]/R, \quad A' = \mathbb{N}[y_1, \ldots, y_{n'}]/R', \quad A'' = \mathbb{N}[z_1, \ldots, z_{n''}]/R'',$$

and let $\phi_1 : A \to A'$ and $\phi_2 : A' \to A''$ be morphisms in $\mathbb{W}$eil.

Following the procedure described in Definition **??**, we write

$$\phi_1(x_i) = \sum_{j=1}^{r_i} y_{ij1} \cdots y_{ijl}, \quad \phi_2(y_i) = \sum_{j=1}^{s_i} z_{ij1} \cdots z_{ijk}$$

and

(7.20) $$(\phi_2\phi_1)(x_i) = \sum_{j=1}^{t_i} z_{ij1'} \cdots z_{ijk'}.$$

Combining the first two formulas, we also have

$$\phi_2(\phi_1(x_i)) = \sum_{j=1}^{r_i} \phi_2(y_{ij1} \cdots y_{ijl})$$

(7.21)

$$= \sum_{j=1}^{r_i} \sum_{j_1=1}^{s_{k_1}} \cdots \sum_{j_l=1}^{s_{k_l}} z_{(ij1)j_1 1} \cdots z_{(ijk)j_k k}.$$

Since the relations that define A'' are all quadratic monomials, we deduce that the final sum in (**??**) must agree with that in (**??**) except for the possible presence of extra monomials that are zero in A'', i.e. that include one of those defining relations. In particular, the sequence of monomials in (**??**) must be a subsequence of those appearing in (**??**). Translating this observation to the functors $\tilde{\phi}$, we deduce that, as functors $\mathcal{S}_{\mathrm{fin},*}^{n''} \to \mathcal{S}_{\mathrm{fin},*}^{n}$, there is a natural equivalence

(7.22) $$\tilde{\phi}_1 \tilde{\phi}_2 \simeq \widetilde{\phi_2 \phi_1} \vee \zeta(\phi_1, \phi_2)$$

where $\zeta(\phi_1, \phi_2)(Z_1, \ldots, Z_{n''})$ is a finite wedge sum of terms each of the form

$$Z_{k_1} \wedge \ldots \wedge Z_{k_q}$$

where $z_{k_i} z_{k_{i'}} = 0$ in A'' for some $i \neq i'$, i.e. $k_i \sim'' k_{i'}$ in the equivalence relation that defines the Weil-algebra A''.

We define $\alpha : \widetilde{\phi_2 \phi_1} \to \tilde{\phi}_1 \tilde{\phi}_2$ to be the inclusion of the wedge summand in the expression (**??**).

Proof of Lemma **??**. The calculation of $\tilde{1}_A$ in part (1) follows immediately from Definition **??**. Part (2) is much more substantial and will occupy the next several pages. We prove that each of the two maps (i) and (ii) is an equivalence, and both of those facts are important later in the paper.

For (i), let $L : \mathcal{S}_{\mathrm{fin},*}^{n} \to \mathcal{C}$ be an A-excisive functor. We have to show that the map

(7.23) $$P_{A''}(L\alpha) : P_{A''}(L\widetilde{\phi_1 \phi_2}) \to P_{A''}(L\tilde{\phi}_2 \tilde{\phi}_1)$$

is an equivalence. The idea here is that the difference between $\widetilde{\phi_1 \phi_2}$ and $\tilde{\phi}_2 \tilde{\phi}_1$, i.e. the factor $\zeta(\phi_1, \phi_2)$ in (**??**), consists of terms that make no contribution after taking the A''-excisive approximation.

By induction on the number of wedge summands in $\zeta(\phi_1, \phi_2)$, we reduce to showing that for any functor $\beta : \mathcal{S}_{\mathrm{fin},*}^{n''} \to \mathcal{S}_{\mathrm{fin},*}^{n}$, the inclusion $\beta \to \beta \vee \xi$ induces an equivalence

$$P_{A''}(L\beta) \to P_{A''}(L(\beta \vee \xi))$$

where
$$\xi(Z_1, \ldots, Z_{n''}) \simeq (*, \ldots, *, Z_{k_1} \wedge \ldots \wedge Z_{k_q}, *, \ldots, *)$$
for some sequence of indices $k_1, \ldots, k_r$ such that $z_{k_i} z_{k_{i'}} = 0$ in A'' for some $i \neq i'$. Equivalently, we show that the collapse map $\beta \vee \xi \to \beta$ induces an equivalence in the other direction.

Writing $L : \mathcal{S}^{n_1}_{\text{fin},*} \times \cdots \times \mathcal{S}^{n_r}_{\text{fin},*} \to \mathcal{C}$ according to the decomposition $A = W^{n_1} \otimes \cdots \otimes W^{n_r}$ of the Weil-algebra A, we can assume without loss of generality that $r = 1$ (i.e. $A = W^n$) since ξ is only non-trivial in one variable. In that case $L : \mathcal{S}^n_{\text{fin},*} \to \mathcal{C}$ is excisive. Applying L to the pushout diagram in $\text{Fun}(\mathcal{S}^{n''}_{\text{fin},*}, \mathcal{S}^n_{\text{fin},*})$ of the form

$$
\begin{array}{ccc}
\beta \vee \xi & \longrightarrow & \beta \\
\downarrow & & \downarrow \\
\xi & \longrightarrow & *
\end{array}
$$

and taking $P_{A''}$ (which preserves pullbacks), we get a pullback square

$$
\begin{array}{ccc}
P_{A''}(L(\beta \vee \xi)) & \longrightarrow & P_{A''}(L\beta) \\
\downarrow & & \downarrow \\
P_{A''}(L\xi) & \longrightarrow & P_{A''}(L(*))
\end{array}
$$

so it is sufficient to show that the bottom map is an equivalence. Replacing $\mathcal{C}$ with the slice ∞-category $\mathcal{C}_{/L(*)}$, we can assume that L is reduced. Our goal is then to show that the functor

$$(7.24) \qquad (Z_1, \ldots, Z_{n''}) \mapsto L(*, \ldots, *, Z_{k_1} \wedge \ldots \wedge Z_{k_q}, *, \ldots, *)$$

has trivial A''-excisive part whenever $z_{k_i} z_{k_{i'}} = 0$ in A'' for some $i \neq i'$.

If we write
$$\mathcal{S}^{n''}_{\text{fin},*} = \mathcal{S}^{n''_1}_{\text{fin},*} \times \cdots \times \mathcal{S}^{n''_s}_{\text{fin},*}$$
according to the decomposition $A'' = W^{n''_1} \otimes \cdots \otimes W^{n''_s}$, then the condition $z_{k_i} z_{k_{i'}} = 0$ implies that the variables Z_{k_i} and $Z_{k_{i'}}$ are in the same factor $\mathcal{S}^{n''_j}_{\text{fin},*}$, and it is sufficient to show that the functor (??) has trivial excisive approximation with respect to that factor. Without loss of generality, we can take $A'' = W^{n''}$ so that we simply have to show that a functor $F : \mathcal{S}^{n''}_{\text{fin},*} \to \mathcal{C}$ of the form (??) has trivial excisive approximation whenever $q \geq 2$.

Since F is reduced in each of the variables Z_{k_i}, it is, by [**?**, 6.1.3.10], q-reduced when viewed as a functor of all of those variables. Since $q \geq 2$, it follows that F has trivial excisive approximation with respect to those variables. It follows from Lemma **??** that the excisive approximation of a functor $\mathcal{S}_{\mathrm{fin},*}^{n''} \to \mathcal{C}$ factors via its excisive approximation with respect to any subset of its variables, so F also has trivial excisive approximation as a functor $\mathcal{S}_{\mathrm{fin},*}^{n''} \to \mathcal{C}$, as desired. This completes the proof that the map (i) is an equivalence.

For (ii), we prove the following: for any $G : \mathcal{S}_{\mathrm{fin},*}^{n_1'} \times \cdots \times \mathcal{S}_{\mathrm{fin},*}^{n_r'} \to \mathcal{C}$ and any Weil-algebra morphism $\phi : A' \to A''$, there is an equivalence

$$(7.25) \qquad P_{A''}(G\tilde{\phi}) \xrightarrow{\sim} P_{A''}(P_{A'}(G)\tilde{\phi})$$

induced by the A'-excisive approximation map $p_{A'} : G \to P_{A'}G$.

It is sufficient to show that excisive approximation with respect to each of the r variables induces an equivalence. Thus we can reduce to the case that $A' = W^{n'}$, in which case $P_{A'} = P_1$. Replacing $\mathcal{C}$ with the slice ∞-category $\mathcal{C}_{G(*)}$ of objects over and under $G(*)$, we can also assume that $\mathcal{C}$ is a pointed ∞-category and that G is reduced, i.e. $G(*) \simeq *$. So we have to show that for any reduced $G : \mathcal{S}_{\mathrm{fin},*}^{n'} \to \mathcal{C}$ the map

$$(7.26) \qquad P_{A''}(G\tilde{\phi}) \to P_{A''}((P_1G)\tilde{\phi})$$

is an equivalence.

We break this proof into two parts. First we use downward induction on the Taylor tower of G to reduce to the case that G is m-homogeneous for some $m \geq 2$. From there we use the specific form of $\tilde{\phi}$, and the fact that ϕ is an algebra homomorphism, to show (**??**) is an equivalence by direct calculation.

To start the induction we apply Lemma **??** which tells us that, since $\tilde{\phi}$ is reduced, the universal n-excisive approximation $p_n : G \to P_nG$ induces an equivalence

$$(7.27) \qquad P_n(G\tilde{\phi}) \xrightarrow{\sim} P_n((P_nG)\tilde{\phi})$$

for any n.

If $n \geq s$, then an A''-excisive functor $\mathcal{S}_{\mathrm{fin},*}^{n_1''} \times \cdots \times \mathcal{S}_{\mathrm{fin},*}^{n_s''} \to \mathcal{C}$ is n-excisive by [**?**, 6.1.3.4]. It follows from (**??**) that p_n induces an equivalence

$$(7.28) \qquad P_{A''}(G\tilde{\phi}) \xrightarrow{\sim} P_{A''}((P_nG)\tilde{\phi}).$$

Next consider the fibre sequences

$$P_mG \to P_{m-1}G \to R_mG$$

provided by [**?**, 6.1.2.4] (Goodwillie's delooping theorem for homogeneous functors) in which $R_m G$ is m-homogenous. We show below that

$$(7.29) \qquad\qquad P_{A''}(H\tilde{\phi}) \simeq *$$

for any functor $H : \mathcal{S}_{\text{fin},*}^{n'} \to \mathcal{C}$ that is m-homogeneous for some $m \geq 2$. Since $P_{A''}$ preserves fibre sequences, we then deduce that for $m \geq 2$

$$P_{A''}((P_m G)\tilde{\phi}) \xrightarrow{\sim} P_{A''}((P_{m-1}G)\tilde{\phi}),$$

which combined with (**??**) implies the desired result (**??**).

So our goal now is (**??**). By [**?**, 6.1.2.9], we can assume that $\mathcal{C}$ is a stable ∞-category, and then Lemma **??** below gives a classification of m-homogeneous functors $\mathcal{S}_{\text{fin},*}^{n'} \to \mathcal{C}$. That result tells us that H is a finite product of terms of the form

$$(7.30) \qquad (Y_1, \ldots, Y_{n'}) \mapsto L(Y_1^{\wedge m_1} \wedge \ldots \wedge Y_{n'}^{\wedge m_{n'}})_{h\Sigma_{m_1} \times \cdots \times \Sigma_{m_{n'}}}$$

for linear $L : \mathcal{S}_{\text{fin},*} \to \mathcal{C}$ and $m = m_1 + \cdots + m_{n'}$. Since $P_{A''}$ preserves finite products, we can reduce our goal (**??**) to the case that H is of the form in (**??**).

The next part of the proof depends essentially on the fact that $\phi : A' \to A''$ is an algebra homomorphism, and we start by describing what that condition implies about the functor $\tilde{\phi}$.

Recall that we have reduced to the case

$$A' = W^{n'} = \mathbb{N}[y_1, \ldots, y_{n'}]/(y_i y_{i'})_{i,i'=1,\ldots,n'}$$

and let us denote the generators of

$$A'' = W^{n_1''} \otimes \cdots \otimes W^{n_s''}$$

as $z_{j,1}, \ldots, z_{j,n_j''}$ for $j = 1, \ldots, s$. Similarly, we denote inputs to the corresponding functor

$$\tilde{\phi} : \mathcal{S}_{\text{fin},*}^{n_1''} \times \cdots \times \mathcal{S}_{\text{fin},*}^{n_s''} \to \mathcal{S}_{\text{fin},*}^{n'}$$

with the notation $Z_{j,k}$.

Since ϕ is an algebra homomorphism, we have, for any i, i':

$$\phi(y_i)\phi(y_{i'}) = \phi(y_i y_i') = \phi(0) = 0.$$

There are no terms in A'' with negative coefficients, so this means that the product of any monomial in $\phi(y_i)$ and any monomial in $\phi(y_{i'})$ must contain a factor of the form $z_{j,k} z_{j,k'}$ for some j, k, k'.

Translating this observation into a statement about the map $\tilde{\phi}$, we see that for all $i, i' = 1, \ldots, n'$:

$$(7.31) \qquad \tilde{\phi}_i(\underline{Z}) \wedge \tilde{\phi}_{i'}(\underline{Z}) \simeq \bigvee Z_{j_1, k_1} \wedge \ldots \wedge Z_{j_t, k_t},$$

a finite wedge sum of terms each of which satisfies $j_l = j_{l'}$ for some $l \neq l'$.

Now consider the functor $H\tilde{\phi} : \mathcal{S}_{\text{fin},*}^{n''} \to \mathcal{C}$ where H is of the form (??) with $m = m_1 + \cdots + m_{n'} \geq 2$. Applying H to $\tilde{\phi}$ that satisfies (??), and since the linear functor L takes finite wedge sums to products, we obtain a decomposition of the form

$$(7.32) \qquad H\tilde{\phi}(\underline{Z}) \simeq \prod L(Z_{j_1, k_1} \wedge \ldots \wedge Z_{j_t, k_t})_{hG}$$

a product of terms where $L : \mathcal{S}_{\text{fin},*} \to \mathcal{C}$ is linear, $j_l = j_{l'}$ for some $l \neq l'$, and G is a subgroup of Σ_m that acts by permuting some of the factors $Z_{j,k}$.

By Lemma ?? each of the terms in (??) is r_j-homogeneous in the variable $\mathcal{S}_{\text{fin},*}^{n''_j}$ where r_j is the number of times that the index j appears in the list $j_1, \ldots, j_t$. We know that there is some j such that $r_j \geq 2$, which implies that each of these terms has trivial A''-excisive approximation. We therefore obtain the desired (??):

$$P_{A''}(H\tilde{\phi}) \simeq *.$$

Putting this calculation together with our earlier induction, we get the equivalence (??) which completes the proof that map (ii) is an equivalence. $\qquad \square$

Lemma 7.33. *Let $\mathcal{C}$ be a stable ∞-category. A functor $H : \mathcal{S}_{\text{fin},*}^k \to \mathcal{C}$ is m-homogeneous if and only if it can be written in the form*

$$H(Y_1, \ldots, Y_k) \simeq \prod_{m_1 + \cdots + m_k = m} L_{(m_1, \ldots, m_k)}(Y_1^{\wedge m_1} \wedge \ldots \wedge Y_k^{m_k})_{h\Sigma_{m_1} \times \cdots \times \Sigma_{m_k}}$$

for a collection of linear (i.e. reduced and excisive) functors

$$L_{(m_1, \ldots, m_k)} : \mathcal{S}_{\text{fin},*} \to \mathcal{C}$$

indexed by the ordered partitions of m into k non-negative integers. The subscript on the right-hand side of this equivalence denotes the (homotopy) coinvariants for the action of the group $\Sigma_{m_1} \times \cdots \times \Sigma_{m_k}$ that permutes each of the smash powers $Y_i^{\wedge m_i}$.

Proof. Suppose H is m-homogeneous. By [**?**, 6.1.4.14], we can write

$$H(\underline{Y}) \simeq M(\underline{Y}, \ldots, \underline{Y})_{h\Sigma_m}$$

where $M : (\mathcal{S}^k_{\mathrm{fin},*})^m \to \mathcal{C}$ is symmetric multilinear. By induction on Lemma **??**, a linear functor $\mathcal{S}^k_{\mathrm{fin},*} \to \mathcal{C}$ is of the form

$$L_1(Y_1) \times \cdots \times L_k(Y_k)$$

and it follows that the symmetric multilinear M can be written as

$$M(\underline{Y}_1, \ldots, \underline{Y}_m) \simeq \prod_{1 \le j_1, \ldots, j_m \le k} L_{j_1, \ldots, j_m}(Y_{1, j_1}, \ldots, Y_{m, j_m})$$

where the symmetric group Σ_m acts by permuting the indexes $j_1, \ldots, j_m$ as well as the inputs of each multilinear functor $L_{j_1, \ldots, j_m} : \mathcal{S}^m_{\mathrm{fin},*} \to \mathcal{C}$.

We therefore get

$$H(\underline{Y}) \simeq \prod_{m_1 + \cdots + m_k = m} L_{1, \ldots, 1, \ldots, k, \ldots, k}(Y_1, \ldots, Y_1, \ldots, Y_k, \ldots, Y_k)_{h \Sigma_{m_1} \times \cdots \times \Sigma_{m_k}}$$

where the index i is repeated m_i times in each term. Finally, by induction on [**?**, 1.4.2.22], multilinear functors $\mathcal{S}^m_{\mathrm{fin},*} \to \mathcal{C}$ factor via the smash product $\wedge : \mathcal{S}^m_{\mathrm{fin},*} \to \mathcal{S}_{\mathrm{fin},*}$ yielding the desired expression.

Conversely, suppose H is of the given form. It is sufficient to show that each functor of the form

$$F(Y_1, \ldots, Y_k) \mapsto L(Y_1^{\wedge m_1} \wedge \ldots \wedge Y_k^{\wedge m_k})$$

with L linear, is m-homogeneous, since the finite product and coinvariants constructions preserve homogeneity of functors with values in a stable ∞-category. It can be shown directly from the definition that F is m_i-excisive in its i-th variable, so that F is m-excisive by [**?**, 6.1.3.4]. To see that F is also m-reduced, we apply [**?**, 6.1.3.24] by directly calculating the relevant cross-effects of F. □

Finally for this section, we note the following compatibility between natural transformations of the form α which will be important for establishing the higher homotopy coherence of our constructions in Section **??**.

Lemma 7.34. *Let* $A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} A_2 \xrightarrow{\phi_3} A_3$ *be morphisms of Weil-algebras. Then the following diagram in* $\mathrm{Fun}(\mathcal{S}^{n_3}_{\mathrm{fin},*}, \mathcal{S}^{n_0}_{\mathrm{fin},*})$ *strictly commutes:*

$$
\begin{array}{ccc}
\widetilde{\phi_3 \phi_2 \phi_1} & \xrightarrow{\alpha_{12,3}} & \widetilde{\phi_2 \phi_1} \tilde{\phi}_3 \\
{\scriptstyle \alpha_{1,23}} \downarrow & & \downarrow {\scriptstyle \alpha_{1,2} \tilde{\phi}_3} \\
\tilde{\phi}_1 \widetilde{\phi_3 \phi_2} & \xrightarrow{\tilde{\phi}_1 \alpha_{2,3}} & \tilde{\phi}_1 \tilde{\phi}_2 \tilde{\phi}_3
\end{array}
$$

Proof. Following through Definition **??** we see that each composite in this diagram is the inclusion of those factors in the wedge sum $\tilde{\phi}_1\tilde{\phi}_2\tilde{\phi}_3$ that correspond to nonzero monomials in the Weil-algebra A_3. □

Basic tangent structure properties. We show in Section **??** that the constructions of Definitions **??**, **??** and **??** extend to a functor, i.e. map of simplicial sets,

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}.$$

But first we check that our definitions so far satisfy the various conditions needed to determine a tangent structure on $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, starting with the observation that T corresponds to an action of the monoidal ∞-category $\mathbb{W}\mathrm{eil}$ on $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$.

Lemma 7.35. *Let $\mathcal{C}$ be a differential ∞-category. Then:*

(1) *there is a natural isomorphism $T^{\mathbb{N}}(\mathcal{C}) \cong \mathcal{C}$ given by evaluation at the unique point in $\mathcal{S}_{\mathrm{fin},*}^0 = *$;*

(2) *for Weil-algebras A, A', there is a natural isomorphism*

$$T^{A'}(T^A(\mathcal{C})) \cong T^{A \otimes A'}(\mathcal{C})$$

given by restricting the isomorphism

$$\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n'}, \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})) \cong \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n+n'}, \mathcal{C})$$

to the subcategories of suitably excisive functors.

Proof. These results follow immediately from the definition of $T^A(\mathcal{C})$ in (**??**). □

We also verify that the foundational pullbacks (**??**) in $\mathbb{W}\mathrm{eil}$ are preserved by the action map $T.$.

Lemma 7.36. *Let A be a Weil-algebra, m, n positive integers, and $\mathcal{C}$ a differentiable ∞-category. Then the following diagram is a pullback of ∞-categories:*

$$
\begin{array}{ccc}
T^{A \otimes W^{m+n}}(\mathcal{C}) & \longrightarrow & T^{A \otimes W^m}(\mathcal{C}) \\
\downarrow & & \downarrow \\
T^{A \otimes W^n}(\mathcal{C}) & \longrightarrow & T^A(\mathcal{C})
\end{array}
$$

where the horizontal maps are induced by the augmentation $W^n \to \mathbb{N}$, and the vertical by $W^m \to \mathbb{N}$.

Proof. To prove this lemma, we should first specify precisely what we mean by a 'pullback of ∞-categories'. For the purposes of this proof, we take that condition to mean that the given diagram is a homotopy pullback in the Joyal model structure on simplicial sets, and hence a pullback in the ∞-category $\mathbb{C}\mathrm{at}_\infty$. (In the proof of Theorem **??** in the next section we show that this diagram is also a pullback in the subcategory $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$.)

Using Lemma **??**(2), and replacing $T^A(\mathcal{C})$ with $\mathcal{C}$, we can reduce to the case $A = \mathbb{N}$, and also to $m = n = 1$, in which case the diagram takes the following form

$$
\begin{array}{ccc}
\mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}^2, \mathcal{C}) & \xrightarrow{\ p_1\ } & \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \\
\Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle p} \\
\mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) & \xrightarrow{\ \ p\ \ } & \mathcal{C}
\end{array}
$$

where $p : \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \to \mathcal{C}$ is given by $L \mapsto L(*)$, and p_1, p_2 are similarly given by evaluating at $*$ in the each variable of $\mathcal{S}_{\mathrm{fin},*}^2$ respectively.

First we argue that the map p is a fibration in the Joyal model structure. This is true for the corresponding projection

$$
\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \to \mathcal{C}
$$

since $* \to \mathcal{S}_{\mathrm{fin},*}$ is a cofibration between cofibrant objects, and the Joyal model structure is closed monoidal. Since $\mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C})$ is a full subcategory of $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C})$ that is closed under equivalences, p is also a fibration by [**?**, 2.4.6.5].

It is now sufficient to show that the induced map

$$
\pi : \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*} \times \mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \to \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C})
$$

is an equivalence of ∞-categories.

To see this, we define an inverse map

$$
\iota : \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \to \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*} \times \mathcal{S}_{\mathrm{fin},*}, \mathcal{C})
$$

that sends a pair (F_1, F_2) of excisive functors $F_i : \mathcal{S}_{\mathrm{fin},*} \to \mathcal{C}$, such that $F_1(*) = F_2(*)$, to the functor

$$
(X_1, X_2) \mapsto F_1(X_1) \times_{F_2(*)} F_2(X_2).
$$

It is simple to check that $\pi\iota \simeq \mathrm{id}$, and it follows from Lemma **??** that $\iota\pi \simeq \mathrm{id}$. $\qquad\square$

The final thing we check in this section is that $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ has finite products given by the ordinary product of quasi-categories, which are preserved by the tangent bundle functor $T = T^W : \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$. This fact implies that the tangent structure on $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ is cartesian in the sense of Definition **??**.

Lemma 7.37. *For differentiable ∞-categories $\mathcal{C}, \mathcal{C}'$, the product $\mathcal{C} \times \mathcal{C}'$ is differentiable, and is the product of $\mathcal{C}$ and $\mathcal{C}'$ in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$. Moreover, the projections determine an equivalence of ∞-categories*

$$T(\mathcal{C} \times \mathcal{C}') \simeq T(\mathcal{C}) \times T(\mathcal{C}').$$

Proof. Since limits and colimits in the product ∞-category are detected in each term, the product $\mathcal{C} \times \mathcal{C}'$ is differentiable. A functor $\mathcal{D} \to \mathcal{C} \times \mathcal{C}'$, with $\mathcal{D}$ differentiable, preserves sequential colimits if and only if each component preserves sequential colimits. It follows that $\mathcal{C} \times \mathcal{C}'$ is the product in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$. Finally, the isomorphism

$$\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}_1 \times \mathcal{C}_2) \cong \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}_1) \times \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}_2)$$

restricts to the ∞-categories of excisive functors since a square in $\mathcal{C}_1 \times \mathcal{C}_2$ is a pullback if and only if it is a pullback in each factor. $\qquad\square$

The vertical lift axiom. The last substantial condition we need in order that the constructions earlier in this section underlie a tangent structure T on the ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ is that T preserves the vertical lift pullback (**??**) in the category $\mathbb{W}\mathrm{eil}$. Applying Definition **??** to the Weil-algebra morphisms that appear in that pullback square, and using Lemma **??**, we reduce to the following result.

Proposition 7.38. *Let $\mathcal{C}$ be a differentiable ∞-category. Then the following diagram is a pullback of ∞-categories:*

$$
\begin{array}{ccc}
T(\mathcal{C}) \times_{\mathcal{C}} T(\mathcal{C}) & \xrightarrow{\ v\ } & T(T(\mathcal{C})) \\
{\scriptstyle p \times p}\big\downarrow & & \big\downarrow{\scriptstyle T(p)} \\
\mathcal{C} & \xrightarrow{\ \ 0\ \ } & T(\mathcal{C})
\end{array}
$$

where v sends the pair (L_1, L_2) of excisive functors to the $(1,1)$-excisive functor $\mathcal{S}_{\mathrm{fin},*}^2 \to \mathcal{C}$ given by

$$(X, Y) \mapsto L_1(X \wedge Y) \times_{L_1(*)=L_2(*)} L_2(Y).$$

Proof. As in the proof of Lemma **??**, our goal is to show that this diagram is a homotopy pullback in the Joyal model structure on simplicial sets, and the argument for that lemma also shows that $T(p)$ is a fibration, so it is sufficient to show that the induced map

$$f : \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \to \mathcal{C} \times_{T\mathcal{C}} T^2(\mathcal{C})$$

is an equivalence of quasi-categories.

Using Lemma **??** again, we can write f as a map

$$f : \mathrm{Exc}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C}) \to \mathrm{Exc}^{1,1*}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C}); \quad L \mapsto [(X, Y) \mapsto L(X \wedge Y, Y)]$$

from the ∞-category of excisive functors $\mathcal{S}^2_{\mathrm{fin},*} \to \mathcal{C}$ to the ∞-category of functors $M : \mathcal{S}^2_{\mathrm{fin},*} \to \mathcal{C}$ that are excisive in each variable individually and reduced in the second variable (in the sense that the map $M(X, *) \xrightarrow{\sim} M(*, *)$ is an equivalence for each $X \in \mathcal{S}_{\mathrm{fin},*}$).

To show that f is an equivalence of quasi-categories, we describe an explicit homotopy inverse

$$g : \mathrm{Exc}^{1,1*}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C}) \to \mathrm{Exc}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C}).$$

For a functor $M : \mathcal{S}^2_{\mathrm{fin},*} \to \mathcal{C}$ we set

(7.39) $$g(M)(X, Y) := M(X, S^0) \times_{M(*, S^0)} M(*, Y)$$

where the map $M(*, Y) \to M(*, S^0)$ used to construct this pullback is induced by the null map $Y \to S^0$ (that maps every point in Y to the basepoint in S^0).

We should show that if M is excisive in each variable and reduced in Y, then $g(M)$ is excisive as a functor $\mathcal{S}^2_{\mathrm{fin},*} \to \mathcal{C}$. Consider a pushout square in $\mathcal{S}^2_{\mathrm{fin},*}$

$$\begin{array}{ccc}
(X, Y) & \longrightarrow & (X_1, Y_1) \\
\downarrow & & \downarrow \\
(X_2, Y_2) & \longrightarrow & (X_0, Y_0)
\end{array}$$

consisting of individual pushout squares in each variable. Applying $g(M)$ to this pushout square we get the square in $\mathcal{C}$ given by

$$\begin{array}{ccc}
M(X, S^0) \times_{M(*, S^0)} M(*, Y) & \longrightarrow & M(X_1, S^0) \times_{M(*, S^0)} M(*, Y_1) \\
\downarrow & & \downarrow \\
M(X_2, S^0) \times_{M(*, S^0)} M(*, Y_2) & \longrightarrow & M(X_0, S^0) \times_{M(*, S^0)} M(*, Y_0)
\end{array}$$

Since M is excisive in each variable, this is a pullback of pullback squares, and hence is itself a pullback. Therefore $g(M)$ is excisive.

We then have, for $L \in \mathrm{Exc}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C})$:

$$\begin{aligned} g(f(L))(X,Y) &= f(L)(X,S^0) \times_{f(L)(*,S^0)} f(L)(*,Y) \\ &= L(X \wedge S^0, S^0) \times_{L(*\wedge S^0, S^0)} L(* \wedge Y, Y) \\ &\simeq L(X, S^0) \times_{L(*,S^0)} L(*,Y). \end{aligned}$$

Consider the diagram

$$\begin{array}{ccccccc} L(X,Y) & \longrightarrow & L(X,*) & \longrightarrow & L(X,S^0) & \longrightarrow & L(X,*) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L(*,Y) & \longrightarrow & L(*,*) & \longrightarrow & L(*,S^0) & \longrightarrow & L(*,*). \end{array}$$

The first and third squares are pullbacks because L is excisive. The composite of the second and third squares is also a pullback. Hence the second square is a pullback by [**?**, 4.4.2.1], and so the composite of the first and second squares is also a pullback, again by [**?**, 4.4.2.1]. This calculation implies that the natural map

$$L \to g(f(L))$$

is an equivalence.

On the other hand we have, for $M \in \mathrm{Exc}^{1,1*}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C})$:

$$\begin{aligned} f(g(M))(X,Y) &= g(M)(X \wedge Y, Y) \\ &= M(X \wedge Y, S^0) \times_{M(*,S^0)} M(*,Y). \end{aligned}$$

We claim that $f(g(M))$ is equivalent to M. First note that M factors via the pointed ∞-category $\mathcal{C}_{M(*,*)}$, of objects over/under $M(*,*)$. We can therefore assume without loss of generality that $\mathcal{C}$ is pointed and that $M(*,*)$ is a null object. Since the map $M(*,Y) \to M(*,S^0)$ is then null, we can write $f(g(M))$ as the product

(7.40) $f(g(M))(X,Y) \simeq \mathrm{hofib}[M(X \wedge Y, S^0) \to M(*,S^0)] \times M(*,Y).$

Now observe that $M \in \mathrm{Exc}^{1,1*}(\mathcal{S}^2_{\mathrm{fin},*}, \mathcal{C})$ can be viewed as a functor

$$\mathcal{S}_{\mathrm{fin},*} \to \mathrm{Exc}_*(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}); \quad X \mapsto M(X,-)$$

from the *pointed* ∞-category $\mathcal{S}_{\mathrm{fin},*}$ to the *stable* ∞-category $\mathrm{Exc}_*(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \simeq \mathcal{S}p(\mathcal{C})$ of linear functors $\mathcal{S}_{\mathrm{fin},*} \to \mathcal{C}$. Any such functor splits off the image of the null object, so we have an equivalence

(7.41) $M(X,Y) \simeq \mathrm{hofib}[M(X,Y) \to M(*,Y)] \times M(*,Y).$

Comparing (**??**) and (**??**) we see that to get an equivalence $f(g(M)) \simeq M$, it is enough to produce an equivalence

$$D(X, Y) \simeq D(X \wedge Y, S^0)$$

where $D : \mathcal{S}^2_{\text{fin},*} \to \mathcal{C}$ is given by

$$D(X, Y) := \text{hofib}(M(X, Y) \to M(*, Y)).$$

This D is given by reducing M in its first variable, and it follows that D is reduced and excisive in both variables, i.e. is multilinear. The equivalence we need is a consequence of the classification of multilinear functors $\mathcal{S}^2_{\text{fin},*} \to \mathcal{C}$ for any ∞-category $\mathcal{C}$ with finite limits.

To be explicit, it follows from [**?**, 1.4.2.22] that the evaluation map

$$\text{Exc}^{1,1}_{*,*}(\mathcal{S}^2_{\text{fin},*}, \mathcal{C}) \to \text{Exc}_*(\mathcal{S}_{\text{fin},*}, \mathcal{C}); \quad D \mapsto D(-, S^0)$$

is an equivalence of ∞-categories, and it is clear that a one-sided inverse is given by

$$\text{Exc}_*(\mathcal{S}_{\text{fin},*}, \mathcal{C}) \to \text{Exc}^{1,1}_{*,*}(\mathcal{S}^2_{\text{fin},*}, \mathcal{C}); \quad F \mapsto F(X \wedge Y).$$

Thus this functor is a two-sided inverse, and we therefore get the desired natural equivalence

$$D(X, Y) \simeq D(X \wedge Y, S^0).$$

This completes the proof that $M \simeq f(g(M))$, and hence the proof that f is an equivalence of quasi-categories. $\square$

8. The Goodwillie tangent structure: formal construction

The constructions of Section **??** contain the basic data and lemmas on which our tangent structure rests, but to have a tangent ∞-category we need to extend those definitions to an actual functor (i.e. map of simplicial sets)

$$T : \mathbb{Weil} \times \mathbb{Cat}^{\text{diff}}_\infty \to \mathbb{Cat}^{\text{diff}}_\infty$$

which provides a strict action of the simplicial monoid $\mathbb{Weil}$ on the simplicial set $\mathbb{Cat}^{\text{diff}}_\infty$. The goal of this section is to construct such a map. We start by giving an explicit description of the ∞-category $\mathbb{Cat}^{\text{diff}}_\infty$.

The ∞-category of differentiable ∞-categories. We recall Lurie's model for the ∞-category of ∞-categories from [**?**, 3.0.0.1].

Definition 8.1. Let $\mathbb{C}\mathrm{at}_\infty$ be the simplicial nerve [**?**, 1.1.5.5] of the simplicial category whose objects are the quasi-categories, and for which the simplicial mapping spaces are the maximal Kan complexes inside the usual functor ∞-categories:

$$\mathrm{Hom}_{\mathbb{C}\mathrm{at}_\infty}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}(\mathcal{C}, \mathcal{D})^\simeq,$$

i.e. the subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose morphisms are the natural equivalences. An n-simplex in $\mathbb{C}\mathrm{at}_\infty$ therefore consists of the following data:

- a sequence of quasi-categories $\mathcal{C}_0, \ldots, \mathcal{C}_n$;
- for each $0 \le i < j \le n$, a map of simplicial sets

$$\lambda_{i,j} : \mathcal{P}_{i,j} \to \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)$$

  where $\mathcal{P}_{i,j}$ denotes the poset of those subsets of

$$\{i, i+1, \ldots, j-1, j\}$$

  that include both i and j, ordered by inclusion;

such that

(1) for each edge e in $\mathcal{P}_{i,j}$ the natural transformation $\lambda_{i,j}(e)$ is an equivalence;
(2) for each $i < j < k$, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P}_{i,j} \times \mathcal{P}_{j,k} & \xrightarrow{\lambda_{i,j} \times \lambda_{j,k}} & \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j) \times \mathrm{Fun}(\mathcal{C}_j, \mathcal{C}_k) \\
\cup \downarrow & & \downarrow \circ \\
\mathcal{P}_{i,k} & \xrightarrow{\quad \lambda_{i,k} \quad} & \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_k).
\end{array}
$$

In particular, a 2-simplex in $\mathbb{C}\mathrm{at}_\infty$ comprises three functors

$$
\begin{array}{ccc}
\mathcal{C}_0 & \xrightarrow{\quad H \quad} & \mathcal{C}_2 \\
& \searrow_{F} \quad \nearrow_{G} & \\
& \mathcal{C}_1 &
\end{array}
$$

together with a natural equivalence $H \xrightarrow{\sim} GF$.

Definition 8.2. Let $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \subseteq \mathbb{C}\mathrm{at}_\infty$ be the maximal simplicial subset whose objects are the differentiable ∞-categories and whose morphisms are the functors that preserve sequential colimits. We refer to $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ as the *∞-category of differentiable ∞-categories*.

Note that $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ is the simplicial nerve of a simplicial category whose objects are the differentiable ∞-categories with simplicial mapping objects

$$\mathrm{Hom}_{\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}_{\mathbb{N}}(\mathcal{C}, \mathcal{D})^{\simeq},$$

the maximal Kan complex of the full subcategory $\mathrm{Fun}_{\mathbb{N}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of sequential-colimit-preserving functors. As is our convention, we do not distinguish notationally between the ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ and this underlying simplicial category.

Differentiable relative ∞-categories. It turns out that $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ is not the most convenient ∞-category on which to build the Goodwillie tangent structure. In this section, we describe another ∞-category, denoted $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, which is equivalent to $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ and which is more amenable.

The objects of $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ are *relative ∞-categories*. A *relative category* is simply a category $\mathcal{C}$ together with a subcategory $\mathcal{W}$ of 'weak equivalences' which contains all isomorphisms in $\mathcal{C}$. Associated to the pair $(\mathcal{C}, \mathcal{W})$ is an ∞-category $\mathcal{C}[\mathcal{W}^{-1}]$ given by formally inverting the morphisms in $\mathcal{W}$. Barwick and Kan showed in [**?**] that any ∞-category can be obtained this way, so that relative categories are yet another model for ∞-categories.

Mazel-Gee [**?**] uses a nerve construction of Rezk to extend the localization construction to 'relative ∞-categories' in which $\mathcal{C}$ and $\mathcal{W}$ are themselves allowed to be ∞-categories. In other words, the Rezk nerve describes a very general 'calculus of fractions' for ∞-categories.

The reason that relative ∞-categories are convenient for us is that we can replace the ∞-category $\mathrm{Exc}^A(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ of A-excisive functors with the *relative ∞-category*

$$(\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C}), P_A\mathcal{E})$$

consisting of the full ∞-category of functors together with the subcategory consisting of those natural transformations that become equivalences on applying the A-excisive approximation P_A.

The benefit of this approach is that we can work directly with the functor ∞-categories $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^n, \mathcal{C})$ without involving the explicit P_A-approximation functor. In particular, it makes the monoidal nature of our construction almost immediate.

Definition 8.3. A *relative ∞-category* is a pair $(\mathcal{C}, \mathcal{W})$ consisting of an ∞-category $\mathcal{C}$ and a subcategory $\mathcal{W} \subseteq \mathcal{C}$ that includes all equivalences in $\mathcal{C}$. (In particular $\mathcal{W}$ contains all the objects of $\mathcal{C}$.)

A *relative functor* $G : (\mathcal{C}_0, \mathcal{W}_0) \to (\mathcal{C}_1, \mathcal{W}_1)$ between relative ∞-categories is a functor $G : \mathcal{C}_0 \to \mathcal{C}_1$ such that $G(\mathcal{W}_0) \subseteq \mathcal{W}_1$.

A *natural transformation* α between relative functors $G, G' : (\mathcal{C}_0, \mathcal{W}_0) \to (\mathcal{C}_1, \mathcal{W}_1)$ is a natural transformation between the functors $G, G' : \mathcal{C}_0 \to \mathcal{C}_1$, that is, a functor $\alpha : \Delta^1 \times \mathcal{C}_0 \to \mathcal{C}_1$ that restricts to G on $\{0\} \times \mathcal{C}_0$ and to G' on $\{1\} \times \mathcal{C}_0$.

We say that a natural transformation α is a *relative equivalence* if for each $X \in \mathcal{C}_0$, the morphism $\alpha_X : G(X) \to G'(X)$ is in the subcategory $\mathcal{W}_1 \subseteq \mathcal{C}_1$. In this case, we also say that the natural transformation α *takes values in $\mathcal{W}_1$*.

Example 8.4. Associated to any ∞-category is a *minimal* relative ∞-category $(\mathcal{C}, \mathcal{E}_\mathcal{C})$ where $\mathcal{E}_\mathcal{C}$ is the subcategory of equivalences in $\mathcal{C}$. A relative functor between minimal relative ∞-categories is just a functor between the underlying ∞-categories, and a relative equivalence between such functors is just a natural equivalence in the usual sense.

We now wish to restrict to those relative ∞-categories whose localization is a *differentiable* ∞-category. In fact, it will be convenient to consider not all such relative ∞-categories, but only those for which the localization is an exact reflective subcategory.

Definition 8.5. We say that a relative ∞-category $(\mathcal{C}, \mathcal{W})$ is *differentiable* if $\mathcal{C}$ is a differentiable ∞-category, and $\mathcal{W}$ is the subcategory of local equivalences for an exact localization functor $\mathcal{C} \to \mathcal{C}$ in the sense of [**?**, 5.2.7]. In other words, there exists an adjunction of differentiable ∞-categories

$$f : \mathcal{C} \rightleftarrows \mathcal{D} : g$$

such that

- f preserves finite limits;
- g is fully faithful and preserves sequential colimits;
- $\mathcal{W}$ is the subcategory of f-equivalences in $\mathcal{C}$, i.e. those morphisms that are mapped by f to an equivalence in $\mathcal{D}$.

We will say that a relative functor $G : (\mathcal{C}_0, \mathcal{W}_0) \to (\mathcal{C}_1, \mathcal{W}_1)$ between differentiable relative ∞-categories is *differentiable* if its underlying functor $G : \mathcal{C}_0 \to \mathcal{C}_1$ preserves sequential colimits.

Our task is now to build an ∞-category whose objects are the differentiable relative ∞-categories, and which is equivalent to $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$. We start by giving a simplicial enrichment to the category of differentiable relative ∞-categories and differentiable relative functors.

Definition 8.6. Let $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ be the simplicial category in which:

- objects are the differentiable relative ∞-categories $(\mathcal{C}, \mathcal{W})$;
- the simplicial mapping object $\mathrm{Hom}_{\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_0))$ is given by the subcategory

$$\mathrm{Fun}_{\mathbb{N}}^{\widetilde{\simeq}}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_0)) \subseteq \mathrm{Fun}(\mathcal{C}_0, \mathcal{C}_1)$$

  whose objects are the differentiable relative functors, and whose morphisms are the relative equivalences.

Proposition 8.7. *There is a Dwyer-Kan equivalence of simplicial categories (i.e. an equivalence in Bergner's model structure [?])*

$$M_0 : \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$$

that sends each differentiable ∞-category $\mathcal{C}$ to the differentiable relative ∞-category $(\mathcal{C}, \mathcal{E}_\mathcal{C})$ where $\mathcal{E}_\mathcal{C}$ is the subcategory of equivalences in $\mathcal{C}$, and on simplicial mapping objects is given by the equivalences (in fact, equalities):

$$\mathrm{Fun}_{\mathbb{N}}^{\widetilde{\simeq}}(\mathcal{C}_0, \mathcal{C}_1) \overset{\sim}{\longrightarrow} \mathrm{Fun}_{\mathbb{N}}^{\widetilde{\simeq}}((\mathcal{C}_0, \mathcal{E}_{\mathcal{C}_0}), (\mathcal{C}_1, \mathcal{E}_{\mathcal{C}_1})).$$

Proof. First note that when $\mathcal{C}$ is a differentiable ∞-category, $(\mathcal{C}, \mathcal{E}_\mathcal{C})$ is a differentiable relative ∞-category; the identity adjunction on $\mathcal{C}$ satisfies the conditions of Definition **??**.

Since M_0 is clearly fully faithful, it remains to show that it is essentially surjective on objects. So let $(\mathcal{C}, \mathcal{W})$ be an arbitrary differentiable relative ∞-category. Then we know that $\mathcal{W}$ is the subcategory of f-equivalences for some adjunction of differentiable ∞-categories

$$f : \mathcal{C} \rightleftarrows \mathcal{D} : g$$

that satisfies the conditions of Definition **??**. The functors f and g both preserve sequential colimits, so determine differentiable relative functors

$$f : (\mathcal{C}, \mathcal{W}) \rightleftarrows (\mathcal{D}, \mathcal{E}_\mathcal{D}) : g$$

which we claim are isomorphisms in the homotopy category of the simplicial category $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$.

The counit of the localizing adjunction (f, g) is an equivalence $\epsilon : fg \overset{\sim}{\longrightarrow} 1_\mathcal{D}$ and therefore also a relative equivalence $fg \overset{\sim}{\longrightarrow} 1_{(\mathcal{D}, \mathcal{E}_\mathcal{D})}$. Hence $fg = 1_{(\mathcal{D}, \mathcal{E}_\mathcal{D})}$ in the homotopy category of $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$.

It remains to produce a relative equivalence between gf and $1_{(\mathcal{C}, \mathcal{W})}$. The unit of the localizing adjunction (f, g) is a natural transformation $1_\mathcal{C} \to gf$ between relative functors $(\mathcal{C}, \mathcal{W}) \to (\mathcal{C}, \mathcal{W})$. To test if this natural transformation is a

relative equivalence, we must show that it becomes an equivalence in $\mathcal{D}$ after applying f. But $f\eta : f \to fgf$ is inverse to the equivalence ϵf, so this is indeed the case. Hence $gf = 1_{(\mathcal{C}, \mathcal{W})}$ in the homotopy category too, and so f and g are isomorphisms as claimed. Thus M_0 is essentially surjective. $\square$

Note that $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$ is enriched in quasi-categories but not in Kan complexes because a relative equivalence is not necessarily invertible. The simplicial nerve of $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$ is therefore not an ∞-category. In order to rectify this problem, we add inverses for the relative equivalences by taking a fibrant replacement for the simplicial mapping objects in $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$.

For this purpose we use an explicit fibrant replacement for the Quillen model structure given by Kan's Ex^∞ functor [**?**]. We do not give a complete definition of the functor $\mathrm{Ex}^\infty : \mathsf{Set}_\Delta \to \mathsf{Set}_\Delta$, but here are the key properties from our point of view. For any simplicial set Y, the simplicial set $\mathrm{Ex}^\infty(Y)$ is a fibrant replacement for Y in the Quillen model structure. So $\mathrm{Ex}^\infty(Y)$ is a Kan complex, and there is a natural map $r_Y : Y \xrightarrow{\sim} \mathrm{Ex}^\infty(Y)$ which is a weak equivalence and cofibration in that model structure.

The vertices of $\mathrm{Ex}^\infty(Y)$ can be identified with the vertices of Y, and the edges of $\mathrm{Ex}^\infty(Y)$ can be identified with zigzags

$$y_0 \to y_1 \leftarrow y_2 \to \cdots \leftarrow y_{2k}$$

of edges in Y, where we identify zigzags of different lengths by including additional identity morphisms on the right. The map r sends an edge $y_0 \to y_1$ of Y to the zigzag

$$y_0 \to y_1 = y_1.$$

The functor Ex^∞ preserves finite products and has simplicial enrichment coming from the composite

$$X \times \mathrm{Ex}^\infty(Y) \xrightarrow{r_X} \mathrm{Ex}^\infty(X) \times \mathrm{Ex}^\infty(Y) \cong \mathrm{Ex}^\infty(X \times Y).$$

When Y is an ∞-category, we can think of the Kan complex $\mathrm{Ex}^\infty(Y)$ as a model for the '∞-groupoidification' of Y given by freely inverting the 1-simplexes.

Definition 8.8. Let $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$ denote the simplicial category whose objects are the differentiable relative ∞-categories, and whose simplicial mapping spaces are the Kan complexes

$$\mathrm{Hom}_{\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1)) := \mathrm{Ex}^\infty \mathrm{Fun}^{\widetilde{\cong}}_{\mathbb{N}}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1))$$

given by applying Ex^∞ to the simplicial mapping objects in $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$. Composition in $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$ is induced by that in $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$ using the fact that Ex^∞ preserves finite products.

There is a canonical functor

$$r : \mathbb{Rel}_0\mathbb{Cat}_\infty^{\mathrm{diff}} \to \mathbb{Rel}\mathbb{Cat}_\infty^{\mathrm{diff}}$$

given by the identity on objects and by inclusions of the form $r_Y : Y \to \mathrm{Ex}^\infty(Y)$ on mapping spaces. Since r_Y is a weak equivalence in the Quillen model structure, r is a Dwyer-Kan equivalence.

By construction $\mathbb{Rel}\mathbb{Cat}_\infty^{\mathrm{diff}}$ is enriched in Kan complexes, and hence its simplicial nerve is a quasi-category which, following our usual convention, we also denote $\mathbb{Rel}\mathbb{Cat}_\infty^{\mathrm{diff}}$.

Corollary 8.9. *There is an equivalence of ∞-categories*

$$M : \mathbb{Cat}_\infty^{\mathrm{diff}} \xrightarrow{\sim} \mathbb{Rel}\mathbb{Cat}_\infty^{\mathrm{diff}}$$

given by composing the map M_0 from Proposition **??** *with the functor $r :$ $\mathbb{Rel}_0\mathbb{Cat}_\infty^{\mathrm{diff}} \xrightarrow{\sim} \mathbb{Rel}\mathbb{Cat}_\infty^{\mathrm{diff}}$ of Definition* **??**.

Proof. Each of M_0 and r is a Dwyer-Kan equivalence, so their composite is a Dwyer-Kan equivalence between fibrant objects in the Bergner model structure. Taking simplicial nerves we get an equivalence of ∞-categories. $\square$

The equivalence M tells us that we can use $\mathbb{Rel}\mathbb{Cat}_\infty^{\mathrm{diff}}$ as a model for the ∞-category of differentiable ∞-categories and sequential-colimit-preserving functors. We show in the next section that $\mathbb{Rel}\mathbb{Cat}_\infty^{\mathrm{diff}}$ admits the required tangent structure, which can then be transferred along M to $\mathbb{Cat}_\infty^{\mathrm{diff}}$ using Lemma **??**.

We conclude this section by giving an explicit description of the simplexes in the quasi-category $\mathbb{Rel}\mathbb{Cat}_\infty^{\mathrm{diff}}$ which will be useful in constructing our tangent structure.

Remark 8.10. An n-simplex λ in the ∞-category $\mathbb{Rel}\mathbb{Cat}_\infty^{\mathrm{diff}}$ consists of the following data:

- a sequence of differentiable relative ∞-categories

$$(\mathcal{C}_0, \mathcal{W}_0), \ldots, (\mathcal{C}_n, \mathcal{W}_n);$$

- for each $0 \leq i < j \leq n$, a functor

$$\lambda_{i,j} : \mathcal{P}_{i,j} \to \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)$$

subject to the following conditions:

(1) for each object $I \in \mathcal{P}_{i,j}$, $\lambda_{i,j}(I) : \mathcal{C}_i \to \mathcal{C}_j$ is a differentiable relative functor;

(2) for each morphism $\iota : I \subseteq I'$ in $\mathcal{P}_{i,j}$, the edge

$$\lambda_{i,j}(\iota) \in \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)_1$$

is a zigzag of relative equivalences in $\mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)$;

(3) for each $i < j < k$, the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{P}_{i,j} \times \mathcal{P}_{j,k} & \xrightarrow{\lambda_{i,j} \times \lambda_{j,k}} & \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j) \times \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_j, \mathcal{C}_k) \\
\cup \downarrow & & \downarrow \mathrm{Ex}^\infty(\circ) \\
\mathcal{P}_{i,k} & \xrightarrow{\lambda_{i,k}} & \mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_k).
\end{array}
$$

Tangent structure on differentiable relative ∞-categories. We now build a tangent structure on the ∞-category $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$ by describing explicitly the corresponding action map

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty \to \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty.$$

In order to make this a strict action of the monoidal quasi-category $\mathbb{W}\mathrm{eil}$, we need to be careful about one point. When we write a functor ∞-category of the form

$$\mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C})$$

we will actually mean the isomorphic simplicial set

$$\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \dots, \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) \dots))$$

with n iterations. It follows that the simplicial sets $\mathrm{Fun}(\mathcal{S}^m_{\mathrm{fin},*}, \mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C}))$ and $\mathrm{Fun}(\mathcal{S}^{m+n}_{\mathrm{fin},*}, \mathcal{C})$ are actually *equal* not merely isomorphic.

With that warning in mind, we start by defining our desired functor T on objects.

Definition 8.11. Let A be a Weil-algebra with n generators, and $(\mathcal{C}, \mathcal{W})$ a differentiable relative ∞-category. We define a relative ∞-category

$$T^A(\mathcal{C}, \mathcal{W}) := (\mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C}), P_A\mathcal{W})$$

where $P_A\mathcal{W}$ is the subcategory of the functor ∞-category $\mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C})$ consisting of those morphisms, i.e. natural transformations $\beta : \Delta^1 \times \mathcal{S}^n_{\mathrm{fin},*} \to \mathcal{C}$, that, after applying the A-excisive approximation functor P_A from Definition **??**, take values in $\mathcal{W}$. That is, for each $X \in \mathcal{S}^n_{\mathrm{fin},*}$, the component $(P_A\beta)_X$ is in $\mathcal{W}$.

Lemma 8.12. *The relative ∞-category $T^A(\mathcal{C}, \mathcal{W})$ is differentiable.*

Proof. Since $\mathcal{C}$ is differentiable, and limits and colimits in a functor ∞-category are calculated objectwise, it follows that $\mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C})$ is also differentiable. It therefore remains to show that $P_A \mathcal{W}$ is determined by a suitable localizing adjunction.

Since $(\mathcal{C}, \mathcal{W})$ is a differentiable relative ∞-category, there is an adjunction of differentiable ∞-categories

$$f : \mathcal{C} \rightleftarrows \mathcal{D} : g$$

satisfying the conditions of Definition **??**. Now consider the pair of adjunctions

$$\mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C}) \underset{\iota}{\overset{P_A}{\rightleftarrows}} \mathrm{Exc}^A(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C}) \underset{g_*}{\overset{f_*}{\rightleftarrows}} \mathrm{Exc}^A(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{D})$$

where the maps f_* and g_* are given by composing with f and g respectively, noting that since these functors preserve finite limits, f_* and g_* preserve A-excisive functors.

We verify that the composed adjunction $(f_* P_A, \iota g_*)$ satisfies the conditions of Definition **??**:

- P_A preserves finite limits by Proposition **??**, and f_* preserves finite limits because f does and those limits are calculated objectwise in $\mathrm{Exc}^A(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C})$ and $\mathrm{Exc}^A(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{D})$ (also by **??**);
- ι is fully faithful and preserves sequential colimits by **??**; g_* is fully faithful because g is, and because $\mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, -)$ preserves fully faithful inclusion; g_* preserves sequential colimits again because of **??**;
- a morphism β in $\mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C})$ is an $f_* P_A$-equivalence if and only if $P_A(\beta)$ is an f_*-equivalence if and only if (since equivalences in the ∞-category $\mathrm{Exc}^A(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{D})$ are detected objectwise) $P_A(\beta)_X$ is an f-equivalence, i.e. in $\mathcal{W}$, for all $X \in \mathcal{S}^n_{\mathrm{fin},*}$; thus the subcategory $P_A \mathcal{W}$ consists precisely of the $f_* P_A$-equivalences.

Thus $(\mathrm{Fun}(\mathcal{S}^n_{\mathrm{fin},*}, \mathcal{C}), P_A \mathcal{W})$ is a differentiable relative ∞-category as claimed. $\square$

Next, we define T on morphisms.

Definition 8.13. Let $\phi : A_0 \to A_1$ be a morphism of Weil-algebras, and let $G : (\mathcal{C}_0, \mathcal{W}_0) \to (\mathcal{C}_1, \mathcal{W}_1)$ be a differentiable relative functor between differentiable relative ∞-categories.

We let $T^\phi(G) : (\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_0}, \mathcal{C}_0), P_{A_0}\mathcal{W}_0) \to (\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_1}, \mathcal{C}_1), P_{A_1}\mathcal{W}_1)$ be the relative functor

$$T^\phi(G) : \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_0}, \mathcal{C}_0) \to \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_1}, \mathcal{C}_1); \quad L \mapsto GL\tilde{\phi}$$

given by composition with the maps of simplicial sets $\tilde{\phi} : \mathcal{S}_{\mathrm{fin},*}^{n_1} \to \mathcal{S}_{\mathrm{fin},*}^{n_0}$ (of Definition **??**) and $G : \mathcal{C}_0 \to \mathcal{C}_1$.

Lemma 8.14. *The functor $T^\phi(G)$ is a differentiable relative functor*
$$T^{A_0}(\mathcal{C}_0, \mathcal{W}_0) \to T^{A_1}(\mathcal{C}_1, \mathcal{W}_1).$$

Proof. We must show that $T^\phi(G)(P_{A_0}\mathcal{W}_0) \subseteq P_{A_1}\mathcal{W}_1$, so consider a morphism of $\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_0}, \mathcal{C}_0)$, i.e. a natural transformation $L \to L'$, such that $(P_{A_0}L)(X) \to (P_{A_0}L')(X)$ is in $\mathcal{W}_0$ for all $X \in \mathcal{S}_{\mathrm{fin},*}^{n_0}$.

First, since G is a relative functor, it follows that
$$G(P_{A_0}L)\tilde{\phi}(Y) \to G(P_{A_0}L)\tilde{\phi}(Y)$$
is in $\mathcal{W}_1$ for any $Y \in \mathcal{S}_{\mathrm{fin},*}^{n_1}$.

Now recall that since $(\mathcal{C}_1, \mathcal{W}_1)$ is differentiable, there is an adjunction
$$f_1 : \mathcal{C}_1 \rightleftarrows \mathcal{D}_1 : g_1$$
satisfying the conditions of Definition **??**. Thus $\mathcal{W}_1$ is the subcategory of f_1-equivalences, and so the map
$$f_1 G(P_{A_0}L)\tilde{\phi} \to f_1 G(P_{A_0}L')\tilde{\phi}$$
is a natural equivalence between functors $\mathcal{S}_{\mathrm{fin},*}^{n_1} \to \mathcal{D}_1$. Since $\mathcal{D}_1$ is differentiable, we can apply P_{A_1} to this map to obtain a natural equivalence
$$P_{A_1}(f_1 G(P_{A_0}L)\tilde{\phi}) \xrightarrow{\sim} P_{A_1}(f_1 G(P_{A_0}L')\tilde{\phi}).$$
Since f_1 preserves both sequential colimits and finite limits, it commutes with the construction P_{A_1} by Lemma **??**. Thus we also have a natural equivalence
$$f_1 P_{A_1}(G(P_{A_0}L)\tilde{\phi}) \xrightarrow{\sim} f_1 P_{A_1}(G(P_{A_0}L')\tilde{\phi}).$$
In other words, the natural map
$$P_{A_1}(G(P_{A_0}L)\tilde{\phi}) \to P_{A_1}(G(P_{A_0}L')\tilde{\phi})$$
takes values in $\mathcal{W}_1$. But, applying the equivalences of (**??**) and (**??**), it follows that
$$P_{A_1}(GL\tilde{\phi}) \to P_{A_1}(GL'\tilde{\phi})$$
takes values in $\mathcal{W}_1$, so that
$$GL\tilde{\phi} \to GL'\tilde{\phi}$$
is in $P_{A_1}\mathcal{W}_1$. So $T^\phi(G)$ is a relative functor as required.

Finally, $T^\phi(G)$ preserves sequential colimits because G does and these colimits are calculated objectwise in the functor ∞-categories. So $T^\phi(G)$ is differentiable. $\qquad\square$

Before moving on to the general case, it is worthwhile also to define T explicitly on 2-simplexes.

Definition 8.15. Let ϕ be a 2-simplex in $\mathbb{W}$eil, that is, a pair of Weil-algebra morphisms

$$A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} A_2.$$

Let λ be a 2-simplex in $\mathbb{R}el\mathbb{C}at_\infty^{\mathrm{diff}}$. According to Remark **??**, λ consists of a diagram of relative functors

$$
\begin{array}{ccc}
(\mathcal{C}_0, \mathcal{W}_0) & \xrightarrow{\quad H \quad} & (\mathcal{C}_2, \mathcal{W}_2) \\
& F \searrow \qquad \nearrow G & \\
& (\mathcal{C}_1, \mathcal{W}_1) &
\end{array}
$$

together with an edge in $\mathrm{Ex}^\infty \mathrm{Fun}(\mathcal{C}_0, \mathcal{C}_2)$, that is, a zigzag

$$\lambda_{0,1,2} : H \to E_1 \leftarrow \cdots \to E_{2k-1} \leftarrow GF,$$

in which each map is a relative equivalence.

We define $T^\phi(\lambda)$ to be the 2-simplex in $\mathbb{R}el\mathbb{C}at_\infty^{\mathrm{diff}}$ consisting of the corresponding diagram of relative functors

$$
\begin{array}{ccc}
T^{A_0}(\mathcal{C}_0, \mathcal{W}_0) & \xrightarrow{\quad T^{\phi_2\phi_1}(H) \quad} & T^{A_2}(\mathcal{C}_2, \mathcal{W}_2) \\
& T^{\phi_1}(F) \searrow \qquad \nearrow T^{\phi_2}(G) & \\
& T^{A_1}(\mathcal{C}_1, \mathcal{W}_1) &
\end{array}
$$

together with the following zigzag of relative equivalences
(8.16)
$$(H(-)\widetilde{\phi_2\phi_1}) \to (E_1(-)\tilde\phi_1\tilde\phi_2) \leftarrow \cdots \to (E_{2k-1}(-)\tilde\phi_1\tilde\phi_2) \leftarrow (GF(-)\tilde\phi_1\tilde\phi_2)$$

between $T^{\phi_2\phi_1}(H)$ and $T^{\phi_2}(G)T^{\phi_1}(F)$, in which each map is induced by the corresponding map in $\lambda_{0,1,2}$, and the first map, in addition, involves the natural transformation

$$\alpha : \widetilde{\phi_2\phi_1} \to \tilde\phi_1\tilde\phi_2$$

of Definition **??**.

Lemma 8.17. *The construction of* $T^\phi(\lambda)$ *in Definition* **??** *produces a 2-simplex in* $\mathbb{R}el\mathbb{C}at_\infty^{\mathrm{diff}}$.

Proof. The only thing left to check is that each of the natural transformations in the zigzag (**??**) is a relative equivalence, i.e. that each of these natural transformations takes values in $P_{A_2}\mathcal{W}_2$, that is, takes values in $\mathcal{W}_2$ after applying P_{A_2}.

Let $\gamma : E \to E'$ be a relative equivalence between relative functors

$$E, E' : (\mathcal{C}_0, \mathcal{W}_0) \to (\mathcal{C}_2, \mathcal{W}_2),$$

i.e. for each $X \in \mathcal{C}_0$, the map $\gamma_X : E(X) \to E'(X)$ is in $\mathcal{W}_2$.

It follows that for every functor $L : \mathcal{S}_{\mathrm{fin},*}^{n_0} \to \mathcal{C}_0$, the map induced by γ

$$EL\tilde{\phi}_1\tilde{\phi}_2 \to E'L\tilde{\phi}_1\tilde{\phi}_2$$

takes values in $\mathcal{W}_2$. A similar argument to that in the proof of Lemma **??** implies that

$$P_{A_2}(EL\tilde{\phi}_1\tilde{\phi}_2) \to P_{A_2}(E'L\tilde{\phi}_1\tilde{\phi}_2)$$

takes values in $\mathcal{W}_2$. Therefore the map induced by γ,

$$E(-)\tilde{\phi}_1\tilde{\phi}_2 \to E'(-)\tilde{\phi}_1\tilde{\phi}_2$$

takes values in $P_{A_2}\mathcal{W}_2$ as required.

This argument shows that each map in the zigzag (**??**) after the first is a relative equivalence. To show that the first map is also a relative equivalence, we note that

$$P_{A_2}(E_1(-)\widetilde{\tilde{\phi}_2\phi_1}) \to P_{A_2}(E_1(-)\tilde{\phi}_1\tilde{\phi}_2)$$

is an equivalence of the type described in (**??**), hence is in $\mathcal{W}_2$. Combined with the previous argument, we deduce that

$$H(-)\widetilde{\tilde{\phi}_2\phi_1} \to E_1(-)\tilde{\phi}_1\tilde{\phi}_2$$

is also a relative equivalence. $\qquad\qquad\square$

We now extend our constructions above to simplexes of arbitrary dimension.

Definition 8.18. Let ϕ be an n-simplex in $\mathbb{W}$eil, that is a sequence of Weil-algebra morphisms

$$A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_n} A_n.$$

Also let λ be an n-simplex in $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ as described in Remark **??**.

We define $T^\phi(\lambda)$ to be the n-simplex in $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ consisting of the differentiable relative ∞-categories

$$T^{A_0}(\mathcal{C}_0, \mathcal{W}_0), \dots, T^{A_n}(\mathcal{C}_n, \mathcal{W}_n)$$

and the functors
$$T^\phi(\lambda)_{i,j} : \mathcal{P}_{i,j} \to \mathrm{Ex}^\infty \, \mathrm{Fun}(\mathrm{Fun}(\mathcal{S}^{n_i}_{\mathrm{fin},*}, \mathcal{C}_i), \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{C}_j))$$

constructed as follows.

Firstly, from the n-simplex ϕ, we construct for $0 \le i < j \le n$ a functor
$$\tilde{\phi}_{i,j} : \mathcal{P}_{i,j} \to \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{S}^{n_i}_{\mathrm{fin},*}).$$

An object of $\mathcal{P}_{i,j}$ is a subset
$$I = \{i = i_0, i_1, \ldots, i_k = j\} \subseteq \{i, i+1, \ldots, j-1, j\}$$

and we let $\tilde{\phi}_{i,j}(I)$ be the composite functor
$$(\phi_{i_k} \overbrace{\cdots \phi_{i_{k-1}+1}}) \cdots (\phi_{i_1} \overbrace{\cdots \phi_{i_0+1}}) : \mathcal{S}^{n_j}_{\mathrm{fin},*} \to \mathcal{S}^{n_i}_{\mathrm{fin},*}.$$

For a morphism e in $\mathcal{P}_{i,j}$, that is an inclusion $I \subseteq I'$ of subsets of $\{i, i+1, \ldots, j-1, j\}$ that include i and j, we have to produce a natural transformation
$$\tilde{\phi}_{i,j}(e) : \tilde{\phi}_{i,j}(I) \to \tilde{\phi}_{i,j}(I') : \mathcal{S}^{n_j}_{\mathrm{fin},*} \to \mathcal{S}^{n_i}_{\mathrm{fin},*}.$$

For example, when e is the inclusion $\{0,2\} \subseteq \{0,1,2\}$, then $\tilde{\phi}_{0,2}(e)$ is required to be a natural transformation
$$\widetilde{\phi_2\phi_1} \to \tilde{\phi}_1\tilde{\phi}_2$$

which we choose to be the map α of Definition **??**. For a general morphism e, the desired map $\tilde{\phi}_{i,j}(e)$ is obtained by (a composite of) generalizations of α to more than two factors. It follows from Lemma **??** that, for inclusions $I \xrightarrow{e} I' \xrightarrow{e'} I''$, we have
$$\tilde{\phi}_{i,j}(e'e) = \tilde{\phi}_{i,j}(e')\tilde{\phi}_{i,j}(e)$$

and so we obtain a functor $\tilde{\phi}_{i,j} : \mathcal{P}_{i,j} \to \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{S}^{n_i}_{\mathrm{fin},*})$ as desired.

From the n-simplex λ, we also have a functor
$$\lambda_{i,j} : \mathcal{P}_{i,j} \to \mathrm{Ex}^\infty \, \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)$$

as described in Remark **??**.

We complete the definition of the n-simplex $T^\phi(\lambda)$ by defining the functor $T^\phi(\lambda)_{i,j}$ to be the composite
$$
\begin{aligned}
\mathcal{P}_{i,j} &\xrightarrow{\langle \tilde{\phi}_{i,j}, \lambda_{i,j} \rangle} \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{S}^{n_i}_{\mathrm{fin},*}) \times \mathrm{Ex}^\infty \, \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j) \\
&\xrightarrow{\langle r, \mathrm{Id} \rangle} \mathrm{Ex}^\infty \, \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{S}^{n_i}_{\mathrm{fin},*}) \times \mathrm{Ex}^\infty \, \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j) \\
&\xrightarrow{\cong} \mathrm{Ex}^\infty(\mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{S}^{n_i}_{\mathrm{fin},*}) \times \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)) \\
&\xrightarrow{\mathrm{Ex}^\infty(c)} \mathrm{Ex}^\infty \, \mathrm{Fun}(\mathrm{Fun}(\mathcal{S}^{n_i}_{\mathrm{fin},*}, \mathcal{C}_i), \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{C}_j))
\end{aligned}
$$

where

$$c : \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{S}^{n_i}_{\mathrm{fin},*}) \times \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j) \to \mathrm{Fun}(\mathrm{Fun}(\mathcal{S}^{n_i}_{\mathrm{fin},*}, \mathcal{C}_i), \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{C}_j))$$

is adjoint to the composition map.

Lemma 8.19. *The construction of Definition ?? defines an n-simplex $T^\phi(\lambda)$ in $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$.*

Proof. We will verify each of the conditions in Remark ??. First note that each $T^{A_i}(\mathcal{C}_i, \mathcal{W}_i)$ is a differentiable relative ∞-category by Lemma ??.

Now consider an object $I = \{i = i_0, i_1, \ldots, i_k = j\} \in \mathcal{P}_{i,j}$. Then $T^\phi(\lambda)_{i,j}(I)$ is the functor

$$\mathrm{Fun}(\mathcal{S}^{n_i}_{\mathrm{fin},*}, \mathcal{C}_i) \to \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{C}_j)$$

given by pre-composition with the functor

$$(\widetilde{\phi_{i_k} \cdots \phi_{i_{k-1}+1}}) \cdots (\widetilde{\phi_{i_1} \cdots \phi_{i_0+1}}) : \mathcal{S}^{n_j}_{\mathrm{fin},*} \to \mathcal{S}^{n_i}_{\mathrm{fin},*}$$

and post-composition with the differentiable relative functor $\lambda_{i,j}(I) : \mathcal{C}_i \to \mathcal{C}_j$. The argument of Lemma ??, with $\tilde{\phi}$ replaced by the composite functor $(\widetilde{\phi_{i_k} \cdots \phi_{i_{k-1}+1}}) \cdots (\widetilde{\phi_{i_1} \cdots \phi_{i_0+1}})$, and G replaced by $\lambda_{i,j}(I)$, implies that $T^\phi(\lambda)_{i,j}(I)$ is a differentiable relative functor. This verifies condition (1) of Remark ?? for our tentative n-simplex $T^\phi(\lambda)$.

Next consider an edge $I \subseteq I'$ in $\mathcal{P}_{i,j}$. Then $T^\phi(\lambda)_{i,j}$ applied to that edge is a zigzag of natural transformations

$$T^\phi(\lambda)_{i,j}(I) \to \cdots \leftarrow T^\phi(\lambda)_{i,j}(I') : \mathrm{Fun}(\mathcal{S}^{n_i}_{\mathrm{fin},*}, \mathcal{C}_i) \to \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{C}_j)$$

induced by natural transformations of the type $\alpha : \widetilde{\phi_2 \phi_1} \to \tilde{\phi}_1 \tilde{\phi}_2$ and a zigzag $\lambda_{i,j}(I) \to \cdots \leftarrow \lambda_{i,j}(I')$. The argument of Lemma ??, slightly generalized, implies that each entry in this zigzag is a relative equivalence. This verifies condition (2) of Remark ??.

It remains to check condition (3), which is a large but easy diagram-chase in the category of simplicial sets, and which follows from the corresponding condition for the n-simplex λ, the naturality of Ex^∞, and the fact that the

following diagrams involving the functors $\tilde{\phi}_{i,j}$ commute:

$$\begin{array}{ccc}
\mathcal{P}_{i,j} \times \mathcal{P}_{j,k} & \xrightarrow{\tilde{\phi}_{i,j} \times \tilde{\phi}_{j,k}} & \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_j}, \mathcal{S}_{\mathrm{fin},*}^{n_i}) \times \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_k}, \mathcal{S}_{\mathrm{fin},*}^{n_j}) \\
\cup \downarrow & & \downarrow \circ \\
\mathcal{P}_{i,k} & \xrightarrow{\tilde{\phi}_{i,k}} & \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_k}, \mathcal{S}_{\mathrm{fin},*}^{n_i}).
\end{array}$$

$\square$

Proposition 8.20. *The construction of T on simplexes in Definition* **??** *gives a well-defined action of the simplicial monoid $\mathbb{W}$eil on the simplicial set $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$.*

Proof. The definition commutes with the simplicial structure so defines a map of simplicial sets
$$T : \mathbb{W}\mathrm{eil} \times \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}.$$
To show that the map T is a strict action map, consider first the case of 0-simplexes. We have to check that
$$T^{A'} T^A(\mathcal{C}, \mathcal{W}) = T^{A' \otimes A}(\mathcal{C}, \mathcal{W}).$$
Recall that we have chosen our functor ∞-categories so that there is an *equality* (not just an isomorphism)
$$\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n'}, \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n}, \mathcal{C})) = \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n+n'}, \mathcal{C})$$
so it is sufficient to show that we have an equality of subcategories $P_{A'}(P_A \mathcal{W}) = P_{A' \otimes A} \mathcal{W}$.

In other words, let $f : L \to L'$ be a natural transformation of functors $\mathcal{S}_{\mathrm{fin},*}^{n+n'} \to \mathcal{C}$. Then f is in $P_{A'}(P_A \mathcal{W})$ if $P_{A'} L \to P_{A'} L'$ takes values in $P_A \mathcal{W}$, i.e. if
$$P_A P_{A'} L \to P_A P_{A'} L'$$
takes values in $\mathcal{W}$. Since $P_A P_{A'} \simeq P_{A' \otimes A}$, this is the case if and only if f is in $P_{A' \otimes A} \mathcal{W}$.

Now let us turn to higher degree simplexes. We have to show that, for n-simplexes ϕ, ϕ' in $\mathbb{W}$eil, and an n-simplex λ in $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, we have
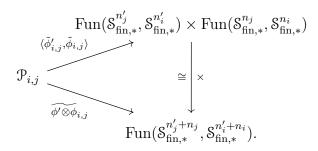$$T^{\phi' \otimes \phi}(\lambda) = T^{\phi'} T^\phi(\lambda))$$
where $\phi' \otimes \phi$ denotes the n-simplex in $\mathbb{W}$eil given by
$$A_0' \otimes A_0 \xrightarrow{\phi_1' \otimes \phi_1} \ldots \xrightarrow{\phi_n' \otimes \phi_n} A_n' \otimes A_n.$$
That is, we have to show that two functors
$$\mathcal{P}_{i,j} \to \mathrm{Ex}^\infty \mathrm{Fun}(\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_i' + n_i}, \mathcal{C}_i), \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_j' + n_j}, \mathcal{C}_j))$$

are equal. This is another large diagram-chase; the key fact is that the following diagram commutes:

$$\text{Fun}(\mathcal{S}_{\text{fin},*}^{n'_j}, \mathcal{S}_{\text{fin},*}^{n'_i}) \times \text{Fun}(\mathcal{S}_{\text{fin},*}^{n_j}, \mathcal{S}_{\text{fin},*}^{n_i})$$

$$\mathcal{P}_{i,j} \xrightarrow{\langle \tilde{\phi}'_{i,j}, \tilde{\phi}_{i,j} \rangle}$$

$$\cong \Big\downarrow \times$$

$$\xrightarrow{\widetilde{\phi' \otimes \phi}_{i,j}}$$

$$\text{Fun}(\mathcal{S}_{\text{fin},*}^{n'_j + n_j}, \mathcal{S}_{\text{fin},*}^{n'_i + n_i}).$$

$\square$

We now transfer the $\mathbb{W}$eil-action on $\mathbb{R}\text{el}\mathbb{C}\text{at}_\infty^{\text{diff}}$ along the equivalence of ∞-categories $M : \mathbb{C}\text{at}_\infty^{\text{diff}} \xrightarrow{\sim} \mathbb{R}\text{el}\mathbb{C}\text{at}_\infty^{\text{diff}}$ of Corollary **??**, using Lemma **??**.

Definition 8.21. Let $T : \mathbb{W}\text{eil}^\otimes \to \text{End}(\mathbb{C}\text{at}_\infty^{\text{diff}})^\circ$ be the monoidal functor given by composing the adjoint of the action map of Proposition **??** with the monoidal equivalence

$$\text{End}(\mathbb{R}\text{el}\mathbb{C}\text{at}_\infty^{\text{diff}})^\circ \simeq \text{End}(\mathbb{C}\text{at}_\infty^{\text{diff}})^\circ$$

associated to the equivalence M of Corollary **??**, as given by Lemma **??**.

We finally have the following result (completing the definition of the Goodwillie tangent structure and the proof of Theorem **??**).

Theorem 8.22. *The monoidal functor*

$$T : \mathbb{W}\text{eil}^\otimes \to \text{End}(\mathbb{C}\text{at}_\infty^{\text{diff}})^\circ$$

*of Definition **??** determines a cartesian tangent structure on the ∞-category $\mathbb{C}\text{at}_\infty^{\text{diff}}$, in the sense of Definition **??**, whose underlying endofunctor and projection are, up to equivalence, as described in **??** and following.*

Proof. To see that T is a tangent structure on $\mathbb{C}\text{at}_\infty^{\text{diff}}$, we have to show that it preserves the foundational and vertical lift pullbacks. This claim follows from Lemma **??** and Proposition **??** with one proviso; we must show that the homotopy pullbacks in the Joyal model structure appearing in those results are pullbacks in the ∞-category $\mathbb{C}\text{at}_\infty^{\text{diff}}$.

We prove that claim by applying a result of Riehl and Verity [**?**, 6.4.12] with $\mathcal{K} = \mathsf{CAT}_\infty$ the ∞-cosmos of ∞-categories. (See [**?**, Ch. 1] for an introduction to the theory of ∞-cosmoses.) We deduce from that result that there is a 'cosmologically-embedded' sub-∞-cosmos $\mathsf{CAT}_\infty^\mathbb{N} \subseteq \mathsf{CAT}_\infty$ whose objects are

the ∞-categories that admit sequential colimits, and whose 1-morphisms are the functors that preserve sequential colimits.

The claim that $\mathsf{CAT}_\infty^\mathbb{N}$ is cosmologically-embedded [**?**, 6.4.3] implies that any square diagram in $\mathsf{CAT}_\infty^\mathbb{N}$ that is a pullback along a fibration in CAT_∞ is also a pullback along a fibration in $\mathsf{CAT}_\infty^\mathbb{N}$. The pullbacks of **??** and **??** fit that bill, and so they determine pullbacks in the corresponding ∞-category $\mathbb{C}\mathrm{at}_\infty^\mathbb{N}$ (of ∞-categories that admit sequential colimits and functors that preserve them), and hence also in the full subcategory $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \subseteq \mathbb{C}\mathrm{at}_\infty^\mathbb{N}$.

Finally, it follows from Lemma **??** that $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ has finite products which are preserved by the tangent bundle functor, so the tangent structure T is cartesian. □

9. Differential objects are stable ∞-categories

Having constructed the Goodwillie tangent structure, we now turn to its initial study, and in this section we look at its differential objects. Since the objects of $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ are *differentiable* ∞-categories, there is a serious danger of confusing the words 'differential' and 'differentiable' in this section.

Recall from [**?**, 1.1.3.4] that an ∞-category $\mathcal{C}$ is *stable* if it is pointed, admits finite limits and colimits, and a square in $\mathcal{C}$ is a pushout if and only if it is a pullback. (In particular, a stable ∞-category admits biproducts which we denote by $\oplus$.) Also recall from [**?**, 6.1.1.7] that a stable ∞-category is differentiable if and only if it admits countable coproducts. We then have the following simple characterization.

Theorem 9.1. *A differentiable ∞-category $\mathcal{C}$ admits a differential structure within the Goodwillie tangent structure if and only if $\mathcal{C}$ is a stable ∞-category.*

Proof. We apply Corollary **??**, so it is sufficient to show that the tangent spaces in $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ are the stable ∞-categories. For any differentiable ∞-category $\mathcal{C}$ and object $X \in \mathcal{C}$, we can identify $T_X\mathcal{C}$ with the ∞-category of excisive functors $\mathcal{S}_{\mathrm{fin},*} \to \mathcal{C}$ that map $*$ to X. We therefore have

$$T_X\mathcal{C} \simeq \mathrm{Exc}_*(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}_{/X})$$

where the right-hand side is the ∞-category of *reduced* excisive functors from $\mathcal{S}_{\mathrm{fin},*}$ to the slice ∞-category $\mathcal{C}_{/X}$. Thus $T_X\mathcal{C}$ is stable by [**?**, 1.4.2.16].

Conversely, if $\mathcal{C}$ is stable, then

$$T_*\mathcal{C} \simeq \mathrm{Exc}_*(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C})$$

which is equivalent to $\mathcal{C}$ by [**?**, 1.4.2.21]. Therefore $\mathcal{C}$ is equivalent to a tangent space and so admits a differential structure by **??**. $\square$

The last part of this proof provides a canonical identification of any stable (differentiable) ∞-category $\mathcal{C}$ with a tangent space, and hence, by Proposition **??**, a canonical differential structure on each such $\mathcal{C}$. This observation allows us to define a cartesian differential structure (see [**?**]) on the homotopy category of stable (differentiable) ∞-categories.

Theorem 9.2. *Let $\mathbb{C}\mathrm{at}^{\mathrm{diff,st}}_{\infty}$ be the full subcategory of $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$ whose objects are the stable differentiable ∞-categories. Then the homotopy category $h\mathbb{C}\mathrm{at}^{\mathrm{diff,st}}_{\infty}$ has a cartesian differential structure in which the monoid structure on an object $\mathcal{C}$ is given by the biproduct functor $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and the derivative of a morphism $F : \mathcal{C} \to \mathcal{D}$ is the 'directional derivative'*

$$\nabla(F) : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$$

given by

$$\nabla(F)(V, X) := D_1(F(X \oplus -))(V).$$

This formula denotes Goodwillie's linear approximation D_1 applied to the functor $F(X \oplus -) : \mathcal{C} \to \mathcal{D}$ and evaluated at object $V \in \mathcal{C}$.

Proof. It is not hard to verify directly that the axioms in [**?**, 2.1.1] hold for $h\mathbb{C}\mathrm{at}^{\mathrm{diff,st}}_{\infty}$, but we deduce this theorem from the results of Section **??** in order to illustrate how the cartesian differential structure on $h\mathbb{C}\mathrm{at}^{\mathrm{diff,st}}_{\infty}$ is related to the Goodwillie tangent structure on $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$.

Recall that Theorem **??** determines a cartesian differential structure on the category $\widehat{h\mathbb{D}\mathrm{iff}}(\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty})$ whose objects are the differential objects in $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$, and whose morphisms are maps in $h\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$ between underlying objects.

We define a functor

$$T_* : h\mathbb{C}\mathrm{at}^{\mathrm{diff,st}}_{\infty} \to \widehat{h\mathbb{D}\mathrm{iff}}(\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty})$$

by sending the stable differentiable ∞-category $\mathcal{C}$ to the differential object in $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$ with underlying object the tangent space

$$T_*\mathcal{C} = \mathrm{Exc}_*(\mathcal{S}_{\mathrm{fin},*}, \mathcal{C}) = \mathcal{S}p(\mathcal{C})$$

and differential structure determined by Proposition **??**. For a functor $F : \mathcal{C} \to \mathcal{D}$, we define $T_*(F)$ to be the morphism in $h\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$ given by the composite

$$T_*\mathcal{C} \xrightarrow[\sim]{\Omega^{\infty}} \mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow[\sim]{\Omega^{\infty}} T_*\mathcal{D}$$

where $\Omega^\infty : T_*\mathcal{C} = \mathcal{S}p(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$ denotes evaluation of a reduced excisive functor at S^0; see [**?**, 1.4.2.21]. This definition makes T_* into a fully faithful embedding since morphisms in both categories are taken from $h\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$.

Given an object of $\widehat{h\mathbb{D}\mathrm{iff}}(\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}})$, i.e. a differential object $\mathcal{D}$ in the Goodwillie tangent structure, we know from Theorem **??** that the underlying ∞-category of $\mathcal{D}$ is stable. The equivalence

$$\Omega^\infty : T_*\mathcal{D} \xrightarrow{\sim} \mathcal{D}$$

is an isomorphism in $\widehat{h\mathbb{D}\mathrm{iff}}(\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}})$, so T_* is essentially surjective on objects. Hence T_* is an equivalence of categories, and we can transfer the cartesian differential structure from Theorem **??** along T_* to $h\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff,st}}$.

It remains to show that this inherited cartesian differential structure is as claimed in the statement of the theorem. To calculate that structure, we first examine the differential structure on an object $\mathcal{C}$ in the tangent category $h\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff,st}}$ determined from that on $T_*\mathcal{C}$ by the equivalence T_*. It can be shown by working through Definition **??** that the relevant monoid structure on $T_*\mathcal{C}$ is given by the fibrewise-product

$$+ : T_*\mathcal{C} \times T_*\mathcal{C} \to T_*\mathcal{C}; \quad (L_1, L_2) \mapsto L_1(-) \times L_2(-).$$

Since this map commutes with evaluation at S^0, we deduce that the relevant monoid structure on $\mathcal{C}$ is the product (and hence biproduct)

$$\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

The map $\hat{p} : T(T_*\mathcal{C}) \to T_*\mathcal{C}$ associated to the differential structure on $T_*\mathcal{C}$ can also be found by working through Definitions **??** and **??**. That map $\hat{p}$ is precisely the map g appearing in the proof of Proposition **??**; see (**??**). We deduce that $\hat{p} : T(T_*\mathcal{C}) \to T_*\mathcal{C}$ is given by

$$\hat{p}(M)(X) = \mathrm{hofib}[M(X, S^0) \to M(*, S^0)]$$

where $M : \mathcal{S}_{\mathrm{fin},*}^2 \to \mathcal{C}$ is excisive in both variables and reduced in its second variable. Transferring back to $\mathcal{C}$ along Ω^∞, we deduce that $\hat{p} : T(\mathcal{C}) \to \mathcal{C}$ is given by

$$\hat{p}(L) := \mathrm{hofib}[L(S^0) \to L(*)].$$

The derivative $\nabla(F)$ of a morphism $F : \mathcal{C} \to \mathcal{D}$ in $h\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff,st}}$ is the composite

$$\mathcal{C} \times \mathcal{C} \xleftarrow[\sim]{\langle p, \hat{p} \rangle} T(\mathcal{C}) \xrightarrow{T(F)} T(\mathcal{D}) \xrightarrow{\hat{p}} \mathcal{D}.$$

To evaluate this composite at $(X, V) \in \mathcal{C} \times \mathcal{C}$ we have to identify an excisive functor $L : \mathcal{S}_{\mathrm{fin},*} \to \mathcal{C}$ such that $L(*) \simeq X$ and $\mathrm{hofib}[L(S^0) \to L(*)] \simeq V$. We can express this functor as

$$L(-) = X \oplus (- \otimes V)$$

where $\otimes$ denotes the canonical tensoring of a pointed ∞-category $\mathcal{C}$ with finite colimits over $\mathcal{S}_{\text{fin},*}$. (The functor $\otimes : \mathcal{S}_{\text{fin},*} \times \mathcal{C} \to \mathcal{C}$ can be constructed using the characterization in [?, 1.4.2.6] of the ∞-category $\mathcal{S}_{\text{fin}}$ of finite spaces.) It follows that

$$\nabla(F)(X, V) = \text{hofib}[P_1(F(X \oplus (- \otimes V)))(S^0) \to F(X)].$$

Since $- \otimes V$ commutes with colimits, it also commutes with the construction of P_1, so that we have

$$\nabla(F)(X, V) = \text{hofib}[P_1(F(X \oplus -))(S^0 \otimes V) \to F(X)]$$

which is precisely $D_1(F(X \oplus -))(V)$ as claimed. $\square$

Theorem **??** is closely related to [?, Cor. 6.6] which is the result that first inspired this paper. Let us briefly discuss the connection.

Definition 9.3. Let $h\mathbb{C}\text{at}^{\text{ab}}$ be the category in which:

- an object is an abelian category;
- a morphism from $\mathcal{A}$ to $\mathcal{B}$ is a pointwise-chain-homotopy class of functors

$$F : \mathcal{A} \to \text{Ch}_+(\mathcal{B})$$

  where the target is the category of non-negatively-graded chain complexes of objects in $\mathcal{B}$, and two such functors F, G are *pointwise-chain-homotopy equivalent* if for each object $A \in \mathcal{A}$ there is a chain-homotopy equivalence $F(A) \simeq G(A)$.

Composition of morphisms is achieved via 'Dold-Kan prolongation' of such an F to a functor $\text{Ch}_+(\mathcal{A}) \to \text{Ch}_+(\mathcal{B})$, see [?, 3.2].

Theorem 9.4 (Bauer-Johnson-Osborne-Riehl-Tebbe [?, 6.6]). *The category $h\mathbb{C}\text{at}^{\text{ab}}$ has a cartesian differential structure in which the monoid structure on an object $\mathcal{A}$ is given by the biproduct $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and the derivative of a morphism $F : \mathcal{A} \to \text{Ch}_+(\mathcal{B})$ is the 'directional derivative' $\nabla(F) : \mathcal{A} \times \mathcal{A} \to \text{Ch}_+(\mathcal{B})$ given by*

$$\nabla(F)(X, V) := D_1(F(X \oplus -))(V)$$

where D_1 denotes the linearization *of a chain-complex-valued functor in the sense of Johnson and McCarthy [?].*

Despite the close similarity, there are slight differences between the context of Theorems **??** and **??** that prohibit a direct comparison. In particular, note that $h\mathbb{C}\text{at}^{\text{ab}}$ is defined using *pointwise* chain-homotopy equivalence, rather

than natural equivalence (though we suspect that [**?**] could have been written entirely in terms of natural chain-homotopy equivalence instead).

Modulo that distinction, we speculate that there is an equivalence of cartesian differential categories between a subcategory of $h\mathbb{C}\mathrm{at}^{\mathrm{ab}}$ (say, on those abelian categories $\mathcal{A}$ that admit countable coproducts and suitably continuous functors) with a subcategory of $h\mathbb{C}\mathrm{at}^{\mathrm{diff,st}}_{\infty}$ (say, on the corresponding stable ∞-categories $N_{dg}\mathrm{Ch}(\mathcal{A})$ given by the differential graded nerves [**?**, 1.3.2.10] of the categories of chain complexes on such $\mathcal{A}$). We do not pursue a precise equivalence of this form here.

10. Jets and n-excisive functors

Goodwillie's notion of excisive functor played a central role in the construction of what we have called the Goodwillie tangent structure on the ∞-category $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$ of differentiable ∞-categories. Our goal in this section is to show that the notions of n-excisive functor, for $n > 1$, are implicit in that tangent structure, so that the entirety of Goodwillie's theory can be recovered from it.

We actually describe how the notion of n-excisive *equivalence*, i.e. the condition that a natural transformation determines an equivalence between n-excisive approximations, relates to the Goodwillie tangent structure. The notion from ordinary differential geometry that corresponds to n-excisive equivalence is that of 'n-jet'. Recall that we say two smooth maps $f, g : M \to N$ between smooth manifolds *agree to order n at* $x \in M$ if $f(x) = g(x)$, and the (multivariable) Taylor series of f and g in local coordinates at x agree up to degree n. The *n-jet at x* of the map f is its equivalence class under the relation of agreeing to order n at x.

We can interpret the Taylor series condition in terms of the standard tangent structure on the category $\mathbb{M}\mathrm{fld}$. Let $T^n_x(M)$ denote the *n-fold tangent space to M at x*, that is, the fibre of the projection map $T^n(M) \to M$ over the point x, where $T^n(M)$ is the n-fold iterate of the tangent bundle functor T. A smooth map $f : M \to N$ then induces a smooth map

$$T^n_x(f) : T^n_x(M) \to T^n_{f(x)}(N) \subseteq T^n(N)$$

i.e. the restriction of $T^n(f)$ to $T^n_x(M)$.

Lemma 10.1. *Let $f, g : M \to N$ be smooth maps. Then f and g have Taylor series at x that agree to degree n if and only if $T^n_x(f) = T^n_x(g)$.*

The main result of this section is an analogue of Lemma **??** that connects the higher excisive approximations in Goodwillie calculus to the Goodwillie tangent structure on $\mathbb{Cat}^{\mathrm{diff}}_\infty$ constructed in Section **??**. We refer the reader to [**?**] for the original theory of n-excisive approximation, and to [**?**, 6.1.1] for the generalization of that theory to (differentiable) ∞-categories.

Definition 10.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a sequential-colimit-preserving functor between differentiable ∞-categories, and suppose that $\mathcal{C}$ admits finite colimits. For an object $X \in \mathcal{C}$, the *n-excisive approximation* to F at X is

$$P_n^X F := P_n(F_{/X}) : \mathcal{C}_{/X} \to \mathcal{D}$$

that is, the n-excisive approximation of the restriction of F to the slice ∞-category $\mathcal{C}_{/X}$ of objects over X.

Theorem 10.3. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be morphisms in $\mathbb{Cat}^{\mathrm{diff}}_\infty$, where $\mathcal{C}$ admits finite colimits. Let $\alpha : F \to G$ be a natural transformation, and let X be an object of $\mathcal{C}$. Then α induces an equivalence*

$$P_n^X \alpha : P_n^X F \xrightarrow{\sim} P_n^X G$$

of n-excisive approximations at X if and only if α induces an equivalence

$$T^n(\alpha)\iota_X : T^n(F)\iota_X \xrightarrow{\sim} T^n(G)\iota_X$$

in the functor ∞-category

$$\mathrm{Fun}(T_X^n(\mathcal{C}), T^n(\mathcal{D})),$$

where $T_X^n(\mathcal{C})$ is the fibre over X of the projection $T^n(\mathcal{C}) \to \mathcal{C}$, and $\iota_X : T_X^n(\mathcal{C} \to T^n(\mathcal{C})$ is the inclusion of that fibre.

Proof. We start by noting that each of the conditions in question implies that $\alpha_X : F(X) \xrightarrow{\sim} G(X)$ is an equivalence: if $P_n^X \alpha$ is an equivalence, then so is

$$P_0^X \alpha \simeq \alpha_X,$$

and if $T^n(\alpha)\iota_X$ is an equivalence, then so is

$$p_\mathcal{D}^n T^n(\alpha)\iota_X \simeq \alpha_X.$$

Replacing $\mathcal{C}$ with $\mathcal{C}_{/X}$ and $\mathcal{D}$ with $\mathcal{D}_{/G(X)}$, we can now reduce to the case that X is a terminal object in $\mathcal{C}$, and F, G are reduced.

In this case, we can identify $T_X^n(\mathcal{C})$ with the ∞-category of functors $L : \mathcal{S}^n_{\mathrm{fin},*} \to \mathcal{C}$ that are excisive in each variable separately and satisfy $L(*, \dots, *) \simeq *$. Then

$$T_X^n(F) : T_X^n(\mathcal{C}) \to T^n(\mathcal{D})$$

corresponds to the map

$$L \mapsto P_{1,\dots,1}(FL).$$

Our goal is therefore to show that $\alpha : F \to G$ induces an equivalence

$$P_n\alpha : P_nF \to P_nG$$

if and only if it induces an equivalence

$$P_{1,\ldots,1}(\alpha L) : P_{1,\ldots,1}(FL) \to P_{1,\ldots,1}(GL)$$

for every reduced and $(1,\ldots,1)$-excisive functor $L : \mathcal{S}^n_{\text{fin},*} \to \mathcal{C}$.

Suppose first that $P_n\alpha$ is an equivalence, and consider the commutative diagram

(10.4)

$$
\begin{array}{ccc}
P_{1,\ldots,1}(FL) & \longrightarrow & P_{1,\ldots,1}((P_nF)L) \\
\downarrow & & \downarrow \\
P_{1,\ldots,1}(P_n(FL)) & \longrightarrow & P_{1,\ldots,1}(P_n((P_nF)L))
\end{array}
$$

The vertical maps are given by n-excisive approximation and are equivalences since being $(1,\ldots,1)$-excisive is a stronger condition than being n-excisive, by [**?**, 6.1.3.4]. The bottom horizontal map is an equivalence by Lemma **??**, so the top map is too. From the assumption that $P_n\alpha : P_nF \to P_nG$ is an equivalence, it therefore follows that $P_{1,\ldots,1}(\alpha L)$ is an equivalence too.

Conversely suppose $P_{1,\ldots,1}(\alpha L)$ is an equivalence for any reduced functor $L : \mathcal{S}^n_{\text{fin},*} \to \mathcal{C}$ that is excisive in each variable. Note that this same condition then holds for any reduced L regardless of it being excisive; that claim follows from the equivalences

$$P_{1,\ldots,1}(FL) \overset{\sim}{\longrightarrow} P_{1,\ldots,1}(F(P_{1,\ldots,1}L))$$

given by (**??**).

Our strategy for showing that $P_n\alpha$ is an equivalence is to use induction on the Taylor tower. Consider the pullback diagram

(10.5)

$$
\begin{array}{ccc}
P_kF & \longrightarrow & P_{k-1}F \\
\downarrow & & \downarrow \\
* & \longrightarrow & R_kF
\end{array}
$$

of [**?**, 6.1.2.4] (Goodwillie's delooping theorem for homogeneous functors), in which $R_kF : \mathcal{C} \to \mathcal{D}$ is k-homogeneous. By induction on k, it suffices to show that α induces an equivalence $R_k\alpha : R_kF \to R_kG$ for each $1 \le k \le n$.

It is clear from (**??**) that the construction R_k naturally takes values in the ∞-category $\mathcal{D}_*$ of pointed objects in $\mathcal{D}$. By [**?**, 6.1.2.11], it is sufficient to restrict also to pointed objects in $\mathcal{C}$, i.e. to show that $R_k\alpha$ is an equivalence of k-homogeneous functors $\mathcal{C}_* \to \mathcal{D}_*$. Such functors naturally factor via $\mathcal{S}p(\mathcal{D}_*)$ by [**?**, 6.1.2.9], so it is sufficient to show that $\Omega R_k\alpha$ is an equivalence, i.e. that α induces an equivalence

$$D_k\alpha : D_kF \to D_kG$$

between the k-th layers of the Taylor towers of functors $\mathcal{C}_* \to \mathcal{D}_*$.

To show that $D_k\alpha$ is an equivalence we show that α induces an equivalence on multilinearized cross-effects. Since $\mathcal{C}_*$ is pointed and admits finite colimits, it has a canonical tensoring over $\mathcal{S}_{\mathrm{fin},*}$, i.e. a functor

$$\otimes : \mathcal{S}_{\mathrm{fin},*} \times \mathcal{C}_* \to \mathcal{C}_*$$

that preserves finite colimits in each variable, and such that $S^0 \otimes Y \simeq Y$ for each $Y \in \mathcal{C}_*$. The functor $\otimes$ can be constructed from the characterization in [**?**, 1.4.2.6] of the ∞-category $\mathcal{S}_{\mathrm{fin}}$ of finite spaces.

Take objects $A_1, \ldots, A_n \in \mathcal{C}_*$ and consider the functor

$$L : \mathcal{S}_{\mathrm{fin},*}^n \to \mathcal{C}_*; \quad (X_1, \ldots, X_n) \mapsto (X_1 \otimes A_1) \vee \ldots \vee (X_n \otimes A_n)$$

where $\vee$ is the coproduct in $\mathcal{C}_*$. Our hypothesis on α implies that $P_{1,\ldots,1}(\alpha L)$ is an equivalence.

Since the functor

$$\mathcal{S}_{\mathrm{fin},*}^n \to \mathcal{C}_*^n; \quad (X_1, \ldots, X_n) \mapsto (X_1 \otimes A_1, \ldots, X_n \otimes A_n)$$

preserves colimits (including the null object) in each variable, it commutes with $P_{1,\ldots,1}$, by [**?**, 6.1.1.30]. It follows that α induces an equivalence

$$P_{1,\ldots,1}(\alpha(- \vee \ldots \vee -))(- \otimes A_1, \ldots, - \otimes A_n).$$

Evaluating at $(S^0, \ldots, S^0)$, we deduce that α induces an equivalence (of functors $\mathcal{C}_*^n \to \mathcal{D}_*$):

(10.6) $$P_{1,\ldots,1}(F(- \vee \ldots \vee -)) \xrightarrow{\sim} P_{1,\ldots,1}(G(- \vee \ldots \vee -)).$$

For any $1 \leq k \leq n$, the k-th cross-effect of F, see [**?**, 6.1.3.20], is the total homotopy fibre of a k-cube whose entries are functors of the form

$$F(- \vee \ldots \vee -)$$

with some subset of its arguments replaced by the null object $*$ in $\mathcal{C}_*$. Since $P_{1,\ldots,1}$ commutes with the construction of that total homotopy fibre, it follows from (**??**) that α induces an equivalence

$$P_{1,\ldots,1}(\mathrm{cr}_k F) \xrightarrow{\sim} P_{1,\ldots,1}(\mathrm{cr}_k F)$$

for all $1 \leq k \leq n$. It follows by [**?**, 6.1.3.23] that α then induces equivalences

$$\mathrm{cr}_k(D_k F) \to \mathrm{cr}_k(D_k G)$$

and hence also, by [**?**, 6.1.4.7], equivalences

$$D_k F \to D_k G$$

for $1 \leq k \leq n$, as required. This completes the proof that $P_n \alpha : P_n F \xrightarrow{\sim} P_n G$ is an equivalence. $\qquad\square$

Remark 10.7. Theorem **??** explains how the notion of n-excisive approximation is related to the Goodwillie tangent structure on $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$. However, it is not quite true to say that this notion is fully encoded in that tangent structure, since the statement of Theorem **??** relies on natural transformations that are not equivalences, and hence are not part of the ∞-category $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$. We can include those natural transformations by replacing $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$ with a corresponding ∞-bicategory $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$.

11. THE $(\infty, 2)$-CATEGORY OF DIFFERENTIABLE ∞-CATEGORIES

The goal of this section is to show that the Goodwillie tangent structure on $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$ extends to a tangent structure, in the sense of Definition **??**, on an ∞-bicategory $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ of differentiable ∞-categories. We start by defining that object.

Definition 11.1. Let $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ be the scaled nerve (see Example **??**) of the simplicial category whose objects are the differentiable ∞-categories, with simplicial mapping objects given by the ∞-categories

$$\mathrm{Hom}_{\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}_\mathbb{N}(\mathcal{C}, \mathcal{D})$$

of sequential-colimit-preserving functors. Since each mapping object is a quasi-category, $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ is an ∞-bicategory.

Our construction of a tangent structure on the ∞-bicategory $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ follows a similar path to that on the ∞-category $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$. We again start with relative differentiable ∞-categories (Definition **??**).

Definition 11.2. Let $\mathbb{R}\mathrm{el}_0 \mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ be the simplicial category in which:

- objects are the differentiable relative ∞-categories $(\mathcal{C}, \mathcal{W})$;
- mapping simplicial sets are the full subcategories

$$\mathrm{Fun}_\mathbb{N}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1)) \subseteq \mathrm{Fun}(\mathcal{C}_0, \mathcal{C}_1)$$

consisting of the differentiable relative functors, i.e. those sequential-colimit-preserving functors $G : \mathcal{C}_0 \to \mathcal{C}_1$ for which $G(\mathcal{W}_0) \subseteq \mathcal{W}_1$.

The scaled nerve of $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ is an ∞-bicategory by Example **??**, but to get an accurate model for $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ we still need to invert the relative equivalences in the simplicial mapping objects, just as we did in constructing the ∞-category $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$ in Definition **??**. We accomplish this inversion by forming the following homotopy pushout of ∞-bicategories.

Definition 11.3. Let $\mathbb{R}\mathrm{el}_1\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ be the scaled simplicial set given by the pushout (in the category of scaled simplicial sets):

(11.4)
$$
\begin{array}{ccc}
\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty & \xrightarrow{\;r\;} & \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty \\
\downarrow & & \downarrow \\
\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty & \longrightarrow & \mathbb{R}\mathrm{el}_1\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty
\end{array}
$$

where the top horizontal map r is described in Definition **??**, and the left-hand map is determined by the inclusions

$$\mathrm{Fun}^{\widetilde{\simeq}}_{\mathbb{N}}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1)) \subseteq \mathrm{Fun}_{\mathbb{N}}((\mathcal{C}_0, \mathcal{W}_0), (\mathcal{C}_1, \mathcal{W}_1)).$$

Both maps are monomorphisms of ∞-bicategories, hence cofibrations in the scaled model structure on $\mathsf{Set}^{\mathsf{sc}}_\Delta$, so the square is a homotopy pushout in that model structure. The scaled simplicial set $\mathbb{R}\mathrm{el}_1\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ is not itself an ∞-bicategory, but we will eventually take a fibrant replacement of it in the scaled model structure to obtain an ∞-bicategory $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ on which to define the Goodwillie tangent structure.

Before that, however, we show that the scaled simplicial sets in (**??**) admit $\mathbb{W}\mathrm{eil}$-actions that are compatible with the inclusions, and hence determine a $\mathbb{W}\mathrm{eil}$-action on the pushout $\mathbb{R}\mathrm{el}_1\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$.

Definition 11.5. We define a map of simplicial sets

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty \to \mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$$

following a similar, but simpler, pattern to that in Definition **??**. Recall from there that an n-simplex ϕ in $\mathbb{W}\mathrm{eil}$ of the form

$$A_0 \xrightarrow{\phi_1} A_1 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_n} A_n$$

determines a collection of functors

$$\tilde{\phi}_{i,j} : \mathcal{P}_{i,j} \to \mathrm{Fun}(\mathcal{S}^{n_j}_{\mathrm{fin},*}, \mathcal{S}^{n_i}_{\mathrm{fin},*}).$$

An n-simplex λ in $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_\infty$ consists of a sequence of differentiable relative ∞-categories

$$(\mathcal{C}_0, \mathcal{W}_0), \ldots, (\mathcal{C}_n, \mathcal{W}_n)$$

and a collection of functors

$$\lambda_{i,j} : \mathcal{P}_{i,j} \to \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)$$

which take values in the subcategories of differentiable relative functors

$$\mathrm{Fun}_{\mathbb{N}}((\mathcal{C}_i, \mathcal{W}_i), (\mathcal{C}_j, \mathcal{W}_j)) \subseteq \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j).$$

We define $T^{\phi}(\lambda)$ to consist of the sequence

$$T^{A_0}(\mathcal{C}_0, \mathcal{W}_0), \ldots, T^{A_n}(\mathcal{C}_n, \mathcal{W}_n)$$

together with the functors $T^{\phi}(\lambda)_{i,j}$ given by the composite

$$(11.6) \qquad \mathcal{P}_{i,j} \xrightarrow{\langle \tilde{\phi}_{i,j}, \lambda_{i,j} \rangle} \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_j}, \mathcal{S}_{\mathrm{fin},*}^{n_i}) \times \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)$$
$$\xrightarrow{\quad c \quad} \mathrm{Fun}(\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_i}, \mathcal{C}_i), \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_j}, \mathcal{C}_j))$$

where

$$c : \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_j}, \mathcal{S}_{\mathrm{fin},*}^{n_i}) \times \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j) \to \mathrm{Fun}(\mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_i}, \mathcal{C}_i), \mathrm{Fun}(\mathcal{S}_{\mathrm{fin},*}^{n_j}, \mathcal{C}_j))$$

is adjoint to the composition map.

Proposition 11.7. *The $\mathbb{W}$eil-action map T of Definition* **??** *agrees with that of Definition* **??** *on the simplicial subset $\mathbb{W}\mathrm{eil} \times \mathbb{R}\mathrm{el}_0 \mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$, and so determines an action*

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{R}\mathrm{el}_1 \mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}} \to \mathbb{R}\mathrm{el}_1 \mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}}$$

of the scaled simplicial monoid $\mathbb{W}\mathrm{eil}$ on the scaled simplicial set $\mathbb{R}\mathrm{el}_1 \mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}}$.

Proof. A similar argument to that of Lemma **??**, but with a simpler argument for part (2) since the relevant edge is no longer a zigzag, implies that Definition **??** defines a scaled morphism

$$T : \mathbb{W}\mathrm{eil} \times \mathbb{R}\mathrm{el}_0 \mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}} \to \mathbb{R}\mathrm{el}_0 \mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}}.$$

A similar argument to that of Proposition **??** implies that T is an action of $\mathbb{W}\mathrm{eil}$ on $\mathbb{R}\mathrm{el}_0 \mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}}$.

The compatibility of Definitions **??** and **??** with respect to the inclusions $\mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j) \subseteq \mathrm{Ex}^{\infty} \mathrm{Fun}(\mathcal{C}_i, \mathcal{C}_j)$ implies that the $\mathbb{W}$eil-actions on $\mathbb{R}\mathrm{el}_0 \mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}}$ and $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$ agree on their common simplicial subset $\mathbb{R}\mathrm{el}_0 \mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$. It follows that these actions determine a $\mathbb{W}$eil-action on the pushout $\mathbb{R}\mathrm{el}_1 \mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}}$ as claimed. $\qquad\square$

Having established an action of $\mathbb{W}\mathrm{eil}$ on $\mathbb{R}\mathrm{el}_1 \mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}}$, we now take a fibrant replacement to obtain a corresponding action on an ∞-bicategory $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}_{\infty}^{\mathrm{diff}}$. For that purpose we introduce the following model structure on the category of scaled simplicial sets with an action of $\mathbb{W}\mathrm{eil}$.

Proposition 11.8. *Let* $\mathsf{Mod}^{\mathsf{sc}}_{\mathbb{W}\mathrm{eil}}$ *be the category of scaled* $\mathbb{W}$*eil-modules, i.e. the category of modules over* $\mathbb{W}\mathrm{eil}$*, viewed as a monoid in* $\mathsf{Set}^{\mathsf{sc}}_{\Delta}$ *with its maximal scaling. Then* $\mathsf{Mod}^{\mathsf{sc}}_{\mathbb{W}\mathrm{eil}}$ *has a model structure in which a morphism is a weak equivalence (or fibration) if and only if the underlying map of scaled simplicial sets is a weak equivalence (or fibration).*

Proof. We apply Schwede-Shipley's result [**?**, 4.1] to the scaled monoid $\mathbb{W}\mathrm{eil}$. By [**?**, 4.2] we must verify that the scaled model structure on $\mathsf{Set}^{\mathsf{sc}}_{\Delta}$ is monoidal with respect to the cartesian product, which is done in [**?**, 2.1.21]. $\qquad\square$

Definition 11.9. Let $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ be the scaled $\mathbb{W}\mathrm{eil}$-module given by a fibrant replacement, in the model structure of Proposition **??**, of the $\mathbb{W}\mathrm{eil}$-action on $\mathbb{R}\mathrm{el}_1\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ described in Proposition **??**.

The scaled simplicial set $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ is an ∞-bicategory, and we can choose the fibrant replacement so that the comparison map

$$\mathbb{R}\mathrm{el}_1\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty} \overset{\sim}{\longrightarrow} \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$$

is a cofibration, hence a monomorphism, of scaled simplicial sets. Altogether we have produced the following diagram of inclusions of ∞-bicategories, with compatible $\mathbb{W}\mathrm{eil}$-actions, which is also a homotopy pushout of $(\infty, 2)$-categories:

(11.10)
$$\begin{array}{ccc} \mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty} & \overset{r}{\rightarrowtail} & \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty} \\ \downarrow & & \downarrow \\ \mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty} & \rightarrowtail & \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty} \end{array}$$

We now show that $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ is a model for the $(\infty, 2)$-category $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ of differentiable ∞-categories described in Definition **??**.

Proposition 11.11. *The composite map*

$$M : \mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty} \overset{M_0}{\longrightarrow} \mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty} \rightarrowtail \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$$

is an equivalence of ∞*-bicategories, where* $M_0 : \mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty} \to \mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ *is the* qCat*-enriched functor given by* $\mathcal{C} \mapsto (\mathcal{C}, \mathcal{E}_{\mathcal{C}})$*.*

Proof. Our strategy is to produce a model for the inclusion $\mathbb{R}\mathrm{el}_0\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty} \to \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ in the context of marked simplicial categories by translating the

homotopy pushout (**??**) back into that world. Consider the diagram of marked simplicial categories

(11.12)
$$
\begin{array}{ccc}
(\mathbb{Rel}_0\mathbb{Cat}^{\mathrm{diff}}_\infty, \simeq) & \longrightarrow & (\mathbb{Rel}_0\mathbb{Cat}^{\mathrm{diff}}_\infty, \mathbb{Rel}_0\mathbb{Cat}^{\mathrm{diff}}_\infty) \\
\downarrow & & \downarrow \\
(\mathbb{Rel}_0\mathbb{CAT}^{\mathrm{diff}}_\infty, \simeq) & \longrightarrow & (\mathbb{Rel}_0\mathbb{CAT}^{\mathrm{diff}}_\infty, \mathbb{Rel}_0\mathbb{Cat}^{\mathrm{diff}}_\infty)
\end{array}
$$

where a pair $(\mathbb{C}, \mathbb{M})$ denotes the simplicial category $\mathbb{C}$ with markings given by those edges in the subcategory $\mathbb{M}$. We use the notation $\simeq$ to denote the natural marking of a qCat-category: the subcategory consisting of all the equivalences in the mapping objects. The horizontal functors in (**??**) are given by the identity map, and the vertical functors by the inclusion $\mathbb{Rel}_0\mathbb{Cat}^{\mathrm{diff}}_\infty \subseteq \mathbb{Rel}_0\mathbb{CAT}^{\mathrm{diff}}_\infty$.

We claim that (**??**) is a homotopy pushout diagram in the model structure on marked simplicial categories described by Lurie in [**?**, A.3.2]. The top horizontal map is a cofibration because it has the left lifting property with respect to acyclic fibrations, and the square is a strict pushout of marked simplicial categories. Since the model structure on marked simplicial categories is left proper by [**?**, A.3.2.4], it follows that (**??**) is a homotopy pushout in that model structure.

Consider the top-right corner of (**??**): there is a marked simplicial functor of maximally marked simplicial categories

$$(\mathbb{Rel}_0\mathbb{Cat}^{\mathrm{diff}}_\infty, \mathbb{Rel}_0\mathbb{Cat}^{\mathrm{diff}}_\infty) \xrightarrow{r} (\mathbb{Rel}\mathbb{Cat}^{\mathrm{diff}}_\infty, \mathbb{Rel}\mathbb{Cat}^{\mathrm{diff}}_\infty) = (\mathbb{Rel}\mathbb{Cat}^{\mathrm{diff}}_\infty, \simeq)$$

given on mapping objects by the map r of Definition **??**. To see that this functor is an equivalence of marked simplicial categories, we note that the maximal marking functor takes an acyclic cofibration of simplicial sets (in the Quillen model structure), such as each r_Y, to an equivalence in the marked model structure. This fact can be checked directly from the definition of marked (cartesian) equivalence in [**?**, 3.1.3.3].

We have now done enough to establish that the homotopy pushout square (**??**) corresponds, under the Quillen equivalence of [**?**, 4.2.7], to the homotopy pushout square (**??**), and hence that the map $\mathbb{Rel}_0\mathbb{CAT}^{\mathrm{diff}}_\infty \to \mathbb{Rel}\mathbb{CAT}^{\mathrm{diff}}_\infty$ can be modelled by the marked simplicial functor

$$(\mathbb{Rel}_0\mathbb{CAT}^{\mathrm{diff}}_\infty, \simeq) \to (\mathbb{Rel}_0\mathbb{CAT}^{\mathrm{diff}}_\infty, \mathbb{Rel}_0\mathbb{Cat}^{\mathrm{diff}}_\infty)$$

that is the identity on the underlying simplicial category. The desired proposition is therefore reduced to showing that the functor

$$M : (\mathbb{CAT}^{\mathrm{diff}}_\infty, \simeq) \to (\mathbb{Rel}_0\mathbb{CAT}^{\mathrm{diff}}_\infty, \mathbb{Rel}_0\mathbb{Cat}^{\mathrm{diff}}_\infty); \quad \mathcal{C} \mapsto (\mathcal{C}, \mathcal{E}_{\mathcal{C}})$$

is an equivalence of marked simplicial categories. Given two differentiable ∞-categories $\mathcal{C}_0, \mathcal{C}_1$, we have

$$\mathrm{Fun}_{\mathbb{N}}(\mathcal{C}_0, \mathcal{C}_1) = \mathrm{Fun}_{\mathbb{N}}((\mathcal{C}_0, \mathcal{E}_{\mathcal{C}_0}), (\mathcal{C}_1, \mathcal{E}_{\mathcal{C}_1})$$

so M is fully faithful. The proof that M is essentially surjective on objects follows by the construction in the proof of Proposition **??** with no changes. $\square$

We now transfer the $\mathbb{W}$eil-action on the ∞-bicategory $\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ along the equivalence $M : \mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty} \xrightarrow{\sim} \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ of Proposition **??**. This transfer requires an ∞-bicategorical version of Lemma **??**.

Lemma 11.13. *Let* $i : \mathbb{X} \xrightarrow{\sim} \mathbb{Y}$ *be an equivalence of* ∞-*bicategories. Then there is an equivalence of monoidal* ∞-*categories*

$$\mathrm{End}_{(\infty,2)}(\mathbb{X}) \simeq \mathrm{End}_{(\infty,2)}(\mathbb{Y})$$

whose underlying functor is equivalent to $i(-)i^{-1}$.

Proof. The method of proof for Lemma **??** applies in exactly the same way, using the fact that the construction $\mathrm{Fun}_{(\infty,2)}(-,-)^{\simeq}$ takes an equivalence of ∞-bicategories (in either of its variables) to an equivalence of ∞-categories. $\square$

Definition 11.14. Let

$$T : \mathbb{W}\mathrm{eil}^{\otimes} \to \mathrm{End}_{(\infty,2)}(\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty})^{\circ}$$

be the monoidal functor obtained by composing the action map

$$\mathbb{W}\mathrm{eil}^{\otimes} \to \mathrm{End}_{(\infty,2)}(\mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty})^{\circ}$$

associated to Definition **??** with the equivalence of monoidal ∞-categories induced, via Lemma **??**, by the equivalence $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty} \xrightarrow{\sim} \mathbb{R}\mathrm{el}\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ of Proposition **??**.

Theorem 11.15. *The map* T *of Definition* **??** *is a tangent structure on the* ∞-*bicategory* $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ *which (up to equivalence) extends that of Theorem* **??** *on the* ∞-*category* $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_{\infty}$.

Proof. To show that T is a tangent structure, we apply Proposition **??**. The only thing remaining to show is that the foundational and vertical lift pullbacks in $\mathbb{W}$eil determine homotopy 2-pullbacks in $\mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$ for each object $\mathcal{C} \in \mathbb{C}\mathrm{AT}^{\mathrm{diff}}_{\infty}$. In the proof of Theorem **??** we showed each such square is a pullback along a fibration in the ∞-cosmos $\mathsf{CAT}^{\mathbb{N}}_{\infty}$, which immediately implies that claim. $\square$

Goodwillie calculus in an ∞-bicategory. In this final section of the paper we show how to use Theorem **??** to define a notion of P_n-equivalence, and hence Taylor tower, in an arbitrary tangent ∞-bicategory $\mathbb{X}$ which, when $\mathbb{X} = \mathbb{C}\mathrm{AT}_\infty^{\mathrm{diff}}$, recovers Goodwillie's theory.

Definition 11.16. Let $(\mathbb{X}, T)$ be a tangent ∞-bicategory that admits a terminal object $*$, and let $x : * \to \mathcal{C}$ be a 1-morphism in $\mathbb{X}$, i.e. a generalized object in $\mathcal{C}$. We say that $\mathcal{C}$ *admits higher tangent spaces* at x if, for each n, there is a homotopy 2-pullback in $\mathbb{X}$ of the form

$$
\begin{array}{ccc}
T_x^n\mathcal{C} & \xrightarrow{\;\iota_x^n\;} & T^n(\mathcal{C}) \\
\downarrow & & \downarrow{\scriptstyle p^n} \\
* & \xrightarrow{\;\;x\;\;} & \mathcal{C}
\end{array}
$$

Here p^n denotes the natural transformation associated to the Weil-algebra morphism given by the augmentation $W^{\otimes n} \to \mathbb{N}$.

Example 11.17. When $\mathbb{X} = \mathbb{C}\mathrm{AT}_\infty^{\mathrm{diff}}$, a morphism $x : * \to \mathcal{C}$ as in Definition **??** is an actual object of a differentiable ∞-category $\mathcal{C}$ which admits the higher tangent spaces $T_x^n\mathcal{C}$ as described in Theorem **??**.

Definition 11.18. Let $(\mathbb{X}, T)$ be a tangent ∞-bicategory, and suppose the object $\mathcal{C}$ in $\mathbb{X}$ admits higher tangent spaces at x. For any $\mathcal{D} \in \mathbb{X}$ and $n \geq 0$, we define the *subcategory of P_n^x-equivalences*

$$
\mathcal{P}_n^x \operatorname{Hom}_\mathbb{X}(\mathcal{C}, \mathcal{D}) \subseteq \operatorname{Hom}_\mathbb{X}(\mathcal{C}, \mathcal{D})
$$

to be the subcategory of morphisms that map to equivalences under the functor

$$
T^n \iota_x^n : \operatorname{Hom}_\mathbb{X}(\mathcal{C}, \mathcal{D}) \to \operatorname{Hom}_\mathbb{X}(T_x^n\mathcal{C}, T^n\mathcal{D}); \quad F \mapsto T^n(F)\iota_x^n.
$$

Lemma 11.19. *Let $(\mathbb{X}, T)$ and $x : * \to \mathcal{C}$ be as in Definition **??**. Then for all $n \geq 1$:*
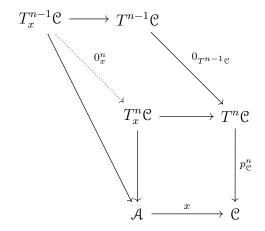
$$
\mathcal{P}_n^x \operatorname{Hom}_\mathbb{X}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{P}_{n-1}^x \operatorname{Hom}_\mathbb{X}(\mathcal{C}, \mathcal{D}).
$$

Proof. It is sufficient to show that, up to natural equivalence, we have a factorization

(11.20)
$$
\begin{array}{ccc}
\operatorname{Hom}_\mathbb{X}(\mathcal{C}, \mathcal{D}) & \xrightarrow{\;T^n \iota_x^n\;} & \operatorname{Hom}_\mathbb{X}(T_x^n\mathcal{C}, T^n\mathcal{D}) \\
& {\scriptstyle T^{n-1}\iota_x^{n-1}}\searrow & \downarrow \\
& & \operatorname{Hom}_\mathbb{X}(T_x^{n-1}\mathcal{C}, T^{n-1}\mathcal{D})
\end{array}
$$

where the vertical map is given by postcomposition with $p_{T^{n-1}\mathcal{D}} : T^n\mathcal{D} \to T^{n-1}\mathcal{D}$ and precomposition with the map $0_x^n : T_x^{n-1}\mathcal{C} \to T_x^n\mathcal{C}$ induced by the bottom homotopy 2-pullback square in the following diagram

Note that $p_{\mathcal{C}}^{n-1} \simeq p_{\mathcal{C}}^n 0_{T^{n-1}\mathcal{C}}$ by uniqueness of the augmentation map $W^{\otimes(n-1)} \to \mathbb{N}$. Finally, the diagram (**??**) commutes since

$$p_{T^{n-1}\mathcal{D}} T^n F 0_{T^{n-1}\mathcal{C}} \simeq T^{n-1}F$$

by that same calculation together with the naturality of T. $\square$

Example 11.21. Let $\mathbb{X} = \mathbb{C}\mathrm{AT}_\infty^{\mathrm{diff}}$, and let x be an object in a differentiable ∞-category $\mathcal{C}$ which admits finite colimits. Then Theorem **??** tells us that

$$\mathcal{P}_n^x \operatorname{Hom}_{\mathbb{C}\mathrm{AT}_\infty^{\mathrm{diff}}}(\mathcal{C}, \mathcal{D}) \subseteq \operatorname{Fun}_\mathbb{N}(\mathcal{C}, \mathcal{D})$$

is precisely the subcategory of P_n^x-equivalences between (sequential-colimit-preserving) functors $\mathcal{C} \to \mathcal{D}$.

We recover Goodwillie's notion of n-excisive functor $\mathcal{C} \to \mathcal{D}$, and hence the Taylor tower, by observing that in $\mathbb{C}\mathrm{AT}_\infty^{\mathrm{diff}}$ the subcategory of P_n^x-equivalences is associated with a left exact localization of $\operatorname{Fun}_\mathbb{N}(\mathcal{C}, \mathcal{D})$. The local objects for that localization are the n-excisive functors. We generalize this observation to give a definition of Taylor tower in an arbitrary tangent ∞-bicategory.

Definition 11.22. Let $\mathbb{X}$ be a tangent ∞-bicategory, and suppose the $\mathcal{C}$ in $\mathbb{X}$ admits higher tangent spaces at x. We can say that $\mathbb{X}$ *admits Taylor towers expanded at x* if, for each $\mathcal{D} \in \mathbb{X}$ and each $n \geq 0$, there is a full subcategory

$$\operatorname{Jet}_x^n(\mathcal{C}, \mathcal{D}) \subseteq \operatorname{Hom}_\mathbb{X}(\mathcal{C}, \mathcal{D}),$$

whose inclusion admits a left adjoint P_n^x, such that the subcategory of P_n^x-equivalences, in the sense of **??**, is equal to the subcategory of morphisms that are mapped to equivalences by P_n^x. We can refer to $\operatorname{Jet}_x^n(\mathcal{C}, \mathcal{D})$ as *the ∞-category of n-jets at x* for morphisms $\mathcal{C} \to \mathcal{D}$ in $\mathbb{X}$.

In that case, by Lemma **??**, we necessarily have

$$\mathrm{Jet}_x^{n-1}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Jet}_x^n(\mathcal{C}, \mathcal{D})$$

and for each 1-morphism $F : \mathcal{C} \to \mathcal{D}$ in $\mathbb{X}$, there is a sequence of morphisms in $\mathrm{Hom}_{\mathbb{X}}(\mathcal{C}, \mathcal{D})$ of the form

$$F \to \cdots \to P_n^x F \to P_{n-1}^x F \to \cdots \to P_0^x F$$

which we can call the *Taylor tower* of F at x. Taking $\mathbb{X}$ to be the Goodwillie tangent structure on the ∞-bicategory $\mathbb{C}\mathrm{AT}_\infty^{\mathrm{diff}}$, we recover Goodwillie's notion of Taylor tower for functors between differentiable ∞-categories.